Any-k Algorithms for Enumerating Ranked Answers to Conjunctive Queries

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We study ranked enumeration for Conjunctive Queries (CQs) where the answers are ordered by a given ranking function (e.g., an ORDER BY clause in SQL). We develop "any-k" algorithms, which, without knowing the number k of desired answers, push down the ranking into joins by carefully ordering the computation of intermediate tuples and avoiding materialization of join answers until they are needed. For this to be possible, the ranking function needs to obey a particular type of monotonicity. Supported ranking functions include the common sum-of-weights, where answers are compared by the sum of input-tuple weights, as well as any commutative selective dioid. Our results extend a well-known unranked-enumeration dichotomy, which states that only free-connex CQs are tractable (under certain hardness hypotheses and for CQs without self-joins). For this class of queries and with n denoting the size of the input, the data complexity of our ranked enumeration approach for the time to the kth CQ answer is $O(n + k \log k)$, which is only a logarithmic factor slower than the $O(n + k)$ unranked-enumeration guarantee.

A core insight of our work is that ranked enumeration for CQs is closely related to Dynamic Programming and the fundamental task of path enumeration in a weighted DAG. We uncover a previously unknown tradeoff, both for this problem and for CQs, under the lens of combined complexity where the query size is not considered a constant: one any-k algorithm has lower complexity when the number of returned answers is small, the other when their number is large. This tradeoff is eliminated under a stricter monotonicity property that we define and exploit for a novel algorithm that asymptotically dominates all previously known alternatives, including Eppstein’s algorithm for sum-of-weights path enumeration. We empirically demonstrate the findings of our theoretical analysis in an experimental study that highlights the superiority of our approach over the join-then-rank approach that existing database systems follow.

1 INTRODUCTION

Joins are an essential building block of queries in relational and graph databases. They are notoriously critical for performance, as they can produce huge intermediate or final results. Enumeration [16, 41, 127] is a query-answering paradigm that circumvents this by returning the answers as a stream as quickly as possible, even if the full output is too large to compute. Among other key problems at the forefront of recent research [136], such as worst-case optimal joins [114], (hyper)tree decompositions [71], and factorized representations [117], ranked enumeration has been identified as an important open problem [30]: enumerate answers in an order determined by a given ranking function. This augments the enumeration paradigm by imposing an order on the output; the top-ranked answers are returned first, followed by lower-ranked ones in quick succession.

Ranked enumeration shares a similar motivation with top-k queries [88], yet has two crucial differences: (1) For top-k, the number of desired answers, k, is provided in advance. In contrast, a ranked enumeration (or "any-k") algorithm needs to perform well for all possible values of k. Any-k algorithms can conceptually be seen as a fusion of top-k [88] and anytime algorithms [149] that gradually improve their results over time. (2) Algorithms for top-k joins, including the celebrated Threshold Algorithm [62], were developed for a “middleware” cost model that accounts only for accesses to external data sources, but not for memory accesses and intermediate-result size [135].

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Our goal are strong guarantees for time and space complexity in the RAM model of computation, for every value $k$. We refer to those as $TT(k)$ (i.e., Time-To-$k$) and $MEM(k)$ respectively.

**Example 1 (4-path query).** Let $R(A, B, W_1)$, $S(B, C, W_2)$, $T(C, D, W_3)$, and $U(D, E, W_4)$ be relations where the last column stores tuple weights. The following query joins the relations and orders the answers by ascending sums of weights.

```sql
SELECT *, R.W1 + S.W2 + T.W3 + U.W4 AS SUMW
FROM R, S, T, U
WHERE R.B = S.B AND S.C = T.C AND T.D = U.D
ORDER BY SUMW ASC
```

If the relations have $n$ tuples each, the query can produce $\Omega(n^3)$ answers in the worst case [15]. Thus, any “join-then-rank” algorithm that first joins the relations and then applies the ranking (e.g., by sorting) needs $\Omega(n^3)$ time to find even the first answer. This is due to the sheer number of possible answers that have to be compared and is true for any join strategy, such as sort-merge, hash-join, or the optimal Yannakakis algorithm [145]. Our any-$k$ algorithms push down the ranking into the join and achieve $TT(k) = O(n + k \log k)$. This means that the top-ranked answer is returned in linear time, and thereafter, for $k$ returned answers, the additional cost is the same as sorting $k$ elements. One may wonder how the situation changes if instead of a 4-path query, we have a 4-cycle (i.e., ‘AND $R.A = U.E$’), or if we project away the A attribute, or if we replace the sum (+) with max. Our work answers all these questions in a principled way.

**Contributions.** (1) We develop a theory of ranked enumeration for Dynamic Programming (DP) that applies to CQs. It reveals the deeper common foundations between isolated prior works that only partially address the problem: $k$-shortest paths [59, 91, 103], ranked retrieval of graph patterns [45, 143], and ranked enumeration for CQs [55, 97]. While interesting in its own right, this general framework allows us to apply our ranked enumeration techniques not only to these problems but also to any problem whose top-ranked solution can be found via DP, such as DNA sequence alignment [50] or Viterbi decoding [128]. The ranking function needs to have certain properties to allow efficient ranked enumeration, which is not surprising given that DP does not apply to all optimization problems. We identify subset-monotonicity (s-monotonicity) as a sufficient condition and show how it is related to algebraic structures called commutative selective dioids [70], which are a special case of the more well-known semirings [77].

(2) For a large class of CQs, our any-$k$ algorithms achieve $TT(k) = O(n + k \log k)$, in data complexity [139], where $n$ is the size of the database and query size is treated as a constant. This is close to the $\Omega(n + k)$ lower bound and the additional logarithmic factor is expected since the $k$ answers are returned sorted. We mainly focus on acyclic CQs because cyclic CQs can be handled by first decomposing them to acyclic ones [71] and then applying our techniques. Since acyclic CQs can have a more general tree structure (compared to the more restrictive path structure of standard DP), we first extend our Dynamic Programming framework to a class of problems that we call Tree-DP (T-DP). and apply it to full acyclic CQs. When the query has projections, an established unranked-enumeration dichotomy [16] states that, under plausible assumptions in fine-grained complexity, the only (self-join-free) CQs that admit $TT(k) = O(n + k)$ are those that are free-connex.¹ We establish that the frontier of tractability remains the same with an s-monotone ranking function (minus logarithmic factors); free-connex CQs can be handled with $TT(k) = O(n + k \log k)$ and no other (self-join-free) CQ admits this running time.

¹The original dichotomy is phrased in terms of preprocessing and delay.
(3) We compare different any-\(k\) algorithms in a more refined analysis of \textit{combined complexity}, where query size is not treated as a constant, and propose a new algorithm that asymptotically dominates all others under a stronger monotonicity property. In particular, we find that the best algorithm based on the Lawler-Murty procedure \cite{103}, which we refer to as \textsc{anyK-part}, is in general asymptotically better than approaches based on the Recursive Enumeration Algorithm (REA) \cite{55, 91}, which we refer to as \textsc{anyK-rec}. However, \textsc{anyK-rec} smartly reuses comparisons and this can pay off as \(k\) increases: there exist inputs for which it produces the full sorted output with lower time complexity than \textsc{anyK-part}, and even faster than sorting the materialized output. We propose \textsc{anyK-part+}, an algorithm that combines the best of both worlds. However, it requires a stricter monotonicity property from the ranking function, which we call \textit{strong-subset-monotonicity} (ss-monotonicity). We give examples of ranking functions with this property, and show how it follows from elementary algebraic properties like cancellation \cite{70}. \textsc{anyK-part+} is asymptotically faster than Eppstein’s algorithm \cite{39} for path enumeration in a DAG, assuming that the returned paths are in an explicit listing representation (i.e., the path size is proportional to the number of its edges) instead of an implicit representation that Eppstein leverages.

(4) We provide the \textit{first empirical study} that directly compares these ranked-enumeration algorithms. We show a tradeoff between \textsc{anyK-part} and \textsc{anyK-rec}, with each one winning for different queries or values of \(k\). As in theory, \textsc{anyK-part+} combines their best qualities and is in most cases close or better than the fastest of the other two. Importantly, our study clearly illustrates the advantage of any-\(k\) for queries with large join output over the join-then-rank approach. Existing database systems follow the latter, and our algorithms outperform them by orders of magnitude.

\textbf{Conference version.} This article is an extended version of an earlier conference paper \cite{133}, significantly extending its scope and depth. \(i\) First, we propose the \textsc{anyK-part+} algorithm that combines the best features of the two previous algorithms (Section 5.3). \(ii\) Second, we include the study of CQs with projections (Section 6.2) and give the corresponding dichotomy result. \(iii\) Third, we provide an analysis of the supported ranking functions, their properties, and how they relate to other monotonicity definitions and algebraic structures (Section 7). \(iv\) Finally, this article makes several other additions, such as a more extensive experimental study that better evaluates the scalability of the approach and includes a comparison with a commercial database system, a more detailed review of the literature that includes a comparison to unranked and lexicographic order enumeration, pseudocode for the algorithms, and proofs of correctness.

The project web page at \url{https://northeastern-datalab.github.io/anyk/} contains code, slides, videos, and further information.

\section{Preliminaries}
We use \([m]\) to denote the set of integers \(\{1, \ldots, m\}\) and \([m]_0\) for \(\{0, \ldots, m\}\).

\subsection{Basic Notions}
\textbf{Graphs and Hypergraphs.} A directed graph \(G(V, E)\) is weighted if a weight function \(w : E \rightarrow W\) assigns weights from a domain \(W\) to its edges. The graph size \(|G| = |V| + |E|\) is the number of its nodes and edges. A path (or walk) \(p\) of length \(\lambda\) is a sequence of nodes \(\langle v_0, \ldots, v_\lambda \rangle\) such that \(\langle v_{i-1}, v_i \rangle \in E, i \in [\lambda]\). Note that the nodes or edges do not have to be distinct. Similarly, a path of length \(\lambda\) in a hypergraph \(\mathcal{H}(V, E)\) is a sequence \(\langle v_0, e_0, v_1, \ldots, e_\lambda, v_\lambda \rangle\) such that \(v_{i-1}, v_i \in e_i, e_i \in E, \forall i \in [\ell]\). The distance between two nodes \(u, v\) is the minimum length of any path with \(u = v_0, v = v_\lambda\). The \textit{diameter} of a hypergraph is the maximum distance between any pair of nodes. If we fix a source node \(s\) and a target node \(t\) we call a \textit{prefix} of a node \(v_i\) any path \((s, v_1, \ldots, v_i)\) to \(v_i\) that starts at \(s\) and \textit{suffix} of \(v_i\) any path \((v_i, v_{i+1}, \ldots, v_\lambda, t)\) from \(v_i\) that ends in \(t\). We use \(\circ\) as a concatenation operator for paths. Given a prefix and a suffix, \((s, v_1, \ldots, v_i) \circ (v_i, v_{i+1}, \ldots, v_\lambda, t)\) is
We also write \( v_i \circ (v_{i+1}, \ldots, v_\lambda, t) \) for some \((v_i, v_{i+1}, \ldots, v_\lambda, t) \in E\). When \( s \) and \( t \) are fixed and it is clear from the context, we sometimes omit \( s, t \) from prefixes or suffixes, i.e., we write \( (v_1, \ldots, v_\lambda) \) instead of \( (s, v_1, \ldots, v_\lambda, t) \).

**Conjunctive Queries (CQs).** A CQ \( Q \) is a first-order formula \( \exists Y (R_1(X_1) \land \ldots \land R_\ell(X_\ell)) \), written as \( Q(Z):=R_1(X_1), \ldots, R_\ell(X_\ell) \) in Datalog notation, where \( Y, Z, X_i, i \in [\ell] \) are lists of variables where \( Y \subseteq \bigcup_{i \in [\ell]} X_i \) and \( Z = \bigcup_{i \in [\ell]} X_i \setminus Y \) if interpreted as sets. Each atom \( R_i(X_i) \) refers to a relation with \( |X_i| \) columns (or attributes). The variables in \( Z \) are called *free* and denoted by \( \text{free}(Q) \), while the rest of the variables \( Y \) are called *existential*. A Boolean CQ has no free variables (i.e., \( Z = \emptyset \)) and only asks for the satisfiability of the formula. In a full CQ, all variables are free, i.e., \( Z = \bigcup_{i \in [\ell]} X_i \). The occurrence of the same variable in different atoms encodes an equi-join condition, implying that the values of the corresponding attributes in a query answer need to be equal. Different atoms can refer to the same relation, in which case we have a *self-join*. A self-join-free query has no self-joins. Without loss of generality, for our algorithms we can assume that (1) CQs are self-join-free since tables can be copied, and (2) selections on individual relations (like \( R(x, 1) \) or \( R(x, x) \)) have been removed in a linear-time preprocessing step.

**Query semantics.** CQs are evaluated over a database \( D \) of relations that draw values from a domain \( \text{dom} \), such as \( \mathbb{N} \). An output tuple or query answer to \( Q \) is a mapping \( \text{free}(Q) \to \text{dom} \) such that the first-order formula is satisfied. The set of all query answers is \( Q(D) \), and we use \( q \in Q(D) \) for a query answer. A semantic evaluation strategy to compute \( Q(D) \) is to (i) materialize the Cartesian product of the \( \ell \) relations, (ii) select tuples that satisfy the equi-joins, and (iii) project on the \( Z \) attributes. A witness \([37]\) of a query answer is a size-\( \ell \) vector of input tuples, one from the relation of each atom, that can join to produce the answer. We denote the set of witnesses of an answer \( q \) by \( \text{wit}(q) \). For full CQs, we can equivalently represent an answer by its unique witness.

**Join trees.** A CQ is associated with a hypergraph where variables form the nodes and atoms form the hyperedges. We say that a CQ is acyclic if its hypergraph is alpha-acyclic \([33]\), which means that we can construct a join tree. A join tree is a rooted tree where the atoms form the nodes and the running intersection property holds: For every variable \( x \), all tree nodes containing \( x \) form a connected subtree. The acyclicity of a CQ can be tested, and a corresponding join tree can be constructed in linear time in the query size by the GYO reduction \([130, 147]\). A CQ that is not acyclic is called cyclic. The diameter \( \text{diam}(Q) \) of a CQ \( Q \) is the diameter of its hypergraph.

**Example 2 (\( \ell \)-path and \( \ell \)-cycle queries).** Let \( E(\text{FROM}, \text{TO}) \) be a relation that stores the directed edges of a graph. The following \( \ell \)-path CQ computes paths of length \( \ell \) between any pair of nodes in the graph: \( Q_{\ell}(x) := E(x_0, x_1), E(x_1, x_2), \ldots, E(x_{\ell-1}, x_\ell) \). These CQs are acyclic; to construct a join tree, we can organize the atoms in a path with \( E(x_0, x_1) \) as the root.

Length-\( \ell \) cycles can be expressed as \( Q_{\ell}(x) := E(x_1, x_2), E(x_2, x_3), \ldots, E(x_\ell, x_1) \). These CQs are cyclic, and no join tree exists; for example, the path with \( E(x_1, x_2) \) as the root does not have the running intersection property; the first and the last atom both contain \( x_1 \) but are disconnected.

### 2.2 Ranked Enumeration

Ranked enumeration assumes a given ranking function that orders the query answers by mapping them to a domain \( W \) equipped with a total order \( \preceq \). We denote by \( \text{min}(S) \) the smallest element in \( S \subseteq W \) according to \( \preceq \), where \( S \) is allowed to be a multiset.

**Weight aggregation.** In this work, we focus on aggregate ranking functions where weights are assigned to input tuples and the mapping to \( W \) is done by aggregating the weights of input tuples forming the witness of an output tuple. A common example is the sum-of-weights case as in Example 1: Real-valued weights are assigned to input tuples and the weight of a (full) CQ answer
is computed by adding up the weights of tuples in its witness. More generally, we are given an input-weight function \( w_1 \) that associates each input tuple with some weight in \( W \) and an aggregate function \( w_A : \mathbb{N}^W \to W \) that returns a unique weight from a multiset of elements in \( W \). Aggregate functions are not sensitive to the order that the input weights are provided, which is captured by the fact that their input is a multiset \([90]\). We further assume that \( w_A(\emptyset) = \min(W) \).

For CQs with projections, a query answer can have multiple witnesses with differing weights. In that case, we adopt the \( \min\)-weight-projection semantics: its weight is the minimum (according to \( \preceq \) or \( \leq \)) among the witnesses. Formally, the weight of \( q \in Q(D) \) is \( w(q) = \min_{(t_1, \ldots, t_\ell) \in \text{wit}(q)} (w_A(\{w_1(t_i) \mid i \in [\ell]\})) \). In Section 6.2, we discuss other possible semantics. For the remainder of this paper, we refer to aggregate ranking functions simply as ranking functions, and, when it is clear from the context, we may use \( w \) instead of \( w_1 \) or \( w_A \).

**Monotonicity.** Let \( \cup \) be the multiset union operator. Our algorithms exploit the following monotonicity properties of common ranking functions:

**Definition 3 (Subset-Monotonicity [97]).** A ranking function \( w \) is \( s \)-monotone if \( w_A(X_1 Y_1) \leq w_A(Y_2) \Rightarrow w_A(X Y_1) \leq w_A(X Y_2) \) for all \( X, Y, X_1, Y_1, Y_2 \in \mathbb{N}^W \).

Intuitively, \( s \)-monotonicity allows us to infer the ranking of complete solutions from the ranking of partial solutions. Next, we define a stronger notion that allows us to rank complete solutions from other related complete solutions. This stronger property allows us to develop an algorithm with the best known asymptotic guarantees in Section 5.3.

**Definition 4 (Strong-Subset-Monotonicity).** A ranking function \( w \) is \( \text{ss}\)-monotone if \( w_A(X_1 Y_1) \leq w_A(X_1 \cup Y_2) \land w_A(X_1) \leq w_A(X_2) \Rightarrow w_A(X_2 Y_1) \leq w_A(X_2 Y_2) \) for all \( X_1, X_2, Y_1, Y_2 \in \mathbb{N}^W \).

Notice that \( \text{ss}\)-monotonicity implies \( s \)-monotonicity by setting \( X_1 = \emptyset \) (recall that \( w_A(\emptyset) = \min(W) \)). The common sum-of-weights case satisfies both properties. An example of a ranking function that is \( s \)-monotone but not \( \text{ss}\)-monotone is given in Example 36. Also, note that the condition \( w_A(X_1) \leq w_A(X_2) \) in our definition does not restrict the set of ranking functions that satisfy \( \text{ss}\)-monotonicity, but expands it. For example, \( \min/\max \) ranking is \( \text{ss}\)-monotone: Although \( \max\{2, x\} \leq \max\{1, x\} \) is not true for all values of \( x \), it is true for \( x \geq 3 \).

**Incremental aggregation.** Another issue related to the ranking function is whether we can compute \( w_A(X \cup Y) \) in \( O(1) \) given two partial weights \( w_A(X) \) and \( w_A(Y) \) (i.e., without performing the aggregation from scratch). Such an aggregate ranking function has been called algebraic \([75]\) or decomposable \([90]\). We adopt the former term. Examples include \( \text{SUM}, \text{COUNT}, \text{MIN}, \) and \( \text{AVERAGE} \). In contrast, holistic functions like \( \text{MEDIAN} \) require \( \omega(1) \) space for intermediate results.

**Problem definition.** The central problem in this paper is the following:

**Definition 5 (Ranked enumeration for CQs).** Given a CQ \( Q \) over a database \( D \) and a ranking function, ranked enumeration returns the query answers \( Q(D) \) one-at-a-time in ascending \( \preceq \) order without duplicates.

As we will see, the above problem can be modeled as an instance of ranked enumeration for Dynamic Programming (DP), which is in turn intimately related to the problem of path enumeration in a DAG. Thus, we also consider the problem of ranked enumeration for such paths. The ranking function there is defined naturally over the edges forming a path \( p: w(p) = w_A(\{w_1(e) \mid e \in p\}) \). Crucially, we consider the variant of the problem where paths have to be enumerated explicitly by returning the list of edges of each path. It is known that just enumerating the weights (or the

\[2\] For simplicity, we assume that the input weights and the answer weights have the same domain, but a generalization is straightforward.
Fig. 1. Time-to-\(k\) (TT(\(k\))) is a more practical measure of success than delay. Figure 1a illustrates how an algorithm \(A\) with minimal TT(\(k\)) can return each answer faster than an algorithm \(B\) even if the maximum delay of \(A\) is higher (between answers 4-5 in the figure). Figure 1b shows that the maximum delay of an algorithm can be lowered by buffering the answers and returning them later, but this slows down the algorithm.

weights and an implicit pointer representation of the paths) can remove the dependency on the path length from the running time, as shown in the seminal work of Eppstein [59].

**Definition 6 (Ranked enumeration of paths).** Given a weighted DAG \(G\) with a source node \(s\) and target node \(t\), and a ranking function, (explicit) ranked enumeration returns the paths from \(s\) to \(t\) one-at-a-time in ascending order represented as lists of edges and without duplicates.

In Section 7, we delve deeper into issues related to the ranking function including the relationship of monotonicity properties to algebraic structures. Until then, we phrase our algorithms using the sum-of-weights model, which is commonly used in standard DP formalism and \(k\)-shortest paths.

### 2.3 Complexity Measures and Hypotheses

**Model of computation.** We consider in-memory computation only and analyze all algorithms in the standard Random Access Machine (RAM) model with uniform cost measure where every operation or memory access costs \(O(1)\). We do not assume any given indexes or sorting in the database relations. In line with previous work [25, 41, 71, 114], we also assume that it is possible to build a lookup table in linear time to support tuple lookups in constant time. In practice, this is virtually guaranteed by hashing.

**Problem parameters.** We use \(n\) to refer to the maximum cardinality of any database relation referenced in \(Q\) and \(r\) for \(|Q(D)|\). The size of a CQ is measured in terms of \(\ell\) (i.e., the number of atoms) and also the maximum arity among atoms, which we denote by \(\alpha\). We use two different notions of complexity for CQs [139]. The first is data complexity, where the size of the query is treated as a constant. The second is the more detailed combined complexity, where query size is treated as a variable. We apply the latter analysis to full acyclic CQs, revealing interesting differences between our algorithms and aiding in the explanation of our experimental results. For this analysis, we further assume that the ranking function is algebraic. This is not a strict requirement and our analysis can easily be extended to cases where this does not hold by keeping track of all the elements even after aggregating them.

**Measures of success.** We measure the complexity of enumeration by the time required until the \(k\)th answer is returned (TT(\(k\)) for Time-To-\(k\)) for all values of \(k\). For a lower bound, note that it takes \(\Omega(n)\) just to look at each input tuple and \(\Omega(k)\) to return \(k\) output tuples. Sorting \(k\) independent items takes \(\Omega(k \log k)\) with a comparison-based algorithm. We denote the special case of the Time-To-Last TT(\(r\)) by TTL. Additionally, we measure the space complexity MEM(\(k\)).

**Time-to-\(k\) versus delay.** The goal of enumeration is to provide each answer as quickly as possible, which is why we adopt TT(\(k\)) as our measure of success. A large body of work on
Any-k Algorithms for Enumerating Ranked Answers to Conjunctive Queries

Enumeration [16, 40, 86, 119, 127] has instead focused on measuring the delay between answers after a preprocessing phase, typically striving for constant delay after linear-time preprocessing \( O(n) \) (in data complexity). This is desirable because it guarantees that every \( k \)th answer is returned in time \( TT(k) = O(n + k) \), which is optimal (without ranking). However, low delay is sufficient but not necessary to achieve low \( TT(k) \). As Figure 1a shows, a lower delay does not necessarily imply a faster algorithm.

Given enough buffer space (not necessarily main memory), the delay of an algorithm can be made uniform across all \( k \) values, e.g., by buffering the answers and returning them at regular intervals. We illustrate this in Figure 1b. Techniques in that spirit have been developed [39, 41, 52, 53] to guarantee low delay for algorithms that already have low \( TT(k) \). From a practical point of view, this is usually undesirable [38, 41] because it slows down the algorithm. Aiming for low delay might still be relevant if (1) a downstream application requires uniform interarrival times and (2) at the same time there is not enough space to buffer the answers. We are currently not aware of any such scenario in practice.

**Hardness hypotheses.** Works in the wider area of enumeration [16, 41, 42, 32] have used certain hypotheses on the hardness of problems to prove lower bounds and achieve dichotomies for CQs. In this work, we extend some of these results to the case of ranked enumeration using the same set of hypotheses. We list these below and we refer the reader to Berkholz et al. [22] for an excellent discussion on their plausibility.

**Hypothesis 1 (sparseBMM [22]).** Two Boolean matrices \( A \) and \( B \), represented as lists of non-zeros, cannot be multiplied in time \( m^{1+o(1)} \), where \( m \) is the number of non-zeros in \( A, B, \) and \( AB \).

The hypothesis states that the above problem cannot be solved in near-linear time in input and output size. As Berkholz et al. [22] argue, this is a weaker (i.e., more plausible) assumption than the original assumption of Bagan et al. [16] that the problem cannot be solved in quadratic time in size.

**Hypothesis 2 (Hyperclique [1, 104]).** For every \( k \geq 2 \), there is no \( O(m \text{ polylog } m) \) algorithm to decide the existence of a \((k+1, k)\)-hyperclique in a \( k \)-uniform hypergraph with \( m \) hyperedges, where a \((k+1, k)\)-hyperclique is a set of \( k+1 \) vertices such that every subset of \( k \) elements is a hyperedge and a \( k \)-uniform hypergraph is one where all hyperedges have size \( k \).

For \( k = 2 \), Hyperclique says that detecting a triangle in a graph cannot be done in quasilinear time. The fastest known algorithm for triangle detection [11] uses fast matrix multiplication and runs in time \( \Omega(n^{4/3}) \), even if the matrix multiplication exponent is 2, that is, the lowest possible.

### 2.4 Known Results on Unranked and Lexicographic Enumeration

We now review a key dichotomy result for unranked enumeration, where there is no ranking function and the task is to enumerate query answers in no particular order. We also touch upon results for the special case of ranking according to lexicographic orders.

**Free-connex CQs.** Bagan et al. [16] show that unranked enumeration for full CQs is always possible with linear-time preprocessing and constant delay in data complexity. When the CQ has existential variables (i.e., projections), then this is only possible (under the hypotheses of the previous section) for the CQs that are free-connex. A CQ is free-connex if it is acyclic and additionally, it remains acyclic if we add an atom that contains all free variables [32]. Trivially, every full acyclic CQ is free-connex. We restate the dichotomy in terms of \( TT(k) \) below:

\[ An example of a function that is in \( m^{1+o(1)} \) is \( m^{1+1/m} \). Note that this is higher than \( O(m \text{ polylog } m) \). \]

\[ The lower bound on \( TT(k) \) stated here is slightly stronger than the one mentioned in the original papers (which bound the delay without logarithmic factors), but is covered by the hypotheses in a straightforward way. \]
We are now in a position to state the main results of this work more formally. The first result is that the factorization order used in the construction of their data structure require an expensive enumeration function is possible with maximum arity \(\alpha\) ranking functions.

**Theorem 7 ([16, 32]).** Let \(Q\) be a CQ. If \(Q\) is free-connex, then unranked enumeration (in arbitrary order) is possible with \(TT(k) = O(n + k)\). Otherwise, if \(Q\) is also self-join-free, then it is not possible with \(TT(k) = O(n + k \log k)\), assuming \textsc{sparseBMM} and \textsc{Hyperclique}.

**Enumeration by lexicographic orders.** A closer look at the algorithm of Bagan et al. [16] reveals that the query answers are actually returned in some lexicographic order of the variables. In a lexicographic order \(\langle x_1, x_2, \ldots \rangle\), two output tuples are first compared on the \(x_1\) value, and if equal then on their \(x_2\) value, and so on. The order of the variables crucially depends on the structure of the query (i.e., it needs to be an alpha elimination order [33]) and there are certain lexicographic orders that cannot be realized by the algorithm. For example, for the full 2-path query \(QP_2(x, y, z) := R_1(x, y), R_2(y, z)\), the lexicographic order \(x, y, z\) is not allowed. Intuitively, this is because after choosing \(x\) in \(R_1\), we need to fix \(y\) to determine the possible options for \(z\) in \(R_2\).

Later work by Bakibayev et al. on factorized databases [17] shows how to achieve constant-delay enumeration according to any lexicographic order. However, lexicographic orders that do not agree with the “factorization order” used in the construction of their data structure require an expensive restructuring operation. For \(QP_2\) with lexicographic order \(x, z, y\), their approach would need to construct a representation of \(O(n^2)\) size, which implies \(TT(k) = O(n^2 + k)\). (see Appendix B).

In our ranked-enumeration framework, any lexicographic order is supported with only an additional logarithmic factor (i.e., \(TT(k) = O(n + k \log k)\)) for any free-connex CQ, including the above example. As we explain in more detail in Section 7.4.1, lexicographic orders are at least as hard as the sum-of-weights case.

### 3 Overview of Results and Outline

We are now in a position to state the main results of this work more formally. The first result concerns the data complexity of ranked enumeration for free-connex CQs, accompanied by a conditional lower bound, similar to the lower bound of unranked enumeration. We note that our approach does not only apply to free-connex CQs, but also to any CQ with weaker guarantees.

**Theorem 8 (Any-\(k\) CQs, Data Complexity).** Let \(Q\) be a CQ. If \(Q\) is free-connex, then ranked enumeration with an s-monotone ranking function is possible with \(TT(k) = O(n + k \log k)\) and \(MEM(k) = O(n + k)\). Otherwise, if it is also self-join-free, then it is not possible with \(TT(k) = O(n + k \log k)\), assuming \textsc{sparseBMM} and \textsc{Hyperclique}.

**Proof Outline.** We start with the simplest case of full path-structured CQs and show the (serial) DP structure of the problem in Section 4. All three algorithms presented in Section 5 achieve the data complexity bounds of the theorem. In Section 6.1, we modify our techniques for tree-structured problems which include all full acyclic CQs. To ease the presentation, we adopt the sum-of-weights ranking but carefully note which properties of the ranking function we use. Free-connex CQs are reduced to full CQs in Section 6.2. In Section 7.3, we generalize to all s-monotone ranking functions, showing algorithm correctness for these cases. \(\square\)

In a more detailed analysis, we show that the \textsc{anyK-part+} algorithm that we develop achieves the best known guarantees in combined complexity for the class of full acyclic CQs and ss-monotone ranking functions.

**Theorem 9 (Any-\(k\) CQs, Combined Complexity).** For a full acyclic CQ \(Q\) with \(\ell\) atoms of maximum arity \(\alpha\), ranked enumeration with an algebraic and s-monotone ranking function is possible with \(TT(k) = O(n\ell \alpha + k(\log k + \ell \alpha))\). Ranked enumeration with an algebraic and ss-monotone ranking function is possible with \(TT(k) = O(n\ell \alpha + k(\log(\min\{k, n^{diam(Q)+1}\}) + \ell \alpha))\). In both cases, \(MEM(k) = O(n\ell \alpha + k\ell \alpha)\).
Proof Outline. The two time bounds are achieved, respectively, by the anyK-part and anyK-part+ algorithms, as we show in Section 5.4 (for path-structured CQs) and in Section 6.1 (for tree-structured CQs). Compared to Theorem 8, proving these time complexity bounds requires (1) the special handling of cases where the ranking function does not have an inverse (Section 7.3) and (2) the assumption that the ranking function is algebraic (which we adopt in Section 5.4).

According to the theorem, an ss-monotone ranking function such as sum-of-weights allows us to reduce the $k \log k$ term in the running time to $k \log(\min\{k, n^{\ell-\Delta \log(Q) + 1}\})$. This means that the higher the diameter of the CQ, the smaller the logarithmic factor. The largest difference is observed for path CQs $Q_{\ell}$ where $\Delta \log(Q) = \ell$, and hence $TT(k) = O(n\ell^\alpha + k \log(\min\{k, n\} + \ell\alpha))$. To see why this is an improvement, note that $k$ can be very large, e.g., $\Omega(n^{\ell/21})$ for $Q_{\ell}$ [15]. In the other extreme, for star queries $Q_{\ell}(x_1, \ldots, x_{2\ell-2}) := R_1(x_1, \ldots, x_{\ell-1}), R_2(x_1, x_1, \ldots, R_\ell(x_1, x_{2\ell-2})$, we have $\Delta \log(Q_{\ell}) = 3$ and hence we obtain $TT(k) = O(n\ell^\alpha + k(\ell \log n + \ell\alpha))$, i.e., an additional factor $\ell$ in the logarithm term. We note that our upper bound is pessimistic, since a high diameter is a sufficient but not necessary condition for our algorithm to reduce the $k \log k$ term in the running time. This ultimately depends on the height of the constructed join tree. For example, the arity-2 version of the star queries $Q_{\ell}(x_0, \ldots, x_{\ell}) := E(x_0, x_1), E(x_0, x_2), \ldots, E(x_0, x_{\ell})$ have $\Delta \log(Q_{\ell}) = 2$, however they admit a join tree that is a path (similarly to the path CQs $Q_{\ell}$) and, as a result, admit the same running time as $Q_{\ell}$.

As a side benefit, our algorithm can also be used for ranked enumeration of paths in a directed acyclic graph and thus applies to a wide class of DP problems.

Theorem 10 (Any-k DP). In a weighted DAG $G$ with $N$ nodes and maximum path length $\ell$, explicit ranked enumeration with an algebraic and ss-monotone ranking function is possible with $TT(k) = O(|G| + k(\log N + \ell))$ and $MEM(k) = O(|G| + k\ell)$.

Proof Outline. The bounds are achieved by the anyK-part+ algorithm and its analysis in Section 5.4. The correctness for ranking functions beyond sum is established in Section 7.3.

Our asymptotic time complexity dominates the previously best-known algorithm due to Eppstein [59] with $TT(k) = O(|G| + k(\log k + \ell))$. We strictly improve over this since $k$ can be as high as $2^{n-2}$ in a DAG and for $k < n$ the first term dominates the running time.

4 PATH-STRUCTURED CQS AND DYNAMIC PROGRAMMING (DP)

We start with the subset of full path-structured CQs. Note that this does not include only the CQs $Q_{\ell}$ (Example 2), but also any CQ whose join tree is a path (i.e., every node in the join tree has at most one child). We show that the problem of computing the single, top-ranked query answer is solvable by Dynamic Programming. As a consequence, the weighted query answers can be represented as weighted paths in a DAG, and their ranked enumeration (Definition 5) can be achieved by ranked enumeration of paths in that DAG (Definition 6).

4.1 Dynamic Programming Formulation

We now describe a framework for Dynamic Programming (DP) as a DAG that captures a wide range of problems [27, 50]. Our ranked enumeration approach described in the next section applies to any problem expressible in this framework.

DP as a DAG. The DAG $G(V, E)$ of a DP problem captures the dependencies between different subproblems. The nodes $V$ represent states, which contain local information for decision making [27].

\footnote{Our algorithm actually achieves $O(|G| + k(\log(\min\{k, n\}) + \ell))$ but asymptotically, this is the same as the bound of the theorem as we explain in Section 5.4.}
Among them, there is a source state \( s \) and a terminal or target state \( t \). In each state \( v \in V \), we have to make a decision that leads to another state \( v' \). These decisions are encoded as edges \( E \). Each decision \( (v, v') \) is associated with a weight (or cost) \( w(v, v') \), thus the DAG is weighted on its edges.

A solution \( \Pi = (v_1, \ldots, v_k) \) to the DP problem is a sequence of \( \lambda \) states that together with \( s \) and \( t \) form an \( s - t \) path in the graph, i.e., \( v_0 = s, v_{k+1} = t \) and \( (v_i, v_{i+1}) \in E, \forall i \in [\lambda + 1]_0 \). Notice that we do not include \( s \) and \( t \) when we write a solution, since they appear in all solutions. The objective function is the total cost of a solution,

\[
w(\Pi) = \sum_{i=0}^{\lambda} w(v_i, v_{i+1}),
\]

and DP finds the minimum-cost solution \( \Pi_1 \). This corresponds to a shortest \( s - t \) path in the DAG.\(^7\)

In our notation, the index denotes the rank, i.e., \( \Pi_k \) is the \( k \)-th best solution.

By serial DP we refer to the special case of a multi-stage graph: The states are partitioned into \( \ell \) stages and every decision from a state of stage \( S_i \) can only lead to a state of stage \( S_{i+1} \). As a result, all solutions have the same size \( \ell \). This is precisely the case for path CQs, as we shall see next.

**Principle of optimality.** \(^{[20]}\) The core property of DP is that a solution can be efficiently derived from solutions to subproblems. In the shortest-path view of DP, the subproblem in \( \Pi \) is the problem of finding the shortest path from \( v \) to \( t \). With \( \Pi_1(v) \) and \( \pi_1(v) = w(\Pi_1(v)) \) denoting a shortest path from \( v \) and its weight respectively, the DP algorithm computes the minimum of the objective function as follows:

\[
\begin{align*}
\pi_1(t) &= 0 \text{ for the target } t \\
\pi_1(v) &= \min_{(v, v') \in E} \{w(v, v') + \pi_1(v')\}, \text{ for } v \in V \setminus \{t\}.
\end{align*}
\]

The cost of the optimal DP solution is then \( \pi_1(s) \), i.e., the weight of the shortest path from \( s \) to \( t \). For convenience, we define the set of paths compared in Eq. (2) (i.e., those that start at \( v \), continue to a neighbor of \( v \), and then reach \( t \) optimally) as \( \text{Choices}_1(v) \). With \( \odot \) denoting concatenation, \( \text{Choices}_1(v) = \{v \odot \Pi_1(v') \mid (v, v') \in E\} \).

**DP algorithm.** Eq. (2) can be computed for all states in time \( O(|V| + |E|) \) “bottom-up”, i.e., in a reverse topological sort of the DAG. To compute \( \text{Choices}_1(v) \) for state \( v \in V \), the algorithm retrieves all decisions \( (v, v') \in E \) from \( v \) to any state \( v' \in V \), looks up \( \pi_1(v') \), and keeps track of the minimum total weight \( w(v, v') + \pi_1(v') \). If no such edge is found, then the weight is set to \( \infty \). When computing \( \pi_1(v) \), the algorithm also adds pointers to keep track of optimal solutions. In this way, the corresponding paths can be reconstructed by tracing the pointers back “top-down” from \( s \).\(^{[27]}\)

Whenever the bottom-up phase encounters a state \( v \) without outgoing edges (i.e., \( \text{Choices}_1(v) = \emptyset \)), then \( v \) and all its adjacent edges can be removed without affecting the solution space. If we apply this to all states, there will be no “dead ends” in the graph; every node or edge reachable from \( s \) will be part of some solution. We use \( \forall \subseteq V \) and \( \exists \subseteq E \) to denote the sets of remaining states and decisions, respectively. Note that the DP algorithm we describe corresponds to variable elimination \(^{[51]}\) with the tropical semiring \(^{[121]}\) and the removal of states and decisions is the same as the semi-join reductions by Yannakakis \(^{[145]}\).

### 4.2 DP Instance for Path CQs

Computing the top-ranked answer to a full path CQ \( Q_P \) can be modeled as an instance of serial DP. To create it, we use the structure of the join tree. Assume that the join tree of \( Q_P \) is a path with the

\(^6\)“Cost” is more common in optimization problems, “weight” in shortest-path problems. We use the latter throughout the paper.

\(^7\)We use the term “shortest” for the path that has the smallest weight, regardless of the number of edges.
node at depth \(i\) referencing the relation \(R_i\), \(i \in [\ell]\). The stages \(S_i, i \in [\ell + 1]_0\) of the DP instance are as follows: (1) atom \(R_i, i \in [\ell]\) corresponds to stage \(S_i\) and each tuple in \(R_i\) maps to a unique state in \(S_i\), (2) there is an edge between \(v \in S_i\) and \(v' \in S_{i+1}\) iff the corresponding input tuples join and the weight of the edge is the weight of the tuple corresponding to \(v'\), (3) there is an edge from \(s\) to each state in \(S_i\) whose weight is the weight of the corresponding \(R_1\)-tuple, and (4) each state in \(S_\ell\) has an edge to \(t\) of weight 0. Clearly, there is a 1:1 correspondence between paths from \(s\) to \(t\) and query answers. Due to the commutativity of sum, the weight of such a path is equal to the weight of the corresponding query answer, regardless of the stage order imposed by the join tree. Consequently, the \(k\)th-best query answer corresponds to the \(k\)th-shortest path in the DP instance.

**Example 11 (Mapping of path CQ to DP).** We use the problem of finding the minimum-weight answer to the 3-path \(Q_{P_3}\) as a running example. Figure 2a shows an example database and the weights assigned to each tuple. The figure also shows a possible join tree that has a path structure with \(R_1\) as the root. The corresponding DP instance is depicted in Figure 2b. It has 5 stages: Three of them correspond to relations and encode input tuples, while the remaining two correspond to the source and terminal nodes \(s\), \(t\). Every edge encodes two joining tuples. For example, node \((1, 1)\) is connected to \((1, 5)\) because \(y\) is a common variable between \(R_1(x, y)\) and \(R_2(y, z)\) and both tuples assign 1 to \(y\). The edges are weighted according to target-tuple weight, e.g., edge \(((1, 1), (1, 5))\) has weight 100 because \(w((1, 5)) = 100\). The DP algorithm visits the stages right-to-left (i.e., from \(R_3\) to \(R_1\)) and computes the minimum weight path to target \(t\) for every node. For example, \(\pi_1((1, 1)) = \min\{100+10, 200+30, 300+40\} = 110\) where 10, 30, 40 are \(\pi_1((1, 4)), \pi_1((1, 5)), \pi_1((1, 6))\), respectively, which have already been computed when we visit \((1, 1)\). For the source node, we have \(\pi_1(s) = 111\) which corresponds precisely to the minimum-weight query answer. A top-down traversal from \(s\) can retrieve the exact edges that were chosen for \(\pi_1(s)\), from which we reconstruct the witness of the top-1 query answer \(((1, 1), (1, 4), (4, 1))\).

**Encoding equi-joins efficiently.** For an equi-join, the naive construction we describe above has \(O(\ell n)\) states and \(O(\ell n^2)\) edges, therefore the DP algorithm would have quadratic time complexity in the number of tuples. This is due to values that are "heavy hitters", i.e., occur with high frequency and result in a dense connection pattern (e.g., value 1 between \(R_1\) and \(R_2\) in Fig. 2b). We reduce the graph size to \(O(\ell n)\) with an equi-join-specific graph transformation illustrated in Fig. 3.

Consider the join between \(R_1\) and \(R_2\), representing stages \(V_1\) and \(V_2\), respectively. For each join-attribute value, the corresponding states in \(R_1\) and \(R_2\) form a fully connected bipartite graph.
Fig. 3. Equi-join representation from $O(n^2)$ to $O(n)$. Intermediate nodes are introduced (in color), which correspond to values of the join attribute(s). Additionally, nodes that have no outgoing edges are removed (equivalent to a semi-join reduction).

For each state, all incoming edges have the same weight, as edge weight is determined by tuple weight. Therefore, we can represent the subgraph equivalently with a single node “in-between” the matching states in $V_1$ and $V_2$, assigning zero weight to edges starting from states in $V_1$ and the target-tuple weight to those leading to states in $V_2$. The transformed representation has only $O(n)$ edges and preserves all the connections that exist in the original graph. Its total size is $O(n^\alpha)$ since the states that correspond to input tuples need to store $O(\alpha)$ attribute values. At its core, our encoding relies on the conditional independence of the non-joining attributes given the join attribute value, a property also exploited in factorized databases [117]. Here, we provide a different perspective on it as a graph transformation of bipartite cliques [63].

This efficient representation can be constructed directly from the input tables and the join tree in $O(n^\alpha)$. Similarly to the DP algorithm, we visit the input relations bottom-up, and for each stage, we create the edges to the previous stage by using a lookup table (i.e., a hash table) on the join attribute(s).

8 If indexes already exist in the database, then they can be exploited here. Note that using a B-tree or a sort-based method will incur an additional logarithmic factor in the analysis.

Example 12 (Eqi-join Transformation). The transformed DP instance for Fig. 2 is shown in Fig. 3. The 6 edges between $R_1$ and $R_2$ for join value 1 are replaced by one intermediate node and 2 + 3 edges. This reduces the number of edges from quadratic to linear. Constructing the representation bottom-up, we first remove (2, 7) because 7 does not appear in the first column of $R_3$, and as a consequence, we also remove (3, 2) and (4, 2) of $R_1$.

5 ANY-k ALGORITHMS FOR DP

We defined a class of DP problems that can be described in terms of a DAG, where every solution is equivalent to a path from $s$ to $t$ in a “reduced” graph $(V, E)$. In addition to the minimum-weight path, ranked enumeration retrieves all paths in weight order (Definition 6). Let $\Pi_k(v)$ be the $k^{th}$-shortest suffix from node $v$ to $t$ and $\pi_k(v)$ its weight (i.e., $\pi_k(v) := w(\Pi_k(v))$). The goal is to return the sequence $\Pi_1(s), \Pi_2(s), \ldots$

First, we explore algorithms that follow two different approaches. The first partitions the solution space to find the next best solution and traces its roots to the works of Hoffman and Pavely [85], Lawler [103], and Murty [110]. We call it ANYK-PART and show that it has the lowest $TT(k)$ for small $k$. The second, which we call ANYK-REC, formulates the problem recursively [21, 58, 91], and,
as we show, it has the lowest TT\(k\) for large \(k\) on certain inputs. Then we extend ANYK-PART with ideas from ANYK-REC to develop ANYK-PART+, a new algorithm that combines the respective advantages of both algorithms to give the best asymptotic TT\(k\) for any \(k\).

5.1 Partitioning-based Algorithm (ANYK-PART)

The ANYK-PART algorithm (1) relies on the Lawler-Murty procedure [103, 110], which is a general ranked-enumeration approach that applies to a wide range of optimization problems, and (2) exploits the DP structure of our problem using the concept of deviations.

**Deviations.** Given the source node \(s\) and the target node \(t\), we can impose a deviation structure on the \(s \to t\) paths.

**Definition 13 (Deviation [85]).** A deviation of a path \(\Pi\) is a path \(\Pi'\) that follows the same edges as \(\Pi\) from \(s\) up to a node \(v_i\), then takes a different edge to a node \(v'_{i+1}\), and then the optimal path \(\Pi_1(v'_{i+1})\) to \(t\). The edge \((v_i, v'_{i+1})\) is called the deviating edge.

Already in 1959, Hoffman and Pavely [85] showed that every path is a deviation of some shorter (or equally short) path. Unfortunately, the precise statement in the original paper is inaccurate in the presence of ties between different paths. We present the formal statement and its proof more precisely and for more general ranking functions:

**Lemma 14.** For an \(s\)-monotone ranking function over the \(s \to t\) paths of a DAG and \(k > 1\), there exists a valid ordering of the paths such that the \(k\)-th-ranked path \(\Pi_k(s)\) is a deviation of some higher-ranked path \(\Pi_j(s)\) (i.e., \(j < k\)).

**Proof.** Let \(\Pi_k(s) = ⟨s, v_1, v_2, …, v_λ, t⟩\). Also, let \(v_i\) be the last node of \(\Pi_k(s)\) with the property that the suffix \(\Pi(v_i) = ⟨v_i, v_{i+1}, …, v_λ, t⟩\) is not the same as \(\Pi_1(v_i)\). Then, \(\Pi_k(s)\) is a deviation of \(\Pi_j(s) = ⟨s, v_1, v_2, …, v_i⟩ △ Π_1(v_i)\) because, from the way we picked \(v_i\), the suffix \(⟨v_{i+1}, …, v_λ, t⟩\) is necessarily optimal. Since \(Π_1(v_i)\) is the optimal suffix from \(v_i\), we have \(w(Π_1(v_i)) ≤ w(Π(v_i))\) and by \(s\)-monotonicity, \(w(Π_j(s)) ≤ w(Π_k(s))\). If the inequality is strict, then \(Π_j(s)\) necessarily appears before \(Π_k(s)\) in the ranked order, i.e., \(j < k\).

Otherwise, their weight is the same and either of them can be ranked higher. For this case, we show that there exists a tie-breaking mechanism so that \(Π_j(s)\) is ranked higher than \(Π_k(s)\). Let \(Π(\nu)\) and \(Π(\nu')\) be two different equal-weight paths starting from some node \(\nu\) and ending at \(t\). Also let \(\nu'\) be their last common node starting from \(\nu\). If either \(\Pi(\nu)\) or \(\Pi(\nu')\) coincides with the optimal suffix \(Π_1(\nu)\), then it is preferred. Otherwise, we can tie-break arbitrarily. By this mechanism, \(Π_j(s)\) has to appear before \(Π_k(s)\) since \(Π_j(s)\) contains \(Π_1(v_i)\) and the two paths coincide up to \(v_i\).

Deviations provide an efficient way to explore the solution space, starting from the best path \(Π_1(s)\) which is already computed by DP. ANYK-PART maintains a priority queue of candidates Cand initialized with \(Π_1(s)\). To produce the next-best path, it pops from Cand, returns the path, and pushes its deviations back to Cand. For the correctness of this algorithm, note that every path can be constructed through deviations starting from \(Π_1(s)\) and Lemma 14 ensures that the order in which they are produced is correct.

**Avoiding duplicates.** A straightforward application of the idea described above that attempts all possible deviations of every path produces duplicate candidates. The Lawler-Murty procedure [103, 110] provides a way to choose deviations in order to avoid duplicates. For a path whose last deviating edge is \((v_i, v_{i+1})\), it does not produce new candidates for: (1) deviating edges from \(s\) up to \(v_i\) and (2) deviating edges to replace \((v_i, v_{i+1})\) that have already been produced with the same prefix.

**Example 15 (Cartesian Product).** Figure 4 depicts how a Cartesian Product of three relations \(R_1, R_2, R_3\) of size 4 is explored according to the Lawler-Murty procedure. Equivalently, this problem...
can be encoded in the DP framework as a 3-stage fully-connected graph together with a source and a target node (see Section 4.1). The figure shows the new candidates that we generate after returning each solution as its children in a tree. Every solution is denoted by three numbers, indicating the rank of the input tuples in each relation. For example, \((1, 2, 3)\) refers to the combination of the best \(R_1\)-tuple with the 2\(^{nd}\)-best \(R_2\)-tuple and the 3\(^{rd}\)-best \(R_3\)-tuple. Clearly, \((1, 1, 1)\) is the best solution. The candidates for the 2\(^{nd}\)-best solution are its three deviations, one for each relation (or stage).

Notice that \((2, 2, 1)\) could potentially be generated by both \((1, 2, 1)\) and \((2, 1, 2)\). To avoid this, the Lawler-Murty procedure produces deviations only in the third stage for \((2, 1, 2)\); this is because its last deviating edge is in the third stage (rule 1). Further, notice that according to our definition, \((1, 1, 1)\) is a deviation of \((1, 2, 1)\) so we have to avoid producing it as a candidate again after returning \((1, 2, 1)\) (rule 2). In our example, this is guaranteed by considering the input tuples in increasing order of their ranks. We will formalize this with the notion of a successor.

**Encoding of candidates.** A convenient and efficient way to encode the candidate paths is to represent them as prefixes up to the last deviating edge. Under this convention, every prefix \(⟨v_1, v_2, \ldots, v_i⟩\) corresponds to the path that we obtain if we expand it optimally up to the terminal node, i.e., \(⟨v_1, v_2, \ldots, v_i⟩ \circ Π_1(\nu)\). This expansion is performed whenever we pop a path from \(\text{Cand}\) so that the algorithm returns the path in an explicit form. A benefit of this encoding is that it makes it easier to identify the deviations that we need to generate in order to avoid duplicates; we simply start from the last edge contained in the prefix (see Figure 4).

**The successor function.** In Example 15, we assumed that the input relations were already sorted, which allowed us, for example, to generate only \((2, 1, 1)\) from \((1, 1, 1)\) in the first stage, and not \((3, 1, 1)\) or \((4, 1, 1)\). In general, when we deviate from an edge \((v_i, v_{i+1})\), we want to generate only the best deviation that we have not yet considered. The possible deviations we can pick are determined by the optimal suffixes of \(\text{Choices}_1(v_i)\). Therefore, we assume a total order on \(\text{Choices}_1(v_i)\) that orders the children \(v'\) of \(v_i\) according to the weight of their optimal suffix \(w(v_i \circ Π_1(v'))\). This order is given by a successor function \(\text{suc}(v_i, v_{i+1})\), which returns the node \(v'_{i+1}\) that is the next-best choice, illustrated in Figure 5. The successor function also helps us to apply rule (2) of the Lawler-Murty procedure, since we visit the deviations in successor order.

We are now in a position to describe the complete \textsc{anyK-part} approach, as given in Algorithm 1. In each iteration, we pop a prefix \(⟨v_1, \ldots, v_{i-1}, v_i⟩\) from \(\text{Cand}\), expand it optimally into \(⟨v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_j⟩\), and then generate the candidates:

\[
\langle v_1, \ldots, v_{i-1}, \text{suc}(v_{i-1}, v_i) \rangle \\
\langle v_1, \ldots, v_{i-1}, v_i, \text{suc}(v_i, v_{i+1}) \rangle \\
\langle v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, \text{suc}(v_{j-1}, v_j) \rangle
\]
Algorithm 1: anyK-part

1. Input: DP problem as a DAG $G(V, E)$ with source $s$ and target $t$
2. Output: solutions in increasing order of weight
3. Execute standard DP to produce for each node $v$: $\Pi_1(v)$ and $\pi_1(v)$
4. //Initialize candidates with top-1.
5. Cand = $\{v_1^*\}$, where $\Pi_1(s) = s \circ \Pi_1(v_1^*)$
6. repeat
7. \hspace{1em} $\langle v_1, \ldots, v_i \rangle$ = Cand.popMin()
8. \hspace{1em} //Expand optimally into full solution.
9. \hspace{2em} $\langle v_1, \ldots, v_i, v_j \rangle$ = $\langle v_1, \ldots, v_i \rangle \circ \Pi_1(v_j)$
10. \hspace{1em} //Generate new candidates by taking deviations.
11. \hspace{2em} for $j$ from $i$ to $\lambda$ do
12. \hspace{3em} Cand.add($\langle v_1, \ldots, v_i, \ldots, v_j - 1, \text{suc}(v_j - 1, v_j) \rangle$)
13. \hspace{2em} output solution $\langle v_1, \ldots, v_i, \ldots, v_j \rangle$
14. until query is interrupted or Cand is empty

Example 16 (anyK-part on Example 11). Consider again the DAG of Example 11. For ease of presentation, Figure 6a shows the same graph without the equi-join transformation which introduces intermediate nodes (Example 12) and with every node identified by the weight of its corresponding tuple. Cand initially contains only one candidate $\langle 1 \rangle$ (Line 6). It corresponds to the shortest path $\langle 1, 100, 10 \rangle$ found by DP. This prefix is popped and expanded in the first iteration of the repeat loop (Line 7), leaving Cand empty for now. As Figure 6b shows, the for loop in Line 12 generates three new deviations as candidates. These are $\langle 2 \rangle$, $\langle 1, 200 \rangle$, $\langle 1, 100, 20 \rangle$. Each one is generated by applying the successor function; for example, we obtain $\langle 1, 200 \rangle$ from $\langle 1, 100 \rangle$ because $\text{suc}(1, 100) = 200$ ($100 + 10 < 200$). From the three new candidates, $\langle 2 \rangle$ is the one with the minimum weight $2 + \pi_1(2) = 112$ and will be the one popped next from Cand. In the fifth iteration, $\langle 1, 200 \rangle$ will be popped. Following the Lawler-Murty procedure, the for-loop (Line 12) for that prefix will be executed only two times, generating only one new candidate since 30 has no successor.
Variants of \textsc{anyK-part}. Different variants of \textsc{anyK-part} arise from different implementations of the successor function. The earlier version of our work \cite{133} presented, analyzed, and compared several variants with ideas that were present in past work\footnote{Some of these variants require a more general definition of the successor function, allowing it to return a set of nodes instead of only one.}. \textsc{Eager} which uses sorting, \textsc{All} based on Yang et. al \cite{143}, \textsc{Lazy} based on Chang et. al \cite{45}, and \textsc{Take2} as the asymptotically best variant. However, \textsc{Take2} is only better in terms of delay instead of the more practically relevant $TT(k)$ (see Section 2.3). A more careful analysis of $TT(k)$ shows that \textsc{Lazy} and \textsc{Take2} have the same complexity. This is a consequence of Lemma 21 that we prove in Section 5.4.

In the present article, we elect to put less emphasis on the differences between these variants and defer a more detailed discussion to Appendix E. In practice, we found a variant which we call \textsc{Quick} to be the best performer and this is the one we show in our experimental evaluation (Section 8). \textsc{Quick} uses the Incremental Quicksort algorithm \cite{120} to sort the children of every node incrementally. It is a randomized algorithm and, in expectation, it achieves the same $TT(k)$ as \textsc{Lazy} and \textsc{Take2}.

To simplify the analysis in Section 5.4, we assume the deterministic \textsc{Lazy} variant. After the bottom-up phase of DP, \textsc{Lazy} constructs a binary heap in linear time for each node that contains one element per outgoing edge. Successor calls are handled by popping from the heap and moving the popped element to a sorted list. As the algorithm progresses, the heap gradually empties out, filling the sorted list, allowing subsequent successor calls to be handled in constant time.

5.2 Recursive-based Algorithm (\textsc{anyK-rec})

\textbf{Generalized principle of optimality.} \textsc{anyK-rec} relies on a generalization of the principle of optimality of DP, which in its usual form states that there exists an optimal path consisting of optimal suffixes. The generalization gives a similar property to the $k^{th}$ path in the ranking. To our knowledge, the first algorithm that uses this principle is due to Bellman and Kalaba \cite{21}, while the principle has been stated more explicitly in later work by Martins et al. \cite{105}.

\textsc{Lemma 17 (Generalized Principle of Optimality).} For an $s$-monotone ranking function over the $s-t$ paths of a DAG $G(V,E)$ and $k \geq 1$, there exists a valid ordering of the paths such that for every node $v$, $\Pi_k(v)$ is equal to $v \circ \Pi_{k'}(v')$ for some $k' \leq k$ and $(v,v') \in E$.

\textbf{Proof.} Let $v'$ be the node after $v$ in $\Pi_k(v)$. Suppose that $\Pi_k(v) = v \circ \Pi_i(v')$ for some $i > k$. Consequently, there has to exist an index $j \in [k]$ such that the path $v \circ \Pi_j(v')$ is not in the...

![Fig. 7. Example 18: Recursive enumeration](image)
Algorithm 2: ANYK-rec

1. **Input:** DP problem as a DAG $G(V, E)$ with source $s$ and target $t$
2. **Output:** solutions in increasing order of weight
3. Execute standard DP to produce for each state $s$: $\Pi(s)$, $\pi_1(s)$, and $\text{Choices}_1(s)$
4. //Assume that $\Pi_0(v)$ for a node $v$ is a special value used for initialization.
5. $k = 0$
6. repeat
7.   output solution $\Pi_{k+1}(s) = \text{next}(\Pi_k(s))$
8.   $k = k + 1$
9. until query is interrupted or $\text{Choices}_k(s)$ is empty
10. //Returns the next best solution starting from $v$.
11. Function $\text{next}(\Pi_k(v))$:
12.   //Base case: Terminal. Assume that null values are ignored throughout.
13.   if $v == t$ then
14.     return null
15.   //If $\Pi_{k+1}(v)$ has been computed by some previous call, it has been stored at node $v$.
16.   if $\Pi_{k+1}(v)$ is not in the sorted list of $v$ or $k == 0$ then
17.     $\Pi_{k+1}(v) = \text{Choices}_k(v).\text{popMin}()$
18.     Store $\Pi_{k+1}(v)$ in the sorted list of $v$
19.     //Replace the popped path with the next one that goes to the same node, computed recursively.
20.     Let $\Pi_{k+1}(v) = v \circ \Pi_{k'}(v')$
21.     $\text{Choices}_{k+1}(v) = \text{Choices}_k(v).\text{add}(v \circ \text{next}(\Pi_k(v')))$
22.   return $\Pi_{k+1}(v)$

top $k$ suffixes from $v$. Since $\Pi_j(v')$ precedes $\Pi_j(v')$ in the ranking of suffixes from $v'$, we have $w(\Pi_j(v')) \leq w(\Pi_j(v'))$ and by $s$-monotonicity, $w(v \circ \Pi_j(v')) \leq w(v \circ \Pi_j(v'))$. If the last inequality is strict, then we have a contradiction, since $v \circ \Pi_j(v')$ should have been in the top $k$ paths from $v$. If their weights are equal, then we can swap them in the ordered list (producing a different valid ordering) so that $\Pi_k(v)$ is equal to $v \circ \Pi_k(v')$. □

For the shortest path (i.e., $k = 1$), we have $k' = 1$, which is the principle of optimality that we saw in Section 4.1. The generalized principle of optimality gives rise to an algorithm that computes lower-ranked suffixes for all nodes, not just the source node $s$. In particular, to compute $\Pi_k(s)$, it is sufficient to compute $\Pi_k(v)$ for every node $v$ [21]. However, it is not necessary [58] since fewer than $k$ suffixes might be needed for some of the nodes. ANYK-rec computes only those that are needed using a recursive call structure proposed by Jiménez et al. [91].

Recursive calls. Recall that the shortest path $\Pi_1(s)$ from the source node $s$ is found as the minimum-weight path in $\text{Choices}_1(s)$. Assume it goes through $v'$. Through which node does the 2nd-shortest path $\Pi_2(s)$ go? It must be either the 2nd-shortest path through $v$, of weight $w(s, v) + \pi_2(v)$, or the shortest path through any of the other nodes adjacent to $s$. In general, the $k$th-shortest path $\Pi_k(v)$ from any node $v$ is determined as the minimum-weight path in some later version of the set of choices $\text{Choices}_k(v) = \{v' \circ \Pi_{k'}(v') \mid (v, v') \in E\}$, for appropriate values of $k'$. Let $\Pi_k(v) = v \circ \Pi_k(v')$. Then the $(k + 1)^{th}$ solution $\Pi_{k+1}(v)$ is found as the minimum over the same set of choices as in iteration $k$, except that $v \circ \Pi_{k'+1}(v')$ replaces $v \circ \Pi_{k'}(v')$. To find $\Pi_{k'+1}(v')$, the same procedure is applied recursively at $v'$ top-down. Intuitively, an iterator-style next call at source node $s$ triggers a chain of such next calls along the path that was found in the previous iteration.
**Data structures.** The sets of choices $\text{Choices}_k(v)$ of every node $v$ is implemented with a priority queue so that we can find the minimum-weight choice efficiently. Additionally, every node maintains a sorted list of suffixes. As the lower-ranked suffixes are computed, they are added to the sorted list so that they can be reused (when the same node is visited via a different prefix). This type of memoization (in addition to that of standard DP) makes the algorithm faster as the value of $k$ increases. Algorithm 2 contains the detailed pseudocode.

**Example 18 (anyK-rec on Example 11).** Consider source node $s$ in Fig. 6a. Since its children are 1 and 2, the shortest path $\Pi_1(s)$ is selected from $\text{Choices}_1(s) = \{ s \circ \Pi_1(1), s \circ \Pi_1(2) \}$. The first next call on node $s$ returns $s \circ \Pi_1(1)$, updating the set of choices for $\Pi_2(s)$ to $\{ s \circ \Pi_2(1), s \circ \Pi_1(2) \}$ as shown in the left box in Fig. 7b. The subsequent next call on $s$ then returns $s \circ \Pi_1(2)$ for $\Pi_2(s)$, causing $s \circ \Pi_1(2)$ in $\text{Choices}_2(s)$ to be replaced by $s \circ \Pi_2(2)$ for $\text{Choices}_3(s)$; and so on. The paths $\Pi_2(1), \Pi_2(2), \ldots$ are themselves constructed in a similar way with pointers to paths that start at nodes 100, 200, and 300, as shown in Fig. 7a.

5.3 ANYK-PART with Memoization (ANYK-PART+)

anyK-rec stores and reuses the order of suffixes $\nu$ visits a node $p$ increases. Algorithm 2 contains the detailed pseudocode.

**Example 18 (anyK-rec on Example 11).** Consider source node $s$ in Fig. 6a. Since its children are 1 and 2, the shortest path $\Pi_1(s)$ is selected from $\text{Choices}_1(s) = \{ s \circ \Pi_1(1), s \circ \Pi_1(2) \}$. The first next call on node $s$ returns $s \circ \Pi_1(1)$, updating the set of choices for $\Pi_2(s)$ to $\{ s \circ \Pi_2(1), s \circ \Pi_1(2) \}$ as shown in the left box in Fig. 7b. The subsequent next call on $s$ then returns $s \circ \Pi_1(2)$ for $\Pi_2(s)$, causing $s \circ \Pi_1(2)$ in $\text{Choices}_2(s)$ to be replaced by $s \circ \Pi_2(2)$ for $\text{Choices}_3(s)$; and so on. The paths $\Pi_2(1), \Pi_2(2), \ldots$ are themselves constructed in a similar way with pointers to paths that start at nodes 100, 200, and 300, as shown in Fig. 7a.

**Leading prefix and followers.** The idea behind anyK-PART+ is to let only the first prefix that visits a node $\nu_m$ explore the solution space as in ANYK-PART. The suffix order discovered by that prefix will be stored in a sorted list $\text{SortedSuff}(\nu_m)$ and reused by all other prefixes that reach $\nu_m$.

**Definition 19 (Leading Prefix).** The prefix $\langle v_1, \ldots, v_m \rangle$ is called the leading prefix of $\nu_m$ if $\langle v_1, \ldots, v_m, \ldots, v_1 \rangle$ is the first solution returned by ANYK-PART (or ANYK-PART+) that contains $\nu_m$.

Recall that after popping a prefix $\langle v_1, \ldots, v_i \rangle$ from $\text{Cand}$, ANYK-PART first expands it optimally into $\langle v_1, \ldots, v_i, \ldots, v_1 \rangle$ and then generates a deviation at every step $v_i \ldots v_i$. In ANYK-PART+, we again traverse the nodes $v_i \ldots v_1$ in order to find the first non-leading prefix $\langle v_1, \ldots, v_i, \ldots, v_m \rangle$. To detect it, we can simply check whether the data structure $\text{SortedSuff}(\nu_m)$ has already been initialized; if it is, then this means that node $\nu_m$ has been visited before by some other prefix. For all prefixes $\langle v_1, \ldots, v_j \rangle$, $j \in [i, m]$, we create deviations using the successor function as in ANYK-PART. For the rest of the prefixes $\langle v_1, \ldots, v_j \rangle$, $j \in [m + 1, \lambda]$, we do not create any deviations. Instead, we use the sorted list $\text{SortedSuff}(\nu_m)$ to directly find the next-best suffix and create a single new candidate for that subspace. We will refer to the solutions generated by appending a suffix from $\text{SortedSuff}(\nu_m)$ to a non-leading prefix as a follower. A follower consists of a prefix and a suffix, together with an annotation $\{ k' \}$ that specifies the rank of the suffix. Whenever we pop a follower $p \circ \Pi_k(\nu)$ with annotation $\{ k' \}$ from $\text{Cand}$, we simply replace it with a new candidate $p \circ \Pi_{k+1}(\nu)$.

The following property is helpful in order to better understand the behavior of the algorithm:

**Observation 1.** For all iterations of ANYK-PART+ except the first one, when we pop from $\text{Cand}$ a prefix $\langle v_1, \ldots, v_i-1, v_i \rangle$ that is not a follower, we have that $\langle v_1, \ldots, v_i-1 \rangle$ is a leading prefix.
Any-k Algorithms for Enumerating Ranked Answers to Conjunctive Queries

Algorithm 3: ANYK-PART+

1. **Input:** DP problem as a DAG \( G(V, E) \) with source \( s \) and target \( t \)
2. **Output:** solutions in increasing order of weight
3. Execute standard DP to produce for each node \( v \): \( \Pi_1(v) \) and \( \pi_1(v) \)
4. //Initialize candidates with top-1.
5. //The priority of a prefix \( (v_1, \ldots, v_i) \) in Cand is the weight of its optimal expansion

\[
\text{w}(\langle s, v_1, \ldots, v_i \rangle) + \pi_1(v_i).
\]

6. \( \text{Cand} = \{[\langle v_i^* \rangle] \text{ where } \Pi_1(s) = s \circ \Pi_1(v_i^*) \}
7. \text{repeat}
8. \text{current = Cand.popMin()}
9. \text{if current = \langle v_1, \ldots, v_i \rangle (not a follower) then}
10. \text{//Expand optimally into full solution.}
11. \text{\langle v_1, \ldots, v_i, v_{j-1} \rangle = \langle v_1, \ldots, v_i \rangle \circ \Pi_1(v_j)
12. \text{//Find first non-leading prefix.}
13. \text{m = min\{j | j \in [i, \lambda] \wedge \text{SortedSuff}(v_j) \text{ initialized} \} or } \lambda \text{ if none}
14. \text{//Generate new candidates by taking deviations until prefix is non-leading.}
15. \text{for } j \text{ from } i \text{ to } m \text{ do}
16. \text{\text{Cand}.add(\langle v_1, \ldots, v_{j-1}, \text{suc}(v_{j-1}, v_j) \rangle)
17. \text{//For the rest, find the 2^{nd} best suffix from the leading prefix.}
18. \text{if } m < \lambda \text{ then}
19. \text{\text{follow}(\langle v_1, \ldots, v_m \rangle, 2)
20. \text{//Update sorted lists.}
21. \text{for } j \text{ from } 1 \text{ to } m - 1 \text{ do}
22. \text{\text{store}(\langle v_1, \ldots, v_j \rangle, \langle v_j, \ldots, v_{\lambda} \rangle)
23. \text{else if current is a follower \( (v_1, \ldots, v_m) \circ \langle v_m, \ldots, v_\lambda \rangle \) with annotation \{k'\} then
24. \text{//Keep following the leading prefix.}
25. \text{\text{follow}(\langle v_1, \ldots, v_m \rangle, k' + 1)
26. \text{//Update sorted lists.}
27. \text{for } j \text{ from } 1 \text{ to } m - 1 \text{ do}
28. \text{\text{store}(\langle v_1, \ldots, v_j \rangle, \langle v_j, \ldots, v_{\lambda} \rangle)
29. \text{output solution \langle v_1, \ldots, v_i, \ldots, v_{\lambda} \rangle
30. \text{until query is interrupted or Cand is empty
31. \text{Procedure follow(\langle v_1, \ldots, v_m \rangle, k):
32. \text{if } \Pi_k(v_m) \notin \text{SortedSuff}(v_m) \text{ then
33. \text{Add } (\langle v_1, \ldots, v_m \rangle, k) \text{ to subscribers of SortedSuff}(v_m)
34. \text{else}
35. \text{Cand}.add(\langle v_1, \ldots, v_m \rangle \circ \Pi_k(v_m) \text{ with annotation } \{k\})
36. \text{Procedure store(\langle v_1, \ldots, v_i \rangle, \langle v_j, \ldots, v_{\lambda} \rangle):}
37. \text{SortedSuff}(v_i).append(\langle v_1, \ldots, v_{\lambda} \rangle)
38. \text{for Subscriber (\langle v'_1, \ldots, v'_i \rangle, k) of SortedSuff(v_i) do}
39. \text{Unsubscribe and Cand}.add(\langle v'_1, \ldots, v'_i \rangle \circ \langle v_i, \ldots, v_{\lambda} \rangle \text{ with annotation } \{k\})

Proof. We prove this by (strong) induction. The base case is the second iteration of the algorithm, where the solution we pop from Cand is necessarily a deviation of the optimal DP path, which necessarily consists of leading prefixes. For the inductive hypothesis, assume that the statement is true for all iterations \( k \leq 2 \) and we prove it for iteration \( k + 1 \). For the sake of contradiction, suppose
that we pop a non-follower \(\langle v_1, \ldots, v_{i-1}, v_i \rangle\) and \(v_{i-1}\) has been visited before by a different prefix. The prefix \(\langle v_1, \ldots, v_{i-1}, v_i \rangle\) was generated in some previous iteration \(k' < k + 1\) by computing the successor \(v_i = suc(v_{i-1}, v'_i)\) where \(v'_i\) was the node following \(v_{i-1}\) in that iteration. If the popped prefix in iteration \(k'\) was \(\langle v_1, \ldots, v'_i \rangle\), then by the inductive hypothesis \(\langle v_1, \ldots, v_{i-1} \rangle\) has to be a leading prefix. Otherwise, \(\langle v_1, \ldots, v'_i \rangle\) was created by optimally expanding a shorter prefix in iteration \(k'\). In that case, since \(\langle v_1, \ldots, v_{i-1} \rangle\) is not a leading prefix, the algorithm would create a follower instead of a deviation, which is a contradiction.

**Observation 1** implies that for a popped prefix \(\langle v_1, \ldots, v_{i-1}, v_i \rangle\), ANYK-PART+ will always generate the deviation \(\langle v_1, \ldots, v_{i-1}, suc(v_{i-1}, v_i) \rangle\).

**Finding the suffix order.** Keeping track of the suffix order for every node is not as straightforward as in ANYK-REC. In ANYK-PART+, we infer it from the output of the algorithm (which consists of complete solutions) using ss-monotonicity. Whenever we return a solution \(\langle v_1, \ldots, v_i \rangle\), then we can break it up into a prefix \(\langle v_1, \ldots, v_j \rangle\) and suffix \(\langle v_j, \ldots, v_i \rangle\) for any \(j \in [\lambda]\) and record the suffix in SortedSuff\((v_j)\). Using ss-monotonicity, we can show that any later solution that passes through \(v_j\) with a different prefix will also follow the same suffix order. We defer the proof of correctness to Section 7.3.

We now discuss how to avoid storing duplicate suffixes. In principle, this can be achieved by attempting to store all possible suffixes obtained by breaking up the solutions returned by the algorithm and checking if these already exist in our sorted lists. We describe a more efficient method that does not require checking for duplicates. The goal is to store a suffix only the first time we encounter it, which is when it is preceded by a leading prefix. There are two cases:

1. **Non-follower:** Suppose that we have returned a solution that is not a follower and let \(\langle v_1, \ldots, v_m \rangle\) be its first non-leading prefix. Then, we store the suffix of the solution for each node \(v_j, j \in [m - 1]\). This is because from the way the algorithm operates, every prefix contained in a leading prefix must be leading.

2. **Follower:** For a follower solution \(\langle v_1, \ldots, v_m, v_1 \rangle \circ \langle v_m, \ldots, v_1 \rangle\), we also know that the prefix \(\langle v_1, \ldots, v_{m-1} \rangle\) is leading from the way we construct followers. Thus, we store the suffix for each node \(v_j, j \in [m - 1]\).

**Delaying candidate generation.** A complication that arises is that followers might not yet have access to the next-best suffix during the iteration in which we pop them. Note that it is guaranteed (by ss-monotonicity) that a follower \(p \circ \Pi_k(v)\) will never need to be popped from \(\text{Cand}\) before \(p^* \circ \Pi_k(v)\) where \(p^*\) is the leading prefix of \(v\). However, it is not necessarily the case that \(p^* \circ \Pi_{k+1}(v)\) will be popped before \(p \circ \Pi_k(v)\). If we have not yet popped \(p^* \circ \Pi_{k+1}(v)\), then \(\Pi_{k+1}(v)\) is not yet in the sorted list SortedSuff\((v)\), so we cannot immediately replace the follower \(p \circ \Pi_k(v)\) with \(p \circ \Pi_{k+1}(v)\). To handle this scenario, we maintain a list of subscribers along with SortedSuff\((v)\). A subscriber is a follower that waits for the next best suffix to be discovered and inserted into the sorted list by the leading prefix. When it does, we push all subscribers to \(\text{Cand}\) together with the newly discovered suffix.

**Algorithm 3** contains the complete pseudocode of ANYK-PART+.

**Example 20** (ANYK-PART+ on Example 11). We return to the running example of Figure 6a. The first iteration of the algorithm is the same as ANYK-PART since all prefixes in the solution are leading, but we also store the suffixes for the popped solution (Line 28). \(\Pi_1(1) = \langle 1, 100, 10 \rangle\) is appended to SortedSuff\((1)\), \(\Pi_1(100) = \langle 100, 10 \rangle\) to SortedSuff\((100)\), and \(\Pi_1(10) = \langle 10 \rangle\) to SortedSuff\((10)\).

In the second iteration, after we expand \(2\) into \(\langle 2, 100, 10 \rangle\), we generate two new candidates \(\langle 3 \rangle\) and \(\langle 2, 200 \rangle\) by using the successor function. In contrast to ANYK-PART, we stop taking successors after node 100 because it has been visited before (Line 13). For all the following nodes, we create
Fig. 8. Complexity of ranked-enumeration algorithms for \(s-t\) path enumeration in a DAG. Best performing algorithms are colored in green. \(|G|\) is the graph size, \(N\) is the number of nodes, \(r\) is the total number of paths, and \(t\) is the maximum length of a path. The ranking function can be subset-monotone (s-monotone), strong-subset-monotone (ss-monotone), or arbitrary. For instances created from path CQs (Section 4.2), \(t\) is the number of atoms, \(N = O(nt)\), with \(n\) being the maximum relation size, and \(|G|\) is also \(O(nt)\). CP is a fully-connected multi-stage graph resulting from a Cartesian Product.

5.4 Complexity Analysis

We now analyze our any-\(k\) algorithms in terms of time and memory. For the analysis, we assume that the problem at hand is \(s-t\) path enumeration in a DAG \(G\), but the same analysis also applies to path-structured CQs as we explained in Section 4.2. In the following, \(N\) is the number of nodes, \(|G|\) is the number of nodes and edges, \(t\) is the maximum length of a path, and \(r\) is the total number of paths. For ANYK-PART and ANYK-PART+, we assume that Cand is implemented by a priority queue with a logarithmic-time pop and constant-time insert [102]. With JoinFirst, we refer to an algorithm that computes all paths and then sorts them with a comparison-based algorithm. For full acyclic CQs, this is equivalent to the Yannakakis algorithm [145] followed by sorting.

For the time to the first solution (\(k = 1\)), all any-\(k\) algorithms execute DP in time \(O(|G|)\). All initializations of data structures such as the priority queues \(\text{Choices}_v(a)\) of anyK-rec or the heaps of the Lazy variant of anyK-part and anyK-part+ also take \(O(|G|)\). Every algorithm pays \(O(t)\) to return a path of size \(O(t)\) in each iteration.

ANYK-PART. Since at most \(t\) candidates are generated in each iteration, \(|\text{Cand}| \leq kt\). Thus, popping the best candidate as well as bulk-inserting all new candidates takes \(O(\log(kt))\). For efficient candidate generation (Line 13 in Algorithm 1), the new candidates do not copy the previous solution prefix, but simply create a pointer to it. Therefore, a new candidate can be created in \(O(1)\). To evaluate the successor function, the Lazy variant may need to pop from a binary heap of size \(O(N)\). We can assume that all the second-best choices are already computed in the preprocessing phase in linear time by popping from every heap once. Since at most one deviation per iteration does not involve the second-best choice from a node, the overall cost of the successor function is \(O(\log N)\) per iteration. Putting it all together, we have \(\text{TT}(k) = O(|G| + k(\log(kt) + \log N + t)) = O(|G| + k(\log k + \log N + t))\).

**Lemma 21.** For all \(N \geq 1\) and \(k \geq 1\), we have \(N + k \log N = O(N + k \log k)\).
Proof. If \( k \geq N \), then \( k \log N \leq k \log k \), thus the statement is obvious. For any \( 1 \leq k \leq N \), it holds that \( N/k \geq \log(N/k) \) and \( \log k \geq 0 \) and therefore
\[
\frac{N}{k} \geq \log N - \log k \geq \log N - 2 \log k
\]
\[
\Rightarrow N \geq k \log N - 2k \log k
\]
\[
\Rightarrow 2(N + k \log k) \geq N + k \log N
\]
This means that there exists an \( a > 0 \) (\( a = 2 \) here) for which \( N + k \log N \leq a(N + k \log N) \) for all values of \( N \), which completes the proof. \( \Box \)

By Lemma 21, the time complexity of ANYK-PART is \( TT(k) = O(|G| + k\log k) \). If \( k \) is equal to the size of the full output \( r \), we obtain \( TTL = O(|G| + r(\log r + \ell)) \), which is the same as JOINFIRST.

**ANYK-REC.** Each next call on source node \( s \) triggers \( O(\ell) \) next calls (at most one per node on the current path). A next call deletes the top choice at the node and replaces it with the next-best suffix through the same child node. With a priority queue, these operations together take time \( O(\log N) \) per node accessed, for a total delay of \( O(\ell \log N) \) between consecutive solutions. In total, we obtain \( TT(k) = O(|G| + k\ell \log N) \). The resulting TTL bound of \( O(|G| + r\ell \log N) \) is tight in the sense that there exist inputs where ANYK-REC runs in \( O(|G| + r\ell \log N) \). However, it does not take into account the effect of memoization that we also exploited in ANYK-PART++; in later iterations many next calls will stop early because the corresponding suffixes \( \Pi_i \) have already been computed by an earlier call. Taking this into account, we can prove the following:

**Proposition 22.** There exists a class of inputs where ANYK-REC has asymptotically lower time-to-last (TTL) complexity than JOINFIRST.

Proof. Regardless of the implementation of JOINFIRST, before it terminates it has to (i) process the input in \( \Omega(|G|) \), (ii) enumerate all solutions in \( \Omega(r \cdot \ell) \) and (iii) use a standard comparison-based sort algorithm on the entire output in \( (r \log r) \). In total, it needs \( \Omega(|G| + r(\log r + \ell)) \).

If ANYK-REC returns the full output, each suffix \( \Pi_i(v) \) of a node \( v \) is inserted into and removed from the priority queue managing choices at \( v \) exactly once. Therefore, the total number of priority queue operations, each costing \( O(\log N) \), is equal to the number of suffixes. Let \( \Pi_i(i) \) denote the number of suffixes starting at nodes which are at distance \( i \) from \( s \). Then the total cost for all priority-queue operations is \( O(\log N \sum_{i=1}^{\ell} \Pi_i(i)) \). If \( \sum_{i=1}^{\ell} \Pi_i(i) = O(\Pi_1(1)) \), then this cost is \( O(r \cdot \log N) \). To see this, note that the set of paths starting at distance-1 nodes is the set of all possible paths, i.e., the full output. Together with preprocessing time and time to assemble each output tuple, the total TTL complexity of ANYK-REC then adds up to \( O(|G| + r(\log N + \ell)) \). To complete the proof, we show inputs where the condition \( \sum_{i=1}^{\ell} \Pi_i(i) = O(\Pi_1(1)) \) holds and in which the running time of JOINFIRST is strictly worse.

Consider multi-stage graphs where \( \ell \) consecutive stages of size \( n \) are fully connected (\( s \) and \( t \) are the single nodes contained in two additional stages respectively). These graphs are obtained from Cartesian Product CQs if we follow the mapping in Section 4.2. The output size is \( r = n^{\ell} \) and the number of suffixes in the first stage is also \( \Pi_1(1) = n^{\ell} \). The ratio between \( \Pi_i(i) \) and \( \Pi_i(i + 1) \) for some stage \( i \in [\ell - 1] \) is \( n \). Therefore, the sum \( \sum_{i=1}^{\ell} \Pi_i(i) \) is a geometric series and the first term \( \Pi_1(1) \) asymptotically dominates. Also note that the running time of JOINFIRST in these instances is \( \Omega(n^{\ell} \cdot \ell \log n) \), which is higher than the complexity \( O(n^\ell (\log n + \ell)) \) of ANYK-REC. \( \Box \)

The lower TTL of ANYK-REC is at first surprising, given that JOINFIRST is seemingly optimized for bulk-computing and bulk-sorting the entire output. The reason why ANYK-REC wins is that it exploits the shared structure of the solutions, which enables the reuse of shared path suffixes, while JOINFIRST uses general-purpose comparison-based sorting.
AnyK-part+. In AnyK-part+, the maintenance of additional data structures for memoization does not incur any additional cost compared to AnyK-part. In any iteration, the first non-leading prefix can be identified in $O(\ell)$ and the lists $\text{Sort tedSuff}(v)$ for nodes $v$ contained in a solution can also be updated in $O(\ell)$. The benefit of memoization is evident in the size of the priority queue $\text{Cand}$. Followers do not increase the size of $\text{Cand}$ because they are replaced by at most one candidate. Non-followers can generate one follower and a number of other deviations through the successor function. We can again assume that the generated follower does not increase the size of $\text{Cand}$ due to the pop that occurs in each iteration. The generated deviations are responsive for increasing the size of $\text{Cand}$, but this can only happen once for each leading prefix (i.e., the first time we visit a node). Since the number of leading prefixes is at most $N$, the size of $\text{Cand}$ is also bounded by $N$, and the cost of popping or inserting elements in $\text{Cand}$ is $O(\log(\min\{N, k\}))$.

Furthermore, the list of subscribers affects the worst-case delay, but not $\text{TT}(k)$ (see Section 2.3). Indeed, since the prefix of every follower is by construction composed of a leading prefix followed by one edge, we can have $O(N)$ subscribers in one list which are all added to $\text{Cand}$ when the next-best suffix becomes available. However, this delayed candidate generation can only improve $\text{TT}(k)$. To see this, compare it with an algorithm that has access to an oracle which immediately gives the next-best suffix without the need of subscriber lists. For each subscriber that we add to $\text{Cand}$ in a later iteration $k$ than we should, there exists a previous iteration $k'$ where we added one less candidate. This means that the number of priority queue insertions up to iteration $k$ is the same as if it had been added to $\text{Cand}$ in iteration $k'$ and all priority queue operations in iterations in-between can only cost less because $\text{Cand}$ is smaller.

Using Lemma 21, we can simplify the complexity of AnyK-part+ to $\text{TT}(k) = O(|G| + k(\log N + \ell))$. This is strictly better than the $\text{TT}(k)$ complexity of both AnyK-part and AnyK-rec, and also matches the improved bound of AnyK-rec that we showed in Proposition 22.

Memory. All algorithms need $O(|G|)$ memory for storing the input. The memory consumption of AnyK-part depends on the size of $\text{Cand}$. Since at most $O(\ell)$ new candidates are generated per iteration, we have $\text{MEM}(k) = O(|G| + k\ell)$. For AnyK-rec, the size of a choice set $\text{Choices}_k(v)$ is bounded by the out-degree of $v$, hence cannot exceed $N$. However, we need to store the suffixes $\Pi_k(v)$, whose number is $O(\ell)$ per iteration, so $\text{MEM}(k) = O(|G| + k\ell)$. AnyK-part+ has the same memory consumption; the size of $\text{Cand}$ is bounded by $N$ but the sorted lists $\text{Sort tedSuff}(v)$ occupy $O(k\ell)$ space. JoinFirst first materializes the output and then sorts it in-place, therefore has $\text{MEM}(k) = O(|G| + r\ell)$, regardless of $k$.

Summary. Figure 8 summarizes the analysis for $\text{TT}(k)$, highlighting the case of $k = N$ (this is a case where $k$ is sufficiently large for the enumeration cost to exceed the preprocessing cost). We also show TTL for cases where the output is sufficiently large (so that enumeration dominates preprocessing), TTL on Cartesian Product inputs where we can see the advantage of AnyK-rec (Proposition 22), and MEM$(k)$. AnyK-part is faster than AnyK-rec in general (e.g., for $\text{TT}(N)$), but AnyK-rec has the edge for large values of $k$ on certain inputs. AnyK-part+ matches the best running times of both and achieves the lowest $\text{TT}(k)$ complexity overall. All any-k algorithms require minimal space, depending only on input size and the number of iterations $k$ times the solution size. JoinFirst requires more memory because it materializes the full output.

6 Extension to General CQs

We extend our ranked enumeration framework from path-structured CQs to general CQs. First, we study DP problems with a tree structure (T-DP), which allows us to handle any acyclic CQ. Then, we discuss how to handle projections for non-full CQs and finally, how our techniques can be applied for cyclic CQs.
6.1 Tree-Based DP (T-DP)

We now consider a class of Dynamic Programming problems where the states (i.e., the nodes) are organized into stages similarly to serial DP and the stages are organized in a tree structure. Compared to the DP framework of Section 4.1, T-DP is more general because it allows tree-structured problems, but also less general because it does not allow arbitrary "jumps" between stages. Both are instances of Non-Serial Dynamic Programming [26] which generalizes the stage-by-stage computation of serial DP; the DP framework of Section 4.1 allows "feedforward loops", while T-DP allows "diverging branches" [61]. We also note that the top-1 solution for T-DP can be phrased in the framework of Functional Aggregate Queries [4].

In T-DP, we have a set of stages $V_1, \ldots, V_t$ that partition the states $V$. The stages are organized in a rooted tree with $V_0 = \{s\}$ as the root stage. We let the leaves of the tree be a set of terminal stages $V_{t+1}, \ldots, V_{t+b}$, each one containing a single terminal node $t_i$, $i \in [\ell+1, \ell+b]$.

For every parent stage $V_p$ and child stage $V_c$, there is a distinct set of decisions (i.e., edges) $E_{pc}$ directed from $V_p$ states to $V_c$ states. Every root-to-leaf path in the stage tree represents an instance of serial DP (Section 4.1). We now extend our approach to such problems.

We establish a tree order that serializes the stages and assume that their indexing follows this order. Non-leaf stages $V_1, \ldots, V_t$ are ordered by a breadth-first traversal of the tree; if the distance from the root to $V_i$ is greater than $V_j$, then $i > j$. Irrespective of their depth, the $t$ leaf nodes are always indexed last in arbitrary order. We now introduce some helpful notation for trees. We define $C(V_p)$ to be the set of indices of child stages of a stage $V_p$ and $pr(V_c)$ as the index of the parent stage of a stage $V_c$. By $\| V_i \|$ we denote all stage indexes in the subtree rooted at $V_i$, excluding $i$. Slightly overloading the notation, we also use $C(v_i) := C(V_i)$ for a state $v_i \in V_i$, and analogously for $pr(v_i)$ and $\| v_i \|$.

**Example 23 (T-DP Instance).** Figure 9b shows an instance of T-DP with 6 “internal” stages $V_1$ to $V_6$, a root stage $V_0$ and 3 terminal stages $V_7$ to $V_9$. Notice that the indexing of stages follows a breadth-first order. If we restrict ourselves to a root-to-leaf path, such as $V_0$ to $V_7$, then the problem degenerates to serial DP (Figure 2). In this example, $\| V_5 \| := \{5, 6, 8, 9\}$, $C(V_3) = \{5, 6\}$ and $pr(V_3) = 1$.

A T-DP solution $\Pi = \langle v_1, \ldots, v_t \rangle$ is a tree with one state per stage that satisfies $(v_p, v_c) \in E_{pc}$, $\forall c \in [\ell + b]$, where $p = pr(v_c)$, $v_0 = s$, and $v_{t+i} = t_{i+1}$, $i \in [b]$. The objective function aggregates the

---

10 Artificial stages can always be introduced to meet this assumption.

11 Notice that as in DP, we do not include the unique root state and the $b$ terminal nodes of the $b$ leaf stages in the solution.
weights of decisions across the entire tree structure:
\[
\mathcal{W}(\Pi) = \sum_{c=1}^{\ell+b} w(\varphi_{pr(c)}, c)
\]  
(3)

**Principle of optimality and T-DP Algorithm.** The optimal solution is computed bottom-up, following the reverse serial ordering of the stages. A bottom-up step for a state \(v\) solves a subproblem which corresponds to finding an optimal subtree \(\Pi(v)\). If \(\mathcal{C}(v) = \{i_1, \ldots, i_d\}\), then that subtree consists of \(v\) and a list of other subtrees rooted at its children \(v_{i_1}, \ldots, v_{i_d}\). To solve a subproblem, we can independently choose the best decision for each child stage. The equations describing the bottom-up phase in T-DP are recursively defined for all states and stages by

\[
\pi_1(v) = 0, \quad \text{for the } b \text{ terminals with } \mathcal{C}(v) = \emptyset
\]

\[
\pi_1(v) = \sum_{c \in \mathcal{C}(v)} \min_{(\varphi_c, c) \in \mathcal{E}_{pc}} \left\{ w(v, \varphi_c) + \pi_1(c) \right\}, \quad \text{for } v \in \mathcal{V}_p, p \in \{1\}
\]  
(4)

Similarly to DP, after the bottom-up phase we get reduced sets of states \(\mathcal{V}_i \subseteq \mathcal{V}_i, \mathcal{E}_{pc} \subseteq \mathcal{E}_{pc}\) and the top-1 solution \(\Pi(v)\) is found by a top-down phase that follows optimal decisions.

Comparing the above with serial DP, we now have multiple terminal states (i.e., leaves in the tree) that are initialized with zero weight, but we still have only one single root node. A minimum-weight solution contains other subtree solutions that themselves achieve minimum weight for their respective subproblems.

The correctness of the T-DP algorithm is well-established; for example, it is a special case of the InsideOut algorithm for Functional Aggregate Queries [4]. We give here an independent proof:

**Proposition 24 (T-DP).** Equation (4) finds a solution that minimizes Eq. (3).

**Proof.** We show by induction on the tree stages in reverse serial order that for all nodes \(v \in \mathcal{V}:\)

\[
\min_{\Pi(v)} \left\{ \sum_{j \in \|v\|} \mathcal{W}(s_{pr(v_j)}, v_j) \right\} = \pi_1(v)
\]

The base case for the (terminal) leaf states follows by definition from Equation (4). For the inductive step, assume that the above holds for all descendant states \(v_d \in \mathcal{V}_d, d \in \|v\|\) of a node \(v\).

Then for any state \(v \in \mathcal{V}_p:\)

\[
\pi_1(v) = \sum_{c \in \mathcal{C}(v)} \min_{(\varphi_c, c) \in \mathcal{E}_{pc}} \left\{ w(v, \varphi_c) + \pi_1(c) \right\} \overset{\text{ind. step}}{=} \min_{(\varphi_c, c) \in \mathcal{E}_{pc}} \left\{ w(v, \varphi_c) + \min_{\Pi(c)} \left\{ \sum_{c' \in \|c\|} \mathcal{W}(\varphi_{pr(c')}, v) \right\} \right\}
\]

\[
\overset{\text{distributivity/ subset–monotonicity}}{=} \min_{(\varphi_{c_1}, \ldots, \varphi_{c_d}) \in \mathcal{E}_{pc}} \left\{ \sum_{c \in \mathcal{C}(v)} w(v, \varphi_{c}) + \sum_{c' \in \|c\|} \mathcal{W}(\varphi_{c'}, c) \right\} = \min_{\Pi(v)} \left\{ \sum_{j \in \|v\|} \mathcal{W}(\varphi_{pr(v_j)}, v_j) \right\}
\]

The important property that allows us to swap the minimization over all solutions in the subtree with the sum over the minimum solution of each child (second-to-last step) is the distributivity of sum over min or, alternatively, the subset-monotonicity of the ranking function.

Since the above holds for any node \(v\), it also holds for the source \(s\), and the statement follows. \(\square\)

6.1.1 **Any-\(k\) for T-DP.** To enumerate lower-ranked solutions for T-DP, we extend the path-based any-\(k\) algorithms for serial DP.

**Changes to ANYK-PART.** ANYK-PART is straightforward to extend to the tree case by following the serialized order of the stages. In particular, the \(i^{th}\) stage in the tree order is treated like the \(i^{th}\) stage in serial DP, except that the sets of choices are determined by the parent-child edges in
the tree. For illustration, assume a tree order as indicated by the stage indices in Figure 9b. Given a prefix \(\langle v_1, v_2, v_3 \rangle\), the choices for \(v_4 \in V_4\) are not determined by \(v_3\) (as they would be in serial DP with stages \(V_1, V_2, \ldots\)), but by \(v_2 \in V_2\), because \(V_2\) is the parent of \(V_4\) in the tree. In general, at stage \(V_c\), we find the successor \(\text{Suc}(v_p, v_c)\) where \(p = pr(v_c)\). Similarly, to optimally expand a prefix \(\langle v_1, \ldots, v_{c-1} \rangle\) by one stage, we append \(v_c\) such that \(\Pi_1(v_c)\) is a subtree of \(\Pi_1(v_p)\). Thus, we can run Algorithm 1 unchanged as long as we define the choice sets based on the parent-child relationships in the tree. Hence the complexity analysis in Section 5.4 still applies as summarized in Figure 8.

**Changes to anyK-rec.** Recall that in serial DP, each node \(v_i\) processes a next call by recursively calling next on the node \(v_{i+1}\) that follows in the solution. In T-DP, the solutions have a tree structure, and therefore, to find the next-best subtree solution at \(v_i\), we have to consider the next-best subtrees for all of its children \(v_c\), for \(c \in C(v_i)\). For example, consider a node \(v_1 \in V_1\) where the children of \(V_1\) are \(V_2\) and \(V_3\). A solution rooted at \(v_1\) consists of two parts: one solution rooted at some node of \(V_2\) and another rooted at \(V_3\). Suppose that the current solution rooted at \(v_1\) contains the 2\textsuperscript{nd}-best solution from \(V_2\) and the 4\textsuperscript{th}-best solution from \(V_3\) and denote that by \([\Pi_2, \Pi_4]\). Then the next-best solution from \(v_1\) could be either \([\Pi_1, \Pi_4]\) or \([\Pi_2, \Pi_3]\). Since any combination of child solutions \([\Pi_{j_1}, \Pi_{j_2}]\) is valid for the parent, the problem is essentially to rank the Cartesian product space of subtree solutions which agree with \(v_1\).

More generally, let \(\Pi_j(v, c)\) be the \(j\textsuperscript{th}\) best subsolution starting from \(v\) and restricted only to a single branch \(c \in C(v)\). \(\Pi_j(v, c)\) consists of state \(v\), then a child \(v_c \in V_c\) and, from there, a list of pointers to other solutions (i.e., subtrees) that have their own ranks \(j_1, \ldots, j_d\). We write that as \(\Pi_j(v, c) = v \circ \left[\Pi_{j_1}(v_c, i_1), \ldots, \Pi_{j_d}(v_c, i_d)\right]\) for \(C(v_c) = \{i_1, \ldots, i_d\}\). For example, in Figure 9b, \(\Pi_k(v_1, 3) = v_1 \circ \left[\Pi_{k_1}(v_3, 5), \Pi_{k_2}(v_3, 6)\right]\) for some ranks \(j_1, j_2\). Notice that this definition matches the one in Section 5.2 for \(|C(s_c)| = 1\) and since we can, without loss of generality, assume that \(V_0\) always has a single child \(V_1\), we have \(\Pi_k(s) = \Pi_k(s, 1)\) for all values of \(k\). A node \(v \in V_p\) maintains one data structure per branch \(c \in C(v)\) for storing and comparing solutions \(\Pi_j(v, c)\). At the beginning of the algorithm, we initialize it as \(\text{Choices}_1(v, c) = \{v \circ \left[\Pi_1(v_c, i_1), \ldots, \Pi_1(v_c, i_d)\right] | (v, v_c) \in E_{pc}, C(v_c) = \{i_1, \ldots, i_d\}\}\). To process a next call, we pop the best solution from the data structure but unlike serial DP, we now have to replace it with more than one new candidate. To compute next of \(\Pi_j(v, c) = v \circ \left[\Pi_{j_1}(v_c, i_1), \ldots, \Pi_{j_d}(v_c, i_d)\right]\), the new candidates are:

\[
v \circ \left[\Pi_{j_1+1}(v_c, i_1), \ldots, \Pi_{j_d}(v_c, i_d)\right]
\]

\[
\vdots
\]

\[
v \circ \left[\Pi_{j_1}(v_c, i_1), \ldots, \Pi_{j_d+1}(v_c, i_d)\right].
\]

While this algorithm is correct, it has two drawbacks. First, there can be up to \(\ell\) children, hence up to \(\ell\) new candidates, and each one could have size (i.e., number of pointers) up to \(\ell\). Thus, to create them we would have to pay \(O(\ell^2)\), which results in an \(O(\ell)\) factor increase in TT(\(k\)) complexity. Second, this candidate generation process creates duplicates, similarly to the issue we described in Section 5.1. We address both of these issues by applying the more efficient candidate generation process of anyK-part for each priority queue \(\text{Choices}_j(v, c)\). As a result, anyK-rec remains the same as the serial DP case for stages with a single child, but behaves similarly to anyK-part when encountering branches. In the extreme case of star queries (where a root stage is directly connected to all leaves), anyK-rec degenerates to anyK-part.

**Changes to anyK-part+.** The expansion and successor-taking phases of anyK-part+ are modified in the same way as anyK-part. Unfortunately, the memoization of the order of suffixes does not extend to trees in a direct way. The reason is that for T-DP, the independence of suffixes from prefixes does not hold. In DP, every prefix \(p\) that arrives at a node \(v\) can be combined with any suffix that starts from \(v\). This is not true in T-DP; a prefix \(\langle v_1, v_2, v_3 \rangle\) may not admit the same
suffixes as a different prefix \(\langle v'_1, v'_2, v'_3 \rangle\) if the parent of \(v_4\) is not \(V_3\). In the example of Figure 9b, the choice of the node from \(v_4\) is restricted by the node chosen at \(V_2\), and different \(V_2\) nodes allow different choices at \(v_4\).

However, we can still use memoization whenever the prefix-suffix independence holds. For example, in Figure 9b, if we fix the choice of nodes in both \(V_2\) and \(V_3\), then the prefixes (the nodes before \(V_2\)) and the suffixes (the nodes after \(V_3\)) are independent. To capture this intuition, we define the following notion:

**Definition 25 (Serial Decomposition).** A serial decomposition of a T-DP instance is a path where every vertex consists of a set of stages such that (1) every stage appears exactly once, (2) adjacent stages in T-DP appear in the same or adjacent vertices in the path, and (3) stage \(V_j\) cannot appear strictly before stage \(V_i\) in the path if \(j > i\). The width of a serial decomposition \(w_{sd}\) is the maximum number of non-terminal stages contained in a vertex.

This notion of width is reminiscent of query width for CQs [47], but with the additional restrictions that we impose here. And compared to path decompositions for CQs [119], the difference here is twofold. First, the serial decomposition is tied to a particular join tree. Second, we do not materialize the bags of the decomposition, but only use the serial decomposition structure to determine which sorted suffix lists to maintain for anyK-part+, as we explain below.

We adapt the suffix memoization of anyK-part+ to the serial T-DP decomposition. In particular, we maintain a list of sorted suffixes (Line 38 in Algorithm 3) for every combination of nodes that belong to stages in the same vertex of the decomposition. Follower generation (Line 19) and suffix storing (Line 22) are only done when the node being processed belongs to the last stage of a decomposition vertex. For all other nodes, we take their successor (Line 16) as in anyK-part.

**Example 26 (anyK-part+ for T-DP).** Consider again the example T-DP instance of Figure 9. A serial decomposition of width 3 is shown in Figure 9c. Note that the terminal stages colored in gray are ignored when computing the width. Suppose that our current solution after expansion is \(\langle v_1, \ldots, v_4 \rangle\) and we are now processing node \(v_2\). Since \(v_2\) is in the same vertex as \(V_3\) in the decomposition, we do not perform any of the additional operations of anyK-part+ and only take the successor \(\langle v_1, \text{suc}(v_1, v_2) \rangle\). When we move on to \(v_3\), we have seen all the nodes of the \(V_2, V_3\) vertex, therefore we check if the current prefix \(\langle v_1, v_2, v_3 \rangle\) is leading in the sense that the combination \(\langle v_2, v_3 \rangle\) is visited for the first time. If not, then we generate a follower solution for the rest of the stages. If it is a leading prefix, then we move on to the next node but also update the sorted list SortedSuff((\(v_2, v_3\))) with \(\langle v_4, \ldots, v_5 \rangle\).

The time complexity of anyK-part+ on T-DP depends on the width of the serial decomposition. In particular, the number of leading prefixes per level of the decomposition is at most \(n^{w_{sd}}\), where \(n\) is the maximum number of nodes in a stage. The size of the priority queue \(\text{Cand}\) is thus bounded by \(O(n^{w_{sd}} - \ell)\) and the cost of priority queue operations is bounded by \(O(\log n^{w_{sd}} + \log \ell)\). Overall, we obtain \(TT(k) = O(n^k + k(\log(\min\{k, n^{w_{sd}}\}) + \ell))\). The memory consumption is the same as the serial DP case since we store less suffixes.

### 6.1.2 Any-k for full acyclic CQs

Given a full acyclic CQ, we can easily map it to a T-DP instance using a join tree. As in serial DP, the stages correspond to relations, and states correspond to input tuples. The connections between states of parent-child stages are created by connecting joining tuples in linear time as in Section 4.2. To achieve the best possible bound in combined complexity (Theorem 9), we cannot afford to try all possible join trees and all possible serial decompositions to find the one that gives the lowest width. Instead, we work with an arbitrary join tree computed by one run of the GYO reduction [116, 130, 147].
The width of the serial decomposition of our join tree will determine the running time of \textsc{anyK-part}, but now we connect this running time to a structural property of the CQ. For any possible join tree, we can pick as the root node the one that maximizes the depth of the tree; this is easy to find in linear time. From the definition of the diameter of a CQ, it easily follows that this maximum depth is at least $\text{diam}(Q)$. For the serial decomposition, we choose one that works "level-by-level"; this means that all stages at depth $i$ from the root are placed together in the $i$th vertex on the path. Since the depth of the tree is at least $\text{diam}(Q)$, it is guaranteed that it has width $w_{sd} \leq \ell - \text{diam}(Q) + 1$.

To derive the final complexity of Theorem 9, we also need to take into account that every state corresponds to a tuple that contains $O(\alpha)$ values from the database. Thus, \textsc{anyK-part} yields $O(n\alpha + k(\log(\min\{k, n^{\ell - \text{diam}(Q) + 1}\}) + \ell\alpha))$.

**Example 27 (Full acyclic CQ to T-DP).** A join tree of the query $Q_T(x, y, z, u, a, p, f, g) \leftarrow R_1(x, y, z), R_2(y, u), R_3(u, a), R_4(y, p), R_5(p, f), R_6(p, g)$ is shown in Figure 9a and the resulting T-DP instance is that of Figure 9b. The connections between $V_1$ nodes and $V_2$ nodes are constructed by introducing intermediate nodes, in the same way as the connections between $R_1$ and $R_2$ in Figure 3. Similarly for $V_1$ and $V_3$. The width of the serial decomposition in Figure 9c is 3 because three non-leaf stages $V_4, V_5, V_6$ are placed in the same vertex. However, the diameter of $Q_T$ is 5 (this is the shortest distance from variable $g$ to variable $a$). Therefore, it is possible to achieve a width up to $\ell - \text{diam}(Q) + 1 = 6 - 5 + 1 = 2$ if we choose a better root; this is indeed the case if we choose $R_6$ as the root of the join tree.

### 6.2 CQs with Projections

So far, we have only considered full CQs, i.e., those that do not have projections. In this section, we first investigate the different possible semantics of ranked enumeration with projections. Then, we show that under the semantics we introduced in Section 2.2, free-connex CQs can be handled as efficiently as full acyclic CQs (in data complexity).

**Alternative ways to define ranked enumeration.** The set of witnesses $\text{wit}(q)$ for a query answer $q$ is not necessarily unique for non-full CQs. Thus, there are at least two reasonable semantics for ranked enumeration over CQs with projections. Consider the 2-path query $Q_{\text{P2E}}(x_1) \leftarrow R_1(x_1, x_2), R_2(x_2, x_3)$ where we want to return only the values of the first variable $x_1$.

Recall that we assume that input weights have been placed on the relation tuples. What do we do if the same value $c_1$ of $x_1$ appears in two different witnesses $((c_1, c_2), (c_2, c_3))$ and $((c_1, c_2'), (c_2', c_3'))$ with weights $w$ and $w'$, respectively? We identify two different semantics:

1. **All-weight-projection semantics:** The first option is to return $c_1$ twice with both weights $w, w'$ in the correct sequence. The corresponding SQL query would be:
   ```sql
   SELECT R1.X1, R1.W + R2.W as Weight
   FROM R1, R2
   WHERE R1.X2 = R2.X2
   ORDER BY Weight ASC
   ```

   In general, for a CQ $Q$, we return the answers and the weights that the corresponding full CQ would return projected on the variables $\text{free}(Q)$.

12 In the case that two answers have the same weight, we still return both of them.
(2) *Min-weight-projection* semantics: The second option, which was, to our knowledge, first proposed by Kimelfeld and Sagiv [97], is to return $c_1$ only once with the best (minimum) of the two weights. This is precisely the definition of answer weights we gave in Section 2.2. In this case, the SQL query is:

```
SELECT Y.X1, Y.Weight
FROM (SELECT R1.X1, MIN(R1.W + R2.W) as Weight
FROM R1, R2
WHERE R1.X2=R2.X2
GROUP BY R1.X1) Y
ORDER BY Y.Weight
```

In general, we define the weight of $q \in Q$ as $w(q) = \min_{(t_1, \ldots, t_ℓ) \in \text{wit}(q)} \sum_{i=1}^ℓ w(t_i)$ and rank the answers by those weights. Each returned answer has the minimum weight over all answers to the corresponding full CQ $Q(x)$ that agree with $q$ on $\text{free}(Q)$. We note that an interpretation of min-weight-projection semantics is that the query has a *group-by clause*, and the aggregation in the group-by is the same as the one used in the ranking function. A simple way to handle min-weight-projection semantics is to apply the projections on the output of the enumeration and discard all lower-ranked duplicates. However, the non-trivial TT($k$) guarantees we have proved do not hold in that case. The reason is that the answers that project to the same values can be as many as $O(r)$ in the worst case, delaying the enumeration of distinct answers. We next discuss a non-trivial extension that can efficiently handle min-weight-projection semantics for free-connex CQs.

**Handling free-connex CQs.** We modify the techniques that have been developed for *unranked* enumeration (Theorem 7) in order to accommodate efficient ranked enumeration under min-weight-projection semantics. Intuitively, unranked enumeration for free-connex CQs works by constructing an appropriate join tree that groups the free variables together. The tree is first swept bottom-up with semi-joins as in the Yannakakis algorithm [145] and then pruned so that only the free variables remain. The answers to the query can then be enumerated as if it were full (without projections). We present a modification of this approach for ranked enumeration under min-weight-projection semantics. Essentially, we replace the semi-joins with our Dynamic Programming framework.

**Example 28 (Free-connex CQ).** Consider the CQ $Q_{FC} : \neg R_1(y_1, y_2), R_2(y_2, y_3), R_3(x_1, y_1, y_4), R_4(x_2, y_3)$. We can verify that it is free-connex if we add an additional hyperedge $R’$ that encompasses all the free variables (Fig. 10a) and then check that the hypergraph is acyclic (e.g., by finding a join tree). Using the algorithm of Brault-Baron [32], we can construct a join tree for $Q_{FC}$ such that a connected subset of nodes $U$ contain all the free variables and no existentially quantified ones (Fig. 10b). In order to achieve that, we have to introduce two additional relations $R'_3, R'_4$ which are projections of $R_3$ and $R_4$ respectively. Given this join tree and a database instance (Fig. 10c), we can construct the T-DP instance $T$ shown in Fig. 10d. The non-terminal stages correspond to the nodes of the join tree and are populated by states (depicted by white circles) that correspond to input tuples. Ranked enumeration directly on $T$ would produce the answers to the full query. Instead, we only run the bottom-up phase that computes the values $π_1(v)$ for all states $v$, shown on the top-right of each state. We proceed by removing the stages that do not belong to $U$ and replacing them with terminal nodes, thereby getting a modified instance $T'$ shown in Fig. 10e. Observe that ranked enumeration on $T'$ will now enumerate the answers to $Q_{FC}$. To get the correct min-weight-projection semantics, we also have to modify the input weights on $T'$. Consider state $(1, 1) \in V_2$ on $T$ that has two branches, one towards $V_4$ and one towards $V_5$. For the first branch, we have to choose between $(1, 1) \in V_4$ and $(2, 1) \in V_5$. The minimum achievable weight is achieved through $(1, 1)$ since $1 + 0 < 2 + 0$. Therefore,
This takes $T$-DP instance $T$ of the full CQ using the join tree of Fig. 10b. For every state $v$, $\pi_1(v)$ is depicted on its top-right.

**Fig. 10.** Example 28: ranked enumeration under min-weight-projection semantics for the free-connex CQ $Q_{FC}(y_1, y_2, y_3, y_4) := R_1(y_1, y_2), R_2(y_2, y_3), R_3(x_1, y_1, y_4), R_4(x_2, y_3)$ on an example database.

when we remove stage $V_4$ in $T'$ and replace it with a terminal stage $V_4' = \{t_4\}$, we set the weight of the edge $((1, 1), t_5)$ to be equal to 1 (i.e., the minimum). The minimum achievable weights are computed from the bottom-up phase on $T$ (see Eq. (4)).

**Lemma 29 (Free-connex CQ to Full CQ).** Given a free-connex CQ $Q$ of size $O(1)$ and a database $D$ of size $O(n)$, we can construct a full CQ $Q'$ of size $O(1)$ and a database $D'$ of size $O(n)$ such that ranked enumeration with an $s$-monotone ranking function under min-weight-projection semantics for $Q$ produces the same answers as ranked enumeration for $Q'$.

**Proof.** Let $y$ be the free variables of $Q$. In a join tree of $Q$, let $\text{var}(u)$ be the set of variables of the corresponding atom of a node $u$. Since the query is free-connex, we can compute a join tree with a connected subset of nodes $U$ that satisfy $\bigcup_{u \in U} \text{var}(u) = y$ in $O(|Q|)$ time using known techniques [22, 32]. To achieve this, additional atoms might be introduced; set the input weights of all tuples materialized from those atoms to 0. Also set the root of the tree to be any node $u \in U$.

Next, from the join tree, construct in a bottom-up fashion a T-DP instance $T$ as in Section 6.1. This takes $O(n)$ time and removes all states that do not participate in any solution. Every solution of $T$ is by construction an answer to the full query $Q_F$ that has the same body as $Q$ (without projections). Given that (1) $U$ contains all the free variables $y$ needed for answering $Q$ and (2) bottom-up consistency with stages not in $U$ has already been enforced, the subtree induced by $U$ is the answer to $Q$.

---

\footnote{By slightly abusing the notation, we say that a stage belongs to $U$ if its corresponding node in the join tree belongs to $U$.}
contains precisely the answers to $Q$. In more detail, create a copy $T'$ of $T$ that only retains the stages that belong to $U$. Complete $T'$ with an artificial starting stage as the root of the tree and terminal stages as the leaves, exactly as in Section 6.1. We argue that there is a 1-to-1 correspondence between the T-DP solutions of $T'$ and the answers to $Q$. First, consider a T-DP solution $\Pi$ of $T'$. It has to contain states that belong to $\forall$ (recall that these are the ones not removed by the bottom-up pass), hence they can reach the terminal states of the original T-DP instance $T$. Thus, there is a way to extend $\Pi$ to a solution to the original state-space $T$, which corresponds to an answer to the full CQ $Q_F$. The values assigned to the variables $y$ constitute an answer to $Q$. Conversely, an answer $q \in Q$ assigns values to the $y$ variables. Since the subset $U$ of the join tree contains precisely the $y$ variables, we can find tuples in the materialized relations of the join tree or equivalently, states in $T'$ that form a T-DP solution using those values.

To get min-weight-projection semantics, we have to make adjustments to the input weights of $T'$. In particular, we set the weights of the edges that reach the additional terminal nodes we introduced in $T'$ according to the weights of $T$ that do not appear in $T'$. Let $V_c$ be a non-leaf stage in $T$ and $V_p = pr(V_c)$ its parent such that $V_c \not\in U$ and $V_p \in U$. Also let $V'_p$ be the copy of $V_p$ in $T'$. The construction of $T'$ added a stage $V'_c = \{t'_j\}$ and decisions $(v'_p, t'_j)$ for all $v'_p \in V'_p$. The weight of the edges to reach $t'_j$ from $v'_p$ is set to be the minimum achievable weight that $v'_p$ could reach in $T$ from the branch that goes to $V_c$: $w(v'_p, t'_j) = \min_{(v_p, v_c) \in E_p} \{w(v_p, v_c) + \pi_1(v_c)\}$. This can be done in time linear in $T$. For a solution $\Pi$ of $T$ and a solution $\Pi'$ of $T'$, let $\Pi' \subset_U \Pi$ if they agree on the subset $U$. By subset-monotonicity, we have that $w(\Pi') = \min_{\Pi \subset_U \Pi} w(\Pi)$ for every solution $\Pi'$ of $T'$.

The total time spent so far is linear in the size of the database $D$. From $T'$ we can easily construct a corresponding CQ $Q'$ and database $D'$ by creating a relation for every stage and a tuple for every state. The weight of a tuples is equal to the weight of the (unique) edge that connects it to its parent plus the weight of the edges that connect it to terminal children stages (i.e those that are leaves), is such exist.

Lemma 29 says that for free-connex CQs and ranking functions that are s-monotone, we can achieve ranked enumeration by first creating a full CQ and a modified database and then applying our any-$k$ algorithms. By following this approach, we obtain the guarantees of Theorem 8. The corresponding lower bound of the theorem is immediate from Theorem 7 since ranked enumeration is strictly harder than unranked enumeration.

### 6.3 Cyclic Queries

Our work mainly targets acyclic CQs, yet the techniques we develop can also be used for cyclic CQs by leveraging (hyper)tree decompositions [74]. The main idea of such decomposition methods is to reduce cyclic CQs to acyclic CQs. Extending the notion of tree decompositions for graphs [123], (hyper)tree decompositions [74] organize the relations into "bags" and arrange those bags into a tree structure. Each decomposition is associated with a width parameter that captures the degree of acyclicity in the query and affects the complexity of subsequent evaluation: smaller width implies lower time complexity. Our approach is orthogonal to the decomposition method used and it adds ranked enumeration capability virtually "for free."

**Decomposition methods.** The state-of-the-art decompositions rely on the submodular width $\text{subw}(Q)$ of a query $Q$ [107]. There exist decomposition methods that run in time $O(f(|Q|)n^{(2+\delta)\text{subw}(Q)})$ for $\delta > 0$ [25] or $O(f_1(|Q|)n^{\text{subw}(Q)}(\log n)^{f_2(|Q|)})$ [6] for query-dependent functions $f$, $f_1$ and $f_2$. For example, $\ell$-cycle CQs $Q_C \ell$ can be decomposed into trees where each bag materializes a relation of size $O(n^{2-1/(\ell+2)})$ [11, 124]. Since this is an active research area, we expect these algorithms to be improved and we believe our framework is general enough to accommodate future decomposition methods. Sufficient conditions for our approach to apply with no additional
preprocessing cost are (1) the full output of \( Q \) is the union of the output produced by the trees in the decomposition and (2) the number of trees depends only on query size \(|Q|\). Both are satisfied by current decompositions.\(^{14}\)

**Applying any-\( k \).** To use our any-\( k \) algorithms on top of an existing decomposition, three issues have to be addressed. First, the decomposition (e.g., based on the submodular width) might create more than one tree. To handle that, we define UT-DP as a union of T-DP problems where a solution to any of the T-DP problems constitutes a valid solution. Thus, we are given a set of \( u \) functions \( F = \{ f^{(i)} \} \), each defined over a solution space \( \Pi^{(i)}, i \in [u] \). The UT-DP problem is then to find the minimum solution across all T-DP instances. The necessary changes to any of our any-\( k \) algorithms are straightforward: We add one more top-level data structure \( \text{Union} \) that maintains the last returned solution of each separate T-DP algorithm in a single priority queue. Whenever a solution is popped from \( \text{Union} \), it gets replaced by the next best solution of the corresponding T-DP problem. The second challenge is that we have to properly compute the tuple weights in the bags of the decomposition. For this, we track the lineage of the relations participating in a bag at the schema level: We only need to know from which input relation a tuple originates and if that relation’s weight values had already been accounted for by another bag. Third, we have to deal with possible output duplicates when a decomposition creates multiple trees. We eliminate the duplicates by maintaining a lookup table that contains all the answers produced so far. Note that many consecutive duplicates can increase the delay of the algorithm, however \( \text{TT}(k) \) is unaffected in data complexity because the number of duplicates for an answer is bounded by \( O(|Q|) \).

## 7 RANKING FUNCTIONS

This section focuses on different aspects related to the ranking functions that are supported by our framework. We explore how algebraic structures can be used to define ranking functions (Section 7.1), how the monotonicity properties we have discussed relate to algebraic properties, as well as other definitions of monotonicity in the literature (Section 7.2), and show in more detail the correctness of our algorithms when we generalize from the sum-of-weights model we have focused on so far to other ranking functions (Section 7.3).

### 7.1 Relationship to Algebraic Structures

In many cases, the aggregate ranking function \( w_A \) is defined through a binary operator such as +, \( \times \) or max. Algebraic structures [70] capture the properties of such operators in an abstract framework. As a prominent example, semirings have been repeatedly shown to be at the core of many efficient algorithms [4, 9, 77]. We now show how these are related to ranking functions and different monotonicity properties.

**Definitions.** A **monoid** is a 3-tuple \((W, \oplus, \bar{0})\) where \( W \) is a non-empty set and \( \oplus : W \times W \rightarrow W \) is a closed binary operation such that:

1. \((x \oplus y) \oplus z = x \oplus (y \oplus z)\) (associativity),
2. \(\bar{0} \in W\) satisfies \(x \oplus \bar{0} = \bar{0} \oplus x = x, \forall x \in W\) (neutral or identity element).

A monoid is commutative if it also satisfies (3) \(x \oplus y = y \oplus x, \forall x, y \in W\) (commutativity).

A **semiring** is a 5-tuple \((W, \oplus, \otimes, \bar{0}, \bar{1})\), where

1. \((W, \oplus, \bar{0})\) is a commutative monoid and \((W, \otimes, \bar{1})\) is a monoid,
2. \(\forall x, y, z \in W:\ (x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)\) (distributivity of \( \otimes \) over \( \oplus \)),
3. \(\forall x \in W : x \otimes \bar{0} = \bar{0}\) (\( \bar{0} \) is absorbing or annihilating for \( \otimes \)).

\(^{14}\)In general, a decomposition might be designed only for Boolean CQs where it can take “shortcuts” because no output tuples are needed, e.g., through (fast) matrix-multiplication [11].
A commutative semiring is one where \((W, \otimes, \bar{1})\) is also commutative. A binary operator \(\oplus\) is called selective iff it always returns one of the two operands, i.e., \(\forall x, y \in W : (x \oplus y = x) \lor (x \oplus y = y)\). A selective (commutative) dioid is a (commutative) semiring in which \(\oplus\) is selective. A selective commutative dioid has a lattice structure if \(\forall x, y \in W : x \otimes y = x \Rightarrow x \otimes y = y\), i.e., \(\otimes\) is also selective and follows the reverse order. An element \(a\) of a commutative monoid \((W, \otimes, \bar{1})\) is called cancellative iff it can be “canceled out” to solve equations, i.e., \(\forall x, y \in W : x \otimes a = y \otimes a \Rightarrow x = y\).

Notice the correspondence of semiring multiplication \(\otimes\) and semiring addition \(\oplus\), there is a corresponding selective dioid where the \(\oplus\) acts as a comparator and \(\otimes\) to both sides of the second equation we get \(a \oplus (b \otimes c) = a \oplus b\) and by associativity, \((a \oplus b) \otimes c = a \oplus c\), hence \(a \oplus c = a\) or \(a \leq_D c\).

For ranked enumeration, a total order on the domain is necessary and it is known that the selective property is enough to induce such an order.

**Lemma 30 ([70]).** For any selective dioid \(D = (W, \oplus, \otimes, \bar{0}, \bar{1})\), the relation \(a \leq_D b \equiv a \oplus b = a\) is a total order. Also, this order is monotonic (or translation-invariant) with respect to \(\otimes\) (i.e., it is a total order that satisfies \(a \leq_D b \Rightarrow a \oplus c \leq_D b \otimes c, \forall a, b, c \in W\)).

**Proof.** First, we prove that \(\leq_D\) is a total order. (1) To show it is transitive, suppose that \(a \leq_D b\) and \(b \leq_D c\) for \(a, b, c \in W\), thus \(a \oplus b = a\) and \(b \oplus c = b\). By adding \(a\) to both sides of the second equation we get \(a \oplus (b \oplus c) = a \oplus b\) by associativity, \(a \oplus b \oplus c = a \oplus c\), hence \(a \oplus c = a\) or \(a \leq_D c\). (2) For antisymmetry, suppose that \(a \leq_D b\) and \(b \leq_D a\) for \(a, b \in W\), thus \(a \oplus b = a\) and \(b \oplus a = b\). By commutativity, \(a \leq_D b\) if \(b \leq_D a\), so it follows that \(a = b\). (3) The fact that it is total (or connected), i.e., \(a \leq_D b\) or \(b \leq_D a\) for all \(a, b \in W\), follows directly from selectivity. (4) Similarly, the fact that it is reflexive follows immediately from \(a \oplus a = a\), hence \(a \leq_D a\).

Next, we prove that the order is monotonic w.r.t. \(\otimes\). For all \(a, b, c \in W\) we have \(a \leq_D b \Rightarrow a \oplus b = a \Rightarrow c \oplus a \oplus b = c \oplus (a \oplus b) = c \oplus a\) and by distributivity, \((c \oplus a) \oplus (c \otimes b) = c \otimes a\), which implies that \(a \oplus c \leq_D b \otimes c\).

The converse is also true [28]: given a commutative monoid equipped with a total order that is monotonic with respect to \(\oplus\), there is a corresponding selective dioid where the \(\oplus\) operator selects the least element according to the order. Therefore, these two perspectives are equivalent.

**Algebraic structures as ranking functions.** A selective commutative dioid can be used as a ranking function where \(\oplus\) acts as a comparator and \(\otimes\) as an aggregator of input weights. The sum-of-weights model corresponds to \(((\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0))\), also called the tropical semiring [121]. Notice the correspondence of semiring multiplication \(\otimes\) to + and semiring addition \(\oplus\) to min. Another semiring we can use is \(((\mathbb{R} \cup \{\infty, -\infty\}, \min, \max, \infty, -\infty))\), where we essentially swap the + operator for max. Ranked enumeration with this semiring prioritizes query answers whose maximum-weight witness is as small as possible (regardless of the sum of weights).

**Definition 31.** Given a commutative selective dioid \(D = (W, \oplus, \otimes, \bar{0}, \bar{1})\), we can define an algebraic ranking function \(w^D\) such that \(w^D_A = \otimes_{x \in X} x\) and \(x \leq y \iff x \oplus y = x\).

**On commutativity.** Commutativity of the product operator \(\otimes\) together with associativity guarantees that the outcome of an aggregation does not depend on the order of the elements. It is required so that the ranking function defined from a selective dioid is an aggregate function, which is order-insensitive (Section 2.2). The reason that this property is important is because different join trees impose a different order of stages in DP, thus a different order of weight aggregation. Without commutativity, the weight of a query answer could be dependent on the chosen join tree.\(^{16}\)

\(^{15}\)To be precise, the aggregate ranking function is distributive [75] or self-decomposable [90] in this case.

\(^{16}\)In the earlier version of this article [133], commutativity was not stated as an explicit requirement. This oversight is not important if the DP structure is fixed and the ranking function order agrees with it, but it is necessary if we want the freedom to choose different join trees.
Example 32 (Non-commutative structures). The prime example of a non-commutative operator is matrix multiplication since for two matrices $A, B$ we could have $AB \neq BA$. However, we are not aware of any non-commutative selective dioid on infinite domains. For example, if we assume that $\otimes$ is matrix multiplication, there is no obvious way to define a selective operator on matrices so that distributivity also holds. Using the Mace4 tool [108], we were able to identify a finite such structure on a domain of 5 elements (see Appendix D).

7.2 Comparison of Monotonicity Properties

In this section, we compare different notions of monotonicity of the ranking function and show how these can be derived from the properties of algebraic structures introduced previously.

Holistic-Monotonicity. A well-known notion of monotonicity is due to Fagin et al. [62]. To avoid ambiguity in the use of the generic term "monotonicity", we use the term "holistic-monotonicity".

Definition 33 (Holistic-Monotonicity [62]). A ranking function $w$ is holistic-monotone if $x_i \preceq x'_i, \forall i \in [\ell]$ implies $w_A(\{x_1, \ldots, x_\ell\}) \preceq w_A(\{x'_1, \ldots, x'_\ell\})$.

Intuitively, a holistic-monotone ranking function will yield a better (or equal) result when all of its components are better (or equal). We note that there is an equivalent definition of this property where only one component changes value: a ranking function $w$ is holistic-monotone if $x_i \preceq x'_i$ implies $w_A(\{x_1, \ldots, x_i, \ldots, x_\ell\}) \preceq w_A(\{x'_1, \ldots, x'_i, \ldots, x_\ell\})$.

Subset-Monotonicity. We remind the reader that s-monotonicity allows us to compare two (sub)sets of elements ($w(Y_1) \preceq w(Y_2)$) to infer their order when they are extended with the same (sub)set $X$ (Definition 3). Compared to holistic-monotonicity, s-monotonicity is a stricter property that fewer ranking functions satisfy. To see this, notice that holistic-monotonicity is implied if $|Y_1| = |Y_2| = 1$. Intuitively, holistic-monotonicity does not allow us to rely on the ordering of larger (than size-one) sets of elements when we assemble larger sets. Indeed, the Threshold Algorithm [62] always makes comparisons either between individual elements or between complete solutions. On the other hand, DP and our any-$k$ algorithms require the ability to meaningfully compare partial solutions (prefixes or suffixes).

Example 34 (Not subset-monotone). Consider as a ranking function the median over the real numbers. It is holistic-monotone and therefore supported by the Threshold Algorithm [62], because substituting any individual element with a smaller one can only decrease the median. However, it is not s-monotone. Consider $L_1 = \{1, 10, 100\}$ and $L_2 = \{19, 20, 21\}$. Then $\text{median}(L_1) = 10 \leq \text{median}(L_2) = 20$. Therefore, $\text{median}(\otimes L_1, L_2) = 20 \neq 10 = \text{median}(L_1) \otimes \text{median}(L_2)$.
If the ranking function is defined via a commutative selective dioid, then s-monotonicity is guaranteed as it follows from the other algebraic properties.

**Lemma 35.** For a commutative selective dioid \( D \), the ranking function \( w^D \) is s-monotone.

**Proof.** For all multisets \( Y_1, Y_2, X \), \( w^D_A(Y_1) \leq w^D_A(Y_2) \Rightarrow (\otimes_{y \in X} x) \otimes (\otimes_{y_1 \in Y_1} y_1) \leq \otimes_{y_2 \in Y_2} y_2 \Rightarrow (\otimes_{y \in X} x) \otimes (\otimes_{y_2 \in Y_2} y_2) \Rightarrow w^D_A(X \uplus Y_1) \leq w^D_A(X \uplus Y_2) \), where \( \leq_D \) is the total order \( a \leq_D b \equiv a + b \) and the second-to-last implication follows from the monotonic property of that order (Lemma 30).

**Strong-Subset-Monotonicity.** The ss-monotonicity property (Definition 4) is even stricter. It enables us to swap a set \( X_1 \) in a comparison \( w_A(X_1 \uplus Y_1) \leq w_A(X_1 \uplus Y_2) \) with a different set \( X_2 \), provided that \( w_A(X_1) \leq w_A(X_2) \). As previously mentioned, s-monotonicity is implied by setting \( X_1 = \emptyset \). The benefit of ss-monotonicity is that it allows us to reuse the relative ranking of \( Y_1 \) and \( Y_2 \) even if we do not directly compare them, but instead compare their extension with \( X_1 \).

We now give some intuition on how ss-monotonicity differs from the weaker s-monotonicity. If we could show that \( w_A(Y_1) \leq w_A(Y_2) \), then ss-monotonicity would be guaranteed by s-monotonicity. As we will show shortly, this is the case for cancellative algebraic structures where we can cancel out \( X_1 \). For example, in the tropical semiring, if \( 1 + 2 \leq 1 + 3 \), then we can swap out 1 with any other element, e.g., \( 5 + 2 \leq 5 + 3 \) because we can infer \( 2 \leq 3 \). But ss-monotonicity may still hold even if \( w_A(Y_1) \not\leq w_A(Y_2) \). An example is the min-max semiring where \( \max(5, 3) \leq \max(5, 2) \) and \( \max(i, 3) \leq \max(i, 2) \) for any \( i \geq 5 \), even though \( 3 \not\leq 2 \). The condition \( i \geq 5 \) in the example is captured by the requirement that \( w_A(X_i) \leq w_A(X_5) \). In terms of our ANYK-PART+ algorithm, this continues to hold as long as we continue to visit lower-ranked partial solutions. We prove in more detail how this property guarantees the correctness of the algorithm in Section 7.3.

**Example 36 (Not Strong-Subset-Monotone).** An example of a ranking function that is s-monotone but not ss-monotone is the one defined by \( ([0, \infty], \min, x, \infty, 1) \). For example, we have \( 0 \times 2 \leq 0 \times 1 \) but \( 5 \times 2 > 5 \times 1 \) even though \( 0 < 5 \). However, if we remove the non-cancellative element 0, then the structure \( ([0, \infty], \min, x, \infty, 1) \) gives us an ss-monotone ranking function.

We now show two algebraic properties sufficient to guarantee ss-monotonicity:

**Lemma 37.** Given a commutative selective dioid \( D \) that is either cancellative or has a lattice structure, the corresponding ranking function \( w^D \) is ss-monotone.

**Proof.** Strong-subset-monotonicity requires that for multisets \( X_1, X_2, Y_1, Y_2 \), if \( w^D_A(X_1 \uplus Y_1) \leq w^D_A(X_1 \uplus Y_2) \) and \( w^D_A(Y_1) \leq w^D_A(Y_2) \), then \( w^D_A(X_2 \uplus Y_1) \leq w^D_A(X_2 \uplus Y_2) \). By the definition of \( w^D_A \), this is equivalent to showing that if \( (x_1 \otimes y_1) \otimes (x_1 \otimes y_2) = x_1 \otimes y_1 \) and \( x_1 \otimes x_2 = x_1 \), then \( x_2 \otimes y_1 \otimes x_2 \otimes y_2 = x_2 \otimes y_2 \).

First, let \( D \) be cancellative. We apply distributivity to get \( x_1 \otimes (y_1 \otimes y_2) = x_1 \otimes y_1 \). If \( x_1 \not= 0 \), we cancel it out to get \( y_1 \otimes y_2 = y_1 \) and by Lemma 30, \( (x_2 \otimes y_1) \otimes (x_2 \otimes y_2) = x_2 \otimes y_2 \). If \( x_1 = 0 \), then because \( x_1 \otimes x_2 = x_1 \) we have \( x_2 = 0 \) and \( (x_2 \otimes y_1) \otimes (x_2 \otimes y_2) = x_2 \otimes y_1 \) because \( x_2 \) is absorptive.

Second, let \( D \) have a lattice structure. Then, we multiply \( (x_1 \otimes y_1) \otimes (x_1 \otimes y_2) = x_1 \otimes y_1 \) by \( x_2 \) to get \( x_2 \otimes (x_1 \otimes y_1) \otimes (x_1 \otimes y_2) = x_2 \otimes x_1 \otimes y_1 \) and by distributivity, \( (x_2 \otimes x_1 \otimes y_1) \otimes (x_2 \otimes x_1 \otimes y_2) = x_2 \otimes x_1 \otimes y_1 \). Then, because \( x_1 \otimes x_2 = x_1 \), we have \( x_1 \otimes x_2 = x_2 \), hence \( (x_2 \otimes y_1) \otimes (x_2 \otimes y_2) = x_2 \otimes y_1 \). □
7.3 Correctness of Any-k Algorithms

In this section, we elaborate on how our algorithms can work with ranking functions that are more general than the sum-of-weights model that we have mostly assumed throughout the paper, as long as these satisfy the monotonicity properties we have discussed.

Correctness by monotonicity. For ANYK-PART and ANYK-REC, the key property that is required for correctness is that the order of suffixes from a node is maintained when those suffixes are grown to longer paths by appending the same prefix in front. This is required in (1) Dynamic Programming (Equations (2) and (4)), (2) deviations for ranking (Lemma 14), (3) the successor function for ANYK-PART (Section 5.1), and (4) the generalized principle of optimality for ANYK-REC (Lemma 17). Fortunately, this property is an immediate consequence of subset-monotonicity.

Lemma 38. For an s-monotone ranking function w, if \( w(\Pi_i(v)) \leq w(\Pi_j(v)) \) for two suffixes \( \Pi_i(v) \), \( \Pi_j(v) \) starting at node \( v \), then \( w(p \circ \Pi_i(v)) \leq w(p \circ \Pi_j(v)) \) for any path \( p \) ending at \( v \).

For ANYK-PART+, we additionally need the property that from the order of complete solutions popped from the priority queue Cand, we can obtain a ranking of suffixes that we reuse for later solutions. This is not the same as Lemma 38 because we do not directly have access to the actual ranking of suffixes. Here, we need the stricter ss-monotonicity property.

Lemma 39. For an ss-monotone ranking function w, if \( w(p_1 \circ \Pi_i(v)) \leq w(p_1 \circ \Pi_j(v)) \) for two suffixes \( \Pi_i(v) \), \( \Pi_j(v) \) starting at node \( v \) and the leading prefix \( p_1 \) of \( v \), then for any path \( p_2 \circ \Pi_i(v) \) where \( p_2 \) is a prefix ending at \( v \) with \( p_2 \neq p_1 \), we have that \( w(p_2 \circ \Pi_i(v)) \leq w(p_2 \circ \Pi_j(v)) \).

Proof. For brevity, we write any \( \Pi_k(v) \) as \( \Pi_k \). Since \( p_1 \) is the leading prefix, we have that \( w(p_1 \circ \Pi_1) \) is smaller or equal than all other s - t paths going though \( v \) for some suffix \( \Pi_1 \). We will also be using a derivative property of strong subset-monotonicity: if \( w(Y_1 \uplus X_1) = w(Y_1 \uplus X_2) \) and \( w(X_1) \leq w(X_2) \), then \( w(Y_2 \uplus X_1) = w(Y_2 \uplus X_2) \). This is proven by rewriting the equality as the conjunction of two inequalities \( \leq \) and \( \geq \) and applying ss-monotonicity twice. We consider four distinct cases.

Case 1: \( w(\Pi_i) \leq w(\Pi_j) \). By s-monotonicity, \( w(p_2 \circ \Pi_i) \leq w(p_2 \circ \Pi_j) \).

Case 2: \( w(p_1) \leq w(p_2) \). Starting from \( w(p_1 \circ \Pi_i) \leq w(p_1 \circ \Pi_j) \) and using ss-monotonicity to replace \( p_1 \) with \( p_2 \), we obtain \( w(p_2 \circ \Pi_i) \leq w(p_2 \circ \Pi_j) \).

Case 3: \( w(\Pi_i) \leq w(\Pi_j), w(p_2) \leq w(p_1), \) and \( w(\Pi_i) \leq w(\Pi_j) \). We apply s-monotonicity to \( w(p_2) \leq w(p_1) \) to get \( w(p_2 \circ \Pi_i) \leq w(p_1 \circ \Pi_1) \). Since \( p_1 \) is leading, we also have \( w(p_1 \circ \Pi_1) \leq w(p_2 \circ \Pi_1) \), therefore \( w(p_1 \circ \Pi_1) = w(p_2 \circ \Pi_1) \). Now applying the equality version of ss-monotonicity to replace \( \Pi_1 \) with \( \Pi_1 \) (recall that \( w(\Pi_1) \leq w(\Pi_i) \)), we obtain \( w(p_1 \circ \Pi_j) = w(p_2 \circ \Pi_j) \). Repeating the same process for \( \Pi_1 \) instead of \( \Pi_j \) (by transitivity, \( w(\Pi_1) \leq w(\Pi_j) \)), we obtain \( w(p_1 \circ \Pi_1) = w(p_2 \circ \Pi_1) \). We now use our initial assumption that \( w(p_1 \circ \Pi_1) \leq w(p_1 \circ \Pi_j) \) and prove \( w(p_2 \circ \Pi_i) = w(p_1 \circ \Pi_i) \leq w(p_1 \circ \Pi_1) = w(p_2 \circ \Pi_1) \).

Case 4: \( w(\Pi_i) \leq w(\Pi_j), w(p_2) \leq w(p_1), \) and \( w(\Pi_j) \leq w(\Pi_i) \). We apply s-monotonicity to \( w(\Pi_j) \leq w(\Pi_i) \) to get \( w(p_2 \circ \Pi_j) \leq w(p_2 \circ \Pi_1) \). Since \( p_1 \) is leading, we also have \( w(p_1 \circ \Pi_1) \leq w(p_1 \circ \Pi_j) \), therefore \( w(p_1 \circ \Pi_j) = w(p_1 \circ \Pi_1) \). Similarly, we apply s-monotonicity to \( w(p_2) \leq w(p_1) \) to get \( w(p_2 \circ \Pi_j) \leq w(p_1 \circ \Pi_j) \). Since \( p_1 \) is leading, we also have \( w(p_1 \circ \Pi_1) \leq w(p_2 \circ \Pi_1) \). Up to this point, we have \( w(p_2 \circ \Pi_j) \leq w(p_1 \circ \Pi_j) = w(p_1 \circ \Pi_1) \leq w(p_2 \circ \Pi_1) \). Therefore, \( w(p_2 \circ \Pi_1) = w(p_2 \circ \Pi_j) \). We now apply the equality version of ss-monotonicity to replace \( \Pi_1 \) with \( \Pi_1 \) (recall that \( w(\Pi_1) \leq w(\Pi_j) \)) to obtain \( w(p_2 \circ \Pi_i) = w(p_1 \circ \Pi_i) \). Similarly to Case 3, we derive \( w(p_2 \circ \Pi_i) = w(p_1 \circ \Pi_i) \leq w(p_1 \circ \Pi_j) = w(p_2 \circ \Pi_j) \).

Lemma 39 shows that the order of suffixes discovered by the leading prefix \( p_1 \) can be reused for other prefixes. Since the ranking function is commutative, it does not distinguish between prefixes
and suffixes and the dual statement also holds: if \( w(p_1 \circ \Pi_i(v)) \leq w(p_2 \circ \Pi_i(v)) \) for \( p_2 \neq p_1 \), then \( w(p_1 \circ \Pi_i) \leq w(p_2 \circ \Pi_i) \). This shows that a subscriber will never be needed before the leading prefix discovers the next-best suffix, establishing the correctness of \textsc{anyK-part}+.

\textbf{Any-k without an inverse.} \textsc{anyK-part} and \textsc{anyK-part}+ compute the weights of \( O(ℓ) \) new candidates that are deviations of the current solution in \( O(ℓ) \) time in every iteration. A careful reader might have noticed that while this is straightforward for sum-of-weights using subtraction, other ranking functions might not have such an inverse operation. In more detail, recall from Section 2.3 that our analysis assumes an algebraic ranking function that allows us to aggregate partial weights \( w(X) \) and \( w(Y) \) into \( w(X \uplus Y) = w(X) \otimes w(Y) \) with some operation \( \otimes \) in \( O(1) \). An inverse, denoted here by \( \otimes \) allows us to reverse such an operation, i.e., \( w(X) = w(X \uplus Y) \otimes w(Y) \).

Commutative monoids with this property are called Abelian groups. Typical examples of monoids that are not groups are the logical conjunction \( \{0, 1, \wedge, 1\} \) and the minimum over reals \( (\mathbb{R}, \min, \infty) \). In general, an inverse allows us to speed up the computation by reusing prior results.

In the context of our algorithms, \textsc{anyK-part} and \textsc{anyK-part}+ can use the inverse operation to calculate from a path \((s, v_1, \ldots, v_ℓ) \circ \Pi_1(v_j)\), the weight of a deviation \((s, v_1, \ldots, v_j') \circ \Pi_1(v_j')\) as \( w((s, v_1, \ldots, v_j) \circ \Pi_1(v_j)) \otimes w(v_{j-1}, v_j) \otimes w(\Pi_1(v_j)) \otimes w(v_{j-1}, v_j') \otimes w(\Pi_1(v_j')) \), i.e., taking the current weight, “subtracting” the suffix weight and “adding” the new suffix weight. We now discuss how to achieve the same \( O(ℓ) \) computation (for \( O(ℓ) \) new candidates) per iteration without an inverse. Note that \textsc{anyK-rec} never uses an inverse since it always constructs solutions by appending one node to a suffix or a list of subtrees (see Line 22 of Algorithm 2).

For DP, the modification that we have to make is that, as we expand a popped solution, we keep track of the weight of every prefix. Using the weight of the prefix, we can compute the weight of a deviation \((s, v_1, \ldots, v_j') \circ \Pi_1(v_j')\) in \( O(1) \) as \( w((s, v_1, \ldots, v_{j-1}) \otimes w(v_{j-1}, v_j) \otimes w(\Pi_1(v_j)) \otimes w(v_{j-1}, v_j') \otimes w(\Pi_1(v_j')) \). For T-DP, the situation is more involved because a suffix of a solution consists of many disconnected subtrees. As a consequence, the new node \( v_j' \) only has access to the optimal weight of its subtree and not the optimal weight of the suffix. For the example of Figure 9, suppose that our current solution is \( \langle s, v_1, v_2, v_3 \rangle \) and we generate the deviation \( \langle s, v_1, v_2, v_3' \rangle \). Then, \( \Pi_1(v_3') \) does not contain the weight of the \( V_2 \)–\( V_4 \) transition because it is in a different subtree. Naively, computing the weight of the optimal suffix for all deviations of an iteration will take \( O(ℓ^2) \). We now describe how to bring this down to \( O(ℓ^2) \). First, recall that the stages are indexed in BFS order. For each node \( v_i \) of the current solution, let first\((v_i)\) be the smallest index of a node at the same tree level and let last\((v_i)\) be the largest. We now define the “sums” \( \text{pref}(v_i) = \bigotimes_{k \in [\text{first}(v_i), i]} \pi_1(v_k) \) and \( \text{suffix}(v_i) = \bigotimes_{k \in [i, \text{last}(v_i)]} \pi_1(v_k) \). These can be computed for every node in the current solution in \( O(ℓ) \). Using these data structures, the weight of a deviation \((s, v_1, \ldots, v_j') \circ \Pi_1(v_j')\) is \( w((s, v_1, \ldots, v_{j-1}) \otimes \text{pref}(v_{i-1}) \otimes \pi_1(v_j') \otimes \text{suffix}(v_{j+i}) \), where \( c \) is the smallest index of a child of \( v_j \) assuming that \( c - 1 \) is at the same level as \( c \) and \( j + 1 \) is at the same level as \( j \). For the example of \( \langle s, v_1, v_2, v_3 \rangle \) in Figure 9 we have \( \langle s, v_1, v_2, v_3' \rangle = w((s, v_1, v_2)) \otimes \text{pref}(v_4) \otimes \pi_1(v_3') \) because \( 3 + 1 \) is not at the same level as \( 3 \).

7.4 Capturing Other Ranking Functions

7.4.1 Lexicographic orders. As discussed in Section 2.4, any lexicographic order is supported in our framework, although with an additional logarithmic factor compared to work on constant-delay enumeration [16, 17]. To be consistent with the rest of the paper that assumes weights on the input tuples, we assume that the lexicographic order is given in terms of the relations rather than the query variables. For this to be well-defined, self-joins need to be de-duplicated and there must exist a total order on the tuples within each relation.

\textbf{Reduction to sum-of-weights.} One way to handle the lexicographic order is to reduce it to the sum-of-weights ranking function by defining appropriate weights on the input. Given two
consecutive tuples $r_i, r_{i+1}$ of some relation $R_j$, we want the weight of $r_{i+1}$ to be sufficiently large, such that any sum $\sum_{j'=1}^{i} w(r_{j'}) + \sum_{j'=1}^{i} w(r_{j'}) > r_j \in R_j$ is larger than any such sum where $w(r_{i+1})$ is replaced by $w(r_i)$. Assuming that all relations have size $n$, this is guaranteed by setting the weight of the $i$th tuple, $i \in [0, n-1]$, in relation $R_j$, $j \in [0, \ell-1]$ to $i \cdot n^{\ell-1-i}$.

**Dioid.** An alternative way to handle a lexicographic order is to define an appropriate commutative selective dioid; however, this approach incurs an additional factor $O(\ell)$ in time and memory. We set the domain of the dioid to $W = \mathbb{N}^\ell$, i.e., each tuple weight is an $\ell$-dimensional integer vector. Without loss of generality, assume that the total order within a relation is represented by natural numbers such that input tuple $r$ is associated with $w'(r) \in \mathbb{N}$. Input tuple $r_j \in R_j$ has weight $w(r_j) = (0, \ldots, 0, w'(r_j), 0, \ldots, 0)$ with zeros except for position $j$ that stores the “local” weight value in $R_j$. The operator $\oplus$ is standard element-wise vector addition; therefore, the weight of a query answer with witness $(r_1, \ldots, r_\ell)$ is $(w'(r_1), \ldots, w'(r_\ell))$. To order two such vectors, the dioid addition $\oplus$ returns the operand that comes first according to the lexicographic order e.g., for $\ell = 2$, $(a, b) \oplus (c, d) = (a, b)$ if $w'(a) < w'(c)$, or $w'(a) = w'(c)$ and $w'(b) < w'(d)$, and $(c, d)$ otherwise. The 0 and 1 elements of the dioid are $(\infty, \ldots, \infty)$ and $(0, \ldots, 0)$, respectively.

### 7.4.2 Attribute vs Tuple weights.

We now discuss how to handle the case where the input weights are assigned to the domain values instead of the tuples. All algorithms and complexity results also apply to that case since there is a simple linear-time reduction from domain-value weights to tuple weights so that the weights of the query answers remain the same. The reduction assigns each variable to one of the atoms that it appears in and then computes the weight of a tuple by aggregating the weights of the attribute values that have weights assigned to them. For the lower bound of Theorem 8, note that the hardness proof relies only on the fact that ranked enumeration is at least as hard as unranked enumeration regardless of the ranking function.

### 8 EXPERIMENTS

Since asymptotic complexity only tells part of the story, we compare all algorithms in terms of actual running time. The code to reproduce the experiments can be found in our project page [https://northeastern-datalab.github.io/anyk/](https://northeastern-datalab.github.io/anyk/).

**Algorithms.** We implement our proposed algorithms in the same Java 11 environment. We compare: (1) **ANYK-PART**, (2) **ANYK-REC**, (3) **ANYK-PART+**, and (4) **JOINFIRST** which computes the full result using the Yannakakis algorithm [145] followed by sorting. Both **ANYK-PART** and **ANYK-PART+** use the **QUICK** variant which relies on Incremental Quicksort [120] to evaluate the successor function. We also evaluate the performance of two database systems: (5) **PSQL** is PostgreSQL 9.5.20, and (6) **SYSTEM X** is a commercial database system.

**Queries.** We conduct our study on **acyclic** queries, since handling cyclic queries is an issue orthogonal to our work (see Section 6.3). We use three types of queries, parameterized by their number of atoms $\ell$.

1. **A path** query $Q_{\text{PF}}(x) := R_1(x_0, x_1), R_2(x_1, x_2), \ldots, R_\ell(x_{\ell-1}, x_\ell)$ joins the relations in a chain and is a case of serial DP (Section 4.2).

2. **A star** query $Q_{\text{SF}}(x) := R_1(x_0, x_1), R_2(x_0, x_2), \ldots, R_\ell(x_0, x_\ell)$, where all relations join on $x_0$. Even though we could construct a path-structured join tree for this query, we instead use a join tree with minimal depth and maximal degree so that it is structurally the opposite of the path query and requires treatment as a tree (T-DP).

3. **A branch** query $Q_{\text{BF}}(x) := R_1(x_0, x_1), R_2(x_1, x_2), \ldots, R_{\ell-1}(x_{\ell-2}, x_{\ell-1}), R_\ell(x_{\ell-2}, x_\ell)$ is similar to a path, but requires treatment as a tree (T-DP) due to a single branch in the join tree. It has a low serial-decomposition width $w_{sd} = 2$ (by placing the last 2 atoms together in the serial decomposition).
Datasets used for experiments with real data.

For real datasets, query atoms refer to the same relation. We set the ranking function to sum.

**Synthetic data.** Our synthetic data generator creates input with regular structure and tunable parameters. We generate relations $R_i(A_{i1}, A_{i2}, W_i)$, $i \geq 1$, where the columns $A_{i1}, A_{i2}$ are used for joins, while $W_i$ contains tuple weights. The join distribution is controlled by the sampling process of the values that populate the $A_{i1}, A_{i2}$ columns. For a *Uniform* distribution, we draw integers from $[0, |\text{dom}|)$ uniformly at random with replacement for a given value $|\text{dom}|$, which defaults to $n/10$. (This means that a tuple joins, in expectation, with 10 others in a joining relation.) For a *Gaussian* distribution, we round to integers the values drawn with a mean of 0 and a given standard deviation, which defaults to $n/10$. Tuple weights are real numbers uniformly drawn from $[0, 10000]$.

**Real Data.** We use real-world networks where the output size of the joins typically exceeds the input size. In *Bitcoin* [99, 100], edges have weights that represent the degree of trust between users. *Twitter* [148] models followership among users as edges. Edge weight is set to the sum of the PageRanks [35] of both endpoints. To control input size, we retain edges between users whose IDs are below a certain threshold. We also use two smaller networks where computing the entire join output is feasible. *Friendship* [7, 101, 109] is created from a student survey in which each participant indicated their best friends. In *Foodweb* [66, 101, 137], an edge indicates that a taxon uses another taxon as food with a given trophic factor. Figure 12 summarizes relevant statistics. Note that the size of our relations $n$ is equal to the number of edges.

**Implementation details.** All experiments are conducted on a machine with Ubuntu Linux 20.04.2, an Intel Xeon E5-2643 CPU, and 128 GB RAM, from which 100GB are allocated to the JVM. Each measurement is the median of at least 20 separate JVM invocations. To avoid the non-deterministic nature of garbage collection, we try to stay below the available memory limit in each experiment. As an optimization to all our algorithms, we initialize their data structures lazily when they are accessed for the first time. For example, in ANYK-rec, we do not create the priority queue Choices$_1(v)$ for a node $v$ until this node is visited for the first time. This can significantly reduce $TT(k)$ for small $k$. Notice that our complexity analysis in Section 5.4 assumes constant-time inserts for priority queues, which is important for algorithms that push more elements than they pop per iteration. This bound is achieved by data structures that are well known to perform poorly in practice [48, 102]. Instead, we use the standard Java library binary heaps.

**Tuning the Database Systems.** For PSQL, following standard methodology [18], we remove the system overhead as much as possible and make sure that the input relations are cached in memory by timing the second of two runs. We turn off fsync, synchronous_commit, full_page_writes, we set bgwriter_delay to the maximum (10 sec), bgwriter_lru_maxpages to 0, checkpoint_timeout to 1 hour and max_wal_size to a large value (1000 GB). We also give shared_buffers and work_mem 32 GB and set the isolation level to the lowest possible (READ UNCOMMITTED). We follow a similar approach for System X, and also tune it for in-memory computation.

**Methodology.** First, we evaluate the different approaches on top- $k$ queries where the value of $k$ is fixed to a relatively small value ($k = 10^3$). In this setting, we vary different parameters such as data size, query size, and join distribution to show the advantage of any- $k$ algorithms even when enumeration is not required. For these experiments, we only show ANYK-PART+ as its performance is similar to ANYK-PART and ANYK-rec because the enumeration phase is dominated.
by the preprocessing (i.e., Dynamic Programming). Then, we move on to a study of any-\(k\) where we compare the different approaches on \(\text{TT}(k)\) as the value of \(k\) changes. We look into the regime where the enumeration continues until the last query answer and also into a regime where relatively fewer answers are returned (\(k = n\)), yet the enumeration cost is not dominated by the preprocessing.

8.1 Results for Fixed \#Answers \(k\)

Figure 13 shows the time taken to retrieve the top \(k = 10^3\) answers for different settings. In the real datasets \textsc{Bitcoin} and \textsc{Twitter}, increasing the size of the data (Figures 13a to 13d) or increasing the size of the query (Figures 13e to 13h) results in the \textsc{JoinFirst} approach becoming infeasible, either running out of memory or exceeding the timeout of 1 hour. \textsc{PSQL} and \textsc{System X} exhibit a similar behavior since they implement a similar evaluation strategy.

We observe a similar effect with synthetic data if we vary the join distribution (Figures 13i to 13l). Increasing the size of the join output, either by decreasing the domain size for the Uniform distribution or by decreasing the standard deviation for the Gaussian distribution (while maintaining a fixed relation size \(n = 10^6\)), results in the other approaches faltering, while any-\(k\) is stable or even faster. On the contrary, a relatively small join output size, e.g., when domain size equals relation size \(n\), favors the \textsc{JoinFirst} strategy and the database systems that implement it. The overhead of any-\(k\) in these cases is not excessive; it remains within a factor of 2 from \textsc{JoinFirst} in all cases.

![Fig. 13. Experiments with fixed \(k = 10^3\).](image-url)
Fig. 14. Experiments with varying \( k \), enumerating all query answers on Synthetic Uniform data (\(|\text{dom}| = n/10\)). The value of \( n \) is chosen so that output size is approximately the same (\( \sim 10^7 \)) for different query sizes.

Fig. 15. Experiments with varying \( k \), enumerating all query answers on real data.

8.2 Results for Varying \#Answers \( k \)

We report the number of query answers returned in ranked order (y-axis) over time (x-axis). In Figure 14, we test the different approaches when all query answers are returned using synthetic data, where we control the input to achieve a join output size in the order of \( 10^7 \). All three any-\( k \) algorithms return the first few million answers much faster in all cases. For path queries with \( \ell > 3 \) (Figures 14b to 14d), \textsc{AnyK-Rec} not only returns the first answers quickly, but also finishes the enumeration before \textsc{JoinFirst}. The advantage becomes more evident as we increase the size of the query because we keep the output size fixed, which results in denser joins where the opportunities for memoization are greater. \textsc{AnyK-Part+} exhibits the same behavior. The situation is different for
star queries (Figures 14e to 14h), where we use a depth-2 join tree and these two algorithms do not exploit the memoization of the suffix orders. ANYK-REC is the slowest for TTL out of the three any-k options, while the overhead of ANYK-PART+ compared to ANYK-PART is not significant. Interestingly, ANYK-PART+ retains the TTL advantage over JOINFIRST for branch queries (Figures 14k and 14l) whose join tree is very close to a path.

We next test the entire output enumeration on the two smaller real datasets. We observe that the TTL advantage over JOINFIRST only occurs for FOODWEB (Figure 15a) and not for the less-dense FRIENDSHIP (Figure 15c). Similarly to the synthetic case, ANYK-REC underperforms for star queries (Figures 15b and 15d). SYSTEM X did not manage to run in-memory for the star query on FOODWEB because of buffer size limits.

Finally, we conduct experiments where the join output is very large and ask only for $k = n$ answers. Figure 16 shows the results for BITCOIN and TWITTER, noting the output size of the join in each case. In this regime, JOINFIRST, PSQL, and SYSTEM X exceed the timeout and are not depicted. ANYK-PART wins over ANYK-REC for these smaller values of $k$. ANYK-PART+ is close to ANYK-PART+ and in some cases overtakes it thanks to its memoization strategy (Figures 16c and 16d).

**Summary.** JOINFIRST is impractical for joins with large output and is outperformed by any-k by orders of magnitude. PSQL and SYSTEM X follow a strategy similar to JOINFIRST. ANYK-REC is usually the fastest for path queries when the enumeration is carried out to the end, and may even beat JOINFIRST for TTL, but does not work as well for non-path-like queries. ANYK-PART is marginally faster when fewer answers are returned. ANYK-PART+ achieves the best of both worlds, having a competitive TTL for all queries, and a similar or better performance to ANYK-PART for small $k$.

9 RELATED WORK

**Top-k joins.** Top-k queries have received significant attention in the database community [8, 10, 19, 36, 46, 88, 89, 122, 131, 132]. Much of that work relies on the value of $k$ given in advance in order to prune the search space. Besides, the cost model introduced by the seminal Threshold Algorithm
Any-k Algorithms for Enumerating Ranked Answers to Conjunctive Queries

(TA) [62] only accounts for the cost of fetching input tuples from external sources. Later approaches such as J* [111], Rank-Join [87], LARA-J* [105], and a-FRPA [65] generalize TA to complex joins like the ones we consider, yet still focus on minimizing the number of accessed input tuples. Thus, they are sub-optimal when accounting for all steps of the computation, including intermediate result size (see Appendix C). In a recent tutorial [135], we examined the relationship between top-k joins and the ranked enumeration paradigm discussed in this work. Bonifati et al. [29] study the impact of thresholds (i.e., LIMIT clauses) in query evaluation, but without ranking.

Optimal join processing. Acyclic Boolean CQs can be evaluated optimally in \(O(n)\) data complexity by the Yannakakis algorithm [145]. The AGM bound [15], a tight upper bound on the output size for full CQs, motivated worst-case optimal join algorithms [14, 112, 114, 115, 140] and was generalized to take into account functional dependencies [73] and degree constraints [5, 6]. The complexity of answering cyclic Boolean CQs has improved over the years thanks to decompositions that achieve ever smaller width-measures [47, 74, 123], such as fractional hypertree width [78] and submodular width [107]. Work has also been done in the direction of achieving instance-optimality [12, 96, 113]. Our approach can leverage prior work on decompositions to obtain the top-ranked answer in the same time bound as the Boolean CQ. We also show that it is possible to achieve better TTL complexity than sorting the output of a join algorithm if the ranking is integrated into the join.

Factorization. Our efficient encoding of the CQ answers as a (T-)DP instance with intermediate nodes in-between bicliques [63] is a type of factorization, which has been studied systematically in the context of factorized databases [18, 94, 117, 118, 119]. The main idea is to represent the query answers compactly while supporting different types of tasks, such as enumeration [93, 119], aggregation [2, 4], training machine learning models [3, 126, 129], uniform sampling [43] and directly accessing ranked answers [42]. We build on that body of work, showing that ranked enumeration can also be performed efficiently for certain ranking functions. Through factorization, CQs with “short” inequality predicates (i.e., between adjacent relations in the join tree) can be reduced to CQs without inequalities over (poly)logarithmically larger relations [134]; this extends the cases where our ranked enumeration techniques apply.

Unranked enumeration. Significant effort has been made to identify the classes of queries whose answers can be enumerated in arbitrary sequence with linear preprocessing and constant delay in data complexity [13, 16, 23, 41, 86, 127, 141]. Much of the focus is on the less-practically-relevant measure of delay (see Section 2.3), sometimes exploring its tradeoff with preprocessing [52, 54] instead of focusing on \(TT(k)\). Constant-delay enumeration is possible for cyclic CQs with a higher preprocessing cost by decomposing them into acyclic CQs [25, 119]. Our work shows that adding ranking to enumeration of CQ answers (with an s-monotone ranking function) only incurs a logarithmic cost factor in data complexity. On the other hand, unranked enumeration has been shown to be tractable for CQs with “long” inequality predicates [142] (i.e., between non-adjacent relations in the join tree), where it is unclear whether ranked enumeration can be performed efficiently.

Ranked enumeration. The ranked enumeration paradigm can be seen as sorting done incrementally. For a set of elements that is given, Paredes and Navarro [120] propose a simple modification to the well-known quicksort algorithm [84] for ranked enumeration, which we leverage as a subprocedure for the Quick variant of ANYK-PART. For graph-pattern queries that are less general than CQs, Chang et al. [45] introduce the idea behind the Lazy variant of ANYK-PART, while Yang et al. [143] describe ALL. The idea of applying our any-k algorithms to cyclic CQs using (multiple) hypertree decompositions appeared in a very preliminary version of this work [144]. The generalization of the REA algorithm [91] from paths to trees is achieved by ANYK-REC, as well as the concurrent work of Deep and Koutris [55]. In comparison with that work, we (1) reveal the general DP structure of the
problem and deep relationships to classic work on $k$-shortest paths, (2) provide a more fine-grained analysis that shows important differences between the algorithms and surprising results such as asymptotically faster time-to-last than sorting, (3) propose an asymptotically better algorithm that also applies to the underlying problem of path enumeration in a DAG, (4) include CQs with projections in our study, and (5) implement and experimentally compare different algorithms. On the other hand, Deep and Koutris [55] define and support monotonicity properties that are sensitive to the structure of the join tree (or the tree decomposition); they allow for example the ranking function $f(x, y) + f(z, u)$ for arbitrary $f$ by always placing $x, y$ together in one bag and $z, u$ together in another bag of the decomposition. For CQs with arbitrary projections (i.e., not necessarily free-connex), Bagan et al. [16] provide an algorithm with linear preprocessing and linear delay. For arbitrary projections and s-monotone ranking functions, Kimelfeld and Sagiv [97] give one with linear preprocessing and polynomial delay. Interestingly, a closer look at these algorithms reveals that the former can support any lexicographic order, while the latter has linear (instead of polynomial) delay. For lexicographic orders and sum-of-weights, these guarantees were later matched by Deep et al. [53]. The algorithm of Kimelfeld and Sagiv [97] uses the Lawler-Murty procedure like ANYK-PART but does not exploit the DP structure of the problem (i.e., the deviations) since that is not possible for arbitrary CQs. ContourJoin [57] is a ranked-enumeration algorithm for binary joins, but, similarly in spirit to top-$k$ joins, does not offer any non-trivial guarantees in the RAM cost model. Bourhis et al. [31] study ranked enumeration for queries in Monadic Second Order (MSO) logic which is more general than CQs, but the word data model is simpler since it cannot describe arbitrary relations. Ranked enumeration under updates has been considered by Berkholz et al. [24] for probabilistic databases.

**k-shortest paths.** The literature is rich in algorithms for finding the $k$-shortest paths [60] with the sum-of-weights model (i.e., the tropical semiring) where typically the graph can be cyclic. Many variants focus on the loopless version where a path cannot visit a node twice [44, 83, 95, 146], a property always satisfied in DAGs. Hoffman and Pavley [85] introduce the idea of deviations (see Section 5.1). Building on that idea, Dreyfus [58] proposes an algorithm that can be seen as a modification to the procedure of Bellman and Kalaba [21]. The Recursive Enumeration Algorithm (REA) [91] uses the same set of equations as Dreyfus, but applies them in a top-down recursive manner. Our ANYK-rec builds upon REA. To the best of our knowledge, prior work has ignored the fact that this algorithm reuses computation in a way that can asymptotically outperform sorting in some cases. In another line of research, Lawler [103] generalizes an earlier algorithm of Murty [110] and applies it to $k$-shortest paths. Aside from $k$-shortest paths, the Lawler-Murty procedure has been widely used for a variety of problems in the database community [69]. This procedure and the Hoffman-Pavley deviations are the main ingredients of ANYK-PART. Eppstein’s algorithm [59] has the best-known complexity $TT(k) = O(N + k \log k)$, which does not depend on the $O(\ell)$ length of returned paths. To achieve that, it returns an $O(1)$ implicit representation of a path, i.e., its weight together with a pointer to the start of the path from which it can be reconstructed. For the task of explicitly enumerating paths in a DAG, an $O(k\ell)$ term is unavoidable and our ANYK-PART achieves the same complexity as Eppstein’s algorithm [59] with a much simpler construction; ANYK-PART+ asymptotically improves upon this. In Appendix F, we further discuss the literature on $k$-shortest paths.

**X+Y.** The open X+Y sorting problem asks whether the pairwise sums of two $n$-size sets can be sorted faster than the naive $O(n^2 \log n)$ algorithm [34, 56, 68, 81]. This corresponds precisely to sorting the answers to $Q_{XY}(x, y) := R_1(x), R_2(y)$. The problem of selection where only the $k$th answer needs to be returned has also been studied for X+Y [67] as well as for more sets $X_1 + X_2 + \cdots + X_m$ [92]. Our work shows that it is possible to improve the logarithmic factor in sorting $X_1 + X_2 + \cdots + X_m$. 
CSPs and Homomorphisms. The connections between conjunctive query evaluation, constraint satisfaction (CSP), and homomorphisms between relational structures are well known [64, 80, 98, 138]. Ranked enumeration using the Lawler-Murty procedure in a way similar to Kimelfeld and Sagiv [97] for CQs has also been proposed for CSPs [72, 76]. Our results apply directly to these problems as well; for example, they generalize minimum-cost homomorphism problems [79, 82] to ranked enumeration of homomorphisms.

10 CONCLUSIONS AND FUTURE WORK

We described a framework for ranked enumeration of answers to full acyclic CQs and, by extension, free-connex CQs and cyclic CQs. More generally, it applies to a wide range of DP problems that can be expressed through semirings, such as DNA sequence alignment [50] or Viterbi decoding [128]. Given that special cases of this problem had been studied in isolation in the past, our general framework aims to avoid piecemeal rediscovery in the future. The algorithms we proposed, such as anyK-part+ which asymptotically dominates all previously known alternatives, directly apply to all these problems and improve the state-of-the-art upper bounds.

Although our work thoroughly resolves ranked enumeration for cases where a DP structure can be identified in a problem, it remains to be seen whether ranked enumeration can be done efficiently in cases where this structure is not clear, as is the case in Unions of CQs [41] or CQs with “long” inequality predicates [142]. Furthermore, our analysis could be extended by going beyond the worst-case and studying the average-case behavior [125] of our algorithms. On the practical side, we showed the advantage of any-k over the traditional DBMS approach, but we have left for future work to determine the best way to integrate our techniques with a DBMS query processor and optimizer. To that end, an external-memory adaptation and analysis of our algorithms might be necessary.

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## A NOMENCLATURE

| Symbol | Definition |
|--------|------------|
| \([m]\) | Integers \(\{1, \ldots, m\}\) |
| \([m]_0\) | Integers \(\{0, \ldots, m\}\) |
| \(D\) | Database instance |
| \(n\) | Maximum size of a relation |
| \(Q\) | Conjunctive Query \(Q(X) := R_1(X_1), \ldots, R_\ell(X_\ell)\) |
| \(X, Y, Z\) | Lists of variables |
| \(x, y, z\) | Variables |
| free\((Q)\) | The free variables of CQ \(Q\) |
| \(R_1, \ldots, R_\ell\) | Relations |
| \(t\) | Number of atoms of a CQ |
| \(\alpha\) | Maximum arity of an atom of a CQ |
| dom | The domain of the relations |
| \(Q(D)\) | Set of query answers of \(Q\) over \(D\) |
| \(q \in Q(D)\) | A query answer |
| \(\oplus, \otimes\) | Operators of a commutative selective dioid |
| diam\((Q)\) | Diameter of the hypergraph of CQ \(Q\) |
| \(G(V, E)\) | Graph with nodes \(V\) and edges \(E\) |
| \(N\) | Number of nodes in a graph |
| \(s, t\) | Source node \(s\), target node \(t\) |
| \(S_1, \ldots, S_\ell\) | Stages of a node-partitioned graph |
| \(w\) | Weight function for input tuples/query answers/graph edges/paths |
| \(\pi_k(v)\) | Weight of \(k^{th}\) shortest path from \(v\) |
| \(\Pi_k(v)\) | \(k^{th}\) shortest path from \(v\) |
| \(\langle v_1, \ldots, v_\lambda \rangle\) | A path of length \(\lambda\) |
| \(\text{suc}(v_i, v_{i+1})\) | Next best child of \(v_i\) after \(v_{i+1}\) according to the optimal weight needed to reach \(t\) |
| Choices\(_s(v)\) | The paths compared by DP at node \(v\): thy go to a child of \(v\) and then optimally to \(t\) |
| \(E_{pc}\) | Edges (decisions) from stage \(V_p\) to stage \(V_c\) in T-DP |
| \(C(V_p), C(v_p)\) | Indexes of children stages of \(V_p\) or \(v_p \in V_p\) in T-DP |
| \(pr(V_c, v_c)\) | Index of parent stages of \(V_c\) or \(v_c \in V_c\) in T-DP |
| \([\Vert V_i \Vert], [\Vert v_i \Vert]\) | Index of all stages inn the subtree of \(V_i\) or \(v_i \in V_i\) excluding \(i\) in T-DP |
| \(w_{sd}\) | Width of serial decomposition of a T-DP problem |

## B ORDER-BY QUERIES IN FACTORIZED DATABASES

Factorized databases (FDBs) [18, 117, 118, 119] support constant-delay enumeration of query answers according to a desired lexicographic order on the attributes [17]. Lexicographic orders are a special case of the ranking function considered in this paper and our approach supports them (see Section 7.4.1) but with logarithmic delay. Here we look closer at the differences between the two approaches for this special case of lexicographic orders and show that our approach can be asymptotically better in certain cases despite the logarithmic delay.

First, we provide a very short description of the main idea behind factorized databases and we refer the reader to the original works for a deeper understanding. To achieve a succinct representation, factorized databases repeatedly apply the distributivity law in an order described by a tree structure whose nodes are the attributes [118]. Intuitively, if \(X\) is the attribute of a node of the tree and \(\text{anc}(X)\) are its ancestor attributes, then every value \(x \in X\) is represented at most once for each combination of values of \(\text{anc}(X)\). D-representations [119] provide further succinctness by making the dependencies of each attribute in the tree explicit. This means that some attributes in \(\text{anc}(X)\)
might not actually determine what the possible $X$ values are. Truly dependent ancestor attributes of a node are denoted as $\text{key}(X)$. Each value $x \in X$ is then represented at most once for each combination of values of $\text{key}(X)$.

Such a factorized representation allows constant-delay unranked enumeration of the query answers. Yet for specific lexicographic orders, there are two conditions that have to be met: (i) the order-by attributes have to be “at the top” of the tree and (ii) the tree order has to agree with the lexicographic order. If the tree order is not in agreement (e.g., we want $A$ before $B$ but $A$ is a child of $B$ in the tree), then the whole representation has to be restructured. The restructuring operation takes an input representation and transforms it to an output representation consistent with the lexicographic order in time linear (ignoring log factors) in the input and output representation sizes. However, the output representation itself could be very large. We next illustrate the simplest example where an ill-chosen lexicographic order results in a quadratic representation for a simple binary join.

**Example 40 (Lexicographic orders).** Consider the 2-path query $Q_{P2}(A,B,C) := R(A,B), S(B,C)$. As usual, $n$ is the maximum number of tuples in a relation. Ideally, we would want to factorize it using a tree that has $B$ as the root and $A, C$ as its children. That way, every $A$ and $C$ value in the query result would be represented independently for each $B$ value. However, for the lexicographic order $A \rightarrow C \rightarrow B$ this factorization is not in agreement since $B$ comes after $A$ and $C$. The only possible tree that satisfies the condition (ii) above is a path from $A$ to $C$ to $B$. Note that the tree with $A$ as the root and $B, C$ as the children is not possible because of the “path condition” in factorized databases: attributes that belong to the same relation ($B$ and $C$ here) are dependent and have to lie in the same root-to-leaf path. In the only valid tree, $\text{key}(B) = \{A, C\}$.

According to Lemma 7.20 in [119], there exist arbitrarily large databases such that the number of $B$ values in the representation is at least $n^{p^*} (B: \text{key}(B))$, where $p^*$ is the fractional edge cover of the query, thus $\Omega(n^2)$.

Figure 17 presents a concrete instance where this happens. For this database, the single $B$-value 1 will be represented once for each combination of $A, C$ values and there are $n^2$ of them. In contrast, our approach begins the enumeration after only linear time preprocessing. Thus in this case, the preprocessing step of FDBs takes $O(n^2)$ after which results can be enumerated in constant time. In contrast, our approach has $\text{TT}(k) = O(n + k \log k)$.

**C  TOP-K JOINS IN OUR COST MODEL**

Consider the database $I_2$ from Fig. 18 with $\ell = 3$ relations and $n = 10$ tuples per relation. The top output tuple is marked in blue; it consists of the lightest tuples from the first $\ell - 1$ relations and the heaviest tuple from $R_\ell$. $J^*$ [111] and Rank-Join [87] access the tuples in the input relations by decreasing weight. Their cost model takes into account only the number of database accesses, hence they try to minimize the depth up to which the sorted relations have to be accessed in order to find the top-k results. In this case, both $J^*$ and Rank-Join will consider the $(n - 1)^{\ell - 1}$ combinations between
### D MINIMAL EXAMPLE OF A NON-COMMUTATIVE SELECTIVE DIOID

In Section 7.1, we mentioned that commutativity of “multiplication” in selective dioids is an important algebraic property for us since non-commutative structures would not allow any order of aggregation, restricting the space of possible join trees for CQs. However, all typical examples of selective dioids on infinite domains are commutative, and it can be difficult to find an example of a non-commutative structure. Figure 19 show such an example on a domain of 5 elements that we identified with the Mace4 tool [108].

### E ANYK-PART VARIANTS

As we discussed in Section 5.1, the specific implementation of the successor function $\text{succ}(v_i, v_{i+1})$ for an edge $(v_i, v_{i+1})$ gives rise to a number of variants that have appeared in prior work. In the conference version of this article [133], we analyzed and compared these in more detail. They can broadly be categorized as “strict”, which follow the successor definition adopted in the main body of our paper, and “relaxed” which follow a more general definition.

#### Strict approaches.

A natural implementation of the successor function returns precisely the next-best choice.

Eager Sort (EAGER): Since a state might be reached repeatedly through different prefixes, it may pay off to pre-sort all choice sets by weight and add pointers from each choice to the next one in sort order. Then $\text{Suc}(v_i, y)$ returns the next-best choice at $v_i$ in constant time by following the next-pointer from $v_{i+1}$.
Any-k Algorithms for Enumerating Ranked Answers to Conjunctive Queries

| Algorithm | \(T(k)\) | Delay \(k\) | MEM \(k\) |
|-----------|----------|-------------|----------|
| EAGER     | \(O(|G| \log |G| + k(\log k + t))\) | \(O(\log k + t)\) | \(O(|G| + kt)\) |
| LAZY      | \(O(|G| + k(\log k + t))\) | \(O(\log k + t + \log N)\) | \(O(|G| + ktf)\) |
| QUICK     | \(O(|G| + k(\log k + t))\) | \(O(\log k + t + \log N)\) | \(O(|G| + ktf)\) |
| ALL       | \(O(|G| + k(\log k + tN))\) | \(O(\log k + tN)\) | \(O(|G| + \min\{kN, r\} t)\) |
| TAKE2     | \(O(|G| + k(\log k + t))\) | \(O(\log k + t)\) | \(O(|G| + ktf)\) |

Fig. 20. Complexity of ANYK-PART variants for ranked-enumeration of s-t paths in a DAG. Best performing algorithms are colored in green. |G| is the graph size, \(N\) is the number of nodes, \(r\) is the total number of paths, and \(t\) is the maximum length of a path. The guarantees of QUICK are randomized and hold in expectation.

Lazy Sort (LAZY): To lower pre-processing cost, we can leverage the approach Chang et al. [45] proposed in the context of graph-pattern search. Instead of sorting a choice set, it constructs a binary heap in linear time. Since all but one of the successor requests in a single repeat-loop execution are looking for the second-best choice, the algorithm already pops the top two choices off the heap and moves them into a sorted list. For all other choices, the first access popping them from the heap will append them to the sorted list that was initialized with the top-2 choices. As the algorithm progresses, the heap of choices gradually empties out, filling the sorted list and thereby converging to EAGER.

QUICK: Using the Incremental Quicksort algorithm [120], we can sort the children of each node incrementally and obtain the same guarantees as LAZY in expectation.

Relaxed approaches. Instead of finding the single true successor of a choice, what if the algorithm could return a set of potential successors? Correctness is guaranteed, as long as the true successor is contained in this set or is already in Cand. (Adding potential successors early to Cand does not affect correctness, because they have higher weight and would not be popped from Cand until it is “their turn.”) This relaxation may enable faster successor finding, but inserts candidates earlier into Cand.

All choices (ALL): This approach is based on a construction that Yang et al. [143] proposed in the context of graph-pattern search. Instead of trying to find the true successor of a choice, all but the top choice are returned by Suc. While this avoids any kind of pre-processing overhead, it inserts \(O(n)\) potential successors into Cand.

TAKE2: This approach aims to keep pre-processing at a minimum (like ALL), but also return a small number of successors fast (like EAGER). To achieve this, we organize each choice set as a binary heap. In this tree structure, the root node is the minimum-weight choice and the weight of a child is always greater than its parent. Function \(\text{Suc}(u_i, u_{i+1})\) returns the two children of \(u_{i+1}\) in the tree. Unlike LAZY, we never perform a pop operation and the heap stays intact for the entire operation of the algorithm; it only serves as a partial order on the choice set, pointing to two successors every time it is accessed. Also note that the true successor does not necessarily have to be a child of node \(u_{i+1}\). Overall, returning two successors is asymptotically the same as returning one and heap construction time is linear [49], hence this approach achieves lower delay.

In Figure 20, we summarize the differences in complexity among these variants. We remind the reader that in the analysis in the main body of this article, we assume the LAZY variant which is simple and achieves the best \(TT(k)\). As we argue in Section 2.3, the lower delay of TAKE2 may not be practically relevant.

F MAP OF RELATED WORK FOR k-SHORTEST PATHS

Figure 21 depicts important ideas and the connections between them in the history of the k-shortest paths problem on a DAG that we study in this work.

\(^{17}\)During each execution of the repeat-loop, only the first iteration of Line 12 looks for a lower choice.
Fig. 21. Map of related work for the problem of $k$-shortest paths on a DAG. Arrows indicate the progression of ideas historically. Within each node, we note the application of these algorithms to different areas. Implicit path enumeration [59] (left) refers to the problem where an $O(1)$ representation of each path is returned and the lower bound of $\Omega(k\ell)$ no longer applies. In this work, we focus on explicit path enumeration (right) where each path has to be returned as a list of edges. Most algorithms in the literature operate on a sum-of-weights model and can be generalized to $s$-monotone ranking functions (middle). The Lawler-Murty procedure [110, 103] only requires an (efficient) sub-routine for solving the top-1 problem (top). Our ANYK-PART+ algorithm requires the stronger ss-monotone property (bottom).