Cumulative Dragging - An Intrinsic Characteristic of Stationary Axisymmetric Spacetime

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Abstract

The Cumulative Drag Index defined recently by Prasanna \cite{1} has been generalised to include the centrifugal acceleration. We have studied the behaviour of the drag index for the Kerr metric and the Neugebauer-Meinel metric representing a self-gravitating rotating disk and their Newtonian approximations. The similarity of the behaviour of the index for a given set of parameters both in the full and approximated forms, suggests that the index characterises an intrinsic property of spacetime with rotation. Analysing the index for a given set of parameters shows possible constraints on them.

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1 Introduction

The phenomenon of rotation plays a very important role in almost all classes of objects that encompass our physical universe. Particularly in the discussion of Inertia, rotational features characterise global effects on local physics as implied by Mach’s principle. Recently, Prasanna [1] has defined a new parameter called the *Cumulative Drag Index* for stationary axisymmetric spacetimes, using the notion of inertial forces within the framework of general relativity. The index, defined for particles in circular orbit along the trajectory on which the centrifugal acceleration is zero, characterises the intrinsic feature of rotation through the drag induced on both co-rotating and counter-rotating particles. However, for practical applications, it would be useful to generalise the drag index to include the centrifugal acceleration also. A few years ago, astronomers discovered two co-spatial stellar disks in the galaxy NGC-4550, one orbiting prograde and the other retrograde with respect to the galactic nucleus, in the core of the Virgo cluster [2,3]. Bicak and Ledvinka [4] have tried to construct sources for the Kerr geometry using counter-rotating thin disks. If one considers the galactic nucleus as a black hole, one can then use the Kerr geometry for the outside and have counter-streaming jets outside the ergo-region. The presence of such co- and counter-rotating particle streams may perhaps be
characterised through the drag index, defined as

\[ C = \frac{(F_{cf} + F_{co} - F_{gr})}{(F_{cf} + F_{co} + F_{gr})}, \tag{1} \]

where \( F_{cf}, F_{co}, \) and \( F_{gr} \) denote, respectively, the centrifugal, the Coriolis and the gravitational accelerations acting on a particle in circular orbit in a stationary, axisymmetric gravitational field. Within the framework of general relativity this definition is unique, when one considers the spacetime expressed in the conformal 3+1 splitting with the four acceleration \( a_i \) being expressible covariantly as [5,1]

\[ a_i = -\nabla_i \phi + \gamma^2 V (n^k \nabla_k \tau_i + \tau^k \nabla_k n_i) + (\gamma V)^2 \tilde{\tau}^k \tilde{\nabla}_k \tilde{\tau}_i. \tag{2} \]

The various quantities on the r.h.s. of eq. (2) are as described below: \( n^i \) is the vector field corresponding to the zero angular momentum observers expressed in terms of the Killing vectors \( \eta^i \) (timelike) and \( \xi^i \) (spacelike) as

\[ n^i = e^{\phi} (\eta^i + \omega \xi^i), \quad \omega = -\langle \eta, \xi \rangle / \langle \xi, \xi \rangle, \tag{3} \]

and \( \phi \) is the scalar potential

\[ \phi = -\frac{1}{2} \ln(-\langle \eta, \eta \rangle - 2\omega \langle \xi, \eta \rangle - \omega^2 \langle \xi, \xi \rangle). \tag{4} \]

\( \tau^i \) is the unit spacelike vector orthogonal to \( n^i \) along the circle depicting the orbit of the particle with a constant speed \( V \), and \( \gamma = 1/\sqrt{1-V^2} \) is the
Lorentz factor. The particle four velocity, $U^i$, is thus expressible as

$$U^i = \gamma (n^i + V \tau^i),$$

and is also equal to $A(\eta^i + \Omega \xi^i)$, with $A$ the redshift factor defined as

$$A^2 = - (\langle \eta, \eta \rangle + 2\Omega \langle \xi, \eta \rangle + \Omega^2 \langle \xi, \xi \rangle)^{-1},$$

(5)

$\Omega$ being the angular velocity, $\Omega \tau^i = e^\phi (\Omega - \omega) \xi^i$. $\tilde{\tau}^i = e^{-\phi} \tau^i$ is the vector defined on the conformally projected 3-space having the positive definite metric $h_{ik} = g_{ik} + n_in_k$ and $\tilde{\nabla}_i$ is the covariant derivative with respect to $\tilde{h}_{ik} = e^{2\phi} h_{ik}$.

As shown earlier, for the metric

$$ds^2 = (g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2) + (g_{rr} dr^2 + g_{\theta\theta} d\theta^2)$$

(6)

the accelerations are given as

Gravitational: $$(F_{g\ell})_i = -\nabla_i \phi = \frac{1}{2} \partial_i \left\{ \ln \left[ \frac{g_{t\phi}^2 - g_{t\phi} g_{\phi\phi}}{g_{\phi\phi}} \right] \right\},$$

(7)

Coriolis: $$(F_{co})_i = \gamma^2 V n^j (\nabla_j \tau_i - \nabla_i \tau_j)$$

$$= -A^2 (\Omega - \omega) g_{\phi\phi} \partial_i (g_{t\phi} / g_{\phi\phi}),$$

(8)

Centrifugal: $$(F_{ct})_i = (\gamma V)^2 \tilde{\tau}^k \tilde{\nabla}_k \tilde{\tau}_i$$

$$= -\frac{A^2 (\Omega - \omega)^2}{2} g_{\phi\phi} \partial_i \left\{ \ln \left[ \frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{tt} g_{\phi\phi}} \right] \right\}.$$
2 Kerr spacetime

Taking now the specific example of Kerr spacetime

\[ ds^2 = -\left(1 - \frac{2mr}{\Sigma}\right)dt^2 - \frac{4mra}{\Sigma} \sin^2 \theta dtd\phi + \frac{B}{\Sigma} \sin^2 \phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \quad (10) \]

where \( B = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \), \( \Delta = r^2 + a^2 - 2mr \) and \( \Sigma = r^2 + a^2 \cos^2 \theta \), and considering a particle in circular orbit on the equatorial plane (\( \theta = \pi/2 \)), it can be seen that the index is

\[ C = \frac{m[r^4(r - 2m) + 2a^2r(r^2 - 8mr + 10m^2) + a^4(r - 6m)]}{\Delta(r^3 + a^2r + 2a^2m)(-m(1 - a\Omega)^2 + \Omega^2r^3)} \quad (11) \]

It is clear that, of the two infinities of the index, one appears at the event horizon (\( \Delta = 0 \)) while the other depends on both \( a \) and \( \Omega \) and appears for a given \( a \) and \( \Omega \) at \( r = [m(1 - a\Omega)^2/\Omega^2]^{1/3} \). Fig. (1) shows the nature of \( C \) at the three locations, \( r_{\text{php}} \) (the prograde photon orbit), \( r_{\text{phr}} \) (the retrograde photon orbit) and \( r_{\text{cfo}} \) (the orbit where the centrifugal force is zero).

While at \( r_{\text{php}} \) the index is positive only for a very small range of \( \Omega \) for counter-rotating particles (\( \Omega < 0 \)), at \( r_{\text{phr}} \) the index is positive for the same range of \( \Omega \) for co-rotating particles only. On the other hand, as was discussed earlier
in [1], at \( r_{\text{cf6}} \), the index is positive for both co- and counter-rotating particles, but again for a very narrow range of values of \( \Omega \) (Fig. (1c)). This change of behaviour of \( C \) at the two photon orbits arises due to the following reason: When the centrifugal force is not zero, the two zeros of the denominator of \( C \) outside the event horizon corresponding to the fixed value of \( r \) and \( a \), are at

\[
\Omega_1 = \frac{\sqrt{m}}{a\sqrt{m} + r^{3/2}} \quad \text{and} \quad \Omega_2 = \frac{\sqrt{m}}{a\sqrt{m} - r^{3/2}},
\]

(12)

and corresponding to these two \( \Omega \) values the numerator of \( C \) factors as

\[
2mr^{5/2}(-2a\sqrt{m} + 3mr^{1/2} - r^{3/2})[4a^2mr - (a^2 + r^2)^2]
\]

(13)

and

\[
2mr^{5/2}(2a\sqrt{m} + 3mr^{1/2} - r^{3/2})[4a^2mr - (a^2 + r^2)^2],
\]

respectively. Thus the zero at \( \Omega_1 \) cancels with the numerator at the prograde photon orbit, while the one at \( \Omega_2 \) cancels with the numerator at the retrograde photon orbit.

For many applications, often one takes the view that the linearized Kerr metric might be sufficient to incorporate the relativistic effects, when the body is slowly rotating. In order to examine this, let us consider the nature of \( C \) under this approximation. The three accelerations acting on a particle in circular orbit, approximated to terms linear in the Kerr parameter \( a \), are given by
\[ F_{gr} = \frac{m}{r^2} \left( 1 - \frac{2m}{r} \right)^{-1}, \]

\[ F_{co} = -\frac{6am\Omega}{(r^2 - 2mr - \Omega^2 r^4)}, \]

\[ F_{cf} = \frac{\Omega(r - 3m)(\Omega^3 r^6 - \Omega r^4 + 2m\Omega r^3 + 4mr - 8am^2)}{r(r - 2m)(r - 2m - \Omega^2 r^3)^2}, \]  \hspace{1cm} (14)

and thus the index is

\[ \ell C = \left[ -m(r - 2m) - 2am\Omega r - \Omega^2 r^3(r - 5m) \right. \]

\[ + 6am\Omega^3 r^3 + \Omega^4 r^6(r - 4m)/(r - 2m) \bigg] \bigg/ \left[ \begin{array}{c} m(r - 2m) - 2am\Omega r - \Omega^2 r^3(r - m) + 6am\Omega^3 r^3 + \Omega^4 r^6 \end{array} \right]. \]  \hspace{1cm} (15)

The very first change one notices is that, neglecting \( a^2 \) and higher order terms in \( a \), moves the infinity at the horizon to \( r = 2m \), as this would now represent the horizon, just like in the static case. Similarly, the orbit where the centrifugal acceleration is zero also coincides with that in the static case, viz., \( r = 3m \), for all \( \Omega \). Fig. (2) shows the index for the linearised version at the two photon orbits and at the orbit on which the centrifugal acceleration is zero. Comparison of these plots with those for the unapproximated \( C \) (Fig. (1)) clearly shows that for particles with angular velocity \( |\Omega| > 0.3 \), the behaviour is exactly same with or without the approximation at the photon orbits whereas at the orbit with \( F_{cf} = 0 \), the similarity is striking for all values of \( \Omega \).

As it would be almost impossible for particles to have a low value of \( \Omega \) close
to photon orbits (as they would be relativistic), the behaviour of the index shows that the linearisation approximation for the forces is amply justified for all practical purposes. However, as a matter of principle one finds that for very low values of $\Omega$ the behaviour of the linearised version differs from that for the exact version, the difference arising mainly because of the centrifugal acceleration being non-zero. If one considers the behaviour of $\ell \mathcal{C}$ as a function of $r$ for fixed $a$ and $\Omega$, it seems to be exactly like for $\mathcal{C}$, the expression without approximation.

Looking at the overall feature of the inertial accelerations it then seems that for understanding the frame dragging coming from rotation, for practical purposes of considering the forces, it may indeed be sufficient to calculate the gravitational, Coriolis and centrifugal acceleration in the linearised approximation, as given in eq. (14).

It is further interesting to consider the Newtonian limit of the accelerations with lowest order corrections in the centrifugal acceleration, as given by:

\[
\mathcal{F}_{gr} = \frac{m}{r^2}, \quad \mathcal{F}_{co} = -\frac{6am\Omega}{r^2},
\]

\[
\mathcal{F}_{cf} = -\Omega^2 (r - 3m) + \frac{4am\Omega}{r^2}.
\]  

(16)

With these expressions the drag index turns out to be

\[
NC = \frac{(r^3 - 3mr^2)\Omega^2 + 2am\Omega + m}{(r^3 - 3mr^2)\Omega^2 + 2am\Omega - m}.
\]  

(17)
Fig. (3) shows the plots of $N\mathcal{C}$ as a function of $\Omega$, for fixed $r$ and $a$ (3a,b) and as a function of $r$ for fixed $a$ and $\Omega$ (3c,d). As the Newtonian approximation can be valid only for larger values of $r$, it is clear that the index is positive for both co- and counter-rotating particles for $|\Omega| > 0.1$, independent of the values of $a$.

3 Disk spacetime

We shall next consider the behaviour of the index for the Neugebauer-Meinel metric [6] for a rigidly rotating disk of dust as given in the discussion of its dragging effects by Meinel and Kleinwächter [7]. The metric components and their first radial derivatives of significance are

$$
\Omega_d^2 g_{\phi\phi} = \frac{\mu}{2} - \left(\frac{z}{1+z}\right)^2, \quad \Omega_d g_{\phi t} = \frac{z}{1+z} - \frac{\mu}{2}, \quad g_{tt} = \frac{\mu}{2} - 1,
$$

$$
\Omega_d^2 r_0 g_{\phi\phi, r} = \frac{1 - z}{1+z}, \quad \Omega_d r_0 g_{\phi t, r} = \frac{z}{1+z}, \quad r_0 g_{tt, r} = -\mu. \quad (18)
$$

where $\Omega_d$ is the angular velocity of the disk, $z$ represents the relative redshift of photons from the centre of the disk measured at infinity, $\mu$ is a parameter defined through the relation $\mu = 2\Omega_d^2 r_0^2(1 + z)^2$ and $r_0$ represents the rim of the disk. Using eqs. 1, 7–9 and 18, the drag index may be obtained as:

$$
\mathcal{C} = \left[\mu^2 (\Omega_d^2 - \Omega^2) + 2\mu \left\{4(\mu - 1)\Omega_d^2 - 5\mu \Omega_d \Omega + \mu \Omega^2\right\} z \right.
$$

$$
+2 \left\{4(15\mu + 9\mu^2)\Omega_d^4 + 3\mu(4 - 5\mu)\Omega_d \Omega + \mu(6\mu - 1)\Omega^2\right\} z^2 \right.
$$

$$
2 \left\{2(15\mu + 4\mu^2)\Omega_d^4 - (8 - 26\mu + 15\mu^2)\Omega_d \Omega + \mu(7\mu - 8)\Omega^2\right\} z^3.
$$
\[(\Omega - \Omega_d)^2(8 - 14\mu + 5\mu^2)z^4]/\left[\mu(\Omega_d - \Omega)(1 + z)\{\Omega_d(1 + z) + \Omega(1 - z)\}\{\mu(1 + z)^2 - 2z^2\}\right], \quad (19)\]

where \(\Omega\) is the angular velocity of the particle.

It is clear from the above expression that the two zeros of the denominator, where the index blows up, correspond to the two circular geodesic orbits, as the sum of the forces acting on the particle is zero for these parameters. While the zero at \(\Omega = \Omega_d\), corresponds to the prograde geodesic, the one at \(\Omega = -\Omega_d(1 + z)/(1 - z)\), corresponds to the retrograde geodesic as also shown by Meinel and Kleinwächter. However, one finds that if \(z > 1\), the second zero occurs at a positive value of \(\Omega\) as shown in the Fig. (4d). Fig. (4) shows the plots of the index for four different values of \(\mu\).

4 Discussion

The presence of centrifugal acceleration does bring in a difference in the behaviour of the drag index at the two photon orbits, with the co-rotating ones having a positive value for a narrow range of \(\Omega\) at the retrograde photon orbit and the counter-rotating ones having a similar feature at the prograde photon orbit. However, if the black hole is slowly rotating \((a \ll 1, a^2\) negligible), then adopting the linearised version of the acceleration changes the behaviour of the index only for very low values of \(\Omega\) \((|\Omega| < 0.3)\) at the photon orbits, whereas
for higher values of $\Omega$ the behaviour resembles that of the full $C$ without any approximation. On the other hand, for given $a$ and $\Omega$, as a function of $r$ the radial distance parameter, the index shows no change with approximation, thus indicating that the drag index signifies something intrinsic to the space-time with rotation, as its behaviour for both co- and counter-rotating particles appears similar, from the point of view of a locally non-rotating observer.

In order to appreciate the significance of the index defined, one can consider its behaviour for the physical case of a rotating disk as depicted in Fig. (4). Comparing with the discussions of Meiner and Kleinwächter, we find that the constraints on the value of $z$ appearing through the parameter $\mu$, is well reflected in Figs. (4a,b,c). While Fig. (4a) shows clearly the existence of two geodesic orbits (pro and retro) for $\mu = 0.1$, Fig. (4b) shows for $\mu = 0.5$, the existence of only a prograde geodesic. In fact it is instructive to compare this with Fig. (1b) which corresponds to $C$ at the retrograde photon orbit for Kerr, and thus conclude that $\mu = 0.5$ corresponds indeed to the last possible retrograde geodesic orbit. Further, from the zeros of the denominator of $C$ (eq. 19), the appearance of the retrograde geodesic corresponds to the particle angular velocity $\Omega = -\Omega_d(1 + z)/(1 - z)$. In fact, for this value of $\Omega$ the numerator of the index factors as

$$8(2\mu - 1)\Omega^2_d z(1 + z)^2(\mu - z - \mu z)/(1 - z)^2,$$  

(20)
which clearly shows that the zero of the denominator cancels with the term 
\((2\mu - 1)\) in the numerator for \(\mu = 1/2\), exactly similar to what happens in 
the case of the Kerr metric. It is obvious that for the case \(z > 1\), i.e., \(\mu > 1.3519\), 
this value of \(\Omega\) becomes positive, the infinity of the index appearing 
for a prograde orbit. However, this orbit turns out to be spacelike \((v > c)\).

Thus, it appears that the counter-streaming particles would have a limitation 
from the point of view of their redshift and this could play an important 
role in the analysis of particle motion in the disk associated with the Virgo 
cluster. From this discussion it follows that given any stationary axisymmetric 
metric representing an astrophysical situation, one can straightaway determine 
constraints on possible physical parameters characterising the geodesic orbits 
through the behaviour of the drag index.

Though Newtonian physics does not directly predict anything regarding the 
nature of spacetime as influenced by rotation, it is amazing to see that the cu-
mulative drag index shows exactly similar behaviour in the Newtonian approx-
imation for the case of the full Kerr geometry as well as for the Neugebauer-
Meinel disk geometry (Fig. (5)), for all values of \(a\) and \(\Omega_d\), either prograde or 
retrograde. Thus it is clear that the cumulative drag index defined above char-
acterises an intrinsic property of spacetime with rotation, which goes beyond 
approximations. The fact that it is positive for both co- and counter-rotating 
particles having reasonable angular velocities, outside the ergo-region of a
black hole or in a self-gravitating disk, clearly supports the possibility of sus-
taining counter-rotating streams. In fact, this analysis points out a constraint
on the $z$ value for counterrotating streams to be $< 0.285$, which might be tested
in the case of streams encountered in the Virgo cluster. Eventhough the drag
index itself does not measure any observable quantity directly, it characterises
intrinsic rotation for stationary, axisymmetric spacetimes, be it in an empty
region (like Kerr geometry) or within a rotating disk (Neugebauer-Meinel class
of solutions), independent of approximations. If the metric potentials depend
upon a directly observable parameter then the behaviour of the index for dif-
ferent combinations of rotational parameters could possibly give constraints
on the physical parameter, which may be measurable. Further, as the free
orbits are defined through the equilibration of the forces acting on a particle,
the index would go to infinity and thus studying the behaviour of the index
in general yields information for orbits both free and otherwise. Thus for the
case of stationary axisymmetric metrics, the location of geodesics for given
angular velocity and rotational parameters may be identified by plotting the
drag index without necessarily solving the equations of motion.

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Figure Captions

**Figure 1:** $\mathcal{C}$ (cdi) as a function of $\Omega$ ($a = 0.5m$) at the prograde photon orbit (a), the retrograde photon orbit (b), and the orbit with $\mathcal{F}_{\text{cf}} = 0$ (c).

**Figure 2:** $\ell \mathcal{C}$ as a function of $\Omega$ ($a = 0.5m$) at the prograde photon orbit (a), the retrograde photon orbit (b), and the orbit with $\mathcal{F}_{\text{cf}} = 0$ (c).

**Figure 3:** $\mathcal{N} \mathcal{C}$ as a function of $\Omega$ for fixed $r$ and $a$ (a, b), and as a function of $r$ for fixed $a$ and $\Omega$ (c, d).

**Figure 4:** $\mathcal{C}$ as a function of $\Omega$ for a rotating disk of dust with (a) $\mu = 0.2$, $\Omega_d = 0.1$, (b) $\mu = 0.5$, $\Omega_d = 0.1$, (c) $\mu = 2$, $\Omega_d = 0.99$ and (d) $\mu = 1.9$, $\Omega = 0.1$.

**Figure 5:** Comparison of $\mathcal{C}$ and $\mathcal{N} \mathcal{C}$ for Kerr (a, b) and the rotating disk (c, d).
Fig. 1. $C$ as a function of $\Omega (a = 0.5m)$ at $r_{\text{php}}$ (a), $r_{\text{phr}}$ (b) and $r_{\text{cfo}}$ (c).
Fig. 2. $\ell C$ as a function of $\Omega$ ($a = 0.5m$) at $r_{\text{php}}$ (a), $r_{\text{phr}}$ (b) and $r_{\text{cfo}}$ (c).
Fig. 3. $N\mathcal{C}$ as a function of $\Omega$ (a, b) and as a function of $r$ (c, d).
Fig. 4. $C$ as a function of $\Omega$ for a rotating disk of dust

(a) (b) (c) (d)
Fig. 5. Comparison of $\mathcal{C}$ and $\mathcal{N}\mathcal{C}$ for Kerr (a, b) and the rotating disk (c, d).