Abstract

GRAPH BURNING asks, given a graph $G = (V, E)$ and an integer $k$, whether there exists $(b_0, \ldots, b_{k-1}) \in V^k$ such that every vertex in $G$ has distance at most $i$ from some $b_i$. This problem is known to be NP-complete even on connected caterpillars of maximum degree 3. We study the parameterized complexity of this problem and answer all questions arose by Kare and Reddy [IWOCA 2019] about parameterized complexity of the problem. We show that the problem is W[2]-complete parameterized by $k$ and that it does not admit a polynomial kernel parameterized by vertex cover number unless $\text{NP} \subseteq \text{coNP}/\text{poly}$. We also show that the problem is fixed-parameter tractable parameterized by clique-width plus the maximum diameter among all connected components. This implies the fixed-parameter tractability parameterized by modular-width, by treedepth, and by distance to cographs. Although the parameterization by distance to split graphs cannot be handled with the clique-width argument, we show that this is also tractable by a reduction to a generalized problem with a smaller solution size.

1 Introduction

Bonato, Janssen, and Roshanbin introduced GRAPH BURNING as a model of information spreading. This problem asks to burn all the vertices in a graph in the following way: we first pick a vertex and set fire to the vertex; at the beginning of each round, the fire spreads one step along edges; at the end of each round, we pick a vertex and set fire to it; the process finishes when all vertices are burned. The objective in the problem is to minimize the number of rounds (including the first one for just picking the first vertex) to burn all the vertices. The minimum number of rounds that can burn a graph $G$ in such a process is the burning number of $G$, which is denoted by $b(G)$. Given a graph $G$ and integer $k$, GRAPH BURNING asks whether $b(G) \leq k$.

In other words, the burning number of $G$ can be defined as the minimum length $k$ of a sequence $(b_0, \ldots, b_{k-1})$ of vertices of $G$ such that every vertex in $G$ has distance at most $i$ from some $b_i$. We call such a sequence a burning sequence. Note that in this definition, $b_i$ is the vertex we set fire in the $(k - i)$th round. Note also that we do not ask the vertices $b_0, \ldots, b_{k-1}$ to be distinct. It is also useful to introduce the generalized neighborhood of vertices. For a vertex $v$ of a graph $G$, let $N_d[v]$ be the set of vertices with distance at most $d$ in $G$. For example, $N_0[v]$ contains only $v$, $N_1[v]$ is just the closed neighborhood of $v$, 

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and $N_2[v] = \bigcup_{u \in N(v)} N[u]$. With this terminology, a sequence $(b_0, \ldots, b_{k-1})$ of vertices of $G = (V, E)$ is a burning sequence of $G = (V, E)$ if and only if $\bigcup_{0 \leq i \leq k-1} N_i[b_i] = V$.

1.1 Previous work

As a model of information spreading, it is important to know how fast the information can spread under the model in the worst case. This question can be answered by finding the maximum burning number of graphs of $n$ vertices. The literature is rich in this direction. In the very fast paper [6, 7], it is shown that $b(G) \leq 2\lceil \sqrt{n} \rceil - 1$ for every connected graph $G$ of order $n$ and conjectured that $b(G) \leq \lceil \sqrt{n} \rceil$ holds. Note that the connectivity requirement is essential here as an edgeless graph of order $n$ needs $n$ rounds. Some improvements of the general upper bound and studies on special cases are done [26, 28, 2, 29, 4, 9, 27, 5, 16], but the conjecture of $b(G) \leq \lceil \sqrt{n} \rceil$ for general connected graphs remains unsettled.

The computational complexity of Graph Burning has been studied intensively as well. It is shown that Graph Burning is NP-complete on trees of maximum degree 3, spiders, and linear forests [1]. The NP-completeness result is further extended to connected caterpillars of maximum degree 3 [21, Section 4.2], which form a subclass of connected interval graphs and connected permutation graphs. On the other hand, Graph Burning admits a 3-approximation algorithm for general graphs [8]; that is, given a graph $G$, the algorithm finds a burning sequence of $G$ of length at most $3 \cdot b(G)$ in polynomial time. Algorithms with approximation factors parameterized by path-length and tree-length are known as well [24]. Bonato and Kamali [8] asked whether the problem is APX-hard. Recently, this question has been answered in the affirmative [30].

Kare and Reddy [25] initiated the study on parameterized complexity of Graph Burning. They showed that Graph Burning on connected graphs is fixed-parameter tractable parameterized by distance to cluster graphs (disjoint unions of complete graphs) and by neighborhood diversity. The parameterized complexity with respect to the natural parameter $k$, the burning number, remained open. Recently, Janssen [23] has generalized the problem to directed graphs and has shown that the directed version is W[2]-complete parameterized by $k$ even on directed acyclic graphs. It was mentioned in [23] that the original undirected version parameterized by $k$ was still open.

1.2 Our results

In the literature, the input graph of Graph Burning is sometimes assumed to be connected (e.g. in [25]). In this paper, however, we do not generally assume the connectivity of input graphs. All positive results in this paper hold on possibly disconnected graphs, while all negative results hold even on connected graphs. Note that in Graph Burning, the disconnected case would be nontrivially more complex than the connected case. For example, while the burning number of a path of $n$ vertices is $\lceil \sqrt{n} \rceil$ [6, 7], Graph Burning is NP-complete on graphs obtained as the disjoint union of paths [4].

Our study in this paper is inspired by Kare and Reddy [25] and Janssen [23]. We generalize the results in [25] and solve all open problems on parameterized complexity in [25]. In Section 2 we present our positive algorithmic results. We show that Graph Burning is fixed-parameter tractable parameterized by clique-width plus the maximum diameter among all connected components. This implies that Graph Burning is fixed-parameter tractable parameterized by modular-width, by treedepth, and by distance to cographs. We also show that Graph Burning is fixed-parameter tractable parameterized by distance to split graphs. The complexity parameterized by distance to cographs and by distance
to split graphs are explicitly asked in [25]. The fixed-parameter tractability parameterized by modular-width generalizes the one parameterized by neighborhood diversity in [25]. In Section 3, we present some negative results. We show that GRAPH BURNING parameterized by the natural parameter \( k \) is W[2]-complete. This settles the main open problem in this line of research. As a byproduct, we also show that GRAPH BURNING parameterized by vertex cover number does not admit a polynomial kernel unless NP \( \subseteq \text{coNP/poly} \). This also answers a question in [25]. See Figure 1 for a summary of the results.

![Figure 1](image_url)  
**Figure 1** Graph parameters and the complexity of GRAPH BURNING. The results on the parameters with dark background are the main results of the paper. Connections between two parameters imply the existence of a function in the one above (being in this sense more general) that lowerbounds the one below. “The maximum diameter among all connected components” is shortened as “max diameter.” The paraNP-completeness parameterized by bandwidth follows from the result by Gupta et al. [21, Section 4.2] who showed that the problem is NP-complete on caterpillars of maximum degree 3, which have bandwidth at most 2.

We assume that the readers are familiar with the basic terms and concepts in the parameterized complexity theory. See some textbooks in the field (e.g., [15, 12]) for definitions. We omit the definitions of most of the graph parameters in this paper as we do not explicitly need them. We only need the definition of the vertex cover number of a graph: it is the minimum integer \( k \) such that their exists a set of \( k \) vertices of the graph (called a vertex cover) such that each edge in the graph has at least one endpoint in the set. We refer the readers to [33] for the definitions of other graph parameters and the hierarchy among them.

## 2 Positive results

We first observe that GRAPH BURNING is expressible as a first order logic (FO) formula of length depending only on \( k \).

The syntax of FO of graphs includes (i) the logical connectives \( \lor, \land, \neg, \leftrightarrow, \Rightarrow \), (ii) variables for vertices, (iii) the quantifiers \( \forall \) and \( \exists \) applicable to these variables, and (iv) the following binary relations: equality of variables, and \( \text{adj}(u, v) \) for two vertex variables \( u \) and \( v \), which means that \( u \) and \( v \) are adjacent. If \( G \) models an FO formula \( \varphi \) with no free variables, then we write \( G \models \varphi \).
The following formula $\text{dist}_{\leq d}(v, w)$ is true if and only if the distance between $v$ and $w$ is at most $d$:

$$\text{dist}_{\leq d}(v, w) := \exists u_0, \ldots, u_d \ (u_0 = v) \land (u_d = w) \land \left( \bigwedge_{0 \leq i < d} (u_i = u_{i+1}) \lor \text{adj}(u_i, u_{i+1}) \right).$$

Clearly, $\text{dist}_{\leq d}(v, w)$ has length depending only on $d$. Now we define the formula $\varphi_k$ such that $G \models \varphi_k$ if and only $(G, k)$ is a yes instance of Graph Burning as follows:

$$\varphi_k := \exists v_0, \ldots, v_{k-1} \forall v \bigvee_{0 \leq i \leq k-1} \text{dist}_{\leq i}(v, v_i).$$

It is known that on nowhere dense graph classes, testing an FO formula $\psi$ is fixed-parameter tractable parameterized by $|\psi|$, where $|\psi|$ is the length of $\psi$ [20]. (See [20] for the definition of nowhere dense graph classes.) Since $\varphi_k$ is an FO formula of length depending only on $k$, the following holds.

Observation 2.1. **Graph Burning** on nowhere dense graphs parameterized by $k$ is fixed-parameter tractable.

For an $n$-vertex graph $G$ of clique-width at most $\text{cw}$ and for a one-sorted monadic-second order logic (MSO$_1$) formula $\psi$, one can check whether $G \models \psi$ in time $O(f(|\psi|, \text{cw}) \cdot n^3)$, where $f$ is a computable function [11, 52]. Since an FO formula is an MSO$_1$ formula and the length of $\varphi_k$ depends only on $k$, we can observe the following fact.

Observation 2.2. **Graph Burning** parameterized by clique-width $+k$ is fixed-parameter tractable.

We extend this observation in a nontrivial way to show the main result of this section. To this end, it is useful to generalize $\varphi_k$ as follows. For nonempty $I = \{i_1, \ldots, i_{|I|}\} \subseteq \{0, \ldots, k-1\}$, let $\varphi_I$ be a formula that means that there are vertices $b_{i_1}, \ldots, b_{i_{|I|}}$ such that $\bigcup_{j \in I} N_{i_j}[b_j] = V$, which can be expressed as follows:

$$\varphi_I := \exists v_{i_1}, \ldots, v_{i_{|I|}} \forall v \bigvee_{1 \leq j \leq |I|} \text{dist}_{\leq i_j}(v, v_{i_j}).$$

Clearly, checking $G \models \varphi_I$ is still fixed-parameter tractable parameterized by clique-width $+k$.

Theorem 2.3. **Graph Burning** is fixed-parameter tractable parameterized by the clique-width of the input graph plus the maximum diameter among all connected components.

Proof. Let $(G, k)$ be an instance of **Graph Burning**, where $G$ has $n$ vertices. Let $C_1, \ldots, C_p$ be the connected components of $G$ and $d_{\text{max}}$ be the maximum diameter of the components. We assume that $k \geq p$ since otherwise $(G, k)$ is a trivial no instance. By Observation 2.2 we can also assume that $d_{\text{max}} \leq k$.

Observe that if a component $C_q$ contains $b_i$ for some $i \geq d_{\text{max}}$, then $N_i[b_i] = V(C_q)$. Hence the problem is equivalent to finding a sequence $(b_0, \ldots, b_{d_{\text{max}}-1})$ that burns as many connected components as possible. That is, we want to find a maximum cardinality subset $\mathcal{C} \subseteq \{C_1, \ldots, C_p\}$ and a sequence $(b_0, \ldots, b_{d_{\text{max}}-1})$ such that $\bigcup_{0 \leq i \leq d_{\text{max}}-1} N_i[b_i] = \bigcup_{C_q \in \mathcal{C}} V(C_q)$. It holds that $p - |\mathcal{C}| \leq k - d_{\text{max}}$ if and only if $(G, k)$ is a yes instance.

We reduce this problem to **Disjoint Sets**. Given a universe $U$, a subset family $\mathcal{S} \subseteq 2^U$, and an integer $t$, **Disjoint Sets** asks whether there are $t$ disjoint subsets in $\mathcal{S}$. Let $U = \{0, 1, \ldots, d_{\text{max}} - 1\} \cup \{c_1, \ldots, c_p\}$ and $\mathcal{S} = \{I \cup \{c_q\} \mid I \subseteq \{0, \ldots, d_{\text{max}} - 1\}, C_q \models$
Theorem 2.7. ▶

Bounded-search tree algorithm finds such a set in time $\Theta(t_{\max \in S}|S| + |U|)^{O(1)}$ \cite{15} DISJOINT r-SUBSETS \cite{[13] BOUNDED RANK DISJOINT SETS}. In our instance, $t \cdot \max_{S \in S} |S| \leq d_{\max}(d_{\max} + 1)$ holds.

For each $q \in \{1, \ldots, p\}$ and for each $I \subseteq \{0, \ldots, d_{\max} - 1\}$, we can test whether $C_q \models \varphi_I$ in time $O(f(d_{\max} + \cw) \cdot n^3)$ for some computable function $f$, where $\cw$ is the clique-width of $G$, because $|\varphi_I|$ depends only on $d_{\max}$. Thus, $S$ can be constructed in time $O(f(d_{\max} + \cw) \cdot n^3 \cdot p \cdot 2^{d_{\max}})$. Since $|S| \leq 2^{d_{\max}} \cdot p$, the last step of solving DISJOINT SETS can be done in time $2^{O(d_{\max}^2)}(2^{d_{\max}} \cdot p + (d_{\max} + p))^{O(1)}$. Since $p \leq n$, the total running time is $g(d_{\max} + \cw) \cdot n^{O(1)}$ for some computable function $g$. This completes the proof. ▶

The definition of modular-width implies that every connected component of a graph of modular-width at most $w$ has both diameter and clique-width at most $w$ \cite{19}. It is known that every connected component of a graph of treedepth at most $d$ has diameter at most $2^{d^4}$ \cite{31} and its clique-width is bounded by a function of treedepth (or even smaller treewidth) \cite{10}. Therefore, Theorem 2.3 implies the fixed-parameter tractability with respect to these parameters.

▶ Corollary 2.4. Graph Burning is fixed-parameter tractable parameterized by modular-width.

▶ Corollary 2.5. Graph Burning is fixed-parameter tractable parameterized by treedepth.

For a graph class $C$ and a graph $G$, the distance from $G$ to $C$ is defined as the minimum integer $k$ such that by removing at most $k$ vertices from $G$, one can obtain a member of $C$.

Let $C$ be a graph class with constants $c$ and $d$ such that each graph in $C$ has clique-width at most $c$ and each connected component of each member of $C$ has diameter at most $d$. Observe that a graph of distance at most $k$ to $C$ has clique-width at most $c \cdot 2^k$ since after a removal of a single vertex, the clique-width remains at least half of the original clique-width \cite{22}. Observe also that each connected component of a graph of distance at most $k$ to $C$ has diameter at most $k + d$. Thus, by Theorem 2.3, Graph Burning is fixed-parameter tractable parameterized by distance to $C$. This observation can be applied immediately to cographs that are known to be the $P_4$-free graphs and the graphs of clique-width at most 2.

▶ Corollary 2.6. Graph Burning is fixed-parameter tractable parameterized by distance to cographs.

Now we consider the distance to split graphs. A graph is a split graph if its vertex set can be partitioned into a clique and an independent set. For a graph $G = (V, E)$, which is not necessarily a split graph, a subset $S \subseteq V$ is a split-deletion set if $G - S$ is a split graph. Then the distance to split graphs from $G$ is equal to the minimum size of a split-deletion set. It is known that the split graphs are exactly the $(2K_2, C_4, C_5)$-free graphs \cite{17}. This characterization implies that when designing an algorithm parameterized by distance $d$ to split graphs we can assume that a split-deletion set of minimum size is given since a standard bounded-search tree algorithm finds such a set in time $5^d \cdot n^{O(1)}$, where the number 5 is the maximum order of the forbidden induced subgraphs.

▶ Theorem 2.7. Graph Burning is fixed-parameter tractable parameterized by distance to split graphs.
**Proof.** Let \((G,k)\) be an instance of \textsc{Graph Burning} and \(S\) be a minimum split-deletion set of \(G = (V,E)\). We denote \(|S|\) by \(s\). Let \((K,I)\) be the partition of \(V-S\), where \(K\) is a clique and \(I\) is an independent set. We further partition \(I\) into \(I_K\), \(I_S\), and \(I_B\) in such a way that \(I_B\) is the set of degree-0 vertices in \(G\), \(I_S\) is the set of vertices in \(I \setminus I_B\) that have neighbors only in \(S\), and \(I_K = I \setminus (I_B \cup I_S)\). Note that each vertex in \(I_S\) has at least one neighbor in \(S\) and each vertex in \(I_K\) has at least one neighbor in \(K\) (and possibly some neighbors in \(S\)).

We first reduce the number of vertices in \(I_S\). For a nonempty subset \(S' \subseteq S\), let \(J_{S'} \subseteq I_S\) be the set of vertices whose neighborhood is exactly \(S'\). Since the pairwise distance between vertices in \(J_{S'}\) is 2, a burning sequence does not pick four or more vertices in \(J_{S'}\). Thus, we can remove all but three vertices in \(J_{S'}\) and obtain an equivalent instance. We apply this reduction to all subsets \(S' \subseteq S\) and denote the reduced subset of \(I_S\) by \(I_S^*\). Note that \(|I_S^*| < 3 \cdot 2^{|S|}\). If \(k \geq 3 + |S| + |I_S^*| + |I_B| = 3 + s + 3 \cdot 2^s + |I_B|\), then \((G,k)\) is a yes instance: we can take all the vertices in \(S \cup I_S^* \cup I_B\) and three vertices in \(K \cup I_K\) and then arbitrarily order them to construct a burning sequence of \(G\). Hence, in the following, we assume that \(k < 3 + s + 3 \cdot 2^s + |I_B|\).

We next remove all vertices in \(I_B\) and obtain an equivalence instance of a slightly generalized problem. Observe that if \((G,k)\) is a yes instance, then \(k \geq |I_B|\) and there is a burning sequence \((b_0,\ldots,b_{k-1})\) of \(G\) such that the set of first \(|I_B|\) vertices \(\{b_0,\ldots,b_{|I_B|}\}\) is \(I_B\); we need to take every isolated vertex into a burning sequence, but even \(b_0\) is good enough to burn an isolated component. In the following, we only consider burning sequences with this restriction. Now the problem is reduced to the one for finding a sequence \((b_{|I_B|},b_{|I_B|+1},\ldots,b_{k-1})\) of vertices in \(V \setminus I_B\) such that \(\bigcup_{i=0}^{k-1} N_i[b_i] = V \setminus I_B\). We denote by \((G',k,|I_B|)\) the obtained instance of the new problem, where \(G' = G[K \cup I_K \cup S \cup I_S^*]\).

To solve the reduced problem, we first guess which vertices in \(S \cup I_S^*\) appear in \((b_{|I_B|},b_{|I_B|+1},\ldots,b_{k-1})\) and where they are placed in the sequence. The number of candidates of such a guess depends only on \(s\) as \(|S \cup I_S^*|^{k-|I_B|} < (s + 3 \cdot 2^s)^{s+3 \cdot 2^s + 3}\). The vacant slots of \((b_{|I_B|},b_{|I_B|+1},\ldots,b_{k-1})\) after the guess tell us which \(b_i\) belongs to \(K \cup I_K\). If there are at most three vertices in \(K \cup I_K\) appear in \((b_{|I_B|},b_{|I_B|+1},\ldots,b_{k-1})\) then we try all \(O(n^3)\) combinations to complete the sequence. Otherwise, we guess from \(O(n)\) candidates the vertex in \(K \cup I_K\) that appears in \((b_{|I_B|},b_{|I_B|+1},\ldots,b_{k-1})\) and has the largest index. Since the index of the guessed vertex in \((b_{|I_B|},b_{|I_B|+1},\ldots,b_{k-1})\) is at least 3 and \(K \cup I_K\) induces a connected split graph, which has diameter at most 3, the guessed vertex in \(K \cup I_K\) burns all vertices in \(K \cup I_K\).

Finally, we fill the positions in \((b_{|I_B|},b_{|I_B|+1},\ldots,b_{k-1})\) that still remain vacant. Let \(X \subseteq \{b_1,\ldots,b_{k-1}\}\) be the set of indices \(i\) for which no vertex is guessed as \(b_i\) so far, and let \(X' = \{|b_1,\ldots,k-1\} \setminus X\). Let \(U = V(G') \setminus \bigcup_{i \in X} N_i[b_i]\). Note that \(U \subseteq S \cup I_S^*\). Our task is to find \(\{b_i : i \in X\}\) in \(K \cup I_K\) such that \(U \subseteq \bigcup_{i \in X} N_i[b_i]\). This task can be seen as an instance \((U',S,p)\) of \textsc{Set Cover}, where \(U' = U \cup X\), \(S = \{\{N_i[v] \cap U\} \cup \{i\} : v \in K \cup I_K, i \in X\}\), and \(p = |X| < k - |I_B| \leq 3 + s + 3 \cdot 2^s\). Since \textsc{Set Cover} parameterized by \(|U'|\) is fixed-parameter tractable \cite{FPT} Lemma 2] and \(|U'| \leq s + 3 \cdot 2^s + p\), the theorem follows. \(\Box\)

### 3 Negative results

This section is devoted to the proofs of the following theorems.

- **Theorem 3.1.** \textsc{Graph Burning} is \(W[2]\)-complete parameterized by \(k\).

- **Theorem 3.2.** \textsc{Graph Burning} does not admit a polynomial kernel parameterized by vertex cover number unless \(\text{NP} \subseteq \text{coNP/poly}\).
We present a reduction from Set Cover to Graph Burning that proves both Theorems 3.1 and 3.2. Given a set \( U = \{u_1, \ldots, u_n\} \), a subset family \( S = \{S_1, \ldots, S_m\} \subseteq 2^U \), and a positive integer \( s \), Set Cover asks whether there exists a subfamily \( S' \subseteq S \) such that \( |S'| \leq s \) and \( \bigcup_{S \in S'} S = U \).

Let \( (U = \{u_1, \ldots, u_n\}, S = \{S_1, \ldots, S_m\}, s) \) be an instance of Set Cover. We construct an equivalent instance \( (G, k = s + 2) \) of Graph Burning. (See Figure 2.) We first construct \( s = k - 2 \) isomorphic graphs \( G_2, \ldots, G_{k-1} \) as follows. For each \( i \in \{2, 3, \ldots, k - 1\} \), the vertex set of \( G_i \) is \( U_i \cup V_i \), where \( U_i = \{u^{(i)}_1, \ldots, u^{(i)}_{n_i}\} \) is a clique and \( V_i = \{v^{(i)}_1, \ldots, v^{(i)}_{m_i}\} \) is an independent set. In \( G_i \), \( u^{(i)}_p \) and \( v^{(i)}_q \) are adjacent if and only if \( u_p \in S_q \). From each \( G_i \), we construct \( H_i \) by adding \( i + 2 \) copies of a path of \( i \) vertices and all possible edges between each vertex in \( V_i \) and one of the degree-1 vertices in each path. We then take the disjoint union of \( H_2, H_3, \ldots, H_{k-1} \) and add \( U \) as a clique. For each \( i \in \{2, 3, \ldots, k - 1\} \) and \( j \in \{1, 2, \ldots, m\} \), we connect \( u^{(i)}_j \) and \( u_p \) with a path of length \( i - 1 \) with \( i - 2 \) new inner vertices. Finally, we attach a vertex \( w \) to a vertex in \( U \), and a path \((x, y, z)\) to the same vertex. We set \( V_0 = \{w\} \) and \( V_1 = \{x, y, z\} \). We denote the constructed graph by \( G \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{The reduction from Set Cover to Graph Burning. The edges in the cliques \( U_2, \ldots, U_{k-1} \), and \( U \) are omitted.}
\end{figure}

\textbf{Lemma 3.3.} \((U, S, s)\) is a yes instance of Set Cover if and only if \((G, k)\) is a yes instance of Graph Burning.

\textbf{Proof.} (\( \Rightarrow \)) Assume that \((U, S, s)\) is a yes instance of Set Cover and \( S' \subseteq S \) is a certificate; that is, \( |S'| \leq s \) and \( \bigcup_{S \in S'} S = U \). We assume without loss of generality that \( |S'| = s \) and \( S' = \{S_2, S_3, \ldots, S_{s+1=k-1}\} \). We set \( b_0 = w \), \( b_1 = y \), and \( b_k = v^{(i)}_k \) for
2 ≤ i ≤ k−1. We show that \( \langle b_0, \ldots, b_{k-1} \rangle \) is a burning sequence of \( G \).

Clearly, \( N[v_0] = V_0 \) and \( N[v_1] = V_1 \). For \( 2 ≤ i ≤ k−1 \), observe that \( N[v_i] = N[v_0] \) includes all the vertices of \( H_i \): the farthest vertices in \( i \)-vertex paths have distance exactly \( i \) from \( v_0(v_i, v_j(i)) = 2 \) for each \( j \neq i \) as \( v_j \) and \( v_i \) share a neighbor (an endpoint of a path of \( i \) vertices); \( \text{dist}(v_i, v_j(i)) ≤ 2 \) for every \( j \) since \( v_i \) has at least one neighbor in the clique \( U_i \). Moreover, \( N[v_i] \) includes all inner vertices of the paths from \( U_i \) to \( U_j \) as \( \text{dist}(v_i, v_j(i)) ≤ 2 \) for every \( j \). Finally, \( u_j \in N[v_i] \) if and only if \( u_j \in S_1 \); if \( u_j \in S_1 \), then \( v_i \) and \( u_j \) are adjacent, and thus \( \text{dist}(v_i, u_j) ≤ 1 + \text{dist}(u_j, v_j(i)) = i \); otherwise, \( v_i \) and \( u_j \) are not adjacent and thus \( \text{dist}(v_i, u_j) ≥ \min\{2 + \text{dist}(u_j, v_j(i)), 1 + \text{dist}(u_{j+p}, u_j)\} ≥ i \).

This implies that \( U \subseteq \bigcup_{2 ≤ i ≤ k−1} N[v_i] \) since \( \bigcup_{S \subseteq S_1} S = U \).

( \( \Leftarrow \) ) Assume that \( (G, k) \) is a yes instance of \textsc{Graph Burning} and \( \langle b_0, \ldots, b_{k-1} \rangle \) is a burning sequence of \( G \).

We first show that \( b_i \in V_i \) for all \( 0 ≤ i ≤ k−1 \). Let \( i \in \{2, \ldots, k−1\} \). Assume that we already know that \( b_j \in V_j \) for \( i + 1 ≤ j ≤ k−1 \). Since there are \( i + 2 \) paths attached to \( V_i \), at least one of them, say \( P_i \), has no vertex in the remaining vertices \( b_0, \ldots, b_i \). The degree-1 vertex in \( P_i \) has distance exactly \( i \) from every vertex in \( V_i \) and distance at least \( i + 1 \) from every vertex not in \( (P_i) \cup V_i \). Hence, \( b_i \in V_i \). Now we know that \( b_i \in V_i \) for \( 2 ≤ i ≤ k−1 \). Let \( u_j \in U \) be the vertex where \( V_0 \) and \( V_1 \) are attached to. For \( 2 ≤ i ≤ k−1 \), we have \( \text{dist}(b_i, u_j) \geq i \), and thus \( N_0[b_i] \) contains no vertex in \( V_0 \cup V_1 \). Since \( N_0[b_0] \) will cover only one vertex, \( b_1 = y \) and \( b_0 = w \) hold.

For \( 2 ≤ i ≤ k−1 \), let \( b_i = v_{h_i} \). Since \( N_0[b_0] = \{w\} \) and \( N_1[b_1] = \{x, y, z\} \), we have \( U \subseteq \bigcup_{2 ≤ i ≤ k−1} N[v_{h_i}] \). As we saw in the only-if case, \( u_j \in N[v_{h_i}] \) if and only if \( u_j \in S_{h_i} \) for \( 1 ≤ j ≤ n \). This implies that \( N[v_{h_i}] \cap U = S_{h_i} \), and thus \( U = \bigcup_{2 ≤ i ≤ k−1} S_{h_i} \). Therefore, the subfamily \( \{S_{h_2}, S_{h_3}, \ldots, S_{h_{k−1}}\} \subseteq S \) of at most \( k−2 \) subsets shows that \( (U, S) \) is a yes-instance of \textsc{Set Cover}.

\textbf{Proof of Theorem 3.3} By Lemma 3.3, the construction of \( (G, k) \) from \( (U, S, s) \) described above is a parameterized reduction from \textsc{Set Cover} parameterized by \( s \) to \textsc{Graph Burning} parameterized by \( k = s + 2 \). Since \textsc{Set Cover} is \( W[2] \)-complete parameterized by \( s \), the \( W[2] \)-hardness follows.

The membership to \( W[2] \) can be shown by the following reduction to \textsc{Set Cover}. Let \( (G = (V, E), k) \) be an instance of \textsc{Graph Burning}. We set \( s = k \), \( U = V \cup \{0, 1, \ldots, k−1\} \), and \( S = \{N[v] \cup \{i\} \mid v \in V, 0 ≤ i ≤ k−1\} \). This is just an undirected version of the proof by Janssen [24], who showed the membership to \( W[2] \) for \textsc{Graph Burning} on directed graphs, and the correctness can be shown in the same way.

\textbf{Proof of Theorem 3.2} The graph \( G \) constructed above has an independent set \( \bigcup_{2 ≤ i ≤ k−1} V_i \). The vertices not belonging to this independent set form a vertex cover of size \( 4 + (k−1)|U| + \sum_{2 ≤ i ≤ k−1}(i + 2)(2i − 2) \), which is a polynomial in \( k = s + 2 \) and \( |U| \). By Lemma 3.3, the construction of \( (G, k) \) from \( (U, S, s) \) described above is a polynomial parameter transformation from \textsc{Set Cover} parameterized by \( |U| + s \) to \textsc{Graph Burning} parameterized by vertex cover number. Since \textsc{Set Cover} parameterized by \( |U| + s \) does not admit polynomial kernels unless \( \text{NP} \subseteq \text{coNP/poly} \), the theorem holds.

The reduction above also shows the \( W[2] \)-hardness parameterized by diameter since the diameter of a connected graph is smaller than the square of its burning number [6, 7]. We further observe that the graph \( G \) in the reduction is \( P_{4(k−3)} \)-free. That is, \( G \) does not contain a path of \( 4k − 3 \) vertices as an induced subgraph. Let \( P \) be an induced path in \( G \). For
i \in \{2, \ldots, k - 1\}, let H_i' be the graph consists of H_i and the paths from U_i to U. Since U is a clique, there are at most two indices i such that P intersects H_i' - U. We can see that |V(P) \cap V(H_i')| \leq 2i + 1 for each i, and thus |V(P)| \leq 2(k - 1) + 1 + 2(k - 2) + 1 = 4k - 4. This implies the W[2]-hardness parameterized by q.

**Corollary 3.4.** Graph Burning on P_q-free graphs is W[2]-hard parameterized by q.

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**References**

1. Stéphane Bessy, Anthony Bonato, Jeannette C. M. Janssen, Dieter Rautenbach, and Elham Roshanbin. Burning a graph is hard. *Discret. Appl. Math.*, 232:73–87, 2017. doi:10.1016/j.dam.2017.07.016
2. Stéphane Bessy, Anthony Bonato, Jeannette C. M. Janssen, Dieter Rautenbach, and Elham Roshanbin. Bounds on the burning number. *Discret. Appl. Math.*, 235:16–22, 2018. doi:10.1016/j.dam.2017.09.012
3. Hans L. Bodlaender, Stéphan Thomassé, and Anders Yeo. Kernel bounds for disjoint cycles and disjoint paths. *Theor. Comput. Sci.*, 412(35):4570–4578, 2011. doi:10.1016/j.tcs.2011.04.039
4. Anthony Bonato, Sean English, Bill Kay, and Daniel Moghbel. Improved bounds for burning fence graphs. *CoRR*, abs/1911.01342, 2019. arXiv:1911.01342
5. Anthony Bonato, Karen Gunderson, and Amy Shaw. Burning the plane. *Graphs Comb.*, 2020. to appear. doi:10.1007/s00373-020-02182-9
6. Anthony Bonato, Jeannette C. M. Janssen, and Elham Roshanbin. Burning a graph as a model of social contagion. In *WAW 2014*, volume 8882 of *Lecture Notes in Computer Science*, pages 13–22, 2014. doi:10.1007/978-3-319-319-319-3-8.2
7. Anthony Bonato, Jeannette C. M. Janssen, and Elham Roshanbin. How to burn a graph. *Internet Math.*, 12(1-2):85–100, 2016. doi:10.1080/15427951.2015.1103339
8. Anthony Bonato and Shahin Kamali. Approximation algorithms for graph burning. In *TAMC 2019*, volume 11436 of *Lecture Notes in Computer Science*, pages 74–92, 2019. doi:10.1007/978-3-030-14812-6_6
9. Anthony Bonato and Thomas Lidbetter. Bounds on the burning numbers of spiders and path-forests. *Theor. Comput. Sci.*, 794:12–19, 2019. doi:10.1016/j.tcs.2018.05.035
10. Dóra G. Czirók and Udi Rotics. On the relationship between clique-width and treewidth. *SIAM J. Comput.*, 34(4):825–847, 2005. doi:10.1137/S0097539703438551
11. Bruno Courcelle, Johann A. Makowsky, and Udi Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory Comput. Syst.*, 33(2):125–150, 2000. doi:10.1007/s002240050009
12. Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015. doi:10.1007/978-3-319-49787-7_1
13. Michael Dom, Daniel Lokshtanov, and Saket Saurabh. Kernelization lower bounds through colors and IDs. *ACM Trans. Algorithms*, 11(2):13:1–13:20, 2014. doi:10.1145/2650261
14. Rodney G. Downey and Michael R. Fellows. Fixed-parameter tractability and completeness II: basic results. *SIAM J. Comput.*, 24(4):873–921, 1995. doi:10.1137/S0097539792228228
15. Rodney G. Downey and Michael R. Fellows. *Parameterized Complexity*. Springer, 1999. doi:10.1007/978-1-4612-0515-9
16. Zahra Rezai Farokhi, Maryam Tahmasbi, Zahra Haj Rajab Ali Tehrani, and Yousof Buali. New heuristics for burning graphs. *CoRR*, abs/2003.09314, 2020. arXiv:2003.09314
17. Stéphane Foldes and Peter L. Hammer. Split graphs. In *The Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing*, volume 19 of *Congressus Numerantium*, pages 311–315, 1977. doi:10.1007/978-3-319-49787-7_1
Fedor V. Fomin, Dieter Kratsch, and Gerhard J. Woeginger. Exact (exponential) algorithms for the dominating set problem. In *WG 2004*, volume 3353 of *Lecture Notes in Computer Science*, pages 245–256, 2004. doi:10.1007/978-3-540-30559-0_21

Jakub Gajarský, Michael Lampis, and Sebastian Ordyniak. Parameterized algorithms for modular-width. In *IPEC 2013*, volume 8246 of *Lecture Notes in Computer Science*, pages 163–176, 2013. doi:10.1007/978-3-642-45450-2_15

Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. *J. ACM*, 64(3):17:1–17:32, 2017. doi:10.1145/3051095

Arya Tanmay Gupta, Swapnil Lokhande, and Kaushik Mondal. NP-completeness results for graph burning on geometric graphs. *CoRR*, abs/2003.07746, 2020. arXiv:2003.07746

Frank Gurski. The behavior of clique-width under graph operations and graph transformations. *Theory Comput. Syst.*, 60(2):346–376, 2017. doi:10.1007/s00224-016-9685-1

Anjeneya Swami Kare and I. Vinod Reddy. Parameterized algorithms for graph burning problem. In *IWOCA 2019*, volume 11638 of *Lecture Notes in Computer Science*, pages 304–314, 2019. doi:10.1007/978-3-030-25005-8_25

Max R. Land and Linyuan Lu. An upper bound on the burning number of graphs. In *WAW 2016*, volume 10088 of *Lecture Notes in Computer Science*, pages 1–8, 2016. doi:10.1007/978-3-319-49787-7_1

Huiqing Liu, Ruiting Zhang, and Xiaolan Hu. Burning number of theta graphs. *Appl. Math. Comput.*, 361:246–257, 2019. doi:10.1016/j.amc.2019.05.031

Dieter Mitsche, Pawel Pralat, and Elham Roshanbin. Burning graphs: A probabilistic perspective. *Graphs Comb.*, 33(2):449–471, 2017. doi:10.1007/s00373-017-1768-5

Dieter Mitsche, Pawel Pralat, and Elham Roshanbin. Burning number of graph products. *Theor. Comput. Sci.*, 746:124–135, 2018. doi:10.1016/j.tcs.2018.06.038

Debajyoti Mondal, N. Parthibhan, V. Kavitha, and Indra Rajasingh. APX-hardness and approximation for the k-burning number problem. *CoRR*, abs/2006.14733, 2020. arXiv:2006.14733

Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity: Graphs, Structures, and Algorithms*. Algorithms and combinatorics. Springer, 2012. doi:10.1007/978-3-642-27875-4

Sang-il Oum. Approximating rank-width and clique-width quickly. *ACM Trans. Algorithms*, 5(1):10:1–10:20, 2008. doi:10.1145/1435375.1435385

Manuel Sorge and Mathias Weller. The graph parameter hierarchy, 2019. URL: https://manyu.pro/assets/parameter-hierarchy.pdf