Simple Applications of Effective Field Theory and Similarity Renormalization Group Methods

Sérgio Szpigel and Robert J. Perry

Department of Physics
The Ohio State University, Columbus, OH 43210

September 17, 2018

Abstract

We use two renormalization techniques, Effective Field Theory and the Similarity Renormalization Group, to solve simple Schrödinger equations with delta-function potentials in one and two dimensions. The familiar one-dimensional delta-function does not require renormalization, but it provides the simplest example of a local interaction that can be replaced by a sequence of effective cutoff interactions that produce controllable power-law errors. The two-dimensional delta-function leads to logarithmic divergences, dimensional transmutation and asymptotic freedom, providing an example of some of the most important renormalization problems in gauge field theories. We concentrate on the power-law analysis of errors in low-energy observables. The power-law suppression of the effects of irrelevant operators is critical to the success of field theory, and understanding them turns renormalization group techniques into powerful predictive tools for complicated problems where exact solutions are not available.
1 Introduction

The theory of renormalization is one of the greatest successes of physics in the twentieth century [1]. Early attempts to marry quantum mechanics with special relativity naturally led to the use of local interactions, maintaining the causal paradigm that has driven physics for centuries while avoiding signals that propagate at speeds greater than that of light. Early encounters with perturbative divergences led some of the best theorists in the world to question the foundations of quantum mechanics, and eventual successes at fitting precise atomic experimental data led to the universal acceptance of renormalization recipes that were acknowledged to make little sense [2]. Initially the perturbative renormalization of QED required theorists to match perturbative expansions in powers of a bare and physical electronic charge [3], but the bare charge clearly diverges logarithmically in QED and the success of an expansion in powers of such a coupling was mysterious at best [4].

The first steps towards making sense of renormalization theory were taken in the 1950’s with the invention of the perturbative renormalization group [5, 6, 7, 8, 9], although serious investigators found the theory was still plagued by non-convergent sums because QED is not asymptotically free. The development of Wilson’s renormalization group formalism [10, 11] and the discovery of asymptotic freedom [12] allowed physicists to produce a logically reasonable picture of renormalization in which perturbative expansions at any high energy scale can be matched with one another, with no necessity to deal with intermediate expansions in powers of a large parameter. But in all theories of interest the running coupling becomes large at some scale and the lattice [13] provides a tool for attacking gauge theories non-perturbatively. This shifted the problems to achieving a continuum limit in which the lattice spacing can be taken to zero implicitly and set the stage for the development of a standard model that rests heavily on asymptotically free field theories.

Although this story took a dramatic turn with the invention of string theories, its main themes survive in essentially all predictive field theory calculations and the control the renormalization group provides has made it possible for physicists to accept the standard model as an effective theory that is accurate over a very large range of scales that covers all currently experimentally accessible energies. It has also led to the acceptance of the development of effective field theories as a fundamental endeavor that goes far beyond the apparently arbitrary introduction of non-local potentials, which has persistently undermined broad interest in attempts to model complicated interactions such as the strong nuclear interaction.

We make no attempt to survey this history, nor are we able to provide a complete review of the two methods illustrated in this article. We take advantage of the fact that the divergences that provided the original impetus leading to renormalization theory result entirely from the use of local interactions. To understand the most important aspects of renormalization theory requires only a background in nonrelativistic quantum mechanics, because as has been long known the divergences of field theory are directly encountered when one tries to impose locality on the Schrödinger equation. In this case the only available interactions that are local at all scales are delta functions and derivatives of delta functions. These divergences can be regulated by the introduction of a cutoff, and the artificial effects of this cutoff must be removed by renormalization. The simplicity of the one-body Schrödinger equation makes it possible to renormalize the theory exactly, disentangling the effects of locality from the complicated many-body effects and symmetry constraints encountered in realistic field theories. There is a large literature on the subject [14]-[31], largely pedagogical.
Effective Field Theory (EFT) is primarily used to replace complicated ‘fundamental’ theories such as QCD with simpler theories such as chiral perturbation theory \[32\]. These simpler effective theories are to be used only at ‘low’ energy scales at which the fundamental degrees of freedom and interactions cannot be resolved, and the scaling analysis that we illustrate makes this limitation clear. EFT \[33\]-\[39\] is motivated by Wilson’s work on the renormalization group \[10, 11\]; however, practitioners typically employ only basic renormalization group tools.

EFT and essentially all renormalization techniques rely on the assumption that physics at a low energy scale and/or long distance scale is insensitive to the details of the underlying physics at a high energy scale and/or short distance scale. Microscopic degrees of freedom and their interactions can be replaced by effective macroscopic degrees of freedom and their effective interactions. Moreover, these effective degrees of freedom can be treated as if they were point-like and their effective interactions are local at the scale of interest. These assumptions have clearly been critical through the entire evolution of physics. However, the detailed procedure for producing effective interactions and the understanding of how errors depend on the parameters in these interactions are relatively new.

One of the most interesting recent applications of EFT is to nuclear physics and the two-nucleon problem in particular \[40\]-\[75\]. Traditionally nuclear theorists have developed nonlocal two-body interactions, sometimes motivated by meson exchange models, that are adjusted to fit scattering phase shifts and properties of the deuteron. These nonlocal interactions are typically parameterized by a few masses and couplings, but there has been no systematic, logically simple procedure for improving interactions and many nonlocal interactions work equally well. In EFT the cutoff is regarded as artificial and parameters are allowed to run with the cutoff to remove its effects as much as possible. There is a systematic procedure for improving the effective interactions by adding operators, and the goal of relating the complicated nucleon-nucleon interaction to QCD may become much more realistic.

Recent work to extend this approach to deal with the three-nucleon problem \[76, 77\] and to explicitly include mesons has uncovered problems that go beyond a straightforward application of EFT. The fact that the two-nucleon interaction is an irrelevant operator, but the scattering phases shifts are large, has led to an understanding that naive dimensional scaling is violated. There is a non-free fixed point that contains this interaction and presumably alters the scaling properties of all operators \[47, 54, 57, 59\]. Condensed matter theorists have long appreciated the importance of understanding the scaling behavior of interactions around non-Gaussian fixed points, but there are few field theory examples that are relevant to nuclear and particle physics. The study of effective nuclear field theories may offer such examples and then drive interesting new developments in EFT.

The similarity renormalization group (SRG) is a very recent development invented by Stan Glazek and Ken Wilson \[78, 79\], and independently by Franz Wegner \[80\]. Most of the applications of the SRG are in light-front field theory \[81\]-\[95\], but there are a growing number of applications in other areas \[96\]-\[108\].

In EFT a cutoff is introduced along with a fixed number of effective interactions. The resultant hamiltonian is used to compute low energy observables, and the strengths (i.e., couplings) of the effective interactions are adjusted to fit a subset of the low energy data. It is sometimes possible to explicitly derive equations that govern how these couplings evolve as the cutoff changes, an approach that parallels the Gell-Mann–Low approach to the renormalization group \[3\], and by studying these equations the scaling behavior of these
Figure 1: Two ways to run a cutoff on free energy. In (a) a cutoff on the magnitude of the energy is lowered from the solid to the dashed lines, with problems resulting from the removed shaded region. In (b) a cutoff on how far off diagonal matrix elements appear is lowered from the dashed to the solid lines.

couplings can be understood. In the SRG, as in Wilson’s original renormalization group formalism \[109, 110], transformations that explicitly run the cutoff are developed. These transformations are the group elements that give the renormalization group its name.

In his earliest work \[109, 110] Wilson exploited a transformation originally invented by Claude Bloch \[111]. It uses a cutoff on the states themselves, and as the cutoff is lowered, states are removed from the Hilbert space. If the hamiltonian is viewed as a matrix, these cutoffs can be seen as limiting the size of this matrix and the transformation reduces this size, as illustrated in Fig. 1a. Wilson introduced a rescaling operation to allow transformed hamiltonians to be compared with initial hamiltonians, despite the fact that they act in different spaces; however, the Bloch transformation is ill-defined and even in perturbation theory it leads to artificial divergences. These divergences come from the small energy differences between states retained and states removed by the transformation, and they appear in the form of small energy denominators in the perturbative expansion of the transformed hamiltonian. These small energy denominator problems led Wilson to abandon the hamiltonian formulation of field theory in favor of path integral formulations, but the virtues of the hamiltonian formulation over the path integral formulation for many problems remains.

The breakthrough provided by the SRG is that the transformations are typically unitary, making them well-defined, and they run a cutoff on energy differences rather than on individual states, as illustrated in Fig. 1b. Again viewing the hamiltonian as a large matrix, these cutoffs limit the off-diagonal matrix elements and as they are reduced the hamiltonian
is forced towards diagonal form. The perturbative expansion for transformed hamiltonians contains no small energy denominators, so the expansion breaks down only when interactions become sufficiently strong, in which case perturbation theory should fail in any case.

Although the SRG has not yet been applied to a wide range of problems, it may be an important new tool both for attacking field theories and non-relativistic many-body problems. As we will illustrate, the effective interactions that arise in the SRG are nearly identical to those encountered in EFT; however, the existence and use of an explicit transformation that lowers the cutoff allows one to directly study scaling behavior before using the hamiltonian to fit data.

When the SRG is used with coupling coherence [112, 113], which we explain in the text, it allows one to construct effective theories with the same number of free parameters as the underlying ‘fundamental’ theory. For the delta-function examples there is one fundamental parameter, the strength of the regulated delta-function as the cutoff is removed. In EFT this single parameter is replaced by an arbitrarily large number of effective parameters that are fixed by data. In the SRG with coupling coherence, there is still one fundamental coupling and all new couplings are fixed perturbative functions of the fundamental coupling. It is the renormalization group flow of the added couplings, and a boundary condition that they vanish when the fundamental coupling is taken to zero, that fixes their dependence on the fundamental coupling.

The examples we use in this article do not illustrate non-Gaussian fixed points, so their scaling properties are driven by naive dimensional analysis. However, we will see that even in these cases scaling behavior of effective hamiltonians derived using a perturbative similarity renormalization group can be very complicated. We will see that errors in EFT scale as inverse powers of the cutoff, which is what one expects since these operators are irrelevant. However, in the perturbative SRG there are additional errors arising from the approximate treatment of the fundamental running coupling and the approximate treatment of the relation between this coupling and the new couplings of irrelevant operators.

The examples we use in this article do not illustrate non-Gaussian fixed points, so their scaling properties are driven by naive dimensional analysis. However, we will see that even in these cases scaling behavior of effective hamiltonians derived using a perturbative similarity renormalization group can be very complicated. We will see that errors in EFT scale as inverse powers of the cutoff, which is what one expects since these operators are irrelevant. However, in the perturbative SRG there are additional errors arising from the approximate treatment of the fundamental running coupling and the approximate treatment of the relation between this coupling and the new couplings of irrelevant operators.

The easiest way to understand errors in the perturbative SRG is by comparison with EFT. The same set of operators are found in the SRG hamiltonian as in the EFT hamiltonian, but the coefficients are approximated using the perturbative SRG equations instead of being fit to data.

In a realistic calculation the marginal coupling, which corresponds to the strength of the regulated delta function, would be fit to data. In order to clearly illustrate the logarithmic errors that result from using the perturbative SRG equations, we approximate this marginal coupling in this article rather than renormalizing it nonperturbatively by fitting data. The strengths of the irrelevant operators, which correspond to derivatives of the regulated delta function, are approximated using expansions in powers of the approximate running coupling that are fixed by coupling coherence. The approximate running coupling differs from the exact running coupling by inverse powers of logarithms of the cutoff, and the error analysis for the binding energy displays the resultant inverse logarithmic errors in addition to power-law errors similar to those seen in EFT. In addition there are errors in the strengths of the irrelevant operators resulting from using a truncated expansion in powers of the running coupling and an approximate running coupling, both of which introduce inverse logarithmic errors in addition to the power-law errors seen for these operators in EFT.

We do not argue that EFT or that SRG is to be preferred. Both allow us to produce effective hamiltonians whose errors can be understood and thereby controlled. In some
problems it will be clear that it is much easier to produce an approximate hamiltonian using EFT, but this requires the calculation of scattering observables and in field theories it may require the use of much of the existing data for a system just to fix the parameters. In the SRG the number of parameters equals the number of parameters in the underlying ‘fundamental’ theory, but the SRG equations become extremely complicated as one goes beyond second order. In addition EFT can be used even when there is no underlying theory with the same degrees of freedom, but coupling coherence should fail in this case. This issue deserves investigation.

There are interesting questions about whether the SRG formalism can be used with coupling coherence when there is no clear connection between the effective theory and any underlying fundamental theory. For example, the number of couplings in chiral perturbation theory far exceeds the number of parameters in QCD. If chiral effective theories are exact representations of QCD in the low energy limit, coupling coherence would lead one to expect that the number of independent couplings may be the same, so that there must be constraints on the coupling in chiral field theories that have not been determined. This article does not explore some of the most interesting questions, but it is intended to clearly lay the groundwork required to initiate such investigations.

2 Effective Field Theory Approach

EFT provides a systematic procedure for improving the approximate description of composite effective degrees of freedom and their dynamics by increasing the number of local effective interactions. The effects of high-energy (short-distance) degrees of freedom are incorporated in effective interactions whose strength is adjusted to fit appropriate low-energy (long-distance) results. The keys to maintaining predictive power in EFT are approximate locality and an expansion in powers of a small ratio of low-to-high energy scales or equivalently, short-to-long distance scales.

We begin by discussing the formulation of an effective description for a system of two non-relativistic particles that interact via a Dirac delta-function potential in D-dimensions, following Lepage’s presentation of effective field theory \[37\]. As mentioned in the Introduction, there is a large literature on delta-function interactions where all of the basic results we need can be found \[14\]-\[31\]. The hamiltonian describing the relative motion of the system is given by

$$H = -\nabla_r^2 - \alpha_0 \delta^{(D)}(r).$$

We use units in which \(\hbar = 1\) and choose the coupling \(\alpha_0\) to be dimensionless. In the \(D = 1\) case the kinetic energy operator is implicitly divided by \(2m = 1\), where \(m\) is the two-particle system reduced mass. In the \(D = 2\) case a mass scale is not necessary, since with a dimensionless coupling the hamiltonian is scale invariant.

First, we define a regularized delta-function potential using a cutoff \(\lambda\) that eliminates contributions from processes involving momentum transfer \(|\mathbf{q}| > \lambda\) (short-distance dynamics):

$$\delta^{(D)}_\lambda(r) = \frac{1}{(\sqrt{2\pi})^D} \lambda^D e^{-\frac{x^2}{2\lambda^2}}.$$  

The cutoff also regulates the interaction at \(r = 0\), avoiding possible divergences in the
calculation of matrix elements. By Fourier transforming we obtain
\[ \tilde{\delta}^{(D)}_{\lambda}(q) = \frac{1}{(2\pi)^D} e^{-\frac{q^2}{2\lambda^2}}, \tag{3} \]
where \( q = p - p' \) is the transferred momentum.

Next, we introduce an effective potential that consists of a series of approximately local contact interactions
\[ V_{\lambda}(r, r') = \delta^{(D)}(r - r') \left[ C_0(\lambda) \delta^{(D)}_{\lambda}(r) + \frac{1}{2\lambda^2} C_2(\lambda) \nabla^2_{r} \delta^{(D)}_{\lambda}(r) + \frac{1}{4\lambda^4} C_4(\lambda) \nabla^4_{r} \delta^{(D)}_{\lambda}(r) + \ldots \right]. \tag{4} \]

In momentum space, the potential consists of the Fourier transform of the regularized delta-function and an infinite number of approximately local effective interactions (non-renormalizable)
\[ V_{\lambda}(q) = \left[ C_0(\lambda) + C_2(\lambda) \frac{q^2}{2\lambda^2} + C_4(\lambda) \frac{q^4}{4\lambda^4} + \ldots \right] e^{-\frac{q^2}{2\lambda^2}}. \tag{5} \]

The effective interactions correspond to the derivatives of the delta-function in momentum space and model the effects of the excluded high-momentum transfer contributions, systematically eliminating the dependence of the observables on the cutoff.

To simplify the calculations, we drop terms such as \( p \cdot p' \), obtaining an equally satisfactory effective potential that is easier to work with because it is completely separable:
\[ V_{\lambda}(p, p') = \left[ C_0(\lambda) + C_2(\lambda) \frac{(p^2 + p'^2)}{2\lambda^2} + C_4(\lambda) \frac{(p^4 + p'^4)}{4\lambda^4} + \ldots \right] e^{-\frac{p^2}{2\lambda^2}} e^{-\frac{p'^2}{2\lambda^2}}. \tag{6} \]

The parameters \( C_i \) in the expansion can be determined order by order by fitting data for low-energy processes. Here we use as “data” the values for the inverse “on-shell” K-matrix, calculated with an exact theory. We follow the method developed by Steele and Furnstahl \[52\], determining the parameters by requiring the EFT calculation to give the same result as the calculation using the exact theory to a given order of accuracy. Instead of using the T-matrix, we iterate the effective potential via the Lippmann-Schwinger equation for the K-matrix, which is real and therefore more convenient for numerical calculations of scattering observables.

The Lippmann-Schwinger equation for the K-matrix with the effective potential, generalized to \( D \) dimensions is given by
\[ K_{\lambda}(p, p'; k) = V_{\lambda}(p, p') + \mathcal{P} \int d^D q \frac{V_{\lambda}(p, q)}{k^2 - q^2} K_{\lambda}(q, p'; k), \tag{7} \]
where \( \mathcal{P} \) denotes the principal value. In the \( D = 2 \) case only S-wave scattering occurs, so the above equation refers to the S-wave potential (angular variable integrated out) and \( d^2 q = q \, dq \). To obtain the K-matrix we solve the Lippmann-Schwinger equation (non-perturbatively), using the technique described in Refs. \[31, 17, 54, 60\], which is suitable for
separable potentials. The effective potential given in Eq. \((8)\) (truncated at order \(p^4\)) can be written in the form

\[
V_{\lambda}(p, p') = e^{-\frac{p^2}{2\lambda^2}} e^{-\frac{p'^2}{2\lambda^2}} \sum_{i,j=0}^{2} p_i^2 \Lambda_{ij} p'^2 j ,
\]

where \(\Lambda_{ij}\) are the matrix elements of

\[
\Lambda = \begin{pmatrix}
C_0(\lambda) & C_2(\lambda)/2\lambda^2 & C_4(\lambda)/4\lambda^4 \\
C_2(\lambda)/2\lambda^2 & C_4(\lambda)/2\lambda^4 & 0 \\
C_4(\lambda)/4\lambda^4 & 0 & 0
\end{pmatrix}.
\]

The solution of the Lippmann-Schwinger equation can then be written in the form

\[
K_{\lambda}(p', p; k) = e^{-\frac{p^2}{2\lambda^2}} e^{-\frac{p'^2}{2\lambda^2}} \sum_{i,j=0}^{1} p_i^2 \tau_{ij}(k) p'^2 j .
\]

The unknown matrix \(\tau(k)\) satisfies the equation

\[
\tau(k) = \Lambda + \Lambda \mathcal{I}(k) \tau(k)
\]

where

\[
\mathcal{I} = \begin{pmatrix}
I_0(k) & I_1(k) & I_2(k) \\
I_1(k) & I_2(k) & I_3(k) \\
I_2(k) & I_3(k) & I_4(k)
\end{pmatrix},
\]

with

\[
I_n(k) = \mathcal{P} \int dq \frac{1}{k^2 - q^2} e^{-\frac{q^2}{2\lambda^2}}.
\]

Solving Eq. \((11)\) for \(\tau\) analytically and substituting the solution in Eq. \((10)\) we obtain the effective K-matrix.

Including only the first operator (with \(C_0\)) in the effective potential we obtain

\[
K^{(0)}_{\lambda}(p, p'; k) = \frac{C_0(\lambda)}{1 - C_0(\lambda) I_0(k)} e^{-\frac{p^2}{2\lambda^2}} e^{-\frac{p'^2}{2\lambda^2}}.
\]

Including the first two operators (with \(C_0\) and \(C_2\)) we obtain

\[
K^{(2)}_{\lambda}(p, p'; k) = \frac{C_0(\lambda) + \frac{C_2(\lambda)}{2\lambda^2} (p^2 + p'^2) + \frac{C_4(\lambda)}{4\lambda^4} [I_2(k) - (p^2 + p'^2) I_1(k) + p^2 p'^2 I_0(k)]}{1 - C_0(\lambda) I_0(k) - 2 \frac{C_2(\lambda)}{2\lambda^2} I_1(k) - \frac{C_4(\lambda)}{4\lambda^4} [I_2(k) I_0(k) - I_1^2(k)]} \times e^{-\frac{p^2}{2\lambda^2}} e^{-\frac{p'^2}{2\lambda^2}}.
\]

We then fit the difference between the effective and exact inverse “on-shell” K-matrix \((p^2 = p'^2 = k^2)\) to an interpolating polynomial in \(k^2/\lambda^2\) to highest possible order,

\[
\Delta \left[ \frac{1}{K} \right] = A_0 + A_2 \frac{k^2}{\lambda^2} + A_4 \frac{k^4}{\lambda^4} + \cdots.
\]
The coefficients $A_i$ are minimized with respect to the variation in the parameters $C_i$ of the effective potential. The number of coefficients that can be minimized is given by the number of parameters appearing in the effective potential. This method is robust and numerically more stable than the matching at discrete momenta used by Lepage, allowing extension to higher orders [52].

By adjusting only the parameter $C_0(\lambda)$ we should eliminate the leading error. As each term is added to the effective potential, followed by the adjustment of the respective parameter, we expect the errors in the “on-shell” K-matrix to be systematically reduced by powers of $k^2/\lambda^2$. Thus, in a log-log plot for $\Delta[1/K]$ we expect to obtain straight lines with slope given by the dominant power of $k/\lambda$ in the error. We know of no rigorous proof of this expectation in EFT, although it may be possible to show that each $A_i$ can be taken to zero with a complete set of operators to the appropriate order in $p^2/\lambda^2$ in the hamiltonian. Note, however, that the number of couplings, $C_i$, grows more rapidly than the number of $A_i$'s and the relation between them is nonlinear. We assume that we can adjust operators of order $(p^2/\lambda^2)^m$ to remove errors in the inverse “on-shell” K-matrix of order $(k^2/\lambda^2)^m$, and all of our results confirm this.

The expansion given by Eq.(16), at best, becomes invalid when the momenta involved are of the same order as the cutoff, where one expects the short distance effects to be directly resolved. This point corresponds to the radius of convergence of the effective theory.

The identification of this power counting for the order of errors implies that the same systematic dependence on momentum is expected in the evaluation of errors for other low-energy observables. It is important to observe that this power counting applies to the predictions for the observables and not to the potential. As new terms are added and the corresponding couplings are adjusted, the lower-order couplings change. This occurs because the truncated potential still contains all orders in $(p/\lambda)^2$ due to the regulating function, so when the Lippmann-Schwinger equation is solved non-perturbatively all powers of momentum are generated and there is no simple linear relationship between powers of momentum in the potential and powers in the scattering observables.

One of our goals is to compare the EFT and the SRG methods. In principle, this could be achieved by comparing the results of the error analysis for low-energy scattering observables obtained in each case. However, the calculation of the K-matrix (or T-matrix) with the renormalized potential obtained using Wegner’s transformation [80] requires the numerical solution of an integral equation involving a non-separable kernel, which is unstable when simple techniques are applied. Since we expect the bound-state energy and the low-energy scattering observables to have similar error scaling (after fixing the effective potential to a given order), we focus on the bound state problem, which is numerically much more stable and allows a satisfactory error analysis. Using the effective potential (truncated at a given order) with the parameters fixed by fitting the inverse “on-shell” K-matrix, we diagonalize the effective hamiltonian numerically, obtaining the bound-state energies. This is implemented by discretizing the Schrödinger equation.

3 Similarity Renormalization Group

In this section we review the general formulation of the SRG developed by Glazek and Wilson [78, 79] and a specific transformation developed by Wegner [80]. The reader may wish to
3.1 Głazek-Wilson Formulation

Consider a system described by a Hamiltonian written in the form

\[ H = h + V , \]  

where \( h \) is the free Hamiltonian and \( V \) is an interaction.

In general, the Hamiltonian can couple states of all energy scales and such couplings can be a source of ultraviolet divergences. The goal of the SRG is to obtain an effective Hamiltonian in which the couplings between high and low-energy states are removed, while avoiding any problems from small energy denominators in effective interactions. The procedure is implemented by a unitary transformation that generates effective interactions that reproduce the effects of the eliminated couplings. The effective Hamiltonian cannot produce ultraviolet divergences at any order in perturbation theory as long as its matrix elements are finite.

In our discussion we will use the basis of eigenstates of the free Hamiltonian,

\[ h| \ i \rangle = \epsilon_i | \ i \rangle . \]  

We start by defining a bare Hamiltonian, \( H_\Lambda \), regulated by a very large cutoff \( \Lambda \) (here with dimensions of energy) on the change in free energy at the interaction vertices,

\[ H_\Lambda \equiv h + V_\Lambda , \] \[ V_\Lambda \equiv f_\Lambda V_\Lambda , \] \[ \overrightarrow{V}_\Lambda \equiv v + H_\Lambda^{ct} , \]

where \( f_\Lambda \) is a “similarity function”, \( \overrightarrow{V}_\Lambda \) is defined as the reduced interaction and \( H_\Lambda^{ct} \) are counterterms that must be determined through the process of renormalization in order to remove \( \Lambda \) dependence in physical quantities.

The similarity function \( f_\Lambda \) regulates the Hamiltonian by suppressing matrix elements between free states with significantly large energy difference and acts in the following way:

\[ < i | f_\Lambda H_\Lambda | j > \equiv \epsilon_i \delta_{ij} + f_\Lambda (\epsilon_i - \epsilon_j) < i | \overrightarrow{V}_\Lambda | j > \equiv \epsilon_i \delta_{ij} + f_{\Lambda \delta_{ij}} V_{\Lambda \delta_{ij}} . \]  

Typically, the similarity function is chosen to be a smooth function satisfying

\( (i) f_\Lambda (\epsilon_i - \epsilon_j) \to 1, \text{ when } |\epsilon_i - \epsilon_j| < \Lambda , \)
\( (ii) f_\Lambda (\epsilon_i - \epsilon_j) \to 0, \text{ when } |\epsilon_i - \epsilon_j| > \Lambda . \)  

In several papers the similarity function has been chosen to be a step function. Although useful for doing analytic calculations, such a choice can lead to pathologies. [87]

The similarity transformation is defined to act on the bare regulated Hamiltonian, \( H_\Lambda \), lowering the cutoff down to a scale \( \lambda \):

\[ H_\Lambda \equiv U(\lambda, \Lambda) H_\Lambda U^\dagger(\lambda, \Lambda) . \]
The renormalized Hamiltonian can be written in the general form

\[
H_\lambda \equiv h + V_\lambda, \tag{25}
\]

\[
V_\lambda \equiv f_\lambda \nabla_\lambda. \tag{26}
\]

The transformation is unitary, so \(H_\Lambda\) and \(H_\lambda\) produce the same spectra for observables. Also, if an exact transformation is implemented, the physical predictions using the renormalized Hamiltonian must be independent of the cutoff \(\lambda\) and \(H^{ct}_\Lambda\) is chosen so that they also become independent of \(\Lambda\) as \(\Lambda \to \infty\).

The unitarity condition is given by:

\[
U(\lambda, \Lambda) U^\dagger(\lambda, \Lambda) \equiv U^\dagger(\lambda, \Lambda) U(\lambda, \Lambda) \equiv 1. \tag{27}
\]

The similarity transformation \(U\) can be defined in terms of an anti-hermitian operator \(T_\lambda\) \((T_\lambda^\dagger = -T_\lambda)\) which generates infinitesimal changes of the cutoff energy scale,

\[
U(\lambda, \Lambda) \equiv \mathcal{T} \exp \left( \int_\lambda^\Lambda T_{\lambda'} \, d\lambda' \right), \tag{28}
\]

where \(\mathcal{T}\) orders operators from left to right in order of increasing energy scale \(\lambda'.\) Using

\[
T_\lambda = U(\lambda, \Lambda) \frac{dU(\lambda, \Lambda)}{d\lambda} = -\frac{dU(\lambda, \Lambda)}{d\lambda} U^\dagger(\lambda, \Lambda), \tag{29}
\]

and the unitarity condition Eq. (27), we can write Eq. (24) in a differential form,

\[
\frac{dH_\lambda}{d\lambda} = [H_\lambda, T_\lambda]. \tag{30}
\]

This is a first-order differential equation, which is solved with the boundary condition \(H_\lambda|_{\lambda \to \Lambda} \equiv H_\Lambda.\) The bare Hamiltonian is typically given by the canonical Hamiltonian plus counterterms that must be uniquely fixed to complete the renormalization.

The operator \(T_\lambda\) is defined by specifying how \(V_\lambda\) and \(h\) depend on the cutoff scale \(\lambda.\) For simplicity in this paper, we demand that \(h\) is independent of \(\lambda,\) although this may not lead to an increasingly diagonal effective hamiltonian in all cases. We also demand that no small energy denominators can appear in the hamiltonian. These constraints are implemented by the conditions

\[
\frac{dh}{d\lambda} \equiv 0, \tag{31}
\]

\[
\frac{dV_\lambda}{d\lambda} \equiv [V_\lambda, T_\lambda]. \tag{32}
\]

To obtain the renormalized Hamiltonian perturbatively, expand

\[
\nabla_\lambda = \nabla_\lambda^{(1)} + \nabla_\lambda^{(2)} + \cdots, \tag{33}
\]

\[
T_\lambda = T_\lambda^{(1)} + T_\lambda^{(2)} + \cdots, \tag{34}
\]

\[
H^{ct}_\lambda = H^{ct}_\lambda^{(2)} + H^{ct}_\lambda^{(3)} + \cdots, \tag{35}
\]

11
where the superscripts denote the order in the original interaction, $V$. A general form of these effective interactions is

$$\nabla^{(i)}_{\lambda} = - \sum_{j,k=1}^{\infty} \delta_{(j+k,i)} \int_{\Lambda}^{\Lambda'} d\lambda' \left[ V^{(j)}_{\lambda'} , T^{(k)}_{\lambda'} \right] + H^{(i),ct}_{\lambda}, \quad (36)$$

for $i = 2, 3, \ldots$, with $\nabla^{(1)}_{\lambda} = v$. For instance, the explicit form of the second-order effective interaction $\nabla^{(2)}_{\lambda}$ is

$$\nabla^{(2)}_{\lambda_{ij}} = \sum_{k} V_{ik} V_{kj} \left( \frac{g^{(\lambda \Lambda)}_{ikj}}{\Delta_{ik}} + \frac{g^{(\lambda \Lambda)}_{jki}}{\Delta_{jk}} \right) + H^{(2),ct}_{\lambda_{ij}}, \quad (37)$$

where

$$g^{(\lambda \Lambda)}_{ikj} \equiv \int_{\Lambda}^{\Lambda'} d\lambda' f^{\lambda'}_{ikj} \frac{df^{\lambda'ki}}{d\lambda'}, \quad (38)$$

$$\Delta_{ij} = \epsilon_{i} - \epsilon_{j}. \quad (39)$$

The counterterms $H^{(n),ct}_{\lambda}$ can be determined order-by-order using the idea of coupling-coherence \[112, 113\]. This is implemented by requiring the hamiltonian to reproduce itself in form under the similarity transformation, the only change being explicit dependence on the running cutoff in the operators and the implicit cutoff dependence in a finite number of independent running couplings. All other couplings depend on the cutoff only through their dependence on the independent couplings. In general, we also demand the dependent couplings to vanish when the independent couplings are taken to zero; i.e, the interactions are turned off. If the only independent coupling in the theory is $\alpha_{\lambda}$, the renormalized hamiltonian can be written as an expansion in powers of this coupling:

$$H_{\lambda} = h + \alpha_{\lambda} O^{(1)} + \alpha_{\lambda}^{2} O^{(2)} + \ldots . \quad (40)$$

In this way, the effective hamiltonian obtained using the similarity transformation is completely determined by the underlying theory. The procedure can be extended to arbitrarily high orders, although it becomes increasingly complex both analytically and numerically.

### 3.2 Wegner Formulation

The Wegner formulation of the SRG \[80\] is defined in a very elegant way in terms of a flow equation analogous to the SRG Equation in the Glazek-Wilson formalism \[78, 79\],

$$\frac{dH_{s}}{ds} = \left[ H_{s}, T_{s} \right]. \quad (41)$$

Here the hamiltonian $H_{s} = h + v_{s}$ evolves with a flow parameter $s$ that ranges from 0 to $\infty$. The flow-parameter has dimensions $1/(\text{energy})^{2}$ and is given in terms of the similarity cutoff $\lambda$ by $s = 1/\lambda^{2}$.

In Wegner’s scheme the similarity transformation in defined by an explicit form for the generator of the similarity transformation, $T_{s} = [H_{s}, H_{0}]$, which corresponds to the choice of a gaussian similarity function with uniform width. In the original formulation, Wegner advocates the inclusion of the full diagonal part of the hamiltonian at scale $s$ in $H_{0}$. For a
perturbative calculation of $H_s$, we can use the free hamiltonian, $H_0 = h$. With this choice, the flow equation for the hamiltonian is given by

$$\frac{dH_s}{ds} = [H_s, [H_s, h]]. \tag{42}$$

The reduced interaction, $V_{sij}$ (the interaction with the gaussian similarity function factored out) is defined by

$$V_{sij} = f_{sij} \nabla_{sij}, \tag{43}$$

$$f_{sij} = e^{-s\Delta^2_{ij}}. \tag{44}$$

Assuming that the free hamiltonian is independent of $s$, we obtain the flow equation for the reduced interaction,

$$\frac{dV_{sij}}{ds} = \sum_k (\Delta_{ik} + \Delta_{jk}) V_{sik} V_{skj} e^{-2s\Delta_{ik}\Delta_{jk}}, \tag{45}$$

where we use $\Delta^2_{ij} - \Delta^2_{ik} - \Delta^2_{jk} = -2\Delta_{ik}\Delta_{jk}$. We should emphasize that this is an exact equation.

To solve this equation we impose a boundary condition, $H_s|_{s \to s_0} \equiv H_{s_0}$. Then, we make a perturbative expansion,

$$\nabla_s = \nabla_s^{(1)} + \nabla_s^{(2)} + \cdots, \tag{46}$$

where the superscript implies the order in the bare interaction $\nabla_{s_0}$. It is important to observe that counterterms are implicit in the bare interaction and can be determined in the renormalization process using coupling coherence.

At first order we have

$$\frac{d\nabla_{sij}^{(1)}}{ds} = 0, \tag{47}$$

which implies

$$\nabla_{sij}^{(1)} = \nabla_{s_0ij}, \tag{48}$$

where $s$ is the final scale. Because of the dimensions of the flow parameter we have $s > s_0$, corresponding to a smaller cutoff. The “no cutoff limit” corresponds to $s_0 \rightarrow 0$.

At second order we have

$$\frac{d\nabla_{sij}^{(2)}}{ds} = \sum_k (\Delta_{ik} + \Delta_{jk}) V_{s_0ik} V_{s_0kj} e^{-2s\Delta_{ik}\Delta_{jk}}. \tag{49}$$

Integrating, we obtain

$$\nabla_{sij}^{(2)} = \frac{1}{2} \sum_k \nabla_{s_0ik} \nabla_{s_0kj} \left( \frac{1}{\Delta_{ik}} + \frac{1}{\Delta_{jk}} \right) \times$$

$$\times \left[ e^{-2s_0\Delta_{ik}\Delta_{jk}} - e^{-2s\Delta_{ik}\Delta_{jk}} \right]. \tag{50}$$

By construction, the Wegner transformation is unitary and avoids small energy denominators. The Wegner transformation is one of the Glazek-Wilson transformations, with the similarity function chosen to be $f_{\lambda ij} = e^{-\Delta^2_{ij}/\lambda^2}$. 

13
### 3.3 Strategy

In our applications of the SRG we use Wegner’s transformation. The renormalized hamiltonian for the non-relativistic delta-function potential in D-dimensions is given by

\[ H_\lambda(p, p') = p^2 \delta^{(D)}(p - p') + e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} \left[ \tilde{V}_\lambda^{(1)}(p, p') + \tilde{V}_\lambda^{(2)}(p, p') + \ldots \right], \quad (51) \]

where

\[ \tilde{V}_\lambda^{(1)}(p, p') = \frac{-\alpha_{\lambda,i}}{(2\pi)^D}, \quad (52) \]
\[ \tilde{V}_\lambda^{(2)}(p, p') = \alpha_{\lambda,i}^2 F_s^{(2)}(p, p'), \quad (53) \]
\[ \tilde{V}_\lambda^{(n)}(p, p') = \alpha_{\lambda,i}^n F_s^{(n)}(p, p'). \quad (54) \]

Here \( \lambda \) is a momentum cutoff (as opposed to the energy cutoff discussed above) related to the flow parameter by \( s = 1/\lambda^4 \). The index \( i \) denotes the order of the calculation for the running coupling.

The renormalized hamiltonian can be used to compute eigenvalues and eigenstates. Since the hamiltonian is derived perturbatively we expect cutoff dependent errors in the observables. Formally, we can regroup the terms in the renormalized hamiltonian and write it as a momentum expansion similar to the one for EFT in Eq. (6). The difference is that the expansion parameters are analytic functions of the running coupling \( \alpha_\lambda \). Expanding the operators \( F_s^{(n)}(p, p') \) in powers of \( p^2/\lambda^2 \) we obtain

\[ F_s^{(n)}(p, p') = z_0 + z_2 \frac{(p^2 + p'^2)}{2\lambda^2} + z_4 \frac{(p^4 + p'^4)}{4\lambda^4} + z_4' \frac{p^2 p'^2}{2\lambda^4} + \ldots, \quad (55) \]

where the \( z_i \)'s are constants. Regrouping the terms we obtain

\[ H_\lambda(p, p') = p^2 \delta^{(D)}(p - p') + e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} \left[ g_0(\alpha_\lambda) + g_2(\alpha_\lambda) \frac{(p^2 + p'^2)}{2\lambda^2} + \ldots \right], \quad (56) \]

where

\[ g_i(\alpha_\lambda) = a_i(\lambda) \alpha_\lambda + b_i(\lambda) \alpha_\lambda^2 + \ldots. \quad (57) \]

For \( D = 2 \) all \( a_i(\lambda), b_i(\lambda), \) etc. are simply constants, while for \( D = 1 \) they are functions of \( \lambda \) because of the implicit mass scale which is hidden when we set \( 2m = 1 \).

We can identify three interdependent sources of errors in the perturbative similarity renormalization group when the hamiltonian given by Eq. (56) is truncated and used to compute a physical quantity:

a) errors introduced by the truncation of the hamiltonian at a given order in \( p^2/\lambda^2 \), which are analogous to the errors in EFT;

b) errors introduced by the truncation of the hamiltonian at a given order in the running coupling \( \alpha_{\lambda,i} \), which correspond to the use of an approximation for the functions \( g_i \);

c) errors introduced by the approximation for the running coupling \( \alpha_{\lambda,i} \).
In the actual calculation using the hamiltonian given by Eq. (51) errors of type (a) do not appear directly because we do not truncate the operators that appear in the hamiltonian. However, errors of type (b) can be understood as coming from approximating the couplings in front of the operators in Eq. (56). In EFT these couplings are fit to data, but we approximate them as in Eq. (57) and this leads to an approximate cancellation of the power-law errors completely removed by these operators in EFT. Errors of type (c) appear in our calculations only because we do not fit the canonical coupling to data at each scale, but fix it at a given scale and evolve it perturbatively from that scale. The strategy we would use for a realistic theory (e.g., QED and QCD) is the following:

1) Obtain the renormalized hamiltonian using the similarity transformation and coupling-coherence, truncating the hamiltonian at a given order in powers of $\alpha_{\lambda,i}$.

2) Fix the coupling $\alpha_\lambda$ by fitting an observable (e.g., a bound-state energy).

3) Evaluate other observables (e.g., scattering phase shifts).

As pointed out before, the evaluation of scattering observables with the similarity hamiltonian with standard techniques is complicated and so in our examples we focus on the bound state errors. We fix the coupling at some scale using a given renormalization prescription and use the flow-equation to obtain the coupling as a function of the cutoff $\lambda$ to a given order. We then perform a sequence of bound-state calculations with better approximations for the hamiltonian such that the errors in the bound-state energy are systematically reduced. Once the sources of errors are identified, it becomes relatively simple to analyze order-by-order how such errors scale with the cutoff $\lambda$. In principle, to completely eliminate the errors proportional to some power $m$ in the momentum expansion we should use the similarity hamiltonian with the exact running coupling (renormalized to all orders) and include the contributions up to $O(p^m/\lambda^m)$ coming from all effective interactions (all orders in $\alpha_\lambda$). Since we use a perturbative approximation for the running coupling and only a finite number of effective interactions, we do not expect the scaling analysis for the errors in SRG to be as simple as in the EFT approach. Some details of this scaling analysis are presented later for the specific examples we work out. We should emphasize that in a realistic calculation we would fit the coupling $\alpha_\lambda$ to an observable. This nonperturbative renormalization eliminates the dominant source of errors we display in SRG calculations in this paper. We choose to renormalize the coupling perturbatively in this paper because the only observable we compute is the single bound state energy of a delta-function potential, and fitting this energy would prevent us from displaying errors.

4 One-Dimensional Delta-function Potential

Consider a system of two non-relativistic particles in one dimension interacting via an attractive Dirac delta-function, contact or zero range potential. The Schrödinger equation in position space (with $2m = 1$ and $\hbar = 1$), separating out the center of mass motion, is given by

$$-\frac{d^2\Psi(x)}{dx^2} - \alpha_0 \delta^{(1)}(x) \Psi(x) = E \Psi(x).$$

(58)
The coupling constant $\alpha_0$ completely fixes the underlying theory.

4.1 Exact Solution

The exact solutions of this equation are familiar to all students of quantum mechanics. There is one bound state, $(E < 0)$ with binding energy $E_0 = -E = \frac{\alpha_0^2}{4}$ and normalized wave-function

$$\Psi(x) = \sqrt{\frac{\alpha_0}{2}} e^{-\frac{\alpha_0}{4}|x|}. \quad (59)$$

In the scattering problem $(E = k^2 > 0)$ the Schrödinger equation is equivalent to a Lippmann-Schwinger equation, whose solution with outgoing waves is given by

$$\Psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \left[ \frac{2ik}{2ik + \alpha_0} \right]. \quad (60)$$

The problem is easily solved in momentum space, also. We do this to lay the groundwork for our first illustration of renormalization techniques. The corresponding Schrödinger equation in momentum space is given by

$$p^2 \Phi(p) - \frac{\alpha_0}{2\pi} \int_{-\infty}^{\infty} dq \ \Phi(q) = E \Phi(p), \quad (61)$$

where $\Phi(p)$ is the Fourier transform of the position space wave-function,

$$\Phi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ \Psi(x) e^{-ipx}. \quad (62)$$

Identifying the integral in Eq. (61) as the position space wave-function at the origin

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq \ \Phi(q) = \Psi(0), \quad (63)$$

we obtain

$$p^2 \Phi(p) - \frac{\alpha_0}{\sqrt{2\pi}} \Psi(0) = E \Phi(p). \quad (64)$$

First, we consider the bound-state problem. By rearranging the terms, we obtain

$$\Phi(p) = \frac{\alpha_0}{\sqrt{2\pi}} \frac{\Psi(0)}{(p^2 + E_0)}. \quad (65)$$

Integrating the above equation in $p$, we obtain

$$1 = \frac{\alpha_0}{2\pi} \int_{-\infty}^{\infty} dp \ \frac{1}{(p^2 + E_0)}. \quad (66)$$

This integral fixes the binding energy, $E_0 = \frac{\alpha_0^2}{4}$. The corresponding bound-state wave-function in momentum space is

$$\Phi(p) = \frac{1}{\sqrt{2\pi}} \frac{2 E_0^2}{(p^2 + E_0)} \cdot (67)$$
Solving for the scattering states we obtain the solution corresponding to outgoing waves

\[ \Phi_k(p) = \delta(p - k) + \frac{\alpha_0}{2\pi} \left[ \frac{2ik}{2ik + \alpha_0} \right] \frac{1}{(p^2 - k^2 - i\epsilon)}, \]

(68)

where \( k = \sqrt{E} \).

All scattering observables can be obtained from the T-matrix, which is given by a Lippmann-Schwinger equation:

\[ T(p, p'; k) = V(p, p') + \int_{-\infty}^{\infty} dq \frac{V(p, q)}{k^2 - q^2 + i\epsilon} T(q, p'; k). \]

(69)

The “on-shell” \( (p^2 = p'^2 = k^2) \) T-matrix for a Dirac delta-function potential is

\[ T_0(k) = -\frac{\alpha_0}{2\pi} \left[ \frac{2ik}{2ik + \alpha_0} \right]. \]

(70)

A more convenient way to treat the scattering problem for our purposes is to use the K-matrix, which satisfies the Lippmann-Schwinger equation

\[ K(p, p'; k) = V(p, p') + \mathcal{P} \int_{-\infty}^{\infty} dq \frac{V(p, q)}{k^2 - q^2} K(q, p'; k). \]

(71)

This equation is similar to that for the T-matrix, except standing-wave boundary conditions are imposed so that the usual \( i\epsilon \) prescription is replaced by the principal value \( \mathcal{P} \). As a result, the K-matrix is real. The on-shell K-matrix is given by

\[ K_0(k) = -\frac{\alpha_0}{2\pi}. \]

(72)

The “on-shell” K-matrix and T-matrix are related by the expression

\[ K_0(k) = -\frac{T_0(k)}{1 - \frac{i\pi}{k} T_0(k)}. \]

(73)

Using either

\[ k \cot \delta_0(k) - ik = -\frac{k^2}{\pi} \frac{1}{T_0(k)}, \]

(74)

or

\[ k \cot \delta_0(k) = -\frac{k^2}{\pi} \frac{1}{K_0(k)}, \]

(75)

the phase-shifts are easily calculated:

\[ \cot \delta_0(k) = \frac{2k}{\alpha_0}. \]

(76)

Although the one-dimensional problem can be solved exactly without regularization and renormalization, for the sake of further comparison with the two-dimensional case and to illustrate the SRG and EFT techniques, it is worth analyzing what happens when a momentum cutoff is introduced. If the integral in Eq. (66) is regulated by a cutoff \( \Lambda \),

\[ 1 = \frac{\alpha_0}{2\pi} \int_{-\Lambda}^{\Lambda} dp \frac{1}{(p^2 + E_0)}, \]

(77)
we obtain a transcendental equation, whose solution gives a cutoff-dependent binding energy:

\[ \frac{\pi}{\alpha_0} \sqrt{E_0(\Lambda)} = \arctg \left( \frac{\Lambda}{\sqrt{E_0(\Lambda)}} \right). \]  

Clearly, the cutoff dependence of the bound state energy is eliminated if we let the coupling run like

\[ \alpha_0 \rightarrow \alpha_\Lambda = \frac{\pi \sqrt{E_0}}{\arctg \left( \frac{\Lambda}{\sqrt{E_0}} \right)}. \]  

For \( \Lambda >> \sqrt{E_0} \) this cutoff dependence can be eliminated perturbatively by using the expansion of \( \alpha_\Lambda \) in powers of \( \sqrt{E_0}/\Lambda \) to a given order:

\[ \alpha_\Lambda = 2\sqrt{E_0} \left[ 1 + \frac{2}{\pi} \frac{\sqrt{E_0}}{\Lambda} + \frac{4}{\pi^2} \frac{E_0}{\Lambda^2} + O \left( \frac{E_0^{3/2}}{\Lambda^3} \right) \right]. \]  

In terms of the “bare” coupling, \( \alpha_0 \), we obtain

\[ \alpha_\Lambda = \alpha_0 \left[ 1 + \frac{1}{\pi} \frac{\alpha_0}{\Lambda} + \frac{1}{\pi^2} \frac{\alpha_0^2}{\Lambda^2} + O \left( \frac{\alpha_0^3}{\Lambda^3} \right) \right]. \]  

Of course \( \alpha_\Lambda \rightarrow \alpha_0 \) when \( \Lambda \rightarrow \infty \).

### 4.2 Effective Field Theory Method

We now apply the EFT approach to the one-dimensional delta-function problem. The power counting scheme in this case is very simple and can be understood easily by analyzing how the errors scale with the cutoff \( \lambda \). For simplicity, we consider the errors in the inverse K-matrix.

As described before, the EFT method is designed to systematically eliminate errors order by order in \( k/\lambda \). Beginning with the leading-order prediction, we use the effective potential with one parameter. Evaluating the difference \( \Delta \left[ 1/K^{(0)} \right] = 1/K_\Lambda^{(0)} - 1/K_0 \) and expanding it in powers of \( k/\lambda \) we obtain:

\[ \Delta \left[ 1/K^{(0)} \right] = \left( \frac{1}{C_0} - \frac{2}{\sqrt{\pi}} \frac{\alpha_0}{\lambda} + \frac{2\pi}{\alpha_0} \right) + \left( \frac{1}{C_0} - \frac{2}{3\lambda} \right) \frac{k^2}{\lambda^2} + O \left( \frac{k^4}{\lambda^4} \right). \]  

Note that the regulator generates only power-law errors, which indicates that there are no divergences as \( \lambda \rightarrow \infty \) and renormalization is required only to remove finite errors. Clearly, the leading order error is eliminated by choosing

\[ C_0 = \frac{\alpha_0/2\pi}{1 - \frac{1}{\sqrt{\pi}} \frac{\alpha_0}{\lambda}}, \]  

and the remaining errors are dominated by the term

\[ O \left( \frac{k^2}{\lambda^2} \right) = \left( \frac{2\pi}{\alpha_0} + \frac{4\sqrt{\pi}}{3\lambda} \right) \frac{k^2}{\lambda^2}. \]  

18
For the next-to-leading order prediction, we use the effective potential with two parameters, obtaining the expansion

$$
\Delta \left[ \frac{1}{K^{(2)}} \right] = A_0 + A_2 \frac{k^2}{\lambda^2} + \mathcal{O} \left( \frac{k^4}{\lambda^4} \right),
$$

(85)

where

$$
A_0 = \frac{-2 \left( 1 + \frac{C_2 \sqrt{\pi}}{2 \lambda} \right)^2 + 2 \left( \frac{2 \sqrt{\pi} C_0}{\lambda} - \frac{C_2^2 \pi}{4 \lambda^2} \right)}{-2 C_0 + \frac{C_2^2 \sqrt{\pi}}{4 \lambda}} + \frac{2 \pi}{\alpha_0},
$$

(86)

$$
A_2 = \frac{8 C_0^2 \sqrt{\pi} + 4 C_0 \lambda \left( 3 + 3 \frac{C_2 \sqrt{\pi}}{2 \lambda} + \frac{5 C_2^2 \lambda}{4 \lambda^2} \right)}{3 \lambda \left( -2 C_0 + \frac{C_2^2 \sqrt{\pi}}{4 \lambda} \right)^2}
$$

$$
- \frac{C_2 \lambda \left( 12 + \frac{33 C_2 \sqrt{\pi}}{2 \lambda} + \frac{15 C_2^2 \lambda}{2 \lambda^2} + \frac{5 C_2^3 \lambda^{3/2}}{4 \lambda^3} \right)}{3 \lambda \left( -2 C_0 + \frac{C_2^2 \sqrt{\pi}}{4 \lambda} \right)^2}.
$$

(87)

The parameters $C_0$ and $C_2$ are determined by solving the system of two coupled non-linear equations given by

$$
A_0 = 0, \quad A_2 = 0.
$$

(88)

One of the solutions gives a parameter $C_0$ that diverges when $\lambda \to \infty$ and so is not acceptable. The other solution,

$$
C_0 = \frac{-27 \sqrt{\pi} \alpha_0 \lambda^2 + 16 \alpha_0^2 \lambda + 12 \lambda^3 \pi}{2 \sqrt{\pi} \left( 4 \alpha_0^2 - 9 \alpha_0 \lambda \sqrt{\pi} + 6 \lambda^2 \pi \right)}
$$

$$
- \frac{\sqrt{6} \lambda^2}{\pi} \left( \alpha_0 - \lambda \sqrt{\pi} \right)^2 \left( 4 \alpha_0^2 - 9 \alpha_0 \lambda \sqrt{\pi} + 6 \lambda^2 \sqrt{\pi} \right),
$$

(89)

$$
C_2 = \frac{-8 \alpha_0^2 \sqrt{\pi} \lambda + 18 \alpha_0 \pi \lambda^2 - 12 \lambda^3 \pi^{3/2}}{\pi \left( 4 \alpha_0^2 - 9 \alpha_0 \lambda \sqrt{\pi} + 6 \lambda^2 \pi \right)}
$$

$$
+ \frac{2 \sqrt{24} \alpha_0^4 \lambda^2 \pi - 102 \alpha_0^3 \lambda^3 \pi^{3/2} + 168 \alpha_0^2 \lambda^2 \pi^2 - 126 \alpha_0 \lambda^5 \pi^{5/2} + 36 \lambda^6 \pi^3}{\pi \left( 4 \alpha_0^2 - 9 \alpha_0 \lambda \sqrt{\pi} + 6 \lambda^2 \pi \right)},
$$

(90)

is such that $C_0 \to -\alpha_0/2\pi$, $C_2 \to 0$ when $\lambda \to \infty$ and leads to the elimination of errors up to $\mathcal{O} \left( \frac{k^2}{\lambda^2} \right)$.

This procedure can be systematically extended to higher orders, although with exponentially increasing algebraic complexity. It is important to observe that the scaling of the parameters, and hence the power-counting scheme, depends on the regularization procedure.
Figure 2: The leading-order and next-to-leading-order EFT expansion parameters for the one-dimensional delta-function potential.

In Fig. 2 we plot the parameters obtained in the leading order and next-to-leading order calculation as a function of the cutoff. Note that when we add the term proportional to $C_2$ the parameter $C_0$ changes. The numerical evaluation here and in what follows is implemented with the underlying $D = 1$ theory fixed by choosing $\alpha_0 = 2$, which corresponds to an exact binding energy $E_0 = 1$.

We proceed by evaluating the errors in the binding energy, using the effective potential with one, two and four parameters adjusted to fit the inverse K-matrix. Instead of solving systems of non-linear coupled equations, we fix the coefficients by applying the minimization procedure described in Section 2, which is simpler numerically and leads to the same results.

We solve the Schrödinger equation numerically for different values of the cutoff $\lambda$. First, we introduce an integration cutoff $\Lambda_0 >> \lambda$, such that

$$p^2 \Phi_\lambda(p) + \int_{-\Lambda_0}^{\Lambda_0} dq \ V_\lambda(p,p') \ \Phi_\lambda(q) = E_\lambda \Phi_\lambda(p) . \quad (91)$$

We discretize the Schrödinger equation (choosing a gaussian quadrature)

$$\sum_{j=1}^{N} w_j \left[ p_i p_j \delta_{ij} \frac{1}{w_j} + V_\lambda(p_i,p_j) \right] \Phi_\lambda(p_j) = E_\lambda \Phi_\lambda(p_i) , \quad (92)$$

obtaining a matrix equation

$$(M - E_\lambda \mathbf{1}) \Phi_\lambda = 0 , \quad (93)$$
where
\[
M_{ij} = w_j \left[ p_i p_j \delta_{ij} \frac{1}{w_j} + V_\lambda(p_i, p_j) \right].
\] (94)

Figure 3: The EFT error in the binding energy for the one-dimensional delta-function using one, two and four parameters. The exact theory is fixed by choosing \( \alpha_0 = 2 \).

Next, the matrix \( M \) is symmetrized, reduced to tridiagonal form and the resulting matrix equation is solved by the QL method. The smallest (negative) eigenvalue corresponds to the bound-state energy. We then refine the result using Richardson extrapolation \(^{114}\) in order to eliminate the dominant errors due to the integration cutoff, which are proportional to \( E_0/\Lambda_0 \). Removing these errors should not be confused with removing \( \lambda \)-dependent errors. Finally, we compare the resulting bound-state energy, \( E_\lambda \), with the exact value.

In Fig. 3 we show in a log-log plot the absolute values for the relative errors in the bound-state energy as functions of \( E_0/\lambda^2 \). As expected, we obtain straight lines with slope given by the dominant power of \( E_0/\lambda^2 \) in the error, in agreement with the scaling analysis.

This result clearly demonstrates the systematic improvement in the accuracy of the effective potential for describing the exact theory in the low-energy regime. We also note that the errors corresponding to each approximation become comparable as \( \lambda^2 \) approaches the binding energy, indicating the breakdown of the effective theory. In this way, the error plot provides a graphical illustration of the radius of convergence, the point where all the error lines converge.
4.3 Similarity Renormalization Group Approach

In the one-dimensional case the canonical hamiltonian in momentum space with a delta-function potential can be written as $H(p, p') = h(p, p') + V(p, p')$, where $h(p, p') = p^2 \delta(p - p')$ corresponds to the free hamiltonian and $V(p, p') = -\alpha_0/2\pi$ corresponds to the Fourier transform of the delta-function potential. This hamiltonian does not produce divergences so an initial cutoff is not required to define the problem.

The flow equation obtained with Wegner’s transformation in terms of matrix elements in the basis of free states is given by

$$\frac{dV_s(p, p')}{ds} = -(p^2 - p'^2)^2 \ V_s(p, p') - \int_{-\infty}^{\infty} dk \ (2k^2 - p^2 - p'^2) \ V_s(p, k) \ V_s(k, p'),$$

(95)

with the initial condition $H_{s=0}(p, p') = H(p, p')$. The reduced interaction $\tilde{V}_s(p, p')$ is defined such that

$$V_s(p, p') = e^{-s(p^2 - p'^2)^2} \ \tilde{V}_s(p, p').$$

(96)

Now, assuming that $h$ is cutoff independent we obtain the flow equation for the reduced interaction,

$$\frac{d\tilde{V}_s}{ds} = -e^{-2s(p^2 + p'^2)} \ int_{-\infty}^{\infty} dk \ (2k^2 - p^2 - p'^2) \ e^{-2s[k^2 - k^2(p^2 + p'^2)]} \times \tilde{V}_s(p, k) \ V_s(k, p').$$

(97)

In order to solve this equation, we employ a perturbative expansion

$$\tilde{V}_s(p, p') = \tilde{V}_s^{(1)}(p, p') + \tilde{V}_s^{(2)}(p, p') + \cdots,$$

(98)

starting with

$$\tilde{V}_s^{(1)}(p, p') = -\frac{\alpha_s}{2\pi}.$$  

(99)

Using coupling-coherence [113] we assume a solution in the form of an expansion in powers of $\alpha_s/2\pi$,

$$\tilde{V}_s(p, p') = \frac{\alpha_s}{2\pi} + \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{2\pi} \right)^n F_s^{(n)}(p, p'),$$

(100)

with the additional condition that the operators $F_s^{(n)}(p, p')$ vanish when $p = p' = 0$ or, equivalently, when $s = 0$. Using the solution Eq. (99) in Eq. (97) we obtain:

$$\frac{d\tilde{V}_s}{ds} = -\frac{1}{2\pi} \ \frac{d\alpha_s}{ds} + \sum_{n=2}^{\infty} \frac{1}{(2\pi)^n} \left[ \ n \ \alpha_s^{n-1} \ \frac{d\alpha_s}{ds} \ F_s^{(n)}(p, p') + \alpha_s^n \ \frac{dF_s^{(n)}(p, p')}{ds} \right]$$

$$= \int_{-\infty}^{\infty} dk \ (2k^2 - p^2 - p'^2) \ e^{-2s[p^2 + k^2 - k^2(p^2 + p'^2)]}$$

$$\times \left[ -\frac{\alpha_s}{2\pi} + \sum_{n=2}^{\infty} \left( \frac{\alpha_s}{2\pi} \right)^n F_s^{(n)}(p, k) \right] \left[ -\frac{\alpha_s}{2\pi} + \sum_{m=2}^{\infty} \left( \frac{\alpha_s}{2\pi} \right)^m F_s^{(m)}(k, p') \right].$$

(101)

This equation is solved iteratively order by order in $\alpha_s/2\pi$. If $\alpha_s/2\pi$ is small the operator $\tilde{V}_s^{(1)}(p, p')$ can be identified as the dominant term in the expansion of $\tilde{V}_s(p, p')$ in powers of
$p$ and $p'$. In fact, since in the limit when $p$ and $p'$ approach zero all $F_s^{(n)}(p,p')$ vanish, the only term left is $-\alpha_s/(2\pi)$, which in the $D = 1$ case corresponds to a relevant operator (even though the coupling is dimensionless, it is multiplied by $2m = 1$). The higher-order terms correspond to irrelevant operators. Thus, at each order $n$ the solution can be viewed as a double expansion obtained by solving the equations for $\alpha_s$ and $F_s^{(n)}(p,p')$ separately.

At second-order we have

$$-\frac{1}{2\pi} \frac{d\alpha_s}{ds} + \frac{1}{(2\pi)^2} \alpha_s^2 \frac{dF_s^{(2)}(p,p')}{ds} = -\alpha_s^2 I_s^{(2)}(p,p') , \quad (102)$$

where

$$I_s^{(2)}(p,p') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk \left( 2k^2 - p^2 - p'^2 \right) e^{-2s[p^2+p'^2+k^4-k^2(p^2+p'^2)]} . \quad (103)$$

The equation for the running coupling, $\alpha_s$, is obtained by taking the limit $(p,p') \to 0$:

$$\frac{1}{2\pi} \frac{d\alpha_s}{ds} = \alpha_s^2 I_s^{(2)}(0,0) , \quad (104)$$

where

$$I_s^{(2)}(0,0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk \left( 2k^2 e^{-2sk^4} = \frac{1}{(2\pi)^2} \Gamma(3/4) \frac{1}{s^{3/4}} . \quad (105)$$

Integrating Eq. (104) we obtain:

$$\alpha_{s,2} = \frac{\alpha_{s0}}{1 - \eta_2 \frac{\alpha_{s0}}{2\pi} (s^{1/4} - s_0^{1/4})} , \quad (106)$$

$$\eta_2 = 4\Gamma(3/4) \frac{23/4}{s^{3/4}} . \quad (107)$$

The extra subscript on $\alpha$ displays the order of the right-hand side of Eq. (104). Thus, knowing the value of $\alpha_{s0}$ for a given $s_0$, Eq. (106) can be used to obtain the running coupling $\alpha_s$ for any value $s$. The choice $s_0 = 0$ corresponds to $\alpha_{s0} = \alpha_0$ ("bare coupling"), such that

$$\alpha_{s,2} = \frac{\alpha_0}{1 - \eta_2 \frac{\alpha_0}{2\pi} s^{1/4}} . \quad (108)$$

In terms of the cutoff $\lambda$ we obtain:

$$\alpha_{\lambda,2} = \frac{\alpha_0}{1 - \eta_2 \frac{\alpha_0}{2\pi} \frac{1}{\lambda}} . \quad (109)$$

The equation for $F_s^{(2)}(p,p')$ is given by:

$$\frac{1}{(2\pi)^2} \frac{dF_s^{(2)}(p,p')}{ds} = I_s^{(2)}(0,0) - I_s^{(2)}(p,p') . \quad (110)$$

Integrating we obtain:

$$F_s^{(2)}(p,p') = (2\pi)^2 \int_{0}^{s} ds' \left[ I_s^{(2)}(0,0) - I_s^{(2)}(p,p') \right]$$

$$= \int_{-\infty}^{\infty} dk \left[ \frac{1}{k^2} \left( 1 - e^{-2sk^4} \right) \right.$$  

$$+ \left[ e^{-2s(k^2-p^2)(k^2-p'^2)} - 1 \right] \left[ \frac{1}{2(k^2-p^2)} + \frac{1}{2(k^2-p'^2)} \right] \right] . \quad (111)$$
The integral over $k$ must be performed numerically.

At third-order we have:

$$
- \frac{1}{2\pi} \frac{d\alpha_s}{ds} + \frac{1}{(2\pi)^2} \alpha_s \frac{dF_s^{(2)}(p, p')}{ds} + \frac{2}{(2\pi)^2} \alpha_s \frac{d\lambda}{ds} F_s^{(2)}(p, p') + \frac{1}{(2\pi)^3} \alpha_s^3 \frac{dF_s^{(3)}(p, p')}{ds} = -\alpha_s^2 I_s^{(2)}(p, p') + \alpha_s^3 I_s^{(3)}(p, p') ,
$$

where

$$
I_s^{(3)}(p, p') = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} dk \left( 2k^2 - p^2 - p'^2 \right) e^{-2s[p^2 p'^2 + k^4 - k^2(p^2+p'^2)]} [F_s^{(2)}(p, k) + F_s^{(2)}(k, p')] .
$$

Taking the limit $p, p' \to 0$ we obtain:

$$
\frac{1}{2\pi} \frac{d\alpha_s}{ds} = \alpha_s^2 I_s^{(2)}(0, 0) - \alpha_s^3 I_s^{(3)}(0, 0) .
$$

For dimensional reasons Eq. (114) takes the form

$$
\frac{d\alpha_s}{ds} = B_2 \alpha_s^2 - B_3 \alpha_s^3 ,
$$

where $B_2$ and $B_3$ can be obtained by evaluating $I_s^{(2)}(0, 0)$ and $I_s^{(3)}(0, 0)$ for $s = 1$. In terms of the cutoff $\lambda$ we obtain:

$$
\frac{d\alpha_s}{\lambda} = B_2 \alpha_s^2 - B_3 \alpha_s^3 .
$$

A simple scaling analysis shows that the similarity function $f_{\lambda ij} = e^{-\Delta^2_{ij}/\lambda^2}$ introduces errors that are expected to scale as inverse powers of $\lambda$. The leading-order errors are proportional to $\sqrt{E_0}/\lambda$ and are expected to be eliminated by using the running coupling renormalized to second-order. The next-to-leading-order errors are proportional to $E_0/\lambda^2$ and are expected to be eliminated by using the running coupling renormalized to third-order and including the second-order effective interaction, since both introduce corrections proportional to $E_0/\lambda^2$, which therefore enter at the same level. Thus, to show the systematic improvement in the perturbative approximation in our study of the one-dimensional delta-function we compute the running coupling to third-order and the operators to second-order. As in the EFT calculation, we choose the exact binding-energy $E_0 = 1$ ($\alpha_0 = 2$) and calculate the bound-state energy in a given approximation by solving the Schrödinger equation numerically. In Fig. 4 we show the running coupling renormalized to second and third-order as a function of the cutoff $\lambda$. In Fig. 5 we show in a log-log plot the absolute value of the relative errors in the binding-energy as a function of the ratio $E^{(0)}/\lambda^2$ using the following approximations for the interaction:
(a) relevant operator with “bare” coupling \( \alpha_0 \),

\[
V_\lambda(p, p') = -\frac{\alpha_0}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{4\lambda^4}};
\]  

(b) relevant operator with running coupling renormalized to second-order (\( \alpha_{\lambda,2} \)),

\[
V_\lambda(p, p') = -\frac{\alpha_{\lambda,2}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{4\lambda^4}};
\]  

(c) relevant operator plus second-order irrelevant operator with running coupling renormalized to third-order (\( \alpha_{\lambda,3}, F^{(2)}_\lambda \)),

\[
V_\lambda(p, p') = \left[ -\frac{\alpha_{\lambda,2}}{2\pi} + \left( \frac{\alpha_{\text{lambda,2}}}{2\pi} \right)^2 F^{(2)}_\lambda(p, p') \right] e^{-\frac{(p^2 - p'^2)^2}{4\lambda^4}};
\]  

(d) relevant operator with running coupling renormalized to third-order (\( \alpha_{\lambda,3} \)),

\[
V_\lambda(p, p') = -\frac{\alpha_{\lambda,3}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{4\lambda^4}};
\]  

(e) relevant operator plus second-order irrelevant operator with running coupling renormalized to third-order, (\( \alpha_{\lambda,3}, F^{(2)}_\lambda \)),

\[
V_\lambda(p, p') = \left[ -\frac{\alpha_{\lambda,2}}{2\pi} + \left( \frac{\alpha_{\text{lambda,2}}}{2\pi} \right)^2 F^{(2)}_\lambda(p, p') \right] e^{-\frac{(p^2 - p'^2)^2}{4\lambda^4}}.
\]  

With Hamiltonian (a) (unrenormalized) the dominant error is proportional to \( \sqrt{E_0}/\lambda \), which is the leading-order error introduced by the regulator. With the renormalized Hamiltonians (b), (c), (d) and (e), for intermediate values of \( \lambda \), we obtain straight lines with slope given by the dominant power of \( E_0/\lambda^2 \) in the error. With Hamiltonian (b) the slope is one, corresponding to an error proportional to \( E_0/\lambda^2 \). With Hamiltonians (c) and (d) there is only a small shift in the errors; there is no significant change in the slope. With Hamiltonian (e) the slope is two, corresponding to an error proportional to \( (E_0/\lambda^2)^2 \). Analogous to the EFT calculation, for small values of \( \lambda \) the lines converge, indicating the breakdown of the perturbative expansion. It is important to note that for very large values of the cutoff \( \lambda \) the scaling behavior might change, since higher-order contributions not included in a given approximation for the Hamiltonian can become important. In our calculations we did not analyze this regime because numerical accuracy limits the maximum range of cutoffs on the integrals.
Figure 4: The SRG running coupling for the one-dimensional delta-function.

Figure 5: The SRG error in the binding energy for the one-dimensional delta-function using various approximations for the SRG hamiltonian. The exact theory is fixed by choosing $\alpha_0 = 2$. 
5 Two-Dimensional Delta-function Potential

We now consider the case of two nonrelativistic particles in two dimensions interacting via an attractive Dirac delta-function potential. The Schrödinger equation for relative motion in position space (with $\hbar = 1$), can be written as:

$$- \nabla^2 \Psi(r) - \alpha_0 \delta^{(2)}(r) \, \Psi(r) = E \, \Psi(r).$$  \hspace{1cm} (123)

Both the delta-function potential in two dimensions and the kinetic energy operator scale as $1/r^2$, therefore, the coupling $\alpha_0$ is dimensionless. As a consequence, the hamiltonian is scale invariant (i.e., there is no intrinsic energy scale) and we can anticipate the presence of logarithmic ultraviolet divergences, analogous to those appearing in QED and QCD. The problem requires renormalization. In this subsection we present the standard method that produces an exact solution analytically, using simple regularization and renormalization schemes \[20\].

5.1 Exact Solution

We start with the Schrödinger equation in momentum space,

$$p^2 \Phi(p) - \frac{\alpha_0}{(2\pi)^2} \int d^2q \, \Phi(q) = E \, \Phi(p),$$  \hspace{1cm} (124)

where $\Phi(p)$ is the Fourier transform of the position space wave-function,

$$\Phi(p) = \frac{1}{2\pi} \int d^2r \, \Psi(r) \, e^{-i p \cdot r}.$$  \hspace{1cm} (125)

As a consequence of scale invariance, if there is any negative energy solution to Eq. (124) then it will admit solutions for any $E < 0$. This corresponds to a continuum of bound states with energies extending down to $-\infty$, so the system is not bounded from below. By rearranging the terms in the Schrödinger equation we obtain

$$\Phi(p) = \frac{\alpha_0}{2\pi} \frac{\Psi(0)}{(p^2 + E_0)},$$  \hspace{1cm} (126)

where $\Psi(0)$ is the position space wave-function at the origin and $E_0 > 0$ is the binding energy.

To obtain the eigenvalue condition for the binding energy, we can integrate both sides of Eq. (126):

$$1 = \frac{\alpha_0}{2\pi} \int_0^{\infty} dp \, p \, \frac{1}{\left(p^2 + E_0\right)}.$$  \hspace{1cm} (127)

The integral on the r.h.s. diverges logarithmically, so the problem is ill-defined.

The conventional way to deal with this problem is renormalization. First, we regulate the integral with a momentum cutoff, obtaining

$$1 = \frac{\alpha_0}{2\pi} \int_0^{\Lambda} dp \, p \, \frac{1}{\left(p^2 + E_0\right)} = \frac{\alpha_0}{4\pi} \ln \left(1 + \frac{\Lambda^2}{E_0}\right),$$  \hspace{1cm} (128)

so that

$$E_0 = \frac{\Lambda^2}{e^{\frac{4\pi}{\alpha_0}} - 1}.$$  \hspace{1cm} (129)
Clearly, if the coupling $\alpha_0$ is fixed then $E_0 \to \infty$ as $\Lambda \to \infty$. In order to eliminate the divergence and produce a finite, well-defined bound state we can renormalize the theory by demanding that the coupling runs with the cutoff $\Lambda$ in such a way that the binding energy remains fixed as the cutoff is removed:

$$\alpha_0 \to \alpha_\Lambda = \frac{4\pi}{\ln \left(1 + \frac{\Lambda^2}{E_0}\right)}.$$  (130)

The dimensionless renormalized running coupling $\alpha_\Lambda$ that characterizes the strength of the interaction is therefore replaced by a new (dimensionful) parameter $E_0 > 0$, the binding energy of the system. This is a simple example of dimensional transmutation: even though the original “bare” hamiltonian is scale invariant, the renormalization procedure leads to a scale that characterizes the physical observables. Note that $E_0$ can be chosen arbitrarily, fixing the energy scale of the underlying (renormalized) theory. It is also interesting to note that the renormalized running coupling $\alpha_\Lambda$ vanishes as $\Lambda \to \infty$ and so the theory is asymptotically free.

This renormalized hamiltonian can be used to compute other observables. The usual prescription for the calculations is to obtain the solutions with the cutoff in place and then take the limit as the momentum cutoff is removed to $\infty$. If an exact calculation can be implemented, the final results should be independent of the regularization and renormalization schemes. As an example, we calculate the scattering wave function,

$$\Phi_k(p) = \delta^{(2)}(p - k) + \frac{\alpha_\Lambda}{2\pi} \frac{\Psi(0)}{(p^2 - k^2 - i \epsilon)},$$  (131)

where $k = \sqrt{E}$. Integrating both sides over $p$ with a cutoff $\Lambda$ in place, we obtain

$$\Psi(0) = \frac{1}{2\pi} \left[ 1 - \frac{\alpha_\Lambda}{4\pi} \ln \left(1 + \frac{\Lambda^2}{-k^2 - i \epsilon}\right) \right]^{-1};$$  (132)

thus,

$$\alpha_\Lambda \Psi(0) = \frac{1}{2\pi} \left[ \frac{1}{4\pi} \ln \left(1 + \frac{\Lambda^2}{E_0}\right) - \frac{1}{4\pi} \ln \left(1 + \frac{\Lambda^2}{-k^2 - i \epsilon}\right) \right]^{-1}.$$  (133)

In the limit $\Lambda \to \infty$ we obtain:

$$\alpha_\Lambda \Psi(0) = \frac{2}{\ln \left(\frac{k^2}{E_0}\right) - i \pi}.$$  (134)

The resulting scattering wave function is then given by

$$\Phi_k(p) = \delta^{(2)}(p - k) + \frac{1}{2\pi} \frac{2}{(p^2 - k^2 - i \epsilon)} \left[ \ln \left(\frac{k^2}{E_0}\right) - i \pi \right]^{-1}.$$  (135)

It is important to note that only S-wave scattering occurs, corresponding to zero angular momentum states. For the higher waves the centrifugal barrier completely screens the delta-function potential and the non-zero angular momentum scattering states are free states.
The same prescription can be used to evaluate the T-matrix or the K-matrix. For the T-matrix, the Lippmann-Schwinger equation with the renormalized potential is given by:

\[ T(p, p'; k) = V(p, p') + \int d^2q \frac{V(p, q)}{k^2 - q^2 + i\epsilon} T(q, p'; k). \]  

(136)

Since only S-wave scattering takes place we can integrate out the angular variable, obtaining

\[ T^{(l=0)}(p, p'; k) = V^{(l=0)}(p, p') + \int_0^\Lambda dq q \frac{V^{(l=0)}(p, q)}{k^2 - q^2 + i\epsilon} T^{(l=0)}(q, p'; k), \]  

(137)

where

\[ V^{(l=0)}(p, p') = -\frac{\alpha\Lambda}{2\pi}. \]  

(138)

The Lippmann-Schwinger equation for the “on-shell” T-matrix is given by:

\[ T^{(l=0)}(k) = -\frac{\alpha\Lambda}{2\pi} - \frac{\alpha\Lambda}{2\pi} T^{(l=0)}(k) \int_0^\Lambda dq q \frac{1}{k^2 - q^2 + i\epsilon}. \]  

(139)

Solving this equation and taking the limit \( \Lambda \to \infty \), we obtain the exact “on-shell” T-matrix:

\[ T_0(k) = -\frac{2}{\ln \left( \frac{k^2}{E_0} \right) - i\pi}. \]  

(140)

Here and in what follows we drop the superscript and use the subscript 0 to denote the exact quantities.

In the same way, the S-wave Lippmann-Schwinger equation for the K-matrix is given by

\[ K(p, p'; k) = V(p, p') + \mathcal{P} \int_0^\Lambda dq q \frac{V(p, q)}{k^2 - q^2} K(q, p'; k), \]  

(141)

and the exact “on-shell” K-matrix is given by

\[ K_0(k) = -\frac{2}{\ln \left( \frac{k^2}{E_0} \right)}. \]  

(142)

The “on-shell” K-matrix and T-matrix are related by

\[ K_0(k) = \frac{T_0(k)}{1 - \frac{i\pi}{2} T_0(k)}. \]  

(143)

Using either

\[ k \cot \delta_0(k) - i k = \frac{2 k}{\pi} \frac{1}{T_0(k)}, \]  

(144)

or

\[ k \cot \delta_0(k) = -\frac{2 k}{\pi} \frac{1}{K_0(k)}, \]  

(145)

we can obtain the exact phase-shifts:

\[ \cot \delta_0 = \frac{1}{\pi} \ln \left( \frac{k^2}{E_0} \right). \]  

(146)
5.2 Effective Field Theory Approach

The EFT power counting scheme in the $D = 2$ case is also simple. Again, we analyze the errors in the inverse K-matrix. For the leading-order prediction, we use the effective potential with one parameter. Evaluating the difference $\Delta \left[ 1/K^{(0)} \right] = 1/K^{(0)} - 1/K_0$ and expanding it in powers of $k/\lambda$ we obtain:

$$\Delta \left[ 1/K^{(0)} \right] = \left[ 1/C_0 - \frac{\gamma}{2} + \frac{1}{2} \ln \left( \frac{\lambda^2}{E_0} \right) \right] + \left[ 1/C_0 - \frac{1}{2} \right] \frac{k^2}{\lambda^2} + O \left( \frac{k^4}{\lambda^4} \right).$$

By choosing

$$C_0(\lambda) = \frac{-2}{\ln \left( \frac{\lambda^2}{E_0} \right) - \gamma},$$

the leading logarithmic error is eliminated and the remaining errors are dominated by the term

$$O \left( \frac{k^2}{\lambda^2} \right) = \frac{1}{2} \left[ \gamma - \ln \left( \frac{\lambda^2}{E_0} \right) - 1 \right] \frac{k^2}{\lambda^2}.$$  

For the next-to-leading order calculation we use the effective potential with two parameters, obtaining the expansion

$$\Delta \left[ 1/K^{(2)} \right] = A_0 + A_2 \frac{k^2}{\lambda^2} + O \left( \frac{k^4}{\lambda^4} \right),$$

where

$$A_0 = \frac{4 + 2 C_2 + \frac{C_2^2}{4}}{2 \left( 2 C_0 - \frac{C_2^2}{4} \right)} - \frac{1}{2} \left[ \gamma - \ln \left( \frac{\lambda^2}{E_0} \right) \right],$$

$$A_2 = \frac{-4 C_0^2 - 2 C_0 \left( 4 + 2 C_2 + \frac{3 C_2^2}{4} \right) + C_2 \left( 8 + 12 C_2 + 3 C_2^2 + \frac{3 C_2^3}{8} \right)}{2 \left( 2 C_0 - \frac{C_2^2}{4} \right)^2}. \quad (152)$$

The parameters $C_0$ and $C_2$ are determined by solving the system of two coupled non-linear equations given by

$$A_0 = 0, \quad A_2 = 0.$$  

The acceptable solution is the one that leads to asymptotic freedom, i.e., $(C_0, C_2) \to 0$ as $\lambda \to \infty$ (for the other solution, $C_0 \to 8$ and $C_2 \to 0$ in this limit). The corresponding expressions for $C_0$ and $C_2$ are algebraically very complicated and so we do not present them here. In Fig. 6 we plot the parameters obtained in the leading order and next-to-leading order calculation as a function of the cutoff. For the numerical evaluation here and in what follows we fix the $D = 2$ theory by choosing the exact binding energy to be $E_0 = 1$.

To evaluate the errors in the binding energy we use the effective potential with one, two and four parameters adjusted to fit the inverse K-matrix (using the same minimization procedure as in the $D = 1$ case).

We solve the Schrödinger equation numerically for different values of the cutoff $\lambda$. Introducing an integration cutoff $A_0 >> \lambda$,

$$p^2 \Phi_\lambda(p) + \int_0^{A_0} dq \ q V_\lambda(p, p') \Phi_\lambda(q) = E_\lambda \Phi_\lambda(p).$$  

30
Discretizing the Schrödinger equation (choosing a gaussian quadrature),
\[
\sum_{j=1}^{N} w_j \left[ p_i p_j \delta_{ij} \frac{1}{w_j} + p_j V_\lambda(p_i, p_j) \right] \Phi_\lambda(p_j) = E_\lambda \Phi_\lambda(p_i) ,
\]
we obtain the matrix equation
\[
(M - E_\lambda 1)\Phi_\lambda = 0 ,
\]
where
\[
M_{ij} = w_j \left[ p_i p_j \delta_{ij} \frac{1}{w_j} + p_j V_\lambda(p_i, p_j) \right] .
\]
The bound-state energy, $E_\lambda$, is obtained by following the same procedure as for the $D = 1$ case and is compared with the exact value.

In Fig. 7 we show in a log-log plot the absolute values for the relative errors in the bound-state energy as a function of $E_0/\lambda^2$. As expected, the result is analogous to the $D = 1$ case: straight lines with slope given by the dominant power of $E_0/\lambda^2$ in the error. Note that the dominant logarithmic errors are completely absorbed by adjusting the parameter $C_0$ and when more parameters are adjusted we obtain power-law improvement.
Figure 7: The EFT error in the binding energy for the two-dimensional delta-function using one, two and four parameters. The exact theory is fixed by choosing $E_0 = 1$.

5.3 Similarity Renormalization Group Approach

In the two-dimensional case the canonical hamiltonian in momentum space with a delta-function potential can be written as

$$H(p, p') = h(p, p') + V(p, p'),$$

(158)

where $h(p, p') = p^2 \delta^{(2)}(p - p')$ corresponds to the free hamiltonian and $V(p, p') = -\alpha_0/(2\pi)^2$ corresponds to the Fourier transform of the delta-function potential.

Integrating out the angular variable, the flow equation obtained with Wegner's transformation in terms of matrix elements in the basis of free states is given by

$$\frac{dV_s(p, p')}{ds} = -(p^2 - p'^2)^2 V_s(p, p') - \int_0^\infty dk \ k \ (2k^2 - p^2 - p'^2) \ V_s(p, k) \ V_s(k, p').$$

(159)

In principle, we can set the boundary condition at $s = 0$ (no cutoff), i.e,

$$H_{s=0}(p, p') = H(p, p') = p^2 \delta^{(1)}(p - p') - \frac{\alpha_0}{2\pi}.$$

(160)

However, the hamiltonian with no cutoff produces logarithmic divergences, requiring renormalization. As we will see, the boundary condition must be imposed at some other point,
leading to dimensional transmutation \[^{113}\]. The reduced interaction \( \bar{V}_s(p, p') \) is defined such that
\[
V_s(p, p') = e^{-s(p^2 - p'^2)^2} \bar{V}_s(p, p') .
\] (161)

Assuming that \( h \) is cutoff independent we obtain the flow equation for the reduced interaction,
\[
\frac{d\bar{V}_s}{ds} = -e^{-2s} p^2 p'^2 \int_0^\infty dk \ k \ (2k^2 - p^2 - p'^2) e^{-2s[k^4 - k^2(p^2 + p'^2)]} \times \bar{V}_s(p, k) \bar{V}_s(k, p') .
\] (162)

As in the \( D = 1 \) case, this equation is solved using a perturbative expansion given by Eq. \(^{115}\), starting with
\[
\bar{V}_s^{(1)}(p, p') = \frac{\alpha_s}{2\pi} .
\] (163)

We assume a coupling-coherent solution in the form of an expansion in powers of \( \alpha_s/2\pi \), satisfying the constraint that the operators \( F^{(n)}_s(p, p') \) vanish when \( p = p' = 0 \),
\[
\bar{V}_s(p, p') = -\frac{\alpha_s}{2\pi} + \sum_{n=2}^{\infty} \left( \frac{\alpha_s}{2\pi} \right)^n F^{(n)}_s(p, p') .
\] (164)

Note that the expansion parameter is \( \alpha_s/2\pi \).

Using the solution Eq. \(^{164}\) in Eq. \(^{162}\) we obtain
\[
\frac{d\bar{V}_s}{ds} = -\frac{1}{(2\pi)^2} \frac{d\alpha_s}{ds} + \sum_{n=2}^{\infty} \frac{1}{(2\pi)^n} \left[ n \alpha_s^{n-1} \frac{d\alpha_s}{ds} F^{(n)}_s(p, p') + \alpha_s^n \frac{dF^{(n)}_s(p, p')}{ds} \right]
\]
\[
= \int_0^\infty dk \ k \ (2k^2 - p^2 - p'^2) e^{-2s[p^2 p'^2 + k^4 - k^2(p^2 + p'^2)]} \times \left[ -\frac{\alpha_s}{2\pi} + \sum_{n=2}^{\infty} \left( \frac{\alpha_s}{2\pi} \right)^n F^{(n)}_s(p, k) \right] \left[ -\frac{\alpha_s}{2\pi} + \sum_{m=2}^{\infty} \left( \frac{\alpha_s}{2\pi} \right)^m F^{(m)}_s(k, p') \right] .
\] (165)

This equation is solved iteratively order-by-order in \( \alpha_s/2\pi \). Again, if \( \alpha_s/2\pi \) is small the operator \( \bar{V}_s^{(1)}(p, p') \) can be identified as the dominant term in the expansion of \( \bar{V}_s(p, p') \) in powers of \( p \) and \( p' \). In the \( D = 2 \) case this operator corresponds to a marginal operator (since the coupling is dimensionless and there is no implicit mass scale). The higher-order terms correspond to irrelevant operators.

At second-order we have
\[
-\frac{1}{2\pi} \frac{d\alpha_s}{ds} + \frac{1}{(2\pi)^2} \alpha_s^2 \frac{dF^{(2)}_s(p, p')}{ds} = -\alpha_s^2 I^{(2)}_s(p, p') ,
\] (166)

where
\[
I^{(2)}_s(p, p') = \frac{1}{(2\pi)^2} \int_0^\infty dk \ k \ (2k^2 - p^2 - p'^2) e^{-2s[p^2 p'^2 + k^4 - k^2(p^2 + p'^2)]}
\]
\[
= \frac{1}{(2\pi)^2} \frac{e^{-2s p^2 p'^2}}{4s} .
\] (167)
The equation for $\alpha_s$ is obtained by taking the limit $(p, p') \to 0$,

$$\frac{1}{2\pi} \frac{d\alpha_s}{ds} = \alpha_s^2 I_s^{(2)}(0, 0), \quad (168)$$

where

$$I_s^{(2)}(0, 0) = \frac{1}{(2\pi)^2} \frac{1}{4s}. \quad (169)$$

Integrating Eq. (168) from $s_0$ to $s$,

$$\alpha_{s, 2} = \frac{\alpha_{s_0}}{1 - \frac{\alpha_{s_0}}{8\pi \ln \left( \frac{s}{s_0} \right)}}. \quad (170)$$

In terms of the cutoff $\lambda$ we obtain

$$\alpha_{\lambda, 2} = \frac{\alpha_{\lambda_0}}{1 + \frac{\alpha_{\lambda_0}}{2\pi \ln \left( \frac{\lambda}{\lambda_0} \right)}}. \quad (171)$$

In principle, knowing the value of $\alpha_{s_0}$ for a given $s_0$ we can determine the running coupling $\alpha_s$ for any $s$. Since we cannot choose $s_0 = 0$ ($\lambda_0 = \infty$) as in the $D = 1$ case, to use Eq. (170) we must specify a renormalization prescription that allows us to fix the coupling at some finite non-zero value of $s_0$. We discuss this issue in detail later in this subsection.

The equation for $F_s^{(2)}(p, p')$ is given by

$$\frac{1}{(2\pi)^2} \frac{dF_s^{(2)}(p, p')}{ds} = I_s^{(2)}(0, 0) - I_s^{(2)}(p, p'). \quad (172)$$

Integrating from $s_0$ to $s$ we obtain

$$F_s^{(2)}(p, p') = \int_{s_0}^{s} ds' \left[ 1 - e^{-2s' p^2 p'^2} \right] \frac{4s'}{4s} \left[ \ln \left( \frac{s}{s_0} \right) - \text{Ei}(-2s p^2 p'^2) + \text{Ei}(-2s_0 p^2 p'^2) \right]. \quad (173)$$

Insisting that $F_s^{(2)}(p, p') = 0$ when $p, p' = 0$ we obtain

$$F_s^{(2)}(p, p') = \frac{1}{4} \left[ \gamma + \ln(2s p^2 p'^2) - \text{Ei}(-2s p^2 p'^2) \right]. \quad (174)$$

At third-order we have

$$-\frac{1}{2\pi} \frac{d\alpha_s}{ds} + \frac{1}{(2\pi)^2} \alpha_s^2 \frac{dF_s^{(2)}(p, p')}{ds} + \frac{2}{(2\pi)^2} \alpha_s \frac{d\alpha_s}{ds} F_s^{(2)}(p, p') + \frac{1}{(2\pi)^3} \alpha_s^3 \frac{dF_s^{(3)}(p, p')}{ds} = -\alpha_s^2 I_s^{(2)}(p, p') + \alpha_s^3 I_s^{(3)}(p, p'), \quad (175)$$

where

$$I_s^{(3)}(p, p') = \frac{1}{(2\pi)^3} \int_{0}^{\infty} dk \int_{0}^{\infty} dk' \int_{0}^{\infty} \frac{d\alpha_s}{ds} \frac{dF_s^{(2)}(p, p')}{ds} \left[ F_s^{(2)}(p, k) + F_s^{(2)}(k, p') \right]. \quad (176)$$
In the limit $p, p' \to 0$ we obtain:

$$\frac{1}{2\pi} \frac{d\alpha_s}{ds} = \alpha_s^2 I_s^{(2)}(0, 0) - \alpha_s^3 I_s^{(3)}(0, 0),$$  \hspace{1cm} (177)

where

$$I_s^{(3)}(0, 0) = \frac{1}{(2\pi)^3} \int_0^\infty dk \, 2k^3 e^{-2sk^4} \left[ F_s^{(2)}(0, k) + F_s^{(2)}(k, 0) \right].$$  \hspace{1cm} (178)

Since $F_s^{(2)}(k, 0) = F_s^{(2)}(0, k) = 0$, the term proportional to $\alpha_s^3$ in Eq. (177) is zero. The equation for $F_s^{(3)}(p, p')$ is given by

$$\frac{1}{(2\pi)^3} \frac{dF_s^{(3)}(p, p')}{ds} = -\frac{1}{\pi} I_s^{(2)}(0, 0) F_s^{(2)}(p, p') + I_s^{(3)}(p, p').$$  \hspace{1cm} (179)

To obtain $F_s^{(3)}(p, p')$ the integrals in $k$ and $s$ must be evaluated numerically.

At fourth-order we obtain

$$-\frac{1}{2\pi} \frac{d\alpha_s}{ds} + \frac{1}{(2\pi)^2} \alpha_s^2 \frac{dF_s^{(2)}(p, p')}{ds} + \frac{2}{(2\pi)^2} \alpha_s \frac{d\alpha_s}{ds} F_s^{(2)}(p, p')$$

$$+ \frac{1}{(2\pi)^3} \alpha_s^3 \frac{dF_s^{(3)}(p, p')}{ds} + \frac{3}{(2\pi)^3} \alpha_s^2 \frac{d\alpha_s}{ds} F_s^{(3)}(p, p') + \frac{1}{(2\pi)^4} \alpha_s^4 \frac{dF_s^{(4)}(p, p')}{ds}$$

$$= -\alpha_s^2 I_s^{(2)}(p, p') + \alpha_s^3 I_s^{(3)}(p, p') + \alpha_s^4 I_s^{(4)}(p, p'),$$  \hspace{1cm} (180)

where

$$I_s^{(4)}(p, p') = \frac{1}{(2\pi)^4} \int_0^\infty dk \, k \left( 2k^2 - p^2 - p'^2 \right) e^{-2sk^4 + k^2(p^2 + p'^2)} \left[ F_s^{(3)}(p, k) + F_s^{(3)}(k, p') + F_s^{(2)}(p, k)F_s^{(2)}(k, p') \right].$$  \hspace{1cm} (181)

In the limit $p, p' \to 0$ we obtain:

$$\frac{1}{2\pi} \frac{d\alpha_s}{ds} = \alpha_s^2 I_s^{(2)}(0, 0) - \alpha_s^4 I_s^{(4)}(0, 0),$$  \hspace{1cm} (182)

where

$$I_s^{(4)}(0, 0) = \frac{1}{2\pi} \int_0^\infty dk \, 2k^2 e^{-2sk^4} \left[ F_s^{(3)}(0, k) + F_s^{(3)}(k, 0) \right].$$  \hspace{1cm} (183)

For dimensional reasons Eq. (182) takes the form

$$\frac{d\alpha_s}{ds} = \frac{B_2}{s} \alpha_s^2 - \frac{B_4}{s} \alpha_s^4,$$  \hspace{1cm} (184)

where $B_2 = \frac{1}{3\pi}$ and $B_4$ can be obtained by evaluating $I_s^{(4)}(0, 0)$ (numerically) for $s = 1$. In terms of the cutoff $\lambda$ we obtain:

$$\frac{d\alpha_\lambda}{d\lambda} = \frac{1}{\lambda^4} B_2 \alpha_\lambda^2 - \frac{1}{\lambda^4} B_4 \alpha_\lambda^4.$$

\hspace{1cm} (185)
Integration of Eq. (182) leads to a transcendental equation which is solved numerically in order to obtain the running coupling $\alpha_{s,4}$.

Qualitatively, the errors are expected to be a combination of inverse powers of $\lambda$ and powers (or inverse powers) of $\ln(\lambda)$ coming from the perturbative expansion in powers of $\alpha_\lambda$ for the coefficients of the irrelevant operators and the perturbative approximation for $\alpha_\lambda$.

As pointed out above, to completely determine the renormalized Hamiltonian at a given order we need to specify the coupling at some scale $\lambda_0$. The simplest way is to choose a value for the ‘exact’ $\alpha_{\lambda_0}$. Formally, this fixes the underlying theory; i.e., if we had the exact Hamiltonian (to all orders) we could obtain the exact values for all observables. However, since the Hamiltonian is derived perturbatively and must be truncated at some order in practical calculations, we can only obtain approximate cutoff-dependent results for the observables. Moreover, in this case the errors cannot be directly evaluated, since the exact values remain unknown. As an example, we calculate the bound-state energy choosing $\alpha_{\lambda_0} = 1.45$ at $\lambda_0 = 100$. In Fig. 8 we show the binding-energy as a function of the cutoff $\lambda$ using the following approximations for the interaction:

(a) marginal operator with coupling ($\alpha_{\lambda_0}$),

$$V_{\lambda}(p, p') = -\frac{\alpha_{\lambda_0}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{4\lambda^4}}.$$  \hspace{1cm} (186)

(b) marginal operator with running coupling renormalized to second-order ($\alpha_{\lambda,2}$),

$$V_{\lambda}(p, p') = -\frac{\alpha_{\lambda,2}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{4\lambda^4}}.$$  \hspace{1cm} (187)

(c) marginal operator plus second-order irrelevant operator with running coupling renormalized to second-order ($\alpha_{\lambda,2}, F_{\lambda}^{(2)}$),

$$V_{\lambda}(p, p') = \left[-\frac{\alpha_{\lambda,2}}{2\pi} + \left(\frac{\alpha_{\lambda,2}}{2\pi}\right)^2 F_{\lambda}^{(2)}(p, p')\right] e^{-\frac{(p^2 - p'^2)^2}{4\lambda^4}}.$$  \hspace{1cm} (188)

(d) marginal operator with running coupling renormalized to fourth-order ($\alpha_{\lambda,4}$),

$$V_{\lambda}(p, p') = -\frac{\alpha_{\lambda,4}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{4\lambda^4}}.$$  \hspace{1cm} (189)

(e) marginal operator plus second-order irrelevant operator with running coupling renormalized to fourth-order $\alpha_{\lambda,4}, F_{\lambda}^{(2)}$),

$$V_{\lambda}(p, p') = \left[-\frac{\alpha_{\lambda,4}}{2\pi} + \left(\frac{\alpha_{\lambda,4}}{2\pi}\right)^2 F_{\lambda}^{(2)}(p, p')\right] e^{-\frac{(p^2 - p'^2)^2}{4\lambda^4}}.$$  \hspace{1cm} (190)

We see that as the approximation is improved the cutoff dependence is reduced. As $\lambda \to \infty$ all curves should approach the same binding-energy, which corresponds to the exact value, and as $\lambda$ becomes small the perturbative approximation breaks down.

A similar prescription is to find $\alpha_{\lambda_0}$ at $\lambda_0$ that produces a given binding-energy, $E_0$. Since the fitting is implemented using a truncated Hamiltonian, this $\alpha_{\lambda_0}$ is an approximation that
Figure 8: The binding energy for the two-dimensional delta-function potential with various approximations for the SRG hamiltonian. The exact theory is fixed by choosing $\alpha_0 = 1.45$ at $\lambda_0 = 100$. 

37
Figure 9: The SRG errors in the binding energy for the two-dimensional delta-function potential using various approximations for the SRG Hamiltonian. The exact theory is fixed by choosing $E_0 = 1$ at $\lambda_0 = 100$.

becomes more accurate if we use a larger $\lambda_0$ and/or include higher order operators. Although in this case we can evaluate the errors, the scaling analysis becomes complicated as $\lambda \to \lambda_0$ because at this point we force the energy to be the exact value and so the error is zero.

As an example, we calculate the bound-state energy when the coupling is fixed at $\lambda_0 = 100$ to give $E_0 = 1$. In Fig. 9 we show the ‘errors’ in the binding energy using the same approximations listed above for the potential. As expected, all error lines drop abruptly to zero when $\lambda \to \lambda_0$, where the running coupling is chosen to fit what we define to be the exact binding energy. Away from this point we can analyze the errors. With the Hamiltonian (a) (unrenormalized) we obtain a dominant error that scales like $\ln (\lambda_0 / \lambda)$, corresponding to the leading order error. With the renormalized Hamiltonian (b) the dominant errors scale like $[\ln (\lambda_0 / \lambda)]^{-2}$, indicating the elimination of the leading order logarithmic errors. With the Hamiltonian (c) there is a small shift, but no significant change in the error scaling. The added irrelevant operator may remove errors of order $(\lambda_0 / \lambda)^2$, but these are smaller than the remaining $[\ln (\lambda_0 / \lambda)]^{-2}$ errors. With Hamiltonians (d) and (e) in the range of intermediate cutoffs ($E_0 << \lambda^2 << \lambda_0^2$) there is also only a shift in the errors. The dips in (d) and (e) correspond to values of $\lambda$ where the binding energy equals the exact value.

This behavior is a perturbative artifact that can be understood in the following way. Consider the Schrödinger equation with potential (d). Rescaling the momenta $p \to \lambda \hat{p}$ we
Figure 10: The SRG running coupling for the two-dimensional delta-function potential obtained with \( \alpha_\lambda \) at \( \lambda_0 = 100 \) fixed to fit \( E_0 = 1 \).

As shown in Fig. 10, the coupling renormalized to fourth-order, \( \alpha_{\lambda_4} \), approximately freezes for small \( \lambda \) and as a consequence the bound-state energy scales like \( E_\lambda \simeq \lambda^2 \times \text{constant} \) eventually becoming equal to the exact value and then deviating again. With the hamiltonian (e) the behavior is similar, with the dip occurring at a different value of \( \lambda \) because of the irrelevant operator. As in the \( D = 1 \) case, for small values of \( \lambda \) the lines converge, indicating the breakdown of the perturbative expansion.

An alternative prescription is to use the potential derived in subsection 5.1 as the starting point for the similarity transformation. We introduce a large momentum cutoff \( \Lambda \), define

\[
\alpha_{s_0=0} = \alpha_\Lambda = \frac{4\pi}{\ln \left( 1 + \frac{\Lambda^2}{E_0} \right)},
\]

and set all irrelevant operators to zero at \( s_0 = 0 \). Note that the coupling \( \alpha_{\lambda_0} \) is fixed at \( \lambda_0 = \infty \) by fitting the exact binding energy. With this definition the similarity hamiltonian with
no similarity cutoff becomes well-defined and we can set all of the similarity transformation boundary conditions at \( s = 0 \). In some sense, the problem becomes similar to the \( D = 1 \) case. The previous derivation remains essentially the same. The only modification is that all integrals over momentum are cut off \( (p \leq \Lambda) \).

At second-order we have

\[
- \frac{1}{2\pi} \frac{d\alpha_{s,\Lambda}}{ds} + \frac{1}{(2\pi)^2} \alpha^2_{s,\Lambda} \frac{dF^{(2)}_{s,\Lambda}(p,p')}{ds} = -\alpha^2_{s,\Lambda} I^{(2)}_{s,\Lambda}(p,p')
\]  

where

\[
I^{(2)}_{s,\Lambda}(p,p') = \frac{1}{(2\pi)^2} \int_{0}^{\Lambda} dk \, (2k^2 - p^2 - p'^2) e^{-2sp^2p'^2+k^4-k^2(p^2+p'^2)}
\]

\[
= \frac{1}{(2\pi)^2} \left[ e^{-2sp^2p'^2} + \frac{1}{4} \alpha^2_{s,\Lambda} \right].
\]

The resulting second-order running coupling and irrelevant operator are given respectively by

\[
\alpha_{s,\Lambda,2} = \frac{\alpha^2_{s,\Lambda}}{1 - \frac{\alpha^2_{s,\Lambda}}{8\pi} \left[ \gamma + \ln(2s\Lambda^4) - \text{Ei}(-2s\Lambda^4) \right]}
\]

and

\[
F^{(2)}_{s,\Lambda}(p,p') = \frac{1}{4} \left[ \gamma + \ln(2s^2p^2p'^2) - \text{Ei}(-2s^2p^2p'^2) \right]
\]

\[
+ \frac{1}{4} \left[ \gamma + \ln(2s\Lambda^4) - \text{Ei}(-2s\Lambda^4) \right]
\]

\[
- \frac{1}{4} \left[ \gamma + \ln \left( (p^2 - \Lambda^2)^2 + (p'^2 - \Lambda^2)^2 - (p^2 - p'^2)^2 \right) - \text{Ei} \left( -s \left[ (p^2 - \Lambda^2)^2 + (p'^2 - \Lambda^2)^2 - (p^2 - p'^2)^2 \right] \right) \right].
\]

At third-order we have

\[
- \frac{1}{2\pi} \frac{d\alpha_{s,\Lambda}}{ds} + \frac{1}{(2\pi)^2} \alpha^2_{s,\Lambda} \frac{dF^{(2)}_{s,\Lambda}(p,p')}{ds} + \frac{2}{(2\pi)^2} \alpha_{s,\Lambda} \frac{d\alpha_{s,\Lambda}}{ds} F^{(2)}_{s,\Lambda}(p,p')
\]

\[
+ \frac{1}{(2\pi)^3} \alpha^3_{s,\Lambda} \frac{dF^{(3)}_{s,\Lambda}(p,p')}{ds} = -\alpha^2_{s,\Lambda} I^{(2)}_{s,\Lambda}(p,p') + \alpha^3_{s,\Lambda} I^{(3)}_{s,\Lambda}(p,p'),
\]

where

\[
I^{(3)}_{s,\Lambda}(p,p') = \frac{1}{(2\pi)^3} \int_{0}^{\Lambda} dk \, (2k^2 - p^2 - p'^2) e^{-2sp^2p'^2+k^4-k^2(p^2+p'^2)}
\]

\[
\times \left[ F^{(2)}_{s}(p,k) + F^{(2)}_{s}(k,p') \right].
\]

In this case,

\[
F^{(2)}_{s,\Lambda}(0,k) = \frac{1}{4} \left[ \gamma + \ln(2s\Lambda^4) - \text{Ei}(-2s\Lambda^4) \right]
\]

\[
= \frac{1}{4} \left[ \gamma + \ln(2s\Lambda^4 - 2sk^2\Lambda^2) - \text{Ei}(-2s\Lambda^4 + 2sk^2\Lambda^2) \right],
\]

40
and so

\[ I_{s,\Lambda}^{(3)}(0,0) = \frac{1}{(2\pi)^3} \frac{1}{4} \int_0^\Lambda dk \, 4k^3 e^{-2sk} \left( \left[ \gamma + \ln(2s\Lambda^4) - \text{Ei}(-2s\Lambda^4) \right] 
- \left[ \gamma + \ln(2s\Lambda^4 - 2sk^2\Lambda^2) - \text{Ei}(-2s\Lambda^4 + 2sk^2\Lambda^2) \right] \right). \]  

(201)

Since \( I_{s,\Lambda}^{(3)}(0,0) \neq 0 \) the term proportional to \( \alpha s,\Lambda \) in Eq. (177) does not vanish. To obtain \( \alpha s,\Lambda,3 \) we evaluate \( I_{s,\Lambda}^{(3)}(0,0) \) and solve Eq. (177) numerically.

The equation for \( F_{s,\Lambda}^{(3)}(p,p') \) is given by

\[ \frac{1}{(2\pi)^3} \frac{dF_{s,\Lambda}^{(3)}(p,p')}{ds} = -\frac{1}{\pi} I_{s,\Lambda}^{(2)}(0,0) F_{s,\Lambda}^{(2)}(p,p') + I_{s,\Lambda}^{(3)}(p,p'). \]  

(202)

To obtain \( F_{s,\Lambda}^{(3)}(p,p') \) the integrals over \( k \) and \( s \) must be evaluated numerically. In the limit \( s\Lambda^4 \to \infty \) with \( s \) fixed at some non-zero value

\[ F_{s,\Lambda}^{(2)}(p,p') \to \frac{1}{4} \left[ \gamma + \ln(2s p^2 p'^2) - \text{Ei}(-2s p^2 p'^2) \right], \]  

(203)

\[ I_{s,\Lambda}^{(3)}(0,0) \to 0, \]  

(204)

and \( \alpha s,\Lambda \) becomes indeterminate, requiring the coupling to be fixed at some \( s_0 \neq 0 \). In this way, we recover the result of the previous prescription.

Although less trivial, this prescription allows a more transparent error analysis. We can also extend the calculation to larger values of the similarity cutoff, \( \lambda \), and analyze the errors in a different scaling regime. In Fig. 11 we show the errors in the binding-energy obtained using the following approximations for the potential with \( \Lambda = 50 \):

(a) marginal operator with coupling (\( \alpha_0 \)),

\[ V_{\lambda,\Lambda}(p,p') = -\frac{\alpha_0}{2\pi} e^{-\frac{(p^2-p'^2)^2}{x^4}}; \]  

(205)

(b) marginal operator with running coupling renormalized to second-order (\( \alpha_{\lambda,2} \)),

\[ V_{\lambda,\Lambda}(p,p') = -\frac{\alpha_{\lambda,2}}{2\pi} e^{-\frac{(p^2-p'^2)^2}{x^4}}; \]  

(206)

(c) marginal operator plus second-order irrelevant operator with running coupling renormalized to second-order (\( \alpha_{\lambda,2}, F_{\lambda}^{(2)} \)),

\[ V_{\lambda,\Lambda}(p,p') = \left[ -\frac{\alpha_{\lambda,2}}{2\pi} + \left( \frac{\alpha_{\lambda,2}}{2\pi} \right)^2 F_{\lambda,\Lambda}^{(2)}(p,p') \right] e^{-\frac{(p^2-p'^2)^2}{x^4}}; \]  

(207)

(d) marginal operator with running coupling renormalized to third-order (\( \alpha_{\lambda,3} \)),

\[ V_{\lambda,\Lambda}(p,p') = -\frac{\alpha_{\lambda,3}}{2\pi} e^{-\frac{(p^2-p'^2)^2}{x^4}}; \]  

(208)
Figure 11: The SRG errors in the binding energy for the two-dimensional delta-function potential using various approximations for the similarity hamiltonian. The exact theory is fixed by regulating the “bare hamiltonian” using a sharp momentum cutoff, $\Lambda$, and letting the bare coupling depend on $\Lambda$ such that the binding energy is fixed. We use $\Lambda = 50$ and $E_0 = 1$.

(e) marginal operator plus second-order irrelevant operator with running coupling renormalized to third-order $\alpha_{\lambda,\Lambda,3}, F_{\lambda}^{(2)}$, 

$$V_{\lambda,\Lambda}(p, p') = \left[ \frac{\alpha_{\lambda,\Lambda,3}}{2\pi} + \left( \frac{\alpha_{\lambda,\Lambda,3}}{2\pi} \right)^2 F_{\lambda,\Lambda}^{(2)}(p, p') \right] e^{-\frac{(p^2-p'^2)^2}{\lambda^2}}. \quad (209)$$

There are clearly two distinct scaling regions when an additional large momentum cutoff $\Lambda$ is placed on the initial matrix and a similarity cutoff is then applied. When the similarity cutoff is larger than $\Lambda$, we see scaling behavior that is similar to EFT. Curves (a), (b), and (d) all have the same slope. None of these hamiltonians contains irrelevant operators, but the marginal coupling differs in each. All results become exact as the similarity cutoff goes to infinity, and these curves are close to one another because the coupling runs little in this region. Curves (c) and (e) show that there is a power-law improvement when irrelevant operators are added, and that once again when the similarity cutoff is larger than $\Lambda$ an improvement in the running coupling makes little difference. These results closely resemble those in EFT and we conclude that it is the leading irrelevant operator that leads to this improvement. Even though the coupling in front of this operator is approximated by the first term in an expansion in powers of the running coupling, the coupling is sufficiently
small that this approximation works well and the operator eliminates most of the leading power-law error in curves (a), (b), and (d).

When the similarity cutoff becomes smaller than \( \Lambda \) we see a crossover to a more complicated scaling regime that resembles the SRG scaling discussed above. The error displayed by curve (a) approaches 100%, while the running coupling introduced in curve (b) reduces the error to an inverse logarithm. Improving the running coupling in curve (d) further reduces the error, and we see that curve (c) crosses curve (d) at a point where improving the running coupling becomes more important than adding irrelevant operators. As above, the best results require us to both improve the running coupling by adding third-order corrections and add the second-order irrelevant operators. In no case do we achieve power-law improvement, because as we have discussed there are always residual inverse logarithmic errors. Had we fit the running coupling to data, as we would do in a realistic calculation, we would obtain power-law improvement and the residual error would be proportional to an inverse power of the cutoff times an inverse power of the logarithm of the cutoff.

In Fig. 12 we show the running coupling at 2nd and 3rd order. Although the 3rd order corrections are small for all \( \lambda \) and vanish when \( \Lambda \to \infty \), the improvement resulting from this correction in Fig. 11 is significant.

The scaling behavior with a large momentum cutoff \( \Lambda \) in place is complicated, but it is fairly straightforward to understand it and to find a sequence of approximations that systematically improve the non-perturbative results. The calculations become increasingly
Figure 13: The SRG running coupling for the two-dimensional delta-function potential renormalized to second-order obtained with $\alpha_{\lambda_0=\infty} \rightarrow \alpha_{\Lambda} = 4\pi/\ln \left(1 + \frac{\Lambda^2}{E_0}\right)$.

complicated, but at each order one must improve the running coupling, or fit it to data, and add higher order irrelevant operators. In a field theory we need to let $\Lambda \rightarrow \infty$ and study the scaling behavior of the theory in the regime where $\lambda \ll \Lambda$. Although we do not display a compete set of figures, in Fig. 13 we show what happens to the running coupling as $\Lambda$ is increased, with the bound state energy fixed at one.

As is evident in the exact solution, as $\Lambda$ increases the coupling decreases. When $\lambda \gg \Lambda$, the coupling runs slowly and stays near its asymptotic value. As $\lambda$ approaches $\Lambda$ the coupling begins to run noticeably, and when $\lambda$ becomes much less than $\Lambda$ the coupling approaches a universal curve that is insensitive to its asymptotic value. Plots of the error in the binding energy for various approximations and different values of $\Lambda$ closely resemble Fig. 11, with two scaling regimes whose boundary is $\lambda = \Lambda$.

We close this section by reminding the reader that in all of these calculations there is only one free parameter. In a realistic calculation we would fit this parameter to a binding energy and we would expect to see residual errors in other observables that is inversely proportional to powers of the cutoff and logarithms of the cutoff.
6 Conclusions

We have illustrated the EFT and SRG methods for producing effective cutoff hamiltonians using the one-dimensional and two-dimensional delta-function potentials. We have shown that the addition of irrelevant operators with couplings tuned to fit low energy scattering observables leads to errors that scale as inverse powers of the cutoff in EFT. We have shown that the SRG with coupling coherence leads to errors that scale as inverse powers of the cutoff and logarithms of the cutoff. EFT typically leads to smaller errors than the SRG, but the cost is an increased number of parameters that must be fit to data. The SRG with coupling coherence requires the same number of parameters as the underlying ‘fundamental’ theory, but the cost is exponentially increasing algebraic complexity to remove errors that contain inverse powers of logarithms of the cutoff.

Our examples do not adequately illustrate the real utility of these renormalization tools, because our examples are easily solved with other methods. EFT can be used when the underlying theory cannot be solved and even when it is not known. All that is required is a knowledge of the degrees of freedom and symmetries that appear at ‘low’ energies. The SRG can also be employed when the underlying theory is not known, but in general it offers no advantages over EFT unless that effective theory is valid over very many energy scales. When the effective theory is the same as the fundamental theory, but with cutoffs introduced, or when it applies over very many scales, it may be possible to fix some of the couplings in terms of the fundamental couplings without using extra data.

7 Acknowledgments

We would like to acknowledge many useful discussions with Brent Allen, Martina Brisudova, Dick Furnstahl, Stan Glazek, Billy Jones, Roger Kylin, Rick Mohr, Jim Steele, and Ken Wilson. This work was supported by National Science Foundation grant PHY-9800964, and S.S. was supported by a CNPq-Brazil fellowship (proc. 204790/88-3).
References

[1] L. M. Brown, “Renormalization,” Springer-Verlag, New York, 1993.
[2] J. Schwinger, “Quantum Electrodynamics,” Dover, New York, 1958.
[3] P. A. M. Dirac, *Theorie du Positron*, (7-eme Conseil du Physique du Solvay: Structure et propiete de noyaux atomiques, Octobre 1933), pp. 203-230, Gauthier-Villars, Paris, 1934.
[4] P.A.M. Dirac, in “Perturbative Quantum Chromodynamics,” (D.W. Duke and J.F. Owens, Eds.), Am. Inst. Phys., New York, 1981.
[5] E. C. G. Stueckelberg and A. Peterman, *Helv. Phys. Acta* 26 (1953), 499.
[6] M. Gell-Mann and F.E. Low, *Phys. Rev.* 95 (1954), 1300.
[7] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, *Doklady* 95 (1954), 497; 96 (1954), 261.
[8] N. N. Bogoliubov and D.V. Shirkov, *Nuovo Cim.* 3 (1956), 845.
[9] N. N. Bogoliubov and D.V. Shirkov, “Introduction to the Theory of Quantized Fields,” Interscience, New York, 1959.
[10] K. G. Wilson and J. B. Kogut, *Phys. Rep.* 12C (1974), 75.
[11] K. G. Wilson, *Rev. Mod. Phys.* 47 (1975), 773.
[12] D. J. Gross and F. Wilczek, *Phys. Rev. Lett.* 30 (1973), 1343; H. D. Politzer, *Phys. Rev. Lett.* 30 (1973), 1346.
[13] K. G. Wilson, *Phys. Rev. D* 10 (1974), 2445.
[14] Ya. B. Zel’dovich, *Soviet Physics JETP* 11 (1960), 594.
[15] F. A. Berezin and L. D. Faddeev, *Sov. Math. Dokl.* 2 (1961), 372.
[16] C. Thorn, *Phys. Rev. D* 19 (1979), 639.
[17] K. Huang, “Quarks, Leptons and Gauge Fields,” World Scientific, Singapore, 1982.
[18] S. Albeverio, F. Gesztesy, R. Hoeg-Krohn and H. Holden, “Solvable Models in Quantum mechanics,” Springer-Verlag, New York, 1988.
[19] C. R. Hagen, *Phys. Rev. Lett.* 64 (1990), 503.
[20] R. Jackiw, in “M. A. B. Beg Memorial Volume,” (A. Ali and P. Hoodbhoy, eds.), World Scientific, Singapore, 1991.
[21] P. Godszinky and R. Tarrach, *Am. J. Phys.* 59 (1991), 70; C. Manuel and R. Tarrach, *Phys. Lett. B* 328 (1994), 113.
[22] J. F. Perez and F. A. B. Coutinho, *Am. J. Phys.* 59 (1991), 52.
[23] L. R. Mead and J. Godines, *Am. J. Phys.* 59 (1991), 935.
[24] C. Manuel and R. Tarrach, *Phys. Lett. B* 301 (1994), 72.
[25] T. J. Fields, K. S. Gupta, and J. P. Vary, *Mod. Phys. Lett. A* **11** (1996), 2233.

[26] K. S. Gupta and S. G. Rajeev, *Phys. Rev. D* **48** (1993), 5940;
R. J. Henderson and S. G. Rajeev, *Int. J. Mod. Phys. A* **10** (1995), 3765;
R. J. Henderson and S. G. Rajeev, *J. Math. Phys.* **38** (1997), 2171.

[27] D. K. Park, *J. Math. Phys.* **36** (1995), 5453.

[28] S. K. Adhikari and T. Frederico, *Phys. Rev. Lett.* **74** (1995), 4572;
S. K. Adhikari, T. Frederico and I. D. Goldman, *Phys. Rev. Lett.* **74** (1995), 487;
S. K. Adhikari and A. Ghosh, *J. Phys. A* **30** (1997), 6553;
C. F. de Araujo, Jr., L. Tomio, S. K. Adhikari and T. Frederico, *J. Phys A* **30** (1997), 4687.

[29] R. M. Cavalcanti, quant-ph/9801033 (1998).

[30] R. J. Henderson and S. G. Rajeev, *J. Math. Phys.* **39** (1998), 749.

[31] D. R. Phillips, S. R. Beane, and T. D. Cohen, *Ann. Phys. (N.Y.)* **263** (1998), 255.

[32] S. Weinberg, *Phys. Rev.* **166** (1968), 1568;
J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* **158** (1984), 142.

[33] S. Weinberg, *Physica A* **96** (1979), 327.

[34] H. Georgi, *Ann. Rev. Nucl. Part. Sci.* **43** (1993), 209.

[35] D. B. Kaplan, Lectures given at 7th Summer School on Nuclear Symmetries, Seattle, Washington, June 1995, nucl-th/9506035.

[36] A. V. Manohar, Lectures given at 35th Internationale Universitatswochen fuer Kern- und Teilchenphysik, Schladming, Austria, March 1996, hep-ph/9606222.

[37] G. P. Lepage, Lectures given at 9th Jorge Swieca Summer School, Sao Paulo, Brazil, Feb. 1997, nucl-th/9706029.

[38] M. B. Wise, Talk given at International Conference on Elastic and Diffractive Scattering, Seoul, Korea, June 1997, hep-ph/9707522.

[39] U. van Kolck, *Nucl. Phys. A* **645** (1999), 273.

[40] C. Ordonez and U. van Kolck, *Phys. Lett. B* **291** (1992), 459.

[41] C. Ordonez, L. Ray and U. van Kolck, *Phys. Rev. C* **53** (1994), 2086.

[42] U. van Kolck, J.L. Friar and T. Goldman, *Phys. Lett. B* **371** (1996), 169.

[43] D. B. Kaplan, *Nucl. Phys. B* **478** (1996), 629.

[44] D. B. Kaplan, *Nucl. Phys. B* **494** (1997), 471.

[45] D. B. Kaplan and A. V. Manohar, *Phys. Rev. C* **56** (1997), 76.

[46] D. R. Phillips and T. D. Cohen, *Phys. Lett. B* **390**, 7 (1997).

[47] K. G. Richardson, M. C. Birse, and J. A. McGovern, (1997), hep-ph/9708433.

[48] U. van Kolck, *Nucl. Phys. A* **631** (1998), 56c.

[49] D. R. Phillips, S. R. Beane, and T. D. Cohen, *Annals Phys. (N.Y.)* **263** (1998), 255.

[50] D. B. Kaplan, M. J. Savage, and M. B. Wise, *Nucl. Phys. B* **534** (1998), 329.
[51] S. R. Beane, T. D. Cohen, and D. R. Phillips, *Nucl. Phys. A* 632 (1998), 445.

[52] J. V. Steele and R. J. Furnstahl, *Nucl. Phys. A* 637 (1998), 46.

[53] A. Ghosh, S. K. Adhikari and B. Talukdar, *Phys. Rev. C* 58 (1998), 1913.

[54] M. C. Birse, Talk given at Caltech/INT Mini Workshop on Nuclear Physics with Effective Field Theory, Caltech, Feb. 1998, nucl-th/9804028.

[55] D. B. Kaplan, Talk given at Caltech/INT Mini Workshop on Nuclear Physics with Effective Field Theory, Caltech, Feb. 1998, nucl-th/9804061.

[56] J. V. Steele, Talk given at Caltech/INT Mini Workshop on Nuclear Physics with Effective Field Theory, Caltech, Feb. 1998, nucl-th/9807023.

[57] M. C. Birse, J. A. McGovern, and K. G. Richardson, (1998), hep-ph/9807302.

[58] T. D. Cohen and J. M. Hansen, *Phys. Lett. B* 440 (1998), 233.

[59] M. C. Birse, J. A. McGovern, and K. G. Richardson, Talk given at 11th Summer School and Symposium on Nuclear Physics, Seoul, Korea, June 1998, hep-ph/9808398.

[60] J. Gegelia, *Phys.Lett. B* 429 (1998), 227.

[61] Ulf-G. Meissner, *Acta Phys. Polon. B* 29 (1998), 2339.

[62] S. R. Beane, *Acta Phys. Polon. B* 29 (1998), 3161.

[63] T. Mehen and I. W. Stewart, (1998), nucl-th/9809093.

[64] Ulf-G. Meissner, Talk given at 8th International Conference on the Structure of Baryons (BARYONS 98), Bonn, Germany, Sept. 1998, nucl-th/9810015.

[65] T. D. Cohen and J. M. Hansen, *Phys. Rev. C* 59 (1999), 13.

[66] D. B. Kaplan, M. Savage, and M. B. Wise, *Phys. Rev. C* 59 (1999), 617.

[67] E. Epelbaoum, W. Glockle, A. Kruger, and Ulf-G. Meissner, *Nucl. Phys. A* 645 (1999), 413.

[68] D. B. Kaplan, M. J. Savage, R. P. Springer, and M. B. Wise *Phys. Lett. B* 449 (1999), 1.

[69] D. B. Kaplan, Talk given at 8th International Conference on the Structure of Baryons (Baryons 98), Bonn, Germany, Sept. 1998, nucl-th/9901003.

[70] T. D. Cohen and J. M. Hansen, (1999), nucl-th/9901065.

[71] U. van Kolck (1999), nucl-th/9902015.

[72] J.-W. Chen, G. Rupak, and M. J. Savage, (1999), nucl-th/9902056.

[73] C. da Rocha, G. Miller and U. van Kolck (1999), nucl-th/9904031.

[74] E. Epelbaum and Ulf-G. Meissner, Talk given at INT Workshop on Nuclear Physics with Effective Field Theory, Seattle, WA, Feb. 1999, nucl-th/9903046.

[75] T. D. Cohen, Talk given at INT Workshop on Nuclear Physics with Effective Field Theory, Seattle, WA, Feb. 1999, nucl-th/9904052.

[76] P. F. Bedaque, H. W. Hammer, and U. van Kolck, *Phys. Rev. Lett.* 82 (1999), 463.

[77] P. F. Bedaque, H. W. Hammer, and U. van Kolck *Nucl. Phys.A* 646 (1999), 444.
[78] S. D. Glazek and K.G. Wilson, Phys. Rev. D 48 (1993), 5863.
[79] S. D. Glazek and K.G. Wilson, Phys. Rev. D 49 (1994), 4214.
[80] F. Wegner, Ann. Physik (Berlin) 3 (1994), 77.
[81] K. G. Wilson, T. S. Walhout, A. Harindranath, W.-M. Zhang, R. J. Perry, and S. D. Glazek, Phys. Rev. D 49 (1994), 6720.
[82] R.J. Perry, in “Proceedings of Hadrons’ 94,” (V. Herscovitz and C. Vasconcelos, eds.), World Scientific, Singapore, 1995.
[83] R. Perry, in “Proceedings of the International Workshop on Light-Cone QCD,” (S. Glazek, ed.), World Scientific, Singapore, 1995.
[84] K. G. Wilson and D. G. Robertson, in “Proceedings of the International Workshop on Light-Cone QCD,” (S. Glazek, ed.), World Scientific, Singapore, 1995.
[85] T. Sugihara and M. Yahiro, Phys. Rev. D 53 (1996), 7239.
[86] M. Brisudová and R. Perry, Phys. Rev. D 54 (1996), 1831.
[87] B. Jones, R. Perry and S. Glazek, Phys. Rev. D 55 (1997), 6561.
[88] B. Jones and R. Perry, Phys. Rev. D 55 (1997), 7715.
[89] M. Brisudová, R. Perry and K. Wilson, Phys. Rev. Lett. 78 (1997), 1227.
[90] W.-M. Zhang, Phys. Rev. D 56 (1997), 1528.
[91] E. L. Gubankova and F. Wegner, Talk given at conference on New Nonperturbative Methods and Quantization on the Light Cone, Les Houches, France, March 1997, hep-th/9708054.
[92] M. Brisudová, S. Szpigel, and R. Perry, Phys. Lett. B 421 (1998),334.
[93] E. L. Gubankova, H.-C. Pauli, and F. J. Wegner, (1998), hep-th/9809143.
[94] R. J. Perry, Talk given at APCTP-RCNP Joint International School on Physics of Hadrons and QCD, Osaka, Japan, Oct. 1998, nucl-th/9901080.
[95] E. L. Gubankova and G. Papp, (1999), hep-th/9904081.
[96] A. P. Szczepaniak, E. S. Swanson, C. R. Ji, and S. R. Cotanch,Phys. Rev. Lett. 76 (1996), 2011.
[97] F. Wegner, Talk given at conference on New Nonperturbative Methods and Quantization on the Light Cone, Les Houches, France, March 1997.
[98] A. P. Szczepaniak and E. S. Swanson Phys. Rev. D 55 (1997), 1578.
[99] A. Mielke, (1998), quant-ph/9803040.
[100] H. J. Pirner and B. Friman, Phys. Lett. B 434 (1998), 231.
[101] S. D. Glazek and K.G. Wilson, Phys. Rev. D 57 (1998), 3558.
[102] B. H. Allen and R. J. Perry, Phys. Rev. D 58 (1998), 125017.
[103] T. S. Walhout, Phys. Rev. D 59 (1999), 065009.
[104] D. G. Robertson, E. S. Swanson, A. P. Szczepaniak, C. R. Ji, and S. R. Cotanch, Phys. Rev. D 59 (1999), 074019.
[105] G. Alexian and E. F. Moreno, Phys. Lett. B 450 (1999), 149.

[106] R. D. Kylin, B. H. Allen and R. J. Perry, (1998), hep-th/9812080.

[107] R. J. Perry and S. Szpigel, Talk given at YITP Workshop on QCD and Hadron Physics, Kyoto, Japan, Oct. 1998, nucl-th/9901079.

[108] S. D. Gázek, (1999), hep-th/9904029.

[109] K. G. Wilson, Phys. Rev. 140 (1965), B445.

[110] K. G. Wilson, Phys. Rev. D2 (1970), 1438.

[111] C. Bloch, Nucl. Phys. 6 (1958), 329.

[112] R. Oehme and W. Zimmermann, Commun. Math. Phys. 97 (1985), 569; R. Oehme, K. Sibold and W. Zimmermann, Phys. Lett. B 147 (1984), 115.

[113] R. J. Perry and K. G. Wilson, Nucl. Phys. B 403 (1993), 587; R. J. Perry, Ann. Phys. (N.Y.) 232 (1994), 116.

[114] G. Dahlquist and A. Bjork, “Numerical Methods,” Prentice-Hall, New Jersey, 1974.

[115] S. Coleman and E. Weinberg, Physical Review D 7 (1973), 1888.
The leading-order and next-to-leading-order EFT expansion parameters for the two-dimensional delta-function potential.