Bogoliubov’s causal perturbative QFT with Hida operators

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Abstract

We will present the axioms of Bogoliubov’s causal perturbative QFT in which the creation-annihilation operators are interpreted as Hida operators. We will shortly present the results that can be achieved in this theory: 1. Removal of UV and IR infinity in the scattering operator, 2. Existence of the adiabatic limit for interacting fields in QED, 3. Proof that charged particles have non-zero mass, 4. Existence of infrared and ultraviolet asymptotics for QED.

keywords: scattering operator; causal perturbative method in QFT; interacting fields; white noise; Hida operators; integral kernel operators; Fock expansion.

1 Introduction

It was the monograph \[1\] where a way to the rigorous formulation of the renormalization method in perturbative QFT was initiated. The causal axioms (I) - (IV) (see below) were formulated in \[1\] for the scattering operator \(S\). This enabled the perturbative QFT to be transferred into the axiomatic path, and

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it was there that the idea of a strict mathematical construction of higher-order contributions \( S_n \) to \( S \) first appeared, which uses only the axioms (I) - (IV), plus, possibly, Ward’s identities, cf. [1], §29.2.

Some freedom, however, concerning the strict mathematical interpretation of the generalized free field operators, their Wick products and the higher order contributions \( S_n \) to \( S \) was left open. Concerning this freedom, it was stated in [1] only a general assumption that these generalized operators should be some kind of operator distributions, which can also be evaluated at the Grassmann valued test functions. The class of generalized operators should include the free fields and their Wick products, and the Wick polynomials in free fields, with the coefficients equal to arbitrary translationally invariant tempered distributions. This class should include the free fields and the Wick product operation, which is well-defined whenever multiplied by any translationally invariant tempered distribution. The class of generalized operators which is sufficient to include the higher order contributions \( S_n \), is characterized in [1] as consisting of Wick polynomials in free fields with the scalar coefficients being tempered translationally invariant distributions. The higher order contributions \( S_n \) are not uniquely determined by the causality axioms (I)-(IV), but only up to a quasi local generalized operators \( \Lambda_n \) supported at the full diagonal. Using this freedom, it was explained in [1], that the scalar coefficients in \( S_n \) (or Green functions), computed with the standard renormalization technique, determine \( S_n \) which are in agreement with the causality axioms (I) - (IV). But concerning the rigorous treatment of \( S_n \), it was indicated in §29.2 only a proof of existence of \( S_n \) based on the axioms (I)-(IV) without giving any explicit mathematically rigorous construction of \( S_n \). A mathematically rigorous and explicit construction of \( S_n \), based solely on the Bogoliubov’s axioms (I) - (IV), and which gives the same scalar coefficients in \( S_n \) as the renormalization method, was given later by Epstein and Glaser [2].

The content of the axioms (I) - (IV) for the \( S \) operator will depend on how we mathematically interpret the free field operators, their Wick products, and the higher-order contributions \( S_n \), as generalized operators. Thus, before giving any mathematically rigorous construction of \( S_n \), based on (I)-(IV), one has to fix the mathematical meaning of these generalized operators. Epstein and Glaser [2] assumed that these operators are operator valued distributions precisely in the Wightman[3] sense. Using this interpretation for the generalized operators, it was shown in [2] that the standard results of the renormalization technique for the computation of \( S_n \) (or rather for the scalar coefficients at the free-field-Wick monomials in \( S_n \)) can be rigorously reconstructed from the axioms (I) - (IV), if we add the assumption (V) of preservation of the singularity degree at zero of the causal coefficient distributions in the computation of their splitting into the retarded and advanced parts. In this way, we arrive at a mathematically consistent formulation of perturbative QFT, without any UV divergences. The work [2] is in fact a continuation of the ideas outlined in [1]. In this approach, instead of the formal (divergent) multiplication by the step theta function in the chronological product and then renormalization removing singular parts in the ill-defined products of distributions by the step function, one uses the inductive
step of Epstein-Glaser in the perturbative approach, which is carried out solely within the axioms (I)-(V). The whole problem of construction of the higher order contributions $S_n$ is thus reduced to the problem of splitting of causal numerical tempered distributions into their retarded and advanced parts. The splitting of a causal distribution is not unique and depends on the so-called singularity degree at zero (in space-time coordinates) of the splitted distribution – freedom corresponding to the non-uniqueness in the ordinary renormalization procedure, and associated with the renormalization group freedom in the approach based on renormalization.

But some IR divergences remained for those QFT with infinite range of interaction (like QED), whenever one wanted to pass to the limit of the intensity of interaction function $g$ to the constant function 1, performed in order to get results with the physical interaction also in the remote part of space-time. This is the celebrated adiabatic limit problem. In some problems of QFT with infinite range of interaction (like QED), passing to this limit is unavoidable.

We propose to improve this Bogoliubov-Epstein-Glaser perturbative QFT by reinterpreting mathematically the generalized operators and regard them not as the operator valued distributions in the Wightman sense, but as the integral kernel operators in the sense of the white noise calculus. This allows us to solve the adiabatic limit problem – impossible to solve when the generalized operators were interpreted as operator valued distributions in the Wightman sense. In this way we arrive at the theory without any UV or IR divergences, which is mathematically consistent.

Our presentation does not give proofs. Nonetheless, it is rigorous, written in the ‘definition-lemma-theorem’ style. We emphasize the essential points, indicate the basic theorems on integral kernel operators and the extension theorems for operators on nuclear spaces which are fundamental in the proof, and try to clarify why the integral kernel operators are effective in the investigation of the adiabatic limit while the operator valued distributions in the Wightman sense are not.

2 Free fields and their Wick products understood as integral kernel operators

In other words, we are using the Hida white noise operators

$$\partial_p^*, \partial_p$$

which respect the canonical commutation or anticommutation relations

$$\left[\partial_p, \partial_q^*\right]_x = \delta(p - q),$$

as the creation-annihilation operators

$$a(p)^+, a(p),$$
of the free fields in the Bogoliubov’s causal perturbative QFT, leaving all the
rest of the theory completely unchanged. I.e. we are using the standard Gelfand
triple
\[ E \subset \mathcal{H} \subset E^* , \]
over the single particle Hilbert space \( \mathcal{H} \) of the total system of free fields
determined by the corresponding standard self-adjoint operator \( A \) in \( \mathcal{H} \) (with some
negative power \( A^{-r} \) being nuclear), and its lifting to the standard Gelfand triple
\[ (E) \subset \Gamma(\mathcal{H}) \subset (E)^* , \]
over the total Fock space \( \Gamma(\mathcal{H}) \) of the total system of free fields with the
Corresponding standard operator \( \Gamma(A) \). Here, \( E \) is the single particle test space in
the total single particle space in the total Fock space, and is equal to the direct
sum of the particular single particle test spaces \( E_1, E_2, \ldots \) of the particular free
fields \( A^{(1)}, A^{(2)}, \ldots \) underlying the QFT in question, which together with their
strong duals \( E_1^*, \ldots \) and the total single particle Hilbert spaces compose Gelfand
triples. (The respective single particle test spaces, the direct summands \( E_i \) of
the corresponding free fields \( A^{(1)}, A^{(2)}, \ldots \), will be denoted by \( E_1, E_2, \ldots \) or
\( E_i, E_i', \ldots \). The plane wave kernels of the free fields will be denoted by \( \kappa^{(1)}_{0,1}, \kappa^{(1)}_{1,0}, \kappa^{(1)}_{1,1}, \kappa^{(1)}_{0,0}, \ldots \)
\[ E \subset \mathcal{H} \subset E^* , \]
\[ E_1 \oplus \ldots \oplus E_N \quad \| \quad \| \quad \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_N \quad \| \quad E_1^* \oplus \ldots \oplus E_N^* , \]
In general the nuclear spaces \( E_1, E_2, \ldots \) are naturally defined as linear spaces of
smooth sections of smooth vector bundles, but are unitary isomorphic (through
a natural unitary isomorphism, defined through the smooth idempotent associ-
ated to the smooth bundle \( E_1, E_2, \ldots \), and which is also continuous in the nuclear
topologies) to the nuclear spaces of test \( C^d_1, C^d_2, \ldots \)-valued functions on the cor-
responding orbits \( \mathcal{O} = \{ p : p \cdot \mathbf{p} = m, p_0 \geq 0 \} \) in momentum space for the fields
of mass \( m \), which can be represented as test function spaces \( E_1 = \mathcal{S}(\mathbb{R}^3; C^d_1), \ldots \)
in massive case, or as \( E_1 = \mathcal{S}_0(\mathbb{R}^3; C^d_1), \ldots \), in massless case \( m = 0 \), and are
restrictions to the corresponding orbits \( \mathcal{O} \) of the Fourier transforms of the scalar,
vector, spinor, \ldots (depending on the kind of field \( A^{(1)}, \ldots \) ) functions lying,
respectively, in the Fourier inverse images of \( \mathcal{S}(\mathbb{R}^4; C^d_1), \ldots \) or \( \mathcal{S}_0(\mathbb{R}^4; C^d_1), \ldots \).
Here \( \mathcal{S}_0(\mathbb{R}^n; C^d) \) is the closed subspace of \( \mathcal{S}(\mathbb{R}^n; C^d) \) of all those functions who’s
all derivatives vanish at zero:
\[ \mathcal{S}_A(\mathbb{R}; C) \subset L^2(\mathbb{R}; C) \subset \mathcal{S}_A(\mathbb{R}; C)^* \]
\[ E \subset \mathcal{H} \subset E^* \]
This means that the space \( E \) consisting of direct sums of restrictions of
the Fourier transforms of space-time test \( C^d \)-valued functions (or rather smooth
sections – their images under a smooth idempotent in case of higher spin charged
fields) to the respective orbits $\mathcal{O}$ in momenta defining the representation of $T_4 \ltimes SL(2, \mathbb{C})$ in the Mackey’s classification (acting in the single particle Hilbert space for each corresponding free field) is given the standard realization with the help of a standard operator $A$ in $L^2(\mathbb{R}^3; \mathbb{C}) \cong \mathcal{H}$ (with standard $A$, i.e. self-adjoint positive, with some negative power of which being nuclear, or trace-class, and with the minimal spectral value greater than 1). It is equal to the the direct sum $\bigoplus_{i=1}^{N} A_i$ of the standard operators $A_i$ corresponding to the single particle Hilbert space of the $i$-th free field. Thus, first we need to construct the standard $A_i$ and the standard Gelfand triples for each of the free fields of the theory. $A = \bigoplus_{i=1}^{N} A_i$ is also standard. It serves for the construction of the standard Gelfand triple over the full single particle Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}) \cong \mathcal{H}$. Namely, with the help of the operator $A$ we construct $E$ as the following projective limit

$$E = \bigcap_{k \in \mathbb{N}} \text{Dom} A^k$$

with the standard realization of $E$ as the countably Hilbert nuclear (in the sense of Grothendieck) test space and its strong (also nuclear) dual as the inductive limit

$$E^* = \bigcup_{k \in \mathbb{N}} \text{Dom} A^{-k}$$

with the Hilbertian defining norms

$$| \cdot |_{k}^\text{df} = |A^k \cdot |_{L^2}, \quad | \cdot |_{-k}^\text{df} = |A^{-k} \cdot |_{L^2}, \quad k = 0, 1, 2, 3, \ldots$$

For any $\Phi$ in $(E)$ or in $(E)^*$ let

$$\Phi = \sum_{n=0}^{\infty} \Phi_n \quad \text{with} \quad \Phi_n \in E^{\otimes n} \quad \text{or, respectively,} \quad \Phi_n \in E^* \otimes n$$

be its decomposition into $n$-particle states of an element $\Phi$ of the test Hida space $(E)$ or in its strong dual $(E)^*$, convergent, respectively, in $(E)$ or in $(E)^*$. We define

$$a(w)\Phi_0 = 0, \quad a(w)\Phi_n = n \pi \hat{\otimes} \Phi_n$$

$$a(w)^+ \Phi_n = w \hat{\otimes} \Phi_n, \quad \text{for each fixed} \quad w \in E^*.$$

**DEFINITION 1.** The Hida operators are obtained when we put here the Dirac delta functional $\delta_{s,p}$ for $w$

$$\partial_{s,p} = a_s(p) = a(\delta_{s,p}), \quad \partial_{s,p}^+ = a_s(p)^+ = a(\delta_{s,p})^+.$$

Let $L(E_1, E_2)$ be the linear space of linear continuous operators $E_1 \rightarrow E_2$ endowed with the natural topology of uniform convergence on bounded sets.
For each fixed spin-momentum point \((s, p)\) the Hida operators are well-defined (generalized) operators
\[
\begin{align*}
a_s(p) & \in L((E), (E)) \subset L((E), (E)^*), \\
a_s(p)^+ & \in L((E)^*, (E)^*) \subset L((E), (E)^*),
\end{align*}
\]
with the last “\(\subset\)” by topological inclusion \((E) \subset (E)^*\). Let \(\phi \in E\) (here \(E\) is the space-time test space \(S\) or \(S^{(0)}\)) and let \(\kappa_{l,m}\) be any \(L(E, \mathbb{C}) = E^*\)-valued distribution
\[
\kappa_{l,m} \in L(E^\otimes (l+m), E^*) = L(E, E^* \hat{\otimes} (l+m)) = E^* \hat{\otimes} (l+m) \otimes E^*,
\]
then we put

**DEFINITION 2.**

\[
\Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi) \overset{df}{=} \sum_{n=0}^{\infty} \kappa_{l,m} \otimes_m (\Phi_{n+m} \otimes \phi)
\]

\[
\Xi_{l,m}(\kappa_{l,m}) \Phi
\]

which for any \(E^*\)-valued distribution \(\kappa_{l,m}\) is a well-defined (generalized) operator
\[
\Xi(\kappa_{l,m}) = \int \kappa_{l,m}(p_1, \ldots, p_l, q_1, \ldots, q_m) \partial_{p_1}^* \cdots \partial_{p_l}^* \partial_{q_1} \cdots \partial_{q_m} \times
\]
\[
\times dp_1 \cdots dp_l dq_1 \cdots dq_m,
\]
\[
\Xi_{l,m}(\kappa_{l,m}) \in L((E) \otimes E, (E)^*) \cong L(E, L((E), (E)^*)).
\]

(for brevity of notation \(p_i\) denote in this formula spin-momentum variables \((s_i, p_i)\) and integrations include summations with respect to spin components \(s_i\).)

\(\Xi_{l,m}(\kappa_{l,m})\) defines integral kernel operator \(\Xi_{l,m}(\kappa_{l,m})\) which is uniquely determined by the condition
\[
\langle \langle \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi), \Psi \rangle \rangle = \langle \kappa_{l,m}(\eta_{\phi, \psi}), \phi \rangle, \quad \Phi, \Psi \in (E), \phi \in E
\]
or, respectively,
\[
\langle \langle \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi), \Psi \rangle \rangle = \langle \kappa_{l,m}(\phi), \eta_{\phi, \psi} \rangle, \quad \Phi, \Psi \in (E), \phi \in E,
\]
depending on \(\kappa_{l,m}\) is regarded as an element of

\(L(E^\otimes (l+m), E^*)\) or, respectively, of \(L(E, E^* \hat{\otimes} (l+m))\).

Here
\[
\eta_{\phi, \psi}(s_1, p_1, \ldots, s_l, p_l, s_{l+1}, p_{l+1}, \ldots, s_{l+m}, p_{l+m})
\]
\[
\overset{df}{=} \langle \langle a_{s_1}(p_1)^+ \cdots a_{s_l}(p_l)^+ a_{s_{l+1}}(p_{l+1}) \cdots a_{s_{l+m}}(p_{l+m}) \Phi, \Psi \rangle \rangle
\]
is the function which always belongs to \(E^\otimes (l+m)\). (Hida, Obata, Saitô)
EXAMPLE. Free fields $\mathbb{A}$ are sums of two integral kernel operators

$$\mathbb{A}(\phi) = \mathbb{A}^{(-)}(\phi) + \mathbb{A}^{(+)}(\phi) = \Xi(\kappa_{0,1}(\phi)) + \Xi(\kappa_{1,0}(\phi))$$

with the integral kernels $\kappa_{l,m}$ represented by ordinary functions:

$$\kappa_{0,1}(\nu, p; \mu, x) = \frac{g_{\nu\mu}}{(2\pi)^3/2p^0(p)} e^{-ip\cdot x}, \quad p = (|p_0(p)|, p), \quad p \cdot p = 0,$$

$$\kappa_{1,0}(\nu, p; \mu, x) = \frac{g_{\nu\mu}}{(2\pi)^3/2p^0(p)} e^{ip\cdot x}, \quad p \cdot p = 0,$$

for the free e.m. potential field $\mathbb{A} = \mathbb{A}$ (in the Gupta-Bleuler gauge) and

$$\kappa_{0,1}(s, p; a, x) = \begin{cases} (2\pi)^{-3/2} u_s(p) e^{-ip\cdot x}, & p = (|p_0(p)|, p), \quad p \cdot p = m^2 \quad \text{if } s = 1, 2 \\ 0 & \text{if } s = 3, 4 \end{cases}$$

$$\kappa_{1,0}(s, p; a, x) = \begin{cases} 0 & \text{if } s = 1, 2 \\ (2\pi)^{-3/2} v_{s-2}(p) e^{ip\cdot x}, & p \cdot p = m^2 \quad \text{if } s = 3, 4 \end{cases}$$

for the free Dirac spinor field $\mathbb{A} = \psi$, and which are in fact the respective plane wave solutions of d’Alembert and of Dirac equation, which span the corresponding generalized eigen-solution sub spaces. Here $g_{\nu\mu}$ are the components of the space-time Minkowski metric tensor, and

$$u_s(p) = \frac{1}{\sqrt{2}} \sqrt{\frac{E(p) + m}{2E(p)}} \begin{pmatrix} \chi_s + \frac{p}{\sigma} \chi_s \\ \chi_s - \frac{p}{E(p) + m} \chi_s \end{pmatrix},$$

$$v_s(p) = \frac{1}{\sqrt{2}} \sqrt{\frac{E(p) + m}{2E(p)}} \begin{pmatrix} \chi_s + \frac{p}{\sigma} \chi_s \\ -(\chi_s - \frac{p}{E(p) + m} \chi_s) \end{pmatrix},$$

where

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E(p) = |p_0(p)|,$$

are the Fourier transforms of the complete system of the free Dirac equation in the chiral representation in which

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \sigma_1, \sigma_2, \sigma_3$$

are the Pauli matrices

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) = \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).$$
LEMMA 1. The free c.m. potential and spinor field operators $A = A, \psi$ are operator-valued distributions, i.e. belong to $L(L(E, L((E), (E))))$, i.e.

\[ \kappa_{0,1}, \kappa_{1,0} \in L(E^*, E^*) = L(E, E) \subset L(E, E^*) \cong E^* \otimes E^*, \]

if and only if

\[ E = S^0(\mathbb{R}^3; \mathbb{C}^4) = S_{A(3)}(\mathbb{R}^3; \mathbb{C}^4), \quad E = S^{00}(\mathbb{R}^4; \mathbb{C}^4) = S_{A(4)}(\mathbb{R}^4; \mathbb{C}^4) \text{ for } A \]

In this case, moreover,

\[ \kappa_{0,1}, \kappa_{1,0} \in L(E, O_M) \]

and for each $\xi \in E$, $\kappa_{0,1}(\xi), \kappa_{1,0}(\xi)$ are smooth having all derivatives bounded.

Recall that here $S^0(\mathbb{R}^n; \mathbb{C}) = S_{A(n)}(\mathbb{R}^n; \mathbb{C})$ is the closed subspace of the Schwartz space $S(\mathbb{R}^n; \mathbb{C})$ of all functions whose all derivatives vanish at zero. $S^{00}(\mathbb{R}^n; \mathbb{C})$ is the Fourier transform inverse image of $S^0(\mathbb{R}^n; \mathbb{C})$. The space $S^0(\mathbb{R}^n; \mathbb{C})$ can be realized as a countably Hilbert nuclear space $S_{A(n)}(\mathbb{R}^n; \mathbb{C})$ associated, in the sense of $\mathbb{R}$, with a positive self adjoint operator $A(n)$ in $L^2(\mathbb{R}^n; d^n\mu)$, with Inf Spec $A(n) > 1$ whose some negative power $[A(n)]^{-r}$, $r > 0$, is of Hilbert-Schmidt class. In order to construct an example of a series of operators $A(n)$, $n = 2, 3, \ldots$, let us consider the Hamiltonian operators

\[ H(n) = -\Delta_{\mathbb{R}^n} + r^2 + 1, \quad r^2 = (p_1)^2 + \ldots + (p_n)^2, \text{ in } L^2(\mathbb{R}^n, d^n\mu), \]

and the following $L^2(\mathbb{R} \times S^{n-1}; dt \times d\mu_{S^{n-1}})$, $L^2(\mathbb{R} \times S^{n-1}; \nu_n(t) dt \times d\mu_{S^{n-1}})$ -spaces on $\mathbb{R} \times S^{n-1}$ with the weights

\[ \nu_n(t) = \frac{(1 + \sqrt{t^2 + 1})^{n-1}}{2^{n-1} (t^2 + 1)^{(n-1)/2}}, \]

and the following unitary operators $U_2 : L^2(\mathbb{R} \times S^{n-1}; \nu_n(t) dt \times d\mu_{S^{n-1}}) \to L^2(\mathbb{R}^n; d^n\mu), U_1 : L^2(\mathbb{R} \times S^{n-1}; dt \times d\mu_{S^{n-1}}) \to L^2(\mathbb{R} \times S^{n-1}; \nu_n(t) dt \times d\mu_{S^{n-1}}), U = U_2 U_1 : L^2(\mathbb{R} \times S^{n-1}; dt \times d\mu_{S^{n-1}}) \to L^2(\mathbb{R}^n; d^n\mu)$ given by the following formulas

\[ U_1 f(t, \omega) = \frac{1}{\sqrt{\nu_n(t)}} f(t, \omega), \quad f \in L^2(\mathbb{R} \times S^{n-1}; dt \times d\mu_{S^{n-1}}), \]

\[ U_2 f(r, \omega) = f(r(t), \omega), \quad t(r) = r - r^{-1}, \quad f \in L^2(\mathbb{R} \times S^{n-1}; \nu_n(t) dt \times d\mu_{S^{n-1}}), \]

where $U_2 f$ is expressed in spherical coordinates $(r, \omega)$ in $\mathbb{R}^n$. We can put

\[ A(n) = U(H(1) \otimes 1 + 1 \otimes \Delta_{S^{n-1}})U^{-1}. \]

Here $\Delta_{\mathbb{R}^n}, \Delta_{S^{n-1}}$ are the standard Laplace operators on $\mathbb{R}^n$ and $S^{n-1}$, with the standard invariant measures $d^n\mu, d\mu_{S^{n-1}}$. In particular

\[ A(3) = -\frac{\partial^2}{r^2} - \frac{\partial^2}{(r^2 + 4)^2} \partial_t + \left[ \frac{r^2 (r^2 + 4)(r^2 - 2)}{4(r^2 + 1)^4} + r^2 + r^{-2} \right] \]
in spherical coordinates.

Let \( \kappa_{0,1}^1, \kappa_{1,0}^1, \kappa_{0,1}^2, \kappa_{1,0}^2, \ldots, \kappa_{0,1}^n, \kappa_{1,0}^n \) be the plain wave kernels of the free fields \( \mathbb{A}^{(1)}, \ldots, \mathbb{A}^{(n)} \). Then

\[
\kappa_{lm} = \kappa_{l_1,m_1}^1 \otimes \ldots \otimes \kappa_{l_n,m_n}^n, \quad l = l_1 + \ldots + l_n, \quad m = m_1 + \ldots + m_n
\]

are the kernels of the Wick (pointwise) product operator

\[
: \mathbb{A}^{(1)}(x) \ldots \mathbb{A}^{(n)}(x) :,
\]

where \( \otimes \) denotes ordinary pointwise product with respect to the space time point \( x \). The kernels \( \kappa_{lm} \) so defined should be symmetrized in Boson spin-momentum variables and antisymmetrized in the Fermion spin-momentum variable \( s \) in order to keep one-to-one correspondence between the kernels and operators. For the (tensor) Wick product operator

\[
: \mathbb{A}^{(1)}(x_1) \ldots \mathbb{A}^{(n)}(x_n) :,
\]

with \( n \) independent space-time variables \( x_j \), we have analogous formula for its kernels \( \kappa_{lm} \), but with the pointwise product \( \otimes \) replaced with ordinary tensor product \( \otimes \). In this case the (tensor) Wick product operator belongs to the class \( L(\mathcal{E}^{\otimes n}, L((\mathcal{E}, (\mathcal{E}^*))) \), irrespectively if the free fields \( \mathbb{A}^{(j)} \) are massive or massless.

We have the following easily verified lemma for the Wick (pointwise) product operator

**LEMMA 2.**

\[
: \mathbb{A}^{(1)} \ldots \mathbb{A}^{(n)} :, \in \begin{cases} L(\mathcal{E}, L((\mathcal{E}, (\mathcal{E}^*))), & \text{if all fields } \mathbb{A}^{(j)} \text{ are massive,} \\ L(\mathcal{E}, L((\mathcal{E}, (\mathcal{E}^*)))), & \text{if some } \mathbb{A}^{(j)} \text{ are massless fields.} \end{cases}
\]

**THEOREM 1.** The standard Wick theorem decomposition holds for the (tensor) product operator

\[
: \mathbb{A}^{(1)}(x) \ldots \mathbb{A}^{(n)}(x) :: \mathbb{A}^{(n+1)}(y) \ldots \mathbb{A}^{(n+k)}(y) :
\]

with the kernels of the decomposition given by the contractions

\[
k_{lm}(\phi \otimes \varphi) = \sum_{k_{l',m'},k_{l'',m'',k}} k_{l',m'}^l(\phi) \otimes k_{l'',m'',k}^l(\varphi)
\]

where in this sum \( k_{l',m'}^l \) and \( k_{l'',m'',k}^l \) range over the kernels respectively of the operators

\[
: \mathbb{A}^{(1)}(x) \ldots \mathbb{A}^{(n)}(x) : \text{ and } : \mathbb{A}^{(n+1)}(y) \ldots \mathbb{A}^{(n+k)}(y) :
\]

and

\[
l' + l'' - k = l, \quad m' + m'' - k = m
\]
and where the contractions $\otimes_k$ are performed upon all $k$ pairs of spin-momentum variables in which the first variable in the pair corresponds to an annihilation operator variable and the second one to the creation operator variable or vice versa. All these contractions are given by absolutely convergent sums/integrals with respect to the contracted variables. After the contraction, the kernels should be symmetrized in Boson spin-momentum variables and antisymmetrized in the Fermion spin-momentum variables in order to keep one-to-one correspondence between the kernels and operators.

Let us give few words of explanation. The pointwise Wick product factors $\mathcal{A}^{(n)} \ldots \mathcal{A}^{(n)}$, and $\mathcal{A}^{(n+1)} \ldots \mathcal{A}^{(n+k)}$, of the last theorem, when evaluated at the test function $\phi \in \mathcal{E}$ and $\varphi \in \mathcal{E}$, are ordinary operators transforming the Hida space $(E)$ into itself, and as such can be composed. In this case the (tensor) product operator is well-defined, and its evaluation at the test function $\phi \otimes \varphi$ is, by definition, understood as equal to the composition

$$\mathcal{A}^{(1)} \ldots \mathcal{A}^{(n)}; (\phi) \otimes \mathcal{A}^{(n+1)} \ldots \mathcal{A}^{(n+k)}; (\varphi).$$

(2)

If among $\mathcal{A}^{(j)}$ in $\mathcal{A}^{(n+1)} \ldots \mathcal{A}^{(n+k)}$; there are massless fields, then, according to lemma 2: $\mathcal{A}^{(n+1)} \ldots \mathcal{A}^{(n+k)}; (\varphi)$ is a generalized operator transforming the Hida space $(E)$ into its dual $(E)\dual$, and the composition (2) is in general meaningless. In this case the four-momentum $p$ in the exponent $e^{\mp ip x}$ defining the plane wave kernels $\kappa_{0,1}, \kappa_{1,0}$ of the massless free field $\mathcal{A}^{(j)}$, ranges over the corresponding massless orbit $O = \{ p : p \cdot p = 0 \}$ with zero component $p_0$ of $p$ being the following function $p_0(p) = |p|$ of the spatial momentum $p$. We replace all the massless exponents $e^{\mp ip x}$ in $\kappa_{0,1}, \kappa_{1,0}$ of the massless fields $\mathcal{A}^{(j)}$ by the massive exponents with mass $\epsilon$ and $p_0(p) = \sqrt{|p|^2 + \epsilon^2}$. After this replacement both operators in (2) become ordinary operators in the Fock space transforming continuously $(E)$ into $(E)$ and the composition (2) is meaningful. Then the limit $\epsilon \to 0$ in (2) defines the evaluation of the (tensor) product operator at the test function $\phi \otimes \varphi$ in case where some $\mathcal{A}^{(j)}$ are massless. In general

$$\mathcal{A}^{(1)} \ldots \mathcal{A}^{(n)}; \mathcal{A}^{(n+1)} \ldots \mathcal{A}^{(n+k)};$$

$$\in \begin{cases} L(E \otimes 2, L((E), (E))), & \text{if all fields } \mathcal{A}^{(j)} \text{ are massive,} \\ L(E \otimes 2, L((E), (E)\dual)), & \text{if some } \mathcal{A}^{(j)} \text{ are massless fields.} \end{cases}$$

where by

$$\mathcal{A}^{(1)} \ldots \mathcal{A}^{(n)}; \mathcal{A}^{(n+1)} \ldots \mathcal{A}^{(n+k)};$$

we have denoted the (tensor) product operator of the Wick (pointwise) products.

Accordingly, in the proof of theorem 1, we proceed in two steps. In the first step, we assume all free fields $\mathcal{A}^{(j)}$ to be massive. In this case both Wick product factors of theorem 1, when evaluated at the test functions $\phi$ and, respectively, $\varphi$, are operators continuously transforming the Hida space $(E)$ into itself, and can be composed. In this case the contractions $\kappa_{\nu,\mu} \otimes_k \kappa_{\nu,\mu}$ have ordinary meaning given by the ordinary dual pairings. In the second step, we consider
the case in which some free fields $A^{(j)}$ are massless. We replace $p_0(p) = |p|$ in the exponent $e^{i p \cdot x}$ defining the plane wave kernels $\kappa_{0,1}, \kappa_{1,0}$ of the massless free field $A^{(j)}$, with the following massive counterpart $p_0(p) = \sqrt{|p|^2 + \epsilon^2}$. Finally, we pass to the limit $\epsilon \to 0$.

2.1 A class of integral kernel operators that allows (tensor) product operation

We are going to show consistency of the Bogoliubov’s causality axioms (I)-(V), with free fields $A^{(j)}$, their pointwise Wick products and the (tensor) products of the pointwise Wick products of free fields, understood as finite sums of integral kernel operators, and finally products of (tensor) products of Wick pointwise products of free fields by tempered translationally invariant distributions, understood as integral kernel operators. The above said operations are sufficient for the axioms as well as for the inductive construction of the scattering operator, determined by the axioms. Then, in order to show the consistency of these axioms we should prove that, indeed, the said operations are meaningful.

We investigate the general class of generalized operators $\Xi'$, equal to finite sums of integral kernel operators

$$\Xi' = \sum_{\nu', m'} \Xi'_{\nu', m'} (\kappa'_{\nu', m'}) \in L(\mathcal{E}, L((E), (E)^*))$$

for which the (tensor) product operation is well-defined and the Wick decomposition through the normal ordering is applicable to their products.

The problem is that in general this class necessary should include the operators of

$$L(\mathcal{E}, L((E), (E)^*))$$

which do not belong to

$$L(\mathcal{E}, L((E), (E))).$$

Indeed the interaction Lagrangian $\mathcal{L}(x)$ is in general equal to a pointwise Wick product $\Xi$ of free fields. In case all free fields $A^{(j)}$ are massive in $\Xi$ the pointwise product belongs to $L(\mathcal{E}, L((E), (E)))$.

But in general the Wick pointwise product $\mathcal{L}(x)$ includes massless free field factors, as is the case e.g. for QED, with $\mathcal{L}(x)$ including the massless e.m. potential field, so that

$$\mathcal{L} \in L(\mathcal{E}, L((E), (E)^*))$$

in this case. Therefore, we proceed, as in the proof of theorem $\Xi$ in two steps. In the first step we replace the zero momentum $p_0(p) = |p|$ in the exponents of the kernels $\kappa_{0,1}, \kappa_{1,0}$ of all massless free fields $A^{(j)}$ of all Wick pointwise products

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of free fields containing massless fields, by \( p_0(p) = \sqrt{|p|^2 + \epsilon^2} \). In the second step, we pass to the limit \( \epsilon \to 0 \).

Using this method we show

**Theorem 2.** The class within which the (tensor) product of generalized operators (understood as finite sums of integral kernel operators with vector valued kernels) is well-defined as a finite sum of integral kernel operators, includes all operators of the form

\[
t(x_1, \ldots, x_n) : W_1(x_1)W_2(x_2) \ldots W(x_k); \quad t \in \mathcal{S}(\mathbb{R}^{4k})^* = \mathcal{S}(\mathbb{R}^4)^* \otimes k,
\]

with translationally invariant \( t \), and \( W_i \) being Wick pointwise products of massless or massive free fields.

For operators \( \Xi', \Xi'' \) in this class there exist \( \epsilon \)-approximations

\[
\Xi'_\epsilon = \sum_{l',m'} \Xi_{l,m}(\kappa'_{l,m}^\epsilon) \in L(\mathcal{E}_l, L((E), (E))))
\]

\[
\Xi''_\epsilon = \sum_{l'',m''} \Xi_{l,m}(\kappa''_{l,m}^\epsilon) \in L(\mathcal{E}_l, L((E), (E))))
\]

(here with \( \mathcal{E}_l \) and \( \mathcal{E}_{l'} \) being some Schwartz spaces of \( \mathbb{C}^{d_i} \)-valued functions) for which

\[
\Xi'_\epsilon \to \Xi', \quad \Xi''_\epsilon \to \Xi''
\]

in

\[
L(\mathcal{E}_{(\epsilon)}, L((E), (E)^*))
\]

and moreover, the Fock decomposition is naturally applicable to their operator composition product \( \Xi \),

\[
\Xi, (\phi \otimes \varphi) \overset{df}{=} \Xi'_\epsilon(\phi) \circ \Xi''_\epsilon(\varphi)
\]

and such that the kernels

\[
\kappa'_{l',m'} \otimes \kappa''_{l'',m''}
\]

of the Wick decomposition of the operator product \( \Xi \), converge, when \( \epsilon \to 0 \), to some kernels

\[
k'_{l',m'} \otimes k''_{l'',m''} \in L(E \otimes (t' + t'' + m' + m'' - 2k), \mathcal{E}_{l'}^* \otimes \mathcal{E}_{l''}^*)
\]

(here, for simplicity of notation, we have assumed all nuclear single particle momentum test spaces equal \( E \), meaning that all involved free fields are assumed to have equal number of components) representing a well-defined finite sum \( \Xi \) of generalized operators. Thus, in this class the product operator \( \Xi \) converges

\[
\Xi \to \Xi \in L(\mathcal{E}_{l'} \otimes \mathcal{E}_{l''}, L((E), (E)^*))
\]

to an operator \( \Xi \) in

\[
L(\mathcal{E}_l \otimes \mathcal{E}_l, L((E), (E)^*))
\]
In practice, we construct the \( \epsilon \)-approximation within the class indicated above, just by replacing the exponents of the free massless field plane wave kernels with exponents in which \( p_0(p) = |p|^2 \) is replaced by \( p_0(p) = (|p|^2 + \epsilon^2)^{1/2} \).

This definition of product can be generalized over a still more general finite sums of integral kernel operators

\[
\Xi' = \sum_{l',m'} \Xi'_{l',m'}(\kappa_{l',m'}') \in L(\mathcal{E}_r, L((E^*)^n)), \quad \mathcal{E}_r = \mathcal{S}(\mathbb{R}^{4k'})
\]

\[
\Xi'' = \sum_{l'',m''} \Xi''_{l'',m''}(\kappa_{l'',m''}'' \in L(\mathcal{E}_r, L((E^*)^n)), \quad \mathcal{E}_r = \mathcal{S}(\mathbb{R}^{4k''}),
\]

provided only they possess the \( \epsilon \)-approximations

\[
\Xi'_r = \sum_{l',m'} \Xi'_{l',m'}(\kappa_{l',m'}') \in L(\mathcal{E}_r, L((E^*)^n)), \quad \mathcal{E} = \mathcal{S}(\mathbb{R}^{4k'})
\]

\[
\Xi''_r = \sum_{l'',m''} \Xi''_{l'',m''}(\kappa_{l'',m''}'' \in L(\mathcal{E}_r, L((E^*)^n)), \quad \mathcal{E} = \mathcal{S}(\mathbb{R}^{4k''}),
\]

with the kernels

\[
\kappa_{l',m'}' \otimes \kappa_{l'',m''}'
\]

of the Wick decompositions of their products converging in the sense defined above. In fact the class of generalized operators, on which product operation is well-defined, can still be extended over infinite sums of integral kernel operators of the class \( \Xi \), provided the infinite sums represent Fock expansions convergent in the sense of \( \mathcal{S} \). In application to causal QFT, where the Lagrangian interaction density operator \( \mathcal{L}(x) \) is equal to a Wick polynomial in free fields of finite degree, the class \( \Xi \), allowing the product operation, is sufficient.

In the first step, indicated above, we need to show that indeed in case all free fields are massive, or in case the exponents of the kernels of the massless fields are replaced by the exponents with \( p_0(p) = (|p|^2 + \epsilon^2)^{1/2} \), the operators \( \Xi', \Xi'' \) of the said class, and their (tensor) product, defined by the composition \( \Xi' \circ \Xi'' \) indeed belong, respectively, to

\[
L(\mathcal{E}_r, \mathcal{L}((E^*)^n)), L(\mathcal{E}_r, \mathcal{L}((E^*)^n)), L(\mathcal{E}_r \otimes \mathcal{E}_r, L((E^*)^n)), L(\mathcal{E}_r \otimes \mathcal{E}_r, L((E^*)^n)), L(\mathcal{E}_r \otimes \mathcal{E}_r, L((E^*)^n))
\]

and (after being evaluated at the test functions) can be composed. In the second step, we need to show that the \( \epsilon \to 0 \) limit of these operators and of their composition defines a finite sum of integral kernel operators, respectively, in

\[
L(\mathcal{E}_r, \mathcal{L}((E^*)^n)), L(\mathcal{E}_r, \mathcal{L}((E^*)^n)), L(\mathcal{E}_r \otimes \mathcal{E}_r, L((E^*)^n)), L(\mathcal{E}_r \otimes \mathcal{E}_r, L((E^*)^n))
\]

We will not present here the proof of theorem \( \Xi \) but mention only that the proof is based on the following two theorems \( \mathcal{S} \) and \( \mathcal{H} \) applied for \( V = \mathcal{E}_r, \mathcal{E}_r, \mathcal{E}_r \otimes \mathcal{E}_r \).

**THEOREM 3** (Hida, Obata [\( \mathcal{H} \)]).  

\[
\Xi(\kappa_{lm}) \in \begin{cases} 
L(V, L((E^*)^n)), \\
L(V, L((E^*)^n)), \\
L(V, L((E^*)^n)), \\
L(V, L((E^*)^n)), \\
L(V, L((E^*)^n))
\end{cases}
\]
if and only if

\[
\begin{align*}
\kappa_{lm} &\in L(E^{\otimes (l+m)}, V^*), \\
\kappa_{lm} \text{ can be extended to a separately cont. map } &E^{\otimes l} \times E^{\otimes m} \rightarrow V^*,
\end{align*}
\]

Let \( E, E_1, F, F_1 \) be t.v.s. such that \( E, F \) are, respectively, dense in \( E_1, F_1 \). Suppose that \( \mathcal{S} \) is a family of bounded subsets of \( E \), with the property that \( \mathcal{S}_1 \) covers \( E_1 \), where \( \mathcal{S}_1 \) denotes the family of the closures, taken in \( E_1 \), of all subsets in \( \mathcal{S} \); analogously let \( \mathcal{F}, \mathcal{F}_1 \) be such families in \( F, F_1 \); finally, let \( G \) be a quasi-complete Hausdorff t.v.s. Under these assumptions, the following extension theorem holds (compare e.g. the proposition 5.4, Chap. III.5.4 in the book [7] of H. H. Schaefer):

**Theorem 4.** Every \((\mathcal{S}, \mathcal{F})\)-hypocontinuous bilinear mapping of \( E \times F \) into \( G \) has a unique extension to \( E_1 \times F_1 \) (and into \( G \)) which is bilinear and \((\mathcal{S}_1, \mathcal{F}_1)\)-hypocontinuous.

### 2.2 Bogoliubov’s causality axioms

Let \( \mathcal{L}(x) \) be the interaction Lagrangian, equal to a pointwise Wick product of free fields, or to a pointwise Wick polynomial in free fields. Let \( g \) be the intensity of interaction scalar function, or more generally, we consider many-component intensity of interaction function \((g, h)\) and a modified interaction Lagrangian \( g\mathcal{L} \) or \( g\mathcal{L} + h\hat{A} \), for any pointwise Wick polynomial \( \hat{A} \) in free fields of even degree in Fermi fields.

Let

\[
g^{\otimes n} \in \mathcal{E}^{\otimes n} \quad \text{or, respectively,} \quad g^{\otimes n} \otimes h^{\otimes k} \in \mathcal{E}^{\otimes n} \otimes (\otimes d\mathcal{E})^{\otimes k}.
\]

For such \( g \) or \((g, h)\) we construct the scattering operator \( S(g) \) or, more generally, \( S(g, h) \), with the modified intensity, \( g \) or \((g, h)\), of interaction as the following series

\[
S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} S_n(g^{\otimes n}), \quad S(g)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} S_n(g^{\otimes n}),
\]

or

\[
S(g, h) = 1 + \sum_{n=1}^{\infty} \sum_{p=0}^{n} \frac{1}{n!} S_{n-p,p}(g^{\otimes (n-p)} \otimes h^{\otimes p})
\]

(denoted also by \( S(g\mathcal{L}) \), \( S(g\mathcal{L})^{-1} \) or, respectively, \( S(g\mathcal{L} + h\hat{A}) \)), with each \( S_n \) or \( S_{n,k} \), understood as integral kernel operators

\[
S_n(g^{\otimes n}) = \sum_{l,m} \int \kappa_{lm} \left( p_1, \ldots, p_l, q_1, \ldots, q_m; g^{\otimes n} \right) \partial_{p_1}^* \cdots \partial_{p_l}^* \partial_{q_1} \cdots \partial_{q_m} \times
\]

\[
\times dp_1 \cdots dp_l dq_1 \cdots dq_m = \int d^4x_1 \ldots d^4x_n S_n(x_1, \ldots, x_n) g(x_1) \cdots g(x_n),
\]

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or

\[ S_{n,k}(g^{\otimes n} \otimes h^{\otimes k}) = \sum_{l,m} \int \kappa_{lm}(p_1, \ldots, p_l, q_1, \ldots, q_m; g^{\otimes n} \otimes h^{\otimes k}) \times \]
\[ \times \partial_{p_1}^* \ldots \partial_{p_l}^* \partial_{q_1} \ldots \partial_{q_m} d^l p_1 \ldots d^l p_l d^m q_1 \ldots d^m q_m \]
\[ = \int d^4x_1 \ldots d^4x_n d^4y_1 \ldots d^4y_k S_{n,k}(x_1, \ldots, x_n, y_1, \ldots, y_k) g(x_1) \ldots g(x_n), \]

with vector-valued distributional kernels \( \kappa_{lm} \) in the sense of Obata\([4]\), with the values in the distributions \( V^* \) over the test nuclear space

\[ V = \mathcal{E}^{\otimes n} \ni g^{\otimes n} \quad \text{or, respectively,} \quad V = \mathcal{E}^{\otimes n} \otimes (\otimes_1^d \mathcal{E})^{\otimes k} \ni g^{\otimes n} \otimes h^{\otimes k}. \]

Let us recall Bogoliubov’s axioms for \( S \). We confine attention to the simpler, but essential, case \( S(g) \) with only the scalar intensity of interaction function \( g \). The more general case \( S(g, h) \) is analogous and brings no essentially new difficulties into the whole analysis.

Bogoliubov’s causality axioms (I)-(V) for \( S(g) \) read

(I)

\[ S(g_1 + g_2) = S(g_2)S(g_1), \quad \text{whenever} \quad \text{supp} \ g_1 \leq \text{supp} \ g_2. \]

(II)

\[ U_{a,\Lambda}S(g)U_{b,\Lambda} = \left. S(T_{b,\Lambda}g) \right|_{a,\Lambda} = g(\Lambda x + b). \]

(III)

\[ \eta S(g)^+ \eta = S(g)^{-1}. \]

(IV)

\[ S_1(x_1) = i\mathcal{L}(x_1) \]

where \( \mathcal{L}(x_1) \) is the interaction Lagrangian density operator.

(V) The value of the retarded part of a vector valued kernel should coincide with the natural formula given by the multiplication by the step theta function on a space-time test function, whenever the natural formula is meaningful for this test function.

The axiom (V) has been added by Epstein and Glaser\([2]\). These axioms should be understood in the order by order sense, and as such, they can be rewritten in the following manner

(I)

\[ S_n(x_1, \ldots, x_n) = S_k(x_1, \ldots, x_k)S_{n-k}(x_{k+1}, \ldots, x_n), \]

whenever \( \{x_{k+1}, \ldots, x_n\} \preceq \{x_1, \ldots, x_k\} \),

(II)

\[ U_{b,\Lambda}S_n(x_1, \ldots, x_n)U_{b,\Lambda}^+ = S_n(\Lambda^{-1}x_1 - b, \ldots, \Lambda^{-1}x_n - b), \]
\[
S_n(x_1, \ldots, x_n) = \eta S_n(x_1, \ldots, x_n)^+ \eta, \\
S_1(x_1) = i \mathcal{L}(x_1),
\]

The singularity degree of the retarded part of a kernel should coincide with the singularity degree of this kernel, for the kernels of the generalized integral kernel and causal operators \( D_{(n)} \) which are equal to linear combinations of products of the generalized operators \( S_k \).

We are going to show that (I)-(V) are meaningful whenever \( S_n \) are understood as integral kernel operators. In order to do it it is sufficient to show that the (tensor) products

\[
S_k(x_1, \ldots, x_k)S_{n-k}(x_{k+1}, \ldots, x_n)
\]

are meaningful as integral kernel operators. Because we start with \( S_1 = \mathcal{L} \) which is equal to a pointwise Wick product of free fields (or pointwise Wick polynomial in free fields), and because the Epstein-Glaser inductive construction of \( S_n \) uses only the mentioned (tensor) product, then indeed the consistency will follow from theorem [2]. As we will see, if we start from the pointwise Wick polynomial \( S_1 = \mathcal{L} \) of free fields, then the inductive step will give operators \( S_n \) of the class of theorem [2].

2.3 Inductive step

Let us recall the inductive step construction for \( S_n \) (the case of \( S_{n,k} \) is analogous). Having given \( \{S_k\}_{k=n-1} \) we want to construct \( S_n(Z, x_n) = S(Z, x_n) \). Here \( X \cup Y = \{x_1, \ldots, x_n\} = Z, X \cap Y = \emptyset \) denote the disjoint subsets of the set \( Z \) of space-time variables \( x_1, \ldots, x_n \). To this end we construct, after Epstein and Glaser[2] or Scharf[8], the following generalized operators

\[
A'_{(n)}(Z, x_n) = \sum_{X \cup Y = Z, X \neq \emptyset} S(X)S(Y, x_n),
\]
\[
R'_{(n)}(Z, x_n) = \sum_{X \cup Y = Z, X \neq \emptyset} S(Y, x_n)\mathcal{S}(X),
\]

and then

\[
A_{(n)}(Z, x_n) = \sum_{X \cup Y = Z} S(X)S(Y, x_n) = A'_{(n)}(Z, x_n) + S(x_1, \ldots, x_n),
\]
\[
R_{(n)}(Z, x_n) = \sum_{X \cup Y = Z} S(Y, x_n)\mathcal{S}(X) = R'_{(n)}(Z, x_n) + S(x_1, \ldots, x_n),
\]

From (I)-(V) it follows that \( A_{(n)} \) and \( R_{(n)} \) have causal supports in space-time variables \( x_1, \ldots, x_n \) and that \( A_{(n)} \) is an advanced and \( R_{(n)} \) a retarded generalized operator, compare [2] or [8]. This means that the support of \( R_{(n)} \), resp. \( A_{(n)} \),
in each variable \(x_1, \ldots, x_{n-1}\) is contained within the forward, resp. past, light cone emerging from \(x_n\). These proofs\(^2\)\(^,\)\(^3\) were performed for \(S_k\) understood as Wightman operator distributions, but remain identical for \(S_k\) understood as integral kernel operators, whenever \(S_k\) are well-defined as integral kernel operators. But that \(S_k\) are indeed well-defined as integral kernel operators can be seen inductively on application of theorem\(^2\).

Therefore

\[
D_{(n)} = R'_{(n)} - A'_{(n)} = R_{(n)} - A_{(n)}
\]

is causally supported

and

\[R_{(n)} - \text{is a retarded part of } D_{(n)} \quad A_{(n)} - \text{is an advanced part of } D_{(n)}\]

so that

\[
S(x_1, \ldots, x_n) = R_{(n)}(x_1, \ldots, x_n) - R'_{(n)}(x_1, \ldots, x_n)
\]

can be computed, as the splitting of causally supported generalized operator into the retarded and advanced part can be computed independently of the axioms (I)-(V). In practice, we apply Wick decomposition to the generalized operator \(D_{(n)}\). All scalar factors multiplying the Wick monomials in this decomposition are causally supported tempered distributions. Thus, computation is reduced to the computation of the splitting of causal scalar tempered distributions into the retarded and advanced parts. As is well-known, such splitting is non-unique if the singularity degree at zero of the splitted distribution is equal or greater than zero (in space-time variables). This reduces the whole problem to the computation of the ultraviolet quasi-asymptotics of the splitted distributions. For the interaction Lagrangians \(\mathcal{L}\) equal to pointwise Wick polynomials in free fields, the ultraviolet (in space-time variables) asymptotics of the causal distributions turns out to be trivial, as the Fourier transforms of the scalar causal distributions which are essential (corresponding to the loop-like divergent Feynman diagrams) are regular function-like distributions, which behave polynomially at infinity. The degree of these asymptotic polynomials gives us the singularity degree of the splitted causal distributions, so that the full theory of quasi-asymptotics of tempered distributions is in fact not needed here.

### 3 Scattering operator and interacting fields

Application of the Hida operators as above converts the free fields, their Wick products \(\mathcal{A}\) and the \(n\)-th order contributions

\[
S_n(g^\otimes n) \quad \text{and} \quad \mathcal{A}_{\text{int}}^{(n)}(g^\otimes n, \phi)
\]

written frequently as

\[
S_n(g) \quad \text{and} \quad \mathcal{A}_{\text{int}}^{(n)}(g, \phi),
\]
to the scattering operator

\[ S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} S_n(g^{\otimes n}), \quad S(g)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \overline{S}_n(g^{\otimes n}), \]

or \( S(g, h) = 1 + \sum_{n=1}^{\infty} \sum_{p=0}^{n} \frac{1}{n!} S_{n-p,p}(g^{\otimes (n-p)} \otimes h^{\otimes p}) \)

(denoted also by \( S(g\mathcal{L}), S(g\mathcal{L})^{-1} \) or, respectively, \( S(g\mathcal{L} + h\mathcal{A}) \)) and to the interacting fields \( \mathcal{A}_{int}(g, \phi) = \int \delta h(x) S(g\mathcal{L} + h\mathcal{A}) \bigg|_{h=0} \phi(x) \),

into the finite sums of generalized integral kernel operators \( \Xi(\kappa_{lm}) = \int \kappa_{lm}(p_1, \ldots, p_l, q_1, \ldots, q_m) \partial_{p_1}^* \cdots \partial_{p_l}^* \partial_{q_1} \cdots \partial_{q_m} \times dp_1 \cdots dp_l \times dq_1 \cdots dq_m \),

e.g. for the contributions \( S_n \):

\[ S_n(g^{\otimes n}) = \sum_{l,m} \int \kappa_{lm}(p_1, \ldots, p_l, q_1, \ldots, q_m; g^{\otimes n}) \partial_{p_1}^* \cdots \partial_{p_l}^* \partial_{q_1} \cdots \partial_{q_m} \times dp_1 \cdots dp_l \times dq_1 \cdots dq_m = \int d^4 x_1 \cdots d^4 x_n S_n(x_1, \ldots, x_n) g(x_1) \cdots g(x_n), \]

with vector-valued distributional kernels \( \kappa_{lm} \) in the sense of Obata\[4\], with the values in the distributions \( V^* \) over the test nuclear space

\[ V = \mathcal{E} \ni \phi, \quad \text{or} \quad V = \mathcal{E}^{\otimes (n-p)} \otimes (\otimes_1^d \mathcal{E})^{\otimes p} \ni g^{\otimes (n-p)} \otimes h^{\otimes p}, \]

or \( V = \mathcal{E}^{\otimes n} \ni g^{\otimes n} \) or, respectively, \( V = \mathcal{E}^{\otimes n} \otimes (\otimes_1^d \mathcal{E}) \ni g^{\otimes n} \otimes \phi \) with

\[ \mathcal{E} = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \].

Each of the 3-dim Euclidean integration \( dp_i \) with respect to the spatial momenta \( p_i \) components \( p_{i1}, p_{i2}, p_{i3} \), also includes here summation over the corresponding discrete spin components \( s_i \in (1, 2, \ldots) \) hidden under the symbol \( p_i \).

The class to which the operators \( S_n \) and \( \mathcal{A}^{(n)}_{int} \) belong, expressed in terms of the Hida test space, depend on the fact if there are massless free fields present in the interaction Lagrange density operator \( \mathcal{L} \) or not. Namely:

**THEOREM 5.**

\[ S_n \in \begin{cases} L(\mathcal{E}^{\otimes n}, L((E), (E))), & \text{if all fields in } \mathcal{L} \text{ are massive,} \\ L(\mathcal{E}^{\otimes n}, L((E), (E)^*)), & \text{if there are massless fields in } \mathcal{L}. \end{cases} \]
and the same holds for $S_{n,p}$ with $\mathcal{E}^\otimes n$ replaced by $\mathcal{E}^\otimes n \otimes (\oplus_{d=1}^d \mathcal{E})^\otimes p$ if both $g$ and $h$, are, respectively, $\mathbb{C}$ and $\mathbb{C}^d$-valued Schwartz test functions. But the same distribution valued kernels of $S_{n,p}$ can be evaluated at the Grassmann-valued test functions $h$, in the sense of Berezin[9], which are used in case we have the modified Lagrangian $gL + h\mathcal{A}$, with any pointwise Wick polynomial $\mathcal{A}$ in free fields which is of odd degree in Fermi fields. So that in this case we have

**THEOREM 6.**

$$S_{n,p} \in \begin{cases} L((E),(E)) \otimes L(\mathcal{E}^\otimes n \otimes \mathcal{E}^p, \mathcal{F}^{p*}), & \text{if all fields in } \mathcal{L} \text{ are massive,} \\ L((E),(E)^*) \otimes L(\mathcal{E}^\otimes n \otimes \mathcal{E}^p, \mathcal{F}^{p*}), & \text{if there are massless fields in } \mathcal{L}, \end{cases}$$

with $\mathcal{F}^{p*}$ being the subspace of grade $p$ of the abstract Grassmann algebra $\oplus_{p=1}^n \mathcal{F}^{p*}$ with inner product and involution in the sense of Berezin. $\mathcal{F}^{p}$ denotes the space of Grassmann-valued test functions $h^p$ of grade $p$ due to Berezin, and replacing ordinary test functions $h^\otimes p$.

Recall that $L(E_1, E_2)$ denotes the linear space of linear continuous operators $E_1 \rightarrow E_2$ endowed with the natural topology of uniform convergence on bounded sets.

### 3.1 Adiabatic limit for interacting fields

Using Hida operators and integral kernel operators as above, we are able to solve positively the adiabatic limit problem. In QED the limit $g \rightarrow 1$ of the $n$-th order contributions $A^{(n)}_{\text{int}}(g), \psi^n_{\text{int}}(g)$ to interacting e.m. potential and the charged massive fields, exists and equal to a finite sum of integral kernel operators with $\oplus_{1}^n \mathcal{E}$-valued kernels in the sense of Obata[4], and belongs in general to

$$L\left(\oplus_{1}^n \mathcal{E}, L((E),(E)^*)\right).$$

This limit exists if and only if the normalization in the splitting of the causal scalar tempered distributions into retarded and advanced parts in the computation of the higher order contributions $S_n$ and $S_{n,k}$ to the scattering operator is “natural”. Moreover, this limit exists if and only if the charged field is massive. *This result can be interpreted as a theoretical proof of the experimentally observed fact that all electrically charged particles are massive.*

We do not present the complete proof but give only an example of higher order contributions which illustrates the role of the choice of the “natural” normalization in the splitting for the existence of this limit and which illustrates why the charged field should necessarily be massive. In the next subsection we will try to explain the essential role of the Hida operators for the existence of the limit at all and why this limit does not exist in general (with massive or massless charged field) if we are using operator valued distributions in the Wightman sense instead of the generalized integral kernel operators with vector valued kernels.
Let $D_0$ be the Pauli-Jordan distribution of the massless scalar field and let $D_0^\gamma$ be its advanced part. Let $\Pi_{\mu\nu}(x_1 - x_2)$ be the vacuum polarization distribution – the coefficient in the second order contribution $S_2(x_1, x_2)$, multiplying the (tensor) Wick product :$A_\mu(x_1)A_\nu(x_2)$: – computed with the help of the splitting into advanced and retarded part in accordance to the inductive step referred to in subsection 2.3, with the corresponding Fourier transforms of $\Pi_{\mu\nu}$ and $\Pi_{\mu\nu}^\gamma$ (‘natural’ normalization in the splitting is assumed)

$$\tilde{\Pi}_{\mu\nu}(p) = (2\pi)^{-4} \left( \frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} \right) \Pi(p),$$

$$\tilde{\Pi}(p) = \frac{1}{16\pi^4} \int_{4m^2}^{\infty} \frac{s+2m^2}{s^2(p^2 - s + i0)^2} \sqrt{1 - \frac{4m^2}{s}} ds,$$  \hspace{1cm} (4)

$$\tilde{\Pi}_{\mu\nu}^\gamma(p) = (2\pi)^{-4} \left( \frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} \right) \Pi(p),$$

$$\tilde{\Pi}^\gamma(p) = \frac{1}{16\pi^4} \int_{4m^2}^{\infty} \frac{s+2m^2}{s^2(p^2 - s + i0)^2} \sqrt{1 - \frac{4m^2}{s}} ds.$$ \hspace{1cm} (5)

The “natural” normalization

$$\tilde{\Pi}(p) \bigg|_{p^2=0} = 0, \quad \tilde{\Pi}(0) = 0,$$ \hspace{1cm} (6)

in the splitting is assumed in [4] and [5]. Let us consider the $n = 2k + 1$-order sub-contribution to interacting e.m. potential field $A_{\mu\nu}(g = 1)$ in the limit $g \to 1$ containing $k$ vacuum polarization graph insertions $\Pi$ in the spinor QED. It can be written as the following repeated convolution

$$D_0^\gamma * \Pi_{\mu\nu}^\gamma \mu_k * \ldots * D_0^\gamma * \Pi_{\mu\nu}^\gamma \mu_1 * D_0^\gamma * \Pi_{\mu\nu}^\gamma \mu_0: \psi^\gamma \gamma \psi:\,$$ \hspace{1cm} (7)

Let $E_1$, $E_2$ be the single particle test spaces of the free fields $\psi^\gamma$, $\psi$ and let $\otimes E$ be the space-time test space. Let $\kappa_{1,0}^\gamma$, $\kappa_{0,1}^\gamma$ be kernels of the Dirac conjugated field $\psi^\gamma$, and $\kappa_{1,0}$, $\kappa_{0,1}$ be kernels of the Dirac field $\psi$. The evaluations $\langle \kappa_{lm}(\xi_1 \otimes \xi_2), \phi \rangle$ of the kernels $\kappa_{lm}$ of the sub-contributions (7) at the test functions $\xi_1 \otimes \xi_2 \otimes \phi \in E_1 \otimes E_2 \otimes (\otimes E)$ are equal to the $\epsilon \to 0$ limits of the integrals

$$\left\langle D_0^\gamma * \Pi_{\mu\nu}^\gamma \mu_k * \ldots * D_0^\gamma * \Pi_{\mu\nu}^\gamma \mu_1 * D_0^\gamma * \Pi_{\mu\nu}^\gamma \mu_0, \phi \right\rangle \times$$

$$\times \Pi_{\mu\nu}^\gamma \mu_k :\phi: \Pi_{\mu\nu}^\gamma \mu_k-1 :\phi: \ldots \Pi_{\mu\nu}^\gamma \mu_1 :\phi: \Pi_{\mu\nu}^\gamma \mu_0 :\phi:$$ \hspace{1cm} (8)

$$p_1, p_2 \in O_{m,0,0,0} = \{p: p \cdot p = m^2, p_0 > 0\}.$$
Recall that \( E_1 = E_2 = S(\mathbb{R}^3) \). \( u^+_s(p) = u_s(p) \), \( u^-_s(p) = u_s(p) \) are the Fourier transforms of the basic solutions of the free Dirac equation, \( \gamma^\mu \) are the Dirac gamma matrices, and finally \( u^+\Sigma \) is the Dirac conjugation of the spinor \( u^+_s(p) \).

The plus sign stands everywhere in \( \pm p_1 \) and in \( u^+_s(p) \) whenever \( (l_1, m_1) = (1, 0) \). The minus sign stands everywhere in \( \pm p_1 \) and in \( u^-_s(p) \) whenever \( (l_1, m_1) = (0, 1) \). Analogously, the plus sign stands everywhere in \( \pm p_2 \) and minus sign in \( u^+_s(p) \) whenever \( (l_2, m_2) = (1, 0) \). The minus sign stands everywhere in \( \pm p_2 \) and plus sign in \( u^-_s(p) \) whenever \( (l_2, m_2) = (0, 1) \).

For the “natural” normalization in the Epstein-Glaser splitting, and in case \( m \neq 0 \), the singularity appearing in the limit

\[
\frac{1}{|p^2 + i\epsilon|^{3/2}} \to \frac{1}{(p^2)^{3/2}} - \text{sgn}(p_0) \frac{i\pi(-1)^s}{k!}\delta(k)(p^2),
\]

is cancelled by the Fourier transform \( \Pi^{av,\mu\nu} \) of \( \Pi^{av,\mu\nu} \), as \( \Pi^{av,\mu\nu} = \left( \frac{p^\mu p^\nu}{p^2} - g^{\mu\nu}\Pi(p) \right) \) with a regular \( \Pi \) in the vicinity of the cone \( p^2 = 0 \), and equal there to \( \Pi(p) = |p^2|^2g_0(p) \) with still regular \( g_0 \) there. Now the freedom in normalization consists here (for Fourier transformed \( \Pi^{\mu\nu} \)) in addition of a polynomial of second degree in \( p \), as the singularity degree at zero of \( \Pi^{\mu\nu} \) is equal to two. In particular we can add a constant term \( g^{\mu\nu} \) in \( \Pi^{av,\mu\nu} \), but this modification will destroy the cancellation of the singularities so that the above \( \epsilon \to 0 \) in \( \Pi \) will no longer exist. Still, in principle (freedom in the splitting), we can add to \( \Pi^{\mu\nu}(p) \), or to \( \Pi^{\mu\nu}(p) \) the term of the form \( f(p^2)g^{\mu\nu} \) with \( f \) which has zero of at least second order at zero. But the kernels of some even order sub contributions

\[
\ldots \left( S_{\text{ret},av} \ast \Sigma_{\text{ret},av} \ast \right) \ldots \ast \psi, \quad \text{and respectively} \quad \ldots \left( D_{0,\text{ret}}^{av,\mu\nu} \ast \Pi^{av,\mu\nu}_{\text{ret},\mu\nu} \ast \right) \ldots \ast A,
\]

we can add to \( \Pi^{\mu\nu}(p) \), or to \( \Pi^{\mu\nu}(p) \) the term of the form \( f(p^2)g^{\mu\nu} \) with \( f \) which has zero of at least second order at zero. But the kernels of some even order sub contributions

\[
\ldots \left( S_{\text{ret},av} \ast \Sigma_{\text{ret},av} \ast \right) \ldots \ast \psi, \quad \text{and respectively} \quad \ldots \left( D_{0,\text{ret}}^{av,\mu\nu} \ast \Pi^{av,\mu\nu}_{\text{ret},\mu\nu} \ast \right) \ldots \ast A,
\]

to interacting fields \( \psi_{\text{int}} \) and, respectively, \( A_{\text{int}} \) are well-defined only with the stronger condition \( \Pi \) put on \( \Pi \).

Because the Fourier transform of the vacuum polarization in QED with massless charged field is not smooth at the cone \( p^2 = 0 \), having the jump \( \theta(p^2) \) there, and this singularity cannot be repaired by any choice of the splitting (addition of any polynomial in momenta of second degree), then we are confronted with the valuation of the distribution

\[
\frac{1}{|v^+ + i\epsilon|^{1/2}} \to \frac{1}{v^+} - \frac{i\pi(-1)^s}{k!}\delta(k)(v),
\]

in single real variable \( v = (p_1 \pm p_2)^2 \) at the “test function” which has the jump-type and \( \sim \frac{1}{\sqrt{v}} \) type singularity at \( v = (p_1 \pm p_2)^2 = 0 \), which, as we know from the distribution theory, is not well-defined, or alternatively: there is no sensible way of definition of the product of the theta function \( \theta(v) \)-distribution (or the \( \frac{1}{\sqrt{v}} \) function-type-distribution) and the derivatives of the Dirac delta distribution \( \delta^{(k)}(v) \).

By the existence of the \( \epsilon \to 0 \) limits in \( \Pi \) defining the vector valued kernels of \( \Pi \), and theorem \( \Pi \) (theorems 3.6 and 3.9 of \( \Pi \) or their generalization to the

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Fermi case or general Fock space) we obtain existence of the sub-contributions (7) as integral kernel operators in the adiabatic limit. But, as we have seen the $\epsilon \to 0$ limits, defining the kernels of (7) in the adiabatic limit for spinor QED with massless Dirac field, do not exist. In general, we have

**THEOREM 7.** For QED with massive charged field, the higher order contributions $\psi_{\text{int}}^{(n)}(g \otimes n)$ and $A_{\text{int}}^{(n)}(g \otimes n)$ to interacting fields $\psi_{\text{int}}$ and $A_{\text{int}}$ in the adiabatic limit $g \to 1$ are well-defined as sums of generalized integral kernel operators with vector valued kernels in the sense of Obata[4],

$$
\lim_{g \to 1} \psi_{\text{int}}^{(n)}(g \otimes n), \lim_{g \to 1} A_{\text{int}}^{(n)}(g \otimes n) \in L\left( \otimes_{1}^{d} \mathcal{E}, L((E),(E)^{*}) \right),
$$

and this is the case only for the “natural” choice in the Epstein-Glaser splitting in the construction of the scattering operator.

But:

**THEOREM 8.** For causal perturbative QED on the Minkowski space-time with the Hida operators as the creation-annihilation operators and with massless charged field, the higher order contributions to interacting fields in the adiabatic limit $g \to 1$ are not well-defined, even as sums of generalized integral kernel operators in the sense of Obata, and for no choice in the Epstein-Glaser splitting in the construction of the scattering operator.

### 3.2 Comparison with the approach based on Wightman’s operator distributions

In the adiabatic limit $g \to 1$ the higher order contributions $A_{\text{int}}^{(n)}(g = 1)$ to interacting fields $A_{\text{int}}$ in QED preserve in general the meaning of the generalized operators – finite sums of integral kernel operators, i.e. continuous maps

$$
\otimes_{1}^{4} \mathcal{E} \ni \phi \xrightarrow{\text{continuously}} A_{\text{int}}^{(n)}(g = 1, \phi) \in L((E),(E)^{*}) \quad \text{case A)}
$$

and only for some exceptional contributions $A_{\text{int}}^{(n)}(g = 1)$, or some of their sub-contributions, we have

$$
\otimes_{1}^{4} \mathcal{E} \ni \phi \xrightarrow{\text{continuously}} A_{\text{int}}^{(n)}(g = 1, \phi) \in L((E),(E)). \quad \text{case B)}
$$

Only in case B) the contribution $A_{\text{int}}^{(n)}(g = 1)$ can be understood as operator valued distributions also in the Wightman sense [3]. Contributions of class A) which are not of class B) cannot be understood as operator valued distributions, and in particular cannot be accounted for within the approach based on operator valued distributions in the Wightman sense. Let us recall that the Hida space $(E)$ contains the so-called fundamental domain $\mathcal{D}_{0}$ used in [3], which consists of all images of the vacuum state under the polynomial expressions in
\( \psi(f_1), \psi^*(f_2), A(f_3), \ldots, \psi(f_{n-1}), \psi^*(f_{n-1}), A(f_n) \), for \( n \in \mathbb{N} \) and \( f_k \) ranging over the Schwartz test functions (if we restrict the arguments \( f_k \) of \( A \) to the subspace \( S^0 \) of the Schwartz space). Contributions of class A) which are not of class B) transform some of the Fock states in \( (E) \) into nonnormalizable states which do not belong to the Fock space, but only to the space \( (E)^* \) dual to the Hida space \( (E) \). In general, the states of \( \mathcal{D}_0 \subset (E) \) are transformed by the contributions of class A) into nonnormalizable states of \( (E)^* \) and cannot represent any operator valued distributions in the Wightman sense. An example of the contribution of type B) is the first order contribution \( A^{(1)}(g = 1) \) to the interacting e.m. potential field.

For example, the first order contribution \( \psi_{\text{int}}^{(1)}(g = 1) \) in the adiabatic limit belongs to class A) but not to class B), and cannot be subsumed within the approach based on Wightman operator distributions. Let \( E_1^{\pm} = S(\mathbb{R}^{3 \times 2}) \) be the positive/negative energy single particle test space of the free Dirac field, \( E_1 = E_1^+ \oplus E_1^- \) – the total single particle test space of the Dirac field, \( (E_2 = S^0(\mathbb{R}^{3 \times 4}) \) the single particle test space of the free e.m. potential field. \( \psi_{\text{int}}^{(1)}(g = 1) \) is a finite sum of well-defined integral kernel operators of class A), which evaluated at a test function \( \phi \in \mathcal{S}(\mathbb{R}^{3 \times 2}) \) is equal to

\[
\psi_{\text{int}}^{(1)}(g = 1; \phi) = \sum_{\nu', s} \int d^3 p' d^3 p \frac{(m + \gamma^a p_a + \gamma^a p'_a) \gamma^\nu u_s(p) \phi(-|p' - p_0(p)|, -p' - p)}{(p' + |(p' - p_0(p)|)} a_{\nu'}(p') d_s(p)
\]

where dots denote the kernels

\[
\kappa_{lm}(\nu', p', s, p) = \frac{(m + \gamma^a p_a + \gamma^a p'_a) \gamma^\nu u_s(p) \phi(|p| + |p_0(p)|, |p|)}{(p' + |(p' - p_0(p)|)}
\]

with the respective \( \pm \) signs in front of the whole expression and in front of the components \( p^a_0(p') = |p'|, p_0(p) = \sqrt{|p|^2 + m^2} \), \( p \) of the momenta in the denominator, correspondingly to the annihilation or the creation operators. Using the elementary estimation

\[
\left| \frac{1}{|(p' + |(p' - p_0(p)|)} \right| > \frac{1}{|p'| + |(p_0(p)|}
\]

we see that the kernels \( \kappa_{lm}(\phi)(\nu', p', s, p) \), regarded as two-particle functions of spin-momenta variables \( (\nu', p', s, p) \) do not belong to the tensor product of single particle Schwartz spaces or even to the two-particle Hilbert spaces having their \( L^2(\mathbb{R}^{3 \times 4} \times \mathbb{R}^{3 \times 2}) \)-norms IR divergent. In particular \( \psi_{\text{int}}^{(1)}(g = 1; \phi) \) acting
on a finite number particle state with smooth Schwartz functions in each spin-momentum variable, lying in the domain $D_0$, gives a nonnormalizable state which, moreover, is not smooth in $(p', p)$. Thus, $\psi^{(1)}_{\text{int}}(g = 1)$ cannot represent any operator valued distribution in the Wightman sense. But, as is easily seen, the above $\kappa_{lm}$, are kernels of well-defined $\bigoplus L^*E^*$-valued distributions which, when integrated with $\xi_2 \otimes \xi_1 \in E_2 \otimes E_1^+ = S(\mathbb{R}^3 \times 4 \times \mathbb{R}^3 \times 2)$ regarded as functions of $(\nu, p', s, p)$, are continuous maps of $\xi_2 \otimes \xi_1$. Thus, by theorem $\exists$, $\psi^{(1)}_{\text{int}}(g = 1)$ is a finite sum of well-defined integral kernel operators in the sense of $[4]$.

In general higher order contributions to interacting fields in the adiabatic limit $g \to 1$ are not well-defined operator valued distributions in the Wightman sense and do not belong to class B) but only to class A), and this is the case only if the normalization in the splitting of causal distributions in the computation of the scattering operator is “natural”. The adiabatic limit $g \to 1$ does not exist in the theory (I)-(V) based on Wightman distributions, so that in particular theorems $[7]$ and $[8]$ of Subsection $[3.1]$ cannot be proved within the approach based on Wightman distributions, contrary to what we have in the approach (I)-(V) based on the integral kernel operators in the sense of $[4]$.

### 3.3 UV and IR asymptotics

For the proof of existence of IR and UV asymptotics of interacting fields, the application of the Hida operators and integral kernel operators is essential. Using the Hida operators in causal perturbative QED we can also compute the UV and IR asymptotics, using the following facts (i)-(iv):

(i) Each higher order contribution $A^{(n)}_{\text{int}}$ to interacting e.m. potential field $A_{\text{int}}$ exists as a generalized integral kernel operator $\Xi(\kappa_{lm})$ in the adiabatic limit $g \to 1$ in Bogoliubov’s causal perturbative QED with Hida operators.

(ii) The UV and IR asymptotics should be $SL(2, \mathbb{C})$ invariant.

(iii) The direct integral decomposition $\int U_\chi d\chi$ of the representation $U$ of $SL(2, \mathbb{C}) \subset T_4 \times SL(2, \mathbb{C})$ acting in the full single particle Hilbert space $\mathcal{H} = \int \mathcal{H}_\chi d\chi$ determines naturally direct integral decomposition $\int \Xi(\kappa_{\chi, lm}) d\chi$ of $\Xi(\kappa_{lm}) = A^{(n)}_{\text{int}}$.

(iv) Decomposition components

$$A_{\chi, \text{int}}(x) = \sum \kappa_{\chi, lm}(\ell_1, k_1, \ldots , \ell_{l+m}, k_{l+m}; x) a_{\chi, \ell_1, k_1}^+ \ldots a_{\chi, \ell_{l+m}, k_{l+m}}$$

of $A_{\text{int}}$ act in the Fock spaces $\Gamma(\mathcal{H}_\chi)$ over the UV-asymptotically homogeneous states of UV-asymptotic homogeneity degree determined by the decomposition parameter $\chi$.

The generalized (discrete) integral kernel operators $[9]$ define the UV asymptotic parts of $A_{\text{int}}$ of UV-asymptotic homogeneity degree determined by the decomposition parameter $\chi$. In order to find the IR asymptotics of $A_{\text{int}}$ we need to compute the IR quasi-asymptotics of the scalar distributions $\kappa_{\chi, lm}(\ell_1, k_1, \ldots , \ell_{l+m}, k_{l+m}; x)$.
in (9) with respect to the semigroup of scaling transformations $S_{\lambda}(x) = \lambda x$, $\lambda > 0$. This is in general non-trivial and uses the full theory of quasiasymptotics of distributions as given in [10, 11, 12]. Collecting all $\kappa_{\chi lm}(\ell_1, k_1, \ldots, \ell_{l+m}, k_{l+m}; x)$ with common IR quasiasymptotic degree we obtain the asymptotic part of $A_{\text{int}}$ with fixed IR asymptotic degree.

Let us give few words of explanation for (i)-(iv). The result (i) itself have been already briefly discussed in previous subsections. The kernels $\kappa_{\chi lm}(\phi)$ of (9) evaluated at the space-time test function $\phi \in \oplus_{l+m} E$, are equal to the Fourier transforms $F[\kappa_{\chi lm}(\phi)](\chi, \ldots, \chi)$ of $\kappa_{\chi lm}(\phi)$, restricted to the diagonal, with the Fourier transform $F$ associated to the decomposition of the action $U \otimes (l+m)$ of $SL(2, \mathbb{C})$ in the full $(l+m)$-particle test space $E \otimes (l+m)$. The existence of the adiabatic limit is essential, because in case we had the interaction $gL$ with the modified intensity $g$ switched on, then the decomposition would depend non-trivially also on $g$.

The problem of classification of all irreducible unitary representations of $SL(2, \mathbb{C})$ has been completely solved by Gelfand and Naimark [18, 13, 14, 15, 16, 17], together with the problem of decomposition of tensor product of any unitary representations of $SL(2, \mathbb{C})$ into irreducible components. But also the representation of $SL(2, \mathbb{C})$ acting in the single particle space of the free e.m. potential field (in the Gupta-Bleuler gauge), although not unitary but Krein-isometric, is decomposable with the decomposition which is determined by the normal scaling operator $S_{\lambda}$ and the respective decomposition components act in the Krein-Hilbert spaces $H_{\chi}$ of homogeneous states of homogeneity degree equal $\chi = -1 + i\nu$, $\nu \in \mathbb{R}$. Therefore the kernels $\kappa_{\chi lm}$ in (9) can indeed be computed explicitly, although the computation is quite involved.

In order to explain the principle on which decomposition of $\Xi(\kappa_{lm})$, and of $\kappa_{lm}(\phi)$ associated with the decomposition of $SL(2, \mathbb{C})$ acting in $E \otimes (l+m)$, is based, let us illustrate it in the simpler case in which we consider the standard unitary action of the translation group (analogue of $SL(2, \mathbb{C})$ group) on the Gelfand triple

$$S(\mathbb{R}; \mathbb{C}) \subset L^2(\mathbb{R}; \mathbb{C}) \subset S(\mathbb{R}; \mathbb{C})^*$$

– the analogue of the Gelfand triple:

$$E \otimes (l+m) \subset H \otimes (l+m) \subset E^* \otimes (l+m).$$

Decomposition of the unitary action $U$ of the translation group into irreducible components $U_{\chi}$ determines the decomposition $L^2(\mathbb{R}) = \int \mathcal{H}_\chi \, d\chi$ into one dimensional Hilbert spaces $\mathcal{H}_\chi = \mathbb{C}$, and with the spectrum of decomposition equal to $\mathbb{R}$ with the spectral measure $d\chi$ of decomposition equal to the ordinary Lebesgue measure. Therefore each $f \in L^2(\mathbb{R})$, and in particular each $f \in S(\mathbb{R})$ has unique decomposition with decomposition components

$$(f)_\chi = \mathcal{F}f(\chi), \quad \chi \in \mathbb{R},$$

where $\mathcal{F}$ is the ordinary Fourier transform, associated to the decomposition of the unitary action of the translation group on $\mathbb{R}$. Decomposition of the elements
$f$ of the nuclear space $S(\mathbb{R})$ determines decomposition of a (decomposable) distribution $F \in S(\mathbb{R}; \mathbb{C})^*$ by the following canonical formula

$$\langle F, f \rangle = \int_{\text{Spec}=\mathbb{R}} F \mathcal{F} f(\chi) \, d\chi = \int_{\text{Spec}=\mathbb{R}} \langle F_\chi, \mathcal{F} f(\chi) \rangle \, d\chi = \int_{\text{Spec}=\mathbb{R}} \langle \mathcal{F} F(\chi), (f)_\chi \rangle \, d\chi, \quad f \in S(\mathbb{R}; \mathbb{C}).$$

We thus see that decomposable $F$ whose decomposition measures $d\chi$ are absolutely continuous with respect to the Lebesgue measure, are precisely those distributions whose Fourier transforms are regular function-like distributions. But more generally, $F$ is decomposable, with any fixed $\sigma$-measure $d\chi$ on $\mathbb{R}$ iff its Fourier transform is equal to a $\sigma$-measure on $\mathbb{R}$.

The same situation we have for decomposable $\kappa_{lm}(\phi)$ which can similarly be decomposed with decomposition associated with the decomposition of $SL(2, \mathbb{C})$ acting in $E^{\otimes (l+m)}$ and which determine decomposition of the associated integral kernel operator. Namely, we put

**Definition 3.** Let

$$\Phi, \Psi \in (E), \quad \Phi = \int \Phi_\chi \, d\chi, \quad \Psi = \int \Psi_\chi \, d\chi$$

be any two elements of the test Hida space with their direct integral decompositions. Let $d\chi$ be a $\sigma$-measure on the spectrum of the decomposition of the representation of $SL(2, \mathbb{C})$ acting in the single particle Hilbert space $\mathcal{H}$. We say that the generalized integral kernel operator $\Xi(\kappa_{lm})$ is equal to the direct integral

$$\Xi(\kappa_{lm}) = \int \Xi_\chi(\kappa_{\chi lm}) \, d\chi$$

of (discrete-) integral kernel operators $\Xi_\chi(\kappa_{\chi lm})$, acting in the Fock spaces over the single particle Gelfand triples $E_\chi \subset \mathcal{H}_\chi \subset E_\chi^*$, if

$$\int \langle \langle \Xi_\chi(\kappa_{\chi lm}(\phi)) \Phi_\chi, \Psi_\chi \rangle \rangle \, d\chi = \langle \langle \kappa_{\chi lm}(\phi), (\eta_{\phi, \psi})_\chi \rangle \rangle = \langle \langle \kappa_{\chi lm}(\phi), \eta_{\phi, \psi} \rangle \rangle = \langle \langle \Xi(\kappa_{lm}(\phi)) \Phi, \Psi \rangle \rangle$$

for all

$$\Phi, \Psi \in (E), \phi \in \mathcal{E}.$$
A quantum theory of classical homogeneous of degree $-1$ e.m. potential field has been developed long time ago by Staruszkiewicz\cite{19,20,21}. The IR asymptotics of $A_{\text{in}}$ of degree $\chi = -1$, computed as above, exists and is non-trivial (nonzero) for QED. It coincides with the theory of Staruszkiewicz\cite{19,20,21} if and only if we add the following assumption: for the total charge operator $Q$, acting in the (asymptotically) homogeneous states of the free fields coupled to the electromagnetic potential, there exist the phase operator $S_0$ which provides, together with $Q$, the spectral realization of the gauge group $U(1)$ in the sense of Connes\cite{22}. This additional assumption emerges from the comparison of the IR (asymptotically) homogeneous part of degree $-1$ with the theory of Staruszkiewicz\cite{19,20,21} and accounts for the universality of the electric charge, which cannot be explained solely within the Bogoliubov’s causal QED with Hida operators, as in principle we can put different coupling constants for different charged fields.

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