RESOLVENT ESTIMATES FOR SPACETIMES BOUNDED BY KILLING HORIZONS

ORAN GANNOT

Abstract. We show that the resolvent grows at most exponentially with frequency for the wave equation on a class of stationary spacetimes which are bounded by non-degenerate Killing horizons, without any assumptions on the trapped set. Correspondingly, there exists an exponentially small resonance-free region, and solutions of the Cauchy problem exhibit logarithmic energy decay under some additional hypotheses.

1. Introduction

1.1. Statement of results. Let \((M, g)\) be a connected \(n + 1\) dimensional Lorentzian manifold of signature \((1, n)\) with connected boundary \(\partial M\), satisfying the following assumptions.

(1) \(\partial M\) is a Killing horizon generated by a complete Killing vector field \(T\), whose surface gravity is a positive constant \(\kappa > 0\) (see Section 2.3 for details),

(2) \(T\) is timelike in \(M^\circ\),

(3) \(M\) is stationary in the sense that there is a compact spacelike hypersurface \(X\) with boundary such that each integral curve of \(T\) intersects \(X\) exactly once.

Consider an operator \(L \in \text{Diff}^2(M)\) such that \(L - \Box_g \in \text{Diff}^1(M)\). Thus we can write

\[
L = \Box_g + \mathcal{W} + \mathcal{V},
\]

where \(\mathcal{W}\) is a smooth vector field and \(\mathcal{V} \in \mathcal{C}^\infty(M)\). In addition, assume the following:

(A) \(\mathcal{W}, \mathcal{V}\) are stationary in the sense that \([\mathcal{W}, T] = 0\) and \(T(\mathcal{V}) = 0\),

(B) \(\mathcal{W}\) is tangent to \(\partial M\),

(C) \(L\) is formally self-adjoint with respect to the volume density.

Identify \(M = \mathbb{R}_t \times X\) under the flow of \(T\). Since \(T\) commutes with \(L\), the composition

\[
P(\omega) = e^{i\omega t}Le^{-i\omega t}
\]

descends to a differential operator on \(X\) depending on \(\omega \in \mathbb{C}\). Fredholm properties of \(P(\omega)\) were first examined in a robust fashion by Vasy \([\text{Vas}]\) using methods of microlocal analysis, and subsequently by Warnick \([\text{War}]\) via physical space arguments.
Here we summarize a simple version of these results, which applies in a strip of fixed width near the real axis. Define the space

$$\mathcal{X} = \{ u \in H^1(X) : P(0)u \in L^2(X) \},$$
equipped with the graph norm. Since $P(\omega) - P(0) \in \text{Diff}^1(X)$, the operator $P(\omega)$ is bounded $\mathcal{X} \to L^2(X)$ for each $\omega \in \mathbb{C}$.

**Proposition 1.1 ([Vas], [War]).** The operator $P(\omega) : \mathcal{X} \to L^2(X)$ is Fredholm of index zero in the half-plane $\{ \text{Im} \ \omega > -\kappa/2 \}$, and is invertible for $\text{Im} \ \omega > 0$ sufficiently large.

The inverse $P(\omega)^{-1} : L^2(X) \to \mathcal{X}$ forms a meromorphic family of operators in $\{ \text{Im} \ \omega > -\kappa/2 \}$, called the resolvent family. Its complex poles are known as resonances, and correspond to nontrivial mode solutions $v = e^{-i\omega t}u$ of the equation $\Box_g v = 0$, where $u \in C^\infty(M)$ satisfies $Tu = 0$. Thus mode solutions with $\text{Im} \ \omega > 0$ grow exponentially in time, whereas those with $\text{Im} \ \omega < 0$ exhibit exponential decay; real resonances correspond to stationary solutions.

Given $\omega_0, C_0 > 0$, define the region

$$\Omega = \{ |\text{Im} \ \omega| \leq e^{-C_0|\text{Re} \ \omega|} \} \cap \{ |\omega| > \omega_0 \}.$$These parameters are fixed in the next theorem, which is the main result of this paper.

**Theorem 1.** There exist $\omega_0, C_0 > 0$ such that $P(\omega)$ has no resonances in $\Omega$. Furthermore, there exists $C > 0$ such that

$$\|P(\omega)^{-1}f\|_{H^1(X)} \leq e^{C|\text{Re} \ \omega|} \|f\|_{L^2(X)} \tag{1.2}$$

for each $f \in L^2(X)$ and $\omega \in \Omega$.

Theorem 1 is also true when $\partial M$ consists of several Killing horizons generated by $T$, each of which has a positive, constant surface gravity. In particular, Theorem 1 applies to any stationary perturbation of the Schwarzschild-de Sitter spacetime (which is bounded by two non-degenerate Killing horizons [Vas, Section 6]) that preserves the horizons and the timelike nature of $T$. Other examples are even asymptotically hyperbolic spaces in the sense of Guillarmou [Gui].

### 1.2. Energy decay

Theorem 1 can be used to prove logarithmic energy decay (necessarily with a loss of derivatives) for the Cauchy problem

$$Lv = 0, \quad v|_X = v_0, \quad Tv|_X = v_1 \tag{1.3}$$

provided the following two additional hypotheses are satisfied.
(1) Solutions of \((1.3)\) are uniformly bounded: there exists \(C > 0\) such that
\[
\| (v(t), Tv(t)) \|_{H^1(X) \times L^2(X)} \leq C \| (v_0, v_1) \|_{H^1(X) \times L^2(X)}
\]
for each \(v \in C^0(\mathbb{R}^+; H^1(X)) \cap C^1(\mathbb{R}^+; L^2(X))\) solving \((1.3)\),

(2) \(\omega = 0\) is not a resonance.

These two conditions are satisfied for instance when \(L = \Box_g + V\) for a strictly positive potential \(V > 0\). Observe that the uniform boundedness hypothesis eliminates possible resonances in \(\{\text{Im} \omega > 0\}\). Furthermore, since \(L\) is formally self-adjoint, the only possible real resonance is \(\omega = 0\) [War, Appendix A.1], hence all resonances lie in \(\{\text{Im} \omega < 0\}\).

**Corollary 1.** If \(L\) satisfies the hypotheses of the previous paragraph, then there exists \(C > 0\) such that
\[
\| (v(t), Tv(t)) \|_{H^1(X) \times L^2(X)} \leq \frac{C}{\log(2 + t)} \| (v_0, v_1) \|_{X \times H^1(X)}
\]
for each \(v \in C^0(\mathbb{R}^+; H^1(X)) \cap C^1(\mathbb{R}^+; L^2(X))\) solving the Cauchy problem \((1.3)\) with initial data \((v_0, v_1) \in X \times H^1(X)\).

Various refinements of Corollary 1 are also available, including pointwise estimates, but for concision these will not be pursued here.

### 1.3. Relationship with previous work.

The analogue of Theorem 1 was first established for compactly supported perturbations of the Euclidean Laplacian in a landmark paper of Burq [Bur1]. Subsequent improvements and simplifications in the asymptotically Euclidean setting are due Burq [Bur2], Vodev [Vod], and Datchev [Dat], while Rodnianski–Tao [RT] considered asymptotically conic spaces.

Most relevant to the setting considered here are the works of Moschidis [Mos] and Cardoso–Vodev [CV]. The former reference shows logarithmic energy decay on Lorentzian spacetimes which may contain Killing horizons, but importantly also contain at least one asymptotically flat end. There, the mechanism of decay is radiation into the asymptotically flat region. In contrast, asymptotically flat ends are not considered in the present paper, but we do allow spacetimes which contain Killing horizons as their only boundary components.

Meanwhile, [CV] applies to a wide class of Riemannian metrics, including those with hyperbolic ends. There is a close connection between asymptotically hyperbolic manifolds and black holes spacetimes, first exploited in the study of resonances by Sá Barreto-Zworski [BZ]. This relationship has attracted a great deal of interest, especially following the paper [Vas] (for a survey of recent developments, see [Zwo2]).
Common to the works described above is the use Carleman estimates in the interior of the geometry, which is then combined with some other (typically more complicated) analysis near infinity. Although the proof of Theorem 1 adopts techniques from [Bur1, Mos, RT], one novelty (and simplifying feature) is that the Carleman estimate employed here is valid up to and including the horizon. In particular, this avoids the use of separation of variables and special function methods [Bur1, Vod], Mourre-type estimates [Bur2], and spherical energies [CV, Dat, Mos, RT].

2. Preliminaries

2.1. Semiclassical rescaling. It is conceptually convenient to rescale the operator by

$$P(z) = h^2 P(h^{-1}z).$$

(2.1)

Thus $\omega = h^{-1}z$, and uniform bounds on $P(z)$ for $\pm z$ in a compact set $[a, b] \subset (0, \infty)$ give high-frequency bounds for $P(\omega)$ as $|\omega| \to \infty$. Theorem 1 is easily seen to be equivalent to the following.

**Theorem 1'**. Given $[a, b] \subset (0, \infty)$, there exist $C, C_1 > 0$ such that

$$\|u\|_{H^1_\delta(X)} \leq e^{C/h}\|P(z)u\|_{L^2(X)}$$

(2.2)

for each $u \in X$ and $\pm z \in [a, b] + ie^{-C_1/h}[-1, 1]$.

The norm on the left-hand side of (2.2) is a semiclassically rescaled Sobolev norm. For detailed expositions on semiclassical analysis, the reader is referred to [Zwo1] and [DZ, Appendix E].

2.2. Stationarity. A tensor on $M$ will be called stationary if it is annihilated by the Lie derivative $L_T$. The definition of stationarity can be extended to $T^*M$ by observing that $T$ lifts to a vector field on $T^*M$ via the identification

$$T^*M = T^*\mathbb{R} \oplus T^*X.$$

Any covector $\varpi \in T^*_qM$ at a point $q = (t, x)$ can be decomposed as $\varpi = \xi + \tau dt$, where $\xi \in T^*_xX$ and $\tau dt \in T^*_t\mathbb{R}$. Thus a function $F \in C^\infty(T^*M)$ is stationary if

$$F(\xi, \tau) = F(\xi + \tau dt)$$

depends only on $\xi \in T^*X$ and $\tau \in \mathbb{R}$. Furthermore, if $\tau = \tau_0$ is fixed, then $F$ induces a function $F(\cdot, \tau_0)$ on $T^*X$. This is compatible with taking Poisson brackets in the sense that for stationary $F_1, F_2 \in C^\infty(T^*M)$, there is equality

$$\{F_1, F_2\}(\xi, \tau_0) = \{F_1(\cdot, \tau_0), F_2(\cdot, \tau_0)\}(\xi).$$

(2.3)

On the left is the Poisson bracket on $T^*M$, and on the right the Poisson bracket on $T^*X$. 
In particular, this discussion applies to the dual metric function \( G \in C^\infty(T^*M) \), whose value at \( \varpi \in T^*M \) is given by

\[ G(\varpi) = g^{-1}(\varpi, \varpi) = g^{\alpha\beta} \varpi_\alpha \varpi_\beta. \]

The semiclassical principal symbol \( p = \sigma_h(P(z)) \) is then given by \( p(\xi, z) = G(\xi - z dt) \).

**Lemma 2.1.** The quadratic form \( \xi \mapsto G(\xi) \) is negative definite on \( T^*X^o \).

**Proof.** The condition \( \tau = 0 \) implies that \( \varpi = \xi + 0 \ dt \) is orthogonal to \( T^\flat \). But \( T^\flat \) is timelike on \( M^o \), whence the result follows. \( \Box \)

If \( \tau_0 \in \mathbb{R} \) is fixed and \( K \subset X^o \) is compact, then by Lemma 2.1 there exist \( c, R > 0 \) such that if \( G(\xi) \geq R \), then

\[ G(\xi + \tau_0 \ dt) \geq cG(\xi) \]

for each \( \xi \in T^*_K X^o \), where the constant \( c \) is locally uniform in \( \tau_0 \). In particular, given a compact interval \( I \subset \mathbb{R} \), the set

\[ \{\xi \in T^*_K X^o : G(\xi + \tau \ dt) = 0 \text{ for some } \tau \in I\} \]

is a compact subset of \( T^*X^o \). This also implies that if \( Q \) is a stationary quadratic form on \( T^*M \), then there exists \( C > 0 \) such that

\[ |Q(\xi + \tau \ dt)| \leq C(1 + |G(\xi + \tau \ dt)|) \]

for each \( \xi \in T^*_K X^o \) and \( \tau \in I \).

### 2.3. Killing horizons and surface gravity.

Recall the hypotheses on \((M, g)\) described in Section 1.1, and set

\[ \mu = g(T, T). \]

The key property of \((M, g)\) is that \( \partial M \) is a Killing horizon generated by \( T \). By definition, this means that \( \partial M \) is a null hypersurface which agrees with a connected component of the set \( \{\mu = 0, T \neq 0\} \). Of course in this case \( T \) is nowhere vanishing. Since orthogonal null vectors are collinear, there is a smooth function \( \kappa : \partial M \to \mathbb{R} \), called the surface gravity, such that

\[ \nabla_g \mu = -2\kappa T \tag{2.4} \]

on \( \partial M \). The non-degeneracy assumption means that \( \kappa > 0 \), and for simplicity it is assumed that \( \kappa \) is in fact constant along \( \partial M \).
2.4. **Decomposing the metric near the boundary.** Let $N$ denote the future pointing unit normal to the level sets of $t$, and define the lapse function $A > 0$ by $A^{-2} = G(dt)$. The shift vector is given by the formula

$$W = T - AN,$$

which by construction is tangent to the level sets of $t$. Let $k$ denote the induced (positive definite) metric on $X$. If $(x^i)$ are local coordinates on $X$, then

$$g = (A^2 - k_{ij}W^iW^j)dt^2 - 2k_{ij}W^idx^jdt - k_{ij}dx^idx^j.$$ 

Inverting this form of the metric gives

$$g^{-1} = A^{-2}(\partial_t - W^i\partial_i)^2 - k^{ij}\partial_i\partial_j.$$

Note that $k(W, W) = A^2 - \mu$, and hence $W \neq 0$ near $\partial M$.

Now use the condition that $\partial M$ is a Killing horizon generated by $T$. The covariant form of (2.4) reads

$$\partial_i\mu = 2\kappa W^jk_{ij}.$$ (2.6)

By assumption $\kappa > 0$, so $W$ is a nonzero inward pointing normal to $X$ along $\partial X$ whose length with respect to $k$ is $A$.

Introduce geodesic normal coordinates $(r, y^A)$ on $X$ near $\partial X$, so $r$ is the distance to $\partial X$ (uppercase indices will always range over $A = 2, \ldots, n$). By construction, $\partial_r$ is an inward pointing unit normal along $\partial X$, so

$$W^r = A, \quad W^A = 0$$ (2.7)

along the boundary. Also by construction, the components of the induced metric in $(r, y^A)$ coordinates satisfy $k^{rr} = 1$ and $k^{rA} = 0$.

**Lemma 2.2.** The function $r$ satisfies $G(dr) = -2\kappa A^{-1}r + r^2C^\infty(M)$.

**Proof.** First observe that $k_{AB}W^AW^B \in r^2C^\infty(M)$ by (2.7), and since $k(W, W) = A^2 - \mu$,

$$A^2 - \mu = (W^r)^2 + k_{AB}W^AW^B.$$ 

Now $\mu$ and $r$ are both boundary defining functions, so $\mu = fr$ for some $f \in C^\infty(M)$, and hence $d\mu = fdr$ on $\partial X$. But on the boundary $\langle W, d\mu \rangle = 2\kappa A^2$ from (2.6), while $\langle W, dr \rangle = W^r = A$ from (2.7). Thus

$$\mu = fr = 2\kappa Ar + r^2C^\infty(M).$$

Plugging this back into the equation for $k(W, W)$ yields

$$(W^r)^2 = A^2 - 2\kappa Ar + r^2C^\infty(M),$$

and therefore $G(dr) = -k^{rr} + A^{-2}(W^r)^2 = -2\kappa A^{-1}r + r^2C^\infty(M)$ as desired.  

2.5. **Reducing the problem.** Observe that the surface gravity depends on the choice of null generator $T$. Consider the rescaled vector field
\[
\hat{T} = T/(2\kappa),
\]
which changes the time coordinate by the transformation $\hat{t} = 2\kappa t$. If $\hat{\mathbf{P}}(\omega)$ is now defined as in (1.1) but replacing $t$ with $\hat{t}$, then
\[
\mathbf{P}(\omega) = \hat{\mathbf{P}}(\omega/(2\kappa)).
\]
It suffices to prove Theorem 1 for $\hat{\mathbf{P}}(\omega)$ then, since rescaling the frequency only changes the constants $\omega_0, C_0, C$. Dropping the hat notation, it will henceforth be assumed that $\kappa = 1/2$.

Next, consider a conformal change $g = f\tilde{g}$, where $f > 0$ is stationary. The operator $L$ can then be written as
\[
L = f^{-1}\square_{\tilde{g}} + (n-1)f^{-2}\nabla_{\tilde{g}} f + \mathcal{W} + \mathcal{V}. \tag{2.8}
\]
Thus we can write $L = f^{-1}\tilde{L}$, where $\tilde{L}$ has the same form as $L$ but with $\tilde{g}$ replacing $g$, provided that the vector field $\nabla_{\tilde{g}} f$ is tangent to $\partial M$. But this follows from the stationarity of $f$, since
\[
g(T, \nabla_{\tilde{g}} f) = 0
\]
and $T$ is normal to $\partial M$. Thus it suffices to prove Theorem 1 with $\tilde{L}$ replacing $L$. Observe that $\partial M$ remains a Killing horizon generated by $T$ with respect to $\tilde{g}$, and the surface gravity is unchanged.

By making a conformal change and dropping the tilde notation, it will henceforth be assumed that
\[
G(dr) = -r. \tag{2.9}
\]
If $(\tau, \rho, \eta_A)$ are dual variables to $(t, r, y^A)$, define a stationary quadratic form $G_0 \in C^\infty(T^*M)$ by
\[
G_0 = -r\rho^2 - 2\rho\tau - k_0^{AB}\eta_A\eta_B. \tag{2.10}
\]
Here $k_0$ is the restriction of $k$ to $\partial M$, which is then extended to a neighborhood of $\partial M$ by requiring that it satisfies $\mathcal{L}_{\partial^\alpha}k_0 = 0$. In the next section, the difference $G - G_0$ will be analyzed.

2.6. **Negligible tensors.** In this section we define a class of tensors which will arise as errors throughout the proof of Theorem 1'.

**Definition 1.** 1) A stationary 1-tensor $F^\alpha\partial_\alpha$ is said to be negligible if its components in a coordinate system $(t, r, y^A)$ satisfy
\[
F^t \in rC^\infty(M), \quad F^r \in r^2C^\infty(M), \quad F^A \in rC^\infty(M).
\]
2) A stationary 2-tensor $H^{\alpha\beta}\partial_\alpha\partial_\beta$ is said to be negligible if its components in a coordinate system $(t, r, y^A)$ satisfy
\[
\begin{aligned}
H^{tt} \in \mathcal{C}^\infty(M),& \quad H^{rr} \in r^2\mathcal{C}^\infty(M),& \quad H^{AB} \in r\mathcal{C}^\infty(M), \\
H^{tA} \in \mathcal{C}^\infty(M),& \quad H^{tr} \in r\mathcal{C}^\infty(M),& \quad H^{rA} \in r\mathcal{C}^\infty(M).
\end{aligned}
\]
Observe that negligibility is invariant under those coordinate changes which leave invariant $(t, r)$. Denote by $N_1$ and $N_2$ all $\mathcal{C}^\infty(T^*M)$ functions of the form $F^\alpha\varpi_\alpha$ and $H^{\alpha\beta}\varpi_\alpha\varpi_\beta$.

Recall the definition of $G_0$ in (2.10). The notion of negligibility is motivated by the fact that
\[
G = G_0 + N_2.
\]
This follows directly from (2.5), (2.7), and (2.9). We will also repeatedly reference the auxiliary functions
\[
Y = (r\rho)^2 + \tau^2, \quad Z = r\rho^2 + \eta^{AB}\eta_A\eta_B.	ag{2.11}
\]
It follows immediately from the Cauchy–Schwarz inequality $2ab < \delta a^2 + b^2 / \delta$ that there exists $C > 0$ satisfying
\[
Z \leq C \left( |G_0| + \tau^2 / r \right). \tag{2.12}
\]
The next two lemmas also follow from judicious applications of the Cauchy–Schwarz inequality and the trivial observation that $(r\rho)^2 = r(r\rho^2)$ is small relative to $r\rho^2$ for small values of $r$.

**Lemma 2.3.** Let $F \in N_1$. Then, for each $\gamma > 0$ there exists $C_\gamma$ such that
\[
r^{-1}|\tau||F| \leq C_\gamma \tau^2 + \gamma Z.
\]
Furthermore, $\rho N_1 \subset N_2$ and $N_1 \cdot N_1 \subset r^2N_2$.

**Lemma 2.4.** Let $H \in N_2$. Then, for each $\gamma > 0$ there exist $C_\gamma, r_\gamma > 0$ such that
\[
|H| \leq C_\gamma Y + \gamma k^{AB}\eta_A\eta_B, \quad |H| \leq C_\gamma \tau^2 + \gamma Z
\]
for $r \in [0, r_\gamma]$.

Now combine Lemma 2.4 with the bound (2.12) and the relation $G = G_0 + N_2$. Thus there exists $R > 0$ and $C > 0$ such that
\[
Z \leq C(|G| + \tau^2 / r) \tag{2.13}
\]
for $r \in [0, R]$.

The next goal is to compute the Poisson brackets $\{G, r\}$ and $\{G, \{G, r\}\}$. To begin, observe that
\[
\{G_0, r\} = -2(r\rho + \tau), \quad \{G_0, \{G_0, r\}\} = 2(r\rho^2 + 2\tau\rho). \tag{2.14}
\]
In order to replace $G_0$ with $G$ we also need to consider the Poisson brackets of functions in $\mathcal{N}_1$ and $\mathcal{N}_2$.

Lemma 2.5. The Poisson bracket satisfies $\{N_2, r\} \subset \mathcal{N}_1$ and $\{N_2, N_1\} \subset \mathcal{N}_2$, as well as $\{G_0, N_1\} \subset \mathcal{N}_2$ and $\{\{G_0, r\}, N_2\} \subset \mathcal{N}_2$. Therefore,

$$\{G, r\} = -2(r\rho + \tau) + \mathcal{N}_1, \quad \{G, \{G, r\}\} = 2(r\rho^2 + 2\tau\rho) + \mathcal{N}_2. \quad (2.15)$$

Furthermore, if $\{G, r\} = 0$, then $\{G, \{G, r\}\} = -2r\rho^2 + \mathcal{N}_2$.

Proof. The first part is a direct calculation, while $(2.15)$ follows from the first part and $(2.14)$. The last statement follows from the inclusion $\rho\mathcal{N}_1 \subset \mathcal{N}_2$. □

3. Carleman estimates in the interior

3.1. Statement of result. In this section we prove a Carleman estimate valid in the interior $X^\circ$, but with uniform control over the exponential weight near $\partial X$.

Recall that $r$ denotes the distance on $X$ to the boundary with respect to the induced metric. Although this function is only well defined a small neighborhood of $X \cap \partial M$, for notational convenience we will assume that $[0, 3]$ is contained in the range of $r$ (otherwise it is just a matter of replacing 3 with $3\varepsilon$ for an appropriate $\varepsilon > 0$).

Proposition 3.1. Given $[a, b] \subset (0, \infty)$, there exists $r_1 \in (0, 1)$ and $\varphi_1, \varphi_2 \in C^\infty(X)$ such that

- on $\{r \leq 1\}$ the functions $\varphi_1, \varphi_2$ are equal and depend only on $r$,
- $\varphi'_i(r) < 0$ is constant on $\{r \leq r_1\}$ for $i = 1, 2$,

with the following property: given a compact set $K \subset X^\circ$ there exists $C > 0$ such that

$$h^{1/2}\|(e^{\varphi_1/h} + e^{\varphi_2/h})u\|_{H^2_h(X)} \leq C\|(e^{\varphi_1/h} + e^{\varphi_2/h})P(z)u\|_{L^2(X)}$$

for each $u \in C^\infty_c(K^\circ)$ and $\pm z \in [a, b]$.

It clearly suffices to prove Proposition 3.1 for the operator $L = \Box_g$, since the lower order terms can absorbed as errors. This uses the hypothesis (A) on $L$ from the introduction, but not (B) or (C). In order to prove Theorem 1′, an additional estimate is needed near the boundary; this is achieved in Section 4 below.

3.2. The conjugated operator. Given $\varphi \in C^\infty(X)$, define the conjugated operator

$$P_\varphi(z) = e^{\varphi/h}P(z)e^{-\varphi/h}.$$

Define $L^2(X)$ with respect to the density $A \cdot dS_X$, where $dS_X$ is the induced volume density on $X$ and $A > 0$ is the lapse function as in Section 2.3. Defining $\text{Re}P_\varphi(z)$ and
\[
\|P\varphi(z)u\|_{L^2(X)}^2 = \|\text{Re} P\varphi(z)u\|_{L^2(X)}^2 + \|\text{Im} P\varphi(z)u\|_{L^2(X)}^2 + i\langle [\text{Re} P\varphi(z), \text{Im} P\varphi(z)]u, u \rangle_{L^2(X)} \tag{3.1}
\]
for \(u \in C^\infty_c(X^\circ)\). The idea is to find \(\varphi\) which satisfies Hörmander’s hypoellipticity condition
\[
\{\text{Re} p\varphi, \text{Im} p\varphi\} > 0 \tag{3.2}
\]
on the characteristic set \(\{p\varphi = 0\}\).

In order to apply the results of Section 2.6 without introducing additional notation, it is convenient to work with the dual metric function \(G\) directly. Define
\[
G\varphi(\varpi) = G(\varpi + i\text{d}\varphi),
\]
so if we assume that \(\tau\) is real, then \(\text{Re} G\varphi(\varpi) = G(\varpi) - G(d\varphi)\), and \(\text{Im} G\varphi(\varpi) = (H_{G\varphi})(\varpi)\). We will then construct \(\varphi\) (viewed as a stationary function on \(M\)) such that
\[
\{\text{Re} G\varphi, \text{Im} G\varphi\} = (H^2_{G\varphi})(\varpi) + (H^2_{G\varphi})(d\varphi) > 0 \tag{3.3}
\]
on \(\{G\varphi = 0\} \cap \{a \leq \pm \tau \leq b\}\). This will imply the original hypoellipticity condition from the discussion surrounding (2.3) and the identifications
\[
p\varphi(\xi, z) = -G\varphi(\xi - zd\tau), \quad z = -\tau.
\]
Note the the dual variable \(\tau\) is now playing the role of a rescaled time frequency.

3.3. Constructing the phase in a compact set. To avoid any undue topological restrictions, we will actually construct two weights \(\varphi_1, \varphi_2\) in the interior, which agree outside a large compact set. This appears already in [Bur1], but we will follow the closely related presentation in [Mos, RT].

**Lemma 3.2.** There exist functions \(\psi_1, \psi_2 \in C^\infty(X)\) with the following properties.

1. \(\psi_1, \psi_2\) have finitely many non-degenerate critical points, all of which are contained in \(\{r > 1\}\).
2. \(\psi_2 > \psi_1\) on \(\{d\psi_1 = 0\}\), and \(\psi_1 > \psi_2\) on \(\{d\psi_2 = 0\}\).
3. The functions \(\psi_1, \psi_2\) are equal and depend only on \(r\) outside of \(\{r \geq 1\}\).

**Proof.** Let \(\zeta \in C^\infty(\{r \geq 1\})\) solve the boundary value problem
\[
\Delta_k \zeta = 1, \quad \zeta|_{\{r=1\}} = 1.
\]
Here \(\Delta_k\) is the non-positive Laplacian with respect to the induced metric \(k\). Since \(\Delta_k \zeta > 0\), the Hessian of \(\zeta\) is positive definite, hence the critical points of \(\zeta\) are non-degenerate, and none of them are local maxima.
In addition, since \( \zeta \) clearly achieves its maximum at each point of \( \{ r = 1 \} \), its outward pointing normal derivative is strictly positive by Hopf’s lemma \([GT, \text{Lemma 3.4}]\). By construction, the outward pointing unit normal is \(-\partial_r\), hence \( \zeta' < 0 \) near \( \{ r = 1 \} \) (for the remainder of the proof, prime will denote differentiation with respect to \( r \)).

Now fix any function \( \bar{\zeta} = \bar{\zeta}(r) \in \mathcal{C}^\infty(\{ r < 2 \}) \) such that \( \bar{\zeta}' < 0 \) everywhere, and \( \bar{\zeta} \geq \zeta \) on their common domain of definition \( \{ 1 \leq r < 2 \} \). Choose a cutoff \( H = H(r) \in \mathcal{C}^\infty(X; [0, 1]) \) such that

\[
H = 1 \text{ for } r < 1 + \gamma, \quad \text{supp } H \subset \{ r < 1 + 2\gamma \},
\]

and \( H' \leq 0 \). Set \( \psi_1 = H\bar{\zeta} + (1 - H)\zeta \), and compute \( \psi_1' = H'(\bar{\zeta} - \zeta) + H\bar{\zeta}' + (1 - H)\zeta' \).

If \( \gamma > 0 \) is sufficiently small, then \( \psi_1' < 0 \) in a neighborhood of \( \text{supp } H \), since the sum of the last two terms is strictly positive on \( \text{supp } H \). On the other hand, outside of such a neighborhood the only critical points of \( \psi_1 \) are those of \( \zeta \).

Let \( p_1, \ldots, p_n \) enumerate the necessarily finite number of critical points of \( \psi_1 \), and choose \( \gamma > 0 \) such that the closed geodesic balls \( B(p_1, \gamma), \ldots, B(p_n, \gamma) \) are mutually disjoint and \( B(p_j, \gamma) \subset \{ r > 1 \} \) for each \( j \). Since \( p_j \) is not a local maximum, for each \( j \) there is a point \( q_j \in B(p_j, r) \) such that

\[
\psi_1(q_j) > \psi_1(p_j).
\]

Now choose a diffeomorphism \( g : X \to X \) which is the identity outside the union of the \( B(q_j, r) \) and exchanges \( p_j \) with \( q_j \). Then, set \( \psi_2 = \psi_1 \circ g \). By construction the only critical points of \( \psi_2 \) are \( q_1, \ldots, q_n \), and furthermore

\[
\psi_2(p_j) > \psi_1(p_j), \quad \psi_1(q_j) > \psi_2(q_j)
\]

for each \( j \). Since outside of \( \{ r > 1 \} \) the functions \( \psi_1 = \psi_2 \) depend on \( r \) only, the proof is complete.

Let \( B_1 \subset \{ r > 1 \} \) be a closed neighborhood of \( \{ d\psi_1 = 0 \} \) such that \( \psi_2 > \psi_1 \) on \( B_1 \), and likewise for \( B_2 \), exchanging the roles of \( \psi_1 \) and \( \psi_2 \). Now define

\[
\varphi_i = \exp(\alpha \psi_i), \quad i = 1, 2,
\]

where \( \alpha > 0 \) is a parameter. The following lemma is a standard computation which is included for the sake of completeness.

**Lemma 3.3.** Given \( \varepsilon > 0 \) and \( \tau_0 > 0 \), there exists \( \alpha > 0 \) such that

\[
\{ \text{Re } G_{\varphi_i}, \text{Im } G_{\varphi_i} \} > 0
\]

on \( (\{ G_{\varphi_i} = 0 \} \cap \{ r \geq \varepsilon \} \cap \{ |\tau| \leq \tau_0 \}) \setminus T^*_{B_1} M \) for \( i = 1, 2 \).
Proof. The subscript \( i = 1, 2 \) will be suppressed. Use the definition (3.4) to compute

\[ H_G \varphi = \alpha e^{\psi} H_G \psi, \quad H_G^2 \varphi = \alpha^2 e^{\psi} (H_G \psi)^2 + \alpha e^{\psi} H_G^2 \psi. \]

Assume that \( G \varphi(\omega) = 0 \). It follows from \( \text{Im} \, G \varphi(\omega) = 0 \) that \( (H_G \varphi)(\omega) = 0 \), and hence \( (H_G \psi)(\omega) = 0 \). Therefore by (3.3),

\[ \{ G - G(d\varphi), H_G \varphi \}(\omega) = \alpha e^{\psi} (H_G^2 \psi)(\omega) + \alpha^3 e^{3\alpha \psi} (H_G^2 \psi)(d\psi) + \alpha^4 e^{3\alpha \psi} |G(d\psi)|^2. \]

Next, use the condition \( \text{Re} \, G \varphi(\omega) = 0 \), which implies that \( G(\omega) = \alpha^2 e^{2\alpha \psi} G(d\psi) \).

By the discussion following Lemma 2.1, there exists \( C > 0 \) such that

\[ |(H_G^2 \psi)(\omega)| \leq C(1 + |G(\omega)|) \]
on \( \{ r \geq \varepsilon \} \cap \{ |\tau| \leq \tau_0 \} \). Thus on the set \( \{ G \varphi = 0 \} \cap \{ r \geq \varepsilon \} \cap \{ |\tau| \leq \tau_0 \} \),

\[ |\alpha e^{\psi} (H_G^2 \psi)(\omega)| + |\alpha^3 e^{3\alpha \psi} (H_G^2 \psi)(d\psi)| \leq C \alpha^3 e^{3\alpha \psi}. \]

On the other hand, as soon as \( d\psi \neq 0 \) the third term \( \alpha^4 e^{3\alpha \psi} |G(d\psi)|^2 \) is positive by Lemma 2.1, and dominates the previous two terms for large \( \alpha > 0 \). Since \( d\psi \neq 0 \) away from \( B \), the proof is complete.

\[ \square \]

3.4. Constructing the phase outside of a compact set. The most delicate part of the argument is the construction of the phase outside of a compact set. Since \( G(dr) = -r \) and \( \varphi \) is a function only of \( r \) in this region,

\[ G \varphi = G + r(\varphi')^2 + i\varphi' H_G r. \]

Now compute the Poisson bracket

\[ \{ \text{Re} \, G \varphi, \text{Im} \, G \varphi \} = \{ G + r(\varphi')^2, \varphi' H_G r \} \]

\[ = \varphi' H_G^2 r + \varphi''(H_G r)^2 - ((\varphi')^3 + 2r(\varphi')^2 \varphi'') \partial_r H_G r. \]

Assume that \( \varphi' < 0 \), in which case \( \text{Im} \, G \varphi = 0 \) is equivalent to \( H_G r = 0 \). The goal is then to arrange negativity of the term

\[ H_G^2 r - ((\varphi')^2 + 2r \varphi' \varphi'') \partial_r H_G r \quad \text{(3.5)} \]
on \( \{ \text{Re} \, G \varphi = 0 \} \). Recall the definition of \( Z \) from (2.11).

Lemma 3.4. There exists \( C > 0 \) and \( R > 0 \) such that \( Z \leq C(r(\varphi')^2 + \tau^2/r) \) on \( \{ \text{Re} \, G \varphi = 0 \} \cap \{ 0 < r \leq R \} \).

Proof. Apply (2.13), using that \( \text{Re} \, G \varphi = 0 \) implies \( G = -r(\varphi')^2 \). \[ \square \]

Putting everything together, it is now easy compute \( H_G^2 r \) on \( \{ G \varphi = 0 \} \) near the boundary.
Lemma 3.5. For each $\delta > 0$ there exists $R_\delta > 0$ such that
\[
|H_G^2r + 2\tau^2/r| \leq \delta(r(\varphi')^2 + \tau^2/r)
\]
on $\{G_\varphi = 0\} \cap \{0 < r \leq R_\delta\}$.

Proof. From the expression (2.15) for $H_G^2r$ and Lemma 2.4, find $C_\gamma > 0$ and $r_{\gamma} > 0$ such that
\[
|H_G^2r + 2r^2| < C_\gamma|\tau|^2 + \gamma Z
\]for $r \in (0, r_{\gamma})$. Now multiply $H_Gr$ by $\rho$, and use that $\rho N_1 \subset N_2$. Therefore by Lemma 2.4, there exists $C_\gamma' > 0$ and $r_{\gamma}' > 0$ such that
\[
|2r\rho^2 + 2\tau\rho| < C_\gamma'|\tau|^2 + \gamma Z
\]
for $r \in (0, r_{\gamma}')$. On the other hand, from $H_Gr = 0$, deduce that $-\tau\rho = \tau^2/r + \tau r^{-1}N_1$.

By Lemma 2.3, there exists $C''_\gamma > 0$ such that
\[
|2\tau\rho + 2\tau^2/r| < C''_\gamma|\tau|^2 + \gamma Z.
\]

Combine (3.6), (3.7), and (3.8) via the triangle inequality with Lemma 3.4 to find that
\[
|H_G^2r + 2\tau^2/r| < 3\gamma C(\tau r(\varphi')^2 + \tau^2/r) + (C_\gamma + C_\gamma' + C''_\gamma) \tau^2
\]
for $r \in (0, \min\{r_{\gamma}, r_{\gamma}', R\})$; here $C > 0$ and $R > 0$ are provided by Lemma 3.4. Finally, choose $\gamma$ sufficiently small depending on $\delta$ and a corresponding $R_\delta > 0$ such that the conclusion of the lemma holds for $r \in (0, R_\delta)$. \hfill \square

Next, observe that $-\partial_\rho H_Gr = 2r + r^2C^\infty(M)$. Given $a > 0$, it follows from (3.5) and Lemma 3.5 that there exists $R_1 > 0$ and $\delta > 0$ such that
\[
(\varphi')^{-1}\{\Re G_\varphi, \Im G_\varphi\} < -a^2/r + 3r(\varphi')^2 + 3r^2\varphi'\varphi'' - \delta
\]
on $\{G_\varphi = 0\} \cap \{0 < r \leq R_1\} \cap \{|\tau| \geq a\}$, provided that $\varphi'' \geq 0$.

Henceforth, assume that $\varphi = \varphi_i$ has been constructed on $\{r \geq R_1\}$ according to Lemma 3.3 (namely take $\varepsilon = R_1$ in Lemma 3.3). Furthermore, by taking $\alpha > 0$ sufficiently large in Lemma 3.3 (but keeping $\alpha > 0$ fixed), it may be assumed that
\[
3(\varphi'(R_1)R_1)^2 > a^2, \quad \varphi''(r) \geq -\varphi'(r)/r \text{ for } r \in [R_1, R_1 + 1].
\]
The following lemma allows one to extend $\varphi$ in such a way that its derivative is controlled; the idea comes from [Bur1, Section 3.1.2], but of course the form of the operator there is quite different.

Lemma 3.6. There exists an extension of $\varphi = \varphi_i$ from $\{r \geq R_1\}$ to $\{r < R_1\}$ such that
\[
\{\Re G_\varphi, \Im G_\varphi\} > 0
\]
on $\{G_\varphi = 0\} \cap \{0 < r \leq R_1\} \cap \{|\tau| \geq a\}$. Furthermore, there exists $r_1 \in (0, R_1)$ such that $\varphi'(r) < 0$ is constant for $r \in [0, r_1]$.\hfill \square
Proof. Motivated by (3.9), consider the differential equation
\[-a^2/r + 3rk^2 + 3r^2kk' = 0, \quad k(R_1) = \varphi'(R_1) < 0.\]

This is a Bernoulli equation whose solution is given by
\[k(r) = -r^{-1}\left((\varphi'(R_1)R_1)^2 + (2/3)a^2 \log(r/R_1)\right)^{1/2}.\]

The solution is certainly meaningful for \(r \in [R_0, R_1]\), where we define \(R_0\) by
\[R_0 = R_1 \exp\left(1/2 - (3/2) (\varphi'(R_1)R_1/a)^2\right).\]

Note that we indeed have \(R_0 < R_1\) by the assumption (3.10). The value \(R_0\) was chosen such that \(k'(R_0) = 0\), and it is easy to see that \(k(r) > 0\) for \(r \in (R_0, R_1]\). In addition, \(k(R_0) < 0\). Let \(\theta = \theta(r)\) be defined on \([0, R_1 + 1]\) by
\[\theta(r) = \begin{cases} 
\varphi'(r), & r \in [R_1, R_1 + 1], \\
k(r), & r \in [R_0, R_1], \\
k(R_0), & r \in [0, R_0].
\end{cases}\]

The function \(\theta\) is strictly negative, and the piecewise continuous function \(\theta'\) satisfies
\[-a^2/r + 3r\theta^2 + 3r\theta\theta' \leq 0\]
for \(r \in (0, R_1 + 1]\). By construction of \(k\) and \(R_0\), the latter statement holds for \(r \in (0, R_1]\), and it is also true for \(r \in (R_1, R_1 + 1]\) by (3.10). Rearranging this,
\[\theta' \geq a^2/(3r^3 \theta) - \theta/r \quad (3.11)\]
for \(r \in (0, R_1 + 1]\).

We now proceed to mollify \(\theta\) in such a way that the hypotheses of the lemma hold. Let \(\eta_\varepsilon(r) = (1/\varepsilon)\eta(r/\varepsilon)\) denote a standard mollifier, where \(\eta \in C_c^\infty((-1, 1))\) has integral one. In addition, choose a cutoff \(H = H(r) \in C^\infty(X; [0, 1])\) such that \(H = 1\) for \(r < R_1 + \gamma\), \(H = 0\) for \(r > R_1 + 2\gamma\), and \(H' \leq 0\). Now define
\[\theta_\varepsilon = (1 - H)\theta + \eta_\varepsilon \ast (H\theta).\]

Clearly \(\theta_\varepsilon\) is smooth, and \(\theta_\varepsilon \rightarrow \theta\) uniformly for \(r \in [0, R_1 + 1]\). Furthermore, there exists \(\varepsilon_0 > 0\) such that if \(\varepsilon \in (0, \varepsilon_0)\), then the following properties are satisfied:

- \(\theta_\varepsilon(r) < 0\) and \(\theta'_\varepsilon(r) \geq 0\) for \(r \in [0, R_1 + 1]\).
- \(\theta_\varepsilon(r) = \varphi'(r)\) for \(r \in [R_1 + 3\gamma, R_1 + 1]\).
- There exists \(r_1 \in (0, R_1)\) such that \(\theta_\varepsilon(r) = k(R_0)\) for \(r \in [0, r_1]\).
Since \( \theta \) is continuous and piecewise smooth,
\[
\theta'_\varepsilon = (1 - H)\theta' - H'\theta + \eta_\varepsilon \ast (H'\theta + H\theta').
\]
Therefore by (3.11),
\[
\theta'_\varepsilon \geq -H'\theta + \eta_\varepsilon \ast (H'\theta) + (1 - H) \left( a^2/(3r^3\theta) - \theta/r \right) + \eta_\varepsilon \ast \left( H \left( a^2/(3r^3\theta) - \theta/r \right) \right)
\]
for \( r \in (0, R_1 + 1) \). The right-hand side converges uniformly to \( a^2/(3r^3\theta) - \theta/r \) for \( r \in [r_1, R_1 + 1] \) since the latter function is continuous there. Since \( \theta_\varepsilon \to \theta \) uniformly for \( r \in [r_1, R_1 + 1] \) as well, given \( \delta > 0 \) there exists \( \varepsilon \in (0, \varepsilon_0) \) such that
\[
-a^2/r + 3r\theta'^2 + 3r^2\theta_\varepsilon \theta'_\varepsilon \leq \delta
\]
for \( r \in [r_1, R_1 + 1] \). This inequality is also true for \( r \in (0, r_1) \), since \( \theta_\varepsilon = k(R_0) \) on that interval. Now extend \( \varphi \) from \( \{r \geq R_1 + 3\gamma\} \) to \( \{r < R_1 + 3\gamma\} \) by the formula
\[
\varphi(r) = \varphi(R_1 + 3\gamma) + \int_{R_1+3\gamma}^{r} \theta_\varepsilon(s) \, ds,
\]
We can now simply replace \( R_1 \) with \( R_1 - 3\gamma \). This completes the proof according to (3.9) by observing that the \( \varphi \) just constructed satisfies \( \varphi''(r) \geq 0 \).

As a remark, if \( \tau \neq 0 \) then the hypoellipticity condition also holds along \( \{r = 0\} \), simply because \( \text{Im } G_\varphi \neq 0 \) in that case. However, since \( \xi \mapsto G(\xi) \) is not elliptic along \( \{r = 0\} \), the hypoellipticity condition alone, stated here in the semiclassical setting, is not sufficient to prove a Carleman estimate — cf. [Hör, Section 8.4]

Now that the phases \( \varphi_1, \varphi_2 \) have been constructed globally, we are ready to finish the proof of Proposition 3.1. Here we come back to the operator \( P_\varphi(z) \) on \( X \). Fix a norm \( | \cdot | \) on the fibers of \( T^*X \) (for instance using the induced metric \( k \)) and let \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \).

**Proof of Proposition 3.1.** Recall that we are given \( [a, b] \subset (0, \infty) \) and a compact set \( K \subset X^\circ \). Without loss, we may assume that \( K = \{r \geq \varepsilon\} \) for some \( \varepsilon > 0 \). Choose \( U_i \subset B^\circ_{r_i} \) such that \( \{\text{Re } p_\varphi, \text{Im } p_\varphi\} \ni \{p_{\varphi_i} = 0\} \subset \{r \geq \varepsilon/2\} \setminus T_{U_i}^*X \), and then let \( \chi_i \in C_c^{\infty}(B^\circ_{r_i}) \) be such that \( \chi_i = 1 \) near \( U_i \). If \( \varphi = \varphi_i \), then
\[
|p_\varphi|^2 + \chi^2 + h\{\text{Re } p_\varphi, \text{Im } p_\varphi\} \geq h \left( M|p_\varphi|^2 + M\chi^2 + \{\text{Re } p_\varphi, \text{Im } p_\varphi\} \right)
\]
for any \( M > 0 \), provided that \( h > 0 \) is sufficiently small. On the other hand, the set \( \{\text{Re } p_\varphi = 0\} \cap \{r \geq \varepsilon/2\} \) is compact by Lemma 2.1, uniformly for \( \pm z \in [a, b] \).

Therefore,
\[
\langle \xi \rangle^{-4} (M|p_\varphi|^2 + M\chi^2 + \{\text{Re } p_\varphi, \text{Im } p_\varphi\}) > 0
\]
near \( T^*X \cap \{ r \geq \varepsilon/2 \} \) for \( M > 0 \) sufficiently large. By (3.1) and the semiclassical Gårding inequality applied to \( e^{\varphi_i/h} u \),

\[
h \| e^{\varphi_i/h} u \|_{H^2_h(X)}^2 \leq C \| e^{\varphi_i/h} P(z) u \|_{L^2(X)}^2 + C \| e^{\varphi_i/h} u \|_{L^2(B_i)}^2
\]

for \( u \in C^\infty_c(K^\circ) \) and \( i = 1, 2 \). Since \( \varphi_1 > \varphi_2 \) on \( B_2 \) and \( \varphi_2 > \varphi_1 \) on \( B_1 \), there is \( \gamma > 0 \) such that

\[
e^{\varphi_i/h} \leq e^{-\gamma/h} (e^{\varphi_1/h} + e^{\varphi_2/h})
\]

on \( B_i \). Now add (3.13) for \( i = 1, 2 \) to absorb the integral over \( B_1 \cup B_2 \) into the left-hand side. \( \square \)

4. Degenerate Carleman estimates near the boundary

4.1. Statement of result. In this section we complement Proposition 3.1 with a result valid up to the boundary. Recall that the phases \( \varphi_1, \varphi_2 \) are equal on \( \{ r \leq 1 \} \). Since we are working near \( \partial X \), we will thus drop the subscript and simply write \( \varphi \).

**Proposition 4.1.** Given \([a, b] \subset (0, \infty)\) there exists \( r_0 > 0 \) and \( C > 0 \) such that

\[
\| e^{\varphi/h} u \|_{H^1_{b,h}(X)} \leq C \left( h^{-1/2} \| e^{\varphi/h} P(z) u \|_{L^2(X)} + e^{\varphi(0)/h} \| u \|_{L^2(\partial X)} \right).
\]

for \( u \in C^\infty_c(\{ r < r_0 \}) \) and \( \pm z \in [a, b] \).

The space appearing on the left-hand side is modeled on the space of vector fields \( V_b(X) \) which are tangent to the boundary; see [Mel]. Thus \( u \in H^1_b(X) \) if \( u \in L^2(X) \) and \( K u \in L^2(X) \) for any \( K \in V_b(X) \). If \( u \in H^1_b(X) \) and \( \text{supp} \ u \subset \{ r < 1 \} \), we can set

\[
\| u \|_{H^1_{b,h}(X)}^2 = \int_X |u|^2 + h^2 |r \partial_r u|^2 + h^2 k^{AB} (\partial_A u \cdot \partial_B \bar{u}) \, dS_X.
\]

Of course away from \( \partial X \) this is equivalent to the full \( H^1_h \) norm. Observe that it is enough to prove Proposition 4.1 for the operator \( L = \Box_g \), since the estimate (4.1) is stable under perturbations \( B \in h\text{Diff}^1_h(X) \) provided that the vector field part of \( B \) is tangent to \( \partial X \). In this case, the latter statement follows from the hypotheses (A), (B), but not (C), made in the introduction.

Proposition 4.1 is proved through integration by parts. A convenient way of carrying out this procedure is by constructing an appropriate multiplier for the wave operator and applying the divergence theorem. This approach to Carleman estimates for certain geometric operators is partly inspired by [AS, IK].

4.2. The divergence theorem. We will use the divergence theorem in the differentiated form

\[
\frac{d}{dt} \int_X g(K, N) \, dS_X + \int_{\partial X} g(K, T) \, dS_{\partial X} = \int_X (\text{div}_g K) A \, dS_X,
\]

(4.2)
valid for any vector field $K$ (see [War, Lemma 3.1] for instance). Here $dS_{\partial X}$ is the volume density on $\partial X$ induced by $k$ (the latter is Riemannian, hence the induced volume density is well defined).

4.3. **Stress-energy tensor.** Given $v \in C^\infty(M)$, let $Q = Q[v]$ denote the usual stress energy tensor associated to $v$ with components

$$Q_{\alpha\beta} = \text{Re} (\partial_\alpha v \cdot \partial_\beta \bar{v}) - (1/2)G(dv)g_{\alpha\beta}.$$ 

This tensor has the property that $(\nabla^\beta Q_{\alpha\beta})^\alpha = \text{Re}(\Box v \cdot S \bar{v})$ for any vector field $S$. Given such a vector field and a function $w$, define the modified vector field $J = J[v]$ with components

$$J^\alpha = Q^\alpha_{\beta} S^\beta + (1/2) w \cdot \partial^\alpha (|v|^2) - (1/2)(\partial^\alpha w)|v|^2.$$ 

The relevant choices in this context are

$$S = \nabla_g r, \quad w = \lambda + (1/2)\Box_g r, \quad \lambda = \lambda(r)$$

(4.3)

where $\lambda = \lambda(r)$ is an undetermined function to be chosen in Lemma 4.4 below. Also, introduce the tensor $\Pi$ with components

$$\Pi^{\alpha\beta} = -\nabla^{\alpha\beta} r - \lambda g^{\alpha\beta}.$$ 

The divergence of $J$ satisfies

$$\text{Re} (\Box_g u \cdot (S \bar{v} + w \bar{v})) = \text{div}_g J + \Pi (dv, d\bar{v}) + (1/2)(\Box_g w)|v|^2,$$

(4.4)

which is verified by a direct calculation.

4.4. **The conjugated operator.** Near $\partial M$, consider the conjugated operator $L_\Phi = e^{\Phi} \Box_g e^{-\Phi}$, where $\Phi = \Phi(r)$. Then, $L_\Phi$ has the expression

$$L_\Phi = \Box_g - 2\Phi' S + ((\Phi')^2 - \Phi'')G(dr) - \Phi' \Box_g r$$

$$= \Box_g - 2\Phi' S + V_0.$$ 

Now $G(dr) = -r$ by assumption, and consequently the potential term $V_0$ satisfies

$$V_0 = r(\Phi'' - (\Phi')^2) - \Phi' \Box_g r.$$ 

Set $V_1 = V_0 - 2\Phi' w$, multiply $L_\Phi v$ by $S \bar{v} + w \bar{v}$, and take the real part to find that

$$\text{Re}(L_\Phi v \cdot (S \bar{v} + w \bar{v})) = \text{Re}(\Box_g v \cdot (S \bar{v} + w \bar{v})) - 2\Phi'|Sv|^2$$

$$+ \text{Re} V_1 v \cdot S \bar{v} + V_0 w|v|^2.$$ 

(4.5)

It is also convenient to write $\text{Re} (V_1 v \cdot S \bar{v})$ as a divergence,

$$\text{Re} (V_1 v \cdot S \bar{v}) = (1/2) \text{div}_g (V_1 |v|^2 S) - (1/2) (S(V_1) + V_1 \Box_g r)|v|^2.$$
In view of this expression, define the vector field \( K = J + (1/2)V_1|v|^2 S \). For future use, also define the modified potential \( V \) by
\[
V = (1/2)(\Box_g w) + V_0 w - (1/2)S(V_1) - (1/2)V_1 \nabla g r + \Phi' w^2. \tag{4.6}
\]

On one hand, integrating the divergence of \( K \) yields boundary integrals; the following special case of this will suffice.

**Lemma 4.2.** Let \( v \in C^\infty(M) \) be given by \( v = e^{-iz/ht} u \), where \( u \) is stationary and \( z \in \mathbb{R} \). Then,
\[
\int_X (\text{div}_g K) A dS_X = -|z/h|^2 \int_{\partial X} |u|^2 dS_{\partial X}. \tag{4.8}
\]

**Proof.** Apply the divergence theorem (4.2). Since \( z \in \mathbb{R} \), the vector field \( K \) is stationary, and hence there is no contribution from the time derivative. As for the integral over \( \partial M \), observe that \( T \) is null and \( S = -T \) on the horizon. Since \( Tv = -i(z/h)v \), it follows that \( g(T, K) = -|Tv|^2 = -|z/h|^2|u|^2 \) on \( \partial M \).

Note that the boundary contribution from Lemma 4.2 has an unfavorable sign, which will account for the boundary term in Proposition 4.1. On the other hand, the divergence of \( K \) can also be expressed in terms of (4.5).

**Lemma 4.3.** If \( \Phi' < 0 \), then the divergence of \( K \) satisfies
\[
(2|\Phi'|)^{-1}|L_\phi v|^2 \geq \text{div}_g K + \Pi(dv, d\bar{v}) - \Phi'|Sv|^2 + V|v|^2, \tag{4.7}
\]
where \( V \) is given by (4.6).

**Proof.** Combine (4.5) with (4.4), and then use the Cauchy–Schwarz inequality to find
\[
\text{Re} \left( L_\phi v \cdot (S\bar{v} + w\bar{v}) \right) \leq (2|\Phi'|)^{-1}|L_\phi v|^2 - \Phi' \left( |Sv|^2 + w^2|v|^2 \right),
\]
recalling that \( \Phi' < 0 \).

### 4.5. Pseudoconvexity

To examine positivity properties of \( \Pi(dv, d\bar{v}) - \Phi'|Sv|^2 \), we establish a certain pseudoconvexity condition. A criterion of this type first appeared in work of Alinhac on unique continuation [Ali], and was also employed in [IK, AS]. Recall that the Poisson bracket is related to the Hessian via the formula
\[
\{G, \{G, f\}\}(\varphi) = 4\varphi_\alpha \varphi_\beta \nabla^{\alpha\beta} f, \tag{4.8}
\]
valid for any \( f \in C^\infty(M) \).

**Lemma 4.4.** There exists \( M, c, R_0 > 0 \), and a function \( \lambda = \lambda(r) \) such that
\[
M\{G, r\}^2 - \{G, \{G, r\}\} - 4\lambda G \geq c \left( (rp)^2 + \tau^2 + k^{AB} \eta_A \eta_B \right) \tag{4.9}
\]
for \( r \in [0, R_0] \).
Proof. Throughout, assume that $M \geq 1$. Let $r \leq (4M)^{-1}$, and define the function $\lambda$ by

$$\lambda = (1/2) - (1 - \delta) r M,$$

where $\delta > 0$ will be chosen sufficiently small. Observe that $1/4 \leq \lambda \leq 1/2$ uniformly in $M \geq 1$ for $r \leq (4M)^{-1}$. Denote the left-hand side of (4.9) by $4 \mathcal{E}$, and the corresponding quantity by $4 \mathcal{E}_0$ if $G$ is replaced with $G_0$. Dividing through by four,

$$\mathcal{E}_0 = M((r \rho)^2 + 2 r \rho \tau + \tau^2) - (1/2)(r \rho^2 + 2 r \tau) - \lambda G_0.$$  

Use the expression for $\lambda G_0$ and the lower bound $\lambda \geq 1/4$ on $\{r \leq (4M)^{-1}\}$ to find that

$$\mathcal{E}_0 \geq M \delta((r \rho)^2 + 2 r \rho \tau) + M \tau^2 + (1/4) k_{AB}^0 \eta_A \eta_B.$$  

Therefore $\mathcal{E}_0 \geq c(M Y + k_{AB}^0 \eta_A \eta_B)$ if $\delta > 0$ is sufficiently small, where recall $Y = (r \rho)^2 + \tau^2$.

Now consider the error $\mathcal{E} - \mathcal{E}_0$ incurred by replacing $G$ with $G_0$. Replacing $M\{G, r\}^2$ with $M\{G_0, r\}^2$ produces an error

$$2M\{G_0, r\}\{G - G_0, r\} + M\{G - G_0, r\}^2.$$  

Using Cauchy–Schwarz on the first term to absorb a small multiple of $M\{G_0, r\}^2$ into $\mathcal{E}_0$ (in other words, changing the constant $c > 0$ in the lower bound for $\mathcal{E}_0$ above) leaves an overall error of the form

$$M (\mathcal{N}_1 \cdot \mathcal{N}_1) \subset (r^2 M) \mathcal{N}_2.$$  

The factor of $r^2 M$ is harmless since $r^2 M \leq (16M)^{-1}$, thus the right-hand side is certainly in $\mathcal{N}_2$ uniformly in $M \geq 1$. Using that $\lambda$ is uniformly bounded in $M \geq 1$ on $\{r \leq (4M)^{-1}\}$, the remaining errors $\lambda(G - G_0)$ and

$$\{G - G_0, \{G - G_0, r\}\} + \{G - G_0, \{G_0, r\}\} + \{G_0, \{G - G_0, r\}\}$$

are also in $\mathcal{N}_2$ by Lemma 2.5, uniformly in $M \geq 1$. Now apply the first bound in Lemma 2.4, choosing $\gamma > 0$ sufficiently small but independent of $M$ so that $\gamma k^{AB} \eta_A \eta_B$ can be absorbed by $c k^{AB} \eta_A \eta_B$ on the right-hand side for $r \in [0, r_\gamma]$. This leaves a large multiple of $Y$, which is then absorbed by $MY$ on the right-hand side by taking $M$ sufficiently large. It then suffices to take $R_0 = \min\{(4M)^{-1}, r_\gamma\}$.

Fix $M > 0$ such that Lemma 4.4 is valid. This fixes the function $\lambda$, and therefore the function $w$ in (4.3). Lemma 4.3 will be applied with the weight $\Phi = \varphi_i / h$, viewed as a stationary function on $M$. In particular, $\Phi' = -C / h$ on $\{r \leq r_1\}$ for some constant $C > 0$ (recall the statement of Proposition 3.1).

Before proceeding, consider the potential term $V$ from Lemma 4.3. Instead of analyzing its positivity properties, we more simply note that for $F' = -C / h$ one has

$$V = f_0 + h^{-1} f_1 + h^{-2} f_2,$$  

(4.11)
where \( f_0, f_1 \in \mathcal{C}^\infty(M) \) and \( f_2 \in r\mathcal{C}^\infty(M) \). The small coefficient of \( f_2 \) means \( V \) can be treated as an error. To be precise, we have the following positivity result for the bulk terms.

**Lemma 4.5.** Given \( a > 0 \), there exists \( c, r_0 \geq 0 \) such that if \( |z| \geq a \), then

\[
\Pi(dv, d\bar{v}) - \Phi'|Su|^2 + V|v|^2 \geq c \left( h^{-2}|u|^2 + |r\partial_r u|^2 + k^{AB}\partial_A u \partial_B \bar{u} \right)
\]

(4.12) on \( \{r \leq r_0 \} \) for each \( v \in \mathcal{C}^\infty(M) \) of the form \( v = e^{-izt/h}u \), where \( u \) is stationary.

**Proof.** Since \( \Phi' = -C/h \), an inequality of the form (4.12) is true for sufficiently small \( h > 0 \) if the term \( V|v|^2 \) is dropped from the left-hand side; this follows from Lemma 4.4 and (4.8). On the other hand, for a potential \( V \) satisfying (4.11), there is clearly \( r_0 > 0 \) such that \( \mathcal{V}|v|^2 \) can be absorbed by \( ch^{-2}|v|^2 \) for \( r \in [0, r_0] \) and \( h > 0 \) sufficiently small. \( \square \)

The proof of Proposition 4.1 is now immediate:

**Proof of Proposition 4.1.** Given \([a, b] \subset (0, \infty)\), apply Lemmas 4.2, 4.3, 4.5 to functions of the form \( v = e^{-izt/h}e^{\varphi/h}u \), where \( \pm z \in [a, b] \) and \( \text{supp } u \subset \{r < r_0\} \). \( \square \)

5. **Proof of Theorem 1**

We prove the equivalent Theorem 1’. Assume that \([a, b] \subset (0, \infty)\) has been fixed. Choose a cutoff function \( \chi \in \mathcal{C}^\infty(X) \) such that

\[
\text{supp } \chi \subset \{r < r_0\}, \quad \chi = 1 \text{ near } \{r \leq r_0/2\}.
\]

Then, apply Proposition 4.1 to \( \chi u \) and Proposition 3.1 to \((1-\chi)u\), where \( u \in \mathcal{C}^\infty(X) \). Since the commutator \([P(z), \chi]\) is supported away from \( \partial X \), the error terms can be absorbed even though the left-hand side is only estimated in the \( H_{b,h}^1 \) norm. Bounding \( e^{\varphi_1/h} + e^{\varphi_2/h} \) from below on the left and from above on the right yields

\[
\|u\|_{H_{b,h}^1(X)} \leq e^{C/h} \left( \|P(z)\|_{L^2(X)} + \|u\|_{L^2(\partial X)} \right)
\]

(5.1) for \( u \in \mathcal{C}^\infty(X) \) and \( \pm z \in [a, b] \). In order to complete the proof of Theorem 1’ the \( H_{b,h}^1 \) needs to be replaced with the \( H_{b,h}^1 \) norm, and the boundary term on the right-hand side must be removed.

The full \( H_{b,h}^1 \) norm can be recovered on the left-hand side using a redshift argument; an elegant and general version is detailed in [War, Section 3.1]. To avoid a lengthy discussion we just cite the relevant conclusion.

**Lemma 5.1 ([War, Theorem 3.8]).** Given \([a, b] \subset (0, \infty)\), there exists \( C > 0 \) such that

\[
\|u\|_{H_{b,h}^1(X)} \leq C \left( \|u\|_{H_{b,h}^1(X)} + h^{-1}\|P(z)u\|_{L^2(X)} \right)
\]

for \( u \in \mathcal{C}^\infty(X) \) and \( \pm z \in [a, b] \).
In fact, Lemma 5.1 is true even if the $H^1_h$ norm on the right-hand side is replaced with the $L^2$ norm. In either case, this clearly allows one to replace the norm on the left-hand side of (5.1) with the $H^1_h$ norm.

In order to estimate the boundary term, we use the hypotheses (B), (C) on $L$. Since $L - \Box_g$ is formally self-adjoint, we conclude that $\mathcal{V} = \bar{\mathcal{V}}$. Applying the divergence theorem 4.2 to the vector field $\bar{v} \nabla_g v - v \nabla_g \bar{v} + |v|^2 \cdot \mathcal{W}$ with $v = e^{-izt/h}u$ then yields Green’s formula

\[
(hz) \int_{\partial X} |u|^2 \, dS_{\partial X} = -\text{Im} \int_X P(z) u \cdot \bar{A} \, dS_X.
\]

There is no boundary contribution coming from $\mathcal{W}$ since we assumed $g(T, W)$ vanishes on $\partial M$. Applying Cauchy–Schwarz to the right-hand side implies that

\[
e^{C/h} \|u\|_{L^2(\partial X)} \leq C'h^{-1/2}e^{2C/h}\|P(z)\|_{L^2(X)} + (1/2)\|u\|_{L^2(X)}
\]

for $\pm z \in [a,b]$. Therefore the boundary term on the right-hand side of (5.1) can be absorbed into the left-hand side at the expense of increasing the constant in the exponent $e^{C/h}$.

This completes the proof of Theorem 1’ in the case when $u \in C^\infty(X)$ and $\pm z \in [a,b]$. By perturbation, this extends to a region $\pm z \in [a,b] + i e^{-C_1/h}[-1,1]$. Simply write

\[
P(z) - P(\text{Re} z) = \text{Im} z \cdot B(z),
\]

where $B(z) \in \text{Diff}_1^1(X)$ is bounded $H^1_h(X) \to L^2(X)$ uniformly for $z \in [a,b]$ (although $B(z)$ is not holomorphic in $z$). Thus the difference can be absorbed into the left-hand side if $|\text{Im} z| \leq e^{-C_1/h}$ for $C_1 > 0$ sufficiently large. Finally, $C^\infty(X)$ is dense in $\mathcal{X}$ (cf. [DZ, Lemma E.47]), so (2.2) is valid for $u \in \mathcal{X}$ as well, thus completing the proof of Theorem 1’.

6. LOGARITHMIC ENERGY DECAY

In this section we outline how Corollary 1 can be deduced from the resolvent estimate (1.2) via semigroup theory. The starting point is that the Cauchy problem (1.3) is associated with a bounded $C^0$-semigroup $e^{-itB}$ on the space $\mathcal{H} = H^1(X) \times L^2(X)$. Since we are interested in solutions to the homogeneous equation $Lv = 0$, there is no loss in multiplying through by the lapse function squared $A^2 > 0$ and assuming that $L$ is of the form

\[
L = L_2 + L_1 \partial_t + \partial_t^2,
\]

where $L_j$ is identified with a differential operator on $X$ of order $j$. The infinitesimal generator is then given by

\[
-iB = \begin{pmatrix} 0 & 1 \\ -L_2 & -L_1 \end{pmatrix}.
\]
Indeed, applying $e^{-itB}$ to initial data in $C_0^\infty(X^o)$ shows that $-iB$ is given by (6.1) in the sense of distributions. Therefore the domain $D(B)$ of $B$ is characterized as those distributions $(v_0, v_1) \in \mathcal{H}$ such that
\[ v_1 \in H^1(X), \quad L_2 v_0 + L_1 v_1 \in L^2(X). \]
Since $L_2$ is proportional to $P(0)$ and $L_1 \in \text{Diff}^1(X)$, this shows that the domain of $B$ is
\[ D(B) = \mathcal{X} \times H^1(X). \]
Furthermore, the spectrum of $B$ in \{Im $\omega > -\kappa/2$\} coincides with poles of $P(\omega)^{-1}$ [War, Lemma 4.2], and the resolvent estimate (1.2) translates into the operator norm bound
\[ \|(B - \omega)^{-1}\|_{\mathcal{H} \to \mathcal{H}} \leq e^{C|\text{Re}\omega|} \]
for $\omega \in \Omega$. The hypotheses of [Bur1, Theorem 3] or [BD, Theorem 1.5] on the logarithmic stabilization of bounded semigroups are therefore satisfied, which yields the bound
\[ \|e^{-itB}(v_0, v_1)\|_{\mathcal{H}} \leq \frac{C}{\log(2 + t)} \|(B - i)(v_0, v_1)\|_{\mathcal{H}} \]
for each $(v_0, v_1) \in \mathcal{X} \times H^1$. Corollary 1 follows since $\|\cdot\|_{\mathcal{X} \times H^1(X)}$ is equivalent to the graph norm on $D(B)$ by the open mapping theorem.

**Acknowledgments**

I would like to thank Georgios Moschidis for carefully explaining parts of his paper [Mos], and Daniel Tataru for a discussion on Carleman estimates.

**References**

[Ali] Serge Alinhac. Unicité du problème de Cauchy pour des opérateurs du second ordre à symboles réels. 34(2):89–109, 1984.

[AS] Spyros Alexakis and Arick Shao. Global uniqueness theorems for linear and nonlinear waves. *Journal of Functional Analysis*, 209(11):3458–3499, 2015.

[BD] Charles Batty and Thomas Duyckaerts. Non-uniform stability for bounded semi-groups on Banach spaces. *Journal of Evolution Equations*, 8(4):765–780, 2008.

[Bur1] Nicolas Burq. Décroissance de l’énergie locale de l’équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta mathematica*, 180(1):1–29, 1998.

[Bur2] Nicolas Burq. Lower bounds for shape resonances widths of long range Schrödinger operators. *American journal of mathematics*, 124(4):677–735, 2002.

[BZ] Antônio Sá Barreto and Maciej Zworski. Distribution of resonances for spherical black holes. *Mathematical Research Letters*, 4:103–122, 1997.

[CV] Fernando Cardoso and Georgi Vodev. Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds. II. *Annales Henri Poincaré*, 3(4):673–691, 2002.

[Dat] Kiril Datchev. Quantitative limiting absorption principle in the semiclassical limit. *Geometric and Functional Analysis*, 3(24):740–747, 2014.
[DZ] Semyon Dyatlov and Maciej Zworski. Mathematical theory of scattering resonances. \texttt{math.mit.edu/~dyatlov/res/}.

[GT] David Gilbarg and Neil S. Trudinger. \textit{Elliptic Partial Differential Equations of Second Order}. Classics in Mathematics. Springer Berlin Heidelberg, 2015.

[Gui] Colin Guillarmou. Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds. \textit{Duke Mathematical Journal}, 129(1):1–37, 2005.

[Hör] Lars Hörmander. \textit{Linear partial differential operators}, volume 116. Springer, 2013.

[IK] Alexandru D Ionescu and Sergiu Klainerman. On the uniqueness of smooth, stationary black holes in vacuum. \textit{Inventiones mathematicae}, 175(1):35–102, 2009.

[Mel] Richard Melrose. \textit{The Atiyah-Patodi-Singer Index Theorem}. Research Notes in Mathematics. Taylor & Francis, 1993.

[Mos] Georgios Moschidis. Logarithmic local energy decay for scalar waves on a general class of asymptotically flat spacetimes. \textit{Annals of PDE}, 2(1):1–124, 2016.

[RT] Igor Rodnianski and Terence Tao. Effective limiting absorption principles, and applications. \textit{Communications in Mathematical Physics}, 333(1):1–95, 2015.

[Vas] András Vasy. Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov). \textit{Inventiones mathematicae}, 194(2):381–513, 2013.

[Vod] Georgi Vodev. Exponential bounds of the resolvent for a class of noncompactly supported perturbations of the Laplacian. \textit{Mathematical research letters}, 7(2):287–298, 2000.

[War] Claude M Warnick. On quasinormal modes of asymptotically anti-de Sitter black holes. \textit{Communications in Mathematical Physics}, 2(333):959–1035, 2015.

[Zwo1] Maciej Zworski. \textit{Semiclassical Analysis}. Graduate studies in mathematics. American Mathematical Society, 2012.

[Zwo2] Maciej Zworski. Mathematical study of scattering resonances. \textit{Bulletin of Mathematical Sciences}, 7(1):1–85, 2017.

\textit{E-mail address}: \texttt{gannot@northwestern.edu}

\textbf{Department of Mathematics, Lunt Hall, Northwestern University, Evanston, CA 60208, USA}