Confidence Intervals of Treatment Effects in Panel Data Models with Interactive Fixed Effects*

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Abstract

We consider the construction of confidence intervals for treatment effects estimated using panel models with interactive fixed effects. We first use the factor-based matrix completion technique proposed by Bai and Ng (2021) to estimate the treatment effects, and then use bootstrap method to construct confidence intervals of the treatment effects for treated units at each post-treatment period. Our construction of confidence intervals requires neither specific distributional assumptions on the error terms nor large number of post-treatment periods. We also establish the validity of the proposed bootstrap procedure that these confidence intervals have asymptotically correct coverage probabilities. Simulation studies show that these confidence intervals have satisfactory finite sample performances, and empirical applications using classical datasets yield treatment effect estimates of similar magnitudes and reliable confidence intervals.

Keywords: Bootstrap, Confidence interval, Treatment effects, Panel data analysis, Interactive effects, Matrix completion

JEL classification: C01, C21, C31, I18

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1 Introduction

Treatment effects of certain policy interventions on economic entities are often of major interest in economic and econometric studies. In panel data models, estimation of and inference on treatment effects can be formulated as missing data problems. For instance, consider a sample of $N$ units with a policy intervention on units $N_0 + 1, \ldots, N$ at time $T_0$, so that the pre-treatment periods are $1, \ldots, T_0$, the post-treatment periods are $T_0 + 1, \ldots, T$, and the control units are $1, \ldots, N_0$. Let $y_{i,t}$ be the potential outcome in the absence of policy intervention for unit $i$ at time $t$, and $y^+_{i,t} = y_{i,t} + \Delta_{i,t}$ be the potential outcome under policy intervention for unit $i$ at time $t$. We are interested in the treatment effect $\Delta_{i,t}$ for $(i, t) \in I_1 = \{N_0 + 1, \ldots, N\} \times \{T_0 + 1, \ldots, T\}$, and it is easy to see that calculating $\Delta_{i,t} = y^+_{i,t} - y_{i,t}$ requires knowing $y_{i,t}$, which is actually unobservable (missing) for $(i, t) \in I_1$.

Many existing methods of estimating $\Delta_{i,t}$ make efforts to construct “counterfactuals” for $y_{i,t}$, $(i, t) \in I_1$ (e.g., Abadie et al., 2010; Hsiao et al., 2012; Bai and Ng, 2021; Athey et al., 2021; among others). The underlying models in counterfactual frameworks can be simplified as $y_{i,t} = \xi_{i,t} + e_{i,t}$, where $\xi_{i,t}$ is the systematic part and $e_{i,t}$ is the idiosyncratic error. Then the predicted values $\{\hat{\xi}_{i,t} : (i, t) \in I_1\}$ serve the role as the counterfactuals for $\{y_{i,t} : (i, t) \in I_1\}$, and the treatment effects are estimated by $\hat{\Delta}_{i,t} = y^+_{i,t} - \hat{\xi}_{i,t}$ for every $(i, t) \in I_1$. According to model specifications, assumptions, and estimation strategies, we can roughly classify the existing methods into the following 4 categories.

The first category is the synthetic control method proposed by Abadie and Gardeazabal (2003) and Abadie et al. (2010). It approximates $y_{N,t}$ by a convex combination of $(y_{1,t}, \ldots, y_{N-1,t})$, which can be obtained from a constrained regression. Generalisations of the synthetic control method can be found in, for example, Amjad et al. (2018), Abadie and L’Hour (2021), Arkhangelsky et al. (2021), Ben-Michael et al. (2021a), Ben-Michael et al. (2021b), Kellogg et al. (2021), and Masini and Medeiros (2021). One can see Abadie (2021) for a comprehensive review. The second category is the panel data approach proposed by Hsiao et al. (2012), which constructs the counterfactual by solving a linear regression of $y_{N,t}$ on $(y_{1,t}, \ldots, y_{N-1,t})$. This approach is later extended by Ouyang and Peng (2015), Li and Bell (2017), and Hsiao et al. (2022). Methods in the third category use the factor model technique (e.g., Bai and Ng, 2002; Bai, 2003; Pesaran, 2006; Bai, 2009; Moon and Weidner, 2015) to construct counterfactuals. Examples include Kim and Oka (2014), Xu (2017), and Bai and Ng (2021). The last category starts from the statistical learning literature on matrix completion (e.g., Candès and Recht, 2009; Candès and Plan, 2010; Mazumder et al., 2010; Gamarnik and Misra, 2016), and a recent example is Athey et al. (2021), which completes the potential outcome matrix via nuclear norm regularisation.

In practice, a mere point estimator of the treatment effect $\Delta_{i,t}$ is not sufficient either theoretically or empirically, and a valid inference is needed, which usually requires the (asymptotic) distribution of $\left(\hat{\Delta}_{i,t} - \Delta_{i,t}\right)$. As $\hat{\Delta}_{i,t} - \Delta_{i,t} = \left(\xi_{i,t} - \hat{\xi}_{i,t}\right) + e_{i,t}$, the idiosyncratic error term $e_{i,t}$ will dominate
the distribution of \( \hat{\Delta}_{i,t} - \Delta_{i,t} \) as \( N_0, T_0 \to \infty \) whenever the systematic part can be consistently approximated. Due to this issue, most studies make inferences on time series average treatment effects as the number of post-treatment periods \( (T - T_0) \to \infty \) (e.g., Hsiao et al., 2012; Li and Bell, 2017; Li, 2020, and Chernozhukov et al., 2021b), or make inferences on cross-sectional average treatment effects as the number of treated units \( (N - N_0) \to \infty \) (e.g., Xu, 2017; Arkhangelsky et al., 2021; Ben-Michael et al., 2021b; Bai and Ng, 2021), or assume the normality of error terms (e.g., Fujiki and Hsiao, 2015; Bai and Ng, 2021). However, these approaches can not simultaneously meet the following needs in empirical studies.

1. Inferences do not rely on any specific distributional assumptions (e.g., normality) on the error terms.
2. Inference are built on small number of post-treatment period, either because the observed post-treatment time span is short or in order to avoid confounding effects from other interventions in longer time span.
3. Inferences are performed not only for the (cross-sectional or time series) average treatment effect, but also for the treatment effect on every treated unit at each post-treatment period.

To the best of our knowledge, there are only few inferential studies simultaneously satisfy the three conditions above. Abadie et al. (2010) use a cross-sectional permutation strategy, where they apply the synthetic control method to every potential control unit to approximate the true distribution of treatment effect estimator under the null hypothesis of \( \Delta_{N,t} = 0 \). Obviously, the validity of this permutation approach relies on the cross-sectional exchangeability of the units. Furthermore, if one wants to construct the confidence intervals of the treatment effects in this fashion, then the equality between distributions of outcomes in the absence of and under intervention up to a location shift is required. Chernozhukov et al. (2021a) apply time series permutation test to the inference on treatment effects. For a given null hypothesis \( H_0 : \Delta_N = \delta_N \), they plug \( \delta_N \) into the observed data matrix to compute the full outcome matrix in the absence of intervention, and then perform time series permutations of the residuals from the treated unit to obtain an approximation of the distribution of the test statistic under \( H_0 \) and the critical value. Confidence intervals can be obtained by grid searching on a set of different values of \( \delta_N \), which may lead to intensive computation. As the latest work, Cattaneo et al. (2021) build prediction intervals for synthetic control methods on conditional prediction intervals and non-asymptotic concentration. Since concentration inequalities only guarantee lower bounds of coverage probabilities, the proposed prediction intervals may suffer from conservativeness.

This paper aims to provide a theoretically non-conservative and computationally simple inferential procedure for the estimated treatment effects that simultaneously satisfy the above conditions, without imposing the cross-sectional exchangeability assumption. To be more specific, we apply bootstrap method to the treatment effect estimators proposed by Bai and Ng (2021). Note that Bai and Ng
(2021)’s model keeps a simple factor structure but is quite general in the sense that it allows for multiple treated units and heterogeneous time of intervention. Bootstrapping is one of most commonly used tools to approximate an unknown distribution via resampling, and is shown to be asymptotically valid under mild conditions. We construct bootstrap-based confidence intervals of treatment effects in panel data models with interactive fixed effects. Specifically, we first apply the factor-based matrix completion technique proposed by Bai and Ng (2021) to obtain point estimators of treatment effects. Then we follow Gonçalves et al. (2017) to approximate the distribution of \( \hat{\Delta}_{i,t} - \Delta_{i,t} \) by (block) wild bootstrap and bootstrap, and use the quantiles of bootstrapped distribution to construct confidence intervals of \( \Delta_{i,t} \) for every \((i, t) \in \mathcal{I}_1\).

Our paper contributes to the literature in the following folds. First, we establish the validity of confidence intervals by proving that they have asymptotically correct coverage probabilities as \((N_0, T_0) \to \infty\), and our proposed confidence intervals do not require any specific distributional assumption on \( e_{i,t} \) and are robust to small number of post-treatment periods. Thus, it is theoretically non-conservative and can meet the three needs for inferential purpose of treatment effects estimation in empirical studies. Second, we extend the bootstrap procedure in Gonçalves et al. (2017) to a more general case. Since Gonçalves et al. (2017) intend to make inference on factor-augmented regression models (Bai and Ng, 2006) where the factors serves as intermediate variables, they only require valid bootstrap approximations of factors. In this paper, we intend to make inference on the factor model (with missing values) itself, and the estimators involve products of estimated factors, their associated loadings, and a rotation matrix. This implies that we need valid bootstrap approximations of factors, loadings, and the rotation matrix, which creates theoretical challenges. Nevertheless, we successfully establish the asymptotic validity of our proposed bootstrap for such a scenario.

The finite sample properties of proposed bootstrap construction of confidence intervals for estimated treatment effects are investigated through Monte Carlo simulation, using different data generating processes (DGPs) for models with or without exogenous covariates, and for models with heteroscedastic errors or serially correlated errors. The simulation results show that the proposed bootstrap procedure works remarkably well for constructing confidence intervals for estimated treatments. Namely, the empirical coverage ratios are quite close to nominal values in all cases we considered regardless of whether the number of unobserved factors are priorly given or estimated from data.

Finally, we revisit the benefits of political and economic integration of Hong Kong with Mainland China in Hsiao et al. (2012), as well as the effectiveness of the California Tobacco Control Program (CTCP) on per capita cigarettes consumption and personal health expenditures in Abadie et al. (2010). We note that the treatment effects estimated using our new method are of similar magnitudes to those in the literature, while our bootstrap procedure provides reliable confidence intervals showing that the impacts of CTCP were significant over time for both per capita cigarettes consumption and personal health expenditures, and that the impacts of political and economic integration of Hong Kong with
Mainland China vanished over time, although significant in the first few years.

The rest of this article is organised as follows. Section 2 establishes the model and the assumptions. Section 3 focuses on the estimation and inference in a model without covariates, and Section 4 deals with a model with covariates. We conduct Monte Carlo simulations in Section 5, and apply our method to classical datasets in Section 6. Section 7 concludes. Mathematical proofs of main results are left to online appendices.

**Notations:** we introduce some notations that are frequently used throughout this paper. \( \mathbb{Z}_+ = \{1, 2, \ldots\} \) is the set of positive integers. \( I_r \) is the \( r \times r \) identity matrix. \( A^\top \) denotes the transpose of matrix \( A \). For a vector \( z \), let \( \| z \| = \sqrt{z^\top z} \) be the Euclidean norm of \( z \). For a matrix \( A \), let \( \| A \| = \sqrt{\text{tr} (A^\top A)} \) be the Frobenius norm of \( A \). \( \mathcal{N} (\mu, \Sigma) \) stands for a normal distribution with mean \( \mu \) and variance \( \Sigma \). And \( \xrightarrow{a.s.} \), \( \xrightarrow{P} \) and \( \xrightarrow{d} \) denote almost sure convergence, convergence in probability and convergence in distribution, respectively. We let \( M \) denote a generic finite positive constant, whose value does not depend on \( N \) or \( T \), and may vary case by case.

### 2 The Model

Suppose there are observations \((y_{i,t}, x_{i,t})\) for \( i = 1, \ldots, N \) and \( t = 1, \ldots, T \). Let the dummy variable \( d_{i,t} \) indicate the \( i \)-th unit’s treatment status at time \( t \) with \( d_{i,t} = 1 \) if under the treatment and \( d_{i,t} = 0 \) if not. The observed data takes the form

\[
Y_{i,t} = y_{i,t} + d_{i,t} \Delta_{i,t}, \tag{1}
\]

where \( y_{i,t} \) is the latent outcome of unit \( i \) at time \( t \) in the absence of treatment, and \( \Delta_{i,t} \) is the treatment effect on unit \( i \) at time \( t \). For the ease of exposition, we assume \( d_{i,t} = 1 \) for all \((i, t)\in I_1\) and \( d_{i,t} = 0 \) for all \((i, t)\in \mathcal{I}\setminus I_1\), where

\[
\mathcal{I} = \{1, \ldots, N\} \times \{1, \ldots, T\}, \quad I_1 = \{N_0 + 1, \ldots, N\} \times \{T_0 + 1, \ldots, T\}
\]

with \( 1 \leq N_0 < N \) and \( 1 \leq T_0 < T \). That is, we assume the last \( N - N_0 \) units are intervened by the treatment at time \( T_0 \). Therefore, the observed matrix of outcomes is

\[
\begin{bmatrix}
y_{1,1} & \cdots & y_{N,1} & y_{N_0+1,1} & \cdots & y_{N,1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
y_{1,T_0} & \cdots & y_{N_0,T_0} & y_{N_0+1,T_0} & \cdots & y_{N,T_0} \\
y_{1,T_0+1} & \cdots & y_{N_0,T_0+1} & y_{N_0+1,T_0+1} + \Delta_{N_0+1,T_0+1} & \cdots & y_{N,T_0+1} + \Delta_{N,T_0+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
y_{1,T} & \cdots & y_{N_0,T} & y_{N_0+1,T} + \Delta_{N_0+1,T} & \cdots & y_{N,T} + \Delta_{N,T}
\end{bmatrix}
\]

Note that the above specification can be generated to allow for heterogeneous intervention time periods by letting \( T_0 \) be the time of the earliest intervention. This is consistent with the definition of \( I_1 \) in Bai and Ng (2021) that \( I_1 \) corresponds to the “largest possible” missing block of the matrix.

We are interested in measuring the treatment effects of the policy intervention for the treated units.
after time $T_0$, which are $\Delta_{i,t}$ for $(i, t) \in I_1$. Note that $\Delta_{i,t}$ measures the difference of the outcomes with and without the intervention of the treatment in the post-treatment periods. Unfortunately, the outcomes with and without the intervention of treatment cannot be simultaneously observed in reality for the same unit at a given time. This is because, once the policy intervention is in effect, then the researchers can only observe $Y_{i,t} = y_{i,t} + \Delta_{i,t}$, not $y_{i,t}$. Thus, in order to estimate $\Delta_{i,t}$, we need to generate the counterfactual of $y_{i,t}$ for $(i, t) \in I_1$, denoted as $\hat{y}_{i,t}$, and thus the treatment effect can be estimated as

$$\hat{\Delta}_{i,t} = Y_{i,t} - \hat{y}_{i,t}, \quad (i, t) \in I_1. \quad [2]$$

Our goal in this paper is to estimate $\Delta_{i,t}$ for $(i, t) \in I_1$ when neither treated units $N_1 = N - N_0$ nor the post treated periods $T_1 = T - T_0$ is large, where the former can be formulated as average treatment effects across units (see Hsiao et al., 2022 for the aggregation of multiple treatment effects), and the latter can be formulated as the average treatment effects across post-treatment periods (see Fujiki and Hsiao, 2015 and Li and Bell, 2017).

Following the literature of treatment effects estimation using panels (e.g., Abadie et al., 2010; Hsiao et al., 2012; and Bai and Ng, 2021), we assume that $y_{i,t}$ is generated by the following panel data model with interactive fixed effects:

$$y_{i,t} = x_{i,t}^T \beta + c_{i,t} + e_{i,t}, \quad c_{i,t} = f_t^T \lambda_i, \quad [3]$$

where $x_{i,t}$ is a $p$-dimensional vector of observed covariates of unit $i$ at time $t$, $e_{i,t}$ is the idiosyncratic error term of unit $i$ at time $t$, $f_t$ is an $r$-dimensional time-variant unobserved factor at time $t$, and $\lambda_i$ is an $r$-dimensional individual specific factor loading of unit $i$, where $r$ is the number of unobserved factors and is usually unknown to researchers.\(^1\)

Let $y_i = (y_{i,1}, \ldots, y_{i,T})^T$, $X_i = (x_{i,1}, \ldots, x_{i,T})^T$, and $e_i = (e_{i,1}, \ldots, e_{i,T})^T$ for every $i \in \{1, \ldots, N\}$. Define $Y = (y_1, \ldots, y_N)$, $F = (f_1, \ldots, f_T)^T$, and $\Lambda = (\lambda_1, \ldots, \lambda_N)^T$. We make the assumptions below.

**Assumption 2.1:** The factors and loadings satisfy the following conditions.

1. $\mathbb{E}\left(\|f_t\|^8\right) \leq M$ for all $t \in \{1, \ldots, T\}$.
2. $\|\lambda_i\| \leq M$ for all $i \in \{1, \ldots, N\}$.
3. There exists an $r \times r$ positive definite matrix $\Sigma_F$, so that

$$\frac{1}{T} \sum_{t=1}^T f_t f_t^T \xrightarrow{p} \Sigma_F, \quad \frac{1}{T_0} \sum_{t=1}^{T_0} f_t f_t^T \xrightarrow{p} \Sigma_F, \quad \frac{1}{T - T_0} \sum_{t=T_0+1}^T f_t f_t^T \xrightarrow{p} \Sigma_F$$

as $T, T_0 \to \infty$.

\(^1\)Even if $r$ is unknown in practice, it can be consistently estimated using the method described in Section 4 and 5 of Bai and Ng (2002) or the method proposed by Alessi et al. (2010), and thus we can treat $r$ as known in the theoretical analyses below. This can be formally justified as follows. Let $\theta$ be a parameter in the model of interest and $\hat{\theta}$ be the estimator of $\theta$ based on estimated number of factors $\hat{r}$. Note that $r$ takes values in $\mathbb{Z}_+$, and then the consistency of its estimator $\hat{r}$ implies that $\mathbb{P}(\hat{r} = r) \to 1$ as $(N, T) \to \infty$. Therefore,

$$\mathbb{P}\left(\hat{\theta} \leq x\right) = \mathbb{P}\left(\hat{\theta} \leq x, \hat{r} = r\right) + \mathbb{P}\left(\hat{\theta} \leq x, \hat{r} \neq r\right) = \mathbb{P}\left(\hat{\theta} \leq x | \hat{r} = r\right) \mathbb{P}(\hat{r} = r) + o(1) = \mathbb{P}\left(\hat{\theta} \leq x | \hat{r} = r\right) + o(1).$$
There exists an $r \times r$ positive definite matrix $\Sigma$, so that

$$\frac{1}{N} \sum_{i=1}^{N} \lambda_i \lambda_i^T \to \Sigma,$$

$$\frac{1}{N_0} \sum_{i=N_0+1}^{N} \lambda_i \lambda_i^T \to \Sigma,$$

$$\frac{1}{N-N_0} \sum_{i=N_0+1}^{N} \lambda_i \lambda_i^T \to \Sigma$$

as $N, N_0 \to \infty$.

(5) The eigenvalues of $\Sigma F \Sigma$ are distinct.

**Assumption 2.2:** The distributions of the idiosyncratic errors have the following properties.

1. The idiosyncratic errors have no cross-sectional dependence, i.e., the sequences $\{e_{1,t}\}_{t=1}^{T}$, $\{e_{2,t}\}_{t=1}^{T}$, ..., $\{e_{N,t}\}_{t=1}^{T}$ are mutually independent.
2. For all $i \in \{1, \ldots, N\}$, the process $\{e_{i,t}\}_{t=1}^{\infty}$ is strictly stationary and ergodic.
3. For all $(i, t) \in I$, the cumulative distribution function of $e_{i,t}$ is everywhere continuous.
4. The error terms $\{e_{i,t} : (i, t) \in I\}$ are independent of the treatment status $\{d_{i,t} : (i, t) \in I\}$.

**Assumption 2.3:** The moment conditions below hold for the full sample indexed by $I$, the balanced subsample indexed by $\{1, \ldots, N_0\} \times \{1, \ldots, T_0\}$, the control subsample indexed by $\{1, \ldots, N_0\} \times \{1, \ldots, T\}$, the pre-treatment subsample indexed by $\{1, \ldots, N\} \times \{1, \ldots, T_0\}$, and the missing subsample indexed by $I_1$.

1. $\mathbb{E}(e_{i,t}) = 0$ and $\mathbb{E}(e_{i,t}^8) \leq M$ for all $(i, t) \in I$.
2. For all $t \in \{1, \ldots, T\}$,
   $$\sum_{s=1}^{T} \left| \mathbb{E}\left( \frac{1}{N} \sum_{i=1}^{N} e_{i,t} e_{i,s} \right) \right| \leq M.$$
3. $$\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} |\mathbb{E}(e_{i,t} e_{i,s})| \leq M.$$
4. For all $(s, t) \in \{1, \ldots, T\}^2$,
   $$\mathbb{E}\left( \left| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( e_{i,s} e_{i,t} - \mathbb{E}(e_{i,s} e_{i,t}) \right) \right|^4 \right) \leq M.$$
5. $$\mathbb{E}\left( \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} f_t e_{i,t} \right\|^2 \right) \leq M.$$
6. For all $t \in \{1, \ldots, T\}$,
   $$\mathbb{E}\left( \left\| \frac{1}{\sqrt{NT}} \sum_{s=1}^{T} \sum_{i=1}^{N} f_s (e_{i,s} e_{i,t} - \mathbb{E}(e_{i,s} e_{i,t})) \right\|^2 \right) \leq M.$$
7. $$\mathbb{E}\left( \left\| \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \sum_{i=1}^{N} f_t \lambda_i^T e_{i,t} \right\|^2 \right) \leq M.$$

**Assumption 2.4:** The factors, loadings and errors satisfy central limit theorems.
(1) For all \( t \in \{1, \ldots, T\} \), there exists an \( r \times r \) positive definite matrix \( \Gamma_t \), so that
\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_i e_{i,t} \overset{d}{\rightarrow} \mathbb{N}(0, \Gamma_t), \quad \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} \lambda_i e_{i,t} \overset{d}{\rightarrow} \mathbb{N}(0, \Gamma_t)
\]
as \( N, N_0 \to \infty \).

(2) For all \( i \in \{1, \ldots, N\} \), there exists an \( r \times r \) positive definite matrix \( \Phi_i \), so that
\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} f_i e_{i,t} \overset{d}{\rightarrow} \mathbb{N}(0, \Phi_i), \quad \frac{1}{\sqrt{T_0}} \sum_{t=1}^{T_0} f_i e_{i,t} \overset{d}{\rightarrow} \mathbb{N}(0, \Phi_i)
\]
as \( T, T_0 \to \infty \).

**Assumption 2.5:** The quantities \( N, T, N_0, T_0 \) satisfy the conditions below.

(1) \( TN_0 > r (T + N_0) \) and \( T_0 N > r (T_0 + N) \).

(2) \( N, T, N_0, T_0 \) are of the same order, i.e.,
\[
\lim_{N, N_0 \to \infty} \frac{N_0}{N} = c_1 \in (0, 1], \quad \lim_{T, T_0 \to \infty} \frac{T_0}{T} = c_2 \in (0, 1], \quad \lim_{N, T \to \infty} \frac{N}{T} = c_3 \in (0, \infty).
\]

**Assumption 2.6:** The conditions below hold for the full sample indexed by \( \mathcal{I} \), the control subsample indexed by \( \{1, \ldots, N_0\} \times \{1, \ldots, T\} \), and the pre-treatment subsample indexed by \( \{1, \ldots, N\} \times \{1, \ldots, T_0\} \).

(1) \( \mathbb{E} \left( \|x_{i,t}\|^8 \right) \leq M \) for every \( (i, t) \in \mathcal{I} \).

(2) \( \inf \{ \mathcal{D}(\mathcal{F}) : \mathcal{F} \in \mathcal{G} \} > 0 \), where
\[
\mathcal{D}(\mathcal{F}) = \left[ \frac{1}{N T} \sum_{i=1}^{N} X_i^\top \left( I_T - \frac{\mathcal{F} \mathcal{F}^\top}{T} \right) X_i \right] - \left[ \frac{1}{T N^2} \sum_{i=1}^{N} \sum_{k=1}^{N} X_i^\top \left( I_T - \frac{\mathcal{F} \mathcal{F}^\top}{T} \right) X_k a_{i,k} \right],
\]
\[
a_{i,k} = \lambda_i^\top \left( \frac{\lambda_i^\top \Lambda}{N} \right)^{-1} \lambda_k,
\]
\[
\mathcal{G} = \left\{ \mathcal{F} \in \mathbb{R}^{T \times r} : \mathcal{F}^\top \mathcal{F}/T = I_r \right\}.
\]

**Assumption 2.7:** The conditions below hold for the full sample indexed by \( \mathcal{I} \), the control subsample indexed by \( \{1, \ldots, N_0\} \times \{1, \ldots, T\} \), and the pre-treatment subsample indexed by \( \{1, \ldots, N\} \times \{1, \ldots, T_0\} \).

(1)
\[
\frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \max_{1 \leq i \leq N} |\mathbb{E}(e_{i,t} e_{i,s})| \leq M.
\]

(2) \( \|\mathbb{E}(e_{i} e_{i}^\top)\|_S \leq M \) for all \( i \in \{1, \ldots, N\} \), where \( \|\cdot\|_S \) is the spectral norm of a matrix.

(3)
\[
\frac{1}{T^2 N} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{u=1}^{T} \sum_{v=1}^{T} \sum_{i=1}^{N} |\text{Cov}(e_{i,t} e_{i,s}, e_{i,u} e_{i,v})| \leq M.
\]

(4)
\[
\frac{1}{T N^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N} \sum_{j=1}^{N} |\text{Cov}(e_{i,t} e_{j,t}, e_{i,s} e_{j,s})| \leq M.
\]
There exists a $p \times p$ positive definite matrix $\Omega$, so that
\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \left[ \left( I_T - F F^T \right) X_i - \frac{1}{N} \sum_{k=1}^{N} a_{i,k} \left( I_T - F F^T \right) X_k \right]^T e_i \overset{d}{\rightarrow} N(0, \Omega).
\]

**Assumption 2.8:** \(\{e_{i,t} : (i,t) \in I\}\) is independent of \(\{f_t\}_{t=1}^T\) and \(\{x_{i,t} : (i, t) \in I\}\).

The moment and convergence conditions in Assumptions 2.1, 2.3, 2.4, 2.6, and 2.7 are used to establish probability bounds and asymptotic distributions. They are quite standard in the literature for panel models with interactive fixed effects, see, for example, Bai (2003), Bai (2009) and Bai and Ng (2021), among others. The distribution and order conditions in Assumptions 2.2, 2.5, and 2.8 mainly work for the validity of our bootstrap procedure.

We make further remarks on some of the assumptions. Assumption 2.2(4) requires the strict exogeneity of treatment status, and one can find similar conditions in Assumption 5 of Hsiao et al. (2012) and Assumption 2 of Xu (2017). Assumption 2.1(3) states that the factors in the pre-treatment subsample and post-treatment subsample have the same asymptotic second sample moment matrices as those in the full sample. Assumption 2.1(4) states that the factor loadings in the control subsample and treated subsample have the same asymptotic second sample moment matrices as those in the full sample. When either \(T - T_0\) or \(N - N_0\) is finite, Assumption 2.1(3) or 2.1(4) can be replaced with equality as in the identifying restriction PC1 on Page 19 of Bai and Ng (2013) to fulfil the condition that either the factors or factor loadings possess the same properties in the full sample as well as in the post-treatment periods or for the treated units.

### 3 Estimation and Inference in a Model without Covariates

To highlight the essence of our approach for estimation and inference of treatment effects, in this section we focus on a special case of Model [3], where there is no covariate (i.e., \(\beta = 0\)) and the model is reduced to a pure approximate factor model:
\[
y_{i,t} = c_{i,t} + e_{i,t}, \quad c_{i,t} = f_t^T \lambda_i. \tag{4}
\]

We will extend our suggested approach to the full model [3] in Section 4.

#### 3.1 Estimation of Treatment Effects

Following Bai and Ng (2021), we consider the factor-based approach to estimate \(y_{i,t}\) for \((i, t) \in I_1\). Let \(Y, F,\) and \(\Lambda\) follow their definitions in Section 2. Since \(y_{i,t}\) is unobserved for all \((i, t) \in I_1\), the south-east block of \(Y\) is missing in reality. Formally, we define two sub-matrices of \(Y\):
\[
Y_{\text{tall}} = \begin{bmatrix}
y_{1,1} & y_{2,1} & \cdots & y_{N_0,1} \\
y_{1,2} & y_{2,2} & \cdots & y_{N_0,2} \\
\vdots & \vdots & \ddots & \vdots \\
y_{1,T} & y_{2,T} & \cdots & y_{N_0,T}
\end{bmatrix}, \quad Y_{\text{wide}} = \begin{bmatrix}
y_{1,1} & y_{2,1} & \cdots & y_{N,1} \\
y_{1,2} & y_{2,2} & \cdots & y_{N,2} \\
\vdots & \vdots & \ddots & \vdots \\
y_{1,T_0} & y_{2,T_0} & \cdots & y_{N,T_0}
\end{bmatrix}. \tag{5}
\]
That is, $Y_{\text{tall}}$ corresponds to the control subsample, and $Y_{\text{wide}}$ corresponds to the pre-treatment subsample. Let $(F_{\text{tall}}, \Lambda_{\text{tall}})$ and $(F_{\text{wide}}, \Lambda_{\text{wide}})$ be the factor and loading matrices associated with $Y_{\text{tall}}$ and $Y_{\text{wide}}$, respectively. It is easy to see that $F_{\text{tall}} = F$, $\Lambda_{\text{wide}} = \Lambda$, $F_{\text{wide}}$ is a sub-matrix formed by the first $T_0$ rows of $F$, and $\Lambda_{\text{tall}}$ is a sub-matrix formed by the first $N_0$ rows of $\Lambda$.

In lieu of Bai and Ng (2021), we can use the algorithm below to estimate the treatment effects $\Delta_{i,t}$ for all $(i, t) \in \mathcal{I}_1$.

Algorithm 3.1: Treatment effect estimation via factor-based matrix completion (without covariates).

1. Perform a singular value decomposition of $\frac{Y_{\text{tall}}}{\sqrt{T N_0}}$, and let $D_{\text{tall}}$ be an $r \times r$ diagonal matrix with the $r$ largest singular values of $\frac{Y_{\text{tall}}}{\sqrt{T N_0}}$ on the diagonal in descending order. Then let $P_{\text{tall}}$ and $Q_{\text{tall}}$ be $T \times r$ and $N_0 \times r$ matrices containing the left and right singular vectors of $\frac{Y_{\text{tall}}}{\sqrt{T N_0}}$ respectively, corresponding to $D_{\text{tall}}$. Compute
   \[
   \hat{F}_{\text{tall}} = \left( \hat{f}_{\text{tall},1}, \ldots, \hat{f}_{\text{tall},T} \right)^T = \sqrt{T} P_{\text{tall}}, \quad \hat{\Lambda}_{\text{tall}} = \left( \hat{\lambda}_{\text{tall},1}, \ldots, \hat{\lambda}_{\text{tall},N_0} \right)^T = \sqrt{N_0} Q_{\text{tall}} D_{\text{tall}}. \quad [6]
   \]

2. Perform a singular value decomposition of $\frac{Y_{\text{wide}}}{\sqrt{T_0 N}}$, and let $D_{\text{wide}}$ be an $r \times r$ diagonal matrix with the $r$ largest singular values of $\frac{Y_{\text{wide}}}{\sqrt{T_0 N}}$ on the diagonal in descending order. Then let $P_{\text{wide}}$ and $Q_{\text{wide}}$ be $T_0 \times r$ and $N \times r$ matrices containing the left and right singular vectors of $\frac{Y_{\text{wide}}}{\sqrt{T_0 N}}$ respectively, corresponding to $D_{\text{wide}}$. Compute
   \[
   \hat{F}_{\text{wide}} = \left( \hat{f}_{\text{wide},1}, \ldots, \hat{f}_{\text{wide},T_0} \right)^T = \sqrt{T_0} P_{\text{wide}}, \quad \hat{\Lambda}_{\text{wide}} = \left( \hat{\lambda}_{\text{wide},1}, \ldots, \hat{\lambda}_{\text{wide},N} \right)^T = \sqrt{N} Q_{\text{wide}} D_{\text{wide}}. \quad [7]
   \]

3. Let $\hat{\Lambda}_{\text{wide,0}}$ be the first $N_0$ rows of $\hat{\Lambda}_{\text{wide}}$. Compute
   \[
   \hat{H}_{\text{miss}} = \hat{\Lambda}_{\text{tall}}^T \hat{\Lambda}_{\text{wide,0}} \left( \hat{\Lambda}_{\text{wide,0}}^T \hat{\Lambda}_{\text{wide,0}} \right)^{-1}, \quad [8]
   \]
   and then let $\hat{C} = \hat{F}_{\text{tall}} \hat{H}_{\text{miss}} \hat{\Lambda}_{\text{wide}}^T$.

4. Let $\hat{c}_{i,t}$ denote the $(t, i)$-th entry of $\hat{C}$, then compute the residuals $\hat{e}_{i,t} = y_{i,t} - \hat{c}_{i,t}$ for $(i, t) \in \mathcal{I} \setminus \mathcal{I}_1$.

5. For $(i, t) \in \mathcal{I}_1$, the variance of $\hat{e}_{i,t}$ is estimated by
   \[
   \hat{\nu}_{i,t} = \frac{1}{T_0} \hat{f}_{\text{tall},t}^T \left( \hat{F}_{\text{tall}}^T \hat{F}_{\text{tall}} T \right)^{-1} \hat{F}_{\text{tall}}^T \hat{F}_{\text{tall}} \hat{f}_{\text{tall},t} + \frac{1}{N_0} \hat{\nu}_{\text{wide},i} \left( \hat{\Lambda}_{\text{wide}}^T \hat{\Lambda}_{\text{wide}} N \right)^{-1} \hat{\Gamma}_i \left( \hat{\Lambda}_{\text{wide}}^T \hat{\Lambda}_{\text{wide}} N \right)^{-1} \hat{\nu}_{\text{wide},i}, \quad [9]
   \]
   where
   \[
   \hat{\Gamma}_i = \frac{1}{N_0} \sum_{j=1}^{N_0} \hat{c}_{j,t}^2 \hat{\nu}_{\text{wide},j} \hat{\nu}_{\text{wide},j}^T, \quad L_{k,i} = \frac{1}{T_0} \sum_{s=k+1}^{T_0} \hat{f}_{tall,s} \hat{e}_{i,s} \hat{e}_{i,s-k} \hat{f}_{tall,s-k}^T, \quad \hat{\Phi}_i = L_{0,i} + \sum_{k=1}^{K} \left( 1 - \frac{k}{K + 1} \right) \left( L_{k,i} + L_{k,i}^T \right), \quad [10]
   \]
with $K \to \infty$ and $\frac{K}{T_0^{1/3}} \to 0$ as $T_0 \to \infty$.

6. For every $N_0 < i \leq N$, compute $\hat{\sigma}_i^2 = \frac{1}{T_0} \sum_{s=1}^{T_0} \hat{e}_{i,s}^2$.

7. For $(i, t) \in I_1$, the estimated treatment effect is $\hat{\Delta}_{i,t} = \hat{\beta}_{i,t} - \hat{e}_{i,t}$, and the standard error of $\hat{\Delta}_{i,t}$ is $\sqrt{\hat{V}_{i,t} + \hat{\sigma}_i^2}$.

**Remark 3.1:** Since $\hat{V}_{i,t} = O_p\left(\frac{1}{T_0} \right)$ for every $(i, t) \in I_1$, it is asymptotically negligible in the standard error of $\hat{\Delta}_{i,t}$. Here we include $\hat{V}_{i,t}$ in the standard error of $\hat{\Delta}_{i,t}$ to improve finite sample performance.

**Remark 3.2:** In the above algorithm, we discuss the estimation for individual treatment effects for each treated unit. If the interest is the average treatment effects (ATE) across treated units or across post-treatment periods, not individual treatment effects, see Hsiao et al. (2022) for discussion of the aggregation of multiple treatment effects across treated units, and see Fujiki and Hsiao (2015) and Li and Bell (2017) for discussion of the average treatment effects over post-treatment periods.

### 3.2 Construction of Confidence Intervals

When neither $N_1$ nor $T_1$ is large, the average treatment effects either across treated units or post-treatment periods may not be a good measure for inferential purpose since it is hard to establish the asymptotic properties of such effects. In order to provide inferential procedure for individual treatment effects especially when neither $N_1$ nor $T_1$ is large, motivated by Gonçalves et al. (2017), we adapt bootstrap method to construct the confidence intervals of $\Delta_{i,t}$ for $(i, t) \in I_1$, without a specific distributional assumption on the error term $e_{i,t}$ or particular quantity on $N_1$ or $T_1$.

**Algorithm 3.2:** Confidence intervals of the treatment effects via bootstrap (without covariates).

1. Apply Algorithm 3.1 to the sample and obtain $\{\hat{c}_{i,t} : (i, t) \in I\}$, $\{\hat{e}_{i,t} : (i, t) \in I\}$, $\{\hat{V}_{i,t} : (i, t) \in I_1\}$, and $\{\hat{\sigma}_i^2 : i = N_0 + 1, \ldots, N\}$.

2. For $b = 1, 2, \ldots, B$:
   
   (1) For $(i, t) \in I \setminus I_1$, let $e^*_{i,t} = u_{i,t}\hat{e}_{i,t}$, where $\{u_{i,t}\}$ are i.i.d. or block i.i.d. from a standard normal distribution $N(0, 1)$ and are independent of the raw sample.

   (2) For $(i, t) \in I_1$, let $e^*_{i,t}$ be independently drawn from a discrete uniform distribution on the set $\{\hat{e}_{i,1} - \bar{e}_i, \hat{e}_{i,2} - \bar{e}_i, \ldots, \hat{e}_{i,T_0} - \bar{e}_i\}$, where $\bar{e}_i = \frac{1}{T_0} \sum_{s=1}^{T_0} \hat{e}_{i,s}$.

   (3) For $(i, t) \in I$, let $y^*_{i,t} = \hat{c}_{i,t} + e^*_{i,t}$. Use $\{y^*_{i,t}\}$ to construct $Y^*_{\text{tall}}$ and $Y^*_{\text{wide}}$ in the same fashion as Equation [5].

   (4) Apply Algorithm 3.1 to the bootstrapped sample $Y^*_{\text{tall}}$ and $Y^*_{\text{wide}}$, and obtain $\{\hat{c}_{i,t}^* : (i, t) \in I_1\}$, $\{\hat{V}_{i,t}^* : (i, t) \in I_1\}$, and $\{((\hat{\sigma}_i^*)^2 : i = N_0 + 1, \ldots, N\}$. 

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(5) For \((i, t) \in I_1\), compute
\[ s_{i,t}^* = \frac{\hat{c}_{i,t}^* - \hat{y}_{i,t}^*}{\sqrt{\hat{V}_{i,t}^* + (\hat{\sigma}_{i,t}^*)^2}} \tag{11} \]

3. In the previous step, we generate \(B\) statistics denoted by \(s_{i,t}^*(1), s_{i,t}^*(2), \ldots, s_{i,t}^*(B)\) for every \((i, t) \in I_1\). Let \(q_{1-\alpha,i,t}\) be the \((1 - \alpha)\) empirical quantile of \(\{s_{i,t}^*(1), s_{i,t}^*(2), \ldots, s_{i,t}^*(B)\}\), and let \(p_{1-\alpha,i,t}\) be the \((1 - \alpha)\) empirical quantile of \(\{\hat{s}_{i,t}^*(1), \hat{s}_{i,t}^*(2), \ldots, \hat{s}_{i,t}^*(B)\}\).

4. For \((i, t) \in I_1\), the equal tailed \((1 - \alpha)\) confidence interval of \(\Delta_{i,t}\) is
\[ \text{EQ}_{1-\alpha,i,t} = \left[ \hat{\Delta}_{i,t} + q_{\alpha/2,i,t}\sqrt{\hat{\sigma}_{i,t}^2}, \hat{\Delta}_{i,t} + q_{1-(\alpha/2),i,t}\sqrt{\hat{\sigma}_{i,t}^2} \right], \tag{12} \]
and the symmetric \((1 - \alpha)\) confidence interval of \(\Delta_{i,t}\) is
\[ \text{SY}_{1-\alpha,i,t} = \left[ \hat{\Delta}_{i,t} - p_{1-\alpha,i,t}\sqrt{\hat{\sigma}_{i,t}^2}, \hat{\Delta}_{i,t} + p_{1-\alpha,i,t}\sqrt{\hat{\sigma}_{i,t}^2} \right]. \tag{13} \]

**Remark 3.3:** As is shown in the proof of Theorem 3.1, the distribution of \((\hat{\Delta}_{i,t} - \Delta_{i,t})\) is dominated by \(\epsilon_{i,t}\) and the effect of \((\hat{c}_{i,t} - c_{i,t})\) is asymptotically negligible. However, the bootstrap procedure above takes \((\hat{c}_{i,t} - c_{i,t})\) into consideration in order to improve the finite sample performances.

**Remark 3.4:** When the error terms \(\{\epsilon_{i,t}\}\) are suspected of having serial correlation, a block wild bootstrap can be used to address this issue (e.g., Gonçalves et al., 2017). That is, a for block width \(b \in \mathbb{Z}_+\), we let \(u_{i,(j-1)b+s} = \bar{u}_{i,j}\) for every \(j \in \mathbb{Z}_+\) and \(s \in \{1, 2, \cdots, b\}\) such that \((j - 1)b + s \leq T_0\) in Step 2(1).

Under Assumptions 2.1–2.5 in this paper, the asymptotic properties of the bootstrapped confidence intervals [12] and [13] are summarized in the following theorem.

**Theorem 3.1:** If Assumptions 2.1–2.5 hold, then
\[ \lim_{N_0,T_0 \to \infty} \mathbb{P}(\Delta_{i,t} \in \text{EQ}_{1-\alpha,i,t}) = 1 - \alpha \quad \text{and} \quad \lim_{N_0,T_0 \to \infty} \mathbb{P}(\Delta_{i,t} \in \text{SY}_{1-\alpha,i,t}) = 1 - \alpha \tag{14} \]
for every \((i, t) \in I_1\).

From Theorem 3.1, we can conclude that the bootstrap confidence intervals will provide a correct coverage for the estimated treatment effects in the post-treatment periods, and thus one could conduct statistical inference about the treatment effects based on the bootstrap confidence intervals. It also worthwhile noting that even if our bootstrap procedure follows the idea of Gonçalves et al. (2017), the proof of bootstrap validity becomes quite complicated due to the differences in purposes and model specifications. Since Gonçalves et al. (2017) intend to make inference on factor-augmented regression models (Bai and Ng, 2006) where the factors serves as intermediate variables, they only require valid bootstrap approximations of factors. In this paper, we intend to make inference on the factor model (with missing values) itself, and the estimators involve products of estimated factors, their associated loadings, and a rotation matrix. This implies that we need valid bootstrap approximations of factors, loadings, and the rotation matrix, which leads to laborious works.
4 Estimation and Inference in a Model with Covariates

In the above section, we discussed the estimation of the treatment effects and associated bootstrap confidence intervals for a pure factor model. Besides the unobserved factors, in practice, the outcome variable could also be affected by some exogenous regressors. To accommodate such a scenario, we extend the factor-based approach to a model with covariates, i.e., Model [3] in Section 2.

4.1 Estimation of Treatment Effects

For a factor model with covariates, i.e., a panel data model with interactive fixed effect, our estimation strategies involve an interactive fixed effect estimation (IFEE), which is described and studied in Bai (2009). If \( \{y_{i,t} : (i,t) \in I\} \) were fully observed, we could directly use IFEE to estimate Model [3]. The general framework of IFEE is summarised in the algorithm to follow.

**Algorithm 4.1:** Interactive fixed effect estimation.

1. Input arguments. \( Y = (Y_1, \ldots, Y_N) \), a \( T \times N \) matrix. \( X = (X_1, \ldots, X_N) \), a \( T \times (pN) \) matrix, where \( X_i \) is a \( T \times p \) matrix for every \( i \).
2. Compute the starting value \( \beta^{(0)} = \left( \sum_{i=1}^{N} X_i^T X_i \right)^{-1} \sum_{i=1}^{N} X_i^T Y_i \). \[15\]
3. Perform the following iteration until \( \| \beta^{(M)} - \beta^{(M-1)} \|_2 < \varepsilon \) for some \( M \in \mathbb{Z}_+ \), where \( \varepsilon \) is a sufficiently small positive number.
   
   (1) Let \( R^{(k)} = \frac{Y - X (I_N \otimes \beta^{(k-1)})}{\sqrt{NT}} \). \[16\]
   (2) Perform a singular value decomposition of the matrix \( R^{(k)} \), and let \( D^{(k)} \) be an \( r \times r \) diagonal matrix with the \( r \) largest singular values of \( R^{(k)} \) on the diagonal in descending order. Then let \( U^{(k)} \) and \( V^{(k)} \) be \( T \times r \) and \( N \times r \) matrices containing the left and right singular vectors of \( R^{(k)} \) corresponding to \( D^{(k)} \).
   (3) Let \( F^{(k)} = \sqrt{T} U^{(k)} \) and \( H^{(k)} = I_T - \frac{F^{(k)} (F^{(k)})^T}{T} \).
   (4) Let \( \beta^{(k)} = \left( \sum_{i=1}^{N} X_i^T H^{(k)} X_i \right)^{-1} \left( \sum_{i=1}^{N} X_i^T H^{(k)} Y_i \right) \). \[17\]
4. Output arguments. \( \beta^{(M)} \), \( F^{(M)} \), and \( \Lambda^{(M)} = \sqrt{N} V^{(M)} D^{(M)} \).

Now we turn to the estimation of treatment effect when \( \{y_{i,t} : (i,t) \in I\} \) are not observable. Let \( Y, Y_{\text{tall}}, Y_{\text{wide}}, X_{\text{tall}}, X_{\text{wide}}, \Lambda, \Lambda_{\text{tall}}, \) and \( \Lambda_{\text{wide}} \) follow their definitions in Sections 2 and 3. Let \( X = (X_1, \ldots, X_N) \), \( X_{\text{tall}} = (X_1, \ldots, X_{N_0}) \), and \( X_{\text{wide}} \) to be a sub-matrix formed by the first \( T_0 \) rows of \( X \).
Then we can use the algorithm below to estimate the treatment effects \( \Delta_{i,t} \) for all \((i, t) \in \mathcal{I}_1 \).

**Algorithm 4.2:** Treatment effect estimation via factor-based matrix completion (with covariates).

1. Apply Algorithm 4.1 to \((Y_{\text{tall}}, X_{\text{tall}})\), and obtain \( \hat{\beta}_{\text{tall}}, \hat{F}_{\text{tall}} = (\hat{f}_{\text{tall},1}, \ldots, \hat{f}_{\text{tall},T})^T \) and \( \hat{\Lambda}_{\text{tall}} = (\hat{\lambda}_{\text{tall},1}, \ldots, \hat{\lambda}_{\text{tall},N_0})^T \).
2. Apply Algorithm 4.1 to \((Y_{\text{wide}}, X_{\text{wide}})\), and obtain \( \hat{F}_{\text{wide}} = (\hat{f}_{\text{wide},1}, \ldots, \hat{f}_{\text{wide},T_0})^T \) and \( \hat{\Lambda}_{\text{wide}} = (\hat{\lambda}_{\text{wide},1}, \ldots, \hat{\lambda}_{\text{wide},N})^T \).
3. Let \( \hat{\Lambda}_{\text{wide},0} \) be the first \( N_0 \) rows of \( \hat{\Lambda}_{\text{wide}} \). Compute
   \[
   \hat{H}_{\text{miss}} = \hat{\Lambda}_{\text{tall}}^T \hat{\Lambda}_{\text{wide},0} \left( \hat{\Lambda}_{\text{wide},0}^T \hat{\Lambda}_{\text{wide},0} \right)^{-1},
   \]
   and then let \( \hat{C} = \hat{F}_{\text{tall}} \hat{H}_{\text{miss}} \hat{\Lambda}_{\text{wide}} \).
4. Let \( \hat{c}_{i,t} \) denote the \((i, t)\)-th entry of \( \hat{C} \), then compute the residuals \( \hat{e}_{i,t} = y_{i,t} - x_{i,t}^T \hat{\beta}_{\text{tall}} - \hat{c}_{i,t} \) for \((i, t) \in \mathcal{I} \setminus \mathcal{I}_1 \).
5. For \((i, t) \in \mathcal{I}_1\), the variance of \( \hat{c}_{i,t} \) is estimated by
   \[
   \hat{V}_{i,t} = \frac{1}{T_0} \hat{F}_{i}^T \left( \frac{\hat{F}_{\text{tall}}^T \hat{F}_{\text{tall}}}{T} \right)^{-1} \hat{F}_{\text{tall}}^T \hat{F}_{i} \left( \frac{\hat{F}_{\text{tall}}^T \hat{F}_{\text{tall}}}{T} \right)^{-1} \hat{f}_{i,t}^2 + \frac{1}{N_0} \hat{\Lambda}_{\text{wide},i}^T \left( \frac{\hat{\Lambda}_{\text{wide},i} \hat{\Lambda}_{\text{wide}}}{N} \right)^{-1} \hat{\Lambda}_{\text{wide},i} \left( \frac{\hat{\Lambda}_{\text{wide},i} \hat{\Lambda}_{\text{wide}}}{N} \right)^{-1} \hat{e}_{i,t}^2,
   \]
   where
   \[
   \hat{\Gamma}_i = \frac{1}{N_0} \sum_{j=1}^{N_0} \hat{c}_{j,t} \hat{\Lambda}_{\text{wide},j} \hat{\Lambda}_{\text{wide},j}^T,
   \]
   \[
   L_{k,i} = \frac{1}{T_0} \sum_{s=k+1}^{T_0} \hat{f}_{\text{tall},s} \hat{c}_{i,s} \hat{e}_{i,s,k} \hat{f}_{\text{tall},s-k}^T
   \]
   \[
   \hat{\Phi}_i = L_{0,i} + \sum_{k=1}^{K} \left(1 - \frac{k}{K+1}\right) \left(L_{k,i} + L_{k,i}^T\right)
   \]
   with \( K \to \infty \) and \( \frac{K}{T_0} \to 0 \) as \( T_0 \to \infty \).
6. For every \( N_0 < i \leq N \), compute \( \hat{\sigma}_i^2 = \frac{1}{T_0} \sum_{s=1}^{T_0} \hat{e}_{i,s}^2 \).
7. For \((i, t) \in \mathcal{I}_1\), the estimated treatment effect is \( \hat{\Delta}_{i,t} = \hat{\gamma}_{i,t} - x_{i,t}^T \hat{\beta}_{\text{tall}} - \hat{c}_{i,t} \), and the standard error of \( \hat{\Delta}_{i,t} \) is \( \sqrt{\hat{V}_{i,t} + \hat{\sigma}_i^2} \).

**Remark 4.1:** Because
   \[
   \hat{\Delta}_{i,t} - \Delta_{i,t} = x_{i,t}^T (\beta - \hat{\beta}_{\text{tall}}) + (c_{i,t} - \hat{c}_{i,t}) + e_{i,t}
   \]
   for every \((i, t) \in \mathcal{I}_1\), the variance of \( \hat{\Delta}_{i,t} \) is comprised of \( \text{Var}(c_{i,t}) \), \( \text{Var}(\hat{c}_{i,t}) \) and \( \text{Var}(\hat{\beta}_{\text{tall}}) \). In spirit of Remark 3.1, we should include an estimate of \( \text{Var}(\hat{\beta}_{\text{tall}}) \) in the standard error of \( \hat{\Delta}_{i,t} \) to improve the finite sample performances. However, by Theorem 3 of Bai (2009), \( \text{Var}(\hat{\beta}_{\text{tall}}) = O\left(\frac{1}{N_0 T_0}\right) \), which is of higher order than \( \text{Var}(\hat{c}_{i,t}) = O\left(\frac{1}{N_0} + \frac{1}{T_0}\right) \) for \((i, t) \in \mathcal{I}_1\). Furthermore, a consistent estimation of \( \text{Var}(\hat{\beta}_{\text{tall}}) \) is quite complicated. As a result of the cost-benefit trade-off, we do not include an
estimate of \( \text{Var} \left( \hat{\beta}_{\text{tall}} \right) \) in the standard error of \( \hat{\Delta}_{i,t} \).

### 4.2 Construction of Confidence Intervals

For the estimated \( \hat{\Delta}_{i,t} \) for \((i, t) \in \mathcal{I}_1\), we shall again use the bootstrap procedure discussed above to construct their confidence intervals.

**Algorithm 4.3:** Confidence intervals of the treatment effects via bootstrap (with covariates).

1. Apply Algorithm 4.2 to the sample and obtain \( \{ \hat{c}_{i,t} : (i, t) \in \mathcal{I} \}, \{ \hat{e}_{i,t} : (i, t) \in \mathcal{I} \}, \{ \hat{\mathcal{V}}_{i,t} : (i, t) \in \mathcal{I}_1 \}, \) and \( \{ \hat{\sigma}_t^2 : i = N_0 + 1, \ldots, N \} \).
2. For \( b = 1, 2, \ldots, B \):
   
   (1) For \((i, t) \in \mathcal{I} \setminus \mathcal{I}_1\), let \( e_{i,t}^* = u_{i,t} \hat{c}_{i,t} \), where \( \{ u_{i,t} \} \) are i.i.d. or block i.i.d. from a standard normal distribution \( N(0, 1) \) and are independent of the raw sample.
   
   (2) For \((i, t) \in \mathcal{I}_1\), let \( e_{i,t}^* \) be independently drawn from a discrete uniform distribution on the set \( \{ \hat{c}_{i,1} - \bar{e}_i, \hat{c}_{i,2} - \bar{e}_i, \ldots, \hat{c}_{i,T_0} - \bar{e}_i \} \), where \( \bar{e}_i = \frac{1}{T_0} \sum_{s=1}^{T_0} \hat{c}_{i,s} \).
   
   (3) For \((i, t) \in \mathcal{I}\), let \( r_{i,t}^* = \hat{c}_{i,t} + e_{i,t}^* \). Use \( \{ r_{i,t}^* \} \) to construct \( R_{\text{tall}}^* \) and \( R_{\text{wide}}^* \) in the same fashion as Equation [5].
   
   (4) Apply Algorithm 3.1 to the bootstrapped sample \( R_{\text{tall}}^* \) and \( R_{\text{wide}}^* \), and obtain \( \{ \hat{c}_{i,t}^* : (i, t) \in \mathcal{I}_1 \}, \{ \hat{\mathcal{V}}_{i,t}^* : (i, t) \in \mathcal{I}_1 \}, \) and \( \{ (\hat{\sigma}_t^*)^2 : i = N_0 + 1, \ldots, N \} \).
   
   (5) For \((i, t) \in \mathcal{I}_1\), compute
   
   \[
   s_{i,t}^* = \frac{\hat{c}_{i,t}^* - r_{i,t}^*}{\sqrt{\hat{\mathcal{V}}_{i,t}^* + (\hat{\sigma}_t^*)^2}}. \tag{[22]}
   \]

3. In the previous step, we generate \( B \) statistics denoted by \( s_{i,t}^*(1), s_{i,t}^*(2), \ldots, s_{i,t}^*(B) \) for every \((i, t) \in \mathcal{I}_1\). Let \( q_{1-\alpha,i,t} \) be the \((1-\alpha)\) empirical quantile of \( \{ s_{i,t}^*(1), s_{i,t}^*(2), \ldots, s_{i,t}^*(B) \} \), and let \( p_{1-\alpha,i,t} \) be the \((1-\alpha)\) empirical quantile of \( \{ |s_{i,t}^*(1)|, |s_{i,t}^*(2)|, \ldots, |s_{i,t}^*(B)| \} \).

4. For \((i, t) \in \mathcal{I}_1\), the equal tailed \((1-\alpha)\) confidence interval of \( \Delta_{i,t} \) is
   
   \[
   \text{EQ}_{1-\alpha,i,t} = \left[ \hat{\Delta}_{i,t} + q_{\alpha/2,i,t} \sqrt{\hat{\mathcal{V}}_{i,t} + \hat{\sigma}_t^2}, \hat{\Delta}_{i,t} + q_{1-(\alpha/2),i,t} \sqrt{\hat{\mathcal{V}}_{i,t} + \hat{\sigma}_t^2} \right], \tag{[23]}
   \]
   
   and the symmetric \((1-\alpha)\) confidence interval of \( \Delta_{i,t} \) is
   
   \[
   \text{SY}_{1-\alpha,i,t} = \left[ \hat{\Delta}_{i,t} - p_{1-\alpha,i,t} \sqrt{\hat{\mathcal{V}}_{i,t} + \hat{\sigma}_t^2}, \hat{\Delta}_{i,t} - p_{\alpha,i,t} \sqrt{\hat{\mathcal{V}}_{i,t} + \hat{\sigma}_t^2} \right]. \tag{[24]}
   \]

**Remark 4.2:** By Equation [21] and in spirit of Remark 3.3, a bootstrap procedure should take the distributions of all the 3 terms \( x_{i,t}^T \left( \beta - \hat{\beta}_{\text{tall}} \right) \), \( (e_{i,t} - \hat{c}_{i,t}) \) and \( e_{i,t} \) into consideration to ensure the finite sample performances. But in Algorithm 4.3, resampled observations are generated by a pure factor model and only the interactive fixed effects \( \{ \hat{c}_{i,t} \} \) are estimated in every bootstrapped sample. Thus the bootstrap procedure only approximates the distribution of \( (c_{i,t} - \hat{c}_{i,t}) + e_{i,t} \), ignoring the effect of \( x_{i,t}^T \left( \beta - \hat{\beta}_{\text{tall}} \right) \). The underlying rationale is also a cost-benefit trade-off mentioned in Remark
4.1. On the one hand, by Bai (2009) and Bai and Ng (2021), we have \((c_{i,t} - \hat{c}_{i,t}) = O_p\left(\frac{1}{\sqrt{N_0 \wedge T_0}}\right)\) and \(x_{i,t}^T(\beta - \hat{\beta}_{tall}) = O_p\left(\frac{1}{\sqrt{N_0 \wedge T_0}}\right)\) as \(N_0, T_0 \to \infty\) for every \((i, t) \in I_1\). On the other hand, the computation of interactive fixed effect estimation is far more intensive than that of estimating a pure factor model. These two facts motivate us to ignore the effect of \(x_{i,t}^T(\beta - \hat{\beta}_{tall})\) in the bootstrap procedure. Furthermore, evidence from Section 5 shows the current bootstrap procedure has already yielded satisfactory finite sample performances.

**Remark 4.3:** If the error terms \(\{e_{i,t}\}\) are suspected of having serial correlation, we suggest to use block wild bootstrap in Step 2(1). See Remark 3.4 for details.

As above, we can establish the validity of the proposed confidence intervals [23] and [24] in the sense that they have asymptotically correct coverage probabilities as \(N_0, T_0 \to \infty\).

**Theorem 4.1:** If Assumptions 2.1–2.8 hold, then

\[
\lim_{N_0, T_0 \to \infty} P\left(\Delta_{i,t} \in EQ_{1-\alpha, i,t}\right) = 1 - \alpha \quad \text{and} \quad \lim_{N_0, T_0 \to \infty} P\left(\Delta_{i,t} \in SY_{1-\alpha, i,t}\right) = 1 - \alpha \quad [25]
\]

for every \((i, t) \in I_1\).

Theorem 4.1 suggests that we can apply the proposed bootstrap procedure to conduct statistical inference for estimated treated effects from a panel with interactive fixed effects model.

5 Simulation Studies

In this section, we conduct several Monte Carlo experiments to investigate the finite sample properties of the proposed confidence intervals.\(^2\)

In the data generating processes below, we assume that the common factors \(\{f_t\}\) are i.i.d. as \(N(0, I_3)\), and the factor loadings \(\{\lambda_i\}\) are also i.i.d. as \(N(0, I_3)\). For the model with covariates, we assume that the covariates \(\{x_{i,t}\}\) are i.i.d. as \(N(0, AA^T)\), where each entry of the \(2 \times 2\) matrix \(A\) is drawn from \(N(0,1)\). Let \(\beta \sim N(0, I_2)\). We consider the following data generating processes.\(^3\)

- **DGP1:** Model without covariates, \(y_{i,t} = f_t^T\lambda_i + e_{i,t}\).
- **DGP2:** Model with covariates, \(y_{i,t} = x_{i,t}^T\beta + f_t^T\lambda_i + e_{i,t}\).

The error term is defined as

\[
e_{i,t} = v_{i,t}\sqrt{\frac{\sigma_i^2}{1 - \rho_i^2}}\quad [26]
\]

where \(v_{i,t} = \rho_i v_{i,t-1} + \varepsilon_{i,t}\) and \(\{\varepsilon_{i,t}\}\) are i.i.d. drawn from \(N(0,1)\). We specify two variance structures of \(\{e_{i,t}\}\).

- **Case 1:** \(\rho_i = 0\) and \(\sigma_i^2 = 1\).

\(^2\)MATLAB codes for simulation are available from the authors upon request.

\(^3\)We also consider AR(1) factors, exponentially distributed errors, other variance structures, and dependence between factors and covariates. These results are available from the authors upon request.
• Case 2: \( \{\rho_i\} \) are i.i.d. drawn from \( \text{Unif}([-0.8, -0.2] \cup [0.2, 0.8]) \), and \( \{\sigma_i^2\} \) are i.i.d. drawn from \( \text{logNormal}(0, 1) \).

Moreover, we consider the following marginal distributions of \( \{v_{i,t}\} \).

• Margin 1: \( v_{i,t} \sim \left[ \chi^2(1) - 1 \right] / \sqrt{2} \).

• Margin 2: \( v_{i,t} \sim (\text{Unif}[-0.5, 0.5]) / \sqrt{12} \).

For demonstration, we assume there is only one treated unit \( i = N \) and 5 post-treatment periods. The treatment effects are assumed to be constants equal to 1, i.e., \( \Delta_{N,t} = 1 \) for \( t = T_0 + 1, \ldots, T_0 + 5 \). The number of control units \( N_0 \in \{30, 50, 100\} \) and the number of pre-treatment periods \( T_0 \in \{20, 40\} \). We construct the 90% and 95%, equal-tailed and symmetric confidence intervals for \( \Delta_{N,t}, \) \( t = T_0 + 1, \ldots, T_0 + 5 \). In Step 2(1) of Algorithms 3.2 and 4.3, we use ordinary wild bootstrap procedure for Case 1, and block wild bootstrap procedure with block width equal to 4 for Case 2. The number of factors is either treated as known or estimated using the method of Bai and Ng (2002). For computational simplicity, a warp-speed method (Giacomini et al., 2013) is applied with 2000 replications for each scenario. We report the coverage rates (in percent) of confidence intervals in Tables 1–8. In these tables, EQ stands for equal-tailed confidence intervals and SY stands for symmetric confidence intervals.

Several interesting findings can be observed from the simulation results. On the first hand, the results in Table 1–4 clearly show that the empirical coverage ratio for treatment effects from a pure factor model is quite close to the nominal values (both 90% and 95% level) when the post-treatment period is short, regardless of whether the idiosyncratic errors are heteroscedastic or serially correlated, or whether the number of unobserved factors is known or estimated from the data. On the other hand, when exogenous covariates are included for treatment effects estimation, Tables 5–8 also show that our proposed bootstrapped confidence intervals are able to provide accurate coverage ratios for the estimated treatment effects in a panel model with exogenous regressors and with heteroscedastic or serially correlated errors. In general, the simulation results confirm the validity of our proposed bootstrap procedure in providing accurate and robust confidence intervals for estimated treatment effects using a panel with interactive fixed effects.
Table 1: Pure factor model with homoscedastic i.i.d. chi-squared errors

| Known number of factors, 90% CI | (T₀, N₀) | (20, 30) | (20, 50) | (20, 100) | (40, 30) | (40, 50) | (40, 100) |
|--------------------------------|----------|----------|----------|----------|----------|----------|----------|
| t                             | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       |
| T₀ + 1                         | 90.30    | 90.85    | 91.55    | 90.55    | 90.70    | 90.90    | 93.05    | 91.45    | 91.60    | 92.05    | 91.85    | 91.45    |
| T₀ + 2                         | 91.35    | 91.00    | 90.85    | 90.55    | 92.55    | 91.45    | 91.15    | 90.95    | 92.85    | 92.90    | 91.35    | 89.90    |
| T₀ + 3                         | 90.50    | 90.30    | 92.95    | 91.90    | 93.30    | 92.15    | 91.50    | 91.45    | 92.75    | 91.10    | 91.90    | 91.50    |
| T₀ + 4                         | 93.15    | 92.35    | 92.05    | 91.15    | 91.00    | 91.65    | 91.80    | 91.40    | 92.40    | 92.35    | 91.85    | 91.40    |
| T₀ + 5                         | 91.55    | 91.65    | 89.85    | 90.85    | 90.20    | 91.30    | 92.35    | 92.60    | 92.55    | 90.95    | 92.05    | 90.35    |

| Known number of factors, 95% CI | (T₀, N₀) | (20, 30) | (20, 50) | (20, 100) | (40, 30) | (40, 50) | (40, 100) |
|--------------------------------|----------|----------|----------|----------|----------|----------|----------|
| t                             | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       |
| T₀ + 1                         | 94.85    | 94.60    | 95.25    | 95.00    | 95.60    | 94.70    | 96.40    | 95.60    | 95.75    | 94.55    | 96.35    | 94.95    |
| T₀ + 2                         | 94.90    | 94.55    | 95.25    | 95.00    | 95.70    | 96.05    | 95.35    | 94.40    | 96.20    | 94.70    | 95.60    | 93.70    |
| T₀ + 3                         | 95.00    | 94.45    | 95.80    | 95.50    | 95.85    | 95.40    | 95.35    | 94.40    | 95.90    | 95.15    | 95.95    | 94.45    |
| T₀ + 4                         | 96.25    | 95.45    | 94.85    | 94.70    | 95.15    | 94.25    | 95.20    | 95.40    | 96.65    | 95.00    | 96.05    | 94.95    |
| T₀ + 5                         | 96.10    | 95.70    | 94.65    | 94.30    | 95.00    | 95.20    | 96.65    | 96.20    | 96.70    | 95.05    | 96.35    | 94.65    |

| Estimated number of factors, 90% CI | (T₀, N₀) | (20, 30) | (20, 50) | (20, 100) | (40, 30) | (40, 50) | (40, 100) |
|-------------------------------------|----------|----------|----------|----------|----------|----------|----------|
| t                                   | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       |
| T₀ + 1                              | 90.30    | 91.05    | 91.35    | 90.65    | 90.65    | 90.85    | 93.30    | 91.75    | 91.65    | 92.05    | 91.85    | 91.45    |
| T₀ + 2                              | 91.35    | 90.95    | 90.65    | 90.80    | 92.35    | 91.50    | 91.10    | 90.85    | 92.90    | 92.85    | 91.35    | 89.90    |
| T₀ + 3                              | 90.60    | 90.50    | 92.35    | 91.80    | 93.30    | 92.15    | 91.35    | 91.35    | 92.70    | 91.25    | 91.90    | 91.50    |
| T₀ + 4                              | 92.55    | 92.70    | 91.80    | 91.05    | 91.00    | 91.75    | 91.70    | 91.20    | 92.45    | 92.35    | 91.85    | 91.40    |
| T₀ + 5                              | 91.70    | 91.65    | 90.65    | 90.60    | 90.20    | 91.20    | 92.35    | 92.80    | 92.50    | 91.05    | 92.05    | 90.30    |

| Estimated number of factors, 95% CI | (T₀, N₀) | (20, 30) | (20, 50) | (20, 100) | (40, 30) | (40, 50) | (40, 100) |
|-------------------------------------|----------|----------|----------|----------|----------|----------|----------|
| t                                   | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       |
| T₀ + 1                              | 94.85    | 94.80    | 95.35    | 95.00    | 95.55    | 94.70    | 96.50    | 95.55    | 95.65    | 94.55    | 96.35    | 94.95    |
| T₀ + 2                              | 95.05    | 94.70    | 95.40    | 94.95    | 95.65    | 96.00    | 95.35    | 94.50    | 96.15    | 94.70    | 95.60    | 93.70    |
| T₀ + 3                              | 94.85    | 94.10    | 96.15    | 95.15    | 95.85    | 95.40    | 95.35    | 94.20    | 95.95    | 95.15    | 95.95    | 94.45    |
| T₀ + 4                              | 95.90    | 95.90    | 94.85    | 94.60    | 95.15    | 94.25    | 95.20    | 95.35    | 96.55    | 95.00    | 96.05    | 94.95    |
| T₀ + 5                              | 95.60    | 95.40    | 94.75    | 94.30    | 95.05    | 95.20    | 96.75    | 96.25    | 96.80    | 94.90    | 96.35    | 94.65    |
Table 2: Pure factor model with homoscedastic i.i.d. uniform errors

### Known number of factors, 90% CI

| $t$   | $(T_0, N_0)$ | $(20, 30)$ | $(20, 50)$ | $(20, 100)$ | $(40, 30)$ | $(40, 50)$ | $(40, 100)$ |
|-------|--------------|------------|------------|------------|------------|------------|------------|
|       |              | EQ         | SY         | EQ         | SY         | EQ         | SY         | EQ         | SY         |
| $T_0 + 1$ | $(20, 30)$   | 91.00      | 90.95      | 89.10      | 89.05      | 92.40      | 92.55      | 91.55      | 91.55      | 92.05      | 92.15      | 91.15      | 90.95      |
| $T_0 + 2$ | $(20, 50)$   | 91.15      | 91.10      | 90.95      | 90.95      | 90.80      | 90.50      | 91.60      | 91.75      | 92.85      | 92.65      | 92.00      | 92.00      |
| $T_0 + 3$ | $(20, 100)$  | 91.80      | 91.85      | 91.45      | 91.40      | 91.45      | 91.50      | 91.60      | 91.60      | 91.60      | 91.70      | 90.20      | 90.50      |
| $T_0 + 4$ | $(40, 30)$   | 90.95      | 90.80      | 91.35      | 91.30      | 92.00      | 92.35      | 92.10      | 92.05      | 91.60      | 91.90      | 91.75      | 91.95      |
| $T_0 + 5$ | $(40, 50)$   | 92.85      | 93.00      | 90.10      | 91.05      | 91.95      | 92.15      | 91.80      | 91.60      | 90.20      | 90.30      | 90.80      | 91.50      |

### Known number of factors, 95% CI

| $t$   | $(T_0, N_0)$ | $(20, 30)$ | $(20, 50)$ | $(20, 100)$ | $(40, 30)$ | $(40, 50)$ | $(40, 100)$ |
|-------|--------------|------------|------------|------------|------------|------------|------------|
|       |              | EQ         | SY         | EQ         | SY         | EQ         | SY         | EQ         | SY         |
| $T_0 + 1$ | $(20, 30)$   | 96.35      | 96.60      | 94.75      | 94.85      | 96.80      | 96.70      | 96.15      | 96.05      | 97.20      | 97.10      | 96.40      | 96.85      |
| $T_0 + 2$ | $(20, 50)$   | 96.90      | 96.90      | 97.20      | 97.30      | 96.45      | 96.65      | 95.85      | 95.80      | 97.45      | 97.70      | 96.60      | 96.60      |
| $T_0 + 3$ | $(20, 100)$  | 96.85      | 96.80      | 96.65      | 96.80      | 97.20      | 97.55      | 96.05      | 96.15      | 97.25      | 97.25      | 96.75      | 96.55      |
| $T_0 + 4$ | $(40, 30)$   | 97.25      | 97.10      | 96.85      | 96.65      | 96.65      | 96.65      | 96.75      | 96.70      | 97.10      | 96.95      | 97.20      | 97.60      |
| $T_0 + 5$ | $(40, 50)$   | 97.55      | 97.20      | 96.20      | 96.30      | 96.45      | 96.55      | 96.00      | 96.00      | 96.65      | 96.60      | 97.30      | 97.25      |

### Estimated number of factors, 90% CI

| $t$   | $(T_0, N_0)$ | $(20, 30)$ | $(20, 50)$ | $(20, 100)$ | $(40, 30)$ | $(40, 50)$ | $(40, 100)$ |
|-------|--------------|------------|------------|------------|------------|------------|------------|
|       |              | EQ         | SY         | EQ         | SY         | EQ         | SY         | EQ         | SY         |
| $T_0 + 1$ | $(20, 30)$   | 90.95      | 90.95      | 89.10      | 89.05      | 92.40      | 92.55      | 91.55      | 91.55      | 92.05      | 92.15      | 91.15      | 90.95      |
| $T_0 + 2$ | $(20, 50)$   | 90.70      | 90.70      | 90.95      | 90.95      | 90.80      | 90.50      | 91.60      | 91.75      | 92.85      | 92.65      | 92.00      | 92.00      |
| $T_0 + 3$ | $(20, 100)$  | 91.75      | 91.80      | 91.45      | 91.40      | 91.45      | 91.50      | 91.60      | 91.60      | 91.60      | 91.70      | 90.20      | 90.50      |
| $T_0 + 4$ | $(40, 30)$   | 90.85      | 90.70      | 91.35      | 91.30      | 92.00      | 92.35      | 92.10      | 92.05      | 91.60      | 91.90      | 91.75      | 91.95      |
| $T_0 + 5$ | $(40, 50)$   | 92.75      | 93.00      | 90.10      | 91.05      | 91.95      | 92.15      | 91.80      | 91.60      | 90.20      | 90.30      | 90.80      | 91.50      |

### Estimated number of factors, 95% CI

| $t$   | $(T_0, N_0)$ | $(20, 30)$ | $(20, 50)$ | $(20, 100)$ | $(40, 30)$ | $(40, 50)$ | $(40, 100)$ |
|-------|--------------|------------|------------|------------|------------|------------|------------|
|       |              | EQ         | SY         | EQ         | SY         | EQ         | SY         | EQ         | SY         |
| $T_0 + 1$ | $(20, 30)$   | 96.30      | 96.60      | 94.75      | 94.85      | 96.80      | 96.70      | 96.15      | 96.05      | 97.20      | 97.10      | 96.40      | 96.85      |
| $T_0 + 2$ | $(20, 50)$   | 96.85      | 96.90      | 97.20      | 97.30      | 96.45      | 96.65      | 95.85      | 95.80      | 97.45      | 97.70      | 96.60      | 96.60      |
| $T_0 + 3$ | $(20, 100)$  | 96.85      | 96.80      | 96.65      | 96.80      | 97.20      | 97.55      | 96.05      | 96.15      | 97.25      | 97.25      | 96.75      | 96.55      |
| $T_0 + 4$ | $(40, 30)$   | 97.20      | 97.05      | 96.85      | 96.65      | 96.65      | 96.65      | 96.75      | 96.70      | 97.10      | 96.95      | 97.20      | 97.60      |
| $T_0 + 5$ | $(40, 50)$   | 97.50      | 97.20      | 96.20      | 96.30      | 96.45      | 96.55      | 96.00      | 96.00      | 96.65      | 96.60      | 97.30      | 97.25      |
Table 3: Pure factor model with heteroscedastic AR(1) chi-squared errors

### Known number of factors, 90% CI

|         | (20, 30) | (20, 50) | (20, 100) | (40, 30) | (40, 50) | (40, 100) |
|---------|----------|----------|-----------|----------|----------|-----------|
|         | EQ       | SY       | EQ        | SY       | EQ       | SY        | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       |
| $T_0 + 1$ | 92.90    | 92.90    | 93.25     | 93.15    | 92.85    | 92.00     | 92.70    | 92.75    | 90.60    | 91.30    | 92.90    | 93.80    |         |         |
| $T_0 + 2$ | 94.55    | 94.70    | 91.15     | 91.15    | 92.45    | 92.20     | 92.35    | 92.65    | 91.10    | 92.20    | 94.45    | 93.50    |         |         |
| $T_0 + 3$ | 93.20    | 92.90    | 90.20     | 90.60    | 93.00    | 93.35     | 91.90    | 91.85    | 91.75    | 91.20    | 91.95    | 91.10    |         |         |
| $T_0 + 4$ | 92.15    | 92.20    | 91.15     | 91.15    | 93.40    | 93.15     | 93.30    | 93.15    | 93.35    | 91.75    | 92.60    | 92.40    |         |         |
| $T_0 + 5$ | 92.20    | 92.00    | 90.10     | 90.55    | 92.65    | 92.25     | 91.15    | 90.95    | 91.10    | 91.50    | 93.05    | 92.30    |         |         |

### Known number of factors, 95% CI

|         | (20, 30) | (20, 50) | (20, 100) | (40, 30) | (40, 50) | (40, 100) |
|---------|----------|----------|-----------|----------|----------|-----------|
|         | EQ       | SY       | EQ        | SY       | EQ       | SY        | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       |
| $T_0 + 1$ | 96.80    | 96.70    | 96.45     | 97.20    | 96.55    | 95.75     | 97.00    | 96.80    | 95.45    | 96.30    | 96.85    | 95.95    |         |         |
| $T_0 + 2$ | 96.95    | 96.80    | 94.70     | 94.55    | 96.20    | 95.60     | 95.90    | 96.10    | 95.05    | 95.40    | 97.65    | 96.40    |         |         |
| $T_0 + 3$ | 96.65    | 96.20    | 94.80     | 94.55    | 97.10    | 96.10     | 96.50    | 96.20    | 94.85    | 95.25    | 96.75    | 95.10    |         |         |
| $T_0 + 4$ | 96.40    | 96.35    | 94.75     | 94.90    | 97.00    | 96.35     | 96.85    | 96.65    | 95.65    | 95.70    | 96.85    | 95.20    |         |         |
| $T_0 + 5$ | 96.50    | 96.35    | 94.95     | 94.50    | 97.45    | 96.15     | 95.75    | 95.75    | 94.55    | 94.05    | 96.25    | 95.30    |         |         |

### Estimated number of factors, 90% CI

|         | (20, 30) | (20, 50) | (20, 100) | (40, 30) | (40, 50) | (40, 100) |
|---------|----------|----------|-----------|----------|----------|-----------|
|         | EQ       | SY       | EQ        | SY       | EQ       | SY        | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       |
| $T_0 + 1$ | 92.95    | 92.55    | 91.85     | 92.80    | 92.25    | 91.95     | 92.60    | 93.05    | 90.90    | 91.10    | 92.70    | 93.40    |         |         |
| $T_0 + 2$ | 92.65    | 92.75    | 90.20     | 90.30    | 92.00    | 91.95     | 92.80    | 92.35    | 91.35    | 91.70    | 93.95    | 93.25    |         |         |
| $T_0 + 3$ | 92.55    | 92.55    | 88.80     | 89.35    | 93.20    | 93.45     | 91.80    | 91.70    | 91.25    | 90.35    | 92.75    | 91.80    |         |         |
| $T_0 + 4$ | 91.75    | 91.70    | 89.60     | 89.60    | 93.50    | 93.30     | 93.20    | 93.10    | 91.25    | 92.35    | 92.65    | 92.00    |         |         |
| $T_0 + 5$ | 92.30    | 92.30    | 89.60     | 89.00    | 91.85    | 92.20     | 91.95    | 91.80    | 91.10    | 91.25    | 92.20    | 91.80    |         |         |

### Estimated number of factors, 95% CI

|         | (20, 30) | (20, 50) | (20, 100) | (40, 30) | (40, 50) | (40, 100) |
|---------|----------|----------|-----------|----------|----------|-----------|
|         | EQ       | SY       | EQ        | SY       | EQ       | SY        | EQ       | SY       | EQ       | SY       | EQ       | SY       | EQ       | SY       |
| $T_0 + 1$ | 96.35    | 96.25    | 96.10     | 96.15    | 95.95    | 95.75     | 96.00    | 95.75    | 95.70    | 95.90    | 96.35    | 95.95    |         |         |
| $T_0 + 2$ | 96.05    | 95.70    | 94.30     | 94.35    | 95.15    | 94.95     | 96.35    | 96.10    | 94.95    | 95.25    | 97.60    | 96.25    |         |         |
| $T_0 + 3$ | 96.85    | 96.95    | 93.45     | 93.55    | 96.55    | 96.05     | 95.85    | 95.55    | 94.40    | 94.75    | 96.55    | 95.75    |         |         |
| $T_0 + 4$ | 95.80    | 95.20    | 93.50     | 93.45    | 97.20    | 96.40     | 96.85    | 96.40    | 95.40    | 95.55    | 96.40    | 95.75    |         |         |
| $T_0 + 5$ | 95.55    | 95.60    | 93.80     | 94.15    | 95.80    | 95.40     | 95.70    | 95.55    | 94.85    | 94.70    | 95.35    | 95.15    |         |         |
Table 4: Pure factor model with heteroscedastic AR(1) uniform errors

**Known number of factors, 90% CI**

| $t$     | $(T_0, N_0)$ | $(20, 30)$  | $(20, 50)$  | $(20, 100)$ | $(40, 30)$  | $(40, 50)$  | $(40, 100)$ |
|---------|--------------|-------------|-------------|-------------|-------------|-------------|-------------|
|         |              | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  |
| $T_0 + 1$ | $(20, 30)$   | 91.30| 91.50| 91.65| 91.95| 95.30| 95.20| 91.50| 91.30| 91.90| 91.85| 92.50| 92.50|
|         | $(20, 50)$   | 91.70| 91.75| 91.50| 91.00| 95.15| 94.90| 92.70| 92.65| 91.70| 91.75| 92.55| 92.90|
|         | $(20, 100)$  | 90.25| 90.50| 91.45| 91.75| 94.80| 94.95| 90.85| 90.65| 91.55| 91.20| 94.70| 94.75|
| $T_0 + 4$ | $(40, 30)$   | 91.70| 91.65| 90.55| 90.65| 94.50| 94.35| 90.40| 90.55| 90.15| 90.00| 92.90| 92.60|
|         | $(40, 50)$   | 92.40| 92.45| 92.10| 91.95| 94.35| 93.95| 93.10| 92.60| 91.35| 91.10| 93.35| 93.35|
| $T_0 + 5$ | $(40, 100)$  | 96.75| 96.80| 96.90| 97.05| 98.65| 98.65| 96.20| 96.35| 97.30| 97.15| 97.85| 97.25|

**Known number of factors, 95% CI**

| $t$     | $(T_0, N_0)$ | $(20, 30)$  | $(20, 50)$  | $(20, 100)$ | $(40, 30)$  | $(40, 50)$  | $(40, 100)$ |
|---------|--------------|-------------|-------------|-------------|-------------|-------------|-------------|
|         |              | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  |
| $T_0 + 1$ | $(20, 30)$   | 96.75| 96.80| 96.90| 97.05| 98.65| 98.65| 96.20| 96.35| 97.30| 97.15| 97.85| 97.25|
|         | $(20, 50)$   | 95.65| 95.60| 97.35| 97.25| 98.60| 98.60| 95.60| 96.45| 95.80| 96.15| 97.80| 97.75|
|         | $(20, 100)$  | 95.35| 95.30| 96.00| 96.05| 98.90| 98.90| 95.80| 95.75| 96.35| 96.65| 98.60| 98.60|
| $T_0 + 4$ | $(40, 30)$   | 95.85| 95.75| 95.10| 95.15| 98.45| 98.50| 95.55| 95.05| 96.35| 96.40| 97.30| 97.10|
|         | $(40, 50)$   | 96.25| 96.20| 96.75| 96.70| 97.80| 98.85| 96.35| 96.55| 96.30| 96.05| 97.85| 98.00|

**Estimated number of factors, 90% CI**

| $t$     | $(T_0, N_0)$ | $(20, 30)$  | $(20, 50)$  | $(20, 100)$ | $(40, 30)$  | $(40, 50)$  | $(40, 100)$ |
|---------|--------------|-------------|-------------|-------------|-------------|-------------|-------------|
|         |              | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  |
| $T_0 + 1$ | $(20, 30)$   | 91.50| 91.65| 90.50| 90.15| 93.95| 93.90| 92.60| 92.35| 91.45| 92.15| 92.05| 91.70|
|         | $(20, 50)$   | 91.95| 91.75| 90.70| 90.60| 93.65| 93.60| 93.15| 93.00| 91.40| 91.70| 91.90| 92.45|
|         | $(20, 100)$  | 91.75| 91.65| 90.90| 90.55| 93.80| 94.15| 90.35| 90.60| 90.85| 90.55| 94.00| 94.00|
| $T_0 + 4$ | $(40, 30)$   | 91.90| 92.20| 89.45| 89.80| 93.75| 93.90| 91.55| 91.70| 90.90| 91.20| 92.95| 92.85|
|         | $(40, 50)$   | 90.85| 91.20| 89.85| 89.85| 93.50| 93.10| 92.95| 93.20| 92.00| 91.90| 92.50| 92.30|

**Estimated number of factors, 95% CI**

| $t$     | $(T_0, N_0)$ | $(20, 30)$  | $(20, 50)$  | $(20, 100)$ | $(40, 30)$  | $(40, 50)$  | $(40, 100)$ |
|---------|--------------|-------------|-------------|-------------|-------------|-------------|-------------|
|         |              | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  | EQ  | SY  |
| $T_0 + 1$ | $(20, 30)$   | 95.95| 96.00| 96.55| 96.40| 98.30| 98.25| 96.55| 96.70| 97.15| 97.10| 97.75| 97.45|
|         | $(20, 50)$   | 96.25| 96.20| 96.10| 96.25| 97.50| 97.50| 97.40| 97.05| 96.50| 96.35| 97.80| 98.05|
|         | $(20, 100)$  | 96.35| 96.00| 95.40| 95.30| 98.35| 98.25| 94.90| 95.30| 95.70| 96.30| 98.20| 98.25|
| $T_0 + 4$ | $(40, 30)$   | 96.25| 96.00| 95.35| 95.65| 98.10| 98.10| 95.80| 95.70| 95.70| 95.90| 97.00| 97.10|
|         | $(40, 50)$   | 96.45| 96.55| 95.85| 95.85| 97.65| 97.50| 96.85| 96.80| 96.50| 96.30| 97.55| 97.60|
Table 5: Factor model with covariates and homoscedastic i.i.d. chi-squared errors

|                  | (T_0, N_0) | (20, 30) |       | (20, 50) |       | (20, 100) |       | (40, 30) |       | (40, 50) |       | (40, 100) |       |
|------------------|------------|----------|-------|----------|-------|-----------|-------|----------|-------|----------|-------|-----------|-------|
|                  | EQ SY      | EQ SY    | EQ SY | EQ SY    | EQ SY | EQ SY     | EQ SY | EQ SY    | EQ SY | EQ SY    | EQ SY | EQ SY     | EQ SY |
| Known number of factors, 90% CI |
| T_0 + 1          | 92.00      | 91.80    | 91.90 | 91.75    | 91.75 | 90.60     | 91.65 | 90.30    | 92.95 | 92.30    | 93.55 | 92.35     |       |
| T_0 + 2          | 93.35      | 91.95    | 91.35 | 91.40    | 91.40 | 91.00     | 93.10 | 91.20    | 92.65 | 91.30    | 92.40 | 91.50     |       |
| T_0 + 3          | 91.75      | 91.35    | 90.75 | 90.60    | 90.70 | 91.20     | 92.70 | 92.00    | 91.60 | 91.00    | 92.60 | 92.25     |       |
| T_0 + 4          | 91.90      | 91.60    | 92.15 | 92.15    | 92.00 | 92.10     | 93.40 | 92.65    | 91.85 | 92.05    | 93.15 | 92.20     |       |
| T_0 + 5          | 91.65      | 91.55    | 90.45 | 90.70    | 91.60 | 90.70     | 91.85 | 91.20    | 93.70 | 91.90    | 93.30 | 90.95     |       |

| Known number of factors, 95% CI |
|---------------------------------|------------|----------|-------|----------|-------|-----------|-------|----------|-------|----------|-------|-----------|-------|
| T_0 + 1                         | 95.80      | 94.80    | 94.80 | 94.50    | 95.55 | 94.80     | 96.40 | 94.70    | 96.50 | 95.70    | 96.60 | 95.50     |       |
| T_0 + 2                         | 96.05      | 95.40    | 96.10 | 94.55    | 95.20 | 94.75     | 96.30 | 95.60    | 96.30 | 94.85    | 96.35 | 95.00     |       |
| T_0 + 3                         | 96.80      | 95.05    | 95.20 | 94.50    | 94.65 | 93.85     | 95.90 | 95.50    | 95.95 | 94.75    | 96.70 | 94.40     |       |
| T_0 + 4                         | 95.85      | 94.60    | 95.25 | 94.65    | 96.50 | 95.05    | 97.15 | 96.10    | 95.85 | 94.65    | 97.15 | 94.90     |       |
| T_0 + 5                         | 95.75      | 94.65    | 94.10 | 94.30    | 95.55 | 94.85    | 96.25 | 95.45    | 97.25 | 95.65    | 96.60 | 95.45     |       |

| Estimated number of factors, 90% CI |
|-----------------------------------|------------|----------|-------|----------|-------|-----------|-------|----------|-------|----------|-------|-----------|-------|
| T_0 + 1                           | 92.40      | 92.70    | 91.80 | 91.45    | 91.75 | 90.60     | 91.60 | 90.50    | 93.05 | 92.25    | 93.55 | 92.35     |       |
| T_0 + 2                           | 92.30      | 91.80    | 91.45 | 91.35    | 91.40 | 90.90     | 92.80 | 91.45    | 92.65 | 91.35    | 92.55 | 91.50     |       |
| T_0 + 3                           | 92.20      | 92.10    | 90.95 | 91.30    | 90.75 | 91.20     | 92.55 | 92.25    | 91.75 | 91.00    | 92.65 | 92.35     |       |
| T_0 + 4                           | 92.25      | 91.75    | 92.25 | 92.10    | 92.00 | 92.10     | 93.45 | 92.70    | 91.95 | 92.10    | 93.15 | 92.20     |       |
| T_0 + 5                           | 91.60      | 91.35    | 90.70 | 90.80    | 91.40 | 90.70     | 91.70 | 91.15    | 93.75 | 92.05    | 93.30 | 90.95     |       |

| Estimated number of factors, 95% CI |
|------------------------------------|------------|----------|-------|----------|-------|-----------|-------|----------|-------|----------|-------|-----------|-------|
| T_0 + 1                            | 96.25      | 95.05    | 94.60 | 94.60    | 95.55 | 94.80     | 96.75 | 94.70    | 96.45 | 95.90    | 96.60 | 95.50     |       |
| T_0 + 2                            | 96.15      | 95.45    | 95.95 | 94.60    | 95.15 | 94.75     | 96.35 | 95.55    | 96.30 | 94.80    | 96.35 | 95.00     |       |
| T_0 + 3                            | 96.75      | 95.60    | 94.90 | 94.45    | 94.70 | 93.85     | 96.25 | 95.30    | 96.05 | 94.75    | 96.65 | 94.45     |       |
| T_0 + 4                            | 95.50      | 94.75    | 95.10 | 94.55    | 96.55 | 95.05    | 97.05 | 96.00    | 95.95 | 94.70    | 97.15 | 94.90     |       |
| T_0 + 5                            | 95.50      | 94.85    | 94.40 | 94.35    | 95.55 | 94.85    | 96.15 | 95.20    | 97.25 | 95.65    | 96.60 | 95.45     |       |
Table 6: Factor model with covariates and homoscedastic i.i.d. uniform errors

|                | (20, 30)       | (20, 50)       | (20, 100)      | (40, 30)       | (40, 50)       | (40, 100)      |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $T_0 + 1$      | EQ SY          | EQ SY          | EQ SY          | EQ SY          | EQ SY          | EQ SY          |
| $T_0 + 2$      | 93.15 92.75    | 93.20 93.15    | 91.90 91.55    | 90.95 91.05    | 92.40 92.20    | 90.25 90.25    |
| $T_0 + 3$      | 90.40 91.10    | 92.65 92.85    | 91.10 90.60    | 91.60 91.60    | 90.95 91.10    | 90.55 90.60    |
| $T_0 + 4$      | 93.35 93.25    | 92.00 91.90    | 91.60 91.35    | 91.85 92.85    | 91.65 91.70    | 92.00 91.95    |
| $T_0 + 5$      | 92.30 93.65    | 88.90 88.85    | 91.55 91.55    | 90.70 90.85    | 92.05 92.15    | 91.35 91.35    |

Known number of factors, 95% CI

|                | (20, 30)       | (20, 50)       | (20, 100)      | (40, 30)       | (40, 50)       | (40, 100)      |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $T_0 + 1$      | EQ SY          | EQ SY          | EQ SY          | EQ SY          | EQ SY          | EQ SY          |
| $T_0 + 2$      | 96.65 97.10    | 96.95 96.80    | 95.60 95.70    | 96.50 96.50    | 96.60 96.55    | 95.55 95.45    |
| $T_0 + 3$      | 96.90 97.05    | 96.80 97.05    | 96.75 96.90    | 97.00 97.05    | 97.50 97.65    | 96.90 97.05    |
| $T_0 + 4$      | 97.05 98.00    | 96.10 96.05    | 96.10 96.25    | 96.80 96.75    | 97.35 97.60    | 96.40 96.25    |
| $T_0 + 5$      | 96.45 96.65    | 97.30 97.30    | 96.30 96.25    | 96.35 96.05    | 96.15 96.25    | 96.95 96.80    |

Estimated number of factors, 90% CI

|                | (20, 30)       | (20, 50)       | (20, 100)      | (40, 30)       | (40, 50)       | (40, 100)      |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $T_0 + 1$      | EQ SY          | EQ SY          | EQ SY          | EQ SY          | EQ SY          | EQ SY          |
| $T_0 + 2$      | 93.15 92.75    | 93.20 93.15    | 91.90 91.55    | 90.95 91.05    | 92.40 92.20    | 90.25 90.25    |
| $T_0 + 3$      | 90.40 91.10    | 92.65 92.85    | 91.10 90.60    | 91.60 91.60    | 90.95 91.10    | 90.55 90.60    |
| $T_0 + 4$      | 93.35 93.25    | 92.00 91.90    | 91.60 91.35    | 91.85 92.85    | 91.65 91.70    | 92.00 91.95    |
| $T_0 + 5$      | 92.30 93.65    | 88.90 88.85    | 91.55 91.55    | 90.70 90.85    | 92.05 92.15    | 91.35 91.35    |

Estimated number of factors, 95% CI

|                | (20, 30)       | (20, 50)       | (20, 100)      | (40, 30)       | (40, 50)       | (40, 100)      |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $T_0 + 1$      | EQ SY          | EQ SY          | EQ SY          | EQ SY          | EQ SY          | EQ SY          |
| $T_0 + 2$      | 97.60 97.65    | 97.35 97.50    | 96.70 96.70    | 97.30 97.35    | 96.85 96.75    | 96.65 96.75    |
| $T_0 + 3$      | 96.65 97.10    | 96.95 96.80    | 95.60 95.70    | 96.50 96.50    | 96.60 96.55    | 95.55 95.45    |
| $T_0 + 4$      | 96.90 97.00    | 96.75 97.05    | 96.75 96.90    | 97.00 97.05    | 97.50 97.65    | 96.90 97.05    |
| $T_0 + 5$      | 97.05 98.00    | 96.10 96.05    | 96.10 96.25    | 96.80 96.75    | 97.35 97.60    | 96.40 96.25    |
Table 7: Factor model with covariates and heteroscedastic AR(1) chi-squared errors

| $t$  | $(T_0, N_0)$ | $(20, 30)$ | $(20, 50)$ | $(20, 100)$ | $(40, 30)$ | $(40, 50)$ | $(40, 100)$ |
|------|--------------|-----------|-----------|-----------|-----------|-----------|-----------|
|      | $(T_0, N_0)$ | EQ SY     | EQ SY     | EQ SY     | EQ SY     | EQ SY     | EQ SY     |
| $T_0 + 1$ | $(20, 30)$ | 92.55 91.75 | 89.50 89.70 | 90.20 90.35 | 92.15 91.60 | 90.20 90.40 | 92.50 93.45 |
| $T_0 + 2$ | $(20, 50)$ | 92.45 91.90 | 88.80 88.80 | 90.75 90.95 | 92.20 91.95 | 91.25 91.80 | 91.70 92.25 |
| $T_0 + 3$ | $(20, 100)$ | 91.40 91.25 | 87.65 87.75 | 91.25 90.90 | 91.90 91.05 | 91.80 91.85 | 92.20 91.85 |
| $T_0 + 4$ | $(40, 30)$ | 89.45 89.50 | 89.25 89.45 | 91.50 91.25 | 90.80 91.15 | 89.70 89.95 | 91.20 91.20 |
| $T_0 + 5$ | $(40, 50)$ | 91.05 90.90 | 88.35 88.30 | 90.30 90.25 | 92.15 92.35 | 90.80 90.45 | 90.95 91.00 |

 Known number of factors, 95% CI

| $t$  | $(T_0, N_0)$ | $(20, 30)$ | $(20, 50)$ | $(20, 100)$ | $(40, 30)$ | $(40, 50)$ | $(40, 100)$ |
|------|--------------|-----------|-----------|-----------|-----------|-----------|-----------|
|      | $(T_0, N_0)$ | EQ SY     | EQ SY     | EQ SY     | EQ SY     | EQ SY     | EQ SY     |
| $T_0 + 1$ | $(20, 30)$ | 96.45 96.05 | 94.90 94.65 | 94.75 93.95 | 96.25 95.75 | 95.70 95.10 | 96.85 96.85 |
| $T_0 + 2$ | $(20, 50)$ | 95.20 95.20 | 94.90 94.50 | 95.90 95.05 | 95.95 95.35 | 95.90 96.40 | 96.20 95.85 |
| $T_0 + 3$ | $(20, 100)$ | 95.20 95.10 | 93.75 94.15 | 94.95 95.25 | 95.60 95.25 | 96.05 95.75 | 96.00 96.15 |
| $T_0 + 4$ | $(40, 30)$ | 93.40 93.55 | 95.05 95.15 | 95.40 95.30 | 95.50 95.20 | 94.90 94.20 | 96.20 95.30 |
| $T_0 + 5$ | $(40, 50)$ | 95.35 95.15 | 93.80 93.80 | 94.40 94.25 | 95.70 95.10 | 95.10 95.25 | 95.70 95.20 |

 Estimated number of factors, 90% CI

| $t$  | $(T_0, N_0)$ | $(20, 30)$ | $(20, 50)$ | $(20, 100)$ | $(40, 30)$ | $(40, 50)$ | $(40, 100)$ |
|------|--------------|-----------|-----------|-----------|-----------|-----------|-----------|
|      | $(T_0, N_0)$ | EQ SY     | EQ SY     | EQ SY     | EQ SY     | EQ SY     | EQ SY     |
| $T_0 + 1$ | $(20, 30)$ | 92.15 92.05 | 89.75 89.85 | 89.35 90.65 | 92.45 92.40 | 90.55 90.45 | 93.00 93.50 |
| $T_0 + 2$ | $(20, 50)$ | 91.85 92.40 | 89.25 89.35 | 90.35 91.10 | 92.00 92.45 | 90.85 92.00 | 92.40 92.10 |
| $T_0 + 3$ | $(20, 100)$ | 92.15 92.05 | 88.00 88.00 | 90.65 90.85 | 92.80 92.20 | 91.25 91.55 | 92.10 91.55 |
| $T_0 + 4$ | $(40, 30)$ | 89.70 89.85 | 88.70 89.20 | 90.20 89.95 | 92.75 92.95 | 89.75 90.30 | 91.05 90.55 |
| $T_0 + 5$ | $(40, 50)$ | 91.45 91.40 | 86.85 87.30 | 89.90 89.80 | 91.10 91.55 | 90.60 90.80 | 91.10 91.05 |

 Estimated number of factors, 95% CI

| $t$  | $(T_0, N_0)$ | $(20, 30)$ | $(20, 50)$ | $(20, 100)$ | $(40, 30)$ | $(40, 50)$ | $(40, 100)$ |
|------|--------------|-----------|-----------|-----------|-----------|-----------|-----------|
|      | $(T_0, N_0)$ | EQ SY     | EQ SY     | EQ SY     | EQ SY     | EQ SY     | EQ SY     |
| $T_0 + 1$ | $(20, 30)$ | 96.60 95.95 | 95.15 95.25 | 94.70 94.75 | 95.95 95.45 | 94.95 94.65 | 97.15 96.55 |
| $T_0 + 2$ | $(20, 50)$ | 95.75 95.80 | 94.85 94.80 | 95.05 94.45 | 95.30 95.30 | 96.50 96.00 | 95.95 95.90 |
| $T_0 + 3$ | $(20, 100)$ | 95.60 95.55 | 93.40 93.25 | 95.05 94.70 | 95.60 95.35 | 95.90 95.90 | 96.35 96.30 |
| $T_0 + 4$ | $(40, 30)$ | 93.85 93.80 | 93.55 93.95 | 94.95 94.90 | 95.95 95.95 | 95.75 94.70 | 95.70 95.05 |
| $T_0 + 5$ | $(40, 50)$ | 95.10 94.95 | 92.00 92.10 | 94.25 93.65 | 95.40 95.20 | 95.05 94.00 | 95.40 95.25 |
Table 8: Factor model with covariates and heteroscedastic AR(1) uniform errors

**Known number of factors, 90% CI**

|        | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ |
|--------|------------------|------------------|------------------|------------------|------------------|------------------|
| $N_0$  | $(20,30)$        | $(20,50)$        | $(20,100)$       | $(40,30)$        | $(40,50)$        | $(40,100)$       |
| $T_0 + 1$ | EQ SY           | EQ SY           | EQ SY           | EQ SY           | EQ SY           | EQ SY           |
|        | 93.70 93.95     | 92.25 92.60     | 89.55 90.60     | 93.05 93.10     | 92.45 92.55     | 90.60 90.50     |
| $T_0 + 2$ | 93.30 93.45     | 92.80 93.20     | 89.50 89.55     | 93.35 93.30     | 92.10 92.05     | 92.30 92.30     |
| $T_0 + 3$ | 94.35 94.35     | 93.95 94.15     | 89.75 89.85     | 93.80 94.00     | 93.45 93.65     | 91.30 91.25     |
| $T_0 + 4$ | 94.25 94.25     | 90.85 91.00     | 92.65 92.55     | 93.30 92.80     | 93.25 93.05     | 93.25 93.30     |
| $T_0 + 5$ | 94.95 94.95     | 90.60 90.60     | 90.30 90.25     | 92.80 93.00     | 91.40 91.40     | 92.00 91.80     |

**Known number of factors, 95% CI**

|        | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ |
|--------|------------------|------------------|------------------|------------------|------------------|------------------|
| $N_0$  | $(20,30)$        | $(20,50)$        | $(20,100)$       | $(40,30)$        | $(40,50)$        | $(40,100)$       |
| $T_0 + 1$ | EQ SY           | EQ SY           | EQ SY           | EQ SY           | EQ SY           | EQ SY           |
|        | 97.50 97.50     | 96.95 96.95     | 95.70 95.85     | 98.20 98.20     | 96.95 96.85     | 95.90 95.60     |
| $T_0 + 2$ | 98.10 98.00     | 97.40 97.65     | 96.05 95.70     | 97.50 97.50     | 96.80 96.80     | 96.50 96.50     |
| $T_0 + 3$ | 97.90 97.85     | 97.25 97.50     | 95.95 95.95     | 96.95 97.00     | 97.25 97.55     | 96.60 96.65     |
| $T_0 + 4$ | 98.15 98.10     | 96.35 96.35     | 96.85 96.75     | 97.50 97.35     | 97.80 97.70     | 97.40 97.50     |
| $T_0 + 5$ | 97.95 98.00     | 96.70 96.70     | 96.00 96.05     | 97.65 97.60     | 97.20 97.05     | 96.70 96.55     |

**Estimated number of factors, 90% CI**

|        | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ |
|--------|------------------|------------------|------------------|------------------|------------------|------------------|
| $N_0$  | $(20,30)$        | $(20,50)$        | $(20,100)$       | $(40,30)$        | $(40,50)$        | $(40,100)$       |
| $T_0 + 1$ | EQ SY           | EQ SY           | EQ SY           | EQ SY           | EQ SY           | EQ SY           |
|        | 92.10 92.30     | 91.45 90.85     | 91.10 91.20     | 93.15 93.30     | 91.80 91.80     | 91.10 90.75     |
| $T_0 + 2$ | 92.40 92.30     | 93.30 93.35     | 89.60 89.55     | 92.10 92.10     | 90.25 90.30     | 92.50 92.85     |
| $T_0 + 3$ | 92.75 92.85     | 91.85 92.25     | 88.90 88.95     | 92.55 92.55     | 93.25 93.25     | 90.50 90.20     |
| $T_0 + 4$ | 93.50 93.40     | 90.25 90.20     | 91.45 91.45     | 93.55 93.90     | 93.10 93.10     | 92.85 92.80     |
| $T_0 + 5$ | 92.25 92.35     | 90.35 90.45     | 90.20 90.00     | 93.45 93.80     | 91.50 91.50     | 91.85 91.65     |

**Estimated number of factors, 95% CI**

|        | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ | $\overline{T}_0$ |
|--------|------------------|------------------|------------------|------------------|------------------|------------------|
| $N_0$  | $(20,30)$        | $(20,50)$        | $(20,100)$       | $(40,30)$        | $(40,50)$        | $(40,100)$       |
| $T_0 + 1$ | EQ SY           | EQ SY           | EQ SY           | EQ SY           | EQ SY           | EQ SY           |
|        | 97.20 97.20     | 95.25 95.60     | 96.25 96.25     | 97.90 97.90     | 95.90 95.90     | 96.15 96.10     |
| $T_0 + 2$ | 97.80 98.15     | 96.90 96.85     | 95.85 95.70     | 96.85 96.80     | 95.85 95.90     | 96.80 96.95     |
| $T_0 + 3$ | 98.10 98.25     | 96.80 96.95     | 94.70 94.70     | 96.85 96.80     | 96.65 96.85     | 95.95 96.40     |
| $T_0 + 4$ | 97.70 97.60     | 95.85 95.80     | 95.75 96.00     | 97.85 97.55     | 97.10 97.10     | 97.25 97.35     |
| $T_0 + 5$ | 96.80 96.80     | 96.75 96.85     | 95.80 95.75     | 97.40 97.45     | 96.50 96.50     | 96.55 96.60     |
6 Empirical Applications

In this section, we re-evaluate the impacts of Hong Kong’s Political and Economic Integration with Mainland China as well as the effects of California’s Tobacco Control Program using our proposed bootstrap procedure.  

6.1 Hong Kong’s Political and Economic Integration with Mainland China Revisited

In this subsection, we revisit the impacts of political and economic integration of Hong Kong with Mainland China, which has been analysed in Hsiao et al. (2012). Since Hsiao et al. (2012) specify a pure factor model without covariates, we apply the methods in Section 3 of this paper to the dataset of Hsiao et al. (2012). For the results about political integration, quarterly real GDP growth rates from 1993Q1 to 1997Q2 of 10 countries and districts are used to form the counter-factual path of Hong Kong from 1997Q3 up to 2003Q4. The 10 countries and districts are Mainland China, Indonesia, Japan, Korea, Malaysia, Philippines, Singapore, Taiwan, Thailand and US. For the analysis of economic integration, quarterly real GDP growth rates of 24 countries and districts from 1993Q1 to 2003Q4 are used to form counter-factual path of Hong Kong from 2004Q1 to 2008Q1.

The results for the impact of political integration and economic integration with Mainland China on Hong Kong’s economic growth are provided in 1 and 2, respectively.

As is shown in Figure 1, the estimated treatment effects of Hong Kong political integration are of similar magnitudes and patterns as those in Hsiao et al. (2012). In Figure 2, the estimated treatment effects of Hong Kong economic integration are positive, as in Hsiao et al. (2012), and confidence intervals cover 0 in most of the periods. This suggests that the impact of political and economic integration of Hong Kong with Mainland China is significant in the first few years after integration, and vanishes afterward. This observation is in general consistent with the insignificant average treatment effects across post-treatment periods in Hsiao et al. (2012).

MATLAB, Python and R codes for application are available from the authors upon request.

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1997 1998 1999 2000 2001 2002 2003 2004 Year
-0.4 -0.3 -0.2 -0.1 0 0.1 0.2 TE of political integration on GDP growth
Hong Kong Political Integration (Wild Boot)
95% CI 90% CI zero

1997 1998 1999 2000 2001 2002 2003 2004 Year
-0.4 -0.3 -0.2 -0.1 0 0.1 0.2 TE of political integration on GDP growth
Hong Kong Political Integration (Block Wild Boot)
95% CI 90% CI zero

Figure 1: Impact of Political Integration with Mainland China on Hong Kong Economic Growth

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4MATLAB, Python and R codes for application are available from the authors upon request.
6.2 California’s Tobacco Control Program Revisited

Now we revisit the effectiveness of CTCP on per capita cigarettes consumption and personal healthcare expenditures using the methods discussed in Section 4 of this article. In November 1988, California passed the Proposition 99, which increased California’s cigarettes tax by 25 cents per pack and earmarked the tax revenue for health and anti-smoking measures. Proposition 99 triggered a wave of local clean-air ordinances in California. Abadie et al. (2010) used the synthetic control method for the period 1970-2000 to show that the California Tobacco Control Program had a significant impact on per capita cigarette consumption for the period 1989-2000, and that its impact continued to be enhanced over time.

Note that in the model and dataset of Abadie et al. (2010), covariates do not change over time, which violates Assumption 2.6(2) of this study. To accommodate time-variant covariates, we use the dataset of Hsiao and Zhou (2019), who also revisit the impact of CTCP on per capita cigarettes consumption and personal healthcare expenditures but use a set of time-variant covariates: per capita GDP obtained from Abadie et al. (2010); poverty rates obtained from the National Census Bureau; educational attainment, defined as the percentage of obtaining college degree of population 25 years and over, obtained from the National Census Bureau.

In our analysis, the number of factors is estimated using the method proposed by Alessi et al. (2010), which shows better performance than Bai and Ng (2002) in this and the next applications. Both ordinary wild bootstrap procedure and block wild bootstrap procedure with block width equal to 3 are considered, and equal tailed confidence intervals are reported. The results for the impact of CTCP on per capita cigarette consumption and personal healthcare expenditures are provided in 3 and 4, respectively.

Figure 3 shows that the estimated treatment effects are of similar magnitudes as those in Abadie et al. (2010); and that the confidence intervals indicate negative and significant impacts over time, consistent with their findings using permutation tests. Figure 4 shows that the estimated treatment
Figure 3: Impact of CTCP on per capita cigarette consumption

Figure 4: Impact of CTCP on personal healthcare expenditures

effects of CTCP on health expenditure are of similar magnitudes as reported in Hsiao and Zhou (2019). Consistent with their findings, confidence intervals indicate that the effects are short-lived. In both figures, we can see that the confidence intervals based on ordinary and block wild bootstrap procedures produce similar results.

7 Conclusion

In this paper, we consider the construction of confidence intervals for treatment effects estimated in panel models with interactive fixed effects, which serves as an alternative inferential approach. We first use the factor-based matrix completion technique proposed by Bai and Ng (2021) for panel models to estimate the treatment effects, and then use bootstrap method to construct confidence intervals of the treatment effects for treated units at each post-treatment period. Our construction of confidence intervals requires neither specific distributional assumptions on the error terms nor large number of post-treatment periods. We also establish the validity of proposed bootstrap procedure that these confidence intervals have asymptotically correct coverage probabilities. Simulation studies show that these confidence intervals have satisfactory finite sample performances, and empirical applications using classical datasets yield treatment effect estimates of similar magnitude and reliable confidence
intervals.

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The online appendices include auxiliary lemmas, technical notes, and the proofs of Theorems 3.1 and 4.1. We first list some results for estimators and the associated residuals in panel models with interactive effects, and then provide the technical properties of bootstrap procedure. Finally, we use the above mentioned results to prove the theorems in the main paper.

Appendix A  Properties of Estimators and Residuals

The lemmas in this section are cited or derived from some of the results in Bai (2003), Bai (2009), Gonçalves and Perron (2014), and Bai and Ng (2021). Note that Assumptions 2.1–2.8 of this paper are sufficient for Assumptions A–G of Bai (2003), Assumptions A–E of Bai (2009), panel factor model relevant conditions of Assumptions 1–5 of Gonçalves and Perron (2014), and Assumptions A–D of Bai and Ng (2021).

A.1 Estimators and Residuals in Section 3

We introduce some notations. Let $\hat{V}_{\text{tall}} = D_{\text{tall}}^2$ and $\hat{V}_{\text{wide}} = D_{\text{wide}}^2$. Define rotation matrices

$$H_{\text{tall}} = \left( \frac{\Lambda_{\text{tall}}^T \Lambda_{\text{tall}}}{N_0} \right) \left( \frac{F_{\text{tall}}^T \hat{F}_{\text{tall}}}{T} \right) \hat{V}_{\text{tall}}^{-1}, \quad H_{\text{wide}} = \left( \frac{\Lambda_{\text{wide}}^T \Lambda_{\text{wide}}}{N} \right) \left( \frac{F_{\text{wide}}^T \hat{F}_{\text{wide}}}{T_0} \right) \hat{V}_{\text{wide}}^{-1}.$$

Lemma A.1: If Assumptions 2.1–2.5 hold, then as $N_0, T_0 \to \infty$,

1. $\hat{V}_{\text{tall}} \overset{P}{\to} V$ and $\hat{V}_{\text{wide}} \overset{P}{\to} V$, where $V$ is the diagonal matrix consisting of the eigenvalues of $\Sigma_A \Sigma_F$, or equivalently, the eigenvalues of $\Sigma_A^{1/2} \Sigma_F \Sigma_A^{1/2}$. 

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(2) $H_{\text{tall}} \xrightarrow{p} \Sigma_{\Lambda}^{1/2}UV^{-1/2}$, $H_{\text{wide}} \xrightarrow{p} \Sigma_{\Lambda}^{1/2}UV^{-1/2}$, $H_{\text{tall}}^{-1} \xrightarrow{p} V^{1/2}U^{\top}\Sigma_{\Lambda}^{-1/2}$, and $H_{\text{wide}}^{-1} \xrightarrow{p} V^{1/2}U^{\top}\Sigma_{\Lambda}^{-1/2}$, where $V$ is the diagonal matrix consisting of the eigenvalues of $\Sigma_{\Lambda}^{1/2}\Sigma_F\Sigma_{\Lambda}^{1/2}$, and $U$ is the corresponding eigenvector matrix such that $\Sigma_{\Lambda}^{1/2}\Sigma_F\Sigma_{\Lambda}^{1/2}U = UV$.

(3) $\hat{f}_{\text{tall},t} - H_{\text{tall}}^{\top}f_t = O_p\left(\frac{1}{\sqrt{N_0}}\right)$ for every $t \in \{1,\ldots, T\}$.

(4) $\hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1}\lambda_i = O_p\left(\frac{1}{\sqrt{T_0}}\right)$ for every $i \in \{1,\ldots, N\}$.

(5) $\hat{H}_{\text{miss}} - H_{\text{tall}}^{-1}H_{\text{wide}} = O_p\left(\frac{1}{N_0 \wedge T_0}\right)$, and $\hat{H}_{\text{miss}} \xrightarrow{p} I_r$.

(6) $\frac{1}{T} \sum_{i=1}^{T} \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}}^{\top}f_t \right\|^2 = O_p\left(\frac{1}{N_0 \wedge T_0}\right)$, and $\frac{1}{T_0} \sum_{i=1}^{T_0} \left\| \hat{f}_{\text{wide},t} - H_{\text{wide}}^{\top}f_t \right\|^2 = O_p\left(\frac{1}{N_0 \wedge T_0}\right)$.

**Proof:** Claim (1) is obtained by Lemma A.3(1) of Bai (2003) (applied to the control subsample and pre-treatment subsample) and Assumption 2.1(3)(4) of this paper. Claim (2) follows from Proposition of Bai (2003) (applied to the control subsample and pre-treatment subsample), Assumption 2.1(3)(4) of this paper, and the continuous mapping theorem. Note that Assumption 2.5(2) implies $\sqrt{N_0}/T \to 0$ and $\sqrt{T_0}/N \to 0$ as $N_0, T_0 \to \infty$. Then applying Theorem 1(i) to the control subsample yields Claim (3), and applying Theorem 2(i) to the pre-treatment subsample yields Claim (4). The first part of Claim (5) is borrowed from Lemma A.1(i) of Bai and Ng (2021), and the second part is by the first part and Claim (2). Claim (6) is obtained by applying Lemma A.1 of Bai (2003) to the control subsample and pre-treatment subsample. \[\square\]

For ease of citation, we state the $C_p$ inequality here. For $k, m \in \mathbb{Z}_+$ and $\{z_j\}_{j=1}^k \subset \mathbb{R}$, we have
\[
\left| \sum_{j=1}^{k} z_j \right|^m \leq k^{m-1} \sum_{j=1}^{k} |z_j|^m.
\]

**Lemma A.2:** If Assumptions 2.1–2.5 hold, then as $N_0, T_0 \to \infty$,
\[
\frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}_{\text{tall},i} - H_{\text{tall}}^{-1}H_{\text{wide}}\hat{\lambda}_{\text{wide},i} \right\|^2 = O_p\left(\frac{1}{T_0}\right).
\]

**Proof:** Let $e_{\text{wide},i} = (e_{i,1}, \ldots, e_{i,T_0})^\top$ for every $i \in \{1,\ldots, N\}$. By the proof of Theorem 2 of Bai (2003) (applied to the pre-treatment subsample), for every $i \in \{1,\ldots, N\}$, we have the following decomposition
\[
\hat{\lambda}_{\text{wide},i} = H_{\text{wide}}^{-1}\lambda_i + b_{1,i} + b_{2,i} + b_{3,i},
\]
where
\[
b_{1,i} = \frac{1}{T_0} \left( \hat{F}_{\text{wide}} - F_{\text{wide}}H_{\text{wide}} \right)^\top e_{\text{wide},i}, \quad b_{2,i} = \frac{1}{T_0} H_{\text{wide}}F_{\text{wide}}^\top e_{\text{wide},i},
\]
\[
b_{3,i} = \frac{1}{T_0} \hat{F}_{\text{wide}}^\top \left( F_{\text{wide}} - \hat{F}_{\text{wide}}H_{\text{wide}}^{-1} \right) \lambda_i.
\]
By the triangle inequality and Cauchy-Schwartz inequality,
\[
\left\| b_{1,i} \right\| = \left\| \frac{1}{T_0} \sum_{s=1}^{T_0} \left( \hat{f}_{\text{wide},s} - H_{\text{wide}}^{\top}f_s \right) e_{i,s} \right\| \leq \frac{1}{T_0} \sum_{s=1}^{T_0} \left\| \hat{f}_{\text{wide},s} - H_{\text{wide}}^{\top}f_s \right\| |e_{i,s}|
\]
Then Markov’s inequality implies that
\[ \frac{1}{NT_0} \sum_{i=1}^{N} T_0 \sum_{s=1}^{T_0} \epsilon_{i,s}^2 \leq \frac{1}{NT_0} \sum_{i=1}^{N} \sum_{s=1}^{T_0} \mathbb{E}(\epsilon_{i,s}^2) \leq M + 1. \]
By Assumption 2.3(1),
\[ \mathbb{E} \left( \left( \frac{1}{NT_0} \sum_{i=1}^{N} \sum_{s=1}^{T_0} \epsilon_{i,s}^2 \right) \right) = \frac{1}{NT_0} \sum_{i=1}^{N} \sum_{s=1}^{T_0} \mathbb{E}(\epsilon_{i,s}^2) = O_P(1). \]

Then Markov’s inequality implies that
\[ \frac{1}{NT_0} \sum_{i=1}^{N} T_0 \sum_{s=1}^{T_0} \epsilon_{i,s}^2 = O_P(1). \]

By Lemma A.1(6) and Assumption 2.5(2),
\[ \frac{1}{N} \sum_{i=1}^{N} \|b_{1,i}\|^2 \leq \left( \frac{1}{NT_0} \sum_{i=1}^{N} \sum_{s=1}^{T_0} \epsilon_{i,s}^2 \right) \left( \frac{1}{NT_0} \sum_{i=1}^{N} T_0 \sum_{s=1}^{T_0} \epsilon_{i,s}^2 \right) = O_P \left( \frac{1}{T_0} \right). \]

From Assumption 2.3(5),
\[ \mathbb{E} \left( \left( \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{T_0} \sum_{s=1}^{T_0} \epsilon_{i,s}^2 \right) \right) \right) = \frac{1}{T_0} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{\sqrt{T_0}} \sum_{s=1}^{T_0} f_s \epsilon_{i,s} \right) \right) \leq M \frac{1}{T_0}. \]

By the properties of matrix norms, the triangle inequality, Lemma A.1(2) and Markov’s inequality,
\[ \frac{1}{N} \sum_{i=1}^{N} \|b_{2,i}\|^2 = \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T_0} H_{\text{wide}} F_{\text{wide}} \right\| \leq \|H_{\text{wide}}\|^2 \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T_0} \sum_{s=1}^{T_0} f_s \epsilon_{i,s} \right\|^2 \right) = O_P \left( \frac{1}{T_0} \right). \]

Applying Lemma B.3 of Bai (2003) to the pre-treatment subsample yields
\[ \frac{1}{T_0} \widehat{F}_{\text{wide}}^T \left( F_{\text{wide}} H_{\text{wide}} - \widehat{F}_{\text{wide}} \right) = O_P \left( \frac{1}{N \wedge T_0} \right). \]

Then by the definition and property of Frobenius norm, Lemma A.1(2), Assumption 2.1(4) and 2.5(2),
\[ \frac{1}{N} \sum_{i=1}^{N} \|b_{3,i}\|^2 \leq \|H_{\text{wide}}\|^2 \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T_0} H_{\text{wide}} \right\|^2 \|H_{\text{wide}}^{-1}\|^2 \|\lambda_i\|^2 \right) \]
\[ = \left\| \frac{1}{T_0} \widehat{F}_{\text{wide}}^T \left( F_{\text{wide}} H_{\text{wide}} - \widehat{F}_{\text{wide}} \right) \right\|^2 \frac{\text{tr}(A^T A)}{N} = O_P \left( \frac{1}{T_0} \right). \]

Therefore, by the properties of matrix norms, the $C_p$ inequality and Lemma A.1(2),
\[ \frac{1}{N} \sum_{i=1}^{N} \left\| H_{\text{wide}} \widehat{\lambda}_{\text{wide},i} - \lambda_i \right\|^2 \leq \|H_{\text{wide}}\|^2 \left( \frac{1}{N} \sum_{i=1}^{N} \|b_{1,i} + b_{2,i} + b_{3,i}\|^2 \right) \]
\[ \leq \|H_{\text{wide}}\|^2 \left( \frac{3}{N} \sum_{i=1}^{N} \|b_{1,i}\|^2 + \|b_{2,i}\|^2 + \|b_{3,i}\|^2 \right) = O_P \left( \frac{1}{T_0} \right). \]
Similarly we can show that
\[ \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| H_{\text{tail}} \widehat{\lambda}_{\text{tail},i} - \lambda_i \right\|^2 = O_P \left( \frac{1}{T_0} \right). \]
By the properties of matrix norms, the $C_p$ inequality, Lemma A.1(2) and Assumption 2.5(2),
\[ \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \widehat{\lambda}_{\text{tail},i} - H_{\text{tail}}^{-1} H_{\text{wide}} \widehat{\lambda}_{\text{wide},i} \right\|^2 \]
\[ \leq 2 \|H_{\text{tail}}^{-1}\|^2 \left( \left( \frac{N}{N_0} \sum_{i=1}^{N} \left\| H_{\text{wide}} \widehat{\lambda}_{\text{wide},i} - \lambda_i \right\|^2 \right) + \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| H_{\text{tail}} \widehat{\lambda}_{\text{tail},i} - \lambda_i \right\|^2 \right) \right) = O_P \left( \frac{1}{T_0} \right), \]
which proves the conclusion.

**Lemma A.3:** If Assumptions 2.1–2.5 hold, then for every $m \in \{1, 2, \ldots, 8\}$, as $N_0, T_0 \to \infty$, 
\[
\begin{align*}
\sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}}^T f_t \right\|^m &= O_P(1), \\
\sum_{t=1}^{T_0} \left\| \hat{f}_{\text{wide},t} - H_{\text{wide}}^T f_t \right\|^m &= O_P(1).
\end{align*}
\]

(2) 
\[
\frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}_{\text{tall},i} - H_{\text{tall}}^{-1} \lambda_i \right\|^m = O_P(1), \\
\frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i \right\|^m = O_P(1).
\]

(3) 
\[
\sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t} \right\|^m = O_P(1), \\
\sum_{t=1}^{T_0} \left\| \hat{f}_{\text{wide},t} \right\|^m = O_P(1).
\]

(4) 
\[
\frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}_{\text{tall},i} \right\|^m = O_P(1), \\
\frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_{\text{wide},i} \right\|^m = O_P(1).
\]

(5) 
\[
\sum_{t=1}^{T} \left| \hat{e}_{i,t} \right|^m = O_P(1) \quad \text{for every } i \in \{1, \ldots, N_0\}.
\]

(6) 
\[
\sum_{t=1}^{T_0} \left| \hat{e}_{i,t} \right|^m = O_P(1) \quad \text{for every } i \in \{1, \ldots, N\}.
\]

(7) 
\[
\sum_{i=1}^{N} \left| \hat{e}_{i,t} \right|^m = O_P(1) \quad \text{for every } t \in \{1, \ldots, T\}.
\]

(8) 
\[
\sum_{i=1}^{N_0} \left| \hat{e}_{i,t} \right|^m = O_P(1) \quad \text{for every } t \in \{1, \ldots, T\}.
\]

(9) 
\[
\sum_{i=1}^{N_0} \sum_{t=1}^{T} \left| \hat{e}_{i,t} \right|^m = O_P(1).
\]

(10) 
\[
\sum_{i=1}^{T_0} \sum_{t=1}^{T} \left| \hat{e}_{i,t} \right|^m = O_P(1).
\]

**Proof:** Firstly, we prove Claims (1) and (2) with \( m = 1 \). By Assumptions 2.1(1)(2) and 2.3(1), we have \( E(\|f_t\|) \leq M + 1 \), \( \|\lambda_i\| \leq M \), and \( E(\epsilon_{i,t}^2) \leq M + 1 \) for every \( t \in \{1, \ldots, T\} \) and every \( i \in \{1, \ldots, N\} \). Note that although Lemma C.1 of Gonçalves and Perron (2014) requires their Assumptions 1–5, only the panel factor model relevant conditions of their Assumptions 1–5 are actually used, and these panel factor model relevant conditions automatically hold under Assumptions 2.1–2.5 of this paper. Then Claims (1) and (2) with \( m = 1 \) are obtained by applying Lemma C.1 (i) and (ii) of Gonçalves and Perron (2014) to the control subsample and pre-treatment subsample.

Claim (1) with \( m = 2 \) follows from Lemma A.1(6) and the fact that \( o_P(1) \) is trivially \( O_P(1) \). For
$m > 2$, we use Lemma A.1(6) and Assumption 2.5(2) to conclude that
\[
\sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}} f_t \right\|^2 = O_p(1) \quad \text{and} \quad \sum_{t=1}^{T_0} \left\| \hat{f}_{\text{wide},t} - H_{\text{wide}} f_t \right\|^2 = O_p(1),
\]
which in turn implies that
\[
\sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}} f_t \right\|^m = O_p(1) \quad \text{and} \quad \sum_{t=1}^{T_0} \left\| \hat{f}_{\text{wide},t} - H_{\text{wide}} f_t \right\|^m = O_p(1)
\]
holds for any integer $m > 2$. Then Claim (1) with $m > 2$ follows from the relationship between $o_p(1)$ and $O_p(1)$. Claim (2) with $m \geq 2$ can be proved analogously using some facts in the proof of Lemma A.2.

For every $m \in \{1, 2, \ldots, 8\}$, Assumption 2.1(1) implies that
\[
E \left( \frac{1}{T} \sum_{t=1}^{T} \left\| f_t \right\|^m \right) = \frac{1}{T} \sum_{t=1}^{T} E \left( \left\| f_t \right\|^m \right) \leq M + 1.
\]
By the $C_p$ inequality, the properties of matrix norms, Markov’s inequality, Lemma A.1(2) and the first part of Claim (1),
\[
\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t} \right\|^m \leq 2^{m-1} \left[ \left\| H_{\text{tall}} \right\|^m \left( \frac{1}{T} \sum_{t=1}^{T} \left\| f_t \right\|^m \right) + \frac{1}{T} \sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}} f_t \right\|^m \right] = O_p(1),
\]
which establishes the first part of Claim (3). The second part and Claim (4) can be proved analogously.

To prove Claims (5)–(10), we perform the following decomposition of $\hat{e}_{i,t}$ for every $(i, t) \in I \setminus I_1$.
\[
\hat{e}_{i,t} = e_{i,t} - \left( \hat{e}_{i,t} - c_{i,t} \right)
\]
\[
= e_{i,t} - \left( \hat{f}_{\text{tall},t} - H_{\text{tall}} f_t \right)^T \hat{H}_{\text{miss}} \left( \hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i \right) - f_t^T H_{\text{tall}} \hat{H}_{\text{miss}} \left( \hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i \right)
\]
\[
- \left( \hat{f}_{\text{tall},t} - H_{\text{tall}} f_t \right)^T \hat{H}_{\text{miss}} H_{\text{wide}}^{-1} \lambda_i - f_t^T \left( H_{\text{tall}} \hat{H}_{\text{miss}} H_{\text{wide}} - I_r \right) \lambda_i. \tag{27}
\]
By the $C_p$ inequality and properties of matrix norms,
\[
\left\| \hat{e}_{i,t} \right\|^m \leq 5^{m-1} \left( \left\| e_{i,t} \right\|^m + \left\| \hat{H}_{\text{miss}} \right\|^m \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}} f_t \right\|^m \left\| \hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i \right\|^m
\]
\[
+ \left\| H_{\text{tall}} \right\|^m \left\| \hat{H}_{\text{miss}} \right\|^m \left\| f_t \right\|^m \left\| \hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i \right\|^m
\]
\[
+ \left\| \hat{H}_{\text{miss}} \right\|^m \left\| H_{\text{wide}}^{-1} \right\|^m \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}} f_t \right\|^m \left\| \lambda_i \right\|^m
\]
\[
+ \left\| H_{\text{tall}} \hat{H}_{\text{miss}} H_{\text{wide}} - I_r \right\|^m \left\| f_t \right\|^m \left\| \lambda_i \right\|^m \right).
\]
Lemma A.1(5) implies that $\hat{H}_{\text{miss}} = O_p(1)$. By Assumptions 2.1(1, 2), 2.3(1) and Markov’s inequality,
\[
\frac{1}{T} \sum_{t=1}^{T} \left\| f_t \right\|^m = O_p(1), \quad \frac{1}{N} \sum_{i=1}^{N} \left\| \lambda_i \right\|^m = O_p(1),
\]
\[
\frac{1}{T} \sum_{t=1}^{T} \left\| e_{i,t} \right\|^m = O_p(1), \quad \frac{1}{N} \sum_{i=1}^{N} \left\| e_{i,t} \right\|^m = O_p(1), \quad \frac{1}{TN} \sum_{t=1}^{T} \sum_{i=1}^{N} \left\| e_{i,t} \right\|^m = O_p(1).
\]
Then Claims (5)–(10) follow from Claims (1)(2), Lemma A.1(2)(5), Assumption 2.5(2), and above facts.

Lemma A.4: If Assumptions 2.1–2.5 hold, then for every $i \in \{1, \ldots, N\}$ and every integer $m \geq 2,$
as \( N_0, T_0 \to \infty, \)

\[
\frac{1}{T_0} \sum_{t=1}^{T_0} |\tilde{e}_{i,t} - e_{i,t}|^m = O_P \left( \frac{1}{T_0} \right).
\]

**Proof:** Firstly consider the case of \( m = 2 \) By Equation [27], the \( C_p \) inequality and the properties of matrix norms,

\[
\frac{1}{T_0} \sum_{t=1}^{T_0} (\tilde{e}_{i,t} - e_{i,t})^2 \leq 4 \| \hat{H}_{\text{miss}} \|^2 \| \hat{\Lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i \|^2 \frac{1}{T_0} \sum_{t=1}^{T_0} \| \hat{f}_{\text{tall},t} - H_{\text{tall}}^T f_t \|^2
\]

\[
+ 4 \| H_{\text{tall}} \|^2 \| \hat{H}_{\text{miss}} \|^2 \| \hat{\Lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i \|^2 \frac{1}{T_0} \sum_{t=1}^{T_0} \| f_t \|^2
\]

\[
+ 4 \| H_{\text{tall}} \hat{H}_{\text{miss}} H_{\text{wide}}^{-1} - I_r \| \| \lambda_i \|^2 \frac{1}{T_0} \sum_{t=1}^{T_0} \| f_t \|^2.
\]

And the conclusion follows from Assumptions 2.1(1)(2), 2.5(2), Markov’s inequality, and Lemma A.1(2)(4)(5)(6).

For the case of \( m > 2, \) the result for the case of \( m = 2 \) implies that

\[
\sum_{t=1}^{T_0} (\tilde{e}_{i,t} - e_{i,t})^2 = O_P(1),
\]

which in turn implies that

\[
\sum_{t=1}^{T_0} |\tilde{e}_{i,t} - e_{i,t}|^m = O_P(1)
\]

for every integer \( m > 2. \)

\[\square\]

### A.2 Estimators and Residuals in Section 4

In this subsection, we show that adding covariates into the panel factor model does not alter the conclusions in Subsection A.1. Now \( \hat{F}_{\text{tall}}, \hat{F}_{\text{wide}}, \hat{\Lambda}_{\text{tall}} \) and \( \hat{\Lambda}_{\text{wide}} \) are estimated by Algorithm 4.1 (interactive fixed effect estimation), and

\[
\hat{V}_{\text{tall}} = \frac{1}{N_0} \hat{\Lambda}_{\text{tall}}^T \hat{\Lambda}_{\text{tall}}, \quad \hat{V}_{\text{wide}} = \frac{1}{N} \hat{\Lambda}_{\text{wide}}^T \hat{\Lambda}_{\text{wide}}, \quad \hat{e}_{i,t} = y_{i,t} - x_{i,t}^T \hat{\beta}_{\text{tall}} - \hat{c}_{i,t}.
\]

Other quantities follow their definitions in Subsection A.1.

**Lemma A.5:** If Assumptions 2.1–2.8 hold, then the results in Lemmas A.1, A.2 and A.3(1)–(4) are true for the estimators in Section 4.

**Proof:** By Remark 5 of Bai (2009), estimation of \( \beta \) does not affect the rates of convergence and the limiting distributions of the estimated factors and loadings, so they are the same as those of a pure factor model. Since Lemmas A.1, A.2 and A.3(1)–(4) are asymptotic properties of the estimated factors and loadings in a pure factor model, they can be naturally extended to a factor model with covariates.

\[\square\]
Lemma A.6: If Assumptions 2.1–2.8 hold, then the results in Lemmas A.3(5)–(10) and A.4 are true for the residuals in Section 4.

Proof: Applying Theorem 1 of Bai (2009) to the control subsample yields
\[
\hat{\beta}_{\text{tall}} - \beta = O_p \left( \frac{1}{\sqrt{N_0 T}} \right).
\]  

Now for every \((i, t) \in T \setminus I_1\), \(e_{i,t}\) admits a decomposition
\[
\hat{e}_{i,t} = e_{i,t} - (\hat{e}_{i,t} - c_{i,t}) - x_{i,t}^T (\hat{\beta}_{\text{tall}} - \beta)
\]
\[
= e_{i,t} - (\hat{f}_{\text{tall},t} - H_{\text{tall},t} f_t)^T \hat{H}_{\text{miss}} \left( \lambda_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i \right) - f_t^T H_{\text{tall}} \hat{H}_{\text{miss}} \left( \lambda_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i \right)
\]
\[
- (\hat{f}_{\text{tall},t} - H_{\text{tall},t} f_t)^T \hat{H}_{\text{miss}} H_{\text{wide}}^{-1} \lambda_i - f_t^T \left( H_{\text{tall}} \hat{H}_{\text{miss}} H_{\text{wide}}^{-1} - I_r \right) \lambda_i - x_{i,t}^T (\hat{\beta}_{\text{tall}} - \beta).
\]

For every \(m \in \{1, 2, \ldots, 8\}\), by the \(C_p\) inequality and properties of matrix norms,
\[
|\hat{e}_{i,t}|^m \leq 6^{m-1} \left( |e_{i,t}|^m + \|\hat{H}_{\text{miss}}\|_m \|\hat{f}_{\text{tall},t} - H_{\text{tall},t} f_t\|_m \|\lambda_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i\|_m 
\right.
\]
\[
+ \|H_{\text{tall}}\|_m \|\hat{H}_{\text{miss}}\|_m \|f_t\|_m \|\lambda_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i\|_m 
\]
\[
+ \|\hat{H}_{\text{miss}}\|_m \|H_{\text{wide}}^{-1}\|_m \|\hat{f}_{\text{tall},t} - H_{\text{tall},t} f_t\|_m \|\lambda_i\|_m 
\]
\[
+ \|H_{\text{tall}} \hat{H}_{\text{miss}} H_{\text{wide}}^{-1} - I_r\|_m \|f_t\|_m \|\lambda_i\|_m + \|x_{i,t}\|_m \|\hat{\beta}_{\text{tall}} - \beta\|_m
\]

Furthermore,
\[
\frac{1}{T_0} \sum_{t=1}^{T_0} (\hat{e}_{i,t} - e_{i,t})^2 \leq 5 \|\hat{H}_{\text{miss}}\|^2 \|\lambda_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i\|^2 \frac{1}{T_0} \sum_{t=1}^{T_0} \|\hat{f}_{\text{tall},t} - H_{\text{tall},t} f_t\|^2
\]
\[
+ 5 \|H_{\text{tall}}\|^2 \|\hat{H}_{\text{miss}}\|^2 \|\lambda_{\text{wide},i} - H_{\text{wide}}^{-1} \lambda_i\|^2 \frac{1}{T_0} \sum_{t=1}^{T_0} \|f_t\|^2
\]
\[
+ 5 \|\hat{H}_{\text{miss}}\|^2 \|H_{\text{wide}}^{-1}\|^2 \|\lambda_i\|^2 \frac{1}{T_0} \sum_{t=1}^{T_0} \|\hat{f}_{\text{tall},t} - H_{\text{tall},t} f_t\|^2
\]
\[
+ 5 \|H_{\text{tall}} \hat{H}_{\text{miss}} H_{\text{wide}}^{-1} - I_r\|^2 \|\lambda_i\|^2 \frac{1}{T_0} \sum_{t=1}^{T_0} \|f_t\|^2
\]
\[
+ 5 \|\hat{\beta}_{\text{tall}} - \beta\|^2 \frac{1}{T_0} \sum_{t=1}^{T_0} \|x_{i,t}\|^2.
\]

Assumption 2.6(1) and Markov’s inequality imply that for every \(m \in \{1, 2, \ldots, 8\}\),
\[
\frac{1}{T} \sum_{t=1}^{T} \|x_{i,t}\|_m = O_P(1), \quad \frac{1}{N} \sum_{i=1}^{N} \|x_{i,t}\|_m = O_P(1), \quad \frac{1}{TN} \sum_{t=1}^{T} \sum_{i=1}^{N} \|x_{i,t}\|_m = O_P(1).
\]

Then the conclusions follow from above results, the arguments in the proofs of Lemmas A.3(5)–(10), A.4, and A.5.

Appendix B  Properties of Bootstrap

As is described in Step 2 of Algorithm 4.3 and explained in Remark 4.2, the bootstrap is conducted within a pure factor framework even for a factor model with covariates (i.e., the model in Section 4). By the construction of resampled observations and Lemma A.5, A.6, it follows that the properties of
bootstrap quantities in Section 4 under Assumptions 2.1–2.8 are identical to those in Section 3 under Assumptions 2.1–2.5. Therefore, unless clarifying assumptions, we do not distinguishing between a pure factor model and a factor model with covariates in this section.

Moreover, since the proof of (ordinary) wild bootstrap can be straightforwardly extended to block wild bootstrap with extra notations, we provide the results for (ordinary) wild bootstrap procedure in what follows to save space.

### B.1 Technical Notes

This subsection introduces some basic concepts and results of bootstrap asymptotic analysis. Let $S_{N,T}$ be the raw sample, *i.e.*, $\{Y_{i,t} : (i,t) \in \mathcal{I}\}$ for a pure factor model and $\{(Y_{i,t}, x_{i,t}) : (i,t) \in \mathcal{I}\}$ for a factor model with covariates. For simplicity of notation, we omit the subscript and just write $S$ in stead of $S_{N,T}$. Define $P^*$ and $E^*$ to be the conditional probability and expectation given $S$, *i.e.*, $P^*(\cdot) = P(\cdot | S)$ and $E^*(\cdot) = E(\cdot | S)$. The conditional variance $\text{Var}^*(\cdot)$ and conditional covariance $\text{Cov}^*(\cdot)$ are defined analogously.

For a bootstrap statistic $A_{N,T}^*$, we say that $A_{N,T}^*$ is of an order $o_{P^*}(1)$ in probability, denoted by $A_{N,T}^* = o_{P^*}(1)$, if and only if for any $\varepsilon > 0$ and $\delta > 0$,\[ \lim_{N,T \to \infty} P^* \left[ |A_{N,T}^*| > \varepsilon \right] = 0. \]

We say that $A_{N,T}^*$ is of an order $O_{P^*}(1)$ in probability, denoted by $A_{N,T}^* = O_{P^*}(1)$, if and only if for any $\delta > 0$, there exists a $0 < M < \infty$ such that\[ \lim_{N,T \to \infty} P^* \left[ |A_{N,T}^*| \geq M \right] = 0. \]

For a sequence of deterministic numbers $\{c_{N,T}\}$,\[ A_{N,T}^* = o_{P^*}(c_{N,T}) \iff \frac{A_{N,T}^*}{c_{N,T}} = o_{P^*}(1), \quad A_{N,T}^* = O_{P^*}(c_{N,T}) \iff \frac{A_{N,T}^*}{c_{N,T}} = O_{P^*}(1). \]

One can also see Pages 2891–2892 and Appendix A.1 of Cheng and Huang (2010) for measure-theoretic definitions of bootstrap stochastic orders and relevant measurability issues.

The lemmas below list some properties of bootstrap stochastic orders that are frequently used in subsequent analyses.

**Lemma B.1:** Let $A_{N,T}^*$ be a bootstrap statistic. If $E^* \left( |A_{N,T}^*| \right) \xrightarrow{P} 0$, then $A_{N,T}^* = o_{P^*}(1)$. If $E^* \left( |A_{N,T}^*| \right) = O_P(1)$, then $A_{N,T}^* = O_{P^*}(1)$.

**Proof:** See the first paragraph on Page 387 of Chang and Park (2003). \hfill \Box

**Lemma B.2:** Suppose $A_{N,T} = O_P(1)$, $a_{N,T} = o_P(1)$, $A_{1,N,T}^* = O_{P^*}(1)$, $A_{2,N,T}^* = O_{P^*}(1)$, $a_{1,N,T}^* = o_{P^*}(1)$, and $a_{2,N,T}^* = o_{P^*}(1)$. Then
1. $A_{1,N,T}^* + A_{2,N,T}^* = O_{P^*}(1)$, and $A_{1,N,T}^* + A_{1,N,T} = O_{P^*}(1)$.
2. $A_{1,N,T}^*A_{2,N,T}^* = O_{P^*}(1)$, and $A_{1,N,T}^*A_{1,N,T} = O_{P^*}(1)$.
3. $a_{1,N,T}^* + a_{2,N,T}^* = o_{P^*}(1)$, and $a_{1,N,T}^* + a_{1,N,T} = o_{P^*}(1)$.
(4) \( a^*_1, N, T a^*_2, N, T = o_{P^*}(1) \), and \( a^*_1, N, T a_1, N, T = o_{P^*}(1) \).

(5) \( A^*_1, N, T + a^*_1, N, T = O_{P^*}(1) \), \( A^*_1, N, T + a_1, N, T = O_{P^*}(1) \), and \( A_1, N, T + a^*_1, N, T = O_{P^*}(1) \).

(6) \( A^*_1, N, T a^*_1, N, T = o_{P^*}(1) \), \( A^*_1, N, T a_1, N, T = o_{P^*}(1) \), and \( A_1, N, T a^*_1, N, T = o_{P^*}(1) \).

**Proof:** These results follow from the last a few lines on Page 1861 of Park (2003), Lemma 1 of Chang and Park (2003), and Lemma 3 of Cheng and Huang (2010).

Now we introduce the definition of conditional convergence in distribution. For a bootstrap statistic \( Z^*_N, T \) and a random element \( Z \) taking values in a metric space \( \mathbb{D} \), we say that \( Z^*_N, T \) conditionally converges in distribution to \( Z \) (or equivalently, the conditional distribution of \( Z^*_N, T \) weakly converges to that of \( Z \)) in probability, denoted by \( Z^*_N, T \Xrightarrow{d} Z \), if and only if for every bounded Lipschitz continuous function \( h : \mathbb{D} \to \mathbb{R} \), we have \( \mathbb{E}^* \left[ h \left( Z^*_N, T \right) \right] \Xrightarrow{P} \mathbb{E} \left[ h(Z) \right] \) as \( N, T \to \infty \). The lemmas below list some properties of conditional convergence in distribution that are frequently used in subsequent analyses.

**Lemma B.3:** If \( Z^*_N, T \Xrightarrow{d} Z \) and \( a^*_N, T = o_{P^*}(1) \), then \( Z^*_N, T = O_{P^*}(1) \) and \( Z^*_N, T + a^*_N, T \Xrightarrow{d} Z \).

**Proof:** See Remark 2 of Chang and Park (2003).

**Lemma B.4:** Suppose that the bootstrap statistic \( Z_N, T \) and random variable \( Z \) take their values in \( \mathbb{R} \), and \( Z^*_N, T \Xrightarrow{d} Z \) as \( N, T \to \infty \). Let \( G^*_N, T \) denote the conditional cumulative distribution function of \( Z^*_N, T \), and \( G \) denote the cumulative distribution function of \( Z \), i.e., \( G^*_N, T(z) = \mathbb{P}^* \left( Z^*_N, T \leq z \right) \) and \( G(z) = \mathbb{P}(Z \leq z) \) for every \( z \in \mathbb{R} \). Then \( G^*_N, T(z) \Xrightarrow{P} G(z) \) as \( N, T \to \infty \) for every \( z \in C_G \), where \( C_G \subset \mathbb{R} \) is the set of all continuous points of \( G \). Moreover, if \( G \) is everywhere continuous on \( \mathbb{R} \), i.e., \( C_G = \mathbb{R} \), then

\[
\sup_{z \in \mathbb{R}} \left| G^*_N, T(z) - G(z) \right| \Xrightarrow{P} 0 \quad \text{as} \quad N, T \to \infty.
\]

**Proof:** For any \( z_0 \in \mathbb{R} \) and any \( C_0 \subset \mathbb{R} \), the distance between \( z_0 \) and \( C_0 \) is measured by

\[
d_e(z_0, C_0) = \inf_{z \in C_0} |z_0 - z|.
\]

For any open set \( C_1 \subset \mathbb{R} \), construct a sequence of functions \( \{h_m\}_{m=1}^\infty \), so that \( h_m(z) = [md_e(z, C_1^c)] \wedge 1 \) for every \( z \in \mathbb{R} \). It is easy to see that \( h_m \) is non-negative, bounded, and Lipschitz continuous with Lipschitz constant \( m \) for every \( m \in \mathbb{Z}_+ \), and \( h_m \uparrow 1_{C_1} \) as \( m \to \infty \). By the monotone convergence theorem,

\[
\lim_{m \to \infty} \mathbb{E} \left[ h_m(Z) \right] = \mathbb{E} \left[ 1_{C_1}(Z) \right] = \mathbb{P}(Z \in C_1).
\]

Hence for any \( \varepsilon > 0 \), there exists an \( m_0 \in \mathbb{Z}_+ \), such that \( \mathbb{E} \left[ h_{m_0}(Z) \right] > \mathbb{P}(Z \in C_1) - \varepsilon / 2 \). Since \( h_{m_0} \leq 1_{C_1} \), it follows that \( \mathbb{P}^* \left( Z^*_N, T \in C_1 \right) \geq \mathbb{E}^* \left[ h_{m_0} \left( Z^*_N, T \right) \right] \) almost surely. By the boundedness and Lipschitz continuity of \( h_{m_0} \) and the definition of conditional convergence in distribution, \( \mathbb{E}^* \left[ h_{m_0} \left( Z^*_N, T \right) \right] \Xrightarrow{P} \mathbb{E} \left[ h_{m_0}(Z) \right] \), which implies that

\[
\lim_{N,T \to \infty} \mathbb{P} \left[ \mathbb{P}^* \left( Z^*_N, T \in C_1 \right) > \mathbb{P}(Z \in C_1) - \varepsilon \right] \geq \lim_{N,T \to \infty} \mathbb{P} \left( \mathbb{E}^* \left[ h_{m_0} \left( Z^*_N, T \right) \right] > \mathbb{P}(Z \in C_1) - \varepsilon \right)
\]
For any \( C_2 \subset \mathbb{R} \), because \( C_2^\circ \) is an open set, the above result implies that

\[
\lim_{N,T \to \infty} \mathbb{P} \left[ \mathbb{P}^* \left( Z_{N,T}^* \in C_2 \right) < \mathbb{P} \left( Z \in C_2 \right) + \varepsilon \right] = \lim_{N,T \to \infty} \mathbb{P} \left[ \mathbb{P}^* \left( Z_{N,T}^* \in C_2 \right) > 1 - \mathbb{P} \left( Z \in C_2 \right) - \varepsilon \right] = 1.
\]

Now consider any Borel set \( C_3 \subset \mathbb{R} \) with \( \mathbb{P} \left[ Z \in \text{cl}(C_3) \setminus \text{int}(C_3) \right] = 0 \), where \( \text{cl}() \) and \( \text{int}() \) denote the closure and interior of a set, respectively. Then it follows that

\[
\lim_{N,T \to \infty} \mathbb{P} \left[ \left| \mathbb{P}^* \left( Z_{N,T}^* \in C_3 \right) - \mathbb{P} \left( Z \in C_3 \right) \right| < \varepsilon \right] = \lim_{N,T \to \infty} \mathbb{P} \left[ \mathbb{P} \left( Z \in \text{int}(C_3) \right) - \varepsilon < \mathbb{P}^* \left( Z_{N,T}^* \in C_3 \right) < \mathbb{P} \left( Z \in \text{cl}(C_3) \right) + \varepsilon \right] 
\geq 1 - \lim_{N,T \to \infty} \mathbb{P} \left[ \mathbb{P}^* \left( Z_{N,T}^* \in C_3 \right) \leq \mathbb{P} \left( Z \in \text{int}(C_3) \right) - \varepsilon \right] 
- \lim_{N,T \to \infty} \mathbb{P} \left[ \mathbb{P}^* \left( Z_{N,T}^* \in C_3 \right) \geq \mathbb{P} \left( Z \in \text{cl}(C_3) \right) + \varepsilon \right] 
\geq 1 - \lim_{N,T \to \infty} \mathbb{P} \left[ \mathbb{P}^* \left( Z_{N,T}^* \in \text{int}(C_3) \right) \leq \mathbb{P} \left( Z \in \text{int}(C_3) \right) - \varepsilon \right] 
- \lim_{N,T \to \infty} \mathbb{P} \left[ \mathbb{P}^* \left( Z_{N,T}^* \in \text{cl}(C_3) \right) \geq \mathbb{P} \left( Z \in \text{cl}(C_3) \right) + \varepsilon \right] = 1,
\]

which implies that \( \mathbb{P}^* \left( Z_{N,T}^* \in C_3 \right) \overset{p}{\to} \mathbb{P} \left( Z \in C_3 \right) \) as \( N,T \to \infty \).

For any \( z \in C_G \), let \( C_4 = (-\infty, z] \), then \( C_4 \) is Borel and satisfies \( \mathbb{P} \left[ Z \in \text{cl}(C_4) \setminus \text{int}(C_4) \right] = 0 \). Therefore,

\[
G_{N,T}^* (z) = \mathbb{P}^* \left( Z_{N,T}^* \in C_4 \right) \overset{p}{\to} \mathbb{P} \left( Z \in C_4 \right) = G(z)
\]
as \( N,T \to \infty \).

If \( G \) is everywhere continuous, i.e., \( C_G = \mathbb{R} \), then for any \( k \in \mathbb{Z}_+ \), there exist \( -\infty = z_0 < z_1 < \cdots < z_k = \infty \), so that \( G(z_i) = i/k \) for every \( i = 1, \ldots, k \). Then for any \( z \in [z_{i-1}, z_i] \),

\[
G_{N,T}^* (z) - G(z) \leq G_{N,T}^* (z_i) - G(z) = G_{N,T}^* (z_i) - G(z) + \frac{1}{k},
\]

\[
G_{N,T} (z) - G(z) \geq G_{N,T}^* (z_{i-1}) - G(z_i) = G_{N,T}^* (z_{i-1}) - G(z_{i-1}) - \frac{1}{k},
\]

which implies that

\[
\sup_{z \in \mathbb{R}} \left| G_{N,T}^* (z) - G(z) \right| \leq \frac{1}{k} + \max_{1 \leq i \leq k} \left| G_{N,T}^* (z_i) - G(z_i) \right|.
\]

For any \( \varepsilon > 0 \), pick a \( k \) such that \( 1/k < \varepsilon/2 \). Since \( \max_{1 \leq i \leq k} \left| G_{N,T}^* (z_i) - G(z_i) \right| \overset{p}{\to} 0 \), we have

\[
\lim_{N,T \to \infty} \mathbb{P} \left( \sup_{z \in \mathbb{R}} \left| G_{N,T}^* (z) - G(z) \right| > \varepsilon \right) \leq \lim_{N,T \to \infty} \mathbb{P} \left( \max_{1 \leq i \leq k} \left| G_{N,T}^* (z_i) - G(z_i) \right| > \frac{\varepsilon}{2} \right) = 0,
\]

and the proof is complete. \( \Box \)

### B.2 Ancillary Results about Bootstrap Samples

Let \( \tilde{F} = (\tilde{f}_1, \ldots, \tilde{f}_T)^T = \tilde{F}_{\text{all}}, \) and \( \tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_N)^T = \tilde{\lambda}_{\text{wide}}\tilde{F}_{\text{miss}}^T \). Analogously to the population version, define \( \tilde{F}_{\text{all}} = \tilde{F} \), \( \tilde{\lambda}_{\text{wide}} = \tilde{\lambda} \), \( \tilde{F}_{\text{wide}} \) to be a sub-matrix formed by the first \( T_0 \) rows of \( \tilde{F} \),
and \( \tilde{A}_{\text{full}} \) to be a sub-matrix formed by the first \( N_0 \) rows of \( \tilde{A} \). Then let \( e_{i,t}^* \) be defined in Step 2(1)(2) of Algorithms 3.2 and 4.3.

Lemmas B.5–B.9 below verify (some of) the bootstrap high level conditions in Gonçalves and Perron (2014) and Gonçalves et al. (2017). Lemma B.5 corresponds to Condition A*(b) of Gonçalves and Perron (2014) and Condition A.1 of Gonçalves et al. (2017).

**Lemma B.5:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 hold for a factor model with covariates, then as \( N_0, T_0 \to \infty \),

1. For every \( t \in \{1, \ldots, T\} \),
   \[
   \sum_{s=1}^{T} \left[ E^* \left( \frac{1}{N_0} \sum_{i=1}^{N_0} e_{i,t}^* e_{i,s}^* \right) \right]^2 = O_p(1).
   \]
2. \[
   \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \left[ E^* \left( \frac{1}{N_0} \sum_{i=1}^{N_0} e_{i,t}^* e_{i,s}^* \right) \right]^2 = O_p(1).
   \]

**Proof:** Since \( \{u_{i,t}\} \) are i.i.d. from \( \mathbb{N}(0,1) \), it follows that \( E( u_{i,t} u_{i,s} ) = 1_{\{s=t\}} \). By the independence between \( \{u_{i,t}\} \) and the raw sample, and Lemma A.3(8),

\[
\sum_{s=1}^{T} \left[ E^* \left( \frac{1}{N_0} \sum_{i=1}^{N_0} e_{i,t}^* e_{i,s}^* \right) \right]^2 = \sum_{s=1}^{T} \left[ \frac{1}{N_0} \sum_{i=1}^{N_0} e_{i,t}^* e_{i,s}^* E( u_{i,t} u_{i,s} ) \right]^2 = \left( \frac{1}{N_0} \sum_{i=1}^{N_0} e_{i,t}^2 \right)^2 = O_p(1).
\]

Moreover, by the \( C_p \) inequality and Lemma A.3(9),

\[
\frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \left[ E^* \left( \frac{1}{N_0} \sum_{i=1}^{N_0} e_{i,t}^* e_{i,s}^* \right) \right]^2 = \frac{1}{T N_0^2} \sum_{t=1}^{T} \left( \sum_{i=1}^{N_0} e_{i,t}^2 \right)^2 \leq \frac{1}{T N_0} \sum_{t=1}^{T} \sum_{i=1}^{N_0} e_{i,t}^4 = O_p(1),
\]
and the proof is complete. \( \square \)

Lemma B.6 corresponds to Condition A*(c) of Gonçalves and Perron (2014) and Condition A.2 of Gonçalves et al. (2017).

**Lemma B.6:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 hold for a factor model with covariates, then as \( N_0, T_0 \to \infty \),

1. For every \( t \in \{1, \ldots, T\} \),
   \[
   \frac{1}{T} \sum_{s=1}^{T} \left[ E^* \left( \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} e_{i,t}^* e_{i,s}^* - E^* \left( e_{i,t}^* e_{i,s}^* \right) \right) \right]^2 = O_p(1).
   \]
2. \[
   \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \left[ E^* \left( \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} e_{i,t}^* e_{i,s}^* - E^* \left( e_{i,t}^* e_{i,s}^* \right) \right) \right]^2 = O_p(1).
   \]

**Proof:** Because \( \{u_{i,t}\} \) are i.i.d. as \( \mathbb{N}(0,1) \) and independent of the raw sample, we have that \( e_{i,t}^* \) and \( e_{j,s}^* \) are conditionally independent given the sample whenever \( i \neq j \) or \( t \neq s \), and \( \text{Var} ( u_{i,t} u_{i,s} ) \leq E \left( u_{i,t}^2 \right) = 3 \). Then by Cauchy-Schwarz inequality, the \( C_p \) inequality, and Lemma A.3(8)(9),

\[
\frac{1}{T} \sum_{s=1}^{T} \left[ E^* \left( \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} e_{i,t}^* e_{i,s}^* - E^* \left( e_{i,t}^* e_{i,s}^* \right) \right) \right]^2 = \frac{1}{T N_0} \sum_{s=1}^{T} \text{Var}^* \left( \sum_{i=1}^{N_0} e_{i,t}^* e_{i,s}^* \right)
\]
\[
\frac{1}{TN_0} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \varepsilon_{i,t}^2 \varepsilon_{i,s}^2 \text{Var} (u_{i,t}u_{i,s}) \leq \frac{3}{N_0} \sum_{i=1}^{N_0} \left[ \frac{1}{T} \sum_{s=1}^{T} \varepsilon_{i,s}^2 \right]
\]

\[
\leq 3 \sqrt{\frac{1}{N_0} \sum_{i=1}^{N_0} \varepsilon_{i,t}^2} \sqrt{\frac{1}{N_0} \sum_{i=1}^{N_0} \left( \frac{1}{T} \sum_{s=1}^{T} \varepsilon_{i,s}^2 \right)} \leq 3 \sqrt{\frac{1}{N_0} \sum_{i=1}^{N_0} \varepsilon_{i,t}^4} \sqrt{\frac{1}{T N_0} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \varepsilon_{i,s}^4} = O_P(1).
\]

To prove the second claim, we use the \(C_p\) inequality and Lemma A.3(9) to conclude that

\[
\frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \mathbb{E}^* \left[ \left( \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} \varepsilon_{i,t} \varepsilon_{i,s}^* - \mathbb{E}^* (\varepsilon_{i,t}^* \varepsilon_{i,s}^*) \right)^2 \right] \leq \frac{3}{T^2 N_0} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \varepsilon_{i,t}^2 \varepsilon_{i,s}^2
\]

\[
= \frac{3}{N_0} \sum_{i=1}^{N_0} \left( \frac{1}{T} \sum_{s=1}^{T} \varepsilon_{i,s}^2 \right)^2 \leq \frac{3}{T N_0} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \varepsilon_{i,s}^4 = O_P(1),
\]

and the proof is complete. \(\square\)

Lemma B.7 corresponds to Condition A.3 of Gonçalves et al. (2017).

**Lemma B.7:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 hold for a factor model with covariates, then as \(N_0, T_0 \to \infty\),

\[
\mathbb{E}^* \left( \left\| \frac{1}{\sqrt{T N_0}} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \tilde{f}_s \left[ \varepsilon_{i,t}^* \varepsilon_{i,s}^* - \mathbb{E}^* (\varepsilon_{i,t}^* \varepsilon_{i,s}^*) \right] \right\|^2 \right) = O_P(1).
\]

**Proof:** We still start from the i.i.d. nature of \(\{u_{i,t}\}\) and consider \(\text{Cov} (u_{i,t}u_{i,s}, u_{j,t}u_{j,q})\). If \(i \neq j\), then \(u_{i,t}u_{i,s}\) and \(u_{j,t}u_{j,q}\) are independent of each other, and hence \(\text{Cov} (u_{i,t}u_{i,s}, u_{j,t}u_{j,q}) = 0\). If \(s \neq q\), then (1) \(u_{i,s}\) and \(u_{j,q}\) are independent of each other; (2) either \(u_{i,t}\) is independent of \(u_{i,s}\), or \(u_{j,t}\) is independent of \(u_{j,q}\). This implies that \(\mathbb{E} (u_{i,t}u_{i,s}u_{j,t}u_{j,q}) = 0\), \(\mathbb{E} (u_{i,t}u_{i,s})\mathbb{E} (u_{j,t}u_{j,q}) = 0\), and hence \(\text{Cov} (u_{i,t}u_{i,s}, u_{j,t}u_{j,q}) = 0\). Now we have established that \(\text{Cov} (u_{i,t}u_{i,s}, u_{j,t}u_{j,q}) \neq 0\) only if \(i = j\) and \(s = q\). By this fact and Cauchy-Schwarz inequality,

\[
\mathbb{E}^* \left( \left\| \frac{1}{\sqrt{T N_0}} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \tilde{f}_s \left[ \varepsilon_{i,t}^* \varepsilon_{i,s}^* - \mathbb{E}^* (\varepsilon_{i,t}^* \varepsilon_{i,s}^*) \right] \right\|^2 \right)
\]

\[
= \mathbb{E}^* \left( \frac{1}{T N_0} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \sum_{j=1}^{T} \sum_{q=1}^{N_0} \tilde{f}_s \tilde{f}_q \left[ \varepsilon_{i,t}^* \varepsilon_{i,s}^* - \mathbb{E}^* (\varepsilon_{i,t}^* \varepsilon_{i,s}^*) \right] \left[ \varepsilon_{j,t}^* \varepsilon_{j,q}^* - \mathbb{E}^* (\varepsilon_{j,t}^* \varepsilon_{j,q}^*) \right] \right)
\]

\[
= \frac{1}{T N_0} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \sum_{j=1}^{T} \sum_{q=1}^{N_0} \tilde{f}_s \tilde{f}_q \text{Cov}^* (\varepsilon_{i,t}^* \varepsilon_{i,s}^*, \varepsilon_{j,t}^* \varepsilon_{j,q}^*)
\]

\[
= \frac{1}{T N_0} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \sum_{j=1}^{T} \sum_{q=1}^{N_0} \tilde{f}_s \tilde{f}_q \varepsilon_{i,t}^* \varepsilon_{i,s}^* \varepsilon_{j,t}^* \varepsilon_{j,q}^* \text{Cov} (u_{i,t}u_{i,s}, u_{j,t}u_{j,q})
\]

\[
= \frac{1}{T N_0} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \sum_{j=1}^{T} \sum_{q=1}^{N_0} \tilde{f}_s \tilde{f}_q \varepsilon_{i,t}^2 \varepsilon_{i,s}^2 \text{Var} (u_{i,t}u_{i,s})
\]

\[
\leq \frac{3}{N_0} \sum_{i=1}^{N_0} \left[ \frac{1}{T} \sum_{s=1}^{T} \tilde{f}_s \tilde{f}_q \varepsilon_{i,t}^2 \varepsilon_{i,s}^2 \right]
\]

\[
\leq \frac{3}{N_0} \sum_{i=1}^{N_0} \left[ \frac{1}{T^2 N_0} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \tilde{f}_s \tilde{f}_q \varepsilon_{i,t}^2 \varepsilon_{i,s}^2 \right]^2.
\]
By the $C_p$ inequality and Cauchy-Schwarz inequality,
\[
\frac{1}{T^2 N_0} \sum_{i=1}^{N_0} \sum_{j=1}^{T} \left( \sum_{s=1}^{T} \tilde{f}_s \tilde{f}_s^t e_{i,s}^2 \right)^2 \leq \frac{1}{T N_0} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \left\| \tilde{f}_s \right\|^4 e_{i,s}^4
\]

\[
\leq \frac{1}{T} \sum_{s=1}^{T} \left\| \tilde{f}_s \right\|^8 \frac{1}{T} \sum_{s=1}^{T} \left( \frac{1}{N_0} \sum_{i=1}^{N_0} e_{i,s}^4 \right)^2 \leq \frac{1}{T} \sum_{s=1}^{T} \left\| \tilde{f}_{\text{full},s} \right\|^8 \frac{1}{T N_0} \sum_{s=1}^{T} \sum_{i=1}^{N_0} e_{i,s}^8.
\]

The desired result follows from Lemma A.3(3)(8)(9).

Lemma B.8 corresponds to Condition A.4 of Gonçalves et al. (2017).

**Lemma B.8:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 hold for a factor model with covariates, then as $N_0, T_0 \to \infty$,
\[
E^* \left( \left\| \frac{1}{\sqrt{T N_0}} \sum_{t=1}^{T} \sum_{i=1}^{N_0} \tilde{f}_i \tilde{\lambda}_i^T e_{i,t}^* \right\|^2 \right) = O_p(1).
\]

**Proof:** From the definition of Frobenius norm and the properties of trace,
\[
\left\| \frac{1}{\sqrt{T N_0}} \sum_{t=1}^{T} \sum_{i=1}^{N_0} \tilde{f}_i \tilde{\lambda}_i^T e_{i,t}^* \right\|^2 &= \text{tr} \left( \frac{1}{T N_0} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \tilde{f}_i \tilde{\lambda}_i^T \tilde{\lambda}_j \tilde{f}_s \tilde{\lambda}_j^T e_{i,t}^* e_{j,s}^* \right)
\]
\[
= \frac{1}{T N_0} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \text{tr} \left( \tilde{f}_i \tilde{\lambda}_i^T \tilde{\lambda}_j \tilde{f}_s \right) e_{i,t}^* e_{j,s}^* = \frac{1}{T N_0} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \tilde{f}_i \tilde{\lambda}_i^T \tilde{\lambda}_j e_{i,t}^* e_{j,s}^*.
\]

Because $\{u_{i,t}\}$ are i.i.d. as $N(0, 1)$, we have $E(\tilde{u}_{i,t} u_{j,s}) = \mathbb{I}_{(i=j)} \mathbb{I}_{(t=s)}$. By the above facts, the properties of matrix norms, and Cauchy-Schwarz inequality,
\[
E^* \left( \left\| \frac{1}{\sqrt{T N_0}} \sum_{t=1}^{T} \sum_{i=1}^{N_0} \tilde{f}_i \tilde{\lambda}_i^T e_{i,t}^* \right\|^2 \right) = E^* \left( \frac{1}{T N_0} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \tilde{f}_i \tilde{\lambda}_i^T \tilde{\lambda}_j e_{i,t}^* e_{j,s}^* \right)
\]
\[
= \frac{1}{T N_0} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \left\| \tilde{f}_i \tilde{\lambda}_i \tilde{\lambda}_j \tilde{f}_s \right\|^2 e_{i,t}^* e_{j,s}^* E(\tilde{u}_{i,t} u_{j,s}) = \frac{1}{T N_0} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N_0} \left\| \tilde{f}_i \right\|^2 \left\| \tilde{\lambda}_i \right\|^2 \tilde{e}_{i,t}^2.
\]

By the $C_p$ inequality and Cauchy-Schwarz inequality,
\[
\frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \tilde{\lambda}_{\text{wide},i} \right\|^2 \tilde{e}_{i,t}^2 \right)^2 \leq \frac{1}{T N_0} \sum_{t=1}^{T} \sum_{i=1}^{N_0} \sum_{t=1}^{T} \sum_{i=1}^{N_0} \left\| \tilde{\lambda}_{\text{wide},i} \right\|^4 \tilde{e}_{i,t}^4
\]
\[
\leq \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \tilde{\lambda}_{\text{wide},i} \right\|^8 \frac{1}{N_0} \sum_{i=1}^{N_0} \left( \frac{1}{T} \sum_{t=1}^{T} \tilde{e}_{i,t}^4 \right)^2 \leq \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \tilde{\lambda}_{\text{wide},i} \right\|^8 \frac{1}{T N_0} \sum_{t=1}^{T} \sum_{i=1}^{N_0} \tilde{e}_{i,t}^8.
\]

The proof is completed by Lemmas A.1(5), A.3(3)(4)(9), and Assumption 2.5(2).

Lemma B.9 corresponds to Condition B*(d) of Gonçalves and Perron (2014) and Conditions A.5, A.6 of Gonçalves et al. (2017).

**Lemma B.9:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 hold
for a factor model with covariates, then as $N_0,T_0 \to \infty$,

1. For every $t \in \{1,\ldots,T\}$,
   \[
   \mathbb{E}^* \left( \left\| \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} \tilde{\lambda}_i e_{i,t}^* \right\|^2 \right) = O_{p^*}(1).
   \]

2. \[
   \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}^* \left( \left\| \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} \tilde{\lambda}_i e_{i,t}^* \right\|^2 \right) = O_{p^*}(1).
   \]

**Proof:** Note that $\mathbb{E} (u_{t,t}u_{j,t}) = I_{\{i=j\}}$ by the i.i.d. $\mathbb{N}(0,1)$ nature of $\{u_{t,i}\}$. By the properties of matrix norms and Cauchy-Schwarz inequality,

\[
\mathbb{E}^* \left( \left\| \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} \tilde{\lambda}_i e_{i,t}^* \right\|^2 \right) = \mathbb{E}^* \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \tilde{\lambda}_i \tilde{\lambda}_j \tilde{e}_{i,t} e_{j,t}^* \right) = \frac{1}{N_0} \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \tilde{\lambda}_i \tilde{e}_{i,t} e_{j,t} \mathbb{E} (u_{i,t}u_{j,t}) = \frac{1}{N_0} \sum_{i=1}^{N_0} \tilde{\lambda}_i^2 \tilde{e}_{i,t}^2 \leq \| \tilde{H}_{\text{miss}} \|^2 \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \tilde{\lambda}_i^4 \tilde{e}_{i,t}^2 \right) \leq \| \tilde{H}_{\text{miss}} \|^2 \sqrt{\frac{1}{N_0} \sum_{i=1}^{N_0} \tilde{\lambda}_i^4} \sqrt{\frac{1}{N_0} \sum_{i=1}^{N_0} \tilde{e}_{i,t}^4}.
\]

Then the first claim follows from Lemmas A.1(5), A.3(4)(8) and Assumption 2.5(2).

To show the second claim, we use the above results, Cauchy-Schwarz inequality and the $C_p$ inequality to conclude that

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}^* \left( \left\| \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} \tilde{\lambda}_i e_{i,t}^* \right\|^2 \right) \leq \| \tilde{H}_{\text{miss}} \|^2 \left( \frac{1}{T N_0} \sum_{t=1}^{T} \sum_{i=1}^{N_0} \tilde{\lambda}_i^4 \tilde{e}_{i,t}^2 \right) \leq \| \tilde{H}_{\text{miss}} \|^2 \sqrt{\frac{1}{N_0} \sum_{i=1}^{N_0} \tilde{\lambda}_i^4} \sqrt{\frac{1}{T N_0} \sum_{t=1}^{T} \tilde{e}_{i,t}^4}.
\]

The proof is completed by Lemmas A.1(5), A.3(4)(9) and Assumption 2.5(2). 

In this paper, more bootstrap high level conditions are used in subsequent contents, and we list them as lemmas below.

**Lemma B.10:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 hold for a factor model with covariates, then for every $m \in \{1,2,\ldots,8\}$, as $N_0,T_0 \to \infty$,

1. \[
\frac{1}{T_0} \sum_{t=1}^{T_0} |e_{i,t}^*|^m = O_{p^*}(1) \quad \text{for every } i \in \{1,\ldots,N\}.
\]

2. \[
\frac{1}{N_0} \sum_{i=1}^{N_0} |e_{i,t}^*|^m = O_{p^*}(1) \quad \text{for every } t \in \{1,\ldots,T\}.
\]
\[
\frac{1}{T_0 N} \sum_{t=1}^{T_0} \sum_{i=1}^{N} |e^*_{i,t}|^m = O_P^*(1) .
\]

**Proof:** Since \( \{u_{i,t}\} \) are i.i.d. as \( \mathbb{N}(0,1) \), we have \( \mathbb{E}(|u_{i,t}|^m) \) is finite for every \( m \in \{1, 2, \ldots, 8\} \). Hence \( \mathbb{E}^*(|e^*_{i,t}|^m) = |\tilde{e}_{i,t}|^m \mathbb{E}(|u_{i,t}|^m) \). The desired results follow from Lemmas A.3(6)(8)(10) and B.1. \( \square \)

**Lemma B.11:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 hold for a factor model with covariates, then as \( N_0, T_0 \to \infty \),

1. For every \( i \in \{1, \ldots, N\} \),

\[
\mathbb{E}^*\left( \left\| \frac{1}{\sqrt{T_0}} \sum_{t=1}^{T_0} \tilde{f}_t e^*_{i,t} \right\|^2 \right) = O_P(1) .
\]

2. \[
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E}^*\left( \left\| \frac{1}{\sqrt{T_0}} \sum_{t=1}^{T_0} \tilde{f}_t e^*_{i,t} \right\|^2 \right) = O_P(1) .
\]

**Proof:** Note that \( \mathbb{E}(u_{i,t} u_{i,s}) = \mathbb{I}_{\{t=s\}} \) by the i.i.d. \( \mathbb{N}(0,1) \) nature of \( \{u_{i,t}\} \). By the properties of matrix norms and Cauchy-Schwarz inequality,

\[
\mathbb{E}^*\left( \left\| \frac{1}{\sqrt{T_0}} \sum_{t=1}^{T_0} \tilde{f}_t e^*_{i,t} \right\|^2 \right) = \mathbb{E}^*\left( \frac{1}{T_0} \sum_{t=1}^{T_0} \sum_{s=1}^{T_0} \tilde{f}_t \tilde{f}_s e^*_{i,t} e^*_{i,s} \right) = \frac{1}{T_0} \sum_{t=1}^{T_0} \sum_{s=1}^{T_0} \tilde{f}_t \tilde{f}_s \tilde{e}_{i,t} \tilde{e}_{i,s} \mathbb{E}(u_{i,t} u_{i,s})
\]

\[
= \frac{1}{T_0} \sum_{t=1}^{T_0} \| \tilde{f}_t \|^2 e^2_{i,t} = \frac{1}{T_0} \sum_{t=1}^{T_0} \| \tilde{f}_{\text{all},t} \|^2 e^2_{i,t} \leq \sqrt{\frac{1}{T_0} \sum_{t=1}^{T_0} \| \tilde{f}_{\text{all},t} \|^4} \sqrt{\frac{1}{T_0} \sum_{t=1}^{T_0} \| \tilde{e}_{i,t} \|^4} .
\]

Then the first claim follows from Lemma A.3(3)(6) and Assumption 2.5(2).

To show the second claim, we use the above results, Cauchy-Schwarz inequality and the \( C_p \) inequality to conclude that

\[
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E}^*\left( \left\| \frac{1}{\sqrt{T_0}} \sum_{t=1}^{T_0} \tilde{f}_t e^*_{i,t} \right\|^2 \right) \leq \frac{1}{T_0 N} \sum_{t=1}^{T_0} \sum_{i=1}^{N} \| \tilde{f}_{\text{all},t} \|^2 e^2_{i,t}
\]

\[
\leq \sqrt{\frac{1}{T_0} \sum_{t=1}^{T_0} \| \tilde{f}_{\text{all},t} \|^4} \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{N} \sum_{i=1}^{N} e^2_{i,t} \right)^2} \leq \sqrt{\frac{1}{T_0} \sum_{t=1}^{T_0} \| \tilde{f}_{\text{all},t} \|^4} \sqrt{\frac{1}{T_0 N} \sum_{t=1}^{T_0} \sum_{i=1}^{N} e^4_{i,t}} .
\]

The proof is completed by Lemma A.3(3)(10) and Assumption 2.5(2). \( \square \)

**B.3 Results about Bootstrap Statistics**

Let \( D_{\text{tail}}^*, \hat{F}_{\text{all},t}, \hat{F}_{\text{tail},t}, \hat{\lambda}_{\text{all},t}, \hat{\lambda}_{\text{tail},t}, \lambda_{\text{wide},t}, \tilde{\lambda}_{\text{wide},t}, \hat{H}_{\text{miss}}, \hat{C}_t^*, \hat{c}_t^*, \hat{e}_t^* \) be the bootstrap analogues of \( D_{\text{tail}}, \tilde{F}_{\text{all},t}, \tilde{F}_{\text{tail},t}, \lambda_{\text{all},t}, \lambda_{\text{tail},t}, \lambda_{\text{wide},t}, \tilde{\lambda}_{\text{wide},t} \), \( H_{\text{miss}}, C_t^* \), \( c_t^* \), \( e_t^* \), \( L_{k,t} \), \( \hat{H}_t \), \( \hat{\Phi}_t \), \( \hat{V}_t \), and \( \hat{\sigma}_t^2 \) respectively. These bootstrap analogues are obtained in steps 2(4) of Algorithms 3.2 and 4.3. Let \( \hat{V}_{\text{all}} = D_{\text{all}}^* \) and \( \hat{V}_{\text{wide}} = D_{\text{wide}}^* \). Moreover, define
the bootstrap version of rotation matrices

\[ H^*_\text{tall} = \left( \frac{\hat{\Lambda}^T_{\text{tall}} \tilde{A}_{\text{tall}}}{N_0} \right) \left( \frac{\hat{F}'_{\text{tall}} \hat{P}^*_{\text{tall}}}{T} \right) \tilde{V}^{*-1}_{\text{tall}}, \quad H^*_\text{wide} = \left( \frac{\hat{\Lambda}^T_{\text{wide}} \hat{A}_{\text{wide}}}{N} \right) \left( \frac{\hat{F}'_{\text{wide}} \hat{P}^*_{\text{wide}}}{T_0} \right) \tilde{V}^{*-1}_{\text{wide}}. \]

**Lemma B.12:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then as \( N_0, T_0 \to \infty \),

1. \( H^*_\text{tall} = H^*_1 + o_p(1) \), and \( H^*_\text{wide} = H^*_2 + o_p(1) \), where both \( H^*_1 \) and \( H^*_2 \) are diagonal matrices with \( \pm 1 \) on diagonals.

2. \( \tilde{V}^*_\text{tall} = V + o_p(1) \), and \( \tilde{V}^*_\text{wide} = V + o_p(1) \), where \( V \) is defined in Lemma A.1.

**Proof:** By the definition and properties of Frobenius norm, Lemma A.3(4), and Assumption 2.5(2),

\[ \left\| \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{\text{wide},i} \hat{\lambda}^T_{\text{wide},i} \right\| \leq \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}_{\text{wide},i} \hat{\lambda}^T_{\text{wide},i} \right\| = \frac{1}{N_0} \sum_{i=1}^{N_0} \sqrt{\text{tr} \left( \hat{\lambda}_{\text{wide},i} \hat{\lambda}^T_{\text{wide},i} \hat{\lambda}_{\text{wide},i} \hat{\lambda}^T_{\text{wide},i} \right)} = \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}_{\text{wide},i} \right\|^2 = O_p(1). \]

Moreover,

\[ \left\| \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{\text{tall},i} \hat{\lambda}^T_{\text{tall},i} \right\|^2 = \text{tr} \left( \frac{1}{N_0} \hat{\Lambda}^T_{\text{tall}} \hat{A}_{\text{tall}} \right) = \text{tr} (I_r) = r. \]

Let \( \epsilon_i = \hat{\lambda}_{\text{tall},i} - H^{-1}_{\text{tall}} H_{\text{wide}} \hat{\lambda}_{\text{wide},i} \) for every \( i \in \{1, \ldots, N_0\} \). Then by the definition and properties of Frobenius norm, and Lemma A.2,

\[ \left\| \frac{1}{N_0} \sum_{i=1}^{N_0} \epsilon_i \epsilon_i^T \right\| \leq \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \epsilon_i \right\|^2 = O_p \left( \frac{1}{T_0} \right). \]

By the triangle inequality, the properties of matrix norms, Cauchy-Schwarz inequality, and the above facts,

\[ \left\| \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{\text{tall},i} \epsilon_i^T \right\| \leq \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}_{\text{tall},i} \right\| \left\| \epsilon_i \right\| \leq \sqrt{\frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}_{\text{tall},i} \right\|^2} \sqrt{\frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \epsilon_i \right\|^2} = O_p \left( \frac{1}{T_0} \right). \]

Combining these results with Lemma A.1(2)(5) yields

\[ \frac{1}{N_0} \hat{\Lambda}^T_{\text{tall}} \hat{A}_{\text{tall}} = \hat{H}_{\text{miss}} \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{\text{wide},i} \hat{\lambda}^T_{\text{wide},i} \right) \hat{H}_{\text{miss}} \]

\[ = \left( H_{\text{tall}}^{-1} H_{\text{wide}} + \hat{H}_{\text{miss}} - H_{\text{tall}}^{-1} H_{\text{wide}} \right) \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{\text{wide},i} \hat{\lambda}^T_{\text{wide},i} \right) \left( H_{\text{tall}}^{-1} H_{\text{wide}} + \hat{H}_{\text{miss}} - H_{\text{tall}}^{-1} H_{\text{wide}} \right)^T \]

\[ = H_{\text{tall}}^{-1} H_{\text{wide}} \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{\text{wide},i} \hat{\lambda}^T_{\text{wide},i} \right) H_{\text{wide}}^T H_{\text{tall}}^{-1} \]

\[ + \left( \hat{H}_{\text{miss}} - H_{\text{tall}}^{-1} H_{\text{wide}} \right) \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{\text{wide},i} \hat{\lambda}^T_{\text{wide},i} \right) H_{\text{wide}} H_{\text{tall}}^{-1} \]

\[ + H_{\text{tall}}^{-1} H_{\text{wide}} \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{\text{wide},i} \hat{\lambda}^T_{\text{wide},i} \right) \left( \hat{H}_{\text{miss}} - H_{\text{tall}}^{-1} H_{\text{wide}} \right)^T \]

\[ + \left( \hat{H}_{\text{miss}} - H_{\text{tall}}^{-1} H_{\text{wide}} \right) \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{\text{wide},i} \hat{\lambda}^T_{\text{wide},i} \right) \left( \hat{H}_{\text{miss}} - H_{\text{tall}}^{-1} H_{\text{wide}} \right)^T. \]
\[H_{\text{tall}}^{-1}H_{\text{wide}} \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \lambda_{\text{wide},i} \lambda_{\text{tall},i}^T \right) H_{\text{wide}}^T H_{\text{tall}}^{-1} + O_p \left( \frac{1}{T_0} \right)\]

\[= \frac{1}{N_0} \sum_{i=1}^{N_0} (\hat{\lambda}_{\text{tall},i} - \epsilon_i) (\hat{\lambda}_{\text{tall},i} - \epsilon_i)^T + O_p \left( \frac{1}{T_0} \right)\]

\[= \frac{1}{N_0} \hat{\Lambda}_{\text{tall}}^T \hat{\Lambda}_{\text{tall}} - \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{\text{tall},i} \epsilon_i^T - \frac{1}{N_0} \sum_{i=1}^{N_0} \epsilon_i \hat{\lambda}_{\text{tall},i} + \frac{1}{N_0} \sum_{i=1}^{N_0} \epsilon_i \epsilon_i^T + O_p \left( \frac{1}{T_0} \right)\]

\[= \hat{V}_{\text{tall}} + O_p \left( \frac{1}{\sqrt{T_0}} \right) + O_p \left( \frac{1}{T_0} \right) = \hat{V}_{\text{tall}} + o_p(1).\]

For the pre-treatment subsample, it follows immediately from Lemma A.1(5) that
\[\frac{1}{N} \hat{\Lambda}_{\text{wide}}^T \hat{\Lambda}_{\text{wide}} = \hat{H}_{\text{miss}} \left( \frac{1}{N} \hat{\Lambda}_{\text{wide}}^T \hat{\Lambda}_{\text{wide}} \right) \hat{H}_{\text{miss}} = \hat{V}_{\text{wide}} + o_p(1).\]

And the desired results follow from the above facts and the proof of Lemma B.1 of Gonçalves and Perron (2014). \[\square\]

**Lemma B.13:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then for any integer \(m \geq 2\), as \(N_0, T_0 \to \infty\),

1. For the control subsample,
\[\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t}^* - H_{\text{tall}}^T \hat{f}_t \right\|^m = O_p \left( \frac{1}{T} \right).\]

2. For the pre-treatment subsample,
\[\frac{1}{T_0} \sum_{t=1}^{T_0} \left\| \hat{f}_{\text{wide},t}^* - H_{\text{wide}}^T \hat{f}_t \right\|^m = O_p \left( \frac{1}{T_0} \right).\]

**Proof:** For \(m = 2\), the first claim follows by applying Lemma 3.1 of Gonçalves and Perron (2014) to the control subsample. Note that Lemma 3.1 of Gonçalves and Perron (2014) relies only on their Conditions A*(b), A*(c), and B*(d), which have been verified by Lemma B.5(2), B.6(2), and B.9(2) of this paper. One can easily see that conclusions of Lemma B.5(2), B.6(2), and B.9(2) also hold for the pre-treatment subsample, which implies that we can apply Lemma 3.1 of Gonçalves and Perron (2014) to the pre-treatment subsample to prove Claim (2).

Now consider \(m > 2\). The case for \(m = 2\) and Assumptions 2.5(2) implies that
\[\sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t}^* - H_{\text{tall}}^T \hat{f}_t \right\|^2 = O_p \left( 1 \right) \quad \text{and} \quad \sum_{t=1}^{T_0} \left\| \hat{f}_{\text{wide},t}^* - H_{\text{wide}}^T \hat{f}_t \right\|^2 = O_p \left( 1 \right),\]

which in turn implies that
\[\sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t}^* - H_{\text{tall}}^T \hat{f}_t \right\|^m = O_p \left( 1 \right) \quad \text{and} \quad \sum_{t=1}^{T_0} \left\| \hat{f}_{\text{wide},t}^* - H_{\text{wide}}^T \hat{f}_t \right\|^m = O_p \left( 1 \right),\]

holds for any integer \(m > 2\). Then the proof is complete. \(\square\)

**Lemma B.14:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then for every \(t \in \{1, \ldots, T\}\), as \(N_0, T_0 \to \infty\),
\[\hat{f}_{\text{tall},t}^* - H_{\text{tall}}^T \hat{f}_t = O_p \left( \frac{1}{T^{1/4}} \right).\]

**Proof:** The proof is completed by applying Lemma 2 of Gonçalves et al. (2017) and the identity
before that lemma to the control subsample. Note that Lemma 2 of Gonçalves et al. (2017) requires their Assumptions 1–2 and Condition A. Assumptions 1–2 of Gonçalves et al. (2017) are implied by Assumptions 2.1–2.5 of this paper, and Condition A of Gonçalves et al. (2017) has been verified by Lemmas B.5–B.9 of this paper.

The convergence rate $T^{-1/4}$ differs from what is implied by Lemma 2 of Gonçalves et al. (2017), i.e., $N_0^{-1/2}$, because we use a looser bound for the term $b^*_t$ defined in the proof of Lemma 2 of Gonçalves et al. (2017). Specifically, by the triangle inequality and Cauchy-Schwarz inequality,

$$\|b^*_t\| \leq \frac{1}{T} \sum_{s=1}^{T} \|f_s\| \|\gamma^*_{s,t}\| \leq \sqrt{\frac{1}{T} \sum_{s=1}^{T} \|f_s\|^2 \sum_{s=1}^{T} \gamma^*_{s,t}^2} \leq \left( \max_{1 \leq s \leq T} \|f_s\| \right) \sqrt{\frac{1}{T} \sum_{s=1}^{T} \gamma^*_{s,t}^2} = O_P \left( T^{1/4} \right) O_P \left( \frac{1}{\sqrt{T}} \right) = O_P \left( \frac{1}{T^{1/4}} \right),$$

and this term turns out to be the dominant term of $\left( \hat{f}^*_{\text{tall},t} - H^*_{\text{tall}} \hat{f}_t \right)$.

For every $i \in \{1, \ldots, N\}$, define $e^*_{\text{wide},i} = (e^*_{i,1}, \ldots, e^*_{i,T_0})^T$.

**Lemma B.15:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then as $N_0, T_0 \to \infty$,

1. For every $i \in \{1, \ldots, N\}$,

$$\frac{1}{T_0} \left( \hat{F}^*_{\text{wide}} - \hat{F}^*_\text{wide} H^*_{\text{wide}} \right)^T e^*_{\text{wide},i} = O_P^* \left( \frac{1}{\sqrt{T_0}} \right).$$

2. \[ \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T_0} \left( \hat{F}^*_{\text{wide}} - \hat{F}^*_\text{wide} H^*_{\text{wide}} \right)^T e^*_{\text{wide},i} \right\|^2 = O_P^* \left( \frac{1}{T_0} \right). \]

**Proof:** By the triangle inequality, Cauchy-Schwarz inequality, and Lemmas B.10(1), B.13(2),

$$\left\| \frac{1}{T_0} \left( \hat{F}^*_{\text{wide}} - \hat{F}^*_\text{wide} H^*_{\text{wide}} \right)^T e^*_{\text{wide},i} \right\| = \left\| \frac{1}{T_0} \sum_{s=1}^{T_0} \left( \hat{f}^*_{\text{wide},s} - H^*_{\text{wide}} \hat{f}_s \right) e^*_{i,s} \right\| \leq \frac{1}{T_0} \sum_{s=1}^{T_0} \left\| \hat{f}^*_{\text{wide},s} - H^*_{\text{wide}} \hat{f}_s \right\| \left\| e^*_{i,s} \right\| \leq \sqrt{\frac{1}{T_0} \sum_{s=1}^{T_0} \left\| \hat{f}^*_{\text{wide},s} - H^*_{\text{wide}} \hat{f}_s \right\|^2} \sqrt{\frac{1}{T_0} \sum_{s=1}^{T_0} e^*_{i,s}^2} = O_P^* \left( \frac{1}{\sqrt{T_0}} \right),$$

which proves the first claim. For the second claim, we use the above facts and Lemmas B.10(3), B.13(2) to conclude that,

$$\frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T_0} \left( \hat{F}^*_{\text{wide}} - \hat{F}^*_\text{wide} H^*_{\text{wide}} \right)^T e^*_{\text{wide},i} \right\|^2 \leq \left( \frac{1}{T_0} \sum_{s=1}^{T_0} \left\| \hat{f}^*_{\text{wide},s} - H^*_{\text{wide}} \hat{f}_s \right\|^2 \right) \left( \frac{1}{T_0 N} \sum_{s=1}^{T_0} \sum_{i=1}^{N} e^*_{i,s}^2 \right) = O_P^* \left( \frac{1}{T_0} \right),$$

and the proof is complete.

**Lemma B.16:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then as $N_0, T_0 \to \infty$, 


(1) For every $i \in \{1, \ldots, N\}$,
\[
\frac{1}{T_0} \tilde{H}_{\text{wide},i}^* \hat{F}_{\text{wide},i}^* = O_{\mathbb{P}^*} \left( \frac{1}{\sqrt{T_0}} \right).
\]

(2) 
\[
\frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T_0} \tilde{H}_{\text{wide},i}^* \hat{F}_{\text{wide},i}^* \right\|^2 = O_{\mathbb{P}^*} \left( \frac{1}{T_0} \right).
\]

**Proof:** Lemma B.11(1) implies that
\[
\frac{1}{T_0} \tilde{F}_{\text{wide},i}^* = \frac{1}{T_0} \sum_{t=1}^{T_0} \tilde{f}_{i,t}^* = O_{\mathbb{P}^*} \left( \frac{1}{\sqrt{T_0}} \right).
\]
and Claim (1) follows from Lemma B.12(1). Furthermore, Lemma B.11(2) implies that
\[
\frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T_0} \tilde{F}_{\text{wide},i}^* \right\|^2 = \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T_0} \sum_{t=1}^{T_0} \tilde{f}_{i,t}^* \right\|^2 = O_{\mathbb{P}^*} \left( \frac{1}{T_0} \right),
\]
and Claim (2) follows by the properties of matrix norms and Lemma B.12(1). \qed

**Lemma B.17:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then as $N_0, T_0 \to \infty$,

(1) For every $i \in \{1, \ldots, N\}$,
\[
\frac{1}{T_0} \hat{F}_{\text{wide}}^* \left( \tilde{F}_{\text{wide}}^* - \hat{F}_{\text{wide}}^* \tilde{H}_{\text{wide}}^* \right) \tilde{\lambda}_i = O_{\mathbb{P}^*} \left( \frac{1}{\sqrt{T_0}} \right).
\]

(2) 
\[
\frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T_0} \hat{F}_{\text{wide}}^* \left( \tilde{F}_{\text{wide}}^* - \hat{F}_{\text{wide}}^* \tilde{H}_{\text{wide}}^* \right) \tilde{\lambda}_i \right\|^2 = O_{\mathbb{P}^*} \left( \frac{1}{T_0} \right).
\]

**Proof:** By the triangle inequality, the properties of matrix norms, Cauchy-Schwarz inequality, Lemmas A.3(3), B.13(2), and Assumption 2.5(2),
\[
\left\| \frac{1}{T_0} \left( \tilde{F}_{\text{wide}}^* - \hat{F}_{\text{wide}}^* \tilde{H}_{\text{wide}}^* \right)^T \tilde{F}_{\text{wide}} \right\| = \left\| \frac{1}{T_0} \sum_{s=1}^{T_0} \left( \tilde{f}_{\text{wide},s}^* - \tilde{H}_{\text{wide}}^* \tilde{f}_s \right) \tilde{f}_s \right\|
\]
\[
\leq \frac{1}{T_0} \sum_{s=1}^{T_0} \left\| \tilde{f}_{\text{wide},s}^* - \tilde{H}_{\text{wide}}^* \tilde{f}_s \right\| \left\| \tilde{f}_s \right\| \leq \sqrt{\frac{1}{T_0} \sum_{s=1}^{T_0} \left\| \tilde{f}_{\text{wide},s}^* - \tilde{H}_{\text{wide}}^* \tilde{f}_s \right\|^2 \frac{1}{T_0} \sum_{s=1}^{T_0} \left\| \tilde{f}_{\text{wide},s} \right\|^2}
\]
\[
= O_{\mathbb{P}^*} \left( \frac{1}{\sqrt{T_0}} \right).
\]

Note that
\[
\frac{1}{T_0} \left( \tilde{F}_{\text{wide}}^* - \hat{F}_{\text{wide}}^* \tilde{H}_{\text{wide}}^* \right)^T \hat{F}_{\text{wide}}^* = -\tilde{H}_{\text{wide}}^* b_i^* + b_2^*,
\]
where
\[
b_1^* = \frac{1}{T_0} \left( \tilde{F}_{\text{wide}}^* - \hat{F}_{\text{wide}}^* \tilde{H}_{\text{wide}}^* \right)^T \tilde{F}_{\text{wide}}^* \tilde{H}_{\text{wide}}^*,
\]
\[
b_2^* = \frac{1}{T_0} \left( \tilde{F}_{\text{wide}}^* - \hat{F}_{\text{wide}}^* \tilde{H}_{\text{wide}}^* \right)^T \left( \tilde{F}_{\text{wide}}^* - \hat{F}_{\text{wide}}^* \tilde{H}_{\text{wide}}^* \right).
\]

By the above result and Lemma B.12(1), $b_1^* = O_{\mathbb{P}^*} \left( 1/\sqrt{T_0} \right)$. By the triangle inequality, the properties of matrix norms, and Lemma B.13(2),
\[
\| b_2^* \| = \left\| \frac{1}{T_0} \sum_{s=1}^{T_0} \left( \tilde{f}_{\text{wide},s}^* - \tilde{H}_{\text{wide}}^* \tilde{f}_s \right) \left( \tilde{f}_{\text{wide},s}^* - \tilde{H}_{\text{wide}}^* \tilde{f}_s \right)^T \right\|.
\]
\[
\leq \frac{1}{T_0} \sum_{i=1}^{T_0} \left\| \tilde{F}_{\text{wide},s} - H_{\text{wide}}^{sT} \tilde{f}_i \right\|^2 = O_p\left( \frac{1}{T_0} \right).
\]

This completes the proof of Claim (1). Moreover, by the properties of matrix norms and Lemmas A.1(5), A.3(4),

\[
\frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T_0} \tilde{F}_{\text{wide}}^{sT} \left( \tilde{F}_{\text{wide}} - \tilde{F}_{\text{wide}} H_{\text{wide}}^{s-1} \right) \tilde{\lambda}_i \right\|^2 \leq \left\| \frac{1}{T_0} \tilde{F}_{\text{wide}}^{sT} \left( \tilde{F}_{\text{wide}} - \tilde{F}_{\text{wide}} H_{\text{wide}}^{s-1} \right) \right\|^2 \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \tilde{\lambda}_i \right\|^2 \right)
\]

\[
\leq \left\| \frac{1}{T_0} \tilde{F}_{\text{wide}}^{sT} \left( \tilde{F}_{\text{wide}} - \tilde{F}_{\text{wide}} H_{\text{wide}}^{s-1} \right) \right\|^2 \left\| \tilde{H}_{\text{miss}} \right\|^2 \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \tilde{\lambda}_{\text{wide},i} \right\|^2 \right) = O_p\left( \frac{1}{T_0} \right),
\]

which proves Claim (2).

Lemma B.18: If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then for every integer \( m \), as \( N_0, T_0 \to \infty \),

\[
\hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{s-1} \tilde{\lambda}_i = O_p\left( \frac{1}{\sqrt{T_0}} \right).
\]

Proof: The result follows immediately from the decomposition

\[
\hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{s-1} \tilde{\lambda}_i = \frac{1}{T_0} \left( \hat{F}_{\text{wide}} - \tilde{F}_{\text{wide}} H_{\text{wide}}^s \right)^T e_{\text{wide},i}
\]

\[
+ \frac{1}{T_0} H_{\text{wide}}^{sT} \tilde{F}_{\text{wide}}^{sT} e_{\text{wide},i} + \frac{1}{T_0} \tilde{F}_{\text{wide}}^{sT} \left( \tilde{F}_{\text{wide}} - \tilde{F}_{\text{wide}} H_{\text{wide}}^{s-1} \right) \tilde{\lambda}_i
\]

[29]

and Lemmas B.15(1), B.16(1), B.17(1).

Lemma B.19: If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then for every integer \( m \geq 2 \), as \( N_0, T_0 \to \infty \),

1. For the pre-treatment subsample,

\[
\frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_{\text{wide},i}^{s} - H_{\text{wide}}^{s-1} \tilde{\lambda}_i \right\|^m = O_p\left( \frac{1}{T_0} \right). \]

2. For the control subsample,

\[
\frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}_{\text{tall},i}^{s} - H_{\text{tall}}^{s-1} \tilde{\lambda}_i \right\|^m = O_p\left( \frac{1}{T} \right). \]

Proof: Firstly consider \( m = 2 \). Then Claim (1) can be easily established by the decomposition [29], the \( C_p \) inequality, and Lemmas B.15(2), B.16(2), B.17(2). One can show that these conclusions also hold for the control subsample, and Claim (2) can be proved analogously.

Now we turn to the case of \( m > 2 \). By the results for the case of \( m = 2 \) and Assumption 2.5(2),

\[
\sum_{i=1}^{N} \left\| \hat{\lambda}_{\text{wide},i}^{s} - H_{\text{wide}}^{s-1} \tilde{\lambda}_i \right\|^2 = O_p(1) \quad \text{and} \quad \sum_{i=1}^{N} \left\| \hat{\lambda}_{\text{tall},i}^{s} - H_{\text{tall}}^{s-1} \tilde{\lambda}_i \right\|^2 = O_p(1),
\]

which in turn implies that

\[
\sum_{i=1}^{N} \left\| \hat{\lambda}_{\text{wide},i}^{s} - H_{\text{wide}}^{s-1} \tilde{\lambda}_i \right\|^m = O_p(1) \quad \text{and} \quad \sum_{i=1}^{N} \left\| \hat{\lambda}_{\text{tall},i}^{s} - H_{\text{tall}}^{s-1} \tilde{\lambda}_i \right\|^m = O_p(1)
\]

for any integer \( m > 2 \). And the proof is completed by using Assumption 2.5(2) again.

Lemma B.20: If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a
factor model with covariates, then for every integer \( m \geq 2 \), as \( N_0, T_0 \to \infty \),
\[
\frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}^{*}_{\text{tall},i} - H_{\text{tall}}^{*-1} H_{\text{wide}}^{*} \hat{\lambda}^{*}_{\text{wide},i} \right\|^{m} = O_{P^*} \left( \frac{1}{T_0} \right).
\]

**Proof:** The conclusion is established by the properties of matrix norms, the \( C_p \) inequality, Lemmas B.12(1), B.19, and Assumption 2.5(2).

**Lemma B.21:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then as \( N_0, T_0 \to \infty \),
\[
\hat{H}_{\text{miss}} - H_{\text{tall}}^{*-1} H_{\text{wide}}^{*} = O_{P^*} \left( \frac{1}{\sqrt{T_0}} \right).
\]

**Proof:** Let \( \epsilon^*_i = \hat{\lambda}^{*}_{\text{tall},i} - H_{\text{tall}}^{*-1} H_{\text{wide}}^{*} \hat{\lambda}^{*}_{\text{wide},i} \) for every \( i \in \{1, \ldots, N_0\} \). By construction,
\[
\hat{H}_{\text{miss}} = \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}^{*}_{\text{tall},i} \hat{\lambda}^{* T}_{\text{tall},i} \right) \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}^{*}_{\text{wide},i} \hat{\lambda}^{* T}_{\text{wide},i} \right)^{-1}
= \left[ \frac{1}{N_0} \sum_{i=1}^{N_0} \left( H_{\text{tall}}^{*-1} H_{\text{wide}}^{*} \hat{\lambda}^{*}_{\text{wide},i} + \epsilon^*_i \right) \hat{\lambda}^{* T}_{\text{wide},i} \right] \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}^{*}_{\text{wide},i} \hat{\lambda}^{* T}_{\text{wide},i} \right)^{-1}
= H_{\text{tall}}^{*-1} H_{\text{wide}}^{*} + \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \epsilon^*_i \hat{\lambda}^{* T}_{\text{tall},i} \right) \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}^{*}_{\text{wide},i} \hat{\lambda}^{* T}_{\text{wide},i} \right)^{-1}.
\]

Moreover,
\[
\frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}^{*}_{\text{wide},i} \hat{\lambda}^{* T}_{\text{wide},i} = H_{\text{wide}}^{*} H_{\text{tall}}^{*-1} \left[ \frac{1}{N_0} \sum_{i=1}^{N_0} \left( \hat{\lambda}^{*}_{\text{tall},i} \hat{\lambda}^{* T}_{\text{tall},i} - \hat{\lambda}^{*}_{\text{tall},i} \epsilon^*_i \hat{\lambda}^{* T}_{\text{tall},i} + \epsilon^*_i \hat{\lambda}^{* T}_{\text{tall},i} + \epsilon^*_i \hat{\lambda}^{* T}_{\text{tall},i} \right) \right] H_{\text{tall}}^{*} H_{\text{wide}}^{*-1}.
\]

By the triangle inequality, the properties of matrix norms, Cauchy-Schwarz inequality, Lemmas B.22(2), B.20, and Assumption 2.5(2),
\[
\left\| \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}^{*}_{\text{tall},i} \epsilon^*_i \hat{\lambda}^{* T}_{\text{tall},i} \right\| \leq \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}^{*}_{\text{tall},i} \right\| \left\| \epsilon^*_i \right\| \leq \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}^{*}_{\text{tall},i} \right\|^2 \right)^{1/2} \left( \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \epsilon^*_i \right\|^2 \right)^{1/2} = O_{P^*} \left( \frac{1}{T_0} \right).
\]

Similarly,
\[
\left\| \frac{1}{N_0} \sum_{i=1}^{N_0} \epsilon^*_i \epsilon^*_i \hat{\lambda}^{* T}_{\text{tall},i} \right\| \leq \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \epsilon^*_i \right\|^2 \leq O_{P^*} \left( \frac{1}{T_0} \right).
\]

Above results imply that
\[
\frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}^{*}_{\text{wide},i} \hat{\lambda}^{* T}_{\text{wide},i} = H_{\text{wide}}^{*} H_{\text{tall}}^{*-1} H_{\text{tall}}^{*} H_{\text{wide}}^{*-1} + O_{P^*} \left( \frac{1}{\sqrt{T_0}} \right).
\]

One can also show that
\[
\left\| \frac{1}{N_0} \sum_{i=1}^{N_0} \epsilon^*_i \hat{\lambda}^{* T}_{\text{wide},i} \right\| = O_{P^*} \left( \frac{1}{\sqrt{T_0}} \right).
\]

Then the desired result follows from Lemma B.12.

**Lemma B.22:** If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then for every \( m \in \{2,3,\ldots,8\} \), as \( N_0, T_0 \to \infty \),
\[
\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t} \right\|^m = O_{P^*} (1), \quad \frac{1}{T_0} \sum_{t=1}^{T_0} \left\| \hat{f}_{\text{wide},t} \right\|^m = O_{P^*} (1).
\]
(2) \[ \frac{1}{N_0} \sum_{i=1}^{N_0} \left\| \hat{\lambda}_{\text{tall},i} \right\|^m = O_P(1), \quad \frac{1}{N} \sum_{i=1}^{N} \left\| \hat{\lambda}_{\text{wide},i} \right\|^m = O_P(1). \]

(3) \[ \frac{1}{T_0} \sum_{t=1}^{T_0} \left| \hat{c}_{t,i}^* \right|^m = O_P(1) \quad \text{for every } i \in \{1, \ldots, N\}. \]

(4) \[ \frac{1}{N_0} \sum_{i=1}^{N_0} \left| \hat{c}_{t,i}^* \right|^m = O_P(1) \quad \text{for every } t \in \{1, \ldots, T\}. \]

(5) \[ \frac{1}{T_0} \sum_{t=1}^{T_0} \left| \hat{c}_{t,i}^* - \hat{c}_{i,t} \right|^m = O_P\left( \frac{1}{T_0} \right) \quad \text{for every } i \in \{1, \ldots, N\}. \]

**Proof:** By the $C_p$ inequality, the properties of matrix norms, and Lemmas A.3(3), B.12(1), B.13,

\[
\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t} \right\|^m = \frac{1}{T} \sum_{t=1}^{T} \left\| H_{\text{tall}}^* \hat{f}_t + \left( \hat{f}_{\text{tall},t} - H_{\text{tall}}^* \hat{f}_t \right) \right\|^m \\
\leq 2^{m-1} \left[ \left\| H_{\text{tall}} \right\| \left( \frac{1}{T} \sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t} \right\|^m \right) + \left( \frac{1}{T} \sum_{t=1}^{T} \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}}^* \hat{f}_t \right\|^m \right) \right] = O_P(1). 
\]

The rest parts of Claims (1) and (2) can be proved analogously.

To verify Claims (3)–(5), we conduct the following decomposition

\[
\hat{c}_{t,i}^* = \hat{e}_{t,i}^* - (\hat{c}_{t,i}^* - \hat{c}_{i,t}) \\
= \hat{e}_{t,i}^* - \left( \hat{f}_{\text{tall},t} - H_{\text{tall}}^* \hat{f}_t \right)^T \tilde{H}_{\text{miss}}^* \left( \hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \hat{\lambda}_i \right) \\
- \left( \hat{f}_{\text{tall},t} - H_{\text{tall}}^* \hat{f}_t \right)^T H_{\text{miss}} \tilde{H}_{\text{miss}}^* H_{\text{wide}}^{-1} \hat{\lambda}_i - \hat{f}_t \left( H_{\text{tall}} \hat{H}_{\text{miss}}^* H_{\text{wide}}^{-1} - I_r \right) \hat{\lambda}_i. \tag{30}
\]

By the $C_p$ inequality,

\[
\left| \hat{c}_{t,i}^* \right|^m \leq 5^{m-1} \left( \left\| \hat{e}_{t,i}^* \right\|^m + \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}}^* \hat{f}_t \right\|^m \right) \left\| \tilde{H}_{\text{miss}}^* \right\|^m \left\| \hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \hat{\lambda}_i \right\|^m \\
+ \left\| \hat{f}_{\text{tall},t} \right\|^m \left\| H_{\text{tall}} \right\|^m \left\| \tilde{H}_{\text{miss}}^* \right\|^m \left\| \hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \hat{\lambda}_i \right\|^m \\
+ \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}}^* \hat{f}_t \right\|^m \left\| \tilde{H}_{\text{miss}}^* \right\|^m \left\| H_{\text{wide}} \right\|^m \left\| \tilde{H}_{\text{miss}} \right\|^m \left\| \hat{\lambda}_{\text{wide},i} \right\|^m \\
+ \left\| \hat{f}_{\text{tall},t} \right\|^m \left\| H_{\text{tall}} \tilde{H}_{\text{miss}}^* H_{\text{wide}}^{-1} - I_r \right\|^m \left\| \tilde{H}_{\text{miss}} \right\|^m \left\| \hat{\lambda}_{\text{wide},i} \right\|^m ,
\]

and

\[
\left| \hat{c}_{t,i}^* - \hat{c}_{i,t} \right|^m \leq 4^{m-1} \left( \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}}^* \hat{f}_t \right\|^m \right) \left\| \tilde{H}_{\text{miss}}^* \right\|^m \left\| \hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \hat{\lambda}_i \right\|^m \\
+ \left\| \hat{f}_{\text{tall},t} \right\|^m \left\| H_{\text{tall}} \right\|^m \left\| \tilde{H}_{\text{miss}}^* \right\|^m \left\| \hat{\lambda}_{\text{wide},i} - H_{\text{wide}}^{-1} \hat{\lambda}_i \right\|^m \\
+ \left\| \hat{f}_{\text{tall},t} - H_{\text{tall}}^* \hat{f}_t \right\|^m \left\| \tilde{H}_{\text{miss}}^* \right\|^m \left\| H_{\text{wide}} \right\|^m \left\| \tilde{H}_{\text{miss}} \right\|^m \left\| \hat{\lambda}_{\text{wide},i} \right\|^m \\
+ \left\| \hat{f}_{\text{tall},t} \right\|^m \left\| H_{\text{tall}} \tilde{H}_{\text{miss}}^* H_{\text{wide}}^{-1} - I_r \right\|^m \left\| \tilde{H}_{\text{miss}} \right\|^m \left\| \hat{\lambda}_{\text{wide},i} \right\|^m .
\]

Recall that $m \in \{2, 3, \ldots, 8\}$. Claims (3) and (5) are established by Lemmas A.1(5), A.3(3), B.10(1), B.12, B.13(1), B.18, B.21 and Assumption 2.5(2). Claim (4) follows from Lemmas A.1(5), A.3(4),
Lemma B.23: If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then \( \hat{\Phi}_i^* = o_{P^*}(T^{1/4}) \) and \( \hat{\Gamma}_i^* = O_{P^*}(1) \) as \( N_0, T_0 \to \infty \).

Proof: For every \( k \in \{1, \ldots, K \} \), by the triangle inequality, the properties of matrix norms, Cauchy-Schwarz inequality, Lemma B.22(1)(3), and Assumption 2.5(2),

\[
\|L_{k,i}^*\| = \left\| \frac{1}{T_0} \sum_{s=k+1}^{T_0} \hat{f}_{tall,s}^* \hat{f}_{tall,s-k}^* \right\| \leq \frac{1}{T_0} \sum_{s=k+1}^{T_0} \|\hat{f}_{tall,s}^*\| \|\hat{f}_{tall,s-k}^*\| \\
\leq \frac{1}{T_0} \sum_{s=1}^{T_0} \|\hat{f}_{tall,s}^*\|^2 \hat{c}_{i,s}^2 \leq \sqrt{\frac{1}{T_0} \sum_{s=1}^{T_0} \|\hat{f}_{tall,s}^*\|^4} \sqrt{\frac{1}{T_0} \sum_{s=1}^{T_0} \hat{c}_{i,s}^4} = O_{P^*}(1).
\]

Then by construction and the triangle inequality,

\[
\|\hat{\Phi}_i^*\| \leq \|L_{0,i}^*\| + 2 \sum_{k=1}^K \|L_{k,i}^*\| \leq (2K + 1) \sqrt{\frac{1}{T_0} \sum_{s=1}^{T_0} \|\hat{f}_{tall,s}^*\|^4} \sqrt{\frac{1}{T_0} \sum_{s=1}^{T_0} \hat{c}_{i,s}^4} = O_{P^*} \left( T^{1/4} \right).
\]

By the triangle inequality, the properties of matrix norms, Cauchy-Schwarz inequality, Lemma B.22(2)(4), and Assumption 2.5(2),

\[
\|\hat{\Gamma}_i^*\| = \left\| \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{\lambda}_{wide,j,i}^* \hat{\lambda}_{wide,j,i}^2 \right\| \leq \frac{1}{N_0} \sum_{i=1}^{N_0} \|\hat{\lambda}_{wide,j,i}^*\|^2 \hat{c}_{j,t}^2 \\
\leq \sqrt{\frac{1}{N_0} \sum_{i=1}^{N_0} \|\hat{\lambda}_{wide,j,i}^*\|^4} \sqrt{\frac{1}{N_0} \sum_{i=1}^{N_0} \hat{c}_{j,t}^4} = O_{P^*}(1).
\]

Then the proof is complete.

\[\square\]

Lemma B.24: If Assumptions 2.1–2.5 hold for a pure factor model or Assumptions 2.1–2.8 for a factor model with covariates, then for every \( i \in \{1, \ldots, N \} \), we have \( \hat{\sigma}_i^2 - \sigma_i^2 = o_{P^*}(1) \) as \( N_0, T_0 \to \infty \).

Proof: By Assumptions 2.2(2), 2.3(1) and the fact that \( \{u_{i,t}\} \) are i.i.d. as \( \mathbb{N}(0,1) \) and independent of the raw sample, \( \mathbb{E} \left( e_{t,i}^2 u_{t,i}^2 \right) = \sigma_i^4 \), \( \mathbb{E} \left( u_{t,i}^4 \right) = 3 \), and \( \mathbb{E} \left( e_{t,i}^2 u_{t,i}^4 \right) \leq 3(M + 1) \). Then by Markov’s inequality, as \( N_0, T_0 \to \infty \),

\[
\frac{1}{T_0} \sum_{t=1}^{T_0} u_{t,i}^4 = O_{P}(1), \quad \frac{1}{T_0} \sum_{t=1}^{T_0} e_{t,i}^2 u_{t,i}^4 = O_{P}(1).
\]

The decomposition [27], the triangle inequality, Cauchy-Schwarz inequality, and Lemmas A.4, A.6 imply that

\[
\frac{1}{T_0} \sum_{t=1}^{T_0} (c_{i,t} - \hat{c}_{i,t})^2 u_{t,i}^4 \leq \sqrt{\frac{1}{T_0} \sum_{t=1}^{T_0} (c_{i,t} - \hat{c}_{i,t})^4} \sqrt{\frac{1}{T_0} \sum_{t=1}^{T_0} u_{t,i}^4} = O_{P} \left( \frac{1}{\sqrt{T_0}} \right).
\]

\[
\frac{1}{T_0} \sum_{t=1}^{T_0} (c_{i,t} - \hat{c}_{i,t}) e_{t,i} u_{t,i}^2 \leq \sqrt{\frac{1}{T_0} \sum_{t=1}^{T_0} (c_{i,t} - \hat{c}_{i,t})^2} \sqrt{\frac{1}{T_0} \sum_{t=1}^{T_0} e_{t,i}^2 u_{t,i}^4} = O_{P} \left( \frac{1}{\sqrt{T_0}} \right).
\]

By Cauchy-Schwarz inequality and Lemmas B.10(1), B.22(5),

\[
\left\| \frac{1}{T_0} \sum_{t=1}^{T_0} e_{t,i}^2 \right\| \left\| \frac{1}{T_0} \sum_{t=1}^{T_0} \left( c_{i,t} - \hat{c}_{i,t}^* \right) \right\| \leq \sqrt{\frac{1}{T_0} \sum_{t=1}^{T_0} e_{t,i}^4} \sqrt{\frac{1}{T_0} \sum_{t=1}^{T_0} \left( c_{i,t} - \hat{c}_{i,t}^* \right)^2} = O_{P^*} \left( \frac{1}{\sqrt{T_0}} \right).
\]
Therefore,
\[
\tilde{\sigma}_i^2 = \frac{1}{T_0} \sum_{t=1}^{T_0} c_{i,t}^2 = \frac{1}{T_0} \sum_{t=1}^{T_0} \left[ e_{i,t}^2 + (\tilde{c}_{i,t} - \tilde{c}_{i,t})^2 \right]
\]
\[
= \frac{1}{T_0} \sum_{t=1}^{T_0} c_{i,t}^2 + \frac{1}{T_0} \sum_{t=1}^{T_0} (\tilde{c}_{i,t} - \tilde{c}_{i,t})^2 + \frac{2}{T_0} \sum_{t=1}^{T_0} e_{i,t}^2 (\tilde{c}_{i,t} - \tilde{c}_{i,t})
\]
\[
= \frac{1}{T_0} \sum_{t=1}^{T_0} c_{i,t}^2 + O_{P^*} \left( \frac{1}{\sqrt{T_0}} \right) = \frac{1}{T_0} \sum_{t=1}^{T_0} [e_{i,t}^2 + (c_{i,t} - \tilde{c}_{i,t})^2] u_{i,t}^2 + O_{P^*} \left( \frac{1}{\sqrt{T_0}} \right)
\]
\[
= \frac{1}{T_0} \sum_{t=1}^{T_0} c_{i,t}^2 u_{i,t}^2 + O_{P^*} \left( \frac{1}{\sqrt{T_0}} \right) = \frac{1}{T_0} \sum_{t=1}^{T_0} c_{i,t}^2 u_{i,t}^2 + O_{P^*} \left( \frac{1}{\sqrt{T_0}} \right).
\]
And the proof is completed by Assumption 2.2(2) and the ergodic theorem.

Appendix C  Proof of Main Results

In this section, we provide the proofs of Theorem 3.1 and Theorem 4.1.

C.1 Proof of Theorem 3.1

For \( (i, t) \in \mathcal{I}_1 \), define
\[
s_{i,t} = \frac{\tilde{c}_{i,t} - y_{i,t}}{\sqrt{\tilde{V}_{i,t} + \tilde{\sigma}_i^2}}.
\]
Since \( \{e_{i,t} : t = 1, 2, \ldots \} \) is strictly stationary, we have \( \text{Var}(e_{i,t}) = \sigma_i^2 \) for all \( t \in \mathbb{Z}_+ \).

**Lemma C.1:** If Assumptions 2.1–2.5 hold, then \( s_{i,t} = \frac{-e_{i,t}}{\sigma_i} + o_P(1) \) as \( N_0, T_0 \to \infty \) for every \( (i, t) \in \mathcal{I}_1 \).

**Proof:** By the decomposition of \( (\tilde{c}_{i,t} - c_{i,t}) \) in Equation [27] and Lemma A.1(2)–(5), it follows that \( \tilde{c}_{i,t} - c_{i,t} = o_P(1) \).

By Cauchy-Schwarz inequality, Assumption 2.3(1), Markov’s inequality, and Lemma A.4,
\[
\left| \frac{1}{T_0} \sum_{t=1}^{T_0} e_{i,s} (c_{i,s} - \tilde{c}_{i,s}) \right| \leq \sqrt{\frac{1}{T_0} \sum_{s=1}^{T_0} e_{i,s}^2} \sqrt{\frac{1}{T_0} \sum_{s=1}^{T_0} (c_{i,s} - \tilde{c}_{i,s})^2} = O_P \left( \frac{1}{\sqrt{T_0}} \right).
\]
Then \( \tilde{\sigma}_i^2 \) admits a decomposition
\[
\tilde{\sigma}_i^2 = \frac{1}{T_0} \sum_{s=1}^{T_0} c_{i,s}^2 = \frac{1}{T_0} \sum_{s=1}^{T_0} [e_{i,s}^2 + (c_{i,s} - \tilde{c}_{i,s})^2]
\]
\[
= \frac{1}{T_0} \sum_{s=1}^{T_0} c_{i,s}^2 + \frac{1}{T_0} \sum_{s=1}^{T_0} (c_{i,s} - \tilde{c}_{i,s})^2 + \frac{2}{T_0} \sum_{s=1}^{T_0} e_{i,s} (c_{i,s} - \tilde{c}_{i,s}) = \frac{1}{T_0} \sum_{s=1}^{T_0} c_{i,s}^2 + O_P \left( \frac{1}{\sqrt{T_0}} \right).
\]
By Assumption 2.2(2) and the ergodic theorem, \( \tilde{\sigma}_i^2 \overset{P}{\to} \sigma_i^2 \) as \( N_0, T_0 \to \infty \).
Theorem 6 of Bai (2003) implies that
\[ \hat{f}_{\text{tall},t} \left( \frac{\hat{F}_{\text{tall}} \hat{F}_{\text{tall}}}{T} \right)^{-1} \hat{\Phi}_t \left( \frac{\hat{F}_{\text{tall}} \hat{F}_{\text{tall}}}{T} \right)^{-1} \hat{f}_{\text{tall},t} = O_p(1), \]
\[ \hat{\lambda}_{\text{wide},i}^T \left( \frac{\hat{\Lambda}_{\text{wide}} \hat{\Lambda}_{\text{wide}}}{N} \right)^{-1} \hat{\Gamma}_t \left( \frac{\hat{\Lambda}_{\text{wide}} \hat{\Lambda}_{\text{wide}}}{N} \right)^{-1} \hat{\lambda}_{\text{wide},i} = O_p(1), \]
and consequently \( \hat{v}_{i,t} = o_p(1) \). Therefore,
\[ s_{i,t} = \frac{(c_{i,t} - c_{i,t}^*) + (c_{i,t} - y_{i,t})}{\sqrt{\sigma_i^2 + \hat{v}_{i,t}^2 + (\hat{\sigma}_i^*)^2 - \sigma_i^2}} = \frac{-c_{i,t} + o_p(1)}{\sqrt{\sigma_i^2 + o_p(1)}} = \frac{-c_{i,t}}{\sigma_i} + o_p(1) \]
as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\).

Lemma C.2: If Assumptions 2.1–2.5 hold, then \( s_{i,t}^* = \frac{c_{i,t}^*}{\sigma_i} + o_p(1) \)
as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\).

Proof: By the decomposition of \( \hat{c}_{i,t} - \hat{c}_{i,t}^* \) in Equation [30] and Lemmas B.12, B.14, B.18, B.21, it follows that \( \hat{c}_{i,t}^* - \hat{c}_{i,t} = o_p(1) \).

By construction,
\[ \hat{v}_{i,t} = \frac{1}{T_0} \hat{f}_{\text{tall},t} \left( \frac{\hat{F}_{\text{tall}} \hat{F}_{\text{tall}}}{T} \right)^{-1} \hat{\Phi}_t \left( \frac{\hat{F}_{\text{tall}} \hat{F}_{\text{tall}}}{T} \right)^{-1} \hat{f}_{\text{tall},t} + \frac{1}{N_0} \hat{\lambda}_{\text{wide},i}^T \left( \frac{\hat{\Lambda}_{\text{wide}} \hat{\Lambda}_{\text{wide}}}{N} \right)^{-1} \hat{\Gamma}_t \left( \frac{\hat{\Lambda}_{\text{wide}} \hat{\Lambda}_{\text{wide}}}{N} \right)^{-1} \hat{\lambda}_{\text{wide},i} \]
Combining this with Lemmas B.12(2), B.23 yields \( \hat{v}_{i,t}^* = o_p(1) \).

By Lemma B.24,
\[ s_{i,t}^* = \frac{(c_{i,t}^* - \hat{c}_{i,t}^*) + (\hat{c}_{i,t}^* - \hat{y}_{i,t})}{\sqrt{\sigma_i^2 + \hat{v}_{i,t}^2 + (\hat{\sigma}_i^*)^2 - \sigma_i^2}} = \frac{-c_{i,t}^* + o_p(1)}{\sqrt{\sigma_i^2 + o_p(1)}} = \frac{-c_{i,t}}{\sigma_i} + o_p(1) \]
as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\).

Let \( G_{e,i} \) and \( G_{s,i} \) be the cumulative distribution functions of \( e_{i,t} \) and \( -\frac{e_{i,t}}{\sigma_i} \), respectively, i.e.,
\[ G_{e,i}(z) = P\left(e_{i,t} \leq z\right), \quad G_{s,i}(z) = P\left(-\frac{e_{i,t}}{\sigma_i} \leq z\right), \quad z \in \mathbb{R}. \]
Let \( G_{e,i}^* \) and \( G_{s,i}^* \) be the conditional cumulative distributions function of \( e_{i,t}^* \) and \( s_{i,t}^* \), respectively, so that \( G_{e,i}^*(z) = P^*(e_{i,t} \leq z) \) and \( G_{s,i}^*(z) = P^*(s_{i,t}^* \leq z) \) for every \( z \in \mathbb{R} \).

For two (unconditional or conditional) distribution functions \( G_1 \) and \( G_2 \), we measure the distance between \( G_1 \) and \( G_2 \) by the Mallows metric that is defined as
\[ d_2(G_1, G_2) = \sqrt{\inf_{\xi \in \Xi(G_1, G_2)} \int_{\mathbb{R}^2} (z_1 - z_2)^2 \, d\xi(z_1, z_2),} \]
where \( \Xi(G_1, G_2) \) is the set of bivariate joint distribution functions with marginal distribution functions \( G_1 \) and \( G_2 \). Let \( \mathcal{Z}(G_1, G_2) = \{ Z \sim \xi : \xi \in \Xi(G_1, G_2) \} \), and \( E_Z \) be the expectation with respect to \( \xi \).
for every \( Z \sim \xi \). Then the Mallows metric can be equivalently expressed as
\[
d_2(G_1, G_2) = \sqrt{\inf_{Z \in \mathcal{Z}(G_1, G_2)} \mathbb{E}_Z \left( |Z_1 - Z_2|^2 \right)}.
\]

Moreover, let \( \rightsquigarrow \) denote weak convergence of distribution functions. We say \( G_{N,T} \rightsquigarrow G \) as \( N, T \to \infty \) if and only if for any bounded Lipschitz continuous function \( h : \mathbb{R} \to \mathbb{R} \),
\[
\lim_{N,T \to \infty} \int_{\mathbb{R}} h(z) \ dG_{N,T}(z) = \int_{\mathbb{R}} h(z) \ dG(z).
\]

One can see Lemma 2.2 (Portmanteau) of van der Vaart (1998) for equivalent characterisations of weak convergence.

**Lemma C.3:** If Assumptions 2.1–2.5 hold, then \( d_2 \left( G_{e,t}^*, G_{e,i} \right) \xrightarrow{P} 0 \) as \( T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\).

**Proof:** Let \( G_{e,i,T_0} \) and \( G_{e,i,T_0} \) be empirical distribution functions so that
\[
G_{e,i,T_0}(z) = \frac{1}{T_0} \sum_{s=1}^{T_0} \mathbb{1} \left\{ \hat{e}_{i,s} \leq z \right\}, \quad G_{e,i,T_0} = \frac{1}{T_0} \sum_{s=1}^{T_0} \mathbb{1} \left\{ e_{i,s} \leq z \right\}, \quad z \in \mathbb{R}.
\]

We firstly show that given almost every realisation of \( S \), the quantity \( z^2 \) is uniformly integrable with respect to \( \{G_{e,i,T_0}\}_{T_0=1}^{\infty} \), i.e.,
\[
\lim_{M \to \infty} \left[ \sup_{T_0 \in \mathcal{Z}_+, \leq M} \int_{\mathbb{R}} z^2 \mathbb{1}(M, \infty) (z^2) \ dG_{e,i,T_0}(z) \right] = 0.
\]
Consider a fixed \( i \in \{N_0 + 1, \ldots, N\} \). By Assumption 2.2(2) and the ergodic theorem,
\[
\mathbb{W}_{T_0,M} \overset{\text{def}}{=} \int_{\mathbb{R}} z^2 \mathbb{1}(M, \infty) (z^2) \ dG_{e,i,T_0}(z) = \frac{1}{T_0} \sum_{s=1}^{T_0} \mathbb{1}(M, \infty) (e_{i,s}) \overset{\text{a.s.}}{\to} \mathbb{E}[\mathbb{1}(M, \infty) (e_{i,s})] \overset{\text{def}}{=} \mathbb{W}_M
\]
as \( T_0 \to \infty \). And by Assumption 2.3(1), \( \mathbb{W}_M \to 0 \) as \( M \to \infty \). Therefore, for any \( \varepsilon > 0 \), there exists \( M_1 > 0 \), such that \( 0 \leq \mathbb{W}_{M_1} < \varepsilon/2 \). Given almost every realisation of \( S \), there exists \( M_2 > 0 \), such that \( \mathbb{W}_{T_0,M_1} - \mathbb{W}_{M_1} < \varepsilon/2 \) for all \( T_0 > M_2 \). Note that \( \mathbb{W}_{T_0,M} \) is decreasing in \( M \) for any fixed \( T_0 \), so \( 0 \leq \mathbb{W}_{T_0,M} < \varepsilon \) holds for all \( T_0 > M_2 \) and \( M \geq M_1 \). Pick \( M_0 = M_1 \vee \left( 1 + \max \left\{ \frac{\hat{e}_{i,1}^2, \ldots, \hat{e}_{i,M_2}^2} \right\} \right) \).

Then
\[
\sup_{T_0 \in \mathcal{Z}_+, \leq M_0} \mathbb{W}_{T_0,M_0} = \left( \max_{1 \leq T_0 \leq M_2} \mathbb{W}_{T_0,M_0} \right) \vee \left( \sup_{T_0 > M_2} \mathbb{W}_{T_0,M_0} \right) = 0 \vee \left( \sup_{T_0 > M_2} \mathbb{W}_{T_0,M_0} \right) < \varepsilon.
\]
By construction, \( G_{e,i,T}^* \overset{\text{a.s.}}{=}_{T_0} G_{e,i,T_0} \) for every \((i, t) \in \mathcal{I}_1 \). The triangle inequality yields
\[
d_2 \left( G_{e,i,T_0}^*, G_{e,i} \right) \leq d_2 \left( G_{e,i,T_0}, G_{e,i,T_0} \right) + d_2 \left( G_{e,i,T_0}, G_{e,i} \right).
\]
The ergodic theorem implies that \( G_{e,i,T_0}(z) \overset{\text{a.s.}}{\to} G_{e,i}(z) \) as \( T_0 \to \infty \) for every \( z \in \mathbb{R} \). By Lemma 2.11 of van der Vaart (1998) and the continuity of \( G_{e,i} \) [which follows from Assumption 2.2(3)],
\[
\sup_{z \in \mathbb{R}} |G_{e,i,T_0}(z) - G_{e,i}(z)| \overset{\text{a.s.}}{\to} 0
\]
as \( T_0 \to \infty \). By Lemma 2.2 of van der Vaart (1998), \( \mathbb{P} \left( G_{e,i,T_0} \rightsquigarrow G_{e,i} \text{ as } T_0 \to \infty \right) = 1 \). We have shown that given almost every realisation of \( S \), the quantity \( z^2 \) is uniformly integrable with respect to \( \{G_{e,i,T_0}\}_{T_0=1}^{\infty} \). Applying Lemma 8.3 of Bickel and Freedman (1981) yields that \( d_2 \left( G_{e,i,T_0}, G_{e,i} \right) \overset{\text{a.s.}}{\to} 0 \).

Let \( J \) be drawn from a discrete uniform distribution on \( \{1, 2, \ldots, T_0\} \). Then conditional on \( S \), we have \( \left( \hat{e}_{i,j} - \hat{e}_i \right) \sim G_{e,i,T_0} \) and \( e_{i,j} \sim G_{e,i,T_0} \) for every \( N_0 < i \leq N \). By the \( C_p \) inequality and Lemma
A.4,
\[
\left[ d_2 \left( \mathbb{G}_{\tilde{e}_{i,t}, T_0}, \mathbb{G}_{e_{i,t}, T_0} \right) \right]^2 \leq \mathbb{E}^* \left[ \left( \tilde{e}_{i,t} - \tilde{e}_i - e_{i,t} \right)^2 \right] = \frac{1}{T_0} \sum_{s=1}^{T_0} \left( \tilde{e}_{i,s} - \tilde{e}_i - e_{i,s} \right)^2
\]
\[
\leq \frac{2}{T_0} \sum_{s=1}^{T_0} (\tilde{e}_{i,s} - e_{i,s})^2 + 2 \left( \tilde{e}_i \right)^2 = \mathbb{P}_\varepsilon \left( \frac{1}{T_0} \right) + 2 \left( \tilde{e}_i \right)^2.
\]
Furthermore, by the \( C_p \) inequality, Assumptions 2.2(2), 2.3(1), the ergodic theorem, and Lemma A.4,
\[
\left( \tilde{e}_i \right)^2 = \left[ \frac{1}{T_0} \sum_{s=1}^{T_0} (\tilde{e}_{i,s} - e_{i,s}) + \frac{1}{T_0} \sum_{s=1}^{T_0} e_{i,s} \right]^2 \leq 2 \left[ \frac{1}{T_0} \sum_{s=1}^{T_0} (\tilde{e}_{i,s} - e_{i,s}) \right]^2 + 2 \left( \frac{1}{T_0} \sum_{s=1}^{T_0} e_{i,s} \right)^2
\]
\[
\leq \frac{2}{T_0} \sum_{s=1}^{T_0} (\tilde{e}_{i,s} - e_{i,s})^2 + \mathbb{P}_\varepsilon(1) = \mathbb{P}(1),
\]
and the proof is complete. □

**Lemma C.4:** If Assumptions 2.1–2.5 hold, then
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{G}_{s_{i,t}}^*(z) - \mathbb{G}_{s_{i,t}}(z) \right| \xrightarrow{P} 0
\]
as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\).

**Proof:** By the proof of Lemma 8.3 of Bickel and Freedman (1981), for every bounded Lipschitz continuous function \( h : \mathbb{R} \to \mathbb{R} \) with Lipschitz constant \( K_L \),
\[
\left| \mathbb{E}^* [h(e_{i,t}^*)] - \mathbb{E} [h(e_{i,t})] \right| = \left| \int_{\mathbb{R}} h(z) \, d\mathbb{G}_{e_{i,t}}^*(z) - \int_{\mathbb{R}} h(z) \, d\mathbb{G}_{e_{i,t}} \right| \leq K_L d_2 \left( \mathbb{G}_{e_{i,t}}^*, \mathbb{G}_{e_{i,t}} \right).
\]
By Lemma C.3 and the definition of conditional convergence in distribution, we have \( e_{i,t}^* \xrightarrow{d} e_{i,t} \) as \( T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\). By Lemmas B.3 and C.2, \( s_{i,t}^* \xrightarrow{d} - \frac{c_{i,t}}{\sigma_i} \sim \mathbb{G}_{s_{i,t}} \) as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\). Furthermore, by Theorem 2.7(ii) of van der Vaart (1998) and Lemma C.1 of this paper, \( s_{i,t} \xrightarrow{d} - \frac{c_{i,t}}{\sigma_i} \) as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\). Since \( \mathbb{G}_{s_{i,t}} \) is everywhere continuous by Assumption 2.2(3), we can use Lemma 2.11 of van der Vaart (1998) and Lemma B.4 of this paper to conclude that
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{G}_{s_{i,t}}(z) - \mathbb{G}_{s_{i,t}}(z) \right| \xrightarrow{P} 0 \quad \text{and} \quad \sup_{z \in \mathbb{R}} \left| \mathbb{G}_{s_{i,t}}^*(z) - \mathbb{G}_{s_{i,t}}(z) \right| \xrightarrow{P} 0
\]
as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\). Applying the triangle inequality yields the result. □

Now we turn to prove Theorem 3.1.

**Proof of Theorem 3.1:** Firstly consider the equal-tailed confidence intervals. By Assumption 2.2(3) and the construction of \( s_{i,t} \), the distribution function \( \mathbb{G}_{s_{i,t}} \) is everywhere continuous, which implies that \( \mathbb{G}_{s_{i,t}}(s_{i,t}) \sim \text{Unif}[0, 1] \). From Lemma C.4, for any \( \varepsilon > 0 \),
\[
\lim_{N_0, T_0 \to \infty} \mathbb{P} \left( \sup_{z \in \mathbb{R}} \left| \mathbb{G}_{s_{i,t}}^*(z) - \mathbb{G}_{s_{i,t}}(z) \right| \geq \frac{\varepsilon}{2} \right) = 0.
\]
Note that
\[
\left\{ \mathbb{G}_{s_{i,t}}(s_{i,t}) \leq \frac{\alpha}{2} - \frac{\varepsilon}{2} \right\} \cap \left\{ \sup_{z \in \mathbb{R}} \left| \mathbb{G}_{s_{i,t}}^*(z) - \mathbb{G}_{s_{i,t}}(z) \right| < \frac{\varepsilon}{2} \right\} \subseteq \left\{ \mathbb{G}_{s_{i,t}}^*(s_{i,t}) \leq \frac{\alpha}{2} \right\}
\]
\[
\subseteq \left\{ \mathbb{G}_{s_{i,t}}(s_{i,t}) \leq \frac{\alpha}{2} + \frac{\varepsilon}{2} \right\} \cup \left\{ \sup_{z \in \mathbb{R}} \left| \mathbb{G}_{s_{i,t}}^*(z) - \mathbb{G}_{s_{i,t}}(z) \right| \geq \frac{\varepsilon}{2} \right\}.
\]
Therefore,
\[
\mathbb{P} \left[ \mathbb{G}_{s_{i,t}}^*(s_{i,t}) \leq \frac{\alpha}{2} \right] \leq \mathbb{P} \left[ \mathbb{G}_{s_{i,t}}(s_{i,t}) \leq \frac{\alpha}{2} + \frac{\varepsilon}{2} \right] + \mathbb{P} \left[ \sup_{z \in \mathbb{R}} \left| \mathbb{G}_{s_{i,t}}^*(z) - \mathbb{G}_{s_{i,t}}(z) \right| \geq \frac{\varepsilon}{2} \right]
\]
\[ I = \frac{\alpha}{2} + \frac{\varepsilon}{2} + o(1), \]

and

\[ \mathbb{P} \left[ \mathcal{G}_{s_{i,t}}^*(s_{i,t}) \leq \frac{\alpha}{2} \right] \geq 1 - \mathbb{P} \left[ \mathcal{G}_{s_{i,t}}^*(s_{i,t}) > \frac{\alpha}{2} - \frac{\varepsilon}{2} \right] - \mathbb{P} \left[ \sup_{z \in \mathbb{R}} \left| \mathcal{G}_{s_{i,t}}^*(z) - \mathcal{G}_{s_{i,t}}(z) \right| \geq \frac{\varepsilon}{2} \right]. \]

Similarly, we have

\[ \frac{\alpha}{2} - \frac{\varepsilon}{2} + o(1). \]

The above facts imply

\[ 1 - \frac{\alpha}{2} - \frac{\varepsilon}{2} + o(1) \leq \mathbb{P} \left[ \mathcal{G}_{s_{i,t}}^*(s_{i,t}) \leq 1 - \frac{\alpha}{2} \right] \leq 1 - \frac{\alpha}{2} + \varepsilon + o(1). \]

Because \( \varepsilon > 0 \) can be arbitrarily small, the desired result follows from the equality

\[ \mathbb{P} (\Delta_{i,t} \in \mathrm{EQ}_{1-\alpha, i,t}) = \mathbb{P} \left( \hat{\mathcal{G}}_{i,t} - q_{1-(\alpha/2), i,t} \sqrt{\hat{\mathcal{Y}}_{i,t} + \widehat{\sigma}^2_i} \leq \mathcal{Y}_{i,t} \leq \hat{\mathcal{G}}_{i,t} - q_{\alpha/2, i,t} \sqrt{\hat{\mathcal{Y}}_{i,t} + \widehat{\sigma}^2_i} \right) \]

\[ = \mathbb{P} (s_{i,t} \leq q_{1-(\alpha/2), i,t}) - \mathbb{P} (s_{i,t} \leq q_{\alpha/2, i,t}) = \mathbb{P} \left( \mathcal{G}_{s_{i,t}}^*(s_{i,t}) \leq 1 - \frac{\alpha}{2} \right) - \mathbb{P} \left( \mathcal{G}_{s_{i,t}}^*(s_{i,t}) \leq \frac{\alpha}{2} \right). \]

The result for the symmetric confidence intervals can be proved analogously. \( \square \)

### C.2 Proof of Theorem 4.1

For \((i, t) \in \mathcal{I}_1\), define

\[ r_{i,t} = y_{i,t} - x_{i,t}^T \beta, \quad \hat{r}_{i,t} = y_{i,t} - x_{i,t}^T \hat{\beta}_{\text{tall}}, \quad s_{i,t} = \frac{\hat{r}_{i,t} - r_{i,t}}{\sqrt{\hat{\mathcal{Y}}_{i,t} + \widehat{\sigma}^2_i}}. \]

**Lemma C.5:** If Assumptions 2.1–2.8 hold, then \( s_{i,t} = -\frac{e_{i,t}}{\sigma_i} + o_p(1) \) as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\).

**Proof:** By Lemmas A.5, A.6 and the proof of Lemma C.1, we can establish that \( \hat{\mathcal{G}}_{i,t} - c_{i,t} = o_p(1) \), \( \sqrt{\hat{\mathcal{Y}}_{i,t}} = o_p(1) \), and \( \widehat{\sigma}^2_i - \sigma^2_i = o_p(1) \) as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\). By Theorem 1 of Bai (2009),

\[ r_{i,t} - \hat{r}_{i,t} = x_{i,t}^T \left( \hat{\beta}_{\text{tall}} - \beta \right) = O_p \left( \frac{1}{\sqrt{T_0}} \right) = o_p(1). \]

Therefore,

\[ s_{i,t} = \frac{(\hat{\mathcal{G}}_{i,t} - c_{i,t}) + (c_{i,t} - r_{i,t}) + (r_{i,t} - \hat{r}_{i,t})}{\sqrt{\sigma^2_i + (\sqrt{\hat{\mathcal{Y}}_{i,t} + \widehat{\sigma}^2_i - \sigma^2_i})}} = -\frac{e_{i,t}}{\sigma_i} + o_p(1) = -\frac{e_{i,t}}{\sigma_i} + o_p(1) \]

as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\), and the proof is complete. \( \square \)

**Lemma C.6:** If Assumptions 2.1–2.8 hold, then \( s_{i,t}^* = -\frac{e_{i,t}}{\sigma_i} + o_p(1) \) as \( N_0, T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\).

**Proof:** Because the bootstrap is conducted within a pure factor structure regardless of the existence of covariates, this lemma follows directly from Lemma C.2. \( \square \)

**Lemma C.7:** If Assumptions 2.1–2.8 hold, then \( d_2 \left( \mathcal{G}_{s_{i,t}}^*, \mathcal{G}_{s,t} \right) \to 0 \) as \( T_0 \to \infty \) for every \((i, t) \in \mathcal{I}_1\).
Proof: The proof is completed by Lemma A.5, A.6 and the proof of Lemma C.3.

Lemma C.8: If Assumptions 2.1–2.8 hold, then
\[
\sup_{z \in \mathbb{R}} \left| G_{s_i,t}^* (z) - G_{s_i,t}(z) \right| \xrightarrow{P} 0
\]
as $N_0, T_0 \to \infty$ for every $(i,t) \in \mathcal{I}_1$.

Proof: The same as the proof of Lemma C.4.

Proof of Theorem 4.1: The proof follows that of Theorem 3.1 and Lemma C.5–C.8.