A NOTE ON MONOGENEITY OF PURE NUMBER FIELDS

LHOUSSAIN EL FADIL

Abstract. Gassert’s paper "A NOTE ON THE MONOGENEITY OF POWER MAPS" is cited at least by 17 papers in the context of monogeneity of pure number fields despite some errors that it contains and remarks on it. In this note, we point out some of these errors, and make some improvements on it.

1. Introduction

Let $K$ be a number field generated by a complex root $\alpha$ of a monic irreducible polynomial $f(x) \in \mathbb{Z}[x]$. Let $\mathbb{Z}_K$ be its ring of integers. It is well known that the ring $\mathbb{Z}_K$ is a free $\mathbb{Z}$-module of rank $n = [K : \mathbb{Q}]$. Let $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ be the index of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$. For any rational prime $p$, if $p$ does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$, then thanks to a well-known theorem of Dedekind, the factorization of the ideal $p\mathbb{Z}_K$ can be directly derived from the factorization of $\overline{f}(x)$ over $\mathbb{F}_p$, where $\overline{f}(x)$ is the reduction of $f(x)$ modulo $p$. Besides in 1894, K. Hensel developed a powerful approach by showing that the primes of $\mathbb{Z}_K$ lying above a prime $p$ are in one-to-one correspondence with monic irreducible factors of $F(X)$ in $\mathbb{Q}_p[X]$. For every prime ideal corresponding to any irreducible factor in $\mathbb{Q}_p[X]$, the ramification index and the residue degree together are the same as those of the local field defined by the irreducible factor. If $(\mathbb{Z}_K : \mathbb{Z}[\theta]) = 1$ for some $\theta \in \mathbb{Z}_K$, then $(1, \theta, \ldots, \theta^{n-1})$ is a power integral bases of $\mathbb{Z}_K$. In such a case, the number field $K$ is said to be monogenic and not monogenic otherwise. The problem of testing the monogeneity of number fields and constructing power integral bases have been intensively studied these last four decades, mainly by Gaál, Nakahara, Pohst, and their collaborators (see for instance [2, 14, 15, 16, 26]). In [13], Funakura, studied the integral bases in pure quartic fields. In [17], Gaál and Remete, calculated the elements of index 1, for which the coefficients with absolute value $< 10^{1000}$ in the integral basis, of pure quartic field generated by $m^\frac{1}{4}$ for $1 < m < 10^7$ and $m \equiv 2, 3 \pmod{4}$. In [1], Ahmad, Nakahara, and Husnine proved that if $m \equiv 2, 3 \pmod{4}$ and $m \not\equiv \pm 1 \pmod{9}$, then the sextic number field generated by $m^\frac{1}{6}$ is monogenic. The same authors showed in [2], that if $m \equiv 1 \pmod{4}$ and $m \not\equiv \pm 1 \pmod{9}$, then the sextic number field generated by $m^\frac{1}{6}$ is not monogenic. In [6], based on prime ideal factorization, El Fadil showed that if $m \equiv 1 \pmod{4}$ or $m \not\equiv 1 \pmod{9}$, then the sextic number field generated by $m^\frac{1}{6}$ is not monogenic. In [18], by applying the explicit form of the index, Gaál and Remete obtained new
results on monogeneity of the number fields generated by \( m^{\frac{1}{n}} \) for \( 3 \leq n \leq 9 \). Recall also that in [19], Gassert studied the integral closedness of \( \mathbb{Z}[\alpha] \), where \( \alpha \) is a complex root of a monic irreducible polynomial \( f(x) = x^n - m \in \mathbb{Z}[x] \). Finally, we cite that in [7, 8, 9, 10], based on Newton polygon techniques, El Fadil studied the monogeneity of pure number fields of degree 24, 12, 36, and \( 2 \cdot 3^k \) respectively. In this paper, we state some comments regarding Gassert’s paper [19], we point out some errors in [19, Proposition 3.4], and so in the proof of [19, Proposition 3.5 and Theorem 1.2], we conclude with Theorem 4.6, which gives sufficient conditions on \( f(x) = x^n - m \) in order to have \( K \) not monogenic, where \( K \) is the number field generated by a complex root of \( f(x) \).

2. Errors in [19] and Comments

(1) In his paper “A NOTE ON THE MONOGENEITY OF POWER MAPS” [19], Gassert introduced the notion of monogenic polynomials as follows: a monic polynomial \( f(x) \in \mathbb{Z}[x] \) is said to be monogenic if it is irreducible and \( K = \mathbb{Q}(\alpha) \) is monogenic, that is the ring of integers \( \mathbb{Z}_K = \mathbb{Z}[\theta] \) for some element \( \theta \in \mathbb{Z}_K \). In [19, Theorem 1.1], he gave sufficient conditions on \( f(x) = x^n - m \in \mathbb{Z}[x] \), in order to have \( f(x) \) is monogenic. He then gave a remark in which he claimed that the hypothesis of irreducibility of \( f(x) \) is not required in [19, Theorem 1.1] because under the hypotheses: \( m \) is a square free integer and \( m^p \neq m \pmod{p^3} \) for every prime integer \( p \) dividing \( n \), \( f(x) \) is \( p \)-Eisenstein for some prime integer \( p \).

(a) This remark is not true, it suffices to consider the following example

\[
f(x) = x^2 + 1 \quad \text{for some non-negative integer } k.
\]

It is clear that \( f(x) \) is not \( p \)-Eisenstein for any prime integer \( p \). In Lemma 4.1 we show that \( f(x) = x^2 + 1 \) is irreducible over \( \mathbb{Q} \) for every non-negative integer \( k \).

(b) Even if the content of [19, Theorem 1.1] was true, it would give only a partial answer to the problem of monogeneity of \( K \). More precisely, it gives sufficient conditions on \( f(x) \) to have \( \mathbb{Z}_K = \mathbb{Z}[\alpha] \). But it does not give any information on the existence of an other integral element \( \theta \) which satisfies \( \mathbb{Z}_K = \mathbb{Z}[\theta] \). As an easy example, for \( f(x) = x^{10} - 10^3 \), by applying [19, Theorem 1.1], \( \mathbb{Z}_K \neq \mathbb{Z}[\alpha] \). For the question, using the above, could we claim that \( K \) is not monogenic? Note that if we replace \( \alpha \) by \( \theta = \frac{x^3}{10^3} \), then \( \theta \in K \). As \( \theta^{10} = 10 \), then \( \theta \) is a root of \( g(x) = x^{10} - 10 \), which is \( 2 \)-Eisenstein. Thus \( \theta \) is a primitive element of \( K \). By applying again [19, Theorem 1.1], we get \( \mathbb{Z}_K \neq \mathbb{Z}[\theta] \).

In our point of view, [19, Theorem 1.1] does not deal completely with the problem of monogeneity of \( K \), but rather, it deals with the problem of integral closedness of \( \mathbb{Z}[\alpha] \). The problem of monogeneity is more hard than that concerning integral closedness, which contributes partially in the study of monogeneity of \( K \). In section 4, we show that [19, Theorem 1.1] characterizes the integral closedness of \( \mathbb{Z}[\alpha] \).

(2) Regarding [19, Theorem 1.2], we split its content in two parts:
(a) Let $p$ be a prime integer which divides $m$. If $\gcd(n, p, \nu_p(m)) = 1$, then 
\[ \nu_p(\text{ind}(f)) = \frac{(n-1)(\nu_p(m)-1)}{2} + d - 1, \]
where $d = \gcd(n, \nu_p(m))$. This part is true. But one can wonder what is the meaning of “$\gcd(n, p, \nu_p(m)) = 1$?”

Under this condition, we have $f(x) = \phi^d$ in $\mathbb{F}_p[x]$ with $\phi = x$ and $N_{\phi}(f) = S$ has a single side of degree $d = \gcd(n, \nu_p(m))$ and $p$ does not divide $d$. In this case, the associated residual polynomial is $f_S(y) = y^d - m_p$ is a separable polynomial over $\mathbb{F}_p = \mathbb{F}_{\phi}$, where $m_p = \frac{m}{\nu_p(m)}$ (mod $p$). Thus according to the terminology of Ore \cite{25,12}, $f(x)$ is $p$-regular. It follows by Theorem 3.2 that $\nu_p(\text{ind}(f)) = \text{ind}_{\phi}(f) = \deg(\phi) \times \text{ind}(S) = \frac{(n-1)(\nu_p(m)-1)+d-1}{2}$ as desired.

(b) The second point is also true. But the proof presented in \cite{19} is not correct because it was based on Proposition 3.5, whose proof in turn was based on Proposition 3.4, which as we will see below is not correct.

(3) Proposition 3.4 is not correct unless we one add some supplementary conditions. In order to show that this result is not correct, we consider the example $f(x) = x^{14} - m$ with $m$ is a square free integer and $\nu_2(1-m) \geq 3$. Then $f(x) = (x^7 - 1)^2$ and $x^7 - 1 = (x + 1) = (x^3 + x^2 + 1)(1 + x)(x^3 + x + 1)$ in $\mathbb{F}_2[x]$. For $\phi = x^3 + x + 1$, we have $f(x) = (x^2 - 4)x^4 + (16 - 4x^2 + 10x^3 - (19 + 4x^2 - 30x^3)\phi^2 + (6 + 16x^2 + 24x^3)\phi + (1 - m - 8x^2 - 4x^3$. So, $N_{\phi}(f) has a single side joining the points $(0, 2), (1, 1),$ and $(4, 0)$. This contradicts the claim of Proposition 3.4, which says that $N_{\phi}(f) = S_1 + S_2$ has two sides joining the points $(0, 3), (1, 1),$ and $(4, 0)$. The error comes from the fact that the author assumed that: for any monic polynomial $\phi \in \mathbb{Z}[x]$, whose reduction is an irreducible factor of $x^t - m$, there exists a polynomial $h(x) \in \mathbb{Z}[x]$ such that $\phi(x)|h(x) = x^t - m$, where $n = p' t$ and $p$ does not divide $t$, which is not correct because even when $\phi$ divides $x^t - m$, it is not necessarily that $\phi$ divides $x^t - m$ in $\mathbb{Z}[x]$.

(4) The comment given by the author after \cite{19} Theorem 1.2], which says “This theorem gives a second proof of Theorem 1.1. Namely, we see that $E_p = 1$ if and only if $m$ is square-free and $m^p \equiv m$ (mod $p^2$) for every primeinteger $p$ dividing $n$” is not correct and can be corrected as follows: “$E_p = 0$ if and only if $m$ is square-free and $m^p \equiv m$ (mod $p^2$) for every primeinteger $p$ dividing $n$”.

(5) More seriously, I can not understand why the author state the comments after Theorem 1.2, on a tower of number fields, despite the fact that the field $K$ is fixed. Moreover this comment is not used anywhere.

(6) The paper finishes with an extra example, Example 3.6, in which the author considered the number field defined by an irreducible polynomial $f(x) = x^{63} - m$ and $\gcd(m, 6) = 1$. The author claimed to calculate the 2-adic and 3-adic valuations of the index, given in two tables. The results are very strange, in fact the $p$-index of a polynomial is always a non-negative integer and can never be a decimal number.

We will give in section 4 two versions improving the statements of both \cite{19} Theorem 1.1, 1.2, show the existence of an error in \cite{19} Proposition 3.5] and in the proof of
Theorem 1.2]. We finish by Theorem 4.6 in which we give sufficient conditions on \( n \) and \( m \) in order to have \( K \) not monogenic.

3. Preliminaries

In order to make clear the technical tools we use in this paper, we recall some fundamental facts on Newton polygon. Let \( K = \mathbb{Q}(\alpha) \) be a number field generated by \( \alpha \) a complex root of a monic irreducible polynomial \( f(x) \in \mathbb{Z}[x] \) of degree \( n \) and \( \mathbb{Z}_K \) its ring of integers of \( K \). It is well known that \( \mathbb{Z}_K \) is a free \( \mathbb{Z} \)-module of rank \( n \). Thus the abelian group \( \mathbb{Z}_K / \mathbb{Z}[\alpha] \) is finite. Its cardinal order is called the index of \( \mathbb{Z}[\alpha] \), and denoted \( \text{ind}(f) = (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \). For a rational prime integer \( p \), if \( p \) does not divide \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \), then a well known theorem of Dedekind says that the factorization of \( p\mathbb{Z}_K \) can be derived directly from the factorization of \( \overline{f(x)} \) in \( \mathbb{F}_p[x] \). In order to apply this theorem in an effective way, one needs a criterion to test whether \( p \) divides or not the index \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \). In 1878, Dedekind proved the well known Dedekind’s criterion (see [5, Theorem 6.1.4]). When Dedekind’s criterion fails, then an algorithm of Guardia, Montes, and Nart [20], using high order Newton polygons can be used. Such an algorithm gives after a finite number of iterations a complete answer on the index \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \), the absolute discriminant \( d_K \), and the factorization of \( p\mathbb{Z}_K \). Let \( p \) be a prime integer and \( f(x) = \prod_{i=1}^r \overline{\phi_i(x)} \) modulo \( p \) the factorization of \( \overline{f(x)} \) into powers of monic irreducible coprime polynomials of \( \mathbb{F}_p[x] \). Recall the well known Dedekind’s criterion. In 1878, Dedekind proved:

**Theorem 3.1.** (Dedekind’s criterion [5, Theorem 6.1.4] and [27])

For a number field \( K \) generated by \( \alpha \) a complex root of a monic irreducible polynomial \( f(x) \in \mathbb{Z}[x] \) and a rational prime integer \( p \), let \( \overline{f(x)} = \prod_{i=1}^r \overline{\phi_i(x)} \) (mod \( p \)) be the factorization of \( \overline{f(x)} \) in \( \mathbb{F}_p[x] \), where the polynomials \( \phi_i \in \mathbb{Z}[x] \) are monic with their reductions irreducible over \( \mathbb{F}_p \) and \( \gcd(\overline{\phi_i}, \overline{\phi_j}) = 1 \) for every \( i \neq j \). If we set \( M(x) = \frac{f(x) - \prod_{i=1}^r \phi_i^l(x)}{p} \), then \( M(x) \in \mathbb{Z}[x] \) and the following statements are equivalent:

1. \( p \) does not divide the index \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \).
2. For every \( i = 1, \ldots, r \), either \( l_i = 1 \) or \( l_i \geq 2 \) and \( \overline{\phi_i(x)}(x) \) does not divide \( M(x) \) in \( \mathbb{F}_p[x] \).

When Dedekind’s criterion fails, that is, \( p \) divides the index \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \) for every generator \( \alpha \) of \( K \), then it is not possible to obtain the prime ideal factorization of \( p\mathbb{Z}_K \) by Dedekind’s theorem. In 1923, Ore developed an alternative approach for obtaining the index \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \), the absolute discriminant, and the prime ideal factorization of the rational primes in a number field \( K \) by using Newton polygons (see [24, 25]). For more details on Newton polygon techniques, we refer to [11, 20]. For any prime integer \( p \), let \( v_p \) be the \( p \)-adic valuation of \( \mathbb{Q}, \mathbb{Q}_p \) its \( p \)-adic completion,
and \( \mathbb{Z}_p \) the ring of \( p \)-adic integers. Let \( \nu_p \) be the Gauss’s extension of \( \nu_p \) to \( \mathbb{Q}_p(x) \); 
\[
\nu_p(P) = \min(\nu_p(a_i), \ i = 0, \ldots, n) \text{ for any polynomial } P = \sum_{i=0}^{n} a_i x^i \in \mathbb{Q}_p[x] \text{ and extended by } \nu_p(P/Q) = \nu_p(P) - \nu_p(Q) \text{ for every nonzero polynomials } P \text{ and } Q \text{ of } \mathbb{Q}_p[x]. \]
Let \( \phi \in \mathbb{Z}_p[x] \) be a monic polynomial whose reduction is irreducible in \( \mathbb{F}_p[x] \), let \( \mathbb{F}_\phi \) be the field \( \frac{\mathbb{F}_p[x]}{(\phi)} \). For any monic polynomial \( f(x) \in \mathbb{Z}_p[x] \), upon the Euclidean division by successive powers of \( \phi \), we expand \( f(x) \) as follows: 
\[
f(x) = \sum_{i=0}^{l} a_i(x)\phi(x)^i, \]
called the \( \phi \)-expansion of \( f(x) \) (for every \( i \), \( \deg(a_i(x)) < \deg(\phi) \)). The \( \phi \)-Newton polygon of \( f(x) \) with respect to \( p \), is the lower boundary convex envelope of the set of points \( \{(i, \nu_p(a_i(x))), a_i(x) \neq 0 \} \) in the Euclidean plane, which we denote by \( N_\phi(f) \). The \( \phi \)-Newton polygon of \( f \), is the process of joining the obtained edges \( S_1, \ldots, S_r \) ordered by increasing slopes, which can be expressed as \( N_\phi(f) = S_1 + \cdots + S_r \). For every side \( S_i \) of \( N_\phi(f) \), the length of \( S_i \), denoted \( l(S_i) \) is the length of its projection to the \( x \)-axis and its height, denoted \( h(S_i) \) is the length of its projection to the \( y \)-axis. Let 
\[
d(S_i) = \text{GCD}(l(S_i), h(S_i)) \]
be the ramification degree of \( S_i \). The principal \( \phi \)-Newton polygon of \( f \), denoted \( N_\phi^+(f) \), is the part of the polygon \( N_\phi(f) \), which is determined by joining all sides of negative slopes. For every side \( S \) of \( N_\phi^+(f) \), with initial point \( (s, u_s) \) and length \( l \), and for every \( 0 \leq i \leq l \), we attach the following residual coefficient \( c_i \in \mathbb{F}_\phi \) as follows:
\[
c_i = \begin{cases} 
0, & \text{if } (s + i, u_{s+i}) \text{ lies strictly above } S \\
\left(\frac{a_{s+i}(x)}{p^{a_{s+i}}}\right) \pmod{(p, \phi(x))}, & \text{if } (s + i, u_{s+i}) \text{ lies on } S,
\end{cases}
\]
where \( (p, \phi(x)) \) is the maximal ideal of \( \mathbb{Z}_p[x] \) generated by \( p \) and \( \phi \). Let \( \lambda = -h/e \) be the slope of \( S \), where \( h \) and \( e \) are two positive coprime integers. Then \( d = l/e \) is the degree of \( S \). Notice that, the points with integer coordinates lying on \( S \) are exactly 
\[
(s, u_s), (s + e, u_s - h), \ldots, (s + de, u_s - dh)
\]
Thus, if \( i \) is not a multiple of \( e \), then \( (s + i, u_{s+i}) \) does not lie in \( S \), and so \( c_i = 0 \). Let 
\[
f_S(y) = t_d y^d + t_{d-1} y^{d-1} + \cdots + t_1 y + t_0 \in \mathbb{F}_\phi[y],
\]
called the residual polynomial of \( f(x) \) associated to the side \( S \), where for every \( i = 0, \ldots, d \), 
\( t_i = c_{ei} \).

Let \( N_\phi^+(f) = S_1 + \cdots + S_r \) be the principal \( \phi \)-Newton polygon of \( f \) with respect to \( p \). We say that \( f \) is a \( \phi \)-regular polynomial with respect to \( p \), if \( f_S(y) \) is square free in \( \mathbb{F}_\phi[y] \) for every \( i = 1, \ldots, r \).

The polynomial \( f \) is said to be \( p \)-regular if \( \overline{f(x)} = \prod_{i=1}^{t} \phi_i^{l_i} \) for some monic polynomials \( \phi_1, \ldots, \phi_t \) of \( \mathbb{Z}[x] \) such that \( \phi_1, \ldots, \phi_t \) are irreducible coprime polynomials over \( \mathbb{F}_p \) and \( f \) is a \( \phi_i \)-regular polynomial with respect to \( p \) for every \( i = 1, \ldots, t \).
The theorem of Ore plays a fundamental key for proving our main Theorems: Let \( \phi \in \mathbb{Z}_p[x] \) be a monic polynomial, with \( \overline{\phi(x)} \) is irreducible in \( \mathbb{F}_p[x] \). As defined in [12, Def. 1.3], the \( \phi \)-index of \( f(x) \), denoted by \( \text{ind}_\phi(f) \), is \( \deg(\phi) \) times the number of points with natural integer coordinates that lie below or on the polygon \( N^+_{\phi}(f) \), strictly above the horizontal axis, and strictly beyond the vertical axis (see Figure 1).

![Figure 1. \( N^+_{\phi}(f) \).](image)

In the example of Figure 1, \( \text{ind}_\phi(f) = 9 \times \deg(\phi) \) and if \( N^+_{\phi}(f) \) has a single side, then \( \text{ind}_\phi(f) = \deg(\phi) \times \text{ind}(S) = \deg(\phi) \times \frac{(l-1)(h-1) + d - 1}{2} \), where \( l \) is the length of \( S \), \( h \) is its height, and \( d = \gcd(l, h) \) (see [20]).

Now assume that \( \overline{f(x)} = \prod_{i=1}^{t} \overline{\phi_i} \) is the factorization of \( \overline{f(x)} \) in \( \mathbb{F}_p[x] \), where every \( \phi_i \in \mathbb{Z}[x] \) is monic polynomial, with \( \overline{\phi_i(x)} \) is irreducible in \( \mathbb{F}_p[x] \), \( \phi_i(x) \) and \( \phi_j(x) \) are coprime when \( i \neq j \) and \( i, j = 1, \ldots, t \). For every \( i = 1, \ldots, t \), let \( N^+_{\phi_i}(f) = S_{i1} + \cdots + S_{ir_i} \) be the principal \( \phi_i \)-Newton polygon of \( f \) with respect to \( p \). For every \( j = 1, \ldots, r_i \), let \( f_{s_{ij}}(y) = \prod_{k=1}^{s_{ij}} \overline{\psi_{ijk}}(y) \) be the factorization of \( f_{s_{ij}}(y) \) in \( \mathbb{F}_p[y] \). Then we have the following index theorem of Ore (see [12 Theorem 1.7 and Theorem 1.9], [11 Theorem 3.9], and [24 pp: 323–325]).

**Theorem 3.2. (Theorem of Ore)**

1. \( \nu_p(\text{ind}(f)) = \nu_p((\mathbb{Z}_K : \mathbb{Z}[\alpha])) \geq \sum_{i=1}^{t} \text{ind}_{\phi_i}(f) \).

The equality holds if \( f(x) \) is \( p \)-regular.

2. If \( f(x) \) is \( p \)-regular, then

\[
p\mathbb{Z}_K = \prod_{i=1}^{t} \prod_{j=1}^{r_i} \prod_{k=1}^{s_{ij}} \mathbb{Z}_{\phi_i}^{e_{ij}}
\]

where \( e_{ij} = l_{ij}/d_{ij} \), \( l_{ij} \) is the length of \( S_{ij} \), \( d_{ij} \) is the ramification degree of \( S_{ij} \), and \( f_{ijk} = \deg(\phi_i) \times \deg(\psi_{ijk}) \) is the residue degree of \( \psi_{ijk} \) over \( p \).
In [20], Guàrdia, Montes, and Nart introduced the notion of \( \phi \)-admissible expansion used in order to treat some special cases when the \( \phi \)-expansion is not obvious. Let

\[
(1) \quad f(x) = \sum_{i=0}^{n} A'_i(x)\phi(x)^i, \quad A'_i(x) \in \mathbb{Z}[x],
\]

be a \( \phi \)-expansion of \( f(x) \), which does not necessarily satisfy \( \deg(A'_i) < \deg(\phi) \). By analogous to the definition of Newton polygon of \( f \), for every \( i = 0, \ldots, n \), let \( u'_i = \nu_p(A'_i(x)) \), and \( N' \) be the lower boundary of the convex envelope of the set of points \( \{(i, u'_i) \mid 0 \leq i \leq n, u'_i \neq \infty \} \). To any \( i = 0, \ldots, n \), we attach the residue coefficient as follows:

\[
c'_i = \begin{cases} 0, & \text{if } (i, u'_i) \text{ lies above } N', \\ \left( \frac{A'(x)}{p^i} \right) \pmod{p^i}, & \text{if } (i, u'_i) \text{ lies on } N'. \end{cases}
\]

Likewise, for any side \( S' \) of \( N' \), we can define the residual polynomial associated to \( S \) and denoted \( f_{S'}(y) \) (similarly to the residual polynomial \( f_{S}(y) \)). We say that the \( \phi \)-expansion \( (1) \) is admissible if \( c'_i \neq 0 \) for each abscissa \( i \) of a vertex of \( N' \) and it is obvious to see that if \( \overline{\phi(x)} \) does not divide \( \left( \frac{A'(x)}{p^i} \right) \), then the \( \phi \)-expansion \( (1) \) is admissible. For more details, we refer to [20].

**Lemma 3.3.** If a \( \phi \)-expansion of \( f(x) \) is admissible, then \( N' = N'_\phi(f) \) and \( c'_i = c_i \). In particular, for any side \( S \) of \( N' \) we have \( R'_A(f)(y) = R_A(f)(y) \) up to multiply by a nonzero coefficient of \( \mathbb{F}_p \).

Again, if \( f(x) \) is not \( p \)-regular, that is theorem of Ore fails, then in order to complete the calcul of index, the factorization of \( f(x) \), and the absolute discriminant of \( K \), Guardia, Montes, and Nart introduced the notion of high order Newton polygon. They showed, thanks to a theorem of index, that after a finite number of iterations this process yields all monic irreducible factors of \( f(x) \), all prime ideals of \( \mathbb{Z}_k \) lying above a prime integer \( p \), the index \( (\mathbb{Z}_k : \mathbb{Z}[\alpha]) \), and the absolute discriminant of \( K \). We recall here some fundamental techniques of Newton polygon of high order. For more details, we refer to [20]. As introduced in [20], a type of order \( r - 1 \) is a data \( t = (g_1(x), -\lambda_1, g_2(x), -\lambda_2, \ldots, g_{r-1}(x), -\lambda_{r-1}, \psi_{r-1}(x)) \), where every \( g_i(x) \) is a monic polynomial in \( \mathbb{Z}_p[x] \), \( \lambda_i \in \mathbb{Q}^\times \), and \( \psi_{r-1}(y) \) is a polynomial over a finite field of \( p^H \) and \( H = \prod_{i=0}^{r-2} f_i \) elements, with \( f_i = \deg(\psi_i(x)) \), satisfying the following recursive properties:

1. \( g_1(x) \) is irreducible modulo \( p \), \( \psi_0(y) \in \mathbb{F}[y] \) (\( \mathbb{F}_0 = \mathbb{F}_p \)) be the polynomial obtained by reduction of \( g_1(x) \) modulo \( p \), and \( \mathbb{F}_1 := \mathbb{F}_0[y]/(\psi_0(y)) \).
2. For every \( i = 1, \ldots, r - 1 \), the Newton polygon of \( p^i \) order, \( N_i(g_{i+1}(x)) \), has a single sided of slope \( -\lambda_i \).
3. For every \( i = 1, \ldots, r - 1 \), the residual polynomial of \( p^i \) order, \( R_i(g_{i+1})(y) \) is an irreducible polynomial in \( \mathbb{F}_i[y] \), \( \psi_i(y) \in \mathbb{F}_i[y] \) be the monic polynomial
determined by $R_i(\psi_{i+1})(y) = \psi_i(y)$ (are equal up to multiplication by a nonzero element of $\mathbb{F}_r$, and $\mathbb{F}_{i+1} = \mathbb{F}_i[y]/(\psi_i(y))$). Thus, $\mathbb{F}_0 \subset \mathbb{F}_1 \subset \cdots \subset \mathbb{F}_r$ is a tower of finite fields.

(4) For every $i = 1, \ldots, r - 1$, $g_{i+1}(x)$ has minimal degree among all monic polynomials in $\mathbb{Z}_p[x]$ satisfying (2) and (3).

(5) $\psi_{r-1}(y) \in \mathbb{F}_{r-1}[y]$ is a monic irreducible polynomial, $\psi_{r-1}(y) \neq y$, and $\mathbb{F}_r = \mathbb{F}_{r-1}[Y]/(\psi_{r-1}(y))$.

Here the field $\mathbb{F}_i$ should not be confused with the finite field of $i$ elements.

As for every $i = 1, \ldots, r - 1$, the residual polynomial of the $i$th order, $R_i(g_{i+1})(y)$ is an irreducible polynomial in $\mathbb{F}_i[y]$, by theorem of the product in order $i$, the polynomial $g_i(x)$ is irreducible in $\mathbb{Z}_p[x]$. Let $\omega_0 = [\nu_p, x, 0]$ be the Gauss’s extension of $\nu_p$ to $\mathbb{Q}_p(x)$.

As for every $i = 1, \ldots, r - 1$, the residual polynomial of the $i$th order, $R_i(g_{i+1})(y)$ is an irreducible polynomial in $\mathbb{F}_i[y]$, then according to MacLane notations and definitions (23), $g_{i+1}(x)$ induces a valuation on $\mathbb{Q}_p(x)$, denoted by $\omega_{i+1} = [e, \omega_i, g_{i+1}, \lambda_i, i+1]$, where $\lambda_i = h_i/e_i$, $e_i$ and $h_i$ are positive coprime integers. The valuation $\omega_{i+1}$ is called the augmented valuation of $\nu_p$ with respect to $\phi$ and $\lambda$ is defined over $\mathbb{Q}_p[x]$ as follows:

$$\omega_{i+1}(f(x)) = \min\{e_{i+1} \omega_i(a_j^{i+1}(x)) + jh_{i+1}, j = 0, \ldots, n_{i+1}\},$$

where $f(X) = \sum_{j=0}^{n_{i+1}} a_j^{i+1}(x)g_{i+1}^j(x)$ is the $g_{i+1}(x)$-expansion of $f(x)$. According to the terminology in (20), the valuation $\omega_r$ is called the $r$th-order valuation associated to the data $t$. For every order $r \geq 1$, the $g_r$-Newton polygon of $f(x)$, with respect to the valuation $\omega_r$ is the lower boundary of the convex envelope of the set of points $\{(i, \mu_i), i = 0, \ldots, n_i\}$ in the Euclidean plane, where $\mu_i = \omega_r(a_i^r(x)g_r^i(x))$. The following are the relevant theorems from Montes-Guardia-Nart’s work (high order Newton polygon):

**Theorem 3.4.** (20, Theorem 3.1)

Let $f \in \mathbb{Z}_p[x]$ be a monic polynomial such that $\bar{f}(x)$ is a positive power of $\bar{\phi}$. If $N_r(f) = S_1 + \cdots + S_g$ has $g$ sides, then we can split $f(x) = f_1 \times \cdots \times f_g$ in $\mathbb{Z}_p[X]$, such that $N_r(f_i) = S_i$ and $R_r(f_i)(y) = R_r(f)(y)$ up to multiplication by a nonzero element of $\mathbb{F}_r$ for every $i = 1, \cdots, g$.

**Theorem 3.5.** (20, Theorem 3.7)

Let $f \in \mathbb{Z}_p[X]$ be a monic polynomial such that $N_r(f) = S$ has a single side of finite slope $-\lambda_r$. If $R_r(f)(y) = \prod_{i=1}^t \psi_i(y)^{n_i}$ is the factorization in $\mathbb{F}_r[y]$, then $f(x)$ splits as $f(x) = f_1(x) \times \cdots \times f_t(x)$ in $\mathbb{Z}_p[x]$ such that $N_r(f_i) = S$ has a single side of slope $-\lambda$, and $R_r(f_i)(y) = \psi_i(y)^{n_i}$ up to multiplication by a nonzero element of $\mathbb{F}_r$ for every $i = 1, \cdots, t$.

In (20, Definition 4, 15), the authors introduced the notion of $r$th-order index of a monic polynomial $f \in \mathbb{Z}[x]$ as follows: For a fixed data

$$t = (g_1(x), -\lambda_1, g_2(x), -\lambda_2, \ldots, g_{r-1}(x), -\lambda_{r-1}, \psi_{r-1}(x)),$$
let $N_r(f)$ be the Newton polygon of $r^{th}$-order with respect to the data $t$ and $\text{ind}_r(f) = f_0 \cdots f_{r-1} \text{ind}(N_r(f))$, where and $\text{ind}(N_r(f))$ is the index of the polygon $N_r(f)$; the number of points with natural integer coordinates that lie below or on the polygon $N^+_0(f)$, strictly above the horizontal line of equation $y = \omega_r(f)$, and strictly beyond the vertical axis. The $r^{th}$-order index of $f$ is defined by

$$\text{ind}_r(f) = \sum_{i \leq t} \text{ind}_i(f).$$

In particular, if $\overline{f(x)}$ is a power of $\overline{\phi(x)}$, then $\text{ind}_1(f) = \text{ind}_0(f)$. In [20, Theorem 4.18], they showed the following index formula which generalizes the theorem of index of Ore:

$$\text{ind}(f) \geq \text{ind}_1(f) + \cdots + \text{ind}_r(f),$$

and the equality holds if and only if $\text{ind}_{r+1}(f) = 0$. Recall that by definition $\text{ind}(N_{r+1}(f)) = 0$ if and only if $N_{r+1}(f)$ has a single side of length 1 or height 1. By [20, Lemma 2.17] (2), if $R_r(f)$ is square free, then the length of $N_r(f)$ is 1. Thus if $R_r(f)$ is square free, then $\text{ind}_{r+1}(f) = 0$, and so the equality $\text{ind}(f) \geq \text{ind}_1(f) + \cdots + \text{ind}_r(f)$ holds.

4. SOME IMPROVEMENTS AND NEW RESULTS

Throughout this section unless otherwise noted $f(x) = x^n - m \in \mathbb{Z}[x]$ is an irreducible polynomial such that $n \geq 2$ and $\nu_p(m) < n$ for every prime integer $p$. Let $K = \mathbb{Q}(\alpha)$ be the number field generated by a complex root $\alpha$ of $f(x)$.

In order to fix the error in the remark of [19, Theorem 1.1], we show the following lemma:

**Lemma 4.1.** Let $k$ be a non-negative integer and $f(x) = x^k + 1 \in \mathbb{Z}[x]$. Then $f(x)$ is irreducible over $\mathbb{Q}$.

**Proof.** Let $\phi = x - 1$. Then $f(x) = \overline{\phi(x)}^k$ in $\mathbb{F}_2[x]$ and $f(x) = \sum_{j=1}^{2^k} \binom{2^k}{j} \phi^j(x) + 2$ is the $\phi$-expansion of $f(x)$. Then $N_0(f) = S$ has a single side of height 1, with respect to the valuation $\nu_2$. Let $g(x) = f(x + 1)$. Then $g(x)$ is 2-Eisenstein. Thus $g(x)$ is irreducible over $\mathbb{Q}$, and so $f(x)$ is irreducible. \qed

Remark that as Dedekind’s criterion characterizes the integral closedness of $\mathbb{Z}[\alpha]$, the converse of [19, Theorem 1.1] holds. So, the following is an improvement on it:

**Theorem 4.2.** $\mathbb{Z}[\alpha]$ is the ring of integers of $K$ if and only if $\nu_p(m^p - m) = 1$ for every prime integer $p$ dividing $n \cdot m$.

Notice also, as it is shown by Corollary 4.3, the statement in Theorem 4.2 does not characterize the monogeneity of $K$. 

Corollary 4.3. Let \( f(x) = x^n - a^v \in \mathbb{Z}[x] \) with \( a \neq \pm 1 \) a square free integer and \( v \) a positive integer which is coprime to \( n \). If for every prime integer \( p \) dividing \( n \), \( v_p(a^{p-1} - 1) = 1 \), then \( K \) is monogenic.

Proof. Since \( \gcd(v, n) = 1 \), let \( i, j \) be two non-negative integers satisfying \( vi - nj = 1 \) with \( i < n \). Let also \( \theta = \frac{a}{\phi} \). Then \( \theta \in K \) and \( \theta^n = \frac{a^n}{\phi} = a \). Since \( a \neq \pm 1 \) is a square free integer, \( g(x) = x^n - a \) is an Eisenstein polynomial, and so is the minimal polynomial of \( \theta \) over \( \mathbb{Q} \). Thus \( \theta \) is a primitive element of \( K \), which is a root of \( g(x) = x^n - a \) with \( a \neq \pm 1 \) is a square free integer. By applying Theorem \([4,2] \), \( Z_K = \mathbb{Z}[\theta] \).

Notice that \([19, \text{Proposition 3.4, 3.5}] \) can be adjusted as follows:

Proposition 4.4. Let \( f(x) = x^n - m \in \mathbb{Z}[x] \) be an irreducible polynomial and \( p \) a prime integer which divides \( n \) and does not divide \( m \). Let \( n = p^t \in \mathbb{Z} \) with \( p \) does not divide \( t \).

Then \( \overline{f(x)} = ((x^t - m)^v)^{p^t} \). Let \( \phi = v_p(m^p - m) \) and \( \theta \in \mathbb{Z}[x] \) be a monic polynomial, whose reduction modulo \( p \) divides \( \overline{f(x)} \). Let us denote \( \theta = \phi(x)Q(x) + R(x) \).

1. If \( v_p(m^p - m) \leq r \) or \( \phi(x) \) divides \( x^t - m \) in \( \mathbb{Z}[x] \), then \( N_\theta^+(f) \) is the lower boundary of the convex envelope of the set of the points \( \{(0, v)\} \cup \{(p^j, r - j), \ j = 0, \ldots, r\} \).

2. If \( v_p(m^p - m) \geq r + 1 \), then \( N_\theta^+(f) \) is the lower boundary of the convex envelope of the set of the points \( \{(0, V)\} \cup \{(p^j, r - j), \ j = 0, \ldots, r\} \) and \( v_p(\text{ind}(f)) = \deg(\phi) \times \sum_{j=1}^{\min_{(r, (m^p - m) - 1)}} p^j \) for some integer \( V \geq r + 1 \).

3. \( v_p(\text{ind}(f)) = \deg(\phi) \times \sum_{j=1}^{\min_{(r, (m^p - m) - 1)}} p^j \).

Proof. Let \( f(x) = (x^t - m + m)^p - m = (\phi(x)Q(x) + R(x) + m)^p - m = \sum_{j=1}^{p} \left( \begin{array}{c} p^j \\ j \end{array} \right) (R(x) + m)^{p^j} - m \). Since \( x^t - m \) is separable over \( \mathbb{F}_{p^t} \), \( \phi \) does not divide \( Q(x) \). Thus if \( R = 0 \), this \( \phi \)-expansion is admissible and so \( N_\phi^+(f) \) is the lower boundary of the convex envelope of the set of the points \( \{(0, v)\} \cup \{(r, r - j), \ j = 0, \ldots, r - 1\} \). If \( R \neq 0 \), then let \( (R(x) + m)^p - m^p = \sum_{k=0}^{p-1} r^k \phi^k(x) \) be the \( \phi \)-expansion of \( (R(x) + m)^p - m^p \). Then

\[
 f(x) = \cdots + \sum_{j=1}^{p} \left( \begin{array}{c} p^j \\ j \end{array} \right) ((R(x) + m)^{p^j - 1}Q^j + r_j)\phi^j(x) + (r_0 + m^p - m) \]

is the \( \phi \)-expansion of \( f(x) \).

Since \( v_p(R) \geq 1 \), \( (R(x) + m)^p - m^p = \sum_{j=1}^{p} \left( \begin{array}{c} p^j \\ j \end{array} \right) m^{p^j - 1}R(x)^j + m^p \), and \( v_p((\sum_{j=1}^{p} \left( \begin{array}{c} p^j \\ j \end{array} \right) m^{p^j - 1}R(x)^j) \geq r - v_p(j) > r + 1 \), we get \( v_p((R(x) + m)^{p^j} - m^p) \geq r + 1 \), and so \( v_p(r_k) \geq r + 1 \) for every \( k \geq 0 \). As for every \( j = 0, \ldots, p^r \), \( v_p((\sum_{j=1}^{p^j} \left( \begin{array}{c} p^j \\ j \end{array} \right) m^{p^j - 1}R(x)^j) \leq r \), \( v_p((\sum_{j=1}^{p^j} \left( \begin{array}{c} p^j \\ j \end{array} \right) ((R(x) + m)^{p^j - 1}Q^j + r_j)) = v_p((\sum_{j=1}^{p^j} \left( \begin{array}{c} p^j \\ j \end{array} \right) ) \) for every \( j = 1, \ldots, p^r \). It follows that if \( v_p(m^p - m) \leq r < r + 1 \), then \( N_\phi^+(f) \) is the lower
boundary of the convex envelope of the set of the points \{0, v_p(m^p - m)\} \cup \{(r, r - j), j = 0, \ldots, r - 1\}. But if \(v_p(m^p - m) \geq r + 1\), then \(N_\phi^+(f)\) is the lower boundary of the convex envelope of the set of the points \{(0, V) \cup \{(r, r - j), j = 0, \ldots, r - 1\}, where \(V = v_p(m^p - m + r_0) \geq \min(v_p(m^p - m), v_p(r_0)) \geq r + 1\). Let \(y = 1, \ldots, \min(v_p(m^p - m) - 1, r)\). As the number of points with positive coordinates \((i, y)\) lying below the polygon \(N_\phi^+(f)\) is \(p^{r-y}\), then \(\text{ind}(N_\phi^+(f)) = \sum_{y=1}^{\min(v_p(m^p - m) - 1, r)} p^y\) and \(\text{ind}_\phi(f) = \deg(\phi) \times \sum_{j=1}^{\min(v_p(m^p - m) - 1, r)} p^{-j}\). Since \(N_\phi^+(f)\) is the join of sides of degree 1, except for \(p = 2\), my be the first one is of degree 2, with associated residual polynomial of \(f(x)\) is \(y^2 + y + 1\), which is irreducible over \(\mathbb{F}_2 = \mathbb{F}_\phi\), the polynomial \(f(x)\) is \(p\)-regular and thus \(v_p(\text{ind}(f)) = \text{ind}_\phi(f) = \deg(\phi) \times \sum_{j=1}^{\min(v_p(m^p - m) - 1, r)} p^{-j}\) as desired. □

An interesting question is "under which weaker conditions on \(n, m, p\) such that \(p\) divides \(m\), we can keep the equality \(v_p(\text{ind}(f)) = \frac{(n-1)(v_p(m) - 1) + \frac{d}{2}}{2}\)? The answer is given by the following proposition:

**Proposition 4.5.** If \(\gcd(p, v_p(m), n) = p\) for some prime integer \(p\), then \(v_p(\text{ind}(f)) > \frac{(n-1)(v_p(m) - 1) + \frac{d}{2}}{2}\).

**Proof.** First by Proposition 4.4 and Theorem 3.2, we have \(v_p(\text{ind}(f)) \geq \frac{(n-1)(v_p(m) - 1) + \frac{d}{2}}{2}\). Moreover if \(\gcd(p, v_p(m), n) = p\). Let \(n = p't\) and \(v_p(m) = p^s u\) with \(p\) does not \(u\). For \(\phi = x, f(x) = \phi(x)\) in \(\mathbb{F}_p[x]\), \(N_\phi(f) = S\) has a single side of slope \(-\lambda_1 = -\frac{m}{p^s - 1}\) and \(f_\phi(y) = (y - m_1)^p\), where \(\nu = \min(s, r)\), we conclude that \(f_\phi(y)\) is not square free, and so we have to use second order Newton polygon. Let \(\lambda_1 = h_1/\epsilon_1\) with \(e_1, h_1\) are two coprime positive integers. According to Nart's notations in [20], let \(\phi_2(x) = x^{e_1} - p^s m^p\) and \(V_2\) be the valuation of second order Newton polygon associated to the data \((x, \lambda, y - m, \phi_2); V_2(\sum_{i=0}^k a_i x^i) = \min(e_1 v_p(a) + i\lambda, i = 0, \ldots, k)\). Let \(f(x) = (x^{p^{s+t}} - p^s m^p + p^s m^p)^{p^s} - m = \sum_{t=1}^{p^s} \left(p^s m^p\right)^{p^{s+t}} - \phi_2(x) + p^s u(m^p - m_p)\) be the \(\phi_2\)-expansion of \(f(x)\). Since \(v_p((m^p - m_p)) \geq 1\), then \(V_2((m^p - m_p)) \geq e_1\). Thus if \(e_1 > 1\), then \(\text{ind}_2(f) \geq 1\), and so \(v_p(\text{ind}(f)) \geq \text{ind}_1(f) + \text{ind}_2(f) > \text{ind}_1(f) = \frac{(n-1)(v_p(m) - 1) + \frac{d}{2}}{2}\). It follows that:

1. If \(s < r\), then \(e_1 = p^{s-t} > p\), and so \(v_p(\text{ind}(f)) > \frac{(n-1)(v_p(m) - 1) + \frac{d}{2}}{2}\).
2. If \(s \geq r\) and \(t > 1\), then \(e_1 = t > 1\) and \(v_p(\text{ind}(f)) > \frac{(n-1)(v_p(m) - 1) + \frac{d}{2}}{2}\).
3. The case \(s \geq r\) and \(t = 1\) is excluded because \(v_p(m)\) is assumed to be less than \(n\). □
The following theorem gives a condition on \( f(x) \) in order to have \( K \) is not monogenic.

**Theorem 4.6.** Let \( n = p^r t, m = p^s u, \) and \( m_p = \frac{m}{p} = u \) with \( p \) does not divide \( tu \). If one of the following holds:

1. \( pt \) odd and \( \nu_p(1 - m) \geq p + 1 \).
2. \( pt \) odd and \( \nu_p(1 + m) \geq p + 1 \).
3. \( p = 3, t \) is even and \( \nu_3(1 + m) \geq 4 \).
4. \( p = 2, 2 \) does not divide \( m, r = 2, \) and \( \nu_2(m^{p-1} - 1) \geq 4 \).
5. \( p = 2, 2 \) does not divide \( m, r \geq 3, \) and \( \nu_2(m^{p-1} - 1) \geq 5 \).
6. \( p \) is odd, \( r = s, t > 1, \gcd(t, p - 1) = 1, r \geq p, \) and \( \nu_p(m_1^p - m_p) \geq p + 1 \).
7. \( p = 2, r = s, r \geq 2, \) and \( \nu_2(m_2 - 1) \geq 4 \).
8. \( p = 2, r = s, r \geq 3, \) and \( \nu_2(m_2 - 1) \geq 5 \).

then \( K \) is not monogenic.

In order to prove Theorem 4.6, we need the following two lemmas. The first one is an immediate consequence of Dedekind’s theorem. The second one follows from the [20 Corollary 3.8].

**Lemma 4.7.** Let \( p \) be rational prime integer and \( K \) a number field. For every positive integer \( f \), let \( P_f \) be the number of distinct prime ideals of \( \mathbb{Z}_K \) lying above \( p \) with residue degree \( f \) and \( N_f \) the number of monic irreducible polynomials of \( \mathbb{F}_p[x] \) of degree \( f \). If \( P_f > N_f \) for some positive integer \( f \), then for every generator \( \theta \in \mathbb{Z}_K \) of \( K \), \( p \) divide the index \( (\mathbb{Z}_K : \mathbb{Z}[\theta]) \).

**Lemma 4.8.** Let \( p \) be a prime integer, \( f(x) \in \mathbb{Z}_p[x] \) a monic polynomial such that \( \overline{f(x)} \) is a power of \( \overline{\phi(x)} \) for some monic polynomial \( \phi \in \mathbb{Z}_p[x] \), whose reduction is irreducible over \( \mathbb{F}_p \), \( N_\phi(f) = S \) has a single side of slope \( -\lambda_1 \), \( f_\psi(y) = \psi^a(y) \) for some monic irreducible polynomial \( \psi \in \mathbb{F}_p[y] \), and \( N_\psi(f) = T \) has a single side of slope \( -\lambda_2 \). Let \( e_1 \) be the smallest positive integer satisfying \( e_1\lambda_i \in \mathbb{Z} \). If \( R_2(f) \) is irreducible over \( \mathbb{F}_2 = \frac{\mathbb{F}_p[y]}{(\psi(y))} \), then \( f(x) \) is irreducible over \( \mathbb{Q}_p \). Let \( v \) be the unique prime ideal of \( \mathbb{Q}_p(\beta) \) lying above \( p \), where \( \beta \) is a root of \( f(x) \). Then \( e(v) = e_1 e_2 \) is the ramification index of \( v \) and \( f(v) = \deg(\phi) \times \deg(\psi) \times \deg(R_2(f)) \) is its residue degree.

**Proof.** of Theorem 4.6

1. Since \( x^t - 1 \) is separable over \( \mathbb{F}_p \), if \( m \equiv 1 \pmod{p} \), then \( \overline{f(x)} = (x^t - 1)^\nu_p = \overline{\phi(x)Q(x)}^\nu_p \in \mathbb{F}_p[x] \), where \( \phi = x - 1 \) and \( Q(x) \in \mathbb{Z}[x] \) with \( \overline{\phi} \) does not divide \( \overline{Q(x)} \) in \( \mathbb{F}_p[x] \). Since \( x - 1 \) divides \( x^t - 1 \) in \( \mathbb{Z}[x] \), by the first part of Proposition 4.4, if \( \nu_p(m - 1) \geq p + 1 \) and \( r \geq p \), then \( N_\phi^+(f) \) has at least \( p + 1 \) sides of degree 1 each one. Thus by Theorem 3.2 there are at least \( p + 1 \) prime ideals of \( \mathbb{Z}_K \) lying above \( p \) of residue degree 1 each one. By Lemma 4.7 and the fact the there are only \( p \)-monic irreducible polynomial of degree 1 in \( \mathbb{F}_p, p \) is a common index divisor of \( K \), and so \( K \) is not monogenic.
(2) For the same argument, if \( pt \) is odd and \( v_p(1 + m) \geq p + 1 \), then for \( \phi = x + 1 \),\( N_{\phi}^+(f) \) has at least \( p + 1 \) sides of degree 1 each one, and so \( p \) is a common index divisor of \( K \).

(3) For the same argument, if \( p = 3 \), \( t \) is even and \( v_3(1 + m) \geq 4 \), then for \( \phi = x^2 + 1 \), \( N_{\phi}^+(f) \) has at least 4 sides of degree 1 each one. Thus by Theorem 5.2 there at least 4 prime ideals of \( \mathbb{Z}_K \) lying above 3 with residue degree 2 each one.

The fact that there are only three monic irreducible polynomial of degree 2 in \( \mathbb{F}_3[x] \), namely, \( x^2 + 1, x^2 + x - 1, \) and \( x^2 - x - 1 \), we conclude that 3 is a common index divisor of \( K \).

(4) For \( p = 2 \) and 2 does not divide \( m \), we have \( \overline{f(x)} = (x^t - 1)^{2^r} = (\overline{\phi(x)Q(x)})^{2^r} \) in \( \mathbb{F}_2[x] \), where \( \phi = x - 1 \) and \( Q(x) \in \mathbb{Z}[x] \) with \( \overline{\phi} \) does not divide \( Q(x) \) in \( \mathbb{F}_2[x] \).

Since \( x - 1 \) divides \( x^t - 1 \) in \( \mathbb{Z}[x] \), by the first point of Proposition 4.4, we have \( N_{\phi}^+(f) \) has at least three sides of degree 1 each one. Thus by Theorem 5.2 there are at least 3 prime ideals of \( \mathbb{Z}_K \) lying above 2 of residue degree 1 each one. Therefore, by Lemma 4.7, 2 is a common index divisor of \( K \), and so \( K \) is not monogenic.

(5) For the last three points, first \( \overline{f(x)} = \phi(x)^{2^r} \) in \( \mathbb{F}_p[x] \) with \( \phi = x \). Since \( N_{\phi}^+(f) = S \) has a single side of slope \( -\lambda = -u/t \) and \( f_5(y) = (y - m_p)^{p^r} \) (because \( r = s \)), we have to use second order Newton polygon techniques. Let \( \lambda_1 = h_1/e_1 \) with \( e_1 \) and \( h_1 \) are two coprime positive integers. Then \( e_1 = t, \phi_2(x) = x - p^m m_p \).

Let \( f(x) = (x^t - p^m m_p + p^m m_p)^{p^r} - m = \sum_{i=1}^{p^r} \binom{p^r}{i} (p^m m_p)^{p^r-i} \phi_2^i(x) + p^{r+m} (m_p^2 - m_p) \) be the \( \phi_2 \)-expansion. Let also \( \omega_2 \) be the valuation of second order Newton polygon associated to \( (x, \frac{v_p(m)}{m}, \psi(y)) \), where \( \psi(y) = y - m_p \).

It follows that:

(a) If \( p \) is odd, \( r \geq p \), and \( v_p(m_p^2 - m_p) \geq p + 1 \), then \( N_2(f) = S_1 + S_2 + \cdots + S_g \), the \( \phi_2 \)-Newton polynomial \( f(x) \) with respect to \( \omega_2 \) has \( g \) sides \( S_1, S_2, \ldots, S_g \) with \( g \geq p + 1 \). Since for every \( l(S_{g-i}) = p^i - p^{i-1} = p^{i-1}(p-1) \) is the length of \( S_{g-i} \) and \( h(S_{g-i}) = t \) is its heigh and \( \gcd(t, p(p-1)) = 1 \), then the side \( S_{g-i} \) is of degree 1 for every \( i = 0, \ldots, p \). By Theorems 3.4 and 3.5 \( f(x) = g(x) \times f_1(x) \times \cdots \times f_{p+1}(x) \) with \( \deg(R_2(f_i)(y)) = 1 \) for every \( i = 1, \ldots, g \). By Hensel’s correspondence, let \( v_i \) be the prime ideal of \( \mathbb{Z}_K \) lying above \( p \) and associated to the factor \( f_i(x) \) for every \( i = 1, \ldots, p + 1 \).

Then by Lemma 4.8 \( f(v_i) = 1 \) for every \( i = 1, \ldots, p + 1 \). Since there is only \( p \) monic irreducible polynomial in \( \mathbb{F}_p[x] \), by Lemma 4.7, \( p \) is a common index divisor of \( K \), and so \( K \) is not monogenic.

(b) If \( p = 2, r = 2, \) and \( v_2(m_p - 1) \geq 4 \), then by analogous to the previous point, \( N_2(f) \) has exactly 4 sides of degree \( \gcd(t, 2) = 1 \) each one. Thus there are 4 prime ideals of \( \mathbb{Z}_K \) lying above 2 with residue degree 1 each one. Since there is only 2 monic irreducible polynomial in \( \mathbb{F}_2[x] \), we conclude 2 is a common divisor index of \( K \), and so \( K \) is not monogenic.

(c) If \( p = 2, r \geq 3, \) and \( v_2(m_p - 1) \geq 5 \), then \( N_2(f) \) has at least 4 sides of which at least 3 are of degree 1 each one. Thus there are at least 3 prime ideals
of $\mathbb{Z}_K$ lying above 2 with residue degree 1 each one. Since there is only 2 monic irreducible polynomial in $\mathbb{F}_2[x]$, we conclude that 2 is a common divisor index of $K$, and so $K$ is not monogenic.

□

5. Examples

(1) Let $n \geq 2$ be an integer and $m = (n^*)^u$, where $n^* = \prod_{p \in I} p$, $I$ is the set of positive prime integers dividing $n$, and $u$ is coprime to $n$. Then $f(x) = x^n - m$ is irreducible over $\mathbb{Q}$. Let $K$ be the pure number field defined by $f(x)$. Then $K$ is monogenic.

(2) Let $f(x) = x^{48} - 528$. Then $f(x)$ is 3-Eisenstein, and so is irreducible over $\mathbb{Q}$. Let $K$ be the pure number field defined by $f(x)$. Since $n = 2^4 \times 3$, $m = 2^4 \times 33$, $m_2 = 33$, and $v_2(m_2 - 1) = 5$, we conclude by Theorem 4.6 (8), that $K$ is not monogenic.

(3) Let $f(x) = x^{135} + 2214$. Then $f(x)$ is 2-Eisenstein, and so is irreducible over $\mathbb{Q}$. Let $K$ be the pure number field defined by $f(x)$. Since $n = 3^3 \times 5$, $m = -3^3 \times (1 + 3^4)$, $m_3 = -(1 + 3^4)$, $v_3(m_3 + 1) = 4$, and $\text{gcd}(5, 3 \cdot 2) = 1$, we conclude by Theorem 4.6 (3), that $K$ is not monogenic.

(4) Let $f(x) = x^{135} - 2214$. Then $f(x)$ is 2-Eisenstein, and so is irreducible over $\mathbb{Q}$. Let $K$ be the pure number field defined by $f(x)$. Since $n = 3^3 \times 5$, $m = 3^3 \times (1 + 3^4)$, $m_3 = (1 + 3^4)$, $v_3(m_3 - 1) = 4$, and $\text{gcd}(5, 3 \cdot 2) = 1$, we conclude by Theorem 4.6 (1), that $K$ is not monogenic.

References

[1] S. Ahmad, T. Nakahara, and S. M. Husnine, Power integral bases for certain pure sextic fields, Int. J. of Number Theory v:10, No 8 (2014) 2257–2265.
[2] S. Ahmad, T. Nakahara, and A. Hameed, On certain pure sextic fields related to a problem of Hasse, Int. J. Alg. and Comput. 26(3) (2016) 577–583.
[3] A. Hameed and T. Nakahara, Integral bases and relative monogeneity of pure octic fields, Bull. Math. Soc. Sci. Math. R épub. Soc. Roum. 58(106) No. 4(2015) 419–433.
[4] M. Bauer, Über die ausserwesentliche Diskriminantenteiler einer Gattung, Math. Ann. 64 (1907) 572–576.
[5] H. Cohen, A Course in Computational Algebraic Number Theory, GTM 138, Springer-Verlag Berlin Heidelberg (1993).
[6] L. El Fadil, On Power integral bases for certain pure sextic fields (To appear in a forthcoming issue of Bol. Soc. Paran. Math.)
[7] L. El Fadil, On Power integral bases for certain pure number fields defined by $x^{24} - m$, Stud. Sci. Math. Hung. 57(3) (2020) 397–407.
[8] L. El Fadil, On Power integral bases for certain pure number fields (To appear in a forthcoming issue of Pub. Math. Deb.)
[9] L. El Fadil, On Power integral bases for certain pure number fields defined by $x^{36} - m$ (To appear in a forthcoming issue of Stud. Sci. Math. Hung.)
[10] L. El Fadil, On Power integral bases for certain pure number fields defined by $x^{2 \cdot 3^k} - m$ (To appear in a forthcoming issue of Acta. Arith.)
[11] L. El Fadil, On Newton polygon’s techniques and factorization of polynomial over henselian valued fields, J. of Algebra and its Appl. 19(10) (2020) 2050188
[12] L. El Fadil, J. Montes and E. Nart, Newton polygons and p-integral bases of quartic number fields, J. Algebra and Appl. 11(4) (2012) 1250073
[13] T. Funakura, On integral bases of pure quartic fields, Math. J. Okayama Univ. 26 (1984) 27–41
[14] I. Gaál, Power integral bases in algebraic number fields, Ann. Univ. Sci. Budapest. Sect. Comp. 18 (1999) 61–87
[15] I. Gaál, Diophantine equations and power integral bases, Theory and algorithm, Second edition, Boston, Birkhäuser, 2019
[16] I. Gaál, P. Olajos, and M. Pohst, Power integral bases in orders of composite fields, Exp. Math. 11(1) (2002) 87–90.
[17] I. Gaál and L. Remete, Binomial Thue equations and power integral bases in pure quartic fields, JP Journal of Algebra Number Theory Appl. 32(1) (2014) 49–61
[18] I. Gaál and L. Remete, Power integral bases and monogeneity of pure fields, J. of Number Theory 173 (2017) 129–146
[19] T.A. Gassert, A note on the monogeneity of power maps, Albanian J. of Math. 11(1) (2017) 3–12
[20] J. Guardia, J. Montes, and E. Nart, Newton polygons of higher order in algebraic number theory, J. trans. of ams 364(1) (2012) 361–416
[21] K. Hensel, Theorie der algebraischen Zahlen, Teubner Verlag, Leipzig, Berlin, 1908.
[22] S. MacLane: A construction for absolute values in polynomial rings. Trans. Amer. Math. Soc. 40 (1936), 363–395
[23] J. Montes and E. Nart, On a theorem of Ore, J. Algebra 146(2) (1992) 318–334
[24] O. Ore, Newtonsche Polygone in der Theorie der algebraischen Korper, Math. Ann., 99 (1928), 84–117
[25] A. Pethö and M. Pohst, On the indices of multiquadratic number fields, Acta Arith. 153(4) (2012) 393–414
[26] R. Dedekind, Über den Zusammenhang zwischen der Theorie der Ideale und der Theorie der höheren Kongruenzen, Göttingen Abhandlungen 23 (1878) 1–23

Faculty of Sciences Dhar El MAHRAZ, P.O. Box 1874 Atlas-Fes, Sidi Mohamed ben Abdellah University, Morocco
Email address: lhoussain.elfadil@usmba.ac.ma