Abstract. We construct a weak conditional expectation from the section $C^*$-algebra of a Fell bundle over a unital inverse semigroup to its unit fibre. We use this to define the reduced $C^*$-algebra of the Fell bundle. We study when the reduced $C^*$-algebra for an inverse semigroup action on a groupoid by partial equivalences coincides with the reduced groupoid $C^*$-algebra of the transformation groupoid, giving both positive results and counterexamples.

1. Introduction

Let $S$ be a unital inverse semigroup. It may act on a space $X$ by partial homeomorphisms, that is, homeomorphisms $U \to V$ for open subsets $U, V$ in $X$. This induces an $S$-action on the $C^*$-algebra $C_0(X)$ by partial isomorphisms, that is, isomorphisms between ideals. We may describe the $S$-action on $X$ through a transformation groupoid $X \rtimes S$, which is an étale, locally compact groupoid, possibly non-Hausdorff. The full inverse semigroup crossed product $C_0(X) \rtimes S$ is canonically isomorphic to the full groupoid $C^*$-algebra $C^*(X \rtimes S)$ (see [12, Theorem 8.5] or [7, Corollary 5.6]). There is an analogous isomorphism $C_0(X) \rtimes r S \cong C^r_*(X \rtimes S)$ for reduced crossed products, which follows from [5, Theorem 4.11] or from one of our main results (see Corollary 8.13). Here we are going to consider more general versions of these reduced crossed product decompositions, allowing a groupoid instead of the space $X$.

The notion of an action of $S$ on a locally compact groupoid $G$ by partial equivalences is defined in [7]. Such an action also has a transformation groupoid $G \rtimes S$, and it induces an $S$-action on $C^*(G)$ by “partial Morita–Rieffel equivalences.” A partial Morita–Rieffel equivalence is the same as a Hilbert bimodule (not necessarily full), so we speak of actions by Hilbert bimodules from now on.

Actions of $S$ on $C^*$-algebras by Hilbert bimodules are equivalent to saturated Fell bundles $(A_t)_{t \in S}$ over $S$. Here the unit fibre $A := A_1$ is the $C^*$-algebra on which the action takes place. The other fibres $A_t$ are Hilbert bimodules over $A$ which, together with the multiplication maps $A_t \otimes_A A_u \to A_{tu}$, describe the action of $S$. The full section $C^*$-algebra $C^*((A_t)_{t \in S})$ of the Fell bundle plays the role of the full crossed product for the action and is also denoted by $A \rtimes S$.

Let $S$ act on a locally compact groupoid $G$ by partial equivalences as in [7]. The full section $C^*$-algebra of the Fell bundle over $S$ that describes the induced action on $C^*(G)$ is identified in [7] with $C^*(G \rtimes S)$, the groupoid $C^*$-algebra of the transformation groupoid. Briefly,

$$C^*(G) \rtimes S \cong C^*(G \rtimes S).$$

Is there a version of this for reduced $C^*$-algebras?

1. Reduced $C^*$-algebra $A := C^r_*(((A_t)_{t \in S})$ of a Fell bundle over an inverse semigroup is defined in [13], and should be the analogue of the reduced crossed

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product for an action of a group or groupoid. The idea is to induce representations of $A$ to representations of $A \rtimes S$ and use only these induced representations to define the $C^*$-norm for $A \rtimes S$. If $S$ is a group, the induction functor comes from a conditional expectation $E : A \rtimes S \to A$. Similarly, such a conditional expectation describes induced of representations for actions of a Hausdorff, locally compact groupoid. For a non-Hausdorff groupoid, however, the conditional expectation takes values in a larger algebra, where we adjoin certain central projections in the enveloping $W^*$-algebra $A''$. For the reduced groupoid $C^*$-algebra $C_r^* (G)$, this is worked out in [20]. The representation of $C_r^* (G) = C_0 (G^0) \rtimes G$ obtained by inducing a faithful representation of $C_0 (G^0)$ need not be faithful any more, unlike in the Hausdorff case.

Inverse semigroup actions behave in many ways like actions of étale groupoids that are possibly non-Hausdorff, so similar problems occur. In [13], induction is only defined for irreducible representations of the coefficient algebra $A$; these induced representations are used to define the reduced crossed product $A \rtimes S$, when $A = C^*_r (G)$ for a locally compact groupoid, it is much more convenient to work with the family of regular representations of $G$ on $L^2 (G^0, \lambda^2)$. Thus, to compare the reduced crossed product $C^*_r (G) \rtimes S$ with $C^*_r (G \rtimes S)$ for an inverse semigroup action on a groupoid, we want to extend the induction process in [13] to arbitrary representations of $A$.

We do this by constructing a weak conditional expectation $E : A \times S \to S$. This produces a $C^*$-correspondence from $A \rtimes S$ to $A''$. Any representation of $A$ extends uniquely to a normal representation of $A''$, which we may tensor with the $C^*$-correspondence to get a representation of $A \rtimes S$. This is the induction functor from the category of representations of $A$ to that of $A \rtimes S$. We let $A \rtimes S$ be the quotient of $A \times S$ that is defined by the $C^*$-seminorm coming from the family of all induced representations; equivalently, we may induce the universal representation of $A$, which gives a faithful representation of $A''$.

Inducing the regular representation $\Lambda_x$ of $C^*_r (G)$ on $L^2 (G^0)$ to $C^*_r (G) \rtimes S$ gives the regular representation of $G \rtimes S$ on $L^2 ((G \times S)^0)$. Hence $C^*_r (G \rtimes S)$ is the image of $C^*_r (G \times S)$ under the induced representation of $\bigoplus_{x \in G^0} \Lambda_x^2$. This always gives a representation of $C^*_r (G) \rtimes S$, so there is a quotient map $C^*_r (G) \rtimes S \to C^*_r (G \rtimes S)$.

We give examples where this representation of $C^*_r (G) \rtimes S$ is not faithful, that is, $C^*_r (G) \rtimes S \neq C^*_r (G \rtimes S)$.

We prove that a representation $\pi$ of $A$ induces a faithful representation of $A \rtimes S$ if the canonical extension of $\pi$ to the $C^*$-subalgebra of $A''$ generated by the image of the weak conditional expectation $E : A \times S \to A''$ remains faithful. As a consequence, our new definition of $A \rtimes S$ using all induced representations is equivalent to the original definition in [13]. And $C^*_r (G) \rtimes S \cong C^*_r (G \rtimes S)$ if $G$ is closed in $G \rtimes S$ or if $G$ is “inner exact”; this exactness property has been studied recently in [1].

We also prove $C^*_r (G, B) \rtimes S \cong C^*_r (G \rtimes S, B)$ for a Fell bundle $B$ over $G \rtimes S$ under similar conditions.

2. Inverse semigroup actions on $C^*$-algebras

Let $S$ be a unital inverse semigroup. That is, $S$ is a monoid and for each $t \in S$ there is a unique $t^* \in S$ with $tt^*t = t$ and $t^*tt^* = t^*$. The map $t \mapsto t^*$ is involutive and satisfies $(tu)^* = u^*t^*$. An element $e$ of $S$ is idempotent if $e^2 = e$. The following results on inverse semigroups are proved, for instance, in [23]. Idempotent elements satisfy $e = e^*$ and commute with each other. Any inverse semigroup is partially ordered by $t \leq u$ if there is an idempotent element $e \in S$ with $t = u e$ or, equivalently, if there is an idempotent element $e \in S$ with $t = e u$. This happens for some $e \in S$ if and only if $t = u t^* t$, if and only if $t = t u t^*$. If
A Hilbert $A, B$-bimodule is an $A, B$-bimodule $H$ with a left, $A$-valued inner product $\langle \xi | \eta \rangle$ and a right, $B$-valued inner product $\langle \xi | \eta \rangle$ for $\xi, \eta \in H$, so that $H$ is a right $B$-module and a left Hilbert $A$-module, and the two inner products are linked by $\xi \cdot \langle \xi | \eta \rangle = \langle \xi | \eta \rangle \cdot \eta$ for all $\xi, \xi, \eta \in H$. Hilbert bimodules are interpreted in $\mathbb{C}$ as partial Morita–Rieffel equivalences. We write $H^*$ for the Hilbert $B, A$-module associated to a Hilbert $A, B$-module $H$ by exchanging the left and right structure. If $\xi \in H$, we denote the corresponding element of $H^*$ by $\xi^* \in H^*$. Let $r(H)$ and $s(H)$ be the ideals in $A$ generated by the left and right inner products of vectors in $H$, respectively. Thus $H$ is a $r(H), s(H)$-imprimitivity bimodule.

**Definition 2.1** ([7]). An action of $S$ on a $C^*$-algebra $A$ by Hilbert bimodules consists of Hilbert $A$-bimodules $H_t$ for $t \in S$ and Hilbert bimodule isomorphisms $\mu_{t, u} : H_t \otimes_A H_u \to H_{tu}$ for $t, u \in S$, such that

(A1) for all $t, u, v \in S$, the following diagram commutes (associativity):

$$
\begin{array}{ccc}
(H_t \otimes_A H_u) \otimes_A H_v & \xrightarrow{\mu_{tu,v}} & H_{tuv} \\
\mu_{t,u} \otimes_A \text{Id}_{H_v} & & \mu_{tu,v}
\end{array}
$$

(A2) $H_1$ is the identity Hilbert $A, A$-bimodule $A$;

(A3) $\mu_{1,t} : H_t \otimes_A A \xrightarrow{\sim} H_t$ and $\mu_{t,1} : A \otimes_A H_t \xrightarrow{\sim} H_t$ for $t \in S$, are the maps defined by $\mu_{1,t}(a \otimes \xi) = a \cdot \xi$ and $\mu_{t,1}(\xi \otimes a) = \xi \cdot a$ for $a \in A, \xi \in H_t$.

If $S$ has no unit, then we define an $S$-action by the same data, subject only to condition (A1).

**Remark 2.2.** Let $S$ be an inverse semigroup, possibly without unit. Let $S^+$ be the inverse semigroup obtained by adding a new unit element $1$ to $S$. An action of $S$ satisfying only (A1) extends uniquely to $S^+$ by choosing $H_1 := A$ and letting $\mu_{1,t}$ and $\mu_{t,1}$ be the multiplication isomorphisms. This automatically satisfies (A1) if $1 \in \{t, u, v\}$, so it gives an action of $S^+$ satisfying (A1) (A3).

**Example 2.3.** An action of $S$ on a $C^*$-algebra $A$ by partial isomorphisms is given by ideals $I_t \subset A$ for idempotent $e \in S$ and $\ast$-isomorphisms $\alpha_t : I_{t^t} \xrightarrow{\sim} I_{tt}$ for $t \in S$ such that $\alpha_{tu}(a) = \alpha_t \circ \alpha_u(a)$ for all $t, u \in S$; this includes the requirement that $\alpha_{tu}(a)$ is defined if and only if $\alpha_t \circ \alpha_u(a)$ is defined, that is, $I_{tu}$ is the set of all $a \in I_{u^t}$ with $\alpha_u(a) \in I_{t^t}$. This gives an action by Hilbert bimodules as in Definition 2.1. Namely, let $H_t := I_{tt}$ with the bimodule structure $a \cdot \xi \cdot b := \alpha_t^{-1}(a) \cdot \xi \cdot b$ and the inner products $\langle \xi_1 | \xi_2 \rangle = \alpha_t(\xi_1^* \xi_2^*)$ and $\langle \xi_1 | \xi_2 \rangle = \xi_1^* \xi_2$. This is indeed a Hilbert bimodule with $r(H) = I_{tt}$, and $s(H) = I_{t^t}$. There are well defined Hilbert bimodule isomorphisms

$$\mu_{t,u} : H_t \otimes_A H_u \xrightarrow{\sim} H_{tu}, \quad \xi \otimes \eta \mapsto \alpha_u^{-1}(\xi \cdot \alpha_u(\eta)),
$$

because

$$
\langle \alpha_u^{-1}(\xi_1 \cdot \alpha_u(\eta_1)) | \alpha_u^{-1}(\xi_2 \cdot \alpha_u(\eta_2)) \rangle_{H_{uu}} = \alpha_u^{-1}(\alpha_u(\eta_1^* \xi_2 \alpha_u(\eta_2)))
= \langle \eta_1^* | \langle \xi_1 | \xi_2 \rangle_{H_t} \cdot \eta_2 \rangle_{H_{uu}},
$$

$$
\langle \alpha_u^{-1}(\xi_1 \cdot \alpha_u(\eta_1)) | \alpha_u^{-1}(\xi_2 \cdot \alpha_u(\eta_2)) \rangle_{H_{uu}} = \alpha_{tu}(\alpha_u^{-1}(\xi_1 \alpha_u(\eta_1))) \alpha_u^{-1}(\xi_2 \alpha_u(\eta_2))^* = \alpha_t(\xi_1 \alpha_u(\eta_1^* \eta_2^*) \xi_2^*) = \langle \eta_1 | \langle \eta_1 | \eta_2 \rangle_{H_t} \cdot \xi_2 \rangle_{H_{uu}}.
$$

This is an action by Hilbert bimodules. Hence these actions generalise actions by isomorphisms. Example 2.1 shows an action by Hilbert modules not of this form.

Actions of $S$ on $C^*$-algebras by Hilbert bimodules are shown in [7] to be equivalent to saturated Fell bundles over $S$ as defined in [13]. Exel’s definition of a (saturated)
Fell bundle in \cite{13} starts with a collection of Banach spaces \((H_t)_{t \in S}\) and requires the existence of multiplications (with linearly dense range) \(H_t \times H_u \to H_{tu}\) and involutions \(H_t \to H_t\). For all \(t, u \in S\) and, in addition, inclusion maps \(j_{u,t}: H_t \hookrightarrow H_u\) for all \(t \leq u\) satisfying a bunch of conditions. The multiplications and involutions can be used to view each \(H_t\) as a Hilbert bimodule over \(A = H_1\), and then this data gives an action of \(S\) on \(A\) in our sense. In the setting of the present paper the existence and properties of the inclusion maps \(j_{u,t}\) follow automatically from \cite{7} Theorem 4.8. We recall briefly how to construct the inclusion maps \(j_{u,t}\) and the involutions, now viewed as isomorphisms of Hilbert bimodules \(J_t: H_t^* \to H_t^*\), from the data in Definition 2.1.

If \(t \leq u\), there is an idempotent element \(e \in S\) with \(te = e\). Since \(e\) is idempotent, there is a unique isomorphism between the Hilbert bimodule \(eH_e\) and an ideal in \(A\) so that the multiplication map \(\mu_{e,e}: eH_e \otimes_A H_e \to H_e\) becomes the usual multiplication in \(A\) (see \cite{7} Proposition 4.6). The inclusion map \(j_{u,t}\) is the composite map
\[
H_t \xrightarrow{\mu_{u,e}} H_u \otimes_A eH_e \xhookrightarrow{} H_u \otimes_A A \cong H_u,
\]
where the last map is the multiplication map in the right \(A\)-module \(H_u\).

For each \(\xi \in H_t\), there is a unique element \(J_t(\xi) \in H_{te}\) with \(\mu_{e,e}(J_t(\xi) \otimes \eta) = \langle \xi \eta \rangle\) for all \(\eta \in H_t\); this defines the involutions \(J_t\), see \cite{7} Theorem 4.8.

We shall need a stronger result about the “intersection” of \(H_t\) and \(H_u\) for \(t, u \in S\). We have
\[
s(H_t) = s(H_{te}) = r(H_{te}) = r(H_{t}^*).
\]
If \(v \leq t\), then the inclusion map \(j_{t,v}\) is a Hilbert bimodule isomorphism
\[
r(H_v) \cdot H_t = H_t \cdot s(H_v) \cong H_v
\]
because \(H' = H \cdot s(H') = r(H') \cdot H\) holds whenever \(H'\) is a Hilbert bimodule contained in another Hilbert bimodule \(H\) by \cite{4} Proposition 4.3.

Hence we get Hilbert bimodule isomorphisms
\[
\theta_{v,t}^* \cdot H_t \cdot s(H_v) \xrightarrow{j_{t,v}} H_v \xrightarrow{J_{v,u}} H_u \cdot s(H_v)
\]
for all \(v, t, u \in S\) with \(v \leq t, u\). Let \(I_{t,u} \triangleleft A\) be the (closed) ideal generated by \(s(H_v)\) for all \(v \leq t, u\). This is contained in \(s(H_t) \cap s(H_u)\), and the inclusion may be strict.

**Lemma 2.5.** There is a unique Hilbert bimodule isomorphism
\[
\theta_{u,t} : H_t \cdot I_{t,u} \xrightarrow{\sim} H_u \cdot I_{t,u}
\]
that restricts to \(\theta_{v,t}^* \cdot H_t \cdot s(H_v)\) for all \(v \leq t, u\). These maps satisfy \(\theta_{u,t}^{-1} = \theta_{t,u}\) for all \(t, u \in S\) and \(\theta_{w,u}(\xi) = \theta_{u,w}(\xi)\) for all \(t, u \in S\) and \(\xi \in H_u \cdot (I_{t,u} \cap I_{w,u})\).

**Proof.** Linear combinations \(\sum_{v \leq t, u} a_v\) with \(a_v \in s(H_v)\) for all \(v\) and only finitely many non-zero \(a_v\) are dense in \(H_{t,u}\). We want to define
\[
\theta_{u,t}(\xi \cdot \sum_{v \leq t, u} a_v) := \sum_{v \leq t, u} \theta_{v,t}^*(\xi \cdot a_v)
\]
for \(\xi \in H_t\), \(a_v \in s(H_v)\). To check that this is well-defined, we first show that inner products are preserved. The left \(r(H_v)\)-module \(H_v \cdot s(H_v)\) is nondegenerate because \(H_t \cdot s(H_v) = H_v = r(H_v) \cdot H_v\). Hence we may write \(\xi \cdot a_v = j_{t,v}(a_v, \xi_v)\) for certain \(a_v \in r(H_v), \xi_v \in H_v\), by the Cohen–Hewitt Factorisation Theorem. For another linear combination \(\sum \eta \cdot b_w\) with \(\eta \in H_t\), \(b_w \in s(H_w)\), and finitely many \(w \leq t, u\),
we compute
\[
\left\langle \sum_{v \leq u, t} \theta^v_{u,t} (\xi \cdot a_u) \right| \sum_{w \leq t, u} \theta^w_{u,t} (\eta \cdot b_w) \right\rangle_{H_u} = \sum_{v, w \leq t, u} \langle \theta^v_{u,t} j_{u,v} (a^v_{u,t} \cdot \xi) \theta^w_{u,t} (\eta \cdot b_w) \rangle_{H_u} = \sum_{v, w \leq t, u} \langle j_{u,v} (a^v_{u,t} \cdot \xi) \theta^w_{u,t} (\eta \cdot b_w) \rangle_{H_u} = \sum_{v, w \leq t, u} \langle j_{u,v} (a^v_{u,t} \cdot \xi) | \eta \cdot b_w \rangle_{H_t} = \left\langle \sum_{u \leq t, u} \xi \cdot a_u \right| \sum_{w \leq u} \eta \cdot b_w \right\rangle_{H_t}.
\]

Since an element of a Hilbert module is determined uniquely by its inner products with other elements of the same Hilbert module, the right hand side in (2.6) does not depend on the chosen decomposition of \( \xi \cdot \sum_{v \leq t, u} a_v \in H_u \cdot I_{t,u}. \) Hence (2.6) well-defines an isometric map from a dense subspace of \( H_t \cdot I_{t,u} \) to \( H_u \cdot I_{t,u}. \) This map extends uniquely to an isometric map \( \theta_{t,u} : H_t \cdot I_{t,u} \to H_u \cdot I_{t,u}. \)

The same construction with \( t \) and \( u \) exchanged gives the map \( \theta_{t,u} : H_u \cdot I_{t,u} \to H_t \cdot I_{t,u}. \) This map is inverse to \( \theta_{u,t} \), so that \( \theta_{u,t} \) is an isomorphism. Since each \( \theta^v_{u,t} \) is \( A \)-bilinear, so is \( \theta_{u,t}. \) Let \( t, u, w \in S \) and \( \xi, \eta \in H_u \cdot (I_{t,u} \cap I_{w,u}). \) The ideal \( I_{t,u} \cap I_{w,u} = I_{t,w} \cdot I_{t,u} \) is the sum of the ideals \( s(H_x) \cdot s(H_f) = s(H_{ef}) \) for idempotent \( e, f \in S \) with \( e \leq t^* u, f \leq w^* u. \) Equivalently, it is the sum of \( s(H_x) \) with \( x \leq t, u, w \). If \( x \leq t, u, w \) and \( \xi \in H_u \cdot s(H_x), \) then
\[
\theta_{w,t}(\theta_{t,u}(\xi)) = j_{x,w} j_{w,t}^{-1} j_{x,t} \theta_{u,t}(\xi) = j_{x,w} j_{w,t}^{-1}(\xi) = \theta_{w,t}(\xi).
\]

This implies \( \theta_{w,t}(\xi) = \theta_{w,t}(\theta_{t,u}(\xi)) \) for linear combinations of such \( \xi \) and hence for all \( \xi \in H_u \cdot (I_{t,u} \cap I_{w,u}). \)

**Definition 2.7.** Let \( (H_t, \mu_{t,u}) \) be an action of \( S \) on a \( C^* \)-algebra \( A \) by Hilbert bimodules. A representation of this action by multipliers of a \( C^* \)-algebra \( D \) is a family of linear maps \( \pi_t : H_t \to \mathcal{M}(D) \) for \( t \in S \) such that
\[
\begin{align*}
(R1) & \quad \pi_{t,u}(\mu_{t,u}(\xi \otimes \eta)) = \pi_t(\xi) \pi_u(\eta) \quad \text{for all } t, u \in S, \xi \in H_t, \eta \in H_u; \\
(R2) & \quad \pi_t(\xi_1^* \pi_t(\xi_2)) = \pi_t(\xi_1 \xi_2) \quad \text{for all } t \in S, \xi_1, \xi_2 \in H_t; \\
(R3) & \quad \pi_t(\xi_1) \pi_t(\xi_2) = \pi_t(\langle \xi_1 | \xi_2 \rangle) \quad \text{for all } t \in S, \xi_1, \xi_2 \in H_t;
\end{align*}
\]
where \( \langle \xi_1 | \xi_2 \rangle \) denote the left and right inner products of \( \xi_1, \xi_2 \in H_t \).

The representation is nondegenerate if \( \pi_t(1)D \) has dense linear span in \( D \), that is, if \( \pi_t \) is a nondegenerate representation of \( A \).

The (full) crossed product \( A \rtimes S \) of the action \( (H_t, \mu_{t,u}) \) is the universal \( C^* \)-algebra for these representations, that is, there is a natural bijection between (nondegenerate) \( * \)-homomorphisms \( A \rtimes S \to \mathcal{M}(D) \) and (nondegenerate) representations of \( (H_t, \mu_{t,u}) \) in \( \mathcal{M}(D) \).

Like the full crossed product \( A \rtimes S \), the full section \( C^* \)-algebra of a Fell bundle is defined by a universal property with respect to representations of the Fell bundle. A representation of a Fell bundle is very close to a representation of the corresponding action \( (H_t, \mu_{t,u}) \). The difference is that representations of the Fell bundle must also be compatible with the maps \( j_{u,t} \) and \( J_{t^*} \), which are part of the data of a Fell bundle over an inverse semigroup. However, this extra data is essentially redundant by [7] Theorem 4.8. We are going to show that any representation of an action is compatible with the maps \( j_{u,t} \) and \( J_{t^*} \) in the appropriate sense. Hence the full section \( C^* \)-algebra of a Fell bundle is the same as the full crossed product of the corresponding action.

By [A3] condition [R1] for \( t = 1 \) and \( u = 1 \) says that the maps \( \pi_t \) are \( A \)-bilinear.

**Lemma 2.8.** Let \( \pi_t : H_t \to \mathcal{M}(D) \) for \( t \in S \) satisfy [R1] and [R2]. Then [R3] is equivalent to the following condition:
(R3′) $\pi_t(H_t)D = \pi_t(r(H_t))D = \pi_t(r(H_{tt^*}))D$ for each $t \in S$.

Proof. First assume [R3]. Recall that $r(H_t) = r(H_{tt^*})$ for all $t \in S$. Since $H_t = r(H_t) \cdot H_t$, we have $\pi_t(H_t)D = \pi_t(r(H_t))\pi_t(H_t)D \subseteq \pi_t(r(H_t))D$. The reverse inclusion follows from [R3]. $\pi_t(H_t)D \supseteq \pi_t(H_t)D$ is an equality. Conversely, assume [R3′]. If $\xi_1, \xi_2, \xi_3 \in H_t$, then

$$\pi_t(\xi_1)\pi_t(\xi_2)^*\pi_t(\xi_3) = \pi_t(\langle\xi_1|\xi_2\rangle)\pi_t(\xi_3).$$

Hence the operators in [R3] are equal on $\pi_t(H_t)D$. Then they are also equal on $\pi_1(\langle\xi_1|\xi_2\rangle) \cdot D$ by [R3′]. Now [R3] follows because

$$\left(\pi_t(\xi_1)\pi_t(\xi_2)^* - \pi_1(\langle\xi_1|\xi_2\rangle)\right) \cdot \left(\pi_t(\xi_1)\pi_t(\xi_2)^* - \pi_1(\langle\xi_1|\xi_2\rangle)\right)^* = 0.$$ 

\[ \square \]

Proposition 2.9. Any representation $(\pi_t)_{t \in S}$ of $(H_t, \mu_{u,t})$ is compatible with the maps $j_{u,t} : H_t \to H_u$ and $J_t : H_{tt^*} \to H_{tt^*}$ in the sense that $\pi_u(j_{u,t}(\xi)) = \pi_t(\xi)$ and $\pi_t(J_t(\xi^*)) = \pi_t(\xi)^*$ for all $t, u \in S$ with $t \leq u$ and $\xi \in H_t$.

Even more, $\pi_u \circ \theta_{u,t}(\xi) = \pi_t(\xi)$ for all $t, u \in S$, $\xi \in H_t \cdot J_{t,u}$.

The full crossed product $A \times S$ is the same as the full section $C^*$-algebra of the Fell bundle associated to the action.

Proof. The full section $C^*$-algebra of a Fell bundle and the full crossed product $A \times S$ are both defined through a universal property for certain representations. To prove their equality, we show that any representation of $(H_t, \mu_{u,t})$ is compatible with the maps $j_{u,t}$ and $J_{t,t}$.

We may assume that $H_e$ for an idempotent element $e \in S$ is an ideal of $A$ with the standard Hilbert bimodule structure and that $\mu_{t,e}$ and $\mu_{e,t}$ are the maps $\xi \otimes a \mapsto \xi \cdot a$ and $a \otimes \xi \mapsto a \cdot \xi$ for all $t \in S$. This normalisation follows from [7] Proposition 4.6 as in the proof of [7] Proposition 3.7. The results in [7] for actions of inverse semigroups on groupoids carry over to actions on $C^*$-algebras because both setups share some basic properties, which suffice for the proofs to go through, compare the proof of [7] Theorem 4.8, which merely refers to the earlier proofs of [7] Propositions 3.7 and 3.9.

Let $t \leq u$ and $e = e^* t = t^* e \in E(S)$, where $E(S) := \{ e \in S \mid e^* e = e \}$. The embedding $j_{u,t} : H_t \to H_u$ is defined by $j_{u,t}(\mu_{u,e}(\xi \otimes a)) = \xi \cdot a$ for $\xi \in H_u$ and $a \in H_e$; this is well-defined because the multiplication map $\mu_{u,e} : H_u \otimes A H_e \to H_t$ is an isomorphism. The conditions for a representation imply

$$\pi_u(j_{u,t}(\mu_{u,e}(\xi \otimes a))) = \pi_u(\xi) \cdot \pi(a), \quad \pi_t(\mu_{u,e}(\xi \otimes a)) = \pi_t(\xi) \cdot \pi_e(a).$$

So we are done if we show that $\pi_e = \pi_t|H_{tt^*}$ for idempotent $e$. For this we take $a, b \in H_e$ and use (R1) to compute:

$$\pi_1(a)\pi_e(b) = \pi_e(ab) = \pi_e(a)\pi_e(b).$$

Hence $\pi_1(a)\xi = \pi_e(a)\xi$ for all $\xi \in \pi_1(H_e)D = \pi_1(H_{tt^*})D$ by Lemma 2.8. Since the image of $\pi_1(a^*) - \pi_e(a^*)$ is contained in $\pi_1(H_e)D = \pi_1(H_{tt^*})D$, this implies $(\pi_1(a) - \pi_e(a))(\pi_1(a)^* - \pi_e(a)^*) = 0$, hence $\pi_1(a) = \pi_e(a)$ for all $a \in H_e$.

The compatibility of $\pi$ with the maps $j_{u,t}$ implies that $\pi_u(\theta_{u,t}^e(\eta)) = \pi_t(\eta)$ for all $v, t, u \in S$ with $v \leq t, u$ and $\eta \in H_t \cdot s(H_e)$. Since this holds for all $v$, we get $\pi_u \circ \theta_{u,t}(\xi) = \pi_t(\xi)$ by the construction of $\theta_{u,t}$.

By definition, $J_t(\xi^*) \in H_{tt^*}$ for $\xi \in H_t$ is the unique element with $\mu_{t,t^*}(J_t(\xi^*) \otimes \eta) = \langle \xi|\eta \rangle$

for all $\eta \in H_t$. Hence

$$\pi_t(\xi)^* \pi_t(\eta) = \pi_t(\langle \xi|\eta \rangle), \quad \pi_t(\mu_{t,t^*}(J_t(\xi^*) \otimes \eta)) = \pi_{t^*}(J_t(\xi^*))\pi_t(\eta).$$

for all $\eta \in H_t$. Hence
Hence $\pi_t(\xi)^* x = \pi_t^*(J_t(\xi^*)) x$ for all $x \in \pi_t(H_{t'})D = \pi_t(H_{t'})D$ by Lemma 2.8.

Since $H_{t'} = H_{t'} \cdot H_{t''}$, we have $\pi_{t'}(H_{t''})^* D = \pi_{t'}((H_{t''})^*)^* \pi_{t'}(H_{t''})^* D \subseteq \pi_{t'}(H_{t''})^* D$.

Hence the image of $\pi_t(\xi) - \pi_{t'}(J_t(\xi^*))$ is contained in $\pi_t(H_{t''})^*D$. Thus $(\pi_t(\xi^* - \pi_{t'}(J_t(\xi^*)^*)) (\pi_t(\xi - \pi_{t'}(J_t(\xi^*))) = 0$, so that $\pi_t(\xi^*) = \pi_{t'}(J_t(\xi^*))$.

We now describe $A \rtimes S$ as the $C^*$-completion of a certain dense *-subalgebra. Let $A \rtimes_{\text{alg}} S$ be the quotient of $\bigoplus_{t \in S} H_t$ by the linear span of $\theta_{u,t}(\xi) \delta_u - \xi \delta_t$ for all $t, u \in S$ and $\xi \in H_t \cdot I_{t,u}$. The multiplication maps $\mu_{t,u}$ and the involutions $J_t$ described above turn this into a *-algebra. Our definition of $A \rtimes_{\text{alg}} S$ is slightly different from the one in [13], where only the linear span of $j_{u,t}(\xi) \delta_u - \xi \delta_t$ for $t, u \in S$ and $\xi \in H_t \cdot H_e$ for all $e \in E(S)$ with $te = we$. For fixed $t, u$, the closure of this subspace in $H_u \oplus H_t$ contains $\theta_{u,t}(\xi) \delta_u - \xi \delta_t$ even if only $\xi \in H_t \cdot I_{t,u}$. Therefore, a linear map or a semi-norm on $\bigoplus_{t \in S} H_t$ that is norm-bounded on each summand and vanishes on $j_{u,t}(\xi) \delta_u - \xi \delta_t$ for $t, u \in S$ with $t \leq u$ and $\xi \in H_t$ still vanishes on $\theta_{u,t}(\xi) \delta_u - \xi \delta_t$ for $t, u \in S$ and $\xi \in H_t \cdot I_{t,u}$.

We will show below that $A \rtimes_{\text{alg}} S$ as defined above embeds into the $C^*$-algebra $A \rtimes S$, which justifies our small change in the definition.

A representation of the action $(H_t)_{t \in S}$ is equivalent to a representation of the *-algebra $A \rtimes_{\text{alg}} S$ by Proposition 2.9. Hence $A \rtimes S$ is the $C^*$-completion of $A \rtimes_{\text{alg}} S$.

Why does the maximal $C^*$-semimorphism of $A \rtimes_{\text{alg}} S$ exist? If $\xi \in H_t$, then $\|\xi\| \leq \|\xi^*\|^{1/2}$ for any $C^*$-semimorphism on $A \rtimes_{\text{alg}} S$. Since $\xi^* \xi \in A$, which is already a $C^*$-algebra, the spectral radius of $\xi^* \xi$ gives a finite upper bound on $\|\xi\|^2$ for any $\xi \in H_t$. This implies a finite upper bound for $\|\xi\|$ for any $\xi \in A \rtimes_{\text{alg}} S$.

In order to define a reduced analogue of $A \rtimes S$, we need a way to induce representations of $A$ to representations of $A \rtimes_{\text{alg}} S$. Then $A \rtimes r S$ is defined as the completion in the $C^*$-semimorphism on $A \rtimes_{\text{alg}} S$ defined by these “regular” representations.

Exel describes an induction process for pure states in [13]. We want, instead, an induction process for all states or, equivalently, for all representations of $A$. By [30, Theorem 6.9], such an induction functor is equivalent to a self-dual (right) Hilbert module over the bidual $A''$ of $A$ with a normal left action of $(A \rtimes S)''$; the normal left action of $(A \rtimes S)''$ is equivalent to a nondegenerate representation of $A \rtimes S$. We shall construct this $C^*$-correspondence from a weak expectation, that is, a normal expectation $E: (A \rtimes S)'' \to A''$; the resulting Hilbert module is the completion of $(A \rtimes S)'$ for the inner product $\langle x_1 | x_2 \rangle := E(x_1^* \cdot x_2)$. To construct this weak expectation, we will extend the action $(H_t, \mu_{t,u})$ on $A$ to an action on the enveloping $W^*$-algebra $A''$.

3. Actions on $W^*$-algebras

A Hilbert module $H$ over a $C^*$-algebra $A$ is self-dual if every bounded $A$-module map $H \to A$ is of the form $\xi \mapsto \langle \eta | \xi \rangle$ for some $\eta \in H$ (see [24]). Any bounded $A$-module map between self-dual Hilbert modules is adjointable. For Hilbert modules over $W^*$-algebras, self-duality is equivalent to compatibility of the action with the weak topologies, see [33].

**Definition 3.1.** Let $S$ be a unital inverse semigroup. An action of $S$ on a $W^*$-module $M$ consists of self-dual Hilbert $M$-bimodules $(H_t)_{t \in S}$ and Hilbert $M$-bimodule isomorphisms

$$\mu_{t,u}: H_t \otimes M H_u \tilde{\to} H_{tu}$$

for $t, u \in S$ satisfying analogues of (A1)–(A3), where $\tilde{\to}$ denotes the weak closure of the tensor product of the two bimodules.
Example 3.2. We shall be mostly interested in the following situation. Let \((\mathcal{H}_t)_{t \in S}\) be an action of \(S\) by Hilbert bimodules on a C*-algebra \(A\), as in the previous section. The bidual \(\mathcal{H}''_t\) of \(\mathcal{H}_t\) is a self-dual Hilbert \(A''\)-bimodule (see \([31]\)). The isomorphism \(\mu_{t,u}\) induces a Hilbert \(A''\)-bimodule isomorphism

\[
\mu_{t,u}^\circ : \mathcal{H}''_t \otimes_{A''} \mathcal{H}''_u \to \mathcal{H}''_{t,u}.
\]

A quick way to see this is to form a linking C*-algebra and its W*-hull:

\[
L_{t,u} := \left( \begin{array}{cc} A & \mathcal{H}''_t \otimes_{A''} \mathcal{H}''_u \\ \mathcal{H}_t & A \end{array} \right), \quad L''_{t,u} = \left( \begin{array}{cc} A'' & \mathcal{H}''_t'' \otimes_{A''''} \mathcal{H}''_u'' \\ \mathcal{H}''_t'' & A'' \\ \mathcal{H}''_u'' & A'' \end{array} \right).
\]

The multiplication in \(L_{t,u}\) restricts to \(\mu_{t,u}\) on the summands \(\mathcal{H}_t, \mathcal{H}_u\), so the multiplication in \(L''_{t,u}\) restricts to \(\mu_{t,u}^\circ\) on \(\mathcal{H}''_t, \mathcal{H}''_u\). The data \((\mathcal{H}''_t, \mu_{t,u}^\circ)\) defined above gives an action of \(S\) on the W*-algebra \(A''\).

We return to the general case. Let \(M\) be a W*-algebra and let \((\mathcal{H}_t, \mu_{t,u})\) be an \(S\)-action on \(M\) by self-dual Hilbert bimodules.

For an idempotent \(e \in E(S)\), \(\mathcal{H}_e \subseteq M\) is a weakly closed, two-sided ideal. So it is of the form \(P_e \cdot M = M \cdot P_e\) for a central projection \(P_e \in M\). For \(t, u \in S\), let \(I_{t,u}\) be the weak closure of \(\sum_{v \leq t,u} s(\mathcal{H}_v)\). Equivalently, \(I_{t,u}\) is the ideal generated by the supremum of the central projections \(P_{e,v}\) for \(v \leq t, u\). In the situation of Example 3.2, \(I_{t,u} \subseteq A''\) is the bidual of the ideal \(I_{t,u}\) in \(A\).

Lemma 3.3. For \(t, u \in S\), there is a unique \(M\)-bimodule map and partial isometry \(\Theta_{u,t} : \mathcal{H}_t \to \mathcal{H}_u\) that extends the maps

\[
\theta_v^u : \mathcal{H}_t \otimes_{\mathcal{H}_u} \theta_v^u(\mathcal{H}_v) \to \mathcal{H}_v \quad \text{for all } v \leq t, u
\]

for all \(v \leq t, u\) and satisfies \(\Theta_{u,t}(\mathcal{H}_t) = \mathcal{H}_t \cdot I_{t,u} = \mathcal{H}_u \cdot I_{t,u}\) and \(\Theta_{u,t}(\mathcal{H}_u) = \mathcal{H}_u \cdot I_{t,u}\). Furthermore, \(\Theta_{u,t} = \Theta_{t,u} \circ j_{t,v} = \Theta_{u,v} \circ j_{t,v}\) for all \(u, v \in S\) with \(v \leq t, v \neq u\), and \(\Theta_{u,t} \circ \Theta_{v,u}(\xi) = \Theta_{u,v}(\xi)\) for all \(t, u, v \in S\) and \(\xi \in \mathcal{H}_v \cdot I_{t,u}\).

Proof. If \(v \leq t, u\), then \(\mathcal{H}_t \otimes_{\mathcal{H}_v} \mathcal{H}_u \subseteq \mathcal{H}_t \cdot s(\mathcal{H}_v) = \mathcal{H}_t \cdot P_{v,v} = P_{v,v} \cdot \mathcal{H}_t\). The projections \(P_{v,v}\) for \(v \leq t, u\) are central and hence commute with each other. Therefore, if \(v_1, \ldots, v_n \leq t, u\), then there is a unique map that extends \(\theta_{v_i}^u\) for \(i = 1, \ldots, n\) and has image \(P_{v_1,v_n} \cdot \mathcal{H}_t\). One way to write it is

\[
\theta_{v_1,\ldots,v_n}^u := \theta_{v_1}^u(P_{v_1,v_1}) + \theta_{v_2}^u(P_{v_2,v_2}(1 - P_{v_1,v_1})) \xi + \cdots + \theta_{v_n}^u(P_{v_n,v_n}(1 - P_{v_1,v_1})(1 - P_{v_2,v_2}) \cdots (1 - P_{v_n-1,v_n-1}) \xi) + \cdots
\]

We have defined partial isometries \(\theta_{F,t}^u\) for each finite set \(F \subseteq S\) with \(v \leq t, u\) for all \(v \in F\). If \(F \not\subseteq F'\), then the partial isometry \(\theta_{F,t}^u\) agrees with \(\theta_{F',t}^u\) on the image of its source projection, and merely extends it to a larger submodule. Such a net of operators has a weak limit, and this limit has the properties required of \(\Theta_{u,t}\) because \(\bigvee_{v \leq t,u} P_{v,v}\) generates \(I_{t,u}\) and is the weak limit of the range projections \(\bigvee_{v \in F} P_{v,v}\) for \(F\) as above. Any bounded operator \(\mathcal{H}_t \to \mathcal{H}_u\) is weakly continuous. If its source projection is \(\bigvee_{v \leq t,u} P_{v,v}\), it is determined uniquely by its restriction to the images of \(\bigvee_{v \in F} P_{v,v}\) for all finite sets \(F\) as above. Hence there is only one operator with the properties required of \(\Theta_{u,t}\).

The operator \(\Theta_{u,t}\) satisfies the conditions that characterise \(\Theta_{u,t}\), so \(\Theta_{u,t} = \Theta_{u,t}\). If \(v \leq t\), the operator \(j_{t,v}\) is an isometry whose range projection \(P_{v,v}\) commutes with the source projection of the partial isometry \(\Theta_{u,t}\). Hence \(\Theta_{u,t} \circ j_{t,v}\) is again a partial isometry. This satisfies the conditions that characterise \(\Theta_{u,t}\) because
the set of $w \leq u, v$ is exactly the set of all $w' = wv^*v = vv^*w$ for $w' \leq t, u$. So $\Theta_{u,t} \circ j_{t,u} = \Theta_{u,u}$ if $t, u, v \in S$ and $v \leq t$. If $\xi \in j_{t,u}(\mathcal{H}_v) = \mathcal{H}_t \cdot \mathcal{S}(\mathcal{H}_w)$ for some $w \in S$ with $w \leq t, v$, then $\xi = j_{t,w}(\xi')$ for $\xi' \in \mathcal{H}_w$, and so

$$
\Theta_{u,t} \Theta_{t,v}(\xi') = \Theta_{u,t} j_{t,v}(\mathcal{H}_t \cdot \mathcal{S}(\mathcal{H}_w)) = \Theta_{u,v}(\xi') = \Theta_{u,t} j_{t,w}(\xi') = \Theta_{u,t}(\xi')
$$

because $\Theta_{u,t} \circ j_{t,u} = \Theta_{u,u}$. The set of $\xi \in \mathcal{H}_t$ with $\Theta_{u,t} \circ j_{t,u}(\xi) = \Theta_{u,t}(\xi)$ is a weakly closed subspace. Since $\mathcal{H}_t \cdot \mathcal{I}_v$ is the weakly closed linear span of $j_{t,u}(\mathcal{H}_w)$ for $w \leq t, v$, the equation $\Theta_{u,t} \circ j_{t,u}(\xi) = \Theta_{u,t}(\xi)$ holds for all $\xi \in \mathcal{H}_t \cdot \mathcal{I}_v$.

\begin{remark}
In general, $\Theta_{u,t} \circ \Theta_{u,v} \neq \Theta_{u,v}$. For instance, $\Theta_{u,u} \circ \Theta_{u,u}$ is the identity map on $\mathcal{H}_u \cdot \mathcal{I}_u$ and zero on its orthogonal complement in $\mathcal{H}_u$, whereas $\Theta_{u,u}$ is the identity on $\mathcal{H}_u$.
\end{remark}

We now define $M \staralg S$ as the quotient of $\bigoplus_{t \in S} \mathcal{H}_t$ by the linear span of $\Theta_{u,t}(\xi)\delta_t - \xi\delta_t$ for all $t, u \in S$ and $\xi \in \mathcal{H}_t \cdot \mathcal{I}_v$. By construction of $\mathcal{I}_u$ and $\Theta_{t,u}$, the linear span of the elements of $\mathcal{H}_t \oplus \mathcal{H}_u$ of the form $j_{u,v}(\xi)\delta_u - j_{v,u}(\xi)\delta_v$ for $\xi \in \mathcal{H}_v$ and $v \leq t, u$ is weakly dense in the space of all $\Theta_{u,t}(\xi)\delta_u - \xi\delta_u$ with $\xi \in \mathcal{H}_t \cdot \mathcal{I}_v$. Therefore, a linear map or a seminorm on $\bigoplus_{t \in S} \mathcal{H}_t$ descends to $M \staralg S$ if it is weakly continuous on each summand and vanishes on $j_{u,v}(\xi)\delta_u - \xi\delta_u$ for $u, v \in S$ with $u \leq v$ and $\xi \in \mathcal{H}_v$.

The map

$$
E: \bigoplus_{t \in S} \mathcal{H}_t \to M, \quad \sum_{t \in S} \xi_t \delta_t \mapsto \sum_{t \in S} \Theta_{1,t}(\xi),
$$

is normal on each summand and vanishes on $j_{u,v}(\eta)\delta_t - \eta\delta_u$ for $t \leq v, \eta \in \mathcal{H}_v$ because Lemma 3.3 gives $\Theta_{1,t} \circ j_{t,u} = \Theta_{1,u}$ if $v \leq t$. Hence [3.5] defines a map $M \staralg S \to M$, which we also denote by $E$. This is an $M$-bimodule map with $E|_M = \Id_M$.

\begin{proposition}
The map $E: M \staralg S \to M$ is a faithful conditional expectation. That is, (1) $E(\mathcal{E}^*) = E(\mathcal{E}^*)$, (2) $E(\mathcal{E}^* \mathcal{E}) \geq 0$ in $M$ for all $\xi \in M \staralg S$, and (3) $E(\mathcal{E}^* \mathcal{E}) = 0$ only if $\mathcal{E} = 0$. Hence $(\xi, \mathcal{E}) \mapsto (\mathcal{E}|_{\mathcal{E}}) := E(\mathcal{E}\mathcal{E})$ defines an $M$-valued inner product on $M \staralg S$.
\end{proposition}

\begin{proof}
Condition (1) holds in general once it holds for $\xi = \xi\delta_t$ with $\xi \in \mathcal{H}_t$ and $t \in S$. In this case, (1) means that $\Theta_{1,t}(\mathcal{E}) = \Theta_{1,t}(\mathcal{E}^*)$. The map $\mathcal{H}_t \ni \xi \mapsto \Theta_{1,t}(\mathcal{E}) \in M$ is also $M$-bilinear and a partial isometry, and it has the properties in Lemma 3.3 that characterise $\Theta_{1,t}$. Hence $\mathcal{H}_t \ni \xi \mapsto \Theta_{1,t}(\mathcal{E})^* = \Theta_{1,t}(\mathcal{E})$ for all $\xi \in \mathcal{H}_t$, as desired.

To prove (2) and (3), fix $\xi \in M \staralg S$. Write $\xi = \sum_{i=1}^n \xi_i \delta_t$ for $t_i \in S, \xi_i \in \mathcal{H}_{t_i}, i = 1, \ldots, n$. First we choose a normal form of this representative of $\xi$. (This normal form becomes unique if we fix some total order $\prec$ on the set $S$ and assume that $t_1 \prec t_2 \prec \cdots \prec t_n$.) First, we split $\xi_i = \Theta_{t_i,t_i}^* \Theta_{t_i,t_i}(\xi_i) + (1 - \Theta_{t_i,t_i}^* \Theta_{t_i,t_i})(\xi_i)$ for $i \geq 2$. Thus

$$
\xi_i \delta_t = \Theta_{t_i,t_i}^* \Theta_{t_i,t_i}(\xi_i) \delta_t + (1 - \Theta_{t_i,t_i}^* \Theta_{t_i,t_i})(\xi_i) \delta_t
$$

in $M \staralg S$ for $i \geq 2$. Replacing $\xi_i \delta_t$ by the right hand side gives a new representative $\xi = \sum_{i=1}^n \xi_{t_i} \delta_{t_i}$ with the extra property $\Theta_{t_i,t_i}(\xi_i) = 0$ for $i = 2, \ldots, n$. Next, we split $\xi_i = \Theta_{t_i,t_i}^* \Theta_{t_i,t_i}(\xi_i) + (1 - \Theta_{t_i,t_i}^* \Theta_{t_i,t_i})(\xi_i)$ for $i \geq 3$ and repeat the normalisation step above. This gives a new representative $\xi = \sum_{i=1}^n \xi''_{t_i} \delta_{t_i}$ with the extra property $\Theta_{t_i,t_i}(\xi''_{t_i}) = 0$ for $i = 3, \ldots, n$; we still have $\Theta_{t_i,t_i}(\xi_i) = 0$ for $i = 2, \ldots, n$.

Continuing this way, we eventually arrive at a new representative $\xi = \sum_{i=1}^n \xi_{t_i} \delta_{t_i}$ with $\Theta_{t_i,t_i}(\xi_i) = 0$ for all $1 \leq i < j \leq n$.

By definition, $E((\xi(t') \delta_t) \cdot \mathcal{E}(\xi) \delta_u) = \Theta_{1,t,u}(\mathcal{E} \cdot \mathcal{E}(\xi))$ for all $t, u \in S, \xi \in \mathcal{H}_t, \mathcal{E} \in \mathcal{H}_u$. We claim that this is equal to $\xi \cdot \mathcal{E}(\xi \cdot \mathcal{E}(\xi))$. If $v \leq 1, t \leq u$, then $tv \leq 1, t \leq u$; conversely, if $v \leq t, u$, then $tv \leq 1, t \leq u$. The maps $e \mapsto te$ and $v \mapsto tv$ are bijective between
Remark 4.2. The module

Definition 4.1. A image of \( A \) by self-dual Hilbert bimodules. The representation of \( A \) a C∗-map of \( S \) these operators form a W∗-algebra

Our normalisation condition \( \Theta_{t,j}(\xi_j) = 0 \) for all \( 1 \leq i < j \leq n \) gives

if \( i < j \). Since \( E(\eta^*) = E(\eta)^* \), this also vanishes for \( i > j \), so

Thus \( E(\xi^*\xi) \geq 0 \) in \( M \), and \( E(\xi^*\xi) = 0 \) in \( M \) only if \( \langle \xi_i|\xi_i \rangle_{\mathcal{H}_t} = 0 \) for all \( i \), that is, only if \( \xi = 0 \).

Proposition 3.6 allows us to complete \( M \rtimes_{\text{alg}} S \) to a Hilbert module \( \ell^2(S,M) \) over \( M \). The left multiplication action of \( M \rtimes_{\text{alg}} S \) on itself extends to a unital left action \( \lambda \) of \( M \rtimes_{\text{alg}} S \) on \( \ell^2(S,M) \) by adjointable operators. This representation is faithful because \( E \) is faithful on \( M \rtimes_{\text{alg}} S \): if \( \lambda(\xi) = 0 \) for \( \xi \in M \rtimes_{\text{alg}} S \), then \( \lambda(\xi)(1) = \xi = 0 \) in \( \ell^2(S,M) \); this gives \( \langle \xi|\xi \rangle = E(\xi^*\xi) = 0 \) and hence \( \xi = 0 \). The module \( \ell^2(S,M) \) need not be self-dual again, so we replace it by its self-dual completion \( \overline{\ell^2}(S,M) \). We still represent \( M \rtimes_{\text{alg}} S \) on \( \overline{\ell^2}(S,M) \) by left multiplication. Since \( \overline{\ell^2}(S,M) \) is self-dual, any bounded \( M \)-linear operator on it is adjointable, and these operators form a W∗-algebra \( \mathbb{B}(\overline{\ell^2}(S,M)) \).

Definition 3.7. The W∗-algebra crossed product \( M \rtimes_{\text{alg}} S \) for an action \( (\mathcal{H}_t,\mu_{t,u}) \) of \( S \) on a W∗-algebra \( M \) by self-dual Hilbert bimodules is defined as the weak closure of \( \lambda(M \rtimes_{\text{alg}} S) \) in the W∗-algebra \( \mathbb{B}(\overline{\ell^2}(S,M)) \).

By construction, \( \lambda \) gives an injective ∗-homomorphism \( M \rtimes_{\text{alg}} S \hookrightarrow M \rtimes_{\text{alg}} S \). The map \( E \) above extends to a faithful conditional expectation \( M \rtimes_{\text{alg}} S \rightarrow M \), namely, \( E(T) = \iota^* \circ T \circ \iota \), where \( \iota: M \rightarrow \overline{\ell^2}(S,M) \) is the inclusion of the summand \( M \rtimes_{\text{alg}} S \).

4. The reduced crossed product and induction

Now we return to the C∗-algebraic case. Let \( (\mathcal{H}_t,\mu_{t,u}) \) be an action of \( S \) on a C∗-algebra \( A \) by Hilbert bimodules. Then \( (\mathcal{H}''_t,\mu''_{t,u}) \) is an action of \( S \) on \( A'' \) by self-dual Hilbert bimodules. The representation of \( A'' \rtimes_{\text{alg}} S \) on \( \ell^2(S,A'') \) restricts to a ∗-homomorphism on \( A \rtimes_{\text{alg}} S \). This extends to a ∗-homomorphism \( A \rtimes S \rightarrow \mathbb{B}(\ell^2(S,A'')) \).

Definition 4.1. The reduced crossed product \( A \rtimes_{\text{alg}} S \) is the image of \( A \rtimes S \) in \( \mathbb{B}(\ell^2(S,A'')) \), the W∗-algebra of adjointable operators on \( \ell^2(S,A'') \).

By construction, \( A \rtimes_{\text{alg}} S \) is contained in the W∗-algebra crossed product \( A'' \rtimes S \).

Remark 4.2. Every adjointable operator on \( \ell^2(S,A'') \) extends uniquely to the self-dual completion \( \ell^2(S,A'') \), and this gives a unital embedding of \( \mathbb{B}(\ell^2(S,A'')) \) into \( \mathbb{B}(\overline{\ell^2}(S,A'')) \). Since \( A \rtimes_{\text{alg}} S \) and hence also the image of \( A \rtimes S \) map the Hilbert submodule \( \ell^2(S,A'') \) into itself by adjointable operators, \( A \rtimes S \) is contained in the image of \( \mathbb{B}(\ell^2(S,A'')) \) in \( \mathbb{B}(\overline{\ell^2}(S,A'')) \). Hence it makes no difference whether we use \( \ell^2(S,A'') \) or \( \overline{\ell^2}(S,A'') \) to define \( A \rtimes S \).

Proposition 4.3. The canonical ∗-homomorphisms from \( A \rtimes_{\text{alg}} S \) to \( A'' \rtimes_{\text{alg}} S \), \( A \rtimes_{\text{alg}} S \), and \( A \rtimes S \) are injective.
Proof. By definition, \( A \rtimes_{\text{alg}} S \) is a quotient of the direct sum \( \bigoplus_{t \in S} \mathcal{H}_t \). Let \( F \subseteq S \) be a finite subset. We are going to prove that the kernel of the map \( \bigoplus_{t \in F} \mathcal{H}_t \rightarrow A'' \rtimes_{\text{alg}} S \) is the linear span \( W_F \) of \( \theta_{t,u}(x) \delta_t - x \delta_u \) for \( t, u \in F \). Therefore, the canonical \(^*\)-homomorphism from \( A \rtimes_{\text{alg}} S \) to \( A'' \rtimes_{\text{alg}} S \) is injective. Then the map from \( A \rtimes_{\text{alg}} S \) to \( A \rtimes S \) is injective because \( A \rtimes S \hookrightarrow A'' \rtimes S \hookrightarrow A'' \rtimes_{\text{alg}} S \). And then the map from \( A \rtimes_{\text{alg}} S \) to \( A \rtimes S \) is injective because the injective map to \( A \rtimes S \) factors through it. Thus everything follows from the above description of the kernel of the map \( \bigoplus_{t \in F} \mathcal{H}_t \rightarrow A'' \rtimes_{\text{alg}} S \).

Let \( F' \) be another finite subset of \( S \) with \( F \subseteq F' \), and assume that \( \xi := \sum_{t,u \in F'} \theta_{t,u}(x_{t,u}) \delta_t - x_{t,u} \delta_u \) with \( x_{t,u} \in \mathcal{H}_u \cdot I_{t,u} \) for all \( t, u \in F' \). It suffices to rewrite \( \xi \) as a sum over \( t, u \in F' \setminus \{ v \} \): if we can always do this, then we may go on and remove the other elements of \( F' \setminus S \) until we bring \( \xi \) into the desired form. Thus we may assume without loss of generality that \( F' = F \cup \{ v \} \).

Since \( \theta_{t,u} = \theta_{u,t}^{-1} \), we may assume that the summands in \( \xi \) containing \( v \) all have \( v \) as the second entry. Any summand with \( t = u = v \) is zero because \( \theta_{v,v} = \text{Id}_{\mathcal{H}_v} \), so we may remove such a summand from our representation of \( \xi \). Let \( F'' \) be the set of \( t \in F' \) for which \( \xi \) contains a summand of the form \( \theta_{t,v}(x_{t,v}) \delta_t - x_{t,v} \delta_v \). Since \( v \notin F'' \), we have \( F'' \subseteq F' \). There is nothing to do if \( F'' \) is empty. So we assume that \( F'' \) is non-empty and pick some \( w \in F'' \). We are going to rewrite \( \xi \) so that \( \xi \) contains a summand for \( (t, v) \) with \( t \in F'' \setminus \{ w \} \) appearing. If we can do this, we may repeat this step and remove all points from \( F'' \), until we arrive at a sum that does not involve \( v \) any more. Thus it suffices to prove that we may reduce \( F'' \) to \( F'' \setminus \{ w \} \).

The \( \delta_v \)-component of \( \xi \) is the sum \( \sum_{t \in F''} x_{t,v} \delta_v \). This must vanish because \( v \notin F' \). Thus \( x_{w,v} = -\sum_{t \in F'' \setminus \{ w \}} x_{t,v} \). This belongs to \( \sum_{t \in F'' \setminus \{ w \}} \mathcal{H}_v \cdot I_{t,v} \) and to \( \mathcal{H}_v \cdot I_{w,v} \). This intersection is \( \mathcal{H}_v \cdot I \) with \( I := I_{w,v} \cap \sum_{t \in F'' \setminus \{ w \}} I_{t,v} = \sum_{t \in F'' \setminus \{ w \}} (I_{w,v} \cap I_{t,v}) \) because the map \( I \mapsto \mathcal{H}_v \cdot I \) is a lattice isomorphism from the lattice of ideals \( I \triangleleft s(\mathcal{H}_v) \) onto the lattice of Hilbert submodules in \( \mathcal{H}_v \). Thus we may rewrite \( x_{w,v} = \sum_{t \in F'' \setminus \{ w \}} x_{w,v,t} \) with \( x_{w,v,t} \in I_{w,v} \cap I_{t,v} \). Then \( \theta_{w,v}(x_{w,v,t}) = \theta_{w,v}(x_{w,v,t}) \delta w - \theta_{w,v}(x_{w,v,t}) \delta v \) by Lemma 2.5, so that

\[
\theta_{w,v}(x_{w,v,t}) \delta w - x_{w,v,t} \delta v = \theta_{w,v}(x_{w,v,t}) \delta w - \theta_{w,v}(x_{w,v,t}) \delta v + \theta_{w,v}(x_{w,v,t}) \delta t - x_{w,v,t} \delta v.
\]

When we substitute this in \( \xi \) for all \( t \in F'' \setminus \{ w \} \), we replace the summand \( \theta_{w,v}(x_{w,v,t}) \delta w - x_{w,v,t} \delta v \) for \( (w, v) \) by summands for \( (w, t) \) and \( (t, v) \) for \( t \in F'' \setminus \{ w \} \). Since \( t, w \) \in \( F' \), this achieves the reduction step that we still need. This finishes the proof that \( \mathcal{W}_F \) is the set of all finite linear combinations of \( \theta_{t,u}(x_{t,u}) \delta_t - x_{t,u} \delta_u \) with \( x_{t,u} \in \mathcal{H}_u \cdot I_{t,u} \) and \( t, u \in S \) that belong to \( \bigoplus_{t \in F'} \mathcal{H}_t \delta_1 \).

Next we prove that \( \mathcal{W}_F \subseteq \bigoplus_{t \in F} \mathcal{H}_t \delta_1 \) is closed in the norm topology for each finite subset \( F \subseteq S \).

We prove this by induction on the size of \( F \). If \( F \) is empty, the assertion is trivial. So let \( |F| \geq 1 \) and pick \( t \in F \). Let \( F' := F \setminus \{ t \} \) and assume that \( \mathcal{W}_{F'} \) is norm closed. Let \( I = \sum_{u \in F'} I_{t,u} \). This is a closed ideal in \( A \), as a sum of finitely many closed ideals. Hence \( \mathcal{H}_t \cdot I \subseteq \mathcal{H}_t \) is closed and contains \( \mathcal{H}_t \cdot I_{t,u} \) for all \( u \in F' \). As in the proof of the claim above, the closure of \( \mathcal{W}_F \) can only contain \( \sum_{u \in F} x_u \delta_u \in \bigoplus_{u \in F} \mathcal{H}_u \) if \( x_t \in \mathcal{H}_t \cdot I \). In that case, we may write \( x_t = x^0_t \cdot a \) with \( x^0_t \in \mathcal{H}_t \), \( a \in I \), and \( \| x^0_t \| = \| x^0_t \|_w \cdot \| a \| < 1 + \varepsilon \) for any \( \varepsilon > 0 \). Moreover, \( a = \sum_{u \in F'} a_u \) with \( a_u \in I_{t,u} \) and \( \| a_u \| < 1 + \varepsilon \) for all \( u \in F' \). Then

\[
\sum_{u \in F} x_u \delta_u - \sum_{u \in F'} (\theta_{t,u}(x^0_t \cdot a_u) \delta_u - x^0_t \cdot a_u \delta_1) \in \bigoplus_{u \in F'} \mathcal{H}_u.
\]
This term no longer involves the summand $H_t$. By our first claim above, this sum belongs to $W_F$.

The argument above shows that $W_F/W_{F'} \cong H_t \cdot I$, where the quotient norm is equivalent to the norm from $H_t$. Since $W_{F'}$ is closed by induction assumption, $W_F$ is an extension of the Banach space $H_t \cdot I$ by the Banach space $W_{F'}$. This implies that $W_F$ is complete in the subspace topology from $\bigoplus_{u \in u} H_u$, so that $W_F$ must be closed in the norm topology.

The range and source submodules of the partial isometries $\Theta_{t,u}: H_u \to H_t$ that appear in the definition of $A'' \rtimes_{\text{alg}} S$ are the weak closures of $H_t \cdot I_{t,u}$ and $H_u \cdot I_{t,u}$, respectively. Therefore, the kernel of the map $\bigoplus_{t \in F} H_t \to A'' \rtimes_{\text{alg}} S$ is contained in the weak closure of $W_F$. By the Hahn–Banach Theorem, a subspace of the Banach space $\bigoplus_{t \in F} H_t$ is weakly closed if and only if it is norm closed: the norm and the weak topology have the same closed convex subsets. We have shown that $W_F$ is norm closed, hence weakly closed. This finishes the proof.

By definition, $\ell^2(S, A'')$ is a $C^*$-correspondence from $A \rtimes_t S$ to $A''$. As explained in [30, p. 65], this $C^*$-correspondence gives a functor from the $W^*$-category of Hilbert space representations of $A$ to that of $A \rtimes_t S$:

**Definition 4.4.** The induction functor $\text{Ind}$ from representations of $A$ to representations of $A \rtimes_t S$ maps a representation $\pi: A \to \mathcal{B}(H)$ to the representation of $A \rtimes_t S$ on $\ell^2(S, A'') \otimes_{\pi''} H$, where $\pi''$ is the unique weakly continuous extension of $\pi$ to $A''$.

The $C^*$-norm on $A \rtimes_t S$ is the supremum of the norms in $\text{Ind} \pi$ for all representations $\pi$ of $A$. We may also take a single faithful representation of $A''$. For instance, the direct sum of the GNS-representations for all states of $\pi$ is equal to the supremum of the norm in $\text{Ind} \pi$, where $\pi$ now runs through the set of GNS-representations for all states on $A$.

But is it enough to take only the irreducible representations as in [3]? Can we even take any faithful representation of $A$? Notice that the direct sum of all irreducible representations of $A$ is always faithful on $A$, but its extension to $A''$ need not be faithful (see [26, 4.3.11]).

To answer the above questions (the first positively, the second negatively), we study the range of the map $E: A \rtimes_t S \to A''$. By definition, $E$ is the restriction to $A \rtimes_t S$ of the conditional expectation $A'' \rtimes_t S \to A''$ constructed in the last section, see [3, 5]. We shall sometimes view $E$ as a map from the full crossed product $A \rtimes S$ to $A''$, and call it the weak conditional expectation associated to the action. The following lemma describes $E$ in the $C^*$-algebraic setting:

**Lemma 4.5.** Let $t \in S$ and let $\theta_{1,t}$ denote the canonical isomorphism $H_t, I_{1,t} \cong I_{1,t}$ as in Lemma [2, 5] View the multiplier algebra of $I_{1,t}$ as a subalgebra of $A''$ in the usual way. The map $E: A \rtimes_t S \to A''$ maps $H_t \subseteq A \rtimes_t S$ into $\mathcal{M}(I_{1,t}) \subseteq A''$. More precisely, $E(\xi \delta_t)$ for $\xi \in H_t$ is the multiplier of $I_{1,t}$, given by $E(\xi \delta_t)(\xi \cdot x)$ for all $x \in I_{1,t}$. If $(u_{t,i})$ is an approximate unit for $I_{1,t}$, then

$$E(\xi \delta_t) = s\text{-lim}_{i} \theta_{1,t}(\xi \cdot u_{t,i}),$$

where $s\text{-lim}$ denotes the limit in the strict topology on $\mathcal{M}(I_{1,t})$.

**Proof.** By definition, $E(\xi) = \Theta_{1,t}(\xi[I_{1,t}])$, where $\Theta_{1,t}$ is the extension (as in Lemma [5, 3]) of the isomorphism $\theta_{1,t}: H_t \to H_1 \cdot I_{1,t} = I_{1,t}$ described in Lemma [2, 5] and $[I_{1,t}]$ is the support projection of the ideal $I_{1,t}$. The ideal $I_{1,t}$ in Lemma [2, 5] is the bidual or, equivalently, the weak closure in $A''$, of the ideal $I_{1,t} \subseteq A$. By construction, $\Theta_{1,t}$ is an isomorphism $\mathcal{H}_{1,t}'' \cdot [I_{1,t}] \cong A'' \cdot [I_{1,t}] \subseteq A''$. 

The isomorphism \( \theta_{1,t} : \mathcal{H}_t \cdot I_{1,t} \to I_{1,t} \) induces an isomorphism of multiplier modules:

\[
\mathcal{M}(\theta_{1,t}) : \mathcal{M}(\mathcal{H}_t \cdot I_{1,t}) := \mathbb{B}(I_{1,t}, \mathcal{H}_t \cdot I_{1,t}) \xrightarrow{(\theta_{1,t})_*} \mathbb{B}(I_{1,t}, I_{1,t}) = \mathcal{M}(I_{1,t}).
\]

There is a canonical map \( \mathcal{H}_t \to \mathcal{M}(\mathcal{H}_t \cdot I_{1,t}) \), sending \( \xi \in \mathcal{H}_t \) to the multiplier \( \hat{\xi} \in \mathcal{M}(\mathcal{H}_t \cdot I_{1,t}) \) given by \( \hat{\xi}(x) := \xi \cdot x \) for \( x \in I_{1,t} \). Thus \( \mathcal{M}(\theta_{1,t})(\hat{\xi}a) = \theta_{1,t}(\xi(a)) \) for all \( a \in I_{1,t} \). The map \( \mathcal{M}(\theta_{1,t}) \) is the unique strictly continuous extension of \( \theta_{1,t} \), and \( \Theta_{1,t} \) is the unique weakly continuous extension of \( \theta_{1,t} \). The obvious embedding \( \mathcal{M}(\mathcal{H}_t) \to \mathcal{M}(\mathcal{H}_t \cdot I_{1,t}) \) is injective because the map \( \pi \) is injective.

Remark 4.7. The weak conditional expectation \( E : A \rtimes_r S \to A'' \) is faithful and is the identity on \( A \). Hence the canonical maps \( \mathcal{H}_t \to A \rtimes_r S \), \( \xi \mapsto \xi \delta_t \), are isometric (this is also proved in [13]), and the same holds for \( A \rtimes S \). These maps form representations of the action \( (\mathcal{H}_t)_{t \in S} \), and turn both \( A \rtimes_r S \) and \( A \rtimes S \) into \( S \)-graded C*-algebras with copies of \( \mathcal{H}_t \) as the subspaces of the grading. The subspaces of a grading over an inverse semigroup that is not a group have non-trivial intersection and so are not linearly independent.

Notation 4.8. Let \( \hat{A} \subseteq A'' \) be the C*-subalgebra generated by \( E(A \rtimes_r S) \subseteq A'' \). We have \( \hat{A} \supseteq A \) because \( E|_{A} = \text{Id}_{A} \). For a representation \( \pi : A \to \mathbb{B}(\mathcal{H}) \), let \( \pi^* \) be the restriction of \( \pi'' : A'' \to \mathbb{B}(\mathcal{H}) \) to \( \hat{A} \).

Definition 4.9. A family of representations \( (\pi_i)_{i \in I} \) of \( A \) is E-faithful if the representation \( \bigoplus_{i \in I} \pi_i \) of \( \hat{A} \) is faithful.

Proposition 4.10. If the family of representations \( (\pi_i)_{i \in I} \) is E-faithful, then the representation \( \bigoplus_{i \in I} \text{Ind } \pi_i \) of \( A \rtimes_r S \) is faithful.

Proof. We may replace the family \( (\pi_i)_{i \in I} \) by the single representation \( \pi = \bigoplus \pi_i \). We may use the \( \hat{A} \)-valued expectation on \( A \rtimes S \) to construct a Hilbert \( \hat{A} \)-module \( \ell^2(S, \hat{A}) \). Then \( \ell^2(S, \hat{A}) \otimes_{\hat{A}} A'' \cong \ell^2(S, A'') \) as correspondences from \( A \rtimes_r S \) to \( A'' \).

The resulting map

\[
\mathbb{B}(\ell^2(S, \hat{A})) \to \mathbb{B}(\ell^2(S, A'')) \quad T \mapsto T \otimes 1_{A''},
\]

is injective because the map \( \hat{A} \to A'' \) is injective. Its image contains the image of \( A \rtimes_{\text{alg}} S \) and hence of \( A \rtimes_r S \). Hence we may as well define \( A \rtimes_r S \) as the C*-subalgebra of \( \mathbb{B}(\ell^2(S, \hat{A})) \) generated by \( A \rtimes_{\text{alg}} S \). Moreover,

\[
\ell^2(S, A'') \otimes_{A''} \pi'' \cong \ell^2(S, \hat{A}) \otimes_{\hat{A}} A'' \otimes_{A''} \pi'' \cong \ell^2(S, \hat{A}) \otimes_{\hat{A}} \pi''|_{\hat{A}} = \ell^2(S, \hat{A}) \otimes_{\hat{A}} \pi.
\]

If \( \pi^* \) is faithful, then the induced representation of \( \mathbb{B}(\ell^2(S, \hat{A})) \) on \( \ell^2(S, \hat{A}) \otimes_{\hat{A}} \pi \) is also faithful. Hence \( \text{Ind } \pi \) is a faithful representation of \( A \rtimes_r S \).

Remark 4.11. It can easily happen that the induced representation \( \text{Ind } \pi \) is faithful although \( \pi \) is not faithful, even without the difficulty of extending from \( A \) to \( \hat{A} \). Consider a finite group \( \Gamma \) with \( n > 1 \) elements and let it act on \( A = C(\Gamma) \). Then \( A \rtimes \Gamma \cong M_n \mathbb{C} \) has only faithful nonzero representations. If \( \pi : A \to \mathbb{C} \) is a character, then \( \pi \) is not faithful, but \( \text{Ind } \pi \) is an irreducible faithful representation of \( A \rtimes \Gamma \).

To turn Proposition 4.10 into a useful criterion, we need to understand \( \hat{A} \) better. We are particularly interested in when \( \hat{A} = A \), that is, when \( E(A \rtimes S) \subseteq A \) so that \( E \) becomes an ordinary conditional expectation \( A \rtimes S \to A \).
5. The commutative case

First we consider the special case where \( A \) is commutative. Our constructions are a minor generalisation of those by Khoshkam and Skandalis in [20]; the only difference is that we also allow groupoid \( C^* \)-algebras twisted by Fell line bundles. Example 2.5 in [20] shows that \( \text{Ind} \pi \) need not be faithful if \( \pi \) is a faithful representation of \( A \). So our problem is non-trivial.

Let \( A = C_0(X) \) be a commutative \( C^* \)-algebra. Let \( \mathcal{H} \) be a Hilbert \( A \)-bimodule. The ideals \( s(\mathcal{H}) \) and \( r(\mathcal{H}) \) correspond to open subsets \( U \) and \( V \) in \( X \), respectively, and \( \mathcal{H} \) is an imprimitivity bimodule between \( C_0(U) \) and \( C_0(V) \). A Hilbert module over \( C_0(V) \) is equivalent to a continuous field of Hilbert spaces over \( V \). Since we want the compact operators on this field of Hilbert spaces to be isomorphic to \( C_0(U) \), hence commutative, this continuous field must be a complex line bundle; equivalently, each fibre has dimension 1. The left action must map \( f \in C_0(U) \) to the operator that multiplies pointwise with the function \( f \circ \alpha^{-1} \) for some homeomorphism \( \alpha : U \to V \). The Hilbert bimodule \( \mathcal{H} \) and the data \( (U, V, L, \alpha) \) determine each other uniquely up to isomorphism. The triple \( (U, V, \alpha) \) is a partial homeomorphism of \( X \). Thus a Hilbert bimodule over \( C_0(X) \) is equivalent to a partial homeomorphism of \( X \) together with a line bundle on its (co)domain.

Now we generalise the above discussion and consider an action of a unital inverse semigroup \( S \) on \( A = C_0(X) \). This is equivalent to a saturated Fell bundle over \( S \) with unit fibre \( A \). Following [5], we now describe these in terms of twisted \( \acute{e} \)tale groupoids, that is, Fell line bundles over \( \acute{e} \)tale groupoids. An action of \( S \) on \( A \) is equivalent to partial homeomorphisms \( \psi_t : D_t_\to D_t \) for certain open subsets \( D_t \subseteq X \) with line bundles \( L_t \) over \( D_t \) for all \( t \in S \), together with suitable multiplication isomorphisms for \( t, u \in S \). These multiplication isomorphisms can only exist if the partial homeomorphisms \( (\psi_t)_{t \in S} \) form an inverse semigroup action on \( X \), that is, \( \psi_t \circ \psi_u = \psi_{tu} \) for all \( t, u \in S \): this is the action on the primitive ideal space of \( A \) induced by an action on \( A \) given by [7, Lemma 6.12]. Hence we may form the transformation groupoid \( X \rtimes S \), which is an \( \acute{e} \)tale, possibly non-Hausdorff, groupoid with object space \( X \).

The complex line bundles \( L_t \) with their multiplication maps are equivalent to a Fell line bundle \( \mathcal{L} \) over the groupoid \( X \rtimes S \), that is, a Fell bundle over \( X \times S \) with only 1-dimensional fibres. This follows from [7, Theorem 6.13], which shows that the action of \( S \) on \( A \) comes from a Fell bundle over the groupoid \( X \rtimes S \) with unit fibre \( A \). For every Hausdorff open subset \( U \subseteq (X \times S)^1 \), the \( C_0 \)-sections of the Fell bundle over \( U \) form a \( C_0(U) \)-linear imprimitivity bimodule between \( C_0(U) \) and itself. As above, this must be the space of sections of a line bundle over \( U \). Since these Hausdorff open subsets cover \( (X \times S)^1 \), we get a line bundle \( \mathcal{L} \) over all of \( (X \times S)^1 \). The multiplication on quasi-continuous sections of the Fell bundle induces the appropriate multiplication between the fibres of \( \mathcal{L} \).

An element of \( \mathcal{H}_t \) is a \( C_0 \)-section of the line bundle \( L_t \) over \( D_t \). We may identify \( \mathcal{H}_t \) with the space of all continuous \( C_0 \)-sections of \( \mathcal{L} \) on the bisection \( (X \times S)_t \) of the arrow space of \( X \times S \) corresponding to \( t \in S \).

A section of the line bundle \( \mathcal{L} \) over \( (X \times S)^1 \) is called quasi-continuous if it is a finite linear combination of \( C_0 \)-sections on Hausdorff, open subsets \( U \) of \( X \times S \), extended by 0 outside \( U \). Let \( \mathfrak{S}(X \times S, \mathcal{L}) \) be the space of quasi-continuous sections. The section \( C^* \)-algebra of the Fell line bundle \( \mathcal{L} \) is defined as the completion of \( \mathfrak{S}(X \times S, \mathcal{L}) \) in the maximal \( C^* \)-seminorm. \( C^* \)-algebras of this type may be viewed as twisted groupoid \( C^* \)-algebras (see [29]); they are the groupoid analogues of twisted group \( C^* \)-algebras.

Elements of \( \mathcal{H}_t \) extended by 0 outside \( (X \times S)_t \) give elements of \( \mathfrak{S}(X \times S, \mathcal{L}) \). The intersection of \( (X \times S)_t \) and \( (X \times S)_u \) for \( t, u \in S \) corresponds to the open
subset $D_{t,u} := \bigcup_{t' \leq t, u} D_{v}$, and $C_0(D_{t,u})$ is the ideal $I_{t,u}$ in $C_0(X)$. Hence the maps $\mathcal{H}_t \to \mathcal{G}(X \times S, \mathcal{L})$ defined above induce a $^*$-isomorphism $A \rtimes_{\text{alg}} \mathcal{S} \to \mathcal{G}(X \times S, \mathcal{L})$ by [2, Proposition B.2]. Hence $A \rtimes S = C^*\{X \times S, \mathcal{L}\}$ (see also [2, Corollary 5.6] and [3, Proposition 2.14]).

The bidual $A''$ of $A$ contains the $C^*$-algebra $B(X)$ of bounded Borel functions on $X$ because any representation of $C_0(X)$ extends uniquely to a representation of $B(X)$. Any ideal in $C_0(X)$ is of the form $C_0(U)$ for an open subset $U \subseteq X$. Its multiplier algebra is $C_0(U)$, and any function in $C_b(U)$, extended by 0 outside $U$, is a Borel function on $X$. Now Lemma 5.2 shows that the image of the weak conditional expectation $E: C_0(X) \rtimes S \to C_0(X)''$ is contained in $B(X)$. So we may as well work in the more concrete subalgebra $B(X) \subseteq C_0(X)''$.

When we identify $A \rtimes_{\text{alg}} \mathcal{S} \cong \mathcal{G}(X \times S, \mathcal{L})$, then the map $E: A \rtimes_{\text{alg}} \mathcal{S} \to A''$ simply restricts sections of $\mathcal{L}$ to the unit fibre $X \subseteq (X \times S)^1$; this gives scalar-valued Borel functions on $X$ through a canonical isomorphism $\mathcal{L}|_X \cong C \times X$ for any Fell line bundle. We can now describe $\tilde{A}$ and decide when $E$ takes values in $A$, that is, when $A = \tilde{A}$:

**Proposition 5.1.** Let $A = C_0(X)$ equipped with an action of a unital inverse semigroup $S$. The $C^*$-algebra $\tilde{A}$ is the $C^*$-subalgebra of $B(X)$ that is generated by functions of the form $f|_{X \cap U}$, extended by zero on $X \setminus (X \cap U)$, for Hausdorff, open subsets $U \subseteq (X \times S)^1$ and $f \in C_c(U)$.

The conditional expectation $E: A \rtimes S \to A''$ takes values in $A$ if and only if the associated transformation groupoid $X \rtimes S$ is Hausdorff.

**Proof.** The first statement about $\tilde{A}$ is clear from our description of the conditional expectation $E$. We have $A = \tilde{A}$ if and only if all functions of the form $f|_{X \cap U}$ for Hausdorff, open subsets $U \subseteq (X \times S)^1$ and $f \in C_c(U)$ are still continuous on $X$. This is the case if and only if $X$ is a closed subset of $(X \times S)^1$. This is equivalent to $(X \times S)^1$ being Hausdorff, see Lemma 5.2.

**Lemma 5.2.** A topological groupoid $G$ is Hausdorff if and only if its unit space $G^0$ is Hausdorff and closed as a subset of $G^1$.

**Proof.** If $G^1$ is Hausdorff, so is the subspace $G^0$. Since $G^0 = \{g \in G^1 \mid g = 1_s(g)\}$ and the map $g \mapsto 1_{s(g)}$ on $G^1$ is continuous, the units $G^0$ form a closed subset of $G^1$ if $G^1$ is Hausdorff.

Conversely, assume that $G^1$ is not Hausdorff. Then there is a net $(g_i)_{i \in I}$ in $G^1$ with two different limit points $g, h \in G^1$. Since the range, inversion and multiplication maps are continuous, $r(g) = \lim_i r(g_i) = r(h)$ and the net $g^{-1}_i \cdot g_i$, which lies in $G^0$, converges both to $g^{-1} \cdot g \in G^0$ and to $g^{-1} \cdot h \neq g^{-1} \cdot g$. If $g^{-1} \cdot h \in G^0$, then $G^0$ is not Hausdorff. Since $g^{-1} \cdot h \notin G^0$, then $G^0$ is not closed in $G^1$.

The same $C^*$-algebra $\tilde{A}$ is used in [20] to define the regular representation of $C_0(X) \rtimes S = C^*(G)$ on a certain Hilbert $\tilde{A}$-module. This coincides with our regular representation. Since $\tilde{A}$ is commutative, it is isomorphic to $C_0(Y)$ for a certain space $Y$. The inclusion map $A \to \tilde{A}$ gives a continuous map $Y \to X$. Since Borel functions are, in particular, functions on $X$, we also get a map $X \to Y$ using evaluation homomorphisms; but this map need not be continuous (see [20]).

Our description of $\tilde{A}$ allows us to characterise when a representation $\pi$ of $C_0(X)$ is $E$-faithful: this means that the resulting representation of $C_0(Y)$ is faithful, that is, its “support” is dense in $Y$. This criterion is already obtained in [20, Corollary 2.11]. Proposition 7.2 below is a noncommutative analogue of this result.
6. Conditional expectation in the noncommutative case

Now let $A$ be an arbitrary $C^*$-algebra, equipped with an action of a unital inverse semigroup $S$ by Hilbert bimodules. We are going to characterise when the conditional expectation $E$ is $A$-valued, that is, $A = \hat{A}$. First we show by an example that $\hat{A}$ may become so complicated that a complete description is not a promising goal. It is easier to describe when $A = \hat{A}$.

Example 6.1. Let $G$ be a group and let $S$ be the inverse semigroup obtained by adding a zero element to $G$. Let $(u_g)_{g \in G}$ be a group representation of $G$ on a Hilbert space $H$. Actually, it is enough to have a group homomorphism to the unitary group in the Calkin algebra $\mathbb{B}(H)/K(H)$, but already ordinary representations lead to rather complicated situations. Let $A = \mathbb{K}(H)^+$, the unitalisation of the $C^*$-algebra of compact operators. The action of $S$ on $A$ is defined by taking $H_0 := \mathbb{K}(H)$ and $H_g := \mathbb{K}(H) \oplus \mathbb{C} \cdot u_g \subseteq \mathbb{B}(H)$ for $g \in G$. We clearly have $H_g H_h = H_{gh}$ for $g, h \in G$, and $H_0 H_2 = H_0 = H_2 H_0$ and $H_0' = H_{g^{-1}}$ for all $g \in G$ as well. Hence we have got an action of $S$ on $A$ by Hilbert bimodules.

The bidual $A''$ is naturally isomorphic to $A'' \cong \mathbb{B}(H) \oplus \mathbb{C}$ because $\mathbb{B}(H) = \mathbb{K}(H)'$, and 1 is a group algebra of $(1, 1, 0) \in (1, 1)$. For $t, u \in S$ with $t \neq u$, there is always a unique element $v \leq t, u$, namely, $v = 0$. Hence $I_{t, u} = \mathbb{K}(H)$ for all $t, u \in S$ with $t \neq u$. Thus $I_{t, u} = \mathbb{B}(H) \oplus 0$, and the projection onto this ideal is $(1, 0) \in \mathbb{B}(H) \oplus \mathbb{C}$. Since $E: \mathbb{H}_g'' \to \mathbb{H}_g''$ is weakly continuous, it must map $k + \lambda u_g \mapsto (k + \lambda u_g, 0) \in \mathbb{B}(H) \oplus \mathbb{C}$ for all $k \in \mathbb{K}(H), \lambda \in \mathbb{C}$, and $g \neq 1$. Thus $\hat{A}$ is the $C^*$-subalgebra of $\mathbb{B}(H) \oplus \mathbb{C}$ generated by $\mathbb{K}(H)$ and the unitaries $u_g$ for $g \in G \setminus \{1\}$ in the first summand and by $(1, 1)$. This gives

$$\hat{A} = (\mathbb{K}(H) + C^*(u_g | g \in G)) \oplus \mathbb{C}.$$

Since any $C^*$-algebra is generated by the unitaries it contains, we may get any $C^*$-algebra containing $\mathbb{K}(H)$ in the first summand.

Let $\text{Prim}(A)$ be the primitive ideal space of $A$. The lattice of ideals in $A$ is isomorphic to the lattice of open subsets of $\text{Prim}(A)$ by [10 Proposition 3.2.2]. The action of $S$ on $A$ by Hilbert bimodules induces an action on $\text{Prim}(A)$ by partial homeomorphisms by [2, Lemma 6.12]. Let $\alpha_t: D_t \to D_t$ for $t \in S$ be the partial homeomorphism of $\text{Prim}(A)$ associated to $t \in S$. The open subsets $D_t$ and $D_t$ of $\text{Prim}(A)$ correspond to the ideals $s(\mathcal{H}_t)$ and $r(\mathcal{H}_t)$ in $A$.

Lemma 6.2. The ideal $I_{t, u}$ for $t, u \in S$ corresponds to the open subset $\bigcup_{v \leq t, u} D_v$ in $\text{Prim}(A)$. A representation $\pi: A \to \mathbb{B}(H)$ maps the central projection $[I_{t, u}] \in A''$ to the orthogonal projection onto the subspace $\pi(I_{t, u}) \cdot \mathcal{H}$.

Proof. Since the bijection between ideals in $A$ and open subsets in $\text{Prim}(A)$ is a lattice isomorphism, the ideal generated by $s(\mathcal{H}_v)$ for $v \leq t, u$ corresponds to the union $\bigcup_{v \leq t, u} D_v \subseteq \text{Prim}(A)$. The second statement holds because $[I] \in A''$ for an ideal $I \triangleleft \hat{A}$ is the weak limit of an approximate unit $(e_n)$ for $I$ and $\pi(e_n)$ converges strongly to the orthogonal projection onto $\pi(I) \cdot \mathcal{H}$.

Proposition 6.3. We have $A = \hat{A}$ if and only if the ideal $I_{t, t}$ is complemented in the larger ideal $s(\mathcal{H}_t)$ for each $t \in S$.

Proof. We use the description of the weak conditional expectation $E: A \times S \to A''$ in Lemma 6.12 and the following computation. Let $a_1, a_2 \in I_{t, t}, \xi_1, \xi_2 \in \mathcal{H}_t$. Then

$$\langle M(\theta_{t, t})(\xi_1) a_1 | M(\theta_{t, t})(\xi_2) a_2 \rangle = \langle \theta_{t, t}(\xi_1 a_1) | \theta_{t, t}(\xi_2 a_2) \rangle = \langle \xi_1 a_1 | \xi_2 a_2 \rangle = a_1^* (\xi_1) a_2.$$

Hence $M(\theta_{t, t})(\xi_1)^* M(\theta_{t, t})(\xi_2) = (\xi_1 | \xi_2)$ holds in $M(I_{t, t}) \subseteq A''$ for all $\xi_1, \xi_2 \in \mathcal{H}_t$. 

Assume first that for each $t \in S$, $I_{1,t}$ is a complemented ideal in $s(\mathcal{H}_t)$, that is, $s(\mathcal{H}_t) = I_1,t \oplus I_{1,t}^\perp$ for some ideal $I_{1,t}^\perp \subset s(\mathcal{H}_t) \subset A$. Since $\mathcal{H}_t$ is a full right module over $s(\mathcal{H}_t)$, we may split $\mathcal{H}_t \cong \mathcal{H}_1^t \oplus \mathcal{H}_2^t$, where $\mathcal{H}_1^t$ and $\mathcal{H}_2^t$ are Hilbert modules over $I_{1,t}$ and $I_{1,t}^\perp$, respectively. Hence $\mathcal{H}_t : I_{1,t} = \mathcal{H}_1^t$, and this is isomorphic to $I_{1,t}$ by $\theta_{1,t}$. The map to $\mathcal{M}(I_{1,t})$ annihilates $\mathcal{H}_2^t$ because $I_{1,t} : I_{1,t}^\perp = 0$. Thus the image of $\mathcal{H}_t$ in $\mathcal{M}(I_{1,t})$ is simply $I_{1,t}$, which is contained in $A$. Since this holds for all $t \in S$ by assumption, we get $E(A_{\text{alg}} S) \subseteq A$ and thus $A = A$ as asserted.

Conversely, assume that $I_{1,t}$ is not complemented in $s(\mathcal{H}_t)$ for some $t \in S$. Then the image of the map $s(\mathcal{H}_t) \to \mathcal{M}(I_{1,t})$ is not contained in $I_{1,t}$; otherwise, the kernel of this map would be a complementary ideal for $I_{1,t}$ in $s(\mathcal{H}_t)$. Hence there is an element $\xi \in \mathcal{H}_t$ such that $\langle \xi | \xi \rangle \in s(\mathcal{H}_t)$ maps to an element of $\mathcal{M}(I_{1,t})$ that does not belong to $I_{1,t}$. Hence $E(\xi)^* E(\xi) = \mathcal{M}(\theta_{1,t})(\xi)^* \mathcal{M}(\theta_{1,t})(\xi) = \langle \xi | \xi \rangle$ does not belong to $I_{1,t}$. Any normal representation of $A''$ that vanishes on $I_{1,t}$ annihilates $E(\xi)^* E(\xi)$ because it belongs to $I_{1,t}$. In particular, the normal extension of a faithful representation of $A/I_{1,t}$ must annihilate $E(\xi)^* E(\xi)$. If $E(\xi)^* E(\xi) / S \in A$, then this implies $E(\xi)^* E(\xi) / I_{1,t}$, which is false. Thus $A \neq A$.

**Corollary 6.4 (14, Theorem 3.15).** The transformation groupoid $X \rtimes S$ of an action of a unital inverse semigroup $S$ on a locally compact Hausdorff space $X$ by partial homeomorphisms $\alpha_t : D_t \to D_t$ is Hausdorff if and only if, for each $t \in S$, the (open) set $D_{1,t} := \bigcup_{e \leq 1,t} D_e$ is closed in $D_t$.

**Proof.** As discussed in Section 5 in the present situation the ideal $I_{1,t}$ corresponds to the open set $D_{1,t}$ defined in the statement. This open set is closed in $D_t$ if and only if the ideal $I_{1,t} = C_0(D_{1,t})$ is complemented in $s(\mathcal{H}) = C_0(D_t)$. The result follows from Propositions 6.3 and 5.1.

Next we reformulate the condition in Proposition 6.3 using the transformation groupoid $G := \text{Prim}(A) \rtimes S$. We may build this as usual for an inverse semigroup action, even if $\text{Prim}(A)$ is not Hausdorff. Its object space is $\text{Prim}(A)$, and it is étale, that is, the range and source maps are local homeomorphisms. Arrows are equivalence classes of pairs $(t, p)$ for $p \in D_t \subseteq \text{Prim}(A)$, where $D_t$ corresponds to the ideal $s(\mathcal{H}_t)$ as above. Two pairs $(t, p)$ and $(t', p')$ are equivalent if $p = p'$ and there is $v \in S$ with $v \leq t, t'$ and $p \in D_v$. There is a unique topology on $(\text{Prim}(A) \times S)^\dagger$ for which $[t, p] \mapsto p$ is a homeomorphism onto $D_v$ for each $t \in S$. The subsets $U_t := \{[t, p] | p \in D_t\}$ form an open covering of $\text{Prim}(A) \rtimes S$ by bisections.

**Theorem 6.5.** The conditional expectation $E$ maps $A \rtimes_t S$ onto $A$ if and only if the subset of units $\text{Prim}(A)$ is closed in the arrow space $\text{Prim}(A) \rtimes S$.

**Proof.** Let $\alpha_t : D_t \to D_t$ be the partial homeomorphisms on $\text{Prim}(A)$ that describe the action of $S$. Let $D_{1,t} := \bigcup_{e \leq 1,t} D_e$ as in Lemma 6.2. For $p \in \text{Prim}(A)$ and $t \in S$, we have $[t, p] = [1, p]$ if and only if $p \in D_{1,t}$.

The ideal $I_{1,t}$ is complemented in $s(\mathcal{H}_t)$ if and only if $D_t = D_{1,t} \cup D_{1,t}^\perp$ for some open subset $D_{1,t}^\perp$ in $\text{Prim}(A)$, namely, the open subset corresponding to the complement of $I_{1,t}$. Of course, $D_{1,t} \cup D_{1,t}^\perp = D_t \setminus I_{1,t}$, so such a decomposition exists if and only if $D_t \setminus I_{1,t}$ is open; equivalently, $D_{1,t}$ is relatively closed in $D_t$. Thus the criterion for $E(A \rtimes_t S) \subseteq A$ in Proposition 6.3 is equivalent to $D_{1,t}$ being relatively closed in $D_t$ for each $t \in S$. The open subsets $D_t \subseteq (\text{Prim}(A) \rtimes S)^\dagger$ form an open covering, and $D_{1,t} = D_t \cap \text{Prim}(A)$. Hence $D_{1,t}$ is relatively closed in $D_t$ for each $t \in S$ if and only if the subset of units $\text{Prim}(A)$ is closed in $(\text{Prim}(A) \rtimes S)^\dagger$.

The theorem above is related to [3, Corollary 4.4].
The existence of a conditional expectation \( A \rtimes_s S \to A \) should be viewed as an analogue for inverse semigroup crossed products of Hausdorffness for groupoid crossed products. By Lemma \[5.2\], a groupoid with Hausdorff object space has Hausdorff arrow space if and only if the set of units is closed. Theorem \[5.5\] involves the same condition for the groupoid \( \text{Prim}(A) \rtimes_s S \), which may have a non-Hausdorff object space. Thus the condition of “having a closed set of units” captures the good features of Hausdorff groupoids in the context of inverse semigroup actions. The next result shows that this property behaves well with respect to equivariant maps.

**Lemma 6.6.** Let \( X \) and \( Y \) be topological spaces with \( S \)-actions by partial homeomorphisms and let \( f : X \to Y \) be an \( S \)-equivariant continuous map. If the units are closed in \( Y \rtimes S \), then the same happens in \( X \rtimes S \).

**Proof.** The map \( f \) induces a continuous functor \( f_* : X \rtimes S \to Y \rtimes S \), mapping the germ of \( t \in S \) at \( x \in X \) to the germ of \( t \) at \( f(x) \). Each \( t \in S \) gives bisections \( D_t^X \subseteq X \rtimes S \) and \( D_t^Y \subseteq Y \rtimes S \). By construction, \( D_t^X = f^{-1}(D_t^Y) \). These bisections give an open covering of the respective arrow spaces. So the unit bisection \( D_1 \) is closed if and only if \( D_1 \cap D_t \) is relatively closed in \( D_t \) for each \( t \in S \). If this holds in \( Y \), then \( D_t^X \cap D_t^Y = f^{-1}(D_t^Y \cap D_t^Y) \) is also relatively closed in \( f^{-1}(D_t^Y) = D_t^X \). \( \square \)

**Example 6.7.** Let \( G \) be a Hausdorff groupoid and let \( S \) be a wide inverse semigroup of bisections of \( G \), so that \( G \cong G^0 \rtimes S \). Let \( A \) be a \( C^* \)-algebra with an action of \( G \) by \( C^* \)-correspondences; that is, \( A \) is the space of \( C_0 \)-sections on \( G^0 \) of a Fell bundle over \( G \). Turn this Fell bundle over \( G \) into an action of \( S \) with an \( S \)-equivariant continuous map \( \text{Prim}(A) \to G^0 \) as in \[7, Theorem 6.13\]. Lemma \[5.2\] shows that the units in \( G = G^0 \rtimes S \) are closed. Hence the units in \( \text{Prim}(A) \rtimes S \) are closed by Lemma \[5.9\]. Thus our conditional expectation \( E \) maps \( A \rtimes G \cong A \rtimes S \) to \( A \). We may also construct the conditional expectation \( A \rtimes G \to A \) directly.

**Example 6.8.** Paterson \[25\] associates a certain locally compact, totally disconnected, possibly non-Hausdorff groupoid \( GP(S) \) to any inverse semigroup \( S \). This is the transformation groupoid for the canonical action of \( S \) on the spectrum of the semilattice \( E = E(S) \) endowed with the totally disconnected Hausdorff topology from the product space \( \{0,1\}^E \). We denote this spectrum by \( \hat{E}_P \), so Paterson’s groupoid is \( GP(S) = \hat{E}_P \rtimes S \). This groupoid has the universal property that there is a natural bijection between actions of \( GP(S) \) on a topological space \( X \) and actions of \( S \) on \( X \) by partial homeomorphisms with clopen domains and codomains (compare \[34, Proposition 5.5\] and \[25, Section 4.3\]). The universal property of \( GP(S) \) follows from that of the universal groupoid \( G(S) := \hat{E} \rtimes S \) constructed in \[6\]. Here \( \hat{E} \) is also the spectrum of \( E(S) \), that is, it is equal to \( \hat{E}_P \) as a set, but it carries a different, non-Hausdorff, topology, which will be explained below in the proof of Proposition \[6.10\]. The space \( \hat{E} \) has the universal property that continuous maps from a topological space \( X \) to \( \hat{E} \) correspond bijectively to semilattice maps (preserving zero and unit) from \( E \) to the lattice of open subsets of \( X \). The map \( X \to \hat{E} \) is continuous as a map to \( \hat{E}_P \) if and only if \( E \) maps into the sublattice of clopen subsets of \( X \).

Call a Hilbert bimodule \( \mathcal{H} \) over a \( C^* \)-algebra \( A \) **complemented** if the ideals \( s(\mathcal{H}) \) and \( r(\mathcal{H}) \) are complemented ideals, that is, \( A \cong s(\mathcal{H}) \oplus I_1 \) and \( A \cong r(\mathcal{H}) \oplus I_2 \) for certain ideals \( I_1, I_2 \subset A \), which are automatically unique. Equivalently, the domain and codomain of the partial homeomorphism of \( \text{Prim}(A) \) associated to \( \mathcal{H} \) are clopen.

Let \( S \) be a unital inverse semigroup, \( A \) a \( C^* \)-algebra, and let \( (\mathcal{H}_t)_{t \in S} \) be an action by complemented Hilbert bimodules. Then the induced action of \( S \) on \( \text{Prim}(A) \) has clopen domains and codomains by assumption. Thus the complemented actions of \( S \) are in bijection with Fell bundles over Paterson’s groupoid \( GP(S) \).
As a consequence, Paterson’s groupoid \( G_P(S) \) is Hausdorff if and only if the conditional expectation on \( A \rtimes S \) takes values in \( A \) for any complemented action, see Example 6.7. Steinberg [14] and Paterson (see [25, Corollary 4.3.1]) characterise when Paterson’s groupoid is Hausdorff: this happens if and only if for all \( t,u \in S \), the set \( \{ v \in S \mid v \leq t,u \} \) is finitely generated as an ordered set, that is, there is a finite set \( F \subseteq S \) such that

\[
\{ v \in S \mid v \leq t,u \} = \{ v \in S \mid v \leq w \text{ for some } w \in F \}.
\]

Since \( G_P(S) \) is a transformation groupoid \( X \rtimes S \), Corollary 6.4 characterises when \( G_P(S) \) is Hausdorff; in this form, this appears in [25, Proposition 4.3.6].

**Example 6.9.** The tight groupoid \( G_{\text{tight}}(S) \) of an inverse semigroup \( S \) is the restriction of Paterson’s groupoid to a certain closed, invariant subset (see [12]). Call an inverse semigroup action on a \( C^* \)-algebra tight if it comes from an action of the tight groupoid (see [7, Theorem 6.13]). Our results show that the tight groupoid of \( S \) is Hausdorff if and only if the weak conditional expectation on \( A \rtimes S \) takes values in \( A \) for each tight action of \( S \) on a \( C^* \)-algebra. Exel and Pardo characterise when the tight groupoid is Hausdorff in [14, Theorem 3.16].

Paterson’s groupoid and the tight groupoid of an inverse semigroup can only account for complemented \( S \)-actions on \( C^* \)-algebras because they have Hausdorff object space. The universal \( S \)-action constructed in [6], which takes place on a certain non-Hausdorff space \( \hat{E} \), allows us to get rid of the assumption on complements; we recall its definition during the proof of the following proposition.

**Proposition 6.10.** Let \( S \) be an inverse semigroup with zero and unit. The following are equivalent:

1. \( S \) is \( E^* \)-unitary: if \( e,t \in S \) satisfy \( e^2 = e \) and \( e \leq t \), then \( e = 0 \) or \( t^2 = t \);
2. if \( e,t \in S \) satisfy \( e \leq 1,t \), then \( e = 0 \) or \( t \leq 1 \);
3. the space of units is closed in the transformation groupoid \( \hat{E} \rtimes S \) for the universal \( S \)-action;
4. the space of units is closed in the transformation groupoid for any zero-preserving \( S \)-action by partial homeomorphisms;
5. the weak conditional expectation \( E \) takes values in \( A \) for any zero-preserving action of \( S \) on a \( C^* \)-algebra by Hilbert bimodules.

In particular, if \( S \) is \( E^* \)-unitary, then the transformation groupoid \( X \rtimes S \) is Hausdorff for any zero-preserving action of \( S \) on a Hausdorff space \( X \).

**Proof.** \((1) \iff (2)\) holds because an element \( e \in S \) of a unital inverse semigroup satisfies \( e^* = e \) if and only \( e \leq 1 \).

Our next goal is to prove \((2) \iff (3)\). First we recall the definition of \( \hat{E} \) and the \( S \)-action on it, see [6]. Elements of \( \hat{E} \) are the characters of \( E = \{ e \in S \mid e^2 = e \} \), that is, functions \( \varphi : E \to [0,1] \) with \( \varphi(ef) = \varphi(e)\varphi(f) \), \( \varphi(1) = 1 \) and \( \varphi(0) = 0 \). The topology is generated by the open subsets

\[
U_e := \{ \varphi \in \hat{E} \mid \varphi(e) = 1 \}
\]

for \( e \in E \). If \( \varphi(e) = \varphi(f) = 1 \) for \( e,f \in E \), then \( \varphi(ef) = 1 \), and if \( \varphi(e) = 0 \) or \( \varphi(f) = 0 \), then \( \varphi(ef) = 0 \). Thus \( U_e \cap U_f = U_{ef} \). Hence the subsets \( U_e \) even form a basis of the topology, and any open subset is the union of the subsets of the form \( U_e \) that it contains. The map sending an open subset \( V \) of \( \hat{E} \) to the set of all \( e \in E \) with \( U_e \subseteq V \) is an isomorphism from the lattice of open subsets in \( \hat{E} \) to the lattice of ideals in \( E \) by [6, Lemma 2.14]; an ideal in \( E \) is a subset \( I \) with \( 0 \in I \) and such that \( e \leq f \) and \( f \in I \) implies \( e \in I \).
The element \( t \in S \) acts on \( \hat{E} \) by the homeomorphism
\[
c_t : U_t \ast \to U_{t^*}, \quad c_t(\varphi)(e) = \varphi(t^* e);
\]
this defines a zero-preserving action of \( S \) on \( \hat{E} \) by partial homeomorphisms, and it is the universal such action on a topological space by [6, Theorem 2.22].

The arrows in \( \hat{E} \times S \) are equivalence classes of pairs \( (t, \varphi) \) with \( t \in S, \varphi \in U_{t^*} \), where \( (t, \varphi) \sim (t', \varphi') \) if \( \varphi = \varphi' \) and there is \( e \in E \) with \( \varphi(e) = \varphi'(e) = 1 \) and \( te = t' e \). The topology is such that the projection map \( [t, \varphi] \mapsto \varphi \) is a local homeomorphism. The subsets \([t] := \{ [t, \varphi] \mid \varphi \in U_{t^*} \}\) form an open cover \( \hat{E} \times S \), and \([t] \) is homeomorphic to \( U_{t^*} \).

By definition, \( [1] \) is the set of units, and \([t] \cap [1] \) is the subset of all \( [t, \varphi] = [1, \varphi] \) with \( \varphi \in \bigcup_{e \leq 1, t} U_e \). Let \( L_t := \{ e \in E \mid e \leq t \} \) and let
\[
L_t^\perp := \{ f \in E \mid f \leq t^* t \text{ and } ef = 0 \text{ for all } e \in L_t \}.
\]
An open subset \( U_f \) is contained in \([t] \setminus [1] \) if and only if \( f \leq t^* t \) and \( U_e \cap U_f = \emptyset \) for all \( e \in L_t \). Since \( U_c \cap U_f = U_{c^*} \), which is only empty if \( ef = 0 \), the open subset \( U_f \) is contained in \([t] \setminus [1] \) if and only if \( f \in L_t^\perp \). The subset \([1] \subseteq \hat{E} \times S \) is closed if and only if \([t] \cap [1] \) is relatively closed in \([t] \) for each \( t \in S \), and if and only if \([t] \setminus [1] \) is open for each \( t \in S \). Since the subsets \( U_f \) form a basis, this happens if and only if \([t] \setminus [1] \) is the union of the subsets \( U_t \) it contains. Thus the units are closed in \( \hat{E} \times S \) if and only if \( \bigcup_{t \in S, t \geq 1} U_e = U_{t^*} \). By [6, Lemma 2.14], this only happens if \( tt^* \in L_t \cup L_t^\perp \). We have \( tt^* \in L_t \) if and only if \( t \leq 1 \), and \( tt^* \in L_t^\perp \) if and only if \( e \leq 1, t \) only for \( e = 0 \). Thus \((2) \iff (3) \)
\[(3) \implies (4) \]
follows from Lemma 6.6 and the universal property of the action of \( S \) on \( E \), see [6, Theorem 2.22]. \((4) \iff (5) \) follows from Theorem 6.5 and Example 6.11 shows an action of \( S \) on a \( \mathbb{C}^\ast \)-algebra \( A \) such that the induced action on \( \text{Prim}(A) \) is the universal action on \( \hat{E} \). Then Theorem 6.5 shows \((5) \iff (3) \) which finishes the proof of the proposition.

Example 6.11. We construct an action of \( S \) by Hilbert bimodules on a graph \( \mathbb{C}^\ast \)-algebra, using their well-understood ideal structure, see [2] or [27]. Our graph has vertex set \( E^* := E \setminus \{0\} \). If \( e, f \in E^* \) satisfy \( e \geq f \), then we put countably many edges \( e \to f \); otherwise there is no edge \( e \to f \). Let \( A(E) \) be the resulting graph \( \mathbb{C}^\ast \)-algebra. Since any vertex receives infinitely many edges \( e \to c \), any subset of the vertex set is “saturated.” Thus the lattice of ideals in \( A(E) \) is isomorphic to the lattice of “hereditary” subsets in \( E^* \) by the main result of [2], see also [11]. A subset \( U \) of \( E^* \) is hereditary if \( e \geq f \) and \( e \in U \) implies \( f \in U \). This means that \( \{0\} \cup U \) is an ideal in \( E \), and these ideals in \( E \) correspond to open subsets of \( \hat{E} \) by [6, Lemma 2.14]. Thus \( \text{Prim}(A(E)) \) and \( \hat{E} \) have isomorphic lattices of open subsets. This implies that they are homeomorphic because both are sober spaces. (Any ideal in \( E \) contains \( 0 \) by convention; this is why we left out \( 0 \in E \) to construct \( A(\hat{E}) \).)

We must still lift the action of \( S \) on \( \hat{E} \) to an action on \( A(E) \). If \( I \subseteq E \) is an ideal, then the corresponding ideal in \( A(E) \) is Morita–Rieffel equivalent to the graph \( \mathbb{C}^\ast \)-algebra of the restriction of the graph above to the vertex set \( I^* = I \setminus \{0\} \subseteq E^* \). If \( t \in S \), then \( e \mapsto te \) maps the subsemilattice \( E_{\leq t^*} \subseteq E \) isomorphically onto \( E_{\leq t^*} \) with inverse \( e \mapsto t^* e \). This is a semilattice isomorphism, that is, it preserves the order relation \( \leq \) and the zero elements. Thus it induces a graph isomorphism between the restrictions of our graphs to \( E_{\leq t^*} \) and \( E_{\leq t^*} \) and thus an isomorphism between the associated graph \( \mathbb{C}^\ast \)-algebras. These are canonically Morita–Rieffel equivalent to the ideals in the graph \( \mathbb{C}^\ast \)-algebra \( A(E) \) corresponding to the ideals \( E_{\leq t^*} \) and \( E_{\leq t^*} \). Hence \( t \) induces a canonical Morita–Rieffel equivalence between these two ideals. This gives an action of \( S \) by Hilbert bimodules on the graph
C*-algebra $A(E)$ that induces the desired action on the open subsets of $\text{Prim}(A)$ and hence on $\text{Prim}(A) \cong \hat{E}$.

An inverse semigroup $S$ is called $E$-unitary if for all $e,t \in S$, the condition $e^2 = e \leq t$ implies $t = t^2$. The inverse semigroup $S$ is $E$-unitary if and only if $S_0 := S \cup \{0\}$ (S with a formal zero added) is $E^*$-unitary. Actions of $S$ on a C*-algebra correspond bijectively to zero-preserving actions of $S_0$, and this correspondence preserves crossed products and weak conditional expectations. So Proposition 6.10 gives the following for $E$-unitary inverse semigroups:

**Corollary 6.12.** Let $S$ be an inverse semigroup with unit, but possibly without zero. The weak conditional expectation has values in $A$ for all actions of $S$ on C*-algebras $A$ if and only if $S$ is $E$-unitary, if and only if the transformation groupoid $X \rtimes S$ has closed units for any action of $S$ on a topological space $X$.

Therefore, if $S$ is $E$-unitary, then $X \rtimes S$ is Hausdorff for any action of $S$ on a Hausdorff space $X$.

7. **Faithful representations of the reduced crossed product**

Any representation $\pi: A \to \mathcal{B}(\mathcal{H})$ extends uniquely to a weakly continuous representation of $A''$, which we may then restrict to a representation $\tilde{\pi}$ of $\hat{A}$. When is $\tilde{\pi}$ faithful? In view of Example 6.1, we only aim for a sufficient condition. Our starting point is Lemma 4.5, the image of the subspace $\mathcal{H}_t \subseteq A \times S$ under the weak conditional expectation $E: A \rtimes S \to A''$ is contained in the multiplier algebra $\mathcal{M}(I_{1,t})$; here we embed $\mathcal{M}(I_{1,t}) \subseteq \mathcal{I}'_{1,t} \subseteq A''$ as before. Hence $\hat{A}$ is contained in the C*-subalgebra of $A''$ generated by $\mathcal{M}(I_{1,t}) \subseteq A''$ for all $t \in S$. Taking even more generators, we let $\mathcal{I}$ be the lattice of ideals generated by $I_{1,t}$ for $t \in S$, that is, we add finite intersections and unions of ideals.

Recall that $[I]$ in $A''$ for an ideal $I \triangleleft A$ denotes the support projection of $I$.

**Lemma 7.1.** Let $I,J$ be ideals of $A$. Then $[I] \cdot [J] = [I \cap J]$ and $[I] \lor [J] = [I + J]$. That is, $I \mapsto [I]$ is a lattice map. In particular, $[I] + [J] = [I + J] + [I \cap J]$.

**Proof.** The supremum $[I] \lor [J]$ and $[I + J]$ act in any representation $\pi: A \to \mathcal{B}(\mathcal{H})$ by the orthogonal projection onto $\pi(I)\mathcal{H} + \pi(J)\mathcal{H} = \pi(I + J)\mathcal{H}$ (see also Lemma 6.2), hence they are equal in $A''$. The assertion $[I] \cdot [J] = [I \cap J]$ is equivalent to $\pi(I)\mathcal{H} \cap \pi(J)\mathcal{H} = \pi(I \cap J)\mathcal{H}$ for any representation $\pi$ of $A$. The inclusion $\subseteq$ is obvious, and $\subseteq$ follows because both $\pi(I)$ and $\pi(J)$ act nondegenerately on $\pi(I)\mathcal{H} \cap \pi(J)\mathcal{H}$, giving $\pi(I)\mathcal{H} \cap \pi(J)\mathcal{H} \subseteq \pi(I)\pi(J)\mathcal{H} = \pi(I \cap J)\mathcal{H}$. \(\square\)

**Proposition 7.2.** Let $\mathcal{I}$ be a lattice of ideals in a C*-algebra $A$ and let $A_{\mathcal{I}} \subseteq A''$ be the C*-subalgebra generated by $\mathcal{M}(I)$ for all $I \in \mathcal{I}$. Let $\pi: A \to \mathcal{B}(\mathcal{H})$ be a representation. The restriction of $\pi''$ to $A_{\mathcal{I}}$ is faithful if $a \in I$ whenever $I,J \in \mathcal{I}$ and $a \in J$ satisfy $I \subseteq J$ and $\pi(a)\pi(J)\mathcal{H} \subseteq \pi(I)\mathcal{H}$.

**Proof.** The C*-algebra $A_{\mathcal{I}}$ is the inductive limit of the subalgebras $A_{\mathcal{F}}$ for finite sublattices $\mathcal{F} \subseteq \mathcal{I}$. Hence a representation on $A_{\mathcal{I}}$ is faithful if and only if it is faithful on $A_{\mathcal{F}}$ for each finite sublattice $\mathcal{F}$. Hence we may assume without loss of generality that the lattice $\mathcal{I}$ is finite. We do this from now on.

For $J \in \mathcal{I}$, let $J^\circ = \sum_{I \in \mathcal{I}, I \subseteq J} I$. Call $J$ irreducible if $J \neq J^\circ$, that is, $J$ is not a sum of strictly smaller ideals in $\mathcal{I}$. We claim that the summands $\mathcal{M}(I)$ for irreducible $I$ already generate $A_{\mathcal{I}}$. If $I \subseteq J$ in $\mathcal{I}$, then $I \in \mathcal{M}(J)$, which gives a unital *-homomorphism $\rho_{I,J}: \mathcal{M}(J) \to \mathcal{M}(I)$ such that $\rho_{I,J}(x) = x \cdot [I]$ in $A''$ for all $x \in \mathcal{M}(J)$. If $J = I_1 + I_2$, then we rewrite $x \in \mathcal{M}(J)$ using Lemma 7.1:

$$x = x[I_1] + x[I_2] - x[I_1 \cap I_2] = \rho_{I_1,J}(x) + \rho_{I_2,J}(x) - \rho_{I_1 \cap I_2,J}(x).$$
Since the right hand side lies in $\mathcal{M}(I_1) + \mathcal{M}(I_2) + \mathcal{M}(I_1 \cap I_2)$, the generators $x \in \mathcal{M}(J)$ are redundant if $J = I_1 + I_2$ for $I_1, I_2 \in \mathcal{I}$ with $I_1, I_2 \neq J$. Since $\mathcal{I}$ is finite, any $J \in \mathcal{I}$ is a finite sum of irreducible ideals in $\mathcal{I}$. Repeating the above decomposition, we see that $A_\mathcal{I}$ is generated by $\mathcal{M}(J)$ for irreducible $J \in \mathcal{I}$.

If $J$ is irreducible, then $J^o < J$ is the maximal element of $\mathcal{I}$ below $J$, and we may decompose any $x \in \mathcal{M}(J)$ as $\rho_{J^o,J}(x)[J^o] + x \cdot ([J] - [J^o])$. The first term in $\mathcal{M}(J^o)$ may be further decomposed using irreducible elements of $\mathcal{I}$ contained in $J^o$. Thus we may replace the generators $x \in \mathcal{M}(J)$ by $x \cdot ([J] - [J^o])$ for irreducible $J \in \mathcal{I}$.

If $I \cap J \neq I$, then $I \cap J \leq J^o$, so that $([I] - [J^o])|J| = 0$ and hence also $([J] - [J^o])([J] - [J^o]) = 0$. By symmetry, the same happens if $I \cap J \neq J$. Hence $([J] - [J^o])([J] - [J^o]) = 0$ whenever $I \neq J$.

Thus $A_\mathcal{I}$ is the orthogonal direct sum of $\mathcal{M}(J)([J] - [J^o])$ for all irreducible $J \in \mathcal{I}$. The representation $\pi''$ is faithful on $A_\mathcal{I}$ if and only if it is faithful on each summand $\mathcal{M}(J)([J] - [J^o])$. Let $x \in \mathcal{M}(J)$ satisfy $\pi''(x[J] - x[J^o]) = 0$, that is, $\pi''(x[J]) = \pi''(x[J^o])$. Then $\pi''(x)(\pi(J)\mathcal{H}) \subseteq \pi(\mathcal{F})\mathcal{H}$ and hence $\pi(xa)(\pi(J)\mathcal{H}) \subseteq \pi(\mathcal{F})\mathcal{H}$ for each $a \in J$. Since $J, J^o \in \mathcal{I}$, the assumption in our proposition gives $xa \in J^o$ for all $a, J$. Thus $\rho(xa)(\pi(J)\mathcal{H}) \subseteq \rho(J^o)\mathcal{H}$, and hence $\pi''(x[J] = x[J^o]$. Thus $\pi''$ is faithful on the summands $\mathcal{M}(J)([J] - [J^o])$, that is, $\pi'' = \pi$ for each representation $\pi$ of $A$.

Theorem 7.3. Let $A$ be a C$^*$-algebra and let $S$ be a unital inverse semigroup acting on $A$ by Hilbert bimodules $(H_t)_{t \in S}$. Let $\mathcal{I}$ be a lattice of ideals in $S$ that contains the ideals $I_t$ for all $t \in S$. By $\pi: A \to \mathcal{B}(\mathcal{H})$ be a representation of $A$ such that, for all $I, J \in \mathcal{I}$ and $a \in J$ with $\pi(a)\pi(J)\mathcal{H} \subseteq \pi(I)\mathcal{H}$, already $a \in I$. Then $\pi$ is E-faithful, so Ind $\pi$ is a faithful representation of $A \rtimes S$.

Proof. Lemma 4.3 shows that $E(\mathcal{H}_t)$ is contained in $\mathcal{M}(I_{t,i})$ for all $t \in S$. Thus $\hat{A}$ is contained in $A_\mathcal{I}$. Proposition 7.2 shows that $\pi$ is E-faithful. Hence Ind $\pi$ is faithful by Proposition 4.10.

Theorem 7.4. The family of all irreducible representations of $A$ is E-faithful.

Proof. We apply Theorem 7.3 to the direct sum $\Pi$ of all irreducible representations and the lattice of all ideals of $A$. Let $I \subset J \subset A$ be ideals and $a \in J$. If $a \not\in I$, then there is an irreducible representation $\pi: J/I \to \mathcal{B}(\mathcal{H}_\pi)$ of $J/I$ with $\pi(a) \neq 0$. This irreducible representation extends to an irreducible representation of $A$, again denoted $\pi$. Since $\pi|_J$ is irreducible and $\pi|_I = 0$, we get $\pi(J)\mathcal{H}_\pi = \mathcal{H}_\pi$ and $\pi(I)\mathcal{H}_\pi = 0$, so $\pi(a)\pi(J)\mathcal{H}_\pi$ is not contained in $\pi(I)\mathcal{H}_\pi$ because $\pi(a) \neq 0$. Since $\pi$ is a direct summand in $\Pi$, the same happens for $\Pi$. Hence $\Pi$ verifies the assumption in Theorem 7.3.

Theorem 7.4 shows that the irreducible representations of $\hat{A}$ already give a faithful representation of $\hat{A}$ and hence of $A \rtimes S$ by induction. In $[13]$ Exel defines $A \rtimes S$ by inducing only irreducible representations of $A$. More precisely, Exel constructs a positive linear functional $\hat{\varphi}$ on $A \rtimes S$ from every pure state $\varphi$ of $A$ and defines $A \rtimes S$ as the image of $A \rtimes S$ in the direct sum of the GNS-representations of $\hat{\varphi}$ for all pure states $\varphi$. The induced functional $\hat{\varphi}: A \rtimes S \to \mathbb{C}$ is computed in $[13]$ Proposition 7.4 as follows. Let $t \in S$ and $\xi \in \mathcal{H}_t$. First assume that $\varphi$ is supported on $s(\mathcal{H}_t)$ for some $e \leq t$. This is equivalent to the existence of a linear functional $\psi \in \mathcal{I}'$ and $b \in I$ with $\varphi(a) = \psi(ab)$ for all $a \in A$. If $\varphi$ is supported on $s(\mathcal{H}_t)$ for some $e \leq t$, then $\varphi(\xi_{\delta_t}) := \lim \varphi(\theta_{t,i}(\xi_{u_{e,i}}))$, where $(u_{e,i})_{e,i} \in S$ is an approximate unit of $s(\mathcal{H}_t) = I_{t,e}$ and $\theta_{t,i}$ is the isomorphism $H_t \to H_{t,i}$ in $[24]$. If $\varphi$ is not supported on $s(\mathcal{H}_t)$ for any $e \leq t$, then $\varphi(\xi_{\delta_t}) := 0$. The
above definition of $\tilde{\varphi}$ is only reasonable if $\varphi$ is pure, and [13] only considers pure states in the definition of the reduced crossed product.

The following proposition shows that this induced state $\tilde{\varphi}$ coincides with $\varphi'' \circ E$, where $\varphi'' : A'' \to \mathbb{C}$ denotes the unique normal extension of $\varphi$. Hence the GNS-representation of $\tilde{\varphi}$ is the induced representation of the GNS-representation of $\varphi$ for any pure state $\varphi$, and so our reduced crossed product is the same one as in [13] by Theorem 7.4.

**Proposition 7.5.** Let $(\mathcal{H}_t)_{t \in S}$ be an action of $S$ on a $C^*$-algebra $A$ by Hilbert bimodules. Let $\varphi \in \mathcal{A}'$ be a bounded linear functional on $A$ and let $t \in S$, $\xi \in \mathcal{H}_t$. Let $(u_{t,i})$ be an approximate unit for the ideal $I_{1,t}$. Then

$$\varphi'' \circ E(\xi \delta_t) = \lim_{i} \varphi(\theta_{1,t}(\xi u_{t,i})), \tag{7.6}$$

If $\varphi$ is supported on $s(\mathcal{H}_e) = I_{1,e}$ for some $e \leq t$, and $(u_{e,i})$ is an approximate unit for $I_{1,e}$, then $\varphi'' \circ E(\xi \delta_t) = \lim_{i} \varphi(\theta_{1,t}(\xi u_{e,i})).$ If $\varphi$ is pure, then $\varphi'' \circ E = \tilde{\varphi}$.

**Proof.** Formula (7.6) follows from (4.6) because bounded linear functionals on $C^*$-algebras are strictly continuous. Assume that $\varphi$ is supported on one of the ideals $I_{1,t} = s(\mathcal{H}_e)$ for $e \leq t$ that generate $I_{1,t}$. Then $\varphi \in I_{1,t}, \xi$ and $b \in I_{1,e}$ with $\phi(a) = \psi(ab)$ for all $a \in A$. Then (7.6) implies

$$\varphi'' \circ E(\xi \delta_t) = \psi(\theta_{1,t}(\xi b)) = \lim_{i} \varphi(\theta_{1,t}(\xi u_{e,i})).$$

If $\varphi$ is pure, then either it is supported on an ideal or it vanishes on this ideal (see [13] Proposition 5.5). Therefore, if $\varphi$ is not supported on any of the ideals $I_{1,e}$ with $e \leq t, t$, then it vanishes on $I_{1,t} = \sum I_{1,e}$. Then $\varphi'' \circ E(\xi \delta_t) = 0$. \hfill $\square$

8. **Iterated crossed products**

We now study reduced crossed products associated to actions of inverse semigroups on groupoids. Let $G$ be a locally compact, locally Hausdorff groupoid with Haar system. Let $S$ be a unital inverse semigroup acting on $G$ by partial equivalences. Let $G \rtimes S$ be the transformation groupoid as defined in [7]. This comes with a canonical $S$-grading, that is, with open subsets $G_t \subseteq G \rtimes S$ for $t \in S$ that satisfy

$$G_t \cdot G_u = G_{tu}, \quad G_t^{-1} = G_{t^{-1}}, \quad G_t \cap G_u = \bigcup_{v \leq t,u} G_v, \quad (G \rtimes S)^1 = \bigcup_{t \in S} G_t.$$

We have $G \cong G_1$, so $G$ is an open subgroupoid of $G \rtimes S$.

For instance, if $G$ is a space, then an action of $S$ on $G$ by partial equivalences is essentially the same as an action on the space $G$ by partial homeomorphisms, and the transformation groupoid is the usual one. If $S$ is a group, then an action of $G$ by topological groupoid automorphisms is an action by equivalences, and $G \rtimes S$ is the semidirect product groupoid. More generally, any group extension $H \to L \to G$ comes from an action by equivalences with $L = H \rtimes S$. These and several other examples are explained in [7]. Recently, we have developed a similar iterated crossed product theorem for actions of possibly non-étale groupoids instead of inverse semigroup actions in [8]. In this context, we also give several more examples that come from inverse semigroup actions. In particular, we observe that actions of a discrete group $G$ on groupoids by equivalences are equivalent to “strongly surjective” $G$-valued cocycles.

Let $B$ be a Fell bundle over $G \rtimes S$. Since $G$ is a subgroupoid of $G \rtimes S$, we may restrict $B$ to a Fell bundle over $G$, which we still call $B$. Let $C^*(G \rtimes S, B)$ and $C^*(G, B)$ be the full section $C^*$-algebras of these Fell bundles. Let $C^*_r(G \rtimes S, B)$ and $C^*_r(G, B)$ be the reduced section $C^*$-algebras; these are defined as follows.

There is a canonical (“left regular”) representation $\Lambda_x$ of $C^*(G, B)$ on the Hilbert
The Disintegration Theorem, if it holds in the case at hand, implies both claims as closure of \( C_\ast \) in \( C \) these sums. Hence the reduced \( C \)-representations of restriction of a regular representation of \( G \) coming from the regular representations at all \( G \) contains the regular representation to the sum of the representations together to a \( G \)-injective, open, continuous map \( s \) on \( S \).

**Proof.** Fix \( x \in G^0 = (G \times S)^0 \) and let \( S_x = \{ t \in S \mid x \in s(G_t) \} \). Equivalently, \( t \in S_x \) if and only if the action of \( t \) on \( G^0/G \) is defined on the orbit of \( x \). Say that \( t \sim_x u \) if there is \( v \in S_x \) with \( v \leq t, u \); this is an equivalence relation on \( S_x \). We have \( t \sim_x u \) if and only if \( [t, x] = [u, x] \) in \( (G \times S)^1 \). For each equivalence class \( h \) in \( S_x/\sim \), pick a representative \( t_h \) and pick \( \eta_h \in G_{t_h} \) with \( s(\eta_h) = x \). Since \( G_{t_h} \) is a partial equivalence of \( G \), the source map induces an injective, open, continuous map \( G_{t_h} \rightarrow G^0 \). Thus the left \( G \)-action gives a \( G \)-equivariant homeomorphism \( G_{t(h)} = (G_{t(h)} x, y \rightarrow y \eta_h) \). These maps piece together to a \( G \)-equivariant homeomorphism \( (G \times S)_x \rightarrow \bigsqcup_{h \in S_x/\sim} G_{s(\eta_h)} \). Thus the restriction of the left regular representation \( \Lambda_x^{G \times S} \) to \( C_\ast(G, B) \) is unitarily equivalent to the sum of the representations \( \bigsqcup_{h \in S_x/\sim} \Lambda_x^{G_{s(\eta_h)}} \). Since \( 1 \in S_x \), this direct sum contains the regular representation \( \Lambda_x^{G} \).

By definition, \( C_\ast(G \times S, B) \) is the completion of \( \mathfrak{S}(G \times S, B) \) in the \( C^\ast \)-seminorm coming from the regular representations at all \( x \in G^0 \). As we just saw, the restriction of a regular representation of \( \mathfrak{S}(G \times S, B) \) to \( \mathfrak{S}(G, B) \) is a sum of regular representations of \( \mathfrak{S}(G, B) \), and every regular representation of \( \mathfrak{S}(G, B) \) occurs in these sums. Hence the reduced \( C^\ast \)-seminorm on \( \mathfrak{S}(G \times S, B) \) restricts to the reduced \( C^\ast \)-seminorm on \( \mathfrak{S}(G, B) \), and the \( \ast \)-homomorphism \( C_\ast(G, B) \rightarrow C_\ast(G \times S, B) \) is faithful. It is nondegenerate because \( \mathfrak{S}(G, B) \cdot \mathfrak{S}(G \times S, B) = \mathfrak{S}(G \times S, B) \). \( \square \)

Let \( C_\ast(G_t, B) \) be the closure of \( \mathfrak{S}(G_t, B) \) in \( C_\ast(G \times S, B) \). The space \( \mathfrak{S}(G_t, B) \) is a pre-Hilbert \( \mathfrak{S}(G, B) \)-\( \mathfrak{S}(G, B) \)-bimodule using the convolution and involution in \( \mathfrak{S}(G \times S, B) \). Thus the closure \( C_\ast(G_t, B) \) becomes a Hilbert bimodule over the closure of \( \mathfrak{S}(G, B) \) in \( C_\ast(G \times S, B) \), which is \( C_\ast(G, B) \) by Lemma 8.1. Moreover,

\[ C_\ast(G_t, B) \otimes_{C_\ast(G, B)} C_\ast(G_u, B) \cong C_\ast(G_{tu}, B) \]

because the convolution map

\[ \mathfrak{S}(G_t, B) \otimes \mathfrak{S}(G, B) \mathfrak{S}(G_u, B) \rightarrow \mathfrak{S}(G_{tu}, B) \]
in $\mathfrak{S}(G \times S, \mathcal{B})$ has dense image for all $t, u \in S$. Thus $C^*_t(G_t, \mathcal{B})$ for $t \in S$ is a saturated Fell bundle over $S$ with unit fibre $C^*_r(G, \mathcal{B})$. Let $C^*_r(G, \mathcal{B}) \rtimes S$ be its reduced section $C^*$-algebra. When is this $C^*$-algebra isomorphic to $C^*_r(G \times S, \mathcal{B})$?

We first attempted to prove this, until we found counterexamples. The problem is related to a possible failure of the following weak exactness property:

**Definition 8.2** (see [1] Definition 2.6). Let $G$ be a locally compact, locally Hausdorff groupoid with Haar system. Let $\mathcal{B}$ be a Fell bundle over $G$. We say that $\mathcal{B}$ is *inner exact* if, for each $G$-invariant open subset $U \subseteq G^0$ and $F := G^0 \setminus G$, the sequence

$$C^*_r(G_U, \mathcal{B}) \rightarrow C^*_r(G, \mathcal{B}) \rightarrow C^*_r(G_F, \mathcal{B})$$

is exact. We call $G$ *inner exact* if the Fell bundle $\mathcal{B}$ describing the canonical (or “trivial”) action of $G$ on $G^0$ is inner exact, that is, if the sequence

$$C^*_r(G_U) \rightarrow C^*_r(G) \rightarrow C^*_r(G_F)$$

is exact for each $G$-invariant open subset $U \subseteq G^0$.

Minimal groupoids are inner exact for all Fell bundles because they have no non-trivial $G$-invariant open subsets $U \subseteq G^0$. The analogue of the sequence (8.3) with full cross-section $C^*$-algebras of Fell bundles is always exact, at least for Hausdorff locally compact groupoids, see [19]. Since amenable groupoids always have isomorphic full and reduced Fell bundle cross-section $C^*$-algebras (see [22]), all Fell bundles over amenable, locally compact, Hausdorff groupoids (with Haar system) are inner exact.

It is easy to find examples of groupoids which are not inner exact and non-Hausdorff (see Example 8.7). Hausdorff examples are also known, but more subtle (see Example 8.10). Inner exactness of $G$ does not imply inner exactness of all Fell bundles $\mathcal{B}$. For instance, any trivial group bundle $G = X \times \Gamma$ is inner exact for the trivial Fell bundle, but not necessarily for all Fell bundles if $\Gamma$ is not exact (see Example 8.8).

### 8.1. Counterexamples

Let $G$ be a groupoid and let $\mathcal{B}$ be a Fell bundle over $G$. Assume that $\mathcal{B}$ is *not* inner exact. Hence there is an open $G$-invariant subset $U \subseteq G^0$ such that the sequence (8.3) is *not* exact. We are going to construct an action of an inverse semigroup $S$ on $G$ with $C^*_r(G, \mathcal{B}) \rtimes S = C^*_r(G \times S, \mathcal{B})$.

The inverse semigroup $S$ and its action on $G$ are embarrassingly trivial. Let $S$ be the inverse semigroup with three elements $0, 1, -1$, with usual number multiplication. Thus $S$ is the group $\mathbb{Z}/2 \cong \{1, -1\}$ with a zero element added. The same inverse semigroup is used in [7] to give an example of an action by partial Morita–Rieffel equivalences that is not equivalent to an action by partial automorphisms. We let both $1$ and $-1$ in $S$ act by the identity automorphism on $G$, and we let $0$ act by the identity on the open subgroupoid $G_U$, where $U$ witnesses the lack of inner exactness. We use the obvious multiplication isomorphisms. We describe the transformation groupoid and its $S$-grading. The transformation groupoid is the quotient of the product groupoid $G \times \mathbb{Z}/2$ by the equivalence relation that is generated by $(g, 1) \sim (g, -1)$ for $g \in G_U$. The $S$-grading is

$$G_1 = \{[g, 1] \mid g \in G^0\}, \quad G_{-1} = \{[g, -1] \mid g \in G^0\}, \quad G_0 = G_1 \cap G_{-1} \cong G_U.$$

The resulting action of $S$ on $A := C^*_r(G, \mathcal{B})$ is trivial on $\mathbb{Z}/2$, and the idempotent $0 \in S$ acts by the ideal $I := C^*_r(G_U, \mathcal{B})$ in $A$. We are going to prove:

**Lemma 8.4.** In the situation above, $C^*_r(G, \mathcal{B}) \rtimes S \ncong C^*_r(G \times S, \mathcal{B})$ whenever inner exactness fails. Explicitly,

$$C^*_r(G, \mathcal{B}) \rtimes S \cong C^*_r(G, \mathcal{B}) \oplus C^*_r(G, \mathcal{B}) \oplus C^*_r(G, \mathcal{B}) \oplus C^*_r(G_F, \mathcal{B}), \quad C^*_r(G \times S, \mathcal{B}) \cong C^*_r(G, \mathcal{B}) \oplus C^*_r(G_F, \mathcal{B}).$$
Although the proof only requires rather trivial actions of $S$, we describe the reduced crossed product $A \rtimes_r S$ for a general $S$-action on a $C^*$-algebra $A$, and we also discuss the weak conditional expectation and the $C^*$-subalgebra of $A'$ that it generates in order to illustrate our theory.

Let $S$ act on a $C^*$-algebra $A$. The idempotents $0, 1$ act by $A_0 = \text{Id}_I$ and $A_1 = \text{Id}_A$ for some ideal $I \subset A$. For theCounterexamples, it is important to choose $I \neq 0$.

The element $-1 \in S$ acts by a full Hilbert bimodule $H_{-1}$. The action also contains isomorphisms of Hilbert bimodules

$$H_{-1} \otimes_A H_{-1} \cong A, \quad I \otimes_A H_{-1} \cong I \cong H_{-1} \otimes_A I;$$

the remaining isomorphisms are canonical. The multiplication isomorphisms above have to be associative as well. Since this implies compatibility with the involution, the two isomorphisms $I \otimes_A H_{-1} \cong I$ and $H_{-1} \otimes_A I \cong I$ determine each other, so we have to specify only one of them.

The Hilbert bimodule $H_{-1}$ and the isomorphism $H_{-1} \otimes_A H_{-1} \cong A$ give a (saturated) Fell bundle over the group $\mathbb{Z}/2$. Any such Fell bundle may be turned into an ordinary action by automorphisms by a stabilisation. The quickest way to see this uses Takesaki–Takai duality. The section algebra of the Fell bundle over $\mathbb{Z}/2$ carries a dual action of the dual group (which is again $\mathbb{Z}/2$), and the crossed product for that dual action carries an action of $\mathbb{Z}/2$ by automorphisms and is $\mathbb{Z}/2$-equivariantly isomorphic to $A \otimes M_2(\mathbb{C})$ by Takesaki–Takai Duality. Hence it is no serious loss of generality to replace the full Hilbert bimodule $H_{-1}$ and the isomorphism $H_{-1} \otimes_A H_{-1} \cong A$ by an automorphism $\alpha$ of $A$ with $\alpha^2 = \text{Id}_A$.

Then $H_{-1} = A$ with the usual right Hilbert module structure and the left action through $\alpha$.

Since $H_{-1} \cong A$ both as a left and a right Hilbert module, there are canonical isomorphisms $H_{-1} \otimes_A I \cong I$ as a right Hilbert module and $I \otimes_A H_{-1} \cong I$ as a left Hilbert module; the first is of the form $a \otimes b \mapsto ab$, the second of the form $a \otimes b \mapsto \alpha(a)b$. Any other such isomorphisms are obtained from these ones by composing with unitary multipliers of $I$. Thus the isomorphisms $H_{-1} \otimes_A I \cong I$ and $I \otimes_A H_{-1} \cong I$ are of the form $a \otimes b \mapsto uab$ and $a \otimes b \mapsto u\alpha(a)b$ for unitary multipliers $u, u'$ of $I$; compatibility with the bimodule structure gives $a|_I^1 = \text{Ad}(u) = \text{Ad}(u')$.

In particular, the ideal $I$ is $\alpha$-invariant. The associativity conditions for a Fell bundle are equivalent to the conditions $u = u'$ and $u^2 = 1$, which are reasonable requests because $\text{Ad}(u^2) = \alpha^2|_I = \text{Id}_A$. Thus after some simplifications, an action of $S$ on $A$ becomes a triple $(I, \alpha, u)$ with $I \subset A$, $\alpha \in \text{Aut}(A)$, $u \in M(I)$, $\alpha^2 = \text{Id}_A$, $\alpha|_I = \text{Ad}(u)$, and $u^2 = 1$.

**Proposition 8.5.** The subset $J := \{ \delta_1 a - \delta_{-1} u a \mid a \in I \}$ in $A$ is a closed ideal and there are canonical isomorphisms

$$A \rtimes S \cong A \rtimes_r S \cong A \rtimes_{\text{alg}} S \cong (A \rtimes_{\alpha} \mathbb{Z}/2)/J.$$

**Proof.** The algebraic crossed product $A \rtimes_{\text{alg}} \mathbb{Z}/2$ consists of elements of the form $\delta_1 a + \delta_{-1} b$, $a, b \in A$, with the usual relations of a crossed product, generated by $\delta_{-1} = \delta_1$ and $\delta_{-1} a \delta_{-1} = \alpha(a)$. It is already a $C^*$-algebra because $\mathbb{Z}/2$ is finite.

The algebraic crossed product $A \rtimes_{\text{alg}} S$ consists of equivalence classes of sums of the form $\delta_1 a + \delta_{-1} b + \delta_0 c$ with $a, b \in A$, $c \in I$, where we divide out the relations $\delta_0 c \equiv \delta_1 c \equiv \delta_{-1} uc$ for all $c \in I$ because $j_{1,0}(c) = c$ and $j_{-1,0}(c) = uc$ for all $c \in I$.

In our case, $j_{1,0} = \theta_{1,0}$ and $j_{-1,0} = \theta_{-1,0}$.

Thus the obvious $\ast$-homomorphism $A \rtimes_{\text{alg}} \mathbb{Z}/2 \to A \rtimes_{\text{alg}} S$ is surjective, and $A \rtimes_{\text{alg}} S$ is the quotient of $A \rtimes_{\text{alg}} \mathbb{Z}/2$ by $J$, forcing it to be a $\ast$-ideal. The map

$$I \to A \rtimes \mathbb{Z}/2, \quad c \mapsto \frac{1}{2}(\delta_1 c - \delta_{-1} uc),$$

provides the necessary isomorphism.
is a \(-\)-homomorphism and \(J\) is its image. So \(J\) is closed and \(A \rtimes_{\text{alg}} S = (A \rtimes_{\text{alg}} \mathbb{Z}/2)/J\) is a \(C^*\)-algebra. Thus the full \(C^*\)-algebra is \(A \rtimes_{\text{alg}} S\) with its quotient norm. So is \(A \rtimes_t S\) because the map \(A \rtimes_{\text{alg}} S \to A \rtimes_t S\) is injective by Proposition 4.3.

We describe the weak conditional expectation \(E\colon A \to A''\) and \(\tilde{A} \subseteq A''\). The ideal \(I_{1,-1}\) is \(I\), and the proof of Proposition 6.3 implies
\[
E(\delta_I a + \delta_{-I} b) = a + ub[I] = a + \alpha(b)u[I] = a + \alpha(b)u
\]
for all \(a, b \in A\), where the products take place in \(A''\). The last step uses \(u^*u = uu^* = [I]\), which holds because \(u \in \mathcal{M}(I)\) is unitary. Notice that \(E|_J = 0\).

The element \(u \in A''\) is a self-adjoint partial isometry with \(u^2 = [I]\). It satisfies \(\alpha(b)uc = E(\delta_I bc) = \alpha(bc)u\) for all \(b, c \in A\), which gives \(uc = \alpha(c)u\) in \(A''\) for \(c \in A\). So elements of the form \(b[I]\) also belong to \(\tilde{A}\). The subset \(\{a(1 - [I]) + bu + c[I] \mid a, b, c \in A\}\) is a \(*\)-subalgebra of \(A''\). We will show below that it is closed; hence this is \(\tilde{A}\).

The elements of the form \(\alpha(1 - [I])\) and \(bu + c[I]\) are orthogonal, and \(\alpha(1 - [I]) = 0\) if and only if \(a = a[I]\), that is, \(a \in I\). Thus \(\{a(1 - [I]) \mid a \in I\}\) gives \(C^*\) isomorphic to \(A/I\) in \(\tilde{A}\). If \(b \in I^\perp = \{a \in A \mid a \cdot b = 0\) for all \(b \in I\), then \(bu = 0\) and \(b[I] = 0\), and vice versa. Hence there is a well-defined map
\[
\varphi\colon A/I^\perp \times_{\alpha} \mathbb{Z}/2 \to A'', \quad [b]_{\delta_I} + [c]_{\delta_{-I}} \mapsto bu + c[I].
\]
This is a \(*\)-homomorphism because \(u^2 = [I]\) and \(ub = \alpha(b)u\) for all \(b \in A\). Hence its image is closed. This image together with the other summand \(A/I\) is \(\tilde{A}\). The \(*\)-homomorphism \(\varphi\) may fail to be injective, for instance, if \(u = [I]\). We do not describe the kernel of \(\varphi\).

After this illustration of our general theory, we return to the proof of Lemma 8.4. Here \(u = 1\) and \(\alpha = \text{Id}\), that is, the only non-trivial aspect of the action is the ideal \(I\). The Fourier isomorphism \(C^*(\mathbb{Z}/2) \cong C^2\) induces an isomorphism
\[
A \rtimes_{\alpha} \mathbb{Z}/2 = A \otimes C^*(\mathbb{Z}/2) \cong A \oplus A, \quad \delta_I a + \delta_{-I} b \mapsto (a + b, a - b).
\]
This isomorphism maps the ideal \(J \subseteq A \rtimes_{\alpha} \mathbb{Z}/2\) to \(0 \oplus I\). Therefore,
\[
(8.6) \quad A \rtimes S = A \rtimes_t S = A \rtimes_{\text{alg}} S \cong A \oplus (A/I)
\]
in this “trivial” special case. This proves the assertion about \(C^*_r(\mathcal{G}, \mathcal{B}) \rtimes_t S\) in Lemma 8.4.

Now we turn to the reduced groupoid \(C^*\)-algebra of \(G \rtimes S\). We use the surjection
\[
\mathfrak{S}(\mathcal{G}, \mathcal{B}) \oplus \mathfrak{S}(\mathcal{G}, \mathcal{B}) \cong \mathfrak{S}(\mathcal{G}, \mathcal{B}) \otimes \mathbb{C}[\mathbb{Z}/2] \cong \mathfrak{S}(G \rtimes \mathbb{Z}/2, \mathcal{B}) \to \mathfrak{S}(G \rtimes S, \mathcal{B}),
\]
where the first isomorphism is the inverse Fourier isomorphism for \(\mathbb{Z}/2\) and the last map is the obvious one, \(a\delta_1 + b\delta_{-1} \mapsto a\delta_1 + b\delta_{-1}\). Thus \(C^*_r(G \rtimes \mathbb{Z}/2)\) is isomorphic to the completion of \(\mathfrak{S}(\mathcal{G}, \mathcal{B}) \oplus \mathfrak{S}(\mathcal{G}, \mathcal{B})\) for the family of regular representations of \(G \rtimes S\), viewed as representations of \(\mathfrak{S}(\mathcal{G}, \mathcal{B}) \oplus \mathfrak{S}(\mathcal{G}, \mathcal{B})\) through the surjection above. If \(x \in U\), then \((G \rtimes S)_x = G_x\). Hence the resulting regular representation is the standard regular representation for \(x\) on one summand \(\mathfrak{S}(\mathcal{G}, \mathcal{B})\) (corresponding to the trivial character on \(\mathbb{Z}/2\)) and kills the other summand. If \(x \in F\), then \((G \rtimes S)_x = G_x \rtimes \mathbb{Z}/2\), and \(\mathbb{Z}/2\) acts freely on the second factor. Hence the resulting regular representation is the standard regular representation for \(x\) on both summands \(\mathfrak{S}(\mathcal{G}, \mathcal{B})\). Thus \(C^*_r(G \rtimes S, \mathcal{B}) \cong C^*_r(\mathcal{G}, \mathcal{B}) \oplus C^*_r(G, \mathcal{B})\). This finishes the proof of Lemma 8.4.

We still have to exhibit examples where inner exactness fails. Here we may use counterexamples produced for other purposes in the literature. The first example comes from [20].
**Example 8.7.** For a discrete group \( \Gamma \), let \( G^\Gamma \) be the (étale) group bundle over \([0, 1]\) with fibres \( G^\Gamma(1) = \Gamma \) and the trivial fibre at \( x \neq 1 \); this is the quotient of the group bundle \([0, 1] \times \Gamma \) by the relation \( (x, \gamma) \sim (x, \gamma') \) for all \( x \in [0, 1], \gamma, \gamma' \in \Gamma \).

We assume that \( \Gamma \) is not amenable, say, a free group. Then

\[
C^*_r(G^\Gamma) \cong C[0, 1] \oplus C^*_r(\Gamma),
\]

see [20], p. 53]. Let \( U = [0, 1) \) and \( F = \{1\} \). Then \( G^\Gamma_U \cong [0, 1) \) and \( G^\Gamma_F = \Gamma \), so \( C^*_r(G^\Gamma_U) \cong C_0([0, 1)) \) and \( C^*_r(G^\Gamma_F) \cong C^*_r(\Gamma) \). Hence \( C^*_r(G^\Gamma) / C^*_r(G^\Gamma_U) \cong C \oplus C^*_r(\Gamma) \neq C^*_r(G^\Gamma_U) \), so inner exactness fails here. Hence the “trivial” action of \( \Gamma \) given by \( U \) satisfies \( C^*_r(G^\Gamma \times S) \neq C^*_r(G^\Gamma) \times_r S \) by Lemma 8.4.

In the above example, the groupoid \( G \) on which \( S \) acts is non-Hausdorff. Now we give counterexamples where \( G \) is a Hausdorff (étale) groupoid, but we also put a rather trivial Fell bundle \( B \) on \( G \).

**Example 8.8.** Let \( \Gamma \) be a non-exact group. Such groups exist by an argument of Gromov (see [15, 16]). The \( C^* \)-algebra \( C^*_r(\Gamma) \) is not exact, that is, there is a \( C^* \)-algebra \( A \) and an ideal \( I \subseteq A \) such that the sequence

\[
I \otimes C^*_r(\Gamma) \to A \otimes C^*_r(\Gamma) \to (A/I) \otimes C^*_r(\Gamma)
\]

is not exact (see [22, Theorem 5.2]). Here and throughout, \( \otimes \) denotes the minimal \( C^* \)-algebra tensor product. Kirchberg and Wassermann [21] show that \( A \) may be chosen to be the section \( C^* \)-algebra of a continuous field \((A_x)_{x \in X}\) over the one-point compactification \( X := N \cup \{\infty\} \) and \( I = C_0(N) \cdot A \). Then \( A/I = A_\infty \) is the fibre at \( \infty \in N \). Let \( G = X \times \Gamma \) be the trivial group bundle over \( X \) with fibre \( \Gamma \) everywhere. Let \( \Gamma \) act trivially on \( A \) and form the Fell bundle \( B \) over \( G \) with fibres \((A_x)_{x \in X}\) from this trivial action. Then inner exactness of \( B \) fails for the open subset \( N \subseteq X \) by construction. Since commutative \( C^* \)-algebras are nuclear, the sequence

\[
C_0(V) \otimes C^*_r(\Gamma) \to C(X) \otimes C^*_r(\Gamma) \to C(X \setminus V) \otimes C^*_r(\Gamma)
\]

is exact for any open subset \( V \subseteq X \). So our groupoid \( G \) is inner exact but a certain Fell bundle \( B \) is not inner exact.

**Example 8.10.** This example is used in [17] as a counterexample for the Baum–Connes conjecture for groupoids. It has also been used recently by Willett [35] as examples of Hausdorff, étale, non-amenable groupoids \( G \) with \( C^*(G) \cong C^*_r(G) \).

Let \( \Gamma \) be a countable discrete group and let \( \Gamma_n \subseteq \Gamma \) be a sequence of normal subgroups of finite index. Let \( F_n := \Gamma / \Gamma_n \) be the quotient groups and let \( \pi_n : \Gamma \to F_n \) be the quotient homomorphisms. Let \( \overline{N} := N \cup \{\infty\} \) be the one-point compactification of \( N \) and let \( G \) be the quotient of \( \overline{N} \times \Gamma \) by the equivalence relation:

\[
(n, g) \sim (m, h) \iff m = n \in N \quad \text{and} \quad \pi_n(g) = \pi_n(h).
\]

We write \([n, g]\) for the equivalence class of \((n, g)\). The resulting étale group bundle \( G \) has fibres \( G_n \cong F_n \) for \( n \in N \) and \( G_\infty \cong \Gamma \). It is Hausdorff if and only if \( \bigcap \Gamma_n = \{1\} \).

If \( \Gamma_x = \Gamma \) for all \( n \), then \( G \) is a (non-Hausdorff) group bundle of the same type as in Example 8.7. However, we want the groupoid \( G \) to be Hausdorff. So we assume that \( \Gamma \) is residually finite and choose the subgroups above with \( \bigcap \Gamma_n = \{1\} \); for instance, \( \Gamma \) may be a free group or \( \text{SL}_n(\mathbb{Z}) \). If \( \Gamma \) is not amenable, then \( G \) is not inner exact because the sequence

\[
C^*_r(G_\mathbb{N}) \to C^*_r(G) \to C^*_r(\Gamma)
\]

is not exact in the middle, not even at the level of \( K \)-theory, see [17].
8.2. Isomorphism of reduced iterated crossed products. Our counterexamples show that there is no isomorphism \( C^*_r(G, \mathcal{B}) \rtimes \iota S \cong C^*_r(G \rtimes S, \mathcal{B}) \) in general. There are, however, many cases where this isomorphism holds. A first result of this nature is proved in [9] for groupoid fibrations of Hausdorff étale groupoids with amenable kernel.

**Theorem 8.11.** Let \( G \) be a locally Hausdorff, locally compact groupoid with Haar system. Let \( S \) be an inverse semigroup acting on \( G \) by partial equivalences. Let \( \mathcal{B} \) be a Fell bundle over the transformation groupoid \( G \rtimes S \) that is inner exact over \( G \). Then \( C^*_r(G, \mathcal{B}) \rtimes \iota S \cong C^*_r(G \rtimes S, \mathcal{B}) \).

**Proof.** First we show that the regular representation \( \Lambda = \bigoplus_{x \in G^0} \Lambda_x \) of \( C^*_r(G, \mathcal{B}) \) on \( L^2(G^x) \) is \( E \)-faithful.

Let \( \mathcal{I} \) be the lattice of ideals of \( C^*_r(G, \mathcal{B}) \) of the form \( C^*_r(G_U, \mathcal{B}) \) for open, \( G \)-invariant subsets \( U \subseteq G^0 \); this is a lattice because \( C^*_r(G_U, \mathcal{B}) \cap C^*_r(G_V, \mathcal{B}) = C^*_r(G_{U \cap V}, \mathcal{B}) \) and \( C^*_r(G_U, \mathcal{B}) + C^*_r(G_V, \mathcal{B}) = C^*_r(G_{U \cup V}, \mathcal{B}) \). The ideals \( I_{1,2} \) in the construction of the conditional expectation \( E \) belong to \( \mathcal{I} \). Thus we may use the criterion in Theorem [7,3] if \( U \subseteq G^0 \) is open and \( G \)-invariant, then

\[
C^*_r(G_U, \mathcal{B}) \cdot \Lambda = \bigoplus_{x \in U} \Lambda_x.
\]

Let \( U \subseteq V \subseteq G^0 \) be open, \( G \)-invariant subsets. Then \( a \in C^*_r(G_V, \mathcal{B}) \) satisfies \( \Lambda(a) \Lambda(C^*_r(G_V, \mathcal{B})) \subseteq \Lambda(C^*_r(G_U, \mathcal{B})) \) if and only if \( \Lambda_x(a) = 0 \) for all \( x \in V \setminus U \). This means that \( a \) is mapped to 0 in \( C^*_r(G_{V \setminus U}, \mathcal{B}) \). Since \( \mathcal{B} \) is inner exact, this implies \( a \in C^*_r(G_U, \mathcal{B}) \). Thus \( \Lambda \) verifies the criterion in Theorem [7,3] and is \( E \)-faithful. Now the following lemma finishes the proof.

**Lemma 8.12.** If the family of regular representations of \( C^*_r(G, \mathcal{B}) \) is \( E \)-faithful, then \( C^*_r(G, \mathcal{B}) \rtimes \iota S \cong C^*_r(G \rtimes S, \mathcal{B}) \).

**Proof.** The isomorphism \( C^*(G, \mathcal{B}) \rtimes \iota S \cong C^*(G \rtimes S, \mathcal{B}) \) is proved in [7]. The sum of the induced representations \( \text{Ind} \Lambda_x \) for \( x \in G^0 \) is a faithful representation of the quotient \( C^*_r(G, \mathcal{B}) \rtimes \iota S \) of \( C^*(G, \mathcal{B}) \rtimes S \) by Proposition [4,10] and our assumption. As in the proof of Lemma [8,1] the induced representation \( \text{Ind} \Lambda_x \) is the regular representation of \( C^*(G, \mathcal{B}) \rtimes \iota S \cong C^*(G \rtimes S, \mathcal{B}) \) on \( L^2((G \rtimes S)^x, \mu^x, \mathcal{B}) \), where \( \mu \) is the unique Haar system on \( G \rtimes S \) extending the given Haar system on \( G \) (see [7, Proposition 5.11]). Hence the reduced norm that gives \( C^*_r(G \rtimes S, \mathcal{B}) \) is defined by the same family of representations that gives the norm on \( C^*_r(G, \mathcal{B}) \rtimes \iota S \).

**Corollary 8.13.** If \( G \) is inner exact, then \( C^*_r(G) \rtimes \iota S \cong C^*_r(G \rtimes S) \).

The criteria above are not optimal. Of course, it suffices to require inner exactness only for the lattice generated by the open \( G \)-invariant subsets of \( G^0 \) corresponding to the ideals \( I_{1,2} \). A more serious limitation of our proof is that we only use \( E(\mathcal{H}_i) \subseteq \mathcal{M}(I_{1,2}) \). For instance, it does not use that \( E(\mathcal{H}_i) \subseteq \mathcal{A} \) if \( t \) is idempotent. This is why the following theorem is not a special case of Theorem [8,11].

**Theorem 8.14.** Let \( G \) be a locally compact, locally Hausdorff groupoid with a action of a unital inverse semigroup \( S \), and let \( \mathcal{B} \) be a Fell bundle over the transformation groupoid \( G \rtimes S \). If \( G \) is closed in \( G \rtimes S \), then the canonical conditional expectation \( C^*_r(G, \mathcal{B}) \rtimes \iota S \to C^*_r(G, \mathcal{B})'' \) takes values in \( C^*_r(G, \mathcal{B}) \) and \( C^*_r(G, \mathcal{B}) \rtimes \iota S \cong C^*_r(G \rtimes S, \mathcal{B}) \).

**Proof.** The conditional expectation \( E \) for the \( S \)-action on \( C^*(G, \mathcal{B}) \) on the dense subalgebra \( \mathcal{S}(G \rtimes S, \mathcal{B}) \subseteq C^*(G, \mathcal{B}) \rtimes \text{alg} S \) simply restricts a function on \( G \rtimes S \) to \( G \). This is a map to \( \mathcal{S}(G, \mathcal{B}) \) if \( G \) is closed in \( G \rtimes S \). (This works also for non-saturated Fell bundles, as considered in [9].) If \( G \) is closed in \( G \rtimes S \), then
$E(C^*_r(G,B)_S) \subseteq C^*_r(G,B)$ and therefore any faithful representation of $C^*_r(G,B)$ is $E$-faithful. Now Lemma 8.12 finishes the proof. \hfill \Box

Remark 8.15. Conversely, if $G$ is not closed in $G \times S$, then $\tilde{A} \neq A$ because we may find $\xi \in G(\times S)$ for which $E(\xi)$ lives on a single bisection of $G$ and is not continuous. And such a function cannot belong to $C^*_r(G,B)$.

Remark 8.16. It may happen that $G$ is not closed in $G \times S$ although $G \times S$ is Hausdorff. For instance, let $S$ be the inverse semigroup of bisections of a non-Hausdorff étale groupoid $H$ with Hausdorff, locally compact unit space $H^0$, and let $G$ be the Čech groupoid of a Hausdorff open cover of the arrow space of $H$ (see [7, Example A.9]). A canonical action of $S$ on $G$ that corresponds to the left translation action of a groupoid on its arrow space is described in [7, Corollary 3.21], and it is shown that the transformation groupoid $G \times S$ is Morita equivalent to the space $H^0$, which is the orbit space of the action of $H$ on $H^1$ by left multiplication. Thus $G \times S$ is a free and proper groupoid, forcing it to have Hausdorff arrow space (see [7, Proposition A.7]). There is a canonical open and continuous functor $G \times S \to H$, such that the arrow space of $G$ is the preimage of the unit subspace $H^0 \subseteq H^1$. Since $H^0 \subseteq H^1$ is open, but not closed, the preimage $G^1 \subseteq (G \times S)^1$ is open, but not closed.

Theorem 8.14 does not apply to this example. But Theorem 8.11 does. Indeed, the Čech groupoid $G$ is amenable, say, because it is étale and its $C^*$-algebra is nuclear (see [4, Theorem 5.6.18]). Hence it is inner exact. The transformation groupoid $G \times S$ is also a Čech groupoid and hence amenable; even more, it is equivalent to the space $H^0$. Hence Theorem 8.11 (or Corollary 8.13) implies $C^*_r(G) \rtimes S \cong C^*_r(G \times S) \cong C^*(G \times S) \sim C_0(H^0)$. The isomorphism $C^*_r(G) \rtimes S \cong C^*(G \times S)$ is shown in [7]. Hence we get $C^*(G) \rtimes S \cong C^*_r(G) \rtimes S$ in this example.

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