An analogue of the Schur-Weyl duality for the automorphisms group of a II\textsubscript{1}-factor

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Abstract

An analogue of the Schur-Weyl duality for the group of automorphisms of the approximately finite dimensional (AFD) II\textsubscript{1}-factor is produced.

Keywords: AFD II\textsubscript{1}-factor, automorphisms group of factor, Schur-Weyl duality.

1 Introduction

Let $M$ be a II\textsubscript{1}-factor with the separable predual $M_*$ and $\text{tr}$ a unique normal trace on $M$ such that $\text{tr}(I) = 1$. The inner product $\langle a, b \rangle = \text{tr}(b^* a)$ makes $M$ a pre-Hilbert space. Denote by $L^2(M, \text{tr})$ its completion. Let $\text{Aut} M$ be the automorphism group of $M$ and $U(M)$ the unitary subgroup of $M$. Every $u \in U(M)$ determines the inner automorphism $\text{Ad} u$ of $M$, $\text{Ad} u(x) = u x u^*$. Denote by $\text{Inn} M$ the subgroup of $\text{Aut} M$ formed by inner automorphisms.

One has a natural unitary representation $\mathfrak{N}$ of $\text{Aut} M$ on the dense subspace $M$ of $L^2(M, \text{tr})$ given by

$$\mathfrak{N}(\theta)x = \theta(x), \quad \theta \in \text{Aut} M, \quad x \in M,$$

which is certainly extendable to a representation on $L^2(M, \text{tr})$. Denote by $\mathfrak{N}_I$ the restriction of $\mathfrak{N}$ to the subgroup $\text{Inn} M$.

$\text{Aut} M$, being embedded as above into the algebra of bounded operators in $L^2(M, \text{tr})$, becomes a topological group under the strong operator topology. The subspace $L_0 = \{ v \in L^2(M, \text{tr}) : \text{tr}(v) = 0 \}$ is $\mathfrak{N}$-invariant: $\mathfrak{N}(\theta)L_0 = L_0$ for all $\theta \in \text{Aut} M$.

**Theorem 1.** The restriction $\mathfrak{N}_I^0$ of the representation $\mathfrak{N}_I$ to the invariant subspace $L_0$ is irreducible.

With an arbitrary II\textsubscript{1}-factor $M$ being replaced in the above settings by the algebra of complex $n \times n$ matrices, Theorem 1 reduces to the well known fact of classical representation theory (see [7], Ch. 3, §17.2, Theorem 2). Thus, in case of the approximately finite dimensional (AFD or hyperfinite) factor $M$, an argument based on approximation of II\textsubscript{1}-factor $M$ by finite dimensional factors is going to be applicable in proving Theorem 1. However, this theorem in its utmost generality requires a new approach.

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Define a diagonal action \( \mathfrak{N}^k \) of Aut \( M \) on \( L^2(M, tr)^{\otimes k} = L^2(M^\otimes k, tr^\otimes k) \) by

\[
\mathfrak{N}^k(\theta) (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = (\mathfrak{N}(\theta)v_1) \otimes (\mathfrak{N}(\theta)v_2) \otimes \cdots \otimes (\mathfrak{N}(\theta)v_k).
\]

Additionally, the symmetric group \( S_k \) acts on \( L^2(M^\otimes k, tr^\otimes k) \) by permutations

\[
^k P(s) (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_{s^{-1}(1)} \otimes v_{s^{-1}(2)} \otimes \cdots \otimes v_{s^{-1}(k)}.
\]  

(1.1)

Since the operators \( \mathfrak{N}^k(\theta) \) and \(^k P(s)\) commute, we obtain a representation \( \mathcal{F} \) of the group Aut \( M \times S_k \). Let \( \mathcal{F}(\theta, s) = \mathfrak{N}^k(\theta) \cdot ^k P(s) \).

Denote by \( \mathfrak{N}_0^k \) and \(^k P_0\) the restrictions of the representations \( \mathfrak{N}^k \) and \(^k P\) to the subspace \( L_0^k \subset L^2(M, tr)^{\otimes k} \).

Recall that the irreducible representations of \( S_k \) are parameterized by the unordered partitions of \( k \). Denote the set of all such partitions by \( \Upsilon_k \). Let \( \lambda \in \Upsilon_k \) and let \( \chi_\lambda \) be the character of the corresponding irreducible representation \( R_\lambda \). Denote by \( \dim \lambda \) the dimension of \( R_\lambda \). The operator

\[
P^\lambda = \frac{\dim \lambda}{k!} \sum_{s \in \mathfrak{S}_k} \chi_\lambda(s) \cdot ^k P(s)
\]

(1.2)

is an orthogonal projection in the centre of the \( w^\ast\)-algebra generated by the operators \( \{ \mathcal{F}(\theta, s) \}_{(\theta, s) \in \text{Aut} M \times S_k} \). Denote by \( \mathcal{F}_0^\lambda \) the representation \( \mathcal{F} \) restricted to the subspace \( H_0^\lambda = P^\lambda(L_0^k) \).

**Theorem 2.** Let \( M \) be an AFD II\(_1\) factor. Then the commutant of the set \( \mathfrak{N}_0^k(\text{Aut} M) \) is generated by \(^k P_0(S_k)\).

**Corollary 3.** The representation \( \mathcal{F}_0^\lambda \) of Aut \( M \times S_k \) is irreducible. With different \( \lambda, \zeta \in \Upsilon_k \), the restrictions of \( \mathcal{F}_0^\lambda \) and \( \mathcal{F}_0^\zeta \) to the subgroup Aut \( M \) are not quasi-equivalent.

Representation \(^k P\) can be extended to a representation \(^k P_{S_k}\) of the symmetric inverse semigroup \( S_k \), which can realize as a semigroup of \( \{0, 1\}\)-matrices \( a = [a_{ij}]_{i,j=1}^k \) with the ordinary matrix multiplication in such a way that \( a \) has at most one nonzero entry in each row and each column. We denote by \( \epsilon_i \) a diagonal matrix \( [a_{pq}] \) such that \( a_{ii} = 0 \) and \( a_{pq} = \delta_{pq} \), if \( p \neq i \) or \( q \neq i \). Of course, \( S_k \subset S_k \). Define operator \(^k P_{S_k}(\epsilon_i)\) on \( L^2(M^\otimes k, tr^\otimes k) \) as follows

\[
^k P_{S_k}(\epsilon_i) (\cdots v_{i-1} \otimes v_i \otimes v_{i+1} \cdots) = \text{tr}(v_i)(\epsilon_i)(\cdots v_{i-1} \otimes 1 \otimes v_{i+1} \cdots).
\]

We set \(^k P_{S_k}(s) = ^k P(s)\), if \( s \in S_k \). Then \(^k P_{S_k}\) is extended to a representation of the semigroup \( S_k \). Using Theorem 2, we prove in section 4 next statement.

**Theorem 4.** If \( M \) is an AFD II\(_1\) factor then the commutant of \( \mathfrak{N}_0^k(\text{Aut} M) \) is generated by \(^k P_{S_k}(S_k)\).

Using the embedding

\[
L^2(M, tr)^{\otimes n} \ni m_1 \otimes \cdots \otimes m_n \mapsto m_1 \otimes \cdots \otimes m_n \otimes 1 \in L^2(M, tr)^{\otimes (n+1)},
\]
we identify $L^2(M, \text{tr})^\otimes_n$ with the subspace in $L^2(M, \text{tr})^\otimes_{(n+1)}$. Denote by $L^2(M, \text{tr})^\otimes_\infty$ the completion of the pre-Hilbert space $\bigcup_{n=1}^\infty L^2(M, \text{tr})^\otimes_n$. It is convenient to consider $\bigcup_{n=1}^\infty L^2(M, \text{tr})^\otimes_n$ as the linear span of the vectors

$$v_1 \otimes \cdots \otimes v_n \otimes I \otimes I \otimes \cdots, \text{ where } v_j \in M.$$ 

At the same time, we will to identify $L^2(M, \text{tr})^\otimes_n$ with the closure of the linear span of all vectors $v_1 \otimes \cdots \otimes v_n \otimes v_{n+1} \otimes \cdots$, where $v_i = I$ for all $i > n$. Define the representation $\mathcal{R}^\otimes_\infty$ of group $\text{Aut } M$ as follows

$$\mathcal{R}^\otimes_\infty(\theta) (v_1 \otimes \cdots \otimes v_n \otimes \cdots) = (\mathcal{R}(\theta)v_1) \otimes \cdots \otimes (\mathcal{R}(\theta)v_n) \otimes \cdots \cdots.$$ 

The infinite symmetric group $\mathfrak{S}_\infty$ acts on $L^2(M, \text{tr})^\otimes_\infty$ by permutations

$$\infty P(s) (v_1 \otimes \cdots \otimes v_n \otimes \cdots) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(n)} \otimes \cdots \cdots, \quad s \in \mathfrak{S}_\infty.$$ 

We prove in section 5 the following statement.

**Theorem 5.** If $M$ is an AFD $\Pi_1$-factor then the commutant of $\mathcal{R}^\otimes_\infty(\text{Aut } M)$ is generated by $b_\mathcal{P}(\mathfrak{S}_\infty)$.

### 2 Proof of Theorem 1

Let $M$ be a $\Pi_1$-factor. Denote by $B\left(L^2(M, \text{tr})\right)$ the algebra of all bounded operators on $L^2(M, \text{tr})$. Recall that a $w^*$-subalgebra $\mathfrak{A} \subset M$ is called masa (maximal Abelian subalgebra) if $(\mathfrak{A}' \cap M) = \mathfrak{A}$, where

$$\mathfrak{A}' = \{ b \in B\left(L^2(M, \text{tr})\right) | \ ba = ab \text{ for all } a \in \mathfrak{A} \}$$

is the commutant of $\mathfrak{A}$. Let $\mathcal{N}(\mathfrak{A}) = \{ u \in U(M) : u\mathfrak{A}u^* = u^*\mathfrak{A}u = \mathfrak{A} \}$ be the normalizer of $\mathfrak{A}$. Let $\mathcal{N}(\mathfrak{A})''$ be the $w^*$-subalgebra generated by $\mathcal{N}(\mathfrak{A})$. A masa $\mathfrak{A}$ is said to be Cartan if $\mathcal{N}(\mathfrak{A})'' = M$.

We need the following claim from [15] (p. 242).

**Proposition 6.** There exists a masa $\mathfrak{A}$ in $M$ and an AFD-subfactor $F$ of $M$ containing $\mathfrak{A}$ such that $\mathfrak{A}$ is a Cartan subalgebra of $M$ and $F' \cap M = CI$.

It is well known that, in the context of latter Proposition, one can readily find the family $\{K_n\}_{n=1}^\infty$ of pairwise commuting $L_2$-subfactors $K_n \subset F$ which generate $F$. Fix a system of matrix units $\{e_{ij}\}_{i,j=1}^\infty \subset K_n$. Denote by $\mathfrak{A}_K$ an Abelian $w^*$-subalgebra generated by $\{e_{11}, e_{22}\}_{r=1}^\infty$. It is easy to check that $\mathfrak{A}_K$ is a Cartan subalgebra in $F$. Since any two Cartan masas $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of $F$ are conjugate, i. e. there exists $\theta \in \text{Aut } F$ such that $\theta(\mathfrak{A}_1) = \mathfrak{A}_2$, we can assume without loss of generality that the masa $\mathfrak{A}$ coincides with $\mathfrak{A}_K$.

Let $E$ be a unique conditional expectation of $M$ onto $\mathfrak{A}$ with respect to $\text{tr}$ [17]. In particular, $E$ is the orthogonal projection of the subspace $L_0$ onto the subspace

$$L_0^\mathfrak{A} = \{ x \in L^2(\mathfrak{A}, \text{tr}) : \text{tr}(x) = 0 \}.$$ 

We claim that $E$ belongs to the $w^*$-algebra generated by $\mathcal{R}(\text{Aut } M)$. To see this, consider a family $\{\Gamma_n\}$ of Abelian finite subgroups of $\text{Aut } M$. Namely, $\Gamma_n$
is generated by the inner automorphisms $\text{Ad} u$, with the unitaries $u$ belonging to the collection $\{ e_{11} - e_{22} \}_{r=1}^n$. Since $\mathfrak{A}$ is a masa in $M$, one has, in view of Proposition 2.2 that
\begin{equation}
\{ e_{11} - e_{22} \}_{r=1}^n = \mathfrak{A}.
\end{equation}
Denote by $E_n$ the orthogonal projection in $L^2(M, tr)$ determined by its values on the dense subset $M \subset L^2(M, tr)$
\begin{equation}
M \ni x \mapsto |\Gamma_n x| - \sum_{\gamma \in \Gamma_n} \gamma(x).
\end{equation}
Since $E_r \geq E_{r+1}$, the sequence $E_r$ converges in the strong operator topology.
Let $\lim_{r \to \infty} E_r = \tilde{E}$. Hence, an application of (2.3) and (2.4) yields
\begin{equation}
\tilde{E}(x) \in \mathfrak{A},
\end{equation}
\begin{equation}
\text{tr}(\tilde{E}(x)) = \text{tr}(x) \quad \text{for all } x \in M,
\end{equation}
\begin{equation}
\tilde{E}(axb) = a\tilde{E}(x)b \quad \text{for all } a, b \in \mathfrak{A}, \quad x \in M.
\end{equation}
Therefore, $\tilde{E}$ is the conditional expectation onto $\mathfrak{A}$. It follows that $\tilde{E} = E$. Thus, in view of (2.4), $E$ belongs to the $w^*$-algebra generated by $\mathfrak{N}(\text{Inn} M)$. Therefore,
\begin{equation}
A' L^0_0 \subset L^0_0 \text{ for all } A' \in (\mathfrak{N}^0_0(\text{Inn} M))'.
\end{equation}
The uniqueness of conditional expectation implies
\begin{equation}
\text{Ad } u \circ E \circ \text{Ad } u^* = E \quad \text{for all } u \in \mathcal{N}(\mathfrak{A}).
\end{equation}
This is to be rephrased by claiming that the action of $\text{Ad} \mathcal{N}(\mathfrak{A})$ leaves invariant $L^0_0$:
\begin{equation}
\text{Ad } u (a) \in L^0_0 \quad \text{for all } a \in L^0_0, \quad u \in \mathcal{N}(\mathfrak{A}).
\end{equation}
Now to prove Theorem 1 it suffices to demonstrate the following:

a) the action of $\mathcal{N}(\mathfrak{A})$, $u \mapsto \text{Ad } u$, leaves no non-trivial closed subspace of $L^0_0$ invariant;
b) the subspace $L^0_0 \subset L_0$ is cyclic with respect to $\mathfrak{N}(\text{Inn} M)$; i. e. the smallest closed subspace, containing $\bigcup_{\theta \in \text{Inn} M} \mathfrak{N}(\theta)L^0_0$, is just $L_0$.

Let us start with proving a). Consider an arbitrary unitary
\begin{equation}
u \in \{ K_1, K_2, \ldots, K_n \}^n,
\end{equation}
to be expanded as
\begin{equation}
u = \sum_{j_1, k_1, j_2, k_2, \ldots, j_n, k_n = 1}^2 u_{j_1 k_1 j_2 k_2 \ldots j_n k_n}^{1} e_{j_1 k_1}^{2} e_{j_2 k_2}^{2} \cdots e_{j_n k_n}^{2}.
\end{equation}
where \( u_{j_1k_1j_2k_2 \ldots j_nk_n} \in \mathbb{C} \). Denote by \( S_2^n \) the group of all bijections of the set \( X_n = \{(i_1,i_2,\ldots,i_n) , \ i_r \in \{1,2\}\} \). Within our current argument, the symmetric group \( S_2^n \) is about to be identified with the subgroup

\[
\{ u \in (K_1,K_2,\ldots,K_n)^{\prime\prime} \cap U(M) : u_{j_1k_1j_2k_2 \ldots j_nk_n} \in \{0,1\} \} \subset \mathcal{N}(\mathfrak{A}),
\]

in terms of the above expansion for \( u \in \{K_1,K_2,\ldots,K_n\}^{\prime\prime} \). It is also convenient to denote by \( 1_n \) the multiindex \( (i_1,i_2,\ldots,i_n) \). Clearly, the collection of vectors \( \{ e_{i_1} = i_1, e_{i_2} = i_2, \ldots, e_{i_n} = i_n \} \) forms an orthogonal basis of the subspace \( \mathfrak{A}_n = \mathfrak{A} \cap \{K_1,K_2,\ldots,K_n\}^{\prime\prime} \).

Let \( \xi_n \) be the orthogonal projection of \( L^2(\mathfrak{A},\mathfrak{tr}) \) onto \( \mathfrak{A}_n \), and consider a bounded operator \( B' \in (\text{Ad} \mathcal{N}(\mathfrak{A}))' \). It is clear that \( nB' \overset{\text{def}}{=} \xi_n B' \xi_n \) belongs to \( (\text{Ad} S_2^n)' \) and

\[
\lim_{n \to \infty} nB' = B' \text{ in the strong operator topology.} \tag{2.7}
\]

Hence, denoting the matrix element \( nB' \xi_{i_n}, \xi_{j_n} \) by \( nB'_{i_n,j_n} \), one has

\[
nB'_{s(i_n),s(j_n)} = nB'_{i_n,j_n} \text{ for all } s \in S_2^n.
\]

Therefore, there exist \( \gamma, \delta \in \mathbb{C} \) such that

\[
nB'_{i_n,j_n} = \begin{cases} 
\gamma, & \text{if } i_n \neq j_n; \\
\delta, & \text{if } i_n = j_n.
\end{cases}
\]

It follows that

\[
nB'\eta = (\delta - \gamma) \eta \text{ for all } \eta \in L_0^\mathfrak{A} \cap \mathfrak{A}_n.
\]

Hence, applying (2.7), we obtain that \( B'\eta = (\delta - \gamma) \eta \) for all \( \eta \in L_0^\mathfrak{A} \). This proves a).

Turn to proving b). It suffices to demonstrate that, given a self-adjoint \( B \in M \) and \( \epsilon > 0 \), there exist \( A \in \mathfrak{A} \) and \( U \in U(M) \) with the property

\[
\| B - UAU^* \| < \epsilon, \quad \text{where } \| \cdot \| \text{ stands for the operator norm.} \tag{2.8}
\]

Choose a positive integer \( n > \| B \| / \epsilon \) and consider the set of reals

\[
\Delta_l = \left\{ r : \frac{2(l-1)\| B \|}{n} - \| B \| < r \leq \frac{2l\| B \|}{n} - \| B \| \right\}
\]

for each \( l = 0,1,\ldots,n \). Let \( E(\Delta_l) \) be the associated spectral projection related to the spectral decomposition of \( B \). Under this setting, with

\[
\alpha_l = \frac{(2l-1)\| B \|}{n} - \| B \|, \quad B_n = \sum_{l=0}^n \alpha_l E(\Delta_l),
\]

we conclude that

\[
\| B - B_n \| \leq \epsilon. \tag{2.9}
\]

One can readily find a family \( (F_l)_{l=0}^n \) of pairwise orthogonal projections in \( \mathfrak{A} \) such that \( \text{tr} (F_l) = \text{tr} (E(\Delta_l)) \). Thus we can also select partial isometries \( u_l \in M \) with the properties \( u_l u_l^* = E(\Delta_l) \) and \( u_l^* u_l = F_l \) for all \( l = 1,2,\ldots,n \). It follows that \( U = \sum_{l=0}^n u_l \) is a unitary operator, and with \( A = \sum_{l=0}^n \alpha_l F_l \) the inequality (2.8) holds.
3 Proof of theorem

Notice first that there exists a family \( \{N_j\}_{j=1}^{\infty} \) of pairwise commuting type \( I_k \) subfactors \( N_j \subset M \) generating \( M \). Let \( M_{i,j} = \left( \{N_j\}_{j=1}^I \right)^{''} \). Fix a system of matrix units \( \{e_{i,j}\}_{i,j=1}^k \subset N_1 \). Denote by \( \mathfrak{A} \) an Abelian \( w^* \)-subalgebra generated by \( \{e_{11}, e_{22}, \ldots, e_{kk}\}_{i=1}^k \). One can reproduce here the argument used at the beginning of Section 2 to demonstrate that \( \mathfrak{A} \) is a Cartan MASA in \( M \).

3.1 The conditional expectation from \( M^{\otimes k} \) onto \( \mathfrak{A}^{\otimes k} \)

It is well known that there exists a unique conditional expectation \( kE \) from the \( \Pi_1 \)-factor \( M^{\otimes k} \) onto the Cartan MASA \( \mathfrak{A}^{\otimes k} \subset M^{\otimes k} \). Recall that \( kE \) is uniquely determined by the following properties (see [14]):

1. \( kE \) is continuous with respect to the strong operator topology and \( kE 1 = I \);
2. \( kE(a_{1}a_{2}) = a_{1}kE(a_{2})a_{2} \) for all \( m \in M^{\otimes k} \) and \( a_{1}, a_{2} \in \mathfrak{A}^{\otimes k} \);
3. \( \text{tr}^{\otimes k}(kEm) = \text{tr}^{\otimes k}(m) \) for all \( m \in M^{\otimes k} \).

We prove below that \( kE \) belongs to \( (\mathfrak{A}^{\otimes k} (\text{Ad} U(M)))^{''} \).

With \( i_{j} = (i_{1}, i_{2}, \ldots, i_{j}) \), let \( e_{i_{j}} \) stand for the minimal projection
\[
1 e_{i_{1}i_{1}} 2 e_{i_{2}i_{2}} \cdots j e_{i_{j}i_{j}}
\]
of the algebra \( M_{i_{j}} \cap \mathfrak{A} \). Let \( ^{nf} \) be the embedding of the finite set
\[
\mathcal{F}_{j} = \{i_{j} = (i_{1}, i_{2}, \ldots, i_{j})\}_{i_{1}, i_{2}, \ldots, i_{j} = 1}^{k-1}
\]
into \( \{n + 1, n + 2, \ldots\} \). Set \( p_{u} = r_{k_{1}} + \sum_{l=1}^{k-1} r_{l+1} \in N_{p} \).

**Lemma 7.** Consider the unitary \( JU_{n} = \sum_{i_{j} \in \mathcal{F}_{j}} e_{i_{j}} \cdot p_{u} \), where \( p = ^{nf} i_{j} \) and \( n > J \). Then for any \( m \in M \) the sequence \( \mathfrak{R} (\text{Ad} (JU_{n})) m \) converges in the weak operator topology so that
\[
\lim_{n \to \infty} \mathfrak{R} (\text{Ad} (JU_{n})) m = E_{j}(m), \quad m \in M_{i_{j}}^{\otimes k} \cap \mathfrak{A}^{\otimes k} \cap M_{i_{j}}.
\]
In particular, \( E_{j} \) belongs to the \( w^{*} \)-algebra generated by \( \mathfrak{R} (\text{Ad} U(M)) \).

**Proof.** Since the algebra \( \bigcup_{Q=1}^{\infty} M_{IQ} \) is dense in \( M \) in the strong operator topology, one can assume without loss of generality that \( m \in M_{1L} \), where \( L > J \). Under this assumption, we have with \( n > L \)
\[
JU_{n} \cdot m \cdot JU_{n}^{*} = \sum_{i_{j} \in \mathcal{F}_{j}} e_{i_{j}} \cdot m \cdot e_{i_{j}}^{*} \cdot p_{u} \cdot q_{u}^{*},
\]
where \( p = ^{nf} i_{j} \), \( q = ^{nf} r_{j} \). Note that with \( i_{j} \neq r_{j} \) one has
\[
\lim_{n \to \infty} p_{u} \cdot q_{u}^{*} = \text{tr} (p_{u} \cdot q_{u}^{*}) I = 0
\]
in the weak operator topology. Therefore, \( \lim_{n \to \infty} JU_{n} \cdot m \cdot JU_{n}^{*} = E_{j}(m) \). \( \square \)
Remark 1. Clearly, $E_J$ is an orthogonal projection in $L^2(M, \text{tr})$. Also, one readily observes that $E_J \geq E_{J+1}$ for all $J$. Hence for any $m \in L^2(M, \text{tr})$ there exists

$$\lim_{J \to \infty} E_J(m) = E(m).$$

In particular,

$$E(m) = E_J(m) \text{ for all } m \in M_{1,J}. \quad (3.11)$$

It is easy to verify that $E$ is the unique conditional expectation of $M$ onto $\mathfrak{A}$ with respect to $\text{tr}$ \cite{14}. On the other hand, 1) - 3) are valid also for the projection $E^\otimes k$. The uniqueness of conditional expectation now implies

$$kE \left( m_1 \otimes m_2 \otimes \cdots \otimes m_k \right) = E(m_1) \otimes E(m_2) \otimes \cdots \otimes E(m_k) \quad (3.12)$$

for all $m_1, m_2, \ldots, m_k \in M$.

Proposition 8. $kE \in (\mathfrak{N}^\otimes k (\text{Ad } U(M)))''$.

Proof. Let $E_J^{\otimes k} \left( m_1 \otimes m_2 \otimes \cdots \otimes m_k \right) \equiv E_J(m_1) \otimes E_J(m_2) \otimes \cdots \otimes E_J(m_k)$. By Lemma \cite{7}

$$E_J^{\otimes k} \in (\mathfrak{N}^\otimes k (\text{Ad } U(M)))'' \quad (3.13)$$

$E_J^{\otimes k}$ is an orthogonal projection in $L^2(M^\otimes k, \text{tr}^\otimes k)$ and $E_J^{\otimes k} \geq E_L^{\otimes k}$ for all $L > J$. It follows that for any $m \in L^2(M^\otimes k, \text{tr}^\otimes k)$ there exists $\lim_{J \to \infty} E_J^{\otimes k}(m) \equiv \tilde{E}(m) \in M^\otimes k \cap (\mathfrak{A}^\otimes k)''$. Therefore, $\tilde{E} \in (\mathfrak{N}^\otimes k (\text{Ad } U(M)))''$. An application of (3.10) allows one to verify that 1) - 3) are valid for $\tilde{E}$. Since $\mathfrak{A}^\otimes k$ is a MASA in $M^\otimes k$, we conclude that $\tilde{E} (M^\otimes k) = \mathfrak{A}^\otimes k$. Therefore, $\tilde{E}$ is a conditional expectation from $M^\otimes k$ onto $\mathfrak{A}^\otimes k$, hence $\tilde{E} = kE = E^\otimes k$ by (3.12). \hfill \Box

3.2 The operators $kE \cdot \mathfrak{N}^\otimes k(u) \cdot kE$ on $L^2(\mathfrak{A}^\otimes k, \text{tr}^\otimes k)$.

With $i_j = \left( i_{j1}, i_{j2}, \ldots, i_{jk} \right)$, $i'_j = \left( i'_{j1}, i'_{j2}, \ldots, i'_{jk} \right)$, denote the partial isometry $\iota'_{i_{j1}, i'_{j1}} \iota'_{i_{j2}, i'_{j2}} \cdots \iota'_{i_{jk}, i'_{jk}} \in M_{1,J}$ by $\iota_{i_j, i'_j}$. Given a collection $\iota x \in M_{1,J}$, $1 \leq l \leq k$, we use below the expansion

$$\iota x = \sum_{i_j, i'_j \in \mathfrak{A}} \iota_{i_j, i'_j, i_j, i'_j} \in M_{1,J}, \text{ where } \iota_{i_j, i'_j} \in \mathbb{C}.$$

In view of \cite{3}, one has

$$kE \left( \iota x \otimes \iota x \otimes \cdots \otimes \iota x \right) = E_J(\iota x) \otimes E_J(\iota x) \otimes \cdots \otimes E_J(\iota x) = \sum_{i_j, i'_j \in \mathfrak{A}} \iota_{i_j, i'_j, i_j, i'_j} \otimes \left( \sum_{i_j, i'_j \in \mathfrak{A}} \iota_{i_j, i'_j} \iota_{i_j, i'_j} \right) \otimes \cdots \otimes \left( \sum_{i_j, i'_j \in \mathfrak{A}} \iota_{i_j, i'_j} \iota_{i_j, i'_j} \right) \quad (3.14)$$

Note that in Subsection 3.1 another notation $\iota_{i_j, i'_j}$ was used for $\iota_{i_j, i'_j}$.

Consider a unitary $u = \sum \iota_{i_j, i'_j} \iota_{i_j, i'_j} \in M_{1,J}$ and a collection $\iota u = \sum \iota_{i_j, i'_j} \iota_{i_j, i'_j} \in M_{1,J} \cap \mathfrak{A}$, $1 \leq l \leq k$, where $\iota_{i_j, i'_j} \iota_{i_j, i'_j} \in \mathbb{C}$. Since

$$kE \left( \mathfrak{N}^\otimes k(\text{Ad } u)(\iota u \otimes \iota u \otimes \cdots \otimes \iota u) \right) = kE \left( u \cdot \iota u \cdot u^* \otimes u \cdot \iota u \cdot u^* \otimes \cdots \otimes u \cdot \iota u \cdot u^* \right),$$
an application of (3.11) and (3.12) yields
\[ l^E (\mathfrak{h} \circ \mathfrak{h} (\text{Ad} u)) (b \oplus b \oplus \cdots \oplus b) = b \oplus b \oplus \cdots \oplus b, \] where
\[ b = \sum_{ij \in J} b_{ij} \cdot e_{ij} \in M_{1, J} \cap \mathfrak{a} \quad \text{and} \quad b_{ij} = \frac{|u_{ij} v_j|^2}{l_{ij}}. \] (3.15)

This way the map
\[ \mu : M_{1, J} \cap U(M) \to M_{1, J}; \quad \sum_{ij \in J} u_{ij} v_j \cdot e_{ij} v_j \mapsto \sum_{ij \in J} |u_{ij} v_j|^2 \cdot e_{ij} v_j. \]
is introduced. It is to be studied and used in what follows.

Note that \(|u_{ij} v_j|^2\) form a doubly stochastic matrix (see Section 3), hence
\[ \sum_{ij \in J} l_{aij} = \sum_{ij \in J} b_{ij} \quad \text{for all } l. \] (3.16)

### 3.2.1 Some properties of the map \( \mu \)

Set \( n = k^2 \). To simplify the notation, it is custom (and really convenient) to identify \( m = \sum_{ij \in J} m_{ij} v_j \cdot e_{ij} v_j \in M_{1, J} \) with the associated matrix \([m_{ij} v_j]\). Let \( M_{1, J}(\mathbb{R}) \) be the subset of real matrices in \( M_{1, J} \). Denote also by \( GL(n, \mathbb{R}) \) the subgroup of all invertible elements of \( M_{1, J}(\mathbb{R}) \). A matrix \( m = [m_{ij} v_j] \in M_{1, J} \) is said to be doubly stochastic if its elements satisfy
\[ m_{ij} v_j \geq 0 \quad \text{for all } i, j, \]
\[ \sum_{j \in J} m_{ij} v_j = 1 \quad \text{for all } i \quad \text{and} \quad \sum_{i \in I} m_{ij} v_j = 1 \quad \text{for all } j. \]

The set of doubly stochastic matrices is a convex polytope known as Birkhoff’s polytope \([2]\). Denote by \( DS_n \) this polytope. Set \( p = [p_{ij} v_j] \), where \( p_{ij} v_j = \frac{1}{n} \) for all \( i, j \). A routine verification demonstrates that \( p \) is a minimal orthogonal projection from \( M_{1, J} \). If \( m = [m_{ij} v_j] \in DS_n \) then
\[ mp = pm = p \quad \text{and} \quad m = p + (I - p)m(I - p). \] (3.17)

A natural method of producing a doubly stochastic matrix is to start with a unitary matrix \( u = [u_{ij}] \) and then to set \( \mu(u) = [u_{ij} v_j]^2 \in DS_n \). The matrices of the form \( \mu(u) \) with \( u \) unitary are called unistochastic.

It is well known that for \( n > 3 \) there are doubly stochastic matrices that are not unistochastic \([5]\).

Let the notation \( G \) stand for the set of those \( g = [g_{ij} v_j] \in GL(n, \mathbb{R}) \) which satisfy \( \sum_{j \in J} g_{ij} v_j = 1 \) for all \( i \) and \( \sum_{i \in I} g_{ij} v_j = 1 \) for all \( j \). The latter relations are obviously equivalent to the vector \( \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \) being invariant under both \( g \) and the transpose \( g^t \) with respect to matrix multiplication, hence \( G \) is a subgroup. One can clearly reproduce (3.17) for \( g \in G \):
\[ g = p + (I - p)g(I - p). \] (3.18)
Consider the one parameter family \( \theta U = [\theta U_{ij}] \) of unitary matrices, where
\[
\theta U_{ij} = \delta_{ij} + \frac{\theta - 1}{n}, \quad \theta \in \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}.
\] (3.19)

Now we are in a position to apply the above idea of the present Section 3.2 in order to introduce the map \( \mu : \text{inn} M \to DS_n \) given by
\[
\text{Ad} U \mapsto \left[ |U_{ij} j^2 | \right], \quad U = [U_{ij}].
\]

An easy calculation demonstrates that
\[
\mu (\theta U) = p + \left( 1 - \frac{|\theta - 1|^2}{n} \right) (I - p).
\] (3.20)

We need the following claim which is proved in Section 6.

**Proposition 9.** With \( \theta \in \mathbb{T} \setminus \{-1, 1\} \) and \( n > 4 \), there exists an open neighborhood \( \mathcal{U} \) of \( U \) such that \( \mu (\mathcal{U}) \) is open in \( G \).

### 3.3 The commutant of \( kE \cdot \mathcal{N}^k \langle \text{Ad} U (M) \rangle \cdot kE \).

Let us start with observing that, in view of (3.12), \( kE (L_0^3) = (L_0^3)^{\otimes k} \). It follows that \( kE \cdot \mathcal{N}^k \langle \text{Ad} U (M) \rangle (L_0^3)^{\otimes k} \subset (L_0^3)^{\otimes k} \). Thus we can view \( kE \cdot \mathcal{N}^k \langle \text{Ad} U (M) \rangle \cdot kE \) as a family of operators on \( (L_0^3)^{\otimes k} \). Finally, let us restrict the representation \( kP \) from \( \mathcal{N}_k \) to the subspace \( (L_0^3)^{\otimes k} \), to be denoted by \( kP_0^3 \).

Let \( N_0 \) be the \( w^* \)-algebra generated by the operators \( kE \cdot \mathcal{N}^k \langle \text{Ad} U (M) \rangle \cdot kE \) in \( (L_0^3)^{\otimes k} \).

**Proposition 10.** \( N_0 \) coincides with \( (kP_0^3 (\mathcal{N}_k))^\prime \).

We need an auxiliary

**Lemma 11.** Let \( kP_p^j \) \((p < J)\) be the conditional expectation of \( M^{\otimes k} \) onto the \( I_N \)-subfactor \( M^{\otimes k} \rangle_p^J \left( \left\{ N_{ij} \right\}_{ij=p}^J \right)^{\otimes k} \) with respect to \( \text{tr}^{\otimes k} \), where \( N = k^J - p + 1 \).

Then \( kP_p^j \) belongs to the \( w^* \)-algebra generated by \( \mathcal{N}^k \) \langle \text{Ad} u \rangle \) with \( u \) spanning the unitary group of \( w^* \)-algebra \( \mathcal{R} \left\langle N_1 N_2 \cdots N_{p-1} N_{j+1} + N_{j+2} \cdots \right\rangle^\prime \).

**Proof.** Notice first that
\[
M^J_{j,p} \cap M = \left\{ N_1 N_2 \cdots N_{p-1} N_{j+1} N_{j+2} \cdots \right\}^\prime.
\] (3.21)

Every \( x \in M \) can be written in the form \( x = \sum_{r,q=1}^N a_{rq} x_{rq}' \), where \( a_{rq} \in M_{pJ} \), \( x_{rq}' \in M_{pJ}^J \). Set \( \mathcal{E}_j^p (x) = \sum_{r,q=1}^N \text{tr} (x_{rq}') a_{rq} \). The uniqueness of conditional expectations implies
\[
kP_p^j \left( \downarrow x \otimes \uparrow x \otimes \cdots \otimes \downarrow x \right) = \mathcal{E}_j^p (\downarrow x) \otimes \mathcal{E}_j^p (\uparrow x) \otimes \cdots \otimes \mathcal{E}_j^p (\downarrow x) \] (3.22)
for any \(1^x, 2^x, \ldots, k^x \in M\). Let \(\{j_l\}\) and \(\{J_l\}\) be two increasing sequences of positive integers with the property
\[
j_{l+1} - j_{l+1} > \max\{J_l, J\} \quad \text{for all } l.
\]
(3.23)

By (3.21), there exists a sequence \(\{U_l\}\) of unitaries from \(M_{p_J}^* \cap M\) such that
\[
U_l \in M_{p_J}^* \cap M_{1, j_{l+1}}, \quad \text{and } \Ad U_l (M_{p_J}^* \cap M_{1, j_l}) \subset M_{j_{l+1}, j_{l+1}}.
\]
(3.24)

Therefore,
\[
\text{w-lim}_{n \to \infty} \Ad U_n (x) = \text{tr}(x)I \quad \text{for each } x \in \bigcup_{r=1}^\infty M_r \cap M_{p_J},
\]
where \(\text{w-lim } x_n\) denote the limit of the sequence \(x_n \in M\) in the weak operator topology. Since \(\bigcup_{r=1}^\infty M_r\) is dense in \(M\) with respect to the strong operator topology, one has
\[
\text{w-lim}_{n \to \infty} \Ad U_n (x) = \text{tr}(x)I \quad \text{for each } x \in M_{p_J}^* \cap M.
\]

Now, in view of the above observations, with \(x = \sum_{r,q=1}^N a_{pq} x_{rq} \in M\), \(a_{rq} \in M_{p_J}\), \(x_{rq} \in M_{p_J}^* \cap M\), one establishes that
\[
\text{w-lim}_{n \to \infty} \Ad U_n (x) = \sum_{r,q=1}^N \text{tr}(x_{rq}) a_{rq} = \mathcal{E}_J^p (x) \in M_{p_J}.
\]

Hence
\[
\text{w-lim}_{n \to \infty} \mathcal{N}^{\otimes k} (\Ad U_n) \left( 1^x \otimes 2^x \otimes \cdots \otimes k^x \right) = \mathcal{E}_J^k (1^x) \otimes \mathcal{E}_J^k (2^x) \otimes \cdots \otimes \mathcal{E}_J^k (k^x).
\]

Now combine the latter with (3.22) and (3.24) to establish the claim of Lemma 11.

\textbf{Proof of Proposition 10.} Note first that the conditional expectations \(k^E\) and \(k^E_J\) commute and
\[
\lim_{J \to \infty} k^E_J = k^E.
\]
(3.25)

To simplify the notation, we substitute below \(F_J\) for \(k^E_J \circ k^E\). The projection \(F_J\) is just the conditional expectation of \(M^{\otimes k}\) onto \(M^{\otimes k}_1 \cap M^{\otimes k}_{1,J}\) with respect to \(\text{tr}^{\otimes k}\). Since \(k^E (L_0^{\otimes k}) \subset (L_0^3)^{\otimes k}\) and \(k^E_J (L_0^{\otimes k}) = L_0^{\otimes k} \cap M^{\otimes k}_1\), one deduces that
\[
F_J (L_0^{\otimes k}) \subset M^{\otimes k}_1 \cap (L_0^3)^{\otimes k} = (M \cap L_0^3)^{\otimes k}.
\]
(3.26)

By Proposition 8 and Lemma 11
\[
F_J \in (\mathcal{N}^{\otimes k} (\Ad U(M)))^*.
\]
(3.27)
We are about to use the notation $T_j(u)$ for the operator $F_j \cdot \mathfrak{N}^{\otimes k}(\text{Ad} u) \cdot F_j$. It follows from (3.26) that

$$T_j(u) \left( M_{1,j}^{\otimes k} \cap (L_0^\mathfrak{A})^{\otimes k} \right) \subset M_{1,j}^{\otimes k} \cap (L_0^\mathfrak{A})^{\otimes k} \quad \text{for each unitary } u \in M_{1,j}. \quad (3.28)$$

The above observations imply that the action of $T_j(u)$ on $M_{1,j}^{\otimes k} \cap (L_0^\mathfrak{A})^{\otimes k}$ is determined by (3.15).

Denote by $\mathfrak{L}$ an auxiliary representation of the general linear group $GL(n, \mathbb{R})$, with $n = k' = |\mathfrak{J}_j|$, which coincides with the natural action of $GL(n, \mathbb{R})$ on the complex $n$-dimensional space $M_{1,j} \cap \mathfrak{A}$; more precisely, with $g = [g_{i,j}]_{1 \leq i,j \leq n} \in GL(n, \mathbb{R})$ one has

$$\mathfrak{L}(g) \left( \sum_{i,j \in \mathfrak{J}_j} a_{i,j} \cdot \mathfrak{e}_{i,j} \right) = \sum_{i,j \in \mathfrak{J}_j} g_{i,j} \mathfrak{e}_{i,j} \cdot a_{i,j}. \quad (3.29)$$

Let us introduce the subgroup $\mathfrak{L}GL(n, \mathbb{R})$ formed by such $g \in GL(n, \mathbb{R})$ that $\mathfrak{L}(g)I = I$ and $\mathfrak{L}(g')I = I$, where the vector $I = \sum_{i,j \in \mathfrak{J}_j} \mathfrak{e}_{i,j}$ is just the unit of the algebra $M_{1,j} \cap \mathfrak{A}$, and the superscript $t$ stands for passage to the transpose. Given a unitary $u = \sum_{i,j \in \mathfrak{J}_j} u_{i,j} \mathfrak{e}_{i,j} \in M_{1,j}$, the matrix $\mu(u) = [u_{i,j} \mathfrak{e}_{i,j}]$ is doubly stochastic. In the case $\mu(u)$ is also invertible one easily deduces from (3.29) that $\mu(u) \in \mathfrak{L}GL(n, \mathbb{R})$, and in view of (3.15) one has

$$T_j(u) = \mathfrak{L}(\mu(u)). \quad (3.30)$$

$\mathfrak{L}GL(n, \mathbb{R})$ is the intersection of stationary subgroups of a vector $I$ with respect to the left action $g \mapsto \mathfrak{L}(g)$ and to the right action $g \mapsto \mathfrak{L}(g')$ on $M_{1,j} \cap \mathfrak{A}$. Hence it is isomorphic to $GL(n - 1, \mathbb{R})$, and

$$\mathfrak{L}(g) \left( M_{1,j} \cap L_0^\mathfrak{A} \right) = M_{1,j} \cap L_0^\mathfrak{A} \quad \text{for all } g \in \mathfrak{L}GL(n, \mathbb{R}). \quad (3.31)$$

By (3.30) and (3.31), the restrictions $T_j^0(u)$ and $\mathfrak{L}_0(g)$ of $T_j(u)$ and $\mathfrak{L}(g)$, respectively, to $M_{1,j} \cap L_0^\mathfrak{A}$ are well defined. We are about to prove that

$$\left\{ T_j^0(u), \ u \in M_{1,j} \cap U(M) \right\}'' = \left\{ \mathfrak{L}_0^{\otimes k} \left( \mathfrak{L}GL(n, \mathbb{R}) \right) \right\}''.$$

Once the latter relation is established, an application of the well known results of classical Schur-Weyl duality (see, for example, [3], Lecture 6) allows one to obtain

$$\left\{ \mathfrak{L}_0^{\otimes k} \left( \mathfrak{L}GL(n, \mathbb{R}) \right) \right\}'' = \left\{ F_j^0 \cdot k\mathfrak{P}^\mathfrak{A} (\mathfrak{S}_k) \ F_j^0 \right\}',$$

and then to deduce that

$$\left\{ T_j^0(u), \ u \in M_{1,j} \cap U(M) \right\}'' = \left\{ F_j^0 \cdot k\mathfrak{P}^\mathfrak{A} (\mathfrak{S}_k) \ F_j^0 \right\}',$$

where $F_j^0$ is the restriction of $F_j$ to $L_0^{\otimes k}$ (see (3.26)).

Now we turn to proving (3.32).

Since, in view of (9\mathfrak{N}^{\otimes k}(\text{Ad} U(M)))'' \subset \left( k\mathfrak{P} (\mathfrak{S}_k) \right)'$ and (3.27) one has $F_j \in (9\mathfrak{N}^{\otimes k}(\text{Ad} U(M)))''$, it follows that

$$F_j^0 \in \mathcal{N}_0 \subset \left( k\mathfrak{P} (\mathfrak{S}_k) \right)'.$$

(3.34)
This implies that for each \( J \) the operators \( F_j^0 \) \( b \mathfrak{p}^\mathfrak{a} (\mathfrak{S}_K) F_j^0 \) determine a unitary representation of \( \mathfrak{S}_K \).

One concludes from Proposition \ref{prop4} that there exists an open neighborhood \( \mathcal{U} \in U(n) \) of \( \mathcal{U} \) such that \( \mu(\mathcal{U}) \) is an open subset in \( tGL(n, \mathbb{R}) \cong GL(n-1, \mathbb{R}) \).

Hence, an application of \ref{prop4} yields

\[
T_j^0(\mathcal{U}) = \mathcal{L}^{\otimes k}_0(\mu(\mathcal{U})) \subset \{ T_j^0(u) \mid u \in M_{1,J} \cap U(M) \}''.
\]

Therefore, with \( \mathcal{U} \cdot \mathcal{U}^{-1} \) being a neighborhood of the identity in \( U(n) \),

\[
\mathcal{L}^{\otimes k}_0(\mu(\mathcal{U}) \cdot \mu(\mathcal{U})^{-1}) \subset \{ T_j^0(u) \mid u \in M_{1,J} \cap U(M) \}''.
\]  \hspace{1cm} \text{(3.35)}

Denote by \( t\mathfrak{gl}(n, \mathbb{R}) \) and \( \mathfrak{gl}(n-1, \mathbb{R}) \) the Lie algebras of \( tGL(n, \mathbb{R}) \) and \( GL(n-1, \mathbb{R}) \), respectively.

A representation \( \mathcal{L}^{\otimes k}_0 \) restricted to the neighborhood \( \mu(\mathcal{U}) \cdot \mu(\mathcal{U})^{-1} \) of unit in \( tGL(n, \mathbb{R}) \cong GL(n-1, \mathbb{R}) \) determines a representation \( \mathcal{L}^{\otimes k}_0 \) of Lie algebra \( t\mathfrak{gl}(n, \mathbb{R}) \cong \mathfrak{gl}(n-1, \mathbb{R}) \) in the \((n-1)^k\)-dimensional vector space \( M_{1,J} \cap (L_0^k)^{\otimes k} \).

By \ref{prop4},

\[
t\mathfrak{gl}(n, \mathbb{R}) \subset \{ T_j^0(u), u \in M_{1,J} \cap U(M) \}''.
\]

This implies \ref{prop4}.

Consider a bounded operator \( B' \in \mathcal{N}_0' \) together with its action on \( (L_0^k)^{\otimes k} \).

It follows from \ref{prop4} that \( F_j^0 B' = B' F_j^0 \). Therefore \( B'_j \overset{\text{def}}{=} F_j^0 B' F_j^0 \) belongs to \( \{ T_j^0(u) \mid u \in M_{1,J} \cap U(M) \}'' \). Let \( R_\lambda, \lambda \in \mathfrak{Y}_k \), be an irreducible representation of \( \mathfrak{S}_k \) and \( \chi_\lambda \) its character. Then the operator \( P_\lambda^0 = \frac{\dim \lambda}{\mathfrak{g}_k} \sum_{s \in \mathfrak{g}_k} \chi_\lambda(s) P^\mathfrak{a}_k(s) \) is an orthogonal projection that belongs to the center of \( (b \mathfrak{p}^\mathfrak{a} (\mathfrak{S}_k))' \).

One can readily find such positive integer \( N \) that for all \( J > N \) one has \( F_j P_\lambda^0 \neq 0 \). Only such \( J \) are to be considered below.

It is clear that \( P_\lambda^0 \in \mathcal{N}_0' \). In view of \ref{prop4},

\[
B'_j = \sum_{g \in \mathfrak{S}_k} c_j(g) F_j^0 b \mathfrak{p}^\mathfrak{a} (g) F_j^0, \text{ where } c_j(g) \in \mathbb{C}, \text{ and}
\]

\[
P_\lambda^0 B'_j = B'_j P_\lambda^0 \text{ for all sufficiently large } J.
\]  \hspace{1cm} \text{(3.36)}

It also follows from \ref{prop4} that

\[
(F_j^0 N_0 F_j^0)' = F_j^0 \{ b \mathfrak{p}^\mathfrak{a} (\mathfrak{S}_k) \}' F_j^0.
\]

Hence, since \( P_\lambda^0 \), which is central in \( (b \mathfrak{p}^\mathfrak{a} (\mathfrak{S}_k))' \) and commutes with \( F_j^0 \in \mathcal{N}_0' \), one has

\[
(P_\lambda^0 F_j^0 N_0 F_j^0 P_\lambda^0)' = F_j^0 P_\lambda^0 \{ b \mathfrak{p}^\mathfrak{a} (\mathfrak{S}_k) \}' P_\lambda^0 F_j^0.
\]

Therefore, \( P_\lambda^0 F_j^0 N_0 P_\lambda^0 F_j^0 \)' is a finite \( \text{Id}_{\dim \lambda} \)-factor for all \( J \) large enough. This implies that the map

\[
F_j^0 P_\lambda^0 \{ b \mathfrak{p}^\mathfrak{a} (\mathfrak{S}_k) \}' P_\lambda^0 F_j^0 \ni A \mapsto F_j^0 A F_j^0 \in F_j^0 P_\lambda^0 \{ b \mathfrak{p}^\mathfrak{a} (\mathfrak{S}_k) \}' P_\lambda^0 F_j^0
\]

is an isomorphism for \( \lambda > N \). Hence an application of \ref{prop4} yields

\[
P_\lambda^0 B'_j = P_\lambda^0 \sum_{g \in \mathfrak{S}_k} c_j(g) F_j^0 b \mathfrak{p}^\mathfrak{a} (g) F_j^0.
\]
Now, using (3.25), after the passage to the limit $\tilde{f} \to \infty$ we obtain

$$P_0^s B' = P_0^s \sum_{g \in \mathcal{G}_k} c_f(g) \ b p_{\alpha}^s (g) \text{ for all } \lambda \in \mathcal{T}_k.$$ 

Therefore, $B' = \sum_{g \in \mathcal{G}_k} c_f(g) \ b p_{\alpha}^s (g) \in (b p_{\alpha}^s (\mathcal{G}_k))''$, which completes the proof of proposition.\(\square\)

### 3.4 The cyclicity of $\mathfrak{N}^\otimes k (\text{Inn} \ M) \left( (\mathfrak{N}^\otimes k) \otimes^k \right)$ in $L_0^\otimes k$.

Denote by $\mathcal{H}$ the closure of the linear span of $\mathfrak{N}^\otimes k (\text{Inn} \ M) \left( (\mathfrak{N}^\otimes k) \otimes^k \right)$ in $L_0^\otimes k$. Our claim to be proved below is that $\mathcal{H}$ coincides with $L_0^\otimes k$.

Let us keep the notation \{\(N^l\)\}_{l=1}^{\infty} introduced at the beginning of Section 3; let also \(\{n_{ij}\}_{i,j=1}^{\infty} \subseteq N_n\) stand for the collection of matrix units of $N_n$. Denote by \(\nu p_i^s, s \in \mathcal{G}_k\), the projection

$$\nu p_i^s (n_{ij} \otimes n_{j2} \otimes \cdots \otimes n_{kk}) \in M^\otimes k \subseteq L^2 (M^\otimes k, tr^\otimes k).$$

Set \(n E_1 = \sum_{s \in \mathcal{G}_k} \nu p_i^s\) and \(n p_2^s = (I - n E_1) \cdot \nu p_i^s\). Proceed with this construction by introducing \(n p_{i+1}^s = (I - n E_i) \cdot \nu p_i^s\) and \(n E_{i+1} = n E_i + \sum_{s \in \mathcal{G}_k} \nu p_{i+1}^s\).

It is clear that the projections \(n p_m^s\) are pairwise orthogonal. Introduce

$$n E_m = \sum_{j=1}^{m} \sum_{s \in \mathcal{G}_k} \nu p_j^s,$$

and \(\tau_i = tr^\otimes k (n E_i)\), which is certainly an increasing sequence. One can readily compute that \(\tau_{i+1} = \tau_i + (1 - \tau_i) \otimes k\), whence

$$\lim_{i \to \infty} tr^\otimes k (n E_i) = 1.$$ 

This implies

$$\sum_{j=1}^{\infty} \sum_{s \in \mathcal{G}_k} \nu p_j^s = I. \tag{3.37}$$

due to faithfulness of the trace $tr^\otimes k$.

**Lemma 12.** Let $A_1, A_2, \ldots, A_k$ be a family of selfadjoint operators in $M_{1J}$. Set $A = A_1 \otimes A_2 \otimes \cdots \otimes A_k$. Then for any pair of positive integers $m, n$ with $n > J$, and any $s \in \mathcal{G}_k$ there exists a unitary $U \in M$ such that $Ad U (A \cdot n p_m^s) \in \mathfrak{A}^\otimes k$.

**Proof.** Note that

$$A \cdot n p_m^s = (I - n E_{m-1}) (B_1 \otimes B_2 \otimes \cdots \otimes B_k),$$

where

$$B_i = A_i \cdot (n + m - 1) \epsilon_{s-1(i)} s^{-1}(i).$$

There exists unitary $U_i \in M_{1J}$ such that

$$U_i A_i U_i^* \in \mathfrak{A} \cap M_{1J}. \tag{3.39}$$

Since $n > J$, the operator $n U_m^* = \sum_{i=1}^{k} U_i \cdot (n + m - 1) \epsilon_{s-1(i)} s^{-1}(i)$ is unitary. By (3.38) and (3.39), $\mathfrak{N}^\otimes k (Ad U_m) (A \cdot n p_m^s) \in \mathfrak{A}^\otimes k$. \(\square\)
Corollary 13. Let $A$ be the same as in Lemma 12. Then $A$ belongs to the closed linear span of the collection of operators $\{ N^{\otimes k}(\text{Ad } u)(A^{\otimes k}) \}_{u \in U(M)}$ with respect to the norm topology of the space $L^2(M^\otimes k, \text{tr}^\otimes k)$.

Proof. One deduces from (3.37) that
\[
A = \sum_{j=1}^{\infty} \sum_{s \in \mathcal{S}_k} A \cdot n_p^s.
\]
Hence, an application of Lemma 12 proves our claim. \qed

3.5 Proof of Theorem 2

Let $\mathfrak{A}$ be a Cartan MASA in $M$ introduced the beginning of section 3. For convenience, we recall the notations used above:

\[
L_0 = \{ v \in L^2(M, \text{tr}) : \text{tr}(v) = 0 \}, \quad L^0_\mathfrak{A} = \{ x \in L^2(\mathfrak{A}, \text{tr}) : \text{tr}(x) = 0 \}.
\]

We denote by $N^{\otimes k}_0$ the restriction of $N^{\otimes k}$ to $L^0_\mathfrak{A}$.

By proposition 10,
\[
(k^E \cdot N^{\otimes k}_0(\text{Ad } U(M)) \cdot k^E)' = (k^P^\mathfrak{A}_0(\mathfrak{S}_k))'', \quad (3.40)
\]

By proposition 11,
\[
(k^E \cdot N^{\otimes k}_0(\text{Ad } U(M)) \cdot k^E)' = (k^P^\mathfrak{A}_0(\mathfrak{S}_k))'', \quad (3.41)
\]

where $k^P^\mathfrak{A}_0$ is a restriction of the representation $k^P$ (see (1.1)) to the subspace $(L^0_\mathfrak{A})^\otimes k$.

Take any operator $B' \in (N^{\otimes k}_0(\text{Ad } U(M)))'$. It follows from Proposition 8 that $k^E \in (N^{\otimes k}_0(\text{Ad } U(M)))''$. Hence, using (3.41), we have
\[
k^E \cdot B' = B' \cdot k^E = k^E \cdot B' \in (k^P^\mathfrak{A}_0(\mathfrak{S}_k))''. \quad (3.42)
\]

It follows from Corollary 13 that the maps
\[
(N^{\otimes k}_0(\text{Ad } U(M)))' \ni X' \overset{\mathcal{G}}{\mapsto} k^E X' \in (N^{\otimes k}_0(\text{Ad } U(M)))', \quad k^E,
\]
\[
(k^P^\mathfrak{A}_0(\mathfrak{S}_k))'' \ni X' \overset{\mathcal{G}}{\mapsto} k^E X' \in (k^P^\mathfrak{A}_0(\mathfrak{S}_k))''
\]

are isomorphisms. Hence, using the equality
\[
(N^{\otimes k}_0(\text{Ad } U(M)))' \ni k^E \overset{3.41}{=} (k^P^\mathfrak{A}_0(\mathfrak{S}_k))'', \quad (3.41)
\]
we get that $B' \in (k^P^\mathfrak{A}_0(\mathfrak{S}_k))''$. Theorem 2 is proven. \qed
4 The Schur-Weyl duality for automorphisms group of factor and the symmetric inverse semigroup

The symmetric inverse semigroup $\mathcal{I}_k$ is formed by all the partial bijections from the set $X_k = \{1, 2, \ldots, k\}$ to itself, with the natural definition of multiplication. An element $b \in \mathcal{I}_m$ is conveniently written as $b = (i_1, i_2, \ldots, i_r \mid j_1, j_2, \ldots, j_r)$, where $\{i_1, i_2, \ldots, i_r\} \subset X_k$, $\{j_1, j_2, \ldots, j_r\} \subset X_k$ and $i_l$ maps to $j_l$. The number $r$ involved here is denoted by rank $b$. There exists a natural involution on $\mathcal{I}_k$: $b^* = (j_1, j_2, \ldots, j_r \mid i_1, i_2, \ldots, i_r)$. Denote by $\text{id}_A \in \mathcal{I}_m$ the partial bijection obtained by restricting the identity map to $A \subset X_k$; introduce also the abbreviation $\epsilon_j = \text{id}_{X_k \setminus \{j\}}$. The subcollection $\{b \in \mathcal{I}_k : \text{rank} b = k\}$ is just the ordinary symmetric group $S_k$.

Let $\{s_i\}_{i=1}^{k-1}$ be the collection of Coxeter generators of $S_k$, where $s_i = (i \ i + 1)$ is the transposition of $i$ and $i + 1$. The following claim is due to L. Popova [11]. A more up-to-date exposition of her results is given in [10].

Theorem 14 (A description of $\mathcal{I}_m$ in the terms of the generators and the relations).

The semigroup $\mathcal{I}_k$ is generated by $\{s_i\}_{i=1}^{k-1}$ and $\epsilon_1$ with the relations as follows:

a) the Coxeter relations for $\{s_i\}_{i=1}^{k-1}$;

b) $s_i \epsilon_1 = \epsilon_1 s_i$ for all $i > 1$;

c) $(s_1 \epsilon_1)^2 = (\epsilon_1 s_1)^2 = \epsilon_1 s_1 \epsilon_1$.

This implies that one can realize $\mathcal{I}_k$ as a semigroup of $\{0, 1\}$-matrices $a = [a_{ij}]$ with the ordinary matrix multiplication in such a way that $a$ has at most one nonzero entry in each row and each column. The matrix $a = [a_{ij}]$, where $a_{11} = 0$ and $a_{ij} = \delta_{ij}$, if $i \neq 1$ or $j \neq 1$, corresponds to $\epsilon_1$ under this realization.

Let $\mathbb{C}[S_k]$ be the complex group algebra of the symmetric group $S_k$. This algebra as well as the group algebra of every finite group, is semisimple. The complex semigroup algebra $\mathbb{C}[\mathcal{I}_k]$ of the inverse symmetric semigroup is semisimple too. Namely, Munn proved the next statement.

Theorem 15 ([6]). The algebra $\mathbb{C}[R_k]$ has the decomposition

$$\mathbb{C}[R_k] = \bigoplus_{l=0}^{k} M_j(\mathbb{C}[S_l]),$$

where $M_j(A)$ is the algebra of all $j \times j$-matrices over an algebra $A$.

Denote by $\Upsilon_m$ the set of all unordered partitions of a positive integer $m \leq k$. It follows from previous theorem that the set of the irreducible representations of the semigroup $R_k$ can be naturally indexed by the set $\bigcup_{m=0}^{k} \Upsilon_m$. 

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4.1 The action of $\mathcal{J}_k$ on $L^2\left(M^{\otimes k}, \text{tr}^{\otimes k}\right)$.

Consider the operators $k\mathcal{P}_{S}(\epsilon_i)$ on $L^2\left(M^{\otimes k}, \text{tr}^{\otimes k}\right)$:

$$k\mathcal{P}_{S}(\epsilon_i)(\cdots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \cdots) = \text{tr}(n_i)(\cdots \otimes v_{i-1} \otimes I \otimes v_{i+1} \otimes \cdots). \quad (4.43)$$

Set also $k\mathcal{P}_{S}(s) = k\mathcal{P}(s)$ with $s \in \mathfrak{S}_k$, see [4]. Theorem [4] implies that $k\mathcal{P}_{S}$ admits an extension to a representation of $\mathcal{J}_k$. One has the following obvious result:

**Proposition 16.** $(\mathfrak{N}^{\otimes k}(\text{Aut } M))^\prime \subset (k\mathcal{P}_{S}(\mathcal{J}_k))'$.

Below we prove the next statement, which is the analogue of Schur-Weyl duality for Aut $M$ and $\mathcal{J}_k$.

**Theorem 17.** $(\mathfrak{N}^{\otimes k}(\text{Aut } M))^\prime = (k\mathcal{P}_{S}(\mathcal{J}_k))'$.

**Remark 2.** The operator $k\mathcal{P}_{S}(\epsilon_i)$ is an orthogonal projection in $L^2(M, \text{tr})^{\otimes k}$ and

$$\prod_{i=1}^{k} (I - k\mathcal{P}_{S}(\epsilon_i)) L^2\left(M^{\otimes k}, \text{tr}^{\otimes k}\right) = \{v \in L^2\left(M^{\otimes k}, \text{tr}^{\otimes k}\right) : k\mathcal{P}_{S}(\epsilon_i)v = 0 \text{ for all } i = 1, 2, \ldots, k\} = L_0^{\otimes k}.$$

Let $\varphi_m(X_k)$ be the collection of all non-ordered $m$-element subsets of $X_k$. With $A \in \varphi_m(X_k)$, let us introduce the pairwise orthogonal projections $k\mathcal{P}_A$ as follows

$$k\mathcal{P}_A = \prod_{j \in X_k \setminus A} k\mathcal{P}_{S}(\epsilon_j) \cdot \prod_{j \in A} (I - k\mathcal{P}_{S}(\epsilon_j)).$$

Hence

$$k\mathcal{P}_{S}(\epsilon_j)k\mathcal{P}_A = 0 \quad \text{for all } j \in A,$$

$$k\mathcal{P}_{S}(\epsilon_j)k\mathcal{P}_A = k\mathcal{P}_A \quad \text{for all } j \in X_k \setminus A. \quad (4.44)$$

Since the projections $k\mathcal{P}_A$ and $k\mathcal{P}_B$ are orthogonal for different $A$ and $B$, then operator $k\mathcal{P}_m = \sum_{A \in \varphi_m(X_k)} k\mathcal{P}_A$ is an orthogonal projection. It is clear that $k\mathcal{P}_m L^2\left(M^{\otimes k}, \text{tr}^{\otimes k}\right) = L_0^{\otimes k}$, $k\mathcal{P}_m L^2\left(M^{\otimes k}, \text{tr}^{\otimes k}\right) = \mathbb{C} I^{\otimes k}$ and

$$\sum_{m=0}^{k} k\mathcal{P}_m L^2\left(M^{\otimes k}, \text{tr}^{\otimes k}\right) = L^2\left(M^{\otimes k}, \text{tr}^{\otimes k}\right).$$

Let $m \leq k$ and let $\mathfrak{S}_m = \{s \in \mathfrak{S}_k : s(j) = j \text{ for all } j \in X_k \setminus X_m\}$, where $X_m = \{1, 2, \ldots, m\} \subset X_k$. Denote by $\chi_{\gamma}$ the character of the irreducible representation $T_\gamma$ of $\mathfrak{S}_m$, corresponding to $\gamma \in \mathbb{Y}_m$, such that its value on the unit is equal to the dimension of $T_\gamma$. Then Young projection

$$P^\gamma = \frac{\dim \gamma}{m!} \sum_{s \in \mathfrak{S}_m} \chi_{\gamma}(s) k\mathcal{P}_{S}(s)$$

$^1\varrho_0(X_k)$ is the unique empty subset.
lies in the center of $\ast$-algebra generated by $k^p H (\mathfrak{S}_m)$. Since $k^p X_m$ belongs to $k^p H (\mathfrak{S}_m)'$, then $k^p X_m = k^p X_m \cdot P^\gamma$ is an orthogonal projection from $k^p H (\mathfrak{S}_m)'$. Denote by $k^p H^\gamma_m$ the closure of the linear span of the set
\[ \{ k^p H (\mathfrak{S}_k) \} \]
with respect to the norm topology of the space $L^2 (M^\otimes k, tr^@ k)$. By proposition \[ \ref{16} \] the $k^p H^\gamma_m$-invariant subspace $k^p H^\gamma_m$ is $\mathfrak{M}^\otimes k (\text{Aut } M)$-invariant too.

### 4.2 Decomposing $\mathfrak{M}^\otimes k$ into factor-components.

Set $k^p H_{X_m} = k^p X_m L^2 (M^\otimes k, tr^@ k)$. By proposition \[ \ref{16} \] $k^p H_{X_m}$ is $\mathfrak{M}^\otimes k$-invariant. Let $\mathfrak{M}^\otimes k_{X_m}$ be the restriction of $\mathfrak{M}^\otimes k$ to $k^p H_{X_m}$. Here $m \leq k$ and we consider $X_m = \{ 1, 2 \ldots, m \}$ as a subset of $X_k$. Clearly, $k^p H_{X_m}$ is invariant under the operators $k^p (s)$, where $s \in \mathfrak{S}_m \subset \mathfrak{S}_k$, and, more generally,
\[ k^p (s) \cdot k^p A \cdot k^p (s^{-1}) = k^p (A) \text{ for all } s \in \mathfrak{S}_k \text{ and } A \in \mathfrak{M} (X_k). \quad (4.45) \]

Consider Young subgroup $\mathfrak{S}_m (k-m) = \{ s \in \mathfrak{S}_k : sX_m = X_m \}$. Let $s_1, s_2, \ldots, s_r$ be a full set of the representatives in $\mathfrak{S}_k$ of the left cosets $\mathfrak{S}_k / \mathfrak{S}_m (k-m)$, where $r = |\mathfrak{S}_k / \mathfrak{S}_m (k-m)|$. Then the projections $k^p s_j (X_m)$ are pairwise orthogonal and
\[ k^p m = \sum_{j=1}^r k^p s_j (X_m). \quad (4.46) \]

By (4.44),
\[ \mathfrak{M}^\otimes k (\theta) \cdot k^p H (s) \cdot k^p m = k^p m \cdot \mathfrak{M}^\otimes k (\theta) \cdot k^p H (s) \quad (4.47) \]
for all $\theta \in \text{Aut } M$ and $s \in \mathfrak{S}_k$. We emphasize again that $k^p X_m \cdot k^p H (s_j) = 0$ for all $j \in X_m$. Therefore,
\[ (k^p X_m \cdot k^p H (s_j))'' = (k^p X_m \cdot k^p H (s_j))'' . \quad (4.48) \]
Let $\gamma \in \mathcal{Y}_m$ be an unordered partition of $m$ and let $\chi_\gamma$ be the character of the corresponding irreducible representation of $\mathfrak{S}_m$. Set
\[ P^\gamma = \frac{\dim \gamma}{m!} \sum_{s \in \mathfrak{S}_m} \chi_\gamma (s) \cdot k^p H (s). \quad (4.49) \]

Since the projections $\{ k^p s_j (X_m) \}_{j=1}^r$ are pairwise orthogonal and
\[ k^p X_m \in \{ k^p H (\mathfrak{S}_m) \}' \text{ then } k^p X_m^\gamma = P^\gamma \cdot k^p X_m \]
is an orthogonal projection from the center of $\ast$-algebra, generated by the operators $k^p H (\mathfrak{S}_m)\otimes k (\text{Aut } M)$ and $k^p X_m \cdot k^p H (\mathfrak{S}_m)$. Therefore, the operator
\[ k^p X_m = \sum_{j=1}^r k^p (s_j) \cdot k^p X_m^\gamma \cdot k^p (s_j^{-1}) \quad (4.50) \]
is an orthogonal projection too. Moreover, the projections \( k_{P_m}^\gamma \) and \( k_{P_m}^\gamma \) are orthogonal for different \( \gamma, \tilde{\gamma} \in \Gamma_m \) and the following equality holds

\[
k_{P_m} = \sum_{\gamma \in \Gamma_m} k_{P_m}^\gamma. \tag{4.51}
\]

The next statement follows from theorem 22.

**Lemma 18.** The family of the operators \( \{ k_{P_m} \} \) define the unitary representation \( k_{P_m} \) of the group \( \mathcal{G}_m \) in the subspace \( k_{P_m} \mathcal{H}_m \) and one has \( (\mathcal{N}_{X_m} \mathcal{H}_m (\mathcal{A} \mathcal{T} M))^\prime = (k_{P_m} \mathcal{H}_m (\mathcal{G}_m))^\prime. \)

Define the representation \( \mathcal{H} \) of the semigroup \( (\mathcal{A} \mathcal{T} M) \times \mathcal{K} \) as follows

\[
k_{(\mathcal{H} \mathcal{T}, s)} = \mathcal{H}^\mathcal{K} (\mathcal{T}) \cdot k_{P_m} (s), \quad \text{where} \quad \mathcal{T} \in (\mathcal{A} \mathcal{T} M), \quad s \in \mathcal{K}. \tag{4.52}
\]

**Lemma 19.** Projection \( k_{P_m}^\gamma \) belongs to \( \mathcal{H}^*-\)algebra \( \mathcal{H}((\mathcal{A} \mathcal{T} M) \times \mathcal{K}) \) and the restriction of \( \mathcal{H} \) to the subspace \( k_{P_m}^\gamma L^2 (\mathcal{M} \mathcal{H}^k, \mathcal{F} \mathcal{T}^k) \) is the irreducible representation of the semigroup \( (\mathcal{A} \mathcal{T} M) \times \mathcal{K}. \)

**Proof.** Let us prove that

\[
k_{P_m}^\gamma \in \mathcal{H}((\mathcal{A} \mathcal{T} M) \times \mathcal{K}) \quad \text{(see (4.50)).} \tag{4.53}
\]

Each \( t \in \mathcal{G}_k \) defines the bijection \( b_t \) of the set \( \{ s_1, s_2, \ldots, s_r \} \), where \( r = \mathcal{G}_k / \mathcal{G}_m (k-m) \).

Hence, since \( k_{P_m}^\gamma = \sum_{j=1}^{(\mathcal{G}_k / \mathcal{G}_m (k-m))} k_{P_m} (s_j) \cdot k_{P_m}^\gamma \cdot k_{P_m} (s_j) \), then

\[
k_{P_m} \cdot k_{P_m}^\gamma \cdot k_{P_m} (t^{-1}) = \sum_{j=1}^{(\mathcal{G}_k / \mathcal{G}_m (k-m))} k_{P_m} (t s_j) \cdot k_{P_m}^\gamma \cdot k_{P_m} (s_j t^{-1} \gamma)
\]

\[
\sum_{j=1}^{(\mathcal{G}_k / \mathcal{G}_m (k-m))} k_{P_m} (b_t (s_j) h_j) \cdot k_{P_m}^\gamma \cdot k_{P_m} (h_j^{-1} (b_t (s_j))^{-1}), \quad \text{where} \quad h_j \in \mathcal{G}_m.
\]

Now, using the equality \( k_{P_m} (h_j) \cdot k_{P_m}^\gamma \cdot k_{P_m} (h_j^{-1}) = k_{P_m}^\gamma \), we obtain

\[
k_{P_m} (t) \cdot k_{P_m}^\gamma \cdot k_{P_m} (t^{-1}) = \sum_{j=1}^{(\mathcal{G}_k / \mathcal{G}_m)} k_{P_m} (b_t (s_j)) \cdot k_{P_m}^\gamma \cdot k_{P_m} ((b_t (s_j))^{-1}).
\]

Since \( b_t \) is the bijection, then

\[
\sum_{j=1}^{(\mathcal{G}_k / \mathcal{G}_m)} k_{P_m} (b_t (s_j)) \cdot k_{P_m}^\gamma \cdot k_{P_m} ((b_t (s_j))^{-1})
\]

\[
= \sum_{j=1}^{(\mathcal{G}_k / \mathcal{G}_m (k-m))} k_{P_m} (s_j) \cdot k_{P_m}^\gamma \cdot k_{P_m} (s_j^{-1}).
\]
Thus
\[ kP(t) \cdot kP_m = kP_m \] for all \( t \in \mathbb{S}_k \).
\[ (4.54) \]

Set \( \mathcal{A}_i = \{ j \in \{1, 2, \ldots, |\mathcal{S}_k / \mathcal{S}_m(k-m)| \} : s_j^{-1}(i) \notin X_m \} \). Since
\[ kP_X = \gamma \cdot kP_X = kP_X \gamma \cdot P \gamma \text{, then, using } (4.44) \text{ and } (4.19), \]
we have
\[ kP(\epsilon_i) \cdot kP_X = kP_X \gamma \cdot kP(\epsilon_i) = \sum_{j \in \mathcal{A}_i} kP(s_j) \cdot kP_X \gamma \cdot kP(s_j^{-1}). \]

Now we conclude from (4.54) that \( kP_X \gamma \in kP(\mathcal{A}_k) \). Hence, applying Proposition 16 we obtain (4.54).

Therefore, the operators \( k\Pi_m(\theta, s) = kP_X \gamma \cdot k\Pi(\theta, s) \), where \( \theta \in \text{Aut } M, s \in \mathcal{A}_k \), define \(*\)-representation of semigroup \( \text{Aut } M \times \mathcal{A}_k \).

Let us prove that \( k\Pi_m \) is an irreducible representation; i.e.
\[ k\Pi_m(\text{Aut } M \times \mathcal{A}_k)' = C \cdot kP_X \gamma. \]

First, we notice that \( kP_X \gamma \in kP_m \), \( kP(\mathcal{A}_k) \subset k\Pi_m(\text{Aut } M \times \mathcal{A}_k)' \). Therefore, if \( B' \in k\Pi_m(\text{Aut } M \times \mathcal{A}_k)' \)
then
\[ B' \cdot kP_X = kP_X \gamma \cdot k\Pi_m(\text{Aut } M \times \mathcal{A}_k)' \cdot kP_X. \]

Hence, applying Lemma 18 we see that
\[ B' \cdot kP_X = c \cdot kP_X \gamma, \]
where \( c \in C \).

Now, using (4.54), we obtain \( B' = B' \cdot kP_X = c \cdot kP_X \gamma. \)

4.3 The proof of Theorem 17
Let \( B' \) lies in \( (\mathcal{H}^{\otimes k} (\text{Aut } M))' \). For the matrix \( \theta U = [\theta U_{ij}, v_j] \) (see (3.19)), we denote by \( \theta U \) an element from \( M_{1, j} \) of the view
\[ \theta U = \sum_{i, j, i' \in \mathcal{J}} \theta U_{ij}, v_{i', v_j}. \]

Let \( a \in M_{1, j} \cap \mathfrak{A} \). Using (3.19) and (3.24), we obtain
\[ kE \circ \mathcal{H}^{\otimes k}(\text{Ad } \theta U)(kP_m(a)) = \left(1 - \frac{1}{n} \right)^m kP_m(a). \]

It follows that
\[ kE \circ \mathcal{H}^{\otimes k}(\text{Ad } \theta U) \circ kE = \sum_{j=0}^{k} \left(1 - \frac{1}{n} \right)^j kE \circ kP_j = \left(1 - \frac{1}{n} \right)^j kE \circ kP_j \circ B' \]

Therefore,
\[ \sum_{j=0}^{k} \left(1 - \frac{1}{n} \right)^j B' \circ kE \circ kP_j = \sum_{j=0}^{k} \left(1 - \frac{1}{n} \right)^j kE \circ kP_j \circ B' \]
Hence, thanks to the relation \( kP_l \circ kP_m = \delta_{lm} kP_l \), we have
\[
\left( 1 - \frac{|\theta - 1|^2}{n} \right)^m kP_l \circ B' \circ kE \circ kP_m
= \sum_{j=0}^{k} \left( 1 - \frac{|\theta - 1|^2}{n} \right)^j kP_l \circ kE \circ kP_j \circ B' \circ kP_m.
\]

Now we conclude from propositions [5] and [16] that
\[
\left( 1 - \frac{|\theta - 1|^2}{n} \right)^m kP_l \circ B' \circ kE \circ kP_m = \left( 1 - \frac{|\theta - 1|^2}{n} \right)^l kP_l \circ kE \circ B' \circ kP_m
\]
and
\[
\left( 1 - \frac{|\theta - 1|^2}{n} \right)^m kP_l \circ B' \circ kE \circ kP_m = \left( 1 - \frac{|\theta - 1|^2}{n} \right)^l kP_l \circ B' \circ kE \circ kP_m.
\]

Therefore, \( kP_l \circ B' \circ kE \circ kP_m = \delta_{lm} kP_m \circ B' \circ kE \circ kP_m \). Now, using the relation \( \sum_{j=0}^{k} kP_j = 1 \), we have
\[
B' \circ kE = kE \circ B' = \sum_{m=0}^{k} kP_m \circ B' \circ kE \circ kP_m.
\]

Hence, applying corollary [13] we conclude
\[
B' = \sum_{m=0}^{k} kP_m \circ B' \circ kP_m. \tag{4.55}
\]

Let us prove that \( B'_m \overset{\text{def}}{=} kP_m \circ B' \circ kP_m \) lies in \( * \)-algebra \( kP_m \overset{bP(\mathscr{A})}{\longrightarrow} kP_m \) (see (4.51) and lemma [18]).

Since \( kP_m \overset{\text{def}}{=} \sum_{A \in \mathcal{P}_m(X_k)} kP_A \), then \( B'_m \overset{\text{def}}{=} \sum_{A, B \in \mathcal{P}_m(X_k)} kP_A \circ B'_m \circ kP_B \). There exist \( s_A, s_B \in \mathcal{S}_k \) such that
\[
s_A(X_m) = A \text{ and } s_B(X_m) = B. \tag{4.56}
\]

Hence, using (4.55), we have
\[
kP_A \circ B'_m \circ kP_B = kP(s_A) \circ kP_{X_m} \circ kP(s^{-1}_B) \circ B'_m \circ kP(s_B) \circ kP_{X_m} \circ kP(s^{-1}_B).
\]

It follows from lemma [18] that \( kP_{X_m} \circ kP(s^{-1}_B) \circ B'_m \circ kP(s_B) \circ kP_{X_m} \) lies in algebra \( kP_{X_m} \overset{bP(\mathcal{S}_m)}{\longrightarrow} kP_{X_m} \). Therefore,
\[
kP_A \circ B'_m \circ kP_B \in \left( bP(\mathscr{A}) \right)^\prime.
\]

Thus \( B' = \sum_{m=0}^{k} \sum_{A, B \in \mathcal{P}_m(X_k)} kP_A \circ B'_m \circ kP_B \) lies in \( \left( bP(\mathcal{S}_k) \right)^\prime \). This completes the proof of Theorem 17.
5 The Schur-Weyl duality for Aut $M$ and the infinite symmetric group

Let $\mathfrak{S}_\infty$ be the group of all bijections of the set $\mathbb{Z}_{>0} = \{1, 2, \ldots\}$. Set $\mathfrak{S}_n = \{s \in \mathfrak{S}_\infty : s(k) = k$ for all $k > n\}$.

Further we will consider $L^2(M, \text{tr})^\otimes_n$ as the subspace of $L^2(M, \text{tr})^\otimes(n+1)$, using the embedding

$$L^2(M, \text{tr})^\otimes_n \ni m_1 \otimes \ldots \otimes m_n \mapsto m_1 \otimes \ldots \otimes m_n \otimes I \in L^2(M, \text{tr})^\otimes(n+1).$$

Let $L^2(M, \text{tr})^\otimes\infty$ be the completion of the pre-Hilbert space $\bigcup_{n=1}^\infty L^2(M, \text{tr})^\otimes_n$.

It is convenient to consider $\bigcup_{n=1}^\infty L^2(M, \text{tr})^\otimes_n$ as the linear span of the vectors $v_1 \otimes \cdots \otimes v_n \otimes I \otimes I \otimes \cdots$, where $v_j \in M$. At the same time, we will to identify $L^2(M, \text{tr})^\otimes_n$ with the closure of the linear span of all vectors $v_1 \otimes \cdots \otimes v_n \otimes v_{n+1} \otimes \cdots$, where $v_j = I$ for all $i > n$. Then the elements $\theta \in \text{Aut } M$ and $s \in \mathfrak{S}_\infty$ act on $L^2(M, \text{tr})^\otimes\infty$ as follows

$$\mathcal{R}^\otimes\infty(\theta)(v_1 \otimes \cdots \otimes v_n \otimes \cdots) = (\mathfrak{R}(\theta)v_1) \otimes \cdots \otimes (\mathfrak{R}(\theta)v_n) \otimes \cdots;$$

$$\mathcal{P}(s)(v_1 \otimes \cdots \otimes v_n \otimes \cdots) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(n)} \otimes \cdots.$$

We now have:

**Theorem 20.** \(\mathcal{R}^\otimes\infty(\text{Aut } M)\)' = \(\{\mathcal{P}(\mathfrak{S}_\infty)\}\)'.

**Proof.** Let $(k l)$ be a transposition that swaps $k$ and $l$. We denote by $\mathfrak{S}_{n, \infty}$ the subgroup $\{s \in \mathfrak{S}_\infty : s(k) = k$ for all $k \in \{1, 2, \ldots, n\}\}$.

Let us prove that

$$L^2(M, \text{tr})^\otimes_n = \{v \in L^2(M, \text{tr})^\otimes\infty : \mathcal{P}(s)v = v$ for all $s \in \mathfrak{S}_{n, \infty}\}. \quad (5.57)$$

Fix any $v \in L^2(M, \text{tr})^\otimes\infty$ such that $\mathcal{P}(s)v = v$ for all $s \in \mathfrak{S}_{n, \infty}$.

Take orthonormal basis $\{e_k\}_{k=0}^\infty$ in $L^2(M, \text{tr})$, where $e_0 = I$ and $e_k \in M$ for all $k$. Denote by $\mathfrak{R}$ a set of all sequences $\mathfrak{r} = \{k_i\}_{i=1}^\infty$, $k_i \in \{0, 1, \ldots\}$ with the property: there exists same natural $N(\mathfrak{r})$ such, that $k_i = 0$ for all $i > N(\mathfrak{r})$. For convenience, we set $N(\mathfrak{r}) = \min \{m : k_i = 0$ for all $i > m\}$. Then the set $\{e_\mathfrak{r} = e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_{N(\mathfrak{r})}} \otimes I \otimes I \otimes \ldots\}_{\mathfrak{r} \in \mathfrak{R}}$ is an orthonormal basis in $L^2(M, \text{tr})^\otimes\infty$. Set

$$v = \sum_{\mathfrak{r} \in \mathfrak{R}} c_\mathfrak{r}(v)e_\mathfrak{r} \text{ where } c_\mathfrak{r}(v) \in \mathbb{C}.\quad (5.57)$$

To prove (5.57) it is sufficient to establish that $c_\mathfrak{r}(v) = 0$ if $N(\mathfrak{r}) > n$.

Consider an orthogonal projection $O_m$ in $L^2(M, \text{tr})^\otimes\infty$ that is defined as follows

$$O_m(e_{k_{m-1}} \otimes e_k \otimes e_{k_{m+1}} \otimes \ldots e_{k_{N(\mathfrak{r})}} \otimes I \otimes I \otimes \ldots) = \text{tr}(e_k)(e_{k_{m-1}} \otimes e_k \otimes e_{k_{m+1}} \otimes \ldots e_{k_{N(\mathfrak{r})}} \otimes I \otimes I \otimes \ldots). \quad (5.58)$$

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It is easily seen that the sequence \( \{ \mathcal{P}(l) \} \) converges in the week operator topology to \( O_m = w - \lim_{l \to \infty} \mathcal{P}(l) \). Therefore,

\[
O_m \in \left( \mathcal{P}(\mathbb{S}_\infty) \right)'' \text{ for all } m, \quad \text{and } O_m v = v \text{ for all } m > n. \quad (5.59)
\]

Hence, applying \( (5.58) \), we have \( c_\varepsilon(v) = 0 \) for all \( \varepsilon \) such that \( N(\varepsilon) > n \). This proves equality \( (5.57) \).

According to \( (5.58) \), we have that the operator \( \mathfrak{P}_{n,N} = O_{n+1}O_{n+2} \cdots O_N \), where \( N > n \) is an orthogonal projection. Since \( \mathfrak{P}_{n,m} \geq \mathfrak{P}_{n,m+1} \) for all \( m > n \), there exists the orthogonal projection \( \mathfrak{P}_n = \lim_{m \to \infty} \mathfrak{P}_{n,m} \). By \( (5.59) \), \( \mathfrak{P}_n \) belongs to \( \left( \mathcal{P}(\mathbb{S}_\infty) \right)'' \). Using \( (5.58) \), we obtain

\[
\mathfrak{P}_n \left( v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes v_{n+1} \otimes \cdots \otimes v_j \otimes \cdots \right) = \left( \prod_{j=n+1}^\infty \right) \left( v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes I \otimes \cdots \otimes I \otimes \cdots \right). \quad (5.60)
\]

Therefore, \( \mathfrak{P}_n \left( L^2 (M, tr) \otimes^n \right) = L^2 (M, tr)^\otimes^n \).

Take operator \( B' \in \{ \mathfrak{N}^\otimes (\text{Aut } M) \}' \). Since projection \( \mathfrak{P}_n \in \left( \mathcal{P}(\mathbb{S}_\infty) \right)'' \) and \( \left( \mathcal{P}(\mathbb{S}_\infty) \right)'' \subset \{ \mathfrak{N}^\otimes (\text{Aut } M) \}' \), then operator \( B'_n = \mathfrak{P}_n B' \mathfrak{P}_n \) belongs \( \{ \mathfrak{N}^\otimes (\text{Aut } M) \}' \), too. It follows from section 4 that

\[
\mathfrak{P}_n \mathfrak{N}^\otimes (\theta) \mathfrak{P}_n = \mathfrak{N}^\otimes (\theta), \quad \theta \in \text{Aut } M,
\]

\[
\mathfrak{P}_n \mathfrak{N}^\otimes (s) \mathfrak{P}_n = \mathfrak{N}^\otimes (s), \quad \text{for all } s \in \mathbb{S}_n,
\]

\[
\mathfrak{P}_n O_i \mathfrak{P}_n = k_{\mathfrak{P}_n} (\epsilon_i), \quad i = 1, 2, \ldots, n.
\]

Hence, applying Theorem 17, we obtain that \( B'_n \) belongs to \( \left( \mathcal{P}(\mathbb{S}_\infty) \right)'' \) (see \( (5.58) \)). Since \( B' = \lim_{n \to \infty} \) in the strong operator topology, operator \( B' \) lies in \( \left( \mathcal{P}(\mathbb{S}_\infty) \right)'' \), too. This completes the proof of Theorem 20.

6 A mapping from unitary to doubly stochastic matrices

Recall that \( n \times n \)-matrix \( P = [P_{ij}] \) is called doubly stochastic if \( \sum_{i=1}^n P_{ij} = 1, \)

\( \sum_{j=1}^n P_{ij} = 1 \) and \( P_{ij} \geq 0 \) for all \( i, j \). The property of \( P \) being doubly stochastic is obviously equivalent to the vector \( \left( \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array} \right) \) being invariant both for \( P \) and the transpose \( P^t \). Let \( DS_n \) stand for the set of all doubly stochastic \( n \times n \) matrices. There exists an orthogonal matrix \( O = [O_{ij}] \) such that for any \( P \in DS_n \) one has \( (OPO^{-1})_{ij} = \delta_{ij} \) and \( (OPO^{-1})_{jl} = \delta_{lj} \) \( (j = 1, 2, \ldots, n) \), where \( \delta_{kl} \) is the Kronecker delta. Let us fix such matrix \( O \).
Lemma 21. Let \( \frac{1}{2}M_n(\mathbb{R}) \) be the set of all real \( n \times n \) matrices of the form
\[
\begin{bmatrix}
\gamma & 0 & 0 & \cdots & 0 \\
0 & a_{22} & a_{23} & \cdots & a_{2n} \\
0 & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix},
\]
Suppose that a doubly stochastic matrix \( P = [P_{ij}] \) has only nonzero entries. Then there exists \( \kappa > 0 \) such that the matrix \( P + O^{-1}BO \) is doubly stochastic for any matrix \( B = [B_{ij}] \in \frac{1}{2}M_n(\mathbb{R}) \) such that \( |B_{ij}| < \kappa \) for all \( i, j \).

By the above Lemma, each double stochastic matrix \( P \) with positive entries is an interior point of \( DS_n \), and the real dimension of the tangent space \( T_p DS_n \) at this point is \( (n - 1)^2 \). In addition, we have a linear one-to-one map between \( T_p DS_n \) and \( \frac{1}{2}M_n(\mathbb{R}) \).

We need in the sequel the obvious claim as follows.

Proposition 22. Let \( U \) be a open subset in \( DS_n \), and \( GL(n, \mathbb{R}) \) stand for the group of real invertible \( n \times n \) matrices. Identify the group \( GL(n - 1, \mathbb{R}) \) with the subgroup \( (O^{-1} \cdot \frac{1}{2}M_n(\mathbb{R}) \cdot O) \cap GL(n, \mathbb{R}) \subset GL(n, \mathbb{R}) \). Then the topological component of the identity in \( GL(n - 1, \mathbb{R}) \) is contained in
\[
\bigcup_{j=1}^{\infty} \left( (U \cap GL(n, \mathbb{R})) \cdot (U \cap GL(n, \mathbb{R}))^{-1} \right)^j.
\]

6.1

Denote by \( U(n) \) a group of unitary \( n \times n \)-matrices. We will consider \( U(n) \) and \( DS_n \) as a real manifolds of the dimension \( n^2 \) and \( (n - 1)^2 \) respectively. Let \( f : U(n) \hookrightarrow DS_n \) be a smooth map and let \( df_u \) be a differential of \( f \) in the point \( u \). Mapping \( df_u \) is the linear operator from the tangent space \( T_u U(n) \) at \( u \) to the tangent space \( T_{f(u)} DS_n \). Function \( f \) is a submersion at a point \( u \in U(n) \) if \( df_u \mid_{T_u U(n)} = T_{f(u)} \) for some \( \kappa > 0 \). In connection with formula \( \text{(3.15)} \) we will find the unitary matrices \( u \) such that the map
\[
U(n) \ni u = [u_{ij}] \mapsto [u_{ij}]^2 \in DS_n \text{ is submersion at the point } u. \tag{6.61}
\]
Hence will follow that there exists the open neighborhood \( U \) of the point \( u \) such that \( \mu(U) \subset DS_n \) is open subset.

We adopt below the results of A. Karabegov \[12\] to make them applicable to proving Proposition \[10\].

Denote by \( SH_n \) the set of all skew-hermitian \( n \times n \)-matrices. It is clear, that the dimension of \( U(n) \), as a real manifold, is equal \( n^2 \). Considering the smooth one parameter family \( U(t) = [U_{kl}(t)] \subset U(n) \) and using the equality \( U(t)^* \cdot U(t) = I_n \), we obtain
\[
U(0)^* \cdot U'(0) + U'(0)^* \cdot U(0) = 0, \text{ where } U'(0) = [U_{kl}'(0)].
\]

Hence
\[
U'(0) \cdot U(0)^* + U(0) \cdot U'(0)^* = 0. \tag{6.62}
\]
This implies that $U'(0) \in T_u(U(n))$ is identified with the skew Hermitian matrix $X = u^* \cdot U'(0) \in T_{U(n)}$ treated as an element of the Lie algebra $\mathfrak{sh}_n$ of $U(n)$. Here $u = [u_{kl}] = U(0)$.

Applying (6.61), we see that $d\mu_u : T_u(U(n)) \rightarrow T_{\mu(u)} \mathcal{D} \mathcal{S}_n$ acts as follows

$$d\mu_u \left( U'(0) \right) = \left[ u_{kl}U_{kl}'(0) + U_{kl}(0)u_{kl}' \right] \in T_{\mu(u)} \mathcal{D} \mathcal{S}_n.$$ 

Let us introduce the operator $d\mu_u : T_u(U(n)) \rightarrow T_{\mu(u)} \mathcal{D} \mathcal{S}_n$ which acts by

$$d\mu_u(A) = d\mu_u(uA), \quad A \in T_u(U(n)), \quad uA \in T_u(U(n)). \quad (6.63)$$

Therefore,

$$d\mu_u \left( u^* U'(0) \right) = \left[ u_{kl}U_{kl}'(0) + U_{kl}(0)u_{kl}' \right] \in T_{\mu(u)} \mathcal{D} \mathcal{S}_n.$$ 

Hence, assuming that all entries of $u = U(0) = [u_{kl}]$ are nonzero, we obtain

$$d\mu_u \left( u^* U'(0) \right) = \left[ \left( U_{kl}'(0) \frac{u_{kl}}{u_{kl}} + \frac{U_{kl}(0)}{u_{kl}} \right) |u_{kl}|^2 \right]. \quad (6.64)$$

Now we can to rewrite the equality (6.62) as follows

$$\sum_{j=1}^n u_{kj} \frac{U_{kl}'(0)}{u_{kj}} a_{lj} + \sum_{j=1}^n u_{kj} \frac{U_{lj}'(0)}{a_{lj}} a_{lj} = 0. \quad (6.65)$$

Consider the family $\mathcal{U} = \{ \mathcal{U}_{kl} \}$ of the unitary matrices, where

$$\mathcal{U}_{kl} = \delta_{kl} + \frac{\theta - 1}{n}, \quad \theta \in \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}.$$ \quad (6.66)

On the space $\mathcal{M}_n$ of all complex $n \times n$-matrices define two inner products

$$(A, B)_{\theta} = \sum_{k,l=1}^n A_{kl}B_{kl} |\mathcal{U}_{kl}|^2, \quad A = [A_{kl}], \quad B = [B_{kl}],$$

$$(A, B)_{\mathbb{T}} = \text{Tr} \left( AB^* \right), \quad \text{where Tr is an ordinary trace on } \mathcal{M}_n.$$ 

Denote by $\mathcal{M}_n^\theta$ and $\mathcal{M}_n^{\mathbb{T}}$ the corresponding Hilbert spaces.

Now we introduce two operators $\mathcal{C}_\theta$ and $\mathcal{D}_\theta$ as follows

$$\mathcal{M}_n^\theta \ni f = [f_{kl}] \overset{\mathcal{C}_\theta}{\rightarrow} Y = [Y_{kl}] \in \mathcal{M}_n^{\mathbb{T}}, \quad \text{where } Y_{kl} = \sum_{j=1}^n \mathcal{U}_{kj}f_{kj}\overline{\mathcal{U}_{lj}};$$

$$\mathcal{M}_n^\theta \ni g = [g_{kl}] \overset{\mathcal{D}_\theta}{\rightarrow} Z = [Z_{kl}] \in \mathcal{M}_n^{\mathbb{T}}, \quad \text{where } Z_{kl} = \sum_{j=1}^n \mathcal{U}_{kj}g_{kj}\overline{\mathcal{U}_{lj}}.$$ 

Hence, using the orthogonality relations between $\mathcal{U}_{k,l}$, can obtain the formulas for the inverse operators

$$(\mathcal{C}_\theta^{-1}Y)_{kj} = \mathcal{U}_{kj}^{-1} \sum_{j=1}^n Y_{kj} \mathcal{U}_{jq} \quad \text{and} \quad (\mathcal{D}_\theta^{-1}Y)_{kj} = \mathcal{U}_{kj}^{-1} \sum_{j=1}^n Y_{kj} \mathcal{U}_{jq}. \quad (6.67)$$
Set \( u = U(0) = \theta U, \) \( X = u^*U'(0), \) \( f_{kj} = \frac{U_{kj}(0)}{u_{kj}} \) and \( \overline{f} = [\overline{f_{kj}}]. \) Then
\[
xu^* = U'(0) \cdot u^* = C_\theta f \quad \text{and} \quad Xu^* = u \cdot U'(0)^* = D_\theta \overline{f}.
\] (6.68)

Hence, applying (6.65), we have
\[
C_\theta f = xu^*, D_\theta \overline{f} = -ux^*.
\] (6.69)

It easy to check that the next statement holds.

**Proposition 23** (Proposition 2.1 [12]). If \( \theta \notin \{-1, 1\} \) then the mappings \( C_\theta \) and \( D_\theta \) are unitary isomorphisms between the Hilbert spaces \( M^\theta_n \) and \( M^\theta_{n^*}. \)

Furthermore, using (6.64) and (6.69), we obtain for \( X = u^*U'(0) \) and \( u = \theta U \)
\[
(\theta d_{\mu U} X)_{kl} = (C^{-1}_\theta (ux^*) - D^{-1}_\theta (ux^*))_{kl} \cdot |u_{kl}|^2.
\] (6.70)

Now we will prove the next statement.

**Theorem 24** (Theorem 5.1 [12]). Let \( u = \theta U, \) where \( \theta \notin \{-1, 1\}. \) Then the dimension of the kernel of the operator \( (C^{-1}_\theta - D^{-1}_\theta) \) is equal to \( 2n - 1. \)

Since the real dimensions of \( T_n U(n) \) and \( T_{\mu_n} DS_n \) are equal \( n^2 \) and \( (n - 1)^2, \) applying (6.71), we obtain the next

**Corollary 25.** If \( \theta \notin \{-1, 1\} \) then the spaces \( d_{\mu U}(T_n U(n)) \) and \( T_{\mu_n} DS_n \) coincide.

**Proof of Theorem 24.** Let \( D_n \) be the set of all diagonal matrices in \( SH_n, \) and let \( K_n \) be a real subspace of \( SH_n, \) generated by \( D_n \) and \( uD_n u^*. \) The ordinary calculations shows that
\[
C^{-1}_\theta \eta = D^{-1}_\theta \eta \quad \text{for all} \ \eta \in K_n \quad \text{and} \quad \dim K_n = 2n - 1.
\] (6.71)

Define the entries of the matrix \( B = [B_{pq}] \) as follows
\[
B_{pq} = \begin{cases} 
0, & \text{if} \ p = q \ or \ (p \notin \{k, l\}) \ \& \ (q \notin \{k, l\}); \\
-1, & \text{if} \ p = k, q = l; \\
1, & \text{if} \ p = l, q = k; \\
\frac{n + \bar{\theta} - 1}{n + \bar{\theta} - 1}(n - 2), & \text{if} \ q = l, p \neq k \ and \ p \neq l; \\
\frac{n + \theta - 1}{n + \theta - 1}(n - 2), & \text{if} \ p = k, q \neq l \ and \ q \neq k; \\
\frac{n + \bar{\theta} - 1}{n + \bar{\theta} - 1}(n - 2), & \text{if} \ q = k, p \neq k \ and \ p \neq l; \\
\frac{n + \theta - 1}{n + \theta - 1}(n - 2), & \text{if} \ p = l, q \neq l \ and \ q \neq k.
\end{cases}
\] (6.72)

Let \( B_n \) be a real subspace of \( SH_n, \) generated by the matrices \( B, \) where \( k, l = 1, 2, \ldots, n. \) By the calculations can be be checked that the subspaces \( K_n \) and \( B_n \) mutually orthogonal and
\[
C^{-1}_\theta \eta = -\frac{n + \bar{\theta} - 1}{n + \theta - 1}D^{-1}_\theta \eta \quad \text{for all} \ \eta \in B_n.
\] (6.73)

It easy to check that the matrices \( B, B, \ldots, (n - 1)B \) are linearly independent. Therefore,
\[
\dim B_n \geq n - 1.
\] (6.74)
Let $O_n$ be one dimensional subspace $\mathbb{R}iO \subset \mathcal{S}H_n$, where $O = [O_{kl}] = [\delta_{kl} - 1]$. By calculations we see that $K_n$ and $B_n$ are orthogonal to $O_n$ and
\[
C^{-1}_\theta O = -\theta \frac{n+\theta-1}{n+\theta-1} D^{-1}_\theta O.
\] (6.75)

Denote by $IS_n$ the real subspace of the matrices $A = [A_{kl}] \in \mathcal{S}H_n$ with the purely imaginary entries such that
\[
A_{kk} = 0 \text{ and } \sum_{l=1}^{n} A_{kl} = 0 \text{ for all } k = 1, 2, \ldots, n.
\] (6.76)

Hence, using (6.67), we obtain
\[
C^{-1}_\theta A = -\theta D^{-1}_\theta A \text{ for all } A \in IS_n.
\] (6.77)

At last we introduce the real subspace $RS_n$ of the matrices $A = [A_{kl}] \in \mathcal{S}H_n$ with the real entries which satisfy (6.76). It follows, by the similar calculations, that
\[
C^{-1}_\theta A = \theta D^{-1}_\theta A \text{ for all } A \in RS_n.
\] (6.78)

Applying (6.76), we obtain
\[
\dim IS_n = \left(\sum_{j=1}^{n-1} (n-j) \right) - n = \frac{n(n-3)}{2}.
\] (6.79)

Analogously,
\[
\dim RS_n = \left(\sum_{j=1}^{n-1} (n-j) \right) - (n-1) = \frac{(n-1)(n-2)}{2}.
\] (6.80)

By the ordinary calculations can to show that subspaces $K_n, B_n, O_n, IS_n, RS_n$ are pairwise orthogonal. Hence, applying (6.71), (6.74), (6.79) and (6.80), we have
\[
\dim (K_n \oplus B_n \oplus O_n \oplus IS_n \oplus IR_n) \geq n^2.
\]

Therefore, $K_n \oplus B_n \oplus O_n \oplus IS_n \oplus IR_n = \mathcal{S}H_n$. Thus any $\Psi \in \mathcal{S}H_n$ can to write as follows $\Psi = \Psi_K + \Psi_B + \Psi_O + \Psi_{IS} + \Psi_{RS}$, where $\Psi_k$ lies in the corresponding orthogonal component. If $\Psi$ lies in kernel of the operator $d\mu_u = (C^{-1}_\theta - D^{-1}_\theta)$ then, using (6.71), (6.73), (6.75), (6.77) and (6.78), we obtain
\[
D_\theta \circ d\mu_u \Psi = \left(-\theta \frac{n+\theta-1}{n+\theta-1} - 1+ \right) \Psi_B + \left(-\theta \frac{n+\theta-1}{n+\theta-1} - 1+ \right) \Psi_O \\
-(\theta + 1) \Psi_{IS} + (\theta - 1) \Psi_{RS}.
\]

Since $\theta \notin \{-1, 1\}$, then $\Psi_B = \Psi_O = \Psi_{IS} = \Psi_{RS} = 0$. Therefore, $\Psi = \Psi_K \in K_n$.

The next statement follows from Corollary 25.

**Corollary 26.** If $\theta \notin \{-1, 1\}$ then $d\mu_u$ is submersion at the point $u = \theta U$. Therefore, there exists an open subset $U$ such that $u \in U$ and $\mu(U)$ is an open subset in $DS_n$.  

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References

[1] Beltita D., Neeb KH., Schur-Weyl Theory for C*-algebras, Mathematische Nachrichten 285 (2012), p. 1170 - 1198.

[2] D. Birkhoff, Tres observaciones sobre el algebra lineal. Univ. Nac. Tucuman Rev, A5 (1946) 147-151

[3] W. Fulton, J. Harris, Representations theory ((A first Course), Springer, 1991, 551pp.

[4] Nessonov N.I., An analogue of Schur–Weyl duality for the unitary group of a $II_1$-factor, Mat. Sb., 2019, Volume 210, Number 3, Pages 162–188.

[5] W. D. Munn, Matrix representations of semigroups, Proc. Cambridge Philos. Soc., 51, 1955, 1-15.

[6] W. D. Munn, The characters of the symmetric inverse semigroup, Proc. Cambridge Philos. Soc., 53, 1957, 13-18.

[7] A. A. Kirillov, Elements of the theory of representations, 2nd ed., Nauka, Moscow 1978, 343 pp.; English transl. of 1st ed., Grundlehren Math. Wiss., vol. 220, Springer-Verlag, Berlin–New York 1976, xi+315 pp.

[8] A. W. Marshall and I. Olkin, Inequalities: Theory of Majorization and Its Applications, (Academic Press, New York, 1979), Chapter 2

[9] C. Grood, A Specht Module Analog for the Rook monoid, The Electronic Journal of Combinatorics 9 (2002), #R2.

[10] East J., Generators and relations for partition monoids and algebras, Journal of Algebra 339 (2011) 1–26

[11] Popova L.M., Defining relations in some semigroups of partial transformations of a finite set, Uchenye Zap. Leningrad Gos. Ped. Inst. 218 (1961) 191–212 (in Russian).

[12] Karabegov A., A mapping from the unitary to double stochastic matrices and symbol on a finite set:arXiv: 0806.2357v1 [math. OA] 14Jun 2008, 13 pp.

[13] Neretin Yu. A., Categories of bistochastic measures, and representations of some infinite-dimensional groups, Russian Acad. Sci. Sb. Math., 75:1 (1993), 197 - 219

[14] Takesaki M., Theory of Operator Algebras, v. I, Springer, 2005, 416 pages.

[15] Sinclair A., Smith R., Finite von Neumann Algebras and Masas, Cambridge University Press, London Mathematical Society, Lecture Notes Series, 351, 2008, 400 pages.

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