Nonlinear sigma model of a spin ladder
containing a static single hole

A. R. Pereira, E. Ercolessi, A.S.T. Pires

Departamento de Física, Universidade Federal de Viçosa
36570-000, Viçosa, Minas Gerais, Brazil

Physics Department, University of Bologna, Via Irnerio 46, I-40126, Bologna, Italy

INFN and CNISM, Bologna, Italy

Departamento de Física, ICEX, Universidade Federal de Minas Gerais
Caixa Postal 702, 30123-970, Belo Horizonte, Minas Gerais, Brazil

Abstract

In this letter we extend the nonlinear $\sigma$ model describing pure spin ladders with an arbitrary number of legs to the case of ladders containing a single static hole. A simple immediate application of this approach to classical ladders is worked out.

PACS numbers: 75.50.Ee; 75.10.Jm; 75.10.Hk; 75.30.Ds

Keywords: magnons; antiferromagnets; impurities.

Corresponding author: A. R. Pereira; e-mail: apereira@ufv.br; Tel.: +55-31-3899-2988, Fax: +55-31-3899-2483.

*E-Mail: apereira@ufv.br
†E-Mail: Elisa.Ercolessi@bo.infn.it
‡E-mail: antpires@fisica.ufmg.br
The interest in low dimensional quantum antiferromagnets has been great ever since Haldane conjectured [1] that integer spin chains have a gap in their excitation spectrum while half-integer spin chains do not. More recently, spin ladders (two or more coupled spin chains) have also attracted much interest, mainly when the effects upon doping are considered. Indeed, when one manages to remove spins from the system (leaving holes behind) the existence of superconductivity is predicted [2] (and experimentally observed [3]). In this letter we would like to study the presence of static holes in spin ladders with an arbitrary number of legs by considering the nonlinear \( \sigma \) model continuum limit of the model.

The continuum limit of pure antiferromagnetic Heisenberg spin ladders with arbitrary number of legs has been derived by several authors [4, 5, 6, 7]. Here we apply the approach of Ref.[4] for studying the case of ladders containing static holes. We start with the Hamiltonian for a pure ladder system with \( n \) legs of length \( Na_0 \) (\( a_0 \) is the lattice constant and \( N \gg n \)) defined as

\[
H = \sum_{a=1}^{n} \sum_{j=1}^{N} \left[ J_a \vec{S}_a(j) \cdot \vec{S}_a(j+1) + \tilde{J}_{a,a+1} \vec{S}_a(j) \cdot \vec{S}_{a+1}(j) \right],
\]

where \( \vec{S}_a(j) \) are the spin operators located in the \( a \)th leg at the position \( j = 1, \ldots, N \), while \( J_a > 0 \) and \( \tilde{J}_{a,a+1} > 0 \) are the antiferromagnetic exchange couplings along the leg and rung respectively. The partition function of the above Hamiltonian in the spin coherent state path-integral representation is given by

\[
Z(\beta) = \int [D\vec{\Omega}] \exp \left\{ iS \sum_{j,a} \omega[\vec{\Omega}_a(j,\tau)] - \int_0^\beta d\tau H(\tau) \right\},
\]

where \( \tau = it \) is the imaginary time variable, \( \omega[\vec{\Omega}_a(j,\tau)] \) is the Berry phase factor and \( H(\tau) \) is obtained by replacing the operator \( \vec{S}_a(j) \) by the classical variable \( S\vec{\Omega}_a(j,\tau) \) in the Hamiltonian (1). To get the continuum limit, it is usual to assume that the dominant
contribution to the path integral comes from paths described by \[1, 8\]

\[
\tilde{\Omega}_a(j, \tau) = (-1)^{a+j} \tilde{\phi}(j, \tau) \left( 1 - \frac{\tilde{I}_a(j, \tau)^2}{S^2} \right)^{1/2} + \frac{\tilde{I}_a(j, \tau)}{S}. \tag{3}
\]

The field \(\tilde{\phi}(j, \tau)\) is supposed to be slowly varying and the fluctuation field \(\tilde{I}_a(j, \tau)\) is supposed to be small (\(\tilde{I}_a(j, \tau)/S << 1\)). The constraint \(\tilde{\Omega}_2^2(j, \tau) = 1\) implies \(\tilde{\phi}^2(j, \tau) = 1\) and \(\tilde{\phi}(j, \tau) \cdot \tilde{I}_a(j) = 0\). Numerical works support the fact that the staggered spin-spin correlation length is much greater than the total width of the ladder \([9, 10]\). Then, assuming that \(\tilde{\phi}(j, \tau)\) depends only on the site index \(j\) along the legs, Dell’Arringa et al. \([4]\) mapped the antiferromagnetic Heisenberg ladder system onto a (1+1) quantum nonlinear \(\sigma\) model

\[
Z_\sigma = \int [D\tilde{\phi}] \exp(i\Gamma[\tilde{\phi}]) \exp \left\{ -\frac{1}{2g} \int_0^\beta d\tau \int dx \left[ \frac{1}{v_s} (\partial_\tau \tilde{\phi})^2 + v_s (\partial_x \tilde{\phi})^2 \right] \right\}, \tag{4}
\]

where \(\Gamma[\tilde{\phi}] = (\theta/4\pi) \int_0^\beta d\tau \int dx \tilde{\phi} \cdot (\partial_\tau \tilde{\phi} \times \partial_x \tilde{\phi})\) (with \(\theta = 2\pi S\)) for \(n\) odd and \(\Gamma[\tilde{\phi}] = 0\) for \(n\) even, reflecting the fact that, for half-spin systems, the excitation spectrum has a gap (is gapless) when \(n\) is even (odd). Besides, the nonlinear \(\sigma\) model parameters, the coupling constant \(g\) and the spin wave velocity \(v_s\), are defined by

\[
g^{-1} = S \left( \sum_{a,b,c} J_{a} L_{b,c}^{-1} \right)^{1/2}, \tag{5}
\]

\[
v_s = S \left( \frac{\sum_{a} J_{a}}{\sum_{b,c} L_{b,c}^{-1}} \right)^{1/2}, \tag{6}
\]

where \(L_{a,b}^{-1}\) is the inverse of the matrix

\[
L_{a,b} = \begin{cases} 4J_a + \tilde{J}_{a,a+1} + \tilde{J}_{a,a-1} & \text{for } a = b \\ \tilde{J}_{a,b} = \tilde{J}_{a,a+1} & \text{for } |a - b| = 1 \end{cases}
\]

(7)

with \(\tilde{J}_{a,a+1} = \tilde{J}_{a-1,a}\) and \(\tilde{J}_{1,0} = \tilde{J}_{n,n+1} = 0\).
Now we consider the system in the presence of static holes (spins removed from the ladder). In two spatial dimensions, one of the simplest way of studying this problem in the continuum limit is through a non-simply connected manifold \[11, 12, 13\]. In this case a disk is removed from the magnetic plane, leaving a hole behind, and this hole is interpreted as a nonmagnetic impurity (or a spin vacancy) since there is no magnetic degrees of freedom insight it. This approach has good qualitative and quantitative agreement with numerical calculations \[11, 12, 13, 14\]. However, in the case of ladders as described by the nonlinear $\sigma$ model given by Eq. \((4)\), we cannot simply remove a disc from the space because the problem becomes essentially one-dimensional. There is no possibility of removing a part of the space without breaking the “effective” lattice. Then, within the above approach a “hole” must affect the exchange interactions $J_a$ and $\tilde{J}_{a,a\pm 1}$. As a consequence, the matrix $L_{a,b}$ becomes $j$-dependent. In such a way that the lattice is not broken for a single defect. It means that the parameters $g$ and $v_s$ are now functions of the position along the ladder. Depending on the number of legs, there are more than one position to put a single vacancy which yields different results. Some examples are shown in Fig.(1).

With the above considerations in mind, our approach for the nonlinear $\sigma$ model describing spin ladders with a static hole centered at $j = x_0$ (along the legs), $a = k$ (along the rungs) are summarized as follows

$$Z_{\sigma,\text{hole}} = \int [D\vec{\phi}] \exp(i\Gamma[\vec{\phi}]) \exp \left( - \int_0^\beta d\tau \int dx L_{\sigma,\text{hole}} \right), \quad (8)$$

with

$$L_{\sigma,\text{hole}} = \int dx \frac{1}{2g_k(x-x_0)} \left[ \frac{1}{v_{s,k}(x-x_0)} (\partial_\tau \vec{\phi})^2 + v_{s,k}(x-x_0) (\partial_x \vec{\phi})^2 \right]. \quad (9)$$

In order to explicitly define the new parameters space dependents $g_k(x-x_0)$ and $v_{s,k}(x-x_0)$...
$x_0$), we first rewrite the $n \times n$ matrix $L_{a,b}$ as follows

\[
L_{a,b} = \begin{pmatrix}
L_{1,1} & L_{1,2} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
L_{2,1} & L_{2,2} & L_{2,3} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & L_{3,2} & L_{3,3} & L_{3,4} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & L_{k-2,k-1} & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & L_{k-1,k-1} & L_{k-1,k} & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & L_{k,k-1} & L_{k,k} & L_{k,k+1} & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & L_{k+1,k} & L_{k+1,k+1} & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & L_{k+2,k+1} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & L_{n,n}
\end{pmatrix}.
\]  

(10)

If we place the vacancy at $(j,a) = (x_0,k)$, the above matrix will be the same for all $j \neq x_0, x_0 - 1$ and equal to (7). For $j = x_0 - 1$ the matrix will be again of rank $n$ with a slightly different coefficient $L_{k,k}$. For $j = x_0$ the matrix has zeroes along the $k-th$ row and the $k-th$ column. Thus we define a new matrix $K_{a,b}$ of order $(n - 1) \times (n - 1)$, which has almost the same elements of the above matrix and without line $k$ and column.
$k$. Explicitly:

$$K_{a,b} = \begin{pmatrix}
L_{1,1} & L_{1,2} & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
L_{2,1} & L_{2,2} & L_{2,3} & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & L_{3,2} & L_{3,3} & L_{3,4} & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & L_{k-2,k-1} & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & K_{k-1,k-1} & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & K_{k+1,k+1} & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & L_{k+2,k+1} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & L_{n,n}
\end{pmatrix}, \quad (11)$$

where $K_{k-1,k-1} = L_{k-1,k-1} - \tilde{J}_{k-1,k}$ and $K_{k+1,k+1} = L_{k+1,k+1} - \tilde{J}_{k+1,k}$. This formula holds only in the region of the spin vacancy ($|x - x_0| \lesssim a_0$). Therefore we may assume that the coefficients $g$ and $v$ are given by:

$$g_k^{-1}(x - x_0) = \begin{cases}
g^{-1} & \text{for } |x - x_0| \gtrsim a_0, \\
S \left( \sum_{a \neq k,b,c} J_a K_{b,c}^{-1} \right)^{1/2} & \text{for } |x - x_0| \lesssim a_0,
\end{cases} \quad (12)$$

$$v_{s,k}(x - x_0) = \begin{cases}
v_s & \text{for } |x - x_0| \gtrsim a_0, \\
S \left( \sum_{b,c} J_s K_{b,c} \right)^{1/2} & \text{for } |x - x_0| \lesssim a_0.
\end{cases} \quad (13)$$

The change in the parameters $g_k(x - x_0)$ and $v_{k,s}(x - x_0)$ in the zone of influence of the vacancy is associated with the discontinuous change in the number of legs in this region (see Fig.(1)). Of course, the field $\vec{\phi}$ must be continuous across the pure and impure regions. Note that, in principle, only the coupling constant and spin wave velocity are (locally) affected by the removed spin. As the Berry phases do not depend on these values, they are not very sensitive to the presence of the defect (see Eq. (8)). It means
that the ground state may not be very affected by the presence of the impurity. These results are in agreement with recent numerical calculations [15], which give evidences that the ground state configuration of the entire ladder system is not changed significantly by the impurity except for the local extraction of the missing bonds. Indeed, the energy cost to remove a spin from a two-leg ladder with spin-1/2 and $J = \tilde{J}$ is $E \approx 1.215 J$ [15], which is almost completely accounted for by the missing energy bonds along the legs (0.350J) and across the rungs (0.455J). Then, the method developed here is a good approximation and can be generalized for ladders containing a low concentration of impurities. Below we give a simple application of this approach. Our example is done for classical spin systems because the calculations are almost direct in this case. Besides, there are also manganese halide compounds [16] that are quasi-one-dimensional and two-dimensional antiferromagnets. Furthermore, these $Mn(II)$ compounds have spin 5/2 so they are also nearly classical and therefore, potential systems to test our results.

At zero temperature, the Hamiltonian of spin ladders possesses, in the classical limit ($S \to \infty$), a minimum given by the antiferromagnetic vacuum solution $\vec{\phi}_0(x) = \hat{z}$, where $\hat{z}$ is an unit vector in the vertical direction. This solution breaks the $O(3)$ invariance of the model down to the subgroup $O(2)$ of rotations around the z-axis. Consequently there should appear two Goldstone modes, which are nothing but spin waves, associated with $\phi_x$ and $\phi_y$. Then a natural first step concerning the impurity systems is to study the interactions between spin waves and holes. In a quantum spin system, the corresponding problem would be the interactions between triplons [17] (which are a triplet of well defined spin-1 magnons) and holes. However, it will be considered in a future work.

Expanding $\vec{\phi}(x,t)$ around the vacuum solution $\vec{\phi}(x,t) = \vec{\phi}_0(x) + \vec{\eta}(x,t)$ and minimizing Hamiltonian [19], one obtains, in the linearized approximation, the following scattering equation $\partial^2_x \vec{\eta}(x,t) - \left[1/v^2_{s,k}(x-x_0)\right] \partial^2_t \vec{\eta}(x,t) = U_k(x)\vec{\eta}(x,t)$, where the scattering potential is $U_k(x) = -\left\{\partial_x \ln[v_{s,k}(x-x_0)/g_k(x-x_0)]\right\} \partial_x$. However, the function
$v_{s,k}(x - x_0)/g_k(x - x_0)$ is constant practically through all space (see Eqs. (12)) while (13) varies only when entering the impurity regions. Therefore, $U_k(x)$ is zero in almost all space and we have magnon solutions for the field equation in the three regions: $(x < x_0 - a_0), (x_0 - a_0 < x < x_0 + a_0)$ and $(x > x_0 + a_0)$. The form of $U_k(x)$ is not explicitly known but its effects can be envisaged using the following simple analysis: if a magnon (for instance, coming from the left) hits the zone of influence of the potential $U_k(x)$ (or the zone of the impurity) at $x_0 - a_0$, then its velocity will be changed from $v_s$ to $v_{k,s} = (\sum_{a \neq k} J_a / \sum_{b,c} K_{b,c}^{-1})^{1/2}$ and after leaving behind this region at $x_0 + a_0$, it will be changed again to $v_s$. Consequently, supposing a plane wave coming from $-\infty$, and assuming that the lowest order effect of $U_k(x)$ is to cause elastic scattering centers for magnons, the solution at $+\infty$ can be approximated by $\vec{\eta}_0 \exp[i(qx - \omega_q t + \delta_{n,k}(q)/2)]$ with frequency $\omega_q = qv_s$, which of course, is precisely the dispersion relation for magnons in the absence of holes. The function $\delta_{n,k}(q)$ may be regarded as a phase-shift (for magnons in a ladder with $n$ legs) which depends on the particular position (leg $k$) of the hole in the spin ladder for a determined wave number $q$. Indeed, after passing the defect, the wave is shifted from the original one due to the different velocity $v_{k,s}$ acquired in the hole region. This phase-shift can be easily estimated considering the difference of paths $v_s \Delta t$ and $v_{k,s} \Delta t$, where the interval $\Delta t = 2a_0/v_{k,s}$ is the time necessary for the wave to leave behind the region of the hole $(x_0 - a_0 < x < x_0 + a_0)$. In the lowest order, it is given by

$$\delta_{n,k}(q) = -4qa_0 \left( \frac{v_s}{v_{k,s}} - 1 \right) = -4qa_0 \left( \frac{\sum_a J_a / \sum_{a \neq k} J_a}{\left( \sum_{b,c} K_{b,c}^{-1} / \sum_{b,c} L_{b,c}^{-1} \right)^{1/2} - 1} \right). \quad (14)$$

For the case of a spin ladder with two legs and $N \to \infty$, the phase-shift does not depend on the position of the vacancy, which can be put at the left ($k = 1$) or right ($k = 2$) leg. For $J_1 = J_2$ and $\tilde{J}_{a,b} = \tilde{J}$, it is given by

$$\delta_{2,1}(q) = \delta_{2,2}(q) = -4qa_0[(1 + \tilde{J}/2J)^{1/2} - 1]. \quad (15)$$
Spin ladders with three or more legs have two different situations. For example, for the case with three legs, if the vacancy is placed at the first or last leg, we have:

\[
\delta_{3,1}(q) = \delta_{3,3}(q) = -4qa_0 \left\{ \left[ \frac{(1 + 3R/4)}{(1 + R/2)(1 + R/12)} \right]^{1/2} - 1 \right\},
\]

(16)

where \( R = \tilde{J}/J \). By the other hand, if the hole is placed at the central leg, the phase shift is

\[
\delta_{3,2}(q) = -4qa_0 \left\{ \left[ \frac{(1 + 3R/4)}{(1 + R/4)(1 + R/12)} \right]^{1/2} - 1 \right\}.
\]

(17)

Finally we discuss the magnon density of states in the impurity spin ladders. To do this, we consider a large system of size \( Na_0 \) and impose periodic boundary conditions on the continuum (magnons) states \( \vec{n}_q(x) \). This periodicity, together with \( \vec{n}_q(x)_{x \rightarrow \pm \infty} \approx \vec{n}_0 \exp[i(qx \pm \delta_{n,k}(q)/2)] \), gives the following condition for the allowed wave vectors:

\[
Na_0 q_m + \delta_{n,k}(q_m) = 2\pi m \quad (m = 0, \pm 1, \pm 2, \ldots).
\]

Clearly, the magnon density of states is changed by the presence of a hole as follows

\[
\varrho_{n,k}(q) = \frac{dm}{dq} = \frac{Na_0}{2\pi} + \frac{1}{2\pi} \frac{d\delta_{n,k}(q)}{dq},
\]

(18)

and so, the change in the density of states \( \Delta\varrho_{n,k}(q) = \varrho_{n,k}(q) - Na_0/2\pi \) is given by

\[
\Delta\varrho_{n,k}(q) = \frac{1}{\pi} \frac{d\delta_{n,k}(q)}{dq}.
\]

(19)

Above we have multiplied by a factor of 2 to take into account the two Goldstone modes.

For a classical two-leg spin ladder one has

\[
\Delta\varrho_{2,1}(q) = \Delta\varrho_{2,2}(q) = -\frac{4a_0}{\pi} [(1 + R/2)^{1/2} - 1].
\]

(20)

For three-leg ladder

\[
\Delta\varrho_{3,1}(q) = \Delta\varrho_{3,3}(q) = -\frac{4a_0}{\pi} \left\{ \left[ \frac{(1 + 3R/4)}{(1 + R/2)(1 + R/12)} \right]^{1/2} - 1 \right\}.
\]

(21)
and

\[ \Delta q_{3,2}(q) = -\frac{4a_0}{\pi} \left\{ \left[ \frac{(1 + 3R/4)}{(1 + R/4)(1 + R/12)} \right]^{1/2} - 1 \right\} \] \hspace{1cm} (22)

In Fig.(2) we plot \( D = \Delta q_{n,k}/a_0 \) as a function of \( R = \tilde{J}/J \) for \( n = 2 \) and \( n = 3 \). In general, the density of states does not depend on the wave-vector \( q \) (since the phase-shift has a linear dependence on \( q \)). For all legs, the limit \( R \to 0 \) implies \( D \to 0 \). As expected, it means that the presence of \( \tilde{J} \) is very important for the phase-shifts as well as for the change in the density of states. In practice, it avoids a broken lattice and leads to \( v_{k,s} \neq 0 \). For a three-leg ladder with the impurity placed at leg 1 or 3, \( \Delta q_{3,1} \) is very small for an appreciable range of \( R \) and becomes positive for \( R > 4 \). It increases considerably for large values of \( R \) and in the limit \( R \to \infty \), \( \Delta q_{3,1} \to 1.27 \). Such a general behavior is also expected for ladders with more than three legs with impurities placed at the external legs (of course, it may have important qualitative changes as, for example, the function is positive for \( R < R_c \) and then becomes positive for \( R > R_c \)). If the vacancy is located at the central leg, the behavior of the three-leg ladder is similar to the previous case but the change in the density of states is much larger. In this case, \( \Delta q_{3,2}(q)/a_0 \) becomes positive only for \( R > 20 \) (not shown in Fig.2). In addition, like \( \Delta q_{3,1}(q)/a_0 \), \( \Delta q_{3,2}(q)/a_0 \to 1.27 \) as \( R \to \infty \). On the other hand, for a two-leg ladder, \( \Delta q_{2,1}(q)/a_0 \) is always negative and its modulus increases monotonically as \( R \) increases. The magnon density of states suffers a very expressive change for large values of \( \tilde{J} \). Therefore, in this circumstance, there is a clear distinction in the magnon spectrum when a hole in the spin ladder is present or absent.

Ladders with a higher number of legs might be approached as well, using for example the Mathematica program. In particular it is interesting to study how the ratio \( w = v_s/v_{k,s} \) changes as we increase the number \( n \) of legs and approach a two-dimensional lattice. We have checked this for the isotropic case \( J = \tilde{J} \) by putting the hole in the center leg of the ladder, getting the following results for \( n = 11, 31, 51, 71, 91 \) respec-
tively: \( w = 1.03358, 1.01029, 1.00605, 1.00428, 1.00331 \). One can immediately see that \( w \) approaches zero as \( n \) goes to infinity. However, the method used here for the phase-shifts is not convenient for two-dimensional (2d) systems because one has to deal with cylindrical waves, which contains an infinite number of angular momentum channels. A more adequate method for the 2d case in the continuum approximation is given in Ref.\[13\]. Thus, we cannot conclude that the influence of a single static hole on the magnon phase shift and density of states becomes negligible in the two-dimensional system.

We also expect that a similar behavior may happen to triplons in spin-1/2 two-leg ladders. Basic differences must appear due to the existence of a gap for these last excitations. Of course, the change in the spin wave density of states has a deep influence on the static and dynamical properties of ladders containing a low concentration of holes as it causes a change in the magnon free energy \[18\].

In summary, we have proposed an approach based on the nonlinear \( \sigma \) model to study static holes in spin ladder systems with an arbitrary number of legs. As an immediate example of application of this method, we have studied the magnon-hole interactions in classical spin ladders and calculated the magnon phase-shifts and the consequent change in the magnon density of states. Such results may also give some insights about the behavior of linearized oscillatory excitations around holes in low-dimensional quantum spin materials. However, the calculations were performed using a long wavelength theory and we have to remark that in experimental situations involving small scales (of order of the lattice size) the application of the nonlinear \( \sigma \) model should be viewed with caution. Future study considering systems containing multiple impurities (which, for adjacent holes, have larger sizes) and applications to quantum ladders are in progress. As in the two-dimensional case \[12, 13\], our results may also be useful to study possible topological excitations interacting with holes in spin ladders.
Acknowledgements

The authors thank CAPES (Brazilian agency) for financial support. This work was partially supported by CNPq (Brazil), the TMR network EUCLID and the Italian MIUR through COFIN projects.
References

[1] F.D.M. Haldane, Phys. Rev. Lett. 50, 1153 (1983).

[2] E. Daggoto and T.M. Rice, Science 271, 618 (1996).

[3] M. Uehara, T. Nagata, J. Akimitsu, H. Takahashi, N. Mori, and K. Kinoshita, J. Phys. Soc. Jpn. 65, 2764 (1996).

[4] S. Dell’Arima, E. Ercolessi, G. Morandi, P. Pieri, and M. Roncaglia, Phys. Rev. Lett 78, 2457 (1997).

[5] G. Sierra, J. Phys. A: Math. Gen. 29, 3299 (1996).

[6] D. Sénéchal, Phys. Rev. B 52, 15319 (1995).

[7] E. Ercolessi, Mod. Phys. Lett. A 18, 2329 (2003).

[8] F.D.M. Haldane, Phys. Rev. Lett. 61, 1029 (1988).

[9] M. Greven, R.J. Birgineau, and U.-J. Wiese, Phys. Rev. Lett. 77, 1865 (1996).

[10] S.R. White, R.M. Noak, and D.J. Scalapino, Phys. Re. Lett. 73, 886 (1994).

[11] A.R. Pereira, S.A. Leonel, P.Z. Coura, and B.V. Costa, Phys. Rev.B 71, 014403 (2005).

[12] F.M. Paula, A.R. Pereira, L.A.S.Mól, Phys. Lett.A 329, 155 (2004).

[13] A.R. Pereira and G.M. Wysin, Phys. Rev. B 73, 214402 (2006).

[14] F.M. Paula, A.R. Pereira, and G.M. Wysin, Phys. Rev. B 72, 094425 (2005).

[15] F. Anfuso and S. Eggert, Europhys. Lett. 73, 271 (2006).

[16] J. de Jongh and A.R. Miedema, Adv. Phys. 23, 1 (1974).
[17] S. Notbohm, P. Ribeiro, D.A. Tennant, K.P. Schmidt, G.S. Uhrig, C. HessR. Klingeler, G. Behr, B. Büchner, M. Reehuis, R.I. Bewley, C.D. Frost, P. Manuel, and R.S. Eccleston, condmat/0608109v1 (2006).

[18] J.F. Currie, J.A. Krumhansl, A.R. Bishop, and S.E Trullinger, Phys. Rev. B 22, 477 (1980).
Figure Captions

Figure 1. Configurations of spin ladders along the z-direction with two and three legs containing a hole. For the case of three-leg ladders, the second and third configurations are equivalents, but the fourth leads to different results.

Figure 2. Change in the density of states $D = \Delta \rho_{n,k}/a_0$ as a function of the rung coupling $R = \tilde{J}/J$ for spin ladders with two-leg (solid line) and for the two possibilities for three-leg: vacancy placed at the central leg (dashed line) and at leg 1 or 3 (dotted line).
