LARGE SPACES BETWEEN THE ZEROS OF THE RIEemann Zeta-Function

S. H. Saker

Abstract. In this paper, we will employ the Opial and Wirtinger type inequalities to derive some conditional and unconditional lower bounds for the gaps between the zeros of the Riemann zeta-function. First, we prove (unconditionally) that the consecutive nontrivial zeros often differ by at least 1.9902 times the average spacing. This value improves the value 1.9 due to Mueller and the value 1.9799 due to Montgomery and Odlyzko. Second, on the hypothesis that the 2\(^k\)-th mixed moments of the Hardy \(Z\)–function and its derivative are correctly predicted by random matrix theory, we derive some explicit formulae for the gaps and use them to establish new (conditional) large gaps.

1. Introduction

The Riemann zeta function \(\zeta(s)\) is defined on \(\{s \in \mathbb{C} : \text{Re}(s) > 1\}\) by the series
\[
\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots, \quad \text{for } \text{Re } s > 1,
\]
which converges in the region described by the Cauchy integral test. There is another representation of \(\zeta\) due to Euler in 1749 which is perhaps more fundamental and which is the reason for the significance of the zeta function and gives analytic expression to the fundamental theorem of arithmetic. This formula is given by
\[
\zeta(s) := \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{for } \text{Re } s > 1,
\]
where the product is taken over all prime numbers. The zeta-function is one of the most studied transcendental functions, having in view its many applications in number theory, algebra, complex analysis, statistics as well as in physics. Another reason why this function has drawn so much attention is the celebrated Riemann conjecture regarding nontrivial zeros which states that all nontrivial zeros of the Riemann zeta function \(\zeta(s)\) lie on the critical line \(\text{Re}(s) = 1/2\). Riemann showed that the zeta-function satisfies a functional equation of the form
\[
\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),
\]
where \(\Gamma\) is the Euler gamma function. By this equation there exist so-called trivial (real) zeros at \(s = -2n\) for any positive integer \(n\) (corresponding to the poles of the appearing Gamma-factors), and all nontrivial (non-real) zeros are distributed symmetrically with respect to the critical line \(\text{Re } s = 1/2\) and the real axis. The significance contribution of the formula (1.1) is the consideration of the zeta-function as an analytic function. We
note from the functional equation that if \( \rho \in \mathbb{Z} \) is a zero of \( \zeta(s) \), so is \( 1 - \rho \), \( \overline{\rho} \), \( 1 - \overline{\rho} \), and \( \zeta(\overline{\rho}) = \overline{\zeta(\rho)} \). Clearly, there are no zeros in the half-plane of convergence \( \Re(s) > 1 \), and it is also known that \( \zeta(s) \) does not vanish on the line \( \Re(s) = 1 \).

The number \( N(t) \) of the non-trivial zeros of \( \zeta(s) \) with ordinate in the interval \([0, T]\) is asymptotically given by the Riemann-von Mangoldt formula (see [13])

\[
N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + O(\log T).
\]

Consequently there are infinitely many nontrivial zeros, all of them lying in the critical strip \( 0 < \Re(s) < 1 \), and the frequency of their appearance is increasing as \( T \to \infty \).

There are three directions regarding the studies of the zeros of the Riemann zeta function. The first direction is concerning with the existence of simple zeros. It is conjectured that all or at least almost all zeros of the zeta-function are simple. For this direction Conrey [7] proved that more than two-fifths of the zeros are simple and on the critical line. This value has been improved by Cheer and Goldston [5] and proved that at least 0.662753 of the zeros are simple assuming the Riemann hypothesis. The second direction is the most important goal, is the determination of the moments of the Riemann zeta function on the critical line and the evaluation of the Riemann zeta function at integers which gives an integral representation of this function. It is important because it can be used to estimate the maximal order of the zeta-function on the critical line, and because of its applicability in studying the distribution of prime numbers and to divisor problems. For more details of this direction, we refer the reader to [10], [20] and [29] and the references cited therein. The third direction is the distribution of the zeros when the Riemann hypothesis is satisfied which is of our interest in this paper. In fact the distribution of zeros of the Riemann zeta-function is of fundamental importance in number theory as well as in physics. In the following, we briefly present some results related to this direction. Assume that \( (\beta_n + i\gamma_n) \) are the zeros of \( \zeta(s) \) in the upper half-plane (arranged in nondecreasing order and counted according to multiplicity) and \( \gamma_n \leq \gamma_{n+1} \) are consecutive ordinates of all zeros. We put

\[
r_n := \frac{(\gamma_{n+1} - \gamma_n)}{(2\pi/\log \gamma_n)},
\]

and define \( \lambda := \limsup_{n \to \infty} r_n \), and \( \mu := \liminf_{n \to \infty} r_n \). The numbers \( \mu \) and \( \lambda \) have received a great deal of attention. As mentioned by Montogomery [24] it would be interesting to see how numerical evidence compare with the above conjectures. It generally conjectured that

\[
\mu = 0, \quad \lambda = \infty.
\]

Now, several results has been obtained, however the failure of Gram’s low (see [14]) indicate that the asymptotic behavior is approached very slowly. Thus the numerical evidence may not be particularly illuminating. In fact, important results concerning the values of \( \lambda \) and \( \gamma \) have been obtained by some authors. Selberg [28] proved that \( 0 < \mu < 1 < \lambda \), and the average of \( r_n \) is 1. Note that \( 2\pi/\log \gamma_n \) is the average spacing between zeros. Fujii [12] also showed that there exist constants \( \lambda > 1 \) and \( \mu < 1 \) such that

\[
\frac{(\gamma_{n+1} - \gamma_n)}{(2\pi/\log \gamma_n)} \geq \lambda, \quad \text{and} \quad \frac{(\gamma_{n+1} - \gamma_n)}{(2\pi/\log \gamma_n)} \leq \mu,
\]

each holds for a positive proportion of \( n \). Mueller [26] obtained \( \lambda > 1.9 \), assuming the Riemann hypothesis. Montogomery and Odlyzko [25] showed, assuming the Riemann
hypothesis, that $\lambda > 1.9799$, and $\mu < 0.5179$. Conrey, Ghosh and Gonek [6] proved that if the Riemann hypothesis is true, then $\lambda > 2.337$, and $\mu < 0.5172$. Conrey, Ghosh and Gonek [8] obtained a new lower bound and proved that $\lambda > 2.68$, assuming the generalized Riemann hypothesis for the zeros of the Dirichlet $L$–functions. Bui, Milinovich and Ng [4] obtained $\lambda > 2.69$, and $\mu < 0.5155$, assuming the Riemann hypothesis. Ng in [27] proved that $\lambda > 3$, assuming the generalized Riemann hypothesis for the zeros of the Dirichlet $L$–functions. Note that any other small values of $\mu$ and large values of $\lambda$ will help in proving the conjecture [13].

Let $\Lambda$ denote the quantity in (1.2) where only zeros $\frac{1}{2} + it_n$ on the critical line, i.e., we define

$$\Lambda := \limsup \frac{t_{n+1} - t_n}{(2\pi/ \log t_n)}.$$ (1.4)

Note that the Riemann hypothesis implies that the $t_n$ corresponded to the positive ordinates of non-trivial zeros of the zeta function, i.e., $N(T) \sim \frac{T}{2\pi} \log T$. The average spacing between consecutive zeros with ordinates of order $T$ is $2\pi/ \log(T)$ which tends to zero as $T \to \infty$. Hall [17] showed that $\Lambda \geq \lambda$, and the lower bound for $\Lambda$ bear direct comparison with such bounds for $\lambda$ dependent on the Riemann hypothesis, since if this were true the distinction between $\Lambda$ and $\lambda$ would be nugatory. Of course $\Lambda \geq \lambda$ and the equality holds if the Riemann hypothesis is true. The behavior of $\zeta(s)$ on the critical line is reflected by the Hardy $Z$–function $Z(t)$ as a function of a real variable, defined by

$$Z(t) = e^{\theta(t)} \zeta\left(\frac{1}{2} + it\right),$$ (1.5)

where $\theta(t) := \pi^{-it/2} \frac{\Gamma(\frac{1}{4} + \frac{1}{2}it)}{\Gamma(\frac{1}{4} + \frac{1}{2}it)}$. It follows from the functional equation (1.1) for $\zeta(s)$ that $Z(t)$ is an infinitely often differentiable and real function for real $t$. Moreover $|Z(t)| = |\zeta(1/2 + it)|$. Consequently, the zeros of $Z(t)$ correspond to the zeros of the Riemann zeta-function on the critical line. The moments $I_k(T)$ of the Hardy $Z$–function $Z(t)$ function and the moments $M_k(T)$ of its derivative are defined by

$$I_k(T) := \int_0^T |Z(t)|^{2k} \ dt, \quad M_k(T) := \int_0^T |Z'(t)|^{2k} \ dt.$$ (1.8)

For positive real numbers $k$, it is believed that $I_k(T) \sim C(k) \ T (\log T)^{k^2}$ and $M_k(T) \sim L(k)T (\log T)^{k^2+2k}$ for positive constants $C_k$ and $L_k$ will be defined later. Keating and Snaith [22] based on considerations from random matrix theory conjectured that

$$I_k(T) \sim a(k)b(k)T (\log T)^{k^2},$$ (1.6)

where $a(k)$ and $b(k)$ are defined by

$$a(k) := \prod_p \left(1 - \frac{1}{p^2}\right) \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}, \quad \text{and} \quad b(k) := \prod_{j=0}^{k-1} j!.$$ (1.7)

To find the lower bound of $\Lambda$ Hall [17] used a Wirtinger-type inequality of Beesack [2] and the moment

$$\int_0^T Z^4(t) dt = \frac{1}{2\pi^2} T \log^4(t) + O(T \log^3),$$ (1.8)

due to Ingham (21) and the moment

$$\int_0^T (Z'(t))^4 dt = \frac{1}{1120\pi^2} T \log^8(t) + O(T \log^7),$$ (1.9)
due to Conrey [9] and proved unconditionally that
\[(1.10)\quad \Lambda \geq \left(\frac{105}{4}\right)^{\frac{1}{3}} = 2.2635.\]

In [16] Hall remarked that Beesack inequality is sharp but it is not optimal for application and proved a new Wirtinger-type inequality and used the moments (1.8)-(1.9) and the moment
\[(1.11)\quad \int_0^T Z^2(t)(Z'(t))^2 dt = \frac{1}{120\pi^2} T \log^6(t) + O(T \log^5),\]
due to Conrey [9], and proved (unconditionally) that \(\Lambda \geq \sqrt{11/2} = 2.3452\). The moments of \(Z(t)\) and its derivative (of mixed powers)
\[(1.12)\quad \int_0^T Z^{2k-2h}(t)(Z'(t))^{2h} dt \sim a(k)b(h,k)T (\log T)^{k^2+2h},\]
has been predicted by Random Matrix Theory (RMT) by Hughes [20] who stated an interesting conjecture on the moments subject to the truth of Riemann’s hypothesis when the zeros are simple. This conjecture includes for fixed \(k > -3/2\) the asymptotes formula of the moments of the higher order of the Riemann zeta function and its derivative. We suppose further that if \(k\) is a fixed positive integer and \(h \in [0, k]\) is an integer then the formula
\[(1.13)\quad \int_0^T Z^{2k-2h}(t)(Z'(t))^{2h} dt \sim a(k)b(h,k)T (\log T)^{k^2+2h},\]
holds. Note that (1.13) has been predicted by Keating and Snaith [22] in the case when \(h = 0\), with wider range \(\text{Re}(k) > -1/2\) and by Hughes [20] in the range \(\min(h, k-h) > -1/2\), \(a(k)\) is a product over the primes and \(b(h, k)\) is rational: indeed for integral \(h\), it is obtained that
\[(1.14)\quad b(h, k) := b(0, k) \left(\frac{(2h)!}{8^h h!}\right) H(h, k),\]
where \(H(h, k)\) is an explicit rational function of \(k\) for each fixed \(h\) and \(b(0, k) = b(k)\) which is defined as in (1.7). The functions \(H(h, k)\) as introduced by Hughes [20] are given in the following table where \(K = 2k\):

| \(H(0, k)\) | \(H(1, k)\) | \(H(2, k)\) |
|------------|------------|------------|
| \(1\)      | \(\frac{1}{K^2-1}\) | \(\frac{1}{(K^2-1)(K^2-9)}\) |
| \(H(3, k)\) | \(\frac{1}{(K^2-1)^2(K^2-25)}\) | \(\frac{1}{(K^2-1)^2(K^2-9)(K^2-25)(K^2-49)}\) |
| \(H(5, k)\) | \(\frac{1}{(K^2-1)^2(K^2-9)^2(K^2-25)(K^2-49)(K^2-81)}\) | \(\frac{1}{(K^2-1)^3(K^2-9)^3(K^2-25)(K^2-49)(K^2-81)(K^2-121)}\) |
| \(H(6, k)\) | \(\frac{1}{(K^2-1)^3(K^2-9)^3(K^2-25)(K^2-49)(K^2-81)(K^2-121)(K^2-169)}\) | \(\frac{1}{(K^2-1)^4(K^2-9)^4(K^2-25)(K^2-49)(K^2-81)(K^2-121)(K^2-169)}\) |

\(\text{Table 1. The values of } H(h, k), \text{ for } h = 0, 1, 2, \ldots, 7 \text{ where } K = 2k.\)

This sequence continuous, and it is believed that both the nominator and denominator are polynomials in \(k^2\), moreover that the denominator is actually (see (11))
\[(1.15)\quad \prod_{a \text{ odd} > 0} \left\{ (K^2 - a^2)^{\alpha(a,h)} ; \alpha(a,h) = \frac{4h}{a + \sqrt{a^2 + 8h}} \right\}.\]
Using the equation (1.14) and the definitions of the functions \( H(h,k) \), we can obtain the values of \( b(0,k)/b(k,k) \) for \( k = 1, 2, \ldots, 7 \). Hall [19] shown that in the case when \( h = 3 \), \( (H(3,k)) \) requires adjustment to fit with (1.15) in that extra factor \( K^2 - 9 \) should be introduced in both the nominator and denominator. I hope also to get the other values of \( H(h,k) \) for \( k \geq 8 \) which will help in deriving new values of \( \Lambda \), since our calculation (at this moment) will stop at \( k = 7 \). The values of \( b(0,k)/b(k,k) \) for \( k = 1, 2, \ldots, 7 \), that we will need in this paper are determined from (1.14) and presented in the following table:

| \( b(0,1) \) | \( b(0,2) \) | \( b(0,3) \) | \( b(0,4) \) |
|-----------|-----------|-----------|-----------|
| 12        | 0.727     | 0.664     | 0.424     |
| 581050229760 | 1140644224898402 | 170968290176011220086 | 5078125 |

Table 2. The Values of the \( b(0,k)/b(k,k) \) for \( k = 1, 2, \ldots, 7 \). Hall in [17, 19] used the moments of mixed powers (1.13) and a new Wirtinger-type inequality designed exclusively for this problem to improve the lower values of \( \Lambda \). In particular Hall [17] proved a the Wirtinger-type inequality

\[
\int_{0}^{\pi} H \left( y'(t)/y(t) \right) y^{2k}(t) dt \geq (2k-1)L \int_{0}^{\pi} y^{2k}(t) dt,
\]

where \( L = L(k,H) \) is determined from the solution of the equation

\[
\int_{0}^{\infty} \frac{G'(u)}{G(u) + (2k-1)L} \frac{du}{u} = k\pi, \quad \text{for } k \in \mathbb{N},
\]

where \( G(u) := uH'(u) - H(u), \ y = y(t) \in C^2[0, \pi] \) and \( y(0) = y(\pi) = 0 \), \( H(u) \) be an even function, increasing, strictly convex on \( \mathbb{R}^+ \) and satisfies \( H(0) = H'(0) = 0 \) and \( uH''(u) \to 0 \) as \( u \to 0 \). The inequality (1.16) is proved by using the calculus of variation which depends on the minimization of the integral on the left hand side subject to the constrains \( y(0) = 0 \) and \( \int_{0}^{\pi} y^{2k}(t) dt = 1 \). Assuming that (1.13) is correctly predicted, Hall employed the inequality (1.16) when

\[
H(u) := \sum_{h=1}^{k} \frac{2k-1}{h} \binom{h}{k} v_h u^{2h}, \quad v_h \geq 0, \quad v_k = 1,
\]

and obtained

\[
\Lambda \geq \sqrt{7533/901} = 2.8915.
\]

The main challenge in [17] was to maximize \( X = \kappa^2 \) (which is not an easy task) where \( X \) satisfies the equation \( 27X^3 + 385\mu X^2 + 10395\theta X - 121275L = 0 \), and \( L \) obtained form the equation

\[
\int_{-\infty}^{\infty} \frac{x^4 + 2\mu x^2 + v}{x^6 + 3\mu x^4 + 3\sigma x^2 + L} dx = \pi.
\]

In [18] Hall employed the generalized Wirtinger inequality (1.16) and simplified the calculation in [17] and converted the problem into one in the classical theory of equations.
involving Jacobi-Schur functions to maximize $X$. Assuming that (1.13) is correctly predicted by RMT, Hall obtained the new values of $\Lambda$ which is listed in the following table:

|   | $\Lambda(3)$ | $\Lambda(4)$ | $\Lambda(5)$ | $\Lambda(6)$ |
|---|-------------|-------------|-------------|-------------|
|   | $\sqrt{7533/901}$ | $3.392272$ | $3.858851$ | $4.2981467$ |

The methods that have been used by Hall to establish the lower bounds of $\Lambda$ are quite complicated and need a lot of calculations as well as the reader should be familiar with calculus of variations and optimization theory. In [30] the authors applied a technique involving the comparison of the continuous global average with local average obtained from the discrete average to a problem of gaps between the zeros of zeta function assuming the Riemann hypothesis. Using this approach, which takes only zeros on the critical line into account, the authors computed similar bounds under assumption of the Riemann hypothesis when (1.13) holds. In particular they showed that for fixed positive integer $r$

\[
\frac{(\gamma_{n+r} - \gamma_n)}{(2\pi r/ \log \gamma_n)} \geq \theta,
\]

holds for any $\theta \leq 4k/\pi r$ for more than $c(\log T)^{-4k^2}$ proportion of the zeros $\gamma_n \in [0, T]$ with a computable constant $c = c(k, \theta, r)$.

In this paper, we will employ some well-known Opial and Wirtinger type inequalities to derive new unconditional lower bound for $\Lambda$ and also establish some explicit formulae for the gaps between the zeros. First, we apply the Wirtinger type inequality due to Brnetić and Pečarić [3] and prove that $\Lambda \geq 1.9902$ (unconditionally) which improves the value 1.9 of Mueller and the value 1.9799 of Montogomery and Odlyzko. Second, assuming that the moments of $Z(t)$ and its derivative are correctly predicted by RMT, we established some new explicit formulae for $\Lambda(k)$ by employing an Opial inequality due to Yang [31] and Wirtinger type inequality due to Agarwal and Pang [1]. As an application, we derived some new conditional series of the lower bounds for $\Lambda(k)$. Our results do not require any additional information from the calculus of variation and optimization theory.

2. Main Results

In this section, we employ some well-known Opial and Wirtinger type inequalities to prove the main results. First, we employ the Wirtinger type inequality due to Brnetić and Pečarić [3] to find a new unconditional lower bound for $\Lambda$. The Wirtinger type inequality due to Brnetić and Pečarić [3] is presented in the following theorem.

**Theorem A.** Assume that $x(t) \in C^1[0, \pi]$ and $x(0) = x(\pi) = 0$, then

\[
\int_0^\pi (x'(t))^{2k} dt \geq \frac{1}{\pi^{2k} I(k)} \int_0^\pi x^{2k}(t) dt, \quad \text{for} \quad k \geq 1,
\]

where

\[
I(k) = \int_0^1 \frac{1}{(t^{1-2k} + (1-t)^{1-2k})} dt.
\]

In the following, we will apply the inequality (2.1) and the moment (1.8) due to Ingham [21] and the moment (1.9) due to Conrey [9] to find the new unconditional value of $\Lambda$. From (2.1), when $k = 2$, we have

\[
\int_0^\pi (x'(t))^4 dt \geq \frac{1}{\pi^4 I(2)} \int_0^\pi x^4(t) dt,
\]
where
\[ I(2) := \int_0^1 \frac{1}{(t^{1-4} + (1-t)^{1-4})} \, dt = \frac{2863}{125000}. \]
By a suitable linear transformation, we can deduce from (2.2) that if \( x(t) \in C^1[a, b] \) and \( x(a) = x(b) = 0 \), then
\[
\int_{a}^{b} \left( \frac{b-a}{\pi} \right)^4 (x'(t))^4 dt \geq \frac{125000}{2863\pi^4} \int_a^b x^{2k}(t) dt
\]
where \( x(a) = x(b) = 0 \).

The following theorem gives the new unconditional value of \( \Lambda \).

**Theorem 2.1.** Let \( \varepsilon(T) \to 0 \) in such a way that \( \varepsilon(T) \log T \to \infty \). Then for sufficiently large \( T \), there exists an interval contained in \([T, (1 + \varepsilon(T))T]\) which is free of zeros of \( Z(t) \) and having length at least
\[
\frac{1}{2\pi} \sqrt{\frac{1000000}{409}} \left\{ 1 + O\left( \frac{1}{\varepsilon(T) \log T} \right) \right\} \frac{2\pi}{\log T}.
\]
Thus
\[
(2.4) \quad \Lambda \geq 1.9902.
\]

**Proof.** We follow the arguments in [16] to prove our theorem. Suppose that \( t_l \) is the first zero of \( Z(t) \) not less than \( T \) and \( t_m \) the last zero not greater than \((1 + \varepsilon)T\) where \( \varepsilon(T) \to 0 \) in such a way that \( \varepsilon(T) \log T \to \infty \). Suppose further that for \( l \leq n < m \), we have
\[
(2.5) \quad L_n = t_{n+1} - t_n \leq \frac{2\pi \kappa}{\log T}.
\]
Applying the inequality (2.3) with \( a = t_n, b = t_{n+1} \) and \( y(t) = Z(t) \), we have
\[
\int_{t_n}^{t_{n+1}} \left( \frac{L_n}{\pi} \right)^4 (Z'(t))^4 dt - \frac{1}{\pi^4} \frac{125000}{2863} \int_a^b Z^4(t) dt \geq 0.
\]
Since the inequality remains true if we replace \( L_n/\pi \) by \( 2\kappa/\log T \), we have
\[
(2.6) \quad \int_{t_n}^{t_{n+1}} \left[ \left( \frac{2\kappa}{\log T} \right)^4 (Z'(t))^4 - \frac{1}{\pi^4} \frac{125000}{2863} Z^4(t) \right] dt \geq 0.
\]
Summing (2.6) over \( n \), and using the moments (1.8)-(1.9), we obtain
\[
\frac{1}{1120\pi^2} \left( \frac{2\kappa}{\log T} \right)^4 T \log^8(T) + O(T \log^7 T)
- \frac{1}{\pi^4} \frac{125000}{2863} \frac{1}{2\pi^2} T \log^4(T) + O(T \log^3 T)
= \frac{(2\kappa)^4}{1120\pi^2} T \log^4(T) + O(T \log^3 T)
- \frac{1}{\pi^4} \frac{125000}{2863} \frac{1}{2\pi^2} (T \log^4 T) + O(T \log^3 T).
\]
Follows the proof of Theorem 1 in [16], we obtain
\[
\kappa^4 \geq \frac{1120}{2^8\pi^4 I(2)} + O(1/\varepsilon(T) \log T).
\]
Then, we have (noting \( \varepsilon(T) \log T \to \infty \) as \( T \to \infty \)) that
\[
\Lambda \geq \frac{1}{2\pi} \sqrt{\frac{125000 \cdot 1120}{2863}} = 1.9902.
\]
The proof is complete.

**Remark 1.** One can easily see that the value \( \Lambda \geq 1.9902 \) improves the value 1.9 of Mueller and the value 1.9799 of Montgomery and Odlyzko.

Next, in the following, we will apply the Opial inequality due to Yang [31] to establish an explicit formula for the lower bounds of \( \Lambda \). The Yang inequality presented in the following theorem.

**Theorem B.** If \( x \) is absolutely continuous on \([a, b]\) with \( x(a) = 0 \) (or \( x(b) = 0 \)), then
\[
(2.7) \quad \int_a^b |x(t)|^m |x'(t)|^n \, dt \leq \frac{n}{m + n} (b - a)^m \int_a^b |x'(t)|^{m+n} \, dt,
\]

The inequality (2.7) has immediate application to the case where \( x(a) = x(b) = 0 \). Choose \( c = (a + b)/2 \) and apply (2.7) to \([a, c]\) and \([c, b]\) and then add to obtain
\[
\int_a^b |x(t)|^m |x'(t)|^n \, dt \\
\leq \frac{n}{m + n} (b - a)^m \left( \int_a^c |x'(t)|^{m+n} \, dt + \int_c^b |x'(t)|^{m+n} \, dt \right) \\
\leq \frac{n}{m + n} (b - a)^m \left( \int_a^b |x'(t)|^{m+n} \, dt \right).
\]
So that if \( x(0) = x(\pi) = 0 \), we have
\[
(2.8) \quad \int_0^\pi |x(t)|^m |x'(t)|^n \, dt \leq \frac{n}{m + n} (\pi/2)^m \int_0^\pi |x'(t)|^{m+n} \, dt.
\]

**Theorem 2.2.** On the hypothesis that the Riemann hypothesis is true and (1.13) is correctly predicted, we have
\[
(2.9) \quad \Lambda \geq \Lambda^*(h, k) = \frac{1}{\pi} \left( \frac{k b(h, k)}{b(h, k)} \right)^\frac{1}{2k} \text{, for } h \neq k \neq 0.
\]

**Proof.** As in the proof of Theorem 2.2 by applying the inequality (2.8) with \( a = t_n \), \( b = t_{n+1} \), \( m = 2k - 2h \), \( n = 2h \), and \( y = Z(t) \), we have
\[
\int_{t_n}^{t_{n+1}} \left( \frac{L_n}{\pi} \right)^{2k} \left( Z'(t) \right)^{2k} \\
\geq \frac{k}{2 h} \left( \frac{2}{\pi} \right)^{2k-2k} \int_{t_n}^{t_{n+1}} \left( \frac{L_n}{\pi} \right)^{2h} |Z(t)|^{2k-2h} \left| Z'(t) \right|^{2h} \, dt.
\]
Since the inequality remains true if we replace \( \frac{L_n}{\pi} \) by \( 2\kappa / \log T \), we have
\[
\int_{t_n}^{t_{n+1}} \left( \frac{2\kappa}{\log T} \right)^{2k} |Z'(t)|^{2k} \geq \int_{t_n}^{t_{n+1}} k \left( \frac{2}{\pi} \right)^{2k-2k} \left( \frac{2\kappa}{\log T} \right)^{2h} |Z(t)|^{2k-2h} |Z'(t)|^{2h} \, dt.
\]
(2.10)

Summing (2.10) over \( n \), using (1.13) we obtain
\[
\left( \frac{2\kappa}{\log T} \right)^{2k} a(k)b(k,k)T \left( \log T \right)^{k^2+2k} \geq \int_{t_n}^{t_{n+1}} k \left( \frac{2}{\pi} \right)^{2k-2k} \left( \frac{2\kappa}{\log T} \right)^{2h} a(k)b(h,k)T \left( \log T \right)^{k^2+2h} \, dt.
\]

This implies that
\[
T \left( \log T \right)^{k^2} \left( (2\kappa)^{2k} a(k)b(k,k) - k \left( \frac{2}{\pi} \right)^{2k-2k} (2\kappa)^{2h} a(k)b(h,k) \right) \geq o(T \left( \log T \right)^{k^2})
\]
whence
\[
k^{2k-2h} \geq \frac{1}{2^{2k-2h}} \frac{k}{h} \left( \frac{2}{\pi} \right)^{2k-2k} \frac{b(h,k)}{b(k,k)} + o(1), \quad \text{(as } T \to \infty).\]

This implies that
\[
\Lambda^{2k-2h}(k) \geq \frac{1}{2^{2k-2h}} \frac{k}{h} \left( \frac{2}{\pi} \right)^{2k-2k} \frac{b(h,k)}{b(k,k)}, \quad h \neq k \neq 0.
\]

which is the desired inequality and completes the proof.

To apply (2.9), we will need the following values of \( b(1,k) \) and \( b(k,k) \) that are determined from (1.14) where \( H(h,k) \) are defined as in Table 1.

\[
\begin{array}{cccccccc}
\hline
k & b(1,2) & b(2,2) & b(1,3) & b(3,3) & b(1,4) & b(4,4) & b(1,5) & b(5,5) \\
\hline
2 & \frac{1}{17} & \frac{1}{81} & \frac{1}{120} & \frac{1}{4064400} & 21946982400 & \frac{1}{2711593609484800000} & 7860533009643200000 & 12834175693844553179200000 \ \\
3 & 1 & 1 & \frac{1}{31} & \frac{1}{1209900} & 21946982400 & \frac{1}{2711593609484800000} & 7860533009643200000 & 12834175693844553179200000 \ \\
4 & 1 & \frac{1}{31} & \frac{1}{1209900} & \frac{1}{2711593609484800000} & 21946982400 & \frac{1}{2711593609484800000} & 7860533009643200000 & 12834175693844553179200000 \ \\
5 & \frac{1}{120} & \frac{1}{1209900} & \frac{1}{2711593609484800000} & \frac{1}{7860533009643200000} & 21946982400 & \frac{1}{2711593609484800000} & 7860533009643200000 & 12834175693844553179200000 \ \\
6 & \frac{1}{120} & \frac{1}{1209900} & \frac{1}{2711593609484800000} & \frac{1}{7860533009643200000} & 21946982400 & \frac{1}{2711593609484800000} & 7860533009643200000 & 12834175693844553179200000 \ \\
7 & \frac{1}{120} & \frac{1}{1209900} & \frac{1}{2711593609484800000} & \frac{1}{7860533009643200000} & 21946982400 & \frac{1}{2711593609484800000} & 7860533009643200000 & 12834175693844553179200000 \ \\
\end{array}
\]

Having an explicit formula for the \( b(h,k) \) and \( b(k,k) \) would via (2.9) help to decide whether the conjecture \( \lambda = \infty \) is true subject to the Riemann hypothesis. Using (2.9) and the values of \( b(1,k) \) and \( b(k,k) \), we have the new lower values \( \Lambda(k) \) for \( k = 2, \ldots, 7 \).
Next in the following, we will apply the Wirtinger inequality due to Agarwal and Pang [1] to establish a new explicit formula for the lower bounds of $\Lambda$. This inequality is presented in the following theorem.

**Theorem C.** Assume that $x(t) \in C^1[0, \pi]$ and $x(0) = x(\pi) = 0$, then

\[
\int_0^\pi (x'(t))^{2k} dt \geq \frac{2 \Gamma (2k + 1)}{\pi^{2k} \Gamma^2 ((2k + 1)/2)} \int_0^\pi x^{2k}(t) dt, \quad \text{for } k \geq 1,
\]

(2.11)

Theorem 2.3. Assuming the Riemann hypothesis and the moment (1.13) is correctly predicted we have

\[
\Lambda(k) \geq \frac{1}{2\pi} \left( \frac{b(0,k)}{b(k,k)} \frac{2 \Gamma(2k + 1)}{\pi^{2k} \Gamma^2 ((2k + 1)/2)} \right)^{\frac{1}{k}}, \quad \text{for } k = 3, 4, \ldots .
\]

(2.12)

**Proof.** To prove this theorem we will employ the inequality (2.11). By a suitable linear transformation, we can deduce from (2.11) that: if $x(t) \in C^1[a, b]$ and $x(a) = x(b) = 0$, then

\[
\int_a^b (\frac{b - a}{\pi})^{2k} (x'(t))^{2k} dt \geq \frac{2 \Gamma (2k + 1)}{\pi^{2k} \Gamma^2 ((2k + 1)/2)} \int_a^b x^{2k}(t) dt, \quad \text{for } k \geq 1.
\]

(2.13)

Since by our assumption (1.13) is correctly predicted by RMT, we have for $k = h$, the moments of the derivative of $Z(t)$

\[
\int_0^T (Z'(t))^{2k} dt \sim a(k)b(k,k)T (\log T)^{k^2 + 2k},
\]

(2.14)

and for $h = 0$, we have the moments of $Z(t)$

\[
\int_0^T Z^{2k}(t) dt \sim a(k)b(0,k)T (\log T)^{k^2}.
\]

(2.15)

Now, follow the proof of [17] by supposing that $t_l$ is the first zero of $Z(t)$ not less than $T$ and $t_m$, the last zero not greater than $2T$. Suppose further that for $l \leq n < m$, we have

\[
L_n = t_{n+1} - t_n \leq \frac{2\pi \kappa}{\log T},
\]

(2.16)

and applying the inequality (2.13), to obtain

\[
\int_{t_n}^{t_{n+1}} \left[ \left( \frac{L_n}{\pi} \right)^{2k} (Z'(t))^{2k} - \frac{2 \Gamma (2k + 1)}{\pi^{2k} \Gamma^2 ((2k + 1)/2)} Z^{2k}(t) \right] dt \geq 0.
\]

Since the inequality remains true if we replace $L_n/\pi$ by $2\kappa/\log T$, we have

\[
\int_{t_n}^{t_{n+1}} \left[ \left( \frac{2\kappa}{\log T} \right)^{2k} (Z'(t))^{2k} - \frac{2 \Gamma (2k + 1)}{\pi^{2k} \Gamma^2 ((2k + 1)/2)} Z^{2k}(t) \right] dt \geq 0.
\]

(2.17)

Summing (2.17) over $n$, applying (2.14) and (2.15), we obtain

\[
a(k)b(k,k) \left( \frac{2\kappa}{\log T} \right)^{2k} T (\log T)^{k^2 + 2k} - \frac{2a(k)b(0,k)}{\pi^{2k} \Gamma^2 ((2k + 1)/2)} T (\log T)^{k^2} \]

\[
= \left( a(k)b(k,k)\kappa^{2k} (2^{2k}) - \frac{2a(k)b(0,k)}{\pi^{2k} \Gamma^2 ((2k + 1)/2)} \right) T (\log T)^{k^2}
\]

\[
\geq O(T \log^k T),
\]
whence, as \( T \to \infty \), we obtain
\[
\kappa^{2k} \geq \frac{a(k)b(0,k)}{2^{2k}a(k)b(k,k)} \frac{2\Gamma (2k + 1)}{\pi^{2k} \Gamma^2 \left( \frac{2k+1}{2} \right)} = \frac{b(0,k)}{2^{2k}b(k,k)} \frac{2\Gamma (2k + 1)}{\pi^{2k} \Gamma^2 \left( \frac{(2k+1)/2}{2} \right)}.
\]
This implies that
\[
\Lambda^{2k}(k) \geq \frac{b(0,k)}{2^{2k}b(k,k)} \frac{2\Gamma (2k + 1)}{\pi^{2k} \Gamma^2 \left( \frac{(2k+1)/2}{2} \right)},
\]
which is the desired inequality. The proof is complete.

Having an explicit formula for the \( b(k,k) \) would via (2.12) help to decide whether the conjecture \( \lambda = \infty \) is true subject to the Riemann hypothesis. Using (2.12) and the values of \( b(0,k)/b(k/k) \) (see Table 2), we have the new lower values of \( \Lambda \)
\[
(2.18)
\begin{array}{cccccc}
\Lambda(3) & \Lambda(4) & \Lambda(5) & \Lambda(6) & \Lambda(7) \\
2.2265 & 2.6544 & 3.0545 & 3.4259 & 3.7676 \\
\end{array}
\]

In the following, we will apply the Wirtinger inequality due Brnetić and Pečarić [3] to establish a new explicit formula for the lower bounds of \( \Lambda \).

**Theorem 2.4.** Assuming the Riemann hypothesis and the moment (1.13) is correctly predicted, we have
\[
(2.19) \quad \Lambda(k) \geq \frac{1}{2\pi} \left( \frac{b(0,k)}{b(k,k)} \right) \frac{1}{I(k)} \frac{1}{\pi^{2k} I(k)} \quad \text{for } k = 3, 4, \ldots.
\]

**Proof.** To prove this theorem, we will employ the inequality (2.1). Proceeding as in Theorem 2.2, we may have
\[
\kappa^{2k} \geq \frac{a(k)b(0,k)}{2^{2k}a(k)b(k,k)} \frac{1}{\pi^{2k} I(k)} = \frac{b(0,k)}{2^{2k}b(k,k)} \frac{1}{\pi^{2k} I(k)} \quad \text{(as } T \to \infty).\]
This implies that
\[
\Lambda^{2k}(k) \geq \frac{b(0,k)}{2^{2k}b(k,k)} \frac{1}{\pi^{2k} I(k)},
\]
which is the desired inequality. The proof is complete.

Again having an explicit formula for the \( b(k,k) \) would via (2.19) help to decide whether the conjecture \( \lambda = \infty \) is true subject to the Riemann hypothesis. To find the new estimation of \( \Lambda(k) \) we need the values of \( I(k) \) for \( k = 3, 4, \ldots, 7 \), which are calculated numerically in the following table:

| \( I(3) \) | \( I(4) \) | \( I(5) \) | \( I(6) \) | \( I(7) \) |
|------------|------------|------------|------------|------------|
| 19.581     | 17.445     | 14.961     | 13.653     | 10.824     |
| 5000001000 | 1000000100 | 1000000100 | 1000000100 | 5000001100 |

Using these values and the values in Table 2, we have by using the explicit formula (2.19) the new estimation of \( \Lambda(k) \) in the following table:

| \( \Lambda(3) \) | \( \Lambda(4) \) | \( \Lambda(5) \) | \( \Lambda(6) \) | \( \Lambda(7) \) |
|----------------|----------------|----------------|----------------|----------------|
| 2.4905         | 2.9389         | 3.3508         | 3.7287         | 4.0736         |
References

[1] R. P. Agarwal and P. Y. H. Pang, Remarks on the generalization of Opial’s inequality, J. Math. Anal. Appl. 190 (1995), 559-557.
[2] P. R. Beesack, Hardy’s inequality and its extensions, Pacific J. Math. 11 (1961), 39-61.
[3] I. Brnetić and J. Pečarić, Some new Opial-type inequalities, Math. Ineq. Appl. 3 (1998), 385-390.
[4] H. M. Bui, M. B. Milinovich and N. Ng, A note on the gaps between consecutive zeros of the Riemann zeta-function, arXiv:0910.2052v1.
[5] A. Y. Cheer and A. D. Goldston, Simple zeros of the Riemann zeta-function, Proc. Amer. Math. Soc. 118 (1993), 356-373.
[6] J. B. Conrey, A. Gosh and S. M. Gonek, A note on gaps between zeros of the zeta function, Bull. London. Math. Soc. 16 (1984), 421-424.
[7] J. B. Conrey, More than two fifth of zeros of the Riemann zeta-function are on the critical line, J. Reine Angew. Math. 399 (1989), 1-22.
[8] J. B. Conrey, A. Gosh and S. M. Gonek, Large gaps between zeros of the zeta function, Mathematica 33 (1986), 212-238.
[9] J. B. Conrey, The fourth moment of derivative of the Riemann zeta function, Quart. J. Math. 2 (1988), 21-36.
[10] D. Cvijović and J. Klinowski, Integral representation of the Riemann zeta function for odd-integer arguments, J. Comp. Appl. Math. 142 (2002), 435-439.
[11] A. Fujii, On the difference between r consecutive ordinates of the Riemann zeta-function, Proc. Japan Acad. 51 (1975), 741-743.
[12] R. Garunkštis and J. Steuding, Simple zeros and discrete moments of the derivative of the Riemann zeta-function, J. Number Theor. 115 (2005), 310-321.
[13] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, 2nd Ed. Cambridge Univ. Press 1952.
[14] R. R. Hall, The behavior of the Riemann zeta function on the critical line, Mathematica 46 (1999), 281-313.
[15] R. R. Hall, A Wirtzinger type inequality and the spacing of the zeros of the Riemann zeta-function, J. Number Theory 93 (2002), 235-245.
[16] R. R. Hall, Generalized Wirtzinger inequalities, random matrix theory, and the zeros of the Riemann zeta-function, J. Number Theory 97 (2002), 397-409.
[17] R. R. Hall, Large spaces between the zeros of the Riemann zeta-function and random matrix theory, J. Number Theory 109 (2004), 240-256.
[18] R. R. Hall, Large spaces between the zeros of the Riemann zeta-function and random matrix theory, II, J. Number Theory 128 (2008), 2836-2851.
[19] C. P. Hughes, Thesis, University of Bristol, Bristol, 2001.
[20] A. E. Ingham, Mean theorems in the theorem of the Riemann zeta-function, Proc. London Math. Soc. 27 (1928), 273-300.
[21] J. P. Keating and N. C. Snaith, Random matrix theory and $\zeta(1/2 + it)$, Comm. Math. Phys. 214 (2000), 57-89.
[22] D. S. Mitinović, J. E. Pečarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publisher, 1993.
[23] H. L. Montgomery, The pair correlation of zeros of the Riemann zeta-function on the critical line, Proc. Sympos. Pure Math. 24, Amer. Math. Soc., Providence, RI, 1973, 181-193.
[24] H. L. Montgomery and A. M. Odlyzko, Gaps between zeros of zeta function, in ”Colloq. Math. Soc. Janos and Bolyai,” Vol. 34, North-Holland, Amsterdam, 1981.
[25] J. Mueller, On the difference between consecutive zeros of the Riemann zeta function, J. Number Theo. 14 (1982), 327-331.
[26] N. Ng, Large gaps between the zeros of the Riemann zeta function, J. Numb. Theo. 128 (2008), 509-556.
[27] A. Selberg, The zeta-function and the Riemann hypothesis, Skand. Math. 10 (1946), 187-200.
[28] J. Steuding, The Riemann zeta-function and moment conjectures from random matrix theory, Math. Slovaca 59 (2009), 323-338.
[29] R. Steuding and J. Steuding, Large gaps between zeros of the zeta function on the critical line and moment conjectures from random matrix theory, Comput. Methods Funct. Theory 8 (2008), 121-132.
[31] G. S. Yang, On a certain result of Z. Opial, Proc. Japan Acad. 42 (1966), 78-83.

Department of Mathematics Skills, PYD, King Saud University, Riyadh 11451, Saudi Arabia. Department of Math., Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

E-mail address: shsaker@mans.edu.eg, mathcoo@py.ksu.edu.sa