The Chiral Phase Transition at High Baryon Density from Nonperturbative Flow Equations

Jürgen Berges †
Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Dirk–Uwe Jungnickel ‡ and Christof Wetterich §
Institut für Theoretische Physik
Universität Heidelberg
69120 Heidelberg

(MIT–CTP–2794, HD–THEP–98–57)

Abstract

We investigate the QCD chiral phase transition at high baryon number density within the linear quark meson model for two flavors. The method we employ is based on an exact renormalization group equation for the free energy. Truncated nonperturbative flow equations are derived at nonzero chemical potential and temperature. Whereas the renormalization group flow leads to spontaneous chiral symmetry breaking in vacuum, we find a chiral symmetry restoring first order transition at high density. Combined with previous investigations at nonzero temperature, the result implies the presence of a tricritical point with long–range correlations in the phase diagram.

† Email: Berges@ctp.mit.edu

‡ Email: D.Jungnickel@thphys.uni-heidelberg.de

§ Email: C.Wetterich@thphys.uni-heidelberg.de

* This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement # DE-FC02-94ER40818 and by the Deutsche Forschungsgemeinschaft.
I. INTRODUCTION

The behavior of QCD at high temperature and baryon density is of fundamental interest and has applications in cosmology, the astrophysics of neutron stars and the physics of heavy ion collisions. Over the past years, considerable progress has been achieved in our understanding of high temperature QCD, where simulations on the lattice and universality arguments played an essential role. Recently, nonperturbative flow equations, based on the Wilsonian formulation of the renormalization group, were derived for an effective linear quark meson model [1–3]. The results of this approach account for both, the low temperature chiral perturbation theory domain of validity as well as the high temperature domain of critical phenomena for two quark flavors [3]. The method may help to shed some light on the remaining pressing questions at high temperature, like the nature of the phase transition for realistic values of the strange quark mass.

Our knowledge of the high density properties of strongly interacting matter is rudimentary so far. There are severe problems to use standard simulation algorithms at nonzero chemical potential on the lattice because of a complex fermion determinant. Different nonperturbative methods as the Wilsonian “exact renormalization group” or Schwinger–Dyson equations seem to present promising alternatives. It is the purpose of this letter to pursue the former approach based on an exact nonperturbative flow equation for a scale dependent effective action $\Gamma_k$ [1], which is the generating functional of the 1PI Green functions in the presence of an infrared momentum cutoff $\sim k$. The renormalization group flow for the average action $\Gamma_k$ interpolates between a given short distance or classical action $S$ and the standard effective action $\Gamma$, which is obtained by following the flow for $\Gamma_k$ to $k = 0$, thus removing the infrared cutoff in the end.

QCD at nonzero baryon density is expected to have a rich phase structure with different possible phase transitions as the density is increased. A prominent example is the liquid–gas nuclear transition which has been studied in low energy heavy ion collisions. Large efforts in ultra–relativistic heavy ion collision experiments focus on a transition at high temperatures and/or densities into what is generally known as the quark–gluon plasma – a new phase of matter in which color is screened rather than confined and chiral symmetry is restored. In addition to the nuclear and quark matter phases a number of possibilities like the formation of meson condensates or strange quark matter have been investigated. Recently, an extensive discussion has focused on the symmetry of the high density ground state, where condensates of quark Cooper pairs may break the color symmetry of QCD spontaneously and lead to a superconducting phase [4,5]. We will concentrate in this work on the chiral phase transition at nonzero baryon number density and apply our methods to an effective linear quark meson model for the low energy degrees of freedom of QCD. We note that the methods we discuss are equally suited to treat quark pair condensation and leave this for future work.

---

1 First calculations indicate that diquark condensates have only little influence on the order parameter associated with chiral symmetry breaking in the vacuum [5].

2 In the context of quark pair formation the renormalization of quark operators in the vicinity of the Fermi surface has been studied in [6].
The linear quark meson model we consider can be viewed as a generalization of the Nambu–Jona-Lasinio model for QCD \cite{7}. Though these approaches do not explain the confinement of quarks, they are adequate for many purposes and their predictions show convincing agreement with the results of real and numerical experiments. Calculations at nonzero baryon density are typically based on a mean field approximation and it was claimed \cite{7} that the order of the chiral phase transition is ambiguous. Recently, the order of this high density transition has raised considerable interest. One can argue on general grounds \cite{5,8} that if two–flavor QCD has a second order transition (crossover) at high temperatures and a first order transition at high densities, then there exists a tricritical point (critical endpoint) with long–range correlations in the phase diagram. The physics around this point is governed by universality and may allow for distinctive signatures in heavy ion collisions \cite{5,8,9}.

In this work it is our main concern to show that in a proper treatment beyond mean field theory the order of the chiral phase transition can be fixed within the models under investigation. For this purpose, a crucial observation is the strong attraction of the flow to partial infrared fixed points \cite{2,3}. The two remaining relevant or marginal parameters can be fixed by the phenomenological values of $f_\pi$ and the constituent quark mass. For two massless quark flavors we find that chiral symmetry restoration occurs via a first order transition between a phase with low baryon density and a high density phase.

We emphasize, nevertheless, that the linear quark meson model captures the low density properties of QCD only incompletely since the effects of confinement are not included. In particular, for a discussion of the liquid–gas nuclear transition the inclusion of nucleon degrees of freedom seems mandatory \cite{10}. On the other hand this model is expected to provide a reasonable description of the high density properties of QCD.

II. LINEAR QUARK MESON MODEL

It is an important property of the average action $\Gamma_k$ that at any scale $k$ only fluctuations with momenta in a small range around $k$ influence its flow with $k$. This implies that for each range of $k$ only those degrees of freedom have to be included which are relevant around $k$. In QCD these may comprise compounds of quarks and gluons. As long as one concentrates only on the chiral properties of QCD one may employ for scales $k$ below a “compositeness scale” $k_\Phi$ a description in terms of quark and (pseudo–)scalar meson degrees of freedom \cite{11,2,3}. All other degrees of freedom are assumed to be integrated out. The scalar and pseudoscalar mesons are described by a complex field $\Phi$ which transforms as $(\mathbf{N}, \mathbf{N})$ under the chiral flavor group $SU_L(N) \times SU_R(N)$ with $N$ the number of light quark flavors. Chiral symmetry breaking to a vector–like subgroup $SU_V(N)$ occurs through a nonvanishing chiral condensate $\langle \Phi^{ab} \rangle = \sigma_0 \delta^{ab}$.

We will restrict our discussion here to two massless quarks. A more realistic treatment would have to include the small up and down quark masses and the finite, but much heavier, strange quark mass. Though their inclusion will change certain quantitative estimates, they most likely do not change the qualitative outcome of the investigation discussed below. In the vacuum the scalar iso–triplet as well as the pseudoscalar singlet (associated with the $\eta'$ meson) contained in $\Phi$ are significantly heavier than the pions and the sigma meson. This property even holds in the symmetric phase where the pions are no longer Goldstone bosons. We take this as a guide also at nonzero density and decouple these degrees of freedom. For
$N = 2$ this can be achieved in a chirally invariant way. This results in a parameterization of $\Phi$ in terms of the $\sigma$-meson field and the three pion field components $\pi^l$ as

$$\Phi = \frac{1}{2} \left( \sigma + i \pi^l \tau_l \right)$$

where $\tau_l$ ($l = 1, 2, 3$) denote the Pauli matrices.

Closed nonperturbative flow equations follow from the exact renormalization group equation by a suitable truncation of the effective average action. We consider here a rather simple truncation of the momentum dependence in terms of standard kinetic terms parameterized by a running meson wave function renormalization constant $Z_{\Phi,k}$ and a scale dependent Yukawa coupling $h_k$. For the meson interaction we consider the most general form of the scalar potential $U_k$ consistent with the symmetries. For two flavors chiral symmetry implies that $U_k$ depends on only one invariant $\rho = \text{Tr}(\Phi^\dagger \Phi)$. The effective potential $U = \lim_{k \to 0} U_k = \lim_{k \to 0} \Gamma_k T/V$ is the relevant quantity for a study of the spontaneous breaking of chiral symmetry and is obtained for constant field configurations. At its minimum $\rho_0$ the effective potential is related to the energy density $\epsilon$, the entropy density $s$, the quark number density $n_q$ and the pressure $p$ by

$$U(\rho_0; \mu, T) = \epsilon - Ts - \mu n_q = -p.$$  

Our truncation precludes any nontrivial momentum dependence of the effective quark and meson propagators or the interactions. In particular, it clearly does not account for confinement effects. At nonzero temperature $T$ and chemical potential $\mu$ associated to the conserved quark number our ansatz for $\Gamma_k$ reads

$$\Gamma_k = \int_0^{1/T} dx_0 \int d^3 x \left\{ i \overline{\psi}^a \left( \gamma^\mu \partial_\mu + \mu \gamma^0 \right) \psi_a + \overline{\Phi} \psi \left[ \frac{1}{2} \gamma^5 \Phi^b - \frac{1}{2} \gamma^5 (\Phi^\dagger)_a \right] \psi_b + Z_{\Phi,k} \partial_\mu \Phi^* \partial^\mu \Phi + U_k(\overline{\rho}, \mu, T) \right\}.$$  

III. FLOW EQUATION FOR THE EFFECTIVE POTENTIAL

The dependence on the infrared cutoff scale $k$ of the effective action $\Gamma_k$ is given by an exact flow equation, which for fermionic fields $\psi$ (quarks) and bosonic fields $\Phi$ (mesons) reads
\[ \frac{\partial}{\partial k} \Gamma_k[\psi, \Phi] = \frac{1}{2} \text{Tr} \left\{ \frac{\partial R_{kB}}{\partial k} \left( \Gamma_k^{(2)}[\psi, \Phi] + R_k \right)^{-1} \right\} - \text{Tr} \left\{ \frac{\partial R_{kF}}{\partial k} \left( \Gamma_k^{(2)}[\psi, \Phi] + R_k \right)^{-1} \right\}. \] (4)

Here \( \Gamma_k^{(2)} \) is the matrix of second functional derivatives of \( \Gamma_k \) with respect to both fermionic and bosonic field components. The first trace on the right hand side of (4) effectively runs only over the bosonic degrees of freedom. It implies a momentum integration and a summation over flavor indices. The second trace runs over the fermionic degrees of freedom and contains in addition a summation over Dirac and color indices. The exact flow equation closely resembles a one–loop equation with the difference that the full inverse propagator \( \Gamma_k^{(2)} \) at the scale \( k \) appears instead of the classical propagator. The infrared cutoff function \( R_k \) has a block substructure with entries \( R_{kB} \) and \( R_{kF} \). Here \( R_{kB} \) denotes the infrared cutoff function for the bosonic fields and we employ an exponential cutoff function

\[ R_{kB} = \frac{Z_{\Phi,k} q^2}{\exp(q^2/k^2) - 1}. \] (5)

With this choice, \( R_{kB} \) acts as an additional mass term \( R_{kB} \simeq Z_{\Phi,k} k^2 \) for the low momentum \( q^2 \ll k^2 \) modes. The infrared cutoff function for the fermions \( R_{kF} \) should be consistent with chiral symmetries. This can be achieved if \( R_{kF} \) has the same Lorentz structure as the kinetic term for free fermions \[ [12]. \]

In presence of a chemical potential \( \mu \) we use

\[ R_{kF} = (\gamma^\mu q_\mu + i\mu \gamma^0) r_{kF}. \] (6)

The effective squared inverse fermionic propagator is then of the form

\[ P_{kF} = [(q_0 + i\mu)^2 + \vec{q}^2](1 + r_{kF})^2 = (q_0 + i\mu)^2 + \vec{q}^2 + k^2 \Theta(k^2 - (q_0 + i\mu)^2 - \vec{q}^2), \] (7)

where the second line defines \( r_{kF} \) and one observes that the fermionic infrared cutoff acts as an additional mass–like term \[ \sim k^2 \] Here the \( \Theta \)–function can be thought of as the limit of some suitably regularized function, e.g. \( \Theta^\epsilon = \exp((q_0 + i\mu)^2 + \vec{q}^2 - k_\Phi^2) / \epsilon + 1 \)–1.

We compute the flow equation for the effective potential \( U_k \) from equation (4) using the ansatz (3) for \( \Gamma_k \) and we introduce a renormalized field \( \rho = Z_{\Phi,k} \overline{\rho} \) and Yukawa coupling \( h_k = Z_{\Phi,k}^{-1/2} \overline{\rho} \). The flow equation for \( U_k \)

\[ \frac{\partial}{\partial k} U_k(\rho; T, \mu) = \frac{\partial}{\partial k} U_{kB}(\rho; T, \mu) + \frac{\partial}{\partial k} U_{kF}(\rho; T, \mu) \] (8)

obtains contributions from bosonic and fermionic fluctuations, respectively.

\[ ^3 \text{The exponential form (3) of the cutoff function } R_{kB} \text{ renders the first term on the right hand side of (4) both infrared and ultraviolet finite. No need for an additional ultraviolet regularization arises in this case. This is replaced by the necessary specification of an initial value } \Gamma_{k_\Phi} \text{ at the scale } k_\Phi. \text{ Here } k_\Phi \text{ is associated with a physical ultraviolet cutoff in the sense that effectively all fluctuations with } q^2 > k_\Phi^2 \text{ are already included in } \Gamma_{k_\Phi}. \text{ A similar property for the fermionic contribution is achieved by the } \Theta \text{–function in (7).} \]
\[
\frac{\partial}{\partial k} U_{kB}(\rho; T, \mu) = \frac{1}{2} T \sum_n \int_{-\infty}^{\infty} \frac{d^3 \vec{q}}{(2\pi)^3} \frac{1}{Z_{\Phi,k}} \frac{\partial R_{kB}(q^2)}{\partial k} \left\{ \frac{3}{P_{kB}(q^2) + U_k'(\rho; T, \mu)} + \frac{1}{P_{kB}(q^2) + U_k'(\rho; T, \mu) + 2\rho U_k''(\rho; T, \mu)} \right\},
\]
\[
\frac{\partial}{\partial k} U_{kF}(\rho; T, \mu) = -8N_c T \sum_n \int_{-\infty}^{\infty} \frac{d^3 \vec{q}}{(2\pi)^3} \frac{k}{P_{kF}(\vec{q})} \frac{\Theta(k^2_{\Phi} - (q_0 + i\mu)^2 - q^2)}{P_{kF}((q_0 + i\mu)^2 + q^2) + h_k^2 \rho/2}.
\]

Here \(q^2 = q_0^2 + q^2\) with \(q_0(n) = 2n\pi T\) for bosons, \(q_0(n) = (2n + 1)\pi T\) for fermions \((n \in \mathbb{Z})\) and \(N_c = 3\) denotes the number of colors. The scale dependent propagators on the right hand side contain the momentum dependent pieces \(P_{kB} = q^2 + Z^{-1}_{\Phi,k} R_k(q^2)\) and \(P_{kF}\) given by (7) as well as mass terms, where \(U_k', U_k''\) denote the first and second derivative of the potential with respect to \(\rho\). The only explicit dependence on the chemical potential \(\mu\) appears in the fermionic contribution (10) to the flow equation for \(U_k\). It is instructive to perform the summation of the Matsubara modes explicitly for the fermionic part. Since the flow equations only involve one momentum integration, standard techniques for one loop expressions apply and we find

\[
\frac{\partial}{\partial k} U_{kF}(\rho; T, \mu) = -8N_c \int_{-\infty}^{\infty} \frac{d^3 \vec{q}}{(2\pi)^4} \frac{k}{\sqrt{q^2 + k^2 + h_k^2 \rho/2}} \left\{ \exp \left[ \frac{1}{(\sqrt{q^2 + k^2 + h_k^2 \rho/2 - \mu})/T + 1} \right] + \exp \left[ \frac{1}{(\sqrt{q^2 + k^2 + h_k^2 \rho/2 + \mu})/T + 1} \right] \right\}
\]

where, for simplicity, we sent \(k_\Phi \to \infty\) in the \(\mu, T\) dependent second integral. This is justified by the fact that in the \(\mu, T\) dependent part the high momentum modes are exponentially suppressed.

For comparison, we note that within the present approach one obtains standard mean field theory results for the free energy if the meson fluctuations are neglected, \(\partial U_{kB}/\partial k \equiv 0\), and the Yukawa coupling is kept constant, \(h_k = h\) in (11). The remaining flow equation for the fermionic contribution could then easily be integrated with the (mean field) initial condition \(U_{k_0}(\rho) = \overline{m}^2_{k_0} \rho\). In the following we will concentrate on the case of vanishing temperature\(^4\). We find (see below) that a mean field treatment yields relatively good estimates only for the \(\mu\)-dependent part of the free energy \(U(\rho; \mu) - U(\rho; 0)\). On the other hand, mean field theory does not give a very reliable description of the vacuum properties which are important for a determination of the order of the phase transition at \(\mu \neq 0\).

\(^4\)For a solution of the flow equations at nonzero temperature and vanishing chemical potential see [3].
IV. RENORMALIZATION GROUP FLOW AT NONZERO CHEMICAL POTENTIAL

In the limit of vanishing temperature one expects and observes a non–analytic behavior of the $\mu$–dependent integrand of the fermionic contribution $[1]$ to the flow equation for $U_k$ because of the formation of Fermi surfaces. Indeed, the explicit $\mu$–dependence of the flow equation reduces to a step function

$$\frac{\partial}{\partial k} U_{kF}(\rho; \mu) = -8N_c \int_{-\infty}^{\infty} \frac{d^3q}{(2\pi)^3} \frac{k\Theta(k^2 - q^2)}{q^2 + k^2 + h_k^2\rho/2}$$

$$+ 4N_c \int_{-\infty}^{\infty} \frac{d^3\tilde{q}}{(2\pi)^3} \frac{k}{\sqrt{\tilde{q}^2 + k^2 + h_k^2\rho/2}} \Theta\left(\mu - \sqrt{\tilde{q}^2 + k^2 + h_k^2\rho/2}\right). \tag{12}$$

The quark chemical potential $\mu$ enters the bosonic part $[9]$ of the flow equation only implicitly through the meson mass terms $U_k'(\rho; \mu)$ and $U_k''(\rho; \mu) + 2\rho U_k''(\rho; \mu)$ for the pions and the $\sigma$–meson, respectively. For scales $k > \mu$ the $\Theta$–function in (12) vanishes identically and there is no distinction between the vacuum evolution and the $\mu \neq 0$ evolution. This is due to the fact that our infrared cutoff adds to the effective quark mass $(k^2 + h_k^2\rho/2)^{1/2}$. For a chemical potential smaller than this effective mass the “density” $-\partial U_k/\partial \mu$ vanishes whereas for larger $\mu$ one can view $\mu = [\tilde{q}^2(k, \rho) + k^2 + h_k^2\rho/2]^{1/2}$ as an effective Fermi energy for given $k$ and $\rho$. A small infrared cutoff $k$ removes the fluctuations with momenta in a shell close to the physical Fermi surface $[\tilde{q}^2(k, \rho) + k^2 + h_k^2\rho/2]^{1/2} - k^2 < q^2 < \mu^2 - h_k^2\rho/2$. Our flow equation realizes the general idea $[13]$ that for $\mu \neq 0$ the lowering of the infrared cutoff $k \to 0$ should correspond to an approach to the physical Fermi surface. For a computation of the meson effective potential the approach to the Fermi surface in (12) proceeds from below and for large $k$ the effects of the Fermi surface are absent. By lowering $k$ one “fills the Fermi sea”.

In vacuum the evolution of $U_k$, the Yukawa coupling $h_k$ and the meson wave function renormalization $Z_{\Phi,k}$ have been computed earlier $[4,5]$. The flow equations for $h_k$ and $Z_{\Phi,k}$ can be derived from (9) using the truncation (3) and one finds that for large enough $k$ their running is well approximated by $[2,5]$ ($k < k_\Phi$)

$$h_k^2 \approx Z_{\Phi,k}^{-1} \simeq -\frac{8\pi^2}{N_c \ln(k/k_\Phi)}. \tag{13}$$

One observes a very strong Yukawa coupling $h_k$ for $k$ in the vicinity of $k_\Phi$. (The Landau pole at $k = k_\Phi$ corresponds to taking $Z_{\Phi,k_\Phi} \to 0$.) This strong Yukawa coupling implies

5If one neglects the mesonic fluctuations one can perform the $k$–integration of the flow equation (12) in the limit of a $k$–independent Yukawa coupling. One recovers (for $k_\Phi^2 \gg k^2 + h^2\rho/2, \mu^2$) mean field theory results except for a shift in the mass, $h^2\rho/2 \to h^2\rho/2 + k^2$, and the fact that modes within a shell of three–momenta $\mu^2 - h^2\rho/2 - k^2 \leq \tilde{q}^2 \leq \mu^2 - h^2\rho/2$ are not yet included. Because of the mass shift the cutoff $k$ also suppresses the modes with $\tilde{q}^2 < k^2$.

6In $[2,5]$ a different variant of the fermionic infrared cutoff function $R_{kF}$ than (7) was used.
a fast approach of running couplings to partial infrared fixed points [2,3]. More precisely, for a rescaled potential \( U_k(\rho)/k^4 = \sum_{n=0}^{\infty} \frac{1}{n!} u_k^{(n)} (\rho/k^2)^n \) the running dimensionless couplings \( u_k^{(n)} \) for \( n \geq 2 \) approach values proportional to powers of the Yukawa coupling \( h_k \), with \( u_k^{(2)}/h_k^2 \approx 1 \), \( u_k^{(3)}/h_k^6 \approx -0.0164 \), \( u_k^{(4)}/h_k^8 \approx 0.0105 \) and similarly for the higher couplings [3]. In consequence, the details of the short distance meson interactions become unimportant and besides the marginal coupling \( h_k \) the only relevant parameter of the model corresponds to the mass parameter \( u_k^{(1)} \). We note that the large value of \( h_k \) is phenomenologically suggested by the comparably large value of the constituent quark mass. The observed fixed point behavior allows us to fix the model by only two phenomenological input parameters and we use \( f_\pi = 92.4 \) MeV and \( 300 \) MeV \( \lesssim M_q \lesssim 350 \) MeV. Previous results for the evolution in vacuum [2,3] show that for scales not much smaller than \( k_f \approx 650 \) MeV chiral symmetry remains unbroken. This holds down to a scale \( k_{\chi SB} \approx 450 \) MeV at which the meson potential \( U_k(\rho) \) develops a minimum at \( \rho_{0,k} > 0 \) thus breaking chiral symmetry spontaneously. Below the chiral symmetry breaking scale running couplings are no longer governed by the partial fixed point. In particular, for \( k \lesssim k_{\chi SB} \) the Yukawa coupling \( h_k \) and the meson wave function renormalization \( Z_{\Phi,k} \) depend only weakly on \( k \) and approach their infrared values.

At \( \mu \neq 0 \) we will follow the evolution from \( k = k_{\chi SB} \) to \( k = 0 \) and neglect the \( k \)-dependence of \( h_k \) and \( Z_{\Phi,k} \) in this range. According to the above discussion the initial value \( U_{k_{\chi SB}} \) is \( \mu \)-independent for \( \mu < k_{\chi SB} \). At this scale \( U_{k_{\chi SB}} \) is well approximated by the polynomial form where the coefficients \( u_{k_{\chi SB}}^{(n)} \), \( n = 2, 3, 4 \), are estimated by their partial infrared fixed point values and coefficients with \( n > 4 \) are neglected. The scalar mass parameter \( u_k^{(1)} \) crosses from positive to negative values at the chiral symmetry breaking scale and thus vanishes at \( k = k_{\chi SB} \). The constant \( u_{k_{\chi SB}}^{(0)} \) is fixed such that the pressure of the physical vacuum is zero. We do not assume a polynomial form for \( U_k(\rho; \mu) \) for \( k < k_{\chi SB} \) and solve the flow equation (8) with (9), (12) numerically as a partial differential equation for the potential depending on the two variables \( \rho \) and \( k \) for given \( \mu \).

In the fermionic part (12) of the flow equation the vacuum and the \( \mu \)-dependent term contribute with opposite signs. This cancelation of quark fluctuations with momenta below the Fermi surface is crucial for the restoration of chiral symmetry at high density. In vacuum, spontaneous chiral symmetry breaking is induced in our model by quark fluctuations which drive the scalar mass term \( U'_k(\rho = 0) \) from positive to negative values at the scale \( k = k_{\chi SB} \). (Meson fluctuations have the tendency to restore chiral symmetry because of the opposite

---

7Nonzero current quark masses, which are neglected in our approximation, result in a pion mass threshold and would effectively stop the renormalization group flow of renormalized couplings at a scale around \( m_\pi \). To mimic this effect one may stop the evolution by hand at \( k_f \approx m_\pi \). We observe that our results are very insensitive to such a procedure which can be understood from the fact that a small infrared cutoff – induced by \( k_f \) or by the nonzero current quark masses – plays only a minor role for a sufficiently strong first order transition. For the results presented in the next section we use \( k_f = 100 \) MeV. The flow of the potential \( U_k \) around its minima and for its outer convex part stabilizes already for \( k \) somewhat larger than \( k_f \). Fluctuations on larger length scales lead to a flattening of the barrier between the minima. The approach to convexity is not relevant for the present discussion.
FIG. 1. The zero temperature effective potential $U$ (in MeV$^4$) as a function of $\sigma \equiv (\rho/2)^{1/2}$ for different chemical potentials. One observes two degenerate minima for a critical chemical potential $\mu_c/M_q = 1.025$ corresponding to a first order phase transition at which two phases have equal pressure and can coexist ($M_q = 316.2$ MeV).

relative sign, cf. (4).) As the chemical potential becomes larger than the effective mass $(k^2 + h^2 \rho/2)^{1/2}$ quark fluctuations with momenta smaller than $q_F^2(\mu, k, \rho) = \mu^2 - k^2 - h^2 \rho/2$ are suppressed. Since $q_F^2$ is monotonically decreasing with $\rho$ for given $\mu$ and $k$ the origin of the effective potential is particularly affected. We will see in the next section that for large enough $\mu$ this leads to a second minimum of $U_{k=0}(\rho; \mu)$ at $\rho = 0$ and a chiral symmetry restoring first order transition.

V. HIGH DENSITY CHIRAL PHASE TRANSITION

In vacuum or at zero density the effective potential $U(\sigma)$, $\sigma \equiv \sqrt{\rho/2}$, has its minimum at a nonvanishing value $\sigma_0 = f_\pi/2$ corresponding to spontaneously broken chiral symmetry. As the quark chemical potential $\mu$ increases, $U$ can develop different local minima. The lowest minimum corresponds to the state of lowest free energy and is favored. In Figure 1 we plot the free energy as a function of $\sigma$ for different values of the chemical potential $\mu = 322.6, 324.0, 325.2$ MeV. Here the effective constituent quark mass is $M_q = 316.2$ MeV. We observe that for $\mu < M_q$ the potential at its minimum does not change with $\mu$. Since (cf. (2))

$$n_q = -\frac{\partial U}{\partial \mu}_{\text{min}}$$

we conclude that the corresponding phase has zero density. In contrast, for a chemical potential larger than $M_q$ we find a low density phase where chiral symmetry is still broken.
FIG. 2. The plot shows $n_q^{1/3}$, where $n_q$ denotes the quark number density as a function of $\mu$ in units of the effective constituent quark mass ($M_q = 316.2$ MeV).

The quark number density as a function of $\mu$ is shown in Figure 2. One clearly observes the non–analytic behavior at $\mu = M_q$ which denotes the “onset” value for nonzero density. From Figure 2 one also notices the appearance of an additional local minimum at the origin of $U$. As the pressure $p = -U$ increases in the low density phase with increasing $\mu$, a critical value $\mu_c$ is reached at which there are two degenerate potential minima. Before $\mu$ can increase any further the system undergoes a first order phase transition at which two phases have equal pressure and can coexist. In the high density phase chiral symmetry is restored as can be seen from the vanishing order parameter for $\mu > \mu_c$. The curvature of the potential at the origin for $\mu \approx \mu_c$ is $\partial^2 U / \partial \sigma^2 (\sigma = 0) \simeq (1.45M_q)^2$. We note that the relevant scale for the first order transition is $M_q$. For this reason we have scaled our results for dimensionful quantities in units of $M_q$.

For the class of quark meson models considered here (with $M_q/f_\pi$ in a realistic range around 3–4) the first order nature of the high density transition has been clearly established. In particular, these models comprise the corresponding Nambu–Jona-Lasinio models where the effective fermion interaction has been eliminated by the introduction of auxiliary bosonic fields. Our treatment, which is based on a nonperturbative evolution equation for the free energy, goes beyond mean field theory and puts it into a more systematic context. For the quark meson model this method has been used in the past to study the chiral phase transition at nonzero temperature, including the critical properties near the second order transition for two massless flavors characteristic of the three dimensional $O(4)$ universality class [3]. In the chiral limit, we therefore find a second order chiral transition at zero chemical potential and a first order chiral transition at zero temperature. By continuity these transitions meet at a tricritical point in the $(\mu,T)$–plane [4,5]. Away from the chiral limit, the second order chiral transition turns into a smooth crossover. The first order line of transitions at low
temperatures now terminates in a critical endpoint with long–range correlations \cite{5,8}. Using the method employed here a precision estimate of the universal critical equation of state of this Ising endpoint has been given in \cite{14}.

The extent to which the transition from nuclear matter to quark matter in QCD differs from the transition of a quark gas to quark matter in the quark meson model or NJL–type models has to be clarified by further investigations. At the phase transition, the quark number density in the symmetric phase $n_{q,c}^{1/3} = 0.52 M_q$ turns out in this model to be not much larger than nuclear matter density, $n_{q,nuc}^{1/3} = 152$ MeV. Here the open problems are related to the description of the low density phase rather than the high density phase. Quite generally, the inclusion of nucleon degrees of freedom for the description of the low density phase shifts the transition to a larger chemical potential and larger baryon number density for both the nuclear and the quark matter phases \cite{10}. An increase in $\mu_c$ results also from the inclusion of vector mesons. This would not change the topology of the phase diagram inferred from our model. Nevertheless, the topology of the QCD phase diagram is also closely connected with the still unsettled question of the order of the high temperature ($\mu = 0$) transition for three flavors with realistic quark masses. If the strange quark mass is too small, or if the axial $U(1)$ symmetry is effectively restored in the vicinity of the transition, then one may have a first order transition at high $T$ which is driven by fluctuations \cite{13}. In this case, critical endpoints only occur if the two first order regions at $T \gg \mu$ and $\mu \gg T$ are disconnected.

Finally, the low–density first order transition from a gas of nucleons or the vacuum to nuclear matter (nuclear gas–liquid transition) can only be understood if the low–momentum fermionic degrees of freedom are described by nucleons rather than quarks \cite{10}. The low–density branch of Figure 2 cannot be carried over to QCD. We emphasize that nucleon degrees of freedom can be included in the framework of nonperturbative flow equations.

**ACKNOWLEDGMENTS**

We thank K. Rajagopal and D. T. Son for helpful conversations.
REFERENCES

[1] C. Wetterich, Phys. Lett. 301B (1993) 90.
[2] D.-U. Jungnickel and C. Wetterich, Phys. Rev. D53 (1996) 5142.
[3] J. Berges, D.-U. Jungnickel and C. Wetterich, Phys. Rev. D to appear, hep-ph/9705474.
[4] M. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. 422B (1998) 247; hep-ph/9804403.
[5] J. Berges, D.-U. Jungnickel and C. Wetterich, Phys. Rev. D to appear, hep-ph/9705474.
[6] M. Alford, K. Rajagopal and F. Wilczek, Phys. Rev. D53 (1996) 5142.
[7] J. Berges and K. Rajagopal, Nucl. Phys. B to appear, hep-ph/9804233.
[8] N. Evans, S.D.H. Hsu and M. Schwetz, hep-ph/9808444; hep-ph/9810514.
[9] T. Schäfer and F. Wilczek, hep-ph/9810509.
[10] J. Berges and K. Rajagopal, Nucl. Phys. B to appear, hep-ph/9806219.
[11] A. Manohar and H. Georgi, Nucl. Phys. B234 (1984) 189.
[12] C. Wetterich, Z. Phys. C48 (1990) 693.
[13] J. Polchinski, Lectures presented at TASI 92, Boulder, CO, June 3–28, 1992 hep-th/9210046.
[14] J. Berges, N. Tetradis and C. Wetterich, Phys. Rev. Lett. 77 (1996) 873; S. Seide and C. Wetterich, cond-mat/9806372.
[15] R. Pisarski and F. Wilczek, Phys. Rev. D29 (1984) 338.