ON SMALL VALUES OF THE RIEMANN ZETA-FUNCTION
ON THE CRITICAL LINE AND GAPS BETWEEN ZEROS

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Abstract. Small values of \(|\zeta(\frac{1}{2}+it)|\) are investigated, using the value distribution results of A. Selberg. This gives an asymptotic formula for

$$\mu\left(\{0 < t \leq T : |\zeta(\frac{1}{2}+it)| \leq c\}\right).$$

Some related problems involving gaps between ordinates of zeros of \(\zeta(s)\) are also discussed.

The aim of this note is to discuss the problem of “small” values of the Riemann zeta-function \(\zeta(s)\) on the critical line \(\Re s = \frac{1}{2}\), and some related problems involving the gaps between the zeros of \(\zeta(s)\). This is in contrast with the so-called “large” values of \(|\zeta(\frac{1}{2}+it)|\) (i.e., values which are \(\geq t^\varepsilon\)), which are extensively discussed in [5]. Since we have (see [5])

$$\int_0^T |\zeta(\frac{1}{2}+it)|^2 \, dt \sim T \log T, \quad \int_0^T |\zeta(\frac{1}{2}+it)|^4 \, dt \sim \frac{T}{2\pi^2} \log^4 T \quad (T \to \infty),$$

this means that \(|\zeta(\frac{1}{2}+it)|\) is small “most of the time”. The problem, then, is to evaluate asymptotically the measure of the subset of \([0,T]\) where \(|\zeta(\frac{1}{2}+it)|\) is “small”.

There are several ways in which one can proceed, and a natural way is the following one. Let \(c > 0\) be a given constant, let \(\mu(\cdot)\) denote measure, and let

$$A_c(T) := \{0 < t \leq T : |\zeta(\frac{1}{2}+it)| \leq c\}.$$
In [8] I raised the question of the asymptotic evaluation of \( \mu(A_c(T)) \). One can tackle this problem by using the limit law

\[
\lim_{T \to \infty} \frac{1}{T} \mu \left( \{ 0 < t \leq T : |\zeta(\frac{1}{2} + it)| \leq e^{y \sqrt{\frac{1}{2} \log \log T}} \} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}u^2} \, du,
\]

where \( y \in \mathbb{R} \) is fixed. This result was proved by A. Laurinčikas [11], who used the fact that

\[
\frac{1}{T} \int_{0}^{T} |\zeta(\frac{1}{2} + it)|^{2k(2\log \log T)^{-1/2}} \, dt = e^{\frac{1}{2}k^2} \left\{ 1 + O\left( (\log \log T)^{-1/4} \right) \right\}
\]

uniformly for \( e^{-\sqrt{\log \log T}} \leq k \leq k_0 \), where \( k_0 \in \mathbb{N} \) is a constant. The proof uses the property that \( e^{\frac{1}{2}k^2} \) is exactly the 2\( k \)–th moment of the distribution function

\[
G(x) = \Phi(\log x) \quad (x > 0), \quad \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} \, du,
\]

so that (2) yields (1) \((G(x) = 0 \text{ for } x \leq 0)\). One does not see, however, how one can obtain (1) from Laurinčikas’ proof in the form which would not give the result only as “lim”, but an asymptotic formula with an error term as \( T \to \infty \). This is because the lognormal law \( G(x) \) is “bad”. It is known from probability theory that the function \( G(x) \) cannot be defined by its moments \( e^{\frac{1}{2}k^2} \). Namely the moments \( e^{\frac{1}{2}k^2} \) are very rapidly increasing, and from this all “bad” consequences follow. To obtain the estimate of the rate of convergence we must consider complex moments, which one may write as

\[
\frac{1}{T} \int_{0}^{T} |\zeta(\frac{1}{2} + it)|^{2\tau(2\log \log T)^{-1/2}} \, dt = e^{-\frac{1}{2}\tau^2} + S_T(\tau) \quad (\tau \in \mathbb{R}),
\]

say. However, the problem of the estimation of the function \( S_T(\tau) \) seems to be very hard.

We shall first show how to use (1) to obtain a weak asymptotic formula for \( \mu(A_c(T)) \). Let \( \varepsilon > 0 \) be fixed. Note that, for \( T \geq T_0(\varepsilon, c) \), we trivially have

\[
e^{-\varepsilon \sqrt{\frac{1}{2} \log \log T}} < c < e^{\varepsilon \sqrt{\frac{1}{2} \log \log T}}.
\]

Therefore, as \( T \to \infty \), (1) gives

\[
\mu(A_c(T)) \leq \mu \left( \{ 0 < t \leq T : |\zeta(\frac{1}{2} + it)| \leq e^{\varepsilon \sqrt{\frac{1}{2} \log \log T}} \} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\varepsilon} e^{-\frac{1}{2}u^2} \, du \cdot T + o(T),
\]
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and

$$\mu(A_c(T)) \geq \mu\left(\{0 < t \leq T : |\zeta(\frac{1}{2} + it)| \geq e^{-\varepsilon \sqrt{\frac{1}{2} \log \log T}}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\varepsilon} e^{-\frac{1}{2}u^2} \, du \cdot T + o(T).$$

But as

$$\int_{-\infty}^{0} e^{-\frac{1}{2}u^2} \, du = \sqrt{2} \int_{0}^{\infty} e^{-x^2} \, dx = \sqrt{\frac{\pi}{2}},$$

it follows that

$$\mu(A_c(T)) = \frac{T}{2} + O(\varepsilon T) + o(T),$$

hence letting $\varepsilon \to 0$ we obtain

$$\mu(A_c(T)) = \frac{T}{2} + o(T) \quad (T \to \infty). \quad (3)$$

Let now $0 < c_1 < c_2$. Since

$$[c_1, c_2] = (0, c_2) \setminus (0, c_1],$$

it follows from (3), as $T \to \infty$, that

$$\mu\left(\{0 < t \leq T : c_1 \leq |\zeta(\frac{1}{2} + it)| \leq c_2\}\right) = \mu(A_{c_2}(T)) - \mu(A_{c_1}(T)) + o(T) = o(T). \quad (4)$$

It turns out that for the above problems one can use Theorem 2 of A. Selberg’s paper [13], which is an asymptotic formula with an error term. Selberg obtained sharper results than Laurinčikas’ before Laurinčikas did, but he published his paper later. Actually Selberg’s paper contains no proofs, but it is hinted at the end that proofs will appear. Also there exists the recent work of D.A. Hejhal [4], which is built on the methods of [13] and complements it. In fact (2.6) of Theorem 2 on p. 374 of Selberg’s paper can be specialized to yield a result sharper than (1), namely

$$\mu\left(\{0 < t \leq T : |\zeta(\frac{1}{2} + it)| \leq e^{y \sqrt{\frac{1}{2} \log \log T}}\right)$$

$$= \Phi(y)T + O\left(T \frac{(\log \log \log T)^2}{\sqrt{\log \log T}}\right), \quad (5)$$

where as before, for $x \in \mathbb{R},$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} \, du.$$
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is the probability integral. Now Hejhal kindly confirmed, by going through Selberg’s unpublished proof, that formula (5) holds uniformly in $y$. Therefore choosing

$$y = \frac{\log c}{\sqrt{\frac{1}{2} \log \log T}}$$

for a given constant $c > 0$, and using the fact that, for $|y| \leq 1$,

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}u^2} du + O\left(\int_{0}^{[y]} e^{-\frac{1}{2}u^2} du\right) = \frac{1}{2} + O(|y|),$$

we obtain from (5)

**THEOREM 1.** We have

(6) $$\mu(A_c(T)) = \frac{T}{2} + O\left(\frac{T(\log \log \log T)^2}{\sqrt{\log \log T}}\right),$$

where as before

$$A_c(T) := \{0 < t \leq T : |\zeta(\frac{1}{2} + it)| \leq c\}.$$  

We also have, for given constants $0 < c_1 < c_2$,

(7) $$\mu\left(\left\{0 < t \leq T : c_1 \leq |\zeta(\frac{1}{2} + it)| \leq c_2\right\}\right) = O\left(\frac{T(\log \log \log T)^2}{\sqrt{\log \log T}}\right).$$

Of course, (7) follows easily from (4) and (6). We note that the formulas (6) and (7), which improve (3) and (4), give a satisfactory solution to the problem of the distribution of “small” values of $|\zeta(\frac{1}{2} + it)|$. The factor $(\log \log \log T)^2$, which appears in (5)–(7), is probably extraneous, but will be very likely difficult to get rid of.

Another way to see how (6) and (7) follow is to apply a result contained in D.A. Hejhal’s work [4], where he successfully deals with zeros of linear combinations of $L$-functions belonging to Selberg’s class [13]. In particular, his equation (4.21), specialized to $\zeta(s)$, says that

(8) $$\mu\left(\{T \leq t \leq 2T : e^a \leq |\zeta(\frac{1}{2} + it)| \leq e^b\}\right) = T \int_{a/\sqrt{\pi \psi}}^{b/\sqrt{\pi \psi}} e^{-\pi v^2} dv + O\left(\frac{T \log^2 \psi}{\sqrt{\psi}}\right)$$

uniformly in $a, b \in \mathbb{R}$, where

$$\psi = \log \log T + O(\log \log \log T).$$
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Therefore the specialization \(a = -\infty, b = \log c\) yields (6), while \(a = \log c_1, b = \log c_2 (0 < c_1 < c_2)\) yields (7).

We shall consider now a problem related to the above one. Let henceforth \(0 < \gamma_1 \leq \gamma_2 \leq \ldots\) denote positive ordinates of complex zeros of \(\zeta(s)\); it is known that \(\gamma_1 = 14.13\ldots\), and all known (> 10^9) zeros are simple and lie on the critical line \(\Re s = \frac{1}{2}\). We define \(\gamma_-(t) = \gamma_n\) if \(\gamma_n \leq t < \gamma_{n+1}\), \(\gamma_+(t) = \gamma_{n+1}\) if \(\gamma_n < t \leq \gamma_{n+1}\), \(\gamma_-(t) = \gamma_+(t) = \gamma_n\) if \(t = \gamma_n\).

\[A(T) = \{ 0 < t \leq T : |\zeta(\frac{1}{2} + it)| \leq \gamma_+(t) - \gamma_-(t) \},\]

\[B(T) = [0, T] \setminus A(T) = \{ 0 < t \leq T : |\zeta(\frac{1}{2} + it)| > \gamma_+(t) - \gamma_-(t) \}.\]

Natural problems are to evaluate asymptotically \(\mu(A(T))\) and \(\mu(B(T))\). We shall prove the following

**THEOREM 2.** We have

(9) \[\mu(B(T)) = T + O\left( T \frac{\log \log \log T}{\sqrt{\log \log T}} \right).\]

**Proof.** We shall first employ a method based on the value distribution result (8). This leads to (9), but with \((\log \log \log T)^2\) in place of \(\log \log \log T\). Then we shall present another approach, which yields the slightly sharper result of Theorem 2. Let

\[C_1(T) := \{ 0 < t \leq T : \gamma_+(t) - \gamma_-(t) < \frac{(\log \log T)^6}{\log T} \},\]

\[C_2(T) := \{ 0 < t \leq T : |\zeta(\frac{1}{2} + it)| \geq \exp(- (\log \log T)^{3/4}) \},\]

and let \(\overline{S}\) denote the complement of \(S\) in \([0, T]\). From (8) with \(a = -\infty, b = -(\log \log T)^{3/4}\) we obtain

\[\mu(\overline{C}_2(T)) \ll T \int_{\frac{1}{2}(\log \log T)^{1/4}}^{\infty} e^{-\pi v^2} dv + T \frac{(\log \log T)^2}{\sqrt{\log \log T}} \ll T \frac{(\log \log T)^2}{\sqrt{\log \log T}},\]

hence

(10) \[\mu(C_2(T)) = T + O\left( T \frac{(\log \log T)^2}{\sqrt{\log \log T}} \right).\]

On the other hand

(11) \[\mu(C_2(T)) = \mu(\overline{C}_1(T) \cap C_2(T)) + \mu(C_1(T) \cap C_2(T)).\]
However we have

\[ \mu(\mathcal{C}_1(T) \cap \mathcal{C}_2(T)) \leq \mu(\mathcal{C}_1(T)) \ll T \frac{\log \log \log T}{\sqrt{\log \log T}}. \]

The second bound in (12) is a consequence of a bound which follows from the following Lemma (weaker results are given in A. Fujii [1], [2] and (without proof) in E.C. Titchmarsh [14, p. 246]).

**Lemma.** Let \( 0 < \gamma_1 \leq \gamma_2 \leq \cdots \) denote imaginary parts of complex zeros of \( \zeta(s) \), and let \( \lambda \geq 2 \). Then there exists a constant \( C > 0 \) such that uniformly

\[ \sum_{T < T + H, \gamma_{n+1} - \gamma_n \geq \lambda \log T} 1 \ll (N(T + H) - N(T)) \exp (-C \lambda) + 1, \]

where \( N(T) \) is the number of zeros of \( \zeta(s) \) with imaginary parts in \((0, T] \), and \( T^a < H \leq T, a > \frac{1}{2} \).

**Proof.** The basic result is the asymptotic formula [15, Theorem 4] of K.-M. Tsang. This says that, for \( T^a < H \leq T, a > \frac{1}{2}, 0 < h < 1 \) and any \( k \in \mathbb{N} \), we have uniformly

\[ \int_T^{T+H} (S(t + h) - S(t))^{2k} dt = \frac{H(2k)!}{(2\pi)^k k!} \log^k (2 + h \log T) \]

\[ + O \left( H(ck)^k \left( k^k + \log^{k-\frac{1}{2}} (2 + h \log T) \right) \right), \]

where \( c > 0 \) is a constant, and as usual \( S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT) \). Thus \( S(T) = O(\log T) \) (see [5] or [14]) and the Riemann–von Mangoldt formula is

\[ N(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + S(T) + \frac{7}{8} + O \left( \frac{1}{T} \right). \]

This gives \( \gamma_{n+1} - \gamma_n \ll 1 \), and also

\[ S(t + h) - S(t) = N(t + h) - N(t) - \frac{h}{2\pi} \log t + O \left( \frac{h^2 + 1}{t} \right). \]

If

\[ \gamma_n < t < \frac{1}{2} (\gamma_n + \gamma_{n+1}), \gamma_{n+1} - \gamma_n \geq \frac{\lambda}{\log T}, T \leq t \leq T + H, h = \frac{\lambda}{2 \log T}, \]

then
then \( N(t + h) - N(t) = 0 \), and \( h \ll 1 \) will hold in view of \( \gamma_{n+1} - \gamma_n \ll 1 \). For \( t \) satisfying (16) we have

\[
|S(t + h) - S(t)| \geq \frac{h}{4\pi} \log t \geq \frac{\lambda}{8\pi},
\]

and (14) will in fact hold for \( 0 < h \ll 1 \). We obtain from (14)

\[
\sum_{T < \gamma_n < \gamma_{n+1} \leq T + H, \gamma_{n+1} - \gamma_n \geq \lambda / \log T} (\frac{\lambda}{8\pi})^{2k} (\gamma_{n+1} - \gamma_n) \ll H(Ak(k + \log \lambda))^k
\]

with suitable \( A > 0 \), which implies that \( (B = (8\pi)^2A) \)

\[
(17) \sum_{T < \gamma_n \leq T + H, \gamma_{n+1} - \gamma_n \geq \lambda / \log T} 1 \ll (N(T + H) - N(T)) \left( Bk \frac{(k + \log \lambda)}{\lambda^2} \right)^k + 1.
\]

We take

\[
k = \left[ \frac{\lambda}{2\sqrt{B}} \right],
\]

and (13) follows from (17) for \( \lambda \geq \lambda_0 (\geq 2) \), while for \( \lambda < \lambda_0 \) the bound in (13) is trivial.

To obtain (12) write

\[
\tilde{C}_1(T) = \bigcup_{k=1}^{\infty} D_k(T), D_k(T) := \{ 0 < t \leq T : V_k(T) \leq \gamma_+(t) - \gamma_-(t) < 2V_k(T) \},
\]

\[
V_k(T) := \frac{2^{k-1}(\log \log T)^6}{\log T}.
\]

Hence with \( \lambda = \lambda(k, T) = 2^{k-1}(\log \log T)^6 \) we have, on using (13),

\[
\mu(D_k(T)) \leq 2V_k(T) \sum_{\gamma_n \leq T, \gamma_{n+1} - \gamma_n \geq \lambda / \log T} 1 \ll T \exp(-2^k (\log \log T)^2),
\]

which gives

\[
\mu(\tilde{C}_1(T)) \ll T \sum_{k=1}^{\infty} \exp(-2^k (\log \log T)^2) \ll T \frac{(\log \log \log T)^2}{\sqrt{\log \log T}},
\]

as asserted.
We therefore have from (10)–(12)
\[ \mu(C_1(T) \cap C_2(T)) = T + O \left( T \frac{(\log \log T)^2}{\sqrt{\log \log T}} \right), \]
and (9) with the error term \( O \left( T \frac{(\log \log T)^2}{\sqrt{\log \log T}} \right) \) follows from
\[ T \geq \mu(B(T)) \geq \mu(C_1(T) \cap C_2(T)), \]
since for \( t \in C_1(T) \cap C_2(T) \) we have
\[ |\zeta(1/2 + it)| > e^{-(\log T)^{3/4}} > \frac{(\log T)^6}{\log T} > \gamma_+ - \gamma_-. \]

To obtain Theorem 2 in the sharper form given by (9), we use a result of A. Perelli and the author [10] (see also [6, Theorem 6.2]) which says that, if \( \psi(T) \) is an arbitrary positive function tending to infinity with \( T \), then for
\[ 0 \leq \lambda \leq (\psi(T) \log T)^{-1/2} \]
we have
\[ \int_0^T |\zeta(1/2 + it)|^\lambda \, dt = T + O(T) \quad (T \to \infty). \]  

For our purposes we need (18) with an \( O \)-term for the error instead of the \( o \)-term. This is given by
\[ \int_0^T |\zeta(1/2 + it)|^\lambda \, dt = T + O \left( \frac{T}{\sqrt{\psi(T)}} \right) + O(\log T). \]

To obtain (19) in place of (18) one has first to note that [6, Lemma 6.7] actually gives
\[ \int_0^T |\zeta(1/2 + it)|^\lambda \, dt \geq T + O(\log T) \]
for any \( \lambda \geq 0 \). For the corresponding upper bound it suffices to note that, in the proof of [6, (6.41)] we obtain, for \( m = [(\log \log T)^{1/2}] \), \( C_1 > 0 \), \( C_2 > 0 \),
\[ (C_1)^{1/4} \frac{m \lambda}{2m} (\log T)^{2m} \leq \exp \left( (C_2 \psi(T) \log \log T)^{-1/2} (\log \log T)^{1/2} \right) \]
\[ = 1 + O \left( \frac{1}{\sqrt{\psi(T)}} \right), \]
which gives then (19), as asserted. Let henceforth
\[
\lambda := \frac{1}{\sqrt{\psi(T) \log \log T}}, \quad \psi(T) := \frac{\log \log T}{9(\log \log \log T)^2}.
\]

On one hand, we have (19), while on the other hand we may write (20)
\[
\int_0^T |\zeta(\frac{1}{2} + it)|^\lambda \, dt = \int_{A(T)} |\zeta(\frac{1}{2} + it)|^\lambda \, dt + \int_{B(T)} |\zeta(\frac{1}{2} + it)|^\lambda \, dt = I_1(T) + I_2(T),
\]
say. For \( I_2(T) \) we use the Cauchy-Schwarz inequality and (19) with \( 2\lambda \) replacing \( \lambda \) to obtain that
\[
I_2(T) \leq (\mu(B(T)))^{1/2} \left( \int_0^T |\zeta(\frac{1}{2} + it)|^{2\lambda} \, dt \right)^{1/2}
\]
(21)
\[
= (\mu(B(T)))^{1/2} \left( T + O \left( \frac{T}{\sqrt{\psi(T)}} \right) \right)^{1/2}.
\]

We have
\[
I_1(T) \leq \sum_{\gamma_n \leq T} (\gamma_{n+1} - \gamma_n)^{\lambda + 1} \ll T \log^{-\lambda} T = T(\log \log T)^{-3}.
\]
(22)

Here we used the bound
\[
\sum_{\gamma_n \leq T} (\gamma_{n+1} - \gamma_n)^\alpha \ll T(\log T)^{1-\alpha} \quad (1 \leq \alpha \leq \alpha_0).
\]
(23)

To obtain (23) we estimate trivially the contribution of \( \gamma_n \) for which \( \gamma_{n+1} - \gamma_n \leq 2/\log T \). The remaining sum is split into subsums where
\[
\frac{2^k}{\log T} < \gamma_{n+1} - \gamma_n \leq \frac{2^{k+1}}{\log T} \quad (k = 1, 2, \ldots),
\]
each of which is estimated by (13), which yields (23).

Therefore we obtain from (15)–(22) a lower bound for \( \mu(B(T)) \) of the form given by (9), and trivially \( \mu(B(T)) \leq T \). This establishes (9).

It is very likely that preceding results hold if the \( \gamma_n \)'s are the ordinates of zeros on the critical line (assuming that the Riemann hypothesis is not true, and \( \zeta(s) \) has zeros lying off the critical line), but in that case the problems are more
difficult. Connected with this is a problem which I posed during the Conference on Elementary and Analytic Number Theory, held in Oberwolfach, March 1994 (see also [7]). This is also related to small values of $|\zeta(\frac{1}{2} + it)|$. Let $\tilde{\gamma}_n$ denote the $n$-th positive zero of $\zeta(\frac{1}{2} + it) = 0$, where possible multiple zeros are counted with their respective multiplicities. Let

$$N_0(T) = \sum_{\tilde{\gamma}_n \leq T} 1, \quad B(T) := N_0(T) - A(T),$$

$$A(T) := \sum_{\tilde{\gamma}_n \leq T, \max_{\tilde{\gamma}_n \leq t \leq \tilde{\gamma}_{n+1}} |\zeta(\frac{1}{2} + it)| \leq \tilde{\gamma}_{n+1} - \tilde{\gamma}_n} 1.$$

The problem is to compare (unconditionally, or under the Riemann hypothesis) $A(T)$ and $B(T)$ to $N_0(T)$ (we know that $T \log T \ll N_0(T) \ll T \log T$). I expect that $B(T) \sim N_0(T)$ (or equivalently $A(T) = o(N_0(T))$) as $T \to \infty$, that is, on the average the maximum between two consecutive zeros on the critical line should be larger than the gap between these zeros. M. Jutila and the author [9] proved that the number of $\tilde{\gamma}_n$’s not exceeding $T$ for which $\tilde{\gamma}_{n+1} - \tilde{\gamma}_n \geq V (>0)$ is uniformly

$$\ll \min(TV^{-2} \log T, TV^{-3} \log^5 T),$$

but unfortunately this bound is not well suited in dealing with the “small gaps”.

Returning to Theorem 2, note that $A(T)$ contains intervals $[\gamma_n, \gamma_{n+1}]$ with $\gamma_n \leq T$ (with the possible exception of one interval), such that

$$\max_{\gamma_n \leq t \leq \gamma_{n+1}} |\zeta(\frac{1}{2} + it)| \leq \gamma_{n+1} - \gamma_n.$$

Then the method of proof of Theorem 2 shows that

$$\sum_{\gamma_n \leq T} ^* (\gamma_{n+1} - \gamma_n) = T + O \left( T \frac{\log \log \log T}{\sqrt{\log \log T}} \right),$$

where $^*$ denotes summation with the conditions

$$\gamma_{n+1} - \gamma_n < \frac{(\log \log T)^6}{\log T}, \quad \max_{\gamma_n \leq t \leq \gamma_{n+1}} |\zeta(\frac{1}{2} + it)| > \gamma_{n+1} - \gamma_n.$$

Now we assume the Riemann hypothesis and apply the Cauchy-Schwarz inequality to the left-hand side of (24). Then by (23) with $\alpha = 2$ we obtain

$$B(T) \gg N_0(T),$$
Small values of $|\zeta(\frac{1}{2} + it)|$

which favours the conjecture that $B(T) \sim N_0(T)$ as $T \to \infty$. Actually the constant in (25) may be explicitly calculated if we use a bound of A. Fujii [3], namely

$$\sum_{\bar{\gamma}_n \leq T} (\bar{\gamma}_{n+1} - \bar{\gamma}_n)^2 \leq 9 \cdot \frac{2\pi T}{\log \frac{T}{2\pi}} \quad (T > T_0).$$

This leads, under RH, to the inequality

$$B(T) \geq (1 + o(1)) \frac{T}{18\pi} \log \left( \frac{T}{2\pi} \right) = \left( \frac{1}{9} + o(1) \right) N_0(T) \quad (T \to \infty).$$

If, in addition to the RH, one assumes the Gaussian Unitary Ensemble Hypothesis, then one can improve the bound in (26) and obtain in fact an asymptotic formula for the sum on the left-hand side of (26). For the details the reader is referred to [5].

Note that $A(T)$ trivially counts the \( \bar{\gamma}_n \)'s for which $\bar{\gamma}_n = \bar{\gamma}_{n+1}$, that is, multiple zeros on the critical line. Hence the conjecture $B(T) \sim N_0(T)$ is stronger than the conjecture that almost all zeros on the critical line are simple (which seems to be independent of the RH). In connection with this it is perhaps natural to consider also

$$D(T) := \sum_{\bar{\gamma}_n < \bar{\gamma}_{n+1} \leq T, \max_{\bar{\gamma}_n \leq t \leq \bar{\gamma}_{n+1}} |\zeta(\frac{1}{2} + it)| \leq \bar{\gamma}_{n+1} - \bar{\gamma}_n} 1,$$

and try to show that

$$D(T) = o(N_0(T)) \quad (T \to \infty),$$

which is implied by $A(T) = o(N_0(T))$. One way to deal with this problem is to note that from Theorem 2 we have, unconditionally,

$$\mu(A(T)) = \sum_{\gamma_n \leq T, \max_{\gamma_n \leq t \leq \gamma_{n+1}} |\zeta(\frac{1}{2} + it)| \leq \gamma_{n+1} - \gamma_n} (\gamma_{n+1} - \gamma_n) \ll T \frac{\log \log \log T}{\sqrt{\log \log T}}.$$

On the other hand, for any $\kappa > 0$,

$$\mu(A(T)) \geq \sum_{\gamma_n \leq T, \max_{\gamma_n \leq t \leq \gamma_{n+1}} |\zeta(\frac{1}{2} + it)| \leq \gamma_{n+1} - \gamma_n, \gamma_{n+1} - \gamma_n > \kappa / \log T} (\gamma_{n+1} - \gamma_n) \geq \frac{\kappa}{\log T} \sum_{\gamma_n \leq T, \max_{\gamma_n \leq t \leq \gamma_{n+1}} |\zeta(\frac{1}{2} + it)| \leq \gamma_{n+1} - \gamma_n, \gamma_{n+1} - \gamma_n > \kappa / \log T} 1.$$
Therefore from (28) we obtain, as $T \to \infty$,

\begin{equation}
\sum_{\gamma_n \leq T, \max_{\gamma_n \leq t \leq \gamma_{n+1}} \left| \zeta\left(\frac{1}{2} + it\right) \right| \leq \gamma_{n+1} - \gamma_n} \leq \frac{1}{\log T} \log N(T) = o(N(T))
\end{equation}

provided that

\begin{equation}
\frac{\log \log \log T}{\kappa \sqrt{\log \log T}} = o(1) \quad (T \to \infty).
\end{equation}

Therefore if we can show that

\begin{equation}
\sum_{0 < \gamma, \gamma' \leq T, 0 < \gamma - \gamma' \leq 2\pi \alpha / \log(T/2\pi)} 1 = o(N(T)) \quad (\kappa = o(1), T \to \infty)
\end{equation}

for $\kappa$ satisfying (30), then from (29) and (31) we obtain, assuming RH, the conjectural relation (27).

However, (31) follows from what is known as the *essential simplicity hypothesis* of zeta zeros. This says that

\begin{equation}
\sum_{0 < \gamma, \gamma' \leq T, \gamma_{n+1} - \gamma_n \leq \kappa / \log T} 1 = o(N(T))
\end{equation}

for $\alpha = o(1), T \to \infty$ together with the relation

\begin{equation}
\sum_{0 < \gamma \leq T} m(\gamma) = (1 + o(1))N(T) \quad (T \to \infty),
\end{equation}

where $\gamma$ and $\gamma'$ denote ordinates of zeta zeros, and $m(\gamma)$ denotes the multiplicity of the zeta zero $\frac{1}{2} + i\gamma$ (assuming RH), which is already counted in the above sum with its multiplicity. Thus (32) says that pairs of different zeros with small gaps are rare, while (33) asserts that almost all zeros are simple. In particular, the essential simplicity hypothesis implies not only (27) but the stronger $A(T) = o(N_0(T))$ as well (under the RH). A discussion of the essential simplicity hypothesis is given by J. Mueller [12]. It is shown there that this hypothesis is, under the RH, equivalent to two other hypotheses involving certain integrals. The relation (31) follows as the limiting case of the Gaussian Unitary Ensemble hypothesis, and it follows also from the limiting case of Montgomery’s pair correlation conjecture that for fixed $\alpha > 0$ and $T \to \infty$

\[\sum_{0 < \gamma, \gamma' \leq T, 0 < \gamma - \gamma' \leq 2\pi \alpha / \log(T/2\pi)} 1 = \left\{ \int_0^\alpha \left(1 - \left(\frac{\sin \pi t}{\pi t}\right)^2\right) dt + o(1) \right\} N(T).\]
Small values of $|\zeta(\frac{1}{2} + it)|$

Thus proving $A(T) = o(N_0(T))$ (or the weaker (27)) assuming only the RH seems to be difficult, while an unconditional proof is certainly out of reach at present.

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