Probabilistic representations of solutions of elliptic boundary value problem and non-symmetric semigroups

Chuan-Zhong Chen
School of Mathematics and Statistics
Hainan Normal University
Haikou 571158, China
czchen@hainnu.edu.cn

Wei Sun
Department of Mathematics and Statistics
Concordia University
Montreal H3G 1M8, Canada
wei.sun@concordia.ca

Jing Zhang
Department of Mathematics and Statistics
Concordia University
Montreal H3G 1M8, Canada
waangel520@gmail.com

Abstract

In this paper, we use a probabilistic approach to show that there exists a unique, bounded continuous solution to the Dirichlet boundary value problem for a general class of second order non-symmetric elliptic operators $L$ with singular coefficients, which does not necessarily have the maximum principle. The theory of Dirichlet forms and heat kernel estimates play a crucial role in our approach. A probabilistic representation of the non-symmetric semigroup $\{T_t\}_{t \geq 0}$ generated by $L$ is also given.

Keywords: Dirichlet boundary value problem, singular coefficient, non-symmetric semigroup, probabilistic representation, Dirichlet form, heat kernel estimate.
1 Introduction and the Main Theorem

In this paper, we will use probabilistic methods to study the Dirichlet boundary value problem for second order elliptic differential operators:

\[
\begin{aligned}
Lu &= 0 \quad \text{in } D, \\
u &= f \quad \text{on } \partial D,
\end{aligned}
\]

where \( D \) is a bounded connected open subset of \( \mathbb{R}^d \). The operator \( L \) is given by

\[
Lu = \frac{1}{2} \sum_{i,j=1}^d \partial_{x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + (c(x) - \text{div} \hat{b}(x))u,
\]

where \( A(x) = (a_{ij}(x))_{i,j=1}^d \) is a Borel measurable, (not necessarily symmetric) matrix-valued function on \( D \) satisfying

\[
\lambda |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \quad \text{for any } \xi = (\xi_i)_{i=1}^d \in \mathbb{R}^d, x \in D
\]

and

\[
|a_{ij}(x)| \leq \frac{1}{\lambda} \quad \text{for any } x \in D, 1 \leq i, j \leq d
\]

for some constant \( 0 < \lambda \leq 1 \); \( b = (b_1, \ldots, b_d)^* \) and \( \hat{b} = (\hat{b}_1, \ldots, \hat{b}_d)^* \) are Borel measurable \( \mathbb{R}^d \)-valued functions on \( D \) and \( c \) is a Borel measurable function on \( D \) satisfying \( |b|^2 \in L^{p_1}(D; dx) \), \( |\hat{b}|^2 \in L^{p_1}(D; dx) \) and \( c \in L^{p_1}(D; dx) \) for some constant \( p > d/2 \). Hereafter we use \( ^* \) to denote the transpose of a vector or matrix, and use \( | \cdot | \) and \( \langle \cdot, \cdot \rangle \) to denote respectively the standard norm and inner product of the Euclidean space \( \mathbb{R}^d \).

In (1.1), \( Lu = 0 \) in \( D \) is understood in the distributional sense:

\[
u \in H^{1,2}(D) \text{ and } \mathcal{E}(u, \phi) = 0 \text{ for every } \phi \in C_0^\infty(D),
\]

where \( H^{1,2}(D) \) is the Sobolev space on \( D \) with norm

\[
\|f\|_{H^{1,2}} := \left( \int_D |\nabla f(x)|^2 dx + \int_D |f(x)|^2 dx \right)^{1/2},
\]

\( C_0^\infty(D) \) is the space of infinitely differentiable functions with compact support in \( D \), and \( (\mathcal{E}, D(\mathcal{E})) \) is the bilinear form associated with \( L \):

\[
\mathcal{E}(u, v) = \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^d \int_D b_i(x) \frac{\partial u}{\partial x_i} v(x) dx
\]

\[
- \sum_{i=1}^d \int_D \hat{b}_i(x) \frac{\partial (uv)}{\partial x_i} dx - \int_D c(x)uv(x) dx,
\]

\[
D(\mathcal{E}) = H^{1,2}_0(D)
\]
with $H^{1,2}_0(D)$ being the completion of $C_0^\infty(D)$ with respect to the Sobolev-norm $\| \cdot \|_{H^{1,2}}$. By setting $a = I$, $b = 0$, $\hat{b} = 0$ and $c = 0$ off $D$, we may assume that the operator $L$ is defined on $\mathbb{R}^d$.

Using probabilistic approaches to solve boundary value problems has a long history. The pioneering work goes back to Kakutani [8], who used Brownian motion to represent the solution of the classical Dirichlet boundary value problem with operator $L = \Delta$, the Laplacian operator. If $\hat{b} = 0$ and $c \leq 0$, then the solution $u$ to problem (1.1) is given by the famous Feynman-Kac formula

$$u(x) = E_x \left[ e^{\int_0^{\tau_D} b(X_s) \, ds} f(X_{\tau_D}) \right], \quad x \in D,$$

where $X = (X_t)_{t \geq 0}$ is the diffusion process associated with the generator $L^b$ given by

$$L^b u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i},$$

and $\tau_D$ is the first exit time of $X$ from $D$. We refer the readers to [5] for the general results obtained in this case.

When $\hat{b} \neq 0$ and $A$ is symmetric, Chen and Zhang [3] used the time reversal of symmetric Markov processes to give an explicit probabilistic representation of the solution to problem (1.1). (Note that the operator $L$ given by (1.2) is the same as that used in [3] if we replace $b$ with $b - \hat{b}$ in (1.2).) We should point out that the $\text{div} \hat{b}$ in (1.2) is just a formal writing since the vector field $\hat{b}$ is merely measurable hence its divergence exists only in the distributional sense. In the remarkable paper [3], Chen and Zhang proved that there exists a unique, bounded continuous weak solution to problem (1.1) without the Markov assumption

$$c - \text{div} \hat{b} \leq 0 \quad \text{in} \quad \mathbb{R}^d,$$

i.e., $\int_{\mathbb{R}^d} c(x) \phi(x) \, dx + \sum_{i=1}^d \int_{\mathbb{R}^d} \hat{b}_i(x) \frac{\partial \phi}{\partial x_i} \, dx \leq 0$ for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^d)$. In [3], Chen and Zhang used essentially the following result due to Meyers [15]:

For every $x_0 \in \mathbb{R}^d$, $R > 0$ and $p > d$, there is a constant $\varepsilon \in (0, 1)$, depending only on $d$, $R$ and $p$, such that if

$$(1 - \varepsilon)I_{n \times n} \leq A(x) \leq I_{n \times n} \quad \text{for a.e.} \ x \in B_R := B(x_0, R),$$

then $\frac{1}{2} \nabla (A \nabla u) = \text{div} f$ in $B_R$ has a unique weak solution in $H_0^{1,p}(B_R)$ for every $f = (f_1, \ldots, f_d) \in L^p(B_R; dx)$. Moreover, there is a constant $c > 0$ independent of $f$ such that

$$\| \nabla u \|_{L^p(B_R; dx)} \leq c \| f \|_{L^p(B_R; dx)}.$$

To apply this result, the diffusion matrix $A$ is assumed to satisfy Condition (1.8) in [3] (see [3] Theorems 3.3 and 4.5)].
In this paper, we will show that there exists a unique, bounded continuous solution to problem (1.1) without assuming Condition (1.8), (1.7) and the symmetry of $A$. Instead of using the above Meyers’s $L^p$-estimate as in [3], we will use the nice two-sided estimates on Dirichlet heat kernels obtained recently by Cho, Kim and Park [6]. To apply [6, Theorem 1.1], we assume from now on that $D$ is a bounded $C^{1,\alpha}$-domain satisfying the connected line condition, where $0 < \alpha \leq 1$ is a fixed constant. (We refer the readers to [6] for the related definitions.) Moreover, we assume that each $a_{ij}$, $1 \leq i, j \leq d$, is Dini continuous which means that there exists a non-decreasing function $\psi : [0, \infty) \to [0, \infty)$ such that for any $x, y \in \mathbb{R}^d$,

$$
\sum_{i,j=1}^{d} |a_{ij}(x) - a_{ij}(y)| \leq \psi(|x - y|) \quad \text{and} \quad \int_0^\infty \frac{\psi(t)}{t} dt < \infty.
$$

In the sequel, we let $X = ((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$ be the Markov process associated with the following (non-symmetric) Dirichlet form

$$
\mathcal{E}^0(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx,
$$

$$
D(\mathcal{E}^0) = H^{1,2}(\mathbb{R}^d).
$$

It is well-known that $X$ is a conservative Feller process on $\mathbb{R}^d$ that has continuous transition density function which admits a two-sided Aronson’s heat kernel estimate. Let $\{\mathcal{F}_t, t \geq 0\}$ be the minimal augmented filtration generated by $X$. By Fukushima’s decomposition (cf. [7, Chapter 5] and [11, Theorem VI.2.5]), we have

$$
X_t = x + M_t + N_t,
$$

where $M_t = (M^1_t, \ldots, M^d_t)^*$ is a martingale additive functional of $X$ with quadratic co-variation

$$
\langle M^1, M^2 \rangle_t = \int_0^t \tilde{a}_{ij}(X_s) ds
$$

and $N_t = (N^1_t, \ldots, N^d_t)^*$ is a continuous additive functional of $X$ locally of zero quadratic variation. Hereafter $\tilde{A} = (\tilde{a}_{ij})_{i,j=1}^{d}$ denotes the symmetrization of $A$, i.e., $\tilde{A} := 1/2(A + A^*)$.

For any vector field $\xi \in L^2(\mathbb{R}^d; dx)$, there exists a unique function $\xi^H \in H^{1,2}(\mathbb{R}^d)$ such that

$$
\int_{\mathbb{R}^d} \langle \xi, \nabla h \rangle dx = -\mathcal{E}_1^0(\xi^H, h), \quad \forall h \in C_0^\infty(\mathbb{R}^d)
$$

(see Lemma 2.2 below). Hereafter $\mathcal{E}_1^0(f, g) := \mathcal{E}^0(f, g) + \gamma \int fg dx$ for any $f, g \in D(\mathcal{E}^0)$ and any constant $\gamma$. We have Fukushima’s decomposition:

$$
\tilde{\xi}^H(X_t) - \tilde{\xi}^H(X_0) = M^\xi_t + N^\xi_t,
$$

(1.10)

where $\tilde{\xi}^H$ is a quasi-continuous version of $\xi^H$. To simplify notation, in the sequel we take $f$ to be its quasi-continuous version $\tilde{f}$ whenever such a version exists.

Now we can state the main theorem of this paper.
Theorem 1.1. Let $d \geq 1$, $D$ be a bounded $C^{1,\alpha}$-domain ($0 < \alpha \leq 1$) satisfying the connected line condition, and $p > d/2$. Suppose that

(i) $A$ satisfies $(1.3)$, $(1.4)$ and each $a_{ij}$, $1 \leq i, j \leq d$, is Dini continuous.

(ii) $|b|^2 \in L^{p^1}(D; dx)$ and $|\hat{b}|^2 \in L^{p^1}(D; dx)$.

(iii) $c \in L^{p^1}(D; dx)$ and $c - \text{div} \hat{b} \leq g$ for some nonnegative function $g \in L^{p^1}(D; dx)$ in the distributional sense.

Then, there exists a constant $M > 0$ depending on $d$, $\lambda$, $\psi$, $D$ and $\|b\|_{L^{p^1}}$ such that whenever $\|g\|_{L^{p^1}} \leq M$, for any $f \in C(\partial D)$, there exists a unique weak solution $u$ to $Lu = 0$ in $D$ that is continuous on $\overline{D}$ with $u = f$ on $\partial D$. Moreover, the solution $u$ admits the following representation:

$$u(x) = E_x \left[ \exp \left( \int_0^T (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2} \int_0^T b^*\tilde{a}^{-1}b(X_s)ds \right) + \int_0^T c(X_s)ds + N_{\tau_D}^{kH} - \int_0^{\tau_D} \hat{b}^H(X_s)ds \right] f(X_{\tau_D}). \quad (1.11)$$

We will give the proof of Theorem 1.1 in Section 2, which consists of three subsections. In Subsection 2.1, we prove the existence of the weak solution and gives its probabilistic representation (1.11). In Subsection 2.2, we prove the continuity of the weak solution. In Subsection 2.3, we prove the uniqueness of the continuous weak solutions. The recently developed Nakao integral for non-symmetric Dirichlet forms (cf. [22] and [2]) will be used in the proof of the uniqueness.

In Section 3, we use some techniques of Section 2 to give a probabilistic representation of the non-symmetric semigroup $\{T_t\}_{t \geq 0}$ generated by $L$ that is defined by (1.2). The obtained result (see Theorem 3.1 below) generalizes the corresponding result of [13] from the case of symmetric diffusion matrix $A$ to the non-symmetric case.

2 Proof of Theorem 1.1

2.1 Proof of the existence of weak solution

We first generalize [5] Theorem 1.1] from the case of symmetric diffusion matrix $A$ to the non-symmetric case. Define

$$L^1 u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + c(x)u.$$ 

Lemma 2.1. Suppose that $D$ is a bounded domain in $\mathbb{R}^d$, $c \leq 0$ and $f \in C(\partial D)$. Then

$$u(x) = E_x \left[ \exp \left( \int_0^T (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2} \int_0^T b^*\tilde{a}^{-1}b(X_s)ds \right) + \int_0^T c(X_s)ds + N_{\tau_D}^{kH} - \int_0^{\tau_D} \hat{b}^H(X_s)ds \right] f(X_{\tau_D}).$$
\[ \int_0^{\tau_D} c(X_s) ds \right) f(X_{\tau_D}) \]

is the unique weak solution of \( L^1 u = 0 \) which is continuous in \( D \) and

\[ \lim_{x \to y, x, y \in D} u(x) = f(y) \]

for \( y \in \partial D \) which is regular for the Laplace operator \((\frac{1}{2} \Delta, D)\).

**Proof.** The proof of Lemma 2.1 is similar to that of [5, Theorem 1.1]. We only point out below the main differences in the argument between the symmetric and the non-symmetric cases.

Denote by \( X_0 \) the part of the process \( X \) on \( D \), that is, \( X_0 \) is obtained by killing the sample paths of \( X \) upon leaving \( D \). It is known that (cf. [1, Theorem 4.4]) the transition density function \( p_0(t, x, y) \) of \( X_0 \) has the upper bound estimate

\[ p_0(t, x, y) \leq \vartheta \frac{\eta^2}{t} e^{-\vartheta |y-x|^2} \eta t, \quad \forall t > 0, \]

for some constants \( \vartheta > 0 \) and \( \eta \in \mathbb{R} \) (2.1) can be used to substitute [5, (2.5)] for the non-symmetric case.

We define

\[ L^0 u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right). \]

Let \( D_1 \) be a bounded subdomain of \( D \) and \( f_1 \in H^{1/2}_0(D) \). By [21], there exists a unique weak solution of \( L^0 u = 0 \) in \( D_1 \) such that \( u - f_1\mid_{D_1} \in H^{1/2}_0(D_1) \). Further, by the famous theorem of Littman, Stampacchia and Weinberger, which holds also for the non-symmetric case (cf. e.g. [13]), we can prove the analog of [5, Theorem 2.1] with the non-symmetric \( A \). By virtue of the Harnack inequality for parabolic equations (cf. [18] and [12]), we can prove that [5, Lemma 2.2] and hence [5, Corollary 2.3 and Theorem 2.4] hold for the non-symmetric case.

Finally, we would like to point out that the exponential martingale \( M_t \) introduced in [5, (3.4)] needs to be replaced with

\[ U_t := \exp \left( \int_0^t (\tilde{a}^{-1} b)(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds \right), \quad t \geq 0 \]

for our non-symmetric case. \( \square \)

**Lemma 2.2.** (i) For any vector field \( \xi \in L^2(\mathbb{R}^d; dx) \), there exists a unique function \( \xi^H \in L^{1,2}(\mathbb{R}^d) \) such that

\[ \int_{\mathbb{R}^d} \langle \xi, \nabla h \rangle dx = -\mathcal{E}_1^0 (\xi^H, h), \quad \forall h \in H^{1,2}(\mathbb{R}^d). \]
(i) If \( \xi_n \) converges to \( \xi \) in \( L^2(\mathbb{R}^d; dx) \) as \( n \to \infty \), then \( \xi_n^H \) converges to \( \xi^H \) in \( H^{1,2}(\mathbb{R}^d) \) as \( n \to \infty \).

(iii) For \( \xi \in C^\infty_0(\mathbb{R}^d) \),

\[
- \int_0^t \text{div} \xi(X_s) ds = N_t^{\xi^H} - \int_0^t \xi^H(X_s) ds, \quad t \geq 0.
\]

Proof. (i) Let \( \xi \in L^2(\mathbb{R}^d; dx) \). We define the map \( \eta : h \in H^{1,2}(\mathbb{R}^d) \mapsto \int_{\mathbb{R}^d} \langle \xi, \nabla h \rangle dx \). By the Riesz representation theorem, there exists a unique \( \xi^0 \in H^{1,2}(\mathbb{R}^d) \) such that

\[
\eta(h) = \tilde{\mathcal{E}}^0_1(\xi^0, h), \quad \forall h \in H^{1,2}(\mathbb{R}^d),
\]

where \( (\mathcal{E}^0, D(\mathcal{E}^0)) \) denotes the symmetric part of the Dirichlet form \( (\mathcal{E}^0, D(\mathcal{E}^0)) \). Thus, by [2, Lemma 2.1], there exists a unique \( \xi^H \in D(\mathcal{E}^0) = H^{1,2}(\mathbb{R}^d) \) such that

\[
\tilde{\mathcal{E}}^0_1(\xi^0, h) = -\mathcal{E}^0_1(\xi^H, h), \quad \forall h \in H^{1,2}(\mathbb{R}^d).
\]

(ii) Suppose \( \xi_n \) converges to \( \xi \) in \( L^2(\mathbb{R}^d; dx) \) as \( n \to \infty \). By (2.5), we get

\[
\|\xi_n^0 - \xi^0\|_{\tilde{\mathcal{E}}^0_1} = \sup_{\|h\|_{\tilde{\mathcal{E}}^0_1} = 1} \tilde{\mathcal{E}}^0_1(\xi_n^0 - \xi^0, h)
\]

\[
= \sup_{\|h\|_{\tilde{\mathcal{E}}^0_1} = 1} \int_{\mathbb{R}^d} \langle \xi_n - \xi, \nabla h \rangle dx
\]

\[
\leq \|\xi_n - \xi\|_{L^2} \sup_{\|h\|_{\tilde{\mathcal{E}}^0_1} = 1} \|h\|_{H^{1,2}}
\]

\[
\to 0 \text{ as } n \to \infty.
\]

Further, by (2.6), we get

\[
\mathcal{E}^0_1(\xi_n^H - \xi^H, \xi_n^H) = \mathcal{E}^0_1(\xi_n^H, \xi_n^H) - \mathcal{E}^0_1(\xi^H, \xi_n^H)
\]

\[
= -\mathcal{E}^0_1(\xi_n^H, \xi_n^H) + \mathcal{E}^0_1(\xi^0, \xi_n^H)
\]

\[
= \tilde{\mathcal{E}}^0_1(\xi^0 - \xi_n^0, \xi_n^H)
\]

\[
\leq \left[ \tilde{\mathcal{E}}^0_1(\xi^0 - \xi_n^0, \xi^0 - \xi_n^0) \right]^{1/2} \left[ \tilde{\mathcal{E}}^0_1(\xi_n^H, \xi_n^H) \right]^{1/2},
\]

\[
\sup_{n \in \mathbb{N}} \mathcal{E}^0_1(\xi_n^H, \xi_n^H) \leq \sup_{n \in \mathbb{N}} \mathcal{E}^0_1(\xi_n^0, \xi_n^0) < \infty,
\]

and

\[
\lim_{n \to \infty} \mathcal{E}^0_1(\xi_n^H - \xi^H, \xi_n^H) = -\lim_{n \to \infty} \tilde{\mathcal{E}}^0_1(\xi_n^0 - \xi^0, \xi_n^H)
\]

\[
= -\lim_{n \to \infty} \int_{\mathbb{R}^d} \langle \xi_n - \xi, \nabla \xi^H \rangle dx
\]

\[
= 0.
\]
Therefore, we obtain by (2.7)-(2.10) that
\[
\lim_{n \to \infty} E^0_1(\xi_n^H - \xi^H, \xi_n^H - \xi^H) = \lim_{n \to \infty} \{E^0_1(\xi_n^H - \xi^H, \xi_n^H) - E^0_1(\xi^H - \xi^H, \xi^H)\} = 0.
\]

(iii) Let \( \xi \in C_0^\infty(\mathbb{R}^d) \). For any \( h \in H^{1,2}(\mathbb{R}^d) \), we have
\[
\lim_{t \to 0} \frac{1}{t} E_{h \cdot dx} \left[ - \int_0^t \text{div} \xi (X_s) ds \right] = - \int_{\mathbb{R}^d} (\text{div} \xi) h dx
\]
\[
= \int_{\mathbb{R}^d} \langle \xi, \nabla h \rangle dx
\]
\[
= - E^0_1(\xi^H, h)
\]
\[
= \lim_{t \to 0} \frac{1}{t} E_{h \cdot dx} \left[ N_t^{\xi^H} - \int_0^t \xi^H(X_s) ds \right].
\]

Therefore, (2.4) holds by [2, Lemma 2.3].

\[\square\]

**Proof of the existence of weak solution and its probabilistic representation.**

We define a family of measures \( \{Q_x, x \in \mathbb{R}^d\} \) on \( \mathcal{F}_\infty \) by
\[
\frac{dQ_x}{dP_x} \bigg|_{\mathcal{F}_t} = U_t, \ t \geq 0,
\]
where \( U_t \) is given by (2.2). Then, under \( \{Q_x, x \in \mathbb{R}^d\} \), \( X \) is a diffusion process on \( \mathbb{R}^d \) with the generator \( L^b \) given by (1.6). Denote by \( E^Q_x \) the expectation with respect to the measure \( Q_x \) for \( x \in \mathbb{R}^d \). **From now on till the end of this section, we fix a constant** \( 0 < \theta < \frac{1}{2} \). **We will show below that there exists a constant** \( M > 0 \) depending on \( d, \lambda, \psi, D \) and \( \|b\|^2_{L^p} \) such that for any \( w \in L^p_{\text{loc}}(\mathbb{R}^d; dx) \) with \( \|w\|_{L^p_{\text{loc}}} \leq M \), we have
\[
\sup_{x \in D} E^Q_x \left[ \int_0^{T_D} |w|(X_s) ds \right] \leq \theta. \tag{2.11}
\]

Since \( |b|^2 \in L^p_{\text{loc}}(D; dx) \) with \( p > d/2 \), \( b \) belongs to the parabolic Kato class \( K_d \) (see [6, Definition 3.1] and cf. also [11, page 640]). By [6, Theorem 4.8], there exists a constant \( c > 0 \) depending on \( d, \lambda, \psi, D \) and \( \|b\|^2_{L^p} \) such that
\[
G_D(x, y) \leq c \begin{cases} 
\frac{1}{|x-y|^{d-2}} \left( 1 \wedge \frac{\rho(x)\rho(y)}{|x-y|} \right), & \text{when } d \geq 3, \\
\log \left( 1 + \frac{\rho(x)\rho(y)}{|x-y|} \right), & \text{when } d = 2, \\
(\rho(x)\rho(y))^{1/2} \wedge \frac{\rho(x)\rho(y)}{|x-y|}, & \text{when } d = 1,
\end{cases} \tag{2.12}
\]
where $G_D(x, y)$ is the Green function of $X$ under \{\(Q_x, x \in D\)\} and $\rho(x) := \text{dist}(x, \partial D)$. We only prove (2.11) when $d \geq 3$. The cases that $d = 1, 2$ can be considered similarly.

Suppose $d \geq 3$. Let $q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then $d - q(d - 2) > 0$. We obtain by (2.12) that

$$E^Q_x \left[ \int_0^{\tau_D} |w|(X_s) ds \right] = \int_D G_D(x, y) |w|(y) dy \leq \int_D \frac{c|w|(y)}{|x - y|^{d-2}} dy \leq c \left( \int_D |w|(y)^p dy \right)^{1/p} \left( \int_D |x - y|^{-q(d-2)} dy \right)^{1/q} \leq c\|w\|_{L^p} \left( \int_0^\varsigma r^{d-q(d-2)-1} dr \right)^{1/q} = \frac{c\varsigma^{d/q-(d-2)}}{[d - q(d - 2)]^{1/q}} \|w\|_{L^p}.$$  

Hereafter $\varsigma$ denotes the diameter of $D$. Set

$$M := \frac{\theta[d - q(d - 2)]^{1/q}}{c\varsigma^{d/q-(d-2)}}.$$

Then $\|w\|_{L^p} \leq M$ implies (2.11). Further, by (2.11) and Khasminskii’s inequality, we get

$$\sup_{x \in D} E^Q_x \left[ \exp \left( \int_0^{\tau_D} |w|(X_s) ds \right) \right] \leq \frac{1}{1 - \theta}.$$  

(2.13)

We define

$$J(x) = \frac{1_{\{|x|<1\}} e^{-\frac{1}{1-|x|^2}}}{\int_{\{|y|<1\}} e^{-\frac{1}{1-|y|^2}} dy}, \quad x \in \mathbb{R}^d.$$  

For $k \in \mathbb{N}$ and $x \in \mathbb{R}^d$, set

$$J_k(x) := k^d J(kx),$$

$$\hat{b}_k(x) := \int_{\mathbb{R}^d} \hat{b}(x - y) J_k(y) dy,$$

$$c_k(x) := \int_{\mathbb{R}^d} c(x - y) J_k(y) dy,$$

$$g_k(x) := \int_{\mathbb{R}^d} g(x - y) J_k(y) dy.$$  

Suppose $\|g\|_{L^{p+1}} \leq M$. Since $c - \text{div} \hat{b} \leq g$ implies that $c_k - \text{div} \hat{b}_k \leq g_k$ for $k \in \mathbb{N}$, we obtain by (2.13) that

$$\sup_{k \in \mathbb{N}} \sup_{x \in D} E^Q_x \left[ \exp \left( \int_0^{\tau_D} (c_k - \text{div} \hat{b}_k)(X_s) ds \right) \right] \leq \frac{1}{1 - \theta}.$$  

(2.14)
Define for \( t \geq 0 \),
\[
Z_t := \exp \left( \int_0^t (\tilde{a}^{-1}b)(X_s)dM_s - \frac{1}{2} \int_0^t b^*\tilde{a}^{-1}b(X_s)ds \right) + \int_0^t c(X_s)ds + N_t^{\hat{b}H} - \int_0^t \hat{b}H(X_s)ds .
\] (2.15)

Then, we obtain by Lemma 2.2, the analog of [7, Corollary 5.2.1(ii)] in the setting of non-symmetric Dirichlet forms, Fatou’s lemma and (2.14) that for any \( x \in D \),
\[
E_x[Z_{\tau_D}] = E^Q_x \left[ \exp \left( \int_0^{\tau_D} c(X_s)ds + N_{\tau_D}^{\hat{b}H} - \int_0^{\tau_D} \hat{b}H(X_s)ds \right) \right] 
\leq \liminf_{k \to \infty} E^Q_x \left[ \exp \left( \int_0^{\tau_D} c_k(X_s)ds + N_{\tau_D}^{\hat{b}_kH} - \int_0^{\tau_D} \hat{b}_kH(X_s)ds \right) \right] 
= \liminf_{k \to \infty} E^Q_x \left[ \exp \left( \int_0^{\tau_D} (c_k - \text{div} \hat{b}_k)(X_s)ds \right) \right] 
\leq \frac{1}{1 - \theta} .
\] (2.16)

For \( k \in \mathbb{N} \), we define
\[
L_ku = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + (c_k(x) - \text{div} \hat{b}_k(x))u .
\]

The bilinear form \((\mathcal{E}_k, D(\mathcal{E}_k))\) associated with \( L_k \) is
\[
\mathcal{E}_k(u, v) = \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^d \int_D b_i(x) \frac{\partial u}{\partial x_i} v(x)dx 
- \sum_{i=1}^d \int_D \hat{b}_{k,i}(x) \frac{\partial (uv)}{\partial x_i} dx - \int_D c_k(x)u(x)v(x)dx ,
\]
\[
D(\mathcal{E}_k) = H^{1,2}_0(D) .
\]

By (2.14), following the argument of [3, Theorem 4.3, pages 1030-1031], we can show that the weak solution to the Dirichlet boundary value problem
\[
\begin{cases}
L_ku = 0 & \text{in } D \\
u = f & \text{on } \partial D 
\end{cases}
\] (2.17)
is given by
\[
u_k(x) = E^Q_x \left[ \exp \left( \int_0^{\tau_D} (c_k - \text{div} \hat{b}_k)(X_s)ds \right) f(X_{\tau_D}) \right] 
= E_x \left[ \exp \left( \int_0^{\tau_D} (\tilde{a}^{-1}b)(X_s)dM_s - \frac{1}{2} \int_0^{\tau_D} b^*\tilde{a}^{-1}b(X_s)ds \right) 
+ \int_0^{\tau_D} (c_k - \text{div} \hat{b}_k)(X_s)ds \right) f(X_{\tau_D}) .
\]
Denote by $v$ the right-hand side of (1.11). We claim that
\[
\lim_{k \to \infty} u_k(x) = v(x), \quad \forall x \in D.
\] (2.18)

In fact, define
\[
W_k := \exp \left( \int_0^{\tau_D} (c_k - \text{div} \hat{b}_k)(X_s)ds \right), \quad k \in \mathbb{N},
\]
\[
W := \exp \left( \int_0^{\tau_D} c(X_s)ds + g_k(X_s)ds \right),
\]
where $g_k(X_s) = N_{\tau_D}^{\hat{b}_k} - \int_0^{\tau_D} \hat{b}_k(X_s)ds$.

By (2.11) and Khasminskii’s inequality, we obtain that for $x \in D$,
\[
\sup_{k \in \mathbb{N}} E^Q_x [W^2_k] = \sup_{k \in \mathbb{N}} E^Q_x \left[ \exp \left( 2 \int_0^{\tau_D} (c_k - \text{div} \hat{b}_k)(X_s)ds \right) \right] \leq \sup_{k \in \mathbb{N}} E^Q_x \left[ \exp \left( 2 \int_0^{\tau_D} g_k(X_s)ds \right) \right] \leq \frac{1}{1 - 2 \theta}.
\] (2.19)

Hence $\{W_k\}$ is uniformly integrable under $Q_x$ for $x \in D$. Therefore, (2.18) holds since $W_k \to W$ in probability as $k \to \infty$.

Finally, we show that $v$ is a weak solution to problem (1.1). By (2.19), we get
\[
\sup_{k \in \mathbb{N}} \|u_k\|_{L^2}^2 = \sup_{k \in \mathbb{N}} \int_D \left( E^Q_x [W_k f(X_{\tau_D})] \right)^2 dx \leq \frac{\|f\|_{L^\infty}^2 |D|}{1 - 2 \theta},
\] (2.20)

where $|D|$ is the Lebesgue measure of $D$. Since $u_k$ is the weak solution to problem (2.17), we have $E_k(u_k, \phi) = 0$ for any $\phi \in C^\infty_0(D)$. Then, $E_k(u_k, \phi) = 0$, $\forall \phi \in H^1(D)$. Thus, we have $E_k(u_k, u_k - u_1) = 0$, which implies that
\[
E_k(u_k, u_k) = E_k(u_k, u_1).
\] (2.21)

Note that $|b|^2$, $|\hat{b}|^2$ and $c$ are in the Kato class. For any $0 < \varepsilon < 1$, there exists a constant $A(\varepsilon) > 1$ such that for $1 \leq i \leq d$ and $\eta \in H^{1,2}(\mathbb{R}^d)$ (cf. [9]),
\[
\int_{\mathbb{R}^d} (b_i^2 + \hat{b}_i^2 + |c|)\eta^2 dx \leq \varepsilon \int_{\mathbb{R}^d} |\nabla \eta|^2 dx + A(\varepsilon) \int_{\mathbb{R}^d} \eta^2 dx.
\] (2.22)

By (2.22), we obtain that for $k \in \mathbb{N}, 1 \leq i \leq d$ and $\eta \in H^{1,2}(\mathbb{R}^d)$,
\[
\int_{\mathbb{R}^d} ((\hat{b}_{k,i})^2 + |c_k|)\eta^2 dx \leq \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} [\hat{b}_i^2(x-y) + |c|(x-y)]J_k(y)dy \right\} \eta^2(x) dx \leq \varepsilon \int_{\mathbb{R}^d} |\nabla \eta|^2 dx + A(\varepsilon) \int_{\mathbb{R}^d} \eta^2 dx.
\] (2.23)
Then, we obtain by (2.21)-(2.23) that for \( k \in \mathbb{N} \),

\[
\frac{\lambda}{2} \| \nabla u_k \|_{L^2}^2 \leq \frac{1}{2} \sum_{i,j=1}^{d} \int_D a_{ij}(x) \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} dx
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{d} \int_D a_{ij}(x) \frac{\partial u_k}{\partial x_i} \frac{\partial u_1}{\partial x_j} dx - \sum_{i=1}^{d} \int_D b_i(x) \frac{\partial u_k}{\partial x_i} u_1(x) dx
\]

\[- \sum_{i=1}^{d} \int_D \hat{b}_{k,i}(x) \frac{\partial u_k}{\partial x_i} u_1(x) dx - \sum_{i=1}^{d} \int_D \hat{b}_{k,i}(x) u_k(x) \frac{\partial u_1}{\partial x_i} dx
\]

\[- \int_D c_k(x) u_k(x) u_1(x) dx + \sum_{i=1}^{d} \int_D b_i(x) \frac{\partial u_k}{\partial x_i} u_k(x) dx
\]

\[+ 2 \sum_{i=1}^{d} \int_D \hat{b}_{k,i}(x) \frac{\partial u_k}{\partial x_i} u_k(x) dx + \int_D c_k(x) u_k^2(x) dx \]

\[\leq \frac{d^2}{2\lambda} \| \nabla u_k \|_{L^2} \| \nabla u_1 \|_{L^2} + 2d A^{1/2}(\varepsilon) \| \nabla u_k \|_{L^2} \| u_1 \|_{H^{1,2}} \]

\[+ d A^{1/2}(\varepsilon) \| \nabla u_1 \|_{L^2} \| u_k \|_{H^{1,2}} + A(\varepsilon) \| u_k \|_{H^{1,2}} \| u_1 \|_{H^{1,2}} \]

\[+ 3d \| \nabla u_k \|_{L^2} \| \nabla u_k \|_{L^2} + A(\varepsilon) \| u_k \|_{L^2}^{2} \]

\[+ (\varepsilon \| \nabla u_k \|_{L^2} + A(\varepsilon) \| u_k \|_{L^2}^{2}) \]  \quad (2.24)

Let \( \varepsilon \) be much smaller than \( \lambda \). Then, we obtain by (2.20) and (2.21) that \( \sup_{k \in \mathbb{N}} \| \nabla u_k \|_{L^2} < \infty \) and thus

\[
\sup_{k \in \mathbb{N}} \| u_k \|_{H^{1,2}} < \infty.
\]

By taking a subsequence if necessary, we may assume that \( u_k \to u_1 \) weakly in \( H^{1,2}(D) \) as \( k \to \infty \) and that its Cesaro mean \( \{ u_k : = \frac{1}{k} \sum_{i=1}^{k} u_i, \; k \geq 1 \} \to v_2 \) in \( H^{1,2}(D) \) as \( k \to \infty \). Clearly \( v_1 = v_2 = v \). Let \( \phi \in C_0^\infty (\bar{D}) \). Note that for \( l \in \mathbb{N} \),

\[
\mathcal{E}_l(u_1, \phi) = \frac{1}{2} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial u_i}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx - \sum_{i=1}^{d} \int_{\mathbb{R}^d} b_i(x) \frac{\partial u_i}{\partial x_i} \phi(x) dx
\]

\[- \sum_{i=1}^{d} \int_{\mathbb{R}^d} \hat{b}_{l,i}(x) \frac{\partial (u_i \phi)}{\partial x_i} dx - \int_{\mathbb{R}^d} c_l(x) u_l(x) \phi(x) dx. \quad (2.25)
\]

By (2.22) and (2.23), we find that (cf. Lemma 2.2(iv))

\[
\lim_{k \to \infty} \frac{1}{2} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial u_k'}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx = \frac{1}{2} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx, \quad (2.26)
\]

\[
\lim_{k \to \infty} \sum_{i=1}^{d} \int_{\mathbb{R}^d} b_i(x) \frac{\partial u_k}{\partial x_i} \phi(x) dx = \sum_{i=1}^{d} \int_{\mathbb{R}^d} b_i(x) \frac{\partial v}{\partial x_i} \phi(x) dx, \quad (2.27)
\]
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{l=1}^{k} \sum_{i=1}^{d} \int_{\mathbb{R}^d} \hat{b}_{l,i}(x) \frac{\partial(u \phi)}{\partial x_i} \, dx = \sum_{i=1}^{d} \int_{\mathbb{R}^d} \hat{b}_i(x) \frac{\partial(v \phi)}{\partial x_i} \, dx, \tag{2.28}
\]
and
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{l=1}^{k} \int_{\mathbb{R}^d} c_l(x) u_l(x) \phi(x) \, dx = \int_{\mathbb{R}^d} c(x) v(x) \phi(x) \, dx. \tag{2.29}
\]
Therefore, we obtain by (1.5) and (2.25)-(2.29) that \( \mathcal{E}(v, \phi) = \lim_{k \to \infty} \frac{1}{k} \sum_{l=1}^{k} \mathcal{E}_l(u_l, \phi) = 0. \)

2.2 Proof of the continuity of weak solution

In this subsection, we will prove that the weak solution \( u \) given by (1.11) is continuous on \( \overline{D} \). It is well-known that any weak solution to the Dirichlet boundary problem (1.1) is locally Hölder continuous in \( D \) (see [16], cf. also [17]). We will show below that \( u \) is continuous at the boundary \( \partial D \). First, we prove an important lemma based on the nice two-sided estimates on Dirichlet heat kernels obtained recently by Cho, Kim and Park [6].

Denote by \( p(t, x, y) \) the transition density function of \( X \) under \( \{Q_x, x \in D\} \). By [6, Theorem 1.1], there exist positive constants \( c_1 \) and \( c_2 \) depending on \( d, \lambda, \psi, D, |||b|||_{L^{p_1}} \) and \( T \) such that for \( x, y \in D \) and \( 0 < t \leq T \),

\[
p(t, x, y) \leq \left( 1 \wedge \frac{\rho(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\rho(y)}{\sqrt{t}} \right) \frac{c_1}{t^{d/2}} e^{-\frac{c_2|x-y|^2}{t}} \tag{2.30}
\]

and

\[
|\nabla_y p(t, x, y)| \leq \frac{c_1}{t^{(d+1)/2}} e^{-\frac{c_2|x-y|^2}{t}}. \tag{2.31}
\]

Suppose \( d \geq 2 \). Let \( p_1 > d \) and \( q_1 > 1 \) satisfying \( \frac{1}{p_1} + \frac{1}{q_1} = 1 \). Then \( q_1 = \frac{p_1}{p_1 - 1} < \frac{d}{d-1} \). We choose \( 0 < \alpha < 1 \) such that

\[
q_1 < \frac{d}{d-\alpha}. \tag{2.32}
\]

Let \( c_3 \) be a constant satisfying

\[
e^{\frac{x}{x}} \geq c_3 |x|^{(d-\alpha)/2}, \quad \forall x \in \mathbb{R}^d. \tag{2.33}
\]

Let \( p_2 > d/2 \) and \( q_2 > 1 \) satisfying \( \frac{1}{p_2} + \frac{1}{q_2} = 1 \). Then \( q_2 = \frac{p_2}{p_2 - 1} < \frac{d}{d-2} \). We choose \( \beta \) satisfying

\[
\frac{d}{2} - 1 < \beta < \frac{d}{2q_2}. \tag{2.34}
\]

Let \( c_4 \) be a constant satisfying

\[
e^{\frac{x}{x}} \geq c_4 |x|^\beta, \quad \forall x \in \mathbb{R}^d. \tag{2.35}
\]
and let $c_5$ be a constant satisfying
\[ e^{\lvert x \rvert} \geq c_5 |x|^{1/4}, \quad \forall x \in \mathbb{R}^d. \]

We denote by $\varsigma$ as above the diameter of $D$. By (2.32) and (2.34), we find that
\[ \int_0^\varsigma r^{d-q_1(d-\alpha)-1}dr < \infty \quad \text{and} \quad \int_0^\varsigma r^{d-2\beta q_2-1}dr < \infty. \]

**Lemma 2.3.** Let $f$ be a vector field on $\mathbb{R}^d$ and $g$ be a function on $\mathbb{R}^d$ such that $f, g \in C^\infty(\mathbb{R}^d)$.

(i) Suppose $d \geq 2$, $p_1 > d$ and $p_2 > d/2$. Then, for $0 < t \leq T$,
\[
\left| \int_{y \in D} p(t, x, y) \text{div} f(y) dy \right| \leq \frac{c_1}{c_2^{(d-\alpha)/2} c_3 t^{(1+\alpha)/2}} \left( \int_0^\varsigma r^{d-q_1(d-\alpha)-1}dr \right)^{1/q_1} \left( \int_{y \in D} |f(y)|^{p_1} dy \right)^{1/p_1}
\]
and
\[
\left| \int_{y \in D} p(t, x, y) g(y) dy \right| \leq \frac{c_1}{c_2^\beta c_4 t^{d/2-\beta}} \left( \int_0^\varsigma r^{d-2\beta q_2-1}dr \right)^{1/q_2} \left( \int_{y \in D} |g(y)|^{p_2} dy \right)^{1/p_2}.
\]

(ii) Suppose $d = 1$. Then, for $0 < t \leq T$,
\[
\left| \int_{y \in D} p(t, x, y) \text{div} f(y) dy \right| \leq \frac{2^{5/8} c_1}{c_2^{1/8} c_3^{1/2} t^{7/8}} \varsigma^{1/4} \left( \int_{y \in D} |f(y)|^2 dy \right)^{1/2}
\]
and
\[
\left| \int_{y \in D} p(t, x, y) g(y) dy \right| \leq \frac{c_1}{t^{1/2}} \int_{y \in D} |g(y)| dy.
\]

**Proof.** We only prove (i). The proof of (ii) is similar so we omit it here.

By (2.31) and (2.33), we get
\[
\left| \int_{y \in D} p(t, x, y) \text{div} f(y) dy \right| = \left| \int_{y \in D} \langle \nabla_y p(t, x, y), f(y) \rangle dy \right| \leq \int_{y \in D} \frac{c_1}{t^{(d+1)/2} e^{c_2 |x-y|^2/t}} |f(y)| dy.
\]
By (2.30) and (2.35), we get

\[ \frac{c_1}{c_2^{(d-\alpha)/2} c_3 t^{(\alpha+1)/2}} \left( \int_{y \in D} \frac{1}{|x-y|^{(d-\alpha)/2}} \right)^{1/q_1} \left( \int_{y \in D} |f(y)|^{p_1} dy \right)^{1/p_1} \]

\[ \leq \frac{c_1}{c_2^{(d-\alpha)/2} c_3 t^{(\alpha+1)/2}} \left( \int_{0}^{\infty} r^{d-q_1(d-\alpha)-1} dr \right)^{1/q_1} \left( \int_{y \in D} |f(y)|^{p_1} dy \right)^{1/p_1}. \]

By (2.30) and (2.35), we get

\[ \left| \int_{y \in D} p(t, x, y) g(y) dy \right| \]

\[ \leq \frac{c_1}{t^{d/2} c_2 |x-y|^2/t^\beta} |g(y)| dy \]

\[ \leq \frac{c_1}{t^{d/2} c_2 |x-y|^2/t^\beta} \left( \int_{y \in D} |g(y)| dy \right) \]

\[ = \frac{c_1}{c_2 c_4 t^{d/2-\beta}} \left( \int_{y \in D} |g(y)| dy \right) \]

\[ \leq \frac{c_1}{c_2 c_4 t^{d/2-\beta}} \left( \int_{y \in D} |x-y|^{2\beta q_2} dy \right)^{1/q_2} \left( \int_{y \in D} |g(y)|^{p_2} dy \right)^{1/p_2} \]

\[ \leq \frac{c_1}{c_2 c_4 t^{d/2-\beta}} \left( \int_{0}^{\infty} r^{d-2\beta q_2-1} dr \right)^{1/q_2} \left( \int_{y \in D} |g(y)|^{p_2} dy \right)^{1/p_2}. \]

Proof of the continuity of weak solution at the boundary.

Note that

\[ u(x) = E_x^Q \left[ \exp \left( \int_{0}^{T_D} c(X_s) ds + \int_{0}^{T_D} \tilde{b} H(X_s) ds \right) f(X_{T_D}) \right] \]

\[ = E_x^Q [f(X_{T_D})] + E_x^Q [f(X_{T_D})(e^{A_{T_D}} - 1)], \]

where \( A_t := \int_{0}^{t} c(X_s) ds + \int_{0}^{t} \tilde{b} H(X_s) ds, t \geq 0. \) By Lemma 2.1 to prove the continuity of \( u \) at \( \partial D \), it suffices to show that

\[ \lim_{x \to y, x \in D} E_x^Q [f(X_{T_D})(e^{A_{T_D}} - 1)] = 0, \quad \forall y \in \partial D. \quad (2.36) \]

For \( t > 0 \), we have

\[ E_x^Q [f(X_{T_D})(e^{A_{T_D}} - 1)] = E_x^Q [f(X_{T_D})(e^{A_{T_D}} - 1); \tau_D \leq t] + E_x^Q [f(X_{T_D})(e^{A_{T_D}} - 1); \tau_D > t]. \]
Therefore, to prove (2.36), it suffices to show that

\[ t > \parallel E^Q_{x} \exp(A_{t_D}) \parallel \]

Then, we obtain by the strong Markov property that for \( x \in D \),

\[
\begin{align*}
& E^Q_x [f(X_{\tau_D})(e^{A_{t_D}} - 1); \tau_D > t] \\
\leq & \|f\|_\infty \{ Q_x(\tau_D > t) + E^Q_x[e^{A_{t_D}}; \tau_D > t] \} \\
\leq & \|f\|_\infty \left\{ Q_x(\tau_D > t) + \frac{E^Q_x[e^{A_{t_D}}; \tau_D > t]}{1 - \theta} \right\}.
\end{align*}
\]

(2.37)

Let \( y \in \partial D \). By Lemma 2.1 following the argument of [5] (2.28), one finds that for every \( t > 0 \),

\[
\lim_{x \to y, x \in D} Q_x(\tau_D > t) = 0.
\]

(2.38)

By Fatou’s lemma and (2.19), we get

\[
\sup_{x \in D} E^Q_x [e^{2A_t}; \tau_D > t] \leq \sup_{x \in D} \sup_{k \in \mathbb{N}} E^Q_x \left[ e^{2 \int_0^{t_D} g_k(X_s) ds} \right] \leq \frac{1}{1 - 2\theta}.
\]

(2.39)

Thus, we obtain by (2.37)-(2.39) that for every \( t > 0 \),

\[
\lim_{x \to y, x \in D} E^Q_x [f(X_{\tau_D})(e^{A_{t_D}} - 1); \tau_D > t] = 0.
\]

Therefore, to prove (2.36), it suffices to show that

\[
\lim_{t \downarrow 0} \sup_{x \in D} E^Q_x [f(X_{\tau_D})(e^{A_{t_D}} - 1); \tau_D \leq t] = 0.
\]

(2.40)

For \( t > 0 \), we obtain by Fatou’s lemma that

\[
\begin{align*}
\sup_{x \in D} & E^Q_x [f(X_{\tau_D})(e^{A_{t_D}} - 1); \tau_D \leq t] \\
\leq & \|f\|_\infty \sup_{x \in D} \liminf_{k \to \infty} E^Q_x \left[ e^{f_0^{t_D}(c_k - \text{div} \hat{b}_k)(X_s) ds} - 1 ; \tau_D \leq t \right] \\
\leq & \|f\|_\infty \left\{ \sup_{x \in D} \limsup_{k \to \infty} E^Q_x \left[ e^{f_0^{t_D} g_k(X_s) ds} - 1; \tau_D \leq t \right] \\
+ & \sup_{x \in D} \limsup_{k \to \infty} E^Q_x \left[ 1 - e^{f_0^{t_D}(c_k - \text{div} \hat{b}_k - g_k)(X_s) ds} ; \tau_D \leq t \right] \right\} \\
\leq & \|f\|_\infty \left\{ \sup_{x \in D} \limsup_{k \to \infty} E^Q_x \left[ e^{f_0^{t_D} g_k(X_s) ds} - 1 \right] \\
+ & \sup_{x \in D} \limsup_{k \to \infty} E^Q_x \left[ 1 - e^{f_0^{t_D}(c_k - \text{div} \hat{b}_k - g_k)(X_s) ds} \right] \right\}.
\end{align*}
\]
By Lemma 2.3 and Khasminskii’s inequality, we get
\[
\lim_{t \downarrow 0} \sup_{x \in D} \sup_{k \in \mathbb{N}} E_x^Q \left[ e^{f_{0 \wedge T_D}^{t \wedge T_D} g_k(X_s) ds} \right] = 1.
\]
Hence, to prove (2.40), we need only show that
\[
\lim_{t \downarrow 0} \inf_{x \in D} \inf_{k \in \mathbb{N}} E_x^Q \left[ e^{f_{0 \wedge T_D}^{t \wedge T_D} (c_k - \text{div} \hat{b}_k)(X_s) ds} \right] \geq 1.
\]
Further, by Jensen’s inequality, we need only show that
\[
\lim_{t \downarrow 0} \sup_{x \in D} \sup_{k \in \mathbb{N}} E_x^Q \left[ \int_{0 \wedge T_D}^{t \wedge T_D} (g_k - c_k + \text{div} \hat{b}_k)(X_s) ds \right] = 0.
\]
By Lemma 2.3 we obtain that
\[
\sup_{x \in D} \sup_{k \in \mathbb{N}} E_x^Q \left[ \int_{0 \wedge T_D}^{t \wedge T_D} (g_k - c_k + \text{div} \hat{b}_k)(X_s) ds \right] \rightarrow 0 \text{ as } t \downarrow 0.
\]

2.3 Proof of the uniqueness of continuous weak solutions

In this subsection, we will prove that there exists a unique continuous weak solution to problem (1.1).

Let \( u_1 \) be a weak solution of problem of (1.1) such that \( u_1 \) is continuous on \( \overline{D} \). We have Fukushima’s decomposition
\[
u_1(X_t) - u_1(X_0) = M_{t}^{u_1} + N_{t}^{u_1} = \int_{0}^{t} \nabla u_1(X_s) dM_s + N_{t}^{u_1}, \quad t < \tau_D. \tag{2.41}
\]
We claim that for \( t < \tau_D \),
\[
N_{t}^{u_1} = - \sum_{i=1}^{d} \int_{0}^{t} b_i(X_s) \frac{\partial u_1}{\partial x_i}(X_s) ds - \int_{0}^{t} u_1(X_s) c(X_s) ds - \int_{0}^{t} u_1(X_s) dN_{t}^{k_H} + \int_{0}^{t} u_1(X_s) \hat{b}^{H}(X_s) ds, \tag{2.42}
\]
where the third term of (2.42) is a Nakao integral (we refer the readers to [2 Definition 2.4] (cf. also [19 Definition 3.1]) for the definition).
Let \( \{D_n\} \) be a sequence of increasing open subsets of \( \mathbb{R}^d \) satisfying \( D = \bigcup_{n \in \mathbb{N}} D_n \) and \( D_n \subset D_{n+1} \) for each \( n \). We choose a sequence \( \{u^{(n)} \subset H^{1,2}_0(D) \cap B_b(D_n)\} \) satisfying \( u_1 = u^{(n)} \) on \( D_n \) for each \( n \). To prove (2.42), it suffices to show that for any \( n \in \mathbb{N} \) and \( t < \tau_{D_n}, \)

\[
N_t^{u(n)} = - \sum_{i=1}^d \int_0^t b_i(X_s) \frac{\partial u(n)}{\partial x_i}(X_s)ds - \int_0^t u(n)(X_s)c(X_s)ds
-
\int_0^t u(n)(X_s)dN^{u}_s + \int_0^t u(n)(X_s)\hat{b}^H(X_s)ds. \tag{2.43}
\]

Denote by \( C_t^{(n)} \) the right hand side of (2.43). Similar to [19, Theorem 2.2] (cf. [7, Lemma 5.4.4]), one finds that to prove (2.43) it suffices to show that for each \( n \),

\[
\lim_{t \downarrow 0} \frac{1}{t} E_{\phi, dx}[N_t^{u(n)}] = \lim_{t \downarrow 0} \frac{1}{t} E_{\phi, dx}[C_t^{(n)}], \quad \forall \phi \in H^{1,2}_0(D_n) \cap B_b(D_n). \tag{2.44}
\]

We fix an \( n \in \mathbb{N} \) and \( \phi \in H^{1,2}_0(D_n) \cap B_b(D_n) \). By (1.5), (1.9) and (2.3), we get

\[
\mathcal{E}^0(u(n), \phi) = \mathcal{E}(u(n), \phi) + \sum_{i=1}^d \int_D b_i(x) \frac{\partial u(n)}{\partial x_i}(x)dx
+
\sum_{i=1}^d \int_D \hat{b}_i(x) \frac{\partial u(n)}{\partial x_i}(x) + \int_D c(x)u(n)(x)\phi(x)dx,
=
\sum_{i=1}^d \int_D b_i(x) \frac{\partial u(n)}{\partial x_i}(x)dx + \int_D c(x)u(n)(x)\phi(x)dx
-
\mathcal{E}_0^0(\hat{b}^H, u(n)\phi). \tag{2.45}
\]

We have

\[
\lim_{t \downarrow 0} \frac{1}{t} E_{\phi, dx}[N_t^{u(n)}] = \lim_{t \downarrow 0} \frac{1}{t} E_{\phi, dx}[u(n)(X_t) - u(n)(X_0) - M_t^{u(n)}]
=
\lim_{t \downarrow 0} \frac{1}{t} \int_D E_x[u(n)(X_t) - u(n)(X_0)]\phi(x)dx
=
- \mathcal{E}^0(u(n), \phi) \tag{2.46}
\]

and

\[
\lim_{t \downarrow 0} \frac{1}{t} E_{\phi, dx} \left[ - \sum_{i=1}^d \int_0^t b_i(X_s) \frac{\partial u(n)}{\partial x_i}(X_s)ds - \int_0^t u(n)(X_s)c(X_s)ds
+
\int_0^t u(n)(X_s)\hat{b}^H(X_s)ds \right]
=
- \sum_{i=1}^d \int_D b_i(x) \frac{\partial u(n)}{\partial x_i}(x)dx - \int_D c(x)u(n)(x)\phi(x)dx
+
\int_D \hat{b}^H(x)u(n)(x)\phi(x)dx. \tag{2.47}
\]
By [2, Remark 2.5], we get
\[
\lim_{t \downarrow 0} \frac{1}{t} E_{\phi \cdot \cdot} \left[ - \int_0^t u^{(n)}(X_s) dN_s^{\hat{H}} \right] = \mathcal{E}^0(\hat{H}, u^{(n)} \phi).
\] (2.48)
Then, (2.44) holds by (2.46)-(2.48). Thus, (2.43) and hence (2.42) hold.

By (2.41) and (2.42), we obtain that for \( t < \tau \),
\[
\begin{align*}
\lim_{t \downarrow 0} \frac{1}{t} E_{\phi \cdot \cdot} \left[ - \int_0^t u^{(n)}(X_s) dN_s^{\hat{H}} \right] &= \mathcal{E}^0(\hat{H}, u^{(n)} \phi).
\end{align*}
\] (2.49)
We now prove that for \( t < \tau \),
\[
d(u_1(X_t)Z_t) = u_1(X_t)Z_t(\tilde{a}^{-1}b)(X_t) dM_t + Z_t \nabla u_1(X_t) dM_t,
\] (2.50)
where \( Z_t \) is defined as in (2.15).

For \( k \in \mathbb{N} \) and \( t > 0 \), we define
\[
V_t^k := \int_0^t \nabla u_1(X_s) dM_s - \sum_{i=1}^d \int_0^t b_i(X_s) \frac{\partial u_1}{\partial x_i}(X_s) ds
\]
\[
- \int_0^t u_1(c_k - \text{div } \hat{b}_k)(X_s) ds
\]
and
\[
Z_t^k := \exp \left( \int_0^t (\tilde{a}^{-1}b)(X_s) dM_s - \frac{1}{2} \int_0^t \hat{b}^* \tilde{a}^{-1}b(X_s) ds \right.
\]
\[
+ \int_0^t (c_k - \text{div } \hat{b}_k)(X_s) ds \right).
\]
Then,
\[
dZ_t^k = Z_t^k(\tilde{a}^{-1}b)(X_t) dM_t + Z_t^k(c_k - \text{div } \hat{b}_k)(X_t) dt.
\]
Note that both \( \{V_t^k\} \) and \( \{Z_t^k\} \) are semi-martingales. Applying Ito’s formula, we obtain that
\[
d(V_t^k Z_t^k) = V_t^k Z_t^k(\tilde{a}^{-1}b)(X_t) dM_t + Z_t^k \nabla u_1(X_t) dM_t
\]
\[
+ Z_t^k(V_t^k - u_1(X_t))(c_k - \text{div } \hat{b}_k)(X_t) dt.
\]
Further, applying Ito’s formula to \( Z_t^k \), we get
\[
d(V_t^k + u_1(X_0)) Z_t^k = Z_t^k(\tilde{a}^{-1}b)(X_t) dM_t
\]
\[
+ Z_t^k(V_t^k - u_1(X_t))(c_k - \text{div } \hat{b}_k)(X_t) dt
\]
\[
+ u_1(X_0) Z_t^k(\tilde{a}^{-1}b)(X_t) dM_t + u_1(X_0) Z_t^k(c_k - \text{div } \hat{b}_k)(X_t) dt
\]
\[
= (V_t^k + u_1(X_0)) Z_t^k(\tilde{a}^{-1}b)(X_t) dM_t + Z_t^k \nabla u_1(X_t) dM_t
\]
\[
+ Z_t^k(V_t^k - u_1(X_t))(c_k - \text{div } \hat{b}_k)(X_t) dt.
\] (2.51)
By [2, Theorem 2.7] and Lemma 2.3(ii), we obtain that \( V_t^k \to u_1(X_t) - u_1(X_0) \) as \( k \to \infty \) for \( t < \tau_D \). Therefore, (2.50) holds by (2.51).

By (2.50), we know that \( \{u_1(X_{t\wedge \tau_D})Z_{t\wedge \tau_D}, t \geq 0\} \) is a \( P_x \)-local martingale for every \( x \in D \). We claim that \( \{Z_{t\wedge \tau_D}, t \geq 0\} \) is \( P_x \)-uniformly integrable for every \( x \in D \). Write
\[
Z_{t\wedge \tau_D} = Z_{\tau_D} 1_{\{\tau_D \leq t\}} + Z_t 1_{\{\tau_D > t\}}.
\]

By (2.16), \( \{Z_{\tau_D} 1_{\{\tau_D \leq t\}}, t \geq 0\} \) is \( P_x \)-uniformly integrable. We now show that \( \{Z_t 1_{\{\tau_D > t\}}, t \geq 0\} \) is \( P_x \)-uniformly integrable. Note that
\[
Z_t 1_{\{\tau_D > t\}} \leq 1_{\{\tau_D > t\}} \exp \left( \int_0^{\tau_D} (\tilde{a}^{-1}b)^*(X_s) dM_s - \frac{1}{2} \int_0^{\tau_D} b^* \tilde{a}^{-1}b(X_s) ds \right) + \int_0^{\tau_D} g(X_s) ds
\]
\[= 1_{\{\tau_D > t\}} Z_{\tau_D}^g.
\]

Hence it suffices to show that \( \{1_{\{\tau_D > t\}}Z_{\tau_D}^g, t \geq 0\} \) is \( P_x \)-uniformly integrable.

By the strong Markov property, we get
\[
1_{\{\tau_D > t\}} E_x[Z_{\tau_D}^g | F_t] = 1_{\{\tau_D > t\}} Z_t^g E_x[Z_{\tau_D}^g] \\
\geq 1_{\{\tau_D > t\}} Z_t^g \inf_{x \in D} E_x[Z_{\tau_D}^g] \\
= 1_{\{\tau_D > t\}} \inf_{x \in D} E_x^Q \left[ \exp \left( \int_0^{\tau_D} g(X_s) ds \right) \right] \\
\geq 1_{\{\tau_D > t\}} Z_t^g.
\] (2.52)

By (2.52) and (2.13), we obtain that \( \{1_{\{\tau_D > t\}}Z_{\tau_D}^g, t \geq 0\} \) is \( P_x \)-uniformly integrable. Therefore \( \{Z_{t\wedge \tau_D}, t \geq 0\} \) is \( P_x \)-uniformly integrable for every \( x \in D \). Since \( u_1 \) is bounded continuous, we find that \( \{u_1(X_{t\wedge \tau_D})Z_{t\wedge \tau_D}, t \geq 0\} \) is a \( P_x \)-martingale for every \( x \in D \). Thus,
\[
u_1(x) = E_x[u_1(X_{t\wedge \tau_D})Z_{t\wedge \tau_D}], \quad \forall x \in D.
\]

Letting \( t \to \infty \), we obtain that
\[
u_1(x) = E_x[f(X_{\tau_D})Z_{\tau_D}], \quad \forall x \in D,
\]
which proves the uniqueness.

\[\square\]

## 3 Probabilistic Representation of Non-symmetric Semigroup

In this section, we will use some techniques of Section 2 to give a probabilistic representation of the non-symmetric semigroup \( \{T_t\}_{t \geq 0} \) associated with the operator \( L \) defined by (1.2). The obtained result (see Theorem 3.1 below) generalizes
Theorem 3.4, which is the first result on the probabilistic representation of semigroups with \( b \neq 0 \), from the case of symmetric diffusion matrix \( A \) to the non-symmetric case. The methods and techniques of this paper can be applied also to some other problems such as the mixed boundary value problem, Dirichlet problem of semilinear elliptic PDEs with singular coefficients, etc. (cf. [4, 23]). We will consider them in future work.

Throughout this section, we let \( D \) be an open subset of \( \mathbb{R}^d \), which need not be bounded. Suppose that \( A(x) = (a_{ij}(x))_{i,j=1}^d \) is a Borel measurable matrix-valued function on \( D \) satisfying (1.3) and (1.4); \( b = (b_1, \ldots, b_d)^* \) and \( \hat{b} = (\hat{b}_1, \ldots, \hat{b}_d)^* \) are Borel measurable \( \mathbb{R}^d \)-valued functions on \( D \) and \( c \) is a Borel measurable function on \( D \) satisfying \( |b|^2 \in L^{p/1}(D; dx), |\hat{b}|^2 \in L^{p/1}(D; dx) \) and \( c \in L^{p/1}(D; dx) \) for some constant \( p > d/2 \). Let \( L \) and \( (\mathcal{E}, D(\mathcal{E})) \) be defined as in (1.2) and (1.5), respectively. Since \( |b|^2, |\hat{b}|^2 \) and \( c \) are in the Kato class, there exists a constant \( \gamma > 0 \) such that \( (\mathcal{E}_\gamma, D(\mathcal{E})) \) is a coercive closed form on \( L^2(D; dx) \) (cf. [13, page 329]). Hence there exits a (unique) strongly continuous semigroup \( \{T_t\}_{t \geq 0} \) on \( L^2(D; dx) \) which is associated with \( (\mathcal{E}, D(\mathcal{E})) \). Denote by \( (\mathcal{L}, D(\mathcal{L})) \) the generator of \( \{T_t\}_{t \geq 0} \). Clearly \( \mathcal{L} \) is formally given by \( L \). Denote by \( \{\hat{T}_t\}_{t \geq 0} \) the dual semigroup of \( \{T_t\}_{t \geq 0} \) on \( L^2(D; dx) \).

We define the Dirichlet form \( (\mathcal{E}^0, D(\mathcal{E}^0)) \) as in (1.9). Let \( X = ((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d}) \) and \( \hat{X} = ((X_t)_{t \geq 0}, (\hat{P}_x)_{x \in \mathbb{R}^d}) \) be the Markov process and dual Markov process associated with the Dirichlet form \( (\mathcal{E}^0, D(\mathcal{E}^0)) \) given by (1.9), respectively. Let \( M_t, (\tilde{a}_{ij})_{i,j=1}^d, v^H, \) etc. be defined the same as in Section 1. Denote by \( m \) the Lebesgue measure \( dx \) on \( \mathbb{R}^d \). Now we can state the main result of this section.

**Theorem 3.1.** For any \( f, g \in L^2(D; dx) \), we have

\[
\int_D f(x)T_tg(x)dx = E_m \left[ f(X_0)g(X_t) \exp \left( \int_0^t (\tilde{a}^{-1}b)^*(X_s)\,dM_s - \frac{1}{2} \int_0^t b^*\tilde{a}^{-1}b(X_s)\,ds \right) \right.
\]

\[
+ \int_0^t c(X_s)\,ds + N_t^{\hat{M}} - \int_0^t \hat{b}^H(X_s)\,ds; t < \tau_D \right].
\]

**Proof.** By (2.1) (cf. [1 Theorem 4.4]), similar to [13 Theorem 2.1], we can prove the following lemma on integrability of functionals of Dirichlet processes.

**Lemma 3.2.** Suppose \( f \in L^{r/1}(D; dx) \) for some \( r > d/2 \) and \( T > 0 \). Then, there exists a constant \( \varrho_1 > 0 \) depending on \( f \), \( r \) and \( T \) such that for any \( 0 \leq t \leq T \),

\[
\sup_{x \in D} E_x \left[ \exp \left( \int_0^t f(X_s)\,ds \right); t < \tau_D \right] \leq \varrho_1 e^{\varrho_1 t},
\]

and

\[
\sup_{x \in D} \hat{E}_x \left[ \exp \left( \int_0^t f(X_s)\,ds \right); t < \tau_D \right] \leq \varrho_1 e^{\varrho_1 t}.
\]
We divide the proof of Theorem 3.1 into three cases.

**Case 1: \( \hat{b} = 0 \).**

For \( g \in \mathcal{B}_b(D) \), we define

\[
P_t g(x) := E_x \left[ \exp \left( \int_0^t (\tilde{a}^{-1} b)(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds + \int_0^t c(X_s) ds \right) g(X_t); t < \tau_D \right].
\]

Clearly \( \{P_t\}_{t \geq 0} \) is a well-defined semigroup. We now show that \( \{P_t\}_{t \geq 0} \) extends to a strongly continuous semigroup on \( L^2(D; dx) \), which will be also denoted by \( \{P_t\}_{t \geq 0} \).

In fact, for any \( g \in L^2(D; dx) \), we obtain by Lemma 3.2 that

\[
\int_D (P_t g(x))^2 dx = \int_D \left( E_x \left[ \exp \left( \int_0^t (\tilde{a}^{-1} b)(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds + \int_0^t c(X_s) ds \right) g(X_t); t < \tau_D \right] \right)^2 dx
\]

\[
\leq \int_D E_x \left[ \exp \left( 2 \int_0^t (\tilde{a}^{-1} b)(X_s) dM_s - 2 \int_0^t b^* \tilde{a}^{-1} b(X_s) ds \right) \right] \cdot E_x \left[ \exp \left( \int_0^t b^* \tilde{a}^{-1} b(X_s) ds + 2 \int_0^t c(X_s) ds \right) g^2(X_t); t < \tau_D \right] dx
\]

\[
= \int_D g^2(x) \hat{E}_x \left[ \exp \left( \int_0^t b^* \tilde{a}^{-1} b(X_s) ds + 2 \int_0^t c(X_s) ds \right); t < \tau_D \right] dx
\]

\[
\leq q_2 e^{q_2 t} \int_D g^2(x) dx
\]

(3.2)

where \( q_2 > 0 \) is a constant independent of \( g \). This gives the existence of the extension of \( P_t \) to \( L^2(D; dx) \). Since \( C_0(D) \) is dense in \( L^2(D; dx) \) and for \( g \in C_0(D) \), \( P_t g(x) \to g(x) \) as \( t \to 0 \), the continuity property of \( P_t \) follows from (3.2).

Define

\[
S_t = \exp \left( \int_0^t (\tilde{a}^{-1} b)(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds + \int_0^t c(X_s) ds \right)
\]

and

\[
\bar{M}_t = \int_0^t (\tilde{a}^{-1} b)(X_s) dM_s.
\]

Then \( S_t = 1 + \int_0^t S_x d\bar{M}_s + \int_0^t S_x c(X_s) ds \). By Ito’s formula, we obtain that for \( u \in D(L) \) and \( t < \tau_D \),

\[
u(X_t)S_t = u(X_0) + \int_0^t S_x d\bar{M}_s + \int_0^t u(X_s)S_x c(X_s) ds + \int_0^t S_x Lu(X_s) ds.
\]
Following the argument of the proof of [13, Theorem 3.2], we can show that \( \{P_t\}_{t \geq 0} \) coincides with \( \{T_t\}_{t \geq 0} \) for this case.

**Case 2:** \( \hat{b} \in C_0^\infty(D) \).

Similar to the proof of [13, Theorem 3.3], we can show that for \( g \in L^2(D; dx) \),

\[
T_t g(x) = E_x \left[ \exp \left( \int_0^t (\tilde{a}^{-1}b)(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s) ds \right.ight.
\]
\[
+ \int_0^t c(X_s) ds - \int_0^t \text{div} \hat{b}(X_s) ds \bigg) g(X_t); \; t < \tau_D \bigg].
\]

The proof of this case is complete by (2.4).

**Case 3:** \( |\hat{b}|^2 \in L^{p,\infty}(D; dx) \).

By Lemma 2.2(ii), we may choose a sequence \( \{\hat{b}_n \in C_0^\infty(\mathbb{R}^d)\} \) such that \( |\hat{b}_n - \hat{b}|^2 \to 0 \) in \( L^{p,\infty}(\mathbb{R}^d; dx) \) and \( \hat{b}_n^H \to \hat{b}^H \) in \( H^{1,2}(\mathbb{R}^d) \) as \( n \to \infty \).

Let \( \{T^n_t\}_{t \geq 0} \) be the semigroup corresponding to the quadratic form \( \mathcal{E} \) with \( \hat{b}_n \) in place of \( \hat{b} \). Then, for \( f, g \in L^2(D; dx) \), we have

\[
\int_D f(x) T^n_t g(x) dx = E_m \left[ f(X_0) g(X_t) \exp \left( \int_0^t (\tilde{a}^{-1}b)^*(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s) ds \right.\right.
\]
\[
+ \int_0^t c(X_s) ds - \int_0^t \text{div} \hat{b}_n(X_s) ds + N^\hat{b}_n t - \int_0^t \hat{b}_n^H(X_s) ds \bigg); \; t < \tau_D \bigg]. \tag{3.3}
\]

By [20, Theorem 1.3], the left-hand side of (3.3) converges to \( \int_D f(x) T_t g(x) dx \) as \( n \to \infty \).

We will prove below that the right-hand side of (3.3) converges to the right-hand side of (3.1) as \( n \to \infty \). Define for \( t \geq 0 \),

\[
Y^n_t = g(X_t) \exp \left( \int_0^t (\tilde{a}^{-1}b)^*(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s) ds \right.\right.
\]
\[
+ \int_0^t c(X_s) ds + N^\hat{b}_n t - \int_0^t \hat{b}_n^H(X_s) ds \bigg), \; n \in \mathbb{N},
\]

and

\[
Y_t = g(X_t) \exp \left( \int_0^t (\tilde{a}^{-1}b)^*(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s) ds \right.\right.
\]
\[
+ \int_0^t c(X_s) ds + N^\hat{b} t - \int_0^t \hat{b}^H(X_s) ds \bigg). \tag{3.4}
\]

23
Then, the right-hand sides of (3.3) and (3.1) equal \( E_{f,m}[\hat{Y}_t^n; t < \tau_D] \) and \( E_{f,m}[Y_t^n; t < \tau_D] \), respectively. To complete the proof, we need only show that \( \{Y_t^n1_{t<\tau_D}\} \) is \( P_{f,m}\)-uniformly integrable. We will establish this below by proving that \( \sup_{n \in \mathbb{N}} E_{f,m}[\hat{Y}_t^n]; t < \tau_D < \infty \).

In fact, we obtain by Cauchy-Schwarz inequality that

\[
E_{f,m}[\hat{Y}_t^n]^2; t < \tau_D
\]

\[
= E_{f,m}\left[ g^2(X_t) \exp\left( 2 \int_0^t (\hat{a}^{-1}b)^*(X_s) dM_s - \int_0^t b^*\hat{a}^{-1}b(X_s) ds \right) + 2 \int_0^t c(X_s) ds + 2 N_t^b - 2 \int_0^t \hat{b}_n(X_s) ds \right]; t < \tau_D \]

\[
= E_{f,m}\left[ g^2(X_t) \exp\left( \frac{1}{2} \int_0^t (\hat{a}^{-1}b)^*(X_s) dM_s - \frac{1}{4} \int_0^t b^*\hat{a}^{-1}b(X_s) ds \right) + \frac{1}{2} \int_0^t c(X_s) ds + 2 N_t^b - 2 \int_0^t \hat{b}_n(X_s) ds \right]
\cdot \exp\left( \frac{3}{2} \int_0^t (\hat{a}^{-1}b)^*(X_s) dM_s - \frac{3}{4} \int_0^t b^*\hat{a}^{-1}b(X_s) ds \right) + \frac{3}{2} \int_0^t c(X_s) ds ; t < \tau_D \]

\[
\leq E_{f,m}\left[ g^4(X_t) \exp\left( \int_0^t (\hat{a}^{-1}b)^*(X_s) dM_s - \frac{1}{2} \int_0^t b^*\hat{a}^{-1}b(X_s) ds \right) + \int_0^t c(X_s) ds + N_t^b - \int_0^t \hat{b}_n(X_s) ds ; t < \tau_D \right]^{1/2}
\cdot E_{f,m}\left[ \exp\left( 3 \int_0^t (\hat{a}^{-1}b)^*(X_s) dM_s \right. \right.
\left. - \frac{3}{2} \int_0^t b^*\hat{a}^{-1}b(X_s) ds + 3 \int_0^t c(X_s) ds \right); t < \tau_D \right]^{1/2}
\]

\[
= \left( \int_D f(x) T_t^n g^4(x) dx \right)^{1/2}
\cdot E_{f,m}\left[ \exp\left( 3 \int_0^t (\hat{a}^{-1}b)^*(X_s) dM_s \right. \right.
\left. - \frac{3}{2} \int_0^t b^*\hat{a}^{-1}b(X_s) ds + 3 \int_0^t c(X_s) ds \right); t < \tau_D \right]^{1/2},
\]

where \( \{T_t^n\}_{t \geq 0} \) is the semigroup corresponding to the quadratic form \( E \) with \( \hat{a} b_n \) in place of \( \hat{a} \). Thus, we obtain by [20, Theorem 1.3] and Lemma 3.2 that

\[
\sup_{n \in \mathbb{N}} E_{f,m}[\hat{Y}_t^n]; t < \tau_D
\]

\[
\leq \sup_{n \in \mathbb{N}} \left( \int_D f(x) T_t^n g^4(x) dx \right)^{1/2} \cdot E_{f,m}\left[ \exp\left( 3 \int_0^t (\hat{a}^{-1}b)^*(X_s) dM_s \right. \right.
\left. - \frac{3}{2} \int_0^t b^*\hat{a}^{-1}b(X_s) ds + 3 \int_0^t c(X_s) ds \right); t < \tau_D \right]^{1/2},
\]

24
\[-\frac{3}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds + 3 \int_0^t c(X_s) ds; \quad t < \tau_D \right]^{1/2} < \infty.\]

Acknowledgments

We acknowledge the support of NSFC (Grant No. 11361021) and NSERC (Grant No. 311945-2013).

References

[1] W. Arendt, A.F.M. Ter Elst, Gaussian estimates for second order elliptic operators with boundary conditions, J. Oper. Theory 38 (1997) 87-130.

[2] C.Z. Chen, L. Ma, W. Sun, Stochastic calculus for Markov processes associated with non-symmetric Dirichlet forms, Sci. China Math. 55 (2012) 2195-2203.

[3] Z.Q. Chen, T.S. Zhang, Time-reversal and elliptic boundary value problems, Ann. Prob. 37 (2009) 1008-1043.

[4] Z.Q. Chen, T.S. Zhang, A probabilistic approach to mixed boundary value problems for elliptic operators with singular coefficients, Proc. Amer. Math. Soc. 142 (2014) 2135-2149.

[5] Z.Q. Chen, Z. Zhao, Diffusion processes and second order elliptic operators with singular coefficients for lower order terms, Math. Ann. 302 (1995) 323-357.

[6] S. Cho, P. Kim, H. Park, Two-sided estimates on Dirichlet heat kernels for time-dependent parabolic operators with singular drifts in $C^{1,\alpha}$-domains, J. Diff. Equat. 252 (2012) 1101-1145.

[7] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet Forms and Symmetric Markov Processes, Second revised and extended edition, Walter de Gruyter, 2011.

[8] S. Kakutani, Two-dimensional Brownian motion and harmonic functions, Proc. Imp. Acad. Tokyo 20 (1944) 706-714.

[9] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1980.

[10] C. Kenig, H. Koch, J. Pipher, T. Toro, A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations, Adv. Math. 153 (2000) 231-298.
[11] P. Kim, R.M. Song, Two-sided estimates on the density of Brownian motion with singular drift, Ill. J. Math. 50 (2006) 635-688.

[12] J. Lierl, L. Saloff-Coste, Parabolic Harnack inequality for time-dependent non-symmetric Dirichlet forms, arXiv:1205.6493v4.

[13] J. Lunt, T.J. Lyons, T.S. Zhang, Integrability of functionals of Dirichlet processes, probabilistic representations of semigroups, and estimates of heat kernels, J. Funct. Anal. 153 (1998) 320-342.

[14] Z.M. Ma, M. Röckner, Introduction to the Theory of (Non-symmetric) Dirichlet Forms, Springer-Verlag, 1992.

[15] N.G. Meyers, An $L^p$-estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Scuola Norm. Sup. Pisa 17 (1963) 189-206.

[16] C.B.Jr. Morrey, Second order elliptic equations in several variables and Hölder continuity, Math Z. 72 (1959) 146-164.

[17] C.B.Jr. Morrey, Multiple Integrals in the Calculus of Variations, Springer-Verlag, 1966.

[18] J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17 (1964) 101-134.

[19] S. Nakao, Stochastic calculus for continuous additive functionals of zero energy, Z. Wahrsch. verw. Gebiete 68 (1985) 557-578.

[20] M. Röckner, T.S. Zhang, Convergence of operator semigroups generated by elliptic operators, Osaka J. Math. 34 (1997) 923-932.

[21] N. Trudinger, Linear elliptic operators with measurable coefficients, Ann. Scuola Norm. Sup. Pisa 27 (1973) 255-308.

[22] A. Walsh, Stochastic integration with respect to additive functionals of zero quadratic variation, Bernoulli 19 (2013) 2414-2436.

[23] T.S. Zhang, A probabilistic approach to Dirichlet problems of semilinear elliptic PDEs with singular coefficients, Ann. Probab. 39 (2011) 1502-1527.