Generalized Tu Formula and Hamilton Structures of Fractional Soliton Equation Hierarchy

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Abstract

With the modified Riemann-Liouville fractional derivative, a fractional Tu formula is presented to investigate generalized Hamilton structure of fractional soliton equations. The obtained results can be reduced to the classical Hamilton hierarchy of ordinary calculus.

Key words: Fractionalized Tu formula; Fractional Hamilton system; Fractional evolutionary equations

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1 Introduction

Nobel Laureate Gerardus’t Hooft once remarked that discrete space-time is the most radical and logical viewpoint of reality. In such discontinuous space-time, fractional calculus plays an important role which can accurately describe many nonlinear phenomena in physics, i.e., Brownian motion, anomalous diffusion, transportation in porous media, chaotic dynamics, physical kinetics and quantum mechanics.

Past decades witness the development of fractional calculus in various fields, such as rheology, quantitative biology, electrochemistry, scattering theory, diffusion, transport theory, probability potential theory and elasticity. For details, see the monographs of Kilbas et al. [1], Kiryakova [2], Lakshmikantham and Vatsala [3], Miller and Ross [4], and Podlubny [5].

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Since Riewe [6] proposed a concept of non-conservation mechanics, fractional conservation laws [7], fractional Lie symmetries [8] and fractional Hamilton systems [9–16] have caught much attention. In recent study, Fujioka et al found that the propagation of optical solitons can be described by an extended nonlinear Schrödinger equation which incorporates fractional derivatives [17, 18].

Searching for new integrable hierarchies of soliton equations is an important and interesting topic in soliton theory. Tu scheme of ordinary calculus [19] is an efficient method to generate integrable Hamilton systems. It took various efficient approaches to have obtained many integrable systems such as AKNS hierarchy, KN hierarchy, Schrödinger system, and so on [20–26]. In order to consider the Hamilton structure of fractional soliton equations, some questions may naturally arise: (1) Can we have a generalized Tu sheme in fractional case? (2) How to define Hamilton equations for fractional soliton hierarchy?

In this study, we start from a Lax pair of fractional order in the sense of the modified Riemann-Liouville's derivative [14] and propose a generalized Tu sheme to investigate the Hamilton structure of fractional soliton evolutionary equations.

### 2 Modified Riemann-Liouville derivative

Generally, there are two kinds of fractional derivatives: local fractional derivatives and nonlocal ones. The most used nonlocal operator is the Caputo derivative which requires the defined functions should be differentiable. The condition is so strict that many engineering problems cannot satisfy, i.e., functions defined on fractal curves, fractional diffusion problem. As a result, the Caputo derivative is not suitable for such problems theoretically.

Several local versions have been proposed: Kolwankar-Gangal’s local fractional derivative [27–29], Chen’s fractal derivative [30, 31], Cresson’s derivative [32], Jumrie’s modified Riemann-Liouville derivative [33] and Parvate’s $F^\alpha$ derivative [34], among which Jumari’s modified R-L derivative is defined as

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha}(f(\xi) - f(0))d\xi, \quad 0 < \alpha < 1.$$  

(1)

Here the derivative on the right-hand side is the Riemann-Liouville fractional derivative. The nonlinear techniques for such fractional differentiable equations can be found in Refs. [8, 35, 36]

We can have following results for Jumarie’s modified Riemann-Liouville (R-L) derivative.

(a) The Leibniz product law

If $f(x)$ is an $\alpha$ order differentiable function in the area of point $x$, from the Rolle-Kolwankar-Jumarie’s Taylor series [31], one can have

$$D_x^\alpha f(x) = \lim_{y \to x} \frac{\Gamma(1+\alpha)(f(y) - f(x))}{(y-x)^\alpha}, \quad 0 < \alpha < 1.$$  

(2)
Assuming \( g(x) \) is an \( \alpha \) order differentiable function, the Leibniz product law can hold

\[
D^\alpha_x (f(x)g(x)) = g(x)D^\alpha_x f(x) + f(x)D^\alpha_x g(x).
\]  

(3)

(b) Integration with respect to \((dx)^\alpha\) (Lemma 2.1 of [37])

\[
a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \xi)^{\alpha-1} f(\xi)d\xi = \frac{1}{\Gamma(\alpha + 1)} \int_0^x f(\xi)(d\xi)^\alpha, 0 < \alpha \leq 1.
\]  

(4)

(c) Generalized Newton-Leibniz Formulation

Assume \( D^\alpha_x f(x) \) is a integrable function in the interval \([a, b]\). Obviously,

\[
\frac{1}{\Gamma(1 + \alpha)} \int_a^b D^\alpha_x f(x)(dx)^\alpha = f(b) - f(a),
\]  

(5)

\[
\frac{1}{\Gamma(1 + \alpha)} \int_a^x D^\alpha_x f(\xi)(d\xi)^\alpha = f(x) - f(a),
\]  

(6)

and

\[
\frac{D^\alpha_x}{\Gamma(1 + \alpha)} \int_a^x f(\xi)(d\xi)^\alpha = f(x).
\]  

(7)

(d) Integration by parts

With the properties (b) and (c), integration by parts for \( \alpha \) order differentiable functions \( f(x) \) and \( g(x) \) can be presented as

\[
\frac{1}{\Gamma(1 + \alpha)} \int_a^b g(x)D^\alpha_x f(x)(dx)^\alpha = g(x)f(x) \big|_a^b - \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(x)D^\alpha_x g(x)(dx)^\alpha.
\]  

(8)

The above properties (a)–(d) can be found in Ref. [33].

(e) Fractional variational derivative

From Jumarie’s variational derivative [14] and Almeida’ fractional variational approach [15], the fractional variational derivative is defined as

\[
\frac{\delta L}{\delta y} = \frac{\partial L}{\partial y} + \sum_{k=1} \frac{(-1)^k}{(\frac{\partial L}{\partial (D^\alpha_x)^ky})},
\]  

(9)

where \( k \) is a positive integer.

(f) From Eq. (2), we can have

\[
D^\alpha_x f(x) = \lim_{h \to 0} \frac{\Gamma(1 + \alpha)(f(x + h) - f(x))}{h^\alpha} = \frac{\Gamma(1 + \alpha)df(x)}{(dx)^\alpha}, 0 < \alpha < 1.
\]  

(10)

As a result, we can find that

\[
f(b) - f(a) = \frac{1}{\Gamma(1 + \alpha)} \int_a^b D^\alpha_x f(x)(dx)^\alpha = \frac{1}{\Gamma(1 + \alpha)} \int_a^b d^\alpha f(x).
\]  

(11)
3 Fractional Hamilton Structure

A number of useful attempts have been made to establish fractional variational principles and Hamilton system [9–16]. Different types of fractional derivatives may lead to different results, for examples, i.e., Baleanu’s fractional Hamilton system with Caputo derivative [10], Riemann-Liouville type Hamilton mechanics [11], Argwal’s Hamilton Formulation with Riesz derivative [12] and Jumairie’s Lagrange formula [14]. In this section, we revisit Jumarie’s fractional Hamilton system.

3.1 A Fractional Exterior Differential Approach

Since Ben Adda proposed the fractional generalization of differential [38, 39], many fractional exterior differential approaches and applications related to different forms of fractional derivatives appeared in open literature [40–42]. A brief review is available in Tarasov’s work [43].

Starting from the total derivative in the integer dimensional space, and assuming $f = f(u, v)$, $u = u(x)$ and $v = v(x)$, where $u, v$ are $\alpha$ order differentiable functions and $f$ is a differentiable function with respect to $u$ and $v$, we obtain the total derivative as follows

$$df = f_u du + f_v dv. \quad (12)$$

Multiplying both sides of Eq. (12) with $\Gamma(1 + \alpha)$, we can get

$$d^\alpha f = f_u d^\alpha u + f_v d^\alpha v = f_u D_x^\alpha u(dx)^\alpha + f_v D_x^\alpha v(dx)^\alpha. \quad (13)$$

On the other hand, if we assume that $u, v$ are differentiable functions and $f$ is a $\alpha$ order differentiable function with respect to $u$ and $v$

$$df = \frac{f_u^{(\alpha)}(du)^\alpha}{\Gamma(1 + \alpha)} + \frac{f_v^{(\alpha)}(dv)^\alpha}{\Gamma(1 + \alpha)}, \quad (14)$$

we can derive a different definition of $d^\alpha$ as follows

$$d^\alpha f = f_u^{(\alpha)}(u_x)^\alpha(dx)^\alpha + f_v^{(\alpha)}(v_x)^\alpha(dx)^\alpha. \quad (15)$$

Thus, we may have two different results for $D_x^\alpha x^2$, $\frac{2\Gamma(1+\alpha)x^{1-\alpha}}{\Gamma(2-\alpha)}$ or $\frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}$, respectively, if we know nothing about the differentiability of $x^2$.

3.2 Fractional Hamilton Equations

We define the fractional functional

$$J[p, q] = \frac{1}{\Gamma(1 + \alpha)} \int [pD_t^\alpha q - H(t, p, q)](dt)^\alpha \quad (16)$$

Then, we can readily derive the fractional Poincare–Cartan 1-form, which reads
\[ \omega = pd^\alpha q - H(dt)^\alpha. \]  

From Eq. (17), we have

\[ d^\alpha \omega = p_t^{(\alpha)}(dt)^\alpha \wedge d^\alpha q + d^\alpha p \wedge d^\alpha q - \frac{\partial H}{\partial p} d^\alpha p \wedge (dt)^\alpha - \frac{\partial H}{\partial q} d^\alpha q \wedge (dt)^\alpha \]

\[ = [p_t^{(\alpha)} + \frac{\partial H}{\partial q}](dt)^\alpha \wedge d^\alpha q + [\frac{\partial H}{\partial p}(dt)^\alpha - d^\alpha q] \wedge d^\alpha p. \]  

The fractional closed condition \( d^\alpha \omega = 0 \) allows us to obtain the following fractional Hamilton equations

\[ D_t^{(\alpha)} q = \frac{\partial H}{\partial p}, \]  

and

\[ D_t^{(\alpha)} p = -\frac{\partial H}{\partial q}. \]  

We must point out, the results Eq. (13), Eq. (15), Eq. (19) and Eq. (20) can be found in Ref. [14].

4 Fractional Tu Formula and Its Application

4.1 A Fractionalized Tu Formula

Set \( A_n = A = (a_{i,j}), \) \( a_{i,j} \in C. \) Assume \( A \) and \( B \in C. \) Define \([A, B] = AB - BA. \) Hence, \( A_n \) is a Lie algebra. The corresponding loop algebra is defined as

\[ \tilde{A}_n = A(n) = A \lambda^n, \quad n \in Z. \]  

Tu Formula is a beautiful identity to generate integral Hamilton equations. In the past decades, many integral Hamilton hierarchies are obtained via this technical [20–26]. Consider the fractional compatibility condition,

\[ \phi_x^{(\alpha)}(x, t) = U \phi, \quad \phi_t^{(\beta)}(x, t) = V \phi, \]  

where the fractional derivative is in the sense of the modified R-L derivative [14, 31], and \( \phi \) is a \( n \)-dimensional function vector.

The compatibility condition of Eq. (22) leads to the generalized zero curvature equation

\[ U_t^{(\beta)} - V_x^{(\alpha)} + [U, V] = 0, \quad [U, V] = UV - VU. \]  

When taking \( \alpha = \beta = 1, \) Eq. (22) reduces to the classical zero curvature equation. Set

\[ U = e_0(\lambda) + \sum_{i=1}^{n} e_i(\lambda)u_i, \quad \{e_i(\lambda), 0 \geq i \leq n\} \subset \tilde{A}_n, \]
where \( u = u(u_1, u_2, ..., u_n)^T \) denotes a vector function. By the gradation of \( \tilde{B}_n \), define \( \text{rank} \lambda = \text{deg}(\lambda) \), then \( \text{rank} (e_0(\lambda)) = \alpha \), \( \text{rank} (e_i(\lambda)) = \alpha_i \), \( 0 \leq i \leq n \) are all known. If we take the ranks of \( u_i \) as \( \alpha - \alpha_i \), \( 1 \leq i \leq n \), then each term in \( U \) is of the same rank \( \alpha \), denoted by \( \text{rank}(U) = \text{rank}(\frac{\partial^\alpha}{\partial x^\alpha}) = \alpha \). \( (25) \)

If a solution of the stationary zero curvature equation

\[-V_x^{(\alpha)} + [U, \ V] = 0, \quad (26)\]

is given by

\[V = \sum_{m \geq 0} V_m \lambda^{-m}, \ (V_m)\lambda = 0, \ m \geq 0. \quad \text{rank} (V_m) \lambda \text{ is assumed to be given so that} \quad \text{rank} (V_m) \lambda = \beta, \ m \geq 0, \ \text{then each term in} \ V \ \text{has the same rank, denoted by} \quad \text{rank}(V) = \text{rank}(\frac{\partial^\beta}{\partial \lambda^\beta}) = \eta. \quad (27)\]

Suppose \( f(A, B) = \text{tr}(AB) \). The following properties can be satisfied

(a) Symmetry relationship

\[f(A, B) = f(B, A); \]

(b) The bilinearity can hold

\[f(c_1 A_1 + c_2 A_2, B) = c_1 f(A_1, B) + c_2 f(A_2, B); \]

(c) In the sense of the local fractional derivative, the gradient \( \nabla_B f(A, B) \) of the functional \( f(A, B) \) is defined by

\[\frac{\partial}{\partial \epsilon} f(A, B + \epsilon C) = f(\delta_B f(A, B), C), \forall A, B, C \in \tilde{A}_n, \quad (28)\]

where \( \delta_B \) is variational derivative with respect to \( B \).

With the fractional variational derivative, we can have the following results,

\[\delta_B f(A, B_x^{k\alpha}) = (-1)^k A_x^{(k\alpha)}, \quad (29)\]

where \( k \) is a positive integer and \( D_x^{k\alpha} = D_x^\alpha \ldots D_x^\alpha \ldots D_x^\alpha \).

(d) Communication relationship

\[f([A, B], C) = f(A, [B, C]), \forall A, B, C \in \tilde{A}_n. \quad (30)\]

Construct a functional

\[W = f(V, U_\lambda) + f(K, V_x^{(\alpha)}) - [U, V], \quad (31)\]

where \( U, V \) meet Eq. (22), \( K \in \tilde{A}_n, \text{rank} \ K = -\text{rank} \ \lambda \).
With the defined fractional variational derivative,
\begin{align*}
\frac{\delta W}{\delta K} = V_x^{(\alpha)} - [U, V], \quad \frac{\delta W}{\delta V} = U_\lambda - K_x^{(\alpha)} + [U, V],
\end{align*}
from the above equations, we can derive
\begin{align*}
[K, V]_x^{(\alpha)} &= [K_x^{(\alpha)}, V] + [K, V_x^{(\alpha)}] \\
&= [U_\lambda + [U, K], V] + [K, [U, V]] \\
&= [U_\lambda, V] + [[U, K], V] + [[V, U], K] \\
&= [U_\lambda, V] + [U, [K, V]].
\end{align*}

We can check \( V' = [K, V] - V_\lambda \) satisfies Eq. (26) and \( \frac{V}{\lambda} \) also satisfies Eq. (26) since \( \text{rank}(Z) = \text{rank}(V_\lambda) = \text{rank}(V) - \text{rank}(\lambda) = \text{rank}(\frac{V}{\lambda}) \). Therefore, if two solutions of Eq. (23), \( V \) and \( V' \) are linearly dependent, we can get \( V' = \frac{\lambda}{\gamma} V \).

Using Eq. (31) again, we can have a fractional trace identity as follows
\begin{align*}
\frac{\delta f(V, U_\lambda)}{\delta u_i} = f(V, \frac{\partial U_\lambda}{\partial u_i}) + f([K, V], \frac{\partial U_\lambda}{\partial u_i}) \\
&= f(V, \frac{\partial U_\lambda}{\partial u_i}) + f(V_\lambda, \frac{\partial U_\lambda}{\partial u_i}) + \frac{\gamma}{\lambda} f(V, \frac{\partial U}{\partial u_i}) \\
&= \frac{\partial}{\partial \lambda} f(V, \frac{\partial U}{\partial u_i}) + f(V_\lambda, \frac{\partial U_\lambda}{\partial u_i}) + (\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma) f(V, \frac{\partial U}{\partial u_i}) \\
&= \lambda^{-\gamma} \frac{\partial}{\partial \lambda} [\lambda^\gamma f(V, \frac{\partial U}{\partial u_i})], \quad 0 \leq i \leq n.
\end{align*}

We must point out, the variational derivative here is defined in the sense of the modified R-L fractional derivative.

### 4.2 Fractional Soliton Hierarchies and Their Hamilton Structures

Recently, Fujioka et al [17] found the propagation of optical solitons can be described by an extended NLS equation which incorporates fractional derivatives. The detailed review can be found in Ref. [18]. In view of this point, we consider fractional AKNS hierarchy starting from the generalized spectral problem
\begin{align*}
\Phi_x^{(\alpha)} = U(\lambda, u) = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix} \Phi, \quad u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.
\end{align*}

Choose a simple subalgebra of \( A_1 \)
\begin{align*}
e_1(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\end{align*}

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equipped with the commutative relations
\[
[e_1(m), e_2(n)] = 2e_2(m+n), \quad [e_1(m), e_3(n)] = -2e_3(m+n), \quad [e_2(m), e_3(n)] = e_1(m+n).
\] (35)

Then, we find that the adjoint representation equation \(V^{(\alpha)}_x = [U, V] = UV - VU\) yields
\[
a^{(\alpha)}_{0x} = qc_0 - rb_0, b_0 = 0, c_0 = 0, \\
a^{(\alpha)}_{ix} = qc_i - rb_i, \\
b^{(\alpha)}_{ix} = -2b_{i+1} - 2qa_i, \\
c^{(\alpha)}_{ix} = 2ra_i + 2c_{i+1}, i \geq 1,
\]

the first few of which reads
\[
a_0 = -1, \quad b_0 = 0, \quad c_0 = 0, \\
a_1 = 0, \quad b_1 = q, \quad c_1 = r, \quad b_2 = -\frac{1}{2}q^{(\alpha)}_x, \\
c_2 = \frac{1}{2}r^{(\alpha)}_x, \quad a_2 = \frac{1}{2}qr.
\]

We can derive the recurrence relationship
\[
\begin{pmatrix}
c_{n+1} \\
b_{n+1}
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
D^\alpha_x - 2rD^{-\alpha}q & 2rD^{-\alpha}r \\
-2qD^{-\alpha}x & -D^\alpha_x + 2qD^{-\alpha}x
\end{pmatrix}
\begin{pmatrix}
c_n \\
b_n
\end{pmatrix}
= L
\begin{pmatrix}
c_n \\
b_n
\end{pmatrix},
\]
(36)

Denoting
\[
(V^{(n)}_+)_x^{(\alpha)} = \sum_{i=0}^{n} a(i)e_1(n-i) + b(i)e_2(n-i) + c(i)e_3(n-i),
\]
\[
V^{(n)}_+ = \lambda^n V - V^{(n)}_+,
\]

Eq. (26) can be written as
\[
-(V^{(n)}_+)_x^{(\alpha)} + [U, V^{(n)}_+] = (V^{(n)}_+)_x^{(\alpha)} - [U, V^{(n)}_+].
\]
(37)

It is easy to verify that the terms on the left-hand side in (37) are of degree \(\geq 0\), while the terms on the right-hand side in Eq. (37) are of degree \(\leq 0\). Thus, we have
\[
-(V^{(n)}_+)_x^{(\alpha)} + [U, V^{(n)}_+] = 2b(n+1)e_2(0) - 2c(n+1)e_3(0).
\]

Taking an arbitrary modified term for \(V^{(n)}_+\) as \(\Delta_n = 0\). Notice \(V^{(n)} = V^{(n)}_+\), it is easy to compute, the zero curvature equation
\[
U^{(\beta)}_t - (V^{(n)}_+)_x^{(\alpha)} + [U, V^{(n)}] = 0,
\]
(38)
which gives rise to

\[
  u^{(\beta)}_t = \begin{pmatrix} q^{(\beta)}_t \\ r^{(\beta)}_t \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} c(n+1) \\ b(n+1) \end{pmatrix} = JL \begin{pmatrix} c(n) \\ b(n) \end{pmatrix} = JL^n \begin{pmatrix} r \\ q \end{pmatrix}.
\]  

(39)

Here \( J \) is Hamiltonian operator.

For \( n = 2 \), we obtain the fractional AKNS equations

\[
  \begin{align*}
  D_\beta q_t &= -\frac{1}{2} D_x^\alpha D_\lambda^\alpha q + q^2 r, \\
  D_\beta r_t &= \frac{1}{2} D_x^\alpha D_\lambda^\alpha r - qr^2.
  \end{align*}
\]

(40)

When \( \alpha = \beta = 1 \), Eq. (40) can be reduced to the classical AKNS system,

\[
  \begin{align*}
  q_t &= -\frac{1}{2} q_{xx} + q^2 r, \\
  r_t &= \frac{1}{2} r_{xx} - qr^2.
  \end{align*}
\]

(41)

In order to use the proposed trace identity, a direct compute leads to

\[
  f(V, U_\lambda) = -2a_n, f(V, \frac{\partial U}{\partial q}) = c_n, f(V, \frac{\partial U}{\partial r}) = b_n.
\]

(42)

As a result,

\[
  \frac{\delta(-2a_n)}{\delta u} = \lambda^{-n-1} \frac{\partial}{\partial \lambda} \lambda^n \begin{pmatrix} c \\ b \end{pmatrix}.
\]

(43)

Compared with the coefficients of \( \lambda^{-n-1} \),

\[
  \frac{\delta(-2a_{n+1})}{\delta u} = (\gamma - n) \begin{pmatrix} c_n \\ b_n \end{pmatrix}.
\]

(44)

Setting \( n = 1 \), we can determine \( \gamma = 0 \) from the initial values. Then we can derive the fractional Hamilton function

\[
  H_n = \frac{2a_{n+1}}{n}, \quad \frac{\delta H_n}{\delta u} = \begin{pmatrix} c_n \\ b_n \end{pmatrix}.
\]

(45)

The generalized evolutionary equations can be given as

\[
  u^{(\beta)}_{tn} = \begin{pmatrix} q^{(\beta)}_{tn} \\ r^{(\beta)}_{tn} \end{pmatrix} = J \frac{\delta H_n}{\delta u}.
\]

(46)

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5 Conclusion

Inspired by the previous work [44], in this study we use a different fractional derivative, modified Riemann-Liouville derivative, establish a fractionalized Tu scheme for fractional differential equations and define a local fractional Hamilton system and derive fractional evolutionary soliton hierarchies.

However, there are still other interesting questions needed to be addressed i.e., physical meaning of fractional soliton which may be related to fractal media, fractional integral coupling method, nonlinear techniques for fractional soliton equations. Such work is under consideration.

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