The classification of radial or totally geodesic ends of real projective orbifolds I: a survey of results

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Abstract.

Real projective structures on n-orbifolds are useful in understanding the space of representations of discrete groups into $\text{SL}(n+1,\mathbb{R})$ or $\text{PGL}(n+1,\mathbb{R})$. A recent work shows that many hyperbolic manifolds deform to manifolds with such structures not projectively equivalent to the original ones. The purpose of this paper is to understand the structures of ends of real projective n-dimensional orbifolds. In particular, these have the radial or totally geodesic ends. Hyperbolic manifolds with cusps and hyper-ideal ends are examples. For this, we will study the natural conditions on eigenvalues of holonomy representations of ends when these ends are manageably understandable. We will show that only the radial or totally geodesic ends of lens type or horospherical ends exist for strongly irreducible properly convex real projective orbifolds under the suitable conditions. The purpose of this article is to announce these results.

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§1. Introduction

In this article, we are interested in real projective structures on orbifolds. Orbifolds are basically objects finitely covered by manifolds. Our orbifolds will satisfy this property. The real projective structures can be considered as torsion-free projectively flat affine connections on orbifolds. Another way to view is to consider these as an immersion from the universal cover $\tilde{\Sigma}$ of an orbifold $\Sigma$ to $\mathbb{R}P^n$ equivariant with respect to a homomorphism
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$h : \pi_1(\Sigma) \to \text{PGL}(n + 1, \mathbb{R})$. These orbifolds have ends. We will study the cases when the ends are of specific type since otherwise it is almost impossible to study. The types that we consider are radial ones, or R-ends, where end neighborhoods are foliated by concurrent projective geodesics. Another type is a totally geodesic ones, or T-ends, when the closure of the end neighborhood can be compactified by an ideal totally geodesic hyper surface in some ambient real projective orbifold.

Kuiper, Koszul, Vey, and Vinberg might be the first people to consider these objects seriously as they are related to proper action of affine groups on affine cones in $\mathbb{R}^n$. We note here, of course, the older study of affine structures on manifolds with many major open questions.

1.0.1. Some recent motivations. Recently, there were many research papers on convex real projective structures on manifolds and orbifolds. (See the work of Goldman [45], Choi [20], [21], Benoist [4], Kim [57], Cooper, Long, Thistlethwaite [36], [37] and so on.) One can see them as projectively flat torsion-free connections on manifolds. Topologists will view each of these as a manifold with a structure given by a maximal atlas of charts to $\mathbb{R}P^n$ where transition maps are projective. Hyperbolic and many other geometric structures will induce canonical real projective structures. (See the numerous and beautiful examples in Sullivan-Thurston [71].) Sometimes, these can be deformed to real projective structures not arising from such obvious constructions. In general, the theory of the discrete group representations and their deformations form very much mysterious subjects still. We can use the results in studying linear representations of discrete groups.

Since the examples are more easy to construct, we will be studying orbifolds, natural generalization of manifolds. The deforming a real projective structure on an orbifold to an unbounded situation results in the actions of the fundamental group on affine buildings which hopefully will lead us to some understanding of orbifolds and manifolds in particular of dimension three as indicated by Cooper, Long, and Thistlethwaite.
1.1. Real projective structures on orbifolds with ends

It was discovered by D. Cooper, D. Long, and M. Thistlethwaite [36], [37] that many closed hyperbolic 3-manifolds deform to real projective 3-manifolds. Later S. Tillmann found an example of a 3-orbifold obtained from pasting sides of a single ideal hyperbolic tetrahedron admitting a complete hyperbolic structure with cusps and a one-parameter family of real projective structure deformed from the hyperbolic one (see [27]). Also, Craig Hodgson, Gye-Seon Lee, and I found a few other examples: 3-dimensional
ideal hyperbolic Coxeter orbifolds without edges of order 3 has at least 6-dimensional deformation spaces in [33].

Crampon and Marquis [40] and Cooper, Long, and Tillmann [38] have done similar study with the finite volume condition. In this case, only possible ends are horospherical ones. The work here studies more general type ends but we have benefited from their work. We will see that there are examples where horospherical ends deform to lens-type ones and vice versa (see also Example 4.6.)

Remark 1.1. A summary of the deformation spaces of real projective structures on closed orbifolds and surfaces is given in [23] and [16]. See also Marquis [64] for the end theory of 2-orbifolds. The deformation space of real projective structures on an orbifold loosely speaking is the space of isotopy equivalent real projective structures on a given orbifold. (See [27] also.)

Also, it seems likely from some examples that these orbifolds with ends deform more easily.

Our main aim is to understand these phenomena theoretically. It became clear from our attempt in [27] that we needed to understand and classify the types of ends of the relevant convex real projective orbifolds. We will start with the simplest ones: radial type ones. But as Mike Davis observed, there are many other types such as ones preserving subspaces of dimension greater than equal to 0. In fact, Cooper and Long found such an example from \( S/\text{SL}(3, \mathbb{Z}) \) for the space \( S \) of unimodular positive definite bilinear forms. Since \( S \) is a properly convex domain in \( \mathbb{RP}^5 \) and \( \text{SL}(3, \mathbb{Z}) \) acts projectively, \( S/\text{SL}(3, \mathbb{Z}) \) is a strongly tame properly convex real projective orbifold by the classical theory of lattices. We will not present any of them here; however, it seems very likely that many techniques here will be applicable.

In [27], we show that the deformation spaces of real projective structures on orbifolds are locally homeomorphic to the spaces of conjugacy classes of representations of their fundamental groups where both spaces are restricted by some end conditions.

It remains how to see for which of these types of real orbifolds, nontrivial deformations exist or not. For example, we can consider examples such as complete hyperbolic manifolds and how to compute the deformation space. From Theorem 1 in [33] with
Coxeter orbifolds, we know that a complete hyperbolic Coxeter orbifold always deforms nontrivially. (See also [18].) S. Ballas [1, 2] also produced some results. We conjecture that maybe these types of real projective orbifolds with R-ends might be very flexible. Of course, we have no real analytical or algebraic means to understand these phenomena yet.

1.2. Our settings

We hope to generalize these theories to noncompact orbifolds with conditions on ends. In fact, we are trying to generalize the class of complete hyperbolic manifolds with finite volumes. These are $n$-orbifolds with compact suborbifolds whose complements are diffeomorphic to intervals times closed $(n - 1)$-dimensional orbifolds. Such orbifolds are said to be strongly tame orbifolds. An end neighborhood is a component of the complement of a compact subset not contained in any compact subset of the orbifold. An end $E$ is an equivalence class of compatible sequences of end neighborhoods. Because of this, we can associate an $(n - 1)$-orbifold at each end and we define the end fundamental group $\pi_1(E)$ as the fundamental group of the orbifold, a subgroup of the fundamental group $\pi_1(O)$. We also put the condition on end neighborhoods being foliated by radial lines or to have totally geodesic ideal boundary. Of course, this is not the only natural conditions, and we plan to explore the other conditions in some other occasions. (We note that a strongly tame orbifold may have nonempty boundary that is compact.)

We studied some such orbifolds of Coxeter type with ends in [18] already. These have convex fundamental polytopes and are easier to understand.

For a strongly tame orbifold $O$, we will require the condition (IE).

(IE) $O$ or $\pi_1(O)$ satisfies the infinite-index end fundamental group condition if $\pi_1(\tilde{E})$ is of infinite index in $\pi_1(O)$ for the fundamental group $\pi_1(\tilde{E})$ of each p-end $\tilde{E}$.

(NA) Let $E$ denote the set of all conjugates of end fundamental group of $\pi_1(O)$. Also, if $\Gamma_{E_1} \cap \Gamma_{E_2}$ is finite for any pair of distinct end fundamental groups $\Gamma_{E_1}$ and $\Gamma_{E_2}$, we say that $O$ or $\pi_1(O)$ satisfies no essential annuli condition or (NA).
We will not need condition (NA) here but will need it in later papers such as [25] and [27].

S. Tillmann studied a complete hyperbolic 3-orbifold obtained from gluing a complete hyperbolic tetrahedron. The one parameter family of deformations exists and can be solved explicitly. Also, I obtained one for a double of Coxeter orbifold based on ideal regular complete hyperbolic tetrahedron. (See Chapter 8 of [27].) Later, Gye-Seon Lee and I computed more examples starting from hyperbolic Coxeter orbifolds (These are not published results.) Assume that these structures are properly convex. In these cases, they have only lens-type R-end by Proposition 4.5.

See Cooper-Long-Tillmann [38], Heusener-Porti [53], and Ballas [1, 2] for some computed 3-manifold examples. Recently, Gye-Seon Lee also computed exact one-parameter families of real projective structures on the figure-eight knot complement and the figure-eight sister knot complement. These all have radial ends. Assume that these structures are properly convex. The ends will correspond to lens-type R-ends or cusp R-ends by Corollary 1.9 of [25] since the computations show that the end satisfies the unit eigenvalue condition of the corollary.

The proper convexity of these types real projective orbifolds of examples will be proved in [27].

However, the convexity of the results was the main question that arose. We will try to answer this.
Also, Cooper, Long, and Tillmann [38] and Crampon and Marquis [40] are studying these types of orbifolds as quotients of convex domains without deforming and hence generalizing the Kleinian group theory for complete hyperbolic manifolds. However, they only study the orbifolds with horospherical types of ends or equivalently finite volume orbifolds. Their work is in a sense dual to this work since we start from orbifolds with real projective structures and deform.

We mention that recently Cooper, Long, and Tillman have produced another set of results concerning orbifolds with more general types of ends. Some of their types of ends overlap with ours. However, the general approaches are different. Our exposition here will not go into their recent work as they will do so in other papers.

In general, the theory of geometric structures on manifolds with ends is not studied very well. We should try to obtain more results here and find what the appropriate conditions are. This question seems to be also related to how to make sense of the topological structures of ends in many other geometric structures such as ones on modelled on symmetric spaces and so on.

Given a vector space $V$, we let $P(V)$ denote the space obtained by taking the quotient space of $\mathbb{R}^{n+1} - \{O\}$ under the equivalence relation $v \sim w$ for $v, w \in \mathbb{R}^{n+1} - \{O\}$ iff $v = sw$, for $s \in \mathbb{R} - \{0\}$.

We let $[v]$ denote the equivalence class of $v \in \mathbb{R}^{n+1} - \{O\}$. For a subspace $W$ of $V$, we denote by $P(W)$ the image of $W - \{O\}$ under the quotient map, also said to be a subspace. Recall that the projective linear group $\text{PGL}(n+1, \mathbb{R})$ acts on $\mathbb{RP}^n$, i.e., $P(\mathbb{R}^{n+1})$, in a standard manner. Let $O$ be a noncompact strongly tame $n$-orbifold where the orbifold boundary is not necessarily empty.

- A real projective orbifold is an orbifold with a geometric structure modelled on $(\mathbb{RP}^n, \text{PGL}(n+1, \mathbb{R}))$. (See [18] and Chapter 6 of [23].)
- A real projective orbifold also has the notion of projective geodesics as given by local charts and has a universal cover $\tilde{O}$ where a deck transformation group $\pi_1(O)$ acting on.
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- The underlying space of $O$ is homeomorphic to the quotient space $\tilde{O}/\pi_1(O)$.
- A real projective structure on $O$ gives us a so-called development pair $(\text{dev}, h)$ where
  - $\text{dev}: \tilde{O} \to \mathbb{R}P^n$ is an immersion, called the developing map, and
  - $h: \pi_1(O) \to \text{PGL}(n+1, \mathbb{R})$ is a homomorphism, called a holonomy homomorphism, satisfying
    \[
    \text{dev} \circ \gamma = h(\gamma) \circ \text{dev} \quad \text{for } \gamma \in \pi_1(O).
    \]
  - The pair $(\text{dev}, h)$ is determined only up to the action
    \[
    g(\text{dev}, h(\cdot)) = (g \circ \text{dev}, gh(\cdot)g^{-1}) \quad \text{for } g \in \text{PGL}(n+1, \mathbb{R})
    \]
    and any chart in the atlas extends to a developing map. (See Section 3.4 of [73].)

Let $\mathbb{R}^{n+1*}$ denote the dual of $\mathbb{R}^{n+1}$. Let $\mathbb{R}P^{n*}$ denote the dual projective space $\mathcal{P}(\mathbb{R}^{n+1*})$. $\text{PGL}(n+1, \mathbb{R})$ acts on $\mathbb{R}P^{n*}$ by taking the inverse of the dual transformation. Then $h$ has a dual representation $h^*$ sending elements of $\pi_1(O)$ to the inverse of the dual transformation of $\mathbb{R}^{n+1*}$.

For an element $g \in \text{PGL}(n+1, \mathbb{R})$, we denote
\[
(1) \quad g \cdot [w] := [(\hat{g}^T)^{-1}(w)] \quad \text{for } [w] \in \mathbb{R}P^{n*}
\]
where $\hat{g}$ is any element of $\text{SL}_+(n+1, \mathbb{R})$ mapping to $g$ and $\hat{g}^T$ the transpose of $\hat{g}$.

The complement of a codimension-one subspace of $\mathbb{R}P^n$ can be identified with an affine space $\mathbb{R}^n$ where the geodesics are preserved. The group of affine transformations of $\mathbb{R}^n$ are is the restriction to $\mathbb{R}^n$ of the group of projective transformations of $\mathbb{R}P^n$ fixing the subspace. We call the complement an affine subspace. It has a geodesic structure of a standard affine space. A convex domain in $\mathbb{R}P^n$ is a convex subset of an affine subspace. A properly convex domain in $\mathbb{R}P^n$ is a convex domain contained in a precompact subset of an affine subspace.

The important class of real projective structures are so-called convex ones where any arc in $O$ can be homotoped with endpoints
fixed to a straight geodesic where \( \text{dev} \) is injective to \( \mathbb{R}P^n \) except possibly at the endpoints. If the open orbifold has a convex structure, it is covered by a convex domain \( \Omega \) in \( \mathbb{R}P^n \). Equivalently, this means that the image of the developing map \( \text{dev}(\tilde{O}) \) for the universal cover \( \tilde{O} \) of \( O \) is a convex domain. \( O \) is projectively diffeomorphic to \( \text{dev}(\tilde{O})/h(\pi_1(O)) \). In our discussions, since \( \text{dev} \) often is an imbedding, \( \tilde{O} \) will be regarded as an open domain in \( \mathbb{R}P^n \) and \( \pi_1(O) \) a subgroup of \( \text{PGL}(n+1, \mathbb{R}) \) in such cases. This simplifies our discussions.

Remark 1.2. We will have the following boundary deformability hypothesis for manageability. Otherwise the research might become to large to handle. Let \( O \) be a strongly tame real projective orbifold. We assume that \( \partial O \) is strictly convex; i.e., each point of \( \partial O \) has a neighborhood mapping to a convex ball with smooth strictly convex boundary under \( \text{dev} \). Then each component of \( \partial O \) can be deformed inward or outward to strictly convex boundary components by arbitrarily small amount since one can find a smooth inward variation of \( \partial O \). The hypersurface remains strictly convex for a short time. (However, we will assume mostly that \( \partial O = \emptyset \) in this paper for convenience and simplicity.)

Since this paper has many topics, we will outline this paper.

1.3. Outline.

In this paper, we will survey some results that the author obtained. There are six parts to expose the work here.

(I) The preliminary review and examples. We discuss some parts on duality and finish our work on complete R-ends (or CA-ends). (This corresponds to the present paper).

(II) We classify properly convex R-ends and T-ends when they satisfy the uniform middle eigenvalue conditions.

(III) We classify nonproperly convex but convex R-ends (NPCC-ends).

The terms will be explained in this paper. The parts will be published later in series of papers. In this paper, we will only survey (I) to (III).

Part I: In Section 2, we go over basic definitions. We discuss ends of orbifolds, convexity, the Benoist theory on convex divisible actions, and so on.
In Section 3 we discuss the dual orbifolds of a given convex real projective orbifold.

In Section 4, we will discuss the ends of orbifolds, covering most elementary aspects of the theory. For a properly convex real projective orbifold, the space of rays in each radial end give us a closed real projective orbifold of dimension $n-1$. The orbifold is convex. The universal cover can be a complete affine space or a properly convex domain or a convex domain that is not properly convex. We call the ends complete, properly convex, or nonproperly convex.

In Section 4.1, we discuss objects associated with radial ends, i.e., R-ends, and examples of ends; horospherical ones, totally geodesic ones, and bendings of ends to obtain more general examples of ends.

In Section 5, we discuss horospherical ends. First, they are complete ends and have holonomy matrices with only unit norm eigenvalues and their end fundamental groups are virtually abelian. Conversely, a complete end in a properly convex orbifold has to be a horospherical end.

We begin the part II:

In Section 6, we start to study the end theory. First, we discuss the holonomy representation spaces. Tubular actions and the dual theory of affine actions are discussed. We show that distanced actions and asymptotically nice actions are dual. We prove that the uniform middle eigenvalue condition implies the existence of the distanced action. The main result here is the characterization of R-ends of weak uniform or uniform middle eigenvalue conditions. That, they are either lens type R-ends or quasi-lens type R-ends. Here, we will classify R-ends satisfying the uniform middle eigenvalue conditions. We also define the quasi-lens type R-ends.

We go to Part III. In Section 7, we discuss the R-ends that are NPCC. First, we show that the end holonomy group for an end $E$ will have an exact sequence

$$1 \to \mathcal{N} \to h(\pi_1(\bar{E})) \to \mathcal{N}_K \to 1$$

where $\mathcal{N}_K$ is in the projective automorphism group $\text{Aut}(K)$ of a properly convex compact set $K$, $\mathcal{N}$ is the normal subgroup of elements mapping to the trivial automorphism of $K$, and $K^o/\mathcal{N}_K$
is compact. We show that $\Sigma_{\hat{E}}$ is foliated by complete affine spaces of dimension $\geq 1$. We prove that an NPCC-end satisfying a weak uniform middle eigenvalue condition is a join or a quasi-joined end.

1.4. Acknowledgements

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§2. Preliminaries

In this paper, we will be using the smooth category: that is, we will be using smooth maps and smooth charts and so on. We explain the material in the introduction again. We will establish that the universal cover $\hat{O}$ of our orbifold $O$ is a domain in $\mathbb{S}^n$ with a projective automorphism group $\Gamma \subset \text{SL}_+(n+1,\mathbb{R})$ acting on it. In this case, $O$ is projectively diffeomorphic to $\hat{O}/\Gamma$. 
2.1. Preliminary definitions.

2.2. Distances used

Definition 2.1. Let $d$ denote the standard spherical metric on $S^n$ (resp. $\mathbb{R}P^n$). Given two compact subsets $K_1$ and $K_2$ of $S^n$ (resp. $\mathbb{R}P^n$), we define the spherical Hausdorff distance $d_H(K_1, K_2)$ between $K_1$ and $K_2$ to be

$$\inf \{ \epsilon > 0 \mid K_2 \subset N_\epsilon(K_1), K_1 \subset N_\epsilon(K_2) \}.$$ 

The simple distance $d(K_1, K_2)$ is defined as

$$\inf \{ d(x, y) \mid x \in K_1, K_2 \}.$$ 

Recall that every sequence of compact sets $\{K_i\}$ in $S^n$ (resp. $\mathbb{R}P^n$) has a convergent subsequence. Also, given a sequence $\{K_i\}$ of compact sets, $\{K_i\} \to K$ for a compact set $K$ if and only if every sequence of points $x_i \in K_i$ has limit points in $K$ only and every point of $K$ has a sequence of points $x_i \in K_i$ converging to it. (These facts can be found in some topology textbooks.)

2.2.1. Topology of orbifolds and their ends. An orbifold $O$ is a topological space with charts modeling open sets by quotients of Euclidean open sets or half-open sets by finite group actions and compatibly patched with one another. The boundary $\partial O$ of an orbifold is defined as the set of points with only half-open sets as models. Orbifolds are stratified by manifolds. Let $O$ denote an $n$-dimensional orbifold with finitely many ends where end neighborhoods are homeomorphic to closed $(n - 1)$-dimensional orbifolds times an open interval. We will require that $O$ is strongly tame; that is, $O$ has a compact suborbifold $K$ so that $O - K$ is a disjoint union of end neighborhoods homeomorphic to closed $(n - 1)$-dimensional orbifolds multiplied by open intervals. Hence $\partial O$ is a compact suborbifold.

An orbifold covering map is a map so that for a given modeling open set as above, the inverse image is a union of modeling open sets that are quotients as above. We say that an orbifold is a manifold if it has a subatlas of charts with trivial local groups. We will consider good orbifolds only, i.e., covered by a simply connected manifold. In this case, the universal covering orbifold $\tilde{O}$ is a manifold with an orbifold covering map $\rho_O : \tilde{O} \to O$. The
group of deck transformations will be denote by \( \pi_1(\mathcal{O}) \) or \( \Gamma \). They act properly discontinuously on \( \hat{\mathcal{O}} \) but not necessarily freely.

By strong tameness, \( \mathcal{O} \) has only finitely many ends \( E_1, \ldots, E_m \), and each end has an end neighborhood diffeomorphic to \( \Sigma_{E_i} \times (0, 1) \). Let \( \Sigma_{E_i} \) here denote the closed orbifold diffeomorphism type of the end \( E_i \), which is uniquely determined. Such end neighborhoods of these types are said to be of the product types.

Each end neighborhood \( U \) diffeomorphic to \( \Sigma_{E_i} \times (0, 1) \) of an end \( E \) lifts to a connected open set \( \hat{U} \) in \( \hat{\mathcal{O}} \) where a subgroup of deck transformations \( \Gamma_{\hat{U}} \) acts on \( \hat{U} \) where \( p_{\hat{\mathcal{O}}}^{-1}(U) = \bigcup_{g \in \pi_1(\mathcal{O})} g(\hat{U}) \).

Here, each component of \( \hat{U} \) is said to a proper pseudo-end neighborhood.

- A pseudo-end sequence is a sequence of proper pseudo-end neighborhoods \( U_1 \supset U_2 \supset \cdots \) so that for each compact subset \( K \) of \( \hat{\mathcal{O}} \) there exists an integer \( N \) so that \( K \cap U_i = \emptyset \) for \( i > N \).
- Two pseudo-end sequences are compatible if an element of one sequence is contained eventually in the element of the other sequence.
- A compatibility class of a pseudo-end sequence is called a pseudo-end, or a p-end, of \( \hat{\mathcal{O}} \). Each of these corresponds to an end of \( \mathcal{O} \) under the universal covering map \( p_\mathcal{O} \).
- For a pseudo-end \( \hat{E} \) of \( \hat{\mathcal{O}} \), we denote by \( \Gamma_{\hat{E}} \) the subgroup \( \Gamma_{\hat{U}} \) where \( U \) and \( \hat{U} \) is as above. We call \( \Gamma_{\hat{E}} \) is called a pseudo-end fundamental group.
- A pseudo-end neighborhood \( U \) of a pseudo-end \( \hat{E} \) is a \( \Gamma_{\hat{E}} \)-invariant open set containing a proper pseudo-end neighborhood of \( \hat{E} \).

2.2.2. Real projective structures on orbifolds. Recall the real projective space \( \mathbb{RP}^n \) is defined as \( \mathbb{R}^{n+1} - \{0\} \) under the quotient relation \( \vec{v} \sim \vec{w} \) iff \( \vec{v} = s\vec{w} \) for \( s \in \mathbb{R} \setminus \{0\} \). We denote by \( [x] \) the equivalence class of a nonzero vector \( x \). The general linear group \( GL(n+1, \mathbb{R}) \) acts on \( \mathbb{R}^{n+1} \) and \( PGL(n+1, \mathbb{R}) \) acts faithfully on \( \mathbb{RP}^n \). Denote by \( \mathbb{R}_+ = \{ r \in \mathbb{R} | r > 0 \} \). The real projective sphere \( S^n \) is defined as the quotient of \( \mathbb{R}^{n+1} - \{0\} \) under the quotient relation \( \vec{v} \sim \vec{w} \) iff \( \vec{v} = s\vec{w} \) for \( s \in \mathbb{R}_+ \). We will also use \( S^n \) as the double cover of \( \mathbb{RP}^n \). Then \( \text{Aut}(S^n) \), isomorphic to the subgroup \( SL_-(n+1, \mathbb{R}) \)
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1, $\mathbb{R}$) of $\text{GL}(n+1, \mathbb{R})$ of determinant $\pm 1$, double-covers $\text{PGL}(n+1, \mathbb{R})$. $\text{Aut}(\mathbb{S}^n)$ acts as a group of projective automorphisms of $\mathbb{S}^n$. A projective map of a real projective orbifold to another is a map that is projective by charts to $\mathbb{R}P^n$. Let $\Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{R}P^n$ be a projection and let $\Pi' : \mathbb{R}^{n+1} - \{O\} \to \mathbb{S}^n$ denote one for $\mathbb{S}^n$.

An infinite subgroup $\Gamma$ of $\text{PGL}(n+1, \mathbb{R})$ (resp. $\text{SL}_\pm(n+1, \mathbb{R})$) is strongly irreducible if every finite-index subgroup is irreducible. A subspace $S$ of $\mathbb{R}P^n$ (resp. $\mathbb{S}^n$) is the image of a subspace with the origin removed under the projection $\Pi$ (resp. $\Pi'$).

A cone $C$ in $\mathbb{R}^{n+1} - \{O\}$ is a subset so that given a vector $x \in C$, $sx \in C$ for every $s \in \mathbb{R}_+$. A convex cone is a cone that is a convex subset of $\mathbb{R}^{n+1}$ in the usual sense. A proper convex cone is a convex cone not containing a complete affine line.

A line in $\mathbb{R}P^n$ or $\mathbb{S}^n$ is an embedded arc in a 1-dimensional subspace. A projective geodesic is an arc developing into a line in $\mathbb{R}P^n$ or to a one-dimensional subspace of $\mathbb{S}^n$. An affine subspace $A^n$ can be identified with the complement of a codimension-one subspace $\mathbb{R}P^{n-1}$ so that the geodesic structures are same up to parameterizations. A convex subset of $\mathbb{R}P^n$ is a convex subset of an affine subspace in this paper. A properly convex subset of $\mathbb{R}P^n$ is a precompact convex subset of an affine subspace. $\mathbb{R}^n$ identifies with an open half-space in $\mathbb{S}^n$ defined by a linear function on $\mathbb{R}^{n+1}$. (In this paper an affine space is either embedded in $\mathbb{R}P^n$ or $\mathbb{S}^n$.)

An $i$-dimensional complete affine subspace is a subset of a projective manifold projectively diffeomorphic to an $i$-dimensional affine subspace in some affine subspace $A^n$ of $\mathbb{R}P^n$ or $\mathbb{S}^n$.

Again an affine subspace in $\mathbb{S}^n$ is a lift of an affine space in $\mathbb{R}P^n$, which is the interior of an $n$-hemisphere. Convexity and proper convexity in $\mathbb{S}^n$ are defined in the same way as in $\mathbb{R}P^n$.

The complement of a codimension-one subspace $W$ in $\mathbb{R}P^n$ can be considered an affine space $A^n$ by correspondence

$$[1, x_1, \ldots, x_n] \to (x_1, \ldots, x_n)$$

for a coordinate system where $W$ is given by $x_0 = 0$. The group $\text{Aff}(A^n)$ of projective automorphisms acting on $A^n$ is identical with the group of affine transformations of form

$$\tilde{x} \mapsto A\tilde{x} + \tilde{b}$$
for a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^n$. The projective geodesics and the affine geodesics agree up to parametrizations.

A subset $A$ of $\mathbb{R}P^n$ or $S^n$ spans a subspace $S$ if $S$ is the smallest subspace containing $A$.

We will consider an orbifold $O$ with a real projective structure: This can be expressed as

- having a pair $(\text{dev}, h)$ where $\text{dev} : \tilde{O} \to \mathbb{R}P^n$ is an immersion equivariant with respect to
- the homomorphism $h : \pi_1(O) \to \text{PGL}(n + 1, \mathbb{R})$ where
  $\tilde{O}$ is the universal cover and $\pi_1(O)$ is the group of deck transformations acting on $\tilde{O}$.

$(\text{dev}, h)$ is only determined up to an action of $\text{PGL}(n + 1, \mathbb{R})$ given by

$$g \circ (\text{dev}, h(\cdot)) = (g \circ \text{dev}, gh(\cdot)g^{-1}) \text{ for } g \in \text{PGL}(n + 1, \mathbb{R}).$$

We will use only one pair where $\text{dev}$ is an embedding for this paper and hence identify $\tilde{O}$ with its image. A holonomy is an image of an element under $h$. The holonomy group is the image group $h(\pi_1(O))$.

Let $x_0, x_1, \ldots, x_n$ denote the standard coordinates of $\mathbb{R}^{n+1}$. The interior $B$ in $\mathbb{R}P^n$ or $S^n$ of a standard ball that is the image of the positive cone of $x_0^2 = x_1^2 + \cdots + x_n^2$ in $\mathbb{R}^{n+1}$ can be identified with a hyperbolic $n$-space. The group of isometries of the hyperbolic space equals the group $\text{Aut}(B)$ of projective automorphisms acting on $B$. Thus, a complete hyperbolic manifold carries a unique real projective structure and is denoted by $B/\Gamma$ for $\Gamma \subset \text{Aut}(B)$.

We also have a lift $\text{dev}' : \tilde{O} \to S^n$ and $h' : \pi_1(O) \to \text{SL}_+(n + 1, \mathbb{R})$, which are also called developing maps and holonomy homomorphisms. The discussions below apply to $\mathbb{R}P^n$ and $S^n$ equally. This pair also completely determines the real projective structure on $O$. We will use this pair as $(\text{dev}, h)$.

Fixing $\text{dev}$, we can identify $\tilde{O}$ with $\text{dev}(\tilde{O})$ in $S^n$ when $\text{dev}$ is an embedding. This identifies $\pi_1(O)$ with a group of projective automorphisms $\Gamma$ in $\text{Aut}(S^n)$. The image of $h'$ is still called a holonomy group.

An orbifold $O$ is convex (resp. properly convex and complete affine) if $\tilde{O}$ is a convex domain (resp. a properly convex domain and an affine subspace.).
A totally geodesic hypersurface \( A \) in \( \tilde{O} \) or \( O \) is a subset where each point \( p \) in \( A \) has a neighborhood \( U \) projectively diffeomorphic to an open or half-open ball where \( A \) corresponds to a subspace of codimension-one.

Given a projective structure where \( \text{dev} : \tilde{O} \to \mathbb{RP}^n \) is an embedding to a properly convex open subset as in this paper, \( \text{dev} \) lifts to an embedding \( \text{dev}' : \tilde{O} \to S^n \) to an open domain \( D \) without any pair of antipodal points. \( D \) is determined up to \( A \).

We will identify \( \tilde{O} \) with \( D \) or \( A(D) \) \( \pi_1(O) \). Then \( \Gamma \) lifts to a subgroup \( \Gamma' \) of \( \text{SL}_\pm(n + 1, \mathbb{R}) \) acting faithfully and discretely on \( \tilde{O} \). There is a unique way to lift so that \( D/\Gamma \) is projectively diffeomorphic to \( \tilde{O}/\Gamma' \). Thus, we also define the p-end vertices of p-R-ends of \( \tilde{O} \) as points in the boundary of \( \tilde{O} \) in \( S^n \) from now on. (see \[27\].)

2.3. Convexity and convex domains

A complete real line in \( \mathbb{RP}^n \) is a 1-dimensional subspace of \( \mathbb{RP}^n \) with one point removed. That is, it is the intersection of a 1-dimensional subspace by an affine space. An affine i-dimensional subspace is a submanifold of \( S^n \) or \( \mathbb{RP}^n \) projectively diffeomorphic to an i-dimensional affine subspace of a complete affine space. A convex projective geodesic is a projective geodesic in a real projective orbifold which lifts to a projective geodesic, the image of whose composition with a developing map does not contain a complete real line. A real projective orbifold is convex if every path can be homotoped to a convex projective geodesic with endpoints fixed. It is properly convex if it contains no great open segment in the orbifold.

In the double cover \( S^n \) of \( \mathbb{RP}^n \), an affine space \( A^n \) is the interior of a hemisphere. A domain in \( \mathbb{RP}^n \) or \( S^n \) is convex if it lies in some affine subspace and satisfies the convexity property above. Note that a convex domain in \( \mathbb{RP}^n \) lifts to ones in \( S^n \) up to the antipodal map \( A \). A convex domain in \( S^n \) not containing an antipodal pair maps to one in \( \mathbb{RP}^n \) homeomorphically. (Actually from now on, we will only be interested in convex domains in \( S^n \).)

**Proposition 2.1.**  
- A real projective n-orbifold is convex if and only if the developing map sends the universal cover to a convex domain in \( \mathbb{RP}^n \) (resp. \( S^n \)).
• A real projective \( n \)-orbifold is properly convex if and only if the developing map sends the universal cover to a properly convex open domain in a compact domain in an affine patch of \( \mathbb{R}P^n \) (resp. \( S^n \)).

• If a convex real projective \( n \)-orbifold is not properly convex and not complete affine, then its holonomy is virtually reducible in \( \text{PGL}(n+1, \mathbb{R}) \) (resp. \( \text{SL}_\pm(n+1, \mathbb{R}) \)). In this case, \( \tilde{O} \) is foliated by affine subspaces \( l \) of dimension \( i \) with the common boundary \( \text{Cl}(l) - l \) equal to a fixed subspace \( \mathbb{R}P^{i-1}_\infty \) (resp. \( S^{i-1}_\infty \)) in \( \text{bd} \tilde{O} \).

\[ \text{Proof.} \] The first item is Proposition A.1 of [17]. The second follows immediately. For the final item, a convex subset of \( \mathbb{R}P^n \) is a convex subset of an affine patch \( A^n \), isomorphic to an affine space. A convex open domain \( D \) in \( A^n \) that has a great open segment must contain a maximal complete affine subspace. Two such complete maximal affine subspaces do not intersect since otherwise a larger complete affine subspace of higher dimension is in \( D \) by convexity. We showed in [15] that the maximal complete affine subspaces foliated the domain. (See also [41].) The foliation is preserved under the group action since the leaves are lower-dimensional complete affine subspaces in \( D \). This implies that the boundary of the affine subspaces is a lower dimensional subspace. These subspaces are preserved under the group action. Q.E.D.

2.3.1. The Benoist theory In late 1990s, Benoist more or less completed the theory of the divisible action as started by Benzecri, Vinberg, Koszul, Vey, and so on in the series of papers [4], [5], [6], [7], [8], [9]. The comprehensive theory will aid us much in this paper.

**Proposition 2.2** (Corollary 2.13 [6]). Suppose that a discrete subgroup \( \Gamma \) of \( \text{SL}_\pm(n, \mathbb{R}) \) (resp. \( \text{PGL}(n, \mathbb{R}) \)) acts on a properly convex \( (n-1) \)-dimensional open domain \( \Omega \) in \( S^{n-1} \) (resp. \( \mathbb{R}P^{n-1} \)) so that \( \Omega/\Gamma \) is compact. Then the following statements are equivalent.

- Every subgroup of finite index of \( \Gamma \) has a finite center.
- Every subgroup of finite index of \( \Gamma \) has a trivial center.
- Every subgroup of finite index of \( \Gamma \) is irreducible in \( \text{SL}_\pm(n, \mathbb{R}) \). That is, \( \Gamma \) is strongly irreducible.
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- The Zariski closure of $\Gamma$ is semisimple.
- $\Gamma$ does not contain an infinite nilpotent normal subgroup.
- $\Gamma$ does not contain an infinite abelian normal subgroup.

Proof. Corollary 2.13 of [5] considers $\text{PGL}(n, \mathbb{R})$ and $\mathbb{R}P^{n-1}$. However, the version for $S^{n-1}$ follows from this since we can always lift a properly convex domain in $\mathbb{R}P^{n-1}$ to one $\Omega$ in $S^{n-1}$ and the group to one in $\text{SL}_\pm(n, \mathbb{R})$ acting on $\Omega$. Q.E.D.

The group with properties above is said to be the group with a trivial virtual center.

Theorem 2.3 (Theorem 1.1 of [6]). Let $n - 1 \geq 1$. Suppose that a virtual-center-free discrete subgroup $\Gamma$ of $\text{SL}_\pm(n, \mathbb{R})$ (resp. $\text{PGL}(n, \mathbb{R})$) acts on a properly convex $(n-1)$-dimensional open domain $\Omega \subset S^{n-1}$ so that $\Omega/\Gamma$ is compact. Then every representation of a component of $\text{Hom}(\Gamma, \text{SL}_\pm(n, \mathbb{R}))$ (resp. $\text{Hom}(\Gamma, \text{PGL}(n, \mathbb{R}))$) containing the inclusion representation also acts on a properly convex $(n-1)$-dimensional open domain cocompactly.

(When $\Gamma$ is a hyperbolic group and $n = 3$, Inkang Kim [57] proved this simultaneously.)

We call the group such as above theorem a vcf-group. By above Proposition 2.2, we see that every representation of the group acts irreducibly.

Proposition 2.4 (Benoist [5]). Assume $n \geq 2$. Let $\Sigma$ be a closed $(n-1)$-dimensional properly convex projective orbifold and let $\Omega$ denote its universal cover in $S^{n-1}$ (resp. $\mathbb{R}P^{n-1}$). Then

- $\Omega$ is projectively diffeomorphic to the interior of a strict join $K_1 \ast \cdots \ast K_b$ where $K_i$ is a properly convex open domain of dimension $n_i \geq 0$ in the subspace $S^{n_i}$ in $S^n$ (resp. $\mathbb{R}P^{n_i}$ in $\mathbb{R}P^n$). $K_i$ corresponds to a convex cone $C_i \subset \mathbb{R}^{n_i+1}$ for each $i$.
- $\Omega$ is the image of $C_1 \oplus \cdots \oplus C_r$.
- The fundamental group $\pi_1(\Sigma)$ is virtually isomorphic to $\mathbb{Z}^{b-1} \times \Gamma_1 \times \cdots \times \Gamma_b$ for $b - 1 + \sum n_i = n$.
- Suppose that each $\Gamma_i$ is hyperbolic or trivial. Each $\Gamma_i$ acts on $K_i$ cocompactly and the Zariski closure is the trivial group or an embedded copy of $\text{SL}(n_i + 1, \mathbb{R}), \text{SL}_\pm(n_i + 1), \text{SO}(n_i, 1)$ or $\text{O}(n_i, 1)$.
(resp. $\text{PGL}(n, \mathbb{R}), \text{PSO}(n, 1), \text{PO}(n, 1))$.

in $\text{SL}_\pm(n, \mathbb{R})$ (resp. $\text{PGL}(n, \mathbb{R})$) and acts trivially on $K_m$ for $m \neq j$.

- The subgroup corresponding to $\mathbb{Z}^{k-1}$ acts trivially on each $K_j$.

**Proof.** First consider the version for $S^n$. The first four items and the last one are from Theorem 1.1. in [5], where the work is done over $\text{GL}(n, \mathbb{R})$. However, we assume that our elements are in $\text{SL}_\pm(n, \mathbb{R})$ and by adding dilatations, we obtain the needed results. The Zariski closure part is obtained by Theorem 1.1 in [4], and Theorem 1.3 of [8].

Let $\tilde{h} : \pi_1(\Sigma_{\tilde{E}}) \to \text{SL}_\pm(n, \mathbb{R})$ be the homomorphism associated with $\tilde{E}$. The first part of the fourth item is also from Theorem 1.1 of [5]. By Theorem 1.1 of [6], the Zariski closure of $\tilde{h}(\pi_1(\tilde{E}))$ is virtually a product $\mathbb{R}^{k-1} \times G_1 \times \cdots \times G_k$ and $G_j, j = 1, \ldots, k_0$, is an irreducible reductive Lie subgroup of $\text{SL}_\pm(V_j)$. Suppose $\Gamma_i$ acts nontrivially on $C_k$ for $k \neq i$. Then elements of Zariski closures $Z_k^i$ of their images commute in $G_k$ and $G_k$ is the centralizer of products of subgroups $Z_k^i$s. Since $G_k$ is an irreducible linear algebraic subgroup as listed above, this is absurd. (We were helped by Benoist in this argument.)

The proof for the $\mathbb{R}P^n$-version follows easily again by the lifting arguments. Q.E.D.

To explain more, $K_i$ could be a point. For some $s, 1 \leq s \leq r$, we could obtain a decomposition where each $K_i$ for $i \geq s$ has dimension $\geq 2$ and $\Gamma_i$ is a hyperbolic group. Then $\Gamma$ is virtually a product of hyperbolic groups and an abelian group that is the center of the group.

§3. The duality of real projective orbifolds

The duality is a natural concept in real projective geometry and it will continue to play an important role in this theory as well.

3.1. The duality

We start from linear duality. Let $\Gamma$ be a group of linear transformations $\text{GL}(n + 1, \mathbb{R})$. Let $\Gamma^*$ be the *affine dual group* defined
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by \( \{g^* g \mid g \in \Gamma \} \). Suppose that \( \Gamma \) acts on a properly convex cone \( C \) in \( \mathbb{R}^{n+1} \) with the vertex \( O \).

An open convex cone \( C^* \) in \( \mathbb{R}^{n+1} \) is dual to an open convex cone \( C \) in \( \mathbb{R}^{n+1} \) if \( C^* \subset \mathbb{R}^{n+1} \) is the set of linear transformations taking positive values on \( \mathbb{C}(C) - \{O\} \). \( C^* \) is a cone with vertex as the origin again. Note \((C^*)^* = C\).

Now \( \Gamma^* \) will acts on \( C^* \). A central dilatational extension \( \Gamma^* \) of \( \Gamma \) by \( Z \) is given by adding a dilatation by a scalar \( s \in \mathbb{R}^+ - \{1\} \) for the set \( \mathbb{R}_+ \) of positive real numbers. The dual \( \Gamma^* \) of \( \Gamma \) is a central dilatation extension of \( \Gamma^* \). Also, if \( \Gamma^* \) is cocompact on \( C \) if and only if \( \Gamma^* \) is on \( C^* \). (See [46] for details.)

Given a subgroup \( \Gamma \) in \( \text{PGL}(n+1, \mathbb{R}) \), a lift in \( \text{GL}(n+1, \mathbb{R}) \) is any subgroup that maps to \( \Gamma \) injectively. Given a subgroup \( \Gamma \) in \( \text{PGL}(n+1, \mathbb{R}) \), the dual group \( \Gamma^* \) is the image in \( \text{PGL}(n+1, \mathbb{R}) \) of the dual of any linear lift of \( \Gamma \).

A properly convex open domain \( \Omega \) in \( \mathbb{P}(\mathbb{R}^{n+1}) \) is dual to a properly convex open domain \( \Omega^* \) in \( \mathbb{P}(\mathbb{R}^{n+1}) \) if \( \Omega \) corresponds to an open convex cone \( C \) and \( \Omega^* \) to its dual \( C^* \). We say that \( \Omega^* \) is dual to \( \Omega \). We also have \((\Omega^*)^* = \Omega \) and \( \Omega \) is properly convex if and only if so is \( \Omega^* \).

We call \( \Gamma \) a divisible group if a central dilatational extension acts cocompactly on \( C \). \( \Gamma \) is divisible if and only if so is \( \Gamma^* \).

Recall \( S^n := S(\mathbb{R}^{n+1}) \). We define \( S^n := S(\mathbb{R}^{n+1}) \).

For an open properly convex subset \( \Omega \) in \( S^n \), the dual domain is defined as the quotient of the dual cone of the cone corresponding to \( C \) in \( S^n \). The dual set is also open and properly convex but the dimension may not change. Again, we have \((\Omega^*)^* = \Omega \).

Given a properly convex domain \( \Omega \) in \( S^n \) (resp. \( \mathbb{R}P^n \)), we define the augmented boundary of \( \Omega \)

\[
(2) \quad \text{bd}^A \Omega := \{ (x, h) \mid x \in \text{bd} \Omega, x \in h, h \text{ is an oriented supporting hyperplane of } \Omega \}\subset S^n \times S^n.
\]

Each \( x \in \text{bd} \Omega \) has at least one oriented supporting hyperspace. An oriented hyperspace is an element of \( S^n \) since it is represented as a linear functional. Conversely, an element of \( S^n \) represent an oriented hyperspace in \( S^n \). (Clearly, we can do this for \( \mathbb{R}P^n \) and the dual space \( \mathbb{R}P^n \)).
Theorem 3.1. Let \( A \) be a subset \( \text{bd}\Omega \). Let \( A' := \Pi_{A^g}^{-1}(A) \) be the subset \( \text{bd}^{A^g}(A) \). Then \( \Pi_{A^g}|A' : A' \rightarrow A \) is a quasi-fibration.

Proof. We take a Euclidean metric on an affine space containing \( \text{Cl}(\Omega) \). The supporting hyperplanes can be identified with unit vectors. Each fiber \( \Pi_{A^g}^{-1}(x) \) is a properly convex compact domain in a sphere of unit vectors through \( x \). We find a continuous unit vector field to \( \text{bd}\Omega \) by taking the center of mass of each fiber with respect to the Euclidean metric. This gives a local coordinate system on each fiber by giving the origin and each fiber is a convex domain containing the origin. Then the quasi-fibration property is clear now. Q.E.D.

Remark 3.2. We notice that for open properly convex domains \( \Omega_1 \) and \( \Omega_2 \) in \( S^n \) (resp. in \( \mathbb{R}P^n \)) we have

\[
\Omega_1 \subset \Omega_2 \quad \text{if and only if} \quad \Omega_2^* \subset \Omega_1^*.
\]

Remark 3.3. Given a strict join \( A \ast B \) for a properly convex compact \( k \)-dimensional domain \( A \) and a properly convex compact \( n - k - 1 \)-dimensional domain \( B \),

\[
(A \ast B)^* = A^* \ast B^*.
\]

This follows from the definition and realizing every linear functional as a sum of linear functionals in the direct-sum subspaces.

An element \((x, h)\) is \( \text{bd}^{A^g}\Omega \) if and only if \( x \in \text{bd}\Omega \) and \( h \) is represented by a linear functional \( \alpha_h \) so that \( \alpha_h(y) > 0 \) for all \( y \) in the open cone \( C \) corresponding to \( \Omega \) and \( \alpha_h(v_x) = 0 \) for a vector \( v_x \) representing \( x \).

Since the dual cone \( C^* \) consists of all nonzero 1-form \( \alpha \) so that \( \alpha(y) > 0 \) for all \( y \in \text{Cl}(C) - \{O\} \). Thus, \( \alpha(v_x) > 0 \) for all \( \alpha \in C^* \) and \( \alpha(v_x) = 0 \). \( \alpha_h \notin C^* \) since \( v_x \in \text{Cl}(C) - \{O\} \) but \( h \in \text{bd}\Omega^* \) as we can perturb \( \alpha_h \) so that it is in \( C^* \). Thus, \( x \) is a supporting hyperspace at \( h \in \text{bd}\Omega^* \). Hence we obtain a continuous map \( D : \text{bd}^{A^g}\Omega \rightarrow \text{bd}^{A^g}\Omega^* \). We define a duality map

\[
D_\Omega : \text{bd}^{A^g}\Omega \leftrightarrow \text{bd}^{A^g}\Omega^*
\]
given by sending \((x, h)\) to \((h, x)\) for each \((x, h) \in \text{bd}^{A^g}\Omega \).

The homeomorphism below will be known as the duality map.
Proposition 3.4. Let $\Omega$ and $\Omega^*$ be dual domains in $\mathbb{R}P^n$ (resp. $\mathbb{R}P^{n*}$).

(i) There is a proper quotient map $\Pi_{Ag}: \partial \Omega \rightarrow \partial \Omega$ given by sending $(x, h)$ to $x$.

(ii) Let a projective automorphism group $\Gamma$ acts on a properly convex open domain $\Omega$ if and only if $\Gamma^*$ acts on $\Omega^*$.

(iii) There exists a duality map $D: \partial \Omega \leftrightarrow \partial \Omega^*$ is a homeomorphism.

(iv) Let $A \subset \partial \Omega$ be a subspace and $A^* \subset \partial \Omega^*$ be the corresponding dual subspace $D(A)$. If a group $\Gamma$ acts on $A$ so that $A/\Gamma$ is compact if and only if $\Gamma^*$ acts on $A^*$ and $A^*/\Gamma^*$ is compact.

Proof. We will prove for $\mathbb{R}P^n$ but the same proof works for $\mathbb{S}^n$. (i) Each fiber is a closed set of hyperplanes at a point forming a compact set. The set of supporting hyperplanes at a compact subset of $\partial \Omega$ is closed. The closed set of hyperplanes having a point in a compact subset of $\mathbb{R}P^{n+1}$ is compact. Thus, $\Pi_{Ag}$ is proper. Clearly, $\Phi_{Ag}$ is continuous, and it is an open map since $\partial \Omega$ is given the subspace topology from $\mathbb{R}P^n \times \mathbb{R}P^{n*}$ with a box topology where $\Phi_{Ag}$ extends to a projection.

(ii) Straightforward. (See Chapter 6 of [46].)

(iii) $D_D$ has the inverse map $D_D^*$. 

(iv) The item is clear from (ii) and (iii). Q.E.D.

Definition 3.1. The two subgroups $G_1$ of $\Gamma$ and $G_2$ of $\Gamma^*$ are dual if sending $g \rightarrow g^{-1,\tau}$ gives us a one-to-one map $G_1 \rightarrow G_2$. A set in $A \subset \partial \Omega$ is dual to a set $B \subset \partial \Omega^*$ if $D: \Pi_{Ag}^{-1}(A) \rightarrow \Pi_{Ag}^{-1}(B)$ is a one-to-one and onto map.

We have $O = \Omega/\Gamma$ for a properly convex domain $\Omega$, the dual orbifold $O^* = \Omega^*/\Gamma^*$ is a properly convex real projective orbifold homotopy equivalent to $O$. The dual orbifold is well-defined up to projective diffeomorphisms. We call $O^*$ a projectively dual orbifold to $O$. Clearly, $O$ is projectively dual to $O^*$.

A point of $\partial \Omega$ is strictly convex if it is not in the interior of a segment $I$ in $\partial \Omega$. 

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Theorem 3.5 (Vinberg). The dual orbifold $\mathcal{O}^*$ is diffeomorphic to $\mathcal{O}$.

Proof. Use here the duality diffeomorphism $\tilde{\mathcal{O}} \to \tilde{\mathcal{O}}^*$ of Vinberg (See [79] and Theorem 6.8 in [46].) Q.E.D.

We call the map the Vinberg duality diffeomorphism.

§4. Ends

Suppose that $\mathcal{O}$ is a strongly tame properly convex real projective orbifold with radial or totally geodesic ends and a universal cover $\tilde{\mathcal{O}}$ with compact boundary $\partial \mathcal{O}$ and some ends. (This will be the universal assumption for this paper.)

An end neighborhood system is a sequence of open sets $U_1, U_2, \ldots$ so that

- $U_i \supset U_{i+1}$ for each $i$
- each $U_i$ is a component of the complement of a compact subset in $\mathcal{O}$ so that $\text{Cl}(U_i)$ is not compact, and
- given each compact set $K$ in $\mathcal{O}$, $U_i \cap K \neq \emptyset$ for only finitely many $i$.

Two such sequences $\{U_1, U_2, \ldots\}$ and $\{U'_1, U'_2, \ldots\}$ are equivalent if

- for each $U_i$ we find $k$ so that $U'_j \subset U_i$ for $j > k$ and
- conversely for each $U'_i$ we find $k'$ such that $U_j \subset U'_i$ for $j > k'$.

An equivalence class of end neighborhoods is said to be an end of $\mathcal{O}$. A neighborhood of an end is one of the open set in the sequence in the equivalence class of the end.

A radial end neighborhood system of $\mathcal{O}$ is the union of mutually disjoint collection of end neighborhoods for all ends where each end neighborhood is of product type and is radially foliated compatibly with the product structure.

Given a component of such a system, the inverse image is a disjoint union of connected open sets. Let $\tilde{U}$ be a component. $\tilde{U}$ is also foliated by lines ending at a common vertex $v_{\tilde{U}}$. $\tilde{U}$ is said to be a proper $p$-end neighborhood. The subgroup $\Gamma_{\tilde{U}}$ acts on $\tilde{U}$ so that $\tilde{U}/\Gamma_{\tilde{U}}$ is homeomorphic to the product end neighborhood. Note that any other component $\tilde{U}'$ is of form $\gamma(\tilde{U})$ for $\gamma \in \Gamma - \Gamma_{\tilde{U}}$ and $\Gamma_{\tilde{U}'} = \gamma \Gamma_{\tilde{U}} \gamma^{-1}$ and $v_{\tilde{U}'} = \gamma(v_{\tilde{U}})$. 
By an abuse of terminology, an open set $\tilde{U}'$ containing $\tilde{U}$ as above and where $\Gamma_{\tilde{U}}$ acts on will be called a \textit{p-end neighborhood} of the p-end vertex $v_{\tilde{U}}$. $\tilde{U}'$ is not required to cover an open set in $\mathcal{O}$. Here, $\tilde{U}'$ may be not be a neighborhood in the topological sense as in the cases of horospherical ends. We call $\Gamma_{\tilde{U}}$ the p-end fundamental group. Up to the $\Gamma$-action, there are only finitely many p-end vertices and p-end fundamental groups. For an end $E$, $\Gamma_U$ is well-defined up to conjugation by $\Gamma$ and we denote it by $\Gamma_{\tilde{E}}$ often for suitable choice of $\tilde{U}$. Its conjugacy class is more appropriately denoted $\Gamma_{\tilde{E}}$.

From now on, we will use the term “p-end” instead of the term “pseudo-end” everywhere.

\textbf{Lemma 4.1.} Suppose that $\mathcal{O}$ is a strongly tame properly convex real projective orbifold with radial or totally geodesic ends and a universal cover $\tilde{\mathcal{O}}$ with $\partial \mathcal{O}$ compact. Let $U$ be an end neighborhood. Let $\tilde{U}$ be the inverse image of the union $U$ of mutually disjoint end neighborhoods. For a given component $U_1$ of $\tilde{U}$, if $\gamma(U_1) \cap U_1 \neq \emptyset$, then $\gamma(U_1) = U_1$ and $\gamma$ lies in the fundamental group $\Gamma_{E'}$ of the p-end $E'$ associated with $U_1$.

\textit{Proof.} This follows since $U_1$ covers an end neighborhood. Q.E.D.

\textbf{Lemma 4.2.} Let $U$ be a p-R-end neighborhood of a p-end vertex $v_{\tilde{E}}$ with $bd U \cap \tilde{\mathcal{O}}$ meeting each open great segment with endpoints $v_{\tilde{E}}$ and $v_{\tilde{E}}^{-}$ uniquely. Suppose that equivalently $Cl(U) \cap \tilde{\mathcal{O}}$ maps into the end neighborhood under the covering map. Then $bd U \cap \tilde{\mathcal{O}}$ covers a compact hypersurface homotopy equivalent to the end orbifold $\Sigma_{\tilde{E}}$ and its end neighborhood and $p_{\mathcal{O}}(U)$ is homeomorphic to $\Sigma_{\tilde{E}} \times \mathbb{R}$.

\textit{Proof.} Let $V'$ be the end neighborhood of $\mathcal{O}$ into which $bd U \cap \tilde{\mathcal{O}}$ or $Cl(U) \cap \tilde{\mathcal{O}}$ maps. Then for a component of the inverse image $V$ of $V'$, we have $U \subset V'$. Since $\Gamma_{\tilde{E}}$ is precisely the set of deck-transformations acting on $V$, $U$ covers $p_{\mathcal{O}}(U)$ in $V'$ with the deck transformation group $\Gamma_{\tilde{E}}$. Also, $bd U \cap \tilde{\mathcal{O}}$ covers the boundary of $p_{\mathcal{O}}(U)$ in $V'$, and hence is a compact hypersurface. Since $V$ is homeomorphic to $\Sigma_{\tilde{E}} \times \mathbb{R}$, the result follows. Q.E.D.
Proposition 4.3. Suppose that \( \mathcal{O} \) is a strongly tame properly convex real projective orbifold with radial or totally geodesic ends, and its developing map sends the universal cover \( \hat{\mathcal{O}} \) to a convex domain. Let \( U' \) be an end neighborhood in \( \mathcal{O} \). Let \( \hat{U} \) be \( p^{-1}(U') \) as above with \( E' \) the p-end in \( \hat{\mathcal{O}} \) associated with a component \( U \) of \( \hat{U} \). Then

- the closure of each component of \( \hat{U} \) contains the p-end vertex \( v_{E'} \) of the leaf of radial foliation in \( \hat{U} \), lifted from \( U \).
- There exists a unique one for each component \( U_1 \) of \( \hat{U} \) associated with a p-R-end \( E' \) of \( \hat{\mathcal{O}} \).
- The subgroup of \( h(\pi_1(\mathcal{O})) \) acting on \( U_1 \) or fixing the p-end vertex \( v_{E'} \) is precisely in the subgroup \( \Gamma_{E'} \).

Definition 4.1. Given a subset \( K \) of a convex domain \( \Omega \) of an affine space \( A^n \) in \( S^n \) (resp. \( \mathbb{R}P^n \)), the convex hull of \( K \) is defined as the smallest convex set containing \( K \) in \( \text{Cl}(\Omega) \subset A^n \) where we required \( \text{Cl}(\Omega) \subset A^n \). The convex hull is well-defined as long as \( \Omega \) is properly convex. Otherwise, it may be not. Often when \( \Omega \) is a properly convex domain, we will take the closure \( \text{Cl}(\Omega) \) instead of \( \Omega \) usually. This does not change the convex hull. (Usually it will be clear what \( \Omega \) is by context but we will mention these.) For \( \mathbb{R}P^n \), the convex hull depends on \( \Omega \) but one can check that the convex hull is well-defined on \( S^n \) as long as \( \Omega \) is properly convex.

We will show later that the p-end neighborhood can be chosen to be properly convex by taking the convex hull of a well-chosen p-end neighborhood in \( \hat{\mathcal{O}} \). However, there is no guarantee that the images of convex ones are disjoint.

Radial ends: We will assume that our real projective orbifold \( \mathcal{O} \) is a strongly tame orbifold and some of the ends are radial. This means that each end has a neighborhood \( U \), and each component \( \hat{U} \) of the inverse image \( p^{-1}(U) \) has a foliation by properly embedded projective geodesics ending at a common point \( v_{U} \in \mathbb{R}P^n \). We call such a point a pseudo-end vertex.
The classification of ends

• The space of directions of oriented projective geodesics through $v_{\tilde{E}}$ forms an $(n-1)$-dimensional real projective space. We denote it by $S^{n-1}_{v_{\tilde{E}}}$ and call it a linking sphere.

• Let $\tilde{\Sigma}_{\tilde{E}}$ denote the space of equivalence classes of lines from $v_{\tilde{E}}$ in $\tilde{U}$ where two lines are regarded equivalent if they are identical near $v_{\tilde{E}}$. $\tilde{\Sigma}_{\tilde{E}}$ projects to a convex open domain in an affine space in $S^{n-1}_{v_{\tilde{E}}}$ by the convexity of $\tilde{O}$. Then by Proposition 2.1 $\tilde{\Sigma}_{\tilde{E}}$ is projectively diffeomorphic to
  - either a complex affine space $A^{n-1}$,
  - a properly convex domain,
  - or a convex but not properly convex and not complete affine domain in $A^{n-1}$.

• The subgroup $\Gamma_{\tilde{E}}$, a pseudo-end fundamental group, of $\Gamma$ fixes $v_{\tilde{E}}$ and acts on as a projective automorphism group on $S^{n-1}_{v_{\tilde{E}}}$. Thus, the quotient $\tilde{\Sigma}_{\tilde{E}}/\Gamma_{\tilde{E}}$ admits a real projective structure of one dimension lower.

• We denote by $\Sigma_{\tilde{E}}$ the real projective $(n-1)$-orbifold $\tilde{\Sigma}_{\tilde{E}}/\Gamma_{\tilde{E}}$. Since we can find a transversal orbifold $\tilde{\Sigma}_{\tilde{E}}$ to the radial foliation in a pseudo-end neighborhood for each pseudo-end $\tilde{E}$ of $O$, it lifts to a transversal surface $\tilde{\Sigma}_{\tilde{E}}$ in $\tilde{U}$.

• We say that a radial pseudo-end $\tilde{E}$ is convex (resp. properly convex, and complete affine) if $\tilde{\Sigma}_{\tilde{E}}$ is convex (resp. properly convex, and complete affine).

Thus, a radial end is either

CA: complete affine,

PC: properly convex, or

NPCC: convex but not properly convex and not complete affine.

Totally geodesic ends: An end is totally geodesic if an end neighborhood $U$ compactifies to an orbifold with boundary in an ambient orbifold by adding a totally geodesic suborbifold $\Sigma_{E}$ homeomorphic to $\Sigma_{E} \times I$ for an interval $I$. The choice of the compactification is said to
be the *totally geodesic end structure*. Two compactifications are equivalent if some respective neighborhoods are projectively diffeomorphic. (One can see in [20] two inequivalent ways to compactify for real projective elementary annulus.) If $\Sigma$ is properly convex, then the end is said to be *properly convex*.

Note that the diffeomorphism types of end orbifolds are determined for radial or totally geodesic ends. We will now say that a radial end is a R-end and a totally geodesic end is a T-end.

**4.0.1. Horospherical domains, lens domains, lens-cones, and so on.** If $A$ is a domain of subspace of $\mathbb{RP}^n$ or $S^n$, we denote by $\text{bd} A$ the topological boundary in the subspace. The closure $\text{Cl}(A)$ of a subset $A$ of $\mathbb{RP}^n$ or $S^n$ is the topological closure in $\mathbb{RP}^n$ or in $S^n$. Define $\partial A$ for a manifold or orbifold $A$ to be the manifold or orbifold boundary. Also, $A^\circ$ will denote the manifold or orbifold interior of $A$.

**Definition 4.2.** Given a convex set $D$ in $\mathbb{RP}^n$, we obtain a connected cone $C_D$ in $\mathbb{R}^{n+1} - \{O\}$ mapping to $D$, determined up to the antipodal map. For a convex domain $D \subset S^n$, we have a unique domain $C_D \subset \mathbb{R}^{n+1} - \{O\}$.

A join of two properly convex subsets $A$ and $B$ in a convex domain $D$ of $\mathbb{RP}^n$ or $S^n$ is defined

$$A \ast B := \{ [tx + (1-t)y] | x, y \in C_D, [x] \in A, [y] \in B, t \in [0,1] \}$$

where $C_D$ is a cone corresponding to $D$ in $\mathbb{R}^{n+1}$. The definition is independent of the choice of $C_D$.

**Definition 4.3.** Let $C_1, \ldots, C_m$ be cone respectively in a set of independent vector subspaces $V_1, \ldots, V_m$ of $\mathbb{R}^{n+1}$. In general, a sum of convex sets $C_1, \ldots, C_m$ in $\mathbb{R}^{n+1}$ in independent subspaces $V_i$, we define

$$C_1 + \cdots + C_m := \{ v | v = c_1 + \cdots + c_m, c_i \in C_i \}.$$ 

A strict join of convex sets $\Omega_i$ in $S^n$ (resp. in $\mathbb{RP}^n$) is given as

$$\Omega_1 \ast \cdots \ast \Omega_m := \cap (C_1 + \cdots + C_m) \ (\text{resp. } \cap'(C_1 + \cdots + C_m))$$

where each $C_i - \{O\}$ is a convex cone with image $\Omega_i$ for each $i$. 
A segment is a properly convex subset of a 1-dimensional subspace of $\mathbb{RP}^n$ or $\mathbb{S}^n$. We will denote by $[x,y]$ if $x$ and $y$ are endpoints. It is uniquely determined by $x$ and $y$. In the following, all the sets are required to be inside an affine subspace $A^n$ and its closure either in $\mathbb{RP}^n$ or $\mathbb{S}^n$.

- A subdomain and a submanifold $K$ of $A^n$ is said to be a horoball if it is strictly convex, and the boundary $\partial K$ is diffeomorphic to $\mathbb{R}^{n-1}$ and $\text{bd} K - \partial K$ is a single point. The boundary $\partial K$ is said to be a horosphere.
- $K$ is lens-shaped if it is a convex domain and $\partial K$ is a disjoint union of two smoothly strictly convex embedded $(n-1)$-cells $\partial_+ K$ and $\partial_- K$.
- A cone is a bounded domain $D$ in an affine patch with a point in the boundary, called an end vertex $v$ so that every other point $x$ has an open segment $[vx]$ in $D$.
- A cone $\{p\} \ast L$ over a lens-shaped domain $L$ in $A^n$, $p \notin \text{Cl}(L)$ is a convex domain so that $\{p\} \ast L = \{p\} \ast \partial_+ L$ for one boundary component $\partial_+ L$ of $L$. A lens is the lens-shaped domain $L$, not determined uniquely by the lens-cone itself.
- We can allow $L$ to have non-smooth boundary that lies in the boundary of $p \ast L$.
  - Each of two boundary components of $L$ is called a top or bottom hypersurfaces depending on whether it is further away from $p$ or not. The top component is denoted by $\partial_+ L$ which can be not smooth. $\partial_- L$ is required to be smooth.
  - A cone over $L$ where $\partial(\{p\} \ast L - \{p\}) = \partial_+ L, p \notin \text{Cl}(L)$ is said to be a generalized lens-cone and $L$ is said to be a generalized lens.
- A quasi-lens cone is a properly convex cone of form $p \ast S$ for a strictly convex open hypersurface $S$ so that

$$\partial(\{p\} \ast S - \{p\}) = S, p \in \text{Cl}(S) - S$$

hold and the space of directions from $p$ to $S$ is a properly convex domain in $\mathbb{S}^{n-1}$. 

(The join above does depend on the choice of cones.)
• A totally-geodesic domain is a convex domain in a hyperspace. A cone-over a totally-geodesic domain $D$ is a union of all segments with one endpoint a point $x$ not in the hyperspace and the other in $D$. We denote it by $\{x\} \ast D$.

Let the radial pseudo-end $\tilde{E}$ have a pseudo-end neighborhood of form $\{p\} \ast L - \{p\}$ that is a generalized lens-cone $p \ast L$ over a generalized lens $L$ where $\partial(p \ast L - \{p\}) = \partial_+ L$ and $\Gamma_{\tilde{E}}$ acts on $L$. A concave pseudo-end neighborhood of $\tilde{E}$ is the open pseudo-end neighborhood in $\tilde{O}$ contained in a radial pseudo-end neighborhood in $\tilde{O}$ that is a component of $\{p\} \ast L - \{p\} - L$ containing $p$ in the boundary. As it is defined, such a pseudo-end neighborhood always exists for a generalized lens pseudo-end.

From now on, we will replace the term “pseudo-end” with “p-end” everywhere.

**Horospherical R-end:** A pseudo-R-end of $\tilde{O}$ is horospherical if it has a horoball in $\tilde{O}$ as a pseudo-end neighborhood, or equivalently an open pseudo-end neighborhood $U$ in $\tilde{O}$ so that $\text{bd} U \cap \tilde{O} = \text{bd} U - \{v\}$ for a boundary fixed point $v$ where the p-end fundamental group properly discontinuously on.

**Lens-shaped R-end:** An R-end $\tilde{E}$ is lens-shaped (resp. totally geodesic cone-shaped, generalized lens-shaped, quasi-lens shaped) if it has a pseudo-end neighborhood that is a lens-cone (resp. a cone over a totally-geodesic domain, a concave pseudo-end neighborhood, or a quasi-lens cone.) Here we require that the pseudo-end fundamental group $\Gamma_{\tilde{E}}$ acts on the lens of the lens-cone.

**Lens-shaped T-end:** A pseudo-T-end of $\tilde{O}$ is of lens-type if it has a lens p-end neighborhood in an ambient orbifold of $\tilde{O}$. A T-end of $\tilde{O}$ is of lens-type if the corresponding pseudo-end is of lens-type.

### 4.1. Examples of ends

We will present some examples here, which we will fully justify later.
The classification of ends

We remark that for the orbifold with the infinite-index end fundamental group condition, the p-end vertices are infinitely many for each equivalence class of vertices.

4.1.1. Examples From hyperbolic manifolds, we obtain some examples of ends. Let \( M \) be a complete hyperbolic manifold with cusps. \( M \) is a quotient space of the interior \( \Omega \) of an ellipsoid in \( \mathbb{RP}^n \) or \( S^n \) under the action of a discrete subgroup \( \Gamma \) of \( \text{Aut}(\Omega) \). Then horoballs are p-end neighborhoods of the horospherical ends.

Suppose that \( M \) has totally geodesic embedded surfaces \( S_1, \ldots, S_m \) homotopic to the ends.

- We remove the outside of \( S_j \)s to obtain a properly convex real projective orbifold \( M' \) with totally geodesic boundary.
- Each \( S_i \) corresponds to a disjoint union of totally geodesic domains \( \bigcup_{j \in J} \tilde{S}_{i,j} \) for a collection \( J \). For each of which \( \tilde{S}_{i,j} \subset \Omega \), a group \( \Gamma_{i,j} \) acts on it where \( \tilde{S}_{i,j}/\Gamma_{i,j} \) is a closed orbifold projectively diffeomorphic to \( S_i \).
- Then \( \Gamma_{i,j} \) fixes a point \( p_{i,j} \) outside the ellipsoid by taking the dual point of \( \tilde{S}_{i,j} \) outside the ellipsoid.
- Hence, we form the cone \( M_{i,j} := \{ p_{i,j} \} \ast \tilde{S}_{i,j} \).
- We obtain the quotient \( M_{i,j}'/\Gamma_{i,j} \) of the interior and attach to \( M' \) to obtain the examples of real projective manifolds with radial ends.

This orbifold is called the hyper-ideal extension of the hyperbolic manifolds as real projective manifolds.

**Proposition 4.4.** Suppose that \( M \) is a strongly tame properly convex real projective orbifold with radial or totally geodesic ends. Suppose that

- the holonomy group of each end fundamental group is generated by the homotopy classes of closed curves about singularities or
- has the holonomy fixing the end vertex with eigenvalues 1 and
- an \( R \)-end \( E \) has a compact totally geodesic properly convex hyperspace in a p-end neighborhood and not containing the p-end vertex.

Then the end \( E \) is of lens-type.
Proof. Let $\tilde{M}$ be the universal cover of $M$ in $\mathbb{S}^n$. It will be sufficient to prove for this case. Let $E$ be an R-end of $M$ with a compact totally geodesic subspace $\Sigma$ in a p-end neighborhood. Then a p-end neighborhood $U$ of $\tilde{E}$ corresponding to $E$ contains the universal cover $\tilde{\Sigma}$ of $\Sigma$.

Since the end fundamental group $\Gamma_{\tilde{E}}$ is generated by closed curves about singularities, and the singularities are of finite order, the eigenvalues of the generators corresponding to the p-end vertex $v_{\tilde{E}}$ equal 1 and hence every element of the end fundamental group has 1 as the eigenvalue at $v_{\tilde{E}}$. Now assume that the holonomy of the elements of the end fundamental group, fixes the p-end vertex with eigenvalues equal to 1.

Then $U$ can be chosen to be the open cone over the totally geodesic domain with vertex $v_{\tilde{E}}$. $U$ is projectively diffeomorphic to the interior of a properly convex cone in an affine subspace $A^n$. The end fundamental group acts on $U$ as a discrete linear group of determinant 1. The theory of convex cones applies, and using the level sets of the Koszul-Vinberg function, we obtain a smooth convex one-sided neighborhood in $U$ (see Lemmas 6.5 and 6.6 of Goldman [46]). Also, the outer one-sided neighborhood can be obtained by a reflection about the plane containing $\tilde{\Sigma}$ and the p-end vertex and some dilatation action so that it is in $\tilde{\mathcal{O}}$. Q.E.D.

Let $S_{3,3,3}$ denote the 2-orbifold with base space homeomorphic to a 2-sphere and three cone-points of order 3.

Proposition 4.5. Let $\mathcal{O}$ be a convex real projective 3-orbifold with radial ends where each end orbifold is homeomorphic to a sphere $S_{3,3,3}$ or a disk with three silvered edges and three vertices of orders 3, 3, 3. Then the orbifold has only lens-shaped R-ends or horospherical R-ends.

Proof. Again, it is sufficient to prove this for the case $\tilde{\mathcal{O}} \subset S^n$. Let $\tilde{E}$ be a p-R-end of type $S_{3,3,3}$ for $\tilde{\mathcal{O}}$. It is sufficient to consider only $S_{3,3,3}$ since it double-covers the disk orbifold. Since $\Gamma_{\tilde{E}}$ is generated by finite order elements fixing a p-end vertex $v_{\tilde{E}}$, every holonomy element has eigenvalue equal to 1 at $v_{\tilde{E}}$. Take a finite-index free abelian group $A$ of rank two. Since $\Sigma_{\tilde{E}}$ is convex, a convex projective torus $T^2$ covers $\Sigma_{\tilde{E}}$ finitely. Therefore, $\Sigma_{\tilde{E}}$ is projectively diffeomorphic either to
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- a complete affine space or
- the interior of a properly convex triangle or
- a half-space

by the classification of convex tori found in many places including [46] and [10] and Proposition 2.1. Since there exists a holonomy automorphism of order 3 fixing a point of \( \tilde{\Sigma}_E \), it cannot be a quotient of a half-space with a distinguished foliation by lines. Thus, the end orbifold admits a complete affine structure or is a quotient of a properly convex triangle.

If \( \Sigma_{\tilde{E}} \) has a complete affine structure, we have a horospherical end for \( E \) by Theorem 5.3. Suppose that \( \Sigma_{\tilde{E}} \) has a properly convex triangle as its universal cover. \( A \) acts with an element \( g' \) with an eigenvalue > 1 and an eigenvalue < 1 as a transformation in \( \text{SL}_\pm(3, \mathbb{R}) \) the group of projective automorphisms at \( S^2 \). \( g' \) fixes \( v_1 \) and \( v_2 \) other than \( v_{\tilde{E}} \) in directions of the vertices of the triangle in the cone. Since the corresponding eigenvalue at \( v_{\tilde{E}} \) is 1 and \( g' \) acts on a properly convex compact domain, \( g' \) has four fixed points and an invariant subspace \( P \) disjoint from \( v_{\tilde{E}} \). That is, \( g' \) is diagonalizable. Since elements of \( A \) commute with \( g' \), so does every other \( g \in A \). The end fundamental group acts on \( P \) as well. We have a totally geodesic R-end and by Proposition 4.4, the end is lens-shaped. (See also [25].) Q.E.D.

Example 4.6 (Lee’s example). Consider the Coxeter orbifold \( \hat{P} \) with the underlying space on a polyhedron \( P \) with the combinatorics of a cube with all sides mirrored and all edges given order 3 but vertices removed. By the Mostow-Prasad rigidity and the Andreev theorem, the orbifold has a unique complete hyperbolic structure. There exists a six-dimensional space of real projective structures on it as found in [33] where one has a projectively fixed fundamental domain in the universal cover.

There are eight ideal vertices of \( P \) corresponding to eight ends of \( \hat{H} \). Each end orbifold is a 2-orbifold based on a triangle with edges mirrored and vertex orders are all 3. Thus, each end has a neighborhood homeomorphic to the 2-orbifold multiplied by \((0,1)\). We can characterize them by a real-valued invariant. Their invariants are related when we are working on the restricted deformation space. (They might be independent in the full deformation space as M. Davis and R. Green observed.)
Then this end can be horospherical or be a radial lens type with a totally geodesic realization end orbifold by Proposition 4.5. When \( P \) is hyperbolic, the ends are horospherical as \( P \) has a complete hyperbolic structure. Numerical experimentations also suggest that we realize totally geodesic \( R \)-ends after deformations.

The following construction is called “bending” and was investigated by Johnson and Millson [54].

**Example 4.7.** Let \( \mathcal{O} \) have the usual assumptions. Let \( E \) be a totally geodesic \( R \)-end with a \( p \)-\( R \)-end \( \tilde{E} \). Let the associated orbifold \( \Sigma_E \) for \( E \) of \( \mathcal{O} \) be a closed 2-orbifold and let \( c \) be a simple closed geodesic in \( \Sigma_{\tilde{E}} \). Suppose that \( E \) has an end neighborhood \( U \) in \( \mathcal{O} \) diffeomorphic to \( \Sigma_E \times (0, 1) \) with totally geodesic \( \partial U \) diffeomorphic to \( \Sigma_E \). Let \( \tilde{U} \) be a \( p \)-end neighborhood in \( \tilde{\mathcal{O}} \) corresponding to \( \tilde{E} \) bounded by \( \tilde{\Sigma}_{\tilde{E}} \) covering \( \Sigma_E \).

Now a lift \( \tilde{c} \) of \( c \) is in an embedded disk \( A' \), covering an annulus \( A \) diffeomorphic to \( c \times (0, 1) \), foliated by lines from \( \mathbf{v}_{\tilde{E}} \). Let \( g_c \) be the deck transformation corresponding to \( \tilde{c} \) and \( c \). Then the holonomy \( g_c \) is conjugate to a diagonal matrix with entries \( \lambda, \lambda^{-1}, 1, 1 \), where \( \lambda > 1 \) and the last 1 corresponds to the vertex \( \mathbf{v} \). We take an element \( k_b \) of \( \text{SL}_+(4, \mathbb{R}) \) of form in this system of coordinates

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & b & 1
\end{pmatrix}
\]

where \( b \in \mathbb{R} \). \( k_b \) commutes with \( g_c \). Let us just work on the end \( E \). We can “bend” \( E \) by \( k_b \):

Then we obtain two copies \( A_1 \) and \( A_2 \) of \( A \) by completing two components of \( U - A \). We take an ambient real projective manifold \( U' \) containing the completion. We can find neighborhoods \( N_1 \) and \( N_2 \) of \( A_1 \) and \( A_2 \) in \( U \) diffeomorphic by a projective map \( \hat{k}_b \) induced by \( k_b \).

We take a disjoint union \( (U - A) \amalg N_1 \amalg N_2 \) and quotient it by identifying elements of \( N_1 \) with elements near \( A_1 \) in \( U - A \) by the identity map and elements of \( N_2 \) with elements near \( A_2 \) in \( U - A \) by the identity also. We then glue back \( N_1 \) and \( N_2 \) by \( k_b \) the real projective diffeomorphism of a neighborhood of \( N_1 \) to that of \( N_2 \).
For sufficiently small $b$, we see that the end is still of lens type and it is not a totally geodesic $R$-end. (This follows since the condition of being a generalized-lens type $R$-end is an open condition. See [25].)

Since $k_b$ fixes a subspace of dimension 2 containing $v_E$ and the geodesic fixed by $g_c$, the totally geodesic subspace is bent. We see that $b > 0$, and we obtain a boundary of an end neighborhood of $E$ bent in a positive manner. The deformed holonomy group acts on a convex domain obtained by bendings of these types everywhere.

For the same $c$, let $k_s$ be given by

\[
\begin{pmatrix}
  s & 0 & 0 & 0 \\
  0 & s & 0 & 0 \\
  0 & 0 & s & 0 \\
  0 & 0 & 0 & 1/s^3
\end{pmatrix}
\]

where $s \in \mathbb{R}_+$. These give us bendings of the second type. (We talked about this in [27].) For $s$ sufficiently close to 1, the property of being of lens-type is preserved and being a radial totally geodesic end. (However, these will be understood by cohomology.)

If $s\lambda < 1$ for the maximal eigenvalue $\lambda$ of a closed curve $c_1$ meeting $c$ odd number of times, we have that the holonomy along $c_1$ has the attacking fixed point at $v_E$. This implies that we no longer have lens-type ends if we have started with a lens-shaped end.

§5. Characterization of complete ends

An ellipsoid is a subset in an affine space defined as a zero locus of a positive definite quadratic polynomial in term of the affine coordinates. Let $E$ be an $i_0$-dimensional ellipsoid $E$ containing the point $v$. A projective conjugate $H_v$ of a parabolic subgroup of $SO(i_0 + 1, 1)$ acting cocompactly on $E - \{v\}$ is called an $i_0$-dimensional cusp group. If the horospherical neighborhood with the $p$-R-end vertex $v$ has the $p$-end fundamental group that is a discrete cocompact subgroup in $H_v$, then we call the $p$-R-end to be of cusp type.

Our first main result classifies CA $p$-R-ends.

**Theorem 5.1.** Let $O$ be a properly convex real projective $n$-orbifold with radial or totally geodesic ends and satisfy (IE). Let
\( \tilde{E} \) be a \( p\)-\( R \)-end of its universal cover \( \tilde{O} \). Then \( \tilde{E} \) is a complete affine \( p\)-\( R \)-end if and only if \( \tilde{E} \) is a cusp \( p\)-\( R \)-end.

**Proof.** Theorem 5.3 implies that a complete end is of cusp type. Since a cusp end is horospherical, Proposition 5.2 implies the converse. Q.E.D.

By an exiting sequence of \( p\)-end neighborhoods \( U_i \) of \( \tilde{O} \), we mean a sequence of \( p\)-end neighborhoods \( \{U_i\} \) so that \( U_i \cap K \neq \emptyset \) for only finitely many \( i \) is for each compact subset \( K \) of \( \tilde{O} \).

By the following proposition shows that we can exchange words “horospherical” with “cuspidal”.

**Proposition 5.2.** Let \( O \) be a properly convex real projective \( n \)-orbifold with radial or totally geodesic ends and satisfy (IE). Let \( \tilde{E} \) be a horospherical end of its universal cover \( \tilde{O} \), \( \tilde{O} \subset S^n \) (resp. \( \subset \mathbb{R}P^n \)) and \( \Gamma_{\tilde{E}} \) denote the \( p\)-end fundamental group.

(i) The space \( \tilde{\Sigma}_{\tilde{E}} := R_{v_\tilde{E}}(\tilde{O}) \) of lines from the endpoint \( v_\tilde{E} \) forms a complete affine space of dimension \( n - 1 \).

(ii) The norms of eigenvalues of \( g \in \Gamma_{\tilde{E}} \) are all 1.

(iii) \( \Gamma_{\tilde{E}} \) is virtually abelian and a finite index subgroup is in a conjugate of a connected Borel subgroup of \( \text{SO}(n, 1) \) of rank \( n - 1 \) in \( \text{SL}_\pm(n + 1, \mathbb{R}) \) or \( \text{PGL}(n + 1, \mathbb{R}) \). And hence \( \tilde{E} \) has a cusp-type.

(iv) For any compact set \( K' \) inside a horospherical end neighborhood, \( O \) contains a smooth convex smooth neighborhood disjoint from \( K' \).

(v) A \( p\)-end vertex of a horospherical \( p\)-end cannot be an endpoint of a segment in \( \text{bd}\tilde{O} \).

**Proof.** We will prove for the case \( \tilde{O} \subset S^n \). The \( \mathbb{R}P^n \)-version follows from this. Let \( U \) be a horospherical \( p\)-end neighborhood with the \( p\)-end vertex \( v_\tilde{E} \). The space of great segments from the \( p\)-end vertex passing \( U \) forms a convex subset \( \tilde{\Sigma}_{\tilde{E}} \) of a complete affine space \( \mathbb{R}^{n-1} \subset S_{E}^{n-1} \) by Proposition 2.1 and covers an end orbifold \( \Sigma_{\tilde{E}} \) with the discrete group \( \pi_1(\tilde{E}) \) acting as a discrete subgroup \( \Gamma_{\tilde{E}} \) of the projective automorphisms so that \( \tilde{\Sigma}_{\tilde{E}}/\Gamma_{\tilde{E}} \) is projectively isomorphic to \( \Sigma_{\tilde{E}} \).

(i) By Proposition 2.1,
• \( \tilde{\Sigma}_E \) is properly convex,
• is foliated by complete affine spaces of dimension \( i_0 \) with the common boundary sphere of dimension \( i_0 - 1 \) and the space of the leaves forms a properly open convex subset \( K^o \) of \( \mathbb{S}^{n-i_0-1} \) or
• is a complete affine space.

Then \( \Gamma_{\tilde{E}} \) acts on \( K^o \) cocompactly but perhaps not discretely. We aim to show that the first two cases do not occur.

Suppose that we are in the second case and \( 1 \leq i_0 \leq n-2 \). This implies that \( \tilde{\Sigma}_E \) is foliated by complete affine spaces of dimension \( i_0 \leq n-2 \).

For each element \( g \) of \( \Gamma_{\tilde{E}} \), a complex or negative eigenvalue of \( g \) in \( \mathbb{C} - \mathbb{R}_+ \) cannot have a maximal or minimal absolute value different from 1: Otherwise by taking the convex hull in \( \tilde{O} \) of \( \{g^m(x)\mid m \in \mathbb{Z}\} \) for a generic point \( x \) of \( U \), we see that \( U \) must be not properly convex. Thus, the largest and the smallest absolute value eigenvalues of \( g \) are positive.

Since \( \Gamma_{\tilde{E}} \) acts on a properly convex subset \( K \) of dimension \( \geq 1 \), an element \( g \) has an eigenvalue > 1 and an eigenvalue < 1 by Benoist [4] as an element of projective automorphism on the great sphere spanned by \( K \). Hence, we obtain the largest norm of eigenvalues and the smallest one of \( g \) in \( \text{Aut}(\mathbb{S}^o) \) both different from 1. Therefore, let \( \lambda_1 > 1 \) be the greatest norm eigenvalue and \( \lambda_2 < 1 \) be the smallest norm one of this element \( g \). Let \( \lambda_{v_{\tilde{E}}} > 0 \) be the eigenvalue of \( g \) associated with \( v_{\tilde{E}} \). These are all positive. The possibilities are as follows

\[
\lambda_1 = \lambda_{v_{\tilde{E}}} > \lambda_2, \quad \lambda_1 > \lambda_{v_{\tilde{E}}} > \lambda_2, \quad \lambda_1 > \lambda_2 = \lambda_{v_{\tilde{E}}}. 
\]

In all cases, at least one of the largest norm or the smallest norm is different from \( \lambda_1 \). Thus \( g \) fixes a point \( x_\infty \) distinct from \( v_{\tilde{E}} \) with the distinct eigenvalue from \( \lambda_0 \). We may assume without loss of generality \( x_\infty \in \text{Cl}(U) \) since \( x_\infty \) is a limit of \( g^i(x) \) for generic point \( x \in U, i \to \infty \) or \( i \to -\infty \). As \( x_\infty \notin U \), we obtain \( x_\infty = v_{\tilde{E}} \) by the definition of the horoballs. This is a contradiction.

The first possibility is also shown to not occur similarly. Thus, \( \tilde{\Sigma}_E \) is a complete affine space.

(ii) If \( g \in \Gamma_{\tilde{E}} \) has a norm of eigenvalue different from 1, then we can apply the second and the third paragraphs above to obtain
a contradiction. We obtain $\lambda_j = 1$ for each norm $\lambda_j$ of eigenvalues of $g$ for every $g \in \Gamma_{\tilde{E}}$.

(iii) Since $\tilde{\Sigma}_{\tilde{E}}$ is a complete affine space, $\tilde{\Sigma}_{\tilde{E}}/\Gamma_{\tilde{E}}$ is a complete affine manifold with the norms of eigenvalues holonomy matrices all equal to 1 where $\Gamma_{\tilde{E}}$ denotes the affine transformation group corresponding to $\Gamma_{\tilde{E}}$. (By D. Fried [42], this implies that $\pi_1(\tilde{E})$ is virtually nilpotent.) The conclusion follows by Proposition 7.21 of [40] (related to Theorem 1.6 of [40]): By the theorem, we see that $\Gamma_{\tilde{E}}$ is in a conjugate of $SO(n,1)$ and hence acts on an $(n-1)$-dimensional ellipsoid fixing a unique point. Since a horosphere has a Euclidean metric invariant under the group action, the image group is in a Euclidean isometry group. Hence, the group is virtually abelian by the Bieberbach theorem.

(iv) We can choose an exiting sequence of p-end horoball neighborhoods $U_i$ where a cusp group acts. We can consider the hyperbolic spaces to understand this.

(v) Suppose that $bd\tilde{O}$ contains a segment $s$ ending at the p-end vertex $v_{\tilde{E}}$. Then $s$ is on an invariant hyperspace of $\Gamma_{\tilde{E}}$. Now conjugating $\Gamma_{\tilde{E}}$ into a parabolic group of $SO(n,1)$ fixing $(1,-1,0,...,0)$. By simple computations, we can find a sequence $g_i \in \Gamma_{\tilde{E}}$ so that $\{g_i(s)\}$ geometrically converges to a great segment. This contradicts the proper convexity of $\tilde{O}$. Q.E.D.

We will now show the converse of Proposition 5.2.

The results here overlap with the results of Crampon-Marquis [40] and Cooper-Long-Tillman [37]. However, the results are more general than theirs and were originally conceived before their papers appeared. We also make use of Crampon-Marquis [40].

**Theorem 5.3.** Let $O$ be a properly convex $n$-orbifold with radial or totally geodesic ends and satisfy (IE). Suppose that $\tilde{E}$ is a complete R-end of its universal cover $\tilde{O}$ in $\mathbb{S}^n$ or in $\mathbb{R}P^n$. Let $v_{\tilde{E}} \in \mathbb{S}^n$ be the p-end vertex with the p-end fundamental group $\Gamma_{\tilde{E}}$. Then

(i) The eigenvalues of elements of $\Gamma_{\tilde{E}}$ have unit norms only.
(ii) A nilpotent Lie group fixing $v_{\tilde{E}}$ contains a finite index subgroup of $\Gamma_{\tilde{E}}$.
(iii) $\tilde{E}$ is horospherical, i.e., cuspidal.

**Proof.** The proof here is for $\mathbb{S}^n$ but it implies the $\mathbb{R}P^n$-version.
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(i) Since $\tilde{E}$ is complete, $\tilde{\Sigma}_{\tilde{E}}$ is identifiable with $\mathbb{R}^{n-1}$. $\Gamma_{\tilde{E}}$ induces $\Gamma'_{\tilde{E}}$ in $\text{Aff}(\mathbb{R}^{n-1})$ that are of form $x \mapsto Mx + b$ where $M$ is a linear map $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ and $b$ is a vector in $\mathbb{R}^{n-1}$. For each $\gamma \in \Gamma_{\tilde{E}}$, we write $\hat{L}(\gamma)$ this linear part of the affine transformation corresponding to $\gamma$.

Suppose that one of the norms of relative eigenvalues of $\hat{L}(\gamma)$ for $\gamma \neq I$ is greater than 1 or less than 1. At least one eigenvalue of $\hat{L}(\gamma)$ is 1 since $\gamma$ acts without fixed point on $\mathbb{R}^{n-1}$. (See [60].)

Now, $\hat{L}(\gamma)$ has a maximal vector subspace $A$ of $\mathbb{R}^{n-1}$ where all norms of the eigenvalues are 1.

Suppose that $A$ is a proper subspace of $\mathbb{R}^{n-1}$. Then $\gamma$ acts on the affine space $\mathbb{R}^{n-1}/A$ as an affine transformation with the linear parts without a norm of eigenvalue equal to 1. Hence, $\gamma$ has a fixed point in $\mathbb{R}^{n-1}/A$, and $\gamma$ acts on an affine subspace $A'$ parallel to $A$.

A subspace $H$ containing $v_{\tilde{E}}$ corresponds to the direction of $A'$ from $v_{\tilde{E}}$. Let $H^+$ be the open half-space of one dimension higher corresponding to directions in $A$ with $bdH^+ \ni v_{\tilde{E}}$ so that $H^+$ is invariant under $\gamma$. For $\gamma$ as a projective transformation fixing the vertex $v_{\tilde{E}}$, the eigenvalue of $\gamma$ corresponding to $v_{\tilde{E}}$ equals the ones for the subspace $H^+$. This equals $\lambda_{v_{\tilde{E}}}$.

There exists a projective subspace $S$ of dimension $\geq 0$ where the points are associated with eigenvalues $\lambda$ where $|\lambda| > \lambda_{v_{\tilde{E}}}$ up to reselecting $\gamma$ to be a nonzero integral power of $\gamma$ if necessary.

Let $S'$ the smallest subspace containing $H$ and $S$. Let $U$ be a p-end neighborhood of $\tilde{E}$. Let $y_1$ and $y_2$ be generic points of $U \cap S'-H^+$ so that $\overline{y_1y_2}$ meets $H$ in its interior.

Then we can choose a subsequence $m_i$, $m_i \to \infty$, so that $\gamma^{m_i}(y_1) \to f$ and $\gamma^{m_i}(y_2) \to \ell_-$ as $i \to +\infty$ unto relabeling $y_1$ and $y_2$ for a pair of antipodal points $f, \ell_- \in S$. This implies $f, \ell_- \in \text{Cl}(\partial)$, and $\partial$ is not properly convex, which is a contradiction. Therefore, the norms of eigenvalues of $\hat{L}(\gamma)$ all equal $\lambda_{v_{\tilde{E}}}$ and $A'$ is the $(n-1)$-dimensional affine subspace $\mathbb{R}^{n-1}$. Thus, the norms of eigenvalues of $\gamma$ all equal to 1 since the product of the eigenvalues equal $\pm 1$.

(ii) Since $\tilde{\Sigma}_{\tilde{E}}/\Gamma_{\tilde{E}}$ is a compact complete affine manifold, a finite index subgroup $F$ of $\Gamma_{\tilde{E}}$ is contained in a nilpotent Lie subgroup acting on $\tilde{\Sigma}_{\tilde{E}}$ by Theorem 3 in Fried [42]. Now, by Malcev, it
(iii) The dimension of $N$ is $n - 1 = \dim \tilde{\Sigma}_E$ by Theorem 3 of [42] again.

Let $U$ be a component of the inverse image of a p-end neighborhood so that $v_{\tilde{E}} \in \text{bd}(U)$. Assume that $U$ is radial. A finite index subgroup $F$ of $\Gamma_{\tilde{E}}$ is in $N$ so that $N/F$ is compact by Malcev's results. $N$ acts on a smaller open set covering a p-end neighborhood by taking intersections under images of it under $N$ if necessary. We let $U$ be this open set from now on. Consequently, $\text{bd}U \cap \tilde{O}$ is smooth.

We will now show that $U$ is a horospherical p-end neighborhood: We identify $v_{\tilde{E}}$ with $[1, 0, \ldots, 0]$. Let $W$ denote the subspace in $S^n$ containing $v_{\tilde{E}}$ supporting $U$. $W$ correspond to the boundary of the direction of $\tilde{\Sigma}_E$ and hence is unique and, thus, $N$-invariant. Also, $W \cap \text{Cl}(\tilde{O})$ is a properly convex subset of $W$.

Let $y$ be a point of $U$. Suppose that $N$ contains sequence $\{g_i\}$ so that

$$g_i(y) \to x_0 \in W \cap \text{Cl}(\tilde{O}) \text{ and } x_0 \neq v_{\tilde{E}};$$

that is, $x_0$ in the boundary direction of $A$ from $v_{\tilde{E}}$. The collection of all such $x_0$ has a properly convex, convex hull $U_1$ in $\text{Cl}(\tilde{O})$ in a subspace $V$ in $W$. The dimension of $V$ is $\geq 1$ as it contains $x_0$.

Again $N$ acts on $V$. Now, $V$ is divided into disjoint open hemispheres of various dimensions where $N$ acts on: By Theorem 3.5.3 of [74], $N$ preserves a flag structure $V_0 \subset V_1 \subset \cdots \subset V_k = V$. We take components of complement $V_{i-1} \subset V_{k-1}$. Let $h_{V_i} := V - V_{i-1}$.

Suppose that $\dim V = n - 1$ for contradiction. Then $h_{V_i} \cap U_1$ is not empty since otherwise, we would have a smaller dimensional $V$. Let $h_V$ be the component of $h_{V_i}$ meeting $U_1$. Since $N$ is unipotent, $h_V$ has an $N$-invariant metric by Theorem 3 of Fried [42].

We claim that the orbit of the action of $N$ is of dimension $n - 1$ and hence locally transitive on $h_{V_i}$: If not, then a one-parameter subgroup $N'$ fixes a point of $h_V$. This group acts trivially on $h_V$ since the unipotent group contains a trivial orthogonal subgroup. Since $N'$ is not trivial, it acts as a group of nontrivial translations on the affine space $H^0$. Then $N'(U)$ is not properly convex. Also,
an orbit of \( N \) is open. Thus, \( N \) acts transitively on \( h_V \) since the orbit of \( N \) in \( h_V \) is closed by the invariant metric on \( h_V \).

Hence, the orbit \( N(y) \) of \( N \) for \( y \in H_V \cap U \) contains a component of \( H_V \). Since \( \Gamma(\tilde{E}) \subseteq \text{Cl}(\tilde{O}) \) and a convex hull in \( \text{Cl}(\tilde{O}) \) is \( \text{Cl}(N(y)) \) where \( N(y) \subset H_V \). Since \( F \Gamma(\tilde{E}) = N \) for a compact subset \( F \) of \( N \), the orbit \( \Gamma(\tilde{E}) \) is within a bounded distance from every point of \( N(y) \). Thus, a convex hull in \( \text{Cl}(H_V) \) is \( N(y) \), and this contradicts the assumption that \( \text{Cl}(\tilde{O}) \) is properly convex (compare with arguments in [40].)

Suppose that the dimension of \( V \) is \( \leq n - 2 \). Let \( H \) be a subspace of dimension 1 bigger than \( \dim V \) and containing \( V \) and meeting \( U \). Then \( H \) is sent to disjoint subspaces or to itself under \( N \). Since \( N \) acts transitively on \( A \), a nilpotent subgroup \( N_H \) of \( N \) acts on \( H \). Now we are reduced to \( \dim V \) by one or more. The orbit \( N_H(y) \) for a limit point \( y \in H_V \) contains a component of \( V_{k-1} \) as above. Thus, \( N_H(y) \) contains the same component, an affine subspace. As above, we have a contradiction to the proper convexity.

Therefore, points such as \( x_0 \in W \cap \text{bd}(\tilde{O}) - \{ v_\tilde{E} \} \) do not exist. Hence for any sequence of elements \( g_i \in \Gamma(\tilde{E}) \), we have \( g_i(y) \to v_\tilde{E} \).

Hence, \( \text{bd}U = (\text{bd}U \cap \tilde{O}) \cup \{ v_\tilde{E} \} \). Since the directions from \( v_\tilde{E} \) to \( \text{bd}U \cap \tilde{O} \) form \( \mathbb{R}^{n-1} \), \( \text{bd}U \) is \( C^1 \) at \( v_\tilde{E} \). Clearly, \( \text{bd}U \) is homeomorphic to an \( (n-1) \)-sphere. Since \( U \) is radial, this means that \( U \) is a horospherical p-end neighborhood.

Q.E.D.

§6. The uniform middle eigenvalue condition and properly convex radial ends

Let \( \tilde{E} \) be a p-end and \( \Gamma(\tilde{E}) \) the associated p-end fundamental group. If every subgroup of finite index of a group \( \Gamma(\tilde{E}) \subset \Gamma \) has a finite center, we say that \( \Gamma(\tilde{E}) \) is a virtual center-free group or a vcf-group. An admissible group is a finite extension of a finite product group \( \mathbb{Z}^{k-1} \times \Gamma_1 \times \cdots \times \Gamma_k \) for trivial or infinite hyperbolic groups \( \Gamma_i \) in the sense of Gromov. If two groups \( G_1 \) and \( G_2 \) have isomorphic finite index subgroup, we write \( G_1 \cong G_2 \). (See Section 2.3.1 for details. In this paper, we will simply use \( \mathbb{Z}^{k-1} \) and \( \Gamma_i \) to denote the subgroup in \( \Gamma(\tilde{E}) \) corresponding to it.) We say that
\( \tilde{E} \) is *virtually non-factorable* if the center is trivial for every finite index subgroup of \( \Gamma_{\tilde{E}} \); otherwise, \( \tilde{E} \) is virtually factorable.

Let \( \Gamma \) be generated by finitely many elements \( g_1, \ldots, g_m \). The *conjugate word length* \( \text{cwl}(g) \) of \( g \in \pi_1(\tilde{E}) \) is the minimum of the word length of the conjugates of \( g \) in \( \pi_1(\tilde{E}) \).

Given a point \( x \) of a 1-dimensional subspace \( l \) of \( \mathbb{R}P^n \) or \( S^n \), let \( \bar{x} \) denote the first coordinate of its homogeneous coordinates of \( l \) where the second coordinate is normalized to 1. Let \( \Omega \) be a convex domain in an affine space \( A \) in \( \mathbb{R}P^n \) or \( S^n \). Let \( [o, s, q, p] \) denote the cross ratio of four points as defined by

\[
\frac{\bar{o} - \bar{q}}{\bar{s} - \bar{q}} \frac{\bar{s} - \bar{p}}{\bar{o} - \bar{p}}.
\]

Define a metric \( d_{O}(p, q) = \log |[o, s, q, p]| \) where \( o \) and \( s \) are endpoints of the maximal segment in \( \Omega \) containing \( p, q \) where \( o, q \) separated \( p, s \). The metric is one given by a Finsler metric provided \( \Omega \) is properly convex. (See [58].) Given a properly convex real projective structure on \( O \), it carries a Hilbert metric which we denote by \( d_{O} \) on \( \tilde{O} \) and hence on \( O \). This induces a metric on \( O \). (Note that even if \( \tilde{O} \) is not properly convex, \( d_{O} \) is still a pseudo-metric that is useful.)

Let \( d_{K} \) denote the Hilbert metric of the interior \( K^o \) of a properly convex domain \( K \) in \( \mathbb{R}P^n \) or \( S^n \). Suppose that a projective automorphism \( g \) acts on \( K \). Let \( \text{length}_K(g) \) denote the infimum of \( \{ d_K(x, g(x)) \mid x \in K^o \} \), compatible with \( \text{cwl}(g) \).

We will learn later that every norm of the eigenvalues \( \lambda_i(g) = 1 \), \( g \in \Gamma_{\tilde{E}} \) if and only if \( \tilde{E} \) is horospherical by Proposition 5.2 and Theorem 5.3. Thus, in these cases, we say that \( \tilde{E} \) satisfies the *uniform middle eigenvalue condition* always.

The following definition applies to properly convex R-ends. However, we will generalize this to NPCC ends in Definition 7.3 in Part III.

*Definition 6.1.* Let \( v_{\tilde{E}} \) be a p-end vertex of a p-R-end \( \tilde{E} \). We assume that \( \Gamma_{\tilde{E}} \) is admissible and the associated real projective orbifold \( \Sigma_{\tilde{E}} \) is properly convex. We assume that \( \Gamma_{\tilde{E}} \) acts on a strict join \( \text{Cl}(\tilde{\Sigma}_{\tilde{E}}) = K := K_1 \ast \cdots \ast K_6 \) in \( S^{n-1}_{\mathbb{Q}_{\tilde{E}}} \) where \( K_j \) is a properly convex compact domain in a projective sphere \( S^{i_j} \) of dimension \( i_j \geq 0 \). Thus, \( \Gamma_{\tilde{E}} \) restricts to a semisimple hyperbolic group \( \Gamma_j \) acting on
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for some $j = 1, \ldots, b_0$ and also contains the central abelian group $\mathbb{Z}^{b_0-1}$. The admissibility implies that $\Gamma_j$ is a hyperbolic group and

$$\Gamma_{\tilde{E}} \cong \mathbb{Z}^{b_0-1} \times \Gamma_1 \times \cdots \times \Gamma_{b_0}.$$  

Let $\tilde{K}_i$ denote the subspace spanned by $v_{\tilde{E}}$ and the segments from $v_{\tilde{E}}$ in the direction of $K_i$. The p-end fundamental group $\Gamma_{\tilde{E}}$ satisfies the uniform middle-eigenvalue condition if each $g \in \Gamma_{\tilde{E}}$ satisfies

$$C^{-1}\text{length}_{K}(g) \leq \log \left( \frac{\bar{\lambda}(g)}{\lambda_{v}(g)} \right) \leq C\text{length}_{K}(g),$$  

for $\bar{\lambda}(g)$ equal to

- the largest norm of the eigenvalues of $g$ which must occur for a fixed point of $\tilde{K}_i$ if $g \in \Gamma_i$

and the eigenvalue $\lambda_{v}(g)$ of $g$ at $v_{E}$.

If we require only $\bar{\lambda}(g) \geq \lambda_{v}(g)$ for $g \in \Gamma_{\tilde{E}}$, and the uniform middle eigenvalue condition for each hyperbolic $\Gamma_i$, then we say that $\Gamma_{\tilde{E}}$ satisfies the weakly uniform middle-eigenvalue conditions.

The definition of course applies to the case when $\Gamma_{\tilde{E}}$ has the finite index subgroup with the above properties.

We give a dual definition:

Definition 6.2. Suppose that $\tilde{E}$ is a properly convex p-T-end. Then let $\Gamma_{\ast\tilde{E}}$ acts on a point $v_{\ast\tilde{E}} \in \mathbb{R}P^{n\ast}$ corresponding to $\tilde{\Sigma}_{\ast\tilde{E}}$ with the eigenvalue to be denoted $\lambda_{v_{\ast\tilde{E}}}$. Let $g^{\ast} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the dual transformation of $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Assume that $\Gamma_{\ast\tilde{E}}$ acts on a properly convex compact domain $K = \text{Cl}(\tilde{\Sigma}_{\ast\tilde{E}})$ and $K$ is a strict join $K := K_1 \ast \cdots \ast K_b$. $\Gamma_i$ is as above. The p-end fundamental group $\Gamma_{\ast\tilde{E}}$ satisfies the uniform middle-eigenvalue condition if it satisfies

- the equation 7 for the largest norm $\bar{\lambda}(g)$ of the eigenvalues of $g$, which must occur for a fixed point of $K_i$ if $g \in \Gamma_i$, and
the eigenvalue $\lambda_{\tilde{E}}(g)$ of $g^*$ in the vector in the direction of $\nu_{\tilde{E}}$.

Here $\Gamma_{\tilde{E}}$ will act on a properly convex domain $K^o$ of lower-dimension and we will apply the definition here. This condition is similar to ones studied by Guichard and Wienhard [51], and the results also seem similar. Our main tools to understand these questions are in Appendix A in [25], and the author does not really know the precise relationship here.)

The condition is an open condition; and hence a “structurally stable one.” (See [25].)

Our main result is:

**Theorem 6.1.** Let $O$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends. and satisfies (IE). Each end fundamental group is virtually isomorphic to a direct product of hyperbolic groups and infinite cyclic groups. Assume that the holonomy group of $O$ is strongly irreducible.

- Let $\tilde{E}$ be a properly convex p-R-end.
  - Suppose that the p-end holonomy group satisfies the uniform middle-eigenvalue condition Then $\tilde{E}$ is of generalized lens-type.
  - Suppose that the each p-end holonomy group satisfies the weakly uniform middle-eigenvalue condition. Then $\tilde{E}$ is of generalized lens-type or of quasi-lens-type.

- If $O$ satisfies the triangle condition or $\tilde{E}$ is virtually factorable or is a totally geodesic R-end, then we can replace the word “generalized lens-type” to “lens-type” in each of the above statements.

- Let $\tilde{E}$ be a totally geodesic end. If $\tilde{E}$ satisfies the uniform middle-eigenvalue condition, then $\tilde{E}$ is of lens-type.

**Remark 6.2 (Duality of ends).** Above orbifold $O = \hat{O}/\Gamma$ has a diffeomorphic dual orbifold $O^*$, where $O^*$ is defined as the quotient of the dual domain $\hat{O}^*$ by the dual group of $\Gamma$ by Theorem 3.5. The ends of $O$ and $O^*$ are in a one-to-one correspondence. Horospherical ends are dual to themselves, i.e., “self-dual”, and properly convex R-ends and T-ends are dual to one another. (See [25].) We will see that properly convex R-ends of generalized lens type are always dual to T-ends of lens type by [25].
Theorem 6.3. Let $\mathcal{O}$ be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends and satisfy (IE). Assume that the holonomy group is strongly irreducible. Assume that the universal cover $\tilde{\mathcal{O}}$ is a subset of $\mathbb{S}^n$ (resp. $\mathbb{R}P^n$). Let $\tilde{S}_E$ be a totally geodesic ideal boundary of a $p$-$T$-end $\tilde{E}$ of $\tilde{\mathcal{O}}$. Then the following conditions are equivalent:

(i) $\tilde{E}$ satisfies the uniform middle-eigenvalue condition.

(ii) $\tilde{S}_E$ has a lens-neighborhood in an ambient open manifold containing $\tilde{\mathcal{O}}$ and hence $\tilde{E}$ has a lens-type $p$-end neighborhood in $\tilde{\mathcal{O}}$.

Remark 6.4 (Self-dual virtually factorable ends). A generalized lens-type virtually factorable properly convex R-end is always totally geodesic by Theorem 2.15 in [25] and Theorem 6.1, the R-end is of lens-type always. The dual end is totally geodesic of lens type since it satisfies the uniform middle eigenvalue condition. The end can be made into a totally geodesic R-end since it fixes a unique point dual to the totally geodesic ideal boundary component and by taking a cone over that point. Thus, the virtually factorable properly convex ends are “self-dual”. Thus, we consider these the best types of cases.

6.1. The characterization of quasi-lens $p$-R-end neighborhoods

This is the last remaining case for the properly convex ends with weak uniform middle eigenvalue conditions. We will only prove for $\mathbb{S}^n$.

Definition 6.3. Let $G$ denote the $p$-end fundamental group satisfying the weak uniform middle eigenvalue condition.

- Let $D$ be the properly convex totally geodesic $n - 2$-dimensional domain so that $U = D \ast v \subset \mathbb{S}^{n-1}$.
- Let $S^1$ be a great circle meeting $\mathbb{S}^{n-1}$ at $v$.
- Extend $G$ to act on $S^1$ as a nondiagonalizable transformation fixing $v$. 

Let $\zeta$ be a projective automorphism acting on $U$ and $\mathbb{S}^1$ so that $\zeta$ commutes with $G$ and restrict to a diagonalizable transformation on $\text{Cl}(D)$ and act as a non-diagonalizable transformation on $\mathbb{S}^1$ fixing $v$ and with largest norm eigenvalue at $v$.

Every element of $G$ and $\zeta$ can be written as a matrix

\[
\begin{pmatrix}
S(g) & 0 \\
0 & \lambda_v(g) \\
0 & \lambda_v(g) \\
0 & \lambda_v(g)
\end{pmatrix}
\]

where $v = [0, \ldots, 1]$. Note that $g \mapsto v(g) \in \mathbb{R}$ is a well-defined map inducing a homomorphism

\[\langle G, \zeta \rangle \to H_1(\tilde{\Gamma}_E) \to \mathbb{R}\]

and hence

\[|v(g)| \leq C \text{cwI}(g)\]

for a positive constant $C$.

**Positive translation condition:** We choose an affine coordinate on a component $I$ of $\mathbb{S}^1 - \{v, v_+\}$. We assume that for each $g \in \langle G, \zeta \rangle$, if $\lambda_v(g) > \lambda_2(g)$ for the largest eigenvalue $\lambda_2$ associated with $\text{Cl}(D)$, then $v(g) > 0$ in equation 8, and

\[\frac{v(g)}{\log \frac{\lambda_v(g)}{\lambda_2(g)}} > c_1 > 0\]

for a constant $c_1$.

**Proposition 6.5.** Suppose that $\langle G, \zeta \rangle$ satisfies the positive translation condition. Then the above $U$ is in the boundary of a properly convex $p$-end open neighborhood $V$ of $v$ and $\langle G, \zeta \rangle$ acts on $V$ properly.

This generalizes the quasi-hyperbolic annulus discussed in [21]. Conversely, we obtain:

**Proposition 6.6.** Let $\mathcal{O}$ be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends and satisfy (IE). Assume that the universal cover $\tilde{\mathcal{O}}$ is a subset of
Suppose that $\pi_1(O)$ is strongly irreducible. Let $\tilde{E}$ be a properly convex radial end satisfying the weak uniform middle eigenvalue conditions but not the uniform middle eigenvalue condition. Then $\tilde{E}$ has a quasi-lens type $p$-end neighborhood.

§7. The uniform middle eigenvalue conditions for NPCC ends

We will now study the ends where the transverse real projective structures are

- not properly convex but
- not projectively diffeomorphic to a complete affine subspace.

Let $\tilde{E}$ be a $p$-R-end of $O$ and let $U$ be the corresponding $p$-end neighborhood in $\tilde{O}$ with the $p$-end vertex $v_{\tilde{E}}$.

The closure $\text{Cl}(\tilde{\Sigma}_{\tilde{E}})$ contains a great $(i_0 - 1)$-dimensional sphere and $\tilde{\Sigma}_{\tilde{E}}$ is foliated by $i_0$-dimensional hemispheres with this boundary by Proposition 2.1. Let $\mathbb{S}^n_{\infty}$ denote the great $(i_0 - 1)$-dimensional sphere in $\mathbb{S}^n_{v_{\tilde{E}}}$ of $\tilde{\Sigma}_{\tilde{E}}$. The space of $i_0$-dimensional hemispheres in $\mathbb{S}^n_{v_{\tilde{E}}}$ with boundary $\mathbb{S}^n_{\infty}$ form a projective sphere $\mathbb{S}^{n-i_0-1}$. The projection

$$\mathbb{S}^n_{v_{\tilde{E}}} - \mathbb{S}^n_{\infty} \rightarrow \mathbb{S}^{n-i_0-1}$$

gives us an image of $\tilde{\Sigma}_{\tilde{E}}$ that is the interior of a properly convex compact set $K$. (See [15] for details. See also [41].)

Let $\mathbb{S}^n_{\infty}$ be a great $i_0$-dimensional sphere containing $v_{E}$ corresponding to the directions of $\mathbb{S}^n_{\infty}$ from $v_{E}$. The space of $(i_0 + 1)$-dimensional hemispheres with boundary $\mathbb{S}^n_{\infty}$ again has the structure of the projective sphere $\mathbb{S}^{n-i_0-1}$, identifiable with the above one. Denote by $\text{Aut}_{\mathbb{S}^n_{\infty}}(\mathbb{S}^n)$ the group of projective automorphisms of $\mathbb{S}^n$ acting on $\mathbb{S}^n_{\infty}$ and fixing $v_{E}$. We also have the projection

$$\Pi_K : \mathbb{S}^n - \mathbb{S}^n_{\infty} \rightarrow \mathbb{S}^{n-i_0-1}$$

giving us the image $K^o$ of a $p$-end neighborhood $U$.

Each $i_0$-dimensional hemisphere $H^{i_0}$ in $\mathbb{S}^n_{v_{E}}$ with $bdH^{i_0} = \mathbb{S}^n_{\infty}$ corresponds to an $(i_0 + 1)$-dimensional hemisphere $H^{i_0+1}$ in $\mathbb{S}^n$ with common boundary $\mathbb{S}^n_{\infty}$ that contains $v_{E}$.
Let $\text{SL}_\pm(n+1, \mathbb{R})_{S^\infty_v}$ denote the subgroup of $\text{Aut}(S^n)$ acting on $S^\infty_v$ and $v_\infty$. The projection $\Pi_K$ induces a homomorphism $\Pi^*: \text{SL}_\pm(n+1, \mathbb{R})_{S^\infty_v} \rightarrow \text{SL}_\pm(n, \mathbb{R})$.

Suppose that $S^\infty_v$ is $h(\pi_1(\tilde{E}))$-invariant. We let $N$ be the subgroup of $h(\pi_1(\tilde{E}))$ of elements inducing trivial actions on $S^{n-\hat{b}-1}$. The above exact sequence

$$1 \rightarrow N \rightarrow h(\pi_1(\tilde{E})) \xrightarrow{\Pi^*_K} N_K \rightarrow 1$$

is so that the kernel normal subgroup $N$ acts trivially on $S^{n-\hat{b}-1}$ but acts on each hemisphere with boundary equal to $S^\infty_v$ and $N_K$ acts faithfully by the action induced from $\Pi^*_K$.

Here $N_K$ is a subgroup of the group $\text{Aut}(K)$ of the group of projective automorphisms of $K$ is called the semisimple quotient of $h(\pi_1(\tilde{E}))$ or $\Gamma_{\tilde{E}}$.

**Theorem 7.1.** Let $\Sigma_{\tilde{E}}$ be the end orbifold of an NPCC p-R-end $\tilde{E}$ of a strongly tame properly convex $n$-orbifold $\tilde{O}$ with radial or totally geodesic ends and satisfy (IE). Let $\tilde{O}$ be the universal cover in $S^n$. We consider the induced action of $h(\pi_1(\tilde{E}))$ on $\text{Aut}(S^{n-1}_{v_\tilde{E}})$ for the corresponding end vertex $v_\tilde{E}$. Then

- $\Sigma_{\tilde{E}}$ is foliated by complete affine subspaces of dimension $i_0$, $i_0 > 0$.
- $h(\pi_1(\tilde{E}))$ fixes the great sphere $S^{i_0-1}_\infty$ of dimension $i_0 - 1$ in $S^{n-1}_{v_\tilde{E}}$.
- There exists an exact sequence

$$1 \rightarrow N \rightarrow \pi_1(\tilde{E}) \xrightarrow{\Pi^*_K} N_K \rightarrow 1$$

where $N$ acts trivially on quotient great sphere $S^{n-\hat{b}-1}$ and $N_K$ acts faithfully on a properly convex domain $\hat{K}$ in $S^{n-\hat{b}-1}$ isometrically with respect to the Hilbert metric $d_K$.

**Proof.** These follow from Section 1.4 of [15]. (See also [41].) Q.E.D.

We denote by $\mathcal{F}$ the foliations on $\Sigma_{\tilde{E}}$ or the corresponding one in $\tilde{\Sigma}_{\tilde{E}}$. 
7.0.1. The main eigenvalue estimations

We denote by $\tilde{\Gamma}_E$ the p-end fundamental group acting on $U$ fixing $\tilde{v}_E$. Denote the induced foliations on $\Sigma_{\tilde{E}}$ and $\tilde{\Sigma}_{\tilde{E}}$ by $\mathcal{F}_{\tilde{E}}$. For each element $g \in \tilde{\Gamma}_E$, we define $\text{length}_K(g)$ to be $\inf\{d_K(x, g(x)) | x \in K^o\}$.

**Definition 7.1.** Given an eigenvalue $\lambda$ of an element $g \in \text{SL}_\pm(n+1, \mathbb{R})$, a $\mathbb{C}$-eigenvector $\tilde{v}$ is a nonzero vector in $\mathbb{R}^{n+1} \cap (\ker(g - \lambda I) + \ker(g - \bar{\lambda} I))$, $\lambda \neq 0, \text{Im}\lambda \geq 0$.

A $\mathbb{C}$-fixed point is the direction of a $\mathbb{C}$-eigenvector.

Any element of $g$ has a Jordan decomposition. An irreducible Jordan-block corresponds to a unique subspace in $\mathbb{C}^{n+1}$, called an elementary Jordan block subspace. We denote by $J_{\mu,i} \subset \mathbb{C}^{n+1}$ for an eigenvalue $\mu \in \mathbb{C}$ for $i$ in an index set. A real elementary Jordan block subspace is defined as $R_{\mu,i} := \mathbb{R}^{n+1} \cap (J_{\mu,i} + J_{\bar{\mu},i}), \mu \neq 0, \text{Im}\mu \geq 0$ of Jordan subspaces with $J_{\mu,i} = J_{\bar{\mu},i}$ in $\mathbb{C}^{n+1}$. We define the real sum of elementary Jordan-block subspaces is defined to be

$$\bigoplus_{i \in I} R_{\mu,i}$$

for a finite collection $I$.

A point $[\tilde{v}], \tilde{v} \in \mathbb{R}^{n+1}$, is affiliated with a norm $\mu$ of an eigenvalue if $\tilde{v} \in \bigoplus_{|\lambda|=\mu, i \in I} R_{\lambda,i}$ for a sum of all real elementary Jordan subspaces $R_{\lambda,i}, \mu = |\lambda|$.

Let $V_i^{l+1}$ denote the subspace of $\mathbb{R}^{n+1}$ corresponding to $S_i^l$. By invariance of $S_i^l$, if

$$\bigoplus_{(\mu,i) \in J} R_{\mu,i} \cap V_i^{l+1} \neq \emptyset$$

for some finite collection $J$, then $\bigoplus_{(\mu,i) \in J} R_{\mu,i} \cap V_i^{l+1}$ always contains a $\mathbb{C}$-eigenvector.

**Definition 7.2.** Let $\Sigma_{\tilde{E}}$ be the end orbifold of a nonproperly convex and p-R-end $\tilde{E}$ of a strongly tame properly convex n-orbifold $O$ with radial or totally geodesic ends. Let $\tilde{\Gamma}_E$ be the p-end fundamental group. We fix a choice of a Jordan decomposition of $g$ for each $g \in \tilde{\Gamma}_E$.  

\[\text{The classification of ends}\]
• Let $\lambda_1(g)$ denote the largest norm of the eigenvalue of $g \in \Gamma^{\tilde{E}}$ affiliated with $\vec{v} \neq 0$, $[\vec{v}] \in S^n - S^n_\infty$, i.e.,

$$\vec{v} \in \bigoplus_{(\mu_1(g), i) \in J} R_{\mu_1(g), i} - V_{\infty}^{\mu_1+1}, |\mu_1| = \lambda_1(g)$$

where $J$ indexes all elementary Jordan subspaces of $\lambda_1(g)$.

• Also, let $\lambda_{n+1}(g)$ denote the smallest one affiliated with a nonzero vector $\vec{v}$, $[\vec{v}] \in S^n - S^n_\infty$, i.e.,

$$\vec{v} \in \bigoplus_{(\mu_{n+1}(g), i) \in J'} R_{\mu_{n+1}(g), i} - V_{\infty}^{\mu_{n+1}+1}, |\mu_{n+1}| = \lambda_{n+1}(g)$$

where $J'$ indexes all real elementary Jordan subspaces of $\lambda_{n+1}(g)$.

• Let $\lambda(g)$ be the largest of the norm of the eigenvalue of $g$ with a $C$-eigenvector $\vec{v}$, $[\vec{v}] \in S^n_\infty$ and $\lambda'(g)$ the smallest such one.

Then for some $l' \geq 1$, $\Gamma^{\tilde{E}}$ is isomorphic to $\mathbb{Z}^{l'-1} \times \Gamma_1 \times \cdots \times \Gamma_r$ up to finite index where each $\Gamma_i$ acts irreducibly on $K_i$ for each $i = 1, \ldots, r$. We assume that $\Gamma_i$ is torsion-free hyperbolic for $i = 1, \ldots, s$ and $\Gamma_i = \{I\}$ for $s + 1 \leq i \leq l'$. We will use the notation $\Gamma_i$ the corresponding subgroup in $\Gamma^{\tilde{E}}$.

Suppose that $K$ has a decomposition into $K_1 \ast \cdots \ast K_{h_0}$ for properly convex domains $K_i$, $i = 1, \ldots, h_0$. Let $K_i, i = 1, \ldots, s$, be the ones with dimension $\geq 2$. $N_K$ is virtually isomorphic to the product

$$\mathbb{Z}^{h_0-1} \times \Gamma_1 \times \cdots \times \Gamma_s$$

where $\Gamma_i$ is obtained from $N_K$ by restricting to $K_i$ and $A$ is a free abelian group of finite rank.

$\Gamma_i \cap N$ is a normal subgroup of $\Gamma_i$. If $\Gamma_i$ is hyperbolic, then this group has to be trivial. Therefore, we obtain that each $\Gamma_i$ for $i = 1, \ldots, s$, is mapped isomorphic to some $\Gamma_i$ in $N_K$ provided $\Gamma_i$ is hyperbolic. Thus, each $K_i$ is a strictly convex domain or a point by the results in [4].

The following definition generalizes Definition 6.1. The two definitions can be stated in the same manner but we avoid doing so here.
Definition 7.3. We also assume that the uniform middle-eigenvalue condition relative to $N$:

$$N_K \cong \mathbb{Z}^l \times \Gamma_1 \times \cdots \times \Gamma_k, l \geq k - 1$$

acts on

$$\text{Cl}(K) = p_1 \ast \cdots \ast p_{l-k-1} \ast K_1 \ast \cdots \ast K_k$$

where $\Gamma_i$ is a hyperbolic group acting on properly convex domain $K_i$ for each $i$, $i = 1, \ldots, k$, and each $p_j$ is a singleton for $j = 1, \ldots, l - k - 1$ with following conditions. Let $\hat{K}_i$ denote the subspace spanned by $\Pi(\hat{K}_i) \cup S_\infty$.

- there exists a constant $C > 0$ independent of $g \in \Gamma_E$ such that

$$C^{-1} \text{length}_K(g) \leq \log \frac{\bar{\lambda}(g)}{\lambda_{v_E}(g)} \leq C \text{length}_K(g)$$

for $\bar{\lambda}(g)$ equal to

- the largest norm of the eigenvalues of $g$.

and for the eigenvalue $\lambda_{v_E}(g)$ of $g$ at $v_E$. We also require that the largest norm eigenvalue must occur for a fixed point of $\hat{K}_i$ if $g \in \Gamma_i$.

If we require only $\bar{\lambda}(g) \geq \lambda_{v_E}(g)$ for $g \in \Gamma_E$, and the uniform middle eigenvalue condition for each hyperbolic group $\Gamma_i$, then we say that $\Gamma_E$ satisfies the weakly uniform middle-eigenvalue conditions.

This definition is simply an extension of one for properly convex end ones since we could have used the degenerate metric on $\tilde{\Sigma}_E$ based on cross-ratios and the definition would agree. In fact, for complete ends, the definition agrees by Proposition 5.2 and Theorem 5.3.

The following proposition is needed to understand to NPCC-ends.

Proposition 7.2. Let $\Sigma_E$ be the end orbifold of a nonproperly convex p-R-end $\tilde{E}$ of a strongly tame properly convex n-orbifold $O$ with radial or totally geodesic ends and satisfying (IE). Suppose that $\tilde{O}$ in $\mathbb{S}^n$ (resp. $\mathbb{R}P^n$) covers $O$ as a universal cover. Let
Γ̃ be the p-end fundamental group satisfying the weak uniform middle-eigenvalue condition. Let \( g \in \Gamma \). Then
\[
\lambda_1(g) \geq \lambda(g) \geq \lambda'(g) \geq \lambda_{n+1}(g)
\]
holds.

7.1. Joins and quasi-joined ends

We will now discuss about joins and their generalizations in depth in this subsection.

For \( \vec{v} \in \mathbb{R}^{i_0} \), we define
\[
(10) \quad N(\vec{v}) := \begin{pmatrix}
I_{n-i_0-1} & 0 & 0 & 0 & \cdots & 0 \\
\vec{0} & 1 & 0 & 0 & \cdots & 0 \\
\vec{c}_1(\vec{v}) & v_1 & 1 & 0 & \cdots & 0 \\
\vec{c}_2(\vec{v}) & v_2 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vec{c}_{i_0+1}(\vec{v}) & \frac{1}{2}||\vec{v}||^2 & v_1 & v_2 & \cdots & 1
\end{pmatrix}
\]
where \( ||\vec{v}|| \) is the norm of \( \vec{v} = (v_1, \cdots, v_{i_0}) \in \mathbb{R}^{i_0} \). We require \( N \) to be a group. The elements of our nilpotent group \( N \) are of this form since \( N(\vec{v}) \) is the product \( \prod_{j=1}^{i_0} \mathcal{N}(e_j)^y \). By the way we defined this, for each \( k, k = 1, \ldots, i_0 \), \( \vec{c}_k : \mathbb{R}^{i_0} \to \mathbb{R}^{n-i_0-1} \) are linear functions of \( \vec{v} \) defined as \( \vec{c}_k(\vec{v}) = \sum_{j=1}^{i_0} \vec{c}_{kj} v_j \) for \( \vec{v} = (v_1, v_2, \ldots, v_{i_0}) \) so that we form a group. (We do not need the property of \( \vec{c}_{i_0+1} \) at the moment.)

We denote by \( C_1(\vec{v}) \) the \((n-i_0-1) \times i_0\)-matrix given by the matrix with rows \( \vec{c}_j(\vec{v}) \) for \( j = 1, \ldots, i_0 \) and by \( C_2(\vec{v}) \) the row \((n-i_0-1)\)-vector \( \vec{c}_{i_0+1}(\vec{v}) \). The lower-right \((i_0+2) \times (i_0+2)\)-matrix is form is called the standard cusp matrix form.

We also need matrices of form
\[
(11) \quad \begin{pmatrix}
S(\vec{g}) & 0 & 0 & 0 \\
0 & \lambda_{v_1}(\vec{g}) & 0 & 0 \\
C_1(\vec{g}) & \lambda_{v_1}(\vec{g}) \vec{v}_1 & \lambda_{v_1}(\vec{g}) \vec{v}_1 & 0 \\
c_2(\vec{g}) & a_7(\vec{g}) & \lambda_{v_1}(\vec{g}) \vec{v}_1 & \lambda_{v_1}(\vec{g}) \vec{v}_1
\end{pmatrix}
\]

Hypothesis 7.3. Let \( G \) be a p-end fundamental group. We assume as in Lemma 10.6 in [26] for \( G \).
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• $G$ acts on the subspace $S^0_{\infty}$ containing $v_{\tilde{E}}$ and the properly convex domain $K''_{m_0}$ in the subspace $S^{n-i_0-2}$ disjoint from $S^0_{\infty}$.

• $N$ acts on these two subspaces fixing every points of $S^{n-i_0-2}$.

We assume $v_{\tilde{E}}$ to have coordinates $[0, \ldots, 0, 1]$. $S^{n-i_0-2}$ contains the standard points $[e_i]$ for $i = 1, \ldots, n-i_0-1$ and $S^0_{\infty}$ contains $[e_i]$ for $i = n-i_0, \ldots, n+1$. Let $H$ be the open $n$-hemisphere defined by $x_{n-i_0} > 0$. Then by convexity of $U$, we can choose $H$ so that $K'' \subset H$ and $S^0_{\infty} \subset Cl(H)$.

By Hypothesis 7.3, elements of $N$ have the forms of equation 10 with

$$G_1(\bar{v}) = 0, \ C_2(\bar{v}) = 0 \text{ for all } \bar{v} \in \mathbb{R}^b$$

and the group $G$ of elements of forms of equation 11 with

$$s_1(g) = 0, \ s_2(g) = 0, \ C_1(g) = 0, \text{ and } C_2(g) = 0.$$ 

We assume further that $O_5(g) = I_{i_0}$.

Again we recall the projection $\Pi_K : S^n - S^0_{\infty} \to S^{n-i_0-1}$. $G$ has an induced action on $S^{n-i_0-1}$ and acts on a properly convex set $K''$ in $S^{n-i_0-1}$ so that $K$ equals a strict join $k * K''$ for $k$ corresponding to a great sphere $S^0_{i_0+1}$.

We define invariants from the form of equation 11

$$\alpha_7(g) := \frac{a_7(g)}{\lambda_{v_{\tilde{E}}}(g)} - \frac{||v_{\tilde{E}}||^2}{2}$$

for every $g \in G$.

$\alpha_7(g^n) = n\alpha_7(g)$ and $\alpha_7(gh) = \alpha_7(g) + \alpha_7(h)$, whenever $g, h, gh \in G$.

Here $\alpha_7(g)$ is determined by factoring the matrix of $g$ into commuting matrices of form
Fig. 4. A figure of quasi-joined p-R-end neighborhood

\[
(12) \begin{pmatrix}
I_{n-i_0-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & I_{i_0} & 0 \\
0 & \alpha_7(g) & 0 & 1
\end{pmatrix} \times
\begin{pmatrix}
S_g & 0 & 0 & 0 \\
0 & \lambda_{v_E}(g) & 0 & 0 \\
0 & \lambda_{v_E}(g)v_E & \lambda_{v_E}(g)O_5(g) & 0 \\
0 & \lambda_{v_E}(g)||v_E^+||_2 & \lambda_{v_E}(g)v_EO_5(g) & \lambda_{v_E}(g)
\end{pmatrix}.
\]

Remark 7.4. We give a bit more explanations. Recall that the space of segments in a hemisphere $H^{k+1}$ with the vertices $v_E^+, v_E^-$ forms an affine space $A^i$ one-dimension lower, and the group $\text{Aut}(H^{k+1})_{v_E}$ of projective automorphism of the hemisphere fixing $v_E$ maps to $\text{Aff}(A^k)$ with kernel $K$ equal to transformations.
of an \((i_0 + 2) \times (i_0 + 2)\)-matrix form

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & I_{i_0} & 0 \\
b & 0 & 1
\end{pmatrix}
\]

(13)

where \(v_\tilde{E}\) is given coordinates \([0, 0, \ldots, 1]\) and a center point of \(H_{i_0+1}^k\) the coordinates \([1, 0, \ldots, 0]\). In other words the transformations are of form

\[
\begin{pmatrix}
1 & x_1 & \vdots & x_{i_0} & x_{i_0+1}
\end{pmatrix}
\begin{pmatrix}
1 \\
x_1 \\
\vdots \\
x_{i_0} \\
x_{i_0+1} + b
\end{pmatrix}
\]

(14)

and hence \(b\) determines the kernel element. Hence \(\alpha_7(g)\) indicates the translation towards \(v_\tilde{E} = [0, \ldots, 1]\).

We define \(G_+\) to be a subset of \(G\) consisting of elements \(g\) so that the largest norm \(\lambda_1(g)\) of the eigenvalue occurs at the vertex \(k\). We also assume that \(\lambda_1(g) = \lambda_{v_\tilde{E}}(g)\) with all other norms of the eigenvalues occurring at \(K''\) is strictly less than \(\lambda_{v_\tilde{E}}(g)\). The second largest norm \(\lambda_2(g)\) of the eigenvalue occurs at the complementary subspace \(K''\) of \(k\) in \(K\). Thus, \(G_+\) is a semigroup.

The condition that \(\alpha_7(g) \geq 0\) for \(g \in G_+\) is said to be the positive translation condition.

Again, we define

\[
\mu_7(g) := \frac{\alpha_7(g)}{\log \frac{\lambda_{v_\tilde{E}}(g)}{\lambda_2(g)}}
\]

where \(\lambda_2(g)\) denote the second largest norm of the eigenvalues of \(g\) and we restrict \(g \in G_+\). The condition \(\mu_7(g) > C_0, g \in G_+\) for a uniform constant \(C_0\) is called the uniform positive translation condition.

Suppose that \(G\) is a p-end fundamental group.
For this proposition, we do not assume $N_K$ is discrete.

**Proposition 7.5.** Let $\tilde{\Sigma}_{\tilde{\mathcal{E}}}$ be the end orbifold of a nonproperly convex $R$-end $\tilde{\mathcal{E}}$ of a strongly tame $n$-orbifold $\mathcal{O}$ with radial or totally geodesic ends. Let $G$ be the $p$-end fundamental group. Let $\tilde{\mathcal{E}}$ be an NPCC $p$-$R$-end and $G$ acts on a $p$-end neighborhood $U$ fixing $v_{\tilde{\mathcal{E}}}$. Let $K, K'', S_k^b$, and $S_k^{b+1}$ be as above. We assume that $K^o/G$ is compact, $K = K'' \ast k$ in $S^{n-b}$ with $k$ corresponding to $S_k^{b+1}$ under the projection $\Pi_K$. Assume that

- $G$ satisfies the weakly uniform middle-eigenvalue condition.
- Elements of $G$ and $N$ are of form of equation 15 with
  \[ C_1(\widetilde{v}) = 0, \quad c_2(\widetilde{v}) = 0, \quad C_1(g) = 0, \quad c_2(g) = 0 \]
  for every $\widetilde{v} \in \mathbb{R}^b$ and $g \in G$.
- $G$ normalizes $N$, and $N$ acts on $U$ and each leaf of $\mathcal{F}$ of $\tilde{\Sigma}_{\tilde{\mathcal{E}}}$.

Then

(i) The condition $\alpha_7 \geq 0$ is a necessary condition that $G$ acts on a properly convex domain in $H$.

(ii) The uniform positive translation condition is equivalent to the existence of properly convex $p$-end neighborhood $U'$ whose closure meets $S_k^{b+1}$ at $v_{\tilde{\mathcal{E}}}$ only.

(iii) $\alpha_7$ is identically zero if and only if $U$ is a join and $U$ is properly convex.
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Definition 7.4. The second case of Proposition 7.5, \( \tilde{E} \) is said to be a quasi-joined p-R-end and \( G \) now is called a quasi-joined end group. An end with an end neighborhood that is covered by a p-end neighborhood of such a p-R-ends is also called a quasi-joined p-R-end.

We will show using Proposition 7.5 that NPCC R-ends satisfying the weak uniform middle eigenvalue conditions are quasi-joins. That is, Theorem 7.6 is shown. These quasi-joined p-R-ends do not satisfy the uniform middle eigenvalue condition essentially because of the existence of the nilpotent Lie group.

Theorem 7.6. Let \( O \) be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends. Assume that the holonomy group of \( O \) is strongly irreducible. Each end fundamental group is admissible. Let \( \tilde{E} \) be an NPCC p-R-end. Suppose that the p-each end holonomy group satisfies the weakly uniform middle-eigenvalue condition. Then \( \tilde{E} \) is a quasi-joined type p-R-end.

We will discuss the idea of the proof: First we discuss the case when \( N_K \) is a discrete. Here, \( N \) is virtually abelian and is conjugate to a discrete cocompact subgroup of a cusp group. Here, the fibres of \( \pi_K \) covers a compact \( i_0+1 \)-dimensional orbifold. The fibers have complete affine structure of dimension \( i_0+1 \). By proper-convexity of \( \tilde{O} \), Theorem 5.1 implies that the fibers are \( i_0+1 \)-dimensional cusps. Hence, there exists a parabolic group as described by \( N \) in equation (10). Then \( \Gamma_{\tilde{E}} \) then virtually normalises the fiber group. By computations involving the normalization conditions, we show that the above exact sequence is virtually split, and we can surprisingly show that the R-ends are of join or quasi-join types.

Then we discuss the case when \( N_K \) is not discrete. Here, there is a foliation by complete affine spaces as above. The leaf closures are compact submanifolds \( V_i \) by the theory of Molino [68] on Riemannian foliations. Each of these are fibers in some singular fibrations. We use some estimate of Proposition 7.2 to show that each leaf is of polynomial growth. This shows that the identity component of the closure of \( N_K \) is abelian and \( \pi_1(V_i) \) for above fiber \( V_i \) is solvable using the work of Carrière [13]. One can then
take the syndetic closure to obtain a bigger group that act transitiely on each leaf. We find a normal cusp group acting on each leaf transitively. Then we show that the end also splits virtually.

Finally for both of these cases, we show that the orbifold has to be reducible by considering the limit actions of some elements in the joined ends. This proves that the joined end does not exist, proving Theorem 6.1.

**Corollary 7.7.** Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends. Assume that the holonomy group of $\mathcal{O}$ is strongly irreducible. Each end fundamental group is virtually isomorphic to a direct product of hyperbolic groups. Suppose that the each end holonomy group satisfies the uniform middle-eigenvalue condition. Then each end is a generalized lens-type $R$-end, an $R$-end of cusp type, or a lens-type $T$-end.

This follows by Theorems 5.1, 5.3, 6.1, and 7.6.

Our work is a “classification” since we will show how to construct lens-type $R$-ends as we shall see from [25] and [26].

**References**

[1] S. Ballas, Deformations of non-compact, projective manifolds, arXiv:1210.8419, math.GT.

[2] S. Ballas, Finite volume properly convex deformations of the figure eight knot, arXiv:1403.331v1 math.GT.

[3] O. Baues, Deformation spaces for affine crystallographic groups, In: Cohomology of groups and algebraic K-theory, 55–129, Adv. Lect. Math. (ALM), 12, Int. Press, Somerville, MA, 2010.

[4] Y. Benoist, Convexes divisibles. I, In Algebraic groups and arithmetic, 339–374, Tata Inst. Fund. Res., Mumbai, 2004.

[5] Y. Benoist, Convexes divisibles. II, Duke Math. J., 120 (2003), 97–120.

[6] Y. Benoist, Convexes divisibles. III, Ann. Sci. Ecole Norm. Sup. (4) 38 (2005), no. 5, 793–832.

[7] Y. Benoist, Convexes divisibles IV : Structure du bord en dimension 3, Invent. math. 164 (2006), 249–278.

[8] Y. Benoist, Automorphismes des cônes convexes, Invent. Math., 141 (2000), 149–193.
The classification of ends

[9] Y. Benoist, Propriétés asymptotiques des groupes linéaires, *Geom. Funct. Anal.* 7 (1997), no. 1, 1–47.
[10] Y. Benoist, Nilvariétés projectives, *Comm. Math. Helv.* 69 (1994), 447–473.
[11] J.-P., Benzécri, Sur les variétés localement affines et localement projectives, Bull. Soc. Math. France 88 (1960) 229–332.
[12] A. Borel, *Linear algebraic group*, Springer Verlag, 2nd edition p.288 + xi, 1991.
[13] Y. Carrièrè, Feuilletages riemanniens à croissance polynômiale, *Comment. Math. Helv.* 63 (1988), 1–20.
[14] R. Canary, D.B.A. Epstein, and P. L. Green, Notes on notes of Thurston, In: *Fundamentals of hyperbolic geometry: selected expositions*, pp. 1–115, London Math. Soc. Lecture Note Ser., 328, Cambridge Univ. Press, Cambridge, 2006.
[15] Y. Chae, S. Choi, and C. Park, Real projective manifolds developing into an affine space, *Internat. J. Math.* 4 (1993), no. 2, 179–191.
[16] S. Choi, Geometric structures on orbifolds and holonomy representations, *Geom. Dedicata* 104 (2004), 161–199.
[17] S. Choi, The convex and concave decomposition of manifolds with real projective structures, *Mémoires SMF*, No. 78, 1999, 102 pp.
[18] S. Choi, The deformation spaces of projective structures on 3-dimensional Coxeter orbifolds, *Geom. Dedicata* 119 (2006), 69–90.
[19] S. Choi, The decomposition and classification of radiant affine 3-manifolds, *Mem. Amer. Math. Soc.* 154 (2001), no. 730, viii+122 pp.
[20] S. Choi, Convex decompositions of real projective surfaces I: π-annuli and convexity, *J. Differential Geom.* 40 (1994), 165–208.
[21] S. Choi, Convex decompositions of real projective surfaces II: Admissible decompositions, *J. Differential Geom.*, 40 (1994), 239–283.
[22] S. Choi, Convex decompositions of real projective surfaces III: For closed and nonorientable surfaces, *J. Korean Math. Soc.* 33 (1996), 1138–1171.
[23] S. Choi, *Geometric structures on 2-orbifolds: exploration of discrete symmetry*, MSJ Memoirs, Vol. 27, 171pp + xii, 2012
[24] S. Choi, The classification of radial ends of convex real projective orbifolds, arXiv:1304.1605
[25] S. Choi, The classification of radial or totally geodesic ends of real projective orbifolds II: properly convex ends, preprint, 2014
[26] S. Choi, The classification of radial or totally geodesic ends of real projective orbifolds III: nonproperly convex convex ends, preprint, 2014
[27] S. Choi, The convex real projective manifolds and orbifolds with radial or totally geodesic ends: the closedness and openness of deformations, arXiv:1011.1060
[28] S. Choi, The deformation spaces of convex real projective orbifolds with radial or totally geodesic ends I: general openness, preprint, 2014
[29] S. Choi, The deformation spaces of convex real projective orbifolds with radial or totally geodesic ends II: relative hyperbolicity, preprint, 2014
[30] S. Choi, The deformation spaces of convex real projective orbifolds with radial or totally geodesic ends III: openness and closedness, preprint, 2014
[31] S. Choi and W.M. Goldman, The deformation spaces of convex $\mathbb{RP}^2$-structures on 2-orbifolds, *Amer. J. Math.* 127 (2005), 1019–1102.
[32] S. Choi and W. M. Goldman, Topological tameness of Margulis spacetimes, arXiv: 1204.5308
[33] S. Choi, C.D. Hodgson, and G.S. Lee, Projective deformations of hyperbolic Coxeter 3-orbifolds, *Geom. Dedicata* 159 (2012), 125–167.
[34] S. Choi and G. Lee, Projective deformations of weakly orderable hyperbolic Coxeter orbifolds, preprint arXiv:1207.3527.
[35] D. Cooper and D.D Long, A generalization of the Epstein-Penner construction to projective manifolds, arXiv:1307.5016
[36] D. Cooper, D. Long, and M. Thistlethwaite, Computing varieties of representations of hyperbolic 3-manifolds into $\text{SL}(4, \mathbb{R})$, *Experimental Math.* 15 (2006), 291–305.
[37] D. Cooper, D. Long, and M. Thistlethwaite, Flexing closed hyperbolic manifolds, *Geom. Topol.* 11 (2007), 2413–2440.
[38] D. Cooper, D. Long, and S. Tillmann, On convex projective manifolds and cusps, Preprint, arXiv:1104.0585.
[39] J. P. Conze and Y. Guivarch, Remarques sur la distalité dans les espaces vectoriels, *C. R. Acad. Sci. Paris* 278 (1974), 1083–1086.
[40] M. Crampon and L. Marquis, Finitude géométrique en géométrie de Hilbert, Preprint arXiv:1202.5442.
[41] J. de Groot and H. de Vries, Convex sets in projective space, *Compositio Math.*, 13 (1958), 113–118.
[42] D. Fried, Distality, completeness, and affine structure, *J. Differential Geometry* 24 (1986), 265–273.
[43] D. Fried and W. Goldman, Three-dimensional affine crystallographic groups, *Adv. Math.* 47 (1983), 1–49.
The classification of ends

[44] D. Fried, W. Goldman, and M. Hirsch, Affine manifolds with nilpotent holonomy, *Comment. Math. Helv.* 56 (1981), 487–523.

[45] W. Goldman, Convex real projective structures on compact surfaces, *J. Differential Geometry*, 31 (1990), 791–845.

[46] W. Goldman, Projective geometry on manifolds, Lecture notes available from the author.

[47] W. Goldman and F. Labourie, Geodesics in Margulis space times, *Ergod. Th. & Dynamic. Sys.* 32 (2012), 643–651.

[48] W. Goldman, F. Labourie, and G. Margulis, Proper affine actions and geodesic flows of hyperbolic surfaces, *Annals of Mathematics* 170 (2009), 1051–1083.

[49] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.* No. 53 (1981), 53–73.

[50] O. Guichard, Sur la régularité Hölder des convexes divisibles, *Erg. Th. & Dynam. Sys.* 25 (2005), 1857–1880.

[51] O. Guichard and A. Wienhard, Anosov representations: domains of discontinuity and applications, *Invent. Math.* 190 (2012), no. 2, 357–438.

[52] D. Heard, C. Hodgson, B. Martelli, and C. Petronio, Hyperbolic graphs of small complexity, *Exp. Math.* 19 no. 2 (2010), 211–236.

[53] M. Heusener and J. Porti, Infinitesimal projective rigidity under Dehn filling, *Geom. Topol.* 15 no.4 (2011), 2017–2071.

[54] D. Johnson and J. Millson, Deformation spaces associated to compact hyperbolic manifolds, In *Discrete groups in geometry and analysis* (New Haven, Conn., 1984), pp. 48–106, Progr. Math., 67, Birkhäuser Boston, Boston, MA, 1987.

[55] V.G. Kac and È.B. Vinberg, Quasi-homogeneous cones, *Math. Zametki* 1 (1967), 347–354.

[56] A. Katok and B. Hasselblatt, *Introduction to modern theory of dynamical systems*, Cambridge University Press 1995.

[57] Inkang Kim, Compactification of strictly convex real projective structures, *Geom. Dedicata* 113 (2005), 185–195.

[58] S. Kobayashi, Projectively invariant distances for affine and projective structures, In: *Differential geometry* (Warsaw, 1979), 127–152, Banach Center Publ., 12, PWN, Warsaw, 1984.

[59] B. Kostant, On convexity, the Weyl group and the Iwasawa decomposition, *Ann. ENS.* 4em sére tome 6 no. 4 (1973), 413–455.

[60] B. Kostant and D. Sullivan, The Euler characteristic of an affine space form is zero, *Bull. Amer. Math. Soc.* 81 (1975), no. 5, 937–938.

[61] J. Koszul, Deformations de connexions localement plates, *Ann. Inst. Fourier (Grenoble)* 18 fasc. 1 (1968), 103–114.
[62] F. Labourie, Flat Projective Structures on Surfaces and Cubic Holomorphic Differentials’, Pure and Applied Mathematics Quaterly 3 no. 4 (2007), 1057–1099, Special Issue: In the Honor of Grisha Margulis, Part 1 of 2.

[63] Jaejeong Lee, A convexity theorem for real projective structures, arXiv:math.GT/0705.3920.

[64] L. Marquis, Espace des modules de certains polyèdres projectifs miroirs, Geom. Dedicata 147 (2010), 47–86.

[65] G. Mess, Lorentz spacetimes of curvature, Geom. Dedicata 126 (2007), 3–45.

[66] P. Molino, Géométrie global des feuilletages riemanniens, Nederl. Akad. Wetensch. Indag. Math. 44 (1982), no. 1, 45–76.

[67] C. Moore, Distal affine transformation groups, Amer. J. Math. 90 (1968) 733–751.

[68] P. Molino, Riemannian foliations, Progress in Mathematics, vol 73, Birkhäuser, Boston, Basel, 1988.

[69] M. S. Raghunathan, Discrete subgroups of Lie groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68, Springer Verlag, Berlin, 1972.

[70] H. Shima, The geometry of Hessian structures, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, xiv+246 pp, 2007.

[71] D. Sullivan and W. Thurston, Manifolds with canonical coordinate charts: some examples, Enseign. Math. (2) 29 no.1–2 (1983) 15–25.

[72] W. Thurston, Geometry and topology of 3-manifolds, available at http://library.msri.org/books/gt3m/.

[73] W. Thurston, Three-dimensional geometry and topology, Princeton University Press, Princeton NJ, 1997.

[74] V.S. Varadarajan, Lie groups, Lie algebras, and their representations, GTM Vol 102, Springer, Berlin, 1972.

[75] J. Vey, Une notion d’hyperbolicitè sur les variétés localement plates, C.R. Acad. Sc. Paris, 266(1968), 622–624.

[76] J. Vey, Sur les automorphismes affines des ouverts convexes sail-lants, Ann. Scuola Norm. Sup. Pisa (3) 24(1970), 641–665.

[77] È.B. Vinberg, Discrete linear groups that are generated by reflections, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 1072–1112.

[78] È.B. Vinberg, Hyperbolic reflection groups, Uspekhi Mat. Nauk 40 (1985), 29–66.

[79] È.B. Vinberg, Homogeneous convex cones, Trans. Moscow Math. Soc. 12 (1963), 340–363.
The classification of ends

[80] A. Yaman, A topological characterization of relatively hyperbolic groups, *J. Reine Angew. Math.* 566 (2004), 41–89.

[81] A. Weil, On discrete subgroups of Lie groups II, *Ann. of Math.* 75 (1962), 578–602.

[82] A. Weil, Remarks on the cohomology of groups, *Ann. of Math.* 80 (1964), 149–157.

[83] D. Witte, Superrigidity of lattices in solvable Lie groups, *Inv. Math.* 122 (1995), 147–193.

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