Novel Phases and Reentrant Melting of Two Dimensional Colloidal Crystals

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We investigate two-dimensional (2d) melting in the presence of a one-dimensional (1d) periodic potential as, for example, realized in recent experiments on 2d colloids subjected to two interfering laser beams. The topology of the phase diagram is found to depend primarily on two factors: the relative orientation of the 2d crystal and the periodic potential troughs, which select a set of Bragg planes running parallel to the troughs, and the commensurability ratio $p = a'/d$ of the spacing $a'$ between these Bragg planes to the period $d$ of the periodic potential. The complexity of the phase diagram increases with the magnitude of the commensurability ratio $p$. Rich phase diagram, with “modulated liquid”, “floating” and “locked floating” solid and smectic phases are found. Phase transitions between these phases fall into two broad universality classes, roughening and melting, driven by the proliferation of discommensuration walls and dislocations, respectively. We discuss correlation functions and the static structure factor in these phases and make detailed predictions of the universal features close to the phase boundaries. We predict that for charged systems with highly screened short-range interactions these melting transitions are generically reentrant as a function of the strength of the periodic potential, prediction that is in accord with recent 2d colloid experiments. Implications of our results for future experiments are also discussed.

I. INTRODUCTION

A. Motivation and background

Two-dimensional (2d) melting and the mathematically related systems, such as for example the normal-to-superfluid and planar paramagnet-to-ferromagnet transitions in films (described by the 2d XY model) are striking examples of the increased importance of thermal fluctuations in low dimensional systems.\textsuperscript{[1,2]} In contrast to their bulk, three- (and higher-) dimensional analogs, where, typically, fluctuations only lead to quantitative modifications of mean-field predictions (e.g., change values of critical exponents), here the effects are qualitative and drastic. Located exactly at the lower-critical dimension ($d_{lc} = 2$), below which the distinction between the high and low temperature phases is erased by fluctuations, two-dimensional melting can proceed via a subtle, two-stage, continuous transition, driven by unbinding of topological defects (dislocations and disclinations). This mechanism, made possible by strong thermal fluctuations, therefore provides an alternative route to a direct first-order melting, argued by Landau’s mean-field analysis\textsuperscript{[3]} to be the exclusive scenario.

Despite of its long history, dating back to the work of Kosterlitz and Thouless\textsuperscript{[4]}, Halperin and Nelson,\textsuperscript{[5]} and Young\textsuperscript{[6]} (KTHNY) (which in turn built on a large body of ideas dating back to Landau and Peierls\textsuperscript{[7]}), interest in 2d melting and related problems persist. On the theoretical side this, in part, is due to the fact that the theory of 2d melting is an unusual example of a nontrivial and quite exotic critical point that lends itself to an asymptotically exact description. Furthermore, the KTHNY class of transitions (2d melting and related disordering of a 2d XY model) provides a rare example of a thermodynamically sharp phase transitions between phases, both of which lack long range order.\textsuperscript{[8]}

Although evidence for defect driven phase transitions has appeared in numerous experiments on liquid crystals\textsuperscript{[9]} and Langmuir-Blodgett films\textsuperscript{[10]}, finding simple model systems which exhibit these phenomena in experiments or simulations has proven to be more controversial. Some system parameters appear to fall in the range in which instead it is discontinuous melting that converts a solid directly into a liquid. However, it appears, that two-stage continuous melting has been recently experimentally observed by Murray et al.\textsuperscript{[11]} and Zahn et al.\textsuperscript{[12]} in beautiful melting experiments on two-dimensional colloids confined between smooth glass plates and super-paramagnetic colloidal systems, respectively. In these experiments, an orientationally quasi-long-range ordered but translationally disordered hexatic phase\textsuperscript{[13]} was observed. This phase, intermediate but thermodynamically distinct from the 2d solid and isotropic liquid, is an important signature of defect driven two-stage melting. In these two-dimensional colloids, particle positions and the associated topological defects can be directly imaged via digital video-microscopy, allowing precise quantitative tests of the theory. Colloids are thus ideal experimental model systems to explore the details of two-dimensional melting and related phenomena, many of which are the focus of the theory presented here.\textsuperscript{[13]}

Soon after the initial development of the theory of two-dimensional melting, theoretical efforts turned to the studies of the effects of substrate, an important ingredient in many physical systems. These studies\textsuperscript{[4,14,15]} uncovered a rich phenomenology stemming from the interplay between the underlying periodic substrate and
a quasi-long-range ordered solid film interacting with it. While many experiments have been undertaken, with a krypton film on a graphite substrate (see e.g. Ref. [15] for a review) being one of the best studied, these systems are far from ideal in exploring this rich phenomenology, because of the lack of substrate tunability; in these systems it is difficult to change the substrate period, dimensionality and pinning strength.

A series of pioneering experiments by Chowdhury, ACKERSON and Clark [16] constituted an important new development. In these studies strongly interacting colloidal particles, confined to two dimensions, were subjected to a one-dimensional periodic potential, induced by the interference fringes from two laser beams crossed in the sample. The light-induced polarization in these micron-size dielectric particles interacts with the laser electric field, leading to a radiation pressure force, [17] directed toward the regions of high laser intensity, the antinodes maxima in the laser standing wave pattern.

One of many interesting phenomena discovered by Chowdhury et al. is the fixed-temperature freezing transition driven by increasing the strength of the laser potential, dubbed “light induced freezing” (LIF). Qualitatively, LIF is due to the suppression of thermal fluctuations transverse to the imposed periodic pinning laser potential. This intuition is also supported by a more quantitative analysis based on Landau’s free energy expansion in the translational order parameters (density Fourier modes) \( \rho G_1 \), with \( \{ G_i \} \) the three smallest reciprocal lattice vectors of a triangular lattice. In the simplest geometry, with e.g., \( G_1 \) commensurate with the laser potential, \( \langle \rho G_1 \rangle \) is trivially induced by the potential even in the liquid phase. Such finite \( \langle \rho G_1 \rangle \) then converts the Landau’s cubic coupling \( \rho G_1 \rho G_2 \rho G_3 \), (which, in mean-field theory, is responsible for melting always being first-order) into a simple upward shift in the melting temperature for the only remaining critical mode \( \psi \equiv \rho G_2 - \rho G_3 \). Not surprisingly the resulting Landau expansion contains only even powers of this complex order parameter \( \psi \), which therefore generically orders via a continuous transition in the XY universality class. Hence, within the mean-field description discussed by Chowdhury et al. [16], one expects to reach a tricritical point upon increasing the light intensity, beyond which the LIF transition becomes continuous.

However, because of the dominant role of thermal fluctuations in two-dimensional systems, such “soft-spin” Landau expansions in order parameter amplitudes (and the related density functional theories) [18] will have difficulties to capture the subtleties of the continuous topological phase transitions possible in these two-dimensional systems. Unfortunately, results from Monte-Carlo simulations are inconclusive. Although earlier simulations [15] claimed to have found a tricritical point at intermediate laser intensities, consistent with density functional theory, recent studies from the same laboratory [20] refute these results. These difficulties are perhaps unsurprising, given that even much larger scale simulations have, so far, failed to completely resolve the nature of 2d melting, even without a periodic external potential. [21]

An alternative (but complementary and in principle equivalent) “hard-spin” defect description (with order parameter amplitude fluctuations represented by defect cores), extended to include a one-dimensional periodic pinning potential may be necessary to correctly capture the rich phenomenology of the early experiments by Chowdhury et al. [16] and the recent ones by Wei et al. [22] and others. [23] Developing such a theoretical framework and exploring its details to interpret these experiments is the goal of the work presented here.

Our interest in this problem was stimulated by the experiments of Wei et al. [22], which extended the light-induced melting experiments to higher laser intensities than those studied in Ref. [14]. One other notable difference is that in contrast to the strong long-range interaction of unscreened charged colloids in highly deionized solution, [14] in the Wei et al. experiments colloidal particles were interacting via a short-ranged Debye potential, with ions in the solution screening the long-ranged Coulomb interaction. In addition to the light-induced freezing observed at low light intensities, the authors of Ref. [22] discovered a reentrant melting phenomena, “light-induced melting” (LIM), driven by the increased strength of the laser-induced one-dimensional periodic potential. As discussed below, this fascinating reentrance phenomena generically emerges from our theoretical analysis in the limit of a short Debye screening length.

The goal of this paper is to investigate the phenomena of two-dimensional melting in the presence of a one-dimensional periodic potential, and to answer many basic questions stimulated by these recent experiments. What is the nature of such melting transition, if not preempted (as it can always be) by the first-order transition? More generally, how is the standard phase diagram for 2d melting on a homogeneous substrate (which includes the 2d crystal, hexpatic and liquid phases) modified by the periodic laser potential? Which of the phases survive the light field and what new ones emerge in its presence? The answers to these and many other questions, provided below, lead to results consistent with experimental observations, and have many testable consequences for the possible future experiments.

### B. Summary of the results

Even in the liquid phase at high temperatures the laser interference fringes, which we choose to run along the x-axis, induce a periodic density modulation in the colloidal liquid. As a consequence the static structure function \( S(q) \) displays Bragg peaks at \( K_n = n(2\pi/d)\vec{y} \), the integer multiples \( n \in \mathbb{Z} \) of the reciprocal lattice vector \( K = (2\pi/d)\vec{y} \) of the imposed one-dimensional periodic
potential with trough spacing $d$. \[24\] The liquid phase density exhibits a finite linear response to such a periodic perturbation with amplitude $U_K$, which is proportional to the input laser intensity $I_{in}$. This is consistent with the observations of Chowdhury et al. \[16\] who found the scattered laser intensity, $I_{out}$, at these directly induced Bragg peaks, to scale as a cube of the input laser power $I_{in}$. \[24\] These explicitly induced features of the modulated liquid persist throughout the phase diagram, with the additional structure emerging as a result of numerous spontaneous symmetry breakings, which we discuss below.

The laser-induced periodic potential also explicitly breaks continuous 2d rotational symmetry down to $\mathbb{Z}_2$ symmetry (rotations by $\pi$). Consequently, the one-dimensional periodic potential induces nematic, square, hexatic and higher orientational harmonics long-range orders, respectively characterized by a $2n$-atic bond orientational order parameter $\psi_{2n} = \langle e^{i2n\theta(r)} \rangle$, which, independent of any other details, are nonzero throughout the phase diagram. Therefore, in particular, the laser potential eliminates the continuous transition from an isotropic liquid to a hexatic liquid phase, expected in two-dimensional liquids in the absence of an external potential. \[13\] This situation is analogous to a ferromagnet in a magnetic field, where the qualitative distinction between paramagnetic and ferromagnetic phases is erased by the external magnetic field, with both phases displaying a finite induced magnetization.

Since the hexatic orientational order is explicitly induced by the laser potential, it must vanish as the laser field is tuned to zero. Analogously to a power-law vanishing of the magnetization with an external magnetic field in a ferromagnet at its critical point, we predict that at low input light intensities, $I_{in}$, the orientational order parameter vanishes as a universal power of $I_{in}$,

$$\psi_6 \sim I_{in}^{1/\delta_{\psi_6}}, \quad (1.1)$$

with $1/\delta_{\psi_6} = 6 \ [26]$ in the liquid phase and

$$1/\delta_{\psi_6} = \frac{6\eta_6}{4 - \eta_6}, \quad (1.2)$$

in the hexatic phase, where $\eta_6$ is the exponent describing the algebraic decay of bond orientational order in the absence of the laser-induced periodic potential. \[8\] We expect $\psi_6$ to approach a nonzero $I_{in}$-independent constant in the solid phase, consistent with the spontaneous long-range hexatic order of the 2d crystal, even in the absence of a periodic potential.

All other details of the phase diagram and the properties of the phases for our system strongly depend on the level of commensurability between the two-dimensional colloidal crystal, in the absence of the laser field, and the one-dimensional periodic potential that it induces. This in turn is determined by two ingredients: (i) the orientation of the triangular colloidal lattice relative to that of the periodic potential troughs, which selects a set of Bragg planes that run parallel to the troughs, (ii) the commensurability ratio of the spacing $a'$ between these Bragg planes to the period $d$ of the laser potential, defined by $p \equiv a'/d$. In this paper we will primarily focus on the commensurate case defined by $p \in \mathbb{Z}$ and defer the rich phenomenology of the incommensurate case and the commensurate-incommensurate transitions to a later publication. \[27\]

For these commensurate densities, independent of the order of commensurability, $p$, at the lowest temperatures we always find that our system freezes into a novel type of a crystal, which we call a “locked floating solid” (LFS). This phase derives its apparently contradictory name from its novel highly anisotropic properties: while the solid is pinned transversely to the troughs of the periodic potential, executing only massive optical phonon-like excitations in that direction, it is able to slide freely along the potential minima with acoustic phonon excitations within the troughs. Upon integrating out the massive $\psi_{gt}$-modes and using standard renormalization group methods \[5\] to eliminate bound dislocation pairs in the LFS phase, we are left with a free energy with temperature and potential strength dependent effective elastic constants,

$$H_{LFS} = \frac{1}{2} \int d^2r \left\{ K_{\text{eff}} (\partial_x u_x)^2 + \mu_{\text{eff}} (\partial_y u_x)^2 \right\}. \quad (1.3)$$

The structure function of LFS is quite unusual. Unlike the high temperature modulated liquid discussed above, the LFS displays a set of delta-function Bragg peaks (reduced by the Debye-Waller factor) located at the multiples of the laser potential reciprocal lattice vector $\mathbf{K} = (2\pi/d)\hat{y}$, which coexist with other spontaneously induced Bragg and quasi-Bragg peaks.

The more detailed properties of the LFS and other phases exhibited by our system, strongly depend on the choice of the infinite set of colloidal crystal orientations relative to the light interference fringes. While we will explore these numerous possibilities in their full generality in the main body of the manuscript, in this subsection we summarize our results only for the simplest orientation studied in the experiments of Refs. \[16\][22], in which the periodic potential troughs run parallel to the primary Bragg planes. \[28\]

Experimentally, we expect our system to display a considerable amount of irreversibility, with the choice of the relative orientation highly dependent on the way the system is taken into the crystal state: if the laser potential is turned on in the liquid phase (field-cooled), the crystal will freeze into the lowest energy orientation consistent with the imposed colloidal density (or the chemical potential) and laser fringe spacing; \[29\] in contrast, in zero-laser-field cooling experiments, an already formed crystal may be unable to reorient significantly, and will therefore lock into a metastable orientation, determined by the plane of the two interfering laser beams.
FIG. 1. Triangular lattice with lattice constant $a$ subject to a periodic potential (maxima indicated by dashed lines) for $pd = a'$ with $a' = \sqrt{3}a/2$ and $p = 2$. Also shown is the low energy dislocation with Burgers vector $b$ parallel to the corrugation of the external potential.

Once we focus on the primary orientation, illustrated for $p = 2$ in Fig. 1, the phenomenology of our system is completely determined by the integer commensurability ratio $p$. As we will show, for commensurate densities, our system admits three phase diagram topologies, corresponding to the three ranges of the values of $p$: (i) $p = 1$, (ii) $1 < p \leq p_c$, and (iii) $p > p_c$, with the critical value of $p_c \approx 3.7$ for the primary orientation.

### 1. Commensurability ratio $p = 1$

For $p = 1$, we find the phase behavior of the 2d colloidal system as summarized by the phase diagram illustrated in Fig. 2.

![Figure 2: Schematic phase diagram for a primary commensurate orientation with commensurability ratio $p = 1$.](image)

In the limit of strong laser potential the particles are confined to a parallel array of equally spaced 1d channels of spacing $d$, illustrated in Fig. 3. If $u_n(x)$ is the particle displacement field along the $n$-th channel, we can write the energy of these weakly coupled one-dimensional rows of particles as

$$H = d \sum_n \int dx \left\{ \frac{1}{2} K \left( \frac{du_n}{dx} \right)^2 - \mu \left( \frac{a}{2\pi d} \right)^2 \cos \left[ \frac{2\pi}{a} (u_{n+1}(x) - u_n(x)) \right] \right\} , \quad (1.4)$$

where $K$ is the bare compressional elastic modulus within each channel and $\mu$ is the microscopic coupling between the channels determining the shear modulus of the 2d system. At high temperatures or weak microscopic coupling $\mu$, the colloid decomposes into an orientationally ordered two-dimensional liquid of decoupled one-dimensional channels. At temperature $T$, the phonon fluctuations within a channel then grow according to

$$\langle |u_n(x) - u_n(0)|^2 \rangle = \frac{k_B T}{dK} x , \quad (1.5)$$

as can be seen from the equipartition theorem. Upon choosing $x$ such that the root mean square phonon fluctuations equal the intrachannel particle spacing $a$, we determine a translational correlation length $\xi_T(T)$, which diverges at low temperatures

$$\xi_T(T) = \frac{Kd}{k_B T} a^2 . \quad (1.6)$$

![Figure 3: Colloidal particles in channels, labeled by $n$, with intrachannel compressional modulus $K$ and interchannel shear coupling $\mu$.](image)
The channels will couple to form a coherent two-dimensional LFS when the effective coupling \( \Delta \xi_T(T) \mu/a(2\pi d)^2 \) between correlated 1d regions of size \( \xi_T(T) \) surpasses the thermal energy \( k_B T \) which decorrelates the 1d channels. We associate this characteristic temperature with the melting temperature \( T_m \) of LFS, which is therefore given by
\[
k_BTm = \text{const.} a^2 \sqrt{\mu K}.
\]

A similar argument leads to the estimate for freezing into the three-dimensional locked floating solid phases discussed by Carraro [22] for rare gas atoms adsorbed into bundles of carbon nanotubes. As we describe in Appendix A, in terms of the weakly coupled model, Eq. 1.1, freezing into LFS takes place at strong coupling \( \mu \), and therefore does not allow a rigorous renormalization group treatment of the transition. Nevertheless an approximate variational treatment is possible and is presented in Appendix A.

Instead, here we take an alternative route to the study of the LFS melting and other transitions by working within a continuum elastic model Eq. 1.3 which is equivalent to the strong coupling (between the channels) limit of the discrete model in Eq. 1.1. Such approach allows a more refined and asymptotically exact renormalization group analysis (presented below), within which we find that for \( p = 1 \) the melting of the LFS phase is in the universality class of the XY model, an is driven by unbinding of dislocation pairs with Burgers vectors \( b = a \hat{x} \) along the troughs of the periodic potential. Consequently, in contrast to the conventional 2d melting transition, at the melting temperature \( T_m \), we predict a universal ratio of the jump in the geometric mean of the long wavelength effective shear and bulk moduli, \( \mu_{\text{eff}}(T_m) \) and \( K_{\text{eff}}(T_m) \) (describing the elasticity of the LFS phase) to \( T_m \),
\[
\frac{\sqrt{K_{\text{eff}}(T_m) \mu_{\text{eff}}(T_m)}}{k_BT_m} = \frac{8\pi}{|b|^2}.
\]

This is in agreement, up to constants of order 1, with the rough estimate of the melting temperature, Eq. 1.5, sketched above, and with the variational method presented in Appendix A. The most striking feature of the \( p = 1 \)-LFS melting transition is the shape of the phase boundary \( T_m(U_K) \), whose universal features guarantee a generically reentrant melting, under conditions such as the experiments of Wei et al. [22]. At low light intensities, i.e., small \( U_K \), we find that the melting curve has a universal, cusp shape:
\[
T_m(U_K) \sim T_m(0) + \ln(k_BT_m(U_K))^{-1/\nu},
\]
with \( \nu \approx 0.36963 \). On the other hand for large \( U_K \), i.e., for \( k_BT_m(U_K)/U_K \ll 1 \), we find that for short-range particle interactions (\( \kappa a \gtrsim 5.8 \)), \( T_m(U_K) \) generically increases with decreasing amplitude \( U_K \) of the periodic modulation, according to
\[
T_m(U_K) = T_m^\infty \left\{ 1 + \frac{5[(\kappa a)^2 - 31]}{64\pi^2} \left( 1 + \frac{13}{3\kappa a} \right) \frac{k_BT_m^\infty}{B^2U_K} \right\}
\]
(1.10)
thus implying reentrant melting for a band of temperatures as a function of potential strength (see Fig. 2). In Eq. 1.10 above, \( \kappa \) is the inverse of the Debye screening length, tunable by adjusting the solution salt concentration, and \( T_m^\infty = T_m(U_K \to \infty) \), which, for the system studied in Ref. [22], we estimate to be approximately \( 1.3T_m(U_K = 0) \).

The structure function for the \( p = 1 \)-LFS, illustrated in Fig. 3 is also quite unusual. In addition to the set of Bragg peaks, directly induced by the laser field, \( S(q) \) also displays an independent set of quasi-Bragg peaks at the off-\( q_0 \)-axis reciprocal lattice vectors \( G \).
\[
S(q) \sim \frac{1}{|q - G|^2 - \eta_G^2},
\]
(1.11)
which distinguishes the LFS from the modulated liquid state. The corresponding density-density correlation function \( C_G(r) = \langle \rho_G(r)\rho_G(0) \rangle \) for reciprocal lattice vectors with \( G_x \neq 0 \) shows a power-law decay
\[
C_G(r) \sim \left( \frac{\mu_{\text{eff}}}{K_{\text{eff}}} \right)^{1/2} x^2 + \left( \frac{K_{\text{eff}}}{\mu_{\text{eff}}} \right)^{1/2} y^2 \right|^{-\eta_G^2/2},
\]
(1.12)
where \( \mu_{\text{eff}} \) and \( K_{\text{eff}} \) are the effective shear and bulk elastic moduli in Eq. 1.3 for the deformations along the troughs (\( x \)-axis) of the periodic potential. The exponent \( \eta_G \) depends on the relative orientation of the colloidal crystal and the troughs. Unlike conventional 2d melting [3], it is universal at the melting transition, and is given by
\[
\eta_G^* \equiv \eta_G(T_m) = (G \cdot b/4\pi)^2,
\]
(1.13)
where \( b \) is the smallest allowed Burgers vector in the trough direction. For the primary orientation, illustrated in Fig. 3 with \( b = a \), the exponent characterizing the algebraic order in the off-\( q_0 \)-axis peaks (see Fig. 3) closest to the \( q_0 \)-axis is \( \eta_G = 1/4 \); for the next row of peaks with \( G_x = 4\pi/a \) we find \( \eta_G = 1 \), consistent with the algebraic decay observed in Ref. [22] (for a more detailed discussion see section VII).

Our analysis also makes exact predictions for the structure function peak amplitudes in the limit of low laser intensity. Similar to the hexatic orientational order parameter \( \psi_6 \), Eq. 1.3, the translational order parameter, defined by \( M_{K_6} \equiv \langle \rho_{K_6} \rangle \), is induced by the periodic potential throughout the phase diagram. However, in contrast to the liquid phase, where it vanishes linearly with \( U_K \), in the crystal phase for \( T < T_m(0) \), we find
\[
M_{K_6} \sim |U_K|^{1/\delta_M},
\]
(1.14)
with \( \delta_M \) defined in analogy with the critical exponent at a ferromagnetic critical point,
\[ 1/\delta_M = \frac{\eta_{K_n}}{4 - \eta_{K_n}} , \]  

(1.15)

and where

\[ \eta_{K_n} = \frac{k_B T}{4 \pi} \frac{3 \mu + \lambda}{\mu (2 \mu + \lambda)} K_n^2 \]  

(1.16)

is the exponent with which the real-space density-density correlation function decays in a 2d crystal without a substrate potential. \[ \Box \] We therefore predict that for \( T < T_m(0) \) the intensity of the on-\( q_y \)-axis Bragg peaks vanishes as \[ \text{an exact} \] power of the input laser intensity \( I_{in} \), according to

\[ I_{out}(K_n) \sim \langle \rho_{K_n} \rangle^2 I_{in} , \]

(1.17a)

\[ \sim I_{in}^{1+2/\delta_M} . \]  

(1.17b)

In contrast, we predict the intensity \( I_{out} \) of the off-axis quasi-Bragg peaks, labeled by a reciprocal wavevector \( \mathbf{G} \), to vanish as

\[ I_{out}(\mathbf{G}) \sim I_{in}^{1+2\hat{\eta}_G/(4-\hat{\eta}_G)} L^{2-(\eta_G-\hat{\eta}_G)} , \]

(1.18)

where

\[ \hat{\eta}_G \equiv \eta_G (1-G_x^2/G^2) , \]  

(1.19)

and \( L \) is the system size.

We can also define the translational correlation length by the widths of the off-\( q_y \)-axis Lorentzian peaks in the structure function. As the melting temperature \( T_m \) is approached from above, given the \( XY \) nature of the \( p = 1 \)-LFS melting transition, we expect the correlation lengths parallel and perpendicular to the troughs to diverge according to

\[ \xi_{x,y} \sim e^{c/[T-T_m]^{1/2}} , \]  

(1.20)

where \( c \) is a constant of order unity.

2. Intermediate commensurability ratios: \( 1 < p < p_c \)

For \( 1 < p < p_c \), the phase diagram, illustrated in Fig. 4, generically includes an additional symmetry-allowed “locked smectic” (LSm) phase.

The LSm distinguishes itself from the modulated liquid by spontaneously breaking the liquid’s discrete translational symmetry by \( \delta \) down to translations by \( pd. \) \[ \Box \] In contrast to the LFS, however, the LSm exhibits only short-range correlations between colloidal positions lying in different troughs, and therefore does not resist shear deformations for displacements along the potential minima. Correspondingly, as illustrated in Fig. 4, the structure function of the LSm phase displays spontaneously induced Bragg peaks at \( K_n/p \), in addition to the Bragg peaks at \( K_n \), directly induced by the laser interference fringes. For \( 1 < p < p_c \), the LFS also displays these spontaneous Bragg peaks on the \( q_y \) axis at \( q = K_n/p \).

FIG. 4. Schematic phase diagram for a primary commensurate orientation with commensurability parameter in the range \( 1 < p < p_c \) (the case \( p = 2 \) is shown here). Thin lines indicate continuous phase transitions. The thick line between the LFS and modulated liquid phase is most likely a first-order phase boundary. \[ \Box \] Insets: Schematic structure functions. As in Fig. 3, the \( x \)'s indicate delta-function Bragg peaks and the shaded circles algebraic peaks.

Symmetry dictates that the freezing of the modulated liquid into the LSm is in the \( p \)-state clock model universality class. Also, similar to the melting of the \( p = 1 \)-LFS, we find that the \( 1 < p < p_c \)-LFS melts into the LSm through a transition in the \( XY \) universality class and will therefore also exhibit the usual Kosterlitz-Thouless phenomenology. \[ \Box \] We have also added in the phase diagram the possibility of a direct transition from the LFS to the modulated liquid at intermediate potential strength. We expect this transition to be different than the LFS-liquid transition for \( p = 1 \). Whereas the \( p = 1 \) transition is in the \( XY \) universality class, for \( 1 < p < p_c \) the LFS-liquid transition is associated with simultaneous loss of \( XY \) and discrete (Ising for \( p = 2 \)) order. Because at this latter transition two unrelated symmetries are simultaneously restored, we expect it to be first order. At the multicritical point, where the liquid, LFS and LSm phases meet, the critical behavior is presumably described by a two-dimensional compressible Ising model (for \( p = 2 \)) \[ \Box \] of the form

\[ H_{1-XY} [u,S] = \int d^2 r \left[ \frac{1}{2} (\nabla S)^2 + \frac{1}{2} r S^2 + v S^4 \right] + H_{LFS}[u] \]

\[ + \int d^2 r (\gamma_x \partial_x u_x + \gamma_y \partial_y u_x) S^2 . \]  

(1.21)

\( S \) is a continuous Ising order parameter, that distinguishes the LSm from the liquid, \( \gamma_x,y \) are “magnetoeelastic” parameters, which couple elastic strain to “mag-
netization” $S$, and where the parameters of the model are tuned to the tricritical point at which both order parameters vanish simultaneously. It would be interesting to study the properties of such tricritical point, which to our knowledge has not been previously explored.

3. Large commensurability ratios and floating phases: $p > p_c$

For these higher values of $p$, the complexity of the 2d colloidal phase diagram (displayed in Fig.5) further increases, allowing two new phases, the “floating solid” (FS) and the “floating smectic” (FSm).

The new phases are distinguished from their “locked” counterpart, the LFS and LSm, by their ability to slide (float) across the troughs of the periodic potential; technically, the periodic potential is irrelevant (in the renormalization group sense) and therefore can be treated perturbatively inside the FS and FSm phases. As illustrated in Fig.6, all the spontaneously induced structure function peaks of these floating phases are quasi-Bragg peaks, and therefore the corresponding density correlation functions display real-space power-law decay, similar to Eq.1.12. Although, in principle, the critical values $p_c^S, p_c^{Sm}$ for the appearance of each of these floating phases are most likely distinct, for simplicity of the presentation we have taken $p_c^S = p_c^{Sm} = p_c$. If in reality these critical values are sufficiently distinct, and $p_c^S < p_c^{Sm}$, then we expect an intermediate range of $p$ values, $p_c^S < p < p_c^{Sm}$, for which no FSm appears.

![Schematic phase diagram for a primary commensurate orientation with commensurability parameter $p > p_c$](image)

FIG. 5. Schematic phase diagram for a primary commensurate orientation with commensurability parameter $p > p_c$ (The case $p = 4$ is shown here). As in Fig.6 the thick line indicates a first order transition. Insets: Schematic structure factors. As in Fig.6, the $x$’s indicate delta-function Bragg peaks and the shaded circles algebraic peaks.

We find that phase transitions between the corresponding locked and floating phases (LFS-FS and LSm-FSm) are in the same universality class as the well-known thermal roughening transition. [33] the dual of the Kosterlitz-Thouless transition, with an identical phenomenology. Similar to the XY-melting LFS-FSm transition discussed above, the melting of the FS into the FSm proceeds via unbinding of dislocation pairs with $x$-directed Burgers vectors. However, because of the presence of massless spectator phonon modes in $y$-direction (transverse to the troughs of the periodic potential), the melting of the FS into the FSm is in a new universality class.

The direct transition from the LFS to the FSm phase is most likely first order. Here the order of the $u_x$-modes changes from quasi-long-range to short-range (via unbinding of type I dislocations, i.e., those with Burgers vector parallel to the troughs of the periodic potential) and for the $u_y$-modes from long-range to quasi-long-range (via depinning from the laser potential, i.e., a roughening transition). If both order parameters become critical at the same point in the phase diagram, which will be the case at the multicritical points where the FS, LFS, and FSm phases meet, we have a phase transition corresponds a simultaneous transition of the KT type and its dual analog.

The remainder of this paper is organized as follows: in Section II we introduce and motivate our model for 2d solids subjected to a 1d periodic potential and discuss the details specific to the experiments on 2d colloids in the laser potential. [22] In Sec III we give a detailed analysis of all the phases which are allowed by symmetry. In particular, the static structure factors and correlation functions are discussed. The mechanisms, dislocation unbinding and soliton proliferation, driving the phase transitions are investigated in Sec IV. In Sec V we derive the universal features of the melting phase boundary, demonstrating that for sufficiently short-range interactions it generically exhibits a reentrant melting observed in the experiments of Wei et al. [23] Some aspects of the response of the translational and bond-orientational order parameter to a small external 1d periodic potential are analyzed in Sec VI using a renormalization group crossover analysis. In Sec VII we elaborate on some implications of our results to experiments and for computer simulations.

II. BASIC INGREDIENTS

A. “Microscopic” model

In the absence of external perturbations, we expect that, at sufficiently low temperatures the 2d colloidal system freezes into a hexagonal 2d crystal illustrated in Fig.6.
associated with the colloidal displacement field \( u \) characterized by fundamental lattice vectors in the elastic strain described by the continuum isotropic elastic Hamiltonian for a particular orientation and magnitude of \( K \) measures the strength of the \( K \)th harmonic of the laser potential, proportional to the input laser intensity \( I_m \).

For the purpose of the discussion in this section we have chosen an “internal” reference frame \((\hat{e}_x, \hat{e}_y)\) where the orientation of the hexagonal lattice is kept fixed. Later, beginning with Sec III, we will switch to a “laboratory frame” \((\hat{x}, \hat{y})\) where the orientation of the laser potential is fixed with \( K \parallel \hat{y} \), i.e. the troughs are running parallel to the \( \hat{x} \)-axis.

B. Commensurability and reciprocal lattice

For a general wave vector \( K \), the periodic (laser) potential — characterized by a plane wave, \( e^{iK \cdot r} \) — will not be commensurate with the hexagonal lattice. Only for a particular orientation and magnitude of \( K \) will the spacing between the potential minima match with the periodicity of the hexagonal lattice. It is this special set of commensurate periodic potentials that is the focus of our work here. The characteristic set of commensurate wave vectors is easy to find since the reciprocal lattice is defined to be the set of all wave vectors \( G \) that yield plane waves with the periodicity of a given Bravais lattice. Hence, commensurability is equivalent to the condition that \( K \) coincides with one of the reciprocal lattice vectors \( G \).

In other words, the planes defined by the minima of the external potential \( \cos(K \cdot r) \) are a superset of the family of lattice planes (Bragg planes) defined by the shortest reciprocal lattice vector, \( G_{\bar{m}} = m_1 G_1 + m_2 G_2 \) with Miller indices \( m_1 \) and \( m_2 \), parallel to the wave vector of the external potential \( K \),

\[
K = p G_{\bar{m}} = p_1 G_1 + p_2 G_2 ,
\]  

Note that here we focus on situations where the colloidal particles are allowed to sit in the minima of the external potential only. More generally, one could also consider

\[
H_K = -U_K \frac{\sqrt{3}}{2} \sum_n \cos \left[ K \cdot (\mathbf{r}_n + u(\mathbf{r}_n)) \right] ,
\]  

where we have focussed on the energetically most important lowest harmonic \( K \) of such a laser-induced potential.
situations where the commensurability parameter is less than 1 with \( p \) a rational fraction. \[3d\]

With \( d = 2\pi/|K| \) being the periodicity of the potential and \( a'_\bar{m} = 2\pi/|G_{\bar{m}}| \) defining the distance between the lattice planes, the commensurability ratio \( p \) is given by

\[
p \equiv \frac{a'_\bar{m}}{d} = \frac{|K|}{|G_{\bar{m}}|} \tag{2.7a}
\]

\[
= \frac{\sqrt{3}a'/2}{d} \left( m_1^2 + m_2^2 - m_1m_2 \right)^{-1/2}. \tag{2.7b}
\]

This allows us to characterize the laser potential by a set of Miller indices \( \bar{m} = (m_1, m_2) \) and the commensurability ratio \( p \), i.e., in summary by a commensurability vector \( \vec{p} = \vec{m} = (p_1, p_2) \). Equivalently, the orientation of the Bragg planes can also be characterized by the shortest direct lattice vector pointing parallel to the troughs of the external potential,

\[
\vec{R}_{\bar{m}} \equiv n_1\vec{e}_1 + n_2\vec{e}_2 \tag{2.8}
\]

with the condition \( \vec{R}_{\bar{m}} \cdot \vec{G}_{\bar{m}} = 0 \), i.e., \( (n_1, n_2) = (m_1, -m_2) \) a set of integers (direct lattice Miller indices), with no common factor complementary to the Miller indices.

In Fig.8 we displayed an example for the simplest set of relative orientations between the periodic potential and the colloidal crystal. In our notation it corresponds to an orientation \( (m_1, m_2) = (1, 0) \) (or equivalently \( (n_1, n_2) = (1, 0) \)) and a commensurability ratio \( p = 3 \), i.e., \( \vec{K} = 3\vec{G}_{(1,0)} = 3\vec{G}_1 \) and a Bragg row spacing \( a'_\bar{m} = a' = \alpha\sqrt{3}/2 \). Because in such \( \bar{m} = (1, 0) \) orientations, it is primary Bragg rows \[28\] that run parallel to the periodic potential troughs, we call these relative orientations “primary”. Aside from the simplicity of these configurations, our interest in them is driven by experiments in Refs. \[16,22\], where a primary \( p = 1 \) orientation was studied.

FIG. 8. 2d hexagonal colloidal crystal in the presence of a commensurate 1d periodic potential with period \( d \), commensurability vector \( \vec{p} = 3(1, 0) \), and potential maxima indicated by solid lines. Dashed lines denote the Bragg rows picked out by the laser potential minima.

In addition to these primary \( \vec{p} = p(1,0) \) configurations, we will also make detailed predictions for the next simplest \( \vec{p} = p(1,-1) \) set of relative lattice–laser potential configurations, illustrated for \( p = 1 \) in Fig.9. We call these orientations “dual-primary”, because they correspond to Bragg rows running perpendicular to a fundamental real space lattice vector with \( \vec{K} = p(\vec{G}_1 - \vec{G}_2) = -\vec{e}_34\pi/a^2 \), rather than to one of the three fundamental reciprocal lattice vectors. In terms of the direct lattice these dual-primary orientations correspond to \( (n_1, n_2) = (1, 1) \) and Bragg row spacing \( a'_\bar{m} = a/2 \).

FIG. 9. 2d hexagonal colloidal crystal in the presence of a commensurate 1d periodic potential with period \( d \), commensurability vector \( \vec{p} = 1(1,1) \), and potential maxima indicated by solid lines. Dashed lines denote the Bragg rows picked out by the laser potential minima.

FIG. 10. 2d hexagonal colloidal crystal in the presence of a commensurate 1d periodic potential with period \( d \), commensurability vector \( \vec{p} = (2, -1) \), and potential maxima indicated by solid lines. Dashed lines denote the Bragg rows picked out by the laser potential minima.
Using the definition of commensurate configurations, Eq. 2.4 in Eq. 2.5 we find that $H_K$ reduces to

$$H_K = -U_K \frac{\sqrt{3}}{2} \sum_n \cos \left[ K \cdot \mathbf{u}(r_n) \right],$$

$$= -U_K a^{-2} \int d^2 r \cos \left[ K \cdot \mathbf{u}(r) \right],$$

where in going from Eq. 2.9a to Eq. 2.9b we went over to a continuum description, an innocuous approximation for the smooth, $|\mathbf{u}(r_{n+1}) - \mathbf{u}(r_n)| \ll a$ distortions, that we study here.

An equivalent “soft-spin” continuum description of the above interaction is in terms of the elementary translational order parameters $\delta \rho_{G_i} \mathbf{e}^{G_i \cdot \mathbf{r}}$ with $i = 1, 2$. The laser-induced periodic potential $h_{\mathbf{K}}(\mathbf{r}) = Re \left[ h_{\mathbf{K}}^0 \mathbf{e}^{G \cdot \mathbf{K} \cdot \mathbf{r}} \right]$, acts like an ordering field on the $\mathbf{p} = \pm (p_1, p_2)$th harmonics of the fundamental order parameters $\rho_{G_i}(\mathbf{r})$, allowing a linear coupling to $\rho_{\pm G}(\mathbf{r}) = \rho_{\pm G}^0 \mathbf{e}^{\pm i G \cdot (\mathbf{r} + \mathbf{u}(\mathbf{r}))}$

$$H_{\mathbf{K}} = -\alpha \int d^2 r \left[ h_{\mathbf{K}}(\mathbf{r}) \rho_{G}(\mathbf{r}) + c.c. \right]$$

$$= -\alpha \rho_{G}^0 h_{\mathbf{K}}^0 \int d^2 r \left[ e^{i(G-K) \cdot \mathbf{r}} e^{iG \cdot \mathbf{u}} + c.c. \right],$$

which is finite at long scales only if the condition Eq. 2.4 is satisfied, in which case it reduces to the expression given in Eq. 2.9b, with the identification $U_K/a^2 = 2 \alpha \rho_{G}^0 h_{\mathbf{K}}^0$. Hence the periodic potential explicitly breaks translational symmetry and therefore induces a finite translational order parameters $\rho_{\pm \mathbf{K}}(\mathbf{r})$ throughout the phase diagram.

C. Broken rotational symmetry and anisotropic elasticity

The imposed 1d periodic potential also explicitly breaks, throughout our phase diagram, the 2d rotational symmetry down to $Z_2$ (Ising) symmetry, corresponding to rotations by $\pi$. We can see this more explicitly by noting that the laser potential $h_{\mathbf{K}}(\mathbf{r}) = Re \left[ h_{\mathbf{K}}^0 e^{iG \cdot \mathbf{K} \cdot \mathbf{r}} \right]$ generates a set of even-rank tensor fields,

$$h_{i_1 \ldots i_{2n}}^{(2n)} = \partial_{i_1} h_{\mathbf{K}}(\mathbf{r}) \partial_{i_2} h_{\mathbf{K}}(\mathbf{r}) \ldots \partial_{i_{2n}} h_{\mathbf{K}}(\mathbf{r}),$$

where in above the overline denotes a spatial average. The lowest order, rank 2, tensor field is given by

$$h_{ij}^{(2)} = \partial_i h_{\mathbf{K}}(\mathbf{r}) \partial_j h_{\mathbf{K}}(\mathbf{r}),$$

$$= \frac{1}{2} \partial K_i K_j,$$

It is clear from their definition, that these laser-generated 2n-rank tensor fields have strengths proportional to $(U_K)^{2n} \propto (I_n)^{2n}$. They act as external ordering fields, which explicitly break rotational invariance (modulo rotations by $\pi$) of our system and therefore induce throughout our phase diagram finite 2n-adic orientational order parameters. These can be characterized by rank 2n symmetric traceless tensors, which are real irreducible representations of the rotation group and are the “angular harmonics” of the lowest order, rank 2, nematic order parameter

$$Q_{ij}^{(2)} = S(\hat{n}_i \hat{n}_j - \frac{1}{2} \delta_{ij}).$$

The unit vector $\hat{n}$, defines the principle axis of the nematic order, and, given Eq. 2.12b, points parallel or perpendicular (depending on the sign of the coupling between $h_{ij}^{(2)}$ and $Q_{ij}^{(2)}$) to the periodic potential wavevector $\mathbf{K}$. In two dimensions these 2n-rank tensor representations are well-known to be equivalent to the one-dimensional complex irreducible representations

$$\psi_{2n} = e^{i2n\theta}.$$  

Since in the presence of these laser-induced ordering fields all $\psi_{2n}$ orientational order parameters are finite throughout our phase diagram, no sharp continuous orientational ordering phase transitions are possible in our system. This is in contrast to the melting of the unperturbed lattice, where a thermodynamically sharp orientational phase transition is allowed between the isotropic and the anisotropic (e.g, hexatic, in a hexagonal lattice) liquids.

Therefore, throughout this paper we confine our attention only to phases and phase transitions that spontaneously break the translational symmetry of the explicitly orientationally ordered, modulated colloidal liquid phase.

The existence of these orientational ordering fields $h_{2n}$ has important consequences to the form of the colloidal crystal elastic energy. To deduce the form of the appropriate elastic Hamiltonian it is instructive to first consider a 2d hexagonal lattice in the absence such explicit symmetry breaking fields. Such state is characterized by a finite value of the hexagonal orientational order parameter $\psi_6$, [3] with the full 2d rotational symmetry broken down to the symmetry of discrete rotations by $2\pi/6$. Nevertheless to a quadratic order in the strain tensor $u_{ij}$, the energy is invariant under a full 2d rotation group.

In the absence of a periodic potential the hexagonal orientational order can be further spontaneously broken down to a lower symmetry. A physically important example is a uniaxially distorted hexagonal 2d crystal of anisotropic, orientationally ordered molecules, as, for instance found in a nematic liquid crystal. Such a system exhibits a spontaneous nematic order parameter $Q_{ij}^{(2)}$, which modifies the isotropic elasticity $H_0$, Eq. 2.4. To a quadratic order in the strain $u_{ij}$ three additional energetic contributions

$$\delta H_0 = \int d^2 r \left[ \alpha_1 u_{ij} Q_{ij}^{(2)} + \alpha_2 (u_{ij} Q_{ij}^{(2)})^2 + \alpha_3 u_{ij} u_{jk} Q_{ki}^{(2)} \right]$$

(2.15)
are allowed. Because the nematic order is induced spontaneously, simultaneous rotations of the lattice degrees of freedom and of the nematic axis (encoded in \(Q_{ij}^{(2)}\)), relative to an arbitrary frame fixed in the lab, is clearly a symmetry of such uniaxially distorted lattice. It is not difficult to show that this rotational freedom allows us to eliminate \(\alpha_1\) coupling linear in \(u_{ij}\)

\[
H_{\alpha_1} = \alpha_1 S \int d^2r \left[ u_{xx} \left( \sin^2 \theta - \frac{1}{2} \right) + u_{yy} \left( \cos^2 \theta - \frac{1}{2} \right) + u_{xy} 2 \sin \theta \cos \theta \right],
\]

by a judicious choice of the rotation angle \(\theta\) and a uniaxial area-preserving distortion

\[
u_i \rightarrow u_i + \phi_i.
\]

It is important to note that this is only possible because the three independent degrees of freedom, \(\theta\), \(\phi_x\) and \(\phi_y\), at our disposal, are sufficient to cancel the three independent linear terms \(u_{xx}, u_{yy}\), and \(u_{xy}\) in \(H_{\alpha_1}\), Eq. 2.16.

Adding \(H_{\alpha_2}\) and \(H_{\alpha_3}\) contributions, Eq. 2.15, to the Hamiltonian of an undistorted hexagonal lattice, Eq. 2.4, we find a general elastic Hamiltonian for a spontaneously uniaxially distorted hexagonal lattice (in the absence of an external potential) \([57]\) is given by

\[
H_0^\alpha = \int d^2r \left[ 2\mu u_{xy}^2 + \frac{1}{2} \lambda_{xx} u_{xx}^2 + \frac{1}{2} \lambda_{yy} u_{yy}^2 + \lambda_{xy} u_{xx} u_{yy} \right],
\]

where we have chosen a coordinate system in which the \(x\) and \(y\) axes coincide with the \(\mathbf{n}\) and \(\hat{z} \times \mathbf{n}\) principal axes of the orientational nematic order parameter \(Q_{ij}^{(2)}\). The two additional elastic constants, a total of four in \(H_0^\alpha\), are consistent with two new couplings \(\alpha_2\) and \(\alpha_3\) allowed by the finite orientational nematic order parameter \(Q_{ij}^{(2)}\). The four independent elastic constants also coincide with the expectation, that with a symmetry between \(x\) and \(y\) broken, the elastic energies associated with the strain tensor components

\[
u_{xx} = \partial_x u_x, \quad \nu_{yy} = \partial_y u_y, \quad \nu_{xy} = \frac{1}{2} (\partial_x u_y + \partial_y u_x)
\]

are clearly independent. Although rotations relative to the orientational uniaxial order is no longer a symmetry of \(H_0^\alpha\), because only the symmetric strain tensor \(u_{ij}\) enters the elastic energy, \(H_0^\alpha\) is still invariant under “atomic” displacements

\[
u = \theta z \times r,
\]

which correspond to global rigid rotations of the 2d solid, by an infinitesimal angle \(\theta\) about the \(z\)-axis. This latter symmetry is clearly present in an anisotropic lattice without an external pinning potential.

In contrast, however, in our system, the 1d periodic potential has a fixed orientation in the laboratory frame. Hence, in addition to the uniaxial lattice anisotropy, such a potential also explicitly breaks symmetry of rotations relative to the lab (and therefore to the periodic potential) frame. It therefore picks out a special coordinate system relative to which the angle \(\theta\) is measured.

As discussed above such external potential acts as an external \(2n\)-rank tensor fields and explicitly breaks the corresponding orientational symmetry. The appropriate elastic energy can be deduced by focusing on the lowest order nematic ordering field \(h_{ij}^{(2)}\). It allows the following additional energetic contributions

\[
H_{h_z} = - \int d^2r \left[ u_{ij} h_{ij}^{(2)} + Q_{ij}^{(2)} h_{ij}^{(2)} \right],
\]

that explicitly break symmetry of rotations relative to the frame picked out by the periodic potential.

For the purposes of classification of the relative orientations discussed in previous subsection, Sec. IIIB, it was more convenient to keep the lattice fixed and to rotate the periodic potential into a particular orientation, uniquely labeled by an integer 2-d Miller index vector \(\vec{p} = (p_1, p_2)\). However, once an orientation has been selected and classified by \(\vec{p}\), to analyze the continuum elastic model and its thermodynamics that follows it is more convenient to work in a coordinate system in which, indeed, the troughs of the 1d periodic potential run along the new \(x\)-axis. For such a choice of a lab coordinate system, \(h_{ij}^{(2)}\)

\[
(2.22)
\]

Using this expressions for \(h_{ij}^{(2)}\) together with \(Q_{ij}\), Eq. 2.13 inside Eq. 2.21 and combining it with \(H_{\alpha_1}\), Eq. 2.16, we find the following symmetry breaking energetic contribution, which, in the presence of a 1d periodic potential must be added to \(H_0^\alpha\), Eq. 2.18

\[
H_{\alpha_1+h_2} = \int d^2r \left[ \alpha_1 S u_{xx} (\sin^2 \theta - \frac{1}{2}) + u_{yy} (\alpha_1 S \cos^2 \theta - \frac{1}{2} h S \cos 2\theta) - \frac{1}{2} h S \cos 2\theta \right],
\]

where \(h = \frac{1}{2} K^2|\vec{h}_K|^2\) and angle \(\theta\) measures the deviation of the nematic axis \(\mathbf{n}\) away from \(\mathbf{K}\) set by the orientation of the periodic potential. While it is still possible to eliminate the terms linear in \(u_{xx}\) and \(u_{yy}\) by a lattice distortion Eq. 2.17, in the presence of the external potential it is no longer possible to shift away the \(u_{xy}\) term. Selecting \(\phi_i\) so as to cancel \(u_{xx}, u_{yy}\) and combining the resulting \(H_{\alpha_1+h_2}\) with \(H_0^\alpha\) we find
\[
H^a = \int d^2 r \left[ 2 \mu u_{xy}^2 + \frac{1}{2} \lambda_{xx} u_{xx}^2 + \frac{1}{2} \lambda_{yy} u_{yy}^2 + \lambda_{xy} u_{xx} u_{yy} + \alpha u_{xy} \sin 2 \theta - \frac{\gamma}{2} \cos 2 \theta \right], \tag{2.24}
\]

where we defined rotational symmetry breaking couplings \( \alpha \equiv \alpha_1 S \) and \( \gamma \equiv h S \). It is clear from the above \( \alpha \) and \( \gamma \) terms in \( H^a \) that, in the absence of strain, \( u_{xy} = 0 \), the energy is minimized by \( \theta = 0 \), corresponding to the nematic axis alignment with \( \mathbf{K} \), imposed by the periodic potential. In the presence of fluctuations \( \theta \) will be small but finite. Expanding \( H^a \), above, in these small fluctuations, we obtain a final form of the elastic Hamiltonian characterizing our system

\[
H_{el} = \int d^2 r \left[ 2 \mu u_{xy}^2 + \frac{1}{2} \lambda_{xx} u_{xx}^2 + \frac{1}{2} \lambda_{yy} u_{yy}^2 + \lambda_{xy} u_{xx} u_{yy} + 2 \alpha u_{xy} \theta + 2 \gamma \theta^2 \right]. \tag{2.25}
\]

Now to complete our derivation we must relate the angle \( \theta \), characterizing the orientation of the nematic order to the elastic \( u_{ij} \) degrees of freedom. We expect that the orientations of the nematic and hexatic order parameters, present in our uniaxially distorted hexagonal lattice, are locked together. Since in the crystalline phase, fluctuations in this bond orientational order are in turn locked to the local rotation angle induced by the phonon displacements, in the Hamiltonian Eq.\( 2\), we can make the well-known identification

\[
\theta = \frac{1}{2} (\partial_x u_y - \partial_y u_x), \tag{2.26}
\]

thereby completing our derivation. We find that the resulting elastic Hamiltonian, which characterizes a hexagonal lattice in the presence of a 1d periodic potential, involves 6 elastic constants. While a similar form was suggested, based on symmetry, by Ostlund and Halperin \cite{37} in their analysis of melting of distorted hexagonal crystal films, the \( \alpha \) term appearing in our Hamiltonian, Eq.\( 2\), was missed in their expression. As illustrated in Fig.\( 11 \), physically, the \( \alpha \) term is present because, with troughs running along the \( x \)-direction, an \( xy \)-strain will bring particles in Bragg planes lying in the troughs out of alignment with the minima of the periodic potential. This generates a torque which attempts to rotate the lattice and improve the alignment.

\[\text{FIG. 11. A } u_{xy} \neq 0 \text{ shear deformation (shown for simplicity for a square lattice) with principle axes along the (1,1) and (1,-1) directions in the xy-plane. In the presence of a trough potential (dashed lines) parallel to the x-direction, the particle array, with axes } x' \text{ and } y', \text{ can lower its interaction energy with the periodic potential by rotating in a clockwise direction to bring the particles into better alignment with the minima in the trough potential.}\]

\[\text{The elastic Hamiltonian } H_{el}, \text{ Eq.}\tag{2.25}, \text{ together with the commensurate pinning potential } H_K, \text{ Eq.}\tag{2.91}, \text{ defines our model 2d colloidal system in the presence of a commensurate periodic laser potential. Our aim, in the remainder of the paper is to analyze the symmetry allowed phases and the nature of the transitions between them embodied in this model.}\]

\[\text{III. SYMMETRY-ALLOWED PHASES}\]

The starting point of our analysis is the model Hamiltonian \( H = H_{el} + H_K \), obtained from combining Eqs.\( 2\) and \( 2\). Here we have chosen (without loss of generality) \( K \) to lie along the \( y \)-axis, i.e., the periodic potential troughs running parallel to the \( x \)-axis, a convention that we will stick to throughout the remainder of the paper. This Hamiltonian admits a rich variety of thermodynamically distinct phases. As discussed in the Introduction, the phase diagrams depend on the commensurability ratio \( p \), or more specifically, in which of the three regimes \( p = 1, 1 < p \leq p_c, \) or \( p > p_c, \) \( p \) actually falls. The complexity of the phase diagram is highest for \( p > p_c \), and so in order to discuss all the phases possible in our system, we focus on these high \( p \) commensurability ratios.

It is convenient to enumerate the five allowed phases starting with the most ordered, which naturally occurs at the lowest temperatures, and proceeding toward the higher temperature disordered phases, by invoking two types of disordering mechanisms, dislocation unbinding
and soliton proliferation. A detailed investigation of these mechanisms is deferred to the subsequent section, where we discuss the nature of these transitions and their hierarchy as a function of temperature and periodic potential strength (laser intensity). Here, we focus on the phases themselves, rather than on their location in the resulting phase diagram. As discussed in detail in the previous section and in the Introduction, the imposed periodic potential explicitly breaks rotational symmetry, and therefore all five phases exhibit true long-range orientational order. This external potential also explicitly breaks continuous translational symmetry along $y$ (with potential troughs taken to run along $x$) down to a discrete symmetry of translations by the period $d$ of the potential. Hence all phases will trivially exhibit long-range order in the corresponding translational order parameter, leading to true delta-function Bragg peaks at the multiples of the reciprocal lattice vector $(2\pi/d)\hat{y}$ in their structure functions.

A. Solid Phases

As in the absence of a periodic potential, the most ordered phase of isotropic particles confined to 2d is a solid. The striking effect of turning on an external 1d periodic potential is that it can lead to two thermodynamically distinct uniaxially distorted hexagonal crystal phases, which we term a “locked floating solid” (LFS) and a “floating solid” (FS). Being crystalline, both of these phases exhibit 2d translational (quasi-long-range) order, and are characterized by a finite shear modulus. These emerge as a result of breaking the translational symmetry $T^y \otimes T^x$ of the “modulated liquid” (ML) (see below), corresponding to independent discrete translations by $d\hat{y}$ and continuous translations along $\hat{x}$, down to 2d discrete translations generated by lattice vectors $e_1$ and $e_2$, Eq.2.1a,2.1b. Although in the presence of thermal fluctuations dislocations will be thermally nucleated, in the solid phases they are confined to finite size dipoles with a zero Burgers “charge”. These, therefore, can be safely integrated out of the partition function with only weak finite renormalization of the elastic constants. Consequently, a purely elastic description in terms of phonon modes $u_x$ and $u_y$ is appropriate in both phases.

The LFS and FS phases differ in the importance of the periodic pinning potential. In the FS, expected to occur at temperatures higher than the LFS, thermal fluctuations in the positions of the colloidal particles are sufficiently large such that at long length scales they average away most of the long scale effects of the periodic potential. In contrast, in the LFS the periodic potential strongly pins the colloidal crystal transversely to its troughs.

1. Floating Solid (FS)

The floating solid can be rigorously differentiated from its locked counterpart as a 2d colloidal crystal phase in which the periodic potential is irrelevant in the renormalization group sense. This implies that at long scales, many, but not all (see below) of the thermodynamic properties of the FS are well described by the elastic Hamiltonian $H_{el}$, Eq.2.2a, with two coupled “massless” $u_x$ and $u_y$ degrees of freedom, and ignoring $H_K$. Therefore, in many ways the FS is qualitatively quite similar to a 2d solid without the periodic pinning potential. In particular, this similarity extends to the lattice displacement correlation functions which are logarithmic in $x$ and $y$. However, these similarities do not extend to all correlation functions, and the periodic pinning potential has important qualitative consequences for the FS phase that distinguish it from an ordinary 2d solid. The “irrelevance” of the periodic potential means only that a perturbative expansion in $U_K$, for a sufficiently small value is convergent. Consequently, average quantities, that are finite at $U_K = 0$ can be well approximated by their $U_K = 0$ values, i.e., working with $H \approx H_{el}$, as is usually done. However, quantities that vanish (or diverge) to this zeroth order, must be evaluated to the next lowest order in $U_K$ to obtain a nontrivial (finite) result.

To illustrate this point, recall that the periodic potential explicitly breaks rotational and translational symmetry, despite its irrelevance in the FS phase. While the former leads to uniaxial anisotropy in the hexagonal lattice, the latter is responsible for the true long-range order in the translational order parameter $\rho_G(r)$, with $G$ integer multiples of the wave vector $K$ characterizing the periodicity of the external potential. In the presence of the periodic potential, even the most disordered modulated liquid phase (see below) displays true long-range translational and orientational order. Clearly then, a more ordered FS will also break these symmetries.

As a concrete example of how the periodic potential affects the FS phase, consider the real-space 2-point correlation function of the translational order parameter

$$\rho_G(r) \equiv e^{iG \cdot u(r)},$$  

(3.1)

defined by

$$C_G(r) \equiv \langle \rho_G(r) \rho_G^*(0) \rangle,$$  

(3.2a)

$$= \langle e^{iG \cdot (u(r)-u(0))} \rangle,$$  

(3.2b)

$$\equiv C_G^{(c)}(r) + \langle \rho_G \rangle \langle \rho_G^* \rangle,$$  

(3.2c)

where in Eq.3.2c $C_G^{(c)}(r)$ is the connected part of $C_G(r)$. The distinguishing feature of the FS phase is the irrelevance of the periodic potential $H_K$. Hence in the limit of a weak laser potential, i.e., small $U_K$, we can compute $C_G(r)$ in a controlled, convergent perturbative expansion in $U_K$. The connected part $C_G^{(c)}(r)$ is nontrivial even to zeroth order in $U_K$, and a standard calculation gives.

$$C_G^{(c)}(r)$$

13
\[ C_G^{(c)}(\mathbf{r}) \sim \frac{1}{|\mathbf{r}|^{3\mu + \lambda}}, \]  
\[ \overline{\eta_G} = \frac{|G|^2 k_B T 3\mu + \lambda}{4\pi \mu 2\mu + \lambda}. \]

For simplicity, we have used the isotropic elastic Hamiltonian, Eq (2.4), in place of the correct six elastic constant anisotropic Hamiltonian \( H_{el} \), Eq (2.25), which leads to a qualitatively similar, but anisotropic power-law decay of spatial correlations. We use long wavelength elastic constants finitely renormalized by thermally excited bound dislocation dipoles.

\[
\langle \rho_G \rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{U_K}{2a^2 k_B T} \right)^n \left( \delta_{G,nK} + \delta_{G,-nK} \right) \prod_{j=1}^{n} \int d^2 r_j \prod_{j=1}^{n} \left( |r_j|/a \right)^{-n\eta_K} \prod_{i<j} \left( |r_i - r_j|/a \right)^{-\eta_K}. 
\]

Here we have used the fact that for \( L \to \infty \)

\[
\langle \exp \left[ i \sum_{\alpha} \mathbf{q}_\alpha \cdot \mathbf{u}(\mathbf{r}_\alpha) \right] \rangle = \exp \left[ \sum_{\alpha < \beta} \mathbf{q}_\alpha \cdot \mathbf{q}_\beta G^{(c)}(\mathbf{r}_\alpha - \mathbf{r}_\beta) \right]. 
\]

for \( \sum_\alpha \mathbf{q}_\alpha = 0 \) and zero otherwise. We have also introduced a phonon connected correlation function \( G^{(c)}(\mathbf{r}) \)

\[
G^{(c)}(\mathbf{r}) \equiv \frac{1}{4} \langle |\mathbf{u}(0) - \mathbf{u}(\mathbf{r})|^2 \rangle_0. 
\]

Averages with elastic Hamiltonian are designated by \( \langle \ldots \rangle_0 \). Upon again approximating \( H_{el} \) by its isotropic form \( H_0 \), Eq (2.4), a straightforward calculation in the limit \( L/a \gg 1, r/a \gg 1 \), gives

\[
G^{(c)}(\mathbf{r}) \approx \frac{\overline{\eta_G}}{G^2} \log(r/a). 
\]

Since \( U_K \) is irrelevant in the floating solid phase the integrals in Eq (3.6) are IR convergent (i.e. for \( L \to \infty \)). The power-laws appearing in the integrand are implicitly understood to be cutoff below the lattice constant \( a \) scale by the obvious behavior (see Eq (3.8) of the phonon correlation function \( \lim_{r \to a} G^{(c)}(\mathbf{r}) = 0 \). Upon performing the spatial integrals, which are dominated by the behavior of the connected phonon correlation function at small distances (UV, lattice cutoff \( a \)), we obtain up to non-universal factors of order 1

\[
\langle \rho_G \rangle \approx \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{U_K}{2a^2 k_B T} \right)^n \left( \delta_{G,nK} + \delta_{G,-nK} \right). 
\]

Hence, as argued above on physical grounds, despite of the irrelevance of the periodic potential, the FS displays true long-range order in the translational order parameter \( \rho_G \), with \( G \) satisfying \( G = \pm nK \), with \( C_G(\mathbf{r}) \) approaching its asymptotic value as a power-law in \( r \).

We can compute the persistent part of \( C_G(\mathbf{r}) \), by calculating

\[
\langle \rho_G \rangle = \langle e^{iG \cdot \mathbf{u}(0)} \rangle, 
\]

in a perturbation theory in \( U_K \), which, because of the irrelevance of the periodic potential is convergent in the FS phase. For \( U_K = 0 \) the translational order parameter vanishes like \( \langle \rho_G \rangle = (L/a)^{-\overline{\eta_G}/2} \) with system size \( L \to \infty \). Upon expanding the Boltzmann weight \( e^{-(H_0 + U_K)/k_B T} \) in a power series in \( U_K \), we find to leading order in \( U_K \) and for \( L \to \infty \)

Other translational order parameters, with \( G \) not satisfying the above condition have pure power-law correlation functions, decaying to zero at long separations. In particular, these include the fundamental translational order parameters \( \rho_G \), which display quasi-long-range order in the FS phase.

Having calculated the translational correlation function \( C_G(\mathbf{r}) \), the structure function

\[
S(\mathbf{q}) = \sum_{\mathbf{r}_m} e^{-i\mathbf{q} \cdot \mathbf{r}_m} C_G(\mathbf{r}_m) 
\]

can now be easily obtained. Using Eqs (3.2), (3.3), and (3.10) and taking advantage of the Poisson summation formula to perform the sum over the lattice sites \( \mathbf{r}_m \), we find

\[
S(\mathbf{q}) \approx \sum_G \frac{1}{|\mathbf{q} - G|^2 - \overline{\eta_G}} + \sum_{n=-\infty}^{\infty} \frac{1}{n!} A_n \delta^{(2)}(\mathbf{q} - nK), 
\]

with

\[
A_n = \frac{1}{(n!)^2} \left( \frac{U_K}{2a^2 k_B T} \right)^{2n}. 
\]

and prime on the summation in Eq (3.12) indicating that the \( n = 0 \) term is excluded.

Equation (3.12) predicts true Bragg peaks (with power-law corrections) at multiples of the periodic potential wave vector \( K \) and pure power-law (quasi-) Bragg peaks at all other reciprocal lattice vectors \( G \), even for those with \( G \parallel K \). Note that in a real physical system, the periodic potential will not in general be a single harmonic as
assumed in our model, Eq.2.9. Hence, we expect that the Bragg peak amplitude $A_n$ observed in experiments will be a sum of terms like those in Eq.3.13 and the square of the amplitude of the $n$th Fourier harmonic, $U_{n\mathbf{K}}$ of the applied periodic potential. This of course will only modify the prefactors in the different contributions to $S(q)$, predicted for the FS in Eq.3.12. We schematically illustrate $S(q)$ for a floating solid in Fig.12 for the commensurability vectors $\tilde{p} = (5, 0)$ and $\tilde{p} = (2, -2)$, respectively, with the $y$-axis chosen to point along $\mathbf{K}$.

![Schematic structure function for the FS phase](image)

FIG. 12. Schematic structure function for the FS phase with the commensurability vector (a) $\tilde{p} = (5, 0)$ and (b) $\tilde{p} = (2, -2)$, illustrating a combination of the quasi- and true Bragg peaks, given by Eq.3.11. Crosses indicate true Bragg peaks and open circles quasi-Bragg peaks.

The set of on-$q_y$-axis quasi-Bragg peaks (indicated by open circles) interleaving the true Bragg peaks (indicated by “x”’s) is the notable feature that distinguishes the FS from its locked counterpart LFS, in which all on-$q_y$-axis peaks are true Bragg peaks.

2. Locked Floating Solid (LFS)

At sufficiently low temperatures, the periodic potential will always be a relevant perturbation, pinning the 2d solid in the direction perpendicular to its troughs. Because of the 1d nature of the pinning potential, the 2d crystal will remain unpinned along the direction of the potential minima and will be able to adjust freely in that direction. To reflect this dual character, we therefore call this phase the “locked floating solid”.

At high laser intensity, such that the bare value of the pinning energy $U_K$ is much larger than the elastic energy $\mu a^2$ for the shortest (and therefore all) wavelength phonon mode, our system is in the strong pinning regime. For a commensurate periodic potential, in this regime, fluctuations in the lattice positions perpendicular to the troughs are small and the periodic potential $H_{\mathbf{K}}$, Eq.2.9 can be safely expanded in powers of the corresponding phonon degree of freedom, $\mathbf{K} \cdot \mathbf{u}$, leading to

\[
H_{\mathbf{K}} \approx \text{const.} + \frac{1}{2} U_K a^2 \int d^2r \left( \mathbf{K} \cdot \mathbf{u}(r) \right)^2, \tag{3.14a}
\]

\[
\approx \text{const.} + \frac{1}{2} U_K a^2 K^2 \int d^2r u_y^2(r). \tag{3.14b}
\]

In contrast, a weak pinning regime, $U_K \ll \mu a^2$, consists of two sets of elastic modes, those with $k < k_c$ and those with $k > k_c$, where $k_c \equiv K/b_s$ is a crossover wavevector for which the elastic energy density $\mu(k,a)^2$ just balances the pinning energy density $U_K(b_s)K^2$ at the same length scale. Since the pinning energy is subdominate to the elastic energy for modes with $k > k_c$, we can simply integrate out these weakly pinned modes perturbatively in $U_K$. This results in an effective strength of the pinning potential given by

\[
U_K(b_s) = U_K b_s^{\nu_\mathbf{K}/2}. \tag{3.15}
\]

After equating this to the corresponding elastic energy $\mu(a/b_s)^2$ we find

\[
b_s = \left( \frac{\mu a^2}{U_K} \right)^{2/(4-\nu_\mathbf{K})}, \tag{3.16}
\]

which, when inserted inside Eq.3.13 leads to

\[
U_K(b_s) = U_K \left( \frac{U_K}{\mu a^2} \right)^{(4-\nu_\mathbf{K})/4}, \tag{3.17a}
\]

\[
= U_K^{4/(4-\nu_\mathbf{K})} / (\mu a^2)^{(4-\nu_\mathbf{K})/(4-\nu_\mathbf{K})}. \tag{3.17b}
\]

Since the $u_y$ fluctuations in the remaining strongly pinned elastic modes are small, the effective pinning potential $H_{\mathbf{K}}$ can once again be safely expanded in powers of $u_y$. Doing this we obtain a result identical to Eq.3.14 but with $U_K$ replaced by $U_K(b_s)$ given in Eq.3.17.

Hence, in both the strongly and weakly pinned regimes, unlike the FS phase, the LFS is characterized at long wavelengths by one acoustic ($u_x$) and one optical ($u_y$) phonon mode, with an effective Hamiltonian

\[
H = H_{el} + \frac{\mu}{2\xi} \int d^2r u_y^2(r). \tag{3.18}
\]

Here, we have introduced a correlation length $\xi$ which, given Eqs.3.14b, 3.17, reads

\[
\xi^{-2}(U_K) = \begin{cases} \frac{U_K^4}{\mu a^2} K^2, & \text{for } \frac{U_K}{\mu a^2} \gg 1, \\ \left( \frac{U_K}{\mu a^2} \right)^{4/(4-\nu_\mathbf{K})} K^2, & \text{for } \frac{U_K}{\mu a^2} \ll 1. \end{cases} \tag{3.19}
\]
At length scales longer than the crossover scale set by \( \xi \), Eq.3.19, we can safely ignore the spatial derivative of \( u_y \) terms, and the LFS is well described by an effective Hamiltonian

\[
H_{LFS} = \frac{1}{2} \int d^2r \left[ B_{yx}(\partial_y u_x)^2 + B_{xx}(\partial_x u_x)^2 + \frac{\mu}{\xi^2} u_y^2 \right],
\]

(3.20)

where

\[
B_{yx} = (\mu + \gamma - \alpha),
\]

(3.21)

\[
B_{xx} = \lambda_{xx}.
\]

(3.22)

We can now compute the translational order parameter correlation function and the structure function that characterize the LFS phase. Repeating first the calculation for the persistent part determined by \( \langle \rho G \rangle \), we immediately find, that in all the phases in the presence of the periodic potential, \( \langle \rho G \rangle \neq 0 \) for \( G = \pm nK \). However, the distinguishing feature of the LFS is that this average is finite for all \( G \) parallel to \( K \), by virtue of the finite pinning length \( \xi \), Eq.3.19. This result can be immediately seen by noting that for \( G \parallel K \), the logarithmically divergent (with \( L \)) \( \langle u^2 \rangle_0 \) correlation function does not appear in \( \langle \rho G \rangle \), where the subscript “0” again represents an average with the elastic Hamiltonian \( H_{el} \) only. Instead we have

\[
\langle \rho G \rangle = e^{-\frac{1}{2}G^2 \langle u^2 \rangle} = \left( \frac{a}{\xi} \right)^{\eta_{G}/2}
\]

(3.23a)

\[
\langle \rho G \rangle = \left( \frac{a}{\xi} \right)^{\eta_{G}/2}
\]

(3.23b)

which only involves the “massive” \( u_y \) degree of freedom, whose logarithmic correlations are cutoff at \( L_c = \xi \) and therefore is finite even in the thermodynamic limit.

We can also obtain the above result via a straightforward matching calculation. The difficulty of computing translational correlation functions in the weakly pinned regime of the LFS phase is that for long length scales \( (\gg \xi) \), despite of the weakness of the pinning potential, a direct perturbative expansion in \( U_K \) is divergent because of its relevance (in the renormalization group sense) inside the LFS phase. The power of the renormalization group is that it allows us to relate this difficult weakly pinned, small \( U_K \) regime to the strongly pinned regime, where \( U_K \) has grown to the magnitude of the elastic energy \( \mu a^2 \), and can therefore be treated as a “mass”, as in Eq.3.14. We can apply this matching procedure to the computation of \( \langle \rho G (U_K) \rangle \), by using a relation between the weakly and strongly pinned regimes, namely

\[
\langle \rho G (U_K) \rangle = b^{-\eta_{G}/2} \langle \rho G (U_K b^{-2} \eta_{G}/2) \rangle
\]

(3.24)

obtained using the scaling dimension of the operator \( \rho G \) and the RG eigenvalue of \( U_K \), both easily extracted from Eq.3.3. Choosing the arbitrary rescaling factor \( b = b_s \) such that \( U_K (b) \) is in the strongly pinned regime, where \( U_K (b_s) = \mu a^2 \), Eq.3.24 becomes

\[
\langle \rho G (U_K) \rangle = \left( \frac{U_K}{\mu a^2} \right)^{\eta_{G}/(4-\eta_{G})} \langle \rho G (\mu a^2) \rangle.
\]

(3.25)

Since the right hand side is in the strong coupling regime, it can be easily computed using the coarse-grained Hamiltonian, Eq.3.18. Doing this we find

\[
\langle \rho G (U_K) \rangle = \left( \frac{U_K}{\mu a^2} \right)^{\eta_{G}/(4-\eta_{G})} \rho_{G}(\mu a^2).
\]

(3.26)

which in the weakly pinned regime is equivalent to the result given in Eq.3.23.

Note that the nontrivial nonlinear power-law response of the translational order parameter to the periodic laser potential, predicted by Eq.3.20, is only a nonanalytic piece of the full response, which includes an analytical background. Hence, although at low temperatures, such as \( \eta_{G}/(4-\eta_{G}) < 1 \), the full response in the \( U_K \rightarrow 0 \) limit is dominated by the nonanalytical part, Eq.3.26, at higher temperatures, the ever-present linear piece of the analytical part will dominate, and experimentally one should instead observe

\[
\langle \rho G (U_K) \rangle \sim U_K.
\]

(3.27)

For our highly anisotropic system, the connected part of the correlation function \( C_G (r) \), is given by

\[
C^{(c)}_G (r) = e^{-\frac{1}{2}[G^2 G_{z z}(r)+2G_{x z}G_{y z}(r)+G_{y y} G_{y y}(r)]},
\]

(3.28)

where \( G_{i j}(r) \equiv \langle (u_i(r)-u_i(0))(u_j(r)-u_j(0)) \rangle \), is the connected phonon correlation function computed with the full Hamiltonian. In the weakly pinned regime, for small length scales, all phonon correlation functions display the usual 2d logarithmic growth, which, in the isotropic approximation, i.e., using Hamiltonian \( H_0 \), Eq.2.4 leads to the power-law correlation for \( C^{(c)}_G (r) \) that we found in Eq.3.3 for the FS phase. However, for length scales longer than \( \xi \), Eq.3.19, while \( G_{z z}(r) \) will continue to grow logarithmically, such growth in \( G_{y y}(r) \) and \( G_{y z}(r) \) will be cutoff by the pinning length \( \xi \). Consequently, in the LFS we find

\[
C^{(c)}_G (r) \approx \left( \frac{a}{r} \right)^{\eta_{G}/2} \left( \frac{a}{\xi} \right)^{\eta_{G}/2},
\]

(3.29)

where

\[
\eta_{G_{z z}} = \frac{G^2_{z z}}{2 \pi} \frac{k_B T}{B_{z z} B_{y z}},
\]

(3.30)

which reduces to \( \eta_{G_{z z}} = \frac{G^2_{z z}}{2 \pi} \frac{k_B T}{\mu a^2 (2 \mu + \lambda)} \) when the effect of the periodic potential on elasticity and renormalizations due to dislocation pairs on the effective elastic coefficients are neglected. A discrete Fourier transform of this correlation function gives the corresponding structure function
\[ S(\mathbf{q}) \approx \sum_{\mathbf{G}} \left[ \frac{B_{\mathbf{G}}}{|\mathbf{q} - \mathbf{G}|^2 - \eta_{\mathbf{G}x}^2} + A_{\mathbf{G}}\delta(2)(\mathbf{q} - \mathbf{G}) \right], \quad (3.31) \]

where the quasi-Bragg peak amplitude \( B_{\mathbf{G}} \) is given by
\[
B_{\mathbf{G}} \propto \left( \frac{a}{\xi} \right) \eta_{\mathbf{G}y},
\]
\[
\propto \begin{cases} 
U_K^{\eta_{\mathbf{G}y}/2}, & \text{for } \frac{U_K}{\mu a} \gg 1, \\
U_K^{\eta_{\mathbf{G}y}/(4 - \eta_{\mathbf{K}})}, & \text{for } \frac{U_K}{\mu a} \ll 1.
\end{cases}
\]
\[
\quad (3.32a)
\]
\[
\quad (3.32b)
\]

and the Bragg peak amplitude \( A_{\mathbf{G}} \)
\[
A_{\mathbf{G}} \propto \delta_{G_x,0} \left( \frac{a}{\xi} \right) \eta_{\mathbf{G}y},
\]
\[
\propto \begin{cases} 
U_K^{\eta_{\mathbf{G}y}/2}, & \text{for } \frac{U_K}{\mu a} \gg 1, \\
U_K^{\eta_{\mathbf{G}y)/(4 - \eta_{\mathbf{K}})}, & \text{for } \frac{U_K}{\mu a} \ll 1.
\end{cases}
\]
\[
\quad (3.33a)
\]
\[
\quad (3.33b)
\]

which is finite if and only if \( \mathbf{G} \) is parallel to \( \mathbf{K} \). As a consequence of the discussion after Eq.\(3.26\) the amplitude \( A_{\mathbf{G}} \) will also have a background analytic in \( U_K \), which in a weak pinning limit scales as \( U_K^2 \) (see Eq.\(3.27\)), and therefore at higher temperatures will dominate over the nonanalytical part predicted in Eq.\(3.33a\).

We illustrate schematically \( S(\mathbf{q}) \) in Fig.\(13\) for the commensurability vectors \( \vec{p} = (5, 0) \) and \( \vec{p} = (2, -2) \), respectively, with the \( y \)-axis chosen to point along \( \mathbf{K} \).

\[ \text{FIG. 13. Schematic structure function for the LFS phase with the commensurability vector (a) } \vec{p} = (5, 0) \text{ and (b) } \vec{p} = (2, -2), \text{ illustrating a combination of the quasi- and true Bragg peaks, given by Eq.\(3.11\).} \]

These predictions for the structure function of the LFS, displaying amplitudes that vanish as nontrivial powers (determined by a continuously varying exponent \( \eta_{G_x} \)) of the periodic potential strength (Eqs.\(3.32a, 3.33b\)) provide the first theoretical explanation for observations of Clark, et al.\[16\].

B. Smectic Phases

As was first pointed out by Ostlund and Halperin\[37\], in uniaxial two-dimensional lattices, dislocations with Burgers vector along and perpendicular to the uniaxial axis will generically have different core energies and will therefore proliferate at different temperatures. This will consequently allow the possibility of a phase that is intermediate between a fully ordered crystal and a completely disordered liquid.

In a commensurate orientation, such that Bragg rows coincide with the periodic potential troughs, we would expect dislocation pairs with Burgers vectors parallel to the potential minima, to unbind first. We call the resulting class of thermodynamically distinct phases, smectics. Their main common characteristic is that they display a finite elastic modulus for shear deformations perpendicular to the Burgers vector of unbound dislocations, but do not resist shear parallel to them, possessing only liquid-like correlations between the corresponding “atomic” rows. Consequently, such 2d smectics display 1d periodicity perpendicular to the Burgers vector of unbound dislocations, and, as illustrated in Fig.\(14\), can be equivalently described as a periodic stack of 1d liquids.

It is important to note, that despite of their name, the smectics discussed here are fundamentally distinct from the smectic phases found in liquid crystal materials and substrate-free smectics discussed in Ref.\[37\].\[39,40\] The most important distinction is that in liquid crystal smectics and those without an underlying pinning substrate, the orientational symmetry is broken spontaneously (uniaxial anisotropy notwithstanding; see Sec.\(II\), leading to a soft Laplacian-curvature (rather than gradient-tension) elasticity, which preserves this underlying symmetry even in the smectic phase, where it is nonlinearly realized.\[39,40\] In fact such substrate-free 2d smectics, because of the softness of their elasticity, are well known to be unstable to thermally-driven unbinding of dislocations, and at scales larger than the distance between these free dislocations are therefore indistinguishable from a nematically-ordered 2d liquid.\[41\] As was recognized by the authors of Ref.\[37\], such thermal instability of substrate-free 2d lattices precludes the existence of thermodynamically distinct intermediate 2d smectic phase in which only one set of Burgers vectors (e.g., along the uniaxial direction) unbind. However, in strong contrast to those rotationally invariant systems, in 2d lattices studied here the periodic (laser) potential explicitly breaks rotational symmetry, binding by a linear potential dis-
location pairs with Burgers vector having components along \( \mathbf{K} \). Consequently, such dislocations remain bound even when those with Burgers vectors perpendicular to \( \mathbf{K} \) unbind and therefore allow the existence of 2d smectic phases that are thermodynamically distinct from a liquid.

Deep in such a smectic phase, the \( u_y(\mathbf{r}) \) phonon field, which (see Fig. 14) describes local fluctuations in the maxima positions of the 1d density wave, is the only remaining important degree of freedom. The ever-present bound dislocation pairs and the density of vacancies and interstitials are “massive” degrees of freedom. They can be easily integrated out, leading only to a finite renormalization of elastic constants for \( u_y \) deformations, and therefore are unimportant in a static theory.

FIG. 14. 2d colloidal smectic phase in the presence of a commensurate 1d periodic potential with period \( d \), commensurability parameter \( p = 3 \), and potential maxima indicated by full horizontal lines. Dashed lines denote the maxima in the smectic density, which are pinned inside the minima of the periodic laser potential.

In close analogy to the translational order parameter of the 2d solid, the smectic is distinguished from a liquid by a finite translational order parameter \( \rho_\mathbf{G} = e^{i\mathbf{G} \cdot \mathbf{u}} \), but with a single (rather than a set, Eq. 2.2) reciprocal vector \( \mathbf{G} = G\mathbf{y} = (2\pi/a)\mathbf{y} \). It is related to the total molecular density via a standard relation

\[
\rho(\mathbf{r}) = \text{Re}[\rho_0 + e^{i\mathbf{G} \cdot \mathbf{r}} \rho_\mathbf{G}(\mathbf{r})],
\]

where \( \rho_0 \) is the mean density of the smectic.

Of course, in the presence of a 1d periodic potential, a smectic is a thermodynamically distinct phase only if \( \mathbf{G} = (2\pi/a)\mathbf{y} \) differs from the wavevector \( \mathbf{K} \) characterizing the external potential and the modulated liquid. Commensurate smectics, which we focus on here, are equivalently characterized by the ratio of their period \( a \) to that of the periodic potential \( d \), with commensurability ratio \( a/d \equiv p \in \mathbb{Z} \). A \( p \)-Smectic then spontaneously breaks the discrete translational symmetry \( T_d \otimes T^x \) of the modulated liquid, with its equal occupancy of each potential minima down to \( T_{d/a} \otimes T^x \), with only every \( p \)-th minima equivalently populated. Clearly then \( p = 1 \) is indistinguishable from the fully disordered modulated liquid.

Above symmetry considerations uniquely specify the Hamiltonian that characterizes the \( p \)-Smectic phase

\[
H_{\text{Sm}} = \int d^2r \left[ \frac{1}{2} \left( B_{xy}(\partial_x u_y)^2 + B_{yy}(\partial_y u_y)^2 \right) - U_K a^{-2} \cos(K u_y(\mathbf{r})) \right],
\]

which, not surprisingly, is an anisotropic scalar Sin-Gordon model in the phonon field \( u_y(\mathbf{r}) \).

Given the form of the Hamiltonian in Eq. 3.35, there is a close similarity between the properties of the smectic and the 2d crystal, studied in the previous section. The quantitative differences between these phases are due to the distinction between the vector \( (u_x, u_y) \) and scalar \( u_y \) nature of elastic degrees of freedom in the 2d solid and smectic, respectively. More specifically, in close analogy to the 2d solid, we find that for a fixed integer commensurability ratio \( p \), there exist a low temperature “locked” and higher temperature “floating smectic” phase. These are distinguished by the importance of the periodic pinning potential, which is relevant (in the RG sense) in the LS\( n \) phase acting as a “mass” for \( u_y \), and irrelevant in the FS\( m \) phase, where for most static properties it can be ignored.

1. Floating Smectic (FS\( m \))

In the “floating smectic phase” (FS\( m \)), thermal fluctuations in the position of the layers are sufficiently large that at long length scales they average away many effects of the periodic pinning potential. Hence, many of the static properties of the FS\( m \) phase can be well described by the Hamiltonian, Eq. 3.35, with \( U_K = 0 \). However, as we discussed in detail in our analysis of the FS phase, despite of the RG irrelevance of the periodic potential, continuous translational symmetry is still explicitly broken by it, which leads to true long-ranged translational order in the smectic order parameter \( \rho_\mathbf{G} \) for \( G \mathbf{y} \) at multiples of the reciprocal lattice vector \( K \mathbf{y} \), characterizing the laser potential.

Calculations that closely parallel those of Sec. III A 1 for the FS, lead to power-law correlations in the connected part of the translational two-point correlation function

\[
C^{(c)}_{\mathbf{G}}(r) \sim \frac{1}{|r|^{\eta_{\text{FSm}}}},
\]

where

\[
\eta_{\text{FSm}} = \frac{k_B T G^2}{2\pi \sqrt{B_{xy} B_{yy}}},
\]

is the exponent characterizing the FS\( m \) phase, in analogy to \( \eta_{\mathbf{G}} \), Eq. 3.34, of the FS.

The disconnected part of the smectic translational correlation function is finite only at \( G = nK \ (n \in \mathbb{Z}) \). The corresponding floating smectic structure function is given
by an expression similar to the FS, Eq. 3.12. The only difference is that η_G of the FS is replaced by η_FSm of the FSm and the summation over G is a sum over integer multiples of 2π/a. Consequently, one expects to see sharp peaks only on the qx-axis, with power-law peaks at G ≠ nK, and true Bragg peaks at G = nK. This FSm structure function is schematically displayed in Fig.15 below.

2. Locked Smectic (LSm)

As the temperature is lowered, the periodic potential becomes relevant, pinning the smectic layers. The resulting “locked smectic” (LSm) phase is characterized by long-range translational order, and, as illustrated in Fig.15, displays true Bragg peaks at all values of the on-qy-axis reciprocal lattice vectors G = n2π/a. At long scales, the effective elastic Hamiltonian that characterizes this phase is simply

\[ H_{LSm} = \frac{1}{2} \frac{\mu}{\xi^2} \int d^2 r \ u_y^2, \]

with ξ given by Eq. 3.19.

3. Modulated Liquid (ML)

The modulated liquid is the most disordered phase, which occurs at highest temperatures and does not spontaneously break any symmetries. It is characterized by a vanishing shear modulus, unbound dislocations, absence of massless Goldstone modes, and a discrete symmetry of translations along the y-axis by periodic potential constant d. The corresponding structure function of this explicitly orientationally ordered phase, illustrated in Fig.16, is a set of true Bragg peaks at multiples of the reciprocal lattice vector K = 2π/d of the periodic potential.

Phase transitions that take place in our system fall into two broad classes: roughening and melting. However, for high values of the commensurability ratio p (p > p_c) these classes are mathematically related to each other by the duality transformations \[ p \mapsto \frac{1}{1-p} \] \[ \eta \mapsto \frac{1}{\eta} \] \[ \tau \mapsto \omega \] \[ K \mapsto 1/K \], and are both examples of the Kosterlitz-Thouless type of transitions, with kinks and dislocations unbinding, respectively. For p < p_c, the roughening transitions are in a different (Ising model...
and other models with a discrete symmetry) universality class.

A. Roughening Transitions

Phase transitions that fall into the roughening transition universality class separate a low temperature ordered phase, in which a potential Goldstone mode is strongly pinned by an external periodic potential, from a quasi-long-range ordered phase, in which the periodic potential is irrelevant in a renormalization group sense. The locked floating solid to floating solid and the locked smectic to floating smectic transitions, discussed in Sec.III fall into this broad universality class, although differ in details that we discuss below.

Despite these small differences the analysis of these transitions are quite similar and can be done via standard perturbative momentum-shell renormalization group (RG) transformation. 

Since the smooth (locked) and rough (floating) phases are distinguished by the relevance and irrelevance of the periodic potential, respectively, we can find the transition temperature by analyzing the behavior of $H_{K}$, Eq.(2.25) as a function of length scale. We separate the phonon field, which for a solid phases is a two component vector and a scalar for a smectic, into the high and small wavevector modes

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}^<(\mathbf{r}) + \mathbf{u}^>(\mathbf{r}), \quad (4.1)$$

and integrate perturbatively in $U_{K}$ the high wavevector part $\mathbf{u}^>(\mathbf{r})$, with nonvanishing Fourier components inside a thin momentum shell

$$\Lambda e^{-\ell} < |\mathbf{q}| < \Lambda. \quad (4.2)$$

We then rescale the lengths and long wavelength part of the fields with

$$H_{el} = \int \frac{d^2q}{(2\pi)^2} \left[ \frac{1}{2} (B_{xx} q_x^2 + B_{yy} q_y^2) |u_x(\mathbf{q})|^2 + \frac{1}{2} (K_{xy} q_x^2 + K_{yy} q_y^2) |u_y(\mathbf{q})|^2 + \frac{1}{2} \delta q_x q_y u_x(\mathbf{q}) u_y(-\mathbf{q}) \right], \quad (4.6)$$

where,

$$B_{xx} \equiv \lambda_{xx}, \quad (4.7a)$$
$$B_{yy} \equiv \mu - \alpha + \gamma, \quad (4.7b)$$
$$K_{yy} \equiv \lambda_{yy}, \quad (4.7c)$$
$$K_{xy} \equiv \mu + \alpha + \gamma, \quad (4.7d)$$
$$\delta \equiv \mu + \lambda_{xy} - \gamma, \quad (4.7e)$$

which, after a simple Gaussian integration leads to

$$\langle u_{y}^{2}\rangle = \int_{q} \frac{k_{B} T}{K_{yy} q_{y}^{2} + K_{xy} q_{x}^{2} - \frac{\delta q_{x} q_{y}}{B_{xx, q_{x}^{2}} + B_{yy, q_{y}^{2}}}}, \quad (4.8)$$

so as to restore the ultraviolet cutoff back to $\Lambda = 2\pi/a$. Because the pinning potential nonlinearity is a periodic function, it is convenient (but not necessary) to take the arbitrary field dimension to be

$$\phi = 0, \quad (4.4)$$

thereby preserving the period $a = 2\pi/\Lambda$ under the renormalization group transformation. Under this transformation the resulting effective Hamiltonian, $H = H_{el} + H_{K}$, can be restored into its original form with effective $\ell$-dependent elastic and $U_{K}$ couplings.

For the periodic pinning potential coupling $U_{K}$, we find in a standard way

$$u_{y}^2 = k_{B} T \int_{q} \frac{d^2q}{(2\pi)^2} K_{y}^{2}\langle u_{y}^{2}\rangle, \quad (4.5)$$

where $\langle u_{y}^{2}\rangle$ is to be computed with the elastic Hamiltonian appropriate to the phase being analyzed, keeping only modes within an infinitesimal momentum shell near the zone boundary $\Lambda$. Hence the nature of the pinning by the substrate potential and the transition temperature obviously depend on the degree of the translational order in the system, i.e. whether the phase is a solid, or a smectic.

1. Locked Floating Solid to Floating Solid Transition

To determine the critical temperature for the LFS-to-FS transition, we compute the $\langle u_{y}^{2}\rangle$ average using the anisotropic elastic Hamiltonian $H_{el}$, Eq.(2.22) describing the 2d solid phase in the presence of a 1d periodic potential. Rewriting $H_{el}$ in terms of Fourier transformed phonon fields $\mathbf{u}(\mathbf{q})$, we find

$$\langle u_{y}^{2}\rangle = \int_{q} \frac{k_{B} T}{2\pi \overline{\mu}} \frac{\ell}{\Lambda}, \quad (4.9)$$

where we have introduced the shorthand notation $\int_{q} \frac{d^2q}{(2\pi)^2}$ for the integral over the momentum shell. In the dilute limit and neglecting effects of the periodic potential on the elastic coefficients this reduces to

$$\langle u_{y}^{2}\rangle = \frac{k_{B} T}{2\pi \overline{\mu}} \frac{\ell}{\Lambda}, \quad (4.9)$$

with $\overline{\mu} = 2\mu(2\mu + \lambda)/(3\mu + \lambda)$. In order to compute $\langle u_{y}^{2}\rangle$ in general we use an elliptical (volume conserving) momentum shell defined by major and minor axes $\Lambda_{x} = \Lambda \sqrt{K_{yy}/K_{xy}}$ and $\Lambda_{y} = \Lambda \sqrt{K_{xy}/K_{yy}}$. We find
where we defined a dimensionless number $c_1$ given by

$$c_1 = \frac{\int_0^{2\pi} d\theta}{2\pi a_x + (a_y - a_x) \sin^2 \theta + a_{xy} \sin^2 \theta}$$

and

$$
\begin{align*}
  a_x &\equiv \frac{B_{xx}}{K_{xy}}, \\
  a_y &\equiv \frac{B_{yy}}{K_{xy}}, \\
  a_{xy} &\equiv \frac{K_{xxy}}{K_{xy}}.
\end{align*}
$$

Upon combining Eq. 4.10 with Eq. 4.11, we find the eigenvalue of the substrate potential to be

$$\lambda_p \equiv 2 - K_y^2 \frac{k_B T c_1}{4\pi \sqrt{K_{yy} K_{xy}}}$$

which after setting

$$\lambda_p(T_{pS}) = 0$$

gives us the depinning transition temperature $T_{pS}$

$$k_B T_{pS} = \frac{8\pi}{c_1} \sqrt{K_{yy} K_{xy}} \left( \frac{d}{2\pi} \right)^2,$$

which separates the LFS and FS phases. In the dilute limit the transition temperature reduces to

$$k_B T_{pS} = 8\pi \left( \frac{d}{2\pi} \right)^2.$$

where for convenience we again used an elliptical momentum shell with axes $\Lambda \sqrt{B_{yy}/B_{xy}}$ and $\Lambda \sqrt{B_{xy}/B_{yy}}$.

After combining this result with Eq. 4.13 we find that translational pinning by the periodic potential is relevant in a floating smectic phase for $T < T_{pSm}$, with $T_{pSm}$ given by

$$k_B T_{pSm} = 8\pi \sqrt{B_{yy} B_{xy}} \left( \frac{d}{2\pi} \right)^2.$$

As discussed in more detail in Sec. V, the elastic moduli in Eqs. 4.13 and 4.19 are functions of the strength of the pinning potential $U_K$, which in turn is proportional to the input laser intensity $I_{in}$. Hence the resulting functions $T_{pS}(I_{in})$ and $T_{pSm}(I_{in})$ in principle determine the LFS-FS and LSm-FSm phase boundaries displayed in Fig. 5 for colloidal densities commensurate with the 1d periodic potential.

B. Dislocation Unbinding Transitions

In the analysis of the preceding Sec.V[A], where we studied a thermal depinning transition within the solid phase, we implicitly assumed that the dislocations that distinguish the 2d solid and the smectic phases from the higher temperature disordered phases remain bound. Hence, these calculations for the pinning transition and Eqs. 4.13, 4.19 remain valid only if they fall below the corresponding dislocation unbinding melting transition temperatures, which we now compute.

1. Locked Floating Solid to Locked Smectic Transition

It is easy to see from the effective Hamiltonian $H_{LFS}$, Eq. 3.20, that the most striking consequence of the 1d periodic potential is that it leads to the LFS phase, in which, the phonon degree of freedom, $u_y$, corresponding to displacements transverse to the potential troughs acquires a “mass”, Eq. 3.14, and as a consequence are effectively suppressed. Therefore, this phonon mode can be safely integrated out, leaving an effective anisotropic 2d XY Hamiltonian, with temperature and potential strength dependent effective elastic constants,

$$H_{LFS} = \frac{1}{2} \int d^2 r (B_{yx} (\partial_y u_x)^2 + B_{xx} (\partial_x u_y)^2),$$

that describes a locked floating solid at scales longer than the correlation length $\xi$ introduced in Sec. III A 2.

Melting of the LFS can be understood in terms of dislocation unbinding. However, in contrast to melting in the absence of an external (e.g., substrate or laser) potential, here only the so-called type I dislocation
pairs (in the notation of Ref. [37]) with Burgers vectors \( \pm \mathbf{b}_1 = \pm b_y \mathbf{e}_x \) (see Sec. II B) aligned parallel to the trough direction (which we continue to take along the \( x \)-axis) can be thermally unbound. In the presence of a periodic potential, oppositely charged dislocations, with Burgers vectors not satisfying the above condition (type II dislocations) are bound by a potential which grows linearly with the separation and therefore cannot thermally unbind. This discussion is consistent with the mapping onto scalar Coulomb gas Hamiltonian, expected to describe logarithmically bound type I dislocations, embodied in the 2d anisotropic XY model Hamiltonian, Eq. (4.24).

Away from the dislocation core, for a commensurate orientation defined by the shortest direct lattice vector pointing parallel to the troughs, \( \mathbf{R}_n = n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 \), labelled by direct lattice Miller indices \( n_1 \) and \( n_2 \) defined by Eq. (4.23), the displacement vector \( \mathbf{u} \) for the active type I dislocation is given by

\[
\mathbf{u} = \frac{b_y}{2\pi} \tan^{-1} \left( \frac{y B_{y,x}^2}{x B_{y,y}^2} \right),
\]

with

\[
\tilde{b}_n = | \mathbf{R}_n | = a \sqrt{n_1^2 + n_2^2 + n_1 n_2}.
\]

Melting of the LFS via unbinding of these defects is identical to the vortex unbinding transition of an anisotropic 2d XY model. A standard calculation leads to the prediction for the LFS melting temperature

\[
k_B T_{\text{LFS} \rightarrow \text{LSm}} = \frac{b_y^2}{8\pi} \sqrt{B_{y,y} B_{y,x}},
\]

and all other concomitant Kosterlitz-Thouless phenomenology. This implies an exponential growths of the translational correlation length

\[
\xi_t \approx ae^\nu(T - T_{\text{LFS} \rightarrow \text{LSm}})^{1/2},
\]

with \( c \) a nonuniversal parameter and a universal ratio of the jump in the geometric mean of the shear and bulk moduli, \( B_{y,x}(T_{\text{LFS} \rightarrow \text{LSm}}) \) and \( B_{x,x}(T_{\text{LFS} \rightarrow \text{LSm}}) \) to \( T_{\text{LFS} \rightarrow \text{LSm}} \).

The resulting high temperature phase is the LSm for low colloidal densities (i.e., high commensurability ratio \( p \)), and a modulated liquid for high densities (\( p \leq 1 \), see below), for which the smectic is indistinguishable from a liquid. Because of the unusually strong growth of the translational correlation length \( \xi_t \), Eq. (4.24), the phenomenology of the LSm-FSm transition that we studied in Sec. IV A 2 will be modified for \( T \rightarrow T^+ \) by a long crossover from the crystal to smectic (or liquid) elasticity.

It is important to note the distinction between this anisotropic 2d XY melting of a LFS into a LSm and an analogous type I melting mechanism of Ostlund and Halperin for melting of uniaxially anisotropic, but substrate-free 2d solids. In the later case, thermal fluctuations destabilize the resulting 2d smectic by further unbinding type II dislocations, asymptotically converting it into a liquid. Here, because of the pinning potential, type II dislocations (e.g., \( \pm \mathbf{b}_{2,3} \) for \( p = (p, 0) \)) remain bound by a linear potential. The resulting LSm phase is therefore distinct from the (orientationally ordered) modulated liquid (in which type II dislocations are also unbound), separated from it by a thermodynamically sharp phase transition.

2. Floating Solid to Floating Smectic Transition

A floating solid can melt continuously via unbinding of the type I dislocations. However, in contrast to the similar melting of a locked floating solid, here the dislocation unbinding in the displacement \( u_x \) proceeds in the presence of another spectator massless phonon mode \( u_y \), which is coupled to it. Consequently, as we will show below, this transition is a nontrivial extension of the Kosterlitz-Thouless theory and to our knowledge is, heretofore unexplored. Once these type I dislocations unbind the most likely resulting phase is the floating smectic.

The phenomenology of the FS-FSm melting transition can be most easily analyzed by the following steps. We (i) introduce dislocation degrees of freedom into the elastic Hamiltonian \( H_{\text{el}} \), Eq. (4.25), (ii) perform a duality transformation to convert the resulting Coulomb gas Hamiltonian into a modified Sine-Gordon model, and (iii) compute the dislocation unbinding temperature by analyzing the resulting dual model.

To execute these standard steps, it is convenient to first perform the following rescalings of spatial coordinates:

\[
x \rightarrow x(B_{x,x}/B_{y,y})^{1/4},
\]

\[
y \rightarrow y(B_{yy}/B_{xx})^{1/4},
\]

which leads to the Hamiltonian

\[
H_{\text{FS}} = \int d^dr \frac{1}{2} \left\{ K_x(\nabla u_x)^2 + c_x(\partial_x u_y)^2 + c_y(\partial_y u_y)^2 + 2\lambda_{xy}(\partial_x u_x)(\partial_y u_y) + 2(\mu - \gamma)(\partial_x u_y)\partial_y u_x \right\}
\]

where we dropped the prime on the rescaled coordinates and defined elastic constants

\[
K_x \equiv \sqrt{B_{x,x} B_{y,y}},
\]

\[
c_x \equiv K_x K_{xy}/B_{xx},
\]

\[
c_y \equiv K_x K_{yy}/B_{yx}.
\]

Because in the presence of dislocations the displacement field \( u_x \) is a multivalued function, it is essential to distinguish the last two terms in Eq. (4.26). In contrast to conventional elastic theory, where dislocations are bound and \( u_x \) is a well-defined function, here these terms cannot be transformed into each other by an integration
by parts. Keeping track of this distinction ensures the proper form for the elastic constants of the resulting smectic phase.

In this new rescaled coordinate system, a type I dislocation located at the origin, with a Burgers vector \( \mathbf{b} = b_\parallel \mathbf{x} \), can be represented by a displacement field

\[
u_s = \frac{\mathbf{b}_\parallel}{2\pi} \tan^{-1} \left( \frac{y}{x} \right),
\]

(4.28)

However, in contrast to the analysis of the melting of the LS phase above, in the presence of a finite \( \partial_y u_y \) deformation, the form of type I dislocation given in Eq.4.28 does not correspond to a relaxed \( u_y \) displacement which minimizes the energy. Consequently, we expect (see Eq.4.34) a bilinear coupling between the dislocation density and the \( u_y \) distortion. For a finite density of dislocations, we define a singular strain \( \nu_s \equiv \nabla u_y^s \) due to a dislocation density \( b(r) \), with the standard relation

\[
\nabla \times \nu_s = \hat{e}_z b(r),
\]

(4.29)

\[
= \hat{e}_z \sum_{r_r} \delta^{(2)}(r-r_r),
\]

(4.30)

\[
= \hat{e}_z b_\parallel n(r),
\]

(4.31)

where the \( \{n_{r_r}\} \) are integer dislocation charges. A general solution to the above equation is given (in Fourier space) by

\[
\nu_s(q) = \frac{iq \times \hat{e}_z}{q^2} b(q) + i q \chi(q),
\]

(4.32)

where \( \chi(q) \) is an arbitrary, single-valued function, which for convenience and without loss of generality we can set to zero. After expressing the gradient of the total displacement field \( \mathbf{u}_r \) in terms of the dislocation part \( \nu_s \) and a single valued phonon field \( \mathbf{u}_q \)

\[
\nabla u_x = \nu_s + \nabla u_x,
\]

(4.33a)

\[
\nabla u_y = \nabla u_y,
\]

(4.33b)

and inserting it into \( H_{FSd} \), we obtain a Hamiltonian that includes both the elastic and dislocation degrees of freedom

\[
H_{FSd} = \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \left\{ \frac{2b_\parallel}{q^2} \left[ \lambda_{xy} q_y^2 + (\gamma - \mu) q_x^2 \right] n(q) u_y(-q) + b_\parallel^2 K_x \frac{|n(q)|^2}{q^2} \right\} + H_{FS}[\mathbf{u}].
\]

(4.34)

After putting the system on the lattice, going to the grand canonical ensemble for dislocations, and adding the dislocation core energy \( E_c \) to account for the energy coming from short length scales, (not included in above analysis), the total partition function is given by

\[
Z = \int [d\mathbf{u}] \sum_{\{n_{r_r}\}} e^{-H_{FSd} - \sum_r E_c n_{r_r}^2},
\]

(4.35)

In above, for convenience we chose to measure all the energies in units of \( k_B T \).

To analyze the dislocation unbinding transition, it is convenient to perform a duality transformation on the above Hamiltonian \( H_{FSd} \). To do this we introduce an auxiliary Gaussian field \( \phi \) to decouple the Coulomb interaction between dislocations and use the Poisson summation formula to perform the summation over the set of lattice integers \( \{n_{r_r}\} \), obtaining

\[
Z = \int [d\phi] [d\mathbf{u}] e^{-H_d},
\]

(4.36)

where,

\[
H_d = \int d^2 r \left\{ \frac{K_x}{2} \left| \nabla \phi \right|^2 - V_V [b_\parallel (\phi + i\theta)] \right\} + H_{FS}[\mathbf{u}].
\]

(4.37)

To obtain \( H_d \), Eq.4.37, above, we defined a field \( \theta(r) \), whose Fourier transform is given by

\[
\theta(q) = \frac{1}{q^2} \left( \lambda_{xy} q_y^2 + (\gamma - \mu) q_x^2 \right) u_y(q),
\]

(4.38)

and used \( V_V(\phi) \) to denote the well-known \( 2\pi \)-periodic Villain potential defined by

\[
e^{-V_V(\phi)} = \sum_{n=-\infty}^{\infty} e^{-E_c n^2 + i n \phi}.
\]

(4.39)

At low fugacity (large core energy), this potential reduces to a cosine function, leading to

\[
H_d = \int d^2 r \left\{ \frac{K_x}{2} \left| \nabla \phi \right|^2 - g \cos [b_\parallel (\phi + i\theta)] \right\} + H_{FS}[\mathbf{u}],
\]

(4.40)

with \( g \equiv 2e^{-E_c} \).

Now the dislocation unbinding transition in the original model of the floating solid is determined by the vanishing of the RG eigenvalue of \( g(\ell) \) cosine nonlinearity in this dual model, defined by

\[
g(\ell) = ge^{(2 - \eta_g/2)\ell},
\]

(4.41)

where \( \eta_g \) is determined by

\[
\eta_g \ell = b_\parallel^2 \left[ \langle \phi^2 \rangle - \langle \theta^2 \rangle \right],
\]

(4.42)

with the right hand side easily computed from the quartic part of the dual Hamiltonian \( H_d \), Eq.4.37. Specifically,

\[
\langle \phi^2 \rangle = K_x \int_0^{(2\pi)^2} \frac{d^2 q}{q^2} \frac{1}{q^2},
\]

(4.43)

\[
= \sqrt{B_{xx} B_{yy}} \ell,
\]

(4.44)
\[ \langle \theta^2 \rangle > = \int \frac{d^2 q}{(2\pi)^2} \frac{(\lambda_{xy} q_y^2 + (\gamma - \mu) q_x^2)^2}{q^4} (|u_y(q)|^2) , \]  
\[ = \int \frac{d^2 q}{(2\pi)^2} \frac{\lambda_{xy} q_y^4 + 2\lambda_{xy}(\gamma - \mu) q_x^2 q_y^2 + (\gamma - \mu)^2 q_x^4}{q^4} , \]  
\[ = \frac{\lambda_{xy} c_2 + \lambda_{xy}(\gamma - \mu) c_3 + (\gamma - \mu)^2 c_4}{2\pi \sqrt{K_{yy} K_{xy}}} , \]  
where,
\[ c_2 = \int_0^{2\pi} \frac{d\theta}{2\pi} \left[ a_x^2 \sin^4 \theta \right] \left[ a_x + (a_y - a_x) \sin^2 \theta \right] \left[ a_x + (a_y - a_x) \sin^2 \theta + a_{xy} \sin^2 \theta \right] \]  
\[ c_3 = \int_0^{2\pi} \frac{d\theta}{2\pi} \left[ a_x^2 \cos^2 \theta \sin^2 \theta \right] \left[ a_x + (a_y - a_x) \sin^2 \theta \right] \left[ a_x + (a_y - a_x) \sin^2 \theta + a_{xy} \sin^2 \theta \right] \]  
\[ c_4 = \int_0^{2\pi} \frac{d\theta}{2\pi} \left[ a_x^2 \cos^4 \theta \right] \left[ a_x + (a_y - a_x) \sin^2 \theta \right] \left[ a_x + (a_y - a_x) \sin^2 \theta + a_{xy} \sin^2 \theta \right] \]  

Upon combining these results, we find that a floating solid melts into a floating smectic at
\[ T_{FS - FSm} = \frac{b_2^2}{8\pi} \left( \frac{B_{xx} B_{yy} - \frac{\lambda_{xy}^2 c_2 + \lambda_{xy}(\gamma - \mu) c_3 + (\gamma - \mu)^2 c_4}{\sqrt{K_{yy} K_{xy}}} \right) , \]  
which reduces to the melting temperature \( T_{LFS - LSm} \), Eq. 4.23 of the LFS in the limit \( K_{xy}, K_{yy} \to \infty \), in which the spectator phonon \( u_y \) mode is frozen out. Not surprisingly, we find that the extra \( u_y \) fluctuations of the FS always suppress the melting temperature of the FS relative to that of the LFS, i.e., for all range of parameters, \( T_{FS - FSm} < T_{LFS - LSm} \).

We now demonstrate that once type I dislocations unbind, the resulting Hamiltonian is that of a floating smectic, described by the Hamiltonian \( H_{FSm} \), given in Eq. 4.35. To see this return to the Hamiltonian \( H_{FSa} \), Eq. 4.34, and note that once dislocations unbind and therefore appear in large densities, the discrete dislocation field \( n_\tau \) can, to a good approximation, be treated as a continuous density \( n(r) \). Within this Debye-Hückel approximation the dislocation degrees of freedom can be easily integrated out of the partition function Eq. 4.35 by replacing the summation over \( n_\tau \) in \( \mathcal{Z} \) by an integration. Simple Gaussian integrations over dislocation density \( n(r) \) and the single valued field \( u_x \) then lead, in the long wavelength limit to an effective floating smectic Hamiltonian
\[ H_{Sm} = \frac{1}{2} \int d^2 r \left\{ \kappa (\partial_x^2 u_y)^2 + B_{xy}(\partial_x u_y)^2 + B_{yy}(\partial_y u_y)^2 \right\} , \]  
where we have restored the original scaling of the spatial coordinates, Eqs. 4.25 and derived the effective elastic constants for the resulting FSm phase
\[ \kappa = \frac{(\mu - \gamma)^2}{\mu + \gamma - \alpha} \left( \frac{2E_c}{b_2^2} \right) , \]  
\[ B_{xy} = \frac{4\mu\gamma - \alpha^2}{\mu + \gamma - \alpha} , \]  
\[ B_{yy} = \frac{\lambda_{yy} - \frac{\alpha^2}{\lambda_{xx}}}{\lambda_{xx}} . \]  

We note that \( B_{xy} \) vanishes as \( \gamma, \alpha \to 0 \), as it must in this rotationally invariant limit, in which one must recover the rotationally invariant 2d liquid crystal smectic elasticity. \[ \text{[38]} \]

Another equivalent but considerably more straightforward way to obtain the smectic Hamiltonian is to note that in the presence of unbound type I dislocations the \( \nabla u_x \), Eq. 4.33a contains both the longitudinal and transverse components, and therefore, despite of its appearance is no longer a conservative vector constrained to be a gradient of a single-valued function. This observation allows us to incorporate unbound type I dislocations into the Hamiltonian \( H_{\partial} \), Eq. 2.23, by the replacement
\[ \nabla u_x \to v , \]  
with \( v \) an arbitrary 2d vector field. Under this substitution \( H_{\partial} \), Eq. 2.23 transforms into
\[ H_{FSd} = \int d^2 r \left\{ \frac{\mu}{2} (\partial_x u_y + v_y)^2 + \frac{\lambda_{xx}}{2} v_x^2 + \frac{\lambda_{yy}}{2} (\partial_y u_y)^2 \right\} \]  
\[ + \lambda_{xy} v_x \partial_y u_y + \frac{\alpha}{2} (\partial_x u_y - v_y)^2 + \frac{\gamma}{2} (\partial_x u_y - v_y)^2 \]  
(4.51)

After performing a simple Gaussian integration over the two independent components of \( v \), we immediately obtain a Hamiltonian for the floating smectic, which in the
long wavelength limit agrees in form and with the expressions for the elastic constants $B_{xy}$ and $B_{yy}$, obtained in Eqs. (4.48) and (4.49).

V. SHAPE OF THE MELTING CURVE

A. Strong pinning limit and reentrant melting

One of the most interesting observations in the colloidal experiments by Wei et al. [24], which in fact stimulated our interest in this problem, is the light-induced reentrant melting. As we shall explicitly demonstrate, this melting reentrance is a generic consequence of short-ranged screened colloidal interactions and thermal fluctuations, and hence should be prevalent in such 2d systems.

To demonstrate the reentrance as a function of laser intensity, we study the shape of the melting curves for the LFS-ML, LFS-LSm and FS-FSm transitions, which we generally denote by $T_m(U_K)$. The common feature of these transitions is that they are all driven by the unbinding of type I dislocations, with $T_m(U_K)$ (see Eqs. (4.23)-(4.47) at least in part determined by the renormalized values of the bulk modulus $B_{xx}$ for compression along the troughs and the corresponding shear modulus $B_{yz}$. Our goal then is to determine how these moduli depend on the potential amplitude $U_K$.

We first note that these melting boundaries $T_m(U_K)$ are constrained by their limiting values

\[ T_m(0) = \frac{b_2 \mu (\mu + \lambda)}{4\pi} , \quad \frac{2\mu + \lambda}{2\mu + \lambda} , \quad (5.1a) \]

\[ T_m(\infty) = \frac{b_2^2 \sqrt{\mu (2\mu + \lambda)}}{8\pi} , \quad (5.1b) \]

where $T_m(0)$ is the well-known result in the absence of an external potential \[ \frac{4\pi}{b_2} \]. In the opposite limit of infinite potential strength, $T_m(\infty)$ is given by Eq. (4.23) with $B_{yz}(U_K \to \infty) \approx 2\mu + \lambda$ and $B_{yy}(U_K \to \infty) \approx \mu$. These results follow from comparing $H_{\text{LFS}}$, Eq. (3.20), with $H_0$, Eq. (4.4), after freezing out the $u_y$ degree of freedom ($u_y = 0$) in $H_0$, as is appropriate in this $U_K \to \infty$ limit. Although in general there is no universal relation between $T_m(0)$ and $T_m(\infty)$, in a dilute colloidal limit, relevant to the experiments of Wei et al. [24], the two Lamé coefficients are equal, $\mu \approx \lambda$, and Eqs. (5.1) reduce to

\[ T_m^{\text{dil}}(0) = \frac{b_2^2 \lambda}{6\pi} , \quad (5.2a) \]

\[ T_m^{\text{dil}}(\infty) = \frac{\sqrt{3} b_2^2 \mu}{8\pi} \approx 1.3 \frac{b_2^2 \mu}{6\pi} . \quad (5.2b) \]

One might have thought that the melting temperature simply increases monotonically with $U_K$ from $T_m(0)$ to $T_m(\infty)$. However, as we will now show explicitly, the $u_y$-mode thermal fluctuations, enhanced as the periodic potential is lowered from infinity, generically increase the melting temperature for $\kappa a \gg 1$. Consequently, the melting curve, $T_m(U_K)$, must have a maximum in this limit, implying reentrant melting for a band of temperatures as a function of the potential amplitude.

The origin of the reentrance effect can be understood on a heuristic level as follows. Clearly, at small $U_K$, we expect that the increase in the strength of the periodic potential suppresses thermal fluctuations in $u_y$, thereby lowering the entropy of the liquid (or the smectic) state, and therefore making freezing into a lattice energetically more costly. This naturally leads to an increase of $T_m(U_K)$ with $U_K$ at low laser intensities. However, for potential strengths $U_K \gg k_B T$, this entropic contribution to the free energy becomes unimportant. In this large $U_K$ limit, the behavior of $T_m(U_K)$ is dominated by a different mechanism having to do with the reduction of the elastic constants with increasing $U_K$ and decreasing temperature. To see this, note that the effective shear modulus $B_{yz}(U_K)$ which enters $T_m(U_K)$ (see Eqs. (4.23) and (4.47)) is determined by the screened Coulomb interaction, $V(r) = V_0 \exp(-\kappa r)/r$, between colloidal particles in neighboring troughs. In order to find an effective shear modulus for the $u_y$-modes, one needs to integrate out the massive modes corresponding to displacements perpendicular to the troughs of the laser potential. This will be the route taken further below. Heuristically, one should get roughly the same result by assuming that the dominant effect comes from the shear modulus $B_{yz}$ and simply averaging the potential over the massive $u_y$ degrees of freedom, which yields

\[ B_{yz}(U_K) \sim \langle \exp(-\kappa |r_{n+1} - r_n|) \rangle_{u_y} , \quad (5.3) \]

where $r_n$ and $r_{n+1}$ are positions of nearest neighbor colloidal particles belonging to the $n$-th and $n+1$-st Bragg planes, running parallel to the laser potential troughs. This gives lowest harmonic order in the fluctuations $u_y$,

\[ B_{yz}(U_K) \sim \langle \exp(-\kappa \alpha_1 - u_y(n+1) - u_y(n)) \rangle \sim \exp(-\kappa \alpha_1^2(n_y^2)) \approx B_{yz}(\infty) e^{\kappa b T/\kappa U_K} . \quad (5.4) \]

with $\alpha$ a dimensionless number of order 1. Such a thermal enhancement of the effective shear modulus $B_{yz}(U_K)$, which decreases as thermal fluctuations in $u_y$ are suppressed by increasing $U_K$, is easy to understand: Even though, in the presence of $u_y$ fluctuations colloidal particles in neighboring troughs spend as much time closer together as further apart, because of the concave form of the interaction potential the enhancement of the effective shear modulus is larger from particles being closer together than the corresponding suppression when they are further apart.

The above simple physical argument for reentrance is supported by detailed microscopic lattice calculations, in which we compute both the effective shear $B_{yz}(U_K)$ and
bulk $B_{xy}(U_K)$ moduli. To do this we start with a microscopic model with a screened repulsive Coulomb interaction $V(r) = V_0 \exp(-\kappa r)/r$, where the screening length $\kappa^{-1}$ is typically much smaller than $a$ and $V_0$ depends on the dielectric constant, $\kappa$ and the sphere radius $\frac{1}{2a}$. Upon integrating out the $u_y$-modes using the screened Coulomb potential to leading order in $k_B T/U_K$ the calculation in Appendix B gives (for orientation $\vec{p} = (1,0)$),

\begin{equation}
B_{yx}(U_K) \approx B_{yx}(\infty) \left\{ 1 + \frac{9(\kappa a)^2}{64\pi^2} \left[ 1 + \frac{17}{3\kappa a} \frac{k_B T}{p^2 U_K} \right] \right\},
\end{equation}

\begin{equation}
B_{xx}(U_K) \approx B_{xx}(\infty) \left\{ 1 + \frac{(\kappa a)^2}{64\pi^2} \left[ 1 - 8v - \frac{23 + 104v}{3\kappa a} \right] \frac{k_B T}{p^2 U_K} \right\},
\end{equation}

where $v = V_0 e^{-\kappa a}/k_B T$, $B_{yx}(\infty) = \frac{2}{3} \kappa b_\infty T \kappa^2$ and $B_{xx}(\infty) = 3\kappa b_\infty$. Lowering the potential strength $U_K$ always increases the shear modulus, whereas the behavior of the compressional modulus depends on the magnitude of $v$ and $\kappa a$. When combined with Eq. (4.23), these expressions imply that the melting temperature increases with decreasing $U_K$ for $\kappa a \geq 5.6$ (in Ref. 22 $\kappa a \approx 10$),

\begin{equation}
T_{LFS-LSm}(U_K) = T_{LFS-LSm}^{\infty} \left\{ 1 + \frac{5[(\kappa a)^2 - 31]}{64\pi^2} \left[ 1 + \frac{13}{3\kappa a} \frac{k_B T_{LFS-LSm}^{\infty}}{p^2 U_K} \right] \right\},
\end{equation}

this implying reentrant melting for a band of temperatures as a function of potential strength observed in experiments and illustrated in Figs. 24 and 3. Clearly given the dependence of the $T_{FS-FSm}$ on the elastic moduli, Eq. (4.24), we expect FS-FSm transition to display reentrance, although quantitative predictions of the size of this reentrant band are much more difficult.

In obtaining Eq. (5.7) we have clearly ignored additional renormalization of the effective elastic constants by phonon nonlinearities and by bound dislocation pairs, which need to be taken into account for more precise estimate of the phase boundary. Based on general structure of Kosterlitz-Thouless-like RG flows, the latter renormalizations generically reduce the elastic moduli and therefore drive the melting temperature down. Since $u_y$ modes fluctuations and therefore the renormalizations that they induce are suppressed by the increasing periodic potential, we expect that $T_m(U_K)$ experiences larger reduction at small $U_K$ than at large $U_K$. The known values for the the potential-free 2d melting and the 2d XY model downward renormalization constrain the extreme $U_K = 0$ and $U_K \to \infty$ ends of the melting curve. Furthermore, since thermal downward renormalization of elastic constants is obviously enhanced with increasing temperature, we expect the suppression of the melting temperature due to these effects to be most pronounced near the maximum in $T_m(U_K)$. Clearly, such a $U_K$-dependent downward renormalization of the elastic constants will generically tend to reduce the range of temperatures over which there is laser-induced reentrant melting. However, these effects are small and we therefore expect reentrant melting to persist even in their presence.

B. Weak pinning: universal shape of the melting curve at small potential strength

In addition to a maximum displayed by the melting curve as a function of laser intensity, we also find that the shape of the melting temperature is universal in the limit of a vanishing periodic potential strength $U_K$. This can be seen most easily from the RG scaling theory applied to the potential-free critical point. More specifically, consider the behavior of the translational correlation length $\xi(t, U_K)$ above the melting transition as a function of $U_K$ and the reduced temperature $t \equiv [T - T_m(U_K = 0)]/T_m(U_K = 0)$. The power of the renormalization group transformation is that it allows us to relate a difficult calculation very close to the transition, where fluctuations are large and perturbation theory is divergent, to a calculation outside of the critical region, where perturbation theory is convergent. Applying this idea to the computation of $\xi(t, U_K)$ we find

\begin{equation}
\xi(t, U_K) = b_* \xi(t(b_*), U_K b_*^{\lambda_K}) ,
\end{equation}

\begin{equation}
= e^{c/t_\nu} \xi(1, U_K e^{\lambda_K/t_\nu}) ,
\end{equation}

where we have chosen the RG rescaling parameter $b_*$ such that the rescaled reduced temperature $t(b_*)$ given by the RG flow equations of Halperin and Nelson is of order unity

\begin{equation}
t(b_*) = 1 .
\end{equation}

$\lambda_K = 2 - \eta_K/2$ is the renormalization group eigenvalue of the 1d periodic potential $U_K$. At the primary potential-free fixed point with $U_K = 0$, we recover the well-known exponential growth of the correlation length $\xi(t, U_K = 0)$ with the exponent $\nu$ given by

\begin{equation}
\nu \approx 0.36963 ,
\end{equation}

where an overbar denotes critical exponents at this fixed point.
The primary critical behavior is unstable for arbitrarily small $U_K$. Hence sufficiently close to the melting temperature $T_m(0)$, the periodic potential always becomes important. This is the case even for the melting of the FS, where it leads to a marginal crossover from a fixed line of isotropic rotationally invariant elasticity to the fixed line characterizing elasticity given by Eq.2.25, where the rotational symmetry is explicitly broken by $U_K$. There, despite the fact that the periodic potential is irrelevant for the translational order parameter, it is always important for the orientational degrees of freedom, since (see Sec.II C) it explicitly breaks orientational symmetry.

In locked phases, it is clear from Eq.5.8b, that for a given small $U_K$, the effects of this weak periodic potential will be felt at $T_m(U_K) > T_m(0)$, such that the $U_K$-dependent argument on the right hand side of Eq.5.8b is large, i.e., grows beyond order $T_m$,

$$U_K \approx k_BT_m e^{-c\lambda K/\ell_m^2}. \quad (5.11)$$

This then predicts a universal cusp for the melting curve $T_m(U_K)$ in the limit $U_K \to 0$ in any phase in which $\lambda K > 0$, i.e., the periodic potential is relevant and the phase is locked ($U_K$) given by

$$T_m(U_K) \sim T_m(0) \left[ 1 + \lambda K \ln(k_BT_m/U_K) \right]^{-1/\pi}, \quad (5.12)$$
as depicted in Figs.2,4 and 5. For floating phases, such as the FS and FSm, where the periodic potential is irrelevant (in the RG sense), we expect the convergent perturbation theory in $U_K$ to lead to a melting temperature $T_m(U_K)$ that instead grows linearly with $U_K$.

VI. RESPONSE OF THE TRANSLATIONAL AND HEXATIC ORDER PARAMETER TO AN EXTERNAL POTENTIAL

In this section we use a renormalization group external scaling analysis to determine the response of the translational order parameter $M_K = \langle \rho_K \rangle$ and the bond orientational order parameter $\psi_6 = \langle e^{i6\theta(r)} \rangle$ to the amplitude $U_K$ of the external laser potential. In the absence of an external potential, $U_K = 0$, there are only algebraic peaks in the static structure function of the crystalline phase and the translational order parameter $M_K \equiv \langle \rho_K \rangle$ vanishes like

$$M_K \sim L^{-\pi_K/2} \to 0 \quad (6.1)$$
as the system size $L \to \infty$, where

$$\pi_K = \frac{k_BT}{4\pi} \frac{3\mu + \lambda}{\mu(2\mu + \lambda)} K^2 \quad (6.2)$$
is the critical exponent of the potential-free case $\mu = 0$. For small values of the external potential $U_K$ we can use standard crossover scaling analysis to determine how the translational order parameter depends on the amplitude of the laser potential. We start from the scaling behavior of the free energy density under a renormalization group transformation

$$f(U_K, T) = e^{-2\ell} f(e^{\lambda K} U_K, T(\ell)) \; , \quad (6.3)$$

where $\lambda K$ is the renormalization group eigenvalue for the periodic potential, and $T(\ell)$ is the renormalized temperature which characterizes the crystalline phase. Since in the free energy density the laser potential $U_K$ couples linearly to $\rho_G$ we have

$$M_K = -\frac{\partial}{\partial U_K} f(U_K, T) \; , \quad (6.4)$$

and dimensional analysis tells us that the exponent of the correlation function $\langle \rho_K(r) \rho_K(0) \rangle \sim r^{-\pi_K}$ is related to $\lambda K$ by

$$\lambda K = 2 - \frac{1}{2} \pi_K \; , \quad (6.5)$$
a result consistent with standard perturbative calculation of $\lambda K$. Hence we get the following scaling relation for the translational order parameter

$$M_K(U_K, T) = e^{-\frac{3}{2} \pi_K} M_K(e^{2-\pi_K/2} U_K, T(\ell)) \; , \quad (6.6)$$

where we expect $T(\ell)$ to approach a finite value as $\ell \to \infty$. Upon choosing $\ell = \ell_s$ such that $e^{2-\pi_K/2} U_K = \mu a^2$, i.e., is comparable to the elastic energy for deformation at the lattice cutoff $a$, we obtain

$$M_K(U_K, T) \sim |U_K|^{\pi_K/(4-\pi_K)} \; . \quad (6.7)$$

For $\pi_K > 2$, $M_K$ vanishes linearly with $U_K$, with a singular correction. In contrast, $M_K$ should always vanish linearly with $U_K$ in the liquid and hexatic [1][2] phases of the unperturbed colloid.

The laser potential will also induce long-range bond orientational order in $\psi_6 = \langle e^{i6\theta(r)} \rangle$. Along similar lines as above, one can show that the bond order parameter $\psi_6$ vanishes linearly with $U_K$ in the liquid, vanishes like a power of $U_K$ in the hexatic phase

$$\psi_6 \sim |U_K|^{\pi_6/(4-\pi_6)} \; , \quad (6.8)$$

where $\pi_6$ is the exponent describing the algebraic decay of bond order, and approaches a nonzero constant as $U_K \to 0$ in the solid phase. [26]

VII. DISCUSSION AND EXPERIMENTAL IMPLICATIONS

A. Melting temperatures and critical commensurability ratios in the dilute limit

One of the interesting predictions of our work is that the LFS-ML, LFS-LSm and FS-FSm transition are all mediated by the unbinding of type-I dislocations with
Burgers vectors parallel to the troughs of the external potential, \( \mathbf{b} = b_y \mathbf{x} \). Consequently, depending on the choice of relative orientation, the periodic potential can be used to suppress the unbinding of a set of dislocations that would otherwise unbind in a “substrate”-free experiments. For example, in the dual-primary orientation shown in Fig.\([7]\), all six fundamental Burgers vectors are confined by a linear potential and therefore cannot unbind entropically. It is therefore the unbinding of nonconfined by a linear potential and therefore cannot unbind entropically. It is therefore the unbinding of nonfundamental dislocations with Burgers vector of charge \( \sqrt{3}a \), illustrated in Fig.\([7]\) that will control the melting transition.

In general, the magnitude \( b_\mathbf{n}^2 = a^2(n_1^2 + n_2^2 + n_3n_2) \) of the lowest energy Burgers vector and hence the melting temperature \( T_m \propto b_\mathbf{n}^2 \) depends strongly on the relative orientation between the 2d solid and the laser potential, e.g., for \( n_2 = 1 \) and \( n_1 = 0, 1, 2, 3 \) one finds \( b_\mathbf{n}^2/a^2 = 1, 3, 7, 13 \). In particular, if one keeps the mean particle spacing \( a \) (i.e., the density) and the potential strength fixed and reduces the spacing \( d \) between the laser troughs (by e.g., varying the angle between the two interfering laser beams) \( d \rightarrow d/\sqrt{3} \) such that one goes from preferred lattice orientation A (\( \mathbf{n}_A = (1, 0) \)) to preferred lattice orientation B (\( \mathbf{n}_B = (1, 1) \)) the melting temperature should increase by a factor of 3 (see vertical arrow in Fig.\([8]\a)). This appears to be consistent with preliminary data of Bechinger et al. [49]. They find that for exactly such a change in trough spacing the onset of light induced freezing at fixed temperature is shifted to smaller laser intensities also by roughly a factor of 3 (see horizontal arrow in Fig.\([8]\a)).

More detailed experimental studies of \( T_m(U_K) \) for various commensurate orientations and trough spacings would clearly be desirable in order to systematically test our predictions for the orientation dependence of the melting transition temperature for the LIF. In performing such studies one must keep in mind considerable irreversibility effects that are expected to plague “zero-laser-field” cooled experiments. In order to avoid dealing with long equilibration times, one would need to warm up into the liquid state, change the laser potential period \( d \) and only then “field-cool” back into the solid.

Since trough spacing \( d \) (controlled by the angle between the interfering laser beams) and laser intensity appear to be convenient experimentally tunable parameters, it is valuable to derive the shape of the melting curve in the \( d-U_K \) plane (for a given temperature and a fixed density of colloidal particles). However, since an arbitrary value of \( d \) will in general not be commensurate with the spacing between a particular fixed set of Bragg planes, a detailed study of incommensurate potentials would need to be done in order to fully understand the behavior as a function of trough spacing \( d \). We hope to discuss some of the ensuing physics in a forthcoming publication [27]. But, for the following we would like to restrict ourselves to values of \( d \) which are commensurate. Hence, strictly speaking, our results will be not be valid for a continuous set of layer spacings but only for a discrete commensurate subset of values. With this precaution in mind we expect the melting curve (for a given.
temperature and particle density) in the $d$-$U_K$ plane to have the shape illustrated in Fig. 13. We note that in the LIF regime the critical potential strength for melting decreases with decreasing distance between the laser fringes, whereas in the LIM regime the critical potential strength increases as the interference fringes become narrower.

Let us now specialize to the dilute limit, $\kappa a \gg 1$, relevant to the experiments of Wei et al. [22]. Then the two Lamé coefficients (characterizing the continuum elastic theory of the hexagonal crystal in the absence of a laser potential) become equal, $\mu \approx \lambda$, and the melting temperature for the LFS reduces to

$$T_m^0 = \mu \frac{b_2^2}{6\pi}, \quad T_m^\infty = \sqrt{3\mu} \frac{b_2^2}{8\pi} \approx 1.3T_m^0,$$

in the limit of zero and infinite potential, respectively. For small values of the commensurability ratio, $p < p_c$, the LFS melts into a modulated liquid or a locked smectic. If $p > p_c$, a floating solid with two soft phonon modes can intervene between the LFS and a modulated liquid or floating smectic phase. As discussed in Sec IV A, the transition from the LFS into the intermediate FS phase is in a roughening universality class where the laser potential becomes irrelevant. In the dilute limit (and neglecting effects of the periodic potential on the elastic coefficients) the corresponding critical temperature is approximately

$$T_{pS}^{\text{dil}} = \frac{3}{\pi} \mu a^2,$$

where $d = a'/p$ with $a' = \sqrt{3}a/(2\sqrt{n_1^2 + n_2^2 + n_1n_2})$ the distance between the Bragg planes parallel to the troughs of the laser potential. Upon combining Eq. 7.3 with Eq. 7.2, the critical commensurability ratio reads

$$p_c^{\text{dil}} = 3 \sqrt{\frac{2}{n_1^2 + n_2^2 + n_1n_2}}.$$  

Note that only for the primary ($p_c^{\text{dil}} = 3\sqrt{3}/2 \approx 3.7$) and for the dual-primary orientation ($p_c^{\text{dil}} = \sqrt{3}/2 \approx 1.2$) is this critical value larger than 1. For any other orientation $p_c$ is less than 1 and hence we expect that there will always be an intervening floating solid phase. A configuration with $\vec{n} = (2,1)$ (see also Fig. 14) and hence $p_c = (3/7)\sqrt{3/2} \approx 0.5$ is likely to be within the range of parameters accessible to experiments with colloidal particles.

For $p > p_c$ there is a roughening transition from a locked floating solid into a uniaxially anisotropic floating solid described by $H_{c2}$, Eq 2.23, which subsequently melts (by unbinding of type I dislocations) into either a modulated liquid or a floating smectic. Since the melting and the roughening transition for a locked smectic phase are respectively given by $T_{mSm} = \frac{a}{3\pi}Ba^2$ and $T_{pSm} = \frac{2}{3}Bd^2$, where $B = \sqrt{B_{xy}B_{yy}}$ and $a$ is the smectic layer spacing, there exists a universal commensurability ratio $p_c' = 4$ above which a floating smectic phase intervenes between a locked smectic or a floating solid and the modulated liquid. This universal value $p_c' = 4$ should be contrasted with the nonuniversal critical commensurability ratio $p_c$ for the existence of the floating solid phase, which depends on the relative magnitude of the elastic constants and strongly on the relative orientation between the colloidal lattice and the 1d periodic potential. Current experiments find it difficult to access large commensurability ratios $p$. We hope that our theoretical results will inspire experimentalists to overcome present obstacles and map out the rich phase diagram shown in Fig. 13.

### B. Phase diagrams as a function of the Debye screening length

Recent Monte-Carlo simulation studies of melting in the presence of a 1d periodic external potential have explored the phase diagram in the parameter space of $U_K/k_B T$ and $\kappa a$ with particle density and temperature fixed. [19,20] Although one might question whether such simulations are in equilibrium with respect to dislocation climb (or even glide), it is important to tabulate the predictions of our defect-mediated melting theory in this parameter space in order to be able to compare with the results of these simulations. In addition, it also seems to be more feasible experimentally to map out the phase diagram as a function of potential strength and particle density.

Adapting our results from Sec III we find the following behavior. Since the melting temperature is proportional to the elastic moduli, which in turn are proportional to the potential strength, for $\kappa a \gg 1$ we expect $T_m$ to display the following dependence on the screening length $T_m \propto (\kappa a)^{2\mu/b_2}$. As an immediate consequence one gets (in the dilute limit) the following implicit equation (see also Eq. 7.1 and 7.2)

$$(\kappa_m^\infty - \kappa_m^0)a \approx 2 \ln \left( \frac{3\kappa_m^\infty}{\kappa_m^0} \right) > 0.$$  

In particular this implies that the difference in the critical values of the inverse screening length at infinite and zero potential strength, $\kappa_m^\infty$ and $\kappa_m^0$, is positive. In the limit $\kappa_m^0a \gg 1$, Eq. 7.5 reduces to $(\kappa_m^\infty - \kappa_m^0)a \approx 2 \ln 1.3 \approx 0.52$. The full solution of Eq. 7.3, together with the asymptotic result is shown in Fig. 14. We find Eq. 7.3 to be consistent with experimental results [49]. It would be interesting to test experimentally the functional dependence of $\kappa_m^\infty$ on $\kappa_m^0$, Fig. 19, predicted here.

The results of Monte-Carlo simulations appear to disagree with experiments and with our predictions from the dislocation-mediated melting theory when compared
for large values of the potential strength. Whereas we find $\kappa_m^{\infty} > \kappa_m^0$, the simulations reported in Ref. [19] show quite the opposite. More recent simulations from the same group [20] seem to refute these earlier results and find in agreement with our theory $\kappa_m^{\infty} - \kappa_m^0 > 0$. Their numerical value for $(\kappa_m^{\infty} - \kappa_m^0)a \approx 1.32$ is, however, more than two times larger than our asymptotic prediction of 0.52. However, because Eq. (7.5) neglects finite renormalization of elastic constants by dislocation dipoles and nonlinear elastic effects, our prediction is an estimate, only accurate up to unknown factors of order 1.

Next we discuss reentrance in the $U_K/k_B T - \kappa_m a$ phase diagram. Upon rewriting Eq. (5.7) we find

$$\frac{U_K}{k_B T} = \frac{\alpha(\kappa_m a)}{T/T_m^{\infty}(\kappa_m a) - 1}$$  \hspace{1cm} (7.6)

with

$$\alpha(\kappa_m a) = \frac{5((\kappa_m a)^2 - 31)}{64 \pi^2} \left( 1 + \frac{13}{3 \kappa_m a} \right).$$  \hspace{1cm} (7.7)

Hence, if $\kappa_m^0 a$ and $\kappa_m^{\infty} a$ are both smaller than the critical value 5.6 for the degree of reentrance, we expect $(\kappa_m a)^{-1}$ to be a monotonically decreasing function of the potential strength, as shown by the dashed line in Fig. 20. If $\kappa_m^0 a$ and $\kappa_m^{\infty} a$ are both larger than the critical value 5.6, we expect reentrant behavior such that with increasing potential strength $(\kappa_m a)^{-1}$ first decreases and reaches a minimum $(\kappa_m^{\min} a)^{-1} < (\kappa_m^{\infty} a)^{-1}$ before it approaches $(\kappa_m^{\infty} a)^{-1}$ as an inverse power of $U_K$ according to Eq. (7.6) (see Fig. 20). This reentrant behavior is consistent with results from experiments of the Konstanz group [22, 49] (see the dashed arrow in Fig. 20, which describes a typical experimental path). It is also similar to what one finds in simulations [19] at small values of the potential strength. However, there are significant differences. First of all, the type of transition is very different. Whereas we discuss a continuous dislocation mediated melting transition, simulations appear to find a first-order transition. Second, as discussed above, the simulations show $\kappa_m^{\infty} < \kappa_m^0$, which is opposite to what our theory predicts. In more recent simulations [20] $\kappa_m a$ is found to increase monotonically with potential strength with no sign for reentrance. This is opposite to what was found in the earlier simulations by the same group [19].

![FIG. 20. Schematic $p = 1$ phase diagram as a function of potential strength $U_K$ and inverse Debye screening length $\kappa$. Solid and dashed curves represent the melting curves for values of $\kappa$ being larger or smaller respectively than the critical value of $\kappa_{crit} a \approx 5.6$.](image)

In summary, we find that our theoretical results are consistent with recent experiments and raise strong doubts on the validity of the Monte-Carlo results to date on melting in a 1d periodic potential. This latter failure of simulations is not completely surprising given difficulties of numerical methods on even larger systems to resolved the nature of 2d melting even without an external potential [2].

C. Static structure factor and pair correlation function

The quantity that is most directly observed in many experiments on colloidal systems and related simulations is the pair correlation function, defined by

$$g(\mathbf{r}) = \frac{V}{N^2} \sum_{ij} \langle \delta(\mathbf{r} - (\mathbf{r}_i - \mathbf{r}_j)) \rangle,$$  \hspace{1cm} (7.8)

where the double sum is over $N$ particles but excludes the diagonal terms where $i = j$. It is related to the static structure factor by

$$g(\mathbf{r}) = \frac{1}{N} \int \frac{d^2q}{2\pi} e^{i\mathbf{q}\cdot\mathbf{r}} S(\mathbf{q}).$$  \hspace{1cm} (7.9)

Neglecting the smooth part of the structure factor and taking into account only the center column of Bragg
peaks and the two neighboring columns of quasi-Bragg peaks with \( \eta_{G_{A,x}}(T_m) = 1/4 \) and \( \eta_{G_{B,x}}(T_m) = 1 \), respectively, one finds for the pair correlation function,

\[
g(x) - 1 = \sum_{G_1} C_{G_1} \cos(G_1 \cdot r) \\
+ r^{-\eta_{G_{A,x}}} \sum_{G_A} C_{G_A} \cos(G_A \cdot r) \\
+ r^{-\eta_{G_{B,x}}} \sum_{G_B} C_{G_B} \cos(G_B \cdot r) ,
\]

where \( \eta_{G_{A,x}} \) are the exponents characteristic for the LFS phase. Note also that according to Eq.1.13 these exponents do only depend on the \( x \)-component of the reciprocal lattice vector \( G_{A,x} \). The amplitudes \( C_{G_{A,x}} \) are proportional to the amplitudes of the corresponding Bragg peaks, \( G_1 = (2\pi/\alpha') \delta_y \), and quasi-Bragg peaks, \( G_{A,x} \) and \( G_{B,x} \) with \( G_{A,x} = G_{0,x}^a = 2\pi/\alpha \) and \( G_{B,x} = 2G_{0,x}^a \) (see Eqs.3.32a–3.33b).

For \( r \parallel \hat{x} \), i.e., looking parallel to the minima of the troughs the sum over the Bragg peaks yields a constant. This simply reflects the effect of the laser potential to induce a periodic modulation of the colloidal particle density with a higher density in the minima of the troughs. Note that this trivially implies that the pair correlation function does not approach unity as \( x \rightarrow \infty \) if \( g(x) \) is normalized with respect to the mean density. Since the amplitudes for the quasi-Bragg peaks decay as a power-law in the strength of the laser potential with an exponent proportional to \( \eta_{G_{A,x}} \), a reasonable approximation for the pair correlation function reads

\[
g(x) - 1 = \text{const.} + g_A \cos(G_{0,x}^a x) x^{-\eta_{G_{A,x}}} \\
+ g_B \cos(2G_{0,x}^a x) x^{-\eta_{G_{B,x}}} .
\]

The relative magnitude of the amplitudes \( g_A \) and \( g_B \) depends on the strength of the laser potential. Whereas \( g_B \) is independent of \( U_K \) (note that the leading quasi Bragg-peak contributing to \( g_B \) has \( G_{B,y} = 0 \)), \( g_A \) vanishes as a nontrivial \( T \)-dependent power law in \( U_K \) for \( U_K/\mu a^2 \ll 1 \) (see Eq.3.32b) increasing the weight of the \( x^{-\eta_{G_{A,x}}} \)-term with increasing potential strength. This prediction should be accessible to experimental verification. Note, that the dependence of the amplitude \( g_A \) on the potential strength may lead to \( U_K \)-dependent effective exponents when one tries to incorrectly fit the experimental data by a single power law. For illustration Fig.21 shows \( g(x) - 1 \) for a special case, where \( \text{const.} = 0, \ G_A = G_B = 1, \ \eta_{G_{A,x}} = 1/4, \ \eta_{G_{B,x}} = 1 \) and all length are measured in units of \( a \). Due to the superposition of the two harmonics with different power law amplitudes the minima are much broader than the maxima of the structure factor, a feature which appears to be present in the data of Ref. 22.

For \( r \parallel \hat{y} \), i.e., looking perpendicular to the minima of the troughs we get

\[
g(y) - 1 = \text{const.'} \cos(2G_{0,y} y) + g_A' \cos(G_{0,y}^a y) y^{-\eta_{G_{A,x}}} \\
+ g_B' y^{-\eta_{G_{B,x}}} 
\]

with \( G_{0,y}^a = 2\pi/\sqrt{3\alpha} \). Hence on top of the periodic density modulation due to the laser potential we have again the algebraic decay from the closest Bragg peaks. For illustration Fig.22 shows the algebraic part of the static structure factor \( f(x,y) = y^{-1/4} \cos(2\pi y/\sqrt{3}) + y^{-1} \) where we have again chosen the amplitudes to be equal and the \( \eta \) exponents equal to their values at the melting temperature, \( \eta_{G_{A,x}} = 1/4, \ \eta_{G_{B,x}} = 1 \). If one would try to fit the
envelope of this function in the regime shown in the graph using a single power law one would find an exponent of 1/2. Hence caution must be exercised in the analysis of the experimental data, and it is essential to take into account both leading and subleading quasi Bragg peaks.

ACKNOWLEDGMENTS

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APPENDIX A: VARIATIONAL THEORY OF THE 2D MELTING TRANSITION IN THE PRESENCE OF A 1D PERIODIC POTENTIAL

In this appendix we study the freezing transition of the modulated liquid in the limit of a strong periodic potential. In such limit the colloidal particles are tightly confined to the troughs of the 1d periodic potential and our system reduces to a weakly coupled array of 1d colloidal liquids. The low energy degrees of freedom of the resulting system are then well characterized by a scalar field \( u_n(x) \) describing particle displacements along the \( n \)-th trough and an effective Hamiltonian

\[
H = \sum_n \int dx \left\{ \frac{1}{2} B \left( \frac{d\phi_n}{dx} \right)^2 - g \cos [\phi_{n+1}(x) - \phi_n(x)] \right\},
\]

(A1)

where for simplicity of notation we have defined rescaled phonon field \( \phi_n(x) \) and elastic couplings \( B \) and \( g \) related to those defined in the Introduction through

\[
\phi_n(x) = \frac{2\pi}{a} u_n(x) ,
\]

(A2a)

\[
B = K d \left( \frac{a}{2\pi} \right)^2 ,
\]

(A2b)

\[
g = \mu d \left( \frac{a}{2\pi d} \right)^2 .
\]

(A2c)

In the Introduction we have used simple qualitative arguments to estimate the colloidal freezing transition temperature. Here we would like to treat this model quantitatively and in more detail. Unfortunately, however, as can be seen from a standard renormalization group analysis weak coupling \( g \) is always irrelevant at long scales, with the effective coupling \( g(\xi_x) \) vanishing at length scale \( \xi_x \) as

\[
g(\xi_x) = g \left( \frac{\xi_x}{a} \right) e^{-\text{const.} (k_B T/B) \xi_x} .
\]

(A3)

Thermal fluctuations, which are especially strong in 1d are responsible for this effective decoupling of the colloidal system into effectively independent one-dimensional liquids. This precludes a description of the freezing transition in weak \( g \) coupling starting from this model. There are two alternatives: One is to study of the melting transition from a complementary strong coupling, fully elastic model with topological defects (dislocations), an approach which lends itself to a rigorous treatment that we undertake in the main part of the paper. Alternatively, an approximate, variational treatment of the model, Eq.(A1) is possible and will be presented in this appendix.

The idea behind a variational approach of a problem is that an approximate free energy

\[
\tilde{F} = \langle H - H_v \rangle_v + F_v
\]

(A4)

is an upper bound for the exact free energy \( F \) corresponding to the Hamiltonian \( H \) of interest and where \( H_v \) any other (the so called variational) Hamiltonian, \( F_v \) is the corresponding free energy and subscript \( v \) on the thermal average indicates that Boltzmann weight with Hamiltonian \( H_v \) is used. The advantage of the variational principle can be taken if the arbitrary variational Hamiltonian \( H_v \) is judiciously chosen to be simple enough, so that thermal averages can be calculated, but at the same time general enough so as to be able approximately capture the physics of the full Hamiltonian \( H \).

Since, unfortunately, our abilities to compute functional integrals do not extend beyond Gaussians, we choose a quadratic form for \( H_v \)

\[
H_v = \sum_n \int dx \left[ \frac{B_x}{2} \left( \frac{d\phi_n}{dx} \right)^2 + \frac{B_y}{2} (\phi_{n+1} - \phi_n)^2 \right] ,
\]

(A5)

with \( B_x \) and \( B_y \) as the effective variational parameters, respectively related to the effective long wavelength bulk and shear moduli, latter given by

\[
\mu = B_y d \left( \frac{2\pi}{a} \right)^2
\]

(A6)

Simple Gaussian averages then lead to the variational free energy density \( \tilde{f}(B_x, B_y) = \tilde{F}(B_x, B_y)/L_2 N_y \)
\[ \tilde{f} = \int_k \left\{ \frac{1}{2} (B - B_x) k_x^2 - B_y (1 - \cos k_y d) \right\} G_v(k) \\
- \frac{1}{2} k_B T \log G_v(k) \right\} \\
- g \exp \left[ - \int_k (1 - \cos k_y d) G_v(k) \right], \quad (A7) \]

where \( L_x \) and \( N_y (= L_y / d) \) are respectively the length and the number of laser potential troughs (i.e., the 2d dimensions of our colloidal system) and \( G_v(k) \) is the Fourier transform of the intra-trough displacement correlation function given by

\[ G_v(k) = k_B T \left( B_x k_x^2 + 2 B_y (1 - \cos k_y d) \right)^{-1}. \quad (A8) \]

To find the upper-bound of the free energy density \( \tilde{f} \), we now minimize \( \tilde{f}(B_x, B_y) \) over the variational parameters \( B_x \) and \( B_y \). Conceptually simple but tedious calculation gives

\[ B_x = B, \quad (A9a) \]

\[ B_y(B, g) = g e^{-k_B T / \pi d (B_y B)^{1/2}}. \quad (A9b) \]

Equation \((A9)\), which determines the behavior of \( B_y \) and therefore the effective shear modulus \( \mu \) as a function of temperature and inter-trough coupling \( g \), illustrated in Fig.23, is the main result of the variational calculation. Simple graphical analysis of Eq.\((A9)\) predicts

\[ B_y(g) = 0, \text{ for } g < g_c, \quad (A10a) \]

\[ B_y(g) \approx g e^{-k_B T / \pi (g B)^{1/2}}, \text{ for } g > g_c, \quad (A10b) \]

where, the critical value of the coupling \( g \) which separates the two solutions for \( B_y \) is given by

\[ g_c = \left( \frac{k_B T_c}{2 \pi} \right)^2 \frac{1}{B}. \quad (A11) \]

FIG. 23. Shear modulus \( \mu \) as a function of the inter-trough coupling \( g \) (at fixed temperature), showing a freezing transition between \( \mu = 0 \) 2d liquid and a \( \mu > 0 \) 2d crystal, and a jump discontinuity at \( g_c \) in the shear modulus.

Combining this with Eq.\((A4)\), we conclude that the transition between the two solutions in Eq.\((A10)\) represents the freezing of a zero shear modulus (\( \mu = 0 \) 2d liquid into a finite shear modulus (\( \mu > 0 \) 2d solid. In terms of the shear modulus \( \mu \) and the bulk modulus \( K \), defined by Eqs.\((A2)\), the corresponding melting transition temperature is given by

\[ k_B T_m = \frac{\alpha^2}{2 \pi \sqrt{K \mu}}, \quad (A12) \]

a value that, up to factors of order 1, is consistent with the asymptotically exact prediction of our strong coupling (elastic model) analysis given in the main text.

APPENDIX B: EFFECTIVE ELASTIC CONSTANTS FOR SCREENED REPULSIVE COULOMB POTENTIAL

To calculate the effective elastic constants in the limit of large through potential, we start from a model with a pair potential given by a screened repulsive Coulomb potential \( V(r) = V_0 a \exp(-\kappa r) / r \), where the screening length \( \kappa^{-1} \) is typically much shorter that the mean particle spacing \( a \). The total potential energy is then given by

\[ \Phi = \frac{1}{2} V_0 a \sum_{(l, l')} \frac{1}{|R_{ll'}|} e^{-\kappa |R_{ll'}|}, \quad (B1) \]

where due to the short range of the potential we can safely restrict summation to nearest neighbors, \((l, l')\). The distance between the colloidal particles numbered \( l \) and \( l' \) can (for a perfect lattice) be decomposed into a distance between the equilibrium positions \( r_l \) and the displacement vectors \( u_l \):

\[ R_{ll'} = r_l - r_{l'} + u_l - u_{l'}, \quad (B2) \]

In the following we restrict ourselves to the primary configurations and write the potential energy as sums over Bragg “planes” (i.e., rows of particles in \( d = 2 \)) indexed by an integer \( r \) and particles within these rows indexed by \( l \),
with

\[ \frac{1}{(a/2 + \Delta u_i)^2 + (d + \Delta h_i)^2} \exp \left[ -\kappa \left[ (a/2 + \Delta u_i)^2 + (d + \Delta h_i)^2 \right]^{1/2} \right] \]

and

\[ \frac{1}{(a/2 + \bar{\Delta} u_i)^2 + (d + \Delta h_i)^2} \exp \left[ -\kappa \left[ (a/2 + \bar{\Delta} u_i)^2 + (d + \Delta h_i)^2 \right]^{1/2} \right] , \]

where the relative intra-valley and inter-valley displacement fields are defined as follows (see Fig. 24):

\[ \delta u_i = u_x(x_i + a, y_i) - u_x(x_i, y_i) , \]

\[ \delta h_i = u_y(x_i + a, y_i) - u_y(x_i, y_i) , \]

and

\[ \Delta u_i = u_x(x_i + a/2, y_i + d) - u_x(x_i, y_i) , \]

\[ \Delta \bar{u}_i = -u_x(x_i - a/2, y_i) + u_x(x_i, y_i) , \]

\[ \Delta h_i = u_y(x_i + a/2, y_i + d) - u_y(x_i, y_i) . \]

FIG. 24. Sketch of two rows of a triangular lattice of colloidal particles in a through potential illustrating two contributions to the effective potential energy. The sum over the lattice sites is done by summing along the valleys; there is one intra-valley nearest neighbor (wiggly line) and two inter-valley nearest neighbors one in the forward direction (\( \Delta u \)) and one in the backward direction (\( \Delta \bar{u} \)) (solid lines).

In the strong pinning limit the laser potential \( H_K \) can be expanded in powers of the phonon fields in the \( y \)-direction,

\[
\beta H_K = \frac{U_K}{k_B T} \sum_i \cos \left( \frac{2\pi}{d} h_i \right) \\
\approx p^2 \frac{8\pi^2}{3} \frac{U_K}{k_B T} \sum_i \left( \frac{h_i}{a} \right)^2 \\
\equiv k_B T w \sum_i \left( \frac{h_i}{a} \right)^2 ,
\]

where we have used \( pd = \sqrt{3}a/2 \). In the following we shall (in order to simplify notation) measure all lengths in units of the mean lattice spacing.

We proceed as follows: (i) first we expand all terms in the total potential energy \( \Phi \) to quadratic order in the out-of-valley displacement fields, (ii) integrate out the massive out-of-valley modes, and (iii) take the continuum limit. Note, that it is only step (i) which explicitly depends on the particular form of the pair potential. For simplicity, we will limit our derivation to the leading order in \( V_0/U_K \) and \( k_B T/U_K \).

Step (i) gives

\[
\beta \Phi[u, h] = \beta(\Phi_1 + \Phi_2 + \Phi_3)
\]

with

\[
\Phi_1[u, h] = v \sum_i \left\{ -\frac{1}{2}(\kappa + 1)h_i^2 + \frac{1}{2}(\kappa^2 + 2\kappa + 2)\delta u_i^2 + \beta_1(\kappa)\delta h_i^2 \delta u_i + \delta_1(\kappa)\delta h_i^2 \delta u_i^2 \right\}
\]

\[
\Phi_2[u, h] = v \sum_i \left\{ -\frac{1}{8}(3\kappa^2 + 5\kappa + 5)\Delta h_i^2 + \frac{1}{8}(\kappa^2 - \kappa - 1)\Delta u_i^2 + \alpha_2(\kappa)\delta h_i \Delta u_i \\
+ \beta_2(\kappa)\Delta h_i^2 \Delta u_i + \gamma_2(\kappa)\Delta h_i \Delta u_i^2 + \delta_2(\kappa)\Delta h_i^2 \Delta u_i^2 \right\}
\]

\[ (B10) \]

\[ (B11) \]

\[ (B12) \]
and \( \Phi_3[u, h] \) obtained from \( \Phi_2[u, h] \) by the replacement \( \Delta u_l \rightarrow \Delta \tilde{u}_l \). Here we have also introduced

\[
\beta_1(\kappa) = \frac{1}{2} (\kappa^2 + 3\kappa + 3), \quad \delta_1(\kappa) = -\frac{1}{4} (\kappa^3 + 5\kappa + 12\kappa + 12),
\]

and

\[
\alpha_2(\kappa) = \frac{\sqrt{3}}{4} (\kappa^2 + 3\kappa + 3), \quad \beta_2(\kappa) = -\frac{1}{16} (3\kappa^3 + 14\kappa^2 + 33\kappa + 33), \quad \gamma_2(\kappa) = \frac{\sqrt{3}}{16} (\kappa^3 + 2\kappa^2 + 3\kappa + 3), \quad \delta_1(\kappa) = \frac{1}{64} (3\kappa^4 + 14\kappa^3 + 55\kappa^2 + 123\kappa + 123).
\]

The dimensionless ratio

\[
v \equiv e^{-\kappa} \frac{V_0}{k_B T}
\]

measures the strength of the pair potential relative to a typical thermal energy. Next we integrate out the massive phonon fields \( h_l \) with a Boltzmann weight given by the external potential \( H_K \),

\[
\exp[-\beta H_{\text{eff}}] = \int [dh] \exp \left[ -u \sum_l h_l^2 - \beta \Phi[u, h] \right],
\]

where \( \int [dh] \) denotes an integration over the \( \{h_l\} \). We find

\[
\beta H_{\text{eff}} = \sum_l \left\{ \delta u_l^2 \left[ \frac{v}{2} (\kappa^2 + 2\kappa + 2) + \frac{v}{w} \delta_1(\kappa) - \frac{v^2}{w} \alpha_2(\kappa) \right] \right. \\
+ \left( \Delta u_l^2 + \Delta \tilde{u}_l^2 \right) \left[ \frac{v}{8} (\kappa^2 - \kappa - 1) + \frac{v}{w} \delta_2(\kappa) \right] \}
\]

In the continuum limit (and reindroducing the scale \( a \)), we have

\[
\delta u_l^2 \rightarrow a^2 (\partial_x u_x)^2
\]

\[
(\Delta u_l^2 + \Delta \tilde{u}_l^2) \rightarrow a^2 \left( \frac{1}{2} (\partial_x u_x)^2 + \frac{3}{2} (\partial_y u_x)^2 \right)
\]

\[
\sum_l \rightarrow \frac{1}{a^2} \int d^2 x
\]

we find finally our desired result, namely

\[
H_{\text{eff}} = \frac{1}{2} \int d^2 r \left[ \mu_{\text{eff}} (\partial_x u_x)^2 + K_{\text{eff}} (\partial_x u_x)^2 \right],
\]

with

\[
\mu_{\text{eff}} \approx \mu_{\text{eff}}^\infty \left\{ 1 + \frac{9(k\kappa)^2}{64\pi^2} \left( 1 + \frac{17}{3k\kappa} \right) \frac{k_B T}{p^2 U_K} \right\},
\]

\[
K_{\text{eff}} \approx K_{\text{eff}}^\infty \left\{ 1 + \frac{(k\kappa)^2}{64\pi^2} \left( 1 - \frac{23 + 104v}{3k\kappa} \right) \frac{k_B T}{p^2 U_K} \right\},
\]

where

\[
\mu_{\text{eff}}^\infty = \frac{3}{8} (\kappa a)^2 - \kappa - 1 \right) V_0 e^{-\kappa a},
\]

\[
K_{\text{eff}}^\infty = \frac{1}{8} \left( 9(k\kappa)^2 + 13k\kappa + 15 \right) V_0 e^{-\kappa a}.
\]

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Experiments on colloids, similar to those described in Ref. [1], with a periodic one dimensional potential provided by lithographic patterning of the confining glass plate, are currently underway at Lucent Technologies (C.-H. Sow and C.M. Murray, private communication).

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The scattered intensity $I_{\text{scatt}}$ is clearly proportional to the product of the scattering cross-section and the input laser intensity $I_{\text{in}}$. Since the former is related to the density-density correlation function, the cross-section itself must be proportional to the square of the input laser intensity $I_{\text{in}}$, which explicitly induces finite density modulation. Consequently $I_{\text{scatt}} \propto I_{\text{in}}^2$.

As explained in detail in Sec. II C, a laser-induced periodic potential, whose strength is proportional to $I_{\text{in}}$, induces a nematic ordering field $h_{ij}$ proportional to $I_{\text{in}}^2$. The cube of the induced nematic order parameter (proportional $(I_{\text{in}}^2)^{1/3} = I_{\text{in}}^{1/3}$) then turn acts as an ordering field on the hexatic order parameter $\phi$.

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A set of primary Bragg planes is defined by those parallel planes of colloidal particles within which all particles are related by multiples of a single fundamental lattice vector. Equivalently, a set of primary Bragg planes can be defined as those planes, which run perpendicular to one of the fundamental reciprocal lattice vectors $G_i$.

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Because an applied periodic potential will generally include the higher harmonics, all of our results for $p = 1$ also directly extend to all rational fractions $1/q$, with $q \in \mathbb{Z}$.

The factor of $1/d^2$ is inserted into the definition of the discrete model purely for convenience, so as to have the interchannel coupling $\mu$ have the same dimensions as the shear modulus $\mu_{\text{eff}}$ of the continuum elastic model in Eq. [3], which then allows its identification as the microscopic shear modulus.

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For $1 < p \leq p_c$, our system in principle admits more exotic locked smectic phases, characterized by invariance under discrete translations by $nd$, where $n \in \mathbb{Z}$, but $n \neq p$. Although these phases are thermodynamically distinct, their properties are similar to those that we consider in the main text. We therefore do not study them any further here.

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It is important to appreciate that even in the FS phase, where the periodic potential is irrelevant in the renormalization group sense, its effects are nontrivial and experimentally observable. For example, in the presence of a periodic potential, even if it is irrelevant, the continuous translational and rotational symmetry is explicitly broken and FS displays true Bragg peaks in its structure function at multiples of $K$. P.G. de Gennes, The Physics of Liquid Crystals (Oxford University Press, London, 1974); P.G. de Gennes and J. Prost, The Physics of Liquid Crystals, 2nd edition (Clarendon Press, Oxford, 1993).

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The choice of the phonon field-rescaling exponent $\phi$ is of course arbitrary. If instead of the convenient $\phi = 0$ choice we left it arbitrary, then to return the Hamiltonian into the form it had before the RG transformation, we would have to also allow the periodic potential wavevector $K$ to flow as $K(\ell) = Ke^{\phi \ell}$. In the end the flow equations for the dimensionless coupling constants would involve $K(\ell)$ and would be independent of the choice of $\phi$.

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The conclusion of a direct LFS-LSm transition is implicitly based on an assumption that the periodic potential that is relevant in the 2d solid phase (LFS) remains relevant (leading to the LSm phase) even when type I dislocation unbind producing a smectic phase. If for a range of parameters $T_{\phi < T_{pSm}}$, then we would instead expect a direct LFS to locked smectic transition, if $T_{\phi > T_{pSm}}$ over some parameter range, so that the periodic potential is irrelevant in the 2d solid (FS) but becomes relevant in the smectic phase (LSm), where type I dislocations are unbound. Because of the nontrivial relation between elastic constants in the solid and smectic phases, we cannot
exclude the exotic possibilities discussed above based on Eqs. 4.13, 4.19 for $T_{pS}$ and $T_{pS\text{sm}}$, respectively.

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