The boundedness of multilinear commutators on locally compact Vilenkin groups

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Abstract. Let $G$ be a locally compact Vilenkin group. In this paper, the authors investigate the boundedness of multilinear commutators of fractional integral operator on Lebesgue spaces on $G$. Furthermore, the boundedness on Hardy spaces are also obtained in this paper.

1. Introduction

The commutators have been studied by many authors for a long time. A well known result which is discovered by Coifman, Rocherg and Weiss ([3], [6], [9]) is that the commutators $[b, T]$ of singular integral operators are bounded on some $L^p(\mathbb{R}^n)(1 < p < \infty)$ if and only if $b \in BMO$, where $[b, T]$ is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$
A natural generalization of the commutator $T^m_b$ is given by

\[(1.1) \quad T^m_b(f)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]^m K(x, y)f(y)dy,\]

where $m \in \mathbb{N}$. It was shown in [1] that it is bounded on $L^p(\omega)(1 < p < \infty)$ when $\omega \in A_p$. And in [2], the authors prove the $L^p(\omega)(1 < p < \infty)$-boundedness for multilinear commutators. And similar results can be found in [10]. This is the motivation of considering the boundedness for multilinear commutators of fractional integral operator on locally compact Vilenkin group $G$.

In order to state our results more precisely we first introduce some notations and definitions.

Throughout this paper, $G$ will denote a bounded locally compact Vilenkin group, that is, $G$ is a locally compact Abelian group containing a strictly decreasing sequence of compact open subgroups $\{G_n\}_{n=-\infty}^{\infty}$ such that

(a) $\cup_{n=-\infty}^{\infty} G_n = G$ and $\cap_{n=-\infty}^{\infty} G_n = \{0\}$;

(b) $\sup\{\text{order}(G_n/G_{n+1}) : n \in \mathbb{Z}\} = B < \infty$.

Choose Haar measure $dx$ on $G$ so that $|G_0| = 1$, where $|A|$ denotes the measure of a measurable subset $A$ of $G$. Let $|G_n| = (m_n)^{-1}$ for each $n \in \mathbb{Z}$. Since $2m_n \leq m_{n+1} \leq Bm_n$, it follows that

$$\sum_{n=k}^{\infty} (m_n)^{-\alpha} \leq c(m_k)^{-\alpha}$$

and

$$\sum_{n=-\infty}^{k} (m_n)^{\alpha} \leq c(m_k)^{\alpha}$$

for any $\alpha > 0, k \in \mathbb{Z}$, where $c$ is a constant independent of $k$. For each $n \in \mathbb{Z}$ we choose elements $z_{l,n} \in G(l \in \mathbb{Z}^+)$ so that the subsets $G_{l,n} := z_{l,n} + G_n$ of $G$ satisfy $G_{k,n} \cap G_{l,n} = \phi$ if $k \neq l$ and $\cup_{l=0}^{\infty} G_{l,n} = G$; moreover, we choose $g_{0,n}$ such that $G_{0,n} = G_n$. We now define the function $d : G \times G \to \mathbb{R}$ by $d(x,y) = 0$ if $x - y = 0$ and $d(x,y) = (m_n)^{-1}$ if $x - y \in G_n \setminus G_{n+1}$. Then $d$ is a metric on $G$ and the topology on $G$ generated by this metric is the same as the original topology on $G$. For $x \in G$, set $|x| = d(x,0)$. Then $|x| = (m_n)^{-1}$ if and only if $x \in G_n \setminus G_{n+1}$. Let $S(G)$ be the space of test functions and $S'(G)$ be the distribution space on $G$. And $\chi_{G_n}$ is the characteristic function of $G_n$. $C$ can be denote various constants.

We also recall the definition of space of bounded mean oscillation. For $x_0 \in G$, set $I_j = x_0 + G_j$, we say a locally integrable function $b$ has bounded
mean oscillation, \( b \in BMO(G) \), if
\[
\sup_{I_j \ni x} \inf_{c \in \mathbb{R}} \left( \frac{1}{|I_j|} \int_{I_j} |f(y) - c| \, dy \right) \leq \infty,
\]
where the supremum is taken over all cosets \( I_j \subset G \) (see[4]). Since the topological nature of \( G \) at any \( x \in G \) is the same as it is at 0, we choose \( x_0 = 0 \) in our article.

2. Main results and proofs

Define multilinear commutator of fractional integral operator \( I_{\alpha} \) as
\[
[b, I_{\alpha}]f(x) = \int_{G} \prod_{j=1}^{m} \left[ b_j(x) - b_j(y) \right] \frac{f(y)}{|x-y|^{1-\alpha}} dy
\]
and maximal function
\[
f^{*}_{\alpha,r}(x) = \sup_{x \in G} |G_k|^\alpha \left( \frac{1}{|G_k|} \int_{G_k} |f(y)|^r \, dy \right)^{1/r}.
\]

Given any positive integer \( m \), for all \( 1 \leq j \leq m \), we denote by \( C^m_j \) the family of all finite subsets \( \sigma = \{ \sigma(1), \ldots, \sigma(j) \} \) of \( \{1, \ldots, m\} \) of \( j \) different elements. For any \( \sigma \in C^m_j \), we associate the complementary sequence \( \sigma' \) given by \( \sigma' = \{1, 2, \ldots, m\} \setminus \sigma \), and
\[
[b_{\sigma}, I_{\alpha}]f(x) = \int_{G} \left[ b_{\sigma(1)}(x) - b_{\sigma(1)}(y) \right] \cdots \left[ b_{\sigma(j)}(x) - b_{\sigma(j)}(y) \right] \frac{f(y)}{|x-y|^{1-\alpha}} dy.
\]
suppose \( \|b_i\|_{BMO} = 1 \) for \( i = 1, 2, \ldots, m \). Let \( p' \) is adjoint index of \( p \). We have the following theorem.

**Theorem 2.1.** Let \( b = (b_1, \ldots, b_m) \), \( b_i \in BMO \), \( 1/q = 1/p - \alpha \), \( 1 < p < 1/\alpha \), \( 0 < \alpha < 1 \), then \([b, I_{\alpha}]\) maps \( L^p(G) \) into \( L^q(G) \).

To prove this theorem, we need the following lemmas.

**Lemma 2.1 (see [7]).** Let \( p > 1, 1/q = 1/p - \alpha, 0 < \alpha < 1 \), then
\[
\|I_{\alpha}f\|_q \leq C\|f\|_p.
\]

**Lemma 2.2 (see [5] and [8]).** Let \( r < p < 1/\alpha \), \( 1/q = 1/p - \alpha \), \( 0 < \alpha < 1 \), then
\[
\|f^{*}_{\alpha,r}\|_q \leq C\|f\|_p.
\]
Lemma 2.3. Let $b = (b_1, \cdots, b_m)$, $b_i \in \text{BMO}$, $0 < \delta < \epsilon < 1$, then there exist a constant $C$ depends on $\delta$, $\epsilon$, and choose $p$ that $1 < p_1 < p < \frac{1}{\alpha}$ such that

$$M^\#_\delta([b, I_\alpha]f)(x) \leq C \left\{ M(I_\alpha f)(x) + \sum_{j=1}^{m-1} M_j([b_{\sigma'} , I_\alpha]f)(x) + M_{\alpha, p_1}(x) \right\}.$$  

Proof. We firstly consider the case of $m = 1$, that is,

$$[b, I_\alpha]f(x) = \int_G [b(x) - b(y)] f(y) \frac{dy}{|x - y|^{1-\alpha}} = [b(x) - \lambda] I_\alpha f(x) - I_\alpha [(b - \lambda)f](x).$$

Fix $x$, let $x \in G_k$, since $||\alpha||^\delta - ||\beta||^\delta \leq |\alpha - \beta|^\delta$ for any $\alpha, \beta \in R$, $0 < \delta < 1$, then we have

$$\left( \frac{1}{|G_k|} \int_{G_k} \left| [b, I_\alpha]f(y) \right|^\delta dy \right)^{1/\delta} \leq \left( \frac{1}{|G_k|} \int_{G_k} \left| [b, I_\alpha]f(y) \right|^\delta dy \right)^{1/\delta}.$$

$$= C \left( \frac{1}{|G_k|} \int_{G_k} \left| [b(y) - \lambda] I_\alpha f(y) \right|^\delta dy \right)^{1/\delta}$$

$$+ C \left( \frac{1}{|G_k|} \int_{G_k} \left| I_\alpha [(b - \lambda)f](y) \right|^\delta dy \right)^{1/\delta} = I + II.$$

choose $\lambda = b_{G_k}$ and $1 < q < \epsilon/\delta$, using Hölder inequality and Jenson inequality, we can deduce that

$$I \leq C \left( \frac{1}{|G_k|} \int_{G_k} |b(y) - \lambda|^\delta q' dy \right)^{1/\delta q'} \left( \frac{1}{|G_k|} \int_{G_k} |I_\alpha f(y)|^\delta q dy \right)^{1/\delta q} \leq C ||b||_{\text{BMO}} M(I_\alpha f)(x).$$

To estimate II, let $f = f_1 + f_2$, $f_1 = f \chi_{G_k}$, then

$$II \leq C \left( \frac{1}{|G_k|} \int_{G_k} |I_\alpha [(b - \lambda)f_1](y)|^\delta dy \right)^{1/\delta}$$

$$+ \left( \frac{1}{|G_k|} \int_{G_k} |I_\alpha [(b - \lambda)f_2](y) - c|^\delta dy \right)^{1/\delta} = III + IV.$$
For III, we have
\[ III \leq C \frac{1}{|G_k|} \int_{G_k} |I_\alpha[(b - \lambda) f]\| dy \]
\[ \leq C \frac{1}{|G_k|} \int_{G_k} \int_{G_k} |b(y) - \lambda| \frac{|f(y)|}{|x - y|^{1-\alpha}} dy dx \]
\[ \leq C \frac{1}{|G_k|^{1-\alpha}} \int_{G_k} |b(y) - \lambda||f(y)|| dy, \]
choose \( p_1 \) such that \( 1 < p < 1/\alpha \), then
\[ III \leq C \left( \frac{1}{|G_k|} \int_{G_k} |b(y) - \lambda|^{p_1} dy \right)^{1 \prime / p_1} |G_k|^\alpha \left( \frac{1}{|G_k|} \int_{G_k} |f(y)|^{p_1} dy \right)^{1 \prime / p_1} \]
\[ \leq C \|b\|_{BMO} f_{\alpha, p_1}(x). \]
Next, we turn to estimate IV. Let
\[ c = (I_\alpha[(b - \lambda) f_2])_{G_k}, \]
then
\[ IV \leq C \frac{1}{|G_k|} \int_{G_k} |I_\alpha[(b - \lambda) f_2](y) - (I_\alpha[(b - \lambda) f_2])_{G_k}| dy \]
\[ = C \frac{1}{|G_k|} \int_{G_k} \int_{G_k \setminus G_k} \left\{ \frac{b(\omega) - \lambda f(\omega)}{|y - \omega|^{1-\alpha}} d\omega \right\} dy \]
\[ = C \frac{1}{|G_k|} \int_{G_k} \left\{ \frac{b(\omega) - \lambda f(\omega)}{|z - \omega|^{1-\alpha}} d\omega \right\} dy \]
\[ \leq C \frac{1}{|G_k|} \int_{G_k} \sum_{i=-\infty}^{k-1} \int_{G_i \setminus G_{i+1}} |b(\omega) - \lambda f(\omega)| \]
\[ \times \left\{ \frac{|y - z|}{|z - \omega|^{2-\alpha}} d\omega dz dy \right\} \]
\[ \leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{1-\alpha}} \int_{G_i} |b(\omega) - \lambda f(\omega)| d\omega \]
\[ \leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{1-\alpha}} \left( \frac{1}{|G_i|} \int_{G_i} |b(\omega) - \lambda|^{p_1} d\omega \right)^{1 \prime / p_1} \]
\[ \times \left( \frac{1}{|G_i|} \int_{G_i} |f(\omega)|^{p_1} d\omega \right)^{1 \prime / p_1} \]
where

$$\frac{1}{|G_n|} \int_{G_n} |b - b_{G_{n+1}}| dx \geq \frac{1}{|G_{n+1}|} \int_{G_{n+1}} |b(x) - b_{G_{n+1}}| dx \leq \frac{|G_{n+1}|}{|G_n|} \|b\|_{BMO} \leq \frac{1}{2} \|b\|_{BMO},$$

for any $n \in \mathbb{Z}$, and

$$|b_{G_i} - b_{G_k}| = |b_{G_i} - b_{G_{i+1}}| + |b_{G_{i+1}} - b_{G_{i+2}}| + \cdots + |b_{G_{k-1}} - b_{G_k}| = \frac{1}{2} (k - i) \|b\|_{BMO}.$$

Therefore, we have proved that

$$M_\lambda^a([b, I_\alpha] f)(x) \leq C \|b\|_{BMO} [M_\epsilon (I_\alpha f)(x) + f_{\alpha, p_i}^*(x)]$$

for $m = 1$.

If $m \geq 2$, let $\lambda = (\lambda_1, \cdots, \lambda_m)$, we have

$$[b, I_\alpha] f(x) = \int_{G} \prod_{j=1}^{m} [(b_j(x) - b_j(y)] \frac{f(y)}{|x - y|^{1 - \alpha}} dy$$

$$= \int_{G} \prod_{j=1}^{m} [(b_j(x) - \lambda_j + \lambda_j - b_j(y)] \frac{f(y)}{|x - y|^{1 - \alpha}} dy$$

$$= \prod_{j=1}^{m} [(b_j(x) - \lambda_j) I_\alpha f(x) + (-1)^m I_\alpha (\prod_{j=1}^{m} (b_j(x) - \lambda_j) f)(x)]$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma \subseteq \alpha} C_{m, j} [b(x) - \lambda_\sigma] \omega_\sigma I_\alpha [f](x),$$
where $C_{m,j}$ are constants depending on $m$ and $j$.

Now, for fixed $x \in G$, for any number $c$ and $G_k \ni x$, since $0 < \delta < 1$, we have

\[
\left( \frac{1}{|G_k|} \int_{G_k} ||[b, I_\alpha]f(y)||^\delta - |c|^\delta \, dy \right)^{1/\delta} \\
\leq \left( \frac{1}{|G_k|} \int_{G_k} |[b, I_\alpha]f(y) - c|^\delta \, dy \right)^{1/\delta} \\
\leq C \left( \frac{1}{|G_k|} \int_{G_k} \left| \prod_{j=1}^m [b_j(y) - \lambda_j I_\alpha(f)(y)] \right| \, dy \right)^{1/\delta} \\
+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_m} \left( \frac{1}{|G_k|} \int_{G_k} |[b(y) - \lambda_\sigma]_{\sigma}[b_{\sigma'}, I_\alpha](f)(y)|^\delta \, dy \right)^{1/\delta} \\
+ C \left( \frac{1}{|G_k|} \int_{G_k} |I_\alpha \left[ \prod_{j=1}^m (b_j - \lambda_j) f \right](y) - c|^\delta \, dy \right)^{1/\delta} \\
= I + II + III.
\]

For the first two parts $I$ and $II$, choose $\lambda_j = (b_j)_{G_k}, j = 1, 2, \ldots, m$, and $1 < q < \delta/\epsilon$, similar to the case of $m = 1$, we can deduce that

\[ I \leq CM_\epsilon(I_\alpha f)(x), \]

and
\[ II \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_m} M_\epsilon \left( [b_{\sigma'}, I_\alpha]f \right)(x). \]

For $III$, let $f = f_1 + f_2$, $f_1 = f \chi_{G_k}$, then

\[ III \leq C \left\{ \frac{1}{|G_k|} \int_{G_k} |I_\alpha \left[ \prod_{j=1}^m (b_j - \lambda_j) f_1 \right](y)|^\delta \, dy \right\}^{1/\delta} \\
+ C \left\{ \frac{1}{|G_k|} \int_{G_k} |I_\alpha \left[ \prod_{j=1}^m (b_j - \lambda_j) f_2 \right](y) - c|^\delta \, dy \right\}^{1/\delta} \\
= IV + V. \]

By Jensen inequality, we have

\[ III \leq C \frac{1}{|G_k|} \int_{G_k} |I_\alpha \left[ \prod_{j=1}^m (b_j - \lambda_j) f_1 \right](y) \, dy \]

\[ \leq C \frac{1}{|G_k|} \int_{G_k} \int_{G_k} \prod_{j=1}^m |b_j(z) - \lambda_j| \frac{|f(z)|}{|y-z|^{1-\alpha}} \, dz \, dy. \]
To estimate \( V \), let

\[
c = \left( I_\alpha \left[ \prod_{j=1}^m (b_j - \lambda_j) f_2 \right] \right)_{G_k},
\]

similar to the case in \( m = 1 \), it can follow that

\[
IV \leq C \frac{1}{|G_k|} \int_{G_k} \left| I_\alpha \left[ \prod_{j=1}^m (b_j - \lambda_j) f_2 \right] (y) - \left[ I_\alpha \left( \prod_{j=1}^m (b_j - \lambda_j) f_2 \right) \right]_{G_k} \right| dy
\]

\[
\leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{2-\alpha}} \int_{G_i} \prod_{j=1}^m |b_j(\omega) - \lambda_j| |f(\omega)| d\omega
\]

\[
\leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{2-\alpha}} \left( \frac{1}{|G_i|} \int_{G_i} \prod_{j=1}^m |b_j(\omega) - \lambda_j|^{p'_i} d\omega \right)^{1/p'_i}
\]

\[
\times \left( \frac{1}{|G_i|} \int_{G_i} |f(\omega)|^{p_1} d\omega \right)^{1/p_1}
\]

\[
\leq C \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|} f_{\alpha, p_1}^*(x) \times \prod_{j=1}^m \left[ \left( \frac{1}{|G_i|} \int_{G_i} |b_j(\omega) - (b_j)_{G_i}|^{mp'_i} d\omega \right)^{1/mp'_i}
\]

\[
+ |(b_j)_{G_i} - (b_j)_{G_k}| \right]
\]

\[
\leq C \sum_{i=-\infty}^{k-1} \frac{m_i}{m_k} (k - i)^m f_{\alpha, p_1}^*(x) \|b\|_{BMO}
\]

\[
\leq C \|b\|_{BMO} f_{\alpha, p_1}^*(x).
\]

Combine with I, II, IV, V, we finish the proof of Lemma 2.3. \( \square \)

**Proof of Theorem 2.1.** We first take it for granted that \( M([b, I_\alpha]f) \in L^q(G) \) and we’ll check this to the end of the proof.
We proceed by induction on $m$. For $m = 1$, by lemma 2.1, 2.2 and 2.3, we have
\[
\| [b, I_{\alpha}] f \|_q \leq C \| M_\delta^l ([b, I_{\alpha}] f) \|_q \\
\leq C \| M_\epsilon (I_{\alpha} f) \|_q + C \| f \|_{p_1} \\
\leq C \| I_{\alpha} f \|_q + C \| f \|_p \\
\leq C \| f \|_p.
\]
Suppose now that for $m - 1$ the theorem is true, and let us prove it for $m$. The same argument as used above and the induction hypothesis give
\[
\| [b, I_{\alpha}] f \|_q \leq C \| (M_\epsilon (I_{\alpha} f)) \|_q + C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \| M_\epsilon ([b_\sigma', I_{\alpha}] f) \|_q + C \| f \|_p \\
\leq C \| f \|_p.
\]
Now let us check that $M ([b, I_{\alpha}] f) \in L^q(G)$. By the boundedness of maximal operator, we only need to prove $[b, I_{\alpha}] f \in L^q(G)$. Suppose for any $j(j = 1, 2, \ldots, m)$, $b_j$ is bounded, since $C_c^\infty(G)$ is dense in $L^p(G)(p > 1)$, we only need to consider the function $f$ with compact support. Suppose $\text{supp } f \subset G_k$, we have
\[
\| [b, I_{\alpha}] f \|_q = \left( \int_{G_k} \| [b, I_{\alpha}] (f) (x) \|^q dx \right)^{1/q} + \left( \int_{G \setminus G_k} \| [b, I_{\alpha}] (f) (x) \|^q dx \right)^{1/q} = A_1 + A_2.
\]
Using the boundedness of $b_1$, we get
\[
A_1 \leq C \left( \int_{G_k} \left( \int_G |b(x) - b(y)| \frac{f(y)}{|x-y|^{1-\alpha}} dy \right)^q dx \right)^{1/q} \\
\leq C \left( \int_{G_k} |I_{\alpha} (f) (x) |^q dx \right)^{1/q} \\
\leq C \| f \|_p.
\]
For the second term, according to the boundedness of $b_i$ and $x \notin G_k$, $y \in G_k$, $|x-y| \sim |x|$, we have
\[
\| [b, I_{\alpha}] f \| \leq C \int_{G_k} \prod_{j=1}^m |b_j (x) - b_j (y)| \frac{|f(y)|}{|x-y|^{1-\alpha}} dy \\
\leq C \int_{G_k} \frac{|f(y)|}{|x-y|^{1-\alpha}} dy
\]
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\[ \leq C \frac{1}{|x|^{1-\alpha}} \int_{G_k} |f(y)| dy \]

\[ \leq C |G_k|^\alpha \left( \frac{1}{|G_k|} \int_{G_k} |f(y)|^{p_1} dy \right)^{1/p_1} \]

\[ \leq C f_{\alpha, p_1}^*(x), \]

Thus,

\[ A_2 \leq C \left( \int_{G \setminus G_k} |f_{\alpha, p_1}^*(x)|^q dx \right)^{1/q} \leq C \| f_{\alpha, p_1}^* \|_q, \]

which is finite by lemma 2.2.

For the general case, we will truncate the symbols \( b_j \) as follows. Denote \( b_N^j = \min\{b_j, N\} \), take into account the fact that \( f \) has compact support, we deduce that any product \( b_{N_1}^j \cdots b_{N_m}^j f \) converges in any \( L^q(G) (q > 1) \) to \( b_{j_1} \cdots b_{j_l} f \) as \( N \to \infty \). By the above discussion, we have

\[ \|[b^N, I_\alpha](f)\|_q \leq C \|b^N\|_{BMO} \|f\|_p. \]

By Fatou’s lemma, we conclude that the theorem holds for this general case. The theorem is proved. \( \square \)

Furthermore, we can discuss the boundedness on Hardy spaces.

**Definition 2.1.** Let \( 0 < p \leq 1 \leq q \leq \infty, p \neq q \), a nonnegative integer \( s \geq [1/p - 1], b = (b_1, \cdots, b_m), b_i \in BMO(G), i = 1, 2, \cdots, m \). A function \( a(x) \in L^p_{loc}(G) \) is said to be a \((p, q, b)\)-atom if it satisfies:

(i) for some \( k \in \mathbb{Z}, \ \text{supp} \ a \subset G_k; \)

(ii) \( \|a\|_q \leq |G_k|^{1/q - 1/p}; \)

(iii) \( \int_{G_k} a(y)dy = \int_{G_k} a(y)\Pi_{i \in \sigma} b_i(y)dy = 0, \) for any \( \sigma \in C^m_j, 1 \leq j \leq m. \)

**Definition 2.2.** The Hardy spaces \( H^p_b(G) \) are defined by

\[ H^p_b(G) = \left\{ f : f \in S'(G), f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, a_k \text{ is } (p, q, b)\text{-atom,} \right. \]

\[ \left. \sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty \right\}, \]

and

\[ \|f\|_{H^p_b(G)} \sim \inf \left\{ \left( \sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} \right\}, \]

where the infimum is taken over all the decomposition of \( f \) as above.
Theorem 2.2. Let $b$ be the same as Theorem 2.1. Let $1/2 < p \leq 1$, $1/q = 1/p - \alpha$, $0 < \alpha < 1$, then $[b, I_\alpha]$ is a bounded operator from $H^p_b(G)$ into $L^q(G)$.

Proof. We only need to prove theorem for the $(p, \infty, b)$-atom. Suppose $\text{supp} \ a \subset G_k$, then

$$\| [b, I_\alpha] a \|_q \leq \left( \int_{G_k} |[b, I_\alpha] a(x)|^q dx \right)^{1/q} + \left( \int_{G \setminus G_k} |[b, I_\alpha] a(x)|^q dx \right)^{1/q} = I + II.$$  

Choose $p_1, q_1$ such that $1 < p_1 < q_1 < \infty$ and $1/q = 1/p - \alpha$, then $q < q_1$. By Hölder inequality and Theorem 2.1, we have

$$I \leq C \left( \int_{G_k} \left| [b, I_\alpha] a(x) \right|^{q_1} dx \right)^{1/q_1} |G_k|^{1/q - 1/q_1},$$

$$\leq C \| a \|_{p_1} |G_k|^{1/q - 1/q_1},$$

$$\leq C \| a \|_\infty |G_k|^{1/p_1 + 1/q - 1/q_1},$$

$$\leq C |G_k|^{1/p_1 - 1/p + 1/q - 1/q_1},$$

$$= C.$$

Note that $x \in G \setminus G_k$, and $y \in G_k$, let $\lambda_j = (b_j)_{G_k}$ for any $j$, then

$$\left| [b, I_\alpha] a(x) \right| = \left| \int_{G_k} \prod_{j=1}^m \left[ b_j(x) - b_j(y) \right] \frac{a(y)}{|x - y|^{1-\alpha}} dy \right|$$

$$= \sum_{j=0}^m \sum_{\sigma \in C_j^m} C |b(x) - \lambda|_\sigma \int_{G_k} \frac{|b(x) - \lambda|_\sigma}{|x - y|^{1-\alpha}} \frac{a(y)}{|x - y|^{1-\alpha}} dy.$$

Using the vanishing moment condition (iii), we get

$$\left| [b, I_\alpha] a(x) \right| \leq \sum_{j=0}^m \sum_{\sigma \in C_j^m} C |b(x) - \lambda|_\sigma \int_{G_k} \frac{|b(x) - \lambda|_\sigma}{|x - y|^{1-\alpha}} \frac{1}{|x - y|^{1-\alpha}} dy$$

$$\leq C \sum_{j=0}^m \sum_{\sigma \in C_j^m} \frac{|G_k|}{|x|^{2-\alpha}} \int_{G_k} \frac{|b(x) - \lambda|_\sigma a(y)}{|y|^{2-\alpha}} dy.$$
Similar to the proof of Theorem 2.1, we deduce that

\[
\begin{align*}
\Pi & \leq C \sum_{j=0}^{m} \sum_{\sigma \in C_j} \int_{G_k} |[b(x) - \lambda]_{\sigma} a(y)| \, dy \\
& \times \left[ \sum_{i=-\infty}^{k-1} \int_{G_i \setminus G_{i+1}} |b(x) - \lambda|_{\sigma}^q \left( \frac{|G_k|}{|x|^{2-\alpha}} \right)^q \, dx \right]^{1/q} \\
& \leq C \sum_{j=0}^{m} \sum_{\sigma \in C_j} \int_{G_k} |[b(x) - \lambda]_{\sigma} a(y)| \, dy \\
& \times \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{2-\alpha}} \left( \int_{G_i \setminus G_{i+1}} |b(x) - \lambda|_{\sigma}^q \, dx \right)^{1/q} \\
& \leq C \|a\|_{\infty} \sum_{j=0}^{m} \sum_{\sigma \in C_j} |G_k| \left( \frac{1}{|G_k|} \int_{G_k} |[b(x) - \lambda]_{\sigma}| \, dy \right)^{1/q} \\
& \times \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{2-\alpha}} |G_i|^{1/q} \left( \frac{1}{|G_i|} \int_{G_i} |[b(x) - \lambda]_{\sigma}|^q \, dx \right)^{1/q} \\
& \leq C \|a\|_{\infty} \|b\|_{BMO} \sum_{i=-\infty}^{k-1} \frac{|G_k|}{|G_i|^{2-\alpha-1/q}} (k-i)^m \\
& \leq C \|b\|_{BMO} \sum_{i=-\infty}^{k-1} \frac{|G_k|^{2-\alpha-1/p}}{|G_i|^{2-\alpha-1/q}} (k-i)^m \\
& \leq C \|b\|_{BMO},
\end{align*}
\]

Here \( m \geq 1, \) \( 1/2 < p \leq 1 \) and \( 1/q = 1/p - \alpha. \)

Therefore, Theorem 2.2 is proved. \( \square \)

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