Exactly solvable quantum mechanical models with infinite renormalization of the wave function

O. Yu. Shvedov
Sub-Dept. of Quantum Statistics and Field Theory, Dept. of Physics, Moscow State University, 119899, Vorobievy Gory, Moscow, Russia

March 27, 2022

Abstract

The main difficulty of quantum field theory is the problem of divergences and renormalization. However, realistic models of quantum field theory are renormalized within the perturbative framework only. It is important to investigate renormalization beyond perturbation theory. However, known models of constructive field theory do not contain such difficulties as infinite renormalization of the wave function. In this paper an exactly solvable quantum mechanical model with such a difficulty is constructed. This model is a simplified analog of the large-$N$ approximation to the $\Phi^4$-$\Phi^2$-model in 6-dimensional space-time. It is necessary to introduce an indefinite inner product to renormalize the theory. The mathematical results of the theory of Pontriagin spaces are essentially used. It is remarkable that not only the field but also the canonically conjugated momentum become well-defined operators after adding counterterms.

1 Introduction

1. An essential feature of realistic models of QFT (such as quantum electrodynamics, Yang-Mills theory etc.) is the property of infinite renormalization of the wave function. This difficulty leads to problems of canonical quantization of the theory. Since the coefficient $z$ of the term $\partial_{\mu}\varphi\partial^\mu\varphi$ of the Lagrangian diverges, the momentum canonically conjugated to the field $\varphi$ should be related with the time derivative of the field $\varphi$ as

$$\pi = z\dot{\varphi}.$$ 

If we believe $\varphi$ to be an operator-valued distribution \cite{1}, its derivative can be also interpreted in the same way. Therefore, the momentum cannot be viewed even as operator distribution because of infinite coefficient $z$.

Infinite renormalization of the wave function is a serious difficulty in the constructive field theory \cite{2,3}. Rigorous construction of mathematical models of QFT have been successful for models with finite $z$ only. The $z = \infty$-case leads to serious difficulties (see, for example, \cite{4}).

\[0\text{e-mail:shvedov@qs.phys.msu.su}\]
This paper deals with the exactly solvable quantum mechanical model with infinite renormalization of the wave function. The Lagrangian of the model is formally written as

\[ L = \frac{\partial^2}{2} - \frac{m^2 Q^2}{2} + \sum_{k=1}^{\infty} \left( \frac{\mu_k^2}{2} - \frac{\Omega_k^2}{2} + g Q \sum_{k=1}^{\infty} \mu_k q_k \right). \]  

(1)

Here \( \mu_k \) are real quantities, while \( \Omega_k, k = 1, \infty \), is an increasing sequence of real positive numbers.

Renormalization properties of the model (1) depend on the large-\( k \) behavior of the sequence \( \mu_k \).

(a) If \( \sum_k \mu_k^2 / \Omega_k < \infty \), the model is quantized in a standard way: one constructs the Hamiltonian, introduces the creation and annihilation operators

\[ q_k = \frac{a_k^+ + a_k^-}{\sqrt{2 \Omega_k}}, \quad \dot{q}_k = i \sqrt{\frac{\Omega_k}{2}} (a_k^+ - a_k^-) \]

with the following commutation relations

\[ [a_k^+, a_m^-] = 0, \quad [a_k^-, a_m^+] = \delta_{km} \]

and shows the obtained Hamiltonian to be a correctly defined self-adjoint operator.

(b) If \( \sum_k \mu_k^2 / \Omega_k = \infty \) but \( \sum_k \mu_k^2 / \Omega_k^2 < \infty \), renormalization of vacuum energy is necessary. The Hamiltonian is a self-adjoint operator with nontrivial domain.

(c) If \( \sum_k \mu_k^2 / \Omega_k^2 = \infty \) but \( \sum_k \mu_k^2 / \Omega_k^2 < \infty \), it is necessary to perform renormalization of \( m^2 \). The vacuum divergences arise. They can be removed by the Faddeev transformation [4].

(d) If \( \sum_k \mu_k^2 / \Omega_k = \infty \) but \( \sum_k \mu_k^2 / \Omega_k^2 < \infty \), there is an additional difficulty: the Stueckelberg divergences [4] arise. They can be removed by the Faddeev-type transformation [6].

(e) If \( \sum_k \mu_k^2 / \Omega_k^2 = \infty \) but \( \sum_k \mu_k^2 / \Omega_k^4 < \infty \), one should perform infinite renormalization of the wave function \( z \).

(f) If \( \sum_k \mu_k^2 / \Omega_k^4 = \infty \), it is necessary to add new counterterms to the Lagrangian.

It will be shown that the expressions

\[ Q(t), \quad z Q(t) - \sum_{k=1}^{\infty} \frac{g \mu_k}{\Omega_k^2} q_k(t) \]  

may be both viewed as operator distributions. Differentiating the second expression, we obtain that the momentum \( P(t) \) canonically conjugated to \( Q(t) \) becomes an operator distribution after adding a counterterm:

\[ P(t) - \sum_{k=1}^{\infty} \frac{g \mu_k}{\Omega_k^2} p_k(t) \]

The model of the type (1) arises in the quantum probability theory [19, 20], in the condensed-matter theory (”polaron model” [21]). It is also an analog of the model \( \Phi \varphi^2 \phi^a \) of a large number of fields which is viewed in the leading order of \( 1/N \)-expansion (see, for example, [8] for more details).

This paper is organized as follows. Section 2 deals with diagonalization of the Hamiltonian. In section 3 renormalization of the model is performed. Section 4 deals with constructing field operators and justifying the hypothesis that expression (2) corresponds to correctly defined operator distributions in the renormalized theory.
2 Investigation of the regularized model

Let us quantize the model (1). Let $\Lambda$ be a positive integer regularization parameter. Perform a substitution $\mu_k \to \mu_k^\Lambda$, where $\mu_k^\Lambda = \mu_k$ at $k < \Lambda$ and $\mu_k^\Lambda = 0$ at $k > \Lambda$. Let $z$ and $m^2$ be also $\Lambda$-dependent, $z_\Lambda$ and $m_\Lambda^2$. Then the Hamiltonian takes the form

$$
H = \sum_{mn} \left( \frac{1}{2} p_m Z_{\Lambda,mn}^{-1} p_n + \frac{1}{2} q_m M_{\Lambda,mn}^2 q_n \right),
$$

where $Z_{\Lambda,mn}$, $m, n = 0, \infty$ and $M_{\Lambda,mn}^2$ are matrices of the form

$$
Z_\Lambda = \begin{pmatrix} z_\Lambda & 0 \\ 0 & 1 \end{pmatrix} \quad M_\Lambda^2 = \begin{pmatrix} m_\Lambda^2 & g \mu^\Lambda \\ g \mu^\Lambda & \Omega^2 \end{pmatrix};
$$

$q_0 \equiv Q$, $p_0 \equiv P$ is a momentum conjugated to $Q$, $p_k$ are momenta conjugated to $q_k$.

Suppose that there exists operators $(Z_{\Lambda}^{-1} M_{\Lambda}^2)^{\pm 1/4}$. After transformation

$$
q_m = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} ((Z_{\Lambda}^{-1} M_{\Lambda}^2)^{-1/4} Z_{\Lambda}^{-1})_{mn} (b_n^+ + b_n^-),
$$

$$
p_m = i \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} (M_{\Lambda}^2 Z_{\Lambda}^{-1})_{mn} (b_n^+ - b_n^-)
$$

the Hamiltonian (3) takes the form

$$
H_{\Lambda} = \sum_{mn=0}^{\infty} b_m^+ [(Z_{\Lambda}^{-1} M_{\Lambda}^2)^{1/2} Z_{\Lambda}^{-1}]_{mn} b_m^- 
$$

up to an additive constant interpreted as vacuum energy which can be removed by renormalization. The canonical commutation relations are written as

$$
[b_m^+, b_n^-] = 0, \quad [b_m^-, b_n^+] = Z_{\Lambda,mn}. 
$$

Choose the Fock representation for the operators $b_m^\pm$. Any state vector can be presented as

$$
\Psi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{k_1...k_n} \psi^{(n)}_{k_1...k_n} b_{k_1}^+ ... b_{k_n}^+ |0 >
$$

where $|0 >$ is a vacuum state, $b_k^- |0 > = 0$, $\psi^{(n)}_{k_1...k_n}$ are functions of $k_1...k_n$ which are symmetric with respect to their transpositions. Relations (6) imply that the inner product can be presented as

$$
(\Phi, \Psi) = \sum_{n=0}^{\infty} \sum_{k_1...k_n p_1...p_n} \psi^{(n)*}_{p_1...p_n} Z_{\Lambda,p_1} ... Z_{\Lambda,p_n} \psi^{(n)}_{k_1...k_n} 
$$

while the Hamiltonian operator acts as

$$
(H_{\Lambda} \psi)^{(n)}_{k_1...k_n} = \sum_{i=1}^{n} \sum_{p_i} (Z_{\Lambda}^{-1} M_{\Lambda}^2)^{1/2} \psi^{(n)}_{k_1...k_{i-1}p_i k_{i+1}...k_n}.
$$

Evolution operator can be written as

$$
(e^{-iH_{\Lambda} t} \psi)^{(n)}_{k_1...k_n} = \sum_{i=1}^{n} \sum_{p_1...p_n} (e^{-i(Z_{\Lambda}^{-1} M_{\Lambda}^2)^{1/2} t})_{k_1 p_1} ... (e^{-i(Z_{\Lambda}^{-1} M_{\Lambda}^2)^{1/2} t})_{k_n p_n} \psi^{(n)}_{p_1...p_n}.
$$

By $\mathcal{P}_\Lambda$ we denote the space of sets $\psi_k$, $k = 0, \infty$ with the indefinite inner product

$$
< \psi, \psi >_\Lambda = (\psi, Z_{\Lambda} \psi).
$$

We see that the state space is the Fock space associated with $\mathcal{P}_\Lambda$:

$$
\mathcal{F}(\mathcal{P}_\Lambda) = \bigoplus_{n=0}^{\infty} \mathcal{P}_\Lambda^{\otimes n},
$$

where $\mathcal{P}_\Lambda^{\otimes n}$ is the $n$-th symmetric tensor degree of the space $\mathcal{P}_\Lambda$ [1]. The evolution operator is

$$
e^{-iH_{\Lambda} t} = \bigoplus_{n=0}^{\infty} (\exp(-i(Z_{\Lambda}^{-1} M_{\Lambda}^2)^{1/2} t))^{\otimes n}.
$$
3 Problem of renormalization

1. There are several ways to renormalize a quantum field theory model. For example, one can first evaluate such vacuum expectations as Green or Wightman functions \([22, 1]\), \(r\)-functions \([23]\) or \(S\)-matrix coefficient functions \([24, 25]\) for the regularized theory and consider the limit \(\Lambda \to \infty\) for these quantities. Then the Wightman reconstruction theorem \([22]\) or its analog can be applied.

In the approach based on the dynamical Hamiltonian equations of motion rather than \(S\)-matrix another way to perform a limit \(\Lambda \to \infty\) has a strong limit as \(\Lambda \to \infty\) has a strong limit as \(\Lambda \to \infty\) can be used. If \(H_{\Lambda}\) is a regularized Hamiltonian acting in the Hilbert space \(\mathcal{H}\), one can try to choose such singular as \(\Lambda \to \infty\) unitary operator \(T_{\Lambda} : \mathcal{H} \to \mathcal{H}\) ("dressing transformation" \([5, 2]\)) that the operator

\[
T_{\Lambda}^+ e^{-iH_{\Lambda}t} T_{\Lambda}
\]

has a strong limit as \(\Lambda \to \infty\). The limit

\[
U(t) = s - \lim_{\Lambda \to \infty} T_{\Lambda}^+ e^{-iH_{\Lambda}t} T_{\Lambda}
\]

(11)
can be interpreted as an evolution operator in the renormalized theory. The difficulty of our case is that different spaces \(\mathcal{F}(\mathcal{P}_{\Lambda})\) are considered at different values of \(\Lambda\). Another essential feature is that \(\mathcal{F}(\mathcal{P}_{\Lambda})\) are not Hilbert spaces but indefinite inner product spaces. Therefore, the requirement (11) should be modified.

We say that renormalization is performed if:

(i) a Hilbert inner product is introduced on \(\mathcal{F}(\mathcal{P}_{\Lambda})\);
(ii) a Pontriagin space \(\mathcal{L}\) ("renormalized state space") is specified;
(iii) an operator \(T_{\Lambda} : \mathcal{L} \to \mathcal{F}(\mathcal{P}_{\Lambda})\) is defined;
(iv) for some operator \(U(t) : \mathcal{L} \to \mathcal{L}\) ("renormalized evolution operator") and any vector \(\Psi = (\psi_0, \psi_1, \ldots, \psi_n, 0, 0, \ldots)\)

\[
||T_{\Lambda} U(t) \Psi - e^{-iH_{\Lambda}t} T_{\Lambda} \Psi|| \to_{\Lambda \to \infty} 0.
\]

(12)

Condition (12) is a modification of condition (11). Its physical meaning is the following. Suppose that \(T_{\Lambda} \Psi\) is chosen to be an initial state in the regularized theory. Then state at time \(t\) can be approximated by the vector \(T_{\Lambda} U(t) \Psi\). The operator \(U(t)\) can be viewed as a renormalized evolution operator.

Note also that relation (12) means that the operator \(e^{-iH_{\Lambda}t} : \mathcal{F}(\mathcal{P}_{\Lambda}) \to \mathcal{F}(\mathcal{P}_{\Lambda})\) tends to \(U(t) : \mathcal{L} \to \mathcal{L}\) in a generalized strong sense \([26, 27]\).

We will choose \(\mathcal{L} = \mathcal{F}(\mathcal{P})\),

\[
T_{\Lambda} = \oplus_{n=0}^{\infty} (\mathcal{P}_{\Lambda})^\otimes n
\]

for some Pontriagin space \(\mathcal{P}\) and some operator \(P_{\Lambda} : \mathcal{P} \to \mathcal{P}_{\Lambda}\).

2. To introduce a Hilbert inner product on \(\mathcal{F}(\mathcal{P}_{\Lambda})\), it is sufficient to introduce it on \(\mathcal{P}_{\Lambda}\). The standard way is the following \([15]\). Let \(e_{\Lambda}\) be such element of \(\mathcal{P}_{\Lambda}\) that \(< e_{\Lambda}, e_{\Lambda} >_{\Lambda} < 0\). Denote by \([e_{\Lambda}]\) the one-dimensional space \(\{\lambda e_{\Lambda} | \lambda \in \mathbb{C}\}\), while \([e_{\Lambda}]^\perp\) is the space of all vectors \(\psi\) such that \(< \psi, e_{\Lambda} >_{\Lambda} = 0\). If the inner product is positively definite on \([e_{\Lambda}]^\perp\), the indefinite inner product space is of the type \(\Pi_1\) \([13, 16]\). We see that it is true for the case \(e_{\Lambda} = (1, 0, \ldots)\), provided that \(z_{\Lambda} < 0\) (this condition will be shown to be satisfied at sufficiently large \(\Lambda\)). The positive definiteness of the inner product on \([e_{\Lambda}]^\perp\) for arbitrary \(e_{\Lambda}\) is a corollary of the general theory of Pontriagin spaces \([13]\).

The Hilbert inner product is introduced as

\[
(f, g)_{e_{\Lambda}} = < f, g >_{\Lambda} + 2 \frac{< f, e_{\Lambda} >_{\Lambda} < e_{\Lambda}, g >_{\Lambda}}{|| e_{\Lambda}, e_{\Lambda} >_{\Lambda} ||}.
\]

(13)
One can notice that \((f, g)_{e_{\Lambda}} = < f, g >_{\Lambda}\) if \(f, g \perp e_{\Lambda}\), \((f, g)_{e_{\Lambda}} = - < f, g >_{\Lambda}\) if \(f, g \in [e_{\Lambda}]\) and \((f, g)_{e_{\Lambda}} = 0\) if \(f \in [e_{\Lambda}], g \perp e_{\Lambda}\). All topologies on \(\mathcal{P}_{\Lambda}\) that correspond to different choices of \(e_{\Lambda}\) are equivalent \[10\]. However, specification of \(e_{\Lambda}\) is important since the convergence requirement \[12\] is formulated in terms of norms \(|| \cdot || \equiv \sqrt{\langle \cdot, \cdot \rangle_{e_{\Lambda}} }\).

3. It seems to be physically reasonable to choose the vector \(e_{\Lambda}\) as an eigenvector of the operator \(Z_{\Lambda}^{-1}M_{\Lambda}^{2}\) entering to the Hamiltonian. Since \(Z_{\Lambda}^{-1}M_{\Lambda}^{2}\) is a Hermitian operator with respect to the inner product \([14]\), it has according to the Pontriagin theorem \[15\] an eigenvector \(e_{\Lambda}\) such that \(< e_{\Lambda}, e_{\Lambda} > > 0\). Let us find its explicit form. Equation \(Z_{\Lambda}^{-1}M_{\Lambda}^{2}e_{\Lambda} = \varepsilon_{\Lambda}e_{\Lambda}\) is rewritten as

\[
m_{\Lambda}^{2}c_{\Lambda} + g\mu_{k}^{\Lambda}\phi_{\Lambda,k} = \varepsilon_{\Lambda}z_{\Lambda}c_{\Lambda},
\]

\[
g\mu_{k}^{\Lambda}c_{\Lambda} + \Omega_{k}^{2}\phi_{\Lambda,k} = \varepsilon_{\Lambda}\phi_{\Lambda,k}.
\]

where \(e_{\Lambda} = (c_{\Lambda}, \phi_{\Lambda})\). Therefore, for \(\phi_{\Lambda,k}\) one has

\[
\phi_{k} = \frac{g\mu_{k}^{\Lambda}c}{\varepsilon - \Omega_{k}^{2}}.
\]

The parameter \(\varepsilon_{\Lambda}\) obeys the following equation

\[
\varepsilon_{\Lambda}z_{\Lambda} - m_{\Lambda}^{2} = g^{2} \sum_{k} (\frac{\mu_{k}^{\Lambda}}{\Omega_{k}^{4}})^{2} \frac{(\mu_{k}^{\Lambda})^{2}}{\varepsilon_{\Lambda} - \Omega_{k}^{2}}.
\]

For vector \([14]\) \(< e_{\Lambda}, e_{\Lambda} > > 0\) if and only if

\[
-b_{\Lambda} \equiv z_{\Lambda} + g^{2} \sum_{k} \frac{(\mu_{k}^{\Lambda})^{2}}{\Omega_{k}^{4}} (\frac{1}{\varepsilon_{\Lambda} - \Omega_{k}^{2}}) < 0.
\]

It follows from the Pontriagin theorem \[15\] that eq.(15) has a (real or complex) solution obeying property \([14]\).

Denote by

\[
z_{\Lambda,R} = z_{\Lambda} + \sum_{k} \frac{(\mu_{k}^{\Lambda})^{2}}{\Omega_{k}^{4}}; \quad m_{\Lambda,R}^{2} = m_{\Lambda}^{2} - \sum_{k} \frac{(\mu_{k}^{\Lambda})^{2}}{\Omega_{k}^{2}}.
\]

the renormalized values of parameters of the theory.

Eq.(15) can be presented in the following form

\[
\sum_{k} \frac{(\mu_{k}^{\Lambda})^{2}}{\Omega_{k}^{2}} \left( \frac{1}{\varepsilon_{\Lambda} - \Omega_{k}^{2}} + \frac{1}{\Omega_{k}^{2}} \right) = z_{\Lambda,R} - m_{\Lambda,R}^{2}/\varepsilon_{\Lambda}.
\]

It is possible to perform a limit \(\Lambda \to \infty\), provided that \(z_{\Lambda}\) and \(m_{\Lambda}^{2}\) are chosen to make \(z_{\Lambda,R}\) and \(m_{\Lambda,R}^{2}\) finite as \(\Lambda \to \infty\):

\[
z_{\Lambda,R} \to z_{R}, \quad m_{\Lambda,R}^{2} \to m_{R}^{2}.
\]

If \(\sum_{k} \mu_{k}^{2}/\Omega_{k}^{4} = \infty\) but \(\sum_{k} \mu_{k}^{2}/\Omega_{k}^{6} < \infty\), the infinite renormalization of the wave function is indeed necessary, while \(z_{\Lambda}\) is negative at sufficiently large \(\Lambda\).

The following cases should be considered.

(1) \(m_{R}^{2} > 0\).

Eq.(15) has a negative solution \(\varepsilon < 0\) obeying condition \([16]\). Hamiltonian system \([3]\) is unstable.

(2) \(m_{R}^{2} < 0, \quad z_{R} > 0\).

There is an alternative. There may be no real solutions of eq.(18) obeying condition \([14]\). There may be also 2 real negative solutions. The smaller one obeys requirement \([14]\). Hamiltonian system \([3]\) is also unstable.
(3) \( m_R^2 \leq 0, z_R < 0 \).

Eq. (18) may have no real solutions obeying condition (16) or may have a real positive solution satisfying requirement (16). The latter case takes place at sufficiently small \( |m_R|^2 \). The Hamiltonian system is stable.

Let us consider the most interesting latter case only. Note that the condition \( z_R < 0 \) arises in investigations of large-\( N \) QED \([28]\).

If \( m_R^2 = 0 \), the formalism of the previous subsection should be slightly modified (the operator \( (Z^{-1}_\Lambda M^2_\Lambda)^{-1/4} \) does not exist). For the simplicity, consider the case \( m_R^2 \neq 0 \) only.

Let us introduce more convenient coordinates on the Pontriagin space \( \mathcal{P}_\Lambda \) in order to remove divergences from Hilbert and indefinite inner products. First of all, present any vector \( \psi \in \mathcal{P}_\Lambda \) as

\[
\psi = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} + c e_\Lambda,
\]

where

\[
e_\Lambda = \begin{pmatrix} 1 \\ \frac{g \mu \Lambda}{\varepsilon - \Omega^2_k} \end{pmatrix}
\]

One finds:

\[
(e_\Lambda, \psi) = -b e_\Lambda + \sum_{k=1}^{\infty} \frac{g \mu \Lambda}{\varepsilon - \Omega^2_k} \varphi_k,
\]

where \( b \) has a limit as \( \Lambda \rightarrow \infty \). Introduce instead of \( c \) new variable \( \alpha = -b^{-1}(e_\Lambda, \psi) \):

\[
\alpha = -b^{-1}(e_\Lambda, \psi) = c - \sum_{k=1}^{\infty} \frac{g \mu \Lambda}{\varepsilon - \Omega^2_k} \varphi_k.
\]

In terms of new variables \( (\alpha; \varphi) \) the inner products (10) and (13) take the form

\[
<\psi, \psi>_{\Lambda} = -b|\alpha|^2 + <\varphi, \varphi>_{\Lambda}
\]

(19)

\[
(\psi, \psi)_{e_\Lambda} = b|\alpha|^2 + <\varphi, \varphi>_{\Lambda}
\]

with

\[
<\varphi, \varphi>_{\Lambda} = (\varphi, \varphi) + \frac{1}{b} \left( \frac{g \mu}{\varepsilon - \Omega^2_k} \varphi \right)^2.
\]

(20)

Formula (20) contains no divergences.

By \( \tilde{\mathcal{P}}_\Lambda \) we denote the Pontriagin space of sets \( (\alpha, \varphi) \) with inner products (13). The introduced isomorphism \( I_\Lambda : \tilde{\mathcal{P}}_\Lambda \rightarrow \mathcal{P}_\Lambda \) has the following form: \( I_\Lambda : (\alpha, \varphi) \mapsto (c, \varphi) \), where

\[
c = \alpha + \frac{1}{b} \sum_{k=1}^{\infty} \frac{g \mu \Lambda}{\varepsilon - \Omega^2_k} \varphi_k,
\]

(21)

\[
\hat{\varphi}_k = \frac{g \mu \Lambda}{\varepsilon - \Omega^2_k} + \varphi_k.
\]

5. It seems to be reasonable to specify the renormalized states by sets \( \psi = (\alpha, \varphi) \). By \( \hat{\mathcal{P}} \) we denote the indefinite inner product space of such sets with inner products

\[
<\psi, \psi> = -b|\alpha|^2 + <\varphi, \varphi> + \frac{1}{b} \left( \frac{g \mu}{\varepsilon - \Omega^2_k} \varphi \right)^2
\]

\[
(\psi, \psi) = b|\alpha|^2 + <\varphi, \varphi> + \frac{1}{b} \left( \frac{g \mu}{\varepsilon - \Omega^2_k} \varphi \right)^2
\]

with \( \varepsilon = \lim_{\Lambda \rightarrow \infty} \varepsilon_\Lambda, b = \lim_{\Lambda \rightarrow \infty} b_\Lambda \). However, \( \hat{\mathcal{P}} \) cannot be viewed as a state space. First, the sequence \( \frac{g \mu \Lambda}{\varepsilon - \Omega^2_k} \) does not belong to \( l^2 \), so that one should impose the conditions on \( \varphi_k \) at \( k \rightarrow \infty \).

For example, one can require \( \Omega \varphi \in l^2 \). Next, the Euclidean space with the inner product \( (\cdot, \cdot) \) is not complete, so that it is necessary to consider the completeness \( \mathcal{P} \) of the space \( \hat{\mathcal{P}} \).
Thus, the renormalized state space $P$ (cf. [24]).
Let $\{(\alpha^{(n)}, \varphi^{(n)})\}$, $\{(\alpha^{(n)}, \varphi^{(n)})\}$ be fundamental sequences. They are equivalent [30] if $||\psi^{(n)} - \psi^{(n)}|| \to n \to \infty 0$. This means that

$$
\alpha^{(n)} - \alpha^{(n)}' \to n \to \infty 0;
||\varphi^{(n)} - \varphi^{(n)}'|| \to n \to \infty 0;
\left(\frac{\varphi^{(n)}}{\alpha^{(n)}} - \frac{\varphi^{(n)}'}{\alpha^{(n)}`}\right) \to n \to \infty 0.
$$

(22)

Furthermore, since the sequence $\{(\alpha^{(n)}, \varphi^{(n)})\}$ is fundamental, sequences $\alpha^{(n)}$, $\varphi^{(n)}$ and $(\frac{\varphi^{(n)}}{\alpha^{(n)}}; \varphi^{(n)})$ are also fundamental. Therefore,

$$
\alpha^{(n)} \to n \to \infty \alpha,
\varphi^{(n)} \to n \to \infty \varphi;
\left(\frac{\varphi^{(n)}}{\alpha^{(n)}}; \varphi^{(n)}\right) \to n \to \infty \xi.
$$

(23)

Thus, two fundamental sequences are equivalent if and only if $\alpha' = \alpha$, $\varphi' = \varphi$, $\xi' = \xi$.

Let us show now that for any set $(\alpha, \xi, \varphi)$ there exists a fundamental sequence obeying conditions (23). Note that any sequence obeying requirements (23) is fundamental. It is sufficient to consider two partial cases:

(i0) $\varphi = 0$;
(ii) $\alpha = \xi = 0$.

Denote $\chi_k = g\mu_k/(\varepsilon - \Omega_k^2)$. For the case (i), set

$$
\alpha^{(n)} = \alpha,
\varphi^{(n)}_k = \chi_k/\sum_{k=0}^{n} |\chi_k|^2, \quad k \leq n.
\varphi^{(n)}_k = 0, \quad k > n.
$$

For the case (ii), it is sufficient to check that the set of all vectors $\varphi \in l^2$ satisfying the relations

$$
(\chi, \varphi) = 0,
$$

(24)

is dense in $l^2$. To prove this property, it is sufficient to notice that any finite vector $\varphi$ can be approximated by a sequence $\varphi^{(n)} \to \varphi$ obeying requirement (24):

$$
\varphi^{(n)}_k = \begin{cases} 
\varphi_k - \frac{\chi_k}{\sum_{k=0}^{n} |\chi_k|^2} \chi_k, & k \leq n; \\
\varphi_k, & k > n
\end{cases}
$$

Thus, the renormalized state space $P$ is a space of sets $(\alpha, \varphi, \xi)$, where $\varphi \in l^2$, $\alpha \in \mathbb{C}$, $\xi \in \mathbb{C}$.

The following inner products are introduced in $P$:

$$
(\psi, \psi) = b|\alpha|^2 + (\varphi, \varphi) + b^{-1}|\xi|^2
< \psi, \psi > = -b|\alpha|^2 + (\varphi, \varphi) + b^{-1}|\xi|^2
$$

(25)

6. Let us construct the mapping $\tilde{\alpha}_A : P \to \tilde{P}_A$ which transforms the renormalized state $(\alpha, \xi, \varphi)$ to the regularized state $(\alpha_A, \varphi_A)$. Choose it in such a way that

$$
\alpha_A \to A \to \infty \alpha, \quad ||\varphi_A - \varphi|| \to A \to \infty 0,
\left(\frac{\varphi^{A}}{\alpha^{A}}; \varphi^{A}\right) \to A \to \infty \xi.
$$

(26)

The mapping $P_A : P \to P_A$ will have the form $P_A = I_A \tilde{P}_A$.

The following proposition is a direct corollary of eqs. (14).

**Proposition 1.** Let $\tilde{P}_A : (\alpha, \xi, \varphi) \mapsto (\alpha_A, \varphi_A)$ be a mapping satisfying requirements (24). Then

$$
||\hat{(\alpha_A, \varphi_A)} - \tilde{P}_A(\alpha, \xi, \varphi)|| \to A \to \infty 0
$$

7
if and only if \((\tilde{\alpha}_\Lambda, \tilde{\varphi}_\Lambda)\) obeys requirements (26).

Proposition 1 tells us that the form of operator \(T_\Lambda\) obeying requirements (26) is not important. By \(Q_\Lambda : \tilde{P}_\Lambda \to P\) we denote the operator \(Q_\Lambda : (\alpha_\Lambda, \varphi_\Lambda) \mapsto (\alpha'_\Lambda, \xi'_\Lambda, \varphi'_\Lambda)\) of the form

\[
\alpha'_\Lambda = \alpha_\Lambda, \quad \varphi'_\Lambda = \varphi_\Lambda, \quad \xi'_\Lambda = \left(\frac{g\mu^\Lambda}{\varepsilon^\Lambda - \Omega^2}, \varphi_\Lambda\right).
\]

Proposition 1 can be reformulated as

**Proposition 1'.** Let \((\alpha_\Lambda, \varphi_\Lambda) \in \tilde{P}_\Lambda\). Then \(|| (\alpha_\Lambda, \varphi_\Lambda) || \to 0\) if and only if \(Q_\Lambda(\alpha_\Lambda, \varphi_\Lambda) \to_{\Lambda \to \infty} 0\).

Eq. (26) also implies

**Proposition 2.**

\[
s - \lim_{\Lambda \to \infty} Q_\Lambda \tilde{P}_\Lambda = 1.
\]

We will also require that

\[
< \tilde{P}_\Lambda \psi, \tilde{P}_\Lambda \tilde{\psi} > \to_{\Lambda \to \infty} < \psi, \tilde{\psi} >, \quad || \tilde{P}_\Lambda || \leq A = \text{const} \quad (28)
\]

for some \(\Lambda\)-independent \(a\).

The explicit form of the mapping \(\tilde{P}_\Lambda : (\alpha, \xi, \varphi) \mapsto (\alpha_\Lambda, \varphi_\Lambda)\) can be chosen as

\[
\varphi_\Lambda = \varphi + \frac{\chi^\Lambda(\xi - (\chi^\Lambda, \varphi))}{(\chi^\Lambda, \chi^\Lambda)}, \quad \alpha_\Lambda = \alpha,
\]

where \(\chi^\Lambda = \frac{g\mu^\Lambda}{\varepsilon^\Lambda - \Omega^2}\). The properties \(\alpha_\Lambda \to \alpha, (\chi^\Lambda, \varphi^\Lambda) \to (\xi, \varphi)\) at \(\Lambda \to \infty\) are evident. Since \(|| \chi^\Lambda || \to \infty\), one has

\[
s - \lim_{\Lambda \to \infty} \frac{\chi^\Lambda}{|| \chi^\Lambda ||} = 0.
\]

It is sufficient then to check that

\[
\lim_{\Lambda \to \infty} \frac{(\chi^\Lambda, \varphi)}{|| \chi^\Lambda ||}, \quad \varphi \in l^2.
\]

For finite sequences \(\varphi\), property (29) is evident. It follows from the standard theorems of functional analysis [31] that property (29) is then satisfied for all \(\varphi \in l^2\).

7. To check property (12), it is convenient to investigate the resolvent of the operator \(Z_\Lambda^{-1}M_\Lambda^2\). To find its explicit form,

\[
(\lambda + Z_\Lambda^{-1}M_\Lambda^2)^{-1} : \begin{pmatrix} c \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} c \\ \phi \end{pmatrix}
\]

one should solve the system of equations

\[
(\lambda z_\Lambda + m^2_\Lambda) c' + g\mu^\Lambda \phi' = z_\Lambda c;
\]

\[
g\mu^\Lambda c' + (\lambda + \Omega^2)\phi' = \phi.
\]

Therefore,

\[
c' = a_\Lambda[z_\Lambda c - g \sum_{k=1}^\infty \frac{\mu^\Lambda_k \phi_k}{\lambda + \Omega^2_k}],
\]

\[
\phi' = \frac{1}{\lambda + \Omega^2} \phi - \frac{g\mu^\Lambda}{\lambda + \Omega^2} c'
\]

with

\[
a_\Lambda = (\lambda z_\Lambda + m^2_\Lambda) - \sum_k \frac{g^2(\mu^\Lambda_k)^2}{\lambda + \Omega^2_k}^{-1}.
\]
Making use of the \((\alpha, \varphi)\)-coordinates, one obtains

\[
\varphi' = \sum_{M=1}^{\infty} \alpha'_M \varphi_M \quad \text{and} \quad \frac{1}{\lambda + \varepsilon} \varphi + \frac{g_{\alpha^2}^L (\varepsilon + \lambda) a_{\lambda}}{\xi (\xi + \Omega^2)^2 (\xi + \Omega^2)} \left( g_{\mu}^A \right.
\]

(eq. (13) is taken into account). Thus, the explicit form of the operator \(I^{-1}_\Lambda (\lambda + Z^{-1}_\Lambda M^2) I : (\alpha, \varphi) \mapsto (\alpha', \varphi')\) is found. This operator is Hermitian at real values of \(\lambda\) with respect to Hilbert and indefinite inner products (19).

8. Investigate now the behavior of the resolvent at \(\Lambda \to \infty\). Consider the operator \(Q_L I^{-1}_\Lambda (\lambda + Z^{-1}_\Lambda M^2) I : (\alpha, \varphi) \mapsto (\alpha', \xi', \varphi')\) which can be presented as

\[
\varphi' = \frac{1}{\lambda + \Omega^2} \varphi - \frac{g_{\alpha^2}^L (\varepsilon + \lambda) a_{\lambda}}{\xi (\xi + \Omega^2)^2 (\xi + \Omega^2)} \xi \varphi + \frac{g_{\mu}^A (\varepsilon + \lambda) a_{\lambda}}{\xi (\xi + \Omega^2)^2 (\xi + \Omega^2)} \xi \varphi \xi \varphi
\]

(eq. (13) is used), where \(\xi = (\frac{g_{\mu}^A}{\xi + \Omega^2}, \varphi)\).

Denote the mapping (30) as \((\lambda + H_{\lambda})^{-1} : (\alpha, \xi, \varphi) \mapsto (\alpha', \xi', \varphi')\). It is a resolvent of a positive self-adjoint operator. We have obtained the following proposition.

**Proposition 3.** The following relation is satisfied:

\[
Q_L I^{-1}_\Lambda (\lambda + Z^{-1}_\Lambda M^2) I = (\lambda + H_{\lambda})^{-1} Q_L
\]

We see that the operator \((\lambda + H_{\lambda})^{-1}\) has a strong limit \((\lambda + H)^{-1}\) being a resolvent of a positive self-adjoint operator. General results of \([32, 26]\) tell us that the following statement is satisfied.

**Proposition 4.** Let \(f\) be a bounded Borel function. Then

\[
s - \lim_{\Lambda \to \infty} f(H_{\Lambda}) = f(H).
\]

Proposition 3 also implies that

**Proposition 5.** For any bounded Borel function \(f : \mathbb{R} \to \mathbb{R}\)

\[
Q_L I^{-1}_\Lambda f(Z^{-1}_\Lambda M^2) I = f(H_{\Lambda}) Q_L
\]

Let us prove relation (12).

**Proposition 6.** For any bounded Borel function \(f : \mathbb{R} \to \mathbb{R}\) and any \(\psi \in \mathcal{P}\)

\[
\|f(Z^{-1}_\Lambda M^2) P_{\Lambda} \psi - P_{\Lambda} f(H) \psi\| \to_{\Lambda \to \infty} 0.
\]

**Proof.** Since \(I_{\Lambda} : \tilde{\mathcal{P}}_{\Lambda} \to \mathcal{P}_{\Lambda}\) is an isomorphism, proposition 1’ implies that relation (3) is satisfied if and only if

\[
\|Q_L I^{-1}_\Lambda f(Z^{-1}_\Lambda M^2) I_{\Lambda} \tilde{\mathcal{P}}_{\Lambda} \psi - Q_L \tilde{\mathcal{P}}_{\Lambda} f(H) \psi\| \to_{\Lambda \to \infty} 0.
\]

It follows from proposition 5 that this relation can be rewritten as

\[
\|f(H_{\Lambda}) Q_L \tilde{\mathcal{P}}_{\Lambda} \psi - Q_L \tilde{\mathcal{P}}_{\Lambda} f(H) \psi\| \to_{\Lambda \to \infty} 0.
\]
It follows from propositions 2 and 4 that \( s - \lim_{\Lambda \to \infty} f(H_{\Lambda})Q_{\Lambda} \tilde{P}_{\Lambda} = f(H) \), i.e.

\[
\| f(H_{\Lambda})Q_{\Lambda} \tilde{P}_{\Lambda} \psi - f(H) \psi \| \to_{\Lambda \to \infty} 0. \tag{32}
\]

Proposition 2 also implies that

\[
\| f(H) \psi - Q_{\Lambda} \tilde{P}_{\Lambda} f(H) \psi \| \to_{\Lambda \to \infty} 0. \tag{33}
\]

Combining eqs.(32) and (33), we obtain relation (31). Proposition 6 is proved.

9. Thus, we have constructed the renormalized state space \( \mathcal{L} = \mathcal{F}(\mathcal{P}) \). The ”one-particle” renormalized space is chosen to be a space of sets \((\alpha, \xi, \varphi)\) with the inner products (24), the Hamiltonian operator is also defined by specifying the resolvent \((\lambda + H)^{-1}\). The evolution operator \( U(t) \) entering to eq.(12) has the form

\[
U(t) = \bigoplus_{n=0}^{\infty} (e^{-iH^{1/2}t}) \otimes^n. \]

4 ”Field” operators

Let us construct now Heisenberg field operators \( Q(t) \), \( q_k(t) \) and their linear combinations in the renormalized theory. According to eqs.(1), they should be expressed via creation and annihilation operators. Let us remind their definition (see, for example, [1]).

The set of all vectors \( \psi \otimes^n \) is a total set in \( \mathcal{P}^{\otimes n} \). By \( b_n^\alpha(\gamma) : \mathcal{P}^{\otimes n} \to \mathcal{P}^{\otimes (n-1)} \), \( b_n^{\pm}(\gamma) : \mathcal{P}^{\otimes (n-1)} \to \mathcal{P}^{\otimes n}, \gamma \in \mathcal{P} \) we denote the linear operators which are uniquely defined from the relations

\[
b_n^\alpha(\gamma)\psi \otimes^{(n-1)} = \sqrt{n}(\gamma, \psi) \psi \otimes^{(n-1)},
\]

\[
b_n^{\pm}(\gamma)\psi \otimes^{(n-1)} = n^{-1/2} \sum_{j=0}^{n-1} \psi \otimes^j \gamma \otimes \psi \otimes^{(n-1-j)}.
\]

Moreover, \( ||b_n^{\pm}(\gamma)|| \leq n^{1/2}||\gamma|| \). By \( b^\pm(\gamma) : \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P}) \) we denote the operators

\[
(b^+(\gamma)\Psi)_n = b_n^+(\gamma)\Psi_{n-1}, \quad b^-(\gamma)\Psi_{n-1} = b_n^+(\gamma)\Psi_n
\]

which are defined on the set of all finite vectors of the Fock space.

**Proposition 7.** The following relations are satisfied:

\[
b^+(\gamma_{\Lambda})T_{\Lambda} = T_{\Lambda}b^+(\gamma), \quad b^-(\gamma_{\Lambda})T_{\Lambda} = T_{\Lambda}b^-(P^+_{\Lambda}\gamma_{\Lambda}),
\]

for arbitrary \( \gamma_{\Lambda} \in \mathcal{P}_{\Lambda}, \gamma \in \mathcal{P} \).

The proof is straightforward.

**Proposition 8.** Let \( \gamma_{\Lambda} \in \mathcal{P}_{\Lambda}, \gamma \in \mathcal{P}, \Psi \) is a finite vector of the renormalized Fock space \( \mathcal{F}(\mathcal{P}) \) and

\[
||\gamma_{\Lambda} - P_{\Lambda}\gamma|| \to_{\Lambda \to \infty} 0. \tag{34}
\]

Then

\[
||b^\pm(\gamma_{\Lambda})T_{\Lambda} - T_{\Lambda}b^\pm(\gamma)\Psi|| \to_{\Lambda \to \infty} 0. \tag{35}
\]

**Proof.** For the creation operator, eq.(33) means that

\[
||b^+(\gamma_{\Lambda} - P_{\Lambda}\gamma)\Psi|| \to_{\Lambda \to \infty} 0.
\]

Let \( \Psi = (\psi_0, \psi_1, ..., \psi_n, 0, 0, ...) \). Then

\[
||b^+(\gamma_{\Lambda} - P_{\Lambda}\gamma)\Psi|| \leq \max(1, ||P_{\Lambda}||^n)||\Psi||n^{1/2}||\gamma_{\Lambda} - P_{\Lambda}\gamma|| \to_{\Lambda \to \infty} 0.
\]
For the annihilation operator, it is necessary to check that
\[
\| T_\Lambda b^{-}(P_\Lambda^{+}\gamma_\Lambda - \gamma)\Psi \| \rightarrow \Lambda \rightarrow \infty 0
\]
for \( \Psi \in \mathcal{P}^{\infty} \). The Banach-Steinhaus theorem \cite{31} implies that it is sufficient to prove property (36) for \( \Psi = \psi^{\otimes n}, \psi \in \mathcal{P} \). This property is correct if and only if
\[
(P_\Lambda^{+}\gamma_\Lambda - \gamma, \psi) \rightarrow \Lambda \rightarrow \infty 0
\]
i.e. \((\gamma_\Lambda, P_\Lambda\psi) \rightarrow \Lambda \rightarrow \infty (\gamma, \psi)\). Eqs. (34) and (28) confirms this property. Proposition is proved.

Propositions 1' and 8 imply the following corollary.

Proposition 9. Let \( \gamma_\Lambda \in \mathcal{P}_\Lambda, \gamma \in \mathcal{P}, \Psi \) is a finite vector of the renormalized Fock space \( \mathcal{F}(\mathcal{P}) \) and
\[
\| Q_\Lambda I_\Lambda^{-1}\gamma_\Lambda - \gamma \| \rightarrow \Lambda \rightarrow \infty 0.
\]
Then property (33) is satisfied.

We see that the operator \( b^{\pm}(\gamma_\Lambda) \) in the regularized theory corresponds to the operator \( b^{\pm}(\gamma) \) in the renormalized theory. One can say \( b^{\pm}(\gamma_\Lambda) \rightarrow b^{\pm}(\gamma) \) in generalized strong sense \cite{26, 27}.

Note that the linear combinations \( \sum_k b^+_k \zeta_k \) and \( \sum_k b^-_k \zeta_k^* \) of the operators \( b^\pm_k \) \cite{4} can be presented as \( b^+(\zeta) \) and \( b^-(\zeta) \) correspondingly with \( \zeta \in \mathcal{P}_\Lambda \).

Consider the linear combination \( \sum_{k=0}^{\infty} q_k(t)\chi_k^\Lambda \) in the regularized theory. It follows from eqs. (4) that
\[
\sum_{k=0}^{\infty} q_k(t)\chi_k^\Lambda = b^+(\gamma_\Lambda^t) + b^-(\gamma_\Lambda^t)
\]
with
\[
\gamma_\Lambda^t = \frac{1}{\sqrt{2}} e^{i(Z_\Lambda^{-1}M_\Lambda^2)^{1/2}t(Z_\Lambda^{-1}M_\Lambda^2)^{-1/4}Z_\Lambda^{-1}\chi_\Lambda}.
\]

Propositions 4, 5 and 9 imply the following statement.

Proposition 10. Let
\[
Q_\Lambda I_\Lambda^{-1}Z_\Lambda^{-1}\chi_\Lambda \rightarrow \Lambda \rightarrow \infty \gamma,
\]
\( \Psi \) is a finite vector of the renormalized Fock space \( \mathcal{F}(\mathcal{P}) \). Then
\[
\| (\sum_{k=0}^{\infty} q_k(t)\chi_k^\Lambda T_\Lambda - T_\Lambda q_\Lambda(\gamma))\Psi \| \rightarrow \Lambda \rightarrow \infty 0
\]
with
\[
q_\Lambda(\gamma) = b^+\left( \frac{1}{\sqrt{2}} e^{iH^{1/2}tH^{-1/4}\gamma} \right) + b^-\left( \frac{1}{\sqrt{2}} e^{iH^{1/2}tH^{-1/4}\gamma} \right)
\]

Let us write down the explicit form of condition (38). Let \( \chi_\Lambda = \left( \begin{array}{c} c_\Lambda \\ \phi_\Lambda \end{array} \right), \gamma = (\alpha, \xi, \varphi), \)
\[
\varphi_\Lambda = \phi_\Lambda - z_\Lambda^{-1}c_\Lambda \frac{g\mu^\Lambda}{\epsilon_\Lambda - \Omega^2}.
\]
Eq. (38) means that
\[
\| \varphi_\Lambda - \varphi \| \rightarrow \Lambda \rightarrow 0 0,
\]
\[
\left( \frac{g\mu_\Lambda}{\epsilon_\Lambda - \Omega\gamma} \right) \rightarrow \Lambda \rightarrow 0 \xi,
\]
\[
z_\Lambda^{-1}c_\Lambda \rightarrow \Lambda \rightarrow 0 \alpha + \xi/b.
\]

Eq. (38) means that
\[
\left( \frac{g\mu_\Lambda}{\epsilon_\Lambda - \Omega\gamma} \right) \rightarrow \Lambda \rightarrow 0 \xi,
\]
\[
z_\Lambda^{-1}c_\Lambda \rightarrow \Lambda \rightarrow 0 \alpha + \xi/b.
\]
We see that the operator \( \sum_{k=0}^{\infty} q_k(t)\gamma_k^\Lambda \) in the regularized theory corresponds to the operator \( q(\gamma) \) in the renormalized theory, provided that requirements (39) are satisfied.
Example 1. Let $\Omega \phi \in l^2$, $\Omega \phi \Lambda \rightarrow \Omega \phi$, $c_\Lambda = 0$. Then the expression $\sum_{k=1}^{\infty} q_k(t) \phi_k$ corresponds to the operator $q(\alpha, \xi, \phi)$ in the renormalized theory, where

$$
\alpha = -\frac{\xi}{b}, \quad \xi = \left( \frac{g\mu}{\Omega(\varepsilon - \Omega^2)}, \Omega \phi \right), \quad \varphi = \phi.
$$

Example 2. Let $c_\Lambda = 1$, $\phi_\Lambda = 0$. Then the expression $\sum_{k=0}^{\infty} q_k(t) \gamma_k^\Lambda$ takes the form $Q(t)$. For this case $\alpha + \xi/\beta = 0$,

$$
\varphi_\Lambda = -z_\Lambda \frac{g\mu^\Lambda}{\varepsilon_\Lambda - \Omega^2},
$$

so that $||\varphi_\Lambda|| \rightarrow \Lambda \rightarrow \infty 0$,

$$
\left( \frac{g\mu^\Lambda}{\varepsilon_\Lambda - \Omega^2}, \varphi_\Lambda \right) = -z_\Lambda^{-1}(-b_\Lambda - z_\Lambda) \rightarrow \Lambda \rightarrow \infty 1
$$

Thus, $\xi = 1$, so that $\alpha = -1/\xi$. We see that

$$
Q(t) \rightarrow q_t(-1/b, 1, 0)
$$

Example 3. Let $c_\Lambda = z_\Lambda$, $\phi_\Lambda = \frac{g\mu^\Lambda}{\varepsilon_\Lambda - \Omega^2}$. Then $\varphi_\Lambda = 0$, $\xi = 0$, $\alpha + \xi/b = 1$. Thus, the expression

$$
z_\Lambda Q(t) + \sum_{k=1}^{\infty} \frac{g\mu^\Lambda_k q_k(t)}{\varepsilon_\Lambda - \Omega^2_k}
$$

corresponds to the operator $q_t(1, 0, 0)$ in the renormalized theory.

Combining examples 1-3, we find that the expressions (2) correspond to correctly defined operators in the renormalized theory.

This work was supported by the Russian foundation for Basic Research, project 99-01-01198.

References

[1] Bogoliubov N N, Logunov A A, Oksak A I and Todorov I T 1987 General Principles of Quantum Field Theory (Moscow, Nauka)

[2] Hepp K 1969 Theorie de la renormalisation (Springer-Verlag)

[3] Glimm J and Jaffe A 1972 Boson quantum field models. In "London 1971, Mathematics of Contemporary Physics", London, pp. 77-143.

[4] Zavialov O I and Sushko V N 1973, In: Statistical Physics and Quantum Field Theory, ed. N.N.Bogoliubov (Moscow: Nauka).

[5] Faddeev L D 1963 Doklady Akad. Nauk SSSR 152 573.

[6] Stueckelberg E C G 1951 Phys.Rev. 81 130.

[7] Maslov V P and Shvedov O Yu 1999 Trudy MIAN (Proceedings of the Mathematical Steklov Institute of the Russian Academy of Sciences) 226 112

[8] Shvedov O Yu, hep-th/0002108

[9] Shvedov O Yu, hep-th/0009035
[10] Berezin F A 1963 Matematicheskii Sbornik 60 425
[11] Zavialov O I 1973 Teor. Mat. Fiz 16 145
[12] Shirokov Yu M 1979 Teor. Mat. Fiz. 41 291
[13] Smirnov V A, Tolokonnikov G K and Shondin Yu G 1983 Fizika Elemetrarnykh Chastits i Atomnogo Yadra (Physics of Elementary Particles and Atomic Nuclei) 14 1030
[14] Shondin Yu G 1985 Teor. Mat. Fiz. 65 24
[15] Pontriagin L S 1944 Izv. AN SSSR, ser. mathematics 8 243
[16] Iokhvidov I S and Krein M G 1956 Proceedings of Moscow Mathematical Society 5 367
[17] Bogner J 1974 Indefinite inner product spaces. (Berlin.:Springer)
[18] Azizov T Ya and Iokhvidov I S 1986 Foundations of the theory of linear operators in the indefinite inner product spaces (Moscow: Nauka).
[19] Hudson R and Parthasaraty K 1984 Comm. Math. Phys. 93 301
[20] Chebotarev A M 1996 Matematicheskie Zametki 68 726
[21] Bogoliubov N N (Jr), Sadovnikov B I and Shumovsky A S 1989 Mathematical Methods of Statistical Mechanics of Model Systems (Moscow: Nauka)
[22] Wightman A S 1956 Phys. Rev. 101 860
[23] Lehman H, Symanzik K and Zimmermann W 1957 Nuovo Cim. 6 319
[24] Bogoliubov N N and Shirkov D V 1959 Introduction to the Theory of Quantized Fields (N.-Y.: Interscience Publishers).
[25] Bogoliubov N N, Medvedev B V and Polivanov M K 1958 Questions of Dispersion Relations Theory (Moscow: Fizmatgiz)
[26] Kato T 1972 Perturbation Theory for Linear Operators (Moscow: Mir)
[27] Trotter E F 1958 Pacific J. of Math. 8 887.
[28] Kirzhnits D and Linde A 1978 Phys. Lett. B73 323.
[29] Shondin Yu G 1999 Matematicheskie Zametki 66 924
[30] Kolmogorov A N and Fomin S V 1989 Elements of Theory of Functions and Functional Analysis (Moscow: Nauka)
[31] Kantorovich L V and Akilov G P 1984 Functional Analysis (Moscow: Nauka)
[32] Faris W S 1975 Self-adjoint operators In: Lect. Notes Math. 433 143.