Non-Regular Likelihood Inference for Seasonally Persistent Processes

Emma J. McCoy\textsuperscript{(1)}
Sofia C. Olhede\textsuperscript{(1)}
David A. Stephens\textsuperscript{(1,2)}

\textsuperscript{(1)} Department of Mathematics, Imperial College London
180 Queens Gate,
London SW7 2AZ,
UK

\textsuperscript{(2)} Department of Mathematics and Statistics, McGill University
805 Sherbrooke St. W
Montreal, QC, H2K 2K2,
Canada

February 5, 2008

Abstract

The estimation of parameters in the frequency spectrum of a seasonally persistent stationary stochastic process is addressed. For seasonal persistence associated with a pole in the spectrum located away from frequency zero, a new Whittle-type likelihood is developed that explicitly acknowledges the location of the pole. This Whittle likelihood is a large sample approximation to the distribution of the periodogram over a chosen grid of frequencies, and constitutes an approximation to the time-domain likelihood of the data, via the linear transformation of an inverse discrete Fourier transform combined with a demodulation. The new likelihood is straightforward to compute, and as will be demonstrated has good, yet non-standard, properties. The asymptotic behaviour of the proposed likelihood estimators is studied; in particular, \( N \)-consistency of the estimator of the spectral pole location is established. Large finite sample and asymptotic distributions of the score and observed Fisher information are given, and the corresponding distributions of the maximum likelihood estimators are deduced. Asymptotically, the estimator of the pole after suitable standardization follows a Cauchy distribution, and for moderate sample sizes, we can use the finite large sample approximation to the distribution of the estimator of the pole corresponding to the ratio of two Gaussian random variables, with sample size dependent means and variances. A study of the small sample properties of the likelihood approximation is provided, and its superior performance to previously suggested methods is shown, as well as agreement with the developed distributional approximations. Inspired by the developments for full likelihood based estimation procedures, usage of profile likelihood and other likelihood based procedures are also discussed. Semi-parametric estimation methods, such as the Geweke-Porter-Hudak estimator of the long memory parameter, inspired by the developed parametric theory are introduced.

**KEYWORDS:** Periodogram; Seasonal persistence; likelihood inference, Whittle likelihood.
1 Introduction

In this paper, we develop likelihood estimation of the parameters of a stationary stochastic process that exhibits seasonal persistence, that is, long memory behaviour associated with a stationary, quasi-seasonal dependence structure. We introduce a new frequency-domain likelihood approximation which is computed using demodulation and which, for the first time, facilitates maximum likelihood estimation. We consider joint estimation of the seasonality and persistence parameters, and establish the asymptotic and large sample properties of the likelihood and its associated maximum likelihood estimators. This is in direct contrast with previously suggested procedures, where the distribution of the estimator of the seasonality parameter could not be established (Giraitis et al., 2001). The estimators are demonstrated to have good small sample properties compared with estimators based on the classic Whittle likelihood, and other non-likelihood derived estimators. Our non-standard asymptotic results rely on the appropriate renormalization of the score and Fisher information, and utilize a parameter-dependent linear transformation of the data. This transformation enables an efficient approximation to the likelihood. The transformation also introduces a number of interesting and non-regular features into the likelihood surface: jumps, local oscillations, and non-regular large sample theory. Despite these issues the large sample theory can be determined, and appropriate finite large sample approximations provided, as will be demonstrated. It transpires that the small sample properties of the estimators are competitive with existing methods, as well be discussed in later sections.

The contributions of this paper thus include new theory for non-regular maximum likelihood problems. In similarly motivated work, Cheng and Taylor (1995) discussed problems associated with maximum likelihood estimation for unbounded likelihoods: in contrast we discuss problems associated with distributions of non-identically distributed, weakly dependent variables with highly compressed and for increasing sample sizes unbounded variances. Given the importance of compressed linear decompositions in modern statistical theory, our work has implications for the distribution of sparseness-inducing transformations much beyond the analysis of seasonal processes and Fourier theory, and forms a contribution to developing methodology for inference of stochastically compressible processes.

One of the concrete and substantive conclusions of our new estimation procedures is illustrated in Figure 1: this figure illustrates that whereas a standard estimation procedure, based on the Whittle likelihood (see Section 1.3), produces estimates that are, on average, biased even in large samples, our new procedure, based on a carefully constructed likelihood (see Sections 2.2 and 3), produces estimators that exhibit no such bias. Full details of this Figure are given in Section 4.1.

1.1 Seasonally Persistent Processes

Stationary time-series models with long range dependence describe a wide range of physical phenomena; see for general discussion Andel (1986) and Gray et al. (1989), and also applications in econometrics (Porter-Hudak, 1990; Gil-Alana, 2002), biology (Beran, 1994) and hydrology (Ooms, 2001). Dependence in a stationary time series is parameterized via the autocovariance sequence, \{\gamma_\tau\}. We are concerned with the estimation of parameters that specify \gamma_\tau under an assumption of seasonal persistence. Specifically, of particular importance is the seasonality of the data characterized by a frequency, \xi, termed the pole, and an associated
Figure 1: Simulated Data: Mean standardized likelihoods for the pole (right) and the long memory parameter (left) over 2000 simulations, with sample size of 1024, and the true values of the long memory parameter and the pole taking the values 0.45 and 1/7, respectively. The vertical solid lines indicate the true values of the parameters. The Demodulated likelihood is noted in equation (16) whilst the discrete Whittle likelihood is noted in equation (8). On average, the demodulated likelihood has its mode at the true values, whereas the Whittle likelihood does not. See Section 4.1 for full details.
degree of dependence, characterized by a persistence (or long memory) parameter $\delta$. Whereas inference for the persistence parameter in the context of poles at frequency zero has been much studied (Beran, 1994), the theoretical behaviour of estimators of the persistence parameter remains largely uninvestigated when the underlying seasonality of the process is unknown.

Let $\{X_t\}$ be a zero-mean, second-order stationary time series with autocovariance (acv) sequence $\gamma_\tau = \text{cov}\{X_t, X_{t+\tau}\} = \mathbb{E}\{X_tX_{t+\tau}\}$, and spectral density function (sdf), $f(\cdot)$,

$$f(\lambda) = \sum_{\tau=-\infty}^{\infty} \gamma_\tau e^{-2\pi \lambda \tau}. \tag{1}$$

The process $\{X_t\}$ exhibits seasonal or periodic persistence if there exist real numbers $H \in (1/2, 1)$ and $\xi \in (0, 1/2)$, and a bounded function $c(\gamma)$ such that

$$\lim_{\tau \to \infty} \frac{\gamma_\tau}{c(\gamma) |\tau|^{2H-2}} = \cos(2\pi \xi \tau),$$

or equivalently if there exist $\beta \in (0, 1)$ and $\xi \in (0, 1/2)$ and a bounded function $c(\lambda)$ such that

$$\lim_{\lambda \to \xi} \frac{f(\lambda) |\lambda - \xi|^\beta}{c(\lambda)} = 1. \tag{2}$$

Following convention, we parameterize the persistence parameter via $\delta = \beta/2$. In line with this definition, a process is considered to be a seasonally persistent process (SPP) if, in a neighbourhood of $\xi$,

$$f(\lambda) = f^\dagger(\lambda) |\lambda - \xi|^{-\beta} + o(|\lambda - \xi|^{-\beta}), \tag{2}$$

where $f^\dagger(\lambda) \equiv c(\lambda) > 0$, $0 < \lambda < \frac{1}{2}$ is bounded above.

Parameters $(\xi, \delta)$ determine the dominant long term behaviour of the process; typically, $\xi$ corresponds to the location of an unbounded but integrable singularity in the sdf. In this paper we consider a parametric family of sdf’s consistent with (2), that is, the parametric model of Giraitis et al. (2001), where

$$f(\lambda) = f_G(\lambda; \xi, \delta, \sigma^2) = \sigma^2 |h(\lambda; \theta)|^2 (1 - 2e^{-2i\pi \lambda} \cos(2\pi \xi) + e^{-4i\pi \lambda})^{-2\delta}, \tag{3}$$

where $h(\lambda; \theta)$ is bounded above and below at $\lambda = \xi$, with some linear process assumptions, given for instance in Hannan (1973); for example, $h(\cdot)$ could be the sdf for a stationary and invertible ARMA process, such as the case for GARMA processes, see Gray et al. (1989). We consider behaviour near the pole in such models by defining $f^\dagger(\lambda)$, where

$$f(\lambda) = f^\dagger(\lambda) |\lambda - \xi|^{-2\delta} = f^\dagger(\lambda; \xi, \delta, \theta) |\lambda - \xi|^{-2\delta}. \tag{4}$$

The results in this paper will also be applicable to nearly non-stationary unit root AR processes, when the roots of the AR process approach unity at a suitable rate in the sample size, this quantifying issues with near unit root processes.

1.2 Estimation for Seasonally Persistent Processes

We consider maximum likelihood estimation of $\xi$ and $\delta$, and denote the true values of these parameters by $(\xi^*, \delta^*)$. Joint estimation of the seasonality and persistence parameters is of importance, as inaccurate estimation of $\xi$ will affect the estimation of $\delta$, and any other
parameters of the sdf – δ quantifies the rate of decay of the dependence, and thus determines the long-term behaviour of the series. Note also that, even in cases where ξ is believed to be known (for calendar data, equal to 1/12, or 1/7, or 1/4 say), there may on occasion be finite sample advantage in estimating ξ rather than using its known value, in terms of estimation of the other parameters of the system. For example, if ξ is regarded as a nuisance parameter, then δ may be more efficiently estimated after conditioning on ξ̂ rather than ξ⋆; see, for example, Robins et al. (1994) and Rathouz et al. (2002) for supporting theory. This issue goes beyond the scope of this paper, but gives further indication that estimation of ξ is intrinsically important.

We will examine inference for the parameters of an SPP based on a realization of the process of length N. Throughout this paper, for convenience and with minimal loss of generality, we will assume N is even, N = 2M say. We establish asymptotic results for these estimators (ξ̂, δ̂), and provide practically useful large sample approximations to the distribution of the estimators. In particular, we define a large sample approximation to the log-likelihood of the periodogram evaluated at a full set of frequencies spaced O(N⁻¹) apart. At a local scale the variational structure of the log-likelihood in ξ remains appreciable over O(N⁻¹) distances; however the magnitude of these variations becomes negligible compared to the total accumulated magnitude of the log-likelihood for increasing sample sizes. We demonstrate that this variation prevents standard likelihood results being valid for the estimator of ξ, although standard asymptotic results can be established for the estimator of the δ, which is in agreement with previous results, see (Hidalgo and Soulier, 2004; Giraitis et al., 2001). We discuss in detail the large sample behaviour of N(ξ̂ − ξ⋆) and establish its approximate large sample distribution, as well as a moderate sample size approximation. Finally we demonstrate that our likelihood-based estimators have good small sample properties on simulated series compared with other, non-likelihood estimators, and consider estimation of the system parameters in a econometric example, using a data set with weekly gasoline sales in the United States, and two meteorological examples, monthly temperature data from a Californian shore-station, and the Southern Oscillation Index data set.

1.3 The Periodogram, Likelihoods and Approximations

We consider a sample from a stationary Gaussian time series, \( X = (X_0, X_1, \ldots, X_{N-1})^\top \), as defined in section 1.1, with covariance matrix \( G_N = G_N(\xi, \delta, \theta, \sigma^2) \) with \((i, j)\)th element \( \gamma_{|i-j|} \). The exact log likelihood, \( \ell_N \), of the finite time-domain sample is given by

\[
2\ell_N (\xi, \delta, \theta, \sigma^2) = 2 \log L_N (\xi, \delta, \theta, \sigma^2) = -N \log (2\pi) - \log |G_N| - X^\top G_N^{-1} X. \tag{5}
\]

This likelihood is often approximated due to the computational complexity associated with the calculation of \( G_N^{-1} \). The standard approximation approach was introduced by Whittle (1951), and the resulting, much studied, discretized approximate likelihood is commonly known as the discrete Whittle likelihood. The Whittle likelihood gives an approximation to the likelihood of the time domain data in the frequency domain via the Fourier coefficients, under assumptions as specified by Beran (1994, p. 109–113, and 116–7). Problems associated with the usage of Whittle's approximation for non-Gaussian and small sample size Gaussian time series has been discussed by Contreras-Cristan et al. (2006).

The final two terms in equation (5) are approximated using results of Whittle (1951) and Grenander and Szegö (1984). It follows that the Whittle likelihood for (ξ, δ) and θ is given
by:

\[ \ell_N^{(W)}(\xi, \theta, \sigma^2) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{I_0(\lambda)}{f(\lambda)} d\lambda, \]  

(6)

where \( I_0(\lambda) \) is the periodogram, defined as the modulus square of the discrete Fourier transform (DFT), \( Z_0(\lambda) \), of the realized time series.

At the Fourier frequencies \( \varphi_j = j/N, j = 0, \ldots, M, \) the periodogram, \( I_0(\varphi_j) \), is given by,

\[ I_0(\varphi_j) = A_0^2(\varphi_j) + B_0^2(\varphi_j) = \frac{1}{N} \left[ \sum_{t=0}^{N-1} X_t^2 + 2 \sum_{t=1}^{N-1} \sum_{s=0}^{t-1} X_t X_s \cos \{2\pi j(t - s)/N\} \right]. \]

(7)

so that

\[ I_0(\varphi_j) = A_0^2(\varphi_j) + B_0^2(\varphi_j) = \frac{1}{N} \left[ \sum_{t=0}^{N-1} X_t^2 + 2 \sum_{t=1}^{N-1} \sum_{s=0}^{t-1} X_t X_s \cos \{2\pi j(t - s)/N\} \right]. \]

For short memory data, the periodogram is an asymptotically unbiased but inconsistent estimator of \( f(\cdot) \) that is commonly used as the basis of more sophisticated estimation procedures. The use of (6) for parameter estimation has been discussed in detail by Walker (1964, 1965) and Hannan (1973) under the assumption that the log spectrum integrates to zero. Hosoya (1974) added a second term of \( \log|f(\lambda)| \) to the integral to deal with more general processes.

For the likelihood in equation (6) to have desirable asymptotic properties, it is assumed that the process is linear, and satisfies certain regularity conditions, thus ensuring good large sample properties of the likelihood based estimators. Note that (6) is an approximation to the log-likelihood of \( X \) based on the periodogram, but that (6) is not a likelihood for the periodogram. The approximation of the likelihood in equation (5) by equation (6), performs well when the process is Gaussian and the covariance of the time series is either rapidly decaying or exactly periodic.

A Riemann approximation to the integral in equation (6) yields the discrete analogue

\[ \ell_N^{(DW)}(\xi, \delta, \theta, \sigma^2) = -\frac{2}{N} \sum_{j=0}^{M} \frac{I_0(\varphi_j)}{f(\varphi_j)}, \]

(8)

and we could also adjust this to allow for more general processes:

\[ \ell_N^{(DW)}(\xi, \delta, \theta, \sigma^2) = -\frac{2}{N} \sum_{j=0}^{M} \frac{I_0(\varphi_j)}{f(\varphi_j)} - \frac{2}{N} \sum_{j=0}^{M} \log|f(\varphi_j)|, \]

(9)

following Hosoya’s proposal. By defining the vector \( C_{2j,2j+1}(A_j, B_j)^\top \), where \( A_j = A_0(\varphi_j) \) and \( B_j = B_0(\varphi_j) \), and \( \Sigma_C \) as the exact covariance of \( C \), we may consider the exact log-likelihood, \( \ell_N^{(f)} \), of the DFT of observed and Gaussian data via:

\[ 2\ell_N^{(f)}(\xi, \delta, \theta, \sigma^2) = -N \log(2\pi) - \log |\Sigma_C| - C^\top \Sigma_C^{-1} C, \]

(10)

in direct analogue with equation (5), acknowledging finite sample effects of the DFT. The difference between this equation and the discrete Whittle likelihood is that it involves the exact covariance matrix, \( \Sigma_C \), of the FFT coefficients. Analysis based on the likelihood of
the Fourier coefficients (in general) involves the inversion of the large, non-sparse covariance matrix, and is thus equally inefficient as the basis of likelihood procedures as equation (5).

Having specified these various likelihood functions that could be used for inference, some justification must be used to motivate their usage. Equation (10) is a natural choice for analysis of seasonal time series, given the compression of the variables of the seasonal effects. We shall use the compression to approximate the likelihood more carefully, acknowledging large finite sample effects related to the compression explicitly.

1.4 Contributions of the Paper

We introduce an approximation to equation (10), and use this as the basis of a maximum likelihood procedure. We focus on the distribution and other properties of the periodogram, given an underlying SPP with sdf \( f(\cdot) \). We focus on Gaussian processes, and do not consider here the non-Gaussian case. However, for other processes, such as those in Brillinger (1975), where asymptotic normality of the DFT holds, our distributional results are still valid.

Specifically, we consider estimation of parameters of spectra with spectral poles away from frequency zero. We consider an adjustment to the standard DFT that simplifies the technical developments of this paper. A simple (but parameter dependent) modification of the choice of grid, conditional on a known spectral pole location, leads to simple approximations to the likelihood of the periodogram at a new set of frequencies spaced at a distance \( O(N^{-1}) \) apart. In particular

1. We propose a new demodulated Whittle discrete likelihood for seasonal processes (sections 2 & 3). We show that the proposed likelihood approximates the distribution of the discrete Fourier transform for any posited value of the true parameters (see Theorem 1). The key idea is to use a different orthogonal transformation of the data conditional on each fixed value of the location of the pole (specification of a compressed representation). This is a non-standard situation.

2. To establish the properties of the likelihood we calculate the large finite sample distribution of the periodogram at the pole itself (Section 2.3).

3. We bound the covariance of the demodulated periodogram at different frequencies spaced \( 1/N \) apart (noted in Section 3), and note its asymptotically negligible contribution to the normalized log-likelihood. Furthermore, the choice of approximation to the likelihood is not everywhere continuous. However, we demonstrate (Section 3) that the discontinuities in the likelihood surface represent a negligible contribution for finite large samples.

4. We prove consistency of the MLEs (see Theorem 2), and determine the large sample first order properties of the score and observed Fisher information (Theorem 3).

5. We determine the asymptotic distribution of the score and observed Fisher information (see Theorem 4) and the asymptotic distribution of the MLEs (see Theorem 5).

6. We give a large finite sample approximation to the distribution of the pole estimator (see Proposition 6).

To derive the appropriate large sample theory, some care is required. It transpires that the score and Fisher information do not exhibit the usual large sample behaviour. Our results
are based on a Taylor expansion of the log-likelihood; we adopt the normalization of the observed Fisher information adopted by Sweeting (1980, 1992). We thus renormalize the observed Fisher information appropriately with a suitable power of $N$. The renormalized score and observed Fisher information converge in law to Gaussian random variables that are asymptotically uncorrelated. The distribution of $\xi$ converges slowly to the asymptotic distribution, and so alternate finite large sample approximations are also given.

These results establish a new large sample theory for seasonally persistent processes, and utilize the data-dependent transformation of the time-domain data that facilitate the computation of the distribution of different random variables for each posited value of the pole, and appropriate normalisation techniques for the score and Fisher information when the data is modelled as highly compressed in the Fourier domain.

1.5 Connections with Recent Work

In connections with other related work, we distinguish between likelihood-based methods and semi-parametric methods for processes exhibiting seasonal persistence. Giraitis et al. (2001) consider fully parametric models, and constrain the maximization over the location to a grid of frequencies spaced $O(N^{-1})$ apart. Hidalgo and Soulier (2004) consider semi-parametric models, and the theoretical properties of the extended Geweke-Porter-Hudak estimator, basing their analysis on estimating the location of the singularity as the Fourier coefficient of the maximum periodogram value in a given frequency interval; in their simulation study, the true location of the singularity is aligned with the Fourier frequency grid. Hidalgo and Soulier (2004) evaluate the Fourier coefficients at the Fourier frequency grid, and restrict the estimate of the location of the pole to a grid of frequencies spaced $O(N^{-1})$ apart. Hidalgo (2005) used semi-parametric methods to estimate the location of the pole, as well as the long memory parameter. By using a two-step procedure he is able to develop large sample theory for the estimator of the singularity, whereas in contrast we focus on full likelihood methods. More recently, Whitcher (2004) used a wavelet packet analysis approach for estimation of seasonally persistent processes.

In terms of asymptotic properties, our rate of convergence matches that of Giraitis et al. (2001). However, in addition, we obtain the large sample distributional results for the estimator of the pole, which they fail to do, having produced a different estimator. Similarly to Giraitis' et al., Beran and Gosh (2000) estimate the location of the pole using the coefficient which maximises the periodogram. Our estimator is again different although asymptotically equivalent with the same rate of convergence, and it has a determinable asymptotic, as well as large finite sample approximate, distribution.

Our work also has a connection with, but is different in spirit from, hidden frequency estimation, in which the seasonal structure is modelled as deterministic, corresponding to a single sinusoid. In this case, the Fourier coefficient which maximizes the periodogram converges to the true coefficient with a faster rate than the convergence of the MLE of the pole. Such rates were improved by secondary analysis, and the corresponding analysis using data tapers, see for example Chen et al. (2000); Hannan (1973, 1986); v. Sachs (1993). Secondary analysis corresponds to partitioning the time series into several groups of data, and using regression to estimate the so-called hidden frequency. Thomson (1990) used multitaper methods to improve the detection of a set of hidden frequencies, and use least squares methods over a given bandwidth. Neither the model we use, nor our proposed inferential method, is equivalent.
to the above mentioned procedures. Secondary analysis can be considered to ‘zoom in’ on local structure near the pole, and may be philosophically related to our procedure, but we implement full likelihood for a full set of Fourier coefficients. Conditionally for each fixed value for the pole, we calculate the distribution of a different set of random variables, but as each set is a linear and orthogonal transformation of the original data, and with a constant and equal Jacobian, this is appropriate.

Finally, we note that the inferential issues are of importance beyond seasonally persistent processes. The inherent non-regularity arises due to a parameter dependent transformation of the time-domain data. Whenever the process is modelled using a suitable parametric linear transformation of the data that will give decomposition coefficients that are non-negligible only for a few sets of indices, our methods will be applicable with some minor modifications. In a more general setting we would write the variances of a set of basis coefficients as satisfying a power-law decay, and we refer to such processes as second order compressive processes. Power-law decay in a suitable basis is an relatively common phenomenon - see for example the discussion in Donoho (2006); Abramovich et al. (2006); Candès and Tao (2004) - and our developments will carry across to this setting if the compression is stochastic rather than deterministic, once the location and decay parameters have been incorporated in the arbitrary basis. Issues of alignment, and/or shift-variance, akin to results that arise for misspecified location of the pole, are very well-documented in other basis expansions (Coifman and Donoho, 1995). Note that the equivalent to the decay parameter discussed by the aforementioned authors will be $p = 1/(2\delta)$. Only for $\delta > 0.25$ are we in their mode of decay, corresponding to extreme regimes of long memory behaviour.

2 Distributional results for the Periodogram

2.1 Large Sample Properties

The large sample properties of the periodogram of seasonally persistent processes were determined in Olhede et al. (2004). We summarize and extend these results below; in particular we compute the statistical properties of the periodogram itself at the pole $\xi$, as this specific Fourier coefficient will contribute substantively to the subsequent likelihood calculation.

Theorem 1 in Olhede et al. (2004) gives the following result concerning the relative bias at frequency $\lambda$, $B_{\lambda,N}(\xi, \delta)$, of the periodogram for all $\lambda \in (-1/2, 1/2), \xi \in (0, 1/2),$

$$B_{\lambda,N}(\xi, \delta) = \begin{cases} 
\text{E} \left\{ \frac{I_0(\lambda)}{f(\lambda)} \right\} & \lambda \neq \xi \\
\text{E} \left\{ \frac{I_0(\xi)}{N^{2\delta} f(\xi)} \right\} & \lambda = \xi 
\end{cases}$$

This notation makes explicit the dependence of the relative bias on $\xi$ and $N$. For frequencies $\varphi_k = k/N, k \in \{0, \ldots, M\}$, we have, for large $N$ and a fixed value of $\xi$, with $\varphi_k \neq \xi$,

$$B_{\varphi_k,N}(\xi, \delta) = \frac{2}{\pi} \int_{-\infty}^{\infty} \left[ \frac{\sin(u/2 - \pi c_N(\xi, \varphi_k))}{u - 2\pi c_N(\xi, \varphi_k)} \right]^2 \frac{2\pi c_N(\xi, \varphi_k)}{u} \, du + o(1), \quad (11)$$

where $c_N(\xi, \varphi_k) = N(\varphi_k - \xi)$ denotes $N$ times the distance between the $k^{th}$ Fourier frequency
and the pole at $\xi$. For the case $\varphi_k = \xi$, the large sample value of $B_{\xi,N}(\xi, \delta)$ is given in Lemma 2.1 in Section 2.4.

For the second order moment properties, let

$$C_{\varphi_k, \varphi_l, N}(u, \xi) = \frac{\sin\{u/2 - \pi c_N(\xi, \varphi_k)\} \sin\{u/2 - \pi c_N(\xi, \varphi_l)\}}{\{u - 2\pi c_N(\xi, \varphi_k)\}\{u - 2\pi c_N(\xi, \varphi_l)\}}$$

$$V_{\varphi_k, \varphi_l, N}(\xi, \delta) = (-1)^{k+l} \frac{2}{\pi} \int_{-\infty}^{\infty} C_{\varphi_k, \varphi_l, N}(u, \xi) \left| \frac{2\pi}{u} \right|^{2\delta} |c_N(\xi, \varphi_k)c_N(\xi, \varphi_l)|^{\delta} du + o(1).$$

Then, for $A_0(\varphi_j), B_0(\varphi_j)$ from (7), Olhede et al. (2004) gives

$$E\{A_0(\varphi_k)A_0(\varphi_l)\} = E\{B_0(\varphi_k)B_0(\varphi_l)\} = \left\{ V_{\varphi_k, \varphi_l, N}(\xi, \delta)/2 + o(1) \right\} \sqrt{f(\varphi_k)f(\varphi_l)}$$

$$E\{A_0(\varphi_k)B_0(\varphi_l)\} = E\{B_0(\varphi_k)A_0(\varphi_l)\} = o(1) \sqrt{f(\varphi_k)f(\varphi_l)}$$

$$= o\left( N^{2\delta} \right) \quad \text{if} \quad c_N(\xi, \varphi_k), c_N(\xi, \varphi_l) = O(1).$$

These results specify the large sample first and second order structure of the periodogram. We now extend these results to the demodulated periodogram described in section 2.2. Note that a direct implication of these results is that the distribution of the periodogram is highly dependent on the distances between the pole $\xi$ and the Fourier frequencies $\{\varphi_k\}$.

### 2.2 The Demodulated Discrete Fourier Transformation

The Discrete Fourier Transform of $\{X_t\}$ is not constrained to be evaluated at $\{\varphi_k\}$, but in fact any $O(N^{-1})$ grid could be considered. This fact leads us to consider demodulation, a grid realignment technique, which for any fixed value of $\xi$ produces a new grid aligned with the pole. Demodulation ensures that the large sample behaviour of the demodulated periodogram is similar to that of the periodogram of a standard long memory process (where $\xi = 0$). Specifically, the large sample bias is the same but the distribution of the periodogram is $\chi^2_2$ rather than a sum of unequally weighted $\chi^2$ random variables (see Hurvich and Beltrao, 1993; Olhede et al., 2004, p. 621).

The Demodulated Discrete Fourier Transform (DDFT) or offset DFT (Pei and Ding, 2004) of a sample of size $N$ from time series $\{X_t\}$ with demodulation via a fixed frequency $\lambda$ is denoted $Z_\lambda$, and is defined for Fourier frequency $\varphi_j$ by

$$Z_\lambda(\varphi_j) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} X_t e^{-2\pi i(\varphi_j + \lambda)t} = A_\lambda(\varphi_j) - iB_\lambda(\varphi_j), \quad j = 0, \ldots, M. \quad (12)$$

The demodulated periodogram at frequency $\varphi_j$ with demodulation via $\lambda$ is denoted $I_\lambda(\varphi_j)$, and is defined via the ordinary periodogram $I_0$ by

$$I_\lambda(\varphi_j) = I_0(\varphi_j + \lambda) = |Z_\lambda(\varphi_j)|^2 = A_\lambda^2(\varphi_j) + B_\lambda^2(\varphi_j).$$

Hence $I_\lambda(\varphi_j)$ is simply the periodogram evaluated at frequency $\varphi_j + \lambda$, or $I_0(\varphi_j + \lambda)$. We will consider evaluating this expression at arbitrary frequency $\varphi$. We define $C_{\lambda(j),\lambda(j+1)} = (A_{\lambda,j}, B_{\lambda,j})^T = (A_\lambda(\varphi_j), B_\lambda(\varphi_j))^T$, in analogue to $C$ in equation (10). For Gaussian data $X$ we then find:

$$C_\lambda = (A_{\lambda,0}, B_{\lambda,0}, A_{\lambda,M}, B_{\lambda,M})^T \overset{\mathcal{D}}{=} \mathcal{N}(0, \Sigma_{C_\lambda}) \quad (13)$$
Note that due to the demodulation, $B_{\lambda,0} \neq 0$ in general, unlike the imaginary component of the DFT at frequency zero. To efficiently formulate the likelihood, we need to explicitly consider the computation of $\Sigma C_\lambda$, the covariance of the DDFT coefficients.

### 2.3 Extending the Olhede et al. (2004) result

The results in Olhede et al. (2004) do not cover the case of demodulation, and to enable calculation of the new likelihood, further results are required. For example $B_{\xi,N}(\xi, \delta)$ needs to be explicitly determined. To minimize the bias in the demodulated periodogram, and simplify the covariance structure, we shift the Fourier grid so that the closest Fourier frequency to the pole in the original grid coincides exactly with the pole in the demodulated version. For a pole at $\xi$, we denote by $j_{0,N}(\xi) = [N\xi]$, where $[x]$ indicates the nearest integer to $x$. We furthermore let $c_N(\xi, \varphi_{j_0,N}(\xi)) = j_{0,N}(\xi) - N\xi$ and specify $\lambda = \lambda_{D,N}(\xi)$ in (12) as $\lambda_{D,N}(\xi) = -c_N(\xi, \varphi_{j_0,N}(\xi))/N$. The approach introduces a new grid of frequencies, namely

$$\lambda_k \equiv \lambda_{k(j),N}(\xi) = \varphi_j + \frac{c_N(\xi, \varphi_{j_0,N}(\xi))}{N} = \xi + \frac{j - j_{0,N}(\xi)}{N} = \xi + \frac{k(j)}{N}. \quad (14)$$

We exclude Fourier frequencies 0 and 1/2, and taking $j = 1, \ldots, M - 1$ we have $k = k(j) = j - j_{0,N}(\xi) \equiv J_1, \ldots, J_2 \equiv -j_{0,N}(\xi), \ldots, M - j_{0,N}(\xi)$. For example, if $N = 16$ and $\xi = 0.15$, then $\lambda_{D,16}(0.15) = 0.025$, $[N\xi] = 2$, $J_1 = -1$ and $J_2 = 5$. Note that for $k(j_2) > k(j_1) \neq 0$, then

$$V_{\lambda_{j_1}(j_2),N}(\xi, \delta) = V_{\varphi_{j_1},\varphi_{j_2};N} \left\{ \frac{j_{0,N}(\xi)}{N}, \delta \right\},$$

so that the covariance properties of the DDFT can be easily determined.

Under this demodulation, the DDFT yields the original periodogram $I_0$ evaluated at frequencies $\lambda_k \equiv \lambda_{k(j)} = \xi + (j - j_{0,N}(\xi))/N$, and takes the form

$$Z_{\lambda D}(\varphi_j) = Z_0(\lambda_{k(j)}) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} X_t e^{-2i\pi\lambda_k t}, \quad k = J_1, \ldots, J_2, \quad (15)$$

so that, for $k = J_1, \ldots, J_2$, $I_{\lambda D}(\varphi_j) = I_0(\lambda_k) = I_0(\xi + k/N)$. The DDFT can be computed efficiently by applying the DFT to the new series $\{Y_t\}$, defined for $t = 0, \ldots, N - 1$, by $Y_t = X_t \exp \{-2i\pi\lambda_{D} t\}$. Demodulation both simplifies the mathematical calculations considerably, and improves estimation of the persistence parameter $\delta$. Naturally the operation is very straightforward to implement. The parameter dependent choice of $\{\lambda_k\}$ will need careful analysis when deriving properties of the parameter estimators.

### 2.4 Expectation of the Periodogram at the Pole

The result in (11) gives the relative bias of periodogram. The expectation of the periodogram is given in the following Lemma.

**Lemma 2.1** The expected value of the periodogram evaluated at the pole $\xi$, after demodulation by $\xi$, is

$$E\{I_0(\xi)\} = (2\pi N)^{2\delta} \{-2f^\dagger(\xi)\Gamma(-1 - 2\delta)\} \cos\{\pi(1/2 + \delta)\} \pi^{-1} + o(1)$$

$$\equiv N^{2\delta} f^\dagger(\xi) B_\xi(\xi, \delta) + o(1) = O(N^{2\delta}).$$
3 Asymptotic Properties of the Likelihood and Estimators

In this section we utilize demodulation, and the large sample approximations described above, to present three theorems that characterize the asymptotic behaviour of the likelihood, the corresponding MLEs for \((\xi, \delta)\) and the associated Fisher information to obtain their large sample properties. Specifically, we establish \(N\)-consistency for the estimator of the location of the pole, thus matching the result of Giraitis et al. (2001).

3.1 Large-sample Likelihood Approximation

For a periodogram demodulated to align the Fourier grid with pole \(\xi\), we have the following asymptotic result.

**Theorem 1** Approximating the Likelihood Function.

*For a Gaussian series from a periodic long memory model as described by (3), where \(f^\dagger(\cdot)\) is twice partially differentiable with respect to \((\xi, \delta)\), the log-likelihood of the discrete Fourier transform can be approximated by

\[
\ell(\xi, \delta, \theta, \sigma^2) = \sum_{j=J_1}^{J_2} \{\log \eta_j - \eta_j I_0(\xi + j/N)\}
\]

accurate to \(o(N)\), where

\[
\eta_j = \frac{|j|^2 \Upsilon\{j \neq 0\}}{B_\xi(\xi, \delta)^\Upsilon\{j = 0\} N^{2\delta} f_j^\dagger}
\]

for \(0 < \delta < 0.5\), where \(\Upsilon\{A\}\) is the indicator function for event \(A\),

\[f_j^\dagger \equiv f^\dagger(\lambda_j) = f^\dagger(\xi + j/N)\]

and \(B_\xi(\xi, \delta)\) is the asymptotic relative bias given by Lemma 2.1.*

**Proof:** See the Appendices A.2-A.4.

**Note I:** The approximation to the likelihood is equivalent to that of independent exponential random variables with rate parameters \(\eta_j\) that depend on \(j\) and \(\delta\) but not on \(\xi\). In equation (17), the function \(B_\xi(\xi, \delta)\) appropriately scales the periodogram contribution from the Fourier frequency aligned with \(\xi\). \(B_\xi(\xi, \delta)\) is monotonically increasing in \(\delta\), with \(\lim_{x \to 0} B_\xi(\xi, x) = 1\), and \(B_\xi(\xi, \delta)\) is bounded away from zero. As the function is monotonic the derivatives of \(B_\xi(\xi, \delta)\) are also bounded away from zero. If \(f^\dagger(\cdot)\) is also bounded away from zero, then the log likelihood is bounded in \(\xi\) and \(\delta\). Thus it is possible to find efficiently the MLEs of \(\xi\) and \(\delta\) numerically.
Note II: This likelihood is not differentiable with respect to $\xi$ at all values of $\xi$; although $\hat{I}_0(\xi + j/N)$ is available in simple form, the dependence of $J_1 = -j_0,N(\xi)$ and $J_2 = M - j_0,N(\xi)$ on $\xi$ renders the overall function discontinuous. However, the discontinuities are $O(1)$ in magnitude, and the log likelihood is uniformly at least $O(N)$, so in fact the discontinuities are negligible, but motivate us to look, in standard fashion, at the $N$-standardized likelihood function $\ell(\xi, \delta, \theta, \sigma^2)/N$. See Appendix A.4 for further details.

Note III: The formulation in Theorem 1 summarizes the data in the frequency domain via the demodulated periodogram for a given $\xi$. We avoid the introduction of the substantial bias and covariance terms found in Olhede et al. (2004), as the demodulated periodogram is perfectly aligned with this singularity. When other demodulations are chosen the likelihood cannot be approximated in such a fashion. Even for frequencies of sufficient distance from any irregular behaviour, the results of Olhede et al. (2004) cannot be applied directly, and to find the approximate Whittle likelihood we additionally need to make assumptions about the spectral density function, and its smoothness (see Dzhaparidze and Yaglom (1983) and Taniguchi and Kakizawa (2000)).

Note IV: The result differs with that of Hurvich and Beltrao (1993) in a number of ways. The ordinates subscripted $j$ and $-j$ in the DFT are no longer complex conjugates, and the likelihood at evaluated at $\lambda_j$ is now approximately $\chi^2_j$ (rather than a mixture of two different $\chi^2$ terms) even for those coefficients closest to the pole. Strictly, the definition for $\eta_j$ in equation (17) has an additional term $V_{\lambda_j,\lambda_k,N}(\xi, \delta)$ for $j, k \in \mathbb{Z}$, but these terms can be bounded appropriately, and thus contribute in a negligible fashion. The bias at the pole reported in Hurvich and Beltrao (1993) is (identically) present in our formulation, but is $o(N)$, and is thus subsumed into the final term – see Hurvich et al. (1998) for relevant supporting arguments.

3.2 Existence and Consistency of the ML estimators

We now use the results of the previous section to construct likelihood-based estimators of $\xi$ and $\delta$ and establish their properties. The following theorem establishes the existence and consistency of the ML estimators derived from the likelihood in Theorem 1.

Theorem 2 Existence and Consistency
For the likelihood of Theorem 1, the ML estimators of $\xi$ and $\delta$, $\hat{\xi}$ and $\hat{\delta}$, exist and are consistent, with convergence rates $N$ and $N^{1/2}$ respectively.

Proof: See Appendix A.5. ■

The $N$-consistency of $\hat{\xi}$ matches the convergence rate of Giraitis et al. (2001). It is unusual to find superconsistent estimators in likelihood based procedures. An intuitive understanding of the rate can be found in the time domain. As we collect $N\xi^*$ full periods of the data the periodicity of the data is determined to an accuracy of $O(N^{-1})$. The reason why this rate is achieved is that the log-likelihood is varying $O(N^{3/2})$ (see proposition 14) over distances in $\xi$ of $O(N^{-1})$ near the value $\xi = \xi^*$. However, the convergence rate is different to that of Chen et al. (2000). The latter model the seasonality as a deterministic seasonal component embedded in
stationary noise. In Chen et al. a regression model is employed to estimate the amplitude and locations of the seasonality, and a rate of $N^{-3/2}$ rather than $N^{-1}$ is achieved. We employ a different model and hence do not expect the same convergence rates as is achieved by Chen et al..

3.3 Properties of The Fisher Information Matrix

**Theorem 3 The Fisher Information.**
For a series from a periodic long memory model as described by (3), for large $N$, the components of the Fisher information

$$\mathcal{F}_N = \begin{pmatrix} \mathcal{F}_{\xi,\xi}^{(N)} & \mathcal{F}_{\xi,\delta}^{(N)} \\ \mathcal{F}_{\xi,\delta}^{(N)} & \mathcal{F}_{\delta,\delta}^{(N)} \end{pmatrix}$$

are given by

$$\mathcal{F}_{\delta,\delta}^{(N)} = \mathcal{F}_{\delta,\delta} N + o(N), \quad \mathcal{F}_{\xi,\delta}^{(N)} = \mathcal{F}_{\xi,\delta}^{(1)} N + o(\log(N)), \quad \mathcal{F}_{\xi,\xi}^{(N)} = \mathcal{F}_{\xi,\xi} N^2 + o(N^2),$$

where $\mathcal{F}_{\delta,\delta}$, $\mathcal{F}_{\xi,\delta}^{(1)}$ and $\mathcal{F}_{\xi,\xi}$ are constants independent of $N$ but are functions of the true values of $\xi$ and $\delta$.

**PROOF:** See Appendices A.6 and A.7. ■

For a full analysis in a regular ML setting, the second order properties of the MLEs can in a general setting be deduced from the above quantities. Large sample properties, specifically consistency and asymptotic variance, may be considered via a Taylor expansion of the log-likelihood, see for example Cheng and Taylor (1995). However, we note that we are not in a standard setting; even if we may expand the log-likelihood near the true value of the parameter, because of the non-standard behaviour of the derivatives of the log-likelihood, the observed Fisher information does not converge to a diagonal matrix with constant entries, but rather the (appropriately standardized) observed Fisher information for $\xi$ converges to a random variable with order one variance. We will discuss the interpretation of the Fisher information in this context, in the appendix. For the derivatives involving the location of the pole, extra terms of magnitude $N$ are introduced and thus the variance of the observed Fisher information in $\xi$ is $O(N^5)$. The magnitude of the variance of the observed Fisher information in $\xi$ implies that a standardization of the random variable must be employed that results in a negligible expectation of the restandardized random variable. We also therefore discuss the large sample theory of the observed Fisher information.

3.4 The Asymptotic Properties of the MLEs

We now consider the use of the Fisher information to determine the asymptotic variance. Consider a Taylor expansion of the score near the true value $\psi^*$ of the parameters $\psi = (\xi, \delta)$ evaluated at the MLE $\hat{\psi}$. We denote the observed Fisher information by $F_N(\psi)$, and let $\psi'$
lie between $\psi$ and $\psi^*$. We denote by $\hat{\ell}(\psi)$ the score in $\psi$, noting that the score is well defined if the log-likelihood is evaluated ignoring the $\xi$ dependence of $J_1$ and $J_2$, see section A-4:

$$\hat{\ell}(\psi) = \begin{pmatrix} \ell_\xi (\psi) \\ \ell_\delta (\psi) \end{pmatrix}.$$  

(18)

Then using a first-order expansion of the log likelihood in the usual way for $N$ sufficiently large, we have the (vector) score function:

$$\hat{\ell}(\psi^*) = \hat{\ell}(\psi) - F_N(\psi^*)(\psi - \psi^*) \iff F_N(\psi^*)(\psi - \psi^*) = \hat{\ell}(\psi^*) - \hat{\ell}(\psi).$$

Thus the difference between $\psi$ and $\psi^*$, when appropriately scaled by the random matrix $F_N(\psi^*)$ corresponds to the value of the score at $\psi^*$ in the usual fashion. The statistical properties of $F_N(\psi^*)$ are not straightforward in this non-regular problem, and require further investigation. Following Sweeting (1992), we define a suitable standardization matrix $B_N$ and the standardized observed Fisher information by

$$B_N = \begin{pmatrix} N^{5/2} & 0 \\ 0 & NF_{\delta,\delta} \end{pmatrix}, \quad W_N = B_N^{-1/2} F_N B_N^{-1/2},$$  

(19)

as the large sample properties of $W_N$ are tractable, and their determination is an important step to finding the large sample properties of the MLE. Specifically, we let

$$B_N^{-1/2} F_N(\psi^*) B_N^{-1/2} B_N^{1/2}(\psi - \psi^*) = B_N^{-1/2} \{ \hat{\ell}(\psi^*) - \hat{\ell}(\psi) \},$$

so that

$$W_N(\psi^*) B_N^{1/2}(\psi - \psi^*) = B_N^{-1/2} \hat{\ell}(\psi^*) = k_N(\psi^*).$$

The latter expression defines the standardized score $k_N(\cdot)$. See the Appendix for a full discussion of these quantities. Note that $B_N$ is the large $N$ approximation to the Fisher information matrix for $\delta$ and corresponds to an appropriate order normalisation for $\xi$, thus $W_N(\psi)$ is the observed Fisher information renormalized by $B_N$.

We note that for $N$ large enough, the expected value of $W_N(\psi^*)$ is the identity matrix for the $\delta$ entry, but the expectation of the first entry of $W_N(\psi^*)$ is $O(1)$ whilst the variance of the first entry is $O(1)$. In a standard setting the expectation is $O(1)$ and the variance $O(1)$.

**Theorem 4 Distribution of the Score and Observed Fisher Information.**

For the likelihood of Theorem 1 the standardized score $k_N(\psi^*)$ and the standardized Observed Fisher information matrix $W_N(\psi^*)$ asymptotically have the following properties:

$$k_N(\psi^*) \xrightarrow{\mathcal{L}} k, \quad W_N(\psi^*) \xrightarrow{\mathcal{L}} W,$$

(20)

where the entries of $k = (k_1, k_2)^\top$ and $W$ are uncorrelated and

$$W_{11} \sim \mathcal{N}(0, 8\pi^4/15), \quad W_{12} = 0, \quad W_{22} = 1$$

$$k_1 \sim \mathcal{N}(0, \pi^2/3), \quad k_2 \sim \mathcal{N}(0, 1).$$

(21)

**Proof:** An outline of the proof given in the Appendix, see Proposition 6, Section A.8.3 and Proposition 9. □

15
Theorem 5 Distribution of the MLE.
For the likelihood of Theorem 1, the ML estimators of \(\xi\) and \(\delta\), \(\hat{\xi}\) and \(\hat{\delta}\), have distributions that for large sample approximately take the form:

\[
N(\hat{\xi} - \xi^*) = \left\{ \frac{N^{5/2}}{-\ell_{\xi\xi}(\psi^*)} \right\} \left\{ \frac{\ell_{\xi}(\psi^*)}{N^{3/2}} \right\} \frac{\xi - \sqrt{5}}{2\sqrt{2}C},
\]

where \(C\) is distributed according to the standard Cauchy distribution, and

\[
\sqrt{N}(\hat{\delta} - \delta^*) \sim AN\left(0, N \left\{ \mathcal{F}_{\delta,\delta}^{(N)} \right\}^{-1}\right).
\]  

An estimator of the asymptotic variance is formed via \(\hat{\mathcal{F}}_{\delta,\delta}^{(N)} = \mathcal{F}_{\delta,\delta}^{(N)}(\hat{\xi}, \hat{\delta})\). The forms of \(\mathcal{F}_{\delta,\delta}^{(N)}(\xi, \delta)\) and \(\mathcal{F}_{\delta,\delta}(\xi, \delta)\) are given in the Appendix.

PROOF: An outline proof is given in the Appendix, see Proposition 9 and section A.8.1.

Note: Giraitis et al. (2001) do not find the limiting distribution of their estimator, \(\hat{\xi}_G\), of \(\xi\). They note that this is an artefact of the maximization over the specified grid. This constraint is not enforced in our approach. Note that \(E(|\hat{\xi} - \hat{\xi}_G|) = O(N^{-1})\) but that \(N|\hat{\xi} - \hat{\xi}_G|\) is not constrained to be zero or even to have a tractable distribution, this result is demonstrated empirically in the simulations. The convergence to the Cauchy for extreme values of \(\delta\) is quite slow, we provide, in the Appendix, a second approximation to the distribution of the renormalized estimator of the pole, via more carefully approximating the dominant contributions to the mean and variance of the numerator and denominator that define the random variable the estimator follows.

To compare the two large sample and asymptotic forms of the distributions, we refer to Figure 2 (a) and (b). As \(\delta\) increases in magnitude it takes longer for the large sample approximation to be close the asymptotic distribution, as is obvious from these plots. For a list of critical values of the distribution see Table 10.

The likelihood of the data changes in magnitude dramatically depending on the value of \(\xi\) and its alignment with the grid of frequencies at which the periodogram is evaluated, determination of the best value of \(\xi\) is pivotal for characterizing the system, and must be the first stage of any analysis. For completeness we now discuss the estimation of the additional parameters, i.e. \(\theta\) and \(\sigma^2\).

3.5 White Noise Variance and Nuisance Parameters

We now consider the estimation of the white noise component and regular spectral component. We model the sdf parametrically by

\[
f(\lambda) = \frac{\sigma^2 |h(\lambda; \theta)|^2}{2 \cos (2\pi \lambda) - 2 \cos (2\pi \xi)} \quad \text{and} \quad f^\dagger(\lambda) = \frac{|h(\lambda; \theta)|^2 \sigma^2 |\lambda - \xi|^{2\delta}}{2 \cos (2\pi \lambda) - 2 \cos (2\pi \xi)}.
\]  

(23)
Differentiating $\ell(\xi, \delta, \theta, \sigma^2)$ from equation (16) with respect to $\sigma^2$ we obtain that:

\[
\frac{\partial \ell(\xi, \delta, \theta, \sigma^2)}{\partial \sigma^2} = \sum_{j=J_1}^{J_2} \left\{ -\frac{1}{\sigma^2} + \frac{\eta_j I_0(\xi + j/N)}{\sigma^2} \right\}
\]

\[
\hat{\sigma}^2 / \sigma^2 = \frac{1}{-J_1 + J_2 + 1} \sum_{j=J_1}^{J_2} \hat{\eta}_j I_0(\xi + j/N)
\]

\[
\hat{\eta}_j = \frac{|j|^{2\hat{\delta}} |2 \cos (2\pi \lambda) - 2 \cos (2\pi \xi)|^{2\hat{\delta}}}{N^{2\hat{\delta}} \sigma^2 |h(\lambda; \theta)|^2 |\lambda - \xi|^{2\hat{\delta}}}. \tag{24}
\]

Thus it follows that (Taylor expanding the other MLEs and using their rates of convergence):

\[
\frac{\hat{\sigma}^2 / \sigma^2}{\sigma^2} = \frac{1}{2(-J_1 + J_2 + 1)} \chi^2_{2(-J_1 + J_2 + 1)} + o(1), \tag{25}
\]

for $J_1, J_2$ sufficiently large. This follows as the estimators of the other parameters of the sdf are nearly unbiased for sufficiently large values of $N$. We note that the covariance of $\hat{\sigma}^2$ with $\hat{\xi}$ and $\hat{\xi}$ can be treated analogously to the results deriving the covariance of $\hat{\xi}$ and $\delta$ or using standard results for $\delta$ and/or $\theta$ and $\hat{\sigma}^2$. Denoting by

\[
\hat{h}(\lambda) = h(\lambda; \theta) \quad \hat{h}_i(\lambda; \theta) = \frac{\partial |h(\lambda; \theta)|^2}{\partial \theta_i} \quad \text{and} \quad \hat{h}_{ik}(\lambda; \theta) = \frac{\partial^2 |h(\lambda; \theta)|^2}{\partial \theta_i \partial \theta_k},
\]

we determine that

\[
\frac{\partial \ell(\xi, \delta, \theta, \sigma^2)}{\partial \theta_i} = -\sum_{j=J_1}^{J_2} \frac{\hat{h}_i(\lambda_j; \theta)}{|h(\lambda_j)|^2} \{1 - \eta_j I_0(\lambda_j)\} = O(N),
\]

and

\[
\frac{\partial^2 \ell(\xi, \delta, \theta, \sigma^2)}{\partial \theta_i \partial \theta_k} = \sum_{j=J_1}^{J_2} \left[ \frac{\hat{h}_{ik}(\lambda_j; \theta)}{|h(\lambda_j)|^4} \{1 - \eta_j I_0(\lambda_j)\} + \frac{\hat{h}_i(\lambda_j; \theta) \hat{h}_k(\lambda_j; \theta)}{|h(\lambda_j)|^2} (1 - 2\eta_j I_0(\lambda_j)) \right],
\]

which is $O(N)$. Thus we find that

\[
\mathbb{E} \left\{ \frac{\partial^2 \ell(\xi, \delta, \theta, \sigma^2)}{\partial \theta_i \partial \theta_k} \right\} = -\sum_{j=J_1}^{J_2} \frac{\hat{h}_i(\lambda_j; \theta) \hat{h}_k(\lambda_j; \theta)}{|h(\lambda_j)|^4} = \mathbf{V}^{-1}_{N,ik}
\]

\[
\mathbb{E} \left\{ \frac{\partial^2 \ell(\xi, \delta, \theta, \sigma^2)}{\partial \theta_i^2} \right\} = -\sum_{j=J_1}^{J_2} \frac{\hat{h}_i^2(\lambda_j; \theta)}{|h(\lambda_j)|^4} = \mathbf{V}^{-1}_{N,ii} = O(N). \tag{26}
\]

This then provides the required score equations. Furthermore using regular ML theory, we have that:

\[
\sqrt{N} \left( \hat{\theta} - \theta \right) \xrightarrow{L} \mathcal{N} \left( \mathbf{0}, \mathbf{V} \right), \tag{27}
\]

where $\mathbf{V}$ contains the Fisher information, and $N^{-1}\mathbf{V}_N \to \mathbf{V}$. This allows us to fit the more general class of GARMA rather than Gegenbauer models, see Gray et al. (1989).
4 Examples

4.1 Analysis of Simulated Data

For our simulation studies we examine the performance of our adjusted Whittle likelihood-based estimators in comparison with those derived from the classic Whittle likelihood. Data were simulated in the time domain using the covariance recursion formulae given in Lapsa (1997) for a seasonally persistent Gegenbauer process with $\xi = 1/7$ corresponding to an weekly cycle in daily data, and $\delta = 0.3, 0.4$ and $0.45$. We generate 2000 replicate series of lengths $N = 1024, 2048, 4096$ and $8192$. Tables 1 to 4 demonstrate the performance of the ML estimators of $\xi$ and $\delta$, in terms of bias, variance, and the relative efficiency ($\sigma_d^2/\sigma_W^2$) of our estimators compared with those derived using the classic Whittle likelihood. For $N = 1024$ the demodulated estimator significantly improves the bias present in the Whittle estimators for both $\xi$ and $\delta$. As $N$ increases and the spacing in the Fourier grid decreases, both estimators for $\xi$ perform well, however, the bias in the Whittle estimator for $\delta$ is still present even for $N = 8192$, and becomes more severe as $\delta$ increases.

To illustrate the problems with the Whittle likelihood for smaller $N$ and large $\delta$, Figure 1 shows the mean conditional likelihoods evaluated at the ML estimates. The improvement gained by demodulation is evident, the scale of the improvement will be dependent on the distance of the pole from the Fourier grid. The plots also demonstrate the discontinuities in the likelihood for $\xi$, as discussed in Section 3.

4.2 U.S. Weekly Crude Oil Imports

The first real data set comprises 756 observations of U.S. Weekly Crude Oil Imports (in millions of barrels per day) from 6th December 1991 to 26th May 2006, downloaded from http://tonto.eia.doe.gov/dnav/pet/hist/wcritis2w.htm. The data were detrended using a linear trend, and are displayed in Figure 3. Periodic behaviour is evident in the raw detrended data.

For these data, we fitted a low order Gegenbauer-ARMA (GARMA) model; the process $\{X_t\}$ is represented as the unique stationary solution of

$$\phi(B)X_t = \theta(B)G_t$$

where $B$ is the backshift operator, and polynomial operators $\phi$ and $\theta$ define an ARMA process in the usual way, and where $\{G_t\}$ is a pure Gegenbauer process as defined by the sdf in equation (3) with $h$ the identity function. We consider at most ARMA(1,1) models, so that $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 + \theta z$ where, under the assumptions of stationarity and invertibility, $|\phi|, |\theta| < 1$. Using standard results, the parametric sdf that we consider takes the form

$$f(\lambda) = \frac{\sigma_e^2}{|1 - 2e^{-2i\pi\lambda}\cos(2\pi\xi) + e^{-4i\pi\lambda}|^2(1 - \phi \cos(2\pi\lambda) + \theta^2)^2}  \left(1 + \theta \cos(2\pi\lambda) + \theta^2\right)$$

In our notation, a GARMA(1,1) model has both $\phi$ and $\theta$ non-zero; for GARMA(1,0), $\theta \equiv 0$, whereas for GARMA(0,1), $\phi \equiv 0$. GARMA(0,0) corresponds to the Gegenbauer model with no ARMA component.
Results: Using numerical methods (the `optim` function in R), each model was fitted using our demodulation approach and also using the standard Whittle likelihood, and the results compared using BIC. The results are presented in Table 5. The best model is the overall is the GARMA(0,1) model fitted under demodulation, indicating that the use of a non-standard Fourier grid can improve the quality of fit, that is, the fit of the model under the standard derivation (we term this the standard Whittle model) is inferior.

For the selected model, the parameter estimates and approximate standard errors are displayed in Table 6.

4.3 Farallon temperature data

The second real data set is a surface temperature series for the shore station at the Farallon Islands, California, United States. Daily temperature data were obtained from the ftp site

ftp://ccsweb1.ucsd.edu/shore/CURRENT_DATA/Temperature/

and formed monthly averages for the period 1960-1996; missing daily quantities were omitted from the monthly averages, whole missing months (there were six in the period of study) were imputed by taking averages for that calendar month across the 37 years of study. In total there were 444 monthly average observations.

We analyze these data in two ways to compare the Whittle maximum likelihood estimates with our demodulation approach. First, we take the 444 data in their entirety, then we perform a second analysis using only the last 440 observations. As the expected annual periodicity would induce a pole in the spectrum at frequency 1/12, and 12 divides 444, the pole will lie at a Fourier frequency when the whole data set is analyzed. However 12 does not divide 440, so for the second analysis, the pole will not lie at a Fourier frequency.

Results: Each of the low order GARMA models were fitted and compared using BIC. The two cases, \( N = 444 \) and \( N = 440 \) were analyzed. The results are presented in Table 7, and the raw time series as well as fitted models are plotted in Figure 4. The model with the highest BIC is, in both cases, the GARMA(1,0), but for the two values of \( N \), the different approaches are favoured in the two cases. For \( N = 444 \), the classic Whittle approach yields a higher log-likelihood, but for \( N = 440 \) the demodulated model performs better, yielding a higher log-likelihood. Parameter estimates from the model are presented in Table 6 for the two values of \( N \).

This data set and analysis illustrates perfectly another of the advantages of using the demodulated likelihood with the bias-adjustment procedure outlined in Section 3 and Theorem 1. In the classic Whittle likelihood, when a Fourier frequency exactly coincides with the pole, the on-the-pole likelihood contribution erases the contribution of that periodogram element. Note first that the omission of a data point from the likelihood causes the likelihood to increase (that is, become less negative) and this explains the higher likelihood value for the classic Whittle likelihood. For the Farallon data set, this omission also leads the remaining periodogram appearing as if it corresponded to a short memory process, hence the low estimated value of \( \delta \) that is essentially no different from zero. The conclusion of such an analysis would be that the underlying process has a pure seasonality at the estimated \( \xi \), in this case \( \xi = 1/12 \), and the seasonally differenced series was essentially a white noise process. However,
seasonal first differencing of the original series leads to a new series that is not a white noise process; in fact the differenced series appears over-differenced. Hence, such a model does not provide an adequate explanation of the data. When $N$ is changed to 440, inferences using the classic Whittle method change dramatically. Note, however, that for the new demodulated likelihood, parameters estimates are closely comparable across different values of $N$.

### 4.4 Southern Oscillation Index

We consider the Southern Oscillation Index (SOI) data analyzed by, for example, Huerta and West (1999). The version of the data we consider has $N = 1668$, the data and fitted spectrum are presented in Figure 5. For this large sample size, the difference between the two approaches is minimal; the BIC values are negligibly different, and the estimates and estimated 95\% intervals are presented in Table 9. In this case, the estimates of the pole position obtained from the likelihood approaches are markedly different from the naive estimate obtained by taking the ordinate corresponding to the maximum of the periodogram (shown as a dotted line in Figure 5(b)).

### 5 Implications for Non-Likelihood Approaches

The results derived in previous sections focus explicitly on likelihood based procedures. However, they motivate the use of adjusted versions of currently existing estimation procedures that improve the performance of those procedures when applied to seasonally persistent series. Given the special role of the location of the singularity when formulating the likelihood, we propose a series of procedures that profit on the simplified distribution that arises by using the demodulation by the (estimated) pole.

#### 5.1 Profile Likelihood

The profile likelihood of $\xi$ is a pseudo-likelihood function given, for each possible $\xi$, by

$$
\ell^P_N(\xi) = \max_{\delta, \theta|\xi} \ell_N(\xi, \delta, \theta).
$$

(28)

$\ell^P_N(\xi)$ may be maximized, yielding a maximum pseudo-likelihood (MPL) estimate of $\xi$, denoted $\hat{\xi}_{Pr}$. Then the values of $\delta$ and $\theta$ which maximize the conditional likelihood given $\xi = \hat{\xi}_{Pr}$, are computed. Specifically, the ML estimate of $\delta$ based on the demodulated likelihood for all values of $\xi \in (0, 1/2)$ is computed. Finally, the estimate $\hat{\delta} = \hat{\delta}(\hat{\xi}_{Pr})$ based on demodulation at $\hat{\xi}_{Pr}$ is obtained.

In many cases the MPL and ML estimators agree closely; in given applications, the MPL approach may potentially be more readily implemented. Note that some care must in generality be used when applying profile likelihood estimation, (see, for example, Berger et al. (1999)), but given the rate of convergence of the MLE of $\xi$ such problems are unlikely to arise.
5.2 A Semi-Parametric Analysis: The Geweke-Porter-Hudak Estimator

The Geweke-Porter-Hudak (GPH) procedure (Geweke and Porter-Hudak, 1983) implements semiparametric estimation of \( \delta \) for the case \( \xi = 0 \) which can be adapted to incorporate a demodulation procedure and profile marginalization. The GPH procedure examines the behaviour of the periodogram on the log scale near frequency zero, and estimates the long memory parameter \( \delta \) using ordinary least squares and a linear regression. We omit full details for brevity, but outline a possible adjustment based on a recent formulation given by Hidalgo and Soulier (2004). It is sufficient to say that the GPH procedure relies on distributional properties of the periodogram near the presumed pole.

In light of the results of earlier sections of this paper, to obtain an improved estimate of the \( \delta \) using GPH we could take two alternative approaches. First, we could adjust the GPH to the demodulated setting, taking the distribution of the periodogram at the singularity fully into account. Alternatively, we could utilize large sample arguments and consider the score function. We consider frequencies indexed \( j \) whose likelihood contributions are influenced by the singularity. The log-likelihood then has three parameters; \( \xi, \delta, \) and \( C = f^\dagger(\xi; \xi, \delta, \theta, \sigma^2) \) which can be treated as a constant, if the \( j \) included are chosen judiciously.

Hidalgo and Soulier (2004, p. 58-59) consider the modified GPH by introducing the following notation:

\[
g(\omega) = -\log \left| 1 - e^{i\omega} \right| \\
g_m = \frac{1}{m} \sum_{k=1}^{m} g(2\pi \varphi_k) \\
s_m^2 = 2 \sum_{k=1}^{m} \left\{ g(2\pi \varphi_k) - \bar{g}_m \right\}^2
\]

where \( m \) periodogram ordinates on either side of the pole are included in the regression. They also define (with slightly different notation) \( a_k = s_m^{-2} \{ g(2\pi \varphi_k) - \bar{g}_m \} \), and define the estimator to be:

\[
\hat{\delta}_{GPH} = \sum_{1 \leq |k| \leq m} a_k \log \{ I_0(\varphi_k + \hat{\xi}_S) \}.
\]  

(29)

Hidalgo and Soulier note that the asymptotic distribution of \( \hat{\xi}_S \) is not known, and that estimation of the pole is an open problem. In their simulation studies, 5000 replications of series length 256, 512 and 1024 are used, with \( \xi = 1/4 \), and thus there is grid alignment with the pole. They implement the GPH procedure, assuming \( \hat{\xi}_S \) is correct on the demodulated periodogram, excluding the contribution from the pole itself. Notice also that they chose \( m = N/4, m = N/8 \) and \( m = N/16 \), i.e \( m = O(N) \).

Having found the distribution of the periodogram at the pole in Lemma 2.1, we can adjust the GPH estimator using a similar profile likelihood approach. Assume that \( \xi \) is known, and consider \( \lambda_k \) such that

\[
\frac{I_0(\lambda_k) |\lambda_k|^{2\delta}}{f^\dagger(\lambda_k)} \sim \chi^2_2.
\]

As \( \xi \) is known we are on the grid, and \( B_{\lambda_k,N}(\xi, \delta) = 1 + O(k^{-1}) \). Thus we may ignore the contributions of the \( O(k^{-1}) \) term, and omit this from the procedure as the terms sum to a negligible contribution. Based on these values of \( k \), least squares is then used to estimate \( \delta \). This requires knowledge of \( \xi \); note that Hidalgo and Soulier estimate \( \xi \) as the Fourier frequency at which the periodogram is maximized, and therefore are restricted to an \( O(N^{-1}) \) grid. In contrast, for any \( \xi \), we demodulate the periodogram by \( \xi \) giving

\[
\log\{ I_0(\lambda_k(\xi)) \} - 2\delta \log(|\lambda_k(\xi)|) - \log(C) \sim \log(\chi^2_2),
\]
where $\lambda_k$ is calculated for the specified $\xi$, not necessarily on the Fourier grid; in practice it is straightforward to use a finer grid over which to do a systematic search. More generally we may allow $\xi$ continuously across the interval $(0,1/2)$, and to use numerical routines, and choose $\hat{\xi}$ to minimize the residual sum of squares after a least squares fit. Not that we can approximate the distribution of the log periodogram accordingly only if we demodulate, as otherwise the distribution is shifted in location by a constant depending on $\xi$. In this case the correlation between the periodogram at frequencies spaced $N^{-1}$ apart is non-negligible, thus necessitating usage of weighted least squares.

6 Discussion

This paper has illustrated the inherent problems with seasonally persistent processes and approximation based on the periodogram. We have demonstrated that realigning the grid of frequencies at which the periodogram is evaluated will simplify the distributional properties and enables us to specify a useful approximation to a likelihood function. Analysis of seasonal persistence will usually be based on frequency domain descriptions. For the usual Fourier grid, the distributional properties of the periodogram are generally not useful for SPPs, if given by previously derived theory Olhede et al. (2004). This paper shows how a small technical adjustment to the DFT to the DDFT alters the distributional properties substantially, making analytic investigation of the properties of the MLEs possible. The theoretical and practical utility of this adjustment is apparently under-appreciated in the literature. Potentially, even for short memory models (with bounded but highly peaked spectra) for moderate values of $N$, there will be an advantage in demodulation.

In this paper, attempts have been made to fill the gaps of current theory. To avoid the problems associated with the location of the singularity, Giraitis et al. (2001) constrained the maximization of the Whittle likelihood to a set of frequencies spaced $O(N^{-1})$ apart, where the likelihood performs well under the assumption that the true value of location of the singularity is constrained to this set. Their important result states that $\hat{\xi} - \xi^* = O_p(N^{-1})$. In fact, this is ensured (informally) by picking a Fourier frequency a distance $C/N$ from the singularity, hence not even necessarily the closest Fourier frequency. In contrast, we have studied the sensitivity of the likelihood of the periodogram to $O(N^{-1})$ perturbations in $\xi$, and found that the estimate of $\delta$ for large finite sample sizes is very sensitive to such variation, thus clarifying that despite the very rapid convergence of $\hat{\xi}$ to $\xi^*$ the potential misalignment of the Fourier grid with the unknown $\xi^*$ must be acknowledged. We also derive the large sample form of the distribution of $\hat{\delta}$ and $\hat{\xi}$, where the latter when re-normalised appropriately has a scaled Cauchy distribution.

Our results relate to frequency domain based analysis at some grid of frequencies. For processes with absolutely convergent autocovariance sequences for large samples, no gain is made by a particular choice of Fourier domain gridding, however for processes with seasonal persistence it is of fundamental importance to chose the correct grid alignment, even in large samples, as this simplifies the distributional results substantively.

Simulated examples show the superiority of our approach in finite sample situations. Furthermore, the methodology has the philosophical advantage of acknowledging the estimation of $\xi$. While other methods do well asymptotically for estimation of the long memory parameter, it is worth noting that for any fixed (maybe large) sample-size, improvements can usually
be found by explicitly considering the estimation of $\xi$ separately. The profile likelihood methods can be simply employed in the extended GPH estimator discussed by Hidalgo and Soulier (2004), extending the ideas to semi-parametric models, and facilitating a tractable analysis.

References

Abramovich, F., Benjamini, Y., Donoho, D. L., and Johnstone, I. M. (2006). Adapting to unknown sparsity by controlling the false discovery rate. *Ann. Statist.*, **34**, 584–653.

Andel, J. (1986). Long memory time series models. *Kybernetika*, **22**, 105–23.

Beran, J. (1994). *Statistics for Long-Memory Processes*. Chapman and Hall, London.

Beran, J. and Gosh, S. (2000). Estimation of the dominating frequency for stationary and nonstationary fractional autoregressive models. *J. Time Ser. Anal.*, **21**, 517–533.

Berger, J. O., Liseo, B., and Wolpert, R. L. (1999). Integrated likelihood methods for eliminating nuisance parameters. *Statistical Science*, **14**, 1–28.

Brillinger, D. (1975). *Time Series, Data Analysis and Theory*. New York, USA: Holt, Rhinehart and Winston.

Candès, E. and Tao, T. (2004). Near optimal signal recovery from random projections: Universal encoding strategies. Technical report, Caltech.

Chen, Z. G., Wu, K. H., and Dahlhaus, R. (2000). Hidden frequency estimation with data tapers. *J. Time Ser. Anal.*, **21**, 113–142.

Cheng, R. C. H. and Taylor, L. (1995). Non-regular maximum likelihood problems. *J. Roy. Statist. Soc. B*, **57**, 3–44.

Coifman, R. R. and Donoho, D. L. (1995). Translation-invariant denoising. In A. Antoniadis and G. Oppenheim (Eds.), *Wavelets and Statistics (Lecture Notes in Statistics, Volume 103)*, pp. 125–150. New York: USA: Springer-Verlag.

Contreras-Cristan, A., Gutierrez-Pena, E., and Walker, S. G. (2006). A note on Whittle’s likelihood. *Comm. Stats. – Sim. Comp.*, **35**, 857–875.

Coursol, J. and Dacunha-Castelle, R. (1982). Remarks on the approximation on the likelihood function of a stationary Gaussian process. *Theory Prob. Appl*, **27**, 162–67.

Donoho, D. L. (2006). Compressed sensing. *IEEE Transactions on Information Theory*, **52**, 1289–1306.

Dzhamparidze, K. O. and Yaglom, A. M. (1983). Spectrum parameter estimation in time series analysis. In P. Krishnaiah (Ed.), *Developments in Statistics*, Volume 4, pp. 1–181. New York: Academic Press.

Geweke, J. and Porter-Hudak, S. (1983). The estimation and application of long memory time series models. *J. Time Ser. Anal.*, **4**(4), 221–238.

Gil-Alana, L. A. (2002). Seasonal long memory in the aggregate output. *Econ. Lett.*, **74**(3), 333–7.
Giraitis, L., Hidalgo, J., and Robinson, P. M. (2001). Gaussian estimation of parametric spectral density with unknown pole. *Ann. Statist.*, **29**, 987–1023.

Gradsteyn, I. S., Ryzhik, I. M., and Jeffrey, A. (1994). *Table of Integrals, Series, and Products*. New York: Academic Press.

Gray, H. L., Zhang, N. F., and Woodward, W. (1989). On generalized fractional processes. *J. Time Ser. Anal.*, **10**, 233–57.

Grenander, U. and Szegö, G. (1984). *Toeplitz Forms and their Applications* (2 ed.). Chelsea Publishing Company, New York.

Hannan, E. J. (1973). Estimation of frequency. *J. Appl. Prob.*, **10**, 510–519.

Hannan, E. J. (1986). A law of the iterated logarithm for an estimate of frequency. *Stochastic Processes And Their Applications*, **22**, 103–109.

Hidalgo, J. (2005). Semiparametric estimation for stationary processes whose spectra have an unknown pole. *Ann. Statist. 33*(4), 1843–1889.

Hidalgo, J. and Soulier, P. (2004). Estimation of the location and exponent of the spectral singularity of a long memory process. *J. Time Ser. Anal. 25*(1), 55–81.

Hosoya, Y. (1974). *Estimation Problems on Stationary Time-Series Models*. Ph. D. thesis, Yale University.

Huerta, G. and West, M. (1999). Priors and component structure in autoregressive time series models. *J. Roy. Statist. Soc. B 61*(4), 881–99.

Hurvich, C. M. and Beltrao, K. (1993). Asymptotics for the low frequency ordinates of the periodogram of a long memory time series. *J. Time Ser. Anal.*, **14**, 455–472.

Hurvich, C. M., Deo, R., and Brodsky, J. (1998). The mean square error of Geweke and Porter-Hudak’s estimator of the memory parameter of a long-memory time series. *J. Time Ser. Anal.*, **19**, 19–46.

Isserlis, L. (1918). On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika*, **12**, 134–139.

Johnson, N. I. and Kotz, S. (1970). *Continuous Univariate Distributions, Vol. 2*. New York, USA: Wiley.

Lapsa, P. (1997). Determination of Gegenbauer-type random process models. *Sig. Proc.*, **63**, 73–90.

Olhede, S. C., McCoy, E. J., and Stephens, D. A. (2004). Large sample properties of the periodogram estimator of seasonally persistent processes. *Biometrika 91*(3), 613–628.

Ooms, M. (2001). A seasonal periodic long memory model for monthly river flows. *Env. Modell. Soft.*, **16**, 559–69.

Pei, S. C. and Ding, J. J. (2004). Generalized eigenvectors and fractionalization of offset dfts and dcts. *IEEE Trans. Sig. Proc.*, **52**, 2032–2046.

Porter-Hudak, S. (1990). An application of the seasonal fractionally differenced model to the monetary aggregates. *J. Amer. Statist. Assoc.*, **85**, 338–344.
Rathouz, P. J., Satten, G. A., and Carroll, R. J. (2002). Semiparametric inference in matched case-control studies with missing covariate data. *Biometrika* 89(4), 905–916.

Robins, J. M., Rotnitsky, A., and Zhao, L. P. (1994). Estimation of regression coefficients when some regressors are not always observed. *J. Amer. Statist. Assoc.*, 89, 846–866.

Robinson, P. M. (1995). Log-periodogram regression of time-series with long-range dependence. *Ann. Statist.*, 23, 1048–72.

Sweeting, T. J. (1980). Uniform asymptotic normality of the maximum likelihood estimator. *Ann. Statist.*, 8, 1375–81.

Sweeting, T. J. (1992). Asymptotic ancillarity and conditional inference for stochastic processes. *Ann. Statist.*, 20, 580–589.

Taniguchi, M. and Kakizawa, Y. (2000). *Asymptotic Theory of Statistical Inference for Time Series*. New York: Springer.

Thomson, D. J. (1990). Time-series analysis of holocene climate data. *Phil. Trans. Roy. Statist. Soc. Lond. A*, 330, 601–616.

v. Sachs, R. (1993). Detecting periodic components in stationary time series by an improved non-parametric procedure. In *Proceedings of the International Conference on Applications of Time Series in Astronomy and Meteorology*, pp. 115–118. University of Padova.

Walker, A. M. (1964). Asymptotic properties of least squares estimates of the spectrum of a stationary non-deterministic time series. *J. Austr. Math. Soc.*, 4, 363–384.

Walker, A. M. (1965). Some asymptotic results for the periodogram of a stationary time series. *J. Austr. Math. Soc.*, 5, 107–128.

Whitcher, B. (2004). Wavelet-based estimation for seasonal long-memory processes. *Technometrics* 46(2), 225–238.

Whittle, P. (1951). *Prediction and regulation by linear least-square methods*. Almquist & Wicksell, Uppsala, Sweden.
Table 1: Demodulated ML estimates of $\xi$

| N   | $\delta$ | bias ($\times 10^{-5}$) | sd($\times 10^{-3}$) | $\sigma^2 / \sigma^2_{\text{ML}}$ | 95% interval          |
|-----|----------|-------------------------|---------------------|---------------------------------|----------------------|
| 1024| 0.30     | 0.8125                  | 1.0218              | 0.7793                          | (0.1406, 0.1453)     |
| 1024| 0.40     | 1.2402                  | 0.5829              | 0.8262                          | (0.1416, 0.1442)     |
| 1024| 0.45     | 0.6699                  | 0.3574              | 0.6424                          | (0.1421, 0.1435)     |
| 2048| 0.30     | -4.4170                 | 0.5355              | 0.7461                          | (0.1414, 0.1439)     |
| 2048| 0.40     | 0.1699                  | 0.3011              | 0.6511                          | (0.1422, 0.1435)     |
| 2048| 0.45     | 0.7178                  | 0.1958              | 0.5286                          | (0.1425, 0.1433)     |
| 4096| 0.30     | 1.1035                  | 0.2770              | 0.9237                          | (0.1422, 0.1436)     |
| 4096| 0.40     | 0.4150                  | 0.1459              | 0.9518                          | (0.1426, 0.1432)     |
| 4096| 0.45     | -0.5254                 | 0.0941              | 1.0443                          | (0.1426, 0.1431)     |
| 8192| 0.30     | 0.9830                  | 0.2031              | 0.9516                          | (0.1424, 0.1433)     |
| 8192| 0.40     | 0.2851                  | 0.1450              | 0.9986                          | (0.1426, 0.1432)     |
| 8192| 0.45     | 0.1254                  | 0.0722              | 1.0283                          | (0.1426, 0.1431)     |

Table 2: Whittle estimates of $\xi$

| N   | $\delta$ | bias($\times 10^{-5}$) | sd($\times 10^{-3}$) | 95% interval          |
|-----|----------|-------------------------|---------------------|----------------------|
| 1024| 0.30     | -2.3159                 | 1.1575              | (0.1406,0.1455)      |
| 1024| 0.40     | -9.8354                 | 0.6413              | (0.1416,0.1445)      |
| 1024| 0.45     | -17.3549                | 0.4460              | (0.1416,0.1436)      |
| 2048| 0.30     | -1.7787                 | 0.6196              | (0.1411,0.1440)      |
| 2048| 0.40     | 3.5435                  | 0.3731              | (0.1421,0.1436)      |
| 2048| 0.45     | 7.6451                  | 0.2694              | (0.1426,0.1436)      |
| 4096| 0.30     | 0.2720                  | 0.2882              | (0.1423,0.1436)      |
| 4096| 0.40     | -1.7299                 | 0.1495              | (0.1426,0.1433)      |
| 4096| 0.45     | -3.4389                 | 0.0920              | (0.1426,0.1431)      |
| 8192| 0.30     | -0.1842                 | 0.2082              | (0.1424,0.1433)      |
| 8192| 0.40     | -0.2633                 | 0.1451              | (0.1426,0.1432)      |
| 8192| 0.45     | -1.1890                 | 0.0712              | (0.1426,0.1431)      |
Table 3: Demodulated ML estimates for $\delta$

| N   | $\delta$ | mean | bias ($\times 10^{-4}$) | sd ($\times 10^{-2}$) | $\sigma^2/I$/$\sigma^2_N$ | 95% interval                  |
|-----|----------|------|--------------------------|------------------------|-----------------------------|-------------------------------|
| 1024| 0.30     | 0.2990 | -9.6364                  | 1.9200                 | 0.8662                      | (0.2642, 0.3370)              |
| 1024| 0.40     | 0.3995 | -4.9333                  | 1.8672                 | 0.7150                      | (0.3600, 0.4334)              |
| 1024| 0.45     | 0.4492 | -8.4182                  | 1.6481                 | 0.5222                      | (0.4136, 0.4773)              |
| 2048| 0.30     | 0.2998 | -1.7333                  | 1.3862                 | 0.8898                      | (0.2709, 0.3267)              |
| 2048| 0.40     | 0.3999 | -0.9515                  | 1.3914                 | 0.6902                      | (0.3715, 0.4249)              |
| 2048| 0.45     | 0.4499 | -0.8000                  | 1.2573                 | 0.4348                      | (0.4246, 0.4736)              |
| 4096| 0.30     | 0.3001 | 1.3333                   | 0.9528                 | 0.9523                      | (0.2818, 0.3200)              |
| 4096| 0.40     | 0.4003 | 2.9515                   | 0.9475                 | 0.8964                      | (0.3812, 0.4188)              |
| 4096| 0.45     | 0.4505 | 5.0612                   | 0.9173                 | 0.7909                      | (0.4316, 0.4663)              |
| 8192| 0.30     | 0.3001 | 1.0872                   | 0.5028                 | 1.0068                      | (0.2912, 0.3124)              |
| 8192| 0.40     | 0.3997 | -2.5643                  | 0.4655                 | 0.9099                      | (0.3906, 0.4088)              |
| 8192| 0.45     | 0.4497 | -3.0083                  | 0.4056                 | 0.9313                      | (0.4414, 0.4576)              |

Table 4: Whittle estimates for $\delta$

| N   | $\delta$ | mean | bias ($\times 10^{-4}$) | sd($\times 10^{-2}$) | 95% interval                |
|-----|----------|------|--------------------------|------------------------|----------------------------|
| 1024| 0.30     | 0.3004 | 4.0909                   | 2.0630                 | (0.2606, 0.3436)            |
| 1024| 0.40     | 0.4076 | 76.1111                  | 2.2083                 | (0.3636, 0.4505)            |
| 1024| 0.45     | 0.4677 | 176.915                  | 2.2807                 | (0.4209, 0.4997)            |
| 2048| 0.30     | 0.3015 | 14.9899                  | 1.4696                 | (0.2717, 0.3293)            |
| 2048| 0.40     | 0.4077 | 76.8081                  | 1.6748                 | (0.3737, 0.4414)            |
| 2048| 0.45     | 0.4680 | 180.012                  | 1.9068                 | (0.4324, 0.4997)            |
| 4096| 0.30     | 0.3004 | 4.0204                   | 0.9764                 | (0.2816, 0.3204)            |
| 4096| 0.40     | 0.4018 | 17.5306                  | 1.0008                 | (0.3816, 0.4205)            |
| 4096| 0.45     | 0.4537 | 36.8571                  | 1.0285                 | (0.4337, 0.4745)            |
| 8192| 0.30     | 0.3003 | 3.3474                   | 0.5011                 | (0.2874, 0.3133)            |
| 8192| 0.40     | 0.4009 | 9.3895                   | 0.4880                 | (0.3900, 0.4131)            |
| 8192| 0.45     | 0.4522 | 21.9531                  | 0.4203                 | (0.4401, 0.4651)            |
Table 5: BIC values for U.S. Petroleum Data

| Method          | Model     | BIC   |
|-----------------|-----------|-------|
| Demodulated     | GARMA(0,0)| 71.246|
|                 | GARMA(1,0)| 32.211|
|                 | GARMA(0,1)| 27.153|
|                 | GARMA(1,1)| 30.921|
| Standard Whittle| GARMA(0,0)| 73.330|
|                 | GARMA(1,0)| 42.794|
|                 | GARMA(0,1)| 38.905|
|                 | GARMA(1,1)| 42.021|

Table 6: U.S. Petroleum data - GARMA(0,1) parameter estimates

|                | $\hat{\xi} \times 10^{-2}$ | $\hat{\delta}$ | $\hat{\theta}$ | $\hat{\sigma}^2$ |
|----------------|-----------------------------|----------------|----------------|-----------------|
| Estimate       | 1.918                       | 0.295          | -0.517         | 0.372           |
| Approx 95 % CI | (1.762,1.956)               | (0.221,0.384)  | (-0.675,-0.360)| (0.337,0.412)  |

Table 7: BIC values for Farallon data.

| Method          | Model     | $N = 444$ BIC | $N = 440$ BIC |
|-----------------|-----------|---------------|---------------|
| Demodulated     | GARMA(0,0)| 274.918       | 272.710       |
|                 | GARMA(1,0)| 257.080       | 254.612       |
|                 | GARMA(0,1)| 266.528       | 262.589       |
|                 | GARMA(1,1)| 262.474       | 259.754       |
| Standard        | GARMA(0,0)| 278.567       | 281.290       |
|                 | GARMA(1,0)| 243.009       | 260.035       |
|                 | GARMA(0,1)| 265.769       | 274.484       |
|                 | GARMA(1,1)| 246.742       | 265.129       |

Table 8: Farallon data - GARMA(1,0) parameter estimates.

|                | $\hat{\xi} \times 10^{-2}$ | $\hat{\delta}$ | $\hat{\phi}$ | $\hat{\sigma}^2$ |
|----------------|-----------------------------|----------------|--------------|-----------------|
| Demodulated N=444 Estimate | 8.358                       | 0.221          | 0.628        | 0.431           |
| 95 % CI        | (8.206,8.438)               | (0.157,0.314)  | (0.558,0.726)| (0.266,0.562)  |
| N=440 Estimate | 8.295                       | 0.234          | 0.644        | 0.401           |
| 95 % CI        | (8.290,8.391)               | (0.156,0.311)  | (0.558,0.728)| (0.252,0.556)  |
| Standard N=440 Estimate | 8.409                       | 0.156          | 0.629        | 0.520           |
| 95 % CI        | (8.222,8.497)               | (0.133,0.305)  | (0.562,0.736)| (0.286,0.594)  |

Table 9: SOI data - GARMA(0,0) parameter estimates

|                | $\hat{\xi} \times 10^{-2}$ | $\hat{\delta}$ | $\hat{\sigma}^2$ |
|----------------|-----------------------------|----------------|-----------------|
| Demodulated    | 2.366                       | 0.237          | 0.782           |
| Approx 95 % CI | (1.452,2.399)               | (0.215,0.254)  | (0.728,0.833)  |
| Standard       | 2.247                       | 0.235          | 0.778           |
| Approx 95 % CI | (1.402,2.381)               | (0.215,0.255)  | (0.730,0.833)  |
Figure 2: Simulated Data: The finite $N$ approximation to the distribution of $N(\hat{\xi} - \xi^*)$ for (a) $\delta = 0.40$ and (b) $\delta = 0.45$. The dotted and dash-dotted curves give the proposed finite large sample approximation for different values of $N$ whilst the solid line gives the Cauchy asymptotic form. It is clear from the plot that for large values of $\delta$ the distribution is quite slow converge to the Cauchy.
Figure 3: U.S. Petroleum Data: Raw data and spectral fits of GARMA(0,0) and GARMA(0,1) models. The GARMA(0,1) model yields a lower BIC value.
Figure 4: Farallon data: Raw data and spectral fits of of GARMA(0,0) and GARMA(1,0) models to the data set with $N = 440$ observations. The fit of the models using the Whittle likelihood are similar, but inferior in BIC terms.
Figure 5: Southern Oscillation Index data: Raw data and spectral fits of GARMA(0,0) model under standard Whittle and demodulation. For comparison, the estimator that takes the maximum periodogram ordinate as the estimate is also displayed.
A Appendix: Proofs

A.1 Expectation of the Periodogram at the Pole

Starting with the same method of calculation as in Olhede et al. (2004, p. 623) we find a large $N$ approximation to the expected value of the periodogram at $\xi$, after demodulation via $\xi$. We have

$$
E \left\{ \frac{I_0(\xi)}{N^{2\delta}f^1(\xi)} \right\} = \frac{1}{N^{2\delta}f^1(\xi)} \int_{-\frac{1}{2N}}^{\frac{1}{2N}} \frac{f^1(\lambda)}{|\lambda - \xi|^{2\delta}} \frac{\sin^2\{\pi N(\lambda - \xi)\}}{N \sin^2\{\pi(\lambda - \xi)\}} d\lambda + o(1)
$$

$$
= \frac{1}{f^1(\xi)} \int_{-\sqrt{N}}^{\sqrt{N}} \frac{f^1(\xi + u/N)}{|u|^{2\delta}} \frac{\sin^2(\pi u)}{N \sin^2(\pi u/N)} \frac{du}{N} + o(1)
$$

$$
= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin^2(\pi u)}{|u|^{2\delta+2}} du + o(1) = \frac{2}{\pi^2} \int_{0}^{\infty} \frac{\sin^2(\pi u)}{u^{2\delta+2}} du + o(1)
$$

$$
= -\Gamma(-1 - 2\delta) \cos\{\pi(1/2 + \delta)\} 2^{2\delta+1} \pi^{2\delta - 1} + o(1) = B_\xi(\xi, \delta) + o(1) \quad (A-1)
$$

This result implicitly defines $B_\xi(\xi, \delta)$. The final line follows from Gradshteyn et al. (1994, §3.823).

From equation (3),

$$
f^1(\xi) = \frac{\sigma^2}{\left\{ 4\pi |\sin(2\pi \xi)| \right\}^{2\delta}} = f^1_0,
$$

as $\lambda \to \xi$. Thus after demodulation, the expectation at the singularity is given by equation (A-1):

$$
E \left\{ I_0(\xi) \right\} = -N^{2\delta} 2\Gamma(-1 - 2\delta) \cos\{\pi\left(\frac{1}{2} + \delta\right)\} \sigma^2 |h(\xi; \theta)|^2 \pi \left\{ 2 |\sin(2\pi \xi)| \right\}^{2\delta} + o\left( N^{2\delta} \right).
$$

A.2 Bounding the Covariance Contributions

Under Gaussianity of the original time series, the DDFT will also be jointly proper complex Gaussian, thus we only need only to approximate for large $N$ the first and second order joint properties of these variables; the zeroth order properties are given in Appendix A.1, in conjunction with the results in Olhede et al. (2004).

We consider the discontinuities of the likelihood of the DDFT coefficients explicitly, and also the effects of ignoring the weak correlation between the Fourier coefficients near the pole (see Robinson (1995)). It is easier to deal with the demodulated sequence only, and so we shall only evaluate the frequency domain quantities at frequencies $\lambda_j$ from (14). Let $I_j = I_0(\lambda_j)$, and take $A_j = A_0(\lambda_j)$ and $B_j = B_0(\lambda_j)$. As we only consider demodulation by $A_D$ we in this section suppress the subscript $D$.

We note that with $i' = i/2$ for $i$ even and $i' = (i - 1)/2$ for $i$ odd, and similarly for $j$, then

$$
(S_{C_\lambda})_{i,j} = \begin{cases} 
\frac{1}{2} B_{\lambda_i, N}(\xi, \delta) f(\lambda_{i'}) & i = j \\
0 & (i-j) \mod 2 = 1 \\
V_{i'-j',N}(\frac{\xi - \sigma^2}{\lambda - \xi}, \delta) \sqrt{f(\lambda_{i'}) f(\lambda_{j'})} + O(1) \sqrt{f(\lambda_{i'}) f(\lambda_{j'})} & (i-j) \mod 2 = 0, i \neq j
\end{cases}
$$

Let $D_\lambda = \text{diag} \left( \sqrt{2 B_{\lambda_1, N}(\xi, \delta)} f(\lambda_{1}) \ldots \sqrt{2 B_{\lambda_{j}, N}(\xi, \delta)} f(\lambda_{j}) \right)$, and let $\tilde{\Sigma}_{C_\lambda} = D_\lambda^{-1} \Sigma_{C_\lambda} D_\lambda^{-1}$.

Twice the log-likelihood based on the sample $C_\lambda$ takes the form:

$$
2l_N^{(f)}(\xi, \delta, \theta, \sigma^2) = -N \log(2\pi) - \log |\Sigma_{C_\lambda}| - C_\lambda^\top \Sigma_{C_\lambda}^{-1} C_\lambda + o(N)
$$

$$
= -N \log(2\pi) - \log |D_\lambda| - C_\lambda^\top D_\lambda^{-2} C_\lambda + R(\theta, \xi, \delta, \sigma^2) + o(N)
$$

$$
= 2l(\xi, \delta, \theta, \sigma^2) + R(\theta, \xi, \delta, \sigma^2) + o(N),
$$

(A-2)
where

\[ R(\theta, \xi, \delta, \sigma^2) = -\log |\Sigma_{C\lambda}| - C^\top_{\lambda} D^{-1}_{\lambda}(\tilde{\Sigma}_{C\lambda}^{-1} - I_2) D^{-1}_{\lambda} C_{\lambda}. \]

Note that \( 2\ell(\xi, \delta, \theta, \sigma^2) = O(N) \). Also, \( \log |\Sigma_{C\lambda}| = \log O(1) \). The latter statement holds as the magnitude of this object can be bounded by considering the trace of the matrix \( \tilde{\Sigma}_{C\lambda} \), and the fact that for \( \log(N) < k < j \) the covariance terms can be bounded by \( k^{-1} \log(j) \) (cf Robinson (1995)). If, for \( \log(N) < k < j \), we consider the terms in the log-likelihood involving \( A_j A_k \) and \( B_j B_k \), then these are \( O(k^{-2} \log^2(j)) \). The higher order terms are obtained by inverting the covariance matrix, and the second derivative coming directly from the order of the contributions. We write \( \mathbb{E} \{ A_j A_k \} = \mathbb{E} \{ B_j B_k \} = T_{jk} k^{-2} \log^2(j), \) for \( T_{jk} = O(1) \) and let \( T = \max_j \max_k |T_{jk}|. \)

When summing the covariance terms we need to split up the terms indexed by negative and positive \( j \) into two sum. Consider one of the two sums, and sum the contributions over indices \( \log(N) < k < j < J = O(N) \), denoting the sum \( R_2 \). To formally derive this for contributions to the left and right of the pole, we can use twice this term, and the order of the contributions are the most important result. Then we note that using Minkowski inequality arguments:

\[
\frac{1}{N} |R_2| \leq \frac{1}{N} \sum_{j=\log(N)}^{J/N} \sum_{k=\log(N)}^{j} \frac{\mathcal{T} \log^2(j)}{k^2} = \frac{\mathcal{T}}{N^2} \sum_{j=\log(N)}^{J/N} \sum_{k=\log(N)}^{j} \frac{\log^2(j)}{k^2} N^2 \frac{\log^2(j)}{N} \frac{1}{k^2} dx + o(1)
\]

\[
= \frac{1}{N} \int_{\log(N)/N}^{J/N} \{ \log^2(x) + 2 \log(x) \log(N) + \log^2(N) \} \left\{ \frac{N}{\log(N)} - \frac{1}{x} \right\} dx + o(1)
\]

\[
= \frac{1}{\log(N)} \left[ \log^3(x) - 2 \log(x) + 1 \right] + 2 \{ \log^2(x) - \log(N) + \log^2(N) x \} \frac{J/N}{\log(N)/N} \frac{J/N}{\log(N)/N} = o(1)
\]

Note that \( A_j B_k \) is for any choice of \( j \) and \( k \), \( o(1) \). Thus as \( (2\ell(\xi, \delta, \theta, \sigma^2))/N \) is \( O(1) \) we can ignore the covariance contributions. Asymptotically, using the likelihood from equation (16) yields equivalent results to using the likelihood constructed from independent exponential random variables with non-equal variances, due to the weak correlation between the Fourier coefficients.

### A.3 Additional Notation

Define \( \psi = (\xi, \delta) \), and denote the true values of the parameters by \( \psi^* \). We suppress the dependence on other parameters, i.e. the dependence on \( \theta \) and \( \sigma^2 \). Consider first expansions of the log-likelihood defined by equation (16), \( \ell(\psi) \). Let

\[
\tilde{I} = \begin{pmatrix} \ell_{\xi, \xi}(\psi) & \ell_{\delta, \xi}(\psi) \\ \ell_{\delta, \xi}(\psi) & \ell_{\delta, \delta}(\psi) \end{pmatrix} = -F_N(\psi), \quad \mathbb{E} \{ F_N(\psi) \} = \mathcal{F}_N(\psi).
\]

denote the matrix of second partial derivatives. Furthermore, it is convenient to introduce additional random variables, required to study the properties of the score and the observed Fisher information. We denote by \( I_j \) and \( \tilde{I}_j \) the quantities \( I_0(\lambda_j) \) and \( \tilde{I}_0(\lambda_j) \) respectively, and by \( I_j(f, N), \tilde{I}_j(f, N) \) and \( \mathcal{I}_j(f, N) \) the standardized periodogram, derivative of the periodogram wrt the \( \xi \) and the second derivative of the periodogram wrt to the \( \xi \), all evaluated on the shifted grid. Then

\[
I_j(f, N) = \begin{cases} \frac{I_0(\lambda_j)}{f(\lambda_j)}, & j \neq 0, \\ \frac{\tilde{I}_0(\lambda_j)}{B_\xi(\xi, \delta) N^{2\delta} f_0^3}, & j = 0 \end{cases}; \quad \tilde{I}_j(f, N) = \begin{cases} \frac{\tilde{I}_0(\lambda_j)}{N f(\lambda_j)}, & j \neq 0, \\ \frac{\tilde{I}_0(\lambda_j)}{B_\xi(\xi, \delta) N^{2\delta + 2} f_0^3}, & j = 0 \end{cases}
\]
These quantities can be written in terms of the real and imaginary part of the DDFT and its derivatives, and so we define for \( j = J_1, \ldots, J_2 \):

\[
A_j = \frac{1}{\sqrt{N}} \sum_t X_t \cos \{2\pi (\xi + j/N)t\} \quad B_j = \frac{1}{\sqrt{N}} \sum_t X_t \sin \{2\pi (\xi + j/N)t\} \\
C_j = \frac{1}{\sqrt{N}} \sum_t tX_t \cos \{2\pi (\xi + j/N)t\} \quad D_j = \frac{1}{\sqrt{N}} \sum_t tX_t \sin \{2\pi (\xi + j/N)t\} \\
E_j = \frac{1}{\sqrt{N}} \sum_t t^2X_t \cos \{2\pi (\xi + j/N)t\} \quad F_j = \frac{1}{\sqrt{N}} \sum_t t^2X_t \sin \{2\pi (\xi + j/N)t\} \\
G_j = \frac{1}{\sqrt{N}} \sum_t t^3X_t \cos \{2\pi (\xi + j/N)t\} \quad H_j = \frac{1}{\sqrt{N}} \sum_t t^3X_t \sin \{2\pi (\xi + j/N)t\},
\]

for \( j = J_1, \ldots, J_2 \), where the sum over \( t \) ranges over \( t = 0, \ldots, N - 1 \). Also, let

\[
A_{j,N}^{(f,N)} = \frac{A_j}{\sqrt{f(\lambda_j)}} \quad B_{j,N}^{(f,N)} = \frac{B_j}{\sqrt{f(\lambda_j)}} \quad C_{j,N}^{(f,N)} = \frac{C_j}{N\sqrt{f(\lambda_j)}} \quad D_{j,N}^{(f,N)} = \frac{D_j}{N\sqrt{f(\lambda_j)}} \\
E_{j,N}^{(f,N)} = \frac{E_j}{N^2\sqrt{f(\lambda_j)}} \quad F_{j,N}^{(f,N)} = \frac{F_j}{N^2\sqrt{f(\lambda_j)}} \quad G_{j,N}^{(f,N)} = \frac{G_j}{N^3\sqrt{f(\lambda_j)}} \quad H_{j,N}^{(f,N)} = \frac{H_j}{N^3\sqrt{f(\lambda_j)}}.
\]

be the corresponding suitably standardized quantities. We shall also derive expressions for the expectation of \( I_j^{(f,N)}, \tilde{I}_j^{(f,N)} \), and \( \tilde{\tilde{I}}_j^{(f,N)} \) and these will be denoted \( B_{\lambda_{j,N}}, \tilde{B}_{\lambda_{j,N}}, \text{ and } \tilde{\tilde{B}}_{\lambda_{j,N}} \), respectively. Their variances take quite complicated forms, and we denote the theoretical constants that give their

\[
\text{their variance using the linear approximation to the variance of } \tilde{I}_1^{(f,N)}. \text{ More details follow later in the text when appropriate. Furthermore, the covariances of the } j\text{th and } k\text{th DDFT coefficients and their derivatives, are denoted by } V_{\lambda_j,\lambda_k,N}, \tilde{V}_{\lambda_j,\lambda_k,N} \text{ and } \tilde{W}_{\lambda_j,\lambda_k,N} \text{ respectively.}
\]

### A.4 Zeroth Order Properties

To acknowledge the dependence of the likelihood on the indices \( J_1 = -j_{0,N}(\xi) + 1 \) and \( J_2 = M - 1 - j_{0,N}(\xi) \), and the fact that these indices depend on \( \xi \), we thus in this section write explicitly \( \ell(\psi, J_1, J_2) \). Note that \( J_1 < 0 \). For any finite value of \( N \) this dependence introduces a discontinuity in the log-likelihood in the form of a jump when the demodulation makes the range to the left decrease by one, and the range on the right increase by one, or vice-versa. This fact is inconvenient for our calculations, as it makes the log-likelihood discontinuous and hence not differentiable. However, it transpires that the magnitude of the discontinuities are of an order that can be ignored for large sample sizes, as will be shown by the first proposition, so that subsequent calculations will be in terms of \( \ell(\psi) \), where \( J_1 \) and \( J_2 \) are treated as fixed with respect to \( \xi \) and of order \( O(N) \).

**Proposition 1** Consider the log-likelihood at \( \xi = \xi' + \Delta \), and assume that \( \xi' \neq 0, 1/2 \). Without loss of generality, assume that \( J_2(\xi' + \Delta) = J_2(\xi') + 1 \), so that \( J_1(\xi' + \Delta) = J_1(\xi') + 1 \). Let

\[
\Lambda_N = \ell(\xi' + \Delta, \delta, J_1 + 1, J_2 + 1) - \ell(\xi' + \Delta, \delta, J_1, J_2)
\]

be the magnitude of the discontinuity introduced by perturbing \( \xi' \). Then

\[
E[\Lambda_N] = O(1) \quad \text{var}[\Lambda_N] = O(1)
\]

and for every \( \epsilon > 0 \)

\[
P\left(N^{-1}|\Lambda_N| \geq \epsilon\right) \rightarrow 0 \quad \text{as } N \rightarrow \infty.
\]

**Proof:** (Sketch) It is straightforward to show that the discontinuities, \( \Lambda_N \), in the likelihood are random quantities with mean and variance that are \( O(1) \), so after standardization it follows from the weak law of large numbers that \( \Lambda_N \overset{P}{\rightarrow} 0 \) and the result follows. Full details are omitted. \( \blacksquare \)
This difference between the log-likelihoods at different values of $\xi$ that induce a change of the grid is $O(1)$. We can therefore apply arguments such as those developed by Coursol and Dacunha-Castelle (1982), to justify the usage of a form of the likelihood which ignores the the jump in the indices, when deriving the properties of using a form of the likelihood that does experience discontinuities as the value of $\xi$ alters. We may from the above calculations note that for large samples it is equivalent to use $\ell (\xi + \Delta, \delta, J_1, J_2)$ or $\ell (\xi + \Delta, \delta)$ in the analysis of the data; see also detailed discussion by Dzhaparidze and Yaglom (1983). For our weak convergence result, we standardize the log-likelihood by a factor of $N^{-1}$ as the log-likelihood terms are both $O(1)$.

The existence of the ML estimators is guaranteed as it is easy to show that the log-likelihood is everywhere bounded on the parameter space. The proof of consistency proceeds very similarly to Giraitis et al. (2001), who assume that the maximisation over $\xi$ is over a grid of frequencies, where is each grid-point is spaced $O(N^{-1})$ apart. This is a sensible choice as the estimation is most often carried out over the Fourier frequency grid via the DFT. Define

$$f_G (\lambda; \xi, \delta) = \sigma^{-2}_e f_G (\lambda; \xi, \delta),$$

and note that this constrains $\log (f_G (\lambda; \xi, \delta))$ to integrate to zero. Giraitis et al. (2001) show strong convergence of the estimated location of the singularity to the point on the grid closest to the true value of the pole, $\xi^*$, using the likelihood defined by equation (8).

The likelihood approximation defined in Theorem 1 cannot be treated identically to the function of $\xi$ defined in (8), as the Fourier transform in the former likelihood is calculated at a different set of frequencies whenever a different value of $\xi$ is picked. However to compare the magnitude of the log-likelihood at $\xi$ and at $\xi^*$ we need to compare likelihood based on different Fourier grids. This may seem problematic, but recall that the DDFT is a linear orthogonal transform, and so both likelihoods may be directly related to the likelihood of the time domain sample whatever grid is used. It is hence suitable to compare the magnitude of the likelihood of the DDFT at different grids. To be able to do this, we introduce some extra notation. Recall the demodulated grid $\lambda_j (\xi) = \xi + j/N$, $j = J_1 \ldots, J_2$. First, define $j_p = j_p (N, \xi, \xi^*) = \arg \min_{j \in \mathbb{Z}} | \xi^* - \lambda_j (\xi) |$. Thus at any value of $N$, when the true value of the pole is $\xi^*$, but the likelihood is evaluated at a grid evenly spaced around $\xi$, this is then the index of the frequency on the grid demodulated by $\xi$ that is closest to $\xi^*$. Thus $| j_p - N \xi | \leq 1/2$, and we define $j_p$ uniquely by taking the least of possible values is the pole is evenly spaced between two demodulated Fourier frequencies. Similarly define $\kappa_0 = \lambda_0 (\xi) = \xi$ and $\kappa_{j_0} = \lambda_{j_0} (\xi) = \xi + j_p/N$. Thus $\kappa_0$ is the demodulated Fourier frequency corresponding to $\xi$ whilst $\kappa_{j_0}$ is the demodulated Fourier frequency closest to $\xi^*$. Note that using the triangle inequality

$$N | \xi - \xi^* | \leq N | \xi - \kappa_{j_0} | + N | \kappa_{j_0} - \xi^* | \leq N | \xi - \kappa_{j_0} | + 1/2. \quad (A-4)$$

A.5 Existence and Consistency Proof

The existence of the ML estimators is guaranteed as it is easy to show that the log-likelihood is everywhere bounded on the parameter space. The proof of consistency proceeds very similarly to Giraitis et al. (2001), who assume that the maximisation over $\xi$ is over a grid of frequencies, where is each grid-point is spaced $O(N^{-1})$ apart. This is a sensible choice as the estimation is most often carried out over the Fourier frequency grid via the DFT. Define

$$\tilde{f}_G (\lambda; \delta, \xi) = \sigma^{-2}_e f_G (\lambda; \xi, \delta),$$

These results for the entire log likelihood at any fixed value of the parameters agree with standard likelihood theory. We shall see that the behaviour of the pole is such that subsequently no result for the estimation of the pole follows as standard likelihood theory would make us anticipate. However, with a suitable standardization, the properties of the MLEs and the likelihood are still tractable.
This allows us to consider the properties of the log-likelihood at the same grid explicitly, as \( P \left( N \left| \xi - \kappa_{j,0} \right| \geq K \right) \geq P \left( N \left| \xi - \xi^* \right| \geq K + 1/2 \right) \). If we establish the result for \( N \left| \xi - \kappa_{j,0} \right| \), we can redefine \( K \) to derive the same result for \( N \left| \xi - \xi^* \right| \). In the vein of Giraitis et al. (2001), to show consistency, we fix \( \epsilon \) and consider choosing \( K \) such that

\[
P \left( N \left| \delta - \delta^* \right|^2 \geq K \right) + P \left( N \left| \hat{\xi} - \xi^* \right| \geq (K + 1) \right) \leq \epsilon \tag{A-5}
\]

Let

\[
u_N(\psi) = N \left| \delta - \delta^* \right|^2 + N \left| \xi - \kappa_{j,0} \right| \psi(\kappa, \delta, \epsilon) \right).
\]

We may obtain a bound for (A-5), \( 2P \left( u_N(\psi) \geq K \right) \), by considering

\[
P \left( N \left| \delta - \delta^* \right|^2 \geq K \right) + P \left( N \left| \hat{\xi} - \kappa_{j,0} \right| \geq K \right) \leq \epsilon.
\tag{A-6}
\]

Define \( \Omega(K) \), a subset of the parameter space \( (\xi, \delta) \), defined for each fixed constant \( K \), by

\[
\Omega(K) = \{ \psi : \xi \in (0, 1/2), \delta \in (0, 1/2), u_N(\psi) \geq K \}.
\]

Let \( \tilde{\psi}^* = (\kappa_{j,0}, \delta^*) \). Analogous to Giraitis et al., we bound (A-6) by

\[
P \left( \sup_{\psi \in \Omega(K)} \left[ \frac{1}{N} \{ \ell(\psi^*) - \ell(\psi) \} \right] \leq 0 \right) = P \left( \sup_{\psi \in \Omega(K)} \left[ \frac{1}{N} \{ \ell(\psi^*) - \ell(\psi) \} / u_N(\psi) \right] \leq 0 \right)
\tag{A-7}
\]

Note that the constant \( B_\xi(\xi, \delta) \) (see equation (A-1)) does not explicitly depend on \( N \) or \( \xi \) (although the bias is computed at a fixed \( \xi \)). Also denote the Kronecker-delta by \( \delta_{ij} \) as usual. Consider first

\[
\ell(\tilde{\psi}^*) - \ell(\psi) \overset{(1)}{=} U_N + T_N - 1 + \frac{1}{f_G(\kappa_0; \delta, \kappa_{j,0})} \frac{I_0(\kappa_0)}{\sigma^2 f_G(\kappa_0; \delta, \kappa_{j,0})} + \frac{I_0(\kappa_{j,0})}{\sigma^2 f_G(\kappa_0; \delta, \kappa_{j,0})} - \frac{I_0(\kappa_{j,0})}{\sigma^2 f_G(\kappa_{j,0}; \delta, \kappa_{j,0})} + \frac{I_0(\kappa_{j,0})}{\sigma^2 f_G(\kappa_{j,0}; \delta, \kappa_{j,0})}
\]

\[
= U_N + T_N - 1 + \delta_{j,0} - \left\{ \frac{I_0(\kappa_0)}{\sigma^2 f_G(\kappa_0; \delta, \kappa_{j,0})} - \frac{I_0(\kappa_{j,0})}{\sigma^2 f_G(\kappa_{j,0}; \delta, \kappa_{j,0})} \right\}
\]

\[
= U_N + T_N + \delta_{j,0} - 1 - I_0(\kappa_0)W_1 - I_0(\kappa_{j,0})(W_2 - W_3),
\]

where

\[
W_1 = -\frac{1}{B_\xi(\xi, \delta)N^2 f^1(\kappa_0, \delta, \kappa_{j,0})}, \quad W_2 = \frac{1}{B_\xi(\xi, \delta)N^2 f^1(\kappa_{j,0}, \delta, \kappa_{j,0})}, \quad W_3 = -\frac{1}{\sigma^2 f_G(\kappa_{j,0}; \delta, \kappa_{j,0})}
\]

where in (1) we have defined \( U_N \) and \( T_N \) as in Giraitis et al. (2001). We can bound the probability in (A-7), in a similar fashion:

\[
P \left( \sup_{\psi \in \Omega(K)} \left| u_N^{-1}U_N \right| + \sup_{\psi \in \Omega(K)} \left| u_N^{-1} \left\{ 1 + I_0(\kappa_0)W_3 \right\} \right| \right.
\]

\[
+ \frac{1}{K} + \sup_{\psi \in \Omega(K)} \left| u_N^{-1}I_0(\kappa_0)W_1 \right| \left. + \sup_{\psi \in \Omega(K)} \left| u_N^{-1}I_0(\kappa_{j,0})W_2 \right| \geq \inf_{\psi \in \Omega(K)} \left| u_N^{-1}T_N \right| \right)
\]

Most terms are the same as in Giraitis et al. (2001), and bound in an identical fashion, apart from \( \left| u_N^{-1}I_0(\kappa_0)W_1 \right| \) and \( \left| u_N^{-1}I_0(\kappa_{j,0})W_2 \right| \). Clearly

\[
E \left\{ \sup_{\psi \in \Omega(K)} \left| u_N^{-1}I_0(\kappa_0)W_1 \right| \right\} = C_2 K^{-1} \quad \text{and} \quad E \left\{ \sup_{\psi \in \Omega(K)} \left| u_N^{-1}I_0(\kappa_{j,0})W_2 \right| \right\} = C_3 K^{-1}.
\]
Hence the result follows, see Theorem 3.1 in Giraitis et al. (2001). The proof follows Giraitis et al.’s and thus for a fixed grid with even spacing from $\xi$, shows that the maximiser in terms of $\xi$ of $\ell(\psi)$ becomes close to the point on the grid closest to $\xi^*$, which by the properties of the grid has to be at most $1/(2N)$ from $\xi^*$. This obviously is not the distance between the maximiser and $\xi^*$ but can be used to show convergence in probability. This strategy lets us avoid dealing with problems in the singularity of the likelihood, as well as the local periodic ripples.

A.6 First Order Properties of Derivatives of the Likelihood

**Proposition 2** For an SPP with parameters $\psi^*$, the expectation of the score evaluated at the $\psi = \psi^*$ is zero, that is $E\{\hat{\ell}(\psi^*)\} = o(N)$.

**Proof:** To deal with the statistical properties, we first note the expectation of the standardized periodogram as given in Olhede et al. (2004), and Lemma 1 in this proof, so that

$$E\{\hat{I}^{(f,N)}_j\} = B_{\lambda_j,N}(\xi,\delta) + o(1) = 1 + O(\log(j)/j) + o(1),$$

by results derived from Robinson (1995). For large $N$, the second order properties of the score is dominated by the $\hat{I}^{(f,N)}_j$ terms, that are distributed like quadratic forms of correlated normal random variables. We start by deriving the expectation of $\hat{I}_j$ in terms of the trigonometrical forms defined in equation (A-3). We find that:

$$E\{\hat{I}_j\} = 4\pi E\{B_jC_j - A_jD_j\} = -2\delta Nf(\lambda_j) \int_{-\infty}^{\infty} u^{-1} \left| \frac{u}{j} \right|^{-2\delta} \frac{\sin^2 \{\pi(j-u)\}}{(\pi(j-u))^2} du + o(N^{2\delta})$$

$$= Nf(\lambda_j) \hat{B}_{\lambda_j,N}(\xi,\delta) + o(N^{2\delta}), \quad (A-8)$$

defining $\hat{B}_{\lambda_j,N}(\xi,\delta)$. From this expression it is obvious that $\hat{B}_{\lambda_j,N}(\xi,\delta) = -\hat{B}_{-\lambda_j,N}(\xi,\delta)$. For large $j$ we have that $\hat{B}_{\lambda_j,N}(\xi,\delta) = O(j^{-1})$, where to derive this result, consider the decomposition

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{\epsilon} + \int_{\epsilon}^{j-\epsilon} + \int_{j-\epsilon}^{j+\epsilon} + \int_{j+\epsilon}^{\infty}.$$ 

As in Robinson (1995) we bound the individual contributions of these integrals. Using identical arguments, we find for $j = 0$,

$$E\{\hat{I}_0\} = 4\pi E\{B_0C_0 - A_0D_0\} = -2(2\delta) N^{1+2\delta} f(\xi) \int_{-\infty}^{\infty} \frac{u^{-1} \sin^2(\pi u)}{|u|^{2\delta} (\pi u)^2} du + o(N^{2\delta}) = o(N^{2\delta})$$

as the integral is zero. Thus when $\xi = \xi^*$,

$$E\{\hat{I}^{(f,N)}_0\} = 0 \quad E\{\hat{I}^{(f,N)}_j\} = \hat{B}_{\lambda_j,N}(\xi,\delta) + o(1),$$

where $\hat{B}_{\lambda_j,N}(\xi,\delta) = O(j^{-1})$, and furthermore note that $\hat{B}_{\lambda_j,N}(\xi,\delta) = -\hat{B}_{-\lambda_j,N}(\xi,\delta)$. Recall the definition of $\eta_j$ from equation (17) in Theorem 1; locally $\eta_j = \eta_{j-1}$. We also define for $j = J_1, \ldots, J_2$

$$R_j^{(1)} = \frac{\partial}{\partial \delta} \log(\eta_j) = -2 \log \left( \frac{N}{j} \right) - \left( \frac{B_{\xi,N}(\xi,\delta)}{B_{\xi,N}(\xi,\delta)} \right)^{\tau(j-0)} - \frac{f^\dagger_{j-\delta}}{f^\dagger_{j}}, \quad S_j^{(1)} = \frac{\partial}{\partial \xi} \log(\eta_j) = -\frac{f^\dagger_{j-\delta}}{f^\dagger_{j}}.$$
We may then write
\[
\ell_\xi(\psi) = \sum_{j=J_1}^{J_2} \left[ S_j^{(1)} \left( 1 - \mathcal{I}_j^{(f,N)} \right) - N \hat{\mathcal{I}}_j^{(f,N)} \right] + o(N),
\]
\[
\ell_\delta(\psi) = \sum_{j=J_1}^{J_2} R_j^{(1)} \left( 1 - \mathcal{I}_j^{(f,N)} \right) + o(N)
\]
\[
E\{\ell_\xi(\psi^*)\} = \sum_{j=J_1}^{J_2} \left( S_j^{(1)} \left( 1 - \mathcal{I}_j^{(f,N)} \right) - N \hat{\mathcal{I}}_j^{(f,N)} \right) + o(N) = o(N),
\]
\[
E\{\ell_\delta(\psi^*)\} = \sum_{j=J_1}^{J_2} R_j^{(1)} \left( 1 - \mathcal{I}_j^{(f,N)} \right) + o(N) = o(N).
\]
Thus \(E\{\ell_\xi(\psi^*)\}\) is \(o(N)\). This characterizes the first order properties of the score functions.

A.7 Second Order Properties of Derivatives of the Likelihood

The following result enables us to determine the properties of the Fisher information. We shall discover that the observed Fisher information does not converge to a point mass, and so far from standard theory ensues.

**Proposition 3** For an SPP with parameters \( \psi = \psi^* \), the Fisher information evaluated at \( \psi^* \) is given by
\[
E\{F_N(\psi^*)\} = \left( \frac{\mathcal{F}_{\xi,\xi}^{(N)}}{\mathcal{F}_{\xi,\delta}^{(N)}} \frac{\mathcal{F}_{\xi,\delta}^{(N)}}{\mathcal{F}_{\delta,\delta}^{(N)}} \right) = \left( \frac{\mathcal{F}_{\xi,\xi} N^2 + o(N^2)}{\mathcal{F}_{\xi,\delta} N + o(N)} \frac{\mathcal{F}_{\xi,\delta}^{(1)} N + o(N)}{\mathcal{F}_{\delta,\delta} N + o(N)} \right)
\]
where \( \mathcal{F}_{\xi,\xi} \) is the expected value of the negative of the second derivative of the log likelihood taken with respect to \( \xi \), and
\[
\mathcal{F}_{\xi,\xi} = \lim_{N \to \infty} \frac{\mathcal{F}_{\xi,\xi}^{(N)}}{N^2},
\]
with \( \mathcal{F}_{\delta,\delta} \) and \( \mathcal{F}_{\xi,\delta}^{(1)} \) similarly defined.

**Proof:** Consider the expectation of the second derivative of the periodogram; we must calculate
\[
E\{\tilde{I}_{0j}\} = 8\pi^2 E\{C_j^2 + D_j^2 - (A_j E_j + B_j F_j)\}.
\]
First, consider \( U_j = C_j - iD_j \), and the standardized version \( U_j^{(f,N)} = C_j^{(f,N)} - iD_j^{(f,N)} \). After some algebra, suitable standardization, and integrating by parts on \( (\xi - 1/\sqrt{N}, \xi + 1/\sqrt{N}) \), with change of variable to \( u \) where \( \xi_1 = \xi + u/N \), we have
\[
E\left\{ U_k^{(f,N)} U_j^{(f,N)\ast} \right\} = \frac{1}{4\pi^2} (-1)^{k-j} \int_{-\infty}^{\infty} \frac{jk}{u^2} \psi(j, k, u) du + o(1)
\]
where
\[
\psi(j, k, u) = \frac{\pi^2 \sin \{\pi(u-j)\} \sin \{\pi(u-k)\}}{\pi \sin \{\pi(u-j)\} \sin \{\pi(u-k)\}} - 2i\pi \frac{\sin \{\pi(u-k)\}}{\pi(u-k)} \frac{\sin \{\pi(u-j)\}}{\pi(u-j)} \frac{\pi \cos \{\pi(u-j)\}}{\pi(u-j)} - \frac{\sin \{\pi(u-j)\}}{\pi(u-j)} \frac{\sin \{\pi(u-k)\}}{\pi(u-k)} \left( \frac{\pi \cos \{\pi(u-k)\}}{\pi(u-k)} - \frac{\sin \{\pi(u-k)\}}{\pi(u-k)} \right).
\]
The calculations are very much in the spirit of Olhede et al. (2004). After some algebra, we have

\[
E \left\{ U_j^{(f, N)} U_k^{(f, N)} \right\} = \frac{1}{8\pi^2} K_{jk} + o(1) = \frac{1}{8\pi^2} \left\{ 2\pi^2 V_{\lambda_j, \lambda_k, N}(\xi, \delta) + W_{\lambda_j, \lambda_k, N}(\xi, \delta) \right\} + o(1),
\]

where \( V_{\lambda_j, \lambda_k, N}(\xi, \delta) \) is defined in Section 2 and we define

\[
W_{\lambda_j, \lambda_k, N}(\xi, \delta) = (-1)^{k-j} 2\int_{-\infty}^{\infty} \left| \frac{u}{j}\right|^{-\delta} \tilde{C}_{\lambda_j, \lambda_k}(u) \, du,
\]

where

\[
\tilde{C}_{\lambda_j, \lambda_k}(u) = \left[ \frac{\partial}{\partial u} \sin\{\pi(u-j)\} \right] \left[ \frac{\partial}{\partial u} \sin\{\pi(u-k)\} \right].
\]

Similarly, after standardization, and some algebra, we obtain

\[
E \left\{ U_j U_k \right\} = E \left\{ C_j C_k \right\} + E \left\{ D_j D_k \right\} + i \left\{ E \left\{ C_j D_k \right\} - E \left\{ C_k D_j \right\} \right\} = N^2 \sqrt{f(\lambda_j) f(\lambda_k)} \frac{1}{8\pi^2} K_{jk},
\]

\[
E \left\{ U_j U_k \right\} = E \left\{ C_j C_k \right\} - E \left\{ D_j D_k \right\} + i \left\{ E \left\{ C_j D_k \right\} + E \left\{ C_k D_j \right\} \right\} = o \left\{ N^2 \sqrt{f(\lambda_j) f(\lambda_k)} \right\}.
\]

Thus, for large \( N \),

\[
E \left\{ C_j C_k \right\} = E \left\{ D_j D_k \right\} = \frac{1}{16\pi^2} N^2 \sqrt{f(\lambda_j) f(\lambda_k)} \left( \Re(K_{jk}) + o(1) \right)
\]

\[
E \left\{ C_j D_k \right\} = -E \left\{ C_k D_j \right\} = \frac{1}{16\pi^2} N^2 \sqrt{f(\lambda_j) f(\lambda_k)} \left( \Im(K_{jk}) + o(1) \right)
\]

and \( E \left\{ C_j D_j \right\} = o(N^2) \). Using similar calculations, we have that

\[
E \left\{ -A_j E_j - B_j F_j \right\} = \frac{1}{8\pi^2} \left\{ -2\pi^2 B_{\lambda_j, N}(\xi, \delta) + \tilde{B}_{\lambda_j, N}(\xi, \delta) - \tilde{C}_{\lambda_j, N}(\xi, \delta) \right\},
\]

where

\[
B_{\lambda_j, N}(\xi, \delta) = \frac{(N-1)^2}{N} \int_{-1/2}^{1/2} f(u) \sin^2 \left\{ N\pi(\xi + j/N - u) \right\} \sin^2 \left\{ \pi(\xi + j/N - u) \right\} \, du
\]

\[
\tilde{B}_{\lambda_j, N}(\xi, \delta) = -\frac{2}{N} \int_{-1/2}^{1/2} f(u) \sin \left\{ N\pi(\xi + j/N - u) \right\} \frac{\partial}{\partial u} \sin \left\{ \pi(\xi + j/N - u) \right\} \, du
\]

\[
\tilde{C}_{\lambda_j, N}(\xi, \delta) = \frac{2}{N} \int_{-1/2}^{1/2} f(u) \left[ \frac{\partial}{\partial u} \sin \left\{ N\pi(\xi + j/N - u) \right\} \right]^2 \, du.
\]

**Case 1** \( [j \neq 0] \) For large \( N \) that looking at the components of this expectation, and standardizing via \( f(\lambda_j) N^2 \) that

\[
B_{\lambda_j, N}(\xi, \delta) = \frac{1}{f(\lambda_j) N^2} \left\{ (N-1)^2 \int_{-1/2}^{1/2} f(u) \sin^2 \left\{ N\pi(\xi + j/N - u) \right\} \sin^2 \left\{ \pi(\xi + j/N - u) \right\} \, du \right\} = B_{\lambda_j, N}(\xi, \delta) + o(1)
\]

This follows directly from OMS. Note that (see Robinson (1995)):

\[
B_{\lambda_j, N}(j, \delta) = 1 + O(\log(j)/j) + o(1),
\]

**Case 1** \( [j \neq 0] \) For large \( N \) that looking at the components of this expectation, and standardizing via \( f(\lambda_j) N^2 \) that

\[
\frac{1}{8\pi^2} \frac{\tilde{B}_{\lambda_j, N}(\xi, \delta)}{f(\lambda_j) N^2} = \frac{1}{f(\lambda_j) N^2} \frac{1}{2\pi^2} \int_{-1/2}^{1/2} \frac{\partial^2 f(u) \sin^2 \left\{ N\pi(\xi + j/N - u) \right\}}{\partial u^2} \sin^2 \left\{ \pi(\xi + j/N - u) \right\} \, du + o(1)
\]
and recalling that
\[
\frac{\partial^2 f(u)}{\partial u^2} = |\xi - u|^{-2\delta} \left\{ \frac{\partial^2 f^1}{\partial u^2} - 4 \frac{\partial f^1}{\partial u} \delta (\xi - u)^{-1} + 2 f^1 \delta (2\delta + 1) (\xi - u)^{-2} \right\}
\]
we have
\[
E \left\{ \frac{1}{8\pi^2} \tilde{B}_{\lambda_j, N}^{(N)}(\xi, \delta) \right\} = \frac{2\delta(2\delta + 1)}{2^4\pi^2} \int_{-\infty}^{\infty} \frac{1}{u^2} \left| \frac{u}{j} \right|^{-2\delta} \sin^2 \left\{ \frac{\pi(j-u)}{\pi(j)} \right\}^2 du + o(1) = \frac{\tilde{B}_{\lambda_j, N}(\xi, \delta)}{2^3\pi^2} + o(1).
\]
Note that the latter integral converges. Note that for \( j \) large
\[
\tilde{B}_{\lambda_j, N}(\xi, \delta) = 2\delta(2\delta + 1) \int_{-\infty}^{\infty} \frac{j^2}{(s+j)^2} \left| \frac{s+j}{j} \right|^{-2\delta} \sin^2(\frac{\pi s}{\pi}) \left( \frac{\pi s}{\pi} \right)^2 ds = O\left( j^{-2} \right),
\]
which decays (Gradshteyn et al., 1994, \S3.821(9)) with increasing \( j \). The derivation of this result resembles that of \( \tilde{B}_{\lambda_j, N}(\xi, \delta) \) but the integration over \( s = 0 \) needs direct appeal to mutatis mutandis of the calculations in Robinson (1995), after the term \( j^{-2} \) has been taken outside the integration. Similarly
\[
\frac{1}{8\pi^2} \tilde{C}_{\lambda_j, N}(\xi, \delta) = \int_{-\infty}^{\infty} \frac{u}{j} \left| \frac{u}{j} \right|^{-2\delta} \left\{ \sin \left\{ \frac{\pi(j-u)}{\pi(j)} \right\} - \cos \left\{ \frac{\pi(j-u)}{\pi(j)} \right\} \frac{\pi(j-u)}{4} \left\{ \frac{\pi(j-u)}{\pi(j)} \right\}^2 \right\} du + o(1)
\]
We note that \( \tilde{C}_{\lambda_j, N}(\xi, \delta) = W_{\lambda_j, N}(\xi, \delta) \). Note that for \( j \) large, mutatis mutandis results from Robinson (1995) bounding the Dirichlet kernel, for the expectation of the periodogram (up to terms \( o(1) \)):
\[
\tilde{C}_{\lambda_j, N}(\xi, \delta) = 2\pi \int_{-\infty}^{\infty} \frac{s+j}{j} \left| \frac{s+j}{j} \right|^{-2\delta} \left\{ \sin \left( \frac{s}{j} \right) - \cos \left( \frac{s}{j} \right) \frac{s}{j} \right\}^2 ds = 2\pi^2 \frac{3}{3} + O\left( \frac{\log(j)}{j} \right) = 2\pi^2 \frac{3}{3} + O\left( \frac{\log(j)}{j} \right),
\]
which tends to a constant for increasing \( j \). We then have that
\[
E \left\{ \tilde{I}_{0j} \right\} = \frac{8\pi^2}{f(\lambda_j) N^2} E \left\{ D_j^2 - A_j E_j - B_j F_j + C_j^2 \right\} = \tilde{B}_{\lambda_j, N}(\xi, \delta) + o(1).
\]
This gives us
\[
E \left\{ \tilde{I}_{0j} \right\} = f(\lambda_j) N^2 \tilde{B}_{\lambda_j, N}(\xi, \delta) + o(N^2).
\]
Case 2 \([j = 0]\) For large \( N \) considering the components of this expectation, and standardizing via \( f^1(\xi) N^{2\delta + 2} \) it follows that
\[
\frac{B_{\xi, N}^{(N)}(\xi, \delta)}{f^1(\xi) N^{2\delta + 2}} = \frac{1}{f^1(\xi) N^{2\delta + 2}} \int_{-1/2}^{1/2} f(u) \sin^2 \left\{ \frac{N\pi(\xi-u)}{\sin^2 \{\pi(\xi-u)\}} \right\} du = B_{\xi}(\xi, \delta) + o(1).
\]
This follows directly from Appendix A, including the definition of \( B_{\xi}(\xi, \delta) \). Again, using the change of variables \( \xi = \xi + u/N \), and a similar series of calculations,
\[
\frac{1}{8\pi^2} \tilde{B}_{\xi, N}^{(N)}(\xi, \delta) = \frac{1}{8\pi^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| u \right|^{-2\delta} (u)^{-1} \left\{ -\frac{\sin^2(\pi u)}{(\pi u)^3} + \frac{\sin(\pi u) \cos(\pi u)}{(\pi u)^2} \right\} du + o(1)
\]
\[
= \frac{1}{8\pi^2} \tilde{B}_{\xi, N}(\xi, \delta) B_{\xi, N}(\xi, \delta) + o(1).
\]
The $B_{\xi,N}(\xi, \delta)$ term has been added to simplify subsequent calculations. The integral converges (to see this Taylor expansion of the integrand at $u = 0$). Finally,

$$\frac{1}{8\pi^2} \int \frac{\mathcal{C}^{(N)}_{\xi,N}(\xi, \delta)}{f^1(\xi) N^{2+2\delta}} \, d\xi = \int_{-\infty}^{\infty} |u|^{-2\delta} \left( -\sin(\pi u) + \cos(\pi u) \frac{\pi}{4} u \right)^2 \frac{du}{4(\pi u)^4} + o(1) = \frac{1}{8\pi^2} \hat{C}_{\xi,N}(\xi, \delta) + o(1).$$

The latter integral also clearly converges. Thus

$$\frac{1}{f^1(\xi) N^{2+2\delta}} E\{ \hat{I}_{00} \} = \frac{8\pi^2}{f^1(\xi) N^{2+2\delta}} E\{ D_0^2 - A_0 E_0 - B_0 F_0 + C_0^2 \} = \tilde{B}_{\xi,N}(\xi, \delta) B_{\xi,N}(\xi, \delta) + o(1),$$

which yields

$$E\{ \hat{I}_{00} \} = \tilde{B}_{\xi,N}(\xi, \delta) B_{\xi,N}(\xi, \delta) f^1(\xi) N^{2+2\delta} + o(N^{2+2\delta}). \quad (A-13)$$

These results enable us to determine the properties of the Fisher Information.

### A.7.1 Asymptotic Properties of the Observed Information Matrix

We define

$$R_j^{(2)} = \frac{\partial^2}{\partial \delta^2} \{ \log(\eta_j) \}, \quad \tilde{R}_j^{(2)} = \frac{1}{\eta_j} \frac{\partial^2}{\partial \delta^2} (\eta_j)$$

so that

$$-\ell_{\delta, \delta} = -\frac{\partial^2 l}{\partial \delta^2} = \sum_j \left\{ R_j^{(2)} - \tilde{R}_j^{(2)} \eta_j I_j \right\}.$$ 

Additionally with

$$T_j = \frac{\partial}{\partial \delta} f_{j,\xi}^1 \quad \tilde{T}_0 = \frac{1}{\eta_j} \frac{\partial}{\partial \delta} \left( \frac{1}{B_{\xi,N}(\xi, \delta) N^{2+2\delta}} f_{0,\xi}^1 \right),$$

$$\tilde{T}_j = \frac{1}{\eta_j} \frac{\partial}{\partial \delta} \left( \frac{1}{|N/j|^{2\delta}} \frac{1}{f_j^1} \right)$$

then we find that

$$-\ell_{\xi, \delta} = -\frac{\partial^2 l}{\partial \xi^2} = \sum_j \left( T_j - \tilde{T}_j \eta_j I_j + \frac{\tilde{T}_j \eta_j I_j}{N} \right).$$

Finally with

$$S_j^{(2)} = \frac{\partial^2}{\partial \xi^2} \log(\eta_j), \quad \tilde{S}_j^{(2)} = \frac{1}{\eta_j} \frac{\partial^2}{\partial \xi^2} (\eta_j), \quad \bar{S}_j^{(2)} = -2 \frac{1}{\eta_j} \frac{\partial^2}{\partial \delta^2} (\eta_j),$$

we have that:

$$-\ell_{\xi, \xi} = -\frac{\partial^2 l}{\partial \xi^2} = \sum_j \left( S_j^{(2)} - \eta_j \tilde{S}_j^{(2)} I_j + N S_j^{(2)} \eta_j I_j + N^2 \eta_j \frac{\tilde{I}_{0j}}{N^2} \right).$$

Then it transpires

$$\mathcal{F}_{\xi,\xi}^{(N)} = \sum_j \left[ S_j^{(2)} - \tilde{S}_j^{(2)} + \tilde{S}(2) E\{ \hat{T}_j^{(f,N)} \} + N^2 E\{ \tilde{T}_j^{(f,N)} \} \right].$$

$$= \frac{f_{i,\xi}^{12}}{f_{0}^{12}} + \sum_{j \neq 0} f_{j,\xi}^{12} + N^2 \tilde{B}_{\xi,N}(\xi, \delta) \sum_{j \neq 0} \left\{ -2 N \tilde{B}_{\delta,N}(\xi, \delta) \frac{f_{j,\xi}^{12}}{f_j^1} + N^2 \tilde{B}_{\delta,N}(\xi, \delta) \right\} + o(N).$$
We can then note that for \( N \) large this sum will be dominated by:

\[
\mathcal{F}_{\xi,\delta}^{(N)} = N^2 \sum_{j=J_1}^{J_2} \sum_{j=J_1}^{J_2} B_{\lambda_j,N}(\xi, \delta) + o(N^2) = \mathcal{F}_{\xi,\delta} N^2 + o(N^2).
\]

Note that \( J_1/N = -\xi + o(1) \) and \( J_2/N = \frac{1}{2} - \xi + o(1) \).

Note that for large values of \( q \) where \( f^\lambda(\lambda) \) admits the representation \( f^\lambda(\lambda) = d_0(\psi) + d_1(\psi)(\lambda - \xi) + d_2(\psi)(\lambda - \xi)^2 \) + \( O((\lambda - \xi)^3) \). After some algebra

\[
\mathcal{F}_{\xi,\delta} = 2 \log(N) \frac{d_{0,\xi}}{d_0} + o\{\log(N)\}
+ \sum_{j=J_1,j\neq 0}^{J_2} \left\{ \frac{d_{0,\xi}}{d_0} + \left( \frac{d_{1,\xi}}{d_0} - \frac{d_{0,\xi} d_{1,\xi}}{d_0 d_2} \right) \left| \frac{j}{N} \right| + o\left( \frac{1}{N} \right) \right\}
+ \left\{ 2 \log \left| \frac{N}{j} \right| + \frac{d_{0,\delta}}{d_0} + \left( \frac{d_{1,\delta}}{d_0} - \frac{d_{0,\delta} d_{1,\delta}}{d_0 d_2} \right) \left| \frac{j}{N} \right| + o\left( \frac{1}{N} \right) \right\}
= 2 \log(N) q_0 + o\{\log(N)\} - 4Nq_0 \log|\xi| + N \frac{q_0 d_{0,\delta}}{2d_0} = \mathcal{F}_{\xi,\delta}^{(1)} N + o\{\log N\},
\]

where \( q_0 = \frac{d_{0,\xi}}{d_0} \) is a suitable constant. Finally tedious, but trivial calculations, based on the Taylor expansion of the function \( f^\lambda(\xi) \) yield

\[
\mathcal{F}_{\delta,\delta} = \frac{B_{\xi,N,\delta}^2(\xi, \delta)}{B_{\xi,N}(\xi, \delta)} + 2 \frac{B_{\xi,N,\delta}(\xi, \delta) f_{0,\delta}}{B_{\xi,N}(\xi, \delta) f_0} + \frac{f_{0,\delta}^2}{f_0^2} + 4 \log(N) \frac{B_{\xi,N,\delta}(\xi, \delta) f_{0,\delta}}{B_{\xi,N}(\xi, \delta) f_{0,\delta}} + 4 \log^2(N)
+ \sum_{j=J_1,j\neq 0}^{J_2} \left( \frac{f_{0,\delta}^2}{f_0^2} + 4 \log \left| \frac{N}{j} \right| \frac{f_{0,\delta}}{f_j^2} + 4 \log^2 \left| \frac{N}{j} \right| \right) + o(N)
= \frac{d_{0,\delta}}{2d_0} N + 2N \left\{ -4 \left( \log|\xi| - 1 \right) \xi \frac{d_{0,\delta}}{d_0} + 4 \left\{ \xi \left( \log^2|\xi| - 2 \log|\xi| + 2 \right) \right\} \right\} + o(N)
= \mathcal{F}_{\delta,\delta} N + o(N).
\]

This proves the large sample properties of the Fisher information matrix. ■

### A.8 Asymptotic Distributions

#### A.8.1 Distributions of Standardized Scores

**Score in \( \delta \):** To determine the properties of the MLEs we need to establish the joint distribution of \( \hat{\ell} \) and \( W_N \), defined in equations (18) and (19). We commence by discussing the first of these quantities. A usual central limit theorem will apply for \( I_\delta(\psi^*) \), and we already noted that \( E(I_\delta(\psi^*)) = 0 \). Furthermore:

\[
\text{var} \{I_\delta(\psi^*)\} = \sum_{j=J_1}^{J_2} B_{\lambda_j,N}(\xi, \delta) + o(N) \equiv \mathcal{F}_{\delta,\delta}^{(N)} + o(N),
\]

- 43
and note that \( F^{(N)}_{\delta, \delta} = F_{\delta, \delta} N + o(N) \). We may make the following note and definition:

\[
\ell_\delta (\psi^*) = \sqrt{N} Z_1 + o(\sqrt{N}), \quad Z_1 \sim N(0, F_{\delta, \delta})
\]

\[
k_{N,2} (\psi^*) = (F_{\delta, \delta} N)^{-1/2} \ell_\delta (\psi^*) \xrightarrow{\mathbb{L}} Z_2, \quad Z_2 \sim \mathcal{N}(0, 1) .
\]

(A-15)

Also we noted in the previous section that

\[ E \{ -\ell_{\delta, \delta} \} = F^{(N)}_{\delta, \delta} = O(N) , \quad \text{var} \{ -\ell_{\delta, \delta} \} = \sum_j \hat{\beta}_j^{(2j)} B_{\lambda_j, N}^2 (\xi) + o(N) = O(N). \]

Thus

\[ W_{N,22} = -\frac{\ell_{\delta, \delta}}{F_{\delta, \delta} N} \xrightarrow{P} 1 , \]

and so we may note that as \( k_{N,2} \) and \( W_{N,22} \) are asymptotically uncorrelated and Gaussian, we find that using Slutsky’s theorem

\[
\sqrt{F_{\delta, \delta} N} (\hat{\delta} - \delta^*) = k_{N,2} (\psi^*) [W_{N,22}]^{-1} \xrightarrow{\mathbb{L}} \mathcal{N}(0, 1), \quad \text{(A-16)}
\]

and from this result we can deduce Theorem 5. The value of \( F^{(N)}_{\delta, \delta} \) and \( F_{\delta, \delta} N \) are given by equations (A-14) and (A-15), respectively.

**Score in \( \xi \):** If the likelihood were sufficiently regular, then the arguments that we used to derive the distribution of \( \sqrt{F_{\delta, \delta} N} (\hat{\delta} - \delta^*) \) could be replicated for \( \xi \) instead of \( \delta \), and the large sample theory would be relative straightforward. However, this is not the case, and we find that for the parameter \( \xi \), the situation is more complicated. The first observation of interest is that we may note that the score is dominated by the derivative of the demodulated periodogram, i.e. \( \hat{T}^{(f,N)}_j \). In fact, with an appropriate standardization of the score we determine that

\[
\frac{1}{N^{3/2}} \ell_{\xi} (\psi^*) = \frac{1}{\sqrt{N}} \left[ \frac{1}{N} \sum_{j=J_3}^{J_2} \left\{ S_j^{(1)} \left( 1 - \eta_j \hat{T}^{(f,N)}_j \right) - \eta_j \hat{T}^{(f,N)}_j \right\} \right] = -\frac{1}{N^{1/2}} \sum_{j=J_3}^{J_2} \hat{T}^{(f,N)}_j + o(1)
\]

(A-17)

where the sum random variable converges in distribution. To be able to determine the large sample properties of this object, we thus need to derive the joint distribution of the random variables \( \{ \hat{T}^{(f,N)}_j \} \). \( \hat{T}^{(f,N)}_j \) is a quadratic form in correlated Gaussian random variables \( A^{(f,N)}_j \), \( B^{(f,N)}_j \), \( C^{(f,N)}_j \) and \( D^{(f,N)}_j \), that make up the standardized derivative of the periodogram. Their joint distribution can be determined from their covariance.

**Proposition 4**

\[
\text{cov} \left\{ \hat{T}^{(f,N)}_j, \hat{T}^{(f,N)}_k \right\} = \frac{5\pi^2}{2} V^2_{\lambda_j, \lambda_k, N} (\xi, \delta) + \frac{1}{2} V^2_{\lambda_j, \lambda_k, N} (\xi, \delta) + \frac{1}{4} V_{\lambda_j, \lambda_k, N} (\xi, \delta) W_{\lambda_j, \lambda_k, N} (\xi, \delta), \ j \neq k,
\]

(A-18)

up to order \( o(1) \), where \( V_{\lambda_j, \lambda_k, N} (\xi, \delta) \) is defined in section 2, \( W_{\lambda_j, \lambda_k, N} (\xi, \delta) \) is defined by eqn (A-9) and \( V_{\lambda_j, \lambda_k, N} (\xi, \delta) \) is given by

\[
V_{\lambda_j, \lambda_k, N} (\xi, \delta) = -2\delta \int_{-\infty}^{\infty} s^{-1} \left| \frac{s^2}{jk} \right|^{-\delta} \sin(\pi j - s) \sin(\pi k - s) \frac{\sin(\pi j - s)}{\pi^2 (j - s)(k - s)} ds.
\]

Note that if \( \log(N) < k < j \) then

\[
V_{\lambda_j, \lambda_k, N} (\xi, \delta) = O\left( \frac{\log(j)}{k} \right), \quad V_{\lambda_j, \lambda_k, N} (\xi, \delta) = O\left( \frac{\log(j)}{k^2} \right).
\]

44
PROOF: Define $V_j = (A_j, B_j, C_j, D_j)\top$ and $V_j^{(f.N)} = \left\{ A_j^{(f.N)} B_j^{(f.N)} C_j^{(f.N)} D_j^{(f.N)} \right\}$ and note that its components are correlated normal random variables. Note that $E\{V_j\} = 0$. We shall derive the final calculations needed to complete the entries of the covariance matrix of this object, namely $\text{cov}\{A_j, C_j\}$, $\text{cov}\{B_j, D_j\}$, $\text{cov}\{B_j, C_j\}$, and $\text{cov}\{A_j, D_j\}$. As above, integrating in the region $(\xi, \pm N^{-1/2})$ after change of variable to $u$ where $\xi = \xi^* + \frac{N}{N}$, we find that the suitably standardized random variates have expectation:

$$E\left\{A_j^{(f.N)} C_j^{(f.N)}\right\} = E\left\{B_j^{(f.N)} D_j^{(f.N)}\right\} = \frac{1}{4} \int_{-\infty}^{\infty} \left| j \right|^2 \frac{\sin^2 [\pi (j - u)]}{(\pi (j - u))^2} du + o(1) = B_{\lambda_j, N}(\xi, \delta)/4 + o(1)$$

and where the terms including the derivative of Féjer’s kernel cancel after a change of variable $u \rightarrow -u$. Also we can note from our calculations of the first differential that

$$E\{A_j D_j\} = -1/2N f(\lambda_j) \tilde{B}_{j,D,N}(\xi, \delta)/(4\pi) + o(N) f(\lambda_j),$$

as the cross-terms contribute terms of lesser order of magnitude for large $N$, and with a change of variable $\xi \rightarrow -\xi$ the terms multiplied by $N - 1$ cancel. We also note that

$$E\{B_j C_j\} = -E\{A_j D_j\} = 1/2N f(\lambda_j) \tilde{B}_{\lambda_j, N}(\xi, \delta)/(4\pi) + o(N) f(\lambda_j),$$

this result characterizing the second order structure of the derivative of the periodogram. This, in combination with OMS and previously derived results yields (up to terms $o(1)f(\lambda_j)$):

$$\text{var}(V_j) = f(\lambda_j) \begin{pmatrix}
\frac{1}{2}B_{\lambda_j, N}(\xi, \delta) & 0 & \frac{1}{2}N B_{\lambda_j, N}(\xi, \delta) & \frac{1}{4\pi} B_{\lambda_j, N}(\xi, \delta) \\
0 & \frac{1}{2}B_{\lambda_j, N}(\xi, \delta) & \frac{1}{4\pi} B_{\lambda_j, N}(\xi, \delta) & \frac{1}{4\pi} N B_{\lambda_j, N}(\xi, \delta) \\
-\frac{1}{4\pi} B_{j,D,N}(\xi, \delta) & \frac{1}{4\pi} B_{j,D,N}(\xi, \delta) & \frac{1}{16\pi^2} N^2 \Re(K_{j,j}) & 0 \\
-\frac{1}{4\pi} B_{j,D,N}(\xi, \delta) & -\frac{1}{4\pi} B_{j,D,N}(\xi, \delta) & 0 & \frac{1}{16\pi^2} N^2 \Re(K_{j,j})
\end{pmatrix}, \quad (A-19)$$

which we shall denote $\Omega_j$. Note that $K_{j,j} = 2\pi^2 B_{\lambda_j, N}(\xi, \delta) + \tilde{C}\lambda_j, N(\xi, \delta)$. We are also interested in the covariance between the terms $\check{T}_k^{(f.N)}$, and thus need to calculate

$$\frac{\text{cov}\{\check{T}_j^{(f.N)}, \check{T}_k^{(f.N)}\}}{(4\pi)^2} = \text{cov}\left\{B_j^{(f.N)} C_j^{(f.N)} - A_j^{(f.N)} D_j^{(f.N)}, B_k^{(f.N)} C_k^{(f.N)} - A_k^{(f.N)} D_k^{(f.N)} \right\}$$

$$= \text{cov}\left\{B_j^{(f.N)} C_j^{(f.N)}, B_k^{(f.N)} C_k^{(f.N)}\right\} - \text{cov}\left\{B_j^{(f.N)} C_j^{(f.N)}, A_k^{(f.N)} D_k^{(f.N)}\right\} - \text{cov}\left\{A_j^{(f.N)} D_j^{(f.N)}, B_k^{(f.N)} C_k^{(f.N)}\right\} + \text{cov}\left\{A_j^{(f.N)} D_j^{(f.N)}, A_k^{(f.N)} D_k^{(f.N)}\right\}$$

Using Isserlis’s theorem (see Isserlis (1918)) for zero-mean Gaussian variates we note that

$$E\{X_1 Y_1 X_2 Y_2\} = E\{X_1 X_2\} E\{Y_1 Y_2\} + E\{X_1 Y_2\} E\{X_2 Y_1\} + E\{X_1 X_2\} E\{Y_1 Y_2\}.$$
Hence we find that cov\( \{ \hat{T}^{(j,N)}_j, \hat{T}^{(j,N)}_k \} \) is equal to

\[
(4\pi)^2 \left[ E \left\{ B_j^{(j,N)} B_k^{(j,N)} \right\} E \left\{ C_j^{(j,N)} C_k^{(j,N)} \right\} + E \left\{ B_j^{(j,N)} C_k^{(j,N)} \right\} E \left\{ B_k^{(j,N)} C_j^{(j,N)} \right\} - E \left\{ B_j^{(j,N)} A_k^{(j,N)} \right\} E \left\{ C_j^{(j,N)} D_k^{(j,N)} \right\} - E \left\{ B_j^{(j,N)} D_k^{(j,N)} \right\} E \left\{ C_j^{(j,N)} A_k^{(j,N)} \right\} \right]
- E \left\{ A_j^{(j,N)} A_k^{(j,N)} \right\} E \left\{ D_j^{(j,N)} D_k^{(j,N)} \right\} + E \left\{ A_j^{(j,N)} D_k^{(j,N)} \right\} E \left\{ D_j^{(j,N)} A_k^{(j,N)} \right\}
\]

\[
= (4\pi)^2 \left\{ \frac{1}{2} V_{\lambda_j,\lambda_k,N}(\xi,\delta) \right\}^2 + (4\pi)^2 \left\{ -\frac{1}{8\pi} \hat{V}_{\lambda_j,\lambda_k,N}(\xi,\delta) \right\} \left\{ -\frac{1}{8\pi} \hat{V}_{\lambda_j,\lambda_k,N}(\xi,\delta) \right\} - o(1)
\]

\[
- (4\pi)^2 \left\{ \frac{1}{4} V_{\lambda_j,\lambda_k,N}(\xi,\delta) \right\}^2 - o(1) - (4\pi)^2 \left\{ \frac{1}{4} V_{\lambda_j,\lambda_k,N}(\xi,\delta) \right\}^2
\]

\[
+ \left\{ \frac{1}{2} \hat{V}_{\lambda_j,\lambda_k,N}(\xi,\delta) \right\} \left\{ \frac{1}{2} \hat{V}_{\lambda_j,\lambda_k,N}(\xi,\delta) \right\}
\]

\[
= \pi^2 \left\{ \frac{5}{4} V_{\lambda_j,\lambda_k,N}(\xi,\delta) + \frac{1}{2} \hat{V}_{\lambda_j,\lambda_k,N}(\xi,\delta) + \frac{1}{4} V_{\lambda_j,\lambda_k,N}(\xi,\delta) \hat{W}_{\lambda_j,\lambda_k,N}(\xi,\delta) + o(1) \right\}
= O \{ k^{-2} \log^2(j) \} + O \{ k^{-4} \log^2(j) \} + O \{ k^{-2} \log^2(j) \} + o(1).
\]

Note that the bound for \( \hat{V}_{\lambda_j,\lambda_k,N} \) follows by arguments, mutatis mutandis, Robinson (1995).

\[\text{(A-20)}\]

**A.8.2 Distribution of the Derivative of the Standardized Periodogram**

We now derive the distribution of \( \hat{T}^{(j,N)}_j \) to be able to determine the distribution of \( \sum_j \hat{T}^{(j,N)}_j \):

**Proposition 5**

\[
\hat{T}^{(j,N)}_j \sim \sum_{k=1}^{4} \gamma_k^{(j)} R_{i,j}^2 + o(1),
\]

where \( \gamma_k^{(j)} \) are the roots of equation

\[
\gamma^4 - \dot{B}_{j,D,N}(\xi,\delta) \gamma^3 + \left\{ \frac{3}{8} \dot{B}_{j,D,N}^2(\xi,\delta) - \frac{1}{4} \dot{B}_{j,N}(\xi,\delta) \dot{C}_{j,N}(\xi,\delta) \right\} \gamma^2 + \frac{1}{4} \ddot{B}_{j,N}(\xi,\delta)
\]

\[
\times \left\{ \frac{B_{j,N}(\xi,\delta) \dot{C}_{j,N}(\xi,\delta)}{2} - 2 \ddot{B}_{j,N}(\xi,\delta) \right\} \gamma
\]

\[
+ \frac{1}{26} B_{j,N}(\xi,\delta) \dot{C}_{j,N}(\xi,\delta) - \frac{1}{26} \ddot{B}_{j,N}(\xi,\delta) \dot{C}_{j,N}(\xi,\delta) \dot{C}_{j,N}(\xi,\delta) \dot{C}_{j,N}(\xi,\delta) \gamma + \frac{1}{26} \ddot{B}_{j,N}(\xi,\delta) = 0,
\]

and \( R_{i,j} \) are independent unit Gaussian variables across \( i \) for each fixed \( j \). This in turn implies

\[
E\{ \hat{T}^{(j,N)}_j \} = \sum_{k=1}^{4} \gamma_k^{(j)} + o(1) \quad \text{var}\{ \hat{T}^{(j,N)}_j \} = 2 \sum_{k=1}^{4} \gamma_k^{2(j)} + o(1).
\]

**Proof:** Firstly note that for a fourth order polynomial with roots \( \left\{ \gamma_k^{(j)} \right\} \) we find that

\[
\prod_{k=1}^{4} \left\{ \gamma - \gamma_k^{(j)} \right\} = \gamma^4 - \gamma^3 \sum_{k=1}^{4} \gamma_k^{(j)} + \gamma^2 \sum_{k < l} \gamma_k^{(j)} \gamma_l^{(j)} - \gamma \sum_{k < l < m} \gamma_k^{(j)} \gamma_l^{(j)} \gamma_m^{(j)} + \gamma \sum_{k < l < m < n} \gamma_k^{(j)} \gamma_l^{(j)} \gamma_m^{(j)} \gamma_n^{(j)}
\]

\[
= \gamma^4 + b_j \gamma^3 + c_j \gamma^2 + d_j \gamma + e_j.
\]

46
Also note that
\[ \sum_{k=1}^{4} \gamma_{k}^{2(j)} = \left( \sum_{k=1}^{4} \gamma_{k}^{(j)} \right)^{2} - 2 \left( \sum_{k} \sum_{l<k} \gamma_{k}^{(j)} \gamma_{l}^{(j)} \right). \]

Thus we find that
\[
E\{ \hat{I}_{j}^{(f,N)} \} = \sum_{k=1}^{4} \gamma_{k}^{(j)} = -b_{j} = \hat{B}_{j,N}(\xi, \delta) + o(1) \tag{A-22}
\]
\[
\text{var}\{ \hat{I}_{j}^{(f,N)} \} = 2 \sum_{k=1}^{4} \gamma_{k}^{2(j)} = 2b_{j}^{2} - 4c_{j} = 2\hat{B}_{j,N}^{2}(\xi, \delta) \quad \text{for } \frac{1}{4} B_{\lambda_{j},N}(\xi, \delta), \quad \frac{3}{8} \hat{B}_{\lambda_{j},N}^{2}(\xi, \delta) - \frac{1}{4} B_{\lambda_{j},N}(\xi, \delta)
\]
\[
\begin{align*}
\hat{C}_{\lambda_{j},N}(\xi, \delta) \bigg) + o(1) &= \frac{1}{2} \hat{B}_{\lambda_{j},N}^{2}(\xi, \delta) + B_{\lambda_{j},N}(\xi, \delta) \hat{C}_{\lambda_{j},N}(\xi, \delta) + o(1), \tag{A-23}
\end{align*}
\]
this giving the full first and second order structure of the standardized derivative, from the quadratic form. Equation (A-22) matches the previously developed results for the expectation of \( \dot{I}_{j}^{(f,N)} \). (A-23) gives a compact expression for the variance. Of some interest is now the difference in magnitude between this quantity and the \( j \)th contribution of \( \mathcal{J}_{\lambda_{j},N}^{(f,N)} / N^{2} \), but this is not sufficient to establish the large sample properties of the distribution, as \( \xi_{j} \), does not converge in probability to a constant if suitably standardized. Note that \( B_{\lambda_{j},N}(\xi, \delta) \) nearly takes the value unity for most \( j \), and for \( j \) small due to the integrand of \( \hat{B}_{\lambda_{j},N}(\xi, \delta) \) being odd near the origin, clearly \( \hat{C}_{\lambda_{j},N}(\xi, \delta) \sim > 0.5 \hat{B}_{\lambda_{j},N}^{2}(\xi, \delta) \). We therefore to derive a compact expression for the properties of \( \dot{I}_{j}^{(f,N)} \) to compare the magnitude of \( \hat{C}_{\lambda_{j},N}(\xi, \delta) \) and \( \hat{B}_{\lambda_{j},N}(\xi, \delta) \) to justify this argument.

To derive eqn (A-21) we use results given in (Johnson and Kotz, 1970, p. 149–188) on quadratic forms. Note that with
\[
T = \begin{pmatrix}
0 & 0 & 0 & -1/2 \\
0 & 0 & 1/2 & 0 \\
0 & 1/2 & 0 & 0 \\
-1/2 & 0 & 0 & 0
\end{pmatrix}
\]
we have
\[
\dot{J}_{j}^{(f,N)} = 4\pi V_{j}^{(f,N)T} TV_{j}^{(f,N)} . \tag{A-24}
\]
Firstly define
\[
\text{var}\{ V_{j}^{(f,N)} \} = \Omega_{j}^{(f,N)} + o(1) = L_{j}L_{j}^{T} + o(1), \tag{A-25}
\]
where \( \Omega_{j}^{(f,N)} \) is the normalized version of eqn (A-19) and where \( L_{j} \) is the lower triangular matrix given by, where, for notational purposes we take \( B_{\lambda_{j},N} = B_{\lambda_{j},N}(\xi, \delta) \) and \( \hat{C}_{\lambda_{j},N} = \hat{C}_{\lambda_{j},N}(\xi, \delta) \ :
\]
\[
L_{j} = \begin{pmatrix}
\sqrt{B_{\lambda_{j},N} / 2} & 0 & 0 & 0 \\
0 & \sqrt{B_{\lambda_{j},N} / 2} & 0 & 0 \\
\sqrt{B_{\lambda_{j},N} / 2} & \hat{B}_{\lambda_{j},N} / 4\pi \sqrt{2B_{\lambda_{j},N}} & 1 / 4\pi & 0 \\
- \hat{B}_{\lambda_{j},N} / 4\pi \sqrt{2B_{\lambda_{j},N}} & \sqrt{B_{\lambda_{j},N} / 2} & 0 & 1 / 4\pi
\end{pmatrix} .
\]
Note that
\[
V_{j}^{(f,N)} \sim \mathcal{N} \left( 0, \Omega_{j}^{(f,N)} \right) + o(1), \tag{A-26}
\]
and thus \( Z_{j} = L_{j}^{-1} V_{j} \sim \mathcal{N} \left( 0, I_{4} \right) \). The quadratic form is then given by (ignoring terms \( o(1) \)):
\[
(4\pi)^{-1} \dot{J}_{j}^{(f,N)} = V_{j}^{(f,N)T} TV_{j}^{(f,N)} = Z_{j}^{T} L_{j}^{T} L_{j} Z_{j} = Z_{j}^{T} M_{j} Z_{j},
\]

47
and thus the distribution of this object depends wholly on the eigenvalue of \( \mathcal{M}_j \). Note that

\[
\mathcal{M}_j = \mathcal{L}_j^T T \mathcal{L}_j = \begin{pmatrix}
\frac{B_{\lambda_j,N}(\xi,\delta)}{8\pi} & 0 & 0 \\
0 & \frac{\hat{B}_{\lambda_j,N}(\xi,\delta)}{8\pi} & \Gamma_j \\
0 & \Gamma_j & 0 \\
\Gamma_j & 0 & 0 \\
\end{pmatrix}
\]

where

\[
\Gamma_j = -\frac{1}{8\pi} \sqrt{\frac{B_{\lambda_j,N}(\xi,\delta)}{2}} \hat{C}_{\lambda_j,N}(\xi,\delta) - \hat{B}_{\lambda_j,N}^2(\xi,\delta).
\]

We are interested in \( 4\pi \mathcal{M}_j \) which has eigenvalues \( \gamma_j^{(j)} \) given as the solution of

\[
\gamma^4 - \hat{B}_{\lambda_j,N}(\xi,\delta)\gamma^3 + \left\{ \frac{3}{8} \hat{B}_{\lambda_j,N}^2(\xi,\delta) - \frac{1}{4} B_{\lambda_j,N}(\xi,\delta) \hat{C}_{\lambda_j,N}(\xi,\delta) \right\} \gamma^2 + \frac{\hat{B}_{\lambda_j,N}(\xi,\delta)}{4} \epsilon_1^{(j)} = 0.
\]

We then note from Johnson and Kotz (1970, p. 151) that if we define new variables \( R_j \) in terms of the orthogonal matrix of eigenvectors of \( \mathcal{M}_j \) and \( Z_j \), they will be \( R_j \sim \mathcal{N}(0, I_4) \), and

\[
\hat{\mathcal{I}}_j^{(f,N)} = 4\pi Z_j^T \mathcal{M}_j Z_j \sim 4 \sum_{k=1}^{4} \gamma_k^{(j)} R_{j,k}^2 + o(1), \quad (A-27)
\]

thus completing the proof of the proposition, and establishing the marginal distribution of \( \hat{\mathcal{I}}_j^{(f,N)} \). □

**Proposition 6** The standardized score function satisfies constraint:

\[
k_{N,1}(\psi^*) = \frac{1}{N^{3/2}} \ell_\xi = K_N + o(1)
\]

\[
K_N \sim \mathcal{N}(0, \sigma_N^2) \quad (A-28)
\]

\[
K_N \Rightarrow \mathcal{Z}_4 \sim \mathcal{N} \left( 0, \frac{\pi^2}{3} \right), \quad (A-29)
\]

where

\[
\sigma_N^2 = \frac{1}{N} \sum_{j=J_1}^{J_2} \left\{ \frac{5\pi^2}{2} \lambda_j,\lambda_k,N(\xi,\delta) \right\} + \frac{1}{N} \sum_{j \neq k} \left\{ \frac{5\pi^2}{2} \lambda_j,\lambda_k,N(\xi,\delta) + \frac{1}{4} \lambda_j,\lambda_k,N(\xi,\delta) \right\}
\]

\[
= \frac{1}{N} \sum_{j=J_1}^{J_2} \hat{C}_{\lambda_j,N}(\xi,\delta) + o(1) \rightarrow \frac{\pi^2}{3}.
\]

**Proof:** PART I (Determining the first and second order properties of \( k_{N,1}(\psi^*) \)): We note that

\[
k_{N,1}(\psi^*) = \frac{1}{N^{3/2}} \ell_\xi = -\frac{1}{\sqrt{\mathcal{N}}} \sum_j \hat{\mathcal{I}}_j^{(f,N)}(\lambda_j) + o(1) = Y_1,N(\psi^*) + o(1),
\]
from equation (A-17), the equation defining the random variable $Y_{1,N}(\psi^*)$. We then note from equations (A-8), (A-23) and (A-18) that:

$$E \{ Y_{1,N}(\psi^*) \} = o(1)$$

$$\text{var} \left\{ \frac{1}{\sqrt{N}} \hat{\beta}_{j,j}^{(f,N)} \right\} = \frac{1}{N} \left\{ \frac{1}{2} B_{\lambda_j,N}(\xi, \delta) + B_{\lambda_j,N}(\xi, \delta) C_{\lambda_j,N}(\xi, \delta) \right\} + o\{N^{-1}\}$$

$$\text{cov} \left\{ \frac{1}{\sqrt{N}} \hat{\beta}_{j,j}^{(f,N)}, \frac{1}{\sqrt{N}} \hat{\beta}_{k,k}^{(f,N)} \right\} = \frac{1}{N} \left\{ \frac{5\pi^2}{2} V_{\lambda_j,N}(\xi, \delta) + \frac{1}{2} V_{\lambda_k,N}(\xi, \delta) + \frac{1}{4} V_{\lambda_j,N}(\xi, \delta) W_{\lambda_k,N}(\xi, \delta) \right\} + o\{N^{-1}\}$$

Thus it follows that:

$$\text{var} \{ Y_{1,N}(\psi^*) \} = \frac{1}{N} \left\{ \sum_j \left( \frac{1}{2} B_{\lambda_j,N}(\xi, \delta) + B_{\lambda_j,N}(\xi, \delta) C_{\lambda_j,N}(\xi, \delta) \right) \right\}$$

$$+ \frac{1}{2N} \sum_{k \neq j} \left[ \hat{V}_{\lambda_k,N}(\xi, \delta) + \hat{V}_{\lambda_k,N}(\xi, \delta) \left\{ 5\pi^2 V_{\lambda_k,N}(\xi, \delta) + \frac{1}{2} W_{\lambda_k,N}(\xi, \delta) \right\} \right] + o(1).$$

Note that

$$\frac{1}{N} \sum_j \left\{ \frac{1}{2} B_{\lambda_j,N}(\xi, \delta) + B_{\lambda_j,N}(\xi, \delta) C_{\lambda_j,N}(\xi, \delta) \right\}$$

$$+ \frac{1}{2N} \sum_{k \neq j} \left[ \hat{V}_{\lambda_k,N}(\xi, \delta) + \hat{V}_{\lambda_k,N}(\xi, \delta) \left\{ 5\pi^2 V_{\lambda_k,N}(\xi, \delta) + \frac{1}{2} W_{\lambda_k,N}(\xi, \delta) \right\} \right]$$

equates to

$$\frac{1}{N} \sum_j C_{\lambda_j,N}(\xi, \delta) + o(1)$$

We arrived at this result using the order of $V_{\lambda_j,N}(\xi, \delta)$, $\hat{V}_{\lambda_j,N}(\xi, \delta)$ and $W_{\lambda_j,N}(\xi, \delta)$ noted in eqn (A-20) and that: $B_{\lambda_j,N}(\xi, \delta) = 1 + O\left\{ \frac{\log(j)}{\ell_{\xi}} \right\}$, $V_{\lambda_j,N}(\xi, \delta) = O\left\{ \frac{\log(j)}{\ell_{\xi}} \right\}$, if $\log(N) < k < j$, $B_{\lambda_j,N}(\xi, \delta) = O\left( \frac{1}{\ell_{\xi}} \right)$, $\hat{V}_{\lambda_j,N}(\xi, \delta) = O\left\{ \frac{\log(j)}{\ell_{\xi}} \right\}$, and $C_{\lambda_j,N}(\xi, \delta) = \frac{2\pi^2}{3} + O\left\{ \frac{\log(j)}{\ell_{\xi}} \right\}$.

We thus have that

$$\text{var} \{ Y_{1,N}(\psi^*) \} = \frac{1}{N} \text{var} \left\{ - \sum_j \hat{\beta}_{j,j}^{(f,N)} \right\} = \frac{1}{N} \left\{ \sum_j \hat{C}_{\lambda_j,N}(\xi, \delta) + o(N) \right\}$$

$$= \frac{2\pi^2}{3N} \times \frac{N}{2} + o(1) = \frac{\pi^2}{3} + o(1). \quad (A-30)$$

Thus to obtain an $O(1)$ random variate we must consider a standardization of $N^{-3/2}\ell_{\xi}$.

PART II (Determining the asymptotic law): In outline, we note:

$$E \left\{ \hat{\beta}_{j,j}^{(f,N)} \right\} = O\left( \frac{1}{j} \right) + o(1) \quad \text{var} \left\{ \hat{\beta}_{j,j}^{(f,N)} \right\} = \frac{2\pi^2}{3} + o\left( \frac{\log(j)}{j} \right) + o(1)$$

$$\text{cov} \left\{ \hat{\beta}_{j,j}^{(f,N)}, \hat{\beta}_{k,k}^{(f,N)} \right\} = O\left( \frac{\log^2(j)}{k^2} \right) + o(1), \quad \log(N) < k < j.$$ 

Now we wish to derive conditional expectations, to be able to derive the stated distributional result.
for $Z_4$. Define for $\log(N) < k < j < N/2$:

$$
\Omega^{(j,N)}_j = \begin{pmatrix}
\frac{1}{2} + O\left\{j^{-1} \log(j)\right\} & 0 & \frac{1}{2} + O\left\{j^{-1} \log(j)\right\} & O\left(j^{-1}\right) \\
0 & \frac{1}{2} + O\left\{j^{-1} \log(j)\right\} & O\left(j^{-1}\right) & \frac{1}{2} + O\left\{j^{-1} \log(j)\right\} \\
\frac{1}{2} + O\left\{j^{-1} \log(j)\right\} & O\left(j^{-1}\right) & \frac{1}{2} + O\left\{j^{-1} \log(j)\right\} & 0 \\
O\left(j^{-1}\right) & \frac{1}{2} + O\left(j^{-1} \log(j)\right) & 0 & \frac{1}{6} + O\left(j^{-1} \log(j)\right)
\end{pmatrix}
$$

$$
\Omega^{(f,N)}_{jk} = \begin{pmatrix}
O\left\{k^{-1} \log(j)\right\} & 0 & O\left\{k^{-1} \log(j)\right\} & O\left\{k^{-2} \log(j)\right\} \\
0 & O\left\{k^{-1} \log(j)\right\} & O\left\{k^{-2} \log(j)\right\} & O\left\{k^{-1} \log(j)\right\} \\
O\left\{k^{-1} \log(j)\right\} & O\left\{k^{-2} \log(j)\right\} & O\left\{k^{-1} \log(j)\right\} & 0 \\
O\left\{k^{-2} \log(j)\right\} & O\left\{k^{-1} \log(j)\right\} & 0 & O\left\{k^{-1} \log(j)\right\}
\end{pmatrix}
$$

Then the full covariance matrix of $\{V^{(f,N)}_j, V^{(f,N)}_k\}$ is given by $\Sigma_{jk} = \begin{pmatrix}
\Omega^{(f,N)}_j & \Omega^{(f,N)}_{jk} \\
\Omega^{(f,N)}_{kj} & \Omega^{(f,N)}_k
\end{pmatrix} + o(1)$, and if we define

$$
\Upsilon_{jk} = \left\{\Omega^{(f,N)}_k - \Omega^{(f,N)}_{kj}(\Omega^{(f,N)}_j)^{-1}\Omega^{(f,N)}_{jk}\right\}^{-1}
$$

then

$$
\Sigma^{-1}_{jk} = \begin{pmatrix}
\Xi_j & \Xi_{jk} \\
\Xi_{kj} & \Xi_k
\end{pmatrix}, \quad \Xi_j = (\Omega^{(f,N)}_j)^{-1} + (\Omega^{(f,N)}_j)^{-1}\Upsilon_{jk}\Omega^{(f,N)}_{kj}(\Omega^{(f,N)}_j)^{-1}.
$$

We may thus deduce that for $\log(N) < k < j < N/2$

$$
E \left\{\tilde{I}^{(f,N)}_j | \tilde{I}^{(f,N)}_k\right\} = O\left(j^{-1}\right) + O\left(j^{-1} k^{-2} \log^2(j)\right) + o(1) \quad \text{(A-31)}
$$

$$
\text{var} \left\{\tilde{I}^{(f,N)}_j | \tilde{I}^{(f,N)}_k\right\} = \frac{2\pi^2}{3} + O\left(j^{-1} \log(j)\right) + O\left(k^{-2} \log^2(j)\right) + o(1). \quad \text{(A-32)}
$$

These results are reminiscent of results obtained for the periodogram itself, thus using arguments in the vein of Hurvich et al. (1998); we argue that for $j$ sufficiently small the sum of the terms over $j$ are of negligible magnitude so that when they are standardized by $N^{-1/2}$, they decay.

In fact if we define $U_j = \tilde{I}^{(f,N)}_j$ and calculate the characteristic function of $\sum_{|j|=l} U_j$, denoted $\phi(t)$, with $l = O\{\log(N)\}$ then

$$
\log(\phi(t)) = \log\left\{E\left(e^{i \frac{t N}{\sqrt{N}} \sum_j U_j}\right)\right\} = \log\left[E\left\{e^{i \frac{t}{\sqrt{N}} \sum_{j=1}^{j-1} U_j} E\left(e^{i \frac{t N}{\sqrt{N}} U_j} | U_{j-1} \ldots\right)\right\}\right]
$$

$$
= \log\left[E\left\{e^{i \frac{t}{\sqrt{N}} \sum_{j=1}^{j-1} U_j} \left(1 + \frac{t}{\sqrt{N}} \left[O\left(\frac{1}{j}\right) + \sum_{k<j} O\left(\frac{\log^2(j)}{k^2}\right)\right]\right)\right\}\right] \quad \text{(A-33)}
$$

$$
- \frac{1}{2} \frac{t^2}{N} - \frac{2\pi^2}{3} + \sum_{k<j} O\left(\frac{\log^2(j)}{k^2}\right)
$$

$$
= \sum_j \log\left(1 + \frac{t}{\sqrt{N}} \left[O\left(\frac{1}{j}\right) + \sum_{k<j} O\left(\frac{\log^2(j)}{k^2}\right)\right]\right)
$$

$$
- \frac{1}{2} \frac{t^2}{N} - \frac{2\pi^2}{3} + \sum_{k<j} O\left(\frac{\log^2(j)}{k^2}\right)
$$

$$
= \frac{it}{\sqrt{N}} \left[O\{\log(N)\} + o(\sqrt{N})\right] - \frac{1}{2} \frac{t^2}{N} \sum_{k<j} O\left(\frac{\log^2(j)}{k^2}\right) + O\left(N^{-3/2}\right)
$$

$$
= \frac{it}{\sqrt{N}} \left[O\{\log(N)\} + o(\sqrt{N})\right] - \frac{1}{2} \frac{t^2}{N} \left\{(J_2 - J_1 + 1 - 2l) \frac{2\pi^2}{3} + o(N)\right\} + O\left(N^{-1/2}\right)
$$

$$
\rightarrow - \frac{1}{2} \frac{t^2}{3} (J_2 - J_1 + 1 - 2l) = - \frac{1}{2} \frac{\pi^2}{3} t^2.
$$
We want the characteristic function of \( N^{-1/2} \sum_{j=1}^{J_2} U_j \). We split this into two parts \( N^{-1/2} \sum_{|j|=l} \) and \( N^{-1/2} \sum_{|j|<l} U_j \), and note that the latter sum converges to the point zero. Thus the sum of the \( \hat{I}_j^{(f,N)} \) will converge to a Gaussian random variable with a zero mean and a variance of \( \frac{\pi}{3} \) or:

\[
k_{N,1}(\psi^*) = K_N + o(1) \sim \mathcal{N} \left( 0, \frac{1}{2} \frac{2\pi^2}{3} \right).
\] (A-34)

In fact, stopping the argument at equation (A-33) and replacing \( 2 \pi^2 / 3 \) by \( \tilde{\mathcal{C}}_{\lambda_j,N}(\xi,\delta) \), we may deduce that

\[
k_{N,1}(\psi^*) = K_N + o(1) \sim \mathcal{AN} \left( 0, \sum_{j=1}^{J_2} \tilde{\mathcal{C}}_{\lambda_j,N}(\xi,\delta) \right).
\] (A-35)

The approximation of eqn. (A-35) may serve as a better approximation to the distribution of \( k_{N,1}(\psi^*) \), rather than the distribution given in eqn (A-34), at moderate values of \( N \). □

A.8.3 Limit behaviour of the Fisher Information

Having established the large sample properties of \( k_{N,1} \) to be able to relate them back to a suitably standardized version of \( \xi \) we must also establish the large sample behaviour of \( [\hat{W}_N]_{11} \) near the true value of the pole. We shall use the same normalizations and local regions as defined by Sweeting (1992), when treating asymptotic ancillarity. Recall that \( \hat{B}_N \) was defined in equation (19), and refer to the notation specified in this section. To be able to do so define the \( B_N^{-1/2} \) neighbourhoods of \( \psi \) by \( \mathcal{N}_N(\psi^*,\epsilon) = \{ \psi \in \Omega: |B_N(\psi - \psi^*)| < \epsilon \} \).

Proposition 7 Define \( \phi^*_N = \{ \psi: \psi = \psi^* + B_N^{-1}\epsilon, \epsilon \in \mathbb{R}^2 \} \). For \( \psi \in \phi^*_N \),

\[
W_N(\psi) \xrightarrow{\xi} W, \quad \text{where} \quad W \sim \begin{pmatrix} W_{11} & 0 \\ 0 & 1 \end{pmatrix},
\] (A-36)

and \( W_{11} \sim N \left( 0, \frac{8\pi^4}{15} \right) \). Furthermore note the finite large sample approximation that for \( \psi = \psi^* \) we have

\[
[W_N(\psi^*)]_{11} = \hat{W}_{N,11} + o(1), \quad \hat{W}_{N,11} \sim \mathcal{N} \left( \sum_j \tilde{B}_{\lambda_j,N}(\xi^*,\delta^*), \frac{\sum_j \hat{\sigma}_j(\psi^*)}{\sqrt{N}} \right)
\] (A-37)

\[
\hat{W}_{N,11} \xrightarrow{\xi} Z_5, \quad Z_5 \sim \mathcal{N} \left( 0, \frac{8\pi^4}{15} \right),
\] (A-38)

where

\[
\hat{\sigma}_j(\psi^*) = \text{var} \left( \hat{I}_j^{(f,N)} \right), \quad \lim_{j \to N} \hat{\sigma}_j(\psi^*) = \frac{16\pi^4}{15}.
\]

Proof: Distribution of \([W_N(\psi^*)]_{11}\) .

We seek to establish the distribution of \( W_N(\psi) \), but intend to start by determining the distribution of \( W_N(\psi^*) \). Most of the entries in the matrix are easily established: we have already specified the distribution of \( [W_N(\psi)]_{12} \) and we may note that \( -\xi,\delta \) when standardized by \( N^{7/4} \), converges to zero (see Proposition 8). This implies that three of the entries of \( W_N(\psi) \) appropriately converge, and the
fourth element needs to be determined, as well as note of the correlation of the four elements need to be considered before the limit is taken. We consider $[W_N(\psi^*)]_{11}$ for large sample sizes. As

$$[W_N(\psi^*)]_{11} = -\frac{1}{N^{5/2}} \ell_{x,i} = \frac{1}{N^{5/2}} \sum_j \left\{ S_j^{(2)} - \eta_j \hat{S}_j^{(2)} I_j + N S_j^{(2)} \eta_j \hat{I}_j + N^2 \eta_j \hat{I}_{0j} \right\},$$

and we note that $I_j, \hat{I}_j$ and $\hat{I}_{0j}$ are quadratic forms in variables $\tilde{V}_j = [A_j, B_j, C_j, D_j, E_j, F_j]^\top$, that are more reasonably treated in terms of the standardized forms, we can note that:

$$[W_N(\psi^*)]_{11} = -\frac{1}{N^{5/2}} \ell_{x,i} (\psi^*) = \frac{1}{N^{1/2}} \sum_j \left\{ \frac{S_j^{(2)}}{N^2} - \frac{\tilde{S}_j^{(2)} \tilde{I}_j}{N^2} + \frac{\tilde{S}_j^{(2)} \tilde{I}_j \tilde{I}^{(f,N)}}{N} + \tilde{I}_j \tilde{I}^{(f,N)} \right\}$$

$$= -\frac{1}{\sqrt{N}} \sum_{j=J_1}^{J_2} \tilde{I}_j^{(f,N)} + o(1) = Y_{2,N} (\psi^*) + o(1). \quad (A-39)$$

As for large $j$, we note that $\tilde{B}_{\lambda_j,N}(\xi, \delta) = O(j^{-2})$, and so we find that:

$$\lim_{N \to \infty} \sum_j \tilde{B}_{\lambda_j,N}(\xi, \delta) \to C_{10} = O(1),$$

and thus,

$$E \{ Y_{2,N} (\psi^*) \} = -N^{-1/2} E \left\{ \sum_{j=J_1}^{J_2} \tilde{I}_j^{(f,N)} \right\} = O(N^{-1/2}). \quad (A-40)$$

We then consider the variance of $Y_{2,N} (\psi^*)$, to determine the properties of this random variable. To find the full properties of $Y_{2,N} (\psi^*)$ we note that it is a quadratic form in the full set $\{ V_j \}$, and replicate our previous treatment of $\{ V_j \}$. It transpires, that the important properties to establish, for a heuristic argument, is the mean and variance of the random variates $\tilde{I}_j^{(f,N)}$. The variates are correlated across $j$, but given the weak correlation, this need not be accounted for, just like in the previous arguments, the combined correlation once suitably renormalized converges to a negligible contribution. After some very lengthy calculations that are not replicated here, we obtain that the variance of $\tilde{I}_j^{(f,N)}$ is given by:

$$\sigma_j^2 = \text{var} \left\{ \tilde{I}_j^{(f,N)} \right\} \quad (A-41)$$

$$= 2^6 \pi^4 \text{var} \left\{ D_j^{(f,N)^2} - A_j^{(f,N)} E_j^{(f,N)} - B_j^{(f,N)} F_j^{(f,N)} + C_j^{(f,N)^2} \right\}$$

$$= 2^6 \pi^4 \left[ \text{var} \left\{ D_j^{(f,N)^2} \right\} + \text{var} \left\{ A_j^{(f,N)} E_j^{(f,N)} \right\} + \text{var} \left\{ B_j^{(f,N)} F_j^{(f,N)} \right\} + \text{var} \left\{ C_j^{(f,N)^2} \right\} \right.$$

$$- 2 \text{cov} \left\{ D_j^{(f,N)^2}, A_j^{(f,N)} E_j^{(f,N)} \right\} - 2 \text{cov} \left\{ D_j^{(f,N)^2}, B_j^{(f,N)} F_j^{(f,N)} \right\} + 2 \text{cov} \left\{ D_j^{(f,N)^2}, C_j^{(f,N)^2} \right\}$$

$$+ 2 \text{cov} \left\{ A_j^{(f,N)} E_j^{(f,N)}, B_j^{(f,N)} F_j^{(f,N)} \right\} - 2 \text{cov} \left\{ A_j^{(f,N)} E_j^{(f,N)}, C_j^{(f,N)^2} \right\} - 2 \text{cov} \left\{ B_j^{(f,N)} F_j^{(f,N)}, C_j^{(f,N)^2} \right\}. \quad \text{Each of these terms is given by}

$$\text{var} \left\{ D_j^{(f,N)^2} \right\} = \text{var} \left\{ C_j^{(f,N)^2} \right\} = \frac{1}{27 \pi^4} \left( 2 \pi^2 B_{\lambda_j,N}(\xi, \delta) + \tilde{C}_{\lambda_j,N}(\xi, \delta) \right)^2 + o(1)$$

$$= \frac{1}{27 \pi^4} (2 \pi^2 + 2 \pi^2 / 3)^2 + o(1) = \frac{1}{18} + o(1).$$

Also

$$\text{var} \left\{ A_j^{(f,N)} E_j^{(f,N)} \right\} = \text{var} \left\{ B_j^{(f,N)} F_j^{(f,N)} \right\} = \frac{1}{4} B_{\lambda_j,N}(\xi, \delta) \left\{ \frac{1}{16} B_{\lambda_j,N}(\xi, \delta) - \frac{1}{16 \pi^2} \tilde{B}_{\lambda_j,N}(\xi, \delta) + \frac{3}{16 \pi^2} \tilde{C}_{\lambda_j,N}(\xi, \delta) + \frac{1}{27 \pi^4} \left( -2 \pi^2 B_{\lambda_j,N} - \tilde{C}_{\lambda_j,N}(\xi, \delta) + \tilde{B}_{\lambda_j,N}(\xi, \delta) \right)^2 + o(1) \right.$$

$$= \frac{1}{20} + \frac{1}{36} + o(1),$$

52
where \( \tilde{C}_{\lambda_j, N}(\xi, \delta) \) is given by

\[
\tilde{C}_{\lambda_j, N}(\xi, \delta) = \frac{1}{16} \left\{ \begin{array}{ll}
B_{\lambda_j, N}(\xi, \delta) - 2 \int_{-\infty}^{\infty} |u|^{25} \frac{1}{\pi \frac{\sin((u-j)\pi)}{u}} \psi_2(j, u) \, du + \int_{-\infty}^{\infty} |u|^{25} \frac{1}{\pi \frac{1}{\pi \frac{\sin((u-j)\pi)}{u}}} \psi_2(0, u) \, du & \text{if } j \neq 0 \\
B_{0, \lambda_j, N}(\xi, \delta) - 2 \int_{-\infty}^{\infty} |u|^{25} \frac{1}{\pi \frac{\sin((u-j)\pi)}{u}} \psi_2(0, u) \, du + \int_{-\infty}^{\infty} |u|^{25} \frac{1}{\pi \frac{1}{\pi \frac{\sin((u-j)\pi)}{u}}} \psi_2(0, u) \, du & \text{if } j = 0
\end{array} \right.
\]

and \( \psi_2(j, u) = 2 \left[ -\cos\{\pi(u-j)/(\pi(u-j))^2 + \sin\{\pi(u-j)/(\pi(u-j))^3\} \right. \). Finally we note that

\[
\text{cov} \left\{ D_j^{(f, N)^2}, A_j^{(f, N)} E_j^{(f, N)} \right\} = \text{cov} \left\{ C_j^{(f, N)^2}, B_j^{(f, N)} F_j^{(f, N)} \right\} = -\frac{B_{\lambda_j, N}}{2^{5\pi^2}} \left\{ \tilde{B}_{\lambda_j, N} + \tilde{C}_{\lambda_j, N}(\xi, \delta) \right\},
\]

plus \( o(1) \) terms where

\[
\tilde{C}_{\lambda_j, N}(\xi, \delta) = \int_{-\infty}^{\infty} \left[ \frac{s}{j} \right]^{-25} s_{-1} \frac{|\sin(\pi(j-s)) - \cos(\pi(j-s)) \pi(j-s)^2}{2 \pi(j-s)^4} \right] ds.
\]

Also

\[
\text{cov} \left\{ C_j^{(f, N)^2}, A_j^{(f, N)} E_j^{(f, N)} \right\} = \text{cov} \left\{ D_j^{(f, N)^2}, B_j^{(f, N)} F_j^{(f, N)} \right\} = \frac{B_{\lambda_j, N}}{2^{6\pi^2}} \left\{ 2 \pi^2 B_{\lambda_j, N} - \tilde{B}_{\lambda_j, N}(\xi, \delta) + \tilde{C}_{\lambda_j, N}(\xi, \delta) \right\} + o(1) = \frac{1}{16} + o(1)
\]

\[
\text{cov} \left\{ D_j^{(f, N)^2}, D_j^{(f, N)^2} \right\} = o(1)
\]

\[
\text{cov} \left\{ A_j^{(f, N)} E_j^{(f, N)}, B_j^{(f, N)} F_j^{(f, N)} \right\} = \delta \tilde{B}_{\lambda_j, N}(\xi, \delta) + o(1) = O(j^{-1}) + o(1).
\]

Combining these results we find that as \( j \rightarrow N \), and \( N \rightarrow \infty \), \( \delta_j^2 \rightarrow 16 \pi^4/15 \approx 104 \). Thus for increasing \( j \) the variance of \( f^{(f, N)}(\psi^*) \) tends to a constant, again the covariance terms will behave like the covariance terms in the score, and the mean is of negligible magnitude. We are thus adding many identically distributed variates with order one variance, and the same weak dependence as before. We can yet again adapt the arguments of Hurvich et al. (1998). The argument will necessarily become very complicated, as we now need to consider a quadratic form in twelve Gaussian correlated variates, and there is no real point in giving the exact details of the argument.

The distribution may for non-negligible values of \( \delta \) be slow to attain, and so for large but more moderate \( N \) we propose to use:

\[
Y_{2, N}(\psi^*) = \tilde{W}_{N, 11} + o(1), \quad \tilde{W}_{N, 11} \sim \mathcal{N} \left( \frac{1}{\sqrt{N}} \sum_j \tilde{B}_{\lambda_j, N}(\xi, \delta), \frac{1}{N} \sum_j \delta_j^2 \right). \tag{A-43}
\]

For large \( N \) we find \( \sqrt{N} \sum_j \tilde{B}_{\lambda_j, N}(\xi, \delta) = o(1) \) whilst

\[
\frac{1}{N} \sum_j \delta_j^2 = \frac{1}{N} \frac{16 \pi^4 N}{15} + o(1) = \frac{8 \pi^4}{15} + o(1),
\]

and so we may note that \( \tilde{W}_{N, 11} \overset{\mathcal{D}}{\rightarrow} Z_5 \), \( Z_5 \sim \mathcal{N}(0, 8 \pi^4/15) \). We furthermore note that as the variance increases linearly with \(|J_2 - J_1|\) the distribution of the second derivative at values of \( j \) near the pole eventually becomes negligible in influence in the random variable \( [W_N(\psi)]_{11} \), and thus the distributional results will also hold for \( [W_N(\psi)]_{11} \) when \( \psi \in \phi_N^\delta \), or

\[
[W_N(\psi)]_{11} \leq [W_N(\psi^*)]_{11} + o(1) \leq Z_5 + o(1). \tag{A-44}
\]

This establishes the distribution of the standardized observed Fisher information of the likelihood. ■

53
However, before we may combine these results to note the distribution of \( N \hat{\xi} \) we must consider the dependence between \( N^{-1} \ell_{\xi}(\psi^*) \) and \( N^{-5/2} \ell_{\xi,i}(\psi) \), which, based on the argument of the distributional equivalence of \( \ell_{\xi,i}(\psi) \), \( \ell_{\xi,i}(\psi^*) \), and the asymptotic Gaussianity of the variables corresponds to bounding the covariance of \( N^{-1} \ell_{\xi}(\psi^*) \) and \( N^{-1} \ell_{\xi,i}(\psi^*) \).

**Proposition 8** The restandardized score in \( \xi \) and the restandardized observed Fisher information in \( \xi \) evaluated at \( \psi^* \) satisfy \( \text{cov} \{k_{N,1}(\psi^*), [W_N(\psi^*)]_{11} \} = o(1) \). We can thus deduce that as \([W_N(\psi)]_{11} \) and asymptotic Gaussianity is valid, asymptotic independence follows.

**Proof:** Due to previous arguments of large sample distributional equivalence, and due to the asymptotic Gaussianity, we need only consider the covariance of \( Y_{1,N}(\psi^*) \) and \( Y_{2,N}(\psi^*) \), and thus start by considering the covariance of the elements that make up these objects. We note that

\[
\tilde{c}_{k,j} = \text{cov} \left\{ j^{(f,N)}, j^{(f,N)} \right\} = (4\pi)(8\pi^2) \text{cov} \left\{ B_k^{(f,N)} C_{k,j}^{(f,N)} - A_k^{(f,N)} D_k^{(f,N)}, D_j^{(f,N)} + C_j^{(f,N)} - A_j^{(f,N)} E_j^{(f,N)} - B_j^{(f,N)} F_j^{(f,N)} \right\}.
\]

We consider the \( j = k \) terms and show that their contribution decays suitably in \( j \): the cross terms will be bounded like in previous arguments, relying for the decay for \( \log(N) \) \( k < j \). Then combining the results of the previous section with Isserlis’s theorem we find that (up to \( o(1) \)):

\[
\tilde{c}_{j,j} = \frac{1}{2} \bar{B}_{\lambda_j,N}(\xi, \delta) \Re(K_{j,j}) + 4\pi^3 \bar{B}_{\lambda_j,N}(\xi, \delta) B_{\lambda_j,N}(\xi, \delta) + \pi^2 \left\{ \bar{B}_{\lambda_j,N}(\xi, \delta) - 4\delta \bar{C}_j^{(2)}(\xi, \delta) / \pi \right\}
+ \frac{1}{2} \left\{ 2\pi^2 B_{\lambda_j,N}(\xi, \delta) - \bar{B}_{\lambda_j,N}(\xi, \delta) - \frac{1}{2} \bar{C}_j^{(2)}(\xi, \delta) \right\} \bar{B}_{\lambda_j,N}(\xi, \delta)
+ \frac{1}{2} \left( \frac{1}{2} B_{\lambda_j,N}(\xi, \delta) - \frac{1}{2} B_{\lambda_j,N}(\xi, \delta) - \frac{1}{2} \bar{C}_j^{(2)}(\xi, \delta) \right) \bar{C}_{\lambda_j,N}(\xi, \delta)
+ \frac{1}{2} \left( 2\pi^2 B_{\lambda_j,N}(\xi, \delta) - \bar{B}_{\lambda_j,N}(\xi, \delta) \right) \bar{C}_{\lambda_j,N}(\xi, \delta) - \frac{1}{2} \bar{B}_{\lambda_j,N}(\xi, \delta) + 4\pi^3 \bar{B}_{\lambda_j,N}(\xi, \delta) B_{\lambda_j,N}(\xi, \delta).
\]

Thus we may deduce \( \tilde{c}_{j,j} = O(j^{-1}) + o(1) \). The cross terms, i.e. \( \tilde{c}_{k,j} \), may be bounded in a standard fashion using the same argument, so that

\[
\text{cov} \{k_{N,1}(\psi^*), [W_N(\psi^*)]_{11} \} = o(1).
\]

We can thus deduce the asymptotic independence of variables \( k_{N,1}(\psi^*) \) and \( [W_N(\psi)]_{11} \). □

**Proposition 9** The large sample distribution of the MLE of \( \xi \) tends to:

\[
N(\hat{\xi} - \xi^*) = \frac{N^{5/2}}{-\ell_{\xi,i}(\psi^*)} N^{-3/2} \ell_{\xi,i}(\psi^*) \rightarrow \frac{\sqrt{5}}{2\pi\sqrt{2}} C,
\]

where \( C \sim \text{Cauchy} \).

**Proof:** To show this result we can simply use Propositions 6, 7 and 8.

**Note on Usage of Asymptotic Form:** We have

\[
C_N = N(\hat{\xi} - \xi^*) = k_{N,1}(\psi^*)/[W_N(\psi)]_{11}.
\]

We note that from equations (A-29) and (A-38), using proposition 8 that

\[
\frac{2^{3/2}}{\sqrt{6}} C_N = \frac{2^{3/2}}{\sqrt{6}} Z_4 + o(1) = \frac{Z_4/\sqrt{\pi^2/3}}{Z_5/\sqrt{8\pi^2/15}} \sim \text{Cauchy}.
\]
Define \( c_1 = \tan(\pi(-\frac{1}{2} + 0.025)) \) and \( c_2 = \tan(\pi(-\frac{1}{2} + 0.975)) \), then

\[
P \left( c_1 \leq \frac{2^{1/2} \sigma_2}{\sqrt{5}} C_N \leq c_2 \right) = P \left( \frac{\sqrt{5}}{2 \pi \sqrt{2}} c_1 \leq C_N \leq \frac{\sqrt{5}}{2 \pi \sqrt{2}} c_2 \right) = 0.95
\]

\[
\therefore P \left( \frac{\sqrt{5}}{2 \pi \sqrt{2}} c_1 \leq \xi \leq \frac{\sqrt{5}}{2 \pi \sqrt{2}} c_2 \right) = 0.95.
\]

Thus a 95% CI is given for \( \xi \) by \( (\hat{\xi} - 3.20/N, \hat{\xi} + 3.20/N) \). This establishes the large sample theory for \( \hat{\xi} \). However, the effect on MLE of low \( j \) contributions decays slowly, and so we provide an additional approximation to the distribution, based on equations (A-28) as well as (A-37).

**Note on Usage of Large Sample Approximation Form** For finite \( N \), as already discussed, it may be more appropriate to approximate the distribution of the two random variables using \( K_N \sim N(\mu_1, \sigma_1^2) \) and \( \tilde{W}_{N,11} \sim N(\mu_2, \sigma_2^2) \) where

\[
\mu_1 = \frac{1}{\sqrt{N}} \sum_{j=1}^{J_2} \hat{B}_{\lambda_j,N}(\xi, \delta) = o(1) \quad \sigma_1^2 = \frac{1}{N} \sum_{j=1}^{J_2} \left\{ \frac{1}{2} \delta^2 \hat{B}_{\lambda_j,N}(\xi, \delta) + B_{\lambda_j,N}(\xi, \delta) \hat{C}_{\lambda_j,N}(\xi, \delta) \right\} + o(1),
\]

\[
\mu_2 = \frac{1}{\sqrt{N}} \sum_{j=1}^{J_2} \hat{B}_{\lambda_j,N}(\xi, \delta) + o(1) \quad \sigma_2^2 = \frac{1}{N} \sum_{j=1}^{J_2} \hat{C}_{\lambda_j,N}(\xi, \delta) + o(1),
\]

where \( \sigma_2^2 \) is given by equation (A-41). With these quantities, we have \( C_1 = K_N/\tilde{W}_{N,11}, C_2 = \tilde{W}_{N,11} \), \( \tilde{W}_{N,11} = C_2 \) and \( K_N = C_1 C_2 \), and find a confidence interval for \( C_1 \), \( \Pr(c_{11} < C_1 < c_{12}) = 1 - \alpha \), assuming that asymptotic independence of \( K_N \) and \( \tilde{W}_{N,11} \) is approximately attained, we have by transformation techniques

\[
f_{C_1,c_2}(c_1,c_2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2} \left\{ \frac{\hat{c}_1^2}{\sigma_1^2} + \frac{(c_2 - \mu_1)^2}{\sigma_2^2} \right\} } |c_2| \quad \therefore f_{C_1}(c_1) = \int_{-\infty}^{c_{11}} \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2} \left\{ \frac{\hat{c}_1^2}{\sigma_1^2} + \frac{(c_2 - \mu_1)^2}{\sigma_2^2} \right\} } |c_2| \, dc_2
\]

\[
= \frac{1}{\sqrt{2\pi}} u(c_1)^{-3/2} \left[ \sqrt{2u(c_1)} e^{-\frac{\mu_1^2}{2\sigma_1^2}} + \sigma_1^2 e^{-\frac{\mu_1^2}{2\sigma_1^2}} \right] \sigma_2 e^{-\frac{\mu_2^2}{2\sigma_2^2}} \left\{ \frac{\sigma_2}{\sqrt{2\sigma_2}} \right\} \, dc_1, \quad c_1 \in \mathbb{R},
\]

\[
\rightarrow \sigma_1 \sigma_2 / \pi u(c_1)^{-1}, \quad c_1 \in \mathbb{R}
\]

as \( \mu_2 \to 0 \), and the distribution becomes a scaled Cauchy distribution. Using this approximation, we may derive CIs for \( \xi \) for a given value \( \delta \) by determining \( c_{11} \) and \( c_{12} \) for that value of \( \delta \) from

\[
P \left( c_{11} < N \left( \hat{\xi} - \xi \right) < c_{12} \right) = P \left( \hat{\xi} - c_{12}/N < \xi < \hat{\xi} - c_{11}/N \right) = 1 - \alpha.
\]

**Long Memory Parameter dependence of the CI’s** The \( \delta \) dependence is implicit in the distribution of \( C_1 \) in equation (A-49), as \( \mu_2, \sigma_1^2 \) and \( \sigma_2^2 \) depend on \( \delta \). Thus a \( (1 - \alpha) \) CI is simply given by \( \hat{\xi} \pm c_{12}/N \). For a real data set, we do not know the true value of \( \delta \), but note that \( \hat{\delta} = \delta^* + Z_2/\sqrt{N\delta} \) where \( Z_2 \sim N(0,1) \), from equation (A-16), as the same central limit argument will be valid for the score evaluated at \( \delta^* \) and \( \hat{\delta} \). We note that \( c_{11} \) and \( c_{12} \) are smooth functions of \( \delta \). Making the dependence on \( \delta \) explicit we find

\[
\left| c_{1k}(\delta^*) - c_{1k}(\hat{\delta}) \right| = N^{-1/2} |v_{1k}^t(\delta^*)| |Z_2|,
\]

(A-50)
Table 10: The quantities necessary to approximate the distribution of $N\hat{\xi}$ using $C_1$.

| $N$  | $\delta$ | 95% interval       | $\mu$  | $\sigma_1^2$ | $\sigma_2^2$ |
|------|----------|-------------------|--------|--------------|--------------|
| 1024 | 0.30     | $\xi \pm 3.17N^{-1}$ | 0.7778 | 3.3413       | 52.9845      |
| 1024 | 0.40     | $\xi \pm 2.90N^{-1}$ | 3.2203 | 3.4503       | 54.9996      |
| 1024 | 0.45     | $\xi \pm 1.43N^{-1}$ | 9.6724 | 3.6872       | 56.5742      |
| 2048 | 0.30     | $\xi \pm 3.18N^{-1}$ | 0.5606 | 3.3180       | 52.5227      |
| 2048 | 0.40     | $\xi \pm 3.00N^{-1}$ | 2.3937 | 3.3779       | 53.6337      |
| 2048 | 0.45     | $\xi \pm 1.96N^{-1}$ | 7.3754 | 3.5085       | 54.6138      |
| 4096 | 0.30     | $\xi \pm 3.19N^{-1}$ | 0.4021 | 3.3051       | 52.2645      |
| 4096 | 0.40     | $\xi \pm 3.10N^{-1}$ | 1.7646 | 3.3377       | 52.8721      |
| 4096 | 0.45     | $\xi \pm 2.41N^{-1}$ | 5.5698 | 3.4091       | 53.4590      |
| 8192 | 0.30     | $\xi \pm 3.20N^{-1}$ | 0.2874 | 3.2981       | 52.1217      |
| 8192 | 0.40     | $\xi \pm 3.14N^{-1}$ | 1.2921 | 3.3157       | 52.4517      |
| 8192 | 0.45     | $\xi \pm 2.41N^{-1}$ | 4.1725 | 3.3544       | 52.7938      |
| $\infty$ | $\delta > 0$ | $\xi \pm 3.20N^{-1}$ | 0      | 3.2899        | 51.9515      |

and so as $N^{-1/2}|C_k'\delta^*||Z_2| \xrightarrow{P} 0$ we can use equation (A-50) with $c_{11}$ and $c_{12}$ calculated at $\delta = \hat{\delta}$. For our simulation study, to reduce the numerical burden of the procedure, we have calculated the CIs at $\delta^*$. This would not be the approach in a real problem, but given the reduced computational cost of a single calculation of $c_{11}$ and $c_{12}$ for real examples, this is not an issue. ■

Finally, we establish that the score in $\delta$ and $\xi$ are uncorrelated, as the off-diagonal terms of the standardized observed Fisher information converge to zero.

**Proposition 10** We have that $\text{cov}\{k_{N,1}(\psi^*), k_{N,2}(\psi^*)\} = o(1)$, and thus we can note that the distributional results follow.

**Proof:** Note that $k_{\xi,N}(\psi^*) = N^{-3/2}l_\xi(\psi^*)$ and $k_{\delta,N}(\psi^*) = F_{\delta,\delta}^{-1/2}N^{-1/2}l_\delta(\psi^*)$. We thus consider $N^{-2}\text{cov}\{l_\xi(\psi^*), l_\delta(\psi^*)\}$. We have

$$\text{cov}\{k_{\xi,N}(\psi^*), k_{\delta,N}(\psi^*)\} = \frac{1}{\sqrt{F_{\delta,\delta}N^2}}\text{cov}\left\{ \sum_j S_j^{(1)} \left\{ 1 - I_{(f,N)}^{(f,N)} \right\} - \hat{I}_j^{(f,N)}, \sum_j R_j^{(1)} \left\{ 1 - I_{(f,N)}^{(f,N)} \right\} \right\},$$
plus o(1) terms. Thus it follows

\[
\text{cov}\{k_{\xi,N}(\psi^*), k_{\delta,N}(\psi^*)\} = \frac{1}{\sqrt{F_{\delta,\delta} N^2}} \text{cov}\left\{-\sum_j S_j^{(f,N)} - \sum_k R_k^{(f,N)} \right\} + o(1)
\]

\[
= \frac{1}{\sqrt{F_{\delta,\delta} N^2}} \left[ \sum_j \sum_k \text{cov}\left\{S_j^{(f,N)}, R_k^{(f,N)}\right\} + o(1) \right]
\]

\[
= \frac{1}{\sqrt{F_{\delta,\delta} N^2}} \left[ 2 \sum_j \sum_{k \leq j} S_j^{(f,N)} R_k^{(f,N)} + o(1) \right]
\]

\[
= \frac{1}{\sqrt{F_{\delta,\delta} N^2}} \sum_j \sum_k R_k^{(f,N)} \text{cov}\left\{\dot{I}_j^{(f,N)}, \dot{I}_k^{(f,N)}\right\} + o(1)
\]

Note that using Isserlis’s theorem (Isserlis, 1918) we have:

\[
\text{cov}\left\{I_j^{(f,N)}, I_k^{(f,N)}\right\} = 4\pi \text{cov}\left\{A_j^{(f,N)^2} + B_j^{(f,N)^2}, B_k^{(f,N)^2} - A_k^{(f,N)} D_k^{(f,N)}\right\}.
\]

For \( j = k \) we have

\[
\text{cov}\left\{\dot{I}_j^{(f,N)}, \dot{I}_j^{(f,N)}\right\} = \frac{\dot{B}_{\lambda_j,N}(\xi, \delta)}{4\pi} \frac{1}{2} B_{\lambda_j,N}(\xi, \delta) + o(1) = O(j^{-1}) + o(1) \rightarrow 0,
\]

for increasing \( j \) and \( N \). The cross-terms \( \text{cov}\left\{I_j^{(f,N)}, I_k^{(f,N)}\right\} \) may be bounded in the usual fashion.

\[
\text{var}\{I_j^{(f,N)}\} = O(1) \quad \text{var}\{\dot{I}_j^{(f,N)}\} = O(1).
\]

Combining these results we find that

\[
\lim_{N \to \infty} \left[ (F_{\delta,\delta})^{-1/2} N^{-2} \text{cov}\left\{\ell_\xi(\psi^*), \ell_\delta(\psi^*)\right\} \right] = 0.
\]