DYNAMICS AND ENERGETICS OF TURBULENT, MAGNETIZED DISK ACCRETION AROUND BLACK HOLES: A FIRST-PRINCIPLES APPROACH TO DISK-CORONA-OUTFLOW COUPLING

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ABSTRACT

We present an analytic description of turbulent, magnetohydrodynamic (MHD) disk accretion around black holes that specifically addresses the relationship between radial and vertical mean field transport of mass, momentum, and energy, thereby complementing and extending numerical simulations. The azimuthal-vertical component of the magnetic stress is fundamental to an understanding of disk-corona-outflow coupling: when it is important for driving the angular momentum transport and mass accretion in the disk, it also has an important influence on the disk-corona-outflow energy budget. The Poynting flux derived from the product of this term with the Keplerian velocity also dominates the Poynting flux into the corona. The ratio of the coronal Alfvén velocity to the Keplerian velocity is an important parameter in disk-corona-outflow physics. If this parameter is greater than unity, then energetically significant winds and Poynting flux into the corona occur. However, significant effects could also occur when this parameter is much less than unity. A limiting solution describing the case of angular momentum transport solely by the vertical-azimuthal stress has the property that all of the accretion power is channeled into a wind, some of which would be dissipated in the corona. More realistic solutions in which there is both radial and vertical transport of angular momentum would have different fractions of the accretion power emitted by the disk and corona, respectively. These results have important implications for existing accretion disk theory and for our interpretation of high-energy emission and nuclear outflows from the central engines of active galactic nuclei and galactic black hole candidates.

Subject headings: accretion, accretion disks — galaxies: active — hydrodynamics — magnetic fields — MHD — quasars: general

1. INTRODUCTION

When the foundations were laid for a standard theory of disk accretion (Pringle & Rees 1972; Shakura & Sunyaev 1973; Novikov & Thorne 1973), two fundamental problems were immediately recognized (Liang & Price 1977; Bisnovatyi-Kogan & Blinnikov 1977; Paczynski 1978): (1) the efficient transport of angular momentum to large radii cannot be attributed to conventional kinematic viscosity, and (2) the observed high-energy spectra and luminosities and the ubiquity of outflow phenomena from accreting black holes imply efficient vertical transport of energy from a relatively cool, dense disk to a hot, tenuous, and unbound corona. While it has been widely accepted that magnetic fields provide the most plausible means of efficiently transporting both angular momentum and energy, the precise nature of this transport has remained unclear until recent numerical simulations demonstrated, unambiguously, that accretion disks work because of magnetohydrodynamic (MHD) turbulence (Balbus & Hawley 1998; Hawley et al. 1999). The turbulence is generated by the magnetorotational instability (MRI; Balbus & Hawley 1991 and references therein), which is driven by the free energy available from the differential rotation of the bulk flow. Notwithstanding these groundbreaking results, however, the nature of vertical energy transport from an accretion disk to a corona and/or outflow still remains an outstanding and contentious issue.

In the context of active galactic nuclei (AGNs), much theoretical effort has recently focused on the physics of accretion disk coronae (Di Matteo et al. 1997a, 1997b; Merloni & Fabian 2001a, 2001b, 2002; Liu et al. 2002), commensurate with the dramatic increase in both the quality and quantity of high-energy observational data. However, there has been little improvement in coupled disk-corona models since the first phenomenological descriptions of Haardt & Maraschi (1991, 1993). Current models (e.g., Merloni & Fabian 2002; Liu et al. 2002) simply replace the fraction of accretion power transferred from the disk to the corona with a Poynting flux quantity estimated from a mean field buoyant velocity and an equipartition, mean field magnetic energy density. While numerical models (e.g., Miller & Stone 2000) do indeed show that turbulent fluctuations in a vertically stratified disk are capable of driving the magnetogravitational modes of the Parker instability (Parker 1955), whether magnetic buoyancy can supply the corona with sufficient power to explain the observed high-energy emission is questionable. Numerical simulations indicate that magnetic buoyancy is an ineffective saturation mechanism for the MRI (Brandenburg et al. 1995; Stone et al. 1996; Miller & Stone 2000), while theoretical models for disk coronae require implausibly ideal buoyancy conditions and limiting accretion conditions (e.g., Merloni 2003). Realistically, the growth of the unstable buoyant Parker modes, which is essentially a wave fluctuation resonance interaction, must
compete against particle fluctuation interactions, which correspond to dissipation of the turbulence and internal heating of the disk.

The production and ubiquity of outflows from accretion disks around black holes also remains a challenging problem, and it is unclear from most theoretical models (e.g., Blandford & Payne 1982; Lovelace et al. 1991; Li et al. 1992; Wardle & Königl 1993; Ustyugova et al. 2000) whether MHD disk turbulence plays a significant role (but see Heinz & Begelman 2000). Nevertheless, numerical simulations of turbulent MHD accretion disks do in fact show that outflows become important in the innermost regions of turbulent accretion disks (e.g., Stone & Pringle 2001; Hawley et al. 2001; Hawley & Balbus 2002). Unfortunately, the numerical models are restricted by their approach: non-Keplerian motions are defined a priori as fluctuating quantities, so that vertical, mean field transport is not self-consistently taken into account. Indeed, the outflows that emerge in some of the simulations (e.g., Hawley & Balbus 2002) are defined as regions where the net radial flow is outward, rather than inward. Furthermore, the neglect of mean vertical angular momentum transport restricts the radial inflow to substantially subsonic speeds (see Balbus & Hawley 1998).

Inevitably, the resulting accretion rates in the numerical models are typically very low (see, e.g., Stone & Pringle 2001; Hawley et al. 2001; Hawley & Balbus 2002). Numerical models are also restricted by computational limitations: simulations that are global and vertically stratified are required to estimate the fraction of accretion power that can be vertically transported, and this is not only computationally prohibitive but also sensitive to numerical dissipation effects, which are difficult to quantify.

Thus, the present status of black hole accretion disk theory is that there is currently no formalism that self-consistently couples turbulent, MHD disk accretion with a magnetically dominant corona in a framework that can accommodate a range of radial inflow and vertical outflow solutions.

In this paper we present the first fully analytic description of a turbulent MHD accretion disk coupled to a corona, self-consistently taking into account both vertical and radial mean-field fluxes of mass, momentum, and energy. By deriving the relevant transport equations from first principles, our formalism provides a nonphenomenological and nonempirical approach to the problem of energy transport to a corona and the associated outflow and the interrelationship of this transport with accretion onto the central black hole. In this treatment, we focus on the effects of turbulent magnetic stresses and the mean magnetic flux density is assumed to be zero, for the sake of clarity. However, our formalism lends itself naturally to the inclusion of nonzero mean magnetic fields, which we intend to explore separately in a subsequent paper. In § 2 we statistically average the equations for a resistive, magnetized gas (summarized in the Appendix) and derive the mean field equations for a turbulent magnetized gas. In § 3 we apply these mean field equations to the dynamics of a geometrically thin accretion disk that is stationary and axisymmetric in the mean, and we expressively examine the implications of vertical mean field transport for the conservation of mass and momentum. In § 4 we utilize the results of § 3 to analyze the total disk energy budget. We conclude with a discussion of the main results in § 5.

2. STATISTICAL AVERAGING: THE MEAN FIELD EQUATIONS

We have summarized the relevant MHD equations for a resistive, viscous and radiative MHD gas in the Appendix. In this section we present a statistical averaging approach to these equations that describes the mean flow in the accretion disk. Appropriate mean values are defined for the dynamical variables, and evolution equations for the mean flow are derived. We consider a nonrelativistic, optically thick gas in which the radiative diffusion approximation holds and the radiation pressure reduces to a scalar.

The independent, unaveraged variables are mass density, \( \rho \); fluid velocity, \( \vec{v} \); gas plus radiation pressure, \( \rho \); gas plus radiation energy density, \( \vec{u} \); gravitational potential, \( \phi \); radiative flux, \( F \); external heat flux, \( Q \); magnetic field, \( B \); electric field, \( \vec{E} \); current density, \( J \); and molecular viscous stress tensor, \( \vec{\tau} \).

The introduction of the last variable, together with the inclusion of resistive terms in the dynamical equations, requires some comment since it is generally assumed that such terms are unimportant in accretion disks and our inclusion of them here may be taken to be controversial. However, we note that, consistent with all other treatments of accretion disks, the molecular viscous and resistive terms involving the mean flow are unimportant since the Reynolds number is extremely large.

Molecular viscosity can, in principle, become important at the high-wavenumber end of a turbulent cascade when the gradient of the turbulent velocity is large enough that viscous dissipation can balance the energy generated on the large scale. Resistivity is important when magnetic fields reconnect or when turbulence produces large gradients. Both reconnection and turbulent magnetic dissipation are also examples of a small-scale, high-wavenumber phenomenon. Undoubtedly, the details of these processes are complicated as fundamental work on the inertial cascade in turbulent MHD gases indicates (e.g., Goldreich & Sridhar 1995, 1997), and we do not consider here the important problem of how turbulence is converted into heat in an accretion disk. Nevertheless, we retain the molecular viscous and resistive terms as representative dissipative terms for a specific purpose, to distinguish the real dissipation in a disk from the terms proportional to the (inner tensor) product of the Reynolds and magnetic stresses with the velocity shear tensor. Although these terms have conventionally been associated with dissipation, a better physical interpretation is that they represent the production of turbulent kinetic and magnetic energy. We support this proposition below (see § 2.5), when we discuss the interpretation of the various forms of the turbulent energy equations.

This apparently minor point, distinguishing between the production of turbulent energy and its dissipation, becomes important when we trace the flow of turbulent energy in a disk. Not only can turbulent energy be dissipated, but it can also be turbulently diffused and/or advected in a wind and then dissipated at a site different from where it is produced. This distinction is unimportant in conventional accretion disk theory where, in effect, dissipation and production are equated in a local “on-the-spot” approximation. However, when we are considering the transport of energy away from the site where it is created, it is important to clearly distinguish between production and dissipation. There are other forms of dissipation that we could consider. For example, Agol & Krolik (1998) examined the effect of photon damping on a turbulent cascade and concluded that it is a more important form of dissipation than other forms. In the treatment that we develop here, the exact form of the dissipation is ultimately unimportant; the dissipative terms, in effect, act as placeholders for whatever dissipative process one may wish to contemplate. As we have indicated, our main purpose in including them is to distinguish between production and dissipation.
So far, a number of the statements in the above paragraph have been made without proof. These assertions are proved below, once we define a suitable statistical averaging procedure for turbulent flow.

2.1. Mean and Fluctuating Variables

In dealing with turbulent incompressible flow, the statistical procedure for averaging the MHD equations is quite straightforward. For the velocity, for example, one defines the mean \( \bar{v}_i \) and fluctuating \( v'_i \) components of velocity by

\[
v_i = \bar{v}_i + v'_i, \quad \langle v'_i \rangle = 0, \tag{1}
\]

where the angle brackets denote an appropriate time or ensemble average. This Ansatz leads to a number of additional terms in the dynamical equations. For example, consider the following expression for the momentum flux, which is used in the momentum equations:

\[
\langle \rho v_i v_j \rangle = \rho \bar{v}_i \bar{v}_j + \rho \langle v'_i v'_j \rangle. \tag{2}
\]

The second term describes the turbulent diffusion of momentum with respect to the mean flow.

This procedure is straightforward in incompressible flow because the density is constant (see, for example, Bradshaw 1976 for unmagnetized gas and Krause & Radler 1980 for application to astrophysical magnetic fields). In compressible flow, however, there are two options. The first option is to adopt the above expression for the mean velocity and express the density similarly:

\[
\rho = \bar{\rho} + \rho', \quad \langle \rho' \rangle = 0. \tag{3}
\]

The mean momentum flux then accumulates three additional components, including a triple correlation:

\[
\langle \rho v_i v_j \rangle = \bar{\rho} \bar{v}_i \bar{v}_j + \rho \langle v'_i v'_j \rangle + \langle \rho' v'_i \rangle \bar{v}_j + \langle \rho' v'_j \rangle \bar{v}_i + \langle \rho' v'_i v'_j \rangle. \tag{4}
\]

In the momentum equations, one of these terms can be eliminated by the continuity equation. Nevertheless, this example of the momentum flux shows that the statistical averaging of the complete set of the compressible MHD equations using this particular decomposition of the density and velocity fields generates a large number of terms. It is for this reason that Favre (1969) proposed a second option in which extensive variables such as the density are averaged in the way but intensive variables such as the velocity are mass averaged. This leads to the Ansatz:

\[
\rho = \bar{\rho} + \rho', \quad \langle \rho' \rangle = 0, \tag{5}
\]

\[
v_i = \bar{v}_i + v'_i, \quad \langle \rho v'_i \rangle = 0. \tag{6}
\]

Note that extensive averages are denoted by a bar and intensive averages are denoted by a tilde. There are two advantages to this approach. First, the momentum flux involves fewer terms:

\[
\langle \rho v_i v_j \rangle = \bar{\rho} \bar{v}_i \bar{v}_j + \rho \langle v'_i v'_j \rangle. \tag{7}
\]

Second, mass, as defined by the mean density, is conserved in the mean flow (see the continuity eq. 10). Favre’s approach is well known in the turbulent fluid dynamics literature and, in astrophysics, has been used in the theory of compressible, turbulent jets (Bicknell 1984, 1986).

In accretion disk theory, a third option has been used by Balbus & Hawley (1998). Velocity components are expressed in terms of fluctuations on the Keplerian velocity; the equations are averaged over a disk height \( h \) and a radius \( \Delta r \gg h \). Some useful physical insights have come from this local approach. However, as a general strategy, we find it unappealing since such an approach cannot readily deal with the case in which there is an additional flow field involved, such as a disk wind. It eventuates, as we show in later sections, that a disk wind is a potentially important aspect of disk-coroona physics so that the Balbus-Hawley approach is overly restrictive for our purposes. Some of the restrictions of the Balbus-Hawley Ansatz have also been pointed out by Balbus & Hawley (1998; see § 5.6.2 of their review in which they refer to an ensemble or time-averaged approach similar to that developed here).

Therefore, given these three options for statistically averaging turbulent flow, we adopt the mass-weighted statistical averaging approach of Favre (1969), generalizing it to include the effects of magnetic fields.

When the flow is steady in the mean, we can think of the averaging process as involving an average over a timescale large compared to the timescales of instabilities and turbulent fluctuations. As we have noted above, a statistical averaging approach is capable of providing valuable insights. In the case of accretion disks we gain useful insights into the energy flow within the system that we have described in qualitative terms above.

In addition to the mass-averaged description of the velocity, the statistical averages of the pressure and magnetic field are represented by

\[
p = \bar{p} + p', \quad \langle p' \rangle = 0, \tag{8}
\]

\[
B_i = \bar{B}_i + B'_i, \quad \langle B'_i \rangle = 0.
\]

We adopt the useful definition for the mean fluid shear

\[
\bar{s}_{ij} = \frac{1}{2} (\bar{v}_{i,j} + \bar{v}_{j,i} - \frac{3}{2} \delta_{ij} \bar{v}_{k,k}). \tag{9}
\]

In this paper we restrict ourselves to the simplest case in which the magnetic field is dominated by its fluctuating component, so that \( \langle B \rangle \equiv \bar{B} \ll \langle B'^2 \rangle^{1/2} \). The more complex case in which a systematic component of the magnetic field is present clearly has important implications for disk accretion solutions, particularly when the field strength is sufficient to quench the MRI and transport angular momentum via large-scale magnetic torques such as those invoked in models for MHD-driven outflows and Poynting flux jets (e.g., Blandford & Payne 1982; Lovelace et al. 1991; Li et al. 1992; Wardle & Königl 1993; Ustyugova et al. 2000). However, it is unclear whether large-scale, organized mean fields can be generated from a highly chaotic underlying flow and whether they can explain collimated outflows associated with accretion disks (see, e.g., the discussion in Heinz & Begelman 2000). The more general case of \( \bar{B} \neq 0 \) is thus deferred for future work. All the mean field equations derived here can, however, be straightforwardly generalized to include terms involving \( \bar{B} \) that are directly analogous to their turbulent counterparts.

The molecular viscous stress tensor can also be written as

\[
t''_{ij} = t''_{ij} + t''_{ij}, \quad \langle t''_{ij} \rangle = 0, \tag{10}
\]

since it can be related to a coefficient of kinematic shear viscosity, \( \nu \), by \( t''_{ij} = 2 \nu \bar{s}_{ij} \) (ignoring
bulk viscosity), where \( s_{ij} \) is the shear tensor defined in equation (A9). In practice, the mean viscous stress, \( \langle t_{ij}^n \rangle = t_{ij} \), usually has a negligible effect on momentum transport (particularly in a high Reynolds number turbulent flow) and is thus usually ignored. However, the fluctuating part of the viscous stress plays an important role in the dissipation of turbulent energy on the smallest scales at the end of a turbulent cascade. Mathematically, this is described by the appearance of the correlation term \( \langle \tilde{t}_{ij} \tilde{v}_{ij} \rangle \) in the mean field internal energy equation (20), derived below. This term represents the mean rate of turbulent viscous heating and is dominated by the high-wavenumber components of the turbulent velocity fluctuations, that is, by the dissipative region (in wavenumber space) of the turbulent cascade. Thus, although we henceforth assume \( \langle t_{ij}^n \rangle = 0 \), we retain \( t_{ij} \) throughout the averaging procedure in order to consistently include stochastic energy dissipation by turbulent viscous stresses.

2.2. Mass Conservation

Statistical averaging of the continuity equation (A1) gives

\[
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial (\bar{\rho} \tilde{v}_i)}{\partial x_j} = 0.
\]  

(10)

Hence, the mass defined by the mean density and velocity is conserved.

2.3. Momentum Conservation

In the case of zero net magnetic flux, the magnetic stresses are

\[
t_{ij} = \frac{B_i^0 B_j^0}{4\pi} - \delta_{ij} \frac{B^2}{8\pi},
\]  

(11)

with ensemble mean

\[
\langle t_{ij} \rangle = \frac{\langle B_i^0 B_j^0 \rangle}{4\pi} - \delta_{ij} \frac{\langle B^2 \rangle}{8\pi}.
\]  

(12)

We also define the Reynolds stresses, \( t_{ij}^R \), by

\[
t_{ij}^R = -\langle \rho' \tilde{v}_i \tilde{v}_j \rangle.\]  

(13)

Statistical averaging of the momentum equation (A2) yields

\[
\frac{\partial (\bar{\rho} \tilde{v}_i)}{\partial t} + \frac{\partial (\bar{\rho} \tilde{v}_i \tilde{v}_j)}{\partial x_j} = -\bar{\rho} \frac{\partial \bar{\rho} \tilde{g}}{\partial x_i} - \bar{\rho} \frac{\partial \tilde{\rho}}{\partial x_i} + \frac{\partial \langle t_{ij}^R \rangle}{\partial x_j}.
\]  

(14)

This equation shows that momentum transport in a fluctuating MHD fluid involves additional, statistically averaged quantities that arise from the turbulent magnetic stresses and the Reynolds stresses. Note that the Reynolds stress is conventionally defined as a mean quantity, whereas, for convenience, we have defined \( t_{ij}^R \) as an unaveraged quantity in order to simplify triple correlation terms that appear in the energy equations derived in § 2.5. Note also that these definitions of the turbulent stresses are opposite in sign to those used, for example, by Balbus & Hawley (1998). Our usage is consistent with the conventional description of this term as a stress.

Often in astrophysical contexts, the terms \( t_{ij}^R \) and \( t_{ij} \) are referred to as viscous and resistive terms since they represent additional forms of momentum and electromagnetic transport. In this paper we consistently refer to these terms as turbulent stresses and reserve the terms viscosity and resistivity for real molecular effects.

2.4. The Induction Equation

Although we assume that the mean magnetic flux density is zero, for completeness, and to make one physical point, we present here the mean field induction equation obtained by statistically averaging equation (A3):

\[
\frac{\partial \tilde{B}_i}{\partial t} + \epsilon_{ijkm} \frac{\partial}{\partial x_j} \left( \tilde{B}_j \tilde{v}_m + \langle \tilde{B}_j' \tilde{v}_{jm}' \rangle \right) = \eta \nabla^2 \tilde{B}_i.
\]  

(15)

This equation forms part of the foundation of classical mean field electrodynamics (e.g., Krause & Radler 1980), which is usually developed under the assumption of an incompressible flow. The contribution from the term involving \( \tilde{B} \tilde{v}_m \) results from compressibility, while the term involving \( \langle \tilde{B}_j' \tilde{v}_{jm}' \rangle \) gives rise to dynamo amplification resulting from mean helicity. As exceptionally clarified by Balbus & Hawley (1998, see § VI-B), when a weak seed field \( B \) is present in a differentially rotating fluid, the fluctuating fields \( \psi' \) and \( B' \) are self-consistently generated by the MRI and thus cannot be prescribed independently of \( B \), which is the underlying assumption of mean field dynamo theory. We do not, therefore, make any assumption in our formalism concerning mean field dynamo amplification.

2.5. Energy

There are a number of different energies to account for in a turbulent MHD fluid: the turbulent magnetic and kinetic energies, the internal energy, and the total (turbulent plus internal plus mechanical) energy. Here we derive the conservation equations describing the evolution of these quantities.

2.5.1. Turbulent Magnetic Energy

The mean field equation for the turbulent magnetic energy density, \( \langle u_B \rangle = \langle B^2 \rangle /8\pi \), is derived by statistically averaging the electromagnetic energy equation (A12), yielding

\[
\frac{\partial \langle u_B \rangle}{\partial t} + \frac{\partial \langle \tilde{B}_j \tilde{v}_j \rangle}{\partial x_j} = \frac{1}{3} \frac{\partial}{\partial x_j} \left( \left[ \langle u_B \rangle \tilde{v}_j + \langle u_{B'} \rangle \tilde{v}_{j'} \right] \right) - \frac{1}{3} \langle u_B \rangle \frac{\partial \langle \tilde{v}_j \tilde{v}_j \rangle}{\partial x_j} - \frac{1}{3} \langle u_{B'} \rangle \frac{\partial \langle \tilde{v}_{j'} \tilde{v}_{j'} \rangle}{\partial x_j} - \eta \frac{\partial^2 \langle \tilde{B}_j \tilde{B}_j \rangle}{\partial x_j^2} - \frac{\langle J^2 \rangle}{\sigma},
\]  

(16)

where the turbulent magnetic stress tensor \( t_{ij}^R \) is defined by equation (11). The first term on the right-hand side of equation (16) is a source term for the production of turbulent magnetic energy as a result of interaction between the turbulent magnetic stresses and the mean fluid shear. The second term represents the interaction between the turbulent magnetic stresses and the fluctuating components of the mean shear. The third and fourth terms describe changes in the turbulent magnetic energy due to compression and expansion in the mean and fluctuating components of the flow, respectively. The last two terms on the right-hand side of equation (16) describe the total work done by resistive forces minus the rate at which turbulent magnetic energy is converted into heat via Joule losses.

2.5.2. Turbulent Kinetic Energy

A similar mean field equation for the turbulent kinetic energy density, \( \langle u_K \rangle = \langle \frac{1}{2} \rho \tilde{v}^2 \rangle \), can be derived by taking the
scalar product of $\mathbf{e}_i$ with the unaveraged momentum equation, equation (A2), statistically averaging this equation, and then subtracting from it the scalar product of $\mathbf{v}_i$ with the statistically averaged momentum equation, equation (14). In symbolic terms, this is $\langle \mathbf{v} \cdot \text{momentum eq.} \rangle - \mathbf{v} \cdot \langle \text{momentum eq.} \rangle$, which is equivalent to $\langle \mathbf{v}' \cdot \text{momentum eq.} \rangle$ and yields

$$
\frac{\partial}{\partial t} (u_K + u_B) + \frac{\partial}{\partial x_i} \left( [u_K + u_B] \frac{\partial \mathbf{v}_i}{\partial x_i} + \left\{ u_K v'_i + \langle u_K u_B \rangle v'_i \right\} \right) = \\
\left\{ \langle t_{ij} \rangle s_{ij} - \left\{ \frac{2}{3} [u_K + u_B] \right\} \frac{\partial}{\partial x_i} \left\{ \langle v'_i \rangle - \langle t^R_{ij} \rangle v'_i \right\} - \left\{ \frac{1}{3} \frac{J^2}{\sigma} \right\} \right\}.
$$

(17)

Note the similarity to the turbulent magnetic energy equation, equation (16), and also to the internal energy equation for an adiabatic $\gamma = 5/3$ gas when the triple correlations are neglected. In particular, there is an analogous source term, $\langle t^R_{ij} \rangle s_{ij}$, describing the rate at which shear in the mean flow does work on the Reynolds stresses. There is also a sink term, $\langle t_{ij} v'_i \rangle$, describing viscous dissipation of turbulent kinetic energy into heat. Note also that the turbulent kinetic energy equation is coupled directly to the turbulent magnetic energy equation via the magnetic term on the right-hand side of equation (17), which also appears explicitly on the right-hand side of equation (16).

2.5.3. Total Turbulent Energy

The magnetic coupling term in equation (17) implies that the turbulent kinetic and magnetic equations can be combined into a single turbulent energy equation:

$$
\frac{\partial}{\partial t} (u_K + u_B) + \frac{\partial}{\partial x_i} \left( [u_K + u_B] \frac{\partial \mathbf{v}_i}{\partial x_i} + \left\{ u_K v'_i + \langle u_K u_B \rangle v'_i \right\} \right) = \\
\left\{ \langle t_{ij} \rangle s_{ij} - \left\{ \frac{2}{3} [u_K + u_B] \right\} \frac{\partial}{\partial x_i} \left\{ \langle v'_i \rangle - \langle t^R_{ij} \rangle v'_i \right\} - \left\{ \frac{1}{3} \frac{J^2}{\sigma} \right\} \right\}.
$$

(18)

where

$$
\langle t_{ij} \rangle = \langle t^R_{ij} \rangle + \langle t^B_{ij} \rangle
$$

is the combined Reynolds and magnetic stress tensor.

The last two terms on the right-hand side of equation (18) represent the dissipation of turbulent kinetic and magnetic energy, whereas the first term on the right represents the production of turbulent energy through the action of the mean shear on the total stresses. The usual approach in many physical applications is to equate the heating rate to the production term. However, the presence of transport terms, as well as terms describing the work done by fluid compression and/or expansion, implies that the steady state production and dissipation of turbulent energy are nonlocal, and therefore the rates cannot in general be equated. Thus, it is not generally correct to simply replace the viscous dissipation term in the internal energy equation with the production term for turbulent energy. The rate of turbulent viscous dissipation in the mean field internal energy equation must be related to the rate of production of turbulent energy self-consistently using equation (18), as demonstrated in the following section.

2.6. Internal Energy

Statistically averaging the internal energy equation (A6) yields

$$
\frac{\partial \tilde{u} \tilde{h}}{\partial t} + \frac{\partial}{\partial x_i} \left( \tilde{u} \tilde{h} + \langle u \rangle \tilde{h} \right) = \\
- \langle \tilde{p} \rangle \langle \mathbf{v} \rangle - \langle \langle F_i \rangle \rangle + \langle \langle \tilde{Q} \rangle \rangle.
$$

(20)

The terms on the left-hand side describe the total rate of change in the gas plus radiation internal energy density as a result of intrinsic temporal variations and advective transport by the mean plus turbulent flow. The terms on the right-hand side of equation (20) describe the work done by compression or expansion in the flow against the gas and radiation pressure, radiative losses, energy exchange from an external heat source or sink, mean field ohmic heating, and turbulent viscous heating. The source terms determine the rate at which energy is converted into random particle energy (some of which is then converted into radiation) and also into bulk kinetic energy. Magnetic energy in particular can be converted directly via Joule heating (usually identified with field line reconnection), as well as via work done by the turbulent flow against the turbulent viscous stresses.

2.7. Turbulent plus Internal Energy

The viscous and Joule dissipation terms appear as sink terms in the turbulent energy equation (18) and as source terms in the internal energy equation (20). Hence, combining these equations eliminates these terms with the following result for the total internal energy of a turbulent MHD fluid:

$$
\frac{\partial}{\partial t} (\tilde{u} + \langle u_K + u_B \rangle) + \frac{\partial}{\partial x_i} \left( \tilde{h} + \langle u_K + u_B \rangle \right) \tilde{v}_i + \langle \rho \tilde{h} \rangle \tilde{v}_i' = \\
\left\{ \langle t_{ij} \rangle \tilde{s}_{ij} - \left\{ \frac{2}{3} \langle u_K + u_B \rangle + \frac{1}{3} u_B \right\} \frac{\partial}{\partial x_i} \left\{ \langle \tilde{v}_i \rangle - \langle t^R_{ij} \rangle \tilde{v}_i \right\} - \left\{ \frac{1}{3} \frac{J^2}{\sigma} \right\} \right\}.
$$

(21)

2.8. Total Energy

Statistically averaging the total energy equation (A18) gives

$$
\frac{\partial}{\partial t} \langle F_i \rangle + \frac{\partial}{\partial x_i} \langle \tilde{F}_i \rangle = 0,
$$

(22)

where the average total energy is

$$
\tilde{u}_{tot} = \frac{1}{2} \tilde{p} \tilde{v}^2 + \tilde{p} \tilde{v} \omega + \langle u \rangle + \langle u_K \rangle + \langle u_B \rangle
$$

(23)

and involves both the components that one expects from the mean flow and the average turbulent kinetic energy, $\langle u \rangle$. We use the second form of the energy flux, equation (A19), to form the mean energy flux,

$$
\langle F_i \rangle = \left( \frac{1}{2} \tilde{p} \tilde{v}^2 + \tilde{p} \tilde{v} \omega + \tilde{h} + \langle u_K \rangle + \langle u_B \rangle \right) \tilde{v}_i + \\
+ \langle \rho \tilde{h} \tilde{v}_i' \rangle + \langle u_K \tilde{v}_i \rangle + \langle u_B \tilde{v}_i \rangle - \langle \tilde{t}_{ij} \rangle \tilde{v}_i - \langle \tilde{t}_{ij} \tilde{v}_i \rangle + \\
+ \langle F_i \rangle - \langle \tilde{t}_{ij} \tilde{v}_i \rangle \frac{\partial}{\partial x_i} \langle \tilde{v}^2_{ij} \rangle.
$$

(24)
The numerous terms in this expression for the mean energy flux, \((P_i^2)\), are all easily interpreted. The first group of five terms represents the energy flux advected by the mean flow and consists of bulk kinetic, gravitational, enthalpy, kinetic, and magnetic terms. The next group of three terms \((\langle \rho h v'_i \rangle + \langle u_k v'_i \rangle + \langle u_B v'_i \rangle)\) represents the turbulent fluxes of enthalpy, turbulent kinetic energy, and turbulent magnetic energy. The next two terms \((- (t_{ij}) \hat{v}_i - (\hat{t}_{ij}) v'_i)\) represent the net work done by the total Reynolds plus magnetic stresses on the mean and turbulent flow. The next two terms \((\langle F'_i \rangle + \langle Q_i \rangle)\) represent the radiative and heat fluxes, respectively.

The last two terms \((- (t_{ij}) \hat{v}_i - (\hat{t}_{ij}) v'_i)\) represent the work done by the viscous and resistive stresses. In most regions of a high Reynolds number flow, they are negligible. Nevertheless, in some regions of high spatial gradients, such as a shock or a reconnection region, they could be comparable to the other terms and their inclusion in equation (24) is logical (see, e.g., Lazarian & Vishniac [1999] for further details on stochastic reconnection). However, when the energy equation is integrated over a volume whose bounding surface is well outside regions of dissipation, the contribution of these two terms to the resulting surface integral of the energy flux is negligible. Therefore, in many circumstances, we can safely ignore these terms in considering the integral form of the energy equation. This point is elaborated in \(\S\) 4 in the context of accretion disks.

The total energy equation describes the net transfer of energy from one component to another and incorporates the work done by the total turbulent stresses through the term \((t_{ij}) \hat{v}_i\), as well as the change in binding energy represented by the terms \(\frac{1}{2} \rho \hat{v}^2 + \rho \phi_G\). The volume dissipative terms \((\hat{t}_{ij} v'_i)\) and \((\hat{t}^2/\sigma)\) disappear because of their equal and opposite contributions in the equations describing production of turbulent kinetic and magnetic energy on one hand and dissipation of that energy into heat on the other. Other terms in the energy equation represent nonlocal effects such as advection and diffusive transport.

Note that the advection terms contain contributions from both turbulent kinetic and magnetic energy, not just enthalpy. Hence, in applications to advective accretion disk models, for example, the magnetic and turbulent kinetic energy should be taken into account especially when the magnetic field is near equipartition or when the turbulent velocities are near transonic.

2.9. Poynting Flux

For future reference (in \(\S\) 4), we also note the mean Poynting flux:

\[
(S_i) = \langle \rho u_i \hat{v}_i \rangle + \langle \rho u_B v'_i \rangle - (\hat{t}^B) v'_i - \langle \hat{t}^B v'_i \rangle. \tag{25}
\]

2.10. Mechanical Energy of the Mean Flow

For completeness, we note that an equation for the mechanical energy can be derived by either taking the scalar product of the momentum equation, equation (14), with the mean velocity \(\hat{v}_i\) or subtracting the total internal energy equation, equation (21), from equation (22), giving

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \rho \hat{v}^2 + \rho \phi_G \right] + \frac{\partial}{\partial x_k} \left( \frac{1}{2} \rho \hat{v}^2 + \rho \phi_G \right) \hat{v}_i + \hat{v}_i \left( \frac{\partial \rho}{\partial x_k} - \frac{\partial (\rho u_j)}{\partial x_j} \right) = 0. \tag{26}
\]

2.11. The Flow of Energy in Accretion Disks

Having presented the various forms of the energy equations, describing the evolution of turbulent kinetic energy, turbulent magnetic energy, internal energy, and total energy, we are now in a position to describe the flow of energy within a turbulent fluid and specifically in accretion disks. These comments serve to justify the comments relating to dissipation and the inclusion of molecular viscous and resistive terms in this treatment.

As we have noted, both of the equations for the turbulent kinetic energy and the turbulent magnetic energy contain terms that describe the production of these quantities. These terms are \(\rho F_i S_{ij}\) and \(\rho F_i \xi_{ij}\), respectively. In each equation, there are also a number of terms operated on by a divergence that represent the flux of quantities that influence the respective turbulent energies. The fluxes are either advective fluxes, that is, fluxes due to the mean motion of the fluid, or turbulent diffusive fluxes, that is, fluxes associated with the diffusive effect of the turbulent velocity. There are also other terms expressing the effect of expansion or compression of the gas. In addition, there are important terms, \(- (t_{ij}) \hat{v}_i\) and \(- (\hat{t}^2/\sigma)\), for the turbulent kinetic energy and \(- (\hat{t}^2/\sigma)\) for the magnetic energy, that describe the dissipation of the turbulent energy via molecular processes.

3. ACCRETION DISK DYNAMICS

We now apply the generalized mean field equations derived in the preceding section to an accretion disk around a black hole. In this section we consider the implications of the radial, azimuthal, and vertical momentum equations. We then apply these results to the energy budget in an accretion disk in \(\S\) 4.

Here we present the full statistically averaged equations in a cylindrical \((r, \phi, z)\) coordinate system for a fluid that is time independent and axisymmetric in the mean \((\langle \partial / \partial t \rangle = \langle \partial / \partial \phi \rangle = 0\). We use Newtonian physics throughout, with the gravitational potential \(\phi_G = -GM(r^2 + z^2)^{-1/2}\). We integrate the mean field conservation equations vertically over an arbitrary disk scale height, \(h = h(r)\). Quantities calculated at the disk surface \((z = \pm h)\) are denoted by a \(\pm\) superscript, \(X^{\pm}\), and we assume reflection symmetry about the disk midplane, so that \(|X^+| = |X^-|\). Midplane values of variables are denoted by \(X_0\). Since we include the effects of a disk wind, we do not identify \(h\) as a hydrostatic scale height, but as a photospheric height, delineating between the disk proper and the transition region leading to a corona, by analogy with the solar atmosphere. We assume that \(h\) is much less than the radius, so that quantities of order \(h/r\) and \(dh/dr\) are neglected. The following comments are useful in the context of the vertical integration process. Let us represent a generic conservation law for mass, momentum, energy, etc., by

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r A_r \right) + \frac{\partial A_z}{\partial z} = S. \tag{27}
\]

Multiplying by \(2\pi r\) and integrating with respect to the vertical coordinate, \(z\), from \(-h\) to \(h\) gives

\[
\int_{-h}^{+h} \frac{\partial (2\pi r A_r)}{\partial r} \, dz + 2\pi r(A^+_r - A^-_r) = \int_{-h}^{+h} 2\pi r S \, dz. \tag{28}
\]
Taking the derivative with respect to $r$ outside of the integral gives
\[
\frac{d}{dr} \int_{-h}^{+h} 2\pi r A_r \, dz = 2\pi r \left( A_r^+ + A_r^- \right) \frac{dh}{dr} + 2\pi r \left( A_r^+ - A_r^- \right)
\]
\[
= \int_{-h}^{+h} 2\pi r S \, dz.
\]  
(29)
Since reflection symmetry about the midplane holds,
\[
|A_r^+| = |A_r^-|, \quad |A_r^+| = |A_r^-|.
\]  
(30)
Generally, there are two surface terms resulting from the integration over disk height, the second and third terms on the left-hand side of equation (29). Unless $|A_r| \gg |A_z|$, the first surface term, which is proportional to $dh/dr = O(h/r)$, is negligible compared to the second, so that the result of the integration over $z$ is
\[
\frac{d}{dr} \int_{-h}^{+h} 2\pi r A_r \, dz + 2\pi r \left( A_r^+ - A_r^- \right) \simeq \int_{-h}^{+h} 2\pi r S \, dz.
\]  
(31)
We believe that the geometrically thin disk approximation (Lynden-Bell & Pringle 1974) provides the most physically plausible disk solutions, given that geometrically thick, advection-dominated disks are found on assumptions (namely, preferential ion heating, negligible electron-ion coupling, and negligible electron heating overall) that are highly idealized, particularly in the presence of MHD turbulence (e.g., Bisnovaty-Kogan & Lovelace 1997, 2000; see also Merloni & Fabian 2002).

Since the MRI is a weak-field instability, it drives subsonic turbulence, with \( \langle \rho v^2 \rangle \ll \langle \rho c_s^2 \rangle \approx \tilde{\rho} v_K^2 \), where
\[
v_K = (GM/r)^{1/2}
\]  
(32)
is the Keplerian velocity and \( c_s = (kT/\mu m_p)^{1/2}(1 + p_{\text{rad}}/p_{\text{gas}})^{1/2} \) is the local sound speed. This includes a contribution from the radiation pressure, \( p_{\text{rad}} \), since numerical simulations (e.g., Turner et al. 2003; Turner 2004) show that the MRI-driven turbulent stresses grow in proportion to \( p_{\text{gas}} + p_{\text{rad}} \) when the optical depth is sufficiently high to couple the gas and photons.

An important distinction between our approach and that adopted in numerical simulation models (see Balbus & Hawley 1998) is that we do not identify the mean and fluctuating fluid velocity components with the Keplerian and non-Keplerian components, respectively. Instead, we adopt the more formal statistical averaging approach (as outlined in §2) of decomposing all velocity components into mean and fluctuating parts. Thus, we explicitly distinguish between mean radial, azimuthal, and vertical motions and their fluctuating counterparts. The fluctuating components are restricted to subsonic speeds. The only restriction we place on the mean fluid velocity components is that they satisfy \( \tilde{v}_r \gg \tilde{v}_\theta \) and \( \tilde{v}_r \) and \( \tilde{v}_\phi \) are independent of \( z \), simplifying the vertical integration. Thus, we are able to self-consistently treat radial and vertical transport of mass, momentum, and energy by the mean fluid velocity fields.

3.1. Mass Transfer
Vertical integration of the mean field continuity equation (10),
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \tilde{\rho} \tilde{v}_r \right) + \frac{\partial \langle \tilde{\rho} \tilde{v}_z \rangle}{\partial z} = 0,
\]  
(33)
gives
\[
\frac{d}{dr} \int_{-h}^{+h} 2\pi r \tilde{\rho} \tilde{v}_r \, dz + 4\pi r \tilde{\rho} \tilde{v}_z^+ = 0.
\]  
(34)
We now introduce the usual definitions for the surface mass density,
\[
\Sigma(r) \equiv \int_{-h}^{+h} \tilde{\rho} \, dz,
\]  
(35)
and mass accretion rate,
\[
\dot{M}_s (r) = 2\pi r \Sigma (-\tilde{v}_r).
\]  
(36)
We also introduce an analogous mass outflow rate,
\[
\dot{M}_w (r) = \int_r^{\infty} 4\pi r \tilde{\rho} \tilde{v}_z^+ \, dr = \int_r^{\infty} 4\pi r \tilde{\rho} \tilde{v}_z^+ \, dr,
\]  
(37)
associated with a mean vertical velocity \( \tilde{v}_z^+ \) at the disk surface, i.e., at the base of a wind.
Using the above definitions, the vertically integrated continuity equation, equation (34), can be written as
\[
\frac{d}{dr} \dot{M}_s (r) = 4\pi r \tilde{\rho} \tilde{v}_z^+ = - \frac{d}{dr} \dot{M}_w (r),
\]  
(38)
implicating that
\[
\dot{M}_s (r) + \dot{M}_w (r) = \dot{M}_s (r) + \dot{M}_w (r) = \text{const} = \dot{M},
\]  
(39)
where \( \dot{M}_s (r) + \dot{M}_w (r) \) is the total mass flux at the innermost stable orbit, \( r_i \). Equation (38) implies that under steady state conditions, the radial mass flux decreases toward small \( r \) at the same rate as the vertical mass flux increases in order to maintain a constant net mass flux, \( \dot{M} \), which is equivalent to the net accretion rate at \( r = \infty \).

3.2. Radial Momentum
The radial component of the mean field momentum equation, equation (14), is
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \tilde{\rho} \tilde{v}_r^2 \right) - \tilde{\rho} \tilde{v}_r \tilde{v}_r = \frac{\partial}{\partial z} (\tilde{\rho} \tilde{v}_z) = - \tilde{\rho} \frac{\partial \tilde{v}_z}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (r \tilde{v}_r) - \langle \tilde{v}_z \rangle \frac{\partial \langle \tilde{v}_z \rangle}{\partial z}.
\]  
(40)
Several of the terms in this equation are negligible compared to the dominant term, \( \tilde{\rho} \tilde{v}_r^2/r \). To determine which terms can be neglected, we first use the mean field continuity equation, equation (33), to simplify the \( \partial/\partial r \) and \( \partial/\partial z \) terms on the left-hand side. Then, evaluating the gravitational term on the right-hand side gives
\[
\tilde{\rho} \tilde{v}_r^2 = \frac{GM \tilde{\rho}}{r^2} + \frac{\partial \tilde{\rho}}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r} (r \tilde{v}_r) - \langle \tilde{v}_z \rangle \frac{\partial \langle \tilde{v}_z \rangle}{\partial z}.
\]  
(41)
The radial gradient terms can be dropped since they are negligible compared to \( \tilde{\rho} \tilde{v}_r^2/r \). However, the vertical gradient in the turbulent stress, \( \partial \tilde{v}_z^2/\partial z \), could be important in a geometrically
thin disk, so this term is retained. Integrating the remaining terms in equation (41) over \( r \) gives

\[
\frac{\Sigma \tilde{v}_z^2}{r} \simeq \frac{GM\Sigma}{r^2} - 2\langle t_{r z} \rangle^+, \tag{42}
\]

which reduces to

\[
\tilde{v}_z^2 \simeq v_K^2 - \frac{2r}{\Sigma} \langle t_{r z} \rangle^+. \tag{43}
\]

Thus, Keplerian rotation prevails in regions where the turbulent \( r z \) stresses on the disk surface are \( \leq h_{av}/r \) times smaller than \( \rho v_K^2 \), where \( h_{av} \) is the density scale height defined below (see eq. [45]). Since the turbulent stresses saturate at subsonic levels, they are unlikely to modify the Keplerian profile. Interestingly, models for flux emergence from the solar photosphere indicate that the Parker instability can enhance the poloidal flux sufficiently to modify the background shear in the rotation velocity, thus giving rise to an effective buoyant shear instability (e.g., Cline et al. 2003). In numerical simulations for accretion disks, however, this does not appear to be the case, with Keplerian rotation profiles emerging in all models (e.g., Hawley & Balbus 2002).

### 3.3. Vertical Momentum

Vertical momentum balance in the disk is obtained from the \( z \)-component of equation (14):

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \tilde{p} \tilde{v}_z \tilde{v}_z \right) + \frac{\partial}{\partial z} \left( \tilde{p} \tilde{v}_z^2 \right) = -\frac{GM\tilde{v}_z}{r^3} - \frac{\partial}{\partial z} \left( \tilde{p} - \langle t_{r z} \rangle \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \langle t_{r z} \rangle \right). \tag{44}
\]

We define a density-averaged disk height and a density-averaged vertical velocity by

\[
h_{av} = 2 \int_0^h \tilde{p} \tilde{v}_z \, dz, \quad \tilde{v}_{z, av} = \frac{2}{\Sigma} \int_0^h \tilde{p} \tilde{v}_z \, dz. \tag{45}
\]

Integration of equation (44) over \( z \) then yields

\[
\frac{1}{4\pi r} \frac{1}{dr} \left( -M_a \tilde{v}_{z, av} \right) + \frac{\partial}{\partial z} \left( \tilde{p} \tilde{v}_z^2 \right) \simeq -\frac{GMh_{av}}{2r^3} \rho_0 - \tilde{p}^+ + \langle t_{r z} \rangle^+ - \langle t_{r z} \rangle_0, \tag{46}
\]

where the \( r z \) stress term has been dropped since its contribution is \( \sim h_{av}/r \) that due to the vertical pressure gradients.

In order to determine the relative importance of the remaining terms, note that the first term on the left-hand side of equation (46) can be written as

\[
\frac{1}{4\pi r} \frac{d}{dr} \left( -M_a \tilde{v}_{z, av} \right) = -\tilde{p}^+ \tilde{v}_{z, av}^+ - \frac{1}{2} \Sigma (-\tilde{v}_z) \frac{d\tilde{v}_{z, av}}{dr}, \tag{47}
\]

where equations (36)–(38), involving the mass fluxes, have been used. The first term on the right-hand side of this equation is \( \tilde{v}_{z, av}/\tilde{v}_z^+ \ll 1 \) times smaller than the \( \tilde{p} \tilde{v}_z^+ \) term in equation (46) and is thus negligible. The relative importance of the second term on the right-hand side of the above equation depends on the radial gradient of \( \tilde{v}_z \); if we reasonably suppose that \( \partial \tilde{v}_z / \partial r \sim \tilde{v}_z / r \), then this term is smaller than the \( \tilde{p} \tilde{v}_z^+ \) term in the vertical balance equation, equation (46), by a factor \( \sim h/r \).

We thus neglect altogether the contribution from the first term on the left-hand side of equation (46) to the overall vertical momentum transport in the disk. Thus, vertical momentum balance in the disk implies a mean vertical outflow from the disk surface given by

\[
\tilde{v}_z^2 \simeq \frac{\tilde{p}_0 - \tilde{p}^+ + \langle t_{r z} \rangle^+ - \langle t_{r z} \rangle_0}{\tilde{p}^+} GMh_{av} \left( \frac{GMh_{av}}{2\tilde{p}^+ r^3} \right). \tag{48}
\]

The following well-known order-of-magnitude estimate for the hydrostatic equilibrium disk height is useful, and we repeat it here for the sake of completeness. Equation (48) with \( \tilde{v}_z = 0 \) and \( \Sigma \sim 2\rho_0 h_{av} \) (\( \rho_0 \) is the central disk density) gives

\[
\frac{h_{av}}{r} \sim \frac{c_0}{v_K}, \tag{49}
\]

where \( c_0 = [\langle \tilde{p}_0 + (B_z^2)/8\pi \rangle/\rho_0]^{1/2} \) is the generalized sound speed at the disk midplane.

The physical situation that we envisage in this paper is that of a disk, more or less in equilibrium, with a wind emanating from its surface. Therefore, to be consistent, we require the vertical velocity within the disk to be less than the disk sound speed. If the vertical velocity were comparable to or greater than the sound speed, then the disk would be in much more of a dynamic state. However, it is of interest to examine the conditions under which this “quasi-equilibrium” assumption may break down. First note that the above equation (48) for the vertical component of velocity is, by itself, insufficient as a means of estimating that quantity. In considering winds, it is always essential to involve the equation of continuity. Thus, consider a disk in which we neglect the mass accretion rate compared to the vertical wind loss. Then, the vertical equation simplifies to

\[
\frac{\partial}{\partial z} \left( \rho \tilde{v}_z^2 \right) = -\frac{GM}{r^3} \rho z - \frac{\partial}{\partial z} \left( \tilde{p} - \langle t_{r z} \rangle \right). \tag{50}
\]

If we assume an approximately isothermal disk with \( \partial \tilde{p} / \partial z \approx c_s^2 \partial \tilde{\rho} / \partial z \) and use the equation of continuity in the form \( \partial \tilde{\rho} / \partial z = -(\rho / \tilde{v}_z) \partial \tilde{v}_z / \partial z \), then we obtain the following equation for the vertical velocity:

\[
\frac{1}{\rho} \frac{\partial \tilde{v}_z}{\partial z} = -\frac{GM\tilde{v}_z}{2r^3} + \frac{\partial}{\partial z} \langle t_{r z} \rangle / \rho (\tilde{v}_z^2 - c_s^2). \tag{51}
\]

Assuming that the magnetic stress dominates the turbulent stress \( \langle t_{r z} \rangle \) and the largest component of the magnetic field is the toroidal component, we have \( \langle t_{r z} \rangle \approx \langle B_z^2 - B_t^2 + B_o^2 \rangle / 8\pi \approx -\langle B_o^2 \rangle / 8\pi \). If we assume that the vertical gradient of \( B_o^2 \) is negative, then this equation admits the possibility of a critical point where \( \tilde{v}_z = \pm c_s \) and \( \rho = \rho_c \) at the location \( z = z_c \) determined by

\[
\rho_c GM \frac{z_c}{r^3} = \frac{\partial \langle B_o^2 \rangle / 8\pi}{\partial z}. \tag{52}
\]

Using the expression for \( h_{av} \) (eq. [49]), Keplerian velocity (eq. [32]), and order-of-magnitude purposes taking \( \partial \langle B_o^2 \rangle / \partial z \sim \langle B_o^2 \rangle / h_{av} \), this condition can be expressed in the form

\[
\frac{z_c}{h_{av}} \approx \left( \frac{\rho_c}{\rho_0} \right)^{-1} \frac{\langle B_o^2 \rangle / \rho_0}{2c_0^2}, \tag{53}
\]
where \(v_{A0}\) is the midplane Alfvén speed. When the sound speed is of order the Alfvén speed, the exponentially vanishing density in an isothermal disk would make it difficult to satisfy equation (53) within a few disk scale heights. That is, the disk vertical velocity would remain subsonic within the disk. However, when radiation pressure starts to dominate and the ratio of Alfvén speed to sound speed decreases, then it may be possible that this equation has a solution for \(z_{e} \ll \text{few times} \ h_{av}\). Hence, increasing radiation pressure at inner radii could cause a departure from the quasi-equilibrium picture that we have adopted. However, recently the importance of photon bubble instabilities in the inner regions of accretion disks has come to be appreciated. Following the work of Gammie (1998) and Blaes & Socrates (2001) showing that magnetized disks are unstable to a photon bubble instability, Begelman (2002) argued that radiation may escape with a super-Eddington luminosity without having a strong effect on the disk equilibrium. Moreover, Blaes & Socrates (2003) showed that the photon bubble instability also applies when the disk is gas pressure dominated. Clearly, these are points that need to be considered in detail in future. For now, we assume a quasi-equilibrium disk.

3.4. Conditions for a Disk Wind

In the context of the specific accretion disk application being considered, a geotropically thin, zero net magnetic flux disk, with turbulence driven by the MRI instability, the thermal pressure cannot drive an energetically significant, large-scale wind, not even for a coronal temperature \(\sim 100\ \text{keV}\). For example, by examining the Bernoulli equation (without magnetic field), it is straightforward to show that the thermal temperature required to drive an outflow in the vicinity of a black hole is \(kT > \frac{1}{2} \gamma (1 - 1) \frac{GM \mu m_p}{r} = \frac{1}{2} \frac{v_{esc}}{c} \sim 10^9 \frac{\gamma}{r} \text{keV}\), where \(\gamma\) is the adiabatic index, \(v_{esc}\) is the escape speed, and \(r_0 = \frac{GM}{c^2} \simeq 1.5 \times 10^{12} M_7\) cm is the gravitational radius of a black hole of mass \(M = 10^7 M_7\) \(M_\odot\).

If the disk luminosity and the corresponding mass accretion rate are super-Eddington, then a radiation-driven outflow driven by the disk is inevitable. However, as noted above, the development of a photon bubble instability means that the disk can remain thin (Begelman 2002).

On the other hand, for sub-Eddington accretion rates, an important driving source for a large-scale wind is the magnetic field. One may derive approximate conditions for the production of a wind, under these circumstances, as follows. Let \(v_{A}\) be the Alfvén speed, with mean value defined by

\[
v_{A} = \left(\frac{B_{\phi}^2}{4\pi \rho}\right)^{1/2},
\]

and consider the vertical component of the energy flux at the disk surface without turbulent diffusion terms:

\[
\left\langle F_{z}^E \right\rangle^+ \simeq \left[\frac{1}{2} \tilde{B}^2 + \phi_G + \tilde{h} + \frac{\left(\tilde{B}^2 + B_{\phi}^2\right)}{4\pi \tilde{\rho}^+} \right] \rho^+ \tilde{v}_z^+ - \left(\frac{\tilde{B}^2}{4\pi \rho}\right)^{1/2} v_{K}.
\]

The azimuthal magnetic field component, \(B_{\phi}\), should dominate, so we set \(\left(\tilde{B}^2 + B_{\phi}^2\right)/4\pi \tilde{\rho}^+ \simeq (\tilde{v}_{\phi}^+)\). There are negative \(\left[\frac{1}{2} \tilde{B}^2 + \phi_G = -\frac{1}{2} (GM/r)\right]\) and positive \(\left(\tilde{h} + \left(\tilde{v}_{\phi}^+\right)\right)\) contributions to the first bracket of terms in the energy flux. The last term (the centrifugal term) makes a positive contribution to the energy flux if the magnetic stress \(\left(\tilde{B}^2 B_{\phi}^2\right)/4\pi < 0\). In a steady state, the energy flux, integrated over the extent of the wind, is conserved, and in order for the wind to escape to infinity, the integrated energy flux should be positive. Neglecting, in the first instance, the centrifugal term and disregarding the specific enthalpy, the energy flux is positive if

\[
v_{A} \gtrsim \frac{1}{\sqrt{2} v_{K}}, \quad (56)
\]

that is, the Alfvén speed at the disk surface should be comparable to, or exceed, the local Keplerian speed.

Now let us suppose that the magnetic stress term is systematically negative and that, owing to the factor of the Keplerian velocity, the term \(-\left(\tilde{B}^2 B_{\phi}^2\right)/4\pi \tilde{\rho}^+ v_{K}/4\pi\) is significantly greater than \(\tilde{h}^+ (\tilde{v}_{\phi}^+)^2\). Then, the condition for the energy flux to be positive is

\[
\frac{-\left(\tilde{B}^2 B_{\phi}^2\right)}{4\pi} v_{K} > \left(\frac{1}{2} \frac{v_{K}^2 + \phi_G}{\tilde{\rho}^+} \right) \tilde{h}^+ \tilde{v}_z^+, \quad (57)
\]

and since \(\phi_G = -v_{K}^2\) and using equation (38) for \(d\dot{M}_w/dr\), this condition becomes

\[
\frac{d\dot{M}_w}{dr} \lesssim -8\pi \left(\frac{\tilde{B}^2 B_{\phi}^2}{4\pi v_{K}}\right), \quad (58)
\]

that is, the existence of a wind in this case implies an upper limit on the wind mass-loss rate.

A more detailed investigation of the conditions for the initiation of disk winds and the corresponding mass-loss limits is beyond the scope of this paper. We note, however, the results of Meier et al. (1997) and Meier (1999), pertaining to the case of a net poloidal field, showing a change in character of disk winds, from loosely collimated to jetlike when the Alfvén speed increases from below \(v_{K}\) to above that parameter. Their results and the above order-of-magnitude estimate suggest that the condition given by equation (56) is an interesting critical value. Note, however, that equation (58) is potentially a weaker condition on the magnetic field and can be expressed as

\[
\frac{(v_{A}^+)^2}{v_{K}^2} \left(\frac{B_{\phi}^2}{B_{\phi}^2}\right) \gtrsim \frac{1}{2} \frac{\tilde{v}_z^+}{\tilde{v}_{K}^+} \sim \frac{h_{av}}{r} \frac{\tilde{v}_z^+}{c_0}. \quad (59)
\]

Given that the vertical wind velocity (at the base of the wind) is likely to be no greater than the disk sound speed and since \(h_{av}/r \ll 1\), the right-hand side of this inequality is much less than unity. Thus, even allowing for the fact that the vertical component of the magnetic field may be significantly less than the magnitude of the magnetic field, the coronal Alfvén speed implied by equation (59) should be considerably less than the local Keplerian speed. Again, given the remarks relating to possible radiation-pressure-dominated zones in \(\tilde{h}\), this conclusion may need to be qualified. Thus, given these estimates and the previous work by Meier and colleagues, it will be of interest to assess the validity of the above wind conditions numerically, especially in the case of a zero net magnetic flux.

Satisfaction of the above conditions (but especially eq. [59]) is possible in principle, even if the magnetic field in the interior of the disk is weak enough for the MRI instability to apply. If the magnetic field is driven into the corona by buoyancy and
does not decrease as rapidly as the square root of the density, then the Alfvén speed in the corona could become quite high. This is, in fact, a feature of the Miller & Stone (2000) simulations, which therefore support the possibility of a disk wind, in the case of zero net magnetic flux, even though a strong wind was not produced in their work.

3.5. Angular Momentum

The azimuthal component of equation (14) is

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \bar{v}_\phi \frac{\partial \bar{v}_\phi}{\partial r} \right) + \frac{\partial}{\partial z} \left( \bar{p} \bar{v}_\phi \bar{v}_z \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \langle \dot{\tau}_{\phi \phi} \rangle \right) + \frac{\partial}{\partial z} \langle \dot{\tau}_{\phi z} \rangle.
\]  

(60)

Let us define the vertically integrated \( r \phi \) stress by

\[
\tau_{r\phi} = \int_{-h}^{+h} \tau_{r\phi} \; dz.
\]  

(61)

Integrating equation (60) over \( z \) and applying the mass continuity given by equation (38) gives

\[
\frac{d}{dr} \left( \dot{M}_d \bar{v}_\phi \bar{r} + 2 \pi r^2 \tau_{r\phi} \right) = \bar{v}_\phi \frac{d \dot{M}_d}{dr} - 4 \pi r^2 \langle \dot{\tau}_{\phi z} \rangle^+.
\]  

(62)

The terms on the left-hand side of this equation describe radial transport of angular momentum due to the mean radial inflow and turbulent MHD stresses, while the terms on the right-hand side describe vertical angular momentum transport due to mass loss in a wind and turbulent stresses on the disk surface. Although we do not explicitly consider the effects of a nonzero mean magnetic field here, we note that equation (62) can be straightforwardly generalized to include electromagnetic torque terms analogous to the turbulent MHD stress terms involving \( \tau_{r\phi} \) and \( \langle \dot{\tau}_{\phi z} \rangle \).

Radially integrating equation (62) gives

\[
\dot{M}_d \bar{v}_\phi \bar{r} - \dot{M}_d \langle r \rangle \bar{v}_\phi \langle r \rangle \bar{r} + 2 \pi r^2 \tau_{r\phi} - 2 \pi r^2 \tau_{r\phi} \langle r \rangle = \int_{r_1}^{r} dr \left( \bar{v}_\phi \frac{d \dot{M}_d}{dr} - 4 \pi r^2 \langle \dot{\tau}_{\phi z} \rangle^+ \right),
\]  

(63)

and using the integrated continuity equation, \( \dot{M}_d + \dot{M}_w = \dot{M} \), this can be conveniently expressed in the form

\[
\dot{M} \bar{v}_\phi \bar{r} + 2 \pi r^2 \tau_{r\phi} - 2 \pi r^2 \tau_{r\phi} \langle r \rangle = \int_{r_1}^{r} dr \left( \dot{M}_w \frac{d}{dr} \langle \bar{v}_\phi \bar{r} \rangle - 4 \pi r^2 \langle \dot{\tau}_{\phi z} \rangle^+ \right),
\]  

(64)

where

\[
\zeta(r) = 1 - \frac{\bar{v}_\phi \langle r \rangle \bar{r}}{\bar{v}_\phi \langle r \rangle \bar{r}}.
\]  

(65)

Note that we do not assume that the \( r \phi \) stress is zero at \( r = r_1 \). While the assumption \( \tau_{r\phi} \langle r \rangle \bar{r} \simeq 0 \) is often made in conventional disk models, its justification rests on the premise (in rotating disk models) that the stresses scale as the disk gas pressure, which falls off dramatically as the matter passes across the marginally stable orbit. When the stresses are turbulent, on the other hand, this assumption is no longer valid. Indeed, numerical simulations (Hawley & Krolik 2002) show that turbulent stresses persist at and inside \( r_1 \). As can be inferred from equation (64), a nonzero \( r \phi \) stress at \( r_1 \) requires an increased torque at large \( r \) to transport the same amount of angular momentum over the same radial distance. The fact that \( \tau_{r\phi} \langle r \rangle \bar{r} \) does not generally vanish also has important implications for energy balance in a turbulent disk, since this stress does work on the rotating disk and the resultant turbulent energy can be dissipated at a comparable rate, thereby providing an additional source of internal heating at \( r_1 \). This is demonstrated explicitly in §4.

An important property of MRI-driven turbulence is that the \( r \phi \) stress has the same sign as \( \dot{\Omega} \) (Balbus & Hawley 1998) and thus, even in the absence of a net vertical angular momentum flux, the \( \tau_{r\phi} \) stress alone can facilitate the outward transport of angular momentum required for accretion to proceed. Whether accretion proceeds at an interesting rate, however, is another issue deserving separate attention. Consider a Keplerian disk in which the mass accretion rate varies as \( \dot{M}_r \propto r \dot{p} \), where \( 0 < p < 1 \) is the mass-loss index used, for example, by Blandford & Begelman (1999; see also Becker et al. 2001).

Substituting into the integrated angular momentum equation (63) yields

\[
\frac{1}{2p + 1} \left[ 1 - \left( \frac{r}{r_1} \right)^{-p/2} \right] \approx \frac{G_{r\phi}(r)}{M_d v_{K} r} \frac{G_{\phi z}(r)}{G_{r\phi}(r)} + \frac{G_{r\phi}(r)}{M_d v_{K} r} \left[ 1 - \frac{G_{\phi z}(r)}{G_{r\phi}(r)} \right],
\]  

(66)

where \( G_{r\phi}(r) = 2 \pi r^2 (- \tau_{r\phi}) \) and \( G_{\phi z}(r) = - \int_{r}^{\infty} 4 \pi r^2 \; dr \langle \dot{\tau}_{\phi z} \rangle^+ \) are the torques associated with the turbulent MHD stresses. This implies that if \( G_{r\phi} \), \( G_{\phi z} \propto r^{-(p+1)/2} \), then

\[
\dot{M}_d v_{K} r \approx (G_{r\phi} + G_{\phi z})(2p + 1),
\]  

(67)

which in turn implies that when vertical transport of angular momentum is taken into account, the mass accretion rate is enhanced by a factor \( (1 + G_{\phi z}/G_{r\phi})(2p + 1) \). Conversely, when the vertical angular momentum flux is neglected, \( v_{K} \) scales as \( \bar{v}_r v_{K} \), a result obtained in numerical models (Hawley & Krolik 2002).

This result can be used to estimate the minimum mass accretion rate at \( r_1 \), where the MRI-driven turbulent stresses saturate at levels comparable to the total pressure, which can include a contribution from the radiation field if the optical depth is sufficiently high to couple the gas and photons (Turner et al. 2003; Turner 2004).

The relation \( \dot{M}_d = 2 \pi r^2 \Sigma(\bar{v}_r) \) can be expressed as

\[
\dot{M}_d = \varepsilon \tau_{r\phi} \frac{r_1}{r_g} \frac{\bar{v}_r}{c} \dot{M}_{\text{edd}},
\]  

(68)

where \( \tau_{r\phi} = \int_{0}^{h} (\sigma_{\text{T}} / m_p) \; \rho \; dz = \frac{1}{4} \Sigma \sigma_{\text{T}} / m_p \) is the Thomson optical depth of the disk over its vertical scale height and \( \dot{M}_{\text{edd}} = 4 \pi G M_M / (c \sigma_{\text{T}}) \approx 0.3 \Sigma e_{\odot}^{-1} M_7 M_7 \; \text{yr}^{-1} \) is the Eddington mass accretion rate for a conversion efficiency \( \varepsilon = 0.1 e_{\odot}^{-1} \).

The conditions \( \bar{v}_r \langle r \rangle \bar{r} \simeq \tau_{r\phi} \langle r \rangle \bar{r} / v_{K} \langle r \rangle \) and \( \dot{M}_d = \text{const.} \), satisfied when angular momentum is solely transported by the turbulent \( r \phi \) stresses, combined with the condition \( v_{K} \langle r \rangle \bar{r} \approx -c_{s}^2 \langle r \rangle \bar{r} / v_{K} \langle r \rangle \), under which the stresses saturate at \( r_1 \), then imply a lower limit to the dimensionless mass accretion rate

\[
\dot{m}_d \equiv \frac{\dot{M}_d}{\dot{M}_{\text{edd}}} \gtrsim 3 \times 10^{-8} \tau_{r\phi}^{1/2} \tau_{r\phi}^{3/2} \Sigma_{\text{T}} \left( \frac{1}{p_{\text{rad}}} \right),
\]  

(69)
where $T_3(r_i) = T(r_i)/10^5$ K is of order unity for accretion onto a supermassive black hole.

This result implies that in nonradiative disks ($p_{na} = 0$), extremely large optical depths ($\tau_T \sim 10^5$) are required for accretion to proceed efficiently (i.e., $\dot{m}_a \geq 0.1$). Indeed, substantially sub-Eddington mass accretion rates are a characteristic property of numerical models of nonradiative disks in which angular momentum is transported solely by $t_{rp}$ (see, e.g., Stone & Pringle 2001; Hawley et al. 2001; Hawley & Balbus 2002). It would thus be of interest to test whether $\dot{m}_a$ can be increased in these models by simply increasing $G$, as predicted by equation (69). In radiative disks, on the other hand, equation (69) predicts that radiation pressure can enhance mass accretion (because the stresses grow as the gas plus radiation pressure when the opacity is not too small). In radiation pressure-dominated disks, however, we also expect radiatively driven disk outflows; therefore, moderate- to high-mass accretion rates cannot solely be attributed to internal $r\phi$ stresses and must be largely attributed to a combination of vertical transport and mean field torques (including large-scale MHD outflows, which are implicit in our equations).

Note, however, that the role of the $G_{\phi z}$ torque in vertical angular momentum transport remains unclear. In principle, $G_{\phi z}$ could be larger than $G_{rp}$ by as much as $r/h$, in which case this surface torque could play a key role in the energetics of thin disks. Consider the condition given by equation (62) for angular momentum conservation. For a Keplerian disk, this implies

$$\frac{\dot{M}_a}{\dot{r}_K} - \left(2\pi r^2 T_{rp}\right) = \frac{-4\pi r^2 \dot{t}_{\phi z}}{M_{\phi z}}.$$  \hspace{1cm} (70)

It is then straightforward to show that the $\phi z$ stress has an important effect on angular momentum transport when

$$\frac{-8\pi r^2 \dot{t}_{\phi z}^+}{\dot{M}_{\phi z}} \sim 1$$  \hspace{1cm} (71)

and that the $\phi z$ stress is important relative to the $r\phi$ stress when

$$\dot{t}_{\phi z}^+ \sim \frac{\dot{h}_{\phi z}}{r}, \dot{t}_{\phi z}^0 = \dot{t}_{\phi z}/\dot{r}_K.$$  \hspace{1cm} (72)

This extraordinary result reflects the fact that $\langle \dot{t}_{\phi z} \rangle^+$ acts over the surface of the disk but $\langle \dot{t}_{\phi z} \rangle^0$ acts only over the height. The simulations reported by Miller & Stone (2000) produce a volume-averaged $\langle \dot{t}_{\phi z} \rangle^+ \sim 0.02\langle \dot{t}_{\phi z} \rangle^0$. Even this low a stress is capable of physically interesting effects, as we argue in the following section when we consider the implications of mass and momentum transport for the energetics of turbulent accretion disks.

4. THE DISK ENERGY BUDGET

We now apply the above deductions from the momentum balance in a magnetized, turbulent accretion disk to calculate the power generated in the disk and emerging from the disk surface as both radiation and mechanical plus nonradiative electromagnetc energy. Dissipation of the latter two components may generate further emissivity in a corona and/or outflow. Our basis for this calculation is the total energy equation (22). Usually in expositions of accretion disk theory, the internal energy equation is used, since this explicitly identifies energy exchange processes, in particular, the transfer of free energy from the Keplerian shear in the bulk flow to the stresses that then dissipate that energy in the disk. In a turbulent MHD disk, this energy transfer is largely attributed to the turbulent energy production term $\langle \dot{t}_{\phi z} \rangle^+ \sim \langle \dot{t}_{\phi z} \rangle^0$, which is related to the rate of viscous dissipation via a turbulent cascade and the rate of bulk heating of the fluid via various additional radial and vertical transport terms. However, the internal energy equation for a turbulent, MHD accretion disk involves many intermediate terms that are cumbersome. The total energy equation, on the other hand, has the advantage that all terms are in total fluxes, which can be integrated straightforwardly to obtain a conservation equation in terms of the net power in each of the available energy fluxes. We therefore prefer to use the total energy equation, which more elegantly describes the contributions of all terms to the overall energy balance in a disk.

4.1. Order-of-Magnitude Estimates

The total energy flux contains a number of terms, and in estimating the relative importance of these, it is useful to deduce order-of-magnitude estimates of the various components of velocity. These are the mean velocity components, $\tilde{v}_r$, $\tilde{v}_\phi$, $\tilde{v}_z = (GM/r)^{1/2}$, and $\tilde{v}_\phi$, and the corresponding fluctuating components, $v_r$, $v_\phi$, $v_z$, and $v_\phi$.

The component $\tilde{v}_r$ is the radial inflow speed associated with the mass accretion rate, viz., $\tilde{v}_r = -M_{\phi z}/2\pi r \Sigma$. An estimate of this quantity depends on which stress dominates equation (70). If the $r\phi$ stresses dominate, then $|\tilde{v}_r| \sim |\dot{t}_{\phi z}|/\Sigma$, implying that

$$\tilde{v}_r \sim \frac{c_s}{\bar{v}_K} \frac{|\dot{t}_{\phi z}|}{\rho_0 c_s^2}.$$  \hspace{1cm} (73)

If the $\phi z$ stresses dominate angular momentum transport, then, approximating the radial derivative in equation (62) by division by $r$, we obtain

$$\tilde{v}_r \sim \frac{c_s}{h_{\phi z}} \frac{|\dot{t}_{\phi z}|}{\rho_0 c_s^2}.$$  \hspace{1cm} (74)

If we take $|\dot{t}_{\phi z}| \sim \rho_0 v^2$ (where $v$ is the magnitude of the turbulent velocity), then since the turbulence is weak ($v^2 \leq c_s^2$), the first estimate of $\tilde{v}_r$ based on the $r\phi$ stresses implies $|\tilde{v}_r|/c_s \ll 1$ for an optically thick disk (see also the estimates in § 3.5). If the $\langle \dot{t}_{\phi z} \rangle^+$ stresses are of similar magnitude to $|\dot{t}_{\phi z}|$, then $\tilde{v}_r$ is larger by a factor of $r/h$ when angular momentum transport by $\langle \dot{t}_{\phi z} \rangle^+$ is important. Thus, if the disk is sufficiently thin that $h_{\phi z}/r \lesssim |\dot{t}_{\phi z}|/|\dot{t}_{\phi z}|$, then the $\phi z$ stresses could increase the radial Mach number $\tilde{v}_r/c_s$ to a value above $c_s/v_K$. So far, simulations of turbulent accretion disks (e.g., Miller & Stone 2000) estimate these stresses to be approximately an order of magnitude smaller than the $\langle \dot{t}_{\phi z} \rangle$, since the MRI preferentially amplifies the $r$- and $\phi$-components of the fluctuating fields. The conclusion from these estimates is that the radial Mach number is likely to be small under most conditions. This means that most terms in the total energy equation involving $\tilde{v}_r$ can be neglected, except those involving multiplication by a large quantity such as $\tilde{v}_r \sim v_K$ or the gravitational potential.

As far as the relative values of kinetic and magnetic stresses are concerned, we rely to some extent, although not exclusively, on numerical results, which show that the turbulent magnetic stresses are a factor of a few higher than the Reynolds stresses.
The fluctuating velocity components $v'_r$ and $v'_z$ are less than the sound speed, so that we neglect the turbulent energy flux terms related to these quantities. However, vertical gradients in turbulent energy flux terms involving $v'_z$ should in general be retained since, in addition to being affected by the MRI, this component of the fluctuating velocity can also be enhanced by the Parker instability and the total rate of vertical energy transport can be nonnegligible in a thin disk. Turbulent perturbations in a magnetized disk become unstable to the gravitational modes of the Parker instability as matter drains down slightly elevated flux tubes, thereby enhancing their buoyant rise to the disk surface. The buoyant velocity field generated in this way can legitimately be considered as a turbulent velocity since regions of rising underdense flux tubes are balanced by falling, overdense regions and the net mass-averaged velocity is zero. We estimate the buoyant velocity of such a flux tube by balancing the buoyant force per unit length corresponding to a density deficit of $-\delta\rho$ with the drag force $\propto C_d h^2$, where $C_d \sim 1$ is the drag coefficient. Thus, we estimate the vertical buoyant velocity of a tube of radius $R_{\text{tube}}$ at a height $z$ to be (e.g., Parker 1979, p. 314)

$$v'_z \approx \left(\frac{\pi}{C_d} \right)^{1/2} \left(\frac{\delta\rho}{\rho} \right)^{1/2} \left(\frac{z R_{\text{tube}}}{h^2} \right)^{1/2} \left(\frac{h_{av}}{r} \right)^{1/2}.$$  

This velocity depends on the density contrast developed within a flux tube, the height of the tube above the midplane, and the tube radius, all of which are uncertain. For modest values of $\delta\rho/\rho$ and $R_{\text{tube}}/h_{av}$, the buoyant velocity can be an appreciable fraction of $c_0$ but is unlikely to be greater than $c_0$. The Miller & Stone (2000) simulations show an average kinematic energy associated with $v'_z$ comparable to that associated with $v'_r$ and $v'_\phi$, consistent with this assertion.

As we have indicated earlier, the vertical mean flow velocity at the base of the corona, $\overline{v}_z$, cannot be estimated without a specific model. Therefore, lacking any detailed knowledge of the magnitude of this velocity component, we retain terms in the vertical energy flux associated with it.

In applying the total energy equation, we also neglect the viscous and resistive terms in the energy flux, $-\langle \tau_{ij} v'_j \rangle + \langle \eta M_{ij} \rangle_{ij}$. As we noted in § 2, these terms could be locally important in, for instance, shocks or reconnection regions, but when the energy equation is integrated over a large volume, their contribution to the surface integral is minor (see, e.g., Hawley & Balbus 2002 for a discussion on resistive heating in turbulent disks). Moreover, it is unlikely that the almost singular surfaces on which these terms are important would intersect substantially with the annular surfaces, $r < r < r + dr, -h < z < h$, over which we integrate the total energy equation. In addition, neither term is important in the high-wavenumber end of a turbulent cascade since they involve less spatial differentiation than the Joule heating and viscous dissipation terms that appear in, say, the internal energy equation. Those terms are locally important but are not required in the integration of the total energy equation.

4.2. Application of the Total Energy Equation

Let us now calculate the net power available from each of the energy flux terms in the statistically averaged equation for total energy conservation, given by equation (22), taking into account the order-of-magnitude estimates deduced in the preceding section. For an axisymmetric, steady state disk,

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \langle F^E_z \rangle \right) + \frac{\partial}{\partial z} \langle F^E_z \rangle = 0,$$  

where $\langle F^E_z \rangle$ and $\langle F^E_r \rangle$ are the radial and vertical components of the total mean energy flux $\langle F^E \rangle$, defined by equation (24). Integration over the disk height yields

$$\frac{d}{dr} \int_{-h}^{+h} 2\pi r \langle F^E_z \rangle dz + 4\pi r \langle F^E_z \rangle^+ \left|_{r=0} \right. = 0,$$  

and this equation is the basis of the calculation of both the differential and total radiative energy flux from the disk.

The dominant terms in the radial component of the total energy flux give

$$F^E_r \simeq \left( \frac{1}{2} \bar{u}^2 + \phi_G \right) \bar{\rho} \bar{v}_r - \langle t_{\phi 0} \rangle \bar{\rho} \bar{\phi}_0 + \langle F_r \rangle.$$  

Upon vertical integration over a thin disk, followed by differentiation with respect to radius, the last two terms become $O(h_{av}/r)$ compared to analogous terms in the vertical energy flux, implying

$$\int_{-h}^{+h} 2\pi r \langle F^E_z \rangle dz \simeq \left( \frac{1}{2} \bar{u}^2 + \phi_G \right) \bar{M}_z - 2\pi r \bar{t}_{\phi 0},$$  

where $T_{\phi 0}$ is the vertically integrated $\bar{\rho} \bar{\phi}$ stress defined by equation (61). The term $-2\pi r \bar{t}_{\phi 0} \simeq -2\pi r \bar{t}_{\phi 0} v_K$ can be eliminated using equation (64) for angular momentum conservation:

$$-2\pi r \bar{t}_{\phi 0} v_K = M_z \bar{v}_z^{(1)}(r) - 2\pi r^2 \bar{t}_{\phi 0}(r) \bar{\Omega}_K(r)$$

$$- \bar{\Omega}_K \int_{r'} \left\{ \bar{M}_z(r') \left( \frac{d}{dr'} [r' \bar{v}_K(r')] - 4\pi r'^2 \langle t_{\phi z} (r') \rangle \right) \right\} dr',$$  

where

$$\bar{v}_z^{(1)}(r) = 1 - \langle r_z / r \rangle^{1/2}.$$  

The vertical component of the energy flux is

$$\langle F^E_z \rangle = \left( \frac{1}{2} \bar{u}^2 + \bar{\rho} + \bar{\phi}_G + \bar{\phi} \bar{h}^2 + \langle u_K \rangle + \langle u_B \rangle + \langle \rho h^2 \rangle + \langle u_K t_z \rangle + \langle u_B t_z \rangle + \langle u_B t_z \rangle + \langle \rho h' t_z \rangle + \langle \rho h' t_z \rangle + \langle F_z \rangle + \langle Q_z \rangle \right) \bar{v}_z^+$$

$$+ \langle \rho h' t_z \rangle + \langle u_K t_z \rangle + \langle u_B t_z \rangle + \langle u_B t_z \rangle + \langle F_z \rangle + \langle Q_z \rangle.$$  

There are some terms in this expression, such as the first two terms representing the vertical flux of kinetic plus gravitational energy in a wind and the term $\langle t_{\phi z} \rangle \bar{v}_z$, that we should obviously keep. There are also some terms in this equation that we can immediately discard. The term $\bar{\rho} \bar{h}^2 \bar{v}_z^+ = (\bar{u}^+ + \bar{\rho}^+) \bar{v}_z^+$ is small at the photosphere-corona boundary. For similar reasons we can also neglect the turbulent enthalpy flux term $\langle \rho h' t_z \rangle$. In view of the order-of-magnitude estimates considered in § 4.1, the remaining terms may be comparable. To simplify the final expression, we combine the remaining terms
involving a vertical velocity component into an energy flux term $\langle \psi_z \rangle^+$, defined by

$$
\langle \psi_z \rangle^+ = \langle (u_K + u_B) - (t_z) \rangle^+ + \langle (u_K + u_B - t_z) \rangle_z^+ - \langle t_{o,z} \rangle^+ .
$$

(83)

With these simplifications, the vertical component of the total energy flux reduces to

$$
4\pi r \langle F_z^E \rangle^+ \approx 4\pi r \left( \frac{1}{2} v_z^2 + \phi_G \right) \hat{\rho}^+ \hat{v}_z^+ - 4\pi r \langle t_{o,z} \rangle^+ v_\phi + 4\pi r \langle \psi_z \rangle^+ + \langle F_z \rangle^+ + \langle Q_z \rangle^+ .
$$

(84)

Substitution of equations (79) and (84) into the disk total energy equation (77), utilizing equation (80), gives the following for the radiative flux from both sides of the disk (after some algebra):

$$
4\pi r \langle F \rangle^+ = \frac{3}{2} \frac{G M \dot{M}}{r^2} \langle \psi_{k1} \rangle (r) - 3\pi r^2 T_{\text{eq}}(r) \frac{\Omega_k}{r} - \frac{3}{2} \frac{\Omega_k}{r}

\times \int_{r_i}^r \left[ \frac{1}{2} M_w(r) v_k(r) - 4\pi r^2 \langle t_{o,z}(r) \rangle^+ \right] dr' - 4\pi r \langle \psi_z \rangle^+ - 4\pi r \langle Q \rangle^+ .
$$

(85)

The first term on the right-hand side of this expression represents the binding energy flux associated with the net mass flux and is familiar from standard disk theory (cf. Shakura & Sunyaev 1973).

The second term represents a correction resulting from a nonvanishing stress at the inner radius. Such a term with the same radial scaling of surface flux ($\propto r^{-3/2}$) has been derived by Agol & Krolik (2000). In that case the stress on the inner boundary is associated with the magnetic stress exerted by the magnetic field of the black hole. In the model we have developed here, there is no magnetic field, and the stress simply relates to residual turbulence as the fluid accretes through the inner boundary.

The third term is related to the flux of kinetic and gravitational energy in the wind; this term is discussed further below.

The fourth term represents the effect of the $\phi z$ turbulent stresses, and its magnetic component is intimately associated with the Poynting flux into the corona; this term is also discussed further below.

The fifth term, $-\langle \psi_z \rangle^+$, represents another component of the wind power associated with the advective part of the Poynting flux and advection of other turbulent quantities.

The sixth term, $-\langle Q_z \rangle^+$, represents the effect of an external heat flux arising from above the disk. Since $\langle Q_z \rangle < 0$, this heating enhances the radiative flux from the disk.

The above form of the radiative flux is convenient for integration since it involves the (constant) total mass accretion rate $\dot{M}$ in the first term and the factor $\zeta_{k1}^{(1)}(r)$ has a simple analytical form. Nevertheless, in order to facilitate comparison of the various terms, the third term may be integrated by parts and partially combined with the first term to give

$$
4\pi r \langle F_z \rangle^+ = \frac{3}{2} \frac{G M \dot{M}}{r^2} \langle \psi_{k1} \rangle (r) - 3\pi r^2 T_{\text{eq}}(r) \frac{\Omega_k}{r} - \frac{3}{2} \frac{\Omega_k}{r}

\times \int_{r_i}^r \left[ \hat{\rho}^+ \hat{v}_z^+ v_k(r') - \langle t_{o,z}(r') \rangle^+ \right] dr' - 4\pi r \langle \psi_z \rangle^+ - 4\pi r \langle Q \rangle^+ ,
$$

(86)

where

$$
\zeta_{k1}^{(2)}(r) = 1 - \left( \frac{r_i}{r} \right)^{1/2} \frac{M_w(r)}{M_w(r_i)} .
$$

(87)

The form given by equation (86) of the radiative flux from the disk more clearly shows the effect of the local accretion rate in the first term and the effect of the angular momentum transported by the wind and the $\phi z$ stress in the third and fourth terms, respectively.

An expression for the total disk radiative luminosity, $L_d$, may be obtained by integrating each term of equation (85) over all disk radii, from $r = r_i$ to $r = \infty$. This also helps to elucidate the significance of the various terms in the expressions for the radiative power from the disk. The result is

$$
L_d = \frac{1}{2} \frac{GM}{r_i} - 2\pi r_i^2 T_{\text{eq}}(r_i) \Omega_k(r_i) \frac{1}{2} \frac{G M \dot{M}_w(r_i)}{r_i}

- \int_{r_i}^\infty \frac{1}{2} \frac{GM dM_w}{r} dr + \int_{r_i}^\infty 4\pi r \langle t_{o,z}(r) \rangle^+ v_k(r) dr

- \int_{r_i}^\infty 4\pi r \langle \psi_z \rangle^+ dr - \int_{r_i}^\infty 4\pi r \langle Q \rangle^+ dr .
$$

(88)

The first and third terms combine to give the familiar expression for the accretion power:

$$
P_a = \frac{1}{2} \frac{GM \dot{M}_w(r_i)}{r_i} .
$$

(89)

In conventional accretion disks, this is the dominant term responsible for the disk luminosity. In the generalized formalism developed here, however, we take into account disk solutions in which $M_a$ decreases toward small $r$ owing to the vertical loss of mass. Thus, for disks with strong wind mass loss, $M_a(r_i)$ is a fraction of the net mass flux $\dot{M}$ and $P_a$ is not necessarily the dominant term.

In considering the effect of a nonvanishing stress at $r_i$, we introduce the parameter

$$
\kappa = \frac{\text{Outward angular momentum flux}}{\text{Inward angular momentum flux}}

= \frac{\text{due to turbulent stresses at } r = r_i}{\text{due to accretion at } r = r_i}

= \frac{-2\pi r_i^2 T_{\text{eq}}(r_i)}{M_a(r_i) r_i^2 \Omega_k(r_i)} = \frac{T_{\text{eq}}(r_i)}{\Sigma(r_i) \Omega_k(r_i) r_i} .
$$

(90)

Note that since $T_{\text{eq}}(r_i) < 0$, consistent with the stresses defined elsewhere in the disk, this parameter is positive.

We now combine the fourth, fifth, and sixth terms on the right-hand side of equation (88) into a single term describing the total power removed from the disk by a wind:

$$
P_w = -\int_{r_i}^\infty \frac{1}{2} \frac{GM}{r} 4\pi r \langle \hat{\rho}^+ \hat{v}_z^+ \rangle^+ dr

- \int_{r_i}^\infty 4\pi r \langle t_{o,z}(r) \rangle^+ v_k(r) dr + \int_{r_i}^\infty 4\pi r \langle \psi_z \rangle^+ dr .
$$

(91)

The leading term in the wind power is negative since the disk material is highly bound; the only way in which the wind power can be positive is for the succeeding terms in $P_w$ to be positive (see the discussion concerning winds below eq. [48]).
We define the net heating rate of the disk due to irradiation and heat conduction from an external corona by

$$P_Q = - \int_{r_i}^{\infty} 4\pi r \langle Q \rangle^+ dr \geq 0. \quad (92)$$

We can now write the total disk luminosity as the following sum:

$$L_d = (1 + 2\kappa) P_a - P_w + P_Q. \quad (93)$$

When vertical energy transport from the disk (i.e., $P_w$) is important, the disk luminosity is reduced as a result of the conservation of energy. Realistically, some of the total power $P_w$ removed from the disk is available to power a corona, and potentially the most important contribution to this term is

$$P_{\phi z} = - \int_{r_i}^{\infty} 4\pi r \langle t_{\phi z} \rangle^+ v_K dr. \quad (94)$$

which represents the integral of part of the Poynting flux over both surfaces of the disk (see eq. [25]). To evaluate the importance of this term; we compare the integrand of equation (94) to the leading term in equation (86) for the disk energy flux. This comparison shows that $P_{\phi z}$ is an important component of the disk energy budget when

$$- \frac{8\pi r^2 \langle B_{\phi z} \rangle^+}{3} \sim 1. \quad (95)$$

We showed in § 3 (see eq. [71]) that $\langle t_{\phi z} \rangle$ has an appreciable effect on the accretion process when

$$- \frac{8\pi r^2 \langle t_{\phi z} \rangle}{M_a v_K} \sim 1. \quad (96)$$

Hence, when this is the case, the power $P_{\phi z}$ is comparable to $P_a$, that is, when the $\phi$ stresses are important for angular momentum transport, they do significant work on the flow.

Of course, the effect of the $\phi$ stress would be unimportant if it fluctuated in sign. However, if the disk has a wind, then one expects $\langle t_{\phi z} \rangle^+ < 0$ as a result of inertia causing magnetic field loops to trail the rotation of the disk. When $\langle t_{\phi z} \rangle^+$ is systematically negative, the disk luminosity is reduced for a given mass accretion rate and a corresponding amount of energy is available to heat the corona. As we have shown in § 3, even a weak stress can be dynamically important and the above shows that it can also be energetically important. Indeed, the importance of $\phi z$ stresses in thin disks is widely appreciated in accretion outflow models. In the Blundell & Payne (1982) and Pudritz & Norman (1983) models, for instance, the entire accretion flow is attributed to outward angular momentum transport by the torque associated with these stresses. In the disk model discussed here, the treatment of disk stresses is unified, with both $r\phi$ and $\phi z$ stresses self-consistently included.

We can also express the condition given by equation (95) in terms of other disk parameters using $M_a \sim 4\pi \rho h_{av} \rho_0 [\tilde{h}_{\phi}]$. For $P_{\phi z}$ to be a significant fraction of the accretion power

$$- \frac{\langle B_{\phi z} \rangle}{4\pi \rho^+ v_K^+} \sim \frac{\rho_0}{\rho^+} h_{av} \frac{\tilde{h}_{\phi}}{r} \sim \frac{\rho_0}{\rho^+} \left( \frac{h_{av}}{r} \right)^2 \left[ \frac{\tilde{h}_{\phi}}{c_0} \right]. \quad (97)$$

If we adopt as representative AGN values $\rho_0/\rho^+ \sim 10^5$, $h_{av}/r \sim 10^{-3}$, this condition effectively implies (cf. eq. [59])

$$\frac{v_K^2 B_{\phi z}^2}{P_{\phi z}^+ v_K^+} \sim \left[ \frac{\tilde{h}_{\phi}}{c_0} \right]. \quad (98)$$

If the radial inflow velocity is subsonic and there is at least a modest vertical field (e.g., $B_{\phi z}^2/B_r^2 \sim 0.1$), then this condition may be satisfied if the coronal Alfvén speed is comparable to the Keplerian speed. However, these estimates are sensitive to the value of $h_{av}/r$, which could vary significantly under the effects of radiation pressure. This is another area in which well-designed numerical simulations could yield some useful insights. It should also be noted that, depending on the details of disk parameters (including $h_{av}/r$ and the radial Mach number), the condition given by equation (97) may be satisfied for much lower values of the Alfvén speed.

At this stage we also consider further the relationship of the power $P_{\phi z}$ to the vertical component of the Poynting flux:

$$\langle S_z \rangle = \langle u_B \tilde{v}_z \rangle + \langle u_B t_z^\theta \rangle - \langle t_{z\phi}^\theta \tilde{v}_z \rangle - \langle t_{z\phi}^\theta \tilde{v}_\phi \rangle - \langle t_{z\phi}^\theta \tilde{v}_\phi \rangle \quad (99)$$

The term

$$\langle S_z^{(1)} \rangle = - \langle t_{z\phi}^\theta \tilde{v}_\phi \rangle \quad (100)$$

in this expression contributes directly to $P_{\phi z}$. (Other terms in $\langle S_z \rangle$ are represented in $\tilde{v}_z$.) The power $P_{\phi z}$ represents the integral of this part of the Poynting flux. It is interesting that the term $\langle t_{z\phi}^\theta \tilde{v}_\phi \rangle$ does not appear in the expression for the local radiative flux, viz., equation (85). Such a term is initially present as the above derivation for the local radiative flux shows. However, it is eliminated by a corresponding opposite term arising from the expression for $T_{\phi z}$. What does remain in the expression for the local radiative flux is a more complicated term involving $Q_K$ multiplied by an integral over $\langle t_{z\phi}^\theta \rangle^+$. The cause of this is that the angular momentum flow is driven by the total stress whereas the Poynting flux is only related to the magnetic part of that stress, as well as the interrelationship between the $r\phi$ stress and the $\phi z$ stress in the angular momentum equation (62).

The term $\int_{2\pi} 4\pi r \tilde{v}_z dr$ involves the integration of a number of terms, all of which could be important in the following. Rather than consider them all here, we consider the representative and probably most important terms:

$$4\pi r \langle S_z^{(2)} \rangle = 4\pi r \left( \langle u_B - t_{z\phi}^\theta \rangle \tilde{v}_z \right) = 4\pi r \left( \frac{B_{z\phi}^2 + B_{\phi z}^2}{4\pi} \tilde{v}_z^+ \right), \quad (101)$$

$$4\pi r \langle S_z^{(3)} \rangle = 4\pi r \langle (u_B - t_{z\phi}^\theta) \tilde{v}_z^+ \rangle = 4\pi r \left[ \langle (B_{z\phi}^2 + B_{\phi z}^2) \tilde{v}_z^+ \rangle \right]. \quad (102)$$

The term $\langle S_z^{(3)} \rangle$ represents the contribution to the Poynting flux from the wind; the term $\langle S_z^{(3)} \rangle$ represents the contribution
to the Poynting flux from turbulent diffusion. Consider the first term,
\[
4\pi r \langle S_z^{(2)} \rangle = 4\pi r \tilde{\rho}^+ v_z^+ \frac{\langle B_r^2 + B_\theta^2 \rangle^+}{4\pi \tilde{\rho}^+} = -\frac{dM_w}{dr} \frac{\langle B_r^2 + B_\theta^2 \rangle^+}{4\pi \tilde{\rho}^+} \sim -\langle v_z^2 \rangle^+ \frac{dM_w}{dr}.
\]
(103)
Comparing this with the leading term in equation (86) for \(4\pi r \langle F_z^2 \rangle\), we have
\[
\langle v_z^2 \rangle^+ = \frac{\langle v_z^+ \rangle^+}{v_K^+} \frac{dM_w}{M_a dr} = \frac{\langle v_z^2 \rangle^+}{v_K^+} \frac{dM_a}{M_a dr}.
\]
(104)
For the power associated with this term to be important, it would seem that both the wind mass-loss rate should be comparable to the mass accretion rate \(r \langle dM_w/dr \rangle \sim M_a\) and the Alfvén speed in the corona should be comparable to the local Keplerian speed. Otherwise, the requirements on each of these factors would probably be excessive.

Similarly, one can compare the diffusive terms to the same leading term in \(4\pi r \langle F_z^2 \rangle\). The diffusive flux is important when
\[
\frac{4\pi r \langle S_z^{(1)} \rangle}{3GMM_a/2r^2} \sim \frac{\langle (B_r^2)^+ / 4\pi \rangle v_z^+}{(3/2)(GMM_a/r^3)} \sim 1.
\]
(105)
This condition is similar in some respects to the one above referring to the systematic velocity component. However, in this case the turbulent velocity is not related to the wind mass-loss rate and one should allow for the possibility that the turbulent velocity can be larger than the mean wind velocity. Again, utilizing \(M_a \sim 4\pi r h\tilde{\rho}_0 [\tilde{\rho} v_z]^{-1}\) in this expression leads to the condition
\[
\frac{\langle v_z^2 \rangle^+}{v_K^+} \geq \frac{\tilde{\rho} v_h}{\tilde{\rho}^+ r v_z^+}.
\]
(106)
As in the treatment of the \(P_{wz}\) term above, with typical AGN values \(\tilde{\rho}_0 / \tilde{\rho}^+ \sim 10^6, h_{\text{av}} / r \sim 10^{-3}\), this condition relies on the ratio of the radial inflow velocity to the turbulent buoyant velocity. Nevertheless, the impression from equation (106) is that the Alfvén speed in the corona should be fairly high with respect to the local Keplerian speed for turbulent diffusion to be important.

4.3. Relative Importance of Poynting Flux Components

We have seen above that the importance of the various terms in the Poynting flux compared to the accretion power generally depends on the ratio of the coronal Alfvén speed to the local Keplerian speed and its relationship to disk and wind parameters. The above relationships give us a good idea of the importance of the various components of the Poynting flux in absolute terms. Further insight is obtained if we compare the various terms with respect to one another; this informs us of the conditions under which each component is likely to dominate. In these comparisons we remind the reader that \(S_z^{(1)}\) is associated with the power \(P_{wz}\), \(S_z^{(2)}\) is associated with the wind advective part of the Poynting flux, and \(S_z^{(3)}\) is associated with the turbulent diffusive part.

The ratio
\[
\frac{S_z^{(2)}}{S_z^{(1)}} \simeq -\frac{\langle (B_r^2)^+ / 4\pi \rangle v_z^+ \tilde{v}_z}{v_K} \sim \frac{\langle (B_\theta^2)^+ \rangle}{v_K} h_{\text{av}} v_z^+.
\]
(107)
We have generally assumed that the vertical component of velocity at the base of the corona is less than the disk sound speed \((\tilde{v}_z \leq c_0)\). If this is the case, the vertical magnetic field would have to be very low \((B_\theta^2 / B_r^2 \leq h_{\text{av}} / r \ll 1)\) in order for the wind advective power to dominate \(P_{wz}\). Following the discussion in § 3.3, if the disk departs from the assumed quasi-equilibrium structure in the inner radiation-dominated zone, then this conclusion may need to be revisited. Despite this qualification, however, the overwhelming impression from equation (107) is that the power \(P_{wz}\) dominates the advective part of the Poynting flux in most regions of the disk.

Similarly, the ratio of the turbulent diffusive power to \(P_{wz}\) is determined by the ratio
\[
\frac{S_z^{(3)}}{S_z^{(1)}} \simeq \frac{\langle (B_r^2)^+ / 4\pi \rangle v_z^+ \tilde{v}_z}{v_K} \sim \frac{\langle (B_\theta^2)^+ \rangle}{v_K} h_{\text{av}} v_z^+.
\]
(108)
Again, given that \(v_z^+ \leq c_0\), the vertical component of the magnetic field would have to be extremely small for the turbulent diffusive term to dominate.

4.4. The Limiting Case of Vertical Transport Only

In a real disk, angular momentum is probably transported by a combination of both radial and vertical fluxes. The limiting case of radial transport only has, of course, been well explored in the past. Let us now investigate the opposite limiting case of vertical transport only, specifically as it affects the estimate of the power of the disk wind. In this limiting case, equation (62) becomes
\[
\frac{d}{dr} (M_e \tilde{v}_z r) = \tilde{v}_z r \frac{dM_a}{dr} - 4\pi r^2 \langle \tilde{v}_{oz} \rangle^+.
\]
(109)
In this limit, the nonkinetic, nongravitational part of the wind power is dominated by \(P_{wz}\) so that
\[
P_w \simeq \int_{r_i}^{\infty} \left( -\frac{1}{2} G M dM_a/dr - 4\pi r^2 \langle \tilde{v}_{oz} \rangle^+ v_K \right) dr.
\]
(110)
Using equation (109), the Poynting flux term can be expressed as
\[
-4\pi r^2 \langle \tilde{v}_{oz} \rangle^+ v_K = \tilde{M}_w v_K \frac{d}{dr} (r v_K).
\]
(111)
Inserting this expression into the integral given by equation (110) for the wind power, one obtains
\[
P_w \simeq \frac{1}{2} G M M_a(r_i) \frac{d}{dr_i},
\]
(112)
i.e., the wind power is equal to the accretion power. The corresponding disk luminosity is
\[
L_d \simeq P_Q,
\]
(113)
i.e., the entire disk luminosity is attributed to external feedback heating from the corona. In this limiting case, some of
the wind power would be dissipated in the corona and the rest
would escape to infinity.

There is a difference between the power in such a wind and
the power in a centrifugally driven wind with a net magnetic
flux threading the disk. For example, the power in a Blandford
& Payne (1982) wind is

\[ P_{w, BP} \propto B_0^2(r_i) r_i^2 \left( \frac{GM}{r_i} \right)^{1/2} \]  (114)

and depends on the net magnetic flux in the disk, as well as the
Keplerian velocity at the inner radius.

In the theory we have developed here, the net flux is zero, so
that the magnetic field plays a different role than that envisaged
in centrifugally driven flows. Since the \( r \phi \) stresses play no role,
the result represented in equation (112) is inevitable. The binding
energy released by the accreting disk material has to be manifest in the power of the wind.

5. SUMMARY AND CONCLUSIONS

In this paper we have established a self-consistent frame-
work for the theory of magnetized, turbulent disk accretion
around black holes. Our formalism establishes such a frame-
work using a robust statistical averaging procedure that ex-
plicitly includes dynamical equations for the evolution of the
magnetic field together with the conservation equations for
mass, momentum, and energy transport. We have paid special
attention to vertical transport of conserved quantities and con-
sistently related such transport to the dynamical structure of the
underlying disk. Although the net magnetic flux is assumed to
be zero, the formalism is nonetheless sufficiently general to
allow the straightforward inclusion of a systematic net mean
field component.

Our main results can be summarized as follows:

1. We have derived a comprehensive set of equations de-
scribing the transport of mass, momentum, internal energy,
turbulent kinetic and magnetic energies, and total energy. The
statistically averaged conservation equations are generally ap-
licable to other subject areas involving turbulent magnetic
fields.

2. We have applied the statistically averaged equations to a
geometrically thin, optically thick accretion disk that is sta-
tionary and axisymmetric in the mean. We have demonstrated
that when vertical transport of mass; radial, vertical, and angular
momentum; and energy is self-consistently treated, the general
equations include additional terms related to a disk wind and
turbulent azimuthal-vertical stresses on the disk surface. We
have shown that the total azimuthal-vertical stress can have
a significant dynamical and energetic effect on the disk even
though it may be numerically small compared to the radial-
azimuthal stress that has dominated a large amount of accretion
disk theory and simulation, to date. This is due to the much larger
surface area over which this stress acts compared to that avail-
able to the \( r \phi \) stress. Note that the importance of this stress has also been realized in accretion outflow models (see Königl &
Pudritz 2000 for a review).

3. We have derived an expression for the radiative lumi-
nosity from the disk photosphere and shown clearly how this
relates to the mechanical power in a wind and to the Poynting
flux. This expression also entails a different spatial distribution
of radiative flux than a standard accretion disk. This in turn
affects the integrated spectrum. Again, we defer the details to
future work.

4. We have discussed the three main sources of Poynting
flux into the corona: a component associated with the product
of the azimuthal-vertical component of the turbulent magnetic
stress, a component associated with wind advection of mag-
netic energy, and a component associated with buoyant tur-
bulent diffusion of magnetic field from the disk into the corona.
The first component probably dominates in most cases even if
the azimuthal-vertical stress is quite small in comparison to the
radial-azimuthal stress. In our analysis of the condition for a
wind and the conditions for a significant Poynting flux into the
corona, the ratio of the coronal Alfvén speed to the Keplerian
velocity emerges as a critical parameter. When this ratio is of
order unity, important magnetic effects are clearly present. How-
ever, there is also the prospect of significant effects when this
parameter is less than unity. This region of parameter space is
currently a relatively unexplored avenue of research in black
hole accretion disks.

5. In the limiting case, when all of the angular momentum
transport is through the vertical-azimuthal stress, we have
shown that the wind power, at the base of the wind, is exactly
equal to the accretion power. Some of this power would be
dissipated in the corona. This is the first time that a coupled
disk-corona model has identified in a physically consistent and
nonphenomenological manner a physical mechanism for the
flux of energy into the corona. This is also the first time that the
power of a disk wind has been dynamically linked to the pro-
cess of accretion. In previous models based on a net magnetic
flux the wind power is related to the strength of the magnetic
field.

6. The existence of a disk wind and that of a disk corona
appear to be inextricably linked. The major influence in power-
ing the corona is the surface stress that is also responsible for
transporting angular momentum vertically away from the disk.

In a realistic disk, we expect that both radial-azimuthal and
azimuthal-vertical stresses would be involved in the transport
of angular momentum, as well as a net mass loss from the
innermost regions. Nevertheless, our limiting solution provides
a good physical basis for the commonly held notion that ac-
cretion power could be channeled into significant coronal emis-
sivity and/or outflow comprised of electromagnetic and
bulk kinetic components. Thus, there is the prospect of ex-
plaining not only the coronal emission from radio-quiet AGNs
and galactic black hole candidates (GBHCs) but also systems
such as ultraluminous X-ray (ULX) sources and analogous
AGN sources such as broad absorption line (BAL) quasars
where large mass outflows (e.g., \( M_{\text{out}} \sim M_{\text{field}} \) are inferred (see,
for example, King & Pounds 2003 and references therein).

There are many aspects of accretion and outflow in disk-
coronal systems that remain to be explored. For example, al-
though we have identified the main source of energy flux into
the corona, the dissipation of this energy and its emergence as
radiation are important areas for future research. In addition, the
effect of radiation pressure in the inner disk structure and its
effect on the relative importance of the different components of
the Poynting flux have been identified as important topics for
future research. Nevertheless, with a self-consistent framework
now established, it is possible to consider further, via specific
models, the complex relationship between disk, corona, and
outflows in a variety of sources and to examine more thoroughly
the conditions for the initiation of a wind and the implications
for the general structure of the immediate environment of ac-
creting black holes.
In this appendix we summarize the standard equations for a resistive, viscous, and radiative MHD gas in a dynamically important gravitational field. We consider only a nonrelativistic, optically thick gas in which the radiative diffusion approximation holds, so that the radiation pressure reduces to a scalar. The independent variables are mass density, \( \rho \); fluid velocity, \( v_i \); gas plus radiation pressure, \( p \); gas plus radiation energy density, \( u \); gravitational potential, \( \phi \); radiative flux, \( F_i \); external heat flux, \( Q_i \); magnetic field, \( B_i \); electric field, \( E_i \); current density, \( J_i \); and viscous stress tensor, \( t_{ij} \).

The fundamental equations are as follows:

1. **Mass continuity:**
   \[
   \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0. \quad (A1)
   \]

2. **Momentum conservation:**
   \[
   \frac{\partial v_i}{\partial t} + \frac{\partial (\rho v_i v_j)}{\partial x_j} = -\frac{\partial \rho \phi}{\partial x_i} + \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \frac{B_i B_j}{4\pi} - \delta_{ij} \frac{B^2}{8\pi} \right) + \frac{\partial t_{ij}^\rho}{\partial x_j}. \quad (A2)
   \]

3. **The induction equation for magnetic flux conservation:**
   \[
   \frac{\partial B_i}{\partial t} + \epsilon_{ijk} \epsilon_{kmn} \frac{\partial}{\partial x_j} (B_i v_m) = \eta \nabla^2 B_i; \quad (A3)
   \]
   which is derived from Maxwell’s equations,
   \[
   J_i = \frac{c}{4\pi} \epsilon_{ijk} \frac{\partial B_k}{\partial x_j}, \quad -\frac{1}{c} \frac{\partial B_i}{\partial t} = \epsilon_{ijk} \frac{\partial E_k}{\partial x_j}; \quad (A4)
   \]
   and a scalar conductivity law,
   \[
   J_i = \sigma \left( E_i + \frac{1}{c} \epsilon_{ijk} v_j B_k \right), \quad (A5)
   \]
   with \( \sigma \) the conductivity and \( \eta = c^2/4\pi \sigma \) the resistivity.

4. **Internal energy:**
   \[
   \frac{\partial u}{\partial t} + \frac{\partial}{\partial x_i} (u v_i) = -p v_{i,i} - F_{i,i} - Q_{i,i} + \frac{J^2}{\sigma} + t_{ij}^\phi v_{i,j}. \quad (A6)
   \]

5. **Electromagnetic energy.** We define the magnetic energy density, \( u_B \), the magnetic stress tensor, \( t_{ij}^B \), and the fluid shear tensor, \( s_{ij} \), by
   \[
   u_B = \frac{B^2}{8\pi}, \quad (A7)
   \]
   \[
   t_{ij}^B = \frac{B_i B_j}{4\pi} - \delta_{ij} \frac{B^2}{8\pi}, \quad (A8)
   \]
   \[
   s_{ij} = \frac{1}{2} \left( v_{i,j} + v_{j,i} - \frac{2}{3} \delta_{ij} v_{k,k} \right). \quad (A9)
   \]

Taking the scalar product of the induction equation, equation (A3), with \( B_i \) gives
   \[
   \frac{\partial u_B}{\partial t} + \frac{\partial}{\partial x_j} (u_B v_j) + \frac{1}{3} u_B v_{k,k} = t_{ij}^B s_{ij} + \frac{\eta}{4\pi} B_i \nabla^2 B_i. \quad (A10)
   \]
   This equation describes the volume rate of change of magnetic energy due to advection, expansion, or compression, shearing, and diffusion of field lines. We write this equation in this particular form so as to emphasize the shear term, which is the source term for

1 Note that this is the viscous stress tensor describing microscopic processes. It does not represent the so-called turbulent viscosity.
magnetic field amplification in a fluid with shearing motions, such as an accretion disk. Equation (A10) also shows the well-known result that in the absence of shear or in the case of an isotropic magnetic field the magnetic energy density evolves similarly to a gas with adiabatic index $\gamma = 4/3$ (i.e., pressure equals $1/4$ times energy density). The diffusion term in equation (A10) satisfies

$$\frac{\eta}{4\pi} B_i \nabla^2 B_j = \frac{J^2}{\sigma} - \frac{\partial^2 \left( \eta B \right)}{\partial x_i \partial x_j},$$

which describes the effects of Joule heating and the diffusion of field lines. Equation (A10) can then be expressed as

$$\frac{\partial u_B}{\partial t} + \nabla \cdot \left( u_B v_i + \eta \frac{\partial B_i}{\partial x_j} \right) = t^{B}_{ij} s_{ij} - \frac{1}{3} v_{B k} u_{B k} - \frac{J^2}{\sigma}.$$

Note that the Poynting flux is

$$S_i = \frac{c}{4\pi} \epsilon_{ijk} E_k B_j = u_B v_i - t^{B}_{ij} v_j + \eta \frac{\partial B_i}{\partial x_j}.$$

Equation (A12) can then be expressed in the more familiar form

$$\frac{\partial u_B}{\partial t} + S_i, i = -J_i E_i = -\epsilon_{ijk} v_j \frac{v_k}{c} J_{B k} - \frac{J^2}{\sigma}$$

describing conservation of electromagnetic energy.

6. Total energy. We define the specific enthalpy of gas plus radiation by

$$h = \frac{u + p}{\rho},$$

and we also define the total energy and corresponding energy flux by

$$u_{\text{tot}} = \frac{1}{2} \rho v^2 + \rho \phi_G + u + u_B,$$

$$F^E_i = \left( \frac{1}{2} \rho v^2 + \rho \phi_G + \rho h \right) v_i + F_i + \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \nabla \cdot \left( \eta B_j \right) \right).$$

The conservation equation for the total energy is obtained by taking the scalar product of the momentum equation, equation (A2), with $v_i$ and then utilizing both the internal energy equation, equation (A6), and the electromagnetic energy equation (A14) to obtain

$$\frac{\partial u_{\text{tot}}}{\partial t} + \frac{\partial F^E_i}{\partial x_i} = 0.$$  

Equation (A17) is the more usual form for the energy flux that explicitly incorporates the Poynting flux. Nevertheless, a more useful form for our purposes is derived by substituting the expression given by equation (A13) for the Poynting flux, giving

$$F^E_i = \left( \frac{1}{2} \rho v^2 + \rho \phi_G + \rho h + u_B \right) v_i - t^{B}_{ij} v_j + F_i + \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \nabla \cdot \left( \eta B_j \right) \right).$$

In this expression, the Poynting flux is mainly replaced by an advection term $u_B v_i$ and a term $t^{B}_{ij} v_j$ representing the rate of work by magnetic stresses on the flow.

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