A shape optimization problem for the first mixed Steklov–Dirichlet eigenvalue

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Abstract
We consider a shape optimization problem for the first mixed Steklov–Dirichlet eigenvalues of domains bounded by two balls in two-point homogeneous space. We give a geometric proof which is motivated by Newton’s shell theorem.

Keywords Spectral geometry · Steklov spectrum · Homogeneous space · Comparison geometry · Trigonometry

Mathematics Subject Classification 58J50 · 35P15 · 43A85

1 Introduction
Let \( M^m \) be a Riemannian manifold of dimension \( m \geq 2 \) and \( \Omega \subset M \) a bounded smooth domain with the boundary \( \partial \Omega \). Let \( \partial \Omega = C_1 \cup C_2 \) where \( C_1 \) and \( C_2 \) are disjoint components. A mixed Steklov–Dirichlet eigenvalue problem is to find \( \sigma \in \mathbb{R} \) for which there exists \( u \in C^\infty(\Omega) \) satisfying

\[
\begin{align*}
\Delta u &= 0 & \text{in } \Omega, \\
u &= 0 & \text{on } C_1, \\
\frac{\partial u}{\partial \eta} &= \sigma u & \text{on } C_2,
\end{align*}
\]

where \( \eta \) is the outward unit normal vector along \( C_2 \). When \( C_1 = \phi \), the problem becomes the Steklov eigenvalue problem introduced by Steklov [24]. We will find a domain maximizing the lowest \( \sigma \) in a class of subsets in \( M \). We call this problem by a shape optimization problem of the first eigenvalue.

The shape optimization problem of the first nonzero Steklov eigenvalue in Euclidean space has been studied since the 1950s. Weinstock [26] considered the case when

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\( M = \mathbb{R}^2 \). He showed that the disk is the maximizer among all the simply connected domains with the same boundary lengths. Recently, Bucur et al. [8] studied this perimeter constraint shape optimization problem in any dimension among all the convex sets and showed that the ball is the maximizer. Without the convexity condition, Fraser and Schoen [12] proved that the ball cannot be a maximizer even among all the smooth contractible domains of fixed boundary volume in \( \mathbb{R}^m, m \geq 3 \). On the other hand, Brock [7] proved in 2001 that the ball is the maximizer among all the smooth domains with fixed domain volume in \( \mathbb{R}^m, m \geq 2 \). Note that he does not need any topological restriction.

These shape optimization problems have been extended to non-Euclidean spaces as well. The first result in this direction was given by Escobar [11] who showed that the first nonzero eigenvalue is maximal for the geodesic disk among all the simply connected domains with fixed domain area in simply connected complete surface \( M^2 \) with constant Gaussian curvature. Binoy and Santhanam [5] extended this result to noncompact rank-one symmetric spaces of any dimension.

Regarding mixed Steklov–Dirichlet eigenvalue problems, it was considered by Hersch and Payne [15]. They considered the problem (1) when \( \Omega \subset \mathbb{R}^2 \) is a doubly connected region bounded by the inner and the outer boundaries, \( C_1 \) and \( C_2 \), respectively. Then, among all the conformally equivalent domains with fixed perimeter of \( C_2 \), the annulus bounded by two concentric circles is the maximizer. Recently, Santhanam and Verma considered connected regions in \( \mathbb{R}^m \) with \( m \geq 3 \) that are bounded by two spheres of given radii and gave the Dirichlet condition only on the inner sphere. Then, the maximizer is obtained by the domain bounded by two concentric spheres [22]. (See also Ftouhi [13].)

The aim of this paper is to extend Santhanam and Verma’s result [22] from Euclidean spaces to two-point homogeneous spaces including \( \mathbb{R}^2 \). The main theorem is as follows. We denote the injectivity radius of \( M \) and the closure of a set \( A \subset M \) by \( \text{inj}(M) \) and \( \text{cl}(A) \), respectively.

**Theorem 1** Let \( M \) be a two-point homogeneous space. Let \( B_1 \) and \( B'_2 \) be geodesic balls of radii \( R_1, R_2 > 0 \), respectively, such that \( \text{cl}(B_1) \subset B'_2 \) and \( R_2 < \text{inj}(M)/2 \). Then, the first mixed Steklov–Dirichlet eigenvalue of the problem

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } B'_2 \setminus \text{cl}(B_1) \\
u &= 0 \quad \text{on } \partial B_1 \\
\frac{\partial u}{\partial \eta} &= \sigma u \quad \text{on } \partial B'_2
\end{align*}
\]  

(\( \eta : \text{the outward unit normal vector along } \partial B'_2 \)) attains maximum if and only if \( B_1 \) and \( B'_2 \) are concentric.

Two-point homogeneous space has similar geometric properties with Euclidean space. For example, for two geodesic balls \( B_3 \) and \( B'_4 \) of radii \( R_1 \) and \( R_2 \), respectively, satisfying \( \text{cl}(B_3) \subset B'_4 \setminus \text{cl}(B_3) \) is isometric to \( B'_2 \setminus \text{cl}(B_1) \) if and only if the distance of the centers of \( B_3 \) and \( B'_4 \) is equal to that of \( B_1 \) and \( B'_2 \). Furthermore, using additional angles, which are not usual Riemannian angles, there are laws of trigonometries and conditions for triangle conditions (for example, see Proposition 1) in two-point homogeneous space.

In Sect. 2, we will briefly review the variational characterization of the mixed Steklov–Dirichlet eigenvalue problem (2) (Sect. 2.1) as well as two-point homogeneous spaces and its trigonometries (Sect. 2.2).
Section 3 is devoted to the proof of the main theorem. We estimate the first eigenvalue by substituting an appropriate test function in the Rayleigh quotient for the variational characterization of the first eigenvalue for \( B_\ell \setminus \text{cl}(B_1) \) [see (3)]. Our test function is the first mixed Steklov–Dirichlet eigenfunction on the domain bounded by concentric balls (Proposition 2).

Before proving main theorem, we obtain some crucial lemmas in Sect. 3.2.1. As a corollary, we give a proof of Newton’s shell theorem for a ball whose radius is less than \( \text{inj}(M)/2 \) in a two-point homogeneous space. (See Corollary 1 and the following Remark.) This theorem was first proved by Newton [20] for \( M = \mathbb{R}^3 \), and it was extended to constant curvature spaces by Kozlov [19] and Izmestiev and Tabachnikov [17].

In Sect. 3.2.2, we prove the main theorem for noncompact rank-one symmetric spaces (noted nCROSSs). We estimate the denominator of the Rayleigh quotient by using a geometric proof motivated by the proof of Newton’s shell theorem (see Corollary 2).

However, the argument of estimation of the numerator of the Rayleigh quotient in Sect. 3.2.2 does not work for compact rank-one symmetric spaces (noted CROSSs) when \( \text{inj}(M)/4 \leq R < \text{inj}(M)/2 \). We overcome this problem by partitioning domains and observing symmetry of a sphere (see Sect. 3.2.3). It is reminiscent of Ashbaugh and Benguria’s work [3] on a shape optimization problem of the first nonzero Neumann eigenvalue for bounded domains with fixed domain volume in \( M = S^n \). They showed that the geodesic ball is the maximizer if we restricted \( \Omega \) to be contained in a geodesic ball of radius \( \text{inj}(M)/2 \). This domain restriction is a refinement of Chavel’s work [10]. But the analogous result is not known if \( M \) is CROSS (see Aithal and Santhanam [2]).

2 Background

2.1 The eigenvalue problem

A mixed Steklov–Dirichlet eigenvalue problem (1) is equivalent to the eigenvalue problem of the Dirichlet-to-Neumann operator:

\[
L : C^\infty(C_2) \longrightarrow C^\infty(C_2)
\]

\[
u \longmapsto \frac{\partial \hat{u}}{\partial \eta},
\]

where \( \hat{u} \) is the harmonic extension of \( u \) satisfying the following:

\[
\begin{cases}
\Delta \hat{u} = 0 & \text{in } \Omega \\
\hat{u} = 0 & \text{on } C_1 \\
\hat{u} = u & \text{on } C_2
\end{cases}
\]

Then, \( L \) is a positive-definite, self-adjoint operator with discrete spectrum (see for instance [1]),

\[
0 < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \cdots \to \infty,
\]

provided that \( C_1 \neq \phi \). We call \( \sigma_k(\Omega) \) by the \( k \)th mixed Steklov–Dirichlet eigenvalue, or simply the \( k \)th eigenvalue. An eigenfunction of \( L \) corresponding to \( \sigma_k(\Omega) \) is called the \( k \)th
mixed Steklov–Dirichlet eigenfunction, or the $k$th eigenfunction. Then, the first eigenvalue $\sigma_1(\Omega)$ is characterized variationally as follows:

$$\sigma_1(\Omega) = \inf \left\{ \int_{\Omega} |\nabla v|^2 \, dv \middle| v \in H^1(\Omega) \setminus \{0\} \text{ and } v = 0 \text{ on } C_1 \right\}. \quad (3)$$

For convenience, we shall call the harmonic extension of the $k$th eigenfunction by the $k$th mixed Steklov–Dirichlet eigenfunction or the $k$th eigenfunction.

### 2.2 Two-point homogeneous spaces and triangle congruence conditions

Three points in a Euclidean space determine a triangle when three points do not lie on a single line. In classical geometry, there are several congruence conditions on triangles and it is determined by lengths of sides and angles. For example, side–angle–side (SAS) congruence is given by two side lengths and the included angle. In two-point homogeneous spaces, analogous properties also hold with additional angles. These facts are obtained by the laws of trigonometries. In this section, we give some information about two-point homogeneous spaces and its congruence conditions of triangles which will be used later. See [6, 16, 27] for more details.

**Definition 1** A connected Riemannian manifold $M$ is called two-point homogeneous space if $x_i, y_i \in M$, $i = 1, 2$ with $\text{dist}(x_1, y_1) = \text{dist}(x_2, y_2)$; there is an isometry $g$ of $M$ such that $g(x_1) = x_2$ and $g(y_1) = y_2$.

In fact, two-point homogeneous spaces are Euclidean spaces or rank-one symmetric spaces. We will call the latter as ROSSs. Furthermore, compact ROSS and noncompact ROSS are denoted by CROSS and nCROSS, respectively. Then, two-point homogeneous spaces with their isotropy representations are classified as in Table 1 (see [16, 27]). Here, $m \geq 1$, $n \geq 2$ and $m = \dim_{\mathbb{R}}M = n \cdot \dim_{\mathbb{R}}K$.

An angle is given by two directions at a point $P$. It is classified by its congruence classes which are given by the orbit space of $U_p M \times U_p M / K$, where $U_p M$ is the unit sphere in the tangent space of $M$ at $P$ and $K$ is the isotropy subgroup of the isometry group $M$ at $P$. The orbit space can be seen by fixing the first component by the action of $K$. More precisely, it is equivalent to an orbit space $U_p M / H$ of an isotropy group $H \subset K$ with respect to a point in $U_p M$. Then, it can be checked that for given $\bar{v}_1 \in U_p M$, $H$-invariant subspaces are $\mathbb{R} \cdot \bar{v}_1, \mathbb{K}' \cdot \bar{v}_1$, and the subspace orthogonal to $\mathbb{K} \cdot \bar{v}_1$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ and $\mathbb{K}'$ is

| $K$ | CROSS | nCROSS | Isotropy representation |
|-----|--------|--------|------------------------|
| $\mathbb{R}$ | $\mathbb{R}^m$ | $\mathbb{S}^m, \mathbb{R}P^m$ | $\mathbb{R}H^n$ | $(O(n), \mathbb{R}^m)$ |
| $\mathbb{C}$ | $\cdot$ | $\mathbb{C}P^n$ | $\mathbb{C}H^n$ | $(U(n), \mathbb{R}^{2n})$ |
| $\mathbb{H}$ | $\cdot$ | $\mathbb{H}P^n$ | $\mathbb{H}H^n$ | $(\text{Sp}(1) \times \text{Sp}(n), \mathbb{R}^{4n})$ |
| $\mathbb{O}$ | $\cdot$ | $\mathbb{O}P^2$ | $\mathbb{O}H^2$ | $(\text{Spin}(9), \mathbb{R}^{16})$ |

Table 1 Two-point homogeneous spaces, $m \geq 1, n \geq 2$
the set of pure imaginary numbers in $\mathbb{K}$. Then, a direction $\vec{v}_2$ is determined up to $H$-action by the following angular invariants (for more details, see [6, 16]):

- $\lambda(\vec{v}_1, \vec{v}_2) = \angle(\vec{v}_1, \vec{v}_2); \ 0 \leq \lambda \leq \pi$,
- $\varphi(\vec{v}_1, \vec{v}_2) = \angle(\vec{v}_1, \mathbb{K} \cdot \vec{v}_2); \ 0 \leq \varphi \leq \frac{\pi}{2},$

where $\angle(\vec{v}_1, \vec{v}_2)$ is the usual (Riemannian) angle and $\angle(\vec{v}_1, \mathbb{K} \cdot \vec{v}_2)$ is the angle between $\vec{v}_1$ and the subspace $\mathbb{K} \cdot \vec{v}_2$. Note that when $\mathbb{K} = \mathbb{R}$, $\lambda = \varphi$ or $\lambda = \pi - \varphi$. Then, angular invariants satisfy the following relations:

$$\lambda(\vec{v}_1, -\vec{v}_2) = \pi - \lambda(\vec{v}_1, \vec{v}_2),$$

(4)

and

$$\varphi(\vec{v}_1, -\vec{v}_2) = \varphi(\vec{v}_1, \vec{v}_2).$$

(5)

Using the previous $H$-invariant decomposition, we can write the metric of ROSS $M$ explicitly. Let $s(r)$ and $c(r)$ be functions defined as follows:

$$s(r) = \begin{cases} 
\sin r & \text{with } 0 \leq r < \pi \text{ if } M = S^m \\
\sinh r & \text{if } M \text{ is nCROSS} 
\end{cases}$$

and

$$c(r) = \begin{cases} 
\cos r & \text{with } 0 \leq r < \frac{\pi}{2} \text{ if } M = \mathbb{C}P^n, \mathbb{Q}P^n, \mathbb{O}P^2 \\
cosh r & \text{if } M \text{ is nCROSS.} 
\end{cases}$$

Then, the metric $(ds)^2$ is given by

$$(ds)^2 = (dr)^2 + (s(r))^2(c(r))^2g + (s(r))^2h,$$

(6)

where $(dr)^2$, $g$, and $h$ are written by $\sigma_1^2$ with the coframe $\sigma_1$ dual to $\vec{v}_1$; $\sigma_1^2 + \cdots + \sigma_k^2$ with coframes $\sigma_1, \ldots, \sigma_k$ dual to orthonormal basis of $\mathbb{K} \cdot \vec{v}_1$; $\sigma_{k+1}^2 + \cdots + \sigma_m^2$ with coframes $\sigma_{k+1}, \ldots, \sigma_m$ dual to the complement orthonormal basis of $\mathbb{R}^m$. Since the density function $\omega$ only depends on distance, we may define $\omega$ as a one-variable function:

$$\omega(r) = (s(r))^{m-1}(c(r))^{k-1}.$$  

Then, the sectional curvature $K_M$ of $M$:

$$\begin{cases} 
1 \leq K_M \leq 4 & \text{if } M \text{ is CROSS} \\
-4 \leq K_M \leq -1 & \text{if } M \text{ is nCROSS.} 
\end{cases}$$

(7)

In particular, $S^m$ and $\mathbb{R}P^n$ have sectional curvature 1. Then, the condition $0 < R_2 < \frac{\text{inj}(M)}{2}$ in Theorem 1 implies:

$$\begin{cases} 
0 < R_2 < \frac{\pi}{4} & \text{if } M = S^m \\
0 < R_2 < \frac{\pi}{4} & \text{if } M = \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n, \mathbb{O}P^2 \\
0 < R_2 & \text{otherwise.} 
\end{cases}$$

(8)
Now, consider a triangle \((PQR)\) in \(M\) with the metric (6), which consists of three distinct points \(P, Q, R \in M\) and three connecting geodesics \(QR, RP, PQ\). The side lengths will be denoted by \(p, q, r\), respectively, and the two angular invariants \(\lambda, \varphi\) determined by the two tangent vectors of geodesic rays \(\bar{PQ}\) and \(\bar{PR}\) at \(P\) will be denoted by \(\lambda(P)\) and \(\varphi(P)\), respectively. Furthermore, we can denote \(\lambda(Q), \varphi(Q), \lambda(R), \varphi(R)\) in an analogous way. Then, it is known that there are congruent conditions of triangles. We introduce some conditions which will be used later. For more conditions, see [6].

**Proposition 1** A triangle \((PQR)\) in \(ROSS\) with the metric (6) is uniquely determined up to isometry as follows:

(a) \(p, q, \lambda(P)\) with \(0 < p, q, r < \pi\) and \(q < p < \frac{\pi}{2}\) if \(M\) is \(\mathbb{S}^m\).

(b) \(p, q, \lambda(P)\) with \(0 < p, q, r < \frac{\pi}{2}\) and \(q < p < \frac{\pi}{4}\) if \(M\) is \(\mathbb{R}P^n\).

(c) \(p, q, \lambda(P), \varphi(P)\) with \(0 < p, q, r < \frac{\pi}{2}\) and \((p - q)(\cos p - \sin q \cos \varphi(P)) > 0\) if \(M\) is \(\mathbb{C}P^n, \mathbb{H}P^n\) or \(\mathbb{O}P^2\).

(d) \(p, q, \lambda(P), \varphi(P)\) with \(0 < p, q, r\) and \(q < p\) if \(M\) is \(nCROSS\).

**Proof** The proof of a can be found in Section VI in [25]. In fact, the condition \(p < \frac{\pi}{2}\) can be replaced by \(p + q < \pi\). The proof of b follows from a. The proofs of c and d can be found in (ix) and (ix’) of Theorem 4 and 4’ in [6].

3 Main proof

Let \(M\) be a \(ROSS\) with the metric (6). Let \(C\) and \(C'\) be the centers of \(B_1\) and \(B_2\), respectively. Define \(B_2\) be the ball of radius \(R_2\), centered at \(C\). See Fig. 1.

3.1 The first eigenfunctions

In this section, we derive an explicit formula for the first mixed Steklov–Dirichlet eigenfunctions in \(B_2 \setminus \text{cl}(B_1)\). Using the following standard argument as in [9, 22], we can show that the first eigenfunction is a function that only depends on the distance from \(X\).

![Fig. 1 Description of \(B_1\) and \(B_2\) (left), and \(B_1\) and \(B_2\) (right)](image-url)
Using separation of variables, a mixed Steklov–Dirichlet eigenfunction $u(r, \theta_1, \ldots, \theta_{m-1})$ in $B_2 \setminus \text{cl}(B_1)$ is obtained by multiplying a Laplacian eigenfunction $f(\theta_1, \ldots, \theta_{m-1})$ on $\partial B_2$ by an appropriate radial function $a(r)$. Here, $(r, \theta_1, \ldots, \theta_{m-1})$ is the polar coordinate in $T_{CM}$. More precisely, we have the following lemma.

**Lemma 1** For given Laplacian eigenfunction $f : S^{m-1} \to \mathbb{R}$, there exists a nonnegative function $a : [R_1, \infty) \to \mathbb{R}^+ \cup \{0\}$ such that the function $u(r, \theta_1, \ldots, \theta_{m-1}) = a(r)f(\theta_1, \ldots, \theta_{m-1})$ is harmonic and $a(R_1) = 0$. Specifically, $u$ is a mixed Steklov–Dirichlet eigenfunction on $B_2 \setminus \text{cl}(B_1)$.

**Proof** Let $a : [R_1, \infty) \to \mathbb{R}$ be a smooth function. Then, we have

$$
\Delta(a(r)f(\theta_1, \ldots, \theta_{m-1})) = (a''(r) + (m - 1)h(r)a'(r) - \lambda(S(r))a(r))f,
$$

(9)

where $S(r)$ is the geodesic sphere of radius $r$, centered at $C$, $\lambda(S(r))$ is the eigenvalue of $f$ on $S(r)$, and $h(r)$ is the mean curvature of the geodesic sphere $S(r)$ that is obtained by trace of the shape operator of $S(r)$ with respect to the inner normal vector times $\frac{1}{m-1}$. Since $(m - 1)rh$ and $r^2\lambda(S(r))$ are analytic (see [4, Proposition 5.3]), 0 is a regular singular point and we can find two linearly independent solutions of the equation (see [14, Theorem 12.1 on p.85]). Thus, there exists $a(r)$ such that $a(R_1) = 0$ and $a \cdot f$ is harmonic. Note that $\lambda(S(r))$ is nonnegative. Then, from (9) with maximum principle, $a(r)$ is a not sign-changing function. Thus, we may assume $a(r)$ is nonnegative in $[R_1, \infty)$. □

Since Laplace eigenfunctions on $S^{m-1}$ are indeed Laplace eigenfunctions on $\partial B_2$ (see [9, Theorem 3.1], or [4, Corollary 5.5]) and it consists of a basis of $L^2(\partial B_2)$, our mixed Steklov–Dirichlet eigenfunctions restrict to $\partial B_2$ become a basis of $L^2(\partial B_2)$. It implies every mixed Steklov–Dirichlet eigenfunction is written by a product of a Laplacian eigenfunction and a radial function. Then, using some computations as in [22, Section 2.1], we can show that the first mixed Steklov–Dirichlet eigenfunction is corresponding to the first Laplacian eigenfunction as the following lemma. Here, we count constant function as the first Laplacian eigenfunction on $S^{m-1}$.

**Lemma 2** Let $a_1 : [R_1, \infty) \to \mathbb{R}^+ \cup \{0\}$ be the nonnegative function that obtained from Lemma 1 when the given Laplacian eigenfunction is constant. Then, we have

$$
a_1'(R_2) \leq \frac{a'(R_2)}{a(R_2)},
$$

and the equality holds if and only if $f$ is constant. Here, we used notations $a, f$ in Lemma 1.

**Proof** From the harmonicity of mixed Steklov–Dirichlet eigenfunctions, we obtained the following equations:

$$
a''_1(r) + \frac{\omega'(r)}{\omega(r)} a'_1(r) = 0
$$

(10)

$$
a''(r) + \frac{\omega'(r)}{\omega(r)} a'(r) - \lambda(S(r))a(r) = 0.
$$

(11)

Then, $(10) \times a - (11) \times a_1) \times \omega(r)$ implies
Note that the equality of the inequality holds if and only if \( \lambda(S(r)) = 0 \) that is \( f \) is constant. Since \( a_1(R_1) = a(R_1) = 0 \), we have

\[
a(R_2)a'_1(R_2) - a'(R_2)a_1(R_2) \leq 0,
\]
or

\[
\frac{a'(R_2)}{a(R_2)} \leq \frac{a'_1(R_2)}{a_1(R_2)}.
\]

We can easily observe that

\[
a'(R_2)
\]
is the Steklov eigenvalue corresponding to \( u \). Now, we can compute the first mixed Steklov–Dirichlet eigenfunction on \( B_2 \setminus \text{cl}(B_1) \) as follows.

Since the first Laplacian eigenfunctions are constants, we obtain the following. We abuse notation \( a_1 \) by \( a \) for convenience.

**Proposition 2** Let \( r_X : B'_2 \to [0, \infty) \) be the distance function from \( X \). Let \( a : [R_1, \infty) \to \mathbb{R} \) be a function defined by

\[
a(r) = \int_{R_1}^r \frac{1}{\omega(t)} \, dt.
\]

Then, the first mixed Steklov–Dirichlet eigenfunction in \( B_2 \setminus \text{cl}(B_1) \) is \( a \circ r_C \) up to constant.

**Proof** By the argument in the paragraph, the first eigenfunction can be written by

\[
a \circ r_C,
\]
where \( a : [R_1, \infty) \to \mathbb{R} \) is a real-valued function. Then, the harmonicity of the eigenfunction implies

\[
0 = \Delta a(r) = a''(r) + \frac{\omega'(r)}{\omega(r)} a'(r) = \frac{1}{\omega(r)} (a'(r) \omega(r))'.
\]

Here, we used \( r \) instead of \( r_C \) for simplicity of notation. With the fact that \( a(R_1) = 0 \) from the boundary condition, we obtain the formula of \( a(r) \) up to constant.

### 3.2 Crucial lemmas and the proof for nCROSS

We begin with two definitions (see Fig. 2).
Definition 2 For given $X \in \mathbb{R}^2$, a vector-valued function $\vec{v}_X : M \setminus \{X\} \to T_X M$ is defined by $P \in M \setminus \{X\}$ and $\vec{v}_X(P) \in T_X M$ such that $\vec{v}_X(P)$ is the unit tangent vector of the geodesic ray $\overline{XP}$ at $X$.

Definition 3 For given $X \in \mathbb{B}_2'$, an angle function with respect to $X$, $\lambda_X : \partial \mathbb{B}_2' \to [0, \pi]$, is a map that assigns to each $P \in \partial \mathbb{B}_2'$ an angle $\lambda(P)$ of the triangle $(PXC)$.

For a given parametrization of $M$ around $X$, we can identify $T_X M$ with $\mathbb{R}^m$. Then, we can give the following definition.

Definition 4 For given $X \in \mathbb{B}_2'$ and a parametrization of $M$ around $X$, a map $\pi_X : \mathbb{S}^{m-1} \subset T_X M \to \partial \mathbb{B}_2'$ is defined by $\pi_X(v) = \exp_X([0, \infty) \cdot v) \cap \partial \mathbb{B}_2'$, i.e., $\pi_X(v)$ is the point of $\partial \mathbb{B}_2'$ in the geodesic emanating from $X$ in $v$ direction.

Note that $\pi_X$ has the inverse map. Thus, for any $P \in \partial \mathbb{B}_2'$, we can find $p_s \in \mathbb{S}^{m-1}$ such that $P = \pi_X(p_s)$. Let $c'_s \in \mathbb{S}^{m-1}$ such that the geodesic ray $\exp_X([0, \infty) \cdot c'_s)$ passes through $C'$. Then, we can define $-p_s \in \mathbb{S}^{m-1}$ such that it is the symmetric points of $p_s$ with respect to $X$. We define $\tilde{p}_s \in \mathbb{S}^{m-1}$ such that it is the symmetric point of $p_s$ with respect to the line passing through $X$ and $c'_s$ in the plane spanned by the vectors $p_s$ and $c'_s$. In addition, $-\tilde{p}_s$ can be defined as the symmetric point of $\tilde{p}_s$ with respect to $X$. Now, we denote $\exp_X(-p_s)$, $\exp_X(\tilde{p}_s)$, and $\exp_X(-\tilde{p}_s)$ by $-P$, $\tilde{P}$, and $-\tilde{P}$, respectively. Figure 3 explains the situation.

Fig. 2 Description of $\vec{v}_X(P)$ and $\lambda_X(P)$.

Fig. 3 Description of $P$, $\tilde{P}$, $-P$, and $-\tilde{P}$. The bigger circle represents $\partial \mathbb{B}_2'$, and the dotted circle represents $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ identified by $\mathbb{R}^m \cong T_X M$ via given parametrization of $M$. 

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3.2.1 Properties of angles and distances

In this section, we prove the lemmas which are essential in the proof of the main proposition in the next section. We prove a lemma about the “symmetric properties” of angles and distances. In addition, we obtain a lemma which is motivated from the concept of solid angle. As a corollary, we introduce Newton’s shell theorem with an infinitesimally thin “shell” in ROSS. We begin with a lemma, which is useful for the lemmas below.

**Lemma 3** A triangle \((PQR)\) in ROSS \(M\) with the metric \((6)\) satisfies:

- (a) If \(M = \mathbb{S}^n\), \(0 < p, q, r < \pi\), and \(p \leq q < \frac{\pi}{2}\), then \(\lambda(P) < \frac{\pi}{2}\).
- (b) If \(M\) is \(\mathbb{R}P^n\), \(\mathbb{C}P^n\), \(\mathbb{H}P^n\), or \(\mathbb{O}P^2\), \(0 < p, q, r < \frac{\pi}{2}\), and \(p \leq q < \frac{\pi}{4}\), then \(\lambda(P) < \frac{\pi}{2}\).
- (c) If \(M\) is \(n\text{CROSS}\), \(0 < p, q, r\), and \(p \leq q\), then \(\lambda(P) < \frac{\pi}{2}\).

**Proof**

(a) Suppose \(\lambda(P) \geq \frac{\pi}{2}\). Using the law of cosines of spherical triangles (see p. 179 in [18]),

\[
\cos p = \cos q \cos r + \sin q \sin r \cos P < \cos q \cos r.
\]

Combining the previous inequality with \(\cos p, \cos q > 0\), we obtain \(\cos r > 0\) and \(\cos p < \cos q\). It implies \(p > q\), contradiction our assumption.

(b) Suppose \(\lambda(P) \geq \frac{\pi}{2}\). Since \(M\) has sectional curvature \(K_M \leq 4\) as in (7), we can apply the triangle comparison theorem (see p. 197 in [18]).

\[
\cos 2p \leq \cos 2q \cos 2r + \sin 2q \sin 2r \cos P < \cos 2q \cos 2r.
\]

Then, by an analogous argument in (a), we obtain a contradiction.

(c) Suppose \(\lambda(P) \geq \frac{\pi}{2}\). Since \(M\) has sectional curvature \(K_M \leq -1\) as in (7), we can apply the triangle comparison theorem (see p. 197 in [18]).

\[
\cosh p \geq \cosh q \cosh r - \sinh q \sinh r \cos P > \cosh q.
\]

Thus, \(p > q\), which contradicts our assumption. \(\Box\)

For \(P \in \partial B'_2\), consider a triangle \((PXC')\) in \(\text{cl}(B'_2)\) defined in the beginning of Section 3, which consists of \(X \in B'_2\), the center \(C'\) of \(B'_2\), \(P\), and geodesics connecting two of them. Then, the next lemma explains relations of distances from \(X\) to \(P, \bar{P}, -P\), and \(-\bar{P}\) and relations of angles at those points (Fig. 4).

![Fig. 4 Illustration of Lemma 4. The circles represent \(\partial B'_2\).](image-url)
Lemma 4 Let $\lambda_X : \partial B'_2 \to [0, \pi]$ be an angle function with respect to $X$ that assigns to each $P \in \partial B'_2$ an angle $\lambda(P)$ of the triangle $(PXC')$. Define $r_X$ as in the Proposition 2. Then, $\lambda_X$ and $r_X$ satisfy the following.

(a) $0 \leq \lambda_X(P) < \frac{\pi}{2}$.

(b) $\lambda_X(P) = \lambda_X(\tilde{P}), r_X(P) = r_X(\tilde{P})$ for all $P \in \partial B'_2$.

(c) $\lambda_X(P) = \lambda_X(-P), r_X(P) \geq r_X(-P)$ for all $P \in \partial B'_2$ satisfying $\angle(\tilde{v}_X(P), \tilde{v}_X(C')) \leq \frac{\pi}{2}$. The equality holds if and only if $\angle(\tilde{v}_X(P), \tilde{v}_X(C')) = \frac{\pi}{2}$.

**Proof** We will prove this lemma when $M = \mathbb{C}P^n, \mathbb{H}P^n$ or $\mathbb{O}P^2$. Then, we have $R_2 < \frac{\sin(M)}{2} = \frac{\pi}{4}$.

(a) Note that $R_2 < \frac{\pi}{4}$ and $|C'X| < |C'P| = R_2$. Then, the statement follows from Lemma 3.

(b) Consider two triangles $(PXC')$ and $(\tilde{P}XC')$. By the constructions of $P$ and $\tilde{P}$, $\lambda(X)$ of $(PXC')$ and $(\tilde{P}XC')$ are identical. The same holds for $\varphi(X)$. Note that the two triangles have the common edge $XC'$ and $|C'P| = |C'\tilde{P}| = R_2$. From the fact that $|C'X| < |C'P| = R_2 < \frac{\pi}{4}$, we have $\sin|C'X| < \cos|C'P|$. Therefore, by Proposition 1, $(PXC')$ and $(\tilde{P}XC')$ are congruent. Then, our statement follows.

(c) Using the fact that $B'_2$ is convex (see p. 148 in [21]), we can define a point $R \in B'_2$ in the complete geodesic containing $X$ and $P$ such that the geodesic meets $CR$ perpendicularly. We claim that $\lambda(XR, XC') \leq \frac{\pi}{2}$. If $X = R$, $\lambda(XR, XC') = \frac{\pi}{2}$. Otherwise, we have $|RC'| < |XC'| < \frac{\pi}{4}$. Then, by Lemma 3 for $(XCR)$, our claim follows. Then, the condition on $P$ in the statement implies $R \in XP$, so $|PR| \leq |PX|$. On the other hand, two triangles $(PRC')$ and $(-PRC')$ are congruent by (4), (5), and Proposition 1 as in the proof of (b). Thus, we obtain that $\lambda_X(P) = \lambda_X(-P)$ and $|PR| = |-PR|$, which imply the desired conclusion.

A slight change in the proof shows it also holds if $M$ is $\mathbb{S}^n, \mathbb{R}P^n$ or nCROSSs.

Now, we will give another lemma that explains that an “infinitesimal area of $\partial B_2$ from $X$” can be calculated by $\lambda_X$ and $r_X$ (Fig. 5).

**Fig. 5** Description of $R$ in the proof of Lemma 5. The dotted circle and the bigger circle represent $\partial B_1$ and $\partial B'_2$, respectively.
Lemma 5 Let $\mu$ be the Lebesgue measure on $S^{m-1}$ and consider the pushforward $\pi_{X#}\mu$ on $\partial B'_2$. Then, for a measurable set $A \subset \partial B'_2$, we have

$$\pi_{X#}\mu(A) = \mu(\pi_X^{-1}(A)) = \int_{\partial B'_2} \frac{\cos \lambda_X}{\omega(r_X)} dS'_2,$$

where $S'_2$ is the induced measure on $\partial B'_2$ from the metric of $M$. Equivalently,

$$dS'_2 = \frac{\omega(r_X)}{\cos \lambda_X} d\pi_{X#}\mu.$$

Proof It is clear that $S'_2$ and $\pi_{X#}\mu$ are $\sigma$-finite and $\pi_{X#}\mu \ll S'_2$ that is to say $(\pi_X)_#\mu$ is absolutely continuous with respect to $S'_2$. Furthermore, $S'_2 \ll \pi_{X#}\mu$. By the Radon–Nikodym theorem, there are functions $f_1$ and $f_2$ on $\partial B'_2$ such that

$$\pi_{X#}\mu(A) = \int_A f_1 dS'_2$$

and

$$dS'_2(A) = \int_A f_2 d\pi_{X#}\mu.$$

Consider a vector field $\mathcal{F}$ on $M \setminus \{X\}$ defined by

$$\mathcal{F}(Y) = \left(\frac{1}{\omega(r_X)} \frac{\partial}{\partial r}\right)(Y),$$

where $\frac{\partial}{\partial r}(Y)$ is the vector in $T_Y M$ obtained by the parallel transport of the unit tangent vector $\tilde{v}_X(Y)$ along $XY$. Then,

$$\text{div}(\mathcal{F}) = \frac{1}{\omega(r_X)} \frac{\partial}{\partial r} \left(\frac{\omega(r_X)}{\omega(r_X)} \cdot \frac{1}{\omega(r_X)}\right) = 0. \quad (12)$$

Let $\mathcal{B}$ be a geodesic ball in $\partial B'_2$ with respect to the induced metric on $\partial B'_2$. We now consider a region $\mathcal{R}$ that is the region in $\text{cl}(B'_2)\setminus B_1$ of the solid cone from $X$ over $\mathcal{B}$. Equivalently,

$$\mathcal{R} = \{\exp_X(t \cdot \tilde{v}_X(Y)) | Y \in \mathcal{B}, R_1 \leq t \leq r_X(Y)\}.$$

Let $B_1 = \mathcal{R} \cap \partial B_1$. Then, applying the divergence theorem to $\mathcal{F}$ on $\mathcal{R}$, we have

$$0 = \int_{\mathcal{R}} \text{div}\mathcal{F} = \int_{\mathcal{B}} \frac{\cos \lambda_X}{\omega(r_X)} dS'_2 - \int_{B_1} \frac{1}{\omega(R_1)} dS_1,$$

where $S_1$ is the measure on $\partial B_1$ induced by the metric of $M$. Combining it with the fact that

$$\int_{B_1} \frac{1}{\omega(R_1)} dS_1 = \mu(\pi_X^{-1}(A)),$$

the first statement is proved for $\mathcal{B}$. Then, by Theorem 4.7 in [23], the first statement is proved. Since $\cos \lambda_X \neq 0$ from Lemma 4, the second argument follows. □
Remark 1 If we extend the domain of \(a(r)\) to \((0, \infty)\) in Lemma 1, the vector field \(\hat{\nabla}\) in the proof is, in fact, \(F(Y) = V(a\circ r_X)(Y)\). Note that \(a\circ r_X\) is harmonic in \(M\setminus\{X\}\). Thus, (12) is obtained without computation.

The following corollary is not necessary for the proof of the main theorem.

Corollary 1 We have
\[
\int_{aB'_2} \frac{\vec{v}_X}{\omega(r_X)} \, ds'_2 = 0.
\]

Proof Using the previous lemma, the left-hand side is equal to
\[
\int_{S^{m-1}} \left( \frac{\vec{v}_X}{\cos \lambda_X} \circ \pi_X \right) \, d\mu.
\]
By Lemma 4, we have
\[
\left( \frac{\vec{v}_X}{\cos \lambda_X} \circ \pi_X(p_s) \right) + \left( \frac{\vec{v}_X}{\cos \lambda_X} \circ \pi_X(-p_s) \right) = 0 \tag{14}
\]
for \(p_s \in S^{m-1}\) (see Fig. 6). Then, this relation gives the desired result.

Remark 2 Note that if \(M = \mathbb{R}^3\), then \(\omega(r) = r^2\). Furthermore, \(\vec{v}_X(\pi_X(p_s))\) is the unit vector from \(X\) to \(P = \pi_X(p_s)\) at \(X\). Thus, the equation becomes Newton's shell theorem, which implies that the net gravitational force of a spherical shell acting on any object inside is zero.

3.2.2 The proof for nCROSS

In this section, we prove the main theorem for nCROSS. We use the fact that the first mixed Steklov–Dirichlet eigenfunction, \(a\circ r_X\), of the annulus \(B_2 \setminus B_1\) is a test function in both of the variational characterizations of \(\sigma_1(B_2' \setminus B_1')\) and \(\sigma_1(B_2 \setminus B_1)\). Substituting the test function into the two Rayleigh quotients, we compare the two denominators and the two numerators in the following two propositions.

Fig. 6 Pictorial explanation of calculation of (14). Each thick arrow represents integrand of (14) at \(p_s\) and \(-p_s\)
Define a map
\[ \int_{\partial B'_2} (a \circ r_C)^2 \, dS'_2 : B'_2 \to \mathbb{R} \]
that assigns to \( X \in B'_2 \)
\[ \int_{\partial B'_2} (a \circ r_X)^2 \, dS'_2. \]
In the following proposition, we show that the function has a minimum value at \( C \) by analyzing the gradient of the function at each \( X \in B'_2 \).

**Proposition 3** We have
\[ \nabla \left( \int_{\partial B'_2} (a \circ r_C)^2 \, dS'_2 \right)(X) \in T_X M. \]

\[ \nabla \left( \int_{\partial B'_2} (a \circ r_C)^2 \, dS'_2 \right)(X) = \begin{cases} -g(X) \cdot \tilde{v}_X(C') & \text{if } X \neq C', \\ 0 & \text{if } X = C', \end{cases} \] (15)

where \( g : B'_2 \setminus \{C'\} \to \mathbb{R}^+ \) is a positive function. Furthermore,
\[ \int_{\partial B'_2} (a \circ r_C)^2 \, dS'_2 \leq \int_{\partial B'_2} (a \circ r_X)^2 \, dS'_2, \] (16)
and equality holds if and only if \( X = C' \).

**Proof** The gradient is calculated at \( X \in B'_2 \), so it does not affect the integration region \( \partial B'_2 \). Then, for \( P \in \partial B'_2 \), \( \nabla (a \circ r_C)^2(X) \in T_X M \). Thus,
\[ -\nabla \left( \int_{\partial B'_2} (a \circ r_C)^2(P) \, dS'_2(P) \right)(X) = \int_{\partial B'_2} \frac{2(a \circ r_X)}{a \circ r_X} \cdot (-\nabla r_C)(P)(X)) \, dS'_2(P). \]

With \( -\nabla (r_C)(P)(X) = \tilde{v}_X(P) \) and Lemma 5, the previous equation is equal to
\[ \int_{S^{m-1}} \left( 2(a \circ r_X) \cdot \frac{\tilde{v}_X}{\cos \lambda_X} \right) \sigma \mu(X). \] (17)

If \( X = C' \), the integral has value 0. Otherwise, we consider the integrand at \( p_s \in \{v|\langle v, c' \rangle \geq 0\} \subset S^{m-1}, \tilde{p}_s, -p_s, \) and \(-\tilde{p}_s \) (see Fig. 7). Note that the condition for \( p_s \) is equivalent to \( \angle(\tilde{v}_X(P), \tilde{v}_X(C')) \leq \frac{s}{2} \). Then, using Lemma 4,
Furthermore, Lemma 4 implies
\[
(a \circ r_X)(P) - (a \circ r_X)(-P) > 0
\]
even if
\[
\angle(\vec{v}_X(P), \vec{v}_X(C')) = \frac{\pi}{2}.
\]
Thus, our integration has a form
\[
g(X) \cdot \vec{v}_X(C')
\]
for some positive function $g$. Note that we actually proved that the gradient of the function has the opposite direction from $X$ to $C'$ (see Fig. 8). It implies our desired inequality (16).

\[\square\]

**Remark 3**

Moreover, Lemma 4 implies $(a \circ r_X)(P) - (a \circ r_X)(-P) > 0$ unless $\angle(\vec{v}_X(P), \vec{v}_X(C')) = \pi/2$. Thus, our integration has a form $g(X) \cdot \vec{v}_X(C')$ for some positive function $g$. Note that we actually proved that the gradient of the function has the opposite direction from $X$ to $C'$ (see Fig. 8). It implies our desired inequality (16).

\[\square\]
In the proof, the function $g$ only depends on the distance between $X$ and $C'$.

The proof is similar to the proof of Corollary 1 if we compare (13) and (17). The difference between the two proofs is the fact that $a$ is an increasing function.

**Corollary 2** We have

$$
\int_{\partial B_2} (a \circ r_C)^2 \, dS_2 \leq \int_{\partial B'_2} (a \circ r_C)^2 \, dS'_2,
$$

where $S_2$ is the measure on $\partial B_2$ induced from the metric of $M$. The equality holds if and only if $B'_2 = B_2$.

**Proof** Note that $B_2$ is a ball of radius $R_2$, centered at $C$. Therefore, we have

$$
\int_{\partial B_2} (a \circ r_C)^2 \, dS_2 = \int_{\partial B'_2} (a \circ r_C)^2 \, dS'_2.
$$

Then, Proposition 3 implies the statement. $\square$

In the following proposition, $(\nabla (a \circ r_X))(Z)$ for $Z \in M \setminus \{X\}$ is the gradient of $a \circ r_X(\cdot) : M \setminus \{X\} \to \mathbb{R}$ at $Z$.

**Proposition 4** We have

$$
\int_{B'_2 \setminus \text{cl}(B_1)} |\nabla (a \circ r_C)|^2 \, dV \leq \int_{B'_2 \setminus \text{cl}(B_1)} |\nabla (a \circ r_C)|^2 \, dV,
$$

where $V$ is the induced measure of $M$. The equality holds if and only if $B'_2 = B_2$.

**Proof** Note that $|\nabla (a \circ r_X)(\cdot)| = |\nabla a| \circ r_C(\cdot)$ and it is easy to check that $|\nabla a(r) = |a'(r)| = \frac{|(\omega(r)}{\omega(r)}$ is a decreasing function since we only consider when $M$ is nCROSS. Then,

$$
\int_{B'_2 \setminus \text{cl}(B_1)} |\nabla (a \circ r_C)|^2 \, dV - \int_{B'_2 \setminus \text{cl}(B_1)} |\nabla (a \circ r_C)|^2 \, dV
$$

$$
= \int_{B_2 \setminus B'_2} |\nabla a(R_2)|^2 \, dV - \int_{B'_2 \setminus B_2} |\nabla a(R_2)|^2 \, dV
$$

$$
\geq \int_{B_2 \setminus B'_2} |\nabla a(R_2)|^2 \, dV - \int_{B'_2 \setminus B_2} |\nabla a(R_2)|^2 \, dV = 0.
$$

To satisfy the equality, $|B'_2 \setminus B_2| = |B_2 \setminus B'_2| = 0$, or $B'_2 = B_2$. $\square$

**Remark 4** We used only the fact that $\omega(r)$ is a concave function in $[0, 2R_2)$. Thus, the proof also applies when $M$ is CROSS and $R_2 < \frac{\eta(M)}{4}$.

Now, we have the following proof of the main theorem when $M$ is a nCROSS.
Proof of Theorem 1 for nCROSS Note that \( u \omega r_C = 0 \) on \( \partial B_1 \). By the variational characterization of \( \sigma_1(B_2' \backslash \text{cl}(B_1)) \),

\[
\sigma_1(B_2' \backslash \text{cl}(B_1)) \leq \frac{\int_{B_2' \backslash \text{cl}(B_1)} |\nabla (u \omega r_C)|^2 \, dV}{\int_{\partial B_2'} (u \omega r_C)^2 \, dS_2}.
\]

By Corollary 2 and Proposition 4, we have

\[
\sigma_1(B_2' \backslash \text{cl}(B_1)) \leq \frac{\int_{B_2' \backslash \text{cl}(B_1)} |\nabla (u \omega r_C)|^2 \, dV}{\int_{\partial B_2'} (u \omega r_C)^2 \, dS_2}.
\]

Since we have shown that \( u \omega r_C \) is the first mixed Steklov–Dirichlet eigenfunction on the annulus \( B_2 \backslash \text{cl}(B_1) \) in Proposition 2, the right-hand side is \( \sigma_1(B_2' \backslash \text{cl}(B_1)) \). It is the desired inequality. In addition, the equality condition is followed from the equality conditions in Corollary 2 and Proposition 4. \( \square \)

**Remark 5** The method of the proof carries over to Euclidean space \( \mathbb{R}^m \).

### 3.2.3 The proof for CROSS

In this section, we modify the proof of Proposition 4 to show that the inequality in this proposition also holds when \( M \) is CROSS and \( R_2 < \frac{\text{inf}(M)}{2} \). Then, using the same argument in Sect. 3.2.2, we can show that the main theorem holds in this situation.

\( B_r(C) \) denotes the ball of radius \( r \), centered at \( C \) and \( d := r_C(C') \) denotes the distance between \( C \) and \( C' \). In addition, let \( \text{Pr}_C : M \backslash \{C\} \to S^{m-1} \subset \mathbb{R}^m \) be the direction of a point in \( M \backslash \{C\} \) with respect to \( C \) in the coordinate of \( M \) we defined in Definition 4 in Sect. 3.2. Then, the difference between the two sides of the inequality in Proposition 4 becomes

\[
\begin{align*}
&\int_{B_2 \backslash \text{cl}(B_1)} \left( \frac{1}{\omega(r)} \right)^2 \, dV - \int_{B_2' \backslash \text{cl}(B_1)} \left( \frac{1}{\omega(r)} \right)^2 \, dV \\
&= \int_{B_2 \backslash B_2'} \left( \frac{1}{\omega(r)} \right)^2 \, dV - \int_{B_2' \backslash B_2} \left( \frac{1}{\omega(r)} \right)^2 \, dV \\
&= \int_{R_2 - d}^{R_2} \int_{\text{Pr}_C((B_2 \backslash B_2') \cap \partial B_{r_1}(C))} \frac{1}{\omega(r_1)} \, d\mu \, dr_1 \\
&\quad - \int_{R_2}^{R_2 + d} \int_{\text{Pr}_C((B_2' \backslash B_2) \cap \partial B_{r_2}(C))} \frac{1}{\omega(r_2)} \, d\mu \, dr_2 \\
&= \int_0^d \left( \int_{\text{Pr}_C((B_2 \backslash B_2') \cap \partial B_{R_2-s}(C))} \frac{1}{\omega(R_2 - s)} \, d\mu \\
&\quad - \int_{\text{Pr}_C((B_2' \backslash B_2) \cap \partial B_{R_2+s}(C))} \frac{1}{\omega(R_2 + s)} \, d\mu \right) \, ds.
\end{align*}
\]

The last equality is obtained by substituting \( r_1 \) and \( r_2 \) by \( R_2 - s \) and \( R_2 + s \) for \( s < d \), respectively. See Fig. 9 for pictorial description of \( (B_2 \backslash B_2') \cap \partial B_{R_2-s}(C) \) and \( (B_2' \backslash B_2) \cap \partial B_{R_2+s}(C) \). Then, the integral becomes nonnegative provided that the following two lemmas hold.
Lemma 6 We have

\[ |\Pr_C((\mathbf{B}_2' \setminus \mathbf{B}_2) \cap \partial B_{R_2-s}(C))| \leq |\Pr_C((\mathbf{B}_2 \setminus \mathbf{B}_2') \cap \partial B_{R_2-s}(C))| \]

for \( s < R_2 \).

Proof Consider \( S \in (\mathbf{B}_2 \setminus \mathbf{B}_2') \cap \partial B_{R_2-s}(C) \). Then, the triangle \((SCC')\) has side lengths

\[ |CC'| = d, |CS| = R_2 - s, |C'S| \geq R_2. \]

Consider the space form \( \mathbb{S}^m_\kappa \) of constant curvature \( \kappa \), where \( \kappa \in \mathbb{R}^+ \) is a constant such that a geodesic ball of radius \( R_2 \) is a hemisphere in \( \mathbb{S}^m_\kappa \). Then, we have

\[ \frac{\pi}{2\sqrt{\kappa}} = R_2, \]

so \( \kappa \) is bigger than the sectional curvature of \( M \). Now, consider a triangle \((S_\kappa C_\kappa C'_\kappa)\) with the same side lengths as \((SCC')\) in \( \mathbb{S}^m_\kappa \). Then, by the triangle comparison theorem (see [18, p. 197]),

\[ \angle SCC' \leq \angle S_\kappa C_\kappa C'_\kappa. \]

Then, it implies the following inequality.

\[
\begin{align*}
|\Pr_C((\mathbf{B}_2 \setminus \mathbf{B}_2') \cap \partial B_{R_2-s}(C))| \\
= |\{\Pr_C(S) | CS = R_2 - s, |C'S| \geq R_2\}| \\
\geq |\{S_\kappa | \langle S_\kappa, C_\kappa \rangle = R_2 - s, |C'_\kappa S_\kappa| \geq R_2\}| \times \frac{1}{s_\kappa(R_2 - s)} \\
= |\{S_\kappa | S_\kappa \in ((\mathbf{B}_2)_\kappa \setminus (\mathbf{B}_2')_\kappa) \cap \partial B_{R_2-s}(C_\kappa)\}| \times \frac{1}{s_\kappa(R_2 - s)},
\end{align*}
\]

where \((\mathbf{B}_2)_\kappa\) and \((\mathbf{B}_2')_\kappa\) are geodesic balls of radius \( R_2 \) in \( \mathbb{S}^m_\kappa \), centered at \( X_\kappa \) and \( C'_\kappa \), respectively, and

Fig. 9 The left and right thick arcs represent \((\mathbf{B}_2 \setminus \mathbf{B}_2') \cap \partial B_{R_2-s}(C)\) and \((\mathbf{B}_2' \setminus \mathbf{B}_2) \cap \partial B_{R_2-s}(C)\), respectively. In addition, the inner circle is \( \partial \mathbf{B}_1 \) and we have \(|CC'| = d\).
By a similar argument, we obtain
\[s_k(r) = \frac{\sin \sqrt{\kappa r}}{\sqrt{\kappa}}.\]

Since
\[\text{Pr}_C((B'_2 \setminus B_2) \cap \partial B_{R_2+s}(C))\]
\[\leq |\{S'_k | S_k \in ((B'_2)_k \setminus (B_2)_k) \cap \partial (B_{R_2+s})_k(C_k)\}| \times \frac{1}{s_k(R_2 + s)}. \tag{19}\]

Since
\[s_k(R_2 - s) = s_k(R_2 + s),\]
and the set
\[\{S_k | S_k \in ((B'_2)_k \setminus (B_2)_k) \cap \partial (B_{R_2+s})_k(C_k)\}\]
is the image of the antipodal map in \(S^m_k\) of
\[\{S'_k | S'_k \in ((B'_2)_k \setminus (B_2)_k) \cap \partial (B_{R_2+s})_k(C_k)\},\]
the right-hand sides of (18) and (19) are equal. Thus, our desired inequality is obtained. \(\square\)

**Lemma 7** We have
\[\omega(R_2 - s) < \omega(R_2 + s)\]
for \(0 < s < R_2\).

**Proof** We begin with \(M = \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n, \mathbb{O}P^2\), which are CROSS except for \(S^m\). Then, \(s < R_2 < \frac{\pi}{4}\). We have two observations of the density function \(\omega(t) = (\sin t)^{m-1}(\cos t)^{k-1}:\)
\[\begin{cases} 
\omega'(t) > 0 \text{ if } t < \arctan \sqrt{\frac{m-1}{k-1}} \\
\omega'(t) < 0 \text{ if } t > \arctan \sqrt{\frac{m-1}{k-1}}
\end{cases}\]
and
\[\omega(t) \leq \omega\left(\frac{\pi}{2} - t\right)\]
for \(t < \frac{\pi}{4}\). The second observation follows from
\[\omega\left(\frac{\pi}{2} - t\right) - \omega(t) = (\cos t)^{m-1}(\sin t)^{k-1} - (\sin t)^{m-1}(\cos t)^{k-1}\]
\[= (\sin t)^{k-1}(\cos t)^{k-1}(\cos t)^{m-k} - (\sin t)^{m-k} > 0.\]

Therefore, if
\[R_2 + s < \arctan \sqrt{\frac{m-1}{k-1}},\]
\[\omega(R_2 - s) < \omega(R_2 + s)\]
the first observation implies
\[ \omega(R_2 - s) < \omega(R_2 + s). \]

Otherwise, the two observations give
\[ \omega(R_2 - s) < \omega(\frac{\pi}{2} - (R_2 - s)) < \omega(R_2 + s). \]

Therefore, the proof for CROSS follows except for $S^m$. The same proof also works for $S^m$ if we replace $\frac{\pi}{4}$ and $\frac{\pi}{2}$ by $\frac{\pi}{2}$ and $\pi$, respectively. \qed

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Agranovich, M.S.: On a mixed Poincaré–Steklov type spectral problem in a Lipschitz domain. Russ. J. Math. Phys. 13, 239–244 (2006)
2. Aithal, A.R., Santhanam, G.: Sharp upper bound for the first non-zero Neumann eigenvalue for bounded domains in rank-1 symmetric spaces. Trans. Am. Math. Soc. 348(10), 3955–3965 (1996)
3. Ashbaugh, M.S., Benguria, R.D.: Sharp upper bound to the first nonzero Neumann eigenvalue for bounded domains in spaces of constant curvature. J. Lond. Math. Soc. (2) 52(2), 402–416 (1995)
4. Bérard-Bergery, L., Bourguignon, J.-P.: Laplacians and Riemannian submersions with totally geodesic fibres. Ill. J. Math. 26, 181–200 (1982)
5. Binoy, Santhanam, G.: Sharp upper bound and a comparison theorem for the first nonzero Steklov eigenvalue. J. Raman. Math. Soc. 29, 133–154 (2014)
6. Brehm, U.: The shape invariant of triangles and trigonometry in two-point homogeneous spaces. Geom. Dedic. 33, 59–76 (1990)
7. Brock, F.: An isoperimetric inequality for eigenvalues of the Stekloff problem. ZAMM Z. Angew. Math. Mech. 81, 69–71 (2001)
8. Bucur, D., Ferone, V., Nitsch, C., Trombetti, C.: Weinstock inequality in higher dimensions. arXiv:1710.04587
9. Castillon, P., Ruffini, B.: A spectral characterization of geodesic balls in non-compact rank one symmetric spaces. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 19(4), 1359–1388 (2019)
10. Chavel, I.: Lowest-eigenvalue inequalities, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979). In: Proc. Sympos. Pure Math., vol. XXXVI, pp. 79–89. American Mathematical Society, Providence (1980)
11. Escobar, J.F.: An isoperimetric inequality and the first Steklov eigenvalue. J. Funct. Anal. 165, 101–116 (1999)
12. Fraser, A., Schoen, R.: Shape optimization for the Steklov problem in higher dimensions. Adv. Math. 348, 146–162 (2019)
13. Ftouhi, I.: Where to place a spherical obstacle so as to maximize the first Steklov eigenvalue. fhhal-02334941 (2019)
14. Hartman, P.: Ordinary differential equations, In: Classics in Applied Mathematics, vol. 38. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2002). Corrected reprint of the second (1982) edition [Birkhäuser, Boston, MA; MR0658490 (83e:34002)], With a foreword by Peter Bates
15. Hersch, J., Payne, L.E.: Extremal principles and isoperimetric inequalities for some mixed problems of Stekloff’s type. Z. Angew. Math. Phys. 19, 802–817 (1968)
16. Hsiang, W.-Y.: On the laws of trigonometries of two-point homogeneous spaces. Ann. Glob. Anal. Geom. 7, 29–45 (1989)
17. Izimestiev, I., Tabachnikov, S.: Ivory’s theorem revisited. J. Integr. Syst. 2(006), 36 (2017)
18. Karcher, H.: Riemannian comparison constructions. Global differential geometry. In: Chern, S.S. (ed.) MAA Stud. Math., vol. 7, pp. 170–222. Mathematical Association of America, Washington (1989)

19. Kozlov, V.V.: Newton and Ivory attraction theorems in spaces of constant curvature. Vestnik Moskov. Univ. Ser. I Mat. Mekh. 68, 43–47 (2000)

20. Newton, I.: Philosophiae Naturalis Principia Mathematica, vol. I. Harvard University Press, Cambridge. Reprinting of the third edition (1726) with variant readings. Assembled and edited by Alexandre Koyré and I, Bernard Cohen with the assistance of Anne Whitman (1972)

21. Petersen, P.: Riemannian geometry, 2nd edn. In: Graduate Texts in Mathematics, vol. 171. Springer, New York (2006)

22. Santhanam, G., Verma, S.: On eigenvalue problems related to the Laplacian in a class of doubly connected domains. arXiv:1803.05750

23. Simon, L.: Lectures on geometric measure theory. In: Proceedings of the Centre for Mathematical Analysis, vol. 3. Australian National University, Centre for Mathematical Analysis, Canberra (1983)

24. Stekloff, W.: Sur les problèmes fondamentaux de la physique mathématique (suite et fin). Ann. Sci. École Norm. Sup. (3) 19, 455–490 (1902)

25. Todhunter, I.: Spherical Trigonometry, for the Use of Colleges and Schools: With Numerous Examples. CreateSpace Independent Publishing Platform (1802), Scotts Valley (2014)

26. Weinstock, R.: Inequalities for a classical eigenvalue problem. J. Ration. Mech. Anal. 3, 745–753 (1954)

27. Wolf, J.A.: Spaces of Constant Curvature, 5th edn. Publish or Perish Inc., Houston (1984)

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