Global existence of solutions to 2-D Navier-Stokes flow with non-decaying initial data in half-plane

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August 2, 2018

Abstract - We investigate the Navier-Stokes initial boundary value problem in the half-plane $\mathbb{R}^2_+$ with initial data $u_0 \in L^\infty(\mathbb{R}^2_+) \cap J^0_0(\mathbb{R}^2_+)$ or with non-decaying initial data $u_0 \in L^\infty(\mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+)$, $p > 2$. We introduce a technique that allows to solve the two-dimesional problem, further, but not least, it can be also employed to obtain weak solutions, as regards the non decaying initial data, to the three-dimensional Navier-Stokes IBVP. This last result is the first of its kind.

Keywords. 2-D Navier-Stokes equations, non-decaying data, global solution

1 Introduction

In this paper we consider the following Navier-Stokes initial boundary value problem

\begin{align}
    u_t + u \cdot \nabla u + \nabla \pi_u &= \Delta u, \quad \nabla \cdot u = 0 \text{ in } (0,T) \times \Omega, \\
    u &= 0 \text{ on } (0,T) \times \partial \Omega, \quad u = u_0 \text{ on } \{0\} \times \Omega,
\end{align}

where the symbol $\Omega$ denotes an exterior domain, $\mathbb{R}^n$ and $\mathbb{R}^n_+$, $n \geq 2$, and by $a \cdot \nabla b$ we mean $(a \cdot \nabla)b$. We look for solutions global in the time to

\footnotesize
\begin{itemize}
    \item The research of P.M. was partially supported by GNFM (INdAM) and by MIUR via the PRIN 2017 “Hyperbolic Systems of Conservation Laws and Fluid Dynamics: Analysis and Applications ”. The research of S.S. was partially supported by JSPS Grant-in-Aid for Scientific Research (B) - 16H03945, MEXT. The latter grant supported a visit of P.M. in Kyoto University. The authors declare no conflicts of interest.
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\end{itemize}

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problem (1) with non decaying initial data. The problem of the existence of solutions to (1) with non decaying data has been considered by several authors and, we think that the first results, where $n \geq 2$, go back to the papers [12, 14, 15, 19, 20]. But the special case of the two-dimensional problem involves a particular interest for the possibility to obtain global existence in the pointwise norm. A natural setting of the problem is the function space $L^\infty((0, T) \times \Omega)$. In this sense a first result is given by Giga, Matsui and Sawada in [15] limited to the Cauchy problem. Subsequently, in [27] Sawada and Taniuchi improve the $L^\infty$-norm of the solutions of [15]. Based on a result by Zelik in [32], a recent contribute given by Gallay in [9] establishes an estimate that up today is the best one:

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq c\|u_0\|_{L^\infty(\mathbb{R}^2)}(1 + c\|u_0\|_{L^\infty(\mathbb{R}^2)}t), \text{ for all } t > 0.$$ 

However all these results concern the Cauchy problem associated to the 2D-Navier-Stokes equations with non decaying data. Subsequently, the problem has been considered in exterior domains. Firstly Abe in [1] gives a result of local existence of the mild solution with initial data $u_0 \in L^\infty(\Omega)$ that can be seen as a weak solution to the Navier-Stokes problem. Then, in [23] Maremonti and Shimizu improve the result by Abe giving the existence and uniqueness of solutions to the Navier-Stokes initial boundary value problem in exterior domains which are defined for all $t > 0$. Actually, these authors are able to prove a smooth extension of the solution determined by Abe. The results contained in [23] can be also seen as a “structure theorem” of the weak solution given in [1]. The result by Maremonti and Shimizu is based on the possibility to reduce the problem to an $L^2$-theory. In the sense that the solution $u$ is seen as the sum of three fields, that is $u = U + W + w$, where $U$ and $W$ are solutions to a linear problem and keep the non decaying character of the initial data, instead $w$ is the solution to a nonlinear perturbed Navier-Stokes with homogenous initial data and suitable force data with compact support. For the field $w$ is applicable the $L^2$-theory (see e.g. [18]). However this approach seems to be unable to work in the case of $\partial \Omega$ not bounded.

More recently, in [2], as particular case of the results by Maremonti and Shimizu, Abe proves global existence in exterior domains by means of the special assumption of $u_0 \in L^\infty(\Omega) \ (\Omega \subseteq \mathbb{R}^2)$ and $\|\nabla u_0\|_2 < \infty$. In the case of the half-plane, he obtains a result under the assumption that the initial data is decaying from the viewpoint of Hardy’s inequality.

Although the geometry of the half-plane, and more in general the one of the half-space, concerns a particular case of the mathematical theory,
it is very interesting in the applications and recall the attention of several authors [6, 7, 8, 17]. Therefore the aim of the present paper is to prove that the result obtained by Maremonti and Shimizu in [23] also holds in the half-plane. In order to state our chief results we introduce some notations.

By the symbol $C_0^0(\Omega)$, we denote the set of all solenoidal vector fields $\varphi \in C_0^0(\Omega)$. By the symbol $J^q(\Omega)$, $q \in (1, \infty)$, we indicate the completion of $C_0(\Omega)$ in $L^q(\Omega)$ Lebesgue space. The symbol $P_q$ indicates the projector from $L^q(\mathbb{R}^2_+)$ into $J^q(\mathbb{R}^2_+)$. We set $J^q_0(\mathbb{R}^2_+)$ := completion of $C_0(\mathbb{R}^2_+)$ in Sobolev space $W^{1,q}(\mathbb{R}^2_+)$. We set $J^q_0(\mathbb{R}^2_+)$ := completion of $C_0(\mathbb{R}^2_+)$ in the space $\dot{W}^{1,q}(\mathbb{R}^2_+)$, that is with respect to the seminorm $\|\nabla \cdot \|_q$.

**Theorem 1.** Let $u_0 \in L^\infty(\mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+)$, $p \in (2, \infty)$. Then there exists a unique solution $(u, \pi_u)$ to problem (1) such that

$$
\text{for all } \eta > 0 \text{ and } T > 0, \ u \in C([0, T) \times \mathbb{R}^2_+) \cap C^2((\eta, T) \times \mathbb{R}^2_+),
$$

(2)

Moreover, up to a function $c(t)$, we get the estimate

$$
|\pi(t, x)| \leq P(t)|x|^\mu, \ \text{for all } t > 0 \text{ and } x \in \mathbb{R}^2_+, \ \text{(3)}
$$

for a suitable $\mu \in (0, 1)$, and, for all $\varepsilon > 0$ and $T > 0$, $P(t) \in C(\varepsilon, T) \cap L^s(0, T)$ for a suitable $s > 1$.

**Remark 1.** Comparing the assumptions made on $u_0$ in [23] with the one of Theorem 1, we point out that other the integrability of $\nabla u_0$, as a consequence of the Sobolev embedding, we assume more regularity for the initial data.

Although it is possible to deduce an estimate of the $\|u(t)\|_\infty$ for all $t > 0$, for the sake of the brevity, we do not give it. Like in the paper [23], the problem of the existence and the bound of the uniform norm of the solutions are two different questions.

Unlike all the quoted results, Theorem 1 enjoys of a quite original proof, which acquires a further interest for its application to the three-dimensional case. The proof is a variant of the one exhibits in the paper [21, 22], where the solutions are decaying in some sense. More precisely, firstly we introduce a finite family of solutions, each is the solution of a Navier-Stokes linearized problem. The first element of the family is the solution to the Stokes problem with non decaying initial data $u_0$. The number of the solutions depends on the exponent $p$ of $\|\nabla u_0\|_p$. If $p = 2$, then we have just
one linear (Stokes) problem, hence the solution \((U, \pi_U)\). Then, in order to solve problem \((1)\), we study the solution \((w, \pi_w)\) to the perturbed (nonlinear) Navier-Stokes problem where the coefficients are \(U\) and \(\nabla U\), the problem has an homogeneous initial data and body force \(U \cdot \nabla U\), that as a matter of course belongs to \(L^2\) (see problem \((57)\)). So a \(L^2\)-theory is applicable to prove the existence of \((w, \pi_w)\). Hence \(u := U + w\) solves \((1)\). If \(p > 2\), we consider the greatest integer floor \(k\) of \(\log_2 \frac{p}{2}\). It is such that \(\frac{p}{2^k} > 2\) and \(\frac{p}{2^{k+1}} \leq 2\). For \(h \in \{1, \ldots, k - 1\}\) we consider \((v^h, \pi_{v^h})\) solution of a corresponding linearized problem (see \((31)\)) where the coefficient are \(v^h, \nabla v^h\), \(\ell = 1, \ldots, h\), the initial data is zero and the force data is the \(v^{h-1} \nabla v^h \in L^{\frac{p}{2h}}(0, T; L^{\frac{p}{2h}}(\mathbb{R}^2_+))\). Since \(\frac{p}{2^k} > \frac{p}{2^{k+1}}\) by construction the last step is a field \((w, \pi_w)\) solution to a nonlinear perturbed Navier-Stokes problem with homogeneous initial data and force data \(F := -v^k \cdot \nabla v^k\) that, recalling the definition of \(k\), belongs to \(L^2(0, T; L^2(\mathbb{R}^2_+))\) (see problem \((64)\)). In this final step we can apply the \(L^2\)-theory which allows to conclude the proof of Theorem 1.

The approach used in the proof of Theorem 1 also allows us to deduce the following result for the 3D-Navier-Stokes Cauchy problem and IBVP in the half-space:

**Theorem 2.** Let \(u_0 \in L^\infty(\Omega) \cap J^p_0(\Omega)\), \(p > 3\) and \(\Omega \equiv \mathbb{R}^3\) or \(\Omega \equiv \mathbb{R}^3_+\). Then there exists a field \(u : (0, \infty) \times \Omega \to \mathbb{R}^3\) which is a solution in the distributional sense to problem \((1)\). Moreover, set \(k\) the greatest integer floor of \(\log_2 \frac{p}{2}\), we get \(u := U + \sum_{\ell=1}^{k+1} v^\ell + w\), where for all \(\eta > 0\) and \(T > 0\), \(U, v^\ell \in C([0, T) \times \Omega) \cap C^2((\eta, T) \times \Omega), U_t, v^\ell_t, \nabla \pi_U, \nabla \pi_{v^\ell} \in C((\eta, T) \times \Omega),\) and \(w \in L^\infty(0, T; J^2(\Omega)) \cap L^2(0, T; J^{1,2}(\Omega))\). Finally, the solution \(u\) is strongly continuous to the initial data, \(\lim_{t \to 0} \|u(t) - u_0\|_\infty = 0\).

Apart of an interesting, but a special, result obtained by Sawada in [26], as far we know, the result of Theorem 2 is the first of its kind. We do not give the proof of Theorem 2. Formally it is quite analogous to the one of the 2D case. The unique difference concerns the last step. Actually, for the field \(w\) we have to employ a Hopf-Leray existence theorem. This last gives to our solution the character of weak solution, and makes the difference with the 2D case, for which the \(L^2\)-theory allows to obtain regular solutions (see e.g. [18]). In the claims of Theorem 2 it has not to surprise that the initial data is assumed continuously with respect to the uniform norm. This is a
consequence of the fact that for all data $u_0 \in C(\Omega)$ at least on some interval $(0, T_0)$ the solution is regular as proved in [14, 19].

The authors would like to conclude the introduction giving special thanks to Professor Yasushi Taniuchi who made our attention to the IBVP for the half-plane problem with non-decaying data.

2 Some notations and preliminaries

By the symbol $\chi$, we denote a smooth positive cutoff function such that $\chi(\rho) = 1$ for $\rho \leq 1$, $\chi(\rho) \in [0,1]$ for $\rho \in [1,2]$ and $\chi(\rho) = 0$ for $\rho \geq 2$. For $R > 0$, we set $\chi_R(x) := \chi(\frac{|x|}{R})$. For $q \neq 2$, we set $l := [2 - \frac{2}{q}]$, and by $\tilde{B}^{2-\frac{2}{q},q}(\Omega) \subset L^{q}(\Omega)$, we mean the set of functions such that

$$
\|u\|_{2-\frac{2}{q},q} := \ll u \gg_{q}^{2-\frac{2}{q}} + \|u\|_{W^{l,q}} < \infty,
$$

where the functional $\ll \cdot \gg_{q}^{2-\frac{2}{q}}$ is given by

$$
\ll u \gg_{q}^{2-\frac{2}{q}} = \left[ \int_{\Omega} |z|^{-2q} \left( \int_{\Omega} |u(x) - u(x+z)|^q dx \right) dz \right]^{\frac{1}{q}}.
$$

For $q = 2$ we set $\tilde{B}^{1,2}(\Omega) := W^{1,2}(\Omega)$.

By the symbol $C^{k,\lambda}(\Omega)$, $k \in \mathbb{N}$ and $\lambda \in (0,1)$, we denote the Hölder’s space of functions continuous differentiable with their derivatives $D^\alpha u$, $|\alpha| \leq k$, and with $D^\alpha u$, $|\alpha| = k \lambda$-Hölder continuous. The norm in $C^{k,\lambda}$ is indicated by $\| \cdot \|_{k,\lambda}$ and Hölder’s seminorm by $[\cdot]_{k,\lambda}^{(\lambda)}$. We use the symbol $[\cdot]_{k,\lambda}^{(\lambda)}$ when there is no confusion about the domain.

Let $q \in [1, \infty)$, let $X$ be a Banach space with norm $\| \cdot \|_X$. We denote by $L^q(a,b; X)$ the set of all function $g : (a,b) \to X$ which are measurable and such that the Lebesgue integral $\int_a^b \|g(\tau)\|_X^q d\tau = \|g\|_{L^q(a,b; X)} < \infty$. As well as, if $q = \infty$ we denote by $L^\infty(a,b; X)$ the set of all function $g : (a,b) \to X$ which are measurable and such that $\text{ess sup}_{t \in (a,b)} \|g(t)\|_X = \|g\|_{L^\infty(a,b; X)} < \infty$. Finally, we denote by $C([a,b]; X)$ (resp. $C(a,b; X)$) the set of functions which are continuous from $[a,b]$ (resp. $(a,b)$) into $X$ and normed with $\sup_{(a,b)} \|g(t)\|_X < \infty$. 
Lemma 1 (Sobolev embedding). Let $f \in \dot{W}^{1,q}(\mathbb{R}^2_+)$, $q \in [1, 2)$. Then there exist a constant $f_0$ such that
\[ \|f - f_0\|_q^{2 - \frac{2}{q}} \leq c \|\nabla f\|_q, \quad (5) \]
where $c$ is independent of $f$. If $f \in \dot{W}^{1,q}_0(\mathbb{R}^2_+)$ then inequality (5) holds with $f_0 = 0$.

Proof. See e.g. [11] Section II.5.

Lemma 2 (Gagliardo-Nirenberg inequality). Let $u \in L^q(\mathbb{R}^2_+)$ with $\nabla^m u \in L^r(\mathbb{R}^2_+)$, $m = 1, 2$. Assume that $\gamma_{tr}(u) = 0$. Then, there exists a constant $c$ independent of $u$ such that
\[ \|\nabla^j u\|_s \leq c \|\nabla^m u\|_r \|u\|_q^{1 - a}, \quad (6) \]
provided that, for $j = 0, 1$ and $m = 1, 2$, $1 \leq q, r \leq \infty$, $\frac{1}{q} = \frac{1}{2} + a(\frac{1}{r} - \frac{1}{2}) + (1 - a)\frac{1}{q}$ with $a \in [\frac{1}{m}, 1]$ for $r \neq 2$, with $a \in [\frac{1}{2}, 1)$ for $r = 2$. Further, if $u \in J^q(\mathbb{R}^2_+)$ and $m = 2$, then, for $r \in (1, \infty)$, we also get
\[ \|\nabla^j u\|_s \leq c \|P_r \Delta u\|_r \|u\|_q^{1 - a}. \quad (7) \]

Proof. Inequality (6) is the well Gagliardo-Nirenberg inequality. Inequality (7) is related to the solenoidal functions and it is consequence of the fact that $\|\nabla^2 u\|_r \leq c \|P_r \Delta u\|_r$ for $r \in (1, \infty)$.

We recall the following well known version of Gronwall inequality:

Lemma 3 (Gronwall lemma). Assume that $\varphi(t), \psi(t), h(t)$ and $k(t)$ are continuous and nonnegative functions on $[0, T]$. Assume that the following integral inequality holds:
\[ \varphi(t) + \int_0^t \psi(\tau)d\tau \leq \int_0^t h(\tau)\varphi(\tau)d\tau + \int_0^t k(\tau)d\tau \quad \text{for all } t \in [0, T]. \]

Then we get
\[ \varphi(t) + \int_0^t \psi(\tau)d\tau \leq \exp\left[\int_0^t h(\tau)d\tau\right] \int_0^t k(\tau)d\tau \quad \text{for all } t \in [0, T]. \]

\[ ^1 \text{We remark that in the case of } m = 2 \text{ inequality (6) also can be given by means of } \Delta \text{ in place of } \nabla^2. \]
3 Stokes problem

In this section we study the following initial boundary value problem:

\[
U_t + \nabla \pi U = \Delta U + G, \quad \nabla \cdot U = 0 \text{ in } (0, T) \times \mathbb{R}^2_+,
\]

\[
U = 0 \text{ on } (0, T) \times \{x_2 = 0\}, \quad U = u_0 \text{ on } \{0\} \times \mathbb{R}^2_+.
\]

In order to discuss problem (8) by means of the Green function with the special assumption \(u_0 \in L^\infty(\mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+),\) we have to premise some results. We start with the following well known result:

**Lemma 4.** Assume that \(u \in L^p_{loc}(\mathbb{R}^2_+)\) with \(\nabla u \in L^p(\mathbb{R}^n_+), p \in [1, \infty).\) Assume that \(u(x_1, 0) = 0\) for all \(x_1 \in \mathbb{R}.\) Then, for all \(\tilde{R} \geq 0,\) we get

\[
\int_{|x| > \tilde{R}} |u(y)|^p |y|^{-p} dy \leq \pi^p \int_{|x| > \tilde{R}} |\nabla u(y)|^p dy.
\]

**Proof.** We reproduce the proof for the sake of the completeness. Introduced a polar coordinate frame \((r, \theta),\) almost everywhere in \(r > 0,\) we get

\[
\int_0^\pi |u(r, \theta)|^p r d\theta \leq \pi^p \int_0^\pi |\frac{\partial u}{\partial \theta}(r, \theta)|^p d\theta.
\]

Recalling that \(\frac{\partial u}{\partial \theta} \leq r |\nabla u|,\) we obtain

\[
\int_0^\pi |u(r, \theta)|^p r^{1-p} d\theta \leq \pi^p \int_0^\pi \left( (\frac{\partial u}{\partial r}(r, \theta))^2 + (\frac{1}{r} \frac{\partial u}{\partial \theta}(r, \theta))^2 \right) \frac{r}{\theta} d\theta.
\]

Integrating this last inequality on \((\tilde{R}, \infty),\) we deduce the thesis. \(\square\)

The following is also a well known result (see e.g. [25] and [2]). Again for the sake of the completeness we reproduce the proof given in [25].

**Lemma 5.** Let \(u_0 \in L^\infty(\mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+), p \geq 2.\) Then there exists a sequence \(\{u^n_0\} \subset L^\infty(\mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+),\) such that, for all \(n \in \mathbb{N},\) \(u^n_0\) has compact support with

\[
\|u^n_0\|_\infty \leq c\|u_0\|_\infty, \text{ for all } n \in \mathbb{N},
\]

\[
\|\nabla u^n_0\|_p \leq \|\nabla u_0\|_p + o(1), \text{ for all } n \in \mathbb{N},
\]

and for all \(R > 0,\) the sequence converges to \(u_0\) in \(L^\infty(B_R \cap \mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+).\)
Proof. We denote by \( \{\chi^n\} \) the sequence of cutoff functions with \( \chi^n := \chi(\frac{x}{n}) \), where \( \chi(\rho) \) is the cutoff function introduced in section 2. Hence, for all \( n \in \mathbb{N} \), the support is the ball \( B_{2n} \) and \( |\nabla \chi^n(x)| \leq \frac{c}{n} \). For all \( n \in \mathbb{N} \), we set \( u^n := u_0 \chi^n \) and we consider the Bogovski problem

\[
\nabla \cdot \tilde{u}^n = -\nabla \cdot u^n = -\nabla \chi^n \cdot u_0 \quad \text{in} \quad (B_{2n} - B_n) \cap \mathbb{R}^2_+,
\]

\[
\tilde{u}^n = 0 \quad \text{on} \quad \partial((B_{2n} - B_n) \cap \mathbb{R}^2_+).
\]

Since \( u_0 \) is divergence free, in problem (11) the compatibility condition is satisfied. Hence there exists at least a solution \( \tilde{u}^n \), and since the domain \( (B_{2n} - B_n) \cap \mathbb{R}^2_+ \) is of homothetic kind, with a constant \( c \) independent of \( n \) we obtain

\[
\|\nabla \tilde{u}^n\|_p \leq c\|u_0 \cdot \nabla \chi^n\|_p \leq cn^{1+\frac{2}{p}}\|u_0\|_\infty;
\]

\[
\|\nabla \tilde{u}^n\|_p \leq c\|\nabla u_0\|_{L^p(|x| \geq n)};
\]

where for the latter estimate we have employed Lemma 4. By the Poincaré inequality and (12) we get

\[
\|\tilde{u}^n\|_p \leq cn\|\nabla \tilde{u}^n\|_p \leq cn^{\frac{2}{p}}\|u_0\|_\infty.
\]

Employing the Gagliardo-Nirenberg inequality, via estimates (12) and (13) we get

\[
\|\tilde{u}^n\|_\infty \leq c\|u_0\|_\infty \quad \text{for all} \quad n \in \mathbb{N}.
\]

We extend \( \tilde{u}^n \) to zero on \( \mathbb{R}^2_+ \). For all \( n \in \mathbb{N} \), the extension is denoted again by \( \tilde{u}^n \). We define \( u^n_0 := \overline{u}^n + \tilde{u}^n \). Hence it follows that the sequence \( \{u^n_0\} \subset L^\infty(\mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+) \). Trivially we get that, for all \( R > 0 \), the sequence \( \{u^n_0\} \) converges in \( L^\infty(B_R \cap \mathbb{R}^2_+) \). Then Lemma 4 estimate (12) ensure the convergence of the sequence \( \{u^n_0\} \) in \( J^p_0(\mathbb{R}^2) \). Actually we get

\[
\|\nabla u_0 - \nabla u^n\|_p \leq \|(1 - \chi^n)\nabla u_0\|_p + \|u_0 \nabla \chi^n\|_p + \|\nabla \tilde{u}^n\|_p \leq c\|\nabla u_0\|_{L^p(|x| > n)}.
\]

We represent the solution to problem (8) by means of the Green function furnished in [28] and in [30], see also [31]. In two-dimensional case the Green
function is defined as follow:

$$G_{11}(t, x, y) := \Gamma(t, x - y) - \Gamma(t, x - y^*) + 4D_x \int_0^x D_{x_1} E(x - z) \Gamma(t, z - y^*) dz$$

$$G_{12}(t, x, y) := 0$$

$$G_{21}(t, x, y) := 4D_x \int_0^x D_{x_1} E(x - z) \Gamma(t, z - y^*) dz$$

$$G_{22}(t, x, y) := \Gamma(t, x - y) - \Gamma(t, x - y^*)$$

(15)

$$G_{*1}(t, x, y) := 4D_x \int_0^x D_{x_1} E(x - z_1 - x_2) \Gamma(t, z_1 - y_1, y_2)$$

$$+ E(x_1 - z_1, x_2) D_{y_2} \Gamma(t, z_1 - y_1, y_2) dz_1$$

(16)

$$G_{*2}(t, x, y) := 0.$$

In formula (15)-(16) function $\Gamma(t, z)$ is the kernel of heat equation and function $E(z)$ is the fundamental solution of Laplace equation and $y^* := (y_1, -y_2)$. We denote by $\Gamma^*(t, x, y) := \Gamma(t, x - y^*)$. We set

$$G_{*r1}(t, x, y) := 4D_x \int_0^x D_{x_1} E(x - z) \Gamma(t, x - y^*) dz, \ r = 1, 2.$$ 

For all $k \in \mathbb{N}$, $h := (h_1, h_2), h_i \in \mathbb{N}, i = 1, 2$, and $l := (l_1, l_2), l_i \in \mathbb{N}, i = 1, 2$, and $\mu > 0$, the following estimates hold:

$$|D_{t,x}^k \Gamma(t, z)| \leq ct^\frac{\mu}{2} (|z|^2 + t)^{-1 - k - \frac{|h_1|}{2}}, \text{ for all } (t, z) \in (0, T) \times \mathbb{R}^2,$$

$$|D_{t,x,y}^k \mathcal{G}_{*r1}^\mu(t, x, y)| \leq ct^{-\frac{l_2}{2}} (x_2 + t)^{-\frac{h_2}{2}} \exp[-\hat{a} \frac{y_2^2}{t}] (|x - y^*|^2 + t)^{-1 - \frac{h_1 + l_1}{2}},$$

$$|D_{t,x,y}^k \mathcal{P}_j(t, x, y)| \leq ct^{-1 - k - \frac{l_2}{2}} \exp[-\hat{a} \frac{y_2^2}{t}] (|x - y^*|^2 + t)^{-1 - \frac{h_1 + h_2 + l_1}{2}},$$

with $c$ and $\hat{a}$ positive constants independent of $t, x$ and $y$.

Via suitable hypotheses for data $u_0$, a solution to the Stokes problem


can be represented in components as

$$U_i(t, x) := \sum_{j=1}^{2} \int_{\mathbb{R}^2_+} G_{ij}(t, x, y) u_0(y) dy =: \mathcal{G}_i[u_0], \quad i = 1, 2,$$

(18)

$$\pi_U(t, x) := \sum_{j=1}^{2} \int_{\mathbb{R}^2} P_{ij}(t, x, y) u_0(y) dy =: \mathcal{P}[u_0].$$

**Theorem 3.** Let $G = 0$ in (8). For all $u_0 \in L^\infty(\mathbb{R}^2_+)$ with null divergence, there exists a unique smooth solution to problem (8) such that

$$\|U(t)\|_\infty \leq c\|u_0\|_\infty, \quad \text{for all } t > 0,$$

$$\|\nabla U(t)\|_\infty \leq c t^{-\frac{1}{2}}\|u_0\|_\infty, \quad \text{for all } t > 0,$$

(19)

with $c$ independent of $u_0$. Moreover, we get

if $u_0 \in L^\infty(\mathbb{R}^2_+)$, then $\lim_{t \to 0} (U(t) - u_0, \varphi) = 0$ for all $\varphi \in \mathcal{C}_0(\mathbb{R}^2_+),$

if $u_0 \in C(\mathbb{R}^2_+)$, then $\lim_{t \to 0} \|U(t) - u_0\|_\infty = 0.$

(20)

Finally, up to a function $c(t)$, for the pressure term $\pi_U$ we get

$$\gamma \in (0, 1), \quad |\pi_U(t, x)| \leq c\|u_0\|_\infty |x|^\gamma t^{-\frac{1}{2}}, \quad \text{for all } (t, x) \in (0, T) \times \mathbb{R}^2.$$

(21)

**Proof.** In the case of $u_0 \in C(\mathbb{R}^2_+)$ see Solonnikov [29], Theorem 1. In the case of $u_0 \in L^\infty(\mathbb{R}^2_+)$, the proof proposed by Solonnikov need some minimum modifications. We have to prove that the solution $U$ given by representation formula (18)$_1$ satisfies the initial condition (20)$_1$. Further, the proof of the uniqueness, proposed by Solonnikov in [29], also works in the case of $u_0 \in L^\infty(\mathbb{R}^2_+)$. Actually the proof is based on an argument of duality, hence the property (20)$_1$ is sufficient in order to prove the reciprocity formula given by the duality. We consider the sequence $\{u^n_0\}$ of Lemma 5. We have that $\|u^n_0\|_\infty \leq c\|u_0\|_\infty$, and $u_0 - u^n_0$ has a support for $|x| > n$, for all $n \in \mathbb{N}$. Hence, choosing $n$ sufficiently large such that $\text{supp } \phi \subset B_n$, we get

$$(U(t) - u_0, \phi) = (U(t) - U^n(t), \phi) + (U^n(t) - u^n_0, \phi),$$

(22)

where $U^n(t)$ is the solution of problem (8) corresponding to the initial data $u^n_0$. Via the representation formula, employing estimates $(17)$_{1,2}$, for all
\( x \in \text{supp} \phi \), we get
\[
|U(t,x) - U^{n+m}(t,x)| = |\mathcal{G}[u_0 - u_0^{n+m}]|
\]
\[
\leq c\|u_0\|_\infty \int_{|y|>n+m} t^{\frac{\mu}{2}} |y|^{-2-\mu} dy + \int_{|y|>n+m} \exp[-\hat{a}^2 \frac{y^2}{t}] |y|^{-2} dy
\]
\[
\leq c\|u_0\|_\infty \left[ t^{\mu} (n+m)^{-\mu} + t^{\frac{\mu}{2}} (n+m)^{-1} \right],
\]
with \( c \) independent of \( t \) and \( n,m \). Hence for the former term of (22) we obtain the estimate
\[
|(U(t) - U^{n+m}(t), \phi)| \leq c\|u_0\|_\infty \left[ t^{\mu} (n+m)^{-\mu} + t^{\frac{\mu}{2}} (n+m)^{-1} \right] \|\phi\|_1, \text{ for all } t > 0.
\]
For the latter term of (22), applying Hölder’s inequality, for \( q \in (1,\infty) \), we get
\[
|(U^n(t) - u_0^n, \phi)| \leq \|U^n(t) - u_0^n\|_q \|\phi\|_q', \text{ for all } t > 0.
\]
Therefore for \( t \to 0 \) we obtain \((20)_1\). The estimate \((21)\) is contained in \([19]\), Theorem 2.1. \(\square\)

**Theorem 4.** For all \( u_0 \in L^\infty(\mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+), \ p \in (1,\infty), \) the solution furnished by Theorem 3 verifies
\[
\|\nabla U(t)\|_p \leq c\|\nabla u_0\|_p, \text{ for all } t > 0,
\]
with a constant \( c \) independent of \( u_0 \).

**Proof.** Firstly we assume \( p \geq 2 \). By virtue of Lemma 5 there exist a sequence \( \{u^n_0\} \subset L^\infty(\mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+) \) converging to \( u_0 \) in \( L^\infty(B_R \cap \mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+) \), where \( u^n_0 := \mathfrak{w}^n + \tilde{u}^n \) has a compact support, and enjoys estimates (10). We denote by \((U^n, \pi_{U^n})\) the sequence of solutions corresponding to \( \{u^n_0\} \), where \( U^n := \mathcal{G}[u^n_0] \) and \( \pi_{U^n} := \mathfrak{P}[u^n_0] \). Recalling that \( u_0 - u^n_0 = (1 - \chi^n)u_0 + \tilde{u}^n \) has a support for \(|x| > n\), by representation formula and estimates (17), for all \( t > 0 \), we get that
\[
|U(t,x) - U^n(t,x)| \leq c\|u_0\|_\infty \left[ t^{\frac{\mu}{2}} \int_{|y|>n} (|x - y|^2 + t)^{-1-\frac{\mu}{2}} dy + c \int_{|y|>n} e^{-\hat{a} \frac{y^2}{t}} (|x - y^*|^2 + t)^{-1} dy \right],
\]
which ensures the uniform convergence on any compact $K$ subset of $\mathbb{R}^2_+$. Analogous is the proof of the following convergence:

for all $t > 0$, and compact $K \subset \mathbb{R}^2_+$:  
$$
\lim_{n \to \infty} \| \nabla U(t) - \nabla U^n(t) \|_{L^\infty(K)} = 0. \quad (24)
$$

From the representation formula of $U^n$ it is not difficult to deduce for the tangential derivative that

$$
i = 1, 2, \quad D_{x_i} U^n_i = \mathfrak{G}_{ij}[D_{y_j} u^n_{0j}], \quad \text{for all } (t, x) \text{ and } n \in \mathbb{N}.
$$

From the equation of the divergence we also get

$$
D_{x_2} U^n_2 = -D_{x_1} U^n_1, \quad \text{for all } (t, x) \text{ and } n \in \mathbb{N}.
$$

Since for all $t > 0$, the kernels are of Calderon-Zigmund kind, via (10), we can deduce uniformly in $t > 0$ and in $n \in \mathbb{N}$

$$
\| D_{x_1} U^n_1(t) \|_p + \| D_{x_2} U^n_2(t) \|_p + \| D_{x_2} U^n_2(t) \|_p \leq c \| \Gamma[\nabla u^n_0](t) \|_p + c \| \nabla u^n_0 \|_p 
$$

$$
\leq c \| \nabla u_0 \|_p + o(1), \quad (25)
$$

where $\Gamma[\nabla u^n_0](t, x) := \int_{\mathbb{R}^2_+} \Gamma(t, x - y) \nabla u_0(y) dy$. We estimate $D_{x_2} U^n_1$. Since $\mathfrak{G}_{12} = 0$, we restrict ourselves to consider only $\mathfrak{G}_{11}[u^n_{01}]$. Since $u^n_0$ has compact support

$$
\text{for all } q \in [1, \infty], \quad \Gamma[u^n_0], \Gamma^*[u^n_0] \in W^{1,q}(\mathbb{R}^2_+), \quad \text{for all } t > 0. \quad (26)
$$

Hence, integrating by parts with respect to $y_1$, we easily get

$$
U^n_1(t, x) = (\Gamma - \Gamma^*)[u^n_{01}](t, x) + 4 \int_0^{x_2} D_{x_1} E(x - z) \Gamma^*[D_{y_1} u^n_{01}](t, z) d z = I^n_1 + I^n_2.
$$

Integrating by parts with respect to $y_2$, we get

$$
D_{x_2} I^n_1(t, x) = (\Gamma + \Gamma^*)[D_{y_2} u^n_{01}].
$$

Hence, it follows

$$
\| D_{x_2} I^n_1(t) \|_p \leq c \| \nabla u_0 \|_p + O(n^{-1 + \frac{2}{p}}) \quad \text{for all } t > 0 \text{ and } n \in \mathbb{N}. \quad (27)
$$
In the case of $I_2^n$, by virtue of (26), we get

$$D_{x_2}I_2^n = -4 \text{P.V.} \int_0^x D_{x_2}^2 E(x-z) \Gamma^* [D_{y_1} u_0^n](t, z) dz$$

$$-4 \text{P.V.} \int \mathbb{R} D_{x_1} E(x_1 - z_1, 0) \Gamma^* [D_{y_1} u_0^n](t, z_1, x_2) dz_1.$$

Applying the Calderon-Zigmund theorem and the properties of heat kernel, we deduce the estimate

$$||D_{x_2}I_2^n(t)||_p \leq c ||\nabla u_0^n||_p + o(1) \text{ for all } t > 0 \text{ and } n \in \mathbb{N}. \quad (28)$$

Collecting estimates (25) and (27)-(28), we deduce that

$$||\nabla U^n(t)||_p \leq c ||\nabla u_0^n||_p + o(1), \text{ for all } t > 0 \text{ and } n \in \mathbb{N}.$$ 

This last estimate, the pointwise convergence ensured by (24) and the Fatou theorem prove (23). If $p \in (1, 2)$, we get $L^\infty(\mathbb{R}_+^2) \cap J_0^p(\mathbb{R}_+^2) \subset L^\infty(\mathbb{R}_+^2) \cap L^q(\mathbb{R}_+^2)$, provided that $q = \frac{2p}{2-p}$. Hence all the the above computations hold without the special approximation $\{u_0^n\}$ of Lemma 5.

We conclude the section recalling the following well known and special result (see e.g. [24]):

**Theorem 5.** Let $u_0 = 0$ in (8). For all $G \in L^r(0, T; L^r(\mathbb{R}_+^2))$, $r \in (1, \infty)$, there exists a unique solution to problem (8) such that $U \in C(0, T; J^{1,r}(\mathbb{R}_+^2))$ with

$$t^{-\frac{2p}{2p-2}} ||U(t)||_r + t^{-\frac{2p}{2p-2}} ||\nabla U(t)||_r \leq c \left[ \int_0^t ||G(\tau)||^r_{L^r} d\tau \right]^\frac{1}{r}, \text{ a.e. in } t \in (0, T),$$

$$\int_0^T \left[ ||U_t(t)||^r_r + ||D^2 U(t)||^r_r + ||\nabla \pi U(t)||^r_r \right] dt \leq c \int_0^T ||G(t)||^r_r dt, \quad (29)$$

where the constant $c$ is independent of $G$. If $u_0 \in C_0^1(\mathbb{R}_+^2)$, then (29)$_2$ becomes

$$\int_0^T \left[ ||U_t(t)||^r_r + ||D^2 U(t)||^r_r + ||\nabla \pi U(t)||^r_r \right] dt \leq c \left[ ||u_0||^r_{L^2} + \int_0^T ||G(t)||^r_r dt \right], \quad (30)$$

with $c$ independent of $u_0$ and $G$. 

4 A linearezed Navier-Stokes problem

For our aims the following initial boundary value problem is crucial:

\[
v_t - \Delta v + \nabla p = - \sum_{\ell=0}^{h} w^\ell \nabla v - v \nabla \sum_{\ell=0}^{h} w^\ell + F, \\
\nabla \cdot v = 0 \quad \text{in} \quad (0,T) \times \mathbb{R}^2_+,
\]

\[
v = 0 \quad \text{on} \quad (0,T) \times \{x_2 = 0\}, \quad v = 0 \quad \text{on} \quad \{0\} \times \mathbb{R}^2_+.
\]

**Assumption 1.** For problem (31) we assume:

i. \( w^0 \in L^\infty((0,\infty) \times \mathbb{R}^2_+) \cap L^\infty(0,\infty; J^p_0(\mathbb{R}^2_+)) \), and \( t^{\frac{Q}{2}} |\nabla w^0(t)|_\infty \leq c \|w^0(0)\|_\infty \).

ii. For all \( \ell = 1, \ldots, h \), and for all \( T > 0 \),

\( w^\ell \in C(0,T; J^{1,\frac{Q}{p}-1}(\mathbb{R}^2_+)) \cap \tilde{L}^{\frac{Q}{p}-1}(0,T; W^{2,\frac{Q}{p}-1}(\mathbb{R}^2_+)) \), where we assume \( p > 2^{\ell+1} \).

iii. \( F \in \tilde{L}^{\frac{Q}{p}}(0,T; L^\frac{Q}{p}(\mathbb{R}^2_+)) \), where we assume \( \frac{p}{2} > 2 \).

We start with the following

**Lemma 6.** Assume i.-ii. for \( w^\ell, \ell = 0, \ldots, h \). Moreover assume that \( v, D^2 v \in \tilde{L}^{\frac{Q}{p}}(0,T; L^\frac{Q}{p}(\mathbb{R}^2_+)) \) with \( v = 0 \) a.e. on \( (0,T) \times \{x_2 = 0\} \) and \( v = 0 \) a.e. on \( \{0\} \times \mathbb{R}^2_+ \). Then, a.e. in \( t \in (0,T) \), it holds that

\[
\|w^\ell(t) \cdot \nabla v(t)\|_{\frac{Q}{p}} \leq c \|w^\ell(t)\|_{\infty} \|D^2 v(t)\|_{\frac{Q}{p}} \left[ \frac{1}{\frac{Q}{p}} \left( \int_0^t \|v_\tau(\tau)\|_{\frac{Q}{p}} d\tau \right) \right]^{\frac{Q}{p}}, \quad (32)
\]

\[
\|v(t) \cdot \nabla w^\ell(t)\|_{\frac{Q}{p}} \leq c \|\nabla w^\ell(t)\|_{\infty} \int_0^t \|v_\tau(\tau)\|_{\frac{Q}{p}} d\tau,
\]

for \( \ell = 0, \ldots, h \), and with \( c \) independent of \( w^\ell, v \) and \( t \).

**Proof.** Since \( p/2^{\ell-1} > 2 \), employing the Gagliardo-Nirenberg inequality, for all \( \ell = 1, \ldots, h \), we get \( (q_\ell := \frac{Q}{2^{\ell-1}}) \)

\[
\|w^\ell\|_{\infty} \leq c \|D^2 w^\ell\|_{q_\ell}^{a} \|w^\ell\|_{q_\ell}^{1-a} < \infty \quad \text{a.e. in} \quad t > 0, \quad a = \frac{1}{q_\ell},
\]

\[
\|\nabla w^\ell\|_{\infty} \leq c \|D^2 w^\ell\|_{q_\ell}^{b} \|w^\ell\|_{q_\ell}^{1-b} < \infty \quad \text{a.e. in} \quad t > 0, \quad b = \frac{1}{2} + \frac{1}{q_\ell}.
\]

We recall that in our hypotheses on \( v \) the following estimates hold

\[
\|v(t)\|_{\frac{Q}{p}} \leq \int_0^t \|v_\tau(\tau)\|_{\frac{Q}{p}} d\tau \quad \text{for all} \quad t \in (0,T),
\]
and
\[ \|\nabla v(t)\|_{\frac{2}{p_0}} \leq c \|D^2 v(t)\|_{\frac{2}{p_0}} \|v(t)\|_{\frac{2}{p_0}}, \text{ a.e. in } t \in (0, T). \] (35)

Applying Holder’s inequality, after employing estimates (34)-(35), estimates (32) easily follow.

**Lemma 7.** Assume i.-ii. for \( w^\ell, \ell = 0, \ldots, h \). Moreover assume that \( v \in C(0, T; J^{1,2}(\mathbb{R}^2_+)) \). Then, a.e. in \( t \in (0, T) \) the following estimates hold with \( q := \frac{p}{2h} \)

\[
\begin{align*}
\|w^\ell(t) \cdot \nabla J_\varepsilon[v(t)]\|_q &\leq c(\varepsilon) \|w^\ell(t)\|_\infty \|\nabla v(t)\|_2, \\
\|J_\varepsilon[v(t)] \cdot \nabla w^\ell(t)\|_q &\leq c(\varepsilon) \|v(t)\|_2 \|\nabla w^\ell(t)\|_\infty,
\end{align*}
\]

(36)

for all \( \ell = 0, \ldots, h \), here \( J_\varepsilon[\cdot] \) is a spatial mollifier and constant \( c(\varepsilon) \) is independent of \( w^\ell, v \) and \( t \).

**Proof.** Estimates (32) hold. Since \( q > 2 \), by virtue of the properties of the mollifier the inequalities (36) hold immediately.

**Lemma 8.** Assume i.-ii. for \( w^\ell, \ell = 0, \ldots, h \). Moreover, assume that \( v \in L^2(0, T; J^{1,2}(\mathbb{R}^2_+)) \) and \( V \in L^2(0, T; L^2(\mathbb{R}^2_+)) \). Then, a.e. in \( t > 0 \), for \( p > 2^\ell \), the following estimates hold

\[
\begin{align*}
|\langle w^0(t) \cdot \nabla J_\varepsilon[v(t)], V(t) \rangle| &\leq \|w^0(t)\|_\infty \|\nabla v(t)\|_2 \|V(t)\|_2, \\
|\langle w^\ell(t) \cdot \nabla J_\varepsilon[v(t)], V(t) \rangle| &\leq c(\varepsilon) \|w^\ell(t)\|_{\frac{p}{2^{p-1}}} \|\nabla v(t)\|_2 \|V(t)\|_2, \\
|\langle J_\varepsilon[v(t)] \cdot \nabla w^0(t), V(t) \rangle| &\leq c(\varepsilon) \|\nabla w^0(t)\|_p \|v(t)\|_2 \|V(t)\|_2, \\
|\langle J_\varepsilon[v(t)] \cdot \nabla w^\ell(t), V(t) \rangle| &\leq c(\varepsilon) \|\nabla w^\ell(t)\|_{\frac{p}{2^{p-1}}} \|v(t)\|_2 \|V(t)\|_2,
\end{align*}
\]

(37)

where \( J_\varepsilon[\cdot] \) is a spatial mollifier.

**Proof.** Applying Holder’s inequality we get

\[
\begin{align*}
|\langle w^0 \cdot \nabla J_\varepsilon[v], V \rangle| &\leq \|w^0\|_\infty \|\nabla v\|_2 \|V\|_2, \\
|\langle w^\ell \cdot \nabla J_\varepsilon[v], V \rangle| &\leq \|w^\ell\|_{\frac{p}{2^{p-1}}} \|\nabla J_\varepsilon[v]\|_{\frac{p}{p-2}} \|V\|_2, \\
|\langle J_\varepsilon[v] \cdot \nabla w^0, V \rangle| &\leq \|\nabla w^0\|_p \|J_\varepsilon[v]\|_{\frac{p}{p-2}} \|V\|_2, \\
|\langle J_\varepsilon[v] \cdot \nabla w^\ell, V \rangle| &\leq \|\nabla w^\ell\|_{\frac{p}{2^{p-1}}} \|J_\varepsilon[v]\|_{\frac{p}{p-2}} \|V\|_2.
\end{align*}
\]

Hence, by virtue of properties of the mollifier, and our hypotheses on \( v \) and \( V \), we deduce (37) a.e. in \( t \in (0, T) \).
**Theorem 6.** Under Assumption 1 for $w^t$ and $F$, there exists a unique solution to problem (31) such that, for all $T > 0$, $v \in C(0,T;J^{1,q}(\mathbb{R}^2_+))$ with $q := \frac{p}{2h}$

$$
t^{-\frac{q-1}{q}}\|v(t)\|_q + t^{-\frac{2}{2q}}\|\nabla v(t)\|_q \leq c \exp\left[\int_0^t g(t)dt\right] \left[\int_0^t \|F(t)\|_q^q dt\right]^\frac{1}{q},$$

for all $t \in [0,T)$, (38)

$$
\int_0^T \left[\|v(t)\|_q^q + \|D^2v(t)\|_q^q + \|\nabla \pi_v(t)\|_q^q\right] \leq c \exp\left[\int_0^T g(t)dt\right] \int_0^T \|F(t)\|_q^q dt,
$$

where the constant $c$ is independent of $w^t$ and $F$, and we set $g(t) := t^{q-1} \sum_{\ell=0}^h [\|w^t(t)\|_\infty^q + \|\nabla w^t(t)\|_\infty^q]$.

**Proof.** We introduce the following approximation problem:

$$v_t - \Delta v + \nabla \pi_v = -\sum_{\ell=0}^h w^t \cdot \nabla J_{\varepsilon}[v] - J_{\varepsilon}[v] \cdot \nabla \sum_{\ell=0}^h w^t + F_{\varepsilon},$$

$$\nabla \cdot v = 0, \text{ in } (0,T) \times \mathbb{R}^2_+,$$

$$v = 0 \text{ on } (0,T) \times \{x_2 = 0\}, \quad v = 0 \text{ on } \{0\} \times \mathbb{R}^2_+,$$

where $J_{\varepsilon}[\cdot]$ is a spatial mollifier and $F_{\varepsilon} := \exp[-\varepsilon|x|^2]F$. Of course, a solution to problem (39) is a pair $(v_{\varepsilon}, \pi_{v_{\varepsilon}})$, for the sake of the simplicity and of the brevity we omit the index $\varepsilon$. For all $\varepsilon > 0$ and $T > 0$, the data $F_{\varepsilon}$ belongs to $L^2(0,T;L^2(\mathbb{R}^2_+))$. Thanks to this firstly we are able to develop a $L^2$-theory for problem (39) depending on $\varepsilon$. Then on the base of this result, we approach the $L^q$-theory of solution to problem (31) based on the $L^q$-theory of the Stokes problem. Employing the Galerkin method, in the way suggested by Heywood in [16], we can easily establish the existence of the Galerkin approximation sequence which satisfies the set of relations

$$\frac{1}{2} \|v\|_2^2 + \int_0^t \|\nabla v\|_2^2 d\tau = -\int_0^t \left(\sum_{\ell=0}^h w^t \cdot \nabla J_{\varepsilon}[v] + J_{\varepsilon}[v] \cdot \nabla \sum_{\ell=0}^h w^t, v\right) d\tau + \int_0^t (F, v) d\tau,$$

$$\frac{1}{2} \|\nabla v\|_2^2 + \int_0^t \|P \Delta v\|_2^2 d\tau = \int_0^t \left(\sum_{\ell=0}^h w^t \cdot \nabla J_{\varepsilon}[v] + J_{\varepsilon}[v] \cdot \nabla \sum_{\ell=0}^h w^t, P \Delta v\right) d\tau - \int_0^t (F, P \Delta v) d\tau,$$
\begin{align*}
\frac{1}{2} \|
abla v(t)\|_2^2 + \int_0^t \| v_t \|_2^2 d\tau &= - \int_0^t (\sum_{\ell=0}^h w^f \nabla J_\varepsilon[v] + J_\varepsilon[v] \cdot \nabla \sum_{\ell=0}^h w^f, v_t) d\tau + \int_0^t (F_\varepsilon, v_t) d\tau .
\end{align*}

Applying Hölder’s inequality, employing estimates (37) with \( V \) substituted by \( v \), and by Lemma 3 from the first relation of the set we easily get the energy inequality

\begin{align*}
\| v(t) \|_2^2 + \int_0^t \| \nabla v(\tau) \|_2^2 d\tau &\leq c \exp \left[ t + c(\varepsilon) \int_0^t \| w^0 \|^2_\infty + \sum_{\ell=1}^h \| w^f \|^2_{2\ell-1} \right] d\tau \\
&+ c(\varepsilon) \int_0^t \left[ \| w^0 \|_p + \sum_{\ell=1}^h \| w^f \|_{2\ell-1} \right] d\tau \int_0^t \| F_\varepsilon \|_2^2 d\tau =: \mathcal{B}(t).
\end{align*}

(40)

Subsequently, from the remaining two relations of the above set, applying Hölder’s inequality, employing estimates (37) with \( V \) substituted by \( \Delta v, v_t \) along the two cases, and via (40) we easily get,

\begin{align*}
\| \nabla v(t) \|_2^2 + \int_0^t (\| v_\tau(\tau) \|_2^2 + \| P \Delta v(\tau) \|_2^2) d\tau &\leq c(\varepsilon) \left[ t \mathcal{B}(t) \sup_{(0,T)} (\| w^0 \|^4_\infty + \| \nabla w^0 \|^2_p \\
&+ \sum_{\ell=1}^h \| w^f \|_{2\ell-1}^4 + \sum_{\ell=1}^h \| \nabla w^f \|_{2\ell-1}^2 ) + \int_0^t \| F_\varepsilon \|^2_2 d\tau \right].
\end{align*}

Now standard arguments related to the Galerkin method allow us to deduce the existence of a pair \((v_\varepsilon, \pi v_\varepsilon)\) solution to problem (39) such that

\begin{align*}
v_\varepsilon &\in C([0, T, J^{1,2}(\mathbb{R}^2_+)) \cap L^2(0, T; W^{2,2}(\mathbb{R}^2_+)), \\
v_{\varepsilon t}, \nabla \pi v_\varepsilon &\in L^2(0, T; L^2(\mathbb{R}^2_+)).
\end{align*}

(41)

Now our task is to prove that the family of solutions \\{v_\varepsilon\} admits a limit for \( \varepsilon \to 0 \) enjoying of the property (38). By virtue of (41), we can apply Lemma 7 for each term of the right hand side of problem (39). Hence, for all \( \varepsilon > 0 \), we can claim that

\begin{align*}
- \sum_{\ell=0}^h w^f \cdot \nabla J_\varepsilon[v] - J_\varepsilon[v] \cdot \nabla \sum_{\ell=0}^h w^f + F_\varepsilon &\in L^q(0, T; L^q(\mathbb{R}^2_+)).
\end{align*}

By virtue of Theorem 5 related to the Stokes problem, for all \( \varepsilon > 0 \) and
For $T > 0$, we can claim that
\[
 s^{-\frac{q-1}{q}} \|v(s)\|_q + s^{-\frac{q-2}{2q}} \|\nabla v(s)\|_q \leq c \left[ \int_0^s \|G(\tau)\|_q^q d\tau \right]^{\frac{1}{q}}, \text{ for all } s \in (0, T),
\]
\[
 \int_0^s \left[ \|v(t)\|_q^q + \|D^2 v(t)\|_q^q + \|\nabla \pi_v(t)\|_q^q \right] dt \leq c \int_0^s \|G(t)\|_q^q dt, \text{ for all } s \in (0, T)
\]
(42)

where $G := -\sum_{\ell=0}^h w^\ell \cdot \nabla J_\varepsilon[v] - J_\varepsilon[v] \cdot \nabla \sum_{\ell=0}^h w^\ell + F_\varepsilon \in L^q(0, T; L^q(\mathbb{R}^n))$, and $c$ is independent of $\varepsilon$ and $T$. Now we look for estimates in $L^q(0, T; L^q(\mathbb{R}^n))$ for function $G$ which are independent of $\varepsilon$. By virtue of Lemma 6 and estimates (33) for $w^\ell$, applying the Cauchy and the Hölder inequality, we get
\[
 \|w^\ell(t) \cdot \nabla J_\varepsilon[v(t)]\|_q^q \leq c t^{q-1} \|w^\ell(t)\|_q^q \int_0^t \|v_\tau(\tau)\|_q^q d\tau + \eta \|D^2 v(t)\|_q^q,
\]
for all $\ell = 0, \ldots, h$. Analogously, we get
\[
 \|J_\varepsilon[v(t)] \cdot \nabla w^\ell(t)\|_q^q \leq c t^{q-1} \|\nabla w^\ell(t)\|_q^q \int_0^t \|v_\tau(\tau)\|_q^q d\tau,
\]
for all $\ell = 0, \ldots, h$. Finally, we easily deduce that
\[
 \|F_\varepsilon(t)\|_q \leq \|F(t)\|_q, \text{ for all } t > 0 \text{ and } \varepsilon > 0.
\]
(45)

Collecting estimates (43)-(45), recalling the definition of $G$, from (42)2, for a suitable $\eta > 0$ in estimate (43), for all $s > 0$, we obtain
\[
 \int_0^s \left[ \|v(t)\|_q^q + \|D^2 v(t)\|_q^q + \|\nabla \pi_v(t)\|_q^q \right] dt \leq c \int_0^s \left[ g(t) \int_0^t \|v_\tau(\tau)\|_q^q d\tau + \|F(t)\|_q^q \right] dt,
\]
(46)

where we have set
\[
 g(t) := t^{q-1} \sum_{\ell=0}^h \left[ \|w^\ell(t)\|_q^q + \|\nabla w^\ell(t)\|_q^q \right].
\]

From (46) an application of Lemma 3 ensures that
\[
 \int_0^s \|v(t)\|_q^q dt \leq c \exp \left[ \int_0^s g(t) dt \right] \int_0^s \|F(t)\|_q^q dt.
\]
(47)
Enclosing the last estimate in the right hand side of (46) a trivial computation furnishes
\[
\int_0^s \left[ \| v_t(t) \|^q_q + \| D^2 v(t) \|^q_q + \| \nabla \pi v(t) \|^q_q \right] dt \leq c \exp \left[ \int_0^s g(t) dt \right] \| F(t) \|^q_q dt.
\]

Estimates (47)-(48) are independent of \( \varepsilon \). Hence, taking into account the definition of \( G \), collecting estimates (43)-(45) and estimate (48), we have proved that
\[
\int_0^s \| G(\tau) \|^q_q d\tau \leq c \exp \left[ \int_0^s g(t) dt \right] \| F(t) \|^q_q dt, s \in (0, T).
\]

Thus, proving that \( \int_0^s g(t) dt \) is finite for all \( s > 0 \), then we have concluded the proof of the theorem. By virtue of the assumption on \( w^0 \) and weight \( t^{q-1} \), the integrability question can be restricted to the cases of \( \ell = 1, \ldots, h \).

By virtue of estimates (33) and assumptions on \( w^\ell \), we have
\[
\| w^\ell \|^q_q \leq c \| D^2 w^\ell \|^q_{A_\ell} \frac{1}{2^{\ell-1}} \| w^\ell \|^q_{B_\ell} \frac{1}{2^{\ell-1}} \leq c \sup_{(0, T)} \| w^\ell \|^q_{B_\ell} \frac{1}{2^{\ell-1}} \| D^2 w^\ell \|^q_{A_\ell} \frac{1}{2^{\ell-1}},
\]
where \( A_\ell := 2qa_\ell = 2^{\ell-h} \) and \( B_\ell := 2q(1 - a_\ell) = 2^{1-h}(p - 2^{\ell-1}) \). Since \( p > 2^{\ell+1} \), we obtain
\[
\int_0^s \| w^\ell \|^q_q dt \leq c \sup_{(0, T)} \| w^\ell \|^q_{B_\ell} \frac{1}{2^{\ell-1}} \int_0^s \| D^2 w^\ell \|^q_{A_\ell} \frac{1}{2^{\ell-1}} dt < \infty \text{ for all } s > 0 \text{ and } \ell = 1, \ldots, h.
\]

Analogously, by virtue of estimates (33)2, we get
\[
\| \nabla w^\ell \|^q_q \leq c \| w^\ell \|^q_{C_\ell} \frac{1}{2^{\ell-1}} \| D^2 w^\ell \|^q_{D_\ell} \frac{1}{2^{\ell-1}} \leq c \sup_{(0, T)} \| w^\ell \|^q_{C_\ell} \frac{1}{2^{\ell-1}} \| D^2 w^\ell \|^q_{D_\ell} \frac{1}{2^{\ell-1}},
\]
where \( C_\ell := q(1 - b_\ell) = \frac{q}{2^\ell} \left( \frac{1}{2} - \frac{2^{\ell-1}}{p} \right) \) and \( D_\ell := qb_\ell = \frac{q}{2^\ell} \left( \frac{1}{2} + \frac{2^{\ell-1}}{p} \right) \). Since \( p > 2^{\ell+1} \) we obtain
\[
\int_0^s \| \nabla w^\ell \|^q_q dt \leq c \sup_{(0, T)} \| w^\ell \|^q_{C_\ell} \frac{1}{2^{\ell-1}} \int_0^s \| D^2 w^\ell \|^q_{D_\ell} \frac{1}{2^{\ell-1}} dt < \infty \text{ for all } s > 0 \text{ and } \ell = 1, \ldots, h.
\]

The theorem is completely proved.
5 The linearized Navier-Stokes IBVP in $J^a$

In order to discuss the uniqueness we have to consider the following linearized Navier-Stokes problem:

\[ \begin{align*}
\phi_t - \Delta \phi + \nabla \pi \phi &= V \cdot \nabla \phi + \phi \cdot (\nabla \sum_{h=1}^{\nu} W^h)^T, \\
\nabla \cdot \phi &= 0 \text{ in } (0, T) \times \mathbb{R}_+^2, \\
\phi &= 0 \text{ on } (0, T) \times \{x_2 = 0\}, \\
\phi &= \phi_0 \text{ on } \{0\} \times \mathbb{R}_+^2, \\
\end{align*} \tag{49} \]

here the symbol $(\nabla b)^T$ means the transpost of tensor $\nabla b$ and $a \cdot (\nabla b)^T = (\frac{\partial b}{\partial x_k} a_i) e_k$. We assume that for all $T > 0$

\[ V \in C((0, T) \times \mathbb{R}_+^2), \quad \text{and } \nabla W^h \in C(0, T; L^{r^h}(\mathbb{R}_+^2)) \quad \text{for some } r^h > 2. \tag{50} \]

The investigation on problem (49) appears very similar to the one of the previous section. Actually there are some different technical aspects that make the difference. Theorem 6 is related to the existence, whereas Theorem 7 given below is related to the uniqueness by duality. Although the proofs are approached by a similar way (that is, initially by means of the $L^2$-theory), we look on the two theorems from a different point of view. As opposed to the previous section, here we discuss the initial boundary value problem with an initial data $\phi_0 \neq 0$ and body force $F = 0$, we study an $L^q$ theory for $q \in (1, 2)$ with the special property (54) (see below Theorem 7).

These are not given in section 4. As well the following lemmas are thought by slightly different way. Hence, in order to make more readable the results of section 4 and of this section, we have thought that it is better to furnish the results in two separated theorems, rather than to state all the results in a unique large theorem.

We start with the following

**Lemma 9.** Let $\phi \in J^{1,2}(\mathbb{R}_+^2)$ and let $\phi \in L^2(\mathbb{R}_+^2)$. Then it holds that

\[ \begin{align*}
|\langle V \cdot J_\varepsilon[\chi_\varepsilon \nabla \phi], \phi \rangle| &\leq \|V\|_{\infty} \|\nabla \phi\|_2 \|\phi\|_2, \\
|\langle J_\varepsilon[\phi] \cdot (\nabla W^h)^T, \phi \rangle| &\leq c \|\nabla \phi\|_2^{a_h} \|\phi\|_2^{1-a_h} \|\nabla W^h\|_{r^h} \|\phi\|_2, \\
\end{align*} \tag{51} \]

with $a_h = \frac{2}{r^h}$ and $c$ independent of $\phi, \phi$. Here $J_\varepsilon[\cdot]$ is a spatial mollifier, and $\chi_\varepsilon(x) := \chi(\varepsilon x)$ is the smooth cutoff function with support $(\frac{1}{\varepsilon}, \frac{2}{\varepsilon})$ ($\chi$ is defined in section 2).
Proof. Applying Hölder’s inequality, employing the properties of the mollifier, and Gagliardo-Nirenberg’s inequality we get

\[ |(V \cdot J_\varepsilon[\chi_\varepsilon \nabla \varphi], \phi)| \leq \|V\|_\infty \|\nabla \varphi\|_2 \|\phi\|_2 \]
\[ |(J_\varepsilon[\varphi] \cdot (\nabla W)^T, \phi)| \leq \|\varphi\|_{L^\infty} \|\nabla W\|_r \|\phi\|_2 \]
\[ \leq c \|\nabla \varphi\|_2 \|\phi\|_2^{1-a} \|\nabla W\|_r \|\phi\|_2, \]

which prove (51) \[\square\]

Lemma 10. Let \( q \leq r_h \) and \( \varphi \in C(0, T; J^q(\mathbb{R}^2_+)) \cap L^q(0, T; J^{1,q}(\mathbb{R}^2_+)) \cap W^{2,q}(\mathbb{R}^2_+) \). We assume that \( \varphi_t \in L^q(0, T; L^q(\mathbb{R}^2_+)) \). Then almost everywhere in \( t \in (0, T) \) it holds that

\[ \|V \cdot J_\varepsilon[\chi_\varepsilon \nabla \varphi]\|_q \leq c \|V\|_\infty \|D^2 \varphi\|_q^{\frac{1}{2}} \left[ \int_0^t \|\varphi_\tau\|_q d\tau + \|\varphi_0\|_q \right]^{\frac{1}{2}} \]
\[ \|J_\varepsilon[\varphi] \cdot (\nabla W^h)^T\|_q \leq c \|\nabla W^h\|_{r^h} \|D^2 \varphi\|_q^{\frac{1}{2}} \left[ \int_0^t \|\varphi_\tau\|_q d\tau + \|\varphi_0\|_q \right]^{\frac{1}{2}}, \]

where \( J_\varepsilon[\cdot] \) is a spatial mollifier, and \( \chi_\varepsilon(x) := \chi(\varepsilon x) \) is the smooth cutoff function with support \((\frac{1}{\varepsilon}, \frac{2}{\varepsilon})\) (\( \chi \) is defined in section 2). The constant \( c \) is independent of \( \varphi \) and \( \varepsilon \).

Proof. We recall the following:

\[ \|\varphi(t)\|_q \leq \|\varphi_0\|_q + \int_0^t \|\varphi_\tau\|_q d\tau, \text{ for all } t \in (0, T). \]

Applying Hölder’s inequality and Gagliardo-Nirenberg’s inequality, we get

\[ \|V \cdot J_\varepsilon[\chi_\varepsilon \nabla \varphi]\|_q \leq c \|V\|_\infty \|D^2 \varphi\|_q^{\frac{1}{2}} \|\varphi\|_q^{\frac{1}{2}}, \text{ a.e. } t \in (0, T). \]

So that easily estimate (53)_1 follows. Analogously applying Hölder’s inequality and Gagliardo-Nirenberg’s inequality we get

\[ \|J_\varepsilon[\varphi] \cdot (\nabla W)^T\|_q \leq \|\nabla W\|_r \|\varphi\|_{L^\infty} \leq c \|\nabla W\|_r \|D^2 \varphi\|_q^{\frac{1}{2}} \|\varphi\|_q^{\frac{1}{2}}, \text{ a.e. in } t \in (0, T). \]

Again we claim that easily estimate (53)_2 follows. \[\square\]
Theorem 7. For all \( \varphi_0 \in C_0(\mathbb{R}^2_+) \) there exists a unique solution to problem (49) such that for all \( T > 0 \)
\[
\varphi \in \bigcap_{1 < q < 2} \left[ C(0, T; J^q(\mathbb{R}^2_+)) \cap L^q(0, T; J^{1,q}(\mathbb{R}^2_+)) \right], \\
D^2 \varphi, \varphi_t, \nabla \pi \varphi \in \bigcap_{1 < q < 2} \left[ L^q(0, T; J^1_q(\mathbb{R}^2_+)) \right].
\]

Proof. We consider the approximation problem
\[
\varphi_t - \Delta \varphi + \nabla \pi \varphi = V \cdot J_\varepsilon[\chi_\varepsilon \nabla \varphi] + J_\varepsilon[\varphi] \cdot (\nabla \sum_{h=1}^{\nu} W^h)^T, \\
\nabla \cdot \varphi = 0 \text{ in } (0, T) \times \mathbb{R}^2_+, \\
\varphi = 0 \text{ on } (0, T) \times \{x_2 = 0\}, \quad \varphi = \varphi_0 \text{ on } \{0\} \times \mathbb{R}^2_+,
\]
where \( J_\varepsilon[\cdot] \) is a spatial mollifier, and \( \chi_\varepsilon(x) := \chi(\varepsilon x) \) is the smooth cutoff function with support \( (\frac{1}{\varepsilon}, \frac{2}{\varepsilon}) \) (\( \chi \) is defined in section 2). In order to obtain the solution to problem (55), we can apply as usual the Galerkin method. Employing estimates (51), where in place of \( \varphi \) we set \( \varphi \) to obtain the energy inequality, then \( \phi = P \Delta \varphi \) and \( \phi = \varphi_t \) to obtain the estimate for \( \|\nabla \varphi\|_2 \), we arrive at the relations:
\[
\frac{1}{2} \|\varphi\|_2^2 + (1 - \eta) \int_0^t \|\nabla \varphi\|_2^2 d\tau \leq \frac{1}{2} \|\varphi_0\|_2^2 + c \int_0^t \left[ \|V\|_\infty^2 + \sum_{h=1}^{\nu} \|\nabla W^h\|_{\mathcal{R}_h^{2/3}}^2 \right] \|\varphi\|_2^2 d\tau,
\]
\[
\frac{1}{2} \|\nabla \varphi\|_2^2 + (1 - \eta) \int_0^t \|P \Delta \varphi\|_2^2 d\tau \leq \frac{1}{2} \|\nabla \varphi_0\|_2^2 + c \int_0^t \left[ \|V\|_\infty^2 \|\nabla \varphi\|_2^2 + \sum_{h=1}^{\nu} \|\nabla W^h\|_{\mathcal{R}_h^{2/3}}^2 \|\nabla \varphi\|_2^{2(1-\alpha_h)} \right] d\tau,
\]
\[
\frac{1}{2} \|\varphi_t\|_2^2 + (1 - \eta) \int_0^t \|\varphi_t\|_2^2 d\tau \leq \frac{1}{2} \|\nabla \varphi_0\|_2^2 + c \int_0^t \left[ \|V\|_\infty^2 \|\nabla \varphi\|_2^2 + \sum_{h=1}^{\nu} \|\nabla W^h\|_{\mathcal{R}_h^{2/3}}^2 \|\nabla \varphi\|_2^{2(1-\alpha_h)} \right] d\tau.
\]
It is known, these estimates allow us to obtain a solution to problem (55) in the \( L^2 \)-setting. Thanks to the properties of mollifier and the previous result of existence in \( L^2 \)-setting, for all \( q \in (1, 2) \), we can consider, not uniformly in \( \varepsilon > 0 \), the right hand side of (55) belonging to \( L^q(0, T; L^q(\mathbb{R}^2_+)) \). Hence by virtue of Theorem 5, for all \( \varepsilon > 0 \) we obtain a solution \((\varphi, \pi \varphi)\) verifying...
also estimate (30) with the right hand side of equation (55) in place of $G$, that is for all $s > 0$ we get
\[
\int_0^s \left[ \|\dot{\varphi}(t)\|_q^q + \|D^2 \varphi(t)\|_q^q + \|\nabla \pi \varphi(t)\|_q^q \right] dt \\
\leq c \left[ \|\varphi_0\|_{2-\frac{q}{q'}}^q + \int_0^s \|V \cdot J_\varepsilon [\chi_\varepsilon \nabla \varphi] + J_\varepsilon [\varphi] \cdot (\nabla \sum_{h=1}^\nu W_h)^T \|_q^q dt \right].
\]

Now we look for estimates of $(\varphi, \pi \varphi)$ in the $L^q$-setting, $q \in (1, 2)$, and uniformly with respect to $\varepsilon$. Applying estimates (53) to the right hand side of (56), we get
\[
\int_0^s \left[ \|\dot{\varphi}(t)\|_q^q + \|D^2 \varphi(t)\|_q^q + \|\nabla \pi \varphi(t)\|_q^q \right] dt \\
\leq c \|\varphi_0\|_{2-\frac{q}{q'}}^q + c \int_0^t \|V\|_\infty^q \|D^2 \varphi\|_q^{q/2} \left[ \int_0^t \|\varphi\|_q^q d\tau + \|\varphi_0\|_q^q \right]^{q/2} dt \\
+ c \sum_{h=1}^\nu \int_0^t \|\nabla W_h\|_{q_h'}^q \|D^2 \varphi\|_q^{q/2} \left[ \int_0^t \|\varphi\|_q^q d\tau + \|\varphi_0\|_q^q \right]^{q/2} dt.
\]

Employing the Cauchy inequality, by means of Hölder’s inequality, we deduce
\[
\int_0^s \left[ \|\dot{\varphi}(t)\|_q^q + \|D^2 \varphi(t)\|_q^q + \|\nabla \pi \varphi(t)\|_q^q \right] dt \\
\leq c \|\varphi_0\|_{2-\frac{q}{q'}}^q + c \sup_{(0, T)} \left[ \|V\|_\infty^{2q} + \sum_{h=1}^\nu \|\nabla W_h\|_{q_h'}^{q_h} \right] \int_0^t \|\varphi\|_q^q d\tau \left[ \int_0^t \|\varphi\|_q^q d\tau \right] dt.
\]

Employing Gronwall’s lemma, we get
\[
\int_0^s \left[ \|\dot{\varphi}(t)\|_q^q + \|D^2 \varphi(t)\|_q^q + \|\nabla \pi \varphi(t)\|_q^q \right] dt \\
\leq c \|\varphi_0\|_{2-\frac{q}{q'}}^q \exp \left[ c \sup_{(0, T)} \left[ \|V\|_\infty^{2q} + \sum_{h=1}^\nu \|\nabla W_h\|_{q_h'}^{q_h} \right] s^q \right].
\]

Since the last inequality is uniform with respect to $\varepsilon > 0$, we have proved that for all $q \in (1, 2)$ there exists a solution $(\varphi, \pi \varphi)$ to problem (55). Now we
prove that, for any pair \( q, \bar{q} \in (1, 2) \), the corresponding solutions to problem with the same initial data \( \varphi_0 \in C_0(\mathbb{R}^2_+) \) coincide. For this goal we denote by \((\psi, \pi_\psi)\) the difference of the solutions \((\varphi_q, \pi_{\varphi_q})\) and \((\varphi_{\bar{q}}, \pi_{\varphi_{\bar{q}}})\) corresponding to the same initial data \( \varphi_0 \in C_0(\mathbb{R}^2_+) \). Since problem (49) is linear, \((\psi, \pi_\psi)\) satisfies the same (49), but with all the homogeneous data. The uniqueness is achieved employing the so called weighted function method in this connection see., e.g. [13].

Multiplying by \( \psi e^{-\mu|x|} \), \( \mu > 0 \), the first equation of (49) related to \( \psi \), setting \( g := e^{-\mu|x|} \), we get

\[
\frac{1}{2} \frac{d}{dt} \| g \psi \|^2_2 + \| g \nabla \psi \|^2_2 = -(\nabla g^2, \nabla \psi \cdot \psi) + (V \cdot \nabla \psi, g^2 \psi) + (\psi \cdot \nabla \nu \sum_{h=1}^{\nu} W_h^2, g^2 \psi) - (\pi \varphi_q \nabla^2 g, \psi) = \sum_{i=1}^{5} J_i(t).
\]

Applying the Cauchy inequality, we get

\[
|J_1(t) + J_2(t)| \leq c(\mu + \sup_{(0,T)} \| V \|^2_\infty) \| g \psi \|^2_2 + \eta \| g \nabla \psi \|^2_2,
\]

Applying Hölder’s inequality, and subsequently the Cauchy inequality, we get

\[
|J_3(t)| \leq \sum_{h=1}^\nu \| \nabla W_h \|_r \| g \psi \|^2_2 r_h
\]

Employing the Gagliardo-Nirenberg inequality, and then the Cauchy inequality, we obtain

\[
|J_3(t)| \leq c \sum_{h=1}^\nu \| \nabla W_h \|_r \| g \psi \|^2_2 \| g \nabla \psi \|^\frac{2}{p} \| g \psi \|^\frac{2}{p} r_h
\]

\[
\leq c \| g \psi \|^2_2 \sum_{h=1}^\nu \left[ \| \nabla W_h \|_r + \| \nabla W_h \|_r^\frac{2}{p} \right] + \eta \| g \nabla \psi \|^2_2.
\]

Finally, we consider the terms with pressure fields. It is enough to argument on single term, the discussion for the other term is analogous. Applying Hölder’s inequality and Lemma 1, we get

\[
|J_4(t)| \leq \mu \| \pi_{\varphi_q} \|_q \| g \|_q \| g \psi \|_2 \leq c \mu^\frac{2}{q}-1 \| \nabla \pi_{\varphi_q} \|_q \| g \psi \|_2.
\]

Collecting the estimates for \( J_i \), choosing \( \eta \) small, we arrive at

\[
\frac{d}{dt} \| g \psi \|^2_2 \leq c \left[ \mu + \sup_{(0,T)} (\| V \|^2_\infty + \sum_{h=1}^\nu [\| \nabla W_h \|_r + \| \nabla W_h \|_r^\frac{2}{p}]) \right] \| g \psi \|^2_2
\]

\[
+ c(\mu^\frac{2}{q}-1 \| \nabla \pi_{\varphi_q} \|_q + \mu^\frac{2}{q}-1 \| \nabla \pi_{\varphi_{\bar{q}}} \|_q).
\]
Integrating the last differential inequality, uniformly in $\mu > 0$, we deduce
\[
\|g\psi\|_2 \leq c \exp t \mathcal{C}(t) \int_0^t (\mu^{\frac{2}{r} - 1} \|\nabla \pi \varphi_q\|_q + \mu^{\frac{2}{r} - 1} \|\nabla \pi \varphi_{\tau q}\|_r) dt,
\]
where we set $\mathcal{C}(t) := c \left[ \mu + \sup_{(0, T)} (\|V\|_2^2 + \sum_{h=1}^n \left[ \|\nabla W^h\|_{r_h} + \|\nabla W^h\|_{r_h}' \right]) \right]$. Hence in the limit for $\mu \to 0$ we prove the uniqueness. \hfill \Box

6 A special IBVP of the perturbed Navier-Stokes equations

In this section we study the following initial boundary value problem:

\begin{align*}
w_t - \Delta w - \nabla \pi_w &= -w \cdot \nabla w - V \cdot \nabla w - w \cdot \nabla V + F, \\
\nabla \cdot w &= 0 \text{ in } (0, T) \times \mathbb{R}^2, \\
\pi_w(t, x) - \pi_w(t, 0) &= j(t) O(|x|^{\gamma + 1}) \text{ a.e. in } t > 0, \text{ for all } x \in \mathbb{R}^2_+,
\end{align*}

where $w \in C([0, T); J^{1, 2}(\mathbb{R}^2_+))$, $w \in C([0, T) \times \mathbb{R}^2_+)$ and $\lim_{t \to 0} \|w(t)\|_\infty = 0$, \hfill (58)

The following result holds:

**Theorem 8.** Assume that $V \in L^\infty((0, T) \times \mathbb{R}^2_+)$, satisfying the divergence free, and, for some $r \geq 2$, $\nabla V \in L^\infty(0, T; L^r(\mathbb{R}^2_+)) \cap L^r(0, T; W^{1, r}(\mathbb{R}^2_+))$. If $F \in L^2(0, T; L^2(\mathbb{R}^2_+)) \cap L^q(0, T; L^q(\mathbb{R}^2_+))$, for some $q > 2$, then there exists a unique solution to problem (57) such that for all $T > 0$

\[
w \in C([0, T]; J^{1, 2}(\mathbb{R}^2_+)) , \quad w_0 \in C([0, T) \times \mathbb{R}^2_+), \quad \lim_{t \to 0} \|w(t)\|_\infty = 0,
\]

\[
\pi_w(t, x) - \pi_w(t, 0) = j(t) O(|x|^{\gamma + 1}) \text{ a.e. in } t > 0, \text{ for all } x \in \mathbb{R}^2_+,
\]

where we have set $\pi_w(t) := \pi_w(t, 0), j(t) \in L^\infty(0, T), \gamma \in (0, 1)$ and $\bar{\sigma} > 2$.

**Proof.** We set $\mathcal{D}(t) := \exp \left[ t + c \int_0^t \|\nabla V\|_{r}^r d\tau \right] \int_0^t \|F\|_2^2 d\tau \cdot \|w(t)\|_2^2 + \int_0^t \|\nabla w\|_2^2 d\tau \leq \mathcal{D}(t), \hfill (59)

\[ \|\nabla w(t)\|^2_t + \int_0^t \left[ \|w_t\|^2 + \|P\Delta w\|^2 \right] d\tau \]

\[ \leq c \exp[D^2(t)] \left[ tD(t) \sup_{(0,T)} (\|V\|_{C^1} + \|\nabla V\|_{L^2(\tau')}^2) + \int_0^t \|F\|^2 d\tau \right], \tag{60} \]

uniformly in \( t > 0 \). Hence we consider as achieved (58)1. We look for the following decomposition of the solution:

\[ w := w^1 + w^2 + w^3 \quad \text{and} \quad \pi_w := \pi_{w^1} + \pi_{w^2} + \pi_{w^3}, \]

where \((w^1, \pi_{w^1})\) is the solution to problem (8) with zero initial data and force data \(G^1 := -(V + w) \cdot \nabla w\), further \((w^2, \pi_{w^2})\) is the solution to problem (8) with zero initial data and \(G^2 := -w \cdot \nabla V\), and finally \((w^3, \pi_{w^3})\) is the solution to problem (8) with zero initial data and \(G^3 := F\). By virtue of estimate (59)-(60), employing the Gagliardo-Nirenberg inequality, we get \(G^1 \in L^8(0, T; L^8(\mathbb{R}^2))\). Hence Theorem 5 ensures the existence of a unique solution such that \(w^1 \in C([0, T); L^8(\mathbb{R}^2)) \cap L^2(0, T; W^2,2(\mathbb{R}^2))\), and by Sobolev embedding theorem \(w^1 \in C([0, T); C(\mathbb{R}^2))\). We also get \(\nabla \pi_{w^1} \in L^8(0, T; L^\infty(\mathbb{R}^2))\). Analogously, by virtue of (59)-(60) under our assumption for \(V\), for some \(r_1 > 2\), we get \(G^2 \in L^{r_1}(0, T; L^{r_1}(\mathbb{R}^2))\). Hence Theorem 5 ensures the existence of a unique solution such that \(w^2 \in C([0, T); J^{1,r_1}(\mathbb{R}^2)) \cap L^{r_1}(0, T; W^{r_1,2}(\mathbb{R}^2))\), and by Sobolev embedding theorem \(w^2 \in C([0, T); C(\mathbb{R}^2))\). We also get \(\nabla \pi_{w^2} \in L^{r_1}(0, T; L^{r_1}(\mathbb{R}^2))\). Finally, by hypotheses on \(F\) we have \(G^3 \in L^q(0, T; L^q(\mathbb{R}^2))\) for some \(q > 2\). Hence Theorem 5 ensures the existence of a unique solution such that \(w^3 \in C([0, T); J^{1,q}(\mathbb{R}^2)) \cap L^q(0, T; W^{2,q}(\mathbb{R}^2))\), and by Sobolev embedding theorem \(w^3 \in C([0, T); C(\mathbb{R}^2))\). We also get \(\nabla \pi_{w^3} \in L^q(0, T; L^q(\mathbb{R}^2))\). As well as for the solutions we get

\[ \lim_{t \to 0} \|w^1(t)\|_\infty = \lim_{t \to 0} \|w^2(t)\|_\infty = \lim_{t \to 0} \|w^3(t)\|_\infty = 0. \tag{61} \]

The difference \((w - w^1 - w^2 - w^3, \pi_w - \pi_{w^1} - \pi_{w^2} - \pi_{w^3})\) is a solution to the Stokes problem with homogenous data. Hence it is easy to prove that, up to a constant for the pressure field, the difference is identically zero. Therefore the decomposition holds and we deduce (58)2 via (61) and (58)3 employing Sobolev embedding theorem and setting \(j(t) := \|\nabla \pi_{w^1}(t)\|_{L^8} + \|\nabla \pi_{w^2}(t)\|_{L^{r_1}} + \|\nabla \pi_{w^3}(t)\|_{L^q}\). Finally, setting \(\gamma := \min\{\frac{1}{2}, 1 - \frac{2}{r_1}, 1 - \frac{2}{q}\}\), as well setting \(\gamma = \min\{\frac{8}{3}, r_1, q\}\), we complete the proof. \(\square\)
7 Proof of Theorem 1

7.1 Existence

We develop the proof distinguishing the following cases for the initial data:

1) $p = 2$,

2) $p \in (2, 4]$,

3) $p > 4$.

In all the cases 1)-2) by $(U, \pi_U)$ we denote the solution to problem (8) assuming data $U(0, x) := u_0$ whose existence is ensured by Theorem 4.

1) In the case $p = 2$, we consider problem (57) with coefficient $V := U$ and data force $F = U \cdot \nabla U$. This data $F \in L^\infty(0, T; L^2(\mathbb{R}^2_+)) \cap L^{\frac{8}{5}}(0, T; L^{\frac{8}{3}}(\Omega))^2$. Denoted by $(w, \pi_w)$ the solution of Theorem 8, setting $u := U + w$ and $\pi_u := \pi_U + \pi_w$ we have proved the existence.

2) In the case of $p \in (2, 4]$, we denote by $(v_1, \pi_{v_1})$ the solution to problem (31) where we assume for data:

$h = 0, \ w^0 := U$ and $\ F := U \cdot \nabla U$.

By the regularity of $U$, for all $T > 0$, we have that $U \cdot \nabla U \in L^p(0, T; L^p(\mathbb{R}^2_+))$. By virtue of Theorem 6, for all $T > 0$, we have $v_1 \in C([0, T]; J^{1, p}(\mathbb{R}^2_+))$. Since $p \in (2, 4]$, we obtain $v_1 \in C(0, T; L^q(\mathbb{R}^2_+))$ for all $q \in [p, \infty]$. Hence we also get that

for all $T > 0$, \quad $v_1 \cdot \nabla v_1 \in L^2(0, T; L^2(\mathbb{R}^2_+)) \cap L^{\frac{8}{5}}(0, T; L^{\frac{8}{3}}(\mathbb{R}^2_+))$. \quad (62)

Actually, if $p \in (2, 4]$, applying Hölder’s inequality, we get

\[ \|v_1 \cdot \nabla v_1\|_2 \leq \|v_1\|_p \|\nabla v_1\|_p \leq \|v_1\|_p \|\nabla v_1\|_p < \infty, \text{ for all } t > 0, \]

\[ \|v_1 \cdot \nabla v_1\|_p \leq \|v_1\|_\infty \|\nabla v_1\|_p \leq \|v_1\|_\infty \|\nabla v_1\|_p < \infty, \text{ for all } t > 0. \]

For the last claim we apply the interpolation of Lebesgue spaces, hence we get

\[ \|\nabla U(t)\|_{\frac{8}{5}} \leq c\|\nabla U(t)\|_{\frac{8}{5}} \|\nabla U(t)\|_2^{\frac{8}{3}} \text{, for all } t > 0. \]

This last estimate and (19) imply $U \cdot \nabla U \in L^{\frac{8}{5}}(0, T; L^{\frac{8}{3}}(\Omega))$. 

---

\(2\) For the last claim we apply the interpolation of Lebesgue spaces, hence we get
The above estimates justify (62). Now we introduce the following perturbed Navier-Stokes problem:

\[ v^2_t - \Delta v^2 + \nabla \pi_{v^2} = -v^2 \cdot \nabla v^2 - \sum_{\ell=0}^{k+1} w^\ell \cdot \nabla v^2 - v^2 \cdot \nabla \sum_{\ell=0}^{k+1} w^\ell + F^1, \]

\[ \nabla \cdot v^2 = 0 \text{ in } (0, T) \times \mathbb{R}^2_+, \]

\[ v^2 = 0 \text{ on } (0, T) \times \{x_2 = 0\}, \quad v^2 = 0 \text{ on } \{0\} \times \mathbb{R}^2_+, \]

where \( w^0 := U, \ w^1 := v^1 \) and \( F^1 := -v^1 \cdot \nabla v^1 \). By virtue of Theorem 8, there exists a unique regular solution \((v^2, \pi_{v^2})\) to problem (63). Hence setting \( u := U + v^1 + v^2 \) and \( \pi_u := \pi_U + \pi_{v^1} + \pi_{v^2} \) we have proved the existence claimed in Theorem 1 for \( p \in (2, 4) \).

3) In the case of \( p > 4 \), we consider the greatest integer floor \( k \) of \( \log_2 \frac{p}{2} \) such that \( \frac{p}{2^k} > 2 \) and \( \frac{p}{2^{k+1}} \leq 2 \). Hence we get \( \frac{p}{2^k} \in (2, 4] \). We consider the finite family of solutions \( \{(v^h, \pi_{v^h})\}_{h=1, \ldots, k+1} \) to problem (31), where we set \( w^0 := U \) with initial data \( u_0 \), and, for all \( \ell = 1, \ldots, k+1 \),

\[ X^\ell := C(0, T; J^{1, \frac{p}{2^\ell}}(\mathbb{R}^2_+)) \cap L^\frac{p}{2^\ell-1}(0, T; W^{2, \frac{p}{2^\ell-1}}(\mathbb{R}^2_+)), \]

\( v^1 \) has as coefficient \( w^0 = U \) and as force data \( F^1 := -U \cdot \nabla U \in L^p(0, T; L^p(\mathbb{R}^2_+)) \), \( v^2 \) has as coefficients \( w^0 = U, \ w^1 = v^1, \) with \( v^1 \in X^1 \), as force data \( F^2 := -v^1 \cdot \nabla v^1 \in L^\frac{p}{2}(0, T; L^\frac{p}{2}(\mathbb{R}^2_+)) \), \( \ldots \ldots \ldots \)

\( v^{k+1} \) has as coefficients \( w^0 = U, \ w^1 = v^1, \ldots, w^k = v^k, \) \( v^\ell \in X^\ell, \ell = 1, \ldots, k, \)

as force data \( F^{k+1} := -v^k \cdot \nabla v^k \in L^\frac{p}{2^k}(0, T; L^\frac{p}{2^k}(\mathbb{R}^2_+)) \).

For each \( h = 1, \ldots, k+1 \) the existence of the pair \((v^h, \pi_{v^h})\) is ensured by Theorem 6. Now, by means of Theorem 8 we can solve the problem

\[ w_t - \Delta w + \nabla \pi_w = -w \cdot \nabla w - \sum_{\ell=0}^{k+1} w^\ell \cdot \nabla w - w \cdot \nabla \sum_{\ell=0}^{k+1} w^\ell + F, \]

\[ \nabla \cdot w = 0 \text{ in } (0, T) \times \mathbb{R}^2_+, \]

\[ w = 0 \text{ on } (0, T) \times \{x_2 = 0\}, \quad w = 0 \text{ on } \{0\} \times \mathbb{R}^2_+, \]
where $w^0 := U$, $w^\ell := v^\ell$, $\ell = 1, \ldots, k + 1$, and $F := -v^{k+1} \cdot \nabla v^{k+1} \in L^2(0, T; L^2(\mathbb{R}_+^2))$. The last claim is consequence of the fact that $\frac{2}{p_2} \in (2, 4]$.

Setting $u := U + w + \sum_{\ell=1}^{k+1} v^\ell$ and $\pi_u := \pi_U + \pi_w + \sum_{\ell=0}^{k+1} \pi_{v^\ell}$, then by construction the pair $(u, \pi_u)$ is a solution to equations (1). The solution satisfies the boundary condition. Since $\lim_{t \to 0} U(t, x) = u_0(x)$ for all $x \in \mathbb{R}_+^2$, in order to prove that $\lim_{t \to 0} u(t, x) = u_0(x)$ for all $x \in \mathbb{R}_+^2$, we can limit ourselves to prove that $\lim_{t \to 0} \|w + \sum_{\ell=1}^{k+1} v^\ell\|_\infty = 0$. Actually, this is a consequence of the estimates (38)$_1$ for the functions $v^\ell$, $\ell = 1, \ldots, k + 1$, and of estimate (58)$_2$ for $w$. The proof of the existence is completed if the regularity properties hold. This is a classical result for solutions to the 2D-Navier-Stokes problem, hence it is omitted.

### 7.2 Uniqueness

We begin recalling that the pressure field $\pi_u$ determined in section 7.1 is given by the sum

$$\pi_u := \pi_U + \sum_{\ell=1}^{k+1} \pi_{v^\ell} + \pi_w.$$ 

For each term we have the following estimates:

iv. for all $\gamma \in (0, 1)$, $|\pi_U(t, x)| = O(|x|^{\gamma})t^{-\frac{\gamma}{2}}$ for all $(t, x) \in (0, T) \times \mathbb{R}_+^2$,

v. for all $\ell = 1, \ldots, k + 1$, almost everywhere in $t > 0$, $\nabla \pi_{v^\ell} \in L^{\frac{2p}{k+1}}(\mathbb{R}_+^2)$ that, via Sobolev embedding, furnishes

$$|\pi_{v^\ell}(t, x) - \pi^0_{v^\ell}(t)| \leq cJ_\ell(t)|x - x^0|^1 - \frac{2^\ell}{p},$$

almost everywhere in $t > 0$ and for all $x \in \mathbb{R}_+^2$, where we set $\pi^0_{v^\ell}(t) := \pi_{v^\ell}(t, x_0)$ and $J_\ell(t) := \|\nabla \pi_{v^\ell}(t)\|_{\frac{2p}{k+1}}$, since $1 - \frac{2^{k+1}}{p} \leq 1 - \frac{2^\ell}{p} < 1$, for all $\ell = 1, \ldots, k + 1$, then, for a suitable constant $c$, we get

$$|\pi_{v^\ell}(t, x) - \pi^0_{v^\ell}(t)| \leq cJ(t)[|x|^{1 - \frac{2^{k+1}}{p}} + 1],$$

almost everywhere in $t > 0$ and for all $x \in R_+^2$, where we have set $J(t) := \sum_{\ell=1}^{k+1} \|\nabla \pi_{v^\ell}(t)\|_{\frac{2p}{k+1}}$ which belongs to $L^{\frac{2}{k+1}}(0, T)$,

vi. finally, for $\pi_w(t, x)$ estimate (58)$_3$ holds.
The following lemma concerns a weighted energy estimate of the kind proved in [13].

**Lemma 11.** Let \((u, \pi_u)\) and \((\overline{u}, \pi_{\overline{u}})\) be two regular solutions to problem (1) assuming the same initial data \(u_0 \in L^\infty(\mathbb{R}^2_+)\). Assume that the fields \(u, \overline{u} \in L^\infty((0, T) \times \mathbb{R}^2_+)\). Assume that there exist \(\rho, \overline{\rho} \in (0, 1)\) such that

\[
\begin{align*}
|\pi_u(t, x) - \pi_u(t)| &\leq J_u(t)(|x| + 1)^{\rho}, \\
|\pi_{\overline{u}}(t, x) - \pi_{\overline{u}}(t)| &\leq J_{\overline{u}}(t)(|x| + 1)^{\overline{\rho}}, \quad t > 0, \quad x \in \mathbb{R}^2_+,
\end{align*}
\]

(65)

where functions \(\pi_u(t), \pi_{\overline{u}}(t), J_u(t), J_{\overline{u}}(t) \in L^1_{loc}[0, T)\). Then, for \(\beta \in (\rho, 1) \cap (\overline{\rho}, 1)\), we get

\[
\|(u(t) - \overline{u}(t))(|x| + 1)^\beta\|_2 < \infty, \text{ for all } t \in (0, T).
\]

(66)

**Proof.** We set \(W := u - \overline{u}\) and \(\pi_W := \pi_u - \pi_{\overline{u}} + \pi_{\overline{u}}(t) - \pi_u(t)\). Denoted by \(\rho_0 := \max\{\rho, \overline{\rho}\}\), by the assumption (65) we have \(|\pi_W(t, x)| \leq (J_u(t) + J_{\overline{u}}(t))(|x| + 1)^{\rho_0}, t > 0, x \in \mathbb{R}^2_+\). We consider the system of the difference \((W, \pi_W)\) written as:

\[
\begin{align*}
W_t - \Delta W + \nabla \pi_W &= -W \cdot \nabla u - \overline{u} \cdot \nabla W, \\
\nabla \cdot W &= 0 \text{ in } (0, T) \times \mathbb{R}^2_+, \\
W &= 0 \text{ on } (0, T) \times \{x_2 = 0\}, \\
W &= 0 \text{ on } \{0\} \times \mathbb{R}^2_+.
\end{align*}
\]

We multiply the first equation by \((|x| + 1)^{-2\beta} \exp[-2\mu|x|]W =: g^2(x)W\). Integrating by parts on \((0, t) \times \mathbb{R}^2_+\) we get the weighted energy relation

\[
\frac{1}{2}\|W(t)g^2\|_2^2 + \int_0^t \|g \nabla W(\tau)\|_2^2 d\tau = \int_0^t \sum_{i=1}^4 I_i(t) dt.
\]

Integrating by parts and applying Hölder’s inequality, we get

- \(|I_1(t)| = |\int_{\mathbb{R}^2_+} \nabla g^2 \cdot \nabla W \cdot W dx| \leq 2(\beta + \mu)\|Wg\|_2\|g\nabla W\|_2,

- \(|I_2(t)| = |\int_{\mathbb{R}^2_+} W \cdot \nabla W \cdot u g^2 dx + \int_{\mathbb{R}^2_+} u \cdot WW \cdot \nabla g^2 dx| \leq \|u\|_\infty \left[\|Wg\|_2\|\nabla Wg\|_2 + 2(\beta + \mu)\|Wg\|_2^2\right],

- \(|I_3(t)| = \frac{1}{2} \int_{\mathbb{R}^2_+} |W|^2 u \cdot \nabla g^2 dx| \leq (\beta + \mu)\|\nabla\|_\infty\|Wg\|_2^2,

- \(|I_4(t)| \leq \int_{\mathbb{R}^2_+} |W| |\nabla W| |g^2| dx| \leq (\beta + \mu)\|\nabla\|_\infty\|Wg\|_2^2,

\]
\[ |I_4(t)| = \left| \int_{\mathbb{R}^2} \pi_w \nabla g^2 \cdot W \, dx \right| \]
\[ \leq 2 \left( \int_{\mathbb{R}^2} + J_u(t) + J_{\pi}(t) \right) \left[ \beta \left\| \frac{|x|+1}{|x|+1}^{\rho_0} g \right\|_2 + \mu \left\| (1+|x|)^{\rho_0} g \right\|_2 \right] \| Wg \|_2. \]

We collect the estimates for \( I_i \), and we estimate the right hand side of the energy relation. After applying the Cauchy inequality, we get
\[
\| W(t)g \|_2 \leq \int_0^t \left[ C(\beta, \mu) + \| u \|_\infty^2 + C(\beta, \mu) \left[ \| u \|_\infty + \| \pi \|_\infty \right] \right] \| Wg \|_2 d\tau \]
\[ + 2 \left[ \beta \left\| \frac{|x|+1}{|x|+1}^{\rho_0} g \right\|_2 + \mu \left\| (1+|x|)^{\rho_0} g \right\|_2 \right] \int_0^t \left( J_u(\tau) + J_{\pi}(\tau) \right) d\tau. \]

Employing Gronwall lemma we get
\[
\| W(t)g \|_2 \leq M(\beta, \mu)H(t, \beta, \mu) \int_0^t \left( J_u(\tau) + J_{\pi}(\tau) \right) d\tau, \quad (67)
\]
where we have set \( H(t, \beta, \mu) := e^0 \left[ C(\beta, \mu) + \| u \|_\infty^2 + C(\beta, \mu) \| u \|_\infty + \| \pi \|_\infty \right] d\tau \), and
\[
M(\beta, \mu) := 2 \left[ \beta \left\| \frac{|x|+1}{|x|+1}^{\rho_0} g \right\|_2 + \mu \left\| (1+|x|)^{\rho_0} g \right\|_2 \right].
\]
Since \( \beta > \rho_0 \), a simple computation furnishes that
\[
M(\beta, \mu) \leq c \| (|x|+1)^{\rho_0-1-\beta} \|_2 =: \overline{M} < \infty, \quad \text{for all } \mu > 0.
\]
Hence, (67) becomes
\[
\| W(t)g \|_2 \leq \overline{M} H(t, \beta, \mu) \int_0^t \left( J_u(\tau) + J_{\pi}(\tau) \right) d\tau.
\]
Since \( H(t, \beta, \mu) \leq H(t, \beta, 1) \), letting \( \mu \to 0 \), via the Beppo Levi theorem, the last estimate leads to the thesis.

Now we are in a position to obtain the uniqueness. We employ a duality argument, which is a variant of the one introduced in [10] for the uniqueness to solution of the Navier-Stokes Cauchy problem. Actually, the result is a consequence of the following:
Lemma 12. Let \((u, \pi_u)\) be the solution to problem (1) furnished by section 7.1. Let \((\overline{u}, \pi_{\overline{u}})\) be a regular solution to problem (1) corresponding to the same data \(u_0\). Assume that \(\overline{u} \in L^\infty((0, T) \times \mathbb{R}_+^2)\). Further assume that \(\pi_{\overline{u}}\) satisfies condition (65)_2. Then, up to a function \(c(t)\), the given solutions coincide on \((0, T)\).

Proof. We denote by \(\hat{V} := u - \overline{u}\) and \(\hat{\pi} := \pi_u - \pi_{\overline{u}}\). The pair \((\hat{V}, \hat{\pi})\) is a solution to the problem

\[
\begin{align*}
\hat{V}_t - \Delta \hat{V} + \nabla \hat{\pi} &= -\overline{u} \cdot \nabla \hat{V} - \hat{V} \cdot \nabla u, \\
\nabla \cdot \hat{V} &= 0 \text{ in } (0, T) \times \mathbb{R}_+^2, \\
\hat{V} &= 0 \text{ on } (0, T) \times \{x_2 = 0\}, \quad \hat{V} = 0 \text{ on } \{0\} \times \mathbb{R}_+^2. \tag{68}
\end{align*}
\]

We start claiming that \(\hat{V}\) enjoys of estimate (66). For this it is enough to verify the hypotheses of Lemma 11. Of course, we can limit ourselves to verify assumption (65)_1. This assumption is a consequence of items iv.-vi. Actually, a simple computation allows us to say that \(\rho = 1 - \frac{2}{p+1}\) in (65)_1.

We also remark that by construction \(\nabla u = \sum_{\ell=1}^{k+3} \nabla v^\ell + \nabla U + \nabla w =: \sum_{h=1}^{k+3} \nabla u^h\), hence each term of the sum belongs to \(C(0, T; L^{r_h}(\mathbb{R}_+^2))\) for a suitable \(r_h > 2\). Thanks to Theorem 7, for a given function \(\varphi_o \in C_0(\mathbb{R}_+^2)\), we can state the existence of the solutions \((\varphi, \pi_\varphi)\) to problem (49) with \(-\overline{u}\) in place of \(V\) and \(\sum_{h=1}^{k+3} \hat{u}^h\) in place of \(W\), where for a fixed \(t > 0\) we have set

\[
\hat{\pi}(\tau, x) := \pi(t - \tau, x) \quad \text{and} \quad \hat{u}^h(\tau, x) := u^h(t - \tau, x), \text{ for all } \tau \in [0, t].
\]

In such a way the pair

\[
\hat{\varphi}(\tau, x) := \varphi(t - \tau, x) \quad \hat{\pi}_\varphi(\tau, x) := \pi_\varphi(t - \tau, x), \text{ for all } \tau \in [0, t],
\]

is a solution to the adjoint problem of (68):

\[
\begin{align*}
\hat{\varphi}_t + \Delta \hat{\varphi} + \nabla \pi_\varphi &= -\overline{u} \cdot \nabla \hat{\varphi} + \hat{\varphi} \cdot \left(\sum_{h=1}^{k+3} \nabla u^h\right)^T, \\
\nabla \cdot \hat{\varphi} &= 0 \text{ in } (0, t) \times \mathbb{R}_+^2, \\
\hat{\varphi} &= 0 \text{ on } (0, T) \times \{x_2 = 0\}, \quad \hat{\varphi} = \varphi_o \text{ on } \{t\} \times \mathbb{R}_+^2. \tag{69}
\end{align*}
\]

We multiply equation (68)_1 by \(\chi_R(x) \hat{\varphi}\). We assume that \(R > \text{diam}\{\text{supp } \varphi_o\}\). Integrating by parts on \((0, t) \times \mathbb{R}_+^2\), we get

\[
(\hat{V}, \varphi_o) = \int_0^t \left[ (\hat{V}_R, \hat{\varphi}_t + \Delta \hat{\varphi}) + (\hat{V}, \nabla \chi_R \cdot \nabla \hat{\varphi}) + 2(\hat{V}, \nabla \chi_R \cdot \nabla \hat{\varphi}) + (\nabla \cdot \hat{\varphi}, \hat{V}_R) + (\nabla \cdot \hat{\varphi}, -\hat{\varphi} \cdot \nabla u, \chi_R) \right] d\tau.
\]
In the previous relation substituting the right hand side of (69), integrating by parts, we get

\[
(\hat{V}, \varphi) = \int_{0}^{t} \left[ (\hat{V}, \Delta \chi_{R} \hat{\varphi}) + 2(\hat{V}, \nabla \chi_{R} \cdot \nabla \hat{\varphi}) + (\nabla \cdot \nabla \chi_{R}, \hat{V} \cdot \hat{\varphi}) 
\right. \\
\left. + (\hat{\pi}, \nabla \chi_{R} \cdot \varphi) + (\varphi, \nabla \chi_{R} \cdot \hat{V}) \right] d\tau = \sum_{i=1}^{5} \frac{\int_{0}^{t} J_{i}(\tau) d\tau.}{(70)}
\]

Since, \( \hat{V}, \hat{\pi} \in L^{\infty}((0, T) \times \mathbb{R}^{2}_{+}) \), and \( \hat{\varphi} \in C(0, T; L^{q}(\mathbb{R}^{2}_{+})) \cap L^{q}(0, T; J^{1,q}(\mathbb{R}^{2}_{+})) \), for all \( q \in (1, 2) \), applying H"older’s inequality, we immediately deduce that

\[
\lim_{R \to \infty} \left| J_{1}(t) + J_{2}(t) + J_{3}(t) \right| = 0, \text{ for all } t \in (0, T).
\]

Since \( \hat{\pi} = \pi_{u} - \pi_{\pi}, \text{ and } \pi_{u} \text{ verifies } (65)_{1} \) with \( \rho = 1 - \frac{2}{p} \) and \( \pi_{\pi} \) verifies \((65)_{2}\) by hypothesis, applying H"older’s inequality, we arrive at

\[
|J_{4}(t)| \leq c \int_{R < |x| < 2R} \left[ |x|^{p-1} |\varphi| + |x|^{2-\beta} |\varphi| \right] dx \leq c(R^{p+1-\rho} \|\varphi\|_{r} + R^{\beta+1-\rho} \|\varphi\|_{\rho}).
\]

Choosing \( r, \beta \) in such a way that the exponent of \( R \) are negative, we get that

\[
\lim_{R \to \infty} |J_{4}(t)| = 0, \text{ for all } t \in (0, T).
\]

Analogously, recalling that \( \hat{V} \) has finite weighted energy (66), applying H"older’s inequality with exponents \( \frac{2}{\beta}, \frac{2}{1-\beta} \) and 2, we get

\[
|J_{5}(t)| = |(\hat{\pi}_{\varphi} - \pi_{\varphi_{0}}, \nabla \chi_{R} \cdot \hat{V})| \\
\leq \|\hat{\pi}_{\varphi} - \pi_{\varphi_{0}}\|_{L^{\frac{2}{1-\beta}}(R < |x| < 2R)} \|\hat{V}\|_{L^{\frac{2}{\beta}}(R < |x| < 2R)} \|x|^{-\beta} \|_{L^{2}(R < |x| < 2R)}.
\]

Here we have introduced the constant \( \pi_{\varphi_{0}} \) of Lemma 1. We remark that the introduction of any constant is allowed by the fact that \( (\nabla \chi_{R}, \hat{V}) = 0 \). As well we remark that for all \( \beta \in (0, 1) \) there exists a \( r \in (1, 2) \) such that \( \frac{2}{\beta} = \frac{2r}{2-r} \). Concurrently made these remarks justify the estimate of \( J_{5} \). Thus, employing (5), the following holds:

\[
|J_{5}(t)| \leq c \|\nabla \hat{\varphi}\|_{r} \|\hat{V}\|_{L^{2}(R < |x| < 2R)}.
\]

Hence we deduce

\[
\lim_{R \to \infty} |J_{5}(t)| = 0, \text{ for all } t \in (0, T).
\]
Collecting the estimates related to $J_i(t)$ we have that for $R \to \infty$ the right hand side of (70) is zero, that proves
\[(\tilde{V}, \varphi_0) = 0, \quad \text{for all } \varphi_0 \in \mathcal{C}_0(\mathbb{R}_+^2).\]
Therefore, function $\tilde{V}$ is the gradient of some $H(t,x)$, which is harmonic for $\tilde{V}$ has divergence free. Hence we can claim that $\tilde{V} \equiv 0$. The uniqueness is proved. \qed

References

[1] Abe K., *Exterior Navier-Stokes flows for bounded data*, Math. Nachr. 1-14 (2016) / DOI 10.1002/mana.201600132.

[2] Abe K., *Global well-psedeness of the two-dimensional exterior Navier-Stokes equations for non-decaying data*, arXiv:1608.06424 (to appear in Arch. Rational Mech. Anal.).

[3] Bae H-O. and Jin B.J., *Existence of strong mild solution of the Navier-Stokes equations in the half space with nondecaying initial data*, J. Korean Math. Soc. 49 (2012), no. 1, 113-138.

[4] Bogovskiǐ M.E., *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, Dokl. Acad. Nauk. SSSR 248 (1979), 1037-1049, English transl., Soviet Math. Dokl., 20 (1979), 1094-1098.

[5] Bogovskiǐ M.E., *Solution of some vector analysis problems connected with operators div and grad*, in “Trudy Seminar S. L. Sobolev,” 80 Akademia Nauk SSSR. Sibirskoe Otdelnie Matematik, Nowosibirsk in Russian (1980), 5-40.

[6] Chang T. and Jin B.J., *Initial and boundary value problem of the unsteady Navier-Stokes system in the half-space with Hölder continuous boundary data*, J. Math. Anal. Appl. 433 (2016), no. 2, 1846-1869.

[7] Chang T. and Jin B.J., *Pointwise decay estimate of Navier-Stokes flows in the half space with slowly decreasing initial value*, Nonlinear Anal. 157 (2017), 167-188.

[8] Chang T. and Jin B.J., *Notes on the space-time decay rate of the Stokes flows in the half space*, J. Differential Equations 263 (2017), no. 1, 240-263.

[9] Gallay Th., *Infinite energy solutions of the two-dimensional Navier-Stokes equations*, arXiv:1411.5156.
[10] Foias C., *Une remarque sur l’unicité des solutions des équations de Navier-Stokes en dimension n*, Bull. Soc. Math. France **89** (1961) 1-8.

[11] Galdi G.P., *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Steady-state Problems*, Second Edition, Springer Monographs in Mathematics. Springer. New-York (2011).

[12] Galdi G.P., Maremonti P. and Zhou Y., *On the Navier-Stokes problem in exterior domains with non decaying initial data*, J. Math. Fluid Mech. **14** (2012) 633-652.

[13] Galdi G.P. and Rionero S., *Weighted energy methods in fluid dynamics and elasticity*, Lecture Notes in Mathem., **1134** Springer-Verlag, Berlin, 1985.

[14] Giga Y., Inui K. and Matsui S., *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data*, Advances in fluid dynamics, 27-68, Quad. Mat., **4**, Dept. Math., Seconda Univ. Napoli, Caserta, 1999.

[15] Giga Y., Matsui S. and Sawada O., *Global existence of two-dimensional navier-Stokes flow with nondecaying initial velocity*, J. math. fluid mech., **3** (2001), 302–315.

[16] Heywood J.G., *The Navier-Stokes equations: on the existence, regularity and decay of solutions*, Indiana Univ. Math. J. **29** (1980), no. 5, 639-681.

[17] Higaki M., *Navier wall law for nonstationary viscous incompressible flows*, J. Differential Equations **260** (2016), no. 10, 7358-7396.

[18] Ladyzhenskaya O.A., *The Mathematical Theory of Viscous Incompressible Flow*, Mathematics and its applications **2**, Golden and Breach science publishers, 184 pages, (1963).

[19] Maremonti P., *Stokes and Navier-Stokes problems in the half-space: existence and uniqueness of solutions non converging to a limit at infinity*, J. Math. Sci. (N.Y.) **159** (2009), no. 4, 486-523.

[20] Maremonti P., *Non-decaying solutions to the Navier-Stokes equations in exterior domains: from the weight function method to the well posedness in $L^\infty$ and in Hölder continuous functional spaces*, Acta Appl. Math. **132** (2014), 411-426.

[21] Maremonti P., *On weak $D$-solutions to the non-stationary Navier-Stokes equations in a three-dimensional exterior domain*, Ann. Univ. Ferrara **60** (2014) 209-223, DOI 10.1007/s11565-013-0199-3.
[22] Maremonti P., *Weak solutions to the Navier-Stokes equations with data in $L(3, \infty)$*, “Mathematics for Nonlinear Phenomena: Analysis and Computation (2015)”, Proceedings in honor of Professor Giga’s 60th birthday, Springer.

[23] Maremonti P. and Shimizu S., *Global existence of solutions to 2-D Navier-Stokes flow with non-decaying initial data in exterior domains*, J. Math. Fluid Mech., (2017) doi.org/10.1007/s00021-017-0348-z.

[24] Maremonti P. and Solonnikov V. A., *On nonstationary Stokes problem in exterior domains*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24** (1997), no. 3, 395-449.

[25] Maremonti P. and Starita G., *On the nonstationary Stokes equations in half-space with continuous initial data*, J. Math. Sci. (N.Y.) **127** (2003), no. 2, 1886-1914.

[26] Sawada O., *A remark on the Navier-Stokes flow with bounded initial data having a special structure*, Hokkaido Math. J., **43** (2014) 1-8.

[27] Sawada O., Taniuchi Y. *A remark on $L^\infty$ solutions to 2-D Navier-Stokes equations*, J. Math. Fluid Mech., **3** (2007), 533–542.

[28] Solonnikov V.A., *Estimates of the solutions of the nonstationary Navier-Stokes system. (Russian) Boundary value problems of mathematical physics and related questions in the theory of functions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **38** (1973), 153-231, (e.t.) J. Soviet Math., **8** (1977), 467-529.

[29] Solonnikov V.A., *On nonstationary Stokes problem and Navier-Stokes problem in a half-space with initial data nondecreasing at infinity*, Function theory and applications, J. Math. Sci. (N.Y.) **114** (2003), no. 5, 1726-1740.

[30] Solonnikov V.A., *On estimates of solutions of the non-stationary Stokes problem in anisotropic Sobolev spaces and on estimates for the resolvent of the Stokes operator*, Russian Math. Surveys, **58** (2003), no. 2, 331-365.

[31] Ukai S., *A solution formula for the Stokes equation in $\mathbb{R}^n_+$*, Theory of nonlinear evolution equations and its applications (Japanese). Sūrikaisekikenkyūsho Kōkyūroku No. 604 (1987), 124-138.

[32] Zelik S., *Infinite energy solutions for damped Navier-Stokes equations in $\mathbb{R}^2$*, J. Math. Fluid Mech., 15 (2013), 717-745.