Sequential Design of Experiments via Linear Programming

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Abstract

The celebrated multi-armed bandit problem in decision theory models the central trade-off between exploration, or learning about the state of a system, and exploitation, or utilizing the system. In this paper we study the variant of the multi-armed bandit problem where the exploration phase involves costly experiments and occurs before the exploitation phase; and where each play of an arm during the exploration phase updates a prior belief about the arm. The problem of finding an inexpensive exploration strategy to optimize a certain exploitation objective is NP-HARD even when a single play reveals all information about an arm, and all exploration steps cost the same.

We provide the first polynomial time constant-factor approximation algorithm for this class of problems. We show that this framework also generalizes several problems of interest studied in the context of data acquisition in sensor networks. Our analyses also extend to switching and setup costs, and to concave utility objectives.

Our solution approach is via a novel linear program rounding technique based on stochastic packing. In addition to yielding exploration policies whose performance is within a small constant factor of the adaptive optimal policy, a nice feature of this approach is that the resulting policies explore the arms sequentially without revisiting any arm. Sequentiality is a well-studied paradigm in decision theory, and is very desirable in domains where multiple explorations can be conducted in parallel, for instance, in the sensor network context.

1 Introduction

The sequential design of experiments is a classic problem first formulated by Wald in 1947 [49]. The study of this problem gave rise to the general field of decision theory; and more specifically, led Robbins [41] to formulate the celebrated multi-armed bandit problem, and Snell [46] and Robbins [41] to invent the theory of optimal stopping. The copious literature in this field is surveyed by Whittle [51, 52].

The canonical problem of sequential design of experiments is best described in the language of the multi-armed bandit problem: There are \( n \) competing options referred to as “arms” (for instance, consider clinical treatments) yielding unknown rewards (or having unknown effectiveness) \( \{p_i\} \). Playing an arm (or testing a treatment on a patient) yields observations that reveal information about the underlying reward or effectiveness. The goal is to sequentially test the treatments (or sequentially play the arms) in order to ultimately choose the “best” one. Such problems are usually studied in a decision theoretic setting, where costs and utilities are associated with actions (testing
a treatment) and outcomes (choosing one treatment finally). The goal of any decision procedure is to come up with a plan for testing the treatments (or playing the arms) and choosing an outcome in order to optimize some criterion based on the costs and utilities. The testing procedure is termed exploration, and choosing the outcome is termed exploitation. The crux of the multi-armed bandit problem, and the reason has been extensively studied, is that it cleanly models the general trade-off between the cost of exploration (or learning more about the state of the system) and the utility gained from exploitation (or utilizing the system).

Various frameworks in decision theory differ in (i) the available information and (ii) optimization criteria for evaluating a decision plan. We now describe the problem we study from the perspective of these design choices. From the perspective of available information, we focus exclusively on the Bayesian setting, first formulated by Arrow, Blackwell and Girshick in 1949 [2]. In this setting, each arm (or treatment) is associated with prior information (specified by distributions) that updates via Bayes’ rule conditioned on the results of the plays (or tests). More formally, we are given a bandit with $n$ independent arms. The set of possible states of arm $i$ is denoted by $S_i$, and the initial state is $\rho_i \in S_i$. When the arm $i$ is played in a state $u \in S_i$, the arm transitions to state $v \in S_i$ w.p. $p_{uv}$ depending on the observed outcome of the play. The initial state models the prior knowledge about the arm. The states in general capture the posterior conditioned on the observations from a sequence of plays (or experiments) starting at the root. The cost of a play depends on whether the previous play was for the same arm or not. If the previous play was for the same arm, the play at $u \in S_i$ costs $c_u$, else it costs $c_u + h_i$, where $h_i$ is the setup cost for switching into arm $i$. Recall that the arms correspond to different treatments or experiments; therefore, this cost models setting up the corresponding experiment. Every state $u \in S_i$ is associated with a reward $r_u$, which is the expected reward of playing in this state (which is of course conditioned on the observations from the plays so far). By Bayes’ rule, the reward of the different states evolve according to a Martingale property: $r_u = \sum_{v \in S_i} p_{uv} r_v$. We present concrete examples of state spaces in Section 2.

From the optimization perspective, our objective is to maximize future utilization. Any policy explores (or tests) the arms for a certain amount of time and subsequently, exploits (or chooses) an arm that yields the best expected posterior (or future) reward. For this objective to be meaningful, we need to constrain the total cost we can incur in exploration before making the exploit decision. A natural example of this is product marketing research, where the entire exploration phase appears before the exploitation phase. Formally, a policy $\pi$ performs a possibly adaptive sequence of plays during the exploration. Since the state evolutions are stochastic, the exploration phase leads to a probability distribution over outcomes, $O(\pi)$. In outcome $o \in O(\pi)$, each arm $i$ is in some final state $u^o_i$. In this outcome $o$ the policy will choose the “best arm” $\max_i r_{u^o_i}$ (or a suitable concave function of the vector $\langle \cdots , r_{u^o_i} , \cdots \rangle$). The expected reward of the policy $\pi$ over the outcomes of exploration, $R(\pi)$ is $\sum_{o \in O(\pi)} q(o, \pi) \max_i r_{u^o_i}$. Let $C(o, \pi)$ denote the cost of the exploration plays made by the policy given an outcome $o$. In the simplest version, we seek to find the policy $\pi$ which maximizes $R(\pi)$ subject to $C(o, \pi) \leq C$ for all $o \in O$. As remarked in [2], this problem is solvable by dynamic programming [11, 13]. However this approach requires computation time polynomial in the joint state space (truncated by the budget constraint) for multiple arms, which is the product of the individual (truncated) state spaces. Unsurprisingly, the problem becomes NP-HARD even when a single play reveals the full information about an arm, and all plays (across different arms) cost the same [27]. Designing a policy which is computationally tractable, at the cost of bounded loss in performance, is the main goal of this paper. We will study the problem from the perspective of approximation algorithms, where we seek to find a provably near optimal

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1Our algorithms also extend to concave costs where the cost of $r$ consecutive play as well as switching out costs, we omit that discussion here.
solution with the restriction that the algorithm must run in time polynomial in the sum of the state spaces. More precisely, we seek an algorithm which would give us an utilization least $OPT/\alpha$ where $OPT = \max_\pi R(\pi)$ subject to $C(o, \pi) \leq C$ for all $o \in \mathcal{O}$; which is denoted as an $\alpha$ approximation. Note that we seek a multiplicative approximation because such a result is invariant under scaling of the rewards (see also the discussion on discount rewards below). Since it is NP-HARD to determine $OPT$, we seek to use a linear program to determine an upper bound $\gamma^* \geq OPT$ and provide an algorithm that achieves $\gamma^*/\alpha$ in the worst case. The added benefit of such an approach is that we have a concrete upper bound $\gamma^*$ for comparison and an algorithm which guarantees $\gamma^*/\alpha$ in the worst case, may have a significantly better (and quantifiable, due to the existence of the upper bound) performance in practice. The interested reader may consult [48] for a review of approximation algorithms.

The necessity of studying this problem is further hastened by the emergence of several applications where the number of arms is large, typically data intensive applications. Examples of this problem arise in “active learning” [38, 42] where the goal is to learn and choose the most discerning hypothesis by sequentially testing the hypotheses on a set of assisted examples; sensor networks [35], where the goal is sensor placement to maximize a utility function such as information gain, based on sequentially collecting a small number of samples; and databases [7], where the goal is to settle upon a possibly long running query execution plan, again based on a few carefully chosen samples.

1.1 Related Models

The future utilization objective is well-known in literature (refer for instance, Berry and Fristedt [12], Chapter 3.6). The unit cost version of this problem is a special case of the infinite horizon discounted multi-armed bandit problem. In the discounted bandit problem, there is an infinite discount sequence $\{\alpha_t \in [0,1] | t = 1, 2, \ldots\}$. Any policy $\pi$ plays an arm at each time step; suppose the expected reward from playing at time $t$ is $R_t(\pi)$. The goal is to design an adaptive policy $\pi$ to maximize $\sum_{t=1}^\infty \alpha_t R_t(\pi)$. The future utilization objective with an exploration budget $C$ corresponds to $\alpha_1 = \alpha_2 = \cdots = \alpha_C = \alpha_{C+2} = \alpha_{C+3} = \cdots = 0$, and $\alpha_{C+1} = 1$. This setting implies the objective is the reward of the arm chosen at the $(C+1)^{st}$ play (exploitation), and only plays of significance for making this choice are the first $C$ plays (exploration). As observed in [12], this problem seems significantly harder computationally than the case where the discount sequence is monotonically decreasing with time. In fact, when the discount sequence is geometric, i.e., $\alpha_t = \beta^t$ for some $\beta < 1$, the celebrated result of Gittins and Jones shows that there exists an elegant greedy optimal solution termed the Gittins index policy [26]; an index policy ranks the arms based solely on their own characteristics and plays the best arm at every step. The Gittins index is suboptimal both the finite horizon setting where $\alpha_t = 1$ for $t \leq C$ and 0 otherwise; as well as the future utilization setting we consider here [38]. Finally, Banks and Sundaram [10] show that no index exists in the presence of switching in/out costs.

Alternatives to the Bayesian formulation are also as old as the original study of Wald [49] and Robbins [11]. These versions do not assume prior information, but instead perform a min-max optimization over possible underlying rewards via a suitably constructed loss or regret measure. As observed in [12, 50], although minmax objectives are more robust, the Bayesian approach is more widely used since it typically requires less samples. Furthermore, the regret criterion naturally forces the optimization to consider the past: What is the minimum loss in the past $N$ trials due to not knowing the true rewards. Note that minimizing regret is not the same as maximizing future utilization, the former being more akin to the finite horizon version with discount sequence $\alpha_t = 1$ for $t \leq C$ and 0 otherwise. Intuitively, in the former, we attempt to minimize the error during the testing process, while in the latter, we do not care about errors in testing, but attempt to ensure
that at the end, we are truly picking the (near) best option for exploitation. Nevertheless, it is natural to ask whether the algorithms suggested in the context of minmax analysis, particularly the seminal works of Lai and Robbins [36], and Auer, Cesa-Bianchi and Fischer [4] (and extended to uniform switching costs in [47, 4]), have good performance guarantees in the future utilization measure. However these are “model free” algorithms, and it is easy to show that for appropriately chosen budget $C$, these algorithms have significantly inferior performance on the future utilization objective as compared to algorithms that use the prior information. This is not surprising because the objectives are different. Similar comments apply to the “experts” problem [18] and subsequent research in adversarial multiarmed bandits [5, 25] where the reward distribution is chosen by an adversary and need not be stochastic.

It is worth pointing out that in the loss function or minmax approach, the loss or regret arises due to lack of information about the rewards. The difficulty in optimizing future utilization in the Bayesian setting arises from the computational aspect. This is quite similar to the differences between the classes of online and approximation algorithms.

1.2 Structure of the Policies

For the future utilization measure, it is worth mentioning that the general structure of the policies are important. Two such classes of policies are noteworthy. The first class is motivated by the stopping time problem, an early example of which is the secretary problem [20]. A policy in this class fixes an ordering of the arms in advance, and samples the arms sequentially, i.e., does not return to previously rejected arm. The benefit of such strategy is that these are often succinct to represent and easy to implement in real hardware from the perspective of control. Another benefit, as the reader would have observed, is that it is easy to model switching/setup costs in such policies; these costs in fact can be generalized so that $r$ consecutive plays have a cost which is concave function in $r$. We define such policies as sequential, because the ordering of the arms is fixed beforehand. Such strategies have been considered in testing between two hypothesis [49], stochastic scheduling [39, 45], stochastic packing [23, 24] and in operator placement in databases [8, 9] – however all except the hypotheses testing results hold for two-level state spaces (or arms with point priors), where a single play reveals complete information about the underlying reward of the arm. (Refer Section 2 for a formal definition.)

The second and more restrictive class of policies performs all the tests (or plays) before observing any of their outcomes. Therefore, the policy has three disjoint successive phases: Test, observe, and select. Such non-adaptive policies are of interest when the observations can be made in parallel, and therefore the final choice can be made quicker. Naturally these strategies are meaningful for two level state spaces, and have thus been found to be of interest in context of sensor networks [35], multihoming networks [11], stochastic optimization [27, 30] and database optimization [7].

For both the above classes, the goal is to show that performance of an algorithm that is restricted to the respective class is not significantly worse compared to an adversary whose strategy is fully adaptive. This is known as the Adaptivity Gap of a strategy. All previous analysis of adaptivity gap was restricted to two level state spaces. This paper provides an uniform framework that extends to both the classes above and applies to multilevel state spaces. It is interesting to note that one of the original goals of Wald [49] in sequential analysis was to explore sequential strategies. Though such strategies are optimal for choosing between two hypothesis, the difficulty in obtaining optimal strategies for testing multiple competing hypotheses was known since that time. The major contribution of this work is to show that in a variety of bandit settings, when we are seeking to optimize any concave function of the posterior probabilities, the adaptivity gap in considering sequential strategies is bounded by a constant. In other words, the performance of a fully adaptive
solution cannot be significantly better than a sequential strategy.

1.3 Problems and Results

We consider three main types of problems in this paper. Recall that there are \( n \) independent arms, each with its own state space \( S_i \); a policy \( \pi \) adaptively explores the arms paying expected cost \( C(\pi) \) before selecting an arm for exploitation based on the observed outcomes. The expected reward of the selected arm over the outcomes of the policy \( \pi \) is denoted \( R(\pi) \).

- **Budgeted (Futuristic) Bandits:** There is a cost budget \( C \). A policy \( \pi \) is feasible if for any sequence of plays made by the policy, the cost is at most \( C \). The goal is to find the feasible policy \( \pi \) with maximum \( R(\pi) \). We have already discussed switching costs. An extension of switching cost is **concave play cost** where the cost of sequential interrupted plays of an arm is concave in the number of plays. This was first hinted at in \([2]\) and the authors explicitly settled on linear costs.

  A generalization of the above problem is **budgeted concave utility bandits** problem where the objective function is an arbitrary concave function of the final rewards of the arms. Examples of such function include choosing the best \( K \) arms, power allocation across noisy channels \([21]\) or optimizing “TCP friendly” network utility functions \([37]\).

- **Model Driven Optimization:** This is a non-adaptive formulation of the above, where the state space \( S_i \) is 2-level and a single play reveals full information about an arm. In such a context, **non-adaptive** strategies are desirable since the plays can be executed in parallel. A feasible non-adaptive policy \( \pi \) chooses a subset of the arms to explore, before seeing the result of any of the plays. There has been a significant number of papers in recent years, specially in the context of sensor networks. Our paper unifies this thread with the bandit framework.

- **Lagrangean (Futuristic) Bandits:** Find the policy \( \pi \) with maximum \( R(\pi) - C(\pi) \). Note that the Lagrangean can be defined on both the adaptive and non-adaptive setting. This is a natural extension of the single-arm optimal stopping time problem.

In this paper, we present a single framework that provides efficient algorithms yielding policies with near-optimal performance for all of the above problems. For the budgeted (futuristic) bandits in the concave cost setting (including switching in/out cost), we show that there exists a sequential strategy that respects the budget, and has objective value at most a factor 4 away from that of the optimal fully-adaptive strategy subjected to the same budget. Section 2 discusses different state spaces. This is presented in Section 3 presents the approximate sequential strategy that respects the budget, for linear utilities (objective function). We also present a bicriteria \( 2(1 + \alpha) \) approximation with the cost constraint relaxed by a factor \( \frac{1}{\alpha} \). In Section 4, we show how the same framework gives a more restricted non-adaptive strategy for 2-level states spaces which is within constant factor of the best adaptive strategy. In contrast, for multi-level state spaces, any non-adaptive strategy has a significant performance loss. We also present a sequential strategy that is a 2 approximation for the Lagrangean Bandits in Section 5. In Section 6, we extend the results in Section 3 to concave utilities with a factor 2 loss of the approximation factor.

Note that constant factor approximations are best possible from the context of **adaptivity gap** of sequential policies as well as **integrality gap** of the linear programming relaxations we use.

**Techniques:** We use a linear programming formulation over the state space of individual arms, and we achieve polynomial sized formulation in the size of each individual state space. This particular
formulation has been used in the past \cite{53,10} and found to be useful in practice. To the best of our knowledge, we present the first analysis of these relaxations in the finite horizon context.

We also bring to bear techniques from stochastic packing literature, particularly the work on adaptivity gaps by Dean, Goemans and Vondrák \cite{23,24,22}. Their results can be viewed as sequential strategies for 2-level state spaces and is similar to the online nature of the policies considered in stochastic scheduling \cite{39,45}, where there is a strong notion of “irrevocable commitment”. While the online notion is related to sequential strategies, they are not the same.

In terms of analysis, our results can be thought of as extending analysis both to arbitrary state spaces as well as for non-adaptive strategies for the 2-level case. Our overall technique can be thought of as “LP rounding via stochastic packing” – finding this connection between finite horizon multi-armed bandits and stochastic packing by designing simple LP rounding policies for a very general class of budgeted bandit problems represents the key contribution of this work.

**Related Work:** Several heuristics had been proposed for the budgeted (futuristic) bandit problem by Schneider and Moore \cite{42} and Madani et al. \cite{38}. The final algorithm that arises from our framework bears resemblance (but is not the same) to the algorithms proposed therein, but as far as we are aware there was no prior analysis of any algorithm in this context. A series of papers \cite{27,55,30} considered the 2-level state spaces (where a single play resolves all information about an arm) for specific problems and presented approximations. The Lagrangean (futuristic) bandit problem with 2-level state space has been considered before in \cite{31}, where a 1.25 approximation is presented. None of those techniques apply for the iterative refinement that is required for multiple level state spaces. Note that most other literature on stochastic packing do not consider refinement of information \cite{33,28}.

Our LP relaxation is well-studied in the context of multi-armed bandit problems \cite{15,53,10} and other loosely coupled systems such as multi-class queueing systems \cite{14,17}; we present the first provable analysis of this formulation. Though LP formulations over the state space of outcomes exist for other stochastic optimization problems such as multi-stage optimization with recourse \cite{34,43,19}, these formulations are based on sampling scenarios. However these problems also do not have a notion of refinement, and are fundamentally different from our setting where the scenarios would be refinement trajectories \cite{32} that are hard to sample.

## 2 Types of State Spaces

Recall that each arm is associated with a state that evolves when the arm is played. The state captures the distributional knowledge about the reward distribution of the arm. Formally, the set of possible states of arm $i$ is denoted by $S_i$, and the initial state is $\rho_i \in S_i$. When the arm $i$ is played in a state $u \in S_i$, the arm transitions to state $v \in S_i$ w.p. $p_{uv}$ depending on the observed outcome of the play. The initial state models the prior knowledge about the arm. The states in general capture the posterior conditioned on the observations from a sequence of plays (or experiments) starting at the root. Every state $u \in S_i$ is associated with a reward $r_u$, which is the expected reward of playing in this state (which is of course conditioned on the observations from the plays so far). By Bayes’ rule, the reward of the different states evolve according to a Martingale property:

$$r_u = \sum_{v \in S_i} p_{uv} r_v.$$

We now present two representative scenarios in order to better motivate the abstract problem formulation. In the first scenario, the underlying reward distribution is deterministic, and the distributional knowledge is specified as a distribution over the possible deterministic values; this implies that the uncertainty about an arm is completely resolved in one play by observing the reward. In the second scenario, the uncertainty resolves gradually over time.
Two-level State Space. A two-level state space models the case where the underlying reward of the arm is deterministic, so that the prior knowledge is a distribution over these values. In this setting, a single play resolves this distribution into a deterministic posterior. Formally, the prior distributional knowledge \( X_i \) is a discrete distribution over values \( \{a_1^i, a_2^i, \ldots, a_m^i\} \), so that \( \Pr[X_i = a_j^i] = p_j^i \) for \( j = 1, 2, \ldots, m \). The state space \( S_i \) of the arm is as follows: The root node \( \rho_i \) has \( r_{\rho_i} = \mathbf{E}[X_i] = \mu_i \). For \( j = 1, 2, \ldots, m \), state \( i_j \) has \( r_{i_j} = a_j^i \), and \( p_{\rho_i, i_j} = p_j^i \). Since the underlying reward distribution is simply a deterministic value, the state space is 2-level, defining a star graph with \( \rho_i \) being the root, and \( i_1, i_2, \ldots, i_m \) being the leaves.

To motivate budgeted bandits in such state spaces, consider a sensor network where the root server monitors the maximum value \([6, 44]\). The probability distributions of the values at various nodes are known to the server via past observations. However, at the current step, probing all nodes to find out their actual values is undesirable since it requires transmissions from all nodes, consuming their battery life. Consider the simple setting where the network connecting the nodes to the server is a one-level tree, and probing a node consumes battery power of that node. Given a bound on the total battery life consumed, the goal of the root server is to maximize (in expectation) its estimate of the maximum value. Formally, each node corresponds to a distribution \( X_i \) with mean \( \mu_i \); the exact value sensed at the node can be found by paying a “transmission cost” \( c_i \). The goal of the server is to adaptively probe a subset \( S \) of nodes with total transmission cost at most \( C \) to maximize the estimate of the largest value sensed, i.e. maximize \( \mathbf{E}[\max(\max_{i \in S} X_i, \max_{i \notin S} \mu_i)] \), where the expectation is over the adaptive choice of \( S \) and the outcome of the probes. The term \( \max_{i \in S} \mu_i \) incorporates the mean of the unprobed nodes into the estimate of the maximum value.

In this context, it is desirable for the sensor node to probe the nodes in parallel, i.e., use a non-adaptive strategy. The question then becomes how good is such a strategy compared to the optimal adaptive strategy. We show positive results for the context of 2-level spaces in Section 4.

Multi-level State Spaces. These are the most general state spaces we consider, and make sense in contexts such as clinical trials where the underlying effectiveness of a treatment is a random variable following a parametrized distribution with unknown parameters. The prior distribution will then be a distribution over possible parameter values. In the clinical trial setting, each experimental drug is a bandit arm, and the goal is to devise a clinical trial phase to maximize the belief about the effectiveness of the drug finally chosen for marketing. Each drug has an effectiveness that is unknown a priori. The effectiveness can be modeled as a coin whose bias, \( \theta \), is unknown a priori – the outcomes of tossing the coin (running a trial) are 0 and 1 which correspond to a trial being ineffective and effective respectively. The uncertainty in the bias is specified by a prior distribution (or belief) on the possible values it can take. Since the underlying distribution is Bernoulli, its conjugate prior is the Beta distribution. A Beta distribution with parameters \( \alpha_1, \alpha_2 \in \{1, 2, \ldots, \} \), which we denote \( B(\alpha_1, \alpha_2) \) has p.d.f. of the form \( c \theta^{\alpha_1 - 1}(1 - \theta)^{\alpha_2 - 1} \), where \( c \) is a normalizing constant. \( B(1,1) \) is the uniform distribution, which corresponds to having no a priori information. The distribution \( B(\alpha_1, \alpha_2) \) corresponds to the current (posterior) distribution over the possible values of the bias \( \theta \) after having observed \( (\alpha_1 - 1) \) 0’s and \( (\alpha_2 - 1) \) 1’s. Given this distribution as our belief, the expected value of the bias or effectiveness is \( \frac{\alpha_1}{\alpha_1 + \alpha_2} \).

The state space \( S_i \) is a DAG, whose root \( \rho_i \) encodes the initial belief about the bias, \( B(\alpha_1, \alpha_2) \), so that \( r_{\rho_i} = \frac{\alpha_1}{\alpha_1 + \alpha_2} \). When the arm is played in this state, the state evolves depending on the outcome observed – if the outcome is 1, which happens w.p. \( \frac{\alpha_1}{\alpha_1 + \alpha_2} \), the child \( u \) has belief \( B(\alpha + 1, \alpha_2) \), so that \( r_u = \frac{\alpha_1 + 1}{\alpha_1 + \alpha_2 + 1} \), and \( p_{\rho_i, u} = \frac{\alpha_1}{\alpha_1 + \alpha_2 + 1} \); if the outcome is 0, the child \( v \) has belief \( B(\alpha_1, \alpha_2 + 1) \), \( r_v = \frac{\alpha_1}{\alpha_1 + \alpha_2 + 1} \), and \( p_{\rho_i, v} = \frac{\alpha_2}{\alpha_1 + \alpha_2 + 1} \). In general, if the DAG \( S_i \) has depth \( C \) (corresponding to playing the arm at most \( C \) times), it has \( O(C^2) \) states. We omit details, since Beta distributions and their
multinomial generalizations, the Dirichlet distributions, are standard in the Bayesian context (refer for instance Wetherill and Glazebrook [50]).

3 Budgeted Bandits

We are given a bandit with $n$ independent arms. The set of possible states of arm $i$ is denoted by $S_i$, and the initial state is $\rho_i \in S_i$. When the arm $i$ is played in a state $u \in S_i$, the arm transitions to state $v \in S_i$ w.p. $p_{uv}$. The reward at a state satisfies $r_u = \sum_{v \in S_i} p_{uv} r_v$. The cost of a play depends on whether the previous play was for the same arm or not. If the previous play was for the same arm, the play at $u \in S_i$ costs $c_u$, else it costs $c_u + h_i$, where $h_i$ is the setup cost for switching into arm $i$. A policy $\pi$ performs a possibly adaptive sequence of plays during the exploration, leading to a probability distribution over outcomes, $O(\pi)$. In outcome $o \in O(\pi)$, each arm $i$ is in some final state $u_i^o$. In this outcome $o$ the policy chooses $\max_i r_{u_i^o}$. The expected reward of the policy $\pi$ over the outcomes of exploration, $R(\pi)$ is $\sum_{o \in O(\pi)} q(o, \pi) \max_i r_{u_i^o}$. Let $C(o, \pi)$ denote the cost of the exploration plays made by the policy given an outcome $o$. In this section, we seek to find the policy $\pi$ which maximizes $R(\pi)$ subject to $C(o, \pi) \leq C$ for all $o \in O$.

We describe the linear programming formulation and rounding technique that yields a $4$-approximation. We note that the formulation and solution are polynomial in $n$, the number of arms, and $m$, the number of states per arm.

3.1 Linear Programming Formulation

Recall the notation from Section 1.3. Consider any adaptive policy $\pi$. For some arm $i$ and state $u \in S_i$, let: (1) $w_u$ denote the probability that during the execution of the policy $\pi$, arm $i$ enters state $u \in S_i$; (2) $z_u$ denote the probability that the state of arm $i$ is $u$ and the policy plays arm $i$ in this state; and (3) $x_u$ denote the probability that the policy $\pi$ chooses the arm $i$ in state $u$ during the exploitation phase. Note that since the latter two correspond to mutually exclusive events, we have $x_u + z_u \leq w_u$. The following LP which has three variables $w_u, x_u, z_u$ for each arm $i$ and each $u \in S_i$. A similar LP formulation was proposed for the multi-armed bandit problem by Whittle [53] and Bertsimas and Nino-Mora [40].

Maximize $\sum_{i=1}^{n} \sum_{u \in S_i} x_u r_u$

$\sum_{i=1}^{n} (h_i z_{\rho_i} + \sum_{u \in S_i} c_u z_u) \leq C$

$\sum_{i=1}^{n} \sum_{u \in S_i} x_u \leq 1$

$\sum_{v \in S_i} z_v p_{vu} = w_u \forall i, u \in S_i \setminus \{\rho_i\}$

$x_u + z_u \leq w_u \forall u \in S_i, \forall i$

$x_u, z_u, w_u \in [0, 1] \forall u \in S_i, \forall i$

Let $\gamma^*$ be the optimal LP value, and $OPT$ be the expected reward of the optimal adaptive policy.

Claim 3.1. $OPT \leq \gamma^*$.

Proof. We show that the $w_u, z_u, x_u$ as defined above, corresponding to the optimal policy $\pi^*$, are feasible for the constraints of the LP. Since each possible outcome of exploration leads to choosing one arm $i$ in some state $u \in S_i$ for exploitation, in expectation over the outcomes, one arm in one state is chosen for exploitation. This is captured by the first constraint. Further, since on each sequence of outcomes (the decision trajectory), the cost of playing and switching into the arm is at
most $C$, over the entire decision tree, the expected cost of switching into the root states $\rho_i$ plus the expected cost of play is at most $C$. This is captured by the second constraint. Note that the LP only takes into account the cost of switching into an arm the very first time this arm is explored, and ignores the rest of the switching costs. This is clearly a relaxation, though the optimal policy might switch multiple times into any arm. However, our rounding procedure switches into an arm at most once, preserving the structure of the LP relaxation.

The third constraint simply encodes that the probability of reaching a state $u \in S_i$ during exploration. It is precisely the probability with which it is played in some state $v \in S_i$, times the probability $p_{iu}$ that it reaches $u$ conditioned on that play. The constraint $x_u + z_u \leq w_u$ simply captures that playing an arm is a disjoint event from exploiting it in any state. The objective is precisely the expected reward of the policy. Hence, the LP is a relaxation of the optimal policy.

3.2 The Single-arm Policies

The optimal LP solution clearly does not directly correspond to a feasible policy since the variables do not faithfully capture the joint evolution of the states of different arms. Below, we present an interpretation of the LP solution, and show how it can be converted to a feasible approximately optimal policy.

Let $(w^*_u, x^*_u, z^*_u)$ denote the optimal solution to the LP. We can assume w.l.o.g. that $w^*_u = 1$ for all $i$. Ignoring the first two constraints of the LP for the time being, the remaining constraints encode a separate policy for each arm as follows: Consider any arm $i$ in isolation. The play starts at state $\rho_i$. The arm is played with probability $z^*_\rho_i$, so that state $u \in S_i$ is reached with probability $z^*_\rho_i p_{\rho_i u}$. This play incurs cost $h_i + c_{\rho_i}$, which captures the cost of switching into this arm, and the cost of playing at the root. At state $\rho_i$, with probability $x^*_\rho_i$, the play stops and arm $i$ is chosen for exploitation. The events involving playing the arm and choosing for exploitation are disjoint. Similarly, conditioned on reaching state $u \in S_i$, with probabilities $z^*_u/w^*_u$ and $x^*_u/w^*_u$, arm $i$ is played and chosen for exploitation respectively. This yields a policy $\phi_i$ for arm $i$ which is described in Figure 1. For policy $\phi_i$, it is easy to see by induction that if state $u \in S_i$ is reached by the policy with probability $w^*_u$, then state $u \in S_i$ is reached and arm $i$ is played with probability $z^*_u$.

The policy $\phi_i$ sets $\mathcal{E}_i = 1$ if on termination, arm $i$ was chosen for exploitation. If $\mathcal{E}_i = 1$ at state $u \in S_i$, then exploiting the arm in this state yields reward $r_u$. Note that $\mathcal{E}_i$ is a random variable that depends on the execution of policy $\phi_i$. Let $R_i, C_i$ denote the random variables corresponding to the exploitation reward, and cost of playing and switching, respectively.

| Policy $\phi_i$: If arm $i$ is currently in state $u$, then choose $q \in [0, w^*_u]$ uniformly at random: |
|---|
| 1. If $q \in [0, z^*_u]$, then play the arm (explore). |
| 2. If $q \in (z^*_u, z^*_u + x^*_u]$, then stop executing $\phi_i$, set $\mathcal{E}_i = 1$ (exploit). |
| 3. If $q \in (z^*_u + x^*_u, w^*_u]$, then stop executing $\phi_i$, set $\mathcal{E}_i = 0$. |

Figure 1: The Policy $\phi_i$.

For policy $\phi_i$, define the following quantities:

1. $P(\phi_i) = E[\mathcal{E}_i] = \sum_{u \in S_i} \Pr[\mathcal{E}_i = 1 \land u] = \sum_{u \in S_i} x^*_u$: Probability the arm is exploited.
2. $R(\phi_i) = E[R_i] = \sum_{u \in S_i} r_u \Pr[\mathcal{E}_i = 1 \land u] = \sum_{u \in S_i} x^*_u r_u$: Expected reward of exploitation.
3. $C(\phi_i) = E[C_i] = h_i z^*_u + \sum_{u \in S_i} c_u z^*_u$: Expected cost of switching into and playing this arm.
Let $\phi$ denote the policy that is obtained by executing each $\phi_i$ independently in succession. Since policy $\phi_i$ is obtained by considering arm $i$ in isolation, $\phi$ is not a feasible policy for the following reasons: (i) The cost $\sum_i C_i$ spent exploring all the arms need not be at most $C$ in every exploration trajectory, and (ii) It could happen that for several arms $i$, $E_i$ is set to 1, which implies several arms could be chosen simultaneously for exploitation.

However, all is not lost. First note that the r.v. $R_i, C_i, E_i$ for different $i$ are independent. Furthermore, it is easy to see using the first two constraints and objective of the LP formulation that $\phi$ is feasible in the following expected sense: $\sum_i E[C_i] = \sum_i C(\phi_i) \leq C$. Secondly, $\sum_i E[E_i] = \sum_i P(\phi_i) \leq 1$. Finally, $\sum_i E[R_i] = \sum_i R(\phi_i) = \gamma^*$. 

Based on the above, we show that policy $\phi$ can be converted to a feasible policy using ideas from the adaptivity gap proofs for stochastic packing problems [23, 24, 22]. We treat each policy $\phi_i$ as an item which takes up cost $C_i$, has size $E_i$, and profit $R_i$. These items need to be placed in a knapsack – placing item $i$ corresponds to exploring arm $i$ according to policy $\phi_i$. This placement is an irrevocable decision, and after the placement, the values of $C_i, E_i, R_i$ are revealed. We need $\sum_i C_i$ for items placed so far should be at most $C$. Furthermore, the placement (or exploration) stops the first time some $E_i$ is set to 1, and uses arm $i$ is used for exploitation (obtaining reward or profit $R_i$).

Since only one $E_i = 1$ event is allowed before the play stops, this yields the "size constraint" $\sum_i E_i \leq 1$. The knapsack therefore has both cost and size constraints, and the goal is to sequentially and irrevocably place the items in the knapsack, stopping when the constraints would be violated. The goal is to choose the order to place the items in order to maximize the expected profit, or the exploitation gain. This is a two-constraint stochastic packing problem. The LP solution implies that the expected values of the random variables satisfy the packing constraints.

We show that the “start-deadline” framework in [22] can be adapted to show that there is a fixed order of exploring the arms according to the $\phi_i$ which yields gain at least $\gamma^*/4$. There is one subtle point – the profit (or gain) is also a random variable correlated with the size and cost. Furthermore, the “start deadline” model in [22] would also imply the final packing could violate the constraints by a small amount. We get around this difficulty by presenting an algorithm GREEDYORDER that explicitly obeys the constraints, but whose analysis will be coupled with the analysis of a simpler policy GREEDYVIOLATE which exceeds the budget. The central idea would be that although the benefit of the current arm has not been “verified”, the alternatives have been ruled out.

### 3.3 The Rounding Algorithm

The GREEDYORDER policy is shown in Figure 2. Note that step 3 ensures that no arm is ever revisited, so that the strategy is sequential. For the purpose of analysis, we first present an infeasible policy GREEDYVIOLATE which is simpler to analyze. The algorithm is the same as GREEDYORDER except for step 2, which we outline in Figure 3.

In GREEDYVIOLATE, the cost budget is checked only after fully executing a policy $\phi_j$. Therefore, the policy could violate the budget constraint by at most the exploration cost $c_{\max}$ of one arm.

**Theorem 3.2.** GREEDYVIOLATE spends cost at most $C + c_{\max}$ and yields reward at least $\frac{OPT}{4}$.

**Proof.** We have $\gamma^* = \sum_i R(\phi_i)$, and $\sum_i P(\phi_i) \leq 1$. We note that the random variables corresponding to different $i$ are independent.

For notational convenience, let $\nu_i = R(\phi_i)$, and let $\mu_i = P(\phi_i) + C(\phi_i)/C$. We therefore have $\sum_i \mu_i \leq 2$. The sorted ordering is decreasing order of $\nu_i/\mu_i$. Re-number the arms according to the sorted ordering so that the first arm played is numbered 1. Let $k$ denote the smallest integer such that $\sum_{i=1}^k \mu_i \geq 1$. By the sorted ordering property, it is easy to see that $\sum_{i=1}^k \nu_i \geq \frac{1}{2} \gamma^*$. 

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Algorithm **GreedyOrder**

1. Order the arms in decreasing order of \( \frac{R(\phi_i)}{P(\phi_i) + \frac{C(\phi_i)}{C_i}} \) and choose the arms to play in this order.

2. For each arm \( j \) in sorted order, play arm \( j \) according to \( \phi_j \) as follows until \( \phi_j \) terminates:
   (a) If the next play according to \( \phi_j \) would violate the budget constraint, then **stop** exploration and **goto** step (3).
   (b) If \( \phi_j \) has terminated and \( E_j = 1 \), then **stop** exploration and **goto** step (3).
   (c) Else, play arm \( j \) according to policy \( \phi_j \) and **goto** step (2a).

3. Choose the **last arm** played in step (2) for exploitation.

**Figure 2:** The **GreedyOrder** policy.

Step 2 (**GreedyViolate**) For each arm \( j \) in sorted order, do the following:

(a) Play arm \( j \) according to policy \( \phi_j \) until \( \phi_j \) terminates.

(b) When the policy \( \phi_j \) terminates execution, if event \( E_j = 1 \) is observed or the cost budget \( C \) is exhausted or exceeded, then **stop** exploration and **goto** step (3).

**Figure 3:** The **GreedyViolate** policy.

Arm \( i \) is reached and played by the policy iff \( \sum_{j<i} E_j = 0 \) and \( \sum_{j<i} C_j < C \). This translates to \( \sum_{j<i} \left( E_j + \frac{C_j}{C_i} \right) < 1 \). Note that \( E[E_j + \frac{C_j}{C_i}] = P(\phi_j) + C(\phi_j)/C = \mu_j \). Therefore, by Markov’s inequality, \( \Pr \left[ \sum_{j<i} \left( E_j + \frac{C_j}{C_i} \right) < 1 \right] \geq \max(0, 1 - \sum_{j<i} \mu_j) \). Note further that for \( i \leq k \), we have \( \mu_i \leq 1 \).

If arm \( i \) is played, it yields reward \( \nu_i \) that directly contributes to the exploitation reward. Since the reward is independent of the event that the arm is reached and played. Therefore, the expected reward of **GreedyViolate** can be bounded by linearity of expectation as follows.

\[
\text{Reward of **GreedyViolate**} = G \geq \sum_{i=1}^{k} (1 - \sum_{j<i} \mu_j) \nu_i
\]

We now follow the proof idea in [22]. Consider the arms \( 1 \leq i \leq k \) as deterministic items with item \( i \) having profit \( \nu_i \) and size \( \mu_i \). We therefore have \( \sum_{i=1}^{k} \nu_i \geq \gamma^*/2 \) and \( \sum_{i=1}^{k} \mu_i \leq 1 \).

Suppose these items are placed into a knapsack of size 1 in decreasing order of \( \frac{\nu_i}{\mu_i} \) with the last item possibly being fractionally placed. This is the same ordering that the algorithm uses to play the arms. Let \( \Phi(q) \) denote the profit when size of the knapsack filled is \( q \leq 1 \). We have \( \Phi(1) \geq \gamma^*/2 \). Plot the function \( \Phi(q) \) as a function of \( q \). This plot connects the points \( \{(0,0), (\mu_1, \nu_1), (\mu_1 + \mu_2, \nu_1 + \nu_2), \ldots, (1, \Phi(1))\} \). This function is concave, therefore the area under the curve is at least \( \frac{\Phi(1)}{2} \geq \gamma^*/4 \). However, the area under this curve is at most

\[
v_1 + v_2(1-\mu_1) + \ldots + v_k(1-\sum_{j<k} \mu_j) \leq G
\]

Therefore, \( G \geq \gamma^*/4 \). Since \( OPT \leq \gamma^* \), \( G \) is at least \( \frac{OPT}{4} \).
Theorem 3.3. The GreedyOrder policy with cost budget $C$ achieves reward at least $\frac{\OPT}{\gamma}$.

Proof. Consider the GreedyViolate policy. This policy could exceed the cost budget because the budget was checked only at the end of execution of policy $\phi_i$ for arm $i$. Now suppose the play for arm $i$ reaches state $u \in S_i$, and the next decision of GreedyViolate involves playing arm $i$ and this would exceed the cost budget. The GreedyViolate policy continues to play arm $i$ according to $\phi_i$ and when the play is finished, it checks the budget constraint, realizes that the budget is exhausted, stops, and chooses arm $i$ for exploitation. Suppose the policy was modified so that instead of the decision to play arm $i$ further at state $u$, the policy instead checks the budget, realizes it is not sufficient for the next play, stops, and chooses arm $i$ for exploitation. This new policy is precisely GreedyOrder.

Note now that conditioned on reaching node $u$ with the next decision of GreedyViolate being to play arm $i$, so that the policies GreedyViolate and GreedyOrder diverge in their next action, both policies choose arm $i$ for exploitation. By the martingale property of the rewards, the reward from choosing arm $i$ for exploitation at state $u$ is the same as the expected reward from playing the arm further and then choosing it for exploitation. Therefore, the expected reward of both policies is identical, and the theorem follows. □

3.4 Bi-criteria Result

Suppose we allow the cost budget to be exceeded by a factor $\alpha \geq 1$, so that the cost budget is $\alpha C$. Consider the GreedyOrder policy where the arms are ordered in decreasing order of $\frac{R(\phi_i)}{\alpha P(\phi_i) + C(\phi_i)/C}$, and the budget constraint is relaxed to $\alpha C$. We have the following theorem:

Theorem 3.4. For any $\alpha \geq 1$, if the cost budget is relaxed to $\alpha C$, the expected reward of the modified GreedyOrder policy is $\frac{\alpha}{2(1+\alpha)}\gamma^*$.

Proof. We mimic the proof of Theorem 3.3 and define $\nu_i = R(\phi_i)$, and let $\mu_i = P(\phi_i) + \frac{1}{\alpha} C(\phi_i)/C$. Note that the LP satisfies the constraint $\sum_i \left( P(\phi_i) + \frac{1}{\alpha} C(\phi_i)/C \right) \leq \frac{1+\alpha}{\alpha}$. We therefore have $\sum_i \mu_i \leq \frac{1}{\alpha + \alpha}$. Let $k$ denote the smallest integer such that $\sum_{i=1}^k \mu_i \geq 1$. By the sorted ordering property, we have $\sum_{i=1}^k \nu_i \geq \frac{\alpha}{1+\alpha} \gamma^*$. The rest of the proof remains the same, and we show that the reward of the new policy, $G$, satisfies: $G \geq \frac{1}{2} \Phi(1)$, and $\Phi(1) \geq \frac{\alpha}{2(1+\alpha)} \gamma^*$. This completes the proof. □

3.5 Integrality Gap of the Linear Program

We now show via a simple example that the linear program has an integrality gap of at least $e/(e - 1) \approx 1.58$. All arms $i = 1, 2, \ldots, n$ have identical 2-level state spaces. Each $S_i$ has $c_0 = 1$, $r_0 = 1/n$, switching cost $h_i = 0$, and two other states $u_0$ and $u_1$. We have $p_{ru_0} = 1 - 1/n$, $p_{ru_1} = 1/n$, $r_{u_0} = 0$, $r_{u_1} = 1$. Set $C = n$, so that any policy can play all the arms. The expected reward of such a policy is precisely $1 - (1 - 1/n)^n \approx 1 - 1/e$. The LP solution will set $z^{*}_p = 1$ and $x^{*}_{u_i} = 1/n$ for all $i$, yielding an LP objective of 1. This shows that the linear program cannot yield better than a constant factor approximation. It is an interesting open question whether the LP can be strengthened by other convex constraints to obtain tighter bounds (refer for instance [22]).

4 Non-adaptive Policies: Bounding the Adaptivity Gap

Recall that a non-adaptive strategy allocates a fixed budget to each arm in advance. It then explores the arms according to these budgets (ignoring the outcome of the plays in choosing the next arm.
to explore), and at the end of exploration, chooses the best arm for exploitation. This is termed an *allocational strategy* in \[38\]. Such strategies are desirable since they allow the experimenter to consider various competing arms in parallel. We show two results in this case: For general state spaces, we show that such a non-adaptive strategy can be arbitrarily worse than the optimal adaptive strategy. On the positive side, we show that for 2-level state spaces, which correspond to deterministic underlying rewards (refer Section \[2\]), a non-adaptive strategy is only a factor 7 worse than the performance of the optimal adaptive strategy.

### 4.1 Lower Bound for Multi-level State Spaces

We first present an example with unit costs where an adaptive strategy that dynamically allocates the budget achieves far better exploitation gain than a non-adaptive strategy. Note that we can ignore switching costs in such strategies.

**Theorem 4.1.** The adaptivity gap of the budgeted learning problem is \(\Omega(\sqrt{n})\). Furthermore, even if we allow the non-adaptive exploration to use \(\gamma > 1\) times the exploration budget, the adaptivity gap remains \(\Omega(\sqrt{n/\gamma})\).

*Proof.* Each arm has an underlying reward distribution over the three values \(a_1 = 0, a_2 = 1/n^9\) and \(a_3 = 1\). Let \(q = 1/\sqrt{n}\). The underlying distribution could be one of 3 possibilities: \(R_1, R_2, R_3\). \(R_1\) is the deterministic value \(a_1\), \(R_2\) is deterministically \(a_2\) and \(R_3\) is \(a_3\) w.p. \(q\) and \(a_2\) w.p. \(1-q\). For each arm, we know in advance that \(\Pr[R_1] = 1-q\), \(\Pr[R_2] = q(1-q)\) and \(\Pr[R_3] = q^2\). Therefore, the knowledge for each arm is a prior over the three distributions \(R_1, R_2, R_3\). The priors for different arms are i.i.d. All \(c_i = 1\) and the total budget is \(C = 5n\).

We first show that the adaptive policy chooses an arm with underlying reward distribution \(R_3\) with constant probability. This policy first plays each arm once and discards all arms with observed reward \(a_1\). With probability at least \(1/2\), there are at most \(2/q\) arms which survive, and at least one of these arms has underlying reward distribution \(R_3\). If more arms survive, choose any \(2/q\) arms. The policy now plays each of the \(2/q\) arms \(2\sqrt{n}\) times. The probability that an arm with distribution \(R_3\) yields reward \(a_3\) on some play is at least once is \(1 - (1-q)^{2/q} \approx \Theta(1)\). In this case, it chooses the arm with reward distribution \(R_3\) for exploitation. Since this happens w.p. at least a constant, the expected exploitation reward is \(\Theta(q)\). Note that this is best possible to within constant factors, since \(\mathbb{E}[R_3] = \Theta(q)\).

Now consider any non-adaptive policy. With probability \(1 - 1/n^{O(1)}\), there are at most \(2\log n\) arms with reward distribution \(R_3\), and at least \(1/(2q)\) arms with reward distribution \(R_2\). Let \(r \gg 2\log n\). The strategy allocates at most \(5r\) plays to at least \(n(1-1/r)\) arms - call this set of arms \(T\). With probability \((1-1/r)^{2\log n} = \Omega(1 - (2\log n)/r)\), all arms with reward distribution \(R_3\) lie in this set \(T\). For any of these arms played \(O(r)\) times, with probability \(1 - O(qr)\), all observed rewards will have value \(a_2\). This implies with probability \(1 - O(qr)\), all arms with distribution \(R_3\) yield rewards \(a_2\), and so do \(\Omega(1/(2q))\) arms with distributions \(R_2\). Since these appear indistinguishable to the policy, it can at best choose one of these at random, obtaining exploitation reward \(q \log n \frac{2\log n}{2(1/q)} = O(q^2 \log n)\). Since this situation happens with probability \(1 - O(\log n/r)\), and with the remaining probability the exploitation reward is at most \(q\), the strategy therefore has expected exploitation reward \(O(q \log n (\frac{1}{r} + q))\). This implies the adaptivity gap is \(\Omega(1/q) = \Omega(\sqrt{n})\) if we set \(r = 1/q\).

Now suppose we allow the budget to be increased by a factor of \(\gamma > 1\). Then the strategy would allocate at most \(5\gamma r\) plays to at least \(n(1-1/r)\) arms. By following the same argument as above, the expected reward is \(O(q \log n (\frac{1}{r} + q\gamma))\). This proves the second part of the theorem. \(\square\)
4.2 Upper Bound for Two-Level State Spaces

We next show that for 2-level state spaces, which correspond to deterministic underlying rewards (refer Section 2), the adaptivity gap is at most a factor of 7.

**Theorem 4.2.** If each state space \( S_i \) is a directed star graph with \( r_{\rho_i} \) as the root, then there is a non-adaptive strategy that achieves reward at least \( \frac{1}{7} \) the LP bound.

**Proof.** In the case of 2-level state spaces, a non-adaptive strategy chooses a subset \( S \) of arms and allocates zero/one plays to each of these so that the total cost of the plays is at most \( C \). We consider two cases based on the LP optimal solution.

In the first case, suppose \( \sum_i r_{\rho_i} x_{\rho_i} \geq \gamma^*/7 \), then not playing anything but simply choosing the arm with highest \( r_{\rho_i} \) directly for exploitation is a 7-approximation.

In the remaining proof, we assume the above is not the case, and compare against the optimal LP solution that sets \( x_{\rho_i} = 0 \) for all \( i \). This solution has value at least \( 6\gamma^*/7 \). For simplicity of notation, define \( z_i = z_{\rho_i} \) as the probability that the arm \( i \) is played. Define \( X_i = \frac{1}{z_i} \sum_{u \in S_i} x_u \) as the probability that the arm is exploited conditioned on being played, and \( R_i = \frac{1}{z_i} \sum_{u \in S_i} x_u r_u \) as the expected exploitation reward conditioned on being played. Also define \( c_i = c_{\rho_i} \). The LP satisfies the constraint: \( \sum_i z_i (\frac{R_i}{z_i} + X_i) \leq 2 \), and the LP objective is \( \sum_i z_i R_i \), which has value at least \( 6\gamma^*/7 \).

A better objective for the LP can be obtained by considering the arms in decreasing order of \( \frac{R_i}{z_i} + X_i \), and increasing \( z_i \) in this order until the constraint \( \sum_i z_i (\frac{R_i}{z_i} + X_i) \leq 1 \) becomes tight. Set the remaining \( z_i = 0 \). It is easy to see \( \sum_i z_i R_i \geq \frac{3}{4} \gamma^* \). At this point, let \( k \) denote the index of the last arm which could possibly have \( z_k < 1 \), and let \( S \) denote the set of arms with \( z_i = 1 \) for \( i \in S \).

There are again two cases.

In the first case, if \( z_k R_k > \gamma^*/7 \), then choosing just this arm for exploitation has reward at least \( \gamma^*/7 \), and is a 7-approximation.

In the second and final case, we have a subset of arms \( \sum_{i \in S} (\frac{R_i}{z_i} + X_i) \leq 1 \), and \( \sum_{i \in S} R_i \geq \frac{3}{4} \gamma^* - \gamma^*/7 = \frac{5}{4} \gamma^* \). If all these arms are played, the expected number of arms that are exploited is \( \sum_{i \in S} X_i \leq 1 \), and the expected reward is \( \sum_{i \in S} R_i \geq \frac{5}{4} \gamma^* \). The proof of Theorem 3.2 can be adapted to show that choosing the best arm for exploitation yields at least half the reward, i.e., reward at least \( \gamma^*/7 \).

5 Lagrangean Version

Recall from Section 1.3 that in the Lagrangean version of the problem, there are no budget constraints on the plays, the goal is to find a policy \( \pi \) such that \( R(\pi) - C(\pi) \) is maximized. Denote this quantity as the profit of the strategy.

The linear program relaxation is below. The variables are identical to the previous formulation, but there is no budget constraint.

\[
\text{Maximize} \quad \sum_{i=1}^{n} \left( \sum_{u \in S_i} (x_u r_u - c_u z_u) - h_i z_{\rho_i} \right) \\
\sum_{i=1}^{n} \sum_{u \in S_i} x_u \leq 1 \\
\sum_{v \in S_i} z_v p_{vu} = w_u \quad \forall i, u \in S_i \setminus \{\rho_i\} \\
x_u + z_u \leq w_u \quad \forall u \in S_i, \forall i \\
x_u, z_u, w_u \in [0, 1] \quad \forall u \in S_i, \forall i
\]
Let $OPT = \text{optimal net profit}$ and $\gamma^* = \text{optimal LP solution}$. The next is similar to Claim 3.1.

**Claim 5.1.** $OPT \leq \gamma^*$. 

From this LP optimum $(w^i_u, x^i_u, z^i_u)$, the policy $\phi_i$ is constructed as described in Figure 1 and the r.v.'s $\mathcal{E}_i, C_i, R_i$ and their respective expectations $P(\phi_i), C(\phi_i)$, and $R(\phi_i)$ are obtained as described in the beginning of Section 3.2. Let r.v. $Y_i = R_i - C_i$ denote the profit of playing arm $i$ according to $\phi_i$. Note that $E[Y_i] = (\sum_{u \in S_i} (x_u r_u - c_u z_u) - h_i z_{\rho_i})$.

The nice aspect of the proof of Theorem 3.2 is that it does not necessarily require the r.v. corresponding to the reward of policy $\phi_i$, $R_i$ to be non-negative. As long as $E[R_i] = R(\phi_i) \geq 0$, the proof holds. This will be crucial for the Lagrangean version.

**Claim 5.2.** For any arm $i$, $E[Y_i] = R(\phi_i) - C(\phi_i) \geq 0$.

**Proof.** For each $i$, since all $r_u \geq 0$, setting $x_{\rho_i} \rightarrow \sum_{u \in S_i} x_u$, $w_{\rho_i} \rightarrow 1$, and $z_u \rightarrow 0$ for $u \in S_i$ yields a feasible non-negative solution. The LP optimum will therefore guarantee that the term $\sum_{u \in S_i} (x_u r_u - c_u z_u) - h_i z_{\rho_i} \geq 0$. Therefore, $E[Y_i] \geq 0$ for all $i$. \hfill $\square$

The GreedyOrder policy orders the arms in decreasing order of $\frac{R(\phi_i) - C(\phi_i)}{P(\phi_i)}$, and plays them according to their respective $\phi_i$ until some $\mathcal{E}_i = 1$.

**Theorem 5.3.** The expected profit of GreedyOrder is at least $OPT/2$.

**Proof.** Let $\mu_i = P(\phi_i)$ and $\nu_i = E[Y_i]$ for notational convenience. The LP solution yields $\sum_i \mu_i \leq 1$ and $\sum_i \nu_i = \gamma^*$. Re-number the arms according to the sorted ordering of $\frac{\nu_i}{\mu_i}$ so that the first arm played is numbered 1.

The event that GreedyOrder plays arm $i$ corresponds to $\sum_{j<i} \mathcal{E}_j = 0$. By Markov’s inequality, we have $\Pr[\sum_{j<i} \mathcal{E}_j = 0] = \Pr[\sum_{j<i} \mathcal{E}_j < 1] \geq 1 - \sum_{j<i} \mu_j$.

If arm $i$ is played, it yields profit $Y_i$. This implies the profit of GreedyOrder is $\sum_i Y_i (1 - \sum_{j<i} \mathcal{E}_j)$. Since $Y_i$ is independent of $\sum_{j<i} \mathcal{E}_i$, and since Claim 5.2 implies $E[Y_i] \geq 0$, the expected profit $\mathcal{G}$ of GreedyOrder can be bounded by linearity of expectation as follows.

$$\mathcal{G} = \sum_i \Pr \left[ \sum_{j<i} \mathcal{E}_j < 1 \right] E[Y_i] \geq \sum_i \nu_i \left( 1 - \sum_{j<i} \mu_j \right)$$

We now follow the proof idea in 22. Consider the arms $1 \leq i \leq n$ as deterministic items with item $i$ having profit $\mu_i$ and size $\mu_i$. We therefore have $\sum_i \nu_i \geq \gamma^*$ and $\sum_i \mu_i \leq 1$. Using the same proof idea as in Theorem 3.2 it is easy to see that $\mathcal{G} \geq \frac{\gamma^*}{2}$. Since $OPT \leq \gamma^*$, $\mathcal{G}$ is at least $\frac{OPT}{2}$. \hfill $\square$

## 6 Concave Utility Functions

The above framework in fact solves the more general problem of maximizing any concave stochastic objective function over the rewards of the arms subject to a (deterministic) packing constraint. Several such examples of concave objective function are given in 37 in the context of optimizing “TCP friendly” network utility functions. In what follows, we extend our arguments in the previous section to develop approximation algorithms for all positive concave utility maximization problems in this exploration-exploration setting. Suppose arm $i$ in state $u \in S_i$ has a value function $g_u(y)$ where $y \in [0, 1]$ denotes the weight assigned to it in the exploitation phase. We enforce the following properties on the function $g_u(y)$:
Concavity. \( g_u(y) \) is an arbitrary positive non-decreasing concave function of \( y \).

Super-Martingale. \( g_u(y) \geq \sum_{v \in S_i} p_{uv} g_v(y) \).

Given an outcome \( o \in \mathcal{O}(\pi) \) of exploration, suppose arm \( i \) ends up in state \( u \), and is assigned weight \( y_i \) in the exploitation phase, the contribution of this arm to the exploitation value is \( g_u(y_i) \). The assignment of weights is subject to a deterministic packing constraint \( \sum_i \sigma_i y_i \leq B \), where \( \sigma_i \in [0, B] \). Therefore, for a given outcome \( o \in \mathcal{O}(\pi) \), the value of this outcome is given by the convex program:

\[
\max \sum_{i=1}^{n} g_u(y_i) \quad \text{s.t.} \quad \sum_{i=1}^{n} \sigma_i y_i \leq B, \forall y_i \in [0, 1]
\]

The goal as before is to design an adaptive exploration phase \( \pi \) so that the expected exploitation value is maximized, where the expectation is over the outcomes \( \mathcal{O}(\pi) \) of exploration and cost of exploration is at most \( C \).

• For the maximum reward problem, \( g_u(y) = r_u y \), \( \sigma_i = 1 \), and \( B = 1 \).

• Suppose we wish to choose the \( m \) best rewards, we simply set \( B = m \). Note that we can also conceive of a scenario where the \( c_i \) correspond to cost of “pilot studies” and each treatment \( i \) requires cost \( \sigma_i \) for large scale studies. This would lead us to a Knapsack type problem where \( \sigma_i \) are now the “sizes”.

6.1 Linear Program

The state space \( S_i \) and the probabilities \( p_{uv} \) are defined just as in Section 1.3. For small constant \( \epsilon > 0 \), let \( L = \frac{\mu}{\epsilon} \). Discretize the domain \([0, 1]\) in multiples of \( 1/L \). For \( l \in \{0, 1, \ldots, L\} \), let \( \zeta_u(l) = g_u(l/L) \). This corresponds to the contribution of arm \( i \) to the exploitation value on allocating weight \( y_i = l/L \). Define the following linear program:

\[
\begin{align*}
\text{Max} & \quad \sum_{i=1}^{n} \sum_{u \in S_i} \sum_{l=0}^{L} x_{ul} \zeta_u(l) \\
\sum_{i=1}^{n} \left( h_i z_{\rho_i} + \sum_{u \in S_i} c_u z_u \right) & \leq C \\
\sum_{i=1}^{n} \sigma_i \left( \sum_{u \in S_i} \sum_{l=0}^{L} l x_{ul} \right) & \leq BL(1 + \epsilon) \\
\sum_{u: u \in D(v) \neq \rho_i} z_u p_{vu} & = w_u \forall i, u \in S_i \setminus \{ \rho_i \} \\
z_u + \sum_{l=0}^{L} x_{ul} & \leq w_u \forall u \in S_i, \forall i \\
w_u, x_{ul}, z_u & \in [0, 1] \forall u \in S_i, \forall i, l
\end{align*}
\]

Let \( \gamma^* \) be the optimal LP value and \( \text{OPT} = \text{value of the optimal adaptive exploration policy} \).

Lemma 6.1. \( \text{OPT} \leq \gamma^* \).

Proof. In the optimal solution, let \( w_u \) denote the probability that the policy reaches state \( u \in S_i \), and let \( z_u \) denote the probability of reaching state \( u \in S_i \) and playing arm \( i \) in this state. For \( l \geq 1 \), let \( x_{ul} \) denote the probability of stopping exploration at \( u \in S_i \) and allocating weight \( y_i \in (\frac{l-1}{L}, \frac{l}{L}] \) to arm \( i \). All the constraints are straightforward, except the constraint involving \( B \). Observe that if the weight assignments \( y_i \) in the optimal solution were rounded up to the nearest multiple of \( 1/L \), then the total size of any assignment increases by at most \( \epsilon B \) since all \( s_i \leq B \). Therefore, this constraint is satisfied. Using the same rounding up argument, if the weight satisfies \( y_i \in (\frac{l-1}{L}, \frac{l}{L}] \), then the contribution of arm \( i \) to the exploitation value is upper bounded by \( \zeta_u(l) \) since the function \( g_u(y) \) is non-decreasing in \( y \). Therefore, the proof follows.
Policy $\phi_i$: If arm $i$ is currently in state $u$, choose $q \in [0, w_u^*]$ u.a.r. and do one of the following:

1. If $q \in [0, z_u^*]$, then play the arm.
2. else Stop executing $\phi_i$.
   - Find the smallest $l \geq 0$ such that $q \leq z_u^* + \sum_{k=0}^l x_{uk}^*$. Set $E_i = \frac{l}{L}$ and $R_i = \zeta_u(l)$.

6.2 Exploration Policy

Let $\langle w_u^*, x_{ul}^*, z_u^* \rangle$ denote the optimal solution to the LP. Assume $w_{\rho_i}^* = 1$ for all $i$. Also w.l.o.g, $z_u^* + \sum_{l=0}^L x_{ul}^* = w_u^*$ for all $u \in S_i$. The LP solution yields a natural (infeasible) exploration policy $\phi$ consisting of one independent policy $\phi_i$ per arm $i$. Policy $\phi_i$ is described in Figure 4.

The policy $\phi_i$ is independent of the states of the other arms. It is easy to see by induction that if state $u \in S_i$ is reached by the policy with probability $w_u^*$, then state $u \in S_i$ is reached and arm $i$ is played with probability $z_u^*$. Let random variable $C_i$ denote the cost of executing $\phi_i$, and let $C(\phi_i) = E[C_i]$. Denote this overall policy $\phi$ – this corresponds to one independent decision policy $\phi_i$ (determined by $\langle w_u^*, x_{ul}^*, z_u^* \rangle$) per arm. It is easy to see that the following hold for $\phi$:

1. $C(\phi_i) = E[C_i] = h_i z_i^* + \sum_{u \in S_i} c_u z_u^*$ so that $\sum_i C(\phi_i) \leq C$.
2. $P(\phi_i) = E[E_i] = \frac{1}{L} \sum_{u \in S_i} \sum_{l=0}^L tx_{ul}^* \Rightarrow \sum_i \sigma_i P(\phi_i) \leq B(1 + \epsilon)$.
3. $R(\phi_i) = E[R_i] = \sum_{u \in S_i} \sum_{l=0}^L x_{ul}^* \zeta_u(l) \Rightarrow \sum_i R(\phi_i) = \gamma^*$.

Algorithm GreedyOrder

1. Order the arms in decreasing order of $\frac{R(\phi_i)}{P(\phi_i) + \frac{1}{B} C(\phi_i)}$.
2. For each arm $j$ in sorted order, play it according to $\phi_j$ as follows until $\phi_j$ terminates:
   a) If the next play would violate the cost constraint, then set $E_j \leftarrow 1$, stop exploration, and goto step (3).
   b) If $\phi_j$ terminates and $\sum_i \sigma_i E_i \geq B$, then stop exploration and goto step (3).
   c) Else, play arm $j$ according to policy $\phi_j$ and goto step (2a).
3. Exploitation: Scale down $E_i$ by a factor of 2.

Figure 5: The GreedyOrder policy for concave functions.

The GreedyOrder policy is presented in Figure 5. We again use an infeasible policy GreedyVio-
late which is simpler to analyze. The algorithm is the same as GreedyOrder except for step (2), where violation of the cost constraint is only checked after the policy $\phi_j$ terminates.

Theorem 6.2. Let $c_{\text{max}}$ denote the maximum cost of exploring a single arm. Then GreedyVio-
late spends cost at most $C + c_{\text{max}}$ and has expected value $\frac{OPT}{8}(1 - \epsilon)$.

Proof. Let $\nu_i = R(\phi_i)$ and let $\mu_i = \frac{2}{B} P(\phi_i) + \frac{1}{B} C(\phi_i)$. The LP constraints imply that $\gamma^* = \sum_i \nu_i$, and $\sum_i \mu_i \leq 2 + \epsilon$. Now using the same proof as Theorem 3.2 we obtain the value $G$
of GreedyViolate according to the weight assignment $E_i$ at the end of Step (2) is at least $\frac{OPT}{4}(1 - \epsilon)$. This weight assignment could be infeasible because of the last arm, so that the $E_i$ only satisfy $\sum_i \sigma_i E_i \leq 2B$. This is made feasible in Step (3) by scaling all $E_i$ down by a factor of 2. Since the functions $g_i(y)$ are concave in $y$, the exploitation value reduces by a factor of $1/2$ because of scaling down.

**Theorem 6.3.** GreedyOrder policy with budget $C$ achieves expected value at least $\frac{OPT}{8}(1 - \epsilon)$.

**Proof.** Consider the GreedyViolate policy. Now suppose the play for arm $i$ reaches state $u \in S_i$, and the next decision of GreedyViolate involves playing arm $i$ and this would exceed the cost budget. Conditioned on this next decision, GreedyOrder sets $E_i = 1$ and stops exploration. In this case, the exploitation value of GreedyOrder for arm $i$ is at least the expected exploitation gain of GreedyViolate for this arm by the super-martingale property of the value function $g$. Therefore, for the assignments at the end of Step (2), the gain of GreedyOrder is at least $\frac{OPT}{4}(1 - \epsilon)$. Since Step (3) scales the $E$'s down by a factor of 2, the theorem follows.

### 7 Conclusions

We studied the classical stochastic multi-armed bandit problem under the future utilization objective in the presence of priors. This model is relevant to settings involving data acquisition and design of experiments. In this problem the exploration phase necessarily precedes the exploitation phase. This makes the problem significantly different from the problems in online optimization, which seeks to minimize regret over the past, because online optimization models problems where exploration and exploitation are simultaneous. The central difficulty of online optimization is the lack of information, whereas the difficulty in optimizing future utilization is computational. In fact the latter is provably NP-Hard. We presented constant factor approximation algorithms that yield sequential policies for several extensions of this basic problem. These algorithms proceed via LP rounding and show a surprising connection to stochastic packing algorithms. We also show that the sequential policy we develop is within constant factor of a fully adaptive solution. Note that a constant factor adaptivity gap result is the best possible.

There are several challenging open questions arising from this work; we mention two of them. First, we conjecture that constructing a (possibly adaptive) strategy for the budgeted learning problem is APX-Hard, i.e., there exists an absolute constant $c > 1$ such that it is NP-Hard to produce a solution which is within factor $c$ times the optimum. Secondly, we have focused exclusively on utility maximization; it would be interesting to explore other objectives, such as minimizing residual information [35].

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