A nonlocal $Q$-curvature flow on a class of closed manifolds of dimension $n \geq 5$

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Abstract

In this paper, we employ a nonlocal $Q$-curvature flow inspired by Gursky-Malchiodi’s work \cite{14} to solve the prescribed $Q$-curvature problem on a class of closed manifolds: For $n \geq 5$, let $(M^n, g_0)$ be a smooth closed manifold, which is not conformally diffeomorphic to the standard sphere, satisfying either Gursky-Malchiodi’s semipositivity hypotheses: scalar curvature $R_{g_0} > 0$ and $Q_{g_0} \geq 0$ not identically zero or Hang-Yang’s: Yamabe constant $Y(g_0) > 0$, Paneitz-Sobolev constant $q(g_0) > 0$ and $Q_{g_0} \geq 0$ not identically zero. Let $f$ be a smooth positive function on $M^n$ and $x_0$ be some maximum point of $f$. Suppose either (a) $n = 5, 6, 7$ or $(M^n, g_0)$ is locally conformally flat; or (b) $n \geq 8$, Weyl tensor at $x_0$ is nonzero. In addition, assume all partial derivatives of $f$ vanish at $x_0$ up to order $n - 4$, then there exists a conformal metric $g$ of $g_0$ with its $Q$-curvature $Q_g$ equal to $f$. This result generalizes Escobar-Schoen’s work [Invent. Math. 1986] on prescribed scalar curvature problem on any locally conformally flat manifolds of positive scalar curvature.

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Later, Hang-Yang in their recent works \([15, 16, 17]\) relaxed the positivity of scalar curvature \(R_{g_0}\) in Gursky-Malchiodi’s assumptions to the Yamabe constant \(Y(g_0) > 0\) and the Paneitz-Sobolev constant \(q(g_0) > 0\), and used variational method to solve the constant \(Q\)-curvature problem on a class of closed manifolds of dimensions other than four. For more background on the \(Q\)-curvature problem, one may refer to \([7, 10, 14, 20]\) and the references therein.

For more than one decade, conformal geometric flows have played important roles in prescribed curvature problems in conformal geometry and have become very powerful tools in such problems comparing with the classical variational methods. In 2003, S. Brendle initiated a negative gradient flow approach in \([6]\) to deal with the prescribed \(Q\)-curvature problem with critical exponential exponent on a closed Riemannian manifold \((M^n, g_0)\). Not long after Brendle’s work, M. Struwe adopted this approach in \([26]\) to study the Nirenberg problem, namely the prescribed Gauss curvature problem. Then Malchiodi and Struwe applied this approach to the prescribed \(Q\)-curvature problem on \(S^4\) in \([23]\), where they made some further developments in Morse theory part. Recently, the author and X. Xu in \([9]\) adopted this method to the perturbation result for the prescribed scalar curvature problem on \(S^n\) with \(n \geq 3\). However, to the author’s knowledge, in the present stage, the presence of only one simple bubble in blow-up analysis is crucial to all the above mentioned works. The closely related topics to the above conformal geometric flows are the Yamabe flow \([4, 5]\) and the fractional Yamabe flow \([21]\). Unfortunately, there still exist some technical difficulties to show the long time existence of positive solutions of such conformal geometric flows of higher order, even though such a strong maximum principle (e.g. for Paneitz operator in some special class of closed manifolds) has been established. After Baird-Fardoun-Regbaoui’\’s work \([2]\) and Gursky-Malchiodi’s \([14]\) emerged, the nonlocal flow comes into play and has shown its power in such prescribed curvature problems.

We are now in position to state our main theorem.

**Theorem 1.1.** Let \((M^n, g_0)\) be a smooth closed manifold of dimension \(n \geq 5\) and not conformally diffeomorphic to the standard sphere \(S^n\). Let \(f\) be a smooth positive function on \(M^n\) and \(x_0\) be a maximum point of \(f\). In the case of either (a) \(n = 5, 6, 7\) or \((M^n, g_0)\) is locally conformally flat, or (b) \(n \geq 8\), Weyl tensor at \(x_0\) is nonzero, suppose either \(R_{g_0} > 0, Q_{g_0} \geq 0\) not identically zero or the Yamabe constant \(Y(g_0) > 0\), the Paneitz-Sobolev constant \(q(g_0) > 0\), \(Q_{g_0} \geq 0\) not identically zero. In addition, assume \(\nabla^{l}_{g_0} f(x_0) = 0\) for \(1 \leq l \leq n - 4\).

Then there exists a conformal metric \(g\) of \(g_0\) with its \(Q_g\)-curvature equal to \(f\).

**Remark 1.1.** Theorem 1.1 is a generalization of Theorem 2.1 in Escobar-Schoen’s work \([11]\) on prescribed scalar curvature problem for any locally conformally flat manifold of dimension at least three, which is not conformally diffeomorphic to the standard sphere.

**Remark 1.2.** One case left open in Theorem 1.1 is that \((M^n, g_0)\) with \(n \geq 8\) is not locally conformally flat and Weyl tensor is zero at any maximum point of \(f\), due to the lack of Positive Mass Theorem for Paneitz operator, which is an obstruction in the construction of initial data of the nonlocal \(Q\)-curvature flow in this case.

The vanishing order condition \((1.4)\) on \(f\) at some maximum point is used to construct some positive initial data of the nonlocal \(Q\)-curvature flow satisfying either semi-positivity hypotheses
(2.1) or (2.2) below together with some restrictions on energy bounds. In dimension five, the condition (1.4) is automatically satisfied.

In section 2, we introduce a nonlocal \( Q \)-curvature flow which is a negative gradient flow of \( E_f[u] \) in some suitable Hilbert space, and a detailed proof for short time existence of the flow is available. In section 3, we show the positivity of \( u(t, \cdot) \) for any time \( t \geq 0 \), as well as some elementary estimates involving \( E_f[u] \) and \( \alpha(t) \) etc. In section 4, the global existence of the nonlocal flow for some special class of initial data is presented. In section 5, we show the positivity of \( u(t, \cdot) \) for any time \( t \geq 0 \), as well as some elementary estimates involving \( E_f[u] \) and \( \alpha(t) \) etc. In section 6, the global existence of the nonlocal flow for some special class of initial data is presented. In section 5, we show the asymptotic convergence of \( \int_M \frac{wP_{g_0}u_t}{|\nabla u_t|^4} \, d\mu_{g_0} \) and the positivities of \( Q \)-curvature, as well as of the scalar curvature under hypotheses (2.1) for time \( t \geq 0 \). Reminiscing about Aubin and Schoen’s dichotomy on the Yamabe problem, in either (a) \( n = 5, 6, 7 \) or \( (M^n, g_0) \) is locally conformally flat, or (b) \( n \geq 8 \), Weyl tensor at some maximum point of \( f \) is nonzero cases, in section 6, we construct initial data satisfying either (2.1) or (2.2), as well as some restrictions on initial energy bounds. Finally, in section 7, we establish sequential convergence of the nonlocal flow meanwhile completing the proof of Theorem 1.1.

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## 2 A non-local \( Q \)-curvature flow

Throughout this paper, suppose \( (M^n, g_0) \) is a smooth closed manifold of dimension \( n \geq 5 \), satisfying either Gursky-Malchiodi’s semipositivity hypotheses:

\[
\text{scalar curvature } R_{g_0} \geq 0, Q_{g_0} \geq 0 \text{ and } Q_{g_0} > 0 \text{ somewhere}; \tag{2.1}
\]

or Hang-Yang’s:

\[
\text{Yamabe constant } Y(g_0) > 0, q(g_0) > 0 \text{ and } Q_{g_0} \geq 0 \text{ is not identically zero.} \tag{2.2}
\]

Under the above assumption (2.1), a strong maximum principle for Paneitz operator \( P_{g_0} \) in [14] plays an important role in the existence of constant \( Q \)-curvature problem on closed manifolds. Define a Paneitz-Sobolev constant by

\[
q(g_0) = q(M^n, g_0) := \inf \left\{ \frac{\int_M wP_{g_0}w \, d\mu_{g_0}}{\left( \int_M w^{\frac{n}{n-4}} \, d\mu_{g_0} \right)^{\frac{4-n}{n}}} : w \in H^2(M^n, g_0) \setminus \{0\} \right\}
\]

which is independent of the selection of the metric in the conformal class of \( g_0 \). The Paneitz-Sobolev constant enjoys an analogous property of Yamabe constant proved by T. Aubin [1]:

**Lemma 2.1.** On a closed Riemannian manifold \( (M^n, g_0) \) of dimension \( n \geq 8 \), suppose there exists \( p \in M^n \) such that the Weyl tensor \( W_{g_0}(p) \neq 0 \), then \( q(g_0) < q(S^n) \).
and Hölder’s inequality, there holds

\[ \|v\|_{\mathcal{P}(g_0)} \leq \frac{\varepsilon}{q(g_0)} \left( \int_{M^n} v P_{g_0} v d\mu_{g_0} \right)^{\frac{2}{q}} \]

and \( \|v\|_{H^2(M^n, g_0)} \) are equivalent. Indeed, by the positivity of Paneitz-Sobolev constant \( q(g_0) \) and Hölder’s inequality, there holds

\[ \int_{M^n} v^2 d\mu_{g_0} \leq \text{vol}(M^n, g_0) \frac{4}{\pi} \left( \int_{M^n} \frac{2n}{n-4} d\mu_{g_0} \right)^{\frac{n-4}{n}} \leq \frac{\text{vol}(M^n, g_0)^{\frac{4}{n}}}{q(g_0)} \int_{M^n} v P_{g_0} v d\mu_{g_0}. \]

From the interpolation Sobolev inequality, given \( 0 < \epsilon < 1 \), one has

\[ \int_{M^n} |\Delta_{g_0} v|^2 d\mu_{g_0} = \int_{M^n} v P_{g_0} v d\mu_{g_0} + \left( \int_{M^n} |\Delta_{g_0} v|^2 d\mu_{g_0} - \int_{M^n} v P_{g_0} v d\mu_{g_0} \right) \leq \int_{M^n} v P_{g_0} v d\mu_{g_0} + \epsilon \int_{M^n} |\Delta_{g_0} v|^2 d\mu_{g_0} + C \epsilon \int_{M^n} v^2 d\mu_{g_0}. \]

Putting the above facts together, one has

\[ \int_{M^n} |\Delta_{g_0} v|^2 d\mu_{g_0} \leq \frac{1}{1 - \epsilon} \int_{M^n} v P_{g_0} v d\mu_{g_0} + C \epsilon \int_{M^n} v^2 d\mu_{g_0} \leq C \int_{M^n} v P_{g_0} v d\mu_{g_0}, \]

which yields

\[ \|v\|_{H^2(M^n, g_0)}^2 \leq C \int_{M^n} v P_{g_0} v d\mu_{g_0}. \]

On the other hand, it is easy to verify the inverted direction of the above inequality.

Let \( f \) be a smooth positive function defined on \( M^n \). Motivated by Baird-Fardoun-Regbaoui [2] and Gursky-Malchiodi [14], we extend their ideas to introduce a nonlocal Q-curvature flow:

\[
\begin{align*}
\frac{\partial g}{\partial t} &= -u + \frac{n-4}{2} P_{g_0}^{-1} (\alpha f |u|^{\frac{n+4}{n-4}}) \quad \text{for } (x, t) \in M^n \times [0, T); \\
u(0, x) &= u_0 \in C^\infty(M^n); \\
\end{align*}
\]

(2.3)

coupled with the constraint function of time \( t \):

\[
\alpha(t) = \frac{2}{n-4} \int_{M^n} u P_{g_0} v d\mu_{g_0}. 
\]

(2.4)
Remark 2.1. From [14] Proposition 2.4 or [16] Lemma 3.2, there exists a Green’s function \(G_{g_0}(p, \cdot)\) of the Paneitz operator \(P_{g_0}\) with pole at \(p \in M^n\), such that \(G_{g_0}(p, \cdot) > 0\) in \(M^n \setminus \{p\}\). Thus, given a function \(f \in C^\infty(M^n)\), the operator of \(P_{g_0}^{-1}(f)\) can be interpreted as the convolution between the Green’s function \(G_{g_0}(p, \cdot)\) of \(P_{g_0}\) and the function \(f\), in other words,

\[
P_{g_0}^{-1}(f)(p) = \int_{M^n} G_{g_0}(p, x) f(x) d\mu_{g_0}(x).
\]

Remark 2.2. Under conformal change of metrics \(g = u^{\frac{4}{n-4}} g_0\), using the conformal covariance (1.2) of the Paneitz operator \(P_g\), we find that its inverse \(P_g^{-1}\) is conformally covariant:

\[
P_g^{-1} = u^{-1} P_{g_0}^{-1}\left(\frac{n+4}{u^{n-4}} \cdot \right),
\]

due to the simple fact that \(\forall \psi \in C^\infty(M^n)\),

\[
u^{-1} P_{g_0}^{-1}\left(\frac{n+4}{u^{n-4}} P_g \psi\right) = u^{-1} P_{g_0}^{-1}\left(P_{g_0}(u \psi)\right) = \psi.
\]

2.1 Short time existence

This subsection is devoted to the study of the short time existence to the flow problem (2.3).

Lemma 2.2. There exists a unique solution in \(C^0([0,T]; C^{4,\lambda}(M^n))\) to the flow problem (2.3) for any \(0 < \lambda < 1\) and \(0 < T \leq \infty\).

Proof. It is sufficient to show the short time existence of the following modified flow problem

\[
\begin{aligned}
\frac{\partial \tilde{u}}{\partial t} &= -\tilde{u} + \frac{n-4}{2} P_{g_0}^{-1}\left( f|\tilde{u}|^{\frac{n+4}{n-4}} \right) \quad \text{for} \ (x, s) \in M^n \times [0,T); \\
\tilde{u}(s = 0, x) &= u_0 \in C^\infty(M^n).
\end{aligned}
\]

(2.5)

Indeed, the modified flow problem (2.5) differs from the original one (2.3) by scalings in time and functions. Set

\[
s(t) = \int_0^t \mu(\tau) d\tau \quad \text{and} \quad u(t, x) = e^{s(t)-t} \tilde{u}(x, s(t))
\]

where

\[
\mu(t) = \frac{2}{n-4} \int_{M^n} \tilde{u} P_{g_0} \tilde{u} d\mu_{g_0}.
\]

By using the expressions of \(\alpha(t)\) and \(\mu(t)\) to get

\[
\alpha(t) = \mu(t) e^{-\frac{8}{n-4}(s(t)-t)},
\]

we conclude that if the short time existence of the above modified flow problem (2.5) is established, so is the flow problem (2.3).
Next we manage to establish the short time existence to (2.5) by the contraction mapping principle. For simplicity we still use \( u \) instead of \( \tilde{u} \) and time variable \( t \) instead of \( s \). Let \( X = C^0([0, T]; C^{4,\lambda}(M^n)) \) for some \( \lambda \in (0, 1) \). For any fixed \( \delta > 0 \), define

\[
X_{\delta} := \{ v \in X; v(0, x) = u_0, \|v - u_0\|_X \leq \delta \}
\]

where \( u_0 \in C^\infty(M^n) \), and put the distance on \( X_{\delta} \) by

\[
\rho(v, w) := \|v - w\|_X \text{ for } v, w \in X_{\delta}.
\]

It is not hard to verify that the space \((X_{\delta}, \rho)\) is a complete metric space. Define a map \( L : X_{\delta} \to X \) by

\[
L(u)(t, x) = u_0(x) - \int_0^t u(\tau, x)d\tau + \frac{n - 4}{2} \int_0^t P_{g_0}^{-1}(f|u|^{\frac{n+4}{n-4}})(\tau, x)d\tau.
\]

Recall that, from the Schauder estimates of elliptic equations, there holds

\[
\|P_{g_0}^{-1}w\|_{C^{4,\lambda}(M^n)} \leq C\|w\|_{C^{\lambda}(M^n)},
\]

where \( C \) depends only on \( n, \lambda \). Then one has

\[
\|P_{g_0}^{-1}(f|u|^{\frac{n+4}{n-4}})(t)\|_{C^{4,\lambda}(M^n)} \leq C\|f|u|^{\frac{n+4}{n-4}}(t)\|_{C^{\lambda}(M^n)}
\]

\[
\leq C\|u(t)\|_{C^{\lambda}(M^n)}^{\frac{8}{n}}\|u(t)\|_{C^{\lambda}(M^n)},
\]

and then

\[
\|L(u)(\cdot, t) - u_0(\cdot)\|_{C^{4,\lambda}(M^n)}
\]

\[
\leq T\|u(t)\|_X + C\int_0^t \|u(\tau)\|_{C^{\lambda}(M^n)}^{\frac{8}{n}}\|u(\tau)\|_{C^{\lambda}(M^n)}d\tau
\]

\[
\leq CT(\|u\|_X + \|u\|_{C^{\lambda}(M^n)}^{\frac{8}{n}})
\]

\[
\leq CT(1 + (\delta + \|u_0\|_{C^{4,\lambda}(M^n)}^{\frac{n+4}{n-4}})).
\]

Thus we have

\[
\|L(u) - u_0\|_X \leq C(f, n, \delta, \lambda)T.
\]

By choosing \( T > 0 \) sufficiently small, it is not hard to verify that the map \( L \) is a contraction on \((X_{\delta}, \rho)\). Then, by the contraction mapping theorem, there exists a unique fixed point of \( L \). Thus, the local well-posedness of the modified flow problem (2.5) is established.

\[\square\]

### 3 Positivity of \( u(t, x) \) and energy estimates

We first need to show that the positivity of \( u \) is preserved along the flow. From now on, we impose some restrictions on initial data, that is \( u_0 \in C^\infty_* \), where

\[
C^\infty_* = \{ w \in C^\infty(M^n); w > 0, P_{g_0}w \geq 0 \}.
\]

It is easy to know \( C^\infty_* \neq \emptyset \) since \( 1 \in C^\infty_* \) in view of the fact \( P_{g_0}1 = \frac{n-4}{2}Q_{g_0} \geq 0 \) due to Gursky-Malchiodi’s semi-positivity hypotheses (2.1) or Hang-Yang’s (2.2).
Lemma 3.1. Let $u$ be a $C^4$-solution to the nonlocal flow equation (2.3) with $u(0, x) = u_0(x) \in C^\infty_*$, then for all $0 \leq t \leq T$, there hold

$$u(x, t) > 0 \text{ and } u \in C^\infty_*.$$  

Proof. A direct computation yields

$$\frac{\partial}{\partial t} P_{g_0}(u) = P_{g_0}(u_t)$$

$$= -P_{g_0} u + \frac{n - 4}{2} \alpha f |u|^{\frac{n+4}{n-4}}$$

$$\geq -P_{g_0} u.$$  

Then, we have

$$P_{g_0} u(t, x) \geq e^{-t} P_{g_0}(u_0) \geq 0,$$

in view of $u_0 \in C^\infty_*$. Thus, under either Gursky-Malchiodi’s (2.1) or Hang-Yang’s (2.2), the strong maximum principle for $P_{g_0}$ (cf. [14] Theorem 2.2 or [16] Proposition 3.1, respectively) gives

$$\frac{n - 4}{2} Q_{g_0} u^{\frac{n+4}{n-4}} = P_{g_0} u(t, x) \geq 0, \quad u(t, x) > 0 \text{ for all } (x, t) \in M^n \times [0, T].$$  

The above fact also implies $u \in C^\infty_*$. This completes the proof. \hfill \Box

Consequently, the flow problem (2.3) turns to

$$\begin{cases}
\frac{\partial u}{\partial t} = -u + \frac{n - 4}{2} P_{g_0}^{-1} (\alpha f u^{\frac{n+4}{n-4}}); \\
u(0, x) = u_0 \in C^\infty_*;
\end{cases} \quad (3.1)$$

where $\alpha(t)$ is given in (2.4). For brevity, let

$$\varphi = -u + \frac{n - 4}{2} P_{g_0}^{-1} (\alpha f u^{\frac{n+4}{n-4}}),$$

and then

$$\frac{\partial}{\partial t} P_{g_0} u = -P_{g_0} u + \frac{n - 4}{2} \alpha f u^{\frac{n+4}{n-4}} = P_{g_0} \varphi. \quad (3.2)$$

From now on, denote by $g(t) = u(t)^{\frac{n-2}{2}} g_0$ the flow metric and $d\mu_g = u(t)^{\frac{2n}{n-2}} d\mu_{g_0}$ the volume form of the flow metric, then $Q$-curvature equation gives

$$P_{g_0} u = \frac{n - 4}{2} Q u^{\frac{n+4}{n-4}} \text{ on } M^n, \quad (3.3)$$

where $Q = Q_g$ is the $Q$-curvature of the flow metric $g(t)$. Define the energy functionals

$$E[u] = \frac{n - 4}{2} \int_{M^n} Q_g d\mu_g = \int_{M^n} u P_{g_0}(u) d\mu_{g_0}$$

$$= \int_{M^n} \left[ (\Delta_{g_0} u)^2 + a_n R_{g_0} |\nabla u|_{g_0}^2 + b_n \text{Ric}_{g_0}(\nabla u, \nabla u) + \frac{n - 4}{2} Q_{g_0} u^2 \right] d\mu_{g_0}$$
and
\[ E_f[u] = \frac{E[u]}{(\int_{M^n} fu^{\frac{2n}{n-4}} d\mu_{g_0})^{(n-4)/n}}. \]

By (2.4) and (3.3), we have
\[ E[u(t)] = \frac{n-4}{2} \int_{M^n} Q d\mu_g \]
and
\[ 0 = \int_{M^n} (\alpha f - Q) d\mu_g = \frac{2}{n-4} \int_{M^n} u P_{g_0} \varphi d\mu_{g_0}. \quad (3.4) \]

Along this flow, the energy \( E[u(t)] \) is preserved for all time \( t \geq 0 \).

**Lemma 3.2.** Along the nonlocal flow (3.1), the energy \( E[u(t)] \) is conserved for any time \( t \geq 0 \).

**Proof.** By the flow equation (3.1) and (2.4), we obtain
\[
\frac{d}{dt} \int_{M^n} u P_{g_0} u d\mu_{g_0} = 2 \int_{M^n} u P_{g_0} (u_t) d\mu_{g_0} \\
= 2 \int_{M^n} u P_{g_0} \left( -u + \frac{n-4}{2} P_{g_0}^{-1} (\alpha f u^{\frac{n+4}{n-4}}) \right) d\mu_{g_0} \\
= -2 \left[ \int_{M^n} u P_{g_0} u d\mu_{g_0} - \frac{n-4}{2} \alpha(t) \int_{M^n} f u^{\frac{2n}{n-4}} d\mu_{g_0} \right] \\
= 0,
\]
which implies the desired assertion. \(\square\)

From Remark 2.2, the flow equation (3.1) is equivalent to
\[ u_t = \frac{n-4}{2} P_{g}^{-1} (\alpha f - Q_g) u, \quad (3.5) \]
that is,
\[ \frac{\partial}{\partial t} g = 2 P^{-1}_g (\alpha f - Q_g) g. \]

Since \( P_g \) is self-adjoint and positive, we can define \( H^2(M^n, g) \) inner product by
\[ \langle \eta, \zeta \rangle_g = \int_{M^n} \eta P_g \zeta d\mu_g, \]
which induces the \( H^2 \)-norm \( \| \cdot \|_{H^2(M^n, g)} \). In this sense, the nonlocal \( Q \)-curvature flow (3.1) is a negative gradient flow of \( E_f[u] \) in the Hilbert space \( H^2(M^n, g) \). Now we pause for a while to give some explanations for the nonlocal \( Q \)-curvature flow. Up to a positive constant, regard \( E_f[u] \) as a functional of the metric \( g \):
\[
Q_f(g) = \frac{\int_M Q_g d\mu_g}{\left( \int_M f d\mu_g \right)^{\frac{2n}{n-4}}}. 
\]
Set 
\[ g_\epsilon = \phi_\epsilon g \]
satisfying 
\[ \phi_\epsilon \big|_{\epsilon = 0} = 1 \quad \text{and} \quad \frac{d\phi_\epsilon}{d\epsilon} \bigg|_{\epsilon = 0} = \phi \in C^\infty(M). \]

Notice that \( P_g \) is invertible under hypotheses (2.1) or (2.2), we obtain 
\[
Q'(f)(g)[\phi] = \frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \frac{\int_M Q_{\phi_\epsilon g} d\mu_{\phi_\epsilon g}}{\left( \int_M f\phi_\epsilon^2 d\mu_g \right)^{\frac{n-4}{n}}} \\
= \frac{2}{n-4} \frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \frac{\int_M \phi_\epsilon^\frac{4}{n} P_g(\phi_\epsilon^\frac{4}{n}) d\mu_g}{\left( \int_M f\phi_\epsilon^2 d\mu_g \right)^{\frac{n-4}{n}}} \\
= \frac{\int_M \phi P_g(1) d\mu_g}{\left( \int_M f d\mu_g \right)^{\frac{n-4}{n}}} - \frac{n-4}{2} \frac{\int_M Q_g d\mu_g}{\left( \int_M f d\mu_g \right)^{\frac{n-4}{n}}} \\
= \frac{n-4}{2} \left( \int_M f \phi_\epsilon^2 d\mu_g \right)^{\frac{4-n}{n}} \left( \int_M \phi(Q_g - \alpha f) d\mu_g \right) \\
= \frac{n-4}{2} \left( \int_M f \phi_\epsilon^2 d\mu_g \right)^{\frac{4-n}{n}} \langle \phi, P_g^{-1}(Q_g - \alpha f) \rangle_g.
\]

**Lemma 3.3.** Along the nonlocal flow (3.1), the energy \( E_f[u(t)] \) is non-increasing for all time \( t \geq 0 \).

**Proof.** By (3.5) and (2.4), we compute 
\[
\frac{d}{dt} E_f[u] = -(n-4) \left( \int_{M^n} f u^{n-4} d\mu_{g_0} \right)^{\frac{4-n}{n}} \int_{M^n} \alpha f u^{-1} u_t d\mu_g \\
= -\frac{(n-4)^2}{2} \left( \int_{M^n} f u^{n-4} d\mu_{g_0} \right)^{\frac{4-n}{n}} \int_{M^n} \alpha f P_g^{-1}(\alpha f - Q) d\mu_g.
\]

From (2.2), (3.3) and (2.4), as well as from (3.4), we notice that 
\[
\int_{M^n} Q P_g^{-1}(\alpha f - Q) d\mu_g = \frac{2}{n-4} \int_{M^n} P_{g_0}(u) P_g^{-1}(\alpha f - Q) d\mu_{g_0} \\
= \frac{2}{n-4} \int_{M^n} P_{g_0}(u) P_{g_0}^{-1}((\alpha f - Q) u^{\frac{n+4}{n-4}}) d\mu_{g_0} \\
= \frac{2}{n-4} \int_{M^n} (\alpha f - Q) d\mu_g = 0.
\]

Thus, we obtain 
\[
\frac{d}{dt} E_f[u] = -\frac{(n-4)^2}{2} \left( \int_{M^n} f u^{n-4} d\mu_{g_0} \right)^{\frac{4-n}{n}} \int_{M^n} (\alpha f - Q) P_g^{-1}(\alpha f - Q) d\mu_g \\
= -\frac{(n-4)^2}{2} \left( \int_{M^n} f u^{n-4} d\mu_{g_0} \right)^{\frac{4-n}{n}} \| P_g^{-1}(\alpha f - Q) \|_{H^2(M^n,g)}^2 \leq 0.
\]

This completes the proof. \( \Box \)
3.1 Some elementary estimates

**Lemma 3.4.** The conformal volume of the flow metric is uniformly bounded above, that is, for all time $t \geq 0$, there exists a positive constant $C_1$ depending on $n, q(g_0)$ and initial energy $E[u_0]$ such that

$$\int_{M^n} u^{\frac{2n}{n-4}} d\mu_{g_0} \leq C_1.$$ 

Moreover, there exists a positive constant $C_2$ depending on $\max_{M^n} f, n, q(g_0)$ and initial energy $E[u_0]$, such that

$$\int_{M^n} f u^{\frac{2n}{n-4}} d\mu_{g_0} \leq C_2.$$ 

**Proof.** By the definition of the Paneitz-Sobolev constant $q(g_0)$ and Lemma 3.2, one obtains

$$q(g_0) \left( \int_{M^n} u^{\frac{2n}{n-4}} d\mu_{g_0} \right)^{\frac{n-4}{n}} \leq E[u(t)] = E[u_0].$$

Thus, it yields

$$\int_{M^n} u^{\frac{2n}{n-4}} d\mu_{g_0} \leq \left( \frac{E[u_0]}{q(g_0)} \right)^{\frac{n}{n-4}} := C_1.$$ 

Moreover, we get

$$\int_{M^n} f u^{\frac{2n}{n-4}} d\mu_{g_0} \leq (\max_{M^n} f) \int_{M^n} u^{\frac{2n}{n-4}} d\mu_{g_0} \leq (\max_{M^n} f) C_1 := C_2.$$ 

This concludes the proof.

Indeed, we manage to show that $\alpha(t)$ is uniformly bounded below and above, as well as is the conformal volume $\int_{M^n} u(t)^{\frac{2n}{n-4}} d\mu_{g_0}$.

**Lemma 3.5.** There exist two uniform positive constants $C_3 = C_3(n, \max_{M^n} f, E[u_0])$ and $C_4 = C_4(n, E[u_0])$ such that

$$0 < C_3 \leq \alpha(t) \leq C_4.$$ 

**Proof.** By the expression (2.4) of $\alpha(t)$ and the conservation of energy $E[u(t)]$ in view of Lemma 3.2, we obtain

$$\alpha(t) \geq \frac{E[u_0]}{C_2} := C_3,$$

where $C_2$ is the positive constant given in Lemma 3.4. On the other hand, from (3.6) one asserts that $\alpha_t \leq 0$, which implies $\alpha(t) \leq \alpha(0) := C_4$. This concludes the proof.

**Lemma 3.6.** For any fixed time $T > 0$, there exists a uniform constant $C$ depending on $n, q(g_0), \max_{M^n} f$ and initial energy $E[u_0]$ such that

$$\left( \frac{n-4}{2} \right)^2 \int_0^T \int_{M^n} (\alpha f - Q) P_g^{-1}(\alpha f - Q) d\mu_g dt = \int_0^T \int_{M^n} \varphi P_{g_0} \varphi d\mu_{g_0} dt \leq C.$$
Proof. From the flow equation (3.1) and Lemma 3.2, one has
\[
\frac{d}{dt} \int_{M^n} \alpha f u^{\frac{2n}{n-4}} d\mu_{g_0} = \frac{2n}{n-4} \int_{M^n} \alpha f u^{\frac{n+4}{n-4}} u_t d\mu_{g_0} + \alpha_t \int_{M^n} f u^{\frac{2n}{n-4}} d\mu_{g_0} = \frac{2n}{n-4} \int_{M^n} \alpha f u^{\frac{n+4}{n-4}} \left[ u + \frac{n-4}{2} P_{g_0}^{-1}(\alpha f u^{\frac{n+4}{n-4}}) \right] d\mu_{g_0} + \frac{2}{n-4} \alpha_t \int_{M^n} u P_{g_0} u d\mu_{g_0} = \frac{2n}{n-4} \int_{M^n} \left[ \frac{n-4}{2} \alpha f u^{\frac{n+4}{n-4}} P_{g_0}^{-1}(\alpha f u^{\frac{n+4}{n-4}}) - \alpha f u^{\frac{2n}{n-4}} \right] d\mu_{g_0} + \frac{2}{n-4} \alpha_t E[u_0].
\]

On the other hand, by the \(Q\)-curvature equation (3.3) and the expression (3.4) of \(\alpha(t)\), we have
\[
\int_{M^n} \varphi P_{g_0} \varphi d\mu_{g_0} = \int_{M^n} \left[ - u + P_{g_0}^{-1}(\frac{n-4}{2} \alpha f u^{\frac{n+4}{n-4}}) \right] P_{g_0} \left[ - u + P_{g_0}^{-1}(\frac{n-4}{2} \alpha f u^{\frac{n+4}{n-4}}) \right] d\mu_{g_0} = \int_{M^n} u P_{g_0} u - \frac{n-4}{2} \alpha f u^{\frac{2n}{n-4}} d\mu_{g_0} + \frac{n-4}{2} \int_{M^n} \left[ \frac{n-4}{2} \alpha f u^{\frac{n+4}{n-4}} P_{g_0}^{-1}(\alpha f u^{\frac{n+4}{n-4}}) - \alpha f u^{\frac{2n}{n-4}} \right] d\mu_{g_0}.
\]

Therefore, by the definition of \(\alpha(t)\), we conclude that
\[
0 = \frac{d}{dt} \int_{M^n} \alpha f u^{\frac{2n}{n-4}} d\mu_{g_0} = \frac{2n}{n-4} \int_{M^n} \varphi P_{g_0} \varphi d\mu_{g_0} + \frac{\alpha_t}{\alpha} E[u_0] \tag{3.6}
\]

Integrating the above equation over \([0, T]\) to obtain
\[
\int_0^T \int_{M^n} \varphi P_{g_0} \varphi d\mu_{g_0} dt = \frac{n-4}{2n} E[u_0] (\log \alpha(0) - \log \alpha(T)) \tag{3.7}
\]

By Lemma 3.4, it yields
\[
\log \alpha(T) \geq \log \left( \frac{2}{n-4} \frac{E[u_0]}{C_2} \right)
\]

for any fixed \(T > 0\). Thus, from (3.7) we conclude that
\[
\int_0^T \int_{M^n} \varphi P_{g_0} \varphi d\mu_{g_0} dt \leq C.
\]

This completes the proof. \(\square\)

As a byproduct of Lemma 3.6, provided that the nonlocal flow globally exists for all time, we obtain
\[
\int_0^\infty \int_{M^n} \varphi P_{g_0} \varphi d\mu_{g_0} dt < \infty. \tag{3.8}
\]

In particular, there exists a sequence \(\{t_j\}_{j=1}^\infty\) with \(t_j \to \infty\), such that
\[
\int_{M^n} \varphi(t_j) P_{g_0}(\varphi(t_j)) d\mu_{g_0} \to 0, \text{ as } j \to \infty. \tag{3.9}
\]
4 Global existence

**Lemma 4.1.** Given any initial data \( u_0 \in C^\infty \), there exists a smooth solution to the nonlocal \( Q \)-curvature flow problem (3.1)-(2.4) for all time \( t \geq 0 \). Moreover, for any fixed time \( T > 0 \), there exist two positive constants \( C \) and \( C' \) depending on \( n, T, f \) and initial data \( u_0 \) such that

\[
\|u(t)\|_{C^0(M^n)} \leq Ce^{C't} \quad \text{for} \quad 0 \leq t \leq T. \tag{4.1}
\]

**Proof.** Let \( \theta > 1 \). From the proof of Lemma 3.1, we obtain that

\[
u(t, x) > 0 \quad \text{and} \quad P_{g_0}u(t, x) > 0\tag{4.2}
\]
as long as the flow exists. Along with this fact, by the equation (3.2) and Lemma 3.5, we have

\[
\frac{d}{dt} \int_{M^n} (P_{g_0}u)^\theta \, d\mu_{g_0} = \theta \int_{M^n} (P_{g_0}u)^{\theta-1} P_{g_0}u_t \, d\mu_{g_0}
= \theta \int_{M^n} (P_{g_0}u)^{\theta-1} \left( -P_{g_0}u + \frac{n-4}{2} \alpha f u^\frac{n+4}{n-4} \right) \, d\mu_{g_0}
\leq -\theta \int_{M^n} (P_{g_0}u)^\theta \, d\mu_{g_0} + C_\theta \int_{M^n} (P_{g_0}u)^{\theta-1} u^\frac{n+4}{n-4} \, d\mu_{g_0}. \tag{4.3}
\]

Using Hölder’s inequality, we estimate

\[
\int_{M^n} (P_{g_0}u)^{\theta-1} u^\frac{n+4}{n-4} \, d\mu_{g_0} \leq \left( \int_{M^n} (P_{g_0}u)^\theta \, d\mu_{g_0} \right)^{\frac{\theta-1}{\theta}} \left( \int_{M^n} u^\frac{n+4}{n-4} \, d\mu_{g_0} \right)^{\frac{1}{\theta}}.
\]

First choose

\[
1 < \theta < \frac{n}{4}
\]

By Hölder’s inequality, one gets

\[
\left( \int_{M^n} u^\frac{(n+4)\theta}{n-4} \, d\mu_{g_0} \right)^{\frac{1}{\theta}} = \left( \int_{M^n} u^\theta u^\frac{4\theta}{n-4} \, d\mu_{g_0} \right)^{\frac{1}{\theta}} \leq \left( \int_{M^n} u^\frac{n\theta}{n-4} \, d\mu_{g_0} \right)^{\frac{1}{\theta}} \left( \int_{M^n} u^\frac{2\theta}{n-4} \, d\mu_{g_0} \right)^{\frac{1}{\theta}}.
\]

By the Sobolev embedding \( W^{4,\theta}(M^n, g_0) \hookrightarrow L^{\frac{n\theta}{n-4}}(M^n, g_0) \) and the basic fact that

\[
\|u\|_{W^{4,\theta}(M^n, g_0)} \approx \|P_{g_0}u\|_{L^{\theta}(M^n, g_0)},
\]

we obtain

\[
\left( \int_{M^n} u^\frac{n\theta}{n-4} \, d\mu_{g_0} \right)^{\frac{n-4}{n\theta}} \leq C_\theta \left( \int_{M^n} (P_{g_0}u)^\theta \, d\mu_{g_0} \right)^{\frac{1}{\theta}}.
\]

Thus, together with Lemma 3.4, one obtains

\[
\int_{M^n} (P_{g_0}u)^{\theta-1} u^\frac{n+4}{n-4} \, d\mu_{g_0} \leq C_\theta \int_{M^n} (P_{g_0}u)^\theta \, d\mu_{g_0}.
\]
Hence, substituting these above facts into (4.3) and using (4.2) to show
\[
\frac{d}{dt} \int_{M^n} (P_{g_0} u)^\theta \, d\mu_{g_0} \leq C_\theta \int_{M^n} (P_{g_0} u)^\theta \, d\mu_{g_0}.
\]
Integrating the above over \((0, t)\) to get
\[
\int_{M^n} (P_{g_0} u)^\theta \, d\mu_{g_0} \leq C_\theta e^{C'_{\theta} t}
\]
for all \(0 \leq t \leq T\). Again using the Sobolev embedding theorem, one gets
\[
\|u\|_{L^{n\lambda}(M^n, g_0)} \leq C_\theta e^{C'_{\theta} t}.
\]
By choosing \(\theta\) sufficiently tending to \(\frac{n}{4}\), we establish that for any \(p > 1\), there holds
\[
\|u(t)\|_{L^p(M^n, g_0)} \leq C_p e^{C'_p t}. \tag{4.4}
\]
Next fix \(\theta = p > \frac{n}{4}\) in (4.3), applying (4.4), Hölder’s and Young’s inequalities to estimate
\[
\begin{align*}
\int_{M^n} (P_{g_0} u)^{p-1} \frac{u_{n-4}}{u^{n-4}} \, d\mu_{g_0} & \leq \left( \int_{M^n} (P_{g_0} u)^p \, d\mu_{g_0} \right)^{\frac{p-1}{p}} \left( \int_{M^n} \frac{u^{(n+4)p}}{u^{n-4}} \, d\mu_{g_0} \right)^{\frac{1}{p}} \\
& \leq \left( C_p e^{C'_p t} \right)^{\frac{1}{p}} \left( \int_{M^n} (P_{g_0} u)^p \, d\mu_{g_0} \right)^{\frac{p-1}{p}} \\
& \leq C_p e^{C'_p t} \left( \int_{M^n} (P_{g_0} u)^p \, d\mu_{g_0} \right)^{\frac{p-1}{p}} \\
& \leq \frac{p}{C_p} \int_{M^n} (P_{g_0} u)^p \, d\mu_{g_0} + C_p e^{C'_p t}.
\end{align*}
\]
Substituting the above back to (4.3) to show
\[
\frac{d}{dt} \int_{M^n} (P_{g_0} u)^p \, d\mu_{g_0} \leq C_p e^{C'_p t}.
\]
Along with the Sobolev embedding theorem, integrating the above over \((0, t)\) to get
\[
\|u(t)\|_{C^\lambda(M^n)} \leq C_p \|P_{g_0} u(t)\|_{L^p(M^n, g_0)} \leq C_p e^{C'_p t},
\]
for all \(0 \leq t \leq T\), where \(\lambda = 4 - \frac{n}{p} \in (0, 1)\). Obviously, the above estimate implies (4.1).

Going back to the flow equation (3.1), through (3.2) and (4.1), we conclude that \(C^{1,\lambda}\)-norm of \(u(t)\) has at most exponential growth in any finite time interval. Therefore, the phenomenon of finite time blow-up is excluded and then the global existence of the flow equation (3.1)-(2.4) is established.
5 Asymptotic behaviors

In this section, we establish asymptotic convergence of \( \int_{M^n} \varphi P_{g_0} \varphi d\mu_{g_0} \), as well as the positivity of \( Q \)-curvature of the flow metric.

**Lemma 5.1.** Let \( u_0 \in C^{\infty}_* \), then there holds \( Q(t,x) > 0 \) for all \( (x,t) \in M^n \times (0, \infty) \). Moreover, under hypotheses (2.1), there holds \( R_g(t,x) > 0 \) for all \( (x,t) \in M^n \times [0, \infty) \).

**Proof.** By the positivities of \( u(t) \), \( \alpha(t) \) and the definition of \( C^{\infty}_* \), integrating (3.2) over \( (0,t) \) to show

\[
\frac{n - 4}{2} Qu_{n-4}^2 = P_{g_0} u(t) = P_{g_0} u_0 + \frac{n - 4}{2} \int_0^t e^{q-t} \alpha(\tau) f u_{n-4}^2(\tau) d\tau > 0,
\]

which implies the first assertion. Next, define

\[
t_\ast := \sup \{ t \in [0, \infty) ; \min_{M^n} R(\cdot, t) = 0 \}.
\]

Notice that the set on the right side is nonempty under Gursky-Malchiodi’s hypotheses (2.1). We claim that \( t_\ast = +\infty \). Suppose \( t_\ast < \infty \). Since \( R_{g_0} > 0 \) together with the positivity of \( Q \), for \( t \in (0, t_\ast) \) we have

\[
R_g(t) \geq 0 \quad \text{and} \quad - \frac{1}{2(n - 1)} \Delta_g R_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n - 1)^2(n - 2)^2} R_g^2 > 0.
\]

By applying strong maximum principle to \( R_g \) at time \( t = t_\ast \), one obtains that \( R_g(t_\ast) > 0 \) or \( R_g(t_\ast) \equiv 0 \). However, in the latter case, it yields that \( Q(t_\ast) = - \frac{2}{(n - 2)^2} | \text{Ric}_g(t_\ast) |^2 \leq 0 \), which contracts the positivity of \( Q(t_\ast) \). Then \( R_g(t_\ast) > 0 \), which also contradicts the definition of \( t_\ast \). From this, the second assertion follows.

From the positivity of the scalar curvature \( R_g \), we obtain some lower bounds of \( u(t) \).

**Lemma 5.2.** Under the hypotheses (2.1), there exists a positive constant \( C \) depending on \( g_0 \) and \( R_{g_0} \) such that

\[
\int_{M^n} u(t)^{\frac{n-2}{n-4}} d\mu_{g_0} \leq C \left( \inf_{M^n} u(t) \right)^{\frac{n-2}{n-4}} \quad \text{and} \quad \left( \int_{M^n} u(t)^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-4}{n-2}} \leq C \inf_{M^n} u(t) \left( \sup_{M^n} u(t) \right)^{\frac{n+2}{n-2}},
\]

for all \( t \geq 0 \).

**Proof.** Under the hypotheses (2.1), by Lemma 5.1 and the scalar curvature equation of the flow metric \( g = u(t)^{\frac{4}{n-4}} g_0 \):

\[
- \frac{4(n - 1)}{n - 2} \Delta_{g_0} u_{n-4}^\frac{n-2}{n-4} + R_{g_0} u_{n-4}^\frac{n-2}{n-4} = R_g u_{n-4}^\frac{n+2}{n-4} > 0,
\]

these two assertions follow from Lemma A.2 and Corollary A.3 in [4], respectively.
Let
\[ F_2(t) = \int_{M^n} \varphi P_{g_0} \varphi d\mu_{g_0}. \]

Notice that \( \alpha(t) \) is non-increasing with respect to time \( t \geq 0 \) along the flow. More precisely, we obtain

**Lemma 5.3.** There exists a positive uniform constant \( C = C(n, \max_{M^n} f, E[u_0]) \) such that
\[ \alpha_t = -\frac{2n}{n-4} \frac{\alpha}{E[u_0]} F_2(t) \leq -CF_2(t) \leq 0 \quad \text{for} \quad t \geq 0. \]

**Proof.** By the equation (3.6) and Lemma 3.5, one has
\[ \alpha_t = -\frac{2n}{n-4} \frac{\alpha}{E[u_0]} F_2(t) \leq -CF_2(t) \leq 0 \]
as desired. \( \square \)

**Lemma 5.4.** There holds
\[ \lim_{t \to \infty} F_2(t) = 0. \]

**Proof.** By (3.1) and (3.2), we have
\[
\frac{1}{2} \frac{d}{dt} F_2(t) = \int_{M^n} \varphi P_{g_0} \varphi d\mu_{g_0} \\
= \int_{M^n} \varphi \frac{\partial}{\partial t} \left( -P_{g_0} u + \frac{n-4}{2} \alpha f u^{\frac{n+4}{n}} \right) d\mu_{g_0} \\
= \int_{M^n} \varphi \left( -P_{g_0} \varphi + \frac{n-4}{2} \alpha_t f u^{\frac{n+4}{n}} + \frac{n+4}{2} \alpha f u^{\frac{s}{n}} \varphi \right) d\mu_{g_0}. \tag{5.1}
\]

By Lemmas 3.4 and 5.3, using Hölder’s inequality and Sobolev embedding, we estimate the second integral by
\[
\left| \int_{M^n} \alpha_t f u^{\frac{n+4}{n}} \varphi d\mu_{g_0} \right| \leq C F_2(t) \left( \int_{M^n} \varphi^{\frac{2n}{n-4}} d\mu_{g_0} \right)^{\frac{n}{n-4}} \left( \int_{M^n} u^{\frac{2n}{n-4}} d\mu_{g_0} \right)^{\frac{n+4}{2n}} \\
\leq C F_2^\frac{3}{2}(t).
\]

By Lemmas 3.4 and 3.5, employing Hölder’s inequality and Sobolev embedding to bound
\[
\left| \int_{M^n} \alpha f u^{\frac{s}{n}} \varphi^2 d\mu_{g_0} \right| \leq C \left( \int_{M^n} u^{\frac{2n}{n-4}} d\mu_{g_0} \right)^{\frac{n}{2}} \left( \int_{M^n} \varphi^{\frac{2n}{n-4}} d\mu_{g_0} \right)^{\frac{n+4}{n}} \\
\leq C F_2(t).
\]

Thus, by (5.1), we obtain that
\[
\frac{d}{dt} F_2(t) \leq C F_2(t)(1 + F_2(t)^\frac{1}{2}). \tag{5.2}
\]
By (3.8), there exists a sequence \( \{t_j\} \) with \( t_j \to \infty \) as \( j \to \infty \), such that \[
\lim_{t \to \infty} F_2(t_j) = 0.
\]

Set \[
H(t) = \int_0^{F_2(t)} \frac{ds}{1 + s^{\frac{2}{n}}} = 2F_2(t)^{\frac{1}{2}} - 2 \log \left(1 + F_2(t)^{\frac{1}{2}}\right),
\]
from which one has \( \lim_{j \to \infty} H(t_j) = 0 \). Integrating (5.2) over \((t_j, t)\) for any \( t \geq t_j \) to show
\[
H(t) \leq H(t_j) + C \int_{t_j}^{t} F_2(\tau) d\tau.
\]

Letting \( j \to \infty \) and by (3.9), we have \( \lim_{t \to \infty} H(t) = 0 \). Then we assert that there exists some uniform constant \( C_0 > 0 \) such that
\[
F_2(t) \leq C_0 \text{ for all time } t \geq 0.
\]

To show this by negation, otherwise there exists a sequence \( \{t_k\} \) with \( t_k \to \infty \) as \( k \to \infty \) such that \( F_2(t_k) > 1 \) for all \( k \in \mathbb{N} \). However,
\[
H(t_k) = \int_0^{F_2(t_k)} \frac{ds}{1 + s^{\frac{2}{n}}} ds \geq \int_0^{1} \frac{ds}{1 + s^{\frac{2}{n}}} ds > 0
\]
which contradicts \( \lim_{t \to \infty} F_2(t) = 0 \). Also it is easy to show \( F_2(t) \leq C_0 H(t) \) for all time \( t \geq 0 \).

Therefore, we conclude that \[
\lim_{t \to \infty} F_2(t) \leq C_0 \lim_{t \to \infty} H(t) = 0.
\]
This completes the proof. \( \square \)

6 Construction of Initial Data

The objective of this section is to construct some positive initial data satisfying either semi-positivity hypotheses (2.1) or (2.2), as well as some restrictions on initial energy bounds. Such initial data are very crucial in establishing sequential convergence of the nonlocal flow in the later part.

Let \( x_0 \in M^n \) be a maximum point of \( f \). It follows from Lee-Parker [22] that there exists a conformal metric \( \tilde{g} = \varphi^{\frac{4}{n-4}} g_0 \) with conformal normal coordinates around \( x_0 \). If \( g_0 \) is locally conformally flat, we choose \( \tilde{g} = \varphi^{\frac{4}{n-4}} g_0 \) flat near \( x_0 \). Since some estimates and computations needed in this paper have been available in [14], we mostly adopt the same notation used in [14] for simplicity. Set
\[
b_n = n(n - 4)(n^2 - 4), \quad c_n = \frac{1}{2(n - 2)(n - 4) \omega_{n-1}},
\]
where \( \omega_{n-1} \) is the volume of the unit \((n-1)\)-sphere.
and
\[ u_\epsilon(x) = \frac{\chi_\delta(x)}{\epsilon^2 + d_\delta(x, x_0)^2} \] where \( \omega_{n-1} = \text{vol}(S^{n-1}, g_{S^{n-1}}) \) and \( \chi_\delta(x) \) is a nonnegative smooth cut off function supported in \( B_{2\delta}(x_0) \) satisfying \( \chi_\delta(x) = 1 \) in \( B_\delta(x_0) \) and \( \chi_\delta(x) = 0 \) outside of \( B_{2\delta}(x_0) \). For brevity, we use \( \nabla \) instead of \( \nabla_{g_\tilde} \).

A direct computation yields the following asymptotics
\[ \int_{M^n} u^n_\epsilon^{-4} d\mu_\tilde = O(\epsilon^{-n}), \quad \int_{M^n} u^{n+4}_\epsilon^{-4} d\mu_\tilde = O(\epsilon^{-4}), \quad \int_{M^n} u^n_\epsilon^{-4} d\mu_\tilde = O(\epsilon^{n-8}), \]
\[ b_n \epsilon^4 \left( \int_{M^n} u^n_\epsilon^{-4} d\mu_\tilde \right)^{\frac{4}{n}} = q(S^n)(1 + O(\epsilon^n)). \]

Since \( \tilde P_g \) is invertible under hypotheses (2.2) or by Proposition 2.3 in [14] under hypotheses (2.1), we consider \( \tilde u_\epsilon \) satisfying
\[ \tilde P_g \tilde u_\epsilon = \frac{b_n \epsilon^4 \chi_\delta(x)}{\epsilon^2 + |x|^2} \] \label{6.1}

Due to the property of Paneitz-Sobolev constant stated in Lemma 2.1 and Positive Mass Theorem (cf. [14] Proposition 2.5 and [19] Theorem 1.1), for technical reasons, we divide the construction of initial data into two cases.

### 6.1 \((M^n, g_0) : n \geq 8 \) and not locally conformally flat at \( x_0 \)

**Lemma 6.1.** Let \((M^n, g_0)\) be a closed manifold of dimension \( n \geq 8 \) and there exists a maximum point \( x_0 \) of \( f \) such that \( W_{g_0}(x_0) \neq 0 \). Suppose either the semipositivity hypotheses (2.1) or (2.2) holds. Moreover, assume
\[ \nabla^l_{g_0} f(x_0) = 0 \quad \text{for} \quad 2 \leq l \leq n - 4, \] \label{6.2}

Then for sufficiently small \( \epsilon > 0 \), there exist a positive function \( u_{0\epsilon} \) and a positive constant \( C_{x_0, f, n} \) such that
\[ E_f[u_{0\epsilon}] \leq \frac{q(S^n)}{\max_{M^n} f} (1 - C_{x_0, f, n} \epsilon^4 |W_{g_0}(x_0)|^2) \quad \text{if} \quad n = 8; \]
\[ E_f[u_{0\epsilon}] \leq \frac{q(S^n)}{\max_{M^n} f} (1 - C_{x_0, f, n} \epsilon^4 |W_{g_0}(x_0)|^2) \quad \text{if} \quad n \geq 9. \]

Moreover, a conformal metric \( \tilde g = u_{0\epsilon}^{\frac{4}{n-4}} g_0 \) enjoys the property that \( Q_{\tilde g} \geq 0 \) and positive somewhere. In addition, under assumptions (2.1), the scalar curvature \( R_{\tilde g} \) is positive.

**Proof.** Define
\[ F_g[u] = \frac{\int_{M^n} u \tilde P_g u d\mu_{\tilde g}}{\left( \int_{M^n} u^{\frac{2n}{n-4}} d\mu_{\tilde g} \right)^{\frac{n-4}{n}}} \]
Also, $\mathcal{F}_{\tilde{g}}[u_\epsilon] = \mathcal{F}_{g_0}[u_\epsilon \varphi]$ by conformal invariance of Paneitz operator. Let $v_\epsilon = \hat{u}_\epsilon - u_\epsilon$, Lemma 4.3 in [14] gives

$$|v_\epsilon| \leq C \log \frac{1}{\epsilon^2 + |x|^2} \quad \text{if } n = 8;$$
$$|v_\epsilon| \leq C (\epsilon^2 + |x|^{\frac{8}{n-2}}) \quad \text{if } n \geq 9. \quad (6.4)$$

The same computations in the proof of Lemma 4.4 in [14] yield

$$\int_{M^n} \hat{u}_\epsilon P_{\tilde{g}} \hat{u}_\epsilon d\mu_{\tilde{g}} = \int_{M^n} u_\epsilon P_g u_\epsilon d\mu_g + 2n\epsilon^4 \int_{M^n} u_\epsilon^{\frac{n+4}{n-4}} v_\epsilon d\mu_{\tilde{g}} + O(1)$$

and

$$\int_{M^n} u_\epsilon P_g u_\epsilon d\mu_g = b_n \epsilon^4 (1 + o(1)) \int_{M^n} u_\epsilon^{\frac{2n}{n-4}} d\mu_{\tilde{g}}.$$

One simple observation is that the covariant derivatives of $f$ with respect to metric $g_0$ at $x_0$ vanish up to order $m \geq 1$, then its covariant derivatives of $f$ with respect to conformal metric $g$ of $g_0$ at $x_0$ vanish up to the same order. Then, from condition (6.2), near $x_0$ there holds

$$f(x) = f(x_0) + \frac{1}{(n-3)!} \nabla_{i_1 \ldots i_{n-3}} f(x_0) x^{i_1} \cdots x^{i_{n-3}} + O(r^{n-2}).$$

In $B_{\delta}(x_0)$, from (6.4) there holds $v_\epsilon \leq C u_\epsilon$, then

$$\left| |u_\epsilon + v_\epsilon|^{\frac{2n}{n-4}} - u_\epsilon^{\frac{2n}{n-4}} - \frac{2n}{n-4} u_\epsilon^{\frac{n+4}{n-4}} v_\epsilon \right| \leq C u_\epsilon^{\frac{n-8}{n-4}} v_\epsilon^2.$$

Putting these facts together, we obtain

$$\int_{M^n} f \hat{u}_\epsilon^{\frac{n+4}{n-4}} d\mu_{\tilde{g}} = \int_{M^n} f |u_\epsilon + v_\epsilon|^{\frac{2n}{n-4}} d\mu_{\tilde{g}}$$

$$= \int_{B_\delta(x_0)} f \left( u_\epsilon^{\frac{2n}{n-4}} + \frac{2n}{n-4} u_\epsilon^{\frac{n+4}{n-4}} v_\epsilon + O(u_\epsilon^{\frac{8}{n-4}} v_\epsilon^2) \right) d\mu_{\tilde{g}} + O(1)$$

$$= f(x_0) \int_{B_\delta(x_0)} \left( u_\epsilon^{\frac{2n}{n-4}} + \frac{2n}{n-4} u_\epsilon^{\frac{n+4}{n-4}} v_\epsilon + O(u_\epsilon^{\frac{8}{n-4}} v_\epsilon^2) \right) d\mu_{\tilde{g}} + O(1)$$

$$+ \int_{B_\delta} O(|x|^{n-3}) \left( u_\epsilon^{\frac{2n}{n-4}} + \frac{2n}{n-4} u_\epsilon^{\frac{n+4}{n-4}} v_\epsilon + O(u_\epsilon^{\frac{8}{n-4}} v_\epsilon^2) \right) d\mu_{\tilde{g}}.$$

Now we set $u_{0\epsilon} = \varphi \hat{u}_\epsilon$. Combining the above estimates and the following asymptotics

$$\int_{M^n} u_\epsilon^{\frac{n+4}{n-4}} v_\epsilon d\mu_{\tilde{g}} = \begin{cases} O(\epsilon^{-4} |\log \epsilon|) & \text{if } n = 8; \\ O(\epsilon^{4-n}) & \text{if } n \geq 9; \end{cases}$$

$$\int_{M^n} u_\epsilon^{\frac{8}{n-4}} v_\epsilon^2 d\mu_{\tilde{g}} = \begin{cases} O(|\log \epsilon|^3) & \text{if } n = 8; \\ O(\epsilon^{8-n}) & \text{if } n \geq 9; \end{cases}$$
we conclude that

\[
\int_{M^n} |x|^{n-3} u_n \frac{2n}{n-4} d\mu_g = O(\varepsilon^{-3});
\]

\[
\int_{M^n} |x|^{n-3} u_n \frac{n+4}{n-4} v_n d\mu_g = \begin{cases} O(\|\log \varepsilon\|^2) & \text{if } n = 8; \\ O(1) & \text{if } n \geq 9; \end{cases}
\]

\[
\int_{M^n} |x|^{n-3} u_n \frac{s}{n-4} v_n^2 d\mu_g = \begin{cases} O(\|\log \varepsilon\|^4) & \text{if } n = 8; \\ O(1) & \text{if } n \geq 9; \end{cases}
\]

From (6.5), (6.3) together with the above estimates, we conclude the first assertion.

By equation (6.1) and conformal invariance of Paneitz operator, one has

\[
P_{g_0}(u_{0\varepsilon}) = P_{\tilde{g}}(\tilde{u}_\varepsilon) \frac{n+4}{n-4} \frac{b_n \varepsilon^4 \chi_\delta(x)}{(\varepsilon^2 + |x|^2)^{\frac{n+4}{2}}} \frac{\frac{n+4}{n-4}}{n-4} \geq 0.
\]

Together with \(Q_{g_0} \geq 0\) not identically zero, under either Gursky-Malchiodi’s hypotheses (2.1) or Hang-Yang’s (2.2), the strong maximum principle for Paneitz operator (cf. [14] Theorem 2.2 or [16] Proposition 3.1, respectively) shows that \(u_{0\varepsilon} > 0\) and then \(\tilde{u}_\varepsilon > 0\). Again by conformal invariance of Paneitz operator, the identity

\[
\frac{n-4}{2} Q_{\tilde{g}} = u_{0\varepsilon} \frac{n+4}{n-4} P_{g_0}(u_{0\varepsilon}) = \frac{b_n \varepsilon^4 \chi_\delta(x)}{(\varepsilon^2 + |x|^2)^{\frac{n+4}{2}}} \frac{\frac{n+4}{n-4}}{n-4} u_{0\varepsilon} \frac{n+4}{n-4},
\]

which implies that \(Q_{\tilde{g}} \geq 0\) is not identically zero. The positivity of \(R_{\tilde{g}}\) under hypotheses (2.1) follows from Lemma 4.6 in [14]. This completes the proof. \(\square\)

6.2 \((\mathbf{M}^n, g_0) : n = 5, 6, 7\) or \(n \geq 8\) locally conformally flat

Lemma 6.2. Let \((\mathbf{M}^n, g_0)\) be a closed manifold of dimensions \(n = 5, 6, 7\) or locally conformally flat. Suppose either the semi positivity hypotheses (2.1) or (2.2) holds. Let \(f\) be a smooth positive function on \(\mathbf{M}^n\). Moreover, for \(n \geq 5\), let \(x_0\) denote a maximum point of \(f\), assume

\[
\text{if } n = 6, \quad \nabla^2_{g_0} f(x_0) = 0;
\]

(6.6)
if \( n = 7 \), \( \nabla_{g_0}^l f(x_0) = 0 \) for \( 2 \leq l \leq 3 \); \( (6.7) \)

if \( n \geq 8 \) and \( g_0 \) is locally conformally flat, \( \nabla_{g_0}^l f(x_0) = 0 \) for \( 2 \leq l \leq n - 4 \). \( (6.8) \)

Then for sufficiently small \( \epsilon > 0 \), there exist a positive function \( u_{0\epsilon} \) and a positive constant \( C_{x_0,f,n} \) such that

\[
E_f[u_{0\epsilon}] \leq \frac{\eta(S^n)}{(\max_{M^n} f)^{n-4}} - C_{x_0,f,n}\epsilon.
\]

Moreover, a conformal metric \( \bar{g} = \frac{4}{u_{0\epsilon}^4} g_0 \) enjoys the property that \( Q_{\bar{g}} \geq 0 \) and positive somewhere. In addition, under assumptions \( (2.1) \), the scalar curvature \( R_{\bar{g}} \) is positive.

**Proof.** We set \( u_{0\epsilon} = \varphi \hat{u}_{\epsilon} \). Let \( G_{\bar{g}}(x_0, \cdot) \) denote the Green’s function with pole at \( x_0 \), from the Positive Mass Theorem for Paneitz operator in dimensions \( n = 5, 6, 7 \) (cf. [14] Proposition 2.5) or \((M^n, g_0)\) is locally conformally flat (cf. [19] Theorem 1.1), \( G_{\bar{g}}(x_0, \cdot) \) has the expansion of \( G_{\bar{g}}(x_0, x) = c_n d_{\bar{g}}(x_0, x)^{4-n} + \alpha_{x_0} + O(r) \),

where \( r = d_{\bar{g}}(x_0, x) \) and \( \alpha_{x_0} > 0 \) is the mass.

**Remark 6.1.** In the latter case, just as pointed out by Humbert-Raulot in their paper [19], under the assumptions of our Theorem 1.1, the condition “\((M^n, g_0)\) is locally conformally flat” can not be weaken to “\( g_0 \) is conformally flat around one (maximum) point \( x_0 \)”, since their proof of Positive Mass Theorem for Paneitz operator involves the standard Schoen-Yau’s Positive Mass Theorem [25] for conformal Laplacian. We wonder whether some more restrictions on Weyl tensor at some maximum point of \( f \) will be helpful to construct initial data in this case or not, for instance, just like the one in [18] for scalar curvature case.

Let \( \beta = \frac{\alpha_{x_0}}{c_n} \), then applying the computations on page 36 in [14] together with conformal invariance of Paneitz operator to show

\[
E[u_{0\epsilon}] = \int_{M^n} \hat{u}_{\epsilon} P_{\bar{g}} \hat{u}_{\epsilon} d\mu_{\bar{g}}
\]

\[
= b_n \epsilon^4 \left( \int_{M^n} u_{\epsilon}^{2n} d\mu_{\bar{g}} + \beta (1 + o(1)) \int_{M^n} u_{\epsilon}^{n+4} d\mu_{\bar{g}} + O(1) \right).
\]

It remains to estimate \( \int_{M^n} f \hat{u}_{\epsilon}^{2n} d\mu_{\bar{g}} \). For some \( 0 < \tilde{\delta} << \delta \), a cut-off function \( \chi_{\tilde{\delta}}(x) \) may similarly defined as \( \chi_{\delta}(x) \) before. Set

\[
\hat{u}_{\epsilon} = \chi_{\tilde{\delta}}(u_{\epsilon} + \beta) + (1 - \chi_{\tilde{\delta}}) \xi_{\bar{g}} \text{ with } \xi_{\bar{g}} = \frac{G_{\bar{g}}(x_0, \cdot)}{c_n},
\]

and rewrite \( \hat{u}_{\epsilon} \) as

\[
\hat{u}_{\epsilon} = \chi_{\tilde{\delta}}(u_{\epsilon} + \beta) + (1 - \chi_{\tilde{\delta}}) \xi_{\bar{g}} + \hat{u}_{\epsilon} - \hat{u}_{\epsilon}.
\]

Since in \( B_{\tilde{\delta}}(x_0) \), \( \beta \) is bounded by \( u_{\epsilon} \), we have

\[
\int_{B_{\tilde{\delta}}(x_0)} (u_{\epsilon} + \beta) \hat{u}_{\epsilon}^{n+4} d\mu_{\bar{g}}
\]
Then, together with in view of Lemma 5.3 in [14], and the following asymptotics hold

\[ f(x) = f(x_0) + \frac{1}{(n-3)!} \nabla_{i_1 \cdots i_{n-3}} f(x_0) x^{i_1} \cdots x^{i_{n-3}} + O(r^{-2}). \]

Then, together with

\[ |\hat{u}_\epsilon - \bar{u}_\epsilon| = o_\delta(1) \]
in view of Lemma 5.3 in [14], we estimate

\[
\begin{align*}
\int_{M^n} f\frac{2n}{n-4} d\mu_{\tilde{g}} &= \int_{B_{\tilde{g}}(x_0)} f(u_\epsilon + \beta) \frac{2n}{n-4} d\mu_{\tilde{g}} + O(1) \\
&= \int_{B_{\tilde{g}}(x_0)} (f(x_0) + \frac{1}{(n-3)!} \nabla_{i_1 \cdots i_{n-3}} f(x_0) x^{i_1} \cdots x^{i_{n-3}} + O(r^{-2}))(u_\epsilon + \beta) \frac{2n}{n-4} d\mu_{\tilde{g}} + O(1).
\end{align*}
\]

Since \( \beta \leq Cu_\epsilon \) in \( B_{\tilde{g}}(x_0) \), then

\[
\begin{align*}
\frac{1}{(n-3)!} \int_{B_{\tilde{g}}(x_0)} \nabla_{i_1 \cdots i_{n-3}} f(x_0) x^{i_1} \cdots x^{i_{n-3}} (u_\epsilon + \beta) \frac{2n}{n-4} d\mu_{\tilde{g}} &= O(1) \int_{B_{\tilde{g}}(x_0)} |x|^{n-3} (u_\epsilon + \beta) \frac{2n}{n-4} d\mu_{\tilde{g}} \\
&= O(1) \left[ \int_{M^n} |x|^{n-3} u_\epsilon \frac{2n}{n-4} d\mu_{\tilde{g}} + \frac{2n}{n-4} \beta \int_{M^n} |x|^{n-3} u_\epsilon \frac{2n}{n-4} d\mu_{\tilde{g}} \\
&\quad + \beta^2 \int_{M^n} O(|x|^2 u_\epsilon^{-\frac{8}{n-4}}) d\mu_{\tilde{g}} + O(1) \right].
\end{align*}
\]

In particular, for dimension \( n = 5 \), more precisely we have

\[
\begin{align*}
\frac{1}{2} \int_{B_{\tilde{g}}(x_0)} \nabla_{ij} f(x_0) x^i x^j (u_\epsilon + \beta) \frac{2n}{n-4} d\mu_{\tilde{g}} &= \frac{1}{2n} \Delta f(x_0) \int_{B_{\tilde{g}}(x_0)} |x|^2 (u_\epsilon + \beta) \frac{2n}{n-4} d\mu_{\tilde{g}} \\
&= \frac{1}{2n} \Delta f(x_0) \left[ \int_{M^n} |x|^2 u_\epsilon^{-\frac{2n}{n-4}} d\mu_{\tilde{g}} + \frac{2n}{n-4} \beta \int_{M^n} |x|^2 u_\epsilon^{-\frac{2n}{n-4}} d\mu_{\tilde{g}} \\
&\quad + \beta^2 \int_{M^n} O(|x|^2 u_\epsilon^{-\frac{8}{n-4}}) d\mu_{\tilde{g}} + O(1) \right].
\end{align*}
\]

and the following asymptotics hold

\[
\int_{M^n} |x|^{n-3} u_\epsilon \frac{2n}{n-4} d\mu_{\tilde{g}} = O(\epsilon^3);
\]
also a similar argument as in (a) yields

\[
\int_{M^n} |x|^{n-3} u^\frac{n+4}{n} d\mu_{\tilde{g}} = \begin{cases} 
O(\epsilon^{-2}) & \text{if } n = 5; \\
O(\epsilon^{-1}) & \text{if } n = 6; \\
O(|\log \epsilon|) & \text{if } n = 7;
\end{cases}
\]

\[
\int_{M^n} |x|^{n-3} u^\frac{8}{n} d\mu_{\tilde{g}} = \begin{cases} 
O(\epsilon^{-1}) & \text{if } n = 5; \\
O(1) & \text{if } n = 6, 7.
\end{cases}
\]

Therefore, putting these facts above together, we obtain

\[
E_f[u_{0k}] = \frac{\int_{M^n} \hat{u}_e P_{g} \hat{u}_e d\mu_{\tilde{g}}}{\left( \int_{M^n} f \hat{u}_e^{\frac{2n}{n-4}} d\mu_{\tilde{g}} \right)^{\frac{n-4}{n}}} = b_n \epsilon^4 \left( \int_{M^n} u^\frac{2n}{n-4} d\mu_{\tilde{g}} \right)^{\frac{4}{n}} \left( 1 + \beta(1 + o(1)) \int_{M^n} u^\frac{n+4}{n} d\mu_{\tilde{g}} + O_{\delta}(\epsilon^n) \right)
\]

\[
= \frac{q(S^n)}{(\max_{M^n} f)^{\frac{n-4}{n}}} \left( 1 - \beta(1 + o(1)) \int_{M^n} u^\frac{n+4}{n} d\mu_{\tilde{g}} + O_{\delta}(\epsilon^n) \right).
\]

(b) In dimension \(n \geq 8\) and \((M^n, g_0)\) is locally conformally flat. From condition (6.8), near \(x_0\), there holds

\[
f(x) = f(x_0) + \frac{1}{(n-3)!} \nabla_{i_1 \ldots i_{n-3}} f(x_0) x^{i_1} \ldots x^{i_{n-3}} + O(p^{n-2}).
\]

Together with the asymptotics

\[
\int_{M^n} |x|^{n-3} u^\frac{2n}{n-4} d\mu_{\tilde{g}} = O(\epsilon^{-3}), \quad \int_{M^n} |x|^{n-3} u^\frac{n+4}{n} d\mu_{\tilde{g}} = O(1), \quad \int_{M^n} |x|^{n-3} u^\frac{8}{n} d\mu_{\tilde{g}} = O(1),
\]

also a similar argument as in (a) yields

\[
E_f[u_{0k}] = \frac{\int_{M^n} \hat{u}_e P_{g} \hat{u}_e d\mu_{\tilde{g}}}{\left( \int_{M^n} f \hat{u}_e^{\frac{2n}{n-4}} d\mu_{\tilde{g}} \right)^{\frac{n-4}{n}}} = b_n \epsilon^4 \left( \int_{M^n} u^\frac{2n}{n-4} d\mu_{\tilde{g}} \right)^{\frac{4}{n}} \left( 1 + \beta(1 + o(1)) \int_{M^n} u^\frac{n+4}{n} d\mu_{\tilde{g}} + O_{\delta}(\epsilon^n) \right)
\]

\[
= \frac{q(S^n)}{(\max_{M^n} f)^{\frac{n-4}{n}}} \left( 1 - \beta(1 + o(1)) \int_{M^n} u^\frac{n+4}{n} d\mu_{\tilde{g}} + O_{\delta}(\epsilon^n) \right).
\]
In conclusion, for dimensions \( n = 5, 6, 7 \) or \( n \geq 8 \) and \((M^n, g_0)\) is not conformally flat near \( x_0 \), we conclude from (6.9) and (6.10) that

\[
E_f[u_0] = \frac{q(S^n)}{\left(\max_{M^n} f\right)^{\frac{n-4}{n}}} (1 - O(\epsilon)).
\]

The proof of the remained assertions is the same as in the proof of Lemma 6.1.

\[
\square
\]

7 Sequential convergence of the nonlocal Q-curvature flow

In this section, we finish the proof of Theorem 1.1 by showing the time sequential convergence of the nonlocal Q-curvature flow (2.3)-(2.4) to a positive solution of Q-curvature equation (1.3).

**Proof of Theorem 1.1.**

Under the assumptions of Theorem 1.1, using initial data which are constructed in Lemmas 6.1 and 6.2, in both (a) \( n = 5, 6, 7 \) or \((M^n, g_0)\) is locally conformally flat, and (b) \( n \geq 8 \), Weyl tensor at \( x_0 \) is nonzero cases, we obtain

\[
E_f[u_0] \leq \frac{q(S^n)}{\left(\max_{M^n} f\right)^{\frac{n-4}{n}}} - \epsilon_0,
\]

for some small \( \epsilon_0 > 0 \). Then by Sobolev inequality on compact manifolds, for any given \( \delta > 0 \) one has

\[
\left( \int_{M^n} u(t)^{\frac{2n}{n-4}} d\mu_{g_0} \right)^{\frac{n-4}{n}} \leq (q(S^n)^{-1} + \delta) \int_{M^n} u(t)^2 P_{g_0} u(t) d\mu_{g_0} + C(M^n, \delta) \int_{M^n} u(t)^2 d\mu_{g_0}
\]

\[
\leq (q(S^n)^{-1} + \delta) E_f[u_0] \left( \int_{M^n} f u(t)^{\frac{2n}{n-4}} d\mu_{g_0} \right)^{\frac{n-4}{n}} + C(M^n, \delta) \int_{M^n} u(t)^2 d\mu_{g_0}
\]

\[
\leq (q(S^n)^{-1} + \delta) \left( \max_{M^n} f \right)^{\frac{n-4}{n}} \left[ \frac{q(S^n)}{\left(\max_{M^n} f\right)^{\frac{n-4}{n}}} - \epsilon_0 \right] \left( \int_{M^n} u(t)^{\frac{2n}{n-4}} d\mu_{g_0} \right)^{\frac{2n}{n-4}} + C(M^n, \delta) \int_{M^n} u(t)^2 d\mu_{g_0}
\]

By choosing \( \delta = \frac{\left(\max_{M^n} f\right)^{\frac{n-4}{n}}}{2q(S^n)} \epsilon_0 \), we obtain

\[
\int_{M^n} u(t)^2 d\mu_{g_0} \geq C \left( \int_{M^n} u(t)^{\frac{2n}{n-4}} d\mu_{g_0} \right)^{\frac{n-4}{n}} \geq C_0 > 0
\]

where we have used the uniform boundedness of the volume of the flow metric by Lemma 3.5.

From Lemmas 3.2, 5.3, 5.4 and Hardy-Littlewood-Sobolev inequality on compact manifolds, there exist sequences of \( \{t_k\} \) with \( t_k \to \infty \) as \( k \to \infty \) and \( \{u_k = u(t_k, \cdot)\} \), such that up to a subsequence as \( k \to \infty \)

\[
\alpha(t_k) \to \alpha_\infty;
\]
\[ u_k \to u_\infty \text{ weakly in } H^2(M^n, g_0) \text{ and strongly in } L^2(M^n, g_0); \]
\[ -u_k + \frac{n-4}{2} \alpha_\infty P_{g_0}^{-1}(f u_k) \to 0 \text{ strongly in } H^2(M^n, g_0). \]

Thus, it also yields \( u_\infty \geq 0 \) and \( u_\infty \) is a strong solution of
\[ u_\infty = \frac{n-4}{2} \alpha_\infty P_{g_0}^{-1}(f u_\infty). \]

By the regularity theory of elliptic equations, one has
\[ P_{g_0} u_\infty = \frac{n-4}{2} \alpha_\infty f u_\infty. \]

Thanks to the estimate (7.1), the strong maximum principle (cf. [14] Theorem 2.2 or [16] Proposition 3.1, respectively) yields \( u_\infty > 0 \). Therefore, we conclude that up to a positive constant, the \( Q \)-curvature of a conformal metric \( u_\infty^{-\frac{4}{n-4}} g_0 \) is equal to \( f \). This completes the proof of the main Theorem 1.1. □

Notes added after submission.

1. After a previous version of this article has been submitted to a journal, the author realized that some arguments of a recent paper [16] are concerned with the left case mentioned in Remark 1.2 or Remark 6.1.

2. A former PhD student of Professor Xingwang Xu, Hong Zhang emailed me for his contribution of this work without attaching his article on 9 January 2015. Some more comments will be given by me only after I see his article of the work of nonlocal \( Q \)-curvature flow.

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