COMPATIBILITY OF SKEIN ALGEBRA AND CLUSTER ALGEBRA ON SURFACES AND APPLICATIONS

HAN-BOM MOON AND HELEN WONG

ABSTRACT. We show the compatibility of two algebras associated with a topological surface with punctures: the cluster algebra of surfaces defined by Fomin, Shapiro, and Thurston, and the classical limit of the generalized skein algebra constructed by Roger and Yang. There is an explicit monomorphism from the former to the latter. As applications, we resolve Roger-Yang’s conjecture on the deformation quantization of the decorated Teichmüller space. We also obtain several new results on the cluster algebra of surfaces. For example, the cluster algebra of a punctured torus is not finitely generated, and it differs from its upper cluster algebra.

1. INTRODUCTION

By a surface \( \Sigma_{g,n} \), we denote a compact Riemann surface of genus \( g \), without boundaries, minus \( n \) punctures. We may associate two ‘algebras of curves’ on \( \Sigma_{g,n} \), but coming from entirely different origins – one from geometric topology, and the other from combinatorial algebra. In this paper, we consider the compatibility of these two algebras.

The first algebra is the curve algebra \( \mathcal{C}(\Sigma_{g,n}) \), which belongs to a family of invariants of surfaces that are related to the Jones polynomial for knots [Jones85] and to the Witten-Reshetikhin-Turaev topological quantum field theory [Wit89, RT91, BHMV95]. The most well-studied of this family is the Kauffman bracket skein algebra of an unpunctured surface [Prz91, Tur91]. It is known to be related to hyperbolic geometry— the skein algebra is the deformation quantization of the \( SL_2 \)-character variety, which contains the Teichmüller space of the surface ([Tur91, Bul97, BFKB99, PS00]). In [RY14], Roger and Yang sought to generalize this relationship between the skein algebra and the Teichmüller space to the case of a punctured surface. For a punctured surface \( \Sigma_{g,n} \), they defined a generalized skein algebra \( \mathcal{S}^q(\Sigma_{g,n}) \) and proposed that it should be the deformation quantization for the decorated Teichmüller space \( \mathcal{T}^d(\Sigma_{g,n}) \) constructed by Penner in [Pen87, Pen92]. The curve algebra \( \mathcal{C}(\Sigma_{g,n}) \) we study in this paper is the classical limit of Roger-Yang’s generalized skein algebra \( \mathcal{S}^q(\Sigma_{g,n}) \) obtained by setting \( q = 1 \).

The second algebra studied in this paper is the cluster algebra \( \mathcal{A}(\Sigma_{g,n}) \) of a surface. Such cluster algebras were observed by [GSV05, FST08] to be interesting examples of the cluster algebras originally introduced by Fomin and Zelevinsky in [FZ02] for studying the total positivity and dual canonical bases in Lie theory. Defined for any punctured surface \( \Sigma_{g,n} \) admitting an ideal triangulation, the cluster algebra \( \mathcal{A}(\Sigma_{g,n}) \) is generated by arcs on
\( \Sigma_{g,n} \) with a *tagging* (of plain or notched) at its endpoints. From the combinatorial perspective, the tagging is needed for \( \mathcal{A}(\Sigma_{g,n}) \) to have the structure of a cluster algebra. However, from the geometric perspective, at least to the authors, the meaning of the tagging was unclear, despite several scattered clues in [FG06, MSW11, AB20]. This paper is based on the authors’ attempt to understand the geometric meaning of the tagging.

1.1. **Main result.** Let \( \Sigma_{g,n} \) be a Riemann surface of genus \( g \) with \( n > 0 \) punctures. We assume that \( \chi(\Sigma_{g,n}) < 0 \), so that \( n \)-punctured spheres with \( n = 1, 2 \) are excluded. When we define the cluster algebra, we also exclude the three-punctured sphere.

**Theorem A.** Let \( \mathcal{A}(\Sigma_{g,n}) \) be the cluster algebra and \( \mathcal{C}(\Sigma_{g,n}) \) be the curve algebra associated to \( \Sigma_{g,n} \). Then there is a monomorphism

\[
\rho : \mathcal{A}(\Sigma_{g,n}) \to \mathcal{C}(\Sigma_{g,n}).
\]

This is not merely the existence statement. As we will discuss in Section 4, we explicitly construct a natural homomorphism. Indeed the definition of \( \rho \) will tell us a geometric interpretation of the tagging, which can be plain or notched (Definition 4.1). The upshot is that “a notch is a vertex class.”

A similar compatibility result for surfaces with boundaries but without punctures were proven by Muller in [Mul16]. In this setup, the vertex classes in the skein algebra and the tagged arcs in the cluster algebra do not exist. Thus, the interpretation of the tagging as a vertex is new in the punctured surface case. On the other hand, Muller’s result covers quantum version of the algebras, but in our setup, we cannot expect a quantum generalization of Theorem A. This is because \( \mathcal{A}(\Sigma_{g,n}) \) does not admit a quantization in the context of cluster algebra (Remark 3.14).

For some experts, perhaps the above compatibility result may not be astonishing, because both algebras have their geometric motivation/intuition from the same decorated Teichmüller space due to Penner. As the readers may see later in Section 4, once the morphism \( \rho \) is defined, the proof of the well-definedness is a routine computation. What we emphasize here is that the reason why Theorem A is interesting is its rich and interesting applications in both the theory of generalized skein algebra and cluster algebra, that we will describe below.

1.2. **Applications to skein algebra and deformation quantization.** One of our initial motivations was to complete the program of Roger and Yang in [RY14] to identify a deformation quantization of Penner’s decorated Teichmüller space.

**Theorem B.** The Roger-Yang generalized skein algebra \( \mathcal{S}^q(\Sigma_{g,n}) \) is the deformation quantization of the decorated Teichmüller space \( \mathcal{T}^d(\Sigma_{g,n}) \).

In [RY14], Roger and Yang introduced a generalized skein algebra \( \mathcal{S}^q(\Sigma_{g,n}) \) as a candidate of the deformation quantization of \( \mathcal{T}^d(\Sigma_{g,n}) \). Their program consists of two steps. First, they showed that \( \mathcal{S}^q(\Sigma_{g,n}) \) is a deformation of quantization of its classical limit \( \mathcal{C}(\Sigma_{g,n}) \). They then proved that there is a Poisson algebra homomorphism \( \Phi : \mathcal{C}(\Sigma_{g,n}) \to \mathcal{C}^\infty(\mathcal{T}^d(\Sigma_{g,n})) \) whose Poisson structures are given by the generalized Goldman bracket
and the Weil-Peterssen form, respectively ([RY14, Theorem 1.2]). Thus, if the Poisson algebra representation is faithful (meaning \(\Phi\) is injective), then \(S_q^d(\Sigma_{g,n})\) can be understood as the quantization of \(T^d(\Sigma_{g,n})\). However, they left the faithfulness as a conjecture ([RY14, Conjecture 3.4]). In Section 5, we prove it by employing Theorem A.

Besides completing the program of Roger and Yang, Theorem A can also be used to show integrality of the algebras.

**Theorem C.** The Roger-Yang curve algebra \(C(\Sigma_{g,n})\) and the generalized skein algebra \(S_q^d(\Sigma_{g,n})\) are domains.

Note that in our earlier paper [MW21], we already showed that Theorems B and C hold when \(n\) is relatively large compared to \(g\) ([MW21, Theorem B]). The proof was based on diagrammatic computation with little theoretical support nor intuition. We find that our proofs here, based on the relationship with cluster algebras established by Theorem A, provides a more satisfactory theoretical reasoning.

Thang Le kindly informed us that with his collaborators, he also proved Theorem C with a completely independent method ([BKL21]). Their proof covers even the case that \(q\) is not a formal variable.

### 1.3. Applications to cluster algebras.

The compatibility in Theorem A implies several consequences in the structure of cluster algebras, too. The key observation is that for \(C(\Sigma_{g,n})\), the finite generation is already established in [BKPW16a].

The first application is the non-finite generation of \(A(\Sigma_{g,n})\) for positive genus. By [Lad13], it is known that \(A(\Sigma_{g,1})\) is not finitely generated for all \(g \geq 1\). We extend this result as the following. We denote the cluster algebra of \(\Sigma_{g,n}\) with \(\mathbb{Z}_2\) coefficient by \(A(\Sigma_{g,n})_{\mathbb{Z}_2}\) (Definition 3.1).

**Theorem D.**

1. For all \(n \geq 3\), \(A(\Sigma_{0,n})\) is finitely generated.
2. For all \(n \geq 1\), \(A(\Sigma_{1,n})\) is not finitely generated.
3. If \(A(\Sigma_{g,2})_{\mathbb{Z}_2}\) is not finitely generated, then \(A(\Sigma_{g,n})\) is not finitely generated for all \(n \geq 2\).

We expect that for any \(g, n \geq 1\), \(A(\Sigma_{g,n})\) is not finitely generated. The above result reduces the question to \(n = 2\) case. We do not have a proof in these cases.

**Conjecture 1.1.** For \(g \geq 2\), \(A(\Sigma_{g,2})\) is not finitely generated.

A second application involves the upper cluster algebra \(U\) (Definition 3.2). It is an algebra that contains \(A\) and is constructed from the same combinatorial data of seed. In many ways, \(U\) behaves better than \(A\), and thus the question of whether \(A = U\) or not has attracted many researchers in the cluster algebra community. For the summary of some known results, see [CLS15, Section 1.2]. For \(A(\Sigma_{g,n})\), when \(n = 1\), it was shown that \(A(\Sigma_{g,1}) \neq U(\Sigma_{g,1})\) by Ladkani ([Lad13]).

Here, we use the curve algebra \(C(\Sigma_{g,n})\) and a variation \(C(\Sigma_{g,n})'\) (Definition 2.10), and obtain an inclusion

\[
A(\Sigma_{g,n}) \subset C(\Sigma_{g,n})' \subset U(\Sigma_{g,n}).
\]
The algebra \( \mathcal{C}(\Sigma_{g,n})' \) is a subalgebra of \( \mathcal{C}(\Sigma_{g,n}) \) generated by the image of \( \mathcal{A}(\Sigma_{g,n}) \) and the Kauffman bracket skein algebra of \( \Sigma_{g,n} \) generated by isotopy classes of loops. We conjecture that \( \mathcal{C}(\Sigma_{g,n})' = \mathcal{U}(\Sigma_{g,n}) \) (Conjecture 6.14). However, to the authors’ knowledge, it is still unknown if the ‘geometric’ subalgebra \( \mathcal{C}(\Sigma_{g,n})' \) generated by tagged arcs and loops coincide with \( \mathcal{U}(\Sigma_{g,n}) \). For the comparison of \( \mathcal{C}(\Sigma_{g,n}) \) and \( \mathcal{U}(\Sigma_{g,n}) \), see Remark 6.13.

We prove

**Theorem E.** If \( \mathcal{A}(\Sigma_{g,n}) \) is not finitely generated, then \( \mathcal{A}(\Sigma_{g,n}) \neq \mathcal{U}(\Sigma_{g,n}) \). In particular, for \( n \geq 1 \), \( \mathcal{A}(\Sigma_{1,n}) \neq \mathcal{U}(\Sigma_{1,n}) \).

Thus, an affirmative answer for Conjecture 1.1 implies the following conjecture as well.

**Conjecture 1.2.** For any \( g, n \geq 1 \), \( \mathcal{A}(\Sigma_{g,n}) \neq \mathcal{U}(\Sigma_{g,n}) \).

On the contrary, when \( g = 0 \), we expect that \( \mathcal{A}(\Sigma_{0,n}) = \mathcal{U}(\Sigma_{0,n}) \) (Conjecture 6.14). For instance, when \( g = 0 \), one can show that loops are also in \( \mathcal{A}(\Sigma_{0,n}) \), by adapting the computation in [BKPW16b] and [ACDHM21] (Remark 6.12).

1.4. **Structure of the paper.** Sections 2 and 3 are review of the definition and basic properties of \( \mathcal{C}(\Sigma_{g,n}) \) and \( \mathcal{A}(\Sigma_{g,n}) \), respectively, and related constructions. In Section 4, we prove Theorem A. The remaining two sections are devoted to applications for the curve algebra (Section 5) and for the cluster algebra (Section 6).

**Acknowledgement.** The authors would like to thank to Hyunkyu Kim, Thang Le, Kyungyong Lee, Gregg Musiker, and Dylan Thurston for valuable conversations. This work was completed while the first author was visiting Stanford University. He gratefully appreciates the hospitality during his visit. The second author is partially supported by grant DMS-1906323 from the US National Science Foundation and a Birman Fellowship from the American Mathematical Society.

2. **The Curve Algebra \( \mathcal{C}(\Sigma_{g,n}) \)**

In this section, we give a formal definition and basic properties of the curve algebra \( \mathcal{C}(\Sigma_{g,n}) \). For details, see [RY14, Section 2.2] and [MW21, Section 2.4].

In this paper, a surface is \( \Sigma_{g,n} := \Sigma_g \setminus V \), where \( \Sigma_g \) is a Riemann surface of genus \( g \) without boundary, and \( V = \{v_1, \ldots, v_n\} \) is a finite set of points in \( \Sigma_g \). We call \( V \) the set of punctures or vertices.

A loop \( \alpha \) on \( \Sigma_{g,n} \) is an immersion of a circle into \( \Sigma_{g,n} \). An arc \( \beta \) in \( \Sigma \) is an immersion of \([0,1] \) into \( \Sigma_g \) such that the image of \((0,1)\) is in \( \Sigma_{g,n} \) and the image of two endpoints are (not necessarily distinct) points in \( V \). The seemingly unnecessary underbar notation will be justified in Section 3.

**Definition 2.1.** Let \( R \) be a commutative ring. The curve algebra \( \mathcal{C}(\Sigma_{g,n})_R \) is the \( R \)-algebra generated by isotopy classes of loops, arcs, \( V = \{v_i\} \), and their formal inverses \( \{v_i^{-1}\} \), modded out by the following relations:
(1) (Skein relation) \[
\begin{array}{c}
\includegraphics[width=2cm]{skein.png}
\end{array}
\]
(2) (Puncture-skein relation) \[
\begin{array}{c}
\includegraphics[width=2cm]{puncture.png}
\end{array}
\]
(3) (Framing relation) \[
\begin{array}{c}
\includegraphics[width=2cm]{framing.png}
\end{array}
\]
(4) (Puncture-framing relation) \[
\begin{array}{c}
\includegraphics[width=2cm]{framing.png}
\end{array}
\]

The multiplication of elements in \(\mathcal{C}(\Sigma_{g,n})_R\) are represented by taking the union of generators (and counted with multiplicity). We allow the empty curve \(\emptyset\) and it is the multiplicative identity. In the relations, the curves are assumed to be identical outside of the small balls depicted, and the \(i\)-th puncture \(v_i\) is depicted in the second relation.

We set \(\mathcal{C}(\Sigma_{g,n}) := \mathcal{C}(\Sigma_{g,n})_\mathbb{Z}\), so that we mean \(R = \mathbb{Z}\) by default. Then \(\mathcal{C}(\Sigma_{g,n})_R = \mathcal{C}(\Sigma_{g,n})_\mathbb{Z} \otimes_\mathbb{Z} R\).

**Remark 2.2.** Note that in the curve algebra originally discussed by Roger-Yang [RY14], they used \(R = \mathbb{C}\), and the vertices \(\{v_i\}\) were treated as coefficients. But for our purpose, it is more natural to think of the vertices as generators of the algebra.

**Remark 2.3.** One might wonder about our choice of coefficient ring \(\mathbb{Z}\), as compared to Roger and Yang’s choice of \(\mathbb{C}\). Clearly, there is a morphism \(\mathcal{C}(\Sigma_{g,n}) \to \mathcal{C}(\Sigma_{g,n})_\mathbb{C}\). In addition, one can adapt the proof of [RY14, Theorem 2.4] by replacing the \(\mathbb{C}\)-coefficient by the \(\mathbb{Z}\)-coefficient to show that \(\mathcal{C}(\Sigma_{g,n})_\mathbb{C}\) (with \(\mathbb{Z}\)-coefficients) has no torsion. Thus, we have an inclusion \(\mathcal{C}(\Sigma_{g,n}) \subset \mathcal{C}(\Sigma_{g,n})_\mathbb{C}\). It follows, for example, that if \(\mathcal{C}(\Sigma_{g,n})_\mathbb{C}\) is an integral domain, then \(\mathcal{C}(\Sigma_{g,n})_\mathbb{Z}\) is also an integral domain.

**Example 2.4.** Let \(\alpha\) be an arc bounding an unpunctured monogon with the vertex \(v\). Then we can compute \(v\alpha\) as follows:

\[
v \left(\begin{array}{c}
\includegraphics[width=2cm]{example.png}
\end{array}\right) = \includegraphics[width=2cm]{example.png} + \includegraphics[width=2cm]{example.png} = -2 + 2 = 0
\]

Since \(v\) is invertible in \(\mathcal{C}(\Sigma_{g,n})\), this shows that any arc bounding an unpunctured monogon is zero in \(\mathcal{C}(\Sigma_{g,n})\).

**Lemma 2.5.** Let \(v\) and \(w\) be two distinct punctures, and \(e\) be an arc connecting \(v\) and \(w\). In addition, let \(\gamma\) be an arc with both ends at \(v\) that bounds a one-punctured monogon containing \(w\). Then \(\gamma = we^2\).

**Proof.**

\[
we^2 = w \left(\begin{array}{c}
\includegraphics[width=2cm]{example.png}
\end{array}\right) = \left(\begin{array}{c}
\includegraphics[width=2cm]{example.png} + \includegraphics[width=2cm]{example.png}
\end{array}\right) = \left(\begin{array}{c}
\includegraphics[width=2cm]{example.png}
\end{array}\right)
\]

since any arc bounding an unpunctured monogon in \(\emptyset\) by the example above. \(\Box\)
2.1. Relationship of \( \mathcal{C}(\Sigma_{g,n}) \) with hyperbolic geometry. Let \( \mathcal{T}^d(\Sigma_{g,n}) \) be the decorated Teichmüller space of \( \Sigma_{g,n} \) constructed by Penner ([Pen87]). It parameterizes all pairs \((m,r)\) where \( m \) is a complete hyperbolic metric \( \Sigma_{g,n} \) and \( r \) is a choice of a horocycle at every puncture of \( \Sigma_{g,n} \). Given such a pair \((m,r)\), one can assign a well-defined length to any loop on \( \Sigma_{g,n} \) and any arc that goes from puncture to puncture on \( \Sigma_{g,n} \). In addition, we set the length of a vertex to be the length of the horocycle around that vertex. These lengths of loops, arcs, and vertices can then be used to define \( \lambda \)-length functions on \( \mathcal{T}^d(\Sigma_{g,n}) \), and it was shown in [Pen87] that the \( \lambda \)-length functions parametrize the ring of \( \mathbb{C} \)-valued \( \mathcal{C}^\infty \) functions on \( \mathcal{T}^d(\Sigma_{g,n}) \). These \( \lambda \)-length functions can be used to define a Poisson structure on \( \mathcal{T}^d(\Sigma_{g,n}) \) induced by the Weil-Petersson form ([Pen92]).

Roger and Yang defined the curve algebra and showed ([RY14, Theorem 1.2]) that there is a Poisson algebra homomorphism

\[
\Phi : \mathcal{C}(\Sigma_{g,n}) \rightarrow \mathcal{C}^\infty(\mathcal{T}^d(\Sigma_{g,n}))
\]

that sends any loop, arc, or vertex to its corresponding \( \lambda \)-length function. One can think of the relations from the curve algebra as designed to mirror the relations from the \( \lambda \)-length functions in \( \mathcal{C}^\infty(\mathcal{T}^d(\Sigma_{g,n})) \). In fact, Roger and Yang conjectured that the curve algebra relations captures all of the relations from \( \mathcal{C}^\infty(\mathcal{T}^d(\Sigma_{g,n})) \), or equivalently, that

**Conjecture 2.6.** [RY14] The Poisson algebra homomorphism \( \Phi \) in \( \text{(2.1)} \) is injective.

Theorem B of this paper proves Roger and Yang’s conjecture in all cases, by appealing to the algebraic properties of \( \mathcal{C}(\Sigma_{g,n}) \) and the following theorem:

**Theorem 2.7 ([MW21, Theorem A]).** If \( \mathcal{C}(\Sigma_{g,n}) \) is an integral domain, then \( \Phi \) is injective.

In previous work [MW21, Theorem B and Section 4], we were able to verify that \( \mathcal{C}(\Sigma) \) is an integral domain when \( \Sigma \) admits a ‘locally planar’ ideal triangulation. In particular, the genus \( g \) and the number of punctures \( n \) should satisfy

\[
n \geq \begin{cases} 
\left\lceil \frac{7 + \sqrt{48g^2}}{2} \right\rceil, & g \neq 2 \\
10, & g = 2.
\end{cases}
\]

In Theorem 5.2 of this paper, we instead use cluster algebras to obtain an independent and unconditional proof of integrality, so that Conjecture 2.6 applies for any \( \Sigma_{g,n} \).

2.2. Relationship of \( \mathcal{C}(\Sigma_{g,n}) \) with Kauffman bracket skein algebra. Roger and Yang’s definition of the curve algebra and the construction of \( \Phi \) in [RY14] was motivated by a search for an appropriate quantization of the decorated Teichmüller space \( \mathcal{T}^d(\Sigma_{g,n}) \). In particular, they wanted to mimic and generalize the set-up of [Tur91, Bul97, BFKB99, PS00] that establishes the Kauffman bracket skein algebra as a quantization of the \( \text{SL}_2 \)-character variety of \( \Sigma_g \) which contains the Teichmüller space as a dense open subspace. Towards this goal, Roger and Yang defined a generalized Goldman bracket for \( \mathcal{C}(\Sigma_{g,n}) \) and used it to define a deformation quantization that we here denote by \( \mathcal{S}^q(\Sigma_{g,n}) \) ([RY14, Theorem 1.1]).

We omit the precise definition of \( \mathcal{S}^q(\Sigma_{g,n}) \), but instead mention some key properties. Firstly, \( \mathcal{S}^q(\Sigma_{g,n}) \) is an \( \mathbb{R}[[q^{\frac{1}{2}}]] \)-algebra generated by arcs, loops, and vertices, and reduces
COMPATIBILITY OF SKEIN ALGEBRA AND CLUSTER ALGEBRA

7 to the usual Kauffman bracket skein algebra in the absence of punctures (so that the puncture-skein and puncture-framing relations can be ignored). For this reason, we will refer to Roger-Yang’s \( S^q(\Sigma_{g,n}) \) as a **skein algebra**. In addition, \( S^q(\Sigma_{g,n}) \) can be identified with \( \mathcal{C}(\Sigma_{g,n}) \) when \( q = 1 \).

As stated in Theorem B, establishment of Conjecture 2.6 implies that \( S^q(\Sigma_{g,n}) \) is indeed a quantization of \( T^d(\Sigma_{g,n}) \), completing Roger and Yang’s original goal. Moreover, many of the results about the curve algebra \( \mathcal{C}(\Sigma_{g,n}) \) have consequences for the skein algebra \( S^q(\Sigma_{g,n}) \). For example, it was proved in [MW21, Theorem C] that if \( \mathcal{C}(\Sigma_{g,n}) \) is an integral domain, then \( S^q(\Sigma_{g,n}) \) is also an integral domain. Hence Theorem C also follows from Theorem 5.2.

Conversely, many results about \( S^q(\Sigma_{g,n}) \) also apply to \( \mathcal{C}(\Sigma_{g,n}) \). The following two theorems about algebraic properties of \( \mathcal{C}(\Sigma_{g,n}) \) were proved for \( S^q(\Sigma_{g,n}) \) with \( \mathbb{C} \)-coefficients, but the same proof works just as well for \( \mathbb{Z} \)-coefficients and with \( q = 1 \).

**Theorem 2.8** ([BKPW16a, Theorem 2.2]). The algebra \( \mathcal{C}(\Sigma_{g,n}) \) is finitely generated.

We now turn to the \( g = 0 \) case. Let \( C \) be a small circle on \( \Sigma_{0,n} \). We may assume that the \( n \) punctures \( \{v_1, \ldots, v_n\} \) lie on \( C \) in the clockwise circular order. Let \( \beta_{ij} \) be the simple arc in the disk bounded by \( C \) that connects \( v_i \) and \( v_j \).

**Theorem 2.9** ([ACDHM21, Theorem 1.1]). The algebra \( \mathcal{C}(\Sigma_{0,n}) \) is isomorphic to

\[
\mathbb{Z}[v_i^{\pm}, \beta_{ij}]_{1 \leq i,j \leq n}/J,
\]

where \( J \) is an ideal generated by

1. \( \beta_{ik} \beta_{jk} = \beta_{il} \beta_{lk} + \beta_{ij} \beta_{kl} \) for any 4-subset \( \{i, j, k, l\} \subset [n] \) in cyclic order;
2. \( \gamma_{ij}^+ = \gamma_{ij}^- \);
3. \( \delta = -2 \),

where \( \gamma_{ij}^{\pm} \) and \( \delta \) are explicit polynomials in the generators (and have a geometric description).

For definitions of \( \gamma_{ij}^{\pm} \) and \( \delta \) and formulas in terms of \( \beta_{ij} \), see [ACDHM21, Section 4].

2.3. **A useful variation of** \( \mathcal{C}(\Sigma_{g,n}) \).

**Definition 2.10.** Let \( \mathcal{C}(\Sigma_{g,n})' \subset \mathcal{C}(\Sigma_{g,n}) \) be the subalgebra generated by the following elements:

1. Isotopy classes of loops;
2. \( \beta, v\beta, w\beta, \) and \( vw\beta \), where \( \beta \) is an arc connecting (possibly non-distinct) vertices \( v \) and \( w \).

For any coefficient ring \( R \), set \( \mathcal{C}(\Sigma_{g,n})'_R := \mathcal{C}(\Sigma_{g,n})' \otimes_{\mathbb{Z}} R \subset \mathcal{C}(\Sigma_{g,n})_R \). Later, we will need a slight extension/variation of Theorem 2.8.

**Theorem 2.11.** For any coefficient ring \( R \), the algebra \( \mathcal{C}(\Sigma_{g,n})'_R \) is finitely generated.
The proof is identical to that of [BKPW16a, Theorem 2.2]. More specifically, one uses a generalized handle decomposition of \( \Sigma_{g,n} \) with a disk removed. The complexity of a curve is defined based on how many times and in what manner a minimal representation of the curve traverses the handles ([BKPW16a, Section 3.1]). By application of skein identities, it is shown that any curve can be recursively written as lower-complexity curves ([BKPW16a, Lemmas 3.1–3.4]). Importantly, none of the skein identities in the recursive steps use the formal inverses of vertices. In particular, the skein identities from [BKPW16a, Lemmas 3.1 and 3.2] involve only undecorated arcs of the form \( \beta \), and those for [BKPW16a, Lemma 3.3] uses arcs of the form \( \beta \) and \( vw\beta \). For [BKPW16a, Lemma 3.4], one identity (first identity on [BKPW16a, p.10]) involves \( v^{-1} \). However, the recursive step comes from substituting it into a previous equation (last identify on [BKPW16a, p.9]), in a term with a factor of \( v \). Because of the cancellation, the recursive step can be written in a form involving only undecorated arcs.

**Remark 2.12.** Later, we will see by Theorem A that the cluster algebra \( A(\Sigma_{g,n}) \) (to be defined in Section 3) can be understood as a subalgebra generated by ‘tagged’ arc classes. On the other hand, the classical limit (\( q = 1 \)) of the original Kauffman bracket skein algebra ([Prz91, Tur91]) is a subalgebra of \( C(\Sigma_{g,n}) \) generated by loop classes. So, one may interpret \( C(\Sigma_{g,n}) \) as the subalgebra of \( C(\Sigma_{g,n}) \) generated by the image of the cluster algebra \( A(\Sigma_{g,n}) \) and the usual Kauffman bracket skein algebra.

### 3. Cluster algebra from surfaces

We review the definition of the cluster algebra \( A(\Sigma_{g,n}) \) constructed from a punctured surface \( \Sigma \), as introduced by Fomin, Shapiro, and Thurston in [FST08].

**3.1. Definition of cluster algebras.** We begin by noting that we will not need the definition of cluster algebras in full generality, which can be found for example in [FZ02]. We will restrict to the case of constant coefficient, skew-symmetric exchange matrix, and no frozen variables. The only minor extension is that we allow more general base ring including finite field, while in many literature a cluster algebra is defined over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) or \( \mathbb{C} \). Essentially the choice of coefficient ring does not significantly impact the theory ([BMRS15, Section 2]).

Let \( R \) be an integral domain. Let \( F \) be a purely transcendental finite extension of \( Q(R) \), the field of fraction of \( R \). A seed is a pair \((x, B)\), where \( x = \{x_1, \ldots, x_m\} \) is a free generating set for \( F \) as a field over \( Q(R) \) and \( B = (b_{ij}) \) is a skew-symmetric \( m \times m \) integral matrix. \( B \) is called the exchange matrix, the set \( x \) is the cluster, and its elements \( x_i \) are the cluster variables of the seed.

For a seed \((x, B)\) and \( k \in \{1, \ldots, m\} \), a mutation in the direction of \( k \) is an operation that produces another seed \( \mu_k(x, B) = (x', B') \) where

1. \( x' = \{x_1', \ldots, x_m'\} \) is such that \( x'_k \) is defined by the exchange relation

   \[
   x_kx'_k = \prod_{b_{jk} > 0} x_j^{b_{jk}} + \prod_{b_{jk} < 0} x_j^{-b_{jk}}
   \]

   and all other cluster variables are identical, so \( x_i = x'_i \) for \( i \neq k \);
(2) $B' = (b'_{ij})$ is defined by

$$b'_{ij} = \begin{cases} 
-b_{ij}, & \text{if } i = k \text{ or } j = k, \\
 b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + |b_{ik}|b_{kj}|), & \text{otherwise}.
\end{cases}$$

Sometimes we notate it as $\mu_k(B) = B'$. It is straightforward to check that a mutation is involutive.

Since a mutation of a seed produces another seed, repeated mutations can be performed following any sequence of indices $1, \ldots, m$. We say that two seeds $(x, B)$ and $(y, C)$ are mutation equivalent and write $(x, B) \sim (y, C)$ if one seed can be obtained from the other by a sequence of mutations.

**Definition 3.1.** The cluster algebra $\mathcal{A}(x, B)$ is the $R$-subalgebra of the ambient field $\mathcal{F}$ generated by

$$\bigcup_{(y, C) \sim (x, B)} y,$$

the cluster variables of seeds that are mutation equivalent to a seed $(x, B)$. Since mutation equivalent seeds produce the same cluster algebra, we write $\mathcal{A}$ instead of $\mathcal{A}(x, B)$ when the choice of initial seed may be safely suppressed. When we need to specify the coefficient ring, we use the notation $\mathcal{A}_R$ for $\mathcal{A}$.

A simplicial complex, called the cluster complex of $\mathcal{A} = \mathcal{A}(x, B)$, is often used to describe the relationships between the cluster variables used to generate it. In particular, the vertices of the cluster complex are the cluster variables $\bigcup_{(y, C) \sim (x, B)} y$ that generate $\mathcal{A}$, and there is a $k$-simplex whenever $k$ cluster variables belong to the same cluster. Thus each seed in a cluster algebra gives rise to a maximal simplex in the cluster complex. The exchange graph is the dual graph, where the vertices are the seeds, and there is an edge between two seeds if they are mutations of each other. So by definition, the exchange graph of a cluster algebra must be an $m$-regular, connected graph.

By the Laurent phenomenon ([FZ02, Theorem 3.1]), for any $x_i \in x$ and an equivalent seed $(y = \{y_1, y_2, \cdots, y_m\}, C) \sim (x, B)$,

$$x_i \in R[y_1^\pm, y_2^\pm, \cdots, y_m^\pm] \subset \mathcal{F}.$$  

**Definition 3.2.** For a cluster algebra $\mathcal{A} = \mathcal{A}(x, B) \subset \mathcal{F}$, the upper cluster algebra $\mathcal{U}$ is defined by

$$\mathcal{U} := \bigcap_{(y, C) \sim (x, B)} R[y_1^\pm, y_2^\pm, \cdots, y_m^\pm] \subset \mathcal{F}.$$  

The Laurent phenomenon tells that $\mathcal{A} \subset \mathcal{U}$. In general they do not coincide. The upper cluster algebra $\mathcal{U}$ behaves better than $\mathcal{A}$; for example, $\mathcal{U}$ is an integrally closed domain if $R$ is ([BMRS15, Lemma 2.1]). However, the computation of $\mathcal{U}$ and the question of whether $\mathcal{A} = \mathcal{U}$ or not are in general difficult. For a partial criterion for $\mathcal{A} = \mathcal{U}$, see [Mul13].
3.2. **Definition of the cluster algebra of a surface.** In this paper, we focus exclusively on cluster algebras associated to a punctured surface \( \Sigma_{g,n} \). The cluster algebra \( \mathcal{A}(\Sigma_{g,n}) \) is essentially the algebra generated by isotopy classes of arcs on the surface \( \Sigma_{g,n} \). Each cluster should come from the arc classes in a maximal compatible set; in other words, the edges of a triangulation should form a cluster. A mutation should correspond to a flip of an edge of the triangulation. Although this idea is sufficient to define \( \mathcal{A}(\Sigma_{g,n}) \), the intuitive picture is not complete as stands, because not every arc in an ordinary triangulation is flippable if the triangulation contains a self-folded triangle. This problem was resolved in [FST08] by introducing tagged arcs.

We begin with a review of ordinary triangulations, and how the data from a single triangulation without a self-folded triangle is sufficient to define a cluster algebra \( \mathcal{A}(\Sigma_{g,n}) \). We then introduce tagged triangulations, which will fully describe the correspondence between cluster variables and tagged arcs. The results in [FST08] also apply to surfaces with boundary and marked points on the boundary, but we do not need that generality here.

3.2.1. **Ordinary Triangulations.** As in Section 2, we denote a punctured surface without boundary by \( \Sigma_{g,n} = \Sigma_g \setminus V \), where \( V = \{v_1, \ldots, v_n\} \). We assume that \( n \geq 1 \), and exclude \( \Sigma_{0,n} \) for \( n \leq 3 \).

Recall an arc of \( \Sigma_{g,n} = \Sigma_g \setminus V \) is an immersion \( \alpha : [0, 1] \to \Sigma_g \) such that \( \alpha \) embeds \((0, 1)\) in \( \Sigma_{g,n} \) and \( \alpha \) takes the endpoints \( \{0, 1\} \) to the punctures \( V \). The set of isotopy classes of arcs connecting two punctures in \( \Sigma_{g,n} \) will be denoted by \( \mathcal{A}(\Sigma_{g,n}) \). Two arcs are said to be **compatible** if they are the same, or if they do not intersect except at the punctures. A maximal collection of distinct, pairwise compatible isotopy classes of arcs forms an **ideal triangulation** \( T \) on \( \Sigma_{g,n} \). The arcs in a triangulation are referred to as **edges**, and the set of edges is denoted by \( E \). Because of maximality, \( E \) separates \( \Sigma_{g,n} \) into a set of triangles, which is denoted by \( T \). Recall that \( n = |V| \), and from now on, we let \( m = |E| \).

A flip is an operation that removes an arc from a triangulation \( T \), replaces it by another compatible arc, so results in another triangulation \( T' \). So \( T \) and \( T' \) share all arcs except one. Note that not every arc in a triangulation is flippable; in particular, the folded edge in a self-folded triangle is not flippable. However, there is a finite sequence of flips that transforms any triangulation into one without self-folded triangles, and more generally, any two triangulations can be connected by finitely many flips.

Let the **arc complex** \( \Delta^o(\Sigma_{g,n}) \) be the abstract simplicial complex where a \( k \)-simplex is a collection of \( k \) distinct, mutually compatible arcs in \( \mathcal{A}(\Sigma_{g,n}) \). Thus each vertex is an isotopy class of an arc, and a maximal simplex corresponds to a triangulation \( T \). Its **dual graph** we denote by \( E^o(\Sigma_{g,n}) \). Equivalently, \( E^o(\Sigma_{g,n}) \) is the graph whose vertices are the ideal triangulations of \( \Sigma_{g,n} \) and two vertices are connected if and only if the ideal triangulations are related by a flip. \( \Delta^o(\Sigma_{g,n}) \) is connected in codimension-one, and \( E^o(\Sigma_{g,n}) \) is connected, with each vertex degree at most \( m \).

3.2.2. **Cluster algebra from an ordinary triangulation.** The combinatorial data from an ordinary triangulation can be encoded using a matrix, which we will define using puzzle
Figure 3.1. Three puzzle pieces and their associated matrix minors

Figure 3.2. The fourth puzzle pieces and their associated matrix minors

pieces. Figure 3.1 shows three “puzzle pieces” which are intended to be glued together along their boundary edges in order to construct triangulations of surfaces. Figure 3.2 depicts a triangulation of the four-punctured sphere $\Sigma_{0,4}$, where the exterior of three self-folded triangles is another triangle, which is not drawn in but which should be understood to be a part of the figure. We sometimes refer to the triangulation in Figure 3.2 as a fourth puzzle piece, even though it is not meant to be glued to any other puzzle piece. The matrix associated to the puzzle pieces are also given in Figures 3.1 and 3.2. Notice that there is one row and column for each edge in the puzzle piece, and all four matrices are skew-symmetric.

As shown in [FST08, Section 4], every triangulation $T$ of $\Sigma_{g,n}$ can be obtained from gluing puzzle pieces of the four types depicted in Figures 3.1 and 3.2. Moreover, there is a well-defined exchange matrix $B = B_T = (b_{ij})$ that is the $m \times m$ matrix whose rows and columns are indexed by the edges of the triangulation, constructed as the sum of all minor matrices obtained from some set of puzzle pieces which can be used to construct $T$. Since an edge of a triangulation can be contained in at most two puzzle pieces, the entries of the exchange matrix must satisfy $-2 \leq b_{ij} \leq 2$ for all $i, j$. We refer the reader to [FST08] for details as well as worked examples.

Observe that the exchange matrix $B_T$ is skew-symmetric, since the minor matrices obtained from the puzzle pieces are skew-symmetric. Thus, we may define the seed from the triangulation $T$ to be the pair $(E_T, B_T)$, where $E_T$ is the set of edges of a triangulation $T$ and $B$ is its exchange matrix.
**Proposition 3.3** ([FST08, Proposition 4.8]). Suppose that the \(k\)-th edge of a triangulation \(T\) is flippable, and let \(T'\) be the result of flipping that edge. Then the exchange matrix for \(T'\) is the exchange matrix for \(T\) mutated in the direction \(k\), i.e., \(B_{T'} = \mu_k(B_T)\).

Since any two triangulations of \(\Sigma_{g,n}\) are related by a sequence of flips, seeds from any two triangulations of \(\Sigma_{g,n}\) are related by a sequence of mutations and hence are mutation equivalent. Hence, we have:

**Definition 3.4.** Let the cluster algebra of \(\Sigma_{g,n}\) be defined as \(A(\Sigma_{g,n}) = A(E_T, B_T)\). Then \(A(\Sigma_{g,n})\) is generated by the edges of triangulations of \(\Sigma_{g,n}\) and hence is independent of the initial choice of triangulation \(T\).

However, the arcs are insufficient to describe all cluster variables. In a cluster algebra, we necessarily are able to mutate along every edge of a triangulation, but when the surface \(\Sigma_{g,n}\) admits a triangulation with self-folded triangles, not every edge is flippable. In other words, for some vertex in \(E^\circ(\Sigma_{g,n})\), the degree might be strictly smaller than \(m\), while the exchange graph of \(A(\Sigma_{g,n})\) has to be \(m\)-regular. So, we can only in general say that \(\Delta^\circ(\Sigma_{g,n})\) is a subcomplex of the cluster complex, and \(E^\circ(\Sigma_{g,n})\) is a subgraph of the cluster algebra’s exchange graph. To fill in this gap, Fomin, Shapiro, and Thurston [FST08] introduced a generalization of ordinary arcs which we describe next.

### 3.2.3. Tagged Triangulations.

**Definition 3.5.** A tagged arc \(\alpha\) on \(\Sigma_{g,n}\) is an arc \(\bar{\alpha}\) on \(\Sigma_{g,n}\) along with one of two decorations, plain or notched, at each of the two ends of \(\bar{\alpha}\) such that:

1. \(\bar{\alpha}\) does not cut a one-punctured monogon;
2. if both ends of the arc are at the same vertex, then they have the same decoration.

The ordinary arc \(\bar{\alpha}\) is the underlying arc of the tagged arc \(\alpha\). The decoration of plain or notched at an end of a tagged arc is referred to as the tag at that end, or at the corresponding vertex. The set of isotopy classes of tagged arcs is denoted by \(A^\circ(\Sigma_{g,n})\). Naturally \(A^\circ(\Sigma_{g,n}) \subset A^\infty(\Sigma_{g,n})\).

Many concepts and constructions for arcs can be extended to tagged arcs. Recall that two ordinary arcs are compatible if, up to isotopy, they are either the same or disjoint except at the vertices.

**Definition 3.6.** If tagged arcs \(\alpha\) and \(\beta\) satisfy the following conditions:

1. the underlying arcs are \(\bar{\alpha}\) and \(\bar{\beta}\) are compatible; and
2. in the case that \(\bar{\alpha} = \bar{\beta}\), then \(\alpha\) and \(\beta\) have the same tag on at least one of the shared vertices;
3. in the case that \(\bar{\alpha} \neq \bar{\beta}\) and they share a vertex \(v\), then \(\alpha\) and \(\beta\) have the same tag at \(v\).

then we say that \(\alpha\) and \(\beta\) are compatible.

It follows from the definition that, if \(\alpha\) and \(\beta\) are compatible tagged arcs whose underlying arcs are not the same but share both vertices, then \(\alpha\) and \(\beta\) must have the same tag at
each vertex. For example, on a one-punctured surface, all compatible arcs share a vertex, and hence all ends of compatible arcs must have the same tag.

**Definition 3.7.** A tagged triangulation $T^{\cap}$ is a maximal collection of compatible, distinct tagged arcs.

If we take the arcs of an ordinary triangulation $T$ and tag all of the ends plainly, then we obtain a tagged triangulation. However, the converse is not true; it is possible that the underlying curves of a tagged triangulation $T^{\cap} = \{\alpha_i\}$ do not form an ordinary triangulation of $\Sigma_{g,n}$. In particular, tagged triangulations may cut out bigons as pictured on the right of Figure 3.3. Because such bigons appear often in tagged triangulations, we have the following language for describing them.

**Definition 3.8.** Let $v$ and $w$ be two distinct vertices. A dangle $d^w_v$ is a bigon with vertices at $v$ and $w$ such that its two boundary arcs are compatible and have different tags at the vertex $v$ (Figure 3.3). The jewel of $d^w_v$ is the vertex $v$ with two distinct tags. An envelope of the dangle $d^w_v$ is the boundary $\gamma^w_v$ of a one-punctured monogon that is based at $w$ and such that it encloses the jewel $v$ and has the same tags at $w$ as on $d^w_v$.

![Figure 3.3](image)

**Figure 3.3.** On the left, the envelope $\gamma^w_v$ encircles its jewel $v$. On the right is the corresponding dangle $d^w_v$, with the taggings necessarily distinct at $v$. In this example, both the tags are plain at $w$, but both could be notched at $w$ instead.

Note that, because the two boundary arcs of a dangle $d^w_v$ are compatible and the tags at the jewel $v$ are different, the tags at the remaining vertex $w$ must be both plain or both notched. In a tagged triangulation, the jewel of a dangle cannot be the endpoint of any other edge besides those of the dangle, and thus the degree of the jewel is two.

Let the tagged arc complex $\Delta^{\cap}(\Sigma_{g,n})$ be the abstract simplicial complex generated by compatible distinct tagged arcs in $A^{\cap}(\Sigma_{g,n})$, and let $E^{\cap}(\Sigma_{g,n})$ be the dual graph of $\Delta^{\cap}(\Sigma_{g,n})$. Equivalently, $E^{\cap}(\Sigma_{g,n})$ is the graph whose vertices are the tagged triangulations of $\Sigma_{g,n}$ and two vertices are connected if and only if the tagged triangulations share all but one edge. An edge of $E^{\cap}(\Sigma_{g,n})$ corresponds to a tagged flip, which we think of as an operation that removes one tagged arc from the tagged triangulation and replaces it with a different compatible tagged arc.

**Proposition 3.9.** [FST08, Proposition 7.10] Let $m$ be the number of edges of an ideal triangulation on $\Sigma_{g,n}$.

When $n \geq 2$, $E^{\cap}(\Sigma_{g,n})$ is an $m$-regular, connected graph. Every edge of a tagged triangulation is flippable and any two tagged triangulations is related by a sequence of tagged flips.
When \( n = 1 \), \( E^\infty(\Sigma_{g,n}) \) is an \( m \)-regular graph with two isomorphic connected components, one where all tags are plain and one where all tags are notched.

It follows that \( \Delta^\infty(\Sigma_{g,n}) \) is also connected when there are at least two punctures and has two isomorphic connected components when there is exactly one puncture. Note that in the case of one puncture, each connected component of \( E^\infty(\Sigma_{g,1}) \) is isomorphic to \( E^\circ(\Sigma_{g,1}) \), and each component of the tagged arc complex \( \Delta^\infty(\Sigma_{g,1}) \) is isomorphic to \( \Delta^\circ(\Sigma_{g,1}) \). For simplicity, we will restrict to the component where all tags are plain in the one puncture case for ease of exposition. With this convention, we have that both \( E^\infty(\Sigma_{g,n}) \) and \( \Delta^\infty(\Sigma_{g,n}) \) are connected in all cases.

The relationship between the ordinary set-up and the tagged one can be described by a map \( \tau : A^\circ(\Sigma_{g,n}) \to A^\infty(\Sigma_{g,n}) \), which we will define using the language of dangles and envelopes from Definition 3.8 and Figure 3.3. If \( e \in A^\circ(\Sigma_{g,n}) \) is not an envelope (that is, it does not cut out a once-punctured monogon), then \( \tau(e) \) is \( e \) tagged plain at both ends. If \( e \) is an envelope based at \( w \) and surrounding \( v \), then \( \tau(e) \) is the unique arc enclosed by \( e \) that connects \( v \) and \( w \) and that is notched at \( v \). For example, in Figure 3.3, \( \tau(e) = e \), but \( \tau \) maps the envelope \( \gamma_w^v \) to the tagged arc on the right.

As shown in [FST08, Section 7], \( \tau \) preserves the compatibility of arcs and provides a way of mapping an ordinary triangulation to a tagged triangulation. In this way, we can understand \( \Delta^\circ(\Sigma_{g,n}) \) as a subcomplex of \( \Delta^\infty(\Sigma_{g,n}) \) (though possibly it is not an induced subcomplex), and \( E^\circ(\Sigma_{g,n}) \) as a subgraph of \( E^\infty(\Sigma_{g,n}) \).

To define the exchange matrix of a tagged triangulation, we again use puzzle pieces, as drawn in Figure 3.4. As before, the fourth puzzle piece by itself is a tagged triangulation of the four-punctured sphere \( \Sigma_{0,4} \). Since it does not have any exterior edge, it cannot be glued with any other puzzle pieces.

**Lemma 3.10.** Any tagged triangulation \( T^\infty \) on \( \Sigma_{g,n} \) is obtained by

1. gluing the tagged puzzle pieces along their boundary edges; and
2. tagging all ends of the glued boundary edges in a compatible way.

**Proof.** For the given tagged triangulation \( T^\infty \), we may think it as a top dimensional simplex in \( \Delta^\infty(\Sigma_{g,n}) \). Take a subsimplex \( S^\infty \subset T^\infty \), by eliminating all dangles. Then at each vertex of \( S^\infty \), the adjacent tagged arcs have the same tag.
Pick a region \( R \subset \Sigma_{g,n} \) bounded by arcs in \( S^\infty \). It is sufficient to show that \( R \) is one of the tagged puzzle pieces. \( R \) is bounded by at most three arcs. Otherwise we can refine the triangulation \( T^\infty \) by introducing a new tagged arc dividing the region \( R \), which violates the maximality of \( T^\infty \). There is no inner vertex \( v \) except the other end of dangles, because otherwise we can insert another compatible tagged edge connecting \( v \) and one of the boundary vertices. If \( R \) has \( k \leq 3 \) boundary arcs, then there are \( 3 - k \) dangles in \( R \), by the maximality of \( T^\infty \). Then Figure 3.4 are the remaining possibilities.

Observe that the four tagged puzzle pieces in Figure 3.4 are the images of the four ordinary puzzle pieces in Figures 3.1 and 3.2 under the map \( \tau \). We define the matrix associated to each tagged puzzle piece as the same one associated to its corresponding ordinary puzzle piece. Note when two tagged arcs have the same underlying arc, their corresponding matrix entries are the same.

**Definition 3.11.** Let \( T^\infty \) be a tagged triangulation with \( m \) edges that is made up of tagged puzzle pieces, and let \( E_{T^\infty} \) be the set of its edges. The exchange matrix \( B_{T^\infty} = (b_{ij}) \) is the \( m \times m \) matrix whose rows and columns are indexed by the edges, constructed as the sum of all minor matrices obtained from the puzzle pieces used to construct \( T^\infty \). The seed from the triangulation \( T^\infty \) is the pair \( (E_{T^\infty}, B_{T^\infty}) \).

**Example 3.12.** Consider the tagged triangulation of \( \Sigma_{0,4} \) shown on the left of Figure 4.7. It is obtained from gluing together two puzzle pieces of Type B. Taking \( e_6 = \alpha \), the exchange matrix for the triangulation on the left is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & -1 & 1 \\
-1 & -1 & 1 & 1 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & 0 
\end{bmatrix}
\]

Mutation of \( \alpha \) produces the triangulation on the right of Figure 4.7, which is by itself the Type D puzzle piece. The mutated exchange matrix \( \mu(B) \) is the one from Figure 3.2.

It is a straightforward calculation to check that the exchange matrix for the tagged triangulation obtained from flipping the \( k \)-th edge of \( T^\infty \) is the exchange matrix for \( T^\infty \) mutated in the direction \( k \).

The following theorem, which is the main result of [FST08], summarizes our discussion so far. In the case \(|V| = 1\), recall that we restricted to the case where all tags are plain, so that \( E^\infty(\Sigma_{g,1}) \) is an \( m \)-regular, connected graph in all cases.

**Theorem 3.13 ([FST08, Theorem 7.11]).** Define the cluster algebra \( A(\Sigma_{g,n}) \) using an initial seed coming from any ordinary or tagged triangulation of \( \Sigma_{g,n} \). Then each seed of \( A(\Sigma_{g,n}) \) comes from a tagged triangulation of \( \Sigma_{g,n} \), and mutation of the seed corresponds to tagged flips of the triangulation. In particular, the cluster complex of \( A(\Sigma_{g,n}) \) is the tagged arc complex \( \Delta^\infty(\Sigma_{g,n}) \) and the exchange graph of \( A(\Sigma_{g,n}) \) is the dual graph \( E^\infty(\Sigma_{g,n}) \) .
**Remark 3.14.** As one can see in Figure 3.2 or Example 3.12, the exchange matrix for a punctured surface is not of full rank. Thus, in contrast to the case of surface with boundaries and without punctures, \( \mathcal{A}(\Sigma_{g,n}) \) does not admit a quantum cluster algebra as its deformation quantization ([BZ05, Proposition 3.3]).

4. **THE HOMOMORPHISM \( \rho : \mathcal{A}(\Sigma_{g,n}) \to \mathcal{C}(\Sigma_{g,n}) \)**

In this section, we prove Theorem A, that there is a monomorphism \( \rho : \mathcal{A}(\Sigma_{g,n}) \to \mathcal{C}(\Sigma_{g,n}) \). After describing \( \rho \), we prove in Proposition 4.3 that it is a well-defined algebra homomorphism, and in Proposition 4.6 that it is injective.

**Definition 4.1.** Let \( \alpha \in \mathcal{A}(\Sigma_{g,n}) \) be a tagged arc with endpoints at the vertices \( v, w \in V \) (which are possibly the same). Let

\[
\rho(\alpha) := \begin{cases} 
\alpha, & \text{if both ends of } \alpha \text{ are plain} \\
v\alpha, & \text{if only the end at } v \text{ of } \alpha \text{ is notched} \\
w\alpha, & \text{if only the end at } w \text{ of } \alpha \text{ is notched} \\
vw\alpha, & \text{if both ends of } \alpha \text{ are notched}
\end{cases}
\]

where \( \alpha \) denotes the underlying arc (Definition 3.5).

**Remark 4.2.**
1. When \( v = w \), both ends of \( \alpha \) must have the same decoration (Definition 3.5). So the formula is \( \rho(\alpha) = \alpha \) if both ends are plain, and \( \rho(\alpha) = v^2\alpha \) if both ends are notched.
2. When there is only one puncture, all endpoints of arcs are tagged plainly. So \( \rho(\alpha) = \alpha \) for edges \( \alpha \) in a once-punctured surface.

By introducing a little more notation, we can write the formula for \( \rho \) more compactly. For a tagged arc \( \alpha \) with an endpoint at \( v \in V \), let

\[
t_v(\alpha) := \begin{cases} 
0 & \text{if } \alpha \text{ is decorated plainly at } v, \\
1 & \text{if } \alpha \text{ is decorated notched at } v.
\end{cases}
\]

Then Definition 4.1 becomes

\[
\rho(\alpha) := v^{t_v(\alpha)} w^{t_w(\alpha)} \alpha,
\]

for an edge \( \alpha \) whose endpoints are \( v \) and \( w \).

**Proposition 4.3.** There is a well-defined algebra homomorphism \( \rho : \mathcal{A}(\Sigma_{g,n}) \to \mathcal{C}(\Sigma_{g,n}) \) that extends Definition 4.1.

**Proof.** Recall that \( \mathcal{A}(\Sigma_{g,n}) \) is generated by the edges of all tagged triangulations of \( \Sigma_{g,n} \), subject to the exchange relations determined by the mutations. \( \rho \) is already defined for all edges of tagged triangulations, and we can extend it uniquely to the polynomial subalgebra of \( \mathcal{F} \) freely generated by the edges of all tagged triangulations of \( \Sigma_{g,n} \). We need to show this map preserves the exchange relations coming from tagged flips along any edge of any tagged triangulation.

With that goal in mind, let \( \alpha \) be an arbitrary edge of an arbitrary tagged triangulation \( \mathcal{T}^{\alpha} \). Let the ends of \( \alpha \) be \( v \) and \( w \) (which are possibly the same). By Lemma 3.10, we
may assume that $T^\omega$ was constructed using tagged puzzle pieces. We split our proof into
techniques: when $\alpha$ is in a dangle and when it is not.

**Step 1.** Assume that $\alpha$ is not in a dangle. Then $\alpha$ must be an edge shared by two tagged
puzzle pieces of type A, B, or C as depicted in Figure 3.4. There are ten cases. In each
case, we will check that the exchange relation from flipping $\alpha$ holds in $C(\Sigma_{g,n})$.

We will be applying the following observation repeatedly. If $\alpha$ and $\alpha'$ are two com-
patible arcs forming a dangle with a jewel $v$ (as in Figure 3.3), then $t_v(\alpha) \neq t_v(\alpha')$ and
$t_v(\alpha) + t_v(\alpha') = 1$. But in all other cases, if $\alpha$ and $\alpha'$ are two compatible arcs that have a
common endpoint at $v$, and $v$ is not the jewel of a dangle, then $t_v(\alpha) = t_v(\alpha')$. In particular,
a tagged triangulation determines a single tagging $t_v$ (independent from $\alpha$) for the vertex
$v$, provided $v$ is not the jewel of a dangle in the triangulation.

Case 1. The arc $\alpha$ is the unique common edge of two puzzle pieces of type A.

The two triangles glued along $\alpha$ form a quadrilateral. Say the edges are $e_1, e_2, e_3, e_4$ in
clockwise order, and $e_1$ and $e_4$ are adjacent to $v$. Figure 4.1 describes the configu-
ration of the arcs, but with the tags suppressed at the four vertices. Let $\alpha'$ be the flip of $\alpha$.
We need to check that $\rho$ preserves the exchange relation $\alpha\alpha' = e_1e_3 + e_2e_4$.

![Figure 4.1](image)

**Figure 4.1.** In Case 1, two type A puzzle pieces are glued along exactly one
deedge $\alpha$. The induced exchange relation from flipping $\alpha$ is $\alpha\alpha' = e_1e_3 + e_2e_4$

Although we have not shown the taggings, we know that on the left $t_v := t_v(\alpha) =
t_v(e_1) = t_v(e_3)$ and $t_w := t_w(\alpha) = t_w(e_2) = t_w(e_3)$, and on the right $t_x := t_x(\alpha') = t_x(e_1) =
t_x(e_2)$ and $t_y := t_y(\alpha') = t_y(e_3) = t_y(e_4)$ by our earlier observation about the compatibility
in the absence of dangles.

By definition of $\rho$, we have $\rho(\alpha\alpha') = \rho(\alpha)\rho(\alpha') = v^{t_v}w^{t_w}x^{t_x}y^{t_y}\overline{\alpha}\overline{\alpha}'$. Similarly, $\rho(e_1e_3) =
\rho(e_2e_4) = v^{t_v}w^{t_w}x^{t_x}y^{t_y}e_3e_4$.

In $C(\Sigma_{g,n})$, we have $\overline{\alpha}\overline{\alpha}' = e_1e_3 + e_2e_4$ by the skein relation (1) in Definition 2.1. Thus
$\rho(\alpha\alpha') = v^{t_v}w^{t_w}x^{t_x}y^{t_y}\overline{\alpha}\overline{\alpha}' = v^{t_v}w^{t_w}x^{t_x}y^{t_y}(e_1e_3 + e_2e_4) = \rho(e_1e_3 + e_2e_4)$.

Case 2. The arc $\alpha$ is one of two common edges of two puzzle pieces of type A.

In this case the two triangles form a one-punctured bigon, as in the left of Figure 4.2. Flipping
$\alpha$ produces the figure on the right, with the tags suppressed for simplicity. If both $\alpha$ and $e_2$ are plain at $w$, then flipping $\alpha$ produces $\alpha'$ notched at $w$ while $e_2$ remains
plain at $w$, as depicted in Figure 4.2. But if both $\alpha$ and $e_2$ are notched at $w$, then flipping
$\alpha$ produces $\alpha'$ plain at $w$ while $e_2$ remains notched at $w$. The taggings at $v$ and $x$ are
unchanged by the flip.
Since the tags are all the same at \( v \), we denote the tagging of any arc ending at \( v \) simply by \( t_v \), and similarly we use \( t_x \) for \( x \). At \( w \), we have \( t_w(\alpha) = t_w(e_2) \), but \( t_w(e_2) \neq t_w(\alpha') \) and \( t_v(\alpha) + t_v(\alpha') = 1 \). So \( \rho(\alpha\alpha') = wv^tx^sz\alpha\alpha' \). Furthermore, note that \( \alpha' = e_2 \), and by the puncture-skein relation in Definition 2.1, we have \( w\alpha\rho e_2 = e_1 + e_3 \). Thus,

\[
\rho(\alpha\alpha') = wv^tx^sz\alpha\rho e_2 = v^tx^s(e_1 + e_3) = v^tx^se_1 + v^tx^se_3 = \rho(e_1 + e_3).
\]

**Case 3.** The arc \( \alpha \) is one of three common edges of two puzzle pieces of type A.

In this case, \( \Sigma_{g,n} = \Sigma_{0,3} \), which is excluded by assumption.

**Case 4.** The arc \( \alpha \) is the unique common edge of two puzzle pieces of type A and B.

The result of gluing the two puzzle pieces is shown in Figure 4.3.

Again by compatibility, we denote the tagging of any arc ending at \( v \), \( w \), and \( x \) by \( t_v \), \( t_w \), and \( t_x \), respectively. Also, exactly one of \( e_4 \) and \( e_5 \) is notched at \( y \). Thus \( \rho(\alpha\alpha') = v^tv^w2t_wx^s\alpha\alpha', \rho(e_1e_4e_5) = v^tv^tx^sz2t_wye_1e_4e_5 \), and \( \rho(e_2e_3) = v^tv^w2t_wx^se_2e_3 \).

In \( C(\Sigma_{g,n}) \), application of a skein relation implies \( \alpha\alpha' = e_1\gamma_y^w + e_2e_3 \), where \( \gamma_y^w \) is the envelope of the dangle \( d_y^w \) (Definition 3.8). Lemma 2.5 further shows \( \gamma_y^w = ye_4^2 \), and since the underlying curves of \( e_4 \) and \( e_5 \) are the same, in fact \( \gamma_y^w = ye_4e_5 \). It follows that

\[
\rho(\alpha\alpha') = v^tv^w2t_wx^s\alpha\alpha' = v^tv^w2t_wx^se_1ye_4e_5 + e_2e_3 = \rho(e_1e_4e_5 + e_2e_3).
\]

**Case 5.** The arc \( \alpha \) is one of two common edges of two puzzle pieces of type A and B. Figure 4.4 shows the two puzzle pieces glued along \( \alpha \).
COMPATIBILITY OF SKEIN ALGEBRA AND CLUSTER ALGEBRA

Figure 4.4. In Case 5, a type A puzzle is glued to a type B puzzle along two edges, and we flip the one labeled $\alpha$. The exchange relation is $\alpha \alpha' = e_1 + e_3 e_4$.

Note that $\alpha$ and $\alpha'$ have different tags at $v$, and $e_3$ and $e_4$ have different tags at $x$. The puncture-skein relation and Lemma 2.5 imply that $v \alpha \alpha' = e_1 + x e_3^2$. Since $e_3 = e_4$, it follows that $\rho(\alpha \alpha') = w^2 w^2 \alpha \alpha' = w^2 w^2 e_1 + x w^2 w^2 e_3 e_4 = \rho(e_1 + e_3 e_4)$.

Case 6. The arc $\alpha$ is the common edge of two puzzle pieces of type A and C. Figure 4.5 shows the two puzzle pieces. Similarly to the previous cases,

$$
\rho(\alpha \alpha') = w^2 w^2 z^2 \alpha \alpha' = w^2 w^2 z^2 (e_1 e_2 + e_2 e_5 e_6) = w^2 w^2 z^2 (e_1 e_2 + e_2 e_5 e_6) = \rho(e_1 e_3 e_4 + e_2 e_5 e_6).
$$

Figure 4.5. In Case 6, a type A puzzle piece is glued to a type C puzzle piece along exactly one arc $\alpha$. The exchange relation is $\alpha \alpha' = e_1 e_3 e_4 + e_2 e_5 e_6$.

Case 7. The arc $\alpha$ is the common edge of two puzzle pieces of type B.

There are two possibilities. The first one is identical to the right figure in Figure 4.5, but where $\alpha'$ plays the role of $\alpha$. The exchange relation is the same as in Case 6, since the cluster mutation is involutive. Thus the argument from Case 6 applies in this case.

The second possibility is the one shown in Figure 4.6. Then

$$
\rho(\alpha \alpha') = v^2 v^2 w^2 \alpha \alpha' = v^2 v^2 w^2 (e_1 e_2 + \gamma x \gamma w) = v^2 v^2 w^2 (e_1 e_2 + (x e_3^2)(y e_6^2))
$$

$$
= v^2 v^2 w^2 (e_1 e_2 + x y e_3 e_4 e_5 e_6) = \rho(e_1 e_2 + e_3 e_4 e_5 e_6).
$$

Case 8. The arc $\alpha$ is one of two common edges of two puzzle pieces of type B.

Two puzzle pieces glue together to produce a triangulation for $\Sigma_{0,4}$. We distinguish between two subcases, as depicted in Figures 4.7 and 4.8.
In subcase I shown in Figure 4.7, we have
\[ \rho(\alpha \alpha') = v w^{2t_w} \alpha \alpha' = w^{2t_w} \left( e_1 - e_2 + e_3 e_4 \right) = \rho(e_1 e_2 + e_3 e_4). \]

Note that \( \alpha \alpha' = \gamma_x^w + \gamma_y^w \) because it is on \( \Sigma_{0,4} \). In subcase II shown in Figure 4.8, we have
\[ \rho(\alpha \alpha') = v^{2t_w} w^{2t_w} \alpha \alpha' = v^{2t_w} w^{2t_w} \left( e_1 - e_2 + e_3 e_4 \right) = v^{2t_w} w^{2t_w} \left( e_1 e_2 + e_3 e_4 \right) = \rho(e_1 e_2 + e_3 e_4 + e_5^2). \]

---

**Figure 4.6.** In Case 7, two puzzle pieces of type B are glued along \( \alpha \). In one xe depicted here, the exchange relation is \( \alpha \alpha' = e_1 e_2 + e_3 e_4 e_5 e_6 \).

**Figure 4.7.** In subcase I of Case 8, two puzzle pieces are glued to produce a triangulation for \( \Sigma_{0,4} \), and the exchange relation from flipping \( \alpha \) is \( \alpha \alpha' = e_1 e_2 + e_3 e_4 \).

**Figure 4.8.** In subcase II of Case 8, again a triangulation for \( \Sigma_{0,4} \) is obtained, and the exchange relation is \( \alpha \alpha' = e_1 e_2 + e_3 e_4 e_5 \). The result of the flip is again a union of two puzzle pieces of type B.

**Case 9.** The arc \( \alpha \) is the common edge of two puzzle pieces of type B and C.

See Figure 4.9. We have
\[ \rho(\alpha \alpha') = w^{4t_w} \alpha \alpha' = w^{4t_w} \left( e_1 - e_2 + e_3 e_4 e_5 \right) = \rho(e_1 e_2 + e_3 e_4 e_7 + e_1 e_4 e_5). \]

**Case 10.** The arc \( \alpha \) is the common edge of two puzzle pieces of type C.

In this situation, the surface must be \( \Sigma_{0,5} \). See Figure 4.10. Then
\[ \rho(\alpha \alpha') = v^{4t_w} \alpha \alpha' = v^{4t_w} \left( e_1 - e_2 + e_3 e_4 e_6 \right) = \rho(e_1 e_2 + e_3 e_4 e_7 + e_1 e_4 e_8). \]

---

**Step 2.** Suppose that \( \alpha \) is on a dangle.
Figure 4.9. In Case 9, two puzzle pieces of type B and C are glued along only one edge $\alpha$. The exchange relation is $\alpha \alpha' = e_1 e_4 e_5 + e_2 e_3 e_6 e_7$.

Figure 4.10. In Case 10, two puzzle pieces of type C are glued along two edges, and the one labeled $\alpha$ is flipped. The exchange relation is $\alpha \alpha' = e_1 e_2 e_5 e_6 + e_3 e_4 e_7 e_8$.

Any dangle must be contained inside one of the tagged puzzle pieces in Figure 3.4. Suppose first that $\alpha$ is notched at the jewel. The mutation of $\alpha$ in a puzzle of type B is the inverse of the flip described in Case 2 and Figure 4.2 (and $\alpha'$ in Figure 4.2 plays the role of $\alpha$). Since the mutation is an involution, the compatibility follows from Case 2. In the case of a puzzle of type C, the mutation is the inverse of the flip in Case 5 and Figure 4.4. In the case of type D, it is the inverse of the flip in subcase I of Case 8 and Figure 4.7. This takes care of all situations where $\alpha$ is on a dangle. If $\alpha$ is tagged plainly at the jewel, then the only difference is that, in the flipped diagram, one needs to change the tagging at the vertex which was the jewel. The rest of the computation is identical. □

Remark 4.4. By tensoring a commutative ring $R$, we obtain

$$\rho_R : A(\Sigma_{g,n})_R \to C(\Sigma_{g,n})_R.$$

We complete the proof of Theorem A by showing that $\rho$ is injective. Indeed, we will show that for any integral domain $R$, $\rho_R$ in Remark 4.4 is injective.

Roger-Yang’s homomorphism $\Phi : C(\Sigma_{g,n}) \to C^\infty(\mathcal{T}^d(\Sigma_{g,n}))$ will factor in our proofs coming up. We here present a slightly different version that we find easier to apply. See [MW21, Section 3] for details.

Lemma 4.5. Let $R$ be an integral domain. Suppose $\mathcal{T}$ is an ideal triangulation of $\Sigma_{g,n}$, and let $E = \{e_i\}_{i=1}^m$ denote its set of edges. Then there is a well-defined homomorphism $\hat{\Phi}_R : C(\Sigma_{g,n})_R \to Q(R)(e_i)$, where $Q(R)$ is the field of fraction of $R$.

Proof. We first consider $R = \mathbb{Z}$ case. The map $\Phi$ sends each arc $e_i$ in $\mathcal{T}$ to the function $\lambda_i$ on the decorated Teichmüller space $\mathcal{T}^d(\Sigma_{g,n})$ that gives the lambda-length of $e_i$. It follows by
[MW21, Lemma 3.3] that \( \Phi \) factors through \( \Phi : \mathcal{C}(\Sigma_{g,n}) \to \mathbb{Z}[\lambda_i^\pm] \). By tensoring a general integral domain \( R \), we obtain a similar map \( \Phi_R : \mathcal{C}(\Sigma_{g,n})_R \to R[\lambda_i^\pm] \).

The decorated Teichmüller space \( T^d(\Sigma_{g,n}) \) is homeomorphic to \( \mathbb{R}^m_{>0} \) and the homeomorphism maps each decorated hyperbolic metric \( (\ell, \tau) \) to the lambda-lengths \( \{\lambda_i\} \) of \( \{e_i\} \) ([Pen87, Theorem 3.1]). Thus, \( T^d(\Sigma_{g,n}) \) is a Zariski-dense semialgebraic set in an \( n \)-dimensional complex torus \( \text{Spec} \mathbb{C}[\lambda_i^\pm] \cong (\mathbb{C}^*)^m \). Therefore, \( \{\lambda_i\} \) is a set of algebraically independent elements. Hence there is a well-defined, canonical isomorphism \( \tau : \mathbb{Z}[\lambda_i^\pm] \cong \mathbb{Z}[e_i^\pm] \) that maps \( \lambda_i \) to \( e_i \). By tensoring \( R \), we obtain \( \tau_R : R[\lambda_i^\pm] \cong R[e_i^\pm] \). Then composition of \( \tau_R \) and \( \Phi_R \) followed by the canonical inclusion \( R[e_i^\pm] \subset Q(R)(e_i) \) yields \( \hat{\Phi}_R \).

**Proposition 4.6.** Let \( R \) be an integral domain. The algebra homomorphism \( \rho_R : \mathcal{A}(\Sigma_{g,n})_R \to \mathcal{C}(\Sigma_{g,n})_R \) is injective.

**Proof.** We fix an ordinary triangulation \( \mathcal{T} \) on \( \Sigma_{g,n} \). Let \( E = \{e_i\} \) be the set of edges in \( \mathcal{T} \). There is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{A}(\Sigma_{g,n})_R & \xrightarrow{\rho_R} & \mathcal{C}(\Sigma_{g,n})_R \\
\iota \downarrow & & \Phi_R \\
Q(R)(e_i) & & \\
\end{array}
\]

Here \( \iota \) is the natural inclusion of the cluster algebra \( \mathcal{A}(\Sigma_{g,n})_R \) into its field of fraction, and \( \rho_R \) is the homomorphism in Remark 4.4. The map \( \hat{\Phi}_R \) is the Roger-Yang homomorphism from Lemma 4.5. For each \( e_i \), we have \( \iota(e_i) = \hat{\Phi}_R \circ \rho(e_i) \). It follows that \( \iota = \hat{\Phi}_R \circ \rho_R \), since all of the elements of \( \mathcal{A}(\Sigma_{g,n})_R \) can be written as a Laurent polynomial with respect to the cluster variables \( e_i \) in a fixed cluster. Since \( \iota \) is injective, \( \rho_R \) must also be injective. \( \square \)

5. **Integrality of \( \mathcal{C}(\Sigma_{g,n}) \) and its implications**

This section is mainly devoted to a proof of integrality of \( \mathcal{C}(\Sigma_{g,n}) \) using the injective homomorphism \( \rho : \mathcal{A}(\Sigma_{g,n}) \to \mathcal{C}(\Sigma_{g,n}) \) and techniques from algebraic geometry, in particular dimension theory. For the definition and basic properties of the dimension of algebraic varieties, see [Eis95, Section 8]. We will need the following lemma from commutative algebra.

Let \( k \) be a field and let \( R \) be a \( k \)-algebra, which is an integral domain. The (Krull) dimension \( \dim R \) of \( R \) is the maximal length \( \ell \) of the strictly increasing chain of prime ideals \( 0 = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_\ell \) of \( R \). For the associated affine scheme \( \text{Spec} R \), its dimension is defined as \( \dim \text{Spec} R = \dim R \).

**Lemma 5.1.** Let \( k \) be a field and let \( R \) be a \( k \)-algebra, which is an integral domain. Let \( Q(R) \) be its field of fractions. Suppose that the transcendental degree \( \text{trdeg}_k Q(R) \) of \( Q(R) \) is \( m \). Then \( \dim \hat{R} \leq m \).

**Proof.** When \( R \) is a finitely generated algebra, the statement is well known ([Eis95, Theorem A, p.221]). We assume that \( R \) is not finitely generated.
Take a chain of prime ideals $0 = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_\ell$ of $R$. For each $1 \leq i \leq \ell$, pick $x_i \in P_i \setminus P_{i-1}$. Let $R'$ be the subalgebra of $R$ generated by $\{x_i\}$, and $Q(R') \subset Q(R)$ be its field of fractions. Since $R'$ is a finitely generated algebra, $\dim R' \leq \text{trdeg}_Q Q(R') \leq \text{trdeg}_R Q(R) = m$.

On the other hand, if we set $P'_i = P_i \cap R'$, the sequence $P'_0 \subset P'_1 \subset P'_2 \subset \cdots \subset P'_k$ is an increasing sequence of prime ideals, and it is strictly increasing as $x_i \in P'_i \setminus P'_{i-1}$. Therefore, $\dim R' \geq \ell$, so we have $\ell \leq m$. This is valid for arbitrary increasing chains of prime ideals, we obtain the desired result. 

We are now ready for the proof of integrality.

**Theorem 5.2.** Suppose that $\chi(\Sigma_{g,n}) = 2 - 2g - n < 0$ and $n > 0$. Then $\mathcal{C}(\Sigma_{g,n})$ is an integral domain.

**Proof.** To start, assume that $\Sigma_{g,n}$ is not a 3-puncture sphere, so that $A(\Sigma_{g,n})$ is defined. As before, we fix an ordinary ideal triangulation $\mathcal{T}$ on $\Sigma_{g,n}$, and let $E = \{e_i\}$ denote the edges of the triangulation.

By [MW21, Lemma 3.2], every element in $\mathcal{C}(\Sigma_{g,n})$ can be written as a rational function (indeed a Laurent polynomial) with respect to the edge classes $\{e_i\}$ in $\mathcal{T}$. In particular, for any $x \in \mathcal{C}(\Sigma_{g,n}) \setminus \rho(A(\Sigma_{g,n}))$, there is a rational function $f(e_i)/g(e_i)$ with respect to $\{e_i\}$, such that $x = f(e_i)/g(e_i)$. Then we can construct a ring extension $A(\Sigma_{g,n})' := A(\Sigma_{g,n})[t]/(t-f/g)$ and an extended homomorphism $\rho' : A(\Sigma_{g,n})' \to \mathcal{C}(\Sigma_{g,n})$, which maps $t \mapsto x$. Since $t = f/g \in Q(e_i)$, $A(\Sigma_{g,n})'$ is also a subring of $Q(e_i)$. We may repeat this procedure and extend the algebra $A(\Sigma_{g,n})'$ further, until the extended map is surjective. Since $\mathcal{C}(\Sigma_{g,n})$ is a finitely generated algebra (Theorem 2.8), this procedure is terminated in finitely many steps. Therefore, we obtain a ring extension $\widetilde{A}(\Sigma_{g,n})$ of $A(\Sigma_{g,n})$ in $Q(e_i)$ and a surjective homomorphism $\tilde{\rho} : \widetilde{A}(\Sigma_{g,n}) \to \mathcal{C}(\Sigma_{g,n})$.

Combined with the Roger-Yang homomorphism from Lemma 4.5 (with $\mathbb{Z}$-coefficient), we have the commutative diagram

$$
\begin{array}{ccc}
A(\Sigma_{g,n}) & \xrightarrow{\rho} & \mathcal{C}(\Sigma_{g,n}) \\
\downarrow & & \downarrow \Phi \\
\widetilde{A}(\Sigma_{g,n}) & \xrightarrow{\tilde{\rho}} & Q(e_i). \\
\end{array}
$$

Note that $\widetilde{A}(\Sigma_{g,n})$ is an integral domain, as it is a subring of $Q(e_i)$. So is $\widetilde{A}(\Sigma_{g,n})_C := \widetilde{A}(\Sigma_{g,n}) \otimes_\mathbb{Z} \mathbb{C} \subset Q(e_i) \otimes_\mathbb{Z} \mathbb{C} = \mathbb{C}(e_i)$. So the associated affine scheme $\widetilde{A}(\Sigma_{g,n})_C$ is integral (irreducible and reduced). If we denote the number of edges in $\mathcal{T}$ by $m$, then $\text{trdeg}_C (\mathbb{C}(e_i)) = m$. Since the field of fractions of $\widetilde{A}(\Sigma_{g,n})_C$ is also $\mathbb{C}(e_i)$, by Lemma 5.1, $\dim \text{Spec} \widetilde{A}(\Sigma_{g,n})_C \leq m$.

Since $\tilde{\rho} : \widetilde{A}(\Sigma_{g,n}) \to \mathcal{C}(\Sigma_{g,n})$ is a surjective homomorphism, so is $\tilde{\rho}_C : \widetilde{A}(\Sigma_{g,n})_C := \widetilde{A}(\Sigma_{g,n}) \otimes_\mathbb{Z} \mathbb{C} \to \mathcal{C}(\Sigma_{g,n})_C$. If we denote $\ker \tilde{\rho}_C = I$, then $\widetilde{A}(\Sigma_{g,n})_C/I \cong \mathcal{C}(\Sigma_{g,n})_C$. Then
Spec $\mathcal{C}(\Sigma_{g,n})_\mathbb{C}$ is a closed subscheme of $\text{Spec } \tilde{\mathcal{A}}(\Sigma_{g,n})_\mathbb{C}$, defined by the ideal $I$. Thus

$$\dim \text{Spec } \mathcal{C}(\Sigma_{g,n})_\mathbb{C} \leq \dim \text{Spec } \tilde{\mathcal{A}}(\Sigma_{g,n})_\mathbb{C} \leq m$$

and if $I$ is nontrivial, then $\dim \text{Spec } \mathcal{C}(\Sigma_{g,n})_\mathbb{C} < \dim \text{Spec } \tilde{\mathcal{A}}(\Sigma_{g,n})_\mathbb{C}$.

Recall from the proof of Lemma 4.5 that $\Phi_i : \mathcal{C}(\Sigma_{g,n})_\mathbb{C} \to \mathbb{C}(e_i)$ is a composition of the map $\mathcal{C}(\Sigma_{g,n})_\mathbb{C} \to \mathbb{C}[\lambda_i^\pm] \to \mathbb{C}[e_i^\pm] \subset \mathbb{C}(e_i)$. Every element in $\mathcal{C}(\Sigma_{g,n})_\mathbb{C}$ can be written as a Laurent polynomial with respect to $\{e_i\}$ ([MW21, Lemma 3.2]), so if we denote $S$ by the multiplicative set of monomials with respect to $\{e_i\}$, then there is a localized morphism $S^{-1}\mathcal{C}(\Sigma_{g,n})_\mathbb{C} \to \mathbb{C}[\lambda_i^\pm]$, which turns out to be an isomorphism ([MW21, Lemma 3.4]). The localization of a ring corresponds to taking an open subset of the associated scheme. Thus $\text{Spec } \mathcal{C}(\Sigma_{g,n})_\mathbb{C}$ has a (Zariski) open subset $\text{Spec } S^{-1}\mathcal{C}(\Sigma_{g,n})_\mathbb{C} \cong \text{Spec } \mathbb{C}[\lambda_i^\pm] \cong \text{Spec } \mathbb{C}[e_i^\pm] \cong (\mathbb{C}^*)^m$. In particular, $\text{Spec } \mathcal{C}(\Sigma_{g,n})_\mathbb{C}$ has an irreducible component, which has an open dense subset isomorphic to the algebraic torus of dimension $m$. Therefore, $\dim \text{Spec } \mathcal{C}(\Sigma_{g,n})_\mathbb{C} \geq m$.

The only possibility is $\dim \text{Spec } \mathcal{C}(\Sigma_{g,n})_\mathbb{C} = m$ and $\ker \tilde{\rho} = I$ is the trivial ideal. Therefore $\mathcal{C}(\Sigma_{g,n})_\mathbb{C} \cong \tilde{\mathcal{A}}(\Sigma_{g,n})_\mathbb{C}$ and hence is an integral domain. Since $\mathcal{C}(\Sigma_{g,n})_\mathbb{C}$ has no torsion (Remark 2.3), $\mathcal{C}(\Sigma_{g,n})_\mathbb{C} \subset \mathcal{C}(\Sigma_{g,n})_\mathbb{C}$ and it is also an integral domain.

Now the only remaining case is $\Sigma_{0,3}$ where $\mathcal{A}(\Sigma_{0,3})$ is undefined. But we may formally set $\mathcal{A}(\Sigma_{0,3}) = \mathbb{Z}[e_i]_{1 \leq i \leq 3}$ and define $\rho : \mathcal{A}(\Sigma_{0,3}) \to \mathcal{C}(\Sigma_{0,3})$ as $\rho(e_i) = \beta_{i,i+1}$ (see Theorem 2.9 for the notation). Then we can follow the same line of the proof to get the same conclusion.

\begin{proof}[Rem 5.3] If $\chi(\Sigma_{g,n}) \geq 0$ (so $g = 0$ and $n = 1, 2$), $\mathcal{C}(\Sigma_{g,n})_\mathbb{C}$ is no longer an integral domain ([ACDHM21, Remark 6.3]).
\end{proof}

\begin{proof}[Proofs of Theorem B and Theorem C] Theorem A of [MW21] states that if $\mathcal{C}(\Sigma_{g,n})_\mathbb{C}$ is an integral domain, $\Phi$ must be injective. Thus, we obtain Theorem B. Theorem C about the integrality of $S^u(\Sigma)$ immediately follows from the integrality of $\mathcal{C}(\Sigma_{g,n})_\mathbb{C}$ by Theorem C of [MW21].
\end{proof}

6. Implications for $\mathcal{A}(\Sigma_{g,n})$

The compatibility of the curve algebra $\mathcal{C}(\Sigma_{g,n})_\mathbb{C}$ and cluster algebra $\mathcal{A}(\Sigma_{g,n})_\mathbb{C}$ provides us new insight to some questions on the structure of cluster algebras. In this section, we investigate two questions regarding the finite generation of $\mathcal{A}(\Sigma_{g,n})$ (Theorem D) and the comparison of $\mathcal{A}(\Sigma_{g,n})$ with $\mathcal{U}(\Sigma_{g,n})$ (Theorem E). We still assume that $\chi(\Sigma_{g,n}) < 0$.

6.1. Non-finite generation for $g \geq 1$. It was observed in [Lad13, Proposition 1.3], following [Mul13, Proposition 11.3], that $\mathcal{A}(\Sigma_{g,1})$ is not finitely generated for all $g \geq 1$. It is plausible to believe that $\mathcal{A}(\Sigma_{g,n})_\mathbb{C}$ is more complicated than $\mathcal{A}(\Sigma_{g,1})$. Thus one may guess that $\mathcal{A}(\Sigma_{g,n})_\mathbb{C}$ is not finitely generated for all $n$. However, the lack of a functorial morphism $\mathcal{A}(\Sigma_{g,n})_\mathbb{C} \to \mathcal{A}(\Sigma_{g,1})$ makes it difficult to prove the non-finite generation of $\mathcal{A}(\Sigma_{g,n})_\mathbb{C}$ in general. We suggest a new approach to resolve this issue, using invariant theory and ‘mod 2 reduction.’
The first key technical ingredient is Nagata’s theorem ([Dol03, Theorem 3.3]) and its extension to arbitrary base ring by Seshadri ([Ses77]). For a finitely generated $k$-algebra $A$, it is not true that its subalgebra $B \subset A$ is finitely generated. However, if $A$ is equipped with a reductive group $G$-action, Nagata’s theorem tells us that the invariant subalgebra $A^G$ is finitely generated. For our purpose, the following consequence of the Seshadri-Nagata’s theorem is handy.

**Lemma 6.1.** Let $k$ be a field. Let $A$ be a finitely generated $\mathbb{Z}^r$-graded $k$-algebra, so $A \cong \bigoplus_{a \in \mathbb{Z}^r} A_a$ such that $A_a A_b \subset A_{a+b}$. Then $A_0$ is finitely generated.

**Proof.** Recall that an affine group scheme $G_{m} := \text{Spec } k[x_i^{\pm}]_{1 \leq i \leq r}$-action on $\text{Spec } A$ is given by a $k$-linear map $A \to A \otimes_k k[x_i^{\pm}]$, which makes $A$ as a comodule under the coalgebra $k[x_i^{\pm}]$. We may set $A_a := \{ r \in A \mid r \mapsto r \otimes \prod x_i^{a_i} \}$. Then it is straightforward to check that the above coalgebra structure is equivalent to a $\mathbb{Z}^r$-grading structure on $A$. Now $A_0 \cong A^{G_m}$, which is finitely generated by [Ses77, Remark 4, p.242].

**Remark 6.2.** The group action in the proof of Lemma 6.1 should be understood as an affine group scheme action, not a set-theoretic one. We will consider the $k = \mathbb{Z}_2$ case. But then the set of $\mathbb{Z}_2$-valued points of $G_{m}^{r} = \text{Spec } \mathbb{Z}_2[x_i^{\pm}]$ has only one point $(1, 1, \cdots, 1)$. Thus, set-theoretically, it is a trivial group.

**Remark 6.3.** Primarily, we will use the contrapositive of Lemma 6.1. If $A_0$ is not finitely generated, then $A$ is not finitely generated.

**Proposition 6.4.** Let $R$ be an integral domain. Then $\mathcal{A}(\Sigma_{g,n})_R$ and $\mathcal{C}(\Sigma_{g,n})_R$ have $\mathbb{Z}^n$-graded ring structure.

**Proof.** We may impose $\mathcal{C}(\Sigma_{g,n})_R$ as a $\mathbb{Z}^n$-graded algebra structure in the following way. Let $V = \{v_i\}$ be the vertex set. For an arc $\alpha$ connecting $v_i$ and another vertex $v_j$ (we allow $i = j$), the grade of $\alpha$ is defined as $e_i + e_j$, where $\{e_i\}$ is the standard basis of $\mathbb{Z}^n$. For any loop, its grade is 0. Finally, the grade of the vertex class $v_i$ is $-2e_i$ (hence the grade of $v_i^{-1}$ is $2e_i$). It is straightforward to check that all skein relations in Definition 2.1 are homogeneous. Thus, it is well-defined.

Since $\mathcal{A}(\Sigma_{g,n})_R \subset \mathcal{C}(\Sigma_{g,n})_R$ is generated by homogeneous elements, $\mathcal{A}(\Sigma_{g,n})_R$ is also a $\mathbb{Z}^n$-graded algebra. □

**Remark 6.5.** For each vertex $v_i$, we may impose a $\mathbb{Z}$-graded algebra structure on $\mathcal{C}(\Sigma_{g,n})_R$ (and on $\mathcal{A}(\Sigma_{g,n})_R$), by composing the grade map with the $i$-th projection $p_i : \mathbb{Z}^n \to \mathbb{Z}$.

The following proposition is proved by Ladkani in [Lad13, Proposition 1.3], over $\mathbb{Z}$ coefficients. The same proof works for arbitrary base ring, but we provide a sketch for the sake of completeness.
Proposition 6.6 (Ladkani). For any integral domain \( R \) and \( g \geq 1 \), \( \mathcal{A}(\Sigma_{g,1})_R \) is not finitely generated.

Proof. By definition when \( n = 1 \), \( \mathcal{A}(\Sigma_{g,1})_R \) is generated by ordinary arcs only, and all exchange relations are homogeneous of degree two with respect to the \( \mathbb{Z} \)-grading in Proposition 6.4. Therefore, a cluster variable cannot be expressed as a polynomial with respect to the other cluster variables. On the other hand, there are infinitely many non-isotopic arc classes on \( \Sigma_{g,1} \), so there are infinitely many cluster variables. Thus, \( \mathcal{A}(\Sigma_{g,1})_R \) cannot be finitely generated. \( \square \)

Proof of (3) of Theorem D. We think of \( \mathcal{A}(\Sigma_{g,n}) \) as a subalgebra of \( \mathcal{C}(\Sigma_{g,n}) \). Thus, instead of arcs and tagged arcs, we will describe all elements as a combination of arcs and vertices.

First of all, observe that to show the non-finite generation of \( \mathcal{A}(\Sigma_{g,n}) \), it is sufficient to show that \( \mathcal{A}(\Sigma_{g,n})_{\mathbb{Z}_2} \) is not finitely generated, as there is a surjective morphism \( \mathcal{A}(\Sigma_{g,n})_\mathbb{Z} \to \mathcal{A}(\Sigma_{g,n})_{\mathbb{Z}_2} \).

We construct a morphism between curve algebras induced from \( \iota : \Sigma_{g,n+1} \to \Sigma_{g,n} \) which forgets a vertex \( v \). With respect to \( v \), note that \( \mathcal{C}(\Sigma_{g,n+1})_R \) and \( \mathcal{A}(\Sigma_{g,n+1})_R \) have a \( \mathbb{Z} \)-graded structure (Remark 6.5). Let \( \mathcal{C}(\Sigma_{g,n+1})_{R,0} \) be the grade 0 subalgebra of \( \mathcal{C}(\Sigma_{g,n+1})_R \).

We claim that when \( R = \mathbb{Z}_2 \), there is a well-defined surjective homomorphism \( \psi : \mathcal{C}(\Sigma_{g,n+1})_{\mathbb{Z}_2,0} \to \mathcal{C}(\Sigma_{g,n})_{\mathbb{Z}_2} \). Indeed, \( \mathcal{C}(\Sigma_{g,n+1})_{\mathbb{Z}_2,0} \) is generated by the following elements:

1. vertex classes \( v_1^+, v_2^+ \cdots v_n^+ \) (except \( v \));
2. loop classes;
3. tagged arcs disjoint from \( v \);
4. \( v\alpha_1\alpha_2 \) where each \( \alpha_1, \alpha_2 \) are arcs connecting \( v \) with other vertices;
5. \( v\beta \) where \( \beta \) is an arc connecting \( v \) and itself.

For each case, by applying a puncture-skein relation, we can find a representative which is disjoint from \( v \). For (1), (2), and (3), this is clear. For (4), by the puncture-skein relation, we can resolve the crossing of \( v\alpha_1\alpha_2 \) to get the sum of two arcs disjoint from \( v \), which we call \( \gamma_1 \) and \( \gamma_2 \). Now if we forget \( v \), then as isotopy classes on \( \Sigma_{g,n} \), we have \( \gamma_1 = \gamma_2 \). Thus \( v\alpha_1\alpha_2 = \gamma_1 + \gamma_2 = 2\gamma_1 = 0 \in \mathcal{C}(\Sigma_{g,n})_{\mathbb{Z}_2} \). The case of (5) is similar. Since we only used the puncture-skein relation, the map \( \psi \) is well-defined. The surjectivity is immediate.

By composition, we obtain a map

\[ \mathcal{A}(\Sigma_{g,n+1})_{\mathbb{Z}_2,0} \to \mathcal{C}(\Sigma_{g,n+1})_{\mathbb{Z}_2,0} \xrightarrow{\psi} \mathcal{C}(\Sigma_{g,n})_{\mathbb{Z}_2} \].

The cluster algebra \( \mathcal{A}(\Sigma_{g,n+1})_{\mathbb{Z}_2,0} \) is generated by multiples of tagged arcs, and the image of them by the map \( \psi \) is still a multiple of tagged arcs on \( \Sigma_{g,n} \). The only exception is a multiple of \( v\beta \), where \( \beta \) is an arc whose both ends are \( v \). (Note that two ends of \( \beta \in \mathcal{A}(\Sigma_{g,n+1}) \), whose underlying curve is \( \beta \), must be tagged in the same way, so \( \beta = \beta \) or \( \beta = v^2\beta \).) In this case, after applying the puncture-skein relation at the endpoint of \( \beta \), \( v\beta \) becomes a multiple of the sum of two loops \( \ell_1 \) and \( \ell_2 \). Once we forget the vertex, then in \( \mathcal{C}(\Sigma_{g,n})_{\mathbb{Z}_2} \) we have \( \ell_1 + \ell_2 = 2\ell_1 = 0 \). In summary, the image of \( \mathcal{A}(\Sigma_{g,n+1})_{\mathbb{Z}_2,0} \) by \( \psi \) is...
still tagged arcs on $\Sigma_{g,n}$. Therefore, if $n \geq 2$, the image is in $A(\Sigma_{g,n})_{\mathbb{Z}_2}$, and we have a morphism $\psi : A(\Sigma_{g,n+1})_{\mathbb{Z}_2,0} \to A(\Sigma_{g,n})_{\mathbb{Z}_2}$.

**Remark 6.7.** On the other hand, when $n = 1$, $A(\Sigma_{g,1})_{\mathbb{Z}_2}$ is generated by ordinary arcs only. Thus $\psi : A(\Sigma_{g,2})_{\mathbb{Z}_2,0} \to C(\Sigma_{g,1})_{\mathbb{Z}_2}$ does not factor through $A(\Sigma_{g,1})_{\mathbb{Z}_2}$ in general.

It is straightforward to check that $\psi : A(\Sigma_{g,n+1})_{\mathbb{Z}_2,0} \to A(\Sigma_{g,n})_{\mathbb{Z}_2}$ is surjective. Therefore, if $A(\Sigma_{g,n})_{\mathbb{Z}_2}$ is not finitely generated, then $A(\Sigma_{g,n+1})_{\mathbb{Z}_2,0}$ is not finitely generated. By Lemma 6.1, $A(\Sigma_{g,n+1})_{\mathbb{Z}_2}$ is not finitely generated, too. \hfill $\square$

When $g = 1$, we can tell more.

**Proof of (2) of Theorem D.** Let $C(\Sigma_{g,n})^+ \subset C(\Sigma_{g,n})$ be a subalgebra generated by arcs, loops, and vertices, but not the inverses of vertices. By Definition 4.1, we know that the homomorphism $\rho : A(\Sigma_{g,n}) \to C(\Sigma_{g,n})$ indeed factors through $C(\Sigma_{g,n})^+$. By taking the tensor product with $\mathbb{Z}_2$, we obtain a homomorphism

$$\rho : A(\Sigma_{g,n})_{\mathbb{Z}_2} \to C(\Sigma_{g,n})^+_{\mathbb{Z}_2}.$$

We have a similar variation for the map $\psi : C(\Sigma_{g,n+1})^+_{\mathbb{Z}_2} \to C(\Sigma_{g,n})^+_{\mathbb{Z}_2}$.

We specialize to $(g, n) = (1, 1)$. For the vertex $v$ that is forgotten by $\iota : \Sigma_{1,2} \to \Sigma_{1,1}$, we impose the associated $\mathbb{Z}$-grading structure on $C(\Sigma_{1,2})_{\mathbb{Z}_2}$, $C(\Sigma_{1,2})^+_{\mathbb{Z}_2}$, and on $A(\Sigma_{1,2})_{\mathbb{Z}_2}$ (Remark 6.5). Consider the composition

$$A(\Sigma_{1,2})_{\mathbb{Z}_2,0} \to C(\Sigma_{1,2})^+_{\mathbb{Z}_2,0} \to C(\Sigma_{1,1})^+_{\mathbb{Z}_2}$$

and denote it by $\psi$.

We claim that the image of $\psi : A(\Sigma_{1,2})_{\mathbb{Z}_2} \to C(\Sigma_{1,1})^+_{\mathbb{Z}_2}$ is $A(\Sigma_{1,1})_{\mathbb{Z}_2}$. Indeed, if we denote the unique vertex by $w$, then the image of $\psi$ is generated by $\beta$ and $w^2 \beta$ for an ordinary arc $\beta$. Applying the puncture-skein relation, we have $w \beta = \gamma_1 + \gamma_2$ for two loops. But any loop in $\Sigma_{1,1}$ is a $(p, q)$-torus knot for two relatively prime integers $p$ and $q$, and $\gamma_1 = \gamma_2$ because they are realized by the same $(p, q)$. Thus, $w \beta = 2 \gamma_1 = 0 \in C(\Sigma_{1,1})^+_{\mathbb{Z}_2}$ and so is $w^2 \beta$. Therefore, the image of $\psi$ is generated by ordinary arcs only, so in $\psi : A(\Sigma_{1,1})_{\mathbb{Z}_2}$ is not finitely generated by Proposition 6.6. Therefore, $A(\Sigma_{1,2})_{\mathbb{Z}_2,0}$ and $A(\Sigma_{1,2})_{\mathbb{Z}_2}$ are not finitely generated by Lemma 6.1. by (3) of Theorem D, $A(\Sigma_{1,n})$ for all $n \geq 1$ are not finitely generated. \hfill $\square$

**Remark 6.8.** Another way to think about the special property of $\Sigma_{1,1}$ is the following. For a fixed triangulation $T$, one may write the vertex class $w$ as a Laurent polynomial with respect to the edges in $T$. An explicit formula can be found, for example, in [MSW11, Definition 5.2]. $\Sigma_{1,1}$ is the only case that $w$ is a multiple of two.

6.2. **Finite generation for $g = 0$.** The situation is entirely different when $g = 0$. The finite generation of $A(\Sigma_{0,n})$ follows immediately from the presentation of $C(\Sigma_{0,n})$ in Theorem 2.9.

**Proof of (1) of Theorem D.** The proof is essentially identical to that of [ACDHM21, Prop 3.2], but for the reader’s convenience, we sketch the proof here.
Recall that, without loss of generality, we assume that the \( n \) punctures lie on a small circle \( C \subset S^2 \), and \( \beta_{i,j} \) is the simple arc connecting \( v_i \) and \( v_j \) in the disk bounded by \( C \).

Let \( \alpha \in \mathcal{A}(\Sigma_{0,n}) \) be a tagged arc. So \( \alpha \) connects two (not necessarily different) punctures. If \( \alpha \) is inside of \( C \), then \( \alpha \) is isotopic to one of \( \beta_{i,j} \) (if \( \alpha \) connects two distinct vertices) or 0 (if two ends of \( \alpha \) are the same). So \( \alpha \) is either zero or one of \( \beta_{i,j}, v_i \beta_{i,j}, v_j \beta_{i,j}, \) or \( v_i v_j \beta_{i,j} \), depending on the tagging.

If \( \alpha \) is outside of \( C \), then we can ‘drag into’ \( \alpha \) and use the puncture-skein relation to break the curve at the vertices. Then we can describe \( \alpha \) as a combination of tagged arcs which meet the outside smaller number of times. Now we may apply induction and get the desired result.

We believe that by the virtue of Theorem 2.9, the following is an interesting and approachable problem.

**Question 6.9.** Find a presentation of \( \mathcal{A}(\Sigma_{0,n}) \).

### 6.3. Comparison with the upper cluster algebra.

We finish this paper with some remarks on the upper cluster algebra \( \mathcal{U}(\Sigma_{g,n}) \). Recall that \( \mathcal{C}(\Sigma_{g,n})' \) is the subalgebra of \( \mathcal{C}(\Sigma_{g,n}) \) generated by isotopy classes of loops, arcs and decorated arcs (Definition 2.10).

**Lemma 6.10.** There are inclusions of algebras

\[
\mathcal{A}(\Sigma_{g,n}) \subset \mathcal{C}(\Sigma_{g,n})' \subset \mathcal{U}(\Sigma_{g,n}).
\]

**Proof.** Theorem A and the fact that the image of \( \rho \) factor through \( \mathcal{C}(\Sigma_{g,n})' \) imply the first inclusion. There are two extra classes of generators of \( \mathcal{C}(\Sigma_{g,n})' \) in \( \mathcal{C}(\Sigma_{g,n})' \setminus \mathcal{A}(\Sigma_{g,n}) \): Loop classes, and \( v\beta \), where \( \beta \) is an arc class with two ends both at \( v \) (Note that \( \beta \) and \( v^2\beta \) are in \( \mathcal{A}(\Sigma_{g,n}) \), if \( n \geq 2 \)). In the latter, by applying the puncture-skein relation, we obtain that \( v\beta \) is a sum of two loop classes. Thus, it is sufficient to check that loop classes are in \( \mathcal{U}(\Sigma_{g,n}) \). For an ordinary triangulation \( T \) with edge set \( E \), it has been proven several times ([FG06, Section 12], [MW13, Theorem 4.2], and [RY14, Theorem 3.22]) that a loop class is a Laurent polynomial with respect to the edges in a triangulation. The case of a tagged triangulation \( T^{\text{tag}} \) is reduced to the case of an ordinary triangulation, by [MSW11, Proposition 3.15].

**Proof of Theorem E.** Suppose that \( \mathcal{A}(\Sigma_{g,n}) = \mathcal{U}(\Sigma_{g,n}) \). Lemma 6.10 implies that \( \mathcal{A}(\Sigma_{g,n}) = \mathcal{C}(\Sigma_{g,n})' \). By Theorem 2.11, \( \mathcal{C}(\Sigma_{g,n})' \) is finitely generated, so is \( \mathcal{A}(\Sigma_{g,n}) \).

**Remark 6.11.** In a recent breakthrough in [GHKK18], for each combinatorial data defining a cluster algebra, Gross, Hacking, Keel, and Kontsevich defined yet another algebra motivated from mirror symmetry, the so-called mid-algebra (\( \text{mid}(V) \) in their terminology). For \( \mathcal{A}(\Sigma_{g,n}) \), the mid-algebra is indeed equal to \( \mathcal{C}(\Sigma_{g,n})' \) and it admits a canonical basis parametrized by the tropical points of the dual cluster variety ([FG06, Section 12]). To the authors’ knowledge, it has not been rigorously proved whether \( \mathcal{C}(\Sigma_{g,n})' = \mathcal{U}(\Sigma_{g,n}) \) or not.

**Remark 6.12.** When \( g = 0 \), all loop classes are generated by tagged arc classes, as proved in [BKPW16b, Proposition 2.2] and as evidenced by Theorem 2.9. Thus, \( \mathcal{A}(\Sigma_{0,n}) = \mathcal{C}(\Sigma_{0,n})' \).
Remark 6.13. If \(n \geq 2\), \(C(\Sigma_{g,n})\) is not a subalgebra of \(U(\Sigma_{g,n})\), because of the vertex classes. For a fixed ordinary triangulation \(\mathcal{T}\) and its edge set \(E = \{e_i\}\), a vertex class \(v\) can be written as a Laurent polynomial with respect to \(E\) (see the proof of [MW21, Lemma 3.2]). However, this is no longer true for a tagged triangulation \(\mathcal{T}^\text{td}\). On the other hand, when \(n = 1\), we do not consider a tagged triangulation, so \(v \in U(\Sigma_{g,n})\) and hence \(C(\Sigma_{g,n}) \subset U(\Sigma_{g,n})\).

Conjecture 6.14. (1) \(C(\Sigma_{g,1}) = U(\Sigma_{g,1})\).

(2) If \(n \geq 2\), \(C(\Sigma_{g,n})' = U(\Sigma_{g,n})\). In particular, if \(n \geq 4\), \(A(\Sigma_{0,n}) = U(\Sigma_{0,n})\).

Remark 6.15. By applying Proposition 6.6, we can also show that \(A(\Sigma_{g,1}) \neq U(\Sigma_{g,1})\). It was shown in [Lad13, Section 2], by constructing an explicit element \(\zeta_p(x, T)\) in \(U(\Sigma_{g,1}) \setminus A(\Sigma_{g,1})\). Within our point of view, we can provide an explicit geometric interpretation of the element: \(\zeta_p(x, T)\) is equal to the vertex class \(v\) (with one exception – for \(\Sigma_{1,1}\), \(2\zeta_p(x, T) = v\)).

References

[AB20] D. Allegretti and T. Bridgeland. The monodromy of meromorphic projective structures. Trans. Amer. Math. Soc. 373 (2020), no. 9, 6321–6367. 2

[ACDHM21] F. Azad, Z. Chen, M. Dreyer, R. Horowitz, and H.-B. Moon. Presentations of the Roger-Yang generalized skein algebra. Algebr. Geom. Topol., 21 (2021), no. 6, 3199–3220. 4, 7, 24, 27

[BHMV95] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the Kauffman bracket. Topology 34 (1995), no. 4, 883–927. 1

[BMRS15] A. Benito, G. Muller, Greg, J. Rajchgot, and K. Smith. Singularities of locally acyclic cluster algebras. Algebra Number Theory 9 (2015), no. 4, 913–936. 8, 9

[BZ05] A. Berenstein and A. Zelevinsky. Quantum cluster algebras. Adv. Math. 195 (2005), no. 2, 405–455. 16

[BKL21] W. Bloomquist, H. Karuo, and T. Le. Degeneration of skein algebras of surfaces and decorated Teichmüller spaces. in preparation. 3

[BKPW16a] M. Bobb, S. Kennedy, H. Wong, and D. Peifer. Roger and Yang’s Kauffman bracket arc algebra is finitely generated. J. Knot Theory Ramifications 25 (2016), no. 6, 1650034, 14 pp. 3, 7, 8

[BKPW16b] M. Bobb, D. Peifer, S. Kennedy, and H. Wong. Presentations of Roger and Yang’s Kauffman bracket arc algebra. Involve 9 (2016), no. 4, 689–698. 4, 28

[Bul97] D. Bullock. Rings of SL_2(C)-characters and the Kauffman bracket skein module Commentarii Mathematici Helvetici 72 (1997), no. 4, 521–542. 1, 6

[BFK99] D. Bullock, C. Frohman, and J. Kania-Bartoszyńska. Understanding the Kauffman bracket skein module. J. Knot Theory Ramifications 8 (1999), no. 3, 265–277. 1, 6

[CLS15] I. Canakci, K. Lee, R. Schiffler. On cluster algebras from unpunctured surfaces with one marked point. Proc. Amer. Math. Soc. Ser. B 2 (2015), 35–49. 3

[Dol03] I. Dolgachev. Lectures on invariant theory. London Mathematical Society Lecture Note Series, 296. Cambridge University Press, Cambridge, 2003. xvi+220 pp. 25

[Eis95] D. Eisenbud. Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995. xvi+785 pp. 22

[FG06] V. Fock and A. Goncharov. Moduli spaces of local systems and higher Teichmüller theory. Publ. Math. Inst. Hautes Études Sci. 103 (2006), 1–211. 2, 28

[FST08] S. Fomin, M. Shapiro, and D. Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. Acta Math. 201 (2008), no. 1, 83–146. 1, 8, 10, 11, 12, 13, 14, 15

[FZ02] S. Fomin and A. Zelevinsky. Cluster algebras. I. Foundations. J. Amer. Math. Soc. 15 (2002), no. 2, 497–529. 1, 8, 9
[GSV05] M. Gekhtman, M. Shapiro, and A. Vainshtein. Cluster algebras and Weil-Petersson forms. *Duke Mathematical Journal*, 127 (2005), no. 2, 291–311.

[GHKK18] M. Gross, P. Hacking, S. Keel, and M. Kontsevich. Canonical bases for cluster algebras. *J. Amer. Math. Soc.* 31 (2018), no. 2, 497–608.

[Jones85] V.F.R. Jones. A new polynomial invariant for links via von Neumann algebras. *Bull. Amer. Math. Soc.* 12 (1985), 103–122.

[Lad13] S. Ladkani. On cluster algebras from once punctured closed surfaces. preprint, arXiv:1310:4454.

[MW21] H.-B. Moon and H. Wong. The Roger-Yang skein algebra and the decorated Teichmüller space. *Quantum Topol.* 12 (2021), no. 2, 265–308.

[Mul13] G. Muller. Locally acyclic cluster algebras. *Adv. Math.* 233 (2013), 207–247.

[Mul16] G. Muller. Skein and cluster algebras of marked surfaces. *Quantum Topol.* 7 (2016), no. 3, 435–503.

[MSW11] G. Musiker, R. Schiffler, and L. Williams. Positivity for cluster algebras from surfaces. *Adv. Math.* 227 (2011), no. 6, 2241–2308.

[MW13] G. Musiker and L. Williams. Matrix formulae and skein relations for cluster algebras from surfaces. *Int. Math. Res. Not. IMRN* 2013, no. 13, 2891–2944.

[Pen87] R. C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.* 113 (1987), 299–339.

[Pen92] R. C. Penner. Weil-Petersson volumes. *J. Differential Geom.* 35 (1992), no. 3, 559–608.

[Prz91] J. Przytycki. Skein modules of 3-manifolds. *Bull. Polish Acad. Sci. Math.* 39 (1991), no. 1-2, 91–100.

[PS00] J. H. Przytycki and A. Sikora. On skein algebras and $SL_2(\mathbb{C})$-character varieties. *Topology* 39 (2000), no. 1, 115–148.

[RT91] N. Reshetikhin and V.G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.* 103 (1991), no. 3, 547–597.

[RY14] J. Roger and T. Yang. The skein algebra of arcs and links and the decorated Teichmüller space. *J. Differential Geom.* 96 (2014), no. 1, 95–140.

[Ses77] C. S. Seshadri. Geometric reductivity over arbitrary base. *Advances in Math.* 26 (1977), no. 3, 225–274.

[Tur91] V. G. Turaev. Skein quantization of Poisson algebras of loops on surfaces. *Ann. Sci. École Norm. Sup. (4)*, 24(6):635–704, 1991.

[Wit89] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.* 121 (1989), no. 3, 351–399.