Signals on the power spectra of a cosmology modeled with Chebyshev polynomials

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Abstract. We present an interacting model with a phenomenological interaction, $\bar{Q}$, between a cold dark matter (DM) fluid and a dark energy (DE) fluid, which takes a time-varying equation of state (EoS) parameter, $\omega_{DE}$. Here, both $\bar{Q}$ and $\omega_{DE}$ are modeled in terms of the Chebyshev polynomials. In a Newtonian gauge and on sub-horizon scales, a set of perturbed equations is obtained when the momentum transfer potential becomes null in the DM rest-frame. This leads to different cases of the interacting model. Then, via a Markov-Chain Monte Carlo (MCMC) method, we constrain such cases by using a combined analysis of geometric and dynamical data. Our results show that in such cases the evolution curves of the structure growth of the matter deviate strongly from the standard model. In addition, we also found that the matter power spectrum is sensitive to $\bar{Q}$. In this way, the coupling modifies the matter scale and generates a slight variation of the turnover point to smaller scales. Likewise, the amplitude of the CMB temperature power spectrum is sensitive the values of $\bar{Q}$ and $\omega_{DE}$ at low and high multipoles, respectively. Here, $Q$ can cross twice the line $Q = 0$ during its background evolution.

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1 Introduction

A number of observations [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51] have indicated that the present universe is undergoing a phase of accelerated expansion, and driven probably by a new form of energy with negative EoS parameter, commonly so-called DE [52]. This energy has been interpreted in various forms and widely studied in [53]. However, within General Relativity (GR) the DE models can suffer the coincidence problem, namely why the DM and DE energy densities are of the same order today. This latter problem could be solved or even alleviated, by assuming the existence of a non-gravitational $\bar{Q}$ within the dark sector, which gives rise to a continuous energy exchange from DE to DM or vice-versa. Currently, there are no physical arguments nor recent observations to exclude $\bar{Q}$ [54,55,56]. Moreover, due to the absence of a fundamental theory to construct $\bar{Q}$, different ansatzes have been widely discussed in [54,55,56]. So, it has been shown in some coupled DE scenarios that $\bar{Q}$ can affect the background evolution of the DM density perturbations and the expansion history of the universe [57]. Thus, $\bar{Q}$ and $\omega_{DE}$ could very possibly introduce new features on the evolution curves of the structure growth of the matter, on the linear matter power spectrum and on the amplitude of the CMB temperature power spectrum at low and high multipoles, respectively [58,59,60].

On the other one, within dark sector we can propose new ansatzes for both $\bar{Q}$ and $\omega_{DE}$, which can be expanded in terms of the Chebyshev polynomials $T_n$, defined in the interval $[-1,1]$ and with a divergence-free $\omega_{DE}$ at $z \rightarrow -1$ [61]. However, that polynomial base was particularly chosen due to its rapid convergence and better stability than others, by giving minimal errors [44,66]. Besides, $\bar{Q}$ could also be proportional to the DM energy density $\rho_{DM}$ and to the Hubble parameter $H$. This new model will guarantee an accelerated scaling attractor and connect to a standard evolution of the matter. Here, $Q$ will be restricted from the criteria exhibit in [67].

The focus of this paper is to investigate the effects of $\bar{Q}$ and $\omega_{DE}$ on the curves of structure growth, on the matter power spectrum and on the CMB temperature power spectrum including the search for a new way to alleviate the coincidence problem. On the other hand, an interacting DE model is discussed, on which we have performed a global fitting, by using an analysis combined of Joint Light Curve Analysis (JLA) type Ia Supernovae (SNe Ia) data [1,2,3,4], including the growth rate of structure formation obtained from redshift space distortion (RSD) data [4,5,6,7,8,9,10,11,12].

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In this work, we consider the spatially flat FRW metric where \( \Omega \) indicates the present day value of the quantity. We also consider that these fluids have EoS parameters corresponding pressures and the energy densities. Here, \( \rho_{DM} \) and \( \rho_{DE} \) are respectively, \( \rho_{r} \), baryonic matter (subscript b), DM (FRW) universe, composed with four perfect fluids-like, and show our conclusions in Sec. VI.

We assume a spatially flat Friedmann-Robertson-Walker (FRW) universe, composed with four perfect fluids-like, the linear matter and CMB temperature power spectra in Sec. III. The constraint method and observational data are presented in Sec. IV. We discuss our results in Sec. V and show our conclusions in Sec. VI.

### 2 Interacting dark energy (IDE) model

We assume a spatially flat Friedmann-Robertson-Walker (FRW) universe, composed with four perfect fluids-like, radiation (subscript r), baryonic matter (subscript b), DM and DE, respectively. Moreover, we postulate the existence of a non-gravitational coupling in the background between DM and DE (so-called dark sector) and two decoupled sectors related to the b and r components, respectively. We also consider that these fluids have EoS parameters \( P_A = \omega_A \rho_A, A = b, r, DM, DE \), where \( P_A \) and \( \rho_A \) are the corresponding pressures and the energy densities. Here, we choose \( \rho_{DM} = \rho_b = 0, \omega_r = 1/3 \) and \( \omega_{DE} \) is a time-varying function. Therefore, the balance equations of our fluids are respectively,

\[
\frac{d\rho_{b}}{dz} - 3H\rho_{b} = 0, \tag{1}
\]
\[
\frac{d\rho_{r}}{dz} - 4H\rho_{r} = 0, \tag{2}
\]
\[
\frac{d\rho_{DM}}{dz} - 3\rho_{DM} (1 + z) = -\frac{Q}{H(1 + z)}, \tag{3}
\]
\[
\frac{d\rho_{DE}}{dz} - 3(1 + \omega_{DE})\rho_{DE} (1 + z) = +\frac{Q}{H(1 + z)}, \tag{4}
\]

where the differentiation has been done with respect to the redshift, \( z \), \( H \) denotes the Hubble expansion rate and the quantity \( Q \) expresses the interaction between the dark sectors. For simplicity, it is convenient to define the fractional energy densities \( \Omega_i \equiv \frac{\rho_i}{\rho_c} \) and \( \Omega_{i,0} \equiv \frac{\rho_{i,0}}{\rho_c,0} \), where the critical density \( \rho_c,0 \equiv 3H^2/8\pi G \) and the critical density today \( \rho_{c,0} = 3H_0^2/8\pi G \) being \( H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1} \) the current value of \( H \). Likewise, we have taken the relation \( \Omega_{b,0} + \Omega_{r,0} + \Omega_{DM,0} + \Omega_{DE,0} = 1 \). Here, the subscript “0” indicates the present day value of the quantity.

In this work, we consider the spatially flat FRW metric with line element

\[
d^2s^2 = -dt^2 + a^2(t)\delta_{ij}dx^idx^j, \tag{5}
\]

where \( t \) represents the cosmic time and “a” represents the scale factor of the metric and it is defined in terms of the redshift \( z \) as \( a = (1 + z)^{-1} \).

Here, we analyze the ratio between the energy densities of DM and DE, defined as \( R \equiv \rho_{DM}/\rho_{DE} \). From Eqs. \( 3 \) and \( 4 \), we obtain

\[
\frac{dR}{dz} = \frac{-R}{(1 + z)} \left( 3\omega_{DE} + \frac{(1 + R)Q}{H\rho_{DM}} \right). \tag{6}
\]

This Eq. leads to

\[
Q = - \left( 3\omega_{DE} + \frac{dR(1 + z)}{dz} \right) \frac{H\rho_{DM}}{1 + R}. \tag{7}
\]

Due to the fact that the origin and nature of the dark fluids are unknown, it is not possible to derive \( Q \) from fundamental principles. However, we have the freedom of choosing any possible form of \( Q \) that satisfies Eqs. \( 3 \) and \( 4 \) simultaneously. Hence, we propose a phenomenological description for \( Q \) as a linear combination of \( \rho_{DM}, H \) and a time-varying function \( I_Q \),

\[
Q \equiv \bar{H}\rho_{DM}I_Q, \quad I_Q \equiv \sum_{n=0}^{\infty} \lambda_n T_n, \tag{8}
\]

where \( I_Q \) is defined in terms of Chebyshev polynomials and \( \lambda_n \) are constant and small \( |\lambda_n| \ll 1 \) dimensionless parameters. This polynomial base was chosen because it converges rapidly, is more stable than others and behaves well in any polynomial expansion, giving minimal errors \([58]\). The first three Chebyshev polynomials are

\[
T_0(z) = 1, \quad T_1(z) = z, \quad T_2(z) = (2z^2 - 1). \tag{9}
\]

From Eqs. \( 8 \) and \( 9 \) an asymptotic value for \( I_Q \) can be found: \( I_Q \rightarrow \infty \) for \( z \rightarrow \infty \), \( I_Q = \lambda_0 - \lambda_2 \) for \( z = 0 \) and \( I_Q \approx \lambda_0 - \lambda_1 + \lambda_2 \) for \( z \rightarrow -1 \).

Similarly, we will focus on an interacting model with a specific ansatz for the EoS parameter, given as

\[
\omega_{DE} \equiv \omega_2 + 2 \sum_{m=0}^{2} \frac{\omega_m T_m}{1 + z^2}. \tag{10}
\]

Within this ansatz a finite value for \( \omega \) is obtained from the past to the future; namely, the following asymptotic values are found:

\[
\omega_{DE} \approx 5\omega_2 \text{ for } z \rightarrow \infty, \quad \omega_{DE} \approx \omega_0 \text{ for } z = 0 \text{ and } \omega_{DE} \approx (5/3)\omega_2 + (2/3)[\omega_0 - \omega_1] \text{ for } z \rightarrow -1. \tag{11}
\]

Therefore, a possible physical description should be studied to explore its properties.

In order to guarantee that \( Q \) may be physically acceptable in the dark sectors \([57]\), we equate the right-hand sides of Eqs. \( 7 \) and \( 4 \), which becomes

\[
\frac{dR}{dz} = \frac{-R}{(1 + z)} \left( I_Q(1 + R) + 3\omega_{DE} \right). \tag{11}
\]

Now, to solve or alleviate of coincidence problem, we require that \( R \) tends to a fixed value at late times. This leads to the condition \( dR/dz = 0 \), which therefore implies two stationary solutions \( R_+ = R(z \rightarrow \infty) = -(1 + 3\omega_{DE}/I_Q) \) and \( R_- = R(z \rightarrow -1) = 0 \). The first solution occurs in the
past and the second one happens in the future. By inserting Eqs. (8) and (10) into Eq. (11), we find that R has no analytical solution, in any case, it is to be solved numerically. Likewise, there are an analytical solution for just \( \bar{\rho}_0, \bar{\rho}_i, \) and \( \bar{\rho}_{DM}, \) respectively, but \( \bar{\rho}_{DE} \) will be obtained from R, as \( \bar{\rho}_{DE} = \bar{\rho}_{DM}/R. \)

Therefore, the first Friedmann equation is given by

\[
E^2 = \frac{\ddot{R}}{R^2} = \Omega_{b,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{DM}(z)(1+R^{-1}),
\]

where have considered that

\[
\Omega_{DM}(z) = (1+z)^3 \Omega_{DM,0} \exp \left( -\frac{z_{\text{max}}}{2} \sum_{n=0}^{2} \lambda_n I_n(z) \right),
\]

\[
\int_{0}^{\infty} T_n(\bar{x}) d\bar{x} \approx \frac{z_{\text{max}}}{2} \sum_{n=0}^{2} \lambda_n I_n(z),
\]

\[
x \equiv \frac{2}{z_{\text{max}}} - 1, \quad a_1 \equiv 1 + \frac{z_{\text{max}}}{2}, \quad a_2 \equiv \frac{z_{\text{max}}}{2},
\]

\[
I_0(\bar{x}) = \frac{2}{z_{\text{max}}} \ln(1+z),
\]

\[
I_1(\bar{x}) = \frac{2}{z_{\text{max}}} \left( \frac{2a}{z_{\text{max}}} - \left( 2 + \frac{z_{\text{max}}}{2} \right) \ln(1+z) \right),
\]

\[
I_2(\bar{x}) = \frac{2}{z_{\text{max}}} \left[ \frac{4a}{z_{\text{max}}} \left( \frac{z}{z_{\text{max}}} - \frac{2}{z_{\text{max}}} - 2 \right) + \left( 1 + \frac{6.8284}{z_{\text{max}}} \right) \frac{1.1716}{z_{\text{max}}} \right],
\]

where \( z_{\text{max}} \) is the maximum value of \( z \) such that \( \bar{x} \in [-1, 1] \) and \( |T_n(\bar{x})| \leq 1 \) and \( n \in [0, 2]. \) If \( Q(z) = 0 \) and \( \omega_{DE} = -1 \) in Eq. (12), the standard ΛCDM model is recovered. Similarly, when \( Q(z) = 0 \) and \( \omega_{DE} \) are nonzero, the ΛDE model is obtained. These non-interacting models have an analytical solution for R.

### 3 IDE in the perturbed universe

#### 3.1 Perturbative equations

In the Newtonian gauge, the perturbed FRW metric becomes \([59,60]\)

\[
d^2 s = - (1 + 2\phi) dt^2 + a^2(t) (1 - 2\gamma) dx^i dx^j,
\]

where \( \phi \) and \( \gamma \) are gravitational potentials, and the four-velocity of fluid A (\( A = \text{DM, DE, b, r} \)) is

\[
U^\mu_A = a^{-1}(1 - \phi, \partial^\mu \nu), \quad U^\mu_A = a(-1 - \phi, \partial^\mu \nu), \quad U^\mu_A = a(-1 - \phi, \partial^\mu \nu),
\]

where \( \nu_A \) is the peculiar velocity potential, and \( \theta_A \) is the velocity perturbation defined as \( \theta_A = -k^2 v_A. \)

The energy-momentum conservation equation of A fluid in interaction is given by \([59,60]\)

\[
\nabla_i T^\mu_{\alpha
u} = Q^\mu_A, \quad Q^\mu_{DM} = -Q^\mu_{DE} \neq 0 = Q^\mu_r = Q^\mu_r,
\]

where \( T^\mu_{\alpha
u} \) is the A-fluid energy momentum tensor.

In general, \( Q^\mu_A \) can be split relative to the total four-velocity \( U^\mu_B \) as \([59,60]\)

\[
Q^\mu_A = Q_A U^\mu_A + F^\mu_A, \quad Q_A = Q_A + \delta Q_A, \quad U^\mu_A F^\mu_A = 0,
\]

\[
Q^\mu_{DM} = -Q^\mu_{DE} \neq 0 = Q^\mu_r = -\delta Q_r,
\]

\[
\delta Q_{DM} = -\delta Q_{DE} \neq 0 = \delta Q_r,
\]

where \( Q_A \) and \( F^\mu_A \) represent the energy and momentum transfer rate, respectively, relative to \( U^\mu_A. \) Likewise, to the first order \( F_A \equiv a^{-1}(0, \partial^\mu f_A) \) where \( f_A \) is a momentum transfer potential and \( Q_A \) represents the interaction term.

From Eqs. (11) and (16), we find \([60]\)

\[
Q_0^A = -a [Q_A(1 + \phi) + \delta Q_A], \quad Q_A^A = a \partial_h [f_A + Q_A A],
\]

with \( \delta Q_{DM} = -\delta Q_{DE} \neq 0 \) and \( \delta Q_{DM} = -\delta Q_{DE}. \)

Here, we have considered that the fluid physical sound speed in the rest-frame of \( c^2_A \equiv \partial P_A/\partial \rho_A \) and the adiabatic sound speed is defined by \( c^2_A \equiv \partial P_A/\partial \rho_A = \omega_A + \left( \frac{\partial \ln \theta_A}{\partial \ln P_A} \right) \rho_A. \) Then, for the adiabatic DM fluid, we take \( c^2_{DM} = \omega_{DM} < 0. \) Instead, for the non-adiabatic DE fluid, \( c^2_{DE} = \omega_{DE} < 0. \) Then, the adiabatic DM fluid physical sound speed for DE is usually considered as \( c^2_{DE} = 1 \) to eliminate possible unphysical instabilities.

Immediately, we have established the simpler physical choice for the momentum transfer potential between the dark sectors, which happens when \( f_A = 0 \) in the rest-frame of either DM or DE \([59]. \) Consequently, this choice allows two different possibilities for \( Q^A_A \) and \( f_A, \) which can be parallel to either the DM or the DE four velocity, respectively. In this work, we focus only on the case \([59,60]\)

\[
Q^\mu_{DE} = \tilde{Q}_{DE} U^\mu_{DM} = -Q^\mu_{DM}, \quad Q^A_A \parallel U^\mu_{DM},
\]

\[
f_{DM} = \frac{Q_A}{k^2(\theta - \delta_{DM})} = -f_{DE}, \quad Q^A_A \parallel U^\mu_{DM},
\]

On the other hand, assuming that \( Q \) depends on the cosmic time through the global expansion rate, then a possible choice for \( \delta H \) can be \( \delta H = 0. \) Likewise, for convenience, we impose that \( \delta Q \ll \delta DM, \) which leads to

\[
\delta Q_{DM} = -H \bar{Q} \rho_{DM} \delta \rho_{DM}.
\]

In a forthcoming article we will extend our study, by considering other relations between \( \delta Q, \delta_{DM} \) and \( \delta H. \) It is beyond the scope of the present paper.

In this work, we are interested in studying the effects of \( \omega_{DE} \) and \( Q \) on the total matter power spectrum and on the CMB temperature power spectrum. For this reason, we consider only the adiabatic perturbations, assume that \( T_3^{\mu\nu} \) is free of anisotropic stress, and the arguments above discussed, we find the evolution equations for the density contrast perturbation \( \delta \) and the velocity perturbations \( \theta_A \) in the IDE model from the general case presented in
Next, we define the growth factor of linear matter perturbations as
\[ \delta \equiv \frac{d \ln \delta_M}{d \ln a}, \quad \delta_M = \Omega_{DM} \delta_{DM} + \Omega_b \delta_b, \]
where \( \delta_M \) is the normalized matter density perturbations. Via the above definition and by considering that \( \delta_b \ll \delta_{DM} \), Eq. (27) can be re-expressed in terms of the redshift for the case \( Q_A^B \| U_{DM}^B \), as
\[ \frac{df}{dz} = \frac{1}{(1+z)} \left[ f^2 + \frac{f}{2} \left( 1 - 3 \omega_M \Omega_{DE} - \Omega_t \right) - \frac{3}{2} \Omega_{DM} \right]. \]
An analytical solution to Eq. (29) is very complicated to obtain, and we need to use numerical methods. For this reason, it is most suitable to approach \( f \) in the form
\[ f = \bar{\Omega}_M^s, \]
where \( \gamma \) is the growth index of the linear matter fluctuations, and in general is a function of the redshift or scale factor. Hence, Eq. (27) can be solved numerically taking into account the conditions at \( z = 0 \): \( \delta_{DM,0} = 1 \) and \( \delta_{b,0} = -\Omega_{M,0} \gamma_0 \), where \( \Omega_{M,0} \) and \( \gamma_0 \) are the values today, and \( \Omega_{M,0} = \Omega_{DM,0} + \Omega_b,0 \). Similarly, by considering the condition \( f_0 = \Omega_{M,0} \gamma_0 \), Eq. (29) can also be solved numerically. On the other hand, the root-mean-square amplitude of matter density perturbations within a sphere of radius \( 8 \, \text{Mpc}^{-1} \) is denoted as \( \sigma_s(z) \) and its evolution is represented by
\[ \sigma_s(z) = \delta_M(z) \sigma_{s,0}, \]
where \( \sigma_{s,0} \) is the normalizations to unity of \( \sigma_s(z) \) today. Thus, the functions \( f \) and \( \sigma_s \) can be combined to obtain \( f \sigma_s \) at different redshifts. From here, we obtain
\[ f(z) \sigma_s(z) = f(z) \delta(z) \sigma_{s,0}. \]

### 3.2 Structure formation

In the Newtonian limit (\( \delta \ll 1 \)) and at sub-horizon scales (\( H^2 \ll k^2 \)), we assume that DE fluid does not contribute in clustering of matter and therefore we could take \( \delta_{DM} \gg \delta_{DE} \approx 0 \). Besides, for simplicity, we also consider that the gravitational potentials \( \phi \) and \( \psi \), satisfy \( \delta_{DM} \gg \phi = \psi \) and \( \phi' = \psi' \approx 0 \).

Since we are only interested in showing the effects of \( \bar{Q} \) and \( \omega_{DE} \) on the evolution of \( \delta_{DM} \) during the matter dominated era, rather than making accurate calculations. Again, for simplicity, we can ignore the contribution of the radiation in our estimations. Due to the arguments above discussed and combining Eqs. (21), (24) and (25), we obtain
\[ \frac{d^2 \delta_{DM}}{dz^2} = -\left( 3 \Omega_{DM} + \Omega_b \right) \frac{d \delta_{DM}}{dz} + \left( 3 \Omega_{DM} + \Omega_b \right) \frac{\delta_{DM}}{dz}. \]
A similar equation can be obtained for \( \delta_b \).
Next, we define the growth factor of linear matter perturbations as
\[ f \equiv \frac{d \ln \delta_M}{d \ln a}, \quad \delta_M = \Omega_{DM} \delta_{DM} + \Omega_b \delta_b, \]

### 3.3 Linear matter power spectrum

The linear matter power spectrum \( P(k, z) \) is
\[ P(k, z) = 2 \pi^2 \delta_{M,0}^2 \left[ \frac{k^2}{2 \Omega_{M,0} \Omega_{DM,0}} \right] T^2(k) \delta_{DM}^2(z), \]
where \( T(k) \) is the transfer function, \( n_s \) is the scalar spectral index of the primordial fluctuation spectrum, \( k \) is the wavenumber and \( \delta_H \) is defined by
\[ \delta_H \approx 1.94 \times 10^{-5} \Omega_{M,0}^{-0.785} - 0.05 \Omega_{M,0} e^{-0.95(n_s - 1)}. \]
In this work, we adopt the fitting formula proposed in [70] that approximates the full transfer function as the sum of the baryon and cold DM contribution on all scales
\[ T(k) = \frac{\Omega_b,0}{\Omega_{M,0}} T_b(k) + \frac{\Omega_{DM,0}}{\Omega_{M,0}} T_{DM}(k). \]
Here, \( T_b(k) \) is the baryon transfer function defined as
\[
T_b(k) = \left[ 1 + \frac{\ln(e + 1.8q)}{1 + (\frac{\ln(e + 1.8q)}{2})^2} \sin(kr) \right]^2 \sin(kr) 
\]
and the shape parameter \( q = \frac{k \Omega_m h}{13.41 k_{eq}} \), \( \alpha_0 = \frac{2.07 k_{eq} q G}{(1 + R_d)^{0.75}} a_{eq} \), \( r_d = \frac{2}{3k_{eq}} \ln \left[ \frac{R_d + H_{eq} + \sqrt{1 + H_{eq}^2}}{1 + \sqrt{R_d^2 + H_{eq}^2}} \right] \), where \( R_d \) is the ratio of the baryon to photon energy density at the drag epoch, \( k_{eq} \) is the wave-number at the drag epoch, and \( a_{eq} \) represents the scale factor at the recombination epoch.

Similarly, the cold DM transfer function, \( T_{DM}(k) \), is defined as \([69,70]\).
\[
T_{DM}(k) = \frac{L_0}{L_0 + C_0 q^2}, \quad L_0 = \ln(5.436 + 1.8q) \]
\[
C_0 = 14.2 + \frac{731}{1 + 62.5q}, \quad q = \frac{k}{H}, \quad (37)
\]
and the shape parameter \( \Gamma \), is given by \([69,70]\)
\[
\Gamma \equiv \Omega_{M,0} h \left( 1 + \frac{\ln(1 + 0.43kr(z_d))}{1 - kr(z_d)} \right) \]
\[
\zeta = 1 - 0.328 \ln(431 \Omega_{M,0} h^2) \nu + 0.38 \ln(22.3 \Omega_{M,0} h^2) \nu^2. \quad (38)
\]

### 3.4 CMB temperature power spectrum.

From Eqs. \([63,67]\), using the results found in \([70]\) and the Limber approximation \([71]\), we have built numerically the CMB temperature power spectrum today as
\[
C_l \alpha B_1 \cos^2 \left( \frac{r_s(z_d)}{D_A} \right) + B_2 \cos \left( \frac{r_s(z_d)}{D_A} \right) T_l(D_A) + B_3 \cos \left( \frac{r_s(z_d)}{D_A} \right) T_2(l, D_A) + B_4 \sin^2 \left( \frac{r_s(z_d)}{D_A} \right) + ..., \quad (39)
\]
where the respective coefficients \( B_1, B_2, B_3 \) and \( B_4 \) are functions of \( H_0, D_A(z_d), \Omega_{M,0}, \delta_H, \delta_M, R_s, k, k_{eq}, k_{silk} \) and \( \tau \). Here, \( D_A(z_d) \) represents the angular distance at the recombination epoch, see Eq. \([67]\), \( R_s \) is the ratio of the baryon to photon energy density at the recombination epoch and \( r_s(z_d) \) is the sound horizon at the recombination epoch, see Eq. \([67]\), and \( \tau \) represents the optical depth.

### 4 Constraint method and observational data

#### 4.1 Constraint method

In general, to constrain the parameter space we build all the necessary codes in the c++ language and use the MCMC method to calculate the best-fit parameters of the \( \Lambda \)CDM, \( \omega \)DE and IDE models, respectively, and their respective parameter space (P main parameters), are given by
\[
P_1 \equiv \left( \Omega_{DM,0}, H_0, \alpha, \beta, M, dM, \gamma_0, \sigma_8 \right),
\]
\[
P_2 \equiv \left( \omega_0, \omega_1, \omega_2, \Omega_{DM,0}, H_0, \alpha, \beta, M, dM, \gamma_0, \sigma_8 \right),
\]
\[
P_3 \equiv \left( \lambda_0, \lambda_1, \lambda_2, \omega_0, \omega_1, \omega_2, \Omega_{DM,0}, H_0, \alpha, \beta, M, dM, \gamma_0, \sigma_8 \right),
\]
where \( \Omega_{DM,0} \) and \( H_0 \) are the DM energy density and the Hubble parameter today, \( \omega_0, \omega_1, \omega_2 \) are dimensionless parameters related to \( \omega \)DE. Similarly, \( \lambda_0, \lambda_1, \lambda_2, \omega_0 \) and \( \omega_1, \omega_2 \) are dimensionless constants related to Q. The nuisance parameters \( \alpha, \beta, M, dM \) and \( \gamma_0, \sigma_8 \) are connected with the global properties of the Supernovas (type Ia), \( \gamma_0 \) and \( \sigma_8 \) today, respectively. The pivot scale of the initial scalar power spectrum \( k_s = 0.045 \text{Mpc}^{-1} \) is assumed. Besides, the constant priors for the model parameters are shown in Table 1. We have also fixed \( \Omega_{B,0} = 0.01(1 + 0.2771 N_{eff}), \) where \( N_{eff} \) represents the effective number of neutrino species \( N_{eff} = 3.04 \pm 0.18 \) and \( \Omega_b = 2.469 \times 10^{-5} h^2 \) were chosen from Table 4 in \([23]\). Similarly, the values of \( \Omega_{B,0} = 0.02230h^{-2} \) and the Gaussian prior on \( n_s = 0.9667 \pm 0.0040 \) were also taken from Table 4 in \([23]\).

Furthermore, the dimensionless parameters such as the ratio of the sound horizon and angular diameter distance, \( \Theta_s \), multiplied by 100, together with the optical depth \( \tau \) and the amplitude of the initial power spectrum \( A_s \), are derived from the parameter space \( P \). In order to have access to the distribution of \( P \), we calculate the overall likelihood \( \mathcal{L} \alpha = \chi^2/2, \) where \( \chi^2 \) is
\[
\chi^2 = \chi^2_{\text{JLA}} + \chi^2_{\text{RSD}} + \chi^2_{\text{BAO}} + \chi^2_{\text{CMB}} + \chi^2_{\text{H}}. \quad (40)
\]

#### 4.2 Observational data

To test the viability of our model and set constraints on \( P \), we use the following data sets:

##### 4.2.1 Join Analysis Luminous data set (JLA).

The Supernovae (SNe Ia) data sample used in this work is the Join Analysis Luminous data set (JLA) \([12,23]\) composed by 740 SNe with high-quality light curves. Here, JLA data include samples from \( z < 0.1 \) to \( 0.2 < z < 1.0 \). The observed distance modulus is modeled by \([12,23]\)
\[
\mu^2_{i,\text{JLA}} = m^2_{B,i} + \alpha x_{i,1} - \beta C_i - M_B, \quad 1 \leq i \leq 740, \quad (41)
\]
where and the parameters \(m_\nu^p\), \(x_1\) and \(C\) describe the intrinsic variability in the luminosity of the SNe. Furthermore, the nuisance parameters \(\alpha\), \(\beta\), \(M\) and \(dM\) characterize the global properties of the light-curves of the SNe and are estimated simultaneously with the cosmological parameters of interest. Then, we defined \(M_B\)

\[
M_B = \begin{cases} M, & \text{if } M_{\text{stellar}} < 10^{10} M_\odot, \\ M + dM, & \text{if otherwise}, \end{cases}
\]

where \(M_{\text{stellar}}\) is the host galaxy stellar mass, and \(M_\odot\) is the solar mass.

On the other hand, the theoretical distance modulus is

\[
\mu^t(z, X) \equiv 5 \log_{10} \left[ \frac{D_L(z, X)}{\text{Mpc}} \right] + 25,
\]

where “th” denotes the theoretical prediction for a SNe at \(z\). The luminosity distance \(D_L(z, X)\), is defined as

\[
D_L(z_{\text{hel}}, z_{\text{CMB}}, X) = (1 + z_{\text{hel}})c \int_0^{z_{\text{CMB}}} \frac{dz'}{H(z', X)},
\]

where \(z_{\text{hel}}\) is the heliocentric redshift, \(z_{\text{CMB}}\) is the CMB rest-frame redshift, \(c = 2.9999 \times 10^5 \text{km/s}\) is the speed of the light and \(X\) represents the model parameters. Thus, we rewrite \(\mu^t(z, X)\) as

\[
\mu^t(z_{\text{hel}}, z_{\text{CMB}}, X) = 5 \log_{10} \left[ (1 + z_{\text{hel}}) \int_0^{z_{\text{CMB}}} \frac{dz'}{E(z', X)} \right] + 52.385606 - 5 \log_{10} (H_0),
\]

Then, the \(\chi^2\) distribution function for the JLA data is

\[
\chi^2_{\text{JLA}}(X) = (\Delta \mu_t)^t (C_{\text{JLA}}^{-1})_{ij} (\Delta \mu_t),
\]

where \(\Delta \mu_t = \mu^t(z, X) - \mu^t_{\text{JLA}}\) is a column vector and \(C_{\text{JLA}}^{-1}\) is the 740 × 740 covariance matrix [3].

4.2.2 Redshift Space Distortion (RSD) data

Represent a compilation of measurements of the quantity \(f_{\sigma_8}\) at different redshifts, and obtained in a model independent way. These data are apparent anisotropies of the galaxy distribution in redshift space due to the differences of the estimates between the redshift observed distances and true distances. Here, \(f\) is combined with the root-mean-square amplitude of matter within a sphere of radius 8 Mpc\(^{-1}\), \(\sigma_8(z)\), in a single quantity. These data were derived from the following galaxy surveys: Pscz, 2dFVVDS, 6dF, 2MASS, BOSS and WiggleZgalaxy, respectively and collected by Mehrabi (see Table in [6]). Then, the standard \(\chi^2\) for this data set is given as [6]

\[
\chi^2_{\text{RSD}}(X) = \sum_{i=1}^{18} \left[ \frac{f_{\sigma_8}^\text{th}(X, z_i) - f_{\sigma_8}^\text{obs}(z_i)}{\sigma(z_i)} \right]^2,
\]

where \(\sigma(z_i)\) is the observed 1σ error, \(f_{\sigma_8}^\text{th}\) and \(f_{\sigma_8}^\text{obs}\) denote the theoretical and observational data, respectively.

4.2.3 BAO data sets

- **BAO I data**: Here, we use a compilation of measurements of the distance ratios \(d_s\) at different redshifts and obtained from different surveys [24,25,26,27,28,29,30,31,32,33,34,35], listed in Table 2. To encode the visual distortion of a spherical object due to the non-Euclidianity of a FRW spacetime, the authors [27,36] constructed a distance ratio \(D_v(z)\)

\[
D_v(z, X) \equiv \frac{1}{H_0} \left[ (1 + z)^2 D_A^2(z) \frac{c^2}{E(z)} \right]^{1/3},
\]

where \(D_A(z)\) is the angular diameter distance given by

\[
D_A(z, X) \equiv c \int_0^z \frac{dz'}{H(z', X)}.
\]

The comoving sound horizon size is defined by

\[
r_s(a) \equiv c \int_0^a \frac{c_s(a') da'}{a'^2 H(a')}.
\]

Then, the \(\chi^2\) distribution function for the JLA data is

\[
\chi^2_{\text{JLA}}(X) = (\Delta \mu_t)^t (C_{\text{JLA}}^{-1})_{ij} (\Delta \mu_t),
\]

where \(\Delta \mu_t = \mu^t(z, X) - \mu^t_{\text{JLA}}\) is a column vector and \(C_{\text{JLA}}^{-1}\) is the 740 × 740 covariance matrix [3].

**BAO II data**: From BOSS DR 9 CMASS sample, Chuang [37] analyzed the shape of the monopole and quadrupole from the two-dimensional two-points correlation function 2d2pCF of galaxies and measured simultaneously \(H, D_A,\)
The errors for the CMB data are contained in $C_{\text{CMB}}^{-1}$ given by

$$C_{\text{CMB}}^{-1} = \begin{pmatrix} +31.032 + 77.773 - 16.796 \\ +77.773 + 2687.7 - 1475.9 \\ -16.796 - 1475.9 + 1323.0 \end{pmatrix}. \quad (64)$$

Considering Eqs. (55), (57), (61) and (63), we can construct the total $\chi^2_{\text{BAO}}$ for all the BAO data, as

$$\chi^2_{\text{BAO}} = \chi^2_{\text{BAO}0} + \chi^2_{\text{BAO}1} + \chi^2_{\text{BAO}1}\text{II} + \chi^2_{\text{BAO}1}\text{IV}. \quad (65)$$

**4.2.4 CMB data**

We use the Planck distance priors data extracted from Planck 2015 results XIII Cosmological parameters. From here, we have obtained the Shift parameter $\bar{R}(z_s)$, the angular scale for the sound horizon at recombination epoch, $l_A(z_s)$, where $z_s$ represents the redshift at recombination epoch $^{23}$.

Hence, the shift parameter $\bar{R}$ is defined by $^{39}$

$$\bar{R}(z_s, \chi) \equiv \sqrt{\Omega_{M,0}} \int_0^{z_s} d\bar{y} \int_0^{\bar{y}} \frac{d\bar{y}}{2 \Omega_{M,0} \bar{y}^{1/2}}. \quad (66)$$

where $E(\bar{y})$ is given by Eq. (12). The redshift $z_s$ is obtained using $^{40}$

$$z_s = 1048 \left[ 1 + 0.00124 (\Omega_{b,0} h^2)^{-0.738} \right] \left[ 1 + g_1 (\Omega_{M,0} h^2)^{1/2} \right], \quad (67)$$

where

$$g_1 = \frac{0.0783 (\Omega_{b,0} h^2)^{-0.238}}{1 + 39.5 (\Omega_{b,0} h^2)^{0.763}}, \quad g_2 = \frac{0.560}{1 + 21.1 (\Omega_{b,0} h^2)^{1.01}}. \quad (68)$$

The angular scale $l_A$ for the sound horizon at recombination epoch is

$$l_A(X) \equiv \pi D_A(z_s, X) / r_s(z_s, X), \quad (69)$$

where $r_s(z_s, X)$ is the comoving sound horizon at $z_s$, and is given by Eq. (52). From $^{29}$, the $\chi^2$ is

$$\chi^2_{\text{CMB}} = \chi^2_{\text{CMB}0} + \chi^2_{\text{CMB}1}\text{II}, \quad (70)$$

where $\Delta x_i = x_i^{\text{th}}(X) - x_i^{\text{obs}}$ is a column vector

$$x_i^{\text{th}}(X) - x_i^{\text{obs}} = \begin{pmatrix} \Delta x_i^{\text{th}}(X) \end{pmatrix}, \quad (71)$$

$\chi^2$ denotes its transpose and $(C_{\text{CMB}}^{-1})_{ij}$ is the inverse covariance matrix $^{11}$ given by

$$C_{\text{CMB}}^{-1} = \begin{pmatrix} +162.48 - 1529.4 + 2.0688 \\ -1529.4 + 2073.23 - 2866.8 \\ +2.0688 - 2866.8 + 53.572 \end{pmatrix}. \quad (72)$$

The errors for the CMB data are contained in $C_{\text{CMB}}^{-1}$.
4.2.5 Hubble data $\dot{H}(z)$

This sample is composed by 38 independent measurements of the Hubble parameter at different redshifts [12] and were derived from differential age $dt$ for passively evolving galaxies with redshift $dz$ and from the two-points correlation function of Sloan Digital Sky Survey. This sample was taken from Table III in [12]. Then, the $\chi^2$ function for this data set is [12]

$$\chi^2(X) = \frac{1}{\sigma^2(z_i)} \sum_{i=1}^{38} \left[ H^{th}(X, z_i) - H^{obs}(z_i) \right]^2,$$

where $H^{th}$ denotes the theoretical value of $\dot{H}$, $H^{obs}$ represents its observed value and $\sigma(z_i)$ is the error.

### Table 3. Shows the $\dot{H}(z)$ data

| $z$ | $\dot{H}(z)$ | $1\sigma$ | Refs. | $z$ | $\dot{H}(z)$ | $1\sigma$ | Refs. |
|-----|-------------|----------|-------|-----|-------------|----------|-------|
| 0.070 | 69.0 | ±19.6 | [33] | 0.570 | 96.8 | ±3.40 | [32] |
| 0.090 | 69.0 | ±12.0 | [33] | 0.593 | 104.0 | ±13.0 | [32] |
| 0.120 | 68.6 | ±26.2 | [33] | 0.600 | 87.9 | ±6.1 | [19] |
| 0.170 | 83.0 | ±8.0 | [33] | 0.680 | 92.0 | ±8.0 | [19] |
| 0.179 | 75.0 | ±4.0 | [33] | 0.730 | 97.3 | ±7.0 | [19] |
| 0.199 | 75.0 | ±5.0 | [33] | 0.761 | 105.0 | ±12.0 | [19] |
| 0.200 | 72.9 | ±29.6 | [33] | 0.875 | 125.0 | ±17.0 | [19] |
| 0.240 | 76.9 | ±2.99 | [33] | 0.880 | 90.0 | ±40.0 | [19] |
| 0.270 | 77.0 | ±14.0 | [33] | 0.900 | 117.0 | ±23.0 | [19] |
| 0.280 | 88.8 | ±36.6 | [33] | 1.037 | 154.0 | ±20.0 | [19] |
| 0.300 | 81.7 | ±6.22 | [33] | 1.300 | 168.0 | ±17.0 | [19] |
| 0.340 | 83.8 | ±3.66 | [33] | 1.363 | 160.0 | ±33.6 | [19] |
| 0.350 | 82.7 | ±9.1 | [33] | 1.430 | 177.0 | ±18.0 | [19] |
| 0.352 | 83.0 | ±14.0 | [33] | 1.530 | 140.0 | ±14.0 | [19] |
| 0.400 | 95.0 | ±17.0 | [33] | 1.750 | 202.0 | ±40.0 | [19] |
| 0.430 | 86.5 | ±3.97 | [33] | 1.965 | 186.5 | ±50.4 | [19] |
| 0.440 | 82.6 | ±7.8 | [33] | 2.300 | 224.0 | ±8.6 | [19] |
| 0.480 | 97.0 | ±62.0 | [33] | 2.340 | 222.0 | ±8.5 | [19] |
| 0.570 | 87.6 | ±7.80 | [33] | 2.360 | 226.0 | ±9.3 | [19] |

### Table 4. Shows the priors on the parameter space.

| Parameters | Constant Priors |
|------------|----------------|
| $\lambda_1$ | $[-1.5 \times 10^{-3}, +1.5 \times 10^{-3}]$ |
| $\lambda_2$ | $[-1.5 \times 10^{-3}, +1.5 \times 10^{-3}]$ |
| $\omega$ | $[0.2, 0.3]$ |
| $\omega_1$ | $[-1.0, +1.0]$ |
| $\omega_2$ | $[0.0, +0.1]$ |
| $\Omega_{M0}$ | $[0.67]$ |
| $H_0$ (km/s/Mpc$^{-1}$) | $[20, 120]$ |
| $\alpha$ | $[0.2, 0.5]$ |
| $\beta$ | $[-2.1, +3.8]$ |
| $M$ | $[-20, -17]$ |
| $dM$ | $[-1.0, +1.0]$ |
| $\gamma_0$ | $[0.2, +1.2]$ |
| $s_{10}$ | $[0.1, +1.65]$ |

5 Results

In this work, we have run eight chains for each of our models on the computer, and the obtained outcomes of the main and derived parameters are presented in Table 5 in which the best estimated parameters with their $1\sigma$ and $2\sigma$ errors are shown. Moreover, the minimum $\chi^2_{\text{min}}$ is 712.3048 for the IDE model, which is smaller in comparison with those obtained in the non-interacting models and the one-dimension probability contours at $1\sigma$ and $2\sigma$ on single parameters are plotted in Fig. 11.

Likewise, from Table 5 and Fig. 11 we notice that the inclusion of CMB and RSD data allow to break the degeneracy among the different parameters of our models, obtaining constraints more stringent on them. When $I_0 = 0$, one finds that the DE model is very close to the IDE model. Due to the two minima obtained in the IDE model (see Table 5), we consider now two different cases to reconstruct $I_0$ in the case 1 is so-called IDE1 with $A_2 > 0$; by contrast, the case 2 is so-called IDE2 with $A_2 < 0$.

From the left above panel of Fig. 2, one can see that the universe evolves from the phantom regime $\omega_{DE} < -1$ to the quintessence regime $\omega_{DE} > -1$, and then it becomes phantom again; and in particular, crosses the phantom divide line $\omega_{DE} = -1$ [22]. The IDE model has two crossing points in $a = 0.4043$ and $a = 0.9894$, respectively. Such a crossing feature $\omega_{DE} = -1$ is favored by the data about at $1\sigma$ error. Then, our fitting results show that the evolution of $\omega_{DE}$ in the DE and IDE models are very close to each other, in particular, they are close to −1 today.

Now, in the right above panel of Fig. 2, we have considered that at early times when DM dominates the universe $I_+$ denotes an energy transfer from DE to DM and $I_-$ denotes an energy transfer from DM to DE. Here,
This change of sign is linked to $\bar{I}$ and remains finite when $a$ from the right below panel of this figure, we note that the ln $R$, exhibits a scaling behavior at early times (keeping $\Theta = 1$). As is apparent, $\bar{I}$ affects violently the coupling affects with $Q$ in the IDE model. By contrast, the situation is opposite to that of the right side of this line. This panel also shows that the background evolution of $\ln R$, exhibits a scaling behavior at early times (keeping constant) but not at the present day. These results significantly alleviate the coincidence problem, but they do not solve it in full. From the right below panel of the Fig. 2 we see that at $\ln a < 0$ the coupling affects violently the background evolution of $\rho_{de}$ in the IDE model. By contrast, the situation is opposite at $\ln a > 0$. Furthermore, the graphs for $\rho_{dm}$ are essentially overlapped during their evolution.

The left above panel of Fig. 3 shows the evolution of the structure growth of the matter, $\sigma_{8}$, along $z$ for the different cosmologies. These curves are comparable with each other at $z < 0.4$ but they deviate one after another at $z > 0.4$. It implies that they are sensitive to the background cosmology. Within the matter era the amplitude of $\sigma_{8}$ in the IDE model is enhanced relative to the $\omega_{de}$ model at $z < 3$, but both are smaller than that found in the $\Lambda$CDM. At $z > 5$, $\bar{Q}$ and $\omega_{de}$ would bring about a large structure formation in the IDE and $\omega_{de}$ scenarios, respectively. Due to the fact that the amount of DM is bigger than the amount of $\omega_{DE}$ at earlier times; therefore, it produces an enhancement on the amplitudes of $\sigma_{8}$ in the IDE model respect to that found in the $\Lambda$CDM, respectively. Our fitting results are consistent at $1\sigma$ error with those reported by $[65,73,74]$. However, our outcomes are smaller. This small discrepancy is due to the ansatz chosen for $\Omega_{0}$ and the data used.

Also, in the left below panel of Fig. 2 we note that $R$ is always positive when both $\Omega_{0}$ and $\omega_{DE}$ are time-varying, and remains finite when $a \rightarrow \infty$. As is apparent, $\Omega$ seems to alleviate the coincidence problem for $\ln a \leq 0$. Likewise, from the right below panel of this figure, we note that the vertical line indicates the moment when $\Omega(1+\omega_{de})/\Omega_{dm}$ and $|\rho_{dm}/\Omega_{0}|$ are equal, see Eq. (4). Here, to the left of this line, $Q$ affects the background evolution of $\rho_{de}$. By contrast, the situation is opposite to that of the right side of this line. This panel also shows that the background evolution of $\ln R$, exhibits a scaling behavior at early times (keeping constant) but not at the present day. These results significantly alleviate the coincidence problem, but they do not solve it in full. From the right below panel of the Fig. 2 we see that at $\ln a < 0$ the coupling affects violently the background evolution of $\rho_{de}$ in the IDE model. By contrast, the situation is opposite at $\ln a > 0$. Furthermore, the graphs for $\rho_{dm}$ are essentially overlapped during their evolution.

Table 5 shows the best-fit values of the cosmological parameters for the studied models with $1\sigma$ and $2\sigma$ errors.

| Parameters | $\times 10^{-4}$ | $\times 10^{-8}$ | $\times 10^{-10}$ | $\times 10^{-12}$ |
|-----------|-----------------|-----------------|-----------------|-----------------|
| $\lambda_{0}$ | N/A | N/A | N/A | N/A |
| $\lambda_{1}$ | N/A | N/A | N/A | N/A |
| $\lambda_{2}$ | N/A | N/A | N/A | N/A |
| $\omega_{a}$ | $-1.0$ | $-1.0364$ | $-0.8244$ | $-0.9265$ |
| $\omega_{b}$ | $N/A$ | $N/A$ | $N/A$ | $N/A$ |
| $\Omega_{DM,0}$ | $+0.2810$ | $+0.2844$ | $+0.0121$ | $+0.0121$ |
| $\Omega_{b,0}$ | $+0.0493$ | $+0.0042$ | $-0.0014$ | $-0.0014$ |
| $H_{0}$ | $+67.1903$ | $+67.1490$ | $+67.1490$ | $+67.1490$ |
| $\alpha$ | $+0.1360$ | $+0.1360$ | $+0.1360$ | $+0.1360$ |
| $\beta$ | $+3.0688$ | $+3.0780$ | $+3.0780$ | $+3.0780$ |
| $\gamma$ | $-0.5764$ | $-0.5764$ | $-0.5764$ | $-0.5764$ |
| $\sigma_{8}$ | $-0.5511$ | $-0.5511$ | $-0.5511$ | $-0.5511$ |
| $\tau$ | $-0.6652$ | $-0.6652$ | $-0.6652$ | $-0.6652$ |
| $\Omega_{0}$ | $1.0479$ | $1.0493$ | $1.0493$ | $1.0493$ |
| $\ln[10^{3.5}A_{s}]$ | $+4.5307$ | $+4.3414$ | $+4.3414$ | $+4.3414$ |

$\chi_{\text{min}}^{2}$ | 737.8851 | 726.0120 | 727.0198 | 727.1856 |

we have found a change from $I_{+}$ to $I_{-}$ and vice versa. This change of sign is linked to $\Omega_{0} = 0$ and is also favored by the data at $1\sigma$ error. The IDE model shows three crossing points in $\alpha = 1.5238$ (IDE1), $\alpha = 0.1512$ (IDE2) and $a = 1.8462$ (IDE2), respectively. The fitting results indicate that $I_{Q}$ is stronger at early times and weaker at later times, namely, $I_{Q}$ remains small today, being $I_{Q,0} = 0.8661 \times 10^{-4} + 0.557 \times 10^{-4}$ for the case IDE1 and $I_{Q,0} = 1.3849 \times 10^{-4} + 0.420 \times 10^{-4}$ for the case IDE2, respectively. These results are consistent at $1\sigma$ error with those reported by $[65,73,74]$. However, our outcomes are smaller. This small discrepancy is due to the ansatz chosen for $\Omega_{0}$ and the data used.

Also, in the left below panel of Fig. 2 we note that $R$ is always positive when both $I_{Q}$ and $\omega_{DE}$ are time-varying, and remains finite when $a \rightarrow \infty$. As is apparent, $\Omega$ seems to alleviate the coincidence problem for $\ln a \leq 0$. Likewise, from the right below panel of this figure, we note that the vertical line indicates the moment when $\Omega(1+\omega_{de})/\Omega_{dm}$ and $|\rho_{dm}/\Omega_{0}|$ are equal, see Eq. (4). Here, to the left of this line, $Q$ affects the background evolution of $\rho_{de}$. By contrast, the situation is opposite to that of the right side of this line. This panel also shows that the background evolution of $\ln R$, exhibits a scaling behavior at early times (keeping constant) but not at the present day. These results significantly alleviate the coincidence problem, but they do not solve it in full. From the right below panel of the Fig. 2 we see that at $\ln a < 0$ the coupling affects violently the background evolution of $\rho_{de}$ in the IDE model. By contrast, the situation is opposite at $\ln a > 0$. Furthermore, the graphs for $\rho_{dm}$ are essentially overlapped during their evolution.

The left above panel of Fig. 3 shows the evolution of the structure growth of the matter, $\sigma_{8}$, along $z$ for the different cosmologies. These curves are comparable with each other at $z < 0.4$ but they deviate one after another at $z > 0.4$. It implies that they are sensitive to the background cosmology. Within the matter era the amplitude of $\sigma_{8}$ in the IDE model is enhanced relative to the $\omega_{DE}$ model at $z < 3$, but both are smaller than that found in the $\Lambda$CDM. At $z > 5$, $\bar{Q}$ and $\omega_{DE}$ would bring about a large structure formation in the IDE and $\omega_{DE}$ scenarios, respectively. Due to the fact that the amount of DM is bigger than the amount of $\omega_{DE}$ at earlier times; therefore, it produces an enhancement on the amplitudes of $\sigma_{8}$ in the IDE model respect to that found in the $\Lambda$CDM, respectively. Our fitting results are consistent at $1\sigma$ error with those reported by $[65,73,74]$. The right above panel of this Figure depicts the evolution of the density matter power for different scenarios at $z = 1.6$. Notice that $P(k)$ in the IDE model is enhanced with respect to that in the $\Lambda$CDM but it is suppressed in relation to that found in the $\omega_{DE}$ scenario. That could be a consequence of the amount of concentrated matter at early times and also the presence of $Q$. Moreover, the vertical line indicates the turnover position, $k_{01}$, which is very close in our models. Also, we notice a series of wiggles on the $P(k)$ due to the coupling between the photons and baryons before recombination; namely, the presence of baryons have left their effect there. These arguments, the IDE and $\omega_{DE}$ can be distinguished from the curves of $\sigma_{8}$, and the structure growth data could break the possible degeneracy between these two models and provides a signature to discriminate them. These outcomes are in correspondence with those found by $[60,62,63]$ at $1\sigma$ error. Likewise, the below panel of this Figure displays the effects of $Q$ and $\omega_{DE}$ on the amplitude of the CMB temperature power spectrum at low multipoles $l < 100$, in where the amplitude of the integrated Sachs-Wolfe effect is deviated in the IDE model respect to that found in the other scenarios. Instead, at high multipoles $l > 100$, $Q$ and $\omega_{DE}$ increase the concentration...
of DM early times, affecting the sound horizon at the end, which shifts to right the values of the acoustic peaks located at $l = n_p \pi D_A(z_\star)/\bar{r}_s(z_\star)$, $n_p = 1, 2, 3...$, and reduced the amplitudes of the first peaks when the studied models are compared. These features can be understood by considering the extra-terms proportionals to $\lambda_0, \lambda_1$ and $\lambda_2$ in
Likewise from Table 5 and Fig. 2, our fitting results show $\omega_{\text{DE}}$ ground evolution. Similarly, $\omega_{\text{DE}}$ an interaction $\bar{I}_Q$.

In this work, we examined an interacting DE model with the Hubble parameter $\bar{H}$, and to a time-varying function, $\bar{I}_Q$ expanded in terms of the Chebyshev polynomials $T_n$, defined in the interval $[-1,1]$. Besides, we also consider a time-varying EoS parameter, $\omega_{\text{DE}}$, expressed in function of those polynomials. These ansatzes have been proposed so that their background evolution are free of divergences at the present time and also at the future time, respectively. In a Newtonian gauge and on sub-horizon scales, a set of perturbed equations is obtained when the momentum transfer potential becomes null in the DM rest-frame. This leads to different cases in the IDE model. Based on a combined analysis of geometric and dynamical probes which include JLA + RSD + BAO + CMB + H data and using the MCMC, we found the best-fit parameters that constrain the background evolution of our models.

We have also considered the perturbed equations for the DM and baryons in the rest-frame of DM. Besides, we have built the theoretical and numerical structures, and in particular, we used the c++ language to show the combined impact of both $\bar{Q}$ and $\omega_{\text{DE}}$ on the evolution of $\bar{R}$, $\bar{\rho}_{\text{DM}}$, $\bar{\rho}_{\text{DE}}$, $\bar{\sigma}_8$, $P(k)$ and $l(l+1)C_l$, respectively.

Likewise from Table 5 and Fig. 2 our fitting results show that $\bar{I}_Q$ can cross twice the line $\bar{Q} = 0$ during its background evolution. Similarly, $\omega_{\text{DE}}$ crosses the line $\omega_{\text{DE}} = -1$ twice as well. These crossing features are favored by the data at $1\sigma$ error. Furthermore, we also notice that $R$ is always positive and remains finite in all our models when $a \to \infty$. Moreover, in the IDE model, $R$ exhibits a scaling behaviour at early times (keeping constant). Then, $\bar{Q}$ seems to alleviate the coincidence problem for ln $a \leq 0$ but it does not solve that problem in full.

On the other hand, from Fig. 3 we found that the evolution curve of $\bar{\sigma}_8$ in the IDE model deviates significantly from that obtained in the $\Lambda$CDM and $\omega_{\text{DE}}$ models. It meant that, the structure formation data could break the possible degeneracy between the IDE and $\omega_{\text{DE}}$ models. In these last two models, several best-fit parameters are very close with each other, therefore, one could then conclude that this detected deviation is brough about mainly by $\omega_{\text{DE}}$, namely, $\bar{\sigma}_8$ is sensitive mainly to the evolution of $\omega_{\text{DE}}$ and depends on its parametrisation. Moreover, the geometric probes favor the existence of an interaction between the dark sectors but the dynamical test constrains its intensity. These effects can be understood by considering the extra-terms proportional to $\bar{I}_Q$ in the DM energy density, (see Eq. (12)), which increases and, in consequence, amplifies the amount of DM at earlier times. That is in accordance with the result found in the previous panel of this Figure and with those found in [60][62][63] at $1\sigma$ error.

6 Conclusions

In this work, we examined an interacting DE model with an interaction $\bar{Q}$ proportional to the DM energy density, to the Hubble parameter $\bar{H}$, and to a time-varying function, $\bar{I}_Q$ expanded in terms of the Chebyshev polynomials $T_n$, defined in the interval $[-1,1]$. Besides, we also consider a time-varying EoS parameter, $\omega_{\text{DE}}$, expressed in function of those polynomials. These ansatzes have been proposed so that their background evolution are free of divergences at the present time and also at the future time, respectively. In a Newtonian gauge and on sub-horizon scales, a set of perturbed equations is obtained when the momentum transfer potential becomes null in the DM rest-frame. This leads to different cases in the IDE model. Based on a combined analysis of geometric and dynamical probes which include JLA + RSD + BAO + CMB + H data and using the MCMC, we found the best-fit parameters that constrain the background evolution of our models.

We have also considered the perturbed equations for the DM and baryons in the rest-frame of DM. Besides, we have built the theoretical and numerical structures, and in particular, we used the c++ language to show the combined impact of both $\bar{Q}$ and $\omega_{\text{DE}}$ on the evolution of $\bar{R}$, $\bar{\rho}_{\text{DM}}$, $\bar{\rho}_{\text{DE}}$, $\bar{\sigma}_8$, $P(k)$ and $l(l+1)C_l$, respectively.

Likewise from Table 5 and Fig. 2 our fitting results show that $\bar{I}_Q$ can cross twice the line $\bar{Q} = 0$ during its background evolution. Similarly, $\omega_{\text{DE}}$ crosses the line $\omega_{\text{DE}} = -1$ twice as well. These crossing features are favored by the data at $1\sigma$ error. Furthermore, we also notice that $R$ is always positive and remains finite in all our models when $a \to \infty$. Moreover, in the IDE model, $R$ exhibits a scaling behaviour at early times (keeping constant). Then, $\bar{Q}$ seems to alleviate the coincidence problem for ln $a \leq 0$ but it does not solve that problem in full.

On the other hand, from Fig. 3 we found that the evolution curve of $\bar{\sigma}_8$ in the IDE model deviates significantly from that obtained in the $\Lambda$CDM and $\omega_{\text{DE}}$ models. It meant that, the structure formation data could break the possible degeneracy between the IDE and $\omega_{\text{DE}}$ models. In these last two models, several best-fit parameters are very close with each other, therefore, one could then conclude that this detected deviation is brough about mainly by $\omega_{\text{DE}}$, namely, $\bar{\sigma}_8$ is sensitive mainly to the evolution of $\omega_{\text{DE}}$ and depends on its parametrisation. Moreover, the geometric probes favor the existence of an interaction between the dark sectors but the dynamical test constrains its intensity. These effects can be understood by considering the extra-terms proportional to $\bar{I}_Q$ in the DM energy density, (see Eq. (12)), which increases and, in consequence, amplifies the amount of DM at earlier times. That is in accordance with the result found in the previous panel of this Figure and with those found in [60][62][63] at $1\sigma$ error.

6 Conclusions

In this work, we examined an interacting DE model with an interaction $\bar{Q}$ proportional to the DM energy density, to the Hubble parameter $\bar{H}$, and to a time-varying function, $\bar{I}_Q$ expanded in terms of the Chebyshev polynomials $T_n$, defined in the interval $[-1,1]$. Besides, we also consider a time-varying EoS parameter, $\omega_{\text{DE}}$, expressed in function of those polynomials. These ansatzes have been proposed so that their background evolution are free of divergences at the present time and also at the future time, respectively. In a Newtonian gauge and on sub-horizon scales, a set of perturbed equations is obtained when the momentum transfer potential becomes null in the DM rest-frame. This leads to different cases in the IDE model. Based on a combined analysis of geometric and dynamical probes which include JLA + RSD + BAO + CMB + H data and using the MCMC, we found the best-fit parameters that constrain the background evolution of our models.

We have also considered the perturbed equations for the DM and baryons in the rest-frame of DM. Besides, we have built the theoretical and numerical structures, and in particular, we used the c++ language to show the combined impact of both $\bar{Q}$ and $\omega_{\text{DE}}$ on the evolution of $\bar{R}$, $\bar{\rho}_{\text{DM}}$, $\bar{\rho}_{\text{DE}}$, $\bar{\sigma}_8$, $P(k)$ and $l(l+1)C_l$, respectively.

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On the other hand, from Fig. 3 we found that the evolution curve of $\bar{\sigma}_8$ in the IDE model deviates significantly from that obtained in the $\Lambda$CDM and $\omega_{\text{DE}}$ models. It meant that, the structure formation data could break the possible degeneracy between the IDE and $\omega_{\text{DE}}$ models. In these last two models, several best-fit parameters are very close with each other, therefore, one could then conclude that this detected deviation is brough about mainly by $\omega_{\text{DE}}$, namely, $\bar{\sigma}_8$ is sensitive mainly to the evolution of $\omega_{\text{DE}}$ and depends on its parametrisation. Moreover, the geometric probes favor the existence of an interaction between the dark sectors but the dynamical test constrains its intensity. These effects can be understood by considering the extra-terms proportional to $\bar{I}_Q$ in the DM energy density, (see Eq. (12)), which increases and, in consequence, amplifies the amount of DM at earlier times. As a result, the growth of structure is significantly affected by $\bar{Q}$ and $\omega_{\text{DE}}$, which induce that the amplitude of $P(k)$ becomes higher in the IDE model than that found in the $\Lambda$CDM model but it is lesser than that found in the $\omega_{\text{DE}}$ model. Moreover, the position of the turnover point in all our models is very close at smaller scales. Likewise, we also notice a series of wiggles on the curve of $P(k)$ due to the coupling between the photons and baryons before of the recombination; namely, the presence of baryons have left their effect in this plot (see Fig. 3). Finally, we also find that the amplitude of the CMB temperature power

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig3.pdf}
\caption{Shows the combined impact of $\bar{Q}$ and $\omega_{\text{DE}}$ on the evolution of the structure growth of the matter $\bar{\sigma}_8$, on the linear matter power spectrum $P(k)$, and on the CMB temperature power spectrum $l(l+1)C_l(2\pi)^{-1}(\mu k)^2$, respectively.}
\end{figure}
spectrum is also sensitive to $Q$ at low and high multipoles. In the IDE model, $Q$ produces a shift of the acoustic peaks to the right and their amplitudes are reduced at high multipoles with respect to the uncoupled models. Besides, at low multipoles the amplitude of the integrated Sachs-Wolfe effect is also affected by $Q$.

The results for the other case when the momentum-transfer potential vanishing in the DE rest-frame will be presented in a future work.

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