Complete invariant geodesic metrics on outer spaces and Jacobian varieties of tropical curves

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November 8, 2012

Abstract

Let Out($F_n$) be the outer automorphism group of the free group $F_n$. It acts properly on the outer space $X_n$ of marked metric graphs, which is a finite-dimensional infinite simplicial complex with some simplicial faces missing. In this paper, we construct complete geodesic metrics and complete piecewise smooth Riemannian metrics on $X_n$ which are invariant under Out($F_n$). One key ingredient is the identification of metric graphs with tropical curves and the use of the tropical Jacobian map from the moduli space of tropical curves to the moduli space of principally polarized tropical abelian varieties.

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*Partially Supported by NSF grant DMS-1104696
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1 Introduction

Let \( F_n \) be the free group on \( n \) generators, \( n \geq 2 \), and \( \text{Out}(F_n) = \text{Aut}(F_n)/\text{Inn}(F_n) \) its outer automorphism group. The group \( \text{Out}(F_n) \) is one of the most basic groups in combinatorial group theory and it has been extensively studied. One reason is that it is related to a basic question: given a finitely generated group \( \Gamma \), how to get all possible generating sets of \( \Gamma \) [27, p. 120]. One key ingredient in understanding the properties of \( \text{Out}(F_n) \) is to use its action on the outer space \( X_n \) of marked metric graphs and the properties of \( X_n \).

According to the celebrated Erlangen program of Klein, an essential part of the geometry of a space is concerned with invariants of the isometries (or symmetries) of the space. Similarly, an essential part of the geometry of a group is to find and understand spaces on which the group acts and preserves suitable additional structures of the space. It is often natural to require that the space is a metric space and the action is by isometries.

In classical geometry, a metric space is a complete Riemannian manifold such as the Euclidean space, the sphere or the hyperbolic space. But it is also important to consider metric spaces which are not manifolds. Examples include Tits buildings and Bruhat-Tits buildings for real and \( p \)-adic semisimple Lie groups. These are simplicial complexes with a natural complete Tits metric so that the groups act isometrically and simplicially on them. Of course, the rich combinatorial structure also makes their geometry interesting.

The outer space \( X_n \) is a finite-dimensional infinite simplicial complex and hence admits a natural simplicial metric \( d_0 \) (Proposition \[23\]), where each \( n \)-simplex is realized as the standard simplex in \( \mathbb{R}^{n+1} \) with vertices \((1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\) and given the metric induced from the metric of \( \mathbb{R}^{n+1} \). Clearly, \( d_0 \) is invariant under \( \text{Out}(F_n) \). Since simplices of \( X_n \) have missing simplicial faces, \( d_0 \) is not a complete metric on \( X_n \).

Since the completeness of metrics is a basic condition which is important for many applications, one natural problem is the following (see [6, Question 2] for more discussion).

**Problem 1.1** Construct complete geodesic metrics on the outer space \( X_n \) which are invariant under \( \text{Out}(F_n) \).

As explained above, solutions to this problem provide geometries of \( \text{Out}(F_n) \) in a certain sense. Another motivation for this problem comes from the analogy with other important groups in geometric group theory: arithmetic subgroups \( \Gamma \) of semisimple Lie groups \( G \) and mapping class groups \( \text{Mod}_{g,n} \) of surfaces of genus \( g \) with \( n \) punctures. A lot of work on \( \text{Out}(F_n) \) is motivated by results obtained for these families of groups.

An arithmetic group \( \Gamma \) acts on the symmetric space \( G/K \), and the mapping class group \( \text{Mod}_{g,n} \) acts on the Teichmüller space \( \mathcal{T}_{g,n} \) of Riemann surfaces of genus \( g \) with \( n \) punctures. The outer space \( X_n \) was introduced as a space on which \( \text{Out}(F_n) \) acts naturally. These actions have played a crucial and similar role in understanding properties of the groups.
Though these three classes of groups and spaces share many common properties, there is one important difference between them, which might have not been emphasized enough: the symmetric spaces $X$ and the Teichmüller space $\mathcal{T}_{g,n}$ are smooth manifolds, but the outer space $X_n$ is not a manifold.

The symmetric space $X = G/K$ admits a complete $G$-invariant Riemannian metric and $\Gamma$ acts isometrically and properly on $X$. It is known that $X$ is a contractible, nonpositively curved Riemannian manifold, and it is an important example of a Hadamard manifold. Furthermore, the quotient $\Gamma \backslash X$ has finite volume.

The Teichmüller space $\mathcal{T}_{g,n}$ admits several complete Riemannian and Finsler metrics such as the Bergman and Teichmüller metrics, and $\text{Mod}_{g,n}$ acts isometrically and properly on them. $\mathcal{T}_{g,n}$ also admits an incomplete negatively curved (Weil-Petersson) metric. For all these natural metrics, the quotient $\text{Mod}_{g,n} \backslash \mathcal{T}_{g,n}$ has finite volume. An interesting feature is that the known complete metrics on $\mathcal{T}_{g,n}$ are not nonpositively curved, but the negatively curved Weil-Petersson metric is incomplete. A natural and folklore conjecture is that $\mathcal{T}_{g,n}$ does not admit any complete nonpositively curved metric which is invariant under $\text{Mod}_{g,n}$.

Though $X_n$ is not a manifold, it is a locally finite simplicial complex and hence is a canonically stratified space with a smooth structure in the sense of [34]. Therefore, the following problem also seems to be natural in view of the above analogy.

**Problem 1.2** Construct piecewise smooth Riemannian metrics on the outer space $X_n$ which are invariant under $\text{Out}(F_n)$ and whose induced length metrics are complete geodesic metrics. Furthermore, the quotient $\text{Out}(F_n) \backslash X_n$ has finite volume.

With respect to the smooth structure on $X_n$ as a canonically stratified space, it is natural to require that the Riemannian metric on $X_n$ is smooth in the sense of [34] Chapter 2. If the second condition of finite volume is satisfied, it might give further justification to the philosophy that $\text{Out}(F_n)$ is an analogue of lattices of Lie groups, which has played a guiding role in understanding $\text{Out}(F_n)$.

In this paper, we construct several invariant complete geodesic metrics and complete piecewise-smooth Riemannian metrics on $X_n$ and hence solve Problem 1.1 and Problem 1.2. The basic idea is to identify metric graphs with tropical curves and use the tropical Jacobian map of tropical curves together with the simplicial metric $d_0$ on the outer space $X_n$. For precise statements, see Theorems 8.1, 9.6, 10.2, and Propositions 11.1, 11.2, 11.4 below. In the process, we also clarify some facts on the moduli space of tropical abelian varieties (Proposition 6.18).

**Remark 1.3** It should be pointed that the tropical Jacobian map, or rather the period map of metric graphs, can be defined directly for compact metric graphs. But without interpretation through tropical curves and their Jacobians, it is a mystery why it is defined in this way. There are infinitely many different ways to define a map from the outer space $X_n$ to the symmetric space $\text{GL}(n, \mathbb{R})/\text{O}(n)$ of positive definite matrices (see Remark 6.3 below). In any case, the connection between metric graphs and tropical curves is interesting and our discussion here also clarifies some questions in tropical geometry. It is expected that the geometry of the outer space will be helpful to the study of tropical curves and their moduli spaces. See Remark 6.19 below.

**Remark 1.4** It is known that $X_n$ admits a non-symmetric geodesic metric, called the Lipschitz (or Thurston) metric. Briefly, the non-symmetric Lipschitz metric is not complete, and its associated (symmetric) is not geodesic. For its definition, some properties and applications, see [14] [2] [1].
Remark 1.5 It seems reasonable to conjecture that $X_n$ does not admit any complete CAT(0)-metric which is invariant under $\text{Out}(F_n)$. For example, it was shown in [5] that the standard spine of $X_n$ does not admit any invariant piecewise-Euclidean or piecewise-hyperbolic metric of nonpositive curvature. It was shown in [16] [7] that $\text{Out}(F_n)$ is not a CAT(0)-group, i.e., it cannot act isometrically and cocompactly on a CAT(0)-space. Since the quotient $\text{Out}(F_n)\backslash X_n$ is not compact, the above result does not exclude the possibility that $X_n$ might admit a CAT(0)-metric which is invariant under $\text{Out}(F_n)$.

Acknowledgments. The idea of using the period map of metric graphs in Equation (3.3) to pull back and define a metric on the outer space was suggested by Enrico Leuzinger, partly motivated by the work [23], and I would like to thank him for sharing this idea. Though the period map fails to be injective (unlike the case of Riemann surfaces), the simple idea of combining the period map with the canonical simplicial metric $d_0$ on the outer space $X_n$ is crucial to overcome this problem. I would also like to thank Mladen Bestvina for helpful information about CAT(0)-metrics on the outer space, and Athanase Papadopoulos for carefully reading a preliminary version of this paper.

2 Definitions and basic facts on $\text{Out}(F_n)$ and the outer space $X_n$

In this section, we recall some basic definitions and results on the outer automorphism group $\text{Out}(F_n)$ and the outer space $X_n$, and relations with arithmetic groups and mapping class groups.

Probably the most basic arithmetic subgroup of a semisimple Lie group is $\text{SL}(n, \mathbb{Z}) \subset \text{SL}(n, \mathbb{R})$. A closely related group is the arithmetic subgroup $\text{GL}(n, \mathbb{Z})$ of a reductive Lie group $\text{GL}(n, \mathbb{R})$, which contains $\text{SL}(n, \mathbb{Z})$ as a subgroup of index 2.

The group $\text{Out}(F_n)$ is related to $\text{GL}(n, \mathbb{Z})$ as follows. Since the abelinization of $F_n$ is $\mathbb{Z}^n$ and $\text{Out}(\mathbb{Z}^n) = \text{Aut}(\mathbb{Z}^n) = \text{GL}(n, \mathbb{Z})$, there is a homomorphism $A : \text{Out}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$. It is known that for every $n \geq 3$, the homomorphism $A : \text{Out}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$ is surjective with an infinite kernel and that when $n = 2$, $A : \text{Out}(F_2) \rightarrow \text{GL}(2, \mathbb{Z})$ is an isomorphism. See [38] p. 2 for references.

Let $S_g$ be a compact oriented surface of genus $g$, $\text{Diff}^+(S_g)$ its group of orientation preserving diffeomorphisms, and $\text{Diff}^0(S_g)$ its identify component. Then the quotient group $\text{Mod}_g = \text{Diff}^+(S_g)/\text{Diff}^0(S_g)$ is the mapping class group of $S_g$.

By the Dehn-Nielson theorem, $\text{Mod}_g = \text{Out}(\pi_1(S_g))$. Since the abelinization of $\pi_1(S_g)$ is $H_1(S_g, \mathbb{Z})$, which is isomorphic to $\mathbb{Z}^{2g}$, we also get a homomorphism $\text{Mod}_g \rightarrow \text{GL}(2g, \mathbb{Z})$. Since elements of $\text{Mod}_g$ also preserve the intersection form on $H_1(S_g, \mathbb{Z})$, which is a symplectic form, the image of $\text{Mod}_g$ lies in $\text{Sp}(2g, \mathbb{Z})$. Therefore, we have a homomorphism

$$j : \text{Mod}_g \rightarrow \text{Sp}(2g, \mathbb{Z}).$$

It is known that $j$ is surjective with an infinite kernel when $g \geq 2$, and $j$ is an isomorphism when $g = 1$, i.e., $\text{Mod}_1 \cong \text{SL}(2, \mathbb{Z})$.

The above discussions establish close relations between $\text{Out}(F_n)$, and the mapping class group $\text{Mod}_g$ and arithmetic groups such as $\text{GL}(n, \mathbb{Z})$ and $\text{Sp}(2g, \mathbb{Z})$. These groups share many common properties, for example, group theoretical properties and cohomological properties. One important approach to study groups is to study their actions on suitable spaces. But it is often difficult to find spaces with finiteness properties.

As mentioned before, an arithmetic subgroup $\Gamma \subset G$ acts isometrically and properly on the symmetric space $X = G/K$, and $X$ is a finite dimensional model of the universal space $G\Gamma$ for proper actions of $\Gamma$. 

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An important space for $\text{Mod}_{g,n}$ to act on is the Teichmüller space $T_{g,n}$, which is the space of marked complex structure of $S_{g,n}$, i.e., marked Riemann surfaces of genus $g$ with $n$ punctures. The action of $\text{Mod}_{g,n}$ on $T_{g,n}$ comes from changing the markings. It is an important result of Teichmüller that the Teichmüller space $T_{g,n}$ is a contractible manifold and that the action is proper and isometric with respect to the Teichmüller metric (and also many other metrics). It is also known that $T_{g,n}$ is a finite dimensional model of $E\text{Mod}_g$.

Motivated by Teichmüller space, Culler and Vogtmann [11] introduced the outer space $X_n$ of marked metric graphs.

A metric graph $(G, \ell)$ is a graph with a positive length $\ell(e)$ assigned to each edge $e$. Note that the fundamental group of a finite graph is a free group. We only consider graphs $G$ with genus equal to $n$, i.e., $\pi_1(G) \cong F_n$ (or rather $H_1(G, \mathbb{Z}) \cong \mathbb{Z}^n$). We also assume that $G$ has no vertex of valence 1 or 2, and no separating edge (or bridge).

Let $R_n$ be the wedge of $n$ circles, also called a rose with $n$ petals. Then a marking on a metric graph $(G, \ell)$ is an homotopy equivalence $h : G \to R_n$, which amounts to a choice of a set of generators of $\pi_1(G)$. A marked metric graph is a metric graph $(G, \ell)$ together with a marking $h$, denoted by $(G, h, \ell)$. Two marked metric graphs $(G, \ell, h), (G', \ell', h')$ are called equivalent if there exists an isometry between $(G, \ell)$ and $(G', \ell')$ that commutes with the markings $h, h'$ up to homotopy.

Usually we normalize (or scale) the metric $\ell$ of each metric graph $(G, \ell)$ so that the sum of lengths of all edges is equal to 1. (In Remark 6.19 we will also use metric graphs whose edge lengths are not normalized.)

Then the outer space (or the reduced outer space) $X_n$ is defined to be the set of equivalence classes of marked metric graphs of total length 1,

$$X_n = \{(G, \ell, h)\} / \sim.$$  

Since elements of $\text{Out}(F_n)$ can be realized by homotopy equivalences of $R_n$, $\text{Out}(F_n)$ acts on $X_n$ by changing the markings of metric graphs.

**Proposition 2.1** The outer space $X_n$ is a locally finite simplicial complex of dimension $3g - 4$, and its simplices are partially open, i.e., some simplicial faces of the simplices are missing.

Briefly, for each marked graph $(G, h)$, let $e_1, \ldots, e_k$ be its edges. Then variation of the edge lengths $\ell(e_1), \ldots, \ell(e_k)$ under the conditions $\ell(e_1), \ldots, \ell(e_k) > 0$, $\sum_{i=1}^k \ell(e_i) = 1$ fills out an open $(k - 1)$-simplex $\Sigma_{(G, h)}$. Its simplicial faces correspond to vanishing of some edge lengths $\ell(e_i)$. For any subset $e_{i_1}, \ldots, e_{i_r}$ of edges, the simplicial face of $\Sigma_{(G, h)}$ defined by $\ell(e_{i_1}) = \cdots = \ell(e_{i_r}) = 0$ is contained in the outer space $X_n$ if and only if no loop of $G$, i.e., a nontrivial element of $\pi_1(G)$, is contained in the union of these edges $e_{i_1}, \ldots, e_{i_r}$. (Note that if an edge length $\ell(e) = 0$, then the edge $e$ is removed from the graph, and if the total length of a loop is equal to zero, the genus of the resulting graph will decrease.)

The outer space $X_n$ is the union of open simplexes $\Sigma_{(G, h)}$, and its maximal dimensional simplexes correspond to trivalent graphs $(G, h)$. There are several ways to put a topology on $X_n$. Probably the most direct way is to glue up the simplicial topology of these simplexes. See [38] for more details and references on this result and several others recalled below.

An important result due to Culler-Vogtmann [11] is

**Theorem 2.2** The outer space $X_n$ is contractible.
Clearly, we can give each $k$-dimensional simplex $\Sigma_{(G,h)}$ the standard simplicial metric by identifying it with the standard simplex in $\mathbb{R}^{k+1}$, $\{(x_1, \cdots, x_{k+1}) \in \mathbb{R}^{k+1} | x_1, \cdots, x_k > 0, x_1 + \cdots + x_{k+1} = 1\}$, whose edges have length $\sqrt{2}$. This metric extends and defines a metric on the closure of $\Sigma_{(G,h)}$ in $X_n$. It is also clear that these metrics are compatible on their common simplicial faces. Therefore, it defines a length function, or a metric on $X_n$, denoted by $d_0$.

**Proposition 2.3** The length metric $d_0$ on $X_n$ is an incomplete metric on $X_n$ which is invariant under $\text{Out}(F_n)$, and its restriction to each simplex $\Sigma_{(G,h)}$ is the length metric induced from a Riemannian metric.

**Proof.** Since all top dimensional simplices in $X_n$ are isometric, it is clear that the restriction to each simplex $\Sigma_{(G,h)}$ is the length metric of the flat Riemannian metric of the ambient Euclidean space $\mathbb{R}^k$, i.e., for every two points in $\Sigma_{(G,h)}$, there is no shortcut by going through neighboring simplices. Since the closure of each open simplex $\Sigma_{(G,h)}$ in $X_n$ is noncompact and its diameter is equal to $\sqrt{2}$, $d_0$ is an incomplete metric. Since the action of $\text{Out}(F_n)$ on $X_n$ preserves the metrics of the simplices, the invariance of $d_0$ under $\text{Out}(F_n)$ is clear.

See [5, §1] for related discussion on how to glue up metrics on simplices.

**Remark 2.4** It is natural to conjecture that the length metric $d_0$ on $X_n$ is a geodesic metric and is also geodesically convex in the sense that for every two points $p, q \in X_n$, there exists a unique geodesic segment connecting them. This is stronger than the result in Theorem 2.2 on the contractibility of $X_n$. If this conjecture is true, then $d_0$ is a good analogue of the Weil-Petersson metric on Teichmüller space. Indeed, though the Weil-Petersson metric of Teichmüller space is incomplete, the Weil-Petersson metric is geodesically convex according to a result of Wolpert. One might try to use this metric to understand elements of $\text{Out}(F_n)$ as the Weil-Petersson metric was used in the paper [13]. On the other hand, the situation is more complicated since axes of fully irreducible outer automorphisms are not unique in general [14] [19] [20].

**Proposition 2.5** Under the action of $\text{Out}(F_n)$ on $X_n$, there are only finitely many orbits of open simplices $\Sigma_{(G,h)}$.

Briefly, since the quotient $\text{Out}(F_n) \backslash X_n$ is the moduli space of metric graphs of genus $n$, the $\text{Out}(F_n)$-orbits of open simplices in $X_n$ correspond to equivalence classes of finite graphs of genus $n$ with only vertices of valence at least 3 and no separating edges. Clearly, there are only finitely many such graphs up to equivalence.

**Proposition 2.6** The group $\text{Out}(F_n)$ acts properly on the outer space $X_n$.

This follows from the fact that for every simplex $\Sigma_{(G,h)}$ in $X_n$, its stabilizer in $\text{Out}(F_n)$ is finite (in fact, isomorphic to a subgroup of the symmetry group $S_N$, where $N$ is the number of vertices of the graph $G$), and $X_n$ is a locally finite simplicial complex.

The combination of the above results suggests the following [39] [25].

**Proposition 2.7** $X_n$ is a universal space for proper actions of $\text{Out}(F_n)$, i.e., for every finite subgroup $H \subset F_n$, the set of fixed points $X_n^H$ of $H$ is nonempty and contractible.
3 A natural approach to define a metric

One natural method to construct an invariant metric on \( X_n \) is to embed \( X_n \) equivariantly into a metric space \((Z,d)\) and then pull back the metric \(d\). If \((Z,d)\) is a complete metric space and the image \(i(X)\) is a complete metric subspace, then the pulled-back metric \(i^*d\) is an invariant complete metric on \(X_n\). On the other hand, even if \((Z,d)\) is a geodesic metric, \(i^*d\) may not be a geodesic metric.

If \(X_n\) were a manifold, and \((Z,d)\) is a Riemannian manifold, then an immersion \(i : X_n \to Z\), not necessary an embedding, will allow us to pull back the Riemannian metric of \(Z\) to obtain a Riemannian metric on \(X_n\) and consequently a Riemannian distance.

One difficulty is that there is no obvious choice of complete metric spaces \((Z,d)\) into which \(X_n\) can be embedded equivariantly. (We note that there is a natural embedding \(X_n \to Y\) into a topological space such that the closure is compact, but the ambient space \(Y\) is not a metric space.

See [12] [38, §1.4] for details and references.)

Instead, we will look for a complete metric space \((Z,d)\) and an equivariant map \(i : X_n \to (Z,d)\) satisfying the condition: for any sequence of points \(x_k \in X_n\) which converges to a boundary point as \(k \to \infty\), i.e., a point in a missing simplex, then for every basepoint \(x_0 \in X_n\), \(d(i(x_k), i(x_0)) \to +\infty\) as \(k \to \infty\). In some sense, we give up the condition of embedding and retain the completeness condition on the image. By combining it with the incomplete simplicial metric \(d_0\) on \(X_n\), we can construct complete geodesic metrics on \(X_n\) which are invariant under \(\text{Out}(F_n)\).

One such map can be constructed by borrowing ideas from Riemann surfaces, Teichmüller spaces, and symmetric spaces. Note that \(\text{GL}(n, \mathbb{R})\) is a reductive Lie group, \(\text{O}(n)\) is a maximal compact subgroup of \(\text{GL}(n, \mathbb{R})\), and the quotient \(\text{GL}(n, \mathbb{R})/\text{O}(n)\) with a \(\text{GL}(n, \mathbb{R})\)-invariant metric is a symmetric space of nonpositive curvature, though not of noncompact type.

**Proposition 3.1** The symmetric space \(X = \text{GL}(n, \mathbb{R})/\text{O}(n)\) can be identified with the space of positive definite quadratic forms, or equivalently, the space of marked compact flat tori of dimension \(n\).

**Proof.** It is clear that \(\text{GL}(n, \mathbb{R})\) acts on the space of positive definite \(n \times n\)-matrices: for \(g \in \text{GL}(n, \mathbb{R})\) and a positive definite matrix \(A\), \(g \cdot A = gAg^t\). By the spectral decomposition of positive definite quadratic forms, we can prove that this action is transitive. Since the stabilizer of the identity matrix is equal to \(\text{O}(n)\), the first statement is proved.

As in the case of marked graphs, a marked flat torus of dimension \(n\) is a compact flat Riemannian torus \(M^n\) with a diffeomorphism \(\varphi : M^n \to \mathbb{Z}^n/\mathbb{R}^n\), where \(\varphi\) is well-defined up to homotopy. To prove the second statement, we note that every positive definite quadratic form on \(\mathbb{R}^n\) defines a flat metric on \(\mathbb{R}^n\) which descends to a flat metric on the torus \(\mathbb{Z}^n/\mathbb{R}^n\). This torus has the marking giving by the identity map \(\mathbb{Z}^n/\mathbb{R}^n \to \mathbb{Z}^n/\mathbb{R}^n\). Conversely, every marked flat torus of dimension \(n\) canonically induces a flat metric on \(\mathbb{Z}^n/\mathbb{R}^n\), which is in turn induced by a unique positive definite quadratic form on \(\mathbb{R}^n\).

Clearly, \(\text{GL}(n, \mathbb{Z})\) acts on the symmetric space \(X\), and the quotient \(\text{GL}(n, \mathbb{Z})\backslash X\) is the isometry classes of compact flat tori of dimension \(n\).

We can define an equivariant map

\[ \Pi : X_n \to X \]

with respect to the natural actions of \(\text{Out}(F_n)\) and \(\text{GL}(n, \mathbb{Z})\), and the homomorphism \(\text{Out}(F_n) \to \text{GL}(n, \mathbb{Z})\) as follows.

For every marked metric graph \((G, \ell, h)\) in \(X_n\), there is a canonical isomorphism \(h_* : H_1(G, \mathbb{Z}) \cong \mathbb{Z}^n\), and hence an isomorphism \(h_* : H_1(G, \mathbb{R}) \to \mathbb{R}^n\) by linear extension. Therefore \(H_1(G, \mathbb{Z})\backslash H_1(G, \mathbb{R})\)
is a marked compact torus. We will use the metric $\ell$ on the graph to define a flat metric on the torus, or equivalently, a positive definite quadratic form $Q$ on the real vector space $H_1(G, \mathbb{R})$. Note that $G$ is of dimension 1 and $H_1(G, \mathbb{R})$ is equal to the set of 1-cycles on $G$. Therefore, it suffices to define a quadratic form $Q$ on the set $C_1(G, \mathbb{R})$ of all 1-chains on $G$ and then to restrict it to the subspace of 1-cycles.

Fix an orientation for every edge $e$ of $G$. Then every 1-chain $\sigma$ on $G$ can be written uniquely as $\sum_{e \in G} a_e e$, where $a_e \in \mathbb{R}$. For every two edges $e, e'$ of $G$, define

$$Q(e, e') = \begin{cases} \ell(e), & \text{if } e = e', \\ 0, & \text{if } e \neq e'. \end{cases}$$

for $x > 0$, we can obtain a positive quadratic form $Q_f$ on $H_1(G, \mathbb{R})$ by defining $Q_f(e, e') = f(\ell(e))$ as in Equation 3.1 and hence an equivariant map $\Pi_f : X_n \to \text{GL}(n, \mathbb{R})/O(n)$. 

**Proposition 3.2** The bilinear quadratic form $Q$ on $C_1(G, \mathbb{R})$, and hence its restriction on the subspace $H_1(G, \mathbb{R})$, is positive definite.

**Proof.** For any nontrivial 1-chain $\sigma$ of $G$, write $\sigma = \sum_{e \in G} a_e e$, where $a_e \neq 0$ for at least one edge. It follows from the definition that

$$Q(\sum_{e \in G} a_e e, \sum_{e \in G} a_e e) = \sum_{e \in G} a_e^2 \ell(e) > 0.$$ 

By extending it linearly, we obtain a quadratic form $Q$ on $C_1(G, \mathbb{R}) \text{ [10 p. 151]}$. 

In terms of the canonical identification $H_1(G, \mathbb{Z}) \cong \mathbb{Z}^n$ and $H_1(G, \mathbb{R}) \cong \mathbb{R}^n$, the quadratic form $Q$ becomes a positive definite matrix of size $n \times n$. It is called the period matrix of the marked metric graph $(G, \ell, h)$. By Proposition 3.1 we obtain a period map

$$\Pi : X_n \to X = \text{GL}(n, \mathbb{R})/O(n), \quad (G, \ell, h) \mapsto Q,$$

which is clearly equivariant with respect to the homomorphism $\text{Out}(F_n) \to \text{GL}(n, \mathbb{Z}).$

By Proposition 3.1 again, this positive definite quadratic form $Q$ induces a flat metric on the marked torus $H_1(G, \mathbb{Z})/H_1(G, \mathbb{R})$. To be consistent with the Jacobian map of tropical curves below, we denote the period map by

$$\Pi : X_n \to X = \text{GL}(n, \mathbb{R})/O(n), \quad (G, \ell, h) \mapsto (H_1(G, \mathbb{Z})/H_1(G, \mathbb{R}), Q),$$

where the image consists of marked compact flat tori.

We will explain below why it is called the period map and compare it with the period (or Jacobian) map of Riemann surfaces.

The definition of the quadratic form $Q$ in Equation 3.1 might look puzzling, and may seem wrong. The reason is that the length of the chain $e$ is equal to $\sqrt{\ell(e)}$, which is not a smooth function of the lengths of the metric graph $(G, \ell)$. From the above proof, it is clear that if we use $\ell(e)^2$ instead of $\ell(e)$ and define $Q(e, e') = \ell(e)^2$, then we still obtain a positive definite quadratic form on the torus $H_1(G, \mathbb{Z})/H_1(G, \mathbb{R})$. This latter definition might look more natural, since the length of a chain $e$ would be equal to $\ell(e)$, which will depend smoothly on the metric $\ell$ of the metric graph $(G, \ell)$.

**Remark 3.3** The proof of Proposition 3.2 shows that for every function $f(x)$ satisfying $f(x) > 0$ for $x > 0$, we can obtain a positive quadratic form $Q_f$ on $H_1(G, \mathbb{R})$ by defining $Q_f(e, e') = f(\ell(e))$ as in Equation 3.1 and hence an equivariant map $\Pi_f : X_n \to \text{GL}(n, \mathbb{R})/O(n)$.
To explain that the function $f(x) = x$ is the right choice, we need to go through tropical geometry. The definition of the quadratic form $Q$ in Equation 3.1 turns out to be the right definition when $(H_1(G, \mathbb{Z}) \setminus H_1(G, \mathbb{R}), Q)$ is viewed as a **principally polarized tropical abelian variety** of a marked tropical curve, when the metric graph $(G, \ell)$ is identified with a tropical curve. Another potentially important point is that $Q$ is a **linear function** when it is restricted to each simplex of the outer space $X_n$ and hence it has the characteristic property of tropical maps in tropical geometry, which might imply that the **tropical Jacobian map is a tropical map** [10, Assumption 3, p. 165]. To explain these, we will describe some basic notions in tropical geometry in the next sections.

Another reason for discussing tropical geometry is that discussion in this paper also clarifies some points on tropical curves and the moduli space of principally polarized tropical abelian varieties (Corollary 6.18). It seems reasonable to believe that the geometry of the outer space $X_n$ will be potentially useful in studying tropical curves and their moduli spaces. See Remark 6.19 below.

To define a metric on $X_n$, there is a further problem with using the period map $\Pi$ in Equation 3.3. When $n \geq 3$, the homomorphism $\text{Out}(F_n) \to \text{GL}(n, \mathbb{Z})$ has an infinite kernel. Since the map $\Pi$ is equivariant with respect to the homomorphism $\text{Out}(F_n) \to \text{GL}(n, \mathbb{Z})$, it cannot be injective. But as we will show below (Corollary 7.10), $\Pi$ is not even injective locally when $n \geq 3$, i.e., the induced quotient map $\Pi : \text{Out}(F_n) \setminus X_n \to \text{GL}(n, \mathbb{Z}) \setminus \text{GL}(n, \mathbb{R}) / O(n)$ is not injective, unlike the period map of Riemann surfaces in Equation 5.3. When it is restricted to each simplex in $X_n$, the period map $\Pi$ is a differential map, but it is not injective or is an immersion in general. Though $\text{GL}(n, \mathbb{R}) / O(n)$ is a symmetric space and has an invariant metric, we cannot use this map to pull back and define a metric on $X_n$.

To overcome this difficulty, one simple and crucial idea is to make use of the simplicial metric $d_0$ on $X_n$ in Proposition 2.3 as well and combine it with the pulled back semimetric by the period map. Several metrics are constructed in this way in §7 and §8. In §9, we also view $d_0$ as an analogue of the Weil-Petersson metric of the Teichmüller space and follow the construction of the McMullen metric on Teichmüller spaces [28] in order to construct another invariant complete geodesic metric on $X_n$.

This brief discussion explains the fruitful analogy between the three kind of spaces: symmetric spaces $X$, the Teichmüller space $T_{g,n}$, and the outer space $X_n$.

## 4 Tropical curves

Metric graphs are closely related to tropical curves. We will define tropical curves and show that there is a 1-1 correspondence between the set of compact metric graphs without vertex of valence 1 or 2 and the set of compact smooth tropical curves.

This relation with tropical curves allows us to explain why the definition of the positive definite quadratic form $Q$ in Equation 3.1 for a metric graph is natural. We also point out in Remark 6.19 that outer spaces can be used to study the moduli space $\mathcal{M}_{g,n}^{\text{trop}}$ of tropical curves.

To make this paper more self-contained and easier for the reader, we have tried to explain some results in tropical geometry starting from the basics. One reason that tropical curves are confusing to some people, in particular to the author of this paper, is that one often uses tropical plane curves as examples of tropical curves. But these curves are **singular in general** and do not correspond bijectively to metric graphs, and only smooth tropical curves correspond to metric graphs. Definitions of abstract smooth tropical curves (and manifolds) basically follow the usual definition of manifolds in the sense that transition functions between local charts are given by $\mathbb{Z}$-affine transformations, but there are several different ways to describe local models and they can be complicated to people who approach tropical curves for the first time. For example, the local
model in [33, §3] is not geometric. In [21, Definition 1.15], a tropical curve is defined as a connected metric graph. We will follow the approach in [31] together with some ideas and statements in [29] [30]. There is another approach in [35].

4.1 The tropical semifield and tropical polynomials

Briefly, the tropical geometry is algebraic geometry over the tropical semifield, and the tropical semifield is \( T = \mathbb{R} \cup \{-\infty\} \) with suitable operations.

It has an addition “\( x + y \) = \( \max\{x, y\} \)”, and \( -\infty \) is the additive zero. It has a multiplication “\( x \cdot y \) = \( x + y \)”, and 0 is the multiplicative identity. Note that \( -\infty \) is also the multiplicative 0, since “\( x \cdot (-\infty) \) = \( -\infty \).

But it does not have a subtraction, since we cannot recover \( x \) from “\( x + y = \max\{x, y\} \)” and \( y \) when \( x < y \).

The division “\( x/y \) = \( x - y \)” is defined when \( y \neq -\infty \). This corresponds to the usual rule that the division is defined when the denominator is not zero.

Once we have addition and multiplication, we can define polynomials. For example, for a polynomial in two variables \( p(x, y) = \"ax^2 + bxy + cy^2\" \), when it is restricted to \( \mathbb{R}^2 \subset T^2 \), it is the maximum of linear functions

\[ p(x, y) = \max\{2x + a, x + y + b, 2y + c\}. \]

4.2 Tropical plane curves

It is well-known that for any polynomial \( p(x, y) \), the equation \( p(x, y) = 0 \) defines a plane curve in \( \mathbb{R}^2 \) (or rather in \( \mathbb{C}^2 \)). Similarly, tropical polynomials define tropical curves in \( T^2 \). We will concentrate on their points inside \( \mathbb{R}^2 \).

Given a tropical polynomial \( p(x, y) \), a direct generalization is to define the associated tropical curve by \( p(x, y) = -\infty \). (Recall that \( -\infty \) is the tropical additive 0). But this has a serious problem of having too few points. For example, for a degree 2 polynomial \( p(x, y) \) above, \( p(x, y) = -\infty \) is equivalent to \( x = -\infty, y = -\infty \).

For a usual polynomial \( p(x, y) \) over \( \mathbb{C} \), the plane curve \( p(x, y) = 0 \) in \( \mathbb{C}^2 \) can also be defined to consist of points \( (x, y) \) where the function \( 1/p(x, y) \) is not regular.

For a tropical polynomial, we can similarly define its tropical plane curve in \( \mathbb{R}^2 \) to consist of points where the quotient “\( 0/p(x, y) \)” = \( -p(x, y) \) is not regular. (Recall that 0 is the tropical multiplicative identity.) See [21, pp. 10-11] for more details.

This means that points of the tropical plane curve of \( p(x, y) \) correspond to points where \( p(x, y) \) is achieved by at least two linear functions. (Note that near a point where two linear functions are equal, \( -p(x, y) \) is not equal to another tropical polynomial, since \( -\max\{f, g\} = \inf\{-f, -g\} \).

The following result is clear from the definition.

**Proposition 4.1** For any tropical polynomial \( p(x, y) \), its tropical plane curve in \( \mathbb{R}^2 \) is a piecewise linear curve with rational slopes.

Though the slope of each line segment is rational, its endpoints are not necessarily rational. For some pictures of plane tropical curves, see [29] [30]. Similarly, a tropical polynomial \( p(x, y, z) \) defines a tropical hyperplane in \( T^3 \), and two tropical polynomials \( p_1(x, y, z), p_2(x, y, z) \) define a tropical subvariety in \( T^3 \) as the intersection of the tropical hyperplanes.

Tropical plane curves are not smooth tropical curves whenever it contains a vertex of valence strictly greater than 3. In order to relate metric graphs to tropical curves, we need to introduce
smooth tropical curves which are not necessarily embedded in some ambient tropical space $\mathbb{T}^n$. In fact, compact tropical curves cannot be embedded in $\mathbb{T}^n$ (which can be seen by using the balancing condition at vertices), and tropical curves in $\mathbb{T}^n$ are affine tropical curves.

4.3 Abstract smooth tropical curves

In this subsection, we introduce the notion of an abstract smooth tropical curve. The basic reference is [31], and the papers [21] [36] are also helpful.

To distinguish tropical curves from graphs, we use $\Gamma$ to denote tropical curves. A smooth tropical curve is a graph $\Gamma$ endowed with a smooth $\mathbb{Z}$-affine structure, which consists of an open covering by compatible $\mathbb{Z}$-affine smooth charts:

1. For each interior point $x$ of an edge $e$, let $U$ be a neighborhood of $x$ homeomorphic to an interval. A $\mathbb{Z}$-affine structure on $U$ is an embedding

$$\phi_U : U \to \mathbb{R}.$$  

2. If $x$ is a vertex of valence $n + 1$, let $e_0, \cdots, e_n$ be the edges connected to $x$. Let $U$ be a neighborhood of $x$ which is contained in the union of these edges and which consists of $n + 1$ intervals. A smooth $\mathbb{Z}$-affine structure on $U$ is an embedding

$$\phi_U : U \to \mathbb{R}^n$$

such that for every edge $e_i$, $\phi_U(U \cap e_i)$ is a line segment with a rational slope, and $\phi_U$ satisfies a balancing and nondegeneracy condition at the vertex $x$. Specifically, let $v_i \in \mathbb{Z}^n$ be the primitive vector in the direction of $\phi_U(U \cap e_i)$ (pointing away from $x$). Then the balancing condition at $x$ is the equation

$$\sum_{i=0}^{n} v_i = 0; \quad (4.1)$$

and the nondegeneracy condition is that any $n$ vectors of the $n + 1$ vectors $v_0, \cdots, v_n$ form a basis of $\mathbb{Z}^n$, which is equivalent to the condition that $v_0, \cdots, v_{n-1}$ form a basis of $\mathbb{Z}^n$ when the balancing condition is satisfied. (One reason for imposing this non-degeneracy condition is that every two bases of $\mathbb{Z}^n$ can be interchanged by some element of $\text{GL}(n, \mathbb{Z})$).

3. The above local charts are compatible as follows. Given two overlapping charts $\phi_1 : U_1 \to \mathbb{R}^{n_1}$, $\phi_2 : U_2 \to \mathbb{R}^{n_2}$, we can choose a common $\mathbb{R}^N$ and inclusions of $\mathbb{R}^{n_1}, \mathbb{R}^{n_2} \subset \mathbb{R}^N$ as linear subspaces

$$\phi_1 : U_1 \to \mathbb{R}^{n_1} \subset \mathbb{R}^N, \phi_2 : U_2 \to \mathbb{R}^{n_2} \subset \mathbb{R}^N,$$

and find a $\mathbb{Z}$-affine linear map $\Phi_{12} : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\phi_1|_{U_1 \cap U_2} = \Phi_{12} \circ \phi_2.$$  

(Recall that a $\mathbb{Z}$-affine linear map is given by $v \in \mathbb{R}^N \mapsto Av + b$, where $A$ is an invertible integral $N \times N$-matrix, i.e., $A \in \text{GL}(n, \mathbb{Z})$, $b \in \mathbb{R}^N$.) Note that the overlap $U_1 \cap U_2$ can be complicated if $U_1$, $U_2$ are large. If they are sufficiently small, then $U_1 \cap U_2$ consists of an open interval since $\Gamma$ is a graph.

The relation between tropical plane curves and smooth tropical curves is explained in the next proposition.
Proposition 4.2 Given a tropical plane curve \( C \), there is a canonical smooth tropical curve \( J \) which can be mapped onto the tropical plane curve \( C \) by a locally \( \mathbb{Z} \)-affine map. The smooth tropical curve \( J \) is called the normalization of the tropical plane curve \( C \).

Suppose a tropical curve \( C \subset \mathbb{R}^2 \) is defined by a tropical polynomial \( p(x, y) \). Then \( C \) is a piecewise-linear curve in the plane such that each edge has a rational slope. When \( C \) contains a vertex \( x_0 \) of valence strictly greater than 3, the primitive integral vectors along all edges of the vertex \( x_0 \) do not satisfy the nondegeneracy condition.

Now we explain how to put a smooth \( \mathbb{Z} \)-affine structure on \( C \). Suppose \( x_0 \in C \) is a vertex of valence \( n+1 \). Let \( e_0, \ldots, e_n \) be all the edges from \( x_0 \). Suppose an edge \( e \) is defined by the equality of two linear functions \( a_1x^jy^k \) and \( a_2x^jy^k \), i.e., \( j_1x + k_1y + a_1 = j_2x + k_2y + a_2 \). Define \( V(e) \) to be the primitive integral vector in the direction of the edge \( e \) (pointing away from the vertex), and the weight \( W(e) \) of the edge to be the greatest common denominator \( \gcd(j_1 - j_2, k_1 - k_2) \).

Lemma 4.3 \((\text{[29, p. 512]})\) Under the above notation, for every vertex \( x_0 \) of a plane tropical curve \( C \), the following balancing condition is satisfied:

\[
\sum_{i=0}^n W(e_i)V(e_i) = 0. \tag{4.2}
\]

**Proof.** Suppose an edge \( e \) out of \( x_0 \) is defined by \( f_1(x, y) = f_2(x, y) \), where \( f_1(x, y) = j_1x + k_1y + a_1 \), and \( f_2(x, y) = k_2x + j_2y + a_2 \). Assume that when we move counter-clockwise around the vertex, \( f_1(x, y) - f_2(x, y) \) changes from positive to negative. Then it can be checked that \( W(e)V(e) = (-(k_1 - k_2), j_1 - j_2) = (k_2 - k_1, j_1 - j_2) \). On the other hand, if \( f_1(x, y) - f_2(x, y) \) changes from negative to positive, then \( W(e)V(e) = (k_1 - k_2, j_2 - j_1) \). (Think of the simplest example of two linear functions \( x, y \) with the vertex at the origin.) For any vertex \( x \) of \( C \), there are linear functions \( f_1(x, y), \ldots, f_k(x, y) \) such that all edges are defined by the equality of two linear functions which achieve the maximum value. These edges divide a small disk with center at \( x \) into cone pieces, and for each cone \( \Omega \), one linear function \( f_i \) is the unique function which takes the maximum value on this cone. When we move counter-clockwise across the two edges \( e_1, e_2 \) of \( \Omega \) defined by \( f_i = f_{i_1} \) and \( f_i = f_{i_2} \) for some \( f_{i_1} \) and \( f_{i_2} \), both possibilities of sign changes of the values \( f_i - f_{i_1} \) and \( f_i - f_{i_2} \) can occur. This means the coefficient of \( x \) (and similarly the coefficient of \( y \)) of \( f_i \) will appear in \( W(e_1)V(e_1) \) and \( W(e_2)V(e_2) \) once as positive and another time as negative. By summing over all edges (or rather all cones), they all cancel out and the sum \( \sum_{i=0}^n W(e_i)V(e_i) \) is zero. (Think of the simplest example of tropical curve defined by the tropical polynomial \( \frac{x + y}{2} \), which has only one vertex at the origin and 3 edges coming out of it.)

Corollary 4.4 For every tropical polynomial \( p(x, y) \), its associated tropical plane curve \( C \) is a piecewise linear curve in \( \mathbb{R}^2 \) with rational slopes such that with respect to a suitable weight \( W(e) \) for each edge \( e \), it satisfies the balancing condition as in Equation \( 4.2 \) at every vertex.

**Remark 4.5** The converse of the above corollary is also true \([29, \text{p. 512}]\). Let \( C \) be a piecewise linear curve in \( \mathbb{R}^2 \) with rational slopes such that with respect to a suitable integral weight for each edge it satisfies the balancing condition in Equation \( 4.2 \) at every vertex. Then there exists a tropical polynomial whose associated tropical curve is \( C \), and the weight of every edge is defined as above. Briefly, this can be proved as in Lemma \( \text{[4.4]} \) by noting that if a finite sequence of numbers \( a_1, \ldots, a_k \) sums to 0, then there exist numbers \( b_1, \ldots, b_k \) such that \( a_i = b_i - b_{i+1} \) for \( i = 1, \ldots, k \).
Consider the following piecewise linear curve \( \Gamma_0 \) in \( \mathbb{R}^n \) with one vertex of valence \( n+1 \) at the origin: the edges out of the origin are rays along the \( n+1 \) vectors: \( v_0 = (1, \cdots, 1), v_1 = (-1, 0, \cdots, 0), \ldots, v_n = (0, \cdots, 0, -1) \). When \( n = 2 \), this is the plane tropical curve defined by the polynomial \( p(x, y) = "x + y + 0" \).

Clearly the edges have rational slopes, and the primitive vectors along the edges are \( v_0, \cdots, v_n \). These integral vectors satisfy the balancing and nondegeneracy conditions and hence \( \Gamma_0 \) is a smooth tropical curve.

**Proposition 4.6** Suppose \( x \) is a vertex of a tropical plane curve \( C \) of valence \( n+1 \). Let \( e_0, \cdots, e_n \) be the edges out of \( x \). For each edge \( e_i \), let \( W(e_i) \) be its weight and \( V(e_i) \) be its primitive integral vector as defined above. Then there exists a unique \( \mathbb{Z} \)-affine linear map \( \pi : \mathbb{R}^n \rightarrow \mathbb{R}^2 \) such that

\[
\pi(0) = x, \quad \pi(v_i) = W(e_i) V(e_i), \quad i = 0, \cdots, n.
\]

**Proof.** The nondegeneracy condition of \( v_0, \cdots, v_n \) implies that \( v_1, \cdots, v_n \) are linearly independent vectors of \( \mathbb{R}^n \) and form a basis of \( \mathbb{R}^n \). This implies that there exists a unique affine linear map \( \pi : \mathbb{R}^n \rightarrow \mathbb{R}^2 \) such that \( \pi(v_i) = W(e_i) V(e_i), \quad i = 1, \cdots, n \), and \( \pi(0) = x \). Since the vectors \( v_0, \cdots, v_n \) and the edges \( e_0, \cdots, e_n \) satisfy the balancing condition, i.e., \( \sum_{i=0}^n v_i = 0 \) and \( \sum_{i=0}^n W(e_i) V(e_i) = 0 \), it follows that \( \pi(v_0) = W(e_0) V(e_0) \). Since \( v_1, \cdots, v_n \) forms a basis of the lattice \( \mathbb{Z}^n \) and \( W(e_1) V(e_1), \cdots, W(e_n) V(e_n) \) are integral vectors, the map \( \pi \) is a \( \mathbb{Z} \)-affine linear map.

Let \( U_0 \) be a small neighborhood of the origin in \( \Gamma_0 \). Then when \( U_0 \) is small enough, \( \pi(U_0) \) is a neighborhood of \( x \), denoted by \( U \), and the restricted map \( \pi_{U_0} : U_0 \rightarrow U \) is a homeomorphism. Its inverse gives a smooth \( \mathbb{Z} \)-affine structure on \( U \):

\[
\phi_U = (\pi_{U_0})^{-1} : U \rightarrow U_0 \subset \mathbb{R}^n.
\]

For each point \( x \in C \) contained in the interior of an edge \( e \), let \( U \) be a neighborhood of \( x \) contained in \( e \). Let \( V(e) \) be the primitive integral vector of \( e \), and \( W(e) \) its weight. Let \( v \) be the unit vector of \( \mathbb{R} \) from 0 to 1. Define a \( \mathbb{Z} \)-affine linear map \( \pi : \mathbb{R} \rightarrow \mathbb{R}^2 \) by

\[
\pi(0) = x, \quad \pi(v) = W(e) V(e).
\]

Then for a suitable neighborhood \( U_0 \) of 0 in \( \mathbb{R} \), \( \pi : U_0 \rightarrow U \) is a homeomorphism, where \( U = \pi(U_0) \). The inverse \( \phi_U = \pi^{-1} : U \rightarrow U_0 \subset \mathbb{R} \) defines a \( \mathbb{Z} \)-affine structure on \( U \). (We note that if we treat \( x \) as a vertex of valence 2, this is a special case of the above general construction for vertices.)

**Proposition 4.7** For any tropical plane curve \( C \), the \( \mathbb{Z} \)-affine structures on neighborhoods of points of \( C \) defined above give the structure of a smooth tropical curve on the graph associated with \( C \). Denote this smooth tropical curve by \( J \).

**Proof.** We need to show that the local charts are compatible. Let \( U_1, U_2 \) be two charts of \( C \). By our choice, \( U_1 \cap U_2 \) is an open interval of an edge \( e \) of \( C \). Under the embeddings \( \phi_1 : U_1 \rightarrow \mathbb{R}^{n_1}, \phi_2 : U_2 \rightarrow \mathbb{R}^{n_2} \), the integral vector \( W(e) V(e) \) along the edge \( e \) is mapped to primitive integral vectors in \( \mathbb{R}^{n_1}, \mathbb{R}^{n_2} \). Embed \( \mathbb{R}^{n_1}, \mathbb{R}^{n_2} \) into a common linear space \( \mathbb{R}^N \). Since every two primitive integral vectors of \( \mathbb{R}^N \) are related by an element \( g \in \text{GL}(n, \mathbb{Z}) \), they are compatible.

**Proof of Proposition 4.7.** By definition, the underlying space of the tropical curve \( J \) in Proposition 4.7 is the same as \( C \), and the identity map from \( J \) to \( C \) is locally a \( \mathbb{Z} \)-affine map. Roughly, the normalization involves both making the edge vectors of every vertex nondenegerate and scaling \( \mathbb{Z} \)-affine maps on the edges according to their weights.
Remark 4.8 From the above discussion, it is clear that the integral structure \( \mathbb{Z}^n \) in \( \mathbb{R}^n \) is used crucially. If all the weights \( W(e) \) are equal to 1, then primitive integral vectors of edges are mapped to primitive integral vectors and the normalization amounts to making the primitive integral vectors of edges nondegenerate by embedding them into a bigger-dimensional linear space.

4.4 Identification between smooth tropical curves and metric graphs

From the definition, it is clear that a smooth tropical curve \( \Gamma \) is topologically a graph. The following results are well-known and often taken as definitions [31, Proposition 1.3] [33, Proposition 3.6] [21, Definition 1.15]. We fill in more details for the convenience of the reader.

Proposition 4.9 Every smooth tropical curve \( \Gamma \) admits a canonical metric and hence becomes a metric graph.

Proof. We will define a metric on a smooth tropical curve \( \Gamma \) using a \( \mathbb{Z} \)-affine structure on charts. It suffices to define lengths of subintervals of the edges. Let \( \phi_U : U \to \mathbb{R} \) be a \( \mathbb{Z} \)-affine chart of an interior point of an edge \( e \). Let \( V \) be an open subinterval contained in \( U \). Define the length of \( V \) to be equal to the length of the image \( \phi_U(V) \) with respect to the standard metric of \( \mathbb{R} \).

If \( x \) is a vertex of valence \( n + 1 \) and \( \phi_U : U \to \mathbb{R}^n \) is a \( \mathbb{Z} \)-affine chart, and \( e_0, \ldots, e_n \) are the edges from \( x \), and \( V(e_i) \) an integral primitive vector on the ray \( \phi_U(U \cap e_i) \), we restrict the standard metric of \( \mathbb{R}^n \) to \( \phi_U(U \cap e_i) \) and scale the metric on it so that the length of the vector \( V(e_i) \) is equal to 1 (i.e., scale the metric on each edge separately so that the length of a primitive integral vector along it is equal to 1). Then for an interval \( V \) of an edge \( e_i \) contained in \( U \), the length of \( V \) is equal to the length of the image \( \phi_U(V) \) with respect to the scaled metric on the ray \( \phi_U(U \cap e_i) \).

In summary, if a directed interval of a smooth tropical curve \( \Gamma \) is mapped to a primitive integral vector under a local \( \mathbb{Z} \)-affine chart, its length is equal to 1. Since these \( \mathbb{Z} \)-affine structures on overlapping charts are compatible, integral primitive vectors in one chart are also integral primitive vectors in another, and the metrics of these charts agree on the overlaps. By gluing these metrics together, we obtain a metric on the tropical curve \( \Gamma \).

The converse is also true.

Proposition 4.10 Every metric graph admits a canonical structure of a smooth tropical curve.

Proof. Let \((G, \ell)\) be a metric graph, and \(x\) is a vertex of valence \(n + 1\). As in the previous subsection, let \(\Gamma_0\) be the tropical curve in \(\mathbb{R}^n\) with a vertex at 0 of valence \(n + 1\), and the edges out of the origin are rays along the vectors \(v_0 = (1, \ldots, 1), v_1 = (-1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, -1)\). Put a metric on \(\Gamma_0\) as in the previous proposition. Map a small neighborhood \(U\) of \(x\) isometrically to a small neighborhood \(U_0\) of 0. This defines a \(\mathbb{Z}\)-affine structure on \(U\).

If \(x \in G\) is an interior point of an edge, and \(U\) is an interval of the edge and contains \(x\), we can map \(U\) isometrically to a neighborhood of the origin of \(\mathbb{R}\). This defines a \(\mathbb{Z}\)-affine structure on \(U\).

For every large integer \(N\), an interval of length \(\frac{1}{N}\) of \(\Gamma\) is mapped to \(\frac{1}{N}\)-multiple of a primitive integral vector in \(\mathbb{R}^n\), where \(\mathbb{R}^n\) depends on the local chart. Using this, it can be checked that these \(\mathbb{Z}\)-affine structures on charts on \(G\) are compatible and hence define a structure of smooth tropical curve on the metric graph \((G, \ell)\).

The basic point of the proof of the above proposition is that a \(\mathbb{Z}\)-affine structure on a one-dimensional manifold is equivalent to a length metric. Because of this proposition, in some places such as [21, Definition 1.15], a tropical curve (or rather a smooth tropical curve) is defined as a metric graph.
Remark 4.11 We can show easily that the two maps in Proposition 4.10 are the inverse of each other. Specifically, start with a smooth tropical curve \(J\), obtain a metric graph \((G, \ell)\) by Proposition 4.9 and then obtain a new tropical curve \(J'\) from \((G, \ell)\) by Proposition 4.10. It can be shown that \(J'\) is isomorphic to \(J\), and the other way of starting with a metric graph can be checked as well.

Remark 4.12 If we start from a tropical plane curve \(C \subset \mathbb{R}^n\), we can directly put a metric on it as follows [20, p. 512]. For every linear segment \(e\) of \(C\), let \(V(e)\) be a primitive integral vector on \(e\), and \(W(e)\) be the weight of \(e\) defined above. The metric on \(e\) is the scaling of the restriction of the standard metric of \(\mathbb{R}^2\) such that the norm of \(V(e)\) is equal to \(1/W(e)\), in particular, the norm of \(V(e)\) is equal to 1 if and only if the weight \(W(e) = 1\). We can check that this metric graph is the same metric graph associated with the normalization \(J\) of \(C\), which is a smooth tropical curve.

This also allows one to construct alternatively a normalization of the tropical plane curve \(C\), i.e., by taking the tropical curve corresponding to the metric graph associated with \(C\).

5 Jacobian varieties of Riemann surfaces and the Jacobian map

Several results on tropical curves have been motivated by results on algebraic curves over \(\mathbb{C}\), or Riemann surfaces. A particularly important one for us is the Jacobian variety of a compact Riemann surface. In this section, we recall in detail the notion of polarized abelian varieties to motivate the corresponding tropical Jacobian variety for a compact smooth tropical curve. We also introduce the Jacobian map and we state the Torelli Theorem for Riemann surfaces.

For a compact Riemann surface \(\Sigma\), we can define its Jacobian variety as follows. Let \(H^{0,1}(\Sigma) = H^0(\Sigma, \Omega)\) be the space of holomorphic 1-forms on \(\Sigma\). It is a complex vector space of dimension \(g\). The first homology group \(H_1(\Sigma, \mathbb{Z})\) is isomorphic to \(\mathbb{Z}^{2g}\) and it can be embedded into the dual space \(H^0(\Sigma, \Omega)^*\) of \(H^0(\Sigma, \Omega)\), \(\pi : H_1(\Sigma, \mathbb{Z}) \to H^0(\Sigma, \Omega)^*, \) by integration: For every 1-cycle \(\sigma\) and a 1-form \(\omega \in H^0(\Sigma, \Omega)\),

\[
\pi(\sigma)(\omega) = \int_\sigma \omega.
\]

The complex torus \(H_1(\Sigma, \mathbb{Z})/H^0(\Sigma, \Omega)^*\) admits a canonical polarization coming from the pairing \(H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}\) and is a principally polarized Abelian variety. It is called the Jacobian variety of \(\Sigma\). To explain this, we discuss the notion of a polarization of an Abelian variety. See [17] for more details.

Recall that a line bundle \(L \to M\) over an algebraic variety \(M\) is called very ample if its global sections, i.e., elements of \(H^0(M, \mathcal{O}(L))\), give an embedding \(M \to \mathbb{P}^N\), or equivalently, if there exists an embedding \(\pi : M \to \mathbb{P}^N\), and \(L = \pi^*H\), where \(H\) is the line bundle over \(\mathbb{P}^N\) corresponding to a hyperplane of \(\mathbb{P}^N\). A line bundle \(L \to M\) over an algebraic variety \(M\) is called ample if for \(k \gg 0\), \(L^k \to M\) is very ample. Such an ample line bundle \(L \to M\) is called a polarization of \(M\). The existence of an ample line bundle over \(M\) is a necessary and sufficient condition for \(M\) to be a projective variety.

The Chern class of an ample line bundle \(L \to M\) is an integral class in \(H^2(M, \mathbb{Z})\) and can be represented by a closed, positive \((1,1)\)-form in the De Rham cohomology \(H^2(M, \mathbb{R})\). Conversely, by the Kodaira embedding theorem, every such positive integral class in \(H^2(M, \mathbb{R})\) is the Chern class of an ample line bundle over \(M\).

Therefore, a polarization of an algebraic variety \(M\) corresponds to a positive integral class in \(H^2(M, \mathbb{R})\). \((1,1)\)-form representing such a class is called a Hodge form.

A complex torus \(M = \Lambda/\mathbb{C}^n\) is an Abelian variety if it is a projective variety, i.e., if it admits a polarization. The existence of a Hodge form on \(\Lambda=\mathbb{C}^n\) depends on the compatibility of the complex structure of \(\mathbb{C}^n\) and the integral structure of \(\mathbb{C}^n\) defined by the lattice \(\Lambda\).
Given a basis $\lambda_1, \ldots, \lambda_{2n}$ of $\Lambda$ and a $\mathbb{C}$-basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$, define the period matrix $\Omega = (\omega_{\alpha i})$ of size $n \times 2n$ by

$$
\lambda_i = \sum_\alpha \omega_{\alpha i} e_\alpha.
$$

By using the fact that every cohomology class can be represented by a differential form which is invariant under translation, one can obtain the following Riemann condition for Abelian varieties [17, p. 306]: A complex torus $M = \Lambda \backslash \mathbb{C}^n$ is an Abelian variety if and only if there exists a basis $\lambda_1, \ldots, \lambda_{2n}$ of $\Lambda$ and a $\mathbb{C}$-basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ such that the period matrix $\Omega$ is of the form

$$
\Omega = (\Delta_\delta, Z),
$$

where $\Delta_\delta$ is a diagonal matrix with positive integral diagonal entries $\delta_1, \ldots, \delta_n$ such that $\delta_1|\delta_2, \ldots, \delta_{n-1}|\delta_n$, and $Z$ is symmetric and $\text{Im} Z$ is positive definite.

More precisely, let $x_1, \ldots, x_{2n}$ be the coordinates of $\mathbb{C}^n$ determined by $\lambda_1, \ldots, \lambda_{2n}$. Then under the above condition on the period matrix, the integral 2-form

$$
\omega = \sum \delta_\alpha dx_\alpha \wedge dx_{n+\alpha}
$$

is positive and hence is a Hodge form, which gives a polarization of the complex torus $\Lambda \backslash \mathbb{C}^n$. If $\delta_1 = \cdots = \delta_n = 1$, the polarization is called principal, and the Abelian variety is called principally polarized.

Conversely, suppose $\omega$ is a Hodge form which is invariant under translation. Then there exists a basis $\lambda_1, \ldots, \lambda_{2n}$ of $\Lambda$ such that $\omega$ is of the form in Equation (5.1). Now computing the form $\omega$ in the complex coordinates determined by $e_1, \ldots, e_n$ shows that the condition that $\omega$ is a positive (1,1)-form is exactly equivalent to the condition on the period matrix $Z$.

When $M = H_1(\Sigma_g, \mathbb{Z}) \backslash H^0(\Sigma_g, \Omega)^*$, for every symplectic basis $a_1, b_1, \cdots, a_g, b_g$ of $H_1(\Sigma_g, \mathbb{Z})$, there exists a normalized basis $\omega_1, \cdots, \omega_g$ of $H^0(\Sigma_g, \Omega)$ such that $\int_{a_i} \omega_j = \delta_{ij}$ [17, p. 231]. (Note that it is not automatic that for every $j$ there exists a holomorphic differential form $\omega_j$ such that the period conditions $\int_{a_i} \omega_j = \delta_{ij}, i=1, \cdots, g$, are satisfied. Once this is known, it is easy to see that different $\omega_j$ are linearly independent forms.)

This implies that with respect to the dual basis of $H^0(\Sigma_g, \Omega)^*$, the period matrix $(\Delta_\delta, Z)$ of the symplectic basis $a_1, b_1, \cdots, a_g, b_g$ is principal, i.e., $\Delta_\delta = I_g$. The Riemann bilinear relation [17, p. 232] implies that the period $Z$ is symmetric and that $\text{Im} Z$ is positive definite. Therefore, the Riemann condition is satisfied by the Jacobian variety $H_1(\Sigma_g, \mathbb{Z}) \backslash H^0(\Sigma_g, \Omega)^*$, and the Jacobian variety is a principally polarized Abelian variety.

Remark 5.1 The Hodge form $\omega$ is a (1,1)-form in $H^2(H_1(\Sigma_g, \mathbb{Z}) \backslash H^0(\Sigma_g, \Omega)^*)$, and hence corresponds to an integral skew-symmetric bilinear form on $H_1(\Sigma_g, \mathbb{Z})$. It turns out to be the intersection form on $H_1(\Sigma_g, \mathbb{Z})$:

$$
H_1(\Sigma_g, \mathbb{Z}) \times H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{Z},
$$

which is a nondegenerate skew-symmetric bilinear form by the Poincare duality.

We will see later that the Chern class of a line bundle over a tropical curve lies in the first cohomology group of the tropical curve and that a polarization corresponds to a positive definite symmetric bilinear form instead of a skew-symmetric bilinear form. One reason is that a Riemann surface is of real dimension 2 and a tropical curve is of real dimension 1.

Let $\mathfrak{h}_g = \{ X + iY \mid X, Y \text{ are real symmetric } g \times g \text{-matrices}, Y > 0 \}$ be the Siegel upper half space. Then the symplectic group $\text{Sp}(2g, \mathbb{R})$ acts holomorphically and transitively on $\mathfrak{h}_g$, and $\mathfrak{h}_g \cong \text{Sp}(2g, \mathbb{R})/U(g)$, a Hermitian symmetric space of noncompact type.
The above discussion shows that for every compact Riemann surface \( \Sigma_g \), its period \( Z \) is in \( h_g \). Different choices of symplectic bases of \( H_1(\Sigma_g, \mathbb{Z}) \) lead to different points in a \( \text{Sp}(2g, \mathbb{Z}) \)-orbit in \( h_g \). This gives a well-defined map

\[
\Pi : \mathcal{M}_g \to \text{Sp}(2g, \mathbb{Z})/h_g, \quad \Sigma_g \mapsto \text{Sp}(2g, \mathbb{Z})Z.
\]  

(5.2)

This is called the period map for Riemann surfaces.

Let \( \mathcal{A}_g \) denotes the moduli space of principally polarized Abelian varieties of dimension \( g \). Then the above discussion shows that \( \mathcal{A}_g \cong \text{Sp}(2g, \mathbb{Z})/h_g \), where a point \( \tau \in h_g \) corresponds to a principally polarized abelian variety \( \mathbb{Z}^n + \tau \mathbb{Z}^n \subset \mathbb{C}^n \), and the period map is equivalent to the Jacobian map:

\[
\Pi : \mathcal{M}_g \to \mathcal{A}_g, \quad \Sigma_g \mapsto H_1(\Sigma_g, \mathbb{Z})/H^0(\Sigma_g, \Omega)^*.
\]  

(5.3)

The Jacobian map is a complex analytic map with respect to the natural complex structures on \( \mathcal{M}_g \) and \( \text{Sp}(2g, \mathbb{Z})/h_g \). It is also a morphism when \( \mathcal{M}_g \) and \( \mathcal{A}_g \) are given the structure of algebraic varieties. The Torelli Theorem \([17]\) p. 359] says that \( \Pi \) is an embedding. The Jacobian map and the Torelli theorem have played a fundamental role in understanding the moduli space \( \mathcal{M}_g \).

**Remark 5.2** The complex torus \( H_1(\Sigma_g, \mathbb{Z})/H_1(\Sigma_g, \mathbb{R}) \) is usually called the Albanese variety of \( \Sigma_g \). Using the duality between \( H_1(\Sigma, \mathbb{R}) \) and \( H^1(\Sigma, \mathbb{R}) \) (and hence also \( H^{0,1}(\Sigma_g) \)), it can be seen that it is isomorphic to the Jacobian variety of \( \Sigma_g \). In general, the Albanese variety of a higher dimensional complex manifold or algebraic variety is a generalization of the Jacobian variety of a projective curve (or a compact Riemann surface).

### 6 Jacobians of tropical curves and metric curves

As mentioned before, metric graphs considered in this paper have no vertices of valence 1 or 2. By the results in Propositions 4.9 and 4.10, metric graphs can be identified with smooth tropical curves. From now on, we will denote metric graphs by \((\Gamma, \ell)\) instead of \((G, \ell)\) to be consistent with the notation for tropical curves.

Recall from Equation 5.1 and Proposition 5.2 that for any compact metric graph \((\Gamma, \ell)\), there is a positive definite quadratic form \( Q \) on the real vector space \( C_1(\Gamma, \mathbb{R}) \) of 1-chains on \( \Gamma \) with \( \mathbb{R} \)-coefficients, which defines an inner product on \( H_1(\Gamma, \mathbb{R}) \) and descends to a flat metric on the torus \( H_1(\Gamma, \mathbb{Z})/H_1(\Gamma, \mathbb{R}) \).

Following the definitions in [24] [10] for (combinatorial) graphs, we have the following definition.

**Definition 6.1** The torus \( H_1(\Gamma, \mathbb{Z})/H_1(\Gamma, \mathbb{R}) \) with the flat metric induced from the quadratic form \( Q \) is called the Albanese torus of the metric graph \((\Gamma, \ell)\), and denoted by \((H_1(\Gamma, \mathbb{Z})/H_1(\Gamma, \mathbb{R}), Q)\).

**Remark 6.2** The definition of the Albanese and Jacobian tori for graphs were introduced in [24] p. 94]. For a graph \( G \), the torus \( H_1(G, \mathbb{Z})/H_1(G, \mathbb{R}) \) with the flat metric where each edge is assigned length 1 is the Albanese torus, and the dual flat torus \( H^1(G, \mathbb{Z})/H^1(X, \mathbb{R}) \) is called the Jacobian torus. These are motivated by, but slightly different from, various notions of Jacobian varieties of algebraic varieties, Kahler manifolds and Riemannian manifolds, which are defined as quotients of cohomology groups. As recalled in the previous section, for a Riemann surface \( \Sigma_g \), the Jacobian variety is a quotient of the dual of the first cohomology, \( H_1(\Sigma_g, \mathbb{Z})/H^0(\Sigma_g, \Omega)^* \subset H^1(\Sigma, \mathbb{C}) \), where \( H^0(\Sigma_g, \Omega) \sim H^{0,1}(\Sigma_g) \subset H^1(\Sigma, \mathbb{C}) \).
In this metric on the space $C_1(\Gamma,\mathbb{R})$ of 1-chains (or the metric of the torus $H_1(\Gamma,\mathbb{Z})\backslash H_1(\Gamma,\mathbb{R})$), the length of an edge $e$ as a 1-chain is equal to $\sqrt{\ell(E)}$. As mentioned before, this makes the definition of the metric $Q$ seem unnatural since its dependence on the metric $\ell$ of the graph $(\Gamma,\ell)$ is not smooth. It might be more tempting to define $Q$ so that the length of the 1-chain $e$ to be $\ell(E)$ in order for it to depend smoothly on the lengths of edges of the graph $(\Gamma,\ell)$. (Of course, for graphs whose edges have normalized length equal to 1, there is no difference in the above two choices.)

In order to explain that the definition of the quadratic form in Equation 3.1 is the right one and that $Q$ gives a principal polarization when $H_1(\Gamma,\mathbb{Z})\backslash H_1(\Gamma,\mathbb{R})$ is given a natural structure of a tropical torus, we will identify $(H_1(\Gamma,\mathbb{Z})\backslash H_1(\Gamma,\mathbb{R}),Q)$ with the tropical Jacobian variety of the tropical curve associated with $(\Gamma,\ell)$. (We note that for all metric graphs $(\Gamma,\ell)$ of genus $g$, $H_1(\Gamma,\mathbb{R})$ are isomorphic, and $H_1(\Gamma,\mathbb{Z})$ are isomorphic lattices. As recalled below, a tropical structure on $H_1(\Gamma,\mathbb{R})$ is determined by another lattice.)

First, we need to define differential forms on a tropical curve, and the polarization of a tropical torus. Then we show that the tropical Jacobian variety of a compact smooth tropical curve has a canonical principal polarization. We will follow [33].

Given a compact smooth tropical curve $J$, a 1-form $\omega$ (i.e., a tropical differential form of degree 1) on $J$ is a collection of constant real differential 1-forms on the edges of $J$ satisfying the balancing condition at every vertex of $J$. Specifically, let $x$ be a vertex of valence $n+1$, and let $U$ be a chart $x$ with a $\mathbb{Z}$-affine structure $\phi_U : U \to \mathbb{R}^n$. Let $e_0,\ldots,e_n$ be the edges out of the vertex $\phi_U(x)$ in $\mathbb{R}^n$, and $V(e_i)$ a primitive integral vector along $e_i$ pointing away from the vertex $\phi_U(x)$. The 1-form $\omega$ defines a constant differential 1-form on each $e_i$ as a smooth differential manifold, and the balancing condition is

$$\sum_{i=0}^{n} \omega(V(e_i)) = 0. \quad (6.1)$$

It is clear that this balancing condition is preserved by $\mathbb{Z}$-linear maps and hence the balancing condition is well-defined.

Let $\Omega(J)$ be the space of global 1-forms on $J$. This is a finite dimensional vector space.

**Lemma 6.3** ([33, §6.1]) If the genus of a compact tropical smooth curve $J$ is equal to $g$, then $\Omega(J)$ is a real vector space of dimension $g$.

**Proof.** Since $J$ is homotopy equivalent to a wedge of $n$ circles, there are $g$ points, $p_1,\ldots,p_g$, contained in the interior of edges of $C$ such that $C - \{p_1,\ldots,p_g\}$ is contractible. They are called breakpoints. From the balancing conditions at the vertices, it is clear that a global 1-form on $J$ is determined by its values on primitive integral tangent vectors of the edges containing these breakpoints $p_1,\ldots,p_g$. It can also be seen that all values at the breakpoints can be achieved by some 1-forms. For example, this is true for the wedge of $n$-circles. By noting that the balancing conditions are preserved by contracting edges of the complement of the breakpoints, we can prove this by induction on the number of edges by starting from the wedge of circles. This proves that $\Omega(J)$ is of dimension $g$.

**Definition 6.4** A 1-form $\omega \in \Omega(J)$ is called integral if it takes an integral value on every integral tangent vector of every edge of $J$.

We note that the notion of integral tangent vectors of each edge of $J$ can be defined using $\mathbb{Z}$-affine local charts and it is well-defined. By the same proof of Lemma 6.3 we obtain the following.
Lemma 6.5 Let \( \Omega_J(J) \) denote the subspace of integral 1-forms on \( J \). Then \( \Omega_J(J) \) is a lattice of the real vector space \( \Omega(J) \).

Let \( \Omega(J)^* \) be the dual space of linear functionals on \( \Omega(J) \). We now define a pairing between \( H_1(J, \mathbb{Z}) \) and \( \Omega(J)^* \), and hence a map

\[
p : H_1(J, \mathbb{Z}) \rightarrow \Omega(J)^*.
\]

(Note that \( p \) stands for period.) For any path \( \gamma : [a, b] \rightarrow J \), the pull back \( \gamma^* \omega \) is a piecewise-constant differential 1-form on \([a, b]\) (with possible discontinuity at the preimage of the vertices), and the integral \( \int_\gamma \omega \) is defined to be \( \int_a^b \gamma^* \omega \). This implies that for any cycle \( \sigma \) and any 1-form \( \omega \), we can define

\[
p(\sigma)(\omega) = \int_\sigma \omega.
\] (6.2)

Clearly, this map \( p : H_1(J, \mathbb{Z}) \rightarrow \Omega(J)^* \) is linear and can be extended to a linear transformation

\[
p : H_1(J, \mathbb{R}) \rightarrow \Omega(J)^*.
\]

Proposition 6.6 The period map \( p : H_1(J, \mathbb{Z}) \rightarrow \Omega(J)^* \) defined above is an embedding, and the linear transformation \( p : H_1(J, \mathbb{R}) \rightarrow \Omega(J)^* \) is an isomorphism.

Proof. If we identify a tropical curve \( \Gamma \) with its associated metric graph \( (\Gamma, \ell) \), and on each edge \( e \) use we the length function \( t \) measured from a vertex as the parameter, then the restriction of a 1-form \( \omega \) to \( e \) is given by \( a dt \), where \( a \) is a constant. At each vertex \( x \) of the graph \( \Gamma \), let \( e_1, \ldots, e_k \) be the edges out of \( x \), and \( t_1, \ldots, t_k \) be the length functions measured from \( x \). Suppose that \( \omega \) is a 1-form, and on each edge \( e_i \), \( \omega = a_i dt_i \). Then the balancing condition at \( x \) for the 1-form \( \omega \) in Equation (6.1) is equivalent to

\[
\sum_{i=1}^k a_i = 0.
\] (6.3)

If we choose an orientation of an edge \( e \), then there is a choice of a length parameter \( t_e \) such that \( \int_e dt_e > 0 \). Fix an orientation of all edges \( e \) of \( \Gamma \) and the compatible choices of the length parameters \( t_e \). For any nontrivial cycle \( \sigma = \sum_e a_e e \), define a 1-form \( \omega = \sum_e a_e dt_e \) on \( \Gamma \), which means that it restricts to \( a_e dt_e \) on each edge \( e \). The compatibility of the edges contained in \( \sigma \) at the vertices to make it a cycle implies that \( \omega \) is a 1-form on \( \Gamma \). Since

\[
p(\sigma)(\omega) = \sum_e a_e^2 \int_e dt_e = \sum_e a_e^2 \ell(e) > 0,
\]

this implies that \( p(\sigma) \) is nonzero, and hence \( p \) is an embedding.

Since the dimensions of \( H_1(J, \mathbb{R}) \) and \( \Omega(J)^* \) are both equal to \( n \), the extended linear transformation is an isomorphism, and the proposition is proved.

Remark 6.7 The above interpretation of 1-forms \( \omega \) on a tropical curve \( \Gamma \) in terms of length parameters of a metric graph \( (\Gamma, \ell) \) gives a different proof of Lemma 6.3. Fix an orientation of edges \( e \) of \( \Gamma \) and the corresponding choices of the length parameters \( t_e \) as above. We can define a linear isomorphism between \( H_1(\Gamma, \mathbb{R}) \) and \( \Omega(\Gamma) \). For every 1-cycle \( \sigma = \sum_e a_e e \), define a 1-form \( \omega = \sum_e a_e dt_e \), which means that the restriction of \( \omega \) to each edge \( e \) is equal to \( a_e \). Conversely, for every 1-form \( \omega = \sum_e a_e dt_e \), we have a 1-cycle \( \sigma = \sum_e a_e e \). Therefore, the dimension of \( \Omega(\Gamma) \) is equal to the dimension of \( H_1(\Gamma, \mathbb{R}) \), which is equal to \( g \).
By Proposition 6.6, the quadratic form \( Q \) on \( H_1(J, \mathbb{R}) \) defines a quadratic form on \( \Omega(J)^* \). The positive definite quadratic form \( Q \) on \( \Omega(J)^* \) gives an isomorphism \( \Omega(J)^* \rightarrow \Omega(J) \), and together with the map \( p : H_1(J, \mathbb{Z}) \rightarrow \Omega(J)^* \) in Proposition 6.6 it gives an embedding

\[
i_Q : H_1(J, \mathbb{Z}) \rightarrow \Omega(J). \tag{6.4}
\]

We emphasize that this embedding \( i_Q \) depends on the positive definite quadratic form \( Q \). The following proposition explains the reason for the choice of value \( Q(e, e) \) for an edge \( e \) in Definition 3.1 of the quadratic form \( Q \) of a compact metric graph.

**Proposition 6.8** In the above notation, the image of the embedding \( i_Q : H_1(J, \mathbb{Z}) \rightarrow \Omega(J) \) is equal to the space of \( \Omega_J(J) \) of integral 1-forms.

**Proof.** We need to show that for any 1-cycle \( \sigma = \sum_e a_e e \) in \( H_1(J, \mathbb{R}) \), the image \( i_Q(\sigma) \in \Omega_J(J) \) if and only if \( \sigma \in H_1(J, \mathbb{Z}) \). For every edge \( e \) of \( \Gamma \), let \( V(e) \) be an integral primitive vector tangent to \( e \). By the definition of the metric on the tropical curve, the norm of \( V(e) \) is equal to \( 1 \). By definition of the integral 1-forms \( \omega \), \( i_Q(\sigma) \) is integral if and only if \( i_Q(\sigma)(V(e)) \) is integral for every edge \( E \), which is equivalent to the fact that \( \int_E i_Q(\sigma) \) is an integral multiple of \( \ell(e) \). On the other hand, by definition,

\[
\int_E i_Q(\sigma) = Q(e, \sigma) = a_e \ell(e).
\]

Therefore, \( i_Q(\sigma) \) is an integral 1-form if and only if \( \sigma \) is an integral 1-cycle, i.e., all coefficients \( a_e \) are in \( \mathbb{Z} \).

**Remark 6.9** If we orient all edges \( e \) of \( \Gamma \) and pick compatible length parameters \( t_e \) for them as in Proposition 6.6, then a 1-form \( \omega \) on \( \Gamma \) is integral if and only if for every edge \( e \), the restriction of \( \omega \) to \( e \) is an integral multiple of \( dt_e \), and the map in Proposition 6.8 can be written down explicitly as follows: For every cycle \( \sigma = \sum_e a_e e \), the restriction of the 1-form \( i_Q(\sigma) \) to the edge \( e \) is equal to \( a_e dt_e \), i.e., \( i_Q(\sigma) = \sum_e a_e dt_e \). As explained in the proof of Proposition 6.8, the definition of the quadratic form \( Q \) on \( H_1(\Gamma, \mathbb{Z}) \) was motivated to obtain this natural equivalence.

Specifically, \( Q \) is the unique positive definite quadratic form which gives an isomorphism \( \Omega(\Gamma) \cong \Omega(\Gamma)^* \) such that once it is composed with the isomorphism \( H_1(\Gamma, \mathbb{R}) \rightarrow \Omega(\Gamma) \) in Remark 6.7, the map \( H_1(\Gamma, \mathbb{Z}) \rightarrow \Omega(\Gamma)^* \) is equal to the period map \( p : H_1(\Gamma, \mathbb{Z}) \rightarrow \Omega(\Gamma)^* \) in Equation (5.2). Since both the period map and the isomorphism in Remark 6.7 are natural, the definition of \( Q \) and the induced isomorphism \( \Omega(\Gamma) \cong \Omega(\Gamma)^* \) are also natural.

Before continuing, we recall the definition of tropical torus according to [33], §5.1. A tropical torus is a real torus \( \Lambda \backslash \mathbb{R}^n \) with a \( \mathbb{Z} \)-affine structure. A \( \mathbb{Z} \)-affine structure on \( \mathbb{R}^n \) is determined by an integral structure, i.e., a lattice \( L \subset \mathbb{R}^n \). Unless specified otherwise, the \( \mathbb{Z} \)-affine structure is determined by the standard lattice \( \mathbb{Z}^n \) in \( \mathbb{R}^n \). Then for any lattice \( \Lambda \subset \mathbb{R}^n \), the torus \( X = \Lambda \backslash \mathbb{R}^n \) has the induced \( \mathbb{Z} \)-affine structure. (Note that \( \Lambda \) is not commensurable with \( \mathbb{Z}^n \) or does not satisfy other relations with \( \mathbb{Z}^n \) in general).

More generally, suppose \( V \) is a real vector space and \( L \subset V \) is a lattice. Then \( V \) has an integral and \( \mathbb{Z} \)-affine structure determined by \( L \). For any lattice \( \Lambda \subset V \), the torus \( \Lambda \backslash V \) is a tropical torus, whose tropical structure is determined by the lattice \( L \).

The subspace \( \Omega_J(J) \) of integral 1-forms defines a dual lattice \( \Omega_J(J)^* \) in \( \Omega(J)^* \), and hence defines a \( \mathbb{Z} \)-affine structure on \( \Omega(J)^* \) and also on the torus \( H_1(J, \mathbb{Z}) \backslash \Omega(J)^* \).

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Remark 6.12 We note that for a complex torus $\Lambda$, affine linear functions with integral slope.

Remark 6.11 Clearly, $H_1(J, R)$ can be mapped isomorphically to $\Omega(J)^*$ by integration as in Proposition 6.6, and $H_1(J, Z)\setminus\Omega(J)^*$ is isomorphic to the Albanese torus $H_1(J, Z)\setminus H_1(J, R)$. Under this isomorphism, the $Z$-affine structure on $H_1(J, Z)\setminus H_1(J, R)$, or rather on $H_1(J, R)$, depends on the positive definite quadratic form $Q$ or the metric $\ell$ of the graph (see Proposition 6.10). In general, it is not the $Z$-affine structure determined by the integral lattice $H_1(J, Z)$ in $H_1(J, R)$.

Besides defining a flat metric on $H_1(J, Z)\setminus\Omega(J)^*$, the positive definite quadratic form $Q$ also defines a polarization, which is defined as follows. Since transition functions between trivializations of a tropical line bundle over different charts of $X$ are affine linear functions with integral slope, a tropical line bundle over $X$ corresponds to an element $H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^* = \text{Aff}$ is the sheaf of affine linear functions with integral slope.

Recall that the sheaf $\mathbb{T}_Z^*$ of locally constant integral 1-forms is defined by the exact sequence

$$0 \to \mathbb{R} \to \mathcal{O}^* \to \mathbb{T}_Z^* \to 0.$$ 

We note that the sheaf $\mathbb{T}_Z^*$ is isomorphic to the locally constant sheaf associated with the abelian group $(\mathbb{Z}^n)^*$. This induces a map

$$c : H^1(X, \mathcal{O}^*) \to H^1(X, \mathbb{T}_Z^*).$$

This is called the Chern class map.

In the case of a complex torus, an ample line bundle $L$ is called a polarization, or rather its first Chern class is called a polarization.

In the tropical case, we call the Chern class $c(L)$ of a tropical line bundle $L$ also a polarization of a tropical torus, according to See [33, §5.1]. It seems more natural to require also that $c(L)$ is positive in the sense defined below.

Now we unwrap the definitions and identify the Chern class of a tropical line bundle of a tropical torus $X$ with a quadratic form. Since $X = \Lambda \setminus \mathbb{R}^n$,

$$H^1(X, \mathbb{T}_Z^*) \cong \Lambda^* \otimes (\mathbb{Z}^n)^* \cong \text{Hom}(\Lambda, (\mathbb{Z}^n)^*).$$

Then a polarization $c(L)$ of a tropical torus $X$ is a linear map $c(L) : \Lambda \to (\mathbb{Z}^n)^*$, which is equivalent to a bilinear form $Q(L) : \Lambda \times \mathbb{Z}^n \to \mathbb{Z}$, which in turn induces a bilinear form $Q(L)$ on $\mathbb{R}^n$, $Q(L) : (\Lambda \otimes \mathbb{R}) \times (\mathbb{Z}^n \otimes \mathbb{R}) \to \mathbb{Z} \otimes \mathbb{R}$.

Remark 6.12 We note that for a complex torus $\Lambda \setminus \mathbb{C}^n$ to admit a polarization, its complex structure of $\mathbb{C}^n$ should be compatible with the integral structure imposed on it by $\Lambda$. Similarly, for a real torus $\Lambda \setminus \mathbb{R}^n$ to be a tropical torus admitting a polarization $c(L)$, the $Z$-affine structure of $\mathbb{R}^n$ imposed by the lattice $\mathbb{Z}^n$ should be compatible with the lattice $\Lambda$ in the sense that $c(L)$ induces a linear map $c(L) : \Lambda \to (\mathbb{Z}^n)^*$.

Proposition 6.13 ([33, §5.1]) For every tropical line bundle $L$ on $X$, the induced bilinear form $Q(L)$ on $\mathbb{R}^n$ is a symmetric bilinear form, and every symmetric bilinear symmetric form $Q$ on $\mathbb{R}^n$ that is integral in the sense that it comes from a linear map $\Lambda \to (\mathbb{Z}^n)^*$ is the Chern class of a tropical line bundle over $\Lambda \setminus \mathbb{R}^n$. 
Proof. In the long exact sequence,
\[ H^1(X, \mathcal{O}^*) \to H^1(X, \mathbb{T}_Z^*) \to H^2(X, \mathbb{R}), \]
if we identify \( H^1(X, \mathbb{T}_Z^*) \cong \Lambda^* \otimes (\mathbb{Z}^n)^* \) and \( H^2(X, \mathbb{R}) \cong \wedge^2(\mathbb{R}^n)^* \), then one can prove that the map \( H^1(X, \mathbb{T}_Z^*) \to H^2(X, \mathbb{R}) \) is the restriction of the skew-symmetry map \( (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \to \wedge^2(\mathbb{R}^n)^* \). Then the proposition is clear.

If the associated symmetric bilinear form \( Q(L) \) of a tropical line bundle \( L \) is positive, the tropical torus \( X \) with the line bundle \( L \) is called a polarized tropical Abelian variety. The index of the image \( c(L)(\Lambda) \) in \((\mathbb{Z}^n)^* \) is called the index of the polarization. If the index is equal to 1, then the polarization \( L \) is called a principal polarization.

Naturally, a tropical abelian variety is defined to be a tropical torus that admits a tropical line bundle whose quadratic form is positive definite.

**Proposition 6.14** The tropical torus \( H_1(J, \mathbb{Z}) \backslash \Omega(J)^* \) with the positive definite quadratic form \( Q \) is a principally polarized tropical abelian variety.

Proof. The quadratic form \( Q \) on \( \Omega(J)^* \) is positive definite and maps \( H_1(J, \mathbb{Z}) \) isomorphically to \( (\Omega_\mathbb{Z}(J)^*)^* = \Omega_\mathbb{Z}(J) \) by Proposition 6.13. Therefore, by Proposition 6.13 it comes from a tropical line bundle over the tropical torus \( H_1(J, \mathbb{Z}) \backslash \Omega(J)^* \), and the induced polarization on \( H_1(J, \mathbb{Z}) \backslash \Omega(J)^* \) is principal.

**Remark 6.15** For a polarized abelian variety \( M \), the Hodge class \( \omega \) defines a non-degenerate skew-symmetric bilinear form on \( H_1(M, \mathbb{Z}) \) and hence a linear map \( i_\omega : H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{Z})^* \). It is principal if and only if the image \( i_\omega(H_1(M, \mathbb{Z})) \) has index 1 in \( H_1(M, \mathbb{Z})^* \). See Remark 5.1 for the case of Jacobian varieties. Therefore, the definition of a principal polarization of a tropical abelian variety is consistent with the definition of a principal polarization for complex abelian varieties.

For applications of tropical Jacobian varieties to metric graphs, we need to identify principally polarized tropical tori with compact flat Riemannian tori.

**Proposition 6.16** There exists a natural 1-1 correspondence between the space \( \mathcal{A}_n^{\text{trop}} \) of principally polarized tropical abelian varieties of dimension \( n \) and the space \( \mathcal{FT}_n \) of compact flat Riemannian tori of dimension \( n \).

Proof. For every principally polarized tropical abelian variety \( (\Lambda \backslash \mathbb{R}^n, Q) \), the positive definite quadratic form \( Q \) defines a flat Riemannian metric on \( \mathbb{R}^n \) and hence also on the compact torus \( \Lambda \backslash \mathbb{R}^n \). Though the integral structure \( \mathbb{Z}^n \) (or \( \mathbb{Z} \)-affine structure) on \( \mathbb{R}^n \) and \( \Lambda \backslash \mathbb{R}^n \) does not show up explicitly in the compact flat torus \( \Lambda \backslash \mathbb{R}^n \), it is uniquely determined by the condition that the positive definite quadratic form \( Q \) has to map \( \Lambda \) isomorphically to \( (\mathbb{Z}^n)^* \), \( \pi_Q : \Lambda \to (\mathbb{Z}^n)^* \). In other words, the lattice \( \mathbb{Z}^n \) in \( \mathbb{R}^n \) is uniquely determined by the pair \((Q, \Lambda)\).

Conversely, for any flat compact Riemannian torus \( \Lambda \backslash \mathbb{R}^n \), we claim that there is a canonical \( \mathbb{Z} \)-affine structure on \( \mathbb{R}^n \) such that \( \Lambda \backslash \mathbb{R}^n \) becomes a principally polarized tropical abelian variety. Let \( Q \) be the quadratic form on \( \mathbb{R}^n \) which defines the flat metric of \( \Lambda \backslash \mathbb{R}^n \). This defines a linear isomorphism \( \pi_Q : \mathbb{R}^n \to (\mathbb{R}^n)^* \) which maps \( \Lambda \) to a lattice \( L^* \) in \((\mathbb{R}^n)^* \), which induces a dual lattice in \( L = (L^*)^* \) in \( \mathbb{R}^n \). The lattice \( L \) defines a \( \mathbb{Z} \)-affine structure on \( \mathbb{R}^n \) and hence a \( \mathbb{Z} \)-affine structure on \( \Lambda \backslash \mathbb{R}^n \), turning \( \Lambda \backslash \mathbb{R}^n \) into a tropical torus. It can be checked directly that the symmetric form \( Q \) is integral in the sense of Proposition 6.13 and hence defines a polarization on the tropical torus \( \Lambda \backslash \mathbb{R}^n \), and it is also a principal polarization by the choice of the \( \mathbb{Z} \)-affine structure.

To determine the moduli space \( \mathcal{A}_n^{\text{trop}} \), we need to identify the space \( \mathcal{FT}_n \).
Proposition 6.17 The moduli space $\mathcal{FT}_n$ of compact flat Riemannian tori of dimension $n$ can be identified with the locally symmetric space $\text{GL}(n, \mathbb{Z})/\text{GL}(n, \mathbb{R})/\text{O}(n)$.

Proof. Let $Q_0$ be the standard positive quadratic form on $\mathbb{R}^n$ which induces the standard inner product $\langle \cdot, \cdot \rangle_0$ on $\mathbb{R}^n$. Then every compact flat torus is isometric to $(\Lambda \setminus \mathbb{R}^n, Q_0)$ for some lattices. A basic point of the geometry of numbers is that every flat torus $(\Lambda \setminus \mathbb{R}^n, Q_0)$ is isometric to $(\mathbb{Z}^n \setminus \mathbb{R}^n, Q)$ for some positive definite quadratic form. In fact, if $A$ is the matrix formed by the column vectors which represent a basis of $\Lambda$, then $Q = A^T A$. It is clear that two flat tori $(\mathbb{Z}^n \setminus \mathbb{R}^n, Q_1)$ and $(\mathbb{Z}^n \setminus \mathbb{R}^n, Q_2)$ are isometric if and only if their quadratic forms $Q_1$ and $Q_2$ are equivalent under $\text{GL}(n, \mathbb{Z})$. Since the space of equivalent classes of positive quadratic forms on $\mathbb{R}^n$ is equal to $\text{GL}(n, \mathbb{Z})/\text{GL}(n, \mathbb{R})/\text{O}(n)$, the proposition is proved.

The combination of the above two propositions gives the following corollary, which was stated as [10, Assumption 2, p. 165] (see also [33, §6.4]).

Corollary 6.18 The moduli space $\mathcal{A}_n^{\text{trop}}$ of principally polarized tropical abelian varieties of dimension $n$ is equal to the locally symmetric space $\text{GL}(n, \mathbb{Z})/\text{GL}(n, \mathbb{R})/\text{O}(n)$:

$$\mathcal{A}_n^{\text{trop}} \cong \text{GL}(n, \mathbb{Z})/\text{GL}(n, \mathbb{R})/\text{O}(n),$$

and $\dim \mathcal{A}_n^{\text{trop}} = \frac{n(n+1)}{2}$.

Remark 6.19 Another assumption made in [10, Assumption 1, p. 164] concerns the existence as a tropical variety of the moduli space $\mathcal{M}_n^{\text{trop}}$ of compact smooth tropical curves of genus $n$. More discussions are given in [4], where $\mathcal{M}_n^{\text{trop}}$ was constructed as a topological space and its connection with the outer space $X_n$ was mentioned [4, §5]. A more concrete outline was given in [30, §3.1]. Certainly, the construction of $\mathcal{M}_n^{\text{trop}}$ as a tropical variety is an important problem in tropical geometry. We provide some details and references to show that outer spaces of metric graphs can be important for this purpose. This might also contribute to the analogy with Teichmüller spaces of Riemann surfaces, for example, to endow the latter with a complex structure) and shows that outer space theory can pay back to the tropical geometry in contrast to the results discussed so far on applications of tropical geometry to metric graphs.

As it is known, the moduli space of Riemann surfaces $\mathcal{M}_g$ is an algebraic variety but not a complex manifold, on the other hand, the Teichmüller space $T_g$ is a complex manifold but not an algebraic variety. Let $\hat{X}_n$ be the unreduced outer space of unnormalized marked metric graphs $(\Gamma, \ell, h)$ of genus $n$, where by the unreduced outer space, we mean that the graphs can contain separating edges, and by unnormalized, we mean that the total sum of edge lengths, $\sum e \ell(e)$, is not required to be 1. For each marked graph $(\Gamma, h)$, there is a simplicial polyhedral cone $\Sigma_{(G,h)} \subset \mathbb{R}^k_{\geq 0}$. When $G$ does not contain any separating edge, the homothety section of the simplicial polyhedral cone $\Sigma_{(G,h)}$ is the simplex $\Sigma_{(G,h)}$ in the outer space $X_n$ defined above in §2.

It is clear that $\hat{X}_n$ is a simplicial polyhedral complex and $\text{Out}(F_n)$ acts on $\hat{X}_n$ by simplicial maps which are also $\mathbb{Z}$-affine maps, and the quotient $\hat{X}_n/\text{Out}(F_n)$ is equal to the moduli space $\mathcal{M}_n^{\text{trop}}$.

The assertion is that $\hat{X}_n$ is a tropical space, or rather is a “tropical manifold” (a space which locally has the structure of a tropical variety), but not a tropical variety, and that $\mathcal{M}_n^{\text{trop}}$ is a tropical orbifold.

According to the definition of tropical varieties [30], locally, $\hat{X}_n$ should be embedded into $\mathbb{R}^N$ as a polyhedral complex with integer slope which satisfies the balancing condition at every polyhedral
face of co-dimension 1. It is clear that each polyhedral cone of \( \hat{X}_n \) has integer slope, but for every polyhedral cone \( F \) of positive codimension, it is not obvious how to embed all polyhedral cones of \( \hat{X}_n \) containing the face \( F \) into some \( \mathbb{R}^N \) as a polyhedral complex with integer slopes satisfying the balancing condition. The idea [11, §3.1] is to note that a marked metric graph \( (\Gamma, \ell, h) \) in the interior of \( F \) has vertices \( v \) of valence at least 4. To move to points of the polyhedral cones which contain \( F \) as a face (i.e., to move to points in a neighborhood of \( (G, \ell, h) \) in \( \hat{X}_n \)), we need to expand such vertices \( v \) into trees with marked boundary points, and to vary the lengths of edges of \( G \) as well. Now trees with marked boundary points correspond to marked rational tropical curves, and their moduli space \( \mathcal{M}_{0,k} \) turns out to be a tropical variety [15, Theorem 3.7] [32]. Since varying the edge lengths of \( G \) clearly traces out a neighborhood of a tropical variety, and since the product of tropical varieties is a tropical variety, this implies that a neighborhood of the marked metric graph \( (G, \ell, h) \) in \( \hat{X}_n \) can be embedded into some \( \mathbb{R}^N \) as a polyhedral complex with integer slope satisfying the balancing condition at every face of codimension 1. Therefore, \( \hat{X}_n \) is a “tropical manifold”.

7 The Torelli Theorem for tropical curves

Let \( \mathcal{M}'_n \) be the moduli space of compact smooth tropical curves of genus \( n \). For every tropical curve \( J \in \mathcal{M}'_n \), by Proposition 6.14 its tropical Jacobian variety \( (H_1(J, \mathbb{Z})\setminus\Omega(J)^*, Q) \) is a principally polarized tropical abelian variety. By Corollary 6.18 we obtain the tropical Jacobian map:

\[
\Pi^{trop} : \mathcal{M}'_n \to \mathcal{A}^{trop}_n = \text{GL}(n, \mathbb{Z})\setminus\text{GL}(n, \mathbb{R})/\text{O}(n), \quad J \mapsto (H_1(J, \mathbb{Z})\setminus\Omega(J)^*, Q). \tag{7.1}
\]

By Proposition 6.6 \((H_1(J, \mathbb{Z})\setminus\Omega(J)^*, Q) \cong (H_1(J, \mathbb{Z})\setminus H_1(J, \mathbb{R}), Q)\), and we also denote the Jacobian of \( J \) by \((H_1(J, \mathbb{Z})\setminus H_1(J, \mathbb{R}), Q)\).

Remark 7.1 To understand better the analogy with the Jacobian (or period) map of Riemann surfaces, we can interpret the quadratic form \( Q \) of a compact tropical curve \( J \) as the period of some normalized 1-forms on \( J \). Choose a basis \( \sigma_1, \ldots, \sigma_g \) of \( H_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^g \). For each 1-cycle \( \sigma_i \), let \( \omega_i \) be the corresponding 1-form as in Remark 6.7. Then the period of \( \omega_j \) on the cycle \( \sigma_i \) is \( \int_{\sigma_i} \omega_j = Q(\sigma_i, \sigma_j) \), i.e., the matrix of the quadratic form \( Q \) with respect to the basis \( \sigma_1, \ldots, \sigma_g \) is the period matrix of the forms \( \omega_1, \ldots, \omega_g \).

As mentioned in §4, the Jacobian \( \Pi : \mathcal{M}_g \to \mathcal{A}_g \) for compact Riemann surfaces is an embedding, but the tropical Jacobian map \( \Pi^{trop} \) is not injective. This is reasonable since the Jacobian variety of a tropical curve \( J \) only depends on the flat metric \( Q \) of the torus \( H_1(J, \mathbb{Z})\setminus H_1(J, \mathbb{R}) \), which depends only on the lengths of 1-cycles in \( J \), but as a metric graph \( (J, \ell) \), \( J \) depends on the lengths \( \ell(e) \) of all edges in \( J \). So some information is lost in passing to the Jacobian variety \((H_1(J, \mathbb{Z})\setminus H_1(J, \mathbb{R}), Q)\). More precise statements are contained in Proposition 7.9 below.

Now we summarize several results from [10] on the exact extent of failure of the tropical Jacobian map \( \Pi^{trop} \).

Definition 7.2 Two metric graphs \((\Gamma, \ell), (\Gamma', \ell')\) are called cyclically equivalent if there exists a bijection between their edges \( \varepsilon : E(\Gamma) \to E(\Gamma') \) such that \( \ell(e) = \ell'(\varepsilon(e)) \), and \( \varepsilon \) induces a bijection between the cycles of \( \Gamma \) and the cycles of \( \Gamma' \). (A cycle is a subgraph that is connected and homotopy equivalent to a circle, but does not contain separating edges.)

Definition 7.3 For \( k \geq 2 \), a graph \( \Gamma \) having at least 2 vertices is said to be \( k \)-edge connected if the graph obtained from \( \Gamma \) by removing any \( k - 1 \) edges is connected.
For example, a graph is 2-edge connected if and only if it is connected and does not contain any separating edge.

To obtain highly edge-connected graphs, we introduce two types of edge contractions:

(A) contract a separating edge,

(B) contract one of two edges which form a separating pair of edges \( \{e_1, e_2\} \), i.e., \( \Gamma - \{e_1\}, \Gamma - \{e_2\} \) are connected, but \( \Gamma - \{e_1, e_2\} \) is disconnected.

**Definition 7.4** The 2-edge connectivization of a connected graph \( \Gamma \) is the 2-edge connected graph \( \Gamma^2 \) which is obtained from \( \Gamma \) by iterating operation (A), i.e., contracting separating edges. A 3-edge connectivization of a connected graph \( \Gamma \) is a 3-edge connected graph \( \Gamma^3 \) which is obtained from the 2-edge connectivization \( \Gamma^2 \) of \( \Gamma \) by iterating operation (B).

Note that there is no unique 3-edge connectivization of a connected graph \( \Gamma \) in general, but all 3-edge connectivizations are cyclically equivalent [10, Lemma 2.3.8], and they retain some information of the original graph.

For this purpose, we need the notion of \( C^1 \)-sets of \( \Gamma \) which corresponds to edges of \( \Gamma^3 \). For every subset \( S \subset E(\Gamma) \) of edges of \( \Gamma \), define a subgraph \( \Gamma - S \) which is obtained by removing the edges in \( S \) but leaving the vertices unchanged. A new, complementary graph \( \Gamma(S) \) is obtained from \( \Gamma \) by contracting every connected component of \( \Gamma - S \) to a single point. (Different connected components are contracted to different points.)

**Definition 7.5** ([10, Definition 2.3.1]) Let \( \Gamma \) be a connected graph that does not contain any separating edge. A subset \( S \subset E(\Gamma) \) is called a \( C^1 \)-set of \( \Gamma \) if \( \Gamma(S) \) is a cycle and \( \Gamma - S \) does not contain any separating edge. Denote the collection of \( C^1 \)-sets of \( \Gamma \) by \( \text{Set}^1(\Gamma) \).

We discuss some examples of \( C^1 \)-sets. If \( S = \{e\} \) and \( \Gamma - \{e\} \) does not contain any separating edge, then \( S \) is a \( C^1 \)-set. For example, suppose that \( \Gamma \) has two vertices connected by more than 3 edges. If \( S \) contains more than one edges, then \( S \) is not a \( C^1 \)-set since \( \Gamma(S) \) is homotopy equivalent to a wedge of more than one circles. In this case, every edge \( E \) is a \( C^1 \)-set. If \( \Gamma \) contains two vertices and two edges, then each edge \( E \) is not a \( C^1 \)-set since \( \Gamma - S \) contains a separating edge, which is, in fact, the only edge of the graph \( \Gamma - S \). More generally, if \( \Gamma \) is homeomorphic to a circle (i.e., all vertices are of valence 2), then the only \( C^1 \)-set \( S \) is the set of all edges due to the second condition that \( \Gamma - S \) does not contain any separating edge.

For a general graph \( \Gamma \), if \( S = \{e_1, e_2\} \) is a separating pair of edges, then \( S \) is a \( C^1 \)-set if and only if \( \Gamma - S \) does not contain any separating edge. The following general fact is true.

**Lemma 7.6** ([10, Corollary 2.3.4]) A graph \( \Gamma \) is a 3-edge connected graph if and only if it is connected and there is a bijection between \( E(\Gamma) \) and \( \text{Set}^1(\Gamma) \), i.e., every edge is a \( C^1 \)-set and every \( C^1 \)-set consists of exactly one edge.

**Proposition 7.7** ([10, Lemma 2.3.8]) Let \( \Gamma \) be a graph, and \( \Gamma^3 \) be a 3-edge connectivization of \( \Gamma \). Then the following holds:

1. The genus of \( \Gamma^3 \) is equal to the genus of \( \Gamma \).

2. There is a canonical bijection between the following three sets: \( \text{Set}^1(\Gamma^3), E(\Gamma^3), \) and \( \text{Set}^1(\Gamma) \).

3. All 3-edge connectivizations of \( \Gamma \) are cyclically equivalent.
We can define similar notions for metric graphs.

**Definition 7.8 ([10, Definition 4.1.7])** A 3-edge connectivization of a metric graph \((\Gamma, \ell)\) is a metric graph \((\Gamma^3, \ell^3)\), where \(\Gamma^3\) is a 3-edge connectivization of \(\Gamma\) and \(\ell^3\) is the length function defined by

\[
\ell^3(e_S) = \sum_{e \in S} \ell(e),
\]

where for each C1-set \(S\) of \(\Gamma\), \(e_S\) is the corresponding edge of \(\Gamma^3\) under the bijection between \(\text{Set}^1(\Gamma)\) and \(E(\Gamma^3)\) in the above proposition.

It follows from the above definition of the metric and the above proposition that all 3-edge connectivizations of a metric graph are cyclically equivalent as metric graphs.

The Torelli Theorem for metric graphs can now be stated.

**Proposition 7.9 ([10, Theorem 4.1.10])** Two metric graphs (or equivalently tropical curves) \(\Gamma\) and \(\Gamma'\) have isomorphic tropical Jacobian varieties if and only if their 3-edge connectivizations \(\Gamma^3\) and \((\Gamma')^3\) are cyclically equivalent.

**Corollary 7.10** When \(n \geq 3\), the tropical Jacobian map

\[
\Pi^{trop} : M^4_n \to A^{trop}_n = \text{GL}(n, \mathbb{Z})\backslash\text{GL}(n, \mathbb{R})/\text{O}(n)
\]

is not injective.

**Proof.** There are several reasons for the failure of the injectivity of \(\Pi^{trop}\). If \(\Gamma\) contains separating edges, then all separating edges are contracted in \(\Gamma^3\). But as metric graphs, the lengths of these separating edges matter. For the outer space of metric graphs, in this paper, we mean the reduced outer space consisting of marked metric graphs with no separating edges (or bridges). This does not cause a problem.

Even for graphs which do not contain any separating edges, the tropical Jacobian map can still fail to be injective. Take a graph \(\Gamma\) such that some C1-set \(S\) of \(\Gamma\) contains at least 2 edges, \(e_1, \ldots, e_k, k \geq 2\). Now in the 3-edge connectivization \(\Gamma^3\), the edge \(e_S\) corresponding to the set \(S\) has length \(\ell^3(e_S) = \ell(e_1) + \cdots + \ell(e_k)\). By Proposition 7.9, the Jacobian variety of \((\Gamma, \ell)\) is the same as the Jacobian variety of its 3-edge connectivization \((\Gamma^3, \ell^3)\), and it depends on the sums \(\sum_{e \in S} \ell(e)\) for C1-sets \(S\). On the other hand, as metric graphs, they depend on the lengths of edges \(\ell(e), e \in S\). There are certainly infinitely many nonisomorphic metric graphs with the same value of \(\ell^3(e_S) = \ell(e_1) + \cdots + \ell(e_k)\).

Some examples of C1-sets that contain more than one element are explained after Definition 7.5 (see also Remark 7.11 below). In particular, we see that for every \(g \geq 3\), there exist metric graphs of genus \(g\) which do not contain any separating edge but contain C1-sets \(S\) with cardinality of at least 2, for example, separating pairs of edges. This proves Corollary 7.10.

**Remark 7.11** To construct a graph \(\Gamma\) which contains a large C1-set \(S\), we start with a graph \(\Gamma_1\) which is homeomorphic to the circle and has \(k \geq 3\) edges. But the valence of every vertex of \(\Gamma_1\) is equal to 2. To overcome this, we attach a connected graph whose vertices have valence of at least 3 at every vertex of \(\Gamma_1\). Denote the new graph by \(\Gamma\). Take \(S\) to consist of the edges of \(\Gamma_1\). Then it can be seen that \(S\) is a C1-set of \(\Gamma\) with \(k\) edges. This explains that there are sets of metric graphs of arbitrarily large dimension which are mapped to a point under the tropical Jacobian map \(\Pi^{trop}\).


8 Tropical Jacobian map and invariant complete geodesic metrics on the outer space

As discussed in the previous section, the tropical Jacobian variety of tropical curves defines the tropical Jacobian map in Equation (7.1):

$$\Pi^{trop} : \mathcal{M}^t_g \rightarrow \mathcal{A}^{trop}_n \cong \text{GL}(n, \mathbb{Z})/\text{GL}(n, \mathbb{R})/O(n).$$

By Propositions 4.9 and 4.10 there is a natural map $X_n \rightarrow \mathcal{M}^t_g$, which factors through the action of $\text{Out}(F_n)$ and gives an embedding

$$\text{Out}(F_n) \backslash X_n \rightarrow \mathcal{M}^t_g$$

and hence a tropical Jacobian map for metric graphs

$$\Pi^{trop} : \text{Out}(F_n) \backslash X_n \rightarrow \text{GL}(n, \mathbb{Z})/\text{GL}(n, \mathbb{R})/O(n).$$

This map can be lifted to the equivariant period map in Equation (3.3)

$$\Pi = \Pi^{trop} : X_n \rightarrow \text{GL}(n, \mathbb{R})/O(n). \quad (8.1)$$

(For simplicity, we denote the period map here and below by $\Pi$ instead of $\Pi^{trop}$.)

This map can be explained more directly as follows. We can fix an isomorphism $H_1(R_n, \mathbb{Z}) \cong \mathbb{Z}^n$, where $R_n$ is the rose with $n$ petals. Then for each marked metric graph $(\Gamma, \ell, h)$ in $X_n$, the marking $h$ induces a canonical isomorphism $h_* : H_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^n$, and hence an isomorphism $h_* : H_1(\Gamma, \mathbb{R}) \cong \mathbb{R}^n$ by linear extension. By Proposition 6.6 we have $\Omega(J)^* \cong H_1(\Gamma, \mathbb{R})$ under the period map in Equation 8.2. This implies that the positive quadratic form $Q$ on $\Omega(J)^*$ induces a positive definite matrix on $\mathbb{R}^n$ with respect to the standard basis and gives a point in $\text{GL}(n, \mathbb{R})/O(n)$. We also note that by Proposition 6.16 the tropical structure and the principal polarization $Q$ on the torus $H_1(\Gamma, \mathbb{Z})/\Omega(\Gamma)^* \cong H_1(\Gamma, \mathbb{Z})/H_1(\Gamma, \mathbb{R})$ are all uniquely determined by the positive definite quadratic form $Q$.

Note that $\text{GL}(n, \mathbb{R})$ is a reductive Lie group, and $\text{GL}(n, \mathbb{R})/O(n)$ is a symmetric space of non-positive curvature. Fix an invariant Riemannian metric on $\text{GL}(n, \mathbb{R})/O(n)$, and let $d_{\text{inv}}$ be the induced Riemannian distance function.

If the map $\Pi$ in Equation 8.1 were injective, then the pull-back metric $\Pi^*d_{\text{inv}}$ would give an $\text{Out}(F_n)$-invariant metric on the outer space $X_n$. Unfortunately, it is not injective due to the infinite kernel of $\text{Out}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$ as pointed out in §2, or by Corollary 7.10 when $n \geq 3$. On the other hand, $\Pi^*d_{\text{inv}}$ is a pseudo-metric, i.e., it satisfies all the conditions of a metric except for the positivity condition: $\Pi^*d_{\text{inv}}(x, y) > 0$ when $x \neq y$.

Recall that $X_n$ admits the canonical $\text{Out}(F_n)$-invariant incomplete simplicial metric $d_0$ (Proposition 2.3). The simple key idea is to combine $\Pi^*d_{\text{inv}}$ with the metric $d_0$. For every two points $x, y \in X_n$, define a distance $d_1(x, y)$ between them by

$$d_1(x, y) = d_0(x, y) + d_{\text{inv}}(\Pi(x), \Pi(y)). \quad (8.2)$$

Proposition 8.1 The function $d_1 : X_n \times X_n \rightarrow \mathbb{R}_{\geq 0}$ is an $\text{Out}(F_n)$-invariant metric on $X_n$.

Proof. Since the map $\Pi$ is equivariant with respect to the homomorphism $\text{Out}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$, $d_{\text{inv}}$ is invariant under $\text{GL}(n, \mathbb{Z})$, and $d_0$ is invariant under $\text{Out}(F_n)$, it is clear that $d_1$ is invariant under $\text{Out}(F_n)$. Since $d_0$ is a metric, it is also clear that $d_1$ is a metric on $X_n$. 

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Proposition 8.2 The metric space \((X_n, d_1)\) is a complete locally compact metric space.

Proof. Fix a basepoint \(y_0 \in X_n\). Since \(X_n\) is a locally finite simplicial complex with some simplicial faces missing, it suffices to prove that for any sequence of marked metric graphs \(y_j = (\Gamma, t_i, h_i)\) in \(X_n\), if \(y_j\) converges to a missing simplicial face of a simplex of \(X_n\), then \(d_1(x_0, y_j) \to +\infty\) as \(j \to +\infty\). We note that when \(y_j\) approaches a missing simplicial face, the length of the shortest loop (or nontrivial cycle) of the metric graph \((\Gamma, t_i, h_i)\) goes to 0 as \(j \to +\infty\). This means that the shortest closed geodesic of the flat tori \(H_1(\Gamma, \mathbb{Z}) \backslash H_1(\Gamma, \mathbb{R}) \cong H_1(R_n, \mathbb{Z}) \backslash H_1(R_n, \mathbb{R}) \cong \mathbb{Z}^n \backslash \mathbb{R}^n\) goes to zero, i.e., the minimum value of the positive definite matrix \(\Pi(y_j) \in \text{GL}(n, \mathbb{R})/O(n)\) on nonzero integral points in \(H_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^n\) goes to 0. By the Mahler compactness theorem for quadratic forms (or lattices), this implies that \(\Pi(y_j)\) is a divergent sequence in the symmetric space \(\text{GL}(n, \mathbb{R})/O(n)\). Since the invariant metric \(d_{inv}\) on the symmetric space \(\text{GL}(n, \mathbb{R})/O(n)\) is complete, \(d(\Pi(y_0), \Pi(y_j)) \to +\infty\) as \(j \to +\infty\). Since \(d_1(x_0, y_j) \geq d_{inv}(\Pi(y_0), \Pi(y_j)\), the proposition is proved.

Remark 8.3 The above result shows an important difference between the tropical period map \(\Pi^{trop}\) and the period map for Riemann surfaces, \(\Pi : \mathcal{T}_g \to \mathfrak{h}_g\), where \(\mathcal{T}_g\) is the Teichmüller space of compact surfaces of genus \(g\) and \(\mathfrak{h}_g = \text{Sp}(2g, \mathbb{R})/U(g)\) is the Siegel upper half space. When we pinch a family of hyperbolic Riemann surfaces along a separating geodesic, i.e., a geodesic which cuts the surface into two connected components, its period matrix does not go to the infinity of \(\mathfrak{h}_g\). Instead it converges to a point in \((\Omega_1, \Omega_2) \in \mathfrak{h}_{g_1} \times \mathfrak{h}_{g_2} \subset \mathfrak{h}_g\), where the limit hyperbolic surface is \(\Sigma_{g_1, 1} \cup \Sigma_{g_2, 1}\), \(g_1 + g_2 = g\), and \(\Omega_1\) is the period of \(\Sigma_{g_1}\) and \(\Omega_2\) is the period of \(\Sigma_{g_2}\). In particular, the period does not detect the punctures in \(\Sigma_{g_1, 1}\) and \(\Sigma_{g_2, 1}\). This implies that the pulled back metric on \(\mathcal{T}_g\) by \(\Pi\) from the invariant metric of the Hermitian symmetric space \(\mathfrak{h}_g\) is not complete. This metric on \(\mathcal{T}_g\) is usually called the Satake metric, probably due to the fact that a compactification of \(\mathcal{M}_g = \text{Mod}_g \backslash \mathcal{T}_g\) was first constructed via \(\Pi\) from the Satake compactification of the Siegel modular variety \(\text{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{h}_g\). We note that metric graphs in the outer space \(X_n\) do not contain separating edges, and the above difficulty is avoided.

Recall that a geodesic segment in a metric space \((X,d)\) is an isometric embedding \(i : (a,b) \to (X,d)\) (sometimes its image is also called a geodesic segment), and \((X,d)\) is called a geodesic metric space if every pair of points are connected by a geodesic segment. For many applications, it is important to obtain geodesic metrics.

For every path connected metric space \((X,d)\), there is a natural length structure \((X,d_{\ell})\) [13, p. 3, 1.4.(a)] [3, §2.3]. Briefly, for every path \(\gamma : [a,b] \to X\), define its length

\[
\ell(\gamma) = \sup_{t_0 < \cdots < t_N} \left\{ \sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i)) \right\},
\]

where the sup is over all partitions \(a = t_0 < t_1 < \cdots < t_N = b\) of the interval \([a,b]\), and for every two points \(x,y \in X\), the length metric \(d_{\ell}(x,y)\) is given by

\[
d_{\ell}(x,y) = \inf_{\gamma} \ell(\gamma),
\]

where the inf is over all paths \(\gamma\) connecting \(x,y\).

In general, a length metric space is not necessarily a geodesic space. Let \(d_{1,\ell}\) be the length metric on \(X_n\) induced from the metric \(d_1\) in Proposition [8,2].

Theorem 8.4 The length metric \(d_{1,\ell}\) on \(X_n\) is an \(\text{Out}(F_n)\)-invariant complete geodesic metric.
Proof. Since $d_1$ is invariant under $\text{Out}(F_n)$, it is clear that $d_{1,\ell}$ is also $\text{Out}(F_n)$-invariant. Since $X_n$ is a locally finite simplicial complex with some simplicial faces missing, and $d_1$ is a complete metric on $X_n$, it is clear that $(X_n, d_1)$ is a complete locally compact metric space. It is also clear from the definition of $d_1$ and the local finiteness of the simplicial complex $X_n$ that for every two points $x, y \in X_n$, $d_{1,\ell}(x, y) < +\infty$. By definition, $d_{1,\ell} \geq d_1$, and hence $(X_n, d_{1,\ell})$ is also a complete locally compact length space. By [8, Proposition 2.5.22], $(X_n, d_{1,\ell})$ is a geodesic length space.

From the above discussion, it is clear that we can also construct other $\text{Out}(F_n)$-invariant complete geodesic metrics on $X_n$. For example, we can define a metric $d_2$ on $X_n$ by

$$d_2(x, y) = \sqrt{d_0(x, y)^2 + d_{\text{inv}}(\Pi(x), \Pi(y))^2},$$

and obtain a corresponding \textit{complete geodesic metric} $d_{2,\ell}$. Similarly, we can define

$$d_\infty(x, y) = \max\{d_0(x, y), d_{\text{inv}}(\Pi(x), \Pi(y))\},$$

and obtain a \textit{complete geodesic metric} $d_{\infty,\ell}$.

As mentioned in the introduction, we would like to construct some complete metrics on $X_n$ which resemble Riemannian metrics, for example, its restriction to each open simplex of $X_n$ is a Riemannian metric. It is not clear whether the above geodesic metrics $d_{1,\ell}, d_{2,\ell}, d_{\infty,\ell}$ on $X_n$ enjoy this property.

In the next two sections, we construct two such metrics, which might offer better analogies with the Riemannian metrics on symmetric spaces and Teichmüller spaces.

9 A complete pseudo-Riemannian metric on the outer space via the tropical Jacobian map

In the previous section, we use the period map of metric graphs in Equation 8.1 to pull back the Riemannian \textit{distance} $d_{\text{inv}}$ on the symmetric space $\text{GL}(n, \mathbb{R})/\text{O}(n)$ in order to define geodesic metrics on $X_n$. In this section, we want to construct geodesic metrics on $X_n$ which are Riemannian metrics on each simplex of $X_n$.

Recall that $X = \text{GL}(n, \mathbb{R})/\text{O}(n)$ is a Riemannian symmetric space of nonpositive curvature. Let $d_{\text{inv}}^2$ be a Riemannian metric tensor on it which is invariant under $\text{GL}(n, \mathbb{R})$. Equivalently, let $\langle \cdot, \cdot \rangle_{\text{inv}}$ be the invariant Riemannian metric on $X$.

When restricted to each open simplex $\Sigma$ of $X_n$, the period map $\Pi : \Sigma \to X$ is a smooth map. It is natural to pull back the Riemannian metric $d_{\text{inv}}^2$ by $\Pi$. But the problem is that $\Pi$ is not an embedding, and not even a local immersion when $n \geq 3$, by Corollary 7.10. (As seen in the proof of Corollary 7.10 it can map subsets of positive dimension of $X_n$ to one point). Therefore, $\Pi^*(d_{\text{inv}}^2)$ is not positive definite, but only semi-positive definite.

To overcome this difficulty, we note that each open simplex $\Sigma$ of $X_n$ has a canonical flat Riemannian metric $\langle \cdot, \cdot \rangle_0$ such that each edge has length $\sqrt{2}$. Let $d_0^2$ be the metric tensor of this flat Riemannian metric.

Define a bilinear form on the tangent bundle of $\Sigma$ by

$$\langle u, v \rangle_{\Sigma} = \langle d\Pi(u), d\Pi(v) \rangle_{\text{inv}} + \langle u, v \rangle_0,$$  \hspace{1cm} (9.1)

where $u, v$ are tangent vectors of $\Sigma$ at any point. Equivalently, its associated symmetric metric tensor is

$$d^2 = \Pi^*(d_{\text{inv}}^2) + d_0^2,$$  \hspace{1cm} (9.2)
Lemma 9.1 The bilinear form $\langle \cdot, \cdot \rangle_\Sigma$ in Equation 9.1 defines a Riemannian metric on every open simplex $\Sigma$ of $X_n$.

Proof. Since $ds^2_0$ is positive definite, and $\Pi^*(ds^2_{inv})$ is semi-positive definite, their sum $ds^2 = \Pi^*(ds^2_{inv}) + ds^2_0$ is positive definite. Since $\Pi$ depends smoothly on the simplicial coordinates of $\Sigma$, this implies that $ds^2$ defines a smooth Riemannian metric on $\Sigma$.

Since $X_n$ is a simplicial complex and is a disjoint union of open simplices $\Sigma$, the metrics $ds^2$ on the open simplices $\Sigma$ give a stratified Riemannian metric on $X_n$. We denote it by $ds^2$ or $\langle \cdot, \cdot \rangle$.

A simplicial complex is a stratified space with a smooth structure in the sense of [34, Chapter 1], and we want to show that the stratified Riemannian metric $ds^2$ gives a smooth Riemannian metric on $X_n$ in the sense of [34, §2.4], or rather more directly it is a piecewise smooth Riemannian metric on $X_n$.

For each simplex $\Sigma$ of $X_n$, let $\mathbb{R}^i$ be the linear space of the same dimension that contains $\Sigma$, or is spanned by $\Sigma$. Let $\overline{\Sigma}$ be the closure of $\Sigma$ in $X_n$, which is contained in $\mathbb{R}^i$ and but still not a closed simplex. A function $f$ or a tensor $T$ on $\overline{\Sigma}$ is called smooth if there exists an open subset $U$ of $\mathbb{R}^i$ which contains $\overline{\Sigma}$ such that $f$ or $T$ can be extended to a smooth function or a tensor on $U$.

Proposition 9.2 For every simplex $\Sigma$ of $X_n$, the Riemannian metric $ds^2$ can be extended to a smooth Riemannian metric on $\overline{\Sigma}$.

Proof. It suffices to prove that for every point $p \in \Sigma$, there is a neighborhood $U_p$ of $p$ in $\mathbb{R}^i$ such that the period map $\Pi : \Sigma \rightarrow X$ extends to a smooth function on $U_p \cup \Sigma \rightarrow X$. Let $(\Gamma, h)$ be a marked graph whose corresponding simplex $\Sigma_{(\Gamma, h)}$ in $X_n$ is $\Sigma$. Then the point $p$ corresponds to a marked graph $(\Gamma, \ell_p, h)$. When $p$ lies on the boundary faces of $\sigma$, some edge lengths $\ell(e)$ are equal to $0$, but for every 1-cycle, its total length is positive. By changing all edge lengths slightly (allowing them to be positive, zero or negative) under the condition that the sum of all edges is equal to $1$, we obtain a neighborhood of $U_p$ in $\mathbb{R}^i$ such that for every 1-cycle of the graph $\Gamma$, its length is positive. Since the period $Q$ of a metric graph $(\Gamma, \ell, h)$ only depends on the lengths of 1-cycles, $Q$ can be extended to $U_p$ and the extended $Q$ is positive definite. This gives the required extension of the period map $\pi$ and the metric $\Pi^*(ds^2_{inv}) + ds^2_0$ to $U_p \cup \Sigma$.

Proposition 9.3 For every two simplices $\Sigma_1, \Sigma_2$ with $\overline{\Sigma_1} \cap \overline{\Sigma_2} \neq \emptyset$, the extended Riemannian metrics $\langle \cdot, \cdot \rangle_{\Sigma_1}$ on $\overline{\Sigma_1}$ and $\langle \cdot, \cdot \rangle_{\Sigma_2}$ on $\overline{\Sigma_2}$ agree on the intersection $\overline{\Sigma_1} \cap \overline{\Sigma_2}$. Therefore, the stratified metric $ds^2$ on $X_n$ is a piecewise smooth Riemannian metric on $X_n$ as a simplicial complex.

Proof. Let $\Sigma_3$ be a common simplicial face of $\overline{\Sigma_1}, \overline{\Sigma_2}$. For every point $p \in \Sigma_3$, let $\langle \cdot, \cdot \rangle_p$ be the inner product associated with the metric tensor $\Pi^*(ds^2_{inv}) + ds^2_0$ on $\Sigma_3$. Then for every tangent vector $v$ to $\Sigma_3$ at $p$, it is clear from the previous proposition that $\langle v, v \rangle_{\Sigma_1} = \langle v, v \rangle_p, \quad \langle v, v \rangle_{\Sigma_2} = \langle v, v \rangle_p, \quad$Therefore, $\langle \cdot, \cdot \rangle_{\Sigma_1}$ on $\overline{\Sigma_1}$ and $\langle \cdot, \cdot \rangle_{\Sigma_2}$ agree on $\Sigma_3$.

With this piecewise smooth Riemannian metric on the stratified space $X_n$, we will introduce a length metric on $X_n$, following the general setup of [34].

Specifically, we consider piecewise smooth curves in $X_n$, i.e., continuous maps $c : [a, b] \rightarrow X_n$ such that there is a partition $a = t_0 < t_1 \cdots < t_n = b$, where each piece $c : [t_{i-1}, t_i]$ is a smooth curve in a simplex $\Sigma_i$. Define the length of $c$ by

$$\ell(c) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \langle c'(t), c'(t) \rangle_{\Sigma_i} dt.$$
For every pair of points \( p, q \in X_n \), define a length function

\[
d_{\ell,R}(p, q) = \inf_c \ell(c),
\]

where \( c \) ranges over all piecewise smooth curves in \( X_n \) connecting \( p \) and \( q \). We note that the subscript \( R \) stands for a Riemannian metric.

**Proposition 9.4** The function \( d_{\ell,R} \) defines a complete, locally compact length metric on \( X_n \) which is invariant under \( \text{Out}(F_n) \) and restricts to a Riemannian distance on each simplex \( \Sigma \) of \( X_n \).

**Proof.** Since \( X_n \) is a locally finite simplicial complex with a piecewise smooth Riemannian metric, and since every two points \( p, q \) of \( X_n \) are connected by a piecewise smooth curve, it is clear that \( d_{\ell,R}(p, q) < +\infty \), and if \( p \neq q \), then \( d_{\ell,R}(p, q) > 0 \), and that \( d_{\ell,R} \) defines the original topology of \( X_n \).

Since the Riemannian metric \( ds^2 \) is invariant under \( \text{Out}(F_n) \), the length function \( d_{\ell,R} \) is also invariant under \( \text{Out}(F_n) \). We need to show that it is complete. This follows from Lemma 9.5 below.

**Lemma 9.5** Let \( c : [0, 1) \to X_n \) be a continuous path such that when \( t \to 1 \) the point \( c(t) \) approaches a missing simplicial face of \( X_n \), i.e., the total length of some cycle (or loop) of the marked metric graph represented by \( c(t) \) goes to 0. Then the image \( \Pi(c(t)) \) goes to the infinity of the symmetric space \( X = \text{GL}(n, \mathbb{R})/\text{O}(n) \) when \( t \to 1 \). In particular, the length of the curve \( \Pi(c(t)) \), and hence of the curve \( c(t) \), \( t \in [0, 1) \), is equal to infinity.

**Proof.** When \( c(t) \) goes to a missing simplicial face of \( X_n \) as \( t \to 1 \), the period matrix \( Q(t) \) of the metric graph corresponding to \( c(t) \) converges to the subspace of semipositive definite singular symmetric matrices. This implies that as \( t \to 1 \), \( \Pi(c(t)) \) goes to the infinity of the symmetric space \( X = \text{GL}(n, \mathbb{R})/\text{O}(n) \). Since the symmetric space \( \text{GL}(n, \mathbb{R})/\text{O}(n) \) is a complete Riemannian manifold, the length of the curve \( \Pi(c(t)), t \in [0, 1) \), is equal to infinity.

When we compute the length of \( c \) using \( \Pi^*ds^2_{\text{inv}} \), it is equal to the length of \( \Pi(c(t)) \). Since the length element of \( ds^2 \) is greater than or equal to \( \Pi^*ds^2_{\text{inv}} \), the length of the curve \( c(t), t \in [0, 1) \), is also equal to infinity.

**Theorem 9.6** The outer space \( X_n \) with the length metric \( d_{\ell,R} \) of the piecewise Riemannian metric \( ds^2 \) is a complete geodesic space, and every two points are connected by a geodesic which is a piecewise smooth curve in \( X_n \).

**Proof.** Since \( X_n \) is locally compact and \( d_{\ell,R} \) is a length distance, it follows from [3] Proposition 2.5.22, [34] Theorem 2.4.15] that \( (X_n, d_{\ell,R}) \) is a geodesic space, i.e., every two points are connected by a geodesic segment. Since \( (X_n, d_{\ell,R}) \) is the length structure induced from a piecewise smooth Riemannian metric of a locally finite simplicial complex, it follows from the usual argument in Riemannian geometry (see [20] for example) that every geodesic segment in \( X_n \) is a piecewise smooth curve.

**Remark 9.7** Once we have endowed \( X_n \) with an invariant piecewise Riemannian metric, it is natural to study some geometric invariants. For example, we can study the growth of balls in \( X_n \), and the spectral theory of \( X_n \) and \( \text{Out}(F_n) \backslash X_n \) with respect to suitable defined Laplace operators. As discussed in [34], there is little known for analysis on general stratified spaces.
10 Another complete pseudo-Riemannian metric on the outer space via the lengths of pinching loops

In this section, we construct another Riemannian metric on the outer space $X_n$ whose asymptotic behaviors are explicitly given. This is motivated by the McMullen metric for Teichmüller space in \[28\].

Fix a sufficiently small $\varepsilon > 0$, for example, $\varepsilon < \frac{1}{6n}$. Define a modified edge length of 1-cycles $\sigma$ of metric graphs $(\Gamma, \ell)$: $\ell_{\varepsilon}(\sigma)$ is a positive smooth function of $\ell(\sigma)$ satisfying the conditions

$$\ell_{\varepsilon}(\sigma) = \begin{cases} 
\varepsilon, & \text{if } \ell(\sigma) \geq 2\varepsilon, \\
\ell(\sigma), & \text{if } \ell(\sigma) \leq \varepsilon.
\end{cases} \quad (10.1)$$

For each simplex $\Sigma$ in $X_n$, define a Riemannian metric by

$$ds_{\varepsilon}^2 = ds_0^2 + \sum_{\sigma \text{ with } \ell(\sigma) < \varepsilon} \left( \frac{d\ell_{\varepsilon}(\sigma)}{\ell_{\varepsilon}(\sigma)} \right)^2, \quad (10.2)$$

where the sum is over all cycles whose lengths are less than $\varepsilon$.

It is clear that for each simplex $\Sigma$, $ds_{\varepsilon}^2$ defines a Riemannian metric. This defines a stratified Riemannian metric on $X_n$. We note that a sequence of marked metric graphs converges to some missing simplicial complexes of $X_n$ if and only if there are cycles (or loops) $\sigma$ whose lengths $\ell(\sigma)$ converge to 0. Such loops are called pinching loops of metric graphs and they correspond to pinching geodesics on hyperbolic surfaces.

By the proof of Proposition 9.2, the length function $\ell(\sigma)$ and hence the modified length function $\ell_{\varepsilon}(\sigma)$ is a smooth function on the closure $\overline{\Sigma}$ in $X_n$, and the Riemannian metric $ds_{\varepsilon}^2$ extends to a smooth Riemannian metric on $\overline{\Sigma}$. Then by the proof of Proposition 9.3, we can prove the following proposition.

**Proposition 10.1** The Riemannian metric $ds_{\varepsilon}^2$ on the stratified space $X_n$ defines an $\text{Out}(F_n)$-invariant piecewise smooth Riemannian metric on $X_n$.

By arguments similar to those of the proofs of Proposition 9.4 and Theorem 9.6, we can prove the following.

**Theorem 10.2** The piecewise smooth Riemannian metric $ds_{\varepsilon}^2$ on the outer space $X_n$ defines a complete geodesic metric $d_{\varepsilon,R,\varepsilon}$ on $X_n$ which is invariant under $\text{Out}(F_n)$ such that every two points are connected by a piecewise smooth geodesic.

11 Finite Riemannian volume of the quotient $\text{Out}(F_n) \backslash X_n$

For a Riemannian manifold, a natural invariant is its volume. As mentioned before, with respect to many natural metrics on $\mathcal{T}_{g,n}$, the quotient $\text{Mod}_{g,n} \backslash \mathcal{T}_{g,n}$ has finite volume. Note that the locally symmetric space $\text{GL}(n,\mathbb{Z}) \backslash \text{GL}(n,\mathbb{R}) / \text{O}(n)$ has infinite volume. Since elements of $\text{GL}(n,\mathbb{Z})$ have determinant equal to $\pm 1$, $\text{GL}(n,\mathbb{Z})$ also acts on $\text{SL}(n,\mathbb{R}) / \text{SO}(n,\mathbb{Z})$, which is a symmetric space of noncompact type, and the quotient space $\text{GL}(n,\mathbb{Z}) \backslash \text{SL}(n,\mathbb{R}) / \text{SO}(n,\mathbb{Z})$ has finite volume. Or equivalently, we note that $\text{GL}(n,\mathbb{Z})$ contains $\text{SL}(n,\mathbb{Z})$ as a subgroup of finite index, and the quotient $\text{SL}(n,\mathbb{Z}) \backslash \text{SL}(n,\mathbb{R}) / \text{SO}(n,\mathbb{Z})$ has finite volume.

If we pursue the analogy between the three pairs of transformation groups $(\text{Out}(F_n), X_n)$, $(\mathcal{T}_g, \text{Mod}_g)$, $(\text{SL}(n,\mathbb{Z}), \text{SL}(n,\mathbb{R}) / \text{SO}(n))$, then we expect that $\text{Out}(F_n)$ is a lattice and the following conjecture seems natural.
**Conjecture 11.1** The volume of \( \text{Out}(F_n) \setminus X_n \) with respect to either of the Riemannian metrics \( ds^2, ds^2_\varepsilon \) defined in the previous section is finite.

By definition, the volume of a simplicial complex with respect to a piecewise Riemannian metric is the sum of volumes of the top dimensional simplices. By Proposition 2.5, there are only finitely many \( \text{Out}(F_n) \)-orbits of simplices. Therefore, it suffices to decide whether for each top dimensional simplex \( \Sigma \) of \( X_n \), the volume of \( \Sigma \) with respect to \( ds^2 \) or \( ds^2_\varepsilon \) is finite.

**Proposition 11.2** When \( n = 2 \), the volume of every 2-simplex \( \Sigma \) in \( X_n \) with respect to the Riemannian metric \( ds^2 \) is finite, and hence the volume of \( \text{Out}(F_n) \setminus X_n \) with respect to the Riemannian metric \( ds^2 \) is finite.

**Proof.** To prove this proposition, we need to compute \( \Pi^*(ds^2_{\text{inv}}) \). Let \( (\Gamma, h) \) be a marked graph corresponding to a 2-dimensional simplex \( \Sigma \) of \( X_2 \). Then \( \Gamma \) has 3 edges \( e_1, e_2, e_3 \) and 2 vertices, with each edge connecting these two vertices. Denote the edge lengths by \( a, b, c \). Then they are characterized by the following conditions: (1) \( a, b, c \geq 0 \), (2) \( a+b+c = 1 \), (3) \( a+b, a+c, b+c > 0 \). In particular, the vanishing of one edge length is allowed, but not two edges simultaneously. Therefore, \( \Sigma \) has 3 missing simplicial faces of dimension 0, i.e., missing vertices, with each corresponding to the vanishing of two edge lengths. It suffices to show that the area of the corner near each missing vertex is zero. Assume that \( a, b = 0 \) at this corner. Let \( \varepsilon \) be a small positive number and define a corner \( \Omega_\varepsilon \) by \( a + b \leq \varepsilon \). The period matrix of the marked metric graph is

\[
Q = \begin{pmatrix} a + b & b \\ b & 1 - a \end{pmatrix},
\]

and its inverse is

\[
Q^{-1} = \frac{1}{ab + (a + b)(1 - a - b)} \begin{pmatrix} 1 - a & -b \\ -b & a + b \end{pmatrix}.
\]

By [37, §4.1.2], the invariant Riemannian metric of the symmetric space \( X = \text{GL}(n, \mathbb{R})/\text{O}(n) \) is given by the metric tensor

\[
ds^2_{\text{inv}} = \text{Tr}((Y^{-1}dY)^2),
\]

where \( Y \in X \) is a symmetric matrix in \( X \). Therefore, the induced Riemannian metric on \( \Sigma \) is \( ds^2 = ds^2_0 + \text{Tr}((Q^{-1}dQ)^2) \). We estimate the growth of \( ds^2 \) when \( a, b \to 0 \). The term \( ds^2_0 \) is bounded and we ignore it. Since \( \frac{1}{ab + (a + b)(1 - a - b)} \) is comparable with \( \frac{1}{a + b} \), and

\[
dQ = \begin{pmatrix} ad + db & db \\ ab & -da \end{pmatrix},
\]

by a direct computation, the area form of the Riemannian metric \( ds^2 \) is bounded from above by a multiple of \( \frac{1}{(a+b)^2} da \wedge db \). By dividing the region \( \Omega_\varepsilon \) into two subregions according to \((I): a \leq b, (II): b \leq a\), we can show easily that the area of \( \Omega_\varepsilon \) with respect to \( ds^2 \) is finite.

**Remark 11.3** In the above estimate, the assumption that \( n = 2 \) was used for the explicit computation. It is conceivable that a more detailed computation will allow one to prove the general case.

Using the explicit form of the metric \( ds^2_\varepsilon \), we can also show that in the corner \( \Omega_\varepsilon \) of a simplex \( \Sigma \) as above, \( ds^2_\varepsilon \) is comparable with \( \frac{(da+db)^2}{(a+b)^2} + da^2 + db^2 \), and that the area form is bounded by a multiple of \( \frac{1}{a+b} ad \wedge db \). Therefore, we can prove that the area of the corner is finite with respect to the Riemannian metric \( ds^2_\varepsilon \) and prove the following result.
**Proposition 11.4** When $n = 2$, the volume of every 2-simplex in $X_n$ with respect to the Riemannian metric $ds^2_ε$ is finite, and hence the volume of $\text{Out}(F_n) \setminus X_n$ with respect to the Riemannian metric $ds^2_ε$ is finite.

**Remark 11.5** When $n = 2$, there is a natural identification of $X_2$ with the Poincare disk $[38]$. It is probably true that the metric $ds^2_ε$ is quasi-isometric to the Poincare metric.

In this paper, we have introduced several geodesic metrics $d_{1,ℓ}, d_{2,ℓ}, d_{∞,ℓ}, d_{ℓ,R}, d_{ℓ,R,ε}$ on $X_n$. We note that $d_{1,ℓ}, d_{2,ℓ}, d_{∞,ℓ}$ are clearly quasi-isometric to each other. The following problem seems to be natural and tempting.

**Problem 11.6** Decide whether all the above $\text{Out}(F_n)$-invariant, complete geodesic metrics $d_{1,ℓ}, d_{ℓ,R}, d_{ℓ,R,ε}$ on $X_n$ are quasi-isometric.

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