A MULTI-PLANK GENERALIZATION OF THE BANG AND KADETS INEQUALITIES

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Abstract. If a convex body in $\mathbb{R}^n$ is covered by the union of convex bodies, multiple subadditivity questions can be asked. The subadditivity of the width is the subject of the celebrated plank theorem of Th. Bang, whereas the subadditivity of the inradius is due to V. Kadets. We adapt the existing proofs of these results to prove a theorem on coverings by certain generalized non-convex “multi-planks”. One corollary of this approach is a family of inequalities interpolating between Bang’s theorem and Kadets’s theorem. Other corollaries include results reminiscent of Davenport’s potato problem, and certain inequalities on the relative width.

1. Introduction

Let $K$ be a convex set in $\mathbb{R}^n$ endowed with the Euclidean norm. We denote $w(K)$ the width of $K$ and $r(K)$ the inradius of $K$. There are two classical results on the subadditivity of $w(\cdot)$ and $r(\cdot)$.

Theorem 1.1 (Th. Bang [5]). If a convex set $K$ is covered by convex sets $C_1, \ldots, C_N$, then

$$\sum_{i=1}^{N} w(C_i) \geq w(K).$$

Theorem 1.2 (V. Kadets [15]). If a convex set $K$ is covered by convex sets $C_1, \ldots, C_N$, then

$$\sum_{i=1}^{N} r(C_i) \geq r(K).$$

If a convex set $K$ sits inside a subspace $L$ of $\mathbb{R}^n$, we use the notation $r(K; L)$ for the inradius of $K$ measured inside $L$.

Definition 1.3. Let $1 \leq k \leq n$. The following quantities will be called the intrinsic inradii of a convex set $K \subset \mathbb{R}^n$.

1. The upper intrinsic inradius of $K$ is defined as

$$r^{(k)}(K) = \min_{\dim L = k} r(K|L; L) = \min_{\dim L = k} r(K + L^\perp),$$

where $L$ runs over the $k$-dimensional subspaces of $\mathbb{R}^n$, and $K|L$ is the orthogonal projection of $K$ onto $L$.

2. The lower intrinsic inradius of $K$ is defined as

$$r_{(k)}(K) = \min_{\dim L = k} \max_{x \in L^\perp} r(K \cap (L + x); L + x),$$

Equivalently, $r_{(k)}(K)$ can be defined via a Kakeya-type property: it is the largest number $r$ such that the open ball of radius $r$ of any $k$-dimensional subspace can be placed in $K$ after a translation.

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Those radii appeared (under different names) in multiple papers, e.g. [6, 7, 12, 13]. Some other notions of successive radii (different from ours) in the context of certain plank problems were considered in [8, 9, 10].

Observe that $r^{(1)}(K) = r_{(1)}(K) = w(K)/2$ and $r^{(n)}(K) = r_{(n)}(K) = r(K)$. It is clear that $r^{(k)}(K) \geq r_{(k)}(K)$, but in general it might happen that this inequality is strict; for instance, this happens for the regular tetrahedron in $\mathbb{R}^3$ and $k = 2$.

The following result, interpolating between Theorem 1.1 and Theorem 1.2, will follow as a corollary of the main theorem, Theorem 3.1.

**Theorem 1.4.** If a convex set $K$ is covered by convex sets $C_1, \ldots, C_N$, then for any $1 \leq k \leq n$,

$$\sum_{i=1}^{N} r^{(k)}(C_i) \geq r_{(k)}(K).$$

Most often Theorem 1.1 is formulated in terms of coverings by planks. A *plank* is the set of all points between two parallel hyperplanes. We will interpret Theorem 1.4 in terms of coverings by certain non-convex “planks” (see Definition 2.2 and Figures 1, 2 for examples) and adapt classical proofs of Theorems 1.1, 1.2 to give a one-page proof of a more general plank theorem (Theorem 3.1).

Another type of corollaries that can be immediately deduced from the main theorem is akin to the Davenport–Alexander potato problem [3]. It tells us that one can arbitrarily apply a commercial pizza cutter to one’s favorite pizza several times and still find a decent size slice.

**Theorem 1.5.** Let us call an $m$-fan ($m \geq 2$) the union of $m$ rays in the plane with the same endpoint and with all angles $\frac{2\pi}{m}$. If the unit disk $B_1$ is partitioned by $m$-fans $S_1, \ldots, S_N$, then there is a piece of inradius at least $\sin \frac{\pi}{m} \frac{\sin \pi / m}{N + \sin \pi / m}$.

Section 2 explains our notion of a generalized non-convex plank and gives several examples. The properties proven there are rather to illustrate the concept and make Definition 2.2 less obscure; none of them are actually used in the proof of the main theorem in Section 3. The main theorem in Section 3 is followed by the proofs of Theorem 1.4 and Theorem 1.5 (together with its higher-dimensional generalizations).

Section 4 discusses to what extent the main theorem generalizes to the case when $\mathbb{R}^n$ is endowed with a non-Euclidean norm, whose unit ball need not be centrally symmetric, generally speaking. The normed counterparts of Theorems 1.1 and 1.2 are widely open questions. The former, known as Bang’s conjecture on relative widths, is solved by K. Ball [4] for the case when the unit ball is centrally symmetric. The latter is far less understood, with some progress towards the case of partitions (instead of coverings) made in [2].

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## 2. Multi-planks

**Definition 2.1.** Given a set $V = \{v^1, \ldots, v^m\}$ of points in $\mathbb{R}^n$, the *anti-Voronoï diagram* (or the *farthest-point Voronoï diagram*) is the partition $\mathbb{R}^n = \bigcup_{j \in [m]} A_{V}^{j}$, where the closed cells $A_{V}^{1}, \ldots, A_{V}^{m}$ are given by

$$A_{V}^{j} = \left\{ x \in \mathbb{R}^n \mid |x - v^j| \geq |x - v^{j'}| \quad \forall j' \in [m] \right\}.$$
In other words, $A^1_V$ consists of all points for which the farthest element of $V$ is $v^j$.

One should notice that all regions $A^j_V$ are convex. Additionally, each cell $A^j_V$ is either unbounded (if $v^j$ is an extreme point of $\text{conv} V$) or empty (otherwise). The unboundedness of non-empty cells follows from the following simple claim: If $x \in A^j_V$ then the entire ray $\{x + t(x - v^j) \mid t \geq 0\}$ lies in $A^j_V$.

In the sequel we use the notation $A^j_{-V}$ for the anti-Voronoï cell of the set $-V = \{-v^1, \ldots, -v^m\}$ corresponding to the point $-v^j$; that is,

$$A^j_{-V} = \left\{ x \in \mathbb{R}^n \mid |x + v^j| \geq |x + v^{j'}| \ \forall j' \in [m] \right\}.$$ 

**Definition 2.2.** Let $V = \{v^1, \ldots, v^m\}$, $m \geq 2$, be a set of points in $\mathbb{R}^n$, such that the smallest ball containing $V$ is centered at the origin. Denote by $r(V)$ its radius.

1. The set

$$P = \mathbb{R}^n \setminus \bigcup_{j \in [m]} (v^j + A^j_{-V})$$

will be called the open centered multi-plank generated by $V$.

2. The closure $\overline{P}$ of $P$ will be called the closed centered multi-plank generated by $V$.

3. A multi-plank generated by $V$ is a translate of $P$ or $\overline{P}$.

In all these cases, the radius $r(V)$ will be called the inradius of a multi-plank (this word choice will be justified by Lemma 2.12). The dimension of the convex hull of $V$ will be called the rank of a multi-plank.

**Example 2.3.** If $V = \{u, -u\}$ for $0 \neq u \in \mathbb{R}^n$, then the corresponding rank 1 (open centered) multi-plank is just the ordinary (open) plank

$$P = \left\{ x \in \mathbb{R}^n \mid -|u|^2 < \langle x, u \rangle < |u|^2 \right\}.$$ 

**Example 2.4.** Let $V = \{v^1, \ldots, v^{n+1}\}$ be a set of affinely independent vectors of length $r$ whose convex hull contains the origin in its interior, so that the smallest ball containing $V$ is $B_r$, the ball of radius $r$ centered at the origin. The corresponding rank $n$ (open centered) multi-plank $P$ can be described as follows. For each $j \in [n + 1]$, draw the tangent hyperplane $H^j$ to the ball $B_r$ at the point $v^j$. Those hyperplanes bound a simplex $S$. Consider the union $F$ of rays with the common endpoint at the origin that intersect the $(n - 2)$-skeleton of $S$. It’s easy to check that $F$ is the fan dividing space into regions $A^j_{-V}$ (which in this case form the ordinary Voronoï diagram of $V$). The multi-plank $P$ looks like a thickened fan $F$, with the widths of its “wings” defined so that $\partial P$ passes through each of the $v^j$. (See Figure 1 for an example.)

**Example 2.5.** Let $V = \{v^1, \ldots, v^{k+1}\}$ be a set of vectors of length $r$ whose convex hull is $k$-dimensional and contains the origin in its relative interior. The corresponding rank $k$ (open centered) multi-plank $P$ is the Minkowski sum of the $k$-dimensional multi-plank generated by $V$ in its affine hull (as in the previous example) with the orthogonal $(n-k)$-dimensional subspace.

The multi-planks as in the examples above (and their closures) will be called simple.

**Lemma 2.6.** Any open (closed) convex set $C$ can be placed inside an open (closed) simple multi-plank of inradius $r^{(k)}(C)$.

**Proof.** Pick $L$ delivering minimum in the definition $r^{(k)}(C) = \min_{\dim L = k} r(C|L; L)$. Let $c + B_r$ be the largest $k$-ball in $C|L \subset L$. Next, pick points $c + v^1, \ldots, c + v^{k+1} \in L$ on the
intersection of (relative) boundaries of $C|L$ and of $c + B_r$, certifying that $r$ was indeed maximal. The plank $c + P$, where $P$ is generated by $V = \{v^1, \ldots, v^{k+1}\}$, as in Example 2.5, will do. □

The rest of this section describes how multi-planks actually look like. This description is not used in Section 3, so the reader interested in the proof of the main theorem and its corollaries may skip right to Section 3.

**Example 2.7.** Let $V$ be the Bang set of the family of planks

$$P_i = \{ x \in \mathbb{R}^n \mid -|u_i|^2 < \langle x, u_i \rangle < |u_i|^2 \};$$

that is, $V$ consists of all combinations $\sum_i \pm u_i$, over all possible sign choices. Let us show that the corresponding to $V$ open centered multi-plank $P$ contains the union $\bigcup P_i$.

Assume $x \notin P$, then $x \in v^j + A_{-V}$ for a certain $v^j = \sum_i \varepsilon_i u_i$, $\varepsilon_i \in \{+1, -1\}$. It means that the farthest from $x - v^j$ element of $-V$ is $-v^j$, that is,

$$\left| (x - v^j) + v^j \right| \geq \left| (x - v^j) + \sum_i \varepsilon_i' u_i \right|,$$

for all $\varepsilon_i' \in \{+1, -1\}$.

Therefore,

$$|x|^2 \geq \left| x + \sum_i (\varepsilon_i' - \varepsilon_i) u_i \right|^2,$$

for all $\varepsilon_i' \in \{+1, -1\}$.

Set all $\varepsilon_i'$ equal to the corresponding $\varepsilon_i$ except one; then we get

$$|x|^2 \geq |x - 2\varepsilon_i u_i|^2 = |x|^2 - 4\varepsilon_i \langle x, u_i \rangle + 4|u_i|^2,$$

for all $i$. This last line implies that $x \notin P_i$, for each $i$; hence, $\bigcup_i P_i \subseteq P$. It turns out that most often the multi-plank $P$ is strictly greater than the union $\bigcup P_i$ (see Figure 2 for an example).
Definition 2.8. Given a set $V = \{v^1, \ldots, v^m\}$ of points in $\mathbb{R}^n$, whose affine hull is the entire $\mathbb{R}^n$, the anti-Delaunay triangulation is a triangulation of $\conv V$ satisfying the full sphere property: for each simplex of the triangulation, the (closed) ball whose boundary passes through the simplex vertices contains the entire $V$.

It is known that the anti-Delaunay triangulation always exists (see, e.g., [1, Section 4]), and is unique provided that no $n + 2$ points lie on a sphere. In the case when the affine hull of $V$ is smaller than $\mathbb{R}^n$, one can define the anti-Delaunay triangulation inside the affine hull of $V$.

Now we give a finer description what a multi-plank looks like. We use the notation $N_T(x)$ for the cone of outer normals of a convex body $T$ at a boundary point $x \in \partial T$; by definition, $N_T(x) = \{\nu \in \mathbb{R}^n \mid \langle \nu, y - x \rangle \leq 0 \ \forall y \in T\}$. If $F$ is a face of $T$, we write $N_T(F)$ for the cone of outer normals at any point from the relative interior of $F$.

Let $V = \{v^1, \ldots, v^m\} \subset \mathbb{R}^n$ be the generating set of a centered open multi-plank $P$. If the rank of $P$ is smaller than $n$, the multi-plank looks like the orthogonal product of a subspace and a lower-dimensional multi-plank. For this reason, we restrict our attention to full rank multi-planks for now.

Consider the anti-Delaunay triangulation $\Sigma$ of $\conv V$ regarded as a simplicial complex. For each top-dimensional cell $\sigma$ of $\Sigma$, let $S_\sigma$ be the translated copy of $\sigma$ such that the origin is equidistant from the vertices of $S_\sigma$. The simplices $S_\sigma$ do not overlap—this follows from the full sphere property of $\Sigma$. Indeed, if $\sigma_1$ and $\sigma_2$ are two anti-Delaunay cells, they need to be pushed apart in order to make their circumspheres concentric.

Let $\tau$ be a cell in $\Sigma$ of dimension greater that 0. For each top-dimensional cell $\sigma$ containing $\tau$, find the corresponding face $T_{\tau,\sigma}$ of $S_\sigma$ (the one that is a translated copy of
\( \tau \). In particular, \( T_{\sigma, \sigma} = S_\sigma \). Consider the following set:

\[
(*) \quad P_\tau = \bigcap_{\sigma \supset \tau} \left( \text{rint} \, T_{\tau, \sigma} + N_{S_\sigma}(T_{\tau, \sigma}) \right),
\]

where the intersection is taken over all top-dimensional cells \( \sigma \) containing \( \tau \). For top-dimensional cells, this definition gives

\[ P_\sigma = \text{int} \, S_\sigma. \]

**Theorem 2.9.** With the notation as above, the multi-plank \( P \) admits the following stratification:

\[
P = \bigcup_{d=1}^{n} \bigcup_{\dim \tau = d} P_\tau,
\]

where the inner union is taken over all cells of \( \Sigma \) of dimension \( d \).

Informally speaking, this description tells us (in the case when no \( n + 2 \) points from \( V \) lie on the same sphere) that locally, near each simplex \( \tau \), each multi-plank \( T \) looks like \( \mathbb{R}^n \setminus \bigcup_{j=0}^{n} (v^j + N_{S_\sigma}(w^j)) \) (see Figure 3).

**Proof of Theorem 2.9.** To begin with, we extend the definition of \( P_\tau \) to the case when \( \tau = v^j \) is a vertex in \( \Sigma \). As before, for each top-dimensional cell \( \sigma \) containing \( v^j \), find the corresponding vertex \( T_{v^j, \sigma} \) of \( S_\sigma \). The stratum corresponding to \( v^j \) is defined as

\[
P_{v^j} = \bigcap_{\sigma \ni v^j} \left( T_{v^j, \sigma} + N_{S_\sigma}(T_{v^j, \sigma}) \right),
\]

where the intersection is taken over all top-dimensional cells \( \sigma \) containing \( v^j \). We claim that \( P_{v^j} \) is nothing else as \( v^j + A^j_{LV} \), the shifted anti-Voronoï cell from Definition 2.2. This is somewhat tedious but straightforward. The set \( v^j + A^j_{LV} \) is defined by the system of inequalities \( |x| \geq |x - v^j + v^{j'}| \), over \( v^{j'} \in V \). In fact, only the vertices \( v^{j'} \) adjacent to \( v^j \) in \( \Sigma \) contribute to this system; this is essentially the duality between the anti-Delaunay triangulation and the anti-Voronoï diagram. Equivalently, one can write those inequalities as

\[
(\bigstar) \quad \left\langle x + \frac{v^{j'} - v^j}{2}, v^{j'} - v^j \right\rangle \leq 0, \quad v^{j'} \text{ adjacent to } v^j.
\]

On the other hand, each set \( T_{v^j, \sigma} + N_{S_\sigma}(T_{v^j, \sigma}) \) is defined by inequalities of the form

\[
(\bigvee) \quad \left\langle x - T_{v^j, \sigma}, T_{v^{j'}, \sigma} - T_{v^j, \sigma} \right\rangle \leq 0, \quad v^{j'} \in \sigma.
\]

Varying \( \sigma \ni v^j \) here, one gets those inequalities for all \( v^{j'} \) adjacent to \( v^j \) in \( \Sigma \). Observe that \( v^{j'} - v^j = T_{v^{j'}, \sigma} - T_{v^j, \sigma}, \) if \( v^j \) and \( v^{j'} \) form an edge in \( \sigma \). Next,

\[
T_{v^j, \sigma} = \frac{T_{v^j, \sigma} + T_{v^{j'}, \sigma}}{2} + \frac{T_{v^j, \sigma} - T_{v^{j'}, \sigma}}{2} = v^j - v^{j'}.
\]

Therefore, \( v^j - v^{j'} \) and \( T_{v^j, \sigma} \) lie in the same hyperplane orthogonal to \( v^j - v^{j'} \); this proves that inequalities \( (\bigstar) \) and \( (\bigvee) \) are equivalent, so \( P_{v^j} = v^j + A^j_{LV} \).

To finish the proof, it suffices to observe that \( \mathbb{R}^n \) is the disjoint union of the strata \( P_\tau \), over faces \( \tau \) in \( \Sigma \) of any dimension. Indeed, for any point \( x \in \mathbb{R}^n \) we can consider
the nearest to $x$ point $y \in \bigcup \sigma S_{\sigma}$ (the union is over top-dimensional cells of $\Sigma$). If $y \in \text{rint} T(\tau, \sigma)$ then it is easy to see that $x \in P_{\tau}$ (and that $P_{\tau}$ is the only stratum containing $x$).

Remark 2.10. Theorem 2.9 remains true in the case when the rank of $P \subset \mathbb{R}^n$ is less than $n$. In this case, the anti-Delaunay triangulation of $V$ should be considered inside the linear hull $L$ of $V$, and in the definition of stratum $(\star)$ the normal cone $N_{S_{\sigma}}(T_{\tau, \sigma})$ gets decomposed as the Minkowski sum of the normal cone in $L$ with $L^\perp$.

Remark 2.11. If $P$ is a centered rank $k$ multi-plank generated by $V$, the stratification of $P$ is defined using the $k$-dimensional anti-Delaunay triangulation of $V$. Let $\rho$ be the smallest radius of a top-dimensional cell of that triangulation. Clearly, $\rho \geq r(P)$, and it might happen that $\rho > r(P)$, if not all vertices of $\text{conv} V$ lie on the sphere of radius $r(P)$.

A direct corollary of Theorem 2.9 is that inside the ball $B_{\rho}$ of radius $\rho$ the multi-plank $P$ can be simplified; namely, $P \cap B_{\rho} = P' \cap B_{\rho}$, where the multi-plank $P'$ is generated by the subset of $V$ of vectors of length $r(P)$. It would be interesting to know whether the
proof of Jiang and Polyanskii [14] of L. Fejes Tóth’s zone conjecture can be retold using this trick in the language of multi-planks.

We use Theorem 2.9 to justify the word “inradius” used in Definition 2.2.

**Lemma 2.12.** Let $P$ be a rank $k$ open multi-plank generated by $V$. The radius $r(V)$ (as in Definition 2.2) is indeed the inradius of $P$; moreover, the upper intrinsic inradii $r^{(k)}(P), \ldots, r^{(n)}(P)$, and the lower intrinsic radii $r^{(k)}(P), \ldots, r^{(n)}(P)$ all equal $r(V)$.

**Proof.** We can assume that $P$ is centered. First we show that the open ball $B_r$ of radius $r = r(V)$ is contained in $P$.

Suppose $x \notin P$, then $x \in v^j + A^j_V$ for some $j$. It means that the farthest from $x - v^j$ element of $-V$ is $-v^j$, that is,

$$|(x - v^j) + v^j| \geq |(x - v^j) + v^j|, \quad \text{for all } v^{j'} \in V.$$ 

Therefore,

$$|x|^2 \geq |x - (v^j - v^{j'})|^2 = |x - v^j|^2 + 2 \langle x - v^j, v^{j'} \rangle + |v^{j'}|^2.$$ 

It is possible to pick $v^{j'} \in \partial B_r$ such that $\langle x - v^j, v^{j'} \rangle \geq 0$, since $B_r$ is the smallest ball containing $V$. For such a choice of $v^{j'}$ one gets

$$|x|^2 \geq |v^{j'}|^2 = r^2,$$

thus proving that $x \notin B_r$.

We have shown that $r^{(n)}(P) \geq r(V)$. Now we need to show that $r^{(k)}(P) \leq r(V)$. In fact, it suffices to show that $r^{(n)}(P) \leq r(V)$, since a rank $k$ multi-plank is the Minkowski sum of a $k$-dimensional multi-plank with the orthogonal subspace.

Let $c + B_\rho$ be the largest open ball contained in $P$, $\rho = r^{(n)}(P)$. Let its center belong to the stratum $P_\tau$ of $P$, where $\tau = \text{conv}\{u^0, \ldots, u^d\}$ is a $d$-dimensional cell of the anti-Delaunay triangulation of conv $V$. The stratum $P_\tau$ can be represented as $P_\tau = s + \text{rint} \tau + R$, where $s \in \mathbb{R}^n$ is a translation vector, and $R$ is a certain closed set of dimension $n - d$, orthogonal to $\tau$. From the stratification result, Theorem 2.9, one can deduce that the sets $s + u^j + R$ are all disjoint from $P$. Hence, the radius $\rho$ does not exceed the shortest among the distances $\text{dist}(c, s + u^j + R) = |\pi_\tau(c - s) - u_j|$, where $\pi_\tau(c - s) \in \text{rint} \tau$ is the orthogonal projection of $c - s$ onto the affine hull of $\tau$. If $\rho > r = r(V)$, then the vertices of $\tau$ are all in $B_{\rho} \setminus (\pi_\tau(c - s) + B_\rho)$, which can be strictly separated from $\pi_\tau(c - s)$ by a hyperplane; this contradicts the fact $\pi_\tau(c - s) \in \text{rint} \tau$. Therefore, $\rho = r^{(n)}(P) \leq r(V)$.

**3. Multi-plank theorem**

Now we are ready to state the main result. The proof follows closely the ideas of Bang and Kadets. Our exposition also makes use of a trick by Bognár [11].

**Theorem 3.1.** If a convex set $K \subset \mathbb{R}^n$ is covered by rank $k$ multi-planks $P_1, \ldots, P_N$, then

$$\sum_{i=1}^N r(P_i) \geq r^{(k)}(K).$$

**Proof.** Every closed multi-plank can be covered by an open one of almost the same inradius; so without loss of generality we assume that the multi-planks are open.

It suffices to consider the case when $K$ is bounded, i.e., $K \subset B_R$ for some $R$. If not, we apply theorem for $K \cap B_R$ and pass to the limit $R \to \infty$; here we use $r^{(k)}(K \cap B_R) \to r^{(k)}(K)$ as $R \to \infty$. 
First we reduce the problem to the case of centered multi-planks and then deal separately with the centered case (this strategy can be traced back to Bognár [11]).

**Step 1.** We think of \( \mathbb{R}^n \) as a coordinate subspace \( H \subset \mathbb{R}^{n+1} \); say, \( H = \{ (x, 0) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n \} \). Now both the set \( K \) and the multi-planks \( P_i \) sit inside \( \mathbb{R}^{n+1} \). Pick a point \( O = (0_n, D) \) very far from the origin; here \( 0_n \) is the origin of \( \mathbb{R}^n \) and \( D \in \mathbb{R} \) is large. For each \( i \), build the cylinder \( C_i = (P_i \cap B_R) + \ell_i \), where \( \ell_i \) is the line passing through \( O \) and through the center of \( P_i \). Those cylinders cover the cone \( \widehat{K} = \text{conv}(K \cup \{O\}) \). There are two statements to check:

1. each \( C_i \) can be covered by a rank \( k \) multi-plank centered at \( O \) and of inradius close to \( r(P_i; H) \) (the proximity depends on \( D \));

2. the intrinsic inradius \( r_{(K)}(\widehat{K}) \) is close to \( r_{(K)}(K; H) \). (The notation \( r_{(K)}(\cdot; H) \) is to specify the ambient space where the inradius is measured.)

For the first one, notice that \( C_i \) splits as the orthogonal product of \( \ell_i \) and of the set \( A(D) \) which is an affine copy of \( P_i \cap B_R \) shrunk negligibly (as long as \( D \) is large) along one direction. The reader can convince themselves that \( A(D) \) can be covered by a scaled copy of \( P_i \cap B_R \) with the homothety coefficient tending to 1 as \( D \to \infty \). This explains the first statement.

For the second claim, fix \( d \in \mathbb{R} \) large enough so that \( r_{(k)}(K; H) = r_{(k)}(K + [0, d]) \); here \( K + [0, d] \) is a shortcut for \( K + [(0_n, 0), (0_n, d)] \subset \mathbb{R}^{n+1} \). Now observe that \( \widehat{K} \cap (K + [0, d]) \) converges to \( K + [0, d] \) in the Hausdorff metric, as \( D \to \infty \). One can check that the function \( r_{(k)}(\cdot) \) is Hausdorff continuous, so we can write

\[
r_{(k)}(K; H) \geq r_{(k)}(\widehat{K}) \geq r_{(k)}(\widehat{K} \cap (K + [0, d])) \to_{D \to \infty} r_{(k)}(K + [0, d]) = r_{(k)}(K; H).
\]

Now, applying the theorem in the centered case, we get the desired inequality with a small error term, which decays as \( D \to \infty \).

**Step 2.** Now we can assume that all the \( P_i \) are centered at the origin. The proof here follows closely the ideas from original papers by Bang and Kadets with certain simplifications. Assume the contrary to the statement of theorem:

\[
\alpha = \frac{r_{(k)}(K)}{\sum_{i=1}^{N} r(P_i)} > 1.
\]

We define the Bang set

\[
X = \left\{ \sum_{i=1}^{N} v_i^{j_i} \mid 1 \leq j_i \leq m_i \right\},
\]

where \( V_i = \{v_i^1, \ldots, v_i^{m_i}\} \) is the generating set of \( P_i \). The strategy of the proof is to show that \( X \) can be covered by a translate of \( K \) (assuming the contrary to the statement of theorem) but at the same time \( X \) does not fit into \( \bigcup P_i \).

**Step 2.1.** The Bang set splits as the Minkowski sum of the generating sets of the multi-planks: \( X = V_1 + \ldots + V_N \). By the definition of the lower intrinsic radius, \( V_i \) can be covered by a translate of \( \frac{r(P_i)}{r_{(k)}(K)} K \) (where the bar denotes closure), hence by a translate of \( \frac{\alpha r(P_i)}{r_{(k)}(K)} K \). Therefore, for some translation vector \( s \in \mathbb{R}^n \),

\[
X = V_1 + \ldots + V_N \subset s + \frac{\alpha r(P_i)}{r_{(k)}(K)} K + \ldots + \frac{\alpha r(P_i)}{r_{(k)}(K)} K = s + K.
\]

**Step 2.2.** Suppose \( X \subset s + \bigcup P_i \), \( s \in \mathbb{R}^n \). Consider the farthest from the origin point in \( X - s \); let it be \( x = -s + \sum_{i=1}^{N} v_i^{j_i} \). Fix \( i \) and consider the family of vectors \( (x - v_i^{j_i}) + v_i^{j_i} \),
over \(1 \leq j_i' \leq m_i\). Since \(x\) is the longest among them, one gets \(x - v_i' \in A_i' V_i\), hence \(x \in v_i' + A_i' V_i \subset \mathbb{R}^n \setminus P_i\). Repeating this over all \(i\), we get a contradiction.

\[\Box\]

\textit{Proof of Theorem 1.4.} If a convex set \(K\) is covered by convex sets \(C_1, \ldots, C_N\), then each \(C_i\) can be replaced by a simple closed multi-plank \(\overline{P}_i\) covering \(C_i\) and having the same rank \(k\) upper intrinsic inradius (see Lemma 2.6). Now Theorem 3.1 implies the desired inequality:

\[
\sum_{i=1}^{N} r^{(k)}(C_i) = \sum_{i=1}^{N} r(\overline{P}_i) \geq r^{(k)}(K).
\]

\[\Box\]

\textit{Proof of Theorem 1.5.} Suppose the contrary, and denote \(r < \frac{\sin \pi / m}{N + \sin \pi / m}\) the radius of the largest disk inscribed in the partition by fans. Pick a number \(\overline{r}\) between \(r\) and \(\frac{\sin \pi / m}{N + \sin \pi / m}\). Then the disk \(B_{1-r}\) is covered by the multi-planks \(P_1, \ldots, P_N\), where \(P_i\) is the \(\overline{r}\)-neighborhood of \(S_i\). The inradius of each multi-plank equals \(\frac{r \sin \pi / m}{N + \sin \pi / m}\), so using Theorem 2.2 one gets the inequality

\[
\sum_{i=1}^{N} r(P_i) = \frac{N \overline{r}}{\sin \pi / m} \geq 1 - r > \frac{N}{N + \sin \pi / m},
\]

contradicting the assumption \(\overline{r} < \frac{\sin \pi / m}{N + \sin \pi / m}\).

\[\Box\]

Theorem 1.5 can be generalized to higher dimensions in the evident way; the only difficulty is to write down the guaranteed inradius in terms of the class of “pizza cutters”. In the examples below, the cutter shape is given by a certain fan \(F\), dividing \(\mathbb{R}^n\) into convex cones so that in the intersection with the unit sphere \(F\) cuts out a bunch of regions all having the same inradius \(\alpha_F\) in the intrinsic sphere metric.

(1) For every regular polytope \(C \subset \mathbb{R}^n\) centered at the origin, one can consider the fan \(F\) consisting of the rays from the origin passing through the \((n - 2)\)-skeleton of \(C\). The regions cut out by \(F\) in the unit sphere are all congruent since \(C\) is regular. For example, in the case of regular simplex, \(\alpha_F = \arccos \frac{1}{n}\).

(2) For every Coxeter hyperplane arrangement \(A\) in \(\mathbb{R}^n\) (that is, the set of hyperplanes passing through the origin and generating a finite reflection group), one can consider the fan \(F\) consisting of the hyperplanes of \(A\). The regions cut out by \(F\) in the unit sphere are all congruent since the reflection group acts transitively on the Weyl chambers. For example, in the case of type \(A_n\) reflection group, \(\alpha_F = \arccos \frac{3}{2(n-1)n(n+1)}\).

(3) The previous two cases are subsumed by the following more general construction. Let \(G\) be a finite subgroup of \(SO(n)\), acting on the unit sphere, and let \(O\) be the \(G\)-orbit of any point from the unit sphere. Then the Voronoï diagram of the set \(O\) gives rise to a fan \(F\). The regions cut out by \(F\) in the unit sphere are all congruent since \(G\) acts transitively on them.

One can repeat literally the same proof as in dimension 2 to obtain the following Davenport-type result: if one cuts the unit ball \(B_1 \subset \mathbb{R}^n\) by \(N\) congruent copies of \(F\), there will be a piece of inradius at least \(\frac{\sin \alpha_F}{N + \sin \alpha_F}\).

\textit{Question 3.2.} Let \(F\) be a sufficiently regular codimension 2 “fan”. For instance, \(F\) can be the union of the rays from the origin passing through the \((n - 3)\)-skeleton of the regular
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n-simplex, n ≥ 3. The cuts are given by N congruent copies of F placed arbitrarily in \( \mathbb{R}^n \). What is the largest radius of an open ball lying in the unit ball and avoiding the cuts?

We finish this section with a brief discussion of the optimality of the main theorem. Theorem 3.1 has many “asymptotic equality cases”, different from trivial equality cases when \( N = 1 \) or \( k = 1 \). For example, the unit disc in the plane can be covered by two multi-planks of radius \( r \) slightly greater than \( 1/2 \), each generated by \( N \gg 1 \) points equidistributed along the circle of radius \( r \). By picking \( N \) sufficiently large one can get \( r \) arbitrarily close to \( 1/2 \). A similar example shows that Theorem 1.5 is asymptotically sharp for each fixed \( m \) and \( N \to \infty \). On the other hand, all those non-trivial asymptotic equality cases involve multi-planks that are not simple. Meanwhile, the proof of Theorem 1.4 only exploits simple multi-planks. Given that, it would be interesting to know how sharp Theorem 1.4 is when, say, \( K \) is not centrally symmetric and \( N > 1 \).

4. Multi-planks in normed spaces

The scheme of the proof of Theorem 3.1 can be repeated to an extent in the setting of a normed space. Let \( \mathbb{R}^n \) be endowed with a norm \( \| \cdot \| \) whose open unit ball is \( B \), an open bounded convex set containing the origin:

\[ \| x \| = \inf \{ r \mid x \in rB \}. \]

We do not require \( B \) to be centrally symmetric, so in general \( \| x \| \neq \| -x \| \) (but the triangle inequality holds). Sometimes the word gauge is used in this context, but we will stick to the word norm.

**Definition 4.1.** Let \( K \) be a convex set in the normed space \( \mathbb{R}^n \), with the unit ball \( B \). Let \( 1 \leq k \leq n \).

1. The **upper intrinsic inradius** \( r^{(k)}_B(K) \) is defined as the largest number \( r \) such that, for any codimension \( k \) subspace \( N \), the homothet \( rB \) can be translated into \( K + N \).
2. The **lower intrinsic inradius** \( r^{(k)}_B(K) \) is defined as the largest number \( r \) such that any \( k \)-dimensional section (passing through the origin) of \( rB \) can be translated into \( K \).

**Definition 4.2.** Let \( V = \{v^1, \ldots, v^m\}, m \geq 2 \), be a set of points in \( \mathbb{R}^n \), such that \( \|v^j\| \leq r \) for all \( j \), and \( V \) cannot be covered by a homothet of \( B \) smaller than \( rB \). Define the **anti-Voronoï cells** as

\[ A^j_{-V} = \left\{ x \in \mathbb{R}^n \mid \| x + v^j \| \geq \| x + v^{j'} \| \ \forall j' \in [m] \right\}, \]

and the open centered multi-plank generated by \( V \) as

\[ P = \mathbb{R}^n \setminus \bigcup_{j \in [m]} (v^j + A^j_{-V}). \]

The number \( r = r^B(P) \) is called the **inradius** of \( P \), and the dimension of the convex hull of \( V \) is called the **rank** of \( P \).

We remark that the cells \( A^j_{-V} \) are no longer convex. See Figure 4 for an example of a rank 1 plank in a non-symmetric norm. In this figure, the unit ball of the norm is depicted in the middle (the origin is marked with a ‘+’ sign), the generating set is \( V = \{v^1, v^2\} \).

The proof of step 1 in Theorem 3.1 falls through in the normed case. It is no longer true that a shifted multi-plank looks similar to the section of a higher-dimensional multi-plank; this is the reason why we have to consider only centered multi-planks. The proof of step 2 is still valid, though, which gives us the following result.
Theorem 4.3. If a convex set $K$ in a normed space $\mathbb{R}^n$ with the unit ball $B$ is covered by rank $k$ (centered) multi-planks $P_1, \ldots, P_N$, then

$$\sum_{i=1}^{N} r_B(P_i) \geq r_{(k)}^B(K).$$

In the case $k = 1$, $K = B$, Theorem 4.2 can be viewed as a result on the subadditivity of relative widths. Take a look at the “bent” plank $P$ in Figure 4: it has the same length intersection with every line parallel to $v^1 - v^2$. In this sense, $P$ has a well-defined “relative width” $r_B(P)$ in this direction. In these terms, Theorem 4.2 says that if $K$ covered by “bent” centered planks then the sum of their “relative widths” is at least 1. This might be reminiscent of Bang’s conjectured inequality on the sum of relative widths: If an open bounded convex set $K$ containing the origin is covered by (conventional straight) planks $P_1, \ldots, P_N$, then

$$\sum_{i=1}^{N} r_{(0)}^B(P_i) \geq 1.$$  

We finish with a strong conjecture subsuming Bang’s conjecture as well as many other subadditivity statements.

Conjecture 4.4. Let $B$ be an open bounded convex set containing the origin, and let a convex set $K$ be covered by convex sets $C_1, \ldots, C_N$. Then for any $1 \leq k \leq n$,

$$\sum_{i=1}^{N} r_{(k)}^B(C_i) \geq r_{(k)}^B(K).$$

References

[1] A. Akopyan, A. Glazyrin, O. R. Musin, and A. Tarasov. The extremal spheres theorem. *Discrete mathematics*, 311(2-3):171–177, 2011.

[2] A. Akopyan and R. Karasev. Kadets-type theorems for partitions of a convex body. *Discrete & Computational Geometry*, 48(3):766–776, 2012.

[3] R. Alexander. A problem about lines and ovals. *The American Mathematical Monthly*, 75(5):482–487, 1968.

[4] K. Ball. The plank problem for symmetric bodies. *Inventiones mathematicae*, 104(1):535–543, 1991.

[5] T. Bang. A solution of the “plank problem”. *Proceedings of the American Mathematical Society*, 2(6):990–993, 1951.

[6] U. Betke and M. Henk. Estimating sizes of a convex body by successive diameters and widths. *Mathematika*, 39(2):247–257, 1992.
A generalization of Steinhaus's theorem. In Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, volume 63, pages 165–176. Springer, 1993.

[8] A. Bezdek and K. Bezdek. A solution of conway’s fried potato problem. Bulletin of the London Mathematical Society, 27(5):492–496, 1995.

[9] A. Bezdek and K. Bezdek. Conway’s fried potato problem revisited. Archiv der Mathematik, 66(6):522–528, 1996.

[10] K. Bezdek. Plank theorems via successive inradii. Discrete Geometry and Algebraic Combinatorics, 625:1, 2014.

[11] M. Bognár. On W. Fenchel’s solution of the plank problem. Acta Mathematica Hungarica, 12(3-4):269–270, 1961.

[12] R. Brandenberg. Radii of regular polytopes. Discrete & Computational Geometry, 33(1):43–55, 2005.

[13] M. Henk and M. A. H. Cifre. Intrinsic volumes and successive radii. Journal of mathematical analysis and applications, 343(2):733–742, 2008.

[14] Z. Jiang and A. Polyanskii. Proof of László Fejes Tóth’s zone conjecture. Geometric and Functional Analysis, 27(6):1367–1377, 2017.

[15] V. Kadets. Coverings by convex bodies and inscribed balls. Proceedings of the American Mathematical Society, 133(5):1491–1495, 2005.

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