LONG AND WINDING CENTRAL PATHS

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Abstract. We disprove a continuous analog of the Hirsch conjecture proposed by Deza, Terlaky and Zinchenko, by constructing a family of linear programs with $3r + 4$ inequalities in dimension $2r + 2$ where the central path has a total curvature in $\Omega(2^r/r)$. Our method is to tropicalize the central path in linear programming. The tropical central path is the piecewise-linear limit of the central paths of parameterized families of classical linear programs viewed through logarithmic glasses. We show in particular that the tropical analogue of the analytic center is nothing but the tropical barycenter, i.e., the maximum of a tropical polyhedron. It follows that unlike in the classical case, the tropical central path may lie on the boundary of the tropicalization of the feasible set, and may even coincide with a path of the tropical simplex method. Finally, our counter-example is obtained as a deformation of a family of tropical linear programs introduced by Bezem, Nieuwenhuis and Rodríguez-Carbonell.

1. Introduction

Since Karmarkar’s seminal work [Kar84], interior-point methods have become indispensable in mathematical optimization. They provide algorithms with a polynomial complexity in the bit model for linear programming. Moreover, interior point methods are also useful for more general convex optimization problems such as semi-definite programming. Path-following interior point methods are driven to an optimal solution along a trajectory called the central path. Thus the performance of an interior point method is tightly linked to the shape of its central path. The purpose of this paper is to apply tools from tropical geometry to study the central paths of linear programs with respect to a logarithmic barrier function.

Tropical geometry can be seen as the (algebraic) geometry on the semiring $(\mathbb{T}; \oplus, \odot)$ where the set $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ is endowed with the operations $a \oplus b = \max(a, b)$ and $a \odot b = a + b$. A tropical variety can be obtained as the limit at infinity of a sequence of classical algebraic varieties depending on one real parameter $t$ and drawn on logarithmic paper, with $t$ as the logarithmic base. This process is known as Maslov’s dequantization [Lit07], or Viro’s method [Vir01]. It can be traced back to the work of Bergman [Ber71]. In a way, dequantization yields a piece-wise linear shadow of classical algebraic geometry. Tropical geometry has a strong combinatorial flavor, and yet it retains a lot of information about the classical objects [IMS09].

The tropical semiring can also be thought of as the image of a non-archimedean field under its valuation map. This is the approach we adopt here. The non-archimedean fields typically used are the field of formal Puiseux series [EKLO00, DY07, RGST05] or the field of generalized Puiseux series with real exponents [Mar10], or larger fields of formal Hahn series [ABCJJ13]. However, since we are aiming at analytic results, matters of convergence play a key role, and this is why dealing with any kind of formal power series is not suitable here. Instead we take the viewpoint of Alessandrini [Ale13] who suggested to study tropicalizations of real semi-algebraic sets via a Hardy field, $\mathbb{K}$, of germs of real-valued functions. The functions $f \in \mathbb{K}$ are definable in some $\alpha$-minimal structure, which ensures a tame topology. In particular, the limit $\lim_{t\to\infty} \log(f(t))/\log(t)$ always exists, and this defines a valuation on $\mathbb{K}$. Furthermore, Alessandrini’s framework is flexible enough to include all power functions into $\mathbb{K}$; this makes the valuation map surjective onto $\mathbb{R} \cup \{-\infty\}$.

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We consider linear programs defined on the Hardy field $\mathbb{K}$. As $\mathbb{K}$ is an ordered field, the basic results of linear programming (Farkas’ lemma, strong duality, etc) still hold true on $\mathbb{K}$. Also, since $\mathbb{K}$ is real closed, the central path of a linear program is well-defined. The elements of $\mathbb{K}$ are real-valued functions. As a result, a linear program over $\mathbb{K}$ encodes a family of linear programs over $\mathbb{R}$, and the central path on $\mathbb{K}$ describes the central paths of this family. The tropical central path is then defined as the image under the valuation map. Thus, the tropical central path is a logarithmic limit of a family of classical central paths. We establish that this convergence is uniform on closed intervals.

The tropical central path has a purely geometric characterization. Applying the valuation map to the feasible region yields a tropical polyhedron. We show that the tropical analytic center is the greatest element of this tropical polyhedron, the tropical equivalent of a barycenter. Thus, the tropical analytic center does not depend on the external representation of the feasible set. Similarly, any point on the tropical central path is the tropical barycenter of the tropical polyhedron obtained by intersecting the values of the feasible region with a tropical sublevel set induced by the objective function. This is in stark contrast with the classical case, where the central path depends on the halfspace description of the feasible set. In this way, Deza, Nematollahi, Peyghami and Terlaky bent the central path of the Klee-Minty cube by adding superfluous halfspaces in its representation, so that it visits a neighborhood of every vertex of the cube.

A maybe surprising feature is that the tropical central path can degenerate to a path taken by the tropical simplex method introduced in [ABGJ13b, ABGJ13a]. We can even provide a quite general sufficient condition under which the tropical central path coincides with the image of a path of the classical simplex method under the valuation map. Consequently, the tropical central path may have the same worst-case behavior as the simplex method.

A main contribution of this paper comes from studying the total curvature of the real central paths arising from lifting tropical linear programs to the Hardy field $\mathbb{K}$. The curvature measures how far a path differs from a straight line. Intuitively, a central path with high curvature should be harder to approximate with line segments, and thus this suggests more iterations of the interior point methods. The total curvature has been studied by Dedieu, Malajovich and Shub [DMS05] via the multihomogeneous Bézout Theorem and by De Loera, Sturmfels and Vinzant [DLSV12] using matroid theory. These two papers provide an upper bound of $O(n)$ on the total curvature averaged over all regions of an arrangement of hyperplanes in dimension $n$. The redundant Klee-Minty cube of [DTZ09] and the “snake” in [DTZ08] are instances which show that the total curvature can be in $\Omega(m)$ for a polytope described by $m$ inequalities. By analogy with the classical Hirsch conjecture, Deza, Terlaky and Zichenco [DTZ08] conjectured that $O(m)$ is also an upper bound for the total curvature. We disprove their conjecture by constructing a family of linear programs with $3r + 4$ inequalities in dimension $2r + 2$ where the central path has a total curvature in $\Omega(2^r/r)$.

**Related Work.** The possible simplex-like behavior of interior point methods was already observed by Megiddo and Shub [MS89] and by Powell [Pow93]. Besides the redundant Klee-Minty cube [DTZ09] and the “snake” [DTZ08], Gilbert, Gonzaga and Karas [GGK01] also exhibited ill-behaved central paths. They showed that the central path can have a “zig-zag” shape with infinitely many turns, on a problem defined in $\mathbb{R}^2$ by non-linear but convex functions. In terms of iteration-complexity of interior-point methods, several worst-case results have been proposed [Ans91, KY91, JY94, Pow93, TY96, BL97]. In particular, Stöer and Zhao [ZS93] showed the iteration-complexity of a certain class of path-following methods is governed by an integral along the central path. This quantity, called the Sonnevend’s curvature, was introduced in [SSZ91]. The tight relationship between the total Sonnevend’s curvature and the
iteration-complexity of interior-points methods have been extended to SDP and symmetric cone programs [KOT13].

Note that Sonnevend’s curvature is a different notion than the geometric curvature we study in this paper. To the best of our knowledge, there is no explicit relation between the geometric curvature and the iteration-complexity of interior-point methods. However, these two notions of curvature share similar properties. In particular, the total geometric curvature and the total Sonnevend’s curvature are both maximal when the number of inequalities is twice the dimension [DTZ08, MT13a]. On the redundant Klee-Minty cube, both the total geometric curvature and the Sonnevend’s curvature are large [MT13a, DTZ09].

2. Preliminaries

We next recall some definitions and results of model theory, referring the reader to [Mar02] for more background.

2.1. Languages and first-order formulae. A language $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$ consists of a set $\mathcal{R}$ of relations, a set $\mathcal{F}$ of functions, and a set $\mathcal{C}$ of constants. Each relation $R$ is equipped with an arity, $n_R$, which is a positive integer. Similarly, each function $F$ also has an arity, denoted as $n_F$. For example, the language of ordered rings is $\mathcal{L}_{or} = (\{<\}, \{+, -, \cdot\}, \{0, 1\})$, where the order relation $<$ and the arithmetic functions $+, -, \cdot$ have arity two.

We shall now describe the (first-order) formulae of a language $\mathcal{L}$. An $\mathcal{L}$-term is either:

- a variable $v_i$, for some $i \geq 1$
- a constant $c \in \mathcal{C}$
- $F(t_1, \ldots, t_{n_F})$ where $F \in \mathcal{F}$ is a function, and $t_1, \ldots, t_{n_F}$ are $\mathcal{L}$-terms

An $\mathcal{L}$-formula is then defined inductively as follows:

- if $t_1$ and $t_2$ are $\mathcal{L}$-terms, then $t_1 = t_2$ is an $\mathcal{L}$-formula
- if $R \in \mathcal{R}$ is a relation, and $t_1, \ldots, t_{n_R}$ are terms, then $R(t_1, \ldots, t_{n_R})$ is an $\mathcal{L}$-formula
- if $\phi$ and $\psi$ are $\mathcal{L}$-formula, then $(\neg \phi), (\phi \land \psi)$ and $(\phi \lor \psi)$ are $\mathcal{L}$-formulae
- if $\phi$ is an $\mathcal{L}$-formula and $v_i$ is a variable, then $\exists v_i \phi$ and $\forall v_i \phi$ are $\mathcal{L}$-formulae

A variable $v_i$ which occurs in a formula $\phi$ without being modified by a quantifier $\exists$ or $\forall$ is said to be free. We shall emphasize the free variables $v_1, \ldots, v_k$ of a formula $\phi$ by writing $\phi(v_1, \ldots, v_k)$. A formula without free variable is called a sentence.

2.2. Structures. Let $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$ be a language. An $\mathcal{L}$-structure $\mathfrak{M}$ consists of a non-empty set $M$ (called the domain of $\mathfrak{M}$) together with an interpretation of the symbols of $\mathcal{L}$ in $M$. A relation $R \in \mathcal{R}$ is interpreted by a subset $S_R \subseteq M^{n_R}$, where a tuple $(x_1, \ldots, x_{n_R})$ satisfies the relation $R$ if $(x_1, \ldots, x_{n_R}) \in S_R$. A function $F \in \mathcal{F}$ is interpreted by a map $M^{n_F} \to M$, and the interpretation of a constant $c \in \mathcal{C}$ is an element of $M$.

The interpretation of the language $\mathcal{L}$ induces an interpretation of the formulae of $\mathcal{L}$ in the structure $\mathfrak{M}$. Every formula $\phi(v_1, \ldots, v_k)$ defines a Boolean function $\phi^{\mathfrak{M}}$ on $M^k$. If $\phi^{\mathfrak{M}}$ is true at $a \in M^k$, we write $\mathfrak{M} \models \phi(a)$. In particular, if $\phi$ is a sentence in $\mathcal{L}$, the function $\phi^{\mathfrak{M}}$ is constant. Thus a sentence $\phi$ defines a statement on $\mathfrak{M}$ which is either true or false. The set of sentences that are true on $\mathfrak{M}$ is called the full theory of $\mathfrak{M}$; it is denoted by $\text{Th}(\mathfrak{M})$. An arbitrary $\mathcal{L}$-structure $\mathfrak{M}$ is a model of the theory $\text{Th}(\mathfrak{M})$ if $\mathfrak{M} \models \phi$ for all $\phi \in \text{Th}(\mathfrak{M})$.

A set $A \subseteq M^k$ is definable (in $\mathfrak{M}$) if there exists an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_k)$ and an element $b \in M$ such that $A = \{a \in M^k \mid \mathfrak{M} \models \phi(a, b)\}$. Given a definable set $A \subseteq M^k$, a map $F : A \to M^l$ is definable if its graph $\{(a, F(a)) \mid a \in A\} \subseteq M^{k+l}$ is a definable set.

An expansion $\mathcal{L}'$ of a language $\mathcal{L}$ is obtained by adding some new relations, functions and constants to $\mathcal{L}$. We define an expansion of an $\mathcal{L}$-structure $\mathfrak{M}$ to be an $\mathcal{L}'$-structure $\mathfrak{M}'$ such that: $\mathcal{L}'$ is an expansion of $\mathcal{L}$, $\mathfrak{M}$ and $\mathfrak{M}'$ have the same domain and the interpretation of the language $\mathcal{L}$ in $\mathfrak{M}$ coincides with the one in $\mathfrak{M}'$. 

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2.3. O-minimal structures and Hardy fields. The $\mathcal{L}_{\text{or}}$-structure of the ordered field of real numbers is denoted by $\bar{\mathbb{R}} = (\mathbb{R}, \{<\}, \{+,-,\cdot\}, \{0,1\})$. Thoughout the following, $\mathcal{L}$ will denote an expansion of the language $\mathcal{L}_{\text{or}}$. Furthermore, $\mathfrak{R}$ will be a $\mathcal{L}$-structure with domain $\mathbb{R}$ (thus an expansion of $\bar{\mathbb{R}}$) that is also o-minimal, which means that any subset of $\mathbb{R}$ definable in $\mathfrak{R}$ is a finite union of points and intervals with endpoints in $\mathbb{R} \cup \{-\infty, +\infty\}$. Under the o-minimality requirement, definable sets and maps are “well-behaved”. For example, the set $\{(x, \sin(1/x)) \mid x > 0\}$ is not definable in an o-minimal structure. We refer the reader to [vdD98] or [Cos00] for more details.

We say that two definable functions $f, g : \mathbb{R} \to \mathbb{R}$ are equivalent, and we write $f \sim g$, if $f(t) = g(t)$ ultimately, i.e., for all $t$ large enough. The germ $f$ of a definable function $f$ is the equivalence class of $f$ for the relation $\sim$. By abuse of notation, $f$ will also denote a representative of the germ $f$.

Let $H(\mathfrak{R}) := \{f \mid f \text{ definable in } \mathfrak{R}\}$ the set of germs of functions definable in $\mathfrak{R}$. Each function symbol $F \in \mathcal{F}$ has a natural interpretation in $H(\mathfrak{R})$, by defining $F(f_1, \ldots, f_n) := \text{the germ of } \text{the definable function } t \mapsto F(f_1(t), \ldots, f_n(t))$. Besides, the set $\mathfrak{R}$ is embedded into $H(\mathfrak{R})$ by identifying each element $a \in \mathbb{R}$ with the constant function with value $a$. This provides an interpretation of the constant symbols of $\mathcal{L}$ in $H(\mathfrak{R})$. Finally, given a relation $R$ of the language $\mathcal{L}$ and $f_1, \ldots, f_n \in H(\mathfrak{R})$, the set $\{t \mid \mathfrak{R} \models R(f_1(t), \ldots, f_n(t))\}$ is definable, and thus consists in a finite union of points and intervals. Hence, $R(f_1(t), \ldots, f_n(t))$ is either ultimately true or ultimately false. This provides an interpretation of $R$ over $H(\mathfrak{R})$.

Consequently, $H(\mathfrak{R})$ has a natural $\mathcal{L}$-structure, which we denote by $\bar{\mathfrak{H}}(\mathfrak{R})$. It follows from [Cos00, Prop. 5.9] that $\bar{\mathfrak{H}}(\mathfrak{R})$ and $\mathfrak{R}$ have the same full theory; see also [Fos10, Lemma 2.2.64]. In other words, the following holds.

**Proposition 1.** Let $\mathfrak{R}$ be an o-minimal $\mathcal{L}$-structure and $\bar{\mathfrak{H}}(\mathfrak{R})$ the natural $\mathcal{L}$-structure of the germs of functions definable in $\mathfrak{R}$. Then, for any $\mathcal{L}$-sentence $\phi$, we have $\mathfrak{R} \models \phi$ if and only if $\bar{\mathfrak{H}}(\mathfrak{R}) \models \phi$.

As an expansion of $\bar{\mathbb{R}}$, the structure $\mathfrak{R}$ satisfies the axioms of the theory of real closed fields. By Proposition 1, $\bar{\mathfrak{H}}(\mathfrak{R})$ satisfies the same axioms. As a result, $H(\mathfrak{R})$ is a real closed field that we will refer to as the Hardy field of structure $\mathfrak{R}$. In particular, $H(\mathfrak{R})$ is an ordered field, and it carries a natural topology induced by the ordering. The standard topology on $\mathbb{R}$ coincides with the subspace topology induced from $H(\mathfrak{R})$.

A structure $\mathfrak{R}$ is polynomially bounded if for any definable function $f : \mathbb{R} \to \mathbb{R}$, there exists a natural number $n$ such that ultimately $|f(t)| \leq t^n$. Miller proved [Mil94b] that in an o-minimal and polynomially bounded expansion of $\mathbb{R}$, if a definable function $f$ is not ultimately zero, then there exists an exponent $r \in \mathbb{R}$ and a non-zero coefficient $c \in \mathbb{R}$ such that

\begin{equation}
\lim_{t \to +\infty} \frac{f(t)}{t^r} = c .
\end{equation}

The set of such exponents $r$ forms a subfield of $\mathbb{R}$, called the field of exponents of the structure $\mathfrak{R}$.

2.4. Logarithmic limits of definable functions. In the following, we will use the structure $\bar{\mathbb{R}}^\mathbb{N}$ which expands $\mathbb{R}$ by adding the family of power functions $(f_r)_{r \in \mathbb{N}}$, where $f_r$ maps a positive number $t$ to $t^r$, and any non-positive number to 0. The structure $\bar{\mathbb{R}}^\mathbb{N}$ is o-minimal, polynomially bounded and its field of exponents is $\mathbb{R}$; see [Mil94a, Mil12]. Another structure with the same properties is $\bar{\mathbb{R}}_{\text{ss}}$, the reals with restricted analytic functions and convergent generalized power series [vdD98].

For the sake of readability, we shall abbreviate $H(\bar{\mathbb{R}}^\mathbb{N})$ by $\mathfrak{K}$. We also use the notation $t^r$ as a shorthand for the germ of the power function $f_r$. The valuation maps any $f \in \mathfrak{K}$ to:

\[
\text{val}(f) := \lim_{t \to +\infty} \log_t |f(t)| ,
\]

where $\log_t(x) = \log(x)/\log(t)$. By (1) the limit above is well-defined. Notice that $\text{val}(f) = -\infty$ if $f(t)$ ultimately vanishes. Since $\bar{\mathbb{R}}^\mathbb{N}$ has $\mathbb{R}$ as its field of exponents, the valuation is a surjective
map from \( \mathbb{K} \) to \( \mathbb{R} \cup \{-\infty\} \). For \( f, g \geq 0 \) we have:

\[
(2) \quad \text{val}(f + g) = \max(\text{val}(f), \text{val}(g)), \quad \text{val}(fg) = \text{val}(f) + \text{val}(g) .
\]

Moreover, if \( f \geq g \geq 0 \), then \( \text{val}(f) \geq \text{val}(g) \). Hence, the valuation map is an order-preserving homomorphism from the semiring of germs of definable functions which are ultimately nonnegative to the tropical semiring. In the sequel, the valuation map will be also be applied, being understood entrywise, to vectors or matrices with entries in the Hardy field \( \mathbb{K} \).

### 2.5. Fields of generalized power series.

Several non-archimedean fields, which differ from the Hardy field \( \mathbb{K} = H(\mathbb{R}) \), have been used in the tropical literature, and it may be useful to review alternative choices. It is common to consider the field \( \mathbb{R}\{\{t\}\} \) of formal Puiseux series with real coefficients \([\text{EKL}06, \text{DY}07, \text{RGST}05]\). It is a technical inconvenience, however, that the valuation map from \( \mathbb{R}\{\{t\}\} \) to \( \mathbb{T} \) is not surjective, as classical Puiseux series have rational exponents. This can be remedied by using the larger field of Hahn series with real coefficients, denoted by \([\mathbb{R}^{\mathbb{R},<}]\) in \([\text{Rib}92]\). An element of this field is a formal series of the form

\[
(3) \quad f = \sum_{\alpha \in \mathbb{R}} a_{\alpha} t^\alpha
\]

with \( a_{\alpha} \in \mathbb{R} \), such that the support \( \{ \alpha \in \mathbb{R} \mid a_{\alpha} \neq 0 \} \) is well ordered. This field is known to be real closed \([\text{Rib}92]\). An alternative to \([\mathbb{R}^{\mathbb{R},<}]\) is the subfield of generalized formal Puiseux series, considered by Markwig \([\text{Mar}10]\), which consists of those series \( f \) such that the support \( \{ \alpha \in \mathbb{R} \mid a_{\alpha} \neq 0 \} \) is either finite or has \( +\infty \) as the only accumulation point. It follows from \([\text{Mar}10]\) that this subfield is also real closed.

Our previous work \([\text{ABGJ}13\text{b}, \text{ABGJ}13\text{a}]\) was developed using formal Hahn series. However, in the present application, we need to work with fields of functions, thinking of \( t \) as a deformation parameter. We note that the subfield of Markwig’s field, consisting of the generalized Puiseux series that are absolutely convergent in a punctured complex disc \( 0 < |t| < r \), actually coincides with the field \( \mathbb{D} \) of generalized Dirichlet series originally considered by Hardy and Riesz \([\text{HR}15]\), already used in the tropical setting in \([\text{ABC}98]\). Classical Dirichlet series can be written as \( \sum_k a_k t^k \), they are obtained from \((3)\) by the change of variable \( t = \exp(s) \), with \( \alpha_k = \log k \). It follows from results of van den Dries and Speissheger \([\text{vdDS}98]\) that the field \( \mathbb{D} \) is real closed, and that the subfield \([\mathbb{R}^{\mathbb{R},<}]\),cvg of \([\mathbb{R}^{\mathbb{R},<}]\) consisting of series that are absolutely convergent in a punctured disk is also real closed. Actually, the elements of the latter field can be identified to the germ of functions of one variable that are definable in the \( \omega \)-minimal structure \( \mathbb{R}_{\text{an},*} \). However, it is enough to work here with the \( \omega \)-minimal structure \( \mathbb{R}^{\mathbb{R}} \), smaller than \( \mathbb{R}_{\text{an},*} \).

### 2.6. Tropicalization of linear programs.

The tropical addition \( \oplus \) extends to vectors and matrices by applying it coordinatewise. Similarly, the tropical multiplication \( \odot \) gives rise to a tropical multiplication of a scalar with a vector and, combined with \( \oplus \), also to a tropical matrix multiplication. A tropical halfspace of \( \mathbb{T}^n \) is the set of points \( x \in \mathbb{T}^n \) which satisfy one tropical linear inequality,

\[
\max(\alpha^+ + x_1, \ldots, \alpha^+_n + x_n, \beta^+) \geq \max(\alpha^- + x_1, \ldots, \alpha^-_n + x_n, \beta^-),
\]

where \( \alpha^+, \alpha^- \in \mathbb{T}^n \) and \( \beta^+, \beta^- \in \mathbb{T} \). A tropical polyhedron is the intersection

\[
\mathcal{P} = \{ x \in \mathbb{T}^n \mid A^+ \odot x \oplus b^+ \geq A^- \odot x \oplus b^- \}
\]

of finitely many tropical halfspaces, where \( A^+, A^- \in \mathbb{T}^{m \times n} \) and \( b^+, b^- \in \mathbb{T}^m \). The tropical semiring is equipped with the order topology, which determines a product topology on \( \mathbb{T}^n \). Note that tropical halfspaces, and so, tropical polyhedra, are closed in this topology.

An analogue of the Minkowski-Weyl Theorem \([\text{GK}11]\) allows one to represent a tropical polyhedron internally in terms of extreme points and rays, meaning that there exist two finite collections of vectors \( v^1, \ldots, v^r \in \mathbb{T}^n \) and \( w^1, \ldots, w^s \in \mathbb{T}^n \) such that \( \mathcal{P} \) can be written as the set
of points of the form

\[ x = \bigoplus_{i=1}^{r} \lambda_i \odot v^i + \bigoplus_{j=1}^{s} \mu_j \odot w^j \]

where \( \lambda_i, \mu_j \in \mathbb{T} \) and \( \bigoplus_{i \in [r]} \lambda_i \) is equal to the tropical unit (the element 0). We shall say that \( \mathcal{P} \) is generated by \( v^1, \ldots, v^r \) and \( w^1, \ldots, w^s \). Note that the tropical polytopes introduced by Develin and Sturmfels [DS04] are obtained by omitting the \( v^i \) terms and by requiring the \( w^j \) to have finite coordinates in the representation (4).

We denote by \( \mathbb{K}_+ \) the set of nonnegative elements of \( \mathbb{K} \), and by \( \mathbb{K}_+^n \) the positive orthant of \( \mathbb{K}^n \). The following fact was already noted by Develin and Yu for tropical polytopes in the sense of Develin and Sturmfels.

**Proposition 2** (Compare with [DY07, Proposition 2.1]). *The image under the valuation map of any polyhedron \( \mathcal{P} \) included in the positive orthant \( \mathbb{K}_+^n \) is a tropical polyhedron of \( \mathbb{T}^n \).*

**Proof.** The Minkowski-Weyl theorem is valid for a polyhedron in any ordered field. Hence, there exists two finite collections of vectors \( v^1, \ldots, v^r \in \mathbb{K}_+^n \) and \( w^1, \ldots, w^s \in \mathbb{K}_+^n \) such that \( \mathcal{P} \) is precisely the set of combinations of the following form

\[ x = \sum_{i=1}^{r} \lambda_i v^i + \sum_{j=1}^{s} \mu_j w^j \]

where \( \lambda_i, \mu_j \in \mathbb{K}_+ \) and \( \sum_{i \in [r]} \lambda_i = 1 \). Since the valuation is a homomorphism from \( \mathbb{K}_+ \) to \( \mathbb{T} \) (see (2)), \( \text{val}(\mathcal{P}) \) is included in the tropical polyhedron \( \mathcal{P} \) generated by the vectors \( \lambda v^1 := \text{val}(v^1) \), \( \lambda v^2 := \text{val}(v^2) \), and \( \lambda w^1 := \text{val}(w^1) \). Conversely, any point in \( \mathcal{P} \) of the form (5) is the image by the valuation of

\[ \sum_{i=1}^{r} \frac{1}{Z} \lambda_i v^i + \sum_{j=1}^{s} \mu_j w^j \]

where \( Z = \sum_{i=1}^{r} t^\lambda_i \) is such that \( \text{val}(Z) = 0 \). \( \Box \)

Conversely, each tropical polyhedron arises as the image under the valuation map of a polyhedron included in \( \mathbb{K}_+^n \). A slightly stronger statement can be established for tropical linear programs. A tropical linear program asks to minimize a tropical linear function \( x \mapsto c^\top \odot x \) on a tropical polyhedron, where \( c \in \mathbb{T}^n \).

**Proposition 3** ([ABGJ13b, Proposition 7]). *Each tropical linear program arises as the image of some Hardy linear program under the valuation map, and this correspondence takes optimal solutions to optimal solutions.***

Actually, the results of [ABGJ13b] were established when the coefficients of the linear program belong to the field of formal generalized Puiseux series [Mar10]. However, the same arguments apply to other non-archimedean real closed fields with residual field \( \mathbb{R} \) that are sent surjectively to \( \mathbb{T} \) by the valuation, including the Hardy field \( \mathbb{K} = H(\mathbb{R}) \); see Section 2.6.

Notice that there may be optimal solutions of a tropical linear program which do not arise as images under the valuation map.

Under tropical genericity conditions, we can directly obtain a halfspace description of \( \text{val}(\mathcal{P}) \) from a halfspace description of \( \mathcal{P} \). Since this is relevant for this paper, we will now describe this in more detail. To ease the connection with the tropical description, assume that \( \mathcal{P} \) is given as the set of \( x \in \mathbb{K}^n \) satisfying linear inequalities of the form \( Ax + b \geq 0 \). We additionally assume that \( \mathcal{P} \) is contained in the positive orthant of \( \mathbb{K}^n \).

We say that the tropicalization of a matrix \( M \in \mathbb{K}^{n \times n} \) is sign non-singular if \( \det(M) \neq 0 \) and all the terms \( \text{sign}(\sigma) \prod_{i \in [n]} M_{\sigma(i)} \) with maximal valuation among those arising in the expansion of

\[ \det M = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i \in [n]} M_{\sigma(i)} \]
share the same sign. The tropicalization of a rectangular matrix $W \in \mathbb{K}^{m \times n}$ is said to be 
*sign generic* if for every square submatrix $M$ of $W$, either the tropicalization of $M$ is sign 
non-singular, or for every permutation $\sigma$, the term $\prod_{i=1}^{n} M_{\sigma(i)}$ vanishes.

For any matrix $M = (M_{ij}) \in \mathbb{K}^{m \times n}$, we denote by $M^+ = (M^+_{ij})$ and $M^- = (M^-_{ij})$ its 
positive and negative parts, i.e., $M^+_{ij} = \max(M_{ij}, 0)$ and $M^-_{ij} = \min(M_{ij}, 0)$. Furthermore, $M_I$ will 
denote the submatrix of $M$ formed by the rows indexed by $I \subset [m]$. Here and below we use the common 
abbreviation $[m] := \{1, 2, \ldots, m\}$.

**Theorem 4** ([ABGJ13b], Theorem 15 and Corollary 16). Suppose that $\mathcal{P} = \{x \in \mathbb{K}^n \mid Ax + b \geq 0\}$ is included in the positive orthant of $\mathbb{K}^n$ and that the tropicalization of $(A, b)$ is sign generic. 
Then, 
\[
\text{val}(\mathcal{P}) = \{x \in \mathbb{T}^n \mid A^+ \circ x \oplus b^+ \geq A^- \circ x \oplus b^-\},
\]
where $(A^+ b^+) = \text{val}(A^+ b^+)$ and $(A^- b^-) = \text{val}(A^- b^-)$. Moreover, for any $I \subset [m]$, we have: 
\[
\text{val}(\{x \in \mathcal{P} \mid A_I x + b_I = 0\}) = \{x \in \text{val}(\mathcal{P}) \mid A_I^+ \circ x \oplus b_I^+ = A_I^- \circ x \oplus b_I^-\}.
\]

### 3. Tropicalizing the Central Path

We consider linear programs of the form:

\[
\text{LP}(A, b, c) \quad \begin{array}{ll}
\text{minimize} & c^\top x \\
\text{subject to} & Ax \leq b, \ x \geq 0, \ x \in \mathbb{R}^n,
\end{array}
\]

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. The dual linear program reads

\[
\begin{array}{ll}
\text{maximize} & -b^\top y \\
\text{subject to} & -A^+ y \leq c, \ y \geq 0, \ y \in \mathbb{R}^m.
\end{array}
\]

The reason for picking this particular form for LP$(A, b, c)$ is that the feasible solutions of both, 
the primal and the dual, are non-negative. This allows to apply our results on linear programs and 
their tropicalization from [ABGJ13b] and [ABGJ13a]. In the following, we shall assume that the polyhedron $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ is bounded with non-empty interior. Given a positive $\mu \in \mathbb{R}$, the *barrier problem* is

\[
\begin{align*}
\text{minimize} & \quad \frac{c^\top x}{\mu} - \sum_{j=1}^n \log(x_j) - \sum_{i=1}^m \log(w_i) \\
\text{subject to} & \quad Ax + w = b, \ x \geq 0, w \geq 0.
\end{align*}
\]

The objective function in (6) is continuous, strictly convex, and it tends to infinity when $(x, w)$ 
tends to the relative boundary of the bounded non-empty convex set $\{(x, w) \in \mathbb{R}^{n+m} \mid Ax + w = b, x \geq 0, w \geq 0\}$. Hence, the problem (6) admits a unique optimum $(x^\mu, w^\mu)$ in the latter set. 
By convexity, this optimum is characterized by the first-order optimality conditions:

\[
\begin{align*}
Ax + w &= b \\
-A^+ y + s &= c \\
w_i y_i &= \mu & \text{for all } i \in [m] \\
x_j s_j &= \mu & \text{for all } j \in [n] \\
x, w, y, s &> 0.
\end{align*}
\]

Thus, for any positive real number $\mu$, there exists a unique solution $(x^\mu, w^\mu, y^\mu, s^\mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ to the system of polynomial equations (7). The *central path* is the image of the map 
$C_{A, b, c} : \mathbb{R}^\geq \rightarrow \mathbb{R}^{2m+2n}$ which sends a positive real number $\mu$ to the vector $(x^\mu, w^\mu, y^\mu, s^\mu)$. The *primal central path* is the projection of the central path onto the $(x, w)$-coordinates. Similarly, 
the *dual central path* is gotten by projecting onto the $(y, s)$-coordinates.
3.1. Dequantization of a definable family of central paths. Let $\mathbb{K} = H(\mathbb{R})$ be the Hardy field of the o-minimal structure $\mathbb{R}$. We consider $A \in \mathbb{K}^{m \times n}, b \in \mathbb{K}^m$ and $c \in \mathbb{K}^n$. Throughout, we will make the following assumption.

**Assumption 5.** The set $\{ x \in \mathbb{K}^n \mid Ax \leq b, x \geq 0 \}$ is bounded with non-empty interior.

Clearly, the latter set is closed. However, in $\mathbb{K}^n$ a closed and bounded set is not necessarily compact.

Under Assumption[3], the central paths of the family of real linear programs $LP(A(t), b(t), c(t))$ are ultimately well-defined. For a fixed real number $M$ let us define the map $C : (M, +\infty) \times \mathbb{R} \to \mathbb{R}^{2m+2n}$ which sends $t \in (M, +\infty)$ and $\lambda \in \mathbb{R}$ to $C(t, \lambda) = C(A(t), b(t), c(t))$. For any $t$ large enough, the map $\lambda \mapsto C(t, \lambda)$ is a parameterization of the central path of $LP(A(t), b(t), c(t))$.

Our goal is to investigate the logarithmic limit

$$(8) \quad C^{\text{trop}} : \lambda \mapsto \lim_{t \to +\infty} \log_t C(t, \lambda) ,$$

where $\log_t$ is applied component-wise. The map $C^{\text{trop}}$ is called the tropical central path of $LP(A, b, c)$. We shall prove the following theorem.

**Theorem 6.** The family of maps $(\log_t C(t, \cdot))_t$ converges uniformly on any closed interval $[a, b] \subseteq \mathbb{R}$ to the tropical central path $C^{\text{trop}}$.

Consider the following linear program over the ordered field $\mathbb{K}$:

$$LP(A, b, c)$$

minimize $c^\top x$

subject to $Ax \leq b$, $x \geq 0$, $x \in \mathbb{K}^n$. 

The problem $LP(A, b, c)$ encodes the family of linear programs $(LP(A(t), b(t), c(t)))_t$. Indeed, if $x$ is feasible for $LP(A, b, c)$, then $x(t)$ is ultimately feasible for $LP(A(t), b(t), c(t))$. Similarly, if $x^*$ is optimal for $LP(A, b, c)$, then $x^*(t)$ is ultimately optimal for $LP(A(t), b(t), c(t))$.

The next lemma shows that the central path of $LP(A, b, c)$ is well-defined, and that it describes the family of central paths of $(LP(A(t), b(t), c(t)))_t$.

**Lemma 7.** For any $\lambda \in \mathbb{R}$, the map $t \mapsto C(t, \lambda)$ is definable in $\mathbb{R}^\mathbb{R}$. Its components are given by the unique solution $(x^\mu, w^\mu, y^\mu, s^\mu) \in \mathbb{K}^{2m+2n}$ of the system of polynomial equations

$$Ax + w = b$$

$$-A^\top y + s = c$$

$$w_i y_i = \mu \quad \text{for all } i \in [m]$$

$$x_j s_j = \mu \quad \text{for all } j \in [n]$$

$$x, w, y, s > 0 ,$$

where $\mu = t^\lambda$.

**Proof.** For an ordered field $K$ and integers $m$ and $n$, consider the following statement:

“For any $A \in K^{m \times n}, b \in K^m$ and $c \in K^n$ which satisfy Assumption[3] and any positive $\mu \in K$, there exists a unique solution $(x', w', y', s') \in K^{2m+2n}$ to the system of polynomial equations $(\mathbb{G})$.”

This is a first-order sentence, $\phi$, which is true in the structure $\bar{K}$, that is for $K = \mathbb{R}$. As $\bar{R}$ is an expansion of $\mathbb{R}$, we have $\bar{R} \models \phi$. Thus, by Proposition[1] the sentence $\phi$ is also true in the structure $\bar{K}(\mathbb{R})$. This means that the induced statement holds in the field $K = \mathbb{K} = H(\mathbb{R})$.

In particular, for any $\lambda \in \mathbb{R}$, it holds for $\mu = t^\lambda \in \mathbb{K}$.

Let $(x^\mu, w^\mu, y^\mu, s^\mu) \in K^{2m+2n}$ be the unique solution of $(\mathbb{G})$ for $\mu = t^\lambda$. Then, for all $t$ large enough, $(x^\mu(t), w^\mu(t), y^\mu(t), s^\mu(t)) \in \mathbb{K}^{2m+2n}$ is a solution of $(\mathbb{G})$ for $A = A(t), b = b(t), c = c(t), \mu = \mu(t)$. Since $(\mathbb{G})$ admits a unique solution, we conclude that $C(t, \lambda) = (x^\mu(t), w^\mu(t), y^\mu(t), s^\mu(t))$ for all $t$ large enough.

$\Box$
Since \( t \rightarrow C(t, \lambda) \) is definable in \( \mathbb{R}^\mathbb{R} \), its image under the (component-wise) valuation map is well-defined, which proves the point-wise convergence of the family \((\log_t C(t, \cdot))_t\). Furthermore, for any \( \lambda \in \mathbb{R} \) we have

\[
\lim_{t \to +\infty} \log_t C(t, \lambda) = \text{val}(x^\mu, w^\mu, y^\mu, s^\mu),
\]

where \( \mu = t^\lambda \), and \((x^\mu, w^\mu, y^\mu, s^\mu)\) is the unique solution of \( \mathbf{9} \).

For fixed \( t \), let \( z_t \) be a component of the map \( \lambda \mapsto \log_t C(t, \lambda) \). To prove uniform convergence, we will use the fact that for all large enough \( t \), the maps \( z_t \) are “almost” 1-Lipschitz.

**Lemma 8.** For \( t \) large enough and any \( \lambda, \lambda' \in \mathbb{R} \), we have:

\[
|z_t(\lambda) - z_t(\lambda')| \leq \log_t(2n + 2m) + |\lambda - \lambda'|.
\]

**Proof.** Let \((x, w, y, s) \in \mathbb{R}^{2m+2n}\) and \((x', w', y', s') \in \mathbb{R}^{2m+2n}\) be two solutions of \( \mathbf{9} \) obtained for two parameters \( \mu = t^\lambda \) and \( \mu' = t^{\lambda'} \). As in [VY96, Lemma 16], by combining the defining equations, we obtain:

\[
\sum_{j=1}^n x_j s'_j + \sum_{j=1}^n x'_j s_j + \sum_{i=1}^m w_i y'_i + \sum_{i=1}^m w'_i y_i = (n + m)(t^\lambda + t^{\lambda'}). \tag{10}
\]

Since the summands on the left-hand side of \((\mathbf{10})\) are all positive, every summand is smaller than \((n + m)(t^\lambda + t^{\lambda'})\). In particular, for any \( j \in [n] \), we have \( x_j s'_j \leq (n + m)(t^\lambda + t^{\lambda'}) \) and \( x'_j s_j \leq (n + m)(t^\lambda + t^{\lambda'}) \). Since \( x_j s_j = t^\lambda \) and \( x'_j s'_j = t^{\lambda'} \), we deduce that:

\[
x_j \leq (n + m)(1 + t^{\lambda - \lambda'}) x'_j
\]
\[
x'_j \leq (n + m)(1 + t^{\lambda' - \lambda}) x_j.
\]

To prove the lemma, it is sufficient to consider \( \lambda \geq \lambda' \). In this case, \( t^{\lambda - \lambda'} \geq 1 \), which implies:

\[
x_j \leq 2(n + m) t^{\lambda - \lambda'} x'_j
\]
\[
x'_j \leq 2(n + m) x_j.
\]

Applying \( \log_t \) to these inequalities yields the conclusion for the components \( x_1, \ldots, x_n \). The same proof readily applies to the other components. \( \square \)

**Proof of Theorem 8.** Let \( z \) be the point-wise limit of the functions \( z_t \) as \( t \) approaches infinity. Consider any closed interval \([a, b] \subset \mathbb{R}\). Let \( \varepsilon > 0 \), and choose a partition \( a = a_1 < a_2 < \cdots < a_k = a_{k+1} = b \) such that \( a_{i+1} - a_i \leq \varepsilon \) for all \( i \in [k] \). Now let \( \lambda \in [a, b] \) and let \( i \) be the index such that \( \lambda \in [a_i, a_{i+1}) \). Then,

\[
|z_t(\lambda) - z(\lambda)| \leq |z_t(\lambda) - z_t(a_i)| + |z_t(a_i) - z(a_i)| + |z(a_i) - z(\lambda)|.
\]

By Lemma 8 we have:

\[
|z_t(\lambda) - z_t(a_i)| \leq \log_t(2n + 2m) + \lambda - a_i \leq \log_t(2n + 2m) + \varepsilon.
\]

Thus, there exists a \( t_\varepsilon \) such that \( |z_t(\lambda) - z_t(a_i)| \leq 2\varepsilon \) for all \( t \geq t_\varepsilon \). Furthermore, Lemma 8 also shows that:

\[
|z(\lambda) - z(a_i)| \leq \lambda - a_i \leq \varepsilon.
\]

Finally, since the functions \( z_t \) converge pointwise to \( z \), there exists a \( t'_\varepsilon \) such that \( |z_t(a_i) - z(a_i)| \leq \varepsilon \) for all \( t \geq t'_\varepsilon \) and all \( i \in [k] \). We conclude that \((z_t)_t\) converges uniformly on \([a, b]\). \( \square \)
3.2. Geometric description of the tropical central path. We now use barrier functions on the Hardy field $H(\mathbb{R})$ to characterize the central path. In order to obtain definable barrier functions, we use the structure $\mathbb{R}_{\exp}$ which expands the ordered real field structure $\mathbb{R}$ by adding the exponential function. The structure $\mathbb{R}_{\exp}$ is o-minimal [vdDM94]. Note that every power function is definable in $\mathbb{R}_{\exp}$, thus the definable functions of $\mathbb{R}$ are also definable in $\mathbb{R}_{\exp}$. As a consequence, the Hardy field $H(\mathbb{R}_{\exp})$ contains $K = H(\mathbb{R})$. The exponential is definable in the structure $\mathcal{S}_y(\mathbb{R}_{\exp})$ of the Hardy field $H(\mathbb{R}_{\exp})$, and thus the logarithm is also definable in this structure. Hence, if $f \in K$ is positive, $\log(f)$ belongs to the ordered field $H(\mathbb{R}_{\exp})$.

Consequently, given $A \in K^{m \times n}, b \in K^n, c \in K$ and $\mu \in K, \mu > 0$, the following optimization problem on $(x, w) \in K^n \times K^m$ is well-defined if the objective function is interpreted in $H(\mathbb{R}_{\exp})$.

(11) \[
\text{minimize } \frac{c^\top x}{\mu} - \sum_{j=1}^n \log(x_j) - \sum_{i=1}^m \log(w_i)
\]
subject to \(Ax + w = b, \ x > 0, \ w > 0\).

Lemma 9. Let \((x^\mu, w^\mu, y^\mu, s^\mu)\) be the unique solution of (9). The point \((x^\mu, w^\mu)\) is the unique solution of (11).

Proof. Let $\mathcal{R}$ be the expansion of the structure $\mathbb{R}_{\exp}$ in which we added a symbol $\log$. The latter is interpreted as the map $x \mapsto \log(x)$ for positive elements $x$ and $x \mapsto 0$ for non-positive elements. The structure $\mathcal{R}$ is still o-minimal, since the sets definable in $\mathcal{R}$ and $\mathbb{R}_{\exp}$ are the same. Given $n, m$, the following statement is a sentence in the language of $\mathcal{R}$.

“For any $A \in K^{m \times n}, b \in K^n$ and $c \in K$ which satisfy Assumption 3 and any positive $\mu \in K$, the optimization problem (11) has a unique solution. It is given by the point $(x', w')$, where $(x', w', y', s')$ is the unique solution of (9).”

We already noted that this sentence is true when $K = \mathbb{R}$, i.e., in the structure $\mathcal{R}$. Since the latter is o-minimal, by Proposition 1 this sentence is also true in $\mathcal{S}_y(\mathcal{R})$, i.e., when $K = H(\mathbb{R}_{\exp})$.

Now if $A, b, c$ and $\mu$ have entries in $K \subset H(\mathbb{R}_{\exp})$, the system (9) admits a unique solution with entries in $K$ by Lemma 7.

Let $C$ be a tropically convex set in $\mathbb{T}^n$, i.e., such that $\lambda \odot u \oplus \mu \odot v \in C$ as soon as $u, v \in C$ and the tropical sum of $\lambda, \mu \in \mathbb{T}$ is equal to the tropical unit 0. In particular, the supremum $\sup(u, v)$ with respect to the partial order of $\mathbb{T}^n$ also belongs to $C$. If in addition $C$ is compact, then the supremum of an arbitrary subset of $C$ is well-defined and belongs to $C$. Consequently, there is a unique element in $C$ which is the coordinate-wise maximum of all elements in $C$. We call it the tropical barycenter of $C$, as it is the mean of $C$ w.r.t. the uniform idempotent measure. In particular if $P$ is a non-empty bounded Hardy polyhedron included in the positive orthant, then it follows from Proposition 2 that $\text{val}(P)$ is a compact tropical polyhedron. So $\text{val}(P)$ has a well-defined tropical barycenter.

Theorem 10. Let \((x^\mu, w^\mu)\) be the point on the primal central path of the Hardy linear program $LP(A, b, c)$ at $\mu \in K$ with $\mu > 0$, and let $\nu$ be that LP’s optimal value. Then $\text{val}(x^\mu, w^\mu)$ is the tropical barycenter of $\text{val}(P^\mu)$ where

\[
\mathcal{P}^\mu := \{(x, w) \in K^{n+m} | Ax + w = b, \ cx \leq \nu + (n + m)\mu, \ x \geq 0, \ w \geq 0\}.
\]

Proof. Let $(x^\mu, w^\mu, y^\mu, s^\mu)$ be a point on the central path. By (9), we have
\[
c^\top x^\mu = (s^\mu)^\top x^\mu - (y^\mu)^\top Ax^\mu = (s^\mu)^\top x^\mu - (y^\mu)^\top (b - w^\mu) = \sum_{j=1}^n s_j^\mu x_j^\mu + \sum_{i=1}^m y_i^\mu w_i^\mu - b^\top y^\mu
\]

\[
= (n + m)\mu - b^\top y^\mu.
\]
Furthermore, $y^\mu$ is a feasible solution of the the dual linear program:

(12) \[
\text{maximize } -b^\top y
\]
subject to $-A^\top y \leq c, \ y \geq 0, \ y \in K^m$.
By Proposition 1, the Weak Duality Theorem [Van08, §5.3] holds on the Hardy field $\mathbb{K}$. Thus $-b^\top y^\mu \leq \nu$. Consequently, $c^\top x^\mu \leq \nu + (n + m)\mu$.

Now by Lemma 3, $(x^\mu, w^\mu)$ is the unique solution of the barrier problem (11). By the discussion above, we can add the constraint $c^\top x \leq \nu + (n + m)\mu$ to the problem (11) without changing its optimal solution. Moreover, adding the constant $-\nu/\mu$ to the objective function still does not change the solution of the problem. Thus $(x^\mu, w^\mu)$ is the unique solution of

$$
\begin{align*}
\text{minimize} & \quad \frac{c^\top x - \nu}{\mu} - \sum_{j=1}^n \log(x_j) - \sum_{i=1}^n \log(w_i) \\
\text{subject to} & \quad Ax + w = b, \quad c^\top x \leq \nu + (n + m)\mu, \quad x > 0, \quad w > 0.
\end{align*}
$$

Let $\mathcal{P}^\mu_{>0}$ be the feasible set of (13) and consider a feasible solution $(x, w) \in \mathcal{P}^\mu_{>0}$. Since $c^\top x - \nu \leq (n + m)\mu$, the term $((c^\top x - \nu)/\mu$ is the germ of a function which is asymptotically $ct^\alpha$ for some $\alpha$, $c \in \mathbb{K}$ with $\alpha \leq 0$. On the other hand, $\log(x_j)$ is asymptotically $\text{val}(x_j)\log(t)$ for any $j \in [n]$. Since $t^\alpha = o(\log(t))$ when $\alpha \leq 0$, the objective value of (13) is asymptotically

$$
- \left( \sum_{j=1}^n \text{val}(x_j) + \sum_{i=1}^n \text{val}(w_i) \right) \log(t).
$$

As a consequence, $\text{val}(x^\mu, w^\mu)$ is the supremum of $\sum_{j=1}^n x_j + \sum_{i=1}^n w_i$ as $(x, w)$ ranges over the set $\text{val}(\mathcal{P}^\mu_{>0})$. Now, let $(x^*, w^*)$ be the tropical barycenter of $\text{val}(\mathcal{P}^\mu)$. Then, $x^*_j \geq \text{val}(x^\mu_j)$ and $w^*_i \geq \text{val}(w^\mu_i)$ for all $i \in [m]$, $j \in [n]$. In particular, $x^*_j > -\infty$ and $w^*_i > -\infty$. It follows that $(x^*, w^*) \in \text{val}(\mathcal{P}^\mu)$, and:

$$
\sum_{j=1}^n \text{val}(x^\mu_j) + \sum_{i=1}^n \text{val}(w^\mu_i) \geq \sum_{j=1}^n x^*_j + \sum_{i=1}^n w^*_i.
$$

We conclude that $\text{val}(x^\mu, w^\mu) = (x^*, w^*)$. \hfill \QED

The analytic center of the polyhedron

$$
\mathcal{P} := \{ (x, w) \in \mathbb{K}^n \mid Ax + w = b, \ x \geq 0, \ w \geq 0 \}
$$

can be defined as the unique minimum point $(x, w)$ of (13), when $c = 0$. Then, the tropical analytic center is defined as the image of the analytic center by the valuation map. By specializing the characterization of the tropical central path to $c = 0$, we get:

**Corollary 11.** The tropical analytic center of the polyhedron $\mathcal{P}$ coincides with the tropical barycenter of the image of this polyhedron by the valuation map. \hfill \QED

Hence, even if the analytic center is an algebraic notion (it depends on the external representation of the set $\mathcal{P}$), the tropical analytic center is, surprisingly, completely determined by the set $\mathcal{P}$. We shall see that the whole tropical central path also has a purely geometric description. We begin with a case where the geometric description can be obtained explicitly from $\text{val}(\mathcal{P})$ and $\text{val}(c)$.

**Corollary 12.** Suppose that the optimal value of $\text{LP}(A, b, c)$ is $\nu = 0$ and that $c$ has nonnegative entries. Then, the tropical central path at $\lambda \in \mathbb{R}$ is the tropical barycenter of the set

$$
\mathcal{P}^\lambda := \{ (x, w) \in \text{val}(\mathcal{P}) \mid \max(x_1 + \text{val}(c_1), \ldots, x_n + \text{val}(c_n)) \leq \lambda \}.
$$

**Proof.** Let $\mu = t^\lambda$. By Theorem 10, the tropical central path at $\lambda$ is the tropical barycenter of $\text{val}(\mathcal{P}^\mu)$. Clearly, $\text{val}(\mathcal{P}^\mu) \subset \mathcal{P}^\lambda$. Thus, we only need to prove that the tropical barycenter $(x^\lambda, w^\lambda)$ of $\mathcal{P}^\lambda$ admits a pre-image by the valuation map which belongs to $\mathcal{P}^\mu$.

By definition, there exists $(x^\lambda, w^\lambda) \in \mathcal{P}$ such that $\text{val}(x^\lambda, w^\lambda) = (x^\lambda, w^\lambda)$. If $c x^\lambda = 0$, then $c x^\lambda \leq (n + m)\mu$ and thus $(x^\lambda, w^\lambda) \in \mathcal{P}^\mu$. Otherwise, the germ $c x^\lambda$ is asymptotically $\alpha t^\beta$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$. Since $c$ and $x^\lambda$ has nonnegative entries, $\alpha > 0$ and we have

$$
\beta = \text{val}(c x^\lambda) = \max(x_1^\lambda + \text{val}(c_1), \ldots, x_n^\lambda + \text{val}(c_n)) \leq \lambda.
$$
If \( \alpha \leq n + m \), then clearly \( cx^* \leq (n + m)t^* = (n + m)\mu \) and thus \( x^* \in \mathcal{P}^\mu \).

We now treat the case \( \alpha > n + m \). Let \((x^*, w^*)\) be an optimal solution of LP\((A, b, c)\). Consider the point:

\[
(x, w) = \frac{1}{\alpha}(x^*, w^*) + \left( 1 - \frac{1}{\alpha} \right) (x^*, w^*) .
\]

As \( \alpha > 1 \), we have \((x, w) \in \mathcal{P}\) by convexity. Moreover, \(cx = \frac{1}{\alpha}cx^*\) since \(cx^* = 0\) by assumption. Thus \(cx\) is asymptotically \(t^\beta\). Since \( \beta \leq \lambda \), we obtain that \(cx \leq (n + m)t^\lambda = (n + m)\mu\), hence that \((x, w) \in \mathcal{P}^\mu\). It remains to show that \(\text{val}(x, w) = (x^\lambda, w^\lambda)\). To this end, observe that \(\text{val}(x, w) \geq (x^\lambda, w^\lambda)\) since \((x^\lambda, w^\lambda)\) and \((x^*, w^*)\) both have nonnegative entries and \(\alpha > 1\). Furthermore, \(\text{val}(x, w) \leq (x^\lambda, w^\lambda)\) as \(\text{val}(x, w) \in \text{val}(\mathcal{P}^\mu) \subset \mathcal{P}^\lambda\). This concludes the proof. \(\square\)

In the general case, the tropical central path still admits a geometric description, but this description involves an optimal solution of the dual of LP\((A, b, c)\).

**Corollary 13.** There exists a pair \((y^*, s^*)\) such that the tropical central path at any \(\lambda \in \mathbb{R}\) is given by the tropical barycenter of the set:

\[
\{(x, w) \in \text{val}(\mathcal{P}) \mid \max(x_1 + s_1^*, \ldots, x_n + s_n^*, w_1 + y_1^*, \ldots, w_m + y_m^*) \leq \lambda \} .
\]

**Proof.** Let \((y^*, s^*)\) be an optimal dual solution and \((x, w) \in \mathcal{P}\). Then, we have:

\[
c^\top x = -b^\top y^* + (s^*)^\top x + (y^*)^\top w .
\]

Furthermore, \(-b^\top y^* = \nu\) by strong duality. Thus,

\[
\mathcal{P}^\mu = \{(x, w) \in \mathcal{P} \mid (s^*)^\top x + (y^*)^\top w \leq (n + m)\mu\} .
\]

Since \((y^*, s^*) \geq 0\), applying the arguments of the proof of Corollary 12 provides the result. \(\square\)

**Example 14.** Consider the Hardy polyhedron of \(\mathbb{K}^2\) defined by:

\[
\begin{align*}
x_1 + x_2 & \leq 2 \\
tx_1 & \leq 1 + t^2x_2 \\
2tx_2 & \leq 1 + t^3x_1 \\
x_1 & \leq t^2x_2 \\
x_1, x_2 & \geq 0 .
\end{align*}
\]

(16)

Its value \(\text{val}(\mathcal{P})\) is the tropical set described by the inequalities:

\[
\begin{align*}
\max(x_1, x_2) & \leq 0 \\
1 + x_1 & \leq \max(0, 2 + x_2) \\
1 + x_2 & \leq \max(0, 3 + x_1) \\
x_1 & \leq 2 + x_2 .
\end{align*}
\]

(17)

The tropical central path on the polyhedron (10) for two objective functions is depicted in Figure 1. The hyperplanes associated with the first four halfspaces in (15) induce an arrangement of the positive orthant \(\mathbb{K}^2_+\). Figure 2 depicts the tropical central paths on the cells of this arrangement for the objective functions \(\min tx_1 + x_2\) and \(\max tx_1 + x_2\). Observe that the central paths trace the arrangement of tropical hyperplanes associated with the tropical halfspaces in (17), as well as the line \(\{(−1 + γ, γ) \mid γ \in \mathbb{R}\}\) associated with the objective function.

4. A TROPICAL CENTRAL PATH CAN DEGENERATE TO A TROPICAL SIMPLEX PATH

In this section, we will restrict our attention to the \(x\) components of the tropical central path. To fix the notation, we consider a Hardy linear program LP\((A, b, c)\), and the polyhedron \(\mathcal{P} = \{(x, w) \in \mathbb{K}^n \mid Ax + w = b, x \geq 0, w \geq 0\}\) from (15). From this viewpoint, the tropical central path may visit the boundary of the (projection on the \(x\)-space) of the set \(\text{val}(\mathcal{P})\). We will show that under some assumptions, the tropical central path lies on the image by the valuation map of the graph of the polyhedron \(\mathcal{P}\).
4.1. Tropical description of the graph of a Hardy polyhedron. When the tropicalization of the extended matrix \((A \ b)\) is sign generic, we have a purely tropical description of the set \(\text{val}(\mathcal{P})\) by Theorem 4. Furthermore, the second part of that result also shows that the images of the faces of \(\mathcal{P}\) under the valuation map also have a tropical description. In particular, this holds for the vertices and the edges of \(\mathcal{P}\). Indeed, in our setting, a vertex of \(\mathcal{P}\) is a point \(x^I \in \mathcal{P}\) which also satisfies \(A_I x + b_I = 0\), where \(I\) is a set of \(n\) inequalities such that \(\det(A_I) \neq 0\).

Using the notation of Theorem 4, \(x \in \text{val}(\mathcal{P})\) is the value of a vertex if and only if it satisfies a system of \(n\) equalities \(A_I x + b_I = 0\) where \(t\)det\((A_I + A_I^-) \neq -\infty\). Recall that the tropical determinant of a square matrix \(M \in \mathbb{T}^{n \times n}\) is

\[
\text{tdet} M := \max_{\sigma \in \text{Sym}(n)} M_{1\sigma(1)} + \cdots + M_{n\sigma(n)},
\]

where \(\text{Sym}(n)\) denotes the set of permutations of \([n]\). Computing \(\text{tdet}(M)\) amounts to solving a linear optimal assignment problem [RGST05].

**Proposition 15.** Consider a Hardy polyhedron \(\mathcal{P} = \{x \in \mathbb{K}^n \mid Ax + b \geq 0\}\) included in the positive orthant such that the tropicalization of \((A \ b)\) is sign generic, and \(A^-\) has at most one
non-zero coefficient in each row. Then the tropical analytic center of \( P \) coincides with the value of a vertex of \( P \).

\textbf{Proof.}\ By Theorem 11, the tropical polyhedron \( \text{val}(P) \) is described by \( \{ x \in T^n \mid A^+ \odot x \oplus b^+ \geq A^- \odot x \oplus b^- \} \). Since \( A^- \) has at most one non-zero coefficient in each row, for every \( i \in [m] \) the tropical inequality \( A^+_i \odot x \oplus b^+_i \geq A^-_i \odot x \oplus b^-_i \) is of the form:

\[
\max(A^+_i + x_1, \ldots, A^+_m + x_m, b^+_i) \geq \max(A^-_i + x_k, b^-_i),
\]

for some \( k \in [n] \).

Let \( x^* \) be the tropical analytic center of \( P \). By Corollary 12, \( x^* \) is the tropical barycenter of \( \text{val}(P) \). By Assumption 5, \( x_j^* \) finite for each \( j \in [n] \). Thus, there must exist an \( i \in [m] \) such that \( A^-_i \neq -\infty \) and

\[
\max(A^+_i + x_1, \ldots, A^+_m + x_m, b^+_i) = A^-_i + x_j^*.
\]

Consequently, \( x^* \) satisfies a set \( I \) of \( n \) equalities, one for each coordinate \( j \in [n] \). By construction we have \( \text{tdet}(A^-_i) \neq \infty \), thus \( \text{tdet}(A^+_i \oplus A^-_i) \neq -\infty \). Consequently, \( x^* \) is the value of a vertex by Theorem 11. \qed

Vertices of \( P \) are connected by edges, which are sets of the form \( \{ x \in P \mid A_K x + b_K = 0 \} \) where \( K \subset [m] \) is of cardinality \( n - 1 \) and \( A_K \) is of rank \( n - 1 \). Under the conditions of Theorem 4 the image of the edges under the valuation map are exactly the sets described by \( \{ x \in \text{val}(P) \mid A^-_K \odot x \oplus b^-_K = A^-_K \odot x \oplus b^-_K \} \) where \( K \subset [m] \) is of cardinality \( n - 1 \) and \( A^-_K \odot A^-_K \) has a maximal minor with non-\( -\infty \) tropical determinant.

\textbf{Proposition 16.}\ Let \( P \) be a Hardy polyhedron which satisfies the conditions of Proposition 15. Consider a linear program of the form:

\[
\text{LP} \quad \min x_k \st \ x \in P,
\]

for some \( k \in [n] \). If the optimal value of \( \text{LP} \) is \( \nu = 0 \), then the tropical central path of \( \text{LP} \) is contained in the image by the valuation map of the graph of \( P \).

\textbf{Proof.}\ By Corollary 12 the point \( x^\lambda \) on the tropical central path at \( \lambda \in \mathbb{R} \) is the tropical barycenter of the tropical polyhedron \( \{ x \in \text{val}(P) \mid x_k \leq \lambda \} \). As in the proof of Proposition 15, for each \( j \in [n] \setminus \{k\} \) the point \( x^\lambda \) must satisfy an equality of the form (19). Thus, \( x^\lambda \) satisfy a set \( K \) of \( n - 1 \) equalities and it is straightforward to check that the minor of \( A^-_K \) formed with the columns indexed by \( [n] \setminus \{k\} \) has a finite tropical determinant. \qed

The latter proposition is illustrated in Figure 1 (left).

5. LONG TROPICAL CENTRAL PATHS AND ORDINARY CENTRAL PATHS WITH HIGH CURVATURE

Bezem, Nieuwenhuis and Rodríguez-Carbonell [BNRC08] constructed a class of tropical linear programs for which an algorithm of Butkovič and Zimmermann [BZ06] exhibits an exponential running time. We lift each of these tropical linear programs to the Hardy field \( \mathbb{K} = H(\mathbb{R}^K) \) which then gives rise to a one-parameter family of ordinary linear programs over the reals. The latter are interesting as their central paths have an unusually high total curvature.

Let \( r \) be any positive integer. We define a linear program, \( \text{LP}_r \), over the Hardy field \( \mathbb{K} \) in the \( 2r + 2 \) variables \( u_0, v_0, u_1, v_1, \ldots, u_r, v_r \) as follows.

\[
\text{LP}_r \quad \min v_0 \st u_0 \leq t, \quad v_0 \leq t^2
\]

\[
v_i \leq t^{1 - \frac{1}{2r}}(u_{i-1} + v_{i-1}) \quad \text{for } 1 \leq i \leq r
\]

\[
u_i \leq tu_{i-1} \quad \text{for } 1 \leq i \leq r
\]

\[
u_i \leq tv_{i-1} \quad \text{for } 1 \leq i \leq r
\]

\[
u_r \geq 0, \nu_r \geq 0
\]
Clearly, the optimal value of $\text{LP}_r$ is $\mathbf{v} = 0$, and an optimal solution is $\mathbf{u} = \mathbf{v} = 0$. It is straightforward to verify that the feasible set is bounded with a non-empty interior. Moreover, the feasible set is contained in the positive orthant and the $3r + 4$ inequalities listed define facets. In particular, the remaining non-negativity constraints $u_i \geq 0$ and $v_i \geq 0$ for $0 \leq i < r$ are satisfied but redundant. We will denote the feasible region of $\text{LP}_r$ as $\mathcal{P}_r$.

Replacing $t$ in $\text{LP}_r$ by any positive real number gives rise to an ordinary linear program. For $t$ sufficiently large the polytope of feasible points is combinatorially equivalent to the polytope of feasible points of the Hardy linear program. Figure 3 shows an example for $r = 1$ and $t = 2$, which is sufficiently large in this case.

![Figure 3. Schlegel diagram for $r = 1$ (and $t = 2$), projected onto the facet $v_1 = 0$; the points are written in $(u_0, v_0, u_1, v_1)$-coordinates](image)

Applying the valuation map of $\mathbb{K}$ to the feasible region of $\text{LP}_r$ yields the tropical polyhedron described by:

\begin{align*}
  u_0 &\leq 1 \\
  v_0 &\leq 2 \\
  v_i &\leq 1 - \frac{1}{2^i} + \max(u_{i-1}, v_{i-1}) \quad \text{for } 1 \leq i \leq r \\
  u_i &\leq 1 + u_{i-1} \quad \text{for } 1 \leq i \leq r \\
  u_i &\leq 1 + v_{i-1} \quad \text{for } 1 \leq i \leq r \\
  u_i &\in \mathbb{T}, \; v_i \in \mathbb{T} \quad \text{for } 0 \leq i \leq r
\end{align*}

(20)

Indeed, one can check that if $(\mathbf{u}, \mathbf{v})$ satisfies the constraints (20), then $\mathbf{u} = (t^{u_0}, \ldots, t^{u_r})$ and $\mathbf{v} = (t^{v_0}, \ldots, t^{v_r})$ is feasible for $\text{LP}_r$.

5.1. The tropical central path. By Corollary 12, the components $u(\lambda), v(\lambda)$ of the tropical central path at $\lambda \in \mathbb{R}$ are given by the greatest element of the tropical polyhedron (20) intersected with the tropical halfspace $v_0 \leq \lambda$. Consequently, $v_i(\lambda)$ is equal to the maximum of $u_{i-1}(\lambda)$ and $v_{i-1}(\lambda)$ translated by $1 - \frac{1}{2^i}$, while $u_i(\lambda)$ follows the minimum of these two variables shifted by 1; see Figure 4. Since the translation offsets differ by $\frac{1}{2^i}$, the components $u_i$ and $v_i$ cross each other $\Omega(2^i)$ times. More precisely, our next results shows that the curve $(u_i(\lambda), v_i(\lambda))$ has the shape of a staircase with $\Omega(2^i)$ steps.

**Proposition 17.** Let $i \in [r]$ and $k \in \{0, \ldots, 2^{i-1} - 1\}$. Then, for all $\lambda$ in the interval $[\frac{4k}{2^i}, \frac{4k+2}{2^i}]$, we have

$$ u_i(\lambda) = i + \lambda - \frac{2k}{2^i} \quad \text{and} \quad v_i(\lambda) = i + \frac{2k + 1}{2^i}, $$
Figure 4. A tropical central path with many segments

while for all $\lambda \in \left[\frac{4k+2}{2^{i}}, \frac{4k+4}{2^{i}}\right]$ we have

$$u_i(\lambda) = i + \frac{2k + 2}{2^{i}} \quad \text{and} \quad v_i(\lambda) = i + \lambda - \frac{2k + 1}{2^{i}}.$$  

Proof. We proceed by a bounded induction on $i \in [r]$. Starting with $i = 1$ and $k = 0$ we consider the tropical central path point at any $\lambda \in [0, 2]$. Our goal is to determine the tropical analytic center. The first variables $(u_0, v_0)$ are only involved in the inequalities $u_0 \leq 1$ along with $v_0 \leq \lambda$. As we are looking for the coordinatewise maximum we deduce that $u_0 = 1$ and $v_0 = \lambda$. Hence, the components $u_1$ and $v_1$ satisfy:

$$v_1 \leq \max(\lambda, 1) + \frac{1}{2}$$
$$u_1 \leq 1 + \lambda$$
$$u_1 \leq 2$$
Thus for $\lambda \in [0, 1]$ the maximum is attained at $u_1 = 1 + \lambda$ and $v_1 = 1 + \frac{1}{2}$. For $\lambda \in [1, 2]$ we have $u_1 = 1 + \frac{1}{2}$ and $v_1 = 1 + \lambda - \frac{1}{2}$. Consequently, the claim holds for $i = 1$.

By induction, suppose the result is verified for $i < r$. We will show that it is also true for $i + 1$. Consider any integer, $k$, in $\{0, \ldots, 2^i - 1\}$. If $k$ is even, let $k' = k/2$. Then, for all $\lambda$ in the interval $[\frac{4k}{2^i+1}, \frac{4k+4}{2^i+1}] = [\frac{4k}{2^i}, \frac{4k+2}{2^i}]$, we have by induction:

$$u_i = i + \lambda - \frac{2k'}{2^i} = i + \lambda - \frac{k}{2^i} \quad \text{and} \quad v_i = i + \frac{2k' + 1}{2^i} = i + \frac{k + 1}{2^i}.$$ 

Thus,

$$u_{i+1} \leq i + 1 + \min \left( \lambda - \frac{k}{2^i}, \lambda - \frac{k}{2^i} \right) \quad \text{and} \quad v_{i+1} \leq i + 1 + \max \left( \lambda - \frac{k+2}{2^i}, \lambda - \frac{k+2}{2^i} \right) - \frac{1}{2^i+1}.$$ 

Separating the cases $\lambda \leq \frac{4k+4}{2^i+1}$ and $\lambda \geq \frac{4k+2}{2^i}$ leads to the desired conclusion.

If $k$ is odd, $k = 2k' + 1$, then for any $\lambda \in [\frac{4k+4}{2^i+1}, \frac{4k+4}{2^i+1}] = [\frac{4k+2}{2^i}, \frac{4k+2}{2^i}]$ we have:

$$u_i = i + \frac{2k' + 2}{2^i} = i + \frac{k + 2}{2^i} \quad \text{and} \quad v_i = i + \lambda - \frac{2k'}{2^i} = i + \lambda - \frac{k + 1}{2^i}.$$ 

Thus,

$$u_{i+1} \leq i + 1 + \min \left( \lambda - \frac{k+1}{2^i}, \lambda - \frac{k+2}{2^i} \right) \quad \text{and} \quad v_{i+1} \leq i + 1 + \max \left( \lambda - \frac{k+1}{2^i}, \lambda - \frac{k+2}{2^i} \right) - \frac{1}{2^i+1}.$$ 

As above, by separating the cases $\lambda \leq \frac{4k+4}{2^i+1}$ and $\lambda \geq \frac{4k+2}{2^i}$ we conclude that the inductive claim holds for $i + 1$. \hfill \Box

**Remark 18.** A similar induction shows that for $\lambda \geq 2$ the tropical central path is at the tropical analytic center, defined by $u_0 = 1, v_0 = 2$ and

$$u_i = i + 1 \quad \text{and} \quad v_i = i + 1 + \frac{1}{2^i} \quad \text{for all} \ 1 \leq i \leq r.$$ 

For $\lambda \leq 0$, the tropical central path is a tropical half-line towards an optimum. We have $u_0(\lambda) = 1, v_0(\lambda) = \lambda$ as well as

$$u_i(\lambda) = i + \lambda \quad \text{and} \quad v_i(\lambda) = i + \frac{1}{2^i} \quad \text{for all} \ 1 \leq i \leq r.$$ 

We will now show that the tropical central path of $\text{LP}_r$ coincides with the image of a path of the simplex method under the valuation map. Our proof is elementary and independent of Proposition 16.

**Proposition 19.** The tropical central path of $\text{LP}_r$ is contained in the image of the vertex-edge-graph of $\mathcal{P}_r$ under the valuation map. The tropical central path at $\lambda \in \mathbb{R}$ is the value of a vertex if and only if $\lambda \geq 2$ or $\lambda = \frac{2k}{2^i}$ for some $k \in \{1, \ldots, 2^i\}$.

**Proof.** We prove the claim by induction on $r$. Suppose that $r = 1$. This situation in four dimensions is depicted in Figure 3. For $\lambda \geq 2$, the tropical central path is at the tropical analytic center of $\text{LP}_1$:

$$u_0 = 1, v_0 = 2, u_1 = 2, v_1 = 5/2.$$ 

This is the value of the vertex $(t, t^2, t^2, t^{5/2} + t^{3/2})$ of the Hardy polyhedron $\mathcal{P}_1$ which is uniquely defined by the conditions

$$(21) \quad u_0 = t, v_0 = t^2, u_1 = tu_0, v_1 = t^{1/2}(u_0 + v_0).$$ 

For $\lambda = 1$ the tropical central path is at the point with coordinates

$$u_0 = 1, v_0 = 1, u_1 = 2, v_1 = 3/2.$$
which corresponds to the vertex \((t, t, t^2, 2t^{3/2})\) of \(\mathcal{P}_1\), the unique solution of:

\[
\begin{align*}
u_0 &= t, \quad v_0 = t, \quad u_1 = t, \quad v_0, v_1 = t^{1/2}(u_0 + v_0) .
\end{align*}
\]

It is straightforward to check that the tropical central path for \(\lambda \in [1, 2]\) is the image by the valuation map of the edge between the vertices \((21)\) and \((22)\). Similarly, for \(\lambda \in ]-\infty, 1]\), the tropical central path:

\[
\begin{align*}
u_0 &= 1, \quad v_0 = \lambda, \quad u_1 = 1 + \lambda, \quad v_1 = 3/2 .
\end{align*}
\]

is the value of the edge between the vertices \((22)\) and \((t, 0, 0, t^{3/2})\) of \(\mathcal{P}_1\) defined by

\[
\begin{align*}
u_0 &= t, \quad v_0 = 0, \quad u_1 = tv_0, \quad v_1 = t^{1/2}(u_0 + v_0) .
\end{align*}
\]

Now suppose that the claim holds for \(r \geq 1\). For \(\lambda \geq 2\), the tropical central path of \(\mathbf{LP}_{r+1}\) is at the analytic center \((u_0, v_0, \ldots, u_r, v_r, u_{r+1}, v_{r+1})\). By Proposition \ref{lem:1} we have \(u_{r+1} = 1 - 1/2^{r+1} + \max(u_r, v_r)\) and \(u_{r+1} = 1 + u_r\). By induction, \((u_0, v_0, \ldots, u_r, v_r)\) is the value of the vertex \((u_0, v_0, \ldots, u_r, v_r)\) of \(\mathbf{LP}_{r}\). The system defining this vertex of \(\mathcal{P}_r\), along with the equalities \(u_{r+1} = t^{1-1/2^{r+1}}(u_r + v_r)\) and \(u_{r+1} = tu_r\) clearly have a unique solution which is feasible for \(\mathbf{LP}_{r+1}\). Thus it defines a vertex of \(\mathcal{P}_{r+1}\). It is straightforward to verify that the valuation map applied to this vertex yields the tropical analytic center. Similarly, the argument above shows that the tropical central path of \(\mathbf{LP}_{r+1}\) is the value of a vertex when \(\lambda = 2k\) for some \(k \in \{1, \ldots, 2^r - 1\}\).

Fix a \(k \in \{1, \ldots, 2^r - 1\}\). Then central path of \(\mathbf{LP}_r\) at \(\lambda \in [\frac{4k}{2^r}, \frac{4k+2}{2^r}] = [\frac{4k}{2^r}, \frac{4k+4}{2^r}]\) is the value of a point on an edge of \(\mathcal{P}_r\). This edge in \(\mathbb{K}^{2r+2}\) defines a 3-dimensional face \(\mathcal{F}\) of \(\mathcal{P}_{r+1}\) in \(\mathbb{K}^{2r+4}\). The intersection of \(\mathcal{F}\) with the three hyperplanes

\[
\begin{align*}
u_{r+1} &= t^{1-1/2^{r+1}}(u_r + v_r), \quad u_{r+1} = tu_r \quad \text{and} \quad u_{r+1} = tu_r,
\end{align*}
\]

yields a vertex of \(\mathcal{P}_{r+1}\), and it can be checked that the value of this vertex is on the tropical central path of \(\mathbf{LP}_{r+1}\) for \(\lambda = \frac{2k}{2^r}\). It follows that the tropical central path of \(\mathbf{LP}_{r+1}\) at \(\lambda \in [\frac{2k}{2^r}, \frac{4k+2}{2^r}]\) and \(\lambda \in [\frac{4k+2}{2^r}, \frac{4k+4}{2^r}]\) corresponds to points on two distinct edges of \(\mathcal{P}_{r+1}\). These two edges are obtained by intersecting \(\mathcal{F}\) with \(v_{r+1} = t^{1-1/2^{r+1}}(u_r + v_r)\) and either \(u_{r+1} = tu_r\) or \(u_{r+1} = tv_r\). It remains to consider \(\lambda \leq \frac{1}{2^{r+1}}\). By induction, the tropical central path of \(\mathbf{LP}_r\), for \(\lambda \leq \frac{1}{2^{r+1}} = \frac{2}{4^{r+1}}\) is the set of values of an edge of \(\mathcal{P}_r\). As above, this edge yields a 3-face \(\mathcal{F}\) of \(\mathcal{P}_{r+1}\). Intersecting \(\mathcal{F}\) with the three hyperplanes \((24)\) yields a vertex whose value is the tropical central path of \(\mathbf{LP}_{r+1}\) at \(\lambda = \frac{2k}{2^r}\). Intersecting \(\mathcal{F}\) with \(v_{r+1} = t^{1-1/2^{r+1}}(u_r + v_r)\) and \(u_{r+1} = tv_r\), yields an edge of \(\mathcal{P}_{r+1}\) whose set of values is the tropical central path at \(\lambda \in [\frac{2}{2^{r+1}}, \frac{4}{2^{r+1}}]\). For \(\lambda \leq \frac{2}{4^{r+1}}\), the tropical central path is the set of values of the edge obtained as the intersection of \(\mathcal{F}\) with \(v_{r+1} = t^{1-1/2^{r+1}}(u_r + v_r)\) and \(u_{r+1} = tu_r\).

\[\square\]

5.2. Curvature analysis. Given any integer \(r \geq 1\), the Hardy linear program \(\mathbf{LP}_r\) gives rise to a family of central paths \((C(t, \cdot))_t\) for ordinary linear programs over the reals. We are interested in the components \((u, v)\) of \(C\) for \(t\) large enough. With the notation of Lemma \ref{lem:2} let \((u^\mu, v^\mu)\) be the \((u, v)\) components of the central path of \(\mathbf{LP}_r\) at \(\mu\). Let us define the map \(\Phi\) as follows:

\[
\begin{align*}
\Phi : \mathbb{R} &\rightarrow \mathbb{R}^{2r+2} \\
\lambda &\mapsto (u^\mu_0(t), v^\mu_0(t), \ldots, u^\mu_r(t), v^\mu_r(t)) \quad \text{where} \quad \mu = t^\lambda .
\end{align*}
\]

Similarly, we denote by \(\Phi_{r+1}^{\text{trop}}\) the projection of the tropical central path onto the \((u, v)\) coordinates:

\[
\begin{align*}
\Phi_{r+1}^{\text{trop}} : \mathbb{R} &\rightarrow \mathbb{R}^{2r+2} \\
\lambda &\mapsto (u_0(\lambda), v_0(\lambda), \ldots, u_r(\lambda), v_r(\lambda)) .
\end{align*}
\]

In particular, \(\Phi_{2r+1}^{\text{trop}}(\lambda) = u_r(\lambda)\) and \(\Phi_{2r+2}^{\text{trop}}(\lambda) = v_r(\lambda)\). The map \(\Phi\) is a parameterization of the projection of a real central path. Thus \(\Phi\) is smooth. For any \(\lambda \geq 0\), the arc length of the path \(\Phi\) between \(\Phi(0)\) and \(\Phi(\lambda)\) is \(\ell(\lambda) := \int_0^\lambda \|\Phi'(\gamma)\|d\gamma\). Let \(\Gamma\) be the parameterization of \(\{\Phi(\lambda) \mid \lambda \geq 0\}\) by its arclength, i.e., \(\Gamma(\ell(\lambda)) := \Phi(\lambda)\) for all \(\lambda \geq 0\). As a consequence, \(\Gamma'(\ell(\lambda)) = \Phi(\lambda)/|\Phi'(\lambda)|\).
Thus, \( \hat{\Phi} \) describes a path on the unit sphere \( S^{2r+1} \subseteq \mathbb{R}^{2r+2} \). The length of the latter path, 
\[
\int_0^{\ell(\lambda)} \| \hat{\Phi}'(\tau) \| \, d\tau,
\]
is the total curvature of \( \Phi \) between \( \Phi(0) \) and \( \Phi(\lambda) \).

By Theorem 6 for any \( \varepsilon > 0 \) we can choose \( t \) large enough so that for any \( \lambda \in \mathbb{R} \) and any \( i \in \{2r+2\} \), the component \( \Phi_i(\lambda) \) of the central path satisfies the following bounds:
\[
t_{\Phi_i^{\text{trop}}(\lambda) - \varepsilon}^{\Phi_i^{\text{trop}}(\lambda) + \varepsilon} \leq \Phi_i(\lambda) \leq t_{\Phi_i^{\text{trop}}(\lambda) + \varepsilon}^{\Phi_i^{\text{trop}}(\lambda) + \varepsilon}.
\]

We shall use these bounds to derive results on the curvature of \( \Phi \). The proofs of Lemma 20 and Theorem 21 below are directly adapted from the proofs of Proposition 7.3 and Corollary 7.2 in [DTZ09].

**Lemma 20.** Consider a segment \([\lambda_1, \lambda_2]\) of the real line, and let \( \varepsilon > 0 \). Suppose that
\[
\begin{align*}
\Phi_i^{\text{trop}}(\lambda_2) - \Phi_i^{\text{trop}}(\lambda_1) & 
\geq 3\varepsilon, \\
\Phi_i^{\text{trop}}(\lambda_2) - \Phi_j^{\text{trop}}(\lambda_1) & 
\geq 3\varepsilon, \\
\Phi_i^{\text{trop}}(\lambda_2) - \Phi_j^{\text{trop}}(\lambda_2) & 
\geq 3\varepsilon
\end{align*}
\]
for some \( i, j \in \{2r+2\} \). Then, there exists a \( l \in [\ell(\lambda_1), \ell(\lambda_2)] \) such that
\[
|\dot{\Phi}(l)| \leq \frac{2}{t-1}.
\]

**Proof.** Since \( \Gamma_j \) is differentiable, by the mean value theorem there exists a \( l \in [\ell(\lambda_1), \ell(\lambda_2)] \) such that 
\[
\Phi_i^{\text{trop}}(\lambda_2) - \Phi_i^{\text{trop}}(\lambda_1) = \Gamma_j(\ell(\lambda_2)) - \Gamma_j(\ell(\lambda_1)).
\]
Thus,
\[
|\dot{\Phi}(l)| = \left| \frac{\Gamma_j(\ell(\lambda_2)) - \Gamma_j(\ell(\lambda_1))}{\ell(\lambda_2) - \ell(\lambda_1)} \right| \leq \frac{\Phi_i^{\text{trop}}(\lambda_2) + \Phi_i^{\text{trop}}(\lambda_1)}{\ell(\lambda_2) - \ell(\lambda_1)} \leq \frac{t_{\Phi_i^{\text{trop}}(\lambda_2) + \varepsilon}^{\Phi_i^{\text{trop}}(\lambda_1) + \varepsilon}}{\ell(\lambda_2) - \ell(\lambda_1)}.
\]
The arc length \( \ell(\lambda_2) - \ell(\lambda_1) \) can be bounded as follows:
\[
\ell(\lambda_2) - \ell(\lambda_1) = \int_{\lambda_1}^{\lambda_2} \| \Phi(\gamma) \| \, d\gamma \geq \int_{\lambda_1}^{\lambda_2} \Phi_i(\gamma) d\gamma = \Phi_i(\lambda_2) - \Phi_i(\lambda_1)
\]
\[
\geq t_{\Phi_i^{\text{trop}}(\lambda_2) - \varepsilon}^{\Phi_i^{\text{trop}}(\lambda_1) + \varepsilon} - t_{\Phi_i^{\text{trop}}(\lambda_2) - \varepsilon}^{\Phi_i^{\text{trop}}(\lambda_1) + \varepsilon} + \Phi_i^{\text{trop}}(\lambda_1) + 2\varepsilon
\]
\[
\geq t_{\Phi_i^{\text{trop}}(\lambda_2) - \varepsilon}^{\Phi_i^{\text{trop}}(\lambda_1) + \varepsilon} (1 - t_{\Phi_i^{\text{trop}}(\lambda_2) - \varepsilon}^{\Phi_i^{\text{trop}}(\lambda_1) + \varepsilon})
\]
It follows that:
\[
|\dot{\Phi}(l)| \leq \frac{t_{\Phi_i^{\text{trop}}(\lambda_2) + \varepsilon}^{\Phi_i^{\text{trop}}(\lambda_1) + \varepsilon}}{\ell(\lambda_2) - \ell(\lambda_1)} \leq \frac{t_{\Phi_i^{\text{trop}}(\lambda_2) - \varepsilon}^{\Phi_i^{\text{trop}}(\lambda_1) + \varepsilon}}{\ell(\lambda_2) - \ell(\lambda_1)} \leq \frac{2}{1 - t - \varepsilon} = \frac{2}{t - \varepsilon}.
\]

**Theorem 21.** For all \( t \) large enough, the total curvature of \( \Phi \) between \( \Phi(0) \) and \( \Phi(2) \) is in \( \Omega(2^r/r) \).

**Proof.** To prove the theorem we shall show that, for any \( k = 0, \ldots, 2^r - 1 \), the total curvature of \( \Phi \) between \( \Phi(\frac{k}{2^r}) \) and \( \Phi(\frac{k+2^r}{2^r}) \) is greater than \( \Omega(1/r) \). Fix such a \( k \). Then, by Proposition 17,
\[
\Phi_{2r+1}^{\text{trop}}(\frac{4k}{2^r}) = r + \frac{2k + 1}{2^r}, \quad \Phi_{2r+2}^{\text{trop}}(\frac{4k}{2^r}) = r + \frac{2k}{2^r},
\]
\[
\Phi_{2r+1}^{\text{trop}}(\frac{4k + 2}{2^r}) = r + \frac{2k + 1}{2^r}, \quad \Phi_{2r+2}^{\text{trop}}(\frac{4k + 2}{2^r}) = r + \frac{2k + 2}{2^r}.
\]
Choosing \( \varepsilon > 0 \) such that \( 3\varepsilon < 2^{-r} \), and applying Lemma 20 with \( i = 2r+1 \) and \( j = 2r+2 \), we obtain a real number, \( l_1 \in [\ell(\frac{4k}{2^r}), \ell(\frac{4k+2}{2^r})] \), such that
\[
|\dot{\Phi}_{2r+1}(l_1)| \leq \frac{2}{t - \varepsilon}.
\]
Since $\|\hat{l}\| = 1$, we deduce that $\sum_{i=1}^{2r+1} \left| \hat{l}_i(l_1) \right|^2 = 1 - \left( \frac{2}{l^2 - 1} \right)^2$. Hence, there exists a $p \in [2n + 1]$ such that:

$$\left| \hat{l}_p(l_1) \right| \geq \left( \hat{l}_p(l_1) \right)^2 \geq \frac{1}{2r + 1} \left( 1 - \left( \frac{2}{l^\varepsilon - 1} \right)^2 \right)$$

Still by Proposition 17, we have:

$$\Phi^{trop}_{2r+2} \left( \frac{4k + 4}{2r} \right) \geq \Phi^{trop}_{2r+2} \left( \frac{4k + 4}{2r} \right) + \frac{1}{2r}$$

Furthermore, it is easy to deduce from Proposition 17 that:

$$\Phi^{trop}_{2r+2} \left( \frac{4k + 4}{2r} \right) \geq \Phi^{trop}_{2r+2} \left( \frac{4k + 4}{2r} \right) + \frac{1}{2r} \quad \text{for all } j = 1, \ldots, 2r + 1$$

In particular, the inequalities above holds for $j = p$. Again by Lemma 20, there exists a $l_2 \in \left[ \ell\left( \frac{4k + 4}{2r} \right), \ell\left( \frac{4k + 4}{2r} \right) \right]$ such that $|\hat{l}_p(l_2)| \leq \frac{2}{l^2 - 1}$.

A lower bound on the total curvature between $l_1$ and $l_2$ is given by the arc length of the geodesic on the sphere between $\hat{l}(l_1)$ and $\hat{l}(l_2)$, which, in turn, is greater than $|\hat{l}(l_1) - \hat{l}(l_2)|$.

Finally:

$$|\hat{l}(l_1) - \hat{l}(l_2)| \geq |\hat{l}_p(l_1) - \hat{l}_p(l_2)| \geq |\hat{l}_p(l_1)| - |\hat{l}_p(l_2)|$$

$$\geq \frac{1}{2r + 1} \left( 1 - \left( \frac{2}{l^\varepsilon - 1} \right)^2 \right).$$

Since $\varepsilon > 0$, we can choose $t$ large enough so that $\|\hat{l}(l_1) - \hat{l}(l_2)\|$ is in $\Omega(1/r)$. \hfill $\square$

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