Model reduction for transport-dominated problems via online adaptive bases and adaptive sampling

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This work presents a model reduction approach for problems with coherent structures that propagate over time such as convection-dominated flows and wave-type phenomena. Traditional model reduction methods have difficulties with these transport-dominated problems because propagating coherent structures typically introduce high-dimensional features that require high-dimensional approximation spaces. The approach proposed in this work exploits the locality in space and time of propagating coherent structures to derive efficient reduced models. First, full-model solutions are approximated locally in time via local reduced spaces that are adapted with basis updates during time stepping. The basis updates are derived from querying the full model at a few selected spatial coordinates. Second, the locality in space of the coherent structures is exploited via an adaptive sampling scheme that selects at which components to query the full model for computing the basis updates. Our analysis shows that, in probability, the more local the coherent structure is in space, the fewer full-model samples are required to adapt the reduced basis with the proposed adaptive sampling scheme. Numerical results on benchmark examples with interacting wave-type structures and time-varying transport speeds and on a model combustor of a single-element rocket engine demonstrate the wide applicability of our approach and the significant runtime speedups compared to full models and traditional reduced models.

Keywords: model reduction, transport-dominated problems, empirical interpolation, nonlinear model reduction, localized model reduction, online adaptive model reduction, sparse sampling, proper orthogonal decomposition

1. Introduction

Model reduction constructs reduced models of large-scale systems of equations in a one-time high-cost offline phase and then uses the reduced models in an online phase to repeatedly compute accurate approximations of the full-model solutions with significantly reduced costs. In projection-based model reduction \[ \text{[46, 2, 6]}, \] a low-dimensional space is constructed that approximates well the high-dimensional solution space of the full model. Then, the full-model equations are projected onto the low-dimensional space and reduced solutions are computed by solving the projected equations. However, solutions of full models that describe transport-dominated behavior, e.g., convection-dominated flows, wave-type phenomena, shock propagation, typically

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exhibit high-dimensional features, which means that no low-dimensional space exists in which the full-model solutions can be approximated well; the Kolmogorov n-width is high. Thus, traditional model reduction methods typically fail for transport-dominated problems [34, 14, 51]. In this work, we exploit that transport-dominated problems typically have a rich structure that is local in nature, which we leverage to derive efficient reduced models.

Model reduction projects the full-model equations on a low-dimensional—reduced—space that is spanned by a set of basis vectors. There are many methods for constructing reduced spaces, including proper orthogonal decomposition (POD) [7, 49], balanced truncation [29, 30], the reduced basis method [41, 53, 17, 24, 46], and interpolatory model reduction methods [23, 2]. In the following, we will be mostly concerned with full-model equations that are nonlinear in the states, where projecting the full-model equations onto a reduced space is typically insufficient to obtain a reduced model that is faster to solve than the full model, because the nonlinear terms entail computations that scale with the dimension of the full-model solution space. One remedy is empirical interpolation, which approximates nonlinear terms by evaluating them at a few, carefully selected interpolation points and approximating all other components via interpolation in a low-dimensional space [5, 22, 10]. We will build on the discrete empirical interpolation method (DEIM) [13, 15], which is the discrete counterpart of empirical interpolation. Note that there are other techniques for nonlinear model reduction, e.g., missing point estimation [3] and the Gauss-Newton with approximated tensors (GNAT) method [12]. All these methods build on the assumption that there is a low-dimensional space in which the full-model solutions can be approximated well, which is violated in case of transport-dominated problems. However, the solutions of transport-dominated problems typically are low dimensional if considered locally in time, which we exploit by approximating the full-model solutions in local low-dimensional spaces that are adapted via low-rank basis updates over time [39, 62]. The basis updates are derived by querying the full model at selected points in the spatial domain. We derive an adaptive sampling scheme that selects where to query the full model to compute the basis updates. Our analysis shows that if the reduced-model residual is local in the spatial domain—which we observe for transport-dominated problems—then only few sampling points are necessary to adapt the local spaces with our adaptive sampling scheme, which makes our model reduction approach computationally efficient.

For reviewing the literature on model reduction for transport-dominated problems, we broadly distinguish between two lines of research. Literature that we categorize into the first line of research aims to transform the full model to recover a low-rank structure that can then be exploited with a reduced space. The work on transformations in the model reduction community seems to have originated from [34], where the transport dynamics are separated from the rest of the dynamics via freezing. In [44, 43], transport maps are constructed that reverse the transport and so recover a low-rank structure. Approaches have been developed that aim to find transformations numerically via, e.g., optimization [8]. Shifted POD [42] recovers the shift due to the transport and applies POD after having reversed the shift. The shift operator introduced in [42] is time dependent and separates different transports to be applicable to, e.g., multiple waves traveling at different wave speeds. In [56, 57], snapshots are transformed to better align them before the reduced bases are constructed. The second line of research on model reduction for transport-dominated problems constructs low-dimensional spaces that explicitly target transport-dominated problems. The work [14] formulates the greedy basis construction methods, developed in the reduced basis community [11, 54], in special norms that are better suited for transport-dominated problems. The work [1] constructs the basis via $L_1$ optimization and the authors of [52, 49] consider time-space discretizations. There is related literature on closure modeling [55, 18, 55, 36] where methods have been developed that construct bases with respect to different norms than the classical $L_2$-optimal POD bases [25, 45]. Then, there are approaches that exploit local structure, such as the approach introduced in [51] that builds on the spatial locality of shock fronts. The work by Carlberg [11] builds on a special type of adaptation that enriches the reduced space during the online phase if an error indicator signals a high error. The dimension of the reduced space grows with each adaptation step, which means that the reduced model becomes computationally more expensive to solve over time. In [21], reduced bases are adapted over time via an auxiliary equation that describes the dynamics of the bases. Under certain conditions, the approach can be seen as an approximation of Lax pairs discussed in [20]. Dynamic
low-rank approximation [27, 38, 32, 31, 32] adapts bases in time similar to our approach; however, the bases in dynamic low-rank approximation approaches are constructed so that they are valid for all parameters in a given parameter domain, which is problematic if the solution manifold contains high-dimensional features because the propagating coherent structure depends on parameters, as in, e.g., [51, Example 2.5]. In contrast, our approach is local in the parameter domain, additionally to being local in time and space. Furthermore, locality in the parameter domain allows our approach to derive basis updates from only few samples of the full model.

This work is organized as follows. Section 2 sets up the problem and gives the problem formulation. Section 3 demonstrates the local structure in transport-dominated problems and proposes an approach to adaptively sample the full model to construct basis updates. Numerical results in Section 4 on benchmark examples and a model combustor of a single-element rocket engine demonstrate that our approach achieves significant speedups and is applicable to a wide range of problems. In particular, the numerical results indicate that our approach faithfully approximates interactions between propagating coherent structures traveling at different speeds. Concluding remarks are provided in Section 5.

2. Preliminaries

We briefly discusses model reduction with empirical interpolation in Section 2.1 and then demonstrate on a toy example in Section 2.2 why these traditional model reduction methods fail for problems exhibiting transport-dominated phenomena.

2.1. Model reduction with empirical interpolation

Consider the system of discretized equations that is obtained after discretizing a partial differential equation (PDE) in space and time

\[ q_k(\mu) = f(q_{k+1}(\mu), \mu), \quad k = 1, \ldots, K, \]

where \( q_k(\mu) \in \mathbb{R}^N \) is the \( N \)-dimensional state at time \( k \) and parameter \( \mu \in \mathcal{D} \), with parameter domain \( \mathcal{D} \). The number of time steps is \( K \in \mathbb{N} \). The function \( f : \mathbb{R}^N \to \mathbb{R}^N \) describes the operators of the discretized PDE and typically is nonlinear in the state \( q_{k+1}(\mu) \). The time discretization is implicit in time, which means that at each time step \( k = 1, \ldots, K \), a potentially nonlinear, large-scale system of equations has to be solved, e.g., with Newton’s method.

To derive a reduced model of the full model [11] with empirical interpolation [5, 13, 15], consider the trajectory at parameter \( \mu \in \mathcal{D} \)

\[ Q(\mu) = [q_1(\mu), \ldots, q_K(\mu)] \in \mathbb{R}^{N \times K}, \]

which is the matrix with the states \( q_1(\mu), \ldots, q_K(\mu) \) as columns. Let the columns of \( U = [u_1, \ldots, u_n] \in \mathbb{R}^{N \times n} \) be the POD basis of dimension \( n \ll N \) obtained from the snapshot matrix

\[ Q = [Q(\mu_1), \ldots, Q(\mu_M)] \in \mathbb{R}^{N \times MK}, \]

with parameters \( \mu_1, \ldots, \mu_M \in \mathcal{D} \). The space spanned by the columns of \( U \) is denoted as \( \mathcal{U} \subset \mathbb{R}^N \) and is a subspace of \( \mathbb{R}^N \). The critical assumption of traditional model reduction is that the singular values of \( Q \) decay fast so that only few basis vectors are necessary to approximate well the columns of \( Q \) in the corresponding space \( \mathcal{U} \). Following QDEIM, introduced in [15], select the interpolation points \( p_1, \ldots, p_n \in \{1, \ldots, N\} \) and define the corresponding interpolation points matrix \( \tilde{P} = [e_{p_1}, \ldots, e_{p_n}] \in \mathbb{R}^{N \times n} \), where \( e_{p_i} \in \mathbb{R}^N \) is the \( p_i \)-th canonical unit vector with entry 1 at the \( p_i \)-th component and entry 0 at all other components. Note that all of the following discussion can be extended in a straightforward way to oversampled empirical interpolation (ODEIM) [38]. Define

\[ \hat{f}(q(\mu); \mu) = (P^T U)^{-1} P^T f(q(\mu); \mu), \]
so that $U\tilde{f}(q(\mu); \mu)$ is the DEIM approximation of $f(q(\mu); \mu)$ at state $q(\mu) \in \mathbb{R}^N$ and parameter $\mu \in \mathcal{D}$. Note that computing $P^T f(q(\mu); \mu)$ typically requires evaluating $f(q(\mu); \mu)$ at the $n$ interpolation points $p_1, \ldots, p_n$ only, see [5, 13, 15]. We overload the notation $\tilde{f}$ in the following so that if we have a reduced state $\tilde{q}(\mu) \in \mathbb{R}^n$, then $\tilde{f}(\tilde{q}(\mu); \mu) = (P^T U)^{-1} P^T f(U\tilde{q}(\mu); \mu)$. The reduced model corresponding to $\tilde{f}$ is

$$\tilde{q}_k(\mu) = \tilde{f}(\tilde{q}_{k+1}(\mu); \mu), \quad k = 1, \ldots, K, \quad (2)$$

with the reduced trajectory $\tilde{Q}(\mu) = [\tilde{q}_1(\mu), \ldots, \tilde{q}_K(\mu)] \in \mathbb{R}^{n \times K}$. Note that we approximate the state and the nonlinear function in the same space $\mathcal{U}$, which is in contrast to the original use of DEIM in [13] and similar to model reduction via missing point estimation [3]. Once a reduced model (2) is constructed in the offline phase, it is solved in the online phase. The one-time high costs of constructing the reduced model are compensated by approximating the full-model solutions with reduced-model solutions for a large number of parameters online, see, e.g., [46, 2, 6, 40] for details on the wide range of outer-loop and many-query applications where such an offline/online splitting is beneficial.

### 2.2. Problem formulation

It has been observed that states of problems with transport-dominated behavior can require DEIM (reduced) spaces $\mathcal{U}$ with high dimensions, see, e.g., [34, 14, 51]. The following toy example illustrates this by demonstrating the slow decay of the singular values of the snapshot matrix of solutions of the advection equation. Let $\Omega = [-1, 1] \subset \mathbb{R}$ be the spatial domain and consider the advection equation

$$\partial_t q(x,t) + \mu \partial_x q(x,t) = 0, \quad x \in \Omega, \quad (3)$$

with periodic boundary conditions $q(1,t) = q(-1,t)$ and time $t \in [0, \infty)$. Set the initial condition to

$$q(x,0) = \frac{1}{\sqrt{\pi 0.02}} \exp \left( - \frac{x^2}{0.0002} \right),$$

which is the probability density function of a normal random variable with standard deviation 0.01 and mean 0, see Figure 1a. Discretize (3) with a second-order upwind scheme and $N = 8192$ inner grid points in the spatial domain and time step size $\delta t = 10^{-6}$ and end time $T = 0.08$. The singular values of the trajectory $Q(\mu)$ are plotted in Figure 1b for $\mu = 10$. According to the decay of the singular values, a DEIM space of more than $n = 200$ dimensions is necessary to approximate the trajectory $Q(\mu)$ with a projection error below.
Figure 2: Advection equation: The plot in (a) shows that the singular values of a local trajectory decay orders of magnitude faster than the singular values of a trajectory that is global in time in this example, which we exploit via online adaptive basis updates. Plot (b) indicates that the squared residual of DEIM approximations of states of transport-dominated problems decays rapidly. We will show that this fast decay of the residual means that basis updates can be derived from only few components—samples—of the residual.

10^{-15} in the Euclidean norm, which is a rather slow decay compared to the fast decay observed in many other problems [6]. This example demonstrates that efficient reduced models of transport-dominated problems will have to exploit structure beyond the classical low-rank structure that traditional model reduction methods rely on.

3. Exploiting local structure via online adaptive basis updates

We propose AADEIM (adaptive bases and adaptive sampling) to exploit local structure to construct online adaptive reduced models of full models that exhibit transport-dominated behavior. We focus on two types of local structure: Local low-rankness that we exploit via online adaptive bases and local coherence that enables updating the bases computationally efficient via few samples from the full model. Section 3.1 describes local low-rankness and local coherence in more detail. Section 3.2 discusses the adaptation of the DEIM basis to exploit local low-rank structure and Section 3.3 shows that only few samples are necessary to derive basis updates if the adapted DEIM spaces are locally coherent. Algorithm 1 in Section 3.4 summarizes our approach and provides practical considerations.

In most of this section, we drop the dependence on the parameter $\mu$ of the function $f$, the state $q_k$ at time step $k$, and the trajectory $Q$, as well as their reduced counterparts. Parametrization of our approach AADEIM is discussed in Section 3.4.

3.1. Local structure in transport-dominated problems

Consider the toy example introduced in Section 2.2. Let $w \in \mathbb{N}$ be a window size and consider the local trajectory $Q_k = [q_{k-w+1}, \ldots, q_k] \in \mathbb{R}^{N \times w}$ at time step $k$, which consists of the $w$ states from time steps $k-w+1$ to time step $k$. Figure 2 compares the singular values of the local trajectory $Q_k$ to the singular values of the global trajectory $Q$ for $w = 500$ and $k = 1$. The results indicate that the singular values of the local trajectory $Q_k$ decay orders of magnitude faster than the singular values of the global trajectory $Q$. We call this behavior—that the singular values of local trajectories decay fast while the singular values of the global trajectory decay slowly—local low-rank structure. By adapting the DEIM spaces, we will exploit this local low-rank structure in Section 3.2. Similar observations about local low-rank structure are exploited in
In Appendix A we analyze analytically an example to demonstrate its low-rank structure.

Let us now consider the local DEIM space $U_k$ of dimension $n = 3$ and the corresponding local interpolation points matrix $P_k$ obtained from the local trajectory $Q_k$. The residual of approximating the state $q_k$ at time step $k = 500$ with DEIM in $U_k$ is

$$r_k = q_k - U_k (P_k^T U_k)^{-1} P_k^T q_k.$$  

Figure 2b shows the decay of the sorted squared components of $r_k$. The decay is fast, which means that there is a high residual locally only, i.e., the residual at a few components dominates whereas at most of the $N = 8192$ components of $r_k$ the residual is low. Intuitively, such a fast decay of the residual means that the basis matrix $U_k$ of the local DEIM space needs to be corrected at a few components only. We will show that such a fast decay of the residual can be the result of a local coherence structure of the local DEIM spaces, which we will make precise and will exploit with adaptive sampling in Section 3.3.

3.2. Exploiting local low-rank structure: Basis updates

To exploit local low-rank structure as described in Section 3.1 we adapt the DEIM spaces and the DEIM interpolation points during the time steps $k = 1, \ldots, K$ in the online phase. The adaptation is initialized with the DEIM basis matrix $U_1$ and interpolation points matrix $P_1$ at time step $k = 1$. Then, at each time step $k = 1, \ldots, K$, the DEIM basis matrix $U_k$ is adapted via an additive low-rank update to $U_{k+1}$. The update is based on ADEIM [39]. The interpolation points matrix $P_{k+1}$ is derived with QDEIM from the adapted basis matrix $U_{k+1}$. In the following description, the DEIM interpolant is adapted at each time step $k = 1, \ldots, K$ for ease of exposition, even though all of the following directly applies to situations where the adaptation is performed at selected time steps only, e.g., every other time step or via a criterion that decides adaptively when to update the DEIM interpolant.

3.2.1. Adaptation with ADEIM

Let $k$ be the current time step and $U_k$ and $P_k$ the DEIM basis matrix and the DEIM interpolation points matrix, respectively. Consider the matrix $F_k \in \mathbb{R}^{N \times w}$ and the coefficient matrix $C_k = (P_k^T U_k)^{-1} P_k^T F_k$. The residual of the DEIM approximation of the columns of $F_k$ is $R_k = U_k C_k - F_k$. Let now $S_k = [e_{s_1}, \ldots, e_{s_m}] \in \mathbb{R}^{N \times m}$ be the sampling points matrix corresponding to the $s_1, \ldots, s_m \in \{1, \ldots, N\}$ samplings points with $m > n$. ADEIM [39] adapts the DEIM basis matrix $U_k$ to $U_{k+1}$ with a low-rank update $\alpha_k \beta_k^T \in \mathbb{R}^{N \times n}$

$$U_{k+1} = U_k + \alpha_k \beta_k^T,$$

with $\alpha_k \in \mathbb{R}^{N \times r}, \beta_k \in \mathbb{R}^{w \times r}$ with rank $r \in \mathbb{N}$. The ADEIM update $\alpha_k \beta_k^T$ is the rank-$r$ matrix that minimizes the residual $U_{k+1} C_k - F_k$ at the sampling points $S_k$ in the Frobenius norm, which means that the ADEIM update $\alpha_k \beta_k^T$ minimizes

$$\| S_k^T ((U_k + \alpha_k \beta_k^T) C_k - F_k) \|_F^2,$$

see [39] for details on how to compute $\alpha_k \beta_k^T$ and the computational costs of computing the update.

Critical for the adaptation is the matrix $F_k$, because ADEIM adapts the space $U_k$ such that the residual of approximating the columns of $F_k$ is minimized at the sampling points. A potentially good choice for the columns of $F_k$ would be the states $q_{k-w+1}, \ldots, q_k$ of the full model at time steps $k - w + 1, \ldots, k$; however, the states of the full model are unavailable because their availability would mean the full model has been solved. Instead, we set the columns of $F_k$ as follows. Let $S_k \in \mathbb{R}^{N \times (N - m)}$ be the complementary sampling points matrix derived from the points $\{1, \ldots, N\} \setminus \{s_1, \ldots, s_m\}$ that have not been selected as sampling points. Let further $\hat{q}_k \in \mathbb{R}_m^N$ be the vector with

$$S_k^T \hat{q}_k = S_k^T f(U_k \hat{q}_k), \quad \tilde{S}_k^T \hat{q}_k = S_k^T U_k (S_k^T U_k) + S_k^T f(U_k \hat{q}_k),$$  

(4)
which means that the components of $\hat{q}_k$ corresponding to the sampling points in $S_k$ are equal to the components of $f(U_k\hat{q}_k)$ and all other components are approximated via DEIM given by $U_k$ and $S_k$. Note that $m > n$ and therefore the Moore-Penrose inverse $(S_k^T U_k)^+$ is used in (6), instead of the inverse when $U_k$ and $P_k$ are used. Note further that $\hat{q}_k$ is the reduced state at time $k$ and that $f$ is the function that defines the full model (1). The matrix $F_k$ that we use in the following for adaptation is

$$F_k = [\hat{q}_{k-w+1}, \ldots, \hat{q}_k].$$

The vectors $\hat{q}_{k-w+1}, \ldots, \hat{q}_k$ serve as surrogates of the full-model states $q_{k-w}, \ldots, q_{k-1}$ to which the DEIM space is adapted, because the full-model states $q_{k+1}$ satisfy $q_k = f(q_{k+1})$ for $k = 1, \ldots, K$, see equation (1).

Computing a vector $\hat{q}_k$ requires evaluating the full-model function $f$ at the sampling points $S_k$.

### 3.2.2. Analysis of the error of the adapted space

We now provide an analysis of the ADEIM adaptation. Let $\mathcal{U}$ and $\mathcal{U}$ be two $n$-dimensional subspaces of $\mathbb{R}^N$. We measure the distance between $\mathcal{U}$ and $\mathcal{U}$ as

$$d(\mathcal{U}, \mathcal{U}) = \|U - UU^T U\|_F^2,$$

where $U$ and $\bar{U}$ are orthonormal basis matrices of $\mathcal{U}$ and $\mathcal{U}$, respectively. The distance $d(\mathcal{U}, \mathcal{U})$ is symmetric and invariant under orthogonal basis transformations, which is true because $d(\mathcal{U}, \mathcal{U}) = n - \|U^T U\|_F^2$ holds. The $n$-dimensional subspace of $\mathbb{R}^N$ to which we want to adapt at iteration $k$ is denoted as $\mathcal{U}_{k+1}$ and the adapted space is $\mathcal{U}_{k+1}$. The following lemma quantifies the reduction of the residual after an ADEIM update and establishes Proposition 1 that bounds the error $d(\mathcal{U}_{k+1}, \mathcal{U}_{k+1})$ of the adapted space $\mathcal{U}_{k+1}$ with respect to the space $\mathcal{U}_{k+1}$.

**Lemma 1.** Let $C_k = (P_k^T U_k)^{-1} P_k^T F_k$ be the coefficient matrix and $R_k = U_k C_k - F_k$ the residual matrix. Let further $S_k$ be the sampling points matrix and $U_{k+1} = U_k + \alpha_k \beta_k^T$ the adapted basis matrix of the adapted space $\mathcal{U}_{k+1}$. Let $\bar{r}$ be the rank of $S_k^T R_k$ and let $r \in \mathbb{N}$ with $r \leq \bar{r}$ be the rank of the update $\alpha_k \beta_k^T$. Then,

$$\|S_k^T (U_{k+1} C_k - F_k)\|_F^2 \leq \|S_k^T R_k\|_F^2 - \sum_{i=1}^r \sigma_i^2,$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the singular values of $S_k^T R_k$.

**Proof.** Follows from [39, Lemma 3.5] by transforming the generalized symmetric positive definite eigenproblem into the symmetric eigenproblem with matrix $(S_k^T R_k)^T (S_k^T R_k)$, which then is equivalent to computing the singular value decomposition of $S_k^T R_k$ so that the squared singular values of $S_k^T R_k$ are equal to the eigenvalues of the generalized eigenproblem. This shows this lemma by using the last identity of [39, proof of Lemma 3.5].

**Proposition 1.** Consider the same setup as in Lemma 1 and additionally assume that $F_k$ has rank $n$ and its columns are in the $n$-dimensional space $\mathcal{U}_{k+1}$. Let further $\mathcal{U}_{k+1}$ be the adapted space derived with the rank-$r$ ADEIM update and sampling points matrix $S_k$. Then, it holds

$$d(\mathcal{U}_{k+1}, \mathcal{U}_{k+1}) \leq \frac{\rho_k^2}{\sigma_{\min}(F_k)}$$

where $\hat{S}_k$ denotes the complementary sampling matrix of $S_k$ and $\sigma_{\min}(F_k)$ is the minimal singular value of $F_k$.

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exists a full-rank matrix \( \tilde{S}_k \) and therefore the smallest singular value where we used \( \sigma \) definition of \( d \) therefore it might be necessary to consider the oblique projection from Lemma 1. Equation (11) holds because of (8). Equation (12) uses \( \rho \). Consider now the norm of the residual with respect to the adapted space \( U_{k+1} \)

\[
\|U_{k+1}C_k - F_k\|_F^2 = \|S_k^T(U_{k+1}C_k - F_k)\|_F^2 + \|\tilde{S}_k^T(U_{k+1}C_k - F_k)\|_F^2.
\]

Equation (13) follows because \( \tilde{S}_k \) is the complementary sampling points matrix of \( S_k \). Equation (11) follows from (10). Equation (12) holds because of (8). Equation (12) uses \( R_k = U_kC_k - F_k \) and that \( \sum_{i=1}^r \sigma_i^2 = \|S_k^TR_k\|_F^2 \). Consider now the projection of the columns of \( F_k \) onto \( U_{k+1} \) and observe that

\[
\|F_k - U_{k+1}U_{k+1}^TF_k\|_F^2 \leq \|U_{k+1}C_k - F_k\|_F^2.
\]

Note that the basis matrix \( U_{k+1} \) obtained with the ADEIM update is not necessarily orthonormal and therefore it might be necessary to consider the oblique projection \( U_{k+1}(U_{k+1}^TU_{k+1})^{-1}U_{k+1}^TF_k \) onto the column span \( U_{k+1} \) of \( U_{k+1} \) in (13); however, the projection error in the Frobenius norm is the same for both projections. With (12) and the definition of \( \rho_k^2 \) follows

\[
\|F_k - U_{k+1}U_{k+1}^TF_k\|_F^2 \leq \rho_k^2.
\]

Let now \( \bar{U}_{k+1} \) be an orthonormal basis matrix of \( \bar{U}_{k+1} \). Since \( \bar{U}_{k+1} \) is spanned by the columns of \( F_k \), there exists a full-rank matrix \( \bar{F}_k \in \mathbb{R}^{n \times w} \) such that \( F_k = \bar{U}_{k+1}\bar{F}_k \). We obtain

\[
\|F_k - U_{k+1}U_{k+1}^TF_k\|_F^2 = \|\bar{U}_{k+1} - U_{k+1}U_{k+1}^TU_{k+1}\|_F^2 \approx \|\bar{U}_{k+1} - U_{k+1}U_{k+1}^TU_{k+1}\|_2^2 \sigma_{\text{min}}^2(F_k) \geq \|\bar{U}_{k+1} - U_{k+1}U_{k+1}^TU_{k+1}\|_F^2 \sigma_{\text{min}}^2(F_k). \quad (15)
\]

where we used \( \sigma_{\text{min}}(\bar{F}_k) = \sigma_{\text{min}}(F_k) \), which holds because \( \bar{U}_{k+1} \) is orthonormal. The matrix \( F_k \) has rank \( n \) and therefore the smallest singular value \( \sigma_{\text{min}}(F_k) > 0 \) is positive. Combining (15) with (14) and the definition of \( d(\bar{U}_{k+1}, U_{k+1}) \) shows (8).

\( \square \)

3.3. Exploiting local coherence: Adaptive sampling

Proposition 1 shows that the choice of the sampling points \( S_k \) influences the bound of the error \( d(\bar{U}_{k+1}, U_{k+1}) \) of the adapted space \( \bar{U}_{k+1} \). In this section, we derive an adaptive sampling strategy that minimizes the upper bound derived in Proposition 1 in case of full-rank ADEIM updates. Then, we show that if our adaptive sampling strategy is used, the error \( d(\bar{U}_{k+1}, U_{k+1}) \) decays at least as fast with the number of sampling points \( m \) as the norm of the DEIM residual, which in turn means that only few sampling points are necessary to adapt the basis if the residual decays fast.
3.3.1. Adaptive sampling strategy based on residual

Following Proposition 1 and the decay factor $\rho_k$ defined in \( \| \hat{S}_k^T R_k \|_F^2 \), we select the sampling points so that $\| \hat{S}_k^T R_k \|_F^2$ is minimized. Note that $\sigma_{\text{min}}(F_k)$ in \( \| \) is independent of the sampling points $\mathbf{S}_k$ and therefore it is sufficient to derive sampling points that lead to a small decay factor $\rho_k$.

Consider the component-wise residual

$$ r_i = \| R_k^T e_i \|_2^2 $$

and let $j_1, \ldots, j_N$ be an ordering of $1, \ldots, N$ such that $r_{j_1} \geq r_{j_2} \geq \cdots \geq r_{j_N}$. Then, select the first $j_1, \ldots, j_m$ components as sampling points and form the corresponding sampling points matrix

$$ \mathbf{S}_k = [e_{j_1}, \ldots, e_{j_m}] . $$

If a full-rank ADEIM update is applied and $k$ is minimized. Note that $\rho_k$ is minimized. We now show that a fast decay in the residual implies a fast decay in the error bound of $d(\mathcal{U}_{k+1}, \mathcal{U}_{k+1})$.

**Proposition 2.** Consider the same setup as in Proposition 1. Let $j_1, \ldots, j_N$ be an ordering of $1, \ldots, N$ such that the component-wise residual defined in (16) decays as

$$ r_{j_i} \leq c_1 e^{-c_2 j_i} , $$

with rate $a > 1$ and constants $c_1, c_2 > 0$. The error of the adapted space $\mathcal{U}_{k+1}$ is bounded as

$$ d(\mathcal{U}_{k+1}, \mathcal{U}_{k+1}) \leq c_3 e^{-c_2 m^a} , $$

if a full-rank ADEIM update is applied and $m$ sampling points are selected via the adaptive sampling (17).

The constant $c_3 = c_1 / ((1 - e^{-c_2}) \sigma_{\text{min}}^2(F_k))$ is independent of $m$. In particular, setting the number of sampling points to

$$ m \geq \min \left\{ N, \left( -\frac{1}{c_2} \log \left( \frac{\epsilon}{c_3} \right) \right)^{1/a} \right\} $$

guarantees $d(\mathcal{U}_{k+1}, \mathcal{U}_{k+1}) \leq \epsilon$ for a threshold $\epsilon > 0$.

**Proof.** In case of a full-rank update, the factor $\rho_k^2$ in (6) is

$$ \rho_k^2 = \| \hat{S}_k^T R_k \|_F^2 , $$

as shown in Proposition 1. Then, we obtain with (18) and the adaptive sampling (17)

$$ \| \hat{S}_k^T R_k \|_F^2 = \sum_{i=m+1}^N \| R_k^T e_i \|_2^2 \leq c_1 \sum_{i=m+1}^N e^{-c_2 i^a} \leq c_1 \sum_{i=m+1}^\infty e^{-c_2 i^a} \leq c_1 \sum_{i=0}^\infty e^{-c_2 (i+(m+1))^a} \leq c_1 e^{-c_2 (m+1)^a} \sum_{i=0}^\infty e^{-c_2 i^a} , $$

where the last inequality holds because $i \geq 0, a > 1$ and thus $(i+(m+1))^a \geq i^a + (m+1)^a$. Using that $a > 1$, we obtain $i^a \geq i$ and $(m+1)^a \geq m^a$ and thus

$$ \| \hat{S}_k^T R_k \|_F^2 \leq c_1 e^{-c_2 (m+1)^a} \sum_{i=0}^\infty e^{-c_2 i^a} \leq c_1 e^{-c_2 m^a} \sum_{i=0}^\infty e^{-c_2 i^a} = c_1 e^{-c_2 m^a} \frac{1}{1 - e^{-c_2}} . $$

Set $c_3 = c_1 / ((1 - e^{-c_2}) \sigma_{\text{min}}^2(F_k))$ to obtain (19) with Proposition 1. Setting $m$ as in (20) shows $d(\mathcal{U}_{k+1}, \mathcal{U}_{k+1}) \leq \epsilon$. 

\( \square \)
3.3.2. Local coherence

Following, e.g., [9], define the coherence of a space $\mathcal{U}$ as

$$\gamma(\mathcal{U}) = \frac{N}{n} \max_{i=1,\ldots,N} \|U^T e_i\|_2^2,$$

where $U$ is an orthonormal basis of $\mathcal{U}$ and $e_i$ is the $i$-th canonical unit vector. Define further the local coherence as

$$\gamma_i(\mathcal{U}) = \frac{N}{n} \|U^T e_i\|_2^2,$$

for $i=1,\ldots,N$, see, e.g., [28, 60]. We refer to [10, 9, 28, 60] for details.

We now show that the rate of the decay of the local coherence of the current DEIM space $\mathcal{U}_k$ and of the space $\mathcal{U}_{k+1}$ to which we want to adapt is reflected in the decay of the bound of the error $d(\mathcal{U}_{k+1}, \mathcal{U}_{k+1})$ of the adapted space $\mathcal{U}_{k+1}$ with respect to the number of sampling points $m$. Thus, we now relate the decay of the error $d(\mathcal{U}_{k+1}, \mathcal{U}_{k+1})$ of the adapted space to a property of the spaces $\mathcal{U}_k$ and $\mathcal{U}_{k+1}$, namely their decay of the local coherence.

**Lemma 2.** Let $j_1,\ldots,j_N$ be an ordering of $\{1,\ldots,N\}$ such that

$$\gamma_{j_i}(\mathcal{U}_k) \leq c_4 \exp(-c_5 a) \quad \text{and} \quad \gamma_{j_i}(\mathcal{U}_{k+1}) \leq \bar{c}_4 \exp(-\bar{c}_5 \bar{a})$$

(22)

holds for $i=1,\ldots,N$, with $c_4, \bar{c}_4, c_5, \bar{c}_5 > 0$ and $a, \bar{a} > 1$. Let the columns of $F_k$ be in $\mathcal{U}_{k+1}$. Define $C_k = (P_k^T U_k)^{-1} P_k^T F_k$, then for the residual $R_k = U_k C_k - F_k$ holds

$$\|R_k^T e_{j_i}\|_2^2 \leq c_7 \|F_k\|_2^2 e^{-\min(c_5,\bar{c}_5)^{\min(a,\bar{a})}}, \quad i = 1, \ldots, N,$$

(23)

with a constant $c_7 > 0$ that is independent of $a, \bar{a}, c_5, \bar{c}_5$.

**Proof.** Denote with $U_{k+1}^{(i)}$, $U_{k+1}^{(j)}$, and $F_k^{(i)}$ the $i$-th row of $U$, $U_{k+1}$, and $F_k$, respectively. The matrix $U_{k+1}$ is an orthonormal basis matrix of $\mathcal{U}_{k+1}$. Let further $\tilde{F}_k \in \mathbb{R}^{n \times \omega}$ be a matrix such that $F_k = \tilde{U}_{k+1} \tilde{F}_k$. Note that $\|F_k\|_2 = \|\tilde{F}_k\|_2$, because $\tilde{U}_{k+1}$ is orthonormal. Then, with $C_k = (P_k^T U_k)^{-1} P_k^T F_k$, follows

$$\|R_k^T e_{j_i}\|_2^2 = \|F_k^{(j_i)} - U_k^{(j_i)} C_k\|_2^2$$

$$= \|F_k^{(j_i)}\|_2^2 - 2 F_k^{(j_i)} (U_k^{(j_i)} C_k)^T + \|U_k^{(j_i)} C_k\|_2^2$$

$$\leq \|F_k^{(j_i)}\|_2^2 + 2 \|F_k^{(j_i)}\|_2 \|U_k^{(j_i)}\|_2 \|C_k\|_2 + \|U_k^{(j_i)}\|_2^2 \|C_k\|_2^2.$$

Now make the following estimate

$$\|F_k^{(j_i)}\|_2^2 \leq \|\tilde{U}_{k+1}^{(j_i)}\|_2^2 \|\tilde{F}_k\|_2 = \|\tilde{U}_{k+1}^{(j_i)}\|_2^2 \|\tilde{F}_k\|_2,$$

because $\|F_k\|_2 = \|\tilde{F}_k\|_2$. Further, we have

$$\|C_k\|_2 \leq \|(P_k^T U_k)^{-1}\|_2 \|F_k\|_2.$$

Set now $\bar{c}_6 = n/N c_4$ and $c_6 = n/N c_4 \|(P_k^T U_k)^{-1}\|_2^2$ to bound

$$\|R_k^T e_{j_i}\|_2^2 \leq c_7 \|F_k\|_2^2 \left( \bar{c}_6 e^{-\bar{c}_5 \bar{a}} + 2 \sqrt{\bar{c}_6} c_6 e^{-\frac{\bar{c}_5}{2} a} + \sqrt{c_6} e^{-c_5 a} \right).$$

Now set $c_7 = c_6 + \bar{c}_6 + 2 \sqrt{c_6 \bar{c}_6}$ to obtain

$$\|R_k^T e_{j_i}\|_2^2 \leq c_7 \|F_k\|_2^2 e^{-\min(c_5,\bar{c}_5)^{\min(a,\bar{a})}},$$

which shows the proposition. \(\square\)

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Since we consider subspaces of finite-dimensional spaces only, i.e., $N$ is finite, the bounds in (22) hold for any subspace by increasing the constants and choosing the rate close to 1. However, Lemma 2 still is meaningful because it shows that the constants and rates that appear in (22) are obtained in the bound of the residual in (23) as well. Thus, the local coherence structure is directly reflected in the decay of the DEIM residual.

The following proposition combines the local coherence structure exploited in Lemma 2 and the adaptive sampling of Proposition 2 to derive the number of sampling points $m$ that are required to achieve $d(\tilde{U}_{k+1}, \tilde{U}_{k+1}) \leq \epsilon$ in probability.

**Proposition 3.** Assume that Lemma 2 applies and consider the same setting as in Proposition 1 except that $F_k = \tilde{U}_{k+1} F_k$ with $\tilde{F}_k$ being an $n \times n$ matrix with independent and identically distributed standard Gaussian entries. Then, $d(\tilde{U}_{k+1}, \tilde{U}_{k+1}) \leq \epsilon$ with probability at least $1 - \delta$ if a full-rank update is applied and

$$m \geq \min \left\{ N, \left( \frac{1}{\min\{c_5, \bar{c}_5\}} \log \left( \frac{c_\delta^2}{c_8} \right) \right)^{1/\min\{n, \bar{a}\}} \right\},$$

(24)

where $c_8$ is a constant independent of $m, a$, and $\bar{a}$.

**Proof.** Since Lemma 2 applies, we have with (21) that

$$\| \bar{S}_k^T R_k \|_F^2 \leq \frac{c_\gamma^2 \|F_k\|^2}{1 - e^{-\min\{c_5, \bar{c}_5\}}} e^{-\min\{c_5, \bar{c}_5\} m^{\min\{n, a\}}}.$$

With Proposition 1 and because a full-rank update is applied, follows

$$d(\tilde{U}_{k+1}, \tilde{U}_{k+1}) \leq \frac{c_\gamma \kappa^2(\tilde{F}_k)}{1 - e^{-\min\{c_5, \bar{c}_5\}}} e^{-\min\{c_5, \bar{c}_5\} m^{\min\{n, a\}}} ,$$

where $\kappa(F_k) = \sigma_{\max}(F_k)/\sigma_{\min}(F_k)$ is the spectral condition number of $F_k$. Since $F_k = \tilde{U}_{k+1} \tilde{F}_k$ with $\tilde{U}_{k+1}$ orthonormal, we have $\kappa(\tilde{F}_k) = \kappa(\tilde{F}_k)$. We now bound the spectral condition number of $\tilde{F}_k$ with high probability by exploiting that $\tilde{F}_k$ has i.i.d. standard Gaussian entries. The work [17, Theorem 1.1] and [4, Theorem 1.1] show that the spectral condition number of $\tilde{F}_k$ is bounded in probability as

$$P[\kappa(\tilde{F}_k) \geq \delta] \leq c\delta^{-1},$$

for $\delta > 0$ and with a positive constant $c$ that depends $n$. Thus, we have $P[\kappa(\tilde{F}_k) \leq c\delta^{-1}] \geq 1 - \delta$ and $P[\kappa^2(\tilde{F}_k) \leq c^2\delta^{-2}] \geq 1 - \delta$, which leads to

$$d(\tilde{U}_{k+1}, \tilde{U}_{k+1}) \leq \frac{c_8}{\delta^2} e^{-\min\{c_5, \bar{c}_5\} m^{\min\{n, a\}}},$$

with probability at least $1 - \delta$ and $c_8 = (c_\gamma c^2)/(1 - e^{-\min\{c_5, \bar{c}_5\}})$. Setting $m$ as in (24) leads to $d(\tilde{U}_{k+1}, \tilde{U}_{k+1}) \leq \epsilon$ with probability $1 - \delta$.

**3.4. Practical considerations and algorithm**

We now provide details on the practical implementation of the AADEIM approach and summarize AADEIM in Algorithm 1.

**3.4.1. Practical considerations**

The AADEIM model is initialized with a DEIM interpolant with basis matrix $U_1$ and $P_1$. We propose to construct $U_1$ and $P_1$ from a local trajectory computed with the full model. This means that the full model is solved for $w_{\text{init}} \in \mathbb{N}$ time steps, with $w_{\text{init}} \ll K$, and the initial DEIM interpolant $(U_1, P_1)$ is constructed.
from the corresponding full-model states. Initializing the AADEIM model with a DEIM interpolant obtained
from full-model states has two benefits. First, no offline phase is necessary to initialize the AADEIM model,
which means that it is unnecessary to develop (e.g., greedy) strategies to build an initial AADEIM model
from full-model states. Initializing the AADEIM model with a DEIM interpolant obtained
Algorithm 1 Adaptive bases and adaptive sampling (AADEIM)

1: procedure AADEIM(q₀, f, μ, n, wᵢₘᵢₜ, w, m, z, r)
2:     Solve full model for wᵢₘᵢₜ time steps Q = solveFOM(q₀, f, μ)
3:     Set k = wᵢₘᵢₜ + 1
4:     Compute n-dimensional POD basis U_k of Q
5:     Compute QDEIM interpolation points p_k = qdeim(U_k) ▷ see appendix
6:     Initialize F = Q[:, k - w + 1 : k - 1] and ̂q_k₋₁ = U_kᵀ Q[:, k - 1]
7:     for k = wᵢₘᵢₜ + 1, . . . , K do
8:         Solve ̂q_k₋₁ = f(̂q_k; μ) with DEIM interpolant (U_k, P_k)
9:         Store Q[:, k] = U_k ̂q_k
10:     if mod(k, z) == 0 | k == wᵢₘᵢₜ + 1 then
11:         Compute F[:, k] = f(Q[:, k]; μ)
12:         R_k = F[:, k - w + 1 : k] − U_k(U_k(p_k[:, i]))⁻¹F[p_k, k - w + 1 : k]
13:         [~, s_k] = sort(sum(R_k, 2), 'descend')
14:         Set ̂s_k = s_k[m + 1 : end] and s_k = s_k[1 : m]
15:     else
16:         Set s_k = s_k₋₁ and ̂s_k = ̂s_k₋₁
17:         Compute F[s_k, k] = f(Q[s_k, k]; μ)
18:         Approximate F[̂s_k, k] = U_k[̂s_k, :](U_k[s_k, :])U_k[̂s_k, :]+F[s_k, k]
19:     end if
20:     Set current window F_k = F[:, k - w + 1 : k]
21:     Adapt [U_k₊₁, p_k₊₁] = adeim(U_k, P_k, F_k[p_k[:, i]], F_k[s_k, :], r) ▷ see appendix
22: end for
23: return Return trajectory Q
24: end procedure

The adaptive sampling strategy as described in Section 3.3.1 requires the residual R_k at all N components.
To avoid computing the residual R_k at all components at each adaptation iteration, we adapt the sampling
points at every z-th iteration instead. Thus, only at every z-th iteration, the residual R_k is computed at all
components to adapt the sampling points, whereas at all other iterations, the residual is computed only
at the sampling points. Note that even if the sampling points are adapted at each iteration k = 1, . . . , K,
and so the residual is computed at each component, it still leads to a lower runtime to compute the ADEIM
update only at m < N sampling points, because the costs of computing the ADEIM update scales linearly
in the number of sampling points m [39].

3.4.2. Algorithm and costs

Algorithm 1 gives details on our AADEIM approach by summarizing time stepping of a reduced model that
uses our adaptive basis updates and adaptive sampling. Helper functions used in Algorithm 1 are given
in Appendix B. Inputs of the algorithm are the initial condition q₀ ∈ ℝᴺ, the full-model function f that
describes the underlying dynamical system, and the parameter μ ∈ D. Parameters of our approach are the
dimension n of the reduced space, the time step wᵢₘᵢₜ until which the full model is solved to initialize the
reduced model, the window size \( w \) for the adaptation, the number of sampling points \( m \), the frequency \( z \) of the sampling points adaptation, and the rank of the update \( r \).

Line 2 solves the full model until time step \( w_{\text{init}} \) to compute the corresponding trajectory \( Q_{w_{\text{init}}} = [q_1, \ldots, q_{w_{\text{init}}}] \in \mathbb{R}^{N \times w_{\text{init}}} \). Lines 3-6 initialize the reduced model by constructing the basis matrix \( U_k \) from \( Q_{w_{\text{init}}} \) and the interpolation points matrix \( P_k \). The loop in line 7 iterates over the time steps \( k = w_{\text{init}} + 1, \ldots, K \) at which the reduced models is solved instead of the full model. Line 8 finds the reduced state \( \tilde{q}_k \) that satisfies the reduced model with respect to \( f_k \) corresponding to the current DEIM interpolant with basis matrix \( U_k \) and interpolation points matrix \( P_k \). The if branch on line 10 decides if either the sampling points are adapted or the sampling points from the previous iteration are reused. If the sampling points are adapted, then the full-model function \( f \) is evaluated at all components and \( S_k \) is derived via the adaptive sampling strategy. Note that \( R_k \) is a MATLAB notation and means that the entries of \( R_k \) are squared. If the sampling points are not adapted, then the full-model \( f \) is evaluated at \( S_{k-1} \) (see line 17) and all other components are approximated (see line 18). The basis and the interpolation points are adapted with ADEIM at line 21. After the time stepping, the trajectory \( \tilde{Q} \in \mathbb{R}^{N \times K} \) is returned. The first \( w_{\text{init}} \) columns in \( \tilde{Q} \) are computed with the full model and the subsequent \( K - w_{\text{init}} + 1 \) columns are computed with the adaptive reduced model.

Let us now consider the computationally expensive steps in Algorithm 1. Solving the full model at line 2 is computationally expensive; however, typically \( w_{\text{init}} \) is chosen much smaller than \( K \), and thus only few time steps are computed with the full model. Furthermore, since \( w_{\text{init}} \ll K \), constructing the POD basis at line 3 is typically cheap as well. Adapting the sampling points at lines 11-14 requires evaluating the full-model function \( f \) at all components. The sampling points are adapted in \( K/z \) iterations, which means that \( K/z \) evaluations of the full-model function \( f \) at all \( N \) components are necessary. Note that evaluating \( f \) is typically significantly cheaper than performing a time step with the full model, because the latter requires solving a nonlinear system of equations, whereas the former typically requires a function evaluation only.

4. Numerical results

This section demonstrates our AADEIM approach on three numerical examples. First, the toy example based on the advection equation introduced in Section 2.2 is revisited in Section 4.1. Second, AADEIM is demonstrated on the Burgers’ equation with a setup that leads to two interacting waves and time-varying viscosity and transport-direction coefficients. Third, we consider a model of a rocket combustor and demonstrate that our AADEIM approach achieves significant speedups in contrast to static reduced models that take even longer to run than the full model. All runtime results are obtained on compute nodes with Intel Xeon E5-1660v4 with 64GB RAM and a MATLAB implementation.

4.1. Advection equation

Consider the same setup as in Section 2.2. Set \( k = 25 \) and \( w = 25 \) and let \( \mathcal{U}_k \) be the \( n = 3 \) dimensional DEIM space derived from \([q_{k-w+1}, \ldots, q_k] \in \mathbb{R}^{N \times w}\) and let \( \mathcal{U}_k \) and \( \mathcal{P}_k \) be the corresponding basis matrix and interpolation points matrix, respectively. Let now \( \hat{\mathcal{U}}_{k+1} \) be the \( n = 3 \) dimensional space derived from \([\mathcal{q}_0, \ldots, \mathcal{q}_{100}] \). Figure 3 shows the local coherence \( \gamma_i(\mathcal{U}_k) \) and \( \gamma_i(\hat{\mathcal{U}}_{k+1}) \) for \( i = 1, \ldots, 1500 \); the local coherence is sorted. The corresponding dashed curve are the bounds as in Lemma 2. Let now \( \hat{F}_k \in \mathbb{R}^{n \times n} \) have i.i.d. standard Gaussian entries and consider \( \mathcal{F}_k = \hat{\mathcal{U}}_{k+1} \hat{F}_k \) and the corresponding residual \( \mathcal{R}_k = \mathcal{F}_k - \mathcal{U}_k (\mathcal{P}_k^T \mathcal{U}_k)^{-1} \mathcal{P}_k^T \mathcal{F}_k \). The row-wise squared norm \( \|R_k^TE_1\|_2^2 \) of the residual \( R_k \) is plotted in Figure 3 together with the bound given by Lemma 2. The results provide evidence that the decay of the residual is inherited from the decay of the local coherence as shown in Lemma 2.

Consider now Figure 3 that shows the error \( d(\mathcal{U}_{k+1}, \hat{\mathcal{U}}_{k+1}) \) of the adapted space \( \hat{\mathcal{U}}_{k+1} \), for ADEIM updates with rank \( r = 1, 2, 3 \), against the number of sampling points \( m \). First, observe that if the rank is \( r < 3 \), then the bound of \( d(\mathcal{U}_{k+1}, \hat{\mathcal{U}}_{k+1}) \) levels off because the singular values \( \sigma_{r+1}, \ldots, \sigma_r \) dominate the decay factor \( \rho_k \). Second, the error \( d(\mathcal{U}_{k+1}, \hat{\mathcal{U}}_{k+1}) \) is bounded by \( \rho_k^2/\sigma_{r+1}^2 \) as proved in Proposition 1. Finally, the
Figure 3: Advection equation: The plot demonstrates that the norm of the rows of the residual $R_k$ is bounded by the decay of the local coherence of the space $U_k$ and $U_{k+1}$ as proved in Lemma 2. The solid curves are the norm of the residual and the local coherence, respectively, and the corresponding dashed curves are the bounds.

Figure 4: Advection equation: The plot demonstrates that the fast decay of the error of the adapted space $d(U_{k+1}, U_{k+1})$ is inherited from the fast decay of the residual. Furthermore, the plot shows that the bounds of the error of the adapted space stop decaying if a low-rank update is applied instead of a full-rank update.
results show that the fast decay of the error of the adapted space with respect to the number of sampling points $m$ is inherited from the fast decay of the residual (shown as the dashed curve in Figure 4), which demonstrates Proposition 2.

4.2. Burgers’ equation with time-varying viscosity

We now apply AADEIM to the Burgers’ equation with time-varying viscosity and a transport direction that changes with time.

4.2.1. Problem setup

Let $\Omega = [-1, 1] \subset \mathbb{R}$ be the spatial domain and let $T = 1.5$ be end time. Consider now the Burgers’ equation

$$\partial_t q(x, t) + \eta(t)q(x, t)\partial_x q(x, t) = \nu(t)\partial_x^2 q(x, t), \quad x \in \Omega,$$

with time $t \in [0, T]$, the solution function $q : \Omega \times [0, T] \to \mathbb{R}$, the time-varying viscosity $\nu : [0, T] \to \mathbb{R}$, and the transport direction $\eta : [0, T] \to \{-1, 1\}$. The viscosity is

$$\nu(t) = \mu (\sin(20\pi t) + \cos(60\pi t) + 2),$$

where $\mu \in D \subset \mathbb{R}$ is the nominal viscosity parameter. The transport direction is

$$\eta(t) = \text{sign} (\sin(20\pi t) + \cos(60\pi t) + 1).$$

The viscosity and transport direction change over time, as shown in Figure 5 for $\mu = 3 \times 10^{-3}$. The PDE \((25)\) is closed with Dirichlet boundary conditions and the initial condition given by

$$q(x, 0) = \begin{cases} 2, & x = -2 \\ 1, & -\frac{1}{2} \leq x \leq -\frac{1}{3} \\ 0, & \text{else} \end{cases}.$$  

The full model is obtained by discretizing the spatial domain $\Omega$ of the PDE \((25)\) on an equidistant grid with $N = 1024$ inner grid points. The time domain is discretized with the implicit Euler method and time step size $\delta t = 5 \times 10^{-5}$. In each time step, Newton’s method is used to solve the corresponding system of
nonlinear equations of the form (1). At each time step, 15 Newton iterations are performed. The solution of the full model for \( \mu = 3 \times 10^{-3} \) is shown in Figure 6.

Figure 7a reports the decay of the singular values corresponding to local trajectories of length \( w = 50 \) at time \( t = 0.77, 1.12, 1.50 \), respectively, and parameter \( \mu = 3 \times 10^{-3} \). The results show an orders of magnitude faster decay of the singular values of the local trajectories than of the global trajectory, which indicates that this problem exhibits a local low-rank structure. Figure 7b shows the decay of the squared residual of approximating the solution at \( t = 0.77, 1.12, 1.50 \) at the corresponding local spaces of dimension \( n = 8 \) of the trajectories for which the singular values are plotted in Figure 7a. A fast decay of the norm of the rows of the residual is observed.

### 4.2.2. Performance of AADEIM

We now compare the runtime and accuracy of static reduced models, our AADEIM models, and the full model. The static reduced models are derived from the trajectories corresponding to the parameters \( \mu \in \{5 \times 10^{-3}, 10^{-3}, 5 \times 10^{-4}\} \), following the procedure outline in Section 2.1. The dimension of the DEIM space is \( n \) and the interpolation points are selected with QDEIM [15]. The AADEIM models are initialized with the first \( w_{init} = 100 \) states computed with the full model. The dimension of the DEIM spaces of the AADEIM models is \( n = 8 \). The interpolation points are selected with QDEIM [15]. The DEIM space \( U_k \) and the DEIM interpolation points \( P_k \) are adapted every other time step. The interpolation points \( P_k \) are adapted by applying QDEIM to the adapted basis. Only updates with rank \( r = 1 \) are applied. The window size is \( w = n + 1 = 9 \). The sampling points are obtained with the adaptive sampling strategy described in Section 3.3. The sampling points are adapted every \( z = 5 \) iteration, except otherwise noted. The error of

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![Figure 6: Burgers' example: The plots show the solution of the full model at different times in the time interval [0, 1.5]. The initial condition (26) leads to two waves propagating to the right, with the waves starting to interact around time \( t = 0.7 \). Note that the transport direction and the viscosity change over time, which cannot be seen in these plots, cf. Figure 5.](image-url)


Figure 7: Burgers’ example: The plots indicate that the singular values of local trajectories (see plot (a)) and the corresponding DEIM residual of the local spaces (see plot (b)) decay fast, cf. Section 3.1.

the static and the AADEIM models are measured in the Frobenius norm

$$\text{err}(\tilde{Q}(\mu)) = \frac{\|\tilde{Q}(\mu) - Q(\mu)\|_F}{\|Q(\mu)\|_F},$$  (27)

where $Q(\mu)$ is the trajectory obtained with the full model at parameter $\mu$ and $\tilde{Q}(\mu)$ is the trajectory obtained either with a static reduced model or an AADEIM model. For computing $\text{err}(\tilde{Q}(\mu))$, only states at time steps $k = 1, \ldots, 1000$ and then for $k > 1000$ at every 50-th time step are taken into account.

Let us first consider parameter $\mu = 3 \times 10^{-3}$. Figure 8a shows the error $\text{err}(\tilde{Q}(\mu))$ of the static reduced models with $n \in \{125, 175, 225\}$ and of the AADEIM models with $m \in \{96, 160, 224, 352, 480, 608\}$ and dimension $n = 8$. The sampling points in the AADEIM models are adapted very $z$-th time step with $z = 5$. The dashed line in Figure 8a marks the runtime of the full model. The plot shows that the static reduced model achieves a speedup compared to the full model in this example; however, more than 100 dimensions are required for the DEIM space to achieve an error below $10^{-1}$. In contrast, the AADEIM model achieves errors below $10^{-1}$ with $n = 8$ dimensions and $m = 96$ sampling points, which leads to about an order of magnitude speedup compared to the full model. The plot in Figure 8a further indicates that increasing the number of sampling points from $m = 96$ to $m = 608$ reduces the error from $10^{-1}$ to $10^{-2}$ without a significant increase of the runtime, which provides evidence that evaluating the full-model function $f$ is significantly cheaper than solving the full model in this example. Figure 8b shows similar behavior of AADEIM for parameter $\mu = 8 \times 10^{-4}$. Figure 9a summarizes the runtime of the static reduced model with dimension $n = 225$ and the AADEIM model with $m = 608$ sampling points, which are required to achieve an error (27) below $10^{-2}$.

Let us now investigate the effect of adapting the sampling points. Figure 9b shows the error (27) of the AADEIM model for parameter $\mu = 3 \times 10^{-3}$ when the sampling points are adapted at every time step, at every 3-th time step, and at every 5-th time step. The number of sampling points $m$ is varied $m = 96, 160, 224, 352, 480, 608$. First, note that the scale of the $x$-axis of the plot is rather small, which means that different numbers of adaptations of the sampling points have a minor effect on the runtime in this example. Second, for $z = 1$, i.e., the sampling points are adapted at each time step, the runtime is almost constant for an increasing number of sampling points. For $z = 1$, the full-model function $f$ is computed at all $N$ components at each time step and therefore the different number of sampling points only affect the costs of the ADEIM adaptation, which is rather low compared to evaluating the full-model function and time stepping. Third, there is almost no difference in the error for adapting the sampling points at every 3-th iteration to adapting at every 5-th iteration, which means that the sampling points are valid over several time steps in this example.
Figure 8: Burgers’ example: The plots show that our AADEIM achieves a speedup of about one order of magnitude compared to the full model. About \( m = 96 \) sampling points out of \( N = 1024 \) are sufficient for our adaptive sampling strategy to obtain an AADEIM model with an error below \( 10^{-1} \). The results in these plots further show that increasing the number of samplings points only slightly increases the runtime, which indicates that computing the full-model function \( f \) to update the DEIM space is computationally cheap in this example.

Figure 9: Burgers’ example: The plot in (a) shows that the AADEIM model achieves a speedup of about one order of magnitude compared to the static and the full model. Plot (b) indicates that adapting the sampling points every 5-th time step, i.e., \( z = 5 \), is sufficient in this example.
4.2.3. Performance of adaptive sampling strategy

We now compare our adaptive sampling strategy to random uniform sampling, which is used in, e.g., [39]. Figure 10 shows the error (27) of the AADEIM model if the sampling points are selected with our adaptive sampling strategy and random uniform sampling of \{1, \ldots, N\}. The random uniform sampling is without replacement. The curves in Figure 10 correspond to \(m \in \{32, 96, 160, 224, 352, 480, 608, 736, N\}\) sampling points. Uniform sampling with \(m = 32\) leads to an unstable model in case of \(\mu = 8 \times 10^{-4}\) and therefore its error is not plotted. The results in Figure 10 indicate that adaptive sampling significantly reduces the error (27) of the AADEIM model compared to uniform sampling. Improvements of up to two orders of magnitude can be observed. Note that both sampling scheme coincide if all points \(m = N\) are selected. Figure 11 reports the error (27) of the AADEIM model for dimensions \(n \in \{4, 6, 8, 10, 12\}\) with \(m \in \{96, 224, 480\}\) sampling points (adaptive sampling strategy). The results indicate that a larger dimension of the DEIM space leads to a lower error only if sufficiently many sampling points are selected for the adaptation. For example, the AADEIM model with dimension \(n = 12\) and \(m = 480\) sampling points achieves an about one order of magnitude lower error than the AADEIM model with the same dimension and \(m = 96\) sampling points.

4.3. Combustion model

In this section, we apply AADEIM to a quasi-1D version of a single-element model rocket combustor. The model we use has been developed in the works [50, 19, 20]. The goal is to approximate the growth of the amplitude of pressure oscillations at a monitoring point, which provides critical insights for designing engines that avoid combustion instabilities and unbounded growth of the amplitude of the pressure oscillations.

4.3.1. Problem setup and full model

Our problem setup and full model follows [58, 54]. The problem setup consists of three parts, namely the oxidizer post, the combustion chamber, and the exit nozzle, see Figure 12. The oxidizer is induced and meets the fuel at the back-step, where it reacts instantaneously. The combustion products exit the chamber through the nozzle. The combustion follows a one-step reaction model

\[
\text{CH}_4 + 2\text{O}_2 \rightarrow \text{CO}_2 + 2\text{H}_2\text{O},
\]
Figure 11: Burgers’ example: The results in these plots indicate that the dimension of the DEIM space and the number of sampling points $m$ need to be traded off with respect to each other. At least in this example, it seems that an increase in the dimension of the DEIM space needs to be accompanied by an increase in the number of sampling points to obtain an AADEIM model with a lower error.

Figure 12: Combustion example: Geometry of the combustion problem with injector, back-step, combustion camber, and nozzle.
Figure 13: Combustion example: The plots report the pressure at the monitoring point (see Figure 12) for four different heat-release parameters. The results indicate that the parameter domain $D = [2, 4.2]$ leads to solutions with significantly different behavior. Note that the solution for $\mu = 3.8$ seems to enter a limit cycle oscillation.
where the fuel is gaseous methane and the oxidizer is a mixture of oxygen and water. Details on the operating conditions are given in Table 2. The governing equations are in conservative form

\[ \partial_t q + \partial_x g = s_A + s_g + s_q(\mu), \]

where \( q = \begin{bmatrix} \rho A \\ \rho v A \\ \rho E A \\ \rho Y_{ox} A \end{bmatrix}, \quad g = \begin{bmatrix} \rho v A \\ \rho E A + (\rho E + p)u A \\ \rho Y_{ox} A \end{bmatrix}, \]

where \( \rho \) is the density, \( v \) is the velocity, \( E \) is the total internal energy, \( Y_{ox} \) is the oxidizer mass fraction, \( A \) is the cross sectional area of the fuel duct, and \( p \) is the pressure. The source terms \( s_A \) and \( s_g \) are given in [58, equation (2)]. The source term \( s_q(\mu) = [0, 0, q'(\mu), 0]^T \) models the heat release, where the parameter \( \mu \in \mathbb{D} = [2, 4.2] \subset \mathbb{R} \) in \( q'(\mu) \) controls the amplification of the heat release, see [58, equation (5)]. Following [58], a steady-state solution is first obtained by ignoring the source term \( s_q(\mu) \). The steady-state solution is then used as initial condition for computing the time-dependent solution that takes the source term \( s_q(\mu) \) into account. The initial condition is perturbed to trigger an instability, which depends on the heat-release parameter \( \mu \). The spatial domain is discretized on 300 equidistant grid points. There are four degrees of freedom at each grid point (density, velocity, energy, mass fraction), and so the full model has a total of \( N = 1200 \) degrees of freedom. Time is discretized with a fourth-order implicit scheme and time step size \( \delta t = 10^{-7} \) with end time \( T = 10^{-1} \). Newton’s method is used to solve the corresponding system of nonlinear equations at each time step. The pressure is monitored at spatial coordinate \( x = 0.3683 \), see Figure 12. Figure 13 shows the pressure at the monitoring point for parameters \( \mu \in \{2, 4, 3.0, 3.8, 4\} \). The solutions converge to a steady state for \( \mu = 2.4 \). A limit cycle oscillation is entered for \( \mu = 3.8 \). A combustion instability is observed for \( \mu = 4.0 \).

### 4.3.2. Performance of AADEIM

We compare static reduced models, our AADEIM models, and the full model. The static reduced model is derived from the trajectories corresponding to the parameters \( \mu \in \{2, 2.4, 2.88, 3.32, 3.76, 4.2\} \), which are the

---

**Figure 14:** Combustion example: Plot (a) shows that the static reduced model is even more expensive than the full model in this example. The AADEIM model achieves a speedup of about 6 compared to the full model. Plot (b) visualizes the speedup as a histogram for \( m = 50 \) sampling points in case of the AADEIM model and dimension 225 in case of the static reduced model, so that both reduced models achieve an error of about \( 10^{-4} \).
six equidistant parameters in the parameter domain $D = [2, 4.2]$. A separate DEIM basis of dimension $n$ is computed for each degree of freedom, see, e.g., [61, Section 2.2]. The DEIM interpolation points are derived with QDEIM, where $n$ points are derived for each of the four DEIM basis and then the union of all four sets of points is used as the set of DEIM interpolation points. The dimension of the AADEIM model is $n = 8$ in the following. The AADEIM model is initialized with the full-model states obtained until $t = 1.6 \times 10^{-4}$, where only every 50-th state is used so that $w_{\text{init}} = 4 \times n = 32$. The sampling points are derived for each degree of freedom separately, then they are ranked by how often each sampling point has been selected, and then the same $m$ sampling points are used for all four bases that have been selected the most. This means that a total of $4 \times m$ components of the full-model residual are computed in each iteration. The basis is adapted at every time step and the sampling points are adapted very other time step ($z = 2$). We measure the error of the pressure at the monitoring point. Let $\hat{y}(\mu) = [y_1(\mu), \ldots, y_K(\mu)]^T \in \mathbb{R}^K$ be the trajectory of the pressure at the monitoring point computed with the full model. Then, we measure the error

$$\text{err}(\hat{y}(\mu)) = \frac{\|\hat{y}(\mu) - y(\mu)\|_2}{\|y(\mu)\|},$$

(28)

where $\hat{y}(\mu)$ is the trajectory computed with either a static or an AADEIM model. The rest of the setup is the same as in Section 4.2.

Figure 14 reports the error (28) of the reduced models and their runtime. The error and runtime of the static reduced model is plotted for $n \in \{60, 70, 80, 90, 100\}$. The results for the AADEIM model are reported for $m \in \{20, 30, 40, 50\}$ and $n = 8$. The parameter is set to $\mu = 3.8$. The AADEIM model achieves a speedup of about 6 compared to the full model in this example. The static reduced model is slower than the full model. Figure 15 shows the speedup of the AADEIM model with $m = 50$ and for a parameter sweep over $\mu \in \{2.4, 3.0, 3.8, 4.0\}$. The dimension of the static reduced model is $n = 225$. The static and the AADEIM model achieve about the same error for all four parameters. Note that the four parameters lead to significantly different behaviors in the solutions, see Figure 13. The AADEIM model achieves a significant speedup compared to the full model, whereas the static reduced model is slower than the full model. Figure 15 demonstrates that our adaptive sampling scheme is orders of magnitude more efficient than uniform sampling without replacement. Uniform sampling requires at least $m = 175$ sampling points per degree of freedom to prevent the Newton method from diverging and to achieve an error (28) of about $10^{-2}$. With our adaptive sampling scheme, our AADEIM approach achieves an error (28) of about $10^{-4}$ with $m = 50$ sampling points per degree of freedom.

5. Conclusions

Our approach AADEIM demonstrates that transport-dominated problems have a rich local structure that can be exploited to construct efficient reduced models. We exploit locality in time via adaptive basis updates and locality in space via adaptive sampling. Our analysis establishes a connection between the local coherence properties of reduced spaces and the number of samples that are required to adapt the basis. The faster the local coherence decays, the fewer samples are required to adapt the reduced spaces. Reduced models built with AADEIM are implicitly parametrized, which means that there is no offline phase to construct reduced models, rather the basis is adapted online to changes in the parameter. Numerical results demonstrated that AADEIM is applicable to a wide range of problems and faithfully approximates behavior that changes significantly with parameters. At the same time, AADEIM achieves significant runtime speedups compared to full and traditional, static reduced models.

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Figure 15: Combustion example: Plot (a) shows that our AADEIM model achieves significant speedups compared to the full model for a parameter sweep over the parameter domain $\mathcal{D} = [2, 4.2]$, where solutions show significantly different behavior, cf. Figure [13]. This provides evidence that our AADEIM approach is robust with respect to changes in the parameter. Plot (b) demonstrates that our adaptive sampling scheme achieves orders of magnitude lower errors than uniform sampling.

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A. Analytic example with local low-rank structure

We analyze analytically an example to demonstrate the local low-rank structure of transport-dominated problems. The example follows [51, Example 2.5]. Consider the function

\[ q(x, t) = \begin{cases} 0, & x \leq t, \\ 1, & x > t, \end{cases} \tag{29} \]

in the spatial domain \( x \in [-5, 5] \) and \( t \in [0, T] \) with \( T = 1 \). There is no \( n \)-dimensional Lagrangian space \( \mathcal{U} \) for which the error \( \sup_{t \in [0, T]} \inf_{q^* \in \mathcal{U}} \| q(\cdot, t) - q^* \|_{L^2(-5, 5)} \) decays faster than linearly in \( 1/\sqrt{n} \), which means there is no space spanned by \( q(\cdot, t_1), \ldots, q(\cdot, t_n) \) with \( n \) pairwise distinct \( t_i \in [0, T] \) with \( i = 1, \ldots, n \) that achieves a faster error decay than \( 1/\sqrt{n} \). Instead of considering the whole time domain \([0, T]\), let us now consider \([0, T/\zeta]\) with \( \zeta > 0 \). Let \( \mathcal{U}_\zeta \) be the space spanned by \( q(\cdot, t_i) \) with \( t_i = ih \) for \( i = 1, \ldots, n \) and \( h = (T/\zeta)/n \). For \( t \in [0, T/\zeta] \), the space \( \mathcal{U}_\zeta \) achieves

\[ \inf_{q^* \in \mathcal{U}_\zeta} \| q(\cdot, t) - q^* \|_{L^2(-5, 5)} = \begin{cases} \sqrt{t_1 - t}, & t < t_1, \\ \sqrt{(t_{i+1} - t)(t - t_i)/(t_{i+1} - t_i)}, & t_i \leq t \leq t_{i+1}, i = 1, \ldots, n - 1, \end{cases} \]

and thus \( \sup_{t \in [0, T/\zeta]} \inf_{q^* \in \mathcal{U}_\zeta} \| q(\cdot, t) - q^* \|_{L^2(-5, 5)} \leq \sqrt{h} = \sqrt{T/(\zeta n)} \). The rate of the error decay can be increased by letting \( \zeta \) depend on \( n \). For example, setting \( \zeta = e^n \) gives \( \sup_{t \in [0, T/\zeta]} \inf_{q^* \in \mathcal{U}_\zeta} \| q(\cdot, t) - q^* \|_{L^2(-5, 5)} \in O(e^{-n}) \), which shows that a local low-rank structure can be recovered if (29) is approximated locally in time.

B. Helper functions for Algorithm 1

**Algorithm 2** Interpolation points selection with QDEIM

1: procedure QDEIM(U)[See reference 15]
2: \([\sim, \sim, P] = qr(U^T', \text{vector'});\)
3: \(P = P[1 : \text{size}(U, 2)]\)
4: return \(P\)
5: end procedure
Algorithm 3 Adaptation with ADEIM

1: procedure ADEIM($U, P, S, F_p, F_s, r$)[See reference [39]]
2: \[ C = U[P, :] \setminus F_p \] \hspace{1cm} \triangleright \text{Coefficients w.r.t. interpolation points}
3: \[ R = U[S, :] (C - F_s) \] \hspace{1cm} \triangleright \text{Residual at sampling points}
4: \[ [\cdot, Sv, Sr] = \text{svd}(R, 0) \] \hspace{1cm} \triangleright \text{Compute SVD of residual}
5: \[ Sv = \text{diag}(Sv) \]
6: \[ (C^T)^+ = \text{pinv}(C^T) \] \hspace{1cm} \triangleright \text{Pseudo inverse of } C^T
7: \[ r = \min([r, \text{length}(Sv)]) \] \hspace{1cm} \triangleright \text{Determine rank of update}
8: \hspace{1cm} \triangleright \text{Apply updates}
9: \hspace{1cm} \triangleright \text{Orthogonalize columns of } U
10: \hspace{1cm} \triangleright \text{Recompute QDEIM interpolation points}
11: \[ P = \text{qdeim}(U) \]
12: end procedure

13: return $U, P$