ON FINSLER SURFACES WITH CERTAIN FLAG CURVATURES

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Abstract. In the present paper, we find out necessary and sufficient conditions for a Finsler surface \((M,F)\) to be Landsbergian in terms of the Berwald curvature 2-forms. We study Finsler surfaces which satisfy some flag curvature \(K\) conditions, viz., \(V(K) = 0\), \(V(K) = -I/F^2\) and \(V(K) = -IK\), where \(I\) is the Cartan scalar. In order to do so, we investigate some geometric objects associated with the global Berwald distribution \(\mathcal{D} := \text{span}\{S,H,V := JH\}\) of a 2-dimensional Finsler metrizable nonflat spray \(S\). We obtain some classifications of such surfaces and show that under what hypothesis these surfaces turn to be Riemannian. The existence of a first integral for the geodesic flow in each case has some remarkable consequences concerning rigidity results. We prove that a Finsler surface with \(V(K) = -I/F^2\) and either \(S(K) = 0\) or \(S(J) = 0\) is Riemannian. Further, a Finsler surface with \(V(K) = -IK\) and \(S(K) = 0\) is Riemannian.

Keywords: Finsler geometry; Sprays; Flag curvature; Rigidity theory; Berwald frame.

MSC 2020: 53C24, 53C60, 58B20, 53B40, 58J60.

1. Introduction

The Finsler surface has been studied, from both local and global point of views, in [2, 3, 4, 9, 10]. In [2], the Berwald frame has been introduced locally by Berwald for a Finsler surface and used to characterize and classify projectively flat Finsler surfaces. Such a frame has been studied and utilized in [11, §9.1]. The Berwald frame \((H,S,V : JH,C)\) has been defined directly for an arbitrary nonflat spray \(S\) and its properties applied to obtain information about the Finsler metrizability of the given spray \(S\) in [5]. The Berwald distribution \(\mathcal{D}\) is a 3-dimensional distribution defined by \(\mathcal{D} = \text{span}\{S,H,V\}\), whose integrability provides a candidate for the Finsler function \(F\).

The classification of Finsler spaces which are Riemannian is one of the fundamental problems which is known as Finsler rigidity theory. The notion of Riemannian sectional curvature is extended to Finsler geometry as the so-called flag curvature \(K\). Also, Finsler geometry has many geometric objectives besides \(K\), like Berwald, Landsberg and Cartan curvatures. In case of Finsler surface, the Cartan (or main) scalar \(I\), Landsberg scalar \(J := S(I)\) and flag curvature completely determine its geometry. It is known that whenever \(I = 0\), the Finsler space is Riemannian [6], while for \(J = 0\), the Finsler surface is Landsbergian. Moreover, the Finsler surface is Berwaldian when \(J = 0\) and \(H(I) = 0\). However, in case of \(K = 0\) or a constant, the Finsler manifold is not necessarily Riemannian [2, 3]. In [1], Akbar-Zadeh proved that “any compact Finsler surface with negative constant flag curvature is Riemannian”. In addition, Finsler manifolds with \(K\) is just a function on \(M\) are called \(K\)-basic Finsler manifolds and Schur’s Lemma shows that these manifolds with \(n \geq 3\) have constant \(K\). However, the 2-dimensional case needs different treatment, see e.g. [8]. It is worth mentioning that there is a strong relation between the Riemannian character of a Finsler surface and the absence of conjugate points, see for instance [9].

The aim of the present paper is to study some rigidity problems in Finsler surfaces. In this direction, we generalize some results of [8, 9, 10]. The Ricci scalar of a Finsler space is given by \(\rho := KF^2\) [4], thus one can work with \(K\) or equivalently \(\rho\). We investigate the Finsler surfaces which satisfy certain flag curvature (Ricci scalar) conditions, namely, i) \(V(\rho) = 0 \iff V(K) = 0\), ii) \(V(\rho) = -I \iff V(K) = -I/F^2\) and iii) \(V(\rho) = -I \rho \iff V(K) = -IK\). One can say that, i) represents the case of \(K\)-basic Finsler surfaces, ii) can be considered as a generalization of i) in the sense that the main scalar \(I\), in general, is a function on \(TM\) not on \(M\) and iii) is equivalent to \(S(J) = 0\), by (3.7), that is \(J\) is a first integral for the geodesic flow. It can be considered as a generalization of Landsberg surface. Finsler surfaces with \(S(J) = 0\) reduce to Landsbergian or Riemannian ones under some conditions, see for example Theorems 5.9 and 5.11.

A key aspect of our work is the use of the Berwald distribution \(\mathcal{D}\) and its Berwald connection. In this paper, we find out, in details, some geometric objects related to the Berwald connection such as the Bianchi identities which describe the relations between some derivatives of the three invariants \(I, J, K\) with respect to Berwald distribution elements. In addition, we compute Berwald curvature, Berwald connection 1-forms, curvature 2-forms and Cartan’s structural equations associated with \(\mathcal{D}\), etc. Then, we use these relations to study the Finsler surfaces which satisfy
the aforementioned curvature conditions. Thereby, we obtain some classifications of such Finsler surfaces.

The paper is organized as follows. We recall some basic facts on the geometry of sprays and Finsler manifolds in §2. We find out some geometric quantities associated with the Berwald distribution using Berwald connection and Cartan’s structural equations in §3. In §4, we give a necessary and sufficient condition for a Finsler surface to be Landsbergian in terms of the Berwald curvature 2-forms. In Landsberg surfaces \((M,F)\), we prove that the function \(F := H(I) + V(J)\) is a first integral for the geodesic flow if and only if \((M,F)\) is Riemannian. In §5, we briefly redo some analysis of \([8]\) in a slightly different way. We show the existence of a first integral \(F\) for the geodesic flow of any Finsler space with \(V(K) = 0\). In addition, we show how Berwald distribution and its associated geometric quantities can be used to derive some rigidity results for Finsler structure defined on compact surfaces. After that, we study the case of \(V(\rho) = -\mathcal{I}\) and show that when either the flag curvature is a first integral for the geodesic flow or the Landsberg scalar \(J\) is a first integral for the geodesic flow, the Finsler surface reduces to Riemannian one. Consequently, a Landsberg surface with \(V(\rho) = -\mathcal{I}\) is Riemannian. Finally, we investigate the case of Finsler surfaces with \(V(K) = -\mathcal{I}K\) as a generalization of \([8]\). In this case, we show that a compact Finsler surface of genus at least one without conjugate points is Riemannian. Finally, we prove that Finsler surface with \(V(K) = -\mathcal{I}K\) and \(S(K) = 0\) is Riemannian.

2. Preliminaries

Now, we recall some definitions from geometry of sprays and Finsler spaces. We refer to \([4, 11]\) for further reading. Let \(M\) be an n-dimensional manifold, \((TM, \pi_M, \mathcal{M})\) be its tangent bundle and \((TM, \pi, M)\) be the subbundle of nonzero tangent vectors. Hereafter, \(x^i, (x^i, y^j)\) and \((x^i, y^j)\) denote the local coordinates on the base manifold \(M\) and the induced coordinates on \(TM\), respectively. The vector 1-form \(J\) on \(TM\) is defined, locally, by \(J = \frac{\partial}{\partial x^i} \otimes dx^i \) is called the almost-tangent structure of \(TM\). The vertical vector field \(C = y^j \frac{\partial}{\partial y^j}\) on \(TM\) is known as the Liouville vector field.

**Definition 2.1.** A vector field \(S \in \mathfrak{X}(TM)\) is said to be a spray on \(M\) if \(JS = C\) and \([C, S] = S\). Locally, \(S\) is given by

\[
S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},
\]

where \(G^i = G^i(x, y)\) are the spray coefficients which are positive homogeneous of degree 2 in \(y\).

Each spray \(S\) induces, using the Frolicher-Nijenhuis formalism, a canonical nonlinear connection \(\Gamma := [J, S]\). The existence of \(\Gamma\) is equivalent to the existence of an \(n\)-dimensional distribution

\[
H : u \in TM \rightarrow H_u \in T_u(TM)
\]

that is supplementary to the vertical distribution which is called the horizontal distribution. Thus, we have

\[
T_u(TM) = H_u(TM) \oplus V_u(TM), \quad \forall u \in TM,
\]

where the corresponding horizontal and vertical projectors, respectively, are given by

\[
h := \frac{1}{2}(Id + [J, S]), \quad v := \frac{1}{2}(Id - [J, S]).
\]

For every \(Z \in \mathfrak{X}(M)\), \(\mathcal{L}_Z\) and \(i_Z\) denote the Lie derivative with respect to \(Z\) and the interior product by \(Z\), respectively. The differential of a function \(h\) is denoted by \(dh\). A vector \(r\)-form on \(M\) is a skew-symmetric \(C^\infty(M)\)-linear map \(T : (\mathfrak{X}(M))^r \rightarrow \mathfrak{X}(M)\). A vector \(r\)-form \(T\) defines two graded derivations \(d_T\) and \(i_T\) of the Grassman algebra of \(M\) such that

\[
d_T := [i_T, d] = i_T \circ d - (-1)^r d \circ i_T,
\]

\[
i_T h = 0, \quad i_T dh = dh \circ T, \quad h \in C^\infty(M).
\]

The Jacobi endomorphism induced by \(S\) is defined by

\[
\Phi = v \circ [S, h] = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j = \left(2 \frac{\partial G^i}{\partial x^j} - S(G^i_j) - G^i_k G^k_j \right) \frac{\partial}{\partial y^i} \otimes dx^j.
\]

\(^1\)From here onwards the Einstein summation convention is in place.
Definition 2.2. A spray $S$ is called isotropic if the Jacobi endomorphism has the form
$$\Phi = \rho J - \beta \otimes C,$$
where $\rho := \text{Tr}(\Phi)$ is called the Ricci scalar and $\beta$ is semibasic 1-form.

Definition 2.3. A smooth Finsler structure on an $n$-dimensional manifold $M$ is a mapping
$$F : TM \to \mathbb{R},$$
such that:
1. $F$ strictly positive on $TM$ and $F(x, y) = 0$ if and only if $y = 0$,
2. $F$ is positively homogenous of degree 1 in the directional argument $y$, that is $\mathcal{L}_c F = F$,
3. The metric tensor $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y_i \partial y_j}$ has maximal rank on $TM$.

The pair $(M, F)$ is called Finsler manifold.

Definition 2.4. A spray $S$ on a manifold $M$ is called Finsler metrizable if there exists a Finsler function $F$ such that the geodesic spray of $F$ is $S$.

Definition 2.5. The Finsler manifold $(M, F)$ is said to be of scalar flag curvature if there exists a function $K \in C^\infty(TM)$ such that the associated Jacobi endomorphism is given by
$$\Phi = K (F^2 J - F dJ F \otimes C).$$
Moreover, if $K$ is constant then $(M, F)$ is called of constant flag curvature.

It follows that a Finsler function $F$ of scalar flag curvature $K$, its geodesic spray $S$ is isotropic with Ricci scalar $\rho = K F^2$ and the semi-basic 1-form $\beta = K F dJ F$. In fact, any 2-dimensional spray is always isotropic [11, Corollary 8.3.11]. When $K$ is a constant, we say that $S$ has a constant flag curvature.

Definition 2.6. Let $\eta$ be a unit speed geodesic, i.e. $\eta'' = S \circ \eta'$, in a Finsler surface $(M, F)$. A vector field $Y \in \mathfrak{X}(M)$ along $\eta$ is called Jacobi field if it satisfies
$$D_X D_K Y + R(Y, X)X = 0,$$
where $D_X Y$ is the covariant differentiation with respect to Chern-Rund connection of $Y$ along $X$ and
$$R(u, v) = K(u) [g_u (u, u)v - g_u (u, v) u], \quad v, u \in TM.$$

Definition 2.7. A point $p \in M$ is said to be conjugate point to $q \in M$ along a geodesic $\eta$ if there exists a nonzero Jacobi field along $\eta$ that vanishes at $p$ and $q$. A Finsler manifold $(M, F)$ is called without conjugate points if no geodesic in $(M, F)$ has conjugate points.

Remark 2.8. The three invariants $I, J, K$ are functions on the unit tangent bundle (or indicatrix bundle, simply, indicatrix) $TM$ [2]. It is worth noting that the main scalar $I$ is a convex geometric invariant describes the shape of each unit tangent circle, the Landsbreg scalar $J = S(I)$ is the rate of change of $I$ along the geodesics and the flag $K$ is a variational invariant which measures the focusing of geodesics. The homogeneity of these invariants are given by

$$C(K) = 0, \quad C(I) = 0, \quad C(J) = J. \quad (2.6)$$

Let us end this section by recalling some interesting rigidity results on Finsler surfaces.

Theorem 2.9. Let $(M, F)$ be a compact Finsler surface of genus at least one without conjugate points. Then, the following are satisfied:

1. If $F$ is a Landsberg metric, then $(M, F)$ is Riemannian [9, Theorem 2].
2. If $S(K) = 0$, then $(M, F)$ has constant flag curvature $K$ [9, Proposition 7.1].
3. If $(M, F)$ has $S(J) = 0$ and $S(K) = 0$, the Finsler surface has a constant flag curvature $K$ and it is Riemannian whenever $K$ nonvanishing [9, Lemma 7.3].

Theorem 2.10. [8] Let $(M, F)$ be a smooth compact connected $K$-basic Finsler surface without conjugate points and genus greater than one. Then, $(M, F)$ is Riemannian.

Theorem 2.11. [7] The flag curvature is a first integral for the geodesic flow, that is $S(K) = 0$, if the Finsler structure is reversible $C^3$ on $TM$ and locally symmetric.

Theorem 2.12. [10] If $(M, F)$ is a connected Landsberg surface with $S(K) = 0$, then $(M, F)$ has a constant flag curvature and is Riemannian whenever $K$ nonvanishing.
3. Berwald frame and Berwald connection on Finsler surfaces

**Definition 3.1.** [5] Let $(M, S)$ be a 2-dimensional smooth manifold $M$ equipped with a nonflat spray $S$. Let $H \in \mathfrak{X}(TM)$ be a positive 2-homogeneous horizontal vector field such that $\beta(H) = 0$, where $\beta$ is semi-basic 1-form defined in Definition 2.2. The regular 3-dimensional distribution given by $\mathcal{D} = \text{span}\{S, H, V = JH\}$ is called Berwald distribution. Berwald frame is a global frame on $TM$ defined by $(H, S, V, C)$.

When the spray is Finsler metrizable, the semi-basic 1-form is given by $\beta = K F dJ F$ and $\mathcal{D}$ is an integrable distribution. Hereafter, we assume that $S$ is a nonflat spray metrizable by a Finsler function $F$. The nonflatness assumption of $S$ that we work with is equivalent to $K \neq 0$.

**Lemma 3.2.** [5] Let $S$ be the geodesic spray of a Finsler function $F$ and let $\mathcal{D}$ be its Berwald distribution. Then, we have
\begin{align*}
(3.1) & \quad H(F^2) = 0, \quad V(F^2) = 0, \\
(3.2) & \quad [S, H] = \rho V, \\
(3.3) & \quad [V, S] = H, \\
(3.4) & \quad [H, V] = S + \mathcal{I} H + J V, \quad \text{where} \quad \mathcal{J} = S(\mathcal{I}).
\end{align*}

**Definition 3.3.** [4, Theorem 3.2.1] The map $D : \mathfrak{X}(TM) \times \mathfrak{X}(TM) \to \mathfrak{X}(TM)$, is given by
\begin{equation}
(3.5) \quad D_X Y = v[hX, vY] + h[vX, hY] + J[vX, \theta Y] + \theta[hX, JY],
\end{equation}
where $\theta$ is the adjoint structure defined by $\theta = h \circ [S, h]$. $D$ is a linear connection on $TM$ which is called Berwald connection.

**Proposition 3.4.** Consider $S$ be the geodesic spray of a Finsler function $F$ and let $(H, S, V, C)$ be its Berwald frame. Then, the nonvanishing components of Berwald connection are the following:
\begin{align*}
D_H H &= \mathcal{J} H, \quad D_H V = \mathcal{J} V, \quad D_V C = V, \quad D_V S = H, \quad D_V H = -S - \mathcal{I} H, \\
D_C H &= H, \quad D_C V = V, \quad D_V V = -C - \mathcal{I} V, \quad D_C S = S, \quad D_C C = C.
\end{align*}

**Proof.** It follows from the facts that $\{S, H\}$ are horizontal vector fields, $\{C, V\}$ are vertical vector fields together with the following relations
\begin{align*}
\theta h &= 0, \quad \theta v = h, \quad h^2 = h, \quad v^2 = v, \quad hv = vh = 0
\end{align*}
and Lemma 3.2 along with
\begin{align*}
(3.6) & \quad [C, H] = H, \quad [C, S] = S, \quad [C, V] = 0.
\end{align*}
For example,
\begin{align*}
D_H H &= v[hH, vH] + h[vH, hH] + J[vH, \theta H] + \theta[hH, JH] = \theta(S + \mathcal{I} H + J V) = \mathcal{J} H. \quad \square
\end{align*}

**Proposition 3.5.** Consider $S$ be a Finsler metrizable spray and let $(H, S, V, C)$ be its Berwald frame. The only nonvanishing components of the Berwald curvatures are given by
\begin{enumerate}
\item hh-curvature $R(S, H)S = -\rho H$, \quad $R(S, H)H = \{S(\mathcal{I}) + \rho \mathcal{I}\} H + \rho S$,
\item hv-curvature $B(V, H)H = F H - 2 \mathcal{J} S$, \quad where \quad $F := H(\mathcal{I}) + V(\mathcal{J})$.
\end{enumerate}
Consequently, $R(S, H)C = -\rho V$, \quad $R(S, H)V = \{S(\mathcal{I}) + \rho \mathcal{I}\} V + \rho C$, \quad $B(V, H)V = F V - 2 \mathcal{J} C$.

**Proof.** It follows from the definition, see [4], of curvature tensor $C$ associated with a connection $D$,
\begin{equation}
(3.6) \quad C(X, Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z
\end{equation}
and the following properties of $D$
\begin{align*}
D_f X Y &= f D_X Y, \quad D_X f Y = f D_X Y + X(f) Y,
\end{align*}
together with Proposition 3.4, (3.2), (3.3) and (3.4). Indeed, we get
\begin{align*}
R(S, H)S &= D_S D_H S - D_H D_S S - D_{[S, H]} S = -D_{\rho V} S = -\rho D_V S = -\rho H, \\
R(S, H)H &= D_S D_H H - D_H D_S H - D_{[S, H]} H = D_S (\mathcal{J} H) - D_{\rho V} H \\
&= \mathcal{J} D_S H + S(\mathcal{J}) H - \rho D_V H = \{S(\mathcal{I}) + \rho \mathcal{I}\} H + \rho S.
\end{align*}
Similarly, the nonvanishing component of $hv$-curvature is the following:

\[
B(V, H)H = D_V D_H H - D_H D_V H - D_{[V,H]} H = D_V (J H) - D_H (-S + I H) + D_{[S + I H + J V]} H
\]

\[
= J D_V H + V(J) H + D_H S + I D_H H + H(I) H + D_S H + I D_H H + J D_V H
\]

\[
= 2 J D_V H + 2 I D_H H + (H(I) + V(J)) H
\]

\[
= -2 J \{S + I H\} + 2 I J H + \{H(I) + V(J)\} H = -2 J S + F H.
\]

The proof is completed by using the fact \[4], \( J(C(X, Y)Z) = C(X, Y)JZ, \quad X, Y, Z \in \mathfrak{X}(TM) \). □

**Remark 3.6.** The function \( F = H(I) + V(J) \) is interesting and has some mysterious properties that proved to be effective and useful in characterization of Finsler surfaces cf. \[8\]. This will be shown during presenting our results.

**Theorem 3.7.** Let \( S \) be a Finsler metrizable spray and \((H, S, V, C)\) be its Berwald frame. Then, we have

\[
S(J) + V(\rho) + I \rho = 0,
\]

\[
S(F) + I V(\rho) + V^2(\rho) = 0, \text{ where } V^2(\rho) := V(V(\rho)).
\]

**Proof.** One can easily check that the only nonvanishing component of the torsion of Berwald connection \( D \) is the \( v(h) \)-torsion, namely, \( v^T(S, H) = -\rho V \). Substituting by \( X = S, Y = V, Z = H, W = V \) in the second Bianchi identity \[4\]: \( \sum_{X, Y, Z} \{(D_X R)(Y, Z) + R(T(X, Y), Z)\}W = 0 \), we obtain

\[
0 = (D_S R)(V, H)V + (D_V R)(H, S)V + (D_H R)(S, V)V + R(T(S, V), H)V
\]

\[
+ R(T(V, H), S)V + R(T(H, S), V)V
\]

\[
= (D_S R)(V, H)V + (D_V R)(H, S)V + (D_H R)(S, V)V.
\]

Using \( (D_X R)(Y, Z)W := D_X (R(Y, Z)W) - R(D_X Y, Z)W - R(Y, D_X Z)W - R(Y, Z)D_X W \) and substituting from Propositions 3.4 and 3.5, we get

\[
0 = -2 \{S(J) + V(\rho) + I \rho\} C + \{S(F) + I V(\rho) + V^2(\rho)\} V.
\]

The proof is completed from the fact that \( C \) and \( V \) are independent vector fields which generate the vertical distribution. □

**Proposition 3.8.** Let \( S \) be a Finsler metrizable spray and \((H, S, V, C)\) be its associated Berwald frame with corresponding dual basis \( \{\eta^i, 1 \leq i \leq 4\} \). The Cartan’s structural equations are

\[
d\eta^1 = -I \eta^1 + \eta^2 \wedge \eta^3 + \eta^2 \wedge \eta^3,
\]

\[
d\eta^2 = -\eta^1 \wedge \eta^3,
\]

\[
d\eta^3 = \rho \eta^1 \wedge \eta^2 - J \eta^1 \wedge \eta^3.
\]

**Lemma 3.9.** Let \( S \) be the geodesic spray of a Finsler function and \((H, S, V, C)\) be its associated Berwald frame with corresponding dual basis \( \{\eta^i, 1 \leq i \leq 4\} \). The trace of the \( hv \) Berwald curvature is given by \( B(V, H) = F \).

**Proof.** It follows from the trace definition, \( B(X, Y) := \text{Tr}(Z \mapsto B(X, Z)Y) = \sum_{\alpha=1}^4 \eta^\alpha(B(X, X_\alpha)Y) \). Indeed, taking into account Proposition 3.5, we get

\[
B(V, H) = \eta^1(B(V, H)H) + \eta^2(B(V, S)H) + \eta^3(B(V, V)H) + \eta^4(B(V, C)H)
\]

\[
= \eta^1(F H - 2 J S) = F.
\]

□

In view of Proposition 3.5 and Lemma 3.9, we obtain:

**Corollary 3.10.** Let \( S \) be the geodesic spray of a Finsler function \( F \) and \((H, S, V, C)\) be its Berwald frame. Then, \((M, F)\) is Berwaldian if and only if it is Landsbergian and the trace \( hv \)-Berwald curvature vanishes identically.

**Proposition 3.11.** Let \( S \) a Finsler metrizable spray and \((H, S, V, C)\) be its Berwald frame with corresponding dual basis \( \{\eta^i, 1 \leq i \leq 4\} \). The following matrix representation of the Berwald connection 1-forms with respect to \((H, S, V, C)\) is given by

\[
\omega = \left[ \begin{array}{cccc}
\eta^1 - I \eta^3 + \eta^4 & -\eta^3 & 0 & 0 \\
\eta^3 & \eta^4 & 0 & 0 \\
0 & 0 & \eta^1 - I \eta^3 + \eta^4 & -\eta^3 \\
0 & 0 & 0 & \eta^3
\end{array} \right].
\]
Proof. It follows from Proposition 3.4 along with the definition [4, §3.7]
\[ \omega^a_b(X) = \eta^a(D_X X_b), \quad 1 \leq a, b \leq 4. \]
For example,
\begin{align*}
\omega^1_1 &= \eta^1(D_H H) \eta^1 + \eta^1(D_S H) \eta^2 + \eta^1(D_V H) \eta^3 + \eta^1(D_C H) \eta^4 \\
&= \eta^1(J H) \eta^1 + \eta^1(-S - I H) \eta^3 + \eta^1(I H) \eta^4 = J \eta^1 - I \eta^3 + \eta^4.
\end{align*}
\[\square\]

Proposition 4.12. Let \( S \) a Finsler metrizable spray and \((H, S, V, C)\) be its Berwald frame with corresponding dual basis \(\{\eta^i, 1 \leq i \leq 4\}\), then the nonvanishing components of the Berwald curvature 2-forms are
\begin{align*}
\Omega^1_1 &= V(\rho) \eta^1 \wedge \eta^2 - F \eta^1 \wedge \eta^3, \\
\Omega^2_1 &= \rho \eta^1 \wedge \eta^2 + 2 J \eta^1 \wedge \eta^3, \\
\Omega^3_1 &= \rho \eta^1 \wedge \eta^2.
\end{align*}
Proof. The curvature 2-forms can be calculated through two ways: (i) using the Carant’s structural equations, (ii) using the curvature tensor. In the first method, it can be calculated by considering (3.9), (3.10) and (3.11) along with
\[ \Omega^1_j := -d\omega^j_1 + \omega^j_2 \wedge \omega^1_1. \]
In the second method, using Proposition 3.5 in \( \Omega^j_1(X, Y) := \eta^r(R(X, Y)X_j) \) and the fact that the matrix of the curvature 2-forms has the following expression [4] \( \begin{pmatrix} \Omega^1_1 & 0 \\
0 & \Omega^1_1 \end{pmatrix} \). In what follows, we compute \( \Omega^1_1 \) which is equal to \( \Omega^3_1 \) while \( \Omega^2_1 \) and \( \Omega^3_1 \) can be calculated by the same way,
\[ \Omega^1_1(H, S) = \eta^3(R(H, S)V) = V(\rho), \quad \Omega^3_1(H, V) = \eta^3(R(H, V)V) = -F, \quad \Omega^3_1(H, C) = \eta^3(R(H, C)V) = 0. \]
This shows that the formula (4.12) is true. \[\square\]

4. Integrability condition for Landsberg Surfaces

Let us start this section with some important applications of the global Berwald frame using Berwald connection that we mentioned in the previous section.

Lemma 4.1. Let \( S \) a Finsler metrizable spray. Then, the dual basis \(\{\eta^i, 1 \leq i \leq 4\}\) of Berwald frame satisfies:
\begin{align*}
d_J \eta^1 &= 0, \\
d_J \eta^2 &= 0, \\
d_J \eta^3 &= -d\eta^1, \\
d_J \eta^4 &= -d\eta^2.
\end{align*}
Proof. Applying (2.3) into the tangent structure \( J \), we get (4.2)
\[ d_J \eta^r = i_J d\eta^r - d_J i_J \eta^r, \quad \text{for } 1 \leq r \leq 4, \]
where \( i_J \) is defined in (2.4). The proof is completed by substituting (3.9), (3.10) and (3.11) into (4.2). \[\square\]

Proposition 4.2. Let \( S \) a Finsler metrizable spray and \((H, S, V, C)\) be its Berwald frame with corresponding dual basis \(\{\eta^i, 1 \leq i \leq 4\}\), then the Berwald curvature 2-forms satisfy the following:
\begin{align*}
d_J \Omega^1_1 &= 0, \\
d_J \Omega^1_2 &= 0, \\
d_J \Omega^3_1 &= -2 J \eta^1 \wedge \eta^2 \wedge \eta^3.
\end{align*}
Proof. In view of (4.12), we have
\begin{align*}
d_J \Omega^1_1 &= (d_J V(\rho)) \eta^1 \wedge \eta^2 + V(\rho) d_J \eta^1 \wedge \eta^2 - V(\rho) \eta^1 \wedge d_J \eta^2 \\
&= -(d_J F) \eta^1 \wedge \eta^2 - F d_J \eta^1 \wedge \eta^3 + F \eta^1 \wedge d_J \eta^3.
\end{align*}
Considering (2.6), we get
\begin{align*}
d_J \eta^r &= d_F \circ J = (d_F \circ J H) \eta^1 + (d_F \circ J S) \eta^2 + (d_F \circ J V) \eta^3 + (d_F \circ J C) \eta^4 \\
&= (d_F \circ V) \eta^1 + (d_F \circ C) \eta^2 = V(F) \eta^1 + C(F) \eta^2 = V(F) \eta^1 + F \eta^2.
\end{align*}
Similarly, \( d_J V(\rho) = V^2(\rho) \eta^1 + 2V(\rho) \eta^2 \). Taking into account (4.1), we obtain
\[ d_J \Omega^3_1 = -2 F \eta^2 \wedge \eta^1 \wedge \eta^3 + F \eta^1 \wedge (\eta^1 - d\eta^1) = 0. \]
In the same way, the other component (3.14) satisfies:
\begin{align*}
d_J \Omega^1_2 &= (d_J \rho) \eta^1 \wedge \eta^2 + \rho (d_J \eta^1) \wedge \eta^2 - \rho \eta^1 \wedge d_J \eta^2 = (V(\rho) \eta^2 + 2 \rho \eta^2) \wedge \eta^1 \wedge \eta^2 = 0.
\end{align*}
Also, using (3.13), we get
\[ d_J \Omega^1_2 = d_J (\rho \eta^2 \wedge \eta^1 + 2 J \eta^1 \wedge \eta^3) = -2 J \eta^1 \wedge \eta^2 \wedge \eta^3. \]
Theorem 4.3. Let $S$ a geodesic spray of a Finsler function $F$ and $(H, S, V, C)$ be its Berwald frame. Then, $(M, F)$ is Landsbergian if and only if the curvature $2$-form $\Omega_2$ is $d\iota$-closed.

Proof. It follows directly from Proposition 4.2 along with the postulate $\mathcal{J} = 0$. \hfill $\square$

The next result classifies Landsberg surfaces in which the trace $hv$-Berwald curvature $\mathcal{J}$ is a first integral of the geodesic flow.

Theorem 4.4. Let $S$ a geodesic spray of Landsberg structure $F$ and $(H, S, V, C)$ be its Berwald frame. Then, the function $\mathcal{J}$ is a first integral of the geodesic flow, that is $\mathcal{J}(\mathcal{J}) = 0$, if and only if $(M, F)$ is Riemannian.

Proof. Let $(M, F)$ be a Landsberg surface, that is $S(\mathcal{I}) = 0$, thereby $\mathcal{J} = H(\mathcal{I})$. Applying the bracket $[S, H]$ to $\mathcal{I}$, we get

$$[S, H](\mathcal{I}) = S(H(\mathcal{I})).$$

But, by using the commutation formula (3.2), we obtain $[S, H](\mathcal{I}) = \rho V(\mathcal{I})$. Hence, we have

$$\mathcal{J}(\mathcal{J}) = \rho V(\mathcal{I}).$$

Putting $\mathcal{J}(\mathcal{J}) = 0$, we get $\rho = 0$ or $V(\mathcal{I}) = 0$. But, $\rho \neq 0$ thus $V(\mathcal{I}) = 0$. Now, $V(\mathcal{I}) = 0$ implies, by (3.3), that

$$H(\mathcal{I}) = [V, S](\mathcal{I}) = V(S(\mathcal{I})) - S(V(\mathcal{I})) = 0.$$

That is, the horizontal covariant derivatives of $\mathcal{I}$ along both $S$ and $H$ vanish and this means that $\mathcal{I}$ is constant along the horizontal distribution. Hence, $(M, F)$ is Berwaldian. In fact, a Berwaldian with nonvanishing flag curvature is Riemannian [2]. Therefore, $(M, F)$ is Riemannian. The converse follows directly, and this completes the proof. \hfill $\square$

5. Finsler surfaces with certain flag curvature

We denote the Berwald connection $1$-form $\omega^1_\alpha$ on the indicatrix $\mathcal{I}M$ by $\alpha$, that is

$$(5.1) \quad \alpha := \omega^1_\alpha = \mathcal{J} \eta^1 - \mathcal{I} \eta^3,$$

consequently, considering the formulae (3.9), (3.11), and (3.7) together with

$$d\mathcal{I} = H(\mathcal{I}) \eta^1 + \mathcal{J} \eta^2 + V(\mathcal{I}) \eta^3,$$

we obtain

$$(5.2) \quad d\alpha := \Omega^1_\alpha = V(\rho) \eta^1 \wedge \eta^2 - \mathcal{F} \eta^1 \wedge \eta^3.$$

Remark 5.1. If the spray $S$ is metrizable by a Riemannian function, thereby the main scalar $\mathcal{I}$ identically vanishes, then the geometric quantities $\mathcal{F}$, $\alpha$ and $d\alpha$ are vanishing. If $\alpha$ vanishes on the horizontal (respectively, vertical) distribution, then $(M, F)$ is Landsbregian (respectively, Riemannian). Also, ker($\alpha$) := $\{X \in \mathfrak{X}(\mathcal{I}M) \mid \alpha(X) = 0\}$ = span{$\mathcal{I}$} on $\mathcal{I}M$, i.e., ker($\alpha$) is an invariant distribution by the Lagrangian planes.

In what follows, we are going to study the Finsler surfaces which satisfy the aforementioned flag curvature (Ricci scalar) conditions. We start with $K$-basic Finsler surfaces.

5.1. The flag curvature satisfies $V(K) = 0$.

Lemma 5.2. Let $S$ be the geodesic spray of a Finsler function $F$ and let $\mathcal{D}$ be its Berwald distribution. Then, $V(\rho) = 0$ if and only if the flag curvature $K$ is a function of $x$ only, that is, the Finsler function is $K$-basic.

Proof. Since the Ricci scalar $\rho = K F^2$ and $V(F) = 0$, which follows from (3.1), thus

$$V(\rho) = 0 \text{ if and only if } V(K) = 0.$$  

Clearly, due to $C(K) = 0$, we get $d_J K = V(K) \eta^1$. Therefore, the assumption $V(K) = 0$ reads that $d_J K = 0$. Hence we have

$$V(\rho) = 0 \iff V(K) = 0 \iff d_J K = 0 \iff d_s K = 0 \iff K \in C^\infty(M). \hfill \square$$

Proposition 5.3. Let $S$ be the geodesic spray of a Finsler function $F$ and let $\mathcal{D}$ be its Berwald distribution. If $V(\rho) = 0$, then the following are satisfied:

(1) The function $\mathcal{F}$ is a first integral of the geodesic flow, that is $\mathcal{J}(\mathcal{F}) = 0$.

(2) $S(\mathcal{J}) = -\mathcal{I} \rho$.

Proof. It follows directly from the identities (3.7) and (3.8). \hfill $\square$
Proposition 5.4. Let \( S \) be the geodesic spray of a Finsler function \( F \) and let \( \mathcal{D} \) be its Berwald distribution. Then, the following assertions are equivalent:

1. The 1-form \( \alpha \) is invariant by the geodesic flow, that is \( \mathcal{L}_S \alpha = 0 \).
2. The Finsler function is \( K \)-basic.
3. The 1-form \( \mathcal{L}_S \alpha \) is closed.

Proof. (1) \( \iff \) (2): It follows from the formulae (5.1) and (5.2) along with the fact that \( \eta^2 \) is the dual of \( S \). Indeed, in the view of (2.4) both \( is(\mathcal{J} \eta^1 - \mathcal{I} \eta^3) \) and \( is(\eta^1 \wedge \eta^3) \) identically vanish. Thus, we have

\[
\mathcal{L}_S \alpha = i_S d\alpha + di_S \alpha = i_S(-V(\rho)\eta^2 \wedge \eta^1 - \mathcal{F} \eta^1 \wedge \eta^3) + di_S(\mathcal{J} \eta^1 - \mathcal{I} \eta^3) = i_S(V(\rho) \eta^1 \wedge \eta^2) = V(\rho) i_S(\eta^1 \wedge \eta^2).
\]

That is,

\[
(5.3) \quad \mathcal{L}_S \alpha = V(\rho) \eta^1.
\]

Then, we obtain \( \mathcal{L}_S \alpha = 0 \iff V(K) = 0 \) by Lemma 5.2.

(2) \( \implies \) (3): Since \( d^2 = 0 \), we have

\[
\mathcal{L}_S d\alpha = (i_Sd + di_S) d\alpha = d(i_S \alpha) = d(V(\rho) \eta^1) = d(V(\rho)) \wedge \eta^1 + V(\rho) d\eta^1.
\]

Substituting by the identity (3.8), the formula (3.9) and \( d(V(\rho)) = H(V(\rho)) \eta^1 + S(V(\rho)) \eta^2 + V(V(\rho)) \eta^3 \), we obtain

\[
(5.4) \quad \mathcal{L}_S \alpha = S(V(\rho)) \eta^2 \wedge \eta^1 + V(\rho) \eta^2 \wedge \eta^3 + S(\mathcal{F}) \eta^1 \wedge \eta^3.
\]

Hence, the postulate \( V(K) = 0 \) gives \( S(\mathcal{F}) = 0 \) by Proposition 5.3. Therefore, \((M, F)\) is \( K \)-basic Finsler surface implies that \( \mathcal{L}_S d\alpha = 0 \).

(3) \( \implies \) (2): Suppose that \( \mathcal{L}_S d\alpha = 0 \), that is \( V(K) = 0 \) and \( S(\mathcal{F}) = 0 \) by using (5.4). In fact, Proposition 5.3 says that \( V(K) = 0 \) implies \( S(\mathcal{F}) = 0 \). Thereby, we can say that \( \mathcal{L}_S d\alpha = 0 \) if and only if \( V(K) = 0 \). \( \square \)

Using Lemma 5.2, we can reformulate Theorem 2.10 as follows.

Theorem 5.5. Let \( S \) be the geodesic spray of a Finsler function \( F \) and let \( \mathcal{D} \) be its Berwald distribution. If \((M, F)\) is a connected compact Finsler surface of genus at least one without conjugate points such that \( V(K) = 0 \), then \((M, F)\) is Riemannian.

Proof. Assume that \( V(K) = 0 \), then by Proposition 5.3, \( S(\mathcal{F}) = 0 \). Applying [9, Theorem 1], we get \( \mathcal{F} \) is constant on \( IM \), that is

\[
d\mathcal{F} = H(\mathcal{F}) \eta^1 + S(\mathcal{F}) \eta^2 + V(\mathcal{F}) \eta^3 = 0.
\]

Consequently, \( H(\mathcal{F}) = V(\mathcal{F}) = 0 \). Now from Stokes Theorem, we have

\[
\int d(\alpha \wedge \eta^2) = \int d\alpha \wedge \eta^2 - \int \alpha \wedge d\eta^2 = 0.
\]

Using the expressions of \( \alpha, d\alpha \), see (5.1), (5.2), and \( d\eta^2 \) from (3.10), we obtain

\[
\int d\alpha \wedge \eta^2 = \int \mathcal{F} \eta^1 \wedge \eta^2 \wedge \eta^3, \quad \int \alpha \wedge d\eta^2 = \int (\mathcal{F} \eta^1 - \mathcal{I} \eta^3) \wedge \eta^3 \wedge \eta^1 = 0.
\]

Therefore,

\[
\int \mathcal{F} \eta^1 \wedge \eta^2 \wedge \eta^3 = 0 \iff \mathcal{F} \text{Vol}(IM) = 0,
\]

where \( \eta^1 \wedge \eta^2 \wedge \eta^3 \) is the volume form of \( IM \). Hence, \( \mathcal{F} = 0 \) on \( IM \), accordingly, \( d\alpha = 0 \) by (5.2). So that, \( \alpha \) is an exact 1-form by Poincare lemma. In other words, there exists a function \( \mu \) such that \( \alpha = d\mu \), where

\[
d\mu = S(\mathcal{I}) \eta^1 - \mathcal{I} \eta^3 = H(\mu) \eta^1 + S(\mu) \eta^2 + V(\mu) \eta^3.
\]

Thus, \( H(\mu) = S(\mathcal{I}), \ S(\mu) = 0 \) and \( V(\mu) = -\mathcal{I} \). Hence, \( \mu \) is a first integral of the geodesic flow. Applying [9, Theorem 1], yields the function \( \mu \) is constant on \( IM \). Consequently, \( d\mu = 0 \), that is \( H(\mu) = 0 \) and \( V(\mu) = 0 \), which means \( \mathcal{I} = 0 \) and \( S(\mathcal{I}) = 0 \). Therefore, \((M, F)\) is Riemannian. \( \square \)
5.2. The flag curvature satisfies $V(K) = -\mathcal{I}/F^2$.

It may be recalled that the condition $V(K) = -\mathcal{I}/F^2$ is equivalent to $V(\rho) = -\mathcal{I}$.

**Lemma 5.6.** Let $S$ be the geodesic spray of a Finsler function $F$ and let $\mathcal{D}$ be its Berwald distribution. If $V(\rho) = -\mathcal{I}$, then we have

1. $S(\mathcal{F}) = \mathcal{I}^2 + V(\mathcal{I})$, $S(\mathcal{J}) = \mathcal{I}(1 - \rho)$.
2. The 1-form $\alpha$ is invariant by the geodesic flow if and only if $(M, F)$ is Riemannian.

**Proof.** (1) Substituting by $V(\rho) = -\mathcal{I}$ in the identities (3.7) and (3.8), we obtain the required.

(2) Plug the condition $V(\rho) = -\mathcal{I}$ into the formula (5.3), we get $\mathcal{L}_S\alpha = -\mathcal{I}\eta^1$. Thus, $\mathcal{L}_S\alpha = 0 \iff \mathcal{I} = 0$ which completes the proof. □

**Theorem 5.7.** Let $S$ be the geodesic spray of a Finsler function $F$ and let $\mathcal{D}$ be its Berwald distribution such that $V(\rho) = -\mathcal{I}$. If either $S(\mathcal{J}) = 0$ or $S(\mathcal{J}) = 0$, then $(M, F)$ is Riemannian.

**Proof.** Suppose that $S(\mathcal{J}) = 0$. By Lemma 5.6 (1), we have $S(\mathcal{J}) = 0$, which is equivalent to $\mathcal{I} (1 - \rho) = 0$. That is, $\mathcal{I} = 0$ or $(1 - \rho) = 0$. Therefore, $\rho = 1$ implies $V(\rho) = 0$ and $V(\rho) = -\mathcal{I}$ gives $\mathcal{I} = 0$. That is, $(M, F)$ is Riemannian.

Applying the commutation formulae (3.2) and (3.3) to $\rho$, we obtain

$$S(H(\rho)) - H(S(\rho)) = \rho V(\rho), \quad S(V(\rho)) - V(S(\rho)) = -H(\rho),$$

Now, assume that $S(\rho) = 0$ and $V(\rho) = -\mathcal{I}$, we get

$$S(H(\rho)) = -\mathcal{I} \rho, \quad H(\rho) = \mathcal{J}.$$ Therefore, $S(\mathcal{J}) = -\mathcal{I} \rho$. By Lemma 5.6 (1), we have $S(\mathcal{J}) = \mathcal{I}(1 - \rho)$. Hence, $-\mathcal{I} \rho = \mathcal{I}(1 - \rho)$, that is, $\mathcal{I} = 0$. □

**Corollary 5.8.** Let $S$ be the geodesic spray of a Landsbergian structure $F$ and let $\mathcal{D}$ be its Berwald distribution such that $V(\rho) = -\mathcal{I}$, then $(M, F)$ is Riemannian.

**Proof.** Suppose that $\mathcal{J} = 0$. Consequently, $S(\mathcal{J}) = 0$. Now, by Theorem 5.7, $(M, F)$ is Riemannian. □

It is worth mentioning that Theorem 5.7 and Corollary 5.8 are related to Theorems 2.9 and 2.12.

5.3. The flag curvature satisfies $V(K) = -\mathcal{I} K$.

It should be noted that the condition $V(K) = -\mathcal{I} K$ is equivalent to $V(\rho) = -\mathcal{I} K$.

**Theorem 5.9.** Let $S$ be the geodesic spray of a Finsler function $F$ and let $\mathcal{D}$ be its Berwald distribution. If $(M, F)$ is a connected compact Finsler surface of genus at least one without conjugate points such that $V(\rho) = -\mathcal{I} \rho$, then $(M, F)$ is Riemannian.

**Proof.** The identity (3.7) when $V(\rho) = -\mathcal{I} \rho$ gives $S(\mathcal{J}) = 0$. Applying [9, Theorem 1], we get $\mathcal{J}$ is constant on $IM$. By Stokes Theorem, we have

$$\int d(\alpha \wedge \eta^1) = 0 \iff \int d\alpha \wedge \eta^1 - \int \alpha \wedge d\eta^1 = 0.$$ Using the expressions of $\alpha$, $d\alpha$ and $d\eta^1$ see (5.1), (5.2) and (3.9), we obtain

$$\int d\alpha \wedge \eta^1 = \int V(\rho) \eta^1 \wedge \eta^2 \wedge \eta^1 - \int \mathcal{F} \eta^1 \wedge \eta^3 \wedge \eta^1 = 0,$$

$$\int \alpha \wedge d\eta^1 = \int (\mathcal{J} \eta^1 - \mathcal{I} \eta^3) \wedge (\eta^2 \wedge \eta^3 - \mathcal{I} \eta^1 \wedge \eta^3) = \int \mathcal{J} \eta^1 \wedge \eta^2 \wedge \eta^3.$$ Consequently, we get

$$\int \mathcal{J} \eta^1 \wedge \eta^2 \wedge \eta^3 = 0 \iff \mathcal{J} \text{Vol}(IM) = 0.$$ Hence, $\mathcal{J} = 0$ on $IM$, that is, $(M, F)$ is Landsbergian. Applying Theorem 2.9 (1), we get $(M, F)$ is Riemannian. □

**Proposition 5.10.** Let $S$ be the geodesic spray of a Finsler function $F$ and let $\mathcal{D}$ be its Berwald distribution such that $V(\rho) = -\mathcal{I} \rho$ and $S(\rho) = 0$, then the trace of $hv$-Berwald curvature identically vanishes.
Proof. Applying the Lie brackets of Berwald frame (3.3) and (3.4) to \( \rho \), yields
\[
S(V(\rho)) - V(S(\rho)) = -H(\rho), \quad H(V(\rho)) - V(H(\rho)) = S(\rho) + \mathcal{I} H(\rho) + \mathcal{J} V(\rho).
\]
Substituting by \( V(\rho) = -\mathcal{I} \rho \) and \( S(\rho) = 0 \) in the last two equations above, gives
\[
H(\rho) = \mathcal{J} \rho, \quad H(\mathcal{I})\rho + 2\mathcal{I} H(\rho) + V(H(\rho)) = \mathcal{I} \mathcal{J} \rho.
\]
Plug \( H(\rho) = \mathcal{J} \rho \) into \( H(\mathcal{I})\rho + 2\mathcal{I} H(\rho) + V(H(\rho)) = \mathcal{I} \mathcal{J} \rho \), we obtain
\[
\mathcal{I} \mathcal{J} \rho = H(\mathcal{I})\rho + 2\mathcal{I} \mathcal{J} \rho + V(\mathcal{J} \rho)
\]
\[
= H(\mathcal{I})\rho + 2\mathcal{I} \mathcal{J} \rho + V(\mathcal{J})\rho - \mathcal{J} \mathcal{I} \rho.
\]
That is, \( H(\mathcal{I})\rho + V(\mathcal{J})\rho = 0 \) which is equivalent to \( \rho \mathcal{F} = 0 \). Thereby, \( \rho = 0 \) or \( \mathcal{F} = 0 \). The proof is completed by looking at Lemma 3.9 and considering \( \rho \neq 0 \) as our Finsler spray is nonflat. \( \square \)

The following result can be considered as a generalization of Theorem 2.9 (3), this is because of a Finsler surface is not necessarily compact of genus at least one without conjugate points.

**Theorem 5.11.** Let \( S \) be the geodesic spray of a Finsler function \( F \) and let \( D \) be its Berwald distribution such that \( V(\rho) = -\mathcal{I} \rho \) and \( S(\rho) = 0 \), then \((M, F)\) is Riemannian.

Proof. Applying the commutation formulae (3.2) and (3.3) to \( \rho \), yields
\[
S(H(\rho)) - H(S(\rho)) = \rho V(\rho), \quad S(V(\rho)) - V(S(\rho)) = -H(\rho).
\]
Plugging \( V(\rho) = -\mathcal{I} \rho \) and \( S(\rho) = 0 \) into the last equations above, we obtain
\[
S(H(\rho)) = -\mathcal{I} \rho^2, \quad \mathcal{J} \rho = H(\rho).
\]
Therefore,
\[
S(H(\rho)) = S(\mathcal{J} \rho) = -\mathcal{I} \rho^2 \iff \rho S(\mathcal{J}) + \mathcal{J} S(\rho) = -\mathcal{I} \rho^2.
\]
But we have \( S(\mathcal{J}) = 0 \) and \( S(\rho) = 0 \), which gives \( \mathcal{I} \rho^2 = 0 \). Therefore, \( \mathcal{I} = 0 \) due to the spray \( S \) is nonflat. \( \square \)

**Acknowledgment.** I would like to express my deep gratitude to Professor I. Bucataru (Alexandru Ioan Cuza University, Romania) and Dr. S. G. Elgendi (Benha University, Egypt) for their useful discussions and comments.

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