Remarks on supersymmetric effective actions and supersymmetry breaking

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Abstract

We discuss some issues related to the definition of different effective actions, in connection with the work on supersymmetric theories by Seiberg and collaborators. We also comment on the possibility of extending this work to broken supersymmetric theories.

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1 Introduction

Recent work by Seiberg and collaborators \[1\] on exact results in supersymmetric (SUSY) field theories depends crucially on the existence of the so-called Wilson effective action. In contrast to the usual one particle irreducible (1PI) effective action, this action, since it has an explicit infra-red cutoff, is free from holomorphic anomalies\[1\]. This enables one to make effective use of the power of holomorphy to constrain the form of the superpotential that is generated by quantum effects, even at the non-perturbative level. As a consequence of this analysis many remarkable properties of these field theories (such as the existence of electro-magnetic duality in non-abelian theories) have been discovered. It is therefore of some importance to discuss the precise (non-perturbative) definition of the Wilson effective action in the context of these theories. In this paper we first discuss problems associated with defining such an action in continuum gauge theories. Next (ignoring the above problem) we investigate how such actions may be defined for composite fields (mesons, and baryons). In the fourth section we discuss questions related to singularities at the origin of moduli space. We conclude with an evaluation of the prospects for extending these methods to broken SUSY theories.

2 Defining a Wilson effective action

The Wilson effective action is obtained by integrating out short distance degrees of freedom of a microscopic theory valid at some short distance scale, down to some low energy scale $\Lambda$. It is then an effective field theory for discussing physics below the scale $\Lambda$. A precise definition was originally given in the context of lattice gauge theory \[2,3\]. Unfortunately lattice methods cannot be easily extended to supersymmetric theories and one would like to have a continuum formulation of the effective action. The first step towards this was taken by Polchinski \[3\] who derived a continuum form of the

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\[1\] This point seems to have been highlighted first by Shifman and Vainshtein \[2\].

\[2\] Recently an explicitly construction of this has been given \[3\].

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Wilson renormalization group equation for scalar $\phi^4$ field theory and used it to prove the renormalizability of the theory. The method depended on explicitly introducing a momentum space cutoff in the kinetic term of the action, and this enables one to derive a differential equation for the Wilson action of the form,

$$\Lambda \frac{d}{d\Lambda} S_W(\phi, \Lambda) = F(S_W, \Lambda) \quad (2.1)$$

The important point about this equation is not so much that it enables one to give a proof of the perturbative renormalization of the theory that is much simpler than the usual one, but that it gives a non-perturbative definition of the theory. In other words regardless of the fact that the derivation of (2.1) depended on the existence of a perturbative formulation of the theory (in so far as one assumes a separation between a kinetic term and an interaction), the equation gives a unique non-perturbative definition of an action at any (“final”) scale $\Lambda$, given the microscopic action at some (“initial”) scale $\Lambda_0$. Thus if we could derive such an equation for SUSY gauge theories we would have a satisfactory object for which the non-perturbative arguments of Seiberg and collaborators will apply.

Unfortunately matters are not so simple as soon as one goes beyond scalar field theory. The problem arises when one tries to define a regulator for gauge theory in order to derive the analog of (2.1). The introduction of higher covariant derivatives by themselves does not help since they do not regularize one loop terms. Thus one is forced to introduce an additional regulator (typically Pauli-Villars) to deal with those. Such a procedure which regularizes all diagrams (including those in which there are divergent one-loop subgraphs) has been given by Warr [6] and has been extended to SUSY and super gravity theories in [7]. However apart from the rather baroque nature of this scheme, it suffers from one serious problem with regard to deriving an equation of the form (2.1). The use of Pauli-Villars regulators requires the introduction of preregulators in order to define each diagram separately, before the cancellation between diagrams involving physical fields and P-V fields takes place. This is done in a fashion that manifestly breaks the gauge invariance. This preregulator must then be first sent to infinity. The problem is that

\[ \text{It is claimed that at the end of the day one has a theory which satisfies the Slavnov-Taylor identities.} \]
the procedure, even if it yields a well-defined gauge (or BRST) invariant perturbation series, cannot be used to derive the analog of [2.1], since it is essentially a diagram by diagram method that is inextricably linked to perturbation theory. It is thus important to devise an alternative regularization that enables one to derive an RG equation.

In the last few years there has been quite a bit of activity on this problem [8, 9, 10, 11]. However none of these attempts seem to be completely satisfactory. In the first part of this work we will derive (in somewhat simpler fashion and using a different regularization method) a flow equation very similar to one derived in [8], [11]. This will then enable us to elucidate the problems that are involved in defining a Wilsonian action.

A 1PI effective action which is invariant under background field gauge transformations may be defined by the equation

\[ e^{-\Gamma(\phi_c)} = \int [d\phi'] e^{-I[\phi_c+\phi']-\phi' \cdot \frac{\delta\Gamma}{\delta\phi_c}} = \int [d\phi'] e^{-I[\phi_c]+\phi'K[\phi_c],\phi'+I_1[\phi_c,\phi'J,\phi']} = e^{-I[\phi_c]} e^{-\frac{1}{2} \text{tr} \ln K[\phi_c] - I_1[\phi_c,\phi',\frac{\delta I}{\delta J}] + \frac{1}{4} J K^{-1} J}. \]  

(2.2)

\( I_1 \) is at least cubic in the field \( \phi' \) and the bar at the end of the equation is an instruction to evaluate the expression with

\[ J = -\frac{\delta\Gamma}{\delta\phi_c} + \frac{\delta I}{\delta\phi_c} \equiv \bar{J}. \]  

(2.3)

The subscript \( \epsilon \) on the measure is an instruction to cut off the functional integral at an ultraviolet cutoff \( \Lambda_\epsilon^2 = \epsilon^{-1} \) and the dot product implies integration over space time. In the case of a gauge theory we need to add (background covariant) gauge fixing terms and the corresponding FP term. Since every term in the functional integral is background gauge invariant so is the effective action. In addition the BRST invariance of the functional integrand and measure implies that \( \Gamma[\phi_c] \) is independent of the gauge fixing.

\(^4\)To keep the notation simple we will use the notation of scalar field theory. When relevant we will discuss the additional complications of gauge theory such as gauge fixing and ghosts.
In order to define a Wilson effective action $\Gamma_L[\phi_c]$ we have to introduce an infra-red cut off $L = \Lambda^{-2}$. A convenient way of doing this while preserving the background gauge invariance, is to modify the background field propagator $K[\phi_c]^{-1}$ using the Schwinger proper time representation,\[
abla_{L}^{-1}[\phi_c] = \int_0^L dte^{-tK[\phi_c]}, \quad \ln K_L = \int_0^L \frac{dt}{t}e^{-tK[\phi_c]}, \quad (2.4)
\]The Wilson action $\Gamma_L$ is then defined by replacing $K \rightarrow K_L$ in (2.2). In a gauge theory this would include the contribution from the gauge fixing term and so $K$ would explicitly depend on the gauge fixing parameter $\xi$ and there would also be a ghost contribution. From (2.2) (with $K \rightarrow K_L$) and (2.4) we see that
\[
\Gamma_{L \rightarrow \infty} = \Gamma, \quad \Gamma_{L \rightarrow 0} = I[\phi_c], \quad (2.5)
\] (ignoring an infinite constant in the second equation). To get a flow equation in this background field formalism we need first to define an object $W_L[\phi_c, J]$ from equation (2.2) by removing the restriction $J = \bar{J}$ (2.8) and of course replacing $K$ by $K_L$. Then differentiating this equation with respect to $\ln L$ and replacing $\phi' \rightarrow -\frac{\delta}{\delta J}$ we easily obtain \( (\dot{=} = L \frac{d}{dT}) \)
\[
\dot{W}_L[\phi_c, J] = tr K_L \left[ \frac{\delta W_L}{\delta \bar{J}} \frac{\delta W_L}{\delta \bar{J}} - \frac{\delta}{\delta \bar{J}} \frac{\delta W_L}{\delta \bar{J}} \right]. \quad (2.6)
\]Since $K_L$ is linearly divergent as $L \rightarrow 0$ the initial condition has to be imposed at $L = \epsilon << |\phi_c|^{-2}$ and we need to keep the $O(\epsilon)$ term so that we have (from the second line of equation (2.2))
\[
W_\epsilon = I[\phi_c] + \frac{1}{2} tr \ln K_\epsilon[\phi_c] - \frac{1}{4} J_\epsilon K_\epsilon^{-1}[\phi_c].J \quad (2.7)
\]The equation then gives us $W_L$. Now the Wilsonian action $\Gamma_L$ is obtained from this by putting $J = \bar{J}$ but since $\bar{J}$ is defined in terms of $\Gamma_L$ (see 5.3) this only gives an implicit definition. To get an explicit version we need to Legendre transform with respect to $J$. Putting $\frac{\delta W_\bar{J}}{\delta \bar{J}} = <\phi'>_J \equiv \phi'_c$ and
\[
\tilde{\Gamma}[\phi_c, \phi'_c] = W_L[\phi_c, J] - J_\epsilon \phi'_c \quad (2.8)
\]
one immediately has from \(\delta \hat{\Gamma} = \hat{W} = \frac{\delta W}{\delta J} = -\left[\frac{\delta \bar{\Gamma} L}{\delta \phi \phi'}\right]^{-1}\) the flow equation,

\[
\dot{\hat{\Gamma}}_L[\phi_c, \phi'_c] = tr \hat{K}_L[\phi_c] \left\{ \phi'_c \phi'_c + \left[\frac{\delta \bar{\Gamma}_L}{\delta \phi'_c \phi'_c}\right]^{-1}\right\},
\]

(2.9)

and the initial condition (2.7) becomes,

\[
\Gamma_\epsilon(\phi_c, \phi'_c) = I[\phi_c] + \frac{1}{2} tr \ln K_\epsilon[\phi_c] + \frac{1}{4} \phi'_c K_\epsilon \phi'_c
\]

(2.10)

In the limit \(L \to \infty\) we have \(\phi'_c = \langle \phi' \rangle, \phi = \langle \phi \rangle, \phi_c = 0\) and \(\bar{\Gamma}[\phi_c, 0] = \Gamma[\phi_c]\) the 1PI effective action. Hence we may define the Wilson effective action as

\[
\Gamma_L[\phi] = \bar{\Gamma}_L[\phi_c, 0]
\]

(2.11)

Thus the equation gives us a Wilson effective action at a low scale \(\Lambda = L^{-1/2}\) given an "initial" action at a high scale \(\Lambda_0 = \epsilon^{-1/2}\).

For a gauge theory the above derivations go through essentially unchanged except for the fact that \(K\) now involves terms from the gauge fixing and also from ghosts. The effective action obtained this way is background gauge invariant. However the introduction of the cutoff \(L\) violates the BRST invariance of the functional integral. The question is, given the background gauge invariance of the effective action, does this matter? Through its dependence on \(K\) the right hand side of the evolution equation is explicitly dependent on \(\xi\) the gauge fixing parameter. With generic initial data we would evolve into an effective action that is \(\xi\) dependent. On the other hand in the limit \(L \to \infty\) the functional integral representation implies that BRST invariance is restored and the 1PI effective action is independent of \(\xi\). It is hard to understand how this comes about from the evolution equation, unless one fine tunes the initial conditions at the physical cutoff in a \(\xi\) dependent manner. This seems to be rather an unsatisfactory state of affairs. Unfortunately we do not know of any procedure for resolving this. Nevertheless we will ignore this gauge fixing dependence in the rest of the paper.

\(^5\)I wish to thank Joe Polchinski for raising this issue.
3 Effective action for composite fields

As constructed above the Wilson action is not suitable for a description of the physics below the confinement scale in a theory such as QCD, where one expects the effective low energy fields to be mesons and baryons. We would need an action that is expressed in terms of these gauge invariant degrees of freedom such as the chiral Lagrangian. So one may define (in a schematic though obvious notation), a generating functional for mesons in the following way\footnote{The extension to baryons can be done in an analogous manner.}:

\[ e^{iW[J]} = \int e^{i\Gamma_{\epsilon}[Q,\bar{Q},V] + iJQ\bar{Q}}[dQd\bar{Q}dV] \int [dM] \delta(M - Q\bar{Q}) \]
\[ = \int [dM] e^{i\bar{\Gamma}_{\epsilon} + iJM}. \quad (3.1) \]

In the above \( Q, \bar{Q} \) are quark and anti-quark fields, \( V \) represents the gauge fields and \( M \) is a meson field. Also \( G_{\epsilon} \) is the Wilson action valid below the scale \( \epsilon^{-1} \) (so it should be identified with \( \Lambda^2 \) of the previous section) and the functional integral is effectively cutoff above that scale. An effective action for mesons is then given by,

\[ e^{i\bar{\Gamma}_{\epsilon}[M]} = \int [dQd\bar{Q}dV] e^{iS_{\epsilon}[Q,\bar{Q},V]} \delta(M - Q\bar{Q}). \quad (3.2) \]

It is important to note that \( \bar{\Gamma}_{\epsilon}[M] \) inherits the ultra-violet cutoff so that in calculating meson processes from it one is only supposed to integrate up to the scale \( \epsilon^{-1} \). Also it is clear that \( \bar{\Gamma} \) has the same global symmetry as \( S \). It is of course expected that \( M \) will in general acquire a non-zero expectation value and the action that describes the degrees of freedom in that vacuum will only exhibit the diagonal \( SU(N_f) \times U(1) \) symmetry. It should also be pointed out here that \( \bar{\Gamma} \) is a Wilsonian action and is not the same as the 2PI effective action \[12\] which is defined by

\[ \Gamma[M_c] = (W[J] - J.M_c)_{M_c = \frac{\delta W}{\delta J}} \simeq \bar{\Gamma}_{\epsilon}[M_c] + O(h), \quad (3.3) \]

the last step following from the saddle point evaluation of the functional integral\[3.1\].
In the SUSY case the action $\bar{\Gamma}$ will satisfy in particular the requirements of holomorphy, so the arguments of [1] as well as that of earlier work [13], [14], [15], should apply to this. In particular this would mean not only that for $N_f > N_c$, $\det M = 0$ but also that (for $N_f \geq N_c$ the classical constraints (for instance $\det M = B\tilde{B}$) are satisfied. Now while the former result is one that is used in [14] to argue that no superpotential is generated for this case, the latter is at variance with the statement in [16] that the quantum moduli space is modified from the classical space, so that for instance in the $N_f = N_c$ case the constraint is changed to $\det M - B\tilde{B} = \Lambda^{2N_c}$. In order to resolve this it seems that one has to redefine the Wilson action for mesons (and baryons). To this end let us first write, (again we explicitly indicate only the meson dependence)

$$e^{-W[J]} = \int e^{-S_L(Q,\tilde{Q},V) - J(Q\tilde{Q})}$$

The functional integral in the above is taken down to some low (non-zero) scale $L^{-\frac{1}{2}}$ and we define the expectation value in the presence of this cutoff by,

$$\frac{\delta W[J]}{\delta J} = <QQ>_{L,J} \equiv M_{L,c}$$

This expression agrees with the actual vacuum expectation value of the theory when we let $L \to \infty$. Now we define the Wilson effective action for the mesons by the Legendre transform.

$$\Gamma_L(M) = (W[J] - J.M)$$

where $J$ is replaced by solving $3.3$ and we have the equation of motion $\frac{\delta \Gamma}{\delta M} = J$. This is a Wilsonian action for finite $L$. To get the true 1MI (2QI) action one needs to take the limit $L \to \infty$ and we have then a functional of the true vacuum expectation value. This will suffer from infra-red problems, but the Wilsonian version defined above has an explicit

7Henceforth we drop the subscripts on $M$, but this object should not be confused with the meson field introduced earlier, which was restricted to be $QQ$.

8An interpretation of the effective action of [13] involving the composite field $S \simeq W^2$ where $W$ is the gauge superfield, as a 2PI effective action has been given earlier in [17].
infra-red cutoff and is therefore expected to satisfy all the holomorphicity restrictions. The arguments of Seiberg et al will therefore apply to this object. Furthermore we see from (3.3) that quantum effects will prevent us from concluding that \( \det M = 0 \) for \( 3N_c > N_f > N_c \) and hence we may write down a superpotential

\[
W = \left( \frac{\det M}{\Lambda_s^{N_f-N_c}} \right)^{\frac{1}{N_f-N_c}} f(L\Lambda_s^2) \tag{3.7}
\]

even in this case (\( \Lambda_s \) is the SQCD scale). It should be noted that this interpretation of \( M \) would appear to resolve the apparent conflict between [14] and [13]. As pointed out in [10] however there are other problems with such an effective action. One is that this superpotential is singular at the origin\(^9\). The other point is that the massless degrees of freedom at the origin of moduli space do not match the quarks and gluons of the original formulation. Equation (3.7) should not therefore be considered as an alternative to the discussion of Seiberg for \( N_f \geq N_c \), nevertheless it is amusing to consider some of its consequences, especially since it can be written down once one adopts (3.4) - (3.6) in contrast to (3.2) as the definition of the effective action. We will have more to say about the origin of moduli space later, but for the time being we will consider the theory with mass terms for mesons (and baryons) so that the minimum is away from the origin. In this case one can still use the above (and its generalizations including baryons) to calculate expressions for the vacuum expectation values. As pointed out in [4] one can consider such terms (for instance a term \( m_i \hat{M}_i \)) as arising from a coupling to a background field (or external source) and then the modification of the field \( M = < Q \bar{Q} >_L \) in the presence of this source is given by the (analog of) the standard equation for the effective action \( \Gamma \) namely \( \frac{\delta \Gamma}{\delta M} = J \). Thus the new expectation value is determined by,

\[
\frac{\delta W}{\delta M_i} = m_i \tag{3.8}
\]

\(^9\)The superpotential for \( N_f < N_c \) is also singular at the origin but in that case the potential is minimized away from the origin at infinity in fact, whereas in the present case it is singular - though finite, at the point where its minimum lies.
with $W$ given by (3.7). This yields (upon taking the limit $L \to \infty$ since in the presence of a non-singular mass matrix $[m]$ there are no infra red singularities) the result of Amati et al [15] that was used in [16].

$$M^i_{c \ i} = <Q_i \bar{Q}^i> = \Lambda^{\frac{3N_c-N_f}{N_c-N_f}} (\det[m])^{\frac{1}{N_c}} [m^{-1}]^i_i$$  (3.9)

As observed by Seiberg [16] taking $m \to 0$ in different directions one can generate any value for the meson expectation values of the massless theory.

Note that even though (3.7) is not defined for $N_f = N_c$ one can take this limit in (3.9) and we get $M = \Lambda^2 (\det[m])^{\frac{1}{N_c}} [m^{-1}]$ so that $\det M = \Lambda^{2N_c}$.

When $N_f > N_c$ we can also define $B_{L,c}^{f_{N_c+1}...f_{N_f}} = \epsilon^{f_1...f_{N_f}} <Q_{f_1}...Q_{f_{N_c}}>_L$ where the RHS is defined as in (3.4, 3.5). Then one can for instance generate terms of the form $W \sim \Lambda^{\# B_{c}^{f_{N_f}}} \tilde{B}_{c}^{f_{N_f}}$ and by adding source terms of the form $bB$ we can generate non-zero expectation values for $B$. The $N_f = N_c$ case is again a limiting one and as with the mesons we argue that even though the superpotential at this point is not defined one can still first compute the expectation values (in the presence of the sources) and then take the limit $N_f \to N_c$.

The cases $N_f = N_c$ and $N_f = N_c + 1$ were discussed extensively in [16]. The arguments depend crucially on the assumption that the constraint on the quantum moduli space is changed (in the $N_f = N_c$ case) from the classical one $\det M - B \tilde{B} = 0$, to the relation $\det M - B \tilde{B} = \Lambda^{2N_c}$. As pointed out earlier this is consistent only with the interpretation of the action given in (3.9). It should also be noted that although with (3.1) we are supposed to integrate down from the infra-red cutoff to get the 1MI (and 1BI) actions, with (3.6) one simply takes the limit $L \to \infty$. In other words with the latter, one is not supposed to compute loops to get the 1PI (or 2PI etc.) actions whereas with the former (the chiral lagrangian for QCD for instance), one computes loops.

The most interesting results of the analysis of Seiberg and collaborators relate to

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10In the above we have put $\frac{f(\infty)}{N_c} \frac{N_c-N_f}{N_c} = 1$ equal to unity. This amounts to a convention for the definition of $\Lambda$. 

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the discussion of the theory at the origin of moduli space. This is the point at which superpotential is singular and it is argued that the description in terms of $M, B$ becomes invalid for $N_f > N_c + 1$. Below we will discuss some questions related to the origin of moduli space.

## 4 The origin of moduli space

Firstly it should be pointed out that since $M = <Q\tilde{Q}> = \left[\frac{\delta^2W}{\delta Q\delta\tilde{Q}}\right]^{-1}$ the point $M = 0$ is a place where the quark inverse propagator blows up. \textsuperscript{11} It is thus possible (as already observed in \cite{10}) that this point is infinitely far away in moduli space. Let us examine these issues more closely.

To begin the discussion let us consider the abelian Higgs model. The model has an Abelian gauge field $A_\mu$ and a complex scalar field $\phi$ with the invariance under the $U(1)$ gauge transformations $A_\mu \rightarrow A_\mu + \partial_\mu \chi$ and $\phi \rightarrow e^{i\chi}\phi$. The degrees of freedom (DOF) in the model are counted as follows: $A_0$ is a Lagrange parameter ($\Pi_0 = 0$) leaving $2 \times (3 + 2)$ phase space dimensions for the three spatial gauge fields and two scalar fields. The Gauss law constraint $\nabla . E = j(\phi, \Pi_\phi)$ and the gauge freedom $A \rightarrow A + \nabla \chi$ then reduce the dimension of phase space by two leaving four as the number of DOF. This may be interpreted as two for the (massless) gauge boson and two for the complex scalar.

Now in this theory it is usually argued that when the potential for $\phi$ has a non-trivial minimum ($V'(\phi_0 \neq 0) = 0$), the gauge symmetry is spontaneously broken; the resulting Goldstone mode is “eaten” by the gauge field which thus acquires a mass and hence has three degrees of freedom with one more coming from real part of the Higgs field. This at least is the classical argument.

Quantum mechanically the argument is somewhat more subtle. The problem is that in a quantum gauge theory one can never really “break” the gauge invariance. The reason is that (at least for compact gauge groups) Elitzur’s theorem \cite{18} tells us that the

\textsuperscript{11}It should be noted that we are always working with a ultra violet cutoff so that products of fields at the same point are well defined.
vacuum expectation value of any local order parameter (which does not have a gauge singlet component so that its integral over the gauge group vanishes) such as $< \phi >$ is zero.\footnote{Strictly speaking this theorem has been proved only for a Euclidean lattice gauge theory. In a Minkowski formulation it would follow from the Gauss law constraint provided that a non-perturbative gauge invariant regularization exists.} Thus if we construct a gauge invariant (1PI or Wilson) effective action $\Gamma[A_c, \phi_c]$ the solution of $\frac{\delta \Gamma[A_c, \phi_c]}{\delta \phi_c} = 0$ is $< \phi > = 0$ which seems to indicate that there is no spontaneous symmetry breaking. However it is possible to reformulate the theory in terms of gauge invariant variables\footnote{The fact that Higgs theories can be expressed in terms of physical gauge invariant variables is an old story and was discussed in connection with the absence of a phase transition between Higgs and confinement phases in theories with fundamental Higgs in\cite{11}.}. One makes the field redefinitions\footnote{This procedure is usually called gauge fixing to the unitary gauge. However it is more appropriate to call it a field redefinition. The discussion later, of the measure should clarify the difference.} $\phi = e^{ig\theta} \rho$ ($\rho > 0$), $A_\mu = G_\mu + \partial_\mu \theta$. Under the gauge transformation $\theta \to \theta + \epsilon$; but the fields $\rho$, $G$ are gauge invariant. Because of gauge invariance the $\theta$ dependence drops out and the classical Lagrangian becomes,

$$L = -\frac{1}{4}((\partial_\mu G_\nu - \partial_\nu G_\mu)^2 - \frac{1}{2}(\partial_\mu \rho)^2 - \frac{1}{2}g^2 \rho^2 G_\mu^2) - V(\rho). \quad (4.1)$$

Obviously when expressed in terms of the physical variables, there is no gauge invariance in the Lagrangian. It is instructive to count the degrees of freedom (DOF) in this formulation. Again $G_0$ is Lagrange multiplier and so the phase space variables are $$(\rho, \Pi_\rho), \ (G, \Pi = E).$$ These variables are unconstrained (the Gauss law which is now $\nabla \cdot E = g^2 G_0 \rho$ involves the Lagrange multiplier) and hence the number of degrees of freedom are again four, but with one coming from the scalar and three from the gauge field.

It should be stressed that this has nothing to do with the shape of the potential $V$. What does depend on the latter is the count of linearized DOF. Unless $V(\rho)$ has a non-trivial minimum this count will not give the correct number of DOF.

What happens in the quantum theory? Here we need to consider the measure. Under our change of variables we get $[dA][d\phi_R][d\phi_I] = [dG][d\theta][\rho d\rho]$. Since the action is independent of $\theta$ the first factor just gives the volume of the gauge group. The last factor exhibits
the typical singularity of polar coordinates and is not in a suitable form for perturbation theory. A further change of variable \( \rho \rightarrow \chi = \rho^2 \) gives a linear measure but now the problem reappears in the kinetic term which becomes \( \frac{1}{8\chi}(\partial\chi)^2 \). The metric on “moduli” space appears to be singular though the distance from any point to the origin is in this case is actually finite (of course here the singularity is just a coordinate singularity because the space is one dimensional). A perturbative evaluation of the functional integral requires the existence of a saddle point \( \chi_0 \neq 0 \). Nevertheless the expression eqn. (2.2) will remain well defined even if (when the external source is set to zero) \( \chi_c = <\chi> = 0 \).

These arguments are easily generalized to non-Abelian Higgs theories. Let us first consider the case of a Higgs field \( \phi \) in the fundamental representation of the gauge group \( \mathcal{G} = SU(N) \). Under gauge transformation \( \phi \rightarrow g\phi \) and \( A \rightarrow gAg^{-1} + gdg^{-1} \) where \( g \in \mathcal{G} \). Again we can redefine the fields by putting \( \phi = \Omega\rho \) (with \( \rho = [\rho, 0...0]^T \) and \( \Omega \in \mathcal{G} \) with \( \rho \) a real and positive field) and \( A = \Omega(\partial + G)\Omega^{-1} \). The gauge transformations are \( \Omega \rightarrow g\Omega \), \( \rho \rightarrow \rho \), \( G \rightarrow G \), and the Lagrangian becomes

\[
L = -\frac{1}{4}trF^G \rho^2 - \frac{1}{2}(D^G\rho)^2 - V(\rho).
\]

(4.2)

In these variables the gauge invariance has been reduced from \( SU(N) \) to \( SU(N-1) \). This fact is again independent of the nature of \( V \). Let us now look at the measure. The original gauge invariant metric on field space is \( \int d^Dx(\delta\phi^\dagger\delta\phi + tr\delta A^2) \). Using \( \Omega^\dagger\Omega = \Omega\Omega^\dagger = 1 \) and the reality of \( \rho \), we find that this becomes \( \int d^Dx(\rho^\dagger\delta\Omega^\dagger\delta\Omega\rho + (\delta\rho)^2 + (\delta G^2)) \). Hence the measure becomes \( \prod_x[d(G/H)(x)]\rho(x)d\rho \prod_{\mu\alpha}\delta B^a_{\mu} \), where the first factor is a volume element on \( SU(N)/SU(N-1) \) which will just give a factor of the volume of the coset in the functional integral. Again as in the Abelian case we have the quantum theory expressed entirely in terms of variables which display a smaller gauge invariance than the original ones but as in the earlier case there is no perturbative description around \( \rho = 0 \).

By repeating the process when there are \( M \) complex scalar fields in the fundamental representation we can express the quantum theory (i.e. the classical lagrangian as well as the measure) completely as a \( SU(N-M) \) invariant theory for \( M < N - 1 \) and as a theory with no gauge invariance for \( M \geq N - 1 \). Again this fact has nothing to do with the
nature of the classical potential.

Consider now a scalar Higgs field in the adjoint representation. We may do the field redefinition \( \Phi = \Omega D \Omega^{-1} \) and \( A = \Omega(\partial + G)\Omega^{-1} \), where \( \Omega e G \) and \( D = \text{diag}[\lambda_1, \ldots, \lambda_N] \), \( (\lambda_i \text{ being the (real) eigenvalue fields of the Hermitian matrix } \Phi) \). As before \( \Omega \) disappears from the Lagrangian but now we are left with a \( U(1)^{N-1} \) symmetry (except on a set of measure zero in the field space where the symmetry is larger). The metric on field space may now be written as

\[
\int d^Dx (tr\delta\Phi^2 + tr\delta A^2) = \int d^Dx \{-2tr[\Omega^{-1}\delta\Omega D]D - D^2(\Omega^{-1}\delta\Omega) + tr\delta G^2\} \quad (4.3)
\]

From this we would again expect a measure of the form \( \sim Vol[SU(N)/U(1)^{N-1}] \prod x D\delta D\delta G \), though it is not clear to us how to establish this precisely. Again one expects that because of the singularity at \( D = 0 \) that perturbation theory about this point is invalid. Nevertheless the complete regulated (for example on a lattice) functional integral expressed as a theory with reduced gauge invariance is perfectly well defined.

Let us now turn to SUSY theories. Consider first \( N = 1 \) SQCD and for simplicity we will take the gauge group to be \( SU(2) \) with two doublet chiral super fields, \( \Phi, \tilde{\Phi} \). The Lagrangian takes the form (in standard superfield notation),

\[
L = \frac{1}{4\pi} \text{Im} \int d^2\theta tr W^\alpha W_\alpha + \int d^4\theta \Phi^\dagger e^V \Phi + \tilde{\Phi} e^{-V^T} \tilde{\Phi} + \int d^2\theta [\Phi, \tilde{\Phi}] + h.c. \quad (4.4)
\]

Under a gauge transformation, we have \( \Phi \to e^{i\Lambda} \Phi, e^V \to e^{-i\Lambda^T} e^V e^{i\Lambda} \) and the gauge invariant measure is obtained from the metric,

\[
||\delta V||^2 + ||\delta \Phi||^2 + ... = \int d^x d^4\theta tr[(e^{-V} \delta e^V)^2 + \delta \Phi^\dagger e^V \delta \Phi + ...] \quad (4.5)
\]

(the ellipses are for the \( \tilde{\Phi} \) terms). This is a highly non-linear metric and so is the associated metric. (In SUSY perturbation theory the measure may be replaced by a linear approximation to it). Let us now try to do the (analogs of the) field redefinitions that we performed in non SUSY theories. Thus we put \( \Phi = e^{-i\omega} \rho, \tilde{\Phi} = e^{-i\omega} \tilde{\rho} \) where the first factor in these expressions is a chiral superfield valued in the gauge group and \( \rho = [\rho, 0]^T \).

Also we put \( e^U = e^{i\omega} e^V e^{-i\omega} \). The gauge transformation is now \( e^{-i\omega} \to e^{-i\Lambda} e^{-i\omega} \) but \( \omega \)
disappears from the Lagrangian and we have a classical theory expressed entirely in terms of gauge invariant variables. Note that $U$ is no longer in Wess Zumino gauge, but (for each generator of the group) the 4 massless degrees of freedom of the vector multiplet combine with the 4 DOF of the chiral multiplet $\omega$ to give the 8 DOF of a massive multiplet. This is just the Higgs effect in a SUSY theory and again as in the case of the non-SUSY Higgs theory discussed above the count of the DOF of the classical non-linear evolution equations have nothing to do with the nature of the potential. Unfortunately in the SUSY case it is not easy to show that the volume of the gauge group factors out. It may be recalled that in the non-supersymmetric case the metric became diagonal because we of the reality of the (gauge invariant) Higgs field $\rho$. In the SUSY case this is necessarily chiral and we cannot make the same argument. Nevertheless it is still reasonable to assume that the qualitative features of the previous discussion (for example the statement that perturbation theory around $\rho = 0$ is singular) may still be expected to be valid. In any case the potential is given by $V = G^{i, j} \frac{\partial W}{\partial \phi^i} \frac{\partial \bar{W}}{\partial \bar{\phi}^j}$ where $G$ is the metric on moduli space and it is invariant under holomorphic field redefinitions and so there are additional singularities (i.e. apart from the singularities of $W$), only if there is a true singularity of the metric as opposed to a coordinate singularity. Our point here is that the singularity at the origin coming from the measure appears to be of the latter kind and hence should not have any physical effect (in contrast to those of the superpotential discussed by Seiberg [1]).

Next we make a remark about $N = 2$ SUSY theories. In these theories there is a $N = 1$ chiral superfield in the adjoint representation, and hence as in the corresponding case of the non-SUSY theory discussed above, (modulo the caveat about measures in the previous case) the $SU(N)$ theory is equivalent to a $U(1)^{N-1}$ theory except that there is no perturbative description around $D = 0$ (here the $D$ is defined by the SUSY generalization of the non SUSY version given in the paragraph above (4.3). As explained in [20] one needs in that case to go to the dual theory, which brings in the issues of monopole condensation and the dual Meissner effect. It appears to us however that this effect, while it has a gauge invariant formulation in these $N = 2$ theories is not easily taken over as an explanation of confinement in QCD where there is no adjoint scalar field to break
the symmetry to $U(1)^2$. The attempts to gauge fix to the maximal Abelian gauge \cite{21} for instance by introducing a term $G^2$ (where $G$ is the gluon field strength) is tantamount to introducing a factor $\frac{0}{0}$ into the path integral (since the integral over the gauge group of any non-trivial irreducible representation of the group is zero) and is thus bound to yield ambiguous results.

\section{Supersymmetry breaking}

Finally let us make some remarks on supersymmetry breaking. We will combine the old observations of Giradello and Grisaru \cite{22} on soft breaking terms, the recent work of Seiberg and Intrilligator \cite{1}, and those of this paper, to investigate to what extent non-perturbative statements can be made in the presence of SUSY breaking.

Consider the SQCD action (4.4) together with the following terms coupling local gauge invariant composite fields to external sources.

$$S_B = \int d^4x \{ J_T^T e^V Q|_D + J_{\tilde T}^T e^{-V^T} \tilde Q|_D + \right.$$

$$\left. + (J_S W_a W^a + J_M Q \tilde Q + J_B Q^{Nc} + J_{\tilde B} \tilde Q^{Nc} + h.c.)|_F \right\}$$

(5.1)

In the above $J_T^T (\tilde T^T)$ are real super fields while the other sources are chiral superfields. Also for simplicity flavor symmetry invariance is preserved.

The computation of correlation functions of gauge invariant composite fields in the SUSY theory are obtained by differentiating the generating functional $W[J]$, defined in (3.4), with respect to the sources and then setting them equal to zero. The correlation functions of (softly) broken theory on the other hand may be obtained by the same generating functional by doing the functional differentiations and then setting

$$J_T = m_T \theta \bar \theta \bar \theta, \quad J_S = m_s \theta, \quad J_M = m_M \theta \theta, \quad etc.$$  (5.2)

\footnote{In using the method of reference \cite{22} we find it unnecessary to include a global SUSY breaking (O’Raffeartaigh) sector as in \cite{23}. This is just as well since in realistic models one expects the soft breaking to be generated from the breaking of local SUSY at the Planck scale.}
The equations
\[
\frac{\delta W}{\delta J_T} = \langle Q^T e^V Q \rangle \equiv T, \quad \frac{\delta W}{\delta J_S} = \langle W^\alpha W_\alpha \rangle \equiv S, \\
\frac{\delta W}{\delta J_M} = \langle \tilde{Q}^T \tilde{Q} \rangle \equiv M, \quad \frac{\delta W}{\delta J_B} = \langle Q^N e \rangle \equiv B
\] (5.3)
together with two more for anti-quarks and anti-baryons) define effective fields in terms of which an effective action for mesons and baryons, and glueballs can be defined. Indeed if \( W \) has been defined in the presence of an infra-red cutoff as in (3.4) then we can again define a Wilsonian effective action \( \Gamma[T, \tilde{T}, S, M, B, \tilde{B}] \) as in (3.6) such that the effective fields (in the presence of the source superfields) are solutions of the equations,
\[
\frac{\delta \Gamma}{\delta T} = J_T, \quad \frac{\delta \Gamma}{\delta S} = J_S, \quad \frac{\delta \Gamma}{\delta M} = J_M, \quad \frac{\delta \Gamma}{\delta B} = J_B, \quad etc. \tag{5.4}
\]
The effective action in the presence of SUSY breaking can then be written as,
\[
\tilde{\Gamma}[T, ... \tilde{B}, J] = \Gamma[T, ... \tilde{B}] + J_T T|_D + ... + (J_{\tilde{B}} \tilde{B}|_F + h.c. \tag{5.5}
\]
with \( J \) being replaced by the values given in (5.2). When the supersymmetry breaking parameters (\( m_T, ... m_B \)) are set to zero and \( T \) is replaced by using the equation of motion \( \frac{\delta \Gamma}{\delta T} = 0 \) giving \( T = a M^\dagger M + b B^\dagger B + etc., \) the effective action should reduce to those derived in the literature. However in the presence of the D-term SUSY breaking involving the real superfield \( T \) very little can be said about the SUSY broken theory. This is because in solving for \( T \) one now has to use \( \frac{\delta \Gamma}{\delta T} = J_T \) and without knowing the dependence of \( \Gamma \) on \( T \) it is not possible to extract the dependence of the SUSY broken action on the symmetry breaking parameters \( M_t, m_{\tilde{T}}. \) Of course one may consider pure F type symmetry breaking and then one would be able to calculate precisely the modifications of the results of Seiberg and collaborators. However as is well known such breaking is not phenomenologically acceptable since for instance they predict squark (and slepton) masses which are smaller than those of quarks (and leptons). One can also assume that the SUSY breaking parameters are small and then the qualitative features of the SUSY limit are preserved [24]. Unfortunately in the real world the SUSY breaking scale is much
larger than the QCD scale. Of course in any case, since there is no chiral scalar superfield description of the proton for instance\footnote{the color antisymmetrization will cause $uud$ to vanish when each is a scalar super field.} there is no smooth limit from the effective theory of the mesons and baryons defined by Seiberg\footnote{For instance with $N_f = N_c$ where the number of massless constituent degrees of freedom in the SUSY limit is matched by the set of $M$'s and $B$'s defined in \cite{1}. Perhaps this implies that the proton composite operator becomes massive.} \cite{1,2} to an effective description in terms of protons, neutrons, pions etc of strongly coupled QCD. A description in terms of the observed baryons and mesons, of the limit where the superpartners decouple, which is smoothly obtained from the SUSY limit, would seem to require additional operators in the original SUSY theory. This fact may have important implications for understanding supersymmetry breaking.

Note added: While preparing this work for publication we came across\footnote{For instance with $N_f = N_c$ where the number of massless constituent degrees of freedom in the SUSY limit is matched by the set of $M$'s and $B$'s defined in \cite{1}. Perhaps this implies that the proton composite operator becomes massive.} in which aspects of supersymmetry breaking are discussed.

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