LOW-DENSITY PARITY-CHECK CODES: COMPARING CLUSTER GRAPH TO FACTOR GRAPH REPRESENTATIONS

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ABSTRACT

We present a comparison study between a cluster and factor graph representation of LDPC codes. In probabilistic graphical models, cluster graphs retain useful dependence between random variables during inference, which are advantageous in terms of computational cost, convergence speed, and accuracy of marginal probabilities. This study investigates these benefits in the context of LDPC codes and shows that a cluster graph representation outperforms the traditional factor graph representation.

Keywords LDPC codes · cluster graphs · factor graphs

1 Introduction

Low-density parity-check (LDPC) codes were first introduced by Gallager in 1962 and rediscovered in 1995 by MacKay [1,2]. These codes are a family of block codes with good theoretical and practical properties and have since been used in a wide variety of wireless communication systems including the 5G new radio (NR) technology standard.

Central to the design of LDPC codes is a sparse matrix that describes dependencies (connections) between information bits and redundant check bits. The matrix translates to a factor graph (also known as a Tanner graph) representation of the code and is used in the encoding and decoding procedures. The factor graph consists of variable nodes, which represent all bits in a codeword, and check nodes that represent the parity check factors. The decoding procedure is applied to the factor graph directly and uses an iterative message passing algorithm that passes messages between variable nodes and check nodes to update the received bit values. The basic method of decoding uses the sum-product algorithm, which is equivalent to the loopy belief propagation (LBP) algorithm used for performing generic inference tasks on probabilistic graphical models (PGMs) [3,4].

Factor graphs are popular throughout the PGM literature, arguably due to their simple construction, which entails a decoupling of all random variables from their factors. However, a joint probability distribution can be factorised into groups of random variables by considering how factors are manipulated during the inference process. This can lead to better-performing PGMs according to [3]. We are reminded of a meditation written by John Donne containing the well-known phrase “No man is an island entire of itself”. In relation to our study, this phrase guards against “individualising” the variable nodes from the full joint distribution, consequently it removes important inter-dependencies between bits that are useful during the decoding procedure. To explore the alternative, we consider a more general probabilistic framework called cluster graphs, which involve joint distributions as messages between nodes instead of univariate distributions.

In general, PGMs with cycles (or loops) are not guaranteed to converge to exact solutions, and interesting problems (such as LDPC codes) are intractable to convert to tree structured graphs. While cluster graphs are not necessarily tree structured, they must satisfy specific structural constraints such as the running interception property (RIP), which ensures accurate inference when using a message passing approach [4]. The layered trees running intersection property (LTRIP) algorithm developed in [5], compiles cluster graphs from a set of input factors and guarantees the RIP that is a step towards tree structured PGMs.
In [5,6], the authors found that cluster graphs outperformed factor graphs in inference tasks related to solving complex Sudoku puzzles. Our study extends the application domain of the LTRIP algorithm, which we use to compile a cluster graph of an LDPC code. We highlight the LDPC code’s performance by investigating the computational benefits and accuracy improvements compared to a factor graph.

**Our contribution:** We view LDPC codes more generally as a PGM and represent it as a cluster graph compiled by a general-purpose algorithm called LTRIP developed in [5]. To the best of our knowledge, this is the first LDPC code represented as a cluster graph, which may be due to a lack of available algorithms that can construct valid cluster graphs. We develop a message passing schedule, since the message order in cluster graphs can influence convergence speed, accuracy, and the computational cost of inference. We demonstrate (1) computational benefits between a cluster graph approach and a standard factor graph approach, (2) convergence benefits, and (3) more accurate marginal probabilities that improve the bit error performance of the LDPC code.

This paper is structured as follows: In Section 2 we explain how LDPC codes are represented in a more general PGM framework. Section 3 compares factor graphs to cluster graphs. A message schedule developed for the cluster graph is discussed in Section 4. The results are shown in Section 5 and finally our conclusions and future work are presented in Section 6.

2 Low-density parity-check codes as PGMs

This section provides an overview of error correction using LDPC codes and introduces a more general representation of LDPC codes using cluster graphs instead of factor graphs. We also explain our message passing approach and end the section with experimental results.

2.1 Error correction with LDPC codes

*Low-density parity-check* (LDPC) codes are a family of linear block codes used as an error correction measure in digital communication systems. A noisy transmission link may cause bit errors that corrupt transmitted messages. LDPC codes use an encoding procedure that adds parity check bits to the message bits that help detect and fix bit errors at the receiver.

A check bit’s value is determined by applying an even parity constraint to a subset of message bits, which ensures that the check bit together with the subset of message bits contain an even number of ones. An LDPC code comprise multiple even parity constraints and some constraints may include other check bits to enhance the code’s overall protection capability. The original message bits together with the encoded check bits is known as a **codeword**. The ratio between the number of message bits $K$ and the total number of codeword bits $N$ is known as the **code rate** $\frac{K}{N}$, which expresses the portion of useful data sent to a receiver. Lower code rate LDPC codes are more robust to bit errors, but exhibit a lower data transfer rate.

From a decoding perspective, a valid codeword must satisfy all parity check constraints originally imposed by the encoder. This can be verified by taking the modulo-2 addition (XOR operation) of a check bit and all its bit dependencies and must equal zero. A group of such checks, called a **syndrome**, can indicate which parity check constraints are not satisfied (typically using matrix notation). Together, the parity check constraints form a sparse system of linear equations that should be solvable using deductive logic. Intuitively, bits that are known to be correct (indicated by the syndrome) can be back-substituted into other parity check constraints to find correct values for other bits. Decoding in this way can be done mathematically using methods such as Gaussian elimination, but is computationally unfeasible for large LDPC codes.

Some bits must be shared between parity check constraints that create necessary associations among constraints, but at the same time may introduce inaccuracies when decoding the received codeword. This is more commonly referred to as **loops** or **cycles** and manifests when a similar set of bits are shared among more than one parity check constraint. A well designed LDPC code does not contain small cycles, and its codewords remain sufficiently separable from one another even after sustaining severe bit errors.

A parity check constraint is better known as a parity check **factor**. An example of such a factor, from a Hamming (7,4) code, with bit dependencies $b_0, b_1, b_2$, and $b_3$ is shown in Table 1. This is known as a discrete table factor. Note that we assign ones (absolute certainty) to the joint states where all bits present an even number of ones, with zero (no possibility) given otherwise. In this case, the check bit is $b_4$ and its value is determined from message bits $b_0, b_1,$ and $b_3$ during encoding.
Table 1: A non-normalised discrete table factor representing a parity check factor.

|   |   |   |   | \(\phi(b_0, b_1, b_2, b_4)\) |
|---|---|---|---|------------------|
| 0 | 0 | 0 | 0 | 1                |
| 0 | 0 | 1 | 1 | 1                |
| 0 | 1 | 0 | 1 | 1                |
| ... | ... | ... | ... | ... |
| 1 | 1 | 1 | 1 | 1                |
| elsewhere | 0 |       |      |                  |

Apart from the received codeword, a decoder also requires knowledge of the codeword construction (i.e., the bit dependencies of all parity check factors) and some iterative message passing algorithm that performs the deductive logic. In this study, we consider a different representation of LDPC codes that is motivated in the following section.

2.2 Representation of LDPC codes

In the channel coding literature, LDPC codes are represented by a bipartite graph called a Tanner graph. A Tanner graph has two distinct sets of nodes with edges between them that express relationships between codeword bits and parity check constraints. We use a Hamming (7,4) block code to illustrate this representation in Figure 1(a). We denote the message bit sequence as \(b_0, ..., b_3\), the parity check bits as \(b_4, ..., b_6\), and the parity check factors as \(\phi(b_1, b_2, b_3, b_5)\), \(\phi(b_1, b_3, b_4, b_6)\), and \(\phi(b_1, b_2, b_4, b_7)\). In a Tanner graph, the codeword bits are called bit nodes (bottom circles) and the parity check factors are called check nodes (top squares). An edge connects a bit node to a check node if that bit is included in the parity check factor. A sparse parity check matrix \(H\) is used to indicate the edge connections. A decoder proceeds by passing messages along the edges of the graph between the bit nodes and check nodes, which makes this representation intuitive and practical from both an encoding and decoding perspective.

A Tanner graph is more or less similar to a factor graph described in PGM literature [4, 7]. A factor graph is an undirected graph with variable nodes and factor nodes, which expresses a non-normalised joint probability distribution (also known as a potential function) as a product of factors. Factor nodes are connected to variables nodes if a factor depends on a variable. Its construction is simply a decoupling of all random variables from their factors. A joint probability distribution can, however, be factorised differently by considering how factors are manipulated during the inference process [4]. We consider a more general probabilistic framework, called cluster graphs, in which a factor graph representation is a special case. The factor graph in Figure 1(b) is presented using cluster graph notation, and is also known as a Bethé graph. Note the structural similarity between graph (a) and (b). Cluster graphs have two types of nodes, a cluster node (ellipse) is a set of random variables, and a sepset (short for “separation set”) node (square) is a set of random variables shared between a pair of clusters. Each parity check cluster contains a discrete table factor with its own scope of variables as shown previously in Table 1.

A cluster graph must satisfy specific structural constraints that ensure accurate inference when using a message passing approach. A key requirement is the running interception property (RIP) [4]. A path between clusters requires clusters that
and sepsets to share a common variable. The RIP then states that any pair of clusters sharing a common variable must have a unique path (i.e., without loops) between those clusters. The Tanner graph in (a) and factor graph in (b) satisfies the RIP due to its star-like topology. Note that all single variable clusters in (b) must contain factors with uniform distributions so that the graph-based representation corresponds to the original product of factors.

For exact inference, a graph needs to have a tree structure (i.e., without any loops whatsoever), which can be constructed for small PGMs (such as the Hamming (7,4) code) using the junction tree algorithm. However, this algorithm is NP-complete in obtaining optimal trees for large PGMs, in our case larger LDPC codes. The cluster graph in (c) is compiled using a general purpose cluster graph construction algorithm termed the layered trees running intersection property (LTRIP) algorithm developed in [5]. This algorithm proceeds in layers by considering each variable in a separate layer. For each variable, it determines an optimal tree structure over all clusters. The sepsets between pairs of clusters are then merged across all layers to form the final sepsets. The resulting cluster graph satisfies RIP and allows richer information content to be shared between clusters, since more that one variable can be embedded in a sepset. The resultant graph structure in general will contain loops, i.e. it is not necessarily a tree structure. Whereas inference on graphs with a tree structure is exact, inference on loopy graphs only approximate the true marginal distributions – we refer the reader to [3] for more detail regarding the LTRIP algorithm.

There are two notable differences between graph (b) and (c). Firstly, the factor graph has more nodes and edges than the cluster graph, which will require more computation during message passing. Secondly, all sepsets in the factor graph are univariate while some sepsets in the cluster graph contain joint probability distributions. Useful correlations between variables may be lost in the factor graph while a cluster graph can preserve them, making messages more informative. We describe our message passing approach in the following section.

2.3 Message passing approach

The sum-product algorithm is the basic decoding algorithm used for decoding LDPC codes [3, 8]. In the PGM literature, the sum-product algorithm is more commonly known as loopy belief propagation (LBP), and is used more generally to perform inference on graphical models. Another decoding approach is the maximum a posteriori probability (MAP) estimate [4], which gives the most likely assignment to all possible codewords. This is not the same as the maximum individual bit marginals. Our study uses a variant of LBP called loopy belief update (LBU), also known as the Lauritzen-Spiegelhalter algorithm [9]. We briefly point out the main differences between LBP and LBU:

- LBU uses cluster beliefs and sepset beliefs to express message passing.
- with LBU, cluster beliefs are updated with messages from all its connections and can be more informative compared to LBP,
- with LBU, sepsets are used to update a target cluster belief, and require only one sepset divide compared to two message divides in LBP.

The next section compares the performance of a factor and cluster graph representation of the Hamming (7,4) code.

3 Contrasting factor graphs to cluster graphs

We discussed the structural differences between the factor graph and cluster graph, and pointed out conceptual advantages of a cluster graph. Using the Hamming (7,4) code, we now consider the computational cost of message passing between the two different representations.

The factor graph in Figure 1(b) has 12 edges emanating from the parity check clusters. During a forward sweep (top to bottom) of message passing, each edge corresponds to a message (sepset belief) and requires $2^4 - 2^1 = 14$ additions to perform marginalisation from a parity cluster belief. The total number of additions required for all edges is $14 \times 12 = 168$. These messages are absorbed into the single variable target clusters, which require $12 \times 2^1 = 24$ multiplications, and completes the forward sweep. The backward sweep (bottom to top) requires $12 \times 2^4 = 192$ multiplications to absorb the updated single variable messages into the parity check clusters (marginalisations are not required due to the univariate sepsets). Message cancellation, for the forward and backward sweep, requires $2 \times (12 \times 2^1) = 48$ multiplications (actually divisions, but counted as multiplications).

A forward sweep (from left to right) for the cluster graph in Figure 1(c) requires $(2^4 - 2^2) + (2^4 - 2^1) + (2^4 - 2^2) = 38$ additions for marginalising the sepset beliefs. Absorbing the sepset beliefs into the target clusters requires $3 \times 2^4 = 48$ multiplications. The backward sweep (right to left) requires the same number of additions and multiplications as the forward sweep. The total number of message cancellation multiplications is $2 \times ((2^2) + (2^1) + (2^2)) = 20$. Table 2 compares the total number of operations required for one iteration of message passing between the factor graph and cluster graph. This indicates a computational advantage for the cluster graph representation.
Table 2: Computational cost difference between a factor graph and cluster graph representation of the Hamming (7,4) code for one iteration of message passing.

|                | Total additions: | Total multiplications: |
|----------------|------------------|------------------------|
| Factor graph   | 168              | 264                    |
| Cluster graph  | 76               | 116                    |

We now compare the error correction capability of the two graphs using binary phase-shift keying (BPSK) signal modulation over an AWGN channel without fading. For this channel model, the received bit values $x_n$ are Gaussian distributed conditioned on $b_n$. We assume a transmitted bit has unit energy and unit noise variance, which fixes the mean and variance parameters of each Gaussian distribution.

The bit error ratio (BER) performance of the factor graph and cluster graph is compared to exact inference in Figure 2(a). The cluster graph performs closer to the optimal BER curve compared to the factor graph. We use the Kullback-Leibler (KL) divergence $D_{KL}(P||Q)$ to evaluate similarity between marginal distributions $P$ obtained from exact inference and approximate marginal distributions $Q$ obtained from the factor and cluster graph respectively. The sum of all KL divergences from a codeword is captured for each codeword and the distribution thereof shown in (b) for each graph. When comparing the median statistics, the marginal distributions obtained from the cluster graph is more similar to the marginal distributions obtained from exact inference as compared to the factor graph. The KL divergence between the cluster graph marginals and exact inference marginals is also less varied compared to the factor graph. We also capture the number of iterations until successful decoding and compare the percentage of instances the decoder requires more than one iteration for successful decoding using a factor graph vs. a cluster graph.

![Figure 2](image-url)

- (a) BER comparison between exact inference, and 25 iterations of message passing in a factor graph and cluster graph.
- (b) Kullback-Leibler measure between exact inference marginals and marginals obtained from a factor graph and cluster graph.
- (c) Percentage of instances requiring more than one iteration for successful decoding using a factor graph vs. a cluster graph.

Figure 2: (a) BER comparison between exact inference, and 25 iterations of message passing in a factor graph and cluster graph, (b) Kullback-Leibler measure between exact inference marginals and marginals obtained from a factor graph and cluster graph, (c) percentage of instances requiring more than one iteration for successful decoding using a factor graph vs. a cluster graph.

4 Message passing schedule

A message passing schedule is an important consideration for loopy graphs, since the message order can influence convergence speed, accuracy, and the computational cost of inference. In loopy graphs, information can propagate from a cluster and continue along a path that eventually ends at the same cluster without traversing the same edge twice. Although not empirically verified here, these feedback loops (or cycles) may reinforce inaccurate cluster beliefs causing self-fulfilling belief updates, which affect the LDPC decoder’s performance. This problem is more prominent in LDPC codes with small feedback loops as described in [8]. Taking this into consideration, our message passing schedule: (1) uses a structured schedule with a fixed computational cost, (2) aims to minimise the effect of loops, and (3) aims to minimise the computational cost of inference.

The schedule is determined by firstly identifying the larger parity check clusters in the graph. Following Figure 3, we select the larger clusters $\phi_0$ (with cardinality 7) and $\phi_3$ (with cardinality 6). The message schedule starts with the selected clusters as initial sources and proceeds by visiting all its neighbouring clusters, which become the immediate next layer of clusters. A set of available clusters are kept to make sure that clusters from previous layers are not revisited.
which helps minimise the effect of loops. We repeat this procedure to add subsequent layers of clusters until all clusters are included. This procedure isolates the initially selected large parity check clusters from the rest of the clusters. The idea is to keep the expensive clusters at the final layer so that the smaller (less expensive) parity clusters, in preceding layers, can resolve most of the uncertainty about the even parity states. When the larger parity clusters get updated, some of the even parity states in their discrete tables may have zero probability, which are removed due to our software implementation. This further reduces a large parity cluster’s computational footprint.

The observed conditional Gaussian clusters are coupled to the parity check clusters in the layer furthest away from the initial isolated group of large clusters – starting with the smallest parity clusters. This avoids expensive computational cost between the observed random variables and the larger parity clusters (even though we only need to multiply them in once). The isolated parity check clusters make up the bottom layer in Figure 3. The smaller parity check clusters in the top layer are given priority in terms of their connectivity to conditional Gaussian clusters. Note that we do not link conditional Gaussian clusters to parity clusters in intermediate layers. We avoid this so that evidence enters the graph from one end (the top) and updates the latent clusters one layer at a time. If the first layer of parity check clusters do not have all the unique bits, the following layers are utilised until all conditional Gaussian clusters are connected.

Once the observed conditional Gaussian clusters updated the first parity cluster layer they are not needed further during message passing. Message passing continues towards the final parity cluster layer, which we refer to as the forward sweep. The backward sweep returns in the opposite direction, which concludes one iteration of message passing.

The following settings and software implementations apply to our inference approach:

- the stopping criterion for inference is if the decoded bit message equals the original bit message, or when a maximum number of iterations is reached,
- all discrete table factors support sparse representations to reduce memory resources,
- zero probability states in discrete tables are removed during inference.

The following section shows the performance comparison between a factor and cluster graph representation of a larger LDPC code.

5 Experimental investigation

The LDPC code used in our study is constructed from the 5G new radio (NR) standard. We use base graph 2 with size \( (42, 52) \) and expansion factor 2 \([10]\). The resultant \( H \) matrix is shortened to a code rate of 0.5, giving \( K = 20 \) message bits and a codeword length of \( N = 40 \) bits. We use BPSK modulation over an AWGN channel and assume a transmitted bit has unit energy and unit noise variance. The cluster graph is compiled from the parity check factors using the LTRIP algorithm, and the message schedule is initialised by clusters with cardinality 8 and 10 (the largest factors), which forms the bottom layer of the PGM (see the example in Section 4).

5.1 Purpose of experiment

The purpose of the experiment is to test whether a cluster graph representation of an LDPC code provides (1) BER performance advantages, and (2) computational advantages in terms of the number of message passing iterations required for successful decoding (which is not necessarily convergence). We compare the cluster graph to a conventional factor graph (or Tanner graph).

We generate random bit messages that are encoded using the \( H \) matrix to produce LDPC packets (or codewords). Bit values in a packet are modulated using BPSK modulation and random noise is added to the resultant signal values by
using a zero mean Gaussian distribution. The channel noise variance can be translated to a rate-compensated SNR given by $\text{SNR}_{\text{dB}} = 10 \log_{10} \left( \frac{E_b}{2Re_c} \right)$, where $R$ is the code rate and $E_b$ the energy per bit [3]. We create random noise between 0 - 8 dB, which consists of 36 equidistant SNR instances. For each instance, 1 million LDPC packets are simulated. Our experiment measures the number of bit errors divided by the total number of transferred bits for each SNR instance. We also capture the number of message passing iterations required to decode a packet. The maximum number of message passing iterations is set to 35 for both the factor graph and cluster graph. The factor and cluster graph receives the exact same encoded and modulated LDPC packets to ensure a like-for-like comparison.

5.2 Results and interpretation

The results are shown in Figure 4. The BER comparison between a factor and cluster graph is shown on the left. The cluster graph outperforms the factor graph over the entire SNR spectrum with a more pronounced difference at higher SNRs. The cluster graph also outperforms the factor graph when comparing the average number of message passing iterations required by the decoder (shown on the right). The difference is more pronounced at lower SNRs when more iterations are required for decoding and the cluster graph maintains a 14% improvement up to $\approx 3$ dB SNR.

![Figure 4: Results showing the BER and message passing iterations comparison between a factor graph and cluster graph representation of an irregular (40,20) LDPC code.](image)

We also note that the cluster graph’s expected behaviour is stable across the entire SNR spectrum, similar to the factor graph.

6 Conclusion and future work

Our view is that cluster graphs play an important role in channel coding research due to its superior performance compared to factor graphs (or Tanner graphs). A factor graph representation of an LDPC code is a specific factorisation of the code’s full joint probability distribution, and a special case when considered more generally as PGMs. Nevertheless, they are intuitive, easy to construct, and turn out to be practically useful. Constructing valid cluster graphs on the other hand is not trivial. However, the authors in [5, 6] developed a general purpose algorithm called LTRIP, which we used to construct a cluster graph of an LDPC code’s parity check factors.

While tree-structured PGMs produce exact marginals (that result in the optimal BER performance), LDPC codes are loopy and the RIP ensures that the approximated marginals are close to the exact marginals (discussed in Section 3). A factor graph representation of LDPC codes also satisfy the RIP however, messages passed between the single variable clusters and parity check clusters are constraint to univariate distributions, which disregard important dependencies between bits. Cluster graphs compiled with the LTRIP algorithm are not limited to univariate messages and contain richer information content that result in faster convergence and better BER performance.

We demonstrated that a cluster graph representation of a Hamming code and LDPC code outperforms its commonly used factor graph representation, and in future work we may look at applying this to other error-correcting codes such as Polar codes.
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