Off-lattice noise reduction and the ultimate scaling of DLA in two dimensions

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Off-lattice DLA clusters grown with different levels of noise reduction are found to be consistent with a simple fractal fixed point. Cluster shapes and their ensemble variation exhibit a dominant slowest correction to scaling, and this also accounts for the apparent “multiscaling” in the DLA mass distribution. We interpret the correction to scaling in terms of renormalized noise. The limiting value of this variable is strikingly small and is dominated by fluctuations in cluster shape. Earlier claims of anomalous scaling in DLA were misled by the slow approach to this small fixed point value.

PACS numbers: 64.60.Ak, 61.43.Hv

I. INTRODUCTION

Since its introduction in 1981, the Diffusion-Limited Aggregation model of Witten and Sander [1] has been a paradigm of self-organised scaling behaviour in irreversible growth. However, even after twenty years, there is still controversy about its scaling properties; many authors have claimed, for example, that DLA clusters do not scale as simple fractals, but instead have various anomalous features. In this paper we give data on DLA clusters with noise reduction which enables us to refute conclusively the basis of these claims of anomalous scaling. We will show that the apparent anomalies arise from a slowly decaying correction to scaling which can be associated with the level of intrinsic growth fluctuations, as suggested in [2]. The analysis of these corrections to scaling gives us considerable insight into the asymptotic behaviour of DLA, i.e. the DLA fixed point.

In (off-lattice) DLA a cluster is rigid and stationary, growing from one seed particle by accretion at first contact of a mobile diffusing hard sphere particles. The diffusing particles are sufficiently dilute so that they can be taken to arrive one at a time. We consider the distribution of where growth (by deposition) occurs at a given cluster size. The average radius of deposition is defined by \( R_{\text{dep}} = \langle r \rangle \), where \( r \) is the the distance of deposition from the center of the cluster. There is no controversy that \( R_{\text{dep}} \propto N^{1/D} \), consistent with a simple fractal of dimension \( D = 1.71 \) for large clusters in two dimensions. However the spread of the deposition radius is thought to show anomalies. Plischke and Rácz [6] introduced the penetration depth, \( \xi \), the standard deviation of radius of deposition of a given cluster, and claimed that it scaled differently from \( R_{\text{dep}} \). More recently, Davidovitch et al. [7] considered the standard deviation of the cluster average radius across the ensemble of clusters, \( \delta R_{\text{eff}} \), and claimed that it was asymptotically negligible compared to the mean. Another anomalous feature that has been claimed of DLA is multiscaling [8]: the fractal dimension of the cluster is said to depend on the distance (relative to the cluster radius) from the center. We will examine these claims using finite size scaling with the help of noise reduction and show that none of them hold. We find that DLA is consistent with simple scaling, and the apparently anomalous scaling can all be explained by a slow correction to scaling.

II. OFF-LATTICE NOISE REDUCTION

Noise reduction for the lattice version of DLA has been introduced [9,10] with the aim of suppressing the shot noise of the individual incoming particles. When growing at lower noise levels, the clusters achieve more asymptotic behavior at smaller sizes: a prime example of this is that the lattice effects show up earlier. These lattice effects on noise reduced clusters (or without noise reduction on very large clusters) are quite strong, so in order to avoid them, any analysis of large scale DLA clusters has to be made off-lattice.

In our version of noise-reduced off-lattice DLA the particles diffuse freely until they contact a particle in the cluster, just as in the original model. However, on contact with a particle of the existing cluster, the diffusing particle is moved into that particle by a factor of \( A \). This means that shallow bumps are added to the cluster, and that we must add \( 1/A \) particles on top of one another to protrude the growth by a particle diameter. A cluster
grown with this method of noise reduction is shown in Fig. 11. Another way to do noise-reduction of this type was introduced by Stepanov and Levitov [9], who generalized the method of iterated conformal maps [10] to add shallow bumps.

III. FINITE SIZE SCALING

Growing clusters at a variety of levels of noise reduction gives us a very clear picture of the finite size scaling effects in DLA. We grew 1000 DLA clusters to 1,000,000 particles with noise reduction levels of $A = 0.3, 0.1, 0.03, 0.01,$ and 4000 clusters with $A = 1$ as well as 25 clusters with $A = 0.001$. At various points in the growth, 100,000 probe particles were fired at each cluster to measure its properties. In the following measurements the center of the cluster was taken naturally as the center of mass of these probe particles (“center of charge”).

Fig. 11. Another way to do noise-reduction of this type was introduced by Stepanov and Levitov [9], who generalized the method of iterated conformal maps [10] to add shallow bumps.

The different levels of noise reduction are all consistent with a universal asymptote, $\Xi = \frac{\xi}{R_{\text{dep}}}$, with varying cluster size at various levels of noise reduction. The abscissa is chosen according to the correction to scaling exponent measured from Fig. 3. The relative penetration depth clearly converges to a non-zero common value. Also shown are curves for clusters grown by the Hastings-Levitov (HL) method with expected equivalence indicated in the legend. The top right panel is a magnification of the asymptotic end of the curves.

A deeper test of the universality of these clusters and their scaling comes from the multipole moments of the

$$Q(N) = Q_\infty(1 + CN^{-\nu})$$

then a plot of $dQ(N)/d\ln(N)$ vs $Q(N)$ should have an intercept on the $Q$ axis of $Q_\infty$ approached with slope $-\nu$, both independent of the magnitude of $C$. Fig. 2 shows this analysis applied to $\Xi$ and this is the basis for the choice of exponent $\nu = 0.33$ for Fig. 3. This provides unbiased evidence that all the different levels of noise reduction approach the same asymptotic value $\Xi_\infty$, consistent with a common correction to scaling exponent. Interestingly the ‘fixed point’ can be approached from either side, corresponding to opposite signs of $C$.

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$$\frac{d\Xi}{d\ln N}$$

plot against $\Xi$. The common dashed asymptote indicates that $\Xi$ has a dominant correction to scaling of the form $\Xi = \Xi_\infty(1 + CN^{-\nu})$ with $\Xi_\infty = 0.121 \pm 0.003$ from the intercept of the plots and $\nu = 0.33 \pm 0.06$ from the slope.

FIG. 2. Behaviour of the relative penetration depth $\Xi = \frac{\xi}{R_{\text{dep}}}$, with varying cluster size at various levels of noise reduction. The abscissa is chosen according to the correction to scaling exponent measured from Fig. 3. The relative penetration depth clearly converges to a non-zero common value. Also shown are curves for clusters grown by the Hastings-Levitov (HL) method with expected equivalence indicated in the legend. The top right panel is a magnification of the asymptotic end of the curves.
growth probability distribution. The \( n \)th multipole moment is given by

\[
M_n = \int dq \,(x + iy)^n
\]

where \( q \) is the probability distribution for where growth will next occur. (Note that \( q \) is equivalent to the charge density on the cluster surface when it is considered to be a conductor held at a fixed potential). The multipole moments for positive \( n \) fully characterise the cluster shape, and can be related invertibly to the Laurent coefficients of its conformal map from the unit circle \([4]\). In practice we measured the \( M_n \) by sampling \( (x + iy)^n \) with non-growing probe particles.

Fig. 4 shows the correction-to-scaling analysis of the corresponding multipole powers

\[
P_n = \frac{|M_n|^2}{R_{\text{eff}}^{2n}}
\]

where we have scaled each \( M_n \) by the appropriate power of the effective (or Laplace \([4]\)) radius, \( R_{\text{eff}} \), which is given by \( \ln R_{\text{eff}} = \int dq \ln(r) \). Each of \( P_2 \) to \( P_5 \) is consistent with having a universal non-zero asymptote, and moreover they are all compatible with a single common correction to scaling exponent 0.33, see Table I. Fig. 5 collects the resulting finite size scaling plots assuming this exponent. Together with the relative penetration depth results, this presents strong evidence for universal asymptotic geometry for DLA clusters, and a universal leading correction to scaling exponent \( \nu = 0.33 \).

| TABLE I. Best fit scaling exponents for \( P_2 \) to \( P_5 \). We have also measured \( P_6 \) to \( P_{10} \). These yield somewhat larger apparent exponents with large statistical errors. |
|---|---|---|---|
| \( P_2 \) | \( P_3 \) | \( P_4 \) | \( P_5 \) |
| \( \nu \) | \( 0.41 \pm 0.08 \) | \( 0.27 \pm 0.06 \) | \( 0.41 \pm 0.12 \) | \( 0.40 \pm 0.12 \) |

In all of the measurements discussed above, the cluster center used was the “center of charge”, natural to a snapshot of the growth. In the following Section, however, we will require to compare data at different cluster sizes where it becomes natural to use a fixed center, namely the cluster “seed”. Accordingly we have also measured the finite size scaling of various lengths with the seed as fixed origin, and in all cases using direct ensemble averages and for clusters with no noise reduction \( (A = 1) \). Using the seed as center also naturally leads to the measurement of penetration depth as the rms spread of deposition radius about its ensemble average:

\[
\xi_0 = \sqrt{\langle r^2 \rangle - \langle r \rangle^2}
\]

rather than computing the variance cluster-by-cluster before averaging, i.e.
IV. MULTISCALING

Now we consider the anomalous scaling claim of multiscaling, when the aggregate has a fractal dimension which depends upon distance from the seed as a fraction of cluster radius. It was proposed in Ref. 3 that the ensemble average of the density of particles \( g_N(r) \) of an \( N \)-particle cluster at distance \( r \) away from the seed obeys

\[
g_N(xR_{\text{gyr}}) = A(x)R_{\text{gyr}}^{-d+D(x)},
\]

where the dimension \( D(x) \) is function of \( x = r/R_{\text{gyr}} \), and the size \( N \) and (average) radius of gyration \( R_{\text{gyr}} \) are of course mutually dependent. Using the above formula at fixed \( x \), one can extract the dimension \( D(x) \) by the scaling with \( R_{\text{gyr}} \):

\[
-d + D(x) = \frac{\partial \ln g_N(R_{\text{gyr}}(x\,R_{\text{gyr}}))}{\partial \ln R_{\text{gyr}}} \bigg|_{x} = \frac{R_{\text{gyr}}}{dR_{\text{gyr}}/dN} \frac{\partial \ln g_N(x\,R_{\text{gyr}}(N))}{\partial N} \bigg|_{x}
\]

Simple fractal scaling would require \( D(x) = D \) independent of \( x \), but the dimension measured this way in Ref. 6 using medium size clusters \( (N = 10^4 \ldots 10^5) \) was observed to be a non-trivial function (see Fig. 8). Others partly confirmed that findings, although with mixed results 11,12.

Now we will repeat the same measurement procedure but instead of direct simulation we use the correction to scaling results of the previous Section, within a scaling function assumption (see below). This turns out to agree quantitatively with the earlier published \( D(x) \) data, but implies that the ultimate behaviour is simple fractal scaling with \( D(x) \to D \) for all \( x \).

Consider the distribution of \( r \), the distance of attaching particles from the seed: as we have seen, this has mean \( R_{\text{dep}} \) and variance \( \xi_0 \). Now we assume that the shape of the probability density function is independent of \( N \):

\[
\frac{1}{\xi_0(N)} h \left( \frac{r - R_{\text{dep}}(N)}{\xi_0(N)} \right),
\]

where \( h \) is a normalized probability density with zero mean and unit variance. After replacing the sum over particles with an integral, for the particle density we get

\[
g_N(r) = \int_0^N \frac{dN'}{\xi_0(N')} h \left( \frac{r - R_{\text{dep}}(N')}{\xi_0(N')} \right).
\]

A similar formula has been suggested in 13. Given that we have already studied \( R_{\text{dep}}(N) \), \( R_{\text{gyr}}(N) \) and \( \xi_0(N) \), the only outstanding quantity to be found is the scaling function \( h \), which we find to be very close to the standard normal distribution, see Fig. 7.

Fig. 8 shows how well \( D(x) \) derived from our finite size scaling results plus a normal distribution for \( h \) agrees with the raw data of Ref. 6. Also shown is what our results imply for the behaviour at larger \( N \), and as \( N \to \infty \) with \( R_{\text{dep}}, R_{\text{gyr}} \) and \( \xi_0 \) approaching pure scaling, \( D(x) \to D \). Thus we conclude that all the apparent reported \( x \)-dependence of \( D(x) \) arises from corrections to scaling, and indeed almost all the effect comes from the relatively large corrections to scaling in \( \xi_0 \). Our new interpretation of this data also resolves a previously noted paradox 4, namely that \( D(x) \) increasing with \( x \) cannot be asymptotic scaling as it would imply some decrease of \( g_N(r) \) with increasing \( N \) at fixed \( r \).
V. SIZE FLUCTUATIONS AND FIXED POINT

We now present an interpretation of the leading correction to scaling, based on new observations from our data and building on earlier work [8]. The amplitude of the leading correction to scaling crosses zero at a common value of noise reduction \( A_f \approx 0.01 \), for all of the plots in Figs. 2 and 5. This suggests that the noise reduction and the correction to scaling are fundamentally related, which can be understood by using the renormalization theory of noise reduction of Barker and Ball [8]. In this view, the cluster is approximated as being at its large \( N \) fixed point if one unit of growth acts as a coarse graining of DLA on finer length scales. This seems to occur if we grow with input noise near \( A_f \). This is equivalent to have \( \delta N/N = \sqrt{A_f} \) for relative fluctuation in the number of particles to advance the growth locally by one particle diameter.

We can also view this in terms of a fixed point for the noise output of the growth, \( \sqrt{A_{out}} = \delta N/N \), in terms of the relative fluctuation in the number of particles to span a fixed radius. Fig. 9 shows our data for the ensemble spread of extremal cluster radius. Since this spread is small, we can infer:

\[
\frac{\delta N}{N} \bigg|_{R_{ext}} = D \frac{\delta R_{ext}}{R_{ext}} \bigg|_{N} = 0.060 \pm 0.005 \tag{11}
\]

from our extrapolated value. Thus we find an asymptotic renormalised noise \( A^* = 0.0036 \pm 0.0006 \). This is in qualitative agreement with our observed value of \( A_f \). Furthermore, Fig. 10 shows how well this vindicates Barker and Ball’s earlier estimates of the fixed point, using our value of \( \nu \) to extrapolate from their finite size calculations. By contrast, the more recent work of Cafiero et al. [14] using a very small scale renormalisation scheme disagrees by two orders of magnitude.

Our interpretation is thus that the renormalized noise is the slow variable which dominates convergence of other
FIG. 10. Estimate of the fixed point value of $A (A^*)$, combined with previous estimates from Barker and Ball (the middle three points) and Cafiero (the rightmost point). The Barker and Ball data is in good agreement with our results, but the Cafiero data disagrees.

quantities to scaling. Our observed input noise value of $A_f \simeq 10^{-2}$ (for the leading correction to scaling to vanish) and the extrapolated fixed point output noise $A^*$ are equal within a factor of order unity, showing the consistency of the picture.

We can take this interpretation a step further to infer that the dominant fluctuations of $R_{\text{ext}}$ determining the noise reduction are fluctuations in cluster shape rather than overall cluster radius. The basis for this is that the logarithmic average radius, $R_{\text{eff}}$, has much smaller spread, asymptotically $\delta R_{\text{eff}}/R_{\text{eff}} = 0.012 \pm 0.001$ compared to $\delta R_{\text{ext}}/R_{\text{ext}} = 0.035 \pm 0.003$. Since $R_{\text{eff}}$ is an average which emphasizes typical size, the larger fluctuations in $R_{\text{ext}}$ which gave us $A^*$ must be attributed to shape. (However, we showed in [3] that $R_{\text{eff}}$ has the same crossover exponent, $\nu$, as the other quantities discussed here.) In this sense DLA clusters are fundamentally stochastic objects with a distribution of shape.

VI. SUMMARY

We believe our work opens the way to a definitive view of DLA in two dimensions, and the extension of this work to three dimensions is in hand. The identification of ‘DLA fixed point behaviour’ is now reasonable, as we have shown the sort of universal limiting amplitudes and correction to scaling exponents associated with such terminology.

Some main areas are outstanding. First, the renormalized noise, $\delta N/N = \sqrt{A^*}$, is not of order unity, as we might expect a priori, and as has been suggested [3]. We do not understand the origin of this small number, and tracing its origin is a central remaining challenge in understanding DLA. Another puzzle which we hope to address in a later paper is why the fractal dimension is comparatively insensitive to the convergence of the renormalized noise.

Also, we need to understand the full scaling of the probability distribution for growth in DLA, corresponding to the harmonic measure of the perimeter. To this end the more expensive cluster growth methods of Hastings and Levitov [10] (HL) are likely to come into their own as they yield the harmonic measure directly. Stepanov and Levitov [9] have already shown some results for HL clusters grown with shallow bumps, corresponding rather closely to our noise reduction technique. The richer, simpler area to explore is the response to anisotropy and its sensitivity to noise. Small DLA clusters appear robust to the intrinsic bias of growing on a square lattice, whereas large clusters (and equivalently noise reduced ones) are driven to grow a four fingered dendrite. The first requirement is a systematic analysis of how this is a relevant perturbation of the isotropic DLA fixed point. Secondly, we might ask whether the anomalous response for small simple DLA clusters is dominated by some other hitherto unsuspected fixed point with much larger noise level. There is another rather neglected nearby fixed point, that of spherical growth, which becomes more pertinent at high noise reduction - where it takes longer to exhibit its instability. We suggest the influence of this fixed point may be responsible for shifting the observed $A_f$ somewhat above $A^*$, and this should be relatively amenable to analytic theory.
ACKNOWLEDGMENTS

NEB would like to thank BP Amoco and EPSRC for the support of a CASE award during this research. ES is supported by the Dutch FOM Foundation and the EU Marie Curie Fellowship. We thank Paul Meakin and Thomas Rage for sending us a computer code which we used for some of the results presented here.

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