ESTIMATES FOR PARAMETERS AND CHARACTERISTICS OF THE CONFINING SU(3)-GLUONIC FIELD IN \( \eta \)-MESON FROM TWO-PHOTON DECAY

YU. P. GONCHAROV

Theoretical Group, Experimental Physics Department, State Polytechnical University
Sankt-Petersburg 195251, Russia

Received 12 April 2007

On the basis of the confinement mechanism earlier proposed by author, the electric form factor of \( \eta \)-meson is nonperturbatively calculated. The latter is then applied to describe electromagnetic decay \( \eta \rightarrow 2\gamma \) which entails estimates for parameters of the confining SU(3)-gluonic field in \( \eta \)-meson. The corresponding estimates of the gluon concentrations, electric and magnetic colour field strengths are also adduced for the mentioned field.

Keywords: Quantum chromodynamics; confinement; mesons.

PACS Nos.: 12.38.-t; 12.38.Aw; 14.40.Aq

1. Introduction and Preliminary Remarks

In Refs. 1–3 for the Dirac-Yang-Mills system derived from QCD-Lagrangian there was found and explored an unique family of compatible nonperturbative solutions which could pretend to describing confinement of two quarks. Applications of the family to description of both the heavy quarkonia spectra\(^4,5,6\) and a number of properties of pions and kaons\(^7\) showed that the confinement mechanism is qualitatively the same for both light mesons and heavy quarkonia.

Two main physical reasons for linear confinement in the mechanism under discussion are the following ones. The first one is that gluon exchange between quarks is realized with the propagator different from the photon one and existence and form of such a propagator is direct consequence of the unique nonperturbative confining solutions of the Yang-Mills equations.\(^2,3\) The second reason is that, owing to the structure of mentioned propagator, quarks mainly emit and interchange the soft gluons so the gluon condensate (a classical gluon field) between quarks basically consists of soft gluons (for more details see Refs. 2, 3) but, because of that any gluon also emits gluons (still softer), the corresponding gluon concentrations rapidly become huge and form the linear confining magnetic colour field of enormous strengths which leads to confinement of quarks. This is by virtue of the fact that just magnetic part of the mentioned propagator is responsible for larger portion of gluon concentrations at large distances since the magnetic part has stronger
Yu. P. Goncharov

infrared singularities than the electric one. Under the circumstances physically non-linearity of the Yang-Mills equations effectively vanishes so the latter possess the unique nonperturbative confining solutions of the Abelian-like form (with the values in Cartan subalgebra of SU(3)-Lie algebra)\(^2,3\) that describe the gluon condensate under consideration. Moreover, since the overwhelming majority of gluons is soft they cannot leave hadron (meson) until some gluon obtains additional energy (due to an external reason) to rush out. So we deal with confinement of gluons as well.

The approach under discussion equips us with the explicit wave functions for every two quarks (meson). The wave functions are parametrized by a set of real constants \(a_j, b_j, B_j\) describing the mentioned nonperturbative confining gluon field (the gluon condensate) and they are nonperturbative modulo square integrable solutions of the Dirac equation in the above confining SU(3)-field and also depend on \(\mu_0\), the reduced mass of the current masses of quarks forming meson. It is clear that under the given approach just constants \(a_j, b_j, B_j, \mu_0\) determine all physical properties of any meson and they should be extracted from experimental data.

Such a program has been to a certain extent advanced in Refs. 4 –7. The aim of the present paper is to continue obtaining estimates for \(a_j, b_j, B_j\) for concrete mesons starting from experimental data on spectroscopy of one or another meson. We here consider \(\eta\)-meson and its electromagnetic decay \(\eta \rightarrow 2\gamma\) whose width amounts to about 40% of the full width of \(\eta\)-meson.\(^8\)

Of course, when conducting our considerations we shall rely on the standard quark model (SQM) based on SU(3)-flavor symmetry (see, e. g., Ref. 8) so in accordance with SQM \(\eta = \sqrt{1/6}(2 ss - uu - dd)\) is a superposition of three quarkonia, consequently, we shall have three sets of parameters \(a_j, b_j, B_j\).

Further we shall deal with the metric of the flat Minkowski spacetime \(M\) that we write down (using the ordinary set of local spherical coordinates \(r, \vartheta, \varphi\) for the spatial part) in the form

\[
\begin{align*}
    ds^2 &= g_{\mu\nu}dx^\mu \otimes dx^\nu = dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) .
\end{align*}
\]

Besides, we have \(|\delta| = |\det(g_{\mu\nu})| = (r^2 \sin \vartheta)^2\) and 0 \(\leq r < \infty\), 0 \(\leq \vartheta < \pi\), 0 \(\leq \varphi < 2\pi\).

Throughout the paper we employ the Heaviside-Lorentz system of units with \(\hbar = c = 1\), unless explicitly stated otherwise, so the gauge coupling constant \(g\) and the strong coupling constant \(\alpha_s\) are connected by relation \(g^2/(4\pi) = \alpha_s\). Further we shall denote \(L_2(F)\) the set of the modulo square integrable complex functions on any manifold \(F\) furnished with an integration measure, then \(L_2^2(F)\) will be the \(n\)-fold direct product of \(L_2(F)\) endowed with the obvious scalar product while \(\dagger\) and \(*\) stand, respectively, for Hermitian and complex conjugation. Our choice of Dirac \(\gamma\)-matrices conforms to the so-called standard representation and is the same as in Ref. 7. At last \(\otimes\) means tensorial product of matrices and \(I_3\) is the unit \(3 \times 3\) matrix.

When calculating we apply the relations 1 GeV\(^{-1}\) \(\approx\) 0.1973269679 fm, 1 s\(^{-1}\) \(\approx\) 0.658211915 \(\times\) 10\(^{-24}\) GeV, 1 V/m \(\approx\) 0.2309956375 \(\times\) 10\(^{-23}\) GeV\(^2\), 1 T \(\approx\) 0.6925075988 \(\times\) 10\(^{-7}\) GeV\(^2\) \(\times\) 10\(^{-24}\) GeV, 1 m \(\approx\) 10\(^{-6}\) GeV\(^2\) \(\times\) 10\(^{-24}\) GeV.\(^2\)
$10^{-15}$ GeV$^2$.

Finally, for the necessary estimates we shall employ the $T_{00}$-component (volumetric energy density) of the energy-momentum tensor for a SU(3)-Yang-Mills field which should be written in the chosen system of units in the form

$$T_{\mu\nu} = -F^a_{\mu\alpha} F^a_{\nu\beta} g^{\alpha\beta} + \frac{1}{4} F^a_{\beta\gamma} F^a_{\alpha\delta} g^{\gamma\delta} g_{\mu\nu} .$$ (2)

2. Specification of Main Relations

As was mentioned above, our considerations shall be based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system (derived from QCD-Lagrangian) studied in details in Refs. 1–3. Referring for more details to those references, let us briefly describe and specify only the relations necessary to us in the present Letter.

One part of the mentioned family is presented by the unique nonperturbative confining solution of the Yang-Mills equations for $A = A_{\mu} dx^\mu = A_a^b \lambda_a dx^b$ ($\lambda_a$ are the known Gell-Mann matrices, $\mu = t, r, \vartheta, \varphi, a=1,...,8$) and looks as follows:

$$A_{1t} \equiv A_t^3 + \frac{1}{\sqrt{3}} A_t^8 = \frac{a_1}{r} + A_1, A_{2t} \equiv -A_t^3 + \frac{1}{\sqrt{3}} A_t^8 = -\frac{a_2}{r} + A_2,$$

$$A_{3t} \equiv -\frac{2}{\sqrt{3}} A_t^8 = \frac{a_1 + a_2}{r} - (A_1 + A_2),$$

$$A_{1\varphi} \equiv A_\varphi^3 + \frac{1}{\sqrt{3}} A_\varphi^8 = b_1 r + B_1, A_{2\varphi} \equiv -A_\varphi^3 + \frac{1}{\sqrt{3}} A_\varphi^8 = b_2 r + B_2,$$

$$A_{3\varphi} \equiv -\frac{2}{\sqrt{3}} A_\varphi^8 = -(b_1 + b_2)r - (B_1 + B_2)$$ (3)

with the real constants $a_j, A_j, b_j, B_j$ parametrizing the family. As has been repeatedly explained in Refs. 2–7, parameters $A_{1,2}$ of solution (3) are inessential for physics in question and we can consider $A_1 = A_2 = 0$. Obviously we have $\sum_{j=1}^{3} A_{jt} = \sum_{j=1}^{3} A_{j\varphi} = 0$ which reflects the fact that for any matrix $T$ from SU(3)-Lie algebra it should be $\text{Tr} T = 0$.

Another part of the family is given by the unique nonperturbative modulo square integrable solutions of the Dirac equation in the confining SU(3)-field of (3) $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ with the four-dimensional Dirac spinors $\Psi_j$ representing the $j$th colour component of the meson, so $\Psi$ may describe relative motion (relativistic bound states) of two quarks in mesons and the mentioned Dirac equation looks as follows

$$i \partial_t \Psi \equiv i \begin{pmatrix} \partial_t \Psi_1 \\ \partial_t \Psi_2 \\ \partial_t \Psi_3 \end{pmatrix} = H \Psi \equiv \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} H_1 \Psi_1 \\ H_2 \Psi_2 \\ H_3 \Psi_3 \end{pmatrix}$$ (4)

where Hamiltonian $H_j$ is

$$H_j = \gamma^0 \left[ \mu_0 - i \gamma^1 \partial_r - i \gamma^2 \frac{1}{r} \left( \partial_\vartheta + \frac{1}{2} \gamma^1 \gamma^2 \right) - i \gamma^3 \frac{1}{r \sin \vartheta} \left( \partial_\varphi + \frac{1}{2} \sin \vartheta \gamma^1 \gamma^3 + \frac{1}{2} \cos \vartheta \gamma^2 \gamma^3 \right) \right]$$
\[ -g\gamma^0 \left( \gamma^0 A_{\mu} + \frac{1}{r \sin \theta} A_{\varphi} \right) \]  

(5)

with the gauge coupling constant \( g \) while \( \mu_0 \) is a mass parameter and one can consider it to be the reduced mass which is equal, e. g., for quarkonia, to half the current mass of quarks forming a quarkonium.

Then the unique nonperturbative modulo square integrable solutions of (4) are (with Pauli matrix \( \sigma_1 \))

\[ \Psi_j = e^{-i\omega_j t} \psi_j \equiv e^{-i\omega_j t} r^{-1} \left( \begin{array}{c} F_{j1}(r) \Phi_j(\theta, \varphi) \\ F_{j2}(r) \sigma_1 \Phi_j(\theta, \varphi) \end{array} \right), \ j = 1, 2, 3 \]  

(6)

with the 2D eigenspinor \( \Phi_j = \left( \begin{array}{c} \Phi_{j1} \\ \Phi_{j2} \end{array} \right) \) of the Euclidean Dirac operator \( D_0 \) on the unit sphere \( S^2 \), while the coordinate \( r \) stands for the distance between quarks. The explicit form of \( \Phi_j \) is not needed here and can be found in Refs. 3, 9, 10. For the purpose of the present Letter we shall adduce the necessary spinors below. Spinors \( \Phi_j \) form an orthonormal basis in \( L_2^2(S^2) \). We can call the quantity \( \omega_j \) relative energy of \( j \)th colour component of meson (while \( \psi_j \) is wave function of a stationary state for \( j \)th colour component) but we can see that if we want to interpret (4) as equation for eigenvalues of the relative motion energy, i. e., to rewrite it in the form \( H\psi = \omega\psi \) with \( \psi = (\psi_1, \psi_2, \psi_3) \) then we should put \( \omega = \omega_j \) for any \( j = 1, 2, 3 \) so that \( H_j\psi_j = \omega_j\psi_j = \omega\psi_j \). Under this situation, if a meson is composed of quarks \( q_1, q_2 \) with different flavours then the energy spectrum of the meson will be given by \( \epsilon = m_{q_1} + m_{q_2} + \omega \) with the current quark masses \( m_{q_k} \) (rest energies) of the corresponding quarks. On the other hand for determination of \( \omega_j \) the following quadratic equation can be obtained\(^1\)\(^-\)\(^2\)\(^3\)

\[
[g^2 a_j^2 + (n_j + \alpha_j)^2] \omega_j^2 - 2(\lambda_j - gB_j)g^2 a_j b_j \omega_j + [(\lambda_j - gB_j)^2 - (n_j + \alpha_j)^2]g^2 b_j^2 - \mu_0^2(n_j + \alpha_j)^2 = 0,
\]  

(7)

that yields

\[
\omega_j = \omega_j(n_j, l_j, \lambda_j) = \frac{\lambda_j g^2 a_j b_j \pm (n_j + \alpha_j) \sqrt{(n_j^2 + 2n_j \alpha_j + \Lambda_j^2)\mu_0^2 + g^2 b_j^2(n_j^2 + 2n_j \alpha_j)}}{n_j^2 + 2n_j \alpha_j + \Lambda_j^2}, j = 1, 2, 3, \]

(8)

where \( a_3 = -(a_1 + a_2), b_3 = -(b_1 + b_2), B_3 = -(B_1 + B_2), \Lambda_j = \lambda_j - gB_j, \alpha_j = \sqrt{\Lambda_j^2 - g^2 a_j^2}, n_j = 0, 1, 2, ..., \) while \( \lambda_j = \pm(l_j + 1) \) are the eigenvalues of Euclidean Dirac operator \( D_0 \) on unit sphere with \( l_j = 0, 1, 2, ... \). It should be noted that in previous papers\(^1\)\(^-\)\(^2\)\(^3\)\(^4\)\(^5\)\(^6\)\(^7\) we used the ansatz (6) with the factor \( e^{-i\omega_j t} \) instead of \( e^{-i\omega_j t} \) but then the Dirac equation (4) would look as \( -i\partial_t \Psi = H\Psi \) and in equation (7) the second summand would have plus sign while the first summand in numerator of (8) would have minus sign. We here return to the conventional form of writing Dirac equation and this slightly modifies the equations (7)–(8). In line with the above we should have \( \omega = \omega_1 = \omega_2 = \omega_3 \) in energy spectrum \( \epsilon = m_{q_1} + m_{q_2} + \omega \) for any meson and this at once imposes two conditions on parameters \( a_j, b_j, B_j \).
when choosing some experimental value for $\epsilon$ at the given current quark masses $m_{q_1}, m_{q_2}$.

Within the given Letter we need only the radial parts of (6) at $n_j = 0$ (the ground state) that are

$$F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left( 1 - \frac{g b_j}{\beta_j} \right), P_j = gb_j + \beta_j,$$

$$F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r} \left( 1 + \frac{g b_j}{\beta_j} \right), \quad Q_j = \mu_0 - \omega_j$$

(9)

with $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2}$ while $C_j$ is determined from the normalization condition $\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{2}$. Consequently, we shall gain that $\Psi_j \in L_2^2(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, as a result, the solutions of (6) may describe relativistic bound states (mesons) with the energy (mass) spectrum $\epsilon$.

It is useful to specify the nonrelativistic limit (when $c \to \infty$) for spectrum (8). For that one should replace $g \to g/\sqrt{\hbar c}, a_j \to a_j/\sqrt{\hbar c}, b_j \to b_j/\sqrt{\hbar c}, B_j \to B_j/\sqrt{\hbar c}$ and, expanding (8) in $z = 1/c$, we shall get

$$\omega_j(n_j, l_j, \lambda_j) = \pm \mu_0 c^2 \left[ 1 \mp \frac{g^2 a_j^2}{2\hbar^2 (n_j + |\lambda_j|)^2} z^2 \right] + \frac{\lambda_j g^2 a_j b_j}{\hbar (n_j + |\lambda_j|)^2} \mp \frac{g^3 B_j a_j^2 f(n_j, \lambda_j)}{\hbar^3 (n_j + |\lambda_j|)^7} z + O(z^2),$$

(10)

where $f(n_j, \lambda_j) = 4\lambda_j n_j (n_j^2 + \lambda_j^2) + \frac{1}{\lambda_j} \left( n_j^4 + 6n_j^2 \lambda_j^2 + \lambda_j^4 \right)$.

As is seen from (10), at $c \to \infty$ the contribution of linear magnetic colour field ($\text{parameters} \ b_j, B_j$) to spectrum really vanishes and spectrum in essence becomes purely Coulomb one (modulo the rest energy). Also it is clear that when $n_j \to \infty$, $\omega_j \to \pm \sqrt{\mu_0^2 + g^2 b_j^2}$.

We may seemingly use (8) with various combinations of signs ($\pm$) before second summand in numerators of (8) but, due to (10), it is reasonable to take all signs equal to plus which is our choice within the Letter. Besides, as is not complicated to see, radial parts in nonrelativistic limit have the behaviour of form $F_{j1}, F_{j2} \sim r^{l_j + 1}$, which allows one to call quantum number $l_j$ angular momentum for $j$th colour component though angular momentum is not conserved in the field (3). So for meson under consideration we should put all $l_j = 0$.

Finally it should be noted that spectrum (8) is degenerated owing to degeneracy of eigenvalues for the Euclidean Dirac operator $D_0$ on the unit sphere $\mathbb{S}^2$. Namely, each eigenvalue of $D_0 \lambda = \pm(l + 1), l = 0, 1, 2, \ldots$, has multiplicity $2(l + 1)$ so we has $2(l + 1)$ eigenspinors orthogonal to each other. Ad referendum we need eigenspinors corresponding to $\lambda = \pm 1 \ (l = 0)$ so here is their explicit form

$$\lambda = -1 : \Phi = \frac{C}{2} \left( e^{i\Phi} \overline{\epsilon} + e^{-i\Phi} \overline{\epsilon} \right) e^{i\varphi/2}, \text{ or } \Phi = \frac{C}{2} \left( e^{i\Phi} \overline{\epsilon} - e^{-i\Phi} \overline{\epsilon} \right) e^{-i\varphi/2},$$
\[ \lambda = 1 : \Phi = \frac{C}{2} \left( e^{-i\frac{\varphi}{2}} \right) e^{i\varphi/2} \text{ or } \Phi = \frac{C}{2} \left( -e^{-i\frac{\varphi}{2}} \right) e^{-i\varphi/2} \]  

(11)

with the coefficient \( C = 1/\sqrt{2\pi} \) (for more details see Refs. 3, 9 and 10).

3. Electric and Magnetic Form factors, Anomalous Magnetic Moment

As has been mentioned in Section 1, we shall analyse electromagnetic decay of \( \eta \)-meson so let us consider what quantities characterizing electromagnetic properties of a meson we could construct within our approach.

Within the present Letter we shall use relations (8) at \( n_j = 0 = l_j \) so energy (mass) of meson under consideration is given by

\[ \mu = 2m_q + \omega \]  

(12)

and, as a consequence, the corresponding meson wave functions of (6) are represented by (9) and (11).

3.1. Choice of quark masses and gauge coupling constant

It is evident for employing the above relations we have to assign some values to quark masses and gauge coupling constant \( g \). In accordance with Ref. 8, at present the current quark masses necessary to us are restricted to intervals 1.5 MeV \( \leq m_u \leq 3 \) MeV, 3.0 MeV \( \leq m_d \leq 7 \) MeV, 95 MeV \( \leq m_s \leq 120 \) MeV, so we take

\[ m_u = (1.5 + 3)/2 \text{ MeV} = 2.25 \text{ MeV}, \quad m_d = (3 + 7)/2 \text{ MeV} = 5 \text{ MeV}, \quad m_s = (95 + 120)/2 \text{ MeV} = 107.5 \text{ MeV}. \]

Under the circumstances, the reduced mass \( \mu_0 \) of (5) will respectively take values \( m_u/2, m_d/2, m_s/2 \). As to gauge coupling constant \( g = \sqrt{4\pi\alpha_s} \), it should be noted that recently some attempts have been made to generalize the standard formula for \( \alpha_s = \alpha_s(Q^2) = 12\pi/[\ln(Q^2/\Lambda^2)](n_f) \) (\( n_f \) is number of quark flavours) holding true at the momentum transfer \( \sqrt{Q^2} \to \infty \) to the whole interval \( 0 \leq \sqrt{Q^2} \leq \infty \). We shall employ one such a generalization \(^{11}\) used in Refs. 12, 13. It looks as follows (\( x = \sqrt{Q^2} \) in GeV)

\[ \alpha(x) = \frac{12\pi}{(33 - 2n_f)} \frac{f_1(x)}{\ln \frac{x^2 + f_2(x)}{\Lambda^2}} \]  

(13)

with

\[ f_1(x) = 1 + \left( \frac{(1 + x)(33 - 2n_f)}{12} \ln \frac{m^2}{\Lambda^2} - 1 \right)^{-1} + 0.6x^{1.3}, \quad f_2(x) = m^2(1 + 2.8x^2)^{-2}, \]

wherefrom one can conclude that \( \alpha_s \to \pi = 3.1415... \) when \( x \to 0 \), i.e., \( g \to 2\pi = 6.2831... \). We used (13) at \( m = 1 \text{ GeV}, \Lambda = 0.234 \text{ GeV}, n_f = 3, \quad x = m_\eta = 547.51 \text{ MeV} \) to obtain \( g = 5.148358007 \) necessary for our further computations at the mass scale of \( \eta \)-meson.
3.2. Electric form factor

For each meson with the wave function \( \Psi = (\Psi_j) \) of (6) we can define electromagnetic current \( J^\mu = \overline{\Psi} (\gamma^\mu \otimes I_3) \Psi = (\Psi^\dagger \Psi, \Psi^\dagger (\alpha \otimes I_3) \Psi) = (\rho, J) \), \( \alpha = \gamma^0 \gamma \). Electric form factor \( f(K) \) is the Fourier transform of \( \rho \)

\[
f(K) = \int \Psi^\dagger \Psi e^{-iK \cdot r} d^3x = \sum_{j=1}^{3} \int \Psi_j^\dagger \Psi_j e^{-iK \cdot r} d^3x = \sum_{j=1}^{3} f_j(K) = \\
\sum_{j=1}^{3} \int (|F_{j1}|^2 + |F_{j2}|^2) \Psi_j^\dagger \Psi_j e^{-iK \cdot r} r^2 d^3x, d^3x = r^2 \sin \theta \, dr \, d\theta \, d\varphi \quad (14)
\]

with the momentum transfer \( K \). At \( n_j = 0 = l_j \), as is easily seen, for any spinor of (11) we have \( \Phi_j^\dagger \Phi_j = 1/(4\pi) \), so the integrand in (14) does not depend on \( \varphi \) and we can consider vector \( K \) to be directed along \( z \)-axis. Then \( K \cdot r = K \cos \theta \) and with the help of (9) and relations (see Ref. 14): \( \int_0^\infty r e^{-pr} dr = \Gamma \left( \frac{1}{p} \right) \), \( \alpha, p > 0, \int_0^\infty r e^{-pr} \frac{\sin \left( \frac{K}{r} \right)}{\cos \left( \frac{K}{r} \right)} \, dr = \Gamma \left( \frac{1}{r} \right) \), \( \Gamma \left( \frac{1}{r} \right) > 0, \alpha > -1, \alpha p > |\text{Im} K|, \Gamma \left( \frac{1}{r} + 1 \right) = \alpha \Gamma \left( \frac{1}{r} \right) \), \( \int_0^\infty e^{-ir} \sin \theta \, d\theta = 2 \sin K / K \), we shall obtain

\[
f(K) = \sum_{j=1}^{3} f_j(K) = \sum_{j=1}^{3} \frac{(2\beta_j)^{2\alpha_j+1}}{6\alpha_j} \sin \left[ 2\alpha_j \arctan \left( \frac{K}{2\beta_j} \right) \right] \frac{K^{2}}{K(2\beta_j)^{2\alpha_j}} \\
= \sum_{j=1}^{3} \frac{1}{3} - 2\alpha_j^2 + 3\alpha_j + 1 \\
6\beta_j^2 \frac{K^2}{6} + O(K^4), \quad (15)
\]

wherefrom it is clear that \( f(K) \) is a function of \( K^2 \), as should be, and we can determine the root-mean-square radius of meson in the form

\[
<r> = \sqrt{\sum_{j=1}^{3} \frac{2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2}} \quad (16)
\]

It is clear, we can directly calculate \( <r> \) in accordance with the standard quantum mechanics rules as \( <r> = \sqrt{\int r^2 \overline{\Psi} \Psi \, d^3x} = \sqrt{\sum_{j=1}^{3} \int r^2 \overline{\Psi}^\dagger \Psi_j \, d^3x} \) and the result will be the same as in (16). So we should not call \( <r> \) of (16) the charge radius of meson – it is just the radius of meson determined by the wave functions of (6) (at \( n_j = 0 = l_j \)) with respect to strong interaction. Now we should notice the expression (15) to depend on 3-vector \( K \). To rewrite it in the form holding true for any 4-vector \( Q \), let us remind that according to general considerations (see, e.g., Ref. 15) the relation (15) corresponds to the so-called Breit frame where \( Q^2 = -K^2 \) [when fixing metric by (1)] so it is not complicated to rewrite (15) for arbitrary \( Q \).
in the form

$$f(Q^2) = \sum_{j=1}^{3} f_j(Q^2) = \sum_{j=1}^{3} \frac{(2\beta_j)^{2\alpha_j+1}}{6\alpha_j} \cdot \sin[2\alpha_j \arctan(\sqrt{|Q^2/(2\beta_j)|})]$$

(17)

which passes on to (15) in the Breit frame.

### 3.3. Magnetic moment, magnetic form factor, anomalous magnetic moment

We can define the volumetric magnetic moment density by

$$\mathbf{m} = \frac{q}{2} (\mathbf{r} \times \mathbf{J}) = \frac{q}{2} \left[ (yJ_z - zJ_y)\hat{i} + (zJ_x - xJ_z)\hat{j} + (xJ_y - yJ_x)\hat{k} \right]$$

with meson charge $q$ and

$$\mathbf{J} = \Psi^\dagger (\alpha \otimes I_3) \Psi.$$ Using (6) we have in the explicit form

$$J_x = \sum_{j=1}^{3} (F_{j1}^* F_{j2} + F_{j2}^* F_{j1}) \frac{\Phi_j^\dagger \Phi_j}{r^2}, \quad J_y = \sum_{j=1}^{3} (F_{j1}^* F_{j2} - F_{j2}^* F_{j1}) \frac{\Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j}{r^2}, \quad J_z = \sum_{j=1}^{3} (F_{j1}^* F_{j2} - F_{j2}^* F_{j1}) \frac{\Phi_j^\dagger \sigma_3 \sigma_1 \Phi_j}{r^2}$$

(18)

with Pauli matrices $\sigma_{1,2,3}$. Magnetic moment of meson is $\mathbf{M} = \int_V \mathbf{m} \, d^3x$, where $V$ is volume of meson (the ball of radius $r$). Then at $n_j = l_j = 0$, as is seen from (9), (11), $F_{j1}^* = F_{j1}$, $F_{j2}^* = -F_{j2}$, $\Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j = 0$ for any spinor of (11) which entails $J_x = J_y = 0$, i.e., $m_z = 0$ while $\int_V m_z \, d^3x = 0$ because of turning the integral over $\varphi$ to zero, which is easy to check. As a result, magnetic moments of mesons with the wave functions of (6) (at $l_j = 0$) are equal to zero, as should be according to experimental data.

We can, however, define magnetic form factor $F(K)$ of meson if noticing that

$$\int_{V} \sqrt{m_x^2 + m_y^2 + m_z^2} d^3x \neq \sqrt{\left( \int_{V} m_x d^3x \right)^2 + \left( \int_{V} m_y d^3x \right)^2 + \left( \int_{V} m_z d^3x \right)^2} = M = 0.$$

Under the circumstances we should define $F(K)$ as the inverse Fourier transform of $\sqrt{m_x^2 + m_y^2 + m_z^2}$

$$\sqrt{m_x^2 + m_y^2 + m_z^2} = \frac{q}{2\mu(2\pi)^3} \int F(K) e^{iK \mathbf{r}} d^3K$$

with meson mass $\mu$ so that

$$F(K) = \frac{2\mu}{q} \int \sqrt{m_x^2 + m_y^2 + m_z^2} e^{-iK \mathbf{r}} d^3x = \mu \int r |J_z| \sin \vartheta e^{-iK \mathbf{r}} d^3x = \frac{\mu}{2\pi} \int \sum_{j=1}^{3} |F_{j1}| |F_{j2}| \sin^2 \vartheta \frac{e^{-iK \mathbf{r}} d^3x}{r}$$

(19)
for any spinor of (11) so the integrand in (19) does not again depend on \( \varphi \). Then \( K \rho = K \cos \vartheta \) and using the relation \( \int_0^\pi e^{-iK \rho \cos \vartheta} \sin^3 \vartheta d\vartheta = 4[\sin (K \rho)/(K \rho)^3 - \cos (K \rho)/(K \rho)^2] \), we shall obtain

\[
F(K) = 4\mu \int_0^\infty r \sum_{j=1}^3 C_j^2 P_j Q_j r^{2\alpha_j} e^{-2b_j r} \left( 1 - \frac{g^2 b_j^2}{\beta_j^2} \right) \left[ \frac{\sin (K \rho)/(K \rho)^3 - \cos (K \rho)/(K \rho)^2}{(K \rho)^2} \right] dr = 
\]

\[
\frac{2\mu}{3} \sum_{j=1}^3 \frac{(2\beta_j)^{2\alpha_j+1}}{\alpha_j} \cdot \frac{P_j Q_j (1 - \frac{g^2 b_j^2}{\beta_j^2})}{(1 - \frac{g b_j}{\beta_j})^2 P_j^2 + (1 + \frac{g b_j}{\beta_j})^2 Q_j^2} \cdot \frac{1}{K^2 (K^2 + 4\beta_j^2)^{\alpha_j}} \cdot \left\{ \frac{\sin [(2\alpha_j - 1) \arctan (K/(2\beta_j))]}{(2\alpha_j - 1) K (K^2 + 4\beta_j^2)^{1/2}} - \cos [2\alpha_j \arctan (K/(2\beta_j))] \right\} = 
\]

\[
\frac{4\mu}{3} \sum_{j=1}^3 \frac{\beta_j}{\alpha_j} \cdot \frac{P_j Q_j (1 - \frac{g^2 b_j^2}{\beta_j^2})}{(1 - \frac{g b_j}{\beta_j})^2 P_j^2 + (1 + \frac{g b_j}{\beta_j})^2 Q_j^2} \cdot \frac{\alpha_j (2\alpha_j + 1)}{6\beta_j^2} \cdot \frac{\alpha_j (4\alpha_j^3 + 12\alpha_j^2 + 11\alpha_j + 3)}{120\beta_j^4} \cdot K^2 + O(K^4) 
\]

(20)

where the fact was used that by virtue of the normalization condition for wave functions we have \( C_j^2 [P_j^2 (1 - g b_j/\beta_j)^2 + Q_j^2 (1 + g b_j/\beta_j)^2] = (2\beta_j)^{2\alpha_j+1}/[3\Gamma(2\alpha_j+1)] \).

It is clear from (20) that \( F(K) \) is a function of \( K^2 \) and we can rewrite (20) for arbitrary 4-vector \( Q \) by the same manner as was done for electric form factor \( f(K) \) in (17).

Now we can define the anomalous magnetic moment density for meson by the relation

\[
m_a = \lim_{K \to 0} \frac{q}{2\mu (2\pi)^3} \int F(K) e^{iK \rho} d^3 K = \frac{q}{2\mu} F(0) \delta(x),
\]

so anomalous magnetic moment is

\[
M_a = \int_V m_a d^3 x \approx \int_{\mathbb{R}^3} m_a d^3 x = \frac{q}{2\mu} F(0) \quad (21)
\]

with

\[
F(0) = \frac{4\mu}{3} \sum_{j=1}^3 \frac{2\alpha_j + 1}{6\beta_j^2} \cdot \frac{P_j Q_j (1 - \frac{g^2 b_j^2}{\beta_j^2})}{(1 - \frac{g b_j}{\beta_j})^2 P_j^2 + (1 + \frac{g b_j}{\beta_j})^2 Q_j^2},
\]

as follows from (20). It is clear that for neutral mesons \( M_a = 0 \) due to \( q = 0 \). It should be noted that a possibility of existence of anomalous magnetic moments for mesons is permanently discussed in literature (see, e.g., Ref. 16 and references therein) though there is no experimental evidence in this direction.
4. Estimates for Parameters of SU(3)-Gluonic Field in $\eta$-Meson

4.1. Basic equations

The question now is how to apply the obtained form factors to electromagnetic decay $\eta \rightarrow 2\gamma$ to estimate parameters of SU(3)-gluonic field in $\eta$-meson. Actually kinematic analysis based on Lorentz- and gauge invariances gives rise to the following expression for width $\Gamma_2$ of the given decay (see, e.g., Ref. 17)

$$\Gamma_2 = \frac{1}{4} \pi \alpha_{em}^2 g_{\eta\gamma\gamma}^2 \mu^3$$

with electromagnetic coupling constant $\alpha_{em} = 1/137.0359895$ and $\eta$-meson mass $\mu = 547.51$ MeV while the information about strong interaction of quarks in $\eta$-meson is encoded in a decay constant $g_{\eta\gamma\gamma}$. Making replacement $g_{\eta\gamma\gamma} = f_P/\mu$ we can reduce (22) to the form

$$\Gamma_2 = \frac{\pi \alpha_{em}^2 \mu f_P^2}{4}$$

with the present-day experimental value $\Gamma_2 \approx 0.510$ keV.\(^8\) We can now notice that the only invariant which $f_P$ might depend on is $Q^2 = \mu^2$, i.e. we should find such a function $F(Q^2)$ for that $F(Q^2 = \mu^2) = f_P$. It is obvious from physical point of view that $F$ should be connected with electromagnetic properties of $\eta$-meson. As we have seen above in Section 3, there are at least two suitable functions for this aim – electric and magnetic form factors. But, as was mentioned, there exist no experimental consequences related to magnetic form factor at present whereas electric one to some extent determines, e.g., an effective size of meson in the form $<r>$ of (16). It is reasonable, therefore, to take $F(Q^2 = \mu^2) = Af(Q^2 = \mu^2)$ with some constant $A$ and electric form factor $f$ of (17) for the sought relation. At last, one should fix the value $A$ to define $f(Q^2 = \mu^2)$. The latter should not differ from 1 too much and considering that for $\pi^0$-meson the value of the corresponding electric form factor at $Q^2 = m_{\pi^0}^2$ is approximately equal to 1 (see Ref. 8), we put $A = 1/9$ which entails $f(Q^2 = \mu^2) \approx 1.343$. Then, denoting the quantities $\mu/(2\beta_j) = x_j$, we obtain the following equation for parameters of the confining SU(3)-gluonic field in $\eta$-meson

$$f(Q^2 = \mu^2) = \sum_{j=1}^{3} f_j(Q^2 = \mu^2) = \sum_{j=1}^{3} \frac{1}{6\alpha_j x_j} \cdot \frac{\sin (2\alpha_j \arctan x_j)}{(1 - x_j^2)^{\alpha_j}} \approx 1.343.$$  (24)

Finally, Eqs. (12) should also be added to (24) and the system obtained in such a way should be solved compatibly.

4.2. Numerical results

The results of numerical compatible solving of Eqs. (12) and (24) are adduced in Tables 1–3 where quantity $<r>$ was computed in accordance with (16).
Table 1: Gauge coupling constant, mass parameter $\mu_0$ and parameters of the confining SU(3)-gluonic field for $\eta$-meson.

| Particle | $g$  | $\mu_0$ (MeV) | $a_1$      | $a_2$      | $b_1$ (GeV) | $b_2$ (GeV) | $B_1$   | $B_2$   |
|----------|------|---------------|------------|------------|-------------|-------------|---------|---------|
| $\eta^{uu}$ | 5.14836 | 1.125         | -0.0328122 | 0.179728   | 0.194979    | 0.119737    | 0.255   | -0.010  |
| $\eta^{dd}$ | 5.14836 | 2.500         | 0.147640   | -0.178707  | 0.305728    | -0.119050   | -0.240  | -0.010  |
| $\eta^{ss}$ | 5.14836 | 53.75         | -0.0141391 | -0.0806779 | 0.252975    | -0.339250   | 0.260   | -0.310  |

Table 2: Theoretical and experimental $\eta$-meson mass and radius.

| Particle | Theoret. $\mu$ (MeV) | Experim. $\mu$ (MeV) | Theoret. $<r>$ (fm) | Experim. $<r>$ (fm) |
|----------|----------------------|----------------------|---------------------|---------------------|
| $\eta^{uu}$ | $\mu = 2m_u + \omega_j(0, 0, 1) = 547.51$ | 547.51               | 0.540243            | –                   |
| $\eta^{dd}$ | $\mu = 2m_d + \omega_j(0, 0, 1) = 547.51$ | 547.51               | 0.542582            | –                   |
| $\eta^{ss}$ | $\mu = 2m_s + \omega_j(0, 0, 1) = 547.51$ | 547.51               | 0.544444            | –                   |

Table 3: Theoretical and experimental $\eta$-meson electric form factor values.

| Particle | Theoret. $f(Q^2 = \mu^2)$ | Experim. $f(Q^2 = \mu^2)$ |
|----------|----------------------------|---------------------------|
| $\eta^{uu}$ | 1.3526                     | 1.343                     |
| $\eta^{dd}$ | 1.3215                     | 1.343                     |
| $\eta^{ss}$ | 1.3086                     | 1.343                     |
One can note that for $K^{\pm}$-mesons experimental estimate for $<r>$ is about 0.560 fm (see Ref. 8), so the values of $<r>$ for $\eta$-meson in Table 2 are reasonable enough since the $\eta$-meson mass does not greatly exceed the one of $K^{\pm}$-mesons.

5. Estimates of Gluon Concentrations, Electric and Magnetic Colour Field Strengths

Now let us remind that, according to Refs. 3, 7, one can confront the field (3) with $T_{00}$-component (volumetric energy density of the SU(3)-gluonic field) of the energy-momentum tensor (2) so that

$$T_{00} = T_{tt} = \frac{E^2 + H^2}{2} = \frac{1}{2} \left( \frac{a_1^2 + a_1 a_2 + a_2^2}{r^4} + \frac{b_1^2 + b_1 b_2 + b_2^2}{r^2 \sin^2 \vartheta} \right) = \frac{A}{r^4} + \frac{B}{r^2 \sin^2 \vartheta}$$

(25)

with electric $E$ and magnetic $H$ colour field strengths and real $A > 0, B > 0$.

To estimate the gluon concentrations we can employ (25) and, taking the quantity $\omega = \Gamma$, the full decay width of a meson, for the characteristic frequency of gluons we obtain the sought characteristic concentration $n$ in the form

$$n = \frac{T_{00}}{\Gamma}$$

(26)

so we can rewrite (25) in the form $T_{00} = T_{00}^{\text{coul}} + T_{00}^{\text{lin}}$ conforming to the contributions from the Coulomb and linear parts of the solution (3). This entails the corresponding split of $n$ from (26) as $n = n_{\text{coul}} + n_{\text{lin}}$.

The parameters of Table 1 were employed when computing and for simplicity we put $\sin \vartheta = 1$ in (25). Also there was used the following present-day full decay width of $\eta$-meson $\Gamma = 1.30$ keV, whereas the Bohr radius $a_0 = 0.529177249 \times 10^5$ fm.$^8$

Table 4 contains the numerical results for $n_{\text{coul}}, n_{\text{lin}}, n, E, H$ for the meson under discussion.

6. Concluding Remarks

As is seen from Table 4, at the characteristic scales of $\eta$-meson the gluon concentrations are huge and the corresponding fields (electric and magnetic colour ones) can be considered to be the classical ones with enormous strengthes. The part $n_{\text{coul}}$ of gluon concentration $n$ connected with the Coulomb electric colour field is decreasing faster than $n_{\text{lin}}$, the part of $n$ related to the linear magnetic colour field, and at large distances $n_{\text{lin}}$ becomes dominant. It should be emphasized that in fact the gluon concentrations are much greater than the estimates given in Table 4 because the latter are the estimates for maximal possible gluon frequencies, i.e. for maximal possible gluon impulses (under the concrete situation of $\eta$-meson). The given picture is in concordance with the one obtained in Refs. 2–7. As a result, the confinement mechanism developed in Refs. 1–3 is also confirmed by the considerations of the present Letter.

It should be noted, however, that our results are of a preliminary character which is readily apparent, for example, from that the current quark masses (as well
Table 4: Gluon concentrations, electric and magnetic colour field strengths in $\eta$-meson.

| $r$ (fm) | $n_{\text{coul}}$ (m$^{-3}$) | $n_{\text{lin}}$ (m$^{-3}$) | $n$ (m$^{-3}$) | $E$ (V/m) | $H$ (T) |
|----------|-------------------------------|-------------------------------|---------------|------------|---------|
| $r_0$    | $0.175430 \times 10^{33}$      | $0.135320 \times 10^{43}$     | $0.097900 \times 10^{12}$ | $0.148152 \times 10^{10}$ |         |
| $r_0$    | $0.161492 \times 10^{49}$      | $0.129833 \times 10^{51}$     | $0.057439 \times 10^{20}$ | $0.145118 \times 10^{14}$ |         |
| $r_0$    | $0.137565 \times 10^{52}$      | $0.378933 \times 10^{52}$     | $0.029441 \times 10^{22}$ | $0.783989 \times 10^{14}$ |         |
| $r_0$    | $0.146492 \times 10^{49}$      | $0.129833 \times 10^{51}$     | $0.057439 \times 10^{20}$ | $0.145118 \times 10^{14}$ |         |
| $r_0$    | $0.137565 \times 10^{52}$      | $0.378933 \times 10^{52}$     | $0.029441 \times 10^{22}$ | $0.783989 \times 10^{14}$ |         |
| $r_0$    | $0.161492 \times 10^{49}$      | $0.129833 \times 10^{51}$     | $0.057439 \times 10^{20}$ | $0.145118 \times 10^{14}$ |         |
as the gauge coupling constant \( g \) used in computation are known only within the certain limits and we can expect similar limits for the magnitudes discussed in the Letter so it is necessary further specification of the parameters for the confining SU(3)-gluonic field in \( \eta \)-meson which can be obtained, for instance, by calculating width of electromagnetic decay \( \eta \to \pi^0 + 2\gamma \) with the help of wave functions of \( \eta \)- and \( \pi^0 \)-mesons discussed above and in Ref. 7. We hope to continue analysing the given problems elsewhere.

Acknowledgment

Author is grateful to Alexandre Deur for sending the formula (13) via e-mail.

References

1. Yu. P. Goncharov, *Mod. Phys. Lett.* A16, 557 (2001).
2. Yu. P. Goncharov, *Phys. Lett.* B617, 67 (2005).
3. Yu. P. Goncharov, in *New Developments in Black Hole Research*, ed. P. V. Kreitler (Nova Science Publishers, 2006), pp. 67–121, Chap. 3, hep-th/0512099.
4. Yu. P. Goncharov, *Europhys. Lett.* 62, 684 (2003).
5. Yu. P. Goncharov and E. A. Choban, *Mod. Phys. Lett.* A18, 1661 (2003).
6. Yu. P. Goncharov and A. A. Bytsenko, *Phys. Lett.* B602, 86 (2004).
7. Yu. P. Goncharov, *Phys. Lett.* B641, 237 (2006).
8. Particle Data Group (W.-M. Yao et.al.), J. Phys. G33, 1 (2006).
9. Yu. P. Goncharov, *Pis’ma v ZhETF* 69, 619 (1999).
10. Yu. P. Goncharov, *Phys. Lett.* B458, 29 (1999).
11. A. Deur, private communication.
12. A. Deur, *Nucl. Phys.* A755, 353 (2005).
13. A. Deur et.al., *Phys. Lett.* B650, 244 (2007).
14. A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series. Elementary Functions* (Nauka, 1981).
15. V. B. Berestezkiy, E. M. Lifshits, L. P. Pitaevskiy, *Quantum Electrodynamics* (Fizmatlit, 2002).
16. A. Samsonov, *Yad. Fiz.* 68, 114 (2005).
17. H.M. Pilkuhn, *Relativistic Particle Physics* (Springer-Verlag, 1979).