Ramsey numbers of graphs
with most degrees bounded in random graphs

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Abstract

For graphs $F$ and $G$, let $F \rightarrow G$ signify that any red/blue edge coloring of $F$ contains a monochromatic $G$. Denote by $\mathcal{G}(N, p)$ the random graph space of order $N$ and edge probability $p$. Using the regularity method, one can show that for any fixed $p \in (0, 1]$, almost all graphs $F \in \mathcal{G}(cn, p)$ have $F \rightarrow G$ for any graph $G$ of order $n$ and all but at most $m$ degrees bounded, where $c$ is an integer depending on $p$ and $m$. Note that $r(K_m, n) \sim 2^m n$ and $r(K_m + K_n) \sim 2^m n$ as $n \to \infty$, for which we investigate the relation between $c$ and $p$. Let $N = \lfloor c2^m n \rfloor$ with $c > 1$ and $p_u, p_\ell = \frac{1}{c2^m}(1 \pm \sqrt{\frac{M \log n}{n}})$, where $M = M(c, m) > 0$. It is shown that $p_u$ and $p_\ell$ are Ramsey thresholds of $K_{m, n}$ in $\mathcal{G}(N, p)$. Namely, almost all $F \in \mathcal{G}(N, p_u)$ and almost no $F \in \mathcal{G}(N, p_\ell)$ have $F \rightarrow K_{m, n}$, respectively. Moreover, it is shown that $p_u$ and $p_\ell$ are (ordinary) upper threshold and lower threshold of $K_m + K_n$ to appear in $\mathcal{G}(N, p/2)$, respectively. We show that $\mathcal{G}(N, p/2)$ can be identified as the set of red (or blue) graphs obtained from $F \in \mathcal{G}(N, p)$ by red/blue edge coloring of $F$ with probability $1/2$ for each color, which leads to the definition of the weak Ramsey thresholds.

Key Words: Ramsey number; Regularity method; Random graph; Ramsey threshold

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1 Introduction

For graphs $F$ and $G$, let $F \rightarrow G$ signify that any red/blue edge coloring of $F$ contains a monochromatic $G$. The Ramsey number $r(G)$ is defined as the minimum $N$ such that $K_N \rightarrow G$.

For two positive functions $f(t)$ and $g(t)$, let us write $f(t) \leq O(g(t))$ or $g(t) \geq \Omega(f(t))$ if $f(t) \leq cg(t)$ for large $t$, where $c > 0$ is a constant, and $f(t) = \Theta(g(t))$ if $\Omega(g(t)) \leq f(t) \leq O(g(t))$. As usual, $f(t) = o(1)$ and $f(t) \sim g(t)$ signify $f(t) \to 0$ and $\frac{f(t)}{g(t)} \to 1$ as $t \to \infty$, respectively.

Let $\Delta(G)$ and $\delta(G)$ be the maximum degree and the minimum degree of a graph $G$, respectively. The following result follows from a remarkable application of Szemerédi’s regularity lemma by Chvátal, Rödl, Szemerédi and Trotter [4].

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Theorem 1 (\cite{4}). Let $\Delta \geq 1$ be a fixed integer. Then there is a constant $c = c(\Delta) > 0$ such that $r(G) \leq cn$ for any graph $G$ of order $n$ with $\Delta(G) \leq \Delta$.

Note that $K_N \to G$ is equivalent to $r(G) \leq N$. The method for the proof is now called the regularity method, which can be easily generalized to have a result for sparse graphs $F$ as $F \to G$ instead of $K_N \to G$. It is more convenient to consider sparse graphs in random graphs. Let $V$ be a labelled vertex set with $|V| = N$. The classic random graph space $\mathcal{G}(N, p)$ was defined by Erdős and Rényi \cite{9}, in which each edge of $K_N$ has a probability $p$ to appear, randomly and independently. For a graph property $P$, if $\Pr[F \in \mathcal{G}(N, p) : F \text{ has } P] \to 1$ as $N \to \infty$, then we say that almost all graphs in $\mathcal{G}(N, p)$ have the property $P$.

Denote by $\mathcal{F}_{\Delta, m}$ the family of graphs $F$, where all but at most $m$ vertices of $F$ have degrees at most $\Delta$.

Theorem 2. Let $\Delta$ and $m$ be positive integers and $p \in (0, 1]$. Then there exists a positive integer $c = c(\Delta, m, p)$ such that almost all graphs $F \in \mathcal{G}(cn, p)$ have $F \to G$, where $G$ is a graph of order $n$ in $\mathcal{F}_{\Delta, m}$.

It is easy to see the constant $c$ is large if $p$ is small. In fact, the constant $c$ in the proof of Theorem 2 is huge. We shall investigate the relation between $c$ and $p$. Since $\mathcal{F}_{\Delta, m}$ is a large family that contains graphs of many types, we shall concentrate on some sub-families.

Let $K_{m, n}$ be the complete bipartite graph on $m$ and $n$ vertices and $K_m + \overline{K}_n$ the graph obtained from $K_m$ by adding new edges to connect $K_m$ and other additional $n$ vertices completely. It is known that $r(K_{m, n}) \sim 2^m n$ as $n \to \infty$, but $r(K_m + \overline{K}_n)$ is much harder to handle. Recently, Conlon \cite{6} proved

$$r(K_m + \overline{K}_n) = (2^m + o(1))n,$$

for which the small term $o(1)$ is shown to be at most $\Theta([\log \log \log n]^{-1/25})$ in Conlon, Fox and Wigderson \cite{7}. The graphs $K_{m, n}$ and $K_m + \overline{K}_n$ are well-known graphs in $\mathcal{F}_{\Delta, m}$.

The Ramsey upper threshold $p_u = p_u(N)$ and Ramsey lower threshold $p_\ell = p_\ell(N)$ of $G$ in $\mathcal{G}(N, p)$ are defined by

$$\lim_{N \to \infty} \Pr[F \in \mathcal{G}(N, p) : F \to G] = \begin{cases} 1 & \text{if } p \geq p_u, \\ 0 & \text{if } p \leq p_\ell. \end{cases}$$

For $N = cr(K_{m, n})$ with $c > 1$, the following result reveals the relation between $c$ and $p$ precisely.

Theorem 3. Let $m \geq 1$ be an integer and $N = \lfloor c 2^m n \rfloor$ with $c > 1$. Then $p_u = \frac{1}{c \sqrt{m}} \left(1 + \frac{\omega(n)}{n}\right)$ and $p_\ell = \frac{1}{c \sqrt{m}} \left(1 - \sqrt{\frac{M \log n}{n}}\right)$ are a Ramsey upper threshold and a Ramsey lower threshold of $K_{m, n}$ in $\mathcal{G}(N, p)$, respectively, where $\omega(n) \to \infty$ and $M = M(c, m) > 0$ is a constant.

Failing to find a reasonable Ramsey upper threshold of $K_m + \overline{K}_n$, we shall weaken the property $F \to G$ by generalizing the deterministic event $F \to G$ to a random event, for which we shall clarify notation.
Let $V$ be a labelled vertex set, and $F$ a fixed graph on $V$. Consider a red/blue edge coloring of $F$ with probability $1/2$ for each color, randomly and independently. Let $F^{1/2}_R$ and $F^{1/2}_B$ be the red and blue graphs of $F$ in such a coloring, which are random graphs as the coloring is random even though $F$ is fixed. Note that different graphs $F$ and $H$ on the same $V$ may yield the same monochromatic graph as $F^{1/2}_R = H^{1/2}_R$ or $F^{1/2}_B = H^{1/2}_B$ with some positive probability. For a fixed graph $L$ on $V$, if $E(L) \subseteq E(F)$, then there is a unique edge coloring of $F$ such that $F^{1/2}_R = L$ that is a random event. If $F \in \mathcal{G}(N,p)$, then $F^{1/2}_R = L$ is a composite event. Intuitively, each edge of $F^{1/2}_R$ for $F \in \mathcal{G}(N,p)$ appears with probability $p/2$, and we shall show that this is true.

Recall that for a fixed graph $L$ on labelled vertex set $V$, the probability
\[ \Pr[F \in \mathcal{G}(N,p) : F = L] = \frac{e(L)}{2} \left(1 - p \right)^{N/2} \left(1 - p \right)^{e(L)}, \]
which is said to be the probability of $L$ to appear in $\mathcal{G}(N,p)$.

**Lemma 1.** Let $V$ be a labelled vertex set with $|V| = N$ and $L$ a graph on $V$. Then
\[ \Pr[F \in \mathcal{G}(N,p) : F^{1/2}_R = L] = \left(\frac{p}{2}\right)^{e(L)} \left(1 - \frac{p}{2}\right)^{N/2} \left(1 - \frac{p}{2}\right)^{e(L)}. \]
Namely, the red graphs $F^{1/2}_R$ form $\mathcal{G}(N,p/2)$ for $F \in \mathcal{G}(N,p)$.

The same argument can be applied for blue graphs $F^{1/2}_B$ as they also form $\mathcal{G}(N,p/2)$.

For a graph $G$ whose vertex set is not necessarily $V$, we define $F^{1/2} \rightarrow G$ for $F \in \mathcal{G}(N,p)$ an event as
\[ \{ F \in \mathcal{G}(N,p) : F^{1/2} \rightarrow G \} = \{ F \in \mathcal{G}(N,p) : G \subseteq F^{1/2}_R \text{ or } G \subseteq F^{1/2}_B \}, \]
and thus
\[ \{ F \in \mathcal{G}(N,p) : G \subseteq F^{1/2}_R \} \subseteq \{ F \in \mathcal{G}(N,p) : F^{1/2} \rightarrow G \} \subseteq \{ F \in \mathcal{G}(N,p) : G \subseteq F^{1/2}_R \} \cup \{ F \in \mathcal{G}(N,p) : G \subseteq F^{1/2}_B \}. \]
Note that if $G \subseteq F^{1/2}_R$ and $G \subseteq F^{1/2}_B$, then $F$ are contained in both of the last two events. In the proof of Lemma 1 we have $\Pr[F^{1/2}_R = L] = \Pr[F^{1/2}_B = L]$ for any fixed graph $L$ on $V$, hence
\[ \Pr[F \in \mathcal{G}(N,p) : G \subseteq F^{1/2}_R] = \Pr[F \in \mathcal{G}(N,p) : G \subseteq F^{1/2}_B], \]
and thus
\[ \Pr[F \in \mathcal{G}(N,p/2) : G \subseteq F] \leq \Pr[F \in \mathcal{G}(N,p) : F^{1/2} \rightarrow G] \leq 2 \Pr[F \in \mathcal{G}(N,p/2) : G \subseteq F], \]
which leads to the following definition.

**Definition 1.** For a graph $G$, we call probability functions $p_u = p_u(N)$ and $p_l = p_l(N)$ a weak Ramsey upper threshold and a weak Ramsey lower threshold of $G$ in $\mathcal{G}(N,p)$, respectively, if
\[ \lim_{N \to \infty} \Pr[F \in \mathcal{G}(N,p/2) : G \subseteq F] = \begin{cases} 1 & \text{for } p \geq p_u, \\ 0 & \text{for } p \leq p_l. \end{cases} \]
Even though \( F \xrightarrow{1/2} G \) for \( F \in \mathcal{G}(N,p) \) is a random event, we do not call the new thresholds as “random Ramsey thresholds” to avoid confusion since “thresholds” have their own random meanings, and we call them “weak Ramsey thresholds” by viewing \( F \xrightarrow{1/2} G \) as a weak form of \( F \to G \). In following result, the weak Ramsey thresholds of \( K_{m,n} \) are exactly its Ramsey thresholds in Theorem 3. It is interesting to find such relation for general graphs.

**Theorem 4.** Let \( m \geq 1 \) be an integer and \( N = \lfloor c2^m n \rfloor \) with \( c > 1 \). Then \( \frac{1}{e^{1/m}} \left( 1 + \omega(n) \right) \) and \( \frac{1}{e^{1/m}} (1 - \sqrt{M \log n / n}) \) are a weak Ramsey upper threshold and a weak Ramsey lower threshold of \( K_{m,n} \) in \( \mathcal{G}(N,p) \), respectively, where \( \omega(n) \to \infty \) and \( M = M(c,m) > 0 \) is a constant.

We give weak Ramsey thresholds of \( K_m + \overline{K}_n \) instead of Ramsey thresholds as follows.

**Theorem 5.** Let \( m \geq 1 \) be an integer and \( N = \lfloor c2^m n \rfloor \) with \( c > 1 \). Then \( \frac{1}{e^{1/m}} \left( 1 + \sqrt{M \log n / n} \right) \) are a weak Ramsey upper threshold and a weak Ramsey lower threshold of \( K_m + \overline{K}_n \) in \( \mathcal{G}(N,p) \), respectively, where \( M = M(c,m) > 0 \) is a constant.

Note that Theorem 3 implies \( r(K_{m,n}) \leq (2^m + o(1))n \) as \( F \to K_{m,n} \) implies \( K_N \to K_{m,n} \) for \( F \) of order \( N = \lfloor c2^m n \rfloor \) and any \( c > 1 \). However, Theorem 5 does not give the similar upper bound for \( r(K_m + \overline{K}_n) \) as it implies only there are monochromatic \( K_m + \overline{K}_n \) in almost all red/blue edge colorings of \( K_N \) instead of any such coloring.

### 2. Outline of the proof of Theorem 2

As the proof of Theorem 2 is similar to the proof of Theorem 1 in [1], so we just outline it.

The edge density of a graph \( F \) of order \( N \) is defined as \( d(F) = e(F) / \binom{N}{2} \), in which \( e(F) \) denotes the number of edges of \( F \). For a disjoint pair \((X,Y)\) of nonempty disjoint vertex subsets \( X \) and \( Y \) of \( V(F) \), let \( e_F(X,Y) \) be the number of edges of \( F \) between \( X \) and \( Y \). The ratio \( d_F(X,Y) = \frac{e_F(X,Y)}{|X||Y|} \) is called the edge density of \( F \) in \((X,Y)\), which is the probability that any pair \((x,y)\) selected randomly from \( X \times Y \) is an edge. If no danger of confusion, we write \( e(X,Y) \) and \( d(X,Y) \) for \( e_F(X,Y) \) and \( d_F(X,Y) \), respectively.

There are many interesting properties of \( \mathcal{G}(N,p) \), see Alon and Spencer [1], Bollobás [2], and Janson, Łuczak and Ruciński [10]. Thomason [14] introduced the notation of jumbled graphs, and Chung, Graham and Wilson [5] showed that many properties are equivalent, which are satisfied by almost all graphs in \( \mathcal{G}(N,p) \). Some equivalent properties imply the existence of the following graphs.

**Lemma 2.** Let \( p \in (0,1) \) be fixed. Then almost all graphs \( F \) in \( \mathcal{G}(N,p) \) have \( d(X,Y) \geq \frac{p}{2} \), where \( X \) and \( Y \) are disjoint vertex sets and \( |X||Y| = \Theta(N^2) \) as \( N \to \infty \).

Let \( \epsilon > 0 \) be a real number. A disjoint pair \((X,Y)\) is called \( \epsilon \)-regular if any \( X' \subseteq X \) and \( Y' \subseteq Y \) with \( |X'| > \epsilon |X| \) and \( |Y'| > \epsilon |Y| \) have \( |d(X,Y) - d(X',Y')| \leq \epsilon \).
The first form of the regularity lemma introduced by Szemerédi was in \cite{12}, and its general form was in \cite{13}. We need the following formulations of the regularity lemma.

**Lemma 3** (Multi-color regularity lemma). For any real $\epsilon > 0$ and any positive integers $\ell$ and $s$, there exists $M = M(\epsilon, \ell, s) > \ell$ such that if the edges of a graph $F$ on $N \geq \ell$ vertices are colored in $s$ colors, all monochromatic graphs have a same partition $C_1, C_2, \ldots, C_k$ with $|C_i| - |C_j| \leq 1$, and $\ell \leq k \leq M$ such that all but at most $\epsilon k^2$ pairs $(C_i, C_j)$, $1 \leq i < j \leq k$, are $\epsilon$-regular in each monochromatic graph.

We shall use Lemma 2 and multi-color regularity lemma for the proof of Theorem 2 that is similar to the proof of Theorem 1 in \cite{4} and thus we omit it.

3 Proofs of Theorem 3 – Theorem 5

Let us prove Lemma 1 first.

**Proof of Lemma 1.** For fixed graphs $F$ and $L$ on the labelled vertex set $V$, if $L \subseteq F$ or equivalently $E(L) \subseteq E(F)$, then

$$\Pr[F_{1/2} = L] = \left(\frac{1}{2}\right)^{e(L)} \left(1 - \frac{1}{2}\right)^{e(F) - e(L)} = \frac{1}{2^{e(F)}},$$

and otherwise $\Pr[F_{1/2} = L] = 0$. Hence, for a fixed graph $L$ on $V$ and $0 < p < 1$, we have

$$\Pr[F \in G(N, p) : F_{1/2} = L] = \sum_{F: L \subseteq F} \Pr[F \in G(N, p)] \Pr[F_{1/2} = L]$$

$$= \sum_{F: L \subseteq F} p^{e(F)} (1 - p)^{(N\choose 2) - e(F)} \frac{1}{2^{e(F)}}$$

$$= (1 - p)^{(N\choose 2)} \left[\frac{p}{2(1 - p)}\right]^{e(L)} \sum_{F: L \subseteq F} \left[\frac{p}{2(1 - p)}\right]^{e(F) - e(L)}$$

$$= (1 - p)^{(N\choose 2)} \left[\frac{p}{2(1 - p)}\right]^{e(L)} \left[1 + \frac{p}{2(1 - p)}\right]^{(N\choose 2) - e(L)}$$

$$= \left(\frac{p}{2}\right)^{e(L)} \left(1 - \frac{p}{2}\right)^{(N\choose 2) - e(L)}$$

as required. The statement for $p = 0$ or $p = 1$ is easy to verify, and the proof is completed. \qed

The rest of this section is devoted to the proofs of Theorem 3 and Theorem 5 for which we shall employ the following Chernoff bound \cite{3, 1, 2}.

**Lemma 4.** Let $X_1, X_2, \ldots$ be mutually independent random variables as

$$\Pr(X_i = 1) = p, \quad \Pr(X_i = 0) = q,$$
for any $i$ and $S_T = \sum_{t=i}^{T} X_i$. Then there exists $\delta_0 = \delta_0(p) > 0$ such that if $0 \leq \delta \leq \delta_0$, it holds

$$\Pr(S_T \geq T(p + \delta)) \leq \exp\{-T\delta^2/(3pq)\},$$

and

$$\Pr(S_T \leq T(p - \delta)) \leq \exp\{-T\delta^2/(3pq)\}.$$  

**Remark.** In following content, we shall write $N = \lfloor c2^m n \rfloor$ for fixed integer $m \geq 1$ and constant $c > 1$, and let $\omega(n)$ be a function with $\omega(n) \to \infty$ slowly such that the associated function $p_u \in [0, 1]$. To simplify the notation in the proofs, we shall admit $c2^m n$ as an integer hence $N = c2^m n$.

**Lemma 5.** Let $p_u = \frac{1}{c^{1/m}}\left(1 + \frac{\omega(n)}{2n}\right)$. Then $p_u$ is a Ramsey upper threshold of $K_{m,n}$ in $\mathcal{G}(N, p)$.

**Proof.** Let $p_0 = \frac{1}{c^{1/m}}\left(1 + \frac{\omega(n)}{2n}\right)$. We claim that almost all graphs $F$ in $\mathcal{G}(N, p_u)$ have $e(F) > p_0 N^2/2$.

Let $T = \frac{N}{2}$ and label all edges of $K_N$ as $e_1, e_2, \ldots, e_T$. For $i = 1, 2, \ldots, T$, define a random variable $X_i$ such that $X_i = 1$ if $e_i$ is an edge of $F \in \mathcal{G}(N, p_u)$ and 0 otherwise. Let $e(F) = \sum_{i=1}^{T} X_i$ be the number of edges of $F$. Note the events that edges appear are mutually independent, so $e(F)$ has the binomial distribution $B(T, p_u)$.

Let $\delta = p_u - p_0 = \frac{\omega(n)}{2c^{1/m}n}$. By the Chernoff bound,

$$\Pr[F \in \mathcal{G}(N, p_u) : e(F) \leq p_0 T] = \Pr[F \in \mathcal{G}(N, p_u) : e(F) \leq T(p_u - \delta)] \leq \exp\{-T\delta^2/(3p_u(1 - p_u))\},$$

in which $3p_u(1 - p_u) \to \frac{3(c^{1/m} - 1)}{c^{1/m}} > 0$ and

$$T\delta^2 \sim \frac{c^22^m n^2}{2} \cdot \frac{\omega^2(n)}{4c^{2/m}n^2} = 2^{2m - 3}c^{2(1 - 1/m)}\omega^2(n) \to \infty.$$  

So the evaluated probability tends to zero, which implies that almost all graphs $F \in \mathcal{G}(N, p_u)$ have $e(F) > p_0 N^2/2$, and the claim follows.

Then, we shall show that

$$\Pr[F \in \mathcal{G}(N, p_u) : F \to K_{m,n}] \to 1.$$  

(1)

From double counting of Kövari, Sós and Turán [11], we know

$$e_x(N; K_{m,n}) \leq \frac{1}{2}[(m - 1)^{1/m}N^{2 - 1/m} + (m - 1)N].$$

It suffices that $2e_x(N; K_{m,n}) < e(F)$ for almost all graphs $F$ in $\mathcal{G}(N, p_u)$. Therefore, by the claim that has been shown, we shall show

$$(n - 1)^{1/m}N^{2 - 1/m} + (m - 1)N < \frac{1}{c^{1/m}}\left(1 + \frac{\omega(n)}{2n}\right)\binom{N}{2}.$$
It suffices to show that
\[
\left( \frac{N}{c2^m} \right)^{1/m} N^{2 - 1/m} + mN \leq \frac{1}{2c^{1/m}} \left( 1 + \frac{\omega(n)}{2n} \right) N(N - 1),
\]
equivalently,
\[
N^2 + 2c^{1/m}mN \leq \left( 1 + \frac{\omega(n)}{2n} \right)(N^2 - N),
\]
which follows by deleting square terms on both sides and noting \( \omega(n) \to \infty \).

We will use a simple fact as follows. For graph properties \( P \) and \( Q \), if \( \Pr(P) \to 1 \) and \( \Pr(Q) \to 1 \) in the same random graph space \( G(N, p) \), then \( \Pr(P \cap Q) \to 1 \) since \( \Pr(P \cap Q) \geq 1 - \Pr(\overline{P}) - \Pr(\overline{Q}) \), where \( \Pr(P) = \Pr[F \in G(N, p) : F \text{ has } P] \).

**Lemma 6.** Let \( p_\ell = \frac{2c^{1/m}}{1 - \sqrt{\frac{M\log n}{n}}} \), where \( M = M(c, m) > 0 \) is a constant. Then \( p_\ell \) is a lower threshold of \( K_{m, n} \) in \( G(N, p) \), and \( 2p_\ell \) is a Ramsey lower threshold of \( K_{m, n} \) in \( G(N, p) \).

**Proof.** Note \((1 - x)^{1/m} = 1 - \frac{1}{m}x + o(x^2)\) for \( x \to 0 \), hence \( 1 - \frac{2}{m}x \leq (1 - x)^{1/m} \leq 1 - \frac{1}{2m}x \) for small \( x > 0 \). By slightly shifting \( M \), it suffices to show that
\[
p_\ell = \frac{1}{2c} \left( 1 - \sqrt{\frac{M\log n}{n}} \right)^{1/m}
\]
is such a lower threshold, i.e., \( \Pr[F \in G(N, p_\ell) : K_{m, n} \subseteq F] \to 0 \).

Let \( U \) be a fixed subset of \( V \) with \(|U| = m\). Let \( v_1, v_2, \ldots, v_{N - m} \) be the vertices outside of \( U \). For each \( i = 1, 2, \ldots, N - m \), define a random variable \( X_i \) such that \( X_i = 1 \) if \( v_i \) is a common neighbor of \( U \) in \( F \) and 0 otherwise. Then \( \Pr(X_i = 1) = p_\ell^m \).

Set \( S_{N - m} = \sum_{i=1}^{N-m} X_i \) that has the binomial distribution \( B(N - m, p_0) \), where \( p_0 = p_\ell^m \). Note that the event \( S_{N - m} \geq n \) means that there is a \( K_{m, n} \) with \( U \) as the part of \( m \) vertices. Hence
\[
\Pr \left[ K_{m, n} \subseteq G(N, p_\ell) \right] \leq \binom{N}{m} \Pr(S_{N - m} \geq n).
\]

We now evaluate the probability \( \Pr(S_{N - m} \geq n) \). Write
\[
n = \frac{N}{c2^m} = (p_0 + \delta)(N - m),
\]
where
\[
\delta = \frac{N}{c2^m(N - m)} - p_0 = \frac{1}{c2^m} \left[ \frac{N}{N - m} - 1 + \sqrt{\frac{M\log n}{n}} \right] = \frac{1}{c2^m} \left[ \frac{m}{N - m} + \sqrt{\frac{M\log n}{n}} \right].
\]

Note \( \frac{m}{N - m} \ll \sqrt{\frac{M\log n}{n}} \). By virtue of Chernoff bound,
\[
\Pr(S_{N - m} \geq n) = \Pr \left[ S_{N - m} \geq (p_0 + \delta)(N - m) \right] \leq \exp \left\{ - (N - m)\delta^2 / (3p_0q_0) \right\}.
\]
Note that $3p_0q_0$ tends a positive constant, and

$$(N - m) \delta^2 \sim N \delta^2 \sim c2^m n \frac{M \log n}{e2^m n} = \frac{M}{e2^m} \log n.$$ 

Taking large $M$ such that $(N - m) \delta^2 / (3p_0q_0) > 2m \log n$ for large $n$, then we have

$$\binom{N}{m} \Pr(S_{N-m} \geq n) \leq \Theta\left(\frac{n^m}{n^{2m}}\right) \to 0,$$

and thus $p_\ell$ is a lower threshold of $K_{m,n}$ in $G(N, p)$.\footnote{Proof of Theorem 3. The proof follows from Lemma 5 and Lemma 6}\footnote{Proof of Theorem 4. The weak Ramsey lower threshold of $K_{m,n}$ is from Lemma 6 and the weak Ramsey upper threshold of $K_{m,n}$ follows from the weak Ramsey upper threshold of $K_m + \overline{K}_n$ in Theorem 5}\footnote{Proof of Theorem 5. Since $p_\ell = \frac{1}{e2^m} (1 - \sqrt{\frac{M \log n}{n}})$ is a weak Ramsey lower threshold of $K_{m,n}$ in $G(N, p)$, it is also a weak Ramsey lower threshold of $K_m + \overline{K}_n$. Then, we shall show that} Now we shall show $2p_\ell$ is a Ramsey lower threshold of $K_{m,n}$ in $G(N, p)$. Recall the proof of the claim in Lemma 5 we are easy to see that almost all graphs $F$ in $G(N, p_\ell)$ have

$$A = \left(p_\ell - \frac{\omega(n)}{2c^{1/m_n}}\right) \binom{N}{2} \leq e(F) \leq \left(p_\ell + \frac{\omega(n)}{2c^{1/m_n}}\right) \binom{N}{2} = B. \quad (2)$$

Denote by $T = 2^{\left(\frac{N}{2}\right)}$. Let us label all graphs on labelled vertex set $V$ as $R_1, R_2, \ldots, R_T$ and set $t = t(n)$ with $1 < t < T$ such that $R_i$ contains no $K_{m,n}$ for $1 \leq i \leq t$ and

$$\sum_{i=1}^{t} \Pr[R_i : R_i \in G(N, p_\ell)] \to 1 \quad (3)$$

as $n \to \infty$. Furthermore, we assume that the number of edges of each $R_i$ with $1 \leq i \leq t$ is in the interval $[A, B]$ described in (2).

We then consider red/blue edge coloring of $F$ from $G(N, 2p_\ell)$ with probability 1/2 for each edge, randomly and independently, for which all the red graphs $F_R$ form the random graph space $G(N, p_\ell)$ by Lemma 1. The same argument can be applied for blue graphs $F_B$ for $F \in G(N, 2p_\ell)$, which form the same space $G(N, p_\ell)$.

The similar proof implies that almost all $F \in G(N, 2p_\ell)$ have $e(F) \in [2A, 2B]$, namely $e(F_R) + e(F_B)$ are in the interval $[2A, 2B]$. Combining this and the fact that both $e(F_R)$ and $e(F_B)$ are in the same interval $[A, B]$, we can further assume that if $F_R = R_i$ for $1 \leq i \leq t$, then there is some $j$ with $1 \leq j \leq t$ such that $F_B = R_j$, namely $R_i$ and $R_j$ are the red and blue graphs obtained from edge coloring of $F$, respectively.

Therefore, for almost all $F \in G(N, 2p_\ell)$, there is a red/blue edge coloring of $F$ such that there is no monochromatic $K_{m,n}$, completing the proof.
\( \frac{1}{c^{1/m}} (1 + \sqrt{\frac{M \log n}{n}}) \) is a weak Ramsey upper threshold of \( K_m + \overline{K}_n \) in \( G(N, p) \), namely almost all graphs \( G(N, p_u/2) \) contain \( K_m + \overline{K}_n \).

The same argument as that in the proof of Lemma 6 tells us that we may assume that

\[
p_u = \left[ \frac{1}{c} \left( 1 + \sqrt{\frac{M \log n}{n}} \right) \right]^{1/m}.
\]

Let \( U \) be a fixed subset of \( V \) with \( |U| = m \). Let \( v_1, v_2, \ldots, v_{N-m} \) be the vertices outside of \( U \). For each \( i = 1, 2, \ldots, N-m \), define a random variable \( X_i(U) \) such that \( X_i(U) = 1 \) if \( v_i \) is common neighbor of \( U \) and 0 otherwise. Then \( \Pr[X_i(U) = 1] = (p_u/2)^m \).

Set \( p_0 = (p_u/2)^m \) and \( S_{N-m} = \sum_{i=1}^{N-m} X_i(U) \). The the event \( S_{N-m} \leq n-1 \) means that \( U \) has at most \( n-1 \) common neighbors. Note that \( S_{N-m} \) has the binomial distribution \( B(N-m, p_0) \) that is independent to \( U \).

We now evaluate the probability \( \Pr(S_{N-m} \leq n-1) \). Write

\[
n - 1 = \frac{N - c2^m}{c2^m} = \left[ \left( \frac{p_u}{2} \right)^m - \delta \right] (N-m) = (p_0 - \delta)(N-m),
\]

where

\[
\delta = \frac{1}{2m} \left[ p_u^m - \frac{N - c2^m}{c(N-m)} \right] = \frac{1}{c2^m} \left[ \sqrt{\frac{M \log n}{n}} + \frac{c2^m - m}{N-m} \right].
\]

By virtue of Chernoff bound,

\[
\Pr(S_{N-m} \leq n-1) = \Pr \left[ S_{N-m} \leq (p_0 - \delta)(N-m) \right] \leq \exp \left\{ - (N-m)\delta^2 / (3p_0 q_0) \right\}.
\]

Note that \( p_0 q_0 \) tends a positive constant, and

\[-(N-m)\delta^2 < -\frac{1}{2} N \frac{M \log n}{c^2 2^m n} = -\log n \cdot M', \]

where \( M' / (3p_0 q_0) > 2m \) if \( M \) is large. Let \( A_U \) be the event that the number of common neighbors of \( U \) in \( F \in G(N, p_u/2) \) is at most \( n-1 \), where \( |U| = m \). Then

\[
\Pr(A_U) = \Pr(S_{N-m} \leq n-1) < \left( \frac{1}{n} \right)^{2m}.
\]

Therefore, we have

\[
\Pr \left[ \bigcup_U A_U \right] \leq \binom{N}{m} \Pr[S_{N-m} \leq n-1] \leq N^m \left( \frac{1}{n} \right)^{2m} \to 0.
\]

Let \( Y = \cap_U A_U \) be the event that each vertex set \( U \) of graph \( F \in G(N, p_u/2) \) has at least \( n \) common neighbors, where \( |U| = m \). The \( \Pr(Y) \to 1 \).

A classic result of Erdős and Rényi [9] tells us that \( \omega(N) / N^{2/(m-1)} \) is an upper threshold of \( K_m \) to appear in \( G(N, p) \), where \( \omega(N) \to \infty \) slowly arbitrarily. Since \( p_u \sim \frac{1}{c^1/r} \gg \frac{\omega(N)}{N^{2/(m-1)}} \), we know that \( p_u \) is an upper threshold of \( K_m \) to appear in \( G(N, p/2) \).
Denote by $Z$ the event that there is a clique $U$ in $F \in \mathcal{G}(N,p_u/2)$ with $|U| = m$. Then $\Pr(Z) \to 1$ and thus $\Pr(Y \cap Z) \to 1$. So almost all graphs in $\mathcal{G}(N,p_u/2)$ contain a $K_m + \overline{K}_n$, and the proof is completed. □

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