Weak existence of a solution to a differential equation driven by a very rough fBm

Davar Khoshnevisan  Jason Swanson
University of Utah  University of Central Florida

Yimin Xiao  Liang Zhang
Michigan State University  Michigan State University

September 5, 2013

Abstract

We prove that if \( f : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous, then for every \( H \in (0, 1/4] \) there exists a probability space on which we can construct a fractional Brownian motion \( X \) with Hurst parameter \( H \), together with a process \( Y \) that: (i) is Hölder-continuous with Hölder exponent \( \gamma \) for any \( \gamma \in (0, H) \); and (ii) solves the differential equation \( dY_t = f(Y_t) \, dX_t \). More significantly, we describe the law of the stochastic process \( Y \) in terms of the solution to a non-linear stochastic partial differential equation.

Keywords: Stochastic differential equations; rough paths; fractional Brownian motion; fractional Laplacian; the stochastic heat equation.

AMS 2000 subject classification: 60H10; 60G22; 34F05.

1 Introduction

Let us choose and fix some \( T > 0 \) throughout, and consider the differential equation

\[
\begin{align*}
\frac{dY_t}{dt} &= f(Y_t) \, dX_t \\
&\quad \quad (0 < t \leq T),
\end{align*}
\]

(\text{DE}_0)

that is driven by a given, possibly-random, signal \( X := \{X_t\}_{t \in [0, T]} \) and is subject to some given initial value \( Y_0 \in \mathbb{R} \) which we hold fixed throughout. The sink/source function \( f : \mathbb{R} \to \mathbb{R} \) is also fixed throughout, and is assumed to be Lipschitz continuous, globally, on all of \( \mathbb{R} \).

It is well known—and not difficult to verify from first principles—that when the signal \( X \) is a Lipschitz-continuous function, then:

\*\*Research supported in part by NSF grant DMS-1307470.
(i) The differential equation \((\text{DE}_0)\) has a solution \(Y\) that is itself Lipschitz continuous;

(ii) The Radon–Nikodým derivative \(dY_t/dX_t\) exists, is continuous, and solves \(dY_t/dX_t = f(Y_t)\) for every \(0 < t \leq T\); and

(iii) The solution to \((\text{DE}_0)\) is unique.

Therefore, the Lebesgue differentiation theorem implies that we can recast \((\text{DE}_0)\) equally well as the solution to the following: As \(\varepsilon \downarrow 0\),
\[
\frac{Y_{t+\varepsilon} - Y_t}{X_{t+\varepsilon} - X_t} = f(Y_t) + o(1),
\]
for almost every \(t \in [0, T]\).\(^1\) Note that \((\text{DE})\) always has an “elementary” solution, even when \(X\) is assumed only to be continuous. Namely, if \(y\) is a solution to the ODE, \(y' = f(y)\), and we set \(Y_t = y(X_t)\), then \(Y_{t+\varepsilon} - Y_t = f(Y_t)(X_{t+\varepsilon} - X_t) + o(|X_{t+\varepsilon} - X_t|)\). Also note that if \(Y\) is a solution to \((\text{DE})\) and \(\xi\) is a process that is smoother than \(X\) in the sense that \(\xi_{t+\varepsilon} - \xi_t = o(|X_{t+\varepsilon} - X_t|)\), then \(Y + \xi\) is also a solution to \((\text{DE})\).

Differential equations such as \((\text{DE}_0)\) and/or \((\text{DE})\) arise naturally also when \(X\) is Hölder continuous with some positive index \(\gamma < 1\). One of the best-studied such examples is when \(X\) is Brownian motion on the time interval \([0, T]\). In that case, it is very well known that \(X\) is Hölder continuous with index \(\gamma\) for any \(\gamma < 1/2\). It is also very well known that \((\text{DE}_0)\) and/or \((\text{DE})\) has infinitely-many strong solutions [36], and that there is a unique pathwise solution provided that we specify what we mean by the stochastic integral \(\int_0^t f(Y_s) \, dX_s\). [consider the integrals of Itô and Stratonovich, for instance].

This view of stochastic differential equations plays an important role in the pathbreaking work [26, 25] of T. Lyons who invented his theory of rough paths in order to solve \((\text{DE}_0)\) when \(X\) is rougher than Lipschitz continuous. Our reduction of \((\text{DE})\) to \((\text{DE}_0)\) is motivated strongly by Gubinelli’s theory of controlled rough paths [17], which we have learned from a recent paper of Hairer [18]. In the present context, Gubinelli’s theory of controlled rough paths basically states that if we could prove \textit{a priori} that the \(o(1)\) term in \((\text{DE})\) has enough structure, then there is a unique solution to \((\text{DE}), \) and hence \((\text{DE}_0)\).

Lyons’ theory builds on older ideas of Fox [12] and Chen [4], respectively in algebraic differentiation and integration theory, in order to construct, for a large family of functions \(X\), “rough-path integrals” \(\int_0^t f(Y_s) \, dX_s\) that are defined uniquely provided that a certain number of “multiple stochastic integrals” of \(X\) are pre specified. Armed with a specified definition of the stochastic integral \(\int_0^t f(Y_s) \, dY_s\), one can then try to solve the differential equation \((\text{DE})\) and/or \((\text{DE}_0)\) pathwise [that is \(\omega\)-by-\(\omega\)]. To date, this program has been particularly successful when \(X\) is Hölder continuous with index \(\gamma \in [1/3, 1]\): When \(\gamma \in [1/3, 1]\):

\[
0 \div 0 := 0
\]

To be completely careful, we might have to define \(0 \div 0 := 0\) in the cases that \(X\) has intervals of constancy. But with probability one, this will be a moot issue for the examples that we will be considering soon.

\[2\]
one uses Young’s theory of integration; \( \gamma = \frac{1}{2} \) is covered in essence by martingale theory; and Errami and Russo [10] and Chapter 5 of Lyons and Qian [24] both discuss the more difficult case \( \gamma \in [\frac{1}{3}, \frac{1}{2}] \). There is also mounting evidence that one can extend this strategy to cover values of \( \gamma \in [\frac{1}{4}, 1] \)—see [1, 2, 3, 6, 7, 16, 31]—and possibly even \( \gamma \in (0, \frac{1}{4}) \)—see the two recent papers by Unterberger [35] and Nualart and Tindel [29].

As far as we know, very little is known about the probabilistic structure of the solution when \( \gamma < \frac{1}{2} \) [when the solution is in fact known to exist]. Our goal is to say something about the probabilistic structure of a solution for a concrete, but highly interesting, family of choices for \( X \) in (DE).

A standard fractional Brownian motion [fBm] with Hurst parameter \( H \in (0, \frac{1}{4}) \)—abbreviated fBm\((H)\)—is a continuous, mean-zero Gaussian process \( X := \{X_t\}_{t \geq 0} \) with \( X_0 = 0 \) a.s. and

\[
E( |X_t - X_s|^2) = |t - s|^{2H} \quad (s, t \geq 0).
\]

Note that fBm\((\frac{1}{2})\) is a standard Brownian motion. We refer to any constant multiple of a standard fractional Brownian motion, somewhat more generally, as fractional Brownian motion [fBm].

Here, we study the differential equation (DE) in the special case that \( X \) is fBm\((H)\) with

\[
0 < H \leq \frac{1}{4}.
\]

It is well known that (1.1) implies that \( X \) is Hölder continuous with index \( \gamma \) for every \( \gamma < H \), up to a modification. Since \( H \in (0, \frac{1}{4}) \), we are precisely in the regime where not a great deal is known about (DE).

In analogy with the classical literature on stochastic differential equations [36] the following theorem establishes the “weak existence” of a solution to (DE), provided that we interpret the little-o term in (DE\(, 0\)), somewhat generously, as “little-o in probability.” Our theorem says some things about the law of the solution as well.

**Theorem 1.1.** Let \( g : \mathbb{R} \to \mathbb{R} \) be Lipschitz continuous uniformly on all of \( \mathbb{R} \). Choose and fix \( H \in (0, \frac{1}{4}) \). Then there exists a probability space \( (\Omega, \mathcal{F}, P) \) on which we can construct a fractional Brownian motion \( X \), with Hurst parameter \( H \), together with a stochastic process \( Y \in \cap_{\gamma \in (0, H)} C^\gamma([0, T]) \) such that

\[
\lim_{\varepsilon \downarrow 0} \sup_{t \in [0, T]} \mathbb{P} \left\{ \frac{Y_{t+\varepsilon} - Y_t}{X_{t+\varepsilon} - X_t} - g(Y_t) > \delta \right\} = 0 \quad \text{for all } \delta > 0.
\]

Moreover, \( Y := \{Y_t\}_{t \in [0, T]} \) has the same law as \( \{\kappa_H u_t(0)\}_{t \in [0, T]} \), where

\[
\kappa_H := \left( \frac{(1 - 2H)\Gamma(1 - 2H)}{2\pi H} \right)^{1/2},
\]

\footnote{In other words, \( X \in \cap_{\gamma \in (0, H)} C^\gamma([0, T]) \) a.s., where \( C^\gamma([0, T]) \) denotes as usual the collection of all continuous functions \( f : [0, T] \to \mathbb{R} \) such that \( |f(t) - f(s)| \leq \text{const} \cdot |t - s|^{\gamma} \) uniformly for all \( s, t \in [0, T] \).}
and \( u \) denotes the mild solution to the nonlinear stochastic partial differential equation,

\[
\frac{\partial}{\partial t} u_t(x) = \frac{1}{2}(\Delta_{\alpha/2} u_t)(x) + \frac{1}{2(1-2H)/2} \cdot \kappa^2_H g(\kappa_H u_t(x)) \dot{W}_t(x),
\]

on \((t, x) \in (0, T] \times \mathbb{R},\) subject to \( u_0(x) \equiv Y_0 \) for all \( x \in \mathbb{R},\) where \( \dot{W} \) denotes a space-time white noise.

The preceding can be extended to all of \( H \in (0, 1/2) \) by replacing, in (3.1) below, the space-time white noise \( \dot{W}_t(x) \) by a generalized Gaussian random field \( \psi_t(x) \) whose covariance measure is described by

\[
\text{Cov}(\psi_t(x), \psi_s(y)) = \frac{\delta_0(t-s)}{|x-y|^\theta},
\]

for a suitable choice of \( \theta \in (0, 1). \) We will not pursue this matter further here since we do not know how to address the more immediately-pressing question of uniqueness in Theorem 1.1. Namely, we do not know a good answer to the following: “What are [necessarily global] non-trivial conditions that ensure that our solution \( Y \) is unique in law”?

Throughout this paper, \( A_q \) denotes a finite constant that depends critically only on a [possibly vector-valued] parameter \( q \) of interest. We will not keep track of parameter dependencies for the parameters that are held fixed throughout; they include \( \alpha \) and \( H \) of (2.16) below, as well as the functions \( g \) [see Theorem 1.1] and \( f \) [see (5.1) below].

The value of \( A_q \) might change from line to line, and sometimes even within the line.

In the absence of interesting parameter dependencies, we write a generic “const” in place of “\( A. \)”

We prefer to write \( \| \cdots \|_k \) in place of \( \| \cdot \|_{L^k(\Omega)} \), where \( k \in [1, \infty) \) can be an arbitrary real number. That is, for every random variable \( Y \), we set

\[
\|Y\|_k := \left\{ \mathbb{E}(|Y|^k) \right\}^{1/k}.
\]

(1.7)

On a few occasions we might write \( \text{Lip}_\varphi \) for the optimal Lipschitz constant of a function \( \varphi : \mathbb{R} \to \mathbb{R} \); that is,

\[
\text{Lip}_\varphi := \sup_{-\infty < x < y < \infty} \frac{\varphi(x) - \varphi(y)}{x - y}.
\]

(1.8)

2 Some Gaussian random fields

In this section we recall a decomposition theorem of Lei and Nualart [23] which will play an important role in this paper; see Mueller and Wu [27] for a related set of ideas. We also work out an example that showcases further the Lei–Nualart theorem.
2.1 fBm and bi-fBm

Suppose that $H \in (0,1)$ and $K \in (0,1]$ are fixed numbers. A standard bifractional Brownian motion, abbreviated as bi-fBm$(H,K)$, is a continuous mean-zero Gaussian process $B^{H,K} := \{B^{H,K}_t\}_{t \geq 0}$ with $B^{H,K}_0 := 0$ a.s. and covariance function

$$\text{Cov} \left( B^{H,K}_t, B^{H,K}_{t'} \right) = 2^{-K} \left( t^{2H} + (t')^{2H} \right)^K - |t - t'|^{2HK}, \quad (2.1)$$

for all $t', t \geq 0$. Note that $B^{H,1}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$. More generally, any constant multiple of a standard bifractional Brownian motion will be referred as bifractional Brownian motion.

Bifractional Brownian motion was invented by Houdré and Villa [19] as a concrete example (besides fractional Brownian motion) of a family of processes that yield natural “quasi–helices" in the sense of Kahane [21] and/or “screw lines" of classical Hilbert-space theory [28, 32]. Some sample path properties of bi-fBm$(H,K)$ have been studied by Russo and Tudor [30], Tudor and Xiao [34] and Lei and Nualart [23]. In particular, the following decomposition theorem is due to Lei and Nualart [23, Proposition 1].

**Proposition 2.1.** Let $B^{H,K}$ be a bi-fBm$(H,K)$. There exists a fractional Brownian motion $B^{HK}$ with Hurst parameter $HK$ and a stochastic process $\xi$ such that $B^{H,K}$ and $\xi$ are independent and, outside a single $P$-null set,

$$B^{H,K}_t = 2^{(1-K)/2}B^{HK}_t + \xi_t \quad \text{for all } t \geq 0. \quad (2.2)$$

Moreover, the process $\xi$ is a centered Gaussian process, with sample functions that are infinitely differentiable on $(0,\infty)$ and absolutely continuous on $[0,\infty)$.

In fact, it is shown in [23, eq.’s (4) and (5)] that we can write

$$\xi_t = \left( \frac{K}{2^{HK}(1-K)} \right)^{1/2} \int_0^\infty 1 - \exp\left(-s(1+K)/2\right) \frac{dW_s}{s^{(1+K)/2}}, \quad (2.3)$$

where $W$ is a standard Brownian motion that is independent of $B^{H,K}$.

2.2 The linear heat equation

Let $\hat{\phi}$ denote the Fourier transform, normalized so that for every rapidly-decreasing function $\phi : \mathbb{R} \to \mathbb{R}$,

$$\hat{\phi}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} \phi(x) \, dx \quad (\xi \in \mathbb{R}). \quad (2.4)$$

Let $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ denote the fractional Laplace operator, which is usually defined by the property that $(\Delta^{\alpha/2} \hat{\phi})(\xi) = |\xi|^{\alpha} \hat{\phi}(\xi)$; see Jacob [20, Vol. II].
Consider the linear stochastic PDE
\[
\frac{\partial}{\partial t}v_t(x) = \frac{1}{2}(\Delta_{\alpha/2}v_t)(x) + \dot{W}_t(x),
\]
(2.5)
where \(v_0(x) \equiv 0\) and \(\dot{W}_t(x)\) denotes space-time white noise; that is,
\[
\dot{W}_t(x) = \frac{\partial^2 W_t(x)}{\partial t \partial x},
\]
(2.6)
in the sense of generalized random fields [15, Chapter 2, §2.4], for a space-time Brownian sheet \(W\).

According to the theory of Dalang [8], the condition
\[
1 < \alpha \leq 2
\]
(2.7)
is necessary and sufficient in order for (2.5) to have a solution \(v\) that is a random function. Lei and Nualart [23] have shown that—in the case that \(\alpha = 2\)—the process \(t \mapsto v_t(x)\) is a suitable bi-fBm for every fixed \(x\). In this section we apply the reasoning of [23] to the present setting in order to show that the same can be said about the solution to (2.5) for every possible choice of \(\alpha \in (1, 2]\).

Let \(p_t(x)\) denote the fundamental solution to the fractional heat operator \((\partial^2/\partial t^2 - \frac{1}{2}\Delta_{\alpha/2})\); that is, the function \((t, x, y) \mapsto p_t(y - x)\) is the transition probability function for a symmetric stable-\(\alpha\) Lévy process, normalized as follows (see Jacob [20, Vol. III]):
\[
\hat{p}_t(\xi) = \exp(-t|\xi|^\alpha/2) \quad (t \geq 0, \xi \in \mathbb{R}).
\]
(2.8)
The Plancherel theorem implies the following: For all \(t > 0\),
\[
\|p_t\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \|\hat{p}_t\|_{L^2(\mathbb{R})}^2 = \frac{1}{\pi} \int_0^\infty e^{-t\xi^\alpha} \, d\xi = \frac{\Gamma(1/\alpha)}{\alpha \pi t^{1/\alpha}}.
\]
(2.9)
Let us mention also the following variation: By the symmetry of the heat kernel, \(\|p_t\|_{L^2(\mathbb{R})}^2 = (p_t \ast p_t)(0) = p_{2t}(0)\). Therefore, the inversion theorem shows that
\[
p_t(0) = \sup_{x \in \mathbb{R}} p_t(x) = \frac{2^{1/\alpha}\Gamma(1/\alpha)}{\alpha \pi t^{1/\alpha}} \quad (t > 0).
\]
(2.10)

Now we can return to the linear stochastic heat equation (2.5), and write its solution \(v\), in mild form, as follows:
\[
v_t(x) = \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \, W(ds \, dy).
\]
(2.11)
It is well known [37, Chapter 3] that \(v\) is a continuous, centered Gaussian random field. Therefore, we combine (2.8), and (2.9), using Parseval’s identity, in order
to see that
\[
\text{Cov}(v_t(x), v_{t'}(x)) = \int_0^{t \wedge t'} ds \int_0^\infty dy \ p_t-s(y)p_{s-t}(y)
= \frac{1}{2\pi} \int_0^{t \wedge t'} ds \int_\infty^{-\infty} d\xi \ \hat{p}_{t-s}(\xi)\hat{p}_{s-t}(\xi)
= \frac{\Gamma(1/\alpha)}{\pi\alpha} \int_0^{t \wedge t'} \left(\frac{t + t' - 2s}{2}\right)^{-1/\alpha} ds.
\] (2.12)

We use the substitution \( r = (t + t' - 2s)/2 \) and note that \((t + t')/2 - (t \wedge t') = |t - t'|/2\) in order to conclude that
\[
\text{Cov}(v_t(x), v_{t'}(x)) = c_\alpha^2 2^{(1-\alpha)/\alpha} \left(|t' + t|^{(\alpha-1)/\alpha} - |t' - t|^{(\alpha-1)/\alpha}\right),
\] (2.13)

where
\[
c_\alpha := \left(\frac{\Gamma(1/\alpha)}{\pi(\alpha - 1)}\right)^{1/2}.
\] (2.14)

That is, we have verified the following:

**Proposition 2.2.** For every fixed \( x \in \mathbb{R} \), the stochastic process \( t \mapsto c_\alpha^{-1}v_t(x) \) is a bi-fBm\((1/2, (\alpha - 1)/\alpha)\), where \( c_\alpha \) is defined in (2.14). Therefore, Proposition 2.1 allows us to write
\[
v_t(x) = c_\alpha 2^{1/(2\alpha)}X_t + R_t \quad (t \geq 0),
\] (2.15)

where \( \{X_t\}_{t \geq 0} \) is fBm\((\alpha - 1)/(2\alpha)\) and \( \{R_t\}_{t \geq 0} \) is a centered Gaussian process that is:

(i) Independent of \( v_\bullet(x)\);

(ii) Absolutely continuous on \([0, \infty)\), a.s.; and

(iii) Infinitely differentiable on \((0, \infty)\), a.s.

**Remark 2.3.** From now on, we choose \( \alpha \) and \( H \) according to the following relation:
\[
\alpha := \frac{1}{1 - 2H} \quad \text{equivalently} \quad H := \frac{\alpha - 1}{2\alpha},
\] (2.16)

so that Dalang’s condition (2.7) is equivalent to the restriction that \( H \in (0, 1/4] \). Propositions 2.1 and 2.2 together show that \( t \mapsto v_t(x) \) is a smooth perturbation of a [non-standard] fractional Brownian motion. In particular, we may compare (1.4) and (2.14) in order to conclude that
\[
\kappa_H = c_\alpha,\] (2.17)

thanks to our convention (2.16). \( \square \)
Remark 2.4. According to (2.3) the process $R_t$ of Proposition 2.2 can be written as

$$R_t = \text{const} \cdot \int_0^\infty \frac{1 - \exp(-st)}{s^{H+(1/2)}} dW_s.$$  
(2.18)

This is a Gaussian process that is $C^\infty$ away from $t = 0$, and its derivatives are obtained by differentiating under the [Wiener] integral. In particular, the first derivative of $R$, away from $t = 0$, is

$$R'_t = \text{const} \cdot \int_0^\infty \exp(-st) s^{H-(1/2)} dW_s \quad (t > 0).$$  
(2.19)

Consequently, $\{R'_q\}_{q>0}$ defines a centered Gaussian process, and Wiener’s isometry shows that $E(|R'_q|^2) = \text{const} \cdot q^{2H-2}$ for all $q > 0$. Therefore,

$$\|R_{t+\varepsilon} - R_t\|_k = A_k \|R_{t+\varepsilon} - R_t\|_2 \leq A_k \int_t^{t+\varepsilon} \|R'_q\|_2 dq,$$

$$= A_k \int_t^{t+\varepsilon} q^{H-1} dq \leq A_k t^{H-1} \varepsilon,$$
(2.20)

uniformly over all $t > 0$ and $\varepsilon \in (0,1)$. 

3 The non-linear heat equation

In this section we consider the non-linear stochastic heat equation

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2} (\Delta_{\alpha/2} u_t)(x) + f(c_\alpha u_t(x)) \dot{W}_t(x)$$  
(3.1)

on $(t,x) \in (0,T] \times R$, subject to $u_0(x) \equiv Y_0$ for all $x \in R$, where $c_\alpha$ was defined in (2.14) and $f : R \to R$ is a globally Lipschitz-continuous function.

As is customary [37, Chapter 3], we interpret (3.1) as the non-linear random evolution equation,

$$u_t(x) = Y_0 + \int_{(0,t) \times R} p_{t-s}(y-x) f(c_\alpha u_s(y)) \, W(ds \, dy).$$  
(3.2)

Dalang’s condition (2.7) implies that the evolution equation (3.2) has an a.s.-unique random-field solution $u$. Moreover, (2.7) is necessary and sufficient for the existence of a random-field solution when $f$ is a constant; see [8]. We will need the following technical estimates.

Lemma 3.1. For all $k \in [2,\infty)$ there exists a finite constant $A_{k,T}$ such that:

$$E \left( |u_t(x)|^k \right) \leq A_{k,T}; \quad \text{and}$$

$$E \left( |u_t(x) - u_t(x')|^k \right) \leq A_{k,T} \left( |x - x'|^{(\alpha-1)k/2} + |t - t'|^{(\alpha-1)k/(2\alpha)} \right),$$

uniformly for all $t,t' \in [0,T]$ and $x,x' \in R$. 

8
This is well known: The first moment bound can be found explicitly in Dalang [8], and the second can be found in the appendix of Foondun and Khoshnevisan [11]. The second can also be shown to follow from the moments estimates of [8] and some harmonic analysis.

Lemma 3.1 and the Kolmogorov continuity theorem [9, Theorem 4.3, p. 10] together imply that $u$ is continuous up to a modification. Moreover, (2.16) and Kolmogorov’s continuity theorem imply that for every $x \in \mathbb{R}$,

$$u_\bullet(x) \in \bigcap_{\gamma \in (0,H)} C^\gamma([0,T]).$$

(3.4)

4 An approximation theorem

The following is the main technical contribution of this paper. Recall that $v$ denotes the solution to the linear stochastic heat equation (2.5), and has the integral representation (2.11).

**Theorem 4.1.** For every $k \in [2, \infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\varepsilon \in (0, 1)$, $x \in \mathbb{R}$, and $t \in [0, T],$

$$E \left( |u_{t+\varepsilon}(x) - u_t(x) - f(c_au_t(x)) \cdot \{v_{t+\varepsilon}(x) - v_t(x)\}|^k \right) \leq A_{k,T} \varepsilon^\mathcal{G}_H,$$

(4.1)

where

$$\mathcal{G}_H := \frac{2H}{1+H}.$$  

(4.2)

**Remark 4.2.** Since $0 < H \leq 1/4$, it follows that

$$\frac{8}{5} \leq \frac{\mathcal{G}_H}{H} < 2. $$

(4.3)

We do not know whether the fraction $8/5 = 1.6$ is a meaningful quantity or a byproduct of the particulars of our method. For us the relevant matter is that (4.3) is a good enough estimate to ensure that $\mathcal{G}_H/H > 1$; the strict inequality will play an important role in the sequel.

Theorem 4.1 is in essence an analysis of the temporal increments of $u_\bullet(x)$. Thanks to (3.2), we can write those increments as

$$u_{t+\varepsilon}(x) - u_t(x) := \mathcal{J}_1 + \mathcal{J}_2,$$

(4.4)

where

$$\mathcal{J}_1 := \int_{(0,t) \times \mathbb{R}} \left[ p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x) \right] f(c_au_s(y)) W(ds \, dy);$$

$$\mathcal{J}_2 := \int_{(t,t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y-x) f(c_au_s(y)) W(ds \, dy).$$

(4.5)

Our proof of Theorem 4.1 proceeds by analyzing $\mathcal{J}_1$ and $\mathcal{J}_2$ separately. Let us begin with the latter quantity, as it is easier to estimate than the former term.
4.1 Estimation of $J_2$

Define 
$$\tilde{J}_2 := f(c_\alpha u_t(x)) \cdot \int_{(t,t+\epsilon) \times \mathbb{R}} p_{t+\epsilon-s}(y-x) W(\text{d}s \text{d}y). \quad (4.6)$$

**Proposition 4.3.** For every $k \in [2, \infty)$ there exists a finite constant $A_{k,T}$ such that for all $\epsilon \in (0,1)$,

$$\sup_{x \in \mathbb{R}} \sup_{t \in [0,T]} E \left( |J_2 - \tilde{J}_2|^k \right) \leq A_{k,T} \epsilon^{2Hk}. \quad (4.7)$$

We split the proof in 2 parts: First we show that $J_2 \approx J_2'$ in $L^k(\Omega)$, where

$$J_2' := \int_{(t,t+\epsilon) \times \mathbb{R}} p_{t+\epsilon-s}(y-x)f(c_\alpha u_s(x)) W(\text{d}s \text{d}y). \quad (4.8)$$

After that we will verify that $J_2' \approx \tilde{J}_2$ in $L^k(\Omega)$. Proposition 4.3 follows immediately from Lemmas 4.4 and 4.5 below and Minkowski’s inequality. Therefore, we will state and prove only those two lemmas.

**Lemma 4.4.** For all $k \in [2, \infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\epsilon \in (0,1)$,

$$\sup_{x \in \mathbb{R}} \sup_{t \in [0,T]} E \left( |J_2 - J_2'|^k \right) \leq A_{k,T} \epsilon^{2Hk}. \quad (4.9)$$

**Proof.** The proof will use a particular form of the Burkholder–Davis–Gundy (BDG) inequality [5, Lemma 2.3]. Since we will make repeated use of this inequality throughout, let us recall it first.

For every $t \geq 0$, let $\mathcal{F}_t^0$ denote the sigma-algebra generated by every Wiener integral of the form $\int_{(0,t) \times \mathbb{R}} \varphi_s(y) W(\text{d}s \text{d}y)$ as $\varphi$ ranges over all elements of $L^2(\mathbb{R}_+ \times \mathbb{R})$. We complete every such sigma-algebra, and make the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ right continuous in order to obtain the “Brownian filtration” $\mathcal{F}$ that corresponds to the white noise $W$.

Let $\Phi := \{\Phi_t(x)\}_{t \geq 0, x \in \mathbb{R}}$ be a predictable random field with respect to $\mathcal{F}$. Then, for every real number $k \in [2, \infty)$, we have the following BDG inequality:

$$\left\| \int_{(0,t) \times \mathbb{R}} \Phi_s(y) W(\text{d}s \text{d}y) \right\|_k^2 \leq 4k \int_0^t \text{d}s \int_{-\infty}^\infty \|\Phi_s(y)\|_k^2. \quad (4.10)$$

10
The BDG inequality (4.10) and eq. (3.2) together imply that
\[
\|J_2 - J'_2\|_{L^k(\Omega)}^2 \\
\leq 4k \int_t^{t+\varepsilon} ds \int_{-\infty}^\infty dy \ |p_t + \varepsilon - s| \|f(c_\alpha u_s(y)) - f(c_\alpha u_s(x))\|_k^2 \\
\leq 4kc_\alpha^2 \text{Lip}_f^2 \cdot \int_t^{t+\varepsilon} ds \int_{-\infty}^\infty dy \ |p_t + \varepsilon - s| \|u_s(y) - u_s(x)\|_k^2 \quad (4.11)
\]
\[
\leq A_{k,T} \int_0^\varepsilon ds \int_{-\infty}^\infty dy \ |p_\varepsilon(y)|^2 \left(\|y\|^{\alpha - 1} \wedge 1\right). \quad (4.12)
\]

The last inequality uses both moment inequalities of Lemmas 3.1. Furthermore, measurability issues do not arise, since the solution to (3.2) is continuous in the time variable \(t\) and adapted to the Brownian filtration \(\mathcal{F}\).

In order to proceed from here, we need to recall two basic facts about the transition functions of stable processes: First of all,
\[
p_s(y) = s^{-1/\alpha} p_1 \left(\|y\|^{1/\alpha}\right) \quad \text{for all } s > 0 \text{ and } y \in \mathbb{R}. \quad (4.12)
\]
This fact is a consequence of scaling and symmetry; see (2.8). We also need to know the fact that \(p_1(z) \leq \text{const} \cdot (1 + |z|)^{-(1+\alpha)}\) for all \(z \in \mathbb{R}\) [22, Proposition 3.3.1, p. 380], whence
\[
p_s(y) \leq \text{const} \times \begin{cases} 
s^{-1/\alpha} & \text{if } |y| \leq s^{1/\alpha}, \\
|y|^{-(1+\alpha)} & \text{if } |y| > s^{1/\alpha}.
\end{cases} \quad (4.13)
\]
Consequently,
\[
\int_0^\varepsilon ds \int_0^1 dy \ |p_s(y)|^2 \left(\|y\|^{\alpha - 1} \wedge 1\right) \\
\leq \text{const} \cdot \left(\int_0^s s^{-2/\alpha} ds \int_0^1 y^{\alpha - 1} dy + \int_0^s s^2 ds \int_{s^{1/\alpha}}^1 y^{-3-\alpha} dy\right) \quad (4.14)
\]
\[
\leq \text{const} \cdot \varepsilon^{2(\alpha - 1)/\alpha}.
\]
We obtain the following estimate by similar means:
\[
\int_0^\varepsilon ds \int_1^\infty dy \ |p_s(y)|^2 \left(\|y\|^{\alpha - 1} \wedge 1\right) \leq \text{const} \cdot \int_0^\varepsilon s^2 ds \int_1^\infty y^{-2-2\alpha} dy \quad (4.15)
\]
\[
= \text{const} \cdot \varepsilon^3 \\
\leq \text{const} \cdot \varepsilon^{2(\alpha - 1)/\alpha},
\]
uniformly for all \(\varepsilon \in (0,1)\). Since \(p_s(y) = p_s(-y)\) for all \(s > 0\) and \(y \in \mathbb{R}\), the preceding two displays and (4.11) together imply that
\[
\|J_2 - J'_2\|_{L^k(\Omega)}^2 \leq \text{const} \cdot \varepsilon^{2(\alpha - 1)/\alpha}. \quad (4.16)
\]
We may conclude the lemma from this inequality, using our convention about \(\alpha\) and \(H\); see (2.16). \(\square\)
In light of Lemma 4.4, Proposition 4.3 follows at once from

**Lemma 4.5.** For all \( k \in [2, \infty) \) there exists a finite constant \( A_{k,T} \) such that uniformly for all \( \varepsilon \in (0, 1) \),

\[
\sup_{x \in \mathbb{R}} \sup_{t \in [0, T]} \mathbb{E} \left( \left| \mathcal{J}'_2 - \tilde{\mathcal{J}}_2 \right|^k \right) \leq A_{k,T} \varepsilon^{2H_k}.
\]  

(4.17)

**Proof.** We apply the BDG inequality (4.10), as we did in the derivation of (4.11), in order to see that

\[
\left\| \mathcal{J}'_2 - \tilde{\mathcal{J}}_2 \right\|_{L^k(\Omega)}^2  
\leq 4k \lambda^2 \text{Lip}_f^2 \int_{t}^{t+\varepsilon} \int_{-\infty}^{\infty} \left| p_{t+s}(y) - u_t(x) \right|^2 \mathbb{E} ds dy \left\| p_{t+s}(y) - u_t(x) \right\|_{L^k(\Omega)}^2
\]

(4.18)

\[
\leq A_{k,T} \int_{t}^{t+\varepsilon} \left| p_{t+s}(y) - u_t(x) \right|^2 \mathbb{E} ds dy \left\| p_{t+s}(y) - u_t(x) \right\|_{L^2(\mathbb{R})}^2
\]

Therefore, (2.9) and a change of variables together show us that the preceding quantity is bounded above by

\[
A_{k,T} \int_{0}^{\varepsilon} s^{(\alpha - 1)/\alpha} (\varepsilon - s)^{-1/\alpha} ds = A_{k,T} \varepsilon^{2(\alpha - 1)/\alpha}.
\]  

(4.19)

The lemma follows from this and our convention (2.16) about the relation between \( \alpha \) and \( H \).

\[ \square \]

### 4.2 Estimation of \( \mathcal{J}_1 \) and proof of Theorem 4.1

Now we turn our attention to the more interesting term \( \mathcal{J}_1 \) in the decomposition (4.5). The following localization argument paves the way for a successful analysis of \( \mathcal{J}_1 \) if \( p_t(x) dx \approx \delta_0(dx) \) when \( t \approx 0 \); therefore one might imagine that there is a small regime of values of \( s \in (0, t) \) such that \( p_{t+s}(y - x) - p_t(x) \) is highly localized [big within the regime, and significantly smaller outside that regime]. Thus, we choose and fix a parameter \( a \in (0, 1) \)—whose optimal value will be made explicit later on in (4.42)—and write

\[ \mathcal{J}_1 = \mathcal{J}_{1,a} + \mathcal{J}'_{1,a}, \]

(4.20)

where

\[
\mathcal{J}_{1,a} := \int_{(0,t-\varepsilon^a) \times \mathbb{R}} \left[ p_{t+s}(y - x) - p_t(x) \right] f(c_\alpha u_s(y)) W(dsdy),
\]

(4.21)

\[
\mathcal{J}'_{1,a} := \int_{(t-\varepsilon^a,t) \times \mathbb{R}} \left[ p_{t+s}(y - x) - p_t(x) \right] f(c_\alpha u_s(y)) W(dsdy).
\]
We will prove that the quantity $J_{1,a}$ is small as long as we choose $a \in (0,1)$ carefully; that is, $J_{1} \approx J'_{1,a}$ for a good choice of $a$. And because $s \in (t - \varepsilon^a, t)$ is approximately $t$, then we might expect that $f(u_s(y)) \approx f(u_t(y))$ [for that correctly-chosen $a$], and hence $J_{1} \approx J''_{1,a}$, where

$$J''_{1,a} := \int_{(t-\varepsilon^a,t) \times \mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)] f(c_{a}u_t(y)) W(\mathrm{d}s \, \mathrm{d}y). \quad (4.22)$$

Finally, we might notice that $p_{t+\varepsilon-s}$ and $p_{t-s}$ both act as point masses when $s \in (t - \varepsilon^a, t)$, and therefore we might imagine that $J_{1} \approx J'_{1,a} \approx J''_{1,a}$, where

$$\widetilde{J}_{1,a} := f(c_{a}u_{t}(x)) \cdot \int_{(t-\varepsilon^a,t) \times \mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)] W(\mathrm{d}s \, \mathrm{d}y). \quad (4.23)$$

All of this turns out to be true; it remains to find the correct choice of $a$ so that the errors in the mentioned approximations remain sufficiently small for our later needs. Recall the parameter $\mathcal{G}_H$ from (4.2). Before we continue, let us first document the end result of this forthcoming effort. We will prove it subsequently.

**Proposition 4.6.** For every $T > 0$ and $k \in [2, \infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\varepsilon \in (0,1)$, $x \in \mathbb{R}$, and $t \in [0,T]$,

$$\mathbb{E}\left( \left| J_{1} - f(c_{a}u_{t}(x)) \cdot \int_{(0,t) \times \mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)] W(\mathrm{d}s \, \mathrm{d}y) \right|^k \right) \leq A_{k,T} \varepsilon^{\mathcal{G}_H k}. \quad (4.24)$$

Thanks to (4.4) and Minkowski’s inequality, Theorem 4.1 follows easily from Propositions 4.3 and 4.6. It remains to prove Proposition 4.6.

We begin with a sequence of lemmas that make precise the various formal appeals to “$\approx$” in the preceding discussion. As a first step in this direction, let us dispense with the “small” term $J_{1,a}$.

**Lemma 4.7.** For all $k \in [2, \infty)$ and $a \in (0,1)$ there exists a finite constant $A_{a,k,T}$ such that uniformly for all $\varepsilon \in (0,1)$,

$$\sup_{x \in \mathbb{R}} \sup_{t \in [0,T]} \mathbb{E}\left( \left| J_{1,a} \right|^k \right) \leq A_{a,k,T} \varepsilon^{[1-a(1-H)]k}. \quad (4.25)$$

**Proof.** We can modify the argument that led to (4.11), using the BDG inequality (4.10), in order to yield

$$\| J_{1,a} \|_{L^k(\Omega)}^2 \leq 4k \int_0^{t-\varepsilon^a} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \left[ p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x) \right]^2 \| f(c_{a}u_s(y)) \|_{L^k(\Omega)}^2 \leq A_{k,T} \int_{\varepsilon^a}^{T} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \left[ p_{s+\varepsilon}(y) - p_{s}(y) \right]^2. \quad (4.26)$$

13
We first bound $\int_{0}^{T} ds$ from above by $e^{T} \cdot \int_{0}^{\infty} e^{-s} ds$, and then apply (2.8) and Plancherel’s formula in order to deduce the following bounds:

$$
\| \mathcal{J}_{1,a} \|_{L^k(\Omega)}^2 \leq A_{k,T} \int_{0}^{\infty} e^{-s} ds \int_{-\infty}^{\infty} d\xi \ e^{-2s|\xi|} \left| 1 - e^{-|\xi|} \right|^2 \\
\leq A_{k,T} \int_{0}^{\infty} e^{-s} ds \int_{0}^{\infty} d\xi \ e^{-2s\xi} (1 + \varepsilon^2 \xi^2) \\
= A_{k,T} \int_{0}^{\infty} (1 + \varepsilon^2 \xi^2) e^{-2\varepsilon \xi^2} \frac{d\xi}{1 + \xi^2}.
$$

(4.27)

since $0 \leq 1 - e^{-z} \leq 1 + z$ for all $z \geq 0$. Clearly,

$$
\int_{0}^{\varepsilon^{-1}/a} (1 + \varepsilon^2 \xi^2) e^{-2\varepsilon \xi^2} \frac{d\xi}{1 + \xi^2} \leq \varepsilon^2 \int_{0}^{\varepsilon^{-1}/a} \xi^2 e^{-2\varepsilon \xi^2} d\xi \\
= \varepsilon^{(a-1)/a} \int_{0}^{1} x^a \exp \left( -\frac{2x^a}{\xi^{1-a}} \right) dx \\
\leq \varepsilon^{(a-1)/a} \int_{0}^{\infty} x^a \exp \left( -\frac{2x^a}{\xi^{1-a}} \right) dx \\
= \text{const.} \cdot \varepsilon^{(2a-a-a)\alpha}/a.
$$

(4.28)

Furthermore,

$$
\int_{\varepsilon^{-1}/a}^{\infty} (1 + \varepsilon^2 \xi^2) e^{-2\varepsilon \xi^2} \frac{d\xi}{1 + \xi^2} \leq \int_{\varepsilon^{-1}/a}^{\infty} e^{-2\varepsilon \xi^2} d\xi \\
\leq \text{const.} \cdot \exp \left( -2\varepsilon^{-1} \right),
$$

uniformly for all $\varepsilon \in (0, 1)$. The preceding two paragraphs together imply that

$$
E \left( \| \mathcal{J}_{1,a} \|^k \right) \leq A_{a,k,T} \varepsilon^{(2a-a-a)k/(2a)},
$$

(4.30)

which proves the lemma, due to the relation (2.16) between $H$ and $\alpha$. \hfill \Box

**Lemma 4.8.** For all $k \in [2, \infty)$ and $a \in (0, 1)$ there exists a finite constant $A_{a,k,T}$ such that uniformly for all $\varepsilon \in (0, 1),$

$$
\sup_{x \in \mathbb{R}} \sup_{t \in [0,T]} E \left( \| \mathcal{J}^{'}_{1,a} - \mathcal{J}^{''}_{1,a} \|^k \right) \leq A_{a,k,T} \varepsilon^{2aHk}.
$$

(4.31)

**Proof.** We proceed as we did for (4.11), using the BDG inequality (4.10), in order to find that

$$
\| \mathcal{J}^{'}_{1,a} - \mathcal{J}^{''}_{1,a} \|_{L^k(\Omega)}^2 \leq A_{k,T} \cdot \int_{0}^{\varepsilon^{-1}/a} s^{(a-1)/a} \| p_{s+\varepsilon} - p_s \|_{L^2(\mathbb{R})}^2 ds \\
= A_{k,T} \varepsilon^{(2a-1)/a} \cdot \int_{0}^{\varepsilon^{-1}/a} r^{(a-1)/a} \| p_{r(1+r)} - p_{r\varepsilon} \|_{L^2(\mathbb{R})}^2 dr,
$$

(4.32)

14
after a change of variables \([r := s/\varepsilon]\). The scaling property (4.12) can be written in the following form:

\[
p_{\tau r}(y) = \varepsilon^{-1/\alpha} p_\tau(y/\varepsilon^{1/\alpha}),
\]

valid for all \(\tau, \varepsilon > 0\) and \(y \in \mathbb{R}\). Consequently,

\[
\|p_{\varepsilon(1+r)} - p_{\tau r}\|_{L^2(\mathbb{R})}^2 = \varepsilon^{-1/\alpha} \cdot \|p_{1+r} - p_r\|_{L^2(\mathbb{R})}^2.
\]

Eq. (2.8) and the Plancherel theorem together imply that

\[
\|p_{1+r} - p_r\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-rt}|z|^\alpha \left(1 - e^{-|z|^\alpha/2}\right)^2 \, dz
\]

\[
\leq \int_0^{\infty} e^{-rz^\alpha} \, dz
\]

\[
= \frac{\Gamma(1/\alpha)}{\alpha r^{1/\alpha}},
\]

for all \(r > 0\). Therefore, (4.32) implies that

\[
\left\| J_{1,a}'' - \hat{J}_{1,a}'' \right\|_{L^k(\Omega)}^2 \leq A_{k,T} \varepsilon^{2(\alpha-1)/\alpha} \cdot \int_0^{s-1} r^{(\alpha-2)/\alpha} \, dr,
\]

which readily implies the lemma. \(\square\)

**Lemma 4.9.** For all \(k \in [2, \infty)\) and \(a \in (0, 1)\) there exists a finite constant \(A_{a,k,T}\) such that uniformly for all \(\varepsilon \in (0, 1), \)

\[
\sup_{x \in \mathbb{R}} \sup_{t \in [0, T]} \mathbb{E} \left( \left| J_{1,a}''(x,t) - \hat{J}_{1,a}''(x,t) \right| \right) \leq A_{k,T} \varepsilon^{2aHk}.
\]

**Proof.** We proceed as we did for (4.11), apply the BDG inequality (4.10), and obtain the following bounds:

\[
\left\| J_{1,a}'' - \hat{J}_{1,a}'' \right\|_{L^k(\Omega)}^2
\]

\[
\leq A_k \int_{t-\varepsilon}^{t} ds \int_{-\infty}^{\infty} dy \left| p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x) \right|^2 \|u_t(y) - u_t(x)\|^2_{L^k(\Omega)}
\]

\[
\leq A_{k,T} \int_0^{s-1} ds \int_{-\infty}^{\infty} dy \left| p_{s+\varepsilon}(y) - p_s(y) \right|^2 \left(|y|^{\alpha-1} \wedge 1 \right)
\]

\[
\leq A_{k,T} \varepsilon \int_0^{s-1} dr \int_{-\infty}^{\infty} dy \left| p_{\varepsilon(r+1)}(y) - p_{\varepsilon r}(y) \right|^2 |y|^{\alpha-1}.
\]

Thanks to the scaling property (4.33), we may obtain the following after a change of variables \([w := y/\varepsilon^{1/\alpha}]\):

\[
\left\| J_{1,a}'' - \hat{J}_{1,a}'' \right\|_{L^k(\Omega)}^2
\]

\[
\leq A_{k,T} \varepsilon^{2(\alpha-1)/\alpha} \int_0^{s-1} dr \int_{-\infty}^{\infty} dw \left| p_{r+1}(w) - p_r(w) \right|^2 |w|^{\alpha-1}.
\]
Next we notice that
\[
\int_0^{\varepsilon^{-1}} \, dr \int_0^\infty \, dw \left| p_r(w) \right|^2 \, w^{\alpha-1} = \int_0^{\varepsilon^{-1}} \, r^{-2/\alpha} \, dr \int_0^\infty \, dw \left| p_1(\frac{w}{r^{1/\alpha}}) \right|^2 \, w^{\alpha-1} \leq \text{const} \cdot \varepsilon^{2(\alpha-1)(-1/\alpha)},
\]
where the last inequality uses the facts that: (i) $\alpha > 1$; and (ii) $p_1(x) \leq \text{const} \cdot (1 + |x|)^{-1-\alpha}$ (see [22, Proposition 3.3.1, p. 380]). Therefore,
\[
\left\| J''_{1,a} - \tilde{J}_{1,a} \right\|_{L^k(\Omega)}^2 \leq A_{k,T} \varepsilon^{2\alpha(\alpha-1)/\alpha},
\]
which proves the lemma, due to the relation (2.16) between $H$ and $\alpha$.

**Proof of Proposition 4.6.** So far, the parameter $a$ has been an arbitrary real number in $(0, 1)$. Now we choose and fix it as follows:
\[
a := \frac{1}{1 + H}.
\]
Thus, for this particular choice of $a$,
\[
1 - a(1 - H) = 2aH = \mathcal{G}_H,
\]
where $\mathcal{G} := 2H/(1 + H)$ was defined in (4.2). Because $\mathcal{G}_H < 2H$ and because of (4.20), Lemmas 4.7, 4.8, and 4.9 together imply that, for this choice of $a$,
\[
E \left( \left| J_{1,a} - \tilde{J}_{1,a} \right|^k \right) \leq A_{k,T} \varepsilon^{\mathcal{G}_H k},
\]
uniformly for all $\varepsilon \in (0, 1)$, $x \in \mathbb{R}$, and $t \in [0, T]$. Thanks to the definition (4.23) of $\tilde{J}_{1,a}$, it suffices to demonstrate the following with the same parameter dependencies as above:
\[
E \left( \left| J_{1,a} - f(c_\alpha u_t(x)) \cdot \Lambda([0, t]) \right|^k \right) \leq A_{k,T} \varepsilon^{\mathcal{G}_H k};
\]
where $\Lambda(Q) := \int_{Q \times \mathbb{R}} \left| p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x) \right| W(ds dy)$ for every interval $Q \subset [0, T]$.

Because $\tilde{J}_{1,a} = f(c_\alpha u_t(x)) \cdot \Lambda([0, t])$, Lemma 3.1 shows that the left-hand side of (4.45)
\[
E \left( \left| \tilde{J}_{1,a} - f(c_\alpha u_t(x)) \cdot \Lambda([0, t]) \right|^k \right) \leq A_{k,T} \varepsilon^{\mathcal{G}_H k}.
\]
Since $\Lambda([0, t - \varepsilon^a])$ is the same as the quantity $J_{1,a}$ in the case that $f \equiv 1$, we may apply Lemma 4.7 to the linear equation (2.5) with $f \equiv 1$ in order to see that
\[
\sqrt{E \left( \left| \Lambda([0, t - \varepsilon]) \right|^{2k} \right)} \leq A_{k,T} \varepsilon^{\mathcal{G}_H k},
\]
which implies (4.45).
5 Proof of Theorem 1.1

We conclude this article by proving Theorem 1.1.

Let us define a Lipschitz-continuous function $f$ by

$$f(x) := \frac{2^H}{\kappa^2 H^2 \sqrt{2}} g(x) \quad (x \in \mathbb{R}),$$

where $\kappa_H$ was defined in (1.4). Let us also define a stochastic process

$$Y_t := c_\alpha u_t(0) \quad (t \geq 0),$$

where the constant $c_\alpha[= \kappa_H]$ was defined in (2.14) and $u$ denotes the solution to the stochastic PDE (3.1). Because of Remark 2.3 and the definition of $f$, we can see that:

(i) $Y_t = \kappa_H u_t(0)$; and

(ii) $u$ solves the stochastic PDE (1.5).

We also remark that $g(x) = c_\alpha 2^{1/(2\alpha)} f(x)$.

We are assured by (3.4) that $Y \in \cap_{r \in (0, H)} C_r([0, t])$, up to a modification [in the usual sense of stochastic processes]. Recall from (2.11) the solution $v$ to the linear SPDE (2.5).

Let $X$ be the fBm($H$) from Proposition 2.2 and choose and fix $t \in (0, T]$. Then

$$\Theta := Y_{t+\varepsilon} - Y_t - g(Y_t)(X_{t+\varepsilon} - X_t)
= Y_{t+\varepsilon} - Y_t - c^{\alpha} 2^{1/(2\alpha)} f(Y_t)(X_{t+\varepsilon} - X_t)
= c_\alpha \left( u_{t+\varepsilon}(0) - u_t(0) - f(c_\alpha u_t(0)) \left[ c_\alpha 2^{1/(2\alpha)} X_{t+\varepsilon} - c_\alpha 2^{1/(2\alpha)} X_t \right] \right)
= c_\alpha \left( u_{t+\varepsilon}(0) - u_t(0) - f(c_\alpha u_t(0)) (v_{t+\varepsilon}(0) - v_t(0)) \right)
+ c_\alpha f(c_\alpha u_t(0)) (R_{t+\varepsilon} - R_t).$$

We proved, earlier in Remark 2.4, that $\|R_{t+\varepsilon} - R_t\|_k \leq A_{k, t} \varepsilon$. Because $f$ is Lipschitz continuous, Hölder’s inequality and (3.3) together imply that $\|c_\alpha f(c_\alpha u_t(0)) (R_{t+\varepsilon} - R_t)\|_k \leq A_{k, t} \varepsilon$, whence we obtain the bound,

$$\sup_{t \in (0, T]} \mathbb{E}(\Theta^2) \leq A_T \varepsilon^{2H},$$

from Theorem 4.1. Since $H_H > H$—see Remark 4.2—the preceding displayed bound and Chebyshev’s inequality together imply that for every $\varepsilon \in (0, 1)$, $\delta > 0$, and $b \in (H, \mathcal{G}_H)$,

$$P \left\{ \frac{Y_{t+\varepsilon} - Y_t}{X_{t+\varepsilon} - X_t} - g(Y_t) > \delta \right\} = P \left\{ \frac{|\Theta|}{|X_{t+\varepsilon} - X_t|} > \delta \right\}
\leq A_T \varepsilon^{2(G_H - b)} + P \left\{ |X_{t+\varepsilon} - X_t| < \frac{\varepsilon^b}{\delta} \right\}.$$
The first term converges to zero as \( \varepsilon \to 0^+ \) since \( b < \mathcal{G}_H \). It remains to prove that the second term also vanishes as \( \varepsilon \to 0^+ \). But since \( X \) is fBm(\( H \)), the increment \( X_{t+\varepsilon} - X_t \) has the same distribution as \( \varepsilon^H Z \) where \( Z \) is a standard normal random variable. Therefore,

\[
\sup_{t \in (0,T]} \text{P} \left\{ |X_{t+\varepsilon} - X_t| < \frac{\varepsilon^b}{\delta} \right\} = \text{P} \left\{ |Z| \leq \frac{\varepsilon^{b-H}}{\delta} \right\},
\]

which goes to zero as \( \varepsilon \to 0^+ \) since \( b > H \).

\[ \square \]

References

[1] Alòs, Elisa, Jorge A. León, and David Nualart. Stochastic Stratonovich calculus fBm for fractional Brownian motion with Hurst parameter less than \( 1/2 \), *Taiwanese J. Math.* 5(3) (2001) 609–632.

[2] Alòs, Elisa, Olivier Mazet, and David Nualart. Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than \( \frac{1}{2} \), *Stoch. Process. Appl.* 86(1) (2000) 121–139.

[3] Burdzy, Krzysztof, and Jason Swanson. A change of variable formula with Itô correction term, *Ann. Probab.* 38(3) (2010) 1817–1869.

[4] Chen, Kuo-Tsai. Integration of paths—a faithful representation of paths by noncommutative formal power series, *Trans. Amer. Math. Soc.* 89 (1958) 395–407.

[5] Conus, Daniel, and Davar Khoshnevisan. Weak nonmild solutions to some SPDEs, *Illinois J. Math.* 54(4) (2010) 1329–1341.

[6] Coutin, Laure, and Nicholas Victoir. Enhanced Gaussian processes and applications, *ESAIM Probab. Stat.* 13 (2009) 247–260.

[7] Coutin, Laure, Peter Fritz, and Nicholas Victoir. Good rough path sequences and applications to anticipating stochastic calculus, *Ann. Probab.* 35(3) (2007) 1172–1193.

[8] Dalang, Robert C. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.’s, *Electron. J. Probab.* 4 no. 6 (1999) 29 pp. (electronic). [Corrigendum: *Electron. J. Probab.* 6 no. 6 (2001) 5 pp.]

[9] Dalang, Robert, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. *A Minicourse in Stochastic Partial Differential Equations* (2006). In: Lecture Notes in Mathematics, vol. 1962 (D. Khoshnevisan and F. Rassoul–Agha, editors) Springer–Verlag, Berlin, 2009.

[10] Errami, Mohammed, and Francesco Russo. \( n \)-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes, *Stoch. Process. Appl.* 104 (2) (2003) 259–299.

[11] Foondun, Mohammad, and Davar Khoshnevisan. Intermittence and nonlinear stochastic partial differential equations, *Electronic J. Probab.*, Vol. 14, Paper no. 21 (2009) 548–568.

[12] Fox, Ralph H. Free differential calculus I. Derivation in the group ring, *Ann. Math.* 57(3) (1953) 547–560.

[13] Fritz, Peter, and Nicholas Victoir. Differential equations driven by Gaussian signals, *Ann. Inst. Henri Poincaré Probab. Stat.* 46(2) (2010) 369–413.

[14] Fritz, Peter K., and Nicholas B. Victoir. *Multidimensional Stochastic Processes as Rough Paths*, Cambridge University Press, Cambridge, 2010.
[15] Gel’fand, I. M. and N. Ya. Vilenkin, N. Ya. *Generalized Functions. Vol. 4*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977].

[16] Gradinaru, Mihai, Francesco Russo, and Pierre Vallois. Generalized covariances, local time and Stratonovich Itô’s formula for fractional Brownian motion with Hurst index $H \geq \frac{1}{4}$, *Ann. Probab.* **31**(4) (2003) 1772–1820.

[17] Gubinelli, M. Controlling rough paths, *J. Funct. Anal.* **216**(1) (2004) 86–140.

[18] Hairer, Martin. Solving the KZ equation, *Ann. Math.* **178**(2) (2013) 559–664.

[19] Houdré, Christian, and José Villa. An example of infinite-dimensional quasi-helix, in: *Stochastic Models* (Mexico City, 2002) pp. 195–201, *Contemp. Math.* **336** Amer. Math. Soc., Providence, 2003.

[20] Jacob, Niels. *Pseudo-Differential Operators and Markov Processes*, Imperial College Press, Volumes II (2002) and III (2005).

[21] Kahane, Jean-Pierre. *Some Random Series of Functions* (second ed.) Cambridge University Press, 1985.

[22] Khoshnevisan, Davar. *Multiparameter Processes*, Springer-Verlag, New York, 2002.

[23] Lei, Pedro, and David Nualart. A decomposition of the bifractional Brownian motion and some applications, *Statistics and Probability Letters* **79**(5) (2009) 619–624.

[24] Lyons, Terry, and Zhongmin Qian. *System Control and Rough Paths*, Oxford University Press, Oxford, 2002.

[25] Lyons, Terry J. The interpretation and solution of ordinary differential equations driven by rough signals, in: *Stochastic Analysis (Ithaca, NY, 1993)*, 115–128, Proc. Sympos. Pure Math. **57**, Amer. Math. Soc. Providence RI, 1995.

[26] Lyons, Terry. Differential equations driven by rough signals. I. An extension of an inequality of L. C. Young, *Math. Res. Lett.* **1**(4) (1994) 451–464.

[27] Mueller, Carl, and Zhixin Wu. A connection between the stochastic heat equation and fractional Brownian motion, and a simple proof of a result of Talagrand, *Electr. Comm. Probab.* **14** (2009) 55–65.

[28] von Neumann, J., and I. J. Schoenberg. Fourier integrals and metric geometry, *Trans. Amer. Math. Soc.* **35** (1941) 226–251.

[29] Nualart, David, and Samy Tindel. A construction of the rough path above fractional Brownian motion using Volterra's representation, *Ann. Probab.* **39**(3) (2011) 1061–1096.

[30] Russo, Francesco, and Ciprian A. Tudor. On bifractional Brownian motion. *Stoch. Process. Appl.* **116**(5) (2006) 830–856.

[31] Russo, Francesco, and Pierre Vallois. Forward, backward and symmetric stochastic integration, *Probab. Theory Related Fields* **97**(3) (1993) 403–421.

[32] Schoenberg, I. J. On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space, *Ann. Math. (2)* **38** (1937) no. 4, 787–793.

[33] Swanson, Jason. Variations of the solution to a stochastic heat equation, *Ann. Probab.* **35**(6) (2007) 2122–2159.

[34] Tudor, Ciprian A., and Yimin Xiao. Sample path properties of bifractional Brownian motion. *Bernoulli* **13**(4) (2007) 1023–1052.

[35] Unterberger, Jérémie. A rough path over multidimensional fractional Brownian motion with arbitrary Hurst index by Fourier normal ordering, *Stoch. Process. Appl.* **120**(8) (2010) 1444–1472.
[36] Yamada, Toshio, and Shinzo Watanabe. On the uniqueness of solutions of stochastic differential equations, *J. Math. Kyoto Univ.* **11** (1971) 155–167.

[37] Walsh, John B. *An Introduction to Stochastic Partial Differential Equations*, in: École d’été de probabilités de Saint-Flour, XIV—1984, 265–439, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986.

Davar Khoshnevisan (davar@math.utah.edu)
Dept. Mathematics, Univ. Utah, Salt Lake City, UT 84112-0090

Jason Swanson (jason@swansonsite.com)
Dept. Mathematics, Univ. Central Florida, Orlando, FL 32816-1364

Yimin Xiao (xiao@stt.msu.edu)
Dept. Statistics & Probability, Michigan State Univ., East Lansing, MI 48824-3416

Lianng Zhang (lzhang81@stt.msu.edu)
Dept. Statistics & Probability, Michigan State Univ., East Lansing, MI 48824-3416