We will show that if $X$ is a tree-complete subspace of $\ell_\infty$, which contains $c_0$, then it does not admit any weakly midpoint locally uniformly convex renorming. It follows that such a space has no equivalent Kadec renorming.

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1. Introduction. It is known that $\ell_\infty$ has an equivalent strictly convex renorming [2]; however, by a result due to Lindenstrauss, it cannot be equivalently renormed in locally uniformly convex manner [10]. In this note, we will show that every tree-complete subspace of $\ell_\infty$, which contains $c_0$, does not admit any equivalent weakly midpoint locally uniformly convex norm. This can be considered as an extension of [1, 8]. Since every strictly convexifiable Banach space with Kadec property admits an equivalent midpoint locally uniformly convex renorming [9], it follows that every subspace of $\ell_\infty$ with the tree-completeness property has no equivalent Kadec renorming. The existence of such a (nontrivial) subspace, which does not contain any copy of $\ell_\infty$, has already been proved by Haydon and Zizler (see [5, 7]).

2. Results. We recall that a norm $\| \cdot \|$ on a Banach space $X$ is said to be midpoint locally uniformly rotund (MLUR) if, whenever $\{x_n\}, \{y_n\}$, and $x$ are in $X$ with $\|x_n\| \to \|x\|$, $\|y_n\| \to \|x\|$, and $\|(x_n + y_n)/2 - x\| \to 0$, we necessarily have $\|x_n - y_n\| \to 0$. If at the end of the last sentence, we replace norm with weak, the definition of weakly midpoint locally uniformly rotund (wMLUR) will be obtained [3]. Let $T$ be the set of all finite (possible empty) strings of 0’s and 1’s. The empty string ( ) is the unique string of length 0; the length $|t|$ of a string $t$ is $n$ if $t \in \{0,1\}^n$. The tree order is defined by $s < t$ if $|s| < |t|$ and $t(m) = s(m)$ for $m \leq |s|$. Each $t \in T$ has exactly two immediate successors, that is, $t0$ and $t1$.

A lattice $L$ is said to be tree-complete if, whenever $\{f_t\}_{t \in T}$ is a bounded disjoint family in $L$, there exists $b \in \{0,1\}^N$, such that $\sum_{n \in N} f_{b|n}$ is in $L$.

Haydon and Zizler [7] constructed a closed linear subspace of $\ell_\infty$ (which is a tree-complete sublattice of $\ell_\infty$) such that it contains $c_0$ but does not contain any subspace isomorphic to $\ell_\infty$. Notice that in this space $X$ every infinite subset $M$ of $N$ has an infinite subset $M_0 \subset M$ such that $1_{M_0} \in X$ [7].

**Theorem 2.1.** Let $X$ be a tree-complete sublattice of $\ell_\infty$. If $X$ contains $c_0$, then $X$ does not admit any equivalent wMLUR renorming.
PROOF. Let $||| \cdot |||$ be an equivalent norm on $X$. We will show that this norm is not wMLUR. Let

\begin{align*}
A(\cdot) &= \{ f \in X : \|f\|_\infty = 1, N \setminus \text{supp}(f) \text{ is infinite} \}, \\
M(\cdot) &= \sup \{|||f||| : f \in A(\cdot) \}, \quad m(\cdot) = \inf \{|||f||| : f \in A(\cdot) \}. \quad (2.1)
\end{align*}

Choose an element $f(\cdot)$ of $X$ such that $|||f(\cdot)||| > (3M(\cdot) + m(\cdot))/4$. Then select two disjoint infinite subsets $N'_0$ and $N'_1$ of $N \setminus \text{supp}(f(\cdot))$ with $1_{N'_i} \in X$ for some $k_i \in N'_i$, define $N_i = N'_i \setminus \{k_i\}$, and let

$$A_i = \{ f \in A(\cdot) : f(n) = f(\cdot)(n) \text{ for each } n \notin N_i \} \quad (i = 0, 1). \quad (2.2)$$

Suppose that for some $t \in T$, with $|t| < n$, $A_t$ is specified. Put

$$M_t = \sup \{|||f||| : f \in A_t \}, \quad m_t = \inf \{|||f||| : f \in A_t \}. \quad (2.3)$$

Let $f_1 \in A_1$ satisfy $|||f_1||| > (3M_t + m_t)/4$ and take two disjoint infinite subsets $N'_{t0}$ and $N'_{t1}$ of $N_t \setminus \text{supp}(f_1)$ with $1_{N'_{ti}} \in X$, put $N_{ti} = N'_{ti} \setminus \{k_{ti}\}$, and define

$$A_{ti} = \{ f \in A_t : f(n) = f_{ti}(n) n \notin N_{ti} \} \quad (i = 0, 1). \quad (2.4)$$

Thus, by induction on $|t|$, we can obtain a family $\{A_t\}_{t \in T}$ of subsets of $X$, a family $\{f_t\}$ of elements of $X$, a family $\{N_t\}$ of infinite subsets of $N$, and a family of integers $\{k_t\}$ with the following properties.

(a) $A_{ti}$ is of the form

$$A_{ti} = \{ f \in A_t : f(n) = f_{ti}(n), \ n \notin N_{ti} \} \quad (i = 0, 1), \quad (2.5)$$

for each $t \in T$.

(b) $k_{ti} \in N_t \setminus N_{ti}$ and $f_t(k_{ti}) = 0$ for $t \in T$ and $i = 0, 1$.

(c) $|||f_t||| > (3M_t + m_t)/4$, where $M_t$ and $m_t$ denote the supremum and infimum of $\{|||f||| : f \in A_t\}$, respectively.

(d) $N_s \subset N_t$ whenever $t < s$ and $N_t \cap N_s = \emptyset$, if $s$ and $t$ are not comparable.

(e) $\text{supp}(f_t - f_s) \subset N_t \setminus N_s$ for $t < s$.

By (e), $\{g_t\}_{t \in T}$, defined by

$$g(\cdot) = f(\cdot), \quad g_{ti} = f_{ti} - f_t \quad (i = 0, 1), \quad (2.6)$$

is a disjoint family of elements of $X$. By the tree-completeness of $X$, there exists some $b \in \{0, 1\}^N$ such that

$$f_b(x) = f(\cdot) + \sum_{n \in N} g_b|n \quad (2.7)$$

is in $X$. Let $\{k_{\alpha(n)}\}$ be a subsequence of $\{k_{b|n}\}$ such that $1_E \in X$, where $E = \{k_{\alpha(1)}, \ k_{\alpha(2)}, \ldots\}$. Let $E_n = \{k_{\alpha(n)}, k_{\alpha(n+1)}, \ldots\}$ and $h_n = 1_{E_n}$. By (a) and (b), $g_{n+1} = f_b + h_{n+1}$ and $g_{n+1} = f_b - h_{n+1}$ are in $A_{b|n}$. Next, select some $\mu \in X^*$, such that $\mu(h_1) = 1$ and $\mu(g) = 0$ for each $g \in c_0$. Clearly, for such an element $\mu$ and each $n \in N$, we have $\mu(h_n) = 1$. By
(a), $2f_b - f \in A_{b|n}$, thus $||2f_b|n - f|| \leq M_{b|n}$ for each $f \in A_{b|n}$ and $n \in N$. It follows that

$$
\frac{(3M_{b|n-1} + m_{b|n-1})}{2} \leq ||2f_b|n|| \leq M_{b|n} + ||f||, \quad \forall f \in A_{b|n},
$$

(2.8)

and so

$$
\frac{(3M_{b|n-1} + m_{b|n-1})}{2} \leq M_{b|n} + m_{b|n} \leq M_{b|n-1} + m_{b|n-1}, \quad \forall n \in N.
$$

(2.9)

Therefore,

$$
M_{b|n} - m_{b|n} \leq M_{b|n} - \frac{(M_{b|n-1} + m_{b|n-1})}{2} \leq M_{b|n-1} - \frac{(M_{b|n-1} + m_{b|n-1})}{2} = \frac{(M_{b|n-1} - m_{b|n-1})}{2}.
$$

(2.10)

The above relations show that

$$
|||g^+_{n+1}||| - |||f_b||| \leq M_{b|n} - m_{b|n} \leq \frac{(M_{b|n-1} - m_{b|n-1})}{2} \leq \frac{(M_{\cdot} - m_{\cdot})}{2n}.
$$

(2.11)

That is $\lim|||g^+_{n}||| = |||f_b||| = \lim|||g^-_{n}|||$. Moreover, $f_b = (g^+_{n} + g^-_{n})/2$. But $\text{weak-lim}\,(g^+_{n} - g^-_{n}) \neq 0$, since $\mu(h_n) = 1$ for each $n \in N$. This shows that $X$ does not admit any wMLUR norm.

It is known that weakly midpoint locally uniformly rotundity of a Banach space $X$ is equivalent to saying that every point of $S(\hat{X})$ is an extreme point of $B(X^{**})$ [11]. It follows that the space considered in Theorem 2.1 has no equivalent norm such that $S(\hat{X})$ is a subset of $B(X^{**})$.

A norm on a Banach space $X$ is said to be strictly convex (rotund) (R) if the unit sphere of $X$ contains no nontrivial line segment. We say that a norm is Kadec if the weak and norm topologies coincide on the unit sphere. Every MLUR Banach space admits Kadec renorming (see [1]). Haydon in [6, Corollary 6.6] gives an example of a Kadec renormable space which has no equivalent R norm. The following result gives an example of a strictly convexifiable space with no equivalent Kadec norm.

**Corollary 2.2.** If a tree-complete subspace $X$ of $\ell_{\infty}$ contains $c_0$, then it does not admit any equivalent Kadec renorming.

**Proof.** It is known that $\ell_{\infty}$ admits an equivalent strictly convex norm (see [4, page 120] or [2]). In [9] it is shown that every R Banach space with the Kadec property admits an equivalent MLUR renorming (see also [3, chapter IV]). Thus the result follows from Theorem 2.1.
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References

[1] G. A. Aleksandrov and I. P. Dimitrov, On the equivalent weakly midpoint locally uniformly rotund renorming of the space $l_\infty$, Mathematics and Mathematical Education (Sunny Beach (Sl"nchev Bryag), 1985), B"lgar. Akad. Nauk, Sofia, 1985, pp. 189–191.
[2] M. M. Day, Normed Linear Spaces, 3rd ed., Springer-Verlag, New York, 1973.
[3] R. Deville, G. Godefroy, and V. Zizler, Smoothness and Renormings in Banach Spaces, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow, 1993.
[4] J. Diestel, Geometry of Banach Spaces—Selected Topics, Lecture Notes in Mathematics, vol. 485, Springer-Verlag, Berlin, 1975.
[5] R. Haydon, A nonreflexive Grothendieck space that does not contain $l_\infty$, Israel J. Math. 40 (1981), no. 1, 65–73.
[6] ———, Trees in renorming theory, Proc. London Math. Soc. (3) 78 (1999), no. 3, 541–584.
[7] R. Haydon and V. Zizler, A new space with no locally uniformly rotund renorming, Canad. Math. Bull. 32 (1989), no. 1, 122–128.
[8] Z. Hu, W. B. Moors, and M. A. Smith, On a Banach space without a weak mid-point locally uniformly rotund norm, Bull. Austral. Math. Soc. 56 (1997), no. 2, 193–196.
[9] B.-L. Lin, P.-K. Lin, and S. L. Troyanski, Characterizations of denting points, Proc. Amer. Math. Soc. 102 (1988), no. 3, 526–528.
[10] J. Lindenstrauss, Weakly compact sets—their topological properties and the Banach spaces they generate, Symposium on Infinite-Dimensional Topology (Louisiana State Univ., Baton Rouge, La, 1967), Ann. of Math. Studies, No. 69, Princeton University Press, Princeton, NJ, 1972, pp. 235–273.
[11] W. B. Moors and J. R. Giles, Generic continuity of minimal set-valued mappings, J. Austral. Math. Soc. Ser. A 63 (1997), no. 2, 238–262.
Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

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