Free Energy Subadditivity for Symmetric Random Hamiltonians

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Abstract

We consider a random Hamiltonian $H : \Sigma \to \mathbb{R}$ defined on a compact space $\Sigma$ that admits a transitive action by a compact group $G$. When the law of $H$ is $G$-invariant, we show its expected free energy relative to the unique $G$-invariant probability measure on $\Sigma$ obeys a subadditivity property in the law of $H$ itself. The bound is often tight for weak disorder and relates free energies at different temperatures when $H$ is a Gaussian process. Many examples are discussed including branching random walks, several spin glasses, random constraint satisfaction problems, and the random field Ising model. We also provide a generalization to quantum Hamiltonians with applications to the quantum SK and SYK models.

1 A General Subadditivity Result

A large part of statistical physics and probability theory is concerned with determining the free energy of a given Hamiltonian. In many models for disorded systems, this Hamiltonian is itself a random function. This paper focuses on free energies of random Hamiltonians obeying rich distributional symmetries.

We begin with a few generalities. Let $\Sigma$ be a compact metric space equipped with reference Borel probability measure $\mu$. We consider a (random) continuous Hamiltonian function $H : \Sigma \to \mathbb{R}$ with law $L$. Define the associated (random) partition function $Z(H)$ and (deterministic) expected free energy $F(L)$ by

$$Z(H) = \int e^{H(\sigma)} \mu(d\sigma); \quad (1.1)$$

$$F(L) = \mathbb{E}[\log Z(H)]. \quad (1.2)$$

$F(L)$ will always be assumed finite.

Our main result requires that $(\Sigma, \mu, H)$ be highly symmetric in a joint sense. Precisely, we require the existence of a continuous and transitive action $G \times \Sigma \to \Sigma$ on $\Sigma$ by a compact group $G$; such an action is said to make $\Sigma$ a homogeneous space for $G$. In particular, each $g \in G$ acts on $\Sigma$ by a homeomorphism $\sigma \mapsto g\sigma$. Since $\Sigma$ and $G$ are compact, there exists a unique $G$-invariant probability measure on $\Sigma$ under which $\sigma$ and $g\sigma$ have the same law for each $g \in G$ (see e.g. [33, Theorem 6.2]). We require $\mu$ to be this $G$-invariant probability measure.

We say that the law $L$ of $H$ is $G$-invariant if for each $g \in G$ the transformed Hamiltonian given by

$$H^g(\sigma) = H(g\sigma), \quad \forall \sigma \in \Sigma$$

also has law $L$. Further, given two Hamiltonian distributions $L_1$ and $L_2$, let $L_1 + L_2$ be the law of the independent sum $H_1 + H_2$ for $(H_1, H_2) \sim L_1 \times L_2$. We can now state our main subadditivity result.

**Theorem 1.1.** Suppose $(\Sigma, \mu, G)$ are as above, and in particular that $\mu$ is the unique probability measure on $\Sigma$ invariant under the transitive action of $G$. Let $L_1, L_2$ be two laws for random Hamiltonians on $\Sigma$ and suppose that $L_1$ is $G$-invariant. Then

$$F(L_1 + L_2) \leq F(L_1) + F(L_2).$$

**Proof.** Consider the random probability measure $\tilde{\mu}$ defined from $H_1$ by

$$\tilde{\mu}(d\sigma) = \frac{e^{H_1(\sigma)} \mu(d\sigma)}{\int e^{H_1(\sigma)} \mu(d\sigma)}.$$
Note that \( \mathbb{E}[\tilde{\mu}(d\sigma)] = \mu(d\sigma) \), i.e. for any bounded, measurable function \( f : \Sigma \to \mathbb{R} \) independent of \( \tilde{\mu} \), we have

\[
\mathbb{E} \left[ \int f(\sigma)\tilde{\mu}(d\sigma) \right] = \int f(\sigma)\mu(d\sigma).
\]

Indeed, (1.3) certainly defines some probability measure \( \mu' \) on the right-hand side. Moreover this probability measure inherits \( \mathcal{G} \)-invariance from \( \mathcal{L}_1 \). The uniqueness result [33, Theorem 6.2] mentioned above now implies \( \mathbb{E}[\tilde{\mu}(d\sigma)] = \mu(d\sigma) \) as claimed. Using this, we obtain

\[
F(\mathcal{L}_1 + \mathcal{L}_2) = \mathbb{E} \left[ \log \int e^{H_1(\sigma)+H_2(\sigma)}\mu(d\sigma) \right] = \mathbb{E} \left[ \log \int e^{H_1(\sigma)}\mu(d\sigma) \right] + \mathbb{E} \left[ \log \int e^{H_2(\sigma)}\mu(d\sigma) \right] = F(\mathcal{L}_1) + F(\mathcal{L}_2),
\]

In the last step we used (1.3) together with Jensen’s inequality. This concludes the proof.

Even without symmetry, a weaker form of the above estimate can be obtained with \( F(\mathcal{L}_2) \) replaced by the simple upper bound

\[
\sup_{\sigma \in \Sigma} \log \mathbb{E}[e^{H_2(\sigma)}].
\]

Indeed by Jensen’s inequality, (1.5) upper-bounds \( \mathbb{E} \left[ \log \int e^{H_2(\sigma)}\tilde{\mu}(d\sigma) \right] \) for any probability measure \( \tilde{\mu} \). In the special case that the marginal distributions of \( H_1(\sigma) \) and \( H_2(\sigma) \) do not depend on \( \sigma \), we similarly find that

\[
F(\mathcal{L}_1) + F(\mathcal{L}_2) \leq \log \mathbb{E}[e^{H_1(\sigma)}] + \log \mathbb{E}[e^{H_2(\sigma)}] = \log \mathbb{E}[e^{H_1(\sigma)+H_2(\sigma)}].
\]

In particular the estimate in Theorem 1.1 is asymptotically tight if \( \mathcal{L}_1 + \mathcal{L}_2 \) satisfies

\[
F(\mathcal{L}_1 + \mathcal{L}_2) \approx \log \mathbb{E}[e^{H_1(\sigma)+H_2(\sigma)}].
\]

This near-equality holds at weak disorder or high temperature in many examples, including some presented in the next section. The right-hand side is often referred to as the annealed free energy of \( H_1 + H_2 \).

Let us also point out that specializing Theorem 1.1 to zero temperature by replacing \( (H_1, H_2) \) with \( (\beta H_1, \beta H_2) \) for large \( \beta \) yields only the trivial bound

\[
\max_{\sigma \in \Sigma} (H_1(\sigma) + H_2(\sigma)) \leq \max_{\sigma \in \Sigma} H_1(\sigma) + \max_{\sigma \in \Sigma} H_2(\sigma).
\]

This bound holds with no symmetry assumption, but for finite \( \beta \) the symmetry conditions in Theorem 1.1 are essential. In fact without symmetry, simple counterexamples exist even on a two-point space \( \Sigma = \{-1, 1\} \) with \( \mu \) uniform and \( H_1 = H_2 \) deterministic. Taking \( H_1(1) = H_2(1) = 0 \) and \( H_1(-1) = H_2(-1) = x \) yields

\[
F(\mathcal{L}_1) + F(\mathcal{L}_2) = 2 \log \left( \frac{1+e^x}{2} \right),
\]

\[
F(\mathcal{L}_1 + \mathcal{L}_2) = \log \left( \frac{1+e^{2x}}{2} \right)
\]

and the latter is easily seen to be strictly larger for all real \( x \neq 0 \).

**Remark 1.2.** Free energy subadditivity in the system size has been established for mean-field spin glasses and sparse graph models obeying certain convexity properties in [48, 11, 49, 53]. These results are fundamentally important as they imply the existence of a limiting asymptotic free energy or ground state value in such models. Theorem 1.1 is in a different spirit as the state space \( \Sigma \) is fixed while the Hamiltonian varies. As demonstrated by the examples in the next section, our result applies somewhat generically and does not require any convexity conditions. On the other hand, Theorem 1.1 is a simpler bound and is usually not tight for low temperatures, while the results of [48, 11, 49] are asymptotically tight at all temperatures. (Indeed, our result only implies an \( \Theta(\beta^2) \) upper bound for free energies at inverse temperature \( \beta \), while \( \Theta(\beta) \) is the correct asymptotic in e.g. mean-field spin glass models.)
2 Examples

We discuss several statistical physics models to which Theorem 1.1 applies. While we focus on models that are important in their own right, many others are easily constructed by e.g. independently summing some of the random Hamiltonians presented below.

2.1 Branching Random Walk

Let \( T_{N,d} \) be a \( d \)-ary rooted tree of depth \( N \), and fix a probability distribution \( \nu \) on \( \mathbb{R} \) with finite exponential moments. For each vertex \( v \in V(T_{N,d}) \), generate an i.i.d. variable \( x_v \sim \nu \). For a leaf \( w \) of \( T_{N,d} \), let \( P(w) = (v_0, v_1, \ldots, v_N = w) \) be the path to \( w \) from the root \( v_0 \). The branching random walk Hamiltonian assigns to each \( w \in V(T_{N,d}) \) the sum

\[
H(w) = \sum_{v \in P(w)} x_v.
\]

(2.1)

This model has been studied in e.g. [19, 52] and is also known as the directed polymer on a tree. A lot of other work including [14, 59, 16] has resulted in a precise understanding of the extreme values, which corresponds to the zero temperature setting.

Here the appropriate transitive symmetry group \( G \) consists of all root-preserving automorphisms of \( T_{N,d} \). It is easy to see that the random Hamiltonian in (2.1) is \( G \)-invariant for any \( \nu \), and that the unique \( G \)-invariant measure on the leaf set \( \Sigma = \partial T_{N,d} = T_{N,d} - T_{N-1,d} \) is uniform. Theorem 1.1 implies that the associated free energy

\[
F_N(\beta) = \mathbb{E} \log \left( \frac{\sum_{w \in \partial T_{N,d}} e^{\beta H(w)}}{d^N} \right)
\]

(2.2)

is a subadditive function of \( \nu \). Moreover Theorem 1.1 is asymptotically tight for small \( \beta \): [17, 19] show that for \( \beta \leq \beta_{\text{crit}}(\mu, d) \), the limiting free energy agrees with the annealed value

\[
F(\beta) = \lim_{N \to \infty} F_N(\beta)/N = \log \mathbb{E} e^{\beta x}
\]

which is clearly additive in \( \nu \).

Let us also point out that the generalized random energy model (GREM) Hamiltonian [31, 32] takes the same form as (2.2), but with the depth \( N \) fixed and the degree \( d \) growing. Here the distribution of \( x_v \) may depend on the depth of \( v \), which doesn’t affect the symmetry used above. It follows that Theorem 1.1 applies also to the GREM free energy.

2.2 Spin Glasses with Gaussian Disorder

Spin glasses give rise to some of the most canonical examples of random Hamiltonians. Theorem 1.1 applies to the following quite general family of Ising spin glasses. Let \( \Sigma_N = \{ -1, 1 \}^N \) and for \( 1 \leq p \leq P \) and each \( (i_1, \ldots, i_p) \in [N]^p \), let

\[
J_{i_1, \ldots, i_p} \sim \mathcal{N}(0, c_{i_1, \ldots, i_p})
\]

be a centered Gaussian with arbitrary variance \( c_{i_1, \ldots, i_p} \geq 0 \). We assume the variables \( J_{i_1, \ldots, i_p} \) are jointly independent and define the Hamiltonian

\[
H_N(\sigma) = \sum_{p=1}^P \frac{1}{N^{(p-1)/2}} \sum_{1 \leq i_1, \ldots, i_p \leq N} J_{i_1, \ldots, i_p} \sigma_{i_1} \cdots \sigma_{i_p}.
\]

(2.3)

To apply Theorem 1.1 to \( H_N \), we take \( G = \mathbb{Z}_2^N \) which acts on \( \Sigma \) by

\[
(g\sigma)_i = (-1)^g_i \sigma_i, \quad i \in [N].
\]

This is a transitive action preserving the law of \( H_N \), and it is easy to see that the \( G \)-invariant measure \( \mu \) is uniform on \( \Sigma \). Moreover the independent sum \( \mathcal{L}_1 + \mathcal{L}_2 \) amounts to adding the vectors \( \vec{c} \) of coefficient variances. For \( \beta \geq 0 \) the associated free energy at inverse temperature \( \beta \) is

\[
F_{N,\beta}(\vec{c}) = \mathbb{E} \log \int e^{\beta H_N(\sigma)} \mu(d\sigma),
\]

for \( \mu \) the uniform measure on \( \Sigma_N \). Of course \( F_{N,\beta}(\vec{c}) = F_N(\beta^2 \vec{c}) \), so we define also \( F_N(\vec{c}) = F_{N,1}(\vec{c}) \). Theorem 1.1 then implies the following.
Corollary 2.1. For entrywise non-negative coefficient variances \( \bar{c} \) and \( \bar{c}' \), we have
\[
F_N(\bar{c} + \bar{c}') \leq F_N(\bar{c}) + F_N(\bar{c}').
\]
In particular for \( \beta_1, \beta_2 \geq 0 \) we have the subadditivity in \( \beta^2 \) bound
\[
F_{N,\sqrt{\beta_1^2+\beta_2^2}}(\bar{c}) \leq F_{N,\beta_1}(\bar{c}) + F_{N,\beta_2}(\bar{c}).
\] (2.4)

The above setup includes many spin glasses, all of which thus satisfy Corollary 2.1. When \( c_1, \ldots, c_p = c \) depends only on \( p \), we recover the well studied mixed \( p \)-spin model (see e.g. [66]) whose covariance
\[
\xi(\sigma^1, \sigma^2) = E[H(\sigma^1)H(\sigma^2)] = \sum_{p=1}^P c_p \left( \frac{\langle \sigma^1, \sigma^2 \rangle}{N} \right)^p
\]
depends only on the overlap \( \langle \sigma^1, \sigma^2 \rangle/N \). In this case we may write \( F_{N,\beta}(\bar{c}) = F_{N,\beta}(\xi) \). Hence we find that for Ising mixed \( p \)-spin models,
\[
F_N(\xi_1 + \xi_2) \leq F_N(\xi_1) + F_N(\xi_2),
\]
\[
F_{N,\sqrt{\beta_1^2+\beta_2^2}}(\xi) \leq F_{N,\beta_1}(\xi) + F_{N,\beta_2}(\xi).
\] (2.5) (2.6)

When \( c_1 = 0 \) and \( \beta \leq \beta_{crit}(c_2, \ldots, c_p) \) is small, the replica-symmetric formula
\[
\lim_{N \to \infty} \frac{F_N(\beta \xi)}{N} = \beta^2 \xi(1)
\]
holds. Thus (2.6) is asymptotically sharp as \( N \to \infty \) for \( \beta_1^2 + \beta_2^2 \leq \beta_{crit}^2 \) in this case. Another notable example of the form (2.3) is the Edwards-Anderson model [39, 64] in which \( (c_{v_1, v_2})_{v_1, v_2 \in V(G)} \) is the adjacency matrix of a finite lattice. We mention that [29] considers the family of Hamiltonians (2.3) and shows that \( F(\bar{c}) \) is coordinate-wise increasing in \( \bar{c} \).

We remark that for mixed \( p \)-spin models, we were surprisingly unable to prove (2.5) in the \( N \to \infty \) limit directly from the (enormously more difficult) Parisi formula. In the appendix we derive from the Auffinger-Chen representation of the Parisi formula the weaker statement that free energy relative to counting measure on \( (-1, 1)^N \) is subadditive in \( \bar{c} \).

Instead of Ising spins \( \Sigma_N = \{ -1, 1 \}^N \), one can also consider a spherical state space or even a product of spheres. Since the Hamiltonian must now be invariant relative to a larger symmetry group, the applications are not quite as general as above. A still broad family of examples comes from setting
\[
\Sigma = \prod_{i=1}^r S^{N_{i-1}}(\sqrt{N_i})
\]
for \( N = \sum_{i=1}^r N_i \), where \( S^{N_{i-1}}(\sqrt{N_i}) \) is the sphere in \( \mathbb{R}^{N_i} \) of radius \( \sqrt{N_i} \). The corresponding transitive symmetry group \( \mathcal{G} = \prod_{i=1}^r O(N_i) \) is a product of orthogonal groups \( O(N_i) \) with the natural action, and the unique invariant probability measure \( \mu \) is the natural choice of uniform measure. Let \( s : [N] \to [r] \) send the \( N_i \) coordinates in the \( S^{N_{i-1}}(\sqrt{N_i}) \) factor above to \( i \in [r] \). It is not difficult to see that Hamiltonians in (2.3) are \( \mathcal{G} \)-invariant if \( c_{v_1, \ldots, v_p} = c_{s(v_1), \ldots, s(v_p)} \). Therefore the subadditivity results in Corollary 2.1 hold for such \( \bar{c} \). These models are known as multi-species spin glasses and are the subject of much recent work [8, 68, 62, 57, 79, 80, 78, 10, 9].

Yet another generalization is to consider pairs \( (\sigma^1, \sigma^2) \in S^{N-1}(\sqrt{N}) \) with the overlap constraint \( \langle \sigma^1, \sigma^2 \rangle = RN \) for some \( R \in [-1, 1] \). Our discussion above goes through essentially unchanged for \( (\sigma^1, \sigma^2) \) on the sphere \( S^{N-1}(\sqrt{N}) \) because all such pairs are related by a simultaneous \( O(N) \) action. For Ising spin glasses, the same is true once \( \mathbb{Z}_2^N \) is replaced by the full symmetry group of the cube which includes also coordinate permutations. In both these cases, the analog of Corollary 2.1 applies to the two-replica Hamiltonian
\[
\bar{H}_N(\sigma^1, \sigma^2) = H_N(\sigma^1) + H_N(\sigma^2).
\] (2.7)

Thus we deduce subadditivity in \( \bar{c} \) for
\[
F_{N,\beta}(\bar{c}) \equiv E \log \int e^{\beta H_N(\sigma^1) + \beta H_N(\sigma^2)} \mu_R(d(\sigma^1, \sigma^2)),
\]
where \( \mu_R \) is the unique \( \mathcal{G} \)-invariant measure on pairs \( (\sigma^1, \sigma^2) \in S^{N-1}(\sqrt{N}) \times S^{N-1}(\sqrt{N}) \) with \( \langle \sigma^1, \sigma^2 \rangle = RN \).
The free energies of such constrained pairs appear in Talagrand’s proof [81] of the Parisi formula and have also been used to study disorder chaos [20, 27, 25, 21, 26, 24]. They are also a special case of the so-called vector spin models [70, 69, 58]. Further generalizations with multiple correlated Hamiltonians and more replicas have been used in relation with the overlap gap property to establish computational barriers against efficiently optimizing $H_N$ [23, 44, 50]—on the sphere, these are still symmetric enough to conclude subadditivity, but we omit the details.

2.3 Orthogonally Invariant Spin Glasses

The orthogonally invariant Sherrington-Kirkpatrick model is a spin glass with dependent couplings studied in [61, 72, 13, 41, 7, 40]. To define it, one chooses a diagonal $N \times N$ matrix $\Lambda_N$ and sets

$$A_N = O_N \Lambda_N O_N^\top$$

for a Haar-random orthogonal matrix $O_N \in O(N)$. The associated random Hamiltonian is $H_N(\sigma) = \langle \sigma, A_N \sigma \rangle$ for $\sigma \in \Sigma_N = \{-1,1\}^N$. We let $F_N(\nu)$ be the corresponding free energy when the entries of $\Lambda$ are drawn i.i.d. from the compactly supported measure $\nu$. Such Hamiltonians are $O(N)$-invariant by definition, and in particular under the natural $\mathbb{Z}_N^2$-action on $\Sigma_N$. Hence the conditions of Theorem 1.1 are met.

In this case, addition of Hamiltonians corresponds to additive free convolution. Indeed if the entries of $\Lambda_{N,i}$ are drawn i.i.d. from $\nu_i$ for $i \in \{1,2\}$, and if $O_{N,1}, O_{N,2} \in O(N)$ are Haar-random and independent, then the sum

$$A_{N,1} + A_{N,2} = O_{N,1} \Lambda_{N,1} O_{N,1}^\top + O_{N,2} \Lambda_{N,2} O_{N,2}^\top$$

is orthogonally invariant with random spectrum converging in probability (in e.g. $W_2$ metric) to the additive free convolution $\nu_1 \boxplus \nu_2$ as $N \to \infty$. Moreover it follows from [13, Proof of Proposition 1.1] that the normalized free energy of the orthogonally invariant SK model is Wasserstein-continuous in the spectrum, leading to the approximate subadditivity relation

$$F_N(\nu_1 \boxplus \nu_2) \leq F_N(\nu_1) + F_N(\nu_2) + o_N(N) \quad (2.8)$$

for the orthogonally invariant SK model. As in the previous subsection, this bound is asymptotically tight in the replica-symmetric phase. Indeed here the limiting free energy $F(\nu) = \lim_{N \to \infty} F_N(\nu)$ agrees with the annealed value, which is given by an integrated $R$-transform of $\nu$ ([13, Theorem 1.2 and Equation (1.8)]) and is therefore additive under free convolution.

A similar construction is possible for tensors as well. For $p \geq 3$ one may construct a random $p$-tensor $A_N^{(p)}$ by starting with a deterministic $p$-tensor $\Lambda_N^{(p)}$ and conjugating on all $p$ “sides” by an independent Haar-random matrix $O_N^{(p)} \in O(N)$. The resulting orthogonally invariant mixed $p$-spin model Hamiltonian

$$H_N(\sigma) = \sum_{p=2}^P \langle A_N^{(p)}, \sigma \otimes_{\sigma} \rangle$$

has law which is $O(N)$-invariant and hence $\mathbb{Z}_N^2$-invariant. Thus for deterministic sequences $(\Lambda_N^{(2)}, \ldots, \Lambda_N^{(P)})$, the subadditivity result Theorem 1.1 still applies. However it is unclear how to extend the clean statement (2.8) beyond matrices; this would probably require a theory of free probability for tensors.

2.4 Random Constraint Satisfaction Problems

In a typical random constraint satisfaction problem, one equips the state space $\Sigma_N = \{-1,1\}^N$ with Hamiltonian

$$H_N(\sigma) = -\sum_{j=1}^M \theta_j(\sigma_{i_1j}, \ldots, \sigma_{i_kj})$$

for $M = \alpha N$ or $M \sim \text{Poisson}(\alpha N)$. Here each $\theta_j : \{-1,1\}^k \to \mathbb{R}_{\geq0}$ is an i.i.d. non-negative function invariant in law under the action of $\mathbb{Z}_N^k$, and the indices $i_j \in [N]$ are i.i.d. as well. For example random $k$-SAT and NAE $k$-SAT can be represented in the above way, where $\theta_j$ equals 0 when the corresponding clause is satisfied and 1 otherwise. These models have been studied in great detail [1, 28, 34, 35, 63].

In this setting, applying Theorem 1.1 shows the free energy is subadditive in the clause density $\alpha$. Specialized to $\beta H_N$ for large $\beta$, this roughly says that the typical solution density is submultiplicative in $\alpha$, which is easy
to see directly. (However technically we cannot set \( \beta = \infty \) in Theorem 1.1 to obtain “hard” constraints as then \( \log(Z) = \log(0) \) holds with positive probability.)

The random Ising perceptron model (see [82, 83, 36]) is similarly \( \mathbb{Z}^N_2 \)-invariant. Here \( k = N \) with deterministic indices \( (i_1, \ldots, i_N) = (1, 2, \ldots, N) \) and the functions take the form \( \theta_{ij}(\sigma) = \varphi(g_i, \sigma) \) for a deterministic function \( \varphi : \mathbb{R} \to \mathbb{R} \) and Gaussian disorder vector \( g_i \sim \mathcal{N}(0, I_N) \). Theorem 1.1 applies also to this model, as well as to the spherical analog where \( \Sigma = S^{N-1}(\sqrt{N}) \) is equipped with uniform measure.

### 2.5 Random Field Ising Model and Spiked Matrices

The random field Ising model (RFIM) was introduced by [51], see also [15, 3, 18, 37]. Here one considers a vertex set such as \( V = [N]^d \) with ferromagnetic nearest neighbor interactions and a random external field \( h \in \mathbb{R}^V \), giving rise to a Hamiltonian with independent two contributions:

\[
H(\sigma) = H_1(\sigma) + H_2(\sigma), \quad \sigma \in \{-1, 1\}^V;
\]

\[
H_1(\sigma) = \sum_{v \in V} h_v \sigma_v, \quad H_2(\sigma) = \sum_{\langle v, w \rangle \in V : \| v - w \| = 1} \sigma_v \sigma_w.
\]

The random field term \( \sum_{v \in V} h_v \sigma_v \) is invariant under the natural \( \mathbb{Z}_2^V \) action of \( \{-1, 1\}^V \) if the values \( (h_v)_{v \in V} \) are jointly independent and each \( h_v \) has negation-invariant law. Since Theorem 1.1 requires only that \( H_1 \) be \( G \)-invariant, this implies an upper bound on the free energy in (2.9) with \( \mu \) the uniform measure on \( \{-1, 1\}^V \).

As the expected free energy of \( H_1 \) is just

\[
\sum_{v \in V} \mathbb{E}[\log \cosh(h_v)]
\]

we obtain an elementary upper bound on the RFIM free energy in terms of the ordinary Ising model free energy. The same technique also gives a simpler proof of the lower bound in [43, Theorem A.1].

Another natural example of a random Hamiltonian which is a sum of two different terms is the spiked matrix or tensor model which has been studied extensively in e.g. [55, 6, 12, 38, 74, 73, 22, 4]. The spiked matrix model is defined by starting with a random matrix \( A \) and adding a random rank 1 spike to obtain \( A + vv^T \). The resulting Hamiltonian is

\[
\hat{H}(\sigma) = \langle \sigma, A\sigma \rangle + \langle \sigma, v \rangle^2.
\]

If for example \( v \) is uniform on the sphere or \( \{-1, 1\}^N \), then the spike is \( O(N) \) or \( \mathbb{Z}_2^N \) invariant. As explained in e.g. [22], the free energy in a spiked model is intimately related with detectability of the spike. In these models as well, Theorem 1.1 applies when either \( A \) or \( v \) obey the requisite symmetry property and gives a simple free energy upper bound separating the different interactions in the Hamiltonian.

### 3 Generalization to Quantum Hamiltonians

A version of our result extends to quantum Hamiltonians. We restrict attention to operators on finite dimensional spaces, though this is probably not essential. Let \( M \) be a random \( N \times N \) Hermitian matrix with law \( \mathcal{M} \). Then the associated quantum partition function and average free energy are given by

\[
Z(M) = \text{Tr}(e^M) / N;
\]

\[
F(M) = \mathbb{E}_{M \sim \mathcal{M}}[\log Z(M)].
\]

In the case that \( M \) is almost surely diagonal in a fixed orthogonal basis \( (v_1, \ldots, v_N) \), \( M \) can be viewed as a classical Hamiltonian taking value \( \lambda_i(M) \) on the state \( \sigma = v_i \). \( Z(M) \) and \( F(M) \) defined above then agree with the classical free energy for the uniform reference measure \( \mu \) on \( \{v_1, \ldots, v_N\} \).

Similarly to before, let \( \mathcal{M}_1 + \mathcal{M}_2 \) denote the law of the independent matrix sum \( M_1 + M_2 \) for \((M_1, M_2) \sim \mathcal{M}_1 \times \mathcal{M}_2 \). We say that \( \mathcal{M} \) is \( G \)-invariant for a compact group \( G \subseteq U(N) \) of unitary transformations if for \( M \sim \mathcal{M} \) independent of any fixed \( g \in G \), we have \( gM \sim \mathcal{M} \).

Our generalization of the symmetry assumption is as follows. With \( \mathbb{E} \in G \) denoting expectation relative to Haar measure, we say that \( G \) is \( M \)-symmetrizing if

\[
\mathbb{E}_{g \in G}[gMg^{-1}] = \frac{\text{Tr}(M)I_N}{N}.
\]

(Here \( \text{Tr}(M) \) is a scalar and \( I_N \) denotes the identity matrix.) For diagonal \( M \), the group \( S_N \subseteq U(N) \) of permutation matrices is \( M \)-symmetrizing, as are all of its transitive subgroups. This recovers the setting of Theorem 1.1.
for finite $\Sigma$. As another example, the subgroup of signed permutation matrices is $M$-symmetrizing for all not-necessary-diagonal $M$. With these definitions in place, we can now state a generalization of Theorem 1.1 to quantum free energies.

**Theorem 3.1.** Suppose $M_1$ is $G$-invariant and $G$ is almost surely $e^{M_1}$-symmetrizing for $M_1 \sim M_1$. Then

$$F(M_1 + M_2) \leq F(M_1) + F(M_2).$$

**Proof.** Recall the Golden-Thompson inequality [45, 84, 42]: for all Hermitian matrices $M_1$ and $M_2$,

$$\text{Tr}(e^{M_1 + M_2}) \leq \text{Tr}(e^{M_1}) + \text{Tr}(e^{M_2}).$$

Using Jensen’s inequality in the final step below, we find:

$$F(M_1 + M_2) = \mathbb{E}\left[ \log \text{Tr}\left( e^{M_1 + M_2} \right)/N \right] = \mathbb{E}\left[ \log \text{Tr}(e^{M_1})/N \right] + \mathbb{E}\left[ \log \frac{\text{Tr}(e^{M_1 + M_2})}{\text{Tr}(e^{M_1})} \right] \leq F(M_1) + \mathbb{E}\left[ \log \text{Tr}(e^{M_1 + M_2})/\text{Tr}(e^{M_1}) \right].$$

Next recall that $e^{gM_1 g^{-1}} = e^{M_1} g^{-1}$ for unitary $g$ and in particular $\text{Tr}(e^{gM_1 g^{-1}}) = \text{Tr}(e^{M_1})$. Using $G$-invariance of $M_1$ in the first step, we find that for any fixed $M_2$,

$$\mathbb{E}^{M_1 \sim M_1} \left[ \frac{\text{Tr}(e^{M_1 + M_2})}{\text{Tr}(e^{M_1})} \right] = \mathbb{E}^{M_1 \sim M_1} \mathbb{E}^{g \in G} \left[ \frac{\text{Tr}(e^{g M_1 g^{-1} + M_2})}{\text{Tr}(e^{g M_1 g^{-1}})} \right] = \mathbb{E}^{M_1 \sim M_1} \mathbb{E}^{g \in G} \left[ \frac{\text{Tr}(e^{g M_1 g^{-1} + M_2})}{\text{Tr}(e^{M_1})} \right].$$

Combining the above displays completes the proof. \hfill \Box

In the next two subsections, we explain how to apply Theorem 3.1 to the quantum SK and SYK models. In Remark 3.7, we briefly explain a more complicated model that subsumes both while still obeying subadditivity.

### 3.1 Application to the Quantum SK Model

The quantum SK model was introduced in [75, 46]. Following rigorous results in [30, 60], its free energy was determined at all temperatures for constant transverse field through a connection to (classical) vector spin glasses in [2]. We consider a variant with Gaussian transverse field. To define the model, we first recall the $2 \times 2$ Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.5)$$

Together with the identity $I_2$, the Pauli matrices form a basis for the $2 \times 2$ Hermitian matrices. Moreover they each square to $I_2$ and satisfy the relations

$$\sigma^x \sigma^y = i \sigma^z = -\sigma^y \sigma^x,$$

$$\sigma^y \sigma^z = i \sigma^x = -\sigma^x \sigma^y,$$

$$\sigma^z \sigma^x = i \sigma^y = -\sigma^y \sigma^z. \quad (3.6)$$

Fix a positive integer $m$, the number of interacting particles. For $i \in [m]$, let $\sigma^x_i : (\mathbb{C}^2)^{\otimes m} \to (\mathbb{C}^2)^{\otimes m}$ be the linear operator which acts by $\sigma^x$ on the $i$-th tensor factor, and by the identity on the others:

$$\sigma^x_i = I_2 \otimes \cdots \otimes \sigma^x \otimes \cdots \otimes I_2.$$
Similarly define $\sigma_i^y$ and $\sigma_i^z$. The quantum SK Hamiltonian with Gaussian transverse field is the random operator $M_m : (\mathbb{C}^2)^{\otimes m} \rightarrow (\mathbb{C}^2)^{\otimes m}$ given by

$$M_m = \frac{\beta}{\sqrt{m}} \sum_{1 \leq i,j \leq m} J_{i,j} \sigma_i^x \sigma_j^x + h \sum_{i=1}^m J_i \sigma_i^z. \quad (3.7)$$

Here $\beta$ and $h$ are constants while as usual $J_{i,j}$ and $J_i$ are i.i.d. standard Gaussians. The corresponding free energy is

$$F_m^{\text{QSK}}(\beta, h) = \mathbb{E} \log \left( \frac{\text{Tr}(e^{M_m})}{2^m} \right).$$

It should be noted that taking $h = 0$ recovers the classical SK model, since then $M_m$ is diagonal in the standard basis. However the transverse field $h \sum_{i=1}^m J_i \sigma_i^z$ behaves differently from a classical external field.

For $i \in [m]$, define the finite group

$$\mathcal{Q}_i = \{ \pm I_2, \pm \sigma_i^x, \pm \sigma_i^y, \pm \sigma_i^z \}$$

of matrices acting on the $i$-th tensor factor. Let $\mathcal{Q} \subseteq \text{End}((\mathbb{C}^2)^{\otimes m})$ be the group generated by all of the $\mathcal{Q}_i$, so that $|\mathcal{Q}| = 2^{2^m+1}$ (there are four choices for the operator in each tensor factor, as well as a global choice of sign). Applying Theorem 3.1 to the quantum SK Hamiltonian with symmetry group $\mathcal{Q}$ yields the following subadditivity result for the quantum SK model.

**Corollary 3.2.** For any $m \geq 1$ and constants $\beta_1, \beta_2, h_1, h_2 \geq 0$, we have

$$F_m^{\text{QSK}}\left(\sqrt{\beta_1^2 + \beta_2^2}, \sqrt{h_1^2 + h_2^2}\right) \leq F_m^{\text{QSK}}(\beta_1, h_1) + F_m^{\text{QSK}}(\beta_2, h_2).$$

**Proof.** It is easy to see that independently summing quantum SK Hamiltonians with parameters $(\beta_1, h_1)$ and $(\beta_2, h_2)$ gives another with parameters $\left(\sqrt{\beta_1^2 + \beta_2^2}, \sqrt{h_1^2 + h_2^2}\right)$, so it suffices to check that the quantum SK model verifies the conditions of Theorem 3.1 with symmetry group $\mathcal{Q}$.

First, note that for any $g \in \mathcal{Q}$, if $M_m$ is as in (3.7), then the relations (3.6) imply

$$gM_mg^{-1} = \frac{\beta}{\sqrt{m}} \sum_{1 \leq i,j \leq m} \epsilon_{i,j}(g) J_{i,j} \sigma_i^x \sigma_j^x + h \sum_{i=1}^m \epsilon_i(g) J_i \sigma_i^z.$$

for deterministic $\epsilon_{i,j}(g), \epsilon_i(g) \in \{-1, 1\}$. Since $J_{i,j}$ and $J_i$ are negation-invariant in law and independent, we find that $gM_mg^{-1}$ has the same law as $M_m$, verifying the $\mathcal{Q}$-invariance condition of Theorem 3.1.

Next we argue that all of $(\mathbb{C}^2)^{\otimes m}$ is symmetrized by $\mathcal{Q}$ in the sense of (3.3). It is not difficult to see that a basis for $\text{End}((\mathbb{C}^2)^{\otimes m})$ consists of the operators

$$M = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_m$$

(3.8)

where $\sigma_i \in \{ I_2, \sigma_i^x, \sigma_i^y, \sigma_i^z \}$ for each $i \in [m]$. Moreover all of these are traceless except the identity $I_2^{\otimes m} = I_{2^m}$ which clearly satisfies (3.3). It remains to show that if $\sigma_i \in \{ \sigma_i^x, \sigma_i^y, \sigma_i^z \}$ for some $i \in [n]$, then $M$ in (3.8) satisfies

$$\mathbb{E}g \in \mathcal{Q}[gMg^{-1}] = 0. \quad (3.9)$$

This in turn follows by considering the effect of conjugation on the $i$-th tensor factor. Indeed since each of $\sigma_i^x, \sigma_i^y, \sigma_i^z$ commutes with half of $\mathcal{Q}_i$ and anti-commutes with the other half, we find that

$$\sum_{g \in \mathcal{Q}_i} g\sigma_i g^{-1} = 0 \quad (3.10)$$

for $\sigma \in \{ \sigma_i^x, \sigma_i^y, \sigma_i^z \}$ which implies (3.9). \qed

### 3.2 Application to the SYK Model

The Sachdev-Ye-Kitaev (SYK) model introduced in [77, 56] is a quantum mechanical model for strongly interacting fermions with applications to holography and black holes [76]. While different from the quantum SK model, it also takes a similar form to classical spin glass Hamiltonians such as (2.3), again with non-commuting variables. We will see that the SYK model also falls under the purview of Theorem 3.1.
To define the SYK model, we work in the involutive \( C_n \)-algebra \( C_n \) generated by \( \chi_1, \ldots, \chi_n \) with relations:

\[
\chi_i = \chi_i^*, \quad i \in [n] \\
\chi_i^2 = 1, \quad i \in [n] \\
\chi_i \chi_j + \chi_j \chi_i = 0, \quad 1 \leq i < j \leq n.
\]  

Its dimension is \( \dim(C_n) = 2^n \), and a basis is given by the set of products

\[
\chi_S = \chi_{s_1} \cdots \chi_{s_k}
\]  

for subsets \( S = \{s_1, \ldots, s_k\} \) with \( 1 \leq s_1 < \cdots < s_k \leq n \).

Being an associative algebra, \( C_n \) acts on itself by left-multiplication. This representation yields a canonical identification of each element \( x \in C_n \) with a linear operator \( M^x : C_n \to C_n \). Fixing a positive even integer \( q \in 2\mathbb{N} \), the SYK Hamiltonian is given by such an operator \( M^x \) for random \( x \). Precisely, we set

\[
x = i^{q/2} \sum_{1 \leq i_1 < \cdots < i_q \leq n} J_{i_1, \ldots, i_q} \chi_{1} \cdots \chi_{q}
\]  

where the disorder variables \( J_{i_1, i_2, \ldots, i_q} \) are i.i.d. centered Gaussians. It is easy to verify using the relations (3.11) that \( M^x \) is Hermitian.

Recalling (3.2), let

\[
F^{\text{SYK}}_{n, q} (\beta) = \mathbb{E} \log \left( \text{Tr}(e^{\beta M^x}) / 2^n \right)
\]  

be the expected free energy of \( M^x \) in (3.13). Our main result for the SYK model follows.

**Corollary 3.3.** For all \( n, q \geq 1 \) with \( q \) even and any \( \beta_1, \beta_2 \geq 0 \),

\[
F^{\text{SYK}}_{n, q} \left( \sqrt{\beta_1^2 + \beta_2^2} \right) \leq F^{\text{SYK}}_{n, q} (\beta_1) + F^{\text{SYK}}_{n, q} (\beta_2).
\]

The proof of Corollary 3.3 is of course based on Theorem 3.1. Let \( G^{\text{SYK}}_n = \{\chi_S\}_{S \subseteq [n]} \) consist of all \( 2^n \) monomials written in sorted order, which is easily seen to be a basis for \( C_n \). \( G^{\text{SYK}}_n \) is not a group, but \( G^{\text{SYK}}_n = G^{\text{SYK}}_n \cup (-G^{\text{SYK}}_n) \) is a group as shown in Proposition 3.6 below. We will apply Theorem 3.1 with \( G = G^{\text{SYK}}_n \). Corollary 3.3 thus reduces to the following two claims.

**Lemma 3.4.** The law of \( M^x \) is \( G^{\text{SYK}}_n \)-invariant.

**Lemma 3.5.** \( G^{\text{SYK}}_n \) is \( M^x \)-symmetrizing for all \( x \in C_n \).

We begin with an easy preliminary result. For any permutation \( \pi \in S_n \) and \( S = \{s_1, \ldots, s_k\} \subseteq [n] \) with \( 1 \leq s_1 < s_2 < \cdots < s_k \leq n \), define

\[
\chi^\pi_S = \chi_{\pi(s_1)} \chi_{\pi(s_2)} \cdots \chi_{\pi(s_k)}.
\]

**Proposition 3.6.** For any \( \pi \in S_n \), the set of \( 2^{n+1} \) monomials \( \{ \pm \chi^\pi_S \}_{S \subseteq [n]} \) coincides with the multiplicative subgroup \( G^{\pi \text{SYK}}_n \subseteq C_n \), and is in particular independent of \( \pi \).

**Proof.** First we show that \( G^{\text{SYK}}_n \) is a group. It is easy to see by using the relations (3.11) to reorder terms that any product of (possibly non-distinct) generators \( \chi_{i_1} \cdots \chi_{i_k} \) takes the form \( \pm \chi_S \) for some \( S \subseteq [n] \), so that \( G^{\text{SYK}}_n \) is closed under multiplication. Similar reasoning reveals that \( \chi_S^\pi \in \{ \chi_S, -\chi_S \} \) for all \( S \subseteq [n] \), so that \( G^{\text{SYK}}_n \) is closed under inverse and is hence a group.

Next, let \( G^{\text{SYK}}_n \subseteq C_n \) denote the set \( \{ \pm \chi^\pi_S \}_{S \subseteq [n]} \). We claim that \( G^{\text{SYK}}_n = G^{\pi \text{SYK}}_n \) if \( \pi' \) is obtained from \( \pi \) by an adjacent transposition, which implies that \( G^{\text{SYK}}_n \) does not depend on the permutation \( \pi \). This in turn follows from the equality of sets \( \{ \chi_{i_j} \chi_j, -\chi_{i_j} \chi_j \} = \{ \chi_j \chi_{i_j}, -\chi_j \chi_{i_j} \} \) for distinct \( i, j \in [n] \), completing the proof. \( \square \)

**Proof of Lemma 3.4.** Fix any monomial \( y = \pm \chi_{j_1} \cdots \chi_{j_m} \) and let

\[
x = i^{q/2} \sum_{1 \leq i_1 < \cdots < i_q \leq n} J_{i_1, \ldots, i_q} \chi_1 \cdots \chi_q
\]

be as in (3.13). Note that \( y^{-1} \in \{y, -y\} \). It follows from the relations (3.11) that

\[
y y^{-1} = i^{q/2} \sum_{1 \leq i_1 < \cdots < i_q \leq n} \varepsilon_{i_1, \ldots, i_q} (y) J_{i_1, \ldots, i_q} \chi_1 \cdots \chi_q
\]

for deterministic constants \( \varepsilon_{i_1, \ldots, i_q} (y) \in \{-1, 1\} \). Since the couplings \( J_{i_1, \ldots, i_q} \) are i.i.d. centered Gaussian, we find that \( x \) and \( y y^{-1} \) have the same law. Since \( x \mapsto M_x \) defines a representation of \( C_n \), \( M^x \) and \( M^y M^{-1}(M^y)^{-1} \) also have the same law. This completes the proof since \( y \in G^{\text{SYK}}_n \) was arbitrary. \( \square \)
Proof of Lemma 3.5. We show that for any linear map \( M : C_n \to C_n \),

\[
\mathbb{E}^{g \in G_{\text{SYK}}} [g M g^{-1}] = \mathbb{E}^{g \in G_{\text{SYK}}} [g M g^{-1}] = \frac{\text{Tr}(M) \cdot I_{2^n}}{2^n}.
\]  \hfill (3.14)

The first equality is clear because \( g M g^{-1} = (-g)M(-g)^{-1} \) for all \( g \in G_{\text{SYK}} \), so we focus on the latter equality. Recall that \( G_{\text{SYK}} \) is a basis for \( C_n \). In this basis, it is easy to see that each \( M^g \) for \( g \in G_{\text{SYK}} \) acts by a signed permutation matrix which is the identity when \( g = 1 \), and otherwise has all diagonal entries 0 hence trace 0. Since (3.14) is clear when \( M \) is the identity, it suffices to establish for all non-identity elements \( x \in G_{\text{SYK}} \) the equality

\[
\sum_{g \in G_{\text{SYK}}} g M^g g^{-1} = 0.
\]  \hfill (3.15)

From the \( S_n \) symmetry guaranteed by Proposition 3.6, we may without loss of generality assume that \( x \) contains the term \( \chi_n \), so that \( x = \chi_{i_1} \cdots \chi_{i_{q-1}} \chi_n \) for \( 1 \leq i_1 < \cdots < i_{q-1} < n \). Next, let \( G_{\text{SYK}}(1) \subseteq G_{\text{SYK}} \) consist of the monomials with no \( \chi_n \) term. Since \( G_{\text{SYK}} = G_{\text{SYK}} + G_{\text{SYK}}(1) \), we find

\[
\sum_{g \in G_{\text{SYK}}} g M^g g^{-1} = \sum_{g \in G_{\text{SYK}}(1)} g M^g g^{-1} + \sum_{g \in G_{\text{SYK}}(1) \subseteq G_{\text{SYK}}} g (M + \chi_n M^g \chi_n) g^{-1} = 0.
\]

In the last step we used the identity \( \chi_n M^g \chi_n = -M^g \) for \( x = \chi_{i_1} \cdots \chi_{i_{q-1}} \chi_n \). This holds because \( q \) is even and \( \chi_n \chi_{i_k} = -\chi_{i_k} \chi_n \) for all \( 1 \leq k \leq q - 1 \). \hfill \( \square \)

Remark 3.7. In the spirit of (2.3), Corollary 3.3 extends to more general formulations of the SYK model. Let \( [Q]_2 = [Q] \cap 2\mathbb{N} \) be the set of positive even integers at most \( Q \) and consider

\[
x = \sum_{q \in [Q]_2} \frac{q}{2} \sum_{1 \leq i_1 < \cdots < i_q \leq n} J_{i_1, \ldots, i_q} \chi_1 \cdots \chi_q \hfill (3.16)
\]

for independent centered Gaussians \( J_{i_1, \ldots, i_q} \) with arbitrary variances \( \bar{c}_i = (c_{i_1, \ldots, i_q})_{i_1, \ldots, i_q} \in [n]^q \subseteq [Q]_2 \). The analog of Corollary 2.4 generalizes to this setting, as indeed Corollary 3.3’s proof extends essentially verbatim.

Further, Corollaries 3.2 and 3.3 admit a common generalization. This is not surprising if one observes that the algebra of linear endomorphisms \( \text{End}(\mathbb{C}^2, \mathbb{C}^2) \) is isomorphic to \( C_2 \). Indeed it follows from the relations (3.6) that an isomorphism is given by extending the map

\[
I_2 \mapsto 1, \quad \sigma^x \mapsto \chi_1, \quad \sigma^y \mapsto i \cdot \chi_1 \chi_2, \quad \sigma^z \mapsto \chi_2.
\]

Therefore the quantum SK Hamiltonians considered previously live in \( C_2^\otimes m \) while SYK Hamiltonians live in \( C_n \). In general, for any positive integers \( m \) and \( (n_1, \ldots, n_m) \), we may consider the algebra \( \bigotimes_{j=1}^m C_{n_j} \) with \( (\chi^j_i)_{i \in [n_j]} \) the generators of \( C_{n_j} \). Then Theorem 3.1 implies analogous free energy subadditivity for the random operators on \( \bigotimes_{j=1}^m C_{n_j} \) given by

\[
M = \sum_{( q_1, \ldots, q_m ) \in [Q]_2^m} \frac{q_j}{2} \sum_{1 \leq i_1 < \cdots < i_{q_j} \leq n_j} J_{i_1, \ldots, i_{q_j}} \prod_{j=1}^m (\chi^j_{i_1} \cdots \chi^j_{i_{q_j}})
\]

where \( J_{ij} \sim \mathcal{N}(0, c_{ij}) \) are independent centered Gaussians with arbitrary variances. The proof is identical to that of Corollary 3.2, with Lemma 3.5 used in the last step on each \( C_{n_j} \) in place of (3.10).

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A Suboptimal Alternate Approach for Mixed $p$-spin Models

Recall from Subsection 2.2 that for a deterministic sequence $(c_2, \ldots, c_P) \in \mathbb{R}_+^{P}$, the corresponding mixed $p$-spin Hamiltonian without external field is given by

$$H_N(\sigma) = \sum_{p=2}^{P} \frac{1}{N^{(p-1)/2}} \sum_{1 \leq i_1, \ldots, i_p \leq N} J_{i_1, \ldots, i_p} \sigma_{i_1} \ldots \sigma_{i_p}$$

for independent Gaussians $J_{i_1, \ldots, i_p} \sim \mathcal{N}(0, c_p)$ with variance $c_p$. The celebrated Parisi formula gives the limiting free energy in this model for the reference measure $\mu$ which is uniform on $\sigma \in \{-1, 1\}^N$.

Surprisingly, we were unable to recover the free energy subadditivity in

$$\xi(x) = \sum_{p=2}^{P} c_p x^p$$

(A.1)
directly from the Parisi formula. In this appendix we use it to show the weaker Corollary A.3 in which free energy is defined relative to counting measure on $\{-1, 1\}^N$ instead of uniform measure. Counting measure is actually more commonly used to define free energy in Ising spin glasses since the corresponding partition function is just the sum $\sum_{\sigma \in \{-1, 1\}^N} H_N(\sigma)$. (Indeed this discrepancy led to some confusion on our part, which motivated us to include this appendix.) In Remark A.4 we discuss why our proof strategy seemingly cannot recover the stronger estimate of Theorem 1.1. It would be interesting to derive free energy subadditivity relative to uniform measure by using the Parisi formula in a different way.

Let $\mathcal{M}_{[0,1]}$ denote the space of increasing and right-continuous functions $\zeta : [0, 1] \rightarrow [0, 1]$. To state the Parisi formula at inverse temperature $\beta = 1$ (without loss of generality since $\beta$ can be absorbed into $\xi$), we define for $\zeta \in \mathcal{M}_{[0,1]}$ the function $\Phi_{\xi, \zeta} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}_{+}$ as the solution to the non-linear PDE:

$$\partial_t \Phi_{\xi, \zeta}(t, x) + \frac{1}{2} \xi''(t) \left( \partial_x \Phi_{\xi, \zeta}(t, x) + \zeta(t) (\partial_x \Phi_{\xi, \zeta}(t, x))^2 \right) = 0$$

(A.2)

$$\Phi_{\xi, \zeta}(1, x) = \log \cosh(x).$$

Existence and uniqueness of solutions are shown in [5, 54] (in fact we will use the formula from [5, Theorem 1] below, with the term log 2 omitted for the main statement with uniform measure). The Parisi functional for the Ising mixed $p$-spin model is defined by

$$P_\xi(\zeta) = \Phi_{\xi, \zeta}(0, 0) - \frac{1}{2} \int_0^1 t \xi''(t) \zeta(t) dt.$$

(A.3)

Finally the Parisi formula [71, 47, 81, 65, 67, 66] states that

$$F(\xi) \equiv \lim_{N \rightarrow \infty} F_N(\xi)/N = \inf_{\zeta \in \mathcal{M}_{[0,1]}} P_\xi(\zeta).$$

We will use the more convenient Auffinger-Chen representation for the Parisi functional which we now describe. Given a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, 1]}$ and adapted standard Brownian $B(t)$, let $\mathcal{D}_{[0,1]}$ denote the space of progressively measurable processes $(u_t)_{t \in [0, 1]}$ such that $|u_t| \leq 1$ holds almost surely for each $t \in [0, 1]$. Define

$$\mathcal{X}_{\xi, \zeta}(u; B) = \mathcal{Y}_{\xi, \zeta}(u; B) - \mathcal{Z}_{\xi, \zeta}(u);$$

$$\mathcal{Y}_{\xi, \zeta}(u; B) \equiv \Phi_{\xi, \zeta} \left( 1, \int_0^1 \zeta(t) \xi''(t) u(t) dt + \int_0^1 \sqrt{\xi''(t)} dB(t) \right),$$

$$\mathcal{Z}_{\xi, \zeta}(u) \equiv \frac{1}{2} \int_0^1 \zeta(t) \xi''(t) u(t)^2 dt.$$

Auffinger-Chen showed the Parisi functional can be represented as a stochastic control problem involving $\mathcal{X}_{\xi, \zeta}$.

**Proposition A.1.** [5, Theorem 3] The function $\Phi_{\xi, \zeta}$ defined in (A.2) satisfies

$$\Phi_{\xi, \zeta}(0, 0) = \max_{u \in \mathcal{D}_{[0,1]}} \mathbb{E}[\mathcal{X}_{\xi, \zeta}(u; B)].$$

We note that the statement in [5] assumes $\mathcal{F}$ is the filtration generated by $B(t)$, but this is not necessary.

---

1Corollary A.3 extends easily to handle an external field, but we omit this for convenience.
Proposition A.2. For mixed p-spin covariances $\xi_1, \xi_2$ as in (A.1) and $\zeta_1, \zeta_2 \in \mathcal{M}_{[0,1]}$, let
\[
\zeta(t) \equiv \frac{\xi_1(t)\xi''_1(t) + \xi_2(t)\xi''_2(t)}{\xi'_1(t) + \xi'_2(t)} \in \mathcal{M}_{[0,1]}.
\] (A.4)

Then we have the inequality
\[
\Phi_{\xi_1+\xi_2,\zeta}(0,0) \leq \Phi_{\xi_1,\zeta_1}(0,0) + \Phi_{\xi_2,\zeta_2}(0,0) + \log(2).
\]

Proof. Let $B_1(t)$ and $B_2(t)$ be independent standard Brownian motions generating together the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$. Define $\zeta$ as in (A.4) and set
\[
B(t) = \sqrt{\xi_1(t) + \xi_2(t)}B_1(t) + \sqrt{\xi_2(t)}B_2(t).
\]

Note that $B(t)$ is also a standard Brownian motion. We choose $u \in \mathcal{D}[0,1]$ to maximize $X_{\xi_1+\xi_2,\zeta}(u; B)$. Proposition A.1 implies that for $i \in \{1, 2\}$, we have
\[
\mathbb{E}[X_{\xi_i,\zeta_i}(u; B_i)] \leq \Phi_{\xi_i,\zeta_i}(0,0).
\] (A.5)

Therefore it suffices to prove that almost surely,
\[
X_{\xi_1+\xi_2,\zeta}(u; B) \leq X_{\xi_1,\zeta_1}(u; B_1) + X_{\xi_2,\zeta_2}(u; B_2) + \log(2).
\] (A.6)

We first have
\[
Z_{\xi_1+\xi_2,\zeta}(u) = Z_{\xi_1,\zeta_1}(u) + Z_{\xi_2,\zeta_2}(u)
\]
yielding the inequality
\[
Y_{\xi_1+\xi_2,\zeta}(u; B) \leq Y_{\xi_1,\zeta_1}(u; B_1) + Y_{\xi_2,\zeta_2}(u; B_2) + \log(2)
\]
follows from the identity
\[
\int_0^1 \zeta(t)(\xi''_1(t) + \xi''_2(t))u(t)dt + \int_0^1 \sqrt{\xi''_1(t) + \xi''_2(t)}dB(t) = \int_0^1 \zeta(t)\xi''_1(t)u(t)dt + \int_0^1 \sqrt{\xi''_1(t)}dB_1(t)
\]
\[
+ \int_0^1 \zeta(t)\xi''_2(t)u(t)dt + \int_0^1 \sqrt{\xi''_2(t)}dB_2(t)
\]
and the easily verified subadditivity of the function $2\cosh(x)$. This concludes the proof. \qed

Corollary A.3. The free energy in the mixed p-spin model satisfies $F(\xi_1 + \xi_2) \leq F(\xi_1) + F(\xi_2) + \log(2)$.

Proof. Choose $\zeta_1, \zeta_2 \in \mathcal{M}_{[0,1]}$ to minimize the respective Parisi functionals, i.e.
\[
F(\xi_i) = P_{\xi_i}(\zeta_i), \quad i \in \{1, 2\}.
\]

Then with $\zeta$ as in (A.4), we have
\[
F(\xi_1 + \xi_2) \leq P_{\xi_1+\xi_2}(\zeta)
\]
\[
= \Phi_{\xi_1+\xi_2,\zeta}(0,0) - \frac{1}{2} \int_0^1 t(\xi''_1(t) + \xi''_2(t))\zeta(t)dt
\]
\[
\leq \Phi_{\xi_1,\zeta_1}(0,0) + \Phi_{\xi_2,\zeta_2}(0,0) + \log(2) - \frac{1}{2} \int_0^1 t(\xi''_1(t)\zeta_1(t) + \xi''_2(t)\zeta_2(t))dt
\]
\[
= P_{\xi_1}(\zeta_1) + P_{\xi_2}(\zeta_2) + \log(2)
\]
\[
= F(\xi_1) + F(\xi_2) + \log(2).
\]

Remark A.4. There is a good reason that our argument above cannot recover the stronger estimate of Theorem 1.1. Since the process $u$ used in proving Proposition A.2 is adapted to the filtration generated by $B(t)$, it cannot agree with any nontrivial process adapted to the filtrations generated by $B_1(t)$ or $B_2(t)$. This means that (A.5) and hence Proposition A.2 essentially never hold with equality. By contrast the bound in Corollary 2.1 does hold with equality at high temperature.

Let us also mention that our proof of Proposition A.2 has a similar spirit to that of [5, Theorem 4], which used Proposition A.1 to establish the strict convexity of the Parisi functional in $\zeta$. In fact the obstruction just outlined resembles their proof that the convexity is strict.
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