On the uniqueness of Barrett’s solution to the fermion doubling problem in Noncommutative Geometry

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Abstract

A solution of the so-called fermion doubling problem in Connes’ Noncommutative Standard Model has been given by Barrett in 2006 in the form of Majorana-Weyl conditions on the fermionic field. These conditions define a $U_{\lambda,\chi}$-invariant subspace of the correct physical dimension, where $U_{\lambda,\chi}$ is the group of Krein unitaries commuting with the chirality and real structure. They require the KO-dimension of the total triple to be 0. In this paper we show that this solution is, up to some trivial modifications, and under some mild assumptions on the finite triple, the only one with this invariance property. We also observe that a simple modification of the fermionic action can act as a substitute for the explicit projection on the physical subspace.

1 Introduction

What are the advantages of the Noncommutative Geometry approach to particle physics with respect to the traditional approach? To answer this question in the most concise way, there are two:

1. Some features which are added by hand in the usual formulation of the Standard Model (SM) pop up naturally in the Noncommutative Standard Model (NCSM): these include the Higgs field and the neutrino mixing term [1].

2. The NCSM is much more constrained.

The constraints alluded to above come in particular from the fact that in the SM one starts with a Lie group, while in the NCSM one starts with a finite-dimensional $*$-algebra. By the Artin-Wedderburn theorem, such an algebra is a direct sum of matrix algebras over the reals, complex numbers or quaternions. There is a much wider selection of simple Lie groups: already at the level of

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1But not only: there are also constraints on the Dirac operator about which we will not talk in this paper. See for instance [2], [3].
the Lie algebra there are four infinite families and five exceptional cases, and one must also take into account the topology of the group. Moreover, and perhaps more importantly, the representation theory of \( \ast \)-algebras is much simpler than that of groups: there is only one irreducible representation of \( M_n(K) \), \( K = \mathbb{R}, \mathbb{C}, \mathbb{H} \), namely the regular one. However, the NCSM also has problems, namely:

1. The unimodularity condition.
2. The definition of the spectral action in the Lorentzian setting.
3. The quadrupling of the fermionic degrees of freedom, hereafter called “fermion doubling” for historical reasons.

The first problem is maybe the most pressing one, but we won’t deal with it in this paper. The second problem can be by-passed: the spectral action can be replaced by the older Connes-Lott action in the Lorentzian setting without trouble [4]. The third problem is the main subject of this work. It has been first stressed in [5]. Its technical origin is the following. The spectral triple of the NCSM is a so-called “almost-commutative manifold”, i.e. it is the tensor product of the spectral triple of the spacetime manifold and a finite spectral triple. The latter includes a finite-dimensional space \( K_F \), a basis of which is \((p,\sigma)\) where \( p \) runs over all elementary particles and \( \sigma \) is a symbol among \( R, L, \bar{R}, \bar{L} \). For instance \( e_L \) is interpreted as an anti-left electron. However, the particle/antiparticle type and parity are already present in the spinor space \( S_x \) at a point of the manifold. The space \( S_x \otimes K_F \) pertaining to the almost-commutative spectral triple of the NCSM thus has four times the expected dimension. It should be noted that the presence of the symbol \( \sigma \) in the first place is necessary in order to have the correct representation of the gauge group on fermions. To solve this problem, one looks for a subspace \( H_x \) of \( S_x \otimes K_F \) of the correct dimension, calls it the “physical subspace” and declares fermion fields to have values in this subspace (a kind of superselection rule). Clearly, the physical subspace must also be invariant under the gauge group.

In 2006, Barrett noticed that a physical subspace could be defined by the conditions

\[
J\Psi = \Psi, \quad \chi\Psi = \Psi
\]

where \( J \) is the real structure and \( \chi \) the chirality of the total triple, henceforth called Majorana-Weyl or Barrett’s conditions. An interesting bonus is that these conditions are consistent only if \( J^2 = 1 \) and \( J\chi = \chi J \). They thus require the total KO-dimension of triple to be 0. In \( 1+3 \) spacetime dimension, this is equivalent to \( J_F^2 = -1 \), where \( J_F \) is the real structure of the finite triple. By what seems to be a happy coincidence, precisely this sign allows for a neutrino mixing term in the action (i.e. the neutrino mixing term must vanish if \( J_F^2 = 1 \)) (for all of this, see [6] and [1]).

Since each one of the conditions [1] divides the dimension by a factor of two, we see that they indeed solve the fermion “doubling” problem. Moreover, the space defined by these conditions is not only invariant by the gauge group of the Standard Model but by the larger group \( U_{J,\chi} \) of Krein unitaries commuting with \( \chi \) and \( J \).
Modifications of Barrett’s conditions immediately come to mind. Namely, one can introduce a phase and a sign, and require

\[ J\psi = e^{i\phi}\psi, \chi\psi = \pm\psi \]  

instead of (1). Now, one can wonder about the uniqueness of Barrett’s solution in KO-dimension 0, up to these trivial modifications, as well as the existence of any solution at all in other KO-dimensions. To our knowledge these questions have never been considered in the literature. In this paper we will show that, with the assumption of invariance under \( U_{J,\chi} \), there exists no other solution than (2), and in particular the KO-dimension has to be 0. We also remark that uniqueness is lost if one only assumes gauge invariance, though we will not fully investigate this much more complex case. Barrett’s solution thus seems to be a satisfactory and almost unique solution to the fermion doubling problem, assuming a very natural invariance. Nevertheless, conditions such as (1) may appear as a feature added by hand to the beautiful construct of Noncommutative Geometry. However, we will observe that instead of projecting the fields on the physical subspace, the resolution of the fermion doubling problem can be achieved by changing the usual fermionic action

\[ S(A, \Psi) = (\Psi, D_A \Psi) \]  

to

\[ S'(A, \Psi) = \frac{1}{8} \left[ (\Psi, D_A \Psi) + (D_A \Psi, \Psi) + (J\Psi, D_A \Psi) + (D_A \Psi, J\Psi) + (\chi\Psi, D_A \Psi) + (D_A \Psi, \chi\Psi) + (\chi J\Psi, D_A \Psi) + (\chi J\Psi, D_A \Psi) \right] \]  

(3)

Using \( S' \) is equivalent to projecting fermionic fields on the physical subspace and then use the traditional \( S \). The action \( S' \) may seem even more natural than \( S \), since it involves all the possible combinations of the background objects with which we can build the action, with only the simplest coefficients. With this point of view, the fermionic fields orthogonal to the physical subspace still exist, but do not participate in any interaction.

In this paper we will assume that the spacetime manifold is four-dimensional with metric signature (1, 3). In this setting a Connes-Lott type noncommutative gauge theory reproducing exactly the bosonic and fermionic actions of the SM can be constructed thanks to a certain finite indefinite triple \([4]\). This triple belongs to a family described in section 2 of this paper, which has 4 free parameters: the signs \( \epsilon_F \) and \( \kappa_F \), and the components \( \eta_R, \eta_L \) of the internal metric. We will show in section 3 that a \( U_{J,\chi} \)-invariant physical subspace exists iff \( \epsilon_F = -1 \), and that in this case it is given by (2). The parameter \( \kappa_F \) is then determined by the requirement to recover a non-trivial fermionic action, as we recall in section 4. In section 5, we will conclude the paper by observing that in the case singled out by the previous considerations, the projection on the physical subspace can be replaced with the use of the action \( S' \).

We stress that in order to write down precise mathematical statements, we are forced to set ourselves in a specific framework with respect to the definition of indefinite spectral triples, while it is a still evolving subject. We will use the one proposed in [7], which will be reviewed in section 2. However, our results are largely independent from the details of the formulation.

2 General setting

We will use the general setting of algebraic backgrounds, proposed in [7]. However, no familiarity with this notion is required, since we recall everything we
A pre-Krein space is a complex vector space $K$ equipped with a non-degenerate indefinite metric $(\cdot,\cdot)$, which is decomposable into the direct sum $K = K^- \oplus K^+$ of a positive and negative definite subspaces. Giving any such decomposition is equivalent to giving a fundamental symmetry $\eta$ which satisfies $\eta^\times = \eta$ (where $\times$ is the adjoint with respect to $(\cdot,\cdot)$), $\eta^2 = 1$, and such that $(\cdot,\cdot)_\eta := (\cdot,\eta \cdot)$ is positive definite. It is said to be $\mathbb{Z}_2$-graded and real if there exists a linear operator $\chi$ (chirality) and an antilinear operator $J$ (graded real structure) such that

$$\chi^2 = 1, \quad J^2 = \epsilon, \quad J \chi = \epsilon'' \chi J, \quad J^\times = \kappa J, \quad \chi^\times = \epsilon'' \kappa'' \chi$$

(4)

where $\epsilon, \kappa, \epsilon'', \kappa''$ are signs ("KO-metric signs"). A fundamental symmetry $\eta$ is said to be compatible with $\chi$ and $\pi$ iff $\chi \eta = \epsilon'' \kappa'' \eta \chi$ and $J \eta = \epsilon \kappa \eta J$

(5)

The pre-Krein space $K$ can be decomposed into even and odd subspaces, $K = K_0 \oplus K_1$, which are the eigenspaces of $\chi$. An operator $A$ which commutes with $\chi$ will respect this decomposition and will be called even. If $A$ anticommutes with $\chi$ it will exchange $K_0$ and $K_1$ and be called odd. Note that if $\epsilon'' \kappa'' = 1$ then $\chi^\times = \chi$ and this implies that $K_0$ and $K_1$ are orthogonal with respect to $(\cdot,\cdot)$. In this case we will say that the Krein product is even. On the contrary if $\epsilon'' \kappa'' = -1$, $K_0$ and $K_1$ are self-orthogonal ($K_i = K_i^\perp$) and we say that the Krein product is odd. These considerations will be of particular importance when we turn to tensor products.

We recall that $\epsilon, \epsilon''$ are given in terms of $n$, an integer modulo 8 called the KO-dimension, by the formulas $\epsilon = (-1)^{(n+2)/8}$, $\epsilon'' = (-1)^{n/2}$, while $\kappa = (-1)^{m(m+2)/8}$, $\kappa'' = (-1)^{m/2}$, where $m$ is another integer modulo 8 called the metric dimension (for more details see [3]). For convenience the values of the signs $\epsilon, \epsilon'', \kappa, \kappa''$ in terms of $m, n$ are gathered in table 1.

| $m, n$ | 0 | 2 | 4 | 6 |
|-------|---|---|---|---|
| $\kappa, \epsilon$ | 1 | -1 | -1 | 1 |
| $\kappa'', \epsilon''$ | 1 | -1 | 1 | -1 |

Table 1: Signs $\epsilon, \epsilon'', \kappa, \kappa''$ in terms of $m, n$.

**Definition 1** An algebraic background $(AB)$ is a tuple $B = (A, K, (\cdot,\cdot), \pi, \chi, J, \Omega^1)$ where:

1. $(K, (\cdot,\cdot), \chi, J)$ is a $\mathbb{Z}_2$-graded real pre-Krein space,

2. $A$ is a $*$-algebra and $\pi$ is a faithful $*$-representation of it by even operators on $K$,

3. the “bimodule of 1-forms” $\Omega^1$ is an $A$-bimodule of odd operators on $K$.

**Remark** In the general case, $A$ is not required to be a $*$-algebra, as explained in [7]. However, since it will always be the case in this paper, we prefer to put it in the axioms. There are also some boundedness conditions on $\pi(A)$ and $\Omega^1$, but we do not need to enter into these details.
Given $B$, one can define its configuration space $\mathcal{D}_B := \{D \in \text{End}(\mathcal{H})|D^\dagger = D, \chi D = -D\chi, JD = DJ, \text{ and } \forall a \in A, [D, \pi(a)] \in \Omega^1\}$. The elements of $\mathcal{D}_B$ are called the compatible Dirac operators. The group of automorphisms of $B$ is $\text{Aut}(B) := \{U \in \text{End}(\mathcal{H})|UU^\dagger = 1, U J = J U, U\chi = \chi U, U\Omega^1 U^{-1} = \Omega^1, U\pi(A)U^{-1} = \pi(A)\}$. Of crucial importance for us is the definition of the tensor product.

**Definition 2** Let $B_1 = (A_1, K_1, D_1, J_1, \chi_1, \Omega_1^1)$ and $B_2 = (A_2, K_2, D_2, J_2, \chi_2, \Omega_2^1)$ be two ABs. Then the tensor product algebraic background $\mathcal{B} = B_1 \otimes B_2$ is defined by:

- $A = A_1 \otimes A_2$ with involution $(a_1 \otimes a_2)^\dagger = a_1^\dagger \otimes a_2^\dagger$ and representation $\pi(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$.
- $K = K_1 \otimes K_2$ tensor product with indefinite product

\[
(\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2) = (\phi_1, \psi_1)(\phi_2, \beta \psi_2)_2
\]

where $\beta = 1$ if $(.,.)_1$ is even, $\chi_2$ if $(.,.)_1$ is odd and $(.,.)_2$ is even, $i\chi_2$ if $(.,.)_1$ and $(.,.)_2$ are both odd.

- $\chi = \chi_1 \otimes \chi_2$,
- $J = J_1 \chi_1^{[J_1]} \otimes J_2 \chi_2^{[J_2]} = J_1 \otimes J_2 \chi_2^{[J_1]}$,
- $\Omega^1 = \Omega_1^1 \otimes \pi(A_2) + \pi(A_1) \otimes \Omega_2^1$.

We stress that in the only case we will be considering, one of the two Krein spaces will be finite-dimensional, so we do not need to be more specific about topological tensor products.

The NCSM in Lorentzian signature is defined thanks to an algebraic background $\mathcal{B}$ which is the tensor product of one coming from a manifold, which we call $B_M$, and a finite one $B_F$. We will now describe these two ABs.

First, the manifold $M$ is a four-dimensional open anti-Lorentzian manifold (West-Coast convention). We suppose that there exists a global tetrad, i.e. a pseudo-orthonormal frame $e = (e_0, \ldots, e_3)$, which is the condition for $M$ to admit a spin structure [5]. Such a spin structure is defined by the trivial spinor bundle $S = M \times S$, with $S = \mathbb{C}^4$, and the choice of the following gamma matrices:

\[
\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, k = 1, 2, 3
\]

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

This choice of $\gamma$ matrices permits to identify $\text{End}(S)$ with the complex Clifford algebra $\mathbb{C}l(1,3)$, and the map $\rho: e_\mu \mapsto \gamma_\mu$ gives an irreducible representation of $\mathbb{C}l(T_x M)$ on $S$. The spinor space $S$ carries a natural Krein product defined by

\[
(\psi, \phi)_S = \psi^\dagger \gamma_0 \phi
\]

The gamma matrices are all self-adjoint with respect to this product, and this property characterizes it up to a non-zero real factor [9]. For any vector $v \in
$T_xM$, the hermitian form $(.,\rho(v)\cdot)$ is positive definite iff $v$ lies in one half of the timelike cone, which we define to be the future cone. This defines a time orientation on $M$.

We still need to define the “local” chirality and real structure $\chi_S$ and $J_S$. The chirality operator $\chi_S$ is none other than the matrix $\gamma^5 = i\gamma^0 \cdots \gamma^3$ which in the chosen (chiral) representation is diag$[J_2,-I_2]$. It satisfies $\chi_S^* = -\chi_S$. The real structure $J_S$ is $\psi \mapsto \gamma_2 \bar{\psi}$, where $\bar{\psi}$ is the complex conjugate of $\psi$ in the chosen basis. One can easily check that $J_S$ anticommutes with gamma matrices and satisfies $J_S^2 = 1$, $J_S^* = -1$. The collection of objects $(S,\rho,\chi_S,H_S,J_S)$ is the algebraic way to define a spin structure on $M$ (see [7] for details).

The neutral component of the spin group Spin$(1,3)^0$ contains by definition the elements $g$ such that:

1. $gg^* = 1$,
2. $J_S g = g J_S$,
3. $\chi_S g = g \chi_S$,
4. $g \Omega^1_S g^{-1} = \Omega^1_S$, where $\Omega^1_S$ is the vector space generated by the gamma matrices.

Now it will turn out to be useful to notice that the last condition is automatically satisfied given the first three in dimension 4.

The AB $B_M$ has algebra $A_M = \check{C}^\infty(M)_c$, which is generated by the constants and the smooth compactly supported real functions. The pre-Krein space $K_M$ consists of smooth spinor fields with compact support equipped with the indefinite product:

$$\langle \phi, \psi \rangle = \int_M (\psi(x), \phi(x))_S dx^0 \cdots dx^3 \quad (9)$$

The chirality $\chi_M$ and charge conjugation $J_M$ are constant and defined by $(\chi_M \Psi)(x) = \chi_S \Psi(x)$, $(J_M \Psi)(x) = J_S \Psi(x)$. We won’t use the bimodules of 1-forms in this paper. For details see [7].

Let us now describe the finite AB. The finite space $K_F$ is identified with $K_0 \otimes I$ where $K_0$ is the vector space generated over $\mathbb{C}$ by all the different fermion species and $I$ is the vector space generated by the four symbols $R,L,\bar{R},\bar{L}$. The space $K_0$ is equipped with a preferred basis given by the symbols of the elementary fermion species (including generations and color: this is a 24-dimensional space). This defines a canonical scalar product for which this basis is orthonormal. Similarly $I$ has preferred basis $(R,\ldots,\bar{L})$ and is equipped with the associated scalar product. These canonical scalar products on $K_0$ and $I$ are both written $(.,.)$, with no risk of confusion. The space $K_F$ is also equipped with a Krein product with fundamental symmetry $\eta_F$. The finite algebra $A_F$ is $\mathbb{C} \oplus \mathbb{R} \oplus M_3(C)$, though this specific form will play no role here. More important is its representation: it is block-diagonal, that is:

$$\pi_F(a)(\phi \otimes \sigma) = \pi_\sigma(a)\phi \otimes \sigma, \forall \phi \in K_0, \sigma = R, L, \bar{R}, \bar{L} \quad (10)$$

The precise definition of $\pi_F$ can be found, with the same notation as we use, in [7] or [10] for instance. It will play no role in this paper. We assume that:

1. $\eta_F = 1 \otimes \eta_l$ with $\eta_l = \text{diag} [\eta_R, \eta_L, \bar{\eta_R}, \bar{\eta_L}]$ a fundamental symmetry on $I,$
2. \( J_F(\phi \otimes \sigma) = \tilde{\phi} \otimes \tilde{\sigma} \), for all \( \phi \in \mathcal{K}_0 \) and \( \sigma = R, L, \tilde{R}, \tilde{L} \), with the convention that \( \tilde{\sigma} = \epsilon_F \sigma \).

**Remark** The first hypothesis ensures that \( \pi_F \) is \( * \)-representation. The second one is necessary in order to recover the correct representation of the gauge group.

From [1] and [2] we easily obtain \( \epsilon_F' = \kappa_F' = -1 \).

It will be useful to have \( J_F \) written in matrix form in the basis \( (R, \tilde{L}, \tilde{R}, L) \). It is:

\[
J_F = 1 \otimes \begin{pmatrix}
0 & 0 & \epsilon_F & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & \epsilon_F & 0 & 1
\end{pmatrix} \circ c.c.
\]

(11)

Using this, we immediately see that in order to have \( J_F \eta_F = \epsilon_F \kappa_F \eta_F J_F \), we must have \( \eta_\tilde{R} = \epsilon_F \kappa_F \eta_R \) and \( \eta_\tilde{L} = \epsilon_F \kappa_F \eta_L \). Thus, to sum up the situation, hypotheses [1] and [2] imply that

\[
\epsilon_F' = \kappa_F' = -1, \quad \eta_R = \epsilon_F \kappa_F \eta_R, \quad \eta_L = \epsilon_F \kappa_F \eta_L
\]

(12)

The Krein product on \( \mathcal{K}_F \) is recovered through:

\[
(\phi \otimes \sigma, \phi' \otimes \sigma')_F = \langle \phi, \phi' \rangle \langle \sigma, \eta_I \sigma' \rangle
\]

(13)

Here again the bimodule of 1-forms will play no role, so we ignore it. Let us now look at \( B = B_M \otimes B_F \), the total AB of the Standard Model. Its algebra is \( \mathcal{A} = \mathcal{C}_c(M, \mathcal{A}_F) \) and its pre-Krein space is \( \mathcal{K} = \mathcal{C}_c(M, S \times \mathcal{K}_F) \). Let us look at the Krein product. It is obtained by integrating a product on \( S \otimes \mathcal{K}_F \), which is, according to [11]:

\[
(\psi \otimes \phi \otimes \sigma, \psi' \otimes \phi' \otimes \sigma') = (\psi, \psi')_S \langle \phi, \phi' \rangle \langle \sigma, \omega \sigma' \rangle
\]

(14)

where \( \omega \) is an effective internal metric given by \( \omega = \chi M \eta_H = \text{diag}[\eta_R, \eta_L, -\eta_R, -\eta_L] \).

The total chirality is

\[
\chi = \chi_M \otimes \chi_F = \chi_M \otimes \text{Id} \otimes \chi_I
\]

(15)

and the real structure is \( J = J_M \chi_M \otimes J_F \chi_F \). In four spacetime dimensions and West-Coast convention, we have \( \epsilon_M = \kappa_M' = 1, \epsilon_M' = \kappa_M = -1 \). Thus \( J = J_M \chi_M \otimes J_F \chi_F \), which can also be written \( J = J_M \otimes J_F \chi_F \). Hence we have

\[
\begin{align*}
J(\Psi \otimes \phi \otimes R) &= J_M \Psi \otimes \phi \otimes \tilde{R} \\
J(\Psi \otimes \phi \otimes \tilde{R}) &= -\epsilon_F J_M \Psi \otimes \phi \otimes R \\
J(\Psi \otimes \phi \otimes L) &= -J_M \Psi \otimes \phi \otimes \tilde{L} \\
J(\Psi \otimes \phi \otimes \tilde{L}) &= \epsilon_F J_M \Psi \otimes \phi \otimes L
\end{align*}
\]

(16)

### 3 Statement and proof of the main result

#### 3.1 Invariance under \( U_{J, \chi} \)

The fermionic action for the NCSM is generally given by

\[
S(A, \Psi) = (\Psi, D_A \Psi)
\]

(17)
where $\Psi$ is an element of $\mathcal{K}$ and $D_A = D + A + JAJ^{-1}$ is a fluctuated Dirac operator. Note that $D_A$ is itself a Dirac operator, with the same commutation relations with $J$ and $\chi$ as $D$, and it can be argued that the true variable of the action is a compatible Dirac operator $D$, belonging to some subspace of $D_B$ which is invariant by $\text{Aut}(\mathcal{B})$ \cite{10}. Note also that in the Euclidean setting the fermionic action can be written $S(D, \Psi) = \langle \Psi, JD\Psi \rangle$ \cite{11}. More generally, it would make sense to suppose the fermionic action to be of the form

$$S(D, \Psi) = \langle \Psi, QD\Psi \rangle$$

where $Q$ is a polynomial in $\chi$ and $J$. Indeed, $\chi$, $J$, as well as the Krein product $(.,.)$ are the only background objects at our disposal, thus \cite{10} is the most general function of the variables $\Psi$ and $D$ that we can write down which is at most of degree 1 in $D$ and quadratic in $\Psi$. Now it is immediate to observe that any action of this form is invariant under the transformation $\Psi \mapsto U\Psi$, $D \mapsto UDU^{-1}$, where $U$ belongs to the group

$$\mathcal{U}_{J,\chi} := \{ U \in \text{End}(\mathcal{K}) | UU^* = 1, U\chi = \chi U, UJ = JU \}$$

Hence we see that the natural invariance group of the fermionic action, is larger than the automorphism group of $\mathcal{B}$. In particular it does not depend on the algebra and bimodule of 1-forms, and will stay the same for models beyond the Standard one which have the same fermion space, such as the $B-L$-extended SM \cite{2} or Pati-Salam \cite{11}.

Now we want to “solve the fermion doubling problem”, that is, to find for each $x \in M$ a real physical subspace $H_x$ of $V := S \otimes \mathcal{K}_F$ which has the correct dimension, i.e. $\dim_{\mathbb{R}}(V)/4$, and to restrict the theory to the space $\mathcal{K}$ of physical fields $x \mapsto \Psi(x) \in H_x$. Of course the space of physical fields has to be invariant under $\text{Aut}(\mathcal{B})$, but in view of the larger group of invariance of the fermionic action, it is very natural to postulate that $\mathcal{K}$ be invariant under $\mathcal{U}_{J,\chi}$ as well.

Let us look at some particular elements of this group.

**Lemma 1** The following operators $U : \Psi \mapsto U\Psi$, $\Psi \in \mathcal{K} = C_{\infty}^\infty(M, S \times \mathcal{K}_F)$ belong to $\mathcal{U}_{J,\chi}$ when $(U\Psi)(x)$ is defined by:

1. $U\Psi(x) = \sqrt{\text{vol}_{\gamma}(x)} \Psi(\theta^{-1}(x))$, where $\theta$ is a diffeomorphism of $M$
2. $U\Psi(x) = (u \otimes \text{Id})\Psi(x)$, where $u \in \text{Spin}(1,3)^0$,
3. $U\Psi(x) = (\text{Id} \otimes u)\Psi(x)$, where $u \in \mathcal{U}_{J,\chi_F}$.

For the proof, see \cite{7}. Using invariance under the elements of the first kind, we see that $H_x$ does not depend on $x$. Hence we want to find a real subspace $H$ of $V = S \otimes \mathcal{K}_F$, of dimension $\frac{1}{4}\dim_{\mathbb{R}}(V)$, which is invariant under the group $G$ of Krein unitary operators on $V$, commuting with $J_x = J_S \otimes J_F \chi_F$ and $\chi_x = \chi_S \otimes \chi_F$. In the following sections we suppress the $x$ altogether, and we write $J, \chi$ instead of $J_x, \chi_x$. With these notations, we are going to prove the following result.

**Theorem 1** There exists a $G$-invariant real subspace $H$ of $V$ of dimension $\dim_{\mathbb{R}}(V)/4$ iff $\epsilon_F = -1$. In that case, there are two families of solutions, each parametrized by $S^1$. They are given by $H = (1 + e^{i\varphi})J(1 \pm \chi)(V)$, $\varphi \in \mathbb{R}$.

Note that the $S^1$-degree of freedom exactly corresponds to the ambiguity in the choice of $J$. 8
3.2 Some preliminaries on Weyl spinors

Recall that, in the chiral basis, the group Spin(1, 3) is represented on \( S \) by the matrices
\[
\begin{pmatrix}
A & 0 \\
0 & \sigma_2 A \sigma_2
\end{pmatrix}
\]
with \( A \in SL_2(\mathbb{C}) \). The spaces of Weyl spinors \( S^+ \) and \( S^- \) contain respectively the vectors \( (\phi, 0) \) and \( (0, \phi) \). They are irreducible \( Spin(1, 3) \)-modules. Over \( \mathbb{C} \) they are the non-isomorphic \((1/2, 0)\) and \((0, 1/2)\)-modules respectively. However, they are isomorphic over \( \mathbb{R} \).

**Lemma 2** The group of real linear automorphisms of the \( Spin(1, 3) \)-modules \( S^\pm \) is \( \mathbb{C}^* \).

**Proof:** Let the map \( K : \mathbb{C}^2 \to \mathbb{C}^2 \) be \( \mathbb{R} \) and \( SL_2(\mathbb{C}) \)-linear. Using the Lie algebra, \( K \) commutes with the traceless matrices
\[
\begin{pmatrix}
0 & i \\
-1 & 0
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]
hence with their product, which is \( i \mathbf{1}_2 \). Thus \( K \) is \( \mathbb{C} \)-linear, and the result follows from Schur’s lemma. \( \square \)

**Lemma 3** The set of \( \mathbb{R} \)-linear \( Spin(1, 3) \)-isomorphisms from \( S^+ \) to \( S^- \) is \( \mathbb{C}^* J_S \).

**Proof:** The real structure \( J_S \) is clearly an isomorphism over \( \mathbb{R} \) between \( S^+ \) and \( S^- \), since \( Spin(1, 3) \) commutes with \( J_S \) by its very definition. The result then follows using the previous lemma. \( \square \)

In the sequel we write \((e_1^+, e_2^+, e_1^-, e_2^-)\) for the canonical (chiral) basis of
\[ S = S^+ \oplus S^- \]

3.3 Proof of the theorem

Writing down the general form of the elements of \( G \) would involve \( 8 \times 8 \) block matrices, and would not turn out to be particularly useful. Instead we will work with special elements. First, \( Spin(1, 3)^0 \otimes 1 \) and \( 1 \otimes U_{J_F, \chi_F} \) are obviously subgroups of \( G \). The general form of elements commuting with \( J_F \) and \( \chi_F \), with the basis order \((R, L, \bar{L}, \bar{R})\) in \( I \) is:
\[
u = \begin{pmatrix}
A & D & 0 & 0 \\
C & B & 0 & 0 \\
0 & 0 & A & \epsilon_F D \\
0 & 0 & \epsilon_F C & B
\end{pmatrix}
\]
(19)
with \( A, B, C, D \in \text{End}(K_0) \). Since \( \eta_F = 1 \otimes [\eta_R, \eta_L, \eta_R, \eta_L] \) in this basis order, \( \nu \) is Krein unitary iff the following conditions are met:
\[
A^\dagger A + \eta_R \eta_L C^\dagger C = 1 \\
A^\dagger D + \eta_R \eta_L C^\dagger B = 0 \\
\eta_R \eta_L D^\dagger D + B^\dagger B = 1
\]
(20)
Note that the 6 conditions boils down to only 3 using \( \eta_R \eta_L = \eta_L \eta_R \).

In the sequel we will consider the subgroup \( G' \) of \( G \) generated by

1. \( Spin(1, 3)^0 \otimes 1 \), and
2. the group $G''$ of endomorphisms $V$ of the form $1 \otimes u$, where $u \in \text{End}(K_F)$ is $[A, B, \tilde{A}, \tilde{B}]$ with $A, B$ unitary matrices of $K_0$, which corresponds to the particular solution $C = D = 0$ of (20).

Let us define the 8 subspaces $H^\pm_\sigma := S^\pm \otimes K_0 \otimes \sigma, \sigma = R, L, \tilde{R}, \tilde{L}$. Note that we consider them as real vector spaces. We clearly have:

$$V = \bigoplus_{\pm, \sigma} H^\pm_\sigma$$

(21)

We call $p^\pm_\sigma$ the projections relative to this decomposition, which are $G'$-linear.

**Lemma 4** The real spaces $H^\pm_\sigma$ are irreducible $G'$-modules.

**Proof:** It is clear that they are stable under Spin(1, 3), and for $1 \otimes u$ of the form (2) we have, for all $\psi \in S^\pm$ and $\phi \in K_0$:

$$(1 \otimes u)\psi \otimes \phi \otimes R = \psi \otimes A\phi \otimes R$$

$$(1 \otimes u)\psi \otimes \phi \otimes L = \psi \otimes B\phi \otimes L$$

$$(1 \otimes u)\psi \otimes \phi \otimes \tilde{R} = \psi \otimes A\phi \otimes \tilde{R}$$

$$(1 \otimes u)\psi \otimes \phi \otimes \tilde{L} = \psi \otimes B\phi \otimes \tilde{L}$$

which shows that $H^\pm_\sigma$ is stable under $1 \otimes u$. Now let $W$ be a non-zero $G'$-submodule of $H^\pm_\sigma$, and let $w \neq 0$ be an element of $W$. In the basis $(e^\pm_1, e^\pm_2)$ of $S^\pm$, the spin group elements are $SL_2(C)$ matrices. We can write $w$ in the form:

$$w = e^\pm_1 \otimes \phi_1 \otimes \sigma + e^\pm_2 \otimes \phi_2 \otimes \sigma$$

(22)

In the + case, acting with the spin group elements $A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ and $A' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we see that

$$w + Ae^\pm_1 \otimes \phi_1 \otimes \sigma + Ae^\pm_2 \otimes \phi_2 \otimes \sigma = e^\pm_2 \otimes \phi_1 \otimes \sigma \in W$$

$$w + A'e^\pm_1 \otimes \phi_1 \otimes \sigma + A'e^\pm_2 \otimes \phi_2 \otimes \sigma = e^\pm_1 \otimes \phi_2 \otimes \sigma \in W$$

In the − case we replace $A$ and $A'$ by $\sigma_2 A\sigma_2$ and $\sigma_2 A'\sigma_2$ respectively, and we obtain the same result. Since $\phi_1$ and $\phi_2$ cannot both vanish, we see that there is a pure tensor in $W$. Now since $S^\pm$ is an irreducible real Spin(1, 3)$^0$-module, we know that the real linear span of $Ae^\pm_1$ when $A$ varies in the spin group is $S^\pm$, and similarly with $Ae^\pm_2$. Thus we see that $W$ contains all vectors of the form $\psi \otimes \phi \otimes \sigma$, where $\psi = \phi_1$ or $\phi = \phi_2$ and $\psi$ is arbitrary in $S^\pm$. Now $K_0$ is an irreducible real module for the unitary group. Consequently, if we act with $1 \otimes U$ as above, we obtain that $\psi \otimes \phi \otimes \sigma \in W$ where $\psi$ and $\phi$ are both arbitrary, and this shows that $W = H^\pm_\sigma$.

**Lemma 5** For every $\sigma$, the $G'$-modules $H^+_\sigma$ and $H^-_\sigma$ are isomorphic. The isomorphisms from $H^+_\sigma$ to $H^-_\sigma$ are of the form $\lambda J_1 \otimes J_F, \lambda \in \mathbb{C}^*$. No other distinct modules among the $H^\pm_\sigma$ are isomorphic.

**Proof:** The given maps are quickly seen to be $\mathbb{R}$-linear isomorphisms. Let us prove that they are $G'$ linear. For this, consider $A \in \text{Spin}(1, 3)^0$ and $u \in \mathcal{U}_{J_F, X_F}$. Then $J_S \otimes J_F$ commutes with $A \otimes 1$ since $J_S$ commutes with $A$, and it also
commutes with $1 \otimes u$ since $J_F$ commutes with $u$. The uniqueness up to a scalar of the above isomorphisms is immediate from the fact that $\text{Aut}(H^+_{\sigma}) = \mathbb{C}^*$, which we now need to prove. This follows from Schur’s lemma once we have proved that these automorphisms are all $\mathbb{C}$-linear. This is proven with the exact same method as in lemma 2. Finally, we must prove that no other isomorphism $H^+_{\sigma}$ exists. It is obvious that $H^+_{\sigma} \neq H^+_{\sigma'}$ if $\sigma' \neq \sigma$ and $\sigma' \neq \bar{\sigma}$, since $1 \otimes u \in G''$ will act by $A$ or $\bar{A}$ in one case and by $B$ or $\bar{B}$ in the other. Now let us suppose $\theta : H^+_{\sigma} \to H^-_{\sigma}$ is a $G'$-linear map. Let us decompose $H^+_{\sigma}$ and $H^-_{\sigma}$ into irreducible $\text{Spin}(1,3)$-modules: $H^+_{\sigma} = \bigoplus_{f} S^+ \otimes f \otimes \sigma$, and $H^-_{\sigma} = \bigoplus_{f} S^- \otimes f \otimes \sigma$, with $f$ running over a basis of elementary fermions. Since $\theta$ is $\text{Spin}(1,3)$-linear, the “matrix elements” $\theta_{f'}$ determined by this decomposition are all $\mathbb{C}$-antilinear by lemma 3. Thus $\theta$ is $\mathbb{C}$-antilinear. But if we now decompose $H^+_{\sigma}$ and $H^-_{\sigma}$ into a sum of irreducible $G''$-modules, we likewise find that $\theta$ is $\mathbb{C}$-linear. Hence $\theta = 0$. Using the same method we find that $H^+_{\sigma}$ and $H^-_{\sigma}$ are not isomorphic.

Gathering the isomorphic summands, we obtain the decomposition of $V$ into isotypical $G'$-components.

$$V = (H^+_R \oplus H^-_R) \oplus (H^+_L \oplus H^-_L) \oplus (H^+_L \oplus H^-_L) \oplus (H^+_R \oplus H^-_R)$$

(23)

Let $W$ be an irreducible $G'$-submodule of $V$. The projections $p^\pm_R$ restricted to $W$ are isomorphisms or vanish. Since they cannot all vanish, we see that $W$ is isomorphic to $H^+_R$, $H^-_L$, $H^+_L$ or $H^-_R$, the four cases corresponding to the isotypical components, and we call them respectively the right-even, left-odd, left-even and right-odd cases.

**Proposition 1** Let $W$ be an irreducible $G'$-submodule of $V$.

1. $W$ is right-even iff there is a pair $\alpha = (\alpha^R, \alpha^L) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $W = W^+_{\alpha^R} := \{\sum \alpha^R \psi_R \otimes \phi \otimes R + \alpha^L J_S \psi_R \otimes \bar{\phi} \otimes \bar{R} \psi \in S^+, \phi \in K_0\}$,

2. $W$ is left-odd iff there is a pair $\alpha = (\alpha^L, \alpha^L) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $W = W^-_{\alpha^L} := \{\sum \alpha^L \psi_L \otimes \phi \otimes L + \alpha^R J_S \psi_L \otimes \bar{\phi} \otimes \bar{L} \psi \in S^-, \phi \in K_0\}$,

3. $W$ is left-even iff there is a pair $\alpha = (\alpha^L, \alpha^L) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $W = W^+_{\alpha^L} := \{\sum \alpha^L \psi_R \otimes \phi \otimes L + \alpha^R J_S \psi_R \otimes \bar{\phi} \otimes \bar{L} \psi \in S^+, \phi \in K_0\}$,

4. $W$ is right-odd iff there is a pair $\alpha = (\alpha^R, \alpha^R) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $W = W^-_{\alpha^R} := \{\sum \alpha^R \psi_L \otimes \phi \otimes R + \alpha^L J_S \psi_L \otimes \bar{\phi} \otimes \bar{R} \psi \in S^-, \phi \in K_0\}$

**Proof:** Let us suppose $W$ is right-even, the other cases being entirely similar. Then we know that all the projections $p^\pm_R$ vanish on $W$, except at least one among $p^+_{\sigma}$ and $p^-_{\sigma}$. Suppose $p^+_{\sigma}$ does not vanish. It is then an isomorphism. Let $f : H^+_R \to W$ be its inverse. Then every $v \in W$ is of the form $f(\sum \psi \otimes \phi \otimes R)$, $\psi \in S^+, \phi \in K_0$. We know by lemma 4 that $p^-_{\sigma} \circ f$ is of the form $\lambda J_S \otimes J_F$, with $\lambda \in \mathbb{C}$. Thus every $v \in W$ is of the form

$$v = f(\sum \psi \otimes \phi \otimes R) = (p^+_R + p^-_{\sigma}) \circ f(\sum \psi \otimes \phi \otimes R) = \sum \psi \otimes \phi \otimes R + \lambda J_S \psi \otimes \bar{\phi} \otimes \bar{R}$$

The pair $(1, \lambda)$ is non-vanishing. If $p^+_R$ vanishes on $W$, then we do the same reasoning with $p^-_{\sigma}$. In the end we obtain a non-vanishing pair such that $W$
has the required form. Conversely every $W$ as in the statement is clearly a $G'$-module isomorphic to $H^\pm_R$.

The 3 other cases are similar, except that we sometimes have to absorb signs coming from $\{0,0\}$ in the definition of the coefficients $\alpha'_\pm$.

The pair $\alpha$ in this proposition is not unique. Let us define the following equivalence relation on $\mathbb{C}^2 \setminus \{(0,0)\}$:

\[(\alpha'_1,\alpha'_2) \sim (\alpha_1,\alpha_2) \iff \exists \lambda \in \mathbb{C}^*, \alpha'_1 = \lambda \alpha_1, \alpha'_2 = \bar{\lambda} \alpha_2 \tag{24}\]

**Lemma 6** Let $W^\pm_\alpha\sigma$ and $W^\pm_\alpha\sigma'$ be two irreducible $G'$-modules. Then $W^\pm_\alpha\sigma = W^\pm_\alpha\sigma'$ iff $\alpha' \sim \alpha$.

**Proof:** We work in the right-even case, the others being similar. Let $\alpha = (a,b)$, $\alpha' = (a',b')$. An element of $W^\pm_\alpha\sigma$ can be uniquely written as:

\[v = \sum_{i,f} (a\lambda_i.f e^+_i \otimes f \otimes R + b\bar{\lambda}_i.J_S e^+_i \otimes f \otimes \bar{R})\]

We thus see that $a' e^+_i \otimes f \otimes R + b' J_S e^+_i \otimes f \otimes \bar{R}$ can be written in this way iff $\exists \lambda \in \mathbb{C}$ such that $a' = \lambda a$ and $b' = \bar{\lambda} b$.

The space $H$ we are looking for is a $G$-module, hence a $G'$-module, and as such it is the direct sum of irreducible sub-modules of the above kind. For dimensional reasons, we need two of them. Now a particular solution of (20) is $u = 1_{\epsilon_{\alpha}} \otimes r$ where $r = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ if $\eta_{R}\eta_{L} = 1$ and $r = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ if $\eta_{R}\eta_{L} = -1$. Hence the elements of $1 \otimes U_{J_F,\chi_F}$ can mix the letters $(R,\bar{L})$ on one hand, and $(L,\bar{R})$ on the other. Thus it is clear that $H$ must be the sum of a left-even/odd and a right-even/odd module. We will now show that the “parity” of the two modules must be opposite. For this, let us introduce a particular class of elements of $G$.

**Lemma 7** The operator $\gamma_{\mu} \otimes T$ is in $G$ iff $\{\chi_F,T\} = 0$, $[J_F,T] = 0$ and $T^\ast T = -(\gamma_{\mu})^2$.

The proof is immediate. Thus $T$ has the block-matrix form

\[
T = \begin{pmatrix}
0 & 0 & \epsilon_F \bar{A} & \bar{B} \\
0 & 0 & \bar{C} & \epsilon_F \bar{D} \\
A & B & 0 & 0 \\
C & D & 0 & 0
\end{pmatrix}
\tag{25}
\]

with

\[
\begin{align*}
\eta_{R}\eta_{\bar{R}}A^\dagger A + \eta_{R}\eta_{L}C^\dagger C &= -(\gamma_{\mu})^2 \\
\eta_{R}A^\dagger B + \eta_{L}C^\dagger D &= 0 \\
\eta_{L}\eta_{R}B^\dagger B + \eta_{L}\eta_{D}D^\dagger D &= -(\gamma_{\mu})^2 \\
\eta_{R}A^\dagger A + \eta_{R}\eta_{L}C^\dagger C &= -(\gamma_{\mu})^2 \\
\eta_{R}A^\dagger B + \eta_{L}C^\dagger D &= 0 \\
\eta_{L}\eta_{R}B^\dagger B + \eta_{L}\eta_{D}D^\dagger D &= -(\gamma_{\mu})^2
\end{align*}
\tag{26}
\]

We immediately see that the 3 last equations are equivalent to the 3 first thanks to (12). A particular solution to (20) is $A = D = 0$, $C = B = 1$ and:
\[ \gamma_\mu = \gamma_0 \text{ if } \eta_R \eta_R = \epsilon_{F\,K\,F} = -1, \]
\[ \gamma_\mu = \gamma_1 \text{ if } \eta_R \eta_R = \epsilon_{F\,K\,F} = 1. \]

We write this solution \( T = \gamma_\mu \otimes 1 \otimes \tau \), where \( \tau \) exchanges the letters \( R \) and \( L \). Hence we have (discarding the \( K_0 \)-factor, which plays no role):

\[ \gamma_\mu \otimes \tau (\alpha^R_+ \psi_R \otimes R + \alpha^R_- J_S \psi_R \otimes \bar{R}) = \alpha_+^R \gamma_\mu \psi_R \otimes L - \alpha_-^R J_S \gamma_\mu \psi_R \otimes \bar{L} \quad (27) \]

Thus we see that if we take \( H \) to be the sum of a right-even and left module, then the left module must be odd, since \( \gamma_\mu \psi_R \in S^- \). Similarly a right-odd module must be associated with a left-even one. Hence we find two possible solutions so far:

\[ H = W^+_{\alpha, R} \oplus W^-_{\beta, L}, \quad \text{or} \]
\[ H = W^-_{\alpha, R} \oplus W^+_{\beta, L}. \]

Suppose we are in the first case. Equation (27) yields conditions on the pairs of complex numbers \( \alpha = (\alpha^R_+, \alpha^R_-) \) and \( \beta = (\alpha^L_-, \alpha^L_+) \): it shows that \((\alpha^L_-, \alpha^L_+)\) and \((\alpha^R_+, -\alpha^R_-)\) define the same left-odd module. We can thus take them to be equal without loss of generality. Thus we have

\[ H = W^+_{(a, b)} \oplus W^-_{(a, -b)} \quad (28) \]

We will now use another particular solution to (26). If \( \epsilon_{F\,K\,F} = 1 \) we can take \( B = C = 0, A = D = 1 \), with \( \mu = 1, 2 \) or 3. If \( \epsilon_{F\,K\,F} = -1 \) we have the same solution but with \( \mu = 0 \). Hence we have the operator \( \gamma_\mu \otimes 1 \otimes \tau \) where this time \( \tau(R) = \bar{R}, \tau(\bar{R}) = \epsilon_F R \) and similarly with \( L \). We obtain \((\gamma_\mu \otimes T)(W^+_{(a, b)}) = W^+_{(-\epsilon_F b, a)}, \) hence we must have \( W^+_{(a, b)} = W^+_{(a, -\epsilon_F b)} \). Thus \((a, b) \sim (-\epsilon_F b, a)\) by lemma 6. This means that there is a \( \lambda \in \mathbb{C}^* \) with \( a = -\lambda e_F b \) and \( b = \lambda a \). We see that \((1 + \epsilon_F |\lambda|^2)b = 0\). Since \( a = b = 0 \) is impossible, we obtain that \( \epsilon_F = -1 \) and \( |\lambda| = 1 \). Hence \( a \) and \( b \) have the same modulus \( r \), and since \( (re^{i\theta}, re^{i\phi}) \sim (1, e^{i(\phi - \theta)}) \), we see that:

\[ H = W^+_{(1, e^{i\phi})} \oplus W^-_{(1, -e^{i\phi})} \quad (29) \]

The elements of \( H \) are then of the form

\[ v = \sum_{\psi_R \in S^+, \phi \in K_0} \psi_R \otimes \phi \otimes R + e^{i\phi} J_S \psi_R \otimes \bar{\phi} \otimes \bar{R} \]
\[ + \sum_{\psi_L \in S^-, \phi' \otimes \phi' \in K_0} \psi_L \otimes \phi' \otimes L - e^{i\phi} J_S \psi_L \otimes \bar{\phi}' \otimes \bar{L} \]
\[ = \sum_{\psi_R \in S^+, \phi \in K_0} \psi_R \otimes \phi \otimes R + e^{i\phi} J(\psi_R \otimes \phi \otimes R) \]
\[ + \sum_{\psi_L \in S^-, \phi' \otimes \phi' \in K_0} \psi_L \otimes \phi' \otimes L + e^{i\phi} J(\psi_L \otimes \phi' \otimes L) \quad (30) \]

Thus \( H = (1 + e^{i\phi} J(\frac{1 - \lambda}{2}))(V) \), and this immediately shows that \( H \) is stable by the whole group \( G \). The odd case is entirely similar and we find \( H = (1 + e^{i\phi} J(\frac{1 + \lambda}{2}))(V) \). This proves the theorem.
4 Generalized Majorana-Weyl conditions and KO-dimension

Remember we had $\epsilon'_F = \kappa''_F = -1$. Consulting the tables we see that $\epsilon_F = -1$ iff the KO-dimension $n_F = 2$. Since the KO-dimension of the manifold $n_M = 6$ we can restate theorem [1] by saying that there exists a $G$-invariant subspace of dimension $\dim_{\mathbb{R}}(V)/4$ iff the KO-dimension of the total algebraic background is 0. This also amounts to say that $\epsilon = \epsilon'' = 1$.

Moreover, introducing an innocuous factor of $1/4$, the space $H$ is $p_\pm(V)$ where $p_\pm = \frac{1 \pm e^{i\phi} J}{2}$. Using $\epsilon = \epsilon'' = 1$ we see that $p_\pm$ is the product of two commuting projectors (the converse is also true). Of course $\frac{1 \pm e^{-i\phi} J}{2}$ projects on the $\pm 1$-eigenspaces of $\chi$, and $\frac{1 \pm e^{i\phi} J}{2}$ projects on the $\pm e^{-i\phi}$-eigenspaces of $J$. Thus, the solution of the fermion quadrupling problem, assuming $U_{J,\chi}$-invariance, is given by the generalized “Majorana-Weyl conditions”

$$
\begin{align*}
\chi \psi &= \pm \psi \\
J \psi &= e^{-i\phi} \psi
\end{align*}
$$

This furnishes a converse to Barrett’s solution with the hypothesis of invariance under $U_{J,\chi}$. Note that without this hypothesis, other solutions exist. Indeed, let $\alpha, \beta, \gamma, \delta$ be four arbitrary complex numbers. Then the real space $H_{\alpha, \beta, \gamma, \delta}$ comprising vectors of the form

$$
\begin{align*}
\alpha \psi_R \otimes \phi \otimes R + \beta J_M \psi_R \otimes \bar{\phi} \otimes \bar{R} + \gamma \psi_L \otimes \phi' \otimes L + \delta J_M \psi_L \otimes \phi' \otimes \bar{L}
\end{align*}
$$

is quickly seen to be invariant under the gauge group of the Standard Model, or for that matter, any model with the same finite space and a finite algebra which does not mix letters $R, L, \bar{R}, \bar{L}$. This space has the correct dimension and could also be taken to be a solution of the fermion quadrupling problem.

Let us close this section be observing that if we insert the Majorana-Weyl condition into the usual form (17) of the fermionic action, we find conditions on $\kappa$ and $\kappa''$ for this action not to vanish. These conditions vary according to whether the fermionic variable are taken to be commuting or anti-commuting. In order to treat the two cases on the same footing, let us set $s = 1$ if the fermion variables commute, and $s = -1$ if they anti-commute. We then find that:

$$
(\psi, D\psi) \neq 0 \Rightarrow \kappa'' = -1 \text{ and } \kappa = s
$$

This is equivalent to say that $m = 6$ if $s = 1$ and $m = 2$ if $s = -1$. Since the metric dimension of the manifold is $m_M = 4$, we find that $m_F = 2$ if $s = 1$ and $m_F = 6$ if $s = -1$. It is interesting to note that this last result is in agreement (modulo 8) with the number of compact dimensions in String Theory.

\footnote{These are not really eigenspaces of the \(\mathbb{R}\)-linear operator $J$, which can only have a $\pm 1$-eigenspaces, these are the space where $J$ and the real operator of multiplication by $e^{-i\phi}$ coincide.}

\footnote{We thank Nadir Bizi for this observation.}
5 Projecton on the physical subspace and the fermionic action

One can see easily that \( \frac{1 + e^{i\varphi}J}{2} \frac{1 + \chi}{2} \) are 4 projectors which sum to 1 and such that the product of any two of them vanish. Let us introduce the following notation:

\[
p_a^b = \frac{1}{4} (1 + ae^{i\varphi}J)(1 + b\chi)
\]

(34)

where \( a, b = \pm 1 \).

The traditional solution of the fermion quadrupling problem is to restrict the field \( \psi \) to have values in the chosen physical subspace, say \( p_1^1 V \). However, it would be equivalent to suppose that \( \psi \) has value in \( V \), but that the fermionic action depends on \( \psi \) only through \( p_1^1 \psi \). In order to formulate this idea more precisely, and to treat commuting and anti-commuting fermionic variables simultaneously, let us introduce the real bilinear form

\[
(\varphi, \psi)_r := \frac{1}{2} ((\varphi, \psi) + (\psi, \varphi))
\]

(35)

If \( s = 1 \) this is just the real part of \( (., .) \), however if \( s = -1 \) this interpretation cannot be maintained since the action has value in a Grassmann algebra. In any case, the usual form of the fermionic action is

\[
S_f(D, \psi) = (\psi, D\psi)_r
\]

(36)

where \( Q \) is any polynomial in \( \chi \) and \( J \). If \( B \) is a real linear operator we will write \( B^+ \) for the adjoint of \( B \) with respect to \( (., .)_r \). Let us note that if \( B \) is \( \mathbb{C} \)-linear then \( B^+ = B^\times \), whereas if \( B \) is antilinear, \( B^+ = sB^\times \), since we have

\[
(\phi, B\psi)_r = \frac{1}{8} ((\phi, B\psi) + (B\psi, \phi))
\]

\[
= \frac{1}{2} ((\psi, B^\times \phi) + (B^\times \phi, \psi))
\]

\[
= s(B^\times \phi, \psi)_r
\]

(37)

Now we see that if we take the action to be

\[
S_f(D, \psi) = (p_1^1 \psi, Dp_1^1 \psi)_r
\]

(38)

we obtain

\[
16S_f(D, \psi) = ((1 + ae^{i\varphi}J)(1 + b\chi)\psi, D(1 + ae^{i\varphi}J)(1 + b\chi)\psi)_r
\]

\[
= (\psi, (1 + bn''\chi)(1 + sace^{i\varphi}J)(1 + ae^{i\varphi}J)(1 - b\chi)D\psi)_r
\]

\[
= 0, \text{ unless } \kappa'' = -1, \kappa s = 1
\]

(39)

We thus find the same conditions as in (33). Moreover, if these conditions are met, then \( J^\times = \kappa J = sJ \), thus we have \( J^+ = J \), while \( \chi^+ = \epsilon'' \kappa'' \chi = -\chi \). We can then deduce that

\[
(p_a^b)_+ = p_a^{s-b}
\]

(40)

Using \( p_a^b D = Dp_a^{s-b} \), we see that the action can also be written

\[
S_f(D, \psi) = (\psi, p_1^{s} D\psi)_r
\]

(41)
which has the form (36). To see how particular this action is with respect to the general form, we first note that thanks to the commutation relations among \(J\) and \(\chi\), as well as \(J^2\) and \(\chi^2\) = constants, the algebra of polynomials \(\mathbb{C}[J, \chi]\) is four dimensional. Moreover, since

\[
\begin{vmatrix}
1 & 1 & e^{i\varphi} & e^{i\varphi} \\
1 & 1 & -e^{i\varphi} & -e^{i\varphi} \\
1 & -1 & e^{i\varphi} & -e^{i\varphi} \\
1 & -1 & -e^{i\varphi} & e^{i\varphi}
\end{vmatrix} \neq 0
\]  

(42)

the projectors \(p^a_b\) form a basis of \(\mathbb{C}[J, \chi]\). Hence \(Q\) can be written as

\[
Q = \sum_{a,b} \pi^b_a p^a_b
\]  

(43)

with \(\pi^b_a \in \mathbb{C}\). Thus we have

\[
(\psi, QD\psi)_r = \sum_{a,b} (\psi, \pi^b_a p^a_b D\psi)_r
\]

\[
= \sum_{a,b} (\psi, \pi^b_a (p^a_b)^2 D\psi)_r
\]

\[
= \sum_{a,b} (p^a_{-b}\psi, \pi^b_a Dp^a_{-b}\psi)_r
\]

\[
= \sum_{a,b} ((p^a_{-b}\psi, \pi^b_a Dp^a_{-b}\psi) + (\pi^b_a Dp^a_{-b}\psi, p^a_{-b}\psi))
\]

\[
= \sum_{a,b} (\pi^b_a (p^a_{-b}\psi, Dp^a_{-b}\psi) + \pi^b_a (Dp^a_{-b}\psi, p^a_{-b}\psi))
\]

\[
= 2 \sum_{a,b} \Re(\pi^b_a) (p^a_{-b}\psi, Dp^a_{-b}\psi)_r
\]  

(44)

Thus the action (38), up to a constant, is obtained when we set the real part of three out of the four numbers \(\pi^b_a\) to 0. The constant can then be absorbed in a redefinition of \(\psi\).

This is a form of fine-tuning, but choosing the traditional action \((\psi, D\psi)\) among all the possible action is also fine-tuning, and we would still have to restrict to the physical subspace. Hence we think that postulating (38) is a more economical solution. Note however that with this solution, the extra degrees of freedom are still there, even if they do not interact with gauge bosons. Let us conclude by remarking that if we take \(e^{i\varphi} = 1\) and develop (41), we obtain a particularly symmetrical formula for the action, which is valid for both \(s = 1\) and \(s = -1\):

\[
S(D, \psi) = \frac{1}{2}[(\psi, D\psi) + (D\psi, \psi) + (D\psi, J\psi) + (J\psi, D\psi) + (\chi\psi, D\psi) + (D\psi, \chi\psi) + (D\psi, \chi J\psi) + (\chi J\psi, D\psi)]
\]  

(45)

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