PRODUCT STRUCTURES IN FLOER THEORY FOR LAGRANGIAN COBORDISMS

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Abstract. We construct a product on the Floer complex associated to a pair of Lagrangian cobordisms. More precisely, given three exact transverse Lagrangian cobordisms in the symplectization of a contact manifold, we define a map $m_2$ by a count of rigid pseudo-holomorphic disks with boundary on the cobordisms and having punctures asymptotic to intersection points and Reeb chords of the negative Legendrian ends of the cobordisms. More generally, to a $(d+1)$-tuple of exact transverse Lagrangian cobordisms we associate a map $m_d$ such that the family $(m_d)_{d≥1}$ are $A_∞$-maps. Finally, we extend the Ekholm-Seidel isomorphism to an $A_∞$-morphism, giving in particular that it is a ring isomorphism.

1. Introduction

1.1. Background. A contact manifold $(Y, \xi)$ is a smooth manifold $Y$ equipped with a completely non-integrable plane field $\xi$ called a contact structure. We consider $\xi$ cooriented, which means that there is a 1-form $\alpha$ such that $\xi = \ker(\alpha)$ and $\alpha \wedge dx \neq 0$. The form $\alpha$ is called a contact form for $(Y, \xi)$. In particular, $Y$ is an odd dimensional manifold. The Reeb vector field $R_{\alpha}$ associated to $(Y, \alpha)$ is the unique vector field on $Y$ satisfying $d\alpha(R_{\alpha}, \cdot) = 0$ and $\alpha(R_{\alpha}) = 1$. In this article, we consider a particular type of contact manifold which is the contactization of a Liouville manifold.

A Liouville domain $(\hat{P}, \hat{\omega}, X)$ is a compact symplectic manifold with boundary equipped with a vector field $X$ satisfying:

1. $\mathcal{L}_X \hat{\omega} = \hat{\omega}$
2. $X$ is pointing outward on $\partial \hat{P}$.

where $\mathcal{L}_X$ is the Lie derivative. Condition (1) can be rewritten $d_X \hat{\omega} = \hat{\omega}$ because $\hat{\omega}$ is closed, and thus this implies that it is an exact form $\hat{\omega} = d\beta$, with $\beta = \iota_X \hat{\omega}$. The 1-form $\beta$ restricted to $\partial \hat{P}$ is a contact form, and the completion of $(\hat{P}, \hat{\omega})$ is the non compact exact symplectic manifold $(P, \omega = d\theta)$ defined by

$$P = \hat{P} \cup_{\partial \hat{P}} ([0, \infty) \times \partial \hat{P})$$

and $\theta$ is equal to $\beta$ on $\hat{P}$, and to $e^\tau \beta|_{\partial \hat{P}}$ on $[0, \infty) \times \partial \hat{P}$, with $\tau$ the coordinate on $[0, \infty)$. The Liouville vector field $X$ on $\hat{P}$ can be extended to the whole $(P, d\theta)$. The manifold $(P, \theta)$ is called a Liouville manifold. The well-known symplectic manifolds $(\mathbb{R}^{2n}, \sum_i dx_i \wedge dy_i)$ and $(T^*M, -d\lambda)$, the cotangent fiber bundle of a smooth manifold $M$ equipped with the standard Liouville form, are examples of Liouville manifolds.

The contactization of a Liouville manifold $(P, d\theta)$ is the contact manifold $(P \times \mathbb{R}, dz + \theta)$ where $z$ is the coordinate on the $\mathbb{R}$-factor. For example, the contactization of $(T^*M, -d\lambda)$ is the 1-jet space $J^1(M)$. From now, we fix a $(2n+1)$-dimensional contact manifold $(Y, \alpha)$ which is the contactization of a $2n$-dimensional Liouville manifold $(P, d\theta)$. Remark that for this special type of contact manifold, the Reeb vector field is $\partial_z$, in particular there are no closed Reeb orbits.

A Legendrian submanifold $\Lambda \subset Y$ is a submanifold of dimension $n$ such that $\alpha|_{T\Lambda} = 0$, which means that for all $x \in \Lambda$, $T_x \Lambda \subset \xi_x$. The Reeb chords of a Legendrian submanifold $\Lambda$
are Reeb-flow trajectories that start and end on \( \Lambda \). Compact Legendrian submanifolds in the contactization of a Liouville manifold generically have a finite number of Reeb chords. These chords correspond to vertical lines which start and end on \( \Lambda \). Let \( \gamma \) be a Reeb chord of length \( \ell \) which starts at a point \( x^- \in \Lambda \) and ends at \( x^+ \in \Lambda \), and let us denote \( \varphi^R_\ell \) the Reeb flow. If \( d_x \gamma = \varphi^R_\ell(T_x \Lambda) \) and \( T_x \Lambda \) intersect transversely, we say that the Reeb chord \( \gamma \) is non-degenerate, and then \( \Lambda \) is called chord generic if all its Reeb chords are non-degenerate. From now, we will only consider compact chord generic Legendrian submanifolds, and we denote by \( \mathcal{R}(\Lambda) \) the set of Reeb chords of \( \Lambda \). If \( \Lambda_1, \Lambda_2, \ldots, \Lambda_d \) are \( d \) Legendrian submanifolds of \( Y \), we can consider the union \( \Lambda = \Lambda_1 \cup \cdots \cup \Lambda_d \). Reeb chords of \( \Lambda \) from \( \Lambda_i \) to itself are called pure Reeb chords while those from \( \Lambda_i \) to \( \Lambda_j \) with \( i \neq j \) are called mixed Reeb chords. We denote by \( \mathcal{R}(\Lambda_i, \Lambda_j) \) the set of Reeb chords from \( \Lambda_i \) to \( \Lambda_j \).

The Lagrangian projection of a Legendrian submanifold \( \Lambda \subset Y = P \times \mathbb{R} \) is the image of \( \Lambda \) under the projection \( \Pi_F : P \times \mathbb{R} \to P \). Reeb chords of \( \Lambda \) are then in bijection with intersection points of \( \Pi_F(\Lambda) \). In the particular case where the contact manifold is the 1-jet space of a manifold \( M \) (i.e. \( Y = J^1(M) = T^*M \times \mathbb{R} \)), the front projection of \( \Lambda \) is the image of \( \Lambda \) under \( \Pi_F : J^1(M) \to M \times \mathbb{R} \). In this case, Reeb chords are in bijection with vertical segments in \( M \times \mathbb{R} \) beginning and ending respectively on points \( c^-, c^+ \in \Pi_F(\Lambda) \), and such that the tangent spaces \( T_{c^-} \Pi_F(\Lambda) \) and \( T_{c^+} \Pi_F(\Lambda) \) are equal.

One natural question when studying Legendrian submanifolds is to understand whether two Legendrian submanifolds \( \Lambda_0, \Lambda_1 \subset Y \) are Legendrian isotopic or not (i.e. is there a smooth function \( F : [0, 1] \times \Lambda \to Y \) such that \( \Lambda \) is a \( n \)-dimensional manifold and \( F(t, \Lambda) \) is a Legendrian submanifold of \( Y \) for all \( t \in [0, 1] \), with \( F(0, \Lambda) = \Lambda_0 \) and \( F(1, \Lambda) = \Lambda_1 \)?)? A lot of work has been achieved in order to answer this question of classification under Legendrian isotopy of Legendrian submanifolds. There exists a lot of Legendrian isotopy invariants, among which the first were the classical ones, namely the smooth isotopy type, the Thurston-Bennequin invariant and the rotation class (see for example \([\text{Etn}], \text{EES05}\)). The development then of non-classical invariants gave new directions in order to understand better Legendrians. One of the first non-classical invariants is a relative version of contact homology \([\text{EGH00}]\) called the Legendrian contact homology. It was defined by Eliashberg in \([\text{Eli98}]\) using pseudo-holomorphic curves techniques. Independently, it was defined combinatorially by Chekanov for Legendrian links in \((\mathbb{R}^3, dz - ydx)\) in \([\text{Che02}]\), and this combinatorial description was generalized in higher dimension by Ekelom, Etnyre and Sullivan \([\text{ENS05a}, \text{EES07}]\). These two definitions were then shown to compute the same invariant, by Etnyre, Ng and Sabloff \([\text{ENS02}]\) in dimension 3, and by Dimitroglou-Rizell \([\text{DK16b}]\) in all dimension. This is a very powerful Legendrian isotopy invariant which gave rise to numerous other invariants, as for example the linearized and “multi-linearized” versions of Legendrian contact homology, using augmentations of the differential graded algebra introduced by Chekanov. Then there are higher algebraic structures on linearized Legendrian contact cohomology that are Legendrian isotopy invariants, as a product structure and an \( A_\infty \) algebra structure (see \([\text{CKE11}]\)), and more generally, there are \( A_\infty \)-categories \( \text{Aug}_-(\Lambda) \) and \( \text{Aug}_+(\Lambda) \), called the augmentation categories of a Legendrian submanifold (see \([\text{BC14}], \text{NRS15}, \text{Subsection 3.2 for } \text{Aug}_-(\Lambda)\)). In parallel to these invariants defined by pseudo-holomorphic curves counts, other types of Legendrian isotopy invariants have been defined, by generating functions techniques. We will not go through these invariants in this article, nevertheless, even if the definition of this two types (pseudo-holomorphic curves vs generating functions) of invariants are really different, they are closely related. Indeed, the existence of a (linear at infinity) generating family for a Legendrian knot \( \Lambda \) in \( \mathbb{R}^3 \) implies the existence of an augmentation such that the linearized contact homology of \( \Lambda \) is isomorphic to the generating family homology of \( \Lambda \) (see \([\text{FR11}]\)). In higher dimension, the relation is not so clear. However, there are parallel results in the Legendrian contact homology side and the generating family homology side, as for example a duality exact sequence (\([\text{EES09}]\)
Definition 1. An exact Lagrangian cobordism from $\Lambda^-$ to $\Lambda^+$, denoted $\Lambda^- \prec_{\Sigma} \Lambda^+$, is a submanifold $\Sigma \subset \mathbb{R} \times Y$ satisfying the following:

1. there exists a constant $T > 0$ such that:
   - $\Sigma \cap (-\infty,-T) \times Y = (-\infty,-T) \times \Lambda^-$,
   - $\Sigma \cap (T,\infty) \times Y = (T,\infty) \times \Lambda^+$,
   - $\Sigma \cap [-T,T] \times Y$ is compact.
2. there exists a smooth function $f : \Sigma \to \mathbb{R}$ such that:
   (a) $e^{\alpha_0} f = df$,
   (b) $f_{(-\infty,-T) \times \Lambda^-}$ is constant,
   (c) $f_{(T,\infty) \times \Lambda^+}$ is constant.

Remark 1. Condition (a) above says by definition that $\Sigma$ is an exact Lagrangian submanifold of $(\mathbb{R} \times Y, d(e^{t}\alpha))$. Moreover, using the fact that $\Sigma$ is a cylinder over $\Lambda^-$ in the negative end and a cylinder over $\Lambda^+$ in the positive end, condition (a) implies that $f$ is constant on each connected component of the negative and the positive ends of $\Sigma$. Conditions (b) and (c) imply that $f$ is in fact globally constant on each end (the constant on the positive end is not necessarily the same as the constant on the negative end). Thus, if $\Lambda^\pm$ are connected, conditions (b) and (c) are automatically satisfied.

We denote by $\overline{\Sigma} := [-T,T] \times \Sigma$ the compact part of the cobordism and the boundary components $\partial_- \overline{\Sigma} = \{-T\} \times \Lambda^-$ and $\partial_+ \overline{\Sigma} = \{T\} \times \Lambda^+$. In the case where $\Sigma$ is diffeomorphic to a cylinder, we call it a Lagrangian concordance from $\Lambda^-$ to $\Lambda^+$ and denote it simply $\Lambda^- \prec \Lambda^+$, and when $\Lambda^- = \emptyset$, $\Sigma$ is called an exact Lagrangian filling of $\Lambda^+$. The existence of a Lagrangian cobordism between two Legendrian submanifolds is in some sense weaker than the existence of a Legendrian isotopy, but if we restrict to the study of Lagrangian concordances, this is not clear. Indeed, a Legendrian isotopy induces a Lagrangian concordance [EG98, Cha10], but if there are concordances $\Lambda_1 \prec \Lambda_2$ and $\Lambda_2 \prec \Lambda_1$, it is not known if it implies that $\Lambda_1$ and $\Lambda_2$ are Legendrian isotopic. Also, as evoked above, some Legendrian isotopy invariants give obstructions to the existence of Lagrangian cobordisms (see for example [Cha10, Ekh12, ST13, CNS16, Pan17], which is absolutely not an exhaustive list). In the same vein, Chantraine, Dimitroglou-Rizell, Ghiggini and Golovko ([CDRG1]) have defined a Floer-type complex associated to a pair of Lagrangian cobordisms, the Cthulhu complex, in order to understand better the topology of a Lagrangian cobordism between two given Legendrians. The goal of this article is to provide a richer algebraic structure associated to Lagrangian cobordisms.

1.2. Results. Let $\Lambda^-_1 \prec_{\Sigma_1} \Lambda^+_1$ and $\Lambda^-_2 \prec_{\Sigma_2} \Lambda^+_2$ be two exact transverse Lagrangian cobordisms, such that the Chekanov-Eliashberg algebras (Legendrian contact homology algebras) $A(\Lambda^-_1)$ and $A(\Lambda^-_2)$ admit augmentations $\varepsilon^-_1$ and $\varepsilon^-_2$ respectively. The Cthulhu complex $(\text{Cth}(\Sigma_1, \Sigma_2), \partial_{\varepsilon^-_1}, \varepsilon^-_2)$ associated to the pair $(\Sigma_1, \Sigma_2)$ is generated by Reeb chords from $\Lambda^+_2$ to $\Lambda^+_1$, intersection points
in $\Sigma_1 \cap \Sigma_2$, and by Reeb chords from $\Lambda^-_2$ to $\Lambda^+_1$. Given $\varepsilon^+_1$ and $\varepsilon^+_2$ augmentations of $A(\Lambda^+_1)$ and $A(\Lambda^+_2)$ respectively induced by $\varepsilon^+_1$ and $\varepsilon^+_2$ (see Section 3.3), the differential of the Cthulhu complex is mapped to a count of rigid pseudo-holomorphic curves with boundary on $\Sigma_1$ and $\Sigma_2$ (see Section 4.1). This complex admits a quotient complex $\mathcal{CF}_{\infty}(\Sigma_1, \Sigma_2)$, generated only by intersection points and Reeb chords from $\Lambda^-_2$ to $\Lambda^+_1$, called the Floer complex of the pair $(\Sigma_1, \Sigma_2)$. The main result of this article is the following:

**Theorem 1.** Let $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ be three transverse exact Lagrangian cobordisms from $\Lambda^-_1$ to $\Lambda^+_i$, for $i = 1, 2, 3$, where $\Lambda^+_i$ are Legendrian submanifolds of $P \times \mathbb{R}$ such that the Chekanov-Eliashberg algebras $A(\Lambda^-_1)$ admit augmentations. Then, for any choice of augmentation $\varepsilon^-_i$ of $A(\Lambda^-_1)$, for $i = 1, 2, 3$, there exists a map:

$$m_2 : \mathcal{CF}_{\infty}(\Sigma_2, \Sigma_3) \otimes \mathcal{CF}_{\infty}(\Sigma_1, \Sigma_2) \to \mathcal{CF}_{\infty}(\Sigma_1, \Sigma_3)$$

which satisfies the Leibniz rule $\partial_{\infty} \circ m_2(-, -) + m_2(\partial_{\infty}, -) + m_2(-, \partial_{\infty}) = 0$.

**Remark 2.** In the case $\Lambda^-_1 = \emptyset$, if $\Sigma_2$ and $\Sigma_3$ are small Hamiltonian perturbations of $\Sigma_1$ such that the pairs $(\Sigma_1, \Sigma_2)$, $(\Sigma_2, \Sigma_3)$ and $(\Sigma_1, \Sigma_3)$ are directed (see Section 3.2), then the homology of the complexes $\mathcal{CF}_{\infty}(\Sigma_1, \Sigma_2)$, $\mathcal{CF}_{\infty}(\Sigma_2, \Sigma_3)$ and $\mathcal{CF}_{\infty}(\Sigma_1, \Sigma_3)$ is isomorphic to the singular cohomology $H^*(\Sigma_1, \partial_+ \Sigma_1)$ and the product $m_2$ corresponds to the cup product via this identification.

Now, Cthulhu homology is invariant by a certain type of Hamiltonian isotopy which permits to displace the Lagrangian cobordisms. This implies the acyclicity of the complex. But the Cthulhu complex is in fact the cone of a map

$$\mathcal{F}^1 : \mathcal{CF}_{\infty}(\Sigma_1, \Sigma_2) \to \mathrm{LCC}^*_{\varepsilon^-_1, \varepsilon^-_2}(\Lambda^+_1, \Lambda^+_2)$$

from the Floer complex to the linearized Legendrian contact cohomology complex generated by Reeb chords from $\Lambda^+_1$ to $\Lambda^+_2$. The acyclicity implies that this map is a quasi-isomorphism. When $\Sigma_1$ is a Lagrangian filling of $\Lambda^+_1$ and $\varepsilon^+_1$ is the augmentation induced by this filling, take $\Sigma_2$ a small perturbation of $\Sigma_1$ such that the pair $(\Sigma_1, \Sigma_2)$ is directed, then the quasi-isomorphism $\mathcal{F}^1$ recovers Ekholm-Seidel isomorphism ([Ekh12, DR16b]). We will show that the map induced by $\mathcal{F}^1$ in homology preserves the product structures, that is to say, the product $m_2$ on Floer complexes is mapped to the product $\mu^2_{\varepsilon^+_2, \varepsilon^+_1}$ of the augmentation category $\mathcal{Aug}_{\infty}(\Lambda^+_1 \cup \Lambda^+_2 \cup \Lambda^+_3)$, where $\varepsilon^+_2, \varepsilon^+_1$ are the augmentation on the algebra $A(\Lambda^+_1 \cup \Lambda^+_2 \cup \Lambda^+_3)$ induced by $\varepsilon^+_1$, $\varepsilon^+_2$ and $\varepsilon^+_3$ (see Section 3.2). More precisely, we have:

**Theorem 2.** Let $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ be three transverse exact Lagrangian cobordisms from $\Lambda^-_1$ to $\Lambda^+_i$, such that the Chekanov-Eliashburg algebras $A(\Lambda^-_1)$ admit augmentations. Then, for any choice of augmentation $\varepsilon^-_i$ of $A(\Lambda^-_1)$ we have:

$$\left[\mu^2_{\varepsilon^+_2, \varepsilon^+_1}(\mathcal{F}^1, \mathcal{F}^1) = \mathcal{F}^1 \circ m_2\right]$$

In the same setting as Remark 2, Theorem 2 implies that the Ekholm-Seidel isomorphism is a ring morphism (see also [EL, Theorem 4]). The product $m_2$ is in fact part of an $A_{\infty}$-structure defined in Section 4. There, we define maps $\{m_d\}_{d \geq 1}$ such that the $m_1$ map is the differential $\partial_{\infty}$. Then we prove the following:

**Theorem 3.** The maps $\{m_d\}_{d \geq 1}$ are $A_{\infty}$-maps, they satisfy for all $d \geq 1$:

$$\sum_{1 \leq j \leq d} m_{d-j+1}(\text{id}^0 \otimes d^{-j-n} \otimes m_j \otimes \text{id}^{\otimes n}) = 0$$
A direct corollary of this theorem is that the product $m_2$ is associative in homology. Finally, once we get this $A_\infty$-structure, we can associate to a Legendrian submanifold $\Lambda$ an $A_\infty$-category of Lagrangian cobordisms that we denote $\mathcal{F}uk_-(\Lambda)$. The objects of this category are triples $(\Sigma, \Lambda^-, e^-)$ such that $\Lambda^-$ is a Legendrian submanifold, $e^-$ is an augmentation of $\mathcal{A}(\Lambda^-)$ and $\Sigma$ is an exact Lagrangian cobordism from $\Lambda^-$ to $\Lambda$. We show then that the map $\mathcal{F}^1$ extends in a family of maps $\mathcal{F} = \{\mathcal{F}^d\}_{d \geq 0}$ which is an $A_\infty$-functor, and we have:

**Theorem 4.** There exists an $A_\infty$-functor $F$: $\mathcal{F}uk_-(\Lambda) \to \text{Aug}_-(\Lambda)$ cohomologically full and faithful.

Remark that Theorem 1 and Theorem 2 are in fact corollaries of Theorem 3 and Theorem 4 respectively. However in this paper we will present first in details the proofs of Theorem 1 and Theorem 2. Finally, in Section 6 we define the $A_\infty$-structure on Floer complexes and prove Theorem 3 and Theorem 4.

The paper is organized as follows. In Section 2 we set up the definition and notations of all types of moduli spaces that are involved in the definition of all maps in the rests of the paper. In Sections 3 and 4 we review the definitions of Legendrian contact homology and Cthulhu homology. In Section 5 we construct the product structure on the Floer complexes and prove Theorem 1 and Theorem 2. Finally, in Section 6 we define the $A_\infty$-structure on Floer complexes and prove Theorem 3 and Theorem 4.

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## 2. Moduli spaces

In this section we describe the different types of moduli spaces of pseudo-holomorphic curves which will be necessary to define the Legendrian contact homology complex, the Cthulhu complex and the product structure. The first three subsections contain some useful material from Sei08 and CDRGG, in order to define the moduli spaces.

### 2.1. Deligne space

Let us denote $\mathcal{R}^{d+1} = \{(y_0, \ldots, y_d) / y_i \in S^1, y_i \in (y_{i-1}, y_{i+1})\}/\text{Aut}(D^2)$ the space of $(d+1)$-tuples of points cyclically ordered on the boundary of the disk $D^2$, where $y_{-1} := y_d$ and $y_{d+1} := y_0$, and denote by $S^{d+1}$ the universal curve:

$$S^{d+1} = \{(z, y_0, \ldots, y_d) / z \in D^2, y_i \in S^1 \text{ et } y_i \in (y_{i-1}, y_{i+1})\}/\text{Aut}(D^2)$$

The projection $\pi: S^{d+1} \to \mathcal{R}^{d+1}$ given by $\pi(z, y_0, \ldots, y_d) = (y_0, \ldots, y_d)$ is a fibration with fiber a disk. For all $r \in \mathcal{R}^{d+1}$ we denote $\hat{S}_r = \pi^{-1}(r)$ and $S_r = \hat{S}_r \setminus \{y_0, \ldots, y_d\}$. 
Given \( r \in \mathcal{R}^{d+1} \), to each marked point \( y_i \) of \( \hat{S}_r \), \( i \geq 1 \), one can associate a neighborhood \( V_i \subset \hat{S}_r \) and a biholomorphism \( \varepsilon_i : (-\infty, 0) \times [0, 1] \to V_i \setminus \{ y_i \} \). For the puncture \( y_0 \), we choose a neighborhood \( V_0 \) and a biholomorphism \( \varepsilon_0 : (0, +\infty) \times [0, 1] \to V_0 \setminus \{ y_0 \} \). These biholomorphisms are called strip-like ends. A universal choice of strip-like ends for \( \mathcal{R}^{d+1} \) corresponds to maps
\[
\varepsilon_0^{d+1} : \mathcal{R}^{d+1} \times (0, +\infty) \times [0, 1] \to S^{d+1}
\]
and
\[
\varepsilon_i^{d+1} : \mathcal{R}^{d+1} \times (-\infty, 0) \times [0, 1] \to S^{d+1}
\]
for \( 1 \leq i \leq d \), such that for all \( r \in \mathcal{R}^{d+1} \), \( \varepsilon_i^{d+1}(r, \cdot, \cdot) \) is a choice of strip-like ends for \( S_r \).

The space \( \mathcal{R}^{d+1} \), for \( d \geq 2 \), admits a compactification which can be described in terms of trees. In fact, we have \( \mathcal{R}^{d+1} = \bigcup_{T} \mathcal{R}^T \) which is a disjoint union over all stable planar rooted trees \( T \) with \( d \) leaves, and with \( \mathcal{R}^T = \bigcup_{|v|} \mathcal{R}^{|v|} \) where the union is over all interior vertices (vertices which are not leaves) \( v_i \) of \( T \). Here, \( |v| \) denotes the degree of the vertex \( v_i \), and recall that a tree is called stable if each interior vertex has degree at least 3. The space \( \mathcal{R}^{d+1} \) corresponds to \( \mathcal{R}^{T_{d+1}} \) where \( T_{d+1} \) is the planar rooted tree with \( d \) leaves and one vertex. Given \( T \) and \( T' \) two stable planar rooted trees with \( d \) leaves, if \( T' \) can be obtained from \( T \) by removing one or several edges (i.e. contracting an edge until the two corresponding vertices are identified), it gives rise to a gluing map:
\[
\gamma^{T, T'} : \mathcal{R}^T \times (-1, 0][Ed^{int}(T)) \to \mathcal{R}^{T'}
\]
where \( Ed^{int}(T) \) is the set of interior edges of \( T \). If \( e \) is an edge from the vertices \( v^- \) to \( v^+ \) to remove of \( T \) to obtain \( T' \), this gluing map consists in gluing the two disks \( S_{v^-} \) and \( S_{v^+} \) along \( e \) with a certain gluing parameter. Let us denote \( \varepsilon_- \) and \( \varepsilon_+ \) the strip-like ends of \( r_{v^-} \) and \( r_{v^+} \) for the marked points connected by \( e \). Given a real \( l_e \in (0, \infty) \), the gluing operation is given by the connected sum
\[
S_{r_{v^-}} \setminus \varepsilon_-((-\infty, l_e) \times [0, 1]) \bigcup S_{r_{v^+}} \setminus \varepsilon_+((l_e, \infty) \times [0, 1])/\sim
\]
where we identify \( \varepsilon_-(l_e - s, t) \sim \varepsilon_+(s, t) \). The map \( \gamma^{T, T'} \) glues each interior edge of \( T \) using the parameter \( \rho_e = e^{-\pi l_e} \in (-1, 0] \) instead of \( l_e \). If \( \rho = 0 \), the edge is not modified (see \cite{Sei08}).

Now suppose that \( S \in S^{d+1} \) is obtained from \( S_{r_1}, S_{r_2}, \ldots, S_{r_k} \) by gluing, then \( S \) admits a thin-thick decomposition. The thin part \( S^{\text{thin}} \) corresponds to strip-like ends of \( S \) and to strips of length \( l_e \) coming from the identification of strip-like ends in the gluing of two disks \( S_{r_{v^-}} \) and \( S_{r_{v^+}} \) along an edge \( e \). The thick part is then \( S \setminus S^{\text{thin}} \). If \( r \in \mathcal{R}^{d+1} \) is in the image of \( \gamma^{T, T_{d+1}} \), then it admits two sets of strip-like ends: one coming from the universal choice on \( \mathcal{R}^{d+1} \) and the other one coming from the universal choice on \( \mathcal{R}^{|v|} \) for all vertices \( v_i \) of \( T \) and the gluing operation. A universal choice of strip-like ends on \( \mathcal{R}^{d+1} \) is said consistent if there exists a neighborhood \( U \subset \mathcal{R}^{d+1} \) of \( \partial \mathcal{R}^{d+1} \) such that the two choices of strip-like ends coincide on \( U \cap \mathcal{R}^{d+1} \).

**Theorem 5.** \cite{Sei08} Lemma 9.3] Consistent universal choices of strip-like ends exist.

**Remark 3.** In the cases \( d = 0, 1 \), a punctured disk in \( S^1 \) is biholomorphic to a half-plane and a punctured disk in \( S^2 \) is biholomorphic to the strip \( Z = \mathbb{R} \times [0, 1] \) with standard coordinates \((s, t)\).

2.2. **Lagrangian labels.** The holomorphic disks we will consider are holomorphic maps from a disk with some marked points removed to the manifold \( \mathbb{R} \times Y \), with boundary on Lagrangian submanifolds of \( \mathbb{R} \times Y \). The choice of Lagrangian submanifolds associated to disks \( S_r \) with \( r \in \mathcal{R}^{d+1} \) is called the Lagrangian label. The boundary of \( S_r \) is subdivided into \( d+1 \) components, we denote by \( \partial_i S_r \) for \( 1 \leq i \leq d+1 \) the part of the boundary between the marked points \( y_{i-1} \) and \( y_i \). The Lagrangian label associates to each component a Lagrangian submanifold as follows.
Let $T$ be a planar rooted tree with $d$ leaves and $\{L_i\}_{i \in I}$ a finite family of Lagrangian submanifolds of $\mathbb{R} \times Y$.

**Definition 2.** A choice of Lagrangian label for $T$ is a locally constant map $L : \mathbb{R}^2 \setminus T \to \{L_i\}$, i.e., it associates to each connected component of $\mathbb{R}^2 \setminus T$ a Lagrangian submanifold in $\{L_i\}_{i \in I}$.

For $r \in \mathbb{R}^{d+1}$, the choice of Lagrangian label on $S_r$ is thus determined by the Lagrangian label for $T_{d+1}$ by associating to $\partial_i S_r$ the Lagrangian submanifold associated to the sector of $\mathbb{R}^2 \setminus T$ containing $\partial_i S_r$. If $L_i$ is the Lagrangian submanifold associated to $\partial_i S_r$, we will denote by $L = (L_1, \ldots, L_{d+1})$ the Lagrangian label for $S_r$. A natural compatibility condition for Lagrangian labels is clearly necessary in order to apply the gluing maps $\gamma^T, T$.

### 2.3. Almost complex structure

In this subsection we recall the different types of almost complex structures that will be useful in order to achieve transversality for moduli spaces. Recall that on a symplectic manifold $(X, \omega)$, an almost complex structure is a map $J : TX \to TX$ such that $J^2 = -1$. We say that $J$ is compatible with $\omega$ (or $\omega$-compatible) if:

1. $\omega(v, Jv) > 0$ for all $v \in TX$ such that $v \neq 0$,
2. $\omega(Ju, Jv) = \omega(u, v)$ for all $x \in X$ and $u, v \in T_x X$.

#### 2.3.1. Cylindrical almost complex structure

Let us go back to the case where the symplectic manifold is the symplectization of a contact manifold $(Y, \alpha)$. An almost complex structure $J$ on $(\mathbb{R} \times Y, d(e^\alpha))$ is cylindrical if:

- $J$ is $d(e^\alpha)$-compatible,
- $J$ is invariant under $\mathbb{R}$-action by translation on $\mathbb{R} \times Y$,
- $J(\partial_t) = R_\alpha$,
- $J$ preserves the contact structure, i.e. $J(\xi) = \xi$.

Following notations of [CDRG], we denote by $J^cyl(\mathbb{R} \times Y)$ the set of cylindrical almost complex structures on $\mathbb{R} \times Y$.

In our setting, the contact manifold is the contactization of a Liouville manifold, $Y = P \times \mathbb{R}$, and recall that a Liouville manifold $P$ can be viewed as the completion of a Liouville domain $(\hat{P}, d\beta)$. An almost complex structure $J_P$ on $P$ is admissible if it is cylindrical on $P \setminus \hat{P}$ outside of a compact subset $K \subset P \setminus \hat{P}$. We denote by $J^{adm}(P)$ the set of admissible almost complex structures on $P$. Now, if $J_P \in J^{adm}(P)$ and $\pi_P : \mathbb{R} \times (P \times \mathbb{R}) \to P$ is the projection on $P$, then there exists a unique cylindrical almost complex structure $\hat{J}_P$ on $\mathbb{R} \times (P \times \mathbb{R})$ such that $\pi_P$ is holomorphic, that is to say $d\pi_P \circ \hat{J}_P = J_P \circ d\pi_P$. Such an almost complex structure is called the cylindrical lift of $J_P$ and we denote by $J^{cyl}_P(\mathbb{R} \times Y)$ the set of cylindrical almost complex structures on $\mathbb{R} \times Y$ which are cylindrical lifts of admissible almost complex structures on $P$.

Let $J^-, J^+ \in J^{cyl}(\mathbb{R} \times Y)$ such that $J^-$ and $J^+$ coincide outside of a cylinder $\mathbb{R} \times K$ where $K \subset Y$ is compact. For all $T > 0$ we consider an almost complex structure $J$ on $\mathbb{R} \times Y$ equals to $J^-$ on $(-\infty, -T) \times Y$, $J^+$ on $(T, \infty) \times Y$ and equals to the cylindrical lift of an admissible complex structure on $P$ in $[-T, T] \times (Y \setminus K)$. The reason for considering such almost complex structures is that transversality holds generically for moduli spaces of Legendrian contact homology with a cylindrical almost complex structure (see Section 3), and that cylindrical lifts of admissible almost complex structures on $P$ are useful to prevent pseudo-holomorphic curves to escape at infinity (the projection on $P \times \mathbb{R}$ must be compact).

We denote by $J^{adm}_{-, +, T}(\mathbb{R} \times Y)$ the set of almost complex structures on $\mathbb{R} \times Y$ described above, and $J^{adm}(\mathbb{R} \times Y) = \bigcup_{J^-, J^+, T} J^{adm}_{-, +, T}(\mathbb{R} \times Y)$. 
2.3.2. Domain dependent almost complex structure. Considering domain dependent almost complex structures is a way to achieve transversality for moduli spaces of pseudo-holomorphic curves. A domain dependent almost complex structure on $\mathbb{R} \times Y$ is the data, for each $r \in \mathcal{R}^{d+1}$, of an almost complex structure parametrized by $S_r$, that is to say a map in $C^\infty(S_r, \mathcal{J}^{adm}(\mathbb{R} \times Y))$. Then, we need some special behavior of the almost complex structure in strip-like ends in order to get some compatibility with the gluing map.

Fix a $r \in \mathcal{R}^{d+1}$, and let $L_1, \ldots, L_{d+1}$ be transverse exact Lagrangian cobordisms in $\mathbb{R} \times Y$ such that $L_i = (L_i, \ldots, L_{d+1})$ is a choice of Lagrangian label for $S_r$. Let $T > 0$ such that all the $L_i$'s are cylindrical out of $L_i \cap [-T, T] \times Y$, and take $J^\pm \in \mathcal{J}^{cyl}(\mathbb{R} \times Y)$.

For each pair $(L_i, L_{i+1})$, we consider a path $J_{L_i,L_{i+1}}^\pm$ for $t \in [0, 1]$ of almost complex structures in $\mathcal{J}^{adm}_{J^-,J^+,T}(\mathbb{R} \times Y)$, such that it is constant near $t = 0$ and $t = 1$. The type of domain dependent almost complex structures we consider are maps

$$J_r : S_r \to \mathcal{J}^{adm}_{J^-,J^+,T}(\mathbb{R} \times Y)$$

such that $J_{r,L}(\varepsilon_i(s,t)) = J_{L_i,L_{i+1}}^\pm$, where $\varepsilon_i$ is a choice of strip-like ends for $S_r$.

Now, consider a universal choice of strip-like ends. A universal choice of domain dependent almost complex structures is the data, for all $r \in \mathcal{R}^{d+1}$ and Lagrangian label $L_r = (L_1, \ldots, L_{d+1})$, of maps $J_{r,L}$ as above that fit into a smooth map

$$J_{d,L} : S^{d+1} \to \mathcal{J}^{adm}_{J^-,J^+,T}(\mathbb{R} \times Y)$$

defined by $J_{d,L}(z) = J_{r,L}(z)$ if $z \in S_r$. Moreover, $J_{d,L}$ must satisfy $J_{d,L}(\varepsilon_i^{d+1}(r,s,t)) = J_{t,L_i,L_{i+1}}^\pm$ where $\varepsilon_i^{d+1}$ is part of the universal choice of strip-like ends.

Again, if $S \in S^{d+1}$ is obtained from $S_{r_1}, S_{r_2}, \ldots, S_{r_\nu}$ by gluing, we need compatibility conditions between the almost complex structure induced by the universal choice and the one induced by the gluing map. The two choices of almost complex structures are said consistent if there exists a neighborhood $U \subset \mathcal{R}^{d+1}$ of $\partial \mathcal{R}^{d+1}$ such that the choice of strip-like ends is consistent, the choices of almost complex structures coincide on the thin parts for each $r \in U$, and for every sequence $\{r^n\}_{n \in \mathbb{N}}$ in $\mathcal{R}^{d+1}$ converging to a point $r \in \partial \mathcal{R}^{d+1}$, the almost complex structures on the thick parts must converge to the almost complex structure on the thick part of $S_r$.

The latter condition on thick part is analogous to the condition on thin parts, the difference is that we ask for convergence of almost complex structures instead of equality because the almost complex structure on thick parts is not fixed, whereas it is on thin parts. Indeed, a universal choice of almost complex structures depends on fixed paths $J_{L_i,L_{i+1}}^\pm$ for each pair of Lagrangian submanifolds.

Theorem 6. [Sei08] Lemma 9.5] Consistent choices of almost complex structures exist.

2.4. Moduli spaces of holomorphic curves. We are now ready to define the moduli spaces we will use in the next sections.

2.4.1. General definition. Let $\Sigma = (\Sigma_1, \ldots, \Sigma_{d+1})$ be a choice of Lagrangian label such that for all $1 \leq i \leq d + 1$, $\Sigma_i$ is an exact Lagrangian cobordism from $\Lambda^-_i$ to $\Lambda^+_i$. We assume that the cobordisms intersect transversely. We consider then a set $A(\Sigma)$ of asymptotics consisting of intersection points in $\Sigma_i \cap \Sigma_j$ for all $1 \leq i \neq j \leq d + 1$, Reeb chords from $\Lambda^+_i$ to $\Lambda^+_j$, and Reeb chords from $\Lambda^-_i$ to $\Lambda^-_j$ for all $1 \leq i,j \leq d + 1$. Let $J$ be an almost complex structure on $\mathbb{R} \times Y$ (we will explain later the properties needed to achieve transversality in each case), and $j$ the standard almost complex structure on the disk $D^2 \subset \mathbb{C}$, which induces an almost complex structure on each $S_r$, $r \in \mathcal{R}^{d+1}$. For $r \in \mathcal{R}^{d+1}$ and $x_0, \ldots, x_d$ in $A(\Sigma)$, we define the moduli
space $\mathcal{M}_{\Sigma}^\epsilon(x_0; x_1, \ldots, x_d)$ as the set of smooth maps:

$$u: (S_r, J) \to (R \times Y, J)$$

satisfying:

1. $du(z) \circ j = J(z) \circ du(z)$, for all $z \in S_r \setminus \partial S_r$,
2. $u(\partial S_r) \subset \Sigma_i$,
3. if $x_0$ is an intersection point then $\lim_{z \to y_0} u(z) = x_i$ and $x_0$ is required to be a jump from $
\Sigma_{d+1}$ to $\Sigma_1$,
4. if $x_i$, $1 \leq i \leq d$, is an intersection point then $\lim_{z \to y_i} u(z) = x_i$,
5. if $x_0$ is a Reeb chord with a parametrization $\gamma_0: [0, 1] \to x_0$, then every $z \in S_r$ sufficiently close to $y_0$ is in $\epsilon_0((0, +\infty) \times [0, 1])$ and we have the condition $\lim_{s \to +\infty} u(\epsilon_0(s, t)) = (+\infty, \gamma_0(t))$. We say that $u$ has a positive asymptotic to $x_0$ at $y_0$.
6. if $x_i$ for $i > 0$ is a Reeb chord with parametrization $\gamma_i: [0, 1] \to x_i$, then either
   - $\lim_{s \to -\infty} u(\epsilon_i(s, t)) = (-\infty, \gamma_0(t))$ and so $u$ has a negative asymptotic to $x_i$ at $y_i$,
   - $\lim_{s \to -\infty} u(\epsilon_i(s, t)) = (-\infty, \gamma_0(1-t))$ and in this case $u$ has a positive asymptotic to $x_i$ at $y_i$.

Then we denote

$$\mathcal{M}_{\Sigma}^\epsilon(x_0; x_1, \ldots, x_d) = \bigsqcup_r \left( \mathcal{M}_{\Sigma}^\epsilon(x_0; x_1, \ldots, x_d)/\text{Aut}(S_r) \right)$$

The moduli space $\mathcal{M}_{\Sigma}^\epsilon(x_0; x_1, \ldots, x_d)$ can be viewed as the kernel of a section of a Banach bundle. The linearization of this section at a point $u \in \mathcal{M}_{\Sigma}^\epsilon(x_0; x_1, \ldots, x_d)$ is a Fredholm operator. Then, the almost complex structure $J$ is called regular if this operator is surjective. In this case, $\mathcal{M}_{\Sigma}^\epsilon(x_0; x_1, \ldots, x_d)$ is a smooth manifold whose dimension is the Fredholm index of the linearized operator. We will denote by $\mathcal{M}_{\Sigma}^\epsilon(x_0; x_1, \ldots, x_d)$ the moduli space of pseudo-holomorphic curves of index $i$ satisfying the conditions (1)-(6) above.

In the following subsections, in order to simplify notations we will not indicate the almost complex structure we use to define the moduli spaces.

2.4.2. Pseudo-holomorphic curves with boundary on a cylindrical cobordism. The moduli spaces of pseudo-holomorphic curves with boundary on a trivial cobordism $R \times \Lambda$ for $\Lambda \subset Y$ a Legendrian submanifold, are useful to define Legendrian contact homology (Section 3.4). We take here a cylindrical almost complex structure on $R \times Y$. The Lagrangian label takes values in a set of only one Lagrangian $R \times \Lambda$ and thus the set of asymptotics $A(R \times \Lambda)$ consists only of Reeb chords of $\Lambda$. If $\gamma, \gamma_1, \ldots, \gamma_d$ are Reeb chords of $\Lambda$, we denote by

$$\mathcal{M}_{R \times \Lambda}(\gamma; \gamma_1, \ldots, \gamma_d)$$

the moduli space of pseudo-holomorphic curves with boundary on $R \times \Lambda$ that have a positive asymptotic to $\gamma$ and negative asymptotics to $\gamma_1, \ldots, \gamma_d$. There is an action of $R$ by translation on the moduli spaces $\mathcal{M}_{R \times \Lambda}(\gamma; \gamma_1, \ldots, \gamma_d)$, and so we denote the quotient by:

$$\tilde{\mathcal{M}}_{R \times \Lambda}(\gamma; \gamma_1, \ldots, \gamma_d) := \mathcal{M}_{R \times \Lambda}(\gamma; \gamma_1, \ldots, \gamma_d)/R$$

2.4.3. Pseudo-holomorphic curves with boundary on a non cylindrical cobordism. In this subsection we describe moduli spaces that are involved in the definition of the algebra morphism induced by an exact Lagrangian cobordism, from the Chekanov-Eliashberg algebra (Section 3.4) of the positive end to the Chekanov-Eliashberg algebra of the negative end (see Section 5.3).
Let $\Lambda^- \prec_{\Sigma} \Lambda^+$ be an exact Lagrangian cobordism (which is not a trivial cylinder) between two Legendrian submanifolds of $Y$. The set of asymptotics we consider here consists of Reeb chords of $\Lambda^-$ and Reeb chords of $\Lambda^+$. If $\gamma^+ \in \mathcal{R}(\Lambda^+)$ and $\gamma^-_1, \ldots, \gamma^-_d \in \mathcal{R}(\Lambda^-)$, we denote by

$$
\mathcal{M}_{\Sigma}(\gamma^+; \gamma^-_1, \ldots, \gamma^-_d)
$$

the moduli space of pseudo-holomorphic curves with boundary on $\Sigma$ and which have a positive asymptotic to $\gamma^+$ and negative asymptotics to $\gamma^-_1, \ldots, \gamma^-_d$. Contrary to the previous case, there is no $\mathbb{R}$-action on the moduli space because the Lagrangian boundary condition is not $\mathbb{R}$-invariant.

2.4.4. **Pseudo-holomorphic curves with boundary on several exact Lagrangian cobordisms.** Finally, we consider moduli spaces of pseudo-holomorphic curves with boundary on the Lagrangians $\Sigma_1, \ldots, \Sigma_{d+1}$, where $\Lambda^- \prec_{\Sigma_i} \Lambda^+_i$. The choice of Lagrangian label $\Sigma$ takes values in $\{\Sigma_1, \ldots, \Sigma_{d+1}\}$ and the set of asymptotics consists of Reeb chords from $\Lambda^+_i$ to $\Lambda^-_j$ for $1 \leq i, j \leq d+1$, and intersection points in $\Sigma_i \cap \Sigma_j$ for $1 \leq i \neq j \leq d+1$. To simplify notations, for Lagrangian labels we will now denote $\mathbb{R} \times \Lambda^+_1 \times \cdots \times \Lambda^+_d = (\mathbb{R} \times \Lambda^+_1 \times \cdots \times \Lambda^+_d)$ and $\Sigma_{i_1, \ldots, i_k} = (\Sigma_{i_1}, \ldots, \Sigma_{i_k})$, for $1 \leq i_1, \ldots, i_k \leq d$. Moreover this label will indicate only the Lagrangians associated to mixed asymptotics, i.e. intersection points and chords from a Legendrian to another one.

In the case $d = 1$, so when we have two cobordisms, the moduli spaces we will consider are those necessary to define the Cthulhu complex. First, there are moduli spaces of pseudo-holomorphic curves with boundary on the cylindrical ends of the cobordisms and with one positive Reeb chord asymptotic. These are moduli spaces of Legendrian contact homology of $\Lambda^+_1 \cup \Lambda^-_2$:

$$
\mathcal{M}_{\mathbb{R} \times \Lambda^+_1, \Lambda^-_2}(\gamma^+_2, \beta_1; \xi^+_2, \beta_2) \quad \text{and} \quad \mathcal{M}_{\mathbb{R} \times \Lambda^-_1, \Lambda^+_2}(\gamma^-_2, \delta_1, \xi^-_2, \delta_2)
$$

for $\gamma^+_2, \xi^+_2, \beta_1, \beta_2 \in \mathcal{R}(\Lambda^+_2, \Lambda^-_1)$, $\gamma^-_2, \xi^-_2, \delta_1, \delta_2 \in \mathcal{R}(\Lambda^-_2, \Lambda^+_1)$, and $\beta_i$ (resp. $\delta_i$) a word of Reeb chords of $\Lambda^+_i$ (resp. $\Lambda^-_i$) for $i = 1, 2$. We will also consider moduli spaces of curves with boundary on the negative cylindrical ends of the cobordisms and with two positive Reeb chord asymptotics:

$$
\mathcal{M}_{\mathbb{R} \times \Lambda^-_1, \Lambda^+_2}(\gamma^-_2, \delta_1, \gamma^-_1, \delta_2)
$$

with $\gamma^-_2 \in \mathcal{R}(\Lambda^-_2, \Lambda^+_1)$ and $\gamma^-_1 \in \mathcal{R}(\Lambda^-_1, \Lambda^+_2)$. Such pseudo-holomorphic curves are called bananas. Finally, we consider moduli spaces of curves with boundary on the compact parts of the cobordisms:

$$
\mathcal{M}_{\Sigma_1}(\gamma^+_2, \delta_1, q, \delta_2), \mathcal{M}_{\Sigma_2}(\gamma^+_2, \delta_1, \xi^-_2, \delta_2),
$$

$$
\mathcal{M}_{\Sigma_1, \Sigma_2}(x^+, \delta_1, q, \delta_2), \mathcal{M}_{\Sigma_1, \Sigma_2}(x^+, \delta_1, \xi^-_2, \delta_2)
$$

with $x^+, q \in \Sigma_1 \cap \Sigma_2$ such that $x^+$ is a jump from $\Sigma_2$ to $\Sigma_1$ and $q$ is a jump from $\Sigma_1$ to $\Sigma_2$.

Let us consider now the case $d = 2$, we have three transverse exact Lagrangian cobordisms $\Sigma_1, \Sigma_2$ and $\Sigma_3$. The moduli spaces we describe here are involved in the definition of the product structure (Subsection 5.1) and in the definition of the order-2 map $\mathcal{F}^2$ of the $A_{\infty}$-functor of Theorem 5.3. Again, first, we have moduli spaces of pseudo-holomorphic curves with boundary on the cylindrical ends of the cobordisms:

$$
\mathcal{M}_{\mathbb{R} \times \Lambda^-_{1,2,3}}(\gamma_{3,1}, \delta_1, c_1, \delta_2, c_2, \delta_3)
$$

where if we denote $\gamma_{i,j}$ a chord in $\mathcal{R}(\Lambda^-_i, \Lambda^-_j)$, we have $c_1 \in \{\gamma_{1,2}, \gamma_{2,1}\}$ and $c_2 \in \{\gamma_{2,3}, \gamma_{3,2}\}$, and again $\delta_i$ is a word of Reeb chords of $\Lambda^-_i$, for $i = 1, 2, 3$, which are negative Reeb chords asymptotics. Remark that, except for the case $c_1 = \gamma_{2,1}$ and $c_2 = \gamma_{3,2}$, such moduli spaces are moduli spaces of curves with two or three positive Reeb chords asymptotics. Then, we consider also moduli spaces of curves with boundary on non cylindrical parts:

$$
\mathcal{M}_{\Sigma_{1,2,3}}(x^+, \delta_1, a_1, \delta_2, a_2, \delta_3)
$$
with \( x^+ \in \Sigma_1 \cap \Sigma_3 \) jump from \( \Sigma_1 \) to \( \Sigma_1 \), \( a_1 \in \{x_1, \gamma_{2,1}\} \) and \( a_2 \in \{x_2, \gamma_{3,2}\} \), with \( x_1 \in \Sigma_1 \cap \Sigma_2 \) jump from \( \Sigma_1 \) to \( \Sigma_2 \), and \( x_2 \in \Sigma_2 \cap \Sigma_3 \) jump from \( \Sigma_2 \) to \( \Sigma_3 \). The curves in these moduli spaces have a positive asymptotic to \( x^+ \) and negative asymptotics to the other punctures. We also consider moduli spaces of curves with a different Lagrangian label:

\[
M_{\Sigma_{1,2}}(x_2; \delta_1, \gamma_1, \delta_1, \gamma_2, \delta_1) \\
M_{\Sigma_{1,3}}(x_2; \delta_1, \gamma_1, \delta_1, \gamma_2, \delta_2) \\
M_{\Sigma_{2,3}}(x_1; \delta_2, \gamma_2, \delta_3, \gamma_1, \delta_1)
\]

The reason for changing the Lagrangian label is mainly conventional. Indeed, we have moduli spaces of curves such that \( x_2 \), which is a jump from \( \Sigma_2 \) to \( \Sigma_3 \) is a positive asymptotic, so the Lagrangian label has to be changed to satisfy the property (3) of the general definition of moduli spaces (Section 2.4.1). Finally, moduli spaces involved in the definition of \( F^2 \) are of the following type:

\[
M_{\Sigma_{1,2}}(\gamma_{1,3}; \delta_1, a_1, \delta_2, a_2, \delta_3)
\]

where \( a_1 \in \{x_1, \gamma_{2,1}\} \) and \( a_2 \in \{x_2, \gamma_{3,2}\} \) as before, and \( \gamma_{1,3}^+ \in \mathcal{R}(\Lambda_1^+, \Lambda_3^+) \).

### 2.5. Action and energy

Consider \( d+1 \) transverse exact Lagrangian cobordisms (\( \Sigma_1, \ldots, \Sigma_{d+1} \)). Recall that by definition, associated to each cobordism there is a function \( f_i: \Sigma_i \rightarrow \mathbb{R} \), primitive of the form \( e^t a|_{\Sigma_i} \), and this function is constant on the cylindrical ends of \( \Sigma_i \). Without loss of generality, we can consider that the constants in the negative ends of the cobordisms are zero, and we denote \( c_i \) the constant for the positive end of \( \Sigma_i \). We also denote \( T > 0 \) and \( \epsilon > 0 \) such that the cobordisms \( \Sigma_i \) are all cylindrical out of \( \Sigma_i \cap ([-T + \epsilon, T - \epsilon] \times Y) \). To each asymptotic, we can associate a quantity called action as follows. For an intersection point \( x \in \Sigma_i \cap \Sigma_j \) with \( i > j \), the action of \( x \) is given by:

\[
a(x) = f_i(x) - f_j(x)
\]

For a Reeb chord \( \gamma \), the length of \( \gamma \) is given by \( \ell(\gamma) := \int_\gamma \alpha \) and then the action of \( \gamma_{i,j}^+ \in \mathcal{R}(\Lambda_i^+, \Lambda_j^+) \) is defined by:

\[
a(\gamma_{i,j}^+) = e^T \ell(\gamma_{i,j}^+) + c_i - c_j
\]

and for a Reeb chord \( \gamma_{i,j}^- \in \mathcal{R}(\Lambda_i^+, \Lambda_j^+) \) we set:

\[
a(\gamma_{i,j}^-) = e^{-T} \ell(\gamma_{i,j}^-)
\]

Remark that Reeb chords have always a positive action whereas intersection points can be of negative action. Then, to a pseudo-holomorphic curve \( u \) in \( M_{\Sigma}(x_0; x_1, \ldots, x_d) \) is associated an energy, which is the analogue of the area for the case of pseudo-holomorphic curves in compact symplectic manifolds. To define it, let \( \chi: \mathbb{R} \rightarrow \mathbb{R} \) be a function such that:

\[
\begin{align*}
\chi(t) = e^t & \quad \text{if } t \in [-T + \epsilon, T - \epsilon] \\
\lim_{t \to +\infty} \chi(t) = e^T \quad \text{if } t \to +\infty \\
\lim_{t \to -\infty} \chi(t) = e^{-T} \quad \text{if } t \to -\infty \\
\chi(t) > 0 & \quad \text{otherwise}
\end{align*}
\]

We define then the \( d(\chi) \)-energy of a pseudo-holomorphic curve \( u: S_r \rightarrow \mathbb{R} \times Y \) by:

\[
E_{d(\chi)}(u) = \int_{S_r} u^* d(\chi)
\]
We have the following very standard result:

**Lemma 1.** $E_{d(\chi_0)}(u) \geq 0$

**Proof.** The $d(\varepsilon \alpha)$-compatibility of the almost complex structure $J$ implies the $d\alpha$-compatibility of the restriction of $J$ to the contact structure $(\xi, (d\alpha)_\xi)$. This permits to show that $E_{d(\chi_0)}(u) = \frac{1}{2} \int_S |du|^2$, where $|v|^2 = d(\chi_0)(v, Jv)$ is strictly positive if $v \neq 0$. □

Now, the energy of a pseudo-holomorphic curve can be expressed by the actions of its asymptotics.

**Proposition 1.** We have the following:

1. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \ldots, \gamma_d)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - \sum_i a(\gamma_i)$,
2. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \ldots, \gamma_d)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - \sum_i a(\gamma_i)$,
3. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) + a(\gamma_2) - a(\delta_1)$,
4. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
5. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
6. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
7. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
8. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
9. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
10. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
11. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
12. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
13. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
14. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
15. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$,
16. if $u \in \mathcal{M}_{\mathbb{S}}(\gamma^+; \gamma_1, \gamma_2, \delta_1)$, then $E_{d(\chi_0)}(u) = a(\gamma^+) - a(\gamma_2) - a(\delta_1)$.

Lemma 1 and Proposition 1 give thus some constraints on the action of asymptotics of pseudo-holomorphic curves. These will be useful in order to cancel some pseudo-holomorphic configurations in Section 5.
2.6. Compactness. When transversality holds, i.e. when the almost complex structure is regular for moduli spaces, these are smooth manifolds which are not necessarily compact. However, they admit a compactification in the sense of Gromov ([Gro85]), by adding broken curves called pseudo-holomorphic buildings. Compactness results together with transversality results imply that the compactification of a moduli space is a compact manifold whose boundary components are in bijection with pseudo-holomorphic buildings arising as degeneration of pseudo-holomorphic curves in the moduli space.

Given again $d + 1$ transverse exact Lagrangian cobordisms $\Lambda^-_i \leadsto \Sigma$, $\Lambda^+_i$, we consider the following Lagrangian labels $\Sigma = (\Sigma_1, \ldots, \Sigma_{d+1})$ and $\mathbb{R} \times \Lambda^\pm = (\mathbb{R} \times \Lambda^+_1, \ldots, \mathbb{R} \times \Lambda^+_{d+1})$. We first recall the definition of pseudo-holomorphic buildings whose components are disks with cylindrical Lagrangian boundary conditions.

**Definition 3.** A pseudo-holomorphic building of height $k$ in $\mathbb{R} \times Y$ with boundary on $\mathbb{R} \times \Lambda^\pm$ is given by a finite number of pairs $B = \{(v_i, \rho_i), i \in I\}$, such that each pair $(v, \rho) \in B$ consists of a pseudo-holomorphic disk $v$ asymptotic to Reeb chords and an integer $1 \leq \rho \leq k$ corresponding to the floor. Moreover, the following conditions must be satisfied:

1. each floor contains at least one disk which is not a trivial strip,
2. for each pair $(v, \rho)$ with $1 < \rho < k$, there exist disks $v^{+1}_1, \ldots, v^{+\rho}$ in the floor $\rho + 1$ and $v^{-1}_1, \ldots, v^{-\rho}$ in the floor $\rho - 1$ such that:
   - each positive asymptotic of $v$ is a negative asymptotic of a disk $v^{+\rho}$,
   - each negative asymptotic of $v$ is a positive asymptotic of a disk $v^{-\rho}$,
   - but the disks $v^{+\rho}$ and $v^{-\rho}$ can also have other asymptotics that those coming from $v$,
3. for each pair $(v, k)$, the negative asymptotics of $v$ are positive asymptotics of some disks in the floor $k - 1$,
4. for each pair $(v, 1)$ the positive asymptotics of $v$ are negative asymptotics of some disks in the floor 2,
5. there is a map $\overline{\pi}: S \to \mathbb{R} \times Y$ defined on some $S \in S^{d+1}$ that is obtained from the domains $(S_i)_{i \in I}$ of the disks $(v_i)_{i \in I}$ in $B$ by the gluing operation (see Section 2.1) with the conditions that $S_i$ can be glued to $S_j$ at a puncture $y$ if:
   - the floors associated to $v_i$ and $v_j$ differ by 1,
   - $v_i$ has a negative asymptotic to a chord $\gamma$ at $y$, and $v_j$ has a positive asymptotic to $\gamma$ at $y$.

**Remark 4.** In the definition above, the glued map $\overline{\pi}$ has positive asymptotics to the positive asymptotics of the disks $(v_i, k) \in B$ and negative asymptotics to the negative asymptotics of the disks $(v_j, 1) \in B$.

We give now a definition of pseudo-holomorphic building whose components are pseudo-holomorphic disks with boundary on Lagrangian cobordisms with cylindrical ends (see [BEH+03] and [Abb14]).

**Definition 4.** A pseudo-holomorphic building of height $k^- \mid k^+$ in $\mathbb{R} \times Y$ with boundary on $\Sigma$ is given by a set of pseudo-holomorphic disks divided in three levels as follows:

- the top level contains a pseudo-holomorphic building of height $k^+$ with boundary on $\mathbb{R} \times \Lambda^+$,
- the bottom level contains a pseudo-holomorphic building of height $k^-$ with boundary on $\mathbb{R} \times \Lambda^-$,
- the central level is an intermediate floor containing pseudo-holomorphic disks $u_i$ with boundary on $\Sigma$ and with asymptotics in $A(\Sigma)$,

so that if we denote $\overline{\pi}_t$ (resp. $\overline{\pi}_b$) the glued map corresponding to the top (resp. bottom) level, the following conditions are satisfied:
(1) each negative Reeb chord asymptotic of $\tau_t$ is a positive Reeb chord asymptotic of a disk in the central level, and reciprocally each positive Reeb chord asymptotic of a disk in the central level is identified with a negative Reeb chord asymptotic of $\tau_t$.

(2) each positive Reeb chord asymptotic of $\tau_b$ is a negative Reeb chord asymptotic for a disk in the central level, and the converse is true.

(3) there is a map $\tau : S \rightarrow \mathbb{R} \times Y$ defined on some $S \in S^{d+1}$ that is obtained from the domains of $\tau_t$, of $\tau_b$, and of the disks in the central level, by the gluing operation. Then $\tau$ has possibly asymptotics to some intersection points, and then it has positive Reeb chords asymptotics to the positive asymptotics of $\tau_t$ and negative Reeb chords asymptotics to the negative asymptotics of $\tau_b$ (see Figure 1).

**Figure 1.** Example of a pseudo-holomorphic building of height 1/1/2.

Given a set of asymptotics $(x_0, x_1, \ldots, x_d)$, denote by $\mathcal{M}_{\Sigma}^{-1}[1][k^+] \times \mathbb{R} \times \Lambda^\pm$ the set of pseudo-holomorphic buildings of height $k^{-1}[1][k^+]$ with boundary on $\Sigma$ such that the corresponding glued map is asymptotic to $(x_0, x_1, \ldots, x_d)$. Moduli spaces of disks with boundary on $\mathbb{R} \times \Lambda^\pm$ described in Subsection 2.4 can be viewed as pseudo-holomorphic buildings of height 1 with boundary on $\mathbb{R} \times \Lambda^\pm$, and moduli spaces of pseudo-holomorphic disks with boundary on non-cylindrical Lagrangian cobordisms can be viewed as pseudo-holomorphic buildings of height 0/1/0 with boundary on $\Sigma$ in other words we have

$$\mathcal{M}_{\Sigma}(x_0; x_1, \ldots, x_d) \subset \mathcal{M}_{\Sigma}^{0}[1][0](x_0; x_1, \ldots, x_d)$$

By Gromov’s compactness, a sequence of pseudo-holomorphic disks $u_s$ in $\mathcal{M}(x_0; x_1, \ldots, x_d)$ admits a subsequence which converges to a pseudo-holomorphic building with boundary on $\mathbb{R} \times \Lambda^\pm$ (resp. $\Sigma$) if the curves $u_s$ have boundary on $\mathbb{R} \times \Lambda^\pm$ (resp. $\Sigma$). The pseudo-holomorphic buildings obtained this way are of two types:
(1) **Stable breaking:** pseudo-holomorphic building such that each component is a curve having at least three mixed asymptotics. For example, a pseudo-holomorphic building in a product
\[ \mathcal{M}(x_0; x_1, \ldots, x_{i-1}, x'_i, x_{i+j}, \ldots, x_d) \]
with \( 1 \leq i \leq d-1 \) and \( j \geq 2 \).

(2) **Unstable breaking:** pseudo-holomorphic building having at least a curve with at most two mixed asymptotics. Such a curve is either a pseudo-holomorphic half-plane (so without mixed asymptotic), or a pseudo-holomorphic strip.

The important result is that the set of buildings asymptotic to \( x_0, x_1, \ldots, x_d \) gives a compactification of the moduli space \( \mathcal{M}(x_0; x_1, \ldots, x_d) \), i.e. the disjoint union
\[ \bigsqcup_{k^{-}, k^{+} \geq 0} \mathcal{M}_{\Sigma}^{k^{-} | k^{+}}(x_0; x_1, \ldots, x_d) \]
is compact.

### 3. Legendrian Contact Homology

Legendrian contact homology is a Legendrian isotopy invariant which has been defined by Chekanov [Che02] and Eliashberg [Eli98] independently. Eliashberg gave a definition of Legendrian contact homology in the setting of Symplectic Field Theory (SFT, see [EGH00]), while Chekanov defined it combinatorially, by a count of certain types of convex polygons with boundary. In fact, it has been proven by Etnyre, Ng and Sabloff [ENS02] that the SFT-version of Legendrian contact homology computes the same invariant as the combinatorial version of Chekanov and its generalization in higher dimension. With this in mind, we recall below the SFT-definition of Legendrian contact homology, which is more in the spirit of this article.

#### 3.1. The differential graded algebra

Given a Legendrian submanifold \( \Lambda \subset P \times \mathbb{R} \), we denote by \( C(\Lambda) \) the \( \mathbb{Z}_2 \)-vector space generated by Reeb chords of \( \Lambda \), and \( \mathcal{A}(\Lambda) = \bigoplus_{i} C(\Lambda)^{\otimes i} \) the tensor algebra of \( C(\Lambda) \), called the *Chekanov-Eliashberg algebra* of \( \Lambda \). There is a grading associated to Reeb chords and defined from the Conley-Zehnder index by the following: if \( \Lambda \) is connected and \( c \in \mathcal{R}(\Lambda) \) then we set \( |c| := \nu_{\gamma_c}(c) - 1 \), where \( \gamma_c \) is a capping path for \( c \). This is a well-defined grading in \( \mathbb{Z} \) modulo the Maslov number of \( \Lambda \) (because of the choice of the capping path) and twice the first Chern class of \( TP \) (because of the choice of a symplectic trivialization of \( TP \) along \( \Pi_P(\gamma_c) \) to compute \( \nu_{\gamma_c}(c) \)), see [EES07] for more details. This induces a grading for each word of Reeb chords in \( \mathcal{A}(\Lambda) \) by \( |b_1 b_2 \ldots b_m| = \sum_i |b_i| \) for Reeb chords \( b_i \). If \( \Lambda \) is not connected and \( c \) is a mixed chord with ends \( c^+ \in \Lambda^+ \) and \( c^- \in \Lambda^- \), where \( \Lambda^\pm \) are connected components of \( \Lambda \), in order to define the grading we choose some points \( p^\pm \in \Lambda^\pm \) and some paths \( \gamma^+_c \subset \Lambda^+ \) and \( \gamma^-_c \subset \Lambda^- \) from \( c^+ \) to \( p^+ \) and from \( p^- \) to \( c^- \) respectively. Then we choose a path \( \gamma^+_- \) from \( p^+ \) to \( p^- \) and so if we denote \( \Gamma_c = \gamma^+_c \cup \gamma^+_- \cup \gamma^-_c \) the concatenation of the three paths, the degree of \( c \) is defined to be \( |c| = \nu_{\gamma_c}(c) - 1 \). The grading of mixed chords depend on the paths \( \gamma^+_- \) but for two mixed chords \( c_1, c_2 \) from \( \Lambda^- \) to \( \Lambda^+ \), the difference \( |c_1| - |c_2| \) does not depend on \( \gamma^+_- \).

Let \( J \) be a cylindrical almost complex structure on \( \mathbb{R} \times Y \). The differential on \( \mathcal{A}(\Lambda) \) is a map \( \partial : \mathcal{A}(\Lambda) \to \mathcal{A}(\Lambda) \) which is defined by a count of pseudo-holomorphic disks in \( \mathbb{R} \times Y \) with
boundary on $\mathbb{R} \times \Lambda$ and asymptotic to Reeb chords. More precisely, if $a \in \mathcal{R}(\Lambda)$:

$$\partial(a) = \sum_{m \geq 0} \sum_{b_1 \cdots b_m \mid b \mid = \mid a \mid - 1} \#\widetilde{\mathcal{M}}_{\mathbb{R} \times \Lambda}(a; b) \cdot b$$

(1)

where $a$ is a positive asymptotic and the chords $b_i$ are negative Reeb chord asymptotics. When $m = 0$ we set $b = 1$. This induces a map on the whole algebra by Leibniz rule.

About transversality results, Dimitroglou-Rizell proved in [DR16a] that generically, a cylindrical almost complex structure on $\mathbb{R} \times Y$ is regular for the moduli spaces $\mathcal{M}_{\mathbb{R} \times \Lambda}(a; b_1, \ldots, b_m)$ which are thus manifolds of dimension

$$\dim \mathcal{M}_{\mathbb{R} \times \Lambda}(a; b_1, \ldots, b_m) = \mid a \mid - \sum \mid b_i \mid$$

and so

$$\dim \widetilde{\mathcal{M}}_{\mathbb{R} \times \Lambda}(a; b_1, \ldots, b_m) = \mid a \mid - \sum \mid b_i \mid - 1$$

This is done by generalizing a result of Dragnev ([Dra04]) to the case of pseudo-holomorphic disks, using the fact that as pseudo-holomorphic curves in the moduli spaces above have only one positive Reeb chord asymptotic, it is always possible to find an injective point. These dimension formula imply that in the definition of the differential (1), this is a mod-2 count of pseudo-holomorphic disks in 0-dimensional moduli spaces. Then, by Gromov’s compactness these 0-dimensional moduli spaces are compact and thus the differential $\partial$ is a well-defined map of degree $-1$. Transversality also holds for almost complex structures that are cylindrical lifts of regular compatible almost complex structures on $P$ (satisfying a technical condition near the intersection points of $\Pi_P(\Lambda)$, see [DR16a, EES07]).

Theorem 7. [Che02, EES05a, EES07, DR16b]

- $\partial^2 = 0$,
- The Legendrian contact homology $LCH_*(\Lambda, J)$ does not depend on a generic choice of cylindrical almost complex structure $J$ and is a Legendrian isotopy invariant.

Legendrian contact homology being generally of infinite dimension, we recall in the next section the linearization process introduced by Chekanov, in order to extract finite dimensional (and so more computable) invariants from Legendrian contact homology.

3.2. Linearization and the augmentation category. We begin this section by recalling the fundamental tool for the linearization: augmentations.

Definition 5. An augmentation for $(\mathcal{A}(\Lambda), \partial)$ is a DGA-map $\varepsilon : \mathcal{A}(\Lambda) \rightarrow \mathbb{Z}_2$ where $\mathbb{Z}_2$ is viewed as a DGA with vanishing differential. In other words, $\varepsilon$ is a map satisfying:

- $\varepsilon(a) = 0$ if $\mid a \mid \neq 0$,
- $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$,
- $\varepsilon \circ \partial = 0$.

A Legendrian submanifold does not necessarily admit an augmentation. Typically, once there is an element of the algebra $a \in \mathcal{A}(\Lambda)$ such that $\partial a = 1$, the third condition in the definition above cannot be satisfied and hence there is no augmentation. For example, loose Legendrians (see [Mur]) do not admit augmentation. In this paper, we will only focus on Legendrians whose Chekanov-Eliashberg algebra can be augmented. So let us consider a Legendrian submanifold $\Lambda \subset \mathbb{R} \times Y$ such that $\mathcal{A}(\Lambda)$ admits an augmentation, then it is possible to associate to $\Lambda$ a new
complex \((C(\Lambda), \partial^{c}_f)\), with \(\partial^{c}_f\) defined on Reeb chords by:

\[
\partial^{c}_f(a) = \sum_{m \geq 0} \sum_{b_0 = b_1, \ldots, b_{m-1}} \sum_{i=1}^{m} \# \mathcal{M}_{\mathbb{R} \times \Lambda}(a; b) \cdot \varepsilon(b_1) \ldots \varepsilon(b_{i-1}) \varepsilon(b_{i+1}) \ldots \varepsilon(b_m) \cdot b_i
\]

In fact, conjugating the differential \(\partial\) by the DGA-morphism \(g_c\) defined on chords by \(g_c(c) = c + \varepsilon(c)\) gives a new differential \(\partial\) on \(\mathcal{A}(\Lambda)\), the differential \(\partial\) twisted by \(\varepsilon\), such that the restriction on \(C(\Lambda)\) can be decomposed as \(\partial^{C(\Lambda)} = \sum_{i \geq 0} \partial_i\), with \(\partial^{C(\Lambda)} : C(\Lambda) \to C(\Lambda)^{\otimes i}\). But the differential \(\partial^{C(\Lambda)}\) does not admit any constant term (i.e. \(\partial_0 = 0\)) due to the properties of \(\varepsilon\), and so \((\partial^{C(\Lambda)})^2 = 0\) implies that \((\partial^{c}_f)^2 = 0\). The homology of the complex \((C(\Lambda), \partial^{c}_f)\) is by definition the Legendrian contact homology of \(\Lambda\) linearized by \(\varepsilon\).

**Theorem 8.** [Che02] The set \(\{LCH^c_*(\Lambda), \varepsilon\}\) of linearized Legendrian contact homologies is a Legendrian isotopy invariant.

This linearization process can be done using two augmentations instead of one (see [BC14]), leading to the bilinearized Legendrian contact homology \(LCH^{c_{1},c_{2}}(\Lambda)\), which is the homology of the complex \((C(\Lambda), \partial^{c_{1},c_{2}}_f)\) with

\[
\partial^{c_{1},c_{2}}_f(a) = \sum_{m \geq 0} \sum_{b_0 = b_1, \ldots, b_{m-1}} \sum_{i=1}^{m} \# \mathcal{M}_{\mathbb{R} \times \Lambda}(a; b) \cdot \varepsilon_1(b_1) \ldots \varepsilon_1(b_{i-1}) \varepsilon_2(b_{i+1}) \ldots \varepsilon_2(b_m) \cdot b_i
\]

The advantage of the bilinearized version in comparison to the linearized one is that it retains some information about the non-commutativity of the Chekanov-Eliashberg DGA. More generally, given \(d + 1\) augmentations \(\varepsilon_1, \ldots, \varepsilon_{d+1}\) of \(\mathcal{A}(\Lambda)\), there is a map

\[
\partial^{c_{1},\ldots,c_{d+1}}_d : C(\Lambda) \to C(\Lambda)^{\otimes d}
\]

such that \(\partial^{c_{1},\ldots,c_{d+1}}_d(a)\) is a sum of words of length \(d\) coming from words in \(\partial a\) to which we keep \(d\) letters and augment the others by \(\varepsilon_1, \ldots, \varepsilon_{d+1}\) in this order (changing the augmentation each time we jump a chord we keep). In all the rest of the article, we will adopt a cohomology point of view, so let us describe the dual maps of the maps \(\partial^{c_{1},\ldots,c_{d+1}}_d\). As the vector space \(C(\Lambda)\) and its dual are canonically isomorphic, by an abuse of notation we will still denote \(C(\Lambda)\) the dual vector space. So the dual of \(\partial^{c_{1},\ldots,c_{d+1}}_d\), denoted \(\mu^{d}_{\varepsilon_{d+1},\ldots\varepsilon_1}(\Lambda, \delta_{1}, b_1, \ldots, b_{d+1})\), is defined by:

\[
\mu^{d}_{\varepsilon_{d+1},\ldots\varepsilon_1}(b_d, \ldots, b_1) = \sum_{a \in \mathcal{R}^{d}_{c\varepsilon_1}(\Lambda)} \sum_{\delta_1, \ldots, \delta_{d+1}} \# \mathcal{M}_{\mathbb{R} \times \Lambda}(a; \delta_1, b_1, \delta_2, \ldots, \delta_d, b_d, \delta_{d+1}) \varepsilon_1(\delta_1) \ldots \varepsilon_{d+1}(\delta_{d+1}) \cdot a
\]

where \(\delta_i\) are words of Reeb chords of \(\Lambda\). In fact, as already explained above for the dual map, the coefficient \(\mu^{d}_{\varepsilon_{d+1},\ldots\varepsilon_1}(b_d, \ldots, b_1, a)\) is computed by looking at all words of length at least \(d\) in \(\partial(a)\) containing the letters \(b_1, \ldots, b_d\) in this order, and augmenting the (possibly) remaining chords between \(b_i\) and \(b_{i+1}\) by \(\varepsilon_{i+1}\), for all \(1 \leq i \leq d\). These maps \(\{\mu^{d}_{\varepsilon_{d+1},\ldots\varepsilon_1}\}_{d \geq 1}\) satisfy the \(A_\infty\)-relations, i.e. for all \(d \geq 1\) and Reeb chords \(b_d, \ldots, b_1\) we have

\[
\sum_{\substack{1 \leq j \leq d \\atop 0 \leq n \leq d-j}} \mu^{d-j+1}_{\varepsilon_{d+1},\ldots\varepsilon_{n+1},\varepsilon_{n+1},\ldots\varepsilon_1}(b_d, \ldots, \mu^{j}_{\varepsilon_{n+j+1},\ldots\varepsilon_{n+1}}(b_{n+j}, \ldots, b_{n+1}), b_n, \ldots, b_1) = 0
\]

and thus the maps \(\{\mu^{d}_{\varepsilon_{d+1},\ldots\varepsilon_1}\}_{d \geq 1}\) are \(A_\infty\)-composition maps of an \(A_\infty\)-category called the augmentation category of \(\Lambda\), denoted \(\text{Aug}_-(\Lambda)\). This category has been defined by Bourgeois and Chantraine in [BC14] as follows:
- \text{Ob}(\text{Aug}(\Lambda)) : \varepsilon \text{ augmentation of } A(\Lambda),
- \text{hom}(\varepsilon_1, \varepsilon_2) = (C(\Lambda), \mu_{\varepsilon_1, \varepsilon_2}) \text{ the bilinearized Legendrian contact cohomology complex},
- the \text{A}_\infty\text{-composition maps are the maps } \mu_{e_{i_1}, \ldots, e_1}^2 \text{ defined above.}

If we look at the full subcategory generated by one object \varepsilon, then we get the \text{A}_\infty\text{-algebra } (C(\Lambda), \{\mu_{d}^2\}_{d \geq 1}) \text{ that appeared first in a work of Civan, Etnyre, Koprowski, Sivek and Walker [CKE+11]}. In fact, the \text{A}_\infty\text{-maps of the augmentation category can be viewed as dual maps of components of the differential of the } (d+1)\text{-copy of } \Lambda \text{ twisted by a particular augmentation. This is a way to show that the } \text{A}_\infty\text{-relations are satisfied, using a bijection between some moduli spaces with boundary on } \Lambda \text{ and some moduli spaces with boundary on the } k\text{-copy of } \Lambda \text{ (see [EES09] Theorem 3.6).}

For } k \geq 1, \text{ the } k\text{-copy of } \Lambda \text{ denoted } \Lambda(\Lambda) \text{ is defined as follows. Set } \Lambda_1 := \Lambda, \text{ and for a small } \varepsilon > 0 \text{ we define } \Lambda_j := \varphi_{(j, -1)}^R(\Lambda) \text{ for } 2 \leq j \leq k, \text{ where } \varphi_t^R \text{ is the Reeb flow (recall } R_\alpha = \partial_t \text{ here). The Legendrian submanifold } \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k \text{ has an infinite number of Reeb chords, so we have to perturb it to turn it into a chord generic Legendrian. Take Morse functions } f_j : \Lambda \to \mathbb{R}, \text{ for } 2 \leq j \leq k, \text{ such that the functions } f_i - f_j \text{ are Morse. Then, identify a small tubular neighborhood of } \Lambda_j \text{ to a neighborhood of the 0-section in } J^1(\Lambda), \text{ and replace } \Lambda_j \text{ by the 1-jet of } f_j \text{ which is by definition the submanifold } J^1(f_j) = \{ (q, \partial_q f_j, f_j(q)) \mid q \in A \} \subset J^1(\Lambda). \text{ We denote this new Legendrian } \Lambda_1 \subset P \times \mathbb{R}. \text{ The } k\text{-copy of } \Lambda \text{ is defined to be the union } \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k. \text{ It is a chord generic Legendrian which has four different types of Reeb chords:}

1. \text{pure chords: } \text{chords of } \Lambda_j, \text{ for all } 2 \leq j \leq k, \text{ and there is a bijection between } \mathcal{R}(\Lambda) \text{ and } \mathcal{R}(\Lambda_j),
2. \text{Morse chords: } \text{mixed chords corresponding to critical points of the functions } f_i - f_j,
3. \text{small chords: } \text{mixed chords (which are not Morse) from } \Lambda_i \text{ to } \Lambda_j \text{ for } i > j, \text{ in bijection with chords of } \Lambda,
4. \text{long chords: } \text{mixed chords (which are not Morse) from } \Lambda_j \text{ to } \Lambda_i \text{ for } i > j \text{ also in bijection with chords of } \Lambda.

\text{Let us denote } C(\Lambda_1, \Lambda_j) \text{ the } \mathbb{Z}_2\text{-vector space generated by Reeb chords from } \Lambda_1 \text{ to } \Lambda_j \text{ which are not Morse. Denote by } C_M(\Lambda_1, \Lambda_j) \text{ the } \mathbb{Z}_2\text{-vector space generated by Morse chords from } \Lambda_j \text{ to } \Lambda_1, \text{ and observe that it is non-zero only if } j < i. \text{ Hence we have the following decomposition:}

\[ C(\Lambda_1, \Lambda_k) = C(\Lambda_1) \bigoplus_{1 \leq i < k} C(\Lambda_i) \bigoplus_{1 \leq j < k} C_M(\Lambda_i, \Lambda_j) \bigoplus_{1 \leq i < j < k} (C(\Lambda_j, \Lambda_1) \oplus C(\Lambda_i, \Lambda_j)) \]

\text{For a chord } a \in \mathcal{R}(\Lambda), \text{ denote by } a_{i,j} \text{ the corresponding Reeb chord in } \mathcal{R}(\Lambda_1, \Lambda_j). \text{ Let } (\varepsilon_1, \ldots, \varepsilon_k) \text{ be augmentations of } \mathcal{A}(\Lambda) \text{ and consider the DGA-morphism } \varepsilon(\Lambda_1, \Lambda_k) \to \mathbb{Z}_2 \text{ defined on Reeb chords by:}

\[ \varepsilon(\Lambda_1, \Lambda_k) = \varepsilon(\Lambda) \]
\[ \varepsilon(\Lambda_1, \Lambda_k) = \varepsilon_i(a) \]
\[ \varepsilon(\Lambda_1, \Lambda_k) = \varepsilon_0 \text{ for } i \neq j \]
\[ \varepsilon(\Lambda_1, \Lambda_k) = 0 \text{ for } e_M \text{ Morse chord} \]

\text{It is shown in [BCL14] that } \varepsilon(\Lambda_1, \Lambda_k) \text{ is an augmentation of } \mathcal{A}(\Lambda_1, \Lambda_k), \text{ that we call } \text{diagonal augmentation induced by } \varepsilon_1, \ldots, \varepsilon_k. \text{ Now, given such an augmentation, we can compute the Legendrian contact homology of } \Lambda(\Lambda) \text{ with the twisted differential } \partial_{\varepsilon(\Lambda)}^0. \text{ Restricted to } C(\Lambda_1, \Lambda_k), \text{ this is the map}

\[ \partial_{\varepsilon(\Lambda)}^0 : C(\Lambda_1, \Lambda_k) \to \bigoplus_{d \geq 1} C(\Lambda_{i_d}, \Lambda_k) \otimes C(\Lambda_{i_{d-1}}, \Lambda_{i_d}) \otimes \cdots \otimes C(\Lambda_1, \Lambda_{i_2}) \]
The dual of each component of this map is then \( \mu^d_{\varepsilon, \varepsilon_1} \). Then, dualizing the relation

\[
(\partial_{\varepsilon}^1)^2 = 0
\]
gives all the \( \Lambda_\infty \)-relations for \( d \leq k - 1 \), i.e. the \( \Lambda_\infty \)-relations for each sequence of objects \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) \). For example, the two first are:

\[
(\mu^1_{\varepsilon, \varepsilon_1})^2 = 0
\]

\[
\mu^2_{\varepsilon, \varepsilon_1} + \mu^2_{\varepsilon, \varepsilon_2, \varepsilon_1}(\mu^1_{\varepsilon, \varepsilon_1} \otimes \text{id}) + \mu^2_{\varepsilon, \varepsilon_2, \varepsilon_1}(\text{id} \otimes \mu^1_{\varepsilon, \varepsilon_1}) = 0,
\]

for all \( 1 \leq i \leq k \).

### 3.3. Morphism induced by a cobordism.

Given an exact Lagrangian cobordism \( \Lambda^- \leadsto \Lambda^+ \), there exists a DGA-map \( \phi_{\Sigma}: \mathcal{A}(\Lambda^+) \to \mathcal{A}(\Lambda^-) \) defined on Reeb chords by

\[
\phi_{\Sigma}(\gamma^+) = \sum_{\gamma_1, \ldots, \gamma_m} \# \mathcal{M}_{\Sigma}(\gamma^+, \gamma_1, \ldots, \gamma_m) \cdot \gamma_1 \cdots \gamma_m
\]

where \( \gamma^+ \in \mathcal{R}(\Lambda^+) \) and \( \gamma_1, \ldots, \gamma_m \in \mathcal{R}(\Lambda^-) \) (see [EHK10]). When \( \Sigma \) is an exact Lagrangian filling of \( \Lambda^+ \) (\( \Lambda^- = \emptyset \)), then the DGA-map \( \phi_{\Sigma} : \mathcal{A}(\Lambda^+) \to \mathbb{Z}_2 \) is an augmentation of \( \mathcal{A}(\Lambda^+) \). In this case, the corresponding linearized Legendrian contact cohomology of \( \Lambda \) is determined by the topology of the filling by the Ekholm-Seidel isomorphism:

**Theorem 9 (EHK12, DR16).** If \( \Lambda \subset Y \) is a \( n \)-dimensional closed Legendrian submanifold which admits a Lagrangian filling \( \Sigma \), then \( H_*(\Sigma) \simeq LCH^*_{\varepsilon}(\Lambda) \), where \( \varepsilon \) is the augmentation of \( \mathcal{A}(\Lambda) \) induced by \( \Sigma \).

This theorem gives a very powerful obstruction to the existence of Lagrangian fillings. For example, once the Legendrian contact cohomology of a Legendrian \( \Lambda \) has a generator of degree strictly less than 0 or strictly higher than \( n \), it means that \( \Lambda \) is not fillable by an exact Lagrangian. More generally, given an augmentation \( \varepsilon^- \) of \( \mathcal{A}(\Lambda^-) \), its pullback by \( \phi_{\Sigma} \) is an augmentation of \( \mathcal{A}(\Lambda^+) \) that we denote \( \varepsilon^+ := \varepsilon^- \circ \phi_{\Sigma} \). This is the order-0 map of a family of maps defining an \( \Lambda_\infty \)-functor \( \Phi_{\Sigma} : \text{Aug}(\Lambda^-) \to \text{Aug}(\Lambda^+) \) as follows (see [BC14]):

- on the objects of the category, \( \Phi_{\Sigma}(\varepsilon^-) = \varepsilon^- \circ \phi_{\Sigma} \),
- for each \((d+1)\)-tuple \((\varepsilon^-_1, \ldots, \varepsilon^-_{d+1})\) of augmentations of \( \mathcal{A}(\Lambda^-) \), there is a map

\[
\Phi_{\Sigma}: \text{hom}(\varepsilon^-_1, \ldots, \varepsilon^-_{d+1}) \to \text{hom}(\varepsilon^+_1, \ldots, \varepsilon^+_{d+1})
\]

defined by

\[
\Phi_{\Sigma}(\gamma^-_1, \ldots, \gamma^-_d) = \sum_{\substack{\gamma^+ \in \mathcal{R}(\Lambda^+) \\delta^-_1 \cdots \delta^-_d \cdots \delta^{d+1}}} \# M_{\Sigma}(\gamma^+; \delta^-_1, \gamma^-_1, \delta^-_2, \ldots, \gamma^-_d, \delta^{d+1}, \varepsilon^+_1, \ldots, \varepsilon^+_{d+1}) \cdot \delta^-_1 \cdots \delta^{d+1} \cdot \gamma^+
\]

The induced functor on cohomology level gives a map on bilinearized Legendrian contact cohomology:

\[
\Phi_{\Sigma} : LCH^*_{\varepsilon_1, \varepsilon_2}(\Lambda^-) \to LCH^*_{\varepsilon_1, \varepsilon_2}(\Lambda^+)
\]

which was shown to be an isomorphism if \( \Sigma \) is a concordance, in [CDRG15].

In the case of the augmentation category \( \text{Aug}_+(\Lambda) \) (defined in [NRS1]), an exact Lagrangian cobordism from \( \Lambda^- \) to \( \Lambda^+ \) also induces a functor \( \tilde{F} : \text{Aug}_+(\Lambda^-) \to \text{Aug}_+(\Lambda^+) \). In particular, this functor was shown to be injective on equivalence classes of augmentations and cohomologically faithful by Yu Pan [Pan17].
4. Floer theory for Lagrangian cobordisms

4.1. The Cthulhu complex. In this section we recall the definition of the Cthulhu complex, a Floer-type complex for Lagrangian cobordisms, defined by Chantraine, Dimitroglou-Rizell, Ghiggini and Golovko in [CDRGG]. Let $\Lambda_1^+ \prec \Sigma_1$, $\Lambda_2^+ \prec \Sigma_2$, $\Lambda_2^- \prec \Sigma_2$ be two transverse exact Lagrangian cobordisms in $\mathbb{R} \times Y$ with $\Lambda_1^-, \Lambda_1^+, \Lambda_2^-, \Lambda_2^+$ Legendrian submanifolds in $Y$. We assume that the Chekanov-Eliashberg algebras $A(\Lambda_1^+)$ and $A(\Lambda_2^-)$ admit augmentations $\varepsilon_1^-$ and $\varepsilon_2^-$ respectively, which induce augmentations $\varepsilon_1^+$ and $\varepsilon_2^+$ of $A(\Lambda_1^+)$ and $A(\Lambda_2^-)$ as we saw previously.

Cthulhu homology is the homology of a graded complex associated to the pair $(\Sigma_1, \Sigma_2)$, denoted $(\text{Cth}(\Sigma_1, \Sigma_2), \partial_{\varepsilon_1^-, \varepsilon_2^-})$, generated by intersection points in $\Sigma_1 \cap \Sigma_2$, Reeb chords from $\Lambda_2^-$ to $\Lambda_1^+$ and Reeb chords from $\Lambda_2^-$ to $\Lambda_1^-$, with shifts in grading:

$$\text{Cth}(\Sigma_1, \Sigma_2) = C(\Lambda_1^+, \Lambda_2^+)[2] \oplus CF(\Sigma_1, \Sigma_2) \oplus C(\Lambda_1^-, \Lambda_2^-)[1]$$

where $CF(\Sigma_1, \Sigma_2)$ denotes the $\mathbb{Z}_2$-vector space generated by intersection points in $\Sigma_1 \cap \Sigma_2$. This is a graded complex. For the grading to be in $\mathbb{Z}$, we assume that $2\mathbb{C}1(P)$ as well as the Maslov classes of $\Sigma_1$ and $\Sigma_2$ vanish (which implies that the Maslov classes of $\Lambda_1^+$ and $\Lambda_2^-$ also vanish).

The grading for Reeb chords is the same as the Legendrian contact homology grading (Section 3.1). For an intersection point $p \in \Sigma_1 \cap \Sigma_2$ the grading is defined to be the Maslov index of a path of graded Lagrangians from $(T_p(\Sigma_1))^\#$ to $(T_p(\Sigma_2))^\#$ in $\text{Gr}^\#(T_p(\mathbb{R} \times Y), d(c^\prime \alpha)_p)$, the universal cover of the Grassmannian of Lagrangian subspaces of $(T_p(\mathbb{R} \times Y), d(c^\prime \alpha)_p)$, see [ScIo8]. Section 11.j] and [CDRGG]. The differential on $\text{Cth}(\Sigma_1, \Sigma_2)$ is a matrix

$$\partial_{\varepsilon_1^-, \varepsilon_2^-} = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & d_{-0} & d_{--} \end{pmatrix}$$

where each component is defined by a count of rigid pseudo-holomorphic disks with boundary on $\Sigma_1$ and $\Sigma_2$ and asymptotic to intersection points and Reeb chords from $\Lambda_2^-$ to $\Lambda_1^+$ as follows:

1. for $\xi_{2,1}^+ \in R(\Lambda_2^+, \Lambda_1^+)$:

$$d_{++}(\xi_{2,1}^+) = \sum_{\gamma_{2,1}^+, \beta_1, \beta_2} \# M_{\mathbb{R} \times \Lambda_1^+}^1(\gamma_{2,1}^+; \beta_1, \xi_{2,1}^+, \beta_2) \varepsilon_1(\beta_1) \varepsilon_2^+(\beta_2) \cdot \gamma_{2,1}^+$$

where the sum is for $\gamma_{2,1}^+ \in R(\Lambda_2^+, \Lambda_1^+)$ and $\beta_i$ words of Reeb chords of $\Lambda_1^+$, for $i = 1, 2$.

The map $d_{++}$ is the restriction to $C(\Lambda_2^+, \Lambda_1^+)$ of the bilinear differential $\mu_{\varepsilon_{2,-}^+, \varepsilon_1^+}$ of the Legendrian contact cohomology of $\Lambda_1^+ \cup \Lambda_2^+$. 

2. for $\xi_{2,1}^- \in R(\Lambda_2^-, \Lambda_1^-)$:

$$d_{--}(\xi_{2,1}^-) = \sum_{\gamma_{2,1}^-, \delta_1, \delta_2} \# M_{\Lambda_1^- \times \mathbb{R}}^1(\gamma_{2,1}^-; \delta_1, \xi_{2,1}^-, \delta_2) \varepsilon_1(\delta_1) \varepsilon_2^-(\delta_2) \cdot \gamma_{2,1}^-$$

$$d_{0-}(\xi_{2,1}^-) = \sum_{x^+ \in \Sigma_1 \cap \Sigma_2} \# M_{\Lambda_2^+ \times \mathbb{R}}^0(\gamma_{2,1}^-; \delta_1, \xi_{2,1}^-, \delta_2) \varepsilon_1(\delta_1) \varepsilon_2^-(\delta_2) \cdot x^+$$

$$d_{-0}(\xi_{2,1}^-) = \sum_{\gamma_{2,1}^-, \delta_1, \delta_2} \# M_{\mathbb{R} \times \Lambda_1^-}^1(\gamma_{2,1}^-; \delta_1, \xi_{2,1}^-, \delta_2) \varepsilon_1(\delta_1) \varepsilon_2^-(\delta_2) \cdot \gamma_{2,1}^-$$

and as for $d_{++}$, the map $d_{--}$ is the restriction of $\mu_{\varepsilon_{2,-}^-, \varepsilon_1^-}$ to $C(\Lambda_1^-, \Lambda_2^-)$.
(3) for \( q \in \Sigma_1 \cap \Sigma_2 \) which is a jump from \( \Sigma_1 \) to \( \Sigma_2 \):

\[
d_{+0}(q) = \sum_{\gamma_2,1} \sum_{\delta_1,\delta_2} \# \mathcal{M}_{\Sigma_1,2}^0 (\gamma_2,1; \delta_1,q,\delta_2) \varepsilon_1^+ (\delta_1) \varepsilon_2^- (\delta_2) \cdot \gamma_2,1
\]

\[
d_{00}(q) = \sum_{\gamma_2,1} \sum_{\delta_1,\delta_2} \# \mathcal{M}_{\Sigma_1,2}^0 (x^+; \delta_1,q,\delta_2) \varepsilon_1^- (\delta_1) \varepsilon_2^- (\delta_2) \cdot x^+
\]

\[
d_{-0}(q) = b \circ d_{-0}^{\Sigma_2,1}(q)
\]

\[
= \sum_{\gamma_2,1} \sum_{\delta_1,\delta_2} \# (\mathcal{M}_{\Sigma_2,1}^0(q; \delta_2,\gamma_1,2,\delta_1') \times \mathcal{M}_{\mathbb{R} \times \Lambda_2}^1 (\gamma_2,1; \delta_1',\gamma_1,2,\delta_2'))
\]

\[
\cdot \varepsilon_1^-(\delta_1') \varepsilon_2^- (\delta_2') \cdot \gamma_2,1
\]

where

- the last sum is for \( \delta_2, \delta_1', \delta_1'' \) words of Reeb chords of \( \Lambda_1 \) such that \( \delta_1' = \delta_1'' \),
- \( \delta_{0}^{\Sigma_2,1} \) is the dual of \( d_{0}^{\Sigma_2,1} : \mathcal{C}(\Lambda_2,\Lambda_1^{-}) \rightarrow \mathcal{C}(\Sigma_2,\Sigma_1) \) with Lagrangian label \((\Sigma_2,\Sigma_1)\),
- \( b : \mathcal{C}(\Lambda_2,\Lambda_1^{-}) \rightarrow \mathcal{C}(\Lambda_1^{-},\Lambda_2^{-}) \) is the map defined by the count of bananas:

\[
b(\gamma_1,2) = \sum_{\gamma_2,1} \sum_{\delta_1,\delta_2} \# \mathcal{M}_{\mathbb{R} \times \Lambda_1}^1 (\gamma_2,1; \delta_1,\gamma_1,2,\delta_2) \cdot \varepsilon_1^- (\delta_1) \varepsilon_2^- (\delta_2) \cdot \gamma_2,1
\]

See Figure 2 for examples of pseudo-holomorphic disks which contribute to the components of \( d_{+} \), except for \( d_{+} \) whose contributing disks are of the same type of those contributing to \( d_{-} \), but with boundary on \( \mathbb{R} \times (\Lambda_1^+ \cup \Lambda_2^+) \).

**Figure 2.** From left to right: schematic picture of pseudo-holomorphic disks contributing to \( d_{+}(\xi_2,1) \), \( d_{0}-(\xi_2,1) \), \( d_{-}-(\xi_2,1) \), \( d_{+0}(q) \), \( d_{00}(q) \) and \( d_{-0}(q) \).

**Remark 5.** The other components of the differential vanish for energy reasons.
When transversality holds, it is again possible to express the dimension of the moduli spaces above by the degree of the asymptotics. In particular, from [CDRGG, Theorem 4.5] we have:

\[
\dim \mathcal{M}_{\mathbb{R} \times \Lambda_1^+} (\gamma^+_2; \beta_1, \xi^+_2, \beta_2) = |\gamma^+_2| + |\xi^+_2| - |\beta_1| - |\beta_2| - 1
\]

\[
\dim \mathcal{M}_{\mathbb{R} \times \Lambda_1^0} (\gamma^-_2; \delta_1, \xi^-_2, \delta_2) = |\gamma^-_2| - |\xi^-_2| - |\delta_1| - |\delta_2| - 1
\]

\[
\dim \mathcal{M}_{\mathbb{R} \times \Lambda_1^-} (\gamma^-_2; \delta_1, \gamma^-_1, \delta_2) = |\gamma^-_2| + |\gamma^-_1| - |\delta_1| - |\delta_2| + 1 - n
\]

This gives that the map \( \phi^-_{\gamma^-_2, \delta^-_2} \) is of degree 1. Without the shifts in grading, we obtain that \( d_{+0} \) is of degree \(-1\), \( d_{+-} \) and \( d_{-0} \) are of degree 0, \( d_{++} \), \( d_{00} \) and \( d_{--} \) are degree 1 maps and \( d_{0-} \) is of degree 2.

The necessary transversality results in order to make the above moduli spaces transversely cut out are given in [CDRGG]. Briefly, as we already saw in the previous section, for Legendrian contact homology-type moduli spaces, cylindrical almost complex structures on \( \mathbb{R} \times Y \) are generically regular. This is also the case for moduli spaces of bananas, since that even if the curves in those spaces have two positive Reeb chords asymptotics, these Reeb chords are distinct, and so the curve is always somewhere injective.

Now, if \( J^k \) are regular for Legendrian contact homology type moduli spaces and banana moduli spaces, then moduli spaces \( \mathcal{M}_{\Sigma} (\gamma^+; \gamma_1, \ldots, \gamma_m) \) are transversely cut out for a generic almost complex structure in \( \mathcal{J}_{adm}^k (\mathbb{R} \times Y) \), using results of [MS12, Chapter 3]. The regularity assumption on \( J^k \) permits in particular to achieve transversality for pseudo-holomorphic curves coming from the gluing of a curve in \( \mathcal{M}_{\Sigma} (\gamma^+; \gamma_1, \ldots, \gamma_m) \) and a curve in \( \mathcal{M}_{\mathbb{R} \times \Lambda^+} (\gamma; \gamma_1, \ldots, \gamma_m) \).

Finally, moduli spaces of the types \( \mathcal{M}_{\Sigma_{+1}} (x^+; \delta_1, q, \delta_2) \) and \( \mathcal{M}_{\Sigma_{-1}} (\gamma^+_2; \delta_1, \gamma^-_2, \delta_2) \) are transversely cut out for a generic domain dependent almost complex structure

\[
J : [0, 1] \to \mathcal{J}_{adm}^k (\mathbb{R} \times Y)
\]

generalizing results of [AD10]. The domain dependence here is just a time-dependence because the domain of a curve is biholomorphic to a strip \( \mathbb{R} \times [0, 1] \) with marked points on the boundary (asymptotic to pure Reeb chords), and we want invariance of the almost complex structure by translation of the \( \mathbb{R} \)-coordinate.

**Theorem 10.** [CDRGG] Given \( \Sigma_1, \Sigma_2 \subset \mathbb{R} \times Y \) exact Lagrangian cobordisms as above,

1. \( \phi^-_{\gamma^-_2, \delta^-_2} = 0 \), and
2. The complex \( \mathcal{C}(\Sigma_1, \Sigma_2, \phi^-_{\gamma^-_2, \delta^-_2}) \) is acyclic.

The first point of this theorem is proven by studying breakings of pseudo-holomorphic curves of index 1 with boundary on \( \Sigma_1 \cup \Sigma_2 \), or of index 2 with boundary on \( \mathbb{R} \times \Lambda_1^+ \cup \mathbb{R} \times \Lambda_2^- \), and two mixed asymptotics. In Section 6.2, we will use the same ideas to prove Theorem 11. The second point of the theorem comes from the fact that it is possible to displace the cobordisms in \( \mathbb{R} \times Y \) such that \( \Sigma_1 \) and \( \Sigma_2 \) no longer have intersection points and such that there are no more Reeb chords from \( \Lambda_1^+ \) to \( \Lambda_2^- \). Briefly, this is done by first wrapping the ends of one of the two cobordisms by a Hamiltonian isotopy in such a way that the complex we get has only intersection points generators (no more Reeb chords) and is canonically isomorphic to the original Cthulhu.
complex. Then, the invariance of the Cthulhu complex by a compactly supported Hamiltonian
isotopy permits to separate the two cobordisms so that there are no more generators, which
implies that the complex vanishes, as well as its homology.

Let us denote \( \partial_{-\infty} = \begin{pmatrix} d_{00} & d_{0-} \\ d_{-0} & d_{--} \end{pmatrix} \) the submatrix of \( \partial_{-\infty_{\Lambda}} \), then

\[
0 = (\partial_{-\infty_{\Lambda}})^2 = \begin{pmatrix} d_{0+}^{2} + & *_{++} \\ 0 & 0 \end{pmatrix}
\]

where \( *_{++} = d_{++} + d_{+0} + d_{+d} + d_{-0} + d_{-d} \) and \( *_{+-} = d_{+-} + d_{++} + d_{+-} + d_{-d} \). So in particular,
\( (C(\Lambda^+_1), d_{++}) \) is a subcomplex of the Cthulhu complex and
\( \{CF_{-\infty}(\Sigma_1, \Sigma_2) := CF(\Sigma_1, \Sigma_2) \oplus C(\Lambda^+_1, \Lambda^+_2)[1], \partial_{-\infty} \} \)
is a quotient complex. Relation (3) implies also that \( d_{+0} + d_{+-} : CF_{-\infty}(\Sigma_1, \Sigma_2) \to C(\Lambda^+_1, \Lambda^+_2) \)
is a chain map, i.e. the Cthulhu complex is the cone of \( d_{+0} + d_{+-} \). This map is in fact a
quasi-isomorphism due to the acyclicity of the Cthulhu complex.

4.2 Hamiltonian perturbations. Given a cobordism \( \Lambda^\pm \approx \Sigma \) in \( (\mathbb{R} \times P \times \mathbb{R}, d(e^t \alpha)) \), we
consider a special type of Hamiltonian isotopies by which we deform \( \Sigma \), in order to extract some
properties of the Cthulhu complex. More precisely, we use a Hamiltonian \( H : \mathbb{R} \times P \times \mathbb{R} \to \mathbb{R} \) that
depends only on the real coordinate in the symplectization, which means that \( H(t, p, z) = h(t) \),
where \( h : \mathbb{R} \to \mathbb{R} \) is a smooth function. The associated Hamiltonian flow is by definition the
flow of the Hamiltonian vector field \( X_H \) defined by \( \iota_{X_H} d(e^t \alpha) = -dH \). We can compute that
\( X_H(t, p, z) = e^{-t}h'(t)\partial_t \) and so the flow \( \Phi_H \) is given by:
\[
\Phi^t_H : \mathbb{R} \times P \times \mathbb{R} \to \mathbb{R} \times P \times \mathbb{R} \\
\Phi^t_H(t, p, z) = (t, p, z + se^{-t}h'(t))
\]

Now, since \( \Sigma \) is an exact Lagrangian cobordism, \( \Phi^t_H(\Sigma) \) is also an exact Lagrangian cobordism.
Indeed, if \( f_\Sigma : \Sigma \to \mathbb{R} \) is the primitive of \( e^t \alpha \) restricted to \( \Sigma \), then we have:
\[
e^t \alpha|_{\Phi^t_H(\Sigma)} = (\Phi^t_H)^*(e^t(dz + \beta)) \\
= e^t(dz + se^{-t}h'(t) + \beta)|_{\Sigma} \\
= e^t(dz + se^{-t}(h'' - h')dt + \beta)|_{\Sigma} \\
= e^{t}|_{\Sigma} + s(h'' - h')dt|_{\Sigma}
\]

So, a primitive of \( e^t \alpha|_{\Phi^t_H(\Sigma)} \) is given by
\[
f_{\Phi^t_H(\Sigma)}(t, p, z) = f_\Sigma(t, p, z) + s(h'' - h')t(t)
\]

In particular, when the function \( h \) is for example the function \( h_D \) below, the primitive \( f_{\Phi^t_H(\Sigma)} \)
given by (4) vanishes on the negative end of \( \Phi^t_H(\Sigma) \). This type of Hamiltonian isotopy is useful
to wrap the cylindrical ends of the cobordisms, and the way to wrap depends on the choice of
the function \( h : \mathbb{R} \to \mathbb{R} \) to define the Hamiltonian. Let us describe here one type of perturbation
(see [CDRGG] for other perturbations). Given \( T > 0 \), we define a function \( h_D : \mathbb{R} \to \mathbb{R} \) by:
\[
h_D(t) = e^t \text{ for } t \leq -T - 1 \\
h_D(t) = 0 \text{ for } t \in [-T, T] \\
h_D(t) = e^t - B \text{ for } t \geq T + 1 \\
h_D'(t) \geq 0 \text{ for } t \in [-T - 1, -T] \cup [T, T + 1]
\]
where $A$ and $B$ are positive constants. Then we denote $H_D$ the corresponding Hamiltonian on $\mathbb{R} \times P \times \mathbb{R}$. Now we look how this Hamiltonian wraps the cylindrical ends of a cobordism. Let $\Sigma_1$ be an exact Lagrangian cobordism and consider its image by the flow at time $\epsilon$, $\Phi^\epsilon_{H_D}(\Sigma_1)$, for a small $\epsilon > 0$. In $[-T,T] \times Y$, the two cobordisms $\Sigma_1$ and $\Phi^\epsilon_{H_D}(\Sigma_1)$ are not at all transverse because they coincide, so we perturb $\Phi^\epsilon_{H_D}(\Sigma_1)$ by a small Hamiltonian isotopy with compact support and denote this new cobordism $\Sigma_D$. The Hamiltonian isotopy used here has compact support in $[-T - \eta , T + \eta] \times Y$ and with $0 < \eta \ll 1$, and for $t$ in the intervals $[-T - \eta , -T + \eta]$ and $[T - \eta , T + \eta]$, it is proportional to the Reeb flow. If the perturbation is small enough, then intersection points of $\Sigma_1$ and $\Sigma_2$ are all contained in $[-T,T] \times Y$ and by the formula, the functions associated to $\Sigma_1$ and $\Sigma_2$ satisfy, for all $t \in [-T,T]$, $f_{\Sigma_2} = f_{\Sigma_1} - A\epsilon$. Thus, every intersection point in $CF(\Sigma_1, \Sigma_2)$ has negative action, we say then that the pair $(\Sigma_1, \Sigma_2)$ is directed. Such a pair of cobordisms satisfy some properties listed in the following proposition:

**Proposition 2.** Let $(\Sigma_1, \Sigma_2)$ be a directed pair of Lagrangian cobordisms such that $\Sigma_2$ is a small perturbation of $\Phi^\epsilon_{H_D}(\Sigma_1)$ as above by a Morse function $f$ on $\Sigma_1$. Let $T > 0$ be such that $\Sigma_i \setminus ([−T,T] \times Y \cap \Sigma_i)$ are cylindrical, and consider a domain dependent almost complex structure $J_t$ in $J^\alpha_{\pm} \cup J^\alpha_0(\mathbb{R} \times Y)$ such that $J^+ \in J^\alpha_0(\mathbb{R} \times Y)$. Assume moreover that $\mathcal{A}(\Lambda_i^-)$ admits augmentations $\epsilon^\pm_1, \epsilon^\pm_2$ which induce augmentations $\epsilon^+_1$ and $\epsilon^+_2$ of $\mathcal{A}(\Lambda_i^+)$, then:

1. there are canonical isomorphisms of the Chekanov-Eliashberg algebras $(\mathcal{A}(\Lambda_i^-), \partial^-_{\epsilon_1}) \simeq (\mathcal{A}(\Lambda_i^+), \partial^+_{\epsilon_2})$ and $(\mathcal{A}(\Lambda_i^+), \partial^+_{\epsilon_1}) \simeq (\mathcal{A}(\Lambda_i^+), \partial^+_{\epsilon_2})$, and so in particular $\epsilon^+_1$ and $\epsilon^+_2$ are augmentations of $\mathcal{A}(\Lambda_i^+)$ under this identification,

2. $LCH^+_{\epsilon_1, \epsilon_2}(\Lambda_i^+, \Lambda_i^\pm) \simeq LCH^+_{\epsilon_1, \epsilon_2}(\Lambda_i^+),$

3. $LCH^+_{\epsilon_1, \epsilon_2}(\Lambda_i^+, \Lambda_i^\pm) \simeq LCH^+_{\epsilon_1, \epsilon_2}(\Lambda_i^+),$

4. if $J_t$ is regular and induced by a Riemannian metric $g$ such that $(f,g)$ is a Morse-Smale pair in a neighborhood of $\Sigma_1$, then $HF_*(\Sigma_1, \Sigma_2) \simeq H_{n+1}(\Sigma_1, \partial^{-}\Sigma_1 ; \mathbb{Z}_2) \simeq H^*(\Sigma_1, \partial^{+}\Sigma_1 ; \mathbb{Z}_2)$.

5. Product structure

5.1. Definition of the product. Let $\Lambda_i^- \prec \Sigma_i, \Lambda_i^+, i = 1, 2, 3$, be three transverse exact Lagrangian cobordisms, and $T > 0$ such that $\Sigma_i \setminus ([−T,T] \times Y \cap \Sigma_i)$ are cylindrical. Recall that the moduli spaces which are useful to define the product are of different types. First, we need moduli spaces of pseudo-holomorphic curves with boundary on the negative cylindrical ends of the cobordisms and with two or three mixed asymptotics:

$$M_{R \times \Lambda_i, \gamma, j} (\gamma_i, \delta_1, \gamma_i, \delta_1), i, j \in \{1, 2, 3\}$$

$$M_{R \times \Lambda_i^+-, \gamma, 1, 2, 3} (\gamma, 1, \gamma, 2, \gamma, 3, \delta_1)$$

$$M_{R \times \Lambda_i^+, \gamma, 1, 2, 3} (\gamma, 1, \gamma, 2, \gamma, 3, \delta_1)$$

$$M_{R \times \Lambda_i^+, \gamma, 1, 2, 3} (\gamma, 1, \gamma, 2, \gamma, 3, \delta_1)$$

$$M_{R \times \Lambda_i^+, \gamma, 1, 2, 3} (\gamma, 1, \gamma, 2, \gamma, 3, \delta_1)$$

$$M_{R \times \Lambda_i^+, \gamma, 1, 2, 3} (\gamma, 1, \gamma, 2, \gamma, 3, \delta_1)$$

Remark that the first one is a moduli space of bananas which already appeared in the definition of the Cthulhu differential.

Then we also need moduli spaces of pseudo-holomorphic curves with boundary in the non-cylindrical parts of the cobordisms, and again with two or three mixed asymptotics:

$$M_{\Sigma_i, \gamma} (\gamma^+, \delta_1, \gamma^+, \delta_2)$$
for \( x^+ \in \Sigma_1 \cap \Sigma_2 \), and

\[
\mathcal{M}_{\Sigma_1,2,3}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3)
\]

\[
\mathcal{M}_{\Sigma_1,2,3}(x^+; \delta_1, x_1, \delta_2, \gamma_3, \delta_3)
\]

\[
\mathcal{M}_{\Sigma_1,2,3}(x^+; \delta_1, \gamma_2, \delta_2, x_2, \delta_3)
\]

\[
\mathcal{M}_{\Sigma_1,2,3}(x^+; \delta_1, \gamma_2, \delta_2, \gamma_3, \delta_3)
\]

for \( x^+ \in \Sigma_1 \cap \Sigma_3 \), and also

\[
\mathcal{M}_{\Sigma_1,3,2}(x_2; \delta_3, \gamma_1, \delta_1, x_1, \delta_2)
\]

\[
\mathcal{M}_{\Sigma_1,3,2}(x_2; \delta_3, \gamma_1, \delta_1, \gamma_2, \delta_2)
\]

\[
\mathcal{M}_{\Sigma_2,1,3}(x_1; \delta_2, \gamma_3, \delta_3, \gamma_1, \delta_1)
\]

We achieve transversality for these moduli spaces using domain dependent almost complex structures. First, remark that moduli spaces of curves with boundary on the negative cylindrical ends above are transversely cut out for a generic almost complex structure in \( \mathcal{J}^{cul}(\mathbb{R} \times \mathbb{Y}) \). Indeed, even if some curves have several positive asymptotics, these are all distinct so it is always possible to find an injective point (same argument as for Legendrian contact homology type moduli spaces).

Now, consider \( J^\pm \in \mathcal{J}^{cul}(\mathbb{R} \times \mathbb{Y}) \) regular almost complex structures for the moduli spaces of curves with boundary on the positive and negative cylindrical ends respectively, with two or three mixed Reeb chords asymptotics and negative pure Reeb chords asymptotics (in fact, we do not need here regularity of \( J^+ \) but in any case it will be useful in Section 5.3). We know that Cthulhu moduli spaces of curves with boundary on non-cylindrical parts of the cobordisms are transversely cut out for a generic time-dependent almost complex structure \( J_t : [0,1] \to \mathcal{J}^{adm}_{J^-,-J^+,T}(\mathbb{R} \times \mathbb{Y}) \). So, let us denote by \( J_t^{\Sigma,j}, \) for \( i,j \in \{1,2,3\}, \) a regular time-dependent almost complex structure for Cthulhu moduli spaces associated to the pair of cobordisms \( (\Sigma_i, \Sigma_j) \), with the convention that \( J_t^{\Sigma_1,\Sigma_2} \) is a constant path. Then, given a consistent universal choice of strip-like ends, we use Seidel’s result [Sei08, Section (9k)] to deduce that a universal domain dependent almost complex structure \( J_{2,\Sigma} : S^3 \to \mathcal{J}^{adm}_{J^-,-J^+,T}(\mathbb{R} \times \mathbb{Y}) \) can be perturb to a regular one for the moduli spaces above with boundary on non-cylindrical parts and three mixed asymptotics.

This means that we can find a regular domain dependent almost complex structure with values in \( \mathcal{J}^{adm}_{J^-,-J^+,T}(\mathbb{R} \times \mathbb{Y}) \) such that all the moduli spaces we have encountered until now are simultaneously smooth manifolds.

**Remark 6.** In all the section, as before, we define maps by a count of rigid pseudo-holomorphic curves. This count will always be modulo 2.

Let us assume that the Chekanov-Eliashberg algebras \( \mathcal{A}(L^+_i), i = 1,2,3, \) admit augmentations \( \varepsilon^+_i \). We want to define a map:

\[
m_2 : CF^+_\infty(\Sigma_2, \Sigma_3) \otimes CF^+_\infty(\Sigma_1, \Sigma_2) \to CF^+_\infty(\Sigma_1, \Sigma_3)
\]

linear in each variable. This map decomposes as \( m_2 = m^0 + m^- \), where \( m^0 \) takes values in \( CF^*(\Sigma_1, \Sigma_3) \) and \( m^- \) takes values in \( CF^*(L^+_1, L^+_3) \). In order to do this, we define these maps on
pairs of generators, which means that we must define the eight following components:

\[ m^0_{00} : CF^*(\Sigma_2, \Sigma_3) \otimes CF^*(\Sigma_1, \Sigma_2) \to CF^*(\Sigma_1, \Sigma_3) \]
\[ m^0_{01} : CF^*(\Sigma_2, \Sigma_3) \otimes CF^*(\Sigma_1, \Sigma_2) \to C^*(\Lambda^+_1, \Lambda^-_1) \]
\[ m^0_{0-} : CF^*(\Sigma_2, \Sigma_3) \otimes C^*(\Lambda^-_1, \Lambda^-_2) \to CF^*(\Sigma_1, \Sigma_3) \]
\[ m_{0-} : CF^*(\Sigma_2, \Sigma_3) \otimes C^*(\Lambda^+_1, \Lambda^-_2) \to C^*(\Lambda^+_1, \Lambda^-_1) \]
\[ m^0_0 : C^*(\Lambda^-_2, \Lambda^-_1) \otimes CF^*(\Sigma_1, \Sigma_2) \to CF^*(\Sigma_1, \Sigma_3) \]
\[ m_{-0} : C^*(\Lambda^-_2, \Lambda^-_1) \otimes CF^*(\Sigma_1, \Sigma_2) \to C^*(\Lambda^-_1, \Lambda^-_2) \]
\[ m^- : C^*(\Lambda^-_2, \Lambda^-_1) \otimes C^*(\Lambda^+_1, \Lambda^-_2) \to CF^*(\Sigma_1, \Sigma_3) \]
\[ m^- : C^*(\Lambda^-_2, \Lambda^-_1) \otimes C^*(\Lambda^+_1, \Lambda^-_2) \to C^*(\Lambda^+_1, \Lambda^-_1) \]

Let us begin by \( m^0 \). We set:

\[ m^0_{00}(x_2, x_1) = \sum_{x^+ \in \delta_i} \# M^0_{\Sigma_1, \Sigma_2} (x^+; \delta_1, x_2, x_2, \delta_3) \epsilon_1^-(\delta_1) \epsilon_2^-(\delta_2) \epsilon_3^-(\delta_3) \cdot x^+ \]
\[ m^0_{0-}(x_2, \gamma_1) = \sum_{x^+ \in \delta_i} \# M^0_{\Sigma_1, \Sigma_2} (x^+; \delta_1, \gamma_1, x_2, x_2, \delta_3) \epsilon_1^-(\delta_1) \epsilon_2^-(\delta_2) \epsilon_3^-(\delta_3) \cdot x^+ \]
\[ m^-_{00}(\gamma_2, x_1) = \sum_{x^+ \in \delta_i} \# M^0_{\Sigma_1, \Sigma_2} (x^+; \delta_1, \gamma_2, x_1, x_2, \delta_3) \epsilon_1^-(\delta_1) \epsilon_2^-(\delta_2) \epsilon_3^-(\delta_3) \cdot x^+ \]
\[ m^-_{0-}(\gamma_2, \gamma_1) = \sum_{x^+ \in \delta_i} \# M^0_{\Sigma_1, \Sigma_2} (x^+; \delta_1, \gamma_1, \gamma_2, \gamma_2, \delta_3) \epsilon_1^-(\delta_1) \epsilon_2^-(\delta_2) \epsilon_3^-(\delta_3) \cdot x^+ \]

where the sums are for \( x^+ \in \Sigma_1 \cap \Sigma_3 \) and for each \( i \in \{1, 2, 3 \} \), \( \delta_i \) is a word of Reeb chords of \( \Lambda^-_1 \). Then, to define \( m^- \) we first introduce intermediate maps. We recall that there is a canonical identification of complexes \( CF_{n+1-1} (\Sigma_6, \Sigma_a) = CF^*(\Sigma_a, \Sigma_6) \) and we denote \( C^*_a (\Lambda^-_a, \Lambda^-_6) \) the dual of \( C^*(\Lambda^-_a, \Lambda^-_6) \). We consider a map:

\[ f^{(2)} : CF_{-\infty} (\Sigma_2, \Sigma_3) \otimes CF_{-\infty} (\Sigma_1, \Sigma_2) \to C_{n-1-\epsilon} (\Lambda^-_3, \Lambda^-_1) \]

defined on each pair of generators by:

\[ f^{(2)} (x_2, x_1) = \sum_{\gamma_1, \delta_1} \# M^0_{\Sigma_1, \Sigma_2} (x_2; \delta_3, \gamma_1, \delta_1, x_1, x_1, \delta_2) \epsilon_1^-(\delta_1) \epsilon_2^-(\delta_2) \epsilon_3^-(\delta_3) \cdot \gamma_1, \delta_1 \]
\[ f^{(2)} (x_2, \gamma_1) = \sum_{\gamma_1, \delta_1} \# M^0_{\Sigma_1, \Sigma_2} (x_2; \gamma_1, \delta_1, \gamma_1, \delta_1, \gamma_2, \delta_2) \epsilon_1^-(\delta_1) \epsilon_2^-(\delta_2) \epsilon_3^-(\delta_3) \cdot \gamma_1, \delta_1 \]
\[ f^{(2)} (\gamma_2, x_1) = \sum_{\gamma_1, \delta_1} \# M^0_{\Sigma_1, \Sigma_2} (x_1; \delta_2, \gamma_2, \delta_3, \gamma_1, \delta_1) \epsilon_1^-(\delta_1) \epsilon_2^-(\delta_2) \epsilon_3^-(\delta_3) \cdot \gamma_1, \delta_1 \]
\[ f^{(2)} (\gamma_2, \gamma_1) = \sum_{\gamma_1, \delta_1} \# M^0_{\Sigma_1, \Sigma_2} (x_1; \gamma_2, \gamma_2, \gamma_1, \delta_1) \epsilon_1^-(\delta_1) \epsilon_2^-(\delta_2) \epsilon_3^-(\delta_3) \cdot \gamma_1, \delta_1 \]

where \( x_2 \in CF(\Sigma_2, \Sigma_3) \), \( x_1 \in CF(\Sigma_1, \Sigma_2) \), \( \gamma_2 \in \mathcal{R}(\Lambda^-_3, \Lambda^-_6) \) et \( \gamma_1 \in \mathcal{R}(\Lambda^-_2, \Lambda^-_1) \). This map is the analogue of the map

\[ \delta^{\Sigma_1}_{-0} : CF_{n+1-\epsilon} (\Sigma_2, \Sigma_1) = CF^*(\Sigma_1, \Sigma_2) \to C_{n-1-\epsilon} (\Lambda^-_2, \Lambda^-_1) \]

with three mixed asymptotics instead of two, where \( \delta^{\Sigma_1}_{-0} \) is the dual of \( \delta^{\Sigma_1}_{-0} : C^*2(\Lambda^-_2, \Lambda^-_1) \to CF^*(\Sigma_2, \Sigma_1) \) (see Section 4). The Lagrangian label being given by the asymptotics, we will now
denote by \( f^{(1)} \) the maps \( \delta^{(1)}_{\Sigma_1} \) and \( \delta^{(1)}_{\Sigma_2} \), which we extend to the whole complex \( CF_\infty(\Sigma_1, \Sigma_2) \) by setting \( f^{(1)}(\gamma_{ij}) = \gamma_{ij} \) for a mixed Reeb chord.

Then we generalize the banana map \( b \) with a map \( b^{(2)} \) defined by a count of pseudo-holomorphic disks with three mixed asymptotics. Let us denote \( \mathcal{C}(\Lambda^-_i, \Lambda^-_j) = C^*(\Lambda_i, \Lambda_j) \oplus C_{n-1-\varepsilon}(\Lambda^-_i, \Lambda^-_j) \), we define:

\[
b^{(2)} : \mathcal{C}(\Lambda^-_2, \Lambda^-_3) \otimes \mathcal{C}(\Lambda^-_1, \Lambda^-_2) \to C^*(\Lambda^-_1, \Lambda^-_3)
\]

by

\[
b^{(2)}(\gamma_{2,3}, \gamma_{1,2}) = \sum_{\gamma_{1,1}, \delta_1} \# \widetilde{M}^{1}_{R \times \Lambda^{-}_{23}}(\gamma_{3,1}; \delta_1, \gamma_{1,2}, \delta_2, \gamma_{2,3}, \delta_3) \epsilon_1^+(\delta_1) \epsilon_2^-(\delta_2) \epsilon_3^-(\delta_3) \cdot \gamma_{3,1}
\]

Remark 7. The map \( b^{(2)} \) restricted to \( C^*(\Lambda^-_2, \Lambda^-_3) \otimes C^*(\Lambda^-_1, \Lambda^-_2) \) corresponds to the product \( \mu^{(2)}_{\Sigma_{21}} \) in the augmentation category \( \text{Aug}_- (\Lambda^-_1 \cup \Lambda^-_2 \cup \Lambda^-_3) \) restricted to this sub-complex, where \( \epsilon_{3,2,1} \) is the diagonal augmentation of \( \mathcal{A}(\Lambda^-_1 \cup \Lambda^-_2 \cup \Lambda^-_3) \) built from \( \epsilon_1^-, \epsilon_2^- \) and \( \epsilon_3^- \) (Section 3.2).

![Figure 3](image-url)

**Figure 3.** Curves contributing to \( b(\gamma_{1,2}), b^{(2)}(\gamma_{2,3}, \gamma_{1,2}), b^{(2)}(\gamma_{3,2}, \gamma_{1,2}) \) and \( b^{(2)}(\gamma_{3,2}, \gamma_{2,1}) \).

Now we define the map \( m^- \) by the following formula. For a pair \( (a_2, a_1) \in CF_\infty(\Sigma_2, \Sigma_3) \otimes CF_\infty(\Sigma_1, \Sigma_2) \), we set:

\[
m^- (a_2, a_1) = b \circ f^{(1)}(a_2, a_1) + b^{(2)}(f^{(1)}(a_2), f^{(1)}(a_1))
\]

More precisely, for each pair of asymptotics, we have:

\[
m^-_0(x_2, x_1) = b \circ f^{(2)}(x_2, x_1) + b^{(2)}(f^{(1)}(x_2), f^{(1)}(x_1))
\]

\[
m^-_0(x_2, \gamma_1) = b \circ f^{(2)}(x_2, \gamma_1) + b^{(2)}(f^{(1)}(x_2), \gamma_1)
\]

\[
m^-_0(\gamma_2, x_1) = b \circ f^{(2)}(\gamma_2, x_1) + b^{(2)}(\gamma_2, f^{(1)}(x_1))
\]

\[
m^-_0(\gamma_2, \gamma_1) = b^{(2)}(\gamma_2, \gamma_1)
\]

Contrary to the definition of \( m^0 \), when at least one input is an intersection point we need to count broken curves instead of just one type of pseudo-holomorphic disk, in order to associate a positive Reeb chord in \( C^*(\Lambda^-_1, \Lambda^-_3) \) to the two inputs. These broken curves have two levels, one
Definition 6. An unfinished pseudo-holomorphic building is a set of pseudo-holomorphic curves coming from a pseudo-holomorphic building such that:

1. It is not a pseudo-holomorphic building (the components cannot be glued),
2. This is necessary to add at least one non-trivial pseudo-holomorphic disk to get a pseudo-holomorphic building.

Thus, unfinished buildings can be viewed as buildings where we have removed some non-trivial components in such a way that this is not a building anymore. On the left of Figure 4, we drew an example of an unfinished pseudo-holomorphic building, whereas on the right the curves do not define an unfinished building. Indeed, by adding a trivial strip $\mathbb{R} \times \gamma$ in the negative cylindrical ends, we get a pseudo-holomorphic building. Thus the map $m^-$ counts unfinished pseudo-holomorphic buildings. On Figures 5, 6, 7 and 8 are schematized the different types of curves and unfinished buildings that contribute to $m^2$.

Remark 8. By [CDRGC, Theorem 4.5] for curves with boundary on three transverse exact Lagrangian cobordisms instead of two, we can express the dimension of the moduli spaces involved in the definition of $m_2$ by the degree of the asymptotics. Then it is not hard to check that $m_2$ is a degree 0 map, with the shift in grading for Reeb chords (see Section 4).

5.2. Proof of Theorem 1. In this section, we prove that $m_2$ satisfies the Leibniz rule:

$$m_2(-, \partial_{-\infty}) + m_2(\partial_{-\infty}, -) + \partial_{-\infty} \circ m_2(-, -) = 0$$

In order to do this, we show that the above relation is satisfied for each pair of generators in $CF_{-\infty}(\Sigma_2, \Sigma_3) \otimes CF_{-\infty}(\Sigma_1, \Sigma_2)$. For example, for $(x_2, x_1) \in CF(\Sigma_2, \Sigma_3) \otimes CF(\Sigma_1, \Sigma_2)$, this
Figure 6. Left: curve contributing to $m_{0-}(x_2, \gamma_1)$; right: curves contributing to $m_{0-}(x_2, \gamma_1)$

Figure 7. Left: curve contributing to $m_{00}(\gamma_2, x_1)$; right: curves contributing to $m_{0}(\gamma_2, x_1)$

Figure 8. Left: curve contributing to $m_{0-}(\gamma_2, \gamma_1)$; right: curve contributing to $m_{-}(\gamma_2, \gamma_1)$.

gives:

\[ m_2(x_2, \partial_{-\infty}(x_1)) + m_2(\partial_{-\infty}(x_2), x_1) + \partial_{-\infty} \circ m_2(x_2, x_1) = 0 \]

\[ \Leftrightarrow m_2(x_2, (d_{00} + d_{-0})(x_1)) + m_2
d_0 + d_{-0}) \circ m^0(x_2, x_1) + (d_{0-} + d_{-}) \circ m^-(x_2, x_1) = 0 \]

\[ \Leftrightarrow \left( m^0(x_2, (d_{00} + d_{-0})(x_1)) + m^0((d_{00} + d_{-0})(x_2), x_1) + d_{00} \circ m^0(x_2, x_1) + d_{0-} \circ m^{-}(x_2, x_1) \right) \]

\[ + \left( m^{-}(x_2, (d_{00} + d_{-0})(x_1)) + m^{-}((d_{00} + d_{-0})(x_2), x_1) \right) \]

\[ + d_{-0} \circ m^0(x_2, x_1) + d_{-} \circ m^{-}(x_2, x_1) \]
and in the last equality the two terms into big brackets must vanish because the first one is an element in \( CF(\Sigma_1, \Sigma_3) \) and the second one is an element in \( C(\Lambda_1^-, \Lambda_2^-) \). Thus, considering each pair of generators we obtain in total eight relations to prove which are the following.

1. For a pair \((x_2, x_1)\) in \( CF(\Sigma_2, \Sigma_3) \otimes CF(\Sigma_1, \Sigma_2)\):

\[
\begin{align*}
(5) \quad & m_{00}^0(x_2, d_{00}(x_1)) + m_{00}^0(d_{00}(x_2), x_1) + d_{00} \circ m_{00}^0(x_2, x_1) \\
& + m_{0-}^0(x_2, d_{-0}(x_1)) + m_{0-}^0(d_{-0}(x_2), x_1) + d_{0-} \circ m_{0-}^0(x_2, x_1) = 0 \\
(6) \quad & m_{00}^0(x_2, d_{00}(x_1)) + m_{00}^-(d_{00}(x_2), x_1) + m_{0-}^-(x_2, d_{-0}(x_1)) \\
& + m_{0-}^-(d_{-0}(x_2), x_1) + d_{0-} \circ m_{00}^0(x_2, x_1) + d_{0-} \circ m_{00}^-(x_2, x_1) = 0
\end{align*}
\]

2. For a pair \((x_2, \gamma_1)\) in \( CF(\Sigma_2, \Sigma_3) \otimes C(\Lambda_1^- \Lambda_2^-)\):

\[
\begin{align*}
(7) \quad & m_{00}^0(x_2, d_{0-}(- \gamma_1)) + m_{0-}^0(d_{00}(x_2), \gamma_1) + d_{00} \circ m_{00}^0(x_2, \gamma_1) \\
& + m_{0-}^0(x_2, d_{-0}(- \gamma_1)) + m_{0-}^0(d_{-0}(x_2), \gamma_1) + d_{0-} \circ m_{0-}^0(x_2, \gamma_1) = 0 \\
(8) \quad & m_{00}^0(x_2, d_{0-}(- \gamma_1)) + m_{0-}^0(d_{00}(x_2), \gamma_1) + m_{0-}^-(x_2, d_{-0}(- \gamma_1)) \\
& + m_{0-}^-(d_{-0}(x_2), \gamma_1) + d_{0-} \circ m_{00}^0(x_2, \gamma_1) + d_{0-} \circ m_{00}^-(x_2, \gamma_1) = 0
\end{align*}
\]

3. For a pair \((\gamma_2, x_1)\) in \( C(\Lambda_2^-, \Lambda_3^-) \otimes CF(\Sigma_1, \Sigma_2)\):

\[
\begin{align*}
(9) \quad & m_{00}^0(\gamma_2, d_{00}(x_1)) + m_{00}^0(d_{00}(\gamma_2), x_1) + d_{00} \circ m_{0-}^0(\gamma_2, x_1) \\
& + m_{0-}^0(\gamma_2, d_{-0}(x_1)) + m_{0-}^0(d_{-0}(\gamma_2), x_1) + d_{0-} \circ m_{0-}^0(\gamma_2, x_1) = 0 \\
(10) \quad & m_{0-}^0(\gamma_2, d_{00}(x_1)) + m_{00}^0(d_{0-}(\gamma_2), x_1) + m_{0-}^0(\gamma_2, d_{00}(x_1)) \\
& + m_{0-}^0(d_{-0}(\gamma_2), x_1) + d_{0-} \circ m_{00}^0(\gamma_2, x_1) + d_{0-} \circ m_{00}^-(\gamma_2, x_1) = 0
\end{align*}
\]

4. For a pair \((\gamma_2, \gamma_1)\) in \( C(\Lambda_2^-, \Lambda_3^-) \otimes C(\Lambda_1^- \Lambda_2^-)\):

\[
\begin{align*}
(11) \quad & m_{00}^0(\gamma_2, d_{0-}(- \gamma_1)) + m_{0-}^0(d_{00}(\gamma_2), \gamma_1) + d_{00} \circ m_{00}^0(\gamma_2, \gamma_1) \\
& + m_{0-}^0(\gamma_2, d_{-0}(- \gamma_1)) + m_{0-}^0(d_{-0}(\gamma_2), \gamma_1) + d_{0-} \circ m_{0-}^0(\gamma_2, \gamma_1) = 0 \\
(12) \quad & m_{0-}^0(\gamma_2, d_{-0}(- \gamma_1)) + m_{0-}^0(d_{-0}(\gamma_2), \gamma_1) + d_{0-} \circ m_{0-}^0(\gamma_2, \gamma_1) = 0
\end{align*}
\]

To obtain these relations, we study the different types of pseudo-holomorphic buildings involved in the definition of each term appearing in the relations. Each curve in these buildings are rigid because the Cthulhu differential and the map \( m_2 \) are defined by a count of rigid configurations. This means that the curves are of index 0 if their boundary is on non-cylindrical Lagrangians, and of index 1 if their boundary is on the negative cylindrical ends of the cobordisms. Compactness and gluing results imply that these broken curves are in bijection with elements in the boundary of the compactification of some moduli spaces. We recall below some properties that must be satisfied by the pseudo-holomorphic buildings we will consider here:

1. each curve in a pseudo-holomorphic building must have positive energy, so for example each component with only Reeb chords asymptotics must have at least one positive Reeb chord asymptotic. For curves with also intersection points asymptotics, as the action is independent of the label, it is possible to have curves with only negative action asymptotics, but in any case the energy must be positive (see Section 2.3 and Subsection 5.2.3 below),

2. each curve has a non negative Fredholm index because of the regularity of the almost complex structure,
(3) the following relation on indices must be satisfied: if $u_1, \ldots, u_k$ are curves forming a pseudo-holomorphic building, the glued solution $u$ has index given by $\text{ind}(u) = \nu + \sum_i \text{ind}(u_i)$, where $\nu$ is the number of intersection points on which the gluing operation has been performed.

Figure 9. Pseudo-holomorphic building contributing to $m_{00}^0(x_2, d_{00}(x_1))$.

5.2.1. Relation (5). The first term appearing in this relation is $m_{00}^0(x_2, d_{00}(x_1))$. For every intersection point $x^+ \in \Sigma_1 \cap \Sigma_3$, the coefficient $\langle m_{00}^0(x_2, d_{00}(x_1)), x^+ \rangle$ is defined by a count of pseudo-holomorphic buildings whose components are two index 0 curves with boundary on $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, which have a common asymptotic at an intersection point $q \in \text{CF}(\Sigma_1, \Sigma_2)$. One curve contributes to $\langle d_{00}(x_1), q \rangle$ and the other contributes to $\langle m_{00}^0(x_2, q), x^+ \rangle$ (see Figure 9, where the numbers in the curves indicate the Fredholm index).

The two curves can be glued together along $q$ and the resulting curve is an index 1 curve in the moduli space $\mathcal{M}^1_{\Sigma_1 \Sigma_2 \Sigma_3}(x^+, \delta_1, x_1, \delta_2, x_2, \delta_3)$. This implies that the holomorphic buildings contributing to $m_{00}^0(x_2, d_{00}(x_1))$ are in the boundary of the compactification of this moduli space. In fact, each term of the Relation (5) is defined by a count of holomorphic buildings whose components can be glued to give a curve in $\mathcal{M}^1_{\Sigma_1 \Sigma_2 \Sigma_3}(x^+, \delta_1, x_1, \delta_2, x_2, \delta_3)$. Thus now we look at all the possible breakings that can occur for a one parameter family of curves in this dimension 1 moduli space. The curve can break on:

1. an intersection point in $\Sigma_1 \cap \Sigma_2$, $\Sigma_2 \cap \Sigma_3$, or $\Sigma_3 \cap \Sigma_1$, giving a pseudo-holomorphic building with one level containing two curves with a common asymptotic at this intersection point,

2. a Reeb chord, giving a building of height $1|1|0$, the central one level containing index 0 curves with boundary on non-cylindrical Lagrangians, the lower level containing an index 1 curve with boundary on $\mathbb{R} \times (\Lambda^-_1 \cup \Lambda^-_2 \cup \Lambda^-_3)$.

Figure 10. $\partial$-breaking of a curve in $\mathcal{M}^1(x^+, \delta_1, x_1, \delta_2, x_2)$. 
Remark 9. In the second case, if the curve breaks on a pure Reeb chord $\gamma \in \mathcal{R}(\Lambda^-_i)$ for $i \in \{1, 2, 3\}$, this is called a $\partial$-breaking (see Figure 13). One component of such a broken curve contributes to $\partial^i(\gamma)$, where $\partial^i$ is the differential of the Legendrian contact homology of $\Lambda^-_i$. We denote by $\mathcal{M}^i(x^+; \delta_1, \delta_2, x_1, \delta_3)$ the union of all the $\partial$-breakings obtained as degeneration of curves in $\mathcal{M}^1_{\Sigma_{123}}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3)$. Now, the Cthuhlu differential and the maps involved in the definition of the product $m_2$ are defined by a count of elements in some moduli spaces of curves with two or three mixed asymptotics and every possible words of pure Reeb chords asymptotics $\delta_i$. Thus, the $\partial$-breakings on a chord $\gamma$ for every possible words of pure chords $\delta_i$ will contain all the curves contributing to $\partial_i(\gamma)$. Then, in the definition of the Cthuhlu differential and the product, pure chords are augmented by $\varepsilon^\gamma_i$ and by definition $\varepsilon^\gamma_i \circ \partial^i = 0$, so this means that the total contribution of $\partial$-breakings vanishes.

The boundary of the compactification of $\mathcal{M}^1_{\Sigma_{123}}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3)$ can be decomposed as follows:

$$
\partial \mathcal{M}^1_{\Sigma_{123}}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3) = \mathcal{M}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3) \bigcup_{p \in \Sigma_{123}} \mathcal{M}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3) \times \mathcal{M}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3)
$$

$$
\bigcup_{q \in \Sigma_{123}} \mathcal{M}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3) \times \mathcal{M}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3)
$$

where the $\delta_1', \delta_2'', \delta_3''$ are words of Reeb chords of $\Lambda^-_i$ such that $\delta_1', \delta_3'' = \delta_i$ for the three first unions, and $\delta_1', \delta_2', \delta_3'' = \delta_i$ in the four last unions where we sum also respectively for:

- $\xi_{1,2} \in \mathcal{R}(\Lambda^-_1, \Lambda^-_2)$, $\xi_{2,1} \in \mathcal{R}(\Lambda^-_2, \Lambda^-_1)$,
- $\xi_{3,1} \in \mathcal{R}(\Lambda^-_3, \Lambda^-_1)$, $\xi_{1,3} \in \mathcal{R}(\Lambda^-_1, \Lambda^-_3)$,
- $\xi_{3,2} \in \mathcal{R}(\Lambda^-_3, \Lambda^-_2)$, $\xi_{2,3} \in \mathcal{R}(\Lambda^-_2, \Lambda^-_3)$,
- $\xi_{3,1} \in \mathcal{R}(\Lambda^-_3, \Lambda^-_1)$, $\xi_{3,2} \in \mathcal{R}(\Lambda^-_3, \Lambda^-_2)$, $\xi_{1,2} \in \mathcal{R}(\Lambda^-_1, \Lambda^-_2)$.

See Figure 14 for a schematic picture of the above pseudo-holomorphic buildings (except the $\partial$-breakings because we do not draw the pure Reeb chords). From this, we can deduce Relation 5. Indeed, there is a one-to-one correspondence between buildings involved in the definition of each term in Relation 5 (from left to right) and buildings in $\partial \mathcal{M}^1_{\Sigma_{123}}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3)$ (from left to right on Figure 14), except that the last term of Relation 5 is defined by a count of the two last types of buildings at the right of the figure. Moreover, $\mathcal{M}^1_{\Sigma_{123}}(x^+; \delta_1, x_1, \delta_2, x_2, \delta_3)$ is a compact 1-dimensional manifold so its boundary consists of an even number of points, hence
Figure 11. Pseudo-holomorphic buildings in the boundary of the compactification of $\mathcal{M}^1(x^+;\delta^-_1, x_1, \delta^-_2, x_2, \delta^-_3)$.

the count of such points vanishes over $\mathbb{Z}_2$ and we get:

$$m^0_0(x_2, d_0(x_1)) + m^0_0(d_0(x_2), x_1) + d_0 \circ m^0_0(x_2, x_1) + m^0_0(x_2, d_0(-x_1)) + m^0_0(d_0(x_2), x_1) + d_0 \circ m^0_0(x_2, x_1) = 0$$

5.2.2. Relation (6). The first term of Relation (6) is $m^0_0(x_2, d_0(x_1))$. For each Reeb chord $\gamma_{3,1} \in \mathcal{R}(\Lambda^-_3, \Lambda^-_1)$, the coefficient $\langle m^0_0(d_0(x_2), x_1), \gamma_{3,1} \rangle$ is defined by a count of unfinished buildings of two types, as we saw in Section 5.1 for the definition of $m^-$. These unfinished buildings have two levels such that the central one is a pseudo-holomorphic building. This means that its components can be glued (see Figure 12). Indeed, the curves in the central level glue together at an intersection point to produce unfinished pseudo-holomorphic buildings respectively in the following products of moduli spaces:

$$\mathcal{M}^1_{\Sigma_{312}}(x_2; \delta_3, \gamma_{1,3}, \delta_1, x_1, \delta_2) \times \mathcal{M}^1_{R \times \Lambda^-_3}(\gamma_{4,1}, \gamma_{1,3}, \xi_3)$$

$$\mathcal{M}^0_{\Sigma_{322}}(x_2; \delta_3, \gamma_{2,3}, \delta_2) \times \mathcal{M}^1_{\Sigma_{21}}(x_1; \beta_2, \gamma_{1,2}, \beta_1) \times \mathcal{M}^1_{R \times \Lambda^-_3}(\gamma_{3,1}, \gamma_{1,2}, \xi_2, \gamma_{2,3}, \xi_3)$$

Similarly to the previous relation, we will study degeneration of curves in these products of moduli spaces. However, these are not the only one we have to consider. Indeed, the second term of Relation (6) is $m^0_0(d_0(x_2), x_1)$, and analogously to the first term, unfinished pseudo-holomorphic buildings contributing to $\langle m^0_0(x_2, d_0(x_1)), \gamma_{3,1} \rangle$ for a Reeb chord $\gamma_{3,1} \in \mathcal{R}(\Lambda^-_3, \Lambda^-_1)$ have a pseudo-holomorphic building central level whose components can be glued. After gluing,
we get unfinished buildings in the following products:

\[
\mathcal{M}_{\Sigma_{312}}(x_2; \delta_3, \gamma_{1,3}, \delta_1, x_1, \delta_2) \times \widetilde{\mathcal{M}}_{\mathbb{R} \times \Lambda^-_1}^1(\gamma_{3,1}; \xi_1, \gamma_{1,3}, \xi_3) \\
\mathcal{M}_{\Sigma_{321}}(x_2; \delta_3, \gamma_{2,3}, \delta_2) \times \mathcal{M}_{\Sigma_{21}}^0(x_1; \beta_2, \gamma_{1,2}, \beta_1) \times \widetilde{\mathcal{M}}_{\mathbb{R} \times \Lambda^-_1}^1(\gamma_{3,1}; \zeta_1, \gamma_{1,2}, \zeta_2, \gamma_{2,3}, \zeta_3)
\]

Observe that the first product type is the same as one we already obtained above, but the second is different. Let us consider now the third term of Relation (6), which is by definition a sum

\[
m_{-0}(x_2, d_{-0}(x_1)) = b \circ f^{(2)}(x_2, d_{-0}(x_1)) + b^{(2)}(f^{(1)}(x_2), d_{-0}(x_1))
\]

The first term of this sum counts unfinished buildings of height \( \sum_2^1 \text{ such that the two upper levels form a pseudo-holomorphic building of height } \sum_1^0 \). The second one, \( b^{(2)}(f^{(1)}(x_2), d_{-0}(x_1)) \), also counts unfinished holomorphic buildings of height \( \sum_2^1 \) but this time the two lower levels are a holomorphic building of height \( \sum_2^0 \) with boundary on \( \mathbb{R} \times \Lambda^- \). Gluing the components of these buildings, we get unfinished buildings in the following products (see Figure 13):

\[
\mathcal{M}_{\Sigma_{312}}(x_2; \delta_3, \gamma_{1,3}, \delta_1, x_1, \delta_2) \times \widetilde{\mathcal{M}}_{\mathbb{R} \times \Lambda^-_1}^1(\gamma_{3,1}; \xi_1, \gamma_{1,3}, \xi_3) \\
\mathcal{M}_{\Sigma_{321}}(x_2; \delta_3, \gamma_{2,3}, \delta_2) \times \mathcal{M}_{\Sigma_{21}}^0(x_1; \beta_2, \gamma_{1,2}, \beta_1) \times \widetilde{\mathcal{M}}_{\mathbb{R} \times \Lambda^-_1}^2(\gamma_{3,1}; \zeta_1, \gamma_{1,2}, \zeta_2, \gamma_{2,3}, \zeta_3)
\]

Again, we already got the first type of product but the second product is a new one we will have to study. Then, the fourth term of Relation (6), \( m_{-0}(d_{-0}(x_2), x_1) \), is symmetric to the third and so counts unfinished buildings such that some levels can be glued to give unfinished buildings in the same products of moduli spaces as above (for study of the third term). The fifth term is \( d_{-0} \circ m^0_0(x_2, x_1) \). The central level of unfinished buildings contributing to this term is

![Figure 13](image-url)
a pseudo-holomorphic building with two curves which glue together at an intersection point in 
\( \Sigma_1 \cap \Sigma_3 \). After gluing, we get unfinished holomorphic buildings in the product:

\[
\mathcal{M}^{1}\Sigma_{312}(x_2; \delta_3, \gamma_1, 3, \delta_1, x_1, \delta_2) \times \mathcal{M}^1_{\mathbb{R} \times \Lambda_3}(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3)
\]

Finally, the last term of the relation, \( d_\Sigma \circ m_{00}(x_2, x_1) \), counts unfinished buildings of height 2[1]0 of two types, such that the two lower levels are holomorphic buildings of height 2 with boundary on \( \mathbb{R} \times \Lambda^- \). Gluing those levels, we get unfinished buildings in the products:

\[
\begin{align*}
\mathcal{M}^{0}_\Sigma\Sigma_{312}(x_2; \delta_3, \gamma_1, 3, \delta_1, x_1, \delta_2) & \times \mathcal{M}^2_{\mathbb{R} \times \Lambda_3}(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3) \\
\mathcal{M}^{0}_\Sigma\Sigma_{32}(x_2; \delta_3, \gamma_2, 3, \delta_2) & \times \mathcal{M}^0_{\Sigma_{21}}(x_1; \beta_2, \gamma_1, 2, \beta_1) \times \mathcal{M}^2_{\mathbb{R} \times \Lambda_{123}}(\gamma_3, 1; \xi_1, \gamma_1, 2, \xi_2, \gamma_2, 3, \xi_3)
\end{align*}
\]

Now, in order to obtain Relation (11), we need to study the boundary of the compactification of each product of moduli spaces appearing above, where live all the broken curves that are involved in the definition of each term of the relation. So, to sum up, we must study the boundary of the compactification of the following products:

\[
\begin{align*}
(13) & \quad \mathcal{M}^1(x_2; \delta_3, \gamma_1, 3, \delta_1, x_1, \delta_2) \times \mathcal{M}^1(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3) \\
(14) & \quad \mathcal{M}^0(x_2; \delta_3, \gamma_1, 3, \delta_1, x_1, \delta_2) \times \mathcal{M}2(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3) \\
(15) & \quad \mathcal{M}^1(x_2; \delta_3, \gamma_2, 3, \delta_2) \times \mathcal{M}0(\gamma_1; \beta_2, \gamma_1, 2, \beta_1) \times \mathcal{M}1(\gamma_3, 1; \xi_1, \gamma_1, 2, \xi_2, \gamma_2, 3, \xi_3) \\
(16) & \quad \mathcal{M}0(x_2; \delta_3, \gamma_2, 3, \delta_2) \times \mathcal{M}0(x_1; \beta_2, \gamma_1, 2, \beta_1) \times \mathcal{M}2(\gamma_3, 1; \xi_1, \gamma_1, 2, \xi_2, \gamma_2, 3, \xi_3) \\
(17) & \quad \mathcal{M}0(x_2; \delta_3, \gamma_2, 3, \delta_2) \times \mathcal{M}0(x_1; \beta_2, \gamma_1, 2, \beta_1) \times \mathcal{M}2(\gamma_3, 1; \xi_1, \gamma_1, 2, \xi_2, \gamma_2, 3, \xi_3)
\end{align*}
\]

In these products, moduli spaces of index 0 curves with boundary on non-cylindrical Lagrangians are compact 0-dimensional manifolds, as well as the quotient of moduli spaces of index 1 curves with boundary on the negative cylindrical ends of the Lagrangian cobordisms. On the other hand, moduli spaces of index 1 curves with boundary on non-cylindrical Lagrangians are non compact 1-dimensional manifolds, as well as the quotient of moduli spaces of index 2 curves with boundary on cylindrical Lagrangians. By compactness results, these 1-dimensional moduli spaces can be compactified and the boundary of the compactification consists of pseudo-holomorphic buildings with rigid components. Thus, we need to describe the followings spaces:

\[
\begin{align*}
(1) & \quad \partial \mathcal{M}^1(x_2; \delta_3, \gamma_1, 3, \delta_1, x_1, \delta_2), \\
(2) & \quad \partial \mathcal{M}^2(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3), \\
(3) & \quad \partial \mathcal{M}^1(x_2; \delta_3, \gamma_2, 3, \delta_2), \\
(4) & \quad \partial \mathcal{M}^3(x_1; \beta_2, \gamma_1, 2, \beta_1), \\
(5) & \quad \partial \mathcal{M}^2(\gamma_3, 1; \xi_1, \gamma_1, 2, \xi_2, \gamma_2, 3, \xi_3)
\end{align*}
\]

where we write \( \mathcal{M}^2 \) instead of \( \mathcal{M}^2 = (\mathcal{M}^2/R) \) to simplify notation for the compactification of the quotient of a moduli space of index 2 curves with boundary on cylindrical Lagrangians. Once we understand the boundaries of the compactified moduli spaces above, we understand all the broken curves appearing as degeneration of unfinished buildings in the products (13), (14), (15), (16) and (17).

1. \( \partial \mathcal{M}^1(x_2; \delta_3, \gamma_1, 3, \delta_1, x_1, \delta_2) \): the different pseudo-holomorphic buildings in this space are listed below, where the unions are, depending on cases, over intersection points \( p \in \Sigma_1 \cap \Sigma_2, \)

\[
q \in \Sigma_2 \cap \Sigma_3, \quad r \in \Sigma_1 \cap \Sigma_3, \quad \text{chords } \xi_{i,j} \in \mathcal{R}(\Lambda_i, \Lambda_j), \quad 1 \leq i \neq j \leq 3, \quad \text{and words of pure chords}
\]
\( \delta_i', \delta_i'', \delta_i''' \) of \( \Lambda^- \) for \( i = 1, 2, 3 \) satisfying \( \delta_1' = \delta_2' = \delta_3' \), or \( \delta_i' \delta_i'' \delta_i''' = \delta_1 \).

\[
\partial \overline{\mathcal{M}}(x_2; \delta_3; \gamma_{1,3}, \delta_1, x_1, \delta_2) = \overline{\mathcal{M}}(x_2; \delta_3; \gamma_{1,3}, \delta_1, x_1, \delta_2) \]

\[
\bigcup_{p, \delta_i', \delta_i''} \mathcal{M}(x_2; \delta_3, \gamma_{1,3}, \delta_i', p, \delta_i'') \times \mathcal{M}(p; \delta_i'', x_1, \delta_2)
\]

\[
\bigcup_{q, \delta_i', \delta_i''} \mathcal{M}(x_2; \delta_3, q, \delta_i'') \times \mathcal{M}(q; \delta_i'', \gamma_{1,3}, \delta_1, x_1, \delta_2)
\]

\[
\bigcup_{r, \delta_i', \delta_i''} \mathcal{M}(x_2; \delta_3, r, \delta_i'', x_1, \delta_2) \times \mathcal{M}(r; \delta_i'', \gamma_{1,3}, \delta_1)
\]

\[
\bigcup_{\xi_2, \delta_i'} \mathcal{M}(x_2; \delta_3, \gamma_{1,3}, \delta_i', \xi_2, \delta_i'', x_1, \delta_2) \times \overline{\mathcal{M}}(\xi_2; \delta_i', \xi_{1,2}, \delta_i'') \times \mathcal{M}(x_1; \delta_i', \xi_{1,2}, \delta_i''', \gamma_{1,3}, \delta_1)
\]

\[
\bigcup_{\xi_2, \delta_i'} \mathcal{M}(x_2; \delta_3, \xi_2, \delta_i'', x_1, \delta_2) \times \overline{\mathcal{M}}(\xi_2; \delta_i', \xi_{1,2}, \delta_i'') \times \mathcal{M}(x_1; \delta_i', \xi_{1,2}, \delta_i''', \gamma_{1,3}, \delta_1)
\]

\[
\bigcup_{\xi_2, \delta_i'} \mathcal{M}(x_2; \delta_3, \xi_2, \delta_i'', x_1, \delta_2) \times \overline{\mathcal{M}}(\xi_{1,3}; \delta_{i''}', \gamma_{1,3}, \delta_i')
\]

\[
\bigcup_{\xi_2, \delta_i'} \mathcal{M}(x_2; \delta_3, \xi_2, \delta_i'', x_1, \delta_2) \times \mathcal{M}(x_1; \delta_i', \xi_{1,2}, \delta_i''') \times \overline{\mathcal{M}}(\xi_{1,3}; \delta_{i''}', \gamma_{1,3}, \delta_i')
\]

See Figure 14 for a schematic picture of the different types of pseudo-holomorphic buildings in \( \partial \overline{\mathcal{M}}(x_2; \delta_3, \gamma_{1,3}, \delta_1, x_1, \delta_2) \).

**Figure 14.** Pseudo-holomorphic buildings in the boundary of \( \overline{\mathcal{M}}(x_2; \delta_3, \gamma_{1,3}, \delta_1, x_1, \delta_2) \).

2. \( \partial \overline{\mathcal{M}}(\gamma_{1,3}; \xi_1, \gamma_{1,3}, \xi_3) \): pseudo-holomorphic buildings appearing as degeneration of index-2 bananas are of two types. We have:

\[
\partial \overline{\mathcal{M}}(\gamma_{1,3}; \xi_1, \gamma_{1,3}, \xi_3) = \overline{\mathcal{M}}(\gamma_{1,3}; \xi_1, \gamma_{1,3}, \xi_3)
\]

\[
\bigcup_{\xi_1, \xi_3} \overline{\mathcal{M}}(\gamma_{1,3}; \xi_1, \gamma_{1,3}, \xi_3) \times \overline{\mathcal{M}}(\xi_3; \xi_1, \gamma_{1,3}, \xi_3)
\]

\[
\bigcup_{\xi_1, \xi_3} \mathcal{M}(\gamma_{1,3}; \xi_1, \gamma_{1,3}, \xi_3) \times \mathcal{M}(\gamma_{1,3}; \xi_3; \xi_1, \gamma_{1,3}, \xi_3)
\]
with again $\xi_{i,j} \in R(\Lambda_i^-, \Lambda_j^-)$, and $\xi_i', \xi_i''$ words of Reeb chords of $\Lambda_i^-$, with $\xi_i' \xi_i'' = \xi_i$ (see Figure 15).

Figure 15. Pseudo-holomorphic buildings in the boundary of $\overline{\mathcal{M}}^2(\gamma_3, 1; \delta_1, \gamma_1, 3, \delta_3)$.

With 1. and 2. above, we can describe all the types of broken curves in the boundary of the compactification of the products (13) and (14). Instead of writing again huge unions of moduli spaces, we drew on Figure 16 schematic pictures of the corresponding unfinished holomorphic buildings. The first seven (from left to right and top to bottom) are in:

(18) $\partial \mathcal{M}^1(x_2; \delta_3, \gamma_1, 3, \delta_1, x_1, \delta_2) \times \mathcal{M}^1(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3)$

and the last two are in:

(19) $\mathcal{M}^0(x_2; \delta_3, \gamma_1, 3, \delta_1, x_1, \delta_2) \times \partial \overline{\mathcal{M}}^2(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3)$

Figure 16. Unfinished pseudo-holomorphic buildings in (18) and (19).
Remark 10. In the bottom of Figure 16 two of the unfinished holomorphic buildings compensate: the leftmost compensate with the rightmost. The first one arises as a degeneration of a curve in $\mathcal{M}^1(x_2; \delta_3, \gamma_1, \delta_1, x_1, \delta_2)$, and the second one as a degeneration of a banana in $\mathcal{M}^2(\gamma_{1,3}; \xi_1, \gamma_{1,3}, \xi_3)$. These two unfinished buildings correspond thus to different geometric configurations because they live in the boundary of the compactification of two different products of moduli spaces. However, these buildings differ only by a trivial strip $\mathbb{R} \times \gamma_{1,1}$ so they contribute algebraically to the same map which is in this case $b \circ \delta \circ f^{(2)}(x_2, x_1)$, where $\delta$ is the dual of $d_{----}$.

In order deduce the algebraic relation that these boundary elements give, we introduce a new map:

$$\Delta^{(2)} : C_{n-1,-e}(\Lambda_3^-, \Lambda_2^-) \times C_{n-1,-e}(\Lambda_2^-, \Lambda_1^-) \to C_{n-1,-e}(\Lambda_3^-, \Lambda_1^-)$$

defined on pairs of generators by:

$$\Delta^{(2)}(\gamma_3, \gamma_1, 1.2) = \sum_{\gamma_{1,2}} #\mathcal{M}^0(\gamma_3, \gamma_1, \delta_1, \gamma_1, \delta_2) \varepsilon_1(\delta_1) \varepsilon_2(\delta_2) \varepsilon_3(\delta_3) \cdot \gamma_{1,3}$$

$$\Delta^{(2)}(\gamma_3, \gamma_1, 2.1) = \sum_{\gamma_{2,1}} #\mathcal{M}^0(\gamma_3, \gamma_1, \delta_1, \gamma_2, \delta_2) \varepsilon_1(\delta_1) \varepsilon_2(\delta_2) \varepsilon_3(\delta_3) \cdot \gamma_{1,3}$$

$$\Delta^{(2)}(\gamma_3, \gamma_2, 1.2) = \sum_{\gamma_{2,1}} #\mathcal{M}^0(\gamma_3, \gamma_2, \delta_1, \gamma_1, \delta_1) \varepsilon_1(\delta_1) \varepsilon_2(\delta_2) \varepsilon_3(\delta_3) \cdot \gamma_{1,3}$$

$$\Delta^{(2)}(\gamma_3, \gamma_2, 2.1) = 0$$

Now, as for Relation (5), the mod-2 count of unfinished pseudo-holomorphic buildings in the products (18) and (19) equals $\partial^4$. On the other hand, these broken curves contribute to some composition of maps we have defined earlier. This implies that the following relation is satisfied:

$$b \circ f^{(2)}(x_2, d_{00}(x_1)) + b \circ f^{(2)}(d_{00}(x_2), x_1) + d_{---} \circ m_{00}(x_2, x_1)$$

$$= b \circ f^{(2)}(x_2, d_{-\cdots}(x_1)) + b \circ f^{(2)}(d_{-\cdots}(x_2), x_1) + b \circ \Delta^{(2)}(f^{(1)}(x_2), f^{(1)}(x_1))$$

$$+ d_{-\cdots} \circ b \circ f^{(2)}(x_2, x_1) = 0$$

where we did not write the term $b \circ \delta \circ f^{(2)}(x_2, x_1)$ as it would appear twice so this vanishes over $\mathbb{Z}_2$ (see Remark 14).

3. $\partial \mathcal{M}^1(x_2; \delta_3, \gamma_{2,3}, \delta_2)$: pseudo-holomorphic buildings in this space are of the following type (see Figure 17).

$$\partial \mathcal{M}^1(x_2; \delta_3, \gamma_{2,3}, \delta_2) = \mathcal{M}^1(x_2; \delta_3, \gamma_{2,3}, \delta_2)$$

$$\bigcup_{q \in \Sigma_2 \times \Sigma_3} \mathcal{M}(x_2; \delta_3, \delta_2', \delta_2'') \times \mathcal{M}(\xi_2; \delta_3', \gamma_{2,3}, \delta_2')$$

$$\bigcup_{\xi_2, \delta_3', \delta_2''} \mathcal{M}(x_2; \delta_3', \xi_2, \delta_2'') \times \mathcal{M}(\xi_2; \delta_3', \gamma_{2,3}, \delta_2')$$

4. $\partial \mathcal{M}^1(x_1; \beta_2, \gamma_{1,2}, \beta_1)$: same types of degenerations as above (case 3.).

5. $\partial \mathcal{M}^1(\gamma_{1,1}; \xi_1, \gamma_{1,2}, \xi_2, \gamma_{2,3}, \xi_3)$: we describe here degenerations of index-2 bananas with three positive Reeb chords asymptotics (see Figure 18 for a schematic picture of the corresponding
where \( \zeta_i' \) and \( \zeta_i'' \) are words of pure chords of \( \Lambda_1^- \) such that \( \zeta_i' \zeta_i'' = \zeta_i \).

By 3., 4. and 5., we can describe all the types of unfinished pseudo-holomorphic buildings in the boundary of the compactification of the products (15), (16) and (17), that is to say, unfinished...
Combining Relations (20) and (21), we get Relation (6): 

\[ m_{00}(d_{00}(x_2), x_1) + m_{00}(d_{00}(x_2), d_{00}(x_1)) + m_{-0}(d_{-0}(x_2), x_1) + m_{-0}(d_{-0}(x_2), d_{-0}(x_1)) + m_\gamma(d_{\gamma}(x_2), d_{\gamma}(x_1)) + d_{-\gamma} \circ m_{00}(d_{00}(x_2), d_{\gamma}(x_1)) \]

where the term \( b \circ \Delta^{(2)}(f^{(1)}(x_2), f^{(1)}(x_1)) \) disappeared because it is at the same time in (20) and (21) so vanishes over \( \mathbb{Z}_2 \).

The corresponding buildings are schematized on Figure 19: the first two (from left to right and top to bottom) are in the boundary of the compactification of (15), the following two are in the boundary of the compactification of (16), and finally the six others are in the boundary of the compactification of (17). As in the preceding case (see Remark 10), several unfinished holomorphic buildings compensate. Indeed, the second and the sixth one, differing by a trivial strip \( R \times \gamma_{3,1} \) contribute to \( b^{(2)}(\delta_{-\gamma} \circ f^{(1)}(x_2), f^{(1)}(x_1)) \), and the fourth and the fifth, differing also by the same type of trivial strip contribute to \( b^{(2)}(f^{(1)}(x_2), \delta_{-\gamma} \circ f^{(1)}(x_1)) \). We obtain this time the relation:

\[
\begin{align*}
\partial \overline{M}^1(x_2; \delta_3, \gamma_{2,3}, \delta_2) & \times M^0(x_1; \beta_2, \gamma_{1,2}, \beta_1) \times M^1(\gamma_{3,1}; \gamma_{1,2}, \gamma_{2,3}, \gamma_3) \\
\bigcup M^0(x_2; \delta_3, \gamma_{2,3}, \delta_2) & \times \partial \overline{M}^1(x_1; \beta_2, \gamma_{1,2}, \beta_1) \times M^1(\gamma_{3,1}; \gamma_{1,2}, \gamma_{2,3}, \gamma_3) \\
\bigcup M^0(x_2; \delta_3, \gamma_{2,3}, \delta_2) & \times M^0(x_1; \beta_2, \gamma_{1,2}, \beta_1) \times \partial \overline{M}^1(\gamma_{3,1}; \gamma_{1,2}, \gamma_{2,3}, \gamma_3)
\end{align*}
\]
5.2.3. Relation (7). This relation is really analogous to Relation (5) except that one of the three mixed asymptotics is a Reeb chord. Each term in (7) counts pseudo-holomorphic buildings whose components can be glued on index-1 disks in the moduli space \( \mathcal{M}^3_{x,23}(x^+; \delta_1, \gamma_1, \delta_2, x_2, \delta_3) \). To determine Relation (7), we have thus to study the broken curves in the boundary of the compactification of this moduli space. This gives:

\[
\begin{align*}
&\partial \mathcal{M}_{\Sigma_{123}}^3(x^+; \delta_1, \gamma_1, \delta_2, x_2, \delta_3) = \mathcal{M}^0(x^+, \delta_1, \gamma_1, \delta_2, x_2, \delta_3) \\
&\bigcup_{p, \delta_1, \delta_2} \mathcal{M}(x^+; \delta_1, p, \delta_2^\prime, x_2, \delta_3) \times \mathcal{M}_{\Sigma_{13}}^3(p; \delta_1^\prime, \gamma_1, \delta_2^\prime) \\
&\bigcup_{q, \delta_1, \delta_2} \mathcal{M}(x^+; \delta_1, \gamma_1, \delta_2, q, \delta_3^\prime) \times \mathcal{M}_{\Sigma_{23}}^3(q; \delta_2^\prime, x_2, \delta_3^\prime) \\
&\bigcup_{r, \delta_1, \delta_2} \mathcal{M}_{\Sigma_{13}}^3(x^+; \delta_1^\prime, r, \delta_2^\prime) \times \mathcal{M}(r; \delta_1^\prime, \gamma_1, \delta_2, x_2, \delta_3^\prime) \\
&\bigcup_{\xi_2, \delta_1, \delta_2} \mathcal{M}(x^+; \delta_1, \gamma_1, \delta_2, \xi_2, \delta_3) \times \tilde{\mathcal{M}}((\xi_2, 1; \delta_1^\prime, \gamma_1, \delta_2) \\
&\bigcup_{\xi_2, \delta_1, \delta_2} \mathcal{M}(x^+; \delta_1, \gamma_1, \delta_2, \xi_2, \delta_3) \times \tilde{\mathcal{M}}((\xi_2, 3; \delta_2^\prime, \delta_3^\prime) \times \mathcal{M}_{\Sigma_{23}}^3(x_2; \delta_3^\prime, \delta_3^\prime, \delta_2^\prime) \\
&\bigcup_{\xi_2, \delta_1, \delta_2} \mathcal{M}(x^+; \delta_1, \gamma_1, \delta_2, \xi_2, \delta_3) \times \tilde{\mathcal{M}}((\xi_2, 3; \delta_2^\prime, \delta_3^\prime) \times \mathcal{M}_{\Sigma_{23}}^3(x_2; \delta_3^\prime, \delta_2^\prime, \delta_2^\prime) \\
&\bigcup_{\xi_2, \delta_1, \delta_2} \mathcal{M}(x^+; \delta_1, \gamma_1, \delta_2, \xi_2, \delta_3) \times \tilde{\mathcal{M}}((\xi_2, 3; \delta_2^\prime, \delta_3^\prime) \times \mathcal{M}_{\Sigma_{23}}^3(x_2; \delta_3^\prime, \delta_2^\prime, \delta_2^\prime)
\end{align*}
\]

where the three first unions are respectively for \( p \in \Sigma_1 \cap \Sigma_2 \), \( q \in \Sigma_2 \cap \Sigma_3 \) and \( r \in \Sigma_1 \cap \Sigma_3 \). On

![Figure 20. Pseudo-holomorphic buildings in the boundary of the compactification of \( \mathcal{M}^3_{x,23}(x^+; \delta_1, \gamma_1, \delta_2, x_2, \delta_3) \).](image)

Figure 20 each broken configuration contributes from left to right to the terms of Relation (7), and so we get:

\[
m^0_0(x_2, d_0\cdot (\gamma_1)) + m^0_0\cdot (d_0, \gamma_1) + d_0 \circ m^0_0(x_2, \gamma_1) \\
+ m^0_0(x_2, d\cdot (\gamma_1)) + m^0_0\cdot (d\cdot (x_2), \gamma_1) + d_0 \cdot m^0_0(x_2, \gamma_1) = 0
\]
5.2.4. Relation (8). Again, to find this relation we argue the same way as for Relation (6). First, let us remark that one term in this relation already vanishes for energy reasons. More precisely, by definition we have:

\[ m_{00}(x_2, d_0 - (\gamma_1)) = b \circ f^{(2)}(x_2, d_0 - (\gamma_1)) + b^{(2)}(f^{(1)}(x_2), f^{(1)} \circ d_0 - (\gamma_1)) \]

but \( b(f^{(1)}(x_2), f^{(1)} \circ d_0 - (\gamma_1)) = 0 \) because such a term would count negative energy curves which is not possible, see Figure 21. Indeed, if there exist pseudo-holomorphic curves \( u \in M^0(\gamma_1, \delta_1, \delta_2) \) and \( v \in M^0(\gamma_1, \delta_2, \delta_3) \), then the energies of these curves are given by (see Section 2.5):

\[
E_{d(\chi_0)}(u) = a(q) - a(\gamma_1) - a(\delta_1) - a(\delta_2)
E_{d(\chi_0)}(v) = -a(q) - a(\gamma_1, 2) - a(\zeta_1) - a(\zeta_2)
\]

The energy of a non-constant pseudo-holomorphic curve is always strictly positive and the action of Reeb chords is also always positive, so the existence of \( v \) implies that \( q \) is an intersection point with a strictly negative action, which then contradicts existence of \( u \). The other terms of

![Figure 21. Impossible breaking.](image)

Relation (8) are defined by a count of unfinished buildings such that some levels are in a pseudo-holomorphic building, and so their components can be glued. After gluing, we get unfinished buildings in the following products of moduli spaces

\[
\begin{align*}
M^1(x_2; \delta_3, \gamma_1, 3, \delta_1, \gamma_1, \delta_2) & \times \overline{M}^1(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3) \\
M^0(x_2; \delta_3, \gamma_1, 1, \delta_1, \gamma_1, \delta_2) & \times \overline{M}^2(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3) \\
M^1(x_2; \delta_3, \gamma_2, 1, \delta_2) & \times \overline{M}^1(\gamma_3, 1; \xi_1, \gamma_1, 2, \zeta_2, \zeta_3) \\
M^0(x_2; \delta_3, \gamma_2, 1, \delta_2) & \times \overline{M}^2(\gamma_3, 1; \xi_1, \gamma_1, 2, \zeta_2, \zeta_3)
\end{align*}
\]

Now, in order to get the relation, we have to find the broken curves in the boundary of the compactification of these products, i.e. broken curves in:

\[
\begin{align*}
& \partial \overline{M}^1(x_2; \delta_3, \gamma_1, 3, \delta_1, \gamma_1, \delta_2) \times \overline{M}^1(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3) \\
& \partial M^0(x_2; \delta_3, \gamma_1, 1, \delta_1, \gamma_1, \delta_2) \times \overline{M}^2(\gamma_3, 1; \xi_1, \gamma_1, 3, \xi_3) \\
& \partial \overline{M}^1(x_2; \delta_3, \gamma_2, 1, \delta_2) \times \overline{M}^1(\gamma_3, 1; \xi_1, \gamma_1, 2, \zeta_2, \zeta_3) \\
& \partial M^0(x_2; \delta_3, \gamma_2, 1, \delta_2) \times \overline{M}^2(\gamma_3, 1; \xi_1, \gamma_1, 2, \zeta_2, \zeta_3)
\end{align*}
\]
We already described $\partial \overline{M}^3(\gamma_3, 1; \xi_1, \gamma_1, \xi_2, \gamma_2, \delta_1, \delta_2)$ and $\partial \overline{M}^3(x_3; \delta_3, \gamma_2, \delta_2)$ in Section 5.2.2 so it remains to study $\partial \overline{M}^3(x_3; \delta_3, \gamma_1, \delta_1, \delta_2)$ and $\partial \overline{M}^3(\gamma_3, 1; \xi_1, \gamma_1, \xi_2, \gamma_2, \delta_3, \delta_3)$. First, we have the following decomposition:

$$
\partial \overline{M}^3(x_3; \delta_3, \gamma_1, \delta_1, \gamma_1, \delta_2) = \overline{M}^3(x_3; \delta_3, \gamma_1, \delta_1, \gamma_1, \delta_2) + \bigcup_{p \in \Sigma_1, \Sigma_2} \mathcal{M}(x_3; \delta_3, \gamma_1, \delta_1, \gamma_1, \delta_2) \times \mathcal{M}(p; \delta_1, \gamma_1, \delta_2)
$$

We already described

Finally, the buildings occurring as degeneration of index-2 bananas with two positive Reeb chord asymptotics and one negative one are of the following type:

$$
\partial \overline{M}^3(\gamma_3, 1; \xi_1, \gamma_1, \xi_2, \gamma_2, \delta_1, \delta_2) = \overline{M}^3(\gamma_3, 1; \xi_1, \gamma_1, \xi_2, \gamma_2, \delta_1, \delta_2) + \bigcup_{\xi_2, \xi_3, \delta_1, \delta_2} \mathcal{M}(\gamma_3, 1; \xi_1, \gamma_1, \xi_2, \gamma_2, \delta_1, \delta_2) \times \overline{M}(\xi_2, \xi_3, \delta_1, \delta_2)
$$

The different types of unfinished buildings corresponding to elements in the products (22), (23), (24) and (25) are schematized on Figure 22. On the top of the figure, unfinished buildings are in (22) and (23), and we did not draw those that compensate by pair in these products because they differ by a trivial strip (see Remark 10). On the bottom of the figure, unfinished buildings are in (24) and (25), where we again did not draw the curves that compensate. However, we can remark that one broken configuration in (22) compensate also with one in (25). Both contribute
algebraically to $b \circ \Delta^{(2)}(f^{(1)}(x_2), \gamma_1)$. The remaining unfinished buildings give the Relation (5):

$$m_{00}(x_2, d_{00}(\gamma_1)) + m^-_{d_{00}(x_2), \gamma_1} + m^-_{00}(x_2, d_{-}(\gamma_1)) + m_{-}(d_{-0}(x_2), \gamma_1) + d_{-0} \circ m^0_{-0}(x_2, \gamma_1) + d_{-} \circ m^0_{-0}(x_2, \gamma_1) = 0$$

5.2.5. Relations (9) and (10). By symmetry, Relations (9) and (10) for a pair $(\gamma_2, x_1)$ are obtained by studying same types of holomorphic curves as for Relations (7) and (8) corresponding to a pair of asymptotics $(x_2, \gamma_1)$. 
5.2.6. Relation (11). Each term of this relation corresponds to a count of broken curves in the boundary of the compactification of $M^1(x^+; \delta_1, \gamma_1, \delta_2, \gamma_2, \delta_3)$, and we have (see Figure 23):

$$\partial \overline{M}^1(x^+; \delta_1, \gamma_1, \delta_2, \gamma_2, \delta_3) = \overline{M}^0(x^+; \delta_1, \gamma_1, \delta_2, \gamma_2, \delta_3)$$

$$\bigcup_{q \in \Sigma_1 \cap \Sigma_2} \overline{M}^0(x^+; \delta_1', q, \delta_3') \times \overline{M}^0(q; \delta_1'', \gamma_1, \delta_2, \gamma_2, \delta_3')$$

$$\bigcup_{q \in \Sigma_1 \cap \Sigma_3} \overline{M}^0(x^+; \delta_1', q, \delta_2', \gamma_2, \delta_3) \times \overline{M}^0(q; \delta_1'', \gamma_1, \delta_2)$$

$$\bigcup_{q \in \Sigma_2 \cap \Sigma_3} \overline{M}^0(x^+; \delta_1', \gamma_1, \delta_2', q, \delta_3') \times \overline{M}^0(q; \delta_1'', \gamma_1, \delta_2, \delta_3)$$

Each term of this relation corresponds to a count of broken curves in the boundary of the compactification of $M^1(x^+; \delta_1, \gamma_1, \delta_2, \gamma_2, \delta_3)$.

5.2.7. Relation (12). The product of two Reeb chords being given by the product $\mu_{\gamma_1,1,1}^2$ in the augmentation category $\mathcal{A}_{\mu=-}(A_1^+ \cup A_2^- \cup A_3^-)$, Relation (12) is satisfied because it is the $A_\infty$-relation for $d=2$ satisfied by the maps $\{\mu^n\}_{n \geq 1}$ (see (2) in Section 3.2). We recall the different kinds of degeneration of a curve in $\overline{M}^2(\gamma_1, \delta_1, \gamma_1, \delta_2, \gamma_2, \delta_3)$ (see Figure 24):

$$\partial \overline{M}^2(\gamma_3, \delta_1, \gamma_1, \delta_2, \gamma_2, \delta_3) = \overline{M}^0(\gamma_3, \delta_1, \gamma_1, \delta_2, \gamma_2, \delta_3)$$

$$\bigcup \overline{M}(\gamma_3, \delta_1', \xi_3, \delta_3') \times \overline{M}(\xi_3, \delta_3', \gamma_1, \delta_2, \delta_3')$$

$$\bigcup \overline{M}(\gamma_3, \delta_1', \xi_2, \delta_2', \gamma_2, \delta_3) \times \overline{M}(\xi_2, \delta_2', \gamma_3, \delta_3')$$

$$\bigcup \overline{M}(\gamma_3, \delta_1', \delta_1, \delta_2', \delta_3') \times \overline{M}(\delta_1', \delta_3', \gamma_3, \delta_3')$$

**Figure 24.** Pseudo-holomorphic buildings in the boundary of $\overline{M}^2(\gamma_3, \delta_1, \gamma_1, \delta_2, \gamma_2, \delta_3)$.
5.3. Proof of Theorem 2 Let us denote $\mathcal{F}^1 := d_{+0} + d_{+-}$ and recall that the acyclicity of the Cthulhu complex implies that $\mathcal{F}_1$ is a quasi-isomorphism. We consider as before three transverse exact Lagrangian cobordisms $\Sigma_1, \Sigma_2$ and $\Sigma_3$ such that the algebras $\mathcal{A}(\Lambda_-)$ admit augmentations. As introduced in Section 2.4.4, we need to consider the new following moduli spaces of curves with boundary on $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ (see Figure 25):

\begin{align*}
(26) & \quad \mathcal{M}_{\Sigma_{123}}(\gamma_{3,1}^+; \delta_1, x_1, \delta_2, x_2, \delta_3) \\
(27) & \quad \mathcal{M}_{\Sigma_{123}}(\gamma_{3,1}^+; \delta_1, \gamma_1, \delta_2, x_2, \delta_3) \\
(28) & \quad \mathcal{M}_{\Sigma_{123}}(\gamma_{3,1}^+; \delta_1, x_1, \delta_2, \gamma_2, \delta_3) \\
(29) & \quad \mathcal{M}_{\Sigma_{123}}(\gamma_{3,1}^+; \delta_1, \gamma_1, \delta_2, \gamma_2, \delta_3)
\end{align*}

with

- $\gamma_{3,i}^+ \in \mathcal{R}(\Lambda_{3,i}^+, \Lambda_{3,i}^-)$, $\gamma_1 \in \mathcal{R}(\Lambda_2^-, \Lambda_2^+)$ and $\gamma_2 \in \mathcal{R}(\Lambda_3^-, \Lambda_3^+)$,
- $x_1 \in \Sigma_1 \cap \Sigma_2$, $x_2 \in \Sigma_2 \cap \Sigma_3$,
- $\delta_i$ are words of Reeb chords of $\Lambda_i^-$, for $i = 1, 2, 3$.

![Figure 25. Examples of curves in the moduli spaces (26), (27), (28), and (29) respectively.](image)

By a count of rigid pseudo-holomorphic disks in these moduli spaces, we introduce a map:

\[ \mathcal{F}^2 : CF_{-\infty}(\Sigma_2, \Sigma_3) \otimes CF_{-\infty}(\Sigma_1, \Sigma_2) \rightarrow C^*(\Lambda_{3,i}^+, \Lambda_{3,i}^+) \]

defined on pairs of generators by:

\begin{align*}
\mathcal{F}^2(x_2, x_1) & = \sum_{\gamma_{3,1}^+, \delta_1} \# \mathcal{M}^0(\gamma_{3,1}^+; \delta_1, x_1, \delta_2, x_2, \delta_3) e_1^-(\delta_1) e_2^-(\delta_2) e_3^-(\delta_3) \cdot \gamma_{3,1}^+ \\
\mathcal{F}^2(x_2, \gamma_1) & = \sum_{\gamma_{3,1}^+, \delta_1} \# \mathcal{M}^0(\gamma_{3,1}^+; \delta_1, \gamma_1, \delta_2, x_2, \delta_3) e_1^-(\delta_1) e_2^-(\delta_2) e_3^-(\delta_3) \cdot \gamma_{3,1}^+ \\
\mathcal{F}^2(\gamma_2, x_1) & = \sum_{\gamma_{3,1}^+, \delta_1} \# \mathcal{M}^0(\gamma_{3,1}^+; \delta_1, x_1, \delta_2, \gamma_2, \delta_3) e_1^-(\delta_1) e_2^-(\delta_2) e_3^-(\delta_3) \cdot \gamma_{3,1}^+ \\
\mathcal{F}^2(\gamma_2, \gamma_1) & = \sum_{\gamma_{3,1}^+, \delta_1} \# \mathcal{M}^0(\gamma_{3,1}^+; \delta_1, \gamma_1, \delta_2, \gamma_2, \delta_3) e_1^-(\delta_1) e_2^-(\delta_2) e_3^-(\delta_3) \cdot \gamma_{3,1}^+
\end{align*}

We have again to study breakings of index-1 pseudo-holomorphic curves in the moduli spaces (26), (27), (28), and (29) in order to prove Theorem 2. For example, we describe below the
Figure 26. Pseudo-holomorphic buildings in $\partial M^1(\gamma_{3,1}^+; \delta_1, x_1, \delta_2, x_2, \delta_3)$ (see Figure 26):

$$\partial \overline{M}^1(\gamma_{3,1}^+; \delta_1, x_1, \delta_2, x_2, \delta_3) = \overline{M}^1(\gamma_{3,1}^+; \delta_1, x_1, \delta_2, x_2, \delta_3)$$

$$\bigcup_{p \in \Sigma_1 \cap \Sigma_2} M(\gamma_{3,1}^+; \delta_1, x_1, \delta_2, x_2, \delta_3) \times M(p; \delta_1', x_1, \delta_2')$$

$$\bigcup_{q \in \Sigma_2 \cap \Sigma_3} M(\gamma_{3,1}^+; \delta_1, x_1, \delta_2', q, \delta_3') \times M(q; \delta_2', x_2, \delta_3')$$

$$\bigcup_{r \in \Sigma_1 \cap \Sigma_3} M(\gamma_{3,1}^+; \delta_1, r, \delta_3') \times M(r; \delta_1', x_1, \delta_2', \delta_3')$$

$$\bigcup_{t \in \Sigma_2 \cap \Sigma_3} M(\gamma_{3,1}^+; \delta_1', x_1, \delta_2', \delta_3') \times M(\xi_{1,2}, \xi_{2,1}', \delta_2', \delta_3') \times M_{\Sigma_3}(x_1; \delta_2', \xi_{1,2}', \delta_3'')$$

$$\bigcup_{t \in \Sigma_2 \cap \Sigma_3} M(\gamma_{3,1}^+; \delta_1', x_1, \delta_2', \delta_3') \times M(\xi_{1,3}^-, \xi_{3,1}^-, \delta_3') \times M_{\Sigma_3}(x_2; \delta_3', \xi_{1,3}^-, \delta_3'')$$

$$\bigcup_{t \in \Sigma_2 \cap \Sigma_3} M(\gamma_{3,1}^+; \delta_1', \xi_{1,3}^-, \delta_3'') \times M(\xi_{3,1}^-, \delta_1', \xi_{3,1}^-, \delta_3'') \times M_{\Sigma_3}(x_2; \delta_3', \xi_{1,3}^-, \delta_3'')$$

$$\bigcup_{t \in \Sigma_2 \cap \Sigma_3} M(\gamma_{3,1}^+; \delta_1', \xi_{1,3}^-, \delta_3'') \times M(\xi_{3,1}^-, \delta_1', \xi_{3,1}^-, \delta_3'') \times M(\Sigma_2; (x_1; \delta_2, \xi_{1,2}', \delta_3'') \times M_{\Sigma_3}(x_2; \delta_3', \xi_{2,3}', \delta_3'')$$

$$\bigcup M^1(\gamma_{3,1}^+; \delta_1, x_1, \delta_2, x_2, \delta_3) \times M(\xi_{1,2}, \xi_{2,1}', \delta_2', \delta_3')$$

$$\bigcup M^0(\xi_{3,1}^+, \delta_1', x_1, \delta_2', \delta_3') \times \prod_{i=1}^s M_{\Sigma_3}(\beta_{1,i}; \delta_1, i) \times \prod_{j=1}^t M_{\Sigma_3}(\beta_{3,j}; \delta_3, j)$$

$$\bigcup M^0(\xi_{1,2}', \delta_2', x_2, \delta_3') \times \prod_{i=1}^l M_{\Sigma_3}(\beta_{1,i}; \delta_1, i) \times \prod_{j=1}^m M_{\Sigma_3}(\beta_{2,j}; \delta_2, j) \times \prod_{i=1}^n M_{\Sigma_3}(\beta_{3,i}; \delta_3, i)$$
where all unions except the last two are also for words of pure Reeb chords $\delta'_1, \delta''_1$ and $\delta'''_1$ of $\Lambda_1^−$ such that $\delta'_1, \delta''_1 = \delta_1$, or $\delta', \delta''', \delta''' = \delta_1$, depending on cases. The second to last union is for:

- $\beta_{1,i} \in \mathcal{R}(\Lambda_1^+) \text{ and } \delta_{1,i} \in \mathcal{R}(\Lambda_1^-)$ for $1 \leq i \leq s$,
- $\beta_{2,i} \in \mathcal{R}(\Lambda_2^+) \text{ and } \delta_{2,i} \in \mathcal{R}(\Lambda_2^-)$ for $1 \leq i \leq t$,
- $\delta''_1$ word of Reeb chords of $\Lambda_1^−$,
- $\delta'_3$ word of Reeb chord of $\Lambda_3^−$.

such that $\delta_1, \ldots, \delta_{1,i}, \delta''_i = \delta_1$ and $\delta'_i \delta_{1,i} \ldots \delta_{3,i} = \delta_3$. The count of curves in the moduli space $\mathcal{M}_{\Sigma}(\beta_{1,i}; \delta_{1,i})$ contribute to the coefficient $\langle \phi_{\Sigma_1}(\beta_{1,i}), \delta_{1,i} \rangle$, where recall that $\phi_{\Sigma_1} : \mathcal{A}(\Lambda_1^+) \rightarrow \mathcal{A}(\Lambda_1^-)$ is the chain map induced by $\Sigma_1$ (see Section 3.3). So the count of curves in

$$\mathcal{M}^1(\gamma_{3,1}^+, \beta_{1,1}, \ldots, \beta_{1,s}, \xi_{3,1}^+, \beta_{3,1}, \ldots, \beta_{3,n}) \times \prod_{i=1}^s \mathcal{M}_{\Sigma_1}(\beta_{1,i}; \delta_{1,i}) \times \prod_{j=1}^t \mathcal{M}_{\Sigma_2}(\beta_{3,j}; \delta_{3,j})$$

contributes to $\langle \mu_{+}^1 \varepsilon_{+}, \varepsilon_1^+ \rangle \langle \xi_{3,1}^+, \gamma_{3,1}^+ \rangle$ because $\varepsilon_{+}^+ = \varepsilon_{-}^- \circ \phi_{\Sigma_1}$. Finally, the last union is for

- $\beta_{1,i} \in \mathcal{R}(\Lambda_1^+) \text{ and } \delta_{1,i} \in \mathcal{R}(\Lambda_1^-)$ for $1 \leq i \leq l$,
- $\beta_{2,i} \in \mathcal{R}(\Lambda_2^+) \text{ and } \delta_{2,i} \in \mathcal{R}(\Lambda_2^-)$ for $1 \leq i \leq m$,
- $\beta_{3,i} \in \mathcal{R}(\Lambda_3^+) \text{ and } \delta_{3,i} \in \mathcal{R}(\Lambda_3^-)$ for $1 \leq i \leq n$,
- $\delta''_1$ word of Reeb chords of $\Lambda_1^−$,
- $\delta'_3$ and $\delta''_3$ words of Reeb chords of $\Lambda_3^−$,
- $\delta'_3 \delta_{1,1} \ldots \delta_{3,n} = \delta_3$.

Again, the count of broken curves in

$$\mathcal{M}(\gamma_{3,1}^+, \beta_{1,1}, \ldots, \beta_{1,s}, \xi_{3,1}^+, \beta_{2,1}, \ldots, \beta_{2,m}, \xi_{3,2}^+, \beta_{3,1}, \ldots, \beta_{3,n}) \times \prod_{i=1}^l \mathcal{M}_{\Sigma_1}(\beta_{1,i}; \delta_{1,i})$$

$$\times \prod_{i=1}^m \mathcal{M}_{\Sigma_2}(\beta_{2,i}; \delta_{2,i}) \times \prod_{i=1}^n \mathcal{M}_{\Sigma_3}(\beta_{3,i}; \delta_{3,i})$$

contributes to the coefficient $\langle \mu_{+}^2 \varepsilon_{+}, \varepsilon_{+} \rangle \langle \xi_{3,1}^+, \gamma_{3,1}^+ \rangle$.

By denoting $d_{+}$ for $\mu_{+}^1 \varepsilon_{+}$, the study of breakings above implies that $\mathcal{F}^2$ satisfies the relation:

$$\mathcal{F}^2(x_2, d_{00}(x_1)) + \mathcal{F}^2(x_2, d_{0-}(x_1)) + \mathcal{F}^2(d_{00}(x_2), x_1) + \mathcal{F}^2(d_{-0}(x_2), x_1) + d_{+} \circ m_{00}(x_2, x_1)$$

$$+ d_{+} \circ m_{-0}(x_2, x_1) + d_{++} \circ \mathcal{F}^2(x_2, x_1) + \mu_{+}^2 \circ \mathcal{F}^2(d_{+0}(x_2), d_{+0}(x_1)) = 0$$

(30)

Analogously, the different types of buildings in $\delta \mathcal{M}_{\Sigma_1}(\gamma_{3,1}^+, \beta_{1,1}, \beta_{2,1}, x_2, x_3)$ are schematized on Figure 27 and this gives for $\mathcal{F}^2$ the relation:

$$\mathcal{F}^2(x_2, d_{0-}(x_1)) + \mathcal{F}^2(x_2, d_{-0}(x_1)) + \mathcal{F}^2(d_{00}(x_2), x_1) + \mathcal{F}^2(d_{-0}(x_2), x_1) + d_{+} \circ m_{00}(x_2, x_1)$$

$$+ d_{+} \circ m_{-0}(x_2, x_1) + d_{++} \circ \mathcal{F}^2(x_2, x_1) + \mu_{+}^2 \circ \mathcal{F}^2(d_{+0}(x_2), d_{+0}(x_1)) = 0$$

(31)

The symmetric relation for the pair $(\gamma_2, x_1)$ of asymptotics is of course also satisfied. Finally, pseudo-holomorphic buildings in $\delta \mathcal{M}_{\Sigma_1}(\gamma_{3,1}^+, \beta_{1,1}, x_2, x_3)$ are schematized on Figure 28.
and thus we get:

\[
\begin{aligned}
&\mathcal{F}^2(\gamma_2, d_0 - (\gamma_1)) + \mathcal{F}^2(\gamma_2, d_+ - (\gamma_1)) + \mathcal{F}^2(d_0, \gamma_1) + \mathcal{F}^2(\gamma_2, \gamma_1) + d_{+0} \circ m^0_+ (\gamma_2, \gamma_1) \\
&+ d_+ \circ m^-_+ (\gamma_2, \gamma_1) + d_{++} \circ \mathcal{F}^2(\gamma_2, \gamma_1) + \mu^2_{\gamma_2, \gamma_1} (d_+ - (\gamma_2), d_+ - (\gamma_1)) = 0
\end{aligned}
\]

(32)

Combining Relations (30), (31) and its symmetric one, and (32), we deduce that \( \mathcal{F}^2 \) satisfies:

![Figure 27. Pseudo-holomorphic buildings in \( \partial \overline{\mathcal{M}}^0(\gamma^+_{3,1}; \delta_1, \gamma_1, \delta_2, x_2, \delta_3) \).](image)

\[
\begin{aligned}
&\mathcal{F}^2(-, \partial_{-\infty}) + \mathcal{F}^2(\partial_{-\infty}, -) + \mathcal{F}^1 \circ m_2 + d_{++} \circ \mathcal{F}^2 + \mu^2_{\gamma_2, \gamma_1} (\mathcal{F}^1, \mathcal{F}^1) = 0
\end{aligned}
\]

The map induced by \( \mathcal{F}^1 \) in homology satisfies then \( \mathcal{F}^1 \circ m_2 + \mu^2_{\gamma_2, \gamma_1} (\mathcal{F}^1, \mathcal{F}^1) = 0 \), and so \( \mathcal{F}^1 \) preserves products in homology.

**Remark 11.** In the case where \( \Lambda^- = \emptyset \) and \( \Sigma_2, \Sigma_3 \) are small Hamiltonian perturbations of \( \Sigma := \Sigma_1 \) such that the pairs \((\Sigma_1, \Sigma_2), (\Sigma_2, \Sigma_3)\) and \((\Sigma_1, \Sigma_3)\) are directed, then \( HF_{-\infty}(\Sigma_i, \Sigma_j) \cong HF(\Sigma_i, \Sigma_j) \) for \( 1 \leq i < j \leq 3 \) (see Proposition 2), and so Theorem 2 tells us that the Ekholm-Seidel isomorphism is a ring isomorphism.
6. An $A_\infty$-structure

The goal of this section is to show that the product structure can be expanded to an $A_\infty$-structure on families of transverse cobordisms. So we want to find operations $m_d$, for $d \geq 1$, satisfying the $A_\infty$-relation. For every pair of transverse exact Lagrangian cobordisms $(\Sigma_1, \Sigma_2)$, the order-1 map is the differential $\partial_{-\infty}$ on $CF_{-\infty}(\Sigma_1, \Sigma_2)$, that we denote now $m_1$. The order-2 map is the product $m_2$ we constructed in Section 5.1 defined for every triple of transverse cobordisms. More generally, for every $(d+1)$-tuple of transverse cobordisms $\Sigma_1, \ldots, \Sigma_{d+1}$, we will construct a map:

$$m_d: CF_{-\infty}(\Sigma_d, \Sigma_{d+1}) \otimes \cdots \otimes CF_{-\infty}(\Sigma_1, \Sigma_2) \to CF_{-\infty}(\Sigma_1, \Sigma_{d+1})$$

such that the family of maps $\{m_d\}_{d \geq 1}$ satisfy for all $d \geq 1$:

$$\sum_{1 \leq j \leq d} m_{d-j+1}(id^\otimes d-j \otimes m_j \otimes id^\otimes n) = 0$$

For each $d \geq 1$, we have $m_d = m_d^0 + m_d^-$ where $m_d^0$ takes values in $CF(\Sigma_1, \Sigma_{d+1})$ and $m_d^-$ takes values in $C^*(\Lambda^-_1, \Lambda^-_{d+1})$. We will define separately those two components.

6.1. Definition of the operations. Let $\Sigma_1, \ldots, \Sigma_{d+1}$, for $d \geq 2$, be transverse Lagrangian cobordisms from $\Lambda^-_i$ to $\Lambda^+_i$ for $i = 1, \ldots, d+1$ such that the algebras $A(\Lambda^-_i)$ admit augmentations $\varepsilon_i$. The map $m_d^0$ is naturally the generalization of $m_d^0$ and is thus defined by a count of rigid pseudo-holomorphic disks with boundary on non-cylindrical parts of the cobordisms, and $d+1$ mixed asymptotics. Indeed, we define:

$$m_d^0: CF_{-\infty}(\Sigma_d, \Sigma_{d+1}) \otimes \cdots \otimes CF_{-\infty}(\Sigma_1, \Sigma_2) \to CF^*(\Sigma_1, \Sigma_{d+1})$$

by

$$m_d^0(a_d, \ldots, a_1) = \sum_{x^+ \in \Sigma_1 \cap \Sigma_{d+1}, \delta_1, \ldots, \delta_{d+1}} \# \mathcal{M}(x^+; \delta_1, a_1, \delta_2, \ldots, a_d, \delta_{d+1}) \cdot \varepsilon^- \cdot x^+$$

where $\delta_i$ are words of Reeb chords of $\Lambda^-_i$ for $1 \leq i \leq d+1$, and the term $\varepsilon^-\cdot x^+$ means that we augment all the pure Reeb chords with the corresponding augmentations, i.e. $\varepsilon^-$ should be replaced by $\varepsilon_1(\delta_1)\varepsilon_2(\delta_2) \cdots \varepsilon_{d+1}(\delta_{d+1})$ in the formula. Also, the choice of Lagrangian label for the moduli spaces involved in the definition of $m_d^0$ is $(\Sigma_1, \ldots, \Sigma_{d+1})$. Now let us define $m_d^-$. As in the case $d=2$, this map is defined by a count of unfinished pseudo-holomorphic buildings, except when all the asymptotics are Reeb chords, and so we define it as a composition of maps. First, consider the map

$$f^{(d)}: CF_{-\infty}(\Sigma_d, \Sigma_{d+1}) \otimes \cdots \otimes CF_{-\infty}(\Sigma_1, \Sigma_2) \to C_{n-1-\cdot}(\Lambda^-_{d+1}, \Lambda^-_1)$$

defined as follows. Let $(a_d, \ldots, a_1)$ be a $d$-tuple of asymptotics with $a_i \in CF_{-\infty}(\Sigma_i, \Sigma_{i+1})$ for all $i = 1, \ldots, d+1$. Assume that at least one $a_i$ is an intersection point, that is to say in $CF(\Sigma_i, \Sigma_{i+1})$, and denote $j$ the largest index such that $a_j$ is an intersection point. We set:

$$f^{(d)}(a_d, \ldots, a_1) = \sum_{\gamma_1, \ldots, \gamma_{d+1} \in \Sigma_1 \cap \Sigma_{d+1}} \# \mathcal{M}(a_d; \delta_{d+1}, a_{j+1}, \ldots, a_d, \delta_{d+1}, \gamma_1, d+1, \delta_1, \ldots, a_j, \delta_j) \cdot \varepsilon^- \cdot \gamma_1, d+1$$

In the case where all the asymptotics are Reeb chords $(\gamma_d, \ldots, \gamma_1)$, with $\gamma_i \in C^*(\Lambda^-_d, \Lambda^-_{d+1})$, we set $f^{(d)}(\gamma_d, \ldots, \gamma_1) = 0$. These maps $f^{(d)}$ are generalizations of the maps $f^{(1)}$ and $f^{(2)}$ defined
in Section 5.1. However, recall that contrary to $f^{(d)}$ for $d \geq 2$, when $\gamma$ is a Reeb chord, we have $f^{(1)}(\gamma) = \gamma$ (and not $f^{(1)}(\gamma) = 0$). Now we generalize the bananas $b^{(1)} := b$ and $b^{(2)}$ with higher order maps. For $j > i$, recall that we denote $\mathcal{C}^*(\Lambda^i_-, \Lambda^j_-) = C_{n-1-\ast} (\Lambda^j_-, \Lambda^i_-) \oplus C^*(\Lambda^i_-, \Lambda^j_-)$. We define for all $d \geq 3$:

$$b^{(d)}(\gamma_d, \ldots, \gamma_1) = \sum_{\delta_1, \ldots, \delta_d, 1} \#M^1(\gamma_d, \gamma_1, \ldots, \gamma_{d+1}) \cdot \mathcal{C}^*(\Lambda^i_-, \Lambda^j_-)$$

where the choice of Lagrangian label is $(\mathbb{R} \times \Lambda^i_-, \ldots, \mathbb{R} \times \Lambda^d_-)$, and the $\delta_i$ are still words of Reeb chords of $\Lambda^i_-$. We generalize the maps $\Delta^{(1)} := \delta_-$ and $\Delta^{(2)}$ (defined in Section 5.2) by:

$$\Delta^{(d)} : \mathcal{C}^*(\Lambda^i_-, \Lambda^j_-) \oplus \cdots \oplus \mathcal{C}^*(\Lambda^i_-, \Lambda^j_-) \rightarrow C_{n-1-\ast}(\Lambda^d_-, \Lambda^i_-)$$

defined as follows. Let $(\gamma_d, \ldots, \gamma_1)$ be a $d$-tuple of Reeb chords. If there is a $1 \leq i \leq d$ such that $\gamma_i \in C_{n-1-\ast} (\Lambda^i_+, \Lambda^j_+)$, then let $j$ be the largest index such that $\gamma_j \in C_{n-1-\ast} (\Lambda^j_+, \Lambda^i_+)$, then we set:

$$\Delta^{(d)}(\gamma_d, \ldots, \gamma_1) = \sum_{\delta_1, \ldots, \delta_d, 1} \#M^1(\gamma_d, \gamma_1, \ldots, \gamma_{d+1}) \cdot \mathcal{C}^*(\Lambda^i_-, \Lambda^j_-)$$

where the Lagrangian label is $(\mathbb{R} \times \Lambda^i_+, \ldots, \mathbb{R} \times \Lambda^j_+, \ldots, \mathbb{R} \times \Lambda^d_+)$. Now, if $(\gamma_d, \ldots, \gamma_1)$ is a $d$-tuple of Reeb chords $\gamma_i \in C^*(\Lambda^i_+, \Lambda^i_+)$, we set $\Delta^{(d)}(\gamma_d, \ldots, \gamma_1) = 0$. Remark that chords from $\Lambda^i_-$ to $\Lambda^j_+$ that are asymptotics of curves in moduli spaces involved in the definition of $\Delta^{(d)}$ are positive asymptotics, while chords from $\Lambda^i_-$ to $\Lambda^j_-$ are negative asymptotics. These maps $\Delta^{(d)}$ are not directly involved in the definition of $m^{-\ast}$ but they will be useful in order to express algebraically the unfinished pseudo-holomorphic buildings appearing in the study of breakings of pseudo-holomorphic curves. We schematized on Figure 29 examples of curves contributing to $f^{(d)}$ and $\Delta^{(d)}$. Now we can finally define the map:

![Figure 29](image-url)
Figure 30. Left: a curve contributing to $f^{(3)}(x_{3,4}, x_{2,3}, \gamma_{2,1})$; right: a curve contributing to $\Delta^{(4)}(\gamma_{4,5}, \gamma_{3,4}, \gamma_{2,3}, \gamma_{2,1})$.

\begin{align*}
m_d^- : CF_{-\infty}^*(\Sigma_d, \Sigma_{d+1}) & \otimes \cdots \otimes CF_{-\infty}^*(\Sigma_1, \Sigma_2) \to C^*(\Lambda_0, \Lambda_{d+1}) \\
\text{by setting} & \\
m_d^- (a_d, \ldots, a_1) &= \sum_{1 \leq j \leq d} b(j) \left( f^{(i_1)}(a_d, ..., a_{d-i_j+1}), \ldots, f^{(i_1)}(a_1, ..., a_1) \right)
\end{align*}

for a $d$-tuple of generators $(a_d, \ldots, a_1)$, and recall the following conventions on the $f^{(d)}$'s in the formula:

\begin{align*}
f^{(1)}(a_i) &= a_i \quad \text{if} \quad a_i = \gamma_{i+1,1} \\
f^{(s)}(\gamma_{i+1,1}, \gamma_{i-1,1}, \ldots, \gamma_{i-s+2,1-i-s+1}) &= 0 \quad \text{for} \quad 1 < s \leq i \leq d
\end{align*}

Remark that, as required, the formulas (33) and (34) in the case $d = 2$ correspond to the product $m_2$.

6.2. Proof of Theorem 3 In order to show the $A_\infty$-relation, again we study breakings of pseudo-holomorphic curves. The $A_\infty$-relation for the maps $\{m_d\}_d$ can be rewritten:

\begin{align*}
&\sum_{1 \leq j \leq d} m_{d-j+1}^0 \left( \text{id}^{\otimes d-j-n} \otimes m_j \otimes \text{id}^{\otimes n} \right) \\
&\quad + \sum_{1 \leq j \leq d} m_{d-j+1}^- \left( \text{id}^{\otimes d-j-n} \otimes m_j \otimes \text{id}^{\otimes n} \right) = 0
\end{align*}

First we start by showing that

\begin{align*}
&\sum_{1 \leq j \leq d} m_{d-j+1}^0 \left( \text{id}^{\otimes d-j-n} \otimes m_j \otimes \text{id}^{\otimes n} \right) = 0
\end{align*}
and then we will prove that
\begin{equation}
\sum_{1 \leq j \leq d} m_{d-j+1}(\text{id}^{\otimes d-j-n} \otimes m_j \otimes \text{id}^{\otimes n}) = 0
\end{equation}

6.2.1. Proof of Relation (37). To show this relation we need to understand the different types of pseudo-holomorphic buildings contributing to the maps in the sum. For a $d$-tuple $(a_d, \ldots, a_1)$ of asymptotics, each term of (37) is either of the form
\begin{equation}
m_{d-j+1}^0(a_d, \ldots, m_j(a_{n+j}, \ldots, a_{n+1}), a_n, \ldots, a_1)
\end{equation}
or
\begin{equation}
m_{d-j+1}^-(a_d, \ldots, m_j(a_{n+j}, \ldots, a_{n+1}), a_n, \ldots, a_1)
\end{equation}
The pseudo-holomorphic buildings contributing to (39) are of height $0|1|0$. The central level of each building contains two curves which have a common asymptotic on an intersection point, and can be glued on a pseudo-holomorphic disk in the moduli space
\begin{equation}
\mathcal{M}^1(x^+; \delta_1, a_1, \delta_2, a_2, \ldots, \delta_d, a_d, \delta_{d+1})
\end{equation}
The same happens for pseudo-holomorphic buildings contributing to the terms in (40). Such a building is of height $1|1|0$ and has components that can be glued on an index-1 curve in the moduli space (41). Indeed, $m^-$ counts unfinished buildings (as soon as one asymptotic at least is an intersection point, otherwise it counts just one banana) of height $1|1|0$ such that the curve with boundary on the negative cylindrical ends is a banana which has for output a positive chord $\gamma_{n+j+1,n+1} \in \mathcal{R}(A_{n+j+1}, A_{n+1})$. The map $m^0$ applied to the remaining asymptotics and $\gamma_{n+j+1,n+1}$ is then given by the count of index-0 pseudo-holomorphic curves in the central level. The unfinished buildings contributing to $m^-$ and the curves contributing to $m^0$ together give a pseudo-holomorphic building, and the corresponding glued curve is an index-1 pseudo-holomorphic curve in (41).

Now, in order to establish Relation (37), we must study the boundary of the compactification of this moduli space. So we consider a 1-parameter family of index-1 curves $u: S_t \to \mathbb{R} \times Y$ in (41), for $r \in \mathcal{R}^{d+1}$, and let us explain the different possible types of degeneration for such a family. Recall first that we can associate to $S_r$ a planar stable rooted tree $T_{d+1}$ with $d$ leaves and one vertex. Let us denote $\rho$ the root.

Given a subset $(y_{n+1}, \ldots, y_{n+j})$ consisting of $j$ successive punctures of $S_r$, stretching the neck between those punctures and the others means

- either deforming $S_r$ to approach a boundary strata of the compactification of the Stasheff associahedra $\mathcal{R}^{d+1}$ for which elements are in bijection with stable rooted trees with $d$ leaves and one interior edge $e$ (stable breaking),
- or, deforming $S_r$ until the corresponding tree is not stable anymore (unstable breaking).

In the first case, the new interior edge $e$ splits the tree $T_{d+1}$ into two subtrees: one with leaves corresponding to the punctures $(y_{n+1}, \ldots, y_{n+j})$ and a root which is an end of $e$; the other subtree has leaves corresponding to the punctures $y_1, \ldots, y_n, y_{n+j+1}, \ldots, y_d$ and the other end of $e$, and its root is the initial root $\rho$ of $T_{d+1}$.

Now, consider a curve $u: S_t \to \mathbb{R} \times Y$ in (41) with mixed asymptotics $a_i \in CF_{-\infty}(\Sigma_i, \Sigma_{i+1})$ for $1 \leq i \leq d$, and $x^+ \in CF(\Sigma_1, \Sigma_{d+1})$. By an abuse of language, given a subset $(a_{n+j}, \ldots, a_{n+1})$ consisting of $j$ successive asymptotics of $(a_d, \ldots, a_1)$, “stretching the neck” between those asymptotics and the others means that we consider a family of pseudo-holomorphic curves obtained by realizing the operation described above on the domain of $u$.

By this stretching the neck procedure, the pseudo-holomorphic curve $u$ can break on (i.e. the asymptotic corresponding to the ends of the new interior edge $e$):
Then, the term we consider is given by a count of unfinished buildings of height 6.2.2.

Proof of Relation
vanishes for energy reasons. So let us assume that at least one appearing in (42). If \( (\alpha_s) \) is a strictly increasing finite sequence of length \( r \leq j \), with \( \alpha_1 = n + 1 \) and \( \alpha_{r+1} = n + j + 1 \). In this case, we get a pseudo-holomorphic building with two levels, such that the central level contains \( r + 1 \) rigid curves. One of them is asymptotic to \( x^+, a_1, \ldots, a_n, \gamma_{n+j+1,n+1}, a_{n+j+1}, \ldots, a_d \) in this cyclic order and contributes to the map \( m^0 \), and each of the others has one negative puncture asymptotic to a chord \( \gamma_{\alpha_s, \alpha_s+1} \) and \( \alpha_{s+1} - \alpha_s \) other asymptotics. Such a curve contributes to the map \( \gamma \left( \alpha_{s+1} - \alpha_s \right) \).

By (1) and (2) above we describe every type of degeneration for a curve in the moduli space \( M^1(x^+; \delta_1, a_1, \delta_2, a_2, \ldots, \delta_d, a_d, \delta_{d+1}) \). These degenerations are pseudo-holomorphic buildings which are in bijection with the elements in the boundary of the compactification of the moduli space. This compactification being a 1-dimensional manifold with boundary, its boundary components arise in pair, which gives 0 modulo 2. This implies Relation \( 38 \).

6.2.2. Proof of Relation \( 38 \). In this section, in order to be less confusing, we do not write the pure chords asymptotics in the moduli spaces anymore. As before, the left-hand side of Relation \( 38 \), with inputs a \( d \)-tuple of asymptotics \( (a_d, \ldots, a_1) \), splits into two sums:

\[
(42) \quad \sum_{1 \leq j \leq d} \sum_{0 \leq n \leq d-j} m_{d-j+1}^-(a_d, \ldots, a_{n+j+1}, m_j^0(a_{n+j}, \ldots, a_{n+1}), a_n, \ldots, a_1)
\]

and

\[
(43) \quad \sum_{1 \leq j \leq d} \sum_{0 \leq n \leq d-j} m_{d-j+1}^+(a_d, \ldots, a_{n+j+1}, m_j^-(a_{n+j}, \ldots, a_{n+1}), a_n, \ldots, a_1)
\]

First, let us look for example the term

\[
b^{(1)} \circ f^{(d-j+1)}(a_d, \ldots, m_j^0(a_{n+j}, \ldots, a_{n+1}), a_n, \ldots, a_1)
\]

appearing in (42). If \( (a_d, \ldots, a_1) \) is a \( d \)-tuple of Reeb chords, then as we already saw this term vanishes for energy reasons. So let us assume that at least one \( a_k \) is an intersection point. Then, the term we consider is given by a count of unfinished buildings of height 1|1|0 with two components in the central level having a common asymptotic at an intersection point, and a banana in the lower level. Gluing the central level components gives an unfinished building in (see Figure 31):

\[
M^1(a_k; a_{k+1}, \ldots, a_d, \gamma_1, a_{d+1}, \ldots, a_{k-1}) \times \overline{M^1(\gamma_{d+1}; \gamma_1, a_{d+1})}
\]
Figure 31. On the top: example of unfinished building contributing to $b^{(1)} \circ f^{(4)}(x_4, x_3, x_2, \gamma_2)$ and the corresponding glued curve. On the bottom: impossible breaking for energy reasons.

Let us take another term of (42), for example:

$$b^{(2)}(f^{(d-j+1)}(a_d, \ldots, a_{n+j+1}), f^{(n+1)}(m_j(a_{n+j}, \ldots, a_{n+1}), a_n, \ldots, a_1))$$

This one counts again unfinished buildings of height $1|1|0$ with the following conditions:

1. Assume $a_d, \ldots, a_{n+j+1}$ are Reeb chords: if $n+j+1 < d$, then $f^{(d-j+1)}(a_d, \ldots, a_{n+j+1}) = 0$, and if $n+j+1 = d$, then $f^{(1)}(a_d) = a_d$. In this latter case, the term above counts unfinished buildings with two components in the central level having a common asymptote to an intersection point, and one banana in the bottom level having two positive Reeb chords asymptotics $\gamma_{1,d}$ (as an input, which is the output of $f^{(n+1)}$) and $\gamma_{d+1,1}$ (output) and $a_d$ as a negative Reeb chord asymptotic.

2. If $a_d, \ldots, a_{n+j+1}$ contains at least one intersection point, and we assume it is $a_{n+j+1}$ (just to simplify the writing of moduli spaces below), we get then an unfinished building with three components in the central level: two of them have a common asymptotic to an intersection point and the other is disjoint from them and has asymptotes $a_d, \ldots, a_{n+j+1}$ and a Reeb chord $\gamma_{n+j+1,d+1}$ (output of $f^{(d-j+1)}$). The bottom level contains a banana with positive Reeb chords which are the output of $f^{(n+1)}$, the chord $\gamma_{n+j+1,d+1}$ and a chord $\gamma_{d+1,1}$ as output.

In the two cases above, we assume that at least one asymptotic among $a_{n+j}, \ldots, a_{n+1}, a_n, \ldots, a_1$ is an intersection point, otherwise the term vanishes for energy reasons. Again, in order to simplify the moduli spaces, assume that $a_1$ is an intersection point. Then, in each case the two components in the central level having a common asymptotic can be glued and thus after gluing we get an unfinished building in:

$$M^1(a_1; a_2, \ldots, a_{d-1}, \gamma_{1,d}) \times \overline{M^1}(\gamma_{d+1,1}; \gamma_{1,d}, a_d)$$
for the case (1) above, and in
\[
\mathcal{M}^1(a_1; a_2, \ldots, a_{n+j}, \gamma_{1,n+j+1}) \times \mathcal{M}^0(a_{n+j+1}; a_{n+j+2}, \ldots, a_d, \gamma_{n+j+1,d+1})
\times \mathcal{M}^1(\gamma_{d+1,1}; \gamma_{1,n+j+1}, \gamma_{n+j+1,d+1})
\]
for the case (2) above. More generally, each term of the sum takes the form
\[
b^{(j)}(f^{(i_1)}) \otimes \cdots \otimes f^{(i_s)}(\text{id} \otimes \eta \otimes \text{id} \otimes \eta \otimes \cdots \otimes \text{id} \otimes \eta)
\]
with \(p + q + r = i_s\). Hence, analogously to the two special terms described above, the unfinished buildings contributing to \(b^{(1)}\) are composed by several rigid curves in the central level so that two of them have a common asymptotic to an intersection point, and one index-1 banana in the bottom level, having for positive input asymptotics the output chords of the maps \(f^{(1)}\), and potentially some negative chords among \(a_1, \ldots, a_d\), and finally a positive Reeb chord asymptotic \(\gamma_{d+1,1}\) as output. These buildings are in the boundary of the compactification of products of moduli spaces of type \(\text{A}\), with possibly more or no (as for \(\text{A}\)) rigid components in the central level. We thus have to study the boundary of the compactification of moduli spaces of index 1 curves with boundary on \(\Sigma_1, \ldots, \Sigma_{d+1}\) and punctures asymptotic to Reeb chords and intersection points (at least one which plays the role of positive puncture) in Floer complexes \(CF_{-\infty}(\Sigma_i, \Sigma_j)\), and one negative puncture asymptotic to a Reeb chord \(\gamma_{1,d+1}\). We will say that these moduli spaces are of type \(A\).

Now, let us describe the different kinds of buildings that contribute to the terms of the sum \(b^{(1)}\). One of the terms of the sum is for example:
\[
b^{(1)} \circ f^{(d-1)}(a_d, \ldots, a_3, b^{(1)} \circ f^{(2)}(a_2, a_1))
\]
which vanishes if \(a_2, a_1\) are both Reeb chords or if \(a_d, \ldots, a_3\) are all Reeb chords, so we assume that at least \(a_1\) and \(a_3\) are intersection points. Such a composition of maps is given by a count of unfinished buildings of height 210 such that the two upper levels form a pseudo-holomorphic building whose components can be glued. After gluing, we get an unfinished building in
\[
\mathcal{M}^1(a_1; a_2, \ldots, a_d, \gamma_{1,d+1}) \times \tilde{\mathcal{M}}^1(\gamma_{d+1,1}; \gamma_{1,d+1})
\]
(see Figure 32) where the first moduli space is of type \(A\) and the second one is a banana. Another

![Figure 32. Unfinished pseudo-holomorphic building contributing to \(b^{(1)} \circ f^{(3)}(x_{4,5}, x_{3,4}, b^{(1)} \circ f^{(2)}(\gamma_{3,2}, x_{1,2}))\) and the corresponding glued curve.](image-url)
term appearing in (43) is for example:

\[ b^{(2)}(f^{(d-2)}(a_d, \ldots, a_3), f^{(1)}(a_2, a_1)) = b^{(2)}(f^{(d-2)}(a_d, \ldots, a_3), b^{(1)}(a_2, a_1)) \]

where the equality comes from the convention (35). The unfinished pseudo-holomorphic buildings contributing to this term are of height \( \geq 1 \). The unfinished pseudo-holomorphic buildings appearing in (43) is for example:

\[ \mathcal{M}^0(a_1; a_2, \gamma_{1,3}) \times \mathcal{M}^0(a_3; a_4, \ldots, a_d, \gamma_{3,d+1}) \times \overline{\mathcal{M}}^2(\gamma_{d+1,1}; \gamma_{1,3}, \gamma_{3,d+1}) \]

The two first moduli spaces are rigid and contribute respectively to the maps \( f^{(2)} \) and \( f^{(d-2)} \), whereas the third moduli space is a moduli space of non rigid bananas.

More generally, each term of (43) is given by a count of unfinished buildings of height \( \geq 1 \) having at least one intersection point (assume this is \( a_k \)). A curve in \( \mathcal{M}^1(a_k; a_{k+1}, \ldots, a_d, \gamma_{1,d+1}, a_1, \ldots, a_{k-1}) \) admits different types of degeneration. For a subset of successive asymptotics \((a_{n+j}, \ldots, a_{n+1})\), the stretching the neck procedure between these asymptotics and the others leads the curve to break on:

1. an intersection point, in this case the broken curve is a pseudo-holomorphic building of height \( 0 \) having at least one intersection point. For index reasons, the building has no other component. This kind of building contributes to

\[ f^{(d-j+1)}(a_d, \ldots, m_{j}\{a_{n+j}, \ldots, a_{n+1}\}, a_n, \ldots, a_1) \]

2. one or several Reeb chords and the pseudo-holomorphic building we get is of height \( \geq 1 \), with possible several index 0 curve in the central level, and one index 1 curve in the bottom level. We have to distinguish two sub-cases. First, assume that the chord \( \gamma_{1,d+1} \) (output of \( f^{(d)} \)) is an asymptotic of a curve in the central level. Then it means that this curve has in particular a negative Reeb chord asymptotic \( \gamma_{n+j+1,n+1} \) which is the positive output of a banana in the bottom level. This banana has possibly negative Reeb chord asymptotics among \((a_{n+j}, \ldots, a_{n+1})\), and possibly other positive Reeb chord asymptotics, which are negative asymptotics for the rigid curves in the central level, having other asymptotics among \((a_{n+j}, \ldots, a_{n+1})\). In other words, we have to consider here all the rigid unfinished buildings which permit to associate to \((a_{n+j}, \ldots, a_{n+1})\) a chord \( \gamma_{n+j+1,n+1} \) and a rigid curve in the central level having in particular the asymptotics \( \gamma_{1,d+1} \) (output) and \( \gamma_{n+j+1,n+1} \). Such pseudo-holomorphic buildings contribute thus to:

\[ f^{(d-j+1)}(a_d, \ldots, m_{j}\{a_{n+j}, \ldots, a_{n+1}\}, a_n, \ldots, a_1) \]

Now assume that in the building, the chord \( \gamma_{1,d+1} \) is an asymptotic of the curve in the bottom level. Then this index-1 curve is a curve that contributes to the map \( \Delta \). Every curve in the central level has in particular a negative Reeb chord asymptotic which is
not in the Floer complexes (chord \( \gamma_{i,k} \) with \( i < k \)), and such a chord is the output of a map \( f \). So, these pseudo-holomorphic buildings contribute to:

\[
\Delta^{(s)}(f^{(i_s)} \otimes \cdots \otimes f^{(i_1)}) \quad \text{with } i_s + \cdots + i_1 = d
\]

and with conventions (35) and (36).

Finally, all these possibilities of breaking give the relation:

\[
\begin{align*}
\sum_{1 \leq j \leq d} \sum_{0 \leq n \leq d-j} f^{(d-j+1)} \left( \text{id}^{\otimes d-j-n} \otimes m_j^n \otimes \text{id}^{\otimes n} \right) + \sum_{1 \leq j \leq d-1} \sum_{0 \leq n \leq d-j} f^{(d-j+1)} \left( \text{id}^{\otimes d-j-n} \otimes m_j^n \otimes \text{id}^{\otimes n} \right) \\
\sum_{1 \leq s \leq d} \Delta^{(s)}(f^{(i_s)} \otimes \cdots \otimes f^{(i_1)}) = 0
\end{align*}
\]

with conventions (35) and (36).

**Index 2 bananas**: \( \partial M^2(\gamma_{d+1,1}; \gamma_1, \ldots, \gamma_d) \). We look now the possible degeneration of index-2 bananas. Consider such a banana with Reeb chord asymptotics \( \gamma_{d+1,1}, \gamma_1, \ldots, \gamma_d \) with \( \gamma_i \in \mathcal{C}^s(\Lambda_i, \Lambda_{i+1}) \). It is a pseudo-holomorphic disk with boundary on the negative cylindrical ends of the cobordisms, so it can break into a pseudo-holomorphic building with boundary on the negative cylindrical ends too, in particular, each component of the building has index at least 1. So, an index 2 banana can only break into a building with two components, which have a common asymptotic to a Reeb chord (which is a positive asymptotic for one component and negative asymptotic for the other). Let us choose a subset \( \gamma_{n+j, \ldots, \gamma_{n+1}} \) of successive asymptotics of the banana. Stretching the neck between these asymptotics and the others gives that the curve can break on:

1. a Reeb chord \( \gamma_{n+1,n+j+1} \) which is not in the Floer complex \( CF_{-\infty}(\Sigma_{n+1}, \Sigma_{n+j+1}) \). We get thus a building with two components. One of them has Reeb chords asymptotics \( \gamma_{n+1,n+j+1}, \gamma_{n+1}, \ldots, \gamma_{n+j} \) with \( \gamma_{n+1,n+j+1} \) negative, and the other disk has Reeb chord asymptotics \( \gamma_{d+1,1}, \gamma_1, \ldots, \gamma_n, \gamma_{n+1,n+j+1}, \gamma_{n+j+1}, \ldots, \gamma_d \) with \( \gamma_{n+1,n+j+1} \) positive. The first one contributes to the coefficient:

\[
\langle \Delta^{(j)}(\gamma_{n+j, \ldots, \gamma_{n+1}}, \gamma_{n+1,n+j+1}) \rangle
\]

which vanishes if \( \langle \gamma_{n+j, \ldots, \gamma_{n+1}} \rangle \) is a \( j \)-tuple of Reeb chords in the Floer complexes by definition of \( \Delta \). The second one contributes to the coefficient:

\[
\langle b^{(d-j+1)}(\gamma_d, \ldots, \gamma_{n+j+1}, \gamma_{n+1,n+j+1}, \gamma_{n+1,n+j+1}, \gamma_{n+j+1}, \ldots, \gamma_1), \gamma_{d+1,1}) \rangle
\]

And so we deduce that those types of pseudo-holomorphic buildings contribute to:

\[
b^{(d-j+1)}(\gamma_d, \ldots, \Delta^{(j)}(\gamma_{n+j, \ldots, \gamma_{n+1}}, \gamma_{n+1,n+j+1}), \gamma_{n+j+1}, \ldots, \gamma_1)
\]

(2) a Reeb chord \( \gamma_{n+j,1,n+1} \in CF_{-\infty}(\Sigma_{n+1}, \Sigma_{n+j+1}) \), and we get again a building with two components which have the same asymptotics as the curves above (with \( \gamma_{n+j,1,n+1} \) instead of \( \gamma_{n+1,n+j+1} \)), but where \( \gamma_{n+j,1,n+1} \) is a positive asymptotic for the first curve and negative for the second one. In this case, the two components of the building are bananas, and so it contributes to

\[
b^{(d-j+1)}(\gamma_d, \ldots, b^{(j)}(\gamma_{n+j, \ldots, \gamma_{n+1}}, \gamma_{n+1,n+j+1}), \gamma_{n+j+1}, \ldots, \gamma_1)
\]

We described above all the types of pseudo-holomorphic buildings in the boundary of the compactification of \( \overline{M}^2(\gamma_{d+1,1}; \gamma_1, \ldots, \gamma_d) \) so we deduce that:

\[
\sum_{1 \leq j \leq d} \sum_{0 \leq n \leq d-j} b^{(d-j+1)}(\text{id}^{\otimes d-j-n} \otimes (b^{(j)} + \Delta^{(j)}) \otimes \text{id}^{\otimes n}) = 0
\]
By combining Relations \((48)\) and \((49)\), we get the following:

\[
\sum_{1 \leq i \leq d} b^{(s)} \left( f^{(i_1)} \otimes \ldots \otimes f^{(i_{k+1})} \otimes \sum_{1 \leq j \leq k} \sum_{0 \leq n \leq l_k-j} f^{(i_k-j+1)} \left( \id^{\otimes i_k-n-j} \otimes m_j^0 \otimes \id^{\otimes n} \right) \otimes \ldots \otimes f^{(i_1)} \right)
\]

\[
+ \sum_{1 \leq s \leq d} \sum_{1 \leq k \leq s} b^{(s)} \left( f^{(i_1)} \otimes \ldots \otimes f^{(i_{s+1})} \otimes \sum_{1 \leq j \leq k} \sum_{0 \leq n \leq l_k-j} f^{(i_k-j+1)} \left( \id^{\otimes i_k-n-j} \otimes m_j^- \otimes \id^{\otimes n} \right) \otimes \ldots \otimes f^{(i_1)} \right)
\]

\[
+ \sum_{1 \leq s \leq d} \sum_{1 \leq j \leq s} b^{(s-j+1)} \left( f^{(i_1)} \otimes \ldots \otimes f^{(i_{s+1})} \otimes \sum_{1 \leq n \leq s} \Delta^{(1)} \left( f^{(i_1)} \otimes \ldots \otimes f^{(i_{s+1})} \right) \otimes \ldots \otimes f^{(i_1)} \right) = 0
\]

where the sum of the first three lines equals zero because of \((48)\), as well as the sum of the two last lines because of \((49)\). The sums in the third and fifth lines are equal so cancel each other (on \(\Z_2\)) and we get:

\[
\sum_{1 \leq s \leq d} \sum_{1 \leq j \leq s} b^{(s-j+1)} \left( f^{(i_1)} \otimes \ldots \otimes f^{(i_{s+1})} \otimes \sum_{1 \leq n \leq s} \Delta^{(1)} \left( f^{(i_1)} \otimes \ldots \otimes f^{(i_{s+1})} \right) \otimes \ldots \otimes f^{(i_1)} \right) = 0
\]

The first sum corresponds to

\[
\sum_{1 \leq j \leq d} \sum_{0 \leq n \leq d-j} m^-_{d-j+1} \left( \id^{\otimes d-j-n} \otimes m_j^0 \otimes \id^{\otimes n} \right)
\]

and the two last sums to

\[
\sum_{1 \leq j \leq d} \sum_{0 \leq n \leq d-j} m^-_{d-j+1} \left( \id^{\otimes d-j-n} \otimes m_j^- \otimes \id^{\otimes n} \right)
\]

because the sum in the third line is equal to the missing terms

\[
b^{(s-j+1)} \left( f^{(i_1)} \otimes \ldots \otimes f^{(i_{s+1})} \otimes m_j^- \otimes \ldots \otimes f^{(i_1)} \right)
\]

in the second line to give the relation. This ends the proof of Theorem 3.
6.3. $A_\infty$-functor. In this subsection, we naturally generalize the maps $F^1$ and $F^2$ in a family of maps $\{F^d\}_{d \geq 1}$ satisfying the $A_\infty$-functor relation which is that for all $d \geq 1$:

$$\sum_{1 \leq j \leq d} F^{d-j+1} \left( \text{id} \otimes d-j-n \otimes m_j \otimes \text{id}^\otimes n \right) + \sum_{1 \leq s \leq d} \mu^s \left( F^{i_s} \otimes \cdots \otimes F^{i_1} \right) = 0 \tag{50}$$

where $\mu^s$ are $A_\infty$-maps of the augmentation category $\text{Aug}_-(\Lambda^+_1 \cup \cdots \cup \Lambda^+_{d+1})$, and we did not write the augmentations in index for the formula to stay readable. The maps $F^d$, for $d \geq 3$, are defined analogously to the maps $F^1$ and $F^2$, by a count of rigid pseudo-holomorphic disks, but with more mixed asymptotics (see for example Figure 33). So we define $F^d$ by

$$F^d \colon CF_{-\infty}(\Sigma_d, \Sigma_{d+1}) \otimes \cdots \otimes CF_{-\infty}(\Sigma_1, \Sigma_2) \to C^*(\Lambda^+_1, \Lambda^+_{d+1})$$

by

$$F^d(a_d, \ldots, a_1) = \sum_{\gamma^+, \delta_1, \ldots, \delta_{d+1} \in \text{R}(\Lambda^+_d, \Lambda^+_1)} \# M^1_{\gamma^+, \delta_1, \ldots, \delta_{d+1}}(\gamma^+, \delta_1, \ldots, \delta_d, a_d, \delta_{d+1}) \cdot \epsilon^- \cdot \gamma^+ \tag{51}$$

where as always the $\delta_i$'s are words of Reeb chords of $\Lambda^+_i$, and again the term $\epsilon^-$ should be replaced by $\prod \epsilon^- \cdot (\delta_i)$. In order to show the $A_\infty$-functor relation, we study degeneration of curves in the moduli space $M^1(\gamma^+, \delta_1, \ldots, \delta_d, a_d, \delta_{d+1})$. An index-1 curve in such a moduli space can break into:

1. A pseudo-holomorphic building of height 0|1|0 such that the central level contains two index 0 curves which have a common asymptotic at an intersection point. These buildings contribute thus to

$$\langle F^{d-j+1}(\text{id} \otimes d-j-n \otimes m_j \otimes \text{id}^\otimes n), \gamma^+ \rangle$$

2. A pseudo-holomorphic building of height 1|1|0 with possibly several index 0 curves in the central level and one index 1 banana in the bottom level (for index reasons, this level can not contain any other non trivial curve). These buildings contribute to

$$\langle F^{d-j+1}(\text{id} \otimes d-j-n \otimes m_j \otimes \text{id}^\otimes n), \gamma^+ \rangle$$

3. A pseudo-holomorphic building of height 0|1|1 with again possibly several index-0 disks in the central level and one index-1 curve in the top level. These buildings contribute to

$$\langle \mu^s(F^{i_s} \otimes \cdots \otimes F^{i_1}), \gamma^+ \rangle$$

As boundary of a 1-dimensional manifold, the sum of all these contributions gives 0 modulo 2, and this implies the relation (50).
6.4. Towards an $A_\infty$-category. In this last section, let us consider the case of non transverse Lagrangian cobordisms, in order to be able to define an $A_\infty$-category of Lagrangian cobordisms $\mathcal{F}_{\text{uk}}(\Lambda)$ associated to a Legendrian submanifold $\Lambda \subset Y$. Objects of this category are triples $(\Sigma, \Lambda^\perp, \varepsilon^\perp)$ where $\Lambda^\perp \subset Y$ is Legendrian, $\varepsilon^\perp$ is an augmentation of $\mathcal{A}(\Lambda^\perp)$ and $\Sigma$ is an exact Lagrangian cobordism from $\Lambda^\perp$ to $\Lambda$. Then, given two such cobordisms $(\Sigma_1, \Lambda_1^\perp, \varepsilon_1^\perp)$ and $(\Sigma_2, \Lambda_2^\perp, \varepsilon_2^\perp)$, we want to set

$$\text{hom}_{\mathcal{F}_{\text{uk}}(\Lambda)}((\Sigma_1, \Lambda_1^\perp, \varepsilon_1^\perp), (\Sigma_2, \Lambda_2^\perp, \varepsilon_2^\perp)) = (C F_{-\infty}(\Sigma_1, \Sigma_2), m_1)$$

but obviously $\Sigma_1$ and $\Sigma_2$ are non transverse because they have the same positive Legendrian end $\Lambda$. We must perturb one to get a transverse pair in order to be able to compute the differential $\partial_{-\infty} = m_1 : C F_{-\infty}(\Sigma_1, \Sigma_2) \to C F_{-\infty}(\Sigma_1, \Sigma_2)$, as well as the map $F^1 : C F_{-\infty}(\Sigma_1, \Sigma_2) \to C(\Lambda)$. More generally, given $d + 1$ objects $(\Sigma_i, \Lambda_i^\perp, \varepsilon_i^\perp)$, for $1 \leq i \leq d + 1$, of $\mathcal{F}_{\text{uk}}(\Lambda)$, we have to perturb them to obtain a $(d + 1)$-tuple of transverse objects and compute the operations $m_d$ and $F^d$. We do the following choice of perturbations.

Let $T > 0$ such that $\Sigma_i \setminus([-T, T] \times Y \cap \Sigma_i)$ are all cylindrical. For $1 \leq i \leq d + 1$, and $\varepsilon > 0$ small, we denote $\Sigma_i^\varepsilon := \Phi_{H^0}^{(i-1)\varepsilon}(\Sigma_i)$ (see Section 3.2). Now, the positive ends of the cobordisms is a cylinder over $d + 1$ parallel copies of $\Lambda$. We choose Morse functions in order to perturb these copies and get the $(d + 1)$-copy of $\Lambda$ as defined in Section 3.2. We still denote $\Sigma_i^\varepsilon$ the cobordisms after perturbation of the Legendrian ends. The family $(\Sigma_i^\varepsilon, \ldots, \Sigma_d^\varepsilon)$ is then a family of transverse objects except maybe in $[-T, T] \times Y$. So now we perturb the compact parts $\Sigma_1^\varepsilon, \ldots, \Sigma_d^\varepsilon$. In order to do so, we equip them with Floer data and perturbation data (see [Sei08]). The maps $m_d$ and $F^d$ on $C F_{-\infty}(\Sigma_1^\varepsilon, \Sigma_d^\varepsilon) \otimes \cdots \otimes C F_{-\infty}(\Sigma_1^\varepsilon, \Sigma_2^\varepsilon)$ are then defined by a count of disks satisfying the equation $du(z) \circ j = J(z) \circ dz$ in the cylindrical ends and the perturbed Floer equation in $[-T, T] \times Y$, i.e. the pseudo-holomorphic curves are asymptotic to Hamiltonian trajectories coming from the Floer data associated to each pair of Lagrangian cobordisms (see [Sei08], Equation (8.9)).

Now, the set of maps $\{F^d\}_{d \geq 0}$ defines an $A_\infty$-functor

$$\mathcal{F} : \mathcal{F}_{\text{uk}}(\Lambda) \rightarrow \mathcal{A}_{\text{ug}}(\Lambda)$$

where $F^0$ is the map on objects given by $F^0(\Sigma, \Lambda^\perp, \varepsilon^\perp) = \varepsilon^\perp \circ \phi_\Sigma$. This functor is cohomologically full and faithful because for each $\Sigma_1, \Sigma_2 \in \text{Ob}(\mathcal{F}_{\text{uk}}(\Lambda))$, the map $F^1 : C F_{-\infty}(\Sigma_1, \Sigma_2) \to C^d(\Lambda_1, \Lambda_2)$ is a quasi-isomorphism. This functor is even a pseudo-equivalence (see [BC14]).

However, this category $\mathcal{F}_{\text{uk}}(\Lambda)$ is by construction dependent on the choice of perturbations we made. But on the other side, by [BC14] Theorem 2.14, two Legendrian isotopic submanifolds have pseudo-equivalent augmentation categories, which means that under pseudo-equivalence, the augmentation category does not depend on the choice of Morse functions made to define the $(d + 1)$-copy. So we have the following diagram, where $\mathcal{P}$ and $\tilde{\mathcal{P}}$ denote two choices of perturbations to compute $\mathcal{F}_{\text{uk}}(\Lambda)$, which contain respectively the choices $\mathcal{P}$ and $\tilde{\mathcal{P}}$ of Morse functions to perturb the $d + 1$ parallel copies of $\Lambda$:

$$\begin{array}{ccc}
\mathcal{F}_{\text{uk}}(\Lambda, \mathcal{P}) & \rightarrow & \mathcal{A}_{\text{ug}}(\Lambda, \mathcal{P}) \\
\downarrow & & \downarrow \\
\mathcal{F}_{\text{uk}}(\Lambda, \tilde{\mathcal{P}}) & \rightarrow & \mathcal{A}_{\text{ug}}(\Lambda, \tilde{\mathcal{P}})
\end{array}$$

The vertical right arrow is a pseudo-equivalence, and the two horizontal arrows are also pseudo-equivalences because they are cohomologically full and faithful functors and every object $\varepsilon$ in $\mathcal{A}_{\text{ug}}(\Lambda)$ is equal to the image object $F^0(\mathbb{R} \times \Lambda, \varepsilon)$ (so in particular these objects are trivially pseudo-isomorphic). By [BC14] Proposition 2.6, pseudo-equivalence is a symmetric relation so there exists a pseudo-equivalence $\mathcal{A}_{\text{ug}}(\Lambda, \tilde{\mathcal{P}}) \rightarrow \mathcal{F}_{\text{uk}}(\Lambda, \tilde{\mathcal{P}})$. Hence, the categories $\mathcal{F}_{\text{uk}}(\Lambda, \mathcal{P})$ and $\mathcal{F}_{\text{uk}}(\Lambda, \mathcal{P})$ are pseudo-equivalent.
In dimension 3, as already evoked, Ng, Rutherford, Shende, Sivek and Zaslow in \cite{NRS} have defined a “$+$” version of the augmentation category, the $A_{\infty}$-category $\operatorname{Aug}_+(\Lambda)$. This category is unital contrary to $\operatorname{Aug}_-(\Lambda)$. The $A_{\infty}$-maps can also be computed by a count of pseudo-holomorphic disks with boundary on the $k$-copy of $\Lambda$, but this time, the Morse perturbations must be really specific, and Morse chords play an essential role. Performing this type of perturbation for triples $(\Sigma_i, \Lambda^-, \varepsilon^-)$ as above, we can define a category $\mathcal{F}uk_+(\Lambda)$, which has the same objects as $\mathcal{F}uk_-(\Lambda)$ but to define the $A_{\infty}$-maps, we proceed as follows. Given $d+1$ objects $(\Sigma_i, \Lambda^-, \varepsilon^-)$, for $1 \leq i \leq d+1$, let $\varepsilon > 0$ small, we denote $\Sigma_i^\varepsilon := \Phi^{(1-i)\varepsilon}_{H_d}(\Sigma_i)$. Now, the positive ends of the cobordisms is a cylinder over $d+1$ parallel copies of $\Lambda$, but this time $\Lambda_1 := \Lambda$ is the top copy while $\Lambda_{d+1} := \varphi^{+\varepsilon}_{T_d}(\Lambda)$ is the bottom copy. We choose Morse functions as in \cite{NRS} to perturb these copies and we still denote $\Sigma_i^\varepsilon$ the cobordisms after perturbation of the Legendrian ends. Then we need to perturb as above the compact parts of the cobordisms to define the maps $m_d$. We get in this case a functor $G : \mathcal{F}uk_+(\Lambda) \to \operatorname{Aug}_+(\Lambda)$ such that $\mathcal{G}$ is a quasi-isomorphism. The category $\operatorname{Aug}_+(\Lambda)$ being unital, $\mathcal{F}uk_+(\Lambda)$ is cohomologically unital and $G$ is a quasi-equivalence.

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