Distributed Saddle Point Problems for Strongly Concave-Convex Functions

Muhammad I. Qureshi and Usman A. Khan

Abstract—In this article, we propose GT-GDA, a distributed optimization method to solve saddle point problems of the form: \( \min_x \max_y \{ F(x, y) := G(x) + \langle y, P x \rangle - H(y) \} \), where the functions \( G(\cdot) \), \( H(\cdot) \), and the coupling matrix \( P \) are distributed over a strongly connected network of nodes. GT-GDA is a first-order method that uses gradient tracking to eliminate the dissimilarity caused by heterogeneous data distribution among the nodes. In the most general form, GT-GDA includes a consensus over the local coupling matrices to achieve the optimal (unique) saddle point, however, at the expense of increased communication. To avoid this, we propose a more efficient variant GT-GDA-Lite that does not incur additional communication and analyze its convergence in various scenarios. We show that GT-GDA converges linearly to the unique saddle point solution when \( G \) is smooth and convex, \( H \) is smooth and strongly convex, and the global coupling matrix \( P \) has full column rank. We further characterize the regime under which GT-GDA exhibits a network topology-independent convergence behavior. We next show the linear convergence of GT-GDA-Lite to an error around the unique saddle point, which goes to zero when the coupling cost \( \langle y, P x \rangle \) is common to all nodes, or when \( G \) and \( H \) are quadratic. Numerical experiments illustrate the convergence properties and importance of GT-GDA and GT-GDA-Lite for several applications.

Index Terms—Decentralized optimization, saddle point problems, constrained optimization, descent ascent methods.

I. INTRODUCTION

Saddle point or min-max problems are of significant practical value in many signal processing and machine learning applications [1], [2], [3], [4], [5], [6], [7], [8], [9]. Applications of interest include but are not limited to constrained and robust optimization, beamforming, weighted linear regression, and reinforcement learning. In contrast to the traditional minimization problems where the goal is to find a global (or a local) minimum, the objective in saddle point problems is to find a point that maximizes the cost in one direction and minimizes it in the other. Consider for example Fig. 1 (left), where we show a simple function landscape \( F : \mathbb{R}^2 \to \mathbb{R} \) that increases in one direction and decreases in the other. Examples of such functions appear in constrained optimization where adding the constraints as a Lagrangian naturally leads to saddle point formulations.

Gradient descent ascent (GDA) methods are popular approaches toward saddle point problems. To find a saddle point of the function in Fig. 1 (left), we would like to maximize \( F \) with respect to the corresponding variable, say \( y \), and minimize \( F \) in the direction, say \( x \). A natural way is to compute the partial gradients \( \nabla_y F \) and \( \nabla_x F \), shown in Fig. 1 (right). Then update the \( y \) estimate moving in the direction of \( \nabla_y F \) and the \( x \) estimate moving opposite to the direction of \( \nabla_x F \).

The arrows, shown in Fig. 2, point towards the next step of GDA dynamics and the method converges to the unique saddle point (red star) under appropriate conditions on \( F \). The extension of this method for convex and strongly concave, and strongly convex and concave objectives is straightforward as it is intuitive that the saddle point \( (x^*, y^*) \in \mathbb{R}^{p_x} \times \mathbb{R}^{p_y} \) is unique such that \( \forall x \in \mathbb{R}^{p_x} \) and \( \forall y \in \mathbb{R}^{p_y} \),

\[
F(x^*, y) \leq F(x^*, y^*) \leq F(x, y^*).
\]

The traditional approaches mentioned above assume that the entire dataset is available at a central location. In many modern
applications [10], [11], [12], [13], however, data is often collected by a large number of geographically distributed devices or nodes, and communicating/storing the entire dataset at a central location is practically infeasible. Distributed optimization methods are often preferred in such scenarios, which operate by keeping data local to each individual device and exploit local computation and communication to solve the underlying problem. Such methods often deploy two types of computational architectures: (i) master/worker networks – where the data is split among multiple workers and computations are coordinated by a master (or a parameter server); (ii) peer-to-peer mesh networks – where the nodes are able to communicate only with nearby nodes over a strongly connected network topology. The topology of mesh networks is more general as it is not limited to hierarchical master/worker architectures.

In this article, we are interested in solving distributed saddle point optimization problems over peer-to-peer networks, where the corresponding data and cost functions are distributed among $n$ nodes, communicating over a strongly connected weight-balanced directed graph. In this formulation, the networked nodes are tasked to find the saddle point of a sum of local cost functions $f_i(x, y)$, where $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^p$. Mathematically, we consider the following problem:

$$\mathbf{P} : \min_{x \in \mathbb{R}^p} \max_{y \in \mathbb{R}^p} F(x, y) = \min_{x \in \mathbb{R}^p} \max_{y \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} f_i(x, y),$$

where each local cost $f_i(x, y)$ is private to node $i$ and takes the form as follows

$$f_i(x, y) := g_i(x) + \langle y, P_i x \rangle - h_i(y).$$

We assume that $G(x) := \frac{1}{n} \sum_{i=1}^{n} g_i(x)$ is convex and $H(y) := \frac{1}{n} \sum_{i=1}^{n} h_i(y)$ is strongly convex, while the global coupling matrix $\mathbf{P} = \frac{1}{n} \sum_{i=1}^{n} P_i \in \mathbb{R}^{p \times p}$ has full column rank. Such problems arise naturally in many sub-fields of signal processing, machine learning, and statistics [14], [15], [16], [17].

### A. Related Work

Theoretical studies on solutions for saddle point problems, in centralized scenarios, have attracted significant research [1], [2], [18], [19]. Recently, saddle point or min-max problems have become increasingly relevant because of their applications in constrained and robust optimization, supervised and unsupervised learning, image reconstruction, and reinforcement learning [5], [6], [7], [9]. Commonly studied sub-classes of saddle point problems of the form $\mathbf{P}$ are when $G(\cdot)$ and $H(\cdot)$ are assumed to be quadratic [14] or strongly convex [20]. In [20], the authors proposed a unified analysis technique for extra-gradient (EG) and optimistic gradient descent ascent (OGDA) methods assuming strongly concave-strongly convex saddle point problems. Furthermore, they discussed the convergence rates for the underlying problem classes and for bilinear objective functions. A more general approach was taken in [15], where $G(\cdot)$ was considered convex but not strongly convex. In [21], the authors established the convergence of OGDA only assuming the existence of saddle points. These are first-order methods with some modification of the vanilla gradient descent (GD). Apart from gradient-based methods, zeroth-order optimization techniques are proposed in [22], [23], [24], [25]. Such methods are useful when gradient computation is not feasible either because the objective function is unknown and the partial derivatives cannot be evaluated for the whole search space, or the evaluation of partial gradients is too expensive. In such cases, Bayesian optimization [23] or genetic algorithms [24], [25] are used. These techniques are usually slower than gradient-based methods.

When the data is distributed over a network of nodes, existing work has mainly focused on minimization problems [26], [27], [28], [29], [30], [31], [32]. Of significant relevance are distributed methods that assume access to a first-order oracle where the early work includes [26], [33], [34]. The performance of these methods is however limited due to their inability to handle the dissimilarity between local and global cost functions, i.e., $\nabla f_j \neq \nabla F$. In other words, linear convergence is only guaranteed but for an inexact solution (with a constant stepsize). To avoid this inaccuracy while keeping linear convergence, recent work [29], [30], [35], [36], [37], [38] propose a gradient tracking technique that allows each node to estimate the global gradient with only local communication; see also [28] for a related method.

On saddle point problems, there is not much progress made towards distributed solutions. Recent work in this regard includes [16], [17], [39], [40], [41], [42], [43], [44], [45]. Two primal-dual sub-gradient methods are proposed in [39] to solve distributed convex minimization problem under constrained sets. Some related work on solving distributed variational inequalities was proposed in [40]. Moreover, [41], [42] discuss extra-step and accelerated methods for distributed saddle point problems. However, majority of the work do not consider heterogeneous data distribution among different nodes. To deal with the dissimilarity between the local and global costs, [16] develops a function similarity metric. Similarly, [44] proposes local stochastic gradient descent ascent using similarity parameters but it is restricted to master/worker networks that are typical in federated learning scenarios. To get rid of the aforementioned similarity assumptions, [43] uses gradient tracking to eliminate this dissimilarity but assumes the functions $G(\cdot)$ and $H(\cdot)$ to be quadratic with a specific structure. Similarly, [45] extends [43] to directed graphs using the ideas from [46].

### B. Main Contributions

In this article, we propose **GT-GDA** and **GT-GDA-Lite** to solve the underlying distributed saddle point problem $\mathbf{P}$. The **GT-GDA** algorithm performs a gradient descent in the $x$ direction and a gradient ascent in the $y$ direction, both of which are combined with a network consensus term along with the communication of coupling matrices $P_i$ with neighbours. **GT-GDA-Lite** is a lighter (communication-efficient) version of **GT-GDA**, which does not require consensus over the coupling matrices and therefore reduces the communication complexity. To address the challenge that arises due to the dissimilarity between the local and global costs, the proposed methods use gradient tracking in both of the descend and ascend updates. To the best of our knowledge, there is no existing work for Problem $\mathbf{P}$ that shows linear convergence when $G(\cdot)$ is convex and $H(\cdot)$ is strongly convex. The main contributions of this article are described next:

**Novel Algorithm:** We propose a novel algorithm that uses gradient tracking for distributed gradient descent ascent updates. Gradient tracking implements an extra consensus update where the networked nodes track the global gradients with the help of local information exchange among the nearby nodes.

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1Note that the problem class $\mathbf{P}$ includes $-H$, which is strongly concave.
Weaker assumptions: We consider the problem class \( P \) such that \( g_i \) and \( h_i \) are smooth, \( G \) is convex, \( H \) is strongly convex, and the coupling matrix \( \mathcal{P} \) has full column rank. We note that the constituent local functions, \( g_i(\cdot) \) and \( h_i(\cdot) \), can be non-convex as we only require convexity on their average. Earlier work [43] that shows linear convergence of distributed saddle point problems is only applicable to specific quadratic functions \( G \) and \( H \), used in reinforcement learning, and does not provide explicit rates. It is noteworthy that the proposed problem \( P \) can be written in the primal form as follows:

\[
\min_{\mathbf{x}} \theta(\mathbf{x}) = \min_{\mathbf{x}} \left\{ H^*(\mathbf{P}\mathbf{x}) + G(\mathbf{x}) \right\}
\]

where \( H^*(\cdot) \) is the conjugate function [47], see Definition 2, of \( H(\cdot) \). We note that because \( G(\cdot) \) is strongly convex, it is enough to ensure that \( \mathcal{P} \) has full column rank to conclude that \( \theta(\cdot) \) is strongly convex [48]. This results in significantly weaker assumptions as compared to the available literature.

Linear convergence and explicit rates: We show that GT-GDA converges linearly to the unique saddle point \((x^*, y^*)\) of Problem \( P \) under the assumptions described above. We note that all these assumptions are necessary for linear convergence even for the centralized case [15]. Furthermore, we evaluate explicit rates for gradient complexity per iteration and provide a regime in which the convergence of GT-GDA is network-independent. We also show linear convergence of GT-GDA-Lite in three different scenarios and establish that the rate is the same as GT-GDA (potentially with a steady-state error) with reduced communication complexity.

Exact analysis for quadratic problems: We provide exact analytic expressions to develop the convergence characteristics of GT-GDA-Lite when \( G \) and \( H \) are in general quadratic forms. With the help of matrix perturbation theory for semisimple eigenvalues, we show that GT-GDA-Lite converges linearly to the unique saddle point of the underlying problem.

C. Notation and Paper Organization

We use lowercase letters to denote scalars, lowercase bold letters to denote vectors, and uppercase letters to denote matrices. We define \( \mathbf{0}_n \) as vector of \( n \) zeros and \( I_n \) as the identity matrix of \( n \times n \) dimensions. For a function \( F(x, y) \), \( \nabla_x F \) is the gradient of \( F \) with respect to \( x \), while \( \nabla_y F \) is the gradient of \( F \) with respect to \( y \). We denote the vector two-norm as \( \| \cdot \| \) and the spectral norm of a matrix induced by this vector norm \( \| \cdot \|_2 \). We denote the weighted vector norm of a vector \( \mathbf{z} \) with respect to a matrix \( C \) as \( \| \mathbf{z} \|_C := \mathbf{z}^\top C \mathbf{z} \) and the spectral radius of \( C \) as \( \rho(C) \). We consider \( n \) nodes interacting over a potentially directed (balanced) graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} := \{1, \ldots, n\} \) is the set of node indices, and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is a collection of ordered pairs \((i, r)\) such that node \( r \) can send information to node \( i \), i.e., \( i \leftarrow r \).

The rest of the article is organized as follows, Section II provides the motivation, with the help of several examples, and describes the algorithms GT-GDA and GT-GDA-Lite. We discuss our main results in Section III, provide simulations in Section IV, the convergence analysis in Section V, and conclude the article with Section VI.

II. MOTIVATION AND ALGORITHM DESCRIPTION

In this section, we provide some motivating applications that take the form of convex-concave saddle point problems. For more applications, see e.g., [49], [50].

A. Some Useful Examples

Distributed constrained optimization: Minimizing an objective function under certain constraints is a fundamental requirement for several applications. For equality constraints, such problems can be written as:

\[
\min_{\mathbf{x}} G(\mathbf{x}), \quad \text{subject to} \quad \mathbf{P}\mathbf{x} = \mathbf{b}.
\]

which has a saddle point equivalent form written using the Lagrangian multipliers \( y \):

\[
\mathcal{L}(\mathbf{x}, y) = G(\mathbf{x}) + \mathbf{y}^\top (\mathbf{P}\mathbf{x} - \mathbf{b})
\]

Assuming zero duality gap, any solution of (2) is a saddle point of the Lagrangian. Hence, it is sufficient to solve for

\[
\mathcal{L}(\mathbf{x}^*, y^*) = \min_{\mathbf{x}} \max_{y} \mathcal{L}(\mathbf{x}, y).
\]

For large-scale problems, the data is distributed heterogeneously and each node possesses its local \( g_i(\cdot), P_i \) and \( b_i \). The network aims to solve (2) such that

\[
G(\mathbf{x}) := \frac{1}{n} \sum_{i=1}^{n} g_i(\mathbf{x}), \quad \mathbf{P} := \frac{1}{n} \sum_{i=1}^{n} P_i, \quad \mathbf{b} := \frac{1}{n} \sum_{i=1}^{n} \mathbf{b}_i.
\]

Then for \( h_i(\mathbf{y}) := \langle \mathbf{b}_i, \mathbf{y} \rangle \) and \( H(\mathbf{y}) := \frac{1}{n} \sum_{i=1}^{n} h_i(\mathbf{y}) \), (2) takes the same form as Problem \( P \).

Distributed beamforming: Constrained optimization is widely used for array signal processing. When the signal is uncorrelated with the interference, the Capon beamformer [4] maximizes the SINR by solving the following problem:

\[
\min_{\mathbf{x}} \mathbf{x}^\top \mathbf{R}_{xx}\mathbf{x}, \quad \text{subject to} \quad \mathbf{s}^\top \mathbf{u} = 1,
\]

where \( \mathbf{x}^\top \) is Hermitian of vector \( \mathbf{x} \), \( \mathbf{R}_{xx} = \mathbb{E}[\mathbf{x}\mathbf{x}^\top H] \) and \( \mathbf{s} \) is the steering vector [3]. Recently, the distributed Capon beamformer is proposed in [51], which essentially solves \( P \).

Distributed weighted linear regression and reinforcement learning: Most applications of weighted linear regression take the form:

\[
\min_{\mathbf{x}} \| \mathbf{P}\mathbf{x} - \mathbf{b} \|^2_{C, 1}.
\]

It can be shown [15] that the saddle point equivalent of (3) is

\[
\min_{\mathbf{x}} \max_{y} \left\{ -\langle \mathbf{y}, \mathbf{b} \rangle - \frac{1}{2} \| \mathbf{y} \|^2_C + \langle \mathbf{y}, \mathbf{P}\mathbf{x} \rangle \right\}.
\]

This signifies the importance of the saddle point formulation, which enables a solution of (3) without evaluating the inverse of the matrix \( C \), thus decreasing the computational complexity. When the local data is distributed, i.e., \( \mathbf{P} := \frac{1}{n} \sum_{i=1}^{n} P_i \), \( h_i(\mathbf{y}) := \langle \mathbf{y}, \mathbf{b}_i \rangle - \frac{1}{2} \| \mathbf{y} \|^2_C \), and \( H(\mathbf{y}) := \frac{1}{n} \sum_{i=1}^{n} h_i(\mathbf{y}) \), the above optimization problem takes the form of Problem \( P \).

In several cases [14], [43], reinforcement learning takes the same form as weighted linear regression. The main objective in reinforcement learning is policy evaluation that requires learning the value function \( V^\pi \), for any given joint policy \( \pi \). The data \( \{s_k, s_{k+1}, r_k\}_{k=1}^N \) is generated by the policy \( \pi \), where \( s_k \) is the state and \( r_k \) is the reward at the \( k \)-th.
time step. With the help of a feature function \( \phi(\cdot) \), which maps each state to a feature vector, we would like to estimate the model parameters \( \phi(\cdot), x \) such that \( V(1) \approx \phi(s, x) \). A well known method for policy evaluation is to minimize the empirical mean squared projected Bellman error, which is essentially weighted linear regression: \( \min_x \| T x - b \|_2^2 \), where \( T := \sum_{k=1}^n (\phi(s_k), b(s_k) - \gamma \phi(s_{k+1})) \), for some discount factor \( \gamma \in (0, 1), C := \sum_{k=1}^n \| \phi(s_k) \|_2^2 \), and \( b := \sum_{k=1}^n r_k \phi(s_k) \).

**Supervised learning:** Classical supervised learning problems are essentially empirical risk minimization. The aim is to learn a linear predictor \( x \) when \( H(\cdot) \) is the loss function to be minimized using data matrix \( P \), and some regularizer \( G(\cdot) \). The problem can be expressed as:

\[
\min_x \left\{ H(T x) + G(x) \right\},
\]

which has the following saddle point formulation: \( \min_x \max_y \{ G(x) + \langle y, T x \rangle - H^*(y) \} \). For large-scale systems, the data \( P \) is geographically distributed among different computational nodes and the local functions \( g_i(\cdot) \) and \( h_i(\cdot) \) are also private. Problem \( P \) can be obtained here by choosing \( T \) as the following:

\[
T := \frac{1}{n} \sum_{i=1}^n P_i, G(x) := \sum_{i=1}^n g_i(x), \quad \text{and} \quad H^*(y) := \sum_{i=1}^n h_i^*(y).
\]

### B. Algorithm Development and Description

In order to motivate the proposed algorithm, we first describe the canonical distributed minimization problem:

\[
\min_x G(x) := \frac{1}{n} \sum_{i=1}^n g_i(x), \quad \text{where} \ G(\cdot) \ \text{is a smooth and strongly convex function. A well-known distributed solution is given by} \ [30, 52]:
\]

\[
x_{i+1} = \sum_{r=1}^n w_{ir} (x_r - \alpha \cdot \nabla g_i(x_r)),
\]

where \( x_r \) is the estimate of the unique minimizer (denoted as \( x^* \) such that \( \nabla g_i(x^*) = \frac{1}{n} \sum_{i=1}^n \nabla g_i(x) = 0 \)) of \( G \) at node \( i \) and time \( k \), and \( w_{ir} \) are the network weights such that \( w_{ir} \neq 0 \), if and only if \( (i, r) \in E \), and \( W = \{ w_{ir} \} \) is primitive and doubly stochastic. Consider for the sake of argument that each node at time \( k \) possesses the minimizer \( x^* \); it can be easily verified that \( x_{k+1} \neq x^* \), because the local gradients are not zero at the minimizer, i.e., \( \nabla g_i(x^*) \neq 0 \). To address this shortcoming of (5), recent work [29, 35, 36, 37, 46] uses a certain gradient tracking technique that updates an auxiliary variable \( y^k \) over the network such that \( y^k \rightarrow \frac{1}{\sum_{i=1}^n} \sum_{i=1}^n \nabla g_i(x^k) \). The resulting algorithm:

\[
x_{i+1} = \sum_{r=1}^n w_{ir} (x^k_r - \alpha \cdot y^k_r),
\]

\[
y_{i+1} = \sum_{r=1}^n w_{ir} (y^k_r + \nabla g_r(x^k_{i+1}) - \nabla g_r(x^k_r)),
\]

converges linearly to \( x^* \) thus removing the bias caused by the local \( \nabla g_r \) versus global gradient \( \nabla G \) dissimilarity.

To deal with data heterogeneity, the proposed method \( \text{GT-GDA} \), formally described in Algorithm 1, uses gradient tracking in both the descend and ascend updates. In particular, there are three main components of the \( \text{GT-GDA} \) method: (i) gradient descent for \( x \) updates; (ii) gradient ascent for \( y \) updates; and (iii) gradient tracking. However, since the coupling matrices \( P_i \)’s are not identical at the nodes, we add an intermediate step to implement consensus on \( P_i \)’s (see Remark 3 for more details). Initially, \( \text{GT-GDA} \) requires random state vectors \( x^0_i \) and \( y^0_i \) at each node \( i \), gradients evaluated with respect to \( x \) and \( y \) and some positive stepsizes \( \alpha \) and \( \beta \) for descent and ascent updates, respectively. At each iteration \( k \), every node computes gradient descent ascent type updates. The state vectors \( x_{k+1}^i \) and \( y_{k+1}^i \) are evaluated by taking a step in the negative (positive) direction of the gradient of the global problem, and then sharing them with the neighboring nodes according to the network topology. It is important to note that \( q^k_i \) and \( w^k_i \) are the global gradient tracking vectors, i.e., \( \nabla q^k_i \rightarrow \nabla x F_i(x, y) \) and \( \nabla w^k_i \rightarrow \nabla y F_i(x, y) \).

**GT-GDA-Lite:** We note that \( \text{GT-GDA} \) implements consensus on the coupling matrices (Step 2), which can result in costly communication when the size of these matrices is large. We thus consider a special case of \( \text{GT-GDA} \) that does not implement consensus on the coupling matrices, namely \( \text{GT-GDA-Lite} \) and characterize its convergence properties for the following three cases:

1. strongly concave-convex problems with different coupling matrices \( P_i \)’s at each node;
2. strongly concave-convex problems with identical \( P_i \)’s;
3. quadratic problems with different \( P_i \)’s at each node.

### III. MAIN RESULTS

Next we provide some definitions followed by the assumptions required to establish the main results.

**Definition 1 (Smoothness and convexity):** A differentiable function \( G : R^p \rightarrow R \) is \( L \)-smooth if \( \forall x, y \in R^p \),

\[
\| \nabla G(x) - \nabla G(y) \| \leq L \| x - y \|
\]

and \( \mu \)-strongly convex if \( \forall x, y \in R^p \),

\[
G(y) + \langle \nabla G(y), x - y \rangle + \frac{\mu}{2} \| x - y \|^2 \leq G(x).
\]

It is of significance to note that if \( G(\cdot) \) is \( L \)-smooth, then it is also \( (L + \xi) \)-smooth, \( \forall \xi > 0 \).

**Definition 2 (Conjugate of a function):** The conjugate of a function \( H : R^p \rightarrow R \) is defined as

\[
H^*(y) := \sup_{x \in R^p} \{ \langle x, y \rangle - H(x) \}, \quad \forall y \in R^p.
\]

Moreover, if \( H(\cdot) \) is closed and convex, then \( [H^*(\cdot)]^* = H(\cdot) \), and if \( H(\cdot) \) is \( L \)-smooth and \( \mu \)-strongly convex, then \( H^*(\cdot) \) is \( \frac{L}{\mu} \)-smooth and \( \frac{1}{L} \)-strongly convex [53].

\[\text{We do not explicitly write } \text{GT-GDA-Lite} \text{ as it is the same as Algorithm 1: GT-GDA but without the consensus (Step 2) on } P_i \text{’s.} \]
Next, we describe the assumptions under which the convergence results of \textbf{GT-GDA} will be developed; note that all of these assumptions may not be applicable at the same time.

**Assumption 1 (Smoothness and convexity):** Each local $g_i$ is $L_1$-smooth and each $h_i$ is $L_2$-smooth, where $L_1, L_2$ are arbitrary positive constants. Furthermore, the global $G$ is convex and the global $H$ is $\mu$-strongly convex.

**Assumption 2 (Quadratic):** The $g_i$’s and $h_i$’s are quadratic functions, i.e.,

$$g_i(x) := x^\top Q_i x + q_i^\top x + q_i,$$

$$h_i(y) := y^\top R_i y + r_i^\top y + r_i,$$

such that $q_i \in \mathbb{R}^{p_i}$, $r_i \in \mathbb{R}^{p_i}$, $q_i, r_i \in \mathbb{R}$, $Q_i \in \mathbb{R}^{p_i \times p_i}$, and $R_i \in \mathbb{R}^{p_i \times p_i}$, $\forall i$. Moreover, for $Q := \frac{1}{n} \sum_{i=1}^{n} Q_i$ and $\mathcal{R} := \frac{1}{n} \sum_{i=1}^{n} R_i$, we assume that $(Q + \mathcal{R})$ is positive definite and $(\mathcal{R} + Q^\top)$ is positive semi-definite.

**Assumption 3 (Full ranked coupling matrix):** The coupling matrix $\mathcal{P} := \frac{1}{n} \sum_{i=1}^{n} P_i$ has full column rank.

**Assumption 4 (Doubly stochastic weights):** The weight matrices $W := \{w_{ik}\}$ associated with the network are primitive and doubly stochastic, i.e., $W1_n = 1_n$ and $1_n^\top W = 1_n^\top$.

We note that Assumption 1 does not require strong convexity of $G$ while Assumptions 1 and 3 are necessary for linear convergence [15]. The primal problem $\min_{x} \theta(x)$, defined in (1), requires poly $(\epsilon^{-1})$ iterations to obtain an $\epsilon$-optimal solution even in the centralized case if we ignore any of the above assumptions. It is important to note that Assumptions 1-3 are not applicable simultaneously; \textbf{GT-GDA} and \textbf{GT-GDA-Lite} are analyzed under different assumptions, clearly stated in each theorem. Next, we define some useful constants to explain the main results. Let $L := \max\{L_1, L_2\}$ and let the condition number of $H(\cdot)$ be $L_2/\mu$. Furthermore, we denote $\kappa := L/\mu \geq L_2/\mu$. The maximum and minimum singular values of the coupling matrices $P_i$’s for all $i$ are defined as $\sigma_M$ and $\sigma_m$, respectively. Moreover, the condition number for the global coupling matrix is denoted by $\gamma := \sigma_M/\sigma_m$.

### A. Convergence Results for \textbf{GT-GDA}

We now provide the main results on the convergence of \textbf{GT-GDA} and discuss their attributes.

**Theorem 1:** Consider Problem P under Assumptions 1, 3, and 4. For a large enough positive constant $c > 0$, assume the stepizes are such that

$$\alpha := \beta := \frac{\mu^2}{c \sigma_M^2},$$

$$\beta := \max \left\{ \frac{\sigma_m^2 (1 - \lambda)^2}{192 \sigma_M^2}, \frac{1}{48 \sigma_M^2}, \frac{1}{384 \kappa L} \right\}.$$

Then \textbf{GT-GDA} achieves an $\epsilon$-optimal solution in

$$O \left( \max \left\{ \gamma^2 \kappa^3, \frac{\sigma_m^2 \sigma_M^2 \kappa}{(1 - \lambda)^2}, \frac{1}{\mu^2}, \frac{1}{\mu^2 (1 - \lambda)^2}, \frac{1}{\gamma^2 \kappa^5} \right\} \log \frac{1}{\epsilon} \right)$$

gradient computations (in parallel) at each node.

**Corollary 1:** Consider Problem P under Assumptions 1, 3, and 4, and \textbf{GT-GDA} with stepizes $\alpha := \overline{\alpha}$, $\beta := \overline{\beta}$ and $\Gamma = \max\{\gamma^2, \sigma_M^2 / L^2\}$. If

$$\kappa \geq \frac{\Gamma}{(1 - \lambda)^2},$$

then \textbf{GT-GDA} achieves an $\epsilon$-optimal solution linearly at a network-independent convergence rate of $O(\gamma^2 \kappa^3 \log \frac{1}{\epsilon})$.

Table I shows the computational complexities of gradient descent and gradient descent ascent methods along with their distributed counterparts. It can be seen that the computational complexity of \textbf{GDA} (centralized) is of the order $O(\kappa^3)$ [15], when the objective function is strongly concave-convex (where we used $L = \max\{L_1, L_2\}$, $\kappa = L/\mu$ and $\gamma = \sigma_M/\sigma_m$). It is typical to lose one order of $\kappa$ in making the algorithm distributed as can be observed in Table I where GD-DGD has $\kappa^2$ dependence. Similar behaviour is found for \textbf{GDA} and \textbf{GT-GDA}.

We now discuss these results in the following remarks.

**Remark 1 (Linear convergence):** \textbf{GT-GDA} eliminates the dissimilarity caused by heterogeneous data at each node using gradient tracking in both of the $x_k$ and $y_k$ updates. Theorem 1 provides an explicit linear rate at which \textbf{GT-GDA} converges to the unique saddle point $(x^*, y^*)$ of Problem P.

**Remark 2 (Network-independence):** Corollary 1 explicitly describes a regime in which the convergence rate of \textbf{GT-GDA} is independent of the network topology. We note that it signifies a relationship between the condition number and the spectral gap $(1 - \lambda)$. For weakly connected graphs, \textbf{GT-GDA} requires a higher value of $\kappa$ to attain network independent convergence.

**Remark 3 (Communication complexity):** At each node, \textbf{GT-GDA} communicates two $p_x$-dimensional vectors, two $p_y$-dimensional vectors, and a $p_y \times p_x$ dimensional coupling matrix, per iteration. In ad hoc peer-to-peer networks, the node deployment may not be deterministic. Let $\omega$ be the expected degree of the underlying (possibly random) strongly connected communication graph. Then the expected communication complexity required for \textbf{GT-GDA} to achieve an $\epsilon$-optimal solution is

$$O \left( \omega p_x p_y \max \left\{ \frac{\gamma^2 \kappa^3}{(1 - \lambda)^4}, \frac{\sigma_m^2 \sigma_M^2 \kappa}{L^2 \mu^2 (1 - \lambda)^4}, \frac{\gamma^2 \kappa^5}{L^2 \mu^2 (1 - \lambda)^4} \right\} \log \frac{1}{\epsilon} \right)$$

scalars per node. We note that $\omega$ is a function of underlying graph, e.g., $\omega = O(1)$ for random geometric graphs and $\omega = O(\log n)$ for random exponential graphs.

\textbf{GT-GDA} converges linearly to the unique saddle point but it requires each node to communicate the local coupling matrix $P_i$ with its neighbors. This incurs additional communication cost for strongly concave-convex local objective functions. Gradient tracking does not account for the discrepancy between local and global coupling matrices as can be seen in Lemma 1. To reduce this communication cost, a finite-time consensus method may be used, see for example [54].

### B. Convergence Results for \textbf{GT-GDA-Lite}

We now discuss \textbf{GT-GDA-Lite} in the context of the aforementioned special cases below.

**Theorem 2 (\textbf{GT-GDA-Lite} for Problem P):** Consider Problem P under Assumptions 1, 3, and 4. If the stepizes $\alpha \in [0, \overline{\alpha}]$ and $\beta \in [0, \overline{\beta}]$, then \textbf{GT-GDA-Lite} converges linearly to an error ball around the unique saddle point.

**Remark 4 (Convergence to an inexact solution):** We note that the speed of convergence for \textbf{GT-GDA-Lite} is of the same order as \textbf{GT-GDA}, however, \textbf{GT-GDA-Lite} converges to an error ball around the unique saddle point, which depends on the size of $\gamma$ (formally defined in Lemma 1). This error $\gamma$ can be eliminated by using identical $P_i$’s at each node or by having

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convergence. The first possibility is considered in the next theorem and the second is explored in GT-GDA.

**Theorem 3 (GT-GDA-Lite for Problem P with same P_i's):** Consider Problem P under Assumptions 1, 3, and 4 and with identical P_i’s at each node. If the stepsizes $\alpha = \pi$ and $\beta = \beta$, then the computational complexity of GT-GDA-Lite to achieve $\varepsilon$-optimal solution is the same as in Theorem 1, whereas the communication cost reduces by a factor of $O(\min\{p_x, p_y\})$.

**Remark 5 (Reduced communication complexity):** We note for GT-GDA-Lite, each node communicates two $p_x$ dimensional vectors and two $p_y$ dimensional vectors per iteration. For large values of $p_x$ and $p_y$, this is significantly less than what is required for GT-GDA, i.e., $O(p_xp_y)$. This makes GT-GDA-Lite more convenient for applications where communication budget is low.

**Theorem 4 (GT-GDA-Lite for quadratically problems):** Consider Problem P under Assumptions 2, 3, and 4 (with different $P_i$’s at the nodes). If the stepsizes $\alpha$ and $\beta$ are small enough, then GT-GDA-Lite converges linearly to the unique saddle point $(x^*, y^*)$ without consensus on $P_i$’s.

**Remark 6 (Exact analysis):** The convergence analysis we provide for the quadratic case is exact. In other words, we do not use the typical norm bounds and derive the error system of equations as an exact LTI system. Using the concepts from matrix perturbation theory for semi-simple eigenvalues, we show that GT-GDA-Lite linearly converges to the unique saddle point of P with quadratic cost functions.

### IV. Simulations

We now provide numerical experiments to compare the performance of distributed gradient descent ascent with GT-GDA and without gradient tracking (D-GDA) and verify the theoretical results. We would like to perform a preliminary empirical evaluation on a linear regression problem. We consider the problem of the form:

$$
\min_{x} \frac{1}{2n} \|P_\lambda x - b\|^2 + \lambda R(x);
$$

and the saddle point equivalent of above problem is

$$
\min_{x} \max_{y} \left\{ \langle y, P_\lambda x \rangle - \langle y, b \rangle - \frac{1}{2} \|y\|^2 + \lambda R(x) \right\}.
$$

Performance characterization using the saddle point form of (8) is common in the literature available on centralized gradient descent ascent [15], [20]. For large-scale problems, when data is available over geographically distributed nodes, decentralized implementation is often preferred. In this article, we consider the network of nodes communicating over strongly connected networks of different sizes and connectivity to extensively evaluate the performance of GT-GDA. Fig. 4 shows two directed exponential networks of $n = 8$ and $n = 32$ nodes. We note that although they are directed, their corresponding matrices $W$ are weight-balanced. To highlight the significance of distributed processing for large-scale problems, we evaluate the simulation results with the networks shown in Fig. 4 and their extensions to $n = 100$ and $n = 200$ nodes.

**Smooth and strongly convex regularizer:** We first consider (9) with smooth and strongly convex regularizer $R(x) := \|x\|^2_2$. Therefore, the resulting problem is strongly-convex strongly-concave. For a peer-to-peer mesh network of $n$ nodes, each node $i$ has its private $b_i \in \mathbb{R}^{p_x}$ and $C_i \in \mathbb{R}^{p_y} \times \mathbb{R}^{p_x}$ such that the average $b := \frac{1}{n} \sum_{i=1}^{n} b_i$ and $C := \frac{1}{n} \sum_{i=1}^{n} C_i$, and $P := \frac{1}{n} \sum_{i=1}^{n} P_i$ has full column rank. We set the dimensions $p_x = 4$, $p_y = 10$, and evaluate the performance of GT-GDA for data generated by a random Gaussian distribution.

We characterize the performance by evaluating the optimality gaps: $\|x^* - x\|_2 + \|y^* - y\|_2$. Fig. 3 represents the comparison of the simulation results of D-GDA and GT-GDA for different sizes of exponential networks ($n = 8, 32, 100$); some shown in Fig. 4. The optimality gap reduces with the increase in the number of iterations. It can be observed that D-GDA (blue curve) converges to an inexact solution because it evaluates gradients with respect to its local data at each step; hence moves towards local optimal. On the contrary, the proposed method GT-GDA (red curve) uses gradient tracking and consistently converges to the unique saddle point of the global problem. We note that each iteration of GT-GDA requires an additional communication cost for exchanging the coupling matrix (see Remark 3 for the exact expression).

**Smooth and non strongly convex regularizer:** Next we use a smooth but non strongly convex regularizer [55]:

$$
R(x) := \sum_{i=1}^{n} \sum_{j=1}^{p_x} \left[ \frac{1}{t_{ij}} \left\{ \log(1 + e^{t_{ij}x_j}) + \log(1 + e^{-t_{ij}x_j}) \right\} \right].
$$

Fig. 5 shows the results for GT-GDA over a network of $n = 32$ and $n = 200$ nodes. It can be seen that GT-GDA converges linearly to the unique saddle point, as its optimality gap decreases, meanwhile D-GDA exhibits a similar convergence rate but settles for an inexact solution due to the heterogeneous nature of data at different nodes.

**Convergence of GT-GDA-Lite:** Next we show the performance of GT-GDA-Lite for (9) with smooth and strongly convex regularizer $R(x) := \|x\|^2_2$. Fig. 6 (left) shows linear convergence of GT-GDA-Lite to the unique saddle point. Similarly, Fig. 6 (right) shows the convergence properties of GT-GDA-Lite for the aforementioned smooth and convex regularizer

| Algorithm | Problem class | Distributed | Computational complexity |
|-----------|---------------|-------------|-------------------------|
| GD        | Strongly convex | No          | $O(\kappa \log \frac{1}{\varepsilon})$ |
| GT-DGD    | Strongly convex | Yes         | $O\left( \frac{\kappa^2}{(1-\lambda^2)\mu_2} \log \frac{1}{\varepsilon} \right)$ |
| GDA       | Strongly concave-convex | No          | $O\left( \frac{\kappa^3}{(1-\lambda)^2} \log \frac{1}{\varepsilon} \right)$ |
| GT-GDA    | Strongly concave-convex | Yes         | $O\left( \max\left\{ \frac{\kappa^3 p_x^2}{(1-\lambda)^2}, \frac{\sigma_2^2 \sigma_4^2 \kappa^5}{2 (1-\lambda)^2}, \gamma^2 \kappa^5 \right\} \log \frac{1}{\varepsilon} \right)$ |
with same and different coupling matrices. It can be observed that the proposed method converges linearly to an error ball around the unique saddle point when the nodes possess different coupling matrices, however, converges to exact solution for the same coupling matrices.

**Network independence and linear speedup:** Now we analyze the convergence of **GT-GDA** considering three types of networks:

1) a circular graph (bad connectivity);
2) an exponential graph; and
3) a complete graph (best connectivity).

For a fixed $\kappa$ and varying $(1 - \lambda)$, Fig 7 (left) shows that the performance of **GT-GDA**. It can be verified that the convergence rate for circular graph is slow but is the same for an exponential graph and a complete graph, which shows network independent convergence rate as claimed in Corollary 1. Finally, we illustrate linear speed-up of **GT-GDA** as compared to its centralized counterpart. We plot the ratio of the number of iterations required to attain an optimality gap of $10^{-14}$ for **GT-GDA** as compared to the centralized GDA method in Fig. 7 (right). The results demonstrate that the performance improves linearly as the number of nodes increases ($n = 8, 16, 32, 100, 200$). We note that for $n$ nodes, the centralized case has $n$ times more data to work with at each iteration and thus has a slower convergence. In distributed setting, the processing is done in parallel which results in a faster overall performance. We emphasize that the implementation of **GT-GDA** requires each node to communicate the coupling matrix $P_i$ and the state variables with its neighbors. However, **GT-GDA-Lite** eliminates the requirement of communicating $P_i$.

**V. CONVERGENCE ANALYSIS**

In this section, our aim is to establish linear convergence of the proposed algorithms to the unique saddle point (under given set of assumptions) for problem class $P$. We first define four global state vectors $x^k, q^k \in \mathbb{R}^{np_x}$, $y^k, w^k \in \mathbb{R}^{np_y}$ that concatenate the local vectors $x^k_i, q^k_i, y^k_i, w^k_i$ for all $i$. We next define the following error quantities with the goal of characterizing their time evolution in order to establish that the error decays to zero:

1) Agreement errors, $\|x^k - W^x_k x^k\|$ and $\|y^k - W^y_k y^k\|$; Note that we define $W^x_\infty := \lim_{k \to \infty} W^x_k = \frac{1}{n} \sum_{i=1}^n 1_n^\top$, $W^y_\infty := \lim_{k \to \infty} W^y_k = \frac{1}{n} \sum_{i=1}^n 1_n^\top$, where $\otimes$ denotes the Kronecker product, and thus each error quantifies how far the network is from agreement;

2) Optimality gaps, $\|x^k - x^*\|$ and $\|y^k - y^*\|$ or $\|\tilde{y}^k - \nabla H^*(P \tilde{y}^k)\|$; Note that $\tilde{x}^k := \frac{1}{n} \sum_{i=1}^n x^k_i$, $\tilde{y}^k := \frac{1}{n} \sum_{i=1}^n y^k_i$, and thus each error quantifies the
discrepancy between the network average and the unique saddle point \((x^*, y^*)\);
3) Gradient tracking errors, \(\|q^k - W_1^k q^k\|^2\) and \(\|w^k - W_2^k w^k\|^2\). Note that these errors quantify the difference between the local and global gradients.

A. Convergence of GT–GDA

The following lemma provides a relationship between the error quantities defined above with the help of an LTI system describing GT–GDA.

Lemma 1: Consider GT–GDA described in Algorithm 1 under Assumptions 1, 3, and 4. We define \(u^k, s^k \in \mathbb{R}^6\) as

\[
\begin{bmatrix}
\|x^k - W_1^k x^k\| \\
\sqrt{n}\|\hat{x}^k - x^k\| \\
L^{-1}\|q^k - W_1^k q^k\| \\
\sqrt{\|y^k - W_2^k y^k\|} \\
\sqrt{\|\hat{y}^k - \nabla H'(P\hat{x}^k)\|} \\
L^{-1}\|w^k - W_2^k w^k\|
\end{bmatrix}, \quad \begin{bmatrix}
\|x^k\| \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]

and let \(N_{\alpha, \beta, k} \in \mathbb{R}^{6 \times 6}\) be such that it has \(\alpha \lambda_k^2 \tau\) and \(\beta \lambda_k^2 \tau\) at the (2, 1) and (1, 5) locations, respectively, and zeros everywhere else. We note that \(\tau := \|P^0 - W_2^k P^0\|\) where \(P^0\) concatenates \(P_3\)'s initially available at each node. For all \(k \geq 0, \alpha > 0, \beta \geq 0, \) and \(\alpha \leq \beta \frac{\mu \sigma}{\sqrt{M}}\), we have

\[
u^{k+1} \leq M_{\alpha, \beta} u^k + N_{\alpha, \beta, k} s^k,\tag{10}
\]
where the system matrix \(M_{\alpha, \beta}\) is defined in Appendix A.

The proof of the above lemma is omitted from here due to space limitations and can be found in the technical report [56]. The main idea behind the proof is to establish bounds on error terms (elements of \(u^k\)) for descent in \(x\) and ascent in \(y\) for a range of stepsizes \(\alpha\) and \(\beta\). It is noteworthy that the coupling between the ascent and descent equations gives rise to additional terms (see \(M_{\alpha, \beta}\) in Appendix A) adding to the complexity of the analysis. Moreover, the analysis requires a careful manipulation of the two stepsizes, unlike the existing approaches. With the help of this lemma, our goal is to establish convergence of GT–GDA and further characterize its convergence rate. To this aim, we first show that the spectral radius \(\rho(M_{\alpha, \beta})\) of the system matrix is less than 1 in the following lemma.

Lemma 2: Consider GT–GDA under Assumptions 1, 3, and 4. For a large enough positive constant \(c > 0\), assume the stepsizes are \(\alpha \in \left(0, \frac{2\sigma}{\sqrt{M}}\right)\) and \(\beta \in (0, \tilde{\beta})\), such that

\[
\tilde{\beta} := \min\left\{s_\alpha^2 \frac{(2\sigma + \lambda \sigma)^2}{182552M_a \sigma^2 L}, \frac{L(1 - \lambda)^2}{465025 \sigma^2} \right\},
\]
then \(\rho(M_{\alpha, \beta}) \leq \eta < 1\), where

\[
\eta := 1 - O\left(\min\left\{\frac{(2\sigma - \lambda \sigma)^2}{182552M_a \sigma^2 L}, \frac{L(1 - \lambda)^2}{465025 \sigma^2} \right\}\right).
\]

Proof: Recall that \(M_{\alpha, \beta}\) is a non-negative matrix. From [57], we know that if there exists a positive vector \(\delta\) and a positive constant \(\eta\) such that \(M_{\alpha, \beta} \delta \leq \eta \delta\), then \(\rho(M_{\alpha, \beta}) \leq \|M_{\alpha, \beta} \|_2 \leq \eta\), where \(\|\cdot\|_2\) is the matrix norm induced by the weighted max-norm \(\|\cdot\|^{\omega}\), with respect to some positive vector \(\delta\). To this end, we first choose \(\gamma := \frac{1 - (1 - \alpha \beta \frac{\mu}{\sqrt{M}})}{\sqrt{\alpha \beta \frac{\mu}{\sqrt{M}}}}\), which is clearly less than 1.

We next solve for a range of \(\alpha, \beta > 0\) and for a positive vector \(\delta = [\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6]^\top\) such that the inequalities in \(M_{\alpha, \beta} \delta \leq [1 - (1 - \alpha \beta \frac{\mu}{\sqrt{M}})\delta]\) hold element-wise. With \(M_{\alpha, \beta}\) in Appendix A, from the first and the fourth rows, we obtain

\[
\alpha \beta \frac{\mu}{\sqrt{M}} + \lambda L = 1 - \lambda,\tag{11}
\]

Similarly, from the second and the fifth rows, we obtain

\[
\alpha \beta \frac{\mu}{\sqrt{M}} + \beta L = 1 - \lambda.\tag{12}
\]

Finally, from the third and the sixth rows, we obtain

\[
\alpha \lambda L + \beta \frac{\mu \lambda}{L} \delta_1 + (\alpha m_1 + \beta m_2) \delta_2 + \beta \lambda \sigma M \delta_6 + (\alpha \lambda M + \beta \sigma \lambda M) \delta_3 + (\alpha \lambda M + \beta \lambda \sigma M) \delta_4 + (\alpha \lambda M + \beta \sigma \lambda M) \delta_5 \leq (1 - \lambda) \delta_3 - \lambda \left(\delta_1 + \frac{\mu \lambda}{L} \delta_4\right) + \beta \lambda L + \alpha \frac{\mu \lambda}{L} \delta_5 \leq (1 - \lambda) \delta_5 - \frac{\lambda \mu \lambda}{L} \delta_1 + \delta_4.\tag{16}
\]

We note that (13)–(16) hold true for some feasible range of \(\alpha\) and \(\beta\) when their right hand sides are positive. Thus, we fix the elements of \(\delta\) (independent of stepsize) as

\[
\delta_1 = \frac{\sigma \mu}{L}, \quad \delta_2 = 4L_2 [1 + l_3], \quad \delta_3 = \frac{1 - \frac{\lambda \mu}{L}}{4}, \quad \delta_4 = \frac{c - 1}{2c(1 + ck)}, \quad \delta_5 = \frac{c_3 + l_3}{2l_3 (1 + c_{l_3} + c_{l_3}),}
\]

where \(c > \frac{2L_2}{\sigma \mu} + \frac{2 \lambda \mu}{\sigma \mu} + 1\) and

\[
l_1 := \frac{c}{\mu} + \frac{\sigma \mu}{L}, \quad l_2 := \frac{c_3 + l_3}{d \sigma \mu}, \quad \eta_3 := \frac{c_3 + l_3}{d \sigma \mu}, \quad d := \frac{1}{2(1 + c_{l_3} + c_{l_3})},
\]

It can be verified that for the above choice of \(\delta\), the right hand sides of (13)–(16) are positive. Next we solve for the range of \(\alpha\) and \(\beta\). It can be verified that (11) and (12) are satisfied when \(\alpha \leq (\frac{1 - \lambda}{2\sigma \mu})^2\),

\[
\beta \leq \frac{(1 - \lambda)^2}{4\sigma \mu} \left(\frac{c - 1}{c_{l_3} + l_3}\right) \quad \text{and} \quad \beta \leq \frac{\alpha (1 - \lambda)}{2\sigma \mu}.
\]

Similarly, the relations (13) and (14) hold for

\[
\alpha \leq \frac{\mu (c - 1)}{2\sigma \mu} \quad \text{and} \quad \beta \leq \frac{1}{2\mu}.
\]

Finally, it can be verified that (15) and (16) hold when

\[
\alpha \leq \min\left\{\frac{\sigma \mu}{384 L^2}, \frac{\sigma \mu}{384 \lambda \sigma M L^2}, \frac{\sigma \mu}{384 \lambda \sigma M L^2}, \frac{d \sigma \mu}{L(1 - \lambda)} \right\},
\]

\[
\beta \leq \min\left\{\frac{\sigma \mu}{192 \sigma M L}, \frac{L(1 - \lambda)^2}{48 \sigma \mu}, \frac{d \sigma \mu}{L(1 - \lambda)} \right\},
\]

and \(\alpha \beta \leq \min\left\{\frac{1 - \lambda}{2\sigma \mu}, \frac{1 - \lambda}{2\sigma \mu}, \frac{1 - \lambda}{2\sigma \mu}\right\}\). The lemma follows by simplifying all the \(\alpha\) and \(\beta\) bounds for some large enough \(c\) with \(\eta = \left(1 - \alpha \beta \frac{\mu}{\sqrt{M}}\right)\).
does not follow the conventional strategies used in the literature on distributed optimization [57], [58], [59]. It requires careful selection of \( \eta \) and evaluation of appropriate bounds on both the stepsizes (\( \alpha \) and \( \beta \)) to ensure convergence. Using the two lemmas above, we are now in a position to prove Theorem 1. It is noteworthy that \( N_{\alpha,\beta,k} \) decays faster than \( M_{\alpha,\beta} \) as it can be verified that \( \lambda \leq \rho(M_{\alpha,\beta}) \). We now show that \( \| u^k \| \to 0 \) and prove Theorem 1.

1) **Proof of Theorem 1:** We first rewrite the LTI system dynamics described in Lemma 1 recursively as

\[
u^k \leq \frac{M_{\alpha,\beta}^k u^0 + \sum_{r=0}^{k-1} M_{\alpha,\beta}^{k-r-1} N_{\alpha,\beta,k}s^r}{1 + \sum_{r=0}^{k-1} M_{\alpha,\beta}^{k-r-1} N_{\alpha,\beta,k}s^r}.
\]

(17)

We now take the norm on both sides such that for some positive constants \( \omega_1, \omega_2, \omega_3 \) and \( \omega_4 \), (17) can be written as

\[
\| u^k \| \leq \| M_{\alpha,\beta}^k u^0 \| + \sum_{r=0}^{k-1} \| M_{\alpha,\beta}^{k-r-1} N_{\alpha,\beta,k}s^r \|
\]

\[
\leq \omega_1 \eta^k + \omega_2 \eta^k \sum_{r=0}^{k-1} \| s^r \|,
\]

where \( \| s^r \| \leq \omega_3 \| u^k \| + \omega_4 \| x^r \| + \omega_5 \| y^r \| \), with the help of some arbitrary norm equivalence constants. It can be verified that for \( \alpha := \omega_4 \| x^r \| + \omega_5 \| y^r \| \),

\[
\| u^k \| \leq \left( \omega_1 + \omega_2 \omega_3 \sum_{r=0}^{k-1} \| u^r \| \right) \eta^k.
\]

Let \( b_k := \sum_{r=0}^{k-1} \| u^r \| \), \( c_k := (\omega_1 + \omega_2) \omega_3 \eta^k \) and \( d_k := \omega_2 \omega_3 \eta^k \). Then the above can be re-written as

\[
u^k = b_{k+1} - b_k \leq (\omega_1 + \omega_2 \omega_3 b_k) \eta^k
\]

\[
\leq b_{k+1} \leq (1 + d_k) b_k + c_k.
\]

For non-negative sequences \( \{b_k\}, \{c_k\} \) and \( \{d_k\} \) related as \( b_{k+1} \leq (1 + d_k) b_k + c_k, \forall k \), such that \( \sum_{k=0}^{\infty} c_k < \infty \) and \( \sum_{k=0}^{\infty} d_k < \infty \), we have that the sequence \( \{b_k\} \) converges and is bounded [60]. Therefore, \( \forall t \in (\eta, 1) \), we have

\[
\lim_{k \to \infty} \frac{\| u^k \|}{\nu^k} \leq \lim_{k \to \infty} \left( \omega_1 + \omega_2 \omega_3 b_k \right) \frac{\eta^k}{\nu^k} = 0,
\]

and there exists a \( \psi > 0 \), such that \( \| u^k \| \leq \psi(\eta + \xi)^k \) for all \( k \), where \( \xi > 0 \) is an arbitrarily small constant. To achieve an e-accurate solution, we need

\[
\| x^k - \mathbf{1}_n \otimes x^r \| + \| y^k - \mathbf{1}_n \otimes y^r \| \leq \epsilon,
\]

\[
\iff \| u^k \| \leq e^{-(1-(\eta+\xi))k} \theta \leq \epsilon,
\]

and the theorem follows. \( \square \)

B. **Convergence of GT-GDA-Lite**

We now establish the convergence of **GT-GDA-Lite** under the corresponding set of assumptions. It can be verified that the LTI system for **GT-GDA-Lite** is similar to the one described in Lemma 1 except that the non-zero elements in \( N_{\alpha,\beta,k} \) are replaced by \( \sigma \tau \) at the (2, 1) location and \( \beta \tau \) at the (1, 5) location (named as \( \tilde{N}_{\alpha,\beta} \)). In the following lemma, we consider the convergence of **GT-GDA-Lite** under least assumptions, i.e., when \( P_i \)'s are not necessarily identical.

1) **Proof of Theorem 2:** Consider **GT-GDA-Lite** under Assumptions 1, 3, and 4. Given the stepsizes \( \alpha \in (0, \overline{\alpha}] \) and \( \beta \in (0, \overline{\beta}] \), with \( \overline{\alpha} \) and \( \overline{\beta} \) defined in Theorem 1, we have

\[
u^k \leq \frac{M_{\alpha,\beta}^k u^0 + \sum_{r=0}^{k-1} M_{\alpha,\beta}^{k-r-1} \tilde{N}_{\alpha,\beta}s^r}{1 + \sum_{r=0}^{k-1} M_{\alpha,\beta}^{k-r-1} \tilde{N}_{\alpha,\beta}s^r}.
\]

(18)

We have already established the fact that \( \rho(M_{\alpha,\beta}) \leq \eta < 1 \) (see Lemma 2). Therefore, the first term disappears exponentially, and the asymptotic response is

\[
\lim_{k \to \infty} \sup \| u^k \| \leq (I_0 - M_{\alpha,\beta})^{-1} \tilde{N}_{\alpha,\beta}s,
\]

(19)

where \( s := [\sup_k \| x^k \|, \sup_k \| y^k \|, 0, 0, 0, 0]^T \). The exact size of the error ball may be evaluated by calculating the norm \( \| (I_0 - M_{\alpha,\beta})^{-1} \tilde{N}_{\alpha,\beta}s \|. \) It is noteworthy that the only two nonzero terms in the vector \( \tilde{N}_{\alpha,\beta}s \) are controllable by the stepsizes \( \alpha \) or \( \beta \) and Theorem 2 follows. \( \square \)

Next, we prove the proof of Theorem 3, which assumes that each node has the same coupling matrix, and establishes convergence of **GT-GDA-Lite**.

2) **Proof of Theorem 3:** Consider **GT-GDA-Lite** under Assumptions 1, 3, and 4, with identical \( P_i \)'s. It can be verified that for stepsizes \( \alpha \in (0, \overline{\alpha}] \) and \( \beta \in (0, \overline{\beta}] \), the LTI system described in (10) reduces to

\[
u^k \leq \frac{M_{\alpha,\beta}^k u^0 + \sum_{r=0}^{k-1} M_{\alpha,\beta}^{k-r-1} \tilde{N}_{\alpha,\beta}s^r}{1 + \sum_{r=0}^{k-1} M_{\alpha,\beta}^{k-r-1} \tilde{N}_{\alpha,\beta}s^r},
\]

(20)

from Lemma 2. \( \square \)

C. **GT-GDA-Lite** for Quadratic Problems

We now consider **GT-GDA-Lite** for quadratic problems where the coupling matrices are not necessarily identical at each node. We show that **GT-GDA-Lite** converges to the unique saddle point without needing consensus. We now define the corresponding LTI system in the following lemma.

**Lemma 3 (GT-GDA-Lite for quadratic problems):** Consider Problem \( P \) under Assumptions 2, 3, and 4 (with different \( P_i \)'s at the nodes). Then the LTI dynamics governing **GT-GDA-Lite** are defined by the following

\[
\tilde{u}^{k+1} = M_{\alpha,\beta} \tilde{u}^k,
\]

(21)

where

\[
\tilde{u}^k := \left[ \begin{array}{c} \mathbf{x}^k - W_{\infty}^k \mathbf{x}^k \\ \mathbf{x}^k - \mathbf{x}^* \\ \mathbf{q}^k - W_{\infty}^k \mathbf{q}^k \\ \mathbf{y}^k - W_{\infty}^k \mathbf{y}^k \\ \mathbf{y}^k - \mathbf{y}^* \\ \mathbf{w}^k - W_{\infty}^k \mathbf{w}^k \end{array} \right],
\]

and the system matrix \( M_{\alpha,\beta} \) is defined in Appendix B.

**Lemma 3** provides an exact analysis of **GT-GDA-Lite** for quadratic problems. The derivation of the above system follows similar arguments as provided in Lemma 1 without the use of norm inequalities; detailed analysis can be found in the technical report [56].

Our aim now is to establish linear convergence of **GT-GDA-Lite** to the exact saddle point. To proceed, we set \( \beta = \alpha, \)
which makes \( \tilde{M}_{\alpha,\beta} \) as a function of \( \alpha \) alone. Thus we define \( \tilde{M}_\alpha := \tilde{M}_{\alpha,\beta=\alpha} \) and \( \tilde{M}_0 := \tilde{M}_{\alpha=0} \) leading to

\[
\tilde{M}_0 := \begin{bmatrix}
\begin{bmatrix}
W_1 & O & O & O & O & O \\
O & I_{p_x} & O & O & O & O \\
O & O & W_1 & O & O & O \\
O & O & O & W_2 & O & O \\
\times & O & O & O & W_2 & \times \\
\end{bmatrix} & O & O & O & O & O \\
\end{bmatrix},
\]

where \( W_1 := W_1 - W_\infty \), \( W_2 := W_2 - W_\infty \). \( O \) are zero matrices of appropriate dimensions, and \( \times \) are “don’t care terms” that do not affect further analysis. Using Schur’s Lemma for determinant of block matrices, we note that, for the given structure of \( \tilde{M}_0 \), the eigenvalues of \( \tilde{M}_0 \) are the eigenvalues of the diagonal block matrices.

Furthermore, we know that \( \rho(W - W_\infty) < 1 \), which implies that \( \rho(W_1) < 1 \) and \( \rho(W_2) < 1 \). Therefore, \( \rho(M_0) = 1 \) and \( M_0 \) has \( p := p_x + p_y \) semi-simple eigenvalues. We would like to show that for sufficiently small positive stepsizes and \( \beta = \alpha \), all eigenvalues of 1 decrease and thus the spectral radius of \( M_{\alpha,\beta} \) becomes less than 1. It is noteworthy that the analysis in the existing literature on distributed optimization [46], [61] is limited to a simple eigenvalue and thus cannot be directly extended to our case. We provide the following lemma to establish the change in the semi-simple eigenvalues with respect to \( \alpha \).

**Lemma 4:** [62] Consider an \( n \times n \) matrix \( M_\alpha \) which depends smoothly on a real parameter \( \alpha \geq 0 \). Fix \( l \in [1, n] \) and let \( \lambda_1 = \cdots = \lambda_l \) be a semi-simple eigenvalue of \( M_0 \), with (linearly independent) right eigenvectors \( y_1, \ldots, y_l \) and (linearly independent) left eigenvectors \( z_1, \ldots, z_l \). Denote by \( \lambda_i(\alpha) \) the eigenvalues of \( M_\alpha \) corresponding to \( \lambda_i \), \( i \in [1, l] \), as a function of \( \alpha \). Then the derivatives \( d\lambda_i(\alpha)/d\alpha \) exist, and \( d\lambda_i(\alpha)/d\alpha|_{\alpha=0} \) is given by the eigenvalues \( [\lambda_i]_l \) of the following \( l \times l \) matrix

\[
S := \begin{bmatrix}
\begin{bmatrix}
z_1^\top & \cdots & z_l^\top \\
M_1^\top y_1 & \cdots & M_1^\top y_l \\
\vdots & \ddots & \vdots \\
z_l^\top & \cdots & z_l^\top \\
M_l^\top y_1 & \cdots & M_l^\top y_l \\
\end{bmatrix}
\end{bmatrix},
\]

where \( M_1 := dM_\alpha/d\alpha|_{\alpha=0} \). Furthermore, the eigenvalue \( \lambda_i(\alpha) \) under perturbation of the parameter \( \alpha \) is given by

\[
\lambda_i(\alpha) = \lambda_i(0) + \alpha[\lambda_i]_1 + o(\alpha), \quad \forall i \in [1, l].
\]

With the help of Lemma 4, we now prove Theorem 4.

**1) Proof of Theorem 4:** We note that \( \tilde{M}_\alpha := \tilde{M}_0 + \alpha M' \), where \( M' := dM_\alpha/d\alpha|_{\alpha=0} \). Furthermore, the left eigenvectors corresponding to \( p \) semi-simple eigenvalues are the rows of the matrix \( U \) and the right eigenvectors are the columns of the matrix \( V \), as defined below:

\[
U := \begin{bmatrix}
O & I_{p_x} & O & O & O & O \\
O & O & I_{p_y} & O & O & O \\
\end{bmatrix} \quad \text{and} \quad V := U^\top.
\]

We would like to ensure that the semi-simple eigenvalues of \( M_0 \) are forced to move inside the unit circle as \( \alpha \) increases. Using Lemma 4, it can be established that the derivatives \( d\lambda_i(\alpha)/d\alpha|_{\alpha=0} \) exist for all \( i \in [1, p] \) and are the eigenvalues \( [\lambda_i]_1 \) of

\[
S := UM'V = \begin{bmatrix}
\begin{bmatrix}
-C \bar{Q} & -\bar{P}^\top \\
\end{bmatrix} & \begin{bmatrix}
\end{bmatrix}
\end{bmatrix}.
\]

Next we define \( a_i := \text{Re}(\lambda_i)_1 \) and \( b_i := \text{Im}(\lambda_i)_1, \forall i \in [1, p] \). From Theorem 3.6 in [2], we know that \( -S \) is positive stable if \( (\bar{Q} + \bar{Q}^\top) \) is positive definite and \( (\bar{R} + \bar{R}^\top) \) is positive semi-definite, i.e., \( \forall i \in [1, p], a_i < 0 \). Furthermore, we know from Lemma 4 that

\[
\lambda_i(\alpha) = \lambda_i(0) + \alpha[\lambda_i]_1 + o(\alpha), \quad (22)
\]

see Theorem 2.7 in [62] for details. For sufficiently small step-size \( \alpha > 0 \), the term \( o(\alpha) \) can be made arbitrarily small as it contains higher order terms. Therefore, we can re-write (22) as

\[
\lambda_i(\alpha) = 1 + \alpha[\lambda_i]_1 = 1 + \alpha(a_i + jb_i), \quad \forall i \in [1, p].
\]

Since the real parts of \( [\lambda_i]_1 \) are \( a_i < 0 \), the semi-simple eigenvalues would move towards the direction of the secants of the unit circle for any \( b_i \), see Fig. 8. We thus obtain that \( \rho(M_\alpha) < 1 \) for sufficiently small stepsize \( \alpha > 0 \) and Theorem 4 follows.

**VI. CONCLUSION**

In this article, we describe first-order methods to solve distributed saddle point problems of the form: \( \min_x \max_y (G(x) + \langle y, P^\top x \rangle) - H(y) \), which has many practical applications. In particular, we assume that the underlying data is distributed over a strongly connected network of nodes such that \( G(x) := \frac{1}{n} \sum_{i=1}^n g_i(x) \), \( H(y) := \frac{1}{n} \sum_{i=1}^n h_i(y) \), and \( P := \frac{1}{n} \sum_{i=1}^n P_i \), where the constituent functions \( g_i, h_i \), and local coupling matrices \( P_i \) are private to each node \( i \). Under appropriate assumptions, we show that \( \text{GT-GDA} \) converges linearly to the unique saddle point of strongly concave-convex problems.

We further provide explicit \( \epsilon \)-complexities of the underlying algorithms and characterize a regime in which the convergence is network-independent. To reduce the communication complexity of \( \text{GT-GDA} \), we propose a lighter (communication-efficient) version \( \text{GT-GDA-Lite} \) that does not require consensus on local \( P_i \)'s and analyze \( \text{GT-GDA-Lite} \) under various relevant scenarios. Finally, we illustrate the convergence properties through numerical experiments.
APPENDIX

A. System Matrix for GT-GDA

To completely describe lemma 1, for \( c > \frac{2L^2}{\sigma_m^2} + \frac{2\sigma_m^2}{\sigma_m^2} + 1 \), we describe the system matrix as follows

\[
M_{\alpha, \beta} := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},
\]

\[
M_{11} := \begin{bmatrix} \lambda & 0 & \alpha L \\ \alpha L_1 & 1 - \alpha \frac{\sigma_2^2}{\lambda} & 0 \\ \lambda + \alpha \lambda L + \beta \frac{\sigma_2^2}{L} & \alpha m_1 + \beta m_2 & \lambda + \alpha \lambda L \end{bmatrix},
\]

\[
M_{12} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ \frac{1}{\lambda} + \alpha + \beta \lambda & \alpha + \beta \lambda & \beta \lambda \end{bmatrix}_{\sigma_M},
\]

\[
M_{21} := \begin{bmatrix} \lambda & 0 & \beta L \\ \alpha \sigma_M \mu & \alpha m_3 & 0 \\ \lambda + \alpha \sigma_M \mu & \beta \lambda L & \lambda + \alpha \lambda L \end{bmatrix},
\]

\[
M_{22} := \begin{bmatrix} 0 & 0 & \beta L \\ \beta m_1 & 0 & 0 \\ \lambda + \alpha \sigma_M \mu & \alpha \lambda L & \lambda + \beta \lambda L \end{bmatrix},
\]

\[
m_1 := \lambda m, \quad m_2 := \lambda \sigma_2^2 (1 + \kappa), \quad m_3 := \frac{m a m}{\mu}, \quad m_4 := \frac{m a m}{\mu}, \quad m_5 := \lambda m, \quad m_6 := L + \sigma_2^2 \mu.
\]

B. System Matrix for GT-GDA-Lite: Quadratic Case

Let \( p := p_x + p_y \), and we define \( \tilde{M}_{\alpha, \beta} \in \mathbb{R}^{(2n+1)p \times (2n+1)p} \) as

\[
\tilde{M}_{\alpha, \beta} := \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix},
\]

\[
\tilde{M}_{11} := \begin{bmatrix} -\alpha \Lambda Q W_{11}^\infty & 0 & -\alpha W_1 \\ -\alpha \left(1 + I_{n p_y}\right) \Lambda_{Q} & I_{p_x} - \alpha Q & 0 \\ 0 & 0 & -\alpha \end{bmatrix},
\]

\[
\tilde{M}_{12} := \begin{bmatrix} 0 & 0 & 0 \\ -\alpha \left(1 + I_{n p_y}\right) \Lambda_{P} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
\tilde{M}_{21} := \begin{bmatrix} \beta \left(1 + I_{n p_y}\right) \Lambda_{P} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
\tilde{M}_{22} := \begin{bmatrix} 0 & 0 & \beta \end{bmatrix},
\]

such that for any collection of matrices \( \{M_1, M_2, \ldots, M_n\} \), we define \( \Lambda_M := \text{diag}(\{M_1, M_2, \ldots, M_n\}) \) as the block diagonal matrix where the \( i \)-th diagonal element is \( M_i \). The rest of the terms used in \( \tilde{M}_{\alpha, \beta} \) are defined below:

\[
\tilde{m}_1 = W_1 [\Lambda_Q ((W_1 - I_{n p_y}) - \alpha W_1^\infty \Lambda_Q) + \beta \Lambda_P W_1^\infty \Lambda_P],
\]

\[
\tilde{m}_2 = W_1 [-\alpha \Lambda_Q W_1^\infty \Lambda_Q + \beta \Lambda_P W_1^\infty \Lambda_P] (I_n \otimes I_{p_x}),
\]

\[
\tilde{m}_3 = W_1 [\Lambda_Q W_1^\infty \Lambda_P], \quad \tilde{m}_4 = W_1 [\beta \Lambda_P W_1^\infty \Lambda_P],
\]

\[
\tilde{m}_5 = W_2 [\beta \Lambda_R W_2^\infty \Lambda_P - \beta \Lambda_P W_2^\infty \Lambda_R], \quad \tilde{m}_6 = W_2 [\beta \Lambda_P W_2^\infty \Lambda_P].
\]

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