Elementary matrix factorizations over Bézout domains

Dmitry Doryn
Calin Iuliu Lazaroiu
Mehdi Tavakol

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Center for Geometry and Physics
Institute for Basic Science
Pohang 37673
Republic of Korea
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Center for Geometry and Physics, Institute for Basic Science, Pohang, Republic of Korea 37673
Max-Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail: doryn@ibs.re.kr, calin@ibs.re.kr, mehdi@mpim-bonn.mpg.de

Abstract: We study the homotopy category $\text{hef}(R,W)$ (and its $\mathbb{Z}_2$-graded version $\text{HEF}(R,W)$) of elementary factorizations, where $R$ is a Bézout domain which has prime elements and $W = W_0W_c$, where $W_0 \in R^\times$ is a square-free element of $R$ and $W_c \in R^\times$ is a finite product of primes with order at least two. In this situation, we give criteria for detecting isomorphisms in $\text{hef}(R,W)$ and $\text{HEF}(R,W)$ and formulas for the number of isomorphism classes of objects. We also study the full subcategory $\text{hef}(R,W)$ of the homotopy category $\text{hmf}(R,W)$ of finite rank matrix factorizations of $W$ which is additively generated by elementary factorizations. We show that $\text{hef}(R,W)$ is Krull-Schmidt and we conjecture that it coincides with $\text{hmf}(R,W)$. Finally, we discuss a few classes of examples.

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Introduction

The study of topological Landau-Ginzburg models $[1,2,3,4]$ often leads to the problem of understanding the triangulated category $\text{hmf}(R,W)$ of finite rank matrix factorizations of an element $W \in R$, where $R$ is a non-Noetherian commutative ring. For example, the category of B-type topological D-branes associated to a holomorphic Landau-Ginzburg pair $(\Sigma,W)$ with $\Sigma$ a non-compact Riemann surface and $W : \Sigma \to \mathbb{C}$ a non-constant holomorphic function has this form with $R = O(\Sigma)$, the non-Noetherian ring of holomorphic functions defined on $\Sigma$. When $\Sigma$ is connected, the ring $O(\Sigma)$ is a Bézout domain (in fact, an elementary divisor domain). In this situation, this problem can be reduced $[5]$ to the study of the full subcategory $\text{hef}(R,W)$ whose objects are the elementary factorizations, defined as those matrix factorizations of $W$ for which
the even and odd components of the underlying supermodule have rank one. In this paper, we
study the category \(\text{hef}(R, W)\) and the full category \(\text{hef}(R, W)\) of \(\text{lmf}(R, W)\) which is additively
generated by elementary matrix factorizations, for the case when \(R\) is a Bézout domain. We say
that \(W\) is \textit{critically-finite} if it is a product of a square-free element \(W_0\) of \(R\) with an element
\(W_c \in R\) which can be written as a finite product of primes of multiplicities strictly greater than
one. When \(W\) is critically-finite, the results of this paper provide a detailed description of the
categories \(\text{hef}(R, W)\) and \(\text{hef}(R, W)\), reducing questions about them to the divisibility theory
of \(R\).

The paper is organized as follows. In Section 1, we recall some basic facts about finite rank
matrix factorizations over unital commutative rings and introduce notation and terminology
which will be used later on. In Section 2, we study the category \(\text{hef}(R, W)\) and its \(\mathbb{Z}_2\)-graded
completion \(\text{HEF}(R, W)\) when \(W\) is any non-zero element of \(R\), describing these categories in
terms of the lattice of divisors of \(W\) and giving criteria for deciding when two objects are
isomorphic. We also study the behavior of these categories under localization at a multiplicative
set as well as their subcategories of primary matrix factorizations. In Section 3, we show that
the additive category \(\text{hef}(R, W)\) is Krull-Schmidt when \(R\) is a Bézout domain and \(W\) is a
critically-finite element of \(R\) and propose a few conjectures about \(\text{lmf}(R, W)\). In Section 4,
we give a formula for the number of isomorphism classes in the categories \(\text{HEF}(R, W)\) and
\(\text{hef}(R, W)\). Finally, Section 5 discusses a few classes of examples. Appendices A and B collect
some information on greatest common divisor (GCD) domains and Bézout domains.

\textit{Notations and conventions.} The symbols \(\bar{0}\) and \(\bar{1}\) denote the two elements of the field \(\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}\),
where \(\bar{0}\) is the zero element. Unless otherwise specified, all rings considered are unital and
commutative. Given a cancellative Abelian monoid \((M, \cdot)\), we say that an element \(x \in M\)
divides \(y \in M\) if there exists \(q \in M\) such that \(y = qx\). In this case, \(q\) is uniquely determined by
\(x\) and \(y\) and we denote it by \(q = y/x\) or \(\frac{y}{x}\).

Let \(R\) be a unital commutative ring. The set of non-zero elements of \(R\) is denoted by \(R^\times = R \setminus \{0\}\),
while the multiplicative group of units of \(R\) is denoted by \(U(R)\). The Abelian categories
of all \(R\)-modules is denoted \(\text{Mod}_R\), while the Abelian category of finitely-generated \(R\)-modules
is denoted \(\text{mod}_R\). Let \(\text{Mod}^{\mathbb{Z}_2}_R\) denote the category of \(\mathbb{Z}_2\)-graded modules and outer (i.e. even)
morphisms of such and \(\text{Mod}^{\mathbb{Z}_2}_R\) denote the category of \(\mathbb{Z}_2\)-graded modules and inner morphisms of
such. By definition, an \(R\)-linear category is a category enriched in the monoidal category \(\text{Mod}_R\)
while a \(\mathbb{Z}_2\)-graded \(R\)-linear category is a category enriched in the monoidal category \(\text{Mod}^{\mathbb{Z}_2}_R\).
With this definition, a linear category is pre-additive, but it need not admit finite bi-products
(direct sums). For any \(\mathbb{Z}_2\)-graded \(R\)-linear category \(\mathcal{C}\), the even subcategory \(\mathcal{C}^0\) is the \(R\)-linear
category obtained from \(\mathcal{C}\) by keeping only the even morphisms.

For any unital integral domain \(R\), let \(\sim\) denote the equivalence relation defined on \(R^\times\) by
association in divisibility:
\[
x \sim y \iff \exists \gamma \in U(R) : y = \gamma x .
\]
The set of equivalence classes of this relation coincides with the set \(R^\times/U(R)\) of orbits for
the obvious multiplicative action of \(U(R)\). Since \(R\) is a commutative domain, the quotient
\(R^\times/U(R)\) inherits a multiplicative structure of cancellative Abelian monoid. For any \(x \in R^\times\),
let \((x) \in R^\times/U(R)\) denote the equivalence class of \(x\) under \(\sim\). Then for any \(x, y \in R^\times\), we have
\((xy) = (x)(y)\). The monoid \(R^\times/U(R)\) can also be described as follows. Let \(G_+(R)\) be the set of non-zero principal ideals of \(R\). If \(x, y\) are elements of \(R^\times\), we have \(\langle x \rangle \langle y \rangle = \langle xy \rangle\),
so the product of principal ideals corresponds to the product of the multiplicative group \(R^\times\)
and makes \(G_+(R)\) into a cancellative Abelian monoid with unit \(\langle 1 \rangle = R\). Notice that \(G_+(R)\)
coincides with the positive cone of the group of divisibility (see Subsection 5.2) \(G(R)\) of \(R\), when
the latter is viewed as an Abelian group ordered by reverse inclusion. The monoids $R^\times/U(R)$ and $G_+(R)$ can be identified as follows. For any $x \in R^\times$, let $(x) \in G_+(R)$ denote the principal ideal generated by $x$. Then $(x)$ depends only on $x$ and will also be denoted by $(x)$. This gives a group morphism $(\cdot) : R^\times/U(R) \to G_+(R)$. For any non-zero principal ideal $I \in G_+(R)$, the set of all generators $x$ of $I$ is a class in $R^\times/U(R)$ which we denote by $(I)$; this gives a group morphism $(\cdot) : G_+(R) \to R^\times/U(R)$. For all $x \in R^\times$, we have $(x) = (x)$ and $(x) = (x)$, which implies that $(\cdot)$ and $(\cdot)$ are mutually inverse group isomorphisms.

If $R$ is a GCD domain (see Appendix A) and $x_1, \ldots, x_n$ are elements of $R$ such that $x_1 \ldots x_n \neq 0$, let $d$ be any greatest common divisor (gcd) of $x_1, \ldots, x_n$. Then $d$ is determined by $x_1, \ldots, x_n$ up to association in divisibility and we denote its equivalence class by $\langle d \rangle$. The principal ideal $\langle d \rangle = \langle (x_1, \ldots, x_n) \rangle \in G_+(R)$ does not depend on the choice of $d$. The elements $x_1, \ldots, x_n$ also have a least common multiple (lcm) $m$, which is determined up to association in divisibility and whose equivalence class we denote by $[x_1, \ldots, x_n] \in R^\times/U(R)$. For $n = 2$, we have:

$$[x_1, x_2] = \frac{(x_1)(x_2)}{(x_1, x_2)}.$$

If $R$ is a Bézout domain (see Appendix B), then we have $\langle x_1, \ldots, x_n \rangle \overset{\text{def}}{=} \langle (x_1, \ldots, x_n) \rangle = (x_1) + \ldots + (x_n)$, so the gcd operation transfers the operation given by taking the finite sum of principal ideals from $G_+(R)$ to $R^\times/U(R)$ through the isomorphism of groups described above.

In this case, we have $\langle x_1, \ldots, x_n \rangle = \langle (x_1, \ldots, x_n) \rangle$. We also have $\langle x_1, \ldots, x_n \rangle = \bigcap_{i=1}^{n} (x_i)$ and hence $[x_1, \ldots, x_n] = \bigcap_{i=1}^{n} (x_i)$. Thus the lcm corresponds to the finite intersection of principal ideals.

1. Matrix factorizations over an integral domain

Let $R$ be an integral domain and $W \in R^\times$ be a non-zero element of $R$.

1.1. Categories of matrix factorizations. We shall use the following notations:

1. MF$(R, W)$ denotes the $R$-linear and $\mathbb{Z}_2$-graded differential category of $R$-valued matrix factorizations of $W$ of finite rank. The objects of this category are pairs $a = (M, D)$, where $M$ is a free $\mathbb{Z}_2$-graded $R$-module of finite rank and $D$ is an odd endomorphism of $M$ such that $D^2 = \text{Wid}_M$. For any objects $a_1 = (M_1, D_1)$ and $a_2 = (M_2, D_2)$ of MF$(R, W)$, the $\mathbb{Z}_2$-graded $R$-module of morphisms from $a_1$ to $a_2$ is given by the inner Hom:

$$\text{Hom}_{\text{MF}(R, W)}(a_1, a_2) = \text{Hom}_R(M_1, M_2) = \text{Hom}_R^0(M_1, M_2) \oplus \text{Hom}_R^1(M_1, M_2),$$

endowed with the differential $\partial_{a_1, a_2}$ determined uniquely by the condition:

$$\partial_{a_1, a_2}(f) = D_2 \circ f - (-1)^\kappa f \circ D_1, \quad \forall f \in \text{Hom}_R^\kappa(M_1, M_2),$$

where $\kappa \in \mathbb{Z}_2$.

2. ZMF$(R, W)$ denotes the $R$-linear and $\mathbb{Z}_2$-graded cocycle category of MF$(R, W)$. This has the same objects as MF$(R, W)$ but morphisms given by:

$$\text{Hom}_{\text{ZMF}(R, W)}(a_1, a_2) \overset{\text{def}}{=} \{ f \in \text{Hom}_{\text{MF}(R, W)}(a_1, a_2) | \partial_{a_1, a_2}(f) = 0 \}.$$
3. BMF$(R, W)$ denotes the $R$-linear and $\mathbb{Z}_2$-graded coboundary category of MF$(R, W)$, which is an ideal in ZMF$(R, W)$. This has the same objects as MF$(R, W)$ but morphism spaces given by:

$$\text{Hom}_{\text{BMF}(R,W)}(a_1,a_2) \overset{\text{def}}{=} \{ \varphi_{a_1,a_2}(f) | f \in \text{Hom}_{\text{MF}(R,W)}(a_1,a_2) \} .$$

4. HMF$(R, W)$ denotes the $R$-linear and $\mathbb{Z}_2$-graded total cohomology category of MF$(R, W)$. This has the same objects as MF$(R, W)$ but morphism spaces given by:

$$\text{Hom}_{\text{HMF}(R,W)}(a_1,a_2) \overset{\text{def}}{=} \text{Hom}_{\text{ZMF}(R,W)}(a_1,a_2)/\text{Hom}_{\text{BMF}(R,W)}(a_1,a_2) .$$

5. The subcategories of MF$(R, W)$, ZMF$(R, W)$, BMF$(R, W)$ and HMF$(R, W)$ obtained by restricting to morphisms of even degree are denoted respectively by mf$(R, W) \overset{\text{def}}{=} \text{MF}^0(R, W)$, zmf$(R, W) \overset{\text{def}}{=} \text{ZMF}^0(R, W)$, bmf$(R, W) \overset{\text{def}}{=} \text{BMF}^0(R, W)$ and hmf$(R, W) \overset{\text{def}}{=} \text{HMF}^0(R, W)$.

The categories MF$(R, W)$, BMF$(R, W)$ and ZMF$(R, W)$ admit double direct sums (and hence all finite direct sums of at least two elements) but do not have zero objects. On the other hand, the category HMF$(R, W)$ is additive, the matrix factorization $\begin{bmatrix} 0 & 1 \\ W & 0 \end{bmatrix}$ being a zero object. Finally, it is well-known that the category hmf$(R, W)$ is triangulated (see [6] for a detailed treatment).

For later reference, recall that the biproduct (direct sum) of MF$(R, W)$ is defined as follows:

**Definition 1.1** Given two matrix factorizations $a_i = (M_i, D_i)$, $(i = 1, 2)$ of $W \in R$, their direct sum $a_1 \oplus a_2$ is the matrix factorization $a = (M, D)$ of $W$, where $M \overset{\text{def}}{=} M^0 \oplus M^1$ and $D \overset{\text{def}}{=} \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$, with:

$$M^\kappa = M_1^\kappa \oplus M_2^\kappa \ \forall \kappa \in \mathbb{Z}_2 \ \text{and} \ u = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}, \ v = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} .$$

Given a third matrix factorization $a_3 = (M_3, D_3)$ of $W$ and two morphisms $f_i \in \text{Hom}_{\text{MF}(R,W)}(a_i, a_3)$ $\overset{\text{Hom}_R(a_i, a_3)}{(i = 1, 2)}$ in MF$(R, W)$, their direct sum of $f_1 \oplus f_2 \in \text{Hom}_{\text{MF}(R,W)}(a_1 \oplus a_2, a_3) = \text{Hom}_R(a_1 \oplus a_2, a_3)$ is the ordinary direct sum of the $R$-module morphisms $f_1$ and $f_2$.

As a consequence, MF$(R, W)$ admits all finite but non-empty direct sums. The following result is elementary:

**Lemma 1.2** The following statements hold:

1. The subcategories ZMF$(R, W)$ and BMF$(R, W)$ of MF$(R, W)$ are closed under finite direct sums (but need not have zero objects).

2. The direct sum induces a well-defined biproduct (which is again denoted by $\oplus$) on the $R$-linear categories HMF$(R, W)$ and hmf$(R, W)$.

3. (HMF$(R, W), \oplus$) and (hmfr$(R, W), \oplus$) are additive categories, a zero object in each being given by any of the elementary factorizations $e_1$ and $e_W$, which are isomorphic to each other in hmf$(R, W)$. In particular, any finite direct sum of the elementary factorizations $e_1$ and $e_W$ is a zero object in HMF$(R, W)$ and in hmf$(R, W)$. 

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1.2. Reduced rank and matrix description. Let \( a = (M, D) \) be an object of \( \text{MF}(R, W) \), where \( M = M^0 \oplus M^1 \). Taking the supertrace in the equation \( D^2 = \text{Wid}_M \) and using the fact that \( W \neq 0 \) shows that \( \text{rk} M^0 = \text{rk} M^1 \). We call this natural number the reduced rank of \( a \) and denote it by \( \rho(a) \); we have \( \text{rk} M = 2 \rho(a) \). Choosing a homogeneous basis of \( M \) (i.e. a basis of \( M^0 \) and a basis of \( M^1 \)) gives an isomorphism of \( R \)-supermodules \( M \cong R_{\rho} |_{\rho} \), where \( \rho = \rho(a) \) and \( R_{\rho} |_{\rho} \) denotes the \( R \)-supermodule with \( \mathbb{Z}_2 \)-homogeneous components \( (R_{\rho} |_{\rho})^0 = (R_{\rho} |_{\rho})^1 = R_{\rho} \).

This isomorphism allows us to identify \( D \) with a square matrix of size \( 2 \rho(a) \) which has block off-diagonal form:

\[
D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix},
\]

where \( u \) and \( v \) are square matrices of size \( \rho(a) \) with entries in \( R \). The condition \( D^2 = \text{Wid}_M \) amounts to the relations:

\[
uv = vu = WI_{\rho}, \tag{1.1}
\]

where \( I_{\rho} \) denotes the identity matrix of size \( \rho \). Since \( W \neq 0 \), these conditions imply that the matrices \( u \) and \( v \) have maximal rank \( 1 \):

\[
\text{rk} u = \text{rk} v = \rho.
\]

Matrix factorizations for which \( M = R_{\rho} |_{\rho} \) form a dg subcategory of \( \text{MF}(R, W) \) which is essential in the sense that it is dg-equivalent with \( \text{MF}(R, W) \). Below, we often tacitly identify \( \text{MF}(R, W) \) with this essential subcategory and use similar identifications for \( \text{ZMF}(R, W) \), \( \text{BMF}(R, W) \) and \( \text{HMF}(R, W) \).

Given two matrix factorizations \( a_1 = (R_{\rho_1} |_{\rho_1}, D_1) \) and \( a_2 = (R_{\rho_2} |_{\rho_2}, D_2) \) of \( W \), write \( D_i = \begin{bmatrix} 0 & v_i \\ u_i & 0 \end{bmatrix} \), with \( u_i, v_i \in \text{Mat}(\rho_i, \rho_i, R) \). Then:

- An even morphism \( f \in \text{Hom}^0_{\text{MF}(R, W)}(a_1, a_2) \) has the matrix form:

\[
f = \begin{bmatrix} f_{00} & 0 \\ 0 & f_{11} \end{bmatrix}
\]

with \( f_{00}, f_{11} \in \text{Mat}(\rho_1, \rho_2, R) \) and we have:

\[
\partial_{a_1,a_2}(f) = D_2 \circ f - f \circ D_1 = \begin{bmatrix} 0 & 0 & v_2 \circ f_{11} - f_{00} \circ v_1 \\ u_2 \circ f_{00} - f_{11} \circ u_1 & 0 \end{bmatrix};
\]

- An odd morphism \( g \in \text{Hom}^1_{\text{MF}(R, W)}(a_1, a_2) \) has the matrix form:

\[
g = \begin{bmatrix} 0 & g_{i0} \\ g_{0i} & 0 \end{bmatrix}
\]

with \( g_{i0}, g_{0i} \in \text{Mat}(\rho_1, \rho_2, R) \) and we have:

\[
\partial_{a_1,a_2}(g) = D_2 \circ g + g \circ D_1 = \begin{bmatrix} 0 & v_2 \circ g_{0i} + g_{i0} \circ u_1 & 0 \\ u_2 \circ g_{i0} + g_{0i} \circ v_1 & 0 \end{bmatrix}.
\]

\(^1\) To see this, it suffices to consider equations (1.1) in the field of fractions of \( R \).
Remark 1.1. The cocycle condition $\delta_{a_1, a_2}(f) = 0$ satisfied by an even morphism $f \in \text{Hom}_{ZMF(R,W)}^0(a_1, a_2)$ amounts to the system:

\[
\begin{cases}
v_2 \circ f_{i1} = f_{00} \circ v_1 \\
u_2 \circ f_{00} = f_{i1} \circ u_1
\end{cases},
\]

which in turn amounts to any of the following equivalent conditions:

\[
f_{i1} = \frac{u_2 \circ f_{00} \circ v_1}{W} \iff f_{00} = \frac{v_2 \circ f_{i1} \circ u_1}{W}.
\]

Similarly, the cocycle condition $\delta_{a_1, a_2}(g) = 0$ defining an odd morphism $g \in \text{Hom}_{ZMF(R,W)}^1(a_1, a_2)$ amounts to the system:

\[
\begin{cases}
v_2 \circ g_{01} + g_{i0} \circ u_1 = 0 \\
u_2 \circ g_{i0} + g_{01} \circ v_1 = 0
\end{cases},
\]

which in turn amounts to any of the following equivalent conditions:

\[
g_{i0} = -\frac{v_2 \circ g_{01} \circ v_1}{W} \iff g_{01} = -\frac{u_2 \circ g_{i0} \circ u_1}{W}.
\]

1.3. Strong isomorphism. Recall that $\text{zmf}(R,W)$ denotes the even subcategory of $ZMF(R,W)$. This category admits non-empty finite direct sums but does not have a zero object.

Definition 1.3 Two matrix factorizations $a_1$ and $a_2$ of $W$ over $R$ are called strongly isomorphic if they are isomorphic in the category $\text{zmf}(R,W)$.

It is clear that two strongly isomorphic factorizations are also isomorphic in $\text{hmf}(R,W)$, but the converse need not hold.

Proposition 1.4 Let $a_1 = (R^{\rho_1, \rho_1}, D_1)$ and $a_2 = (R^{\rho_2, \rho_2}, D_2)$ be two matrix factorizations of $W$ over $R$, where $D_i = \begin{bmatrix} 0 & v_i \\ u_i & 0 \end{bmatrix}$. Then the following statements are equivalent:

(a) $a_1$ and $a_2$ are strongly isomorphic.

(b) $\rho_1 = \rho_2$ (a quantity which we denote by $\rho$) and there exist invertible matrices $A, B \in \text{GL}(\rho, R)$ such that one (and hence both) of the following equivalent conditions is satisfied:

1. $v_2 = Av_1B^{-1}$,
2. $u_2 = Bu_1A^{-1}$.

Proof. $a_1$ and $a_2$ are strongly isomorphic if and only if there exists $U \in \text{Hom}_{\text{zmf}(R,W)}^0(a_1, a_2)$ which is an isomorphism in $\text{zmf}(R,W)$. Since $U$ is an even morphism in the cocycle category, we have:

\[
UD_1 = D_2U.
\]

The condition that $U$ be even allows us to identify it with a matrix of the form $U = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, while invertibility of $U$ in $\text{zmf}(R,W)$ amounts to invertibility of the matrix $U$, which in turn means that $A$ and $B$ are square matrices (thus $\rho_1 = \rho_2 = \rho$) belonging to $\text{GL}(\rho, R)$. Thus relation (1.2) reduces to either of conditions 1. or 2., which are equivalent since $v_1u_1 = u_1v_1 = WI_{\rho}$ and $u_2v_2 = v_2u_2 = WI_{\rho}$. □
1.4. Critical divisors and the critical locus of $W$.

**Definition 1.5** A divisor $d$ of $W$ which is not a unit is called critical if $d^2|W$.

Let:

$$\mathcal{C}(W) \overset{\text{def}}{=} \{ d \in R \mid d^2|W \}$$

be the set of all critical divisors of $W$. The ideal:

$$\mathfrak{I}_W \overset{\text{def}}{=} \cap_{d \in \mathcal{C}(W)} \langle d \rangle$$

(1.3)

is called the critical ideal of $W$. Notice that $\mathfrak{I}_W$ consists of those elements of $R$ which are divisible by all critical divisors of $W$. In particular, we have $(W) \subset \mathfrak{I}_W$ and hence there exists a unital ring epimorphism $R/(W) \rightarrow R/\mathfrak{I}_W$.

**Definition 1.6** A critical prime divisor of $W$ is a prime element $p \in R$ such that $p^2|W$. The critical locus of $W$ is the subset of $\text{Spec}(R)$ consisting of the principal prime ideals of $R$ generated by the critical prime divisors of $W$:

$$\text{Crit}(W) \overset{\text{def}}{=} \{ (p) \in \text{Spec}(R) \mid p^2|W \} .$$

1.5. Critically-finite elements. Let $R$ be a Bézout domain. Then $R$ is a GCD domain, hence irreducible elements of $R$ are prime. This implies that any factorizable element of $R$ has a unique prime factorization up to association in divisibility.

**Definition 1.7** A non-zero non-unit $W$ of $R$ is called:

- non-critical, if $W$ has no critical divisors;
- critically-finite if it has a factorization of the form:

$$W = W_0W_c \text{ with } W_c = p_1^{n_1} \cdots p_N^{n_N} ,$$

(1.4)

where $n_j \geq 2$, $p_1, \ldots, p_N$ are critical prime divisors of $W$ (with $p_i \not\sim p_j$ for $i \neq j$) and $W_0$ is non-critical and coprime with $W_c$.

Notice that the elements $W_0$, $W_c$ and $p_i$ in the factorization (1.4) are determined by $W$ up to association, while the integers $n_i$ are uniquely determined by $W$. The factors $W_0$ and $W_c$ are called respectively the non-critical and critical parts of $W$. The integers $n_i \geq 2$ are called the orders of the critical prime divisors $p_i$.

For a critically-finite element $W$ with decomposition (1.4), we have:

$$\text{Crit}(W) = \{ (p_1), \ldots, (p_N) \} \text{ and } \mathfrak{I}_W = \langle W_{\text{red}} \rangle ,$$

where:

$$W_{\text{red}} \overset{\text{def}}{=} p_1^{\lfloor n_1 \rfloor} \cdots p_N^{\lfloor n_N \rfloor}$$

is called the reduction of $W$. Notice that $W_{\text{red}}$ is determined up to association in divisibility.

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2 I.e. an element of $R$ which has a finite factorization into irreducibles.

3 The notation $\lfloor x \rfloor \in \mathbb{Z}$ indicates the integral part of a real number $x \in \mathbb{R}$. 

Elementary matrix factorizations over Bézout domains
1.6. Two-step factorizations of W. Recall that a two-step factorization (or two-step multiplicative partition) of W is an ordered pair \((u, v) \in R \times R\) such that \(W = uv\). In this case, the divisors \(u\) and \(v\) are called W-conjugate. The transpose of \((u, v)\) is the ordered pair \((v, u)\) (which is again a two-step factorization of \(W\)), while the opposite transpose is the ordered pair \(\sigma(u, v) = (-v, -u)\).

This defines an involution \(\sigma\) of the set \(\text{MP}_2(W)\) of two-step factorizations of \(W\). The two-step factorizations \((u, v)\) and \((u', v')\) are called similar (and we write \((u, v) \sim (u', v')\)) if there exists \(\gamma \in U(R)\) such that \(u' = \gamma u\) and \(v' = \gamma^{-1} v\). We have \(\sigma(u, v) \sim (v, u)\).

**Definition 1.8** The support of a two-step factorization \((u, v)\) of \(W\) is the principal ideal \((u, v) \in G_+(R)\).

Let \(d\) be a gcd of \(u\) and \(v\). Since \(W = uv = d^2 u_1 v_1\) (where \(u_1 \overset{\text{def}}{=} u/d, v_1 \overset{\text{def}}{=} v/d\)), it is clear that \(d\) is a critical divisor of \(W\). Notice that the opposite transpose of the two step factorization \((u, v)\) has the same support as \((u, v)\).

1.7. Elementary matrix factorizations.

**Definition 1.9** A matrix factorization \(a = (M, D)\) of \(W\) over \(R\) is called elementary if it has unit reduced rank, i.e. if \(\rho(a) = 1\).

Any elementary factorization is strongly isomorphic to one of the form \(e_v \overset{\text{def}}{=} (R^{1|1}, D_v)\), where \(v\) is a divisor of \(W\) and \(D_v \overset{\text{def}}{=} \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}\), with \(u \overset{\text{def}}{=} W/v \in R\). Let \(\text{EF}(R, W)\) denote the full subcategory of \(\text{MF}(R, W)\) whose objects are the elementary factorizations of \(W\) over \(R\). Let \(\text{HEF}(R, W)\) denote respectively the cocycle and total cohomology categories of \(\text{EF}(R, W)\).

We also use the notations \(\text{zef}(R, W) \overset{\text{def}}{=} \text{ZEF}^0(R, W)\) and \(\text{hef}(R, W) \overset{\text{def}}{=} \text{HEF}^0(R, W)\). Notice that an elementary factorization is indecomposable in \(\text{zmf}(R, W)\), but it need not be indecomposable in the triangulated category \(\text{hmf}(R, W)\).

The map \(\Phi : \text{ObEF}(M, W) \to \text{MP}_2(W)\) which sends \(e_v\) to the ordered pair \((u, v)\) is a bijection. The suspension of \(e_v\) is given by \(\Sigma e_v = e_u = (R^{1|1}, D_{-u})\), since:

\[
D_{-u} = \begin{bmatrix} 0 & -u \\ -v & 0 \end{bmatrix}.
\]

In particular, \(\Sigma e_v\) corresponds to the opposite transpose \(\sigma(u, v)\) and we have:

\[
\Phi \circ \Sigma = \sigma \circ \Phi.
\]

Hence \(\Sigma\) preserves the subcategory \(\text{EF}(M, W)\) of \(\text{MF}(R, W)\) and the subcategories \(\text{HEF}(R, W)\) and \(\text{hef}(R, W)\) of \(\text{HMF}(R, W)\) and \(\text{hmf}(R, W)\). This implies that \(\text{HEF}(R, W)\) is equivalent with the graded completion \(\text{gr}_{\Sigma \text{hef}}(R, W)\). We thus have natural isomorphisms:

\[
\text{Hom}^i_{\text{HEF}(R, W)}(e_{v_1}, e_{v_2}) \cong R \text{Hom}_{\text{hef}(R, W)}(e_{v_1}, \Sigma e_{v_2}) = \text{Hom}_{\text{hef}(R, W)}(e_{v_1}, e_{-u_2}) ,
\]

\[
\text{Hom}^i_{\text{HEF}(R, W)}(e_{v_1}, e_{v_2}) \cong R \text{Hom}_{\text{hef}(R, W)}(\Sigma e_{v_1}, e_{v_2}) = \text{Hom}_{\text{hef}(R, W)}(e_{-u_1}, e_{v_2}) , \tag{1.5}
\]

for any divisors \(v_1, v_2\) of \(W\), where \(u_1 = W/v_1\) and \(u_2 = W/v_2\).
Definition 1.10 The support of an elementary matrix factorization $e_v$ is the ideal of $R$ defined through:

$$\text{supp} \,(e_v) \overset{\text{def.}}{=} \text{supp} \,(\Phi(e_v)) = \langle v, W/v \rangle.$$ 

Notice that this ideal is generated by any gcd $d$ of $v$ and $W/v$ and that $d$ is a critical divisor of $W$.

We will see later that an elementary factorization is trivial iff its support equals $R$.

Definition 1.11 Two elementary matrix factorizations $e_{v_1}$ and $e_{v_2}$ of $W$ are called similar if $v_1 \sim v_2$ or equivalently $u_1 \sim u_2$. This amounts to existence of a unit $\gamma \in U(R)$ such that $v_2 = \gamma v_1$ and $u_2 = \gamma^{-1} u_1$.

Proposition 1.12 Two elementary factorizations $e_{v_1}$ and $e_{v_2}$ are strongly isomorphic iff they are similar. In particular, strong isomorphism classes of elementary factorization are in bijection with the set of those principal ideals of $R$ which contain $W$.

Proof. Suppose that $e_{v_1}$ and $e_{v_2}$ are strongly isomorphic. By Proposition 1.4, there exist units $x, y \in U(R)$ such that $v_2 = xv_1y^{-1}$ and $u_2 = yu_1x^{-1}$, where $u_i \overset{\text{def.}}{=} W/v_i$. Setting $\gamma \overset{\text{def.}}{=} xy^{-1}$ gives $v_2 = \gamma v_1$ and $u_2 = \gamma^{-1} u_1$, hence $e_{v_1}$ and $e_{v_2}$ are similar. Conversely, suppose that $e_{v_1} \sim e_{v_2}$. Then there exists a unit $\gamma \in U(R)$ such that $v_2 = \gamma v_1$ and $u_2 = \gamma^{-1} u_1$. Setting $x = \gamma$ and $y = 1$ gives $v_2 = xv_1y^{-1}$ and $u_2 = yu_1x^{-1}$, which shows that $e_{v_1}$ and $e_{v_2}$ are strongly isomorphic upon using Proposition 1.4. The map which sends the strong isomorphism class of $e_v$ to the principal ideal $(v)$ gives the bijection stated. \qed

It is clear that $e_{v_1}$ and $e_{v_2}$ are similar iff the corresponding two-step factorizations $(v_1, u_1)$ and $(v_2, u_2)$ of $W$ are similar. Since any strong isomorphism induces an isomorphism in $\text{hef}(R, W)$, it follows that similar elementary factorizations are isomorphic in $\text{hef}(R, W)$.

1.8. The categories $\text{HEF}(R, W)$ and $\text{hef}(R, W)$. Let $\text{EF}(R, W)$ denote the smallest full $R$-linear subcategory of $\text{MF}(R, W)$ which contains all objects of $\text{EF}(R, W)$ and is closed under finite direct sums. It is clear that $\text{EF}(R, W)$ is a full dg subcategory of $\text{MF}(R, W)$. Let $\text{HEF}(R, W)$ denote the total cohomology category of $\text{EF}(R, W)$. Let $\text{hef}(R, W) \overset{\text{def.}}{=} \text{HEF}^{\text{def.}}(R, W)$ denote the subcategory obtained from $\text{HEF}(R, W)$ by keeping only the even morphisms. Notice that $\text{hef}(R, W)$ coincides with the smallest full subcategory of $\text{hmf}(R, W)$ which contains all elementary factorizations of $W$.

2. Elementary matrix factorizations over a Bézout domain

Throughout this section, let $R$ be a Bézout domain and $W$ be a non-zero element of $R$.

2.1. The subcategory of elementary factorizations. Let $v_1, v_2$ be divisors of $W$ and $e_1 \overset{\text{def.}}{=} e_{v_1}$, $e_2 \overset{\text{def.}}{=} e_{v_2}$ be the corresponding elementary matrix factorizations of $W$. Let $u_1 \overset{\text{def.}}{=} W/v_1$, $u_2 = W/v_2$. Let $a$ be a gcd of $v_1$ and $v_2$. Define:

$$b \overset{\text{def.}}{=} v_1/a \ , \ c \overset{\text{def.}}{=} v_2/a \ , \ d \overset{\text{def.}}{=} \frac{W}{abc} \ , \ d' \overset{\text{def.}}{=} a/s \ , \ d'' \overset{\text{def.}}{=} d/s \ ,$$

(2.1)
where $s$ is a gcd of $a$ and $d$. Then $a = a's$ and $d = d's$ with $(a', d') = (1) = (b, c)$ and $W = abcd = s^2a'bc'd'$. In particular, $s$ is a critical divisor of $W$. Moreover:

\[ v_1 = ab = sa'b \quad , \quad v_2 = ac = sa'c \quad , \quad u_1 = cd = scd' \quad , \quad u_2 = bd = sbd' \]  

(2.2)

and we have:

\[ (d) = (u_1, u_2) \quad , \quad (s) = (v_1, v_2, u_1, u_2) \]  

(2.3)

Notice the following relations in the cancellative monoid $R^\times / U(R)$:

\[ (v_1, v_2) = (sa') \quad , \quad (u_1, u_2) = (s)(d') \quad , \quad (u_1, v_1) = (s)(a', c)(b, d') \quad , \quad (u_1, v_2) = (s)(c) \quad , \quad (u_2, v_1) = (s)(b) \quad , \quad (u_2, v_2) = (s)(a', b)(c, d') \]  

(2.4)

In this notation:

\[ D_{v_1} = \begin{bmatrix} 0 & v_1 \\ u_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ cd & 0 \end{bmatrix} = s \begin{bmatrix} 0 & a'b \\ cd' & 0 \end{bmatrix} \quad \text{and} \quad D_{v_2} = \begin{bmatrix} 0 & v_2 \\ u_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ac \\ bd' & 0 \end{bmatrix} = s \begin{bmatrix} 0 & a'c \\ bd & 0 \end{bmatrix} \]  

For $f \in \text{Hom}_{MF(R,W)}^0(e_1, e_2) = \text{Hom}_{R}^0(R^{1|1}, R^{1|1})$ and $g \in \text{Hom}_{MF(R,W)}^1(e_1, e_2) = \text{Hom}_{R}^1(R^{1|1}, R^{1|1})$, we have:

\[ \varphi_{e_1,e_2}(f) = (cf_{11} - f_{00}b) \begin{bmatrix} 0 & a \\ d & 0 \end{bmatrix} \quad \text{and} \quad \varphi_{e_1,e_2}(g) = (ag_{01} + g_{10}d) \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \]  

(2.5)

**Remark 2.1.** Relations (2.1) and (2.4) imply the following equalities in the cancellative monoid $R^\times / U(R)$:

\[ (s) = \frac{(u_1, v_2)}{(u_1, v_2), \frac{v_2}{(v_1, v_2)}} = \frac{(u_2, v_1)}{(u_2, v_1), \frac{v_1}{(v_1, v_2)}} \]  

(2.6)

### 2.1.1. Morphisms in $\text{HEF}(R, W)$

Let $\text{Mat}(n, R^\times / U(R))$ denote the set of square matrices of size $n$ with entries from the multiplicative semigroup $R/U(R)$. Any matrix $S \in \text{Mat}(n, R^\times / U(R))$ can be viewed as an equivalence class of matrices $A \in \text{Mat}(n, R^\times)$ under the equivalence relation:

\[ A \sim_n B \quad \text{iff} \quad \forall i, j \in \{1, \ldots, n\} : \exists q_{ij} \in U(R) \text{ such that } B_{ij} = q_{ij}A_{ij} \]  

(2.7)

**Proposition 2.1** With the notations above, we have:

1. $\text{Hom}_{ZMF(R,W)}^0(e_1, e_2)$ is the free $R$-module of rank one generated by the matrix:

\[ \epsilon_0^0(v_1, v_2) \overset{\text{def}}{=} \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \in \begin{bmatrix} (v_2) & 0 \\ (v_1) & 0 \end{bmatrix} = \epsilon_0^0(v_1, v_2) \]

where the matrix $\epsilon_0^0(v_1, v_2) \in \text{Mat}(2, R/U(R))$ in the right hand side is viewed as an equivalence class under the relation (2.7).

2. $\text{Hom}_{ZMF(R,W)}^1(e_1, e_2)$ is the free $R$-module of rank one generated by the matrix:

\[ \epsilon_1^1(v_1, v_2; W) \overset{\text{def}}{=} \begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix} \in \begin{bmatrix} 0 & (v_2) \\ -(u_1) & 0 \end{bmatrix} = \epsilon_1^1(v_1, v_2; W) \]

and we have $\epsilon_1^1(v_1, v_2; W) = \epsilon_1^1(v_2, v_1; W)$ in $\text{Mat}(2, R/U(R))$. 
Proof. Relations (2.4) imply:
\[
\begin{align*}
\frac{(v_2)}{v_1, v_2} &= (c), & \frac{(v_1)}{v_1, v_2} &= (b), & \frac{(v_2)}{u_1, v_2} &= (v_1), & \frac{(v_2)}{u_2, v_1} &= (a'), & \frac{(u_1)}{u_1, v_2} &= (d'), & \frac{(u_2)}{u_2, v_1} &= (d') .
\end{align*}
\] (2.8)

These relations show that \(\epsilon(v_1, v_2)\) and \(\epsilon_1(v_1, v_2; W)\) belong to the equivalence classes \(\epsilon_0(v_1, v_2)\) and \(\epsilon_1(v_1, v_2; W)\) and that we have \(\epsilon_1(v_1, v_2; W) = \epsilon_1(v_2, v_1; W)\).

For an even morphism \(f : e_1 \to e_2\) in \(\text{MF}(R, W)\), the first equation in (2.5) shows that the condition \(\delta_{e_1, e_2}(f) = 0\) amounts to:
\[f_{11}c - f_{00}b = 0 .\]

Since \(b\) and \(c\) are coprime, this condition is equivalent with the existence of an element \(\gamma \in R\) such that \(f_{00} = \gamma c\) and \(f_{11} = \gamma b\). Thus:
\[f = \gamma \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} = \gamma \epsilon_0(v_1, v_2) .\] (2.9)

On the other hand, the second equation in (2.5) shows that an odd morphism \(g : e_1 \to e_2\) in \(\text{MF}(R, W)\) satisfies \(\delta_{e_1, e_2}(g) = 0\) iff:
\[ag_{01} + dg_{10} = 0 .\]

Since \(a'\) and \(d'\) are coprime, this condition is equivalent with the existence of an element \(\gamma \in R\) such that \(g_{10} = \gamma a'\) and \(g_{11} = -\gamma d'\). Thus:
\[g = \gamma \begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix} = \gamma \epsilon_1(v_1, v_2; W) .\] (2.10)

\(\square\)

**Proposition 2.2** Let \(v_i\) be as in Proposition 2.1. Then \(\text{Hom}^0_{\text{HMF}(R, W)}(e_1, e_2)\) and \(\text{Hom}^1_{\text{HMF}(R, W)}(e_1, e_2)\) are cyclically presented cyclic \(R\)-modules generated respectively by the matrices \(\epsilon_0(v_1, v_2)\) and \(\epsilon_1(v_1, v_2; W)\), whose annihilators are equal to each other and coincide with the following principal ideal of \(R\):
\[\alpha_W(v_1, v_2) \overset{\text{def}}{=} \langle v_1, u_1, v_2, u_2 \rangle = \langle s \rangle .\]

**Proof.** Let \(f \in \text{Hom}^0_{\text{ZMF}(R, W)}(e_1, e_2)\). Then \(f\) is exact iff there exists an odd morphism \(g \in \text{Hom}^1_{\text{MF}(R, W)}(e_1, e_2)\) such that:
\[f = \delta_{e_1, e_2}(g) = (ag_{01} + dg_{10}) \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} .\]

Comparing this with (2.9), we find that \(f\) is exact if and only if \(s \in (a, d)\) divides \(\gamma\). This implies that the principal ideal generated by the element:
\[s \in ((v_1, v_2), (u_1, u_2)) = (v_1, u_1, v_2, u_2)\]
is the annihilator of \(\text{Hom}^0_{\text{ZMF}(R, W)}(e_1, e_2)\).

On the other hand, an odd morphism \(g \in \text{Hom}^1_{\text{ZMF}(R, W)}(e_1, e_2)\) is exact iff there exists an even morphism \(f \in \text{Hom}^0_{\text{MF}(R, W)}(e_1, e_2)\) such that:
\[g = \delta_{e_1, e_2}(f) = (f_{11}c - f_{00}b) \begin{bmatrix} 0 & a \\ -d & 0 \end{bmatrix} .\]

Comparing with (2.10) and recalling that \((b, c) = (1)\), we find that \(g\) is exact iff \((a, d)|\gamma\). Hence the annihilator of \(\text{Hom}^1_{\text{HMF}(R, W)}(e_1, e_2)\) coincides with that of \(\text{Hom}^0_{\text{HMF}(R, W)}(e_1, e_2)\). \(\square\)
Remark 2.2. Since $s$ is a critical divisor of $W$, we have $\mathcal{I}_W \text{Hom}_{\text{HEF}(R,W)}(e_1,e_2) = 0$, where $\mathcal{I}_W$ denotes the critical ideal of $W$ defined in (1.3). In particular, $\text{HEF}(R,W)$ can be viewed as an $R/\mathcal{I}_W$-linear category.

Let $\text{Div}(W) = \{d \in R \mid d|W\}$ and consider the function $\alpha_W : \text{Div}(W) \times \text{Div}(W) \to G_+(R)$ defined in Proposition 2.2. This function is symmetric since $\alpha_W(v_1,v_2) = \alpha_W(v_2,v_1)$. Let $1_{G(R)} = (1) = R$ denote the neutral element of the group of divisibility $G(R)$, whose group operation we write multiplicatively.

**Proposition 2.3** The symmetric function $\alpha_W(v_1,v_2)$ is multiplicative with respect to each of its arguments in the following sense:

- For any two relatively prime elements $v_2$ and $\tilde{v}_2$ of $R$ such that $v_2 \tilde{v}_2$ is a divisor of $W$, we have:
  $$\alpha_W(v_1,v_2 \tilde{v}_2) = \alpha_W(v_1,v_2)\alpha_W(v_1,\tilde{v}_2)$$  \tag{2.11}

  and $\alpha_W(v_1,v_2) + \alpha_W(v_1,\tilde{v}_2) = 1_{G(R)}$, where $+$ denotes the sum of ideals of $R$.

- For any two relatively prime elements $v_1$ and $\tilde{v}_1$ of $R$ such that $v_1 \tilde{v}_1$ is a divisor of $W$, we have:
  $$\alpha_W(v_1 \tilde{v}_1,v_2) = \alpha_W(v_1,v_2)\alpha_W(\tilde{v}_1,v_2)$$  \tag{2.12}

  and $\alpha_W(v_1,v_2) + \alpha_W(\tilde{v}_1,v_2) = 1_{G(R)}$, where $+$ denotes the sum of ideals of $R$.

**Proof.** To prove the first statement, we start from relation (2.6), which allows us to write:

$$\alpha_W(v_1,v_2 \tilde{v}_2) = \left(\frac{(u_1,v_2 \tilde{v}_2)}{(u_1,v_2 \tilde{v}_2,(v_2 \tilde{v}_2),(v_1,v_2)}\right),$$ \tag{2.13}

where $u_1 = W/v_1$. Recall that the function $(-,r)$ is multiplicative on relatively prime elements for any $r \in R^\times$, i.e. $(xy,r) = (x,r)(y,r)$. Thus:

$$(u_1,v_2 \tilde{v}_2) = (u_1,v_2)(u_1,\tilde{v}_2), \quad (v_1,v_2 \tilde{v}_2) = (v_1,v_2)(v_1,\tilde{v}_2).$$ \tag{2.14}

The second of these relations gives $\frac{(v_2 \tilde{v}_2)}{(v_1,v_2 \tilde{v}_2)} = \frac{(v_2)}{(v_1,v_2)} \frac{(\tilde{v}_2)}{(v_1,v_2)}$, Notice that $\frac{(v_2)}{(v_1,v_2)}, \frac{(\tilde{v}_2)}{(v_1,v_2)} = (1)_{G(R)}$ since $v_2$ and $\tilde{v}_2$ are coprime. Hence:

$$\left(\frac{(u_1,v_2 \tilde{v}_2)}{(u_1,v_2 \tilde{v}_2,(v_2 \tilde{v}_2),(v_1,v_2)}\right) = \left(\frac{(u_1,v_2 \tilde{v}_2)}{(u_1,v_2 \tilde{v}_2,(v_2 \tilde{v}_2),(v_1,v_2)}\right) = \left(\frac{(u_1,v_2 \tilde{v}_2)}{(v_2 \tilde{v}_2),(v_1,v_2)}\right)\left(\frac{(u_1,v_2 \tilde{v}_2)}{(v_1,v_2)}\right),$$

where in the last equality we used the first relation in (2.14) and noticed that $(u_1,v_2)$ and $(u_1,\tilde{v}_2)$ are coprime (since $(v_2,\tilde{v}_2) = (1)$), which allows us to use similar-multiplicativity of the function $(-,r)$ for $(r) = \frac{(v_2 \tilde{v}_2)}{(v_1,v_2 \tilde{v}_2)}$ and for $(r) = \frac{(\tilde{v}_2)}{(v_1,v_2)}$. Since $(v_2,\tilde{v}_2) = (1)$, we have $\left(\frac{(u_1,\tilde{v}_2)}{(v_1,\tilde{v}_2)}\right) = (1)_{G(R)}$.

Thus:

$$\left(\frac{(u_1,v_2 \tilde{v}_2)}{(v_1,v_2,\tilde{v}_2)}\right) = \left(\frac{(v_2)}{(v_1,v_2)}\right)\left(\frac{(u_1,v_2 \tilde{v}_2)}{(v_1,v_2)}\right)\left(\frac{(\tilde{v}_2)}{(v_1,v_2)}\right).$$

Using this and the first equation of (2.14) in the expression (2.13) gives relation (2.11). The second statement now follows from the first by symmetry of $\alpha_W$. \qed
2.1.2. Isomorphisms in HEF($R, W$).

We start with a few lemmas.

**Lemma 2.1.** Let $s, x, y, z$ be four elements of $R$. Then the equation:

$$s(g_1x + g_2y) + g_3z = 1 \tag{2.15}$$

has a solution $(g_1, g_2, g_3) \in R^3$ iff $(s(x, y), z) = (1)$.

**Proof.** Let $t$ be a gcd of $x$ and $y$. We treat each implication in turn:

1. Assume that $(g_1, g_2, g_3) \in R^3$ is a solution. Then $t$ divides $g_1x + g_2y$, so there exists $g_4 \in R$ such that $g_1x + g_2y = g_4t$. Multiplying both sides with $s$ and using (2.15), this gives $stg_4 + g_3z = 1$, which implies $(st, z) = (1)$.
2. Assume that $(st, z) = (1)$. Then there exist $g_3, g_4 \in R$ such that:

$$stg_4 + g_3z = 1 \ . \tag{2.16}$$

Since $(t) = (x, y)$, the Bézout identity shows that there exist $\tilde{g}_1, \tilde{g}_2 \in R$ such that $\tilde{g}_1x + \tilde{g}_2y = t$. Substituting this into (2.16) shows that $(g_1, g_2, g_3)$ satisfies (2.15), where $g_1 \overset{\text{def.}}{=} \tilde{g}_1g_4$ and $g_2 = \tilde{g}_2g_4$. \hfill $\square$

**Lemma 2.2.** Let $s, a', b, c, d'$ be five elements of $R$ such that $(a', d') = (1)$. Then the system of equations:

$$\begin{align*}
bcg - s(a'bg_1 + cd'g_2) &= 1 \\
bcg - s(a'ch_1 + bd'h_2) &= 1
\end{align*} \tag{2.17}$$

has a solution $(g, g_1, g_2, h_1, h_2) \in R^5$ iff $a', b, c, d'$ are pairwise coprime and $(bc, s) = (1)$.

**Proof.** Consider the two implications in turn.

1. Assume that (2.17) has a solution $(g, g_1, g_2, h_1, h_2) \in R^5$. By Lemma 2.1, we must have $(bc, s(a'bg_1 + cd'g_2)) = (1)$ and $(bc, s(a'ch_1 + bd'h_2)) = (1)$. This implies $(bc, s) = (1)$ and $(b, c) = (1)$. If a prime element $p \in R$ divides $(a'bg_1 + cd'g_2)$, then it divides both $a'bg$ and $cd'g_2$, hence $p|c$ or $p|b$ since $(a', d') = (1)$. Thus $p|bc$, which contradicts the fact that $bc$ and $s(a'bg_1 + cd'g_2)$ are coprime. It follows that we must have $(a'bg_1 + cd'g_2) = (1)$. Similarly, the second equation implies that we must have $(a'ch_1 + bd'h_2) = (1)$. Since $(a', d') = (1)$ and $(b, c) = (1)$, the last two conditions imply that $a', b, c, d'$ must be pairwise coprime.
2. Conversely, assume that $a', b, c, d'$ are pairwise coprime and $(bc, s) = (1)$. Following the strategy and notations of the previous lemma, we first solve the equation $bcg - sg_4 = 1$ for $g$ and $g_4$ using the Bézout identity. Using the same identity, we solve the system:

$$\begin{align*}
a'b\tilde{g}_1 + cd'\tilde{g}_2 &= 1 \\
a'c\tilde{h}_1 + bd'\tilde{h}_2 &= 1
\end{align*} \tag{2.18}$$

obtaining the solution $(g, g_4\tilde{g}_1, g_3\tilde{g}_2, g_4\tilde{h}_1, g_4\tilde{h}_2)$ of (2.17). \hfill $\square$
Proposition 2.4 With the notations (2.1), we have:
1. $e_1$ and $e_2$ are isomorphic in $\text{hef}(R, W)$ iff $a', b, c, d'$ are pairwise coprime and $(bc, s) = (1)$.
2. An odd isomorphism between $e_1$ and $e_2$ in $\text{HEF}(R, W)$ exists iff $a', b, c, d'$ are pairwise coprime and $(a'd', s) = (1)$.

Proof. 1. Proposition 2.1 gives:

$$\text{Hom}_{\text{ZEF}(R, W)}(e_1, e_2) = R\begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \quad \text{and} \quad \text{Hom}_{\text{ZEF}(R, W)}(e_2, e_1) = R\begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}.$$ 

Two non-zero morphisms $f_{12} = \alpha \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \in \text{Hom}_{\text{ZEF}(R, W)}(e_1, e_2)$ and $f_{21} = \beta \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \in \text{Hom}_{\text{ZEF}(R, W)}(e_2, e_1)$ (where $\alpha, \beta \in R^\times$) induce mutually inverse isomorphisms in $\text{hef}(R, W)$ iff:

$$f_{21}f_{12} = 1 + \delta_{e_1, e_1}(g) \quad f_{12}f_{21} = 1 + \delta_{e_2, e_2}(h)$$

for some $g, h \in \text{End}_{R}^1(R^{[1]}).$ These conditions read:

$$\begin{bmatrix} a\beta bc - s(a'b\delta_1 + g_{10} cd') & 1 \\ a\beta bc - s(a'c\delta_1 + h_{10} bd') & 1 \end{bmatrix} = 1.$$ (2.19)

Since this system has the form (2.17), Lemma 2.2 shows that it has solutions iff $a', b, c, d'$ are pairwise coprime and $(bc, s) = (1)$.

2. Proposition 2.1 gives $\text{Hom}_{\text{ZEF}(R, W)}^1(e_1, e_2) = \text{Hom}_{\text{ZEF}(R, W)}^1(e_2, e_1) = R\begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix}.$ Two non-zero odd morphisms $g_{12} = \alpha \begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix} \in \text{Hom}_{\text{ZEF}(R, W)}^1(e_1, e_2)$ and $g_{21} = \beta \begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix} \in \text{Hom}_{\text{ZEF}(R, W)}^1(e_2, e_1)$ (with $\alpha, \beta \in R^\times$) induce mutually inverse isomorphisms in $\text{HEF}(R, W)$ iff:

$$g_{21}g_{12} = 1 + \delta_{e_1, e_1}(f) \quad g_{12}g_{21} = 1 + \delta_{e_2, e_2}(q)$$

for some $f, q \in \text{End}_{R}^0(R^{[1]}).$ This gives the equations:

$$\begin{bmatrix} a\beta a'd' - s(a'c f_0 + f_1 bd) & 1 \\ 0 & 0 \end{bmatrix} = 1$$

which amount to the system:

$$\begin{cases} a\beta a'd' - s(a'c f_0 + f_1 bd) = 1 \\ a\beta a'd' - s(a'q_{10} + q_{11} cd') = 1 \end{cases}.$$ (2.20)

This system again has the form (2.17), as can be seen by the substitution of the quadruples $(b, c, a', d') := (a', d', b, c).$ As a consequence, it has a solution iff $a', b, c, d'$ are pairwise coprime and $(a'd', s) = (1).$  \[ \square \]
Proposition 2.7 Let $m$ be a morphism. $Hom_m x = y$, and let $v$ and $u$ be modules $Hom_v x = y$. The identity endomorphism of $u$ can be transported to its suspension: $c = u W/v$.

Proof. Let $s = (u, v)$. The isomorphism follows from Proposition 2.4 for $d = b = v/s$, $c = -u/s$.

\[
\begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & bs \\ cs & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & cs \\ bs & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix},
\]

since $(ad, s) = (1)$. □

Remark 2.3. An odd isomorphism in $HEF(R, W)$ between $e_v$ and $\Sigma e_v e = -u$ can also be obtained more abstractly by transporting the identity endomorphism of $e_v$ through the isomorphism of $R$-modules $Hom^1(e_v, e = -u) \simeq Hom^1(e_v, \Sigma e_v)$ which results by taking $v_1 = v$ and $v_2 = -u$ in the first line of (1.5). Since $e = -u$ is similar to $e$, the Proposition implies that $e_v$ and $e_u$ are oddly isomorphic. When $v = 1$, both $e_v = e_1$ and $e_u = e_W$ are zero objects and we have $Hom_{HMF(R,W)}(e_1, e_W) = Hom_{HMF(R,W)}(e_1, e_W) = \{0\}$, so the odd isomorphism is the zero morphism.

Proposition 2.8. Let $W = W_1 W_2$ with $(W_1, W_2) = (1)$ and let $v$ be a divisor of $W_1$. Then $e_{W_2 v} \simeq HMF(R, W) e_v$.

Proof. Let $u_0 \overset{\text{def}}{=} \frac{W_1}{v}$. Setting $v_1 = v$, $u_1 = \frac{W}{v} = W_2 u_0$, $v_2 = W_2 v$ and $u_2 = \frac{W}{u_2} = u_0$, we compute:

\[
a \in (v_1, v_2) = (v), \quad b = \frac{v_1}{a} \in (1), \quad c = \frac{v_2}{a} \in (W_2), \quad d \in \frac{W}{v_1, v_2} = \left(\frac{W_1}{v}\right),
\]

\[
s \in (u_1, v_1, u_2, v_2) = (u_0, v), \quad a' = a/s \in \frac{v}{(u_0, v)}, \quad d' = d/s \in \frac{W_1}{(v)(u_0, v)} = \frac{u_0}{(u_0, v)}.
\]

It is clear that $a', b, c, d'$ are mutually coprime and that $(s, bc) = (1)$. □

2.1.3. The composition of morphisms in $HEF(R, W)$.

Proposition 2.8. Given three divisors $v_1$, $v_2$ and $v_3$ of $W$, we have the following relations:

\[
\begin{align*}
\epsilon_0(v_2, v_3)\epsilon_0(v_1, v_2) & = \frac{(v_2)(v_1, v_3)}{(v_1, v_2)(v_2, v_3)}\epsilon_0(v_1, v_3) \\
\epsilon_0(v_2, v_3)\epsilon_1(v_1, v_2; W) & = \frac{(v_2)(u_1, v_3)}{(u_1, v_2)(v_2, v_3)}\epsilon_1(v_1, v_3; W) \\
\epsilon_1(v_2, v_3; W)\epsilon_0(v_1, v_2) & = \frac{(v_2)(v_1, u_3)}{(u_1, v_2)(u_2, v_3)}\epsilon_1(v_3, v_1; W) \\
\epsilon_1(v_2, v_3; W)\epsilon_1(v_1, v_2; W) & = -\frac{(u_1)(v_1, v_3)}{(u_2, v_3)(u_1, v_2)}\epsilon_0(v_1, v_3).
\end{align*}
\]
Proof. Given three divisors \( v_1, v_2 \) and \( v_3 \) of \( W \), we have:

\[
\epsilon_0(v_2, v_3) \epsilon_0(v_1, v_2) = \begin{bmatrix}
\frac{v_2 v_3}{(v_1, v_2^2)(v_2, v_3)} & 0 \\
0 & \frac{v_1 v_2}{(v_1, v_2)(v_2, v_3)}
\end{bmatrix} = \frac{v_2(v_1, v_3)}{(v_1, v_2)(v_2, v_3)} \epsilon_0(v_1, v_3)
\]

where we used the identity:

\[
[a, b, c](a, b, c)(c, a) = (a)(b)(c)(a, b, c)
\]

This establishes the first of equations (2.21). The remaining equations follow similarly. \( \square \)

**Corollary 2.9** Let \( v \) be a divisor of \( W \) and \( u = W/v \). Then:

1. The \( R \)-algebra \( \text{End}_{ZMF}(R,W)(e_v) \) is isomorphic with \( R \).
2. We have an isomorphism of \( \mathbb{Z}_2 \)-graded \( R \)-algebras:

\[
\text{End}_{ZMF}(R,W)(e_v) \cong \frac{R[\omega]}{(u^2 + t)},
\]

where \( \omega \) is an odd generator and \( t \in \frac{[u,v]}{[u,v]} \). In particular, \( \text{End}_{ZMF}(R,W)(e_v) \) is a commutative \( \mathbb{Z}_2 \)-graded ring.

**Proof.** For \( v_1 = v_2 = v \), we have \( \alpha_W(v,v) = \langle u, v \rangle \). Proposition 2.1 gives:

\[
\epsilon_0(v,v) \overset{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \epsilon_1(v,v) \overset{\text{def}}{=} \begin{bmatrix} 0 & \langle v \rangle \\ \langle u, v \rangle & 0 \end{bmatrix}
\]

and we have:

\[
\epsilon_0(v,v)^2 = \epsilon_0(v,v)
\]

\[
\epsilon_0(v,v) \epsilon_1(v,v; W) = \epsilon_1(v,v; W) \epsilon_0(v,v) = \epsilon_1(v,v)
\]

\[
\epsilon_1(v,v; W)^2 = -\left[ \frac{v}{u, v} \right] \epsilon_0(v,v)
\]

which also follows from Proposition 2.8. Setting \( \omega = e_1(v,v; W) \), these relations imply the desired statements upon using Proposition 2.1. \( \square \)

**Corollary 2.10** Let \( v \) be a divisor of \( W \) and \( u = W/v \). Then:

1. The \( R \)-algebra \( \text{End}_{HMF}(R,W)(e_v) \) is isomorphic with \( R/\langle d \rangle = R/\langle u,v \rangle \), where \( d \in \langle u,v \rangle \).
2. We have an isomorphism of \( \mathbb{Z}_2 \)-graded \( R \)-algebras:

\[
\text{End}_{HMF}(R,W)(e_v) \cong \frac{(R/\langle d \rangle)[\omega]}{(u^2 + t)},
\]

where \( \omega \) is an odd generator, \( d \in \langle u,v \rangle \) and \( t \in \frac{[u,v]}{[u,v]} \). In particular, \( \text{End}_{ZMF}(R,W)(e_v) \) is a supercommutative \( \mathbb{Z}_2 \)-graded ring.

**Proof.** The same relations as in the previous Corollary imply the conclusion upon using Proposition 2.2. \( \square \)

**Corollary 2.11** An elementary matrix factorization \( e_v \) is a zero object of \( hmf(R,W) \) iff \( (u,v) = (1) \), where \( u = W/v \).

**Proof.** The \( R \)-algebra \( \text{End}_{HMF}(R,W)(e_v) \cong R/t \) vanishes iff \( (u,v) = (1) \). \( \square \)
2.2. Localizations. Let $S \subset R$ be a multiplicative subset of $R$ containing the identity $1 \in R$ and $\lambda_S : R \to R_S$ denote the natural ring morphism from $R$ to the localization $R_S = S^{-1}R$ of $R$ at $S$. For any $r \in R$, let $r_S \defeq \lambda_S(r) = \frac{r}{1} \in R_S$ denote its extension. For any $R$-module $N$, let $N_S = S^{-1}N = N \otimes_R R_S$ denote the localization of $N$ at $S$. For any morphism of $R$-modules $f : N \to N'$, let $f_S \defeq f \otimes_R \text{id}_{R_S} : N_S \to N'_S$ denote the localization of $f$ at $S$. For any $\mathbb{Z}_2$-graded $R$-module $M = M^0 \oplus M^1$, we have $M_S = M_S^0 \oplus M_S^1$, since the localization functor is exact. In particular, localization at $S$ induces a functor from the category of $\mathbb{Z}_2$-graded $R$-modules to the category of $\mathbb{Z}_2$-graded $R_S$-modules.

Let $a = (M, D)$ be a matrix factorization of $W$. The localization of $a$ at $S$ is the following matrix factorization of $W_S$ over the ring $R_S$:

$$a_S \defeq (M_S, D_S) \in \text{MF}(R_S, W_S).$$

It is clear that this extends to an even dg functor $\text{loc}_S : \text{MF}(R, W) \to \text{MF}(R_S, W_S)$, which is $R$-linear and preserves direct sums. In turn, this induces dg functors $\text{ZMF}(R, W) \to \text{ZMF}(R_S, W_S)$, $\text{BMF}(R, W) \to \text{BMF}(R_S, W_S)$, $\text{HMF}(R, W) \to \text{HMF}(R_S, W_S)$ and $\text{hmf}(R, W) \to \text{hmf}(R_S, W_S)$, which we again denote by $\text{loc}_S$. We have $\text{loc}_S(a) = a_S$ for any matrix factorization $a$ of $W$ over $R$.

**Proposition 2.12** The functor $\text{loc}_S : \text{hmf}(R, W) \to \text{hmf}(R_S, W_S)$ is a triangulated functor. Moreover, the strictly full subcategory of $\text{hmf}(R, W)$ defined through:

$$K_S \defeq \{ a \in \text{Ob}[\text{hmf}(R, W)] \mid a_S \simeq_{\text{hmf}(R_S, W_S)} 0 \}$$

is a triangulated subcategory of $\text{hmf}(R, W)$.

**Proof.** It is clear that $\text{loc}_S$ commutes with the cone construction (see [6] for a detailed account of the latter). It is also clear that the subcategory $K_S$ is closed under shifts. Since any distinguished triangle in which two objects vanish has the property that its third object also vanishes, $K_S$ is also closed under forming triangles. □

**Proposition 2.13** For any matrix factorizations $a, b$ of $W$, there exists a natural isomorphism of $\mathbb{Z}_2$-graded $R_S$-modules:

$$\text{Hom}_{\text{HMF}(R_S, W_S)}(a_S, b_S) \simeq_{R_S} \text{Hom}_{\text{HMF}(R, W)}(a, b)_S.$$

**Proof.** Follows immediately from the fact that localization at $S$ is an exact functor from $\text{Mod}_R$ to $\text{Mod}_{R_S}$. □

2.3. Behavior of $\text{hef}(R, W)$ under localization.

**Lemma 2.14** The following statements are equivalent for any elements $s, r$ of $R$:

1. $(s, r) = (1)$
2. The class of $s$ modulo the ideal $\langle r \rangle$ is a unit of the ring $R/\langle r \rangle$.

**Proof.** We have $(s, r) = (1)$ iff there exist elements $a, b \in R$ such that $as + br = 1$. In turn, this is equivalent with the condition $\bar{a}s = \bar{1}$ in the ring $R/\langle r \rangle$, where $\bar{x} = x + \langle r \rangle$ denotes the equivalence class of an element $x \in R$ modulo the ideal $\langle r \rangle$. □
Consider the multiplicative set:
\[ S_W \overset{\text{def.}}{=} \{ s \in R \mid (s, W) = (1) \} . \]
Since \( 0 \notin S_W \), the localization \( R_S = S^{-1}R \) of \( R \) at any multiplicative set \( S \subset S_W \) is a sub-ring of the field of fractions \( K \) of \( R \):
\[ R_S = \{ \frac{r}{s} \mid r \in R, s \in S \} \subset K . \]
In particular, \( R_S \) is an integral domain.

**Proposition 2.15** Let \( S \) be any multiplicative subset of \( R \) such that \( S \subset S_W \). Then the localization functor \( \text{loc}_S : \text{hmf}(R, W) \to \text{hmf}(R_S, W_S) \) restricts to an \( R \)-linear equivalence of categories between \( \text{hmf}(R, W) \) and \( \text{hmf}(R_S, W_S) \).

**Proof.** Since \( \text{loc}_S \) preserves the reduced rank of matrix factorizations, it is clear that it restricts to a functor from \( \text{hmf}(R, W) \) to \( \text{hmf}(R_S, W_S) \). Given two elementary factorizations \( e_{v_1}, e_{v_2} \in \text{Ob}[\text{hmf}(R, W)] \), let \( r \in (v_1, v_2, W/v_1, W/v_2) \). By Proposition 2.13, we have:
\[ \text{Hom}_{\text{hmf}(R, W)}(e_{v_1}, S) \simeq \text{Hom}_{\text{hmf}(R_S, W_S)}(e_{v_1}, e_{v_2})S . \quad (2.23) \]
Let \( s \) be any element of \( S \). Since \( S \) is a subset of \( S_W \), we have \( (s, W) = (1) \) and hence \( (s, r) = (1) \) since \( r \) is a divisor of \( W \). By Lemma 2.14, the image \( \bar{s} = s + (r) \) is a unit of the quotient ring \( R/\langle r \rangle \), hence the operator of multiplication with \( s \) is an isomorphism of the cyclic \( R \)-module \( \text{Hom}_{\text{hmf}(R, W)}(e_{v_1}, e_{v_2}) \simeq R/\langle r \rangle \). Thus every element of \( S \) acts as an automorphism of this module, which implies that the localization map \( \text{Hom}_{\text{hmf}(R, W)}(e_{v_1}, e_{v_2}) \to \text{Hom}_{\text{hmf}(R_S, W_S)}(e_{v_1}, e_{v_2})S \) is an isomorphism of \( R \)-modules (where \( \text{Hom}_{\text{hmf}(R, W)}(e_{v_1}, e_{v_2})S \) is viewed as an \( R \)-module by the extension of scalars \( R \to R_S \)). Combining this with (2.23) shows that the restriction \( \text{loc}_S : \text{hmf}(R, W) \to \text{hmf}(R_S, W_S) \) is a full and faithful functor.

Now let \( e_x \) be an elementary factorization of \( W_S \) corresponding to the divisor \( x \) of \( W_S = W/1 \) in the ring \( R_S \). Let \( y = W_S/x \in R_S \). Write \( x = v/s \) and \( y = u/t \) with \( x, y \in R \) and \( s, t \in S \) chosen such that \( (v, s) = (u, t) = (1) \). Then the relation \( xy = W_S \) amounts to \( uv = stW \). Since \( S \) is a subset of \( S_W \), we have \( (s, W) = (t, W) = (1) \). Thus \( st \mid uv \), which implies \( s \mid u \) and \( t \mid u \) since \( (v, s) = (u, t) = (1) \). Thus \( v = v_1 t \) and \( u = u_1 s \) with \( u_1, v_1 \in R \) and we have \( u_1 v_1 = W \). This gives \( x = \gamma v_1 \) and \( y = \gamma^{-1} u_1 \), where \( \gamma \overset{\text{def.}}{=} t/s \) is a unit of \( R_S \). It follows that \( e_x \) is similar to the elementary matrix factorization \( e_{v_1} \) of \( W_S \) over \( R_S \), and hence isomorphic to the latter in the category \( \text{hmf}(R_S, W_S) \) by Proposition 2.5. Since \( u_1 \) and \( v_1 \) are divisors of \( W \) satisfying \( u_1 v_1 = W \), we can view \( e_{v_1} \) as an elementary factorization of \( W \) over \( R \) (it lies in the image of the functor \( \text{loc}_S \)). This shows that any object of \( \text{hmf}(R_S, W_S) \) is even-isomorphic with an object lying in the image of the restricted localization functor, hence the latter is essentially surjective. \( \square \)

### 2.4. Behavior of HEF(R, W) under multiplicative partition of W
For any divisor \( W_1 \) of \( W \), let \( \text{HEF}_{W_1}(R, W) \) denote the full subcategory of \( \text{HEF}(R, W) \) whose objects are those elementary factorization \( e_v \) of \( W \) for which \( v \) is a divisor of \( W_1 \).

**Proposition 2.16** Let \( e_1 \) and \( e_2 \) be as above. Consider elements of \( R \) chosen as follows:
\[ s_1 \in (u_1, v_1) = (s)(a', c)(b, d'), \quad s_2 \in (u_2, v_2) = (s)(a', b)(c, d') , \]
\[ u'_1 \overset{\text{def.}}{=} u_1/s = cd', \quad u'_2 \overset{\text{def.}}{=} u_2/s = bd' , \quad v'_1 \overset{\text{def.}}{=} v_1/s = a'b , \quad v'_2 \overset{\text{def.}}{=} v_2/s = a'c , \]
\[ x(e_1) \in (s, v'_1) = (s, a'c) , \quad y(e_1) \in (s, u'_1) = (s, cd') , \]
\[ x(e_2) \in (s, v'_2) = (s, a'c) , \quad y(e_2) \in (s, u'_2) = (s, bd') . \quad (2.24) \]
Then:

1. $e_1$ and $e_2$ are isomorphic in $\operatorname{hcf}(R,W)$ iff:
   
   $$(i) \quad (s_1) = (s_2) \quad \text{and} \quad ((x(e_1)), (y(e_1))) = ((x(e_2)), (y(e_2))) \quad \text{as ordered pairs of elements in } R^\times/U(R).$$

2. $e_1$ and $e_2$ are isomorphic in $\operatorname{HEF}(R,W)$ iff:

   $$(i) \quad (s_1) = (s_2) \quad \text{and} \quad \{(x(e_1)), (y(e_1))\} = \{(x(e_2)), (y(e_2))\} \quad \text{as unordered pairs of elements in } R^\times/U(R).$$

Notice that $(s_1) = (s_2)$ implies $(s_1) = (s) = (s_2)$, with $s$ defined in (2.3).

Proof.

1. Assume that $e_1 \simeq_{\operatorname{hcf}(R,W)} e_2$. By Proposition 2.4, part 1, to such pair of elementary factorizations we can associate four pairwise coprime divisors $a', b, c, d'$ of $W$ such that $v_1 = a'b$, $u_1 = d'cs$, $v_2 = d'cs$, $u_2 = d'bs$ and together with the equality $(bc, s) = (1)$. Thus $(s_1) = (v_1, u_1) = (a'b, d'cs) = (s)$, since $a'b$ and $d'c$ are coprime. Similarly, $(s_2) = (s)$. The equality $(bc, s) = (1)$ is equivalent to $(b, s) = (1)$ and $(c, s) = (1)$. Using this, we compute:

   $$(x(e_1)) = (s, v_1) = (s, a'b) = (s, a') = (s, a'c) = (s, v_2') = (x(e_2))$$

(2.27)

Acting similarly, we also find $(y(e_1)) = (s, d') = (y(e_2))$. Thus (2.25) holds.

Now assume that (2.25) is satisfied for two elementary factorizations $e_1$ and $e_2$. Let $s \in (s_1) = (s_2)$ and define $a', b, c, d'$ as before, following (2.1). By the very construction, $(b, c) = (1)$ and $(a', d') = (1)$. We first show that $s_1 \sim s_2$ all $a', b, c, d'$ are pairwise coprime. Indeed, if we assume that $p|(a', b)$ then $s_1 \sim s_2$ implies:

$$(s)(a', c)(b, d') = (s_1) = (s_2) = (s)(a', b)(c, d')$$

Since $p$ divides the right hand side, it should divide $(a', c)(b, d')$ and $(c, b) = (1)$ and $(a', d') = (1)$. Thus $p \in U(R)$. In much the same way we prove that other pairs from $a', b, c, d'$ are coprime.

Condition (ii) in (2.25) reads:

$$(s, a'b) = (x(e_1)) = (x(e_2)) = (s, a'c)$$

If $p|b$ and $p|s$ then $p|(s, b)$ and thus $p|(s, a'b)$. By the equality above, we also have $p|(s, a'c)$ and hence $p|a'c$. But $b$ is coprime with both $a'$ and $c$, thus $p \in U(R)$. Similarly, $p|c$ and $p|s$ implies $p \in U(R)$. Thus $(bc, s) = (1)$. Note that $(y(e_1)) = (y(e_2))$ is now automatically satisfied. Proposition 2.4, part 1 implies that $e_1 \simeq_{\operatorname{hcf}(R,W)} e_2$.

2. Assume $e_1 \simeq_{\operatorname{HEF}(R,W)} e_2$. If the isomorphism is even, then it comes from the isomorphism in $\operatorname{hcf}(R,W)$ and part 1 above already proves that (2.25) and thus also (2.26). Thus we can assume that the isomorphism is odd. We will prove that $(s_1) = (s_2)$ and $(x(e_1)) = (y(e_2))$, $(x(e_2)) = (y(e_1))$. Applying Proposition 2.4, part 2, we obtain $a', b, c, d'$ pairwise coprime and $s$ such that $(s, a'd') = (1)$. Then $(s_1) = (s) = (s_2)$ similarly to part 1 above. Using $(s, a'd') = (1)$, we also compute:

$$(x(e_1)) = (s, v_1) = (s, a'b) = (s, b) = (s, d'b) = (s, u_2') = (y(e_2))$$
and also \((x(e_2)) = (y(e_1))\). Thus (2.26).

Assume now that (2.26) is satisfied. Since the statement for even morphisms is covered by (2.25), we only need to consider the situation \((x(e_1)) = (y(e_2))\) and \((x(e_2)) = (y(e_1))\). As in part 1, \((s_1) = (s_2)\) implies that \(a', b, c, d'\) are pairwise coprime. Condition (ii) reads:

\[
(s, a'b) = (x(e_1)) = (y(e_2)) = (s, d'b) \ .
\]

If we assume that \(p | a'\) and \(p | s\) then \(p | (s, a')\) and \(p | (s, a'b)\). The equality implies \((p | d'b)\). Since \(a'\) is coprime with both \(d'\) and \(b\), we obtain \(p \in U(R)\). Similarly \((d', s) = (1)\) and thus \((a'd', s) = (1)\). Proposition 2.4, part 2 implies that \(e_1 \simeq_{\text{HEF}(E,W)} e_2\) by an odd isomorphism.

\[
\Box
\]

**Proposition 2.17** Let \(W_1\) and \(W_2\) be divisors of \(W\) such that \(W = W_1 W_2\) and \((W_1, W_2) = (1)\).

Then there exist equivalences of \(R\)-linear \(\mathbb{Z}_2\)-graded categories:

\[
\text{HEF}(R, W_1) \simeq \text{HEF}_{W_1}(R, W) \ , \ \text{HEF}(R, W_2) \simeq \text{HEF}_{W_2}(R, W) \ .
\]

which are bijective on objects.

**Proof.** For any divisor \(v\) of \(W_1\), let \(e'_v = (R^{v_1}, D'_v)\) and \(e_v = (R^{v_1}, D_v)\) be the corresponding elementary factorizations of \(W_1\) and \(W\), where:

\[
D'_v = \begin{bmatrix} 0 & v \\ W/v & 0 \end{bmatrix} , \ D_v = \begin{bmatrix} 0 & v \\ W/v & 0 \end{bmatrix} .
\]

For any two divisors \(v_1, v_2\) of \(W_1\) and any \(\kappa \in \mathbb{Z}_2\), we have \(W/v_1 = W_2 W/v_2\) and \((v_1, W_2) = (1)\). Thus \((v_1, v_2, W_1/v_1, W_1/v_2) = (v_1, v_2, W/v_1, W/v_2)\). By Proposition 2.2, this gives:

\[
\text{Ann}(\text{Hom}^0_{\text{HEF}(R, W_1)}(e'_v, e'_v)) = \text{Ann}(\text{Hom}^0_{\text{HEF}(R, W)}(e_v, e_v)) , \ \forall \kappa \in \mathbb{Z}_2 .
\]

On the other hand, the modules \(\text{Hom}^0_{\text{HEF}(R, W_1)}(e'_v, e'_v)\) and \(\text{Hom}^0_{\text{HEF}(R, W_1)}(e_v, e_v)\) are generated by the same element \(e'_0(v_1, v_2)\) while \(\text{Hom}^1_{\text{HEF}(R, W_1)}(e'_v, e'_v)\) and \(\text{Hom}^1_{\text{HEF}(R, W_1)}(e_v, e_v)\) are generated by the elements \(e_1(v_1, v_2; W_1)\) and \(e_1(v_1, v_2; W)\), respectively. Hence the functor which maps \(e'_v\) to \(e_v\) for any divisor \(v\) of \(W_1\) and takes \(e'_0(v_1, v_2)\) to \(e_0(v_1, v_2)\) and \(e_0(v_1, v_2; W)\) to \(e_1(v_1, v_2; W)\) for any two divisors \(v_1, v_2\) of \(W\) is an \(R\)-linear equivalence from \(\text{HEF}(R, W_1)\) to \(\text{HEF}_{W_1}(R, W)\). A similar argument establishes the equivalence \(\text{HEF}(R, W_2) \simeq \text{HEF}_{W_2}(R, W)\).

\[
\Box
\]

2.5. **Primary matrix factorizations.** Recall that an element of \(R\) is called primary if it is a power of a prime element.

**Definition 2.18** An elementary factorization \(e_v\) of \(W\) is called primary if \(v\) is a primary divisor of \(W\).

Let \(\text{HEF}_0(R, W)\) denote the full subcategory of \(\text{HEF}(R, W)\) whose objects are the primary factorizations of \(W\).
**Proposition 2.19** Let $W = W_1 W_2$ be a factorization of $W$, where $W_1$ and $W_2$ are coprime elements of $R$. Then there exists an equivalence of $R$-linear $\mathbb{Z}_2$-graded categories:

$$\text{HEF}_0(R,W) \simeq \text{HEF}_0(R,W_1) \lor \text{HEF}_0(R,W_2),$$

where $\lor$ denotes the coproduct of $\text{Mod}_R$-enriched categories.

**Proof.** Let $\text{HEF}_{0,W_i}(R,W)$ denote the full subcategory of $\text{HEF}_0(R,W)$ whose objects are the primary factorizations $e_v$ of $W$ for which $v$ is a (primary) divisor of $W_i$. Since $W = W_1 W_2$ and $(W_1, W_2) = (1)$, a primary element $v \in R$ is a divisor of $W$ iff it is either a divisor of $W_1$ or a divisor of $W_2$. Hence $\text{Ob}_{HEF_0(R,W)} = \text{Ob}_{HEF_{0,W_1}(R,W)} \sqcup \text{Ob}_{HEF_{0,W_2}(R,W)}$. For any primary divisors $v_1$ and $v_2$ of $W$ and any $\kappa \in \mathbb{Z}_2$, we have:

$$\text{Hom}_{\text{HEF}(R,W)}^\kappa(e_{v_1}, e_{v_2}) \simeq R/\langle d \rangle \simeq \begin{cases} 
\text{Hom}_{\text{HEF}(R,W_1)}^\kappa(e_{v_1}, e_{v_2}) & \text{if } v_1 | W_1 \& v_2 | W_1 \\
\text{Hom}_{\text{HEF}(R,W_2)}^\kappa(e_{v_1}, e_{v_2}) & \text{if } v_1 | W_2 \& v_2 | W_2 \\
0 & \text{if } v_1 | W_1 \& v_2 | W_2 \text{ or } v_1 | W_2 \& v_2 | W_1
\end{cases},$$

where $d \in (v_1, v_2, W/W_1, W/W_2)$ and in the third case we used the fact that $v_1 | W_1$ and $v_2 | W_2$ or $v_1 | W_2$ and $v_2 | W_1$ implies $(v_1, v_2) = (1)$ since $W_1$ and $W_2$ are coprime. This shows that $\text{HEF}_0(R,W) = \text{HEF}_{0,W_1}(R,W) \lor \text{HEF}_{0,W_2}(R,W)$. By Proposition 2.17, we have $R$-linear equivalences $\text{HEF}_{0,W_1}(R,W) \simeq \text{HEF}_0(R,W_1)$ which are bijective on objects. This implies the conclusion. \(\square\)

**Definition 2.20** A reduced multiplicative partition of $W$ is a factorization:

$$W = W_1 W_2 \ldots W_n$$

where $W_1, \ldots, W_n$ are mutually coprime elements of $R$.

**Corollary 2.21** Let $W = W_1 \ldots W_n$ be a reduced multiplicative partition of $W$. Then there exists a natural equivalence of $R$-linear categories:

$$\text{HEF}_0(R,W) \simeq \lor_{i=1}^n \text{HEF}_0(R,W_i).$$

**Proof.** Follows immediately from Proposition 2.19. \(\square\)

Let $e_v$ be a primary matrix factorization of $W$. Then $v = p^i$ for some prime divisor $p$ of $W$ and some integer $i \in \{0, \ldots, n\}$, where $n$ is the order of $p$ as a divisor of $W$. We have $W = p^n W_1$ for some element $W_1 \in R$ such that $p$ does not divide $W_1$ and $u = p^{n-1} W_1$. Thus $(u,v) = (p^{\min(i,n-i)})$.

**Definition 2.22** The prime divisor $p$ of $W$ is called the prime locus of $e_v$. The order $n$ of $p$ is called the order of $e_v$ while the integer $i \in \{0, \ldots, n\}$ is called the size of $e_v$.

Let $R$ be a Bézout domain and $p \in R$ be a prime element. Fix an integer $n \geq 2$ and consider the quotient ring:

$$A_n(p) \overset{\text{def}}{=} R/\langle p^n \rangle.$$

Let $m_n(p) = p A_n(p) = \langle p \rangle / \langle p^n \rangle$ and $k_p = R/\langle p \rangle$.

**Lemma 2.23** The following statements hold:
1. The principal ideal \( \langle p \rangle \) generated by \( p \) is maximal.
2. The primary ideal \( \langle p^n \rangle \) is contained in a unique maximal ideal of \( R \).
3. The quotient \( A_n(p) \) is a quasi-local ring with maximal ideal \( m_u(p) \) and residue field \( k_p \).
4. \( A_n(p) \) is a generalized valuation ring.

**Proof.**

1. Let \( I \) be any ideal containing \( \langle p \rangle \). If \( \langle p \rangle \neq I \), then take any element \( x \in I \setminus \langle p \rangle \). Then we have the proper inclusion \( \langle p \rangle \subsetneq \langle p, x \rangle \). Since \( R \) is a Bézout domain, the ideal \( \langle p, x \rangle \) is generated by a single element \( y \). We have \( y|p \), so \( y \) is a unit of \( R \) since \( p \) is prime. Since \( y \) belongs to \( I \), this gives \( I = R \). Thus \( \langle p \rangle \) is a maximal ideal.

2. Let \( m \) be a maximal ideal of \( R \) containing \( \langle p^n \rangle \). Then \( p^n \in m \), which implies \( p \in m \) since \( m \) is prime. Thus \( \langle p \rangle \subseteq m \), which implies \( m = \langle p \rangle \) since \( \langle p \rangle \) is maximal by point 1. This shows that \( R/\langle p^n \rangle \) has a unique maximal ideal, namely \( \langle p \rangle/\langle p^n \rangle \).

3. Since \( R \) is Bézout and \( \langle p^n \rangle \) is finitely-generated, the quotient \( R/\langle p^n \rangle \) is a Bézout ring (which has divisors of zero when \( n \geq 2 \)). By point 2. above, \( R/\langle p^n \rangle \) is also a quasi-local ring.

4. Follows from [7, Lemma 1.3 (b)] since \( R \) is a valuation ring. \( \square \)

Recall that an object of an additive category is called **indecomposable** if it is not isomorphic with a direct sum of two non-zero objects.

**Proposition 2.24** Let \( e_i \) be a primary factorization of \( W \) with prime locus \( p \), order \( n \) and size \( i \). Then \( e_i \) is an indecomposable object of \( \text{hmf}(R,W) \) whose endomorphism ring \( \text{End}_{\text{hmf}(R,W)}(e_i) \) is a quasi-local ring isomorphic with \( A_{\text{min}(i,n-i)}(p) \).

**Proof.** We have \( \text{End}_{\text{hmf}(R,W)}(e_i) = R/\langle u, v \rangle = R/\langle p^{\min(i,n-i)} \rangle \) by Corollary 2.10. This ring is quasi-local by Lemma 2.23. Since quasi-local rings have no nontrivial idempotents, it follows that \( e_i \) is an indecomposable object of \( \text{hmf}(R,W) \). \( \square \)

**Lemma 2.25** Let \( v_1 \) and \( v_2 \) be two divisors of \( W \) which are mutually coprime. Then \( \text{Hom}_{\text{hmf}(R,W)}(e_{v_1}, e_{v_2}) = 0 \).

**Proof.** Let \( u_i := W/v_i \). Then \( (v_1, v_2, u_1, u_2) = (1) \) since \( (v_1, v_2) = (1) \). Thus \( (v_1, v_2, u_1, u_2) = R \) and the statement follows from Proposition 2.2. \( \square \)

**Proposition 2.26** Let \( p \) be a prime divisor of \( W \) of order \( n \) and \( i \in \{1, \ldots, n\} \). Then:

\[
\Sigma e_{p^i} \cong_{\text{hmf}(R,W)} e_{p^{n-i}}.
\]

**Proof.** Let \( W_1 \overset{\text{def}}{=} p^n, W_2 \overset{\text{def}}{=} W/p^n \) and \( v \overset{\text{def}}{=} p^i, u \overset{\text{def}}{=} W/v = p^{n-i}W_2 \). We have \( \Sigma e_{p^i} = \Sigma e_v = e_{-u} \cong_{\text{hmf}(R,W)} e_u \). Since \( p^{n-i} \) is a divisor of \( W_1 \) and \( (W_1, W_2) = 1 \), Proposition 2.7 gives \( e_u = e_{p^{n-i}W_2} \cong_{\text{hmf}(R,W)} e_{p^{n-i}} \). \( \square \)
3. The additive category $\text{hef}(R,W)$ for a Bézout domain and critically-finite $W$

Let $R$ be a Bézout domain and $W$ be a critically-finite element of $R$.

**Proposition 3.1** Let $e_v$ be an elementary factorization of $W$ over $R$ such that $v = \prod_{i=1}^{n} v_i$, where $v_i \in R$ are mutually coprime divisors of $W$. Then there exists a natural isomorphism in $\text{hmf}(R,W)$:

$$e_v \simeq_{\text{hmf}(R,W)} \bigoplus_{i=1}^{n} e_{v_i}$$

In particular, an elementary factorization $e_v$ for which $v$ is finitely-factorizable divisor of $W$ is isomorphic in $\text{hmf}(R,W)$ with a direct sum of primary factorizations.

**Proof.** Let $d$ be any divisor of $W$. By Proposition 2.2, we have isomorphisms of $R$-modules:

$$\text{Hom}_{\text{hmf}(R,W)}(ed, e_v) \simeq_{R} R/\alpha_W(d, v_i) \quad \text{and} \quad \text{Hom}_{\text{hmf}(R,W)}(ed, e_v) \simeq_{R} R/\alpha_W(d, v).$$

Since $v_i$ are mutually coprime, Proposition 2.3 gives $\alpha_W(d, v) = \prod_{i=1}^{n} \alpha_W(d, v_i)$, where $\alpha_W(d, v_i)$ are principal ideals generated by mutually coprime elements. The Chinese remainder theorem gives an isomorphism of $R$-modules:

$$R/\alpha_W(d, v) \simeq_{R} \bigoplus_{i=1}^{n} [R/\alpha_W(d, v_i)]$$

Combining the above, we conclude that there exist natural isomorphisms of $R$-modules:

$$\varphi_d : \text{Hom}_{\text{hmf}(R,W)}(ed, e_v) \overset{\sim}{\longrightarrow} \text{Hom}_{\text{hef}(R,W)}(ed, \bigoplus_{i=1}^{n} e_{v_i}),$$

where we used the fact that $\text{Hom}_{\text{hmf}(R,W)}(ed, \bigoplus_{i=1}^{n} e_{v_i}) \simeq_{R} \bigoplus_{i=1}^{n} \text{Hom}_{\text{hmf}(R,W)}(ed, e_{v_i})$. This implies that the functors $\text{Hom}_{\text{hef}(R,W)}(-, e_v)$ and $\text{Hom}_{\text{hef}(R,W)}(-, \bigoplus_{i=1}^{n} e_{v_i})$ are isomorphic. By the Yoneda lemma, we conclude that there exists a natural isomorphism $e_v \simeq_{\text{hef}(R,W)} \bigoplus_{i=1}^{n} e_{v_i}$. □

Recall that a *Krull-Schmidt category* is an additive category for which every object decomposes into a finite direct sum of objects having quasi-local endomorphism rings.

**Theorem 3.2** The additive category $\text{hef}(R,W)$ is Krull-Schmidt and its non-zero indecomposable objects are the non-trivial primary matrix factorizations of $W$. In particular, $\text{hef}(R,W)$ is additively generated by $\text{hef}_0(R,W)$.

**Proof.** Suppose that $W$ has the decomposition (1.4). Any elementary factorization $e_v$ of $W$ corresponds to a divisor $v$ of $W$, which must have the form $v = v_0 p_{s_1}^{l_1} \cdots p_{s_m}^{l_m}$, where $1 \leq s_1 < \ldots < s_m \leq N$ and $1 \leq l_i \leq n_{s_i}$, while $v_0$ is a divisor of $W_0$. Applying Proposition 3.1 with $v_i = p_{s_i}^{l_i}$ for $i \in \{1, \ldots, m\}$, we find $e_v \simeq_{\text{hmf}(R,W)} \bigoplus_{i=0}^{m} e_{v_i}$, where we defined $u_0 = W/v_0 = W_0^{p_1^{l_1}} \cdots p_N^{l_N}$ and $u_i = W/v_i = W_0^{p_1^{l_1}} \cdots p_{s_i-1}^{l_{s_i-1}} p_{s_i}^{-l_i} \cdots p_N^{l_N}$ for $i \in \{1, \ldots, m\}$. We have $(u_0, v_0) = (u_0, W_0/v_0)$. Since $W_0 = u_0 v_0$, it follows that $(u_0, v_0)^2 | (W_0)$. Since $W_0$ has no critical divisors, we must have $(u_0, v_0) = (1)$ and hence $e_{v_0} \simeq_{\text{hmf}(R,W)} 0$. For $i \in \{1, \ldots, m\}$, we have $(u_i, v_i) = p_{s_i}^{l_i}$, where
\[ \mu_i \overset{\text{def.}}{=} \min(l_i, n_{s_i} - l_i). \] Thus \( e_{v_0} \) is primary of order \( \mu_{s_i} \) when \( \mu_{s_i} \geq 1 \) and trivial when \( \mu_i = 0 \). This gives a direct sum decomposition:

\[ e_{v_0} \overset{\text{hmf}(R,W)}{\simeq} \bigoplus_{i \in \{1,\ldots,m|l_i \leq n_{s_i}-1\}} e_{\pi_i^{l_i}} \overset{\text{hef}(R,W) \ overset{\oplus}{\simeq}}{\simeq} \bigoplus_{i \in \{1,\ldots,m|l_i < n_{s_i}\}} e_{\pi_i^{l_i}}, \]

where all matrix factorizations in the direct sum are primary except for \( e_{v_0} \). If \( l_i = n_{s_i} \) for all \( i \in \{1,\ldots,m\} \), then the sum in the right hand side is the zero object of \( \text{hmf}(R,W) \). We conclude that any elementary matrix factorization decomposes into a finite direct sum of primary matrix factorizations. On the other hand, any matrix factorization of \( W \) decomposes as a finite direct sum of elementary factorizations and hence also as a finite direct sum of primary factorizations whose prime supports are the prime divisors of \( W \). By Proposition 2.24, every primary matrix factorization has a quasi-local endomorphism ring. \( \square \)

**Theorem 3.3** Suppose that \( R \) is a Bézout domain and \( W \) has the decomposition (1.4). Then there exists an equivalence of categories:

\[ \text{hef}(R,W) \overset{\nu}{\simeq} \bigoplus_{i=1}^{N} \text{hef}(R,p_{ni}^{n_i}) \],

where \( \nu \) denotes the coproduct of additive categories.

**Proof.** Theorem 3.2 and Proposition 3.1 imply that \( \text{hef}(R,W) \) is additively generated by the additive subcategories \( \text{hef}(R,p_{ni}^{n_i}) \sim \text{hef}(R,p_{ni}^{n_i}) \), where we used Proposition 2.17. These categories are mutually orthogonal by Lemma 2.25. \( \square \)

### 3.1. A conjecture

Consider the inclusion functor:

\[ \iota : \text{hef}(R,W) \rightarrow \text{hmf}(R,W) \]

**Conjecture 3.4** The inclusion functor \( \iota \) is an equivalence of \( R \)-linear categories.

Conjecture 3.4 and Theorem 3.2 imply:

**Conjecture 3.5** Let \( R \) be a Bézout domain and \( W \) be a critically-finite element of \( R \). Then \( \text{hmf}(R,W) \) is a Krull-Schmidt category.

In [5], we establish Conjecture 3.4 for the case when \( R \) is an elementary divisor domain. This shows that Conjecture 3.4 is implied by the still unsolved conjecture [8] that any Bézout domain is an elementary divisor domain. Some recent work on that conjecture can be found in [9].

### 4. Counting elementary factorizations

In this section, we give formulas for the number of isomorphism classes of objects in the categories \( \text{HEF}(R,W) \) and \( \text{hef}(R,W) \) when \( W \) is critically-finite.
4.1. Counting isomorphism classes in $\text{HEF}(R, W)$. Let $W = W_0 W_c$ be a critically-finite element of $R$, where $W_0 \in R$ is non-critical and $W_c = p_1^{n_1} \ldots p_r^{n_r}$ with prime $p_j \in R$ and $n_j \geq 2$ (see Definition 1.7). Let $\text{Heff}(R, W)$ denote the set of isomorphism classes of objects in the category $\text{HEF}(R, W)$. We are interested in the cardinality:

$$N(R, W) \overset{\text{def}}{=} |\text{Heff}(R, W)|$$

of this set. In this subsection, we derive a formula for $N(R, W)$ as a function of the orders $n_i$ of the prime elements $p_i$ arising in the prime decomposition of $W_c$. The main result of this subsection is Theorem 4.12 below.

Lemma 4.1 The cardinality $N(R, W)$ depends only on the critical part $W_c$ of $W$.

Proof. Let $W = puv$ with a divisor $p$ coprime with both $u$ and $v$. Taking $b = p$ and $c = 1$ in Proposition 2.4 gives:

$$
\begin{pmatrix}
0 & pu & 0 \\
0 & v & 0 \\
u & 0 & 0
\end{pmatrix}
\overset{\text{HEF}(R, W)}{\sim}
\begin{pmatrix}
0 & v & 0 \\
0 & pu & 0 \\
u & 0 & 0
\end{pmatrix}.
$$

(4.1)

Together with Corollary 2.5, this implies $N(R, W) = N(R, W_c)$. □

From now on, we will assume that $W \in R$ is fixed and is of the form:

$$W = W_c = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}.$$  

(4.2)

To simplify notations, we will omit to indicate the dependence of some quantities on $W$.

Definition 4.2 Let $T$ be a non-empty set. A map $f : \text{ObEF}(R, W) \to T$ is called an elementary invariant if $f(e_1) = f(e_2)$ for any $e_1, e_2 \in \text{ObEF}(R, W)$ such that $e_1 \overset{\text{HEF}(R, W)}{=} e_2$. An elementary invariant $f$ is called complete if the map $f : \text{Heff}(R, W) \to T$ induced by $f$ is injective.

To determine $N(R, W)$, we will construct a complete elementary invariant. Let:

$$I \overset{\text{def}}{=} \{1, \ldots, r\},$$

where $r$ is the number of non-associated prime factors of $W$, up to association in divisibility.

Similarity classes of elementary factorizations and normalized divisors of $W$. Let $\text{HEF}_{\text{sim}}(R, W)$ be the groupoid having the same objects as $\text{HEF}(R, W)$ and morphisms given by similarity transformations of elementary factorizations and let $\text{Heff}_{\text{sim}}(R, W)$ be its set of isomorphism classes. Since the similarity class of an elementary factorization $e_v$ is uniquely determined by the principal ideal $\langle v \rangle$ generated by the divisor $v$ of $W$, the map $e_v \to \langle v \rangle$ induces a bijection:

$$\text{Heff}_{\text{sim}}(R, W) \simeq \text{Div}(W),$$

where:

$$\text{Div}(W) \overset{\text{def}}{=} \{\langle v \rangle, | v| W\} = \{\langle v \rangle, v \in R : W \in \langle v \rangle\}$$

is the set of principal ideals of $R$ containing $W$. Let:

$$\text{Div}_1(W) \overset{\text{def}}{=} \{\prod_{i \in I} p_i^{k_i} | \forall i : k_i \in \{0, \ldots, n_i\}\},$$

(4.3)
be the set of normalized divisors of $W$. The map $v \mapsto \langle v \rangle$ induces a bijection between $\text{Div}_1(W)$ and $\text{Div}(W)$. Indeed, any principal ideal of $R$ which contains $W$ has a unique generator which belongs to $\text{Div}_1(W)$, called its normalized generator. Given any divisor $v$ of $W$, its normalization $v_0$ is the unique normalized divisor $v_0 \in \text{Div}_1(W)$ such that $\langle v \rangle = \langle v_0 \rangle$. Given two divisors $t, s$ of $W$, their normalized greatest common divisor is the unique normalized divisor $(t, s)_1$ of $W$ which generates the ideal $Rt + Rs$. The set of exponent vectors of $W$ is defined through:

$$A_W \overset{\text{def}}{=} \prod_{i=1}^{r} \{0, \ldots, n_i\}.$$  

The map $A_W \ni k = (k_1, \ldots, k_r) \mapsto \prod_{i \in I} p_i^{k_i} \in \text{Div}_1(W)$ is bijective, with inverse $\mu : \text{Div}_1(W) \to A_W$ given by:

$$\mu(v) = (\text{ord}_{p_1}(v), \ldots, \text{ord}_{p_r}(v)) = (k_1, \ldots, k_r) \text{ for } v = \prod_{i \in I} p_i^{k_i} \in \text{Div}_1(W).$$

Combining everything, we have natural bijections:

$$\text{Heff}_{\text{sim}}(R, W) \simeq \text{Div}(R, W) \simeq \text{Div}_1(R, W) \simeq A_W.$$  

Remark 4.1. In Proposition 2.16, the quantity $s_1$ was an arbitrary element of the class $(u_1, v_1)$ for $e_{e_1}$. For a fixed critically-finite $W$, we have a canonical choice for this quantity, namely the normalized gcd of $u_1$ and $v_1$. Thus we define $s(e_1) = (u_1, v_1)_1$. The two definitions are connected by the relation $(s_1) = (s(e_1))$. Below we introduce “normalized” quantities $x(e), y(e)$ which belong to the same classes in $R^\times/U(R)$ as the quantities $x$ and $y$ defined in Section 2. The results of Section 2 hold automatically for these normalized choices.

Given $t \in \text{Div}(W)$, its index set is the subset of $I$ given by:

$$I(t) \overset{\text{def}}{=} \text{supp } \mu(t_0) = \{i \in I \mid p_i t\}.$$  

Notice that $I(t)$ depends only on the principal ideal $(t)$, that in turn depends only on the class $(t) \in R^\times/U(R)$. This gives a map from $\text{Div}(W)$ to the power set $\mathcal{P}(I)$ of $I$. Note that $(t) = (1)$ iff $I(t) = \emptyset$.

The essence and divisorial invariant of an elementary factorization. Consider an elementary factorization $e$ of $W$ and let $v(e) \in \text{Div}_1(W)$ be the unique normalized divisor of $W$ for which $e$ is similar to $e_v$. Let $u = u(e) \overset{\text{def}}{=} W/v \in \text{Div}_1(W)$ and let $s(e) \overset{\text{def}}{=} (v, u)_1 \in \text{Div}_1(W)$ be the normalized greatest common divisor of $v$ and $u$. Let $I_s(e) \overset{\text{def}}{=} I(s(e))$. Let $m_i(e) \overset{\text{def}}{=} \text{ord}_{p_i}(s(e))$ and $m(e) = \mu(s(e)) = (m_1(e), \ldots, m_n(e))$. Then $I_s(e) = \text{supp } m(e)$ and:

$$s(e) = \prod_{i \in I_s(e)} p_i^{m_i(e)}. \quad (4.4)$$

Let $v'(e) \overset{\text{def}}{=} v/s(e)$ and $u'(e) \overset{\text{def}}{=} u/s(e)$. Then $(v'(e), u'(e)) = (1)$ and $W = v(e)u(e) = v'(e)u'(e)s(e)^2$. Define:

$$x(e) \overset{\text{def}}{=} (s(e), v'(e))_1, \quad y(e) \overset{\text{def}}{=} (s(e), u'(e))_1$$

and:

$$I_x(e) \overset{\text{def}}{=} I(x(e)) , \quad I_y(e) \overset{\text{def}}{=} I(y(e)).$$
Notice that \((x(e), y(e))_1 = 1\), thus \(I_x(e) \cap I_y(e) = \emptyset\). Defining \(v''(e) \overset{\text{def.}}{=} v'(e)/x(e)\) and \(u''(e) \overset{\text{def.}}{=} u'(e)/y(e)\), we have:

\[ W = x(e)y(e)v''(e)u''(e)s(e)^2, \]

where \(v''(e), u''(e)\) and \(s(e)\) are mutually coprime. Moreover, we have:

\[
\text{ord}_{p_i}(v'(e)) = n_i - 2m_i(e) \quad \forall i \in I_x(e) \quad \text{and} \quad \text{ord}_{p_i}(u'(e)) = n_i - 2m_i(e) \quad \forall i \in I_y(e),
\]

which implies:

\[
\begin{align*}
\text{ord}_{p_i} x(e) &= \max(m_i(e), n_i - 2m_i(e)) \quad \text{for} \quad i \in I_x(e), \\
\text{ord}_{p_i} y(e) &= \max(m_i(e), n_i - 2m_i(e)) \quad \text{for} \quad i \in I_y(e).
\end{align*}
\]

Notice that \(\text{ord}_{p_i} x(e)y(e) \equiv n_i \text{ mod } 2\) for \(i \in I_x \cup I_y\) if \(3m_i < n_i\).

**Definition 4.3** The essence \(z := z(e)\) of an elementary factorization \(e\) of \(W\) is the normalized divisor of \(W\) defined through:

\[
z(e) \overset{\text{def.}}{=} \prod_{I_x(e)} p_i^{m_i(e)}, \quad \text{where} \quad I_z(e) \overset{\text{def.}}{=} I(s)\backslash(I_x(e) \cup I_y(e)).
\]

An elementary factorization \(e\) is called essential if \(z(e) = 1\), i.e. if \(I_z(e) = \emptyset\).

The divisor \(s\) defines \(x, y\) and \(z\) uniquely by (4.6) and (4.7). These 3 divisors in turn also define \(s\) uniquely that can be seen by inverting the max functions above:

\[
\text{ord}_{p_i} s = m_i = \begin{cases} 
\text{ord}_{p_i} x(e) & \text{if } i \in I_x(e) \text{ and } 3\text{ord}_{p_i} x(e) \geq n_i \\
(n_i - \text{ord}_{p_i} x(e))/2 & \text{if } i \in I_x(e) \text{ and } 3\text{ord}_{p_i} x(e) < n_i \\
\text{ord}_{p_i} y(e) & \text{if } i \in I_y(e) \text{ and } 3\text{ord}_{p_i} y(e) \geq n_i \\
(n_i - \text{ord}_{p_i} y(e))/2 & \text{if } i \in I_y(e) \text{ and } 3\text{ord}_{p_i} y(e) < n_i \\
\text{ord}_{p_i} z(e) & \text{if } i \in I_z(e)
\end{cases}
\]

The fundamental property of an essential factorization of \(e\) is the equality of sets \(I_x(e) = I_x(e) \cup I_y(e)\), which will allow us to compute the number \(N_0(R, W)\) of isomorphism classes of such factorizations (see Proposition 4.11 below). Then \(N(R, W)\) will be determined by relating it to \(N_0\) for various reductions of the potential \(W\).

Notice that the essence \(z(e)\) is a critical divisor of \(W\) and that we have \((z(e), v'(e))_1 = (z(e), u'(e))_1 = 1\). Since \(W = v(e)u(e) = v'(e)u'(e)s(e)^2\), this gives:

\[
\text{ord}_{p_i} W = n_i = 2m_i(e) = 2\text{ord}_{p_i} z(e) \quad \text{for any} \quad i \in I_z(e).
\]

**Definition 4.4** The divisorial invariant of an elementary factorization \(e\) of \(W\) is the element \(h(e)\) of the set \(\text{Div}_1(W) \times \text{Sym}^2(P(I))\) defined through:

\[
h(e) = (s(e), \{I_x(e), I_y(e)\})
\]

This gives a map \(h : EF(R, W) \to \text{Div}_1(W) \times \text{Sym}^2(P(I))\).
We have already given a criterion for two elementary factorizations of $W$ to be isomorphic in Proposition 2.16. There exists another way to characterize when two objects of $\text{HEF}(R, W)$ (and also of $\text{hef}(R, W)$) are isomorphic, which will be convenient for our purpose.

**Proposition 4.5** Consider two elementary factorizations of $W$. The following statements are equivalent:

1. The two factorizations are isomorphic in $\text{HEF}(R, W)$ (respectively in $\text{hef}(R, W)$).
2. The two factorizations have the same $(s, \{x, y\})$ (respectively same $(s, x, y)$).
3. The two factorizations have the same divisorial invariant $(s, \{I_x, I_y\})$ (respectively same $(s, I_x, I_y)$).

In particular, the divisorial invariant $h : \text{ObEF}(R, W) \rightarrow \text{Div}_1(W) \times \text{Sym}^2(P(I))$ is a complete elementary invariant.

**Proof.** The equivalence between 1. and 2. follows from Proposition 2.16. Indeed, the proposition shows that for two isomorphic factorizations $e_1$ and $e_2$ the corresponding $s_1$ and $s_2$ are similar: $(s_1) = (s_2)$ in the notations of Section 2. We compute $(s_2) = (u_2, v_2) = ((u_2)_1, (v_2)_1) = ((u(e_2), v(e_2))) = (s(e_2))$ with the last $s(e_2)$ defined in $\text{Div}_1$ by (4.4). Similarly $(s_1) = (s(e_1))$. By the very definition of $\text{Div}_1$ we have $(s(e_1)) = (s(e_2))$ implies $s(e_1) = s(e_2)$.

The implication $2. \Rightarrow 3.$ is obvious. Thus it suffices to prove that 3. implies 2. For this, let $e_1$ and $e_2$ be the two elementary factorizations of $W$. Assume that $s(e_1) = s(e_2)$ and $\{I_x(e_1), I_y(e_1)\} = \{I_x(e_2), I_y(e_2)\}$ and let $s := s(e_1) = s(e_2) = \prod_{i \in I(s)} p_i^{m_i}$. Consider the case $I_x(e_1) = I_x(e_2)$ and $I_y(e_1) = I_y(e_2)$. Applying (4.6) to $v = v(e_1)$ and $v = v(e_2)$ and using the relations $m_i(e_1) = \text{ord}_{p_i} s = m_i(e_2)$ gives:

$$x(e_1) = x(e_2) \quad \text{and} \quad y(e_1) = y(e_2).$$

When $I_x(e_1) = I_y(e_2)$ and $I_x(e_2) = I_y(e_1)$, a similar argument gives $x(e_1) = y(e_2)$ and $y(e_1) = x(e_2)$. \hfill \Box

**Proposition 4.6** The map $z : \text{ObEF}(R, W) \rightarrow \text{Div}_1(W)$ which gives the essence of an elementary factorization is an elementary invariant.

**Proof.** Let $e_1$ and $e_2$ be two factorizations of $W$ which are isomorphic in $\text{HEF}(R, W)$. By Proposition 4.5, we have $s(e_1) = s(e_2)$ and $\{I_x(e_1), I_y(e_1)\} = \{I_x(e_2), I_y(e_2)\}$. Hence:

$$I(z(e_1)) = I(s(e_1)) \setminus (I_x(e_1) \cup I_y(e_1)) = I(s(e_2)) \setminus (I_x(e_2) \cup I_y(e_2)) = I(z(e_2)).$$

(4.10)

Applying (4.9) for $e = e_1$ and $e = e_2$ gives $\text{ord}_{p_i} z(e_1) = \text{ord}_{p_i} z(e_2)$ for any $i \in I(z(e_1)) = I(z(e_2))$. Thus $z(e_1) = z(e_2)$. \hfill \Box

The essential reduction of an elementary factorization. For any normalized critical divisor $z$ of $W$, let $\text{HEF}_{\zeta}(R, W)$ denote the full subcategory of $\text{HEF}(R, W)$ consisting of those elementary factorizations whose essence equals $z$ and let $\text{HEF}_{\zeta}(R, W)$ be its set of isomorphism classes. Then $\text{HEF}_1(R, W)$ consists of the isomorphism classes of essential factorizations.
Definition 4.7 The essential reduction of an elementary factorization \( e := e_v \) of \( W \) is the essential elementary factorization of \( W/z(e)^2 \) defined through:

\[
\text{essred}(e) \overset{\text{def}}{=} e_{v/z(e)}.
\]

This gives a map \( \text{essred} : \text{ObEF}(R,W) \to \text{ObHEF}_1(R,W/z(e)^2) \).

To see that \( \text{essred} \) is well-defined, consider the elementary factorization \( \tilde{e} = e_{v/z(e)} \):

\[
\tilde{W} \overset{\text{def}}{=} W/z(e)^2 = u(\tilde{e})v(\tilde{e}),
\]

where \( v(\tilde{e}) = v(e)/z(e) \) and \( u(\tilde{e}) = u(e)/z(e) \). We compute:

\[
s(\tilde{e}) \overset{\text{def}}{=} (v(\tilde{e}), u(\tilde{e}))_1 = (v(e)/z(e), u(e)/z(e))_1 = s(e)/z(e)
\]

and \( v'(\tilde{e}) \overset{\text{def}}{=} v(\tilde{e})/s(\tilde{e}) = v'(e), u'(\tilde{e}) \overset{\text{def}}{=} u(\tilde{e})/s(\tilde{e}) = u'(e) \). Thus \( x(\tilde{e}) = x(e) \) and \( y(\tilde{e}) = y(e) \). By (4.7) applied to \( \tilde{e} \) and \( e \), we derive \( I_\tilde{e}(\tilde{e}) = I_e(\tilde{e}) \cap (I_\tilde{e}(\tilde{e}) \cup I_y(\tilde{e})) = I_z(e) \cup I_s(e) \setminus (I_x(e) \cup I_y(e)) = \emptyset \), which implies \( z(\tilde{e}) = 1 \). Hence \( \text{essred}(e) \) is an essential elementary matrix factorization of \( \tilde{W} \). Also notice the relation:

\[
(z(e), W/z(e)^2) = (1),
\]

which follows from the fact that \( z(e) \) is coprime with \( v'(e) \) and \( u'(e) \).

Lemma 4.8 For any critical divisor \( z \) of \( W \) such that \( (z, W/z^2) \sim 1 \), the map \( \text{essred}_z : \text{Hef}_z(R,W) \to \text{Hef}_1(R,W/z^2) \).

Proof. We perform the proof in two steps:

1. Let \( e_1 := e_{v_1} \) and \( e_2 := e_{v_2} \) be two elementary factorizations of \( W \) such that \( z(e_1) = z(e_2) = z \) and let \( v_1 = v(e_1), v_2 = v(e_2) \). Define also \( v_3 \overset{\text{def}}{=} v_1/z \) and \( v_4 \overset{\text{def}}{=} v_2/z \). To show that \( \text{essred}_z \) is well-defined, we have to show that \( e_1 \simeq_{\text{Hef}(R,W)} e_2 \) implies that the two essential elementary factorizations \( e_3 := e_{v_3} \) and \( e_4 := e_{v_4} \) of \( \tilde{W} \overset{\text{def}}{=} W/z^2 \) are isomorphic in \( \text{HEF}(R,\tilde{W}) \). For this, we compute:

\[
s(e_1) \overset{\text{def}}{=} (v_1, u_1)_1 = (z(e_1) \cdot v_3, z(e_1) \cdot u_3)_1 = z(e_1) \cdot (v_3, u_3)_1 = z(e_1) \cdot s(e_3).
\]

Thus:

\[
x(e_1) \overset{\text{def}}{=} (s(e_1), v'_1)_1 = (z(e_1) \cdot s(e_3), z(e_1) v_3/z(e_1) s(e_3))_1 = (s(e_4), v'_3)_1 = x(e_3).
\]

The third equality above holds since \( (z(e_1), v'_1)_1 = 1 \) and thus \( (z(e_1), v_3)_1 = 1 \). Similarly, we have \( s(e_2) = z(e_2) \cdot s(e_4) \) and we find \( y(e_1) = y(e_3) \) as well as \( x(e_2) = x(e_4) \) and \( y(e_2) = y(e_4) \). By Proposition 4.5, the condition \( e_1 \simeq_{\text{Hef}(R,W)} e_2 \) implies \( s(e_1) = s(e_2) = s \) and \( I(s(e_1)) = I(s(e_2)) \), thus \( z(e_1) = z(e_2) = z \). If \( (s(e_1), \{x(e_1), y(e_1)\}) = (s(e_2), \{x(e_2), y(e_2)\}) \), then \( (s(e_3), \{x(e_3), y(e_3)\}) = (s(e_4), \{x(e_4), y(e_4)\}) \). Thus \( e_3 \simeq e_4 \).

2. Let \( z \) be a critical divisor of \( W \) such that \( (z, W/z^2) = (1) \). For any essential elementary factorization \( e_v \) of \( \tilde{W} \), the elementary factorization \( e_{zv} \) of \( W \) is an object of \( \text{HEF}_z(R,\tilde{W}) \) and we have \( \text{essred}(e_{zv}) = e_v \). This shows that \( \text{essred}_z \) is surjective. Now let \( e_3 \) and \( e_4 \) be two essential elementary factorizations of \( \tilde{W} \overset{\text{def}}{=} W/z^2 \) which are isomorphic in \( \text{HEF}(R,\tilde{W}) \).
Let \( v_1 \stackrel{\text{def}}{=} zv_3 \) and \( v_2 \stackrel{\text{def}}{=} zv_4 \). To show that \( \text{essred}_z \) is injective, we have to show that the two elementary factorizations \( e_1 \defeq e_{v_1} \) and \( e_2 \defeq e_{v_2} \) of \( W \) are isomorphic in \( \text{HEF}(R,W) \). For this, notice that \( (s(e_3), \{x(e_3), y(e_3)\}) = (s(e_4), \{x(e_4), y(e_4)\}) \) by Proposition 4.5. This implies \( (s(e_1), \{x(e_1), y(e_1)\}) = (s(e_2), \{x(e_2), y(e_2)\}) \), with \( z(e_1) = z(e_2) = z \). Hence \( e_1 \) and \( e_2 \) are isomorphic in \( \text{HEF}(R,W) \) by the same proposition. \( \square \)

**A formula for \( N(R,W) \) in terms of essential reductions.** Let:

\[
S \defeq \text{im} \, h \subset \text{Div}_1(W) \times \text{Sym}^2(\mathcal{P}(I)) .
\] (4.12)

The degrees of the prime factors \( p_i \) in the decomposition (4.2) of \( W \) define on \( I = \{1, \ldots, r\} \) a \( \mathbb{Z}_2 \)-grading given by:

\[
I^0 \defeq \{ i \in I \mid n_i \text{ is even} \} , \quad I^1 \defeq \{ i \in I \mid n_i \text{ is odd} \} .
\] (4.13)

Let:

\[
r^0 \defeq |I^0| , \quad r^1 \defeq |I^1| .
\]

Since \( I = I^0 \sqcup I^1 \), we have \( r = r^0 + r^1 \). Any non-empty subset \( K \subset I \) is endowed with the \( \mathbb{Z}_2 \)-grading induced from \( I \). For any critical divisor \( z \) of \( W \), we have \( z^2 | W \), which implies \( I(z) \subset I^0 \). For any subset \( J \subset I^0 \), define:

\[
z_J \defeq \prod_{i \in J} p_i^{n_i/2} ,
\] (4.14)

which is a normalized critical divisor of \( W \) satisfying \( (z_J, W/z_J^2)_{\text{ef}} = 1 \). Also define:

\[
S_J \defeq h(\text{Heff}_{z_J}(R,W)) \subset S .
\]

and:

\[
N_J(R,W) \defeq |S_J| .
\] (4.15)

Since \( h \) is a complete elementary invariant, we have \( N_{\emptyset}(R,W) = |h(\text{Heff}_1(R,W))| = |\text{Heff}_1(R,W)| \). Moreover, Lemma 4.8 gives:

\[
N_J(R,W) = N_{\emptyset}(R,W/z_J^2) .
\] (4.16)

**Proposition 4.9** We have:

\[
N(R,W) = \sum_{J \subset I^0} N_{\emptyset}(R,W/z_J^2) .
\] (4.17)

**Proof.** Follows immediately from Lemma 4.8 and the remarks above. \( \square \).

**Computation of \( N_{\emptyset}(R,W) \).** Notice that \( z_{\emptyset} = 1 \). Since \( h \) is a complete elementary invariant, we have \( N_{\emptyset}(R,W) = |S_{\emptyset}| \), where:

\[
S_{\emptyset} = \{ h(e) \mid e \in \text{ObHEF}(R,W) : z(e) = 1 \} .
\]

We will first determine the cardinality of the set:

\[
S_{\emptyset,k} \defeq \{ h(e) \mid e \in \text{ObHEF}(R,W) : z(e) = 1 \text{ and } |I_s(e)| = k \} .
\]

We have:

\[
S_{\emptyset} = \bigsqcup_{k=1}^r S_{\emptyset,k} .
\]
Lemma 4.10 For $k \geq 1$, we have:

$$|S_{\emptyset,k}| = 2^{k-1} \cdot \sum_{K \subseteq I, |K| = k} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor .$$

(4.18)

Proof. Consider a subset $K \subseteq I(s)$ of cardinality $|K| = k$. Since $s^2|W$, we have:

$$1 \leq m_i(\alpha_i) \leq \left\lfloor \frac{(n_i - 1)}{2} \right\rfloor \quad \forall i \in I(s).$$

There are $\prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor$ different possibilities for $s$ such that $I(s) = K$. We can have several elements $(s, (x, y))$ of $S_{1, k}$ with the same $s$ since $x$ and $y$ can vary. This is where the coefficient $2^{k-1}$ in front of (4.18) comes from, as we now explain. Fixing the set $I(s)$ with $|I(s)| = k$, we have a set $\mathcal{P}(I(s))$ of $2^k$ partitions $I(s) = I_x \sqcup I_y$ as disjoint union of 2 sets. These can be parameterized by the single subset $I_v \subseteq I(s)$ since $I_u = I(s) \setminus I_v$. Define:

$$S_{\emptyset,k,s} = \{h(e) \mid e \in \text{ObHEF}(R, W) : z(e) = 1 \text{ and } |I_s(e)| = k \text{ and } s(e) = s\} .$$

Consider the surjective map

$$\Psi : \mathcal{P}(I(s)) \to S_{\emptyset,k,s}$$

which sends a partition $\beta = (I_1, I_2)$ of $I(s)$ to the element $\alpha = (s, \{I_1, I_2\})$. The preimage $\Psi^{-1}(h(e))$ of an element $h(e) \in S_{\emptyset,k,s}$ consist of two elements : $(I_1, I_2)$ and $(I_2, I_1)$. Thus the map is 2:1. This holds for every $K$ with $|K| = k$. Comparing the cardinalities of $\mathcal{P}(I(s))$ and $S_{\emptyset,k,s}$, we find:

$$|S_{\emptyset,k,s}| = |\mathcal{P}(I(s))|/2 = 2^k/2 = 2^{k-1} .$$

This holds for any $s$ with $I(s) = K$, where $K \subseteq I$ has cardinality $k$. Since $S_{\emptyset,k} = \sqcup s S_{\emptyset,k,s}$ and since the cardinality $|S_{\emptyset,k,s}|$ does not depend on $s$, we find:

$$|S_{\emptyset,k}| = \sum_s |S_{\emptyset,k,s}| = 2^{k-1} \sum_{K \subseteq I, |K| = k} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor .$$

\[ \square \]

Proposition 4.11 With the definitions above, we have:

$$N_\emptyset(R, W) = |S_\emptyset| = 1 + \sum_{k=1}^r 2^{k-1} \sum_{K \subseteq I, |K| = k} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor .$$

(4.19)

Proof. Since $S_\emptyset = \sqcup_{k=1}^r S_{\emptyset,k}$, the the previous lemma gives:

$$|S_\emptyset| = 1 + \sum_{k=1}^r |S_{\emptyset,k}| = 1 + \sum_{k=1}^r 2^{k-1} \sum_{K \subseteq I, |K| = k} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor .$$

(4.20)

The term $1$ in front corresponds to the unique element $(1, \{\emptyset, \emptyset\})$ of $S$. \[ \square \]
Computation of $N(R,W)$. The main result of this subsection is the following:

**Theorem 4.12** The number of isomorphism classes of $\text{HEF}(R,W)$ for a critically-finite $W$ as in (4.2) is given by:

$$N(R,W) = 2^{r^0} + \sum_{k=0}^{r^1} 2^{r^0 + k - 1} \sum_{K \subseteq I \atop |K^1|=k} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor .$$  \hspace{1cm} (4.21)

**Proof.** Combining Proposition 4.9 and Proposition 4.11, we have:

$$N(R,W) = \sum_{J \subseteq I^0} \left( 1 + \sum_{k=1}^{r-j} 2^{k-1} \sum_{K \subseteq I^0 \setminus J \atop |K|=k} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor \right),$$  \hspace{1cm} (4.22)

where $j \equiv |J|$. We will simplify this expression by changing the summation signs and applying the binomial formula.

Since $r_0 = |I^0|$ and $J \subseteq I^0$, we have $j \leq r_0$. For fixed $j$, we have $(r_0 \choose j)$ different subsets $J \subseteq I^0$ of this cardinality. The contribution to $N_0(R,\tilde{W})$ of any such $J$ has the free coefficient 1. Then the free coefficient of $N(R,W)$ is:

$$r^0 \sum_{j=0}^{r^0} \left( \frac{r^0}{j} \right) = 2^{r^0} .$$  \hspace{1cm} (4.23)

For the other coefficients of (4.22), we consider a subset $K \subseteq I$ as in (4.19). Such an index set $K = K^1 \sqcup K^0 \subseteq I$ of cardinality $k = k^1 + k^0$ appears in $N_0(R,\tilde{W})$ if $K^0 \subseteq I^0 \setminus J$. The coefficient of $\prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor$ in $N(R,W)$ is $2^{k-1}$ for any of $\left( \frac{r^0 - k^0}{j} \right)$ choices of $J$. It follows that this coefficient in $N(R,W)$ is:

$$\sum_{j=0}^{r^0 - k^0} 2^{k-1} \left( \frac{r^0 - k^0}{j} \right) = 2^{r^0 + k^1 - 1} 2^{r^0 - k^0} = 2^{r^0 + k^1 - 1} .$$  \hspace{1cm} (4.24)

Together with (4.23) and (4.24), relation (4.22) gives:

$$N(R,W) = 2^{r^0} + \sum_{\emptyset \subseteq K^0 \subseteq I \atop |K^0|=k^0} 2^{r^0 + k^1 - 1} \prod_{i \in K^1} \left\lfloor \frac{n_i - 1}{2} \right\rfloor,$$

which is equivalent to (4.21). \hspace{1cm} $\square$

The two examples below illustrate how the coefficients behave for $r = 2$ and $r = 3$.

**Example 4.1.** Let $W = p_1^n p_2^m$ for prime elements $p_1, p_2 \in R$ such that $(p_1) \neq (p_2)$ and $n, m \geq 2$ with odd $n$ and $m$. Then:

$$N(R,W) = 1 + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{m-1}{2} \right\rfloor .$$

**Example 4.2.** Consider $W = p_1^n p_2^m p_3^k$ for primes $p_1, p_2, p_3 \in R$ which are mutually non-associated in divisibility and orders $n, m, k \geq 2$ subject to the condition that $n$ and $m$ are even while $k$ is
odd. Then:
\[
N(R, W) = 2 + 2 \left\lfloor \frac{n - 1}{2} \right\rfloor + 2 \left\lfloor \frac{m - 1}{2} \right\rfloor + \left\lfloor \frac{k - 1}{2} \right\rfloor + 4 \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lfloor \frac{m - 1}{2} \right\rfloor + 2 \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lfloor \frac{k - 1}{2} \right\rfloor + 2 \left\lfloor \frac{m - 1}{2} \right\rfloor \right. \]  
\[
+ \left. 2 \left\lfloor \frac{n - 1}{2} \right\rfloor \right\rfloor + 2 \left\lfloor \frac{m - 1}{2} \right\rfloor \right\rfloor + 2 \left\lfloor \frac{k - 1}{2} \right\rfloor \right\rfloor + 2 \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lfloor \frac{m - 1}{2} \right\rfloor + 2 \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lfloor \frac{k - 1}{2} \right\rfloor + 2 \left\lfloor \frac{m - 1}{2} \right\rfloor \left\lfloor \frac{k - 1}{2} \right\rfloor . \]  
(4.25)

### 4.2 Counting isomorphism classes in hef(R, W)

We next derive a formula for the number of isomorphism classes in the category hef(R, W) for a critically-finite W (see Theorem 4.21 below). Since the morphisms of hef(R, W) coincide with the even morphisms of HEF(R, W), the number of isomorphism classes of hef(R, W) is larger than N(R, W). The simplest difference between the two cases arises from the fact that suspension does not preserve the isomorphism class of an elementary factorization in the category hef(R, W). Let \( \hat{H}_{ef} (R, W) \) be the set of isomorphism classes of objects in hef(R, W) and:
\[
\hat{N}(R, W) = |\hat{H}_{ef} (R, W)| .
\]

**Lemma 4.13** The cardinality \( \hat{N}(R, W) \) depends only on the critical part \( W_c \) of W.

**Proof.** The proof is identical to that of Lemma 4.1.

**Definition 4.14** Let T be a non-empty set. A map \( f : \text{ObEF}(R, W) \to T \) is called an even elementary invariant if \( f(e_1) = f(e_2) \) for any \( e_1, e_2 \in \text{ObEF}(R, W) \) such that \( e_1 \cong_{\text{hef}(R, W)} e_2 \). An even elementary invariant \( f \) is called complete if the map \( \hat{f} : \hat{H}_{ef} (R, W) \to T \) induced by \( f \) is injective.

As in the previous subsection, we will compute \( \hat{N}(R, W) \) by constructing an even complete elementary invariant.

**Definition 4.15** The even divisorial invariant of an elementary factorization \( e \) of W is the element \( \hat{h}(e) \) of the set \( \text{Div}_1(W) \times P(I)^2 \) defined through:
\[
\hat{h}(e) = (s(e), I_x(e), I_y(e)) .
\]

This gives a map \( \hat{h} : \text{EF}(R, W) \to \text{Div}_1(W) \times P(I)^2 \).

**Lemma 4.16** The even divisorial invariant \( \hat{h} : \text{ObEF}(R, W) \to \text{Div}_1(W) \times P(I)^2 \) is a complete even elementary invariant.

**Proof.** By Proposition 4.5, two elementary factorizations of W are isomorphic in hef(R, W) iff they have the same \( (s, x, y) \), which in turn is equivalent with coincidence of their even elementary invariants. □

Using the essence \( z(e) \) defined in (4.7), each elementary factorization \( e_a \) of W determines an elementary factorization \( \text{essred}(e_a) = e_{v/z(e)} \) of \( \tilde{W} \) \( = W/z(e)^2 \) (see Definition 4.7). For any normalized critical divisor \( z \) of W, let hef\(_z\) (R, W) denote the full subcategory of hef(R, W) consisting of those elementary factorizations whose essence equals \( z \) and let \( \hat{H}_{ef\_z} (R, W) \) be its set of isomorphism classes.
Lemma 4.17 For any critical divisor \( z \) of \( W \) such that \( (z, W/z^2) = (1) \), the map \( \text{essred} \) induces a well-defined bijection \( \tilde{\text{essred}} : \text{Heff}_z(R, W) \sim \text{Heff}_1(R, W/z^2) \).

Proof. The proof is almost identical to that of Lemma 4.8, but taking into account that in \( \text{heff}(R, W) \) we deal only with the even morphisms of \( \text{HEF}(R, W) \). □

Let:
\[
\check{S} \overset{\text{def}}{=} \text{im} \, h \subset \text{Div}_1(W) \times \mathcal{P}(I)^2 .
\]

For a subset \( J \subset I^0 \), let:
\[
\check{S}_J \overset{\text{def}}{=} h(\text{Heff}_{z_J}(R, W)) \subset \check{S} ,
\]
where \( z_J \) was defined in (4.14). Define \( \tilde{N}_J(R, W) :\overset{\text{def}}{=} |\check{S}_J| \) and \( \tilde{N}_0(R, W) = |\check{h}(\tilde{\text{Heff}}_1(R, W))| = |\tilde{\text{Heff}}_1(R, W)| \), where the last equality holds since \( \check{h} \) is a complete even elementary invariant. We can again compute \( \tilde{N}(R, W) \) in terms of \( \tilde{N}_0(R, W) \):

Proposition 4.18 We have:
\[
\tilde{N}(R, W) = \sum_{J \subset I^0} \tilde{N}_0(R, W/z^2_J) .
\]

Proof. Follows from Lemma 4.17. □

Define:
\[
\check{S}_0 = \{ \check{h}(e) \mid e \in \text{ObHEF}(R, W) \text{ and } z(e) = 1 \}
\]
and:
\[
\check{S}_{0,k} \overset{\text{def}}{=} \{ \check{h}(e) \mid e \in \text{ObHEF}(R, W) , \ z(e) = 1 \text{ and } |I_s(e)| = k \} .
\]

Lemma 4.19 For \( k \geq 1 \), we have:
\[
|\check{S}_{0,k}| = 2^k \cdot \sum_{K \subset I, i \in K \atop |K|=k} \prod_{\frac{n_i-1}{2}} .
\]

Proof. The proof is similar to that of Lemma 4.10. Consider a subset \( K \subset I(s) \) of cardinality \( k \). As in Lemma 4.10, there are \( \prod_{i \in K} \left[ \frac{n_i-1}{2} \right] \) different possibilities for \( s \) such that \( I(s) = K \). Fixing the set \( I(s) \) with \( |I(s)| = k \), we have a set \( \mathcal{P}(I(s)) \) of \( 2^k \) partitions \( I(s) = I_x \sqcup I_y \). Define:
\[
\check{S}_{0,k,s} = \{ \check{h}(e) \mid e \in \text{Obhef}(R, W) , \ z(e) = 1 , \ |I_s(e)| = k \text{ and } s(e) = s \} .
\]
The map which sends a partition \( \beta = (I_1, I_2) \) of \( I(s) \) to the element \( \alpha = (s, I_1, I_2) \) is a bijection. We compute:
\[
|\check{S}_{0,k,s}| = |\mathcal{P}(I(s))| = 2^k .
\]
This holds for any \( s \) with \( I(s) = K \), where \( K \subset I \) has cardinality \( k \). Since \( \check{S}_{0,k} = \sqcup_s \check{S}_{0,k,s} \) and since the cardinality \( |\check{S}_{0,k,s}| \) does not depend on \( s \), we find:
\[
|\check{S}_{0,k}| = \sum_s |\check{S}_{0,k,s}| = 2^k \cdot \sum_{K \subset I, |K|=k} \prod_{i \in K} \left[ \frac{n_i-1}{2} \right] .
\]
□
An immediate consequence is the following:

**Proposition 4.20** With the definitions above, we have:

\[
\tilde{N}_0(R, W) = |S_0| = \sum_{k=0}^{r} 2^k \sum_{K \subseteq I} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor ,
\]  \hspace{1cm} (4.29)

We are now ready to compute \(\tilde{N}(R, W)\).

**Theorem 4.21** The number of isomorphism classes of the category \(\text{hef}(R, W)\) for a critically-finite \(W\) as in (4.2) is given by:

\[
\tilde{N}(R, W) = \sum_{k=0}^{r} \sum_{K \subseteq I, |K|=k} 2^{r^0 + k} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor ,
\]  \hspace{1cm} (4.30)

**Proof.** Using Proposition 4.18 and Proposition 4.20, we write:

\[
\tilde{N}(R, W) = \sum_{J \subseteq I^0} \left( \sum_{k=0}^{r-j} 2^k \sum_{K \subseteq I \setminus J, |K|=k} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor \right),
\]  \hspace{1cm} (4.31)

where \(j \overset{\text{def}}{=} |J|\). Consider a subset \(K \subseteq I\) as in (4.29). Such an index set \(K = K^1 \cup K^0 \subseteq I\) of cardinality \(k = k^1 + k^0\) appears in \(\tilde{N}_0(R, \tilde{W})\) if \(K^0 \subseteq I^0 \setminus J\). The coefficient of \(\prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor\) in \(\tilde{N}(R, \tilde{W})\) is \(2^k\) for any of \(\binom{r^0 - k^0}{j}\) choices of \(J\). It follows that this coefficient in \(\tilde{N}(R, W)\) is:

\[
\sum_{j=0}^{r^0 - k^0} 2^k \binom{r^0 - k^0}{j} = 2^{r^0 + k^1} 2^{r^0 - k^0} = 2^{r^0 + k^1}.
\]  \hspace{1cm} (4.32)

Together with (4.32), relation (4.31) yields (4.30). \(\square\)

### 5. Some examples

In this section, we discuss a few classes of examples to which the results of the previous sections apply. Subsection 5.1 considers the ring of complex-valued holomorphic functions defined on a smooth, non-compact and connected Riemann surface, which will be discussed in more detail in a separate paper. Subsection 5.2 considers rings arising through the Krull-Kaplansky-Jaffard-Ohm construction, which associates to any lattice-ordered Abelian group a Bézout domain having that ordered group as its group of divisibility. Subsection 5.3 discusses Bézout domains with a specified spectral poset, examples of which can be produced by a construction due to Lewis.
5.1. Elementary holomorphic factorizations over a non-compact Riemann surface. Let \( \Sigma \) be any non-compact connected Riemann surface (notice that such a surface need not be algebraic and that it may have infinite genus and an infinite number of ends). It is known that the cardinal Krull dimension of \( \mathcal{O}(\Sigma) \) is independent of \( \Sigma \) and is greater than or equal to \( 2^\aleph_0 \) (see \([10,11]\)).

The following classical result (see \([12,13]\)) shows that the \( \mathbb{C}\text{-algebra} \) of holomorphic functions entirely determines the complex geometry of \( \Sigma \).

**Theorem 5.1 (Bers)** Let \( \Sigma_1 \) and \( \Sigma_2 \) be two connected non-compact Riemann surfaces. Then \( \Sigma_1 \) and \( \Sigma_2 \) are biholomorphic iff \( \mathcal{O}(\Sigma_1) \) and \( \mathcal{O}(\Sigma_2) \) are isomorphic as \( \mathbb{C}\)-algebras.

A Bézout domain \( R \) is called **adequate** if for any \( a \in R^\times \) and any \( b \in R \), there exist \( r, s \in R \) such that \( a = rs \), \( (r, b) = (1) \) and any non-unit divisor \( s' \) of \( s \) satisfies \( (s', b) \neq (1) \). It is known that any adequate Bézout domain \( R \) is an elementary divisor domain, i.e. any matrix with elements from \( R \) admits a Hermite normal form. The following result provides a large class of examples of non-Noetherian adequate Bézout domains:

**Theorem 5.2** For any smooth and connected non-compact Riemann surface \( \Sigma \), the ring \( \mathcal{O}(\Sigma) \) is an adequate Bézout domain and hence an elementary divisor domain.

*Proof.* The case \( \Sigma = \mathbb{C} \) was established in \([8,14]\). This generalizes to any Riemann surface using \([11,12]\). Since \( \mathcal{O}(\Sigma) \) is an adequate Bézout domain, it is also a \( PM^* \) ring\(^4\) \([8]\) and hence \([15]\) an elementary divisor domain. The fact that \( \mathcal{O}(\Sigma) \) is an elementary divisor domain can also be seen as follows. Guralnick \([16]\) proved that \( \mathcal{O}(\Sigma) \) is a Bézout domain of stable range one. By \([17,18]\), this implies that \( \mathcal{O}(\Sigma) \) is an elementary divisor domain. \( \square \)

The prime elements of \( \mathcal{O}(\Sigma) \) are those holomorphic functions \( f : \Sigma \to \mathbb{C} \) which have a single simple zero on \( \Sigma \). This follows, for example, from the Weierstrass factorization theorem on non-compact Riemann surfaces (see \([19, \text{Theorem 26.7}]\)). A critically-finite element \( W \in \mathcal{O}(\Sigma) \) has the form \( W = W_0W_c \), where \( W_0 : \Sigma \to \mathbb{C} \) is a holomorphic function with (possibly infinite) number of simple zeroes and no multiple zeroes while \( W_c \) is a holomorphic function which has only a finite number of zeroes, all of which have multiplicity at least two. All results of this paper apply to this situation, allowing one to determine the homotopy category \( \text{hef}(R,W) \) of elementary D-branes (and to count the isomorphism classes of such) in the corresponding holomorphic Landau-Ginzburg model \([3,4]\) defined by \((\Sigma,W)\).

5.2. Constructions through the group of divisibility. Recall that the group of divisibility \( G(R) \) of an integral domain \( R \) is the quotient group \( K^\times/U(R) \), where \( K \) is the quotient field of \( R \) and \( U(R) \) is the group of units of \( R \). This is a partially-ordered Abelian group when endowed with the order induced by the \( R\)-divisibility relation, whose positive cone equals \( R^\times/U(R) \). Equivalently, \( G(R) \) is the group of principal non-zero fractional ideals of \( R \), ordered by reverse inclusion. Since the positive cone generates \( G(R) \), a theorem due to Clifford implies that \( G(R) \) is a directed group (see \([20, \text{par. 4.3}]\)). It is an open question to characterize those directed Abelian groups which arise as groups of divisibility of integral domains. It is known that \( G(R) \) is totally-ordered iff \( R \) is a valuation domain, in which case \( G(R) \) is order-isomorphic with the value group of \( R \) and the natural surjection of \( K^\times \) to \( G(R) \) gives the corresponding valuation. Moreover, a theorem due to Krull \([21]\) states that any totally-ordered Abelian group arises as

\(^4\) A \( PM^* \)-ring is a unital commutative ring \( R \) which has the property that any non-zero prime ideal of \( R \) is contained in a unique maximal ideal of \( R \).
the group of divisibility of some valuation domain. It is also known\(^5\) that \(R\) is a UFD iff \(G(R)\) is order-isomorphic with a (generally infinite) direct sum of copies of \(\mathbb{Z}\) endowed with the product order (see [20, Theorem 4.2.2]).

An ordered group \((G, \leq)\) is called lattice-ordered if the partially ordered set \((G, \leq)\) is a lattice, i.e. any two element subset \(\{x, y\} \subset G\) has an infimum \(\inf(x, y)\) and a supremum \(\sup(x, y)\) (these two conditions are in fact equivalent for a group order); in particular, any totally-ordered Abelian group is lattice-ordered. Any lattice-ordered Abelian group is torsion-free (see [22, p. 10] or [23, 15.7]). The divisibility group \(G(R)\) of an integral domain \(R\) is lattice-ordered iff \(R\) is a GCD domain \([20]\). In particular, the group of divisibility of a Bézout domain is a lattice-ordered group.

When \(R\) is a Bézout domain, the prime elements of \(R\) are detected by the lattice-order of \(G(R)\) as follows. Given any Abelian lattice-ordered group \((G, \leq)\) and any \(x \in G\), let \(\uparrow x \overset{\text{def}}{=} \{y \in G | x \leq y\}\) and \(\downarrow x \overset{\text{def}}{=} \{y \in G | y \leq x\}\) denote the up and down sets determined by \(x\). A positive filter of \((G, \leq)\) is a filter of the lattice \((G_+, \leq)\), i.e. a proper subset \(F \subset G_+\) having the following two properties:

1. \(F\) is upward-closed, i.e. \(x \in F\) implies \(\uparrow x \subset F\)
2. \(F\) is closed under finite meets, i.e. \(x, y \in F\) implies \(\inf(x, y) \in F\).

Notice that \(\uparrow x\) is a positive filter for any \(x \in G_+\). A positive filter \(F\) of \((G, \leq)\) is called:

(a) principal, if \(G_+ \setminus F\) is a semigroup, i.e. if \(x, y \in G_+ \setminus F\) implies \(x + y \in G_+ \setminus F\).

(b) principal, if there exists \(x \in F\) such that \(F = \uparrow x\).

If \(R\) is a Bézout domain with field of fractions \(K\) and group of divisibility \(G = K^\times / U(R)\), then the natural projection \(\pi : K^\times \to G\) induces a bijection between the set of proper ideals of \(R\) and the set of positive filters of \(G\), taking a proper ideal \(I\) to the positive filter \(\pi(I \setminus \{0\})\) and a positive filter \(F\) to the proper ideal \(\{0\} \cup \pi^{-1}(F)\) (see [24,25]). This correspondence maps prime ideals to prime positive filters and non-zero principal ideals to principal positive filters. In particular, the prime elements of \(R\) correspond to the principal prime positive filters of \(G\).

The following result shows (see [7, Theorem 5.3, p. 113]) that any lattice-ordered Abelian group is the group of divisibility of some Bézout domain, thus allowing one to construct a very large class of examples of such domains using the theory of lattice-ordered groups:

**Theorem 5.3 (Krull-Kaplansky-Jaffard-Ohm)** If \((G, \leq)\) is a lattice-ordered Abelian group, then there exists a Bézout domain \(R\) whose group of divisibility is order-isomorphic to \((G, \leq)\).

For any totally ordered group \(G_0\), the result of Krull mentioned above gives a valuation ring whose divisibility group is order-isomorphic to \(G_0\). This valuation ring can be taken to be the group ring \(k[G_0]\), where \(k\) is a field together with the following valuation on the field of fractions \(k(G_0)\):

\[
v \left( \sum_{i=1}^{m} a_i X_{g_i} / \sum_{j=1}^{n} b_j X_{h_j} \right) = \inf(g_1, \ldots, g_m) - \inf(h_1, \ldots, h_n),
\]

where it is assumed that all coefficients appearing in the expression are non-zero. Lorenzen [26] proved that every lattice-ordered group can be embedded into a direct product of totally ordered groups with the product ordering. This embedding is used by Kaplansky and Jaffard to construct the valuation domain \(R\) of Theorem 5.3. By the result of Lorenzen, there exists

\(^5\) Notice that a UFD is a Bézout domain if and only if it is Noetherian if it is a PID (see Appendix B).
a lattice embedding \( f : G \rightarrow H \) defined as \( \prod_{\gamma \in \Gamma} G_{\gamma} \), where \( G_{\gamma} \) is a totally ordered group for all \( \gamma \in \Gamma \) and \( H \) has the product ordering. Let \( Q = k\langle \{ X_g : g \in G \} \rangle \) be the group field with coefficients in a field \( k \) with the set of formal variables \( X_g \) indexed by elements of \( G \). There is a valuation \( \varphi : Q^\times \rightarrow H \). The integral domain \( R \) is the domain defined by this valuation, i.e. \( R \) defined as \( \{ 0 \} \cup \{ x \in Q^\times : \varphi(x) \geq 0 \} \). It is proved by Ohm that the divisibility group of \( R \) is order-isomorphic to \( G \). Combining this with the results of the previous sections, we have:

**Proposition 5.4** Let \( G \) be any lattice-ordered group and consider the ring \( R \) constructed by Theorem 5.3. Assume further that \( W \) is a critically-finite element of \( R \). Then statements of Theorems 3.2, 3.3 and 4.12 hold.

The simplest situation is when the lattice-ordered group \( G \) is totally ordered, in which case \( R \) is a valuation domain. Then a proper subset \( F \subset G_+ \) is a positive filter if it is upward-closed, in which case the complement \( G \setminus F \) is non-empty and downward-closed. If \( G \setminus F \) has a greatest element \( m \), then \( G \setminus F = \downarrow m \) and \( F = (\uparrow m) \setminus \{ m \} \). If \( G \setminus F \) does not have a greatest element, then \((G \setminus F, F)\) is a Dedekind cut of the totally-ordered set \((G, \leq)\). For example we can take \( G \in \{ \mathbb{Z}, \mathbb{Q}, \mathbb{R} \} \) with the natural total order:

- When \( G = \mathbb{Z} \), the Bézout domain \( R \) is a discrete valuation domain and thus a PID with a unique non-zero prime ideal and hence with a prime element \( p \in R \) which is unique up to association in divisibility. In this case, any positive filter of \( \mathbb{Z} \) is principal and there is only one prime filter, namely \( \uparrow 1 = \mathbb{Z}_+ \setminus \{ 0 \} \). A critically-finite potential has the form \( W = W_0 p^k \), where \( k \geq 2 \) and \( W_0 \) is a unit of \( R \).

- When \( G = \mathbb{Q} \), there are two types of positive filters. The first have the form \( F = (\uparrow q) \setminus \{ q \} = (q, +\infty) \cap \mathbb{Q} \), with \( q \in \mathbb{Q}_{\geq 0} \), while the second correspond to Dedekind cuts and have the form \( F = [a, +\infty) \cap \mathbb{Q} \) with \( a \in \mathbb{R}_{>0} \). In particular, a positive filter is principal iff it has the form \( F = [q, +\infty) \cap \mathbb{Q} \) with \( q \in \mathbb{Q}_{>0} \). A principal positive filter cannot be prime, since \( \mathbb{Q}_+ \setminus F = [0, q) \cap \mathbb{Q} \) is not closed under addition for \( q > 0 \). Hence the Bézout domain \( R \) has no prime elements when \( G = \mathbb{Q} \).

- When \( G = \mathbb{R} \), any proper subset of \( \mathbb{R}_+ \) has an infimum and hence positive filters have the form \( F = (a, +\infty) \) or \( F = [a, +\infty) \) with \( a \in \mathbb{R}_{\geq 0} \), the latter being the principal positive filters. No principal positive filters can be prime, so the corresponding Bézout domain has no prime elements.

We can construct more interesting examples as follows. Let \((G_i, \leq_i)_{i \in I}\) be any family of lattice-ordered Abelian groups, where the non-empty index set \( I \) is arbitrary. Then the direct product group \( G \) defined as \( \prod_{i \in I} G_i \) is a lattice-ordered Abelian group when endowed with the product order \( \leq \):

\[
(g_i)_{i \in I} \leq (g'_i)_{i \in I} \iff \forall i \in I : g_i \leq_i g'_i .
\]

Let \( \text{supp}(g) \) defined as \( \{ i \in I | g_i \neq 0 \} \). The direct sum \( G^0 \) defined as \( \oplus_{i \in I} G_i = \{ g = (g_i) \in G | \text{supp}(g) < \infty \} \) is a subgroup of \( G \) which becomes a lattice-ordered Abelian group when endowed with the order induced by \( \leq \). For any \( x = (x_i)_{i \in I} \in G \), we have \( \uparrow_G x = \prod_{i \in I} \uparrow x_i \) while for any \( x^0 \in G^0 \), we have \( \uparrow_{G^0} x^0 = \oplus_{i \in I} \uparrow x^0_i \), where \( \uparrow_G \) and \( \uparrow_{G^0} \) denote respectively the upper sets computed in \( G \) and \( G^0 \). Hence:

1. The principal positive filters of \( G \) have the form \( F = \prod_{i \in I} F_i \), where:
   1. each \( F_i \) is a non-empty subset of \( G_i^+ \) which either coincides with \( \uparrow 0_i \) or is a principal positive filter of \((G_i, \leq_i)\)
2. at least one of $F_i$ is a principal positive filter of $(G_i, \leq_i)$.

Such a principal positive filter $F$ of $G$ is prime iff the set $G_i^+ \setminus F_i$ is empty or a semigroup for all $i \in I$. In particular, the principal prime ideals of the Bezout domain $R$ associated to $G$ by the construction of Theorem 5.3 are in bijection with families (indexed by $I$) of principal prime ideals of the Bezout domains $R_i$ associated to $G_i$ by the same construction.

The non-zero principal prime ideals of $R$ are in bijection with families $(J_i)_{i \in S}$, where $S$ is a non-empty subset of $I$ and $J_i$ is a non-zero principal prime ideal of $R_i$ for each $i \in S$.

II. The principal positive filters of $G^0$ have the form $F^0 = \oplus_{i \in I} F_i$, where $(F_i)_{i \in I}$ is a family of subsets of $F_i \subseteq G_i^+$ such that the set $\text{supp } F = \{i \in I | F_i \neq \phi, 0_i\}$ is finite and non-empty and such that $F_i$ is a principal positive filter of $(G_i, \leq_i)$ for any $i \in \text{supp } F$. Such a principal positive filter $F^0$ of $G^0$ is prime iff $F_i$ is prime in $(G_i, \leq_i)$ for any $i \in \text{supp } F$. In particular, the non-zero principal prime ideals of the Bezout domain $R^0$ associated to $G^0$ by the construction of Theorem 5.3 are in bijection with finite families of the form $J_{i_1}, \ldots, J_{i_n}$ ($n \geq 1$), where $i_1, \ldots, i_n$ are distinct elements of $I$ and $J_{i_k}$ is a non-zero principal prime ideal of the Bezout domain $R_{i_k}$ associated to $G_{i_k}$ by the same construction.

It is clear that this construction produces a very large class of Bezout domains which have prime elements and hence to which Proposition 5.4 applies. For example, consider the direct power $G = \mathbb{Z}^I$ and the direct sum $G = \mathbb{Z}^{(I)}$, endowed with the product order. Then the Bezout domain $R^0$ associated to $\mathbb{Z}^{(I)}$ is a UFD whose non-zero principal prime ideals are indexed by the finite non-empty subsets of $I$. On the other hand, the non-zero principal prime ideals of the Bezout domain $R$ associated to $\mathbb{Z}^I$ are indexed by all non-empty subsets of $I$. Notice that $R$ and $R^0$ coincide when $I$ is a finite set.

5.3. Constructions through the spectral poset. Given a unital commutative ring $R$, its spectral poset is the prime spectrum $\text{Spec}(R)$ of $R$ viewed as a partially-ordered set with respect to the inclusion between prime ideals. Given a poset $(X, \leq)$ and two elements $x, y \in X$, we write $x \ll y$ if $x < y$ and $x$ is an immediate neighbour of $y$, i.e. if there does not exist any element $z \in X$ such that $x < z < y$. It was shown in [27] that the spectral poset of any unital commutative ring satisfies the following two conditions, known as Kaplansky’s conditions:

I. Every non-empty totally-ordered subset of $(\text{Spec}(R), \leq)$ has a supremum and an infimum (in particular, $\leq$ is a lattice order).

II. Given any elements $x, y \in \text{Spec}(R)$ such that $x < y$, there exist distinct elements $x_1, y_1$ of $\text{Spec}(R)$ such that $x \leq x_1 < y_1 \leq y$ and such that $x_1 \ll y_1$.

It is known [28, 29] that these conditions are not sufficient to characterize spectral posets. It was shown in [30] that a poset $(X, \leq)$ is order-isomorphic with the spectral poset of a unital commutative ring iff $(X, \leq)$ is profinite, i.e. iff $(X, \leq)$ is an inverse limit of finite posets; in particular, any finite poset is order-isomorphic with a spectral poset [29].

A partially ordered set $(X, \leq)$ is called a tree if for every $x \in X$, the lower set $\downarrow x = \{y \in X | y \leq x\}$ is totally ordered. The following result was proved by Lewis:

Theorem 5.5 [29] Let $(X, \leq)$ be a partially-ordered set. Then the following statements are equivalent:

(a) $(X, \leq)$ is a tree which has a unique minimal element $\theta \in X$ and satisfies Kaplansky’s conditions I. and II.
(b) \((X, \leq)\) is isomorphic with the spectral poset of a Bézout domain.

Moreover, \(R\) is a valuation domain iff \((X, \leq)\) is a totally-ordered set.

An explicit Bézout domain \(R\) whose spectral poset is order-isomorphic with a tree \((X, \leq)\) satisfying condition (a) of Theorem 5.5 is found by first constructing a lattice-ordered Abelian group \(G\) associated to \((X, \leq)\) and then constructing \(R\) from \(G\) as in Theorem 5.3. The lattice-ordered group \(G\) is given by [29]:

\[
G = \{ f : X^* \to \mathbb{Z} | |\text{supp}(f)| < \infty \}
\]

where \(X^* \overset{\text{def}}{=} \{ x \in X : \exists y \in X : y \preceq x \}\) and \(\text{supp} f \overset{\text{def}}{=} \{ x \in X^* | f(x) \neq 0 \}\) is a tree when endowed with the order induced from \(X\). The group operation is given by pointwise addition. The lattice order on \(G\) is defined by the positive cone:

\[
G_+ \overset{\text{def}}{=} \{ f \in G | f(x) > 0 \ \forall x \in \minsupp(f) \} = \{ f \in G | f(x) \geq 1 \ \forall x \in \minsupp(f) \}
\]

where the order on \(Z\) is the natural order and the minimal support of \(f \in G\) is defined through:

\[
\minsupp(f) \overset{\text{def}}{=} \{ x \in \supp(f) | \forall y \in X^* \text{ such that } y < x : f(y) = 0 \}
\]

Notice that \(f \in G_+\) if \(\minsupp(f) = \emptyset\) (in particular, we have \(0 \in G_+)\). The lattice-ordered Abelian group \(G\) has the property that the set of its prime positive filters\(^6\) (ordered by inclusion) is order-isomorphic with the tree obtained from \((X, \leq)\) by removing the minimal element \(\theta\) (which corresponds to the zero ideal of \(R\)). Explicitly, the positive prime filter \(F_x\) associated to an element \(x \in X \setminus \{ \theta \}\) is defined through [29, p. 432]:

\[
F_x \overset{\text{def}}{=} \{ f \in G_+ | \exists y \in \minsupp(f) : y \leq x \} = \{ f \in G_+ | \minsupp(f) \cap (\downarrow x) \neq \emptyset \}
\]

By Lemma 2.23, a principal prime ideal of a Bézout domain is necessarily maximal. This implies that the prime elements of \(R\) (considered up to association in divisibility) correspond to certain maximal elements of the tree \((X, \leq)\). Notice, however, that a Bézout domain can have maximal ideals which are not principal (for example, the so-called “free maximal ideals” of the ring of complex-valued holomorphic functions defined on a non-compact Riemann surface \(\Sigma\) [11]). For any maximal element \(x\) of \(X\) which belongs to \(X^*\), let \(1_x \in G\) be the element defined by the characteristic function of the set \(\{ x \}\) in \(X^*\):

\[
1_x(y) \overset{\text{def}}{=} \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \in X^* \setminus \{ x \} \end{cases}
\]

Then \(\supp(1_x) = \minsupp(1_x) = \{ x \}\) and \(1_x \in G_+ \setminus \{ 0 \}\). Notice that \(1_x \in F_x\).

**Proposition 5.6** Let \((X, \leq)\) be a tree which has a unique minimal element and satisfies Kaplansky’s conditions I. and II. and let \(R\) be the Bézout domain determined by \((X, \leq)\) as explained above.

(a) For each maximal element \(x\) of \(X\) which belongs to \(X^*\), the principal positive filter \(\uparrow 1_x\) is prime and hence corresponds to a principal prime ideal of \(R\). Moreover, we have:

\[
\uparrow 1_x = \{ f \in G_+ | \supp(f) \cap \downarrow x \neq \emptyset \}
\]

\(^6\) Called “prime V-segments” in [29].
On the other hand, we have \( \text{minsupp} (x, y) \neq 0 \) holds. The second statement follows from the results of the previous sections. To prove the first statement, let \( x \) be a maximal element of \( X \) which belongs to \( X^* \). We have:

\[
\uparrow 1_x = \{ f \in G_+ \mid f - 1_x \in G_+ \} = \{ f \in G_+ \mid f(y) > 1_x(y) \forall y \in \text{minsupp} (f - 1_x) \}.
\]

(5.6)

On the other hand, we have \( \text{minsupp} (f - 1_x) = \{ y \in X^* \mid f(y) \neq 1_x(y) \& \forall z \in X^* \text{ such that } z < y : f(z) = 0 \} \). Since \( x \) is maximal, any element \( z \in X^* \) for which there exists \( y \in X^* \) such that \( z < y \) satisfies \( z \neq x \) and hence \( 1_x(z) = 0 \). This gives:

\[
\text{minsupp} (f - 1_x) = \{ y \in X^* \mid f(y) \neq 1_x(y) \& \forall z \in X^* \text{ such that } z < y : f(z) = 0 \}
\]

\[
= \{ \text{minsupp} (f) \cup \{ x \} \mid f \in Q_x \} \& \{ \text{minsupp} (f) \setminus \{ x \} \mid f \in G_+ \setminus Q_x \},
\]

where:

\[
Q_x \equiv \{ f \in G_+ \mid f(x) \neq 1 \& \forall z \in X^* \text{ such that } z < x : f(z) = 0 \} = A_x \cup B_x,
\]

with:

\[
A_x \equiv \{ f \in G_+ \mid \forall z \in X^* \text{ such that } z \leq x : f(z) = 0 \} = \{ f \in G_+ \mid \text{supp} (f) \cap \downarrow x = \emptyset \}
\]

\[
B_x \equiv \{ f \in G_+ \mid x \in \text{minsupp} (f) \& f(x) > 1 \} \subset F_x \subset G_+ \setminus A_x.
\]

This gives:

\[
\uparrow 1_x = (G_+ \setminus Q_x) \cup B_x = (G_+ \setminus A_x) \cup B_x = G_+ \setminus A_x = \{ f \in G_+ \mid \text{supp} (f) \cap \downarrow x \neq \emptyset \}
\]

(5.7)

which establishes (5.4). Notice that \( G_+ \setminus (\uparrow 1_x) = A_x \) is a semigroup, so \( \uparrow 1_x \) is a prime principal positive filter and hence it corresponds to a principal prime ideal of \( R \). Also notice that \( F_x \subset \uparrow 1_x \).

Consider an element \( f \in \uparrow 1_x \). Then the non-empty set \( S_f(x) \equiv \text{supp} (f) \cap \downarrow x \) is totally ordered (since \( X \) is a tree and hence \( \downarrow x \) is totally ordered). By Kaplansky’s condition I., this set has an infimum which we denote by \( x_f = \inf S_f(x) \); notice that \( x_f \in \downarrow x \). For any \( y \in X^* \) with \( y < x_f \), we have \( y \notin S_f(x) \) and hence \( f(y) = 0 \). Hence if \( x_f \) belongs to \( S_f(x) \) (i.e. if \( S_f(x) \) has a minimum), then \( x_f = \min S_f(x) \) is an element of \( \text{minsupp} (f) \cap \downarrow x \) and in this case we have \( f \in F_x \). Conversely, given any element \( f \in F_x \), it is easy to see that the totally-ordered set \( \text{minsupp} (f) \cap \downarrow x \) must be a singleton, hence \( \text{minsupp} (f) \cap \downarrow x = \{ x_f \} \) for a unique element \( x_f \in S_f(x) \). This element must be a minimum (and hence an infimum) of the totally-ordered set \( S_f(x) \) if \( x_f \) belongs to \( \text{minsupp} (f) \). We conclude that (5.5) holds.

\( \Box \)

Remark 5.1. Statement (a) of Proposition 5.6 allows us to construct particularly critically-finite elements of \( R \) as follows. For each maximal element of \( X \) which belongs to \( X^* \), let \( p_x \) be prime element of \( R \) which generates the principal prime ideal corresponding to the principal prime positive filter \( \uparrow 1_x \) (notice that \( p_x \) is determined up to association in divisibility). For any finite collection \( x_1, \ldots, x_N \) (\( N \geq 1 \)) of maximal elements of \( X \) which belong to \( X^* \) and any integers \( n_1, \ldots, n_N \) such that \( n_j \geq 2 \) for each \( j \in \{ 1, \ldots, N \} \), the element \( W = \prod_{j=1}^{N} p_{x_j}^{n_j} \in R \) is critically-finite.
The following statement will be used in the construction of some examples below:

**Proposition 5.7** Let \((S, \leq)\) be a well-ordered set. Then \((S, \leq)\) is a tree with a unique minimal element. Moreover, \((S, \leq)\) satisfies Kaplansky’s conditions I. and II. iff \(S\) has a maximum.

**Proof.** Since \(S\) is well-ordered, it is totally ordered and has a minimum, therefore it is a tree with a unique minimal element. Given \(x, y \in S\) such that \(x < y\), we have \(x \leq x_1 < y \leq y_1 \leq y\), where \(x_1 \overset{\text{def}}{=} \min\{s \mid x < s \leq y\}\) and \(y_1 \overset{\text{def}}{=} \min\{s \mid x_1 < s \leq y\}\). Thus \(S\) satisfies Kaplansky’s condition I. Any non-empty totally-ordered subset \(A \subseteq S\) has a minimum since \(S\) is well-ordered. Moreover, \(A\) has a supremum (namely \(\min\{s \in S \mid \forall x \in A : x < s\}\)) iff it has an upper bound. Hence \(S\) satisfies Kaplansky’s condition II. iff every non-empty subset of \(S\) has an upper bound, which amounts to the condition that \(S\) has a greatest element. \(\Box\)

**Remark 5.2.** Every element of \(S\) (except a possible greatest element) has an immediate successor (upper neighbor). In particular, \(S\) has a maximal element iff it has a maximum \(M\), which in turn happens iff the order type \(\alpha\) of \(S\) is a successor ordinal. In this case, \(M\) has a predecessor iff \(\alpha\) is a double successor ordinal, i.e. iff there exists an ordinal \(\beta\) such that \(\alpha = \beta + 2\).

**Example 5.8** Consider the tree \(T\) whose underlying set is the set \(\mathbb{N} = \mathbb{Z}_{\geq 0}\) of non-negative integers together with the following partial order: \(0 < n\) for every \(n \in \mathbb{N}\) and there is no further strict inequality; notice that any maximal vertex \(n \in \mathbb{N}^\ast = \mathbb{Z}_{>0}\) has an immediate lower neighbor, namely \(0\). This corresponds to a countable corolla, i.e. a tree rooted at \(0\) and with an edge connecting the root to \(n\) for every \(n \in \mathbb{N}^\ast\) (and no other edges). By Proposition 5.6, each maximal vertex \(n \in \mathbb{N}^\ast\) corresponds to a principal prime ideal of the associated Bézout domain.

**Example 5.9** We can make the previous example more interesting by replacing the edges of \(T\) with a tree. For each \(x \in \mathbb{N}^\ast\), consider a tree \(T_x\) with a unique root (minimal element) \(r_x \in T_x\) and which satisfies Kaplansky’s conditions I. and II. Consider the tree \(T\) obtained by connecting 0 to \(r_x\) for \(x \in \mathbb{N}^\ast\). Then \(T\) has a unique minimal element (namely \(0\)) and satisfies Kaplansky’s conditions I. and II. By Proposition 5.6, those maximal elements of each of the trees \(T_x\) which have an immediate lower neighbor correspond to prime elements of the associated Bézout domain \(R\). We obtain many examples of Bézout domains by varying the trees \(T_x\):

1. Assume that for every \(x \in \mathbb{N}^\ast\), the tree \(T_x\) is reduced to the single point \(r_x = x\). Then we recover Example 5.8.

2. For any element \(x \in \mathbb{N}^\ast\), consider a finite tree \(T_x\) and let \(\Sigma_x\) be the set of maximal elements of \(T_x\). Then \(T^\ast = T \setminus \{0\}\) and any maximal element of \(T\) different from 0 has an immediate lower neighbor. The corresponding Bézout domain \(R\) has a principal prime ideal for every element of the set \(\bigcup_{x \in \mathbb{N}^\ast} \Sigma_x\).

3. For each \(x \in \mathbb{N}^\ast\), consider a well-ordered set \(S_x\) which has a maximum \(m_x\) and denote the minimum element of \(S_x\) by \(r_x\). By Proposition 5.7, we can take \(T_x = S_x\) in the general construction above, thus obtaining a tree \(T\) and a corresponding Bézout domain \(R\). Let \(U \subseteq \mathbb{N}^\ast\) be the set of those \(x \in \mathbb{N}^\ast\) for which \(S_x\) is a double successor ordinal. Then each element of \(U\) corresponds to a principal prime ideal of \(R\).
A. GCD domains

Let $\mathbb{R}$ be an integral domain and $U(\mathbb{R})$ its multiplicative group of units. For any finite sequence of elements $f_1, \ldots, f_n \in \mathbb{R}$, let $\langle f_1, \ldots, f_n \rangle$ denote the ideal generated by the set $\{f_1, \ldots, f_n\}$. An element $u \in \mathbb{R}$ is a unit iff $\langle u \rangle = \mathbb{R}$. Two elements $f, g \in \mathbb{R}$ are called associated in divisibility (we write $f \sim g$) if there exists $u \in U(\mathbb{R})$ such that $g = uf$. This is equivalent with the condition $\langle f \rangle = \langle g \rangle$. The association relation is an equivalence relation on $\mathbb{R}$.

Definition A.1 An integral domain $\mathbb{R}$ is called a GCD domain if any two elements $f, g$ admit a greatest common divisor (gcd).

Let $\mathbb{R}$ be a GCD domain. In this case, the gcd of two elements $f, g$ is determined up to association and the corresponding equivalence class is denoted by $(f, g)$. Any two elements $f, g$ of $\mathbb{R}$ also admit a least common multiple (l.c.m.), which is determined up to association and whose equivalence class is denoted by $[f, g]$. By induction, any finite collection of elements $f_1, \ldots, f_n$ admits a gcd and lcm, both of which are determined up to association and whose equivalence classes are denoted by:

\[ (f_1, \ldots, f_n) \text{ and } [f_1, \ldots, f_n]. \]

Remark A.1. Any irreducible element of a GCD domain is prime, hence primes and irreducibles coincide in a GCD domain. In particular, any element of a GCD domain which can be factored into primes has unique prime factorization, up to permutation and association of the prime factors.

B. Bézout domains

Let $\mathbb{R}$ be a GCD domain. We say that the Bézout identity holds for two elements $f$ and $g$ of $\mathbb{R}$ if for one (equivalently, for any) gcd $d$ of $f$ and $g$, there exist $a, b \in \mathbb{R}$ such that $d = af + bg$. This amounts to the condition that the ideal $\langle f, g \rangle$ is principal, namely we have $\langle f, g \rangle = \langle d \rangle$.

Definition B.1 An integral domain $\mathbb{R}$ is called a Bézout domain if any (and hence all) of the following equivalent conditions hold:

- $\mathbb{R}$ is a GCD domain and the Bézout identity holds for any two non-zero elements $f, g \in \mathbb{R}$.
- The ideal generated by any two elements of $\mathbb{R}$ is principal.
- Any finitely-generated ideal of $\mathbb{R}$ is principal.

More generally, a Bézout ring is a unital commutative ring $\mathbb{R}$ which has the property that its finitely-generated ideals are principal. Hence a Bézout domain is a Bézout ring which is an integral domain. The following well-known statement shows that the Bézout property is preserved under quotienting by principal ideals:

Proposition B.2 Let $\mathbb{R}$ be a Bézout ring and $I$ be a finitely-generated (hence principal) ideal of $\mathbb{R}$. Then $\mathbb{R}/I$ is a Bézout ring.

If $\mathbb{R}$ is a Bézout domain and $f_1, \ldots, f_n \in \mathbb{R}$, then we have $\langle f_1, \ldots, f_n \rangle = \langle d \rangle$ for any $d \in (f_1, \ldots, f_n)$ and there exist $a_1, \ldots, a_n \in \mathbb{R}$ such that $d = a_1f_1 + \ldots + a_nf_n$. The elements $f_1, \ldots, f_n$...
are called coprime if \((f_1, \ldots, f_n) = (1)\), which amounts to the condition \(\langle f_1, \ldots, f_n \rangle = R\). This happens iff there exist elements \(a_1, \ldots, a_n \in R\) such that \(a_1 f_1 + \ldots + a_n f_n = 1\). Notice that every Bézout domain is integrally closed [31].

Remark B.1. Bézout domains coincide with those Prüfer domains which are GCD domains. Since any Prüfer domain is coherent, it follows that any Bézout domain is a coherent ring.

The following result characterizes finitely-generated projective modules over Bézout domains:

**Proposition B.3** [7] Every finitely-generated projective module over a Bézout domain is free.

In particular, finitely-generated projective factorizations over a Bézout domain coincide with finite-rank matrix factorizations.

### B.2. Examples of Bézout domains.

The following rings are Bézout domains:

- Principal ideal domains (PIDs) coincide with the Noetherian Bézout domains. Other characterizations of PIDs among Bézout domains are given below.
- Any generalized valuation domain is a Bézout domain.
- The ring \(O(\Sigma)\) of holomorphic complex-valued functions defined on any\(^7\) smooth connected non-compact Riemann surface \(\Sigma\) is a non-Noetherian Bézout domain. In particular, the ring \(O(\mathbb{C})\) of entire functions is a non-Noetherian Bézout domain.
- The ring \(A\) of all algebraic integers (the integral closure of \(\mathbb{Z}\) inside \(\mathbb{C}\)) is a non-Noetherian Bézout domain which has no prime elements.

### B.3. The Noetherian case.

The following is well-known:

**Proposition B.4** Let \(R\) be a Bézout domain. Then the following statements are equivalent:

- \(R\) is Noetherian
- \(R\) is a principal ideal domain (PID)
- \(R\) is a unique factorization domain (UFD)
- \(R\) satisfies the ascending chain condition on principal ideals (ACCP)
- \(R\) is an atomic domain.

### B.4. Characterizations of Bézout domains.

**Definition B.5** Let \(R\) be a commutative ring. The Bass stable rank \(\text{bsr}(R)\) of \(R\) is the smallest integer \(n\), such that for any collection \(\{a_0, a_1, \ldots, a_n\}\) of generators of the unit ideal, there exists a collection \(\{\lambda_1, \ldots, \lambda_n\}\) in \(R\) such that the collection \(\{a_i - \lambda_i a_0 : 1 \leq i \leq n\}\) also generate the unit ideal. If no such \(n\) exists, then \(\text{bsr}(R) \equiv \infty\).

\(^7\) Notice that \(\Sigma\) need not be algebraic. In particular, \(\Sigma\) can have infinite genus and an infinite number of ends.
**Definition B.6** A unital commutative ring $R$ is called a Hermite ring (in the sense of Kaplan-sky) if every matrix $A$ over $R$ is equivalent with an upper or a lower triangular matrix.

The following result is proved in [32, Theorem 8.1]

**Theorem B.7** [32] Let $R$ be a Bézout domain. Then $\text{bsr}(R) \leq 2$. Moreover, $R$ is a Hermite ring.

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