Nested Bethe Ansatz for RTT–Algebra $A_n$

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Abstract

This paper continues our recent studies on the algebraic Bethe ansatz for the
RTT–algebras of sp$(2n)$ and $o(2n)$ types. In these studies, we encountered the
RTT–algebras which we called $A_n$. The next step in our construction of the Bethe
vectors for the RTT-algebras of type sp$(2n)$ and $o(2n)$ is to find the Bethe vectors
for the RTT–algebra $A_n$. This paper deals with the construction of the Bethe
vectors of the RTT–algebra $A_n$ using the Bethe vectors of the RTT–algebra $A_{n-1}$.

1 Introduction

In studying the algebraic Bethe ansatz for the RTT–algebras of type sp$(2n)$ and $o(2n)$
[1 2], we discovered some the RTT–algebras which we called $A_n$. The main result of
these works is the assertion that for the construction of eigenvalues and eigenvectors of
the transfer–matrix of the RTT–algebras of type sp$(2n)$ and $o(2n)$ it is enough to find
eigenvalues and eigenvectors for the RTT–algebra $A_n$.

In this work, we deal with the nested Bethe ansatz for the RTT–algebra $A_n$. We show
how to construct eigenvectors for the RTT–algebra $A_n$ by using eigenvectors of the RTT–
algebra $A_{n-1}$.

Note, that the RTT–algebra $A_{n-1}$ is not the RTT–subalgebra $A_n$. However, $A_n$
contains two the RTT–subalgebras $A^{(+)}_n$ and $A^{(-)}_n$, which are of type gl$(n)$. The RTT–algebras
$A^{(\pm)}_{n-1}$ are already the RTT–subalgebras of $A^{(\pm)}_n$. As we will see later, we can construct
some eigenvectors for the RTT–algebras $A_n$ as Bethe vectors of the RTT–algebras $A^{(\pm)}_n$, i.e.
as the Bethe vectors for the RTT–algebras of the type gl$(n)$. Our result for such
eigenvectors is the same as for the nested Bethe ansatz for the RTT–algebras of gl$(n)$,
which can be found in [3]. In this sense, our construction is a certain generalization of
the nested Bethe ansatz for the RTT–algebras of type gl$(n)$.

The proofs of many claims are only a suitable, but long adjustment of the Yang–Baxter
and the RTT–equations. We have included them in Appendix for better clarity of the
main text.

2 The RTT–algebra $A_n$

We denote $E^i_k$ and $E^{-i}_{-k}$, where $i, k = 1, \ldots, n$, the matrices $(E^i_k)^r_s = (E^{-i}_{-k})^{r}_{-s} = \delta^i_k \delta^r_s$.

Then the relations $E^i_k E^r_s = \delta^i_s E^r_k$, $\sum_{i=1}^n E^i_i = I_+$ and $\sum_{i=1}^n E^{-i}_{-i} = I_-$ apply.
The RTT–algebra $\mathcal{A}_n$ is an associative algebra with a unit that is generated by the elements $T^i_k(x)$ and $T^{-i}_k(x)$, where $i, k = 1, \ldots, n$. If we introduce the monodromy matrix $T(x) = T^{(+)}(x) + T^{(-)}(x)$, where

$$T^{(+)}(x) = \sum_{i,k=1}^{n} E^i_k \otimes T^i_k(x), \quad T^{(-)}(x) = \sum_{i,k=1}^{n} E^{-i}_k \otimes T^{-i}_k(x),$$

the commutation relations between generators are defined by the RTT–equation

$$R_{1,2}(x,y)T_1(x)T_2(y) = T_2(y)T_1(x)R_{1,2}(x,y), \quad (1)$$

where $R$–matrix is $R(x, y) = R^{(+,+)}(x, y) + R^{(+,-)}(x, y) + R^{(-,+)}(x, y) + R^{(-,-)}(x, y)$,

$$R^{(+,+)}(x, y) = \frac{1}{f(x, y)} \left( I_+ \otimes I_+ + g(x, y) \sum_{i,k=1}^{n} E^i_k \otimes E^k_i \right),$$

$$R^{(+,-)}(x, y) = I_+ \otimes I_- - k(x, y) \sum_{i,k=1}^{n} E^i_k \otimes E^{-i}_k,$$

$$R^{(-,+)}(x, y) = I_- \otimes I_+ - h(x, y) \sum_{i,k=1}^{n} E^{-i}_k \otimes E^i_k,$$

$$R^{(-,-)}(x, y) = \frac{1}{f(x, y)} \left( I_- \otimes I_- + g(x, y) \sum_{i,k=1}^{n} E^{-i}_k \otimes E^{-k}_i \right),$$

$$g(x, y) = \frac{1}{x - y}, \quad f(x, y) = \frac{x - y + 1}{x - y},$$

$$h(x, y) = \frac{1}{x - y + n - \eta}, \quad k(x, y) = \frac{1}{x - y + \eta}$$

and $\eta$ is any number. For $\eta = -1$ we obtain the RTT–algebra connected with the RTT–algebra of $\text{sp}(2n)$ type and for $\eta = 1$ the RTT–algebra connected with the RTT–algebra of $\text{o}(2n)$ type.

By direct calculation, it can be verified that this $R$–matrix satisfies the Yang–Baxter equation

$$R_{1,2}(x,y)R_{1,3}(x,z)R_{2,3}(y,z) = R_{2,3}(y,z)R_{1,3}(x,z)R_{1,2}(x,y) \quad (2)$$

and has the inverse $R$–matrix

$$(R(x,y))^{-1} = (R^{(+,+)}(x,y))^{-1} + (R^{(+,-)}(x,y))^{-1} + (R^{(-,+)}(x,y))^{-1} + (R^{(-,-)}(x,y))^{-1}$$

where

$$(R^{(+,+)}(x,y))^{-1} = \frac{1}{f(x,y)} \left( I_+ \otimes I_+ + g(y, x) \sum_{i,k=1}^{n} E^i_k \otimes E^k_i \right),$$

$$(R^{(+,-)}(x,y))^{-1} = I_+ \otimes I_- - h(y, x) \sum_{i,k=1}^{n} E^i_k \otimes E^{-i}_k,$$

$$(R^{(-,+)}(x,y))^{-1} = I_- \otimes I_+ - k(y, x) \sum_{i,k=1}^{n} E^{-i}_k \otimes E^i_k,$$

$$(R^{(-,-)}(x,y))^{-1} = \frac{1}{f(x,y)} \left( I_- \otimes I_- + g(y, x) \sum_{i,k=1}^{n} E^{-i}_k \otimes E^{-k}_i \right).$$

Therefore, it defines the RTT–algebra that we denote by $\mathcal{A}_n$.

The explicit form of commutation relations between generators of the RTT–algebra $\mathcal{A}_n$ is given in the Appendix.
It is easily seen that the RTT–equation (1) can be written as

\[ R_{1,2}^{(e_1,e_2)}(x,y)T_1^{(e_1)}(x)T_2^{(e_2)}(y) = T_2^{(e_2)}(y)T_1^{(e_1)}(x)R_{1,2}^{(e_1,e_2)}(x,y), \]  

(3)

where \( e_1, e_2 = \pm \). From this form of the RTT–equation it is clear that in the RTT–algebra \( \mathcal{A}_n \) there are two RTT-subalgebras \( \mathcal{A}_n^{(+)} \) and \( \mathcal{A}_n^{(-)} \), which are generated by the elements \( T_k^i(x) \) and \( T_{-k}^i(x) \), where \( i, k = 1, \ldots, n \).

Using the RTT–equation (3), it is possible to show that in the RTT–algebra \( \mathcal{A}_n \) the operators

\[ H^{(+)}(x) = \text{Tr} T^{(+)}(x) = \sum_{i=1}^{n} T_i^r(x), \quad H^{(-)}(x) = \text{Tr} T^{(-)}(x) = \sum_{i=1}^{n} T_{-i}^r(x) \]

mutually commute.

We deal with the representations of the RTT–algebra \( \mathcal{A}_n \) on the vector space \( \mathcal{W} = \mathcal{A}_n \omega \), where \( \omega \) is a vacuum vector for which the relations

\[
T_k^i(x)\omega = 0 \quad \text{for} \quad 1 \leq i < k \leq n, \quad T_k^i(x)\omega = \lambda_i(x)\omega \\
T_{-k}^i(x)\omega = 0 \quad \text{for} \quad 1 \leq i < k \leq n, \quad T_{-k}^i(x)\omega = \lambda_{-i}(x)\omega
\]

hold. Our goal is to find in the vector space \( \mathcal{W} \) common eigenvectors of the operators \( H^{(\pm)}(x) \).

In the RTT–algebra \( \mathcal{A}_n \) there are two RTT–subalgebras \( \mathcal{A}^{(+)} = \mathcal{A}_{n-1}^{(+)} \) and \( \mathcal{A}^{(-)} = \mathcal{A}_{n-1}^{(-)} \) of \( \text{gl}(n-1) \) type, which are generated by the elements \( T_k^i(x) \) and \( T_{-k}^i(x) \), where \( i, k = 1, \ldots, n-1 \).

First, we will consider the subspace \( \mathcal{W} \) generated by the elements \( \mathcal{A}^{(+)} \mathcal{A}^{(-)} \omega \).

**Proposition 1.** The relations

\[ T_n^i(x)w = T_{-n}^i(x)w = 0, \quad T_n^i(x)w = \lambda_n(x)w, \quad T_{-n}^i(x)w = \lambda_{-n}(x)w \]  

(4)

hold for any \( w \in \mathcal{W} \) and \( i = 1, 2, \ldots, n-1 \).

**Proof.** First, we consider the space \( \mathcal{W}^{(-)} = \mathcal{A}^{(-)} \omega \subset \mathcal{W} \). To prove relation (4) for \( w = w^{(-)} \in \mathcal{W}^{(-)} \), it is sufficient to show that if (4) is valid for \( w^{(-)} \), it also applies to \( T_{-i}^s(y)w^{(-)} \), where \( r, s = 1, \ldots, n-1 \). From the commutation relations we get for \( i = 1, \ldots, n \) and \( r, s = 1, \ldots, n-1 \)

\[ T_{-n}^i(x)T_{-s}^r(y) = T_{-s}^r(y)T_{-n}^i(x) + g(x,y)T_{-i}^r(y)T_{-n}^s(x) - g(x,y)T_{-i}^r(x)T_{-s}^n(y) \]

It follows that for any \( w^{(-)} \in \mathcal{W}^{(-)} \) we have

\[ T_{-i}^s(x)w^{(-)} = 0 \quad \text{for} \quad i = 1, \ldots, n-1, \quad T_{-n}^i(x)w^{(-)} = \lambda_{-n}(x)w^{(-)}. \]

For any \( r, s = 1, \ldots, n-1 \), the commutation relations give

\[ T_n^r(x)T_{-s}^r(y) = T_{-s}^r(y)T_n^r(x), \]

which proves that \( T_n^r(x)w^{(-)} = \lambda_n(x)w^{(-)} \) for any \( w^{(-)} \in \mathcal{W}^{(-)} \).
For any $i, r, s = 1, \ldots, n - 1$ the relations
\[
T^r_n(x)T^{-r}_s(y) = T^{-r}_s(y)T^r_n(x) - \delta^{i,r}h(y,x)\sum_{p=1}^{n-1} T^{-p}_s(y)T^p_n(x) - \delta^{i,r}h(y,x)T^{-n}_s(y)T^n_n(x).
\]
hold. Since for every $w \in \tilde{\mathcal{W}}$ we have
\[
T^{-n}_s(y)T^n_n(x)w = \lambda_n(x)T^{-n}_s(y)w = 0,
\]
we see that for every $w \in \tilde{\mathcal{W}}$ and $i = 1, \ldots, n - 1$ we have $T^i_i(x)w = 0$.

Since $\tilde{\mathcal{W}} = \tilde{A}^{(+)}\tilde{\mathcal{W}}^{(-)}$, it is sufficient to show that if $\tilde{\mathcal{W}}^{(-)}$ holds for $w$, it also holds for $T^r_s(y)w$, where $r, s = 1, \ldots, n - 1$.

For $i = 1, \ldots, n$ a $r, s = 1, \ldots, n - 1$ we have the commutation relation
\[
T^i_i(x)T^r_s(y) = T^r_s(y)T^i_i(x) + g(y,x)T^i_i(x)T^r_s(y) - g(y,x)T^r_s(y)T^i_i(x),
\]
from which we can easily see that for any $w \in \tilde{\mathcal{W}}$
\[
T^i_i(x)w = 0 \quad \text{for} \quad i = 1, \ldots, n - 1, \quad T^n_n(x)w = \lambda_n(x)w
\]
holds.

The relation $T^{-n}_n(x)w = \lambda_n(x)w$ results from the fact that for every $r, s = 1, \ldots, n - 1$ we have
\[
T^{-n}_n(x)T^r_s(y) = T^r_s(y)T^{-n}_n(x).
\]
For $i, r, s = 1, \ldots, n - 1$ we use
\[
T^{-i}_i(x)T^r_s(y) = T^r_s(y)T^{-i}_i(x) - \delta_{i,s}h(x,y)\sum_{p=1}^{n-1} T^p_s(y)T^{-n}_s(x) - \delta_{i,s}h(x,y)T^n_s(y)T^{-n}_s(x),
\]
which implies that $T^{-i}_i(x)w = 0$ for $i = 1, \ldots, n - 1$ and for any $w \in \tilde{\mathcal{W}}$. \hfill \Box

**Proposition 2.** The space $\tilde{\mathcal{W}}$ is invariant with respect to $\tilde{A}^{(+)}$ and $\tilde{A}^{(-)}$.

**Proof:** Obviously, the space $\tilde{\mathcal{W}}$ is invariant for the $\tilde{A}^{(+)}$ action.

To show that the space $\tilde{\mathcal{W}}$ is invariant to the action of the algebra $\tilde{A}^{(-)}$, we will use for $i, k, r, s = 1, \ldots, n - 1$ the commutation relations
\[
T^{-i}_k(x)T^r_s(y) - \delta_{k,s}k(y,x)\sum_{p=1}^{n-1} T^{-i}_p(x)T^p_s(y) - \delta_{k,s}k(y,x)T^{-i}_k(x)T^r_s(y) = T^r_s(y)T^{-i}_k(x) - \delta^{i,r}k(y,x)\sum_{p=1}^{n-1} T^p_s(y)T^{-p}_k(x) - \delta^{i,r}k(y,x)T^n_s(y)T^{-n}_k(x).
\]

From Proposition 1 it follows that if we restrict these relations to subspace $\tilde{\mathcal{W}}$, we get
\[
T^{-i}_k(x)T^r_s(y) - \delta_{k,s}k(y,x)\sum_{p=1}^{n-1} T^{-i}_p(x)T^p_s(y) = T^r_s(y)T^{-i}_k(x) - \delta^{i,r}k(y,x)\sum_{p=1}^{n-1} T^p_s(y)T^{-p}_k(x).
\]

If we multiply these equations by $\left(\delta^k_a\delta^s_b - \delta^{k,s}\delta_{a,b}\tilde{h}(x,y)\right)$, where
\[
\tilde{h}(x,y) = \frac{1}{x-y+(n-1)\eta},
\]

and sum them over $k, s$ from 1 to $n - 1$ and rename the indices, we find that the relations

$$T_{-k}(x)T_s(y) = T_s(y)T_{-k}(x) - \delta_{i,r}k(y, x) \sum_{p=1}^{n-1} T_{p}(y)T^{-p}_{-k}(x) - \delta_{k,s}\tilde{h}(x, y)\sum_{p=1}^{n-1} T_{p}(y)T^{-p}_{-q}(x)$$

are true on the space $\tilde{W}$.

The invariance of the space $\tilde{W}$ with respect to the action $\tilde{A}^0$ can be proven by induction according to numbers of the factors $T_k(x)$ in the vectors $w \in \tilde{W}$.

**Proposition 3.** If we define

$$\tilde{T}^{(+)}(x) = \sum_{i,k=1}^{n-1} E^k_i \otimes T^i_k(x), \quad \tilde{T}^{(-)}(x) = \sum_{i,k=1}^{n-1} E^{-k}_i \otimes T^{-i}_k(x)$$

the commutation relations for $T^k_i(x)$ and $T^{-k}_i(x)$, where $i, k = 1, \ldots, n - 1$, reduced to the space $\tilde{W}$ can be written in the form of the RTT-equation

$$\tilde{R}^{(\epsilon_1, \epsilon_2)}_{1, 2}(x, y)\tilde{T}^{(\epsilon_1)}_{1}(x)\tilde{T}^{(\epsilon_2)}_{2}(y) = \tilde{T}^{(\epsilon_2)}_{2}(y)\tilde{T}^{(\epsilon_1)}_{1}(x)\tilde{R}^{(\epsilon_1, \epsilon_2)}_{1, 2}(x, y)$$

where $\epsilon_1, \epsilon_2 = \pm$ and

$$\tilde{R}^{(+, +)}_{1, 2}(x, y) = \frac{1}{f(x, y)} \left( \tilde{I}_+ \otimes \tilde{I}_+ + g(x, y) \sum_{i,k=1}^{n-1} E^k_i \otimes E^i_k \right),$$

$$\tilde{R}^{(-, -)}_{1, 2}(x, y) = \frac{1}{f(x, y)} \left( \tilde{I}_- \otimes \tilde{I}_- + g(x, y) \sum_{i,k=1}^{n-1} E^{-i}_k \otimes E^{-i}_k \right),$$

$$\tilde{R}^{(\epsilon_1, -)}_{1, 2}(x, y) = \tilde{I}_+ \otimes \tilde{I}_- - k(x, y) \sum_{i,k=1}^{n-1} E^i_k \otimes E^{-i}_k,$$

$$\tilde{R}^{(-, \epsilon_1)}_{1, 2}(x, y) = \tilde{I}_- \otimes \tilde{I}_+ - \tilde{h}(x, y) \sum_{i,k=1}^{n-1} E^{-i}_k \otimes E^i_k,$$

$$\tilde{I}_+ = \sum_{k=1}^{n-1} E^k_k, \quad \tilde{I}_- = \sum_{k=1}^{n-1} E^{-k}_k, \quad \tilde{h}(x, y) = \frac{1}{x - y + n - 1 - \eta}.$$
If we restrict them on the space $\tilde{\mathcal{W}}$, we obtain according to Proposition 1

$$T_k^i(x)T_{-s}^j(y) - \delta^{i,r}k(x,y)\sum_{p=1}^{n-1} T_k^p(x)T_{-p}^j(y) = T_{-s}^j(y)T_k^i(x) - \delta_{k,s}k(x,y)\sum_{p=1}^{n-1} T_{-p}^s(y)T_p^i(x)$$

$$T_{-k}^i(x)T_{-s}^j(y) - \delta_{k,s}k(y,x)\sum_{p=1}^{n-1} T_{-p}^i(x)T_{-p}^j(y) = T_{-s}^j(y)T_{-k}^i(x) - \delta^{i,r}k(y,x)\sum_{p=1}^{n-1} T_{-s}^p(y)T_{-p}^i(x).$$

The first of these commutation relations is

$$\tilde{R}_{1,2}^{(+,-)}(x,y)\tilde{T}_1^{(-)}(x)\tilde{T}_2^{(+)}(y) = \tilde{T}_2^{(-)}(y)\tilde{T}_1^{(+)}(x)\tilde{R}_{1,2}^{(+,-)}(x,y).$$

The second equality can be written using matrices in the form

$$\tilde{T}_1^{(-)}(x)\tilde{T}_2^{(+)}(y)\left[\tilde{1}_- \otimes \tilde{1}_+ - k(y,x)\sum_{i,k=1}^{n-1} E_{-k}^i \otimes E_{k}^i\right] =$$

$$= \left[\tilde{1}_- \otimes \tilde{1}_+ - k(y,x)\sum_{i,k=1}^{n-1} E_{-k}^i \otimes E_{k}^i\right]\tilde{T}_2^{(+)}(y)\tilde{T}_1^{(-)}(x).$$

And since

$$\tilde{R}_{1,2}^{(-,+)}(x,y)\left[\tilde{1}_- \otimes \tilde{1}_+ - k(y,x)\sum_{i,k=1}^{n-1} E_{-k}^i \otimes E_{k}^i\right] =$$

$$= \left[\tilde{1}_- \otimes \tilde{1}_+ - k(y,x)\sum_{i,k=1}^{n-1} E_{-k}^i \otimes E_{k}^i\right]\tilde{R}_{1,2}^{(-,+)}(x,y) = \tilde{1}_- \otimes \tilde{1}_+$$

this relation is equivalent to the RTT-equation

$$\tilde{R}_{1,2}^{(-,+)}(x,y)\tilde{T}_1^{(-)}(x)\tilde{T}_2^{(+)}(y) = \tilde{T}_2^{(-)}(y)\tilde{T}_1^{(+)}(x)\tilde{R}_{1,2}^{(-,+)}(x,y).$$

The following theorem immediately follows from Proposition 3.

**Theorem 1.** The action of the operators $T_k^i(x)$ and $T_{-k}^i(x)$, where $i, k = 1, \ldots, n-1$, in the space $\mathcal{W}$ forms the RTT–algebra $\mathcal{A}_{n-1}$.

### 3 General form of common eigenvectors of $H^{(+)}(x)$ and $H^{(-)}(x)$

Let $\vec{v} = (v_1, v_2, \ldots, v_p)$ and $\vec{w} = (w_1, w_2, \ldots, w_Q)$ be ordered sets of mutually different numbers. We will search for a general shape of the common eigenvectors $H^{(+)}(x)$ and $H^{(-)}(x)$ in the form

$$\mathcal{B}(\vec{v}, \vec{w}) = \sum_{\substack{k_1, \ldots, k_p = 1 \ r_1, \ldots, r_Q = 1 \ k_1, \ldots, k_p = 1}}^{n-1} T_k^m(v_1) \cdots T_k^m(v_p) T_{-r_1}^{m_1}(w_1) \cdots T_{-r_Q}^{m_Q}(w_Q) \Phi_{\vec{r}_1, \ldots, \vec{r}_Q}^{k_1, \ldots, k_p},$$

where $\Phi_{\vec{r}_1, \ldots, \vec{r}_Q}^{k_1, \ldots, k_p} \in \tilde{\mathcal{W}}$. 

We will consider \((n - 1)\)-dimensional spaces \(\mathcal{V}_+\) and \(\mathcal{V}_-\) with the base \(e_k\) and \(e_r\) and denote \(f^k\) and \(f^{-r}\) their dual base in dual spaces \(\mathcal{V}^*_+\) and \(\mathcal{V}^*_-\).

Let us define
\[
\mathbf{b}^{(+)}(v) = \sum_{k=1}^{n-1} f^k \otimes T^m_k(v) \in \mathcal{V}^*_+ \otimes \mathcal{A}_n
\]
\[
\mathbf{b}^{(-)}(w) = \sum_{r=1}^{n-1} e_{-r} \otimes T^{-m}_r(w) \in \mathcal{V}_- \otimes \mathcal{A}_n
\]
and denote
\[
\mathbf{b}_{1^{*},...,p^{*}}^{(+)}(\vec{v}) = \mathbf{b}_1^{(+)}(v_1)\mathbf{b}_2^{(+)}(v_2)\ldots \mathbf{b}_p^{(+)}(v_p) \in \mathcal{V}_1^* \otimes \mathcal{V}_2^* \otimes \ldots \otimes \mathcal{V}_p^* \otimes \mathcal{A}_n
\]
\[
\mathbf{b}_{1,...,Q}^{(-)}(\vec{w}) = \mathbf{b}_1^{(-)}(w_1)\mathbf{b}_2^{(-)}(w_2)\ldots \mathbf{b}_Q^{(-)}(w_Q) \in \mathcal{V}_-^1 \otimes \mathcal{V}_-^2 \otimes \ldots \otimes \mathcal{V}_-^Q \otimes \mathcal{A}_n.
\]

Explicitly, we have
\[
\mathbf{b}_{1^{*},...,p^{*}}^{(+)}(\vec{v}) = \sum_{k_1,...,k_P=1}^{n-1} f^{k_1} \otimes f^{k_2} \otimes \ldots \otimes f^{k_P} \otimes T^m_{k_1}(v_1)T^m_{k_2}(v_2)\ldots T^m_{k_P}(v_P)
\]
\[
\mathbf{b}_{1,...,Q}^{(-)}(\vec{w}) = \sum_{r_1,...,r_Q=1}^{n-1} e_{-r_1} \otimes e_{-r_2} \otimes \ldots \otimes e_{-r_Q} \otimes T^{-m}_{r_1}(w_1)T^{-m}_{r_2}(w_2)\ldots T^{-m}_{r_Q}(w_Q).
\]

If we introduce \(\Phi \in \mathcal{V}_1 \otimes \ldots \otimes \mathcal{V}_P \otimes \mathcal{V}^*_1 \otimes \ldots \otimes \mathcal{V}^*_Q \otimes \tilde{\mathcal{W}}\)
\[
\Phi = \sum_{k_1,...,k_P=1}^{n-1} \sum_{r_1,...,r_Q=1}^{n-1} e_{k_1} \otimes \ldots \otimes e_{k_P} \otimes f^{-r_1} \otimes \ldots \otimes f^{-r_Q} \otimes \Phi_{k_1,k_2,...,k_P}^{r_1,r_2,...,r_Q} = \sum_{k,r} e_{k} \otimes f^{-r} \otimes \Phi_{k,r}^{r^{-}},
\]
where
\[
\Phi_{k_1,k_2,...,k_P}^{r_1,r_2,...,r_Q} = \Phi_{k,r}^{r^{-}}, \in \tilde{\mathcal{W}},
\]
\[
e_{k} = e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_P} \in (\mathcal{V}_+)^\otimes_P,
\]
\[
f^{-r} = f^{-r_1} \otimes f^{-r_2} \otimes \ldots \otimes f^{-r_Q} \in (\mathcal{V}_+)^\otimes_Q,
\]
the assumed shape of the eigenvectors can be written as
\[
\mathfrak{B}(\vec{v}, \vec{w}) = \left\langle \mathbf{b}_{1^{*},...,p^{*}}^{(+)}(\vec{v})\mathbf{b}_{1,...,Q}^{(-)}(\vec{w}), \Phi \right\rangle.
\]

### 4 Bethe vectors and Bethe condition

Our goal is to write the action of the operators \(T^+_n(x), T^-_n(x), \tilde{T}^{(+)}\) and \(\tilde{T}^{(-)}\) on the assumed form of the Bethe vectors using the operators that act only on \(\Phi\). These actions are explicitly given in Lemma 5 of Appendix. Here we list only their consequences.

For \(\vec{v} = (v_1, v_2, \ldots, v_P)\) we introduce a set \(\overline{\mathcal{V}} = \{v_1, v_2, \ldots, v_P\}\), denote
\[
\overline{v}_k = (v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_P), \quad \nu_k = \overline{\mathcal{V}} \setminus \{v_k\},
\]
\[
F(x; \overline{v}) = \prod_{v_k \in \overline{\mathcal{V}}} f(x, v_k), \quad F(\overline{v}; x) = \prod_{v_k \in \overline{\mathcal{V}}} f(v_k, x).
\]
and define

\[ \hat{T}_{0;1\ldots P;1\ldots Q}^{(+)}(x; \vec{v}; \vec{w}) = \hat{R}_{0;1\ldots P}^{(+\ldots +)}(x; \vec{v}; \vec{w}) \hat{T}_0^{(+)}(x) \hat{R}_{0;1\ldots P}^{(+\ldots +)}(x; \vec{v}; \vec{w}) = \sum_{i,k=1}^{n-1} E_k^i \bigotimes \hat{T}_k^{i}(x; \vec{v}; \vec{w}) \]

\[ \hat{T}_{0;1\ldots P;1\ldots Q}^{(-)}(x; \vec{v}; \vec{w}) = \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x; \vec{w}; \vec{v}) \hat{T}_0^{(-)}(x) \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x; \vec{w}; \vec{v}) = \sum_{i,k=1}^{n-1} E_k^{-i} \bigotimes \hat{T}_k^{-i}(x; \vec{v}; \vec{w}) , \]

where

\[ \hat{R}_{0;1\ldots P}^{(+\ldots +)}(x; \vec{v}) = \hat{R}_{0;1\ldots P}^{(+\ldots +)}(x, v_P) \ldots \hat{R}_{0;1\ldots P}^{(+\ldots +)}(x, v_2) \hat{R}_{0;1\ldots P}^{(+\ldots +)}(x, v_1) \]

\[ \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x; \vec{w}) = \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w_P) \ldots \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w_2) \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w_1) \]

\[ \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x; \vec{w}) = \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w_P) \ldots \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w_2) \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w_1) \]

\[ \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x; \vec{w}) = \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w_P) \ldots \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w_2) \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w_1) \]

\[ \hat{R}_{0;1\ldots P}^{(+\ldots +)}(x, v) = \frac{1}{f(x, v)} \left( \hat{I}_{+} \bigotimes \hat{I}_{+} + g(x, v) \sum_{i,k=1}^{n-1} \mathbf{E}_k^i \bigotimes \mathbf{E}_k^i \right) \]

\[ \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w) = \hat{I}_{+} \bigotimes \hat{I}_{-} - \hat{h}(w, x) \sum_{r,s=1}^{n-1} \mathbf{E}_r^s \bigotimes \mathbf{F}_-^r \]

\[ \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, v) = \hat{I}_{-} \bigotimes \hat{I}_{+} - \hat{h}(x, v) \sum_{i,k=1}^{n-1} \mathbf{E}_k^{-i} \bigotimes \mathbf{E}_k^i \]

\[ \hat{R}_{0;1\ldots P}^{(-\ldots -)}(x, w) = \frac{1}{f(w, x)} \left( \hat{I}_{0;1\ldots P} \bigotimes \hat{I}_{-} - \hat{h}(w, x) \sum_{r,s=1}^{n-1} \mathbf{E}_r^s \bigotimes \mathbf{F}_-^r \right) . \]

The main results of this paper are the following three Theorems.

**Theorem 2.** Let \( \Phi = \sum_{k,r} \mathbf{E}_k^r \bigotimes \mathbf{F}_r^{-k} \Phi_{k,r}^r \), where \( \Phi_{k,r}^r \in \mathcal{W} \) is an common eigenvector of the operators

\[ \hat{H}^{(+)}(x; \vec{v}; \vec{w}) = \text{Tr}_0 \left( \hat{T}_{0;1\ldots P;1\ldots Q}^{(+)}(x; \vec{v}; \vec{w}) \right) = \sum_{i=1}^{n-1} \hat{T}_i^{(+)i}(x; \vec{v}; \vec{w}) \]

\[ \hat{H}^{(-)}(x; \vec{v}; \vec{w}) = \text{Tr}_0 \left( \hat{T}_{0;1\ldots P;1\ldots Q}^{(-)}(x; \vec{v}; \vec{w}) \right) = \sum_{i=1}^{n-1} \hat{T}_i^{(-)i}(x; \vec{v}; \vec{w}) \]

with eigenvalues \( \mu^{(+)i}(x; \vec{v}; \vec{w}) \) and \( \mu^{(-)i}(x; \vec{v}; \vec{w}) \). If the Bethe conditions

\[ \lambda_n(v_{\ell}) F(\vec{v}_{\ell}; \vec{v}_{\ell}) F(\vec{w}; v_{\ell} - n + 1 + \eta) = \mu^{(+)}(v_{\ell}; \vec{v}; \vec{w}) F(\vec{v}_{\ell}; \vec{v}_{\ell}) \]

\[ \lambda_{-n}(w_{s}) F(w_{s}; \vec{w}_{s}) F(w_{s} + n - 1 - \eta; \vec{v}) = \mu^{(-)}(w_{s}; \vec{v}; \vec{w}) F(\vec{w}_{s}; w_{s}) \]

are fulfilled for any \( v_{\ell} \in \mathcal{V} \) and \( w_{s} \in \mathcal{W} \), the vector

\[ \mathbf{B}(\vec{v}; \vec{w}) = \left\langle \mathbf{a}_{k,r}^{(+)}(\vec{v}) \mathbf{b}_{k,s}^{(-)}(\vec{w}), \Phi \right\rangle \]

is a common eigenvector of the operators \( H^{(+)}(x) \) and \( H^{(-)}(x) \) with eigenvalues

\[ E^{(+)}(x; \vec{v}; \vec{w}) = \lambda_n(x) F(\vec{v}; x) F(\vec{w}; x - n + 1 + \eta) + \mu^{(+)}(x; \vec{v}, \vec{w}) F(x; \vec{v}) \]

\[ E^{(-)}(x; \vec{v}; \vec{w}) = \lambda_{-n}(x) F(x; \vec{w}) F(x + n - 1 - \eta; \vec{v}) + \mu^{(-)}(x; \vec{v}, \vec{w}) F(\vec{w}; x) \].

**Proof:** According to Proposition 1, we have \( T_{n}^{+}(x) \Phi = \lambda_n(x) \Phi \) and \( T_{-n}^{-}(x) \Phi = \lambda_{-n}(x) \Phi \).
Using Lemma 5 and the relations $\text{Tr}_0 \hat{R}^{(\pm,-)}_{0*,s_+} = \hat{I}_{s_+}$ and $\text{Tr}_0 \hat{R}^{(-,+)}_{0*,\ell_+} = \hat{I}_{\ell_+}$ we obtain

\[
H^{(+)}(x) \langle b^{(+)}_{1,...,p, \ell}(\vec{v}) b^{(-)}_{1,...,Q_1(\vec{w}), \Phi} \rangle = \\
= \left( \lambda_n(x) F(\vec{v}; x) F(\vec{w}; x - n + 1 + \eta) + \mu^{(+)}(x; \vec{v}, \vec{w}) F(x, \vec{v}) \right) \\
\langle b^{(+)}_{1,...,p, \ell}(\vec{v}) b^{(-)}_{1,...,Q_1(\vec{w}), \Phi} \rangle - \\
- \sum_{v \in \mathcal{P}} g(v, x) (\lambda_n(v) F(\vec{v}; v) F(\vec{w}; v - n + 1 + \eta) - \mu^{(+)}(v; \vec{v}, \vec{w}) F(v, \vec{v})) \\
\langle b^{(+)}_{1,...,p, \ell}(\vec{v}) b^{(-)}_{1,...,Q_1(\vec{w}), \Phi} \rangle - \\
- \sum_{w \in \mathcal{W}} h(w, x) (\lambda_n(w) F(w; \vec{v}) F(w; \vec{w}) - \mu^{(-)}(w; \vec{v}, \vec{w}) F(\vec{w}, w)) \\
\langle b^{(+)}_{1,...,p, \ell}(\vec{v}) b^{(-)}_{1,...,Q_1(\vec{w}), \Phi} \rangle.
\]

This immediately proves the statement of Theorem 2.

\[\square\]

**Theorem 3.** The operators $\hat{T}_k^{(+)}(x; \vec{v}, \vec{w})$ and $\hat{T}_k^{-}(x; \vec{v}, \vec{w})$ are for any $\vec{v}$ and $\vec{w}$ generators of the RTT-algebra of $A_{n-1}$ type.

**Proof:** We have to show that for any $\vec{v}$, $\vec{w}$ and $\epsilon, \epsilon' = \pm$ the relation

\[
\hat{R}^{(\epsilon,\epsilon')}_{0,0'}(x, y) \hat{T}^{(\epsilon)}_{0;1,...,P;1',...,Q'}(x; \vec{v}, \vec{w}) \hat{T}^{(\epsilon')}_{0;1,...,P;1',...,Q'}(y; \vec{v}, \vec{w}) = \\
= \hat{T}^{(\epsilon)}_{0;1,...,P;1',...,Q'}(y; \vec{v}, \vec{w}) \hat{T}^{(\epsilon')}_{0;1,...,P;1',...,Q'}(x; \vec{v}, \vec{w}) \hat{R}^{(\epsilon,\epsilon')}_{0,0'}(x, y).
\]

is valid. Since

\[
\hat{T}^{(\epsilon)}_{0;1,...,P;1',...,Q'}(x; \vec{v}, \vec{w}) = \hat{R}^{(\epsilon,-)}_{0;1,...,Q'}(x; \vec{w}) \hat{T}^{(\epsilon)}_{0;1,...,P;1',...,Q'}(x; \vec{v}, \vec{w}) \hat{R}^{(\epsilon,+)}_{0,0'}(x, y)
\]

and $\hat{T}^{(\epsilon)}_0(x)$ satisfies the RTT-equation, it is enough to show that

\[
\hat{R}^{(\epsilon,\epsilon')}_{0,0'}(x, y) \hat{R}^{(\epsilon,-)}_{0;1,...,Q'}(x; \vec{w}) \hat{R}^{(\epsilon',-)}_{0;1,...,Q'}(y; \vec{w}) = \hat{R}^{(\epsilon',-)}_{0;1,...,Q'}(y; \vec{w}) \hat{R}^{(\epsilon,-)}_{0;1,...,Q'}(x; \vec{w}) \hat{R}^{(\epsilon,\epsilon')}_{0,0'}(x, y)
\]

\[
\hat{R}^{(\epsilon,\epsilon')}_{0,0'}(x, y) \hat{R}^{(\epsilon,+)}_{0;1,...,P;1',...,Q'}(x; \vec{v}) \hat{R}^{(\epsilon',+)}_{0;1,...,P;1',...,Q'}(y; \vec{v}) = \hat{R}^{(\epsilon',+)}_{0;1,...,P;1',...,Q'}(y; \vec{v}) \hat{R}^{(\epsilon,+)}_{0;1,...,P;1',...,Q'}(x; \vec{v}) \hat{R}^{(\epsilon,\epsilon')}_{0,0'}(x, y)
\]

hold. According to the definitions, we have

\[
\hat{R}^{(\epsilon,-)}_{0;1,...,Q'}(x; \vec{w}) = \hat{R}^{(\epsilon,-)}_{0;1,...,Q'}(x, w_1) \hat{R}^{(\epsilon,-)}_{0;2,...,Q'}(x, w_2) \ldots \hat{R}^{(\epsilon,-)}_{0;Q',...,Q'}(x, w_Q)
\]

\[
\hat{R}^{(\epsilon,+)}_{0;1,...,P;1',...,Q'}(x; \vec{v}) = \hat{R}^{(\epsilon,+)}_{0;1,...,P;1',...,Q'}(x, v_P) \ldots \hat{R}^{(\epsilon,+)}_{0;Q',...,Q'}(x, v_1) \hat{R}^{(\epsilon,+)}_{0;Q',...,Q'}(x, v_1).
\]
and a theorem then follows from the Yang–Baxter equations

\[
\hat{R}_{0,0}^{(\epsilon,\epsilon')} (x, y) \hat{R}_{0,1^-}^{(\epsilon,\epsilon^-)} (x, w) \hat{R}_{0,1^+}^{(\epsilon',\epsilon^-)} (y, w) = \hat{R}_{0,0}^{(\epsilon',\epsilon')} (x, y) \hat{R}_{0,1^-}^{(\epsilon',\epsilon^-)} (x, w) \hat{R}_{0,0}^{(\epsilon,\epsilon')} (x, y) ,
\]

\[
\hat{R}_{0,0}^{(\epsilon,\epsilon')} (x, y) \hat{R}_{0,1^+}^{(\epsilon,\epsilon^+)} (x, v) \hat{R}_{0',1^+}^{(\epsilon',\epsilon^+)} (y, v) = \hat{R}_{0,0}^{(\epsilon',\epsilon')} (x, y) \hat{R}_{0',1^+}^{(\epsilon',\epsilon^+)} (x, v) \hat{R}_{0,0}^{(\epsilon,\epsilon')} (x, y) .
\]

The following Theorem shows that

\[
\hat{\Omega} = e_{n-1} \otimes \ldots \otimes e_{n-1} \otimes f^{-n+1} \otimes \ldots \otimes f^{-n+1} \otimes \omega
\]

is a vacuum vector for the representation of the RTT–algebra \(A_{n-1}\), which is generated by \(\hat{T}_k^i(x; \bar{v}; \bar{w})\) and \(\hat{T}^{-i}_k(x; \bar{v}; \bar{w})\).

**Theorem 4.** For the vector \(\hat{\Omega}\) and \(i, k = 1, \ldots, n - 1\)

\[
\hat{T}_k^i(x; \bar{v}; \bar{w}) \hat{\Omega} = 0 \quad \text{for} \quad i < k ,
\]

\[
\hat{T}^{-i}_k(x; \bar{v}; \bar{w}) \hat{\Omega} = 0 \quad \text{for} \quad k < i
\]

\[
\hat{T}_i(x; \bar{v}; \bar{w}) \hat{\Omega} = \nu_i(x; \bar{v}; \bar{w}) \hat{\Omega}
\]

\[
\hat{T}^{-i}_i(x; \bar{v}; \bar{w}) \hat{\Omega} = \nu^{-i}_i(x; \bar{v}; \bar{w}) \hat{\Omega}
\]

where

\[
\nu_i(x; \bar{v}; \bar{w}) = \lambda_i(x) F(\nu; x + 1)
\]

\[
\nu_{n-1}(x; \bar{v}; \bar{w}) = \lambda_{n-1}(x) F(x - n + 1 + \eta; \bar{w})
\]

\[
\nu^{-i}_i(x; \bar{v}; \bar{w}) = \lambda^{-i}(x) F(x - 1; \bar{w})
\]

\[
\nu^{-n+1}_i(x; \bar{v}; \bar{w}) = \lambda^{-n+1}(x) F(\nu; x + n - 1 - \eta)
\]

are valid.

**Proof:** If we write

\[
\hat{R}_{0,1^+}^{(\epsilon,\epsilon^+)} (x, v) = \sum_{a,b,q=1}^{n-1} R_{b,q}^{a,p} (x, v) E_a^b \otimes E_p^q, \quad R_{b,q}^{a,p} (x, v) = \frac{\delta^a_b \delta^p_q + g(x, v) \delta^a_q \delta^p_b}{f(x, v)}
\]

\[
\hat{R}_{0,1^-}^{(\epsilon,\epsilon^-)} (x, w) = \sum_{c,d,r,s=1}^{n-1} R_{c,s}^{d,r} (x, w) E_c^d \otimes F_r^s, \quad R_{c,s}^{d,r} (x, w) = \delta^d_c \delta^r_s - \hat{h}(w, x) \delta^{cr} \delta_{d,s},
\]

\[
\hat{R}_{0,1^+}^{(-,\epsilon^+)} (x, v) = \sum_{a,b,q=1}^{n-1} R_{-b,q}^{a,p} (x, v) E_a^b \otimes E_p^q, \quad R_{-b,q}^{a,p} (x, v) = \delta^a_b \delta^p_q - \hat{h}(x, v) \delta^{ap} \delta_{b,q}
\]

\[
\hat{R}_{0,1^-}^{(-,\epsilon^-)} (x, w) = \sum_{c,d,r,s=1}^{n-1} R_{-d,s}^{c,r} (x, w) E_c^d \otimes F_r^s, \quad R_{-d,s}^{c,r} (x, w) = \frac{\delta^c_d \delta^r_s + g(w, x) \delta^c_s \delta^r_d}{f(w, x)},
\]
we obtain
\[ \hat{T}_k(x; \vec{v}; \vec{w})\hat{\Omega} = \]
\[ = R_{d_1, s_1}^{i, n+1}(x, w_1) R_{d_2, s_2}^{i, n+1}(x, w_2) \ldots R_{d_{Q_s-1}, s_{Q_s-1}}^{i, n+1}(x, w_{Q_s-1}) R_{d_{Q_s-1}}^{i, n+1}(x, w_{Q_s}) \]
\[ \quad \quad \quad \quad \quad \quad \quad R_{a_p, p}^{i, n+1}(x, v_P) R_{a_{p-1}, p}^{i, n+1}(x, v_{P-1}) \ldots R_{a_1, n-1}^{i, n+1}(x, v_2) R_{a_1}^{i, n-1}(x, v_1) \]
\[ e_{p_1} \otimes e_{p_2} \otimes \ldots \otimes e_{p_{n-1}} \otimes e_p \otimes \]
\[ \otimes f^{-s_1} \otimes f^{-s_2} \otimes \ldots \otimes f^{-s_{Q_s-1}} \otimes f^{-s_Q} \otimes T_{a_p}^{d_Q}(x)\omega \]
\[ \hat{T}_k(x; \vec{v}; \vec{w})\hat{\Omega} = \]
\[ = R_{d_1, s_1}^{i, n+1}(x, w_1) R_{d_2, s_2}^{i, n+1}(x, w_2) \ldots R_{d_{Q_s-1}, s_{Q_s-1}}^{i, n+1}(x, w_{Q_s-1}) R_{d_{Q_s-1}}^{i, n+1}(x, w_{Q_s}) \]
\[ \quad \quad \quad \quad \quad \quad \quad R_{a_p, p}^{i, n+1}(x, v_P) R_{a_{p-1}, p}^{i, n+1}(x, v_{P-1}) \ldots R_{a_1, n-1}^{i, n+1}(x, v_2) R_{a_1}^{i, n-1}(x, v_1) \]
\[ e_{p_1} \otimes e_{p_2} \otimes \ldots \otimes e_{p_{n-1}} \otimes e_p \otimes \]
\[ \otimes f^{-s_1} \otimes f^{-s_2} \otimes \ldots \otimes f^{-s_{Q_s-1}} \otimes f^{-s_Q} \otimes T_{a_p}^{d_Q}(x)\omega \]
Since \( R_{d_s, n-1}^{i, n+1}(x, w) = \delta_{d_s}^{i, n-1} \) for \( 1 \leq i \leq n - 1 \), we have
\[ \hat{T}_k(x; \vec{v}; \vec{w})\hat{\Omega} = F_{a_p, p}^{i, n+1}(x, v_P) R_{a_{p-1}, n-1}^{i, n+1}(x, v_{P-1}) \ldots R_{a_1, n-1}^{i, n+1}(x, v_2) R_{a_1}^{i, n-1}(x, v_1) \]
\[ e_{p_1} \otimes e_{p_2} \otimes \ldots \otimes e_{p_{n-1}} \otimes e_p \otimes \]
\[ \otimes f^{-n+1} \otimes f^{-n+1} \otimes \ldots \otimes f^{-n+1} \otimes f^{-n+1} \otimes T_{a_p}^{d_Q}(x)\omega \]
As \( T_{a_p}^{d_Q}(x)\omega = 0 \) for \( a_p > i \), this expression is nonzero only for \( a_p \leq i < n - 1 \). In this case \( R_{a_p, n-1}^{i, n+1}(x, v_P) = \frac{1}{F(x, v_P)} \delta_{a_p}^{i, n-1} \). Therefore, we have
\[ \hat{T}_k(x; \vec{v}; \vec{w})\hat{\Omega} = \frac{1}{F(x, \vec{v})} e_{n-1} \otimes e_{n-1} \otimes \ldots \otimes e_{n-1} \otimes e_{n-1} \otimes \]
\[ \otimes f^{-n+1} \otimes f^{-n+1} \otimes \ldots \otimes f^{-n+1} \otimes f^{-n+1} \otimes T_k(x)\omega \]
and so
\[ \hat{T}_k(x; \vec{v}; \vec{w})\hat{\Omega} = 0 \quad \text{for} \quad k > i \]
\[ \hat{T}_i(x; \vec{v}; \vec{w})\hat{\Omega} = \frac{\lambda_i(x)}{F(x, \vec{v})} \hat{\Omega} = \lambda_i(x) F(x, \vec{v}) \hat{\Omega} = \lambda_i(x) F(x, \vec{v}) \hat{\Omega} \]
If \( i = k = n - 1 \) we have
\[ \hat{T}_{n-1}^{n-1}(x; \vec{v}; \vec{w})\hat{\Omega} = \]
\[ = R_{d_1, s_1}^{n-1, n+1}(x, w_1) R_{d_2, s_2}^{n-1, n+1}(x, w_2) \ldots R_{d_{Q_s-1}, s_{Q_s-1}}^{n-1, n+1}(x, w_{Q_s-1}) R_{d_{Q_s-1}}^{n-1, n+1}(x, w_{Q_s}) \]
\[ \quad \quad \quad \quad \quad \quad \quad R_{a_p, p}^{n-1, n+1}(x, v_P) R_{a_{p-1}, p}^{n-1, n+1}(x, v_{P-1}) \ldots R_{a_1, n-1}^{n-1, n+1}(x, v_2) R_{a_1}^{n-1, n-1}(x, v_1) \]
\[ e_{n-1} \otimes e_{n-1} \otimes \ldots \otimes e_{n-1} \otimes e_{n-1} \otimes \]
\[ \otimes f^{-s_1} \otimes f^{-s_2} \otimes \ldots \otimes f^{-s_{Q_s-1}} \otimes f^{-s_Q} \otimes T_{a_p}^{d_Q}(x)\omega \]
Since \( R_{n-1, n-1}^{n, p}(x, v) = \delta_{n-1}^{n, p} \) we obtain
\[ \hat{T}_{n-1}^{n-1}(x; \vec{v}; \vec{w})\hat{\Omega} = \]
\[ = R_{d_1, s_1}^{n-1, n+1}(x, w_1) R_{d_2, s_2}^{n-1, n+1}(x, w_2) \ldots R_{d_{Q_s-1}, s_{Q_s-1}}^{n-1, n+1}(x, w_{Q_s-1}) R_{d_{Q_s-1}}^{n-1, n+1}(x, w_{Q_s}) \]
\[ e_{n-1} \otimes e_{n-1} \otimes \ldots \otimes e_{n-1} \otimes e_{n-1} \otimes \]
\[ \otimes f^{-s_1} \otimes f^{-s_2} \otimes \ldots \otimes f^{-s_{Q_s-1}} \otimes f^{-s_Q} \otimes T_{n-1}^{d_Q}(x)\omega . \]
The conditions $T_{n-1}^{d_Q}(x) \omega = 0$ for $d_Q < n-1$ and $T_{n-1}^{-1}(x) \omega = \lambda_{n-1}(x) \omega$ lead to the equations

$$
\hat{T}_{n-1}^{-1}(x; \bar{v}; \bar{w})\hat{\Omega} = \lambda_{n-1}(x)R_{d_1,-s_1}^{d_1,-n+1}(x, w_1)R_{d_2,-s_2}^{d_2,-n+1}(x, w_2) \ldots R_{d_{Q-1}+1,-s_{Q-1}}^{d_{Q-1}+1,-n+1}(x, w_{Q-1})R_{d_{Q-1}+1,-s_{Q}}^{d_{Q-1}+1,-n+1}(x, w_{Q})
$$

$$
e_n \otimes e_n \otimes \ldots \otimes e_n \otimes e_n \otimes \otimes f^{-s_1} \otimes f^{-s_2} \otimes \ldots \otimes f^{-s_{Q-1}} \otimes f^{-s_{Q}} \otimes \omega
$$

However, $R_{n-1}^{d,Q,n}(x, w) = (1 - \hat{h}(w, x)) \delta_{n-1} \delta_{n-1}$ and so

$$
\hat{T}_{n-1}^{-1}(x; \bar{v}; \bar{w})\hat{\Omega} = \lambda_{n-1}(x)F(x - n + 1 + \eta; \omega) \delta_{n-1} \delta_{n-1}
$$

Since $R_{-a_p}^{-a_p}(x, v) = \delta_{n-1} \delta_{n-1}$ for $1 \leq k < n-1$ we have

$$
\hat{T}_{-k}^{-1}(x; \bar{v}; \bar{w})\hat{\Omega} = \frac{1}{F(\bar{w}; x)} e_n \otimes e_n \otimes \ldots \otimes e_n \otimes e_n \otimes \otimes f^{-n+1} \otimes f^{-n+1} \otimes \ldots \otimes f^{-n+1} \otimes f^{-n+1} \otimes \hat{T}_{-k}^{-1}(x) \omega
$$

For $k < d_Q$ the relation $T_{-k}^{-1}(x) \omega = 0$ holds. Therefore, this expression is nonzero for $1 \leq d_Q \leq k < n-1$ only. But in this case $R_{-d_{Q-1}+1,-s_{Q}}^{d_{Q-1}+1,-n+1}(x, w_{Q}) = \frac{1}{f(w_{Q}, x)} \delta_{d_{Q}-1} \delta_{s_{Q}}$. By repeatedly using this relationship, we get

$$
\hat{T}_{-k}^{-1}(x; \bar{v}; \bar{w})\hat{\Omega} = \frac{\lambda_{-i}(x)}{F(\bar{w}; x)} \hat{\Omega} = \lambda_{-i}(x)F(x - 1; \bar{w})\hat{\Omega}
$$

The relations $T_{-i}^{-1}(x) \omega = 0$ for $k < i$ and $T_{-i}^{-1}(x) \omega = \lambda_{-i}(x) \omega$ lead to the equations

$$
\hat{T}_{-i}^{-1}(x; \bar{v}; \bar{w})\hat{\Omega} = \frac{\lambda_{-i}(x)}{F(\bar{w}; x)} \hat{\Omega} = \lambda_{-i}(x)F(x - 1; \bar{w})\hat{\Omega}
$$

For $i = k = n-1$ we have

$$
\hat{T}_{-n+1}^{-1}(x; \bar{v}; \bar{w})\hat{\Omega} = \frac{1}{F(\bar{w}; x)} e_{p_1} \otimes e_{p_2} \otimes \ldots \otimes e_{p_{p-1}} \otimes e_{p_p} \otimes \otimes f^{-s_1} \otimes f^{-s_2} \otimes \ldots \otimes f^{-s_{Q-1}} \otimes f^{-s_{Q}} \otimes \hat{T}_{-a_p}^{-1}(x) \omega
$$

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Since \( R_{-d,-s}^{n+1, n+1}(x, w) = \delta_d^{n-1} \delta_s^{n-1} \), we obtain

\[
\tilde{T}_{n+1}^{-1}(x; \tilde{v}; \tilde{w}) \tilde{\Omega} =
\]

\[
= R_{-a_p, p}^{-1, pp}(x, v_p) R_{-a_p-1, pp-1}(x, v_{p-1}) \ldots R_{-a_2, p_2}(x, v_2) R_{-a_1, p_1}(x, v_1)
\]

\[
e_p \otimes e_p \otimes \ldots \otimes e_{p_{p-1}} \otimes e_{p_p} \otimes
\]

\[
\otimes f^{-n+1} \otimes f^{-n+1} \otimes \ldots \otimes f^{-n+1} \otimes f^{-n+1} \otimes T_{-a_p}^{-1}(x) \omega
\]

The conditions \( T_{-a_p}^{-1}(x) \omega = 0 \) for \( a_P < n - 1 \) and \( T_{n+1}^{-1}(x) \omega = \lambda_{n+1}(x) \omega \) lead to

\[
\tilde{T}_{n+1}^{-1}(x; \tilde{v}; \tilde{w}) \tilde{\Omega} =
\]

\[
= \lambda_{n+1}(x) R_{-a_P, n-1, n-1}(x, v_p) R_{-a_P-1, n-1, n-1}(x, v_{p-1}) \ldots R_{-a_2, n-1, n-1}(x, v_2) R_{-a_1, n-1, n-1}(x, v_1)
\]

\[
e_p \otimes e_p \otimes \ldots \otimes e_{p_{p-1}} \otimes e_{p_p} \otimes
\]

\[
\otimes f^{-n+1} \otimes f^{-n+1} \otimes \ldots \otimes f^{-n+1} \otimes f^{-n+1} \otimes \omega
\]

However, \( R_{-a_1, n-1}^{-1}(x, v) = (1 - \tilde{h}(x, v)) \delta_a^{n-1} \delta_{n-1} = f(v, x + n - 1 - \eta) \delta_a^{n-1} \delta_{n-1} \), and so

\[
\tilde{T}_{n+1}^{-1}(x; \tilde{v}; \tilde{w}) \tilde{\Omega} = \lambda_{n+1}(x) F(\tilde{v}; x + n - 1 - \eta) \tilde{\Omega}
\]

\[\square\]

These three theorems show that to find the Bethe vectors \( \mathfrak{B}(\tilde{v}; \tilde{w}) \) for the RTT–algebra \( \mathcal{A}_n \), it is sufficient to find the Bethe vectors for the RTT–algebra \( \mathcal{A}_{n-1} \) that is generated by the operators \( \tilde{T}_k^{-1}(x; \tilde{v}; \tilde{w}) \), where \( i, k = 1, \ldots, n-1 \), and that has a vacuum vector \( \tilde{\Omega} \).

## 5 Conclusion

The paper describes the construction of eigenvectors for the representations of the RTT–algebra \( \mathcal{A}_n \) by using the highest weight vectors for the representation of the RTT–algebra \( \mathcal{A}_{n-1} \). We meet these RTT–algebras \([1, 2]\) while studying the algebraic Bethe ansatz for the RTT–algebras of \( \text{sp}(2n) \) and \( \text{o}(2n) \) types.

In the special cases, when \( \tilde{v} \) or \( \tilde{w} \) is an empty set, our construction is known as the algebraic nested Bethe ansatz, which was formulated in \([3]\). So our construction of the Bethe vectors is a generalization of the algebraic nested Bethe ansatz to the RTT–algebra of \( \mathcal{A}_n \) type.

For the RTT–algebra of \( \mathcal{A}_2 \) type we get from theorems 2, 3 and 4 the Bethe vectors

\[\mathfrak{B}_2(\tilde{v}; \tilde{w}) = T_1^2(\tilde{v}) T_2^{-1}(\tilde{w}) \omega\]

and the Bethe conditions

\[\lambda_2(v_\ell) F(\tilde{v}_\ell; v_\ell) F(\tilde{w}; v_\ell - 1 + \eta) = \lambda_1(v_\ell) F(v_\ell - 1 + \eta; \tilde{w}) F(v_\ell; \tilde{v}_\ell)\]

\[\lambda_{-2}(w_s) F(w_s; \tilde{w}_s) F(w_s + 1 - \eta; \tilde{v}) = \lambda_{-1}(w_s) F(\tilde{v}; w_s + 1 - \eta) F(\tilde{w}_s; w_s),\]

which we found for this algebra and \( \nu = -1 \) in \([4]\).
For higher $n$ it is possible by means of Theorems 2, 3 and 4 step-by-step to decrease value $n$ and thus obtain an explicit form of the Bethe vectors. For the RTT–algebra of $\text{gl}(n)$ type this procedure leads to trace-formula [5]. We intend to publish a similar explicit form of the Bethe vectors for the RTT–algebras $\mathcal{A}_n$, of $\text{sp}(2n)$ and $\text{o}(2n)$ types in the near future.

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Appendix

A1 Commutation relations in the RTT–algebra $\mathcal{A}_n$

The RTT–equation for the RTT–algebra $\mathcal{A}_n$ leads to the commutation relations

\[
T^i_k(x) T^r_s(y) + g(x, y) T^r_k(x) T^i_s(y) = T^r_s(y) T^i_k(x) + g(x, y) T^r_k(y) T^i_s(x)
\]

\[
T^{-i}_k(x) T^{-r}_s(y) + g(x, y) T^{-r}_k(x) T^{-i}_s(y) = T^{-r}_s(y) T^{-i}_k(x) + g(x, y) T^{-r}_k(y) T^{-i}_s(x)
\]

\[
T^i_k(x) T^{-r}_s(y) - \delta^{ir} k(x, y) \sum_{p=1}^n T^p_k(x) T^{-p}_s(y) = T^{-r}_s(y) T^i_k(x) - \delta_{k,s} T^i(x, y) \sum_{p=1}^n T^{-r}_p(y) T^i_p(x)
\]

\[
T^{-i}_k(x) T^r_s(y) - \delta^{ir} h(x, y) \sum_{p=1}^n T^p_k(x) T^r_p(s) = T^r_s(y) T^{-i}_k(x) - \delta_{k,s} T^r(x, y) \sum_{p=1}^n T^{-i}_p(y) T^r_p(x)
\]

\[
T^i_k(x) T^r_s(y) + g(x, y) T^i_s(x) T^r_k(y) = T^r_s(y) T^i_k(x) + g(x, y) T^i_s(y) T^r_k(x)
\]

\[
T^{-i}_k(x) T^{-r}_s(y) + g(x, y) T^{-i}_s(x) T^{-r}_k(y) = T^{-r}_s(y) T^{-i}_k(x) + g(x, y) T^{-i}_s(y) T^{-r}_k(x)
\]

\[
T^i_k(x) T^{-r}_s(y) - \delta_{k,s} h(x, y) \sum_{p=1}^n T^p_i(x) T^{-r}_p(y) = T^{-r}_s(y) T^i_k(x) - \delta^{ir} h(y, x) \sum_{p=1}^n T^{-r}_s(y) T^i_p(x)
\]

\[
T^{-i}_k(x) T^r_s(y) - \delta_{k,s} T^i(x, y) \sum_{p=1}^n T^{-r}_p(y) T^i_p(x) = T^r_s(y) T^{-i}_k(x) - \delta^{ir} k(y, x) \sum_{p=1}^n T^r_p(y) T^{-i}_p(x)
\]

A2 Action of the operators $T_{\pm n}^\pm(x)$ and $\tilde{T}(\pm)(x)$ on the Bethe vectors

First, we will rewrite the commutation relations using the operators action for $P = Q = 1$. 
Lemma 1. In the RTT–algebra $\mathcal{A}_n$ the following relations are true:

$$
T_n^n(x)\langle b_{1}^{+}(v), e_k \rangle = f(v, x)\langle b_{1}^{+}(v), e_k \rangle T_n^n(x) - g(v, x)\langle b_{1}^{+}(x), e_k \rangle T_n^n(v)
$$

$$
T_{-n}^{-n}(x)\langle b_{1}^{-}(w), f^{-r} \rangle = f(x, w)\langle b_{1}^{-}(w), f^{-r} \rangle T_{-n}^{-n}(x) - g(x, w)\langle b_{1}^{-}(x), f^{-r} \rangle T_{-n}^{-n}(w)
$$

$$
\hat{T}_0^{(+)}(x)\langle b_{1}^{+}(v), e_k \rangle = f(x, v)\langle b_{1}^{+}(v), \hat{T}_0^{(+)}(x)R_{0_{+, 1_{+}}}(x, v)(I_{0_{+}} \otimes e_k) \rangle -
- g(x, v)\langle b_{1}^{+}(x), \hat{T}_0^{(+)}(v)R_{0_{+, 1_{+}}}(I_{0_{+}} \otimes e_k) \rangle
$$

$$
\hat{T}_0^{(-)}(x)\langle b_{1}^{-}(w), f^{-r} \rangle = f(x, w)\langle b_{1}^{-}(w), \hat{T}_0^{(-)}(x)(I_{-} \otimes f^{-r}) \rangle -
- g(x, w)\langle b_{1}^{-}(x), \hat{T}_0^{(-)}(w)(I_{-} \otimes f^{-r}) \rangle
$$

$$
T_{-n}^{n}(x)\langle b_{1}^{+}(w), f^{-r} \rangle = \frac{\hat{h}(w, x)}{\hat{h}(x, v)}\langle b_{1}^{+}(w), f^{-r} \rangle T_{n}^{n}(x) +
+ \frac{\hat{h}(w, x)}{\hat{h}(x, v)} T_{0}(b_{1}^{+}(x), \hat{T}_{1_{+}}^{(-)}(x)R_{0_{+, 1_{+}}}(I_{+} \otimes e_k) \rangle
$$

$$
\hat{T}_0^{(-)}(x)\langle b_{1}^{+}(v), e_k \rangle = \langle b_{1}^{+}(v), \hat{T}_0^{(-)}(x)(I_{-} \otimes e_k) \rangle -
- \hat{h}(w, x)\langle b_{1}^{+}(x), \hat{T}_{1_{+}}^{(-)}(w)(I_{+} \otimes f^{-r}) \rangle T_{-n}^{-n}(w)
$$

where

$$
\hat{P}_{0_{+, 1_{+}}}^{(+, +)} = \hat{R}_{0_{+, 1_{+}}}^{(+, +)}(x, x) = \sum_{i,k=1}^{n_{-1}} E_i^{k} \otimes E_i^{k}, \quad \hat{P}_{0_{+, 1_{+}}}^{(+, -)} = \sum_{r,s=1}^{n_{-1}} E_r^{s} \otimes F_{-s}^{r},
$$

and

$$
\hat{P}_{0_{-, 1_{-}}}^{(-, +)} = \hat{R}_{0_{-, 1_{-}}}^{(-, -)}(w, w) = \sum_{i,k=1}^{n_{1}} E_i^{k} \otimes E_i^{k}, \quad \hat{P}_{0_{-, 1_{-}}}^{(-, -)} = \sum_{r,s=1}^{n_{1}} E_r^{s} \otimes F_{-s}^{r},
$$

and $\hat{P}_{1_{+, 1_{+}}}^{(+, +)}$ and $\hat{P}_{1_{-, 1_{-}}}^{(-, -)}$ are the linear mappings $\hat{P}_{1_{-, 1_{-}}}^{(+, +)} : \mathcal{V}_{1_{-}} \rightarrow \mathcal{V}_{1_{-}}, \hat{P}_{1_{-, 1_{-}}}^{(-, -)} : \mathcal{V}_{1_{+}} \rightarrow \mathcal{V}_{1_{+}}$, defined by

$$
\hat{P}_{1_{+, 1_{+}}}^{(+, +)}f^{-r} = e_r, \quad \hat{P}_{1_{+, 1_{+}}}^{(-, -)}e_k = f^{-k}.
$$

Proof: The first two equations are only otherwise written commutation relations

$$
T_n^n(x)T_k^n(v) = f(v, x)T_k^n(v)T_n^n(x) - g(v, x)T_k^n(x)T_n^n(v),
$$

and the third and fourth equations are the matrix notation of the commutation relations

$$
T^n_s(x)T^n_k(v) = T^n_k(v)T^n_s(x) + g(x, v)T^n_n(x)T^n_s(x) - g(x, v)T^n_s(x)T^n_n(v),
$$

and

$$
T^{-n}_s(w)T^{-n}_k(x) = T^{-n}_k(x)T^{-n}_s(w) + g(w, x)T^{-n}_n(x)T^{-n}_s(x) - g(w, x)T^{-n}_s(x)T^{-n}_n(w).
$$

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To prove the fifth relation, we first use the commutation relation

\[ T_n^a(x)T_n^{-r}(w) = T_n^{-r}(w)T_n^a(x) - k(x, w) \sum_{p=1}^{n} T_n^{-r}(w)T_p^n(x) = \]

\[ = \left(1 - k(x, w)\right)T_n^{-r}(w)T_n^a(x) - k(x, w) \sum_{p=1}^{n-1} T_n^{-r}(w)T_p^n(x). \]  

If we sum the commutation relations

\[ T^{-r}_{-k}(w)T_k^n(x) = T_k^n(x)T^{-r}_{-k}(w) - h(w, x) \sum_{p=1}^{n} T_p^n(x)T^{-r}_{-p}(w) \]

over \( k = 1, \ldots, n - 1, \) we find that

\[ \sum_{p=1}^{n-1} T^{-r}_{-p}(w)T_p^n(x) = \left(1 - (n - 1)h(w, x)\right)\sum_{p=1}^{n-1} T_p^n(x)T^{-r}_{-p}(w) - (n - 1)h(w, x)T_n^n(x)T^{-r}_{-n}(w) \]

When we substitute this equality into (6), we get

\[ T_n^n(x)T^{-r}_{-n}(w) = \left(1 + \tilde{h}(w, x)\right)T^{-r}_{-n}(w)T_n^n(x) + \tilde{h}(w, x) \sum_{p=1}^{n-1} T_p^n(x)T^{-r}_{-p}(w) \]  

which is another notation of the fifth relationship.

To prove the sixth relation, we use the commutation relations

\[ T^{-n}_{-n}(x)T_k^n(v) = \left(1 - k(v, x)\right)T_k^n(v)T^{-n}_{-n}(x) - k(v, x) \sum_{p=1}^{n-1} T_k^n(v)T^{-p}_{-n}(x). \]  

If we sum the commutation relations

\[ T_i^n(v)T^{-i}_{-n}(x) = T^{-i}_{-n}(x)T_i^n(v) - h(x, v)T^{-n}_{-n}(x)T_i^n(v) - h(x, v) \sum_{p=1}^{n-1} T^{-n}_{-n}(x)T_k^p(v), \]

over \( i = 1, \ldots, n - 1, \) we obtain

\[ \sum_{p=1}^{n-1} T_i^n(v)T^{-p}_{-n}(x) = \left(1 - (n - 1)h(x, v)\right)\sum_{p=1}^{n-1} T^{-p}_{-n}(x)T_k^p(v) - (n - 1)h(x, v)T^{-n}_{-n}(x)T_k^n(v). \]

When we substitute this relation into (8), we get

\[ T^{-n}_{-n}(x)T_k^n(v) = \left(1 + \tilde{h}(x, v)\right)T_k^n(v)T^{-n}_{-n}(x) + \tilde{h}(x, v) \sum_{p=1}^{n-1} T^{-p}_{-n}(x)T_k^p(v) \]  

which can be written in the form shown in Lemma.

To prove the seventh and eighth relationships, we first use the commutation relations

\[ T_k^n(x)T^{-r}_{-n}(w) = T^{-r}_{-n}(w)T_k^n(x) - \delta^{i,r}h(w, x)T^{-n}_{-n}(w)T_k^n(x) - \delta^{i,r}h(w, x) \sum_{p=1}^{n-1} T^{-p}_{-n}(w)T_k^p(x) \]

\[ T^{-s}_{-n}(x)T_k^n(v) = T_k^n(v)T^{-s}_{-n}(x) - \delta_{k,s}h(x, v)T^{-n}_{-n}(v)T_k^n(x) - \delta_{k,s}h(x, v) \sum_{p=1}^{n-1} T^{-n}_{-n}(v)T_k^p(v). \]
Using equations (9) and (7), we obtain

\[ T^r_k(x)T^{-r}_{-n}(w) = T^{-r}_{-n}(w)T^r_k(x) - \delta^{i,r}\tilde{h}(w, x)\sum_{p=1}^{n-1} T^{-p}_{-n}(w)T^p_k(x) - \delta^{i,r}\tilde{h}(w, x)T^n_k(x)T^{-n}_{-n}(w) \]

\[ T^{-r}_{-s}(v)T^n_k(v) = T^n_k(v)T^{-r}_{-s}(v) - \delta_{k,s}\tilde{h}(x, v)\sum_{p=1}^{n-1} T_p^n(v)T^{-r}_{-p}(x) - \delta_{k,s}\tilde{h}(x, v)T^{-r}_{-n}(x)T^n_k(v) \]

which are other notations of the last two equations of Lemma.

Using Lemma 1, it is relatively easy to find members in which \( x \) is exchanged with the first component of the vectors \( \vec{v} \) and \( \vec{w} \). The members, in which \( x \) is interchanged with other components of these vectors, can be found by switching the corresponding component to the first place of vectors.

The following Lemma gives a suitable notation of the commutation relations that we use.

**Lemma 2.** In the RTT–algebra \( \mathcal{A}_n \) the following relations apply:

\[
\begin{align*}
\left\langle b_1^{(+)}(x) b_2^{(+)}(y), e_i \otimes e_k \right\rangle & = \left\langle b_2^{(+)}(y) b_1^{(+)}(x), \hat{R}^{(+,+)}_{1^+_r,2^+_s}(x, y)(e_i \otimes e_k) \right\rangle \\
\left\langle b_1^{(-)}(x) b_2^{(-)}(y), f^r \otimes f^s \right\rangle & = \left\langle b_2^{(-)}(y) b_1^{(-)}(x), \hat{R}^{(-,-)}_{1^-_r,2^-_s}(y, x)(f^r \otimes f^s) \right\rangle \\
\left\langle b_1^{(+)}(x) b_2^{(-)}(y), e_i \otimes e_k \right\rangle & = \left\langle b_2^{(+)}(x) b_1^{(+)}(y), \hat{R}^{(+,-)}_{1^+_r,2^-_s}(e_i \otimes e_k) \right\rangle \\
\left\langle b_1^{(-)}(x) b_2^{(+)y}, 1^+_r \otimes f^r \right\rangle & = \left\langle b_2^{(-)}(x) b_1^{(-)}(y), \hat{R}^{(-,+)_{1^-_r,2^-_s}}_{1^+_r,2^-_s}(f^r \otimes e_k) \right\rangle
\end{align*}
\]

where

\[
\begin{align*}
\hat{R}^{(\cdot, \cdot)}_{1^+_r,2^-_s}(x, y) & = \frac{1}{f(y, x)} \left( \hat{I}^+ \otimes \hat{I}^- + g(y, x) \sum_{r,s=1}^{n-1} F^{-r}_{-s} \otimes F^{-s}_{-r} \right) \\
\hat{R}^{(\cdot, \cdot)}_{1^+_r,2^-_s} & = \hat{R}^{(\cdot, \cdot)}_{1^-_r,2^+_s}(x, x) = \sum_{r,s=1}^{n-1} F^{-r}_{-s} \otimes F^{-s}_{-r}
\end{align*}
\]

**Proof:** The first and second relations are the transcripts of the commutation relations

\[
\begin{align*}
f(x, y)T^n_k(x)T^n_k(y) & = T^n_k(y)T^n_k(x) + g(x, y)T^n_k(y)T^n_k(x) \\
f(y, x)T^{-r}_{-s}(x)T^{-s}_{-n}(y) & = T^{-s}_{-n}(y)T^{-r}_{-s}(x) + g(y, x)T^{-r}_{-n}(y)T^{-s}_{-n}(x),
\end{align*}
\]

the fifth relation is an otherwise written commutation relation

\[
T^n_k(x)T^{-r}_{-n}(y) = T^{-r}_{-n}(y)T^n_k(x)
\]

and the other equations are the identities.
To write the operators' action $T_{\pm n}^{\pm}(x)$ and $\tilde{T}(\pm)(x)$ on the Bethe vectors with the general $\vec{v}$ and $\vec{w}$, we prefer to introduce

$$b_{k_1,\ldots,k_p}^{(+)}(x;\vec{v}_k) = b_{k_1}^{(+)}(x)b_{k_2,\ldots,k_p}^{(+)}$$

$$b_{r,\ldots,Q}^{(-)}(x;\vec{w}_r) = b_{r}^{(-)}(x)b_{r+1,\ldots,Q}^{(-)}$$

$$b_{k_1,\ldots,k_p}^{(+)}(\vec{v}_k) = b_{k_1}^{(+)}(v_1)\cdots b_{k_{p-1}}^{(+)}(v_{k-1})b_{k_p}^{(+)}(v_{k+1})\cdots b_{p}^{(+)}(v_{p})$$

$$b_{r,\ldots,Q}^{(-)}(w_r) = b_{r}^{(-)}(w_1)\cdots b_{r+1}^{(-)}(w_{r-1})b_{r+1}^{(-)}(w_{r+1})\cdots b_{Q}^{(-)}(w_Q)$$

$$\hat{R}_{1,\ldots,k}^{(+)}(\vec{v}) = \hat{R}_{1,\ldots,k}^{(+)}(v_1, v_2, \ldots, v_{k-1}, v_k)$$

$$\hat{R}_{1,\ldots,k}^{(-)}(w_r, w_{r+1}) = \hat{R}_{1,\ldots,k}^{(-)}(w_r, w_{r+1})$$

$$\bar{T}_{0,1,\ldots,n}^{(+)}(x;\vec{v}) = \bar{T}_{0,1,\ldots,n}^{(+)}(x;\vec{v})\hat{R}_{0,1,\ldots,n}^{(+)}(x;\vec{v})$$

$$\bar{T}_{0,1,\ldots,n}^{(-)}(w) = \bar{T}_{0,1,\ldots,n}^{(-)}(w)\hat{R}_{0,1,\ldots,n}^{(-)}(w)$$

**Lemma 3.** For any $\vec{v}$ and $\vec{w}$ the relations

$$T_{n}^{+}(x)\left\langle b_{1,\ldots,n}^{(+)}(\vec{v}), e_{\vec{k}}\right\rangle = F(\vec{v}; x)\left\langle b_{1,\ldots,n}^{(+)}(\vec{v}), e_{\vec{k}}\right\rangle T_{n}^{+}(x)$$

$$- \sum_{v_k \in \vec{v}} g(v_k, x) F(\vec{v}_k; v_k)\left\langle b_{k_1,\ldots,k_p}^{(+)}(\vec{v}_k), e_{\vec{k}}\right\rangle T_{n}^{+}(v_k)$$

$$T_{-n}^{+}(x)\left\langle b_{1,\ldots,n}^{(-)}(\vec{v}), f^{-\vec{k}}\right\rangle = F(x, \vec{v})\left\langle b_{1,\ldots,n}^{(-)}(\vec{v}), f^{-\vec{k}}\right\rangle T_{-n}^{+}(x)$$

$$- \sum_{w_r \in \vec{w}} g(w_r, x) F(w_r; \vec{w}_r)\left\langle b_{r_1,\ldots,r_p}^{(-)}(w_r), e_{\vec{k}}\right\rangle T_{-n}^{+}(w_r)$$

$$\tilde{T}_{0}^{+}(x)\left\langle b_{1,\ldots,p}^{(+)}(\vec{v}), e_{\vec{k}}\right\rangle = F(x, \vec{v})\left\langle b_{1,\ldots,p}^{(+)}(\vec{v}), e_{\vec{k}}\right\rangle$$

$$- \sum_{v_k \in \vec{v}} g(v_k, x) F(v_k; \vec{v}_k)\left\langle b_{k_1,\ldots,k_p}^{(+)}(v_k), e_{\vec{k}}\right\rangle$$

$$\tilde{T}_{0}^{(-)}(x)\left\langle b_{1,\ldots,q}^{(-)}(\vec{w}), f^{-\vec{k}}\right\rangle = F(\vec{v}; x)\left\langle b_{1,\ldots,q}^{(-)}(\vec{w}), f^{-\vec{k}}\right\rangle$$

$$- \sum_{w_r \in \vec{w}} g(w_r, x) F(w_r; \vec{w}_r)\left\langle b_{r_1,\ldots,r_p}^{(-)}(w_r), e_{\vec{k}}\right\rangle$$

hold in the RTT–algebra $\mathcal{A}_n$.

**Proof:** We can prove these statements by induction over the number of elements $P$ and $Q$ of the sets \(\vec{v}\) and \(\vec{w}\).

For $P = Q = 1$ these relations are proven in Lemma 1.

We will assume that the statement is valid for $P$ and $Q$ and denote $\vec{v} = (v_1, \ldots, v_P, v_{P+1})$, $\vec{w} = (w_1, \ldots, w_Q, w_{Q+1})$, $\vec{k} = (k_1, \ldots, k_P, k_{P+1})$ and $\vec{r} = (r_1, \ldots, r_Q, r_{Q+1})$. According to
the induction assumption and Lemma 1, we have

\[ T_n(x) \left< b_{1^*,...,n}(\vec{v}), e_k \right> = T_n(x) \left< b_1(v_1), e_k \right> \left< b_{2^*,...,n}(\vec{v}), e_k \right> = \]
\[ = f(v_1, x) \left< b_1(v_1), e_k \right> T_n(x) \left< b_{2^*,...,n}(\vec{v}), e_k \right> - g(v_1, x) \left< b_1(v_1), e_k \right> T_n(x) \left< b_{2^*,...,n}(\vec{v}), e_k \right> = \]
\[ = F(\pi, x) \left< b_1(s, \vec{v}), e_k \right> T_n(x) - g(v_1, x) F(\pi, v_1) \left< b_1(v_1), e_k \right> T_n(x) - \]
\[ - \sum_{v_k \in \pi_1} F(\pi_1; k, v_1) \left[ g(v_k, x) f(v_1, x) \left< b_1(v_1) b_{k^*,...,n}(\vec{v}), e_k \right> - g(v_k, v_1) g(v_1, x) \left< b_1(v_1) b_{k^*,...,n}(\vec{v}), e_k \right> \right] T_n(v_k), \]
\[ T_n(x) \left< b_{1^*,...,n}(\vec{v}), f^{-r} \right> = T_n(x) \left< b_{1^*,...,n}(w_1), f^{-r} \right> \left< b_{2^*,...,n}(\vec{v}), f^{-r} \right> = \]
\[ = F(\pi, x) \left< b_{1^*,...,n}(\vec{v}), f^{-r} \right> T_n(x) - g(x, w_1) F(\pi, w_1) \left< b_{1^*,...,n}(\vec{v}), f^{-r} \right> T_n(w_1) - \]
\[ - \sum_{w_r \in \pi_1} F(\pi_1; r, w_1) \left[ g(x, r) f(x, w_1) \left< b_{1^*,...,n}(w_1) b_{r^*,...,n}(\vec{v}), e_k \right> - g(x, w_1) g(w_1, r) \left< b_{1^*,...,n}(w_1) b_{r^*,...,n}(\vec{v}), e_k \right> \right] T_n(w_r), \]
\[ T_0(x) \left< b_{1^*,...,n}(\vec{v}), e_k \right> = T_0(x) \left< b_{1^*,...,n}(\vec{v}), e_k \right> = \]
\[ = F(\pi, x) \left< b_{1^*,...,n}(\vec{v}), e_k \right> - g(x, v_1) F(\pi, v_1) \left< b_{1^*,...,n}(\vec{v}), e_k \right> - \]
\[ - \sum_{v_k \in \pi_1} g(x, v_k) f(v_1, x) F(\pi; v_1, k, v_1) \left< b_{1^*,...,n}(v_1) b_{k^*,...,n}(\vec{v}), e_k \right> + \]
\[ + \sum_{v_k \in \pi_1} g(x, v_1) g(v_1, x) F(\pi; v_1, k, v_1) \left< b_{1^*,...,n}(v_1) b_{k^*,...,n}(\vec{v}), e_k \right> = \]
\[ T_0(x) \left< b_{1^*,...,n}(\vec{v}), f^{-r} \right> = F(\pi, x) \left< b_{1^*,...,n}(\vec{v}), f^{-r} \right> = \]
\[ \left< b_{1^*,...,n}(\vec{v}), f^{-r} \right> - g(x, v_1) F(\pi, v_1) \left< b_{1^*,...,n}(\vec{v}), f^{-r} \right> - \]
\[ - \sum_{w_r \in \pi_1} g(x, r) f(x, w_1) F(\pi_1; r, w_1) \left< b_{1^*,...,n}(w_1) b_{r^*,...,n}(\vec{v}), f^{-r} \right> + \]
\[ + \sum_{w_r \in \pi_1} g(x, w_r) g(w_r, x) F(\pi_1; r, w_r) \left< b_{1^*,...,n}(w_r) b_{r^*,...,n}(\vec{v}), f^{-r} \right>. \]
If we use in the first two equations the relations

\[
\left\langle b_{1}^{(+)}(v_{1})b_{k,*,2,...,(p+1),*}^{(+)}(x; \vec{v}_{1,k}), \hat{R}_{2,...,k}^{(+,+)}(\vec{v}_{1})e_{k} \right\rangle =
\begin{align*}
&= \left\langle b_{k,*,1,...,(p+1),*}^{(+)}(x; \vec{v}_{k}), \hat{R}_{1_{+,k+}}^{(+,+)}(v_{1}, x)\hat{R}_{2,...,k}^{(+,+)}(\vec{v}_{1})e_{k} \right\rangle \\
&= \left\langle b_{k,*,1,...,(p+1),*}^{(+)}(x; \vec{v}_{k}), \hat{R}_{1_{+,k+}}^{(+,+)}\hat{R}_{2,...,k}^{(+,+)}(\vec{v}_{1})e_{k} \right\rangle \\
\end{align*}
\]

that result from Lemma 2, and compare the results with the first two relations of the proven Lemma, we can see that it is enough to show for any \( v_{k} \in \mathcal{V} \) and any \( w_{r} \in \mathcal{W} \) the equalities

\[
g(v_{k}, x) f(v_{1}, v_{k}) \hat{R}_{1_{+,k+}}^{(+,+)}(v_{1}, v_{k}) = g(v_{k}, x) f(v_{1}, x) \hat{R}_{1_{+,k+}}^{(+,+)}(v_{1}, x) - g(v_{k}, v_{1}) g(v_{1}, x) \hat{R}_{1_{+,k+}}^{(+,+)}; \]

\[
g(x, w_{r}) f(w_{r}, w_{1}) \hat{R}_{1_{+,r+}}^{(-,-)}(w_{r}, w_{1}) =
\begin{align*}
&= g(x, w_{r}) f(x, w_{1}) \hat{R}_{1_{+,r+}}^{(-,-)}(x, w_{1}) - g(x, w_{1}) g(w_{1}, w_{r}) \hat{R}_{1_{+,r+}}^{(-,-)}. \\
\end{align*}
\]

However, this is equivalent to the identities

\[
g(v_{k}, x) g(v_{1}, v_{k}) = g(v_{k}, x) g(v_{1}, x) - g(v_{k}, v_{1}) g(v_{1}, x)
\]

\[
g(x, w_{r}) g(w_{r}, w_{1}) = g(x, w_{r}) g(x, w_{1}) - g(x, w_{1}) g(w_{1}, w_{r}).
\]

To prove the third and fourth equality of Lemma, we use the relations

\[
\begin{align*}
\left\langle b_{1}^{(+)}(v_{1})b_{k,*,2,...,(p+1),*}^{(+)}(x; \vec{v}_{1,k}), \hat{R}_{2,...,k}^{(+,+)}(\vec{v}_{1}) \hat{P}_{k:0,2,...,p+1}(\vec{v}_{1}) \hat{R}_{0_{+,1}}^{(+,+)}(x, v_{1}) e_{k} \right\rangle &=
\begin{align*}
&= \left\langle b_{k,*,1,...,(p+1),*}^{(+)}(x; \vec{v}_{k}), \hat{R}_{1_{+,k+}}^{(+,+)}(v_{1}, x) \hat{R}_{2,...,k}^{(+,+)}(\vec{v}_{1}) \hat{P}_{k:0,2,...,p+1}(\vec{v}_{1}) \hat{R}_{0_{+,1}}^{(+,+)}(x, v_{1}) e_{k} \right\rangle \\
&= \left\langle b_{k,*,1,...,(p+1),*}^{(+)}(x; \vec{v}_{k}), \hat{R}_{1_{+,k+}}^{(+,+)} \hat{R}_{2,...,k}^{(+,+)}(\vec{v}_{1}) \hat{P}_{k:0,2,...,p+1}(\vec{v}_{1}) \hat{R}_{0_{+,1}}^{(+,+)} e_{k} \right\rangle,
\end{align*}
\]

\[
\begin{align*}
\left\langle b_{1}^{(-)}(w_{1})b_{r,2,...,Q+1,(x; \vec{w}_{1,r}), \hat{R}_{1_{+,r+}}^{(-,-)}(x, w_{1}) \hat{R}_{2,...,r+}^{(-,-)}(\vec{w}_{1}) \hat{P}_{r:0,2,...,Q+1}(x, w_{1}) \hat{R}_{r_{+,2}}^{(-,-)}(x, w_{1}) e_{r} \right\rangle &=
\begin{align*}
&= \left\langle b_{r,2,...,Q+1}(x; \vec{w}_{r}), \hat{R}_{1_{+,r+}}^{(-,-)}(x, w_{1}) \hat{R}_{1_{+,r+}}^{(-,-)}(\vec{w}_{1}) \hat{P}_{r:0,2,...,Q+1}(\vec{w}_{1}) \hat{R}_{r_{+,2}}^{(-,-)} e_{r} \right\rangle \\
&= \left\langle b_{r,2,...,Q+1}(x; \vec{w}_{r}), \hat{R}_{1_{+,r+}}^{(-,-)} \hat{R}_{r_{+,2}}^{(-,-)}(\vec{w}_{1}) \hat{P}_{r:0,2,...,Q+1}(\vec{w}_{1}) \hat{R}_{r_{+,2}}^{(-,-)} e_{r} \right\rangle,
\end{align*}
\]

\[
\begin{align*}
\left\langle b_{1}^{(-)}(x)b_{r,2,...,Q+1,(w; \vec{w}_{1,r}), \hat{R}_{1_{+,r+}}^{(-,-)}(w, x) \hat{R}_{2,...,r+}^{(-,-)}(\vec{w}_{1}) \hat{P}_{r:0,2,...,Q+1}(w, x) \hat{R}_{r_{+,2}}^{(-,-)}(w, x) e_{r} \right\rangle &=
\begin{align*}
&= \left\langle b_{r,2,...,Q+1}(w; \vec{w}_{r}), \hat{R}_{1_{+,r+}}^{(-,-)}(w, x) \hat{R}_{1_{+,r+}}^{(-,-)}(\vec{w}_{1}) \hat{P}_{r:0,2,...,Q+1}(\vec{w}_{1}) \hat{R}_{r_{+,2}}^{(-,-)} e_{r} \right\rangle \\
&= \left\langle b_{r,2,...,Q+1}(w; \vec{w}_{r}), \hat{R}_{1_{+,r+}}^{(-,-)} \hat{R}_{r_{+,2}}^{(-,-)}(\vec{w}_{1}) \hat{P}_{r:0,2,...,Q+1}(\vec{w}_{1}) \hat{R}_{r_{+,2}}^{(-,-)} e_{r} \right\rangle.
\end{align*}
\]

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that follow from Lemma 2. Then we get the equalities

\[
\mathbf{T}_0^{(+)}(x) \langle b_{1,\ldots,(P+1)^*}(v), e_k \rangle = F(x, v) \langle b_{1,\ldots,(P+1)^*}(v), \hat{T}_{0;1,\ldots,P+1}^{(+)}(x; \tilde{v}_1) e_k \rangle - g(x, v_1) F(v_1, \tilde{v}_1) \langle b_{1,\ldots,(P+1)^*}(x; \tilde{v}_1), \hat{T}_{1;0,1,\ldots,P+1}^{(+)}(v) e_k \rangle
\]

\[- g(x, v_1) F(v_1, \tilde{v}_1) \langle b_{1,\ldots,(P+1)^*}(x; \tilde{v}_1), \hat{T}_{1;0,1,\ldots,P+1}^{(+)}(v) e_k \rangle - \sum_{v_k \in \tau_1} F(v_k, \tilde{v}_1) g(v_1, v_k) \langle b_{1,\ldots,(P+1)^*}(x; \tilde{v}_k), \hat{R}_{1,k}^{(+)}(v_1) \hat{R}_{2,k}^{(+)}(\tilde{v}_1) \hat{T}_{k;0,1,\ldots,P+1}^{(+)}(v) e_k \rangle - g(x, v_1) g(v_1, v_k) \langle b_{1,\ldots,(P+1)^*}(x; \tilde{v}_k), \hat{R}_{1,k}^{(+)}(v_1) \hat{R}_{2,k}^{(+)}(\tilde{v}_1) \hat{T}_{k;0,1,\ldots,P+1}^{(+)}(v) e_k \rangle \]

\[
\mathbf{T}_0^{(-)}(x) \langle b_{1,\ldots,Q+1}(\tilde{w}), f^{-r} \rangle = F(x, v) \langle b_{1,\ldots,Q+1}(\tilde{w}), \hat{T}_{0;1,\ldots,Q+1}^{(-)}(x; \tilde{w}) f^{-r} \rangle - g(w_1, 1) F(v_1, w_1) \langle b_{1,\ldots,Q+1}(x; \tilde{w}_1), \hat{T}_{1;0,1,\ldots,Q+1}^{(-)}(\tilde{w}) f^{-r} \rangle - \sum_{w_r \in \tau_1} F(\tau_{1,r}, w_1) g(w_r, v_1) \langle b_{r,1,\ldots,Q+1}(x; \tilde{w}_r), \hat{R}_{1,r}^{(-)}(w_1) \hat{R}_{2,r}^{(-)}(\tilde{w}_1) \hat{T}_{r;0,2,\ldots,(Q+1)^*}(\tilde{w}_1) f^{-r} \rangle - g(w_r, v_1) g(w_1, x) \langle b_{r,1,\ldots,Q+1}(x; \tilde{w}_r), \hat{R}_{1,r}^{(-)}(w_1) \hat{R}_{2,r}^{(-)}(\tilde{w}_1) \hat{T}_{r;0,2,\ldots,(Q+1)^*}(\tilde{w}_1) f^{-r} \rangle \]

If we show that the relations

\[
g(x, v_k) f(v_k, v_1) \hat{R}_{1,k}^{(+)}(v) \hat{T}_{k;0,1,\ldots,P+1}^{(+)}(v) = g(x, v_k) f(v_k, v_1) \hat{R}_{1,k}^{(+)}(v) \hat{T}_{k;0,1,\ldots,P+1}^{(+)}(v) - g(x, v_1) g(v_1, v_k) \hat{R}_{1,k}^{(+)}(v) \hat{T}_{k;0,1,\ldots,P+1}^{(+)}(v) \hat{R}_{2,k}^{(+)}(\tilde{v}_1) \hat{T}_{k;0,1,\ldots,P+1}^{(+)}(v) \hat{R}_{0,1,\ldots,P+1}^{(+)}(v)
\]

\[
g(w_r, x) f(w_r, x) \hat{R}_{1,r}^{(-)}(w) \hat{T}_{r;0,1,\ldots,Q+1}^{(-)}(w) = g(w_r, x) f(w_r, x) \hat{R}_{1,r}^{(-)}(w) \hat{T}_{r;0,1,\ldots,Q+1}^{(-)}(w) - g(w_r, w_1) g(w_1, x) \hat{R}_{1,r}^{(-)}(w) \hat{T}_{r;0,2,\ldots,(Q+1)^*}(w_1) \hat{T}_{r;0,2,\ldots,(Q+1)^*}(w_1)
\]

are valid for any \( v_k \in \tau_1 \) and \( w_r \in \tau_1 \), the third and fourth equality in Lemma will hold. Since, by definition

\[
\hat{T}_{k;0,1,\ldots,P+1}^{(+)}(v) = \mathbf{T}_0^{(+)k}(v_1) \hat{R}_{0,1,\ldots,P+1}^{(+)}(v_1; \tilde{v})
\]

\[
\hat{T}_{k;0,2,\ldots,P+1}^{(+)}(v_1) = \mathbf{T}_0^{(+)k}(v_1) \hat{R}_{0,2,\ldots,P+1}^{(+)}(v_1; \tilde{v})
\]

\[
\hat{T}_{r;0,1,\ldots,Q+1}^{(-)}(w_1) = \mathbf{R}_0^{(-)}(w_r) \hat{R}_{0,1,\ldots,Q+1}^{(-)}(w_r; \tilde{w}) \hat{T}_0^{(-)}(w_r)
\]

\[
\hat{T}_{r;0,2,\ldots,(Q+1)^*}(w_1) = \mathbf{R}_0^{(-)}(w_r) \hat{R}_{0,2,\ldots,(Q+1)^*}(w_r; \tilde{w}) \hat{T}_0^{(-)}(w_r)
\]
it suffices to show that
\[
g(x, v_k) f(v_k, v_1) \hat{R}_{1, k}^{(+)(+)}(v_1; \bar{v}) \hat{R}_{0, 1, \ldots, P + 1}^{(+)(+)}(v_k; \bar{v}) =
\]
\[
= g(x, v_k) f(x, v_1) \hat{R}_{1, k}^{(+)(+)}(v_1, x) \hat{R}_{0, 2, \ldots, P + 1}^{(+)(+)}(v_1; \bar{v}) \hat{R}_{0, 1, \ldots, P + 1}^{(+)(+)}(x, v_k) -
\]
\[
- g(x, v_1) g(v_1, v_k) \hat{R}_{1, k}^{(+)(+)}(v_1; \bar{v}) \hat{R}_{0, 2, \ldots, P + 1}^{(+)(+)}(v_k; \bar{v}) \hat{R}_{0, 1, \ldots, P + 1}^{(+)(+)}(x, v_1) =
\]
\[
g(w_r, x) f(w_1, x) \hat{R}_{1, r}^{(-)(-)}(w_1, x) \hat{R}_{0, 1, \ldots, (Q + 1)}^{(-)(-)}(w_1; \bar{w_1}) \hat{R}_{0, 0, \ldots, (Q + 1)}^{(-)(-)}(w_r; \bar{w_1}) -
\]
\[
- g(w_r, w_1) g(w_1, x) \hat{R}_{1, r}^{(-)(-)}(w_1, x) \hat{R}_{0, 1, \ldots, (Q + 1)}^{(-)(-)}(w_1; \bar{w_1}) \hat{R}_{0, 0, \ldots, (Q + 1)}^{(-)(-)}(x, w_r)
\]
are true.

If we use the definitions of the products of R–matrices, we find that it is enough to show
\[
g(x, v_k) f(v_k, v_1) \hat{R}_{1, k}^{(+)(+)}(v_1, v_k) \ldots \hat{R}_{k - 1, k}^{(+)(+)}(v_{k - 1}, v_k) =
\]
\[
= g(x, v_k) f(x, v_1) \hat{R}_{1, k}^{(+)(+)}(v_1, x) \hat{R}_{2, k}^{(+)(+)}(v_2, v_k) \ldots \hat{R}_{k - 1, k}^{(+)(+)}(v_{k - 1}, v_k) =
\]
\[
- g(x, v_1) g(v_1, v_k) \hat{R}_{1, k}^{(+)(+)}(v_1, v_k) \hat{R}_{2, k}^{(+)(+)}(v_2, v_k) \ldots \hat{R}_{k - 1, k}^{(+)(+)}(v_{k - 1}, v_k) =
\]
\[
g(w_r, x) f(w_1, x) \hat{R}_{1, r}^{(-)(-)}(w_1, x) \hat{R}_{2, r}^{(-)(-)}(w_2, w_k) \ldots \hat{R}_{r - 1, r}^{(-)(-)}(w_{r - 1}, w_k) =
\]
\[
- g(w_r, w_1) g(w_1, x) \hat{R}_{1, r}^{(-)(-)}(w_1, x) \hat{R}_{2, r}^{(-)(-)}(w_2, w_k) \ldots \hat{R}_{r - 1, r}^{(-)(-)}(w_{r - 1}, w_k) =
\]

When we use the Yang–Baxter equations
\[
\hat{R}_{r + k}^{(+)(+)}(v_k, v_r) \hat{R}_{0, k}^{(+)(+)}(v_r, v_k) = \hat{R}_{0, r}^{(+)(+)}(v_r, v_k) \hat{R}_{r + k}^{(+)(+)}(v_k, v_r)
\]
\[
\hat{R}_{s + r}^{(-)(-)}(w, w_s) \hat{R}_{0, s}^{(-)(-)}(w_s, w) = \hat{R}_{0, s}^{(-)(-)}(w_s, w) \hat{R}_{s + r}^{(-)(-)}(w, w_s)
\]
sufficient to show that it is enough to prove the relations
\[
g(x, v_k) f(v_k, v_1) \hat{R}_{1, k}^{(+)(+)}(v_1, v_k) \hat{R}_{0, 1, \ldots, P + 1}^{(+)(+)}(v_k, v_1) =
\]
\[
= g(x, v_k) f(x, v_1) \hat{R}_{1, k}^{(+)(+)}(v_1, x) \hat{R}_{0, 2, \ldots, P + 1}^{(+)(+)}(v_1, x) =
\]
\[
- g(x, v_1) g(v_1, v_k) \hat{R}_{1, k}^{(+)(+)}(v_1; \bar{v}) \hat{R}_{0, 2, \ldots, P + 1}^{(+)(+)}(v_k; \bar{v}) \hat{R}_{0, 1, \ldots, P + 1}^{(+)(+)}(x, v_1) =
\]
\[
g(w_r, x) f(w_1, x) \hat{R}_{1, r}^{(-)(-)}(w_1, x) \hat{R}_{0, 2, \ldots, (Q + 1)}^{(-)(-)}(w_1, x) \hat{R}_{0, 1, \ldots, (Q + 1)}^{(-)(-)}(x, w_r) =
\]
\[
- g(w_r, w_1) g(w_1, x) \hat{R}_{1, r}^{(-)(-)}(w_1, x) \hat{R}_{0, 2, \ldots, (Q + 1)}^{(-)(-)}(w_1, x) \hat{R}_{0, 1, \ldots, (Q + 1)}^{(-)(-)}(x, w_r) =
\]
which can be verified by direct calculation.

**Lemma 4.** For any \(\vec{v}, \vec{w}\) and \(\vec{k}, \vec{r}\) the following relations are true:

\[
T_n^-(x) \langle b_{1,\ldots,v}^{(-)}(\vec{w}), f^{-\vec{r}} \rangle = F(x) + 1 + \eta \langle b_{1,\ldots,v}^{(-)}(\vec{w}), f^{-\vec{r}} \rangle T_n^-(x) + \\
\sum_{w_s \in \vec{w}} h(w_s, x) F(\vec{w}_s; w_s) T_0 \left( \langle b_{s_x}^{(-)}(x) b_{1,\ldots,v}^{(-)}(\vec{w}_s), \vec{w}_s \rangle \right) \\
T_n^-(x) \langle b_{1,\ldots,v}^{(+)}(\vec{v}), f^{\vec{k}} \rangle = F(x) - 1 - \eta \langle b_{1,\ldots,v}^{(+)}(\vec{v}), f^{\vec{k}} \rangle T_n^-(x) + \\
\sum_{v_x \in \vec{v}} \tilde{h}(x, v_x) F(\vec{v}_x; \vec{v}) T_0 \left( \langle b_{v_x}^{(+)}(x) b_{1,\ldots,v}^{(-)}(\vec{v}_x), \vec{v}_x \rangle \right) \\
T_n^+(x) \langle b_{1,\ldots,v}^{(\vec{k})}(\vec{w}), f^{-\vec{r}} \rangle = \langle b_{1,\ldots,v}^{(\vec{k})}(\vec{w}), T_n^+(x; \vec{w}) f^{-\vec{r}} \rangle - \\
\sum_{w_s \in \vec{w}} h(w_s, x) F(\vec{w}_s; w_s) \langle b_{s_x}^{(+)}(x) b_{1,\ldots,v}^{(-)}(\vec{w}_s), \vec{w}_s \rangle \\
T_n^+(x) \langle b_{1,\ldots,v}^{(-)}(\vec{v}), f^{\vec{k}} \rangle = \langle b_{1,\ldots,v}^{(-)}(\vec{v}), T_n^+(x; \vec{v}) f^{\vec{k}} \rangle - \\
\sum_{v_x \in \vec{v}} \tilde{h}(x, v_x) F(\vec{v}_x; \vec{v}) \langle b_{v_x}^{(-)}(x) b_{1,\ldots,v}^{(+)}(\vec{v}_x), \vec{v}_x \rangle \\
\text{where} \\
\hat{T}_{0;1,\ldots,Q^+}(x; \vec{w}) = \hat{R}_{0;1,\ldots,Q^+}(x; \vec{w}) \hat{T}_0^+(x), \\
\hat{T}_{0;1,\ldots,P^+}(x; \vec{v}) = \hat{T}_0^+(x) \hat{R}_{0;1,\ldots,P^+}(x; \vec{v}).

**Proof:** These statements can be proven by induction according to the number of elements \(P\) and \(Q\) of the sets \(\vec{v}\) and \(\vec{w}\). For \(P = 1\) and \(Q = 1\), these statements are proved in Lemma 3.

Assume that these statements hold for \(P\) and \(Q\) and denote \(\vec{v} = (v_1, \ldots, v_{P+1}), \vec{w} = (w_1, \ldots, w_{Q+1}), \vec{k} = (k_1, \ldots, k_{P+1})\) and \(\vec{r} = (r_1, \ldots, r_{Q+1})\).

To show the first statement, we use the equality

\[
T_n^+(x) \langle b_{1,\ldots,v}^{(-)}(\vec{w}), f^{-\vec{r}} \rangle = T_n^+(x) \langle b_{1,\ldots,v}^{(-)}(w_1), f^{-\vec{r}_1} \rangle \\
= \tilde{h}(w_1, x) \langle b_{1,\ldots,v}^{(-)}(w_1), f^{-\vec{r}_1} \rangle T_n^+(x) \langle b_{1,\ldots,v}^{(-)}(\vec{w}_1), f^{-\vec{r}_1} \rangle \\
+ \tilde{h}(w_1, x) T_0 \langle b_{1,\ldots,v}^{(+)}(x), \tilde{R}_{0;1,\ldots,Q^+}(\vec{w}_1) f^{-\vec{r}_1} \rangle T_0^-(w_1) \langle b_{1,\ldots,v}^{(-)}(\vec{w}_1), f^{-\vec{r}_1} \rangle.
\]
which results from Lemma 1. Using the induction assumption and Lemma 3, we will get

\[ T_n(x) \langle b_{1,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \rangle = F(\tilde{w}; x - n + 1 + \eta) \langle b_{1,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \rangle T_n(x) + \]

\[ + h(w_1, x) F(\tilde{w}_1; w_1) T_0 \left( \langle b_{1,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \rangle \right) \]

\[ \sum_{w_s \in \mathcal{W}_1} \tilde{h}(w_1, x) \tilde{h}(w_s, x) F(\tilde{w}_1, w_s) T_0 \left( \langle b_{1,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \rangle \right) \]

\[ - \sum_{w_s \in \mathcal{W}_1} \tilde{h}(w_1, x) g(w_s, w_1) F(\tilde{w}_1; w_s) T_0 \left( \langle b_{1,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \rangle \right) \]

When we use Lemma 2 and the relationship \( \hat{P}_{1,\ldots,Q+1}^{-} = \hat{I}_{1,\ldots,Q+1}^{+} \), we obtain the relation

\[ \langle b_{0,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \rangle = \]

\[ \langle b_{0,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \rangle \]

So it is enough to show that for any \( s = 2, \ldots, Q + 1 \) we have

\[ \tilde{h}(w_s, x) f(w_1, w_s) \hat{R}_{1,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \]

\[ = \]

\[ \sum_{w_s \in \mathcal{W}_1} \tilde{h}(w_1, x) \tilde{h}(w_s, x) \hat{R}_{1,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \]

It follows from the definitions of the operators that in order to prove a statement, it is sufficient to prove the relation

\[ \tilde{h}(w_s, x) f(w_1, w_s) \hat{R}_{1,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \]

By direct calculation it is possible to show that the Yang–Baxter equation

\[ \hat{R}_{1,\ldots,Q+1}^{-}\tilde{w}, f^{-\tilde{r}} \]
holds and by its repeated use we find that for the proof of the first statement it is sufficient to prove the relation
\[
\tilde{h}(w_s, x) f(w_1, w_s) \hat{R}^{(-, -)}_{1_s, s^*} (w_s, w_1) \hat{R}^{(-, -)}_{0_s, 1^*} (w_s, w_1) \hat{R}^{(-, -)}_{0_s, s^*} = \\
= \frac{\hat{h}(w_1, x)}{\tilde{h}(w_1, x)} \tilde{h}(w_s, x) \hat{R}^{(-, -)}_{0_s, s^*} - \tilde{h}(w_1, x) g(w_s, w_1) \hat{R}^{(-, -)}_{1_s, s^*} \hat{R}^{(-, -)}_{0_s, 1^*} \hat{R}^{(-, -)}_{0_s, s^*}.
\]

But this relationship is equivalent to the identity
\[
\tilde{h}(w_s, x) f(w_1, w_s) = \frac{\hat{h}(w_1, x)}{\tilde{h}(w_1, x)} \tilde{h}(w_s, x) - \tilde{h}(w_1, x) g(w_s, w_1).
\]

To prove the second relationship, we use Lemmas 1 and 3, from which it follows
\[
T^{-n}_n(x) \langle b^{(+)}_{1, \ldots, (P+1)^*}, (\tilde{v}), e_k \rangle = T^{-n}_n(x) \langle b^{(+)}_{1, \ldots, (P+1)^*}, (\tilde{v}), e_k \rangle = \\
= F(x + n - 1 - \eta; v) \langle b^{(+)}_{1, \ldots, (P+1)^*}, (\tilde{v}), e_k \rangle T^{-n}_n(x) + \\
+ \tilde{h}(x, v_1) F(v_1; \bar{v}) T_0 \left( \langle b^{(+)}_{1, \ldots, (P+1)^*}, (\tilde{v}), e_k \rangle \hat{R}^{(-, +)}_{0, 1, \ldots, (P+1)} (v_1) + \tilde{h}(x, v_1) F(v_1; \bar{v}) T_0 \left( \langle b^{(+)\ldots (P+1)^*}, (\tilde{v}), e_k \rangle \hat{R}^{(-, +)}_{0, 1, \ldots, (P+1)} (v_1) \right) - \\
- \sum_{v_1 \in \bar{v}} \tilde{h}(x, v_1) g(v_1, v_1) F(v_1; \bar{v}) T_0 \left( \langle b^{(+)\ldots (P+1)^*}, (v_1; \bar{v}), e_k \rangle \hat{R}^{(-, +)}_{0, 1, \ldots, (P+1)} (v_1) \right).
\]

According to Lemma 2,
\[
\langle b^{(+)\ldots (P+1)^*}, (v_1; \bar{v}), e_k \rangle = \langle b^{(+)\ldots (P+1)^*}, (v_1; \bar{v}), e_k \rangle T^{-n}_n(x) + \\
+ \tilde{h}(x, v_1) F(v_1; \bar{v}) T_0 \left( \langle b^{(+)\ldots (P+1)^*}, (v_1; \bar{v}), e_k \rangle \hat{R}^{(-, +)}_{0, 1, \ldots, (P+1)} (v_1) \right) - \\
- \sum_{v_1 \in \bar{v}} \tilde{h}(x, v_1) g(v_1, v_1) F(v_1; \bar{v}) T_0 \left( \langle b^{(+)\ldots (P+1)^*}, (v_1; \bar{v}), e_k \rangle \hat{R}^{(-, +)}_{0, 1, \ldots, (P+1)} (v_1) \right).
\]

and so
\[
T^{-n}_n(x) \langle b^{(+)\ldots (P+1)^*}, (\tilde{v}), e_k \rangle = F(x + n - 1 - \eta; v) \langle b^{(+)\ldots (P+1)^*}, (\tilde{v}), e_k \rangle T^{-n}_n(x) + \\
+ \tilde{h}(x, v_1) F(v_1; \bar{v}) T_0 \left( \langle b^{(+)\ldots (P+1)^*}, (\tilde{v}), e_k \rangle \hat{R}^{(-, +)}_{0, 1, \ldots, (P+1)} (v_1) \right) - \\
- \sum_{v_1 \in \bar{v}} \tilde{h}(x, v_1) g(v_1, v_1) F(v_1; \bar{v}) T_0 \left( \langle b^{(+)\ldots (P+1)^*}, (\tilde{v}), e_k \rangle \hat{R}^{(-, +)}_{0, 1, \ldots, (P+1)} (v_1) \right).
\]
Therefore, it is sufficient to show that the relation

\[ \tilde{h}(x, v_\ell) f(v_\ell, v_1) \widehat{R}^{(+,+)}_{1,\ell}(v_1) \widehat{T}^{(+,+)}_{\ell;0,1,\ldots,p+1}(v_\ell) = \]

\[ = \frac{\tilde{h}(x, v_1)\tilde{h}(x, v_\ell)}{h(x, v_1)} \tilde{h}(x, v_\ell) \widehat{R}^{(+,+)}_{2,\ell}(v_1) \widehat{T}^{(+,+)}_{\ell;0,2,\ldots,p+1}(v_\ell) - \]

\[ -\tilde{h}(x, v_1) g(v_1, v_\ell) \widehat{R}^{(+,+)}_{1,\ell}(v_1) \widehat{T}^{(+,+)}_{\ell;0,1,\ldots,p+1}(v_\ell) \]

is valid for any \( \ell = 2, \ldots, P + 1 \).

When we use the definitions \( \widehat{R}^{(+,+)}_{1,\ell}(v_1), \widehat{R}^{(+,+)}_{2,\ell}(v_1), \widehat{T}^{(+,+)}_{\ell;0,1,\ldots,p+1}(v_\ell) \) and \( \widehat{T}^{(+,+)}_{\ell;0,2,\ldots,p+1}(v_\ell) \), we find that to prove the statement, it is enough to show the equality

\[ \tilde{h}(x, v_\ell) f(v_\ell, v_1) \widehat{R}^{(+,+)}_{1,\ell}(v_1) \ldots \widehat{R}^{(+,+)}_{\ell-1,\ell}(v_\ell-1, v_\ell) \widehat{R}^{(+,+)}_{0,\ell}(v_\ell, v_1) = \]

\[ = \frac{\tilde{h}(x, v_1)\tilde{h}(x, v_\ell)}{h(x, v_1)} \tilde{h}(x, v_\ell) \widehat{R}^{(+,+)}_{2,\ell}(v_1) \ldots \widehat{R}^{(+,+)}_{\ell-1,\ell}(v_\ell-1, v_\ell) \widehat{R}^{(+,+)}_{0,\ell}(v_\ell, v_1) - \]

\[ -\tilde{h}(x, v_1) g(v_1, v_\ell) \widehat{R}^{(+,+)}_{1,\ell}(v_1) \ldots \widehat{R}^{(+,+)}_{\ell-1,\ell}(v_\ell-1, v_\ell) \widehat{R}^{(+,+)}_{0,\ell}(v_\ell, v_1) \]

By repeatedly using the Yang–Baxter equation

\[ \widehat{R}^{(+,+)}_{k,\ell}(v_k, v_\ell) \widehat{R}^{(+,+)}_{0,\ell}(v_\ell, v_1) = \widehat{R}^{(+,+)}_{0,k}(v_\ell, v_k) \widehat{R}^{(+,+)}_{0,\ell}(v_\ell, v_1), \]

which can be verified by direct calculation, we find that to prove the statement it is enough to prove the relation

\[ \tilde{h}(x, v_\ell) f(v_\ell, v_1) \widehat{R}^{(+,+)}_{1,\ell}(v_1) \ldots \widehat{R}^{(+,+)}_{\ell-1,\ell}(v_\ell-1, v_\ell) \widehat{R}^{(+,+)}_{0,\ell}(v_\ell, v_1) = \]

\[ = \frac{\tilde{h}(x, v_1)\tilde{h}(x, v_\ell)}{h(x, v_1)} \tilde{h}(x, v_\ell) \widehat{R}^{(+,+)}_{0,\ell}(v_\ell, v_1) \]

which is equivalent to the identity

\[ \tilde{h}(x, v_\ell) f(v_\ell, v_1) = \frac{\tilde{h}(x, v_1)\tilde{h}(x, v_\ell)}{h(x, v_1)} \tilde{h}(x, v_\ell) - \tilde{h}(x, v_1) g(v_1, v_\ell). \]

Assuming that the third statement holds for \( Q \), we get by Lemmas 1 and 3

\[ \mathcal{T}^{(+)}_{0}(x) \left< b_{1,\ldots,Q+1}(\vec{w}), f^{-\vec{r}} \right> = \left< b_{1,\ldots,Q+1}(\vec{w}), T^{(+)}_{0,1,\ldots,Q+1}(x; \vec{w}) f^{-\vec{r}} \right> - \]

\[ -\tilde{h}(w_1, x) F(w_1; \vec{w}_1) \left< b_{1,\ldots,Q+1}(\vec{w}_1), \tilde{R}^{(+,+)}_{1,\ldots,Q+1}(\vec{w}_1) f^{-\vec{r}} \right> T^{-n}(w_1) + \]

\[ - \sum_{w_\ell \in \vec{w}_1} \tilde{h}(w_\ell, x) F(w_\ell; \vec{w}_1) \left< b_{\ldots,Q+1}(\vec{w}_1), \tilde{R}^{(+,+)}_{1,\ldots,Q+1}(\vec{w}_1) f^{-\vec{r}} \right> T^{-n}(w_\ell) + \]

\[ + \sum_{w_\ell \in \vec{w}_1} \tilde{h}(w_\ell, x) g(w_\ell, w_1) F(w_\ell, \vec{w}_1) \left< b_{1,\ldots,Q+1}(\vec{w}_1), \tilde{R}^{(+,+)}_{1,\ldots,Q+1}(\vec{w}_1) f^{-\vec{r}} \right> T^{-n}(w_\ell) \]
According to Lemma 2,

\[
\left\langle b_1^{(+)}(x)b_{s;2\ldots, Q+1}(w_1; \tilde{w}_1, s)\hat{R}_1^{(+,\ldots, -)}\hat{R}^{(+,\ldots, -)}_{0,\ldots, 1^*}(\tilde{w}_1) f^{-r}\right\rangle = \\
= \left\langle b_{s}^{(+)}(x) b_{1,\ldots, s; Q+1}(w_1; \tilde{w}_s)\hat{P}^{(+,\ldots, -)}\hat{R}^{(+,\ldots, -)}_{s; 0,\ldots, 1^*} \hat{R}^{(+,\ldots, -)}_{2,\ldots, s^*}(\tilde{w}_1) f^{-r}\right\rangle
\]

and so

\[
\tilde{T}_0^{(+)}(x)\left\langle b_{1,\ldots, Q+1}(\tilde{w}), f^{-r}\right\rangle = \left\langle b_{1,\ldots, Q+1}(\tilde{w}), \tilde{T}_0^{(+)}(x; \tilde{w}) f^{-r}\right\rangle - \\
- \tilde{h}(w_1, x) F(w_1; \tilde{w}_1) \left\langle b_{s}^{(+)}(x) b_{2,\ldots, Q+1}(\tilde{w}_1), \hat{P}^{(+,\ldots, -)}\hat{R}^{(+,\ldots, -)}_{0,\ldots, 1^*}(\tilde{w}_1) f^{-r}\right\rangle T^{-n}(w_1) - \\
- \sum_{w_s \in \tilde{w}_1} \tilde{h}(w_s, x) F(w_s; \tilde{w}_1) \left\langle b_{s}^{(+)}(x) b_{1,\ldots, Q+1}(w_1; \tilde{w}_s), \hat{P}^{(+,\ldots, -)}\hat{R}^{(+,\ldots, -)}_{0,\ldots, 1^*} \hat{R}^{(+,\ldots, -)}_{2,\ldots, s^*}(\tilde{w}_1) f^{-r}\right\rangle T^{-n}(w_s) + \\
+ \sum_{w_s \in \tilde{w}_1} \tilde{h}(w_1, x) g(w_1, w_s) F(w_s; \tilde{w}_1) \left\langle b_{s}^{(+)}(x) b_{1,\ldots, Q+1}(w_1; \tilde{w}_s), \hat{P}^{(+,\ldots, -)}\hat{R}^{(+,\ldots, -)}_{0,\ldots, 1^*} \hat{R}^{(+,\ldots, -)}_{2,\ldots, s^*}(\tilde{w}_1) f^{-r}\right\rangle T^{-n}(w_s)
\]

Therefore, it is enough to show that for any \( s = 2, \ldots, Q + 1 \) we have

\[
\tilde{h}(w_s, x) F(w_s, w_1)\hat{R}^{(+,\ldots, -)}_{0,\ldots, 1^*} \hat{R}^{(+,\ldots, -)}_{1^*,\ldots, s^*}(w_s, w_1) = \\
\tilde{h}(w_s, x) \hat{R}^{(+,\ldots, -)}_{0,\ldots, 1^*}(x, w_1) - \tilde{h}(w_1, x) g(w_1, w_s) \hat{R}^{(+,\ldots, -)}_{0,\ldots, 1^*} \hat{R}^{(+,\ldots, -)}_{2,\ldots, s^*}(w_1, \tilde{w}_s)
\]

If we use the definitions of these mappings, we find that these relations are equivalent to identity

\[
\tilde{h}(w_s, x) g(w_s, w_1) + \tilde{h}(w_s, x) \tilde{h}(w_1, x) + \tilde{h}(w_1, x) g(w_1, w_s) = 0.
\]

To prove the fourth statement, we use the relation

\[
\tilde{T}_0^{(-)}(x)\left\langle b_{1,\ldots, (P+1)^*}(\tilde{v}), e_{\ell} \right\rangle = \left\langle b_{1,\ldots, (P+1)^*}(\tilde{v}), \tilde{T}_0^{(-)}(x; \tilde{v}) e_{\ell} \right\rangle - \\
- \tilde{h}(x, v_1) F(\tilde{v}_1; v_1) \left\langle b_{2,\ldots, (P+1)^*}(\tilde{v}_1) b_{1}^{(-)}(x), \hat{P}^{(-,\ldots, -)}\hat{R}^{(-,\ldots, -)}_{0,\ldots, 1^*} e_{\ell}\right\rangle T^{-n}(v_1) - \\
- \sum_{v_\ell \in \tilde{v}_1} \tilde{h}(x, v_\ell) F(\tilde{v}_1, v_\ell) \left\langle b_{1,\ldots, e,\ldots, (P+1)^*}(\tilde{v}_\ell) b_{1}^{(-)}(x), \hat{P}^{(-,\ldots, -)}\hat{R}^{(-,\ldots, -)}_{0,\ldots, 1^*} e_{\ell}\right\rangle T^{-n}(v_\ell) + \\
+ \sum_{v_\ell \in \tilde{v}_1} \tilde{h}(x, v_1) g(v_\ell, v_1) F(\tilde{v}_1, v_\ell) \left\langle b_{e,\ldots, 2,\ldots, (P+1)^*}(v_1; \tilde{v}_1, \ell) b_{1}^{(-)}(x), \hat{P}^{(-,\ldots, -)}\hat{R}^{(-,\ldots, -)}_{0,\ldots, 1^*} e_{\ell}\right\rangle T^{-n}(v_\ell),
\]

which follows from Lemma 1 and the inductive assumption. According to Lemma 2, we have

\[
\left\langle b_{1,\ldots, (P+1)^*}(\tilde{v}_1, \ell) b_{1}^{(-)}(x), \hat{P}^{(-,\ldots, -)}\hat{R}^{(-,\ldots, -)}_{0,\ldots, 1^*} \hat{R}^{(-,\ldots, -)}_{2,\ldots, \ell}(\tilde{v}_1) e_{\ell}\right\rangle = \\
= \left\langle b_{1,\ldots, (P+1)^*}(\tilde{v}_\ell) b_{1}^{(-)}(x), \hat{P}^{(-,\ldots, -)}\hat{R}^{(-,\ldots, -)}_{0,\ldots, 1^*} \hat{R}^{(-,\ldots, -)}_{2,\ldots, \ell}(\tilde{v}_1) e_{\ell}\right\rangle
\]
and so
\[
\begin{align*}
\mathbf{T}_0^{(-)}(x)\langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v}),(e_{\tilde{k}})\rangle &= \langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v}),(\mathbf{T}_0^{(-)}_{0,1,...,P+1}(x;\tilde{v})e_{\tilde{k}})\rangle - \\
- \tilde{h}(x,v_1)F(\overline{\mathbf{v}},v_1)\langle b^{(+)}_{2^*};(P+1)^*,(\tilde{v}_1)b^{(-)}_{1}(x),\mathbf{T}_0^{(-)}_{1^*,1^*,0^*}(e_{\tilde{k}})T_n^a(v_1) - \\
- \sum_{v_\ell \in \mathbf{v}} \tilde{h}(x,v_\ell)F(\overline{\mathbf{v}},v_\ell)\langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v}_\ell)b^{(-)}_{\ell}(x),
\begin{aligned}
\mathbf{\hat{R}}^{(-)}_{1^*,\ell^* + \mathbf{e}_+} &\mathbf{\hat{R}}^{(-)}_{0^*,\ell^* + \mathbf{e}_+} \mathbf{\hat{R}}^{(+)}_{1^*,\ell^* + \mathbf{e}_+} \mathbf{\hat{R}}^{(+)}_{0^*,\ell^* + \mathbf{e}_+} (x,v_\ell)e_{\tilde{k}}T_n^a(v_\ell) + \\
+ \sum_{v_\ell \in \mathbf{v}} \tilde{h}(x,v_\ell)g(v_\ell,v_1)F(\overline{\mathbf{v}},v_\ell)\langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v}_\ell)b^{(-)}_{\ell}(x),
\begin{aligned}
\mathbf{\hat{R}}^{(-)}_{1^*,\ell^* + \mathbf{e}_+} &\mathbf{\hat{R}}^{(-)}_{0^*,\ell^* + \mathbf{e}_+} \mathbf{\hat{R}}^{(+)}_{1^*,\ell^* + \mathbf{e}_+} \mathbf{\hat{R}}^{(+)}_{0^*,\ell^* + \mathbf{e}_+} (x,v_\ell)e_{\tilde{k}}T_n^a(v_\ell)
\end{aligned}
\end{aligned}
\end{align*}
\]
Therefore, it is enough to show that equality
\[
\begin{align*}
\tilde{h}(x,v_\ell)g(v_\ell,\ell)\mathbf{\hat{R}}^{(-)}_{1^*,\ell^* + \mathbf{e}_+} (x,v_\ell) = \\
= \tilde{h}(x,v_\ell)\mathbf{\hat{R}}^{(-)}_{0^*,\ell^* + \mathbf{e}_+} (x,v_\ell) - \tilde{h}(x,v_1)g(v_\ell,v_1)\mathbf{\hat{R}}^{(+)}_{1^*,\ell^* + \mathbf{e}_+} (x,v_\ell)
\end{align*}
\]
holds for any \(\ell = 2, \ldots, P + 1\). And if we use the definitions of the involved operators, we find that this equality is equivalent to the relation
\[
\tilde{h}(x,v_\ell)g(v_\ell,\ell) + \tilde{h}(x,v_\ell)h(x,v_1) + \tilde{h}(x,v_1)g(v_\ell,v_1) = 0,
\]
which can be easily verified.

\textbf{Lemma 5.} For any \(\tilde{v}, \tilde{w}, \tilde{k}\) and \(\tilde{r}\) the relations
\[
\begin{align*}
T_n^a(x)\langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v})b^{(-)}_{1}(\tilde{w}),e_{\tilde{k}} \otimes f^{-}\rangle &= \\
= F(\overline{\mathbf{v}},x)F(\overline{\mathbf{w}},x - n + 1 + \eta)\langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v})b^{(-)}_{1}(\tilde{w}),e_{\tilde{k}} \otimes f^{-}\rangle T_n^a(x) - \\
- \sum_{v_\ell \in \mathbf{v}} g(v_\ell,\ell)F(\overline{\mathbf{v}},v_\ell)F(\overline{\mathbf{w}},v_\ell - n + 1 + \eta)\langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v}_\ell)b^{(-)}_{1}(\tilde{w}),e_{\tilde{k}} \otimes f^{-}\rangle T_n^a(v_\ell) + \\
+ \sum_{w_s \in \mathbf{w}} \tilde{h}(w_s,x)F(\overline{\mathbf{w}},w_s)\mathbf{T}_0^{(+)}\langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v})b^{(-)}_{1}(\tilde{w}),e_{\tilde{k}} \otimes f^{-}\rangle
\end{align*}
\]
\[
\begin{align*}
\mathbf{T}_0^{(+)}(x)\langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v})b^{(-)}_{1}(\tilde{w}),e_{\tilde{k}} \otimes f^{-}\rangle &= \\
= F(x,\overline{\mathbf{v}})\langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v})b^{(-)}_{1}(\tilde{w}),\mathbf{T}_0^{(+)}_{0,1,...,P+1,1^*};(P+1)^*,(\tilde{v})\overline{\mathbf{w}}e_{\tilde{k}} \otimes f^{-}\rangle - \\
- \sum_{v_\ell \in \mathbf{v}} g(x,v_\ell)F(v_\ell,\overline{\mathbf{v}})\langle b^{(+)}_{1^*};(P+1)^*,(\tilde{v}_\ell)b^{(-)}_{1}(\tilde{w}),e_{\tilde{k}} \otimes f^{-}\rangle - \\
- \sum_{w_s \in \mathbf{w}} \tilde{h}(w_s,x)F(\overline{\mathbf{w}},w_s)\mathbf{T}_0^{(-)}(x;\tilde{v},\overline{\mathbf{w}})e_{\tilde{k}} \otimes f^{-}\rangle T^-_{n}(w_s)
\end{align*}
\]
From Lemmas 3 and 4 we get

$$T_n(x)\langle b^{(+)}_{1,\ldots,p,\ldots} (\bar{v}) b^{(-)}_{1,\ldots,Q} (\bar{w}), e_k \otimes f^{-}\rangle =$$

$$= F(\pi; x) F(x - n + 1 + \eta; \bar{v}) \langle b^{(+)}_{1,\ldots,p,\ldots} (\bar{v}) b^{(-)}_{1,\ldots,Q} (\bar{w}), e_k \otimes f^{-}\rangle T_n(x) -$$

$$- \sum_{w_s \in \bar{w}} g(w_s, x) F(w_s; \bar{w}) F(x - n + 1 + \eta; \bar{v}) \langle b^{(+)}_{1,\ldots,p,\ldots} (\bar{v}) b^{(-)}_{1,\ldots,Q}(\bar{w}), e_k \otimes f^{-}\rangle T_n(x),$$

$$\widehat{R}_{1,\ldots,s,s}^{(-),s}(\bar{w}) e_k \otimes f^{-}\rangle T_n(w_s) +$$

$$+ \sum_{w_s \in \bar{w}} F(w_s; w_s) \widehat{T}_0 (x, \pi) F(w_s; \bar{v}) F(\bar{w}; \pi) T_n(x) -$$

$$- \sum_{w_s \in \bar{w}} g(w_s, x) F(\bar{w}; \bar{v}) F(x; \bar{w}) F(w_s; \bar{v}) F(\pi; x) \langle b^{(+)}_{1,\ldots,p,\ldots} (\bar{v}) b^{(-)}_{1,\ldots,Q}(\bar{w}), e_k \otimes f^{-}\rangle T_n(v)$$

hold.

**PROOF:** In proving this Lemma, we will often use the equation

$$\langle b^{(+)}_{1,\ldots,p,\ldots}(\bar{v}) b^{(-)}_{1,\ldots,Q}(\bar{w}), e_k \otimes f^{-}\rangle =$$

$$= \langle b^{(+)}_{1,\ldots,p,\ldots}(\bar{v}), e_k \rangle \langle b^{(-)}_{1,\ldots,Q}(\bar{w}), f^{-}\rangle = \langle b^{(-)}_{1,\ldots,Q}(\bar{w}), f^{-}\rangle \langle b^{(+)}_{1,\ldots,p,\ldots}(\bar{v}), e_k \rangle,$$

which follows from the relation $b^{(+)}_{1,\ldots,p,\ldots}(x) b^{(-)}_{1,\ldots}(y) = b^{(-)}_{1,\ldots}(y) b^{(+)}_{1,\ldots}(x)$. We calculate action of the operators $T_{\pm n}(x)$ and $T_0^{(\pm)} (x)$ on the element $\langle b^{(+)}_{1,\ldots,p,\ldots}(\bar{v}) b^{(-)}_{1,\ldots,Q}(\bar{w}), e_k \otimes f^{-}\rangle$ in both orders and then we get assertion of Lemma 5 by comparing these expressions.

From Lemmas 3 and 4 we get

$$T_n(x)\langle b^{(+)}_{1,\ldots,p,\ldots} (\bar{v}) b^{(-)}_{1,\ldots,Q} (\bar{w}), e_k \otimes f^{-}\rangle = T_n(x) \langle b^{(+)}_{1,\ldots,p,\ldots} (\bar{v}), e_k \rangle \langle b^{(-)}_{1,\ldots,Q} (\bar{w}), f^{-}\rangle =$$

$$= F(\pi; x) F(\bar{v}; x) - n + 1 + \eta) \langle b^{(+)}_{1,\ldots,p,\ldots} (\bar{v}) b^{(-)}_{1,\ldots,Q}(\bar{w}), e_k \otimes f^{-}\rangle T_n(x) -$$

$$- \sum_{v \in \bar{v}} g(v, x) F(v; \pi) F(\bar{v}; v) F(x - n + 1 + \eta; \bar{v}) \langle b^{(+)}_{1,\ldots,p,\ldots} (\bar{v}) b^{(-)}_{1,\ldots,Q}(\bar{w}), e_k \otimes f^{-}\rangle T_n(v) +$$

$$\widehat{R}_{1,\ldots,\ell}^{(+),\ell}(\bar{v}) e_k \otimes f^{-}\rangle T_n(v) +$$

$$+ \sum_{w_s \in \bar{w}} F(w_s; \bar{w}) \langle b^{(+)}_{1,\ldots,p,\ldots}(\bar{v}) b^{(-)}_{1,\ldots,Q}(\bar{w}), e_k \rangle \langle b^{(-)}_{1,\ldots,Q}(\bar{w}), f^{-}\rangle =$$

$$= \langle b^{(+)}_{1,\ldots,p,\ldots}(\bar{v}) b^{(-)}_{1,\ldots,Q}(\bar{w}), e_k \rangle \langle b^{(-)}_{1,\ldots,Q}(\bar{w}), f^{-}\rangle,$$
In the first term the vectors \( \vec{v} \) and \( \vec{w} \) do not change, in the second the components \( v_\ell \) and \( x \) are interchanged and the third contains expressions in which \( w_\ell \) and \( x \) are interchanged.

On the other hand, we also have

\[
T_n(x) \langle b_{1,\ldots,p,}^{(+)}(\vec{v}) b_{1,\ldots,Q}^{(-)}(\vec{w}), e_k \otimes f^{-r} \rangle = T_n(x) \langle b_{1,\ldots,Q}^{(-)}(\vec{w}), f^{-r} \rangle \langle b_{1,\ldots,p,}^{(+)}(\vec{v}), e_k \rangle =
\]

\[
= F(\vec{v}; x) F(\vec{w}; x - n + 1 + \eta) \langle b_{1,\ldots,p,}^{(+)}(\vec{v}) b_{1,\ldots,Q}^{(-)}(\vec{w}), e_k \otimes f^{-r} \rangle T_n(x) -
\]

\[
- \sum_{v_\ell \in \mathbb{W}} F(\vec{v}_\ell; v_\ell) \left( g(v_\ell, x) F(\vec{w}; x - n + 1 + \eta) \langle b_{1,\ldots,p,}^{(+)}(\vec{v}) b_{1,\ldots,Q}^{(-)}(\vec{w}), e_k \otimes f^{-r} \rangle \right) -
\]

\[
- \sum_{w_\ell \in \mathbb{W}} \tilde{h}(w_\ell, v_\ell) F(\vec{w}_\ell; w_\ell) T_0 \left( b_{1,\ldots,\tilde{\ell},\ldots,p,}^{(+)}(\vec{v}) b_{1,\ldots,\tilde{\ell},\ldots,Q}^{(-)}(w_\ell) b_{1,\ldots,\tilde{\ell},\ldots,Q}^{(-)}(w_\ell) b_{1,\ldots,\tilde{\ell},\ldots,Q}^{(-)}(w_\ell), \tilde{\Pi}_s^{(+,-)} \tilde{R}_s^{(+,-)}(\vec{w}), e_k \otimes f^{-r} \right) T_r(x) +
\]

\[
+ \sum_{w_\ell \in \mathbb{W}} \tilde{h}(w_\ell, x) F(\vec{w}_\ell; w_\ell) T_0 \left( b_{1,\ldots,\tilde{\ell},\ldots,p,}^{(+)}(\vec{v}) b_{1,\ldots,\tilde{\ell},\ldots,Q}^{(-)}(w_\ell), \tilde{\Pi}_s^{(+,-)} \tilde{R}_s^{(+,-)}(\vec{w}), e_k \otimes f^{-r} \right)
\]

In this expression the vectors \( \vec{v} \) and \( \vec{w} \) do not change in the first term, in the second \( x \) is changed by \( v_\ell \) and in the third \( x \) and \( w_\ell \) are interchanged. If we compare these two expressions, we get the first statement of Lemma 5.

We get the second relation when we compare the equalities

\[
\tilde{T}_0^{(+)}(x) \langle b_{1,\ldots,p,}^{(+)}(\vec{v}) b_{1,\ldots,Q}^{(-)}(\vec{w}), e_k \otimes f^{-r} \rangle = \tilde{T}_0^{(+)}(x) \langle b_{1,\ldots,p,}^{(+)}(\vec{v}), e_k \rangle \langle b_{1,\ldots,Q}^{(-)}(\vec{w}), f^{-r} \rangle =
\]

\[
= F(\vec{v}; \vec{v}) \langle b_{1,\ldots,p,}^{(+)}(\vec{v}) b_{1,\ldots,Q}^{(-)}(\vec{w}), \tilde{T}_0^{(+)}(x) \rangle \langle b_{1,\ldots,p,}^{(+)}(\vec{v}), e_k \otimes f^{-r} \rangle -
\]

\[
- \sum_{v_\ell \in \mathbb{W}} F(\vec{v}_\ell; v_\ell) \left( g(v_\ell, x) \tilde{T}_0^{(+)}(x) \langle b_{1,\ldots,p,}^{(+)}(\vec{v}) b_{1,\ldots,Q}^{(-)}(\vec{w}), e_k \otimes f^{-r} \rangle \right) -
\]

\[
- \sum_{w_\ell \in \mathbb{W}} \tilde{h}(w_\ell, v_\ell) F(\vec{w}_\ell; w_\ell) \left( \tilde{T}_0^{(+)}(x) \langle b_{1,\ldots,p,}^{(+)}(\vec{v}) b_{1,\ldots,Q}^{(-)}(\vec{w}), e_k \otimes f^{-r} \rangle \right) T_r(x) -
\]

\[
+ \sum_{w_\ell \in \mathbb{W}} \tilde{h}(w_\ell, x) F(\vec{w}_\ell; w_\ell) \left( \tilde{T}_0^{(+)}(x) \langle b_{1,\ldots,p,}^{(+)}(\vec{v}) b_{1,\ldots,Q}^{(-)}(\vec{w}), e_k \otimes f^{-r} \rangle \right)
\]

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The third equality is obtained by comparing the equalities

\[ T_{-n}^-(x) \langle b_{1^n,\ldots,p^n}^+(\vec{v}) b_{1^n,\ldots,Q}^-(\vec{w}), e_{\vec{k}} \otimes f^{-r} \rangle = T_{-n}^-(x) \langle b_{1^n,\ldots,p^n}^+(\vec{v}), e_{\vec{k}} \rangle \langle b_{1^n,\ldots,Q}^-(\vec{w}), f^{-r} \rangle = \]

\[ = F(x; \vec{w}) F(x + n - 1 - \eta; \vec{v}) \langle b_{1^n,\ldots,p^n}^+(\vec{v}) b_{1^n,\ldots,Q}^-(\vec{w}), e_{\vec{k}} \otimes f^{-r} \rangle T_{-n}^-(w) + \]

\[ + \sum_{vL \in V} \hat{h}(x, vL) F(vL; \vec{v}) T_{-n}^-(x) \langle b_{1^n,\ldots,p^n}^+(\vec{v}) b_{1^n,\ldots,Q}^-(\vec{w}), e_{\vec{k}} \rangle \langle b_{1^n,\ldots,Q}^-(\vec{w}), f^{-r} \rangle \]

\[ T_{-n}^-(x) \langle b_{1^n,\ldots,p^n}^+(\vec{v}) b_{1^n,\ldots,Q}^-(\vec{w}), e_{\vec{k}} \otimes f^{-r} \rangle = T_{-n}^-(x) \langle b_{1^n,\ldots,Q}^-(\vec{w}), e_{\vec{k}} \rangle \langle b_{1^n,\ldots,p^n}^+(\vec{v}), f^{-r} \rangle = \]

\[ = F(x; \vec{w}) F(x + n - 1 - \eta; \vec{v}) \langle b_{1^n,\ldots,Q}^-(\vec{w}) b_{1^n,\ldots,p^n}^+(\vec{v}), e_{\vec{k}} \otimes f^{-r} \rangle T_{-n}^-(w) + \]

\[ + \sum_{vL \in V} \hat{h}(x, vL) F(vL; \vec{v}) F(x; \vec{w}) T_{-n}^-(x) \langle b_{1^n,\ldots,p^n}^+(\vec{v}) b_{1^n,\ldots,Q}^-(\vec{w}), e_{\vec{k}} \rangle \langle b_{1^n,\ldots,Q}^-(\vec{w}), f^{-r} \rangle \]

\[- \sum_{vL \in V} \sum_{vR \in V} g(x, w) \hat{h}(w, vR) F(vR; \vec{v}) F(w; \vec{w}) - \]

\[ \sum_{w \in \vec{w}} \sum_{v \in \vec{v}} \hat{h}(w, v) F(w; \vec{w}) F(v; \vec{v}) - \]

\[ \sum_{w \in \vec{w}} \sum_{v \in \vec{v}} \hat{h}(w, v) F(v; \vec{v}) F(w; \vec{w}) - \]

\[ \sum_{w \in \vec{w}} \sum_{v \in \vec{v}} g(x, w) \hat{h}(w, v) F(v; \vec{v}) F(w; \vec{w}) \]
and the fourth relation is the result of equalities

\[
\tilde{T}^{-1}_0(x)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle = \tilde{T}^{-1}_0(x)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), e^{\gamma}_k \right\rangle b^{(-)}_1, Q(w), f^{-\sigma}\right\rangle = \\
= F(\bar{w}; x)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle - \\
- \sum_{u_s \in \mathcal{M}} g(w_s, x) F(\bar{w}; w_s)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w_i; w_s), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle - \\
- \sum_{u_s \in \mathcal{M}} \sum_{u_t \in \mathcal{M}} \tilde{h}(x, v_t)\tilde{h}(w_s, v_t)F(\bar{w}; w_s)F(\bar{w}_t; v_t) \\
\text{Tr}_0\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle \\
- \sum_{u_t \in \mathcal{M}} \tilde{h}(x, v_t)F(\bar{w}; v_t)F(\bar{w}_t; v_t) - n + 1 + \eta)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle - \\
- \sum_{u_s \in \mathcal{M}} \sum_{u_t \in \mathcal{M}} \tilde{h}(x, v_t) F(\bar{w}_t; v_t)F(\bar{w}; w_s)F(\bar{w}_t; v_t) - \\
+ \sum_{u_t \in \mathcal{M}} \sum_{u_s \in \mathcal{M}} g(w_s, x)\tilde{h}(w_s, v_t)F(\bar{w}_t; v_t)F(\bar{w}; w_s) \\
b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(x; w_s), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle T^n(v_t) + \\
- \sum_{u_s \in \mathcal{M}} \sum_{u_t \in \mathcal{M}} \tilde{h}(x, v_t) F(\bar{w}_t; v_t)F(\bar{w}; w_s)F(\bar{w}_t; v_t) \\
b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(x; w_s), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle T^n(v_t) \\
\tilde{T}^{-1}_0(x)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle = \tilde{T}^{-1}_0(x)\left\langle b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle = \\
= F(\bar{w}; x)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle - \\
- \sum_{u_s \in \mathcal{M}} g(w_s, x) F(\bar{w}; w_s)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w_i; w_s), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle - \\
- \sum_{u_t \in \mathcal{M}} \tilde{h}(x, v_t)F(\bar{w}; v_t)F(\bar{w}_t; v_t) - n + 1 + \eta)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle - \\
- \sum_{u_s \in \mathcal{M}} \sum_{u_t \in \mathcal{M}} \tilde{h}(x, v_t) F(\bar{w}_t; v_t)F(\bar{w}; w_s)F(\bar{w}_t; v_t) - \\
+ \sum_{u_t \in \mathcal{M}} \sum_{u_s \in \mathcal{M}} g(w_s, x)\tilde{h}(w_s, v_t)F(\bar{w}_t; v_t)F(\bar{w}; w_s) \\
b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(x; w_s), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle T^n(v_t) + \\
- \sum_{u_s \in \mathcal{M}} \sum_{u_t \in \mathcal{M}} \tilde{h}(x, v_t) F(\bar{w}_t; v_t)F(\bar{w}; w_s)F(\bar{w}_t; v_t) \\
b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(x; w_s), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle T^n(v_t) \\
\tilde{T}^{-1}_0(x)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle = \tilde{T}^{-1}_0(x)\left\langle b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle = \\
= F(\bar{w}; x)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle - \\
- \sum_{u_s \in \mathcal{M}} g(w_s, x) F(\bar{w}; w_s)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w_i; w_s), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle - \\
- \sum_{u_t \in \mathcal{M}} \tilde{h}(x, v_t)F(\bar{w}; v_t)F(\bar{w}_t; v_t) - n + 1 + \eta)\left\langle b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(w), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle - \\
- \sum_{u_s \in \mathcal{M}} \sum_{u_t \in \mathcal{M}} \tilde{h}(x, v_t) F(\bar{w}_t; v_t)F(\bar{w}; w_s)F(\bar{w}_t; v_t) - \\
+ \sum_{u_t \in \mathcal{M}} \sum_{u_s \in \mathcal{M}} g(w_s, x)\tilde{h}(w_s, v_t)F(\bar{w}_t; v_t)F(\bar{w}; w_s) \\
b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(x; w_s), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle T^n(v_t) + \\
- \sum_{u_s \in \mathcal{M}} \sum_{u_t \in \mathcal{M}} \tilde{h}(x, v_t) F(\bar{w}_t; v_t)F(\bar{w}; w_s)F(\bar{w}_t; v_t) \\
b^{(+)\ldots, p^{(*)}}_1(v), b^{(-)}_1, Q(x; w_s), e^{\gamma}_k \otimes f^{-\sigma}\right\rangle T^n(v_t)