On the Optimality of Ali-Niesen Decentralized Coded Caching Scheme With and Without Error Correction

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Abstract—The decentralized coded caching was introduced in [1]—the coded caching scheme with minimum number of transmissions for a given cache placement scheme. The main contributions of this paper are as follows.

• Optimality of the delivery scheme in [2] is proved for decentralized placement for the number of files not less than the number of users using results from index coding (Section III).
• Optimal error correcting delivery scheme is found for coded caching problems with decentralized placement [2] (Section IV).
• Closed form expression for worst case rate is found for decentralized delivery scheme in the presence of finite number of transmission errors (Section V).

In this paper \(\mathbb{F}_q\) denotes the finite field with \(q\) elements, where \(q\) is a power of a prime, and \(\mathbb{F}_q^n\) denotes the set of all nonzero elements of \(\mathbb{F}_q\). The notation \([K]\) is used for the set \(\{1, 2, ..., K\}\) for any integer \(K\). For a \(K \times N\) matrix \(L\), \(L_i\) denotes its \(i\)th row. For a set \(S \subseteq [K]\), \(L_S\) denotes the \(|S| \times N\) matrix obtained from \(L\) by deleting the rows of \(L\) which are not indexed by the elements of \(S\). We denote \(e_i = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{F}_q^n\) as the unit vector having a one at the \(i\)th position and zeros elsewhere.

A linear \([n, k, d]_q\) code \(C\) over \(\mathbb{F}_q\) is a \(k\)-dimensional subspace of \(\mathbb{F}_q^n\) with minimum Hamming distance \(d\). The vectors in \(C\) are called codewords. A matrix \(G\) of size \(k \times n\) whose rows are linearly independent codewords of \(C\) is called a generator matrix of \(C\). A linear \([n, k, d]_q\) code \(C\) can thus be represented using its generator matrix \(G\) as, \(C = \{yG : y \in \mathbb{F}_q^k\}\). Let
\( N_q[k, d] \) denote the length of the shortest code over \( \mathbb{F}_q \) which has dimension \( k \) and minimum distance \( d \).

\[ \text{II. Preliminaries and Background} \]

In this section we review the basic results from error correcting index coding \([11]\) which are later used in this paper to show the optimality of the decentralized scheme in \([2]\) and to obtain an optimal error correction scheme for the same. We also revisit the decentralized scheme \([2]\) in brief and error correcting coded caching terminologies from \([8]\).

A. Index Coding Problem

The index coding problem with side information was introduced in \([12]\). A single source has \( n \) messages \( x_1, x_2, \ldots, x_n \) where \( x_i \in \mathbb{F}_q \), \( \forall i \in [n] \). There are \( K \) receivers, \( R_1, R_2, \ldots, R_K \). Each receiver possesses a subset of messages as side information. Let \( \mathcal{X}_i \) denote the set of indices of the messages belonging to the side information of receiver \( R_i \). The map \( f : [K] \rightarrow [n] \) assigns receivers to indices of messages demanded by them. Receiver \( R_i \) demands the messages \( x_{f(i)} \), \( f(i) \notin \mathcal{X}_i \) \([11]\). The source knows the side information available to each receiver and has to satisfy the demand of each receiver in minimum number of transmissions. An instance of index coding problem can be completely characterized by a side information hypergraph \([13]\). Given an instance of the index coding problem, finding the best scalar linear index code is equivalent to finding the min-rank of the side information hypergraph \([11]\), which is known to be an NP-hard problem in general \([14]–[16]\).

An index coding problem with \( K \) receivers and \( n \) messages can be represented by a hypergraph \( \mathcal{H}(V, E) \), where \( V = [n] \) is the set of vertices and \( E \) is the set of hyperedges \([13]\). Vertex \( i \) represents the message \( x_i \) and each hyperedge represents a receiver. In \([11]\), the min-rank of a hypergraph \( \mathcal{H} \) corresponding to index coding problem \( \mathcal{I} \) over \( \mathbb{F}_q \) is defined as,

\[
\kappa(\mathcal{I}) \triangleq \min\{\text{rank}_{\mathbb{F}_q}(\{v_i + e_{f(i)}\})_{i \in [K]} : v_i \in \mathbb{F}_q^n, v_i \notin \mathcal{X}_i\},
\]

where \( v_i \notin \mathcal{X}_i \) denotes that \( v_i \) is the subset of the support of \( \mathcal{X}_i \); the support of a vector \( u \in \mathbb{F}_q^n \) is defined to be the set \( \{i \in [n] : u_i \neq 0\} \). This min-rank defined above is the smallest length of scalar linear code for the problem. A linear index code of length \( N \) can be expressed as \( X_L \), where \( L \) is an \( n \times N \) matrix and \( X = [x_1 \ x_2 \ldots \ x_n] \). The matrix \( L \) is said to be the matrix corresponding to the index code.

For an undirected graph \( \mathcal{G} = (V, E) \), a subset of vertices \( S \subseteq V \) is called an independent set if \( \forall u, v \in S, \{u, v\} \notin E \). The size of a largest independent set in the graph \( \mathcal{G} \) is called the independence number of \( \mathcal{G} \). Dau et al. in \([11]\) extended the notion of independence number to the case of directed hypergraph corresponding to an index coding problem. For each receiver \( R_i \), define the sets

\[
\mathcal{Y}_i \triangleq [n] \setminus \{f(i)\} \cup \mathcal{X}_i.
\]

and

\[
\mathcal{J}(\mathcal{I}) \triangleq \bigcup_{i \in [K]} \{f(i)\} \cup \mathcal{Y}_i : \mathcal{Y}_i \subseteq \mathcal{Y}_i \}.
\]

A subset \( H \) of \([n]\) is called a generalized independent set in \( \mathcal{H} \) if every nonempty subset of \( H \) belongs to \( \mathcal{J}(\mathcal{I}) \). The size of the largest independent set in \( \mathcal{H} \) is called the generalized independence number and is denoted by \( \alpha(\mathcal{I}) \). It is proved in \([8]\) that for any index coding problem,

\[
\alpha(\mathcal{I}) \leq \kappa(\mathcal{I}).
\]

The quantities \( \alpha(\mathcal{I}) \) and \( \kappa(\mathcal{I}) \) decide the bounds on the optimal length of error correcting index codes. The error correcting index coding problem with side information was defined in \([11]\). An index code is said to correct \( \delta \) errors if after receiving at most \( \delta \) transmissions in error, each receiver is able to decode its demand. A \( \delta \)-error correcting index code is represented as \( (\delta, \mathcal{I}) \)-ECIC. An optimal linear \( (\delta, \mathcal{I}) \)-ECIC over \( \mathbb{F}_q \) is a \( (\delta, \mathcal{I}) \)-ECIC over \( \mathbb{F}_q \) of the smallest possible length \( N_q[\mathcal{I}, \delta] \). Lower and upper bounds on \( N_q[\mathcal{I}, \delta] \) were established in \([11]\). The lower bound is known as the \( \alpha \)-bound and the upper bound is known as the \( \kappa \)-bound. The length of an optimal linear \( (\delta, \mathcal{I}) \)-ECIC over \( \mathbb{F}_q \) satisfies

\[
N_q[\alpha(\mathcal{I}), 2\delta + 1] \leq N_q[\mathcal{I}, \delta] \leq N_q[\kappa(\mathcal{I}), 2\delta + 1].
\]

The \( \kappa \)-bound is achieved by concatenating an optimal linear classical error correcting code and an optimal linear index code. Thus for any index coding problem, if \( \alpha(\mathcal{I}) \) is same as \( \kappa(\mathcal{I}) \), then concatenation scheme would give optimal error correcting index codes \([17]–[20]\).

B. Error Correcting Coded Caching Scheme

Error correcting coded caching scheme was proposed in \([8]\). The server is connected to \( K \) users through a shared link which is error prone. The server has access to \( N \) files \( X_1, X_2, \ldots, X_N \), each of size \( F \) bits. Every user has an isolated cache with memory \( MF \) bits, where \( M \in [0, N] \). A prefetching scheme is denoted by \( \mathcal{M} \). During the delivery phase, only the server has access to the database. Every user demands one of the \( N \) files. The demand vector is denoted by \( d = (d_1, \ldots, d_K) \), where \( d_i \) is the index of the file demanded by user \( i \). The number of distinct files requested in \( d \) is denoted by \( N_c(d) \). During the delivery phase, the server informed of the demand \( d \), transmits a function of \( X_1, \ldots, X_N \), over a shared link. Using the cache contents and the transmitted data, each user \( i \) needs to reconstruct the requested file \( X_{d_i} \), even if \( \delta \) transmissions are in error.

For the \( \delta \)-error correcting coded caching problem, a communication rate \( R(\delta) \) is achievable for demand \( d \) if and only if there exists a transmission of \( R(\delta)F \) bits such that every user \( i \) is able to recover its desired file \( X_{d_i} \), even after at most \( \delta \) transmissions in error. Rate \( R^*(d, \mathcal{M}, \delta) \) is the minimum achievable rate for a given \( d, \mathcal{M} \) and \( \delta \). The average rate \( R^*(\mathcal{M}, \delta) \) is defined as the expected minimum average rate given \( \mathcal{M} \) and \( \delta \) under uniformly random demand. Thus \( R^*(\mathcal{M}, \delta) = \mathbb{E}_d[R^*(d, \mathcal{M}, \delta)] \).

The average rate depends on the prefetching scheme \( \mathcal{M} \). The minimum average rate \( R^*(\delta) = \min_{\mathcal{M}} R^*(\mathcal{M}, \delta) \) is the minimum rate of the delivery scheme over all possible
The rate-memory trade-off for average rate is finding the minimum average rate \( R^*(\delta) \) for different memory constraints \( M \). Another quantity of interest is the peak rate, denoted by \( R^{\text{w}}(M, \delta) \), which is defined as \( R^{\text{w}}(M, \delta) = \max \delta R_\delta(d, M, \delta) \). The minimum peak rate is denoted as \( R^{\text{w}}(\delta) = \min_M R^{\text{w}}(M, \delta) \).

C. Ali-Niesen Decentralized Scheme

In this subsection, we briefly present the decentralized coded caching scheme in \([2]\). We denote the decentralized prefetching scheme as \( M_D \). During the placement phase, each user independently caches a subset of \( \frac{MN}{K} \) bits of each file, chosen uniformly at random. Hence, each bit of a file is cached by a specific user with a probability \( M/N \). The actions of the placement procedure effectively partition each file \( X_i \) into \( 2^K \) subfiles of the form \( X_{i,S} \), where \( S \subseteq [K] \), denotes the bits of \( X_i \) that are stored exclusively in the cache memories of users in \( S \). Moreover, for large file size \( F \), by the law of large numbers \( |X_{i,S}| \approx (MN)^{|S|}(1 - M/K)^{|S|}F \). Two delivery procedures are proposed in \([2]\), of which the one which is used for the \( N \geq K \) regime is as follows. For \( S \subseteq [K] \) and \( |S| = s \), the server transmits \( \bigoplus_{k \in S} V_k, S \setminus \{k\} \) for \( s = K, K - 1, \ldots, 1 \). Here \( V_{k,S} \) denotes the bits of file \( X_{db} \) requested by user \( k \) cached exclusively at users in \( S \). This delivery scheme achieves the rate

\[
R(M_D, 0) = \left(1 - \frac{MN}{K}\right) \frac{N}{M} \left(1 - \frac{M}{K}\right)^{K},
\]

for \( N \geq K \) or \( M \geq 1 \) regime.

D. Equivalent Index Coding Problems of a Coded Caching Problem

For a fixed prefetching \( M \) and for a fixed demand \( d \), the delivery phase of a coded caching problem is an index caching problem \([1]\). In fact, for fixed prefetching, a coded caching scheme consists of \( N^K \) parallel index coding problems one for each of the \( N^K \) possible user demands. Thus finding the minimum achievable rate for a given demand \( d \) is equivalent to finding the min-rank of the equivalent index coding problem induced by the demand \( d \).

Consider an index coding problem with \( \alpha(\mathcal{I}) = \kappa(\mathcal{I}) \). For this problem, the optimal construction of error correcting index code is by concatenation of a smallest length index code with an optimal error correcting code. For problems with \( \alpha(\mathcal{I}) \neq \kappa(\mathcal{I}) \), the optimal construction of error correcting index codes is unknown. Thus concatenation scheme for the construction of optimal error correcting index code may not be optimal in general, which is summarized as follows.

- For index coding problems with \( \alpha(\mathcal{I}) \neq \kappa(\mathcal{I}) \), the concatenation scheme is not proven to be optimal even if the minimum length index code is known.
- For index coding problems with \( \alpha(\mathcal{I}) = \kappa(\mathcal{I}) \), if an optimal index code is not known, then concatenating a non-optimal index code with an optimal error correcting code is not optimal.
- For index coding problems with \( \alpha(\mathcal{I}) = \kappa(\mathcal{I}) \) and if an optimal index code is known, then concatenating it with an optimal error correcting code is optimal.

Hence, if for some problems, we have an optimal index code and if for such problems, \( \alpha(\mathcal{I}) = \kappa(\mathcal{I}) \), then the concatenation scheme is optimal. In our work, we consider all the index coding problems corresponding to the worst case demands of Ali-Niesen decentralized scheme for \( N \geq K \). For all these index coding problems, we find closed form expression for \( \alpha(\mathcal{I}) \). The number of bits transmitted in Ali-Niesen delivery scheme turns out to be same as the number of bits corresponding to \( \alpha(\mathcal{I}) \). Since \( \alpha(\mathcal{I}) \leq \kappa(\mathcal{I}) \) in general \([3]\), we get \( \alpha(\mathcal{I}) = \kappa(\mathcal{I}) \) for all the corresponding index coding problems. Hence, for all these problems, concatenation of Ali-Niesen delivery scheme with an optimal error correcting code gives an optimal error correcting delivery scheme.

The length of an optimal linear \((\delta, \mathcal{I})\)-ECIC over \( \mathbb{F}_q \) satisfies \([2]\). Whatever be the combinations of index codes and error correcting codes being tried, the length of ECIC should be greater than or equal to the \( \alpha \)-bound. The constructions of error correcting delivery schemes used in this paper are in such a way that their lengths meet the \( \alpha \)-bound with equality and thus are optimal.

It is shown in the Example 4.8 of \([11]\) that the inequality can be strict in general. In particular, it follows that mere application of an optimal length error-correcting code on top of an optimal index code may fail to provide us with an optimal linear ECIC. This example is reproduced here for convenience.

**Example 2.1:** Let field size \( q = 2 \) and number of messages, \( n = 5 \). Let \( \delta = 2 \) errors need to be corrected and let the demand of \( i \)th receiver be \( x_i \), for \( 1 \leq i \leq 5 \). Let the side information sets be given as \( x_1 = \{2, 5\} \), \( x_2 = \{1, 3\} \), \( x_3 = \{2, 4\} \), \( x_4 = \{3, 5\} \) and \( x_5 = \{1, 4\} \). The set \( J(\mathcal{I}) \) for this problem is given by

\[
J(\mathcal{I}) = \{\{1\}, \{1, 3\}, \{1, 4\}, \{1, 3, 4\}, \{2\}, \{2, 4\}, \{2, 5\}, \{2, 4, 5\}, \{3\}, \{1, 3\}, \{3, 5\}, \{1, 3, 5\}, \{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{5\}, \{2, 5\}, \{3, 5\}, \{2, 3, 5\}\}.
\]

For this problem it can be calculated that \( \alpha(\mathcal{I}) = 2 \). Also, for this problem, min-rank, \( \kappa(\mathcal{I}) = 3 \). From code tables in \([2]\), we have \( N_2(2, 5) = 8 \) and \( N_2(3, 5) = 10 \). Hence, \( 8 \leq N_2(\mathcal{I}, \delta) \leq 10 \). Using a computer search, the authors of \([10]\) have found that the optimal length \( N_2(\mathcal{I}, 2) = 9 \). Here the optimal length of the ECIC lies strictly between the \( \alpha \)-bound and the \( \kappa \)-bound.

Consider the decentralized prefetching scheme \( M_D \). The index coding problem induced by the demand \( d \) for the decentralized prefetching is denoted by \( \mathcal{I}(M_D, d) \). The corresponding generalized independence number and min-rank are represented as \( \alpha(M_D, d) \) and \( \kappa(M_D, d) \) respectively. Also, the demand vector \( d_{\text{worst}} \) corresponds to the case when all the demanded files are distinct.
III. Optimality of the Ali-Niesen Decentralized Scheme

In this section we prove that the decentralized scheme in [2] is optimal for \( N \geq K \) using results from index coding. Moreover, the results presented in this section are used to construct optimal error correcting delivery scheme for coded caching problem with decentralized prefetching in Section IV.

The theorem below gives a lower bound for \( \alpha(M_D, d_{\text{worst}}) \) which is used to prove the optimality of the Ali-Niesen decentralized scheme.

**Theorem 3.1:** For the index coding problem \( \mathcal{I}(M_D, d_{\text{worst}}) \) corresponding to a coded caching problem with Ali-Niesen decentralized prefetching for \( N \geq K \) and the worst case demand \( d_{\text{worst}} \),

\[
\alpha(M_D, d_{\text{worst}}) = \kappa(M_D, d_{\text{worst}}) = (1 - M/N)\frac{N}{M}(1 - (1 - M/N)^K)F.
\]

Thus, the decentralized scheme in [2] is optimal for \( N \geq K \).

**Proof:** Worst case demand scenario is considered here. Without loss of generality we can assume that the demand vector is \( d = (1, 2, \ldots, K) \). The corresponding index coding problem \( \mathcal{I}(M_D, d_{\text{worst}}) \) can be viewed to be consisting of \( NF \) messages each of one bit and \( KF \) receivers demanding \( KF \) bits corresponding to the first \( K \) files. We construct a set \( B(I) \), whose elements are messages of the index coding problem such that the set of indices of the messages in \( B(I) \) forms a generalized independent set. The set \( B(I) \) is constructed as

\[
B(I) = \bigcup_{i \in [K]} \{ X_{i,S} : 1, 2, \ldots, i \notin S \}.
\]

Let \( |B(I)| \) be the set of indices of the messages in \( B(I) \). The claim is that \( |H(I)| \) is a generalized independent set. Each message in \( B(I) \) is demanded by one receiver. Hence all the subsets of \( |B(I)| \) of size one are present in \( J(I) \). Consider any set \( C = \{ X_{i,S_1}, X_{i_2,S_2}, \ldots, X_{i_k,S_k} \} \subseteq |B(I)| \) where \( i_1 \leq i_2 \leq \ldots \leq i_k \). Consider the message \( X_{i,S} \). The receiver demanding this message does not have any other message in \( C \) as side information. Thus indices of messages in \( C \) lie in \( J(I) \). Thus any subset of \( H(I) \) lies in \( J(I) \). Since \( H(I) \) is a generalized independent set, we have,

\[
\alpha(M_D, d_{\text{worst}}) \geq |H(I)|. \quad \text{Note that } |H(I)| = |B(I)|.
\]

There are \( \binom{K}{1}^n \) subfiles of the form \( X_{n,S} \) such that \( |S| = i \) in \( B(I) \). Hence, the number of bits of file \( X_n \) of the form \( X_{n,S} \) such that \( |S| = 1 \) in \( B(I) \) is \( \binom{K}{1}^n (M/N)^i (1 - M/N)^{K-i}F \). Thus,

\[
|B(I)| = \sum_{n=1}^{K} \sum_{i=0}^{K-n} \binom{K-n}{i} (M/N)^i (1 - M/N)^{K-i}N
= \sum_{n=1}^{K} \sum_{i=0}^{K-n} \binom{K-n}{i} (M/N)^i (1 - M/N)^{K-n-i}
(1 - M/N)^n F
= \sum_{n=1}^{K} (1 - M/N)^n F.
\]

The last equality follows from the binomial expansion \( (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \). The expression \( \sum_{n=1}^{K} (1 - M/N)^n \) is sum of a geometrical progression. Thus,

\[
|B(I)| = \frac{(1 - M/N)(1 - (1 - M/N)^K)}{(1 - (1 - M/N))} F
= \frac{(1 - M/N)N}{M(1 - (1 - M/N)^K)} F.
\]

Thus, \( \alpha(M_D, d_{\text{worst}}) \geq (1 - M/N)\frac{N}{M}(1 - (1 - M/N)^K)F \). Also, from the achievable scheme in [2], it follows that, \( \kappa(M_D, d_{\text{worst}}) \leq (1 - M/N)\frac{N}{M}(1 - (1 - M/N)^K)F \). The rate achieved by the scheme in [2] thus meets the lower bound, which proves its optimality. Hence the statement of the theorem follows.

Two examples are given below to illustrate the construction of generalized independent set for the index coding problems corresponding to coded caching problem with decentralized placement.

**Example 3.2:** Consider a coded caching problem with \( N = 2 \) files and \( K = 2 \) users each with a cache of size \( M \in [0, 2] \). In the placement phase, each user caches a subset of \( MF/2 \) bits of each file independently at random. Thus both the files \( X_1 \) and \( X_2 \) are effectively partitioned into four subfiles:

\[
X_1 = (X_{1,\phi}, X_{1,(1)}, X_{1,(2)}, X_{1,(1,2)}) \quad \text{and} \quad X_2 = (X_{2,\phi}, X_{2,(1)}, X_{2,(2)}, X_{2,(1,2)}).
\]

For large enough file size \( F \), we have with high probability for \( i \in [2] \),

\[
|X_{i,\phi}| = (M/2)^2 F;
|X_{i,(1)}| = |X_{i,(2)}| = (M/2)(1 - M/2) F \quad \text{and} \quad |X_{i,(1,2)}| = (M/2)^2 F.
\]

Let the demand vector be \( d = (1, 2) \). Let the corresponding index coding problem be \( \mathcal{I}(M_D, d) \). A generalized independent set can be constructed for this problem following the procedure in the proof of Theorem 3.1 as

\[
B(I) = \{ X_{1,\phi}, X_{1,(2)}, X_{2,\phi} \}.
\]

From this, \( |B(I)| = 2(1 - M/2)^2 F + (M/2)(1 - M/2) F \). Thus \( \alpha(M_D, d) \geq 2(1 - M/2)^2 F + (M/2)(1 - M/2) F \). From [2], we have the number of bits transmitted is exactly \( 2(1 - M/2)^2 F + (M/2)(1 - M/2) F \). Hence \( \kappa(M_D, d) \leq 2(1 - M/2)^2 F + (M/2)(1 - M/2) F \). Hence \( \alpha(M_D, d) = \kappa(M_D, d) \). The set \( B(I) \) can be constructed for the two possible combinations of distinct demands and are shown in Table I. For both the cases, the cardinality of \( B(I) \) turns out to be same, which is equal to \( \kappa(M_D, d) \).

| Demand \( d \) | \( B(I) \) |
|------------|-------------|
| (1, 2)     | \{ \( X_{1,\phi}, X_{1,(2)}, X_{2,\phi} \) \} |
| (2, 1)     | \{ \( X_{2,\phi}, X_{2,(2)}, X_{1,\phi} \) \} |

**Table I**

Generalized independent sets of \( \mathcal{I}(M_D, d_{\text{worst}}) \) for different demands for Example [2].
Example 3.3: Consider a coded caching problem with $N = 4$ files and $K = 3$ users. Let $M = 1$. In the placement phase, each user caches a subset of $MF/N = F/4$ bits of each file independently at random. Thus each file is effectively partitioned as follows:

$$X_i = (X_{i,0}, X_{i,1}, X_{i,2}, X_{i,3})$$

for $i \in [4]$. For sufficiently large $F$, the number of bits corresponding to each of these subfiles is given as:

$$|X_{i,0}| = \frac{27}{64} F,$$

$$|X_{i,1}| = |X_{i,2}| = |X_{i,3}| = \frac{9}{64} F,$$

$$|X_{i,2}| = |X_{i,3}| = |X_{i,4}| = \frac{3}{64} F$$

and

$$|X_{i,0}| = \frac{1}{64} F.$$

Let the demand vector be $d = (1, 2, 3)$. A generalized independent set of the corresponding index coding problem can be constructed for following the procedure in the proof of Theorem 3.1 as

$$B(T) = \{X_{1,0}, X_{1,2}, X_{1,3}, X_{2,0}, X_{2,3}, X_{3,0}\}.$$

Hence, $|B(T)| = \frac{111}{64} F$. Thus $\alpha(M_D, d) \geq \frac{111}{64} F$. From [2], we have the number of bits transmitted is $\frac{111}{64} F$. Hence $\kappa(M_D, d) \leq \frac{111}{64} F$. Hence $\alpha(M_D, d) = \kappa(M_D, d)$. The set $B(T)$ can be constructed for all the possible combinations of distinct demands and are shown in Table II. For all the cases, the cardinality of $B(T)$ turns out to be same, which is equal to $\kappa(M_D, d)$.

IV. OPTIMAL ERROR CORRECTING DELIVERY SCHEME FOR ALI-NIESEN DECENTRALIZED PREFETCHING

For the worst case demand, we have proved in Theorem 3.1 that $\alpha(M_D, d_{\text{worst}}) = \kappa(M_D, d_{\text{worst}})$. Hence for this case, the optimal linear error correcting delivery scheme can be constructed by concatenating the delivery scheme in [2] with an optimal error correcting code which corrects the required number of errors. Based on this we give an expression for the worst case rate for decentralized prefetching in the theorem below.

Theorem 4.1: For a coded caching problem with Ali-Niesen decentralized prefetching for $N \geq K$,

$$R_{\text{worst}}^*(M_D, \delta) = \frac{N_q[\kappa(M_D, d_{\text{worst}}), 2\delta + 1]}{F},$$

where $\kappa(M_D, d_{\text{worst}}) = (1 - M/N)(\frac{N}{M}(1 - (1 - M/N)K))F$.

Proof: From Theorem 3.1 we have for any index coding problem corresponding to coded caching with decentralized placement for $N \geq K$, $\alpha(M_D, d_{\text{worst}}) = \kappa(M_D, d_{\text{worst}})$. Hence by [2], the $\alpha$ and $\kappa$ bounds become equal for all these index coding problems. Thus, the optimal number of transmissions required for $\delta$ error corrections in those index coding problems is $N_q[\kappa(M_D, d_{\text{worst}}), 2\delta + 1]$. Hence the statement of theorem follows.

\[ G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \]

Concatenating this $[6, 3, 3]_2$ code with the decentralized trans-
missions give rise to six transmissions given by:

\[
T_1 \equiv X_{1,(2)} \oplus X_{2,(1)}, \\
T_2 \equiv X_{1,\phi}, \\
T_3 \equiv X_{2,\phi}, \\
T_4 \equiv X_{1,(2)} \oplus X_{2,(1)} \oplus X_{1,\phi}, \\
T_5 \equiv X_{1,(2)} \oplus X_{2,(1)} \oplus X_{2,\phi} \text{ and } \\
T_6 \equiv X_{1,\phi} \oplus X_{2,\phi}.
\]

Decoding is done by syndrome decoding for error correcting index codes proposed in [11]. The number of bits involved in each transmission is \( F/4 \). Hence, the rate of transmission is \( 3/2 \). For zero error correcting scenario the rate corresponding to \( M = 1 \) was \( 3/4 \).

**Example 4.3:** Consider the decentralized coded caching problem with \( N = 4 \) and \( K = 3 \) considered in Example 3.3. Consider the case when \( M = 1 \). We use the decentralized delivery scheme [2] for this example. For simplicity we consider that for the corresponding index coding problem, each index coding message is of \( F/64 \) bits. The transmissions given in [2] are:

\[
X_{1,(2,3)} \oplus X_{2,(1,3)} \oplus X_{3,(1,2)}, \\
X_{1,(2)} \oplus X_{2,(1)}, \\
X_{2,3) \oplus X_{3,(1)}, \\
X_{1,\phi}, X_{2,\phi} \text{ and } X_{3,\phi}.
\]

The total number of bits transmitted is \( \frac{11F}{64} \). If each index coding message is considered consisting of \( F/64 \) bits, the min-rank \( \kappa(MD_{\text{dcorr}}) = 111 \). If \( \delta = 1 \) error need to be corrected, the optimal scheme is to concatenate these transmissions with a classical error correcting code of optimal length. From [21], we have \( N_2[111,3] = 118 \). Hence, concatenating \([118,111,3]_2\) code with the decentralized transmissions give rise to optimal error correcting delivery scheme. Decoding is done by syndrome decoding for error correcting index codes proposed in [11].

**V. CONCLUSION**

We considered the decentralized coded caching problem in [2] and proved that for \( N \geq K \), the delivery scheme in [2] is optimal using the results from index coding. Also, since the \( \alpha \) and \( \kappa \) bounds meet for the corresponding index coding problems, the concatenation of decentralized delivery scheme with an optimal classical error correcting code which corrects the required number of errors is optimal. A good direction of future work would be to find the optimality of other coded caching schemes by finding the generalized independence numbers for the corresponding index coding problems. For the case of non-optimal schemes, the gap from optimality can be measured if the generalized independence number is found out.

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