Inverse problem for a subdiffusion equation with the Caputo derivative
O.T. Muhiddinova

Abstract. The article investigates an inverse problem of determining the right-hand side of a subdiffusion equation with Caputo fractional derivative whose elliptic part has the most general form and is defined on an $N$-dimensional torus $\mathbb{T}^N$. The Fourier method is used to prove theorems on the existence and uniqueness of the classical solution of the initial-boundary value problem and on the unique reconstruction of the unknown right-hand side of the equation. Requirements for the initial function and for the additional condition are established under which the classical Fourier method can be applied to the inverse problem under consideration.

Keywords: Subdiffusion equation, the Caputo derivatives, Fourier method, inverse and initial-boundary value problem.

MSC (2010): 35K35; 35C10.

1 Introduction

It is well known that the interest in the study of inverse problems for equations of mathematical physics is due to the importance of their applications in various branches of mechanics, seismology, medical tomography, and geophysics. It should be noted that inverse problems are well studied for differential equations of integer order. A lot of research is also devoted to inverse problems of determining the right-hand side of subdiffusion equations (see, e.g., [1] - [14] and the bibliography therein). The paper [14] studied the inverse problem of determining the right-hand side of the subdiffusion equation with Caputo fractional derivatives whose elliptic part is an arbitrary elliptic operator with discrete spectrum and proved the existence and uniqueness of a generalized solution by the Fourier method. In the paper [14], conditions for $\varphi(x)$ and $\Psi(x)$ are assumed to be defined in the operator’s domain. The other papers, including [9] - [12], take the Caputo derivative as the fractional derivative. In the paper [15], authors studied the inverse problem of determining the right-hand side of the subdiffusion equation with Riemann–Liouville fractional derivatives whose elliptic part is an elliptic operator of an arbitrary order defined in bounded domain with sufficiently smooth boundary.

In this article we investigate an inverse problem of determining the right-hand side of a subdiffusion equation with Caputo fractional derivative whose elliptic
part has the most general form and is defined on an $N$-dimensional torus $\mathbb{T}^N$. The Fourier method is used to prove theorems on the existence and uniqueness of the classical solution of the initial-boundary value problem and on the unique reconstruction of the unknown right-hand side of the equation. Requirements for the initial function and for the additional condition are established under which the classical Fourier method can be applied to the inverse problem under consideration.

Let us move on to precise descriptions of the objects of study.

The fractional derivative in the sense of Caputo of order $0 < \rho < 1$ of the function $f(t)$ defined on $[0, \infty)$ has the form (see, for example, [16], p. 14)

$$D^\rho_t f(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{d\tau}{(t-\tau)^\rho} f(\tau), \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\sigma)$ is Euler’s gamma function. If in this definition we interchange differentiation and fractional integration, then we get the definition of the Riemann-Liouville derivative:

$$\partial^\rho_t f(t) = \frac{1}{\Gamma(1-\rho)} \frac{dt}{dt} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^\rho}, \quad t > 0.$$

Note that if $\rho = 1$, then fractional derivatives coincides with the ordinary classical derivative of the first order: $\partial_t f(t) = D_t^1 f(t) = \frac{d}{dt} f(t)$.

Let $\mathbb{T}^N$ be $N$-dimensional torus: $\mathbb{T}^N = (-\pi, \pi]^N$, $N \geq 1$ and $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ be a homogeneous elliptic symmetric positive differential operator with constant coefficients.

**Problem:** Find the functions $\{u(x, t), f(x)\}$ from the class: A function $f(x) \in C(T^N)$ and a function $u(x, t) \in C(T^N \times [0, T])$ with the properties $D^\rho_t u(x, t), A(x, D)u(x, t) \in C(T^N \times (0, T])$ that satisfies the following differential equation $(0 < \rho \leq 1)$

$$D^\rho_t u(x, t) + A(D)u(x, t) = f(x), \quad x \in \mathbb{T}^N, \quad t > 0; \quad (1.1)$$

and the initial

$$u(x, 0) = \varphi(x), \quad x \in \mathbb{T}^N; \quad (1.2)$$

and additional conditions

$$u(x, T) = \Psi(x), \quad x \in \mathbb{T}^N. \quad (1.3)$$

Instead of the boundary conditions we will require that the desired functions $u(x, t)$ and $f(x)$ are $2\pi$-periodic in each variable $x_j$. We will also assume that the given functions $\varphi(x)$ and $\Psi(x)$ are $2\pi$-periodic in each variable $x_j$ as well.
2 Main result

To formulate the main result of this paper we recall some definitions and statements. By $A$ we denote the operator $A(D)$ defined on $2\pi$-periodic functions in $C^m(\mathbb{R}^N)$. The closure $\hat{A}$ of such an operator in $L_2(\mathbb{T}^N)$ is self-adjoint. The operator $\hat{A}$ has a complete in $L_2(T^N)$-orthonormal system of eigenfunctions $\{(2\pi)^{-N/2}e^{inx}\}$ corresponding to the eigenvalues $A(n)$, $n \in (\mathbb{Z}^N)$. Therefore, by virtue of the von Neumann theorem, for each $\tau \geq 0$ the operator $\hat{A}^\tau$ acts by the rule

$$\hat{A}^\tau g(x) = \sum_{n \in \mathbb{Z}^N} A^\tau(n) g_n e^{inx},$$

where the $g_n$ are the Fourier coefficients of the function $g \in L_2(T^N)$ in the trigonometric system $\{(2\pi)^{-N/2}e^{inx}\}$. The domain of this operator is determined from the condition $\hat{A}^\tau g(x) \in L_2(T^N)$ and has the form

$$D(\hat{A}^\tau) = \{g \in L_2(T^N) : \sum_{n \in \mathbb{Z}^N} A^{2\tau}(n)|g_n|^2 < \infty\}. \tag{2.1}$$

To define the domain of the operator $\hat{A}^\tau$ in terms of Sobolev spaces, recall the definition of these spaces (see, e.g., [17]): a function $g \in L_2(T^N)$ is said to belong to the Sobolev space $L^a_2(T^N)$ with a real number $a > 0$, if the norm

$$||g||^2_{L^a_2(T^N)} = \sum_{n \in \mathbb{Z}^N} (1 + |n|^2)^a |g_n|^2. \tag{2.2}$$

is finite. In the case of noninteger $a$ this space is also referred to as the Liouville space.

It can readily be verified that there exist constants $c_1$ and $c_2$ such that one has the estimates

$$c_1 (1 + |n|^2)^{\tau m} \leq 1 + A^{2\tau}(n) \leq c_2 (1 + |n|^2)^{\tau m}.$$

Consequently, comparing the expressions (2.1) and (2.2), we see that $D(\hat{A}^\tau) = L^{\tau m}_2(T^N)$.

First, let us construct a formal solution of the inverse problem (1.1) - (1.3). According to the Fourier method, we look for a solution of Problem (1.1) - (1.3) in the form of the following series:

$$u(x,t) = \sum_{n \in \mathbb{Z}^N} T_n(t)e^{inx}, \tag{2.3}$$

here the functions $T_n(t)$ are solutions of the Cauchy type problem

$$D_t^\rho T_n + A(n)T_n = f_n(t), T_n(0) = \varphi_n. \tag{2.4}$$
Here and everywhere below we define by $g_n$ the Fourier coefficients of the function $g(x)$. Note that the unknowns in problem (2.4) are also the coefficients $f_n$. To determine these numbers, from (1.3) we obtain the additional condition

$$T_n(T) = \Psi_n. \quad (2.5)$$

The unique solution of problem (2.4) has the form (see, for example, [16], p. 16 or [18], p. 173)

$$T_n(t) = \varphi_n E_{\rho, 1}(-A(n)t^\rho) + f_n \int_0^t \xi^{\rho - 1} E_{\rho, \rho}(-A(n)\xi^\rho)d\xi, \quad (2.6)$$

where $E_{\rho, \mu}$ - is the Mittag-Leffler function

$$E_{\rho, \mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k + \mu)}$$

and $\Gamma(k)$ - is the Euler’s gamma function.

For the integral one has the formula (see, for example, [18], formula (4.4.4))

$$\int_0^t \xi^{\rho - 1} E_{\rho, \rho}(-A(n)\xi^\rho)d\xi = t^\rho E_{\rho, \rho + 1}(-A(n)t^\rho). \quad (2.7)$$

and we have

$$T_n(t) = \varphi_n E_{\rho, 1}(-A(n)t^\rho) + f_n t^\rho E_{\rho, \rho + 1}(-A(n)t^\rho). \quad (2.8)$$

To determine the coefficient $f_n$, by virtue of the additional condition (2.5) we obtain the equation

$$T_n(T) = \varphi_n E_{\rho, 1}(-A(n)T^\rho) + f_n T^\rho E_{\rho, \rho + 1}(-A(n)T^\rho) = \Psi_n.$$ 

Consequently,

$$f_n = \frac{\Psi_n}{T^\rho E_{\rho, \rho + 1}(-A(n)T^\rho)} - \frac{\varphi_n E_{\rho, 1}(-A(n)T^\rho)}{T^\rho E_{\rho, \rho + 1}(-A(n)T^\rho)}. \quad (2.9)$$

Thus, we have constructed the formal series (see (2.3) and (2.8))

$$u(x, t) = \sum_{n \in \mathbb{Z}^N} \varphi_n E_{\rho, 1}(-A(n)t^\rho)e^{inx} + \sum_{n \in \mathbb{Z}^N} f_n t^\rho E_{\rho, \rho + 1}(-A(n)t^\rho)e^{inx}, \quad (2.10)$$

for the solution of the inverse problem, where the numbers $f_n$ are determined using formula (2.9) and

$$f(x) = \sum_{n \in \mathbb{Z}^N} \frac{\Psi_n}{T^\rho E_{\rho, \rho + 1}(-A(n)T^\rho)}e^{inx} - \sum_{n \in \mathbb{Z}^N} \frac{\varphi_n E_{\rho, 1}(-A(n)T^\rho)}{T^\rho E_{\rho, \rho + 1}(-A(n)T^\rho)}e^{inx}. \quad (2.11)$$
Obviously, the function (2.10) and (2.11) are formal solutions.

Now let us state the following theorem on the existence of a solution of the inverse problem on the torus.

**Theorem 2.1.** Let \( \phi \in L^2_T(T^N) \) and \( \Psi \in L^{\tau+1}_T(T^N) \), where \( \tau > \frac{N}{2} \). Then there exists a unique periodic solution \( \{u(x,t), f(x)\} \) of the inverse problem (1.1) - (1.3) which can be represented in the form of the series (2.10) and (2.11).

### 3 Uniqueness

**Theorem 3.1.** (on uniqueness). Let the function \( \varphi(x) \) and \( \Psi(x) \) be continuous in \( T^N \). Then there can exist only one classical solution \( \{u(x,t), f(x)\} \) of the inverse problem (1.1) - (1.3).

**Proof.**

Assume that under the assumptions in the theorem there are two pair of solutions, \( \{u_1(x,t), f_1(x)\} \) and \( \{u_2(x,t), f_2(x)\} \). Let us prove that \( u(x,t) = u_1(x,t) - u_2(x,t) \equiv 0 \) and \( f(x) = f_1(x) - f_2(x) \equiv 0 \). Since the problem under consideration is linear, we have the following problem for determining the functions \( u(x,t) \) and \( f(x) \):

\[
D^\rho_t u(x,t) + A(D)u(x,t) = f(x), \quad x \in T^N, \quad t > 0; \tag{3.1}
\]

\[
u(x,0) = 0, \quad x \in T^N, \tag{3.2}
\]

\[
u(x,T) = 0, \quad x \in T^N. \tag{3.3}
\]

Here, instead of boundary conditions, we require the functions we are looking for to be periodic with respect to each variable.

Let \( u(x,t) \) be a solution of problem (3.1) - (3.3). Consider the following function

\[
w_n(t) = (2\pi)^{-N/2} \int_{T^N} u(x,t) \cdot e^{inx} dx. \tag{3.4}
\]

If we differentiate this equality under the integral and take into account equation (3.1), we obtain the following equality

\[
D^\rho_t w_n(t) = (2\pi)^{-N/2} \int_{T^N} D^\rho_t u(x,t) \cdot e^{inx} dx =
\]

\[
= -(2\pi)^{-N/2} \int_{T^N} A(D)u(x,t) \cdot e^{inx} dx + (2\pi)^{-N/2} \int_{T^N} f(x) \cdot e^{inx} dx, \quad t > 0.
\]
Integrating here by parts and taking into account that the operator \( \hat{A} \) is self-adjoint, we obtain the following equality
\[
D_t^\rho w_n(t) = -(2\pi)^{-N/2} \int_{T^N} u(x,t) A(D) \cdot e^{inx} dx + f_n
\]
\[
= -A(n) \cdot (2\pi)^{-N/2} \int_{T^N} u(x,t) \cdot e^{inx} dx + f_n;
\]
\[
D_t^\rho w_n(t) = -A(n) w_n(t) + f_n, t > 0.
\]
Now, taking into account initial condition (3.2), we conclude that the function \( w_n(t) \) is the solution of the following Cauchy problem:
\[
D_t^\rho w_n(t) + A(n) w_n(t) - f_n = 0, t > 0; \ w_n(0) = 0.
\]
The solution of this problem has the form (see, for example, [16] p. 16 or [18], p. 173)
\[
w_n(t) = f_n \int_0^t \xi^{\rho-1} E_{\rho, \rho}(- A(n) \xi^\rho) d\xi,
\]
For the integral one has the formula (see, for example, [18], formula (4.4.4))
\[
\int_0^t \xi^{\rho-1} E_{\rho, \rho}(- A(n) \xi^\rho) d\xi = t^\rho E_{\rho, \rho+1}(- A(n)t^\rho) . \tag{3.5}
\]
Therefore, using the additional condition (3.3), we obtain \( w_n(T) = f_n T^\rho E_{\rho, \rho+1}(- A(n)T^\rho) = 0 \) and since \( 0 < E_{\rho, \rho+1}(- A(n)T^\rho) \leq 1 \), we conclude that \( f_n = 0 \) for all \( n \). In turn, this implies that \( w_n(t) \equiv 0 \) for all \( n \). Consequently, by virtue of the completeness of the system of eigenfunctions \( \{(2\pi)^{-N/2} e^{inx}\} \), we have \( f(x) = 0 \) and \( u(x,t) = 0 \) for all \( x \in T^N \) and \( t > 0 \). Thus, we have proved the uniqueness of solution of the inverse problem (1.1)-(1.3).

## 4 Existence

The proof of existence is based on the following lemma, which is, in fact, corollary of the classical Sobolev embedding theorem (see M.A. Krasnosel’skii et.al., [20], p. 453).

**Lemma 4.1.** Let \( \tau > |\alpha| |m| + \frac{N}{2m} \). Then for any \( |\alpha| \leq m \), the operator \( D^\alpha \left( \hat{A} + I \right)^{-\tau} \) maps (completely) continuously from \( L_2(T^N) \) into \( C(T^N) \), moreover the following estimate holds true:
\[
\left\| D^\alpha \left( \hat{A} + I \right)^{-\tau} g \right\|_{C(T^N)} \leq C \| g \|_{L_2(T^N)} \tag{4.1}
\]
where we denote by $I$ the unit operator in $L_2(T^N)$.

**Proof.** Since the embedding theorem one has

$$\left\| D^{\alpha} \left( \hat{A} + I \right)^{-\tau} g \right\|_{C(T^N)} \leq C \left\| D^{\alpha} \left( \hat{A} + I \right)^{-\tau} g \right\|_{L_2^2(T^N)},$$

for any $a > \frac{N}{2}$. Therefore it is sufficient to prove the inequality

$$\left\| D^{\alpha} \left( \hat{A} + I \right)^{-\tau} g \right\|_{L_2^2(T^N)} \leq C \|g\|_{L_2(T^N)}.$$

But this is a consequence of the estimate

$$\sum_{n \in \mathbb{Z}^N} |g_n|^2 |n|^{2|\alpha|} (1 + A(n))^{-2\tau} (1 + |n|^2)^a \leq C \sum_{n \in \mathbb{Z}^N} |g_n|^2,$$

that is valid for $\frac{N}{2} < a \leq \tau m - |\alpha|$.

Now we show how the existence of the solution of problem (1.1)-(1.3) can be derived from this lemma.

Let the assumptions of Theorem 2.1 be satisfied. First, let us prove the absolute and uniform convergence of the series (2.11) for the function $f$. Set

$$f_j^1(x) = \sum_{n \in \mathbb{Z}^N} \frac{\Psi_n}{T^\rho E_{\rho,\rho+1}(-A(n)T^\rho)} e^{inx}, \quad (4.2)$$

$$f_j^2(x) = \sum_{n \in \mathbb{Z}^N} \frac{\varphi_n E_{\rho,1}(-A(n)T^\rho)}{T^\rho E_{\rho,\rho+1}(-A(n)T^\rho)} e^{inx}. \quad (4.3)$$

The condition of the theorem implies the convergence of the series

$$\sum_{n \in \mathbb{Z}^N} (A(n) + 1)^{2(\tau+1)} |\Psi_n|^2 \leq C_{\Psi} < \infty,$$

for some $\tau > \frac{N}{2m}$ and the convergence of the series

$$\sum_{n \in \mathbb{Z}^N} (A(n) + 1)^{2\tau} |\varphi_n|^2 \leq C_{\varphi} < \infty.$$

for the function $\varphi$.

Since $(\hat{A} + I)^{-\tau} e^{inx} = (A(n) + 1)^{-\tau} e^{inx}$, for $f_j^1$ we have

$$f_j^1(x) = (\hat{A} + I)^{-\tau} \sum_{n \in \mathbb{Z}^N} \frac{\Psi_n (A(n) + 1)^\tau}{T^\rho E_{\rho,\rho+1}(-A(n)T^\rho)} e^{inx}.$$
Then, applying the lemma we obtain

\[ \|f_1^j\|_{C(T^N)} = \|\hat{A} + I\|^{-\tau} \sum_{n \in \mathbb{Z}^N} \frac{\Psi_n (A(n) + 1)^\tau}{T^\rho E_{\rho,\rho+1}(-A(n)T^\rho)} e^{inx} \|_{C(T^N)} \leq \]

\[ \leq C\| \sum_{n \in \mathbb{Z}^N} \frac{\Psi_n (A(n) + 1)^\tau}{T^\rho E_{\rho,\rho+1}(-A(n)T^\rho)} e^{inx} \|_{L^2(T^N)}. \]  \hspace{1cm} (4.4)

Based on this, by virtue of the orthonormality of the system \{\((2\pi)^{-N/2}e^{inx}\)\}, we have

\[ \|f_1^j\|^2_{C(T^N)} \leq C\sum_{n \in \mathbb{Z}^N} \left| \frac{\Psi_n (A(n) + 1)^\tau}{T^\rho E_{\rho,\rho+1}(-A(n)T^\rho)} \right|^2. \]

Let us give estimates for the Mittag-Leffler function required in the sequel. For large \(t\) one has the asymptotic estimate (see e.g., \[16\], p. 13, recall that \(0 < \rho \leq 1\))

\[ E_{\rho,\rho+1}(-t) = \frac{1}{t}(1 + O\left(\frac{1}{t}\right)), \quad t > 1, \]  \hspace{1cm} (4.5)

and for arbitrary complex numbers \(\mu\) we have the estimate

\[ 0 < |E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}, \quad t > 0. \]  \hspace{1cm} (4.6)

Let us get back to estimating the norm of the function \(f_1^j\). By virtue of the estimate (4.5), we have

\[ \|f_1^j\|^2_{C(T^N)} \leq C\sum_{n \in \mathbb{Z}^N} (A(n) + 1)^{2(\tau+1)} |\Psi_n|^2 \leq C\psi < \infty. \]  \hspace{1cm} (4.7)

Applying the same argument as above to the function \(f_2^j\), we arrive at the inequality

\[ \|f_2^j\|_{C(T^N)} \leq C\| \sum_{n \in \mathbb{Z}^N} \frac{\varphi_n (A(n) + 1)^\tau E_{\rho,\rho+1}(-A(n)T^\rho)}{T^\rho E_{\rho,\rho+1}(-A(n)T^\rho)} e^{inx} \|_{L^2(T^N)}. \]  \hspace{1cm} (4.8)

Hence, by virtue of the orthonormality of the system \{\((2\pi)^{-N/2}e^{inx}\)\}, we have

\[ \|f_2^j\|^2_{C(T^N)} \leq C\sum_{n \in \mathbb{Z}^N} \left| \frac{\varphi_n (A(n) + 1)^\tau E_{\rho,\rho+1}(-A(n)T^\rho)}{T^\rho E_{\rho,\rho+1}(-A(n)T^\rho)} \right|^2. \]

Further, applying the estimates (4.5) and (4.6), we obtain

\[ \|f_2^j\|^2_{C(T^N)} \leq \frac{C}{T^\rho} \sum_{n \in \mathbb{Z}^N} (A(n) + 1)^{2\tau} |\varphi_n|^2 \leq \frac{C}{T^\rho} C\psi < \infty. \]  \hspace{1cm} (4.9)
The estimates \( |4.7| \) and \( |4.9| \) imply the uniform convergence of series \( |2.11| \), defining the function \( f(x) \). On the other hand, the sums in \( |4.4| \) and \( |4.8| \) converge under any permutation of their terms, because these terms are mutually orthogonal. This implies the absolute convergence of series \( |2.11| \).

Let us proceed to considering the series in \( |2.10| \) for the function \( u \). To this end, using the definition of the coefficients \( f_n \), we study the following three sums:

\[
\begin{align*}
  u^1_j(x,t) &= \sum_{n \in \mathbb{Z}^N} \varphi_n E_{\rho,1}(1-A(n)t^\rho)e^{inx}, \quad (4.10) \\
  u^2_j(x,t) &= \sum_{n \in \mathbb{Z}^N} \Psi_n t^\rho E_{\rho,\rho+1}(1-A(n)t^\rho)e^{inx}, \quad (4.11) \\
  u^3_j(x,t) &= \sum_{n \in \mathbb{Z}^N} \varphi_n E_{\rho,1}(1-A(n)T^\rho)T^\rho E_{\rho,\rho+1}(1-A(n)t^\rho)e^{inx}. \quad (4.12)
\end{align*}
\]

Let \( |\alpha| = m \) and \( \delta > 0 \). We need to show that each of these sums converges uniformly and absolutely in the domain \( \mathbb{T}^N \times [\delta, T] \) after the operators \( D^\alpha \) and \( D_t^\rho \) are term-by-term applied to the sum.

We have

\[
  u^1_j(x,t) = (\hat{A} + I)^{-\tau-1} \sum_{n \in \mathbb{Z}^N} (A(n) + 1)^{\tau+1} \varphi_n E_{\rho,1}(1-A(n)t^\rho)e^{inx}.
\]

Therefore, by the lemma, one has the estimate

\[
  \|D^\alpha u^1_j\|_{C(T^N)} = \|D^\alpha (\hat{A}+I)^{-\tau-1} \sum_{n \in \mathbb{Z}^N} (A(n) + 1)^{\tau+1} \varphi_n E_{\rho,1}(1-A(n)t^\rho)e^{inx}\|_{C(T^N)} \leq \\
  \leq C\| \sum_{n \in \mathbb{Z}^N} (A(n) + 1)^{\tau+1} \varphi_n E_{\rho,1}(1-A(n)t^\rho)e^{inx}\|_{L_2(T^N)}. \quad (4.13)
\]

Using the orthonormality of the system \( \{(2\pi)^{-N/2}e^{inx}\} \), we have

\[
  \|D^\alpha u^1_j\|^2_{C(T^N)} \leq C \sum_{n \in \mathbb{Z}^N} |(A(n) + 1)^{\tau+1} \varphi_n E_{\rho,1}(1-A(n)t^\rho)|^2 \leq \\
  \leq Ct^{-2\rho} \sum_{n \in \mathbb{Z}^N} (A(n) + 1)^{2\tau} |\varphi_n|^2 \leq C\delta^{-2\rho} C_\varphi.
\]

Here the last inequality has been obtained based on the estimate \( |4.6| \).

This implies the uniform convergence of the sum \( |4.10| \) differentiated with respect to the variables \( x_n \) in the closed cylinder \( (x,t) \in \mathbb{T}^N \times [\delta, T] \). On the other hand, the sum \( |4.13| \) converges under any permutation of its terms, because these terms are mutually orthogonal. This implies the absolute convergence of the sum \( |4.10| \) differentiated with respect to the variables \( x_n \).
Repeating a similar argument, we establish that the same assertions also hold true for the sums (4.11) and (4.12), determining the functions $u_2(x, t)$ and $u_3(x, t)$.

Now it remains to verify the validity of term-by-term application of the operator $D_\rho^\alpha$ to the series (2.10) defining the solution $u(x, t)$. It can readily be verified that

$$D_\rho^\alpha \sum_{n \in \mathbb{Z}^N} T_n(t)e^{inx} = -\sum_{n \in \mathbb{Z}^N} A(n)T_n(t)e^{inx} + \sum_{n \in \mathbb{Z}^N} f_n(t)e^{inx} =$$

$$-A(x, D)(\hat{A}+I)^{-\tau-1}\sum_{n \in \mathbb{Z}^N} (A(n)+1)^{\tau+1}T_n(t)e^{inx}+(\hat{A}+I)^{-\tau}\sum_{n \in \mathbb{Z}^N} (A(n)+1)^{\tau}f_ke^{inx}.$$  

The absolute and uniform convergence of the last two sums has already been proved above.

The fact that the functions (2.10) and (2.11) satisfy all conditions (1.1)- (1.3) of the inverse problem is obvious. Theorem has thus been proved completely.

Acknowledgements

The author convey thanks to R. R. Ashurov for discussions of these results.

References

1. Zhang Y., Xu X. Inverse source problem for fractional differential equations, Inverse Probl., 2011, vol. 27, no. 3, pp. 31–42.

2. Furati K.M., Iyiola O.S., Kirane M. An inverse problem for a generalized fractional diffusion, Appl. Math. Comput., 2014, vol. 249, pp. 24–31.

3. Ismailov M.I., Cicek M. Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions, Appl. Math. Model., 2016, vol. 40, pp. 4891–4899.

4. Kirane M., Malik A.S. Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time, Appl. Math. Comput., 2011, vol. 218, pp. 163–170.

5. Kirane M., Samet B., Torebek B.T. Determination of an unknown source term and the temperature distribution for the subdiffusion equation at the initial and final data, Electron. J. Differ. Equat., 2017, vol. 217, pp. 1–13.

6. Nguyen H.T., Le D.L., Nguyen V.T. Regularized solution of an inverse source problem for a time fractional diffusion equation, Appl. Math. Model., 2016, vol. 40, pp. 8244–8264.

7. Torebek B.t., Tapdigoglu R. Some inverse problems for the nonlocal heat equation with Caputo fractional derivative, Math. Methods Appl. Sci., 2017, vol. 40, pp. 6468–6479.
8. Heymans N., Podlubny I. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, Rheol. Acta, 2006, vol. 45, pp. 765–771.

9. Li Z., Liu Y., Yamamoto M. Initial–boundary value problem for multi-term time-fractional diffusion equation with positive constant coefficients, Appl. Math. Comput., 2015, vol. 257, pp. 381–397.

10. Rundell W. Zhang Z. Recovering an unknown source in a fractional diffusion problem, J. Comput. Phys., 2018, vol. 368, pp. 299–314.

11. Asl N.A., Rostamy D. Identifying an unknown time-dependent boundary source in time-fractional diffusion equation with a non-local boundary condition, J. Comput. Appl. Math., 2019, vol. 335, pp. 36–50.

12. Sun L., Zhang Y., Wei T. Recovering the time-dependent potential function in a multi-term time-fractional diffusion equation, Appl. Numer. Math., 2019, vol. 135, pp. 228–245.

13. Malik S.A., Aziz S. An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions, Comput. Math. Appl., 2017, vol. 3, pp. 7–19.

14. Ruzhansky M., Tokmagambetov N., Torebek B.T. Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations, J. Inverse Ill-Posed Probl., 2019, vol. 27, pp. 891–911.

15. R.R. Ashurov, A.T. Mukhiddinova Inverse Problem of Determining the Heat Source Density for the Subdiffusion Equation ISSN 0012-2661, Differential Equations, 2020, Vol. 56, No. 12, pp. 1550–1563.

16. Pskhu, A.V. Uravneniya v chastnykh proizvodnykh drobnogo poriadka (Fractional Partial Differential Equations), Moscow: Nauka, 2005.

17. Alimov, Sh.A., Ashurov, R.R., and Pulatov, A.K., Multiple series and Fourier integrals, Itogi Nauki Tekh., Ser.: Sovrem. Probl. Mat. Fundam. Napravleniya, 1989, vol. 42, pp. 7–104.

18. Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogozin, S.V. Mittag-Leffler functions, related topics and applications. Springer. 2014.

19. Agmon S. On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, Commun. Pure Appl. Math., 1962, vol. 15, no. 2, pp. 119–143.

20. Krasnosel’skii M.A., Zabreyko P.P., Pustylnik E.I., Sobolevski P.E., Integral’nye operatory v prostranstvakh summiruemyh funktsii (Integral Operators in Spaces of Integrable Functions), Moscow: Nauka, 1966.

21. Alimov Sh.A., Fractional degrees of elliptic operators and isomorphism of classes of differentiable functions, Differ. Uravn., 1972, vol. 8, no. 9, pp. 1609–1626.
Muhiddinova O.T.,

Institute of Mathematics named after V.I. Romanovsky, the Academy of Sciences of the Republic of Uzbekistan, University street, 4, Tashkent, 100174, Uzbekistan

e-mail: oqila1992@mail.ru
This figure "frog.jpg" is available in "jpg" format from:

http://arxiv.org/ps/2204.00355v1