QGLBT for polytopes

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Abstract

We extend the assertion of the Generalized Lower Bound Theorem (GLBT) to general polytopes under the assumption that their low dimensional skeleton is simplicial, with partial results for the general case. We prove a quantitative version of the GLBT for general polytopes, and use it to give a topological necessary condition for polytopes to have vanishing toric $g_k$ entry. As another application of the QGLBT we prove a conjecture of Kalai on $g$-numbers for general polytopes approximating a smooth convex body.

1 Introduction

The well known $g$-theorem [BL80, Sta80, McM70] characterizes the face numbers of simplicial polytopes, and in particular says that the $g$-numbers are non-negative. The Generalized Lower Bound Theorem (GLBT) [MW71, MN13] characterizes the simplicial polytopes attaining equality $g_k = 0$, being exactly the

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(k − 1)-stacked polytopes. For general polytopes \( P \), the toric \( g \)-vector, introduced by Stanley \cite{Sta87}, is computed from the face poset of the polytope, and coincides with the \( g \)-vector in the simplicial case. Karu \cite{Kar04} proved non-negativity of the toric \( g \)-vector by showing it computes the dimension of the primitive cohomology of the associated Combinatorial Intersection Homology module \( IH(P) \) (introduced in \cite{BBFK02, BL03}) w.r.t. a Lefschetz element. The following problem naturally arise:

**Problem 1.1.** Given \( 1 \leq k \leq d/2 \), for which \( d \)-polytopes \( P \) does toric \( g_k(P) = 0 \)?

We provide a few results in this direction. A polytope is \( k \)-simplicial if all its \( k \)-dimensional faces are simplices. The following generalizes the GLBT:

**Theorem 1.2.** Let \( P \) be a \((2k − 1)\)-simplicial \( d \)-polytope with \( g_k(P) = 0 \) \((k \leq d/2)\). Let \( \Delta \) be the collection of geometric \( d \)-simplices whose \((k − 1)\)-skeleton is a subcomplex of the face complex of \( P \). Then \( \Delta \) is a triangulation of \( P \).

This result is tight: note that the pyramid over \( P \) satisfies \( g(\text{Pyr}(P)) = g(P) \). Thus by taking a \((d − (2k − 1))\)-fold pyramid \( Q = \text{Pyr} \circ \text{Pyr} \circ \cdots \circ \text{Pyr} \) over any simplicial \((2k − 1)\)-polytope \( P \), we have \( g_k(Q) = 0 \), \( Q \) is \((2k − 2)\)-simplicial, and the clique complex over the \((k − 1)\)-skeleton of \( Q \) is typically a strict subset of \( Q \).

The key point in the proof of Theorem 1.2 is noticing that, up to degree \( k \), the module \( IH(P) \) is isomorphic to the corresponding Stanley-Riesner (quotient) ring, and then use either of the proofs of the GLBT, \cite{MN13} or \cite{Adi17}, via crystallization or propagation resp.

The only \( d \)-polytope with \( g_1(P) = 0 \) is the \( d \)-simplex. The simplest open case then is to characterize 2-simplicial 4-polytopes \( P \) with \( g_2(P) = 0 \). One reduces to the prime case, namely when \( P \) has no missing tetrahedra \( T \), as otherwise \( P = P_1 \cup T P_2 \) and each \( g_2(P_i) = 0 \), \( i = 1, 2 \). Paffenholz and Werner \cite{PW06} constructed examples of such polytopes with \( n \) vertices, for each \( n \geq 13 \). We give a structural condition that all such polytopes must satisfy, given by restricting the following
theorem to \( k = 2 \). Let \( P_W \) denote the polytopal complex whose faces are all faces of a polytope \( P \) whose vertexset is contained in \( W \), where \( W \) is a subset of the vertexset \( V(P) \) of \( P \). Say a simplex \( \sigma \) is a missing simplex in \( P \) if \( \partial \sigma \) is a subcomplex of \( \partial P \) and \( \sigma \) is not a face of \( P \).

**Theorem 1.3.** Let \( P \) be a \( d \)-polytope with \( g_k(P) = 0 \), \( d \geq 2k \geq 2 \), and let \( W \) denote any subset of the vertices of \( P \). Then the embedding of \( P_W \) in \( \partial P \setminus P_{V(P)\setminus W} \) induces the zero map on the \((k - 1)\)th rational homology groups. In particular, any missing \( k \)-simplex in \( P \) is contained in a facet of \( P \).

Aiming for a characterization, does the converse also hold? For \( k = 2 \) this reads as:

**Problem 1.4.** Let \( P \) be a polytope of dimension \( \geq 4 \). If for any subset of the vertices of \( P \), denoted by \( W \), the embedding of \( P_W \) in \( \partial P \setminus P_{V(P)\setminus W} \) induces the zero map on the first rational homology groups, then \( g_2(P) = 0 \).

Theorem 1.3 is inspired by, and extends, Kalai’s result [Kal87, Kal94] that when such \( P \) is simplicial it has no missing \( k \)-simplices\(^1\). More generally, as we shall see, the following extension of the quantitative GLBT (QGLBT) by Adiprasito [Adi17] to the toric case gives a lower bound on \( g_k(P) \) in terms of topological Betti numbers of induced subcomplexes of \( P \). Specifically,

**Theorem 1.5.** Let \( P \) be a \( d \)-polytope, and \( W \) any subset of its vertices \( V = V(P) \). Let \( k \leq \frac{d}{2} \). Then the induced simplicial subcomplex \( P_W \) satisfies

\[
\alpha_{k-1}(P_W) \leq g_k(P).
\]

Here \( \alpha_{k-1}(P_W) \) denotes the dimension of the image of \( H_{k-1}(P_W; \mathbb{R}) \to H_{k-1}(\partial P \setminus P_{V\setminus W}; \mathbb{R}) \).

We use Theorem 1.5 to extend the recent proof of Kalai’s conjecture on simplicial polytopes approximating smooth convex bodies [ANS16a] to the non-simplicial case.

\(^1\)The case \( d = 2k > 4 \) was not worked out by Kalai; later Nagel [Nag08, Cor.4.8] proved this case as well.
Corollary 1.6. Let $K$ be a smooth convex body in $\mathbb{R}^d$, and $(P_n)$ a sequence of $d$-polytopes such that $P_n \to K$ in the Hausdorff metric. Then for any $1 \leq k \leq d/2$, $g_k(P_n) \to \infty$ as $n \to \infty$.

Further, if $K$ has a $C^2$ boundary, and a $d$-polytope $P$ is $\epsilon$-close to $K$ for some small enough $\epsilon > 0$, then $g_k(P) = \Omega(\epsilon^{-\frac{d-1}{2}})$.

Outline: in Section 2 we recall the construction of $IH(P)$ and prove Theorem 1.2, in Section 3 we prove the QGLBT, namely Theorem 1.5, and deduce from it Theorem 1.3 and Corollary 1.6.

2 Intersection cohomology for general fans

In order to work in the context of general polytopes, we use the Barthel-Brasselet-Fieseler-Kaup [BBFK02] and Karu [Kar04] construction of the equivariant intersection cohomology sheaf, constructed inductively on the $i$th skeleton, by iteratively applying the Lefschetz theorem to faces of dimension $i - 1$. In particular, the equivariant sheaf $L(P)$ of a polytope $P$ is constructed as a subspace of $L(sdP)$, where $sdP$ is the simplicial polytope whose boundary complex is the derived subdivision of the boundary complex of $P$. The stalk over a proper face $\sigma$ of $P$ in $L(P)$ is a free module over the primitive elements with respect to the operation of the Lefschetz element induced on $IH(\sigma)$, the (non-equivariant) intersection cohomology of $\sigma$. It follows in particular that low-degree intersection cohomology depends more on the simplicial structure than higher degrees. Specifically, for a geometric simplicial complex $\Delta$ in $\mathbb{R}^d$ denote by $A(\Delta)$ the quotient of the Stanley-Riesner ring of $\Delta$ over $\mathbb{R}$ by the ideal generated by the $d$ elements of degree 1 corresponding to the embedding of the vertices of $\Delta$ in $\mathbb{R}^d$. Then,

Remark 2.1. We adopt Karu’s abuse of notation and do not adopt the natural grading for intersection cohomology arising from toric geometry (where we would naturally only have intersection cohomology in even degrees) and in-
stead use degrees coming from the underlying model for the intersection ring, the Stanley-Reisner ring.

**Proposition 2.2.** If the \((2k - 1)\)-skeleton \(X_{\leq 2k-1}\) of a geometric polyhedral complex \(X\) is simplicial, then \(IH^i(X) \cong A^i(X_{\leq 2k-1})\) for every \(i \leq 2k\).

This proposition allows us to prove Theorem 1.2 by following either of the proofs [MN13] and [Adi17].

**Proof of Theorem 1.2.** Recall \(\Delta\) denotes the simplicial complex consisting of all subsets \(\sigma\) of vertices of the \(d\)-polytope \(P\) all whose subsets of size \(\leq k\) are simplices in \(\partial P\). Denote by \(\Delta_W\) the induced subcomplex of \(\Delta\) on the vertex set \(W\), and by \(V(F)\) the vertices of a face \(F\) of \(P\). We show the following three properties for any \(i\)-face \(F\) of \(P\):

(A) \(\Delta_{V(F)}\) is Cohen-Macaulay of dimension \(\dim F\).

We can assume that \(F\) is of dimension at least \(2k\). It follows immediately from flabbiness of the intersection cohomology sheaf that \(g_j(F) \leq g_j(P)\) for all \(j\) [BM99, Bra06]; hence the claim follows at once from [Adi17, Cor. 4.7] for \(\Delta_{V(F)}\), which is indeed applicable using Proposition 2.2 and Karu’s hard Lefschetz for polytopes [Kar04].

(B) \(\Delta_{V(F)}\) is a geometric complex embedded in \(F\).

This is a result of McMullen, namely the argument in the proof of [McM04, Thm.4.1]; see also [BD14, Prop.3.4] and [MN13, Lem.4.2].

(C) Finally, conclude that \(\Delta_{V(F)}\) triangulates \(F\).

For this we need that the geometric realization of \(\Delta_{V(F)}\) contains the boundary of \(F\), which we know by the induction hypothesis (for \(i = 2k\) this is the data that \(P\) is \((2k - 1)\)-simplicial). But every Cohen-Macaulay subcomplex of \(\mathbb{R}^i\) and of dimension \(i\) is a ball.

For \(i = d\) we obtain that \(\Delta\) is a geometric triangulation of \(P\), as desired. □
3 Quantitative generalized lower bound theorem and applications

3.1 Induced cohomology

We shall predominantly need a notion of topology of induced subcomplexes. Let $X$ be a strongly regular CW complex, namely the open cells in $X$ are embedded and the intersection of closures of any two cells is the closure of a cell in the boundary of both. Let $W$ be a subset of the vertexset of $X$. We denote by $X_W$ the collection of those faces whose vertices are subsets of $W$. Then $\alpha_{k-1}(P_W)$ denotes the dimension of the image of $H_{k-1}(X_W; \mathbb{R})$ in $H_{k-1}(X \setminus X_{V(X)\setminus W}; \mathbb{R})$.

The following lemma is folklore.

**Lemma 3.1.** If $\Delta$ is a simplicial complex, then $\Delta \setminus \Delta_{V(\Delta)\setminus W}$ is homotopy equivalent, by a deformation retraction, to $\Delta_W$.

The standard argument to prove it gives for polytopal complexes the following:

**Lemma 3.2.** Let $P$ be a polytope and $W \subseteq V = V(P)$. Then $\partial P \setminus P_{V\setminus W}$ deformation retracts onto $\partial P \cap \text{conv} W$; the latter contains $P_W$ but may not be homotopy equivalent to it.

This motivates the following definition, in view of [ANS16b, Fact 3.1(4)]:

**Definition 3.3.** A strongly regular CW complex $X$ is called resolution $i$-chordal if $\alpha_i(P_W) = 0$ for every subset $W$ of the vertices of $X$.

The first part of Theorem 1.3 can be rephrased as $\partial P$ being resolution $(k-1)$-chordal.

**Remark 3.4.** In fact, if $P$ is a $d$-polytope with $g_k(P) = 0$ ($d \geq 2k \geq 2$), then $\partial P$ is resolution $(i-1)$-chordal for any $k \leq i \leq d - k$. To see this recall that
Kalai showed that $g_k(P) = 0$ implies $g_{k+1}(P) = \ldots = g_{\frac{d}{2}}(P) = 0$, see [Bra06]; now apply Theorem 1.3 to $k \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$. The case $\frac{d}{2} < i \leq d - k$ follows by naturality of Alexander duality w.r.t. inclusions. Indeed, assume by contradiction that the map $H_{i-1}(P_\mathcal{W}; \mathbb{R}) \to H_{i-1}(\partial P \setminus P_\mathcal{V}; \mathbb{R})$ were nonzero, then the map $H^{d-1-i}(P_\mathcal{V}\setminus\mathcal{W}; \mathbb{R}) \to H^{d-1-i}(\partial P \setminus P_\mathcal{W}; \mathbb{R})$ would be nonzero, and as we work with field coefficients also the map $H_{d-1-i}(P_\mathcal{V}\setminus\mathcal{W}; \mathbb{R}) \to H_{d-1-i}(\partial P \setminus P_\mathcal{V}\setminus\mathcal{W}); \mathbb{R})$ would be nonzero, a contradiction. This is tight, as shown by $W$ the vertexset of a missing simplex of dimension at least $d - k$, which indeed exists in simplicial $(k - 1)$-stacked $d$-polytopes.

3.2 Proof of Theorem 1.5

With the structure of intersection cohomology given in Section 2, we conclude the proof of Theorem 1.5. While it is possible to prove the theorem in the same way as in [Adi17], this is a little cumbersome as the ”support” of a Chow cohomology class is a little tricky to phrase in the intersection ring. Instead, we give a proof that focuses on an argument similar to [Kal87] and the appendix of [ANS16a], where we proved the same under the assumption that lower-dimensional cohomologies vanish.

Let $Q$ denote a simplicial polytope and let $W$ denote a subset of vertices of $Q$, which we may identify with a set of prime divisors. Assume that the closed neighborhood of the induced subcomplex $Q_\mathcal{W}$, namely the subcomplex consisting of all faces of $P$ that are contained in a face containing a vertex from $W$, is a regular neighborhood of $Q_\mathcal{W}$.

Define $\mathcal{I} := \text{ann}\langle x_w : w \in W \rangle \subset A(Q)$, and let $\mathcal{I}_{k-1}$ denote the ideal generated by elements of $\mathcal{I}$ of degree $\leq k - 1$. Set $A(Q_\mathcal{W}) := A(Q)/\mathcal{I}$ and $\overline{A}(Q_\mathcal{W}) := A(Q)/\mathcal{I}_{k-1}$. Then $A(Q) \to \overline{A}(Q_\mathcal{W}) \to A(Q_\mathcal{W})$, with the latter two agreeing in degrees $\leq k - 1$. 

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Then, by definition, we have an injection

\[ A(Q_W) \hookrightarrow \bigoplus_{w \in W} A(st_w Q) \]

where \( A(st_w Q) \) denotes the quotient ring of \( A(Q) \) corresponding to the closed star of \( w \) in \( Q \).

Consider \( \ell \) the class of an ample divisor in \( A(Q) \). Then we have a diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & A^{k-1}(Q_W) \\
& & \downarrow \ell \\
0 & \longrightarrow & A^k(Q_W)
\end{array}
\]

\[
\begin{array}{ccc}
& \bigoplus_{w \in W} A^{k-1}(st_w Q) & \\
& \downarrow \ell & \\
& \bigoplus_{w \in W} A^k(st_w Q) &
\end{array}
\]

where the second vertical map \( \ell \), and therefore also the first, is an injection. Finally, we have from the second page of the Ishida spectral sequence (see [Ish80, Secs. 4, 5] and [Oda91]) an isomorphism\(^2\)

\[ (H^{k-1}(Q_W))^{(\ell)} \cong \ker [\overline{A}^k(Q_W) \longrightarrow \bigoplus_{w \in W} A^k(st_w Q)] \]

so that we also obtain an injection of \( (H^{k-1}(Q_W))^{(\ell)} \) into \( \overline{A}^k(Q_W) / \ell \overline{A}^{k-1}(Q_W) \).

In particular,

**Lemma 3.5.** Under the above conditions, \( H^{k-1}(Q_W) \) injects into \( \overline{A}^k(Q_W) / \ell \overline{A}^{k-1}(Q_W) \).

We now go back to \( P \). Subdivide it barycentrically twice to obtain \( P'' \). Then the corresponding subdivision \( P''_{W''} \) of \( P_W \) is an induced subcomplex satisfying the regular neighborhood condition for the previous lemma. Therefore, \( H^{k-1}(P_W) \) injects into \( \overline{A}^k(P''_{W''}) / \ell \overline{A}^{k-1}(P''_{W''}) \).

Now, the cokernel of the pullback inclusion of \( IH(P) \) to \( A(P'') \) is generated by the images of the Gysin maps (see [Ful98, Section 6.5]); where for a single arrow in the sequence of stellar subdivisions \( P = P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_m = P'' \),

\[ \text{For the reader inclined towards commutative more than homological algebra, this also follows easily as in the work of Novik and Swartz [NS09, Thm. 2.2] on Buchsbaum modules.} \]

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with \( v_i \) the vertex in \( P_i \) and not in \( P_{i-1} \), this image is the image of the composition \( IH(\text{lk}_{v_i} P_i) \to IH(P_i) \to IH(P') \cong A(P') \), with the first arrow being the Gysin injection and the second arrow a pullback map. Thus, these images correspond to cohomologically trivial cycles in \( H^{k-1}(\partial P \setminus P V \setminus W) \). Hence, we conclude an injection of the image of \( H_{k-1}(P W; \mathbb{R}) \) in \( H_{k-1}(\partial P \setminus P V \setminus W; \mathbb{R}) \) into

\[
IH^k(P) / \ell(IH^{k-1}(P)).
\]

Remark 3.6. The reader may have noticed that in the last step we seem to have lost some accuracy without explanation, from an injection of \( (H^{k-1}(Q W))^{(d)} \) to injection of only one copy of \( H_{k-1}(P W; \mathbb{R}) \) into the coprimitive classes of the Chow ring. This is explained by the use of independence as cohomology cycles that we relied on to tell coprimitive classes apart. In general, it cannot be avoided: For instance, the join of an \( n \)-gon with a triangle forms a simplicial 4-polytope \( P \) with \( g_2(P) = 1 \), and \( IH^2(P) / \ell IH^1(P) \) is generated by the induced cohomology class of the cycle of length \( n \).

### 3.3 Applications: proofs of Theorem 1.3 and Corollary 1.6.

**Proof of Theorem 1.3.** That the image of \( H_{k-1}(P W; \mathbb{Q}) \) in \( H_{k-1}(\partial P \setminus P V \setminus W; \mathbb{Q}) \) is zero is immediate from the QGLBT, Theorem 1.5 (passing from real to rational coefficients using the universal coefficients theorem); namely \( \partial P \) is resolution \((k - 1)\)-chordal. If \( P W \) were a missing \( k \)-simplex in \( P \) not contained in any facet, then \( \partial P \cap \text{conv} W = P W \), but then the image above in homology would have been nonzero (in fact, an isomorphism), a contradiction.

**Proof of Corollary 1.6.** When the convex body \( K \) is smooth, let \( b \) be bigger then any given constant, and when \( K \) has a \( C^2 \) boundary, let \( b = \Omega(\epsilon^{-\frac{1}{d+1}}) \) when \( P \) is \( \epsilon \)-close to \( K \) for small enough \( \epsilon > 0 \). With Theorem 1.5 (QGLBT) and Lemma 3.2 at hand, we proceed exactly as in [ANS16a] and find \( b \) copies \( \gamma_1, \ldots, \gamma_b \) of the \((k - 1)\)-sphere in \( \partial K \), and \( 0 < \epsilon' < \epsilon \) (where \( P \) is \( \epsilon \)-close to \( K \), for some \( \epsilon \) small enough) such that:
(i) the $\epsilon$-neighborhoods $\gamma_i + \epsilon$, $1 \leq i \leq b$, are pairwise disjoint in $\partial K$, and each $\gamma_i + \epsilon$ deformation retracts to $\gamma_i$;

(ii) if $v, u \in V(P)$ such that $v \in \gamma_i + \epsilon'$ and $u \in \gamma_j + \epsilon'$ for some $i \neq j$, then no proper face of $P$ contains both $v$ and $u$;

(iii) for $W_i := V(P) \cap \gamma_i + \epsilon'$, the following inclusions hold

$\pi_P \gamma_i \subset P_{W_i} \subseteq \partial P \cap \text{conv} W_i \subseteq \gamma_i + \epsilon$

where $\pi_P$ denotes the closest point projection to $P$.

We conclude the proof by noticing that (i) and (iii) imply $\alpha_{k-1}(P_{W_i}) \geq 1$, and thus, by (ii), for $W = \bigcup_i W_i$, $\alpha_{k-1}(P_W) = \sum_{i=1}^b \alpha_{k-1}(P_{W_i}) \geq b$. $\square$

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