ON BINOMIAL COEFFICIENTS
OF REAL ARGUMENTS

Tatiana I. Fedoryaeva

Sobolev Institute of Mathematics, Novosibirsk, Russia
E-mail: fti@math.nsc.ru

Abstract. As is well-known, a generalization of the classical concept of the factorial $n!$ for a real number $x \in \mathbb{R}$ is the value of Euler’s gamma function $\Gamma(1+x)$. In this connection, the notion of a binomial coefficient naturally arose for admissible values of the real arguments.

By elementary means, it is proved a number of properties of binomial coefficients $(\binom{r}{\alpha})$ of real arguments $r, \alpha \in \mathbb{R}$ such as analogs of unimodality, symmetry, Pascal’s triangle, etc. for classical binomial coefficients. The asymptotic behavior of such generalized binomial coefficients of a special form is established.

Keywords: factorial, binomial coefficient, gamma function, real binomial coefficient.

MSC 2020: 05A10, 11B65

INTRODUCTION

We study binomial coefficients of real arguments. The aim of the investigation is to obtain by elementary methods analogs of the basic properties well known for the classic binomial coefficients. Such properties are of independent interest and, in addition, can simplify the work with binomial coefficients of the form $\binom{n}{m}$ with integer non-negative arguments $n$ and $m$, given essentially by real values with considered rounding to an integer (when, for example, floor and ceiling functions for a real number are used, etc.). So, for example, the properties of unimodality and symmetry allow passing from such binomial coefficients $\binom{n}{m}, 0 \leq m \leq n$ to ”close” real binomial coefficients of the form $\binom{\alpha}{\alpha}$, $\alpha \in (-1, r + 1)$ and vice versa. This approach simplifies the evaluation of expressions with discrete binomial coefficients with integer arguments of the specified form.
Note that the binomial coefficients of the form \( \binom{r}{n} \), where \( r \in \mathbb{R} \) and \( n \in \mathbb{N} \), can be defined in the standard way as
\[
\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}.
\]
This approach was discussed in [4], where a numerous number of identities for such binomial coefficients is given. In [3], D. Fowler studied the graph of the function \( \binom{r}{\alpha} \) of two real variables \( r \) and \( \alpha \), various slices of this graph were constructed using a computer and their analysis was carried out. It is also indicated there an explicit expression for the binomial coefficient \( \binom{n}{\alpha} \), where \( n \) is a non-negative integer, through elementary functions (see Proposition 2 in Section 2). On the basis of this representation, Stuart T. Smith investigated the binomial coefficients of the form \( \binom{n}{z} \) with complex variable \( z \in \mathbb{C} \) and fixed natural number \( n \in \mathbb{N} \), a number of properties of such a function of complex argument \( z \) is established in [6]. In particular, the derivatives of the first and second orders are calculated, and for the real argument \( z \), increasing and decreasing intervals, zeros of the function, etc are found. It is also noted there the nontriviality of the function investigation \( \binom{n}{\alpha} \) of real variable \( \alpha \) exactly on the interval \( \alpha \in (-1, n+1) \), in contrast to the domain outside this interval. In particular, the increasing and decreasing of this function was established rather difficult.

In this paper, we prove by elementary means a number of properties of the binomial coefficients \( \binom{r}{\alpha} \) of real arguments \( r, \alpha \in \mathbb{R}, \alpha \in (-1, r+1) \) (analogs of the properties of unimodality, symmetry, Pascal’s triangle, etc. for discrete binomial coefficients), which may be useful in further research (see, for example, [1]).

1. Preliminary information

The article uses the generally accepted concepts and notation of real analysis [2], as well as the standard concepts of combinatorial analysis [4]. Denote by \((a, b)\) the open real interval between the numbers \( a, b \in \mathbb{R} \), \( o(1) \) is an infinitesimal function in a neighborhood of \( \infty \), \( n! \) is the factorial of non-negative integer \( n \), i.e. \( n! = n(n-1)\cdots2\cdot1 \), and wherein we define \( 0! = 1 \), \( \binom{n}{m} \), where \( 0 \leq m \leq n \), is the (standard) binomial coefficient (with non-negative integer arguments \( n, m \) ), i.e.
\[
\binom{n}{m} = \frac{n!}{m!(n-m)!}.
\]
To denote asymptotic equality of real-valued functions \( f(x) \) and \( g(x) \) as \( x \to \infty \), we use the notation \( f(x) \sim g(x) \), which by definition means that \( f(x) = g(x)(1+r(x)) \) in some neighborhood of \( \infty \), where \( r(x) = o(1) \), or, equivalently (for functions positive in some neighborhood of \( \infty \))
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.
\]

The standard approach is considered, according to which the concept of factorial for non-negative integers extends to real (and even complex) numbers by the gamma function \( \Gamma(\alpha) \). We will use its definition in the following Euler-Gauss form [2, p. 393–394, 812]
\[
\Gamma(\alpha) = \lim_{n \to \infty} \frac{(n-1)! n^\alpha}{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}, \quad \alpha \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}.
\]
such a limit exists for any specified value of \( \alpha \) (see, for example, [5] or [2, p. 393]). In view of the problem statement, we do not consider extensions of the gamma function outside its standard domain of definition. Note that when defining the gamma function in the form of the Euler integral of the second kind

\[
\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx,
\]

converging for \( \alpha > 0 \), we obtain an equivalent definition on the interval \((0, \infty)\) [2, p. 811]. The gamma function \( \Gamma(\alpha) \) is continuous and has continuous derivatives of all orders on \((0, \infty)\), has no real roots, and it is positive on \((0, \infty)\). For any non-negative integer \( n \) the following equality holds

\[
\Gamma(1 + n) = n!,
\]  

(2)

moreover, the next reduction formula is valid [2, p. 394]

\[
\Gamma(1 + \alpha) = \alpha \Gamma(\alpha) , \quad \alpha \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}.
\]  

(3)

As a generalization of the discrete binomial coefficient \( \binom{n}{m} \), the binomial coefficient of real arguments is defined as follows (see, for example, [3])

\[
\binom{r}{\alpha} = \frac{\Gamma(1 + r) \Gamma(1 - \alpha)}{\Gamma(1 + \alpha) \Gamma(1 - r - \alpha)}.
\]  

(5)

Note that if \( r \in (-1, +\infty) \) and \( \alpha \in (-1, r + 1) \), the binomial coefficient \( \binom{r}{\alpha} \) is defined correctly by the equality (5).

2. Binomial coefficients \( \binom{r}{\alpha} \) for \( r, \alpha \in \mathbb{R} \)

**Theorem 1** (properties of the binomial coefficient of real arguments). Let \( r \in (-1, +\infty) \) and \( \alpha \in (-1, r + 1) \). Then

(i) \( \binom{r}{0} > 0 \), \( \binom{r}{0} = 1 \) and \( \binom{r}{r} = 1 \);

(ii) \( \binom{0}{\alpha} = \begin{cases} 1, & \text{if } \alpha = 0, \\ \frac{\sin \pi \alpha}{\pi \alpha}, & \text{if } \alpha \neq 0 \text{ and } \alpha \in (-1, 1); \end{cases} \)

(iii) \( \binom{r}{r-\alpha} = \binom{r}{\alpha} \);

(iv) \( \binom{r}{r} = \binom{r-1}{r-1} \), if \( r \in (0, +\infty) \) and \( \alpha \in (0, r) \);

(v) binomial coefficient \( \phi(\alpha) = \binom{r}{\alpha} \) is strictly increasing on the interval \((-1, \frac{r}{2}]\) and strictly decreasing on the interval \([\frac{r}{2}, r + 1)\);

(vi) binomial coefficient \( \psi(r) = \binom{r}{\alpha} \) is strictly increasing for \( \alpha > 0 \), strictly decreasing for \(-1 < \alpha < 0 \) and \( \psi(r) \equiv 1 \) if \( \alpha = 0 \).

**Proof.** Statement (i) follows from the relations (2), (5).

Prove (ii). If \( \alpha = 0 \), the required equality follows from (i). Further, we assume that \( \alpha \neq 0 \). Using the relations (2)–(5), we obtain

\[
\binom{0}{\alpha} = \frac{\Gamma(1)}{\Gamma(1 + \alpha) \Gamma(1 - \alpha)} = \frac{1}{\alpha \Gamma(\alpha) \Gamma(1 - \alpha)} = \frac{\sin \pi \alpha}{\pi \alpha}.
\]
Note that if \( \alpha \in (-1, r + 1) \), then \( r - \alpha \in (-1, r + 1) \). Therefore, the binomial coefficient \( \binom{r}{\alpha} \) is defined and the required equality from (iii) is satisfied due to (5). It is also easy to prove (iv) from (3) and (5).

Prove statement (v). Let \( \alpha, \beta \in (-1, r + 1) \). From (1) we obtain

\[
\frac{\Gamma(1 + \alpha) \Gamma(1 + r - \alpha)}{\Gamma(1 + \alpha) \Gamma(1 + r - \alpha)} = \lim_{n \to \infty} \frac{(n - 1)! (n - 1)! n^{2 + r}}{\prod_{i=1}^{n} (\alpha + i) (r - \alpha + i)}. 
\]

Hence,

\[
\frac{\phi(\alpha)}{\phi(\beta)} = \frac{\Gamma(1 + \beta) \Gamma(1 + r - \beta)}{\Gamma(1 + \alpha) \Gamma(1 + r - \alpha)} = \lim_{n \to \infty} \frac{n^{\alpha}}{\prod_{i=1}^{n} \delta_i(\alpha, \beta)}, \text{ where }
\]

\[
\delta_i(\alpha, \beta) = \frac{(\alpha + i)(r - \alpha + i)}{(\beta + i)(r - \beta + i)}. 
\]

Note that \( \delta_i(\alpha, \beta) > 0 \) for every \( \alpha, \beta \in (-1, r + 1) \) and \( i = 1, \ldots, n \). It is also easy to prove that

\[
\delta_i(\alpha, \beta) \geq 1 \iff f(\alpha) \geq f(\beta), \quad (7)
\]

where \( f(x) = -x^2 + xr \) and parabola \( f(x) \) is strictly increasing on \( (-\infty, \frac{1}{2}] \) as well as strictly decreasing on \( [\frac{1}{2}, \infty) \). Moreover, it is directly established that

\[
\delta_1(\alpha, \beta) = 1 + \varepsilon(\alpha, \beta), \quad \text{where } \varepsilon(\alpha, \beta) = \frac{(r - \alpha - \beta)(\alpha - \beta)}{(\beta + 1)(r - \beta + 1)}. \quad (8)
\]

Let \( -1 < \beta < \alpha < \frac{1}{2} \). Then \( f(\alpha) > f(\beta) \) and \( \varepsilon(\alpha, \beta) > 0 \). By virtue of (7), we have \( \delta_i(\alpha, \beta) \geq 1 \) for every \( i = 1, \ldots, n \). Hence, from (6) and (8) we obtain

\[
\frac{\phi(\alpha)}{\phi(\beta)} \geq \delta_1(\alpha, \beta) = 1 + \varepsilon(\alpha, \beta) > 1.
\]

Similarly, if \( \frac{1}{2} < \beta < \alpha < r + 1 \), then \( f(\alpha) < f(\beta) \) and \( \varepsilon(\alpha, \beta) < 0 \). Therefore, \( 0 < \delta_i(\alpha, \beta) < 1, i = 1, \ldots, n \) and

\[
\frac{\phi(\alpha)}{\phi(\beta)} \leq \delta_1(\alpha, \beta) = 1 + \varepsilon(\alpha, \beta) < 1.
\]

Prove statement (vi). In view of statement (i), we can assume that \( \alpha \neq 0 \). Let \( r < r' \). Note that \( 1 + r + i > 0, 1 + r' + i > 0 \) and \( \alpha/(1 + r + i) < 1, \alpha/(1 + r' + i) < 1 \) for every \( i \geq 0 \). From (1) we obtain

\[
\frac{\Gamma(1 + r)}{\Gamma(1 + r - \alpha)} = \left( 1 - \frac{\alpha}{1 + r} \right) \lim_{n \to \infty} n^{\alpha} \prod_{i=1}^{n-1} \left( 1 - \frac{\alpha}{1 + r + i} \right).
\]

Hence, \( \Gamma(1+r)/\Gamma(1+r-\alpha) < \Gamma(1+r')/\Gamma(1+r'-\alpha) \) for \( \alpha > 0 \) (and the reverse strict inequality holds for \( \alpha < 0 \)). In view of (5), we conclude \( \psi(r) < \psi(r') \) (respectively \( \psi(r) > \psi(r') \) for \( \alpha < 0 \)).

**Proposition 1.** Let \( r \) takes real values, \( \alpha \in \mathbb{R} \) does not depend on \( r \) and \( 0 < \alpha < 1 \). Then the following asymptotic equality is valid as \( r \) tends to infinity

\[
\binom{r}{ra} \sim \sqrt{\frac{1}{2\pi(1-\alpha)r}} \left( \frac{1}{\alpha} \right)^{ar} \left( \frac{1}{1-\alpha} \right)^{(1-\alpha)r}.
\]
Proposition 2. The coefficients of real arguments. Its justification is based on the properties of the gamma function and binomial transformations the asymptotic equality (9) from (10).

Proof. The required equality is proved in Theorem 1 for $n = 0$, and for $\alpha \in \{0, 1, \ldots, n\}$ it follows from the property of the gamma function (2). Let now $n \geq 1$ and $\alpha \notin \{0, 1, \ldots, n\}$. Suppose that $n - i - \alpha \in \{0, -1, -2, \ldots\}$ for some $i \in \mathbb{N}$ and $0 \leq i \leq n - 1$. Then $\alpha - (n - i) \in \{0, 1, 2, \ldots\}$. Hence, $\alpha \in \mathbb{N}$ and therefore $\alpha \in \{0, 1, \ldots, n\}$, got a contradiction. Thus, $n - i - \alpha \notin \{0, -1, -2, \ldots\}$ for every $i = 0, 1, \ldots, n - 1$. By virtue of the reduction formula (3), we have

$$\Gamma(1 + n - \alpha) = (n - \alpha)\Gamma(n - \alpha) = \ldots = \prod_{i=0}^{n-1}(n - i - \alpha)\Gamma(1 - \alpha).$$

Since $\alpha \in \mathbb{R} \setminus \{0, \pm 1, \pm 2, \ldots\}$, from (3)-(5) we obtain the required expression for the binomial coefficient $\binom{n}{\alpha}$.
References

[1] T.I. Fedoryaeva, *Logarithmic asymptotic of the number of central vertices of almost all n-vertex graphs of diameter k*, Siber. Electr. Math. Reports, to appear.

[2] G.M. Fikhtengol’ts, *Course of Differential and Integral Calculus Volume 2*, Fizmatlit, Moscow, 2003. ISBN 5-9221-0157-9

[3] D. Fowler, *The Binomial Coefficient Function*, The American Mathematical Monthly, Vol. 103, No. 1 (Jan., 1996), pp. 1–17. DOI.org/10.2307/2975209 Zbl 0857.05003

[4] R.L. Graham, D.E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994. Zbl 0836.00001

[5] J.L.W.V. Jensen and T.H. Gronwall, *An Elementary Exposition of the Theory of the Gamma Function*, Annals of Mathematics, Mar., 1916, Second Series, Vol. 17, No. 3 (Mar., 1916), pp. 124–166. DOI.org/10.2307/2007272 Zbl 46.0563.02

[6] S.T. Smith, *The binomial coefficient \( \binom{n}{x} \) for arbitrary \( x \)*, Online Journal of Analytic Combinatorics, December 2020. https://hosted.math.rochester.edu/ojac/vol15/176.pdf Zbl 1468.11069