Mobile vector soliton in a spin–orbit coupled spin-1 condensate

Sandeep Gautam and S K Adhikari

Instituto de Física Teórica, Universidade Estadual Paulista—UNESP, 01.140-070 São Paulo, São Paulo, Brazil

E-mail: sandeepgautam24@gmail.com and adhikari44@yahoo.com

Received 6 February 2015
Accepted for publication 16 February 2015
Published 20 March 2015

Abstract

We study the formation of bound states and three-component bright vector solitons in a quasi-one-dimensional spin–orbit-coupled hyperfine spin $f = 1$ Bose–Einstein condensate using numerical solution and variational approximation of a mean-field model. In the antiferromagnetic domain, the solutions are time-reversal symmetric, and the component densities have multi-peak structure. In the ferromagnetic domain, the solutions violate time-reversal symmetry, and the component densities have single-peak structure. The dynamics of the system are not Galilean invariant. From an analysis of Galilean invariance, we establish that the single-peak ferromagnetic vector solitons are true solitons and can move maintaining constant component densities, whereas the antiferromagnetic solitons cannot move with constant component densities.

Keywords: spinor condensate, spin–orbit coupling, vector solitons

(Some figures may appear in colour only in the online journal)

1. Introduction

A bright soliton is a self-reinforcing solitary wave that can traverse at a constant velocity without changing its shape due to a cancelation of the non-linear and dispersive interactions. The various systems in which solitons have been studied include water waves, non-linear optics, Bose–Einstein condensates (BECs), etc [1]. Solitons have been observed by manipulating the non-linear interaction near a Feshbach resonance [2] in a BEC of $^7$Li [3] and $^{85}$Rb [4]. Solitons have also been studied in binary BECs [5].

In a neutral spinor BEC with a nonzero hyperfine spin $f$, there is no spin–orbit (SO) coupling between the spin of the atoms and their center-of-mass motion [6]. However, a synthetic SO coupling can be realized in a spinor BEC by controlling the atom-light interaction leading to the generation of artificial Abelian and non-Abelian gauge potentials coupled to the atoms [7]. Solitons have been extensively studied in spinor BECs without SO coupling [8]. An SO coupling with equal Rashba [9] and Dresselhaus [10] strengths was realized experimentally by Raman dressing two atomic spin states with a pair of lasers [11]. In that study, the SO coupling between two of the three spin components of the $f = 1$ state $5S_{1/2}$ of $^{87}$Rb—the so-called pseudospin-1/2 state—was considered. There are other experimental studies on SO-coupled spinor BECs [12].

Solitonic structures in SO-coupled pseudospin-1/2 [13, 14] and spin-1 BECs [15] have been investigated theoretically. In this letter, we study two types of three-component vector solitons in an SO-coupled spin-1 BEC in a quasi-one-dimensional trap [16] with multi-peak or single-peak structure using a mean-field coupled Gross–Pitaevskii (GP) equation. A spin-1 spinor BEC is characterized by two interaction strengths, namely $c_0 \propto (a_0 + 2a_2)/3$ and $c_2 \propto (a_2 - a_0)/3$, where $a_0$ and $a_2$ are $s$-wave scattering lengths in total spin $f_{tot} = 0$ and 2 channels respectively [17]. For $c_2 > 0$ (antiferromagnetic) the multi-peak structure emerges, whereas for $c_2 < 0$ (ferromagnetic) the single-peak structure emerges. We use a variational method to determine the bright soliton solutions for the SO-coupled trapped BEC in each of the two domains. The appropriate variational ansatz in each of the domains is constructed using the solutions of the SO-coupled single particle Hamiltonian. The variational analysis provides
the necessary and sufficient conditions which \( c_0 \) and \( c_2 \) must satisfy to obtain a stable bright soliton. We compare the variational results with the numerical solution of the GP equation.

In [15], only antiferromagnetic multi-peak solitons for \( c_2 > 0 \) were identified as the bright solitons in a three-component spin-1 SO-coupled BEC. These solitons are time-reversal symmetric, but are not true vector solitons as they cannot propagate maintaining the shape of the individual components. We demonstrate that this system can support ferromagnetic single-peak solitons for \( c_2 < 0 \), provided that \( c_0 + c_2 < 0 \). These solitons break the time-reversal symmetry of the Hamiltonian. Nevertheless, they are shown to be true vector solitons as they can propagate with a constant velocity maintaining the shape of the individual components.

2. Spin–orbit-coupled BEC in quasi-1D trap

We consider an SO-coupled spinor condensate in a quasi-1D trap in which the trapping frequencies along the \( y \) and \( z \) axes (\( \omega_y \) and \( \omega_z \)) are much larger than that along the \( x \) axis (\( \omega_x \)) [16]. The single particle Hamiltonian of the condensate with equal strengths of Rashba [9] and Dresselhaus [10] SO-couplings in such a quasi-1D trap is [18]

\[
H_0 = \frac{p_x^2}{2m} + V(x) + y \gamma \Sigma_x, \tag{1}
\]

where \( p_x = -i\hbar \partial / \partial x \) is the momentum operator along \( x \) axis, \( V(x) = m_0 \omega_x^2 x^2 / 2 \) is the harmonic trapping potential along \( x \) axis, and \( \Sigma_x \) is the irreducible representation of the \( x \) component of the spin matrix:

\[
\Sigma_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2}
\]

This SO-coupling is distinct from a previous coupling [19, 20] used in the study of a quasi-1D BEC.

Using the single particle model Hamiltonian (1) and considering interactions in the Hartree approximation, a quasi-1D [16] spin-1 BEC can be described by the following set of three coupled mean-field differential equations for the wavefunction components \( \psi_j \) [17, 21]

\[
\begin{align*}
\frac{i}{\hbar} \frac{\partial \psi_{\pm 1}}{\partial \xi} &= \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) + c_0 \rho \right) \psi_{\pm 1} - \frac{i}{\sqrt{2}} \hbar \gamma \frac{\partial \psi_0}{\partial x} \\
&+ c_2 (\rho_{\pm 1} + \rho_0 - \rho_{\mp 1}) \psi_{\pm 1} + c_2 \rho_{\pm 1} \psi_{\mp 1}, \\
\frac{i}{\hbar} \frac{\partial \psi_0}{\partial \xi} &= \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) + c_0 \rho \right) \psi_0 - \frac{i}{\sqrt{2}} \hbar \gamma \frac{\partial \psi_{\pm 1}}{\partial x} \\
&+ \frac{\partial \psi_{\pm 1}}{\partial x} + c_2 (\rho_{\pm 1} + \rho_0 - \rho_{\mp 1}) \psi_0 + 2 c_2 \rho_{\pm 1} \psi_{\mp 1}, \tag{3}
\end{align*}
\]

where \( c_0 = 2 \hbar^2 (a_0 + a_2) / (3m \ell_x^2) \), \( c_2 = 2 \hbar^2 (a_2 - a_0) / (3m \ell_x^2) \), \( a_0 \) and \( a_2 \) are the \( s \)-wave scattering lengths in the total spin \( j_{tot} = 0 \) and 2 channels, respectively. \( \rho_j = |\psi_j|^2 \) with \( j = 1, 0, -1 \) are the component densities, \( \rho(x) = \sum_{j=-1}^{1} \rho_j \) is the total density, and \( L_z = \sqrt{\hbar (\omega_{z0} \ell_z)} \) with \( \omega_{z0} = \sqrt{\omega_x \omega_z} \) is the oscillator length in the transverse \( y-z \) plane. For the sake of simplicity, let us transform equations (3) and (4) into dimensionless form using

\[
\tilde{\tau} = \omega_x \tau, \quad \tilde{x} = \frac{x}{L}, \quad \phi_{\pm 1, 0}(\tilde{x}, \tilde{\tau}) = \frac{1}{\sqrt{\omega_x \ell_x N}} \psi_{\pm 1, 0}(x, \tau), \tag{5}
\]

where \( L_x = \sqrt{\hbar (\omega_{z0} \ell_x)} \) is the oscillator length along \( x \) axis, and \( N \) is the total number of atoms:

\[
\begin{align*}
\frac{i}{\hbar} \frac{\partial \psi_{\pm 1}}{\partial \tilde{\tau}} &= \left( -\frac{\hbar^2}{2 \omega_x \ell_x^2} + \tilde{V}(\tilde{x}) + \tilde{c}_0 \tilde{\rho} \right) \psi_{\pm 1} - \frac{i}{\sqrt{2}} \frac{\partial \psi_0}{\partial \tilde{x}} \\
&+ c_2 (\tilde{\rho}_{\pm 1} + \tilde{\rho}_0 - \tilde{\rho}_{\mp 1}) \psi_{\pm 1} + c_2 \tilde{\rho}_{\pm 1} \psi_{\mp 1}, \\
\frac{i}{\hbar} \frac{\partial \psi_0}{\partial \tilde{\tau}} &= \left( -\frac{\hbar^2}{2 \omega_x \ell_x^2} + \tilde{V}(\tilde{x}) + \tilde{c}_0 \tilde{\rho} \right) \psi_0 - \frac{i}{\sqrt{2}} \frac{\partial \psi_{\pm 1}}{\partial \tilde{x}} \\
&+ \frac{\partial \psi_{\pm 1}}{\partial \tilde{x}} + c_2 (\tilde{\rho}_{\pm 1} + \tilde{\rho}_0 - \tilde{\rho}_{\mp 1}) \psi_0 + 2 c_2 \psi_{\pm 1} \psi_{\mp 1}, \tag{6}
\end{align*}
\]

where \( \tilde{V} = \tilde{x}^2 / 2, \; \tilde{\rho} = \tilde{c}_0 L_x^2 / (\omega_{z0} \ell_x), \; \tilde{\rho}_0 = 2 N (a_0 + a_2) \ell_x^2 / (3 \ell_x^2), \; \tilde{\rho}_{\pm 1} = 2 N (a_2 - a_0) \ell_x^2 / (3 \ell_x^2), \; \tilde{\rho}_{\pm 1} = \tilde{\rho}_0 \tilde{c}_2^j \psi_j^2 \) with \( j = 1, 0, -1 \), and \( \tilde{c}_2 = 2 \hbar^2 (a_2 - a_0) / (3m \ell_x^2) \). The total density is now normalized to unity, i.e. \( \int_{-\infty}^{\infty} \tilde{\rho}(\tilde{x}) d \tilde{x} = 1 \). We present the scaled variables without tildes in the rest of the letter for notational simplicity. For a non-interacting trapless system \( V(x) = c_0 = c_2 = 0 \), there are two linearly independent solutions of the SO-coupled set of equations (6) and (7) with the lowest energy \( E_{\text{min}} = -N \gamma^2 / 2 \):

\[
\begin{align*}
\Phi_1 &= e^{i \gamma x / 2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \\
\Phi_2 &= e^{-i \gamma x / 2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \tag{7}
\end{align*}
\]

where wave functions \( \Phi_1 \) and \( \Phi_2 \) are normalized to unity.

Hence, the most general solution of equations (6) and (7) for a non-interacting trapless system with a fixed density \( n \) is given by the linear superposition of \( \sqrt{n} \Phi_1 \) and \( \sqrt{n} \Phi_2 \):

\[
\begin{align*}
\sqrt{n} \Phi_1 &= \begin{pmatrix} \rho_1 \\ \rho_0 \\ \rho_{-1} \end{pmatrix}, \\
\sqrt{n} \Phi_2 &= \begin{pmatrix} \alpha_1 \rho_1 + \alpha_2 \rho_{-1} \\ \beta_0 \rho_1 \\ \beta_{-1} \rho_{-1} \end{pmatrix}, \tag{8}
\end{align*}
\]

where \( |\alpha_1|^2 + |\alpha_2|^2 = 1 \) to ensure that \( \Phi \) is normalized to unity.

The energy of the BEC in scaled units is

\[
E = N \int_{-\infty}^{\infty} \left( \frac{1}{2} \sum_{j=-1}^{1} \frac{d \rho_j}{d \xi}^2 - \frac{i \gamma}{\sqrt{2}} (\rho_1^* \Phi_1 + \rho_{-1}^* \Phi_{-1}) \frac{d \rho_0}{d \xi} \\
- \frac{i \gamma}{\sqrt{2}} \rho_0 (\rho_{-1} \Phi_1 + \rho_{1} \Phi_{-1}) + c_2 |\Psi_0|^2 \right) d \xi, \tag{9}
\]

where \( \Psi_0 \) is spin–density vector, whose three components \( F_s, F_y, F_z \) are defined as...
\[
F_x = \frac{1}{\sqrt{2}} [\phi_0(\phi_1^* + \phi_2^*) + \phi_0^*(\phi_1 + \phi_2)],
\]
\[
F_y = \frac{i}{\sqrt{2}} [\phi_0(\phi_1^* - \phi_2^*) + \phi_0^*(\phi_1 - \phi_2)],
\]
\[
F_z = |\phi|^2 - |\phi|^2.
\]
Hence, for the SO-coupled Hamiltonian with its general solution given by (9), we get
\[
F_x = n(\alpha_1^2 - |\alpha_1|^2),
\]
\[
F_y = F_z = 0.
\]

Also, the magnetization \( M = \int F_z dz \) for minimum energy solutions of the single-particle SO-coupled Hamiltonian.

Now, let us switch on the interactions; the interaction energy per particle for the uniform system is [21]
\[
\epsilon_{\text{int}} = \left[ \frac{c_0}{2} n + \frac{c_2}{2n} |F|^2 \right],
\]
\[
= \left[ \frac{c_0}{2} n + \frac{c_2}{2} n(|\alpha_1|^2 - |\alpha_1|^2) \right].
\]
If \( c_2 > 0 \), then the BEC is in the antiferromagnetic or polar phase, and the minimum of \( \epsilon_{\text{int}} \) corresponds to \( |\alpha_1| = |\alpha_2| = 1/\sqrt{2} \) leading to \( |F|/n = 0 \). In this case, the wave function (9) is time-reversal symmetric. On the other hand, for \( c_2 < 0 \), the BEC is in the ferromagnetic phase, and \( \epsilon_{\text{int}} \) can be minimized if \( |\alpha_1| = 1, |\alpha_2| = 0 \) or \( |\alpha_1| = 0, |\alpha_2| = 1 \), which leads to \( |F|/n = 1 \). This corresponds to the wave functions (8) apart from a multiplying phase factor. These states are degenerate, violate the time-reversal symmetry and are mutually connected by the time-reversal operator. These are the only two distinct structures which emerge as the ground states in the SO-coupled quasi-1D BECs. In a quasi-two-dimensional BEC with Rashba or Dresselhaus SO coupling, there is a circular degeneracy in the energy eigen functions of the single particle Hamiltonian [22]. Hence, depending upon the interaction parameters, more than two plane waves can superpose resulting in different types of lattice structures in ground state density profiles [23].

3. Bright solitons

3.1. Stationary bright solitons

Stationary bright solitons can emerge as the ground state of a spinor BEC with attractive interactions [13, 15]. We use a variational method to determine the bright soliton solutions of equations (6) and (7). As has been discussed in section 2, an SO-coupled spinor BEC can have two types of ground states depending upon the sign of \( c_2 \). This necessitates the use of two different variational ansatz in these two domains.

**Antiferromagnetic phase** (\( c_2 > 0 \)): here we consider the following variational ansatz to determine the shape of the soliton
\[
\Phi_{\text{var}} = \frac{\sqrt{\alpha}}{2} \left( \frac{\pm \cos(\gamma x)}{\mp \sqrt{2} i \sin(\gamma x)} \right) \text{sech}(\alpha x),
\]
where \( \alpha \) is a variational parameter and characterizes the width and the strength of the bright soliton. The ansatz (17) corresponds to the wave function (9) with \( \alpha_1 = \alpha_2 = \pm 1/\sqrt{2} \) multiplied by the localized spatial soliton \( \sqrt{\alpha_2^2 \text{sech}(\alpha x)} \) instead of \( \sqrt{\alpha} \). As the two solutions (8) are degenerate, and a mixing between them is allowed, the soliton profile could have a multi-peak structure. Noting that in the \( c_2 > 0 \) domain, for \( |\alpha_1| = |\alpha_2| = 1/\sqrt{2} \), one can have other choices for the variational ansatz like
\[
\Phi_{\text{var}} = \frac{\sqrt{\alpha}}{4} \left( \frac{e^{i\gamma x} \mp i e^{-i\gamma x}}{-\sqrt{2}(e^{i\gamma x} \mp i e^{-i\gamma x})} \right) \text{sech}(\alpha x),
\]
and then \( c_2 < 0 \). Hence, the SO-coupled spin-1 spinor BEC can support an antiferromagnetic bright soliton defined by (17) or (18) and (19), provided that \( c_0 < 0 \) and \( c_2 > 0 \). From (17) and (21) it is evident that the wavefunction of the bright soliton is independent of the strength of spin–exchange interactions \( c_2 \). This is expected since for \( c_2 > 0 \), there is no contribution to the energy from the \( c_2 \)-dependent term of the SO-coupled spinor BEC.

**Ferromagnetic phase** (\( c_2 < 0 \)): here we consider the following variational ansatz
\[
\Phi_{\text{var}} = \frac{\sqrt{\alpha}}{2 \sqrt{2}} \left( \frac{1}{\sqrt{2}} \right) \text{sech}(\alpha x),
\]
where \( \alpha \) is again, a variational parameter characterizing the width and the strength of the bright soliton. This variational ansatz corresponds to \( \alpha_1 = 1, \alpha_2 = 0 \) in (9) multiplied by the localized bright soliton \( \sqrt{\alpha_2^2 \text{sech}(\alpha x)} \) instead of \( \sqrt{\alpha} \). In this case the soliton will have a single peak. Also, the ansatz like \( -\Phi_{\text{var}} \) or \( \pm i \Phi_{\text{var}} \) are equally reasonable choices and correspond to \( \alpha_1 = -1, \alpha_2 = 0 \) and \( \alpha_1 = \pm i, \alpha_2 = 0 \), respectively, in (9). Substituting (22) in (10), the energy of the soliton is

\[
E = \frac{N}{6} (-3\gamma^2 + \sigma^2 + \alpha c_0).
\]
\[ E = \frac{1}{6}(-3\gamma^2 + \sigma^2 + \sigma c_0 + \sigma c_2). \]  

The minima of this energy occurs at
\[ \sigma = -\frac{1}{2}(c_0 + c_2), \]  

provided \( c_0 + c_2 < 0 \). Hence the SO-coupled spinor BEC can have a ferromagnetic soliton defined by (22) and (24), provided \( c_2 < 0 \) and \( c_0 + c_2 < 0 \). In this case, unlike in the case of an antiferromagnetic soliton, the bright soliton profile is sensitive to both \( c_0 \) and \( c_2 \).

3.2. Moving bright solitons

If \( \Phi \) is the static bright solitonic solution of the coupled equations (6) and (7), then the Galilean invariance of these equations ensures that a soliton moving with velocity \( v \) is defined as
\[ \Phi_d(x, t) = \Phi(x - vt, t)e^{i\sigma \cdot (x - vt)/v}, \]  

where \( \sigma \) characterizes the width and the strength of the soliton. The breakdown of the Galilean invariance of the SO-coupled equation can be explicitly seen by using the transformation \( x' = x + vt, \ t' = t \), where \( v \) is the velocity of the unprimed coordinate system with respect to the primed coordinate system, then the wavefunction \( \Phi \) of (6) and (7) should transform to \( \Phi_d \) as
\[ \Phi(x, t) = \Phi_d(x', t')e^{-i\sigma \cdot (x' - vt')/v}. \]  

Now, substituting (26) in (6) and (7) and using \( \partial \phi / \partial x = \partial \phi / \partial x' \) and \( \partial \phi / \partial t = \partial \phi / \partial t' + \nu \partial \phi / \partial x' \), we obtain
\[ i \frac{\partial \Phi_d(x', t')}{\partial t'} = \left[ -\frac{1}{2} \frac{\partial^2}{\partial x'^2} - \gamma \Sigma_j \left( \frac{1}{\sigma} \frac{\partial}{\partial x'} + \nu \right) \right] \Phi_d(x', t'). \]  

where the terms proportional to \( c_0 \) and \( c_2 \) have been suppressed for the sake of simplicity in addition to a \( \sigma \)-dependent additive term which does not contribute to the dynamics. The presence of the extra term \( -\gamma \Sigma_j \phi(x', t') \) on the right-hand side of (27) shows that the SO-coupled Hamiltonian is no longer Galilean invariant and the SO-coupled soliton solution of the GP equation will depend on its velocity \( \nu \). The SO-coupled equation (27), in the absence of trap and interactions, has the solutions \( \Phi_1 \) and \( \Phi_2 \) of equation (8) with energies \( E = -N(\gamma^2/2 - \gamma v) \) and \( E = -N(\gamma^2/2 + \gamma v) \), respectively. For \( \nu = 0 \), the two solutions (8) were degenerate, and this degeneracy has been removed in the case of the SO-coupled moving solitons. In the antiferromagnetic phase, a multi-peak solution was possible through a mixture of two degenerate solutions (8) for \( \nu = 0 \). For a nonzero \( \nu \), the degeneracy is removed and such a mixing is not possible. This means that the multi-peak soliton cannot propagate with a constant velocity maintaining its shape and energy. For the moving multi-peak soliton profile, the variational analysis of section 3.1 will no longer be valid. In the ferromagnetic phase, as a mixing between the two degenerate solutions is not allowed, one can only have a single-peak soliton which can propagate with a constant velocity maintaining its shape, and the variational analysis presented in section 3.1 remains valid.

4. Results and conclusions

We numerically solve the coupled equations (6) and (7) using the split-time-step Crank–Nicolson method [24, 25] with real- and imaginary-time propagations. The ground state is determined by solving (6) and (7) using imaginary-time propagation, which neither conserves norm nor magnetization. Both norm and magnetization can be fixed by transforming the wave-function components as
\[ \phi(x, \tau + \text{d} \tau) = d_j \phi(x, \tau), \]  

where \( d_j \) are defined as [20, 26]
\[ d_0 = \frac{\sqrt{1 - \mathcal{M}^2}}{\sqrt{N_0 + 4(1 - \mathcal{M}^2)N_0N_1 + \mathcal{M}^2N_0^2}}, \]  
\[ d_1 = \frac{1 + \mathcal{M} - c_0^2N_0}{2N_1}, \]  
\[ d_{-1} = \frac{1 - \mathcal{M} - c_0^2N_0}{2N_{1}}, \]

and here \( N_j = \int |\phi_j(x, \tau)|^2 \text{d}x \). These normalizations ensure simultaneous conservation of norm and magnetization after each iteration in imaginary time. The spatial and time steps used in the present work are \( \Delta x = 0.05 \) and \( \Delta t = 0.000125 \), respectively.

We consider an SO-coupled spin-1 spinor BEC of 10000 \(^{23}\text{Na} \) or \(^{87}\text{Rb} \) atoms trapped in a harmonic trapping potential with \( a_s/(2\pi) = 20 \) Hz and \( a_s/(2\pi) = 15.8/(2\pi) = 2.919 \) Hz. Similarly, the scattering lengths of \(^{23}\text{Na} \) or \(^{87}\text{Rb} \) atoms trapped in a harmonic trapping potential with \( a_s/(2\pi) = 2.646 \) nm and \( a_s/(2\pi) = 7.76 \) nm are \( a_s/(2\pi) = 2.646 \) nm and \( a_s/(2\pi) = 7.76 \) nm, respectively. The oscillator lengths for \(^{23}\text{Na} \) with these parameters are \( l_0 = 4.69 \mu m \) and \( l_{xy} = 1.05 \mu m \), whereas those for \(^{87}\text{Rb} \) are \( l_0 = 2.41 \mu m \) and \( l_{xy} = 0.54 \mu m \). We use these values of \( l_0 \) for writing the dimensionless GP equations (6) and (7) for the trapped states, whereas for solitons \( l_0 = 4.69 \mu m \) in this letter. The scattering lengths of \(^{23}\text{Na} \) in total spin states \( J = 0 \) and \( 2 \) channels are \( a_{0} = 2.646 \text{ nm}, a_{2} = 2.919 \text{ nm}, \) respectively [26], resulting in \( c_0 = 241.28 \) and \( c_2 = 7.76 \). Similarly, the scattering lengths of \(^{87}\text{Rb} \) are \( a_{0} = 5.387 \text{ nm} \) and \( a_{2} = 5.313 \text{ nm} \) [26], leading to \( c_0 = 885.71 \) and \( c_2 = 4.09 \). In imaginary time propagation, we use a real Gaussian function multiplied by the solution of the single-particle SO-coupled Hamiltonian as the initial input for the component wavefunctions, i.e.

\[ \Phi_{\text{initial}} = \frac{e^{-\sigma^2/2}}{\sqrt{\sqrt{\pi}} \left( \sqrt{2} |\alpha_1| + \sqrt{2} |\alpha_2| \right)} \left( \frac{\alpha_1e^{i\gamma x} + \alpha_2e^{-i\gamma x}}{\alpha_1e^{i\gamma x} + \alpha_2e^{-i\gamma x}} \right). \]  

where \( |\alpha_1| = |\alpha_2| = 1/\sqrt{2} \) for \(^{23}\text{Na} \) and \( |\alpha_1| = 1, |\alpha_2| = 0 \) for \(^{87}\text{Rb} \). Hence, by using different values of \( |\alpha_1| \) and \( |\alpha_2| \) in (32), one can obtain different solutions corresponding to the
same density distribution and energy. For example, the two ground state solutions with $M = 0$ for $^{23}\text{Na}$ obtained by using $\alpha_1 = \alpha_2 = 1/\sqrt{2}$ and $\alpha_1 = 1/\sqrt{2}, \alpha_2 = -1/\sqrt{2}$ are shown in figures 1(a) and (b), respectively. In figures 1(a) and (b), only the non-zero real ($\mathcal{R}$) and imaginary ($\mathcal{I}$) parts of the component wavefunctions are shown. In these two cases, wavefunctions are either purely real or imaginary and not complex. On the other hand, the component wavefunctions in the ground state solution for $^{23}\text{Na}$ obtained by using $\alpha_1 = 1/\sqrt{2}, \alpha_2 = i/\sqrt{2}$ are complex with non-zero real and imaginary parts. The real and imaginary parts of the component wavefunctions in this case are shown in figures 1(c) and (d), respectively. The multi-peak density profile corresponding to these three solutions presented in figures 1(a)–(d) is the same and is shown in figure 1(g). The multi-peak nature of the solution in this case is consistent with analytic results obtained in section 2. The multi-peak solution effectively leads to a weak phase separation between $\rho_{\pm 1}$ and $\rho_0$. Here, weak phase separation implies that there are no local minima in the total density profile [27]. This is in contrast to the strong phase separation possible with the model of the SO coupling discussed in [19, 20], where a notch appears in the total density profile at the interface separating the components when $\gamma$ exceeds a critical value. The solutions illustrated in figures 1(a)–(d) are time-reversal symmetric. Similarly, the real and imaginary parts of the complex ground state solution with $M = 0$ for $^{87}\text{Rb}$ obtained with $\alpha_1 = 1/\sqrt{2}, \alpha_2 = -1/\sqrt{2}$ in equation (32) are shown in figures 1(e) and (f), which lead to the single-peak density distribution of figure 1(h). The solution presented in figures 1(e), (f), and (h) violates time-reversal symmetry, as there are two degenerate solutions in this case connected by the time-reversal operation.

In order to obtain the bright solitons in SO-coupled spinor BECs, we take $V(x) = 0$ in (6) and (7) and consider two cases: (a) $c_0 < 0, c_2 > 0$ and (b) $c_0 + c_2 < 0, c_2 < 0$. In case (a), we
consider $c_0 = -1.2$, $c_2 = 0.3$, and $\gamma = 1$. The numerically and variationally obtained bright solitons, defined by (17) and (21) with $M = 0$, are shown in figure 2(a). The multi-peak solution in this case is time-reversal symmetric. In case (b), we consider $c_0 = -1.5$, $c_2 = -0.3$, and $\gamma = 1$. The numerical and variational solutions, defined by (22) and (24), in this case are shown in figure 2(b).

The single-peak solution in this case breaks time-reversal symmetry of the Hamiltonian. It is evident from figure 2 that there is an excellent agreement between the numerical and variational results.

In order to study the dynamics of the moving solitons, we first generate the stationary solitons numerically using imaginary-time propagation for both antiferromagnetic and ferromagnetic interactions. In order to set these solitons into motion with a constant velocity $v = 0.2$, we multiply the wavefunction components for the stationary soliton with $\exp(i0.2x)$, and then use real-time propagation to study its evolution. We observe that in the case of the antiferromagnetic soliton, there are spin–mixing dynamics due to which the component densities are not conserved as the soliton moves. This is evident from figure 3(a) and its inset, which show the dynamics of the antiferromagnetic soliton initially located at $x = -10$ and the spin–mixing dynamics, respectively; the interaction parameters are the same as those in figure 2(a). At $t = 0$ the soliton is set into motion at a constant velocity. As the soliton moves, component densities keep on changing without any change in the total density. On the other hand, if one starts with the ferromagnetic soliton at $t = 0$, the component densities and hence the total density do not change while the soliton is moving. This is shown in figure 3(b) for the soliton initially located at $x = -10$ and with the same interaction parameters as in figure 2(b). This is consistent with the analytic results of section 3.2.

5. Summary

We study the generation and propagation of a vector soliton with three components in an SO-coupled spin-1 BEC with either antiferromagnetic or ferromagnetic interactions. In the antiferromagnetic case, the solutions are time-reversal symmetric and the component densities have multi-peak structure. In the ferromagnetic case, the solutions violate time-reversal symmetry and the component densities have single-peak structure. The GP equation for this system is not Galilean invariant. From an analysis of the Galileian invariance of this equation, we establish that the single-peak ferromagnetic SO-coupled solitons can move with constant component densities and are true solitons, whereas the multi-peak antiferromagnetic SO-coupled solitons change the component densities during motion.

Acknowledgments

This work is financed by the Fundação de Amparo à Pesquisa do Estado de São Paulo (Brazil) under Grants Nos. 2013/07213-0, 2012/00451-0 and also by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (Brazil).
References

[1] Kivshar Y S, Malomed B A 1989 Rev. Mod. Phys. 61 763
Abdullaev F K, Gammal A, Kamchatnov A M and Tomio L 2005 Int. J. Mod. Phys. B 19 3415
[2] Inouye S et al 1998 Nature 392 151
[3] Strecker K E, Partridge G B, Truscott A G, Hulet R G 2002 Nature 417 150
Khaykovich L, Schreck F, Ferrari G, Bourdel T, Cubizolles J, Carr L D, Castin Y and Salomon C 2002 Science 256 1290
[4] Cornish S L, Thompson S T and Wieman C E 2006 Phys. Rev. Lett. 96 170401
[5] Pérez-García V M and Beitia J B 2005 Phys. Rev. A 72 033620
Adhikari S K 2005 Phys. Lett. A 346 179
Adhikari S K 2005 Phys. Rev. A 72 053608
Salasnich L and Malomed B A 2006 Phys. Rev. A 74 053610
Salasnich L and Malomed B A 2006 Phys. Rev. A 74 053610
[6] Li Y, Martone G I and Stringari S arXiv:1410.5526
Galitski V and Spielman I B 2013 Nature 494 49
Osterloh K, Baig M, Santos L, Zoller P and Lewenstein M 2005 Phys. Rev. Lett. 95 010403
Ruseckas J, Juzeliūnas G, Öhberg P and Fleischhauer M 2005 Phys. Rev. Lett. 95 010404
Juzeliūnas G, Ruseckas J and Dalibard J 2010 Phys. Rev. A 81 053403
Lan Z and Öhberg P 2011 Rev. Mod. Phys. 83 1523
Zhang X-F, Biao L and Zhang S-G 2013 Laser Phys. 23 105501
[8] Zhang X-F, Biao L and Zhang S-G 2013 Laser Phys. 23 105501
Ieda J, Miyakawa T and Wadati M 2004 Phys. Rev. Lett. 93 194102
Li L, Li Z, Malomed B A, Mihalache D and Liu W M 2005 Phys. Rev. A 72 033611
Zhang W, Münstercapriolü Ö E and You L 2007 Phys. Rev. A 75 043601
Dąbrowska-Wüster B J, Ostrovskaya E A, Alexander T J and Kivshar Y S 2007 Phys. Rev. A 75 023617
Doktorov E V, Wang J and Yang J 2008 Phys. Rev. A 77 043617
Xiong B and Gong J 2010 Phys. Rev. A 81 033618
Szankowski P, Tripenbach M, Infeld E and Rowlands G 2010 Phys. Rev. Lett. 105 125302
Mobarak M and Pelster A 2013 Laser Phys. Lett. 10 115501

Topic O et al 2010 Laser Phys. 20 1156
Guileumus M, Julia-Díaz B, Melle-Messeguer M and Polls A 2010 Laser Phys. 20 1163
[9] Bychkov Y A and Rashba E I 1984 J. Phys. C 17 6039
[10] Dresselhaus G 1955 Phys. Rev. 100 880
[11] Lin Y-J, Jiménez-García K and Spielman I B 2011 Nature 471 83
[12] Aidelburger M et al 2011 Phys. Rev. Lett. 107 255301
Fu Z, Wang P, Chai S, Huang L and Zhang J 2011 Phys. Rev. A 84 043609
Zhang J-Y et al 2012 Phys. Rev. Lett. 109 115301
Qu C, Hamner C, Gong M, Zhang C and Engels P 2013 Phys. Rev. A 88 021604
[13] Xu Y, Zhang Y and Wu B 2013 Phys. Rev. A 87 013614
[14] Salasnich L and Malomed B A 2013 Phys. Rev. A 87 063625
Salasnich L, Cardoso W B and Malomed B A 2014 Phys. Rev. A 90 033629
Cao S, Shan C-J, Zhang D-W, Qin X and Xu J 2015 J. Opt. Soc. Am. B 32 201
[15] Liu Y-K and Yang S-J 2014 Eur. Phys. Lett. 108 30004
[16] Salasnich L, Parola A and Reatto L 2002 Phys. Rev. A 65 043614
[17] Ohmi T and Machida K 1998 J. Phys. Soc. Japan 67 1822
Ho T L 1998 Phys. Rev. Lett. 81 742
[18] Zhai H 2012 Int. J. Mod. Phys. B 26 1230001
[19] Gautam S and Adhikari S K 2015 Phys. Rev. A 91 013624
[20] Gautam S and Adhikari S K 2014 Phys. Rev. A 90 043619
[21] Kawaguchi Y and Ueda M 2012 Phys. Rep. 520 253
[22] Wang C, Gao C, Jian C-M and Zhai H 2010 Phys. Rev. Lett. 105 160403
[23] Ruokokoski E, Huhtamäki J A M and Möttönen M 2012 Phys. Rev. A 86 051607
Xu Z F, Kawaguchi Y, You L and Ueda M 2012 Phys. Rev. A 86 033628
[24] Wang H 2011 J. Comput. Phys. 230 6155
Wang H 2014 J. Comput. Phys. 274 473
[25] Muruganandam P, Adhikari S K 2009 Comput. Phys. Commun. 180 1888
Vudragovic D, Vidanovic I, Balaz A, Muruganandam P and Adhikari S K 2012 Comput. Phys. Commun. 183 2021
[26] Bao W, Lim F Y 2008 SIAM J. Sci. Comput. 30 1925
Lim F Y and Bao W 2008 Phys. Rev. E 78 066704
[27] Ao P and Chui S T 1998 Phys. Rev. A 58 4836

7