On the Representation Number of Bipartite Graphs

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Word-representable graph

Definition
A simple graph $G = (V, E)$ is said to be a *word-representable graph* if it can be presented by a word over $V$. Any two vertices $a$ and $b$ are adjacent if and only if $abababa...$ or $bababab...$ appear in the word $w$.

$w = 413423124$ represents the graph $C_4$.

- **41414** represents 1 and 4 are adjacent.
- **1331** represents that 1 and 3 are non-adjacent vertices.
**k-word-representable graph**

- If every letters in \( w \) appear exactly \( k \) times then \( w \) is said to be \( k \)-uniform.

**Definition**

A graph \( G \) is said to be a *k-word-representable graph* if there exist a \( k \)-uniform word that represents it.

**Figure 2**: A Graph \( G \).

- \( 123312 \) represents \( G \).
- \( G \) is a 2-word-representable graph.
Proposition (Kitaev and Pyatkin (2008))

A $k$-word-representable graph $G$ is also $(k + 1)$-word-representable graph. In particular, each word-representable graph has infinitely many word-representants.
Definition

The minimal $k$ such that a graph $G$ is a $k$-word-representable graph is called the *representation number* of $G$ and is denoted by $\mathcal{R}(G)$.

- $123$ represents $K_3$.
- Any complete graph, $K_n$, on $n$ vertices can be represented by a permutation. So, they have representation number 1.

Figure 3: Graph $K_3$. 

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Permutationally representable graph

Definition

A graph $G = (V, E)$ is said to be a \emph{permutationally representable graph} if it can be represented by a word of the form $w = p_1 p_2 \ldots p_k$, where $p_i$ is a permutation of $V$, for all $i$ where $1 \leq i \leq k$. We say $G$ is \emph{permutationally $k$-representable}.

Figure 4: Cycle $C_4$.

- $1324 \ 3142$ represents the graph $C_4$.
- $C_4$ is permutationally 2-representable.
**Permutation representation number**

**Definition**

The minimal $k$ such that a graph $G$ is a permutationally $k$-representable graph is called the *permutation representation number of $G$*, and it is denoted by $\mathcal{R}^p(G)$.

**Remark**

- For any permutationally representable graph $G$, $\mathcal{R}(G) \leq \mathcal{R}^p(G)$.
- For all $n \geq 1$, $\mathcal{R}^p(K_n) = 1$. 
Example

\[
\begin{array}{c}
\text{Figure 5 : } K_{3,4} \\
\end{array}
\]

Remark

- Any complete bipartite graph \( K_{m,n} \), \( \mathcal{R}^p(K_{m,n}) = 2 \).
Semi-transitive Orientation

Definition

A graph $G = (V, E)$ is said to be semi-transitive if it admits a semi-transitive orientation, i.e., $G$ admits an acyclic orientation such that for any directed path $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n \forall v_i, \in V$ where $1 \leq i \leq n$, either

- there is no edge from $v_1$ to $v_n$, or

- if there is an edge from $v_1$ to $v_n$ then there are edges from $v_i$ to $v_j$, $\forall 1 \leq i < j \leq n$. 
Figure 6: Semi-transitive orientation of $G$. 

$G$

$\phi(G)$
Theorem (Halldórsson et al. (2016))

A graph is word-representable if and only if it admits a semi-transitive orientation.

Theorem (Halldórsson et al. (2016))

Each non-complete word-representable graph $G$ on $n$ vertices is $2(n - \mathcal{K}(G))$-word-representable, where $\mathcal{K}(G)$ is the size of the maximum clique in $G$.

Corollary

If $G$ is non-complete word-representable on $n$ vertices, then $\mathcal{R}(G) \leq 2(n - \mathcal{K}(G))$. 
The recognition problem for word-representable graph is decidable.

In Halldórsson et al. (2016), it is established that the recognition problem for word-representable graph is NP-complete.
Classes of graphs which are word-representable

- All graphs on atmost 5-vertices are word-representable.
- Complete graphs.
- Comparability graphs.
- There are classes of graphs that contain both word-representable and non-word-representable graphs: 4-colourable graphs, perfect graphs, etc.
Comparability graphs

Definition

A graph \( G = (V, E) \) is a comparability graph if it admits a transitive orientation, i.e, an orientation in which if \( \vec{xy} \) and \( \vec{yz} \) are directed edges, then \( \vec{xz} \) is a directed edge, for all \( x, y, z \in V \).

\[ H \]

\[ \phi(H) \]

\textbf{Figure 7 :} Transitive orientation of \( H \).
Theorem (Kitaev and Seif (2008))

A graph is permutationally representable if and only if it is a comparability graph.

- Every comparability graph corresponds to a partially ordered set (poset).
- In Yannakakis (1982), the problem of finding the dimension of a poset is NP-hard.

Result (Kitaev and Lozin (2015))

A comparability graph $H$ is permutationally $k$-representable if and only if the poset induced by this graph has dimension at most $k$. 
Definition

Crown graphs are graphs which are obtained from a complete bipartite graph by removing a perfect matching.

Figure 8: $K_{3,3}$

Figure 9: $H_{3,3}$
Theorem (Halldorsson et al. (2011))

The permutation representation number of a crown graph $H_{n,n}$ is $n$.

Theorem (Glen et al. (2018))

The representation number of a crown graph $H_{n,n}$ is $\left\lceil \frac{n}{2} \right\rceil$.

Conjecture (Glen et al. (2018))

Every bipartite graph on $n$ vertices has representation number at most $\left\lceil \frac{n}{4} \right\rceil$. 
Main Result

- Every bipartite graph is a comparability graph.
- The class of bipartite graph is precisely the class of comparability graphs that are isomorphic to the Hasse diagram of the corresponding posets.
- We devise an algorithmic procedure that works in polynomial time to construct a word representing permutationally a given bipartite graph.
Relabeling Algorithm

1. Given a bipartite graph $G = (V, E)$ where $V = V_1 \cup V_2$, consider the set, say $V_1$, with minimum number of vertices.
2. Check whether the graph is a complete bipartite graph. If yes, exit.
3. Else, from the set $V_1$ choose a vertex with at least one non adjacent vertex and relabel it first.
4. Relabel the rest of vertices in $V_1$ and with respect to the relabeling done in $V_1$, we relabel the vertices in $V_2$ accordingly.
5. We then create permutations (linear orders) with respect to the vertices in $V_1$.
6. Concatenate all the permutations and relabel the vertices to their original label.
7. The word produced represents the bipartite graph $G$, permutationally.
1. Given a bipartite graph $G$, suppose $V_1 = \{a_1, \ldots, a_m\}$ and $V_2 = \{b_1, \ldots, b_n\}$.

Figure 10: A bipartite graph $G$
2. If \( N(a) = V_2 \) for all \( a \in V_1 \), then consider the following word and exit:

\[
w = a_1 a_2 \cdots a_m b_1 b_2 \cdots b_n a_m a_{m-1} \cdots a_1 b_n b_{n-1} \cdots b_1.
\]

3. Else, choose \( a \) in \( V_1 \) such that \( V_2 \setminus N(a) \neq \emptyset \) and label it as \( c_1 \).

Choosing \( a \) to be \( a_4 \).
4. Relabel the remaining vertices of $V_1$ as $c_2, \ldots, c_m$, arbitrarily.
5. Relabel the vertices in $V_2$ such that the vertices which are not adjacent to $c_1$ are relabeled first, then the vertices not adjacent to $c_2$ and so on. Lastly, we relabel the remaining vertices in $V_2$. 

![Diagram of vertices and edges]

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6. Create a list of permutations of the vertices $c_1, \ldots, c_{m+n}$ as per the following:

$$w_1 = c_m c_{m-1} \cdots c_2 c_{m+1} c_{m+2} \cdots c_{m+k_1} c_1 c_{m+k_1+1} c_{m+k_1+2} \cdots c_{m+n}$$

![Graph Diagram]

$$w_1 = c_4 c_3 c_2 c_5 c_6 c_7 c_1 c_8 c_9$$
For $i = 2$ to $m$,

If $V_2 \setminus N(c_i) \neq \emptyset$, then set

$$w_i = c_1 \cdots c_{i-1} c_{i+1} \cdots c_m \text{Dec}(V_2 \setminus N(c_i)) c_i \text{Dec}(N(c_i));$$

else, set $w_i = \varepsilon$. 

$$W_2 = c_1 c_3 c_4 c_6 c_5 c_2 c_9 c_8 c_7$$
$$W_3 = \varepsilon$$
$$W_4 = \varepsilon$$
7. If \( N(c) = V_2 \) for some \( c \in V_1 \) then set
\[
\omega_{m+1} = c_1 c_2 \cdots c_m c_{m+n} \cdots c_{m+1};
\]
ext else, set \( \omega_{m+1} = \varepsilon \).

\[ W_5 = c_1 c_2 c_3 c_4 c_9 c_8 c_7 c_6 c_5 \]
8. Concatenate the permutations \( w_1, w_2, \ldots w_m, w_{m+1} \) to form
\[ w' = w_1 w_2 \cdots w_m w_{m+1} \]
which permutationally represents the relabeled graph of \( G \).

\[ w' = \overbrace{c_4 c_3 c_2 c_5 c_6 c_7 c_1 c_8 c_9}^{c_1 c_3 c_4 c_6 c_5 c_2 c_9 c_8 c_7} \overbrace{c_1 c_2 c_3 c_4 c_9 c_8 c_7 c_6 c_5}^{c_1 c_2 c_3 c_4 c_9 c_8 c_7 c_6 c_5} \]

9. Replacing the original labels of the vertices of \( G \) in the word \( w' \).

\[ a_1 a_3 a_2 b_5 b_4 b_1 a_4 b_3 b_2 a_4 a_3 a_1 b_4 b_5 a_2 b_2 b_3 b_1 a_4 a_2 a_3 a_1 b_2 b_3 b_1 b_4 b_5 \]

Remark

*The relabeling algorithm works in \( O(mn) \).*
Theorem (Correctness of the relabeling algorithm)

The word $w$ generated by the relabeling algorithm represents the bipartite graph $G$.

Proof: Suppose $G = (V_1 \cup V_2, E)$ such that $m = |V_1| \leq |V_2| = n$. Let $G'$ represent the relabeled graph. We prove that $a$ and $b$ are adjacent in $G'$ if and only if $a$ and $b$ alternate in the word $w'$.

- **Case 1:** $a$ and $b$ are adjacent vertices in $G'$.
  - $w_1 = c_mc_{m-1} \cdots c_2c_{m+1}c_{m+2} \cdots c_{m+k_1}c_1c_{m+k_1+1}c_{m+k_1+2} \cdots c_{m+n}$
  - If $w_i \neq \varepsilon$ for $i = 2$ to $m$,
    $w_i = c_1 \cdots c_{i-1}c_{i+1} \cdots c_{m}\text{Dec}(V_2 \setminus N(c_i))c_i\text{Dec}(N(c_i))$
  - If $w_{m+1} \neq \varepsilon$
    $w_{m+1} = c_1c_2 \cdots c_mc_{m+n} \cdots c_{m+1}$
  else, set $w_{m+1} = \varepsilon$.

In each $w_i$, the subword $ab$ is produced.
Case 2: $a$ and $b$ are non-adjacent vertices in $G'$.
We deal this case in three subcases. In each case, we identify two linear orders (permutations): one with the subword $ab$ and other with the $ba$.

Subcase 2.1: $a, b \in V_1$.
- In $w_1$, if $a = c_i$ and $b = c_j$ with $i < j$ then $ba$ is a subword.
  $$w_1 = c_mc_{m-1} \cdots c_2c_{m+1}c_{m+2} \cdots c_{m+k_1}c_1c_{m+k_1+1}c_{m+k_1+2} \cdots c_{m+n}$$
- In $w_j$, if $w_j \neq \varepsilon$ then $ab$ is a subword.
  $$w_j = c_1 \cdots c_{j-1}c_{j+1} \cdots c_m \text{Dec}(V_2 \setminus N(c_j))c_j\text{Dec}(N(c_j))$$
- Else, $w_{m+1} \neq \varepsilon$ then $ab$ is a subword.
  $$w_{m+1} = c_1c_2 \cdots c_mc_{m+n} \cdots c_{m+1}$$
Subcase 2.2: $a \in V_1$ and $b \in V_2$.

- If $a = c_1$ and $b = c_j$, then $ba$ is a subword of
  \[ w_1 = c_m c_{m-1} \cdots c_2 c_{m+1} c_{m+2} \cdots c_{m+k_1} c_1 c_{m+k_1+1} c_{m+k_1+2} \cdots c_{m+n} \]
  The graph has no isolated vertices, $\exists c_k \in V_1$ such that $b \in N(c_k)$, 
  \[ w_k = c_1 \cdots c_{k-1} c_k c_{k+1} \cdots c_m \text{Dec}(V_2 \setminus N(c_k)) c_k \text{Dec}(N(c_k)) \]
  $ab$ is a subword.

- If $a = c_i$ and $b = c_j$, then $ab$ is a subword of $w_1$. Since $a$ and $b$ are not adjacent,
  \[ w_i = c_1 \cdots c_{i-1} c_{i+1} \cdots c_m \text{Dec}(V_2 \setminus N(c_i)) c_i \text{Dec}(N(c_i)) \]
  $ba$ is a subword.
**Subcase 2.3:** $a, b \in V_2$. If $a = c_i$ and $b = c_j$ with $i < j$. Then $ab$ is a subword of $w_1$.

$$w_1 = c_mC_{m-1} \cdots c_2c_{m+1}c_{m+2} \cdots c_{m+k_1}c_1c_{m+k_1+1}c_{m+k_1+2} \cdots c_{m+n}$$

The graph has no isolated vertices, $\exists c_k \in V_1$ such that $a \in N(c_k)$.

- If $N(c_k) = V_2$ then $w_{m+1} \neq \varepsilon$, $ba$ is a subword.
  $$w_{m+1} = c_1c_2 \cdots c_mc_{m+n} \cdots c_{m+1}$$

- Else, $w_k \neq \varepsilon$ then $ba$ is a subword.
  $$w_k = c_1 \cdots c_{k-1}c_{k+1} \cdots c_m\text{Dec}(V_2 \setminus N(c_k))c_k\text{Dec}(N(c_k))$$
Theorem

Let $m$ be the size of the smallest set in the bipartition of a bipartite graph $G$, the permutation representation number $\mathcal{R}^p(G) \leq m$. Consequently, $\mathcal{R}(G) \leq m$.

Corollary

Let $\{a_{i_1}, \ldots, a_{i_k}\} \subseteq V_1$ be the set of vertices each of which is adjacent to all vertices of $V_2$ in a bipartite graph $G$, then $\mathcal{R}^p(G) \leq m - k + 1$.

Remark

The relabeling algorithm proposed for bipartite graphs has a scope to extend it for comparability graphs, in general.
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Thank you.