Differentiability of excessive functions of one-dimensional diffusions and the principle of smooth fit

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Dedicated to the memory of a dear friend and colleague
Professor Esko Valkeila (1951-2012)

Abstract

The principle of smooth fit is probably the most used tool to find solutions to optimal stopping problems of one-dimensional diffusions. It is important, e.g., in financial mathematical applications to understand in which kind of models and problems smooth fit can fail. In this paper we connect - in case of one-dimensional diffusions - the validity of smooth fit and the differentiability of excessive functions. The basic tool to derive the results is the representation theory of excessive functions; in particular, the Riesz and Martin representations. It is seen that the differentiability may not hold in case the speed measure or the representing measure of the excessive function has atoms.

As an example, we study optimal stopping of sticky Brownian motion. It is known that the validity of the smooth fit in this case depends on the value of the discounting parameter (when the other parameters are fixed). We decompose the size of the jump in the derivative of the value function into two factors. The first one is due to the atom of the representing measure and the second one due to the atom of the speed measure.

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1 Introduction

Let \( X = (X_t)_{t \geq 0} \) be a one-dimensional diffusion process in the sense of Itô and McKean [15] living on an interval \( I \subseteq \mathbb{R} \), i.e., \( X \) is a time-homogeneous strong Markov process with continuous sample paths. As usual, the notations \( P_x \) and \( E_x \) are used for the probability measure and the expectation operator, respectively, associated with \( X \) when initiated from \( x \in I \). The lifetime of \( X \) is defined as \( \zeta := \inf\{t : X_t \notin I\} \) and we set \( X_t = \Delta \) for \( t \geq \zeta \), where \( \Delta \) is a fictitious state – the so called cemetery state. Recall that a measurable function \( f : I \cup \{\Delta\} \mapsto \mathbb{R}_+ \cup \{\infty\} \) is called \( \alpha \)-excessive, \( \alpha \geq 0 \), if for all \( x \in I \) the following two conditions hold:

\[
\begin{align*}
E_x(e^{-\alpha t} f(X_t)) &\leq f(x), \quad \forall t > 0, \quad (1) \\
\lim_{t \downarrow 0} E_x(e^{-\alpha t} f(X_t)) &= f(x), \quad (2)
\end{align*}
\]

where, by convention, \( f(\Delta) = 0 \). An alternative and equivalent definition is obtained by replacing \((1)\) and \((2)\) by

\[
\beta E_x \left( \int_0^\zeta e^{-(\alpha+\beta)t} f(X_t) \, dt \right) \leq f(x), \quad \forall \beta > 0, \quad (3)
\]

\[
\lim_{\beta \uparrow +\infty} E_x \left( \int_0^\zeta e^{-(\alpha+\beta)t} f(X_t) \, dt \right) = f(x). \quad (4)
\]

We refer to Dynkin [11] Vol. II for results on excessive functions in general and in particular for one-dimensional diffusions. For excessive functions in the framework of the potential theory of Markov processes, see Blumenthal and Getoor [2] and Chung and Walsh [5]. Excessive functions being descendants of superharmonic functions have, hence, deep roots in the classical potential theory and constitute also fundamental concept in the theory of Markov processes.

Our main motivation for the present study comes, however, from the theory of optimal stopping where excessive functions play a crucial role. Indeed, given a continuous non-negative (reward) function \( g \) the optimal
stopping problem with the underlying process $X$ is to find a (value) function $V$ and an (optimal) stopping time $\tau^*$ such that

$$V(x) := \sup_{\tau \in \mathcal{M}} \mathbb{E}_x(e^{-\alpha\tau} g(X_\tau)) = \mathbb{E}_x(e^{-\alpha\tau^*} g(X_{\tau^*})), \quad (5)$$

where $\mathcal{M}$ denotes the set of all stopping times with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by $X$. The fundamental result due to Snell and Dynkin (see Shiryayev [28] and Peskir and Shiryayev [24] for details and references), is that $V$ is the smallest $\alpha$-excessive function dominating $g$ and an optimal stopping time is given by

$$\tau^* = \inf\{t \geq 0 : X_t \in \Gamma\},$$

where $\Gamma := \{x : V(x) = g(x)\}$ is the so called stopping region. Therefore, a good knowledge of excessive functions is a key to a deeper understanding of optimal stopping.

Although our focus is on applications in optimal stopping we wish to point out that excessive functions can also be used, e.g., to condition and/or to kill a process in some particular desirable way. Such conditionings of the underlying process are called excessive transforms or Doob’s $h$-transforms due to Doob’s pioneering work [9]. We refer also to McKean [18], Dynkin [12], and Meyer et al. [19] for early seminal papers. The theory of $h$-transforms in a general setting is discussed in Chung and Walsh [5] Chapter 11. Moreover, a fairly recent problem arising from financial mathematics is to construct for a given process $X$ a martingale having the same distribution as $X$ at a fixed time or at a random time, see, e.g., Cox et al. [6], Hirsch et al. [14], Ekström et al. [13] and Noble [21] and references therein. In particular, Klimmek [16] exploits explicitly $h$-transforms to find the solution of the problem for a random (exponential) time.

These and other applications in mind – and also per se – we offer in this paper firstly a discussion on continuity and differentiability properties of excessive functions of one-dimensional diffusions and secondly applications to optimal stopping with an example. Our approach utilizes the Riesz and Martin representations which are valid in their strongest and most explicit forms for one-dimensional regular diffusions.

In the next section we give the Riesz and Martin representations with needed prerequisites and also present some examples. An immediate implication of the Riesz representation is then the continuity of excessive functions, see Proposition 2.8. The continuity is implicitly stated already in Salminen [26]. In Dayanik and Karatzas [8] and in Peskir and Shiryayev
the continuity is proved for a special class of excessive functions, that is, for value functions in optimal stopping problems. Their proofs utilize the properties of the value functions and the concavity, as in Dynkin and Yuschkevitch [10]. The advantage of the present approach is that it yields the result with full generality. We also shortly list - for general interest - some other basic potential theoretical and related results for one-dimensional regular diffusions. In particular, it is seen that all additive functionals are continuous. This is also pointed out in [3] p. 28 but with a slightly different explanation.

In the third section the differentiability properties of excessive functions are investigated. The Riesz representation allows us to derive conditions for differentiability with respect to any increasing continuous function $F$, see Theorem 3.1. This extends the result in [26] where differentiability with respect to the scale function is studied. We also represent the jump of the derivative of an excessive function as the sum of two terms: the first one is induced by the representing measure and the second one by the speed measure.

These results are then used in the fourth section to study the principle of smooth fit in optimal stopping of one-dimensional diffusions. Our contribution hereby is to demonstrate – using the results in Section 3 – that the proof of the condition for the smooth fit with respect to the scale as presented in [26] can be rewritten – changing mainly only the notation – to a proof of the condition for the smooth fit in the ordinary sense as given in Peskir [23] and Samee [27], see also [24, p. 160] (e.g. when studying the case where the scale function is not differentiable at the stopping point). We conclude by analyzing the smooth fit property in an optimal stopping problem where the underlying process is a sticky Brownian motion. It is known, see Crocce and Mordecki [7], that if the optimal stopping point is the sticky point then the smooth fit typically fails. Our results enhance the understanding of this phenomenon by giving an explicit form for the jump of the derivative of the value function in this case.

2 Riesz and Martin representations

We start with by introducing more notation and recalling some basic facts. Let $l \geq -\infty$ and $r \leq +\infty$ denote the left and the right, respectively, end point of $I$ which is an interval of any kind. Recall that $I$ is the state space of $X$. The notations $m$ and $S$ are used for the speed measure and the scale function, respectively. Moreover, let $\mathcal{G}$ denote the generalized differential
operator associated with $X$ and $\tau_y$ the first hitting time of $y \in I$, that is,

$$\tau_y := \inf\{t : X_t = y\}.$$ 

We assume that $X$ is regular (cf. Dynkin [11] Vol. II p.121), that is,

$$\mathbb{P}_x(\tau_y < \infty) > 0, \quad \forall x, y \in I$$

in other words, no matter where $X$ starts there is a positive probability to hit any point in $I$. This means that a regular diffusions do not have absorbing points and exit points and entrance points are not included in $I$. Moreover, a consequence of the regularity is that there does not exist non-empty polar sets (for this notion, see [2, p. 79]).

**Remark 2.1.** The above definition of regularity differs from another often used definition in which (6) is assumed to hold for all $x \in (l, r)$ and $y \in I$ (see, e.g., Revuz and Yor [25] p. 300). According to this latter definition we could have a regular diffusion with $I = [l, r]$ and $l$ and $r$ absorbing. As demonstrated below in Example 2.9 such a diffusion has discontinuous excessive functions – the case we want to exclude.

As showed in Itô and McKean [15, p. 124] the Laplace transform of $\tau_y$ can be expressed for $\alpha > 0$ as

$$\mathbb{E}_x(e^{-\alpha \tau_y}) = \begin{cases} 
\frac{\psi_\alpha(x)}{\psi_\alpha(y)}, & x \leq y, \\
\frac{\varphi_\alpha(x)}{\varphi_\alpha(y)}, & x \geq y,
\end{cases}$$

where $\psi_\alpha$ and $\varphi_\alpha$ are continuous, positive, increasing and decreasing, respectively, solutions of the generalized differential equation

$$\mathcal{G}u = \alpha u. \quad (7)$$

Imposing appropriate boundary conditions determine $\psi_\alpha$ and $\varphi_\alpha$ uniquely up to a multiplicative constant. The Wronskian $\omega_\alpha$ - a constant - is defined as

$$\omega_\alpha := \psi^+_\alpha(x)\varphi_\alpha(x) - \psi_\alpha(x)\varphi^+_\alpha(x)$$

$$= \psi^-_\alpha(x)\varphi_\alpha(x) - \psi_\alpha(x)\varphi^-_\alpha(x).$$
where the superscripts $^+$ and $^−$ denote the right and left derivatives with respect to the scale function, i.e., for $u = ψ_α$ or $ϕ_α$

\[
\begin{align*}
 u^+(x) &:= \frac{d^+ u}{dS}(x) := \lim_{\delta \to 0^+} \frac{u(x + \delta) - u(x)}{S(x + \delta) - S(x)}, \\
 u^−(x) &:= \frac{d^− u}{dS}(x) := \lim_{\delta \to 0^+} \frac{u(x - \delta) - u(x)}{S(x - \delta) - S(x)},
\end{align*}
\]

cf. (19) and (20) below and recall that the scale function of a diffusion is continuous. It is well-known (see [15, p. 150]) that

\[
G_α(x, y) := \begin{cases} 
 w^{-1}_α ψ_α(x) φ_α(y), & x \leq y, \\
 w^{-1}_α ψ_α(y) φ_α(x), & x \geq y,
\end{cases}
\]

serves as a resolvent kernel (also called the Green function) of $X$, i.e., for any Borel subset $A$ of $I$

\[
\mathbb{E}_x \left( \int_0^\zeta e^{-\alpha t} 1_A(X_t) \, dt \right) = \int_A G_α(x, y) m(dy),
\]

Using Theorem 12.4 in Dynkin [11] it is fairly straightforward to check that for every fixed $y$ the function $x \mapsto G_α(x, y)$ is $α$-excessive (see [26, p. 89]). Since $(x, y) \mapsto G_α(x, y)$ is symmetric it follows that $X$ is self-dual with respect to the speed measure, that is

\[
< f, G_α g >_m = < G_α f, g >_m,
\]

where

\[
< f, g >_m := \int_I f(x)g(x)m(dx), \quad G_α f(x) := \int_I G_α(x, y)f(y)m(dy),
\]

with $f$ and $g$ bounded Borel measurable functions satisfying appropriate integrability conditions. For the concept of duality and related topics, see Kunita and Watanabe [17], Blumenthal and Getoor [2] and Chung and Walsh [5]. We wish to apply the Riesz representation theorem, see [2, p. 272], and remark that the assumptions for its validity as presented in [2] Chapter VI (see also [17, Theorem 2 p. 505]) are satisfied. An important assumption is that $X$ has a dual process which is standard in the sense of the definition in ibid. p. 45. Clearly, $X$ is standard and since $X$ is self-dual the needed assumption is fulfilled. Notice also that (2.1) and (2.2) in [2, p. 265-266] hold.
Theorem 2.2. (The Riesz representation) Let $\alpha > 0$ and $u$ an $\alpha$-excessive function of the regular one-dimensional diffusion $X$. It is assumed that $u$ is locally integrable with respect to $m$. Then there exist an $\alpha$-harmonic function $h_{\alpha}$ and a Radon measure $\sigma_u$ on $I$ such that $u$ can be represented uniquely as

$$u(x) = \int_I G_\alpha(x,y) \sigma_u(dy) + h_{\alpha}(x).$$  \hspace{1cm} (10)$$

Remark 2.3. (i) The Riesz representation holds also for $\alpha = 0$ when $X$ is transient. The Green function when $\alpha = 0$ has a similar structure as in case $\alpha > 0$ but now the corresponding functions $\varphi_0$ and $\psi_0$ express the hitting probabilities instead of the Laplace transforms of the hitting distributions. In Section 3 we discuss shortly the special case in which the diffusion is not killed inside the state space $I$.

(ii) The assumption on the local integrability is superfluous in case of one-dimensional diffusions. Indeed, assuming that the $\alpha$-excessive function $u \neq +\infty$ choose a point $x$ such that $u(x) < +\infty$. From (3) we have

$$\beta \int_I G_{\alpha+\beta}(x,y) u(y) m(dy) = \beta \mathbb{E}_x \left( \int_0^\zeta e^{-(\alpha+\beta)t} u(X_t) dt \right) \leq u(x).$$  \hspace{1cm} (11)$$

The local integrability of $u$ follows now easily from the explicit form of $G_{\alpha+\beta}$ and the continuity of $\psi_{\alpha+\beta}$ and $\varphi_{\alpha+\beta}$.

(iii) The $\alpha$-harmonicity of $h_{\alpha}$ means that for all compact subset $A$ of $I$ it holds

$$h_{\alpha}(x) = \mathbb{E}_x \left( e^{-\alpha \tau_A} h_{\alpha}(X_{\tau_A}) \right),$$  \hspace{1cm} (12)$$

where

$$\tau_A := \inf \{ t : X_t \notin A \}.$$
Theorem 2.4. (The Martin representation) Let $u$ be an $\alpha$-excessive function of the one-dimensional diffusion $X$ and $x_0 \in I$ a point such that $u(x_0) = 1$. Then $u$ can be represented uniquely as

$$u(x) = \int_{[l,r]} \frac{G_\alpha(x,y)}{G_\alpha(x_0,y)} \nu^o_u(dy) + \frac{\varphi_\alpha(x)}{\varphi_\alpha(x_0)} \nu^o_u([l]) + \frac{\psi_\alpha(x)}{\psi_\alpha(x_0)} \nu^o_u(\{r\}),$$  \hspace{1cm} (13)

where $\nu^o_u$ is a probability measure on $[l,r]$ characterized via

$$\nu^o_u([x,r]) = \frac{\psi_\alpha(x_0)}{\omega_\alpha} \left( \varphi_\alpha(x) u^+(x) - u(x) \varphi_\alpha^+(x) \right), \quad x \geq x_0,$$  \hspace{1cm} (14)

$$\nu^o_u([l,x]) = \frac{\varphi_\alpha(x_0)}{\omega_\alpha} \left( u(x) \psi_\alpha^-(x) - \psi_\alpha(x) u^-(x) \right), \quad x \leq x_0.$$  \hspace{1cm} (15)

Conversely, given a probability measure $\mu$ on $[l,r]$ and $x_0 \in I$ then the right hand side of (13) when putting $\nu^o_u = \mu$ defines an $\alpha$-excessive function.

Remark 2.5. The expression on the right hand side of (13) is well defined since $\varphi_\alpha$ and $\psi_\alpha$ are positive on $I$. Notice that the probability measure $\nu^o_u$ is defined on the closure of $I$; also in case $l = -\infty$ and/or $r = +\infty$. In fact, $[l,r]$ is the so-called Martin compactification of $I$.

Combining (13) with (10) yields a characterization of the $\alpha$-excessive functions in the Riesz representation. This together with other relationships between the two representations are discussed in the next

Corollary 2.6. Let $u$ and $h_\alpha$ be as in Theorem 2.2. Then there exist $c_1 \geq 0$ and $c_2 \geq 0$ such that $h_\alpha = c_1 \varphi_\alpha + c_2 \psi_\alpha$. The Riesz and the Martin representing measures of $u$ are connected via the identity

$$\sigma_u(A) = \int_A \frac{1}{G_\alpha(x_0,y)} \nu^o_u(dy),$$  \hspace{1cm} (16)

where $A$ is a Borel subset of $I$. Moreover, $u$ has the unique representation

$$u(x) = \int_{[l,r]} G_\alpha(x,y) \sigma_u(dy) + \hat{h}_\alpha(x),$$  \hspace{1cm} (17)

where

$$\hat{h}_\alpha(x) := c'_1 \varphi_\alpha(x) + c'_2 \psi_\alpha(x)$$

with $c'_1 \geq 0$ and $c'_2 \geq 0$.

Next example highlights the difference of the Riesz and Martin representations via the fact that $\psi_\alpha$ and/or $\varphi_\alpha$ could be potentials, that is, not $\alpha$-harmonic.
Example 2.7. Let $X$ be a Brownian motion reflected at 0 and killed at 1. Hence, $I = [0, 1)$ and it is readily checked that we may take

$$\varphi_\alpha(x) = \text{sh}((1 - x)\sqrt{2\alpha}) \quad \text{and} \quad \psi_\alpha(x) = \text{ch}(x\sqrt{2\alpha}).$$

Notice that $\varphi_\alpha(1) = 0$ and $\psi'_\alpha(0+) = 0$ which, in fact, are the appropriate boundary conditions to characterize $\varphi_\alpha$ and $\psi_\alpha$, respectively. For this process, $\psi_\alpha$ is $\alpha$-harmonic but $\varphi_\alpha$ is not. Standard computations show that both $\psi_\alpha$ and $\varphi_\alpha$ satisfy the $\alpha$-harmonicity condition (12) for intervals of the form $[a, b]$, $0 < a < b < 1$. However, when $a = 0$ the condition fails for $\varphi_\alpha$. Indeed, putting $A = [0, b]$, $0 < b < 1$ we have

$$E_x(e^{-\alpha \tau_A} \varphi_\alpha(X_{\tau_A})) = E_x(e^{-\alpha \tau_{\tau_A}} \varphi_\alpha(X_{\tau}))$$

$$= \varphi_\alpha(a) E_x(e^{-\alpha \tau_{\tau_A}})$$

$$= \varphi_\alpha(b) \frac{\psi_\alpha(x)}{\psi_\alpha(b)} \neq \varphi_\alpha(x).$$

Consequently, the Riesz representation of $\varphi_\alpha$ does not have the $\alpha$-harmonic part and, hence,

$$\varphi_\alpha(x) = \int_{[0, 1)} G_\alpha(x, y) \sigma_\varphi(dy).$$

It follows by the uniqueness of the representing measure that $\sigma_\varphi$ is a multiple of the Dirac measure at 0.

Next we prove the continuity of excessive functions which is an important stepping stone to the differentiability studied in Section 3.

Proposition 2.8. For a one-dimensional regular diffusion all $\alpha$-excessive functions are continuous.

Proof. Let $u$ be an $\alpha$-excessive function. Substituting the explicit form of the Green kernel in the representation (17) yields

$$u(x) = w^{-1}_\alpha \varphi_\alpha(x) \int_{(l, x]} \psi_\alpha d\sigma_u + w^{-1}_\alpha \psi_\alpha(x) \int_{(x, r]} \varphi_\alpha d\sigma_u + \hat{h}_\alpha(x)$$

$$= w^{-1}_\alpha \varphi_\alpha(x) \int_{(l, x]} \psi_\alpha d\sigma_u + w^{-1}_\alpha \psi_\alpha(x) \int_{(x, r]} \varphi_\alpha d\sigma_u + \hat{h}_\alpha(x),$$

from which evoking the continuity of $\varphi_\alpha$ and $\psi_\alpha$ it easily follows that

$$\lim_{\varepsilon \to 0^+} u(x + \varepsilon) = \lim_{\varepsilon \to 0^+} u(x - \varepsilon) = u(x).$$

\qed
Example 2.9. To stress the importance of the regularity as defined in (6) we give an example showing that if an end point of $I$ is absorbing then there exist discontinuous excessive functions. Let $X$ denote a Brownian motion on $\mathbb{R}^+$ absorbed at 0 and consider the function

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\
\frac{1}{2}, & \text{if } x = 0. \end{cases}$$

Since $\mathbb{P}_0(X_t = 0) = 1$ for all $t \geq 0$ we have for $x > 0$

$$\mathbb{E}_x(f(X_t)) = \mathbb{P}_x(t < \tau_0) + \frac{1}{2} \mathbb{P}_x(t \geq \tau_0)$$

$$= 1 - \frac{1}{2} \mathbb{P}_x(t \geq \tau_0)$$

$$\leq 1 = f(x),$$

and, for $x = 0$,

$$\mathbb{E}_0(f(X_t)) = \frac{1}{2} = f(0).$$

Clearly, also (2) holds. Consequently, $f$ is a discontinuous excessive function.

We conclude this subsection by pointing out some important properties of one-dimensional diffusions which can be deduced from the potential theoretical generalities using the explicit form of the Green kernel and the continuity of the excessive functions:

- Firstly, since $X$ is self dual and for all $y \in I$ the function $x \mapsto G_\alpha(x, y)$ is bounded and continuous it follows from (4.11) p. 290 in [2] that every point in $I$ is regular, i.e.,

$$\mathbb{P}_x(\tau_x^+ = 0) = 1 \quad \forall x \in I,$$

where

$$\tau_x^+ := \inf\{t > 0 : X_t = x\}.$$  

Consequently, by ibid. (3.13) p. 216, $X$ possesses at every point $x \in I$ a local time $(L_t^{(x)})_{t \geq 0}$.

- Secondly, again due to the self duality together with the continuity of the sample paths, it holds that all additive functionals of $X$ are continuous, see ibid. p. 289. In particular, $t \mapsto L_t^{(x)}$ is a.s. continuous.

- Thirdly, recall that an excessive function $f$ is called regular for $X$ if $t \mapsto f(X_t)$ is continuous on $[0, \zeta)$ (see ibid. p. 287-288). Hence, from Proposition 2.8 it follows via the continuity of the sample paths that all (finite) excessive functions for $X$ are regular.
3 Differentiability

For an increasing continuous function \( F : I \mapsto \mathbb{R} \) and an arbitrary \( \alpha \)-excessive function \( u \) we introduce the one sided derivatives of \( u \) with respect to \( F \):

\[
\frac{d^+ u}{dF}(x) := \lim_{\delta \to 0^+} \frac{u(x + \delta) - u(x)}{F(x + \delta) - F(x)}, \tag{19}
\]

\[
\frac{d^- u}{dF}(x) := \lim_{\delta \to 0^+} \frac{u(x - \delta) - u(x)}{F(x - \delta) - F(x)}. \tag{20}
\]

for every \( x \in I \) for which the limits on the right hand sides exist and are finite. We say that \( u \) is \( F \)-differentiable at \( x \in I \) if

\[
\frac{d^+ u}{dF}(x) = \frac{d^- u}{dF}(x).
\]

Our basic result gives conditions for the \( F \)-differentiability of an arbitrary \( \alpha \)-excessive function \( u \). Recall from the Riesz representation that there exists a Radon measure \( \sigma_u \) such that (10) holds.

**Theorem 3.1.** Let \( u \) and \( F \) be as above and assume that functions \( \psi_\alpha \) and \( \varphi_\alpha \) are for \( \alpha > 0 \) \( F \)-differentiable at a point \( z \in I \). Then the left and the right \( F \)-derivative of \( u \) exist at \( z \) and satisfy

\[
\frac{d^- u}{dF}(z) - \frac{d^+ u}{dF}(z) \geq 0. \tag{21}
\]

Moreover, \( u \) is \( F \)-differentiable at \( z \) if and only if \( \sigma_u(\{z\}) = 0 \).

**Proof.** Since functions \( \psi_\alpha \) and \( \varphi_\alpha \) are assumed to be \( F \)-differentiable at \( z \) we may, without loss of generality, take \( c'_1 = c'_2 = 0 \) in (17), and, hence, \( u \) has the representation

\[
u(z) = w^{-1}_\alpha \varphi_\alpha(z) \int_{(l,z]} \psi_\alpha d\sigma_u + w^{-1}_\alpha \psi_\alpha(z) \int_{(z,r)} \varphi_\alpha d\sigma_u. \tag{22}\]

Using (22) it is seen after som simple manipulations that for \( \delta > 0 \)

\[
\frac{u(z + \delta) - u(z)}{F(z + \delta) - F(z)} = w^{-1}_\alpha \psi_\alpha(z + \delta) - \varphi_\alpha(z) \int_{(l,z]} \psi_\alpha d\sigma_u
\]

\[
+ w^{-1}_\alpha \psi_\alpha(z + \delta) - \psi_\alpha(z) \int_{(z,r]} \varphi_\alpha d\sigma_u
\]

\[
+ w^{-1}_\alpha J(z, \delta),
\]

\[
11
\]
where
\[
J(z, \delta) := \frac{\phi_\alpha(z + \delta) \int_{(z, z + \delta]} \psi_\alpha d\sigma_u - \psi_\alpha(z + \delta) \int_{(z, z + \delta]} \varphi_\alpha d\sigma_u}{F(z + \delta) - F(z)}.
\]

Since \(\psi_\alpha\) and \(\varphi_\alpha\) are increasing and decreasing, respectively, it holds
\[
R(z, \delta) \leq J(z, \delta) \leq 0
\]

Evoking that \(\sigma_u\) is a measure and \(\varphi_\alpha\) and \(\psi_\alpha\) are assumed to be \(F\)-differentiable at \(z\) we obtain
\[
\lim_{\delta \downarrow 0} R(z, \delta) = (d\varphi_\alpha dF(z) \psi_\alpha(z) - \varphi_\alpha(z + \delta) \varphi_\alpha(z)) \lim_{\delta \downarrow 0} \sigma_u((z, z + \delta])
\]
\[
= 0.
\]

Consequently,
\[
\lim_{\delta \downarrow 0} J(z, \delta) = 0.
\]

It follows that \(u\) has the right \(F\)-derivative given by
\[
\frac{d^+ u}{dF}(z) = w_\alpha^{-1} \left( \int_{(l, z]} \psi_\alpha d\sigma_u + \frac{d\psi_\alpha}{dF}(z) \int_{(z, r)} \varphi_\alpha d\sigma_u \right). \tag{23}
\]

Analogous calculations yield for the left \(F\)-derivative
\[
\frac{d^- u}{dF}(z) = w_\alpha^{-1} \left( \int_{(l, z]} \psi_\alpha d\sigma_u + \frac{d\psi_\alpha}{dF}(z) \int_{[z, r)} \varphi_\alpha d\sigma_u \right). \tag{24}
\]

Hence, we have
\[
\frac{d^- u}{dF}(z) = \frac{d^+ u}{dF}(z)
\]
\[
= w_\alpha^{-1} \left( \int_{(l, z]} \psi_\alpha d\sigma_u + \frac{d\psi_\alpha}{dF}(z) \int_{[z, r)} \varphi_\alpha d\sigma_u \right)
\]
\[
\geq 0
\]
and this completes the proof. \(\square\)
Remark 3.2. Choosing $F$ equal to the scale function yields (3.7) Corollary in [26]. In fact, the idea of the proof is the same as in [26]. However, therein the proof is based explicitly on the Martin representation. Notice that taking $F = S$ in (25) yields
\[ u^-(z) - u^+(z) = \sigma_u\{z\}, \]
which differs from the formula in the proof of (3.7) Corollary in [26] due to the different normalizations of the representing measure in the Riesz and Martin representations. Notice also that taking $F(x) = x$ gives, of course, a condition for the differentiability in the usual sense.

We study next the differentiability of 0-excessive functions. Hence, it is assumed that $X$ is transient and, moreover, that the killing measure is identically zero. Then $\lim_{t \to \zeta} X_t = l$ or $r$ with probability 1. As stated in Remark 1 after Theorem 2.2 the Riesz representation holds also for the 0-excessive functions. In fact, in [2] only the case with $\alpha = 0$ is discussed in detail. The differentiability of 0-excessive functions can be analyzed similarly as was done in Theorem 3.1 for $\alpha$-excessive functions. Therefore, we formulate the result as a corollary. For simplicity, it is assumed that the boundary condition at a regular boundary point is killing. Then the Green function can be written as (see [15, p. 130], and [3, p. 20])

\[
G_0(x,y) = \int_0^\infty p(t;x,y) \, dt
\]

\[
= \begin{cases} 
\lim_{a \downarrow l, b \uparrow r} \frac{(S(x) - S(a)(S(b) - S(y)))}{S(b) - S(a)}, & \text{if } x \leq y, \\
\lim_{a \downarrow l, b \uparrow r} \frac{(S(y) - S(a)(S(b) - S(x)))}{S(b) - S(a)}, & \text{if } x \geq y.
\end{cases} 
\]

Corollary 3.3. Let $X$ be a transient diffusion as introduced above and $u$ a 0-excessive function of $X$. Assume that the scale function $S$ of $X$ is differentiable at a given point $z$. Then $u$ has the left and the right derivative at $z$ and it holds
\[
\frac{d^-u}{dx}(z) - \frac{d^+u}{dx}(z) \geq 0.
\]
Moreover, $u$ is differentiable at $z$ if and only if $\sigma_u\{z\} = 0$.

Proof. Consider formula (17) in case $\alpha = 0$ and $-\infty < S(l) < S(r) < +\infty$. Since $l$ and $r$ are assumed to be killing boundaries we have
\[
\varphi_0(x) = S(r) - S(x) \quad \text{and} \quad \psi_0(x) = S(x) - S(l).
\]
Consequently, we may assume, without loss of generality, that in the representation of \( u \) in (17) \( \hat{h}_0 \equiv 0 \). Hence,

\[
\begin{align*}
  u(z) &= \frac{S(r) - S(z)}{S(r) - S(l)} \int_{(t,z]} (S(y) - S(l)) \sigma_u(dy) \\
  &\quad + \frac{S(z) - S(l)}{S(r) - S(l)} \int_{(z,r)} (S(r) - S(y)) \sigma_u(dy).
\end{align*}
\]

(29)

The cases \( S(l) = -\infty, S(r) < \infty \) and \( S(l) > -\infty, S(r) = \infty \) can be handled similarly; we leave the details to the reader. Formula (29) corresponds (22) and the proof can be continued similarly as was done after (22) but taking \( F(x) = x \).

Next theorem extends formula (26) for diffusions having sticky points.

Theorem 3.4. Let \( u \) be an \( \alpha \)-excessive function of the diffusion \( X \). Then it holds

\[
\begin{align*}
  u^-(z) - u^+(z) &= \sigma_u(\{z\}) - \alpha(u(z)),
\end{align*}
\]

(32)

where \( u^+ (u^-) \) denotes the right (left) derivative with respect to the scale function.

Proof. Due to (30) and (31) we may assume without loss of generality that \( u \) has the Riesz representation

\[
\begin{align*}
  u(z) &= \int_{(l,r]} G_\alpha(z,y) \sigma_u(dy).
\end{align*}
\]

(33)

By similar calculations as in the proof of Theorem 3.1 taking therein \( F \equiv S \) we obtain (cf. (23) and (24))

\[
\begin{align*}
  u^+(z) &= w_\alpha^{-1} \left( \varphi^+_\alpha(z) \int_{(z,r]} \psi_\alpha d\sigma_u + \psi^+_\alpha(z) \int_{(z,r]} \varphi_\alpha d\sigma_u \right)
\end{align*}
\]

(34)
and
\[ u^-(z) = w^{-1}_\alpha \left( \varphi^-_\alpha(z) \int_{(l,z)} \psi_\alpha d\sigma_u + \psi^-_\alpha(z) \int_{[z,r)} \varphi_\alpha d\sigma_u \right). \] (35)

Subtracting (34) from (35) yields
\[ u^-(z) - u^+(z) = \omega^{-1}_\alpha \left( \varphi^-_\alpha(z) - \varphi^+_\alpha(z) \right) \int_{(l,z)} \psi_\alpha d\sigma_u \]
\[ + \omega^{-1}_\alpha \left( \psi^-_\alpha(z) - \psi^+_\alpha(z) \right) \int_{(z,r)} \varphi_\alpha d\sigma_u \]
\[ + \omega^{-1}_\alpha \psi^-_\alpha(z) \varphi_\alpha(z) \sigma_u(\{z\}) - \omega^{-1}_\alpha \varphi^+_\alpha(z) \psi_\alpha(z) \sigma_u(\{z\}). \] (36)

Using (30), (31) and noticing that
\[ \psi^-_\alpha(z) \varphi_\alpha(z) - \varphi^+_\alpha(z) \psi_\alpha(z) = \psi^-_\alpha(z) \varphi_\alpha(z) - \varphi^-_\alpha(z) \psi_\alpha(z) \]
\[ + \psi_\alpha(z) \left( \varphi^-_\alpha(z) - \varphi^+_\alpha(z) \right) \]
\[ = \omega_\alpha + \psi_\alpha(z) \left( \varphi^-_\alpha(z) - \varphi^+_\alpha(z) \right). \]

Identity (36) can be written as follows
\[ u^-(z) - u^+(z) = -\alpha m(\{z\}) \left( \omega^{-1}_\alpha \varphi_\alpha(z) \int_{(l,z)} \psi_\alpha d\sigma_u \right) \]
\[ + \omega^{-1}_\alpha \psi_\alpha(z) \int_{(z,r)} \varphi_\alpha d\sigma_u \]
\[ + \omega^{-1}_\alpha \varphi_\alpha(z) \psi_\alpha(z) \sigma_u(\{z\}) + \omega^{-1}_\alpha \omega_\alpha \sigma_u(\{z\}) \]
\[ = -\alpha m(\{z\}) \int_{(l,r)} G_\alpha(z,y) \sigma_u(dy) + \sigma_u(\{z\}) \]
\[ = -\alpha m(\{z\}) u(z) + \sigma_u(\{z\}) \]
by (33), and the proof is complete. \( \square \)

4 Application to optimal stopping

4.1 Smooth fit

Probably the most used method to solve optimal stopping problems (with infinite horizon) for one-dimensional diffusions is based on the principle of smooth fit. This principle says that the value function \( V \) as defined in (5) meets the reward function \( g \) smoothly at the boundary points of the stopping
region \( \Gamma := \{ x : V(x) = g(x) \} \), i.e., \( v'(x) = g'(x) \) at the boundary points in case \( g' \) exists. The idea of the method is to guess the form of \( \Gamma \) and to find its boundary points using the continuity and the differentiability of the proposed value function. After this, a verification theorem (see, e.g., Øksendal [22, p. 215 Theorem 10.4.1]) is needed to show that the proposed value is indeed the right one.

In [26] a criterion for the validity of the smooth fit (with respect to the scale function) is derived. This criterion can be extracted from Theorem 4.1 below by choosing \( F \) equal to \( S \), the scale function. A condition for the smooth fit with respect to “usual” differentiation is obtained by taking \( F \) to be the identity mapping. The criterion holds also for \( \alpha = 0 \) (transient case with general killing measure). We formulate the result for a left boundary point of \( \Gamma \); obviously there is a similar result for a right boundary point.

**Theorem 4.1.** Let \( z \) be a left boundary point of \( \Gamma \), i.e., \([z, z + \varepsilon_1) \subset \Gamma \) and \((z - \varepsilon_2, z) \subset \Gamma^c\) for some positive \( \varepsilon_1 \) and \( \varepsilon_2 \). Let \( F \) be a continuous and increasing function and assume that the reward function \( g \) and the functions \( \varphi_\alpha \) and \( \psi_\alpha \), \( \alpha \geq 0 \), are \( F \)-differentiable at \( z \). Then the value function \( V \) in (5) is \( F \)-differentiable at \( z \) and the smooth fit with respect to \( F \) holds:

\[
\frac{d^+ V}{dF}(z) = \frac{d^- V}{dF}(z) = \frac{dg}{dF}(z). \tag{37}
\]

**Proof.** Since \( V > g \) on \( \Gamma^c \) and \( V = g \) on \( \Gamma \) we have

\[
\frac{d^+ V}{dF}(z) = \lim_{\delta \to 0^+} \frac{V(z + \delta) - V(z)}{F(z + \delta) - F(z)} = \lim_{\delta \to 0^+} \frac{g(z + \delta) - g(z)}{F(z + \delta) - F(z)}
\]

\[
= \frac{d^+ g}{dF}(z) = \frac{dg}{dF}(z)
\]

and

\[
\frac{d^- V}{dF}(z) = \lim_{\delta \to 0^+} \frac{V(z - \delta) - V(z)}{F(z - \delta) - F(z)} = \lim_{\delta \to 0^+} \frac{V(z) - V(z - \delta)}{F(z) - F(z - \delta)}
\]

\[
\leq \lim_{\delta \to 0^+} \frac{g(z) - g(z - \delta)}{F(z) - F(z - \delta)}
\]

\[
= \frac{d^- g}{dF}(z) = \frac{dg}{dF}(z).
\]

Consequently,

\[
\frac{d^- V}{dF}(z) \leq \frac{d^+ V}{dF}(z),
\]
and, hence, (21) yields
\[ \frac{d^- V}{dF}(z) = \frac{d^- V}{dF}(z), \]
proving the claim.

Specializing to transient diffusions without killing inside the state space and applying Corollary 3.3 yields the following result which is the contents of Theorem 2.3 in [23], see also [24] section 9.1.

**Corollary 4.2.** Let \( X \) be a transient diffusion as introduced in Corollary 3.3. Let \( z \) be a point such that \( g(z) = V(z) \). If the reward function \( g \) and the scale function \( S \) are differentiable at \( z \) then the smooth fit holds at \( z \):
\[ \frac{d^+ V}{dx}(z) = \frac{d^- V}{dx}(z) = \frac{dg}{dx}(z). \] (38)

**4.2 Example: Sticky Brownian motion**

In this section we study an optimal stopping problem when the underlying process is a sticky Brownian motion with drift \( \mu \leq 0 \). We let \( X = (X_t)_{t \geq 0} \) denote this process and, by definition, we take it to be sticky at 0. The speed measure and the scale function of \( X \) are given for \( \mu < 0 \) by
\[ m(dx) = 2e^{2\mu x} dx + 2c\delta_{\{0\}}(dx), \quad \text{and} \quad S(x) = \frac{1}{2\mu}(1 - e^{-2\mu x}), \]
respectively, where \( \delta_{\{0\}} \) denotes the Dirac measure at 0 and the stickyness parameter \( c \) is positive. In case, \( \mu = 0 \) put \( S(x) = x \), i.e., \( X \) is in natural scale. The infinitesimal operator associated with \( X \) (see Ito and Mckean [15, p. 111-112]) is given for \( x \neq 0 \) by
\[ G = \frac{1}{2} \frac{d^2}{dx^2} + \mu \frac{d}{dx} \]
and defined by continuity at 0, that is, \( Gf(0) = Gf(0^+) = Gf(0^-). \) The domain is taken to be
\[ \mathcal{D} := \{ f : f \in C^2_b(\mathbb{R}), Gf \in C_b(\mathbb{R}), f^+(0) - f^-(0) = 2c Gf(0) \}. \]
Notice that in our case \( S'(0) = 1 \), and, hence, for instance, \( f^+(0) = f'(0^+) \).

To find the fundamental solutions \( \psi_\alpha \) and \( \varphi_\alpha \) associated with \( X \) recall that the unique positive (up to a multiplicative constants) increasing and decreasing solutions the ODE
\[ \frac{1}{2} u''(x) + \mu u'(x) = \alpha u(x) \]
are given by
\[ \psi_\alpha^0(x) = e^{(\theta-\mu)x} \quad \text{and} \quad \varphi_\alpha^0(x) = e^{-(\theta+\mu)x}. \]
respectively, where \( \theta := \sqrt{2\alpha + \mu^2} \). Consequently, we should find constants \( A, B, C, \) and \( D \) such that
\[ \psi_\alpha(x) := \begin{cases} \psi_\alpha^0(x), & x \leq 0, \\ A\psi_\alpha^0(x) + B\varphi_\alpha^0(x), & x \geq 0, \end{cases} \]
and
\[ \varphi_\alpha(x) := \begin{cases} C\varphi_\alpha^0(x) + D\varphi_\alpha^0(x), & x \leq 0, \\ \varphi_\alpha^0(x), & x \geq 0, \end{cases} \]
are continuous (at 0) and, moreover, satisfy the condition (cf. (30) and (31))
\[ u'(0^+) - u'(0-) = 2c\alpha u(0). \quad (39) \]
Straightforward calculations show that
\[ \psi_\alpha(x) = \begin{cases} e^{(\theta-\mu)x}, & x \leq 0, \\ (1 + \gamma)e^{(\theta-\mu)x} - \gamma e^{-(\theta+\mu)x}, & x \geq 0, \end{cases} \quad (40) \]
and
\[ \varphi_\alpha(x) = \begin{cases} (1 + \gamma)e^{-(\theta+\mu)x} - \gamma e^{(\theta-\mu)x}, & x \leq 0, \\ e^{-(\theta+\mu)x}, & x \geq 0, \end{cases} \quad (41) \]
where \( \gamma := c\alpha/\theta \). We remark that these expressions coincide in case \( \mu = 0 \) with the formulas in [3, p. 123].

We study now the OSP as given in (5) with \( g(x) = (1 + x)^+ \)
\[ V(x) := \sup_{\tau \in \mathcal{M}} E_x \left( e^{-\alpha \tau}(X_{\tau+} + 1)^+ \right) = E_x \left( e^{-\alpha \tau^+}(X_{\tau^+} + 1)^+ \right), \quad (42) \]
where \( X \) is the sticky Brownian motion introduced above.

**Proposition 4.3.** In case \( \alpha = 0 \) the problem is equivalent with the corresponding problem for ordinary Brownian motion with drift. The smooth fit holds and the optimal stopping time is \( \tau^* := \inf\{t : X_t \geq (1 - 2|\mu|)/2|\mu|\} \).
Proof. The value function of the problem is the smallest 0-excessive majorant of the reward function. Recall that the Green function in case $\alpha = 0$ and there is no killing inside $I$ is determined by linear combinations of the scale function (see (27) for an example). Since the scale functions of $X$ and the ordinary Brownian motion with drift are equal it follows from the Martin representation that the classes of 0-excessive functions for these processes are identical. Consequently, in the considered OSPs the value functions and the optimal stopping times are equal. The solution of the latter problem was found already by Taylor [29], see also, e.g., [20], [28, p. 124-5], and [26], and, herefrom, it is clearly seen that the smooth fit holds.

Remark 4.4. Another explanation/proof of Proposition 4.3 is that making a state sticky in BM does not change the probabilities of hitting points. Since in case $\alpha = 0$ it does not "cost to wait" the problems with or without the sticky point have the same solutions.

We specialize now to case $\mu = 0$. It is proved in [7] for $c = 1$ that the smooth fit does not hold when the discounting parameter $\alpha$ is in the interval $[\alpha_1, \alpha_2)$, where $\alpha_1 = (\sqrt{1 + 4c} - 1)^2/8c$ and $\alpha_2 = 1/2$. We wish to study this phenomenon via the representing measure of the value function. Consider the following functions defined for $x \neq -1, 0$

$$s(x) := \varphi_\alpha(x)g'(x) - \varphi'_\alpha(x)g(x)$$

$$= \begin{cases} 0, & x < -1, \\ e^{-\sqrt{2\alpha}x}((1 + x)\sqrt{2\alpha} + 1) \\ + c\sqrt{2\alpha}((1 + x)\sqrt{2\alpha}\text{ch}(\sqrt{2\alpha}x) + \text{sh}(\sqrt{2\alpha}x)), & -1 < x < 0, \\ e^{-\sqrt{2\alpha}x}((1 + x)\sqrt{2\alpha} + 1), & 0 < x, \end{cases}$$

and

$$t(x) := g(x)\psi'_\alpha(x) - g'(x)\psi_\alpha(x)$$

$$= \begin{cases} 0, & x < -1, \\ e^{\sqrt{2\alpha}x}((1 + x)\sqrt{2\alpha} - 1), & -1 < x < 0, \\ e^{\sqrt{2\alpha}x}((1 + x)\sqrt{2\alpha} - 1) \\ + c\sqrt{2\alpha}((1 + x)\sqrt{2\alpha}\text{ch}(\sqrt{2\alpha}x) - \text{sh}(\sqrt{2\alpha}x)), & 0 < x. \end{cases}$$

Notice that these functions are multiples of expressions in (14) and (15), of the Martin representing measure if on the RHS we use $g$ instead of $u$. It is straightforward to check the following properties of $s$ and $t$:

(s1) $x \mapsto s(x)$ is decreasing for $x > -1$, 

19
(s2) \( \lim_{x \to +\infty} s(x) = 0, \)

(s3) \( \lim_{x \uparrow 0} s(x) = \sqrt{2\alpha} + 2\alpha c, \quad \lim_{x \downarrow 0} s(x) = \sqrt{2\alpha} + 1. \)

(t1) \( x \mapsto t(x) \) is increasing for \( x > -1, \)

(t2) \( \lim_{x \downarrow -1} t(x) = -e^{-\sqrt{2\alpha}} < 0, \quad \lim_{x \to +\infty} t(x) = +\infty, \)

(t3) \( \lim_{x \uparrow 0} t(x) = \sqrt{2\alpha} - 1, \quad \lim_{x \downarrow 0} t(x) = \sqrt{2\alpha} - 1 + 2\alpha c. \)

Let \( x^* \) denote the unique solution (if it exists) of the equation \( t(x) = 0 \) for \( x > -1, x \neq 0; \) in case there is no solution we put \( x^* = 0. \) Let \( x_\alpha > \max\{0, x^*\} \) and define

\[
\nu^\alpha_g((x, +\infty]) := \frac{\psi_\alpha(x_\alpha)}{w_\alpha g(x_\alpha)} s(x), \quad x \geq x_\alpha,
\]

and

\[
\nu^\alpha_g((-\infty, x]) := \begin{cases} 
0, & x \leq x^*, \\
\frac{\varphi_\alpha(x_\alpha)}{w_\alpha g(x_\alpha)} t(x), & x^* < x \leq x_\alpha.
\end{cases}
\]

where \( w_\alpha = 2\sqrt{2\alpha} + 2\alpha c. \) From the properties of \( s \) and \( t \) it is seen that these definitions induce a Borel measure on \( \mathbb{R}. \) Using the definition of the Wronskian \( w_\alpha \) we obtain

\[
\nu^\alpha_g((-\infty, x_\alpha)) + \nu^\alpha_g((x_\alpha, +\infty]) = 1.
\]

Therefore, setting \( \nu^\alpha_g(\{x_\alpha\}) = 0 \) makes \( \nu^\alpha_g \) a probability measure. Notice also that

\[
\nu^\alpha_g(\{-\infty\}) = \lim_{x \to -\infty} \nu^\alpha_g((-\infty, x]) = 0.
\]

and

\[
\nu^\alpha_g(\{+\infty\}) = \lim_{x \to +\infty} \nu^\alpha_g((x, +\infty]) = 0.
\]

The probability measure \( \nu^\alpha_g \) yields via the representation formula (13) the \( \alpha \)-excessive function

\[
V_\alpha(x) := \begin{cases} 
g(x^*) g(x_\alpha)^{-1} \psi_\alpha(x), & x \leq x^*, \\
\frac{1}{g(x_\alpha)} g(x), & x \geq x^*.
\end{cases}
\]

In this context we call \( \nu^\alpha_g \) the Martin representing measure of \( V_\alpha. \) Clearly, the function \( x \mapsto V^*(x) := g(x_\alpha)V_\alpha(x) \) does not depend on \( x_\alpha. \) We conclude with the following
Proposition 4.5. The function $V^*$ is the value function of OSP (42), i.e., $V^*$ is the smallest $\alpha$-excessive majorant of $g$. The optimal stopping time is $\tau^* = \inf\{t : X_t \geq x^*\}$. In particular, for $\alpha \in [\alpha_1, \alpha_2]$ with $\alpha_1 = (\sqrt{1 + 4\epsilon} - 1)^2/8\epsilon$ and $\alpha_2 = 1/2$ it holds that $x^* = 0$, \[ \frac{d^-V^*(0)}{dx} - \frac{d^+V^*(0)}{dx} = \sqrt{2\alpha} - 1 \leq 0, \] (45) and the Riesz representing measure has an atom at 0: \[ \sigma_{V^*}(\{0\}) = \sqrt{2\alpha} - 1 + 2\epsilon. \] (46) In case, $\alpha = 1/2$ the smooth fit holds and $\sigma_{V^*}(\{0\}) > 0$. If $\alpha = \alpha_1$ the smooth fit fails and $\sigma_{V^*}(\{0\}) = 0$.

Proof. By the construction, the function $V^*$ is $\alpha$-excessive. To prove that $V^*$ is a majorant of $g$ is straightforward and elementary from the explicit expressions. For a more sophisticated proof, notice that on $(-1, x^*)$ the function $q(x) := g(x)/\psi_\alpha(x)$ is increasing since $q'(x) = -t(x)/\psi^2_\alpha(x) > 0$ on $(-1, x^*)$ (and for $x \neq 0$ if $0 < x^*$). Consequently, for $x < x^*$ \[ \frac{g(x)}{\psi_\alpha(x)} < \frac{g(x^*)}{\psi_\alpha(x^*)} \Leftrightarrow g(x) < V^*(x). \]

Assume next that there exists an $\alpha$-excessive majorant $\hat{V}$ smaller than $V^*$. Consider first the case where the equation $t(x) = 0$ has a unique root on $(-1, +\infty) \setminus \{0\}$. We let, as above, $x^*$ denote this root. Since $\hat{V}$ is assumed to be an $\alpha$-excessive majorant of $g$ smaller than $V^*$ it holds that $\hat{V}(x) = V^*(x) = g(x)$ for $x \geq x^*$. Consequently, the Martin representing measures of $\hat{V}$ and $V^*$ are equal on $[x^*, +\infty)$ and given by (14) and (15). However, because $t(x^*) = 0$ the representing measure of $\hat{V}$ does not put mass on $[-\infty, x^*]$. Hence, the representing measures of $\hat{V}$ and $V^*$ are equal and so, by the uniqueness of the Martin representation, $\hat{V} = V^*$. In case $t$ does not have a zero on $(-1, +\infty) \setminus \{0\}$ the Martin representing measure of $V^*$ has an atom at $\{0\}$ given by \[ \nu_{V^*}(\{0\}) = c^* t(0+), \] (47) where $c^*$ is a non-negative constant given explicitly in (50). Since the representing measures of $V^*$ and $\hat{V}$ are equal on $(0, +\infty)$ and it is assumed that $\hat{V} \neq V^*$ we must have \[ \nu_{V^*}(\{0\}) > \nu_{\hat{V}}(\{0\}) \geq 0 \]
and
\[ \nu_{V^*}(\{0\}) = \nu_{\hat{V}}([-\infty, 0]) > 0. \]

Consider now the Martin representations of \( V^* \) and \( \hat{V} \) for \( x < 0 \):

\[
V^*(x) = \int_{(\infty, -\infty)} \frac{G_\alpha(x, y)}{G_\alpha(x_0, y)} \nu_{V^*}(dy)
\]

and

\[
\hat{V}(x) = \int_{(\infty, -\infty)} \frac{G_\alpha(x, y)}{G_\alpha(x_0, y)} \nu_{\hat{V}}(dy) + \varphi_\alpha(x) \nu_{\hat{V}}([-\infty])
\]

respectively. We show that, in fact, \( \hat{V}(x) > V^*(x) \) for all \( x < 0 \) contradicting the assumption that \( \hat{V} \) is smaller than \( V^* \). Indeed, for \( x < 0 \)

\[
\hat{V}(x) - V^*(x) = \int_{(\infty, x]} \frac{\psi_\alpha(y)\varphi_\alpha(x)}{\psi_\alpha(y)\varphi_\alpha(x_0)} \nu_{\hat{V}}(dy) + \int_{[x, 0]} \frac{\psi_\alpha(x)\varphi_\alpha(y)}{\psi_\alpha(y)\varphi_\alpha(x_0)} \nu_{\hat{V}}(dy)
\]

\[
+ \frac{\varphi_\alpha(x)}{\varphi_\alpha(x_0)} \nu_{\hat{V}}([-\infty]) - \frac{\psi_\alpha(x)\varphi_\alpha(0)}{\psi_\alpha(0)\varphi_\alpha(x_0)} \nu_{V^*}(\{0\}).
\]

Using that \( \varphi_\alpha(0) = \psi_\alpha(0) = 1 \) and \( \varphi_\alpha(x_0) > 0 \) it is seen that for \( x < 0 \)

\[
\hat{V}(x) - V^*(x) > 0 \quad (48)
\]

is equivalent with

\[
\varphi_\alpha(x) \nu_{\hat{V}}([-\infty, x]) + \psi_\alpha(x) \int_{[x, 0]} \frac{\varphi_\alpha(y)}{\psi_\alpha(y)} \nu_{\hat{V}}(dy) - \psi_\alpha(x) \nu_{V^*}(\{0\}) > 0.
\]

Since \( y \mapsto \varphi_\alpha(y)/\psi_\alpha(y) \) is decreasing (48) holds if

\[
\varphi_\alpha(x) \nu_{\hat{V}}([-\infty, x]) + \psi_\alpha(x) \nu_{\hat{V}}([x, 0]) - \psi_\alpha(x) \nu_{V^*}(\{0\}) > 0. \quad (49)
\]

Observing that

\[
\nu_{V^*}(\{0\}) = \nu_{\hat{V}}([-\infty, x]) + \nu_{\hat{V}}((x, 0])
\]

it is seen that (49) is true if for all \( x < 0 \)

\[
\varphi_\alpha(x) - \psi_\alpha(x) > 0,
\]

and to check this is elementary from (40) and (41) or follows directly from the monotonicity. This completes the proof that \( V^* \) is the smallest \( \alpha \)-excessive majorant of \( g \).
It remains to prove (45) and (46). Letting \( x \to 0^+ \) in (43) it is seen (cf. (47)) that

\[ \nu_{V^*}(\{0\}) = c^* t(0^+) = g(x_o) \nu_0^0(\{0\}) = \frac{\varphi_0(x_o)}{\omega_\alpha} (\sqrt{2\alpha} - 1 + 2\alpha c). \]  \hspace{1cm} (50)

Using (16) we find the atom of the Riesz representing measure of \( V^* \)

\[ \sigma_{V^*}(\{0\}) = \frac{1}{G_\alpha(x_o, 0)} \nu_{V^*}(\{0\}) = \sqrt{2\alpha} - 1 + 2\alpha c. \]

It follows from (32) (and can also be checked directly from (44)) that

\[
\begin{align*}
\frac{d^- V^*}{dx}(0) - \frac{d^+ V^*}{dx}(0) &= \sigma_{V^*}(\{0\}) - 2\alpha m(\{0\}) V^*(0) \\
&= \sqrt{2\alpha} - 1 + 2\alpha c - 2\alpha c \\
&= \sqrt{2\alpha} - 1 \leq 0,
\end{align*}
\]

as claimed.

\[ \square \]

Figure 1: Function \( x \mapsto t(x) \), \( \alpha = 0.25 \), \( c_1 = 1 \), \( x^* = 0 \).
Figure 2: Function $x \mapsto t(x)$, $\alpha = 0.1$, $c_1 = 1$, $x^* > 0$.

Figure 3: Function $x \mapsto t(x)$, $\alpha = 0.6$, $c_1 = 1$, $x^* < 0$.

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