A theory of nice triples and a theorem due to O.Gabber

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Abstract

In a series of papers [Pan0], [Pan1], [Pan2], [Pan3] we give a detailed and better structured proof of the Grothendieck–Serre’s conjecture for semi-local regular rings containing a finite field. The outline of the proof is the same as in [P1], [P2], [P3]. If the semi-local regular ring contains an infinite field, then the conjecture is proved in [FP]. Thus the conjecture is true for regular local rings containing a field.

The present paper is the one [Pan0] in that series. Theorem 1.2 is one of the main result of the paper. The proof of the latter theorem is completely geometric. It is based on a theory of nice triples from [PSV] and on its extension from [P]. The theory of nice triples is inspired by the Voevodsky theory of standard triples [Voe].

Theorem 1.2 yields an unpublished result due to O.Gabber (see the theorem 1.1=the theorem 3.1).

1 Main results

Let $R$ be a commutative unital ring. Recall that an $R$-group scheme $G$ is called reductive, if it is affine and smooth as an $R$-scheme and if, moreover, for each algebraically closed field $\Omega$ and for each ring homomorphism $R \rightarrow \Omega$ the scalar extension $G_\Omega$ is a connected reductive algebraic group over $\Omega$. This definition of a reductive $R$-group scheme coincides with [SGA3] Exp. XIX, Definition 2.7. A well-known conjecture due to J.-P. Serre and A. Grothendieck (see [Se] Remarque, p.31), [Gr1] Remarque 3, p.26-27, and [Gr2] Remarque 1.11.a]) asserts that given a regular local ring $R$ and its field of fractions $K$ and given a reductive group scheme $G$ over $R$, the map

$$H^1_{\text{ét}}(R, G) \rightarrow H^1_{\text{ét}}(K, G),$$

induced by the inclusion of $R$ into $K$, has a trivial kernel. If $R$ contains an infinite field, then the conjecture is proved in [FP].

For a scheme $U$ we denote by $A^1_U$ the affine line over $U$ and by $P^1_U$ the projective line over $U$. Let $T$ be a $U$-scheme. By a principal $G$-bundle over $T$ we understand a principal

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$G \times_U T$-bundle. We refer to [SGA3, Exp. XXIV, Sect. 5.3] for the definitions of a simple simply-connected group scheme over a scheme and a semi-simple simply-connected group scheme over a scheme.

**Theorem 1.1.** Let $k$ be a finite field. Let $\mathcal{O}$ be the semi-local ring of finitely many closed points on a $k$-smooth irreducible affine $k$-variety $X$ and let $K$ be its field of fractions. Let $G$ be a simply-connected reductive group scheme over $k$. Then the map

$$H^1_{et}(\mathcal{O}, G) \to H^1_{et}(K, G),$$

induced by the inclusion $\mathcal{O}$ into $K$, has trivial kernel.

The latter theorem is an unpublished theorem due to O. Gabber.

**Theorem 1.2.** Let $k$ be a field. Let $\mathcal{O}$ be the semi-local ring of finitely many closed points on a $k$-smooth irreducible affine $k$-variety $X$ and let $K$ be its field of fractions. Let $G$ be a reductive group scheme over $k$. Let $G$ be a principal $G$-bundle over $U$ trivial over the generic point of $U$. Then there exists a principal $G$-bundle $\tau h$ over the affine line $A^1_U \simeq \text{Spec } \mathcal{O}[t]$ and a monic polynomial $h(t) \in \mathcal{O}[t]$ such that

(i) the $G$-bundle $\tau h$ is trivial over the open subscheme $(A^1_U)_h$ in $A^1_U$ given by $h(t) \neq 0$;

(ii) the restriction of $\tau h$ to $\{0\} \times U$ coincides with the original $G$-bundle $\tau$.

(iii) $h(1) \in \mathcal{O}$ is a unit.

If the field $k$ is infinite a stronger result is proved in [PSV, Thm. 1.2]. Theorem 1.2 is easily derived from Theorem 1.3 (Geometric).

**Theorem 1.3** (Geometric). Let $X$ be an affine $k$-smooth irreducible $k$-variety, and let $x_1, x_2, \ldots, x_n$ be closed points in $X$. Let $U = \text{Spec } \mathcal{O}_{x_1, x_2, \ldots, x_n}$ and $f \in k[X]$ be a non-zero function vanishing at each point $x_i$. Then there is a monic polynomial $h \in \mathcal{O}_{x_1, x_2, \ldots, x_n}[t]$, a commutative diagram of schemes with the irreducible affine $U$-smooth $Y$

$$\begin{array}{ccc}
(A^1 \times U)_h & \xrightarrow{\tau h} & Y_h := Y_{f(h)} \xrightarrow{(p_X)_h} X_f \\
\text{inc} & & \text{inc} \\
(A^1 \times U) & \xrightarrow{\tau} & Y \xrightarrow{p_X} X
\end{array}$$

and a morphism $\delta : U \to Y$ subjecting to the following conditions:

(i) the left hand side square is an elementary distinguished square in the category of affine $U$-smooth schemes in the sense of [MV, Defn. 3.1.3];

(ii) $p_X \circ \delta = \text{can} : U \to X$, where $\text{can}$ is the canonical morphism;

(iii) $\tau \circ \delta = i_0 : U \to A^1 \times U$ is the zero section of the projection $\text{pr}_U : A^1 \times U \to U$;

(iv) $h(1) \in \mathcal{O}[t]$ is a unit.
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2 Proof of Theorem 1.2

Proof of Theorem 1.2. The $U$-group scheme $G$ is defined over the base field $k$. We may and will suppose that the principal $G$-bundle $G'$ is the restriction to $U$ of a principal $G$-bundle $G'$ on $X$, and the restriction of $G'$ to an principal open subset $X_f$ is trivial. If $U = \text{Spec}(\mathcal{O}_{x_1, \ldots, x_n})$, then we may and will suppose that $f$ vanishes at each point $x_i$.

Theorem 1.3 (=[P, Thm. 1.2]) states that there are a monic polynomial $h \in \mathcal{O}_{x_1, \ldots, x_n}[t]$, a commutative diagram (1) of schemes with the irreducible affine $U$-smooth $Y$, and a morphism $\delta : U \to Y$ subjecting to conditions (i) to (iv) from Theorem 1.3.

Now take the monic polynomial $h \in \mathcal{O}_{x_1, \ldots, x_n}[t]$ as the desired polynomial and construct the desired principal $G$-bundle on $A^1 \times U$ as follows.

Take the pull-back $p^*_X(G')$ of $G'$ to $Y$. The restriction of $p^*_X(G')$ to $Y_h$ is trivial, since the restriction of $G'$ to $X_f$ is trivial. Take now the trivial $G$-bundle over the principal open subset $(A^1 \times U)_h$ and glue it with $p^*_X(G')$ via an isomorphism over $Y_h$. This way we get a principal $G$-bundle $G_t$ over $A^1 \times U$. Clearly, the monic polynomial $h$ and the principal $G$-bundle on $A^1 \times U$ are the desired ones. 

3 Simply-connected case of a theorem due to Gabber

An unpublished theorem due to Gabber states particularly that if the base field $k$ is finite, then the Grothendieck–Serre conjecture is true for any reductive group scheme $G$ over $k$. The main aim of the present section is to recover that result in the simply-connected case.

Theorem 3.1. Let $k$ be a finite field and let $R$ be a regular local ring containing $k$, and let $K$ be its field of fractions. Given a simply-connected reductive group scheme $G$ over $k$, the map

$$H^1_{\text{et}}(R, G) \to H^1_{\text{et}}(K, G),$$

induced by the inclusion of $R$ into $K$, has a trivial kernel.

Proof. The case of a general regular local ring containing $k$ is easily reduced to the case, when $R$ is the semi-local ring of a finitely many closed points on an affine $k$-smooth variety $X$. Moreover we may and will suppose that $G$ is a simple simply-connected $k$-group.

The $k$-group scheme $G$ is defined over $k$ and $k$ is finite, and $G$ is a simple simply-connected $k$-group. Hence, $G$ contains a $k$-Borel subgroup scheme. Particularly, $G$ is isotropic. The bundle $G_t$ and the monic polynomial $h$ from Theorem 1.3 satisfy the hypotheses of Theorem [PSV Thm.1.3]. Thus the principal $G$-bundle $G_t$ from Theorem
$1.2$ is trivial. Hence the restriction of $\mathcal{G}_t$ to $\{0\} \times U$ is trivial. From the other side the latter restriction coincides with the original $G$-bundle $\mathcal{G}$. Hence the original $G$-bundle $\mathcal{G}$ is trivial.

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