Point vortices in the plane: positive-dimensional configurations.

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Abstract. The problem of constructing and classifying stationary equilibria of point vortices in the plane is studied. An ordinary differential equation that enables one to find positions of point vortices with circulations $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ in stationary equilibrium is obtained. A necessary condition of an equilibrium existing is derived. The case of point vortex systems consisting of $n + 2$ point vortices with $n$ vortices of circulation $\Gamma_1$ and two vortices of circulations $\Gamma_2 = a\Gamma_1$ and $\Gamma_3 = b\Gamma_1$, where $a$ and $b$ are integers, is considered in detail. The properties of polynomial solutions of the corresponding ordinary differential equation are investigated. A set of positive–dimensional equilibrium configurations is found. A continuous free parameter is presented in the coefficients of corresponding polynomial solutions. These free parameters affect the positions of the roots and hence the vortex positions. Stationary equilibrium that could be derived from each other by rotation, extension, parallel translation is considered as equivalent. All found configurations seem to be new.

1. Introduction
The motion of $M$ point vortices with circulations (or strengths) $\Gamma_1$, $\ldots$, $\Gamma_M$ at positions $z_1$, $\ldots$, $z_M$ is governed by the Helmholtz’s equations

$$\frac{dz_k}{dt} = \frac{1}{2\pi i} \sum_{j=1}^{M} \frac{\Gamma_j}{z_k - z_j}, \quad k = 1, \ldots, M, \quad (1)$$

where the prime means that the case $j = k$ is excluded and the symbol $*$ stands for complex conjugation. This system induces complicated dynamics and is not integrable in the case $M > 3$. Nevertheless some particular “motions” including stationary, translating, and rotating equilibria are successfully studied. A convenient approach applicable to such types of motion is the so–called “polynomial method”. According to this method, polynomials with roots at vortex positions are introduced. In 1964 Tkachenko obtained a differential equation satisfied by generating polynomials of stationary vortex arrangements with equal in absolute value circulations [1]. Now this equation is known as the Tkachenko equation. A simpler derivation of the Tkachenko equation was proposed by Aref [2]. Further the polynomial method was generalized to the case of vortices with generic choice of circulations [3, 4, 5]. In addition it was shown that the polynomial method can be successfully used to construct rotating equilibria and configurations of collapse and scattering [6].
2. Equation of stationary equilibria of point vortices

Consider a system of \( M \) point vortices with the circulations \( \Gamma_\gamma, \gamma = 1, \ldots, N \) from equations (1). We set \( \frac{dz^*}{dt} = 0 \) in the equations of motion (1) and subdivide the vortices into groups according to the values of circulations. Suppose vortices at positions \( a_1^{\gamma}, \ldots, a_l^{\gamma} \) have circulations \( \Gamma_\gamma, \gamma = 1, \ldots, N \).

\[
\sum_{\gamma=1}^{N} \sum_{i=1}^{l_\gamma} \frac{\Gamma_\gamma}{a_i^{\gamma} - a_i^0} = 0, \quad i_0 = 1, \ldots, l_\gamma, \quad \gamma_0 = 1, \ldots, N \tag{2}
\]

A convenient tool for analyzing such a situation is to introduce polynomials with roots at the positions of vortices (2). In our case, we have

\[
P_\gamma(z) = \prod_{i=1}^{l_\gamma} (z - a_i^{\gamma}), \quad \gamma = 1, \ldots, N \tag{3}
\]

Thus, we see that roots of the polynomial \( P_\gamma(z) \) give positions of vortices with circulation \( \Gamma_\gamma \).

Furthermore, the following approach is proposed which consists in introducing new functions \( \tilde{P}(z) \) and \( \tilde{Q}(z) \), according to the rules:

\[
\tilde{P}(z) = \prod_{\gamma=1}^{N} P_\gamma^{\frac{\Gamma_\gamma(\Gamma_\gamma+1)}{2}}(z), \quad \tilde{Q}(z) = \prod_{\gamma=1}^{N} P_\gamma^{\frac{\Gamma_\gamma(\Gamma_\gamma-1)}{2}}(z) \tag{4}
\]

Using the equalities

\[
\frac{d^2}{dz^2} \ln \left\{ \tilde{P}(z)\tilde{Q}(z) \right\} + \left( \frac{d}{dz} \ln \left\{ \frac{\tilde{P}(z)}{\tilde{Q}(z)} \right\} \right)^2 = 0 \tag{5}
\]

present one can (5) in the more convenient form

\[
\tilde{P}_{zz}\tilde{Q} - 2\tilde{P}_z\tilde{Q}_z + \tilde{P}\tilde{Q}_{zz} = 0 \tag{6}
\]

Equation (6) is known as the Tkachenko equation. There exists necessary condition of an equilibrium in the arrangements of vortices:

\[
N \sum_{\gamma=1}^{N} l_\gamma \Gamma_\gamma^2 - \left[ \sum_{\gamma=1}^{N} l_\gamma \Gamma_\gamma \right]^2 = 0 \tag{7}
\]

Using the polynomial method we derive the ordinary differential equation

\[
(z^2 - 1) \frac{d^2}{dz^2} w(z) + 2((a + b)z + (b - a)) \frac{d}{dz} w(z) + 2abw(z) = 0 \tag{8}
\]

describing stationary equilibria of point vortex systems consisting of \( n + 2 \) point vortices with \( n \) vortices of circulation \( \Gamma_1 \) and two vortices of circulations \( \Gamma_2 = a\Gamma_1 \) and \( \Gamma_3 = b\Gamma_1 \), where \( a \) and \( b \) are integers. In (8) \( w(z) \) is a polynomial of degree \( n \).

\[
w_n(z) = C_n z^n + C_{n-1} z^{n-1} + C_{x-2} z^{n-2} + \ldots + C_1 z + C_0, \tag{9}
\]

We set \( C_n = 1 \). Thus, \( w(z) \) is unitary polynomial.
3. Theorem on the existence of a polynomial solution of equation (8)
Suppose that solution of equation (8) has the form:

\[ w(z) = \ldots + C_nz^n + C_{n-1}z^{n-1} + \ldots + C_{k+1}z^{k+1} + C_kz^k + C_{k-1}z^{k-1} + \ldots + C_1z + C_0 + C_{-1}z^{-1} + C_{-2}z^{-2} + \ldots \]  

Substituting the series (10) in equation (8) and taking into account the summands for different degrees \( z \), we get:

\[ \ldots + B_nz^n + B_{n-1}z^{n-1} + \ldots + B_{k+1}z^{k+1} + B_kz^k + B_{k-1}z^{k-1} + \ldots + B_1z + B_0 + B_{-1}z^{-1} + \ldots = 0 \]  

Coefficient at \( z^k \) takes the form:

\[ B_k = C_k(2ab + 2k(a + b) + k(k - 1)) + C_{k+1}2(k + 1)(b - a) - C_{k+2}(k + 1)(k + 2) = 0 \]  

Consider a natural number \( n \). We assume that the solution of the equation is a polynomial of degree \( n \). Let \( C_n = 1 \). Taking into account that \( w(z) \) is a polynomial of degree \( n \) and coefficients \( C_{n+1} = 0 \) and \( C_{n+2} = 0 \), we obtain:

\[ B_n = 2ab + 2n(a + b) + n(n - 1) = 0 \]  

Let us to prove the following lemma:

**Lemma 1.** The Diophantine equation (13) is a necessary condition for the existence of a polynomial solution of equation (8) of degree \( n \).

We set \( a \) is free parameter in (13) and find the dependence of \( b \) on \( a \). Further consider all the coefficients \( B_k \) from \( k = n \) to \( k = 0 \) consistently, and express at each step \( C_k \) through the two preceding factors \( C_{k+1} \) and \( C_{k+2} \). If \( k < 0 \) then \( C_{k+1} \), \( C_{k+2} \) are equal 0. Eventually, expression (10) is a polynomial of degree \( n \) and equation (8) has one polynomial solution.

**Lemma 2.** Solution of the equation (8) has not roots at the points \( z = \pm 1 \).

**Proof.** The proof is by reductio ad absurdum. Consider \( z = 1 \). Assume that the point \( z = 1 \) is root of a polynomial of multiplicity \( r \in \mathbb{N} \).

\[ w(z) = (z - 1)^r \psi(z), \quad \psi(1) \neq 0 \]  

Twice differentiating \( w(z) \) with respect to \( z \) we substitute the result into equation (8). Presenting similar terms with respect to the degrees of expression \((z - 1)\). Finding the Taylor series in a neighborhood of the point \( z = 1 \), we obtain:

\[ 2r(r - 1)\psi(1) + 4br\psi(1) = 2r\psi(1)((r - 1) + 2b) = 0 \]  

If \( r = 0 \) then the condition of equality to zero is always satisfied. If \( r \neq 0 \), we have:

\[ r = 1 - 2b \]  

\( r \) is positive integer (\( r \) is degree of multiplicity of the root of a polynomial). If \( b \) has a half-integer negative value then condition (16) is true. We do not consider such cases because we have an
interest in cases when \( b \) is positive integer. Thus, only one case suits us: \( r = 0 \). Contradiction is obtained as it follows that \( z = 1 \) isn’t polynomial root \( w(z) \).

Similar arguments are valid when \( z = -1 \). The lemma is proved.

**Lemma 3.** If the polynomial is a solution of the equation (8), then all its roots are simple.

**Proof.** Suppose \( w_n(z) \) is the solution of equation (8). Let \( w_n(z) \) be given by the form (9).

Expanding the polynomial \( w_n(z) \) by the Taylor formula at the point \( z_0 \). \( z_0 \) is assumed equal to one of the roots of the given polynomial. We construct a general formula for the derived degree \( q \) of equation (8). Everyone can verify the validity of the formula for \( q = k - 2 \) by induction:

\[
(z^2 - 1)w^{(k)}(z) + 2((k - 2 + a + b)z + 2(b - a))w^{(k-1)}(z) + ((k - 2)((k - 3) + 2(a + b)) + 2ab)w^{(k-2)}(z) = 0 \quad k \geq 2
\]

(17)

The proof is by reduxitio ad absurdum. Assume that \( z_0 \) is not simple root.

\[
w(z_0) = 0 \quad w'(z_0) = 0. \tag{18}
\]

Each successive derivative of the polynomial (9) can be expressed through the two preceding ones using the formulas (17). It follows from (18) and Lemma 2 that all subsequent derivatives of \( w_k \) will be zero. We see that all coefficients in the Taylor expression (3.9) are equal to zero. As a result, function \( w_n \) is equal to zero identically. This contradiction proves the lemma. Hence, all the roots of the polynomial are simple.

It is a well-known fact [10] that two linearly independent solutions of an ordinary differential equation of the second order

\[
y'' + p(x)y' + q(x)y = 0 \tag{19}
\]

are related by:

\[
y_2(x) = y_1(x) \int e^{-\int p(x)dx} \frac{dx}{y_1^2(x)} \tag{20}
\]

The general solution of the equation (19) has the form:

\[
y = C_1y_1 + C_2y_2, \quad \text{where} \quad C_1, C_2 \text{ are constants} \tag{21}
\]

It is a special case when the equation (8) has two polynomial solutions. In this case, we get an infinite number of stationary equilibrium of point vortices. It follows from (21). We prove the following theorem.

**Theorem.** If the Diophantine equation

\[
n(n - 1) + 2(a + b)n + 2ab = 0 \tag{22}
\]

with fixed integer values of the parameters \( a \) and \( b \) has two integer solutions, then the general solution of ordinary differential equation (8) is polynomial.

**Proof.** Representing the equation (8) in the form (19)

\[
w''(z) + \frac{(2(a + b)z + 2(b - a))}{(z^2 - 1)}w'(z) + \frac{2ab}{z^2 - 1}w(z) = 0 \tag{23}
\]
and substituting (20) in this equation, we get:

$$w_2(z) = w_1(z) \int (z + 1)^{2a} (z - 1)^{2b} \frac{dz}{w_1^2(z)} \quad (24)$$

We want to show that there exists the polynomial solutions of equation (8). This solution is a polynomial of a lesser degree (of two polynomial solutions).

We must prove that the solution $w_2(z)$ has no singularity. Consider expression (24). All the roots of the polynomial $w_1(z)$ are simple. (It follows from Lemma (3)) Consequently, the generalized Laurent series in a neighborhood of $z_0$ has the form

$$w_1^2 = K^2(z - z_0)^2, \quad K \neq 0 \quad (25)$$

Substituting the series (25) into (24):

$$w_2 = K(z - z_0) \int \left( \frac{\kappa_{-2}}{(z - z_0)^2} + \frac{\kappa_{-1}}{(z - z_0)} + \kappa_0 + \ldots \right) dz \quad (26)$$

Represent in the form:

$$w_2 = K(z - z_0) \left( -\frac{\tilde{\kappa}_{-2}}{z - z_0} + \tilde{\kappa}_{-1} \ln(z - z_0) + \ldots \right) \quad (27)$$

The form of the expression shows that the solution $w_2(z)$ can have poles. We apply the analysis by the theory of residues. This shows that we obtain a rational function after integrating. This function becomes a polynomial when multiplied by $w_1(z)$ (This follows from the formula (24)). Consequently, the equation (8) has a polynomial solution. The theorem is proved.

4. Classification of configurations

In this paper, we study the problem of finding, researching and classifying stationary equilibrium of point vortices in the plane. We considered the case when stationary equilibrium of point vortices consists of $n$ point vortices with circulations $\Gamma_2 = a\Gamma_1$ and another one with circulations $\Gamma_3 = b\Gamma_1$, where $a$ and $b$ are integers. The values of circulations are found for which the number of nonequivalent stationary equilibrium of point vortices is infinite. Some examples are given in Tab.1

As a consequence of this theorem we see that a continuous free parameter is present in the coefficients of corresponding polynomial solutions whenever the condition of the theorem is valid. These free parameters affect the positions of the roots and hence the vortex positions. Using this theorem we find new positive–dimensional equilibrium configurations. Until recently only few sets of such configurations have been known.

Some examples are given in Fig.1.

$$w_2^{(-5,-3)}(z) = 7z^2 + 4z + 1; \quad (28)$$

$$w^{(-5,-3)}_{15}(z) = 91 - 182z + 455z^2 - 455z^3 + 91z^4 + 273z^5 - 221z^6 - 65z^7 + \left( \frac{455}{3} \right)z^8 - \left( \frac{1547}{33} \right)z^9 + \left( \frac{273}{11} \right)z^{10} + \left( \frac{35}{11} \right)z^{11} - 5z^{12} + z^{13} + z^{14};$$

Thus, the general solution of the equation (8) (where $a = -5, \; b = -3$ ) has the form :

$$W^{(-5,-3)}_{15}(z) = C_1(7z^2 + 4z + 1) + C_2(91 - 182z + 455z^2 - 455z^3 + 91z^4 + 273z^5 - 221z^6 -$$

$$- 65z^7 + \left( \frac{455}{3} \right)z^8 - \left( \frac{91}{3} \right)z^9 - \left( \frac{1547}{33} \right)z^{10} + \left( \frac{273}{11} \right)z^{11} + \left( \frac{35}{11} \right)z^{12} - 5z^{13} + z^{14});$$
These results can be summarized as follows. Varying the values of the variables $C_1$ and $C_2$ in a polynomial (4) we get that configurations pass from one to another.

Table 1. The results of the search for parameter values $a, b, n$ in the case of infinite number of nonequivalent stationary equilibrium of point vortices. $n_1$ - the degree of the smallest polynomial is a solution of equation (8), $l$ - the number of pairs of coefficients with an infinite number of nonequivalent stationary equilibrium, $n_2$ - the degree of the largest polynomial is a solution of equation (8)

| $n_1$ | $l$ | $a$ | $b$ | $n_2$ | the general solution of equation (8) | number of configurations |
|-------|-----|-----|-----|-------|-------------------------------------|-------------------------|
| 2     | 1   | -5  | -3  | 15    | $C_1 w_2^{(-5,-3)} + C_2 w_{15}^{(-5,-3)}$ | $\infty$               |
| 3     | 2   | -9  | -4  | 24    | $C_1 w_3^{(-9,-4)} + C_2 w_{24}^{(-9,-4)}$ | $\infty$               |
|       |     | -6  | -5  | 20    | $C_1 w_3^{(-6,-5)} + C_2 w_{20}^{(-6,-4)}$ | $\infty$               |
| 4     | 2   | -14 | -5  | 35    | $C_1 w_4^{(-14,-5)} + C_2 w_{35}^{(-14,-5)}$ | $\infty$               |
|       |     | -9  | -6  | 27    | $C_1 w_4^{(-9,-6)} + C_2 w_{27}^{(-9,-6)}$ | $\infty$               |
| 5     | 2   | -20 | -6  | 48    | $C_1 w_5^{(-20,-6)} + C_2 w_{48}^{(-20,-6)}$ | $\infty$               |
|       |     | -10 | -8  | 32    | $C_1 w_5^{(-10,-8)} + C_2 w_{32}^{(-10,-8)}$ | $\infty$               |
| 6     | 2   | -27 | -7  | 63    | $C_1 w_6^{(-27,-7)} + C_2 w_{63}^{(-27,-7)}$ | $\infty$               |
|       |     | -13 | -9  | 39    | $C_1 w_6^{(-13,-9)} + C_2 w_{39}^{(-13,-9)}$ | $\infty$               |
| 7     | 3   | -35 | -8  | 80    | $C_1 w_7^{(-35,-8)} + C_2 w_{80}^{(-35,-8)}$ | $\infty$               |
|       |     | -21 | -9  | 54    | $C_1 w_7^{(-21,-9)} + C_2 w_{54}^{(-21,-9)}$ | $\infty$               |
|       |     | -14 | -11 | 44    | $C_1 w_7^{(-14,-11)} + C_2 w_{44}^{(-14,-11)}$ | $\infty$               |
| 8     | 4   | -44 | -9  | 99    | $C_1 w_8^{(-44,-9)} + C_2 w_{99}^{(-44,-9)}$ | $\infty$               |
|       |     | -26 | -10 | 65    | $C_1 w_8^{(-26,-10)} + C_2 w_{65}^{(-26,-10)}$ | $\infty$               |
|       |     | -20 | -11 | 55    | $C_1 w_8^{(-20,-11)} + C_2 w_{55}^{(-20,-11)}$ | $\infty$               |
|       |     | -17 | -12 | 51    | $C_1 w_8^{(-17,-12)} + C_2 w_{51}^{(-17,-12)}$ | $\infty$               |

5. Conclusion
In this article we have studied the problem of finding any stationary configurations of point vortices with circulations $\Gamma_1, \Gamma_2, \Gamma_3$ in the plane. We considered the case when stationary equilibrium of point vortices consists of $n$ point vortices with circulations $\Gamma_2 = a\Gamma_1$ and another one with circulations $\Gamma_3 = b\Gamma_1$, where $a$ and $b$ are integers. The values of circulations have been found for which the number of nonequivalent stationary equilibrium of point vortices is infinite.
**Figure 1.** Plots of vortex positions in stationary equilibrium. Circles denote vortices with circulation $\Gamma_1$, squares denote vortices with circulation $\Gamma_2$ and rhombus denote vortices with circulation $\Gamma_3$.

(a) $z(n)$, $n=63$, $C_1=10$, $C_2=1$,
(b) $z(n)$, $n=63$, $C_1=10^{11}$, $C_2=10^{11}$,
(c) $z(n)$, $n=63$, $C_1=10$, $C_2=10^{12}$,
(d) $z(n)$, $n=63$, $C_1=10$, $C_2=10^{14}$

Some examples are given in Tab.1. The obtained equation is investigated. In particular, the existence of a polynomial solution is proved (Theorem 1).

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