Exact Chiral Invariance at Finite Density on Lattice

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Abstract. A new lattice action is proposed for the overlap Dirac matrix with nonzero chemical potential. It is shown to preserve the full chiral invariance for all values of lattice spacing exactly. It is further demonstrated to arise in the domain wall formalism by coupling the chemical potential only to the physically relevant wall modes.

1. Introduction
A fundamental aspect of the phase diagram of Quantum Chromo Dynamics (QCD) in the $T$-$\mu_B$ plane is the critical point, where $T$ and $\mu_B$ denote the the temperature and baryonic chemical potential respectively. Theoretical [1, 2, 3] as well as experimental searches for locating it are currently going on [4]. Its discovery would be exciting in many ways. Apart from becoming a new milestone in our understanding of the nature of the strongly interacting matter, it would also be unique compared to the other known phase diagrams in the way theory and experiment compliment each other in locating the critical point in it. It is desirable to predict the existence of the QCD critical point (or rule it out) starting from first principles, employing only the QCD Hamiltonian. Lattice QCD is the best suited tool for such an exploratory exercise in view of its success in other non-perturbative aspects.

The prime reason for expecting the critical point are the chiral symmetries QCD has due to the quark mass spectra. Our world has two light quarks, having masses much smaller than $\Lambda_{QCD}$, and a moderately heavy quark, making it a ‘2+1’ flavour QCD. All attempts to explore nonzero $\mu_B$ on the lattice use the staggered quarks. They have a $U(1)_V \times U(1)_A$ chiral symmetry on the lattice, and a corresponding order parameter, the chiral condensate. Staggered quarks, however, break the flavour and spin symmetry on the lattice, and have no flavour singlet axial anomaly. In the continuum limit of $a \to 0$, they have 4 quark ‘flavours’ of the same mass. The ‘2+1’ flavour QCD is approximated by taking square root and fourth root of the staggered quark determinant. Whether this can restore the correct flavour symmetry as well as the chiral anomaly in the continuum limit has been a controversial issue. Moreover, it has been argued the ‘rooting problem’ becomes even worse [5] at nonzero $\mu_B$. Since the QCD critical point is expected [6] to exist if only if one has two light flavours, and the flavour singlet anomaly is mildly temperature dependent [7], as in our world, it appears desirable to improve upon them. The
overlap Dirac fermions [8], or the closely related domain wall fermions [9], offer such a possibility to improve. Indeed, the overlap quarks have all the symmetries of the continuum QCD and also have an index theorem [10] as well, raising the hope that even the anomaly effects could be well treated. Unfortunately, adding the chemical potential turns out to be nontrivial for them. Bloch and Wettig [11] made a proposal to do so but it violates [12] the exact chiral invariance on the lattice as does the simple addition [13] of a baryon number term. In this talk I report on our alternative action [14] which does have exact and continuum-like chiral invariance on lattice for any value of the lattice spacing and any chemical potential. Its chiral order parameter permits, in principle, the task of mapping out the $T-\mu_B$ phase diagram, assuming that the algorithmic developments can handle the fermion sign problem well.

2. Exact Chiral Invariance for $\mu \neq 0$

We begin by noting that the massless continuum QCD action can be written in a form where the chiral symmetry is manifest in terms of the fields appearing in the action and ask [14] if this can be imitated on lattice. The continuum QCD action at finite density in an explicit chiral

\[ S_{QCD} = \int d^3x \, d\tau [\bar{\psi}_L(D + \mu \gamma^4)\psi_L + \bar{\psi}_R(D + \mu \gamma^4)\psi_R - F_{\mu\nu} F_{\mu\nu}/4] , \] (1)

where $\psi_L = (1 - \gamma_5)\psi/2$ and $\psi_R = (1 + \gamma_5)\psi/2$ with $\bar{\psi}_L = \bar{\psi}(1 + \gamma_5)/2$ and $\bar{\psi}_R = \bar{\psi}(1 - \gamma_5)/2$. The last term in the action is for gluons and will play no role in our discussion below. We assume that some usual convenient lattice form for it has been chosen and focus on the lattice fermionic action only. We propose that addition of chemical potential on the lattice be done in such an explicit chiral symmetry preserving manner as well, opting therefore for the overlap quarks on the lattice.

The overlap quarks have all the symmetries of the continuum QCD but also have a nonlocal action:

\[ S_F = \sum_{n,m} \bar{\psi}_n aD_{ov,nm} \psi_m , \] (2)

where the sum over $n$ and $m$ runs over all the space-time lattice sites, $a$ is the lattice spacing, and the overlap Dirac matrix $D_{ov}$ is defined by $aD_{ov} = 1 + \gamma_5 sgn(\gamma_5 D_W)$. $sgn$ denotes the sign function. $D_W$ is the standard Wilson-Dirac matrix on the lattice but with a negative mass term $M \in (0, 2)$:

\[ aD_W(x,y) = (4 - M)\delta_{x,y} - \sum_{i=1}^{4} [U_i^\dagger(x-i)\delta_{x-i,y} \frac{1 + \gamma_i}{2} + \frac{1 - \gamma_i}{2} U_i(x) \delta_{x+i,y}] . \] (3)

The overlap Dirac matrix satisfies the Ginsparg-Wilson relation [15], $\{\gamma_5, D\} = aD\gamma_5 D$ and has exact chiral symmetry on lattice. The corresponding infinitesimal chiral transformations [16] are

\[ \delta\psi = i\alpha\gamma_5(1 - aD_{ov})\psi \quad \text{and} \quad \delta\bar{\psi} = i\alpha\bar{\psi}\gamma_5 . \] (4)

The generators of the transformation in Eq.(4) satisfy $\gamma_5^2 = 1$ and $\gamma_5^2 \equiv [\gamma_5(1 - aD_{ov})]^2 = 1$. One therefore defines the left-right projections for quark fields as $\psi_L = (1 - \gamma_5)\psi/2$ and $\psi_R = (1 + \gamma_5)\psi/2$, leaving the antiquark field decomposition as in the continuum. Such a decomposition is commonly done for writing chiral gauge theories [17] on the lattice and is possible since $\psi$ and $\bar{\psi}$ are independent fields in the Euclidean field theory. In analogy with the Eq.(1) of continuum QCD, the action for the overlap quarks in presence of nonzero chemical
The domain wall action of [18] for a single flavour of quark, i.e.,
\[ S = \sum_{n,m} [\bar{\psi}_{n,L} (aD_{ov} + a\mu \gamma^4)_{nm} \psi_{m,L} + \bar{\psi}_{n,R} (aD_{ov} + a\mu \gamma^4)_{nm} \psi_{m,R}] \]  
\[ = \sum_{n,m} \bar{\psi}_n [(1 - a\mu \gamma^4/2) aD_{ov} + a\mu \gamma^4]_{nm} \psi_m. \]  

It is easy to verify that this action is invariant under the chiral transformation Eq.(4) for all values of \( a\mu \) and \( a \) and it reproduces the continuum action in the limit of \( a \to 0 \) with \( \mu \to \mu/M \) scaling. In order to obtain the order parameter for checking if the symmetry is spontaneously broken, one obtains the order parameter valid for all \( T \) and \( \mu \) on the lattice by taking a derivative of the log of the partition function with respect to \( am \) as,
\[ \langle \bar{\psi}\psi \rangle = \lim_{am \to 0} \lim_{\eta \to \infty} \langle \text{Tr} \frac{1}{aK_{ov} + am + a\mu \gamma^4} \rangle. \]  

where \( K_{ov} = D_{ov}(1 - aD_{ov}/2)^{-1} \), such that \( \{ \gamma_5, K_{ov} \} = 0 \). Although the discussion above is for a single flavour of quark, i.e., \( U(1)_L \times U(1)_R \) symmetry, its generalization to \( N_f \) flavours is straightforward. Indeed, since it relies only on the spin-structure, the flavour index as well as the corresponding generator matrices just carry through.

3. Physical Picture

Domain wall fermions are known to be akin to the overlap fermions, and may provide a physical picture for introduction of \( \mu \) in the way above. Moreover, full QCD simulations with them are easier, so it is also practically useful to check how the above exact chiral invariance can be preserved. The domain wall action of [18] for \( \mu = 0 \), is
\[ S = \sum_{x,x',s,s'=1} N_5 \ a^4 \bar{\psi}(x,s) \left[ am \ \delta_{x,x'} \left( \delta_{s,1} \delta_{s',N_5} P_+ + P_- \delta_{s,N_5} \delta_{s',1} \right) - \delta_{x,x'} (P_- \delta_{s,s+1} + P_+ \delta_{s',s-1}) + (a_5 D_W(x,x') + \delta_{x,x'}) \delta_{s,s'} \right] \psi(x',s'), \]  

where \( P_\pm = (1 \mp \gamma_5)/2 \) and \( N_5 \), \( a_5 \) are the number of sites and the lattice spacing in the fifth direction respectively. The physically relevant 4D fermion field is identified with the fermion fields at the boundaries of the fifth dimension as,
\[ \Psi(x) = P_- \bar{\psi}(x,1) + P_+ \bar{\psi}(x,N_5), \quad \bar{\Psi}(x) = \bar{\psi}(x,1) P_+ + \bar{\psi}(x,N_5) P_- . \]  

These are the lowest energy field configurations of the five dimensional theory and these correspond to the four dimensional chiral fermions in the limit \( m \to 0 \) and \( N_5 \to \infty \). In order to get a four dimensional fermion determinant representing the dynamics of the physical quark fields, one has to integrate out all the fermion degrees of freedom from the five dimensional action and remove the contribution of the higher energy configurations. The latter are the bulk modes since these are delocalized in the fifth dimension even when \( N_5 \to \infty \). Integrating out the fermion degrees of freedom is conveniently done by recasting the domain wall action in terms of the fields \( \eta_i \) localized on four dimensional branes existing at each site \( i \) along the fifth dimension [19]. These are related to the 5D fermion fields in an asymmetric way. The neighbouring pairs of \( \eta \) are related by a transfer matrix, \( T = (1 + a_5 H_W P_+)^{-1} (1 - a_5 H_W P_-) \) with \( H_W = \gamma_5 D_W \). Integrating over the \( \eta \) fields successively one obtains a determinant which is a functional of this transfer matrix:
\[ D^{(5)}(ma) = \det \left[ P_- - maP_+ - T^{-N_5} (P_+ - maP_-) \right] . \]
It has the contribution of all the fermions in the five dimensional theory. Choosing \( ma = 1 \), one obtains contribution from all but the lowest energy configurations which is divided out to obtain the physical quark determinant. Taking first \( a_5 \to 0 \) limit and then \( N_5 \to \infty \), the ratio of determinants turns out to be the determinant of the traditional overlap Dirac matrix.

To get correct physics at finite \( \mu \) one needs to construct the number density corresponding to the physical degrees of freedom. We proposed [14] that the following term be added to the domain wall action in Eq. (8) for nonzero density:

\[
\sum_{x,x'} \sum_{s,s'} a^4 \overline{\psi}(x,s) a \mu \gamma_4 \delta_{x,x'} (\delta_{s,1} \delta_{s',1} P_- + P_+ \delta_{s,N_5} \delta_{s',N_5}) \psi(x',s').
\]

We then follow the same procedure outlined above for the \( \mu = 0 \) case and rewrite the full action in terms of the \( \eta \)-fields. The \( \mu \)-dependent term then becomes as below:

\[
\mu a \left[ \bar{\eta}_1 (a_5 H_W P_- - 1)^{-1} \gamma_4 P_- \eta_1 - \bar{\eta}_{N_5} (a_5 H_W P_- - 1)^{-1} \gamma_4 P_+ \eta_1 \right] \quad (11)
\]

It is important to note that unlike the Bloch-Wettig case [20], the chemical potential in our proposal is coupled only to the physical fermions which are strongly localized on the boundaries of the fifth dimension, hence our transfer matrix continues to remain the same, i.e., \( \mu \)-independent. Moreover since we did not couple \( \mu \) to the number density of the bulk modes, the pseudo-fermion action remains the same as in the \( \mu = 0 \) case. Integrating over the \( \eta \) fields one obtains the physical fermion determinant in a ratio form similar to that for \( \mu = 0 \) with the \( \det D^{(5)}(ma) \) in the numerator generalised to,

\[
\det D^{(5)}(ma, \mu) = \det \left[ P_- - ma P_+ + a \mu (a_5 H_W P_- - 1)^{-1} \gamma_4 P_- - T^{-N_5} \left( P_+ - ma P_- - a \mu (a_5 H_W P_+ + 1)^{-1} \gamma_4 P_+ \right) \right]. \quad (12)
\]

and the pseudo-fermion determinant in the denominator remaining unchanged. On taking the limits \( a_5 \to 0 \) and \( N_5 \to \infty \), the ratio of determinant simplifies to \( \det[D_{ov} + (1 - D_{ov}/2)(ma + a \mu \gamma_5)] \), where both the dimensional parameters \( \mu \) and \( m \) have been scaled by a factor of \( 1/M \) due to the tree level wavefunction renormalization [21]. A little algebra shows that \( \gamma^4 \) can be commuted through in the determinant above to yield the same overlap matrix of Eq. (6) with exact chiral symmetry on the lattice.

The action in Eq.(6) leads to an overlap fermion determinant which is identical to that in the recent work [22] with fermionic sources in the overlap formalism of [23]. The main difference is, however, in the necessity of sources in [22] to define the chiral symmetry. Indeed, the chiral symmetry transformation there is local, defined as rotation of the sources, while our Eq.(4) is nonlocal, defined as the rotation of quark fields. The left-right symmetry is in-built in the formalism there whereas we needed to introduce the left-right projections in form of \( L \)- and \( R \)-fields to do so.

It is easy to verify that our \( D_{ov}(a \mu) \), defined above, is not \( \gamma_5 \)-hermitian. In general, therefore, it is not clear whether it may be diagonalizable. Noting that an \( M \)-scaling was essential in the continuum limit of our \( D_{ov} \) with \( a \mu / M \) governing the density dependent term, it is easily seen that for any arbitrary value of \( \mu \) the density term can be made small by choosing small enough \( a \). Hence it suffices to study the chiral anomaly at \( \mathcal{O}((a \mu / M)^2) \). The leading term in such an expansion in \( a \mu / M \), is \( D_{ov}(0) \) which is \( \gamma_5 \)-hermitian. Its eigenvalues come in complex pairs \((\lambda, \lambda^*)\), with the corresponding eigenvectors related by a \( \gamma_5 \)-rotation. Using these properties, we showed that the chiral anomaly arises as usual, and is governed by the number of zero modes of \( D_{ov}(0) \) even on the lattice. In particular, the contribution of the eigenvectors for the eigenvalue
two, i.e., the ‘doublers’, to the chiral anomaly is identically zero while that of the pairs \((\lambda, \lambda^*)\) cancels mutually. Using first order perturbation theory in \(a\mu/M\), it is easy to show that the latter statement remains unchanged even for nonzero \(\mu\) on a fine enough lattice for our proposal while the former is true for any \(a\mu/M\). Since zero modes continue to be eigenvalues of \(\gamma_5\), all one needs to do is to compute the trace of \(\gamma_5\) in a new basis, rotated by a unitary transformation. After some algebra, one can show that the chiral anomaly remains unchanged on the lattice for arbitrary \(\mu\) using our proposal to introduce a chemical potential for the overlap quarks, exactly as in the continuum.

4. Summary
In conclusion, we used the analogy with continuum QCD to demand a manifest chiral invariance in terms of \(L\) and \(R\)-fields on the lattice to obtain \(aD_{\text{on}} + a\mu\gamma^4(1 - aD_{\text{on}}/2)\) as the exact chiral invariant form of the overlap Dirac matrix for any \(\mu\) and/or lattice spacing. We wrote down the exact chiral order parameter on the lattice for chiral symmetry restoration in the entire \(T - \mu_B\) plane. The chiral anomaly remains unaffected for our proposal for small enough lattice spacing \(a\), just as in continuum. We provided a physically appealing interpretation of our proposal by explicitly writing down the corresponding domain wall fermion action at nonzero chemical potential and taking the limit of infinite fifth dimension.

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