Localized Flow-Based Clustering in Hypergraphs

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ABSTRACT

Local graph clustering algorithms are designed to efficiently detect small clusters of nodes that are biased to a localized region of a large graph. Although many techniques have been developed for local clustering in graphs, very few algorithms have been designed to detect local clusters in hypergraphs, which better model complex systems involving multiway relationships between data objects. In this paper we present a framework for local clustering in hypergraphs based on minimum cuts and maximum flows. Our approach extends previous research on flow-based local graph clustering, but has been generalized in a number of key ways. First of all, we demonstrate how to incorporate recent results on generalized hypergraph s-t cut problems. This allows us to accommodate a wide range of different hypergraph cut functions, which can assign different penalties based on how each hyperedge is split across different clusters. Furthermore, our algorithm comes with a number of attractive theoretical properties in terms of recovering nodes sets with low hypergraph conductance and hypergraph normalized cut scores. Finally, and most importantly, our method is strongly-local, meaning that its runtime depends only on the size of an input set. In practice this allows our method to quickly find localized clusters without exploring an entire input hypergraph. We demonstrate the power of our method in local cluster detection experiments on an Amazon product hypergraph and a Stack Overflow question hypergraph. Although both datasets involve millions of nodes, millions of edges, and a large average hyperedge size, we are able to detect local clusters in a matter of a few seconds or a few minutes, depending on the size of the cluster.

1 INTRODUCTION

Graphs are a common mathematical abstraction for modeling pairwise interactions between objects in a dataset. A standard task in graph-based data analysis is to identify well-connected clusters of nodes, which share more edges with each other than the rest of the graph [30]. For example, detecting clusters in a graph is a standard way to identify communities [14], predict class labels in machine learning applications [7], or segment an image [18]. A standard model for such clusters are ratio cut objectives, which measure the ratio between the number of edges leaving a cluster (the cut) and some notion of the cluster’s size (e.g., the number of edges or nodes in the cluster); common ratio cut objectives include conductance, sparsest cut, and normalized cut. Ratio cut objectives are intimately related to spectral clustering techniques, with the latter providing approximation guarantees for ratio cut objectives (such as conductance) via so-called Cheeger inequalities [12]. In some cases, these ratio cut objectives are optimized over an entire graph to solve a global clustering or classification task [18]. In other situations, the goal is to find sets of nodes that have a small ratio cut and are localized to a certain region of a large graph [3, 4, 27, 34].

Recently, there has been a surge of hypergraph-based methods for machine learning and data mining [1, 2, 5, 11, 23, 25, 37, 39], as hypergraphs are better able to model multiway relationships in data. Common examples of multiway relationships include academic researchers’ co-authoring papers, retail products that are co-purchased by a shopper, or sets of online products or services that are reviewed by the same person. Due to superior modeling ability, there are many hypergraph generalizations of many graph-cut objectives, including hypergraph variants of ratio cut objectives like conductance and normalized cut [6, 8, 9, 23, 39].

Nevertheless, there are numerous challenges in extending graph-cut techniques to the hypergraph setting, and current methods for hypergraph-based learning are much less developed than their graph-based counterparts. One major challenge in generalizing graph cut methods is that the concept of a cut hyperedge—how to define it and how to penalize it—is more nuanced than the concept of a cut edge. While there is only one way to separate the endpoints of an edge into two clusters, there are several ways to split up a set of three or more nodes in a hyperedge. Many objective functions model cuts with an all-or-nothing penalty function, which assigns the same penalty to any way of splitting up the nodes of the hyperedge (and a penalty of zero if all nodes in the hyperedge are placed together) [16, 17, 21] (Fig. 1a). However, a common practical heuristic is a clique expansion, which replaces each hyperedge with a weighted clique in a graph [6, 16, 23, 25, 39, 40] (Fig. 1b). The advantage is that graph methods can be directly applied, but this heuristic actually penalizes cut hyperedges differently than the all-or-nothing model. Another downside is that for hypergraph with large hyperedges, clique expansion produces a very dense graph.
Solving an all-or-nothing cut or applying clique expansion are only two specific models for higher-order relationships. And if one uses a ratio cut model for clusters in a hypergraph, how to penalize a cut hyperedge may depend on the application. Along these lines, inhomogeneous hypergraphs model every possible way to split up a hyperedge to assign different penalties [23]; however, these more sophisticated models are still approximated with weighted clique expansions. Recent research has provided a framework for exactly solving hypergraph s-t cut problems under similar notions of generalized hyperedge splitting functions [33] (Fig. 1c). However, these techniques have not been applied to ratio cut objectives or to solve practical machine learning and data mining problems.

There is also little work on localized hypergraph clustering, i.e., methods for finding well-connected sets of nodes that are biased towards a given region of a hypergraph. Existing specialized approaches [10, 22, 38] use random-walk-based local graph clustering methods [3] on clique expansions. An alternative to random-walk-based techniques are flow-based methods [4, 20, 27, 34], which have strong runtime and ratio cut quality guarantees. These methods solve maximum s-t flow problems (equivalently, minimum s-t cut problems) as a subroutine. Despite the success of these methods for graphs, they have not been extended to the hypergraph setting.

The present work: flow-based local hypergraph clustering. Here, we develop a flow-based framework for local hypergraph clustering based on minimizing localized ratio cut objectives. Our framework takes in a set of input nodes and solves a sequence of hypergraph s-t cut problems to return a well-connected cluster of nodes that has a high overlap with the input set, where “well-connected” is formalized with a ratio-cut-style objective that can incorporate a wide range of hypergraph cut functions, including the all-or-nothing penalty or the clique-expansion-penalty.

Unlike clique expansion techniques, we do not simply reduce to an existing graph technique. Instead, our flow-based framework uses minimum hypergraph s-t cut computations. Thus, we can leverage recent results on minimum s-t hypergraph cut algorithms [33] to exactly solve localized hypergraph ratio cut objectives for generalized notions of hypergraph cuts. The s-t hypergraph cut solver does reduce to solving graph s-t cut problems, but importantly, the hypergraph ratio-cut is exactly optimized. Our implementation can also make use of high-performance maximum s-t flow solvers.

Our method comes with new guarantees in terms of hypergraph conductance and normalized cut. The conductance results are a generalization of previous guarantees under the graph setting for hypergraph cuts. The normalized cut guarantees are the first of their kind, and, since graphs are a special case of hypergraphs, we also get a new result for the graph setting as a bonus. We show examples for which our new theory provides tighter guarantees than obtained by applying existing results for approximate recovery of low-conductance sets, even in the graph setting.

A major feature of our methods is that they run in strongly-local time, meaning that the runtime is dependent only on the size of the input set, rather than the entire hypergraph. Therefore, we can find optimal clusters without even seeing the entire hypergraph, making our algorithms remarkably scalable in theory and practice.

We demonstrate our method on large real-world hypergraphs, detecting product categories in an Amazon dataset and identifying questions of a similar topic on Stack Overflow. Given a small set of seed nodes, our method can pick out clusters with thousands of nodes from hypergraphs with millions of nodes and hyperedges and large average hyperedge size, often within a few seconds. We show that our methods are much more accurate than heuristics based on refining neighborhoods of a seed set or running existing graph clustering methods on clique expansions.

2 PRELIMINARIES AND RELATED WORK

We start with preliminaries on graph and hypergraph cut problems.

2.1 Background: Local Conductance in Graphs

Let G = (V, E) be an undirected graph and w_{ij} ≥ 0 be the weight for edge (i, j) ∈ E. The degree of a node v is \( \deg(v) = \sum_{u \in N_v} w_{uv} \), where N_v is the set of nodes sharing edge with v. A common graph clustering objective is conductance, defined for \( S \subseteq V \) by

\[
\text{cond}_G(S) = \frac{\text{cut}(S)}{\min(\text{vol}(S), \text{vol}(\bar{S}))},
\]

where \( \text{vol}(S) = \sum_{v \in S} \deg(v) \) is the volume of nodes in \( S \), and \( \text{cut}(S) = \sum_{i \in S, j \in \bar{S}} w_{ij} \), which equals \( \text{cut}(S) \) by definition. We use \( \bar{S} \) to denote the complement set of \( S \). A related objective, which differs by at most a factor of two from conductance, is normalized cut [18]:

\[
\text{ncut}_G(S) = \frac{\text{cut}(S)}{\text{vol}(S)} + \frac{\text{cut}(\bar{S})}{\text{vol}(\bar{S})} = \frac{\text{vol}(V) - \text{cut}(S)}{\text{vol}(S)\text{vol}(\bar{S})}.
\]

Conductance and normalized cut are both NP-hard to minimize [36]. However, localized variants of conductance can be minimized in polynomial time with repeated maximum s-t flow computations [4, 20, 27, 34, 35]. For example, given a set of nodes \( R \subseteq V \) with \( \text{vol}(R) \subseteq \text{vol}(\bar{R}) \), the following objective can be minimized in polynomial time with such approaches [20]:

\[
\min_{S \subseteq R} \text{cond}_G(S).
\]

In other words, given a region defined by a reference set \( R \), one can find the minimum conductance subset of \( R \) in polynomial time, even though minimizing conductance over an entire graph is NP-hard. A more general local conductance objective is

\[
\text{local-cond}_{R, \epsilon}(S) = \frac{\text{cut}(S)}{\text{vol}(S \cap R) - \epsilon \text{vol}(S \cap \bar{R})}.
\]

Objective (4) is minimized over all sets of nodes for which the denominator is positive, in order to avoid trivial outputs. The denominator rewards sets \( S \) with a high overlap with \( R \). The penalty for including nodes outside \( R \) is controlled by the locality parameter \( \epsilon \). As \( \epsilon \to \infty \), minimizing Eq. 4 under the constraint that the denominator is positive becomes equivalent to minimizing the objective in Eq. (3). Andersen and Lang [4] showed how to minimize this objective for \( \epsilon = \text{vol}(R) / \text{vol}(\bar{R}) \), and faster algorithms were later developed for when \( \epsilon \gg \text{vol}(R) \) [27, 34]. All of these algorithms repeatedly solve maximum flow problems on an auxiliary graph.

Minimizing objective (4) can also provide cluster quality guarantees in terms of standard conductance (Eq. (1)). For example, the optimal set for (4) with \( \epsilon = \text{vol}(R) / \text{vol}(\bar{R}) \) has conductance within a small factor of the conductance of any set with a certain amount of overlap with the reference set \( R \) [4]. There are several related results for other values of \( \epsilon \) and objective variations [27, 34, 35].
2.2 Background: Generalized Hypergraph Cuts

We now consider a hypergraph \( H = (V, E) \), where each edge \( e \in E \) is a subset of the nodes in \( V \) (an undirected graph is then the special case that \( |e| = 2 \) for all \( e \in E \)). A hyperedge \( e \in E \) is cut by a set \( S \subset V \) if \( e \cap S \neq \emptyset \) and \( e \cap S \neq \emptyset \), i.e., the hyperedge spans more than one cluster. We denote the set of edges cut by \( S \) as \( \partial S \). The most common way to generalize graph cut penalties to hypergraphs is to assign no penalty if a hyperedge \( e \in E \) is not cut, but assign a fixed-weight scalar penalty of \( w_e \) for any way of cutting \( e \). Inhomogeneous hypergraphs generalize this by associating a weight function with each edge, rather than a scalar [23]. In this model, every distinct way of separating the nodes of a hyperedge can have its own penalty. Recent research considered a related notion of splitting functions for hyperedges, in the context of hypergraph s-t cut problems [33].

We review the splitting function terminology here. In order to encode generalized cut penalties, each edge \( e \in E \) is associated with a splitting function \( w_e : A \subseteq e \Rightarrow \mathbb{R}_{\geq 0} \) that maps each subset \( A \subseteq e \) to a nonnegative splitting penalty. If a hyperedge \( e \in E \) comes with a scalar weight \( w_e \), this can be incorporated directly into the splitting function \( w_e \). By definition, splitting functions are required to be symmetric and penalize only cut hyperedges. Formally:

\[
\begin{align*}
    w_e(A) &= w_e(e \setminus A) \quad (5) \\
    w_e(e) &= w_e(\emptyset) = 0. \quad (6)
\end{align*}
\]

A splitting function is submodular if for all \( A \subseteq e \) and \( B \subseteq e \),

\[
    w_e(A) + w_e(B) \geq w_e(A \cup B) + w_e(A \cap B). \quad (7)
\]

A splitting function is cardinality-based if it depends only on the number of nodes on each side of a split:

\[
    w_e(A) = w_e(B) \quad \text{whenever } |A| = |B|. \quad (8)
\]

Given a splitting function for each hyperedge, the generalized hypergraph cut penalty for a set \( S \subset V \) is given by

\[
\begin{align*}
    \text{cut}_H(S) &= \sum_{e \in E} w_e(e \cap S). \quad (9)
\end{align*}
\]

(By the symmetry constraint in Eq. (5), \( \text{cut}_H(S) = \text{cut}_H(\overline{S}) \).) The generalized hypergraph s-t cut objective is then:

\[
\begin{align*}
    \text{minimize} \quad & \text{cut}_H(S) \\
    \text{subject to} \quad & s \in S, t \in \overline{S},
\end{align*}
\]

where \( s \) and \( t \) are designated source and sink nodes. Under the all-or-nothing splitting penalty, the hypergraph s-t cut problem is solvable in polynomial time via reduction to a directed graph s-t cut problem [21]. In more generality, if all splitting functions are submodular, the hypergraph s-t cut problem is equivalent to minimizing a sum of submodular functions. This can be solved using general submodular function minimization [15, 28, 31] or speciality solvers for sums of submodular functions [13, 19, 24, 32]. Recently, Veldt et al. [33] showed that when every splitting function is cardinality-based (Eq. (8)), the hypergraph s-t cut can be solved via reduction to a graph s-t cut problem if and only if all splitting functions are submodular. Cardinality-based submodular splitting functions will be the focus of our models.

2.3 Hypergraph Ratio Cut Objectives

Given this generalized framework for hypergraph cuts, we present definitions for hypergraph conductance and normalized cut, which subsume existing hypergraph ratio cut objectives. Let \( \mathcal{H} = (V, E) \) be a hypergraph. For \( S \subset V \), the hypergraph volume of \( S \) is

\[
    \text{vol}_H(S) = \sum_{e \in S} d_e,
\]

where \( d_e = \sum_{i \in e} w_e(i) \) is the hypergraph degree of \( e \) [23]. We define the hypergraph conductance of a set \( S \subset V \) to be

\[
    \text{cond}_H(S) = \frac{\text{cut}_H(S)}{\min\{\text{vol}_H(S), \text{vol}_H(\overline{S})\}},
\]

and similarly, the hypergraph normalized cut is defined as

\[
    \text{ncut}_H(S) = \frac{\text{cut}_H(S)}{\text{vol}_H(S)} + \frac{\text{cut}_H(\overline{S})}{\text{vol}_H(\overline{S})}. \quad (12)
\]

When \( \mathcal{H} \) is a graph, these reduce to the definitions of conductance (1) and normalized cut (2) in graphs. The hypergraph conductance in Eq. (11) has been used with the all-or-nothing splitting function [6, 8, 9]. Eq. (12) generalizes the version of hypergraph normalized cut from Zhou et al. [39], which is the special case of

\[
    w_e(A) = \frac{w_e}{|e|} \cdot |A| \cdot |e \setminus A| \quad \text{for all } A \subseteq e.
\]

(Here, \( w_e \) is a scalar weight associated with a hyperedge \( e \in E \).) Eq. (12) also corresponds to the definition of hypergraph normalized cut considered by Li and Milenkovic [23].

3 HYPERGRAPH LOCAL CONDUCTANCE

We now define our new localized hypergraph ratio cut objectives. Let \( \mathcal{H} = (V, E) \) be a hypergraph and \( R \) a set of input nodes. We define a function \( \Omega_{R, \epsilon} \), which measures the overlap between \( R \) and another set of nodes \( S \), parameterized by some \( \epsilon \geq \text{vol}_H(R) / \text{vol}_H(\overline{R}) \).

\[
    \Omega_{R, \epsilon}(S) = \frac{\text{vol}_H(S \cap R)}{\epsilon \text{vol}_H(\overline{R})} - \frac{\epsilon \text{vol}_H(S \setminus R)}{\text{vol}_H(R)}. \quad (13)
\]

This is our hypergraph analog to the denominator in Eq. 4. To find a good cluster of nodes in \( \mathcal{H} \) "near" \( R \), we minimize

\[
    \text{HLC}_{R, \epsilon}(S) = \min_{\Omega_{R, \epsilon}(S) < 1} \epsilon \Omega_{R, \epsilon}(S) \quad \text{if } \Omega_{R, \epsilon}(S) > 0 \quad \text{and } \epsilon \Omega_{R, \epsilon}(S) \quad \text{otherwise}, \quad (14)
\]

which we call hypergraph \((R, \epsilon)\)-localized conductance. When \( R \) and \( \epsilon \) are clear from context, we refer to (14) as HLC (hypergraph localized conductance), and denote its value by \( \text{HLC}(S) \). This objective reduces to the graph case when the hypergraph is a graph.

In this section, we show to minimize HLC, given access to a minimum hypergraph s-t cut solver and consider cases where such solvers can be easily implemented with standard graph s-t. Section 4 shows how to optimize the procedure outlined in this section scalable (more formally, have strongly-local runtime guarantees), and Section 5 adapts these results to provide bounds on the hypergraph conductance and normalized cut objectives.

3.1 Minimizing the HLC Objective

We now provide a procedure that can minimize HLC, given polynomially many queries to a solver for \( \text{cut}_H \). Section 3.2 then considers cases where the solver itself requires polynomial time. Let \( \mathcal{H} = (V, E) \) be the original input graph and \( R \) the input set. We minimize HLC by repeatedly solving hypergraph minimum s-t cut
Algorithm 1 Hypergraph local conductance minimization.

Input: $\mathcal{H}$, $R$, $\epsilon \geq \frac{\text{vol}_{\mathcal{H}}(R)}{\text{vol}_{\mathcal{H}}(\bar{R})}$, $\text{cut}_{\mathcal{H}}$.

Set $\alpha = \text{HLC}(R)$ and $S = R$.

do
  Update $S_{best} \leftarrow S$ and save $a_0 \leftarrow \alpha$
  $S \leftarrow \arg \min_{\mathcal{H}} \text{H-st-Cut}_{\alpha}(S)$
  $\alpha \leftarrow \text{HLC}(S)$
while $\alpha < a_0$

Return: $S_{best}$

problems on an extended hypergraph $\mathcal{H}_{ext}$, parameterized by some $\alpha \in (0, 1)$. We construct $\mathcal{H}_{ext}$ as follows:

- Keep all nodes and hyperedges in $\mathcal{H}$ with their original splitting functions
- Introduce a source node $s$ and sink node $t$.
- For each $r \in R$, add an edge $(s, r)$ with weight $\alpha d_r$.
- For each $j \in R$, add an edge $(t, j)$ with weight $\alpha d_j$.

By construction, $\mathcal{H}_{ext}$ contains hyperedges from $\mathcal{H}$ (including two or more nodes) as well as simple pairwise edges attached to source and sink nodes. Each hyperedge $e$ in $\mathcal{H}$ is associated with a splitting function $w_e$. For the edges adjacent to $s$ and $t$, we use the standard cut penalty: 0 if the edge is not cut, otherwise the penalty is equal to the weight of the edge. For any set of nodes $S \subseteq V$, the value of the hypergraph cut $S \cup \{s\}$ in $\mathcal{H}_{ext}$ is given by

$$\text{H-st-cut}_{\alpha}(S) = \text{cut}_{\mathcal{H}}(S) + \alpha \text{vol}_{\mathcal{H}}(S \cap R) + \alpha \text{vol}_{\mathcal{H}}(S \cap \bar{R}).$$ (15)

Choosing $S = \emptyset$ gives an upper bound of $\alpha \text{vol}_{\mathcal{H}}(R)$ on the minimum cut score. Thus, if the minimizer $S^*$ for Eq. (15) has a cut score strictly less than $\alpha \text{vol}_{\mathcal{H}}(R)$, then $S^*$ must be nonempty, and we can rearrange (15) to show that

$$\text{HLC}(S^*) = \frac{\text{cut}_{\mathcal{H}}(S^*)}{\text{vol}_{\mathcal{H}}(S^* \cap R) - \alpha \text{vol}_{\mathcal{H}}(S^* \cap \bar{R})} < \alpha.$$ (16)

Thus, for any $\alpha \in (0, 1)$, to find out if there is a nonempty $S \subseteq V$ with HLC value less than $\alpha$, it suffices to solve a generalized hypergraph $s$-$t$ cut problem. Algorithm 1 gives a procedure for minimizing HLC, based on repeatedly solving objective (15) for smaller values of $\alpha$ until no more improvement in the HLC objective is possible.

### 3.2 A New Hyperedge Splitting Function

Algorithm 1 repeately solves the H-st-cut objective (15) in an auxiliary hypergraph $\mathcal{H}_{aux}$. For submodular splitting functions, this can be done in polynomial time with methods for minimizing sums of submodular functions [13, 19, 24, 32]. For the more restrictive class of cardinality-based submodular splitting functions, we only need to solve directed graph $s$-$t$ cut problems [33]. Implementations of such solvers are readily available and perform well in practice.

As an example, we present a new class of cardinality-based splitting functions that depends on a single tunable integer parameter $\delta \geq 1$, which we use for our numerical experiments for its modeling capability and computational appeal:

$$w_e(A) = \min\{\delta, |A|, |e\cup A|\} \text{ for any } A \subseteq e.$$ (17)

We call this the $\delta$-linear threshold splitting function, since the penalty is linear in terms of the small side of the cut, up until a maximum penalty of $\delta$. In other words, in a split hyperedge, we incur a unit cost for adding another node to the small side of the cut, up until we reach $\delta$ such nodes. The $\delta = 1$ case is equivalent to the unweighted all-or-nothing cut. For large enough $\delta$, $\delta$-linear threshold is the linear hyperedge splitting penalty, which is equivalent to applying a star expansion to the hypergraph [40]. Choosing different values for $\delta$ interpolates between these common splitting functions, which enables the detection of different types of cut sets in a hypergraph and provides the data modeler with flexibility.

We now show how to efficiently optimize $s$-$t$ hypercut cuts for this splitting function using graph $s$-$t$ cuts. Let $\mathcal{H} = (V, E)$ be a hypergraph in which all hyperedges are endowed with a $\delta$-linear threshold splitting function. A minimum $s$-$t$ cut problem in $\mathcal{H}$ can be reduced to a minimum $s$-$t$ cut problem in a new directed graph $G_{\mathcal{H}}$ by replacing each $e \in E$ with the following gadget:

- Introduce two auxiliary nodes $v'_e$ and $v''_e$.
- Create a directed edge from $v'_e$ to $v''_e$ with weight $\delta$.
- For each $v \in e$, add directed edges $(v, v'_e)$ and $(v''_e, v)$, both with weight 1.

In any minimum $s$-$t$ cut solution in $G_{\mathcal{H}}$, the auxiliary nodes of $e$ will be arranged in a way that leads to a minimum possible cut. If $e$ is cut and $A \subseteq e$ is placed on the source side of the cut, then the cut penalty in $G_{\mathcal{H}}$ will remove all directed paths from $A$ and $e \setminus A$ by taking the smaller penalty among three options: cutting the middle edge $e'$, cutting all edges from $A$ to $v''_e$, or all edges from $v'_e$ to $A$. Thus, the cut penalty in $G_{\mathcal{H}}$ at this gadget will exactly equal the original hypergraph splitting penalty (17). Our approach for modeling the $\delta$-linear threshold function is related to recent techniques for modeling general cardinality-based submodular splitting functions [33]. However, our approach requires fewer auxiliary nodes and directed edges. Later in our experiments section, we will see how minimizing the HCL objective with the $\delta$-linear threshold penalty and different values of $\delta$ will enable us to efficiently detect better ground truth clusters in a large hypergraph.

Finally, we provide an upper bound on the number of minimum $s$-$t$ cut problems Algorithm 1 must solve if the $\delta$-linear threshold penalty is applied.

**Theorem 3.1.** Let $S_i \subseteq V$ be the set returned after the $i$th iteration of Algorithm 1. Then the value of $\text{cut}_{\mathcal{H}}(S_i)$ strictly decreases until the last iteration. Thus, if the $\delta$-linear threshold penalty is used, the cut value must decrease by at least one in each iteration, for a maximum of $\text{cut}_{\mathcal{H}}(R)$ iterations.

**Proof.** We want to show that $\text{cut}_{\mathcal{H}}(S)$ strictly decreases in each pass of the while loop (except the last) in Algorithm 1. To simplify notation, we will drop the terms $\mathcal{H}$, $R$ and $\epsilon$ from function subscripts, since these are fixed and clear from context. Consider a pair of consecutive $s$-$t$ cut solutions where there is a strict improvement in HLC score. Starting with some value $a_{j-1}$, let $S_j = \text{argmin } \text{H-st-cut}_{a_{j-1}}(S)$, and $a_j = \text{HLC}(S)$, with $a_j < a_{j-1}$. Let $S_{j+1} = \text{H-st-cut}_{a_j}(S)$ be the set obtained in the next pass through the while loop, with $a_{j+1} = \text{HLC}(S_{j+1})$. Since we are assuming that HLC improves in both of these steps, we have $a_{j+1} < a_j < a_{j-1}$.
Now observe that
\[ \text{H-st-cut}_{\alpha_j}(S_j) \]
\[ = \text{cut}(S_j) + \alpha_j \cdot \text{vol}(S_j \cap R) + \alpha_j \cdot \text{vol}(S_j \cap R) \]
\[ = \text{cut}(S_j) - \alpha_j \cdot \text{vol}(S_j \cap R) + \alpha_j \cdot \text{vol}(S_j \cap R) + \alpha_j \cdot \text{vol}(R) \]
\[ = \text{cut}(S_j) - \alpha_j \cdot \Omega(S_j) + \alpha_j \cdot \text{vol}(R) \]
\[ = \alpha_j \cdot \text{vol}(R) + \Omega(S_j)(\alpha_j - \alpha_j - 1) \]
\[ = \alpha_j \cdot \text{vol}(R) + \Omega(S_j)(\alpha_j - 1). \]

The same essential steps show that
\[ \text{H-st-cut}_{\alpha_j}(S_{j+1}) = \alpha_j \cdot \text{vol}(R) + \Omega(S_{j+1})(\alpha_j - 1). \]

We know that \( \text{H-st-cut}_{\alpha_j}(S_j) \leq \text{H-st-cut}_{\alpha_j}(S_{j+1}) \), since \( S_j \) is the optimal \( s \)-t cut solution for parameter \( \alpha_j - 1 \). This implies that
\[ \Omega(S_j)(\alpha_j - 1) \leq \Omega(S_{j+1})(\alpha_j - 1), \]
which in turn means that \( \Omega(S_{j+1}) < \Omega(S_j) \), since \( (\alpha_j - 1) < (\alpha_j - 1) < 0 \). Finally, because the HLC score and its denominator \( \Omega \) decrease when going from \( S_j \) to \( S_{j+1} \), it must also be the case that \( \text{cut}(S_{j+1}) < \text{cut}(S_j) \). Thus, until the last step of Algorithm 1, the cut function is strictly decreasing.

In most cases this bound is loose—in nearly all of our experiments, the algorithm converges in 2–5 iterations. Similar results can also be developed for using non-integer \( \delta \), though we omit the details.

4 STRONGLY-LOCAL ALGORITHM

Recall that \( r \) is a locality parameter, and implicitly controls the types of sets \( S \subseteq V \) for which HLC \( (S) < \infty \). If \( r \) is large, then \( \Omega_{R_j}(S) < 0 \) for many sets that do not share a significant enough overlap with \( R \). In local clustering applications where \( \text{vol}(R) \) is very small with respect to the size of the input hypergraph, there is little benefit to be gained by considering sets \( S \) that share little overlap with \( R \). In these settings, it is preferable to explore only a small region nearby \( R \), rather than exploring every node and hyperedge in the hypergraph. Thus, it is natural to choose a larger value of \( r \) and compute output low hypergraph conductance sets overlapping with \( R \). Ideally, we want to avoid even looking at the entire hypergraph.

We can formalize this idea via strong locality. A local clustering algorithm is strongly-local if its runtime depends only on the size of the input set and not the entire input hypergraph. In contrast, Algorithm 1 is weakly-local, meaning that its output set is biased towards a certain region of the hypergraph, but its runtime may still depend on the size of the entire hypergraph. In this section, we give a strongly-local variant of Algorithm 1 for the case where \( \text{vol}(R) \approx \frac{1}{\text{vol}(R)} \) and when \( r \approx \frac{\text{vol}(R)}{\text{vol}(R)} \) is treated as a small constant. Our strongly-local procedure generalizes previous strongly-local methods for minimizing local conductance in graphs [27, 34]. In order to manage the hypergraph setting, we combine previous techniques for local minimum \( s \)-t cut computations in graphs with existing techniques for converting hypergraph \( s \)-t cut problems into graph \( s \)-t cut problems [21, 33].

4.1 Making the Procedure Strongly-Local

In order to solve the hypergraph \( s \)-t cut objective (15) in strongly-local time, we must avoid forming all of the hypergraph \( \mathcal{H}_a \). We instead begin with a sub-hypergraph \( L \) of \( \mathcal{H}_a \), which we call the local hypergraph, and alternate between the following two steps:

1. Solve a hypergraph minimum \( s \)-t cut problem on \( L \).
2. Grow the subgraph \( L \) based on the \( s \)-t cut solution.

The algorithm will proceed until a convergence criterion is satisfied, at which point the subgraph growth mechanism in Step 2 will stop and the algorithm will output the minimum \( s \)-t cut solution for \( \mathcal{H}_a \).

Algorithm Terminology. Let \( \mathcal{H} = (V, E) \) be the hypergraph over which we wish to minimize the HLC objective. For any node \( v \in V \), let \( E(v) = \{ e \in E : v \in e \} \), and define \( E(S) = \bigcup_{v \in S} E(v) \) for any set \( S \subseteq V \). Let \( \mathcal{H}_a = (V \cup \{s, t\}, E \cup E^{st}) \) be defined as in Section 3, where \( E^{st} \) is the edge set involving the source \( s \) and sink \( t \). For \( v \in V \), let \( e_{L, v}^{st} \) denote its terminal edge in \( \mathcal{H}_a \) (recall that nodes in \( R \) are connected to \( s \) and nodes in \( R \) are connected to \( t \)), and let \( E^{st}(S) \) denote the set of edges between nodes in \( S \subseteq\{s, t\} \) for any set \( S \).

Our goal is to find a minimum \( s \)-t cut of \( \mathcal{H}_a \) without forming all of it explicitly. In order to do so, we assume that we have an oracle function that allows us to quickly access \( E(v) \) for any node \( v \in V \).

Note that from \( E(v) \) we can extract the neighborhood of \( v \) in \( \mathcal{H} \):

\[ N(v) = \{ u \in V \mid \exists e \in E \text{ such that } u, v \in e \} \]

For a set \( S \), we define \( \mathcal{N}(S) = \bigcup_{v \in S} N(v) \).

The Local Hypergraph. Let \( L = (V_L \cup \{s, t\}, E_L \cup E^{st}_L) \) denote the local hypergraph, a growing subgraph of \( \mathcal{H}_a \) over which we will repeatedly solve minimum \( s \)-t cut problems. We initialize \( L \) to contain all nodes and neighbors of \( R \), i.e., \( V_L = \mathcal{R} \cup \mathcal{N}(R) \), and add the terminal edge for each of these nodes to \( E^{st}_L \). The set \( E_L \) is initialized to \( E(R) \), the set of hyperedges containing at least one node from \( R \). As the algorithm progresses, \( L \) will grow to include more and more of the nodes and edges from \( \mathcal{H}_a \), always maintaining that \( V_L \subseteq V \), \( E_L \subseteq E \), and \( E^{st}_L \subseteq E^{st} \). For a set \( S \subseteq V_L \), let \( L-s-t-cut(S) \) denote the value of the \( s \)-t cut \( S \cup \{s, t\} \) in \( L \). Because \( L \) is a sub-hypergraph of \( \mathcal{H}_a \), at every stage of the algorithm we will have that

\[ L-s-t-cut(S) \leq H-st-cut_a(S) \quad \text{for all } S \subseteq V_L. \]

By carefully selecting new nodes and edges from \( \mathcal{H}_a \) to add to \( L \), we will be able to show that the minimum \( s \)-t cut solution in \( L \) will converge to the minimum \( s \)-t cut solution in \( \mathcal{H}_a \), even without ever fully forming \( \mathcal{H}_a \).

Two Alternating Steps. After initializing \( L \), we repeat two steps until convergence: (1) compute a minimum \( s \)-t cut in \( L \), and (2) grow \( L \) based on the solution to the cut problem. In order to grow \( L \) at each iteration, we keep track of which nodes from \( R \) have had their edge to the sink cut by some minimum \( s \)-t cut solution in \( L \) during a previous iteration. Whenever a node \( v \in R \) has its terminal edge cut for the first time, we expand the local hypergraph by adding all neighbors and hyperedges adjacent to \( v \) in \( \mathcal{H}_a \):

- Update \( V_L \leftarrow V_L \cup N(v) \).
- Update \( E_L \leftarrow E_L \cup E(v) \) and \( E^{st}_L \leftarrow E^{st}_L \cup e_{L, v}^{st} \).

At this point, we say that node \( v \) has been explored, and we maintain a set of nodes \( X \) that have been explored at any point during the algorithm. Since \( R \) is already contained in \( L \), any new nodes we add to the local hypergraph will be from \( R \). Algorithm 2 shows pseudo-code for the overall procedure.

Convergence and Locality. The algorithm terminates when, after a minimum \( s \)-t cut computation, no new edges need to be explored.
We use the term graph-reducible with the bounds from Theorem 4.2, the graph reduction of the we use the Algorithm 2 Strongly-Local Min s-t cut solver

\begin{algorithm}
\textbf{Input:} \( R, \epsilon, \sigma \), and access to \( E(v) \) for any \( v \) in hypergraph \( H \). \\
\textbf{Output:} \( \text{Min } s-t \text{ cut solution } S \) for \( H_{\sigma} \), objective (15) \\
// Initialize local graph \( L \) \\
\( V_L \leftarrow R \cup N(R), E_L^{st} \leftarrow E^{st}(V_L), E_L \leftarrow E(R) \) \\
\( X \leftarrow \emptyset, N \leftarrow \emptyset \) \\
do \\
// \textbf{Step 1:} Solve a minimum } s-t \text{ cut problem on } L. \\
\( S_L = \text{argmin}_{S \subseteq V_L} \text{L-st-cut}(S) \) \\
\( N = S_L \cap R \cap V \setminus X \) (nodes around which to expand \( L \)) \\
// \textbf{Step 2:} Grow \( L \). \\
\( V_L \leftarrow V_L \cup N(N), E_L \leftarrow E_L \cup E(N), E_L^{st} \leftarrow E^{st}_L \cup E^{st}(N) \) \\
\( X \leftarrow X \cup N \).
\\
while \( N \neq \emptyset \) \\
Return \( S_L \)
\end{algorithm}

At this point, we can guarantee that the minimum \( s-t \) cut set in \( L \) is the minimum \( s-t \) cut set of the entire graph \( G_{\sigma} \).

**Theorem 4.1.** The set \( S \) returned by Algorithm 2 minimizes objective (15), the minimum } s-t \text{ cut objective on } H_{\sigma}.

Furthermore, the following theorem shows that under reasonable conditions, the growth of the local hypergraph will be bounded in terms of the volume of \( R \). Thus, our computations are strongly-local. We use the term graph-reducible to refer to any hypergraph cut function for which the hypergraph } s-t \text{ cut problem can be reduced to an equivalent } s-t \text{ cut problem in a directed graph.

**Theorem 4.2.** Suppose we have use a cardinality-based submodular splitting function scaled with minimum non-zero penalty \( 1 \) (e.g., } \( \delta \)-linear threshold) and that no nodes in \( R \) are isolated. Then the local hypergraph \( L \) will have at most \( \frac{2}{3}(1+1/\epsilon)\text{vol}_H(R) \) hyperedges, and the number of nodes and terminal edges will both be at most \( k \text{vol}_H(R)(1+1/\epsilon) \), where \( k \) is the maximum size hyperedge in \( H_{\sigma} \).

Proofs for Theorems 4.1 and 4.2 are included in the appendix. The minimum value on the splitting function is just a scaling issue, and the cardinality-based submodular restriction lets us bound set sizes by volumes of those sets. The assumption that \( R \) have no isolated nodes is minor; these nodes could be removed from the graph in a pre-processing step and have no effect on the objective.

### 4.2 Runtime Analysis

Theorem 4.2 gives strongly local runtimes for Algorithm 2 when we use the } \( \delta \)-linear threshold splitting function with } \( \delta \geq 1 \), since it is a cardinality-based submodular splitting function where the minimum non-zero penalty is \( 1 \). This immediately implies the same runtime result for the all-or-nothing penalty (\( \delta = 1 \)). We implement Algorithm 2 for the } \( \delta \)-linear threshold penalty by replacing each hyperedge } \( e \) with a small directed graph as outlined in Section 3.2. If } \( k \) is the maximum size hyperedge, this graph reduction introduces } \( 2k+1 \) directed edges for each hyperedge that appear in the local hypergraph. Combining this with the bounds from Theorem 4.2, the graph reduction of the largest local hypergraph } \( L \) will have at most } \( k\text{vol}_H(R)(1+1/\epsilon)+3(1+1/\epsilon)\text{vol}_H(R) = O(k\text{vol}_H(R)(1+1/\epsilon)) \) nodes and } \( k\text{vol}_H(R)(1+1/\epsilon)+(2k+1)\frac{2}{3}(1+1/\epsilon)\text{vol}_H(R) = O(k\text{vol}_H(R)(1+1/\epsilon)) \) edges. For a graph } \( G = (V, E) \), there is an } \( s-t \text{ cut algorithm with runtime } O(||E||) \) [29]. In theory, we can use this to solve the hypergraph } \( s-t \text{ cut problem with } \delta \text{-linear threshold penalties on the largest local hypergraph } L \text{ in time } O(k^2\text{vol}_H(R)^2(1+1/\epsilon)^2). \) The local hypergraph grows by at least one hyperedge (i.e., } \( (2k+1) \) directed edges) each step, so we need to solve } \( O(k\text{vol}_H(R)(1+1/\epsilon)) \) } \( s-t \text{ cut problems, for an overall runtime of } O(k^3\text{vol}_H(R)^3(1+1/\epsilon)^3). \)

Using high-performance max-flow/min-cut solvers, the runtime of our algorithm is much faster in practice — on hypergraphs with millions of nodes and edges, roughly a few seconds for small } \( R \) and a few minutes for large } \( R \). Nevertheless, this serves as a proof of concept that as long as } \( 1/\epsilon \) is independent of the size of the input hypergraph (e.g., } \( \epsilon = 1 \) is always a valid choice), then our overall procedure for minimizing localized ratio cuts in hypergraphs is strongly-local. Furthermore, if the maximum hyperedge size } \( k \) is a constant, our asymptotic runtime is the same as the runtime for strongly-local clustering algorithms in graphs [34, 35], which are also effective in practice. Finally, although we focused our analysis on the } \( \delta \)-linear threshold penalty used for our experiments, we can achieve a similar runtime for any cardinality-based submodular splitting function (with a slightly worse dependence on } \( k \).

### 5 Ratio Cut Improvement Guarantees

The HLC objective is meaningful in its own right, and we use it in our experiments; however, understanding the relationship between HLC and more standard ratio cut objectives that do not inherently depend on } \( R \) and } \( \epsilon \) is also useful. To this end, we derive guarantees satisfied by Algorithm 1 and the HLC objective in terms of hypergraph conductance and normalized cut. Our theory shows that the algorithm output has a ratio cut score that is nearly as good as any other set of nodes that have some overlap with } \( R \).

Our results in this section are for a fixed input hypergraph } \( H \), so we drop the subscript } \( H \) to simplify notation. Throughout this section, let } \( \epsilon_0 = \text{vol}(R)/\text{vol}(R) < 1 \) denote the minimum value of the locality parameter. Setting } \( \epsilon = \epsilon_0 \) gives the best cut improvement guarantees, which is always a valid choice. However, we prove results for more general parameter settings, since, as discussed in Section 4, there are locality and runtime benefits for setting } \( \epsilon > \epsilon_0 \). Proofs for theorems in this section can be found in the appendix.

### 5.1 Hypergraph Conductance Guarantees

We first generalize previous conductance improvement guarantees for local graph clustering [4, 34] to our hypergraph objective.

**Theorem 5.1.** Let } \( S^* \) be the set returned by Algorithm 1 for some } \( \epsilon \in (\epsilon_0, \epsilon_0+1), \) and let } \( \mu = \epsilon - \epsilon_0 \geq 0. \) 

(1) For any } \( T \subseteq R \), } \( \text{cond}(S^*) \leq \text{cond}(T). \)

(2) For any set } \( T \) satisfying } \( \text{vol}(T) \leq \text{vol}(T) \) and 

\[
\frac{\text{vol}(T \cap R)}{\text{vol}(T)} \geq \frac{\text{vol}(R)}{\text{vol}(V)} + \gamma \frac{\text{vol}(R)}{\text{vol}(V)},
\]  

(20) 

for some } \( \gamma \in (\mu, 1) \), we have that 

\[
\text{cond}(S^*) \leq \frac{1}{\gamma - \mu} \text{cond}(T).
\]
5.2 Hypergraph Normalized Cut Guarantees

Given that conductance and normalized cut differ by at most a factor of two, we can immediately translate the Theorem 5.1 into bounds for normalized cut. For example, we can conclude that Algorithm 1 returns a set with normalized cut that is within a factor two of the minimum normalized cut subset of \( R \). Our next theorem, however, shows that much better guarantees by directly developing bounds for hypergraph normalized cut. This demonstrates how the applicability of our algorithmic framework transcends its relationship with conductance, as it can be used to find sets that also satisfy strong guarantees for other common objectives.

**Theorem 5.2.** Let \( S^* \) be the set returned by Algorithm 1 for some \( \epsilon \in (\epsilon_0, \epsilon_0 + 1) \), and let \( \mu = \epsilon - \epsilon_0 \geq 0 \). For any set \( T \subset V \) satisfying \( \text{vol}(T) \leq \text{vol}(\bar{T}) \) and

\[
\frac{\text{vol}(T \cap R)}{\text{vol}(T)} \geq \frac{\text{vol}(\bar{T} \cap R)}{\text{vol}(\bar{T})} + \beta
\]

for some \( \beta \in (2\mu/(1+2\mu), 1) \), we have that

\[
\text{ncut}(S^*) \leq \frac{1}{\beta} \text{ncut}(T).
\]

Specifically, when \( \epsilon = \epsilon_0 \), \( \text{ncut}(S^*) \leq (1/\beta) \text{ncut}(T) \).

The overlap assumptions (20) and (21) in Theorems 5.1 and 5.2 differ. The assumption in Theorem 5.1 matches previous local graph clustering results [4, 34] and measures how much \( R \) overlaps with a set \( T \). In contrast, assumption (21) provides a more intuitive measure of how much more \( R \) overlaps with \( T \) than it does with \( \bar{T} \). This is the first application of this type of overlap assumption for cut improvement — graph or hypergraph. We next give a simple example for how this overlap assumption and Theorem 5.2 provide meaningful new normalized cut improvement guarantees, even in the well-studied graph setting.

**Example.** Consider a hypergraph (or graph) \( \mathcal{H} = (V, E) \) containing a low-conductance target set \( T \) with \( \text{vol}(T) = \text{vol}(\bar{T}) = \text{vol}(V)/2 \). Assume that we do not know all of \( T \), but we have access to a set \( R \) constituting half the volume of \( T \), i.e., \( R \subset T \) with \( \text{vol}(R) = \text{vol}(\bar{R})/2 \). Let \( S^* \) be the set returned by Algorithm 1 when \( \epsilon = \text{vol}(R)/\text{vol}(\bar{R}) \). First, we apply Theorem 5.1 to bound \( \text{cond}(S^*) \), where assumption (20) holds with \( \gamma = 1/3 \):

\[
\frac{\text{vol}(R \cap T)}{\text{vol}(T)} = \frac{1}{2} \leq \frac{1}{4} + \frac{1}{4} = \frac{\text{vol}(R)}{\text{vol}(\bar{V})} + \frac{\text{vol}(\bar{R})}{\text{vol}(\bar{V})} \Rightarrow \text{cond}(T) \leq 2 \text{cond}(S^*) \leq 2 \text{cut}(S^*) \leq 3 \text{cond}(T) \leq 6 \text{cut}(T).
\]

On the other hand, assumption (21) is satisfied with \( \beta = 1/2 \), so Theorem 5.2 guarantees that \( \text{ncut}(S^*) \leq 2 \text{cut}(T) \). This is significantly tighter than combining the bound from Theorem 5.1 and the relationship between normalized cut and conductance. This result demonstrates that although HLC is presented as a localized variant of conductance, there are also situations in which we can obtain even better set recovery guarantees in terms of normalized cut than conductance. To summarize, our approach returns meaningful results in terms of more than just one clustering objective.

6 EXPERIMENTS

We call running Algorithm 1 with Algorithm 2 as a subroutine HyperLocal, since it operates on hypergraphs and has a strongly-locally runtime. Next, we apply HyperLocal to identify clusters corresponding to question topics on Stack Overflow and to detect categories of products in an Amazon review hypergraph.

6.1 Algorithms and Implementation Details

We implement HyperLocal in Julia, using a push-relabel implementation of the maximum s-t flow method for the underlying s-t cut problems. All experiments were run on a laptop with 8 GB of RAM and a 2.2 GHz Intel Core i7 processor. An implementation of our algorithm and code for our experimental results are available online at https://github.com/nveldt/HypergraphFlowClustering.

**Neighborhood Baselines.** If \( R_s \) is a set of seed nodes, let \( N(R_s) \) be its one-hop neighborhood. In the hypergraphs we consider, the one-hop neighborhood of a seed set is often quite large. We design two baselines for returning a cluster nearby a set of seeds. \( \text{TopNeighbors} \) orders nodes in \( N(R_s) \) based on the number of hyperedges that each \( v \in N(R_s) \) shares with at least one node from \( R_s \) and outputs the top \( k \) such nodes. Similarly, \( \text{BestNeighbors} \) orders each node \( v \in N(R_s) \) by the fraction of hyperedges incident to \( v \) that are also incident to at least one node from \( R_s \) and outputs the top \( k \). In our experiments, we choose \( k \) to be equal to ground truth cluster sizes, which provides an additional advantage to these baselines.

**Clique Expansion + FlowSeed Baselines.** FlowSeed [35] is a flow-based method for solving localized conductance (4) in graphs. For one baseline, we convert an input hypergraph to a graph and then run FlowSeed with the same input set \( R \) and locality parameter \( \epsilon \) as we use for Hyperlocal. We consider two types of expansion: replacing a hyperedge \( \epsilon \) with an unweighted clique, and replacing a hyperedge with a clique where each edge has weight 1/|\( \epsilon \)|. These are representative of existing clique expansion techniques for (local) hypergraph clustering [22, 38, 39].

6.2 Question Topics on Stack Overflow

HyperLocal is able to identify clusters of questions associated with the same topic on Stack Overflow. We represent each question as a node and define hyperedges as the set of questions answered by a single user. Tags indicate sets of questions on the same topic (e.g., “julia”, “netsuite”, “common-lisp”), which we use as ground truth cluster labels (many questions have multiple tags). The resulting hypergraph has 15,211,989 nodes and 1,103,243 edges, with an average hyperedge size of 23.7. The dataset has 56,502 tags. We consider the 45 tags that have between 2000 and 10000 questions and a hypergraph conductance score below 0.2 under the all-or-nothing penalty. Thus, we focus on sets of tags that can reasonably be viewed as modestly-sized clusters in the dataset.

**Experimental Setup.** We assume that HyperLocal is given a small subset of seed nodes \( R_s \) from the target cluster and a rough idea of the cluster’s size. HyperLocal is designed to find good clusters by refining a moderately-sized reference set, so we use BestNeighbors to grow \( R_s \) into an initial reference set \( R \supset R_s \) that can be refined
Table 1: Average runtime in seconds, precision, recall, and F1 scores across 45 target clusters that correspond to question topics on Stack Overflow. Top F1 is the number of times out of 45 that a method/set obtained the top F1 score (including ties). Runtimes for last three rows are negligible. UCE and WCE indicate unweighted and weighted clique expansions.

| Method     | runtime | pr  | re  | f1 | Top F1 |
|------------|---------|-----|-----|----|--------|
| HyperLocal | 25.0    | 0.69| 0.47| 0.53| 29     |
| UCE + FlowSeed | 32.8 | 0.3 | 0.58 | 0.4 | 1      |
| WCE + FlowSeed | 32.9 | 0.3 | 0.58 | 0.4 | 1      |
| BestNeighbors | –     | 0.49| 0.49| 0.49| 11     |
| TopNeighbors | –      | 0.45| 0.45| 0.45| 6      |
| R          | –      | 0.3 | 0.6 | 0.4 | 1      |

Figure 2: F1 scores for each of 45 clusters in the StackOverflow hypergraph. For FlowSeed, we show the best result from the weighted or unweighted clique expansion; we also show the best of BN/TN. HyperLocal has the highest mean F1 score, and outperforms all methods in all cases where at least one method has an F1 score above 0.6.

by HyperLocal. We ensure HyperLocal finds a cluster that strictly contains $R_s$ by adding infinite weight edges from $R_s$ to the source node in all the underlying minimum $s$-$t$ cut problems, following prior approaches for local graph clustering [35].

For each target cluster $T$, we randomly select 5% of $T$ as a seed set $R_s$, and use BestNeighbors to grow $R_s$ by an additional $2|T|$ nodes. This produces a reference set input $R$ for HyperLocal. We set $\epsilon = 1.0$ and use the $\delta$-linear threshold penalty with $\delta = 5000$. This large threshold tends to produce good results (see Section 6.4 for details), and the threshold is meaningful as some hyperedges contain tens of thousands of nodes.

We run TopNeighbors and BestNeighbors, outputting the top $|T|$ nodes in the ordering defined by each. Therefore, we give these methods an advantage by assuming that they know the exact size of the target cluster. We also run FlowSeed on unweighted and weighted clique expansions, using the same parameters as HyperLocal. In order to use clique expansion without densifying the graph too much and running into memory issues, we first discard hyperedges with 50 or more nodes (around 8% of all hyperedges).

Results. Table 1 reports the performance of each method, averaged across all 45 target clusters. HyperLocal has the highest average F1 score overall and obtains the best F1 scores on many more clusters compared to other methods. Figure 2 visualizes the F1 scores for individual clusters.

6.3 Detecting Amazon Product Categories

In our next experiment we use HyperLocal to quickly detect clusters of retail products with the same category (e.g., “Appliances”, “Software”) from a large hypergraph constructed from Amazon product review data [26]. We build the hypergraph by defining hyperedges to be sets of retail products (nodes) reviewed by the same person. The resulting hypergraph has 2,268,264 nodes and 4,285,363 hyperedges, with an average hyperedge size just over 17. We use product category labels as ground truth cluster identities and consider the 9 smallest clusters, each of which represents only a very small fraction of nodes in the hypergraph (Table 2).

Experimental Setup. We use $\epsilon = 1.0$ and the standard all-or-nothing cut penalty, i.e., $\delta = 1$ for the $\delta$-linear threshold penalty. Unlike the Stack Overflow dataset, this smaller $\delta$ tends to work well (see Section 6.4). For the six smallest clusters (under 200 nodes), we use $|R_s| = 10$ random seed nodes and use BestNeighbors to grow an initial cluster $R$ with $R_s$ plus 200 additional nodes for HyperLocal to refine. For the two clusters closer to 1000 nodes, we use 50 seed nodes, which we grow by another 2000 nodes using BestNeighbors. The largest two clusters have around 5000 nodes. For these, we extract a random subset of 200 nodes and use BestNeighbors to add 10000 neighbors to form $R$.

Results. We compare HyperLocal to BestNeighbors and TopNeighbors in terms of F1 detection scores. (We also attempted to run FlowSeed, but were unable to perform a clique expansion on the hypergraph due to memory constraints, as the expanded graph becomes too dense even after removing all hyperedges with 50 nodes or more.) Table 2 reports F1 detection scores, averaged across 5 different trials with different random seed sets. In all cases, HyperLocal substantially improves upon the baselines.

To test robustness, we ran numerous additional experiments on the smallest five clusters while varying the locality parameter $\epsilon$ in $\{10^{-3}, 10^{-2}, 10^{-1}, 1, 10\}$ and reference set size $|R| \in \{200, 300, 500\}$. In all cases, we obtained results similar to those in Table 2. Regarding runtime, HyperLocal takes between a few seconds and a few minutes, depending on the target cluster size. This is remarkably fast considering that the method is repeatedly finding minimum $s$-$t$ cuts in a hypergraph with millions of nodes and hyperedges, where the mean hyperedge size is above 17.

6.4 Varying Splitting Functions

We additionally consider how different hypergraph splitting functions affect the output solution. We ran HyperLocal with the $\delta$-linear threshold splitting function for varying $\delta$ for a handful of clusters from both datasets, measuring the target cluster recovery F1 score (Fig. 3). We tested both integer and non-integer values of $\delta$, and found no meaningful difference in runtime. For the Stack Overflow hypergraph, HyperLocal’s performance plummets for $\delta$ near 100 and performance is maximized for very large $\delta$ (Fig. 3a). In contrast, for the Amazon hypergraph, the maximum F1 for each cluster is almost always obtained when $\delta = 1.0$ (Fig. 3b). This suggests that if one has access to a subset of ground truth clusters in a
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7 PROOFS FOR THEOREMS

7.1 Proof of Theorem 4.1

Proof. For any \( v \in S \), the set of nodes and edges that are adjacent to \( v \) in \( L \) is exactly the same as the set of nodes and edges that are adjacent to \( v \) in \( \mathcal{H}_t \). The reason is that when the algorithm terminates (\( S_t = S_{L-1} \) for some \( L \)), any \( v \) in the output has either been explored (or is in \( R \)) and had its neighbors added to \( L \). We deduce, therefore, that \( \text{L-st-cut}_t(S) = \text{H-st-cut}_t(S) \). Since the function \( \text{L-st-cut}_t \) is a lower bound on \( \text{H-st-cut}_t \) in general,

\[
\text{L-st-cut}_t(S) = \text{H-st-cut}_t(S) \geq \min_A \text{H-st-cut}_t(A) = \text{L-st-cut}_t(S).
\]

Thus, \( S = \arg\min_A \text{H-st-cut}_t(A) \).

7.2 Proof of Theorem 4.2

Proof. While Algorithm 2 does not rely on explicitly applying graph reduction techniques nor computing maximum \( s \)-\( t \) flows, our proof will rely on the existence of both, as well as on a basic understanding of the minimum cut, maximum flow theorem.

**Implicit graph \( s \)-\( t \) cuts.** Let \( L_I = (V_I \cup \{s, t\}, E_I \cup E_{s,t}^I) \) be the local hypergraph over which we solve a minimum \( s \)-\( t \) cut in the \( i \)th iteration of the algorithm, where \( V_I \subseteq V \), \( E_I \subseteq E \), and \( E_{s,t}^I \subseteq E_{s,t} \). Let \( S_i \) be the minimum \( s \)-\( t \) cut set for \( L_i \), and let \( N_i \subseteq S_i \) be the set of nodes that are explored in the \( i \)th iteration, i.e., nodes whose terminal edges are cut for the first time in iteration \( i \). Since we assume the hypergraph cut function to be graph reducible, for each \( L_i \) there exists a graph \( G_i \) with the same set of nodes \( V_i \cup \{s, t\} \) plus potentially other auxiliary nodes, such that the minimum \( s \)-\( t \) cut value in \( G_i \) is the minimum \( s \)-\( t \) cut value in \( L_i \). Formally, let \( S'_i \) be the minimum \( s \)-\( t \) cut set in \( G_i \) (excluding \( s \) itself), so that \( S_i = S'_i \cap V_i \). In other words, if we exclude auxiliary nodes, the minimum \( s \)-\( t \) cut set in \( G_i \) is the minimum \( s \)-\( t \) cut set in \( L_i \).

**Bounding set sizes.** Let \( S \) be any subset of vertices such that \( S \) contains no isolated nodes. Let \( E(S, S) \) denote the set of hyperedges that are completely contained inside \( S \). We have the following bounds:

\[
|\partial S| \leq \text{vol}_{G}(S) \quad (23)
\]

\[
|S| \leq \text{vol}_{G}(S) \quad (24)
\]

\[
E(S, S) \leq \frac{\text{vol}_{G}(S)}{2} \quad (25)
\]

These bounds use the theorem assumptions on the splitting function; the fact that the minimum weight is one means that the volume of a node is equal to its degree. The first bound is tight whenever every edge that is cut by \( S \) contains exactly one node from \( S \). The second is tight when every node in \( S \) has degree one. Bound (25) follows from the fact that every hyperedge is of size at least 2, and therefore each hyperedge that is completely contained \( S \) is made up of at least two nodes from \( S \). For \( k \)-uniform hypergraphs, \( E(S, S) \leq \frac{\text{vol}_{G}(S)}{2} \), though we use the bound (25) so that we can apply our results more generally.

Assume Algorithm 2 terminates after iteration \( t \), so that \( L_t \) is the largest local hypergraph formed. Let \( P \) denote the set of nodes that were explored at some point during the algorithm.

\[
P = \bigcup_{i=1}^{t} N_j.
\]

We will prove later that the volume of \( P \) can be bounded as follows:

\[
\text{vol}_{G}(P) \leq \frac{\text{vol}_{G}(R)}{\epsilon}.
\]

For now, we assume this to be true and use it to prove the bounds given in the statement of the theorem.

Let \( Q \) denote the set of nodes in \( L_t \) that were never explored. The size of this set can be bounded as follows:

\[
|Q| \leq (k-1)(|\partial R| + |\partial P|).
\]

This bound will often be quite loose in practice. However, in theory it is possible for a hyperedge in \( L_t \) to contain only one node from \( R \cup P \), and \( (k-1) \) nodes from the set \( Q \). We bound the total number of nodes in \( L_t \) with help from Eqs. (23), (24), and (26):

\[
|V_t| = |R| + |P| + |Q| \leq \text{vol}_{G}(R) + \text{vol}_{G}(P) + (k-1)(|\partial R| + |\partial P|)
\]

\[
\leq \text{vol}_{G}(R)(1 + 1/\epsilon) + (k-1)(\text{vol}_{G}(R) + \text{vol}_{G}(P))
\]

\[
\leq k\text{vol}_{G}(R)(1 + 1/\epsilon).
\]

Note also that \( |V_t| \) is the exact number of terminal edges in \( L_t \), so we also have a bound on the number of terminal edges.

We can bound the number of hyperedges \( \text{E}_L \) in \( L_t \) above by \( |E(R, R)| + |E(P, P)| + |\partial R| + |\partial P| \). Note that any hyperedge that includes a node from \( Q \) is accounted for by the terms \( |\partial R| \) and \( |\partial P| \). We again use bounds (23), (24), (25) and (26) to bound the number of hyperedges in terms of \( \text{vol}(R) \):

\[
|\text{E}_L| \leq |E(R, R)| + |E(P, P)| + |\partial R| + |\partial P|
\]

\[
\leq \frac{3}{2} \left( \text{vol}_{G}(R) + \text{vol}_{G}(P) \right) \leq \frac{3}{2} \left( 1 + \frac{1}{\epsilon} \right) \text{vol}_{G}(R).
\]

The last step of the proof is to show the volume bound on \( P \) in (26), which we do by proving the existence of a maximum \( s \)-\( t \) flow on a graph reduction of \( L_t \) with certain properties.

**Bounding \( P \) with an implicit maximum flow argument.** The min-cut max-flow theorem states that the value of the minimum \( s \)-\( t \) cut in a graph \( G \) is equal to the maximum \( s \)-\( t \) flow value in \( G \). An edge is saturated if the value of the flow on an edge equals the weight of the edge, which always upper bounds the flow. Given a set of edges \( C \) defining a minimum \( s \)-\( t \) cut \( G \), any maximum \( s \)-\( t \) flow \( F \) in \( G \) must saturate all edges in \( C \). If any edge in \( C \) were not saturated by \( F \), then the flow value would be strictly less than the cut value, contradicting the optimality of either \( F \) or \( C \). We will use this understanding of the min-cut max-flow theorem to prove the existence of a flow that saturates all edges of explored nodes in the local hypergraph.

The set \( N_t \) is made up of all nodes whose edge to the sink is cut in \( G_t \) when we compute a minimum \( s \)-\( t \) cut. We know that even if we do not compute it explicitly, there exists some flow \( F_t \) in \( G_t \) that saturates all edges between \( N_t \) and the sink \( t \). In the next iteration, \( N_{t+1} \) is the set of nodes whose edges to the sink are cut for the first time. One way to compute a maximum \( s \)-\( t \) flow \( F_{t+1} \) in \( G_{t+1} \) is to start with \( F_t \), the maximum \( s \)-\( t \) flow in \( G_t \), and then find new augmenting flow paths until no more flow can be routed from \( s \) to \( t \). We can assume without loss of generality that the terminal
edges of $N_1$ remain saturated by $F_2$, since there can be no net gain from reversing the flow on a saturated edge to the sink. As a result, the flow $F_2$ will saturate the terminal edges of $N_1$ as well as all terminal edges of $N_2$, which are cut by the minimum $s$-$t$ cut in $G_2$. Continuing this process inductively, we note that in the $i$th iteration there exists some flow $F_i$ that saturates all the terminal edges to the sink that have been cut by some $s$-$t$ cut in a previous iteration. In other words, there exists some flow $F_i$ in $G_i$ that saturates the terminal edge of every node in $P = \bigcup_{i=1}^{f} N_i$. Recall that

$$\text{Weight of terminal edges of } P = \sum_{v \in P} \alpha v = \alpha \text{vol}_H(P)\,.$$  (29)

Finally, observe that the minimum $s$-$t$ cut score in $G_t$ is bounded above by $\text{vol}_H(R)$, since this is the weight of edges adjacent to the source. This provides an upper bound on the weight of $P$'s terminal edges, implying the desired bound on the volume of $P$:

$$\alpha \text{vol}_H(P) \leq \alpha \text{vol}_H(R) \quad \Rightarrow \quad \text{vol}_H(P) \leq \frac{\text{vol}_H(R)}{\varepsilon}.$$  (30)

7.3 Proof of Theorem 5.1

Proof. We first prove that $\Omega_{R,(S)}(S) \leq \text{HLC}(S)$ for any set $S$, which implies that $\text{cond}(S) \leq \text{HLC}(S)$:

$$\Omega_{R,(S)}(S) = \text{vol}(S \cap R) - \varepsilon \text{vol}(S \setminus R) \leq \text{vol}(S \cap R) + \text{vol}(S \setminus R) = \text{vol}(S)$$

$$\text{HLC}(S) \geq \text{cond}(S),$$

where the last equality uses the definition of $\varepsilon$ and the final inequality uses $\varepsilon = 0.01$. Finally, for any $T \subseteq R$ where $\text{vol}(R) \leq \text{vol}(\bar{R})$, $\text{HLC}(T) = \text{cond}(T)$, which gives the first theorem statement: $\text{cond}(S) \leq \text{HLC}(S) \leq \text{HLC}(T) = \text{cond}(T)$.

For the second statement, if $\Omega_{R,(T)}(T) \leq 0$, then $\text{HLC}(T) = \infty$ and the result is trivial. Assume then that $\Omega_{R,(T)}(T) > 0$. Because $\text{HLC}(S) \leq \text{HLC}(T)$, the result will hold if we can prove that $\text{HLC}(T) \leq \frac{1}{1-\mu} \text{cond}(T)$, which is true as long as $\Omega_{R,(T)}(T) \geq (\gamma - \varepsilon) \text{vol}(T)$. We prove this by applying assumption (20):

$$\text{vol}(T) \geq (1 + \varepsilon)\left(\text{vol}(T) - \frac{\text{vol}(\bar{R})}{\text{vol}(V)}\right)$$

$$\text{vol}(T) \geq (1 + \varepsilon)\left(\text{vol}(T) - \frac{\text{vol}(\bar{R})}{\text{vol}(V)}\right) - \varepsilon$$

$$= (1 + \frac{\text{vol}(\bar{R})}{\text{vol}(V)}) \cdot \frac{\gamma \text{vol}(\bar{R})}{\text{vol}(V)} + \mu \text{vol}(\bar{R}) = \text{vol}(V) - \frac{\text{vol}(\bar{R})}{\text{vol}(V)} \geq (\gamma - \mu) \text{vol}(V).$$

7.4 Proof of Theorem 5.2

Proof. Assume throughout that we deal only with sets $S$ satisfying $\Omega_{R,(S)}(S) > 0$, to avoid trivial cases. We begin by defining

$$g(S) = \text{vol}(\bar{R}) \cdot \text{vol}(S \cap R) - \text{vol}(R) \cdot \text{vol}(S \setminus \bar{R}).$$  (31)

Dividing every term in $g(S)$ by $\text{vol}(\bar{R})$ gives

$$g(S)/\text{vol}(\bar{R}) = \text{vol}(S \cap R) - \varepsilon \text{vol}(S \setminus \bar{R}).$$  (32)

This allows us to re-write the HLC objective as

$$\text{HLC} = \text{vol}(\bar{R}) \cdot \frac{\text{cut}(S)}{g(S) - \mu \text{vol}(\bar{R}) \cdot \text{vol}(S \setminus \bar{R})}.$$  (33)

Applying a few steps of algebra produces another useful characterization of the function $g$:

$$g(S) = \text{vol}(\bar{R}) \cdot \text{vol}(S \cap R) - \text{vol}(R) \cdot \text{vol}(S \setminus \bar{R}) = \text{vol}(R) \cdot \text{vol}(S \cap R) - \text{vol}(R) \cdot \text{vol}(S \setminus \bar{R}) = \text{vol}(S \cap R) - \text{vol}(S \setminus \bar{R}).$$  (34)

This characterization of $g$ allows us to see that for any $S \subseteq V$:

$$\text{vol}(S) \cdot \text{vol}(S) \geq \text{vol}(S \cap R) \cdot \text{vol}(S \setminus \bar{R}) \geq g(S) \geq (g(S) - \varepsilon \text{vol}(\bar{R}) \cdot \text{vol}(S \setminus \bar{R}).$$  (35)

We use this to upper bound $\text{cut}(S)$ in terms of $\text{HLC}(S)$:

$$\text{vol}(\bar{R}) \cdot \text{cut}(S) \leq \text{vol}(S \setminus \bar{R})(36) \text{vol}(S \cap R) \geq g(S) \geq g(S) - \varepsilon \text{vol}(\bar{R}) \cdot \text{vol}(S \setminus \bar{R}).$$  (35)

Next we need to prove a lower bound on the normalized cut score of $T$. We again use the characterization of $g$ given in (34), this time in conjunction with property (21), satisfied by $T$, to see that

$$g(T) \geq \beta \text{vol}(T) \cdot \text{vol}(T).$$  (37)

Property (21) also implies that

$$\frac{1}{1-\mu} \text{vol}(T \cap R) \geq \text{vol}(T) \leq (1 - \beta) \text{vol}(T).$$

Combining this with $\text{vol}(\bar{R}) \leq \text{vol}(V) \leq 2 \text{vol}(\bar{T})$ produces

$$\text{vol}(\bar{R}) \cdot \text{vol}(T \cap R) \leq 2(1 - \beta) \text{vol}(T) \cdot \text{vol}(T).$$  (38)

Inequalities (37) and (38) together imply that

$$\text{vol}(\bar{R}) \cdot \text{vol}(T \cap R) \geq (\gamma - \mu) \text{vol}(V) \cdot \text{vol}(T \cap R) \geq (\gamma - \mu) \text{vol}(T) \cdot \text{vol}(T).$$  (39)

Finally, we put together the bound (36), the characterization of the HLC objective given in (33), and inequality (39), to see that

$$\text{vol}(\bar{R}) \cdot \text{vol}(V) \cdot \text{cut}(S) \leq \text{HLC}(S) \leq \text{HLC}(T)$$

$$= \text{vol}(\bar{R}) \cdot \frac{\text{cut}(T)}{g(T) - \mu \text{vol}(\bar{R}) \cdot \text{vol}(T \cap R)}$$

$$\leq \text{vol}(\bar{R}) \cdot \frac{1}{\beta - 2\mu(1 - \beta)} \cdot \frac{\text{cut}(T)}{\text{vol}(T) \cdot \text{vol}(T)}$$

$$= \frac{\text{vol}(\bar{R}) \cdot \text{vol}(V) \cdot \text{cut}(T)}{\beta - 2\mu(1 - \beta)}.$$  (36)

Dividing through by $\text{vol}(\bar{R}) \cdot \text{vol}(V)$ yields the desired bound on normalized cut.  

\[ \square \]