Unfoldings of maps, the first results on stable maps, and results of Mather-Yau/Gaffney-Hauser type in arbitrary characteristic.

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Abstract. Consider the (formal/analytic/algebraic) map-germs $\text{Maps}(X, (k^p, o))$. Let $\mathcal{G}$ be the group of right/contact/left-right transformations. I extend the following (classical) results from the real/complex-analytic case to the case of arbitrary field $k$.

- A separable unfolding is locally trivial iff it is infinitesimally trivial.
- An unfolding is locally versal iff it is infinitesimally versal.
- The criterion of factorization of map-germs in zero characteristic. When is the map $X \to (k^p, o)$ $\mathcal{G}$-equivalent to the composition $X \to \tilde{X} \to (k^p, o)$ with $X \cong \tilde{X} \times (k^p, o)$?
- Criteria of trivialization of unfoldings over affine base.
- Fibration of $X$-orbits into $\mathcal{G}$-orbits.
- A map is locally stable iff it is infinitesimally stable.
- Stable maps are unfoldings of their genotypes.
- Stable maps are determined by their local algebras.
- Results of Mather-Yau/Scherk/Gaffney-Hauser type. How does the module $T^\mathcal{G}f$, or related algebras, determine the $\mathcal{G}$-equivalence type of $f$?

1. Introduction

1.1. Let $k \in \mathbb{R}, \mathbb{C}$ and consider $k$-analytic map-germs $(k^p, o) \to (k^p, o)$. These are studied up to the right ($\mathcal{R}$), contact ($\mathcal{K}$), and left-right ($\mathcal{A}$) equivalences. Among the cornerstones of Singularity Theory are the finite determinacy, the theory of unfoldings and the theory of stable maps. (See e.g. [A.G.V], [A.G.L.V], [Gr.Lo.Sh], [Martinet], [Mo. N.B].) An additional line of research was around the question “How is a map determined by its critical/singular/instability locus?” One way to answer this goes via the classical results of [Mather-Yau], [Gaffney-Hauser], [Scherk].

The classical approach relied heavily on vector fields integration. Initially numerous results could not be extended to the case “$k$ is any field”, not even to the zero characteristic case. The study of $\mathcal{R}, \mathcal{K}$ equivalences over an arbitrary field began in [Gre.Kr.90]. Many results on $\mathcal{R}, \mathcal{K}$ are available by now, see e.g. [B.G.K.22], [B.K.16], [Gr.Ph.19], [Greuel.18] for further references. But the $\mathcal{A}$-case was untouched.

Let $k$ be any field, let $R_X$ denote the quotient ring of formal power series, $k[[x]]/J$, resp. analytic, $k[x]/J$, resp. algebraic, $k[x]/J$, see [21]. Accordingly we have the (formal/analytic/algebraic) scheme-germ, $X := \text{Spec}(R_X)$. Consider the (formal/analytic/algebraic) map-germs, $\text{Maps}(X, (k^p, o))$. The groups $\mathcal{R}, \mathcal{K}, \mathcal{A}$ act on $\text{Maps}(X, (k^p, o))$. In [Kerner.21] I have studied the group orbits, $\mathcal{G}f$, and their (image) tangent spaces, $T_{\mathcal{G}}f$. I have obtained various “linearization” results of type “$T_{\mathcal{G}}f$ vs $\mathcal{G}f$”.

The current paper is the next step. I construct the theory of unfoldings for $\text{Maps}(X, (k^p, o))$, the beginning of the theory of stable maps, and prove several local Torelli-type theorems (the Mather-Yau/Gaffney-Hauser theorems in zero and positive characteristic).

1.2. The structure/contents of the paper. $R_X$ is one of $k[[x]]/J$, $k[x]/J$, $k[x]/J$ and $\mathcal{G} \in \mathcal{R}, \mathcal{K}, \mathcal{A}$. §2 sets the notations for the rings, maps of spaces, groups and their tangent spaces. For all the other definitions and results I refer to [Kerner.21].

Note: through this paper $T_{\mathcal{G}}f$ denotes the extended tangent space. (Classically one writes $T_{\mathcal{G}^0}f$, $T_{\mathcal{R}}f$, $T_{\mathcal{A}}f$.) The classical tangent space is $T_{\mathcal{G}^0}f$, for the filtration $\mathfrak{m}^n : R_X^{ep}$ on the space of maps. §3 sets the basic notions of unfoldings (over an arbitrary field): the pullback and $\mathcal{G}$-equivalence, the (infinitesimal) $\mathcal{G}$-triviality, the (infinitesimal) $\mathcal{G}$-versality and the (infinitesimal) stability. These are copied verbatim from the classical case.

A remark (to avoid any confusion):

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2.1. Rings and germs.

§ In the unfoldings one deforms only the map, not the source (i.e. \( X \) remains constant), unlike e.g. Mond-Montaldi.\[1\].

§ Except for \[1, 5\] we consider the germs of unfoldings, the unfolding base being \((\mathbb{k}', o)\). Therefore (unlike e.g. [Gr.Ng.16] or [Gr.Lo.Sh] pg.234) we work with versal unfoldings and do not introduce complete unfoldings.

§ treats the triviality of unfoldings.

§.1 gives examples showing that the classical Thom-levine criterion cannot hold in positive characteristic. Such pathological unfoldings are “\( \mathcal{G} \)-inseparable”. The relation to the (in)separability of the group orbit map is explained in \[4\].

§ contains the Thom-levine theorem, “A separable unfolding is locally trivial iff it is infinitesimally-trivial, i.e. \( \partial_t f_t \in T_{\mathcal{G}(t)} f_t \)”. (To repeat, in zero characteristic all unfoldings are separable.)

§ gives an application of this triviality criterion to the factorization, in char\((k) = 0\):

i. If the germ \( X \) admits \( r \) vector fields that are linearly independent at the origin then \( X \cong \tilde{X} \times (\mathbb{k}', o) \).

ii. Moreover, suppose a map \( f : X \to (\mathbb{k}^p, o) \) “does not see” these vector fields, i.e. \( T_{\mathcal{G}} f = T_{\mathcal{X}(o)} f \), resp. \( T_{\mathcal{X}} f = T_{\mathcal{X}(o)} f + T_{\partial} f \). (See \[2,2\].) Then \( f \) factorizes as \( X \to \tilde{X} \to (\mathbb{k}^p, o) \), i.e. is \( \mathcal{G} \)-equivalent to the pullback of a map \( \tilde{X} \to (\mathbb{k}^p, o) \).

Here part a. is well known for \( \mathbb{C} \)-analytic germs, but the algebraic version (over \( k(\langle x \rangle / j) \)) seems new. Part b. seems new in all cases.

§ is about the rank of a map \( f : X \to (\mathbb{k}^p, o) \) and the corresponding “preliminary form”, splitting \( f \) into its linear part and the part of order \( \geq 2 \).

§ The unfolding trivializations of \[1, 2\] are local, they hold over the germ of parameter space \((\mathbb{k}'_!, o)\).

In the \( k \)-analytic case one gets the trivialization over small balls, \( o \in \text{Bal}_r \subseteq \mathbb{k}' \). In the algebraic case \((R_X = k(\langle x \rangle / j))\) one gets the trivialization over étale neighborhoods of \( o \in \mathbb{k}' \). But in some cases one works over a global base, e.g. \( R_X \) is one of \( k[t][\langle x \rangle / j], k[t] / (\langle x \rangle / j), k[t](\langle x \rangle / j) \). Then one wants the global trivialization, over the whole \( k' \). This is possible if one restricts to the filtration-unipotent subgroup \( \mathcal{G}(1) < \mathcal{G} \). In \[1, 5\] we prove: If \( \partial_t f_t \in T_{\mathcal{G}(1)} f_t \) then the (separable) unfolding \((f_t, t)\) is globally trivial.

The weaker condition \( \partial_t f_t \in T_{\mathcal{G}(1)} f_t \) implies the weaker statement: all the fibres are \( \mathcal{G} \)-equivalent.

§ treats the (local) versality of unfoldings.

§ gives the pre-normal form: any unfolding \((f_t, t)\) is formally \( \mathcal{G} \)-equivalent to the unfolding \((f_o + \sum a_j(t) v_j, t)\), where \( \{v_\bullet\} \) go to the generators of the \( k \)-vector space \( T_{\mathcal{G}} f_o \).

§ As a simple corollary one gets: an unfolding is versal iff it is infinitesimally versal.

§ Recall that the orbit of contact group, \( \mathcal{K}f \), is essentially larger than that of the left-right group, \( \mathcal{A}f \). Hence several old questions: “How does a \( \mathcal{A} \)-orbit split into \( \mathcal{A} \)-orbits?”, “When do these orbits coincide?”, “For which maps is \( \mathcal{A}f \subset \mathcal{K}f \) an open dense subset?”. In \[5, 3\] we address the local version: How does a \( \mathcal{K} \)-trivial unfolding split into a family of \( \mathcal{A} \)-unfolding?

§ treats the global version: “How does a \( \mathcal{K} \)-orbit split into a family of \( \mathcal{A} \)-orbits?”.

§ extends several results of Mather on stable maps to the maps \( X \to (\mathbb{k}^p, o) \), in arbitrary characteristic.

§.1 A map is locally stable iff it is infinitesimally stable, i.e. \( T_{\mathcal{A}} f = R_X^{\text{op}} \).

§.2 Stable maps are unfoldings of their genotypes, i.e. each stable map is \( \mathcal{A} \)-equivalent to \((f_o + \sum u_i v_i, u)\), where the elements \( \{v_\bullet\} \) generate the vector space \((x) \cdot T^1 f_o \).

§.3 Stable maps are determined by their local algebras. Namely, two stable maps are \( \mathcal{A} \)-equivalent iff their genotypes are \( \mathcal{A} \)-equivalent.

§ extends the classical results of Mather-Yau, Gaffney-Hauser,\[85\], Scherk,\[83\] to the case of \( k \) of zero characteristic, and establishes the corresponding versions in positive characteristic. For isolated hypersurface singularities \((p = 1, R_X = k[[x]] \) and \( \mathcal{G} \in \mathcal{A} \) this was done in [Gr.Ph.17]. Our versions \((p \geq 1, R_X \) one of \( k[[x]] / j, k(\langle x \rangle / j), k(\langle x \rangle / j) \), and \( \mathcal{G} \in \mathcal{A}, \mathcal{K}, \mathcal{A} \)) do not assume that the singularity is isolated. Moreover, the char\((k) > 0\) version is essentially stronger than that of [Gr.Ph.17].

2. Notations and conventions

Below we recall only the main notions. See [Kerner.21] \$2,3\] for the full exposition.

2.1. Rings and germs.

i. In this paper \( R_X \) is one of the rings \( k[[x]] / j, k(\langle x \rangle / j), k(\langle x \rangle / j) \). Here:

- \( k \) is any field, for \( k(\langle x \rangle / j) \) we assume \( k \) to be normed and complete with respect to its norm.
• $x = (x_1, \ldots, x_n)$ and $J \subseteq (x)^2$.
We denote the images of $\{x_i\}$ in $R_X$ by the same letters, this causes no confusion. E.g. the maximal ideal $m \subset R_X$ is generated by $\{x_i\}$, sometimes we write $m = (x)$.

When $J \neq 0$ we always assume the following jet$_0$-condition. Take a derivation $\xi \in \text{Der}_X(R_X)$, present it as $\sum c_i(x)\partial_{x_i}$. Suppose $\xi$ vanishes at the origin to the second order, i.e. $c_2(x) \in (x)^2 \subset R_X$. If $R_X$ is regular, i.e. $J = 0$, then the map $\Phi : x \to x + \xi(x)$ defines a coordinate change, $\Phi \in \text{Aut}_k(R_X)$. In the non-regular case the map $x \to x + \xi(x)$ is not an automorphism of $R_X$.

**jet$_0$ assumption:** any derivation $\xi \in \text{Der}^{(1)}_k(R_X)$ induces an automorphism $x \to x + \xi(x) + h(x)$, where $h(x)$ is a higher order term in the following sense: $(h(x)) \subseteq ((x_1^\\ast(x)))^2 \subset R_X$.

This jet$_0$-assumption holds if $\text{char}(k) = 0$, [B.G.K.22]. But in positive characteristic this is a non-trivial condition.

ii. The source of a map is the scheme-germ $X := \text{Spec}(R_X)$. The target is the germ $(k^p, o) := \text{Spec}(R_Y)$, where $R_Y$ is $k[[y]]$, resp. $k\{y\}$, resp. $k(y)$. Fix some local coordinates in the target $(k^p, o)$, then the space of maps $\text{Maps}(X, (k^p, o))$ is identified with the $R_X$-module $m \cdot R_X^{\text{pp}}$.

Given a map $f : X \to (k^p, o)$ take its dual $f^* : R_Y \to R_X$ and $f'^* : R_Y^{\text{pp}} \to R_X^{\text{pp}}$. For a submodule $\Lambda_Y \subset R_Y^{\text{pp}}$ we distinguish between the image $f^*(\Lambda_Y) \subset R_X^{\text{pp}}$ and the pullback $f'^*(\Lambda_Y) := R_X \times f'^*(\Lambda_Y) \subset R_X^{\text{pp}}$.

iii. To work with deformations/unfoldings we take $\mathbb{K}$ as one of the rings $k[[t]]$, $k\{t\}$, $k(t)$, here $t = (t_1, \ldots, t_r)$. Accordingly we extend the $k$-algebras $R_X, R_Y$ to the $\mathbb{K}$-algebras, $R_X, t$ is one of $\mathbb{K}[[x]]/\langle x \rangle$, $\mathbb{K}\{x\}/\langle x \rangle$, and $R_Y$, $t$ is one of $\mathbb{K}\{y\}$, $\mathbb{K}\{y\}$, $\mathbb{K}(y)$.

iv. We often use the Nakayama lemma over a local ring: if $M = C + m \cdot M$ (for a finitely generated $M$-module and its subset $C$) then $M = C$.

v. The nested version of Artin approximation, [Ronel.18, §5.2].

2.2. **The groups $\mathcal{G} \circ \text{Maps}(X, (k^p, o))$ and tangent spaces $T_{\mathcal{G}}$.**

i. The group of right equivalences consists of the coordinate changes in the source, $\mathcal{B} := \text{Aut}_X := \text{Aut}_k(R_X)$, the action $\mathcal{B} \circ \text{Maps}(X, (k^p, o))$ is given by $f \to \Phi_X(f) := f \circ \Phi_X^{-1}$.

Similarly the group of left equivalences, $\mathcal{L} := \text{Aut}_k(R_Y) \circ \text{Maps}(X, (k^p, o))$, acts by $f \to \Phi_Y(f)$.

Recall that the orbits of the contact group, $\mathcal{K} := \mathcal{G} \times \mathcal{B}$, coincide with the orbits of a much smaller, $\mathcal{K}^\text{fin} := \text{GL}(p, R_X) \times \text{Aut}_X$, see [Kerner.21, §3.2].

When working with families, i.e. $R_{X,t}, R_{Y,t}$, are $\mathbb{K}$-algebras, we take the $\mathbb{K}$-linear automorphisms, $\text{Aut}_{X,t} := \text{Aut}_k(R_{X,t}), \text{Aut}_{Y,t} := \text{Aut}_k(R_{Y,t})$.

The tangent spaces to the groups $\mathcal{B}, \mathcal{K}, \mathcal{A}$ are defined (and studied) in [Kerner.21, §3]. As we mentioned in [17,2] $T_{\mathcal{G}}$ denotes the extended tangent space:

$T_{\mathcal{G}} := \text{Der}_X f, \quad T_{\mathcal{L}} := f^*\left(R_Y^{\text{pp}}\right), \quad T_{\mathcal{B}} := f^*\left(R_X^{\text{pp}}\right), \quad T_{\mathcal{A}} := T_{\mathcal{B}} + f^* \cdot R_X^{\text{pp}}, \quad T_{\mathcal{G}} := T_{\mathcal{B}} + T_{\mathcal{L}} + T_{\mathcal{A}}$.

ii. Take the filtration $I^\ast \cdot R_X^{\text{pp}}$ on $\text{Maps}(X, (k^p, o)) \cong m \cdot R_X^{\text{pp}}$. We get the associated filtrations, the submodules $T_{\mathcal{G}} = T_{\mathcal{G}}(-1) \supseteq T_{\mathcal{G}}(-0) \supseteq T_{\mathcal{G}}(1) \supseteq \cdots$ and the submodules $T_{\mathcal{G}} := T_{\mathcal{G}}(0) \supseteq T_{\mathcal{G}}(0) \supseteq T_{\mathcal{G}}(1) \supseteq \cdots$.

Here:

$T_{\mathcal{G}}(0) := \{\xi \in \text{Der}_X \mid \xi(I^\ast) \subseteq I^{i+j} \ast \forall \ast \geq 0\}, \quad T_{\mathcal{A}}(0)sf := f^*((y)^{i+j} \ast R_Y^{\text{pp}}), \quad T_{\mathcal{B}}(0)sf := T_{\mathcal{B}}(0)sf + f^* \cdot R_X^{\text{pp}}$.

E.g. if $R_X$ regular (i.e. $J = 0$), and $I = m$, then the space $T_{\mathcal{G}}(0)$ is the “classical” tangent space.

iii. Take a $\mathcal{K}$-finite map $f \in m \cdot R_X^{\text{pp}}$. Fix some elements $\{v_i\}$ in $m \cdot R_X^{\text{pp}}$ that go to a basis of the $k$-vector space $m \cdot T_{\mathcal{G}}(0)sf$. Thus $\text{Span}_k \{v_i\} + T_{\mathcal{G}}sf + f \cdot R_X^{\text{pp}} + k^p = R_X^{\text{pp}}$. The following presentation is used often.

**Lemma 2.1.** a. $\text{Span}_{\text{R}_X} \{v_i\} + T_{\mathcal{G}}sf = R_X^{\text{pp}}$.

b. If $(f) \subseteq m^2 \subset R_X$ then $\text{Span}_{\text{R}_X} \{v_i\} + T_{\mathcal{G}}sf + T_{\mathcal{G}}sf \cap m = m \cdot R_X^{\text{pp}}$.

**Proof.** a. We have $(\text{Span}_{\text{R}_X} \{v_i\} + T_{\mathcal{G}}sf + f \cdot R_X^{\text{pp}}) / T_{\mathcal{G}}sf = R_X^{\text{pp}} / T_{\mathcal{G}}sf \subset R_X^{\text{pp}} / T_{\mathcal{G}}sf$. Consider $R_X^{\text{pp}} / T_{\mathcal{G}}sf$ as an $R_Y$-module. It is finitely generated, because $f$ is $\mathcal{K}$-finite, [Kerner.21, §3]. Therefore (by Nakayama over $R_Y$) we get $\text{Span}_{\text{R}_X} \{v_i\} + T_{\mathcal{G}}sf / T_{\mathcal{G}}sf = R_X^{\text{pp}} / T_{\mathcal{G}}sf$. Hence $\text{Span}_{\text{R}_X} \{v_i\} + T_{\mathcal{G}}sf = R_X^{\text{pp}}$.

Now b. follows from a.
2.3. Changing the base field. Let $R_X$ be one of $k[[x]]/J$, $k[x]/J$, $k(x)/J$ and take a (ny) field extension $k \mapsto \bar{K}$. Take the ring extension (in the formal case) $R_{X,K} := \bar{K}[[x]]/J$.

Given $\mathcal{G} \in \mathcal{R}, \mathcal{X}, \mathcal{A}$ we get the group $\mathcal{G}_K$, e.g. $\mathcal{G}_K := \text{Aut}_K(R_{X,K})$. Take the filtration $M_* := I^* \cdot R_{X,K}^{\mathbb{Z}_p}$.

Lemma 2.2. Let $R_X = k[[x]]/J$ for $\mathcal{G} = \mathcal{A}$ or $R_X = k[x]/J$, $k(x)/J$ for $\mathcal{G} \in \mathcal{R}, \mathcal{X}$. Take two maps $f_0, f_1 \in m \cdot R_{X,K}^{\mathbb{Z}_p}$.

1. Suppose $\text{char}(k) = 0$. Let $j \geq 1$. If $f \not\in \mathcal{G}_k$ then $f \not\sim f_0$.

2. Suppose either the algebraic closure $\bar{k}$ is uncountable or (char$(k) = 0$ and $\sqrt{t} = m$). Let $j \geq 1$.

If $f_0 \not\in \mathcal{G}_k$ then $f_0 \not\sim f_1$.

Proof. It is enough to consider only the case of $R_X = k[[x]]/J$. For the rings $k(x)/J$, $k(x)/J$ and the groups $\mathcal{G} \in \mathcal{R}, \mathcal{X}$ one invokes the Artin approximation, §2.1

1. Suppose $g_k f_0 = f_1$ for some $g_k \in \mathcal{G}_k^{(1)}$. Present this group element as the exponential from the tangent space, $g_k = e^{\xi_k}$, here $\xi_k \in T_{\mathcal{G}_k} := \mathcal{G} \otimes T_{\mathcal{G}_1}$. (See e.g. [B.G.K.22 §3.2].) One has $\text{ord}(\xi_k(f)) = \text{ord}(f) + d$ for some $d \geq 1$. Therefore $e^{\xi_k} f - f = \xi_k f \in \mathcal{G} \otimes M_{\text{ord}(f) + d + 1}$. As $e^{\xi_k} f, f \in R_{X,K}^{\mathbb{Z}_p}$, we get $\xi_k f \in R_{X,K}^{\mathbb{Z}_p} + \mathcal{G} \otimes M_{\text{ord}(f) + d + 1}$. Pass to the quotient vector space, (3)

$$[\xi_k f] \in R_{X,K}^{\mathbb{Z}_p} + \mathcal{G} \otimes M_{\text{ord}(f) + d + 1} \subset \mathcal{G} \otimes R_{X,K}^{\mathbb{Z}_p} / \mathcal{G} \otimes M_{\text{ord}(f) + d + 1}.$$  

Recall the general fact: for a vector subspace $V_k \subset W_k$ one has $(\mathcal{G} \otimes V_k) \cap W_k = V_k$. Therefore $\xi_k f = (I_{\mathcal{G}_1} f) + \mathcal{G} \otimes M_{\text{ord}(f) + d + 1}$. Thus we can expand $\xi_k = \xi_d + \xi_{d+1}$, where $\xi_d \in T_{\mathcal{G}_1}, \xi_{d+1} \in \mathcal{G} \otimes T_{\mathcal{G}_1}$, and $\text{ord}(\xi_d f) = \text{ord}(f) + d$, $\text{ord}(\xi_{d+1} f) \geq \text{ord}(f) + d + 1$. Replace $f$ by $e^{-\xi_d} f$ and iterate.

Eventually we get the infinite product $g := \lim_j (e^{\xi_1} \ldots e^{\xi_1}) \in \mathcal{G}^{(1)}$. We claim: this limit exists. Indeed, $\xi_d f \in T_{\mathcal{G}_1} f \cap I^d \cdot R_{X,K}^{\mathbb{Z}_p}$. Then by the Artin-Rees type property, [Kerner 21 §6], we get: $\xi_d f \in T_{\mathcal{G}_1} f$, with $\lim_d j d = \infty$.

Altogether we have $f = g \cdot f$. This proves the statement for $R_X = k[[x]]/J$.

2. (We show only the case of $\mathcal{A}$-equivalence, the other cases are similar.) We should resolve the condition $\Phi_X \circ f_0 \circ \Phi_X = f_1$. Take the Taylor expansions, $\Phi_X(x) = \sum_{|m| \geq 1} C_m(x)^m$ and $\Phi_Y(y) = \sum_{|y| \geq 1} C_m(y)^m$. Here $m$ is the multi-index, and $\{C_m(x)\}, \{C_m(y)\}$ are unknowns. By comparing the coefficients of the monomials, the condition $\Phi_X \circ f_0 \circ \Phi_X = f_1$ (and $\Phi_X(J) = J \subset k[[x]]$) is transformed into a countable system of polynomials equations, $P_j \{C_m(x)\}, \{C_m(y)\} = 0$.

Each polynomial here is in a finite number of variables.

This system is solvable over $\bar{k}$. We claim: each finite subsystem of $\{P_j \{C_m(x)\}, \{C_m(y)\} = 0\}$ is solvable over $\bar{k}$. Indeed, this subsystem defines a subscheme in an affine space over $\bar{k}$. If this finite subsystem is not solvable then this subscheme has no points over $\bar{k}$. By Hilbert Nullstellensatz we get: the defining ideal of this subscheme is the whole polynomial ring of the affine space. And then this finite subsystem can have no solutions over $\bar{k}$.

Finally we apply the assumptions on $k$.

- Suppose $k$ is uncountable. In this case a countable system of polynomial equations over $\bar{k}$ is solvable iff each finite subsystem is solvable, [Popescu-Rond 19 Theorem 5]. Thus the solvability over $\bar{k}$ implies that over $k$.

- If char$(k) = 0$ and $\sqrt{t} = m$, then it is enough to take only finite expansions in $\Phi_X(x) = \sum_{|m| \geq 1} C_m(x)^m, \Phi_Y(y) = \sum_{|y| \geq 1} C_m(y)^m$. Resolving the corresponding equations one gets $f \not\sim f_0$ with $f \not\in \mathcal{G}_k$. Now apply part 1.

Remark 2.3. The first statement fails in positive characteristic. For example, let $f(x) = x^p \in k[x]$, where char$(k) = p$. Then $x^p \not\in \mathcal{A} R x^p + ax^{2p}$ for any $a \in k$. On the other hand, suppose the Frobenius morphism is non-surjective (i.e. the field $k$ is non-perfect). Then $x^p \not\in \mathcal{A} x^p + ax^{2p}$ for any $a \in k \setminus k^p$.

3. The basic notions of unfolding

Let $R_X$ be one of $k[[x]]/J, k[x]/J, k(x)/J$, accordingly take $k$ and $R_{X,A}$, see §2.1
3.1.

**Definition 3.1.** An unfolding of a map \( f : X \to (k^p, o) \) is the map \( F : X \times (k^r, o) \to (k^p, o) \times (k^r, o) \) of the form \( F(x, t) = (f_t(x), t) \), i.e. an element \((f_t, t) \in (m + (t)) \cdot R^{\infty}_{X,t}\), satisfying \( f_0 = f \).

Take the actions \( \mathcal{R}, \mathcal{L} \circ m \cdot R^{\infty}_{X} \). Accordingly we have the actions \( \mathcal{R}_t, \mathcal{L}_t \circ (m + (t)) \cdot R^{\infty}_{X,t} \), where
\[
(4) \quad \mathcal{R}_t := \{ \Phi_{X,t} | \Phi_{X,o} = Id_X \} < \text{Aut}_K(R_{X,t}), \quad \mathcal{L}_t := \{ \Phi_{Y,t} | \Phi_{Y,o} = Id_Y \} < \text{Aut}_K(R_{Y,t}).
\]

Geometrically we have the coordinate changes \( \mathcal{R}_t \circ X \times (k^r, o) \) and \( \mathcal{L}_t \circ (k^{p+r}, o) \) that restrict to identities on the central fibres \( X \times \{ o \}, k^r \times \{ o \} \), and preserve all the \( t = \text{const} \) slices, \( X \times \{ t \}, (k^p, o) \times \{ t \} \).

Equivalently: the elements of \( \mathcal{R}_t, \mathcal{L}_t \) are unfoldings of identity maps, \((x, t) \to (x, t)\) and \((y, t) \to (y, t)\).

Similarly one extends the groups \( \mathcal{A}, \mathcal{K}, \mathcal{A}^{lin}_t = GL(p, R_X) \times \mathcal{R} \) to:
\[
(5) \quad \mathcal{A}_t := \mathcal{X}_t \times \mathcal{R}_t \circ (x, t) \cdot R^{\infty}_{X,t}, \quad \mathcal{K}_t := \mathcal{A}_t \times \mathcal{A}^{lin}_t \circ (x, t) - \text{trivial}. \quad \mathcal{X}_t := GL(p, R_X) \times \mathcal{R}_t \circ (x, t) \cdot R^{\infty}_{X,t}.
\]

These groups act on families of maps, \( \mathcal{G}_t \circ \text{Maps}(X \times (k^r, o), (k^{p+r}, o)) \). Here the groups \( \mathcal{C}_t, GL(p, R_X) \) preserve the origin of \((k^p, o)\) (for each \( t \)). The action \( \mathcal{A}_t \) (resp. \( \mathcal{L}_t \)) does not preserve the origin of \((k^n, o)\) (resp. \((k^p, o)\)), i.e. \( \Phi_{X,t}(x) \notin (x) \) and \( \Phi_{Y,t}(y) \notin (y) \). To preserve the origin(s) one considers the subgroups \( \mathcal{G}_t^{(o)} \) and subspaces \( T^{(o)}_{\mathcal{A}_t} \) for the filtration \((x)^*\), see \([22]\).

The actions of \( \mathcal{A}_t, \mathcal{X}_t \times \mathcal{R}_t \) are t-linear. The actions of \( \mathcal{A}_t, \mathcal{L}_t, \mathcal{A}_t \) are not t-linear (neither additive nor k-multilinear).

For \( R_X = k[x] \) (with \( k \in R, C \)) or \( C^\infty(R^n, o) \), we get the classical unfolding notions. The map is \((k^n, o) \xrightarrow{T} (k^p, o)\) and its t-unfolding is \((k^n \times k^r, o) \xrightarrow{T} (k^p \times k^r, o), (x, t) \to (f_t(x), t)\).

The (extended) tangent spaces to these groups are:
\[
(6) \quad T_{g_t} := \text{Der}_K(R_X, t), \quad T_{g_t} := \text{Der}_K(R_{Y,t}), \quad T_{\mathcal{A}_t} := T_{\mathcal{L}_t} \oplus T_{g_t}, \quad T_{\mathcal{X}_t} := T_{\mathcal{A}_t} \oplus \text{Mat}_{p \times p}(R_X, t).
\]

Accordingly we have the image tangent spaces \( T\mathcal{G}_t f_t \subseteq \mathcal{R}_{X,t}^{\infty} \) and \( T\mathcal{G}_t F \subseteq \mathcal{R}_{X,t}^{\infty} \).

The standard notions of unfoldings are introduced (verbatim) from the classical case, \([A.C.Y.], [A.C.L.V.2], [Mo. N.B. 4,5], [Martinet], Chapter XIV\]. Let \( \mathcal{I} \) be one of \( \mathcal{R}, \mathcal{K}, \mathcal{A} \), accordingly \( \mathcal{G}_t \in \mathcal{R}_t, \mathcal{K}_t, \mathcal{A}_t \).

**Definition 3.2.** 1. Two unfoldings \( F, \tilde{F} \in (m + (t)) \cdot R_{X,t}^{\infty} \) of \( f \in m \cdot R_X^{\infty} \) are called \( \mathcal{G} \)-equivalent if \( g_t \cdot f_t = \tilde{f}_t \) for some \( g_t \in \mathcal{G}_t \).

2. \( a \) An unfolding \( F \) is called \( \mathcal{G} \)-trivial if it is \( \mathcal{G} \)-equivalent to the constant unfolding, i.e. \( F \sim (f_0, t) \).

3. An unfolding \( F \) is called infinitesimally-\( \mathcal{G} \)-trivial if \( \text{Span}_k(\partial_{t_1} f_t, \ldots, \partial_{t_n} f_t) \subseteq T\mathcal{G}_t f_t \).

4. A map \( f \in \text{Maps}(X, (k^p, o)) \) is called \( \mathcal{A} \)-stable if all its unfoldings are \( \mathcal{A} \)-trivial.

A map \( f \in \text{Maps}(X, (k^p, o)) \) is called infinitesimally-\( \mathcal{G} \)-stable if \( T\mathcal{G} f = R^{\infty}_{X,t} \).

5. The pull-back of an unfolding \((f_t(x), t) \in (m + (t)) \cdot R_{X,t}^{\infty} \) via the map \( \phi : (k^r_t, o) \to (k^r, o) \) is the unfolding \((f_t(x), t) \in (m + (t)) \cdot R_{X,t}^{\infty} \). Here \( t \) is the map \( \phi^\ast \) for the map \( \phi^\ast : k^r_t \to k^r \).

An unfolding \( F \in (m + (t)) \cdot R_{X,t}^{\infty} \) of \( f \) is called \( \mathcal{G} \)-versal if any other unfolding of \( f \) is \( \mathcal{G} \)-equivalent to a pullback of \( F \).

An unfolding \( F \in (m + (t)) \cdot R_{X,t}^{\infty} \) of \( f \) is called infinitesimally-\( \mathcal{G} \)-versal if
\[
\text{Span}_k(\partial_{t_1} f_t|_{t=0}, \ldots, \partial_{t_n} f_t|_{t=0}) + T\mathcal{G} f_0 = R^{\infty}_{X,t}.
\]

Namely, the elements \( \partial_{t_1} f_t|_{t=0}, \ldots, \partial_{t_n} f_t|_{t=0} \) are sent to generators of the vector space \( T^1_{\mathcal{G}} f := R^{\infty}_{X,t} / T\mathcal{G} f_0 \).

**Remark 3.3.** i. The stability notions (in part 3) are introduced for \( \mathcal{A} \)-equivalence only. They are not useful for \( \mathcal{R}, \mathcal{K} \) equivalences. Indeed, any infinitesimally \( \mathcal{R} \) or \( \mathcal{K} \)-stable map is necessarily equivalent to \( f(x) = (x_1, \ldots, x_p) \). (Thus in particular \( p \leq n \).)

ii. These definitions are over local rings. Geometrically one works in the infinitesimal neighborhood of \( o \in X \). For the ring \( k[x]/j \) one can pass (in the standard way) to small neighborhoods. E.g., suppose the unfolding \( F \) is trivial, i.e. \( F \sim (f_0, t) \). Then there exist small balls, \( Ball_\varepsilon \subseteq X \) and \( o \in Ball_\varepsilon \subseteq k^r \), with \( 0 < \delta \ll \varepsilon \), and an analytic family of elements \( \{g_t \in \mathcal{G}_t\}_{t \in Ball_\delta} \), satisfying (in \( Ball_\varepsilon \)): \( g_t f_t = \tilde{f}_t \) for each \( t \in Ball_\delta \).
For the ring $k(x)/j$ the small neighborhoods are the étale covers. If $F$ is $\mathcal{G}$-trivial then there exists an étale map $U \xrightarrow{\phi} k_{r_1}$, with $\phi(U)$ a Zariski open neighborhood of $o \in k^r$, such that the pullback $\phi^*(F) = (f_t(i), i)$ is (globally) trivial over $U$.

iii. Take two maps $f_o, f_t \in Maps(X, (k^p, o))$ and their unfoldings $F, \tilde{F}$. If $F \simeq \tilde{F}$ (as unfoldings) then $f_o \simeq \tilde{f}_o$.

3.2.

Lemma 3.4. The conditions of (infinitesimal) $\mathcal{G}$-triviality, (infinitesimal) $\mathcal{A}$-stability, (infinitesimal) $\mathcal{G}$-versality are preserved under the $\mathcal{G}$-equivalence of unfoldings.

Namely, if $F \simeq \tilde{F}$, and $F$ has one of these properties, then so does $\tilde{F}$.

Proof. The invariance of the triviality/stability/versality under $\mathcal{G}$-equivalence is tautological.

- (infinitesimal $\mathcal{G}$-triviality) We prove: the condition $\partial_t f_t \in T_{\mathcal{G}_t} f_t$ is preserved under $\mathcal{G}_t$-transformations, i.e. $\partial_t(g \cdot f_t) \in T_{\mathcal{G}_t}(g \cdot f_t)$ for any $g \in \mathcal{G}_t$.

Suppose $J = 0$. Below $f$ is a power series in $x$, while $f'$ denotes the $x$-derivative.

- The $R$-case. We should prove: $\partial_t(\Phi_{X,t}(f_t)) \in T_{\mathcal{G}_t}(\Phi_{X,t}(f_t))$. First we verify: $\Phi_{X,t}(T_{\mathcal{G}_t} f_t) = T_{\mathcal{G}_t}(\Phi_{X,t}(f_t))$. Then the statement.

- The $\mathcal{A}$-case. Assuming $\partial_t f_t \in T_{\mathcal{G}_t} f_t$ we have:

\begin{equation}
\partial_t U \cdot (f_t \circ \Phi_{X,t}) = (\partial_t U) \cdot (f_t \circ \Phi_{X,t}) + U \cdot \partial_t f_t \circ \Phi_{X,t} + U \cdot f'_t \circ \Phi_{X,t} \cdot \partial_t \Phi_{X,t} \in T_{\mathcal{A}_t}(U \cdot (f_t \circ \Phi_{X,t})).
\end{equation}

- The $\mathcal{G}$-case. Assuming $\partial_t f_t \in T_{\mathcal{G}_t} f_t$ we have:

\begin{equation}
\partial_t(\Phi_{Y,t} \circ f_t \circ \Phi_{X,t}) = (\partial_t \Phi_{Y,t})(f_t \circ \Phi_{X,t}) + \Phi'_{Y,t}(f_t \circ \Phi_{X,t}) \cdot \partial_t f_t \circ \Phi_{X,t} + \Phi'_{Y,t}(f_t \circ \Phi_{X,t}) \cdot f'_t \circ \Phi_{X,t} \cdot \partial_t \Phi_{X,t} \in T_{\mathcal{G}_t}(\Phi_{Y,t} \circ f_t \circ \Phi_{X,t}) + T_{\mathcal{G}_t} f_t \circ \Phi_{X,t} \cdot T_{\mathcal{G}_t} \Phi_{X,t} \subset T_{\mathcal{A}_t}(\Phi_{Y,t} \circ f_t \circ \Phi_{X,t}).
\end{equation}

For $J \neq 0$ one repeats these arguments for representatives of $f_t, \Phi_{X,t}$. Namely, let $S$ be one of $k[[x]], k(x)$ and take a representative $f_t \in (x, t) \cdot S^C_{x,t}$ of $f_t \in R^C_{X,t}$. The group $Aut_K(R_t)$ lifts to $Aut_{K,t}(S_{x,t})$, these are automorphisms that preserve the ideal $J$. The tangent space is now $T_{\mathcal{A}_t} := D_{\log}(J)$.

- (infinitesimal $\mathcal{A}$-stability) $T_{\mathcal{A}_t} f_t \circ \Phi_{X,t} = f'_t \circ \Phi_{X,t} \cdot T_{\mathcal{A}_t} \Phi_{X,t} = f'_t \circ \Phi_{X,t} \cdot T_{\mathcal{A}_t} f_t \circ \Phi_{X,t} = T_{\mathcal{A}_t} (f_t \circ \Phi_{X,t})$.

- (infinitesimal $\mathcal{G}$-versality) The verification is again the chain rule, as in the equations (7), (8), (9). □

Take a map $f_o \in (x) \cdot R^p_X$ and its deformation $f_t \in (x, t) \cdot R^p_{X,t}$. Let $\mathcal{G} \in \mathcal{R}, \mathcal{A}, \mathcal{A}'$, accordingly $\mathcal{G}_t \in \mathcal{G}_t, \mathcal{A}_t, \mathcal{A}'_t$. Let $\Lambda \subset R^p_{X,t}$ be a $k$ or $R_Y$-submodule, finitely generated by $\{v_i\}$. Associate to $\Lambda$ the submodule $\Lambda_t := K(\{v_i\})$, resp. $R_{Y,t}(v_i)$. □

Lemma 3.5. The “algebraic lemma of unfolding”, e.g. [Martinet pg. 193]

Suppose $T_{\mathcal{G}_t} f_o + \Lambda = R^p_{X,t}$, where $\Lambda \subset R^p_{X,t}$ is a finite-dimensional $k$-vector subspace or a finitely-generated $R_Y$-submodule. Then $T_{\mathcal{G}_t} f_t + \Lambda_t = R^p_{X,t}$.

Proof. We have the obvious presentation $R^p_{X,t} = T_{\mathcal{G}_t} f_t + (t) \cdot R^p_{X,t} + \Lambda_t$. Thus $R^p_{X,t}/T_{\mathcal{G}_t} f_t = (t) \cdot R^p_{X,t}/T_{\mathcal{G}_t} f_t + \Lambda_t + T_{\mathcal{G}_t} f_t/T_{\mathcal{G}_t} f_t$. Note: $K[[t]] \otimes R^p_{X,t}/T_{\mathcal{G}_t} f_t \cong R^p_{X,t}/T_{\mathcal{G}_t} f_t$, the later quotient being finitely generated over $k$ (resp. $R_Y$). Therefore $R^p_{X,t}/T_{\mathcal{G}_t} f_t$ is finitely generated by Weierstraß finiteness, [Kerner 21, §2]. Finally, Nakayama over $K$ (resp. $R_Y$) gives $R^p_{X,t}/T_{\mathcal{G}_t} f_t = \Lambda_t + T_{\mathcal{G}_t} f_t/T_{\mathcal{G}_t} f_t$. □

4. Triviality of unfolding

4.1. As in the classical case, $R_X = \mathbb{C}(x)$, one wants to establish Thom-Levine’s criterion, “triviality vs infinitesimal triviality”. This does not hold in full generality because of the positive characteristic obstructions.

Example 4.1. Let $k$ be a field of characteristic $p$ and take $R_X = k[[x]], \mathcal{G} = \mathcal{R}$.
i. The unfolding $f_t(x) = x^n + t^p x$ is non-trivial, even though $\partial_t f_t = 0 \in T_{\mathcal{G}_t} f_t$. Note that $f_t$ is obtained by the $t \to t^p$ base-change from $x^n + tx$.

The unfolding $f_t(x) = x^n + t^p x^p + t x^d \in \mathbb{k}[[t, x]]$, for $p < n < d$, is non-trivial, with $\partial_t f_t \in T_{\mathcal{G}_t} f_t$. And this is not a $t \to t^p$ base change of another unfolding. Here one can take $d \gg n$, i.e. to ensure $\partial_t F \in T_{\mathcal{G}_t} F$ for $j \gg 1$.

ii. Consider the unfolding $f_t(x) = x^p + t^p x^p + t x^d \in \mathbb{k}[[t, x]]$, with $gcd(d, p) = 1$. Then $\partial_t f_t \in x \cdot T_{\mathcal{G}_t} f_t$, but $f_t$ is not an $\mathcal{G}$-trivial unfolding. Indeed, for a coordinate change $x \to x + x \cdot h(t, x)$ one has $f_t(x + x \cdot h(t, x)) = x^p + x^p \cdot h(t, x)^p + x^p \cdot (1 + h(t, x))^p + t x^d$. If this coordinate change trivializes $f_t$ then we must have $ord_x (x^p \cdot h(t, x)^p) > p$. But then necessarily $ord_x (x^p \cdot h(t, x)^p) = ord(x^p + t x^d, t) = p + d$, contradicting the divisibility $p|ord_x (x^p \cdot h(t, x)^p)$.

4.2. Infinitesimal vs local triviality of unfoldings. In view of example 4.1 we introduce:

**Definition 4.2.** An unfolding of the map $f_o \in \mathfrak{m} \cdot R_X^{\mathcal{G}_t}$ is called $\mathcal{G}$-inseparable if $f_t(x) \neq f_o(x) + t^d \cdot f_d(x) + (t^{d+1}) \cdot R_X^{\mathcal{G}_t}$, where $char(k) \mid d$ and $f_d \not\in T_{\mathcal{G}_t} f_o$.

Thus in zero characteristic all unfoldings are separable. The name “separable” is due to the relation to separability of group orbit map, see §3.

**Theorem 4.3.** Let $R_X$ be one of $\mathbb{k}[x]/j, \mathbb{k}[x]/j, \mathbb{k}(x)/j$, see §2.1. Let $\mathcal{G} \in \mathcal{P}, \mathcal{K}, \mathcal{A}$. Take an unfolding $F(x, t) = (f_t(x), t)$, for $t = (t_1, \ldots, t_r) \in (\mathbb{k}^r, o)$.

1. If $F$ is trivial then it is infinitesimally trivial, i.e. $\partial_{t_i} f_1, \ldots, \partial_{t_r} f_t \in T_{\mathcal{G}_t} f_t$.

2. Suppose $F$ is infinitesimally-$\mathcal{G}$-trivial. For $char(k) > 0$ assume that $F$ is $\mathcal{G}$-separable. Then:
   • (for $\mathcal{G} = \mathcal{P}, \mathcal{K}$) $F$ is trivial.
   • (for $\mathcal{G} = \mathcal{A}$) $F$ is formally-$\mathcal{A}$-trivial. Moreover:
     - if $char(k) = 0$ and $R_X = \mathbb{k}(x)/j$, then the unfolding $F$ is $\mathcal{A}$-trivial.
     - if the map $f_0 \in R_X^{\mathcal{A}_t}$ is $\mathcal{A}_t$-finitely determined (as an element of $R_X^{\mathcal{A}_t}$), then the unfolding $F$ is $\mathcal{A}$-trivial.

**Proof.** First we reduce the cases of $r$-parameters to the one-parameter unfolding.

• (Part 1.) Consider $(f_t, t_r)$ as a one-parameter unfolding of the map $f_t|_{t_r = 0} \in R_X^{\mathcal{G}_t, t_1, \ldots, t_r-1}$. Here $R_X^{\mathcal{G}_t, t_1, \ldots, t_r-1}$ is the algebra over $\mathbb{K} = \mathbb{k}[[t_1 \ldots t_r-1]]$, resp. $\mathbb{k}\{t_1 \ldots t_{r-1}\}$, resp $\mathbb{k}(t_1 \ldots t_{r-1})$. This unfolding is still trivial, therefore (assuming the case $r = 1$) it is infinitesimally trivial, $\partial_{t_r} f_t \in T_{\mathcal{G}_t} f_t$. Repeating this for all $t_i$ we get the statement.

• (Part 2.) Consider $(f_t, t_r)$ as a one-parameter unfolding of the map $f_t|_{t_r = 0} \in R_X^{\mathcal{G}_t, t_1, \ldots, t_r-1}$. This unfolding is still infinitesimally trivial, $\partial_{t_r} f_t \in T_{\mathcal{G}_t} f_t$. Therefore (assuming the case $r = 1$) it can be trivialized. Namely, $g^{(r)}_t f_t(x)$ depends on the variables $(t_1, \ldots, t_{r-1}, x)$ only, for an element $g^{(r)}_t \in \mathcal{G}_t$ that acts as identity on $t_1, \ldots, t_{r-1}$. Iterate this argument to get the full trivialization: $(g^{(1)}_t \cdot \cdots \cdot g^{(r)}_t) f_t(x) = f_o(x)$.

Therefore, below we consider only one-parameter unfoldings.

The proof of part 1 is characteristic-free. For part 2 we get an additional proof in $char(k) = 0$ case.

1. **The $\mathcal{G}$-case.** Take the trivialization: $f_t = \Phi_{X, t}(f_o)$ for $\Phi_{X, t} \in Aut_K(R_X, t)$ with $\Phi_{X, o} = Id$, see §2.1(iii). We should prove: $\partial_t f_t \in T_{\mathcal{G}_t} f_t$.

Define the operator $\xi_X := \partial_t - \Phi_{X, t} \circ \partial_t \circ \Phi_{X, t}^{-1}$. We claim: this is a $t$-linear derivation, $\xi_X \in Der_K(R_X)$. First observe that the action $\xi_X \circ R_X$ is well defined. Moreover, this action is $\mathbb{k}$-linear. As $\Phi_{X, t}$ is $\mathbb{k}$-linear, we get $\xi_X(\mathbb{k}) = 0$. Thus $\xi_X \in End_K(R_X)$. Now we verify the Leibniz rule. For any $a, b \in R_X$ one has $\xi_X(ab) = \partial_t(ab) - \Phi_{X, t} \circ \partial_t \circ \Phi_{X, t}^{-1}(ab) = \xi_X(a)b + a \xi_X(b)$. Thus $\xi_X \in Der_K(R_X) = T_{\mathcal{G}_t}$.

Finally we observe: $T_{\mathcal{G}_t} f_t \ni \xi_X(f_t) = \partial_t f_t - \Phi_{X, t} \circ \partial_t(f_o) = \partial_t f_t$.

**The $\mathcal{K}$-case.** (As always, we replace $\mathcal{K}$ by $\mathcal{K}^{un}$.) Suppose $f_t = (U \cdot f_o) \circ \Phi_{X, t}$ for some $\Phi_{X, t} \in Aut_K(R_X, t)$ with $\Phi_{X, o} = Id$, and $U \in GL(p, R_X)$ with $U|_o = I$. As in the $\mathcal{G}$-case we define the operator $\xi_X := \partial_t - \Phi_{X, t} \circ \partial_t \circ \Phi_{X, t}^{-1}$. As in the $\mathcal{G}$-case this is a derivation, $\xi_X \in Der_K(R_X)$.

Now we verify:

$$\xi_X(f_t) = \partial_t f_t - \Phi_{X, t} \circ \partial_t(U \cdot f_o) = \partial_t f_t - \Phi_{X, t}(\partial_t U \cdot U^{-1}) \cdot f_t.$$  

Therefore $\partial_t f_t \in T_{\mathcal{G}_t} f_t$. 

(10)
• The $\mathcal{A}$-case. Present the trivial unfolding in the form \( f_t = (\Phi_{Y,t}^{-1}, \Phi_{X,t}^{-1})(f_0) \). Define the operator \( \Psi \circ (m + (l)) \cdot R^{\mathcal{A}}_{X,t} \) as follows:

\[
(11) \quad \Psi(h) := (\Phi_{Y,t}^{-1})^{-1}|_h \cdot \Phi_{X,t}^{-1} \circ \frac{\partial}{\partial t} \circ (\Phi_{Y,t}, \Phi_{X,t})(h).
\]

Here \( \Phi_{Y,t} \) is the derivative operator and \( (\Phi_{Y,t})^{-1} \) is its inverse. We observe: \( \Psi(f_t) = 0 \in R^{\mathcal{A}}_{X,t} \).

Now we check the \( \Psi \)-action on \( \mathbb{K} \). Take \( h(t) \in \mathbb{K}^{\mathcal{A}} \) then:

\[
(12) \quad \Psi(h(t)) = (\Phi_{Y,t}^{-1})^{-1}|_{h(t)} \cdot \frac{\partial}{\partial t} \Phi_{Y,t}(h(t)) = (\Phi_{Y,t}^{-1})^{-1}|_{h(t)} \cdot \left[ (\partial_t \Phi_{Y,t})_{|h(t)} + \Phi_{Y,t}|_{h(t)} \cdot \frac{\partial h(t)}{\partial t} \right] = \xi_Y(y)|_{h(t)} + \frac{\partial h(t)}{\partial t} \in R^{\mathcal{A}}_{Y,t}.
\]

Here \( \xi_Y := (\Phi_{Y,t}^{-1})^{-1} \cdot \partial_t \Phi_{Y,t} \in T_{\mathcal{A}} \). Therefore \((\Psi - \xi_Y - \frac{\partial}{\partial t})(\mathbb{K}^{\mathcal{A}}) = 0 \). Moreover, for \( h \in (m + (l))R^{\mathcal{A}}_{X,t} \) one has:

\[
(13) \quad (\Psi - \xi_Y - \frac{\partial}{\partial t})h|_{(x,t)} = (\Phi_{Y,t}^{-1})^{-1}|_{h} \cdot \Phi_{X,t}^{-1} \circ \frac{\partial}{\partial t} \Phi_{Y,t} h - \xi_Y(h) - \frac{\partial h}{\partial t} = \Phi_{X,t}^{-1} \circ \frac{\partial}{\partial t} \Phi_{Y,t} h - \frac{\partial}{\partial t} h = h' \cdot (\Phi_{X,t}^{-1} \circ \frac{\partial}{\partial t} \Phi_{Y,t} \in T_{\mathcal{A}} h). \]

We thus get the derivation \( \xi_X := \Psi - \xi_Y - \frac{\partial}{\partial t} \in T_{\mathcal{A}} \). Finally, \( 0 = \Psi(f_t) = (\xi_Y + \xi_X + \partial_t)f_t \) gives:

\[
\partial_t f_t = - (\xi_Y + \xi_X)f_t \in T_{\mathcal{A}} f_t.
\]

2. The case \( \text{char}(k) = 0 \), $\mathcal{F} \not\subset \mathcal{A}$. First we establish the statement in the formal case, \( R_{X,t} = k[[x,t]]/j \).

• The $\mathcal{A}$-case. Suppose \( \partial_t f_t = \xi_X f_t \). Differentiation again in \( \xi_X \) for a derivation \( \xi_X \in \text{Der}_{k[[t]]}(R_{X,t}) \). Then \((\partial_t - \xi_X)f_t = 0 \) for each \( j \geq 1 \).

Extend the ring \( R_{X,t} \) by a new formal variable \( \tilde{t} \) to \( R_{X,t,\tilde{t}} = k[[x,t,\tilde{t}]]/j \). Then this we ring we have:

\[
e^{\tilde{t} (\partial_{\tilde{t}} - \xi_X)}f_t = f_t. \] (Note the derivation \( \tilde{t} \cdot (\partial_{\tilde{t}} - \xi_X) \) is filtration-nilpotent, see [Kerner.21, §2.1]) Therefore one has the identity:

\[
e^{-\tilde{t} \partial_{\tilde{t}}} \cdot e^{\tilde{t} (\partial_{\tilde{t}} - \xi_X)} f_t = e^{-\tilde{t} \partial_{\tilde{t}}} f_t = f_{t-\tilde{t}}.
\]

Now we use the Baker-Campbell-Hausdorff formula, §2.1.ix of [Kerner.21]: \( e^{-\tilde{t} \partial_{\tilde{t}}} \cdot e^{\tilde{t} (\partial_{\tilde{t}} - \xi_X)} \equiv e^{\sum \tilde{t} \cdot p_l(\partial_{\tilde{t}} - \xi_X)} \). Note that the commutator of derivations is a derivation (of first order).

Therefore \( e^{-\tilde{t} \partial_{\tilde{t}}} \cdot e^{\tilde{t} (\partial_{\tilde{t}} - \xi_X)} = e^{\tilde{t} \xi_X} \), for a derivation \( \xi_X \in \text{Der}_{k[[t]]}(R_{X,t,\tilde{t}}) \). In particular, \( \tilde{t} \cdot \xi_X = \tilde{t} \cdot T_{\mathcal{A},t,\tilde{t}} \).

Therefore we get the coordinate change \( e^{\tilde{t} \xi_X} \in \mathcal{B}_{X,t,\tilde{t}} \), which satisfies: \( e^{\tilde{t} \xi_X} f_t = f_{t-\tilde{t}} \). This equality is an identity in \( R_{X,t,\tilde{t}} \). We can restrict this identity to \( \tilde{t} = t \), as \( \xi_X \in k[[t]] \)-linear. We get the trivialization: \( e^{\tilde{t} \xi_X} f_t = f_{t-\tilde{t}} \). Note that \( e^{\tilde{t} \xi_X} \in \mathcal{B}_t \), i.e. is the unfolding of identity Id in \( \mathcal{B}_t \).

• The $\mathcal{X}$-case. Suppose \( \partial_t f_t = \xi_X f_t + u \cdot f_t \in T_{\mathcal{X},t,\tilde{t}} \), where \( \xi_X \in \text{Der}_{k[[t]]}(R_{X,t,\tilde{t}}) \), \( u \in \text{Mat}_{p \times p}(R_{X,t,\tilde{t}}) \). As in the \( \mathcal{A} \)-case we get \( e^{\tilde{t} (\partial_{\tilde{t}} - \xi_X - u)} f_t = f_{t-\tilde{t}} \) in the ring \( R_{X,t,\tilde{t}} \). Then, as before, we have \( e^{-\tilde{t} \partial_{\tilde{t}} \cdot e^{\tilde{t} (\partial_{\tilde{t}} - \xi_X - u)} f_t = f_{t-\tilde{t}} \). Using the BCH-formula, as before, we get the identity:

\[
e^{\tilde{t} \xi_X} f_t = f_{t-\tilde{t}}, \text{ where } \xi_X \in \text{Der}_{k[[t,\tilde{t}]]}(R_{X,t,\tilde{t}}) \text{ and } u \in \text{Mat}_{p \times p}(R_{X,t,\tilde{t}}).
\]

In particular \( \tilde{t} \cdot (\xi_X - u) \in \tilde{t} \cdot T_{\mathcal{X},t,\tilde{t}} \). As before, one substitutes \( \tilde{t} = t \) to get the trivialization: \( e^{\tilde{t} \xi_X} f_t = f_{t-\tilde{t}} \). Here \( e^{\tilde{t} \xi_X} f_t = f_{t-\tilde{t}} \in \mathcal{X}_t \).

• The $\mathcal{A}$-case cannot be addressed by verbatim the same argument for various reasons. For example, the condition \( \partial_t f_t = \xi_X f_t + \xi_Y f_t \in T_{\mathcal{A}} f_t \) does not imply \( \partial_t - \xi_X - \xi_Y \) is nilpotent in \( \mathcal{A} \). By the algebraic Thom-Levine property, [Kerner.21, §3.6], we get: \( e^{\tilde{t} \xi_X} f_t = e^{\tilde{t} \xi_X} f_t \). (Note that both \( \tilde{t} \partial_t - \xi_X \) and \( \tilde{t} \xi_Y \) are nilpotent derivations for the filtration \( (x,t,\tilde{t}) \in R_{X,t,\tilde{t}} \). Thus \( f_t = e^{-\tilde{t} \xi_X} f_t \). As in the \( \mathcal{A} \)-case we get:

\[
f_{t+\tilde{t}} = e^{\tilde{t} \partial_{\tilde{t}}} \circ e^{-\tilde{t} \xi_X} f_t = e^{\xi_X} f_t, \quad \text{for some } \xi \in \text{Der}_{k[[t,\tilde{t}]]}(R_{X,t,\tilde{t}}).
\]
The identity \( f_{t+i} = e^{t \xi_X} \circ e^{t \xi_Y} f_t \) is a power series in \( t \), \( t \), and does not involve any \( t, t \)-derivatives. Substituting \( t = -t \) we get \( f_o = e^{t \xi_{i=t}} \circ e^{t \xi_Y} f_t \), i.e. the trivialization.

We have constructed the trivializations in the formal case, \( R_{r,t} = \mathcal{K}[x,t]/j \). In the analytic case, \( R_{r,t} = \mathcal{K}[x,t]/j \), it is enough to remark that the BCH formula gives analytic derivations, \( \xi_X, \xi_Y \), see [Kerner,21 §2.1.i]. (Note again, that the derivations \( t(\partial_t - \xi_X), \xi_Y \) are filtration-nilpotent.) Thus the constructed trivializations for \( \mathcal{A}, \mathcal{K}, \mathcal{A} \) are analytic.

If \( r \notin \mathcal{K} \), then \( \mathcal{G} \) is not assumed to have an isolated critical/singular/instability point.

We give a corollary of Mather-Yau/Gaffney-Hauser type.

**Remark 4.5.**

1. The trivializations in part 2 of the theorem involve the automorphism \( \Phi \).
2. The general case, char \( \mathcal{K} \), \( \mathcal{K} \).

**Example 4.4.**

If char \( \mathcal{K} = 0 \) then all the unfoldings are \( \mathcal{G} \)-separable.

**Remark 4.5.**

1. The trivializations in part 2 of the theorem involve the automorphism \( \Phi_{X,t} \in Aut_{\mathcal{T}}(R_{r,t}) \).
2. One can impose various restrictions on the trivializations, accordingly modifying the statements.

We give a corollary of Mather-Yau/Gaffney-Hauser type.

**Corollary 4.6.**

Let \( \mathcal{G} = \mathcal{K} \) with \( R_{r,t} \in \mathcal{K}[x,t]/j \), \( \mathcal{K}[x,t]/j \), \( \mathcal{K}[x,t]/j \), see [22] i or \( \mathcal{G} = \mathcal{A} \) with \( R_{r,t} = \mathcal{K}[x,t]/j \). For char \( \mathcal{K} > 0 \) suppose the unfolding \( F(x,t) = (f_t(x), t) \) is \( \mathcal{G} \)-separable. If \( T_{\mathcal{G}} f_{t} = T_{\mathcal{G}} f_{o} \subseteq R_{r,t}^{\mathcal{G}} \), then \( F \) is \( \mathcal{G} \)-trivial.

Proof. We have: \( f_t \in T_{\mathcal{G}} f_{t} = T_{\mathcal{G}} f_{o} \). Expand in \( t \)-powers, \( f_t(x) = \sum_{d \geq 0} t^d f_d(x) \). Then one gets: \( f_d \in T_{\mathcal{G}} f_{o} \) for each \( d \geq 0 \). Therefore at the formal level (for \( R_{r,t} = \mathcal{K}[x,t]/j \)) one has: \( \partial_t f_t \in T_{\mathcal{G}} f_{o} = T_{\mathcal{G}} f_{t} \). By part 2 of theorem [4.3] we get: \( F \) is formally \( \mathcal{G} \)-trivial, i.e. \( f_t \in T_{\mathcal{G}} f_{o} \).
For $R_{X,t} = k[x,t]/f$, $k(x,t)/f$ and $\mathcal{G} = \mathcal{X}$ one applies the Artin approximation, §2.11.

4.3. An application: factorization of space-germs and map-germs. The map of spaces $X \overset{f}{\to} Y$ defines the germ of (not locally trivial) fibration $\mathcal{F}_f$, by $X = \bigsqcup_{y \in Y} f^{-1}(y)$.

Two fibrations are called equivalent, $\mathcal{F}_f \sim \mathcal{F}_g$ if there exist automorphisms $\Phi_X \circ X$, $\Phi_Y \circ Y$ satisfying $\Phi_X(f^{-1}(y)) = f^{-1}(\Phi_Y(y))$. Therefore one gets: $f \sim \tilde{f}$ iff $\mathcal{F}_f \sim \mathcal{F}_{\tilde{f}}$.

Recall the classical factorization statement. Suppose a $C$-analytic germ admits $r$ vector fields that are linearly independent at the origin. Then $(X,x) \cong (C^r,0) \times (X,z)$. See e.g. [Fischer §2.12].

We establish (in zero characteristic) the $\mathcal{R}$, $\mathcal{X}$, $\mathcal{A}$-versions of this statement.

**Definition 4.7.** Let $(R_X,m)$ be a local $k$-algebra, with $k \cong R_x/m$.

1. The value of the derivation $\xi \in \text{Der}_X$ at the origin is the $k$-linear map $m/\text{m}^2 \overset{\xi}{\to} R_x/m$. (Thus $\xi|_0 \neq 0$ iff $\xi(m) \not\subseteq \text{m}$.)

2. The set of values of vector fields at the origin is the $k$-vector subspace $\text{Der}_X|_o := \text{Span}_k(\xi|_o, \xi \in \text{Der}_X) \subseteq \text{Hom}_k(m/\text{m}^2, R/m)$. Thus $\dim_k(\text{Der}_X|_o)$ is the maximal number of derivations linearly independent at $o \in X$.

We observe:

- $\text{Der}_X|_o$ is invariant under $\text{Aut}_X$-transformations, i.e. preserved by the coordinate changes with unit linear part.
- $\dim(\text{Der}_X|_o)$ is preserved by any coordinate change, $\text{Aut}_X$.

**Theorem 4.8.** Let $S$ be one of $k[[x]]$, $k\{x\}$, $k(x)$, where $\text{char}(k) = 0$. Let $R_X = S/j$ with $J \subseteq (x)^2$. Suppose $r := \dim(\text{Der}_X|_o) > 0$.

1. Then $X \cong \tilde{X} \times (k^r,o)$, i.e. the corresponding $k$-algebras are isomorphic,

\[ k[[x]]/j \cong \tilde{R}[[z]], \quad k\{x\}/j \cong \tilde{R}\{z\}, \quad k(x)/j \cong \tilde{R}(z). \]

Here $\tilde{R} = k[[\tilde{x}]]/J_{\tilde{x}}$, resp. $k\{\tilde{x}\}/J_{\tilde{x}}$, resp. $(k(x)/J_{\tilde{x}}$, with $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n-r)$, $J_{\tilde{x}} \subseteq (\tilde{x})^2$, and $z = (z_1,\ldots,z_r)$.

2. Moreover, take a map $X \overset{f}{\to} (k^p,o)$ and the filtration $m^* \subseteq R_X$.

- (R) If $T_{\mathcal{G}}f = T_{\mathcal{G}(0)}f$ then $f$ factorizes as $X \to \tilde{X} \overset{f|_{\tilde{X}}}{{\sim}} (k^p,o)$. Namely, $f$ is $\mathcal{R}$-equivalent to a pullback of $f|_{\tilde{X}=o}$.

- (X) If $T_{\mathcal{X}}f = T_{\mathcal{X}(0)}f$ then the subscheme $V(f) \subseteq X$ factors into $V(f|_{\tilde{X}=o}) \times (k^r,o) \subseteq \tilde{X} \times (k^r,o)$. Namely, $f$ is $\mathcal{X}$-equivalent to a pullback of $f|_{\tilde{X}=o}$.

- (A) Suppose $R_X = k[x]/j$ or $f$ is $\mathcal{A}$-finitely determined. If $T_{\mathcal{A}}f = T_{\mathcal{A}(0)}f + T_{\mathcal{A}(0)}f$ then the fibration factorizes, $(X,F_f) \cong (\tilde{X},F_f|_{\tilde{X}=o}) \times (k^r,o)$. Namely, $f$ is $\mathcal{A}$-equivalent to a pullback of $f|_{\tilde{X}=o}$.

**Proof.**

1. Take a derivation $\xi \in \text{Der}_X$ that does not vanish at the origin, i.e. $\xi|_0 \neq 0$, i.e. $\xi(m) \not\subseteq m$. Applying a coordinate change, i.e. $\text{Aut}_X$, one can assume $\xi(x_n) = 1$. Therefore we have the presentation $\xi = \frac{\partial}{\partial x_n} - \sum_{i=1}^{n-1} a_i(x) \frac{\partial}{\partial x_i}$. Fix some generators $J = (q_1,\ldots,q_l) \subseteq S$. Then for the vector $q \in S^{\otimes l}$ one has $\delta_{x_n}q \in \text{Jac}_{x_1,\ldots,x_{n-1}}(q) + (q_1,\ldots,q_l) \cdot S^{\otimes l}$. By theorem 1.3 $q(x_1,\ldots,x_n)$ is a $\mathcal{X}$-trivial unfolding of $q(x_1,\ldots,x_n-1,0)$. Namely: $q(x_1,\ldots,x_n) \overset{GL(l,S) \times \text{Aut}_k(S)}{\cong} q(x_1,\ldots,x_n-1,0) \in S^{\otimes l}$. Therefore $R_X \cong (k[[x]]/J_{\tilde{x}})[[z]]$, and similarly for the analytic/algebraic cases.

In the case $r = 1$ this proves the statement. For $r > 1$ one starts from the rectified germ $\tilde{X} \times (k^1,o)$. Then $\text{Der}_{X \times (k^1,o)} \supseteq \frac{\partial}{\partial x_n}$, this vector field does not vanish at $o$. By the assumption there exist other derivations $\xi_1,\ldots,\xi_{r-1}$, linearly independent of $\frac{\partial}{\partial x_n}$ at the origin. One can assume: $\xi_j(x_n) = 0$ for $j = 1,\ldots,r$. And then repeat this procedure for $\tilde{X}$.

2. We start from the factorized germ, $\tilde{X} \times (k^r,o)$. Note that the condition $T_{\mathcal{A}}f = T_{\mathcal{A}(0)}f$ is preserved under the $\text{Aut}_X$-coordinate change, see lemma 3.4.

(\mathcal{R}) The assumption $T_{\mathcal{A}}f = T_{\mathcal{A}(0)}f$ gives:

\[ \text{Span}_{R_X}[\partial_{x_1}f,\ldots,\partial_{x_r}f] \subseteq (\tilde{x},z) \cdot \text{Span}_{R_X}[\partial_{x_1}f,\ldots,\partial_{x_r}f] + \text{Der}_{X}^{(0)}(f). \]
By Nakayama we get: \( \text{Span}_{R_X}\{\partial_1, \ldots, \partial_r, f\} \subseteq \text{Der}_{X}^{(0)}(f) \). By theorem 13 the unfolding \( f(\tilde{x}, z) \) of \( f(\tilde{x}, o) \) is \( \mathcal{R} \)-trivial. Its trivialization is a \( z \)-linear automorphism of \( R_X \). Thus it preserves the factorized form \( \tilde{X} \times (k', o) \) and gives the statement.

(\( \mathcal{X} \)) The assumption \( T_{\mathcal{X}} f = T_{\mathcal{X}(0)} f \) is now:

\[
(18) \quad \text{Span}_{R_X}\{\partial_1, \ldots, \partial_r, f\} \subseteq (z, \tilde{x}) \cdot \text{Span}_{R_X}\{\partial_1, \ldots, \partial_r, f\} + \text{Der}_{X}^{(0)}(f) + (f) \cdot R_X^{\oplus}. 
\]

By Nakayama we get: \( \text{Span}_{R_X}\{\partial_1, \ldots, \partial_r, f\} \subseteq \text{Der}_{X}^{(0)}(f) + (f) \cdot R_X^{\oplus} \). By theorem 13 the unfolding \( f(\tilde{x}, z) \) of \( f(\tilde{x}, o) \) is \( \mathcal{X} \)-trivial. Its trivialization preserves the factorized form \( \tilde{X} \times (k', o) \) and gives the statement.

(\( \mathcal{A} \)) The assumption is now:

\[
(19) \quad \text{Span}_{R_X}\{\partial_1, \ldots, \partial_r, f\} \subseteq (z, \tilde{x}) \cdot \text{Span}_{R_X}\{\partial_1, \ldots, \partial_r, f\} + \text{Der}_{X}^{(0)}(f) + T_{\mathcal{X}(0)} f. 
\]

By Nakayama over \( R_X \), see [21]iv, we get: \( \text{Span}_{k}\{\partial_1, \ldots, \partial_r, f\} \subseteq \text{Der}_{X}^{(0)}(f) + T_{\mathcal{X}(0)} f \). By theorem 13 the unfolding \( f(\tilde{x}, z) \) of \( f(\tilde{x}, o) \) is \( \mathcal{A} \)-trivial. Its trivialization preserves the factorized form \( \tilde{X} \times (k', o) \) and gives the statement.

In part 2 of this theorem we can take the filtration \( (\tilde{x}, z_1, \ldots, z_j) \in R_{k' \times (k', o)} \) for some \( j \leq r \). We get (for \( \mathcal{R} \)): if \( T_{\mathcal{R}} f = T_{\mathcal{R}(0)} f \) then \( f \) factorizes as \( X \to \tilde{X} \times (k', o) \to (k', o) \). (And similarly in the \( \mathcal{X}, \mathcal{A} \)-cases.) The proof is the same. We state this explicitly.

**Corollary 4.9.** If \( (\partial_1 - \xi_{2,2,2,\ldots,2}) f = 0 \), where \( \xi_{2,2,2,\ldots,2} \in \text{Der}_{X}^{(0)}(f) + m \cdot \text{Span}_{R_X}\{\partial_{22}, \ldots, \partial_r\} \), then \( f \) is \( \mathcal{A} \)-equivalent to the pullback of a map \( \tilde{X} \to (k', o) \).

This is the classical “geometric lemma of deformations”, e.g. [Martinet] pg.192] (for \( C^\infty \)-case).

### 4.4. The rank of a map.
Given a map \( F \in m \cdot R_X^{\oplus} \) we evaluate its tangent image \( T_{\mathcal{X}} F \subseteq R_X^{\oplus} \) at the origin, i.e. take the image \( R_X/m \cdot T_{\mathcal{X}} F \subseteq R_X^{\oplus} \) and \( k' \). This is a k-vector subspace.

**Definition 4.10.** The rank of \( F \) is the dimension \( \text{dim}_k(R_X/m \cdot T_{\mathcal{X}} F) \).

If \( \text{rank}(F) = r \) then, in particular, \( \text{dim}_k(\text{Der}_X|_o) \geq r \). Thus, (assuming \( \text{char}(k) = 0 \)) by theorem 4.8 we get: \( X \cong \tilde{X} \times (k'_o, o) \).

**Lemma 4.11.**

1. Take a map \( F \in m \cdot R_X^{\oplus(p+r)} \) of rank \( r \). Suppose either \( \text{char}(k) = 0 \) or \( J = 0 \). Then \( F \sim (f(\tilde{x}) + h(\tilde{x}, u), u) \), where \( u = (u_1, \ldots, u_n) \) and \( f(\tilde{x}) \in (\tilde{x})^2 \cdot R_X^{\oplus} \) and \( h(\tilde{x}, u) \in (u) \cdot (\tilde{x}) \cdot R_X^{\oplus} \).

2. Here the \( \mathcal{A} \)-type of \( f \) is not uniquely determined due to the residual \( \mathcal{A} \)-equivalence:

\[
(\text{f + h(x, u), u) \sim (f + h(x, u(\tilde{u} + q(f))), \tilde{u})}
\]

for any element \( q(y) \in (y) \cdot R_Y^{\oplus} \) and the corresponding parameter change \( \tilde{u} = u + q(f) - q(f + h(x, u)) \).

In particular, part 2 gives: \( (f + h(x, u), u) \sim (f + h(x, u(q(f))), (x)(\tilde{u}), \tilde{u}) \).

**Proof.**

1. Using the subgroup \( GL(p, k) \subset \mathcal{L} \) we can ensure: the last \( r \) components of \( F \) are of order 1 (and their linear parts are independent), while the other components of \( F \) are of order \( \geq 2 \). By an \( \mathcal{R} \)-transformation on the last \( r \) components, we get \( F \sim ((u, x)^2, u) \). Then by \( \mathcal{L} \) we achieve the claimed form \( f(\tilde{x}) + (\tilde{x}) \cdot (u, u) \).

2. The relation \( \tilde{u} = u(u) \) is invertible. Rewrite it as \( u(\tilde{u}) = \tilde{u} - q(f) + q(f + h(x, u)) \). Then:

\[
(20) \quad (f + h(x, u), u) = (f + h(x, u(\tilde{u})), u(\tilde{u})) \overset{\mathcal{L}}{\sim} (f + h(x, u(\tilde{u})), \tilde{u} - q(f)) \overset{\mathcal{L}}{\sim} (f + h(x, u(\tilde{u} + q(f)), \tilde{u})).
\]

**Example 4.12.**

i. Suppose \( p = 1 \), i.e. the map is of corank\( =1 \). By Morse lemma, [Kerner 21] §4], \( F \sim (f(\tilde{x}) + Q_2(z) + h(\tilde{x}, u), u) \), where \( f(\tilde{x}) \in (\tilde{x})^2 \), while \( Q_2(z) \) is a non-degenerate quadratic form.

ii. Suppose \( h(x, u) = h - u \), where \( h \in \text{Mat}_{p \times p}(k) \). Take \( q(y) := y \), where \( q \in \text{Mat}_{p \times p}(k) \). We get \( \tilde{u} = u - q \cdot h \cdot u \). Thus \( u = [I - q \cdot h]^{-1} \cdot \tilde{u} \). Then part 2 gives:

\[
(f + h \cdot u, u) \sim (f + h \cdot [I - q \cdot h]^{-1}(\tilde{u} + q \cdot f), \tilde{u}) = ([I - h \cdot q]^{-1} f + h \cdot [I - q \cdot h]^{-1} \tilde{u}, \tilde{u}).
\]

In particular, if \( (h \cdot u \) spans the vector space \( (x) \cdot T^1_{\mathcal{X}} f \), then \([I - h \cdot q]^{-1} f - f \) spans \( (x) (f) \cdot R_X^{\oplus} \mod T_{\mathcal{X}} f \). Then also \( h \cdot [I - q \cdot h]^{-1} \) spans \( (x) \cdot T^1_{\mathcal{X}} f \). Thus any two \( \mathcal{A} \)-stable maps that are \( \mathcal{X} \)-equivalent are also \( \mathcal{A} \)-equivalent.
4.5. Trivializing unfoldings over non-local bases. Suppose an unfolding is infinitesimally trivial. Theorem 4.3 ensures the local triviality of the unfolding over the base \((K^r, o)\). In the analytic case, \(R_X = k[[x]]/J\), one gets the trivialization over a small ball, for \(\|t\| \ll 1\). In the henselian case, \(R_X = k(x)/\mathcal{J}\), one gets the trivialization over an étale neighborhood.

Take an unfolding with the affine base, \(F = (f_t(x), t)\) with \(t \in k^r\). The ring \(R_{X,t}\) is now one of \(k[[t]][x]/J\), \(k[t][x]/J\), \(k[t][x]/J\), i.e. power series in \(x\) whose coefficients are polynomials in \(t\). Take the corresponding group \(\mathcal{G}_t \in \mathcal{R}, \mathcal{K}, \mathcal{A}\). These transformations are \(t\)-global.

Accordingly we want the \(t\)-global trivialization criteria. These are obstructed in two ways.

- The \(\mathcal{G}_t\)-transformations do not preserve the origin. Therefore, e.g. for \(R_X = \mathbb{C}\{x\}\), when taking non-small \(t\), one runs out of the ball of convergence in \(\mathbb{C}^n\). Therefore one should restrict to the subgroup \(\mathcal{G}_t(0) \leq \mathcal{G}_t\) of elements that preserve the origins of \(X\) and \((k^p, o)\).
- The filtered-nilpotent vector fields can be “integrated”, i.e. there exists a map \(T_{\mathcal{G}_t(1)} \to \mathcal{G}_t(1)\) that approximates the classical exponential, \([\text{B.G.K.22}]\) §3. But not every element \(\xi \in T_{\mathcal{G}_t(0)}\) induces an element \(g_\xi \in \mathcal{G}_t(0)\) satisfying: \(\text{ord}(g_\xi - 1d - \xi) > \text{ord}(\xi)\). For example, for \(tx\partial_x \in T_{\mathcal{G}_t(0)}\) the corresponding \(\mathcal{R}\)-transformation would be of the type \(x \to x + tx + (x)^2\). But this is not invertible for \(t = 1\).

Therefore below we restrict to the subgroup \(\mathcal{G}_t^{(1)} < \mathcal{G}_t\). For the subgroup \(\mathcal{G}_t^{(0)} < \mathcal{G}_t\) our results are weaker, we can only compare particular fibres of the family \(\{f_t\}\).

4.5.1. Trivialization for the subgroup \(\mathcal{G}_t^{(1)} < \mathcal{G}_t\).

Lemma 4.13. Let \(R_{X,t} = k[[t]][x]/J\), where \(k\) is a field of zero characteristic. Let \(\mathcal{G} \in \mathcal{R}, \mathcal{K}, \mathcal{A}\). Any infinitesimally \(\mathcal{G}^{(1)}\)-trivial unfolding is \(t\)-globally \(\mathcal{G}^{(1)}\)-trivial.

Proof. We use the trivializations constructed in the proof of theorem 4.3 equations (14), (15), (16). We should only verify: the derivation \(\zeta_X\) constructed via \(e^{-i\partial_t} \circ e^{i(\partial_t + \xi_X)} = e^{i\xi_X}\) belongs to \(T_{\mathcal{G}_t^{(1)}}\). In particular, in its \((t,x)\)-expansion all the coefficients of \(x\)-monomials should be polynomials in \(t\) (rather than just power series).

Indeed, by the BCH formula, \([\text{Kerner.21}]\) §2.1, \(\zeta_X = \sum \partial_t^j p_l(\partial_t, \partial_t - \xi_X)\), where \(p_l\) is a homogeneous polynomial expressible via repeated commutators of \(\partial_t\) and \(\xi_X\). Observe:

- \(T_{\mathcal{G}_t^{(1)}} \leq T_{\mathcal{G}_t^{(1+j)}}\).
- For each \(\eta \in T_{\mathcal{G}_t^{(j)}}\) we have \([\partial_t, [\partial_t, \ldots, [\partial_t, \eta]]]_{T_{\mathcal{G}_t^{(j+1)}}}\), for the \(\partial_t\)-commutator repeated sufficiently many times. This holds because the coefficient of every \(x\)-monomial is a polynomial in \(t\).

Therefore every summand that appears in \(\sum \partial_t^j p_l(\partial_t, \partial_t - \xi_X)\) belongs to \(T_{\mathcal{G}_t^{(j)}}\), and moreover, \(p_l(\partial_t, \partial_t - \xi_X)\) \(T_{\mathcal{G}_t^{(j)}}\) for each \(j\) and a corresponding \(l \gg 1\). Altogether, \(\zeta_X \in T_{\mathcal{G}_t^{(1)}}\). 

4.5.2. Comparison of fibres for the subgroup \(\mathcal{G}^{(0)} \leq \mathcal{G}_t\).

Proposition 4.14. Let \(R_X\) be one of \(k[[x]]/J, k(x)/J, k(x)/J\), with \(k\) either \(\mathbb{R}\) or alg.closed field of zero characteristic. For \(\mathcal{G} = \mathcal{A}\) take \(R_X = k[[x]]/J\). Take an unfolding \(F = (f_t(x), t) \in R_X^{\mathbb{P}^*}\). Suppose \(\sqrt{t} = m\). If \(\partial_t f_t \in T_{\mathcal{G}_t^{(0)}} f_t\) then \(f_0 \sim f_t\) for every \(t_0 \in k^r\).

Proof. 

Step 1. We prove the statement for \(k \in \mathbb{R}, \mathbb{C}\), and the ring \(k(x)/J\). Take a path \(t \mapsto t_o\) (in \(k\)). Along this path (at every point on it) we have \(\partial_t f_t \in T_{\mathcal{G}_t^{(0)}} f_t\). By theorem 4.3 we get: \(F\) is \(\mathcal{G}\)-trivializable on a small ball \(\mathcal{B}(t')\). More precisely, the trivialization is done by a representative of an element of \(\mathcal{G}\). In fact \(F\) is \(\mathcal{G}^{(0)}\)-trivializable, see remark 4.3ii.

Cover the path \(t \mapsto t_o\) by such balls, and choose a finite subcover. Then we can pass from \(f_{t_0}\) to \(f_{t_o}\) by a finite number of \(\mathcal{G}^{0}\) elements, or rather by their representatives. All these representatives preserve the origin, and are defined on the same (small) balls in \(k^2, k^p\). Thus their product is defined, and we get: \(f_o \sim f_{t_o}\).

Step 2. (The general case, \(k = \bar{k}\).) Let \(\partial_t f_t = \xi(f_t)\) for \(\xi \in T_{\mathcal{G}_t^{(0)}}\) and \(R_{X,t} = k[[t]][x]/J\). Pass to the finite jets \(k[[t]][x]/J + (x)^s\). Note that \(\mathcal{G}_t^{(0)}\) and \(T_{\mathcal{G}_t^{(0)}}\) act on this quotient. We still have the condition \(\partial_t f_t = [\xi([f_t])]\). Let \(\{C_*\}\) be the (finite) collection of \(k\)-coefficients in the Taylor
expansion of $[f_i]$ and $[\xi]$. Then $\mathbb{Q}(C_\bullet) \supseteq \mathbb{Q}$ is a finite extension of fields. Therefore one can embed $j : \mathbb{Q}(C_\bullet) \hookrightarrow \mathbb{C}$.

We still have $\partial_j(f_t) = j(\xi) \cdot j(f_t)$, now over the ring $\mathbb{C}[[t]][j(J)] + (x) \cong \mathbb{C}[t][x]/j(J) + (x)^d$.

Therefore, by part 1, we get $j(f_0) \not\sim j(f_0)$, Consider this as the equivalence $[f_0] \not\sim j(f_0)$ over some field extension of $k$. By lemma [2.2] we get the equivalence over $k = k$. (Now the polynomial system to resolve is finite, thus we do not need the assumption “$k$ is uncountable”.)

Returning to $k[[t]]/j(J)$, we get (for every $d \geq 1$) an element $g_d \in \mathcal{G}$ satisfying $f_0 \not\sim f_0 \mod (x)^d$. Applying this iteratively, we can present this via the product, $f_{t_n} = g_d \cdot g_1 \cdot f_0$, with $g_j \in \mathcal{G}(j)$. This product converges formally. Therefore $f_0 \sim f_0$ are $\mathcal{G}$-equivalent.

For $\mathcal{B}, \mathcal{X}$ one applies the Artin approximation.

\textbf{Remark 4.15.} i. In the first part of this proof we needed the normed field $k$ to be connected and locally compact. This forces $k$ to be $\mathbb{R}$ or $\mathbb{C}$, [Wieslaw].

ii. The statement does not hold over fields that are not $\mathbb{R}$-closed. Even the t-local version (over $k[[t]][x]$) fails. For example, $f_t := (1 + t)x^d \in k[[t]][x]$ satisfies $\partial_t f_t \in T_\mathcal{G} f_t$. But to achieve $f_0 \sim f_1$, at least for $t \ll 1$, one needs the property $\sqrt{1 + t} \in k$ for $t \ll 1$.

5. THE PRE-NORMAL FORM OF UNFOLDING AND VERACITY

5.1. Let $R_X = k[[x]]/J$, see [2.2] and fix a map $f_0 \in m \cdot R_X^{ep}$, not necessarily $\mathcal{G}$-finite. We get the $k$-vector space $T_\mathcal{G} f_0 := R_X^{(p-1)} f_0$, possibly of infinite dimension. Fix (any) elements $\{v_\bullet\} \subset R_X^{ep}$ whose images formally generate the $k$-vector space $T_\mathcal{G} f_0$. Namely, any element of $T_\mathcal{G} f_0$ is presentable as some elements $c_\bullet v_\bullet$, with $c_\bullet \in k$, and this sum converges in $R_X$. For $\mathcal{G} = \mathcal{X}$ we always assume $\{v_\bullet\} \subset m \cdot R_X^{ep}$.

\textbf{Definition 5.1.} The $\mathcal{G}$-pre-normal form of an unfolding of $f_0$ is the unfolding $(f_0(x) + \sum a_j(t) \cdot v_j(x), t)$, where $a_j(t) \in (t) \cdot k[[t]]$.

\textbf{Lemma 5.2.} Any unfolding $F(t, x) = (f_t(x), t) \in R_X(t, x)$ is (formally) $\mathcal{G}$-equivalent to its pre-normal form. Moreover, this $\mathcal{G}$-equivalence preserves the image of $\text{Span}_k[\partial_t f_t | t = 0] \subset T_\mathcal{G} f_0$, i.e.

\begin{equation}
T_\mathcal{G} f_0 + \text{Span}_k[\partial_t f_t | t = 0] \subset T_\mathcal{G} f_0 + \text{Span}_k[\sum \partial_t a_j(t) | t = 0] \subset T_{\mathcal{G}_p} f_0
\end{equation}

\textbf{Proof.} The transition to the pre-normal form is inductive. Using the generators $\{v_\bullet\}$ we can present $F(t, x) = (f_0 + \xi(t) f_0 + \sum a_j(t) v_j(t), t)$. Here $1 \leq d < \infty$ and $0 \neq \xi(t) f_0 \in (t)^4 \cdot T_\mathcal{G} f_0$, and $a_j(t) \in (t) \subset k[[t]]$. Take a group element $g_0 \in G$ of the form $g = 1d - \xi(t) + (t)^d$. This is ensured by the jet$_0$-assumption on $R_X$, see [2.2]. Then $g_0(F) = (f_0 + \xi(t) f_0 + \sum a_j(t) v_j(t), t)$, for some new power series $\{a_j(t)\}$. Iterate this process. Note: $a_j(t) - a_j(t) \in (t)^d$.

The infinite product limit$_{d \rightarrow \infty}(g_0 \cdot g_1)$ converges (formally) to an element $g \in \mathcal{G}$. Indeed, at $d$’th step we do not change the t$^d$-elements with $i < d$.

Altogether, the unfolding $gf_0$ is in the pre-normal form. Finally we verify equation [2.1].

\begin{itemize}
  \item $g_1$ changes $\text{Span}_k[\partial_t f_t | t = 0]$ only by an element of $T_{\mathcal{G}} f_0$. Hence $g_1$ preserves $T_{\mathcal{G}} f_0 + \text{Span}_k[\partial_t f_t | t = 0]$.
  \item For $d \geq 2$ the element $g_d$ does not affect $\partial_t f_t | t = 0$.
\end{itemize}

\textbf{Remark 5.3.} i. The pre-normal form is usually far from being unique. E.g. suppose $gf_0 = f_0$ for some $1d \neq g \in \mathcal{G}$. Then the (\mathcal{G}-equivalent) unfoldings $f_1, g_1$ have different pre-normal forms.

ii. Suppose the ring is $R_X = k[[x]]/J$, the group is $\mathcal{G} \in \mathcal{B}, \mathcal{X}$ and $f_1$ is a $\mathcal{G}_f$-finite map. Then the pre-normal form can be achieved over $R_X$, not just formally. Indeed, we should resolve the condition $g \cdot f_1 = f_0 + \sum a_j(t) v_j$, with the unknowns $g, \{a_\bullet\}$. Here $g$ can depend on $x, t$, but $a_\bullet$ should depend on $t$ only. This is a (finite) system of $k(x, t)$-equations. The lemma ensures a formal solution $\hat{g}, \{\hat{a}_\bullet(t)\}$, over $R_X$. Apply the nested Artin approximation to achieve an ordinary solution, $g, \{a_\bullet(t)\}$.

iii. In the $\mathcal{X}$-case another presentation of pre-normal form is often useful. Suppose $f$ is \mathcal{X}-finite. Fix some elements $\{v_\bullet\} \subset m \cdot R_X^{ep}$ that go to a basis of $m \cdot T_{\mathcal{X}} f_0$. Then $R_X^{ep} = \text{Span}_{R_X}(v_\bullet) + T_{\mathcal{X}} f_0$, see lemma [2.1]. And then, lemma [5.2] gives: any unfolding of $f_0$ is formally $\mathcal{X}$-equivalent to $(f_0 + \sum a_j(t, f_0) \cdot v_j, t)$, for some $a_j(t, y) \in (t) \subset k[[t, y]]$.

\textbf{Example 5.4.} i. Take a map $f : (k^n, o) \rightarrow (k^n, o)$ of corank one. Present it as an unfolding, $f = (u, f_0(x) + (x)(u))$, here $u = (u_1, \ldots, u_{n-1})$. Then $f \sim (u, x^d + \sum_{i=1}^{d-1} x^i a_i(u))$, with $a_j(u) \in (u)$. 

5.2. Versatility vs infinitesimal versatility. Let \( G = \mathcal{A} \) with \( R_X = \mathbb{k}[x]/J \) or \( G \in \mathcal{R}, \mathcal{K} \) with \( R_X \) one of \( \mathbb{k}[x]/J, \mathbb{k}(x)/J, \) or \( \text{char}(\mathbb{k}) = 0 \) with \( G \in \mathcal{R}, \mathcal{K}, \mathcal{A} \) and \( R_X = \mathbb{k}(x)/J \).

Theorem 5.5. 1. Let a (finite) tuple \( \{v_\bullet\} \) in \( R_X^{\mathcal{P}} \) generate \( T_g^o f \). For \( G \in \mathcal{R}, \mathcal{K} \) the unfolding \( (f_0 + \sum t_j v_j, t) \) is \( G \)-versal. For \( G = \mathcal{A} \) the unfolding \( (f_0 + \sum t_j v_j, t) \) is formally \( \mathcal{A} \)-versal.

2. An unfolding \( F \) is \( G \)-versal if it is infinitesimally \( G \)-versal.

3. \( \dim_g T_g^o f \) is the minimal number of parameters in a \( \mathcal{G} \)-versal unfolding.

Proof.

1. Start from a(ny) unfolding \( F = (f_t(x), t) \in R_X^{\mathcal{P}+} \). We should resolve the condition \( g(f_t) = f_0 + \sum a_j v_j \), with the unknowns \( g \in G_t \) and \( a_j \in \mathbb{k}[t] \), resp. \( \mathbb{k}(t) \). The pre-normal form, lemma 5.2, ensures the formal solution. In the \( \mathcal{R}, \mathcal{K} \)-cases and \( R_X = \mathbb{k}(x)/J \) one uses remark 5.3 and corollary 1.9.

For \( \text{char}(\mathbb{k}) = 0 \) we give another proof for \( R_X = \mathbb{k}[x]/J \), the direct generalization of [Martinet, Chapter XIV]. Extend the ring by a new variable \( \bar{t} \) and extend the unfolding, \( F := (f_{\bar{t}}(x, t), t) \), with \( f_{\bar{t}}(x) := f_t(x) + \sum t_j v_j \). We prove: \( F \) is induced from the unfolding \( (f_0 + \sum t_j v_j, t) \). By lemma 3.5 we get: \( \mathcal{R}, \mathcal{K} \)-versal unfolding.

By lemma 3.5, \( \sum a_j v_j \), with the unknowns \( a_j \in \mathbb{k}[t] \), resp. \( \mathbb{k}(t) \). Therefore we have \( \partial_i f_i = \sum \partial_i f_t + \operatorname{Span}_{\mathbb{k}}(\{v_\bullet\}) \). By corollary 1.9, one has \( f_t \sim f_i \).

2. The part \( \Leftarrow \). Let \( \mathbb{K} = \mathbb{k}[t] \) for \( G \in \mathcal{A} \) and \( \mathbb{K} = \mathbb{k}[t], \mathbb{k}(t) \) for \( G \in \mathcal{R}, \mathcal{K} \).

By part 1, \( F \) is \( G \)-equivalent to \( F := (f_0 + \sum a_j v_j, t) \), with some coefficients \( a_j(t) \in \mathbb{K} \).

Moreover, \( F \) is infinitesimally-versal, by lemma 3.4. Thus the elements \( \partial_i(\sum a_j(t)v_j)|_{t_0} \), \( i = 1, \ldots, \omega \), generate the vector space \( T_{f_0} \). Thus then the map \( t \to \{a_j(t)\} \) is full since the unfolding \( (f_0 + \sum t_j v_j, t) \) is a pullback of \( F \). Hence \( F \) is versal.

3. For any unfolding \( F = (f_t(x), t) \), the number of parameters is at least \( \dim_g \operatorname{Span}_{\mathbb{K}}(\partial_i f_t, \ldots, \partial_r f_t)|_{t_0} \geq \dim T_{f_0} \).

Example 5.6. i. This theorem holds also for the ring \( \mathbb{k}[x]/J \) in positive characteristic, provided \( f_0 \) is \( \mathcal{A} \)-equivalent to a polynomial map. In this case one chooses \( \{v_\bullet\} \) as polynomials in \( x \) and works in the ring \( \mathbb{k}(x)/J \).

ii. For \( R_X = \mathbb{R}[x], \mathbb{C}[x], C^\infty(\mathbb{R}^n, o) \) this is a classical theorem. See e.g. [Martinet, Chapter XIV] (for \( \mathcal{A} \)-equivalence in \( C^\infty \)-case), [Damon84, Theorem 9.3] (for \( \mathcal{A} \)-equivalence in all cases), pg. 143 of [Mo, N.B., or Theorem 1.16, pg.238 of [Gr,Lo,Sh] (for \( \mathcal{K} \)-equivalence of \( \mathbb{C}[x] \)). We remark: the formal statement is simple, but the \( \mathbb{k}[x] \), \( C^\infty \)-cases are non-trivial.

5.3. Fibration of \( \mathcal{K} \)-trivial unfoldings into \( \mathcal{A} \)-unfoldings. Let \( f_t \) be a \( \mathcal{K} \)-trivial family. To understand the \( \mathcal{A} \)-types appearing in this family we should take a slice transversal to the subspace \( T_{f_0} \cap T_{\mathcal{K}} f_0 \subseteq T_{f_0} \). Take the quotient \( T_{f_0} / T_{f_0} \cap T_{\mathcal{K}} f_0 \). Both parts contain \( T_{f_0} \), hence the standard isomorphism of \( \mathcal{R} \)-modules gives:

\[
T_{f_T f_0} / T_{f_0} \cap T_{f_T f_0} \cong (x) \cdot (f_0) \cdot R_X^{\mathcal{P}} / T_{f_0} \cap (x) \cdot (f_0) \cdot R_X^{\mathcal{P}} = (x) \cdot (f_0) \cdot R_X^{\mathcal{P}} + (y) \cdot T_{f_T f_0} / T_{f_0}.
\]

Assume \( f_0 \) is \( \mathcal{K} \)-finite, and fix some elements \( \{v_\bullet\} \subseteq (x) \cdot (f_0) \cdot R_X^{\mathcal{P}} \) whose images generate the \( \mathcal{R} \)-module \( (x) \cdot (f_0) \cdot R_X^{\mathcal{P}} / T f_0 \cap (x) \cdot (f_0) \cdot R_X^{\mathcal{P}} \).

Evidently any unfolding of type \((x_0 + (t) \cdot \operatorname{Span}_{R_X}(v_\bullet), t)\) is \( \mathcal{K} \)-trivial.

Definition 5.7. The \( \mathcal{A} \)-pre-normal form of a \( \mathcal{K} \)-trivial unfolding of the map \( f_0 \in (x) \cdot R_X^{\mathcal{P}} \) is the unfolding of type \((f_0 + (t) \cdot \operatorname{Span}_{R_X}(v_\bullet), t)\).
Lemma 5.8. Let $R_X$ be one of $k[[x]]/J$, $k[x]/J$, $k(x)/J$. If $\text{char}(k) > 0$ then assume $J = 0$. Any $\mathcal{X}$-trivial unfolding $(f_t(x), t)$ of $f_0 \in (x) \cdot R_{X,u}^{\mathcal{X}}$ is formally-$\mathcal{X}$-equivalent to its $\mathcal{A}$-pre-normal form.

Moreover, if the map $f_t \in (x) \cdot R_{X,u}^{\mathcal{X}}$ is $\mathcal{A}$-finitely determined then the unfolding $(f_t(x), t)$ is $\mathcal{A}$-equivalent to its pre-normal form.

In this sense the unfolding $(f_0 + \sum \tau_j v_j, t)$ is versal among all the $\mathcal{X}$-trivial unfoldings.

Proof. As the unfolding $(f_t(x), t)$ is $\mathcal{X}$-trivial, we can assume (by an $\mathcal{A}$-transformation): $(f_t(x)) = (f_0(x)) \subset R_{X,t}$. Then $R_{X,t}^{\mathcal{X}} \ni f_t(x) \sim GL(p, R_{X,t})$, we can assume:

$$f_t \sim f_0 \in (t) \cdot (x) \cdot (f_0) \cdot R_{X,t}^{\mathcal{X}}.$$

Therefore $f_t = f_0 - t(\xi_X(x) + \xi_Y(y)|f_0) \in (t) \cdot \text{Span}_{R_{X,t}}\{v_s\}$, for some derivations $\xi_X \in T_{\mathcal{X}}, \xi_Y \in T_{\mathcal{X}}$.

If $R_X$ is regular (i.e. $J = 0$) then define the coordinate change $\Phi_{X,t}$ by $x \to x + t\xi_X(x)$. In the general case define $\Phi_{X,t}$ as the extension $x \to x + t\xi_X(x) + (t)^2$ of the map $x \to x + \xi_X(x)$. (Our assumptions ensure the jet$_0$-condition.) We remark that $\Phi_{X,t}$ does not necessarily preserve the origin of the source, i.e. $\Phi_{X,t}(x) \neq (x) \subset R_{X,t}$.

Define $\Phi_{Y,t} \in L_t$ by $y \to y + t\xi_Y(y)$. It does not necessarily preserve the origin of the target, $\Phi_{Y,t}(y) \neq (y) \subset R_{Y,t}$. Altogether we get: $\Phi_{X,t}^{-1} \circ \Phi_{Y,t}^{-1}(f_t) \sim f_0 \in (t) \cdot \text{Span}_{R_{X,t}}\{v_s\} + (t)^2 \cdot T_{\mathcal{X}} f_0$.

Iterate this process to get the formal $\mathcal{A}$-equivalence, $(f_t, t) \sim (f_0 + (t) \cdot \text{Span}_{R_{X,t}}\{v_s\}, t)$.

Suppose $f_t$ is $\mathcal{A}$-finitely determined. We have proved: $f_0 + \text{Span}_{R_{X,t}}\{v_s\} \sim f_t + (t)^d$ for $d \gg 1$. Now invoke the finite determinacy.

Example 5.9. ($p = 1$) Take a map $f_0 : (k^n, o) \to (k^1, o)$, $f(x) \in (x)^2$. Suppose $f \in (x)^2$ is “almost weighted homogeneous” in the following sense: $(x) \cdot f \subseteq \text{Jac}(f) + \text{Span}_{\mathcal{X}}(f^2, \ldots)$. Then in equation 22 one get $T_{\mathcal{X}} f_0 f_t = 0$. Therefore any $\mathcal{X}$-trivial unfolding of $f$ is also (formally) $\mathcal{A}$-trivial.

5.4. Fibration of $\mathcal{X}$-orbits into $\mathcal{A}$-orbits. Assume $\text{char}(k) = 0$ or $J = 0$. Take a map of rank $r$, present it as $F = \{f(x, u), \in R_{X,u}^{\mathcal{X}(p+r)}\}$, where $f(x, u) \in (x) \cdot (x, u) \in R_{X,u}^{\mathcal{X}}$ (see lemma 4.1). Here $R_{X,u}$ is one of $k[x, u]/J$, $k[x, u]/J$, $k[x, u]/J$. Accordingly $R_{Y,u}$ is one of $k[y, u], k[y, u], k[y, u]$.

Suppose the map $f_0(x) = f(x, o) \in (x)^2 \cdot R_{X,u}^{\mathcal{X}}$. Fix some elements $\{v_s\} \subset (x) \cdot R_{X,u}^{\mathcal{X}}$ that go to a basis of $(x) \cdot T_{\mathcal{X}} f_0$. By the direct check: these elements go also to the basis of $(x) \cdot T_{\mathcal{X}} F$. By the above, $f_0$ is finitely determined. Then in equation 22 one get $T_{\mathcal{X}} f_0 f_t = 0$. Therefore any $\mathcal{X}$-trivial unfolding of $f$ is also (formally) $\mathcal{A}$-trivial.

Proposition 5.10. Suppose $k$ is an infinite field. Then:

$$\mathcal{X}F \cap (\mathcal{A} \text{-finitely determined}) \subseteq \mathcal{A}(\{f\} + \text{Span}_{R_{Y,u}}((y, u) \cdot \{v_s\}), u).$$

Proof. Suppose a map $\tilde{F} \in \mathcal{X}F$ is $\mathcal{A}$-finitely determined. Then to bring it to the claimed form it is enough to work over $R_{X,u} = k[x, u]/J$ and $R_{Y,u} = k[y, u]$.

Step 1. By an $\mathcal{A}$-transformation we can assume $(\tilde{F}) = (F) \subset R_{X,u}$. Then, by lemma 4.11 we can assume $\tilde{F} = \{f(x, u), \in R_{X,u}^{\mathcal{X}(p+r)}\}$, with $f, \tilde{f} \in (x) \cdot (x, u) \cdot R_{X,u}^{\mathcal{X}}$. Here $(f, \tilde{f}, u) \subset R_{X,u}$. Apply $GL(p, k[[u]])$ to get: $\tilde{f} - f \in (x) \cdot (f, u) \cdot R_{X,u}^{\mathcal{X}}$.

By lemma 2.1 we have $(x) \cdot R_{X,u}^{\mathcal{X}} = T_{\mathcal{X}} f_0 + T_{\mathcal{X}(o)} f_0 + \text{Span}_{R_{Y,u}}\{v_s\}$. Therefore

$$\begin{equation}
(x, u) \cdot R_{X,u}^{\mathcal{X}} = T_{\mathcal{X}} f + (y, u) \cdot T_{\mathcal{X}} f + \text{Span}_{R_{Y,u}}\{v_s\}.
\end{equation}$$

Hence $\tilde{f} - f \in \text{Span}_{R_{Y,u}}\{(y, u) \cdot \{v_s\}\} + (f, u) T_{\mathcal{X}} f + (y, u)^2 \cdot T_{\mathcal{X}} f$.

As $\tilde{F}$ is $\mathcal{A}$-finitely determined, it is enough to prove: $\tilde{F} + h_d \in \mathcal{A}(f + \text{Span}_{R_{Y,u}}\{(y, u) \cdot \{v_s\}), u)$ for some $h_N \in (x, u)^N \cdot R_{X,u}^{\mathcal{X}(p+r)}$, with $N \gg 1$. Therefore it is enough to prove:

$$\begin{equation}
\tilde{F} \in \mathcal{A}(f + \text{Span}_{R_{Y,u}}\{(y, u) \cdot \{v_s\}), u \cdot (x, u)^N \cdot R_{X,u}^{\mathcal{X}(p+r)} \text{ for } N \gg 1.
\end{equation}$$

Step 2. Start from $\tilde{f} = f + t g$, where $g \in (x) \cdot (f, u) \cdot R_{X,u}^{\mathcal{X}}$, and $t$ is an indeterminate. The transition to the form of (23) is done inductively. By equation (23) we can present $g \in \text{Span}_{R_{Y,u}}\{(y, u) \cdot \{v_s\} + (f, u)^d \cdot T_{\mathcal{X}} f + (y, u)^{d+1} \cdot T_{\mathcal{X}} f$ for some $d \geq 1$. 
• Inductive step for the case of a regular ring, $R_{X,u} = k[[x,u]]$. Define the coordinate change $\Phi_X \in \mathcal{D}$ by $x \to x + t(f,u)^d \cdot c \cdot \xi(x)$ and $u \to u$. (Here $c \in R_{X,u}$ is an unknown.)

Note: this coordinate change is filtration-unipotent, i.e. $\Phi_X - Id$ is filtration-nilpotent for $(x,u) \in R_{X,u}$. Accordingly

$$\Phi_X(f + tg) - f \in t(f,u)^d(c + c(g))T_{\mathcal{D}} f + t^2 \cdot (f,u)^{d+1} \cdot (c)^2 R_{X,u}^{\text{rep}} + t^2(f,u)^d \cdot (c) \cdot (x) R_{X,u}^{\text{rep}}.$$  

Expand $c = \sum c_{m,n} x^m \cdot u^n$, with multi-indices $m, n$. Here $\{c_{m,n}\} \in k[[t]]$ are unknowns. Then we get:

$$\Phi_X(f + tg) - f \in t(f,u)^d \cdot \sum m, n \left( c_{m,n} + t \cdot H_{m,n}(\{c_s\}) \right) \cdot x^m \cdot u^n \cdot T_{\mathcal{D}} f +$$

$$t^2(\{c_s\})^2 \cdot \text{Span}_{R_Y}(y) \cdot \{q_s\} + t(\{c_s\}) \cdot \text{Span}_{R_Y}(y)^{d+1}\{q_s\}.$$  

In the infinite summation $\sum m, n$ we need only the finite part (by the finite determinacy).

Thus we get a finite system of polynomial equations,

$$\{c_{m,n} + t \cdot H_{m,n}(\{c_s\}) = 0\}_{m,n}.$$  

Apply IFT$_1$ to get the solution $c_{m,n}(t) \in k[[t]]$. For these coefficients the transformation $\Phi_X$ satisfies:

$$\Phi_X(f + tg) - f \in (x,u)^N \cdot R_{X,u}^{\text{rep}} + \text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\} + t(\{y,u\})^{d+1} T_{\mathcal{D}} f.$$  

Apply a (filtration-unipotent) $\mathcal{L}$-transformation to eliminate the term $t(\{y,u\})^{d+1} T_{\mathcal{D}} f$. We get:

$$\Phi_Y \Phi_X(f + tg) - f \in (x,u)^N \cdot R_{X,u}^{\text{rep}} + \text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\} + t(\{y,u\})^{d+1} \cdot (x) R_{X,u}^{\text{rep}}.$$  

Now expand $(x) R_{X,u}^{\text{rep}}$ as in Step 1, and iterate. In at most $N$ steps one gets equation (24).

• Inductive step for the the general case, $R_{X,u} = k[x,u]/J$. The previously found transformation, $\Phi_X : x \to x + t(f,u)^d \cdot c \cdot \xi(x)$ and $u \to u$, is not necessarily an automorphism of $R_{X,u}$, as it does not preserve the ideal $J$. However, using the jet$_0$ assumption, §2.1 we can adjust it to an automorphism: $\Phi_X : x \to x + t(f,u)^d \cdot c \cdot \xi(x) + t^2(f,u)^d+1$ and $u \to u$. Applying this $\Phi_X$ we get again equation (29).

This completes the induction step in the general case.

Iterating this induction step one gets

$$\Phi_Y \Phi_X(f + tg) - f \in (x,u)^N R_{X,u}^{\text{rep}} + \text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\}, u \text{ for } N \gg 1.$$  

By the finite determinacy we conclude: $\Phi_Y \Phi_X(f + tg) - f \in \text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\}, u$.

Step 3. We have proved: $(f + tg, u) \in \mathcal{A}(\{f\} + \text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\}, u)$ for $t$ an indeterminate. The key step was the solvability of the finite polynomial system (24) in variables $\{c_{m,n}\}$. It defines a closed algebraic subscheme in the (finite dimensional) affine space, $Z \subset \text{Spec}(k[c_s]) \times k_1$. The projection of the algebraic germ $(Z,o) \to (k_1^1, o)$ is submersive. In particular, as $k$ is infinite, the image of $Z$ is an infinite subset of $k_1^1$.

Therefore $(f + tg, u) \in \mathcal{A}(\{f\} + \text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\}, u)$ for an infinite set of values of $t \in k_1^1$.

Step 4. Inside the space $	ext{Maps}(X, (k^p, o))$ we study the intersection of the line $(f + tg)_t \in \text{Jet}(\mathcal{A}(\{f\} + \text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\}, u))$ by the finite determinacy we can pass to the finite jets, $\text{jet}_N(R_{X,u}) := R_{X,u}(u,x)^N$. This is a finite-dimensional affine space. The transformations $\Phi_X, \Phi_Y$ of Step 2 were unipotent. Thus we have the algebraic, unipotent action of $\mathcal{A}(1)$ on the affine space $(R_{X,u}(u,x)^N)^{(p+r)}$. By Kostant-Rosenlicht theorem (see e.g. [F-S, R. Theorem 2.11]) its orbits are Zariski-closed. Moreover, $(f, u) + (\text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\}, 0)$ is a linear subspace, and its $\mathcal{A}(1)$-orbit is Zariski closed as well.

By Step 3 the intersection of the line $(f + tg)_t \in \text{Jet}(\mathcal{A}(\{f\} + \text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\}, u)$ is infinite. Therefore the whole line lies inside $\mathcal{A}(1)(\{f\} + \text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\}, u)$. Namely, $(f + tg, u) \in \mathcal{A}(1)(\{f\} + \text{Span}_{R_Y,u}(y) \cdot \{y^\bullet\}, u)$ for every $t \in k_1^1$.

6. Stable maps

Let $R_X$ be one of $k[x]/J$, $k(x)/J$, $k(x)/J$, see §2.1.
6.1. Stability vs infinitesimal stability. Take a map \( f : X \to (k^p, o) \).

**Proposition 6.1.** 1. If \( f \) is stable then it is infinitesimally stable, i.e. \( T_\varphi f = R_X^{\varphi p} \), i.e. \( T_\varphi f = 0 \).

2. (If char\( (k) > 0 \) then assume: \( J = 0 \) and \( k \) is infinite.) If \( f \) is infinitesimally stable then it is stable.

**Proof.**

1. Take a perturbation \( v \in R_X^{\varphi p} \) and the unfolding \( f_t = f_o + t \cdot v \). As \( f \) is stable, this unfolding is trivial. By theorem 4.3 we get: \( v \in T_\varphi f \). The t = 0 part of this condition is \( v \in T_\varphi f \). Therefore \( T_\varphi f = R_X^{\varphi p} \).

2. Take an unfolding \( F(x, t) = (f_t(x), t) \) of \( f = f_o \). As \( T_\varphi f = R_X^{\varphi p} \) one gets \( T_\varphi f = R_X^{\varphi p} \). In particular the unfolding \( F \) is a\( \varphi \)-separable. Moreover, \( f_o \) is a\( \varphi \)-finitely determined as an element of \( R_X^{\varphi p} \), see \([\text{Kerner.21}] \) \S7. By lemma [5.3] we get \( T_\varphi f \in R_X^{\varphi p} \).

Therefore \( f_t \in T_\varphi f \). Now apply theorem [4.3] to conclude: the unfolding \( F \) is a\( \varphi \)-trivial.

**Example 6.2.** For \( R_X = k[x] \) with \( k \in \mathbb{R}, \mathbb{C} \) and for \( R_X = C^\infty(\mathbb{R}^n, o) \), this is the classical lemma \([\text{Mather}] \), see also Theorem 3.2 (pg. 62) of \([\text{Mo. N.B.}] \).

6.2. Stable maps are unfoldings of their genotypes. Take a map \( F : \tilde{X} \to (k^{p+r}, o) \) of rank \( r \). If \( \text{char}(k) > 0 \) then we assume: \( k \) is infinite and \( J = 0 \) (i.e. the germ \( \tilde{X} \) is smooth). Then (lemma [4.11]) \( \tilde{X} \cong X \times (k^t_t, o) \) and \( F(x, t) = (f(x) + c(x, t), t) \), for some \( f \in (x)^2 \cdot R_X^{\varphi p} \) and \( c(x, t) \in (x) \cdot (t) \cdot R_X^{\varphi p} \).

**Theorem 6.3.** \( F \) is stable iff \( F \) is a\( \varphi \)-equivalent to the unfolding \( (f + \sum_{j=1}^{r'} t_j v_j, t) \), where \( t = (t_1, \ldots, t_r) \), \( r \geq r' \), \( f \in (x)^2 \cdot R_X^{\varphi p} \) is a\( \mathcal{K} \)-finite, and \( \{v_1, \ldots, v_{r'}\} \in (x) \cdot R_X^{\varphi p} \) go to a set of generators of the \( k \)-vector space \((x) \cdot T_{\varphi}^f \).

In this case the map \( f \) is called “the genotype” of \( F \), see e.g. \([\text{AGL.VI}] \) III.1.7.

We emphasize: the generating tuple \( \{v_i\} \) can be chosen arbitrarily, and is not necessarily minimal.

**Proof.** \( \iff \) By proposition 6.1 it is enough to prove: \( T_\varphi f = R_X^{\varphi p} \). We have \( \text{Span}_{R_r} \{v_i\} + T_\varphi f = R_X^{\varphi p} \), cf. (2.1).

Fix some generators \( \{\xi_j\} \) of \( \text{Der}_k(R_X) \), then the generators of \( \text{Der}_k(R_X, t) \) are \( \{\xi_j\} \) and \( \{\partial_t\} \). The submodule \( T_\varphi f \subseteq R_X^{\varphi p} \) is generated by the matrix with columns \( \{\xi_j F\}, \{\partial_t F\} \). We write these columns (as rows):

\[
\{\xi_j f + \sum t_i \xi_j q_i, 0, \ldots, 0\}^t \in_{j=1,\ldots,n} \{q_{j-p} \cdot e_1 + e_j\} \in_{j=p+1,\ldots,p+r'} \in_{e_p+1,\ldots,e_p+r}.
\]

Here \( \{e_p\} \) is the standard basis of \( k^{p+r} \).

By the direct check: \( R_X^{\varphi p} \oplus 0 \subseteq T_\varphi f \). And then \( 0 \oplus R_X^{\varphi p} \subseteq T_\varphi f \). Altogether: \( T_\varphi f \mid_{t=0} = R_X^{\varphi p} \).

**Example 6.4.** In the C-analytic case, \( R_X = \mathbb{C}[x] \), this is Theorem 7.2 of \([\text{Mo. N.B.}] \), for \( C^\infty \)-case see \([\text{Martinelli}] \) pg.200.

**Corollary 6.5.** Any \( \mathcal{K} \)-finite map \( f \in R_X^{\varphi p} \) admits a stable unfolding. Moreover, for a fixed number of parameters the stable unfolding is unique up to a\( \varphi \)-equivalence.

For \( R_X = \mathbb{C}[x] \) this is Proposition 7.2 of \([\text{Mo. N.B.}] \).
Given an ideal \(I \subseteq (x)^2 \subset R_X\) fix its generators, \(I = R_X\{g_\bullet\} \subset R_X\). This defines the map \(g : X \to (k^p, o)_f\). Define the Tjurina number of an ideal as \(\tau(I) := \text{codim}_x \mathcal{g} := \dim_k T^1_x \mathcal{g}\). (This number does not depend on the choice of the generators.)

**Corollary 6.6.** For every ideal \(I \subseteq (x)^2 \subset R_X\) with \(\tau(I) < \infty\) there exists a stable germ \(f \in R^\mathcal{p}_{X,I}\) such that \((I, t) = (f) \subset R_X, t\).

For \(R_X = \mathbb{C}\{x\}\) this Corollary 7.2 of [Mo. N.B.].

### 6.3. Stable maps are determined by their local algebra.

**Proposition 6.7.** (If char\((k) > 0\) then assume: \(J = 0\) and \(k\) is an infinite field.) Two stable maps are \(\mathcal{A}\)-equivalent iff they are \(\mathcal{X}\)-equivalent.

Recall, the stable maps are equivalent as maps, not as unfoldings. See example ?? in [Mo. N.B.].

*Proof.* (We prove the non-trivial direction, \(\Leftarrow\).) Stable maps are finitely determined, [Kerner.21 §], therefore we take \(R_X = \mathbb{k}[x]/j\).

By theorem 6.3 we can take one map in the form \(F = (u, f + \sum u_j v_j)\). Here \(f(x) \in (x)^2 \cdot R^\mathcal{p}_X\) is \(\mathcal{X}\)-finitely determined, \(u = (u_1, \ldots, u_r)\), while \(\{v_\bullet\}\) are sent to generators (not necessarily a basis) of the \(\mathbb{k}\)-vector space \((x) \cdot T^1_x f\).

By proposition 5.10 we can take the second map in the form \(\tilde{F} = (u, f(x) + \sum c_j(f, u)v_j)\), where \(c_j \in (y, u) \cdot \mathbb{k}[y, u]\). As \(\tilde{F}\) is infinitesimally stable, the map \(u \rightarrow \tilde{c}(u, o)\) is invertible. Applying the \(\mathcal{R}_u\)-transformation \(u_j + c_j(f, u) \sim u\) we get: \(\tilde{F} \sim (u + c(f, u), f(x) + \sum u_j v_j(x))\). Then by \(\mathcal{L}\)-transformation we get to \(\tilde{F} \sim (u + q(u, x), f(x) + \sum u_j v_j(x))\), where \(q(u, x) \in (u) \cdot (x) \cdot R^\mathcal{p}_{x,o}\). Now again an \(\mathcal{R}\) transformation on \(u\) gives \(\tilde{F} \sim (u, f(x) + \sum u_j v_j(x) + (u) \cdot (x))\). Therefore \(\tilde{F}\) is an unfolding of \(f\). Finally, bring \(\tilde{F}\) to the pre-normal form, theorem 6.3. \(\blacksquare\)

### 7. Results of Mather-Yau/Scherk/Gaffney-Hauser type

It is well known that the mapping \(f : (k^n, o) \to (k^p, o)\) is determined (up to \(\mathcal{R}, \mathcal{X}, \mathcal{A}\)-equivalence) by its “behaviour” at the critical/singular/instability locus. One way to obtain a precise statement was given in [Mather-Yau] for \(\mathcal{X}\)-equivalence of functions \((p = 1)\), in [Scherk.83] for \(\mathcal{R}\)-equivalence of functions, and extended in [Gaffney-Hauser.85] to \(\mathcal{X}\) and \(\mathcal{A}\)-equivalences of maps \((p \geq 1)\). The initial proofs were \(\mathbb{C}\)-analytic. The first extension to zero/positive characteristic was done for \(p = 1, \mathcal{R}, \mathcal{X}\) in [Gr.Ph.19] (in the case of isolate critical point). We establish the general statements/strengthen the known results.

#### 7.1. The \(\mathcal{X}\)-version in zero characteristic.

Consider modules over rings, \(M_j \in \text{mod-}R_j\), with the following notion of isomorphism: \(M_1 \sim M_2\) if \(M_1 \cong \phi^* M_2\) for an isomorphism of rings \(\phi : R_1 \to R_2\).

Below we consider the module \(T^1_x f\) up such isomorphisms. For \(p = 1\) the data “\(T^1_x f\) up to isomorphism” is equivalent to the data of \(k\)-algebra \(T^1_x f \cong R^\mathcal{X}/(f) + \text{Jac}(f)\).

Let \(R_X\) be one of \(\mathbb{k}[x]/J, \mathbb{k}\{x\}/J, \mathbb{k}(x)/J\), with char\((k) = 0, k = \mathbb{k}\). Take a map \(f : X \to (k^p, o)\).

**Theorem 7.1.**

1. The \(\mathcal{X}\)-type of \(f\) is determined by the isomorphism type of \(T^1_{\mathcal{X}_X}(f)\).

2. If \(f\) is \(\mathcal{X}\)-finite then the \(\mathcal{X}\)-type of \(f\) is determined by the isomorphism type of \(T^1_{\mathcal{X}_X} f\).

More precisely, if \(T^1_{\mathcal{X}_X} f_0 \sim T^1_{\mathcal{X}_X} f_1\) (in the sense as above), then \(f_0 \sim f_1\). (And similarly for part 2.)

*Proof.* The proofs of parts 1,2 are the same except for Step 3i. To simplify notations we work mostly with \(T^1_X\) and \(T^1_{\mathcal{X}_X}\).

**Step 1.**

i. It is enough to establish the formal case, \(R_X = \mathbb{k}[x]/J\). For the statements over \(\mathbb{k}(x)/J, \mathbb{k}(x)/J\) one applies Artin approximation.

ii. Given an isomorphism \(T^1_{\mathcal{X}_X} f_0 \cong \phi^* T^1_{\mathcal{X}_X} f_1\) coming from an isomorphism \(\phi : R_X \cong R_X\), replace \(f_1\) by \(f_1 \circ \phi^{-1}\) to get an isomorphism of \(R_X\)-modules \(T^1_{\mathcal{X}_X} f_0 \cong T^1_{\mathcal{X}_X} f_1\). After an \(R_X\)-linear automorphism of the module \(R^\mathcal{X}\) we can assume: \(T^1 f_0 = T^1 f_1 \subset R^\mathcal{X}\).

Similarly, in the first part we can assume: \(T^1_{\mathcal{X}_X} f_0 = T^1_{\mathcal{X}_X} f_1 \subset R^\mathcal{X}\).

**Step 2.** Extend the ring, \(R_X := R_X[t]\), note that \(R_X[t]\) is non-local. Take the unfolding \(f_t := f + t(f_1 - f_0)\). We have the tangent spaces \(T^1_{\mathcal{X}_X}, T^1_{\mathcal{X}_X} f_t\) and \(T^1_{\mathcal{X}_X} f_t\), see 4.10.

We claim: \(\partial_t f_t|_{t_o} \in T^1_{\mathcal{X}_X} f_t|_{t_o}\) for \(t_o \in k^1 \setminus \{finite\ set\}\). (For the first part: \(\partial_t f_t|_{t_o} \in T^1_{\mathcal{X}_X} f_t|_{t_o}\) for \(t_o \in k^1 \setminus \{finite\ set\}\).) More precisely, we claim: \(\partial_t f_t \in T^1_{\mathcal{X}_X} f_t\) over the factor ring \(k[t][g^{-1}]\), for some polynomial \(0 \neq g \in k[t]\).
Obviously $\partial_t f_t = f_1 - f_0 \in T_{X_1} f_1 + T_{X_1} f_0 = T_{X_1} f_0$. Thus it is enough to prove: $T_{X_1} f_0 = T_{X_1} f_{t_0}$ for all $t_0 \in k^1$ except for a finite set.

i. We have: $T_{X_1} f_1 = T_{X_1} f_0 + t (T_{X_1} f_0 + T_{X_1} f_1) = T_{X_1} f_0$.

ii. Similarly: $T_{X_1} f_0 \subseteq T_{X_1} f_1 + t$; $T_{X_1} (f_1 - f_0) \subseteq T_{X_1} f_1 + t \cdot T_{X_1} f_0$. These $R_{X_1}$-modules are finitely generated. Localize at the ideal $(x, t)$ and apply Nakayama over the local ring $(R_{X_1})_{(x, t)}$. We get: $(T_{X_1} f_0)_{(x, t)} \subseteq (T_{X_1} f_1)_{(x, t)}$.

iii. Similarly one has: $T_{X_1} f_1 \subseteq T_{X_1} f_1 + (1 - t) \cdot T_{X_1} (f_1 - f_0) \subseteq T_{X_1} f_1 + (1 - t) \cdot T_{X_1} f_1$. Therefore $(T_{X_1} f_0)_{(x, t)} = (T_{X_1} f_1)_{(x, t)} \subseteq (T_{X_1} f_0)_{(x, t)}$.

iv. Take the quotient module $M := T_{X_1} f_0 T_{X_1} f_1$. Thus $M$ is finitely generated over $R_X [t]$. Its localizations at two points vanish, $M_{(t, x)} = 0$ and $M_{(1 - t, x)} = 0$. Therefore its support, $\text{Supp}(M) \subseteq \text{Spec}(R_X [t])$, does not contain the points $V(t, x), V(1 - t, x) \in \text{Spec}(R_X [t])$. But $\text{Supp}(M)$ is a Zariski-closed subset. Therefore $\text{Supp}(M) \cap V(x) \subset k_1$ is a finite set of points.

Altogether, we have proved: $\partial_t f_{t_0} \in T_{X_1} f_{t_0}$ (resp. $\partial_t f_{t_0} \in T_{X_1} f_{t_0}$) for $t_0 \in k^1 \setminus \{\text{finite set}\}$.

Step 3. i. (The case $k = \mathbb{C}$.) We have the unfolding $f_t$ over $C$, and $\partial_t f_{t_0} \in T_{X_1} f_{t_0}$ for $t_0 \in k^1 \setminus \{\text{finite set}\}$. Take a path $\gamma : 0 \sim 1$ in $C$ that avoids this finite set. By theorem 4.3 the unfolding $f_t$ is locally $\mathcal{H}$ (resp. $\mathcal{H}(0)$)-trivial at each point of the path. The $\mathcal{H}(0)$-trivialization (in the first case) preserves the origin of this finite set. The $\mathcal{H}$-trivialization (in the second case) preserves the origin of $X$ as $V(f_0), V(f_1)$ have isolated singularities.

As the path is compact we take a finite cover $\gamma = \bigcup_i U_i$, such that $f_t$ is trivial on each $U_i$.

And by the connectedness of the path we get: $f_0 \sim f_1$, resp. $f_0 \sim f_1$.

ii. (The general case, $k = \mathbb{C}$, char$(k) = 0$.) Pass to the finite jets, $\text{jet}_d R_X := R_X / (x)^{d+1}$. Then for $t \in k^1 \setminus \{\text{finite set}\}$ we have: $\partial_t \text{jet}_d f_t = \text{jet}_d \partial_t f_t \in T_{X_1} f_1 \subseteq T_{X_1} (\text{jet}_d R_X) \cdot \text{jet}_d f_t$.

Explicitly, we have $\partial_t \text{jet}_d f_t = \text{jet}_d \xi (\text{jet}_d f_t) + \text{jet}_d (U) \cdot \text{jet}_d f_t$. This holds for almost all $t \in k^1$, i.e. over $k[t][g^{-1}]$.

Taylor-expand the elements $\text{jet}_d \xi, \text{jet}_d U, \text{jet}_d (J)$ up to order $d$. Let $\{ \mathcal{C}_t \}$ be the (finite) set of the coefficients. Then $Q(\mathcal{C}_t)$ is a finite field extension of $Q$. Therefore this extension can be (re-)embedded, $\epsilon : Q(\mathcal{C}_t) \hookrightarrow \mathbb{C}$.

Over $\mathbb{C}$ we still have: $\partial_t \text{jet}_d (f_t) \in T_{X_1(\epsilon(\text{jet}_d R_X))} \sim \text{jet}_d (f_t)$ for almost all $t \in k^1$. By part i. we get equivalence over $\mathbb{C}$: $\text{jet}_d (f_0) \sim \text{jet}_d (f_1)$, resp. $\text{jet}_d (f_0) \sim \mathcal{H}(0)(\epsilon(\text{jet}_d R_X)) \text{jet}_d (f_1)$. Then theorem 2.2 gives equivalence over $k$: $\text{jet}_d f_0 \sim \mathcal{H}(0)(\epsilon(\text{jet}_d R_X)) \text{jet}_d f_1$.

This holds for each $d \gg 1$. Therefore the condition $f_t \in \mathcal{H}(0)(f_0)$, which is an implicit function equation, has an order-by-order solution. By Pfister-Popescu theorem, [Pfister-Popescu,75], we get the equivalence $f_0 \sim f_1$ over $k[\epsilon]/\mathbb{J}$.

Remark 7.2. i. The statement does not hold if $k$ is not algebraically closed. E.g. suppose $\sqrt[d]{a} \notin k$, for some $d \geq 4$, and compare $x_1^d + x_2^d$ to $x_1^d + a \cdot x_2^d$.

ii. If the singularity is not “of isolated type” then the $\mathcal{H}$-type of $f$ is not determined by the module $T_{X_1} f$. See [Gaffney-Hauser,85], §4. Recall the standard example. Let $f_t (x, y) = x y (x - y) (x - t - y)$. Here $T_{X_1} f_t = (x^2, x^2 y, y^2)$, independent of $t$. This family is trivialized by the coordinate change $\phi : (x, y, z) \to (x, y, z + t)$, which does not preserve the origin. Thus $\phi$ is not an automorphism of the local ring $k[[x, y, z]]$. And no automorphism of $k[[x, y, z]]$ can trivialize this family, as any automorphism will preserve the cross-ratio of the four planes of $V(f_t) \subset (k^3, o)$.

iii. Instead of the module $T_{X_1} f$ we could take the $R_X$-module $T_{X_1} f$, getting the similar statement.

7.2. The $\mathcal{H}$-version in zero characteristic.

Below we consider $T_{X_1} f$ as a mixed $(R_Y, R_X)$-module. We write $T_{X_1} f \sim T_{X_1} f$ if there exists an isomorphism of algebras $(\Phi_X, \Phi_Y)$ (see the diagram) that induces the isomorphism of $R_Y$-modules $\Phi_Y : T_{X_1} f \sim T_{X_1} f$.

Example 7.3. i. If $f \sim \tilde{f}$, i.e. $\tilde{f} = \Phi_Y \circ f \circ \Phi_X$, then $T_{X_1} f \sim T_{X_1} \tilde{f}$. Indeed, $T_{X_1} (\tilde{f} \circ \Phi_X^{-1}) = R_{X_1} / T_{X_1} (\Phi_Y f) + T_{X_1} (\Phi_Y f) = R_{X_1} / \Phi_Y T_{X_1} f + T_{X_1} f \sim T_{X_1} f$.

ii. Suppose $T_{X_1} f \sim T_{X_1} \tilde{f}$. Using the corresponding morphism of algebras we get $T_{X_1} (\Phi_Y f \circ \Phi_X) = T_{X_1} \tilde{f}$, i.e. $T_{X_1} (\Phi_Y f \circ \Phi_X) = T_{X_1} \tilde{f} \subset R_{X_1}$. 

Theorem 7.4. Let $R_X = k[[x]]/I$, with $k = \bar{k}$ and $\text{char}(k) = 0$. Suppose $\sqrt{I} = m$.

1. The $\mathcal{A}$-type of $f$ is determined by the mixed module type of $T_{1,0}^{\mathcal{A}(0)} f$.

2. If $f$ is $\mathcal{A}$-finite and not stable then the $\mathcal{A}$-type of $f$ is determined by the mixed module type of $T_{1,0}^{\mathcal{A}} f$.

More precisely, if $T_{1,0}^{\mathcal{A}(0)} f_0 \sim T_{1,0}^{\mathcal{A}(0)} f_1$, then $f_0 \sim f_1$. (And similarly in case 2.)

Proof. By the example above we can assume $T_{1,0}^{\mathcal{A}(0)} f_0 = T_{1,0}^{\mathcal{A}(0)} f_1 \subset R_X^{\mathcal{A} \text{-type}}$ (resp. $T_{1,0}^{\mathcal{A}} f_0 = T_{1,0}^{\mathcal{A}} f_1 \subset R_X^{\mathcal{A} \text{-type}}$). The proof is similar to the $\mathcal{K}$-case and is inductive. We apply a sequence of transformations $g_d \in \mathcal{A}(0)(\text{jet}_d(R_X))$ and verify $\text{jet}_d f^g \sim \text{jet}_d f$. Moreover, the transformation can be chosen to satisfy $\text{jet}_{d-1}(g_d) = 1d$.

Fix any $d \geq 1$ and replace $R_X$ by the Artinian ring $R_X/\mathcal{A}^{d+1}$. Assume $\text{jet}_{d-1}(f_0) = \text{jet}_{d-1}(f_1)$.

**Step 1.** As in the $\mathcal{K}$-case we define $f_t := f_0 + t(f_1 - f_0)$ and $R_{X,t} := R_X \otimes \bar{k}[t]$, $T_{d,t} := T_{d} \otimes \bar{k}[t]$. If $R_{X,t}$ is determined by the $\mathcal{A}$-type of $\mathcal{A} \text{-type}$ of theorem 4.3 to get: $(T_{d,t}) f_t \subset (1 - t) \cdot T_{d,t} f_0 + t \cdot T_{d,t} f_1 \subset T_{d,t} f_1$. In addition

$$T_{d,t} f_t \subset \{ \sum_{i,j \geq 0} q_0(f_0) \cdot v_j(f_1) \} \subset T_{d,t} f_0 + T_{d,t} f_1 \subset T_{d,t} f_0.$$ (Note that $R_X$ is Artinian now.) And similarly $T_{d,t} f_t \subset T_{d,t} f_1$.

- As in the $\mathcal{K}$-case we have: $T_{d,t} f_t \subset T_{d,t} f_t + t \cdot T_{d,t} f_0$. These modules are finitely generated over $R_{Y,t}$, because $R_X$ is Artinian. Localize at $(y,t)$ to get: $(T_{d,t} f_0)_{(t,y)} \subset (T_{d,t} f_t)_{(t,y)}$. Similarly one gets $(T_{d,t} f_1)_{(t,y)} \subset (T_{d,t} f_t)_{(t,y)}$.

- As in the $\mathcal{K}$-case take the quotient $M := T_{d,t} f_0 / T_{d,t} f_1$. This is an $R_{Y,t}$-module. Thus: $M_{(t,y)} = 0$, and $M_{(1-t,y)} = 0$. Hence $\text{Supp}(M) \subset V(y) \subset \text{Spec}(R_{Y,t})$ is a finite subset.

**Step 2.**

i. **(The case $k = \mathbb{C}$.)** As in the $\mathcal{K}$-case take a path inside $C_t \cong V(y) \subset (C_y, o) \times (C_t, o)$ avoiding the finite subset $\text{Supp}(M)$. Along this path one has $\partial_i f_t \in T_{d,t} f_t$. Now use part of theorem 4.3 to get $f_0 \sim f_1$. Note that the ring $R_X$ is Artinian, therefore the formal and ordinary equivalence coincide.

For part 2 we remark that the $\mathcal{A}$-trivialization preserves the origins of $(k^n, o)$, $(k^p, o)$, as $f_0, f_1$ are $\mathcal{A}$-finite, i.e. their instability locus is just one point.

ii. **(The general case, $k = \bar{k}, \text{char}(k) = 0$.)** By the same Lefschetz-type arguments (and lemma 2.2) we get $f_0 \sim f_1$.

Now combine these constructed transformations to get the (normal) group element $g := \lim_{d \to \infty} (g_d \cdots g_1) \in \mathcal{A}(0)$ (resp. $g := \lim_{d} (g_d \cdots g_1) \in \mathcal{A}$), satisfying $g f = f$.

### 7.3. The case of arbitrary characteristic.

Let $R_X$ be one of $k[[x]]/I$, $k(x)/I$, $k(x)/I$, here $k$ is any field. If $\text{char}(k) > 0$ then we assume the jet condition of 2.3. Take a map $f : (k^n, o) \to (k^p, o)$, thus $0 \neq f \in m \cdot R_X^{\mathcal{A} \text{-type}}$. The order of $f$ is the largest $\text{ord}(f) \in \mathbb{N}$ satisfying $(f) \subseteq m^{\text{ord}(f)}$. If $\text{ord}(f) \leq 2$ then we take $m^{\text{ord}(f) - 2} = R_X$.

**Theorem 7.5.**

1. **($\mathcal{K}$-case)** Suppose an ideal $a \subseteq m$ satisfies: $a^2 \cdot m^{\text{ord}(f) - 2} \cdot R_X^{\mathcal{A} \text{-type}} \subseteq m \cdot a \cdot T_{\mathcal{A}} f + \text{m}(f) \cdot R_X^{\mathcal{A} \text{-type}}$. Then the $\mathcal{K}$-type of $f$ is determined by the $\mathcal{K}$-algebra $R_X(f) + a \cdot a_{\mathcal{A}}$.

2. **($\mathcal{A}$-case, $k$-finite) ** Suppose an ideal $a \subseteq m$ satisfies: $a^2 \cdot m^{\text{ord}(f) - 2} \cdot R_X^{\mathcal{A} \text{-type}} \subseteq a \cdot T_{\mathcal{A}} f + f^e(y)^2 \cdot T_{\mathcal{A}} f$. Then the $\mathcal{A}$-type of $f$ is determined by the $\mathcal{A}$-algebra $R_X f^e(y) + a^2$.

3. **($\mathcal{R}$-case)** Suppose an ideal $a \subseteq m$ satisfies: $a^2 \cdot m^{\text{ord}(f) - 2} \cdot R_X^{\mathcal{A} \text{-type}} \subseteq m \cdot a \cdot T_{\mathcal{A}} f$. Then the $\mathcal{R}$-type of $f$ is determined by the $\mathcal{R}$-algebra $R_X / a \cdot a_{\mathcal{A}}$. 


Here is the explicit form of the statement. Fix some maps $f, \tilde{f} \in \mathfrak{m} \cdot R_X^{\mathbb{Q}_p}$ and ideals $a^f, a^{\tilde{f}} \subseteq \mathfrak{m}^2$.

- $\mathcal{E}$ Suppose $(a^f)^2 \cdot m^{ord(f)} - 2 \cdot R_X^{\mathbb{Q}_p} \subseteq \mathfrak{m} \cdot a^f \cdot T_x f + \mathfrak{m} \cdot (f) \cdot R_X^{\mathbb{Q}_p}$ and $(a^{\tilde{f}})^2 \cdot m^{ord(\tilde{f})} - 2 \cdot R_X^{\mathbb{Q}_p} \subseteq \mathfrak{m} \cdot a^{\tilde{f}} \cdot T_x \tilde{f} + \mathfrak{m} \cdot (\tilde{f}) \cdot R_X^{\mathbb{Q}_p}$. If the $k$-algebras are isomorphic, $R_X(f) + a^f \cdot a_{\mathfrak{m}}^f \cong R_X(\tilde{f}) + a^{\tilde{f}} \cdot a_{\mathfrak{m}}^{\tilde{f}}$, then $f \mathrel{\sim} \tilde{f}$.

- $\mathcal{A}$ Suppose $(a^f)^2 \cdot m^{ord(f)} - 2 \cdot R_X^{\mathbb{Q}_p} \subseteq \mathfrak{m} \cdot a^f \cdot T_x f + f^#(y)^2 \cdot T_x f$ and $(a^{\tilde{f}})^2 \cdot m^{ord(\tilde{f})} - 2 \cdot R_X^{\mathbb{Q}_p} \subseteq \mathfrak{m} \cdot a^{\tilde{f}} \cdot T_x \tilde{f} + f^#(y)^2 \cdot T_x \tilde{f}$. If the $k$-algebras are isomorphic, $R_X(f^#(y)) + (a^f)^2 \cong R_X(\tilde{f}^#(y)) + (a^{\tilde{f}})^2$, then $f \mathrel{\sim} \tilde{f}$.

- $\mathcal{R}$ Suppose $(a^f)^2 \cdot m^{ord(f)} - 2 \cdot R_X^{\mathbb{Q}_p} \subseteq \mathfrak{m} \cdot a^f \cdot T_x f$ and $(a^{\tilde{f}})^2 \cdot m^{ord(\tilde{f})} - 2 \cdot R_X^{\mathbb{Q}_p} \subseteq \mathfrak{m} \cdot a^{\tilde{f}} \cdot T_x \tilde{f}$. Suppose the $k[f]$-algebra $R_X[a^f, a_{\mathfrak{m}}^f]$ is isomorphic to the $k[\tilde{f}]$-algebra $R_X[a^{\tilde{f}}, a_{\mathfrak{m}}^{\tilde{f}}]$. Then $f \mathrel{\sim} \tilde{f}$.

We do not assume isolated singularities or that the germ $V(f) \subset X$ is a complete intersection.

**Proof.**

1. Given an isomorphism of $k$-algebras $R_X(f) + a^f \cdot a_{\mathfrak{m}}^f \mathrel{\sim} R_X(\tilde{f}) + a^{\tilde{f}} \cdot a_{\mathfrak{m}}^{\tilde{f}}$ take its representative $\phi : R_X \to R_X$. Thus $\phi$ is invertible (hence an isomorphism of $k$-algebras) and sends $(f) + a^f \cdot a_{\mathfrak{m}}^f$ to $(\tilde{f}) + a^{\tilde{f}} \cdot a_{\mathfrak{m}}^{\tilde{f}}$. Therefore after a coordinate change (\mathcal{R}-equivalence) we can assume:

$$(33) \quad (f) + a^f \cdot a_{\mathfrak{m}}^f = (\tilde{f}) + a^{\tilde{f}} \cdot a_{\mathfrak{m}}^{\tilde{f}} \subset R_X.$$  

Note that the assumption $a^f \cdot m^{ord(f)} - 2 \cdot R_X^{\mathbb{Q}_p} \subseteq \mathfrak{m} \cdot a^f \cdot T_x f + m(f) \cdot R_X^{\mathbb{Q}_p}$ is preserved under $\mathcal{E}$-equivalence, see [Kerner.21 §5].

By the initial assumption $a^f, a^{\tilde{f}} \subseteq \mathfrak{m}^2$. Then equation (33) gives: $rank(f) = rank(\tilde{f})$. By $\mathcal{E}$-equivalence we can assume $f = (x_1, \ldots, x_r)$, with $(f) \subseteq (x_{r+1}, \ldots, x_n)$. (And similarly for $\tilde{f}$.) Therefore the whole question is reduced to the case $(f) = (\tilde{f}) \subseteq \mathfrak{m}^2$.

Now apply $GL(p, R_X)$ transformations to $f$ and $\tilde{f}$ to get: $f - \tilde{f} \in (m(f) + a^f \cdot a_{\mathfrak{m}}^f) \cdot R_X^{\mathbb{Q}_p}$. Finally, by [Kerner.21 §5], we have $\mathcal{E} \supseteq \{ f \} + (m(f) + a^f \cdot a_{\mathfrak{m}}^f) \cdot R_X^{\mathbb{Q}_p}$. In particular, $f \in \mathcal{E}$.

2. As in the $\mathcal{E}$-case we lift the isomorphism $R_X(f^#(y)) + a^f \cdot a^f \mathrel{\sim} R_X(f^#(y)) + a^{\tilde{f}} \cdot a^{\tilde{f}}$ to an isomorphism of $k$-algebras $\phi : R_X \mathrel{\sim} R_X$ sending $f^#(y) + (a^f)^2$ to $\tilde{f}^#(y) + (a^{\tilde{f}})^2$. Then by a coordinate change (\mathcal{A})

we can assume $f^#(y) + (a^f)^2 = \tilde{f}^#(y) + (a^{\tilde{f}})^2 \subset R_X$. Therefore $\tilde{f}^#(y) \in f^#(y) + (a^f)^2$. Applying $GL(p, k)$ transformations to $f$ we can assume: $f - \tilde{f} \in f^#(y)^2 T_x f + (a^f)^2 \cdot R_X^{\mathbb{Q}_p}$. Finally, by [Kerner.21 §7], we have $\mathcal{A} \supseteq \{ f \} + (f^#(y)^2 \cdot T_x f + (a^f)^2 \cdot R_X^{\mathbb{Q}_p}$.

In particular, $f \in \mathcal{A}$.

3. We have the isomorphism $R_X[a^f, a_{\mathfrak{m}}^f] \cong R_X[a^{\tilde{f}}, a_{\mathfrak{m}}^{\tilde{f}}]$, compatible with the isomorphism $k[f] \cong k[\tilde{f}]$. The later isomorphism (after an automorphism of $k[\tilde{f}]$) can be taken as $f_i \to \tilde{f}_i$. Then $R_X[a^f, a_{\mathfrak{m}}^f] \cong R_X[a^{\tilde{f}}, a_{\mathfrak{m}}^{\tilde{f}}]$ becomes an isomorphism of $k[f]$-algebras.

Lift it to an isomorphism $R_X \mathrel{\phi} R_X$. Therefore by a coordinate change (\mathcal{R}-equivalence) we can assume: $a^f \cdot a_{\mathfrak{m}}^f = a^{\tilde{f}} \cdot a_{\mathfrak{m}}^{\tilde{f}} \subset R_X$. Moreover, we have: $\tilde{f}_i = \tilde{f}_i \cdot 1 \in f_i + a^f \cdot a_{\mathfrak{m}}^f \cdot R_X^{\mathbb{Q}_p}$. By [Kerner.21 §4] we get $\mathcal{R} \supseteq \{ \tilde{f}_i \} + a^f \cdot a^\tilde{f}_i \cdot R_X^{\mathbb{Q}_p}$. In particular, $\tilde{f} \in \mathcal{R}$.

**Example 7.6.** Take $p = 1$ and assume $J = 0$, i.e. $X \cong (k^n, o)$.

i. (The $\mathcal{E}$-case.) Suppose $Jac(f) \cdot \mathfrak{m} \cdot a + m(f) \cdot (f) \geq a^2 \cdot m^{ord(f)} - 2$. Then the $\mathcal{E}$-type of $f$ is determined by the $k$-algebra $R_X(f) + a \cdot Jac(f)$. The singularity can be non-isolated, we do not assume $\sqrt{a} = m$.

As a particular case take $a = m^d$ with $d \geq 2$. Suppose $Jac(f) \cdot m^{d+1} + m(f) \cdot (f) \geq m^{2d+ord(f)}$. Then the $\mathcal{E}$-type of $f$ is determined by the $k$-algebra $R_X(f) + m^d \cdot Jac(f)$. Compare this to [Gr.Ph.17. Theorem 2.2] for $(R_X = k[[x]])$:

If $Jac(f) \cdot m^d \geq m(f) \cdot (f) \geq m^{2d+ord(f)}$ then $\mathcal{E}$-type of $f$ is determined by the $k$-algebra $R_X(f) + m^d \cdot Jac(f)$. Their assumption implies (is stronger than) the condition $Jac(f) \cdot m^{d+1} + m(f) \cdot (f) \geq m^{2d+ord(f)}$, which is much stronger than ours.

ii. (The $\mathcal{R}$-case.) Suppose $Jac(f) \cdot m \cdot a \geq a^2 \cdot m^{ord(f)} - 2$. Then the $\mathcal{R}$-type of $f$ is determined by the $k[f]$-algebra $R_X[a, Jac(f)]$. Again, we do not assume $\sqrt{a} = m$.

As a particular case take $a = m^d$ with $d \geq 2$. Suppose $Jac(f) \cdot m^{d+1} \geq m^{2d+ord(f)}$. Then the $\mathcal{R}$-type of $f$ is determined by the $k$-algebra $R_X[m^d, Jac(f)]$. This strengthens (and extends) [Gr.Ph.17. Theorem 2.4] for $(R_X = k[[x]])$.
The $R, \mathcal{R}$-cases for $p > 1$ and the $\mathcal{A}$-case are new.

**Remark 7.7.** i. One would like a stronger statement, e.g. for $p = 1$ of the form “the $\mathcal{X}$-type is determined by the $k$-algebra $R(f) + m^d \cdot \text{Jac}(f)$, where $d$ depends on $k$ and $\dim(R_X)$, but not on $f$”.

This is impossible due to the following example:

(34) $\text{char}(k) = p, R_X = k[[x, y]]$ and $f(x, y) = x^{p+1} + y^{2N+1}, \tilde{f}(x, y) = f(x, y) + x^py^pd, \text{ for } N > pd$.

Here $\text{Jac}(f) = \text{Jac}(\tilde{f})$ and $m^d \cdot \text{Jac}(\tilde{f}) \supset m^d \cdot x^p \supset x^py^pd$ for any $j \leq d$. Therefore $(f) + m^d \cdot \text{Jac}(\tilde{f}) = (\tilde{f}) + m^d \cdot \text{Jac}(g)$ for $j \leq d$. But $f \not \cong g$, e.g. because the monomial $x^py^pd$ lies under the Newton diagram of $f$.

ii. (For $p = 1$) Recall that for $R = \mathbb{C}\{x\}$ the $R$-type of $f$ is not determined just by the $k$-algebra structure of $R_f / \text{Jac}(f)$. Moreover, the $\mathcal{X}$-type is not determined by the $k$-algebra $R_f / \text{Jac}(f)^d$, for any $d$.

Indeed, suppose $f \not \cong \tilde{f}$, but $f \cong \tilde{f}$ via $\tilde{f}(x) = c \cdot f(\phi(x))$, with $c \in k$. (An explicit example is $f_{i}(x_1, x_2) = x_1^2 + x_2^5 + t \cdot x_2^2 x_1^3$, see [Gr.Lo.Sh] pg.133.) Then $\text{Jac}(f_{i})^d = \text{Jac}(f_{\tilde{i}})^d$ for all $d \geq 1$.

**Appendix A. Separability of unfoldings**

Below we assume $k = \bar{k}$.

To an unfolding $F = (f_t(x, u), t)$ we associate the element $[f_t] \in T_{g_{t}}f_t$. Define the group action

$$G := \text{Aut}_{k', o}) := \text{Aut}_{k'}(k \otimes R_{X,t}^{\otimes p}) \text{ by } q(x, t) \rightarrow q(x, \phi(t)).$$

This action preserves the embedding $T_{g_{t}}f_t \subseteq R_{X,t}^{\otimes p}$ and hence descends to the action $G \otimes T_{g_{t}}f_t$. We get the group orbit $[Gf_{t}] \subseteq T_{g_{t}}f_t$ and the orbit map $G \rightarrow T_{g_{t}}f_t$ by $g \rightarrow [gf_{t}]$.

Take an ideal $b \subset R_{X,t}$ satisfying $\sqrt{b} = (x, t)$ and pass to the finite jets, $\text{jet}_b(R_{X,t}) := R_{X,t}/b$. This is a finite-dimensional $k$-vector space. Similarly one has $\text{jet}_b(R_{X,t}^{\otimes p}), \text{jet}_b(T_{g_{t}}f_t) := R_{X,t}^{\otimes p}/b \cdot R_{X,t}^{\otimes p}$ and $\text{jet}_b(G) := \text{Aut}_{k}(k[[t]][k[[t]]] \otimes b)$. We get the (regular) action of the affine algebraic group on the affine space, $\text{jet}_b(G) \otimes \text{jet}_b(T_{g_{t}}f_t)$.

**Definition A.1.** (Suppose $k = \bar{k}$.) The orbit map $G \rightarrow T_{g_{t}}f_t$ is called separable if all its finite jets, $\{\text{jet}_b(G) \rightarrow \text{jet}_b(T_{g_{t}}f_t)\}_{b}$, are separable as morphisms onto their images.

(The morphism $\text{jet}_b(f_t)$ is called separable if the corresponding field extensions are separable, see e.g. [E-S.R].)

Take the tangent space, $T_G := (t) \cdot \text{Der}_{k'}(k)$, and its finite jets, $\{\text{jet}_bT_G\}_{b}$. Take the map from the jet(image tangent space) to the tangent space of the jet-orbit, $\text{jet}_b(T_{g_{t}}f_t) \rightarrow T_{g_{t}}\text{jet}_b(Gf_{t})$. We write $T_Gf_t \rightarrow [T(Gf_t)] \subseteq T_{g_{t}}f_t$ if the surjectivity holds for all the finite jets. Recall the general fact: the map $f_t : G \rightarrow T_{g_{t}}f_t$ is separable (as a morphism onto its image) iff $T_Gf_t \rightarrow [TGf_t]$. (Namely, the surjectivity holds for all the finite jets.)

**Definition A.2.** The unfolding $F$ is called separable if the map $G \rightarrow T_{g_{t}}f_t$ is separable.

If a field $k$ is not algebraically closed then we call $F$ separable if $\bar{k} \otimes F$ is separable.

**Example A.3.** i. For $\text{char}(k) = 0$ the group orbit map is always separable. Hence any unfolding is separable.

ii. The (in)separability is preserved under the $G_t$-equivalence. For example, for $G = R$ and any $b$ we have the commutative diagram of algebraic varieties

$$\begin{array}{cccc}
\text{jet}_b(G) \rightarrow & \downarrow & \text{jet}_b(T_{g_{t}}f_t) \\
\downarrow f_t \circ \phi & \rightarrow & \text{jet}_bT_{g_{t}}f_t \circ \phi & = \text{jet}_bT_{g_{t}}f_t
\end{array}$$

iii. A trivial unfolding is separable. Indeed, we can take $F = (f_0, t)$, then $G \rightarrow T_{g_{t}}f_t$ is the zero map. Hence $[T_Gf_t] = 0$. And thus trivially $T_Gf_t \rightarrow [TGf_t]$.

**Lemma A.4.** An unfolding $F$ is inseparable iff $f_{i} \otimes f_0 + t^d f_d + \cdots$, where $\text{char}(k) \mid d$ and $f_d \not \in T_g f_0$.

**Proof.** We can assume $f = f_0 + t^d f_d + \cdots$ with $\text{char}(k) \mid d$ and $f_d \not \in T_g f_0$. Then $T_Gf_t \subseteq t^{d+1} \cdot R_{X,t}^{\otimes p}$. Thus for $b = (t^{d+1}, (x)^{N})$ we get $\text{jet}_b(T_Gf_t) = 0$. But $Gf_t \ni f_0 + c t^d f_d + \cdots$ for any $c \in k$. Therefore the subvariety $[\text{jet}_b(Gf_t)] \subseteq \text{jet}_bT_{g_{t}}f_t$ is of positive dimension. Thus $T(\text{jet}_bGf_t) \neq 0$. Therefore the map $T_Gf_t \rightarrow [TGf_t] \subset T_{g_{t}}f_t$ cannot be surjective.
Suppose \( f \sim f_o + t^d f_d + \cdots \) with \( \text{char} (k) \nmid d \) and \( f_d \notin T_{\text{reg}} f_o \). Then \( \text{jet}_b T_G f_t = \text{Span}_{k[[t]]} \{ t^{d-1} f_d + \cdots \} \) and \( T_{\text{jet}_b} (G f_t) = \text{Span}_{k[[t]]} \{ t^{d-1} f_d + \cdots \} \). Hence \( F \) is separable.

Otherwise for each \( d \) we get: \( f \sim f_o + t^d f_d + \cdots \) with \( f_d \in T_{\text{reg}} f_o \). Then \( F \) is formally trivial. Trivialize it to get: \( (f_o, t) \) is separable.

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