ON THE SUPERCRITICAL MEAN FIELD EQUATION ON PIERCED DOMAINS

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Abstract. We consider the problem

\[
(P_\varepsilon) \quad \Delta u + \lambda \frac{e^u}{\int_{\Omega \setminus B(\xi, \varepsilon)} e^u} = 0 \text{ in } \Omega \setminus B(\xi, \varepsilon), \quad u = 0 \text{ on } \partial (\Omega \setminus B(\xi, \varepsilon)),
\]

where \( \Omega \) is a smooth bounded open domain in \( \mathbb{R}^2 \) which contains the point \( \xi \). We prove that if \( \lambda > 8\pi \), problem \((P_\varepsilon)\) has a solutions \( u_\varepsilon \) such that

\[
u_{\varepsilon}(x) \to \frac{8\pi + \lambda}{2} G(x, \xi) \text{ uniformly on compact sets of } \Omega \setminus \{\xi\}
\]
as \( \varepsilon \) goes to zero. Here \( G \) denotes Green’s function of Dirichlet Laplacian in \( \Omega \). If \( \lambda \not\in 8\pi \mathbb{N} \) we will not make any symmetry assumptions on \( \Omega \), while if \( \lambda \in 8\pi \mathbb{N} \) we will assume that \( \Omega \) is invariant under a rotation through an angle \( \frac{8\pi^2}{\lambda} \) around the point \( \xi \).

1. Introduction

Our paper concerns the mean field equation

\[
\begin{cases}
\Delta u + \lambda \frac{e^u}{\int_D e^u} = 0 & \text{in } D, \\
u = 0 & \text{on } \partial D,
\end{cases}
\]

when \( \lambda \in \mathbb{R} \) and \( D \) is a smooth bounded domain of \( \mathbb{R}^2 \). It is well known that solutions to \((1.1)\) are nothing but the critical points of the functional \( J_\lambda : H_0^1(D) \to \mathbb{R} \) defined by

\[
J_\lambda(u) := \frac{1}{2} \int_D |\nabla u|^2 - \lambda \ln \int_D e^u.
\]

This variational problem arises from Onsager’s vortex model for turbulent Euler flows. In this interpretation, \( -u/\lambda \) is the stream function in the infinite vortex limit, whose canonical Gibbs measure and partition function are finite if \( \lambda < 8\pi \). In this situation Caglioti et al. \[3\] and Kiessling \[9\] proved the existence of a minimizer.
of $J_\lambda$. Indeed, a corollary of the classical Moser-Trudinger inequality \[11\]
\[
\frac{1}{2} \int_D |\nabla u|^2 \geq \frac{1}{8\pi} \ln \int_D e^{8\pi u} \quad \text{for any } u \in H^1_0(D)
\]
implies the compactness and coercivity properties for $J_\lambda$ if $\lambda < 8\pi$. Therefore, problem \[(1.1)\] has at least a solution for any $\lambda < 8\pi$.

When $\lambda \geq 8\pi$ the situation turns out to be more complicated, because in general $J_\lambda$ is no longer compact and coercive. This is a supercritical case for the Moser-Trudinger inequality. The existence of solutions of \[(1.1)\] actually depends on the geometry of the domain. For example, problem \[(1.1)\] always has a solution whenever $\lambda \notin 8\pi \mathbb{N}$ and $D$ is not simply connected, as established by Chen and Lin \[6,7\] using a degree argument.

If $\lambda \in 8\pi \mathbb{N}$ the existence of solutions to problem \[(1.1)\] is a delicate issue, because it does not depend only on the topology, but also on the geometry of the domain. Indeed, when $\lambda = 8\pi$ problem \[(1.1)\] has no solutions if $D$ is a ball and has at least one solution if $D$ is a long and thin rectangle as showed by Caglioti et al. \[4\]. Moreover, for $D$ simply connected and $\lambda = 8\pi$, Chang, Chen and Lin \[5\] gave necessary and sufficient conditions for \[(1.1)\] to have a solution. Moreover, if $\lambda \leq 8\pi$, they proved that \[(1.1)\] has at most one solution. This result has been recently extended for not simply connected domains by Bartolucci and Lin \[2\]. To our knowledge, a part from a recent result on some convex domains obtained by Bartolucci and De Marchis \[1\] indicated there are no further results concerning the resonant supercritical case $\lambda \in 8\pi \mathbb{N}$ and $\lambda > 8\pi$.

In this paper we prove that if $\lambda > 8\pi$ and $D$ has a small hole and is suitably symmetric, then the problem \[(1.1)\] has at least one solution. More precisely, we consider the problem
\[
\begin{cases}
\Delta u + \lambda \int_{\Omega_\epsilon} e^u \left( \int_{\Omega_\epsilon} e^u \right)^{-1} = 0 & \text{in } \Omega_\epsilon, \\
u = 0 & \text{on } \partial \Omega_\epsilon,
\end{cases}
\]

where $\lambda > 8\pi$ is a positive parameter and $\Omega_\epsilon := \Omega \setminus B(\xi, \epsilon)$, $\epsilon$ is a small positive number, $\xi \in \Omega$ and $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$ such that
\[(1.3)\]
\[
(i) \text{ if } \lambda \notin 8\pi \mathbb{N}, \text{ we will not require any symmetry assumptions on } \Omega,
\]

\[
(ii) \text{ if } \lambda \in 8\pi \mathbb{N}, \text{ i.e., } \lambda = 8\pi \kappa \text{ for some } \kappa \in \mathbb{N}, \text{ we will assume that } \\
\Omega \text{ is } \kappa-\text{symmetric with respect to the point } \xi, \text{ i.e., } \\
x \in \Omega - \xi \text{ if and only if } R_\kappa x \in \Omega - \xi, \text{ where } R_\kappa := \begin{pmatrix}
\cos \frac{\pi}{\kappa} & \sin \frac{\pi}{\kappa} \\
-\sin \frac{\pi}{\kappa} & \cos \frac{\pi}{\kappa}
\end{pmatrix}
\]

Our main result reads as follows.

**Theorem 1.1.** Let $\lambda > 8\pi$ and assume \[(1.3)\]. If $\epsilon$ is small enough, problem \[(1.2)\] has a solution $u_\epsilon$ such that as $\epsilon$ goes to zero
\[
u_\epsilon(x) \to \frac{8\pi + \lambda}{2} G(x, \xi) \text{ uniformly on compact sets of } \Omega \setminus \{\xi\}.
\]
Here

\begin{equation}
G(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} + H(x, y), \quad x, y \in \Omega 
\end{equation}

is the Green’s function of Dirichlet Laplacian in \( \Omega \) and \( H(x, y) \) is its regular part. The function \( H(x, x) \) is the Robin’s function of the domain \( \Omega \).

It is not clear if the symmetry assumption \((1.3)\) when \( \lambda \in 8\pi\mathbb{N} \) can be removed. Indeed, let us consider the simple case \( \lambda = 8\pi \) and \( \Omega_\epsilon := B(0, 1) \setminus B(\xi, \epsilon) \). If \( \xi = 0 \), then problem \((1.1)\) has a radial solution (see for example Caglioti et al. \[4\]). On the other hand if \( \xi \neq 0 \) the problem \((1.1)\) has no solutions (see Bartolucci and Lin \[2\]). We point out that 0 is the unique critical point of the Robin’s function in the ball \( B(0, 1) \).

This result suggests that existence of solutions in the pierced domain \( \Omega_\epsilon \) depends on the mutual position of the center of the hole \( \xi \) and the critical points of the Robin’s function of the domain \( \Omega \). Indeed, in our situation, if \( \lambda = 8\pi\kappa \) for some \( \kappa \in \mathbb{N} \), we find a solution provided the domain \( \Omega \) is symmetric with respect to the center of the hole, namely the point \( \xi \). We point out that also in this case, the center of symmetry is a critical point of the Robin’s function.

A couple of questions naturally arise.

**Question 1.** Does problem \((1.1)\) have a solution if \( \lambda \in \{16\pi, 24\pi, 32\pi, \ldots\} \) when \( \xi \) is not a critical point of the Robin’s function of the domain \( \Omega \)?

**Question 2.** Does problem \((1.1)\) have a solution if \( \lambda \in \{16\pi, 24\pi, 32\pi, \ldots\} \) when \( \xi \) is a critical point of the Robin’s function of the domain \( \Omega \), but \( \Omega \) is not symmetric with respect to it?

The argument we use to find the solution relies on a simple contraction mapping argument. We set \( \alpha := \frac{\lambda}{4\pi} \) and we look for a solution to problem \((1.2)\) whose shape resembles the bubble

\begin{equation}
\begin{aligned}
\end{equation}

which solve the singular Liouville problem (see Prajapat and Tarantello \[12\])

\begin{equation}
-\Delta w = |x|^\alpha - 2e^w \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^\alpha - 2e^w(x) dx < +\infty.
\end{equation}

If \( \alpha \) is not an even integer, namely \( \lambda \not\in 8\pi\mathbb{N} \), the linear operator \( L_\lambda \) introduced in \((A.3)\) is substantially invertible, while if \( \alpha \) is an even integer, namely \( \lambda \in 8\pi\mathbb{N} \), we have to look for a solution to problem \((1.2)\) in the space of symmetric functions according to \((A.2)\), where the linear operator \( L_\lambda \) is substantially invertible. Therefore, a direct contraction mapping argument is enough to catch the solution.

2. The Ansatz

For the sake of simplicity, we will assume \( \xi = 0 \).

Let us introduce the projection \( P_\epsilon \) of a function \( u \) into \( H^1_0(\Omega_\epsilon) \), i.e.,

\begin{equation}
\begin{aligned}
(2.1) \quad \Delta P_\epsilon u = \Delta u \quad \text{in} \quad \Omega_\epsilon, \quad P_\epsilon u = 0 \quad \text{on} \quad \partial \Omega_\epsilon.
\end{aligned}
\end{equation}

**Lemma 2.1.** Assume \( \delta \) and \( d\beta \) are of the same order for some \( d > 0 \) and \( \beta \in (0, 1) \). It holds true that

\begin{equation}
P_\epsilon w_\delta^\alpha(x) = w_\delta^\alpha(x) - \ln 2\alpha^2\delta^\alpha + 4\pi\alpha H(x, 0) - \gamma_{\delta, \epsilon} \alpha G(x, 0) + O(\delta^\alpha) + O(\epsilon)
\end{equation}
where

\[(2.2)\]  
\[\gamma_{\delta,\epsilon}^\alpha := \frac{\ln \frac{1}{(\delta^\alpha + \epsilon^\alpha)^2} + 4\pi \alpha H(0, 0)}{2\pi \ln \frac{1}{\epsilon} + H(0, 0)}.\]

Proof. The function

\[\rho(x) := P_\epsilon w_\delta^\alpha(x) - \left[ w_\delta^\alpha(x) - \ln 2\alpha^2 \delta^\alpha + 4\pi \alpha H(x, 0) - \gamma_{\delta,\epsilon}^\alpha G(x, 0) \right]\]

solves \(-\Delta \rho = 0\) in \(\Omega_\epsilon\). Moreover, it is easy to check that

\[\rho(x) = -\ln \left(\frac{1}{\delta^\alpha + |x|^\alpha} + \frac{1}{|x|^{2\alpha}}\right) = O(\alpha) \text{ if } x \in \partial \Omega_\epsilon\]

and

\[\rho(x) = O(\epsilon) \text{ if } x \in \partial B(0, \epsilon)\]

Indeed on \(\partial B(0, \epsilon)\), one has

\[\rho(x) = \gamma_{\delta,\epsilon}^\alpha (H(x, 0) - \alpha \ln \epsilon)\]

and our assumption on \(\delta\) ensures that \(\gamma_{\delta,\epsilon}^\alpha = O(1)\). Therefore, the claim follows by the maximum principle. \(\square\)

We look for a solution to (1.2) as

\[(2.3)\]  
\[u_\epsilon := P_\epsilon w_\delta^\alpha(x) + \phi_\epsilon(x),\]

where

\[(2.4)\]  
\[\alpha = \frac{\lambda}{4\pi}\]

and the concentration parameter are chosen so that (see (2.2))

\[(2.5)\]  
\[\gamma_{\delta,\epsilon}^\alpha = 2\pi(\alpha - 2),\]

namely

\[(2.6)\]  
\[2 \ln(\delta^\alpha + \epsilon^\alpha) - (\alpha - 2) \ln \epsilon = 2\pi(\alpha + 2)H(0, 0).\]

Let us point out that by (2.6) we deduce the rate of the concentration parameter with respect to the size of the hole

\[(2.7)\]  
\[\delta \sim e^{\frac{\alpha + 2}{\alpha} \pi H(0, 0) \epsilon^{-\frac{\alpha - 2}{2\pi}}}.\]

We point out that the choice of the \(\alpha\) and \(\delta\)'s made in (2.4) and (2.5) is motivated by the need that the error term defined in (3.1) goes to zero as \(\epsilon\) goes to zero. In particular, the choice of \(\delta\) made in (2.5) together with Lemma 2.1 ensure that

\[(2.8)\]  
\[P_\epsilon w_\delta^\alpha(x) = w_\delta^\alpha(x) - \ln 2\alpha^2 \delta^\alpha + 4\pi \alpha H(x, 0) - 2\pi(\alpha - 2)G(x, 0) + O(\epsilon^{\sigma_1})\]

where \(\sigma_1 := \min\left\{\frac{\alpha - 2}{2}, 1\right\}\).

In particular, it holds true that

\[(2.9)\]  
\[P_\epsilon w_\delta^\alpha(x) = 2\pi(\alpha + 2)G(x, 0) + o(1)\] uniformly on compact sets of \(\Omega \setminus \{0\}\).

The rest term \(\phi_\epsilon\) belongs to the space \(H\) defined as follows:

\[(2.10)\]  
\[H := \begin{cases} 
H_0^1(\Omega_\epsilon) & \text{if } \lambda \not\in 4\pi \mathbb{N}, \\
\left\{ \phi \in H_0^1(\Omega_\epsilon) : \phi(x) = \phi(\Re_\alpha x), x \in \Omega_\epsilon \right\} & \text{if } \lambda \in 4\pi \mathbb{N}, 
\end{cases}\]

where \(\Re_\alpha\) is defined in (1.3).
In the following, we will denote by
\[ \|u\|_p := \left( \int_{\Omega_e} |u(x)|^p dx \right)^{\frac{1}{p}} \quad \text{and} \quad \|u\| := \left( \int_{\Omega_e} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} \]
the usual norms in the Banach spaces \( L^p(\Omega_e) \) and \( H^1_0(\Omega_e) \), respectively.

3. Estimate of the error term

In this section we will estimate the following error term:

\[ (3.1) \quad \mathcal{R}_\epsilon(x) := \Delta P_\epsilon w^\alpha_\delta(x) + \lambda \frac{e^{P_\epsilon w^\alpha_\delta(x)}}{\int_{\Omega_e} e^{P_\epsilon w^\alpha_\delta(x)} dx}. \]

**Lemma 3.1.** Let \( \mathcal{R}_\epsilon \) as in (3.1). There exist \( p_0 > 1 \) and \( \epsilon_0 > 0 \) such that for any \( p \in (1, p_0) \) and \( \epsilon \in (0, \epsilon_0) \) we have

\[ (3.2) \quad \|\mathcal{R}_\epsilon\|_{L^p} = O(\epsilon^{\sigma_p}) \quad \text{where} \quad \sigma_p := \frac{(\alpha - 2)(2 - p)}{2\alpha p}. \]

**Proof.** Now, using (2.8) we can compute

\[
\begin{align*}
\int_{\Omega_e} e^{P_\epsilon w^\alpha_\delta(x)} dx &= \int_{\Omega_e} \frac{|x|^{\alpha - 2}}{(\delta^\alpha + |x|^\alpha)^2} e^{2\pi(\alpha + 2)H(x,0) + O(\epsilon^\sigma_1)} dx \\
&= \int_{\Omega_e} \frac{|x|^{\alpha - 2}}{(\delta^\alpha + |x|^\alpha)^2} e^{2\pi(\alpha + 2)H(0,0) + O(|x|) + O(\epsilon^\sigma_1)} dx \\
&= \int_{\Omega_e} \frac{|x|^{\alpha - 2}}{(\delta^\alpha + |x|^\alpha)^2} e^{2\pi(\alpha + 2)H(0,0)} dx + O\left( \int_{\Omega_e} \frac{|x|^{\alpha - 2}}{(\delta^\alpha + |x|^\alpha)^2} (|x| + \epsilon^\sigma_1) dx \right) \\
&= \frac{1}{\delta^\alpha} \int_{\Omega_e} \frac{|y|^{\alpha - 2}}{(1 + |y|^\alpha)^2} e^{2\pi(\alpha + 2)H(0,0)} dy + O\left( \frac{1}{\delta^\alpha} \int_{\Omega_e} \frac{|y|^{\alpha - 2}}{(1 + |y|^\alpha)^2} (\delta|y| + \epsilon^\sigma_1) dy \right) \\
&= \frac{1}{\delta^\alpha} \left( \frac{2\pi}{\alpha} e^{2\pi(\alpha + 2)H(0,0)} + O(\epsilon^{\sigma_2}) \right)
\end{align*}
\]

where \( \sigma_2 := \min \left\{ \frac{(\alpha + 1)(\alpha - 2)}{2}, \frac{\alpha + 2}{2} \right\} \). Indeed

\[
\begin{align*}
\int_{\Omega_e} \frac{|y|^{\alpha - 2}}{(1 + |y|^\alpha)^2} dy &= \int_{\mathbb{R}^2} \frac{|y|^{\alpha - 2}}{(1 + |y|^\alpha)^2} dy - \int_{\mathbb{R}^2 \setminus \Omega_e} \frac{|y|^{\alpha - 2}}{(1 + |y|^\alpha)^2} dy + \int_{B(0,\epsilon/\delta)} \frac{|y|^{\alpha - 2}}{(1 + |y|^\alpha)^2} dy \\
&= \frac{2\pi}{\alpha} + O\left( \delta^{\alpha + 1} \right) + O\left( \left( \frac{\epsilon}{\delta} \right)^2 \right).
\end{align*}
\]
By (3.3) we deduce that
\[
(3.5) \quad \frac{1}{\int_{\Omega_x} e^{P_{\sigma}w_\sigma^\alpha(x)} dx} = \delta^\alpha \left( \frac{\alpha}{2\pi} e^{-2\pi(\alpha+2)H(0,0)} + O(\epsilon^{\sigma_2}) \right).
\]

Therefore, we can compute
\[
R_\epsilon(x) = -|x|^{\alpha-2} e^{w_\sigma^\alpha(x)} + \frac{\lambda}{\int_{\Omega_x} e^{P_{\sigma}w_\sigma^\alpha(x)} dx} e^{P_{\sigma}w_\sigma^\alpha(x)} \quad \text{(we use (2.8))}
\]
\[
= |x|^{\alpha-2} e^{w_\sigma^\alpha(x)} \left[ -1 + \frac{\lambda}{2\alpha^2 \delta^\alpha \int_{\Omega_x} e^{P_{\sigma}w_\sigma^\alpha(x)} dx} e^{2\pi(\alpha+2)H(x,0)+O(\epsilon^{\sigma_1})} \right] \quad \text{(we use (3.5))}
\]
\[
= |x|^{\alpha-2} e^{w_\sigma^\alpha(x)} \left[ -1 + \frac{\lambda}{4\pi^2 \alpha} (1 + O(\epsilon^{\sigma_2})) e^{2\pi(\alpha+2)[H(x,0)-H(0,0)]+O(\epsilon^{\sigma_1})} \right]
\]
\[
\quad \text{(we use the mean value theorem and the choice of } \alpha \text{ in (2.4)})
\]
\[
= |x|^{\alpha-2} e^{w_\sigma^\alpha(x)} \left\{ -1 + \frac{\lambda}{4\pi^2 \alpha} [1 + O(\epsilon^{\sigma_2})] \left[ 1 + O(|x|) + O(\epsilon^{\sigma_1}) \right] \right\}
\]
\[
\quad \text{(we use the choice of } \alpha \text{ made in (2.4))}
\]
\[
= |x|^{\alpha-2} e^{w_\sigma^\alpha(x)} [O(|x|) + O(\epsilon^{\sigma_1})]
\]
because $\sigma_1 = \min \{\sigma_1, \sigma_2\}$. Finally, we get
\[
\int_{\Omega_x} |R_\epsilon(x)|^p dx = O \left( \int_{\Omega_x} \left( \frac{|x|^{\alpha-1}}{(\delta^\alpha + |x|^{\alpha})^2} \right)^p dx \right) + O \left( \int_{\Omega_x} \left( \epsilon^{\sigma_1} \frac{|x|^{\alpha-2}}{(\delta^\alpha + |x|^{\alpha})^2} \right)^p dx \right)
\]
\[
\quad \text{(we scale } x = \delta y)\]
\[
= O \left( \delta^{2-p} \int_{\mathbb{R}^2} \left( \frac{|y|^{\alpha-1}}{(1 + |y|^{\alpha})^2} \right)^p dx \right) + O \left( \epsilon^{p\sigma_1} \int_{\mathbb{R}^2} \left( \frac{|y|^{\alpha-2}}{(1 + |y|^{\alpha})^2} \right)^p dx \right)
\]
\[
\quad \text{(we use (2.7) and we take } p \text{ close enough to 1)}
\]
\[
= O \left( e^{(\alpha-2)p/2} \right) + O \left( \epsilon^{p\sigma_1} \right) = O \left( e^{(\alpha-2)p/2} \right),
\]
because $(\alpha-2)p/2 = \min \left\{ \frac{(\alpha-2)p}{2}, p\sigma_1 \right\}$ if $p$ is close enough to 1. That proves our claim. \hfill \square

4. The linear theory

It is useful to introduce the Banach spaces
\[
(4.1) \quad L_\alpha(\mathbb{R}^2) := \left\{ u \in W^{1,2}_{loc}(\mathbb{R}^2) : \left\| \frac{|y|^{\alpha-2}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\}
\]
and
\[
(4.2) \quad H_\alpha(\mathbb{R}^2) := \left\{ u \in W^{1,2}_{loc}(\mathbb{R}^2) : \left\| \nabla u \right\|_{L^2(\mathbb{R}^2)} + \left\| \frac{|y|^{\alpha-2}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\},
\]
endowed with the norms
\[ \|u\|_{L_\alpha} := \left\| \frac{|y|^{\alpha-2}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} \quad \text{and} \quad \|u\|_{H_\alpha} := \left( \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \left\| \frac{|y|^{\alpha-2}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)}^2 \right)^{1/2}. \]

It is important to point out the compactness of the embedding \( i_\alpha : H_\alpha(\mathbb{R}^2) \hookrightarrow L_\alpha(\mathbb{R}^2) \) (see, for example, [10]).

Let us consider the linear operator
\[ (4.3) \quad \mathcal{L}_\epsilon(\phi) := -\Delta \phi - \lambda \frac{e^{P_n w_n}}{\Omega} \int_{\Omega} e^{P_n w_n(x)} dx \phi_n + \lambda \frac{e^{P_n w_n}}{\Omega} \int_{\Omega} e^{P_n w_n(x)} \phi_n(x) dx. \]

Let us study the invertibility of the linearized operator \( \mathcal{L}_\epsilon \).

**Proposition 4.1.** For any \( p > 1 \) there exist \( \epsilon_0 > 0 \) and \( c > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \) and for any \( \psi \in L^p(\Omega_\epsilon) \) there exists a unique \( \phi \in W^{2,2}(\Omega_\epsilon) \cap H \) solution of
\[ \mathcal{L}_\epsilon(\phi) = \psi \text{ in } \Omega_\epsilon, \quad \phi = 0 \text{ on } \partial \Omega_\epsilon, \]
which satisfies
\[ \|\phi\| \leq c |\ln \epsilon| \|\psi\|_p. \]

**Proof.** We argue by contradiction. Assume there exist \( p > 1 \), sequences \( \epsilon_n \to 0 \), \( \psi_n \in L^p(\Omega_n) \) and \( \phi_n \in W^{2,2}(\Omega_n) \) such that
\[ (4.4) \quad -\Delta \phi_n - \lambda \frac{e^{P_n w_n}}{\Omega_n} \int_{\Omega_n} e^{P_n w_n(x)} dx \phi_n + \lambda \frac{e^{P_n w_n}}{\Omega_n} \int_{\Omega_n} e^{P_n w_n(x)} \phi_n(x) dx = \psi_n \text{ in } \Omega_n, \quad \phi_n = 0 \text{ on } \partial \Omega_n, \]
where \( \Omega_n := \Omega_{\epsilon_n}, \ P_n := P_{\epsilon_n}, \ w_n := w_{\delta_n}^\alpha \), the parameters \( \delta_n \) being in (2.6) and
\[ (4.5) \quad \|\phi_n\| = 1 \quad \text{and} \quad |\ln \epsilon_n| \|\psi_n\|_p \to 0. \]

We set \( \pi_n(x) := |x|^{\alpha - 2} e^{w_n(x)} \) and rewrite (4.4) by using (3.1):
\[ (4.6) \quad -\Delta \phi_n - \pi_n \phi_n + \frac{1}{\lambda} \pi_n \int_{\Omega_n} \pi_n \phi_n(x) dx = \psi_n + \rho_n \text{ in } \Omega_n, \quad \phi_n = 0 \text{ on } \partial \Omega_n, \]
where
\[ \rho_n(x) := R_n(x) \phi_n(x) + \frac{1}{\lambda} \left[ (R_n(x) + \pi_n(x)) \int_{\Omega_n} R_n(x) \phi_n(x) dx + R_n \int_{\Omega_n} \pi_n(x) \phi_n(x) dx \right]. \]

By Lemma 3.1 and (4.5) we deduce that
\[ (4.7) \quad \|\rho_n\|_p = O(e^\sigma) \quad \text{for some } \sigma > 0. \]

We define \( \hat{\phi}_n(y) := \phi_n(\delta_n y) \) with \( y \in \hat{\Omega}_n := \frac{\Omega}{\delta_n} \). It solves
\[ (4.8) \quad -\Delta \hat{\phi}_n - \pi \hat{\phi}_n + \frac{1}{\lambda} \pi \int_{\Omega_n} \pi(y) \hat{\phi}_n(y) dy = \delta_n^2 (\psi_n(\delta_n y) + \rho_n(\delta_n y)) \text{ in } \hat{\Omega}_n, \quad \hat{\phi}_n = 0 \text{ on } \partial \hat{\Omega}_n, \]
where \( \pi(y) := 2\alpha^2 \frac{|y|^{\alpha-2}}{(1 + |y|^\alpha)^2}. \)
Step 1. We will show that \( \hat{\phi}_n \to \hat{\phi} \) weakly in \( H_\alpha(\mathbb{R}^2) \) and strongly in \( L_\alpha(\mathbb{R}^2) \) with
\[
\hat{\phi}(y) - \frac{1}{\lambda} \int_{\mathbb{R}^2} \pi(y)\hat{\phi}(y)dy = a \frac{1 - |y|^\alpha}{1 + |y|^\alpha} \quad \text{for some } a \in \mathbb{R}.
\]

First of all we claim that each \( \hat{\phi}_n \) is bounded in the space \( H_\alpha(\mathbb{R}^2) \) defined in (4.2). We remark that by scaling
\[
\int_{\hat{\Omega}_n} |\nabla \hat{\phi}_n(y)|^2 dy = \delta_n^2 \int_{\hat{\Omega}_n} |(\nabla \phi)(\delta_n y)|^2 dy = \int_{\hat{\Omega}_n} |\nabla \phi(x)|^2 dx = 1.
\]
Assume by contradiction that
\[
\|\hat{\phi}_n\|_{L_\alpha(\mathbb{R}^2)}^2 = \int_{\hat{\Omega}_n} \pi(y)\hat{\phi}_n(y)^2 dy \to +\infty \quad \text{as } n \to +\infty.
\]
Then, if we introduce the normalized sequence \( \hat{\phi}_n^* := \frac{\hat{\phi}_n}{\|\hat{\phi}_n\|_{L_\alpha(\mathbb{R}^2)}} \) we have (up to a subsequence) that
\( \hat{\phi}_n^* \to \hat{\phi}^* \) weakly in \( L_\alpha(\mathbb{R}^2) \).

If we multiply (4.8) by \( \hat{\phi}_n \) we deduce
\[
0 \leq \int_{\hat{\Omega}_n} \pi(y)\hat{\phi}_n^2(y)dy - \frac{1}{\lambda} \left( \int_{\hat{\Omega}_n} \pi(y)\hat{\phi}_n(y)dy \right)^2 = 1 + o(1).
\]
The R.H.S follows by (4.5) and (4.7), while the L.H.S. follows by Hölder’s inequality and the choice of \( \alpha \) in (2.4), namely
\[
\frac{1}{\lambda} \left( \int_{\hat{\Omega}_n} \pi(y)\hat{\phi}_n(y)dy \right)^2 \leq \frac{1}{\lambda} \left( \int_{\hat{\Omega}_n} \pi(y)dy \right) \left( \int_{\hat{\Omega}_n} \pi(y)\hat{\phi}_n^2(y)dy \right) = \frac{4\pi\alpha}{\lambda} \left( \int_{\hat{\Omega}_n} \pi(y)\hat{\phi}_n^2(y)dy \right).
\]
Next, we divide (4.10) by \( \|\hat{\phi}_n\|_{L_\alpha(\mathbb{R}^2)}^2 \), we pass to the limit and we get
\[
\frac{1}{\lambda} \left( \int_{\mathbb{R}^2} \pi(y)\hat{\phi}^*(y)dy \right)^2 = 1,
\]
because the constant function \( 1 \in L_\alpha(\mathbb{R}^2) \) and \( \hat{\phi}_n^* \to \hat{\phi}^* \) weakly in \( L_\alpha(\mathbb{R}^2) \). On the other hand, we divide (4.8) by \( \|\hat{\phi}_n\|_{L_\alpha(\mathbb{R}^2)} \), we pass to the limit and we get
\[
-\pi(y)\hat{\phi}^*(y) + \frac{1}{\lambda} \pi(y) \int_{\mathbb{R}^2} \pi(y)\hat{\phi}^*(y)dy = 0.
\]
(We use the fact that the constant function \( 1 \in L_\alpha(\mathbb{R}^2) \) and \( \hat{\phi}_n^* \to \hat{\phi}^* \) weakly in \( L_\alpha(\mathbb{R}^2) \).) Finally, by (4.12) we immediately deduce that \( \hat{\phi}^* \) is a constant function and by the choice of \( \alpha \) in (2.4) we get that either \( \hat{\phi}^* \equiv 0 \) or \( \hat{\phi}^* \equiv 1 \), which contradicts (4.11).

Therefore, each \( \hat{\phi}_n \) is bounded in the space \( H_\alpha(\mathbb{R}^2) \) defined in (4.2) and (up to a subsequence)
\[
\hat{\phi}_n(y) \to \hat{\phi} \quad \text{weakly in } H_\alpha(\mathbb{R}^2) \quad \text{and strongly in } L_\alpha(\mathbb{R}^2).
\]
So we pass to the limit into (4.8) and we deduce that \( \hat{\phi} \in H \) (see (2.10)) is a solution to the equation
\[
-\Delta \hat{\phi} = \pi \hat{\phi} - \frac{1}{\lambda} \int \pi(y) \hat{\phi}(y) dy \text{ in } \mathbb{R}^2.
\]

Then the function \( \phi_0(y) := \hat{\phi}(y) - \frac{1}{\lambda} \int \pi(y) \hat{\phi}(y) dy \) is a solution in the space \( H \) defined in (2.10) to the linear problem \(-\Delta \phi = \pi \phi_0 \) in \( \mathbb{R}^2 \). By Theorem A.1 we get our claim.

**Step 2.** We will show that \( a = 0 \) in (4.9) and then either \( \hat{\phi} \equiv 0 \) or \( \hat{\phi} \equiv 1 \) in \( \mathbb{R}^2 \).

First of all, we introduce the function
\[
Z(y) := \frac{1 - |y|^{\alpha}}{1 + |y|^{\alpha}} \quad \text{and} \quad Z_n(x) := Z \left( \frac{x}{\delta_n} \right) = \frac{\delta^{\alpha} - |x|^{\alpha}}{\delta^{\alpha} + |x|^{\alpha}}.
\]

We know that \( Z_n \) solves (see Theorem A.1)
\[
-\Delta Z_n = \pi_n Z_n \quad \text{in } \mathbb{R}^2.
\]

Let \( P_n Z_n \) be its projection onto \( H^{1}_0(\Omega_n) \) (see (2.1)), i.e.,
\[
-\Delta P_n Z_n = \pi_n Z_n \quad \text{in } \Omega_n, \quad P_n Z_n = 0 \quad \text{on } \partial\Omega_n.
\]

By maximum principle (see also Lemma 2.1) we deduce that
\[
P_n Z_n(x) = Z_n(x) + 1 - \frac{G(x,0)}{2\pi \ln \frac{1}{\epsilon_n} + H(0,0)} + O(\epsilon^\sigma),
\]
for some \( \sigma > 0 \). Set \( \gamma_n := 2\pi \ln \frac{1}{\epsilon_n} + H(0,0) \).

We are going to show that
\[
\lim_n \int_{\Omega_n} G(x,0) \pi_n(x) \phi_n(x) dx - \frac{1}{\lambda} \int_{\Omega_n} G(x,0) \pi_n(x) dx \int_{\Omega_n} \pi_n(x) \phi_n(x) dx = 0.
\]

We multiply (4.6) by \( \gamma_n P_n Z_n \) and (4.13) by \( \gamma_n \phi \). If we subtract the two equations obtained, we get
\[
\gamma_n \int_{\Omega_n} \pi_n(x) \phi_n(x) (P_n Z_n(x) - Z_n(x)) dx - \frac{\gamma_n}{\lambda} \int_{\Omega_n} \pi_n(x) P_n Z_n(x) dx \int_{\Omega_n} \pi_n(x) \phi_n(x) dx
\]
\[
= \gamma_n \int_{\Omega_n} (\psi_n(x) + \rho_n(x)) P_n Z_n(x) dx,
\]
which implies together with (4.14)
\[
\gamma_n \int_{\Omega_n} \pi_n(x) \phi_n(x) \left[ 1 - \frac{G(x,0)}{\gamma_n} + O(\epsilon^\sigma) \right] dx
\]
\[
- \frac{\gamma_n}{\lambda} \int_{\Omega_n} \pi_n(x) \left[ Z_n(x) + 1 - \frac{G(x,0)}{\gamma_n} + O(\epsilon^\sigma) \right] dx \int_{\Omega_n} \pi_n(x) \phi_n(x) dx
\]
\[
= \gamma_n \int_{\Omega_n} (\psi_n(x) + \rho_n(x)) P_n Z_n(x) dx.
\]
and so
\[
\gamma_n \int_{\Omega_n} \pi_n(x) \phi_n(x) \left[ 1 - \frac{1}{\lambda} \int_{\Omega_n} \pi_n(x) (Z_n(x) + 1) \, dx \right] 
\]
(4.16)
\[-\int_{\Omega_n} G(x, 0) \pi_n(x) \phi_n(x) \, dx + \frac{1}{\lambda} \int_{\Omega_n} G(x, 0) \pi_n(x) \int_{\Omega_n} \pi_n(x) \phi_n(x) \, dx = o(1),
\]
because of (4.15), (4.17) and the fact that \(\gamma_n \sim |\ln \epsilon_n|\). Estimate (4.15) follows by (4.16) once we prove that
\[
\gamma_n \int_{\Omega_n} \pi_n(x) \phi_n(x) \left[ 1 - \frac{1}{\lambda} \int_{\Omega_n} \pi_n(x) (Z_n(x) + 1) \, dx \right] = o(1).
\]
(4.17)
We have that
\[
\frac{1}{\lambda} \int_{\Omega_n} \pi_n(x) (Z_n(x) + 1) \, dx = \frac{1}{\lambda} \int_{\Omega_n} \pi(y) (Z(y) + 1) \, dy \\
= \frac{1}{\lambda} \int_{\mathbb{R}^2} \pi(y) (Z(y) + 1) \, dy - \frac{1}{\lambda} \int_{\mathbb{R}^2 \setminus \Omega_n} \pi(y)(Z(y) + 1) \, dy
\]
(4.18)
\[= 1 + O \left( \epsilon_n^{-2} \right).
\]
Indeed, a straightforward computation leads to
\[
\int_{\mathbb{R}^2} \pi(y) Z(y) \, dy = \int_{\mathbb{R}^2} 2\alpha^2 \frac{|y|^{\alpha-2} \left( 1 - |y|^{\alpha} \right)}{(1 + |y|^{\alpha})^2} \, dy = 0,
\]
(4.19)
\[
\int_{\mathbb{R}^2} \pi(y) \, dy = \int_{\mathbb{R}^2} 2\alpha^2 \frac{|y|^{\alpha-2}}{(1 + |y|^{\alpha})^2} \, dy = 4\pi \alpha = \lambda,
\]
(4.20)
because of the choice of \(\alpha\) in (2.4) and
\[
\int_{\mathbb{R}^2 \setminus \Omega_n} \pi(y) (Z(y) + 1) \, dy = O \left( \delta_n^{2\alpha} \right) = O \left( \epsilon_n^{\alpha-2} \right)
\]
because of (2.7). Finally, (4.17) follows by (4.18) taking into account (4.5) and the fact that \(\gamma_n \sim |\ln \epsilon_n|\).

Finally, we can show that \(a = 0\) in (4.9). By (4.15) we get
\[
\int_{\Omega_n} \left[ \frac{1}{2\pi} \ln |x| + H(x, 0) \right] \pi_n(x) \phi_n(x) \, dx \\
- \frac{1}{\lambda} \int_{\Omega_n} \left[ \frac{1}{2\pi} \ln |x| + H(x, 0) \right] \pi_n(x) \, dx \int_{\Omega_n} \pi_n(x) \phi_n(x) \, dx = o(1)
\]
and scaling $y = \delta_n x$ we deduce

$$
\frac{1}{2\pi} \left[ \int_{\hat{\Omega}_n} \ln |y| \pi(y) \phi_n(y) dy - \frac{1}{\lambda} \int_{\hat{\Omega}_n} \ln |y| \pi(y) \int_{\hat{\Omega}_n} \pi(y) \phi_n(y) dy \right]
$$

$$
= - \left( \frac{1}{2\pi} \ln \delta_n + H(0, 0) \right) \int_{\hat{\Omega}_n} \pi(y) \phi_n(y) dy \left( 1 - \frac{1}{\lambda} \int_{\hat{\Omega}_n} \pi(y) dy \right) + o(1)
$$

(4.21) $\quad = o(1),$

because using the choice of $\alpha$ in (2.4) and arguing as in (4.18) it holds true that

$$
1 - \frac{1}{\lambda} \int_{\hat{\Omega}_n} \pi(y) dy = O (\delta_n^{\alpha}) = O \left( \epsilon_n^{2\alpha - 2} \right).
$$

On the other hand, by (4.9) we can assume that the weak limit of $\hat{\phi}_n$ reads as $\hat{\phi} = aZ + b$, for some constants $a$ and $b$, so we pass to the limit on the L.H.S. of (4.21) and we get

$$
\int_{\hat{\Omega}_n} \ln |y| \pi(y) \phi_n(y) dy - \frac{1}{\lambda} \int_{\hat{\Omega}_n} \ln |y| \pi(y) \int_{\hat{\Omega}_n} \pi(y) \phi_n(y) dy
$$

$$
= \int_{\mathbb{R}^2} \ln |y| \pi(y) (aZ(y) + b) dy - \frac{1}{\lambda} \int_{\mathbb{R}^2} \ln |y| \pi(y) \int_{\mathbb{R}^2} \pi(y) (aZ(y) + b) dy + o(1)
$$

(4.22)

$$
= a \int_{\mathbb{R}^2} \ln |y| \pi(y) Z(y) dy + o(1) = -4a\pi + o(1),
$$

because of (4.20), (4.19) and

(4.23) $\quad \int_{\mathbb{R}^2} \ln |y| \pi(y) Z(y) dy = \int_{\mathbb{R}^2} 2\alpha^2 \frac{|y|^{\alpha - 2}}{(1 + |y|^\alpha)^2} \frac{1 - |y|^{\alpha}}{1 + |y|^\alpha} \ln |y| dy = -4\pi.$

as a straightforward computation proves. Combining (4.21) and (4.22) we deduce that $a = 0$.

Finally, if $a = 0$ by (4.9) using the choice of $\alpha$ made in (2.4), we immediately deduce that $\hat{\phi}$ is a constant function whose possible values are 0 or 1. That concludes the proof.
Step 3. We will show that a contradiction arises!

We multiply equation (4.8) by \( \hat{\phi}_n \) and we get

\[
1 = \int_{\Omega_n} \pi(y) \hat{\phi}_n^2(y) dy - \frac{1}{\lambda} \left( \int_{\Omega_n} \pi(y) \hat{\phi}_n(y) dy \right)^2 + o(1)
\]

\[
= \int_{\mathbb{R}^2} \pi(y) \hat{\phi}^2(y) dy - \frac{1}{\lambda} \left( \int_{\mathbb{R}^2} \pi(y) \hat{\phi}(y) dy \right)^2 + o(1)
\]

(because \( \phi_n \to 0 \) strongly in \( L_\alpha(\mathbb{R}^2) \))

\[
= 0 \quad \text{if either} \quad \hat{\phi}(y) \equiv 0 \quad \text{or} \quad \hat{\phi}(y) \equiv 1 \quad \text{(because of the choice of} \quad \alpha \quad \text{in (2.4))}
\]

and a contradiction arises! \( \square \)

5. A CONTRACTION MAPPING ARGUMENT AND THE PROOF OF THE MAIN THEOREM

First of all we point out that \( P_{\epsilon}w^\alpha_{\delta} + \phi_{\epsilon} \) is a solution to (1.2) if and only if \( \phi_{\epsilon} \) is a solution of the problem

\[
(5.1) \quad \mathcal{L}_\lambda(\phi) = \mathcal{N}_\epsilon(\phi) + \mathcal{R}_\epsilon \quad \text{in} \quad \Omega_{\epsilon}
\]

where the error term \( \mathcal{R}_\epsilon \) is defined in (3.1), the linear operator \( \mathcal{L}_\lambda \) is defined in (4.3) and the higher order term \( \mathcal{N}_\epsilon \) is defined as

\[
(5.2) \quad \mathcal{N}_\epsilon(\phi) := \lambda \left[ \int_{\Omega_{\epsilon}} e^{P_{\epsilon}w^\alpha_{\delta}(x) + \phi_{\epsilon}(x)} dx - \int_{\Omega_{\epsilon}} e^{P_{\epsilon}w^\alpha_{\delta}(x)} dx - \int_{\Omega_{\epsilon}} e^{P_{\epsilon}w^\alpha_{\delta}(x)} dx \phi_{\epsilon} dx \right].
\]

**Proposition 5.1.** There exist \( p_0 > 0 \), \( \epsilon_0 > 0 \) and \( R_0 > 0 \) such that for any \( p \in (1, p_0) \), \( \epsilon \in (0, \epsilon_0) \) and \( R \geq R_0 \) there exists a unique solution \( \phi_{\epsilon} \in H \) to the equation

\[
(5.3) \quad \Delta(P_{\epsilon}w^\alpha_{\delta} + \phi_{\epsilon}) + \lambda \frac{e^{P_{\epsilon}w^\alpha_{\delta}(x) + \phi_{\epsilon}(x)} dx}{\int_{\Omega_{\epsilon}} e^{P_{\epsilon}w^\alpha_{\delta}(x) + \phi_{\epsilon}(x)} dx} = 0 \quad \text{in} \quad \Omega_{\epsilon}, \phi = 0 \quad \text{on} \quad \partial \Omega_{\epsilon}
\]

and (see (3.2))

\[
\| \phi_{\epsilon} \| \leq R e^{\sigma_\epsilon} |\ln \epsilon|.
\]

**Proof.** As a consequence of Proposition 4.1, we conclude that \( \phi \) is a solution to (6.3) if and only if it is a fixed point for the operator \( \mathcal{T}_\epsilon : H \to H \), defined by

\[
\mathcal{T}_\epsilon(\phi) = (\mathcal{L}_\epsilon)^{-1} (\mathcal{N}_\epsilon(\phi) + \mathcal{R}_\epsilon),
\]

where \( \mathcal{L}_\epsilon, \mathcal{N}_\epsilon \) and \( \mathcal{R}_\epsilon \) are defined in (3.1), (5.2) and (3.1), respectively.

Let us introduce the ball \( \mathcal{B}_{\epsilon,R} := \{ \phi \in H : \| \phi \| \leq R e^{\sigma_\epsilon} |\ln \epsilon| \} \). We will show that \( \mathcal{T}_\epsilon : \mathcal{B}_{\epsilon,R} \to \mathcal{B}_{\epsilon,R} \) is a contraction mapping provided \( \epsilon \) is small enough and \( R \) is large enough.

Let us prove that \( \mathcal{T}_\epsilon \) maps the ball \( \mathcal{B}_{\epsilon,R} \) into itself, i.e.,

\[
(5.4) \quad \| \phi \| \leq R e^{\sigma_\epsilon} |\ln \epsilon| \implies \| \mathcal{T}_\epsilon(\phi) \| \leq R e^{\sigma_\epsilon} |\ln \epsilon|.
\]
By Lemma 5.1 (where we take \( h = N_\epsilon(\phi) + R_\epsilon \)), by (5.6) and by Lemma 3.1 we deduce that:
\[
\|T_\epsilon(\phi)\| \leq c \ln \epsilon \left( \|N_\epsilon(\phi)\|_p + \|R_\epsilon\|_p \right) \leq c \ln \epsilon \left( \epsilon^{\sigma'_p}\|\phi\|^2 + \epsilon^{\sigma_p} \right) \leq R\epsilon^{\sigma_p}\ln \epsilon
\]
provided \( p \) is close enough to 1, \( R \) is suitably large and \( \epsilon \) is small enough. That proves (5.5).

Let us prove that \( T_\epsilon \) is a contraction mapping, i.e., there exists \( \ell > 1 \) such that
\[
\|\phi\| \leq R\epsilon^{\sigma_p}\ln \epsilon \implies \|T_\epsilon(\phi_1) - T_\epsilon(\phi_2)\| \leq \ell\|\phi_1 - \phi_2\|.
\]

By Lemma 5.1 (where we take \( \psi = N_\epsilon(\phi_1) - N_\epsilon(\phi_2) \)) and by (5.7), we deduce that:
\[
\|T_\epsilon(\phi)\| \leq c \ln \epsilon \left( \|N_\epsilon(\phi_1) - N_\epsilon(\phi_2)\|_p \right)
\]
\[
\leq c\epsilon^{\sigma'_p}\ln \epsilon\|\phi_1 - \phi_2\| (\|\phi_1\| + \|\phi_2\|) \leq \ell\|\phi_1 - \phi_2\|
\]
for some \( \ell < 1 \), provided \( p \) is close enough to 1, \( R \) is suitably large and \( \epsilon \) is small enough. That proves (5.5).

**Lemma 5.1.** There exist \( p_0 > 1 \) and \( \epsilon_0 > 0 \) such that for any \( p \in (1, p_0), \epsilon \in (0, \epsilon_0) \) and \( R > 0 \) we have for any \( \phi, \phi_1, \phi_2 \in B_{\epsilon,R} := \{ \phi \in H^1_0(\Omega) : \|\phi\| \leq R\epsilon^{\sigma_p}\ln \epsilon \} \)
\[
(5.6) \quad \|N_\epsilon(\phi)\|_p = O \left( \epsilon^{\sigma'_p}\|\phi\|^2 \right)
\]
and
\[
(5.7) \quad \|N_\epsilon(\phi_1) - N_\epsilon(\phi_2)\|_p = O \left( \epsilon^{\sigma'_p}\|\phi_1 - \phi_2\| (\|\phi_1\| + \|\phi_2\|) \right),
\]
for some \( \sigma'_p > 0 \).

**Proof.** Since (5.6) follows by (5.7) choosing \( \phi_2 = 0 \), we only prove (5.7). We point out that
\[
N_\epsilon(\phi) = \lambda \left[ f(\phi) - f(0) - f'(0)(\phi) \right], \quad \text{where} \quad f(\phi) := \frac{e^{P,w_t^2+\phi} - e^{P,w_t^2}(x) + \phi(x)dx}{\Omega},
\]
Therefore, we apply the mean value theorem, we set \( \phi_0 = \theta\phi_1 + (1 - \theta)\phi_2 \) and \( \phi_\eta = \eta\phi_1 + (1 - \eta)\phi_2 \) for some \( \theta, \eta \in [0,1] \)
\[
N_\epsilon(\phi_0) - N_\epsilon(\phi_2) = \lambda \left[ f(\phi_1) - f(\phi_2) - f'(0)\phi_1 + \phi_2 \right] = \lambda \left[ f'(\phi_0) - f'(0) \right] \phi_1 - \phi_2
\]
(5.8) 
\[
= \lambda f''(\phi_\eta)[\phi_\theta, \phi_1 - \phi_2]
\]
where
\[
f''(u)[\phi, \psi] = \int_{\Omega_x} e^u(\phi \psi) - \frac{e^u}{2} \phi \int_{\Omega_x} e^u \psi - \frac{e^u}{2} \psi \int_{\Omega_x} e^u \phi
\]
\[
- \frac{e^u}{2} \int_{\Omega_x} e^u \phi \psi + 2 \frac{e^u}{3} \phi \int_{\Omega_x} e^u \psi - e^u \phi \int_{\Omega_x} e^u \psi.
\]

We use Hölder’s inequalities
\[
\|uv\|_p \leq \|u\|_{pr}\|v\|_{ps}, \quad \frac{1}{r} + \frac{1}{s} = 1 \quad \text{or} \quad \|vw\|_p \leq \|u\|_{pr}\|v\|_{ps}\|w\|_{pt}, \quad \frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1.
\]
and we get

$$
\|f''(u)[\phi, \psi]\|_p \leq \frac{\|e^u\|_{p'} \|\phi\|_{p}\|\psi\|_{p't}}{\|e^u\|_1^2} + 2 \frac{\|e^u\|_{p'}^2 \|\phi\|_{p}\|\psi\|_{p'}}{\|e^u\|_1^2} + 2 \frac{\|e^u\|_{p'}^2 \|\phi\|_{p}\|\psi\|_{p'}}{\|e^u\|_1^2} \frac{\|\phi\|_{p}\|\psi\|_{p'}}{\|e^u\|_1^2}
$$

(5.9)

because $L^{p'}(\Omega_\epsilon) \hookrightarrow L^p(\Omega_\epsilon)$ for any $r \geq 1$ and $L^q(\Omega_\epsilon) \hookrightarrow H^1_0(\Omega_\epsilon)$ for any $q > 1$. Now, we put together (5.8) and (5.9) with

(5.10)

$$
\|e^{P_s w^\alpha}\|_q = O \left( \delta^{\frac{2}{q}-(2+\alpha)} \right) \text{ for any } q \geq 1.
$$

Moreover, (5.3) implies that

(5.11)

$$
\|e^{P_s w^\alpha}\|_1 \geq C_0 \frac{1}{\delta^\alpha} \text{ for some } C_0 > 0.
$$

On the other hand, using the estimate $|e^a - 1| \leq |a|$ for any $a \in \mathbb{R}$ we have

(5.12)

$$
\|e^{P_s w^\alpha+\phi}\|_q - e^{P_s w^\alpha} \|_q = \left( \int_{\Omega_\epsilon} \left| e^{P_s w^\alpha+\phi} - e^{P_s w^\alpha} \right|^q dx \right)^{1/q}
$$

$$
= \left( \int_{\Omega_\epsilon} \left| e^{P_s w^\alpha} \right|^q |\phi|^q dx \right)^{1/q}
$$

$$
\leq \left( \int_{\Omega_\epsilon} \left| e^{P_s w^\alpha} \right|^q |\phi|^q dx \right)^{1/q}
$$

(because of the estimate $|e^a - 1| \leq |a|$ for any $a \in \mathbb{R}$ )

$$
\leq \left\| e^{P_s w^\alpha} \right\|_{q_s} |\phi|_{q_t} \text{ (we use Hölder’s estimate with } \frac{1}{q_s} + \frac{1}{q_t} = 1)
$$

$$
\leq c_0 \delta^{\frac{2}{q_t}-(2+\alpha)} |\phi| \text{ (because of (5.10) and the fact that } L^{q_t}(\Omega_\epsilon) \hookrightarrow H^1_0(\Omega_\epsilon))
$$

(5.13)

$$
\leq c_0 \delta^{\frac{2}{q_t}-(2+\alpha)} e^{\sigma \rho |\ln \epsilon|} \text{ (because } \phi \in B_{\epsilon,R}).
$$

In particular, if $q = 1$ we get

(5.14)

$$
\|e^{P_s w^\alpha+\phi}\|_1 - e^{P_s w^\alpha} \|_1 = O \left( \delta^{\frac{2}{q_t}-(2+\alpha)} e^{\sigma \rho |\ln \epsilon|} \right) \text{ for any } s > 1.
$$

By (5.10) and (5.12) we get

(5.15)

$$
\|e^{P_s w^\alpha+\phi}\|_q - e^{P_s w^\alpha} \|_q = O \left( \delta^{\frac{2}{q}-(2+\alpha)} e^{\sigma \rho |\ln \epsilon|} \right)
$$
and by (5.11) and (5.14) taking into account (2.7) and choosing $s$ close enough to 1, we get

$$\|e^{Pr}w_\alpha + \phi_\alpha\|_1 \geq \frac{c_0}{s^\alpha} - c\delta s^{2-2(p+s)} e^{\sigma_r} |\ln \epsilon| \geq \frac{1}{\delta^\alpha} \left( c_0 - ce^{\frac{\alpha-2}{2\alpha}(2-2-2)} + \sigma_r |\ln \epsilon| \right) \geq \frac{c_0}{2\delta^\alpha}. $$

Finally, by (5.15) with $q = pr$ and (5.16) taking into account (2.7) and choosing $\alpha = 2$ if $p, r, s$ are close enough to 1.

where the exponent $\sigma_r := \frac{\alpha-2}{2\alpha} \left( \frac{2}{pr} - 2 \right) + \sigma_p > 0$ if $p, r, s$ are close enough to 1.

Now, we can conclude the proof. By (5.8), (5.9) and (5.17) we get

$$\|\mathcal{N}_\epsilon(\phi_1) - \mathcal{N}_\epsilon(\phi_2)\|_p \leq c \|f''(\phi)\|_{\|\phi_\theta, \phi_1 - \phi_2\|} \leq c \|\|e^{Pr}w_\alpha + \phi_\alpha\|_{pr} \|\phi_\theta\| \|\phi_1 - \phi_2\|$$

$$\leq e^{\sigma_r}|\ln \epsilon| (\|\phi_1\| + \|\phi_2\|) \|\phi_1 - \phi_2\|$$

that proves our claim.

*Proof of Theorem 1.1 completed.* The existence of a solution $u_\epsilon = P_{\epsilon}w_\alpha + \phi_\epsilon$ follows directly by Proposition 5.1. The asymptotic shape of the solution $u_\epsilon$ as $\epsilon$ goes to zero follows by (2.9).

**Appendix A**

In the study of the linear theory we used in a crucial way the following results.

**Theorem A.1.** Let $\phi$ a solution to the equation

$$(A.1) \quad -\Delta \phi = 2\alpha^2 \frac{|y|^{\alpha-2}}{(1 + |y|^{\alpha})^2} \phi \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla \phi(y)|^2 dy < +\infty.$$ 

If $\alpha = 2\kappa$ for some $\kappa \in \mathbb{N}$, we also require that $\phi$ be $\kappa$-symmetric with respect to the origin, i.e.,

$$(A.2) \quad \phi(y) = \phi(R\kappa y) \text{ for any } y \in \mathbb{R}^2,$$

where $R\kappa$ is defined in (1.3). Then

$$\phi(y) = \frac{1 - |y|^\alpha}{1 + |y|^\alpha} \text{ for some } \gamma \in \mathbb{R}.$$ 

*Proof.* If $\alpha$ is an even integer, del Pino-Esposito-Musso in [8] proved that all the bounded solutions to (A.1) are a linear combination of the following functions (which are written in polar coordinates):

$$(A.3) \quad \phi_0(y) := \frac{1 - |y|^\alpha}{1 + |y|^\alpha}, \quad \phi_1(y) := \frac{|y|^\frac{\alpha}{2}}{1 + |y|^\alpha} \cos \frac{\alpha}{2} \theta, \quad \phi_2(y) := \frac{|y|^\frac{\alpha}{2}}{1 + |y|^\alpha} \sin \frac{\alpha}{2} \theta.$$ 

We observe that $\phi_0$ always satisfies (A.2), while the functions $\phi_1$ and $\phi_2$ do not satisfy (A.2). When $\alpha$ is not an even integer, the situation is easier, since only the function $\phi_0$ generates the set of solutions to the linear equation (A.1). In [10] it was proved that any solution $\phi$ of (A.1) is actually a bounded solution. That concludes the proof. \qed
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