On the second boundary value problem for special Lagrangian curvature potential equation

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Abstract
This is a sequel to (Huang and Ou in J Geom Anal 27:2601–2617, 2017) and (Chong et al. in On the second boundary value problem for Lagrangian mean curvature equation. arXiv:1808.01139), which study the second boundary problem for special Lagrangian curvature potential equation. As consequences, we obtain the existence and uniqueness of the smooth uniformly convex solution by the method of continuity through carrying out a-priori estimate on the solutions.

Keywords Special Lagrangian curvature operator ·Boundary defining function · Legendre transformation

Mathematics Subject Classification Primary 53C44; Secondary 53A10

1 Introduction
The first author and his coauthors were concerned with special Lagrangian equations

\[ F_{\tau}(\lambda(D^2 u)) = c, \quad \text{in} \quad \Omega, \quad (1.1) \]

associated with the second boundary value problem

\[ Du(\Omega) = \hat{\Omega}, \quad (1.2) \]
for given $F_\tau$, $\Omega$ and $\tilde{\Omega}$. Specifically, $\Omega$, $\tilde{\Omega}$ are uniformly convex bounded domains with smooth boundary in $\mathbb{R}^n$ and

\[
F_\tau(\lambda) := \begin{cases} 
\frac{1}{2} \sum_{i=1}^{n} \ln \lambda_i, & \tau = 0, \\
\frac{\sqrt{a^2+1}}{ab} \sum_{i=1}^{n} \ln \frac{\lambda_i + a - b}{\lambda_i + a + b}, & 0 < \tau < \frac{\pi}{4}, \\
-\sqrt{2} \sum_{i=1}^{n} \frac{1}{1 + \lambda_i}, & \tau = \frac{\pi}{4}, \\
\frac{\sqrt{a^2+1}}{b} \sum_{i=1}^{n} \arctan \frac{\lambda_i + a - b}{\lambda_i + a + b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\
\sum_{i=1}^{n} \arctan \lambda_i, & \tau = \frac{\pi}{2}.
\end{cases}
\]

Here $a = \cot \tau$, $b = \sqrt{|\cot^2 \tau - 1|}$, $x = (x_1, x_2, \ldots, x_n)$, $u = u(x)$ and $\lambda(D^2 u) = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of Hessian matrix $D^2 u$ according to $x$. The Eq. (1.1) comes from [20], where Warren established the calibration theory for spacelike Lagrangian submanifolds in $(\mathbb{R}^{2n}, g_\tau)$ (for general calibration theory for Lagrangian submanifolds in Riemannian and pseudo-Riemannian manifolds, see Harvey and Lawson [8] and Mealy [13]).

We present here the main results regarding the solvability of solutions to (1.1) and (1.2). Brendle and Warren [1] proved the existence and uniqueness of the solution by the elliptic method in the case of $\tau = \frac{\pi}{2}$. The first author [10] considered the longtime existence and convergence of the solutions to the second boundary problem of Lagrangian mean curvature flow and also obtained existence results in this case. As far as $\tau = 0$ is concerned, the Eq. (1.1) can be transformed into Monge–Ampère equation. Some results had already been given in the literature for the second boundary value problem of Monge–Ampère equation. Delanoë [6] obtained a unique smooth solution in dimension 2 if both domains are uniformly convex. Later his result was extended to any dimension by Caffarelli [2] and Urbas [17]. Using the parabolic method, Schnürer and Smoczyk [14] also arrived at the existence of solutions to (1.1) for $\tau = 0$. For further studies on the rest of $0 \leq \tau \leq \frac{\pi}{2}$, see [4, 11, 12] and the references therein.

The aim of this paper is to study special Lagrangian curvature potential equation

\[
\sum_{i=1}^{n} \arctan \kappa_i = c, \quad \text{in} \quad \Omega, \tag{1.3}
\]

in conjunction with the second boundary value problem

\[
Du(\Omega) = \tilde{\Omega}, \tag{1.4}
\]

where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of the graph $\Gamma = \{(x, u(x)) : x \in \Omega\}$ and $c$ is a constant to be determined, and $\Omega$, $\tilde{\Omega}$ are uniformly convex bounded domains with smooth boundary in $\mathbb{R}^n$. It shows that a very nice geometric interpretation of special Lagrangian curvature operator $F(\kappa_1, \ldots, \kappa_n) \triangleq \sum_{i=1}^{n} \arctan \kappa_i$ in [15]. The background of the so-call special Lagrangian curvature potential equation (1.3) can be seen in the literature [9, 15].

Now we state our main theorem.
Theorem 1.1 Suppose that \( \Omega, \hat{\Omega} \) are uniformly convex bounded domains with smooth boundary in \( \mathbb{R}^n \). Then there exist a uniformly convex solution \( u \in C^\infty(\hat{\Omega}) \) and a unique constant \( c \) solving (1.3) and (1.4), and \( u \) is unique up to a constant.

To obtain the existence result we use the method of continuity by carrying out a-priori estimate on the solutions to (1.3) and (1.4).

The rest of this article is organized as follows. The next section is to present the structural conditions for the operator \( F \) with the related operator \( G \) and the fundamental formulas for the principal curvatures of the graph \( \Gamma \). By the preliminary knowledge we implement the strictly oblique estimate in Sect. 3 and then in Sect. 4 we obtain the \( C^2 \) estimate by considering the operator on manifold as same as Urbas’ work [19]. Finally we give the proof of the main theorem by the continuity method in Sect. 5.

2 Preliminary

In this section, we first state some facts about the special Lagrangian curvature operator

\[
F(\kappa_1, \ldots, \kappa_n) := \sum_{i=1}^{n} \arctan \kappa_i,
\]

on the positive cone

\[
\Gamma_n^+ := \{(\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n : \kappa_i > 0, \ 1 \leq i \leq n \}.
\]

These properties are trivial and can be found in [11].

Lemma 2.1 Based on the above definition, the following properties are established.

(i) \( F \) is a smooth symmetric function of the principal curvatures \( \kappa_1, \ldots, \kappa_n \) defined on \( \Gamma_n^+ \) and satisfying

\[
0 < F(\kappa_1, \ldots, \kappa_n) < \frac{n\pi}{2}, \ \forall(\kappa_1, \ldots, \kappa_n) \in \Gamma_n^+,
\]

\[
\frac{\partial F}{\partial \kappa_i} > 0, \ 1 \leq i \leq n, \ on \ \Gamma_n^+,
\]

and

\[
\left( \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j} \right) \leq 0, \ on \ \overline{\Gamma_n^+}.
\]

(ii) For any \( (\mu_1, \ldots, \mu_n) \in \Gamma_n^+ \), denote

\[
\kappa_i = \frac{1}{\mu_i}, \ 1 \leq i \leq n,
\]

and

\[
\tilde{F}(\mu_1, \ldots, \mu_n) := -F(\kappa_1, \ldots, \kappa_n).
\]

Then

\[
\left( \frac{\partial^2 \tilde{F}}{\partial \mu_i \partial \mu_j} \right) \leq 0, \ on \ \overline{\Gamma_n^+}.
\]
For any $s_1 > 0, s_2 > 0$, define
\[
\Gamma^+_{s_1, s_2} = \left\{ (\kappa_1, \ldots, \kappa_n) \in \Gamma^+_n : 0 \leq \min_{1 \leq i \leq n} \kappa_i \leq s_1, \max_{1 \leq i \leq n} \kappa_i \geq s_2 \right\}.
\]

Then there exist positive constants $\Lambda_1$ and $\Lambda_2$, depending on $s_1$ and $s_2$, such that for any $(\kappa_1, \ldots, \kappa_n) \in \Gamma^+_{s_1, s_2}$,
\[
\Lambda_1 \leq \sum_{i=1}^n \frac{\partial F}{\partial \kappa_i} \leq \Lambda_2,
\]
and
\[
\Lambda_1 \leq \sum_{i=1}^n \frac{\partial F}{\partial \kappa_i} \kappa_i^2 \leq \Lambda_2.
\]

It follows from [3, 7] that we can state various geometric quantities associated with the graph of $u \in C^2(\Omega)$. In the coordinate system Latin indices range from 1 to $n$ and indicate quantities in the graph. We use the Einstein summation convention for repeated indices. $u_i, u_{ij}, u_{ijk} \ldots$ denote the all derivatives of $u$ according to $x_i, x_j, x_k \ldots$ The induced metric on
\[
\Gamma \triangleq \{(x, u(x)) : x \in \Omega\},
\]
is given by
\[
g_{ij} = \delta_{ij} + u_i u_j,
\]
and its inverse is
\[
g^{ij} = \delta_{ij} - \frac{u_i u_j}{1 + |Du|^2}.
\]
The second fundamental form is denoted as
\[
h_{ij} = \frac{u_{ij}}{\sqrt{1 + |Du|^2}}.
\]
The principal curvatures of $\Gamma$ are the eigenvalues of the second fundamental form $[h_{ij}]$ relative to $[g_{ij}]$, i.e., the eigenvalues of the mixed tensor $[h^j_i \equiv h_{ik} g^{kj}]$. By [3] we remark that they are the eigenvalues of the symmetric matrix
\[
a_{ij} = \frac{1}{v} b^{jk} u_{kl} b^{lj}, \tag{2.1}
\]
where $v = \sqrt{1 + |Du|^2}$ and $[b^{ij}]$ is the positive square root of $[g^{ij}]$ taking the form
\[
b^{ij} = \delta_{ij} - \frac{u_i u_j}{v(1 + v)}.
\]
Explicitly we have shown
\[
a_{ij} = \frac{1}{v} \left\{ u_{ij} - \frac{u_i u_j u_{il}}{v(1 + v)} - \frac{u_j u_i u_{il}}{v(1 + v)} + \frac{u_i u_j u_k u_{kl}}{v^2(1 + v)^2} \right\}. \tag{2.2}
\]
Then by (2.1) one can deduce that
\[
u_{ij} = v b_{ik} a_{kl} b_{lj} \tag{2.3}
\]
where $b_{ij}$ is the inverse of $b^{ij}$ expressed as

$$b_{ij} = \delta_{ij} + \frac{u_i u_j}{1 + v}.$$

We immediately have the following crucial lemma outlined above.

**Lemma 2.2** Let $\kappa_1(x), \ldots, \kappa_n(x)$ be the principal curvatures of $\Gamma = \{(x, u(x)) : x \in \Omega\}$ at $x$. Suppose that $u \in C^2(\tilde{\Omega})$ is a uniformly convex solution of (1.3) and (1.4), then there exists $M_1 > 0$ and $M_2 > 0$ depending only on $\Omega$ and $\tilde{\Omega}$ such that

$$\min_{1 \leq i \leq n} \kappa_i(x) \leq M_1, \quad \max_{1 \leq i \leq n} \kappa_i(x) \geq M_2,$$

for any $x \in \tilde{\Omega}$.

**Proof** From $Du(\Omega) = \tilde{\Omega}$, we obtain

$$\int_\Omega \det D^2u(x) dx = |\tilde{\Omega}|.$$

Denote $\lambda_1(x), \ldots, \lambda_n(x)$ be the eigenvalues of $D^2u$ at $x \in \Omega$, then there exists $\bar{x} \in \tilde{\Omega}$ such that

$$\prod_{i=1}^n \lambda_i(\bar{x}) = \det D^2u(\bar{x}) = \frac{|\tilde{\Omega}|}{|\Omega|} = \theta_0.$$

By (2.1), we obtain

$$\prod_{i=1}^n \kappa_i(\bar{x}) = \det \left( \frac{1}{v} b^{ik} b^{kj} \right) \det D^2u(\bar{x}) = \det \left( \frac{1}{v} b^{ik} b^{kj} \right) \prod_{i=1}^n \lambda_i(\bar{x}).$$

According to the boundary condition (1.4), there exist positive constants $\sigma_1, \sigma_2$ depending only on $\Omega$ and $\tilde{\Omega}$, such that

$$\sigma_1 \theta_0 \leq \prod_{i=1}^n \kappa_i(\bar{x}) \leq \sigma_2 \theta_0.$$

By (1.3), for any $x \in \tilde{\Omega}$ we obtain

$$F \left( \min_{1 \leq i \leq n} \kappa_i(x), \ldots, \min_{1 \leq i \leq n} \kappa_i(x) \right) \leq F (\kappa_1(x), \ldots, \kappa_n(x))$$

$$= F (\kappa_1(\bar{x}), \ldots, \kappa_n(\bar{x}))$$

$$\leq \arctan \left( \min_{1 \leq i \leq n} \kappa_i(\bar{x}) \right) + \frac{(n - 1)\pi}{2},$$

and

$$F \left( \max_{1 \leq i \leq n} \kappa_i(x), \ldots, \max_{1 \leq i \leq n} \kappa_i(x) \right) \geq F (\kappa_1(x), \ldots, \kappa_n(x))$$

$$= F (\kappa_1(\bar{x}), \ldots, \kappa_n(\bar{x}))$$

$$\geq \arctan \left( \max_{1 \leq i \leq n} \kappa_i(\bar{x}) \right).$$
Hence
\[
\arctan \left( \min_{1 \leq i \leq n} \kappa_i(x) \right) \leq \frac{\arctan[\sigma_2 \theta_0] \frac{1}{2} + \frac{(n-1) \pi}{2}}{n} < \frac{\pi}{2},
\]
and
\[
\arctan \left( \max_{1 \leq i \leq n} \kappa_i(x) \right) \geq \frac{\arctan[\sigma_1 \theta_0] \frac{1}{n}}{n} > 0.
\]
Therefore, we get the desired results. \(\square\)

By Lemma 2.2, the points \((\kappa_1, \ldots, \kappa_n)\) are always in \(\Gamma_{M_1, M_2}^+\) under the problem (1.3) and (1.4). Then we have the following corollary.

**Corollary 2.1** Suppose that \(u \in C^2(\overline{\Omega})\) is a uniformly convex solution of (1.3) and (1.4), then there exist \(\Lambda_1 > 0\) and \(\Lambda_2 > 0\) depending only on \(\Omega\) and \(\Omega\), such that \(F\) satisfies the structural conditions
\[
\Lambda_1 \leq \sum_{i=1}^{n} \frac{\partial F}{\partial \kappa_i} \leq \Lambda_2,
\]
and
\[
\Lambda_1 \leq \sum_{i=1}^{n} \frac{\partial F}{\partial \kappa_i} \kappa_i^2 \leq \Lambda_2.
\]

Denote \(A = [a_{ij}]\) and \(F[A] = \sum_{i=1}^{n} \arctan \kappa_i\), where \(\kappa_1, \ldots, \kappa_n\) are the eigenvalues of the symmetric matrix \([a_{ij}]\). Then the properties of the operator \(F\) are reflected in Lemma 2.1. It follows from Lemma 2.1 (i) that we can show that
\[
F_{ij}[A] \xi_i \xi_j > 0, \quad \text{for all } \xi \in \mathbb{R}^n - \{0\},
\]
where
\[
F_{ij}[A] = \frac{\partial F[A]}{\partial a_{ij}}.
\]
From [19] we see that \([F_{ij}]\) diagonal if \(A\) is diagonal, and in this case
\[
[F_{ij}] = \text{diag} \left( \frac{\partial F}{\partial \kappa_1}, \ldots, \frac{\partial F}{\partial \kappa_n} \right).
\]
If \(u\) is convex, by (2.1) we deduce that the eigenvalues of the matrix \([a_{ij}]\) must be in \(\Gamma_n^+\). Then Lemma 2.1 (i) implies that
\[
F_{ij,kl}[A] \eta_i \eta_j \eta_k \eta_l \leq 0,
\]
for any real symmetric matrix \([\eta_{ij}]\), where
\[
F_{ij,kl}[A] = \frac{\partial^2 F[A]}{\partial a_{ij} \partial a_{kl}}.
\]
According to Eqs. (1.3) and (2.2), we consider the fully nonlinear elliptic differential equation of the type
\[
G(Du, D^2u) = F[A] = c. \quad (2.4)
\]
As in [19], differentiating this once we have
\[ G_{ij} u_{ijk} + G_i u_{ik} = 0, \]  
(2.5)
where we use the notation
\[ G_{ij} = \frac{\partial G}{\partial r_{ij}}, \quad G_i = \frac{\partial G}{\partial p_i}, \]
with \( r \) and \( p \) representing for the second derivative and gradient variables respectively. In order to obtain relevant estimates for the problem (1.3)–(1.4), we need to recall some expressions for the derivatives of \( G \) from the calculations already done in [19]. In fact, there holds
\[ G_{ij} = F_{kl} \frac{\partial a_{kl}}{\partial r_{ij}} = \frac{1}{v} b^{ik} F_{kl} b^{lj}, \]  
(2.6)
and
\[ G_i = F_{kl} \frac{\partial a_{kl}}{\partial p_i} = F_{kl} \frac{\partial}{\partial p_i} \left( \frac{1}{v} b^{kp} a^{ql} \right) u_{pq}. \]
A simple calculation yields
\[ G_i = -\frac{u_i}{v^2} F_{kl} a_{kl} - \frac{2}{v} F_{kl} a_{lm} b^{ik} u_m. \]

The explicit expression for \( T_G = \sum_{i=1}^n G_{ii} \) is the trace of a product of three matrices by (2.6), so it is invariant under orthogonal transformations. Hence, to compute \( T_G \), we may assume for now that \( [a_{ij}] \) is diagonal. With the aid of (1.4), we obtain that \( Du \) is bounded and the eigenvalues of \( [b^{ij}] \) are bounded between two controlled positive constants. Since (2.6) as Lemma 2.2, it follows that there exist positive constants \( \sigma_1, \sigma_2 \) depending only on \( \Omega_1 \) and \( \tilde{\Omega}_1 \), such that
\[ \sigma_1 T \leq T_G \leq \sigma_2 T, \]
where \( T = \sum_{i=1}^n F_{ii} \).

By making use of Lemma 2.2, (2.3) and (2.6), there exist two positive constants \( \sigma_3, \sigma_4 \) depending only on \( \Omega \) and \( \tilde{\Omega} \), such that
\[ \sigma_3 \sum_{i=1}^n \frac{\partial F}{\partial \kappa_i} \lambda_i^2 \leq \sum_{i=1}^n \frac{\partial G}{\partial \lambda_i} \lambda_i^2 \leq \sigma_4 \sum_{i=1}^n \frac{\partial F}{\partial \kappa_i} \lambda_i^2, \]
where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of Hessian matrix \( D^2 u \) at \( x \in \Omega \). Therefore, by Corollary 2.1 we have
\[ \sigma_3 \Lambda_1 \leq \sum_{i=1}^n \frac{\partial G}{\partial \lambda_i} \lambda_i^2 \leq \sigma_4 \Lambda_2. \]  
(2.7)

The concavity of \( F \) and the positive definiteness of \( [F_{ij} a_{ij}] \) imply that \( [F_{ij} a_{ij}] \) is bounded, i.e.,
\[ 0 < F_{ij} a_{ij} = \sum_{i=1}^n \frac{\partial F}{\partial \kappa_i} \kappa_i \leq F(\kappa_1, \ldots, \kappa_n) \leq \frac{n\pi}{2}. \]  
(2.8)
Thus \( G_i \) is bounded.

In order to get the strict obliqueness estimate, we will use the Legendre transformation of \( u \) which is the convex function \( u^* \) on \( \tilde{\Omega} = Du(\Omega) \) defined by
\[ u^*(y) = x \cdot Du(x) - u(x), \]
and

\[ y = Du(x). \]

It follows that

\[ \frac{\partial u^*}{\partial y_i} = x_i, \quad \frac{\partial^2 u^*}{\partial y_i \partial y_j} = u^{ij}(x), \]

where \([u^{ij}] = [D^2 u]^{-1}\). In view of (1.4) and (2.4), the function \(u^*\) in \(\tilde{\Omega}\) satisfies that

\[ G^*(y, D^2 u^*) \triangleq -G(y, [D^2 u^*]^{-1}) + \frac{n\pi}{2} = \frac{n\pi}{2} - c, \quad (2.9) \]

and

\[ Du^*(\tilde{\Omega}) = \Omega. \quad (2.10) \]

Hence, we rewrite Eq. (2.9) as

\[ F^*[a^*_ij] \triangleq F^*(\mu_1, \ldots, \mu_n) = -F(\mu_1^{-1}, \ldots, \mu_n^{-1}) + \frac{n\pi}{2} = \frac{n\pi}{2} - c. \quad (2.11) \]

where \(\mu_1, \ldots, \mu_n\) are the eigenvalues of the matrix \([a^*_ij]\) given by

\[ a^*_ij = \sqrt{1 + |y|^2} b^*_i k b^*_j l, \]

where

\[ b^*_i j = \delta_i j + \frac{y_i y_j}{1 + \sqrt{1 + |y|^2}}. \]

The inverse matrix \([b^{*ij}]\) of \([b^*_ij]\) is given by

\[ b^{*ij} = \delta_i j - \frac{y_i y_j}{\sqrt{1 + |y|^2}(1 + \sqrt{1 + |y|^2})}. \]

Thus,

\[ G^*(y, D^2 u^*) = F^*[a^*_ij] = \frac{n\pi}{2} - c. \quad (2.12) \]

The Legendre transformation of \(u\) which is the convex function \(u^*\) on \(\tilde{\Omega} = Du(\Omega)\) satisfies the Eq. (2.11). The points \((\mu_1, \ldots, \mu_n)\) are also always in \(\Gamma^{+}_{M_1, M_2}\) by \([a^*_ij] = [a^*ij]^{-1}\). Due to (1.3) and (2.11), it can be shown easily that \(F^* = F\). Then \(F^*\) also satisfies the structural conditions of Lemma 2.1 (iii). In the following we need to examine the bound properties of the coefficients for the linearized operator according to (2.9). On account of \(y \in \tilde{\Omega}\), an argument similar to that used above show that there exist two controlled positive constants \(\sigma_5, \sigma_6\), such that

\[ \sigma_5 T^* \leq T^*_G \leq \sigma_6 T^*, \]

where \(T^* = \sum_{i=1}^n F^*_ii\) and \(T^*_G = \sum_{i=1}^n G^*_ii\). Taking derivatives to (2.9), we conclude that

\[ G^*_ij u^*_ijkl + G^*_yk = 0, \quad (2.13) \]

where

\[ G^*_ij = \sqrt{1 + |y|^2} b^*_i k F^*_kl b^*_lj, \]

and

\[ G^*_yk = \frac{\partial G^*}{\partial y_k} = F^*_ii \frac{\partial}{\partial y_k} \left( \sqrt{1 + |y|^2} b^*_i p b^*_ql \right) u^*_pq \]

\[ = \frac{y_k}{1 + |y|^2} F^*_ii a^*_ij + 2F^*_ii a^*_im (Dk b^*_ip) b^*_pm. \]

By making use of \(F^* = F\), as same as (2.8), we can show that \(G^*_yk\) is also bounded.
3 The strict obliqueness estimate

In order to utilize the continuity method we first need to carry out the strict obliqueness estimate. To do this, it is convenient to restate the boundary condition (1.4). That is, given the bound uniformly convex domain $\Omega$ with smooth boundary in $\mathbb{R}^n$, there exists so-called boundary defining function as follows and the construction process can be seen in [17].

**Definition 3.1** A smooth function $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is called the defining function of $\Omega$ if

$$\Omega = \{ p \in \mathbb{R}^n : h(p) > 0 \}, \quad |Dh|_{\partial\Omega} = 1,$$

and there exists $\theta > 0$ such that for any $p = (p_1, \ldots, p_n) \in \Omega$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$,

$$\frac{\partial^2 h}{\partial p_i \partial p_j} \xi_i \xi_j \leq -\theta |\xi|^2.$$

Therefore, the diffeomorphism condition $Du(\Omega) = \tilde{\Omega}$ in (1.4) is equivalent to

$$h(Du) = 0, \quad x \in \partial \Omega.$$

(3.1)

Then (1.3) and (1.4) can be rewritten as

$$\begin{cases}
  F[A] = c, & x \in \Omega, \\
  h(Du) = 0, & x \in \partial \Omega,
\end{cases}$$

(3.2)

where $A$ is denoted by (2.2). This is an oblique boundary value problem of second order fully nonlinear elliptic equation. We also denote $\beta = (\beta^1, \ldots, \beta^n)$ with $\beta^i := h p_i(Du)$, and $\nu = (\nu_1, \ldots, \nu_n)$ as the unit inward normal vector at $x \in \partial \Omega$. The expression of the inner product is

$$\langle \beta, \nu \rangle = \beta^i \nu_i.$$

**Remark 3.2** In the following estimates, we have used the facts that $\Omega$ and $\tilde{\Omega}$ are uniformly convex domains with smooth boundary.

**Lemma 3.3** (See J. Urbas [17] ) Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ be the unit inward normal vector of $\partial \Omega$. If $u \in C^2(\bar{\Omega})$ with $D^2 u \geq 0$, then there holds $h p_k(Du) \nu_k \geq 0$.

**Lemma 3.4** Assume that $[A_{ij}]$ is semi-positive real symmetric matrix and $[B_{ij}]$, $[C_{ij}]$ are two real symmetric matrices. Then

$$2A_{ij} B_{jk} C_{ki} \leq A_{ij} B_{ik} B_{jk} + A_{ij} C_{ik} C_{jk}.$$

**Proof** Denote $A = [A_{ij}]$, $B = [B_{ij}]$, $C = [C_{ij}]$. We see that $A(B - C)(B - C)$ is a semi-positive real symmetric matrix. Then

$$\text{Trace}(A(B - C)(B - C)) \geq 0.$$

Then we obtain

$$\text{Trace}(ABC + ACB) \leq \text{Trace}(ABB + ACC).$$

By direct computations, one can deduce that

$$A_{ij} B_{jk} C_{ki} + A_{ij} C_{jk} B_{ki} \leq A_{ij} B_{ik} B_{jk} + A_{ij} C_{ik} C_{jk},$$

by making use of $B = B^T$, $C = C^T$. It’s easy to see that $A_{ij} B_{jk} C_{ki} = A_{ij} C_{jk} B_{ki}$, then we obtain the desired results. ⊓⊔

$\text{Springer}$
Now, we can present

**Lemma 3.5** If \( u \) is a uniformly smooth convex solution of (3.2), then the strict obliqueness estimate

\[
\langle \beta, v \rangle \geq \frac{1}{C_1} > 0,
\]

holds on \( \partial \Omega \) for some universal constant \( C_1 \), which depends only on \( \Omega \) and \( \tilde{\Omega} \).

**Remark 3.6** Without loss of generality, in the following we set \( C_1, C_2 \cdots \) to be constants depending only on \( \Omega \) and \( \tilde{\Omega} \).

**Proof** The proof of this is similar in the parts to the proof of [17], but it is different that elliptic operators and barrier functions are given in this paper. Define

\[
\omega = \langle \beta, v \rangle + \tau h(Du),
\]

where \( \tau \) is a positive constant to be determined. Let \( x_0 \in \partial \Omega \) such that

\[
\langle \beta, v \rangle(x_0) = h_p(Du(x_0))v_k(x_0) = \min_{\partial \Omega} \langle \beta, v \rangle.
\]

By rotation, we may assume that \( v(x_0) = (0, \ldots, 0, 1) \). Applying the above assumptions and the boundary condition (3.1), we find that

\[
\omega(x_0) = \min_{\partial \Omega} \omega = h_{p_n}(Du(x_0)).
\]

By the smoothness of \( \Omega \) and its uniform convexity, we extend \( v \) smoothly to a tubular neighborhood of \( \partial \Omega \) such that in the matrix sense

\[
(v_{kl}) := (D_k v_l) \leq -\frac{1}{C_2} \text{diag}(1, \ldots, 1, 0), \tag{3.4}
\]

where \( C_2 \) is a positive constant. By Lemma 3.3, we see that \( h_{p_n}(Du(x_0)) \geq 0 \).

At the minimum point \( x_0 \) one may show that

\[
0 = \omega_r = h_{p_n} p_k u_{kr} + h_{p_k} v_{kr} + \tau h_{p_k} u_{kr}, \quad 1 \leq r \leq n - 1. \tag{3.5}
\]

We assume that the following key result

\[
\omega_n(x_0) > -C_3, \tag{3.6}
\]

holds which will be proved later, where \( C_3 \) is a positive constant depending only on \( \Omega \) and \( \tilde{\Omega} \). We observe that (3.6) can be rewritten as

\[
h_{p_n} p_k u_{kn} + h_{p_k} v_{kn} + \tau h_{p_k} u_{kn} > -C_3. \tag{3.7}
\]

Multiplying \( h_{p_n} \) on both sides of (3.7) and \( h_{p_k} \) on both sides of (3.5) respectively, and summing up together, one gets

\[
\tau h_{p_k} h_{p_l} u_{kl} \geq -C_3 h_{p_n} - h_{p_k} h_{p_l} v_{kl} - h_{p_k} h_{p_n} p_l u_{kl}.
\]

Combining (3.4) and

\[
1 \leq r \leq n - 1, \quad h_{p_k} u_{kr} = \frac{\partial h(Du)}{\partial x_r} = 0, \quad h_{p_k} u_{kn} = \frac{\partial h(Du)}{\partial x_n} \geq 0, \quad -h_{p_n p_n} \geq 0,
\]

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we have
\[ \tau h_{pk} h_{pl} u_{kl} \geq -C_3 h_{pn} + \frac{1}{C_2} |Dh|^2 - \frac{1}{C_2} h_{pn}^2 \geq -C_4 h_{pn} + \frac{1}{C_4} - \frac{1}{C_4} h_{pn}^2, \]
where \( C_4 = \max\{C_2, C_3\} \). Now, to obtain the estimate \( \langle \beta, \nu \rangle \) we divide \(-C_4 h_{pn} + \frac{1}{C_4} - \frac{1}{C_4} h_{pn}^2\) into two cases at \( x_0 \).

Case (i). If
\[ -C_4 h_{pn} + \frac{1}{C_4} - \frac{1}{C_4} h_{pn}^2 \leq \frac{1}{2}, \]
then
\[ h_{pk} (Du)_v = h_{pn} \geq \sqrt{\frac{1}{2} + \frac{C_4^4}{4} - \frac{C_4^2}{4}}. \]

It means that there is a uniform positive lower bound for \( \min_{\partial \Omega} \langle \beta, \nu \rangle \).

Case (ii). If
\[ -C_4 h_{pn} + \frac{1}{C_4} - \frac{1}{C_4} h_{pn}^2 > \frac{1}{2}, \]
then we know that there is a positive lower bound for \( h_{pk} h_{pl} u_{kl} \).

Let \( u^* \) be the Legendre transformation of \( u \), then by (2.12) \( u^* \) satisfies
\[ \begin{align*}
G^*(y, D^2 u^*) &= \frac{n\pi}{2} - c, \quad y \in \tilde{\Omega}, \\
h^*(Du^*) &= 0, \quad y \in \partial \tilde{\Omega},
\end{align*} \]
where \( h^* \) is the defining function of \( \Omega \). That is,
\[ \Omega = \{ \tilde{\rho} \in \mathbb{R}^n : h^*(\tilde{\rho}) > 0 \}, \quad |Dh^*|_{\partial \Omega} = 1, \quad D^2 h^* \leq -\tilde{\rho} I, \]
where \( \tilde{\rho} \) is a positive constant. Here we emphasize that the structural conditions \( G^* \) as same as \( G \) by (2.9) and (2.11). The unit inward normal vector of \( \partial \Omega \) can be represented as \( \nu = Dh^* \). By the same token, \( \tilde{\nu} = Dh \), where \( \tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \ldots, \tilde{\nu}_n) \) is the unit inward normal vector of \( \partial \tilde{\Omega} \).

Let \( \hat{\tilde{\rho}} = (\tilde{\rho}_1^1, \ldots, \tilde{\rho}_n^n) \) with \( \tilde{\rho}^k := h^*_{pk}(Du^*) \). Using the representation as the works of [11, 12, 14], we also define
\[ \tilde{\omega} = \langle \tilde{\beta}, \tilde{\nu} \rangle + \tilde{\tau} h^*(Du^*), \]
in which
\[ \langle \tilde{\beta}, \tilde{\nu} \rangle = \langle \beta, \nu \rangle, \]
and \( \tilde{\tau} \) is a positive constant to be determined. Denote \( y_0 = Du(x_0) \). Then
\[ \tilde{\omega}(y_0) = \omega(x_0) = \min_{\partial \hat{\Omega}} \tilde{\omega}. \]

Using the same methods, under the assumption of
\[ \tilde{\omega}_n(y_0) \geq -C_5, \]
(3.9)
we obtain the positive lower bounds of \( h_{pk}^* h_{pi}^* u_{kl}^* \) or

\[
h_{pk}(Du)v_k = h_{pk}^*(Du^*)\tilde{v}_k = h_{pn}^* \geq \sqrt{\frac{1}{2} + \frac{C_6^4}{4} - \frac{C_6^2}{2}}.
\]

On the other hand, one can easily check that

\[
h_{pk}^* h_{pi}^* u_{kl}^* = \nu_i \nu_j u_{ij}.
\]

Then by the positive lower bounds of \( h_{pk}^* h_{pi}^* u_{kl}^* \) and \( h_{pk}^* h_{pi}^* u_{kl}^* \), the desired conclusion can be obtained by

\[
\langle \beta, \nu \rangle = \sqrt{h_{pk}^* h_{pi}^* u_{ij} \nu_i \nu_j}.
\]

For details of the proof of the above formula, see [19]. It remains to prove the key estimates (3.6) and (3.9).

At first we give the proof of (3.6). By (2.5), Lemma 3.4 and the boundedness of \( G_i \), we have

\[
\begin{align*}
\mathcal{L} \omega &= G_{ij} \partial_{ij} + G_i \partial_i \omega \\
&= G_{ij} u_{il} u_{jm} (h_{pk} p_i p_m v_k + \tau h_{pi} p_m) \\
&\quad + 2G_{ij} h_{pk} p_i u_{il} v_{kj} + G_{ij} h_{pk} v_{kij} + G_i h_{pk} v_{ki} \\
&\leq (h_{pk} p_i p_m v_k + \tau h_{pi} p_m + \delta_{im}) G_{ij} u_{il} u_{jm} + C_7 T_G + C_8,
\end{align*}
\]

(3.10)

where \( \mathcal{L} = G_{ij} \partial_{ij} + G_i \partial_i \) and

\[
2G_{ij} h_{pk} p_i u_{li} v_{kj} \leq G_{ij} u_{im} u_{mj} + C_7 T_G.
\]

Since \( D^2 h \leq -\theta I \), we may choose \( \tau \) large enough depending on the known data such that

\[
(h_{pk} p_i p_m v_k + \tau h_{pi} p_m + \delta_{im}) < 0.
\]

Consequently, we deduce that

\[
\mathcal{L} \omega \leq C_9 T_G \text{ in } \Omega.
\]

(3.11)

by the convexity of \( u \).

Denote a neighborhood of \( x_0 \) in \( \Omega \) by

\[
\Omega_r := \Omega \cap B_r(x_0),
\]

where \( r \) is a positive constant such that \( u \) is well defined in \( \Omega_r \). In order to obtain the desired results, it suffices to consider the auxiliary function

\[
\Phi(x) = \omega(x) - \omega(x_0) + \sigma \ln(1 + kh^*(x)) + A|x - x_0|^2,
\]
where \( \sigma, k \) and \( A \) are positive constants to be determined. By \( h^* \) being the defining function of \( \Omega \) and \( G_i \) being bounded one can show that

\[
\mathcal{L}(\ln(1 + kh^*)) = G_{ij} \left( \frac{kh^*_i}{1 + kh^*} - \frac{kh^*_j}{1 + kh^*} \right) + G_i \frac{kh^*_i}{1 + kh^*}
\]

\[
\leq G_{ij} \frac{kh^*_i}{1 + kh^*} - G_{ij} \eta_i \eta_j + G_i \eta_i
\]

\[
\leq \left( -\frac{k\tilde{\theta}}{1 + kh^*} + C_{10} - C_{11} |\eta - C_{12} I|^2 \right) T_G
\]

\[
\leq \left( -\frac{k\tilde{\theta}}{1 + kh^*} + C_{10} \right) T_G,
\]

where \( \eta = \left( \frac{kh^*_1}{1 + kh^*}, \frac{kh^*_2}{1 + kh^*}, \ldots, \frac{kh^*_n}{1 + kh^*} \right) \). By taking \( r \) to be small enough such that we have

\[
0 \leq h^*(x) = h^*(x) - h^*(x_0)
\]

\[
\leq \sup_{\Omega_r} |Dh^*||x - x_0|
\]

\[
\leq r \sup_{\Omega} |Dh^*| \leq \frac{\bar{\theta}}{3C_{10}}.
\]

By choosing \( k = \frac{2C_{10}}{\bar{\theta}} \) and applying (3.13) to (3.12) we obtain

\[
\mathcal{L}(\ln(1 + kh^*)) \leq -C_{10} T_G.
\]

Combining (3.11) with (3.14), a direct computation yields

\[
\mathcal{L}(\Phi(x)) \leq (C_9 - \sigma C_{10} + 2A + 2AC_{13}) T_G.
\]

On \( \partial \Omega \), it is clear that \( \Phi(x) \geq 0 \). Because \( \omega \) is bounded, then it follows that we can choose \( A \) large enough depending on the known data such that on \( \Omega \cap \partial B_r(x_0) \),

\[
\Phi(x) = \omega(x) - \omega(x_0) + \sigma \ln(1 + kh^*) + Ar^2
\]

\[
\geq \omega(x) - \omega(x_0) + Ar^2 \geq 0.
\]

Let

\[
\sigma = \frac{C_{10} + 2A + 2AC_{13}}{C_9}.
\]

Consequently,

\[
\begin{cases}
\mathcal{L}\Phi \leq 0, & x \in \Omega_r, \\
\Phi \geq 0, & x \in \partial \Omega_r.
\end{cases}
\]

According to the maximum principle, it follows that

\[
\Phi|\Omega_r \geq \min_{\partial \Omega_r} \Phi \geq 0.
\]

By the above inequality and \( \Phi(x_0) = 0 \), we have \( \partial_n \Phi(x_0) \geq 0 \), which gives the desired estimate (3.6).
Finally, we are turning to the proof of (3.9). The proof of (3.9) is similar to that of (3.6). Define
\[ \tilde{L} = G^*_{ij} \partial_{ij}. \]

From (2.13) and \( G^*_{yk} \) being bounded, we get
\[
\begin{align*}
\tilde{L} \tilde{\omega} &= G^*_{ij} u^*_li u^*_mj (h^*_{qk, qm} \tilde{v}_k + \tilde{\tau} h^*_{qk, qm}) + 2G^*_{ij} h^*_{yq, qm} u^*_li \tilde{v}_k j \\
&\quad - G^*_{yk} (h^*_{qk, qm} \tilde{v}_m + \tilde{\tau} h^*_{qk}) + G^*_{ij} h^*_{qk} \tilde{v}_{kij} \\
&\leq (h^*_{qk, qm} \tilde{v}_k + \tilde{\tau} h^*_{qk, qm} + \delta_{lm}) G^*_{ij} u^*_li u^*_jm + C_{14} T^*_G + C_{15}(1 + \tilde{\tau}),
\end{align*}
\]
where
\[
2G^*_{ij} h^*_{yq, qm} u^*_li \tilde{v}_k j \leq \delta_{lm} G^*_{ij} u^*_li u^*_jm + C_{14} T^*_G,
\]
by Lemma 3.4. Since \( D^2 h^* \leq -\bar{\theta} I \), we only need to choose \( \tilde{\tau} \) sufficiently large depending on the known data such that
\[
h^*_{qk, qm} \tilde{v}_k + \tilde{\tau} h^*_{qk, qm} + \delta_{lm} < 0.
\]

Therefore,
\[
\tilde{L} \tilde{\omega} \leq C_{16} T^*_G, \tag{3.15}
\]
by the convexity of \( u^* \).

Denote a neighborhood of \( y_0 \) in \( \tilde{\Omega} \) by
\[ \tilde{\Omega}_\rho := \tilde{\Omega} \cap B_\rho(y_0), \]
where \( \rho \) is a positive constant such that \( \tilde{v} \) is well defined in \( \tilde{\Omega}_\rho \). In order to obtain the desired results, we consider the auxiliary function
\[ \Psi(y) = \tilde{\omega}(y) - \tilde{\omega}(y_0) + \tilde{k} h(y) + \tilde{A} |y - y_0|^2, \]
where \( \tilde{k} \) and \( \tilde{A} \) are positive constants to be determined. It is easy to check that \( \Psi(y) \geq 0 \) on \( \partial \tilde{\Omega} \). Now that \( \tilde{\omega} \) is bounded, it follows that we can choose \( \tilde{A} \) large enough depending on the known data such that on \( \tilde{\Omega} \cap \partial B_\rho(y_0) \),
\[ \Psi(y) = \tilde{\omega}(y) - \tilde{\omega}(y_0) + \tilde{k} h(y) + \tilde{A} \rho^2 \geq \tilde{\omega}(y) - \tilde{\omega}(y_0) + \tilde{A} \rho^2 \geq 0. \]

It follows from \( D^2 h \leq -\theta I \) that
\[ \tilde{L}(\tilde{k} h(y) + \tilde{A} |y - y_0|^2) \leq (-\tilde{k} \theta + 2\tilde{A}) T^*_G. \]
Then by (3.15) and choosing \( \tilde{k} = \frac{2\tilde{A} + C_{14}}{\theta} \) we obtain
\[ \tilde{L} \Psi(y) \leq 0. \]

Consequently,
\[
\begin{cases}
\tilde{L} \Psi \leq 0, & y \in \tilde{\Omega}_\rho, \\
\Psi \geq 0, & y \in \partial \tilde{\Omega}_\rho.
\end{cases}
\]
The rest of the proof of (3.9) is the same as (3.6). Thus the proof of (3.3) is completed. \( \square \)

**Remark 3.7** The above detail proof involves that the estimate is independent on the upper bound of \( T_G \) and \( T^*_G \), i.e., we need not the upper bound of \( \sum_{i=1}^{n} \frac{\partial F}{\partial k_i} \) and the lower bound of \( \sum_{i=1}^{n} \frac{\partial F}{\partial k_i} k_i^2 \) in Lemma 2.1.
Remark 3.8 In the obliqueness estimation we note that these barriers are clearly constructed using the positive definiteness of \([G_{ij}]\) and \([G^{*}_{ij}]\) as well as the boundedness of \(G_{i}\) and \(G^{*}_{iy}\).

4 The second derivative estimate

Before deriving the global \(C^{2}\) estimate, we first introduce a useful definition that provides a basic connection between (3.2) and (3.8) and which will be useful for the sequel.

Definition 4.1 We say that \(u^{*}\) in (3.8) is a dual solution to (3.2).

To carry out the global \(C^{2}\) estimate, we use the following strategy that is to reduce the \(C^{2}\) global estimate of \(u\) and \(u^{*}\) to the boundary. By differentiating the boundary condition \(h(Du) = 0\) in any tangential direction \(\varsigma\), we have

\[
u_{\beta\varsigma} = h_{\nu_{\beta}k}u_{k\varsigma} = 0. \quad (4.1)
\]

However, the second order derivative of \(u\) on the boundary is controlled by \(u_{\beta\varsigma}, u_{\beta\beta}\) and \(u_{\varsigma\varsigma}\).

We now give the arguments as in [17] and one can see there for more details. At \(x \in \partial \Omega\), any unit vector \(\xi\) can be written in terms of a tangential component \(\varsigma(\xi)\) and a component in the direction \(\beta\) by

\[
\xi = \varsigma(\xi) + \frac{\langle v, \xi \rangle}{\langle \beta, v \rangle} \beta, \quad \beta^T := \beta - \langle \beta, v \rangle v.
\]

Since (3.3), we see that in fact

\[
|\varsigma(\xi)|^2 = 1 - \left(1 - \frac{|\beta^T|^2}{\langle \beta, v \rangle^2}\right)\langle v, \xi \rangle^2 - 2\langle v, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, v \rangle} \leq 1 + C_{17} v_{\varsigma}^2 - 2\langle v, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, v \rangle} \leq C_{18}. \quad (4.2)
\]

Denote \(\varsigma := \frac{\varsigma(\xi)}{|\varsigma(\xi)|}\), then we combine (4.1), (4.2) and (3.3) to obtain

\[
u_{\xi\xi} = |\varsigma(\xi)|^2 u_{\varsigma\varsigma} + 2|\varsigma(\xi)| \frac{\langle v, \xi \rangle}{\langle \beta, v \rangle} u_{\beta\varsigma} + \frac{\langle v, \xi \rangle^2}{\langle \beta, v \rangle^2} u_{\beta\beta} = |\varsigma(\xi)|^2 u_{\varsigma\varsigma} + \frac{\langle v, \xi \rangle^2}{\langle \beta, v \rangle^2} u_{\beta\beta} \leq C_{19} (u_{\varsigma\varsigma} + u_{\beta\beta}), \quad (4.3)
\]

where \(C_{19}\) depends only on \(\Omega, \tilde{\Omega}\). Therefore, we only need to estimate \(u_{\beta\beta}\) and \(u_{\varsigma\varsigma}\) respectively.

Lemma 4.2 If \(u\) is a smooth uniformly convex solution of (3.2), then there exists a positive constant \(C_{20}\) such that

\[
0 \leq \sup_{\partial \Omega} u_{\beta\beta} \leq C_{20}. \quad (4.4)
\]
Proof Let $\tilde{h} = h(Du)$ and denote $L = G_{ij} \partial_{ij} + G_i \partial_i$ as (3.10). We arrive at

\[
L \tilde{h} = G_{ij} \partial_{ij} \tilde{h} + G_i \partial_i \tilde{h} = h_{pk} G_{ij} u_{ijk} + h_{pk} G_i u_{ik} u_{ji} h_{pk} \partial_i \geq -C_{21},
\]

by (2.5) and (2.7) for some positive constant $C_{21}$ depending only on the known data. One can define the auxiliary function as follows

\[
\varpi = \sigma \ln(1 + k h^*(x)).
\]

By the computations in (3.14), there exist some constants $\sigma$ and $k$ such that

\[
L \varpi \leq L \tilde{h}, \quad \text{on } \Omega.
\]

It follows from $\varpi|_{\partial \Omega} = \tilde{h}|_{\partial \Omega} = 0$ and the maximum principle that

\[
\varpi \geq \tilde{h}, \quad \text{on } \Omega.
\]

We observe that

\[
\varpi|_{\partial \Omega} = \tilde{h}|_{\partial \Omega} = 0.
\]

Then we arrive at

\[
\varpi \geq \tilde{h} = u_{\beta \beta}, \quad \text{on } \partial \Omega.
\]

\[\square\]

Lemma 4.3 If $u$ is a smooth uniformly convex solution of (3.2), then we have

\[
\sup_{\partial \Omega} |D^2 u| \leq C_{22} \left( 1 + \sup_{x \in \partial \Omega, \xi \in T_{\Omega}, |\xi| = 1} u_{\xi \xi} \right).
\]

Proof By virtue of (4.3) and (4.4) we see that for any direction $\xi$, on $\partial \Omega,$

\[
u_{\xi}^{\xi} = |\xi(\xi)|^2 u_{\xi \xi} + 2 |\xi(\xi)|{\langle v, \xi \rangle \over \langle \beta, v \rangle} u_{\beta \xi} + {\langle v, \xi \rangle^2 \over \langle \beta, v \rangle^2} u_{\beta \beta}
\]

\[
\leq C_{18} u_{\xi \xi} + C_{23}.
\]

Hence the desired result follows immediately.

\[\square\]

And secondly, estimating the remaining second derivative $u_{\xi \xi}$ of the function $u$ is somewhat complicated. This estimate can be obtained by performing some computations as same as [19] on the graph $\Gamma$ using the local orthonormal frame fields.

In a neighbourhood of any point of $\Gamma$, there exists a local orthonormal frame field $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$ on $\Gamma$. We denote covariant differentiation on $\Gamma$ in the direction $\hat{e}_i$ by $\nabla_{\hat{e}_i}$. Let

\[
\mu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}},
\]

denotes the upwards pointing unit normal vector field to $\Gamma$ and $[h_{ij}]$ denotes the second fundamental form of $\Gamma$, so that for the frame $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$ we have

\[
h_{ij} = \langle D_{\hat{e}_i} \hat{e}_j, \mu \rangle,
\]

\[\square\] Springer
where $D$ denotes the usual gradient operator on $\mathbb{R}^{n+1}$.

In order to control the global second derivative by its value on the boundary we will need
the following differential inequalities for the mean curvature.

**Lemma 4.4** If $u$ is a smooth uniformly convex solution of (1.3), then we have

$$F_{ij} \nabla_i \nabla_j H \geq 0,$$

where $H = \sum_{i=1}^n h_{ii}$ is the mean curvature of $\Gamma$.

**Proof** First, we need to deal with $F_{ij} \nabla_i \nabla_j h_{ll}$ in the similar way as in [18]. Applying $\Delta = \nabla_i \nabla_i$ to $F[A] = c$, we have

$$F_{ij,rs} \nabla_i h_{rs} \nabla_i h_{ij} + F_{ij} \nabla_i h_{ij} = 0.$$  

The next procedure in the proof is to calculate $\nabla_i \nabla_i h_{ij}$ by using the Codazzi equations which tell us that $\nabla_i h_{ij}$ is symmetric in all indices, together with the standard formula for handling covariant derivatives, and Gauss equations

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk},$$

where $R_{ijkl}$ denotes the curvature tensor on $\Gamma$. Direct computations yield

$$\nabla_i \nabla_i h_{ij} = \nabla_i \nabla_i h_{ji}$$
$$= \nabla_i \nabla_i h_{ji} + R_{lijm} h_{ml} + R_{lijm} h_{mj}$$
$$= \nabla_i \nabla_i h_{ll} + h_{ij} h_{im} h_{ml} - h_{im} h_{ij} h_{ml} + h_{ll} h_{im} h_{mj} - h_{im} h_{ij} h_{mj}$$
$$= \nabla_i \nabla_i h_{ll} + h_{ll} h_{im} h_{mj} - h_{im} h_{ij} h_{ml},$$

so that, we obtain

$$F_{ij} \nabla_i \nabla_j h_{ll} = -F_{ij,rs} \nabla_i h_{rs} \nabla_j h_{ij} - F_{ij} h_{im} h_{mj} h_{ll} + F_{ij} h_{ij} h_{im} h_{lm}.$$  

Thus summing over $l = 1, \ldots, n$ in (4.6) we have

$$F_{ij} \nabla_i \nabla_j H = -F_{ij,rs} \nabla_i h_{rs} \nabla_j h_{ij} - F_{ij} h_{im} h_{mj} H + F_{ij} h_{ij} h_{im} h_{lm}.$$  

The first term on the right hand side of (4.7) is non-negative due to the concavity of $F$. Therefore, one only need to consider

$$P := F_{ij} h_{ij} h_{lm} h_{lm} - H F_{ij} h_{im} h_{mj},$$

being non-negative everywhere.

Without loss of generality, we may assume that $\hat{e}_1, \ldots, \hat{e}_n$ have been chosen such that $[h_{ij}]$ is diagonal at the point at which we are computing with eigenvalues $\kappa_1, \ldots, \kappa_n$. Hence
we finally conclude that
\[
P = \left( \sum_{i=1}^{n} \frac{\partial F}{\partial \kappa_i} \kappa_i \right) \left( \sum_{j=1}^{n} \kappa_j^2 \right) - \left( \sum_{j=1}^{n} \frac{\partial F}{\partial \kappa_j} \kappa_j^2 \right).
\]
\[
= \sum_{i,j=1}^{n} \left( \frac{\partial F}{\partial \kappa_i} \kappa_i \kappa_j^2 - \frac{\partial F}{\partial \kappa_j} \kappa_j \kappa_i^2 \right)
\]
\[
= \frac{1}{2} \sum_{i,j=1}^{n} \left( \frac{1}{1 + \kappa_j^2} - \frac{1}{1 + \kappa_i^2} \right) (\kappa_i - \kappa_j) \kappa_i \kappa_j
\]
\[
= \frac{1}{2} \sum_{i,j=1}^{n} \frac{1}{1 + \kappa_j^2} \frac{1}{1 + \kappa_i^2} (\kappa_j + \kappa_j) (\kappa_i - \kappa_j) \kappa_i \kappa_j
\]
\[
\geq 0,
\]
because the last estimate is positive by the convexity of \( \Gamma \). The desired conclusion follow from the above. \( \square \)

**Lemma 4.5** If \( u \) is a smooth uniformly convex solution of (1.3), then we have
\[
\sup_{\Gamma'} H \leq \sup_{\partial \Gamma} H. \tag{4.8}
\]

**Proof** Using (4.5) and the maximum principle, the estimate (4.8) follows immediately. \( \square \)

**Remark 4.6** Since the processes of proving Lemmas 4.4 and 4.5 use only the concavity of \( F \), these conclusions are also valid for concave operators in general.

In order to control the global second derivatives of \( u \) on the boundary, we adopt the following notation. For any unit vector \( \xi \in \mathbb{R}^n \), one can define
\[
\tilde{\xi} = \frac{(\xi, D\xi u)}{\sqrt{1 + |D\xi u|^2}}. \tag{4.9}
\]
then \( \tilde{\xi} \) is a unit tangent vector to \( \Gamma \). So that (4.9) gives a one to one correspondence from \( \mathbb{R}^n \) to the tangent space \( T_x \Gamma \) at \( (x, u(x)) \). One can recall the explicit expressions of the normal curvature \( k(\tilde{\xi}) \) of \( \Gamma \) in the direction \( \tilde{\xi} \) as follows
\[
k(\tilde{\xi}) = \frac{D\xi \xi u}{\sqrt{1 + |Du|^2(1 + |D\xi u|^2)}}. \tag{4.10}
\]
For the above details, we refer to the computations of [16], Sect.2.

The following equation will be useful later in this section.

**Lemma 4.7** If \( u \) is a smooth uniformly convex solution of (1.3), then for any direction \( \zeta \) in \( \mathbb{R}^{n+1} \), we have
\[
F_{ij \nabla_i \nabla_j \langle \mu, \zeta \rangle} + F_{ij} h_{ik} h_{jk} \langle \mu, \zeta \rangle = 0, \tag{4.11}
\]
where
\[
\mu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}.
\]
Proof By similar proof as [18], we may assume that the vector fields \( \hat{e}_1, \ldots, \hat{e}_n \) and \( \mu \) have been extended in a \( C^2 \) fashion so they form an orthonormal frame in a neighbourhood of the point in \( \mathbb{R}^{n+1} \) at which we are computing. Since \( \mu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}} \), a direct calculation yields

\[
\langle D_{\hat{e}_j} \mu, \mu \rangle = 0,
\]
and then we arrive at

\[
D_{\hat{e}_j} \langle \mu, \zeta \rangle = \langle D_{\hat{e}_j} \mu, \zeta \rangle + \langle D_{\hat{e}_j} \zeta, \mu \rangle = -h_{jk} \langle \hat{e}_k, \mu \rangle \tag{4.12}
\]

by \( h_{jk} = \langle D_{\hat{e}_j} \hat{e}_k, \mu \rangle \). We now calculate \( \nabla_i \nabla_j \langle \mu, \zeta \rangle \). First, we recall the formula

\[
\nabla_j f = D_{\hat{e}_j} f, \quad \text{for } f \in C^\infty,
\]
and

\[
\nabla_i U_j = D_{\hat{e}_i} U_j - \Gamma^k_{ij} U_k, \quad \text{for } U_j = \langle U, \hat{e}_j \rangle,
\]
where \( U \) is a vector field and \( \Gamma^k_{ij} \) are the coefficients of connection on \( \Gamma \). Then

\[
\nabla_i \nabla_j \langle \mu, \zeta \rangle = \nabla_i D_{\hat{e}_j} \langle \mu, \zeta \rangle = D_{\hat{e}_i} D_{\hat{e}_j} \langle \mu, \zeta \rangle - \Gamma^k_{ij} D_{\hat{e}_k} \langle \mu, \zeta \rangle,
\]
where \( \Gamma^k_{ij} = \langle D_{\hat{e}_i} \hat{e}_j, \hat{e}_k \rangle \). Hence

\[
\nabla_i \nabla_j \langle \mu, \zeta \rangle = -D_{\hat{e}_i} [h_{jk} \langle \hat{e}_k, \zeta \rangle] + \Gamma^k_{ij} h_{kl} \langle \hat{e}_l, \zeta \rangle
= -D_{\hat{e}_i} h_{jk} \langle \hat{e}_k, \zeta \rangle - h_{jk} \langle D_{\hat{e}_i} \hat{e}_k, \mu \rangle \tag{4.13}
- h_{jk} \langle D_{\hat{e}_i} \hat{e}_k, \zeta \rangle + \Gamma^k_{ij} h_{kl} \langle \hat{e}_l, \zeta \rangle
= -h_{jk} \langle D_{\hat{e}_i} \hat{e}_k, \zeta \rangle - \Gamma^l_{ik} h_{jk} \langle \hat{e}_l, \zeta \rangle - h_{jk} h_{ik} \langle \mu, \zeta \rangle + \Gamma^k_{ij} h_{kl} \langle \hat{e}_l, \zeta \rangle
= -\nabla_i h_{jk} \langle \hat{e}_l, \zeta \rangle - h_{jk} h_{ik} \langle \mu, \zeta \rangle
= -\nabla_i h_{jk} \langle \hat{e}_l, \zeta \rangle - h_{jk} h_{ik} \langle \mu, \zeta \rangle.
\]
In the last four lines of the above, we used the fact that \( \Gamma^k_{ik} = -\Gamma^k_{il} \), the standard formula

\[
\nabla_i h_{jl} = D_{\hat{e}_i} h_{jl} - \Gamma^k_{il} h_{jk} - \Gamma^k_{ij} h_{kl},
\]
and the Codazzi equations. Furthermore, since \( F[A] = c \), then \( F_{ij} \nabla_i h_{ij} = 0 \), i.e., \( F_{ij} \nabla_i \langle \hat{e}_i, \zeta \rangle = 0 \). By substituting (4.13) into the equation we obtain the desired result. □

The following pre-knowledge was learned from [19] for calculating the geometric quantity which satisfies the useful differential inequality on manifold. By (4.10) we can make an assumption of

\[
k(\xi) \mu_{n+1} = \sup_{x \in \partial \Omega, \xi \in T_x \partial \Omega, |\xi| = 1} \frac{D_{\xi} u}{(1 + |D_{\xi} u|^2)}.
\]
restricted to directions \( \xi \) which are tangential to \( \partial \Omega \) at \( x \). Without loss of generality, we may take \( x \) to be the origin and it is often convenient to choose coordinate systems such that \( \xi = e_1 \triangleq (1, 0, \ldots, 0) \). Then we have

\[
\tilde{e}_1 = \frac{(e_1, D_1 u)}{\sqrt{1 + |D_1 u|^2}}.
\]
Let $\varsigma$ be a smooth unit tangent vector field on $\partial \Omega \cap B_{\rho}(0)$ for some $\rho > 0$ such that $\varsigma(0) = e_1$ and $\varsigma$ can be smoothly extended to $\Omega \cap \overline{B_{\rho}(0)}$ with $|\varsigma(x)| \equiv 1$ for any $x \in \Omega \cap \overline{B_{\rho}(0)}$ where $\rho$ depends only on $\Omega$.

One can lift $\varsigma$ to a vector field on $\Gamma_{1}$ by (4.10) and then we obtain a unit tangent vector field $\tilde{\varsigma}$ on $\Gamma_{1} \cap (B_{\rho}(0) \times \mathbb{R})$ such that $\tilde{\varsigma}$ is tangential to $\partial \Gamma_{1}$ on $\Gamma_{1} \cap \partial \Gamma$.

By making use of Gram-Schmidt orthogonalization to the basis $\{\tilde{\varsigma}, \tilde{e}_2, \ldots, \tilde{e}_n\}$ where $\tilde{e}_j = (e_j, D_j u) \sqrt{1 + |D_j u|^2}$ ($j = 2, 3, \ldots, n$), we get a local orthonormal frame field on $\Gamma_{1}$ denoted by $\{\hat{e}_1 \equiv \tilde{\varsigma}, \hat{e}_2, \ldots, \hat{e}_n\}$.

Under the local orthonormal frame field $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$, it induces the second fundamental form $[h_{ij}]$ according to $\Gamma_{1}$. Thus

$$k(\hat{e}_1) = h_{11}. \quad (4.14)$$

With the aid of the representation in (4.10), we get

$$\frac{h_{11}}{\mu_{n+1}} \bigg|_{\Gamma_{\rho} \cap \partial \Gamma} \leq \frac{h_{11}(0)}{\mu_{n+1}(0)}. \quad (4.15)$$

Considering $W \triangleq \frac{h_{11}}{\mu_{n+1}}$, we can deduce that $W$ satisfies the following graceful differential inequality on $\Gamma_{\rho}$ in order to estimate $h_{11}$ on the boundary.

**Lemma 4.8** If $u$ is a smooth uniformly convex solution of (1.3), then we have

$$F_{ij} \nabla_i \nabla_j W + b_i \nabla_i W \geq 0, \quad \text{on} \quad \Gamma_{\rho},$$

where $b = (b_1, b_2, \ldots, b_n)$ is a bounded vector field.

**Proof** We directly compute

$$\nabla_j W = \frac{\mu_{n+1}\nabla_j h_{11} - h_{11} \nabla_j \mu_{n+1}}{\mu_{n+1}^2},$$

and

$$\nabla_i \nabla_j W = \frac{\nabla_i \nabla_j h_{11}}{\mu_{n+1}} - \frac{h_{11} \nabla_i \nabla_j \mu_{n+1}}{\mu_{n+1}^2} - \frac{\nabla_j h_{11} \nabla_i \mu_{n+1}}{\mu_{n+1}^2} - \frac{\nabla_i h_{11} \nabla_j \mu_{n+1}}{\mu_{n+1}^2} + 2 \frac{h_{11} \nabla_i \mu_{n+1} \nabla_j \mu_{n+1}}{\mu_{n+1}^3} - \frac{\nabla_i \mu_{n+1} \nabla_j W}{\mu_{n+1}} - \frac{\nabla_j \mu_{n+1} \nabla_i W}{\mu_{n+1}}.$$

Furthermore, from Eq. (4.11), we have

$$F_{ij} \nabla_i \nabla_j \mu_{n+1} + F_{ijk} \nabla_k \mu_{n+1} = 0. \quad (4.16)$$
Therefore, we have

\[
F_{ij} \nabla_i \nabla_j W + F_{ij} \left( \frac{\nabla_i \mu_{n+1} \nabla_j W}{\mu_{n+1}} + \frac{\nabla_j \mu_{n+1} \nabla_i W}{\mu_{n+1}} \right) = \frac{F_{ij} \nabla_i h_{11}}{\mu_{n+1}} - \frac{F_{ij} h_{11} \nabla_i \mu_{n+1}}{\mu_{n+1}^2}
\]

\[
= -\frac{F_{ij,rs} \nabla_1 h_1 \nabla_1 h_{ij}}{\mu_{n+1}} + \frac{h_{11} F_{ij} h_{ik} h_{jk} \mu_{n+1}}{\mu_{n+1}^2} - \frac{F_{ij} h_{lm} h_{mj} h_{11} - F_{ij} h_{ij} h_{1m} h_{1m}}{\mu_{n+1}} \geq \frac{F_{ij} h_{ij} h_{1m} h_{1m}}{\mu_{n+1}} \geq 0,
\]

by (4.6), (4.16) and the concavity of \(F\). Next, by (4.12) we have

\[
b_i = 2F_{ij} \frac{\nabla_j \mu_{n+1}}{\mu_{n+1}} = -2F_{ij} h_{jk} \frac{\langle \hat{e}_k, e_{n+1} \rangle}{\mu_{n+1}}.
\]

The convexity of \(\Gamma\) and (2.8) imply that \([F_{ij} h_{jk}]\) is bounded. Therefore, \(b_i\) is bounded. The proof of the lemma is finished. \(\square\)

**Lemma 4.9** If \(u\) is a smooth uniformly convex solution of (3.2), then we get

\[
0 \leq \sup_{x \in \partial \Omega, \xi \in T_x \partial \Omega, |\xi| = 1} u_{\xi \xi} \leq C_{24}.
\]

**Proof** By the previous assumption, it is enough that we estimate \(u_{11}(0)\). Without loss of generality, we assume that \(W(0) \geq 1\). One can set

\[
Z = \frac{W}{W(0)},
\]

and define

\[
\mathcal{L} = F_{ij} \nabla_i \nabla_j + b_i \nabla_i.
\]

By Lemma 4.8 and using (4.15), we have

\[
\mathcal{L}Z \geq -C_{25}, \quad \text{on} \quad \Gamma_\rho,
\]

\[
|Z| \leq C_{26}, \quad \text{on} \quad \partial \Gamma_\rho,
\]

\[
Z \leq 1, \quad \text{on} \quad \Gamma_\rho \cap \partial \Gamma,
\]

\[
Z(0) = 1. \quad (4.17)
\]

We consider the barrier function

\[
\Phi(X) = 1 + \sigma \ln(1 + kh^*(x)) + A|x|^2,
\]

for \(X = (x, u(x)) \in \Gamma\). As same as the details in the proof of (3.14), there exist constants \(\sigma\), \(k\), \(A\) depending on the known data such that we get

\[
\mathcal{L} \Phi \leq -C_{25}, \quad \text{on} \quad \Gamma_\rho,
\]

\[
\Phi \geq C_{26}, \quad \text{on} \quad \partial \Gamma_\rho,
\]

\[
\Phi \geq 1, \quad \text{on} \quad \Gamma_\rho \cap \partial \Gamma,
\]

\[
\Phi(0) = 1. \quad (4.18)
\]
Combining (4.17) and (4.18), the maximum principle implies that
\[ \Phi - \Phi(0) \geq Z - Z(0), \quad \text{on} \quad \Gamma_\rho. \]
So that we obtain
\[ D_\beta Z(0) \leq D_\beta \Phi(0) \leq C_{27}. \]
Then
\[ D_\beta W(0) \leq C_{27} W(0). \quad (4.19) \]
Recalling (4.10) and (4.14) we obtain
\[ W = \frac{D_{\xi \xi} u}{1 + |D_\xi u|^2} \quad \text{on} \quad \Gamma_\rho, \]
for a smooth tangent vector field \( \xi \) on \( \partial \Omega \) with \( \xi(0) = e_1 \).
At \( X = (0, u(0)) = (0, 0) \), from (4.19) a direct computation shows that
\[ \frac{D_{11\beta} u}{1 + |D_1 u|^2} - 2 \frac{D_{11} u}{(1 + |D_1 u|^2)^2} D_1 u D_1 \beta u \leq C_{27} D_{11} u, \]
Thus, by (4.1) and the second boundary condition, we obtain
\[ D_{11\beta} u \leq C_{28} D_{11} u, \quad (4.20) \]
On the other hand, differentiating \( h(Du) = 0 \) twice in the direction \( e_1 \) at 0, we have
\[ h_{\partial_1} D_{11k} u + h_{\partial_1 p_1} D_{k1} u D_{11} u = 0. \]
Let \( M = D_{11} u(0) \). From Definition 3.1 the concavity of \( h \) yields that
\[ h_{\partial_1} D_{11k} u = -h_{\partial_1 p_1} D_{k1} u D_{11} u \geq \theta M^2. \]
Combining it with \( h_{\partial_1} D_{11k} u = D_{11\beta} u \), and using (4.20) we obtain
\[ \theta M^2 \leq C_{28} M. \]
Then we get the upper bound of \( M = D_{11} u(0) \) and thus the desired result follows. \( \square \)

**Remark 4.10** Since the procedure of proving Lemmas 4.8 and 4.9 uses the concavity and inverse concavity of \( F \) as well as the upper and lower bounds of \( \sum_{i=1}^n \frac{\partial F}{\partial \kappa_i}, \sum_{i=1}^n \frac{\partial^2 F}{\partial \kappa_i \kappa_i^2} \), i.e., operator \( F \) satisfies the structural conditions in Lemma 2.1 and Corollary 2.1, these conclusions also hold for operators that satisfy the above structural conditions in general.

**Lemma 4.11** If \( u \) is a smooth uniformly convex solution of (1.3), then we have
\[ \sup_{\Omega} |D^2 u| \leq C_{29} \sup_{\partial \Omega} |D^2 u|. \quad (4.21) \]

**Proof** According to the representation (4.10) of the normal curvature and the definition of the mean curvature, it follows the convexity of \( \Gamma \) that
\[ \frac{1}{C_{30}} D_{\xi \xi} u \leq H \leq C_{30} |D^2 u|, \]
for any direction \( \xi \) and some constant \( C_{30} \) depending only on \( \Omega \) and \( \Omega \). Thus from (4.8) we obtain
\[ \sup_{\Omega} \frac{1}{C_{30}} D_{\xi \xi} u \leq \sup_{\Gamma} H \leq \sup_{\partial \Gamma} H \leq \sup_{\partial \Omega} C_{30} |D^2 u|. \]
This completes the proof of (4.21). \( \square \)
In terms of Lemmas 4.3, 4.9, 4.11, we see that

**Lemma 4.12** If $u$ is a smooth uniformly convex solution of (1.3) and (1.4), then

$$\max_{\bar{\Omega}} |D^2 u| \leq C_{31}.$$ 

In the following, we describe the positive lower bound of $D^2 u$. For (1.3), in consider of the Legendre transformation of $u$, the function $u^*$ satisfies (2.9) and (2.10) where the structural condition $G^*$ as same as $G$ according to (2.11). Repeating the proof of Lemma 4.12, we have

**Lemma 4.13** If $u$ is a smooth uniformly convex solution of (1.3) and (1.4), then the function $u^*$ satisfies

$$\max_{\bar{\Omega}} |D^2 u^*| \leq C_{32}.$$ 

By Lemmas 4.12 and 4.13, we conclude that

**Lemma 4.14** If $u$ is a smooth uniformly convex solution of (1.3) and (1.4), then

$$\frac{1}{C_{33}} I \leq D^2 u(x) \leq C_{33} I, \quad x \in \bar{\Omega},$$

where $I$ is the $n \times n$ identity matrix.

**Remark 4.15** The proof of Lemma 4.14 also shows that if the operator $F$ satisfies the structural conditions in Lemma 2.1 and Corollary 2.1, then the solution of the equation for the second boundary value problem has the corresponding second derivative estimate.

## 5 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. There are two key lemmas when we start the proof. Through the discussion on the uniqueness in the fifth section of [1], we conclude that

**Lemma 5.1** If $u \in C^\infty(\bar{\Omega})$ are uniformly convex solutions of (1.3) and (1.4), then $u$ is unique up to a constant.

**Proof** Suppose that $(u, c)$ and $(\hat{u}, \hat{c})$ are solutions of (1.3) and (1.4). We argue by contradiction. Assume that the Lemma is false, then $u - \hat{u}$ is not a constant. Further we may assume that $\hat{c} \leq c$. Write $q = \hat{u} + s(u - \hat{u})$, then a straightforward calculation shows that

$$G(Du, D^2 u) - G(D\hat{u}, D^2 \hat{u}) = \int_0^1 \frac{d}{ds} \left[ G(sD\hat{u} + (1 - s)D\hat{u}, sD^2 u + (1 - s)D^2 \hat{u}) \right] ds$$

$$= \int_0^1 \frac{d}{ds} \left[ G(D\hat{u} + s(Du - \hat{u}), D^2 \hat{u} + s(D^2 u - D^2 \hat{u})) \right] ds$$

$$:= A^{ij}(u - \hat{u})_{ij} + B^i (u - \hat{u})_i,$$

where we denote

$$A^{ij} = \int_0^1 \nabla_{q_{ij}} G(sDu + (1 - s)D\hat{u}, sD^2 u + (1 - s)D^2 \hat{u}) ds$$
and
\[ B^i = \int_0^1 \nabla_{q_i} G(s Du + (1 - s) D\hat{u}, s D^2 u + (1 - s) D^2 \hat{u}) \, ds. \]

Clearly, \((A^{ij})\) is positive definite for all \(x \in \Omega\). Moreover, we have
\[
A^{ij} D_{ij}^2 (u - \hat{u}) + B^i \nabla_i (u - \hat{u}) = G(Du, D^2 u) - G(D\hat{u}, D^2 \hat{u}) = c - \hat{c} \geq 0
\]
for all \(x \in \Omega\). The following proof is similar as [1]. By the maximum principle, the function \(u - \hat{u}\) attains its maximum at a point \(x_0 \in \partial \Omega\). By the Hopf Lemma (see [7], Lemma 3.4) and \(u - \hat{u}\) is not a constant, there exists a real number \(\vartheta > 0\) such that
\[
\nabla u(x_0) - \nabla \hat{u}(x_0) = \vartheta \nabla h^*(x_0).
\]

Using Brendle–Warren’s Proposition 2.11 in [1], we obtain
\[
\langle \nabla u(x_0) - \nabla \hat{u}(x_0), \nabla (\nabla u(x_0)) \rangle = \vartheta \langle \nabla h^*(x_0), \nabla h(\nabla u(x_0)) \rangle > 0.
\]

On the other hand, we have
\[
\langle \nabla u(x_0) - \nabla \hat{u}(x_0), \nabla h(\nabla u(x_0)) \rangle \leq h(\nabla u(x_0)) - h(\nabla \hat{u}(x_0)) = 0
\]
since \(h\) is concave. This is a contradiction. Therefore, the function \(u - \hat{u}\) is constant and \(c = \hat{c}\). \qed

Next, by Lemma 5.2 in [5], we obtain

**Lemma 5.2** If \(u \in C^2(\bar{\Omega})\) is a uniformly convex solution of (1.3) and (1.4), then \(u \in C^\infty(\bar{\Omega})\).

By the continuity method, we now show the following

**Proof of Theorem 1.1** For each \(t \in [0, 1]\), set
\[ G^t(Du, D^2 u) = G(t Du, D^2 u), \]
then for any \(t \in [0, 1]\), one can consider the equation
\[
\begin{cases}
G^t(Du, D^2 u) = c(t), & x \in \Omega, \\
Du(\Omega) = \hat{\Omega},
\end{cases}
\]
which is equivalent to
\[
\begin{cases}
G^t(Du, D^2 u) = c(t), & x \in \Omega, \\
h(Du) = 0, & x \in \partial \Omega.
\end{cases}
\]

For any \(t \in [0, 1]\), (5.1) can be written as
\[
G^t(Du, D^2 u) = F(a^t_{ij}) = \sum_{i=1}^n \arctan \kappa_i^t = c(t),
\]
where
\[
a^t_{ij} = \frac{1}{v^t} \left\{ u_{ij} - \frac{\kappa^2 u_i u_j}{v^t(1 + v^t)} - \frac{\kappa^2 u_i u_j u_k u_k}{v^t(1 + v^t)^2} + \frac{\kappa^3 u_i u_j u_k u_l u_k u_k}{(v^t)^2(1 + v^t)^2} \right\},
\]
or
\[
a^t_{ij} = \frac{1}{v^t} b^t_{ij} u_k b^t i j k l,
\]
and \( \kappa'_1, \ldots, \kappa'_n \) are the eigenvalues of \( [a'_{ij}] \), \( v' = \sqrt{1 + t^2|Du|^2} \) as well as

\[
b'^{ij} = \delta_{ij} - \frac{t^2 u_i u_j}{v'(1 + v')}.\]

According to the relationship Eq. (5.4) of \( a'_{ij} \) and \( u_{kl} \), the points \( (\kappa'_1, \ldots, \kappa'_n) \) are always in \( \Gamma_{M_3,M_4}^+ \) under the problem (5.1). Thus there exist \( \Lambda_3 > 0 \) and \( \Lambda_4 > 0 \) depending only on \( \Omega \) and \( \tilde{\Omega} \) such that

\[
\Lambda_3 \leq \sum_{i=1}^{n} \frac{\partial F}{\partial \kappa_i} \leq \Lambda_4, \quad \Lambda_3 \leq \sum_{i=1}^{n} \frac{\partial F}{\partial \kappa_i} (\kappa'_i)^2 \leq \Lambda_4.
\]

On the other hand, in order to consider the relation between the operator \( G_t \) and the principal curvature \( \kappa_1, \ldots, \kappa_n \) of the graph \( \Gamma \), we rewrite (5.3) as

\[
F_t := F(a'_{ij}).
\]

According to (2.3) and (5.4) we can obtain

\[
a'_{ij} = \frac{v'}{v^t} b'^{ij} b_{pk} a_{kl} b_{lq} b'^t_{qj}. \quad (5.5)
\]

By the relationship Eq. (5.5) of \( a'_{ij} \) and \( a_{kl} \), it is clear that

\[
\frac{\partial F_t}{\partial a_{kl}} = \frac{\partial F_t}{\partial a'_{ij}} \frac{\partial a'_{ij}}{\partial a_{kl}} = \frac{v'}{v^t} b'^{ip} b_{pk} \frac{\partial F_t}{\partial a'_{ij}} b_{lq} b'^t_{qj}.
\]

Therefore, from the boundary condition (1.4), it follows that both \( b'^{ip} b_{pk} \) and \( b_{lq} b'^t_{qj} \) are positive definite and have upper and lower bounds, i.e., there exist \( \Lambda'_3 > 0 \) and \( \Lambda'_4 > 0 \) depending only on \( \Omega \) and \( \tilde{\Omega} \) such that

\[
\Lambda'_3 \leq \sum_{i=1}^{n} \frac{\partial F_t}{\partial \kappa_i} \leq \Lambda'_4, \quad \Lambda'_3 \leq \sum_{i=1}^{n} \frac{\partial F_t}{\partial \kappa_i} (\kappa'_i)^2 \leq \Lambda'_4.
\]

Moreover, since \( F \) is concave and inverse concave operator for \( \kappa'_1, \ldots, \kappa'_n \), \( F_t \) is concave and inverse concave operator for \( \kappa_1, \ldots, \kappa_n \). Therefore, from the Remark 4.15, it is clear that the solution of Eq. (5.1) has the corresponding second derivative estimate. Then there exists some positive constant \( C \) depending only on \( \Omega \) and \( \tilde{\Omega} \) such that the solution \( u \) of Eq. (5.1) satisfies

\[
\frac{1}{C} I \leq D^2 u(x) \leq CI, \quad x \in \tilde{\Omega}. \quad (5.6)
\]

As a result, in the following we can show the existence of the solution of the second boundary value problem using the continuity method.

By Brendle–Warren’s Theorem in [1], (5.1) is solvable if \( t = 0 \). According to Lemma 5.2, we define the closed subset

\[
\mathcal{B}_1 := \left\{ u \in C^{2,\alpha}(\tilde{\Omega}) : \int_{\Omega} u = 0 \right\}.
\]
in $C^{2,\alpha}(\tilde{\Omega})$ and
\[ \mathcal{B}_2 := C^\alpha(\tilde{\Omega}) \times C^{1,\alpha}(\partial \Omega). \]
Moreover, we define a map from $\mathcal{B}_1 \times \mathbb{R}$ to $\mathcal{B}_2$ by
\[ \mathcal{F}^t := (G^t(Du, D^2u) - c(t), h(Du)). \]
Then the linearized operator $D\mathcal{F}_t^t(\mathcal{B}_1 \times \mathbb{R})$ to $\mathcal{B}_2$ is given by
\[ D\mathcal{F}_t^t(\mathcal{B}_1 \times \mathbb{R}) = \left( \sum_{ij} G_{ij}^t(Du, D^2u) \partial_{ij} w - \sum_i G_i^t(Du, D^2u) \partial_i w - a, \right. \] \[ \left. h(Du) \partial_i w \right). \]
Following the same proof as Theorem 6.31 in [7], we obtain that $D\mathcal{F}_t^t$ is invertible for any $(u, c)$ satisfying (5.2) and $t \in [0, 1]$.

Write
\[ I := \{ t \in [0, 1] : (5.1) \text{has at least one uniformly convex solution} \}. \]
Since $0 \in I$, $I$ is not empty. We claim that $I = [0, 1]$, which is equivalent to the fact that $I$ is not only open, but also closed. It follows from Theorem 6.31 in [7] again and Theorem 17.6 in [7] that $I$ is open. So we only need to prove that $I$ is a closed subset of $[0, 1]$.

That $I$ is closed is equivalent to the fact that for any sequence $\{t_k\} \subset I$, if $\lim_{k \to \infty} t_k = t_0$, then $t_0 \in I$. For $t_k$, denote $(u_k, c(t_k))$ solving
\[ \begin{cases} G_{ik}(Du_k, D^2u_k) = c(t_k), & x \in \Omega, \\ Du_k(\Omega) = \tilde{\Omega}. \end{cases} \]
It follows from the estimates (5.6) and the proof of Lemma 5.2 in [5] that
\[ \|u_k\|_{C^{2,\alpha}(\tilde{\Omega})} \leq C, \] where $C$ is independent of $t_k$. Since
\[ |c(t)| = |G^t(Du, D^2u)| \leq n\pi, \]
by Arzela–Ascoli Theorem we know that there exists $\hat{u} \in C^{2,\alpha}(\tilde{\Omega})$, $\hat{c} \in \mathbb{R}$ and a subsequence of $\{t_k\}$, which is still denoted as $\{t_k\}$, such that letting $k \to \infty$,
\[ \begin{cases} \|u_k - \hat{u}\|_{C^2(\tilde{\Omega})} \to 0, \\ c(t_k) \to \hat{c}. \end{cases} \]
Since $(u_k, c(t_k))$ satisfies
\[ \begin{cases} G_{ik}(Du_k, D^2u_k) = c(t_k), & x \in \Omega, \\ h(Du_k) = 0, & x \in \partial \Omega. \end{cases} \]
Letting $k \to \infty$, we arrive at
\[ \begin{cases} G_{ik}(D\hat{u}, D^2\hat{u}) = \hat{c}, & x \in \Omega, \\ h(D\hat{u}) = 0, & x \in \partial \Omega. \end{cases} \]
Therefore, $t_0 \in I$, and thus $I$ is closed. Consequently, $I = [0, 1]$. By Lemma 5.1 we know that the solution of (5.1) is unique up to a constant.

Then we complete the proof of Theorem 1.1. \qed
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