Bypassing the Groenewold–van Hove Obstruction on $\mathbb{R}^{2n}$: A New Argument in Favor of Born–Jordan Quantization

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Abstract
There are known obstructions to a full quantization of $\mathbb{R}^{2n}$ in the spirit of Dirac’s approach, the most known being the Groenewold–van Hove no-go result. We show, following a suggestion of S. K. Kauffmann, that it is possible to construct a well-defined quantization procedure by weakening the usual requirement that commutators should correspond to Poisson brackets. The weaker requirement consists in demanding that this correspondence should only hold for Hamiltonian functions of the type $T(p) + V(q)$. This reformulation leads to a non-injective quantization of all observables $H \in \mathcal{S}'(\mathbb{R}^{2n})$ which, when restricted to polynomials, is the rule proposed by Born and Jordan in the early days of quantum mechanics.

1 Introduction

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The problem of quantization harks back to the early years of quantum mechanics when physicists were confronted to ordering problems (see, in this context, the well-documented reviews by Ali and Englis [1] and Castellani [6]). In the present paper we will deal more specifically with what is sometimes called “canonical quantization”, which is a procedure for finding the quantum analogue of a classical theory (in this case Hamiltonian mechanics), while attempting to preserve the formal structure, such as symmetries, of the classical theory, to the greatest extent possible. By definition a canonical quantization of $\mathcal{S}'(\mathbb{R}^{2n})$ is a continuous linear map

$$\text{Op} : \mathcal{S}'(\mathbb{R}^{2n}) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$$

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associating to each $H \in \mathcal{S}'(\mathbb{R}^{2n})$ a continuous linear operator
\[
\hat{H} = \text{Op}(H) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)
\]
and such that some additional properties hold, for instance:

(CQ1) $\text{Op}(q_j^r)\psi(q) = q_j^r\psi(q)$ and $\text{Op}(p_j^r)\psi(q) = (-i\hbar \partial_{q_j})^*\psi(q)$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$ (hence in particular $\text{Op}(1) = I_d$);

(CQ2) (Born’s CCR) $[\hat{q}_j, \hat{p}_k] = i\hbar \delta_{jk}$ for all $j, k$;

(CQ3) When $H \in \mathcal{S}'(\mathbb{R}^{2n})$ is real then $\hat{H}$ is a symmetric operator defined on the dense subspace $\mathcal{S}(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$.

We notice that the usual Weyl quantization satisfies the axioms above; it is also the preferred quantization in physics since it is in a sense the one which sticks the closest to the classical structures (it allows to preserve the covariance of Hamiltonian mechanics under linear canonical transformations; for detailed accounts see [13, 15, 24]). In addition to the axioms above, one has tried, following Dirac’s program [10], to require that commutators correspond (up to a constant) to Poisson brackets:

\[
[\text{Op}(H), \text{Op}(K)] = i\hbar \text{Op}(\{H, K\}). \quad (1)
\]

The rub comes from the seminal papers by Groenewold [21] and van Hove [26, 27], who showed that this “commutator $\leftrightarrow$ Poisson bracket” correspondence cannot hold for all observables (see [3, 20, 22] for detailed discussions). In fact, one shows, after some preparatory work involving Poisson algebras of polynomials that one is led to a contradiction. In fact, assuming $n = 1$ one sets out to quantize $q^2p^2$. Using the rules $\text{Op}(q^r) = (\text{Op}(q))^r$ and $\text{Op}(p^r) = (\text{Op}(p))^r$, the application of (1) to the trivial identity

\[
q^2p^2 = \frac{1}{9}\{q^3, p^3\} = \frac{1}{3}\{q^2p, p^2q\}
\]
leads to the conflicting formulas

\[
\text{Op}(q^2p^2) = \frac{1}{9}\text{Op}\{q^3, p^3\} = (\hat{q})^2(\hat{p})^2 - 2i\hbar \hat{q}\hat{p} - \frac{2}{3}\hbar^2 \quad (2)
\]
and

\[
\text{Op}(q^2p^2) = \frac{1}{3}\text{Op}\{q^2p, p^2q\} = (\hat{q})^2(\hat{p})^2 - 2i\hbar \hat{q}\hat{p} - \frac{1}{3}\hbar^2; \quad (3)
\]
one concludes that there is thus no quantization satisfying Dirac’s correspondence for all monomials.

In the present work we show that these difficulties can be overcome (in a physically satisfactory way) if one relaxes the general Dirac correspondence and replaces it with a weaker condition, suggested by Kauffmann [23], namely that (1) only holds for Hamiltonian functions which are of the type “generalized
kinetic energy plus potential” $T(p) + V(q)$. We will see that this weaker assumption allows to construct a quantization procedure for all tempered distributions on $\mathbb{R}^{2n}$ which, when restricted to monomials $q^r p^s$, is that proposed by Born and Jordan \cite{BornJordan} and which we have extensively studied \cite{CrispinoFuchs,Fuchs,CrispinoFuchs2,Fuchs2}. This result is thus another argument in favor of Born–Jordan quantization. (We notice that the idea of by-passing the Groenewold–van Hove obstruction by some means is not quite new, see Gotay’s paper \cite{Gotay}).

Using the notation $X = \{0\} \times \mathbb{R}^n$ and $X^* = \mathbb{R}^n \times \{0\}$ we will show that

\begin{equation}
\text{[Op}(T), \text{Op}(V)] = i\hbar \text{Op}([T, V]) \quad \text{for all } T \in C^\infty(X^*) \text{ and } V \in C^\infty(X) \text{ that are } S'((\mathbb{R}^n)^*);
\end{equation}

The axiom (BJQ1) will be referred to as the reduced Dirac condition; using the linearity of the Poisson bracket, it is equivalent to the axiom:

\begin{equation}
\text{[H}, \text{K]} = i\hbar \text{Op}([H, K]) \quad \text{for all } H, K \in C^\infty(X) \oplus C^\infty(X^*) \text{ that are in } S'((\mathbb{R}^n)^*)
\end{equation}

where $C^\infty(X) \oplus C^\infty(X^*)$ is the space of all functions $V(x) + T(p)$.

The rule (BJQ1bis) in particular applies to all Hamiltonians of the physical type “kinetic energy + potential”. Notice that (BJQ1) implies that for all integers $r, s > 0$ we have

\begin{equation}
\text{Op}(\{q^r_j, p^s_j\}) = i\hbar rs \text{Op}(q^{r-1}_j p^{s-1}_j)
\end{equation}

and $\text{Op}(\{q^r_j, p^s_k\}) = 0$ if $j \neq k$.

## 2 Quantization of Monomials

We will use the following commutation relations valid for all operators $\hat{q}$ and $\hat{p}$ satisfying the CCR $[\hat{q}, \hat{p}] = i\hbar$:

\begin{equation}
[(\hat{q})^r, (\hat{p})^s] = \text{sih} \sum_{j=0}^{r-1} (\hat{q})^{r-1-j} (\hat{p})^{s-1} (\hat{q})^j = ri\hbar \sum_{j=0}^{s-1} (\hat{p})^{s-1-j} (\hat{q})^{r-1} (\hat{p})^j
\end{equation}

(we will give a proof of this equality in the Appendix).

**Lemma 1** Let $r \geq 0$ be an integer. We have

\begin{equation}
\hat{q}_j^r = (\hat{q}_j)^r, \quad \hat{p}_j^r = (\hat{p}_j)^r
\end{equation}

\begin{equation}
\hat{q}_j \hat{p}_j = \frac{1}{2} (\hat{q}_j \hat{p}_j + \hat{p}_j \hat{q}_j)
\end{equation}

**Proof.** It is sufficient to assume $n = 1$ and $r > 0$. We have

\begin{equation}
[\text{}^{r+1}(\hat{q}), (\hat{p})] = i\hbar \text{Op}([q^{r+1}, p]) = i\hbar (r+1) (\hat{q})^r
\end{equation}
hence, using the second equality (5),

\[(\hat{q})^r = \frac{1}{i\hbar (r + 1)} [i(\hat{q})^{r+1}, \hat{p}] = (\hat{q})^r.\]

The formula \(\hat{p}^r = (\hat{p})^r\) is proven by a similar argument, writing \(\hat{q}, (\hat{p})^r+1\) = \(i\hbar\)\(\text{Op}((\hat{q}, p_{r+1}))\). To prove (7) it suffices to note that, since \(\{q_j^2, p_k^2\} = 4q_jp_j\)
we have, using the commutation formula (5),

\[\text{Op}(qp) = \frac{1}{4i\hbar}[(\hat{q})^2, (\hat{p})^2] = \frac{1}{2}(\hat{qp} + \hat{pq}).\]

Let us now show that formula (4) allows, as claimed in the introduction, an unambiguous quantization of monomials in the \(q_j, p_k\) variables.

We recall [18, 11, 25] that the Born–Jordan quantization of a monomial \(q_jp_k\) is given by the equivalent formulas

\[\text{Op}_{BJ}(q_jp_k) = \frac{1}{r+1} \sum_{\ell=0}^{r} (\hat{q}_j)^{r+1-\ell}(\hat{p}_j)^{s}(\hat{q}_j)\ell \tag{8}\]

\[\text{Op}_{BJ}(q_jp_k) = \frac{1}{s+1} \sum_{\ell=0}^{s} (\hat{p}_j)^{r-\ell}(\hat{q}_j)^{s}(\hat{p}_j)\ell. \tag{9}\]

**Proposition 2** We have for all integers \(r, s \geq 0\)

\[\text{Op}(q_jp_k) = \text{Op}_{BJ}(q_jp_k).\]

**Proof.** It is sufficient to consider the case \(n = 1\); we write \(q = q_1\) and \(p = p_1\). Taking the commutation formulas (5) into account we can rewrite the definitions (8) and (9) as

\[\text{Op}_{BJ}(q_j^r p_k^s) = \frac{1}{i\hbar (r + 1)(s + 1)} [i(\hat{q})^{r+1}, (\hat{p})^{s+1}]. \tag{11}\]

We have

\[q_j^r p_k^s = \frac{1}{(r+1)(s+1)} \{q_j^{r+1}, p_k^{s+1}\}\]

and hence, using the axiom (GQ1),

\[\text{Op}(q_j^r p_k^s) = \frac{1}{i\hbar (r + 1)(s + 1)} [i(\hat{q})^{r+1}, (\hat{p})^{s+1}]; \tag{12}\]

the identity (10) follows using formula (11). ■

Notice that formula (12) is interesting per se: it shows that the Born–Jordan quantization of a polynomial in the position and momentum variables can be expressed as a linear combination of commutators.
3 Quantization of $e^{i\hat{q}(q_0q + p_0p)}$

From now on we assume that $\hat{q}_i$ and $\hat{p}_j$ are the usual operators “multiplication by $q_j$” and $-i\hbar \partial_{x_j}$ (condition (CQ1)). The result below is essential because it is the key to the quantization of arbitrary observables.

Lemma 3 Let $X(q_0) = e^{i\hat{q}q}$ and $Y(p_0) = e^{i\hat{p}p}$. Let $\text{Op}$ be an arbitrary quantization satisfying the axiom (CQ1). We have

$$\text{Op}(X(q_0)) = e^{i\hat{q}q} \quad \text{and} \quad \text{Op}(Y(p_0)) = e^{i\hat{p}p}$$

(13)

that is

$$\text{Op}(X(q_0))\psi(q) = e^{i\hat{q}q}\psi(q), \quad \text{Op}(Y(p_0))\psi(q) = \psi(q + p_0).$$

(14)

Proof. It is sufficient to consider the case $n = 1$, we write again $q = q_1$ and $p = p_1$. Expanding the exponential $e^{i\hat{q}q}$ in a Taylor series we have, in view of the continuity of $\text{Op}$ and using the first equation (CQ1)

$$\text{Op}(X(q_0))\psi(q) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i}{\hbar} q_0 \right)^k \psi(q) = e^{i\hat{q}q}\psi(q).$$

Similarly, using the second equation (CQ1),

$$\text{Op}(Y(p_0))\psi(q) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i}{\hbar} p_0(-i\hbar \partial_q) \right)^k \psi(q) = \psi(q + p_0).$$

Let us apply the result above to a quantization of Weyl’s characteristic function $M(q_0, p_0) = e^{i\hat{q}q + i\hat{p}p}$; using the notation above $M(q_0, p_0) = X(q_0) \otimes Y(p_0)$. We will see that $\hat{M}(q_0, p_0) = \hat{T}_q(q_0) \otimes \hat{Y}(p_0)$. In fact:

Proposition 4 In what follows $\text{Op}$ is a quantization satisfying the reduced Dirac condition (BJQ1). (i) The operator $\hat{M}(q_0, p_0) = \text{Op}(M(q_0, p_0))$ is given by the formula

$$\hat{M}(q_0, p_0) = \text{sinc} \left( \frac{2\pi \hbar t}{\hbar} \right) e^{i\hat{q}(q_0q + p_0p)}$$

(15)

where $\text{sinct} = \sin t/t$ if $t \neq 0$, $\text{sinc} 0 = 1$. (ii) We have $\hat{M}(q_0, p_0) = 0$ for all $(q_0, p_0) \in \mathbb{R}^{2n}$ such that $p_0q_0 \neq 0$ and $p_0q_0 \in 2\pi\hbar \mathbb{Z}$.

Proof. (i) If $q_0 = 0$ or $p_0 = 0$ the result is obvious. Assume $p_0q_0 \neq 0$. The reduced Dirac rule (GQ1) yields

$$[\hat{X}(q_0), \hat{Y}(p_0)] = i\hbar \text{Op}(\{e^{i\hat{q}q}, e^{i\hat{p}p}\})$$

$$= \frac{1}{i\hbar} p_0 q_0 \text{Op}(e^{i\hat{q}q + i\hat{p}p})$$

$$= \frac{1}{i\hbar} p_0 q_0 \hat{M}(q_0, p_0)$$
that is
\[
\hat{M}(q_0, p_0) = \frac{i\hbar}{p_0 q_0} (\hat{X}(q_0)\hat{Y}(p_0) - \hat{Y}(p_0)\hat{X}(q_0))
\]
\[
= \frac{i\hbar}{p_0 q_0} (e^{i\hbar q_0\hat{p}} e^{i\hbar p_0\hat{q}} - e^{i\hbar p_0\hat{p}} e^{i\hbar q_0\hat{q}}).
\]

In view of the Baker–Campbell–Hausdorff formula
\[
e^{A+B} = e^{-\frac{i}{\hbar}[A,B]}e^{A}e^{B} = e^{\frac{i}{\hbar}[A,B]}e^{B}e^{A}
\]
valid for all operators \(A\) and \(B\) commuting with \([A, B]\) we have
\[
e^{\frac{i}{\hbar}p_0\hat{q}} e^{\frac{i}{\hbar}q_0\hat{p}} = e^{-\frac{i}{\hbar}p_0q_0} e^{\frac{i}{\hbar}(q_0\hat{p} + p_0\hat{q})}
\]
\[
e^{\frac{i}{\hbar}p_0\hat{p}} e^{\frac{i}{\hbar}q_0\hat{q}} = e^{-\frac{i}{\hbar}p_0q_0} e^{\frac{i}{\hbar}(q_0\hat{p} + p_0\hat{q})}
\]
and hence
\[
\hat{M}(q_0, p_0) = \frac{i\hbar}{p_0 q_0} (e^{-\frac{i}{\hbar}p_0q_0} - e^{\frac{i}{\hbar}p_0q_0}) e^{\frac{i}{\hbar}(q_0\hat{p} + p_0\hat{q})}
\]
which is formula (15). (ii) is obvious.

### 4 The Case of Arbitrary Observables

Let \(H\) be an element of \(S(\mathbb{R}^{2n})\); let \(\mathcal{H}\) be the Fourier transform of \(H\), defined by
\[
\mathcal{H}(q_0, p_0) = (\frac{1}{2\pi})^n \int H(q, p) e^{-\frac{i}{\hbar}(q_0q + p_0p)} d^nq.
\]
in view of the Fourier inversion formula we have
\[
H(q, p) = (\frac{1}{2\pi})^n \int \mathcal{H}(q_0, p_0) e^{\frac{i}{\hbar}(q_0\hat{q} + p_0\hat{p})} d^nq_0 d^nq_0
\]
Let \(\text{Op}\) be any quantization; by continuity and linearity we have
\[
\text{Op}(H) = (\frac{1}{2\pi})^n \int \mathcal{H}(q_0, p_0) \text{Op}(e^{\frac{i}{\hbar}(q_0\hat{q} + p_0\hat{p})}) d^nq_0 d^nq_0.
\]
Viewing the integral as a distribution bracket, this formula extends to arbitrary \(H \in S'(\mathbb{R}^{2n})\) by continuity, noting that \(S(\mathbb{R}^{2n})\) is dense in \(S'(\mathbb{R}^{2n})\). This formula shows that every quantization is uniquely determined by its action of the exponentials \(e^{\frac{i}{\hbar}(q_0\hat{q} + p_0\hat{p})}\) (for a much more general context, see Bergeron and Gazeau [?]). For instance, if \(\text{Op}(e^{\frac{i}{\hbar}(q_0\hat{q} + p_0\hat{p})}) = e^{\frac{i}{\hbar}(q_0\hat{p} + p_0\hat{q})}\) we get the usual Weyl quantization of \(H\) of the observable \(H\) \cite{12,13,24}. Suppose now that
\[
\text{Op}(e^{\frac{i}{\hbar}(q_0\hat{q} + p_0\hat{p})}) = \hat{M}(q_0, p_0)
\]
where \(\hat{M}(q_0, p_0)\) is defined by formula (15). For \(H \in S(\mathbb{R}^{2n})\) and \(\psi \in S(\mathbb{R}^n)\) we have
\[
\hat{H}\psi(q) = (\frac{1}{2\pi})^n \int \mathcal{H}(q_0, p_0) \hat{M}(q_0, p_0)\psi(q) d^nq_0 d^nq_0.
\]
Rewriting (17) as a distributional bracket
\[ \hat{H}\psi = \left( \frac{1}{2\pi\hbar} \right)^n \langle \mathcal{H}(\cdot,\cdot), \hat{\mathcal{M}}(\cdot,\cdot)\psi \rangle \quad (18) \]
we can extend the definition of \( \hat{H} \) to arbitrary \( H \in \mathcal{S}'(\mathbb{R}^{2n}) \) noting that \( \hat{\mathcal{M}}(q_0,p_0)\psi \in \mathcal{S}(\mathbb{R}^n) \).

Choosing \( H = 1 \) we have \([H(\cdot,\cdot), \hat{\mathcal{M}}(\cdot,\cdot)\psi] = (2\pi\hbar)^n \psi \) and \( \text{Op}(1) = I_d \) in view of (18).

Part (ii) of Proposition 4 shows that the correspondence \( H \mapsto \hat{\mathcal{H}} \) is not injective: we have
\[ \text{Op}(H + \sum_{(q_j,p_j) \in \Lambda} c_j e^{i(q_jq + p_jp)}) = \text{Op}(H) \]
where \( \Lambda \) is any finite lattice in \( \mathbb{R}^{2n} \) consisting of points \((q_j,p_j)\) such that \( q_jp_j = 2N\pi\hbar \) for an integer \( N \neq 0 \). The correspondence \( H \mapsto \hat{\mathcal{H}} \) is however surjective: for every \( \hat{\mathcal{H}} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)) \) there exists (a non-unique) \( H \in \mathcal{S}'(\mathbb{R}^{2n}) \) such that \( \hat{\mathcal{H}} = \text{Op}(H) \). The proof of this property is difficult and technical (it relies on the Paley–Wiener theorem and the theory of division of distributions), and we refer to our recent paper [9] with Cordero and Nicola for a detailed treatment of this issue.

There remains to show that Axiom (GQ3) (symmetry on a dense subspace) is verified. In fact:

**Proposition 5** If \( H \in \mathcal{S}'(\mathbb{R}^{2n}) \) is a real distribution, then \( \langle \hat{\mathcal{H}}\psi, \phi \rangle = \langle \psi, \hat{\mathcal{H}}\phi \rangle \) for all test functions \( \phi, \psi \in \mathcal{S}(\mathbb{R}^n) \).

**Proof.** Returning to integral notation for clarity, we begin by remarking that (18) can be rewritten as
\[ \hat{\mathcal{H}}\psi(q) = \left( \frac{1}{2\pi\hbar} \right)^n \int \mathcal{H}(q_0,p_0)\Theta(q_0,p_0)e^{i(q_0\hat{q} + p_0\hat{p})}\psi(q)d^nq_0d^np_0 \quad (19) \]
where the Cohen kernel \( \Theta \) is given by
\[ \Theta(q_0,p_0) = \text{sinc} \left( \frac{p_0q_0}{2\pi\hbar} \right) . \]

Operators of the type (19) with arbitrary Cohen kernel \( \Theta \in \mathcal{S}'(\mathbb{R}^{2n}) \) are well-known in the literature and one proves ([8], §4.7) that the formal adjoint of \( \hat{\mathcal{H}} \) is given by
\[ \hat{\mathcal{H}}^\dagger\psi(q) = \left( \frac{1}{2\pi\hbar} \right)^n \int \mathcal{H}^*(q_0, -p_0)\Theta^*(-q_0, -p_0)e^{i(q_0\hat{q} + p_0\hat{p})}\psi(q)d^nq_0d^np_0 \quad (20) \]
In the present case we have \( \Theta^*(-q_0, -p_0) = \Theta(q_0,p_0) \) hence \( \hat{\mathcal{H}}^\dagger = \hat{\mathcal{H}} \) requires that \( \mathcal{H}^*(-q_0, -p_0) = \mathcal{H}(q_0,p_0) \), which holds if and only if \( H \) is real. \( \blacksquare \)
5 Discussion and Conclusion

As follows from the Groenewold–van Hove obstruction the general Dirac requirement

\[ \{\hat{H}, \hat{K}\} = i\hbar \text{Op}(\{H, K\}) \]  

(21)
is not compatible with a full-blown quantization; with some hindsight this can be understood as follows: the notion of Poisson bracket is intimately related to the symplectic structure underlying Hamiltonian mechanics (this is pretty obvious when one works on a symplectic manifold \((M, \omega)\) since the Poisson bracket is not defined \textit{ex nihilo}, but by contracting the symplectic form \(\omega\) with the Hamiltonian fields \(X_H\) and \(X_K\): \(\{H, K\} = i_{X_K}i_{X_H}\omega\). One could therefore say that, in a sense, Dirac’s condition \(21\) tries very hard to force quantum mechanics to mimic Hamiltonian mechanics by imposing symplectic covariance \([16]\).

Now, it is reasonably well known (see \([15]\) and the references therein) that the only quantization enjoying such full symplectic covariance is the Weyl correspondence \([8, 12, 13, 24]\). But the Weyl correspondence does not satisfy the general Dirac condition \(21\), as already follows from the conflicting formulas \(2\) and \(3\). Also, our restriction of \(21\) to Hamiltonians of the type \(H(q, p) = T(p) + V(q)\) shows why the symplectic covariance properties of Born–Jordan quantization are limited to linear symplectomorphisms of the type \((q, p) \mapsto (p, -q)\) or \((q, p) \mapsto (L^{-1}q, (L^{-1})^T p)\): these are the only, symplectic automorphisms \(S\) (together with their products) for which \(H \circ S\) is again of the type above (see \([15, 19, 18]\)).

Our results also makes clear that there can’t be any canonical quantization satisfying Dirac’s condition \(21\) in full generality, that is for all functions \(H\) and \(K\): if such a quantization existed, then it would hold in particular for \(H, K \in C^\infty(X) \oplus C^\infty(X^*)\). But then this quantization is that of Born–Jordan, for which \(21\) does not hold for arbitrary \(H\) and \(K\). Notice that this argument actually gives a new proof of the Groenewold–van Hove result.

A last remark: we have chosen to implement the Dirac correspondence rule \(21\) on a specific subspace of observables, those of the type \(T(p) + V(x)\); these do not form an algebra. It is not clear whether this space of observables is a maximal one, nor is it clear whether one could recover some other quantization schemes by changing this space of observables. We will come back to these delicate questions in the future.

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APPENDIX

Let us prove formula (5). We begin by noting that the equalities

\[
[(\hat{q})^r, (\hat{p})^s] = si\hbar \sum_{j=0}^{r-1} (\hat{q})^{r-1-j} (\hat{p})^{s-1}(\hat{q})^j \\
[(\hat{q})^r, (\hat{p})^s] = r\hbar \sum_{j=0}^{s-1} (\hat{p})^{s-1-j} (\hat{q})^{r-1}(\hat{p})^j
\]

are equivalent. In fact, swapping \(\hat{q}\) and \(\hat{p}\) in (22) amounts to changing the bracket \([\hat{q}, \hat{p}] = i\hbar\) into \([\hat{p}, \hat{q}] = -i\hbar\) so that

\[
[(\hat{p})^r, (\hat{r})^s] = -si\hbar \sum_{j=0}^{r-1} (\hat{p})^{r-1-j} (\hat{q})^{s-1}(\hat{p})^j;
\]

swapping \(r\) and \(s\) then yields (23), taking into account the antisymmetry of the commutator bracket. Let us prove (23) by induction on the integer \(s \geq 1\). Let \(s = 1\); then \([[(\hat{q})^r, \hat{p}] = (\hat{q})^r \hat{p} - \hat{p}(\hat{q})^r\] and we have, by repeated use of \([\hat{q}, \hat{p}] = i\hbar\)

\[
\hat{p}(\hat{q})^r = \hat{p}q(\hat{q})^{r-1} = \hat{q}\hat{p}(\hat{q})^{r-1} - i\hbar(\hat{q})^{r-1} \\
= (\hat{q})^r\hat{p} - r\hbar(\hat{q})^{r-1}
\]

that is \([[(\hat{q})^r, \hat{p}] = r\hbar(\hat{q})^{r-1}\] which proves (23) in this case. Let now \(s\) be an arbitrary integer \(\geq 2\) and assume that

\[
[[(\hat{q})^r, (\hat{p})^{s-1}] = r\hbar \sum_{j=0}^{s-2} (\hat{p})^{s-2-j} (\hat{q})^{r-1}(\hat{p})^j.
\]

We then have

\[
[[(\hat{q})^r, (\hat{p})^s] = (\hat{q})^r(\hat{p})^s - (\hat{p})^s(\hat{q})^r \\
= (\hat{q})^r(\hat{p})^{s-1}\hat{p} - (\hat{p})^{s-1}\hat{p}(\hat{q})^r \\
= (\hat{q})^r(\hat{p})^{s-1}\hat{p} - (\hat{p})^{s-1}((\hat{q})^r\hat{p} - r\hbar(\hat{q})^{r-1}) \\
= [[(\hat{q})^r, (\hat{p})^{s-1}]\hat{p} + r\hbar(\hat{p})^{s-1}(\hat{q})^{r-1}.
\]

In view of assumption (23) this is

\[
[[(\hat{q})^r, (\hat{p})^s] = r\hbar \sum_{j=0}^{s-2} (\hat{p})^{s-2-j} (\hat{q})^{r-1}(\hat{p})^j + r\hbar(\hat{p})^{s-1}(\hat{q})^{r-1} \\
= r\hbar \sum_{j=0}^{s-1} (\hat{p})^{s-1-j} (\hat{q})^{r-1}(\hat{p})^j
\]

which completes the proof.
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