Finite orbits for rational functions

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Abstract

Let $K$ be a number field and $\phi \in K(z)$ a rational function. Let $S$ be the set of all archimedean places of $K$ and all non-archimedean places associated to the prime ideals of bad reduction for $\phi$. We prove an upper bound for the length of finite orbits for $\phi$ in $\mathbb{P}_1(K)$ depending only on the cardinality of $S$.

Introduction

Let $K$ be a number field and $R$ its ring of integers. With every rational function $\phi \in K(z)$ we associate in the canonical way a rational map $\Phi : \mathbb{P}_1 \to \mathbb{P}_1$ defined over $K$. For every point $P \in \mathbb{P}_1(K)$ we call its forward orbit under $\Phi$ (or simply orbit) the set $O_{\phi}(P) = \{\Phi^n(P) \mid n \in \mathbb{Z}_{\geq 0}\}$, where $\Phi^n$ is the $n$-th iterate of $\Phi$ and $\Phi^0(P) = P$. If $O_{\phi}(P)$ is a finite set one says that $P$ is a pre-periodic point for $\Phi$. This definition is due to the following fact: if $O_{\phi}(P)$ is finite then there exist two integers $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{> 0}$ such that $\Phi^n(P) = \Phi^{n+m}(P)$. In this case one says that $\Phi^n(P)$ is a periodic point for $\Phi$. If $m$ is the smallest positive integer with the above property, then one says that $m$ is the period of $P$. If $P$ is a periodic point then its orbit is called a cycle.

It is not difficult to prove that every polynomial in $\mathbb{Z}[x]$ has cycles in $\mathbb{Z}$ of length at most 2 and every finite orbit has cardinality at most 6. For a fixed finite set $S$ of valuations of $K$, containing all the archimedean ones, Narkiewicz in [11] has shown that if $\Phi$ is a monic polynomial with coefficients in the ring of $S$-integers $R_S$ (see the definition at the beginning of the next section), then the length of its cycles in $K$ is bounded by a function $B(R_S) = C^{S(3S+2)}$, for an absolute constant $C$. Note that the bound depends only on the cardinality of $S$. The value of $B(R_S)$ has been diminished by Pezda in [13]. Indeed, the main result of Pezda [13] Theorem 1], which concerns polynomial maps in local rings, combined with the estimate given in [11] Theorem 4.7] on the height of the $|S|$-th rational prime,
gives rise to the following inequality

\[ B(R_S) \leq [12|S| \log(5|S|)]^{2[K:Q]+1}. \]  

(1)

Later Narkiewicz and Pezda in [12] extended [13, Theorem 1] to finite orbits so including pre-periodic points. By considering the limit in (1) and the Evertse’s bound proved in [6] for the number of \( S \)-unit non-degenerate solutions to linear equations in three variables, the result of Narkiewicz and Pezda [12, Theorem 1] states that the length of a finite orbit in \( K \) for a monic polynomial with coefficients in \( R_S \) is at most

\[ \frac{1}{3} [12|S| \log(5|S|)]^{2[K:Q]+1} (31 + 2^{103}|S|) - 1. \]

R. Benedetto has recently obtained a different bound, again for polynomial maps, but his bound also depends on the degree of the map. He proved in [2] that if \( \phi \in K[z] \) is a polynomial of degree \( d \geq 2 \) which has bad reduction at \( s \) primes of \( K \), then the number of pre-periodic points of \( \phi \) is at most \( O(s \log s) \). The big-\( O \) constant is essentially \( (d^2 - 2d + 2)/\log d \) for large \( s \). Benedetto’s proof relies on a detailed analysis of \( p \)-adic Julia sets.

In the present paper we will generalize to finite orbits for rational maps the result of Narkiewicz and Pezda [12] obtained for polynomial maps. We will study the same semigroup of rational maps studied in [4], namely: we fix an arbitrary finite set \( S \) of places of \( K \) containing all archimedean ones and consider the rational maps with good reduction outside \( S \). We recall the definition of good reduction for a rational map at a non zero prime ideal \( p \) (for the details see [10] or [4]): a rational map \( \Phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1 \), defined over \( K \), has good reduction at a prime ideal \( p \) if there exists a rational map \( \tilde{\Phi}: \mathbb{P}_1 \rightarrow \mathbb{P}_1 \), defined over \( K(p) \), such that deg \( \Phi = \deg \tilde{\Phi} \) and the following diagram

\[
\begin{array}{ccc}
\mathbb{P}_1,K & \xrightarrow{\Phi} & \mathbb{P}_1,K \\
\sim & & \sim \\
\mathbb{P}_1,K(p) & \xrightarrow{\tilde{\Phi}} & \mathbb{P}_1,K(p)
\end{array}
\]

is commutative, where \( \sim \) is the reduction modulo \( p \). In other words, an endomorphism \( \Phi \) of \( \mathbb{P}_1 \) defined over \( K \) has good reduction at \( p \) if \( \Phi \) can be written as \( \Phi([X : Y]) = [F(X, Y), G(X; Y)] \), where \( F \) and \( G \) are homogeneous polynomials of the same degree, with coefficients in the local ring \( R_p \) of \( R \) at \( p \), and such that the resultant \( \text{Res}(F, G) \) of polynomials \( F \) and \( G \) is a \( p \)-unit in \( R_p \). Note that, from this definition, a rational map on \( \mathbb{P}_1(K) \) associated to a polynomial in \( K[z] \) has good
reduction outside $S$ if and only if its coefficients are $S$-integers and its leading coefficient is an $S$-unit.

In this paper we prove:

**Theorem 1.** Let $K$ be a number field. Let $S$ be a finite set of cardinality $s$ of places of $K$ containing all the archimedean ones. There exists a number $c(s)$, depending only on $s$, such that the length of every finite orbit in $\mathbb{P}_1(K)$, for rational maps with good reduction outside $S$, is bounded by $c(s)$. We can choose $c(s)$ equal to

$$e^{10^{12}(s + 1)^8(\log(5(s + 1)))^8}.$$  \hfill (2)

The proof of Theorem 1 uses two non-elementary facts: the first is [9, Corollary B] where Morton and Silverman proved that if $\Phi$ is a rational map of degree $\geq 2$ which has bad reduction only at $t$ prime ideals of $K$ and $P \in \mathbb{P}_1(K)$ is a periodic point with minimal period $n$, then the inequality

$$n \leq [12(t + 2)\log(5(t + 2))]^{[K:Q]}$$ \hfill (3)

holds. The second one is the theorem proved by Evertse, Schlickewei and Schmidt in [7] on the number of non-degenerate solutions $(u_1, \ldots, u_n) \in \Gamma$ to equation $a_1u_1 + \ldots + a_nu_n = 1$ where $\Gamma$ is a given subgroup of $(\mathbb{C}^*)^n$ of finite rank and the $a_i's$ are given non-zero elements of $K$. For $n = 2$ and $a_1 = a_2 = 1$ we use the upper bound proved by Beukers and Schlickewei in [3]. The main point to obtain the estimate of Theorem 1 is the fact that the upper bounds in the theorems in [7] and [3] only depend on the rank of $\Gamma$. From Theorem 1 we easily deduce the following result concerning finite orbits for rational maps contained in a given finitely generated semigroup of endomorphisms of $\mathbb{P}_1$:

**Corollary 1.** Let $\mathcal{F}$ be a finitely generated semigroup of endomorphisms of $\mathbb{P}_1$ defined over a number field $K$. There exists a uniform upper bound $C$ which bounds the length of every finite orbit in $\mathbb{P}_1(K)$ for any rational map in $\mathcal{F}$. Furthermore it is possible to give an explicit bound for $C$ in terms of a set of generators of $\mathcal{F}$.

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1 Proofs

In all the present paper we will use the following notation:

- $K$ a number field;
- $R$ the ring of integers of $K$;
- $\mathfrak{p}$ a non zero prime ideal of $R$;
- $v_\mathfrak{p}$ the $\mathfrak{p}$-adic valuation on $R$ corresponding to the prime ideal $\mathfrak{p}$ (we always assume $v_\mathfrak{p}$ to be normalized so that $v_\mathfrak{p}(K^*) = \mathbb{Z}$);
- $S$ a fixed finite set of places of $K$ of cardinality $s$ including all archimedean places;
- $R^*_S := \{ x \in K \mid v_\mathfrak{p}(x) \geq 0 \text{ for every prime ideal } \mathfrak{p} \notin S \}$ the ring of $S$-integers;
- $R^*_S^* := \{ x \in K^* \mid v_\mathfrak{p}(x) = 0 \text{ for every prime ideal } \mathfrak{p} \notin S \}$ the group of $S$-units.

Let $P_1 = [x_1 : y_1]$ and $P_2 = [x_2 : y_2]$ be points in $\mathbb{P}_1(K)$. Using the notation of [10] we will denote by

$$\delta_\mathfrak{p}(P_1, P_2) = v_\mathfrak{p}(x_1 y_2 - x_2 y_1) - \min\{v_\mathfrak{p}(x_1), v_\mathfrak{p}(y_1)\} - \min\{v_\mathfrak{p}(x_2), v_\mathfrak{p}(y_2)\}$$

the $\mathfrak{p}$-adic logarithmic distance between the points $P_1, P_2$; note that $\delta_\mathfrak{p}(P_1, P_2)$ is independent of the choice of the homogeneous coordinates, i.e. it is well defined.

We will use the two following propositions contained in [10]:

**Proposition 1.** [10] Proposition 5.1

$$\delta_\mathfrak{p}(P_1, P_3) \geq \min\{\delta_\mathfrak{p}(P_1, P_2), \delta_\mathfrak{p}(P_2, P_3)\}$$

for all $P_1, P_2, P_3 \in \mathbb{P}_1(K)$. □

**Proposition 2.** [10] Proposition 5.2 Let $\Phi : \mathbb{P}_1(K) \to \mathbb{P}_1(K)$ be a rational map defined over $K$. Then

$$\delta_\mathfrak{p}(\Phi(P), \Phi(Q)) \geq \delta_\mathfrak{p}(P, Q)$$

for all $P, Q \in \mathbb{P}_1(K)$ and all prime ideals $\mathfrak{p}$ of good reduction for $\Phi$. □

With $(Q_{-m}, \ldots, Q_0, \ldots, Q_{n-1})$ we always represent a finite orbit for a rational map $\Psi$ in which the 0-th term $Q_0$ is a $n$-th periodic point for $\Psi$. Moreover, for all indexes $i \geq -m$, $Q_{i+1} = \Psi(Q_i)$ holds, bearing in mind that $Q_n = Q_0$. We will use the following remark which is a direct consequence of the previous two propositions.
Remark 1. Let \((Q_m, \ldots, Q_0, \ldots, Q_{n-1})\) be a finite orbit in \(\mathbb{P}_1(K)\) for a rational map \(\Psi\) with good reduction outside \(S\); then for all integers \(a, b\) with \(-m \leq a \leq n-1, b \geq 0, k \geq 0\) and for every prime ideal \(\mathfrak{p} \notin S\)

\[
\delta_{\mathfrak{p}}(Q_a, Q_{a+kb}) \geq \min\{\delta_{\mathfrak{p}}(Q_a, Q_{a+b}), \delta_{\mathfrak{p}}(Q_{a+b}, Q_{a+2b}), \ldots, \delta_{\mathfrak{p}}(Q_{a+(k-1)b}, Q_{a+kb})\} = \delta_{\mathfrak{p}}(Q_a, Q_{a+b}).
\]

Proof. It is a direct application of the triangle inequality (Proposition 1) and Proposition 2. In fact the \(b\)-th iterate of \(\Psi\) has good reduction at every prime ideal \(\mathfrak{p}\) of \(S\), therefore

\[
\delta_{\mathfrak{p}}(Q_{a+lb}, Q_{a+(l+1)b}) = \delta_{\mathfrak{p}}(\Psi^b(Q_{a+(l-1)b}), \Psi^b(Q_{a+lb})) \geq \delta_{\mathfrak{p}}(Q_{a+(l-1)b}, Q_{a+lb})
\]

for all indexes \(0 < l \leq k\). □

In the first version of this paper, in Theorem 1 we proved an upper bound of the form \(c(s, h)\) also depending on the class number \(h\) of the ring \(R_S\). Indeed we worked with a set \(\mathfrak{S}\) of places of \(K\) containing \(S\) such that the ring \(R_\mathfrak{S}\) was a principal ideal domain. From a simple inductive argument it results that it is possible to choose \(\mathfrak{S}\) such that \(|\mathfrak{S}| \leq s + h - 1\). From some suitable applications of Proposition 1 and Proposition 2 we obtained some equations in two and three \(\mathfrak{S}\)-units and by using the upper bounds proved by Evertse in [5] and [6] we deduced a bound in Theorem 1. Following the useful suggestions made by the anonymous referee we shall use, instead of the classical \(\mathfrak{S}\)-unit equation theorem, the refined result of Evertse, Schlickewei and Schmidt [7] (and of Beukers and Schlickewei [3] for \(n = 2\)) leading to an upper bound in Theorem 1 depending only on the cardinality of \(\mathfrak{S}\), even if \(R_\mathfrak{S}\) is not a principal ideal domain. Now we state the last two quoted theorems and then we present the referee’s suggestion to use these results.

Let \(L\) be a number field. Let \((L^*)^n\) be the \(n\)-fold direct product of \(L^*\), with coordinatewise multiplication \((x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_1y_1, \ldots, x_ny_n)\) and exponentiation \((x_1, \ldots, x_n)^l = (x_1^l, \ldots, x_n^l)\). We say that a subgroup \(\Gamma\) of \((L^*)^n\) has rank \(r\) if \(\Gamma\) has a free subgroup \(\Gamma_0\) of rank \(r\) such that for every \(x \in \Gamma\) there is \(m \in \mathbb{Z}_{>0}\) with \(x^m \in \Gamma_0\).

Theorem A [3] Let \(L\) be a number field and let \(\Gamma\) be a subgroup of \((L^*)^2\) of rank \(r\). Then the equation

\[
x + y = 1 \quad \text{in} \quad (x, y) \in \Gamma
\]

has at most \(2^{8r+1}\) solutions. □
Theorem B [7] Let $L$ be a number field, let $n \geq 3$ and let $a_1,\ldots,a_n$ be non zero elements of $L$. Further, let $\Gamma$ be a subgroup of $(L^*)^n$ of rank $r$. Then the equation
\[ a_1x_1 + \ldots + a_nx_n = 1 \quad \text{in } (x_1,\ldots,x_n) \in \Gamma \]
has at most $e^{(6n)^{3(r+1)}}$ solutions such that $\sum_{i \in I} a_i x_i \neq 0$ for each non empty subset $I \subset \{1,\ldots,n\}$. □

Let $a_1,\ldots,a_h$ be a full system of representatives for the ideal classes of $R_S$. For each $i \in \{1,\ldots,h\}$ there is an $S$-integer $\alpha_i \in R_S$ such that $a_i = \alpha_i R_S$. (5)

Let $L$ be the extension of $K$ given by
\[ L = K(\zeta, \sqrt[1]{\alpha_1}, \ldots, \sqrt[1]{\alpha_h}) \] (6)
where $\zeta$ is a primitive $h$-th root of unity. Of course if $h = 1$ then $L = K$. Let us define the following subgroups of $L^*$
\[ \sqrt{K}^* := \{ a \in L^* \mid \exists m \in \mathbb{Z}_{>0} \text{ with } a^m \in K^* \} \]
and
\[ \sqrt{R_S} := \{ a \in L^* \mid \exists m \in \mathbb{Z}_{>0} \text{ with } a^m \in R_S^* \}. \]

Let $S$ denote the set of places of $L$ lying above the places in $S$ and denote by $R_S$ and $R_S^*$ the ring of $S$-integers and the group of $S$-units, respectively, in $L$. By definition it is clear that $R_S^* \cap \sqrt{K}^* = \sqrt{R_S}$ and so it follows that $\sqrt{R_S}$ is a subgroup of $L^*$ of rank $s-1$. With the just stated notation we prove the following:

**Proposition 3.** Let $L$ and $S$ be as above. Let $\Phi$ be a rational map from $\mathbb{P}_1$ to $\mathbb{P}_1$ defined over $L$, having good reduction at all prime ideals outside $S$. Let
\[ \{P_{-m}, \ldots, P_{-1}, P_0\} \] (7)
be a set of $m+1$ distinct points of $\mathbb{P}_1(L)$ such that $\Phi(P_i) = P_{i+1}$ for all $i \in \{-m,\ldots,-1\}$ and $\Phi(P_0) = P_0$. Further, suppose that $P_i = [x_i : y_i]$ for all indexes $i \in \{-m,\ldots,0\}$, where $x_i,y_i \in L$ such that

1. $x_0 = 0, y_0 = 1$;
2. $x_i R_S + y_i R_S = R_S$ for all indexes $i \in \{-m,\ldots,0\}$;
3. $x_i y_j - x_j y_i \in \sqrt{K}^*$ for any distinct indexes $i, j \in \{-m,\ldots,0\}$. 

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Then \( m < e^{10^{12}} - 2 \).

The proof of this proposition will be a direct consequence of the following three lemmas.

**Lemma 1.** With the same hypothesis of Proposition \( \text{[3]} \) let \( P_{l-k}, \ldots, P_{l-1}, P_l \) be distinct points of the orbit \( \text{[7]} \) such that for every prime ideal \( p \notin S \)

\[
\delta_p(P_{l-i}, P_0) = \delta_p(P_l, P_0) \text{ for every index } 0 \leq i \leq k. \tag{8}
\]

Then \( k < 2^{16} \).

**Proof.** For every prime ideal \( p \notin S \) and for any two indexes \( k \geq i > j \geq 0 \) from Proposition \( \text{[1]} \) and condition \( \text{[8]} \) it follows that

\[
\delta_p(P_{l-i}, P_{l-j}) \geq \min\{\delta_p(P_{l-i}, P_0), \delta_p(P_{l-j}, P_0)\} = \delta_p(P_l, P_0). \tag{9}
\]

Moreover, since \( P_n = P_0 \) for all \( n \geq 0 \), by applying Remark \( \text{[1]} \) to the orbit \( (P_{-m}, \ldots, P_{-1}, P_0) \) with \( a = l - i, b = i - j \) and \( k = (m + 1) \), where \( m \) is the maximum integer such that \( l - i + m(i - j) < 0 \), it follows that

\[
\delta_p(P_{l-i}, P_0) \geq \min\{\delta_p(P_{l-i}, P_l), \delta_p(P_{l-j}, P_{l+1 + 2j}), \ldots, \delta_p(P_{l-i+m(i-j)}, P_0)\}
= \delta_p(P_{l-i}, P_{l-j}).
\]

By the last inequality, \( \text{[8]} \) and \( \text{[9]} \) we have that

\[
\delta_p(P_{l-i}, P_{l-j}) = \delta_p(P_l, P_0). \tag{10}
\]

Note that by condition \( (2) \)-Proposition \( \text{[3]} \)

\[
\delta_p(P_i, P_j) = v_p(x_i y_j - x_j y_i) \tag{11}
\]

for all indexes \( i, j \in \{-m, \ldots, 0\} \) and every prime ideal \( p \notin S \). Since \( P_0 = [0 : 1] \), from \( (3) \)-Proposition \( \text{[3]} \) it follows that \( x_{l-i} \in \sqrt{K^*} \) and so, by \( \text{[11]} \), condition \( \text{[8]} \) is equivalent to \( x_{l-i}R_S = x_i R_S \), for every index \( 0 \leq i \leq k \). Hence

\[
u_{l-i} := \frac{x_{l-i}}{x_l} \in R_S^* \cap \sqrt{K^*} = \sqrt{R_S^*} \tag{12}
\]

and \( P_{l-i} = [x_l : y_{l-i}/u_{l-i}] \). Furthermore, again from \( (3) \)-Proposition \( \text{[3]} \) combined with \( \text{[10]} \) and \( \text{[12]} \) we deduce that

\[
u_{l-i,j} := \frac{x_{l-i} y_{l-j} - x_{l-j} y_{l-i}}{x_l u_{l-i} u_{l-j}} = \frac{y_{l-i}}{u_{l-i}} - \frac{y_{l-j}}{u_{l-j}} \in R_S^* \cap \sqrt{K^*} = \sqrt{R_S^*} \tag{13}
\]
for all distinct indexes $i, j \in \{0, \ldots, k\}$. In particular, either $k \in \{0, 1\}$ or we have a system of three equations

$$\begin{align*}
y_l - y_{l-1}/u_{l-1} &= u_{l-1,i} \\
y_l - y_{l-i}/u_{l-i} &= u_{l-i,i} \\
y_{l-1}/u_{l-1} - y_{l-i}/u_{l-i} &= u_{l-i,i-1} \
\end{align*}$$

(14)

The first one is obtained from (13) substituting $j = 0$ and $i = 1$ and the two other ones with $j = 0, j = 1$ and $i$ an arbitrary index $k \geq i \geq 2$ (recall that $u_l = 1$).

We deduce from (14) the following linear relation:

$$u_{l-1,i} + u_{l-i,l-1} = u_{l-i,j},$$

so $(u_{l-1,i}/u_{l-i,j}, u_{l-i,l-1}/u_{l-i,j}) \in \sqrt{R_S^*} \times \sqrt{R_S^*}$ is a solution of the equation $u + v = 1$. Note that the group $\sqrt{R_S^*} \times \sqrt{R_S^*}$ has rank $2(s - 1)$ therefore, by Theorem A (Beukers and Schlickewei [3]) with $\Gamma = \sqrt{R_S^*} \times \sqrt{R_S^*}$, there are at most $2^{8(2s-2)+1} = 2^{16s-8}$ possibilities for $(u_{l-1,i}/u_{l-i,j}, u_{l-i,l-1}/u_{l-i,j})$. Now from (14) it follows that

$$\frac{y_{l-i}}{u_{l-i}} = y_l - \frac{u_{l-i,j}}{u_{l-i,j}}u_{l-1,i}.$$ 

Thus the set of points $\{P_{l-i} = [x_l : y_{l-i}/u_{l-i}] | k \geq i \geq 2\}$ has cardinality bounded by $2^{16s-8}$ so $k \leq 2^{16s-8} + 1 < 2^{16s}$.

The next step is to prove an upper bound, which depends only on $s$, for the number of points $P_{-i}$ of (17) such that $x_{-i}R_S \neq x_{-i+1}R_S$. We need two lemmas.

We say that a $S$-integer $T$ is representable in two essentially different ways as sum of two elements of $\sqrt{R_S^*}$ if there exist

$$u_1, u_2, v_1, v_2 \in \sqrt{R_S^*}$$

such that $\{u_1, u_2\} \neq \{v_1, v_2\}$ and $T = u_1 + u_2 = v_1 + v_2$. (15)

Lemma 2. The cardinality of the set of non zero principal ideals of $R_S$

$$\{T \cdot R_S \mid T \text{ satisfies (15)}\}$$

is bounded by $e^{18\gamma(3s-2)}$.

Proof. Let $T \in R_S/\{0\}$ be written as $T = u_1 + u_2 = v_1 + v_2$ which satisfies the condition in (15). Therefore the left term of equation

$$\frac{u_1}{v_1} + \frac{u_2}{v_1} - \frac{v_2}{v_1} = 1$$

has no vanishing subsums. Now, applying Theorem B (Evertse, Schlickewei and Schmidt [7]) with $n = 3$ and $\Gamma = \sqrt{R_S^*} \times \sqrt{R_S^*} \times \sqrt{R_S^*}$ we obtain that the principal ideal

$$T \cdot R_S = v_1 \left(1 + \frac{v_2}{v_1}\right) \cdot R_S$$

has at most $e^{18\gamma(3s-2)}$ possibilities. \hfill \Box
Remark 2. By previous lemma, we can choose a set $\mathcal{I}$ of $\mathbf{S}$-integers, with cardinality at most $e^{18^{9}9^{3}e^{2}}$, such that every non zero $\mathbf{S}$-integer with the property (15) is representable as $uT$, where $u \in \mathbb{R}_S^{*}$ and $T \in \mathcal{I}$.

Lemma 3. With the same hypothesis of Proposition 3, if there exist five distinct points $P_{n_5} = [x_{n_5} : y_{n_5}], P_{n_4} = [x_{n_4} : y_{n_4}], P_{n_3} = [x_{n_3} : y_{n_3}], P_{n_2} = [x_{n_2} : y_{n_2}], P_{n_1} = [x_{n_1} : y_{n_1}]$ of the orbit (7), with $n_5 < n_4 < n_3 < n_2 < n_1 < 0$, then $x_{n_1}/x_{n_2}$ is a non zero $\mathbf{S}$-integer that is representable, in two essentially different ways, as sum of two elements of $\sqrt{R_S^{*}}$.

Proof. Since $\Phi(P_0) = P_0 = [0 : 1]$, from Proposition 2, considering $\Phi^{n-n_j}$, $P = P_{n_j}$ and $Q = P_0$, it follows that $x_{n_j}y_{n_j} \in R_S$ for all couple of integers $j \geq i$. Therefore there exist four non zero $\mathbf{S}$-integers $T_1, T_2, T_3, T_4$ such that

$$x_{n_i} = T_i x_{n_{i+1}} \text{ for all } i \in \{1, 2, 3, 4\}$$

and so for every couple of distinct indexes $1 \leq i < j \leq 5$

$$x_{n_i} = T_i \cdot \ldots \cdot T_{j-1} x_{n_j}. \quad (16)$$

By Remark 1 we have that

$$\delta_{\Phi}(P_{n_j}, P_0) \geq \min\{\delta_{\Phi}(P_{n_j}, P_{n_i}), \delta_{\Phi}(P_{n_j}, P_{2n_j}), \ldots, \delta_{\Phi}(P_{m-n_j-(m-1)n_j}, P_0)\} = \delta_{\Phi}(P_{n_i}, P_{n_j})$$

for a suitable integer $m$, so it follows that $(x_{n_j}y_{n_j} - x_{n_i}y_{n_i})/x_{n_j}$ and by identity (16)

$$y_{n_j} - T_i \cdot \ldots \cdot T_{j-1} y_{n_j} = \frac{x_{n_j}y_{n_j} - x_{n_i}y_{n_i}}{x_{n_j}} \in \mathbb{R}_S^{*} \cap \sqrt{K^{*}} = \sqrt{R_S^{*}}. \quad (17)$$

(Recall that, by conditions (1) and (3) in the hypothesis of Proposition 3, the $\mathbf{S}$-integers $x_{n_j}y_{n_j} - x_{n_i}y_{n_i}$ and $x_{n_j}$ belong to $\sqrt{K^{*}}$.) From (17) we obtain:

$$y_{n_1} - T_1 y_{n_2} = v_1, \quad (18)$$
$$y_{n_2} - T_2 y_{n_3} = v_2, \quad (19)$$
$$y_{n_3} - T_1 T_2 y_{n_4} = v_3, \quad (20)$$
$$y_{n_4} - T_3 y_{n_5} = v_4, \quad (21)$$
$$y_{n_2} - T_2 T_3 y_{n_4} = v_5, \quad (22)$$
$$y_{n_1} - T_1 T_2 T_3 y_{n_5} = v_6, \quad (23)$$
$$y_{n_2} - T_2 T_3 T_4 y_{n_5} = v_7, \quad (24)$$
$$y_{n_1} - T_1 T_2 T_3 T_4 y_{n_5} = v_8, \quad (25)$$
$$y_{n_3} - T_3 T_4 y_{n_5} = v_9, \quad (26)$$
$$y_{n_4} - T_4 y_{n_5} = v_{10}. \quad (27)$$

9
where \( v_i \in \sqrt{R_S} \) for all indexes \( 1 \leq i \leq 10 \).

From (18), (20) and (19) we obtain

\[
T_1 = \frac{v_3}{v_2} - \frac{v_1}{v_2}.
\]

From (23), (18) and (22) we obtain

\[
T_1 = \frac{v_6}{v_5} - \frac{v_1}{v_5}.
\]

From (25), (18) and (24) we obtain

\[
T_1 = \frac{v_8}{v_7} - \frac{v_1}{v_7}.
\]

Now we finish proving that among (28), (29), (30) there exist at least two distinct representations of \( T_1 \) as sum of two elements of \( \sqrt{R_S} \). From (24), (19) and (26) we obtain that

\[
T_2 = \frac{v_7}{v_7} - \frac{v_2}{v_7};
\]

therefore \( v_7 \neq v_7 \) and so

\[
\left\{ \frac{v_3}{v_2}, -\frac{v_1}{v_2} \right\} = \left\{ \frac{v_6}{v_5}, -\frac{v_1}{v_5} \right\} \Rightarrow -\frac{v_1}{v_2} = \frac{v_3}{v_2}.
\]

From (24), (22) and (27) we obtain that \( T_2T_3 = \frac{v_7}{v_7} - \frac{v_2}{v_7} \); therefore \( v_7 \neq v_5 \) and so

\[
\left\{ \frac{v_6}{v_5}, -\frac{v_1}{v_5} \right\} = \left\{ \frac{v_8}{v_7}, -\frac{v_1}{v_7} \right\} \Rightarrow -\frac{v_1}{v_5} = \frac{v_8}{v_7}.
\]

From (31) and (32) it follows that

\[
\left\{ \frac{v_3}{v_2}, -\frac{v_1}{v_2} \right\} = \left\{ \frac{v_6}{v_5}, -\frac{v_1}{v_5} \right\} = \left\{ \frac{v_8}{v_7}, -\frac{v_1}{v_7} \right\} \Rightarrow -\frac{v_1}{v_2} = -\frac{v_1}{v_5}.
\]

But this is not possible since, from (22), (19) and (21), \( T_2 = \frac{v_7}{v_4} \neq 0 \) holds. \( \square \)

**Proof of Proposition 3.** The set \( \{P_{i_1}, \ldots, P_{i_r}, P_{i_1}\} \) of all points \( P_{i_r} \) of the orbit (7) such that \( x_{i_r}R_S \neq x_{i_{r-1}}R_S \) has cardinality equal to \( r + 1 \leq 4 + e^{18^9(3s-2)} \). Indeed, if such five points do not exist, we have finish; otherwise for every index \( i_{r-2} < i_r \leq i_1 \) we apply the previous lemma with \( n_1 = -1, n_2 = i_r, n_3 = i_{r-2}, n_4 = i_r, n_5 = i_r \) obtaining that \( x_{i_r}x_{i_r}^{-1} = uT \) where \( T \in \mathcal{T} \) (the set chosen in Remark 2) and \( u \) is a suitable \( S \)-unit. Therefore

\[
P_{i_r} = [x_{i_r}/T : uy_{i_r}].
\]

In this way we have proved that \( r \) is bounded by \( 3 + |\mathcal{T}| \). Now, by Lemma 1 it is clear that it is possible to choose as upper bound for \( m \) the number

\[
\left( 4 + e^{18^9(3s-2)} \right) \left( 2^{16s} + 1 \right) < e^{10^{12s}} - 2.
\]

\( \square \)
Proof of Theorem \[1\] The bound \[2\] holds for finite orbit length for all rational maps of degree 1, i.e. automorphisms of \( \mathbb{P}_1(K) \). Indeed every pre-periodic point for a bijection is a periodic point. Thus we have to study only the cycle lengths. If a point of \( \mathbb{P}_1(K) \) is a periodic point for an automorphism \( \Psi \in \text{PGL}_2(K) \), with period \( n \geq 3 \), then \( \Psi^n \) is the identity map of \( \mathbb{P}_1(K) \). The order of an element of \( \text{PGL}_2(K) \) is bounded by \( 2 + 4[K : \mathbb{Q}]^2 \), so from \( 2s \geq [K : \mathbb{Q}] \) it results that
\[
n \leq 2 + 16s^2 < c(s). \tag{33}
\]

Now we consider rational maps of degree \( \geq 2 \) with good reduction outside \( S \). We reduce to the hypothesis of Proposition \[3\] Let \((Q_{-i}, \ldots, Q_0, \ldots, Q_{n-1})\) be a finite orbit in \( \mathbb{P}_1(K) \), for a rational map \( \Psi: \mathbb{P}_1 \to \mathbb{P}_1 \) defined over \( K \) with good reduction outside \( S \), including \((Q_0, \ldots, Q_{n-1})\) as a cycle for \( \Psi \). We can associate a finite orbit in which the cycle consists of one single point (i.e. a fixed point). Indeed, the tuple \((Q_{-i}, \ldots, Q_0, Q_0)\) is an orbit for \( \Psi^n \) and \( Q_0 \) is a fixed point. We set \( m := \left\lceil \frac{n}{l} \right\rceil \). Of course \( \Psi^n \) can be viewed as an endomorphism of \( \mathbb{P}_1 \) defined over \( L \) (the extension of \( K \) defined in \( [6] \)). For every index \( i \in \{0, \ldots, m\} \), let \( Q_{-i:n} = [l_i : t_i] \) be a representation of \( Q_{-i:n} \) in \( S \)-integral homogeneous coordinates.

Recall that \( \{a_1, \ldots, a_m\} \) is a full system of representatives for the ideal classes of \( R_S \) and that the \( \alpha_i \)'s are the \( S \)-integers verifying \( [5] \). Let \( b_i \in \{a_1, \ldots, a_m\} \) be the representative of the ideal \( t_iR_S + l_iR_S \). Let \( \beta_i \in \{\alpha_1, \ldots, \alpha_m\} \) be such that \( b_i^h = \beta_iR_S \). Hence there exists \( \lambda_i \in K^* \) satisfying \((t_iR_S + l_iR_S)^h = \lambda_i^h \beta_iR_S \). As suggested by the referee, in \( L \), we define
\[
t'_i := \frac{t_i}{\lambda_i \sqrt[2]{\beta_i}}, \quad l'_i := \frac{l_i}{\lambda_i \sqrt[2]{\beta_i}}. \tag{34}
\]
It is clear that \( t'_i, l'_i \) are elements of \( \sqrt[2]{K^*} \) such that
\[
(t'_iR_S + l'_iR_S) = R_S. \tag{35}
\]
Furthermore, for any two distinct indices \( i, j \)
\[
(t'_i l'_j - t'_j l'_i)^h = \frac{(t_i l_j - t_j l_i)^h}{\lambda_i^h \lambda_j^h \beta_i \beta_j} \in K^*.
\]

By \( (35) \) with \( i = 0 \) there exist \( r_0, s_0 \in R_S \) such that \( r_0 t'_0 + s_0 l'_0 = 1 \). Define the matrix
\[
A = \begin{pmatrix} l'_0 & -t'_0 \\ r'_0 & s'_0 \end{pmatrix}
\]
and further define \( x_i, y_i \) by
\[
\begin{pmatrix} x_i \\ y_i \end{pmatrix} = A \begin{pmatrix} t'_i \\ l'_i \end{pmatrix}
\]
for all } i \in \{0, \ldots, m\}. \text{ If we now set } P_i := [x_i : y_i] \text{ for all } i \in \{0, \ldots, m\} \text{ and } \Phi := [A] \circ \Psi^n \circ [A]^{-1}, \text{ where } [A] \text{ is the automorphism of } \mathbb{P}_1 \text{ induced by } A, \text{ then, by Proposition } m = \left\lfloor \frac{l}{n} \right\rfloor < e^{10^{12}s} - 2 \text{ and so } l < n(e^{10^{12}s} - 1). \text{ Therefore the orbit } (Q_j, \ldots, Q_{-1}, Q_0, \ldots, Q_{n-1}) \text{ has cardinality bounded by } ne^{10^{12}s}. \text{ Since in } S \text{ there are at most } s - 1 \text{ prime ideals and } 2s \geq [K : \mathbb{Q}], \text{ the inequality (2) becomes}

\[ n \leq \left[ 12(s + 1) \log(5(s + 1)) \right]^{8s} \]

and so the theorem is proved.

\[ \square \]

**Proof of Corollary** \[ \square \] Choose a finite set of generators of } F\text{. Each of these generators has at most finitely many prime ideals of } R \text{ of bad reduction. So there is a finite set } S \text{ of prime ideals such that each of the chosen generators, and therefore each of the elements of } F, \text{ has good reduction outside } S. \text{ We conclude by applying Theorem } \[ \square \]

**References**

[1] APOSTOL T Introduction to Analytic Number Theory. Springer-Verlag (1976), New York.

[2] R. BENEDETTO, Preperiodic points of polynomials over global fields, (2005) [ArXiv:math.NT/0506480].

[3] F. BEUKERS AND H.P. SCHLICKEWEI, The equation } x+y = 1 \text{ in finitely generated groups, Acta Arith. LXXVIII.2 (1996), 189-199.

[4] J. K. CANCI, Cycles for rational maps of good reduction outside a prescribed set, Monatsh. Math., to appear.

[5] J. H. EVERTSE, On equations in } S \text{-units and the Thue-Mahler equation, Invent. Math. 75 (1984), 561-584.

[6] J. H. EVERTSE, The number of solutions of decomposable form equations, Invent. Math. 122, (1995), 559-601.

[7] J. H. EVERTSE, H.P. SCHLICKEWEI AND W. M. SCHMIDT, Linear equations in variables which lie in a multiplicative group, Ann. Math. 155, (2002), 807-836.

[8] S. LANG, Algebra-Revised third edition, Springer-Verlag, GTM 211, 2002
[9] P. Morton and J.H. Silverman, *Rational Periodic Points of Rational Functions*, Inter. Math. Res. Notices 2 (1994), 97-110.

[10] P. Morton and J.H. Silverman, *Periodic points, multiplicities, and dynamical units*, J. Reine Angew. Math. 461 (1995), 81-122.

[11] W. Narkiewicz, *Polynomial cycles in algebraic number fields*, Colloq. Math. 58 (1989), 151-155.

[12] W. Narkiewicz and T. Pezda, *Finite polynomial orbits in finitely generated domain*, Monatsh. Math. 124 (1997), 309-316.

[13] T. Pezda, *Polynomial cycles in certain local domains*, Acta Arith. LXVI:1 (1994), 11-22.

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