ON THE RATIONAL TYPE OF MOMENT-ANGLE COMPLEXES

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ABSTRACT. In this note, it will be shown that the only moment-angle complexes $Z(K; (D^2; S^1))$ which are rationally elliptic are those which are a products of odd spheres and a disk.

1. Introduction

Félix and Halperin showed in [7] that there is a dichotomy for simply-connected finite CW-complexes $X$. Recall their definition:

Either

(1) $\pi_*(X) \otimes Q$ is a finite $Q$-vector space, in which case $X$ is called rationally elliptic, or

(2) $\pi_*(X) \otimes Q$ grows exponentially, in which case $X$ is called rationally hyperbolic.

Next, recall the definition of $Z(K; (D^2; S^1))$ from [1], [6].

Let $(D^2, S^1)$ be the pair of a 2-disk and its boundary circle, and $K$ be a finite abstract simplicial complex with $n$ vertices. Then $Z(K; (D^2; S^1))$ is a subspace of $(D^2)^n$ defined as the union over all simplices $\sigma \in K$ of subspaces of $(D^2)^n$

$$D(\sigma) = \{(x_1, ..., x_n) | x_i \in S^1 \text{ if } i \not\in \sigma\},$$

$Z(K; (D^2; S^1))$ is a 2-connected finite CW-complex. In this note we obtain:

Theorem 1.1. The only moment-angle complexes $Z(K; (D^2; S^1))$ which are rationally elliptic, are those which are a product of odd spheres and a disk.

After an inquiry from one of the authors, A. Berglund supplied an alternative proof of Theorem 1.1 [4].

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2. Minimal non-faces and abstract simplicial complexes

Let \([n]\) denote an abstract set of vertices \(\{v_1, ..., v_n\}\).

The next definition specifies an abstract simplicial complex in terms of its “missing” faces.

**Definition 2.1.** A family \(M = \{m_1, ..., m_k\}\) of subsets of \([n]\) satisfying:

1. \(|m_i| > 1\)
2. \(m_i \not\subset m_j\) for any \(i \neq j \in \{1, 2, ..., k\}\) is called a set of minimal non-faces.

Let \(\nu = \bigcup_{i=1}^{k} m_i\). Associated to \(M\) are two abstract simplicial complexes.

\[
K(M, [n]) = \{ \sigma \subset [n] \mid m_i \not\subset \sigma \text{ for all } i = 1, ..., k \}
\]

\[
K(M, \nu) = \{ \sigma \subset \nu \mid m_i \not\subset \sigma \text{ for all } i = 1, ..., k \}
\]

which agree if \(\nu = [n]\). If \(|\nu| < n\), \(K(M, \nu)\), is called the reduced simplicial complexes corresponding to \(M\). The empty set \(\emptyset\) is considered to be in both simplicial complexes.

Recall that in an abstract simplicial complex \(K\) a minimal non-face is a sequence \(Q = (v_1, ..., v_q)\) so that \(Q \not\in K\), but every proper subsequence of \(Q\) is a simplex of \(K\).

**Remark 2.2.** If \(K\) is an abstract simplicial complex with \(n\) vertices, and \(M\) is its set of minimal non-faces, then there is a homeomorphism of underlying simplicial complexes \(K \rightarrow K(M, [n])\).

Next recall that the join of two disjoint simplicial complexes \(K_1\) and \(K_2\) denoted by \(K_1 * K_2\) is defined by:

\[
K_1 * K_2 = \{ \sigma_1 \cup \sigma_2 \mid \sigma_1 \in K_1, \sigma_2 \in K_2 \}
\]

**Proposition 2.3.** Let \(|\nu| < n\) and set \(n' = |\nu|\), then there is a simplicial isomorphism:

\[
K(M, [n]) \rightarrow K(M, \nu) * \Delta^{n-n'-1}
\]

where \(\Delta^{n-n'-1}\) is the simplex with \(n - n'\) vertices

\[
[n] - \nu = \{v_{i_1}, ..., v_{i_{n-n'}}\}
\]

**Proof:** The sets \(\nu\) and \([n] - \nu\) are disjoint, so \(K(M, \nu) * \Delta^{n-n'}\) as sets in \([n]\), hence \(\sigma \rightarrow \sigma_1 \cup \sigma_2\) where \(\sigma_1 = \sigma \cap K(M, \nu), \sigma_2 = \sigma \cap K(M, \nu)\).

Now \(\Delta^{n-n'-1}\) is a simplex as every subset of it is a simplex of \(K(M, [n])\). \(\square\)
Definition 2.4. Given $M$ as in Definition 2.1, a graph $G(M)$ is defined with its vertices the $m_i$ and an edge joining $m_i$ and $m_j$ if $m_i \cap m_j \neq \emptyset$.

Let $G(M) = \{C_1(M), \ldots, C_l(M)\}$ be the connected components of $G(M)$. Let $M_i \subset M$ be the set of $m_j$ in $C_i(M)$.

Proposition 2.5. There is a simplicial isomorphism

$$K(M, \nu) \xrightarrow{\sim} K(M_1, \nu_1) \ast K(M_2, \nu_2) \ast \cdots \ast K(M_l, \nu_l).$$

Proof: Let $\sigma \in K(M, \nu)$, let $\sigma_i$ be the part of $\sigma$ that lies in $K_i$. It is a simplex in $K_i$. Then $\sigma = \sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_l$, where $\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_l \in K(M_1, \nu_1) \ast \cdots \ast K(M_l, \nu_l)$.

Conversely given $\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_l \in K(M_1, \nu_1) \ast \cdots \ast K(M_l, \nu_l)$ it is a simplex of $K(M, \nu)$ since it is no divisible by any of the $m_i \in M$. \qed

Proposition 2.6. Let $m \in M$ be a minimal non-face, then $K(m, |m|)$ is isomorphic to the boundary of a simplex, $\partial \Delta(|m| - 1)$.

Proof: Every proper subsequence of $m$ is a simplex in $K(m, |m|)$, and then isomorphic to the boundary of $\Delta(|m| - 1)$, $\partial \Delta(|m| - 1)$. \qed

Corollary 2.7. If the minimal non-faces of $M$ are pairwise disjoint, then there is a simplicial isomorphism

$$K(M, \nu) \cong \partial \Delta(|m_1| - 1) \ast \cdots \ast \partial \Delta(|m_k| - 1)$$

where $|M| = k$.

Given a subset $I \subset [n]$, denote by $K_I(M, \nu)$ the full subcomplex of $K(M, \nu)$, i.e.

$$K_I(M, \nu) = \{\sigma \cap I \mid \sigma \in K(M, \nu)\}$$

and let $M_I = \{m \in M \mid m \subset I\}$.

Corollary 2.8. There is an equality

$$K_I(M, n) = K(M_I, |I|)$$

Proof: Let $\tau \in K_I(M, n)$, $\tau = \sigma \cap I$, where $\sigma \in K(M, n)$, so $n \not\subset \sigma$ for all $m \in M$, then since $\sigma \cap I \subset \sigma$, $m \not\subset \sigma \cap I$ for all $m \in M$, so $\sigma \cap I \in K(M, n)$ and has all its vertices in $I$. Hence $n \not\subset \sigma \cap I$ for all $m \in M_I$, and has all its vertices in $I$, thus, $\sigma \cap I \in K(M_I, |I|)$, and thus $K_I(M, n) \subset K(M_I, |I|)$.

Now let $\tau \in K(M_I, |I|)$, then for $m \in M_I$, $m \not\subset \tau$ and for $m \in M - M_I$, $m$ contains vertices
in \([n] - I\), so \(m \not\subset \tau\) for all \(m \in M\), and hence \(\tau \in K(M, n)\) with all its vertices in \(I\), i.e., \(\tau \in K_I(M, n)\). Thus \(K(M_I, |I|) \subset K_I(M, n)\) and hence \(K_I(M, n) = K(M_I, |I|)\). \(\square\)

3. Detecting a wedge of rational spheres

The case not covered by Corollary 2.7 is addressed next.

**Proposition 3.1.** Suppose that \(M\) contains at least one pair \((m_i, m_j)\) with \(m_i \cap m_j \neq \emptyset\). Then \(K(M, \nu)\) contains a full subcomplex \(K_I(M, \nu)\), such that every pair of minimal non-faces in \(M_I\) intersect.

**Proof:** Let \(P = \{(m_i, m_j) \mid i \neq j\text{ and }m_i \cap m_j \neq \emptyset\}\). Let \(\bar{m}_0, \bar{m}_1 \in P\) be such that \(|\bar{m}_0 \cup \bar{m}_1|\) is minimal. It may be not unique. Let \(I = \bar{m}_0 \cup \bar{m}_1\). It is unclear how many classes are in \(M_I\), but it will follow by induction that \(M_I\) has all pairwise intersecting pairs.

Suppose \(\bar{m}_0, ..., \bar{m}_j \in M_I\), such that \(\bar{m}_i \cup \bar{m}_k = I\) and all pairs intersect.

Suppose \(\bar{m}_{j+1} \in M_I\). Then because \(\bar{m}_i \cup \bar{m}_k = I\) and \(\bar{m}_{j+1} \subset I\), \(\bar{m}_{j+1}\) contains vertices of \(\bar{m}_i\) and \(\bar{m}_k\) for every pair, and hence intersects \(\bar{m}_i\) and \(\bar{m}_k\), for every pair. Also \(\bar{m}_{j+1} \cup \bar{m}_k = I\) by minimality of \(I\). Thus \(\bar{m}_0, ..., \bar{m}_{j+1}\) is a family of intersecting pairs and \(M_I\) consists of intersecting pairs. \(\square\)

4. The dichotomy for moment angle complexes

The following properties of moment angle complexes may be found in [1], [6].

\[(4.1)\quad Z(K_1 \ast K_2; (D^2, S^1)) \cong Z(K_1; (D^2, S^1)) \times Z(K_2; (D^2, S^1))\]

\[(4.2)\quad Z(\Delta^k; (D^2, S^1)) \cong D^{2k+2}\]

\[(4.3)\quad Z(\partial \Delta^k; (D^2, S^1)) \cong S^{2k+1}\]

\[(4.4)\quad \text{If } K_I \text{ is a full subcomplex of } K \text{ then } Z(K_I; (D^2, S^1)) \text{ is a retract of } Z(K; (D^2, S^1))\]

**Proposition 4.1.** Suppose \(M\) consist of pairwise intersecting pairs then \(\bar{H}^*(Z(K(M, n); (D^2, S^1)))\) is a trivial ring.
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**Proof:** Recall the following definition from [2]:

\[ \mathcal{A}^*(K; (D^2, S^1)) = \bigoplus_I H^*(\hat{Z}(K_I; (D^2, S^1))) \]

with product given by a pairing,

\[ H^*(\hat{Z}(K_J)) \otimes H^*(\hat{Z}(K_L)) \rightarrow H^*(Z(K_{J\cup L})) \]

denoted by \( u \ast v \). It was proven in [?] that

\[ H^*(Z(K)) \cong \mathcal{A}(K; (D^2, S^1)) \]

as maps, and in the case \((D^2, S^1)\), \( u \ast v = 0 \) if \( J \cap L \neq \emptyset \) for every pair \((u, v)\).

It will be checked next that the product \( u \ast v = 0 \) for all \( u, v, \) and all \( J \) and \( L \).

If \(|M_L| = 0\), then \( Z(K(M_L, v)) \cong (D^2)^t \).

Suppose \( M_I \) and \( M_L \) contain minimal non-faces, then each \( m \in M_J \) intersects every \( m' \in M_L \).

Then \( J \cap L \neq \emptyset \), and \( \mathcal{A}^*(K; (D^2, S^1)) \) is a trivial ring isomorphic to \( \hat{H}^*(Z(K; (D^2, S^1))) \) which is thus a trivial ring.

The following Theorem is due to Berglund [3], and Berglund and Jöllenbeck [5].

**Theorem 4.2.** If \( \hat{H}^*(Z(K; (D^2, S^1))) \) is a trivial ring, then \( Z(K; (D^2, S^1)) \) is a rationally a wedge of spheres.

The next Theorem follows.

**Theorem 4.3.** If \( Z(K) \) is not homotopy equivalent to a product of odd spheres, then \( Z(K) \) is rationally hyperbolic.

**Proof:** Notice that \( M(K) \) has an intersecting pair, and hence an \( M_I(K) \), which consist of intersecting pair. By the alone proportion \( Z_I(K) \) is rationally a wedge of spheres. But \( Z_I(K) \) is a retract of \( Z(K) \), and hence since \( \pi_*(Z_I(K)) \otimes \mathbb{Q} \) is a free Lie algebra, which is a direct summand of \( \pi_*(Z(K)) \otimes \mathbb{Q} \), it follows that \( Z(K) \) is rationally elliptic. \( \square \)

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