HOLOMORPHIC FUNCTIONS AND REGULAR QUATERNIONIC FUNCTIONS ON THE HYPERKÄHLER SPACE $\mathbb{H}$

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ABSTRACT. Let $\mathbb{H}$ be the space of quaternions, with its standard hypercomplex structure. Let $\mathcal{R}(\Omega)$ be the module of $\psi$-regular functions on $\Omega$. For every unitary vector $p$ in $\mathbb{S}^2 \subset \mathbb{H}$, $\mathcal{R}(\Omega)$ contains the space of holomorphic functions w.r.t. the complex structure $J_p$ induced by $p$. We prove the existence, on any bounded domain $\Omega$, of $\psi$-regular functions that are not $J_p$-holomorphic for any $p$. Our starting point is a result of Chen and Li concerning maps between hyperkähler manifolds, where a similar result is obtained for a less restricted class of quaternionic maps. We give a criterion, based on the energy-minimizing property of holomorphic maps, that distinguishes $J_p$-holomorphic functions among $\psi$-regular functions.

Key words: Quaternionic regular functions, hypercomplex structure, hyperkähler space

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1. INTRODUCTION

Let $\mathbb{H}$ be the space of quaternions, with its standard hypercomplex structure given by the complex structures $J_1$, $J_2$ on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by $i$ and $j$. Let $J_1^*$, $J_2^*$ be the dual structures on $T^*\mathbb{H}$.

We consider the module $\mathcal{R}(\Omega) = \{ f = f_1 + f_2j \mid J_2^*(df) = 0 \text{ on } \Omega \}$ of left $\psi$-regular functions on $\Omega$. These functions are in a simple correspondence with Fueter left regular functions, since they can be obtained from them by means of a real coordinate reflection in $\mathbb{H}$. They have been studied by many authors (see for instance Sudbery[8], Shapiro and Vasilevski[9] and Nôno[5]). The space $\mathcal{R}(\Omega)$ contains the identity mapping and any holomorphic mapping $(f_1, f_2)$ on $\Omega$ defines a $\psi$-regular function $f = f_1 + f_2j$. This is no more true if we replace the class of $\psi$-regular functions with that of regular functions. The definition of $\psi$-regularity is also equivalent to that of $q$-holomorphicity given by Joyce[2] in the setting of hypercomplex manifolds.

For every unitary vector $p$ in $\mathbb{S}^2 \subset \mathbb{H}$, $\mathcal{R}(\Omega)$ contains the space $\text{Hol}_p(\Omega, \mathbb{H}) = \{ f : \Omega \to \mathbb{H} \mid df + p(J_p(df)) = 0 \text{ on } \Omega \}$ of holomorphic functions w.r.t. the complex structure $J_p = p_1J_1 + p_2J_2 + p_3J_3$ on $\Omega$ and to the structure induced on $\mathbb{H}$ by left-multiplication by $p$ ($J_p$-holomorphic functions on $\Omega$).

We show that on every domain $\Omega$ there exist $\psi$-regular functions that are not $J_p$-holomorphic for any $p$. A similar result was obtained by Chen and Li[1] for the larger class of $q$-maps between hyperkähler manifolds.

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This result is a consequence of a criterion (cf. Theorem 2 of $J_\rho$-holomorphy, which is obtained using the energy-minimizing property of $\psi$-regular functions (cf. Proposition 11 and ideas of Lichnerowicz [4] and Chen and Li [1]).

In Sec. 4.4 we give some other applications of the criterion. In particular, we show that if $\Omega$ is connected, then the intersection $Hol_\rho(\Omega, \mathbb{H}) \cap Hol_\rho(\Omega, \mathbb{H})$ (p \neq \pm p') contains only affine maps. This result is in accord with what was proved by Sommese [7] about quaternionic maps (cf. Sec. 3.2 for definitions).

2. Fueter-regular and $\psi$-regular functions

2.1. Notations and definitions. We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$. Let $\Omega$ be a bounded domain in $\mathbb{H} \simeq \mathbb{C}^2$. A quaternionic function $f = f_1 + f_2j \in C^1(\Omega)$ is (left) regular on $\Omega$ (in the sense of Fueter) if

\[ Df = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0 \quad \text{on} \; \Omega. \]

Given the “structural vector” $\psi = (1, i, j, -k)$, $f$ is called (left) $\psi$-regular on $\Omega$ if

\[ D'f = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} - k \frac{\partial f}{\partial x_3} = 0 \quad \text{on} \; \Omega. \]

We recall some properties of regular functions, for which we refer to the papers of Sudbery [3], Shapiro and Vasilevski [6] and Nôno [5]:

1. $f$ is $\psi$-regular if and only if $Df_1 = D\overline{f}_2$, $D\overline{f}_1 = -Df_2$.
2. Every holomorphic map $(f_1, f_2)$ on $\Omega$ defines a $\psi$-regular function $f = f_1 + f_2j$.
3. The complex components are both holomorphic or both non-holomorphic.
4. Every regular or $\psi$-regular function is harmonic.
5. If $\Omega$ is pseudoconvex, every complex harmonic function is the complex component of a $\psi$-regular function on $\Omega$.
6. The space $R(\Omega)$ of $\psi$-regular functions on $\Omega$ is a right $\mathbb{H}$-module with integral representation formulas.

2.2. $q$-holomorphic functions. A definition equivalent to $\psi$-regularity has been given by Joyce [2] in the setting of hypercomplex manifolds. Joyce introduced the module of $q$-holomorphic functions on a hypercomplex manifold. On this module he defined a (commutative) product. A hypercomplex structure on the manifold $\mathbb{H}$ is given by the complex structures $J_1, J_2$ on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by $i$ and $j$. Let $J_1^*, J_2^*$ be the dual structures on $T^*\mathbb{H}$. In complex coordinates

\[
\begin{align*}
J_1^*dz_1 &= i\;dz_1, & J_1^*dz_2 &= i\;dz_2 \\
J_2^*dz_1 &= -d\bar{z}_2, & J_2^*dz_2 &= d\bar{z}_1 \\
J_3^*dz_1 &= i\;d\bar{z}_2, & J_3^*dz_2 &= -i\;d\bar{z}_1
\end{align*}
\]

where we make the choice $J_3^* = J_1^*J_2^* \Rightarrow J_3 = -J_1J_2$.

A function $f$ is $\psi$-regular if and only if $f$ is $q$-holomorphic, i.e.

\[ df + iJ_1^*(df) + jJ_2^*(df) + kJ_3^*(df) = 0. \]
In complex components $f = f_1 + f_2j$, we can rewrite the equations of $\psi$-regularity as

$$\overline{\partial}f_1 = J^*_q(\partial f_2).$$

3. Holomorphic maps

3.1. Holomorphic functions w.r.t. a complex structure $J_p$. Let $J_p = p_1J_1 + p_2J_2 + p_3J_3$ be the complex structure on $\mathbb{H}$ defined by a unit imaginary quaternion $p = p_1i + p_2j + p_3k$ in the sphere $S^2 = \{p \in \mathbb{H} \mid p^2 = -1\}$. It is well-known that every complex structure compatible with the standard hyperkähler structure of $\mathbb{H}$ is of this form. If $f = f^0 + if^1 : \Omega \to \mathbb{C}$ is a $J_p$-holomorphic function, i.e. $df^0 = J^*_p(df^1)$ or, equivalently, $df + iJ^*_p(df) = 0$, then $f$ defines a $\psi$-regular function $f = f^0 + pf^1$ on $\Omega$. We can identify $\tilde{f}$ with a holomorphic function

$$\tilde{f} : (\Omega, J_p) \to (\mathbb{C}_p, L_p)$$

where $\mathbb{C}_p = \langle 1, p \rangle$ is a copy of $\mathbb{C}$ in $\mathbb{H}$ and $L_p$ is the complex structure defined on $T^*\mathbb{C}_p \simeq \mathbb{C}_p$ by left multiplication by $p$.

More generally, we can consider the space of holomorphic maps from $(\Omega, J_p)$ to $(\mathbb{H}, L_p)$

$$\text{Hol}_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = \text{Ker}\overline{\partial}_p$$

(the $J_p$-holomorphic maps on $\Omega$) where $\overline{\partial}_p$ is the Cauchy-Riemann operator w.r.t. the structure $J_p$

$$\overline{\partial}_p = \frac{1}{2}(d + pJ^*_p \circ d).$$

For any positive orthonormal basis $\{1, p, q, pq\}$ of $\mathbb{H}$ ($p, q \in S^2$), the equations of $\psi$-regularity can be rewritten in complex form as

$$\overline{\partial}_p f_1 = J^*_q(\partial f_2),$$

where $f = (f^0 + pf^1) + (f^2 + pf^3)q = f_1 + f_2q$. Then every $f \in \text{Hol}_p(\Omega, \mathbb{H})$ is a $\psi$-regular function on $\Omega$.

Remark 1. 1) The identity map is in $\text{Hol}_i(\Omega, \mathbb{H}) \cap \text{Hol}_j(\Omega, \mathbb{H})$, but not in $\text{Hol}_k(\Omega, \mathbb{H})$.
2) $\text{Hol}_{-p}(\Omega, \mathbb{H}) = \text{Hol}_p(\Omega, \mathbb{H})$
3) If $f \in \text{Hol}_p(\Omega, \mathbb{H}) \cap \text{Hol}_{p'}(\Omega, \mathbb{H})$, with $p \neq \pm p'$, then $f \in \text{Hol}_{p''}(\Omega, \mathbb{H})$ for every $p'' = \frac{ap + b}{\overline{ap} + \overline{b}}$.
4) $\psi$-regularity distinguishes between holomorphic and anti-holomorphic maps: if $f$ is an anti-holomorphic map from $(\Omega, J_p)$ to $(\mathbb{H}, L_p)$, then $f$ can be $\psi$-regular or not. For example, $f = \tilde{z}_1 + \bar{z}_2j \in \text{Hol}_j(\Omega, \mathbb{H}) \cap \text{Hol}_k(\Omega, \mathbb{H})$ is a $\psi$-regular function induced by the anti-holomorphic map

$$\tilde{(z_1, z_2)} : (\Omega, J_1) \to (\mathbb{H}, L_i),$$

while $(\bar{z}_1, 0) : (\Omega, J_1) \to (\mathbb{H}, L_i)$ induces the function $g = \bar{z}_1 \notin \mathcal{R}(\Omega)$.

3.2. Quaternionic maps. A particular class of $J_p$-holomorphic maps is constituted by the quaternionic maps on the quaternionic manifold $\Omega$. Sommese\cite{S} defined quaternionic maps between hypercomplex manifolds: a quaternionic map is a map

$$f : (X, J_1, J_2) \to (Y, K_1, K_2)$$

that is holomorphic from $(X, J_1)$ to $(Y, K_1)$ and from $(X, J_2)$ to $(Y, K_2)$. 
In particular, a quaternionic map
\[ f : (\Omega, J_1, J_2) \rightarrow (\mathbb{H}, J_1, J_2) \]
is an element of \( \text{Hol}_1(\Omega, \mathbb{H}) \cap \text{Hol}_j(\Omega, \mathbb{H}) \) and then a \( \psi \)-regular function on \( \Omega \). Sommese showed that quaternionic maps are affine. They appear for example as transition functions for 4-dimensional quaternionic manifolds.

4. Non-holomorphic \( \psi \)-regular maps

A natural question can now be raised: can \( \psi \)-regular maps always be made holomorphic by rotating the complex structure or do they constitute a new class of harmonic maps? In other words, does the space \( \mathcal{R}(\Omega) \) contain the union
\[ \bigcup_{p \in \mathbb{S}^2} \text{Hol}_p(\Omega, \mathbb{H}) \]
properly?

Chen and Li \[1\] posed and answered the analogous question for the larger class of \( q \)-maps between hyperkähler manifolds. In their definition, the complex structures of the source and target manifold can rotate independently. This implies that also anti-holomorphic maps are \( q \)-maps.

4.1. Energy and regularity. The energy (w.r.t. the euclidean metric \( g \)) of a map \( f : \Omega \rightarrow \mathbb{C}^2 \simeq \mathbb{H} \), of class \( C^1(\overline{\Omega}) \), is the integral
\[ E(f) = \frac{1}{2} \int_{\Omega} \|df\|^2 dV = \frac{1}{2} \int_{\Omega} \langle g, f^* g \rangle dV = \frac{1}{2} \int_{\Omega} \text{tr}(J_C(f)J_C(f)^T) dV, \]
where \( J_C(f) \) is the Jacobian matrix of \( f \) with respect to the coordinates \( \bar{z}_1, z_1, \bar{z}_2, z_2 \).

Lichnerowicz \[4\] proved that holomorphic maps between Kähler manifolds minimize the energy functional in their homotopy classes. Holomorphic maps \( f \) smooth on \( \overline{\Omega} \) minimize energy in the homotopy class constituted by maps \( u \) with \( u|_{\partial \Omega} = f|_{\partial \Omega} \) which are homotopic to \( f \) relative to \( \partial \Omega \).

From the theorem, functions \( f \in \text{Hol}_p(\Omega, \mathbb{H}) \) minimize the energy functional in their homotopy classes (relative to \( \partial \Omega \)). More generally:

Proposition 1. If \( f \) is \( \psi \)-regular on \( \Omega \), then it minimizes energy in its homotopy class (relative to \( \partial \Omega \)).

Proof. We repeat arguments of Lichnerowicz, Chen and Li. Let \( i_1 = i, i_2 = j, i_3 = k \) and let
\[ K(f) = \int_{\Omega} \sum_{\alpha=1}^{3} \langle J_\alpha, f^* L_{i_\alpha} \rangle dV, \quad I(f) = \frac{1}{2} \int_{\Omega} \|df + \sum_{\alpha=1}^{3} L_{i_\alpha} \circ df \circ J_\alpha \|^2 dV. \]
Then \( K(f) \) is a homotopy invariant of \( f \) and \( I(f) = 0 \) if and only if \( f \in \mathcal{R}(\Omega) \). A computation similar to that made by Chen and Li \[1\] gives
\[ E(f) + K(f) = \frac{1}{4} I(f) \geq 0. \]
From this the result follows immediately. \( \square \)
4.2. A criterion for holomorphicity. We now come to our main result. Let $f : \Omega \to \mathbb{H}$ be a function of class $C^1(\Omega)$.

**Theorem 2.** Let $A = (a_{\alpha\beta})$ be the $3 \times 3$ matrix with entries

$$a_{\alpha\beta} = -\int_\Omega (J_\alpha, f^*L_\beta) dV.$$ 

Then

1. $f$ is $\psi$-regular if and only if $\mathcal{E}(f) = \text{tr}A$.
2. If $f \in \mathcal{R}(\Omega)$, then $A$ is real, symmetric and

$$\text{tr}A \geq \lambda_1 = \max\{\text{eigenvalues of } A\}.$$ 

It follows that $\det(A - (\text{tr}A)I_3) \leq 0$.
3. If $f \in \mathcal{R}(\Omega)$, then $f$ belongs to some space $\text{Hol}_p(\Omega, \mathbb{H})$ if and only if $\mathcal{E}(f) = \text{tr}A = \lambda_1$ or, equivalently, $\det(A - (\text{tr}A)I_3) = 0$.
4. If $\mathcal{E}(f) = \text{tr}A = \lambda_1$, $X_p = (p_1, p_2, p_3)$ is a unit eigenvector of $A$ relative to the largest eigenvalue $\lambda_1$ if and only if $f \in \text{Hol}_p(\Omega, \mathbb{H})$.

4.3. The existence of non-holomorphic $\psi$-regular maps. The criterion can be applied to show that on every domain $\Omega$ in $\mathbb{H}$, there exist $\psi$-regular functions that are not holomorphic.

**Example 1.** Let $f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$. Then $f$ is $\psi$-regular, but not holomorphic, since on the unit ball $B$ in $\mathbb{C}^2$, $f$ has energy $\mathcal{E}(f) = 6$ and the matrix $A$ of the theorem is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$ 

Therefore $\mathcal{E}(f) = \text{tr}A > 2 = \lambda_1$.

In the preceding example, the Jacobian matrix of the function has even rank, a necessary condition for a holomorphic map. In the case when the rank is odd, the non-holomorphicity follows immediately. For example, $g = z_1 + \bar{z}_1 + \bar{z}_2j$ is $\psi$-regular (on any $\Omega$) but not $J_p$-holomorphic, for any $p$, since $\text{rk}J_c(f)$ is odd.

**Example 2.** The linear, $\psi$-regular functions constitute a $\mathbb{H}$-module of dimension 3 over $\mathbb{H}$, generated e.g., by the set $\{z_1 + z_2j, z_2 + z_1j, \bar{z}_1 + \bar{z}_2j\}$. An element

$$f = (z_1 + z_2j)q_1 + (z_2 + z_1j)q_2 + (\bar{z}_1 + \bar{z}_2j)q_3$$

is holomorphic if and only if the coefficients $q_1 = a_1 + a_2j$, $q_2 = b_1 + b_2j$, $q_3 = c_1 + c_2j$ satisfy the 6th-degree real homogeneous equation

$$\det(A - (\text{tr}A)I_3) = 0$$

obtained after integration on $B$. The explicit expression of this equation is given in the Appendix. So “almost all” (linear) $\psi$-regular functions are non-holomorphic.

**Example 3.** A positive example (with $p \neq i, j, k$). Let $h = \bar{z}_1 + (z_1 + \bar{z}_2)j$. On the unit ball $h$ has energy 3 and the matrix $A$ is

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}.$$
then $\mathcal{E}(h) = \text{tr} A$ is equal to the (simple) largest eigenvalue, with unit eigenvector $X = \frac{1}{\sqrt{3}}(1, 0, 2)$. It follows that $h$ is $J_p$-holomorphic with $p = \frac{1}{\sqrt{3}}(i + 2k)$, i.e. it satisfies the equation

$$df + \frac{1}{2}(i + 2k)(J_1^p + 2J_3^p)(df) = 0.$$ 

**Example 4.** We give a quadratic example. Let $f = |z_1|^2 - |z_2|^2 + \bar{z}_1z_2j$. $f$ has energy 2 on $B$ and the matrix $A$ is

$$A = \begin{bmatrix} -2/3 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}$$

Then $f$ is $\psi$-regular but not holomorphic w.r.t. any complex structure $J_p$.

**4.4. Other applications of the criterion.** 1) If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$ for two $\mathbb{R}$-independent $p, p'$, then $X_p, X_{p'}$ are independent eigenvectors relative to $\lambda_1$. Therefore the eigenvalues of the matrix $A$ are $\lambda_1 = \lambda_2 = -\lambda_3$.

If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H}) \cap Hol_{p''}(\Omega, \mathbb{H})$ for three $\mathbb{R}$-independent $p, p', p''$ then $\lambda_1 = \lambda_2 = \lambda_3 = 0 \Rightarrow A = 0$ and therefore $f$ has energy 0 and $f$ is a (locally) constant map.

2) If $\Omega$ is connected, then $Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H}) (p \neq \pm p')$ contains only affine maps (cf. Sommese[7]).

We can assume $p = i$, $p' = j$ since in view of property 3) of Remark 1 we can suppose $p$ and $p'$ orthogonal quaternions and then we can rotate the space of imaginary quaternions. Let $f \in Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H})$ and $a = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} \\ \frac{\partial f_2}{\partial z_1} \\ \bar{a}_1 \bar{a}_2 \end{pmatrix}$, $b = \begin{pmatrix} \frac{\partial f_2}{\partial z_2} \\ \frac{\partial f_1}{\partial z_2} \\ \bar{a}_1 \bar{a}_2 \end{pmatrix}$. Since $f \in Hol_i(\Omega, \mathbb{H})$, the matrix $A$ is obtained after integration on $\Omega$ of the matrix

$$\begin{bmatrix} |a|^2 + |b|^2 & 0 & 0 \\ 0 & 2\text{Re}(a, b) & -2\text{Im}(a, b) \\ 0 & -2\text{Im}(a, b) & -2\text{Re}(a, b) \end{bmatrix}$$

where $\langle a, b \rangle$ denotes the standard hermitian product of $\mathbb{C}^2$.

Since $f \in Hol_j(\Omega, \mathbb{H})$, we have $\int_\Omega \text{Im}(\langle a, b \rangle)dV = 0$ and $\int_\Omega |a - b|^2dV = 0$. Therefore $a = b$ on $\Omega$. Then $a$ is holomorphic and anti-holomorphic w.r.t. the standard structure $J_1$. This means that $a$ is constant on $\Omega$ and $f$ is an affine map with linear part of the form

$$(a_1z_1 - \bar{a}_2z_2) + (a_2z_1 + \bar{a}_1z_2)j$$

i.e. the right multiplication of $q = z_1 + z_2j$ by the quaternion $a_1 + a_2j$.

3) We can give a classification of $\psi$-regular functions based on the dimension of the set of complex structures w.r.t. which the function is holomorphic. Let $\Omega$ be connected. Given a function $f \in \mathcal{R}(\Omega)$, we set

$$\mathcal{J}(f) = \{ p \in \mathbb{S}^2 \mid f \in Hol_p(\Omega, \mathbb{H}) \}.$$ 

The space $\mathcal{R}(\Omega)$ of $\psi$-regular functions is the disjoint union of subsets of functions of the following four types:

(i) $f$ is $J_p$-holomorphic for three $\mathbb{R}$-independent structures $\quad \Rightarrow f$ is a constant and $\mathcal{J}(f) = \mathbb{S}^2$.

(ii) $f$ is $J_p$-holomorphic for exactly two $\mathbb{R}$-independent structures $\quad \Rightarrow f$ is a $\psi$-regular, invertible affine map and $\mathcal{J}(f)$ is an equator $\mathbb{S}^1 \subset \mathbb{S}^2$. 


(iii) \( f \) is \( J_p \)-holomorphic for exactly one structure \( J_p \) (up to sign of \( p \)) \( \Rightarrow \mathcal{J}(f) \)
is a two-point set \( \mathbb{S}^0 \).

(iv) \( f \) is \( \psi \)-regular but not \( J_p \)-holomorphic w.r.t. any complex structure \( \Rightarrow \mathcal{J}(f) = \emptyset \).

We will return in a subsequent paper to the application of the criterion to the study of \( \psi \)-biregular functions, which are invertible \( \psi \)-regular functions with \( \psi \)-regular inverse (see Królikowski and Porter[3] for the class of biregular functions). This class contains as a proper subset the invertible holomorphic maps.

5. Sketch of proof of Theorem 2

If \( f \in \mathcal{R}(\Omega) \), then \( \mathcal{E}(f) = -\mathcal{K}(f) = trA \). Let

\[
\mathcal{I}_p(f) = \frac{1}{2} \int_\Omega \| df + L_p \circ df \circ J_p \|^2 dV.
\]

Then we obtain, as in Chen and Li[1]

\[
\mathcal{E}(f) + \int_\Omega (J_p, f^* L_p) dV = \frac{1}{4} \mathcal{I}_p(f).
\]

If \( X_p = (p_1, p_2, p_3) \), then

\[
XAX^T = \sum_{\alpha,\beta} p_\alpha p_\beta a_{\alpha \beta} = -\int_\Omega \left( \sum_\alpha p_\alpha J_\alpha, f^* \sum_\beta p_\beta L_{i\beta} \right) dV
\]

\[
= -\int_\Omega (J_p, f^* L_p) dV = \mathcal{E}(f) - \frac{1}{4} \mathcal{I}_p(f).
\]

Then \( trA = \mathcal{E}(f) = XAX^T + \frac{1}{4} \mathcal{I}_p(f) \geq XAX^T \), with equality if and only if \( \mathcal{I}_p(f) = 0 \) i.e if and only if \( f \) is a \( J_p \)-holomorphic map.

Let \( M_\alpha (\alpha = 1, 2, 3) \) be the matrix associated to \( J_p^* \) w.r.t. the basis \( \{d\bar{z}_1, dz_1, d\bar{z}_2, dz_2\} \).

The entries of the matrix \( A \) can be computed by the formula

\[
a_{\alpha \beta} = -\int_\Omega (J_\alpha, f^* L_{i\beta}) dV = \frac{1}{2} \int_\Omega tr(B_\alpha^T C_\beta) dV
\]

where \( B_\alpha = M_\alpha J_\mathcal{C}(f)^T \) for \( \alpha = 1, 2 \), \( B_\alpha = -M_\alpha J_\mathcal{C}(f)^T \) for \( \alpha = 3 \) and \( C_\beta = J_\mathcal{C}(f)^T M_\beta \) for \( \beta = 1, 2, 3 \).

A direct computation shows how from the particular form of the Jacobian matrix of a \( \psi \)-regular function it follows the symmetry property of \( A \).

APPENDIX

We give the explicit expression of the 6\textsuperscript{th}-degree real homogeneous equation satisfied by the complex coefficients of a linear \( J_p \)-holomorphic \( \psi \)-regular function.
\[ a_1 b_1 b_2 \overline{a}_1 \overline{c}_1 \overline{c}_2 - a_2 b_1 b_2 \overline{a}_2 \overline{c}_1 \overline{c}_2 - a_1 b_2^2 \overline{a}_2 \overline{c}_1 \overline{c}_2 - b_1 \overline{a}_1 \overline{a}_2 \overline{b}_1 \overline{c}_1 \overline{c}_2 - b_2 \overline{a}_2^2 \overline{b}_1 \overline{c}_1 \overline{c}_2 - b_1 \overline{a}_1^2 \overline{b}_2 \overline{c}_1 \overline{c}_2 - b_2 \overline{a}_1 \overline{a}_2 \overline{b}_2 \overline{c}_1 \overline{c}_2 - a_2 b_1 b_2 \overline{a}_1 \overline{c}_2^2 - a_2 b_2^2 \overline{a}_2 \overline{c}_2^2 + b_1 \overline{a}_1^2 \overline{b}_1 \overline{c}_2^2 + b_2 \overline{a}_1 \overline{a}_2 \overline{b}_1 \overline{c}_2^2 = 0 \]

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