PERIODIC DISTRIBUTIONS AND PERIODIC ELEMENTS IN MODULATION SPACES

JOACHIM TOFT AND ELMIRA NABIZADEH

Abstract. We characterize periodic elements in Gevrey classes, Gelfand-Shilov distribution spaces and modulation spaces, in terms of estimates of involved Fourier coefficients, and by estimates of their short-time Fourier transforms. If \( q \in [1, \infty) \), \( \omega \) is a suitable weight and \((E_0^q)'\) is the set of all \( E \)-periodic elements, then we prove that the dual of \( M_{\omega}^{\infty} \cap (E_0^q)' \) equals \( M_{(1/\omega)}^{\infty} \cap (E_0^q)' \) by suitable extensions of Bessel's identity.

0. Introduction

A fundamental issue in analysis concerns periodicity. For example, several problems in the theory of partial differential equations and in signal processing involve periodic functions and distributions. In such situations it is in general possible to discretize the problems by means of Fourier series expansions of these functions and distributions.

We recall that if \( f \) is a smooth 1-periodic function on \( \mathbb{R}^d \), then \( f \) is equal to its Fourier series

\[
\sum_{\alpha \in \mathbb{Z}^d} c(\alpha) e^{2\pi i \langle \cdot, \alpha \rangle},
\]

(0.1)

where the Fourier coefficients \( c(\alpha) \) can be evaluated by the formula

\[
c(f, \alpha) = c(\alpha) = \int_{[0,1]^d} f(x) e^{-2\pi i (x, \alpha)} \, dx.
\]

(Our investigations later on involve functions and distributions with more general periodics. See also [17], and Sections 1 and 2 for notations.) By the smoothness of \( f \) it follows that for every \( N \geq 0 \), there is a constant \( C_N \geq 0 \) such that

\[
|c(\alpha)| \leq C_N \langle \alpha \rangle^{-N},
\]

(0.2)

and it follows from Weierstrass theorem that the series \( (0.1) \) is uniformly convergent (cf. e.g. [17] Section 7.2)).

2010 Mathematics Subject Classification. primary: 42B05, 42B35, 46F99, 46Exx secondary: 46B40.
Assume instead that $f$ is a 1-periodic distribution on $\mathbb{R}^d$, and let $\phi$ be compactly supported and smooth on $\mathbb{R}^d$ such that
\[ \sum_{k \in \mathbb{Z}^d} \phi(\cdot - k) = 1. \quad (0.3) \]
Then $f$ is a tempered distribution and is still equal to its Fourier series (0.1) in distribution sense. The Fourier coefficients for $f$ are uniquely defined and can be computed by
\[ c(f, \alpha) = c(\alpha) = \langle f, \phi e^{-i(\cdot, \alpha)} \rangle, \quad (0.4) \]
and satisfy
\[ |c(\alpha)| \leq C(\alpha)^N, \quad (0.5) \]
for some constants $C$ and $N$ which only depend on $f$. (Cf. e.g. [17, Section 7.2]. See also [29] for an early approach to formal Fourier series expansions.)

The conditions (0.2) and (0.5) are not only necessary but also sufficient for a formal Fourier series expansion (0.1) being smooth respectively a tempered distribution. Hence, by a unique extension of Parseval’s identity
\[ (f, \phi)_1 \equiv \sum_{\alpha \in \mathbb{Z}^d} c(f, \alpha) \overline{c(\phi, \alpha)} = \int_{[0,1]^d} f(x) \overline{\phi(x)} \, dx \quad (0.6) \]
on smooth 1-periodic functions on $\mathbb{R}^d$, it follows that the dual of the set of smooth 1-periodic functions on $\mathbb{R}^d$ is the set of all 1-periodic tempered distributions on $\mathbb{R}^d$.

Some investigations of periodicity in the framework of ultra-differentiability have also been performed. More precisely, let $s > 0$ and $\mathcal{E}^1_{s}(\mathbb{R}^d)$ be the set of all 1-periodic functions in the Gevrey class $\mathcal{E}_{s}(\mathbb{R}^d)$ of Roumieu type. (Our investigations later on also involve Gevrey classes of Beurling type.) It is proved in [19] by Pilipović that a smooth 1-periodic function $f$ on $\mathbb{R}^d$ belongs $\mathcal{E}^1_{s}(\mathbb{R}^d)$, if and only if its Fourier coefficients satisfy
\[ |c(\alpha)| \leq C e^{-r|\alpha|^t}, \]
for some constants $C > 0$ and $r > 0$ which are independent of $\alpha$.

Due to straight-forward extensions of Parseval’s identity it follows that the dual $(\mathcal{E}^1_{s})'(\mathbb{R}^d)$ of $\mathcal{E}^1_{s}(\mathbb{R}^d)$ can be identify with all expansions (0.1) such that for every constants $r > 0$ there is a constant $C > 0$ such that
\[ |c(\alpha)| \leq C e^{r|\alpha|^t}. \]
In the case $s \geq 1$, it seems to be shown by Gorbačuk and Gorbačuk in [12,13], and commented in [19] that the set of such formal Fourier series expansions coincide with the set of 1-periodic Gelfand-Shilov distributions $(\mathcal{S}^t)'(\mathbb{R}^d)$ when $t > 1$. 2
The previous properties have been extended and explained in different ways, see e.g. [3, Theorem 2.3] and [4,8,10,23–25] by Da sgupta, Fischer, Garetto, Ruzhansky and Turunen. For example, in [4], it is shown that characterizations of the previous types also hold on more general manifolds, e.g. compact ones. Here we remark that such (global) characterizations of Gevrey spaces and ultradistributions in terms of Fourier coefficients are used to prove the well-posedness and estimates for solutions to wave equations for Hörmander’s sums of squares in [10].

The aim of the paper is obtain analogous and other characterizations for periodic functions in Gevrey classes, and for periodic ultradistributions. Especially we characterize periodic Gelfand-Shilov distributions and periodic elements in modulation spaces, in terms of estimates of their Fourier coefficients. At the same time we deduce an integral formula for evaluating the Fourier coefficients, and which involve the short-time Fourier transforms of the involved periodic distributions. Finally we show that the duals of the periodic functions in Gevrey classes can be identified with suitable classes of periodic Gelfand-Shilov distributions, and characterize elements in these classes by suitable estimates on the short-time Fourier transforms of the involved functions. In contrast to earlier contributions, our characterizations hold when the Gevrey parameters belong to the interval $(0,\infty)$ instead of the sub interval $[1,\infty)$.

In Section 2 we deduce other characterizations of $\mathcal{E}_s^1(\mathbb{R}^d)$ and $(\mathcal{E}_s^1)'(\mathbb{R}^d)$. For example let $\phi$ be a non-zero element in the Gelfand-Shilov space $\mathcal{S}_s^t(\mathbb{R}^d)$. Then we show that $f \in \mathcal{E}_s^1(\mathbb{R}^d)$, if and only if $f$ is 1-periodic ultra-distribution and that its short-time Fourier transform $V_\phi f$ satisfies

$$|V_\phi f(x,\xi)| \leq Ce^{-r|\xi|^s}$$

for some constants $C > 0$ and $r > 0$. In the same way we show that $f \in (\mathcal{E}_s^1)'(\mathbb{R}^d)$, if and only if $f$ is 1-periodic ultra-distribution and for every $r > 0$ there is a constant $C > 0$ such that

$$|V_\phi f(x,\xi)| \leq Ce^{r|\xi|^s}.$$

At the same time we show (for any $s > 0$) that $(\mathcal{E}_s^1)'(\mathbb{R}^d)$ may in canonical ways be identified with the set of periodic elements in the Gelfand-Shilov distribution space $(\mathcal{S}_s^t)'(\mathbb{R}^d)$, provided $t > 0$ satisfies $s + t \geq 1$.

An ingredient in the proofs of these properties is the formula

$$(f,\psi)_1 = \|\phi\|_{L^2}^{-2} \int_{[0,1]^d} \left( \int_{\mathbb{R}^d} (V_\phi f)(x,\xi) (V_\phi \psi)(x,\xi) d\xi \right) dx.$$  \hspace{1cm} (0.7)

proved in Section 2 when evaluating the form in (0.6). By letting $\psi = e^{2\pi i \langle \cdot, \alpha \rangle}$, it follows by straight-forward computations that (0.7) takes
the form
\[ c(f, \alpha) = \|\phi\|_{L^2([0,1]d)}^{-2} \int_{[0,1]^d} \left( \int_{\mathbb{R}^d} (V_\phi f)(x, \xi) \hat{\phi}(\alpha - \xi) e^{-2\pi i (x, \alpha - \xi)} d\xi \right) dx. \]

Here the integrand belongs to \( L^1([0,1]^d \times \mathbb{R}^d) \) due to the deduced characterizations of \((\mathcal{E}_s^t)'(\mathbb{R}^d)\).

It seems to be difficult to find the previous formulae in the literature. When using (0.4) to compute the Fourier coefficients, it is essential that \( \phi \) satisfies (0.3). For these reasons it is difficult to carry over (0.4) to the Gevrey or Gelfand-Shilov situation when \( s \) above is less than 1, since it is difficult to find \( \phi \in \mathcal{S}_s(\mathbb{R}^d) \) which satisfies (0.3).

In Section 3 we characterize periodic distributions in modulation spaces. In particular we deduce that if \( q \in (0, \infty) \) and \( \omega(x, \xi) = \omega_0(\xi) \) is a suitable weight on \( \mathbb{R}^d \), then the 1-periodic elements in the modulation spaces \( M_{\infty,q}(\mathbb{R}^d) \) and \( W_{\infty,q}(\mathbb{R}^d) \) agree and are equal to the set of formal Fourier series expansions in (0.1) such that
\[
\{ c(\alpha)\omega_0(\alpha) \}_{\alpha \in \mathbb{Z}^d} \in \ell^q.
\]
In particular we extend Proposition 2.6 in [21] and Proposition 5.1 in [22] to involve more general weights and permit \( q \) to be in the broader interval \((0, \infty]\) instead of \([1, \infty)\).

In the last part of Section 3 we apply these results to deduce that if \( q \in [1, \infty) \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \), then the dual of \( M_{(1/\omega)}^{\infty,q}(\mathbb{R}^d) \cap (\mathcal{E}_s^t)'(\mathbb{R}^d) \) is equal to \( M_{(1/\omega)}^{\infty,q}(\mathbb{R}^d) \cap (\mathcal{E}_s^t)'(\mathbb{R}^d) \) through suitable extensions of the form \((\cdot, \cdot)_1\) on \( \mathcal{E}_s^t(\mathbb{R}^d) \times \mathcal{E}_s^t(\mathbb{R}^d) \).

1. Preliminaries

In this section we recall some basic facts. We start by discussing Gelfand-Shilov spaces and their properties. Thereafter we recall some properties of modulation spaces and discuss different aspects of periodic distributions.

1.1. Gelfand-Shilov spaces and Gevrey classes. Let \( 0 < s, t \in \mathbb{R} \) be fixed. Then the Gelfand-Shilov space \( \mathcal{S}_s^t(\mathbb{R}^d) \) (Σ_s^t(\mathbb{R}^d)) of Roumieu type (Beurling type) with parameters \( s \) and \( t \) consists of all \( f \in C^\infty(\mathbb{R}^d) \) such that
\[
\|f\|_{\mathcal{S}_s^t} = \sup_{h > 0} \frac{|x^\beta D^\alpha f(x)|}{h^{\alpha + \beta}|\alpha!s^\beta t^\beta}}
\]
is finite for some \( h > 0 \) (for every \( h > 0 \)). Here the supremum should be taken over all \( \alpha, \beta \in \mathbb{N}^d \) and \( x \in \mathbb{R}^d \). We equip \( \mathcal{S}_s^t(\mathbb{R}^d) \) (Σ_s^t(\mathbb{R}^d)) by the canonical inductive limit topology (projective limit topology) with respect to \( h > 0 \), induced by the semi-norms in (1.1).
For any \( s, t, s_0, t_0 > 0 \) such that \( s > s_0, t > t_0 \) and \( s + t \geq 1 \) we have
\[
\mathcal{S}_s^{s_0} (\mathbb{R}^d) \hookrightarrow \Sigma_s (\mathbb{R}^d) \hookrightarrow \mathcal{S}_t (\mathbb{R}^d) \hookrightarrow \mathcal{F} (\mathbb{R}^d),
\]
\begin{equation}
\mathcal{F}' (\mathbb{R}^d) \hookrightarrow (\mathcal{S}_t)' (\mathbb{R}^d) \hookrightarrow (\Sigma_s)' (\mathbb{R}^d) \hookrightarrow (\mathcal{S}_s^{s_0})' (\mathbb{R}^d),
\end{equation}
with dense embeddings. Here and in what follows we use the notation \( A \hookrightarrow B \) when the topological spaces \( A \) and \( B \) satisfy \( A \subseteq B \) with continuous embeddings. The space \( \Sigma_s (\mathbb{R}^d) \) is a Fréchet space with seminorms \( \| \cdot \|_{\Sigma_s^h}, h > 0 \). Moreover, \( \Sigma_s (\mathbb{R}^d) \neq \{0\} \), if and only if \( s + t \geq 1 \) and \( (s, t) \neq \left( \frac{1}{2}, \frac{1}{2} \right) \), and \( \mathcal{S}_t (\mathbb{R}^d) \neq \{0\} \), if and only if \( s + t \geq 1 \).

The Gelfand-Shilov distribution spaces \( (\mathcal{S}_t)' (\mathbb{R}^d) \) and \( (\Sigma_s)' (\mathbb{R}^d) \) are the dual spaces of \( \mathcal{S}_t (\mathbb{R}^d) \) and \( \Sigma_s (\mathbb{R}^d) \), respectively. As for the Gelfand-Shilov spaces there is a canonical projective limit topology (inductive limit topology) for \( (\mathcal{S}_t)' (\mathbb{R}^d) \) \( (\Sigma_s)' (\mathbb{R}^d) \). For convenience we set
\[
\mathcal{S}_s = \mathcal{S}_s^s, \quad \mathcal{S}_s' = (\mathcal{S}_s^s)', \quad \Sigma_s = \Sigma_s^s \quad \text{and} \quad \Sigma_s' = (\Sigma_s^s)'.
\]

From now on we let \( \mathcal{F} \) be the Fourier transform which takes the form
\[
(\mathcal{F} f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)e^{-i(x,\xi)} \, dx
\]
when \( f \in L^1(\mathbb{R}^d) \). Here \( (\cdot, \cdot) \) denotes the usual scalar product on \( \mathbb{R}^d \). The map \( \mathcal{F} \) extends uniquely to homeomorphisms on \( \mathcal{F}'(\mathbb{R}^d) \), from \( (\mathcal{S}_t)'(\mathbb{R}^d) \) to \( (\mathcal{S}_s)'(\mathbb{R}^d) \) and from \( (\Sigma_s)'(\mathbb{R}^d) \) to \( (\Sigma_t)'(\mathbb{R}^d) \). Furthermore, \( \mathcal{F} \) restricts to homeomorphisms on \( \mathcal{F}(\mathbb{R}^d) \), from \( \mathcal{S}_s(\mathbb{R}^d) \) to \( \mathcal{S}_t(\mathbb{R}^d) \) and from \( \Sigma_s(\mathbb{R}^d) \) to \( \Sigma_t(\mathbb{R}^d) \), and to a unitary operator on \( L^2(\mathbb{R}^d) \).

Gelfand-Shilov spaces can in convenient ways be characterized in terms of estimates the functions and their Fourier transforms. More precisely, in [15] it is proved that if \( f \in \mathcal{S}'(\mathbb{R}^d) \) and \( s, t > 0 \), then \( f \in \mathcal{S}_s^t(\mathbb{R}^d) \) \( \{ f \in \mathcal{S}_s^t(\mathbb{R}^d) \} \), if and only if
\begin{equation}
|f(x)| \lesssim e^{-r|x|^\frac{1}{t}} \quad \text{and} \quad |\hat{f}(\xi)| \lesssim e^{-r|\xi|^\frac{1}{s}},
\end{equation}
for some \( r > 0 \) (for every \( r > 0 \)). Here and in what follows, \( A \lesssim B \) means that \( A \leq cB \) for a suitable constant \( c > 0 \). We also set \( A \asymp B \) when \( A \lesssim B \) and \( B \lesssim A \).

Gelfand-Shilov spaces and their distribution spaces can also be characterized by estimates of short-time Fourier transforms, (see e.g. [16], [28]). More precisely, let \( \phi \in \mathcal{S}_0(\mathbb{R}^d) \) be fixed. Then the short-time Fourier transform \( V_\phi f \) of \( f \in \mathcal{S}_s^t(\mathbb{R}^d) \) with respect to the window function \( \phi \) is the Gelfand-Shilov distribution on \( \mathbb{R}^{2d} \), defined by
\[
V_\phi f(x, \xi) = \mathcal{F}(f \phi(\cdot - x))(\xi).
\]
If \( f, \phi \in \mathcal{S}_0(\mathbb{R}^d) \), then it follows that
\[
V_\phi f(x, \xi) = (2\pi)^{-\frac{d}{2}} \int f(y)\overline{\phi(y - x)}e^{-i(y, \xi)} \, dy.
\]
Remark 1.1. Let \( s_1, s_2, t_1, t_2 > 0 \). Then the Gelfand-Shilov space \( S_{s_1, s_2}^{t_1, t_2}(\mathbb{R}^{2d}) \) \( (\Sigma_{s_1, s_2}^{t_1, t_2}(\mathbb{R}^{2d})) \) is the set of all \( F \in C^\infty(\mathbb{R}^{2d}) \) such that
\[
||x_1^{a_1} x_2^{a_2} \partial_x^{b_1} \partial_x^{b_2} F||_{L^\infty} \lesssim h^{a_1 + \alpha_1 + b_1 + b_2 + t_1 t_2 s_1 s_2}
\]
for some \( h > 0 \) (for every \( h > 0 \)).

We also let \( (S_{s_1, s_2}^{s_1, s_2}(\mathbb{R}^{2d})) \) \( (\Sigma_{s_1, s_2}^{s_1, s_2}(\mathbb{R}^{2d})) \) be corresponding duals (distribution spaces).

By [26, Theorem 2.3] it follows that the definition of the map \( (f, \phi) \mapsto V_\phi f \) from \( \mathcal{S}(\mathbb{R}^{d}) \times \mathcal{S}(\mathbb{R}^{d}) \) to \( \mathcal{S}(\mathbb{R}^{2d}) \) is uniquely extendable to a continuous map from \( (\mathcal{S}_s^{t}(\mathbb{R}^{d})) \times (\mathcal{S}_s^{t}(\mathbb{R}^{d})) \) to \( (\mathcal{S}_{s,t}^{s,t}(\mathbb{R}^{2d})) \), and restricts to a continuous map from \( \mathcal{S}_s^{t}(\mathbb{R}^{d}) \times \mathcal{S}_s^{t}(\mathbb{R}^{d}) \) to \( \mathcal{S}_{s,t}^{s,t}(\mathbb{R}^{2d}) \).

The same conclusion holds with \( \Sigma_t \) and \( \Sigma_{t,s} \) in place of \( \Sigma_s \) and \( \Sigma_{s,t} \), respectively, at each place.

The following properties characterize Gelfand-Shilov spaces and their distribution spaces in terms of estimates of short-time Fourier transform.

Proposition 1.2. Let \( s, t > 0 \) be such that \( s + t \geq 1 \). Also let \( \phi \in \Sigma_t^{s}(\mathbb{R}^{d}) \setminus \{0\} \) \( (\phi \in \Sigma_t^{s}(\mathbb{R}^{d}) \setminus \{0\}) \) and \( f \) be a Gelfand-Shilov distribution on \( \mathbb{R}^{d} \). Then \( f \in \Sigma_t^{s}(\mathbb{R}^{d}) \) \((f \in \Sigma_t^{s}(\mathbb{R}^{d}))\), if and only if
\[
|V_\phi f(x, \xi)| \lesssim e^{-r(|x|^\frac{1}{t} + |\xi|^\frac{1}{s})}, \tag{1.4}
\]
for some \( r > 0 \) (for every \( r > 0 \)).

Proposition 1.3. Let \( s, t > 0 \) be such that \( s + t \geq 1 \). Also let \( \phi \in \Sigma_t^{s}(\mathbb{R}^{d}) \setminus \{0\} \) \( (\phi \in \Sigma_t^{s}(\mathbb{R}^{d}) \setminus \{0\}) \) and \( f \) be a Gelfand-Shilov distribution on \( \mathbb{R}^{d} \). Then \( f \in (\Sigma_t^{s}(\mathbb{R}^{d})) \) \((f \in (\Sigma_t^{s}(\mathbb{R}^{d}))\), if and only if
\[
|V_\phi f(x, \xi)| \lesssim e^{r(|x|^\frac{1}{t} + |\xi|^\frac{1}{s})}, \tag{1.5}
\]
for every \( r > 0 \) (for every \( r > 0 \)).

We note that if \( s = t = \frac{1}{2} \) in Propositions 1.2 and 1.3 then it is not possible to find any \( \phi \in \Sigma_t^{s}(\mathbb{R}^{d}) \setminus \{0\} \). Hence, these results give no information in the Beurling case for such choices of \( s \) and \( t \).

A proof of Proposition 1.2 can be found in e.g. [16] (cf. [16, Theorem 2.7]) and a proof of Proposition 1.3 in the general situation can be found in [28]. See also [2] for related results.

In Section 2 we deduce analogous characterizations for periodic functions and distributions.

Remark 1.4. The short-time Fourier transform can also be used to identify elements in \( \mathcal{S}(\mathbb{R}^{d}) \) and in \( \mathcal{S}'(\mathbb{R}^{d}) \). In fact, if \( \phi \in \mathcal{S}(\mathbb{R}^{d}) \setminus \{0\} \) and \( f \) is a Gelfand-Shilov distribution on \( \mathbb{R}^{d} \), then the following is true:

(1) \( f \in \mathcal{S}(\mathbb{R}^{d}) \), if and only if for every \( N \geq 0 \), it holds
\[
|V_\phi f(x, \xi)| \lesssim \langle (x, \xi) \rangle^{-N};
\]
(2) \( f \in \mathcal{S}'(\mathbb{R}^d) \), if and only if for some \( N \geq 0 \), it holds
\[
|V_\phi f(x,\xi)| \lesssim ((x,\xi))^N.
\]
(Cf. [14, Chapter 12].)

Next we consider Gevrey classes on \( \mathbb{R}^d \). Let \( s \geq 0 \). For any compact set \( K \subseteq \mathbb{R}^d \), \( \phi > 0 \) and \( f \in C^\infty(\mathbb{R}^d) \) let
\[
\|f\|_{K,\phi,s} \equiv \sup_{\alpha \in \mathbb{N}^d} \left\| \partial^\alpha f \right\|_{L^\infty(K)} h|\alpha|_1^s.
\]
(1.6)

The Gevrey class \( \mathcal{E}_s(K) \) \( (\mathcal{E}_{0,s}(K)) \) of order \( s \) and of Roumieu type (of Beurling type) is the set of all \( f \in C^\infty(K) \) such that (1.6) is finite for some (for every) \( h > 0 \). We equip \( \mathcal{E}_s(K) \) \( (\mathcal{E}_{0,s}(K)) \) by the inductive (projective) limit topology supplied by the seminorms in (1.6). Finally if \( \{K_j\}_{j \geq 1} \) is an exhausted sets of compact subsets of \( \mathbb{R}^d \), then let
\[
\mathcal{E}_s(\mathbb{R}^d) = \text{proj lim } \mathcal{E}_s(K_j) \quad \text{and} \quad \mathcal{E}_{0,s}(\mathbb{R}^d) = \text{proj lim } \mathcal{E}_{0,s}(K_j).
\]

In particular,
\[
\mathcal{E}_s(\mathbb{R}^d) = \bigcap_{j \geq 1} \mathcal{E}_s(K_j) \quad \text{and} \quad \mathcal{E}_{0,s}(\mathbb{R}^d) = \bigcap_{j \geq 1} \mathcal{E}_{0,s}(K_j).
\]

It is clear that \( \mathcal{E}_s(\mathbb{R}^d) \) contains all trigonometric polynomials, which is not the case for \( \mathcal{E}_{0,s}(\mathbb{R}^d) \).

1.2. Modulation spaces. We consider a general class of modulation spaces (cf. [7]), and begin with discussing general properties for the involved weight functions. A weight on \( \mathbb{R}^d \) is a positive function \( \omega \in L^\infty_{\text{loc}}(\mathbb{R}^d) \) such that \( 1/\omega \in L^\infty_{\text{loc}}(\mathbb{R}^d) \). A usual condition on \( \omega \) is that it should be moderate, or \( v \)-moderate for some positive function \( v \in L^\infty_{\text{loc}}(\mathbb{R}^d) \). This means that
\[
\omega(x+y) \lesssim \omega(x)v(y), \quad x,y \in \mathbb{R}^d.
\]
(1.7)
We note that (1.7) implies that \( \omega \) fulfills the estimates
\[
v(-x)^{-1} \lesssim \omega(x) \lesssim v(x), \quad x \in \mathbb{R}^d.
\]
(1.8)
We let \( \mathcal{P}_E(\mathbb{R}^d) \) be the set of all moderate weights on \( \mathbb{R}^d \).

It can be proved that if \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \), then \( \omega \) is \( v \)-moderate for some \( v(x) = e^{r|x|} \), provided the positive constant \( r \) is large enough (cf. [15]). In particular, (1.8) shows that for any \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \), there is a constant \( r > 0 \) such that
\[
e^{-r|x|} \lesssim \omega(x) \lesssim e^{r|x|}, \quad x \in \mathbb{R}^d.
\]

We say that \( v \) is submultiplicative if \( v \) is even and (1.7) holds with \( \omega = v \). In the sequel, \( v \) and \( v_j \) for \( j \geq 0 \), always stand for submultiplicative weights if nothing else is stated.
Definition 1.5. Let \( r \in (0, 1], v \in \mathcal{P}_E(\mathbb{R}^d) \) and let \( \mathcal{B} \subseteq L^r_{\text{loc}}(\mathbb{R}^d) \) be a quasi-Banach space. Then \( \mathcal{B} \) is called \( v \)-invariant on \( \mathbb{R}^d \) if the following is true:

1. \( x \mapsto f(x + y) \) belongs to \( \mathcal{B} \) for every \( f \in \mathcal{B} \) and \( y \in \mathbb{R}^d \).
2. There is a constant \( C > 0 \) such that \( \|f_1\|_{\mathcal{B}} \leq C\|f_2\|_{\mathcal{B}} \) when \( f_1, f_2 \in \mathcal{B} \) are such that \( |f_1| \leq |f_2| \). Moreover,
   \[
   \|f(\cdot + y)\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{B}}v(y), \quad f \in \mathcal{B}, \ y \in \mathbb{R}^d.
   \]

The quasi-Banach spaces in the previous definition is usually a mixed quasi-normed Lebesgue space, given in Definition 1.6 below. Here \( S_d \) is the set of permutations on \( \{1, \ldots, d\} \),

\[
\max \mathbf{q} = \max(q_1, \ldots, q_d) \quad \text{and} \quad \min \mathbf{q} = \min(q_1, \ldots, q_d),
\]

when \( \mathbf{q} = (q_1, \ldots, q_d) \in (0, \infty]^d \).

Here we also let \( E \) be a non-degenerate parallelepiped in \( \mathbb{R}^d \). That is, there is a basis \( e_1, \ldots, e_d \) of \( \mathbb{R}^d \) such that

\[
E = \{ x_1e_1 + \cdots + x_de_d; (x_1, \ldots, x_d) \in \mathbb{R}^d, \ 0 \leq x_k \leq 1, \ k = 1, \ldots, d\}.
\]

The corresponding lattice, dual parallelepiped and dual lattice are given by

\[
\Lambda_E = \{ j_1e_1 + \cdots + j_de_d; (j_1, \ldots, j_d) \in \mathbb{Z}^d \},
\]

\[
E' = \{ \xi_1e_1' + \cdots + \xi_de_d'; (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \ 0 \leq \xi_k \leq 1, \ k = 1, \ldots, d\},
\]

and

\[
\Lambda'_E = \Lambda_{E'} = \{ \iota_1e_1' + \cdots + \iota_de_d'; (\iota_1, \ldots, \iota_d) \in \mathbb{Z}^d \},
\]

respectively, where \( e_1', \ldots, e_d' \) satisfies

\[
\langle e_j, e_k' \rangle = 2\pi\delta_{jk} \quad \text{for every} \quad j, k = 1, \ldots, d.
\]

Evidently, \( e_1', \ldots, e_d' \) is a basis of \( \mathbb{R}^d \). It is called the dual basis of \( e_1, \ldots, e_d \). We observe that there is a matrix \( T_E \) with \( e_1, \ldots, e_d \) as the image of the standard basis, and that the image of the standard basis of \( T_{E'} = 2\pi(T_E^{-1})^t \) is given by \( e_1', \ldots, e_d' \).

Definition 1.6. Let \( E \) be a non-degenerate parallelepiped in \( \mathbb{R}^d \), \( E' \) be the dual parallelepiped spanned by the ordered set \( \mathcal{O}_0 = (e_1', \ldots, e_d') \) in \( \mathbb{R}^d \), \( \mathbf{q} = (q_1, \ldots, q_d) \in (0, \infty]^d \), \( r = \min(1, \mathbf{q}) \) and \( \tau \in S_d \). If \( a \in \ell_0'(\Lambda_E) \) and \( f \in L^r_{\text{loc}}(\mathbb{R}^d) \), then

\[
\|a\|_{\ell^d_{\mathcal{B}_\tau}} \equiv \|b_{d-1}\|_{\ell^d(\mathbb{Z})} \quad \text{and} \quad \|f\|_{L^r_{\mathcal{B}_\tau}} \equiv \|g_{d-1}\|_{L^r(\mathbb{R})}
\]
where \( b_k(l_k) \) and \( g_k(z_k), l_k \in \mathbb{Z}^{d-k} \) and \( z_k \in \mathbb{R}^{d-k}, k = 0, \ldots, d - 1 \), are inductively defined as

\[
\begin{align*}
\quad b_0(j_1, \ldots, j_d) & \equiv \left| a(j_1 e_{r(1)} + \cdots + j_d e_{r(d)}) \right|
\quad g_0(x_1, \ldots, x_d) & \equiv \left| f(x_1 e_{r(1)} + \cdots + x_d e_{r(d)}) \right|,
\quad b_k(l_k) & \equiv \| b_{k-1}(\cdot, l_k) \|_{\ell^q_k(\mathbb{Z})},
\end{align*}
\]

and

\[
\quad g_k(z_k) \equiv \| g_{k-1}(\cdot, z_k) \|_{\ell^q_k(\mathbb{R})}, \quad k = 1, \ldots, d - 1.
\]

The space \( \ell^q_{\tilde{C}_0, \tau}(\Lambda'_{\tilde{F}}) \) consists of all \( a \in \ell^q_0(\Lambda'_{\tilde{F}}) \) such that \( \|a\|_{\ell^q_{\tilde{C}_0, \tau}} \) is finite, and \( L^q_{\tilde{C}_0, \tau}(\mathbb{R}^d) \) consists of all \( f \in L^q_{loc}(\mathbb{R}^d) \) such that \( \|f\|_{L^q_{\tilde{C}_0, \tau}} \) is finite.

**Definition 1.7.** Let \( \omega, v \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate, \( \mathcal{B} \) be a \( v \)-invariant space on \( \mathbb{R}^{2d} \), and let \( \phi \in \mathcal{S}_{1/2}(\mathbb{R}^d) \setminus \{0\} \). Then the modulation space \( M(\omega, \mathcal{B}) \) consists of all \( f \in \mathcal{S}'_{1/2}(\mathbb{R}^d) \) such that

\[
\|f\|_{M(\omega, \mathcal{B})} \equiv \|V_\phi f \cdot \omega\|_{\mathcal{B}}
\]

is finite.

The theory of modulation spaces has developed in different ways since they were introduced in [6] by Feichtinger. (Cf. e.g. [7, 9, 14, 27].) For example, by [9, 27] it follows that if \( \mathcal{B} \) in Definition 1.7 is a mixed quasi-normed space of Lebesgue type and \( \phi \in M^r_v(\mathbb{R}^d) \setminus \{0\} \), then \( M(\omega, \mathcal{B}) \) is a quasi-Banach space. Moreover, \( f \in M(\omega, \mathcal{B}) \) if and only if \( V_\phi f \cdot \omega \in \mathcal{B} \), and different choices of \( \phi \) give rise to equivalent quasi-norms in (1.9). We also note that for any such \( \mathcal{B} \), then

\[
\Sigma_1(\mathbb{R}^d) \subseteq M(\omega, \mathcal{B}) \subseteq \Sigma_1'(\mathbb{R}^d).
\]

Now let \( \phi \in \mathcal{S}_{1/2}(\mathbb{R}^d) \setminus \{0\}, r \in (0, 1], \omega, v \in \mathcal{P}_E(\mathbb{R}^d), \mathcal{B} \subseteq L^q_{loc}(\mathbb{R}^d) \) be a \( v \)-invariant quasi-Banach space. We are especially interested in the modulation spaces \( M^\infty(\omega, \mathcal{B}) \) and \( W^\infty(\omega, \mathcal{B}) \), which are defined as the sets of all \( f \in \mathcal{S}'_{1/2}(\mathbb{R}^d) \) such that

\[
\|f\|_{M^\infty(\omega, \mathcal{B})} \equiv \left\| \left( \text{ess sup}_{x \in \mathbb{R}^d} |V_\phi f(x, \cdot)| \cdot \omega \right) \right\|_{\mathcal{B}}
\]

respective

\[
\|f\|_{W^\infty(\omega, \mathcal{B})} \equiv \text{ess sup}_{x \in \mathbb{R}^d} \left( \|V_\phi f(x, \cdot) \cdot \omega\|_{\mathcal{B}} \right)
\]

are finite. By straight-forward computations it follows that

\[ M^\infty(\omega, \mathcal{B}) \hookrightarrow W^\infty(\omega, \mathcal{B}). \]
1.3. Classes of periodic elements. We shall mainly view three aspects on periodicity. First we consider spaces of periodic Gévre functions and their duals. Thereafter we focus (formal) spaces of Fourier series expansions. Finally we consider periodic Gelfand-Shilov distributions. In Section 2 we show that these different approaches lead to the same type of spaces.

Let \( s, t \in \mathbb{R}_+ \) be such that \( s + t \geq 1 \), \( f \in (\mathcal{S}_t^s)'(\mathbb{R}^d) \) and let \( E \) be a non-degenerate parallellepiped in \( \mathbb{R}^d \). Then \( f \) is called \( E \)-periodic or \( \Lambda_E \)-periodic if \( f(x + j) = f(x) \) for every \( x \in \mathbb{R}^d \) and \( j \in \Lambda_E \).

The sets of periodic elements in \((\mathcal{S}_t^s)'(\mathbb{R}^d)\) and \((\Sigma_t^s)'(\mathbb{R}^d)\) are denoted by \((\mathcal{S}_t^E)'(\mathbb{R}^d)\) and \((\Sigma_t^E)'(\mathbb{R}^d)\), respectively.

We note that for any \( \Lambda_E \)-periodic function \( f \in C^\infty(\mathbb{R}^d) \), we have

\[
f = \sum_{\alpha \in \Lambda_E'} c(f, \alpha) e^{i \langle \cdot, \alpha \rangle}, \tag{1.10}
\]

where \( c(f, \alpha) \) are the Fourier coefficients given by

\[
c(f, \alpha) \equiv |E|^{-1}(f, e^{i \langle \cdot, \alpha \rangle})_{L^2(E)}.
\]

For any \( s \geq 0 \) and non-degenerate parallellepiped \( E \subseteq \mathbb{R}^d \) we let \( \mathcal{E}_s^E(\mathbb{R}^d) \) and \( \mathcal{E}_s(\mathbb{R}^d) \) be the sets of all \( E \)-periodic elements in \( \mathcal{E}_s(\mathbb{R}^d) \) and in \( \mathcal{E}_s(\mathbb{R}^d) \), respectively. Evidently,

\[
\mathcal{E}_s^E(\mathbb{R}^d) \simeq \mathcal{E}_s(\mathbb{R}^d / \Lambda_E) \text{ and } \mathcal{E}_s^E(\mathbb{R}^d) \simeq \mathcal{E}_s(\mathbb{R}^d / \Lambda_E),
\]

which is a common approach in the literature. The duals of \( \mathcal{E}_s^E(\mathbb{R}^d) \) and \( \mathcal{E}_s(\mathbb{R}^d) \) are denoted by \((\mathcal{E}_s^E)'(\mathbb{R}^d)\) and \((\mathcal{E}_s)'(\mathbb{R}^d)\), respectively.

In Section 3 we shall characterise spaces of periodic elements given in the following definition.

**Definition 1.8.** Let \( E \in \mathbb{R}^d \) be a non-degenerate parallellepiped, \( \omega \) be a weight on \( \mathbb{R}^d \) and let \( \mathcal{B} \) be a quasi-Banach space continuously embedded in \( L^\infty_{\text{loc}}(\Lambda_E') \). Then \( \mathcal{E}^E(\omega, \mathcal{B}) \) consists of all \( f \in (\mathcal{E}_s^E)'(\mathbb{R}^d) \) such that \( \|f\|_{\mathcal{E}^E(\omega, \mathcal{B})} \equiv \|\{c(f, \alpha) \omega(\alpha)\}_{\alpha \in \Lambda_E'}\|_{\mathcal{B}} \) is finite.

Next we introduce suitable spaces of formal Fourier series expansions. For any \( r \in \mathbb{R} \) and \( s > 0 \), we let \( \mathcal{G}_s^{E_r}(\mathbb{R}^d) \) be the set of all formal expansions

\[
f = \sum_{\alpha \in \Lambda_E'} c(\alpha) e^{i \langle \cdot, \alpha \rangle}, \tag{1.10'}
\]

such that

\[
\|f\|_{\mathcal{G}_s^{E_r}} \equiv \sup_{\alpha \in \Lambda_E'} |c(\alpha) e^{r|\alpha|^d}| \text{ is finite}.
\]
is finite. Then $\mathcal{G}_{s,r}^E(\mathbb{R}^d)$ is a Banach space under the norm $\| \cdot \|_{\mathcal{G}_{s,r}^E}$. We let

$$
\mathcal{G}_{s,r}^E(\mathbb{R}^d) = \text{ind lim}_{r > 0} \mathcal{G}_{s,r}^E(\mathbb{R}^d), \quad \mathcal{G}_{0,s}^E(\mathbb{R}^d) = \text{proj lim}_{r > 0} \mathcal{G}_{s,r}^E(\mathbb{R}^d),
$$

and

$$(\mathcal{G}_{s,r}^E)'(\mathbb{R}^d) = \text{proj lim}_{r > 0} \mathcal{G}_{s,r}^E(\mathbb{R}^d), \quad (\mathcal{G}_{0,s}^E)'(\mathbb{R}^d) = \text{ind lim}_{r < 0} \mathcal{G}_{s,r}^E(\mathbb{R}^d).$$

We also let $\mathcal{G}_{0,0}^E(\mathbb{R}^d)$ be the set of all constant functions on $\mathbb{R}^d$, $\mathcal{G}_{0,s}^E(\mathbb{R}^d)$ be the set of all expansions in (1.10)′ such that all but finite numbers of $c(\alpha)$ are zero, and we let $(\mathcal{G}_{0}^E)'(\mathbb{R}^d)$ be the set of all formal expansions of the form (1.10)′ (cf. [29]).

The topology of $(\mathcal{G}_{0}^E)'(\mathbb{R}^d)$ is defined through the semi-norms

$$
\|f\|_{[N]} \equiv \sup_{\alpha \in \mathcal{N}_E, |\alpha| \leq N} |c(\alpha)|, \quad f = \sum_{\alpha \in \mathcal{N}_E} c(\alpha)e^{i(\cdot, \alpha)},
$$

in which $(\mathcal{G}_{0}^E)'(\mathbb{R}^d)$ becomes a Fréchet space. The set $\mathcal{G}_{0}^E(\mathbb{R}^d)$ is the union of finite-dimensional spaces of trigonometric polynomials with canonical topologies, and $\mathcal{G}_{0}^E(\mathbb{R}^d)$ is equipped with the inductive limit topology of these vector spaces.

Evidently, if $f \in \mathcal{G}_{s}^E(\mathbb{R}^d)$ or $f \in \mathcal{G}_{0,s}^E(\mathbb{R}^d)$ for some $s \geq 0$ is given by (1.10), then $\sum_{\alpha \in \mathcal{N}_E} |c(\alpha)|$ is convergent, and we may identify $f$ by a continuous $E$-periodic function.

If $s \geq 0$, $f \in (\mathcal{G}_{s}^E)'(\mathbb{R}^d)$ and $\phi \in \mathcal{G}_{s,r}^E(\mathbb{R}^d)$ or $f \in (\mathcal{G}_{0,s}^E)'(\mathbb{R}^d)$ and $\phi \in \mathcal{G}_{0,s}^E(\mathbb{R}^d)$, then we set

$$(f, \phi)_E = \sum_{\alpha \in \mathcal{N}_E} c(f, \alpha)\overline{c(\phi, \alpha)}$$

and

$$
(f, \phi)_E = \sum_{\alpha \in \mathcal{N}_E} c(f, \alpha)c(\phi, \alpha),
$$

and it follows that the duals of $\mathcal{G}_{s}^E(\mathbb{R}^d)$ and $\mathcal{G}_{0,s}^E(\mathbb{R}^d)$ can be identified by $(\mathcal{G}_{s}^E)'(\mathbb{R}^d)$ and $(\mathcal{G}_{0,s}^E)'(\mathbb{R}^d)$ respectively. We also note that by the identification of $\mathcal{G}_{s}^E(\mathbb{R}^d)$ as subspace of $E$-periodic continuous functions, the form $(\cdot, \cdot)_E$ on $\mathcal{G}_{s,r}^E(\mathbb{R}^d)$ extends uniquely to a scalar product on $L^2(E)$ and that

$$
\|f\|_E = |E|^{-\frac{1}{2}} \|f\|_{L^2(E)},
$$

where $|E|$ is the volume of $E$.

In Section 2 we show that

$$
\mathcal{E}_{s}^E(\mathbb{R}^d) = \mathcal{G}_{s}^E(\mathbb{R}^d), \quad \mathcal{E}_{0,s}^E(\mathbb{R}^d) = \mathcal{G}_{0,s}^E(\mathbb{R}^d),
$$

(1.11)

$$
(\mathcal{E}_{s}^E)'(\mathbb{R}^d) = (\mathcal{G}_{s}^E)'(\mathbb{R}^d) = (\mathcal{S}_{s}^{E,s})'(\mathbb{R}^d),
$$

(1.12)

and

$$
(\mathcal{E}_{0,s}^E)'(\mathbb{R}^d) = (\mathcal{G}_{0,s}^E)'(\mathbb{R}^d) = (\Sigma_{s}^{E,s})'(\mathbb{R}^d).
$$

(1.13)
and the Fourier coefficients. We also deduce a convenient formula for computing estimates on the short-time Fourier transforms of the involved functions

\[ \langle f, \phi \rangle = (2\pi)^d \sum_{\alpha \in E} c(f, \alpha) \hat{\phi}(-\alpha). \]  

(1.14)

If instead \( f \in (G^E_s)'(R^d) \), then the map which takes \( \phi \in \mathcal{S}^*_s(R^d) \) into the right-hand side of (1.14), defines an element in \( (\mathcal{S}^*_t)'(R^d) \) since

\[ |c(f, \alpha)| \lesssim e^{r_2|\alpha|^{\frac{1}{2}}} \quad \text{and} \quad |\hat{\phi}(\xi)| \lesssim e^{-r_1|\xi|^{\frac{1}{2}}} \]

for every \( r_2 > 0 \) and some \( r_1 > 0 \). Similar arguments hold with \( G_{0,s}^E \) and \( \Sigma^*_t \) in place of \( G_s^E \) and \( \Sigma^*_t \) (at each place).

This shows that any \( f \) in \( G_s^E \) (in \( G_{0,s}^E \)) can be identified as an element in \( (\mathcal{S}^*_t)'(R^d) \) (in \( (\Sigma^*_t)'(R^d) \)) and that the mappings which take \( G_s^E(R^d) \) and \( G_{0,s}^E(R^d) \) into \( (\mathcal{S}^*_t)'(R^d) \) and \( (\Sigma^*_t)'(R^d) \), respectively, are continuous.

2. CHARACTERIZATIONS OF PERIODIC FUNCTIONS AND DISTRIBUTIONS

In this section we show that (1.11)–(1.13) hold. At the same time we deduce characterizations of such spaces in terms of suitable estimates on the short-time Fourier transforms of the involved functions and distributions. We also deduce a convenient formula for computing the Fourier coefficients.

In the first result we show that (1.11)–(1.13) hold.

**Theorem 2.1.** Let \( E \subseteq R^d \) be a non-degenerate parallelepiped, and let \( s, t > 0 \) be such that \( s + t \geq 1 \). Then the following is true:

1. if \( f \in (G^E_s)'(R^d) \) \( (f \in (G^E_{0,s})'(R^d)) \) is given by (1.10) and \( \phi \in \mathcal{S}^*_s(R^d) \) \( (\phi \in \Sigma^*_t(R^d)) \), then (1.11) holds;

2. the equalities in (1.11) and (1.12) hold true. If in addition \( (s, t) \neq (\frac{1}{2}, \frac{1}{2}) \), then (1.13) holds true.

In Theorem 2.1 it is understood that in (2) we interpret the elements in \( (G^E_s)'(R^d) \) \( ((G^E_{0,s})'(R^d)) \) as elements in \( (\mathcal{S}^*_t)'(R^d) \) \( ((\Sigma^*_t)'(R^d)) \), which is possible in view of Remark 1.9.

The next result shows that the form \( (\cdot, \cdot)_E \) can be obtained in terms of suitable integrals of short-time Fourier transforms.

**Theorem 2.2.** Let \( E \subseteq R^d \) be a non-degenerate parallelepiped, \( s, t > 0 \) be such that \( s + t \geq 1 \), \( f \in (E^E_s)'(R^d) \), \( \psi \in E^E_s(R^d) \) and \( \phi \in \mathcal{S}^*_t(R^d) \setminus 0 \). Then

\[ (x, \xi) \mapsto (V_\psi f)(x, \xi)(\overline{V_\psi \psi})(x, \xi) \in L^1(E \times R^d) \]  

(2.1)

and

\[ (f, \psi)_E = (\|\phi\|_{L^2(E)}^2)^{-1} \int_{E} \left( \int_{R^d} (V_\psi f)(x, \xi)(\overline{V_\psi \psi})(x, \xi) \, d\xi \right) \, dx \]  

(2.2)
The same holds true with $\mathcal{E}_s^E$ and $\Sigma_t^s$ in place of $\mathcal{E}_{0,s}^E$ and $\mathcal{S}_t^s$.

**Remark 2.3.** Let $f$ and $\phi$ be the same as in Theorem 2.2. Then it follows from (2.2) that

$$c(f, \alpha) = (f, e^{i(\cdot, \alpha)})_E$$

$$= (\|\phi\|_{L^2(\mathbb{R}^d)}|E|)^{-1} \int_E \left( \int_{\mathbb{R}^d} (V_\phi f)(x, \xi) \hat{\phi}(\alpha - \xi)e^{-i(x, \alpha - \xi)} d\xi \right) dx.$$

We also have the following characterizations of periodic ultra-distributions in terms of short-time Fourier transforms, analogous to Propositions 1.2 and 1.3.

**Theorem 2.4.** Let $E \subseteq \mathbb{R}^d$ be a non-degenerate parallelepiped, $s > 0$, $f$ be an $E$-periodic Gevrey distribution on $\mathbb{R}^d$, and let $\phi \in \mathcal{S}_t^s(\mathbb{R}^d) \setminus 0$ for some $t \geq 0$. Then the following is true:

1. $f \in (\mathcal{E}_s^E)'(\mathbb{R}^d)$ if and only if $|V_\phi f(x, \xi)| \lesssim e^{r|\xi|^s}$ for every $r > 0$ (for some $r > 0$).

2. $f \in \mathcal{E}_s^E(\mathbb{R}^d)$ if and only if $|V_\phi f(x, \xi)| \lesssim e^{-r|\xi|^s}$ for some $r > 0$ (for every $r > 0$).

The identities (1.11) and the first two equalities in (1.12) and (1.13) in Theorem 2.1 also hold for $s = 0$, which is a consequence of the following result.

**Proposition 2.5.** Let $s \geq 0$ and let $E \subseteq \mathbb{R}^d$ be a non-degenerate parallelepiped. Then (1.11) and the first equalities in (1.12) and (1.13) hold. In particular the map

$$f \mapsto \sum_{\alpha \in \Lambda_E^s} c(f, \alpha)e^{i(\cdot, \alpha)}$$

(2.3)

is a homeomorphism from $\mathcal{E}_s^E(\mathbb{R}^d)$ to $\mathcal{G}_s^E(\mathbb{R}^d)$ and from $\mathcal{E}_{0,s}^E(\mathbb{R}^d)$ to $\mathcal{G}_{0,s}^E(\mathbb{R}^d)$, and extend uniquely to homeomorphisms from $(\mathcal{E}_s^E)'(\mathbb{R}^d)$ to $(\mathcal{G}_s^E)'(\mathbb{R}^d)$ and from $(\mathcal{E}_{0,s}^E)'(\mathbb{R}^d)$ to $(\mathcal{G}_{0,s}^E)'(\mathbb{R}^d)$.

Proofs of (1.11) in Theorem 2.1 in the case $s > 0$ can be found in e.g. [3, 12, 19]. In order to be self-contained we here present a proof including this part as well.

**Proof.** We only prove the first equality in (1.11). The second one follows by similar arguments and is left for the reader. The first equalities in (1.12) and (1.13) are then immediate consequences of (1.11) and duality.

First we consider the case when $s > 0$. Assume that $f \in \mathcal{E}_s^E(\mathbb{R}^d) \setminus 0$ and $c(f, \alpha) = 0$ for every $\alpha$. Then Bessel’s equality gives

$$\sum_{\alpha \in \Lambda_E^s} |c(f, \alpha)|^2 = |E|^{-1} \int_E |f(x)|^2 dx > 0, \quad f \in L^2(E),$$
and it follows that \( c(f, \alpha) \neq 0 \) for at least one \( \alpha \in \Lambda'_E \). Hence the right-hand side of (2.3) is non-zero as an element in \( G^E_s(\mathbb{R}^d) \), and the injectivity follows.

Next we show that

\[
G^E_s(\mathbb{R}^d) \subseteq \mathcal{E}_s^E(\mathbb{R}^d) \quad \text{and} \quad G^E_{0,s}(\mathbb{R}^d) \subseteq \mathcal{E}_{0,s}^E(\mathbb{R}^d). \tag{2.4}
\]

Suppose that \( f \) is given by (1.10), where \( |c(f, \alpha)| \lesssim e^{-r|\alpha|^\frac{1}{s}} \) when \( \alpha \in \Lambda'_E \), for some \( r > 0 \). Then \( f \) is \( E \)-periodic and smooth, and

\[
\|\partial^\beta f\|_{L^\infty} \lesssim \sup_\alpha (|\alpha| |\beta| e^{-r|\alpha|^\frac{1}{s}})
\]

for some \( r > 0 \). The embeddings \( G^E_s(\mathbb{R}^d) \subseteq \mathcal{E}_s^E(\mathbb{R}^d) \) and \( G^E_{0,s}(\mathbb{R}^d) \subseteq \mathcal{E}_{0,s}^E(\mathbb{R}^d) \) follow if we prove that

\[
|\alpha| |\beta| e^{-r|\alpha|^\frac{1}{s}} \lesssim h^{|\beta|} \beta!^s, \tag{2.5}
\]

for some \( h \approx \frac{1}{r} \). In order to show (2.5) we consider

\[
g(t) = tk e^{-rt^\frac{1}{s}}, \quad t > 0.
\]

Then

\[
g'(t) = \left(kt^{k-1} - \frac{rt^{k+\frac{1}{s}-1}}{s}\right) e^{-rt^\frac{1}{s}}
\]

is equal to 0, if and only if \( t = t_0 = \left(\frac{ks}{r}\right)^s \) in which \( g \) attains its maximum. Hence

\[
0 < g(t) \leq g(t_0) = \left(\frac{ks}{r}\right)^k e^{-ks},
\]

and Stirling’s formula gives

\[
g(t) \lesssim \left(\frac{s}{r}\right)^{sk} k!^s = h_1^k k!^s, \quad \text{where} \quad h_1 = \left(\frac{s}{r}\right)^s. \tag{2.6}
\]

By (2.6) we now get

\[
|\alpha| |\beta| e^{-r|\alpha|^\frac{1}{s}} \lesssim h_1^{|\beta|} \beta!^s \lesssim (d^s h_1)^{|\beta|} \beta!^s,
\]

which gives (2.5) and thereby (2.4).

In order to prove the opposite embedding we let \( f \in \mathcal{E}_s^E(\mathbb{R}^d) \). Since smooth periodic functions agree with their Fourier series expansions with absolutely convergent Fourier series, (1.10) holds with

\[
c(f, \alpha) = |E|^{-1} \int_E f(x)e^{-i(x, \alpha)} \, dx.
\]

By differentiations we get

\[
\alpha^\beta |c(f, \alpha)| = |c(f^{(\beta)}), \alpha)| \leq Ch^{|\beta|} \beta!^s,
\]
which gives
\[ |c(f, \alpha)| \leq C \prod_{j=1}^{d} g_{\alpha_j}(\beta_j) \]  
(2.7)

where \( g_0(t) = 1 \) and
\[ g_k(t) = \frac{h^t_{\text{st}}}{k^d}, \quad t \geq 0, \]
when \( k \geq 1 \) is an integer. If \( k \geq 1 \), then \( g_k'(t) = 0 \) exactly for
\[ t = t_0 = \frac{k^+}{h^+}, \]
in which \( g_k \) attains its global maximum. By straightforward computations we get
\[ g_k(t) \leq g_k(t_0) = e^{-\frac{1}{\pi_1} h_1^+}, \quad h_1 \equiv \frac{h^+}{s}. \]

By letting \( k = \alpha_j \) and \( t = \beta_j \) in the last estimate, (2.7) gives
\[ |c(f, \alpha)| \leq C \prod_{j=1}^{d} e^{-\frac{1}{\pi_1} h_2^{[\alpha_j]^+}}, \]
for some \( h_2 \) which is proportional to \( h_1 \). This shows that equalities hold in (2.4), and the result follows.

It remains to consider the case when \( s = 0 \). First assume that \( f \in \mathcal{G}_E^E(\mathbb{R}^d) \). Then for some integer \( N \geq 0 \) we have
\[ f(x) = \sum_{\alpha \in \Lambda'_E,N} c(\alpha) e^{i \langle x, \alpha \rangle}, \]
where
\[ \Lambda'_E,N = \{ \alpha \in \Lambda_E; |\alpha| \leq N \} \]
For every \( \beta \in \mathbb{N}^d \) we get
\[ |\partial^\beta f(x)| \lesssim \max_{\alpha \in \Lambda'_E,N} |c(\alpha)\alpha^\beta| \lesssim N^{|eta|}, \]
which implies that \( f \in \mathcal{E}_0^E(\mathbb{R}^d) \), and we have shown that \( \mathcal{G}_E^E(\mathbb{R}^d) \subseteq \mathcal{E}_0^E(\mathbb{R}^d) \).

Assume instead that \( f \in \mathcal{E}_0^E(\mathbb{R}^d) \). Then \( f \) is \( E \)-periodic, smooth and \( |\partial^\beta f(x)| \lesssim h^{|eta|} \), for some \( h > 0 \). This implies that \( f \) is given by (1.10).

By differentiations and Bessel’s equality we get
\[ \sum_{\alpha \in \Lambda'_E} |c(f, \alpha)\alpha^\beta|^2 \asymp \|\partial^\beta f\|_{L^2(\mathcal{E})}^2 \lesssim h^{2|eta|}. \]
This gives
\[ \sup_{\beta \Lambda'_E} \left| c(f, \alpha) \left( \frac{\alpha}{h} \right) \beta \right| < \infty, \]
which implies that $c(f,\alpha) = 0$ when $|\alpha_j| > h$ for some $j = 1, \ldots, d$. That is, the right-hand side of (1.10) must be a finite sum. Hence $f \in \mathcal{G}^E_s(R^d)$, and (1.11) follows.

By (1.11) it follows that the duals $(\mathcal{E}^E_s)'(R^d)$ and $(\mathcal{E}^E_{0,s})'(R^d)$ of $\mathcal{E}^E_s(R^d)$ and $\mathcal{E}^E_{0,s}(R^d)$ agree with $(\mathcal{G}^E_s)'(R^d)$ and $(\mathcal{G}^E_{0,s})'(R^d)$ through the forms $\langle \cdot, \cdot \rangle_E$ and $\langle \cdot, \cdot \rangle_E$. □

The following lemma is needed for the proof of the second equalities in (1.12) and (1.13).

**Lemma 2.6.** Let $E \subseteq R^d$ be a non-degenerate parallelepiped, and $s, t > 0$ be such that $s + t \geq 1$. Then
\[
(\mathcal{E}^E_s)'(R^d) \hookrightarrow (\mathcal{S}^E_t)'(R^d),
\]
and if in addition $(s, t) \neq (1, 1)$, then
\[
(\mathcal{E}^E_{0,s})'(R^d) \hookrightarrow (\mathcal{S}^E_t)'(R^d).
\]

**Proof.** Let $f \in (\mathcal{E}^E_s)'(R^d) \ (f \in (\mathcal{E}^E_{0,s})'(R^d))$ be equal to 0 in $(\mathcal{S}^E_t)'(R^d)$ $((\Sigma^E_t)'(R^d))$. Since the continuity is already proved, it remains to show that $f$ is equal to 0 in $(\mathcal{E}^E_s)'(R^d)$ $((\mathcal{E}^E_{0,s})'(R^d))$.

Let $\alpha_0 \in \Lambda_E$ and $\phi_{\varepsilon, \alpha_0} \in \mathcal{S}^E_t(R^d)$ $(\phi_{\varepsilon, \alpha_0} \in \Sigma^E_t(R^d))$ be such that
\[
\hat{\phi}_{\varepsilon, \alpha_0} = \hat{\varphi}(\varepsilon^{-1}(\xi + \alpha_0))
\]
for some function $\varphi$ which satisfies $\hat{\varphi}(0) = 1$. Then
\[
0 = \langle f, \phi_{\varepsilon, \alpha_0} \rangle = \sum_{\alpha \in \Lambda_E} c(f, \alpha) \hat{\varphi}(\varepsilon^{-1}(\alpha_0 - \alpha)) \rightarrow c(f, \alpha_0)
\]
as $\varepsilon \to 0$. Hence $c(f, \alpha) = 0$ for every $\alpha$, which shows that $f = 0$ in $(\mathcal{E}^E_s)'(R^d)$ $((\mathcal{E}^E_{0,s})'(R^d))$. □

In order to prove the opposite embeddings to (2.8) and (2.9) we need the following propositions, which are at the same time main ingredients in the proof of Theorem 2.4. They show that periodic Gelfand-Shilov distributions and periodic elements in Gevrey classes can be characterized by suitable estimates of short-time Fourier transforms.

**Proposition 2.7.** Let $E \subseteq R^d$ be a non-degenerate parallelepiped, $s, t \geq 0$ be such that $s + t > 1$, $f$ be a $E$-periodic Gelfand-Shilov distribution on $R^d$, and let $\phi \in \mathcal{S}^E_t(R^d) \setminus 0$ $(\phi \in \Sigma^E_t(R^d) \setminus 0)$. If $f \in \mathcal{E}^E_s(R^d)$ $(f \in \mathcal{E}^E_{0,s}(R^d))$, then
\[
|V_{\phi,f}(x, \xi)| \lesssim e^{-r|\xi|^\frac{1}{t}},
\]
for some $r > 0$ (for every $r > 0$).

**Proposition 2.8.** Let $E \subseteq R^d$ be a non-degenerate parallelepiped, $s, t \geq 0$ be such that $s + t > 1$, $f$ be an $E$-periodic Gelfand-Shilov distribution on $R^d$, and let $\phi \in \mathcal{S}^E_t(R^d) \setminus 0$ $(\phi \in \Sigma^E_t(R^d) \setminus 0)$. Then the following conditions are equivalent:
\( f \in (S^s_t)'(\mathbb{R}^d) \) (\( f \in (\Sigma^s_t)'(\mathbb{R}^d) \));

(2) \(|V_\phi f(x, \xi)| \lesssim e^{r|\xi|^\frac{1}{s}}\) for every \( r > 0 \) (for some \( r > 0 \)).

**Proof of Proposition 2.7.** We only prove the assertion in the Roumeeu case. The Beurling case follows by similar arguments and is left for the reader.

For some \( r > 0 \) we have
\[
|\hat{\phi}(\xi)| \lesssim e^{-r|\xi|^\frac{1}{s}}.
\]
Assume that \( f \in \mathcal{E}_s^E(\mathbb{R}^d) \). Then \( f \) is given by (1.10), where
\[
|c(\alpha)| \lesssim e^{-2r|\alpha|^\frac{1}{s}}
\]
for some \( r > 0 \) which is independent of \( \alpha \in \Lambda_E' \) and \( \xi \in \mathbb{R}^d \). Hence, for some \( r > 0 \) we have
\[
|V_\phi f(x, \xi)| \leq \sum_{\alpha \in \Lambda_E} |c(\alpha)\phi(e^{i\langle \cdot, \alpha \rangle})(x, \xi)|
\]
\[
= \sum_{\alpha \in \Lambda_E} |c(\alpha)\hat{\phi}(\xi - \alpha)| \lesssim \sum_{\alpha \in \Lambda_E} e^{-r|\alpha|^\frac{1}{s}} e^{-r(|\alpha|^\frac{1}{s} + |\xi - \alpha|^\frac{1}{s})}
\]
\[
\lesssim \sup_{\eta \in \mathbb{R}^d} (e^{-r(|\eta|^\frac{1}{s} + |\xi - \eta|^\frac{1}{s})}) \leq e^{-rc|\xi|^\frac{1}{s}},
\]
for some \( c > 0 \) which only depends on \( s \). This gives the result. \( \square \)

**Proof of Proposition 2.8.** Again we only prove the assertion in the Roumeeu case, leaving the Beurling case for the reader.

Assume that (1) holds. By Proposition 1.3 we get
\[
|V_\phi f(x, \xi)| \lesssim e^{r(|x|^\frac{1}{s} + |\xi|^\frac{1}{s})}
\]
for every \( r > 0 \). Since \( f \) is \( E \)-periodic, it follows that the same holds true for the map \( x \mapsto |V_\phi f(x, \xi)| \), and the previous estimate gives
\[
|V_\phi f(x, \xi)| = |V_\phi f(x + j, \xi)| \lesssim e^{r(|x + j|^\frac{1}{s} + |\xi|^\frac{1}{s})}
\]
for every \( j \in \Lambda_E \). By taking the infimum over all \( j \in \Lambda_E \) we get
\[
|V_\phi f(x, \xi)| \lesssim e^{r|\xi|^\frac{1}{s}}
\]
for every \( r > 0 \), and (2) follows.

If instead (2) holds, then
\[
|V_\phi f(x, \xi)| \lesssim e^{r(|x|^\frac{1}{s} + |\xi|^\frac{1}{s})}
\]
for every \( r > 0 \), and Proposition 1.3 shows that \( f \in (S^s_t)'(\mathbb{R}^d) \). This gives the result. \( \square \)
Remark 2.9. Let \( f \) and \( \psi \) be the same as in Theorem 2.2. Then \( (2.1) \) holds in view of Propositions 2.8 and 2.7, which implies that the right-hand side of \( (2.2) \) makes sense.

From these properties and \( (1.3) \) it follows that

\[
\sum_{\alpha, \beta \in \Lambda'_{E}} |c(f, \alpha)c(\psi, \beta)\hat{\phi}(\xi - \alpha)\hat{\phi}(\xi - \beta)| < \infty
\]

for every fixed \( \xi \in \mathbb{R}^d \).

Proof of Theorem 2.2. Again we only prove the result in the Roumieu case, leaving the Beurling case for the reader.

By straight-forward computations we get

\[
(V_{\phi}f)(x, \xi) = e^{-i(x, \xi)} \sum_{\alpha \in \Lambda'_{E}} c(f, \alpha)\hat{\phi}(\alpha - \xi)e^{i(x, \alpha)}.
\]

Hence Remark 2.9 and Weierstrass and Fubbini’s theorems give

\[
\int_{E} \left( \int_{\mathbb{R}^d} (V_{\phi}f)(x, \xi)(V_{\psi}f)(x, \xi) \, d\xi \right) \, dx
\]

\[
= \int_{\mathbb{R}^d} \left( \int_{E} \sum_{\alpha, \beta \in \Lambda'_{E}} c(f, \alpha)c(\psi, \beta)\overline{\hat{\phi}(\alpha - \xi)}\hat{\phi}(\beta - \xi) e^{i(x, \alpha - \beta)} \, d\xi \right) \, dx
\]

\[
= \int_{\mathbb{R}^d} \left( \sum_{\alpha, \beta \in \Lambda'_{E}} c(f, \alpha)c(\psi, \beta)\overline{\hat{\phi}(\alpha - \xi)}\hat{\phi}(\beta - \xi) \int_{E} e^{i(x, \alpha - \beta)} \, dx \right) \, d\xi
\]

\[
= |E| \int_{\mathbb{R}^d} \left( \sum_{\alpha \in \Lambda'_{E}} c(f, \alpha)c(\psi, \alpha)|\hat{\phi}(\alpha - \xi)|^2 \right) \, d\xi
\]

Since

\[
\sum_{\alpha \in \Lambda'_{E}} |c(f, \alpha)\overline{c(\psi, \alpha)}| \int_{\mathbb{R}^d} |\hat{\phi}(\alpha - \xi)|^2 \, d\xi = \|\phi\|_{L^2}^2 \sum_{\alpha \in \Lambda'_{E}} |c(f, \alpha)\overline{c(\psi, \alpha)}| < \infty,
\]

an other application of Weierstrass theorem now gives

\[
\int_{\mathbb{R}^d} \left( \sum_{\alpha \in \Lambda'_{E}} c(f, \alpha)c(\psi, \alpha)|\hat{\phi}(\alpha - \xi)|^2 \right) \, d\xi
\]

\[
= \sum_{\alpha \in \Lambda'_{E}} c(f, \alpha)c(\psi, \alpha) \int_{\mathbb{R}^d} |\hat{\phi}(\alpha - \xi)|^2 \, d\xi
\]

\[
= \|\phi\|_{L^2}^2 \sum_{\alpha \in \Lambda'_{E}} c(f, \alpha)c(\psi, \alpha) = \|\phi\|_{L^2}^2 (f, \psi)_E,
\]
and the result follows by combining these equalities.

In the following definition we assign any element in \((S_t^{E,s})'(\mathbb{R}^d)\) \(((\Sigma_t^{E,s})'(\mathbb{R}^d))\), an element in \((E_s^{E})'(\mathbb{R}^d)\) \(((E_{0,s}^{E})'(\mathbb{R}^d))\). Then we prove that the latter element agrees with the former one as element in \((S_t^{E,s})'(\mathbb{R}^d)\) \(((\Sigma_t^{E,s})'(\mathbb{R}^d))\), which will give the last part of Theorem 2.1.

**Definition 2.10.** Let \(E \subseteq \mathbb{R}^d\) be a non-degenerate parallelepiped, \(s,t > 0\) be such that \(s + t \geq 1\), \(f \in (S_t^{E,s})'(\mathbb{R}^d)\) \((f \in (\Sigma_t^{E,s})'(\mathbb{R}^d))\), and let \(\phi \in S_t(\mathbb{R}^d) \setminus 0\) \((\phi \in \Sigma_t(\mathbb{R}^d) \setminus 0)\).

1. The Fourier coefficient \(c(f,\alpha)\) for \(f\) of order \(\alpha \in \Lambda_E'\) is given by (2.2).
2. The Fourier series of \(f\) with respect to \(E\) is given by

\[
FS_E(f) \equiv \sum_{\alpha \in \Lambda_E'} c(f,\alpha) e^{i\langle \cdot, \alpha \rangle}.
\]

Evidently by (1.3) and Proposition 2.8 and definitions it follows that \(c(f,\alpha)\) in Definition is well-defined. Hence \(FS_E(f)\) in Definition 2.10 exists as an element in \((E_s^{E})'(\mathbb{R}^d)\).

Theorem 2.1 is an immediate consequence of (2.8), (2.9) and the following result.

**Proposition 2.11.** Let \(E \subseteq \mathbb{R}^d\) be a non-degenerate parallelepiped, \(s,t > 0\) be such that \(s + t \geq 1\), and let \(f \in (S_t^{E,s})'(\mathbb{R}^d)\) \((f \in (\Sigma_t^{E,s})'(\mathbb{R}^d))\). Then the following is true:

1. \(FS_E(f) \in (E_s^{E})'(\mathbb{R}^d)\) \((FS_E(f) \in (E_{0,s}^{E})'(\mathbb{R}^d))\).
2. \(FS_E(f) = f\) as elements in \((S_t^{E,s})'(\mathbb{R}^d)\) \(((\Sigma_t^{E,s})'(\mathbb{R}^d))\).

We need some preparations for the proof. First we recall that the usual properties on tensor products also hold for Gelfand-Shilov distributions. More precisely, the following result follows by similar arguments as the proof of [17, Theorem 5.1.1]. The details are left for the reader.

**Lemma 2.12.** Let \(s,t > 0\) and \(f_j \in (S_t^{i,s})'(\mathbb{R}^{d_i})\) \((f_j \in (\Sigma_t^{i,s})'(\mathbb{R}^{d_i}))\), \(j = 1,2\). Then there is a unique \(f \in (S_t^{i,s})'(\mathbb{R}^{d_1+d_2})\) \((f \in (\Sigma_t^{i,s})'(\mathbb{R}^{d_1+d_2}))\) such that

\[
\langle f, \phi_1 \otimes \phi_2 \rangle = \langle f_1, \phi_1 \rangle \langle f_2, \phi_2 \rangle \quad \text{when} \quad \phi_j \in S_t^{i,s}(\mathbb{R}^{d_j}) \,(\phi_j \in \Sigma_t^{i,s}(\mathbb{R}^{d_j})), \, j = 1,2.
\]

If \(\phi \in S_t^{i,s}(\mathbb{R}^{d_1+d_2})\) \((\phi \in \Sigma_t^{i,s}(\mathbb{R}^{d_1+d_2}))\),

\[
\psi_1(x_1) = \langle f_2, \phi(x_1, \cdot) \rangle \quad \text{and} \quad \psi_2(x_2) = \langle f_1, \phi(\cdot, x_2) \rangle,
\]

then \(\psi_j \in S_t^{i,s}(\mathbb{R}^{d_j})\) \((\psi_j \in \Sigma_t^{i,s}(\mathbb{R}^{d_j})), \, j = 1,2,\) and

\[
\langle f, \phi \rangle = \langle f_1, \psi_1 \rangle = \langle f_2, \psi_2 \rangle.
\]

We recall that \(f\) in Lemma 2.12 is called the tensor product of \(f_1\) and \(f_2\) and is usually denoted by \(f_1 \otimes f_2\).

We also have the following.
Lemma 2.13. Let $s, t > 0$, $E \subseteq \mathbb{R}^d$ be a non-degenerate parallelepiped, $f \in (S^s_t)(\mathbb{R}^d)$ ($f \in (\Sigma^s_t)(\mathbb{R}^d)$) and let $\phi_0, \psi \in S^s_t(\mathbb{R}^d)$ ($\phi_0, \psi \in \Sigma^s_t(\mathbb{R}^d)$). Then the following is true:

1. $\sup_{x \in \mathbb{R}^d} \left( \sum_{k \in \Lambda_E} | \langle f, \phi_0(\cdot - x + k) \psi \rangle | \right) < \infty$.

2. $\sum_{k \in \Lambda_E} \phi_0(\cdot + k) \psi$ converges in $S^s_t(\mathbb{R}^d)$ ($\Sigma^s_t(\mathbb{R}^d)$).

Proof. We only prove the result in the Roumieu case, leaving the Beurling case for the reader.

Let $\phi = \overline{\phi_0}$. We have

$$|\phi(y - x + k)| \lesssim e^{-r_0|y-x+k|^t} \quad \text{and} \quad |\psi(y)| \lesssim e^{-r_0|y|^t}$$

for some $r_0 > 0$, which gives

$$|\phi_0(y - x + k)\psi(y)| \lesssim e^{-r_0(|y-x+k|^t + |y|^t)} \lesssim e^{-c|y-k|^t + |y|^t} \quad (2.11)$$

for some $c \in (0, 1)$ which only depends on $t$. Moreover,

$$|\mathcal{F}(\phi_0(\cdot - x + k)\psi(y))(\eta)| = |(V_\eta \psi)(x - k, \eta)| \lesssim e^{-r_0(|x-k|^t + |\eta|^t)} \quad (2.12)$$

in view of Proposition 1.2.

Since the topology of $S^s_t(\mathbb{R}^d)$ can be obtained through the semi-norms

$$\phi \mapsto \|\phi e^{r\cdot|\cdot|^t}\|_{L^\infty} + \|\widehat{\phi} e^{r\cdot|\cdot|^t}\|_{L^\infty}, \quad r > 0,$$

(2.11) and (2.12) give

$$|\langle f, \phi_0(\cdot - x + k) \psi \rangle| \lesssim \|\phi_0(\cdot - x + k)\psi e^{c|\cdot|^{1/2}}\|_{L^\infty} + \|\mathcal{F}(\phi_0(\cdot - x + k)e^{c|\cdot|^{1/2}}\|_{L^\infty} \lesssim e^{-c|y-k|^t}.$$ 

Hence,

$$\sup_{x \in \mathbb{R}^d} \left( \sum_{k \in \Lambda_E} | \langle f, \phi_0(\cdot - x + k) \psi \rangle | \right) \lesssim \sup_{x \in \mathbb{R}^d} \left( \sum_{k \in \Lambda_E} e^{-c|y-k|^t} \right) < \infty,$$

and (1) follows.

(2) It is clear that $\psi_0 = \sum_{k \in \Lambda_E} \phi_0(\cdot + k) \psi$ is well-defined and smooth. Let

$$\Lambda_{E,N} = \{ k \in \Lambda_E : |k| < N \}; \quad \Omega_{E,N} = \Lambda_E \setminus \Lambda_{E,N},$$

and let

$$\psi_N = \psi_0 - \sum_{k \in \Lambda_{E,N}} \phi_0(\cdot + k) \psi.$$

We shall prove that $\psi_N \to 0$ in $S^s_t(\mathbb{R}^d)$, as $N$ tends to $\infty$. 


For some $r_0 > 0$ we have
\[ |\psi_N e^{r_0|\cdot|^\frac{1}{r}}|^{1/2} \leq \sum_{k \in \Omega_{E,N}} |\phi_0(\cdot + k)||\psi| e^{r_0|\cdot|^\frac{1}{r}} \]
which tends to 0 as $N$ tends to $\infty$. Moreover,
\[ |\hat{\psi}_N(\eta)| \lesssim \sum_{k \in \Omega_{E,N}} |V_\phi \psi(-k, \eta)| \lesssim \sum_{k \in \Omega_{E,N}} e^{-r_0(|k|^\frac{1}{r} + |\eta|^\frac{1}{r})} , \]
by Proposition 1.12. Hence
\[ |\hat{\psi}_N(\eta)e^{r_0|\eta|^\frac{1}{r}/2}| \lesssim \sum_{k \in \Omega_{E,N}} e^{-r_0(|k|^\frac{1}{r} + |\eta|^\frac{1}{r})/2} \]
which tends to 0 as $N$ tends to $\infty$. Consequently,
\[ \lim_{N \to \infty} \left( \|\psi_N e^{r_0|\cdot|^\frac{1}{r}}\|_{L^\infty} + \|\hat{\psi}_N e^{r_0|\cdot|^\frac{1}{r}}\|_{L^\infty} \right) = 0 , \]
for some $r_0 > 0$, which shows that $\psi_N \to 0$ in $S^r_0(\mathbb{R}^d)$ as $N \to \infty$ and the result follows.

Proof of Proposition 2.11. We only prove the result in the Roumieu case. The Beurling case follows by similar arguments and is left for the reader.

(1) We have
\[ |\xi + \eta|^\frac{1}{r} \leq c(|\xi|^\frac{1}{r} + |\eta|^\frac{1}{r}) , \quad |\hat{\phi}(\xi)| \lesssim e^{-r_0|\xi|^\frac{1}{r}} \quad \text{and} \quad |V_\phi f(x, \xi)| \lesssim e^{r|\xi|^\frac{1}{r}} , \]
for some $c \geq 1$ which only depends on $s$, for some $r_0 > 0$, and for every $r > 0$. By choosing $r < (c + 1)^{-1}r_0$ we get
\[ |c(f, \alpha)| \lesssim \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_\phi f(x, \xi)||\hat{\phi}(\alpha - \xi)| \, d\xi \right) \, dx \]
\[ \lesssim \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} e^{r|\xi|^\frac{1}{r}} e^{-r_0|\alpha - \xi|^\frac{1}{r}} \, d\xi \right) \, dx \]
\[ \times \int_{\mathbb{R}^d} e^{r(|\xi|^\frac{1}{r} - c|\alpha - \xi|^\frac{1}{r})} e^{-r|\alpha - \xi|^\frac{1}{r}} \, d\xi \]
\[ \leq e^{rc|\alpha|^\frac{1}{r}} \int_{\mathbb{R}^d} e^{-r|\alpha - \xi|^\frac{1}{r}} \, d\xi \sim e^{rc|\alpha|^\frac{1}{r}} . \]
Since $rc > 0$ can be made arbitrary close to 0 we get $|c(f, \alpha)| \lesssim e^{r|\alpha|^\frac{1}{r}}$ for every $r > 0$, and (1) follows.
Let $f_0 = \mathcal{F}_E(f), \phi \in \mathcal{S}_s^s(\mathbb{R}^d) \setminus 0$ and let $\psi \in \mathcal{S}_s^s(\mathbb{R}^d)$. Then

\[
(2\pi)^{-\frac{d}{2}} \|\phi\|_{L^2(\mathbb{R}^d)}|E| \langle f_0, \psi \rangle = \sum_{\alpha \in \Lambda_E} c(\alpha, f) \hat{\psi}(-\alpha)
\]

\[
= \sum_{\alpha \in \Lambda_E} \int_E \left( \int_{\mathbb{R}^d} (V\phi f)(x, \xi) \hat{\phi}(\alpha - \xi) e^{-i(x, \alpha - \xi)} d\xi \right) dx
\]

\[
= \int_E \left( \int_{\mathbb{R}^d} (V\phi f)(x, \xi) F(x, \xi) d\xi \right) dx, \tag{2.13}
\]

where

\[
F(x, \xi) = \sum_{\alpha \in \Lambda_E} \hat{\phi}(\alpha - \xi) \hat{\psi}(-\alpha) e^{-i(x, \alpha - \xi)}.
\]

In the last equality in (2.13) we have used the fact

\[
\sum_{\alpha \in \Lambda_E} \int_E \left( \int_{\mathbb{R}^d} |(V\phi f)(x, \xi)| |\hat{\phi}(\alpha - \xi)| d\xi \right) dx |\hat{\phi}(-\alpha)| < \infty,
\]

which implies that we may interchange orders of summations and integrations.

We shall rewrite $F(x, \xi)$. By straightforward computations we get

\[
\hat{\phi}(\alpha - \xi) \hat{\psi}(-\alpha) = (2\pi)^{-\frac{d}{2}} \mathcal{F}((\phi e^{i(\cdot, \xi)}) \ast \hat{\psi})(\alpha),
\]

and Poisson’s summation formula gives

\[
F(x, \xi) = (2\pi)^{-\frac{d}{2}} e^{i(x, \xi)} \sum_{\alpha \in \Lambda_E} \mathcal{F}((\phi e^{i(\cdot, \xi)}) \ast \hat{\psi})(\alpha) e^{-i(x, \alpha)}
\]

\[
= (2\pi)^{-\frac{d}{2}} |E| e^{i(x, \xi)} \sum_{k \in \Lambda_E} ((\phi e^{i(\cdot, \xi)}) \ast \hat{\psi})(k - x)
\]

\[
= (2\pi)^{-\frac{d}{2}} |E| \sum_{k \in \Lambda_E} \mathcal{F}(\hat{\psi}(\cdot - k))(x, \xi). \tag{2.14}
\]

A combination of (2.13) and (2.14) leads to

\[
\|\phi\|_{L^2(\mathbb{R}^d)}^2 \langle f_0, \psi \rangle = \left( \sum_{k \in \Lambda_E} \int_{\mathbb{R}^d} V\phi f(x, \xi)(\mathcal{F}(\hat{\psi}(\cdot - k))(x, \xi) d\xi \right) dx.
\]

By Fourier inversion formula we get

\[
\int_{\mathbb{R}^d} (V\phi f)(x, \xi)(\mathcal{F}(\hat{\psi}(\cdot - k))(x, \xi) d\xi = \langle f, |\phi(\cdot - x + k)|^2 \psi \rangle.
\]

Hence

\[
\|\phi\|_{L^2(\mathbb{R}^d)}^2 \langle f_0, \psi \rangle = \int_{E} \sum_{k \in \Lambda_E} \langle f, |\phi(\cdot - x + k)|^2 \psi \rangle dx,
\]

\[
\]
and we shall use Lemma 2.12 and 2.13 to reformulate the right-hand side.

By Lemma 2.13 (1) and Lebesgue’s theorem we have

\[
\int \sum_{k \in \Lambda_E} \langle f, \phi(\cdot - x + k) |^2 \rangle \psi \, dx = \sum_{k \in \Lambda_E} \int_E \langle f, \phi(\cdot - x + k) |^2 \rangle \psi \, dx,
\]

and letting

\[
\psi_k(x, y) = |\phi(y - x + k)|^2 \psi(y) \in \mathcal{S}_t^s(\mathbb{R}^{2d}),
\]

it follows from Lemma 2.12 that

\[
\int \langle f, \phi(\cdot - x + k) |^2 \rangle \psi \, dx = \langle \chi_E \otimes f, \psi_k \rangle = \langle f, \phi_0(\cdot + k) \psi \rangle,
\]

where

\[
\phi_0(y) = \int_E |\phi(y - x)|^2 \, dx \in \mathcal{S}_t^s(\mathbb{R}^d).
\]

Here \( \chi_E \) is the characteristic function of \( E \).

Let \( \Lambda_{E,N} \) be the same as in Lemma 2.13. A combining the identities above and Lemma 2.13 gives

\[
\|\phi\|_{L^2(\mathbb{R}^d)}^2 \langle f_0, \psi \rangle = \lim_{N \to \infty} \sum_{k \in \Lambda_{E,N}} \langle f, \phi_0(\cdot + k) \psi \rangle
\]

\[
= \lim_{N \to \infty} \left\langle f, \left( \sum_{k \in \Lambda_{E,N}} \phi(\cdot + k) \right) \psi \right\rangle = \left\langle f, \left( \sum_{k \in \Lambda_E} \phi(\cdot + k) \right) \psi \right\rangle
\]

\[
= \|\phi\|_{L^2(\mathbb{R}^d)}^2 \langle f, \psi \rangle,
\]

and the result, and thereby Theorem 2.1 follow. \( \square \)

**Proof of Theorem 2.4** The assertion (1) and one part of (2) are immediate consequences of Propositions 2.8 and 2.7 and Theorem 2.1. We need to show that (2.10) for some \( r > 0 \) (every \( r > 0 \)) is sufficient that \( f \in \mathcal{E}_t^E(\mathbb{R}^d) \) (\( f \in \mathcal{E}_0^E(\mathbb{R}^d) \)).

We only consider the Roumieu case. The Beurling case follows by similar arguments and is left for the reader.

Suppose that \( |V_{\alpha}f(x, \xi)| \lesssim e^{-2r|\xi|^\frac{1}{2}} \) for some \( r > 0 \). Then \( f \in \mathcal{E}_t^E(\mathbb{R}^d) \) due to (1), and hence has a Fourier series expansion with coefficients \( c(f, \alpha) \). By Remark 2.3 and the fact that \( \phi \in \mathcal{S}_t^s(\mathbb{R}^d) \) we
have
\[
|c(f, \alpha)| = |(f, e^{i\langle \cdot, \alpha \rangle})_E| \\
\leq (\|\phi\|^2_{L^2(\mathbb{R}^d)}|E|)^{-1} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_\phi f(x, \xi)||\hat{\phi}(\alpha - \xi)| d\xi \right) dx \\
\lesssim \int_{\mathbb{R}^d} e^{-r|\xi|^s} e^{-r(|\xi|^s + |\alpha - \xi|^s)} d\xi \\
\lesssim e^{-rc|\alpha|^\frac{1}{s}} \int_{\mathbb{R}^d} e^{-r|\xi|^s} d\xi \\
\lesssim e^{-rc|\alpha|^\frac{1}{s}}
\]
for some \(c > 0\) which only depends on \(s > 0\). This implies that \(f \in G^E_{s}(\mathbb{R}^d) = \mathcal{E}^E_{s}(\mathbb{R}^d)\), and the result follows.

\[\square\]

Remark 2.14. Evidently Theorem 2.1 (1) is true also when \(\phi \in \mathcal{S}(\mathbb{R}^d)\) and \(f \in (\mathcal{E}^E_{s})' \cap \mathcal{S}'(\mathbb{R}^d)\).

It also follows from the proof of Theorem 2.2 that the conclusions of that theorem is also true when in the case when \(f \in (\mathcal{E}^E_{s})' \cap \mathcal{S}'(\mathbb{R}^d)\), \(\psi \in (\mathcal{E}^E_{s})' \cap C^\infty(\mathbb{R}^d)\) and \(\phi \in \mathcal{S}(\mathbb{R}^d)\). The details are left for the reader.

Remark 2.15. Let \(f\) be a Gelfand-Shilov distribution on \(\mathbb{R}^d\) and let \(\phi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}\). Then the following conditions are equivalent:

1. \(f \in \mathcal{S}(\mathbb{R}^d);\)
2. \(|f(x)| \lesssim \langle x \rangle^{-N} \) and \(|\hat{f}(\xi)| \lesssim \langle \xi \rangle^{-N} \) for every \(N \geq 0;\)
3. \(|V_\phi f(x, \xi)| \lesssim \langle (x, \xi) \rangle^{-N} \) for every \(N \geq 0.\)

We also have that \(f \in \mathcal{S}'(\mathbb{R}^d)\) if and only if
\[
|V_\phi f(x, \xi)| \lesssim \langle (x, \xi) \rangle^N
\]
for some \(N \geq 0\) (cf. e.g. [1], [14] or Remark 1.3 in [28]).

Now assume that \(f\) is an \(E\)-periodic Gelfand-Shilov distribution on \(\mathbb{R}^d\). From the previous characterizations it follows by similar arguments as for the proof of Theorem 2.4 that the following is true:

1. \(f \in \mathcal{S}'(\mathbb{R}^d)\) if and only if
   \[
   |V_\phi f(x, \xi)| \lesssim \langle \xi \rangle^N
   \]
   for some \(N \geq 0.\)
2. \(f \in C^\infty(\mathbb{R}^d)\) if and only if
   \[
   |V_\phi f(x, \xi)| \lesssim \langle \xi \rangle^{-N}
   \]
   for every \(N \geq 0.\)
Example 2.16. Suppose \( E \subseteq \mathbb{R}^d \) is a rectangle, and that \( f(t, x), (t, x) \in \mathbb{R} \times \mathbb{E} \) satisfies the heat equation

\[
\partial_t f = \Delta_x f, \quad f(0, x) = f_0(x), \quad \text{where}
\]

\[
f_0(x) = \sum_{\alpha \in A_E} c(\alpha) e^{i(x, \alpha)}
\]

(2.15)

is a fixed element in \((\mathcal{E}_0^E)'(\mathbb{R}^d)\). We are interested to find well-posedness properties in the framework of the spaces \( \mathcal{E}_s^E(\mathbb{R}^d) \) and \((\mathcal{E}_s^E)'(\mathbb{R}^d)\) when \( s \geq 0 \), and the spaces \( \mathcal{E}_{0,s}^E(\mathbb{R}^d) \) and \((\mathcal{E}_{0,s}^E)'(\mathbb{R}^d)\) when \( s > 0 \).

The formal solution is given by

\[
f(t, x) = \sum_{\alpha \in A_E} c(\alpha) e^{-|\alpha|^2 t} e^{i(x, \alpha)}.
\]

By Theorem 2.1 it follows that the following is true:

1. If \( 0 \leq s < \frac{1}{2} \), then the map \( (t, f_0) \mapsto f(t, \cdot) \) is continuous from \( \mathbb{R} \times \mathcal{E}_s^E(\mathbb{R}^d) \) to \( \mathcal{E}_s^E(\mathbb{R}^d) \). Moreover, if \( f_0 \in \mathcal{E}_s^E(\mathbb{R}^d) \), then \( t \mapsto f(t, \cdot) \) is smooth from \( \mathbb{R} \to \mathcal{E}_s^E(\mathbb{R}^d) \). The same holds true with \((\mathcal{E}_s^E)'(\mathbb{R}^d)\) in place of \( \mathcal{E}_s^E(\mathbb{R}^d) \) at each occurrence;

2. If \( 0 < s \leq \frac{1}{2} \), then the map \( (t, f_0) \mapsto f(t, \cdot) \) is continuous from \( \mathbb{R} \times \mathcal{E}_{0,s}^E(\mathbb{R}^d) \) to \( \mathcal{E}_{0,s}^E(\mathbb{R}^d) \). Moreover, if \( f_0 \in \mathcal{E}_{0,s}^E(\mathbb{R}^d) \), then \( t \mapsto f(t, \cdot) \) is smooth from \( \mathbb{R} \to \mathcal{E}_{0,s}^E(\mathbb{R}^d) \). The same holds true with \((\mathcal{E}_{0,s}^E)'(\mathbb{R}^d)\) in place of \( \mathcal{E}_{0,s}^E(\mathbb{R}^d) \) at each occurrence;

3. If \( s = \frac{1}{2} \) and \( f_0 \in (\mathcal{E}_s^E)'(\mathbb{R}^d) \), then \( f(t, \cdot) \in \mathcal{E}_s^E(\mathbb{R}^d) \) when \( t > 0 \), and \( f(t, \cdot) \in (\mathcal{E}_{0,s}^E)'(\mathbb{R}^d) \) when \( t < 0 \).

3. Periodic elements in modulation spaces

In this section we show that \( E \)-periodic elements in the modulation spaces \( M^\infty(\omega, \mathcal{B}) \) and \( W^\infty(\omega, \mathcal{B}) \) agree with \( \mathcal{E}_s^E(\omega, \mathcal{B}) \) in Definition 1.8 for suitable \( \omega \) and \( \mathcal{B} \).

More precisely we have following extension of [21, Proposition 2.6].

Theorem 3.1. Let \( E \subseteq \mathbb{R}^d \) be a non-degenerate parallelepiped, \( \mathcal{O}_0 \) be as in Definition 1.8, \( q \in (0, \infty]^d, \tau \in \mathbb{S}_d \), and let \( \omega \in \mathcal{P}_{E}(\mathbb{R}^d) \). Also let

\[
\mathcal{B} = L^q_{\mathcal{O}_0, \tau}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{B}_0 = \ell^q_{\mathcal{O}_0, \tau}(\mathcal{N}_E)
\]

Then

\[
\mathcal{E}_E(\omega, \mathcal{B}_0) = M^\infty(\omega, \mathcal{B}) \bigcap \mathcal{E}_0^E(\mathbb{R}^d) = W^\infty(\omega, \mathcal{B}) \bigcap (\mathcal{E}_0^E)'(\mathbb{R}^d).
\]

We note that compactly supported as well as periodic elements in modulation spaces have been investigated in different contexts. For example, Theorem 3.1 is related to [22, Proposition 5.1].
Proof. It is clear from the definitions that $M^\infty(\omega, \mathcal{B}) \subseteq W^\infty(\omega, \mathcal{B})$ (see also [27]). Hence it suffices to prove

$$W^\infty(\omega, \mathcal{B}) \cap (\mathcal{E}_0^E)'(\mathbb{R}^d) \subseteq M^\infty(\omega, \mathcal{B}) \cap (\mathcal{E}_0^E)'(\mathbb{R}^d).$$

(3.1)

For every $f \in W^\infty(\omega, \mathcal{B}) \cap (\mathcal{E}_0^E)'(\mathbb{R}^d)$, we have

$$f = \sum_{\iota \in \Lambda} \sum_{j \in \Lambda_E} (V_\psi f)(j, \iota) \phi(\cdot - j) e^{i(\cdot, \iota)},$$

and

$$\|f\|_{W^\infty(\omega, \mathcal{B})} \asymp \sup_{j \in \Lambda_E} \|(V_\psi f)(j, \cdot)\|_{\mathcal{B}_1}, \quad \mathcal{B}_1 = \ell^q_{C_0, \tau}(\Lambda)$$

for some $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$, $\psi \in M^*_r(\mathbb{R}^d) \setminus 0$ and sufficiently dense lattice $\Lambda \subseteq \mathbb{R}^d$, which are fixed (cf. [27, Theorem 3.7]). We may assume that $\Lambda_0 \subseteq \Lambda$.

By using the $E$-periodicity of $f$, we get by straight-forward computations that

$$(V_\psi f)(j, \iota) = e^{-i(j, \iota)}(V_\psi f)(0, \iota),$$

which gives

$$(V_\psi f)(x, \xi) = \sum_{\iota \in \Lambda} \sum_{j \in \Lambda_E} (V_\psi f)(0, \iota)(V_\psi \phi)(x - j, \xi - \iota)e^{-i(j, \xi)}.$$ 

In particular, by Proposition 1.2 we get

$$|V_\psi f(x, \xi)| \lesssim \sum_{\iota \in \Lambda} \sum_{j \in \Lambda_E} |(V_\psi f)(0, \iota)| e^{-R(|x - j| + |\xi - \iota|)},$$

for every $R > 0$, giving that

$$|(V_\psi f)(x, \xi)| \lesssim \sum_{\iota \in \Lambda} |(V_\psi f)(0, \iota)| e^{-R|\xi - \iota|}$$

for every $R > 0$. By (1.3) and Remark 2.3 we get

$$|c(f, \alpha)| \lesssim \int_E \left( \int_{\mathbb{R}^d} |(V_\psi f)(x, \xi)| |\hat{\phi}(\alpha - \xi)| \, d\xi \right) \, dx \lesssim \sum_{\iota \in \Lambda} |(V_\psi f)(0, \iota)| \int_{\mathbb{R}^d} e^{-R|\xi - \iota|} |\hat{\phi}(\alpha - \xi)| \, d\xi \lesssim \sum_{\iota \in \Lambda} |(V_\psi f)(0, \iota)| e^{-R|\alpha - \iota|}.$$
for every $R > 0$. By choosing $R$ large enough it follows that $v(x) \lesssim e^{R|x|/2}$. For such $R$ we have

$$\|f\|_{\mathcal{L}^p(\omega,\mathcal{B}_0)} = \left\|\{c(f, \alpha)\omega(\alpha)\}_{\alpha \in \Lambda_E}\right\|_{\mathcal{B}_0} \lesssim \|(V_{\phi}f)(0, \cdot)\omega|*(e^{-R|\cdot|}v)\|_{\mathcal{B}_0},$$

$$\lesssim \|(V_{\phi}f)(0, \cdot)\omega|*(e^{-R|\cdot|}v)\|_{\mathcal{B}_1} \leq \|(V_{\phi}f)(0, \cdot)\omega|e^{-R|\cdot|}v\|_{C(\Lambda)} \times \|(V_{\phi}f)(0, \cdot)\omega\|_{\mathcal{B}_1} = \sup_{j \in \Lambda_E} \|(V_{\phi}f)(j, \cdot)\omega\|_{\mathcal{B}_1} \approx \|f\|_{W^\infty(\omega,\mathcal{B})}. \quad (3.2)$$

Here $*$ is the discrete convolution with respect to $\Lambda$, and the second inequality follows from the fact that $\Lambda'_E \subseteq \Lambda$. This gives the first embedding in (3.1).

In order to prove the second embedding in (3.1) we observe that

$$(V_{\phi}f)(x, \xi) = \sum_{\alpha \in \Lambda_E} c(f, \alpha)\hat{\phi}(\alpha - \xi)e^{i(x,\alpha - \xi)}.$$ 

Let

$$f_0(\beta) = \sup_{x \in \mathbb{R}^d} \left( \sup_{\xi \in \beta + \mathbb{E}'} |V_{\phi}f(x, \xi)| \right), \quad \beta \in \Lambda'_E$$

and

$$\kappa(\beta) = \sup_{\xi \in \beta + \mathbb{E}'} |\hat{\phi}(\xi)|, \quad \beta \in \Lambda'_E.$$ 

Then $\|f\|_{M^\infty(\omega,\mathcal{B})} \approx \|f_0 \cdot \omega\|_{\mathcal{B}_0}$, and for every $R > 0$ we have

$$\|f\|_{M^\infty(\omega,\mathcal{B})} \lesssim \left\|\sum_{\alpha \in \Lambda_E} |c(f, \alpha)|\|\kappa(\alpha - \cdot)\omega\|_{\mathcal{B}_0}\right\|_{\mathcal{B}_0} \lesssim \left\|c(f, \alpha)\omega|*(e^{-R|\cdot|}v)\right\|_{\mathcal{B}_0} \leq \|c(f, \cdot)\omega\|_{\mathcal{B}_0}\|e^{-R|\cdot|}v\|_{L_{\min(1,r)}} \approx \|c(f, \cdot)\omega\|_{\mathcal{B}_0}.$$ 

This gives the result. \qed

3.1. **Duality properties of** $M^\infty(\omega,\mathcal{B}) \cap (\mathcal{E}_0^E)'(\mathbb{R}^d)$. We begin with the following duality result.

**Theorem 3.2.** Let $E \subseteq \mathbb{R}^d$ be a non-degenerate parallelepiped, $\mathcal{O}_0$ be as in Definition 1.6, $q \in [1, \infty]^d$, $\tau \in \mathcal{S}_d$, and let $\omega \in \mathcal{P}_E(\mathbb{R}^d)$. Then the following is true:

1. The form $(\cdot, \cdot)_E$ from $\mathcal{E}_0^E(\mathbb{R}^d) \times \mathcal{E}_0^E(\mathbb{R}^d)$ to $C$ extends to a continuous map from

$$\mathcal{E}_0^E(\omega, \mathcal{T}_{\mathcal{O}_0,\tau}'(\Lambda'_E)) \times \mathcal{E}_0^E(1/\omega, \mathcal{T}_{\mathcal{O}_0,\tau}(\Lambda'_E))$$

27
to $C$. If in addition $\min q > 1$ or $\max q < \infty$, then the extension is unique.

(2) if $\max q < \infty$, then the dual of $\mathcal{E}^E(\omega, \ell^q_{\mathcal{O}_0, \tau}(\mathcal{L}_E))$ can be identified by $\mathcal{E}^E(1/\omega, \ell^q_{\mathcal{O}_0, \tau}(\mathcal{L}_E))$ through the form $(\cdot, \cdot)_E$.

Proof. By the definitions we may identify $\mathcal{E}^E(\omega, \ell^q_{\mathcal{O}_0, \tau}(\mathcal{L}_E))$ and the form $(\cdot, \cdot)_E$ with $\ell^q_{\mathcal{O}_0, \tau}(\mathcal{L}_E)$ and the form $(\cdot, \cdot)_{\mathcal{E}^E(\omega, \ell^q_{\mathcal{O}_0, \tau}(\mathcal{L}_E))}$. The result now follows from the fact that similar properties hold true for mixed normed Lebesgue spaces.

Corollary 3.3. Let $E \subseteq \mathbb{R}^d$ be a non-degenerate parallelepiped, $\mathcal{O}_0$ be as in Definition 3.1, $\omega \in [1, \infty)^d$, $\tau \in S_d$, and let $\omega \in \mathcal{P}(\mathbb{R}^d)$. Then the form $(\cdot, \cdot)_E$ from $\mathcal{E}^E(\mathbb{R}^d) \times \mathcal{E}^E(\mathbb{R}^d)$ to $C$ extends uniquely to a continuous map from $M^\infty(\omega, L^q_{\mathcal{O}_0, \tau}(\mathbb{R}^d)) \cap (\mathcal{E}^E)'(\mathbb{R}^d) \times M^\infty(1/\omega, L^q_{\mathcal{O}_0, \tau}(\mathbb{R}^d)) \cap (\mathcal{E}^E)'(\mathbb{R}^d)$ to $C$, and the dual of $M^\infty(\omega, L^q_{\mathcal{O}_0, \tau}(\mathbb{R}^d)) \cap (\mathcal{E}^E)'(\mathbb{R}^d)$ can be identified by $M^\infty(1/\omega, L^q_{\mathcal{O}_0, \tau}(\mathbb{R}^d)) \cap (\mathcal{E}^E)'(\mathbb{R}^d)$ through this form.

In particular, if $q \in [1, \infty)$ and $\omega_0(x, \xi) = \omega(\xi)$, then the dual of $M^\infty_{(\omega_0)}(\mathbb{R}^d) \cap (\mathcal{E}^E)'(\mathbb{R}^d)$ can be identified by $M^\infty_{(1/\omega_0)}(\mathbb{R}^d) \cap (\mathcal{E}^E)'(\mathbb{R}^d)$.

Proof. The result follows by combining Theorems 3.2 with 3.1. \qed

References

[1] J. Chung, S.-Y. Chung, D. Kim, Characterizations of the Gelfand-Shilov spaces via Fourier transforms, Proc. Amer. Math. Soc. 124 (1996), 2101–2108.

[2] E. Cordero, S. Pilipović, L. Rodino, N. Teofanov Quasianalytic Gelfand-Shilov spaces with applications to localization operators, Rocky Mt. J. Math. 40 (2010), 1123-1147.

[3] A. Dasgupta, M. Ruzhansky Gevrey functions and ultradistributions on compact Lie groups and homogeneous spaces, Bull. Sci. Math. 138 (2014), 756–782.

[4] A. Dasgupta, M. Ruzhansky Eigenfunction expansions of ultradifferentiable functions and ultradistributions, Trans. Amer. Math. Soc. 368 (2016), 8481–8498.

[5] S. J. L. Eijndhoven Functional analytic characterizations of the Gelfand-Shilov spaces $S^p_\omega$, Nederl. Akad. Wetensch. Indag. Math. 49 (1987), 133–144.

[6] H. G. Feichtinger Modulation spaces on locally compact abelian groups, Technical report, University of Vienna, Vienna, 1983; also in: M. Krishna, R. Radha, S. Thangavelu (Eds) Wavelets and their applications, Allied Publishers Private Limited, NewDelhi Mumbai Kolkata Chennai Nagpur Ahmedabad Bangalore Hyderabad Lucknow, 2003, pp. 99–140.

[7] H. G. Feichtinger Modulation spaces: Looking back and ahead, Sampl. Theory Signal Image Process. 5 (2006), 109–140.

[8] V. Fischer, M. Ruzhansky Quantization on nilpotent Lie groups, Birkhäuser, Progress in Mathematics 314, Boston, 2016.

[9] Y. V. Galperin, S. Samarah Time-frequency analysis on modulation spaces $M_{p,q}^\alpha$, $0 < p, q \leq \infty$, Appl. Comput. Harmon. Anal. 16 (2004), 1–18.

[10] C. Garetto, M. Ruzhansky Wave equation for sums of squares on compact Lie groups, J. Differential Equations 258 (2015), 4324–4347.
[11] I. M. Gelfand, G. E. Shilov, Generalized functions, II-III, Academic Press, New York London, 1968.
[12] V. I. Gorbačuk On Fourier series of periodic ultradistributions, Ukrainian Math. J. 34 (1982), 144–150. (Russian)
[13] V. I. Gorbačuk, M. L. Gorbačuk Trigonometric series and generalized functions, Dokl. Adad. Nauk SSSR 257 (1981), 799–804.
[14] K. H. Gröchenig Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001.
[15] K. Gröchenig Weight functions in time-frequency analysis in: L. Rodino, M. W. Wong (Eds) Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis, Fields Institute Comm., 52 2007, pp. 343–366.
[16] K. Gröchenig, G. Zimmermann Spaces of test functions via the STFT J. Funct. Spaces Appl. 2 (2004), 25–53.
[17] L. Hörmander The Analysis of Linear Partial Differential Operators, vol I–III, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1983, 1985.
[18] S. Pilipović Generalization of Zemanian spaces of generalized functions which have orthonormal series expansions, SIAM J. Math. Anal. 17 (1986), 477D484.
[19] S. Pilipović Structural theorems for periodic ultradistributions, Proc. Amer. Math. Soc. 98 (1986), 261–266.
[20] S. Pilipović Tempered ultradistributions, Boll. U.M.I. 7 (1988), 235–251.
[21] M. Reich A non-analytic superposition result on Gevrey-modulation spaces, Diploma thesis, Technische Universität Bergakademie Freiberg, Germany, Angewandte Mathematik Registration list 51765.
[22] M. Ruzhansky, M. Sugimoto, J. Toft, N. Tomita Changes of variables in modulation and Wiener amalgam spaces, Math. Nachr. 284 (2011), 2078–2092.
[23] M. Ruzhansky, V. Turunen On the toroidal quantization of periodic pseudo-differential operators, Numerical Functional Analysis and Optimization 30 (2009), 1098–1124.
[24] M. Ruzhansky, V. Turunen Quantization of pseudo-differential operators on the torus, J. Fourier Anal. Appl. 16 (2010), 943–982.
[25] M. Ruzhansky, V. Turunen Pseudo-differential operators and symmetries. Background analysis and advanced topics, Pseudo-Differential Operators. Theory and Applications 2, Birkhäuser Verlag, Basel, 2010.
[26] J. Toft The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators, J. Pseudo-Differ. Oper. Appl. 3 (2012), 145–227.
[27] J. Toft Gabor analysis for a broad class of quasi-Banach modulation spaces in: S. Pilipović, J. Toft (eds), Pseudo-differential operators, generalized functions, Operator Theory: Advances and Applications 245, Birkhäuser, 2015, 249–278.
[28] J. Toft Images of function and distribution spaces under the Bargmann transform, J. Pseudo-Differ. Oper. Appl. (Appeared online 2016).
[29] Z. Zieleźny On formal trigonmetrical series, Studia Math. 24 (1964), 305–310.

DEPARTMENT OF MATHEMATICS, LINNÉUS UNIVERSITY, VÄXJÖ, SWEDEN
E-mail address: joachim.toft@lnu.se

DEPARTMENT OF MATHEMATICS, LINNÉUS UNIVERSITY, VÄXJÖ, SWEDEN
E-mail address: elmira.nabizadeh.extern@lnu.se