ON THE UNIPOTENCE OF AUTOEQUIVALENCES OF TORIC COMPLETE INTERSECTION CALABI-YAU CATEGORIES

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Abstract. We identify a class of autoequivalences of triangulated categories of singularities associated with Calabi-Yau complete intersections in toric varieties. Elements of this class satisfy relations that are directly linked to the toric data.

0. Introduction

Let $X$ be a smooth projective variety over $\mathbb{C}$, and $D_1, \ldots, D_k$ with $k \leq \dim(X)+1$ effective divisor classes satisfying

$$D_1 \cap D_2 \cap \cdots \cap D_k = 0.$$  

For each $i = 1, \ldots, k$, we consider the autoequivalence of the bounded derived category of coherent sheaves, $D^b(X)$, given by tensoring with the line bundle $\mathcal{O}(D_i)$, i.e. for $A \in D^b(X)$,

$$\mathcal{M}_i(A) = A \otimes \mathcal{O}(D_i).$$

Choosing for each $i$ a generic section $s_i$ of $\mathcal{O}(D_i)$, we let $\mathcal{N}(s_i)$ be the endofunctor of $D^b(X)$ sending $A$ to

$$\mathcal{N}(s_i)(A) = \text{Cone}(s_i : A \to \mathcal{M}_i(A)).$$

Since as a consequence of eq. (1), the complete intersection of $(s_1, \ldots, s_k)$ is empty, the associated Koszul complex $\mathcal{K}(s_1, \ldots, s_k)$ is exact. Therefore,

$$\mathcal{N}(s_1) \circ \mathcal{N}(s_2) \circ \cdots \circ \mathcal{N}(s_k)(A) \cong 0$$

for any object $A$ in $D^b(X)$. When pushed to K-theory, and denoting the image of $\mathcal{M}_i$ by $M_i : K(D^b(X)) \to K(D^b(X))$, we obtain the relation

$$\prod_{i=1}^k (M_i - \text{id}) = 0.$$

In this paper we obtain generalizations of these formulas in a situation coming from toric geometry and of interest in mirror symmetry. We now summarize the ideas involved.

Let $\mathbb{P}_{\hat{\Sigma}}$ be a toric Calabi-Yau variety with fan $\hat{\Sigma}$. Toric varieties birationally equivalent to $\mathbb{P}_{\hat{\Sigma}}$ may be constructed by suitable modifications of $\hat{\Sigma}$. We here think of modifications $\hat{\Phi}$ obtained by crossing a face in the secondary fan, giving rise to blowdowns, blowups, or flops. Following physics terminology, we shall refer to the various $\hat{\Phi}$ as different "phases" of the secondary fan. For each such modification

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of \( \hat{\Sigma} \), we have an equivalence of triangulated categories \( D^b(\mathbb{P}_{\hat{\Sigma}}) \cong D^b(\mathbb{P}_S) \). The corresponding functors have been described explicitly by van den Bergh \[1\] and Kawamata \[2\]. Clearly then, autoequivalences of \( D^b(\mathbb{P}_{\hat{\Sigma}}) \) induce autoequivalences of \( D^b(\mathbb{P}_S) \), and vice-versa. However, certain elements of \( \text{Aut}(D^b(\mathbb{P}_{\hat{\Sigma}})) \) and relations among them are easier to see in one description than in the other, which is going to be the main theme of our paper.

The primary interest, however, is in studying compact Calabi–Yau manifolds \( X \) that are complete intersection in \( \mathbb{P}_\Sigma \) when the latter is a smooth, toric Fano. (This is not the most general case to which our methods apply, but one of much interest.) So let \( P_1, P_2, \ldots, P_\ell \) be \( \ell \) anti-effective divisor classes of \( \mathbb{P}_\Sigma \), with \( \sum_a P_a = K_{\mathbb{P}_\Sigma} \).

Choose generic sections \( G_a \) of \( \mathcal{O}_{\mathbb{P}_\Sigma}(-P_a) \) for \( a = 1, 2, \ldots, \ell \), and consider

\[
X := \{ G_1 = 0, G_2 = 0, \ldots, G_\ell = 0 \} \subset \mathbb{P}_\Sigma.
\]

We recall some preliminaries about the derived category \( D^b(X) \) in section \[1\]. In section \[2\] we give a description of \( D^b(X) \) in terms of a singularity category that will be useful for later purposes. We replace \( \mathbb{P}_\Sigma \) with a larger combinatorial object \( \hat{\mathbb{P}}_\Sigma \) that may be thought of as the total space of the bundle \( \bigoplus_{a=1}^\ell \mathcal{O}(P_a) \rightarrow \mathbb{P}_\Sigma \), with fibers equipped with an additional grading called \( R \)-grading, a terminology from physics. Let \( p_a \) be the fibre coordinate on \( \mathcal{O}(P_a) \). Then the function, a.k.a superpotential or Landau–Ginzburg potential,

\[
W = \sum_{a=1}^\ell p_a G_a,
\]

is the homological device that reduces \( D^b(\mathbb{P}_\Sigma) \) to \( D^b(X) \): The singularity category of \( W \) (namely, the triangulated category of the singularity \( W : \mathbb{P}_\Sigma \rightarrow \mathbb{C} \), in the sense of \[3\] \[4\]) is equivalent to \( D^b(X) \) (see Theorem \[2\]).

Then, for each modification \( \hat{\Phi} \) of \( \hat{\Sigma} \) in the above sense, one may construct a triangulated category, \( \mathcal{C}_{\hat{\Phi}} \), as a quotient of singularity categories (Definition \[4\]),

\[
\mathcal{C}_{\hat{\Phi}} = \frac{D_{sg}(\text{gr} \hat{S})}{D_{sg}(\text{tor} \hat{S})}.
\]

This is done in section \[3\]. The category \( D_{sg}(\text{gr} \hat{S}) \) depends on \( W \), but is independent of \( \hat{\Phi} \), whereas the subcategory \( D_{sg}(\text{tor} \hat{S}) \) of torsion modules depends additionally on \( \hat{\Phi} \). In fact, however, all \( \mathcal{C}_{\hat{\Phi}} \) are equivalent as triangulated categories to the fixed category \( \mathcal{C}_\Phi \cong D^b(X) \) (Theorem \[3\]).

Working now in a fixed phase \( \Phi \), we associate in section \[4\] to any toric divisor class \( D \in \text{Pic}(\mathbb{P}_{\hat{\Sigma}}) \) an automorphism \( \mathcal{M}_D^\Phi \) of \( \mathcal{C}_\Phi \) that can be thought of as tensoring with a line bundle. We emphasize that the equivalences amongst the \( \mathcal{C}_{\hat{\Phi}} \) will not identify the \( \mathcal{M}_D^\Phi \) with each other as \( \hat{\Phi} \) varies. (For a familiar example, see comments below.)

To describe the relations analogous to \( (2) \) among the \( \mathcal{M}_D^\Phi \), we recall that the toric divisors \( D_1, \ldots, D_n \) of \( \mathbb{P}_{\hat{\Sigma}} \) are in one-to-one correspondence with the set of one-dimensional cones \( \Sigma(1) = (v_1, \ldots, v_n) \) of \( \Sigma \). We denote the canonical sections of \( \mathcal{O}(D_i) \) by \( x_i, i = 1, \ldots, n \). For \( \mathbb{P}_{\hat{\Sigma}} \), this list is extended by (the pullbacks of) the divisors \( P_a =: D_{a+a}, \) for \( a = 1, \ldots, \ell \), with canonical section \( p_a =: x_{n+a} \). This extended list is in one-to-one correspondence with the set \( \hat{\Sigma}(1) \) of one-dimensional
cones \((\hat{v}_i)_{1 \leq i \leq n + \ell}\) of \(\hat{\Sigma}\), and contains \(\hat{\Phi}(1)\) for any \(\hat{\Phi}\). We denote the \(R\)-grading of \(x_i\) by \(r_i\).

\[
 r_i = \begin{cases} 
 0, & 1 \leq i \leq n, \\
 2, & n + 1 \leq i \leq n + \ell. 
\end{cases}
\]

Our main result will follow straightforwardly from these definitions in section 4.

**Theorem 1.** For each toric divisor \(D_i\), with canonical section \(x_i\) of \(\mathcal{O}(D_i)\), define an endofunctor \(N_{\hat{\Phi}}\) of \(C_{\hat{\Phi}}\) by

\[
 N_{\hat{\Phi}}(x_i)(-): \text{Cone}(x_i : - \rightarrow - \otimes \mathcal{O}(D_i)[r_i]).
\]

Then for each subset \(I \subset \{1, \ldots, n + \ell\}\) such that the corresponding set of edges \((\hat{v}_i)_{i \in I}\) is not contained in any cone of \(\hat{\Phi}\), we have the relation

\[
 \bigotimes_{i \in I} N_{\hat{\Phi}}(x_i) \cong 0.
\]

A simple consequence is the following

**Corollary 1.** Let \(M^D_{\hat{\Phi}}\) be the automorphism induced by \(\mathcal{M}^D_{\hat{\Phi}}\) on the \(K\)-theory \(K(C_{\hat{\Phi}})\). Then for each \(I\) as above,

\[
 \prod_{i \in I}(M^D_{\Phi} - \text{id}) = 0.
\]

Our original motivation for this work was to understand the generalization of an old observation of Kontsevich. We let \(Y\) be the family of Calabi–Yau manifolds that is mirror to \(X\) according to Batyrev’s construction. Let \(B\) be the base of the family after removing the singular fibers (and possibly more). Monodromies around loops in \(B\) induce symplectic transformations (generalized Dehn twists) that can be lifted to autoequivalences of the symplectic category (the Fukaya category \(\text{Fuk}(Y)\), where \(Y\) is a generic fiber of \(\mathcal{Y}\)). Via the homological mirror symmetry (HMS) conjecture, \(\text{Fuk}(Y) \cong D^b(X)\), one is led to expect the existence of a monodromy representation \(\rho: \pi_1(B) \rightarrow \text{Aut}(D^b(X))\) that has attracted some attention over the years.

When \(X\) is the quintic threefold, and \(Y\) its mirror family, we may model \(B\) as \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\). Note that the point at \(\infty\) does not correspond to a singular threefold, but to one with an additional automorphism of order 5. Somewhat surprisingly, this symmetry is only realized projectively at the categorical level. Indeed, if \(\gamma_\infty\) is a path around \(\infty\), the categorical monodromy \(\mathcal{M}_\infty = \rho(\gamma_\infty) \in \text{Aut}(D^b(X))\) (modulo HMS conjecture) satisfies the relation

\[
 (\mathcal{M}_\infty)^5 \cong (-)[2]
\]

To fully appreciate this remarkable relation, we rewrite this in more familiar terms using \(\gamma^{-1}_\infty = \gamma_0 \circ \gamma_1\). According to HMS for the quintic threefold, the monodromy \(\gamma_0\) around 0 in \(\mathbb{P}^1\) corresponds to the autoequivalence \(\mathcal{M}_0\) of tensoring with the line bundle \(\mathcal{O}_X(1)\), whilst monodromy \(\gamma_1\) (around the conifold) is realized as twist \(T_{\mathcal{O}_X}\) by the structure sheaf [5]. These transformations of \(D^b(X)\) can be checked to satisfy [5, Chapter 7.1.4]

\[
 (\mathcal{M}_0 \circ T_{\mathcal{O}_X})^5 \cong (-)[2],
\]

which is the way in which [3] is often quoted. (Some hypersurfaces in weighted projective space are treated in [7].)
Our results give a uniform treatment of such relations for the general class of Calabi–Yau complete intersections in toric varieties. We elaborate on this point of view, together with some other applications, in section 5.

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1. Preliminaries

Consider $\mathbb{P}_\Sigma$, a complete, smooth toric variety defined by a fan $\Sigma$ in the lattice $N \cong \mathbb{Z}^{n-k}$. Let $v_i$ for $i = 1, \ldots, n$ be the generators of the one-dimensional cones of $\Sigma$. We briefly recall the construction of $\mathbb{P}_\Sigma$ from $\Sigma$ in terms of homogeneous coordinates $(x_1, \ldots, x_n)$. First, if $M = N^\ast$ is the dual lattice and $\text{Pic}(\mathbb{P}_\Sigma) \cong \mathbb{Z}^k$ the Picard lattice, the exact sequence

\[ M \xrightarrow{v^*} \mathbb{Z}^n \xrightarrow{w} \text{Pic}(\mathbb{P}_\Sigma) \]

induces a $\mathbb{Z}^k$-grading on the homogeneous coordinate ring through the $(\mathbb{C}^\times)^k$-action $(\mathbb{C}^\times)^k \ni \lambda : (x_1, \ldots, x_n) \mapsto (\lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n)$. Second, if by a (Batyrev) primitive collection we mean a collection of generators $v_i$ that generates none of the cones in $\Sigma$, whereas any proper subset of it does, we define the exceptional set as the union

\[ Z_\Sigma = \bigcup_p Z_{I_p}, \]

where $p$ indexes all primitive collections, and $Z_{I_p} = \cap_{i: v_i \in I_p} \{x_i = 0\}$. Then, the toric variety is the quotient

\[ \mathbb{P}_\Sigma = \mathbb{C}^n / Z_\Sigma. \]

Given $\mathbb{P}_\Sigma$, we consider complete intersections $X \subset \mathbb{P}_\Sigma$ defined by transversal polynomials $G_a$ of $\mathbb{Z}^k$-degree $d_a$ for $a = 1, \ldots, \ell$. We require $X$ to be a Calabi–Yau variety, i.e. $\sum_{i=1}^n w_i = \sum_{a=1}^\ell d_a$.

Switching to the algebraic description, let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the $\mathbb{Z}^k$-graded coordinate ring associated with the toric data, and $J_\Sigma = (\prod_{v_i \in I_p} x_i)^{\sigma} | \sigma \in \Sigma$ the Cox ideal, whose vanishing locus is $Z_\Sigma = \text{V}(J_\Sigma)$. The complete intersection ring is denoted by $S = R/(G_1, \ldots, G_\ell)$, and the image of the Cox ideal along $R \to S$ by the same symbol, $J_\Sigma$. The following definitions involving $J_\Sigma$ can be made over $R$ or over $S$.

Let $\text{gr}S$ be the abelian category of graded $S$-modules. The morphisms of $\text{gr}S$ are the module homomorphisms of degree 0. For a given $S$-module $A$, and $q \in \mathbb{Z}^k$, a shift in degree is denoted by $A(q)$. We call an $S$-module $J_\Sigma$-torsion if it is annihilated by $(J_\Sigma)^m$ for some positive integer $m$.

Definition 1. Let $D^b(\text{gr}S)$ be the bounded derived category of graded $S$-modules and $D^b(\text{tor}_\Sigma S)$ the full triangulated subcategory of graded $J_\Sigma$-torsion $S$-modules, i.e. any object is isomorphic to a complex of $J_\Sigma$-torsion modules. The quotient category,

\[ D^b(\text{qgr}_\Sigma S) = \frac{D^b(\text{gr}S)}{D^b(\text{tor}_\Sigma S)}, \]
is defined by localization along the multiplicative system of morphisms \( s \) that fit into distinguished triangles

\[
A \xrightarrow{s} B \longrightarrow C \longrightarrow A[1],
\]

where \( A, B \in D^b(\text{gr} S) \) and \( C \in D^b(\text{tor} S) \).

A generalization of Serre’s correspondence identifies the abelian category of coherent sheaves on \( X \subset \mathbb{P}_\Sigma \) with the abelian category of graded \( S \)-modules modulo \( J_\Sigma \)-torsion modules. This correspondence extends to the derived categories,

\[
D^b(X) \cong D^b(\text{qgr} S).
\]

When we subsequently consider the derived category of coherent sheaves on \( X \), we will stick to its algebraic description in terms of modules over rings, that is, to \( D^b(\text{qgr} S) \).

Since any graded \( S \)-module has a projective resolution (though possibly unbounded from below), the derived category \( D^b(\text{gr} S) \) can be defined to be the homotopy category of projective complexes \( K^- \text{(gr} S) \) with quasi-isomorphisms inverted. Here, the minus index indicates that the complexes may be unbounded to the left, but with bounded cohomology. Namely, in the homotopy category of graded \( S \)-modules all quasi-isomorphisms are homotopy equivalences and therefore \( D^b(\text{gr} S) \) is in fact \( K^- \text{(gr} S) \).

By taking the quotient \[\text{bigr} \text{eq} \], all objects of the full subcategory \( D^b(\text{tor} S) \) become zero objects in \( D^b(\text{gr} S) \). Over \( R \), an important class of zero objects is given by the Koszul complexes associated with the primitive collections \( \mathcal{I}_p \).

\[
K_p(R) := \mathcal{K}_p = \bigotimes_{i|v_i \in \mathcal{I}_p} \text{Cone}(x_i : R \to R(w_i)).
\]

Indeed, the Koszul complex \( K_p(R) \) is nothing but the resolution of a \( J_\Sigma \)-torsion \( R \)-module, hence \( K_p(R) \cong 0 \) in \( D^b(\text{qgr} S) \). Over the complete intersection ring \( S \) an even shorter regular sequence \( \mathcal{S}_{I_p} \subset \{x_i\}_{i|v_i \in \mathcal{I}_p} \) may be the resolution of a \( J_\Sigma \)-torsion \( S \)-module. For \( G_1, \ldots, G_\ell \) generic, \( \mathcal{S}_{I_p} \) is just a subset of \( \{x_i\}_{i|v_i \in \mathcal{I}_p} \), with corresponding set of divisor classes \( \{D_j\}_j \). In general, we can choose representatives \( s_j \) of that shorter list of divisor classes, such that \( \cap_j \{s_j = 0\} \cap \cap_a \{G_a = 0\} \) has support on \( Z_{I_p} \). Thus, we have

**Lemma 1.** Let \( S \) be the complete intersection ring associated with the regular sequence \( (G_1, \ldots, G_\ell) \). For a primitive collection \( \mathcal{I}_p \) of the smooth toric fan \( \Sigma \), we denote by \( \mathcal{S}_{I_p} \) a regular sequence as described above. We let \( w_j \) be the degree of \( s_j \).\n
Then the Koszul complex

\[
\mathcal{K}(\mathcal{S}_{I_p}; S) = \bigotimes_j \text{Cone}(s_j : S \to S(w_j)),
\]

is a zero object in \( D^b(\text{qgr} S) \). \( \square \)

The goal of this work is to understand autoequivalences of \( D^b(\text{qgr} S) \cong D^b(X) \) and relations among them that can be directly traced back to the toric data, and more specifically to the secondary fan of \( X \subset \mathbb{P}_\Sigma \). To this end, we need a realization of \( D^b(\text{qgr} S) \) that has a natural generalization to all other maximal cones in the secondary fan. This alternative construction begins with realizing \( X \) as the critical locus \( \{dW = 0\} \) of a holomorphic function \( W \) on the total space of a certain
holomorphic vector bundle over \( \mathbb{P}_\Sigma \). This then leads us to consider the singularity category associated with \( W \) [3 4].

2. The singularity category for complete intersections

We append the data of the \( \ell \) polynomials \( G_a \) to the toric data of \( \mathbb{P}_\Sigma \) to obtain an enhanced fan \( \hat{\Sigma} \) in \( \hat{N}_\mathbb{R} \), where \( \hat{N} = N \oplus \mathbb{Z}^\ell \), cf. [10 Chapter 5]. Explicitly, if \( v_i \) has coordinates \( (v_i^1, \ldots, v_i^{n-k}) \) with respect to some basis of \( N \), the generators \( \hat{v}_i \) of the one-dimensional cones of \( \hat{\Sigma} \) are as follows. For \( i = 1, \ldots, n \) the coordinates of \( \hat{v}_i \) are \( (v_i^1, \ldots, v_i^{n-k}, u_i^1, \ldots, u_i^\ell) \), where the \( u_i^a \)'s are chosen to satisfy \( \sum_{i=1}^n v_i^au_i^a = d_a \). For \( a = 1, \ldots, \ell \), \( \hat{v}_{n+a} \) is given by a vector with 1 at the \((n+a)\)-th position and 0 else. Let \( \hat{\omega} \) subsume the vectors \( (w_1, \ldots, w_n, -d_1, \ldots, -d_\ell) \). We will sometimes use \( w_{n+a} := -d_a \). Then the exact sequence (4) is extended to

\[
\hat{M} \xrightarrow{\hat{\omega}} \mathbb{Z}^{n+\ell} \xrightarrow{\hat{w}} \text{Pic}(\mathbb{P}_\Sigma) = \text{Pic}(\mathbb{P}_\Sigma) .
\]

The (Calabi–Yau) toric variety of the enhanced fan \( \hat{\Sigma} \) is given by

\[
\mathbb{P}_\Sigma = \mathbb{C}^{n+\ell}/(\mathbb{C}^\times)^k \cong \text{Tot}(\hat{\omega}_{a=1}^\ell O(-d_a) \to \mathbb{P}_\Sigma) .
\]

Note that the exceptional set \( Z_\Sigma \) is just the pull-back of \( Z_\Sigma \) along \( \mathbb{C}^{n+\ell} \to \mathbb{C}^n \).

We denote by \( p_a \) the homogeneous coordinate associated with \( \hat{v}_{n+a} \) in \( \hat{\Sigma} \). Let \( \check{R} = \mathbb{C}[x_1, \ldots, x_n, p_1, \ldots, p_\ell] \) be the \( \mathbb{Z}^\ell \)-graded homogeneous coordinate ring associated with the fan \( \hat{\Sigma} \). Similarly, \( J_\Sigma \) be the Cox ideal for \( \Sigma \). For what follows it is necessary to introduce an additional \( \mathbb{Z} \)-grading, called R-grading, on \( \check{R} \). The R-grading of the \( x_i \) is 0, that of the \( p_a \) is 2. For a given \( \check{R} \)-module \( A \), a shift in R-grading by \( 2r \) is denoted by \( A_{(2r)} \).

The holomorphic function \( W \) on \( \mathbb{P}_\Sigma \) is built from the regular sequence \( (G_1, \ldots, G_\ell) \) and the auxiliary coordinates \( p_a \) as

\[
W = \sum_{a=1}^\ell p_a G_a .
\]

\( W \) is a polynomial of degree 0 and R-grading 2. Letting \( \check{S} = \check{R}/(W)\check{R} \), we have the isomorphism of graded rings, \( S \cong \check{S}/(G_1, p_1, \ldots, G_\ell, p_\ell) \check{S} \), which geometrically corresponds to the embedding of the complete intersection \( X \) in the toric Calabi–Yau variety \( \mathbb{P}_\Sigma \) as the critical locus of \( W \),

\[
X = \{dW = 0\} \subset \mathbb{P}_\Sigma .
\]

This follows on account of the transversality of the polynomials \( G_1, \ldots, G_\ell \), because \( dW \) vanishes iff \( p_a = 0 \) and \( G_a = 0 \) for \( a = 1, \ldots, \ell \).

Following [3 4], we now introduce the singularity category associated with the polynomial \( W \).

**Definition 2.** Let \( \mathcal{D}^b(\text{gr}\check{S}) \) be the derived category of graded \( \check{S} \)-modules, and \( \text{Perf}(\text{gr}\check{S}) \) the full triangulated subcategory of perfect complexes, i.e. bounded complexes of free modules. The singularity category is the quotient

\[
D_{sg}(\text{gr}\check{S}) := \frac{\mathcal{D}^b(\text{gr}\check{S})}{\text{Perf}(\text{gr}\check{S})} .
\]
Let \( \sigma \) denote the corresponding functor by cutting off the finite piece with positive \( R \)-degrees, in any homological degree. We mod out by perfect complexes, the two-periodic part determines the morphisms uniquely up to homotopy. On the representatives \( (6) \) a chain map \( \cdots A_0 \langle -2 \rangle \xrightarrow{g_A} A_1 \langle -2 \rangle \xrightarrow{f_A} A_0 \xrightarrow{g_A} A_1 \) with \( \text{rk} A_0 = \text{rk} A_1 \). The underlined module is in homological degree 0.

Notice that for any \( A' \in D_{\text{sg}}(\text{gr} \hat{S}) \), the shift in \( R \)-grading by \(-2\) in \( (6) \) is due to the \( R \)-grading of \( W \). In fact, the complex of free \( \hat{S} \)-modules \( (6) \) can be lifted to a sequence of free \( \hat{R} \)-modules so that the homomorphisms compose as \( f_A g_A = W \cdot \text{id}_{A_0} \) and \( g_A f_A = W \cdot \text{id}_{A_1} \), where \( A_j = \hat{A}_j \otimes_{\hat{R}} \hat{S} \) for \( j = 0, 1 \). This also leads to Orlov’s result in \( [4] \) that \( D_{\text{sg}}(\text{gr} \hat{S}) \) can equivalently be described by the triangulated category of matrix factorizations of \( W \) over the ring \( \hat{R} \).

The morphisms in the singularity category are given by chain maps modulo homotopy and modulo chain maps that factor through perfect complexes, the two-periodic part determines the morphisms uniquely up to homotopy. On the representatives \( (6) \) a chain map is a commutative diagram

\[
\begin{array}{cccc}
\cdots & A_0 \langle -2 \rangle & A_1 \langle -2 \rangle & A_0 \langle -2 \rangle \\
\downarrow \psi_0 & \downarrow \psi_1 & \downarrow \psi_0 & \downarrow \psi_1 \\
\cdots & B_0 \langle -2 \rangle & B_1 \langle -2 \rangle & B_0 \langle -2 \rangle \\
\end{array}
\]

For the following purposes, a useful representative of the object \( A' \in D_{\text{sg}}(\text{gr} \hat{S}) \) in \( D^b(\text{gr} \hat{S}) \) is constructed by continuing the complex \( (6) \) periodically by \( 2r \) steps to the right, so that the \( R \)-degrees of \( A_0 \{2r\} \) and \( A_1 \{2r\} \) are all positive, and then cutting off the finite piece with positive \( R \)-degrees, in any homological degree. We denote the corresponding functor by

\[
\sigma_{\leq 0} : D_{\text{sg}}(\text{gr} \hat{S}) \longrightarrow D^b(\text{gr} \hat{S}).
\]

Let \( \pi_{\text{sg}} : D^b(\text{gr} \hat{S}) \longrightarrow D_{\text{sg}}(\text{gr} \hat{S}) \), then clearly we have \( A' \cong \pi_{\text{sg}} \sigma_{\leq 0} A' \) for any object \( A' \in D_{\text{sg}}(\text{gr} \hat{S}) \). It follows from two-periodicity that a free resolution in the image of \( \sigma_{\leq 0} \), say \( B' \); satisfies the relation

\[
\sigma_{\leq 0} \pi_{\text{sg}} \{ B' \{2r\} \} = B \{2r\}, \quad \text{for} \quad r = 0, 1, 2, \ldots .
\]

In fact, the subcategory of such objects is a full triangulated subcategory of \( D^b(\text{gr} \hat{S}) \).

**Lemma 2.** Let \( D_{\leq 0} \) be the full triangulated subcategory of objects satisfying \( (9) \). Then the adjoint pair of functors,

\[
D_{\text{sg}}(\text{gr} \hat{S}) \xrightarrow{\sigma_{\leq 0}} D_{\leq 0} \subset D^b(\text{gr} \hat{S}),
\]

is an equivalence of triangulated categories.
Proof. We know already that $A' \cong \pi_{sg} \sigma_{\leq 0} A'$ for any object $A' \in D_{sg}(\text{gr} \hat{S})$, and by definition $B' \cong \sigma_{\leq 0} \pi_{sg} B'$ for any $B' \in D_{\leq 0}$. It remains to check isomorphism of morphisms: Because of two-periodicity a chain map between objects in $D_{\leq 0}$ cannot factor through a perfect complex, and hence

$$\text{Hom}_{sg}(\pi_{sg} A', \pi_{sg} B') \cong \text{Hom}_{D_{\leq 0}}(A', B'),$$

for $A', B' \in D_{\leq 0}$.

\begin{align*}
\text{Proposition 1.} \quad &\text{The triangulated categories } D_{sg}(\text{gr} \hat{S}) \text{ and } D^b(\text{gr} S) \text{ are equivalent.} \\
\text{Proof.} \quad &\text{Using Lemma 2 it remains to construct an adjoint pair of functors,}
\end{align*}

$$D^b(\text{gr} R) \leftarrow D^b(\text{gr} S) \xrightarrow{E} D_{\leq 0} \subset D^b(\text{gr} \hat{S})$$

that implements the equivalence.

The functor $E$ is provided by Eisenbud’s work \[11\] and is constructed in two steps. For the first step we include $D^b(\text{gr} S)$ as full triangulated subcategory in $D^b(\text{gr} R)$ by considering $S$-modules as $R$-modules which are annihilated by the regular sequence $(G_1, \ldots, G_t)$. For an object $A \in D^b(\text{gr} S)$ we take the (minimal) $R$-free resolution $P(A)$. Let $P(A)$ be the free $R$-module in the complex $P(A)$ and $d_0$ the differential.

The second step constructs from $P(A)$ a half-infinite complex in $D_{\leq 0}$. Theorem 7.1 of \[11\] allows us to introduce auxiliary endomorphisms $d_n : P(A)[2|n|] \to P(A)$ for $n \in (\mathbb{Z}_{\geq 0})^t$ and $|n| = \sum_{a=1}^t n_a$, which have homological degree 1 as well as degree $\sum_a n_a d_0$ and satisfy

\begin{equation}
\sum_{m, |m| \leq |n|} d_m d_{n-m} = G_a \cdot \text{id}_{P(A)}[-2],
\end{equation}

for $|n| > 1$.

Here $e_a = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 at the $a$th position, and we set $d_0 = d_0$.

Define the free $\hat{S}$-module $\hat{P}(A) = (\mathbb{C}[p_1, \ldots, p_t] \otimes_{\mathbb{C}} P(A))/(W)$. Then $E(A) \in D_{\leq 0}$ is the $\hat{S}$-module

$$\bigoplus_{r=0}^{\infty} \hat{P}(A)[-2r][2r]$$

together with the differential $d = \sum_n p^n \otimes d_n$. Notice that the latter has degree 0 and preserves the $R$-grading. In view of \[11\], $d^2 = W \cdot \text{id}[-2]$, which is zero on $\hat{S}$-modules. $E(A)$ satisfies condition \[9\], and is therefore an object in $D_{\leq 0}$. In fact, the object is determined by $d_0$ uniquely up to isomorphism, since finding a solution to the recursive relations \[10\] admits the freedom, $d \to UdU^{-1}$ for $U = \text{id} + \sum_{n>0} p^n u_n$.

The functor $\omega$ is defined by

$$\omega : D_{\leq 0} \to D^b(\text{gr} S) \subset D^b(\text{gr} R),$$

$$A' \mapsto (\sigma_{\geq 0} A') \otimes \hat{S}/(p_1, \ldots, p_t) \hat{S}.$$
From the definitions of $E$ and $\omega$ we have
\[ \omega E(A) = P(A) \cong A, \]
for any $A \in D^b(\text{gr} S) \subset D^b(\text{gr} R)$. Conversely, for $B' \in D_{\leq 0}$, since $\omega(B')$ is the R-grading 0 component of $B'$ and $E$ reconstructs the auxiliary endomorphisms in $B'$, we also have
\[ E\omega(B') \cong B'. \]

It remains to show that $\omega$ and $E$ are adjoint functors, that is
\[ \text{Hom}_{D_{\leq 0}}(E(A), B') = \text{Hom}_{\text{gr} S}(A, \omega(B')). \]
for $A \in D^b(\text{gr} S)$ and $B' \in D_{\leq 0}$. We use that $D^b(\text{gr} S)$ is full in $D^b(\text{gr} R)$ and write the right-hand side as $\text{Hom}_{\text{gr} R}(P(A), \omega(B'))$. For the left-hand side write a morphism in $\text{Hom}_{D_{\leq 0}}(E(A), B')$ as $\psi = \sum_{n \in \mathbb{Z} > 0} p^n \psi_n$. Just as the differential $d$, the morphism $\psi$ is determined (up to isomorphisms) by its R-grading 0 component $\psi_0$, hence the left-hand side is also isomorphic to $\text{Hom}_{\text{gr} R}(P(A), \omega(B'))$. \qed

**Definition 3.** Let $D^b(\text{tor}_\Sigma \hat{\Phi})$ be the full triangulated subcategory of $D^b(\text{gr} \hat{S})$ consisting of graded $J_\Sigma$-torsion modules, and consider its image in $D_{\text{sg}}(\text{gr} \hat{S})$, that is $D_{\text{sg}}(\text{tor}_\Sigma \hat{\Phi}) := \pi_{\text{sg}} D^b(\text{tor}_\Sigma \hat{\Phi})$. This is a full triangulated subcategory of $D_{\text{sg}}(\text{gr} \hat{S})$.

We define the quotient category
\[ D_{\text{sg}}(\text{gr} \Sigma \hat{\Phi}) := \frac{D_{\text{sg}}(\text{gr} \hat{S})}{D_{\text{sg}}(\text{tor}_\Sigma \hat{\Phi})}. \]

**Theorem 2.** The image of the torsion subcategory $D_{\text{sg}}(\text{tor}_\Sigma \hat{\Phi})$ under the equivalence $\omega_{\leq 0} = \omega_{\Sigma} : D_{\text{sg}}(\text{gr} \hat{S}) \to D^b(\text{gr} S)$ is $D^b(\text{tor}_\Sigma \hat{\Phi})$, and, consequently,
\[ D_{\text{sg}}(\text{gr} \Sigma \hat{\Phi}) \cong D^b(\text{gr} \Sigma \hat{\Phi}). \]

**Proof.** Recalling that $Z_\Sigma$ is the pull-back of $Z_{\hat{\Phi}}$ we find from the definition of $\omega$ that it maps $J_\Sigma$-torsion complexes to $J_\Sigma$-torsion complexes, hence $\omega_{\leq 0}(D_{\text{sg}}(\text{tor}_\Sigma \hat{\Phi})) \subseteq D^b(\text{tor}_\Sigma \hat{\Phi})$. Also, from the definition of $D_{\text{sg}}(\text{tor}_\Sigma \hat{\Phi})$, we know that $\pi_{\text{sg}}(D^b(\text{tor}_\Sigma \hat{\Phi})) \subseteq D_{\text{sg}}(\text{tor}_\Sigma \hat{\Phi})$. Applying $\omega_{\leq 0}$, we obtain $\omega_{\leq 0} \circ \pi_{\text{sg}}(D^b(\text{tor}_\Sigma \hat{\Phi})) = D^b(\text{tor}_\Sigma \hat{\Phi}) \subseteq \omega_{\leq 0}(D_{\text{sg}}(\text{tor}_\Sigma \hat{\Phi}))$, and therefore $D^b(\text{tor}_\Sigma \hat{\Phi}) \cong \omega_{\leq 0}(D_{\text{sg}}(\text{tor}_\Sigma \hat{\Phi}))$. \qed

### 3. Phases of triangulated categories

The realization of the derived category $D^b(X) \cong D^b(\text{gr} Y)$ through the singularity category $D_{\text{sg}}(\text{gr} Y)$ in Theorem 2 motivates us to define singularity categories for each maximal cone of the secondary fan of $\mathbb{P}_\Sigma$. We shall label the latter by the toric fan $\hat{\Phi}$.

The corresponding toric variety is
\[ \mathbb{P}_\hat{\Phi} = \frac{\mathbb{C}^{n+\ell} - Z_{\hat{\Phi}}}{(\mathbb{C}^*)^\ell}. \]

Note that in general $\hat{\Phi}$ contains fewer one-dimensional cones than $\Sigma$, i.e. fewer than $n + \ell$. This is taken into account in (11) by the exceptional set being the union,
\[ Z_{\hat{\Phi}} = Z_{\hat{\Phi}}^{\text{prim}} \cup Z_{\Sigma(1)} \setminus \hat{\Phi}(1). \]
Here $Z^{prim}_\Phi$ is given in terms of Batyrev’s primitive collections for the fan $\hat{\Phi}$, and

$$Z_{\Sigma(1)\setminus \Phi(1)} = \bigcup_{\hat{\alpha}, \in \Sigma(1)\setminus \Phi(1)} \{ x_i = 0 \}.$$  

Note that

$$\mathbb{P}_\Phi = \frac{(\mathbb{C}^{n+\ell-r} - Z^{prim}_\Phi) \times (\mathbb{C}^\times)^r}{(\mathbb{C}^\times)^k} \cong \frac{(\mathbb{C}^{n+\ell-r} - Z^{prim}_\Phi)}{(\mathbb{C}^\times)^{k-r} \times G_\Phi}$$

where $r$ is the number of one-dimensional cones in $\hat{\Sigma}(1)\setminus \hat{\Phi}(1)$ and $G_\Phi$ is a finite group.

To construct quotient singularity categories in every maximal cone $\hat{\Phi}$, it is convenient to start with the graded coordinate rings $\hat{\mathbb{R}} = \mathbb{C}[x_1, \ldots, x_n, p_1, \ldots, p_\ell]$ and $\hat{\mathbb{S}} = \hat{\mathbb{R}}/(W)\hat{\mathbb{R}}$, with the degree and R-grading as before. In a maximal cone $\hat{\Phi}$ of the secondary fan consider the ideal $J_\Phi = \langle \prod_{j \in \Sigma(1)\setminus \Phi(1)} x_j | x_j \in \Phi \rangle$. Its vanishing locus is the exceptional set $Z_\Phi$. We say that a graded $\hat{\mathbb{S}}$-module is $J_\Phi$-torsion if it is annihilated by $J_\Phi^{\otimes_m}$ for some positive integer $m$.

**Definition 4.** Let $\hat{\mathbb{S}} = \hat{\mathbb{R}}/(W)\hat{\mathbb{R}}$, and $D^b(gr \hat{\mathbb{S}})$ as well as $D_{sg}(gr \hat{\mathbb{S}})$ be as in Definition 2. Take any maximal cone $\hat{\Phi}$ in the secondary fan of $\mathbb{P}_\Sigma$. Let $D^b(tor \hat{\mathbb{S}})$ be the full triangulated subcategory of $J_\Phi$-torsion modules in $D^b(gr \hat{\mathbb{S}})$, and $D_{sg}(tor \hat{\mathbb{S}})$ its image in $D_{sg}(gr \hat{\mathbb{S}})$. Then we define the quotient

$$\mathcal{C}_\Phi = D_{sg}(qgr \hat{\mathbb{S}}) := \frac{D_{sg}(gr \hat{\mathbb{S}})}{D_{sg}(tor \hat{\mathbb{S}})}.$$  

**Remark 1.** For every irreducible component $Z_{Z_\Phi}$ of the exceptional set $Z_\Phi$, the associated Koszul complex

$$K_\Phi(z) := K(\{ x_i \}_{i \in I_p} ; \hat{\mathbb{S}}) = \bigotimes_{i \in I_p} \text{Cone}(x_i : \hat{\mathbb{S}} \to \hat{\mathbb{S}}(w_i) \{ r_i \}).$$

is an object in $D^b(tor \hat{\mathbb{S}})$. Here, $r_i$ is the R-degree of $x_i$, i.e. $r_i = 0$ for $i = 1, \ldots, n$ and $r_{n+a} = 2$ for $a = 1, \ldots, \ell$. Furthermore, for any object $A$ of $D^b(gr \hat{\mathbb{S}})$, the tensor product $A \otimes \hat{\mathbb{S}} K_\Phi$ is in $D^b(tor \hat{\mathbb{S}})$ and via $\pi_{sg}$ in $D_{sg}(tor \hat{\mathbb{S}})$.

We have introduced the quotient singularity categories separately for each maximal cone. The following result relates them.

**Theorem 3.** For any pair of neighboring maximal cones, $\hat{\Phi}_1$ and $\hat{\Phi}_2$, in the secondary fan of $\mathbb{P}_\Sigma$, with Calabi–Yau complete intersection $X \subset \mathbb{P}_\Sigma$, there is a family $\{ F_m^{\hat{\Phi}_1, \hat{\Phi}_2} \}_{m \in \mathbb{Z}}$ of equivalences of the corresponding quotient singularity categories,

$$F_m^{\hat{\Phi}_1, \hat{\Phi}_2} : D_{sg}(qgr_{\hat{\Phi}_1}) \cong D_{sg}(qgr_{\hat{\Phi}_2}).$$

**Proof.** Let $\mathbb{P}_{\hat{\Phi}_1}$ and $\mathbb{P}_{\hat{\Phi}_2}$ be the (Calabi–Yau) toric varieties in the neighboring maximal cones. We let $T \in \text{Pic}(\mathbb{P}_\Sigma)^*$ be the primitive dual vector characterizing the face between the two adjacent maximal cones by the condition $T(v) = 0$ for all

---

1Note that the union may include some of the auxiliary coordinates $x_{n+a} = p_a$. 
v in the face. The exceptional sets $Z_1$ and $Z_2$ in the geometric construction of $\mathbb{P}_{\hat{\Phi}_1}$ and $\mathbb{P}_{\hat{\Phi}_2}$ are related by [12, Chapter 4.5]

\[
Z_1 = Z_+ \cup (Z_1 \cap Z_2), \quad Z_2 = Z_- \cup (Z_1 \cap Z_2),
\]

where $Z_{\pm} = \cap_{i|T(w_i) \geq 0} \{x_i = 0\}$. We let $\text{tor}_{\hat{\Phi}_+} \hat{S}$, $\text{tor}_{\hat{\Phi}_-} \hat{S}$, and $\text{tor}_{\hat{\Phi}_{12}} \hat{S}$ denote the categories of torsion modules associated with $Z_+$, $Z_-$, and $Z_1 \cap Z_2$, respectively.

By the Calabi–Yau condition, $\sum_{i=1}^{n+\ell} w_i = 0$, we may define

\[
\sigma := \sum_{i|T(w_i) > 0} T(w_i) = - \sum_{i|T(w_i) < 0} T(w_i).
\]

For $m \in \mathbb{Z}$ let $K^m(\text{gr} \hat{S}) \subset K^-(\text{gr} \hat{S})$ be the homotopy category generated from invertible modules $\hat{S}(q)$ satisfying (cf. [12])

\[
m \leq T(q) < m + \sigma.
\]

(13)

Denote the associated singularity category by $D^m_{\text{sg}}(\text{gr} \hat{S})$ and consider

\[
D_{\text{sg}}(\text{gr} \hat{S}) \xrightarrow{\omega_+} D^m_{\text{sg}}(\text{gr} \hat{S}) \xrightarrow{\pi_-} D_{\text{sg}}(\text{gr} \hat{S}),
\]

where $D_{\text{sg}}(\text{gr} \hat{S})$ are the quotients of $D_{\text{sg}}(\text{gr} \hat{S})$ by $D_{\text{sg}}(\text{tor} \hat{S})$. The functors $\pi_{\pm}$ are the projections of $D_{\text{sg}}(\text{gr} \hat{S})$ to the quotient categories applied to the subcategory $D^m_{\text{sg}}(\text{gr} \hat{S})$. Comparing (13) with $Z_{\pm}$, we find that the objects in $D^m_{\text{sg}}(\text{gr} \hat{S})$ can not be torsion. Also, the objects of $D^m_{\text{sg}}(\text{gr} \hat{S})$ generate $D_{\text{sg}}(\text{gr} \hat{S})$, so that $\pi_{\pm}$ are bijective on the set of objects. $D^m_{\text{sg}}(\text{gr} \hat{S})$ containing no torsion objects also means that there are no non-trivial extensions, and the functors $\pi_{\pm}$ are fully faithful, hence equivalences. The inverses $\omega_{\pm}$ take the unique (up to homotopy) representative of an isomorphism class in $D_{\text{sg}}(\text{gr} \hat{S})$ that satisfies the restriction (13).

Finally, taking at each step in (14) the quotient by $D_{\text{sg}}(\text{tor} \hat{S})$ we find that the functor in the theorem is given by the compositions

\[
D_{\text{sg}}(\text{gr} \hat{S}) \xrightarrow{\omega_+} D^m_{\text{sg}}(\text{gr} \hat{S}) \xrightarrow{\pi_-} D_{\text{sg}}(\text{gr} \hat{S}).
\]

A combination of Theorems 2 and 3 was proved for the hypersurface case by Orlov in [3]. Van den Bergh [1] and Kawamata [2] stated analogous results for the derived categories $D^b(\mathbb{P})$ of Calabi–Yau toric varieties using noncommutative crepant resolutions. They build a tilting module, say $P$, out of a generating set of invertible modules $\hat{S}(q)$ satisfying (13), and use the derived category of the noncommutative endomorphism algebra, $D^b(\text{End}(P))$, to show the equivalence of $D^b(\mathbb{P})$ and $D^b(\mathbb{P})$.

Remark 2. In string theory the objects of the quotient singularity categories correspond to boundary condition of certain two-dimensional supersymmetric field theories. A physics derivation of the functor (12) was given in [12]. Therein it was found that the choice of the integer $m$ corresponds to a choice of a homotopy class of paths connecting limit points corresponding to $\hat{\Phi}_1$ and $\hat{\Phi}_2$ in the moduli space $B$ of the mirror of $X$. 


4. AUTOEQUIVALENCES: THE MAIN FORMULA

In this section we study elements of the group $\text{Aut}_\hat{\Phi}$ of autoequivalences of $D_{\text{sg}}(qgr_{\hat{\Phi}}\hat{S})$. We begin in the phase $\hat{\Phi} = \hat{\Sigma}$.

Immediate elements of $\text{Aut}_\hat{\Sigma}$, the group of automorphisms of $D^b(X) \cong D^b(qgr_{\hat{\Sigma}}\hat{S})$, are given by the shift functors $[n]$, for $n \in \mathbb{Z}$, and the twists $\mathcal{M}^q$ by the modules $S(q)$ for any $q \in \mathbb{Z}^k \cong \text{Pic}(\mathbb{P}_{\hat{\Sigma}})$. For $A \in D^b(qgr_{\hat{\Sigma}}\hat{S})$,

$$\mathcal{M}^q : A \mapsto A \otimes_S S(q).$$

Together with the automorphisms $\text{Aut}(X)$ of $X$, that is, the graded ring automorphisms of $S$ modulo $(\mathbb{C}^*)^k$, these generate a subgroup $\mathcal{A}_0(X) \cong \text{Aut}(X) \times \text{Pic}(\mathbb{P}_{\hat{\Sigma}}) \times \mathbb{Z}$ of $\text{Aut}(D^b(X))$.

Because of Theorem 2 the group of automorphisms $\text{Aut}(D^b(X))$, and in particular its subgroup $\mathcal{A}_0(X)$, also acts on the quotient singularity category $D_{\text{sg}}(qgr_{\hat{\Sigma}}\hat{S})$. The latter category appears to have an additional functor of twisting by $\hat{S}\{2r\}$. Recall however from [7] that a shift in $R$-grading by $\{2r\}$ is equal to the shift functor $\{2r\}$, hence does not introduce a new autoequivalence.

Let $s$ be an element of $S$ with degree $w$. Introduce on $D^b(qgr_{\hat{\Sigma}}\hat{S})$ the endofunctor

$$\mathcal{N}(s) : A \mapsto \text{Cone}\left(s : A \to \mathcal{M}^w(A)\right).$$

**Proposition 2.** For every primitive collection $\mathcal{I}_p$ of $\Sigma$ choose a regular sequence $\mathcal{S}_{I_p}$ of elements $s_j \in S$, as in Lemma 1. Then,

$$\bigcirc_{s_j \in \mathcal{S}_{I_p}} \mathcal{N}(s_j)(A) \cong 0,$$

for any $A \in D^b(qgr_{\hat{\Sigma}}\hat{S})$.

**Proof.** By Lemma 1 the Koszul complex $\mathcal{K}(\mathcal{S}_{I_p}; S)$ is isomorphic to the zero object in $D^b(qgr_{\hat{\Sigma}}\hat{S})$. The same is true for the tensor product $A \otimes_S \mathcal{K}(\mathcal{S}_{I_p}; S)$ for any object $A$. This is nothing but the left-hand side of (15) since $\mathcal{N}(s_j)(A) = A \otimes_S \text{Cone}\left(s : S \to S(w)\right)$. □

For any other maximal cone $\hat{\Phi}$ of the secondary fan, there are twist autoequivalences acting on $D_{\text{sg}}(qgr_{\hat{\Phi}}\hat{S})$ as well. We set

$$\mathcal{M}^q_{\hat{\Phi}} : A' \mapsto A' \otimes_S \hat{S}(q) \quad \text{for} \quad A' \in D_{\text{sg}}(qgr_{\hat{\Phi}}\hat{S}).$$

For any element $s \in \hat{S}$ with degree $w$ and $R$-grading $2r$, let

$$\mathcal{N}_{\hat{\Phi}}(s) : A' \mapsto \text{Cone}\left(s : A' \to \mathcal{M}^w_{\hat{\Phi}}(A')\{2r\}\right).$$

Notice the shift in homological degree and recall that $\{2r\} \cong \{2r\}$ on $D_{\text{sg}}(qgr_{\hat{\Phi}}\hat{S})$. Then, we have

**Proposition 3.** Let $\hat{\Phi}$ be the fan associated to a maximal cone in the secondary fan of $\mathbb{P}_{\hat{\Sigma}}$. Consider an arbitrary object $A'$ in the quotient singularity category $D_{\text{sg}}(qgr_{\hat{\Phi}}\hat{S})$. Then, for every primitive collection $\mathcal{I}_p$ of $\hat{\Phi}$,

$$\bigcirc_{i \in \mathcal{I}_p} \mathcal{N}_{\hat{\Phi}}(x_i)(A') \cong 0,$$

and for every $\hat{\nu}_i \in \hat{\Sigma}(1)\backslash \hat{\Phi}(1)$ associated with $x_i \in \hat{S}$,

$$\mathcal{N}_{\hat{\Phi}}(x_i)(A') \cong 0, \quad \text{or equivalently,} \quad \mathcal{M}^w_{\hat{\Phi}}(A') \cong (A')[-2r_i].$$

**Proof.** Using Remark 1 the proof is similar to the proof of Proposition 2. □
Remark 3. In the maximal cone \( \hat{\Sigma} \) we may write \( \mathcal{M}^g = \omega \sigma_{<0} \mathcal{M}_{\Sigma, \pi_{sg}} E \). Using this equivalence, the relations of Proposition 2 are in general stronger than those of Proposition 3. One may sometimes obtain similarly strong relations also in the other phases, namely whenever the exceptional set \( Z_\Phi \) admits the solution of \( dW = 0 \) to set \( p_a = G_a = 0 \) for (at least) one \( a = 1, \ldots, \ell \). Then, we may work over the ring \( \hat{S}/(p_a, G_a)\hat{S} \). In fact, Proposition 2 applies because over \( \mathbb{P}_\Sigma \), \( dW = 0 \) admits \( p_a = G_a = 0 \) for all \( a = 1, \ldots, \ell \).

Theorem 3 relates the singularity categories in neighbouring maximal cones \( \hat{\Phi}_1 \) and \( \hat{\Phi}_2 \). The associated twists are only partly independent autoequivalences. In fact, condition (13), which defines the functor \( F_{m}^{\hat{\Phi}_2 \hat{\Phi}_1} : \mathcal{C}_{\hat{\Phi}_1} \to \mathcal{C}_{\hat{\Phi}_2} \), implies that

\[
\mathcal{M}^q_{\hat{\Phi}_2} \circ F_{m}^{\hat{\Phi}_2 \hat{\Phi}_1} \cong F_{m+T(q)}^{\hat{\Phi}_2 \hat{\Phi}_1} \circ \mathcal{M}^q_{\hat{\Phi}_1}.
\]

In particular, the twists with \( T(q) = 0 \) commute with the functor \( F_{m}^{\hat{\Phi}_2 \hat{\Phi}_1} \) (for a fixed integer \( m \)), which says that they correspond to the same autoequivalence,

\[
\mathcal{M}^q_{\hat{\Phi}_1} \cong \mathcal{M}^q_{F_{m}^{\hat{\Phi}_2 \hat{\Phi}_1}} := (F_{m}^{\hat{\Phi}_2 \hat{\Phi}_1})^{-1} \mathcal{M}^q_{\hat{\Phi}_2} \circ F_{m}^{\hat{\Phi}_2 \hat{\Phi}_1}, \quad \text{for } T(q) = 0.
\]

On the other hand, if \( T(q) \) does not vanish the composition,

\[
(F_{m}^{\hat{\Phi}_2 \hat{\Phi}_1})^{-1} F_{m+T(q)}^{\hat{\Phi}_2 \hat{\Phi}_1} \cong \mathcal{M}^q_{F_{m}^{\hat{\Phi}_2 \hat{\Phi}_1}} \mathcal{M}^{-q}_{\hat{\Phi}_1},
\]

is a non-trivial autoequivalence of \( \mathcal{C}_{\hat{\Phi}_1} \).

For any \( \Phi \) let \( F_{\Phi} : \mathcal{C}_{\hat{\Sigma}} \to \mathcal{C}_{\hat{\Phi}} \) be a composition of functors of Theorem 3. Then the twists in \( \hat{\Phi} \) act on \( \mathcal{C}_{\hat{\Sigma}} \cong D^b(X) \) via \( \mathcal{M}^q_{F_{\Phi}} = F_{\Phi}^{-1} \circ \mathcal{M}^q_{\hat{\Phi}} \circ F_{\Phi} \). By combining Proposition 3 with Theorem 3 we obtain

**Theorem 4.** Let \( X \) be a smooth Calabi–Yau complete intersection in a toric variety \( \mathbb{P}_\Sigma \), and \( D_{sg}(\text{qgr}_S) \) (\( \cong D^b(\text{qgr}_S) \)) its quotient singularity category. Then for every maximal cone \( \Phi \) in the secondary fan of \( \mathbb{P}_\Sigma \), the autoequivalences \( \mathcal{M}^q_{\Phi} \) induce an action of the Picard lattice \( \text{Pic}(\mathbb{P}_\Sigma) \) on \( D_{sg}(\text{qgr}_S) \), subject to the following relations. For any object \( A \in D_{sg}(\text{qgr}_S) \) and every primitive collection \( \mathcal{I}_p \) of \( \hat{\Phi} \),

\[
\bigcirc_{i \in \mathcal{I}_p} \mathcal{N}_{F_{\Phi}}(F_{\Phi}(x_i))(A') \equiv 0,
\]

and for every \( \hat{\varphi} \in \hat{\Sigma}(1) \backslash \hat{\Phi}(1) \) and associated element \( x_i \in \hat{S} \),

\[
\mathcal{N}_{F_{\Phi}}(F_{\Phi}(x_i))(A') \equiv 0, \quad \text{or equivalently,} \quad \mathcal{M}^{\omega}_{F_{\Phi}}(A') \cong (A')[-2r_1].
\]

**Proof.** Writing for any \( A' \in \mathcal{C}_{\hat{\Sigma}} \)

\[
\mathcal{N}_{F_{\Phi}}(F_{\Phi}(s))(A') := \text{Cone}(F_{\Phi}(s) : A' \longrightarrow \mathcal{M}^{\omega}_{F_{\Phi}}(A')[-2r]) =
\]

\[
= F_{\Phi} \text{Cone}(s : F_{\Phi}^{-1}(A') \longrightarrow \mathcal{M}^{\omega}_{F_{\Phi}}F_{\Phi}^{-1}(A')) =
\]

\[
= F_{\Phi} \mathcal{N}_{\Phi}(s)F_{\Phi}^{-1}(A'),
\]

the claim follows immediately from Proposition 3. \( \square \)
5. Applications

5.1. Monodromy representations — more about Aut(\(D^b(X)\)).

According to Theorem 4, we have an action of the toric Picard lattice of the complete intersection \(X\) on \(D^b(X)\) for every phase \(\Phi\) in the secondary fan of \(\mathbb{P}_\Sigma\), generalizing the simple twisting by line bundles for \(\Phi = \Sigma\). As is evident from the relations (18) and (19), this action can be very different for each \(\Phi \neq \Sigma\).

In terms of the monodromy representation \(\pi_1(B) \to \text{Aut}(D^b(X))\), mentioned in the introduction, the action of \(\text{Pic}(\mathbb{P}_\tilde{\Sigma})\) obtained from \(\Phi\) is a categorical lift of the monodromies around the boundary divisor of \(B\) corresponding to the maximal cone \(\Phi\) of the secondary fan. This point of view fits in nicely with the interpretation of the equivalence between the various \(\mathcal{C}_\Phi\)'s as depending on the homotopy class of a path in \(B\). According to Remark 2 the composition (17) for \(T(q) = 1\) is the monodromy along a path around the discriminant locus of \(B\), and therefore corresponds to a Seidel–Thomas twist [5], cf. also [13, 14] and [12, Chapter 10.5].

5.2. Proof of K-theory formula.

The projection to K-theory splits the cones in Proposition 3, which implies Corollary 1.

When interpreted in terms of monodromy representations (see previous subsection), the relations in K-theory can also be obtained by studying the analytic continuation of periods of the mirror variety, as in [15]. (See also [16] for a recent study.) Our results provide the precise categorical lift of the monodromies and the relations between them.

5.3. Some examples.

Consider the list of 14 Calabi–Yau complete intersections \(X(d_1 \ldots d_\ell)\) in weighted projective spaces \(\mathbb{P}^{3+\ell}(w_0 \ldots w_{3+\ell})\) with rank 1 Picard lattice, that is \(\text{Pic}(X) \cong \mathbb{Z}\) (cf. [17, 18]):

| \(X(5)\) | \(X(2, 4)\) | \(X(2, 12)\) | \(X(6)\) | \(X(3, 3)\) | \(X(4, 6)\) | \(X(8)\) | \(X(3, 4)\) | \(X(6, 6)\) | \(X(10)\) | \(X(2, 6)\) | \(X(2, 2, 3)\) | \(X(4, 4)\) | \(X(2, 2, 2, 2)\) |
| \(\subset \mathbb{P}^4(11111)\) | \(\subset \mathbb{P}^2(11111)\) | \(\subset \mathbb{P}^5(111111)\) | \(\subset \mathbb{P}^4(11112)\) | \(\subset \mathbb{P}^5(111111)\) | \(\subset \mathbb{P}^5(111223)\) | \(\subset \mathbb{P}^4(11114)\) | \(\subset \mathbb{P}^5(111112)\) | \(\subset \mathbb{P}^5(112233)\) | \(\subset \mathbb{P}^4(111225)\) | \(\subset \mathbb{P}^5(111113)\) | \(\subset \mathbb{P}^6(111111)\) | \(\subset \mathbb{P}^5(111122)\) | \(\subset \mathbb{P}^7(1111111)\) |

The polynomial ring associated to the toric data is \(\hat{R} = \mathbb{C}[x_0, \ldots, x_{3+\ell}, p_1, \ldots, p_\ell]\).

The secondary fan has two maximal cones, say \(\Sigma\) and \(\Xi\), where the first shall correspond to the complete intersection itself. The respective (Cox) ideals are

\[
J_\Sigma = \langle \prod_{i=0}^{3+\ell} x_i \rangle, \quad \text{and} \quad J_\Xi = \langle \prod_{a=1}^\ell p_a \rangle.
\]

For the following let \(\mathcal{M} = \mathcal{M}_\Sigma^1\) and \(\mathcal{L} := \mathcal{M}_\Xi^1\) abbreviate the autoequivalences of twisting by \(\hat{S}(1)\) in the two associated singularity categories.

In the category \(\mathcal{C}_\Sigma\):

For \((G_1, \ldots, G_\ell)\) generic, according to Proposition 2, a regular sequence \(S_T\) of four elements, say \(\underline{x} = (x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3})\), gives rise to a complex isomorphic to the zero
object,

\[ A[4] \xrightarrow{\varphi} \bigoplus_{b=0}^{3} \mathcal{M}^{w_{ib}}(A)[3] \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} \bigoplus_{b=0}^{3} \mathcal{M}^{w_{i-b}}(A)[1] \xrightarrow{\varphi} \mathcal{M}^{w}(A) \cong 0, \]

where \( w = \sum_{b=0}^{3} w_{ib} \). In K-theory for each regular sequence \( S_I \) the relation becomes \( \prod_{b=0}^{3}(M^{w_{ib}} - \text{id}) = 0. \)

As an example, for \( X(10) \subset \mathbb{P}^4(11125) \) the K-theory relations are

\[
(M - \text{id})^2(M^2 - \text{id})(M^5 - \text{id}) = 0, \\
(M - \text{id})^3(M^2 - \text{id}) = 0, \\
(M - \text{id})^3(M^5 - \text{id}) = 0.
\]

It is easy to verify that for each complete intersection in the above list, the relations imply the well-known result that the autoequivalence \( M \) is maximally unipotent, that is

\[ (M - \text{id})^4 = 0. \]

In the category \( \hat{\Sigma} \):

Proposition 3 tells us that for \( p = (p_1, \ldots, p_\ell) \) the complex

\[ L^d(A) \xrightarrow{P} \oplus_a L^{d-a} (A)[1] \xrightarrow{P} \cdots \xrightarrow{P} \oplus_a L^{d_a}(A)[\ell-1] \xrightarrow{P} A[\ell], \]

is isomorphic to the zero object. Here, \( d = \sum_a d_a \). On the level of K-theory this becomes

\[ \prod_{a=1}^\ell (L^{d_a} - \text{id}) = 0. \]

Back in \( \Sigma \):

Theorem 4 uses the functor \( F_{\hat{\Sigma}} : \Sigma \rightarrow \hat{\Sigma} \) to map the relation (20) to a relation for the autoequivalence \( L_{F_{\hat{\Sigma}}} = F_{\hat{\Sigma}}^{-1} L F_{\hat{\Sigma}} \) on \( \Sigma \). Although the relation for \( L_{F_{\hat{\Sigma}}} \) is a straightforward consequence of (20), we stress that it is highly non-trivial, even more so, if we "forget" that \( L_{F_{\hat{\Sigma}}} \) in fact comes from an action of the Picard lattice in another phase.

We illustrate this for the complete intersection \( X = X(3,3) \) of two cubics in \( \mathbb{P}^5 \), applying the autoequivalence \( L_{F_{\hat{\Sigma}}} \) on the structure sheaf \( \mathcal{O} \) of \( X \). Let \( \Omega \) be the
pull-back of the cotangent bundle of $\mathbb{P}^5$ to $X$. We obtain

\[
\begin{align*}
\mathcal{L}_{F_{\mathbb{P}}}^1(\mathcal{O}) &= \Omega(1), \\
(\mathcal{L}_{F_{\mathbb{P}}}^1)^2(\mathcal{O}) &= \wedge^2 \Omega(2), \\
(\mathcal{L}_{F_{\mathbb{P}}}^1)^3(\mathcal{O}) &= \mathcal{O}[2] \oplus \wedge^3 \Omega(3), \\
(\mathcal{L}_{F_{\mathbb{P}}}^1)^4(\mathcal{O}) &= \Omega(1)[3] \oplus \wedge^4 \Omega(4), \\
(\mathcal{L}_{F_{\mathbb{P}}}^1)^5(\mathcal{O}) &= \wedge^2 \Omega(2)[4] \oplus \wedge^5 \Omega(5), \\
(\mathcal{L}_{F_{\mathbb{P}}}^1)^6(\mathcal{O}) &= \mathcal{O}[4] \oplus \wedge^3 \Omega(3)[5],
\end{align*}
\]

where the arrows are canonical elements in Ext$^1(\mathcal{O}, \mathcal{O})$. In fact, the object in the last line is isomorphic to

\[
(\mathcal{L}_{F_{\mathbb{P}}}^1)^6(\mathcal{O}) = \mathcal{O}[4] \oplus \wedge^3 \Omega(3)[5],
\]

which confirms the relation for $\mathcal{L}_{F_{\mathbb{P}}}$, when applied to $\mathcal{O}$,

\[
(\mathcal{L}_{F_{\mathbb{P}}}^1)^6(\mathcal{O}) \cong (\mathcal{L}_{F_{\mathbb{P}}}^1)^3(\mathcal{O})[2] \oplus \mathcal{O}[3].
\]
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