Numerical Modeling of Solutions of the Applied Problems for the Poisson’s Equation

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Abstract. Many problems of continuum physics, applied physics, electrostatics, heat engineering, and structural mechanics are described by boundary value problems for the Poisson equation. The article is devoted to the construction of a numerical algorithm for solving plane boundary value problems for the Poisson equation. The algorithm is based on the transition to a polyharmonic equation, which is solved by the method of linear boundary elements. We consider two boundary value problems with different boundary conditions: Dirichlet boundary conditions and Neumann boundary conditions. To construct a numerical solution using the linear boundary element method, the boundary of the area is replaced by an inscribed polygon. The boundary conditions are satisfied at the middle (control) points of the elements. A polyharmonic equation is reduced to a system of linear integral equations, which can be reduced to a system of linear algebraic equations. Test numerical examples confirming the effectiveness of the proposed algorithm are given.

1. Introduction

Many important mathematical models of continuum physics, electrostatics, thermal engineering, structural mechanics are described by boundary value problems for the Poisson’s equation. In particular, applications in thermal engineering are important to study of the patterns of heat transfer in order to provide the necessary thermal conditions in buildings and structures. The methodology proposed in this paper can be applied to modelling of the solutions of problems of continuum physics, specifically the theory of torsion of a prismatic rod and problems of plate bending. These mathematical models are important in the study of the durability of building structures and materials, and also in the study of issues of the structural mechanics (strength, rigidity and stability of buildings and structures).

In the fundamental monograph of I.N. Vekua [1] a detailed theoretical study of linear differential equations of elliptic type is given, and also information about the works of other authors is presented. The analytical solution of such equations using the apparatus of almost periodic functions was previously considered in [2]. Analytical solutions are expressed by complex integral representations through special functions or by contour singular integrals, that presents significant difficulties for calculations. Specific calculations can be performed mainly for some special areas or particular analytical solutions.

In [3] an algorithm for the numerical solution of boundary value problems for polyharmonic equation is proposed; the problem is reduced to a system of linear algebraic equations for the values of
auxiliary functions at control points. A series of numerical results confirms the efficiency and high accuracy of calculations. In this work the generalization of the numerical boundary element method [3] for the Poisson’s equation with a known right-hand side in a plane domain $D$

$$\Delta u(x,y) = f(x,y), \quad (x,y) \in D$$  \hspace{1cm} (1.1)

is given.

Referring to the results of I N Vekua [1], we leave aside the existence and uniqueness theorems, and we focus on the construction of a numerical algorithm.

2. The problems statements

The desired is the function $u(x, y)$ that satisfies the Poisson’s equation (1.1) in a plane domain $D$, it is continuous with its partial derivatives in a closed domain $\bar{D} = D \cup \partial D$, and satisfies one of the following conditions on a smooth boundary $\partial D$:

1) Dirichlet condition:

$$u|_{\partial D} = g_0(s), \quad s \in \partial D,$$  \hspace{1cm} (2.1)

2) Neumann condition:

$$\frac{\partial u}{\partial n}|_{\partial D} = g_1(s), \quad s \in \partial D,$$  \hspace{1cm} (2.2)

where $\frac{\partial}{\partial n}$ is the normal derivative operator, $g_0(s), g_1(s)$ are the given real functions of the arc coordinate $s$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is two-dimensional Laplace operator.

3. The transition to the boundary value problem for polyharmonic equation

If the function $f(x, y)$ is a polyharmonic function of some order $n - 1$ it follows from equation (2.1) that $u$ is the polyharmonic function of $n$ order and satisfies the equation

$$\Delta^n u = 0.$$  \hspace{1cm} (3.1)

In particular, a polynomial of some degree is a polyharmonic function of some order.

In the simplest case, if the function $f$ is the constant (this case has many practical applications),
the function $u$ obviously satisfies the biharmonic equation, for the solution of which two boundary conditions are necessary. Then, as the second condition for finding it, the equality

$$\Delta u = f$$

can be used.

Similarly, if the function $f(x, y)$ is polyharmonic of some order $n - 1$, then as the missing $n - 1$ boundary conditions for the $n$-harmonic equation it is not difficult to obtain the following equalities using differentiation:

$$\Delta u = f(x,y), \quad \Delta^2 u = \Delta f(x,y), \quad \ldots, \quad \Delta^{n-1} u = \Delta^{n-2} f(x,y).$$  \hspace{1cm} (3.2)

In particular, when implementing a numerical algorithm, the piecewise polynomial approximation of the function $f$ on the domain $D$ can be applied [4]. In the linear boundary element method, the domain $D$ is approximated by some polygon, which can be divided into triangular sub-domains. On the each sub-domain, we can construct the corresponding Newton polynomial of the minimum degree.
for a given number of sub-domains. The approximating polynomial can be further applied to an approximate calculation of partial derivatives to obtain the missing boundary conditions (3.2).

A similar approach was previously used to solve the problems of a thin plate bending [5] - [8], the mathematical model of which is described using a non-uniform biharmonic equation.

4. The approximation of the boundary of the domain and of the boundary integral

To construct a numerical solution using the linear boundary element method, the boundary of the domain \( D \) is replaced by a polygon \( C \) with \( N \) sides (elements), the corners of the polygon lie on the boundary \( \partial D \) and called nodes. If there are corner points on the boundary \( \partial D \), they are aligned with the nodes. The boundary conditions are satisfied at the middle (control) points of the elements. The estimate of the error of such approximation of the boundary is given in [9].

Next, to reduce the entries, we introduce a complex variable \( z = x + iy \). Let us number the nodes \( z_j (j = 1, N+1) \), \( z_i = z_{N+1} \) and control points \( Z_k (k = 1, N) \), and then for some control point \( k = 1, N \), the length of the boundary element, the value of the arc coordinate and the outward normal are determined by the formulas

\[
Z_k = \frac{z_{k+1} + z_k}{2}, \quad h_k = |z_{k+1} - z_k|, \quad S_k = \sum_{n=1}^{k} h_n - \frac{h_k}{2}, \quad n_k = -i \frac{z_{k+1} - z_k}{h_k}.
\]  

(4.1)

We consider two functions \( u(z) \) and \( g(z, z') \), \( z, z' \in C \), the first function is continuous on the boundary, while the function \( g(z, z') \) can have an integrable feature at the point \( z_k \). The integral of the product of two functions along the contour is determined by the formula about the average, erroneous order \( O\left(\max(h_m)\right)\):

\[
J(Z_k) = \oint_C u(s) g(Z_k, s) ds \approx \sum_{m=1}^{N} u(Z_m) \int_{h_m} g(Z_k, s) ds.
\]  

(4.2)

The approximate value of the integral (4.2) can be written as a product of matrices

\[
J = MU
\]  

(4.3)

The components of the matrix \( M \) are integrals over elements \( h_m, (m = 1, N) \).

Polyharmonic equation \( \Delta^n u = 0 \) can be reduced to a system of linear integral equations [10]:

\[
u u + \sum_{j=0}^{n-1} \oint_C \delta u_{j+p} H_p ds - \sum_{j=0}^{n-1} \oint_C \delta q_{j+p} G_p ds = 0, \quad (j = 0, n-1),
\]  

(4.4)

where

\[
u u = \Delta' u, \quad q_j = \frac{\partial u_j}{\partial n}, \quad \Delta' G_p (r) = G(r), \quad H_p = \frac{\partial G_p}{\partial n}.
\]  

(4.5)

Since the contour is smooth at the control point \( Z_k \), then the multiplier \( \varepsilon = 1/2 \); in the inner point \( \varepsilon = 1 \). The third equality (4.5) is an ordinary differential equation of the form

\[
\left( 1 - \frac{d}{dr} \frac{dG_j(r)}{dr} \right)' = G(r).
\]

The solution can be obtained by direct integration:
5. The discretization of the system of integral equations

Using the approximate representation (4.2) or (4.3), we can write system (4.4) as a system of linear algebraic equations for the discrete values of the function \( u_j(s) \) and the normal derivative \( q_j(Z_k) = n_k \cdot \Delta u_j(Z_k) \) at the control points:

\[
\frac{1}{2} u_j(Z_k) + \sum_{p=0}^{n_j-1} \sum_{m=0}^{N} \left( u_{j+p} (Z_m) \int_{h_m} H_p(Z_k,s) ds - q_{j+p} (Z_m) \int_{h_m} G_p(Z_k,s) ds \right) = 0,
\]

or in the form of \( n \) matrix equations

\[
\frac{1}{2} U_j + \sum_{p=0}^{n_j-1} \left( H_p U_{j+p} - G_p Q_{j+p} \right) = 0, \quad (j = 0, n-1). \tag{5.1}
\]

Equations (5.1) are \( nN \) equations for the \( 2nN \) components \( U_{j,k} \) and \( Q_{j,k} \), \( (k = 1, N) \) which are the elements of the matrix-columns \( U_j \) and \( Q_j \) and are defined as \( U_{j,k} = u_j(Z_k) \), \( Q_{j,k} = q_j(Z_k) \).

Suppose that the function \( f(z) \) is continuous and differentiable in a closed domain \( \overline{D} \) and it is a known polyharmonic function of order \( n-1 \). Then by acting on equation (1.1) \( n-1 \) times with the Laplace operator, we obtain a polyharmonic equation of \( n \)-order of the form (3.1), which can be reduced to a system of linear algebraic equations (5.1). To solve this system, it is necessary to specify the missing \( nN \) values of the elements of the matrix-columns \( U_j \) and \( Q_j \). These values are determined by the boundary conditions (2.1) and (3.2) or (2.2) and (3.2) depending on the type of the problem:

1) Dirichlet problem: elements of the matrices \( U_0, U_1, U_2, \ldots, U_n \) are known;
2) Neumann problem: elements of the matrices \( Q_0, U_1, U_2, \ldots, U_n \) are known.

The components of matrices and vectors are the values of the corresponding functions at \( N \) control points.

By solving the system (5.1), the value of the function \( u(z) \) at an arbitrary interior point of the domain \( D \) can be determined from equation (4.4) with \( \varepsilon = 1 \).

6. Test examples

6.1. The Dirichlet problem in two-dimensional domain

To compare a numerical solution with an analytic function, we consider the polyharmonic function \( u(x,y) = x^3 (x^2 - 5y^2) \), which is a solution of the equation

\[
\Delta u = 10x^3 - 30xy^2. \tag{6.1}
\]

We solve equation (6.1) numerically in a circular ring, if on the boundary

\[
\partial D_1: \begin{cases} x = a \cos s, \\ y = -a \sin s; \end{cases} \quad \partial D_2: \begin{cases} x = b \cos s, \\ y = b \sin s; \end{cases} \quad s \in [0, 2\pi), \quad b > a
\]

the Dirichlet condition is given.

It is possible to reduce equation (6.1) to the biharmonic equation
with boundary conditions

\[ u \bigg|_{x_1} = a^5 \cos^5 s - 5 \cdot a^5 \sin^2 s \cdot \cos^3 s, \quad u \bigg|_{x_2} = b^5 \cos^5 s - 5 \cdot b^5 \sin^2 s \cdot \cos^3 s, \]

\[ \Delta u \bigg|_{x_1} = 10a^3 \cos^3 s - 30 \cdot a^3 \sin^2 s \cdot \cos s, \quad \Delta u \bigg|_{x_2} = 10b^3 \cos^3 s - 30 \cdot b^3 \sin^2 s \cdot \cos s. \]

Using the method described above, we obtain a system of linear matrix equations

\[
\begin{cases}
(0.5E + H_0)U_1 - G_0Q_1 = 0, \\
(0.5E + H_0)U_0 - G_0Q_0 + H_1U_1 - G_1Q_1 = 0, 
\end{cases}
\]

(6.2)

relatively unknown vectors \( Q_0, Q_1 \). The solution of equations (6.2) is

\[ Q_1 = (G_0)^{-1}(0.5E + H_0)U_1, \]

\[ Q_0 = (G_0)^{-1}(0.5E + H_0)U_0 + H_1U_1 - G_1(0.5E + H_0)U_1. \]

The value of the desired function \( u \) at an arbitrary point of the domain may be found by formula (4.4) with \( \varepsilon = 1 \).

Numerical solution was found for circular ring with inner and outer radii \( a = 4, b = 5 \). On Figure 1 the results of the analytical and numerical (\( N = 50 \)) solutions on the contour

\[ x = 4.5 \cos s, \quad y = 4.5 \sin s, \quad s \in [0, 2\pi), \]

are presented.

Figure 1. Analytical (solid line) and numerical (points) solutions of the Dirichlet problem for equation (6.1).

6.2. The Neumann problem in one-dimensional domain

The next example is to solve the Poisson’s equation

\[ \Delta u(x, y) = 20(x^2 y + xy^3) \]

(6.3)
in a closed domain, if on the boundary of the domain, which is a circle of unit radius, the Neumann condition is given:

\[ \frac{\partial u}{\partial n} \bigg|_{\partial D} = 3\sin 2s \left(1 - \frac{1}{2} \sin^2 2s\right), \quad s \in [0, 2\pi). \]
The exact solution of equation (6.3) is the function \( u = x^5 + xy^5 \).

The right side of equation (6.3) is biharmonic function. To find the numerical solution it is possible to obtain a third-order polyharmonic equation \( \Delta^3 u = 0 \) with boundary conditions
\[
\frac{\partial^3 u}{\partial n^3} = 3\sin 2s \left( 1 - \frac{1}{2} \sin^2 2s \right), \quad \Delta u|_{s=0} = 10\sin 2s, \quad \Delta^2 u|_{s=0} = 120\sin 2s, \quad s \in [0, 2\pi).
\]

This equation is reduced to solving a system of linear algebraic equations:
\[
\begin{align*}
(0.5E + H_0)U_0 + G_1Q_1 + G_2Q_2 &= G_0Q_0 + H_1U_1 + H_2U_2, \\
G_0Q_1 + G_1Q_2 &= (0.5E + H_0)U_1 + H_1U_2, \\
G_0Q_2 &= (0.5E + H_0)U_2.
\end{align*}
\]

For comparison, the analytical and numerical (\( N = 50 \)) solutions are built on a circle \( x = 0.75\cos s, \quad y = 0.75\sin s, \quad s \in [0, 2\pi) \), the results are presented on Figure 2.

![Figure 2](image_url)

**Figure 2.** Analytical (solid line) and numerical (points) solutions of the Neumann problem for equation (6.3).

7. **Conclusion**
The stated algorithm of calculation can be applied to a wide class of similar systems of equations which have many applications in the continuum mechanics [11–14]. The effectiveness of the method is confirmed by comparison of numerical results and analytical solutions on test examples.

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