A damage localisation method based on higher order spatial derivatives of displacement and rotation fields

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Abstract. This paper presents the development of a damage localisation method based on low and higher order spatial derivatives of displacement and rotation fields. The method relies on the Timoshenko beam theory, the Hamilton’s principle and the Ritz method, allowing the computation of an approximate analytical solution of natural frequencies, displacement and rotation fields. Since these fields are expressed analytically, through a series expansion, the spatial derivatives are also obtained analytically. Besides the usual curvature difference and damage index, which are based on comparisons of second order spatial derivative of the displacement field of a beam in the undamaged and damaged states, other damage indicators are proposed. These new indicators rely on spatial derivatives of rotation fields. The indicators are applied to the localisation of various cases of damages in beams and a comparison among them is carried out. It is found that, for relatively thick beams, the values of the indicators based on rotation fields spatial derivatives are, in general, lower than those based on displacement fields. It is also observed that these differences are negligible for beams with high length to thickness ratio.

1. Introduction
Several methods for structural damage identification, based on vibration characteristics, have been proposed over the past decades [1, 2]. The main idea behind them is that whenever there is damage in a structure, its stiffness, mass and damping properties change, therefore changing its dynamic behaviour.

The mode shape curvature method, firstly developed by Pandey et al. [3], allows, under certain conditions, the localisation of damage. In this work the curvatures of beams are computed using a second order central finite difference. However, if sparse and noisy displacement fields are used, this differentiation scheme can lead to false localisations of damage. This problem is mainly due to the intrinsic error of the finite difference numerical technique using sparse data and the amplification of noise present in the measurements. A method also based on mode shapes curvatures is the damage index method proposed by Stubbs et al. [4], which relates the second spatial derivative of the undamaged and damaged displacement fields in each segment or element of the structure.

These two damage identification methods and similar ones, such as the frequency response functions (FRF) curvatures method [5], have been applied and improved over the years (e.g. [6]–[12]). However, the curvatures are usually computed based on a differentiation of a relatively
sparse displacement field and, therefore, the damage indicators generally describe the state of damage in areas of the structures. In other words, the damage localisation is performed element-by-element or segment-by-segment. Moreover, the beams are analysed using the Euler-Bernoulli kinematic assumptions, therefore limiting the applicability to thin beams.

The development of a damage localisation method based on spatial derivatives of displacement and rotation fields is presented in this paper. Since this method relies on the Timoshenko beam theory, the Hamilton’s principle and the Ritz method, the displacement and rotation fields are described by a series expansion. Therefore, instead of applying a numerical technique to obtain the derivatives of these fields, the differentiations are performed analytically. Furthermore, this allows the damage localisation to be performed in a point-by-point fashion or continuously. Also, and since the method is based on the Timoshenko kinematic assumptions, thick beams can be analysed. Due to these kinematic assumptions, the displacement field spatial derivative of relatively thick beams is not equal to their rotation field [13].

This paper also describes the application of the usual mode shape curvature difference indicator [3] and the damage index [4] to the damage identification of simply supported beams. Three other damage indicators, based on third order spatial derivatives of displacement fields and first and third orders spatial derivatives of rotation fields, are presented and applied. The application of these indicators for damage localisation in relatively thick beams shows that those based on rotation fields spatial derivatives are lower than the ones based on displacement fields spatial derivatives.

2. Methodology

This section contains a description both of undamaged and damaged beam models. The mode shape curvature difference indicator and the damage index are also presented, along with the description of four novel damage indicators.

2.1. Undamaged and damaged beam models

The Hamilton’s principle for the free vibration of an undamaged beam according to the Timoshenko theory is written, in terms of the displacement $w$ and rotation $\phi$ fields and their spatial derivatives, as follows [14]:

$$
\int_{t_1}^{t_2} \int_0^L \left[ \rho A \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} + \rho A \frac{h^2}{12} \frac{\partial \phi}{\partial t} \frac{\partial \delta \phi}{\partial t} - EI \frac{\partial \phi}{\partial x} \frac{\partial \delta \phi}{\partial x} - GAK_s \left( \frac{\partial w}{\partial x} + \phi \right) \left( \frac{\partial \delta w}{\partial x} + \delta \phi \right) \right] dx \, dt = 0
$$

(1)

with $\delta w(t_1) = \delta w(t_2) = \delta \phi(t_1) = \delta \phi(t_2) = 0$. In this Equation $h$, $A$ and $I$ are, respectively, the beam thickness, area and second moment of area, while $\rho$, $E$, $G$ and $K_s$ are the mass density, the Young’s modulus, the shear modulus and the shear correction factor, respectively. Note that by setting $\partial w/\partial x = -\phi$ the distortion $\gamma_{xz} = 2\epsilon_{xz}$ becomes zero and the Euler-Bernoulli kinematic assumptions are recovered (Figure 1). Also note that in the Timoshenko theory the flexural rigidity $EI$ is no longer associated with the second derivative of the displacement field, but with the first derivative of the rotation field.

The Ritz method leads to a $2N$ parameter periodic analytical solution, where the displacement and rotation fields are given by a series of $N$ terms [14]

$$
\begin{align*}
    w(x, t) &= \sum_{n=1}^{N} W_n w_n(x) e^{i\omega_n t}, \quad \phi(x, t) = \sum_{n=1}^{N} \Phi_n \varphi_n(x) e^{i\omega_n t},
\end{align*}
$$

(2)

respectively, where $\omega_n$ is the natural frequency associated with the $n$-th mode shape and $W_n$ and $\Phi_n$ are coefficients, known as the Ritz coefficients [14].
The assumed or approximation functions, $w_n(x)$ and $\varphi_n(x)$, must be such that they: (1) are continuous, (2) satisfy the homogeneous form of the specified essential (geometric) boundary conditions and (3) define sets that are linearly independent and complete [14]. An eigenvalue problem is obtained by substituting Equation (2) into (1) and performing the necessary mathematical operations

$$[K][Q] = [M][Q][\omega^2]$$  \hspace{1cm} (3)

where $[K]$ and $[M]$ are matrices related to the stiffness and inertia, respectively, and $[Q]$ and $[\omega^2]$ are matrices containing the Ritz coefficients and the natural frequencies. This last matrix is diagonal, while $[Q]$ is a full matrix. Matrices $[K]$ and $[M]$ are partitioned by grouping the terms $w_i(x)w_j(x)$, $w_i(x)\varphi_j(x)$, $\varphi_i(x)\varphi_j(x)$ and their derivatives, such that

$$[K] = \begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{12}]^T & [K_{22}] \end{bmatrix} \quad \text{and} \quad [M] = \begin{bmatrix} [M_{11}] & [0] \\ [0] & [M_{22}] \end{bmatrix}$$  \hspace{1cm} (4)

with the elements given by

$$(K_{11})_{ij} = \int_0^L GAK_s \frac{\partial w_i(x)}{\partial x} \frac{\partial w_j(x)}{\partial x} dx,$$

$$(K_{12})_{ij} = \int_0^L GAK_s \frac{\partial w_i(x)}{\partial x} \varphi_j(x) dx$$ \hspace{1cm} (5)

$$(K_{22})_{ij} = \int_0^L \left[ EI \frac{\partial \varphi_i(x)}{\partial x} \frac{\partial \varphi_j(x)}{\partial x} + GAK_s \varphi_i(x) \varphi_j(x) \right] dx,$$  \hspace{1cm} (6)

$$(M_{11})_{ij} = \int_0^L \rho Aw_i(x)w_j(x) dx,$$

$$(M_{22})_{ij} = \int_0^L \rho A h^2 \frac{k^2}{12} \varphi_i(x) \varphi_j(x) dx$$  \hspace{1cm} (7)

In this work, the assumed functions $w_n(x)$ and $\varphi_n(x)$ for a simply supported beam are defined by

$$w_n(x) = \sin(\lambda_n x) \quad \text{and} \quad \varphi_n(x) = \lambda_n \cos(\lambda_n x), \quad \text{with} \quad \sin(\lambda_n L) = 0$$  \hspace{1cm} (8)

Note that that for beams with high length to thickness aspect ratios, i.e. thin beams, the Euler-Bernoulli solution is recovered, since the assumed functions of the rotation field are the derivatives of assumed functions of the displacements field.
Considering that the beam has a crack of depth \( p \) and width \( c \) located between coordinates \( x_1 \) and \( x_2 \) (Figure 2), Hamilton’s principle now reads

\[
\int_{t_1}^{t_2} \int_{0}^{L} \left[ \rho A \frac{\partial w}{\partial t} \frac{\partial w}{\partial t} + \rho \tilde{A} h^2 \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial t} \right] + EI \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} - GAKs \left( \frac{\partial w}{\partial x} + \phi \right) \left( \frac{\partial \delta w}{\partial x} + \delta \phi \right) \right] \, dx \, dt
\]

\[
- \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left[ \rho A \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} + \rho \tilde{A} h^2 \frac{\partial \phi}{\partial t} \frac{\partial \delta \phi}{\partial t} \right] - E \tilde{I} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} - GAKs \left( \frac{\partial w}{\partial x} + \phi \right) \left( \frac{\partial \delta w}{\partial x} + \delta \phi \right) \right] \, dx \, dt = 0
\]

where \( \tilde{A} = A(1 - p/h) \) and \( \tilde{I} = I(1 - p/h)^3 \). A very similar approach for the modelling of damaged thin composite plates is presented by Li et al. [15] and Qu et al. [16]. The application of the Ritz method yields the eigenvalue problem

\[
[\tilde{K}] [\ddot{Q}] = [\tilde{M}] [\ddot{Q}] [\ddot{\omega}]^2
\]

where

\[
(\tilde{K})_{ij} = (K_{11})_{ij} - \int_{x_1}^{x_2} GAK_s \frac{\partial w_i(x)}{\partial x} \frac{\partial w_j(x)}{\partial x} \, dx, \quad (\tilde{K}_{12})_{ij} = (K_{12})_{ij} - \int_{x_1}^{x_2} GAK_s \frac{\partial w_i(x)}{\partial x} \varphi_j(x) \, dx
\]

\[
(\tilde{K}_{22})_{ij} = (K_{22})_{ij} - \int_{x_1}^{x_2} \left[ E \tilde{I} \frac{\partial \varphi_i(x)}{\partial x} \frac{\partial \varphi_j(x)}{\partial x} + GAK_s \varphi_i(x) \varphi_j(x) \right] \, dx,
\]

\[
(\tilde{M})_{ij} = (M_{11}) - \int_{x_1}^{x_2} \rho \tilde{A} w_i(x) w_j(x) \, dx, \quad (\tilde{M}_{22})_{ij} = (M_{22}) - \int_{x_1}^{x_2} \rho \tilde{A} h^2 \varphi_i(x) \varphi_j(x) \, dx.
\]

Since the damaged beam is also simply supported, the computation of these matrices is performed considering the same assumed functions, \( w_n(x) \) and \( \varphi_n(x) \), that were used for the undamaged beam and matrices \([\tilde{K}]\) and \([\tilde{M}]\) are also symmetric. However, although for simply supported beams \([K_{11}]\), \([K_{12}]\), \([K_{22}]\), \([M_{11}]\) and \([M_{22}]\) are diagonal matrices, the same is not true for the submatrices \([\tilde{K}_{11}]\), \([\tilde{K}_{12}]\), \([\tilde{K}_{22}]\), \([\tilde{M}_{11}]\) and \([\tilde{M}_{22}]\). It is worth noticing that this model, contrary to most damage models found in the literature, besides the changes in rigidities that are due to the presence of damage, also takes into account the changes in inertial properties. These changes are accounted for in the second term on the r.h.s of the two expressions in (13).

The solution of the eigenvalue problem in (10) yields, respectively, the damaged displacement and rotation fields:

\[
\tilde{w}(x, t) = \sum_{n=1}^{N} \tilde{W}_n w_n(x) e^{i\bar{\omega}_n t}, \quad \tilde{\phi}(x, t) = \sum_{n=1}^{N} \tilde{\Phi}_n \varphi_n(x) e^{i\bar{\omega}_n t}
\]

By comparing the spatial derivatives of these fields with those of the fields in (2), one is able, under certain conditions, to locate the damage. The series in (2) and (14) are determined by substitution of the entries of matrices \([Q]\) and \([\tilde{Q}]\), respectively, which are computed by solving the eigenvalue problems (3) and (10).
2.2. Damage localisation method

The different damage indicators used in this work are described by the following four Equations, where \( CD, DI, MCD \) and \( MDI \) stand for Curvature Difference, Damage Index, Modified Curvature Difference and Modified Damage Index, respectively:

\[
CD(p, i, x) = \left| \frac{\partial^p \tilde{w}_i(x)}{\partial x^p} - \frac{\partial^p w_i(x)}{\partial x^p} \right|
\]

\[
DI(p, i, x) = \left[ \left( \frac{\partial^p \tilde{w}_i(x)}{\partial x^p} \right)^2 + \sum_{k=1}^{NP} \left( \frac{\partial^p \tilde{w}_i(x_k)}{\partial x^p} \right)^2 \right]^{\frac{1}{2}} - \left[ \left( \frac{\partial^p \tilde{w}_i(x)}{\partial x^p} \right)^2 + \sum_{k=1}^{NP} \left( \frac{\partial^p w_i(x_k)}{\partial x^p} \right)^2 \right]^{\frac{1}{2}}
\]

\[
MCD(p, i, x) = \left| \frac{\partial^p \tilde{\phi}_i(x)}{\partial x^p} - \frac{\partial^p \phi_i(x)}{\partial x^p} \right|
\]

\[
MDI(p, i, x) = \left[ \left( \frac{\partial^p \tilde{\phi}_i(x)}{\partial x^p} \right)^2 + \sum_{k=1}^{NP} \left( \frac{\partial^p \tilde{\phi}_i(x_k)}{\partial x^p} \right)^2 \right]^{\frac{1}{2}} - \left[ \left( \frac{\partial^p \tilde{\phi}_i(x)}{\partial x^p} \right)^2 + \sum_{k=1}^{NP} \left( \frac{\partial^p \phi_i(x_k)}{\partial x^p} \right)^2 \right]^{\frac{1}{2}}
\]

with \( p \) denoting the order of the spatial derivative, \( i \) the mode shape, \( x \) the coordinate where these indicators are computed and \( NP \) is the number of points where measurements are taken. One can identify the mode shape curvature difference indicator \([3]\) in Equation (15) and the damage index \([4]\) in Equation (16) with \( p = 2 \). The behaviour and experimental validation of mode shapes higher order derivatives have been reported recently \([17, 18]\). The \( MCD \) and \( MDI \) indicators with \( p = 1 \) were proposed and applied to experimental data in \([13]\). Since the displacement and rotation fields are defined by a series expansion, all the differentiations in Equations (15) trough (18) are performed analytically.

3. Numerical simulations

In order to study the damage model and to compare the ability of the different indicators to locate damage, simply supported beams with \( E = 210 \text{ GPa}, \nu = 0.3, \rho = 7800 \text{ kg/m}^3, K_s = 5/6, \text{ length } L = 1 \text{ m and various widths } b = 2h \) and thicknesses \( h \) are studied. Several cases of simulated damage, located at mid-span, are considered here, with the cracks having a width \( c = h/10 \) and varying depths \( p \) (Figure 2).

3.1. Damage model

The computed damaged frequencies show a dependency on the number of terms in the series of Equation (14), i.e. on the order of matrices \([\tilde{K}]\) and \([\tilde{M}]\). Figure 3 illustrates this dependency, which becomes larger as the damage increases. A study of the influence of inertial changes on the frequencies is depicted in Figure 4. One sees that the difference in frequencies with and without accounting for this changes, i.e. by including or not including the second term in the two expressions of (13) in the computations, increases as the crack depth increases. This difference is higher for frequencies associated with modes where the damage is not located at a node, such as modes 1 and 3. Since all the differences between frequencies are lower than 1\%, the influence of the inertial changes can be considered as small.
3.2. Damage localisation

The following damage localisations were performed based on computations with matrices of order 200 × 200 and p = 2, p = 4 in Equations (15) and (16) and p = 1, p = 3 in Equations (17) and (18). In other words, localisations are based on second and fourth order derivatives of the displacement fields \( w \) and \( \tilde{w} \) and first and third order derivatives of the rotation fields \( \phi \) and \( \tilde{\phi} \), respectively. The mode shape considered is the first and therefore \( i = 1 \).

Figures 5, 6, 7, and 8 show the localisation results of damages in beams with different aspect ratios \( L/h \) and crack depths \( p \) using the damage indicators described. The crack simulated at mid-span is clearly noticeable in all these Figures. One sees that the values of the modified indicators \( MCD \) and \( MDI \) are lower than the values of \( CD \) and \( DI \) for thick beams, i.e. for beams with \( L/h = 10 \) (Figures 5a) and c), 6a) and c), 7a) and c), and 8a) and c)). This trend is independent of the crack depth. For relatively thin beams these differences are negligible (Figures 5b) and d), 6b) and d), 7b) and d), and 8b) and d)). By comparing the values at \( x = L/2 \) of indicators based on low order derivatives and higher order derivatives (\( CD(2,1,x) \) versus \( MCD(1,1,x) \), \( DI(2,1,x) \) versus \( MDI(1,1,x) \) versus \( MDI(3,1,x) \) for thick beams, i.e. for \( L/h \) approaches 100, for a negligible value.

The differences in the values of the indicators, based on displacement fields and rotation fields, (\( CD \) versus \( MCD \) and \( DI \) versus \( MDI \)) at coordinate \( x = L/2 \) for beams with several aspect ratios are depicted in Figure 9. This Figure shows that the absolute differences between the indicators \( CD \) and \( MCD \) are higher than the differences between the indicators \( DI \) and \( MDI \). For the indicators \( CD(2,1,L/2) \) versus \( MCD(1,1,x) \) and \( CD(4,1,L/2) \) versus \( MCD(3,1,x) \) these differences are positive and decrease as the aspect ratios increase. The same applies for the differences of indicators \( DI(2,1,L/2) \) and \( MDI(1,1,L/2) \). Therefore, for relatively thick beams, the values of the indicators based on spatial derivatives of rotation fields are lower than those based on spatial derivatives of displacement fields. On the other hand, the differences between the indicators \( DI(4,1,L/2) \) and \( MDI(3,1,L/2) \) are negative for certain values of \( L/h \). Nevertheless, the differences of all indicators converge, as \( L/h \) approaches 100, for a negligible value.

Only damage localisation results using the first mode are reported here. However, it should be noted that for modes where the cracks are located in nodal points, the localisation is not accomplished successfully. A similar problem is reported in [10, 19], among others.
4. Conclusions
Several damage indicators based on low and higher order spatial derivatives of displacement and rotation fields are presented and applied to damage localization in beams. These fields are obtained using the Timoshenko beam theory, the Hamilton’s principle and the Ritz method. Contrary to most methods found in the literature, the derivatives of these fields are computed analytically, since they are defined by a series expansion. The number of terms included in this series, i.e. the order of the eigenvalue problem, influences the computed damaged frequencies, since for a large damage a large number of terms must be used. The influence of the inertial changes on the modal characteristics seems negligible, but it increases as the damage increases. It was found that the indicators based on higher order derivatives present values that are higher than those presented by indicators based on low order derivatives. A comparative study of the indicators based on spatial rotation fields with the indicators based on spatial derivatives of displacement fields, shows that, for thick beams, the former are in general lower than the latter. However, for thin beams, this difference is unnoticeable and there is a convergence of values using indicators based on displacement and rotation fields as the beam length to thickness ratio increases. It remains a challenge to obtain experimental data from thick beams and to apply higher order derivatives to these data. Work is under way to tackle such a challenge, namely by using shearography techniques, which are full field measurement techniques.
Figure 6. Indicators $DI(2, 1, x)$ and $MDI(1, 1, x)$.

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Figure 7. Indicators $CD(4, 1, x)$ and $MCD(3, 1, x)$.

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Figure 8. Indicators $DI(4, 1, x)$ and $MDI(3, 1, x)$.

Figure 9. Differences in the values of indicators at coordinate $x = L/2$. 