Abstract
The type A colored Tverberg theorem of Blagojević, Matschke, and Ziegler (Theorem 1.2) provides optimal bounds for the colored Tverberg problem, under the condition that the number of intersecting rainbow simplices is a prime number. We extend this result to an optimal, type A colored Tverberg theorem for multisets of colored points, which is valid for each prime power \( r = p^k \), and includes Theorem 1.2 as a special case for \( k = 1 \). One of the principal new ideas is to replace the ambient simplex \( \Delta^N \), used in the original Tverberg theorem, by an “abridged simplex” of smaller dimension, and to compensate for this reduction by allowing vertices to repeatedly appear a controlled number of times in different rainbow simplices. Configuration spaces, used in the proof, are combinatorial pseudomanifolds which can be represented as multiple chessboard complexes. Our main topological tool is the Eilenberg-Krasnoselskii theory of degrees of equivariant maps for non-free actions.

1 Introduction
The following result is known as the topological Tverberg theorem, [BSS, Œ, M03].

**Theorem 1.1.** Let \( r = p^k \) be a prime power, \( d \geq 1 \), and \( N = (r - 1)(d + 1) \). Then for every continuous map \( f : \Delta^N \to \mathbb{R}^d \), defined on an \( N \)-dimensional simplex, there exist disjoint faces \( \Delta_1, \ldots, \Delta_r \) of \( \Delta^N \) such that

\[
f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset.
\]

It is known [BFZ2] that the condition on \( r \) is essential, and that if \( r \) is not a prime power the topological Tverberg theorem may fail in general.

The following relative of Theorem 1.1 is sometimes referred to as the Optimal colored Tverberg theorem [BMZ], see also the review paper [Ž17] where it is classified as a Type A colored Tverberg theorem.
Theorem 1.2. ([BMZ]) Let \( r \geq 2 \) be a prime, \( d \geq 1 \), and \( N := (r - 1)(d + 1) \). Let \( \Delta^N \) be an \( N \)-dimensional simplex with a partition (coloring) of its vertex set into \( d + 2 \) parts,

\[
V = [N + 1] = C_0 \uplus \cdots \uplus C_d \uplus C_{d+1},
\]

with \( |C_i| = r - 1 \) for \( i \leq d \) and \( |C_{d+1}| = 1 \). Then for every continuous map \( f : \Delta^N \to \mathbb{R}^d \), there are \( r \) disjoint “rainbow simplices” \( \Delta_1, \ldots, \Delta_r \) of \( \Delta^N \) satisfying

\[
f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset
\]

where by definition a face \( \Delta \) of \( \Delta^N \) is a rainbow simplex if and only if \( |\Delta \cap C_j| \leq 1 \) for each \( j = 0, \ldots, d + 1 \).

Note that Theorem 1.2 does not include Theorem 1.1 as a special case. Indeed, we need a stronger condition in the colored Tverberg theorem, where \( r \) is a prime rather than a prime power. It remains an interesting question if this condition on \( r \) can be relaxed.

Our main new result (Theorem 1.3) is valid for each prime power \( r = p^k \), and includes Theorem 1.2 as a special case for \( k = 1 \). One of the guiding ideas in the proof is to replace the simplex \( \Delta^N \) (used in both Theorems 1.1 and 1.2) by a simplex of smaller dimension, and to compensate this by allowing its vertices to appear a controlled number of times in different faces \( \Delta_i \).

**Theorem 1.3.** Let \( r = p^k \) be a prime power, \( d \geq 1 \), and \( N := k(p - 1)(d + 1) \). Let \( \Delta^N \) be an \( N \)-dimensional simplex whose vertices are colored by \( d + 2 \) colors, meaning that there is a partition \( \text{Vert}(\Delta^N) = C_0 \cup C_1 \cup \cdots \cup C_d \cup C_{d+1} \) into \( d + 2 \) monochromatic subsets. We also assume that:

1. Each of the colored sets \( C_0, \ldots, C_d \) has \( (p - 1)k \) vertices. The vertices in each \( C_i \) are assigned multiplicities, as prescribed by the vector \((p^{k-1}, \ldots, p, 1)^{\times (p - 1)} \in \mathbb{N}^{k(p-1)}\).

2. The (exceptional) color class \( C_{d+1} \) contains a single vertex with multiplicity one.

(Altogether there are \( N + 1 = k(p - 1)(d + 1) + 1 \) vertices. Counted with multiplicities, their total number is \( (r - 1)(d + 1) + 1 \).)

We claim that under these conditions for any continuous map \( f : \Delta^N \to \mathbb{R}^d \) there exist \( r \) (not necessarily disjoint or even different) faces \( \Delta_1, \ldots, \Delta_r \) of \( \Delta^N \) such that:

(A) \( f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset \).

(B) The number of occurrences of each vertex of \( \Delta^N \) in all faces \( \Delta_i \), does not exceed the prescribed multiplicity of that vertex.

(C) All faces \( \Delta_i \) are multicolored or rainbow simplices, in the sense that their vertices have different colors, \((\forall i)(\forall j) |\text{Vert}(\Delta_i) \cap C_j| \leq 1 \).

Theorem 1.3 has an alternative formulation where the rainbow simplices \( \Delta_i \) are faces of the original simplex \( \Delta^{(r-1)(d+1)} \), rather than the faces of the abridged simplex \( \Delta^{k(p-1)(d+1)} \) (used in Theorem 1.3). While Theorem 1.3 is more intuitive and has a clear geometrical meaning, an advantage of Theorem 1.4 is that all simplices \( \Delta_i \) are distinct and the result is closer in form to Theorems 1.1 and 1.2.
Theorem 1.4. Assuming that $r = p^k$ is a prime power and $r' = r - 1$ let

$$K = K_{r', r', ..., r', 1} \cong [r'] \ast [r'] \ast \cdots \ast [r'] \ast \ast 1 = [r'] \ast (d+1) \ast \ast 1$$

be a $(d+1)$-dimensional simplicial complex on a vertex set $V = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_{d+1}$, divided into $(d + 2)$ color classes where $|V_0| = |V_1| = \cdots = |V_d| = r' = r - 1$ and $|V_{d+1}| = 1$. Assume that $f : K \rightarrow \mathbb{R}^d$ is an $L$-collapse map (Definition 3.3), meaning that $f = \hat{f} \circ \alpha$ for some map $\hat{f} : K_{k(p-1), k(p-1), ..., k(p-1), 1} \rightarrow \mathbb{R}^d$ where

$$\alpha : K_{r', r', ..., r', 1} \rightarrow K_{k(p-1), k(p-1), ..., k(p-1), 1}$$

is the simplicial map arising from a choice of a $L$-collapse map $\theta : [r'] \rightarrow [k(p-1)]$. Then there exist $r$ pairwise vertex disjoint simplices ($r$ vertex-disjoint rainbow simplices) $\Delta_1, \ldots, \Delta_r$ in $K$ such that

$$f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset.$$ (1.1)

The overall organization of the paper is as follows. The proof of Theorem 1.3 (and its equivalent form, Theorem 1.4) is given in Section 4. The role of chessboard complexes and their generalizations, as configuration spaces for theorems of Tverberg type, is briefly reviewed in 2. In Section 2 we formulate our main topological result of Borsuk-Ulam type (Theorem 2.1), used in the proof of Theorem 1.3 and its companion (Theorem 2.2), about degrees of maps from multiple chessboard complexes.

In Section 3 we develop the theory of multiple chessboard complexes in the generality needed for applications in the proofs of Theorems 1.3, 1.4 and 2.2. The focus is on multiple chessboard complexes which turn out to be pseudomanifolds (Sections 3.1 and 3.2).

In the final section (Section 6) we outline the proof of [KB, Theorem 2.1] (slightly extended to the case of pseudomanifolds), as one of the central results illustrating the Eilenberg-Krasnoselskii comparison principle for degrees of equivariant maps, in the case of non-free group actions.

2 Chessboard complexes and equivariant maps

The central role of chessboard complexes, as proper configuration spaces for colored Tverberg problem and its relatives, was recognized in [ŽV92] almost thirty years ago. To the present day these complexes remain, together with their generalizations (the multiple chessboard complexes) in the focus of research in this area of geometric combinatorics.

Recall that the (standard) chessboard complex $\Delta_{p,q}$ is the complex of all non-attacking placements of rooks in a $(p \times q)$-chessboard (a placement is non-attacking if it is not allowed to have more than one rook in the same row or in the same column). More generally, the multiple chessboard complex $\Delta_{p,q}^{A,B}$ (see Definition ?), where $A \in \mathbb{N}^p$ and $B \in \mathbb{N}^q$, arises if we allow more than one rook in each row (each column), where their precise number is determined by vectors $A$ and $B$. 
Central results in this area are the *Topological type A colored Tverberg theorem* (Theorem 2.2 in [BMZ]) and the *Topological type B colored Tverberg theorem* [ZV92, ZV94]. Both of these results are obtained by applications of the *Configuration Space/Test Map scheme* involving chessboard complexes.

The associated *test maps* are respectively (2.1) (in the Type A case) and (2.2) (for the Type B result).

\[
\begin{align*}
  f &: (\Delta_{r,r-1})^{*d} \ast [r] \xrightarrow{\mathbb{Z}/r} W_r^{\oplus d}. & (2.1) \\
  f &: (\Delta_{r,2r-1})^{*(k+1)} \xrightarrow{\mathbb{Z}/r} W_r^{\oplus d} & (2.2)
\end{align*}
\]

Both theorems are consequences of the corresponding Borsuk-Ulam-type statements claiming that in the either case the $\mathbb{Z}_r$-equivariant map $f$ must have a zero if $r$ is a prime number.

The following theorem extends (2.1) and serves as a basis for a new Type A topological Tverberg theorem, which extends (in a natural way) the result of Blagojević, Matschke and Ziegler to the prime power case.

**Theorem 2.1.** Let $G = (\mathbb{Z}_p)^k$ be a $p$-toral group of order $r = p^k$. Let $\Delta^{1:1}_{k(p-1),p^k}$ be the multiple chessboard complex (based on a $k(p-1) \times p^k$ chessboard), where $\mathbb{I} = (1, \ldots, 1) \in \mathbb{R}^{p^k}$ and $\mathbb{L} = (p^{k-1}, p^{k-2}, \ldots, p, 1)^{(p-1)} \in \mathbb{R}^{k(p-1)}$. Let $\partial \Delta_{[p^k]} \cong S^{p^k-2}$ be the boundary of a simplex with $p^k$ vertices. Then there does not exist a $G$-equivariant map

\[
  f : (\Delta^{1:1}_{k(p-1),p^k})^{*(d+1)} \ast [p^k] \longrightarrow (\partial \Delta_{[p^k]})^{*(d+1)} \cong (S^{p^k-2})^{*(d+1)} \cong S^{(p^k-1)(d+1)-1}.
\]

Theorem 2.1 is a consequence of the following theorem about degrees of equivariant maps.

**Theorem 2.2.** Let $G = (\mathbb{Z}_p)^k$ be a $p$-toral group of order $r = p^k$. Let $\Delta^{1:1}_{k(p-1),p^k}$ be the multiple chessboard complex (based on a $k(p-1) \times p^k$ chessboard), where $\mathbb{I} = (1, \ldots, 1) \in \mathbb{R}^{p^k}$ and $\mathbb{L} = (p^{k-1}, p^{k-2}, \ldots, p, 1)^{(p-1)} \in \mathbb{R}^{k(p-1)}$. Let $\partial \Delta_{[p^k]} \cong S^{p^k-2}$ be the boundary of a simplex with $p^k$ vertices. Then $\deg(f) \neq 0 \pmod{p}$ for any $G$-equivariant map

\[
  f : (\Delta^{1:1}_{k(p-1),p^k})^{*(d+1)} \longrightarrow (\partial \Delta_{[p^k]})^{*(d+1)} \cong (S^{p^k-2})^{*(d+1)} \cong S^{(p^k-1)(d+1)-1}.
\]

### 3 Chessboard pseudomanifolds

Following [JVZ-1, JVZ-2], a multiple chessboard complex $\Delta^{\mathbb{Z}_p:1}_{m,n} = \Delta^{k_1,\ldots,k_n,1_j,\ldots,l_m}_{m,n}$ is an abstract simplicial complex with vertices in $[m] \times [n]$, where the simplices have at most $k_i$ elements in the row $[m] \times \{i\}$ and at most $l_j$ elements in each column $\{j\} \times [n]$).

We shall be mainly interested in complexes $\Delta^{\mathbb{Z}_p:1}_{m,n} = \Delta^{1,\ldots,1_j,\ldots,l_m}_{m,n}$ where at most one rook is permitted in each of the rows of the chessboard $[m] \times [n]$.

The group $S_n$, permuting the rows of the chessboard $[m] \times [n]$, acts on the multiple chessboard complex $\Delta^{\mathbb{Z}_p:1}_{m,n}$. Moreover, the simplicial map $C_g : \Delta^{1:1'}_{m',n} \rightarrow \Delta^{1:1}_{m,n}$, associated to a “collapse map” $\theta : [m'] \rightarrow [m]$ (Section 3.1), is $S_n$-equivariant.
Proposition 3.1. The multiple chessboard complex $\Delta_{m,n}^{1:L}$ is a pseudomanifold if
\[
    n = l_1 + l_2 + \cdots + l_m + 1 .
\]
More precisely, the links of simplices of codimension 1 and 2 are spheres of dimensions 0 and 1, while in codimension 3 may appear both 2-spheres and 2-dimensional tori $T^2$.

Proof: Let $S \in \Delta_{m,n}^{1:L}$ and let $s_i := |S \cap \{(i) \times [n]\}|$. The link $\text{Link}(S)$ is clearly isomorphic to the multiple chessboard complex $\Delta_{m,n}^{1:T}$ where $T = (t_1, \ldots, t_m)$ and $t_i := l_i - s_i$. (Here we allow that $t_j = 0$ for some $j \in [m]$.) The proof is completed by an explicit description of all multiple chessboard complexes that arise as links of simplices in codimension $\leq 3$.

If $\text{codim}(S) = 1$ then there exists $j_0$ such that $l_{j_0} = s_{j_0} + 1$ and $l_j = s_j$ for each $j \neq j_0$. The condition $3.1$ guarantees that $t_{j_0} = 2$, which together with $t_j = 0$ for $j \neq j_0$ implies $\text{Link}(S) \cong \Delta_{1,2} \cong S^0$.

If $\text{codim}(S) = 2$ then there are two possibilities. Either (I) there exists $j_0$ such that $l_{j_0} = s_{j_0} + 2$ and $l_j = s_j$ for each $j \neq j_0$, or (II) there exists $j_0 \neq j_1$ such that both $l_{j_0} = s_{j_0} + 1, l_{j_1} = s_{j_1} + 1$ and $l_j = s_j$ for each $j \neq j_0, j_1$. In the first case $\text{Link}(S) \cong \partial \Delta_{[3]} \cong S^1$, while in the second $\text{Link}(S) \cong \Delta_{2,3} \cong S^1$.

If $\text{codim}(S) = 3$ then the number of non-zero entries in the vector $T = (t_1, \ldots, t_m)$ is 1, 2 or 3. In the first case $\text{Link}(S) \cong \partial \Delta_{[4]} \cong S^2$. In the second case $\text{Link}(S) \cong \Delta_{2,4}^{3:L}$, where $A = (2, 1)$, hence $\Delta_{2,4}^{3:L} \cong S^2$.

Finally, in the third case $\text{Link}(S) \cong \Delta_{3,4} \cong T^2$. \hfill $\Box$

Proposition 3.2. The pseudomanifold $\Delta_{m,n}^{1:L}$ is always orientable. It has a fundamental class $\tau \in H_d(\Delta_{m,n}^{1:L}; \mathbb{Z}) \cong \mathbb{Z}$ where $d = \dim(\Delta_{m,n}^{1:L}) = m - 1$. A permutation $g \in S_n$ reverses the orientation (changes the sign of $\tau$) if and only if $g$ is odd.

As a consequence the $S_n$-pseudomanifolds $\Delta_{m,n}^{1:L}$ and $\Delta_{m',n}^{1:L'}$ are concordant in the sense that each $g \in S_n$ either changes the orientation of both of the complexes if none of them.

Proof: Let $C_\theta : \Delta_{m,n}^{1:L} \to \partial \Delta_{[n]}$ be the collapse map associate to the constant map $\theta : [m] \to [1]$ (Definition 3.3). In other words $C_\theta$ is the map induced by the projection $[m] \times [n] \to [1] \times [n]$ of chessboards, where a simplex $S \in \Delta_{m,n}^{1:L}$ is mapped to a simplex $S' \in \partial \Delta_{[n]}$ if and only if
\[
    \left( \forall i \in [n] \right) \left( \{i\} \times [m] \right) \cap S \neq \emptyset \Leftrightarrow i \in S' .
\]
Let $\hat{S}$ be the simplex $S \in \Delta_{m,n}^{1:L}$ oriented by listing its vertices in the increasing order of rows. Note that if $C_\theta(S) = S' \in \partial \Delta_{[n]}$ then $C_\theta(\hat{S}) \cong \hat{S}'$.

Choose an orientation $\mathcal{O}'$ on the sphere $\partial \Delta_{[n]}$ and use this orientation to define, via the collapse map $C_\theta$, an orientation $\mathcal{O}$ on $\Delta_{m,n}^{1:L}$. More explicitly, an ordered simplex $\hat{S}$ is positively oriented with respect to $\mathcal{O}$ if and only if $\hat{S}'$ is positively oriented with respect to the orientation $\mathcal{O}'$. It is not difficult to check that $\mathcal{O}$ is indeed and orientation on the pseudomanifold $\Delta_{m,n}^{1:L}$ which has all the properties listed in Proposition 3.2. \hfill $\Box$
3.1 Hierarchy of pseudomanifolds $\Delta^{1;L}_{m,n}$

We already know (Proposition 3.1) that $\Delta^{1;L}_{m,n}$ is an orientable pseudomanifold, provided $n = l_1 + \ldots + l_m + 1$. If $L = 1 \in \mathbb{N}^m$ then $\Delta^{1;L}_{m,n} = \Delta_{n-1,n}$ is a standard chessboard complex [BLVZ], while in the case $m = 1$ the complex $\Delta^{1;L}_{m,n} \cong \partial \Delta_{[n]}$ is the boundary sphere $\partial \Delta_{[n]} \cong S^{n-2}$ of the simplex $\Delta_{[n]} := 2^n$.

The pseudomanifolds $\Delta^{1;L}_{m,n}$ form a poset category where the complexes $\Delta_{n-1,n}$ and $\partial \Delta_{[n]}$ play the role of the initial and terminal object. The morphisms in this category are the $\theta$-collapse maps $C_\theta$, described in the following definition.

**Definition 3.3.** Assuming $m \geq m'$, choose an epimorphism $\theta : [m] \to [m']$. Let $\hat{\theta} : [m] \times [n] \to [m'] \times [n]$ be the associated map of chessboards where $\hat{\theta}(i,j) = (\theta(i), j)$. We say that a sequence $B = (b_1, \ldots, b_{m'})$ is obtained by a $\theta$-collapse from a sequence $A = (a_1, \ldots, a_m)$ if $b_i = \sum_{\theta(j)=i} a_j$. Define $C_\theta : \Delta^{1;A}_{m,n} \to \Delta^{1;B}_{m',n}$ as the induced map of multiple chessboard complexes where $C_\theta(S) := \hat{\theta}(S)$, for each simplex $S \in C_\theta : \Delta^{1;A}_{m,n}$.

If $m = l_1 + l_2 + \ldots + l_{m'}$ is interpreted as the cardinality of a multiset $A = \{a_1^{l_1}, \ldots, a_m^{l_m}\}$, with signature $L = (l_1, \ldots, l_{m'})$ and set-cardinality $m'$, both $\theta$ and $C_\theta$ (and other related maps) are often referred to as $\mathcal{L}$-collapse maps.

3.2 Degree of the collapse map $C_\theta$

In the following proposition we calculate the degree of the map $C_\theta$.

**Proposition 3.4.** The degree of the map $C_\theta : \Delta^{1;A}_{m,n} \to \Delta^{1;B}_{m',n}$ is,

$$\deg(C_\theta) = \binom{B}{A} = \frac{b_1! b_2! \ldots b_{m'}!}{a_1! a_2! \ldots a_m!}. \tag{3.2}$$

In the special case when $m' = 1$ we obtain that the degree of the map $C_\theta$ is the multinomial coefficient,

$$\deg(C_\theta) = \frac{(a_1 + a_2 + \ldots + a_m)!}{a_1! a_2! \ldots a_m!} \tag{3.3}$$

and in the special case $a_1 = a_2 = \ldots = a_m = 1$ (3.2) reduces to the formula,

$$\deg(C_\theta) = b_1! b_2! \ldots b_{m'}!. \tag{3.4}$$

**Proof:** Each simplicial map $C_\theta : \Delta^{1;A}_{m,n} \to \Delta^{1;B}_{m',n}$ is non-degenerate in the sense that it maps bijectively the top dimensional simplices of $\Delta^{1;A}_{m,n}$ to top dimensional simplices of $\Delta^{1;B}_{m,n}$. Moreover, it is an orientation preserving map so in order to calculate the degree of $C_\theta$ it is sufficient to calculate the cardinality of the preimage $C_\theta^{-1}(c_0)$ of the barycenter $c_0$ of a chosen top dimensional simplex of $\Delta^{1;B}_{m,n}$.

Since the degree is multiplicative it is sufficient to establish formula (3.4). A simple calculation shows that the cardinality of the set $C_\theta^{-1}(c_0)$ is, in the case of a map $C_\theta : \Delta_{n-1,n} \to \Delta^{1;B}_{m',n}$; indeed given by the formula (3.4). $\square$
4 Proof of Theorem 1.3

By convention $\Delta = \Delta_C$ is a simplex spanned by a set $C$, in particular $\Delta^N \cong \Delta_C$ where $C = \text{Vert}(\Delta^N) = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_d \sqcup C_{d+1}$.

Recall that a set $S \subset C$ (and the corresponding face $\Delta_S \subseteq \Delta_C$) is called a rainbow set (rainbow face) if $|S \cap C_i| \leq 1$ for all $i = 0, 1, \ldots, d+1$. It follows that the set of all rainbow simplices is a subcomplex of $\Delta_C$ which has a representation as a join of 0-dimensional simplicial complexes:

$$ R = \text{Rainbow} := C_0 \ast C_1 \ast \cdots \ast C_d \ast C_{d+1} \subset \Delta_C. \quad (4.1) $$

By assumption $|C_i| = m := k(p - 1)$ for $i = 0, 1, \ldots, d$ and $|C_{d+1}| = 1$, or more explicitly $C_i = \{c^i_{a,\beta}\} (0 \leq \alpha \leq k - 1; 1 \leq \beta \leq p - 1)$ for all $0 \leq i \leq d$, and $C_{d+1} = \{c_0\}$. Theorem 1.3 claims that for each continuous map $f : \Delta_C \to \mathbb{R}^d$ there exist rainbow faces $\Delta_1, \ldots, \Delta_r \in R$ such that:

1. Vertex $c_0$ appears in at most one of the faces $\Delta_i$;
2. For all $i, \alpha, \beta$ the vertex $c^i_{\alpha,\beta}$ may appear in not more than $p^\alpha$ faces $\Delta_1, \ldots, \Delta_r$;
3. $f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset$.

An $r$-tuple $(\Delta_1, \ldots, \Delta_r)$ of rainbow simplices is naturally associated to the join $\Delta_1 \ast \cdots \ast \Delta_r \in R^r$. Our immediate objective is to identify the subcomplex $R^r \subseteq R^r$ which collects all $r$-tuples $(\Delta_1, \ldots, \Delta_r)$ satisfying conditions (1) and (2).

By assumption $\Delta_{i,\nu} := \Delta_i \cap C_\nu$ is either empty or a singleton, for each rainbow simplex $\Delta_i$; a moment’s reflection reveals that the union $\cup\{\Delta_{i,\nu}\}_{i=1}^\nu$ is a simplex in $\Delta^1_{k,\nu}$, for $0 \leq \nu \leq d$ and a simplex in $[r] = [p^k]$ if $\nu = d + 1$. It immediately follows that

$$ R^r \subseteq (\Delta^1_{k(p-1),p^k})^{(d+1)} \ast [r]. $$

Let $f : R \to \mathbb{R}^d$ be the restriction of the map $f : \Delta_C \to \mathbb{R}^d$. The corresponding map defined on the $r$-tuples of rainbow simplices, satisfying conditions (1) and (2) is the map

$$ \hat{f} : (\Delta^1_{k(p-1),p^k})^{(d+1)} \ast [r] \longrightarrow (\mathbb{R}^d)^{sr}. $$

By composing with the projection $(\mathbb{R}^d)^{sr} \to (\mathbb{R}^d)^{sr}/D$ (where $D \cong \mathbb{R}^d$ is the diagonal) and the embedding $(\mathbb{R}^d)^{sr}/D \hookrightarrow (W_r)^{\oplus (d+1)}$, we finally obtain a map

$$ \hat{f} : (\Delta^1_{k(p-1),p^k})^{(d+1)} \ast [r] \longrightarrow (W_r)^{\oplus (d+1)} $$

which has a zero in a simplex $(\Delta_1, \ldots, \Delta_r)$ if and only if $f(\Delta_1) \cap \cdots \cap f(\Delta_r) \neq \emptyset$. Since the sphere $S((W_r)^{\oplus (d+1)}) \cong (S(W_r))^{(d+1)}$ is equivariantly homeomorphic to $(\partial \Delta_{[r]})^{(d+1)}$ a zero exists by Theorem 2.1 which concludes the proof of Theorem 1.3. \hfill \Box
5 Proof of Theorem 2.2

We are supposed to show that the degree \( \deg(f) \) of each \( G \)-equivariant map

\[
f : (\Delta_{k(p-1),p}^{1,\mathbb{L}})^{*(d+1)} \longrightarrow (\partial\Delta_{[p^k]}^{1,\mathbb{L}})^{*(d+1)} \cong (S^{p^k-2})^{*(d+1)} \cong S^{(p^k-1)(d+1)-1}
\]

where \( G = (\mathbb{Z}_p)^k \) is a \( p \)-toral group, is non-zero modulo \( p \). Following the Comparison principle for equivariant maps (Section 6) we should:

(A) Exhibit a particular map (5.1) such that \( \deg(f) \neq 0 \) modulo \( p \);
(B) Check if the conditions of Theorem 6.2 are satisfied.

The following proposition provides the needed example for the first part of the proof.

Proposition 5.1. The \( \theta \)-collapse map

\[
C_{\theta} : \Delta_{k(p-1),p}^{1,\mathbb{L}} \longrightarrow \partial\Delta_{[p^k]}^{1,\mathbb{L}} \cong \Delta_{1,r'}^{1,\mathbb{L}},
\]

where \( \theta : [k(p-1)] \rightarrow [1] \) is a constant map, has a non-zero degree modulo \( p \).

Proof: We calculate the degree of the map (5.2) by applying the formula (3.3). Recall that \( \mathbb{L} = (p^{k-1},p^{k-2},\ldots,p,1)^{*(p-1)} \in \mathbb{R}^{k(p-1)} \) so in this case

\[
\deg(C_{\theta}) = \frac{(p^k - 1)!}{[(p^{k-1})!(p^{k-2})!\ldots!p!1!]^{p-1}}.
\]

The well-known formula for the highest power of \( p \) dividing \( m! \) is

\[
\text{ord}_p(m!) = \left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \ldots.
\]

By applying this formula we obtain

\[
\text{ord}_p((p^k - 1)!) = \text{ord}_p((p^k)!) - k = p^{k-1} + p^{k-2} + \cdots + 1 - k
\]

and by applying the same formula to the denominator of (5.3) we obtain exactly the same quantity.

In light of Proposition 5.1 the map

\[
(C_{\theta})^{*(d+1)} : (\Delta_{k(p-1),p}^{1,\mathbb{L}})^{*(d+1)} \longrightarrow (\partial\Delta_{[p^k]}^{1,\mathbb{L}})^{*(d+1)} \cong (S^{p^k-2})^{*(d+1)} \cong S^{(p^k-1)(d+1)-1}
\]

has a non-zero degree \( \deg((C_{\theta})^{*(d+1)}) = (\deg(C_{\theta}))^{d+1} \) modulo \( p \), which completes part (A) of the proof.

For the second part we begin with the observation that \( \partial\Delta_{[r]} (r = p^k) \) is \( S_r \)-equivariantly homeomorphic to the unit sphere \( S(W_r) \) in the standard \( S_r \)-representation \( W_r := \{ x \in \mathbb{R}^r \mid \quad \}

$x_1 + \cdots + x_r = 0$. As a consequence, for each subgroup $H \subseteq S_r$ the corresponding fixed point set $\partial \Delta^H_{[r]} \cong S(W_r)^H = S(W_r^H)$ is also a sphere.

The action of $H$ decomposes $[r]$ into orbits $[r] = O_1 \sqcup \cdots \sqcup O_t$. From here easily follows a combinatorial description of the fixed point set $\partial \Delta^H_{[r]}$. A point $x \in \partial \Delta_{[r]}$, with barycentric coordinates $\{\lambda_i\}_{i=1}^r$, is fixed by $H$ if and only if the barycentric coordinates are constant in each of the orbits. Summarising, $\partial \Delta^H_{[r]}$ is precisely the boundary of the simplex with vertices $\{o_i\}_{i=1}^t$, where $o_i$ is the barycenter of the face $\Delta_i \subset \Delta_{[r]}$.

Let $\Delta_{m,r}^{3,\mathbb{L}}$ be a multiple chessboard complex, where $\mathbb{L} = (l_1, \ldots, l_m)$ and $m = k(p - 1)$. It is not difficult to see that the barycenter $b_{i,j}$ of (geometric realization of) the simplex $\{i\} \times O_j$ is in the fixed point set $(\Delta_{m,r}^{3,\mathbb{L}})^H$ if and only if $|O_j| \leq l_i$.

More generally, a point $x$ is in $(\Delta_{m,r}^{1,\mathbb{L}})^H$ if and only if it can be expressed as a convex combination

$$x = \sum_{(i,j) \in S} \lambda_{i,j} b_{i,j}$$

where $S$ is a subset of $[m] \times [t]$ satisfying

1. If $(i, j), (i', j) \in S$ then $i = i'$;
2. $(\forall i \in [m]) \sum\{|O_j| \mid (i, j) \in S\} \leq l_i$.

The $\theta$-collapse map $C_\theta$, where $\theta : [m] \to [1]$ is the constant map, maps $(\Delta_{m,r}^{3,\mathbb{L}})^H$ to $\partial \Delta^H_{[r]}$. Moreover $C_\theta(b_{i,j}) = o_j$ and, in light of (1) and (2), the simplex with vertices $\{b_{i,j}\}_{(i,j) \in S}$ is mapped bijectively to a face of $\partial \Delta^H_{[r]}$. The following inequality is an immediate consequence,

$$\dim((\Delta_{m,r}^{1,\mathbb{L}})^H) \leq \dim(\partial \Delta^H_{[r]}).$$

From here and [5,4] we obtain the inequality

$$\dim((\Delta_{k(p-1),p^k}^{3,\mathbb{L}})^{(d+1)}H) \leq \dim((\partial \Delta_{[p^k]}^{(d+1)})^H$$

which finishes the proof of part (B) and concludes the proof of the theorem. \qed

**Remark 5.2.** It follows from the part (B) of the proof of Theorem 2.2 that $\Delta = (\Delta_{m,r}^{3,\mathbb{L}})^H$ is also a “chessboard complex”. Indeed, $S \subseteq [m] \times [t]$ is a simplex in $\Delta$ if and only if $S$ has at most one rook in each row $[m] \times \{j\}$ and the total weight of the set $(\{i\} \times [t]) \cap S$ is at most $l_i$, where the weight of each element $(i, j)$ is $|O_j|$.

It follows that $\Delta$ can be classified as a complex of the type $\Delta_{m,t}^{1,\mathbb{L}}$ (cf. [JVZ-1, Definition 2.3]), where $\mathcal{L}$ is family of threshold (simplicial) complexes.

## 6 Comparison principle for equivariant maps

The following theorem is proved in [KB] (Theorem 2.1 in Section 2). Note that the condition that the $H_i$-fixed point sets $S^{H_i}$ are locally $k$-connected for $k \leq \dim(M^{H_i}) - 1$ is
automatically satisfied if $S$ is a representation sphere. So in this case it is sufficient to show that the sphere $S^{H_i}$ is (globally) $(\dim(M^{H_i}) - 1)$-connected which is equivalent to the condition 

$$\dim(M^{H_i}) \leq \dim(S^{H_i}) \quad (i = 1, \ldots, m).$$

**Theorem 6.1.** Let $G$ be a finite group acting on a compact topological manifold $M = M^n$ and on a sphere $S \cong S^n$ of the same dimension. Let $N \subset M$ be a closed invariant subset and let $(H_1), (H_2), \ldots, (H_k)$ be the orbit types in $M \setminus N$. Assume that the set $S^{H_i}$ is both globally and locally $k$-connected for all $k = 0, 1, \ldots, \dim(M^{H_i}) - 1$, where $i = 1, \ldots, k$. Then for every pair of $G$-equivariant maps $\Phi, \Psi : M \to S$, which are equivariantly homotopic on $N$, there is the following relation

$$\deg(\Psi) \equiv \deg(\Phi) \pmod{\text{GCD}\{|G/H_1|, \ldots, |G/H_k|\}}. \quad (6.1)$$

The proof of the following extension of Theorem 6.1 to manifolds with singularities doesn’t require new ideas. By a *singular topological manifold* we mean a topological manifold with a codimension 2 singular set. In particular Theorem 6.2 applies to pseudomanifolds $\Delta^1_{m,n}$, introduced in Section 3.

**Theorem 6.2.** Let $G$ be a finite group acting on a compact “singular topological manifold” $M = M^n$ and on a sphere $S \cong S^n$ of the same dimension. Let $N \subset M$ be a closed invariant subset and let $(H_1), (H_2), \ldots, (H_k)$ be the orbit types in $M \setminus N$. Assume that the set $S^{H_i}$ is both globally and locally $k$-connected for all $k = 0, 1, \ldots, \dim(M^{H_i}) - 1$, where $i = 1, \ldots, k$. Then for every pair of $G$-equivariant maps $\Phi, \Psi : M \to S$, which are equivariantly homotopic on $N$, there is the following relation

$$\deg(\Psi) \equiv \deg(\Phi) \pmod{\text{GCD}\{|G/H_1|, \ldots, |G/H_k|\}}. \quad (6.2)$$

**Proof:** Following into footsteps of the proof of Theorem 6.1 (see [KB, Theorem 2.1]) we define a $G$-equivariant map

$$f_0 : (M \times \{0,1\}) \cup (N \times [0,1]) \to B \setminus \{O\} \quad (6.3)$$

where $B = \text{Cone}(S)$ is a cone over the sphere $S$ (with the apex $O$), $\Psi$ and $\Phi$ are restrictions of $f_0$ on $M \times \{0\}$ (respectively $M \times \{1\}$) and the restriction of $f_0$ on $N \times [0,1]$ is a homotopy between $\Psi|_N$ and $\Phi|_N$.

If $f : M \times [0,1] \to B$ is a $G$-equivariant extension of $f_0$ then ([KB Lemma 2.1]) \text{deg}(f) = \pm(\text{deg}(\Psi) - \text{deg}(\Psi)) and the relation (6.2) will follow if

$$\text{deg}(f) = \sum_{i=1}^{m} a_i \cdot |G/H_i| \quad (6.4)$$

for some integers $a_i \in \mathbb{Z}$.

The proof of the following lemma ([KB, Lemma 2.2]) is quite general, in particular it holds for “singular topological manifolds”.

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**Lemma 6.3.** There exists a $G$-equivariant extension $f : M \times [0,1] \to B$ of the map $f_0$ satisfying the following conditions:

$(\alpha)$ $K = f^{-1}(O) = \bigcup_{j=1}^{m} T_j$ where $T_u \cap T_v = \emptyset$ for $u \neq v$;

$(\beta)$ $T_j = G(K_j)$ for a compact set $K_j$;

$(\gamma)$ $K_j = H_j(K_j)$ is $H_j$-invariant;

$(\delta)$ $g(K_j) \cap h(K_j) = \emptyset$ if $gh^{-1} \notin H_j (j = 1, \ldots, m)$.

The proof of Theorem 6.2 is completed as in [KB, Section 2.1.3] by observing that “singular topological manifolds” also have absolute and relative fundamental classes.

More explicitly, if $F_j$ is the restriction $f$ to a sufficiently small neighborhood of $K_j$ then

$$\deg(f) = \sum_{j=1}^{m} \deg(F_j).$$

By the same argument as in [KB] we deduce from Lemma 6.3 that $\deg(F_j) = a_j \cdot |G/H_j|$ for some $a_j \in \mathbb{Z}$, and the relation (6.2) is an immediate consequence. □

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