Fixed point action and topological charge for SU(2) gauge theory

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We present a theoretically consistent definition of the topological charge operator based on renormalization group arguments. Results of the measurement of the topological susceptibility at zero and finite temperature for SU(2) gauge theory are presented.

1. INTRODUCTION

Instantons in the QCD vacuum can explain the $U_A(1)$ problem, and they could be responsible for the low energy hadron and glueball spectrum and the chiral phase transition as well \cite{1}. Instantons have been studied on the lattice for over a decade but these studies have produced about as much controversy as physical results. The problems are three-fold: 1) Instantons are not conserved by topology on the lattice - small instantons can "fall through" the lattice. 2) Most lattice actions are not scale invariant - the lattice action of an instanton depends on its size. 3) The definition of the topological charge is problematic - the geometric definition erroneously identifies dislocations as instantons, while the algebraic definition relies on some sort of smoothing algorithm like cooling or APE-smearing and the topological susceptibility has both additive and multiplicative renormalization. In figure 1 the action of a single smooth instanton calculated with the Wilson action is plotted as a function of its size $\rho$. ($S_I = 8\pi^2$ is the continuum instanton action.) The dotted vertical line indicates where the instanton disappears from the lattice according to the geometric definition and the dashed line is the value of the topological charge evaluated with the simplest algebraic $F\tilde{F}$ operator, the twisted plaquette. The scale violation of the Wilson action is well known but it is interesting to note that the algebraic definition gives $Q < 0.8$ if $\rho < 2.0$ even for very smooth instantons.

Fixed point actions are, by construction, scale invariant, solving problem 2). In addition, the renormalization group equation of fixed point actions offer a theoretically consistent definition of the topological charge. Since fixed point actions are 1-loop perfect but do not follow exactly the renormalized trajectory, this method does not solve 1).

In this paper we describe the renormalization group definition of the topological charge $Q_{\text{RG}}$ and discuss our numerical results for the topological susceptibility both at zero and at finite temperatures.
temperature. Further details and an extensive reference list can be found in Ref. \[2\].

2. FIXED POINT ACTIONS AND THE DEFINITION OF $Q_{RG}$

Fixed point (FP) actions of SU(N) gauge theories are 1-loop perfect actions corresponding to the $\beta = \infty$ fixed point of a renormalization group transformation. The defining renormalization group equation is

$$S^{FP}(V) = \min_{\{U\}} (S^{FP}(U) + T(U, V)),$$  \hspace{1cm} (1)

where $U$ is the original link variable, $V$ is the blocked link variable and $T(U, V)$ is the blocking kernel that defines the transformation. Eqn. \[3\] defines, in a unique way, a fine lattice $\{U\}$ for every coarse configuration $\{V\}$ but it is not a simple interpolation of the coarse lattice. The fine lattice $\{U\}$ blocks into $\{V\}$ under the renormalization group transformation but it is special among the many configurations that block into the original coarse lattice as it is the smoothest one. Indeed, for typical coarse configurations with correlation length of a few lattice spacings and plaquette expectation value about 0.5 out of one, the plaquette expectation value on the fine configuration is typically between 0.95 and 1.0. We will refer to the transformation of Eqn. \[3\] as “inverse blocking”.

It can be proven that the inverse blocking does not change the topological properties of the original lattice, but it changes the scale of any topological objects by a factor of two, i.e. if the coarse lattice had certain number of instantons and anti-instantons, they will be present on the fine lattice but with size doubled. Because the fine lattice is very smooth, the instantons are easily identified and one can avoid most of the problems related to the definition of the topological charge. We define the charge of a coarse configuration as the charge of the inverse blocked configuration measured with any reliable method. In the following we use the geometric definition

$$Q_{RG}(V) = Q_{geom}(U).$$  \hspace{1cm} (2)

The minimization equation gives the value of the fixed point action on the coarse lattice as well

$$S^{FP}(V) = S^{FP}(U(V)) + T(U(V), V).$$  \hspace{1cm} (3)

As it is much easier to parametrize the FP action on the fine configuration, we use this relation in testing the scale invariance of the FP action. In figure 1 the solid vertical line corresponds to the boundary $Q_{RG} = 0 \rightarrow Q_{RG} = 1$ and the diamond symbols show the value of the action calculated with the FP action according to Eqn. \[3\]. Unlike the plaquette action, the FP action is consistent with the theoretical continuum value, independent of the size of the instanton as long as the instanton is present, $Q_{RG} = 1$. That demonstrates not only the scale invariance of the FP action but the validity of the definition of $Q_{RG}$.

We would like to emphasize that the topological properties of any configuration are a dynamical question; they are defined with respect to a given action. Our definition works with a FP action and it cannot be automatically applied to configurations generated with any other action.

3. THE TOPOLOGICAL SUSCEPTIBILITY AT T=0

We have measured the topological susceptibility using $Q_{RG}$ with an approximate SU(2) FP action. The action consists of several powers of two loops, the plaquette and the perimeter-six loop $(x,y,z,-x,-y,-z)$. Its coefficients are tabulated in Ref. \[2\].

This 8 parameter FP action shows almost perfect scaling properties for couplings weaker than the $N_T = 2$ finite temperature phase transition. Since the inverse blocking transformation is quite computer intensive, we have used relatively small lattices and concentrated on the strong coupling region comparing identical physical volume measurements to see scaling or the lack of it. We use the parameter $r_0$, as defined through the force, by $r_0^2 F(r_0) = -1.65$ to set the scale in our calculation. We calculate the dimensionless ratio of susceptibility times the fourth power of $r_0$

$$\chi r_0^4 = \langle Q^2 \rangle (\frac{r_0}{L})^4$$  \hspace{1cm} (4)
on lattices with approximately identical physical volumes characterized by the quantity $L/r_0$.

![Figure 2](image-url)  
Figure 2. Scaling test for topological charge, as a function of lattice spacing. The three lattice sizes (in units of $r_0$) of 1.6, 1.9, and 3.2 are shown by diamonds, crosses, and a square, respectively.

Figure 2 summarizes our results. We plot the quantity $\chi r_0^4$ versus the lattice spacing $a/r_0$ for physical volumes $L/r_0 \sim 1.6, 1.9$ and 3.2. The three data points corresponding to $L/r_0 \approx 1.6$, which span a range in lattice spacing $a = 0.4r_0 - 0.2r_0$ is consistent with scaling. Apparently a lattice with $a = 0.4r_0$ is fine enough to support the physically relevant instantons.

Making the assumption that the largest volume is large enough to approximate infinity, and making the assumption that the pure gauge SU(2) $r_0$ is equal to the phenomenological SU(3) number 0.5 fm, our value $\chi r_0^4 = 0.12(2)$ corresponds to $\chi = (235(10)$ MeV$)^4$. Evaluating the Witten-Veneziano formula with the known physical masses of the appropriate mesons, and $N_f = 3$, this susceptibility yields an $\eta'$ mass of 1520 MeV.

4. THE TOPOLOGICAL SUSCEPTIBILITY AT $T \neq 0$

Not much is known about the role of instantons and the topological susceptibility at finite temperature. At $T < T_c$ one expects a very weak $T$ dependence while at very high temperatures ($T > T_c/3$) the instantons are exponentially suppressed.

In our preliminary study we performed a series of simulations at a fixed $\beta = 1.7$ value corresponding to $N_T \sim 6.4$. We kept the spatial volume fixed at $N_s = 8$ and varied the temporal direction between $N_T = 8$ and $N_T = 2$. Figure 3 shows the susceptibility as a function of $T/T_c$. The dashed line is the high temperature prediction of Yaffe and Pisarski while the solid line is a very simple model prediction following Shuryak’s picture assuming that instantons with diameter approximately equal to the inverse temperature are suppressed. The data points agree surprisingly well with this simple two-parameter model prediction.

![Figure 3](image-url)  
Figure 3. Topological susceptibility at finite temperature. The dashed line line is the high temperature prediction of Yaffe and Pisarski, the solid line is a simple finite volume model prediction.

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REFERENCES

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