Parallel Stochastic Asynchronous Coordinate Descent: Tight Bounds on the Possible Parallelism

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Several works have shown linear speedup is achieved by an asynchronous parallel implementation of stochastic coordinate descent so long as there is not too much parallelism. More specifically, it is known that if all updates are of similar duration, then linear speedup is possible with up to \(\Theta(\sqrt{n}/L_{\text{res}})\) processors, where \(L_{\text{res}}\) is a suitable Lipschitz parameter. This paper shows the bound is tight for essentially all possible values of \(L_{\text{res}}\).

∗Part of the work done while this author was working at Max-Planck Institute for Informatics, Saarland Informatics Campus, and also during his two visits to Courant Institute, NYU in the summers of 2017 and 2018. This author would like to acknowledge Singapore NRF 2018 Fellowship NRF-NRFF2018-07 and MOE AcRF Tier 2 Grant 2016-T2-1-170.

†The work of Richard Cole and Yixin Tao was supported in part by NSF Grant CCF-1527568.
1 Introduction

This paper considers an asynchronous parallel implementation of stochastic coordinate descent applied to composite convex functions $F : \mathbb{R}^n \to \mathbb{R}$ of the form $F(x) = f(x) + \sum_{k=1}^{n} \Psi_k(x_k)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function, and each $\Psi_k : \mathbb{R} \to \mathbb{R}$ is a univariate convex function, but may be non-smooth. It asks:

If one seeks linear speedup, what is the limit on the possible parallelism?

The existing lower bound on the possible parallelism shows linear speedup occurs when $q = \Theta(\sqrt{nL_{\max}}/L_{\text{res}})$, where $q$ an upper bound on how many other updates a single update can overlap, and $L_{\max}$ and $L_{\text{res}}$ are Lipschitz parameters defined in Section 2. In particular, if the length of the updates varies by at most a factor of $c$ and $p$ is the number of processors at hand, then $q \leq c(p-1)$.

Main result The existing lower bound is tight.

The following function family, $f : \mathbb{R}^n \to \mathbb{R}$, will be used to demonstrate the result:

$$f(x) = \left(1 - \frac{\epsilon}{2}\right) \cdot \sum_{i=1}^{n} (x_i)^2 + \frac{\epsilon}{2} \cdot \left(\sum_{i=1}^{n} x_i\right)^2,$$  \hspace{1cm} (1)

for any $\epsilon$ satisfying $\Theta\left(\frac{1}{\sqrt{n}}\right) \leq \epsilon \leq \frac{1}{9}$. As we shall see, $L_{\max} = 1$ and $L_{\text{res}} = \Theta(\epsilon \sqrt{n})$ for this function family. For this function family, $\Psi_k \equiv 0$ for all $k$.

Recall that the performance of a sequential stochastic coordinate descent algorithm is expressed in terms of its convergence rate. On strongly convex functions, it has a linear convergence rate, meaning that each update reduces the expected value of the difference $F(x) - F(x^*)$ by a $(1 - \alpha/n)$ multiplicative factor, for some constant $\alpha > 0$, where $x^*$ denotes an optimum of the function; and on convex functions it will achieve a $\Theta(1/T)$ convergence rate. To achieve linear speedup means that the same convergence rate holds, up to constant factors (i.e. the $\alpha$ might be reduced by a constant but no more, and likewise for the constant in the $\Theta(1/T)$).

The convex optimization problem we consider is to find an (approximate) minimum point of a composite convex function $F : \mathbb{R}^n \to \mathbb{R}$ as specified above. Such functions occur in many data analysis and machine learning problems, such as linear regression (e.g., the Lasso approach to regularized least squares [9]) where $\Psi_k(x_k) = |x_k|$, logistic regression [6], ridge regression [7] where $\Psi_k(x_k)$ is a quadratic function, and Support Vector Machines [2] where $\Psi_k(x_k)$ is often a quadratic function or a hinge loss (essentially, $\max\{0, x_k\}$).

Liu et al. [4] gave the first bound on the parallel performance of asynchronous stochastic coordinate descent on composite convex functions, showing linear speedup when $q = O(\sqrt{nL_{\max}}/L_{\text{res}})$ assuming a consistent read model, where $L_{\text{res}}$ is another Lipschitz parameter defined in Section 2. In the consistent read model all the coordinates a processor reads when performing a single update on one coordinate may have some delay, but they must appear simultaneously at some moment. To make this more precise, we view the updates as committing at integer times $t = 1, 2, \ldots$, and we will write $x^t$ to be the value of $x$ after the update at time $t$. The consistent read model requires the vector of $x$ values used by the update at time $t$ to be of the form $x^{t-c}$ for some $c \geq 1$. Allowing inconsistent reads means that the $x$ values used by the update at time $t$ can be any collection of the form $(x_1^{t-c_1}, \ldots, x_n^{t-c_n})$, where each $c_j \geq 1$ and the $c_j$’s can be distinct.
Note that the bound on $q$ is ensuring the asynchrony is bounded, and so we call it $q$-bounded asynchrony. Some requirement of this sort is unavoidable, otherwise there could be updates of arbitrarily long duration, which, when they commit, could undo an arbitrary amount of progress.

We note that $L_{\text{res}} = L_{\text{res}}$ for the function family $f$ we will be analyzing in this paper. In fact, as we shall see, $L_{\text{res}}$ is equal to $L_{\text{res}}$ on all functions $F$ whose smooth part has the form $x^T A x + b \cdot x + c$, where $A$ is an $n \times n$ matrix, and $b$ is an $n$-vector.

Consistent reads create a substantial constraint on the asynchrony, and so subsequent works sought to avoid this assumption. Liu and Wright \[3\] did so, achieving linear speedup for $q = O(n^{1/4} \sqrt{L_{\text{max}}}/\sqrt{L_{\text{res}}})$, i.e. the square root of the previous bound. There remained several constraints on the possible asynchrony, in addition to the $q$-bounded asynchrony, as pointed out by Mania et al. \[5\] and subsequently by Sun et al. \[8\]. The latter works also gave analyses removing some or all of these constraints, but at the cost of reducing the bound on $q$. Finally, Cheung et al. \[1\] gave an analysis achieving linear speedup for $q = O(\sqrt{n}L_{\text{max}}/L_{\text{res}})$, with the only constraint being the $q$-bounded asynchrony.

In this paper we show the bound in \[1\] is tight for essentially all possible values of $L_{\text{max}}$, $L_{\text{res}}$, and $L_{\text{res}}$ for the function family \[3\].

2 Notation

We let $x^*$ denote a minimum point of $F$ and $X^*$ denote the set of all minimum points of $F$.

**Definition 1.** The function $f$ is $L$-Lipschitz-smooth if for any $x, \Delta x \in \mathbb{R}^n$, $\|\nabla f(x + \Delta x) - \nabla f(x)\| \leq L \cdot \|\Delta x\|$. For any coordinates $j, k$, the function $f$ is $L_{jk}$-Lipschitz-smooth if for any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$, $|\nabla_{k} f(x + r \cdot e_{j}) - \nabla_{k} f(x)| \leq L_{jk} \cdot |r|$; it is $L_{\text{res}}$-Lipschitz-smooth if $\|\nabla f(x + r \cdot e_{j}) - \nabla f(x)\| \leq L_{\text{res}} \cdot |r|$. Finally, $L_{\text{max}} := \max_{j} L_{jj}$ and $L_{\text{res}} := \max_{k} \left( \sum_{j=1}^{n} (L_{kj})^2 \right)^{1/2}$.

The difference between $L_{\text{res}}$ and $L_{\text{res}}$ In general, $L_{\text{res}} \geq L_{\text{res}}$. $L_{\text{res}} = L_{\text{res}}$ when the rates of change of the gradient are constant, as for example in quadratic functions such as $x^T A x + b x + c$. We refer the reader to \[1\] for a discussion of why $L_{\text{res}}$ is needed in general for the analysis in \[1\].

**The update rule** Recall that in a standard coordinate descent, be it sequential or parallel and synchronous, the update rule, applied to coordinate $j$, first computes the accurate gradient $g_j := \nabla_j f(x^{t-1})$, and then performs the update given below.

$$x^t_j \leftarrow x^{t-1}_j + \arg \min_d \{ g_j \cdot d + \Gamma d^2/2 + \Psi_j(x^{t-1}_j + d) - \Psi_j(x^{t-1}_j) \} \equiv x^{t-1}_j + \tilde{d}_j(g_j, x^{t-1}_j)$$

and $\forall k \neq j$, $x^t_k \leftarrow x^{t-1}_k$, where $\Gamma \geq L_{\text{max}}$ is a parameter controlling the step size.

This update rule is motivated by the following reasoning. Recall that $f(x + d \cdot e_j) \leq f(x) + \langle \nabla f(x), d \cdot e_j \rangle + \frac{1}{2} L_{jj} d^2$, and so $F(x + d \cdot e_j) \leq f(x) + g_j \cdot d + \frac{1}{2} \Gamma d^2 + \Psi_j(x^{t-1}_j + d) - \Psi_j(x^{t-1}_j)$. Consequently, minimizing the expression on the RHS gives the smallest possible bound on $F$ based on this calculation.

However, in an asynchronous environment, an updating core (or processor) might retrieve outdated information $\hat{x}$ instead of $x^{t-1}$, so the gradient the core computes will be $\hat{g}_j \equiv \hat{g}_j := \nabla_j f(\hat{x})$, instead of the accurate value $\nabla_j f(x^{t-1})$. The ensuing update rule, which is naturally motivated by
its synchronous counterpart, is
\[ x_j^t \leftarrow x_j^{t-1} + \hat{d}_j(g_j, x_j^{t-1}) = x_j^{t-1} + \Delta x_j^t \quad \text{and} \quad \forall k \neq j, \ x_k^t \leftarrow x_k^{t-1}. \] (2)

We let \( k_t \) denote the coordinate being updated at time \( t \).

**Scaling** Sometimes, rather than have one value of \( \Gamma \geq \Lmax \) for all coordinates, a value \( \Gamma_j \geq \Ljj \) is used for the \( j \)-th coordinate, typically with \( \Gamma_j / \Ljj \) being a fixed value for all \( j \). Alternatively and equivalently, one could rescale the coordinates so that each \( \Ljj = 1 \). The resulting \( \Lmax = 1 \) also, and we would now be using a common value of \( \Gamma \) for the updates of all the coordinates. Note that this may affect the value \( \Lres \).

### 2.1 The Stochastic Asynchronous Coordinate Descent (SACD) Algorithm

The coordinate descent process starts at an initial point \( x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \). Multiple cores then iteratively update the coordinate values, and for our upper bound analysis we assume that at each time, there is exactly one coordinate value being updated, which we can do, as we are choosing the asynchronous schedule.

**Algorithm 1: SACD Algorithm.**

**INPUT:** The initial point \( x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \).

**Multiple processors use a shared memory. Each core iteratively repeats the following six-step procedure, with no global coordination among them:**

1. Choose a coordinate \( j \in [n] \) uniformly at random.
2. Retrieve coordinate values \( \tilde{x} \) from the shared memory.
3. Compute the gradient \( \nabla_j f(\tilde{x}) \).
4. Request a write lock on the memory that stores the value of the \( j \)-th coordinate.
5. Retrieve the most updated \( j \)-th coordinate value, then update it using rule (2).
6. Release the lock acquired in Step 4.

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Even if the core had retrieved the value of the \( j \)-th coordinate from the shared memory in Step 2, the core needs to retrieve it again here, because it needs the most updated value when applying update rule (2).

In the event that the function \( F \) has no non-smooth part, i.e. \( F(x) = f(x) \) for all \( x \), there is no need for a lock, as it suffices to make an atomic add of the value computed in Step 3 to the current value of the coordinate.

Note also that delays due to locking are rare; they only occur when two updates to the same coordinate are overlapping and further they seek to perform Steps 4–6 at more or less the same time. For any one update, a 2-way conflict will occur with probability at most \( q/n \leq 1/\sqrt{n} \). Similarly, a \( c \)-way conflict will occur with probability at most \( q^c/n^c \leq 1/n^{c/2} \).

In the sequel we use the term “with high probability” to mean that the event it is referring to happens with probability at least \( 1 - \frac{1}{e} n^{-\varepsilon} \); \( \lg z = \log_2 z \), while \( \ln z = \log_e z \), where \( e = 2.71828 \cdot \cdot \cdot \).

### 3 The Result

In order to state what we mean when we claim that our asynchronous coordinate descent does not converge, we introduce the notion of stalling. We let \( b_0, \nu > 0 \) be suitable parameters, which we
specify later. We require \( \epsilon \leq \frac{1}{2} \) and \( \Gamma = 2 \). In fact, analogous results can be shown for all \( \Gamma \geq 1 \), but it requires some changes to the constants, and is left to the interested reader.

**Definition 2.** An asynchronous schedule of the coordinate descent is said to stall for \( t \) updates if throughout these \( t \) steps, for all but \( b_0 \) coordinates which we call special coordinates, for \( \alpha := \frac{1}{\Gamma} \) and \( \nu := \frac{3\alpha}{1-3\epsilon} \leq \frac{1}{\Gamma^2} \),

- every non-special coordinate with initial value 1 has a value in either the range \([1 - \alpha, 1 - \alpha + \nu] \) or the range \([1 - \nu, 1] \);
- every non-special coordinate with initial value -1 has a value in either the range \([-1, -1 + \nu] \) or the range \([- (1 - \alpha) - \nu, -(1 - \alpha)] \);
- each special coordinate with original value 1 has a value in the range \((\epsilon, 1 - \alpha] \); and
- each special coordinate with original value -1 has a value in the range \([- (1 - \alpha), -\epsilon] \).

We will show:

**Theorem 1.** For each \( \epsilon \geq 1 \), if \( \frac{96}{\epsilon(1-\epsilon)} + 2145 + 768c \ln n \leq q \leq \sqrt{n} \) and \( \epsilon \leq \frac{1}{6} \), i.e. for \( L_{res} = L_{res} \in [\Theta(1), \Theta(\sqrt{n})] \), then with probability at least \( 1 - 1/n^c \), there is an asynchronous schedule for which the coordinate descent stalls for at least \( n^c \) updates when applied to the function \( f \) in \( \Omega \), where \( b_0 = [1 + \ln 5 + c \ln n] \).

**The update schedule** For our construction, we choose the start and commit times of the updates to be in the same order. The updates will start at times \( t = 1, 2, \ldots \) and will conclude at times \( t + q + 1, t + q + 2, \ldots \). The update that starts at time \( t \) therefore overlaps updates that begin at times \( t - q, t - q + 1, \ldots, t - 1, t + 1, t + 2, \ldots, t + q \), i.e. with \( 2q \) other updates. We will group the updates in phases of \( q/2 \) consecutive updates. The updates in phase \( i \), i.e. with start times \( \frac{q}{2}(i - 1) + \{1, 2, \ldots, q/2\} \) will only read values from updates prior to time \( \frac{q}{2}(i - 1) \), i.e. from earlier phases, and further the only potentially available updates they might not read are those in the preceding phase, i.e. those with start times \( \frac{q}{2}(i - 2) + \{1, 2, \ldots, q/2\} \). The choice of what is read is done separately for each update, which we can do in the inconsistent read model but not in the consistent read model. Note that we have reset \( q \) to \( 2q \) (actually, to \( 2(q - 1) \)) in the definition of the number of other updates that can overlap any single update.

### 4 The Analysis

Recall that we are considering the following function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \):

\[
    f(x) = \frac{1 - \epsilon}{2} \cdot \sum_{i=1}^{n} (x_i)^2 + \frac{\epsilon}{2} \cdot \left( \sum_{i=1}^{n} x_i \right)^2.
\]

Note that \( f \) can be rewritten as \( \frac{1}{\lambda} x^T [(1 - \epsilon) \cdot I + \epsilon \cdot J] x \), where \( I \) is the identity matrix and \( J \) is the all-one matrix. As is well-known, both \( I \) and \( J \) are positive semi-definite, and thus \( f \) is a convex function. Clearly, \( f \) has a unique minimizer: the origin.

Observe that

\[
    \nabla_j f(x) = (1 - \epsilon) \cdot x_j + \epsilon \cdot \sum_{i=1}^{n} x_i.
\]
Consequently, $L_{\text{max}} = 1$, $L_{\text{res}} = \sqrt{1 + (n - 1)\epsilon^2} = \Theta(\epsilon \sqrt{n})$, and $L = 1 + (n - 1)\epsilon$. We set $\Gamma = 2$.

Since there is no univariate component, we may assume the updates are made by atomic additions.

We begin by describing our construction, but leave the choice of the values of several parameters to the analysis which follows.

There will be four invariants we want to maintain with high probability. The arguments for maintaining the first two invariants are direct. For the others, we will turn to a related biased random walk to demonstrate that they too can be maintained with high probability. The precise values of the parameters $b_0, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2$ we introduce below will be determined later. The parameters $b_0 - b_5$ will be providing bounds on certain types of events, and these bounds will hold with high probability. We let $\mathcal{E}_i$ denote the event that the bound $b_i$ holds, and $\mathcal{E}$ denote the event $\bigcap_{0 \leq i \leq 5} \mathcal{E}_i$.

1. For simplicity, we assume $n$ and $q$ are integer multiples of 4. We analyze the updates in phases of $q/2$ consecutive updates. The starting point $x^0$ for the computation has exactly half of its coordinate values equal to +1, and the other half equal to −1. Let $J_1$ denote the set of coordinates with initial value +1, and $J_{-1}$ denote the set comprising the remaining coordinates.

2. The first $q/2$ updates all read the most recent update to their coordinate based on the start time ordering, and for the other coordinates, read their original value, i.e., the value in $x^0$. Let $R$ denote the set of coordinates which are updated during this period.

We define a set of special coordinates, denoted by $S$, as follows:

- Coordinates which are updated two or more times during this period are included in $S$. Let $S'$ denote the set comprising these coordinates.
- We add to $S$ a minimal set of coordinates in $R \setminus S'$, such that after these additions to $S$, $|S \cap J_1| = |S \cap J_{-1}|$.

We will show that with high probability the number of special coordinates is bounded by $b_0$ (see Corollary 3); we call this event $\mathcal{E}_0$.

We will maintain the following invariant:

**Invariant 1.** Conditioned on $\mathcal{E}$, the value of a special coordinate with original value 1 will always lie in the range $(\epsilon, 1 - \alpha]$, while the value of a special coordinate with original value −1 will always lie in the range $[-(1 - \alpha), -\epsilon]$.

Our update rule ensures that Invariant 1 holds for every updated coordinate after each of the first $q/2$ updates. Subsequent updates to special coordinates will leave their values essentially unchanged and will maintain Invariant 1.

3. Conditioned on $\mathcal{E}$, the value of any non-special coordinate $j$ will be maintained at around one of the two values: $x^0_j$ or $(1 - \alpha) \cdot x^0_j$. More precisely, if $x^0_j = 1$, their values will always be of the form $1 + \delta_j$ or $(1 - \alpha) + \delta_j$, where $\delta_j \in [2\nu, \nu]$, and symmetrically, if $x^0_j = -1$, their values will be of the form $-1 + \delta_j$ or $-(1 - \alpha) + \delta_j$. (In fact, the value of $\delta_j$ may depend on the time $t$ of the update; however, we suppress this notational detail.) There are two refinements to this rule:

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$L$ equals the maximum eigenvalue of the Hessian for $f$; the eigenvector corresponding to the maximum eigenvalue is the all-one vector.
coordinates that have never been updated, and coordinates that have been updated just once during the first \( q/2 \) updates will have \( \delta_j = 0 \), i.e., their values are either exactly \( x_j^0 \) or \((1 - \alpha) \cdot x_j^0\). We call the values \( \pm(1 - \alpha) \), \( \pm 1 \) the *ideal* values of the coordinates.

A non-special coordinate \( j \in J_1 \) with current value \((1 - \alpha) + \delta_j \) is called an N-coordinate and otherwise it is a P-coordinate. A non-special coordinate \( j \in J_{-1} \) with current value \(-(1 - \alpha) - \delta_j \) is called a P-coordinate and otherwise it is an N-coordinate. \( m_N \) and \( m_P \) will denote the numbers of N-coordinates and P-coordinates respectively, and \( N \) and \( P \) will denote the sets of N-coordinates and P-coordinates respectively. We will show that with high probability, after the first \( k \) updates, \( |m_N - m_P| \leq b_1 \) (see Lemma 5). We call this event \( \mathcal{E}_1 \).

Let \( C \) denote the set of all \( n \) coordinates. We create a minimal sized set \( B \) so that after the first \( q/2 \) updates, \( C - (S \cup B) \) has equal numbers of N-coordinates and P-coordinates. We let \( b_1 \) denote the size of \( B \). WLOG, suppose that there are more N-coordinates than P-coordinates in \( C \setminus S \); the other case will be symmetric. Then note that \(|N \cap J_{-1}| - |P \cap J_1| = |N \cap J_1| - |P \cap J_{-1}|\). Thus we can put equal numbers of N-coordinates from \( J_1 \) and \( J_{-1} \) into \( B \). We call the set \( A := C \setminus (S \cup B) \) the set of available coordinates. Note that the set \( B \) (and hence also \( A \)) will be updated over time; in particular, we will keep removing coordinates from \( B \), but never add a coordinate to it. But for the first phase, every coordinate is viewed as available.

Subsequently, whenever we update an N-coordinate in \( B \), if it becomes a P-coordinate, then we remove the updated coordinate and an N-coordinate of the opposite ideal value from \( B \), thus restoring them to the set of available coordinates.

As in the first phase, we will not read any of the updates from the current phase. We coin the name base-line gradient for the gradient that results if we read the most recent updates from the previous phases. We will further control the computed gradients by not reading some of the updated values from the preceding phase for the available coordinates. More specifically, we may read either the first update to an available coordinate in the previous phase, or only the most recent value from two phases back. Not reading (“ignoring”) the previous phase update for one such coordinate can change the value of the computed gradient by at most \( \pm \epsilon(\alpha + 2\nu) \), and hence change the value of the coordinate being updated by at most \( \pm \frac{\epsilon(\alpha + 2\nu)}{1 - (1 - 2\epsilon)} = \pm \frac{\nu}{\alpha} \), as \( \nu = \frac{3\alpha}{1(1 - 3\epsilon)} \).

Then, assuming we can ignore sufficiently many coordinates, we can ensure that each non-special coordinate always takes on a desired value (one of \( \pm((1 - \alpha) + \delta), \pm(1 + \delta) \), where \( \delta \in \pm(\frac{\nu}{\alpha}, \nu) \)).

With high probability, we will maintain the following bounds on the sum of the \( \delta_j \) terms.

**Invariant 2.** \[ \sum_{j \in A} \delta_j \leq b_2 \nu. \quad b_2 \text{ will be specified later.} \]

We let \( \mathcal{E}_2 \) denote the event that the bound in Invariant 2 holds for all \( t \leq n^\epsilon \).

4. Updates at times \( t > q/2 \):

- Let \( a_p \) denote the number of coordinates in \( A \cap P \) and \( a_N \) the number of coordinates in \( A \cap N \). We want to maintain the following invariant:

**Invariant 3.** \[ |a_p - a_N| \leq b_3. \quad b_3 \text{ will be specified later.} \]

Again, we let \( \mathcal{E}_3 \) denote the event that the bound in Invariant 3 holds for all \( t \leq n^\epsilon \).

For the second phase of updates (updates \( q/2 + 1 \) to \( q \)) Invariants 1–3 ensure that for each coordinate \( j \) (which can be special or not), the base-line gradient along coordinate \( j \) before an update at time \( t \) always lies within

\[ (1 - \epsilon) \cdot x_j^{t-1} \pm \frac{1}{2} \epsilon \alpha (b_0 + b_1). \quad (4) \]
In the second phase, our asynchrony model allows a core to ignore up to \((1 - \alpha)\). This yields a contribution of \(\alpha\epsilon\) to the gradient for this pair of coordinates. Next, the special coordinates all lie in the range \([-\frac{1}{2}(1 + \epsilon), -\epsilon]\), or \([\epsilon, \frac{1}{2}(1 + \epsilon)]\), and as \(|S \cap J_1| = |S \cap J_{-1}|\), they too can be paired, so each pair makes a contribution of at most \(\frac{1}{2}(1 - \epsilon)\epsilon = \alpha\epsilon\) to the gradient.

For later phases of updates, the base-line gradient lies within

\[
(1 - \epsilon) \cdot x_j^{t-1} \pm \epsilon \left[ \frac{1}{2} \alpha(b_0 + b_1 + b_3) + b_2 \nu \right].
\]  

(5)

To see this we consider two further terms. First, by Invariant 2, the difference between the sum of the ideal values of the coordinate and their actual value is at most \(2b_2\nu\), contributing a total of \(2b_2\nu\) to the gradient. Now, we need only consider ideal values. Again, suppose WLOG that the coordinates in \(B\) are from \(P\). For the terms involving \(b_0\) and \(b_1\) we argue as in the previous paragraph. The argument used for the \(b_1\) term also applies to the coordinates contributing to the difference \(|a_p - a_N|\), yielding a contribution of at most \(\frac{1}{2} \alpha b_3\).

- We will ensure that each update changes the value of a coordinate by a value in \([-\alpha - 2\nu, -\alpha + 2\nu] \cup [-\nu, \nu] \cup [\alpha - 2\nu, \alpha + 2\nu]\). This will allow us to keep the coordinate in the range specified in Definition 2. An update is left-moving (resp. right-moving) if it reduces (resp. raises) the value of a non-special coordinate by roughly \(\alpha\).

We will maintain the following invariant.

**Invariant 4.** Conditioned on \(E\), in Phase 1 there are at least \(\lambda_1 q\) left-moving updates, and another \(\lambda_1 q\) right-moving updates, each of them to a coordinate which is updated exactly once in that phase. And in subsequent phases there are at least \(\lambda_2 q\) left-moving updates and at least \(\lambda_2 q\) right-moving updates to available coordinates, where these are the first updates to these coordinates in this phase.

As an adversary, we will want to be able to choose the update to be either left-moving, right-moving, or neither (of course, for a coordinate in \(P\) a right-moving update is never an option, and analogously for a coordinate in \(N\)). Later, we will specify how to make this choice, so as to maintain Invariants 3 and 4. For a special coordinate, we always choose the computed gradient to be in the range \(\pm \nu\) so that Invariant 1 is maintained.

- In the second phase, our asynchrony model allows a core to ignore up to \(\lambda_1 q\) of the left-moving updates described in Invariant 3. By ignoring all of these updates, the gradient computed by the core will be larger than the base-line gradient by at least \(\lambda_1 q \cdot \epsilon \cdot (\alpha - 2\nu) \geq \frac{1}{2} \lambda_1 q \epsilon \alpha\), if \(\alpha \geq 4\nu\), i.e., if \(\epsilon \leq \Gamma/12 = 1/6\). Symmetrically, by ignoring all right-moving updates, the computed gradient can be smaller than the base-line gradient by at least \(\frac{1}{2} \lambda_1 q \epsilon \alpha\).

Suppose non-special coordinate \(j\) is being updated. If its value is \(1 + \delta_j\), the desired update is to either a value \(1 + \delta_j'\) or to \(-1 + \alpha + \delta_j' = \frac{1}{2}(1 + \epsilon) + \delta_j'\), where \(\delta_j, \delta_j' \in [\pm \frac{3}{2} \nu, \nu]\). Thus, we need

\[
\frac{1}{2} \lambda_1 q \epsilon \alpha \geq (1 - \epsilon)(1 + \delta_j) + \frac{1}{2} \epsilon \alpha (b_0 + b_1).
\]  

(6)

While if its value is \(-1 + \alpha + \delta_j = \frac{1}{2}(1 + \epsilon) + \delta_j\), the desired updated value is either \(-1 + \alpha + \delta_j'\), or \(1 + \delta_j'\). Then we need

\[
\frac{1}{2} \lambda_1 q \epsilon \alpha \geq (1 - \epsilon) \left[ \frac{1}{2}(1 + \epsilon) - \delta_j \right] + \Gamma \left( \frac{1}{2}(1 - \epsilon) + \delta_j + \delta_j' \right) + \frac{1}{2} \epsilon \alpha (b_0 + b_1).
\]  

(7)
The same bounds apply for negative valued non-special coordinates. Finally, for special coordinates, (7) suffices.

As \( \Gamma = 2 \), the bound
\[
\frac{1}{2} \lambda_1 q \epsilon \alpha \geq \frac{3}{2} + 3 \nu + \frac{1}{2} \epsilon \alpha (b_0 + b_1)
\] (8)
covers both (6) and (7).

We repeat this analysis for the later phases. For a coordinate with ideal value \( \pm 1 \), we obtain
the constraint
\[
\frac{1}{2} \lambda_2 q \epsilon \alpha \geq (1 - \epsilon)(1 + \delta_j) + \epsilon \left( \frac{1}{2} \alpha (b_0 + b_1 + b_3) + b_2 \nu \right).
\] (9)

And for a coordinate with ideal value \( \pm (1 - \alpha) \) we obtain
\[
\frac{1}{2} \lambda_2 q \epsilon \alpha \geq (1 - \epsilon) \left( \frac{1}{2} (1 + \epsilon) - \delta_j \right) + \Gamma \left( \frac{1}{2} (1 - \epsilon) + \delta_j' + \delta_j \right) + \epsilon \left( \frac{1}{2} \alpha (b_0 + b_1 + b_3) + b_2 \nu \right).
\] (10)

Again, for special coordinates, (10) suffices.

The bound
\[
\frac{1}{2} \lambda_2 q \epsilon \alpha \geq \frac{3}{2} + 3 \nu + \epsilon \left( \frac{1}{2} \alpha (b_0 + b_1 + b_3) + b_2 \nu \right)
\] (11)
covers both (9) and (10).

Before delving into the derivation of precise bounds on the parameters, we point out that we are going to show the following:

- \( \lambda_1, \lambda_2 \) are \( \Omega(1) \); and
- the term \( (b_0 + b_1) \) in (8) and the term \( \left( \frac{1}{2} \alpha (b_0 + b_1 + b_3) + b_2 \nu \right) \) in (11) can be bounded by \( o(q) \) or by a small positive constant times \( q \).

With these in hand, it is not hard to see that \( q = \Omega(1/\epsilon) \) suffices to satisfy both (8) and (11), and completes the construction.

### 4.1 Determining \( b_0 \), a High Probability Bound on the Number of Special Coordinates

We say a character in a string \( \sigma \) is \textit{reappearing} if it has already occurred to the left of its current location. For example in the string \( abaa \), the second and third \( a \)'s are reappearing characters.

We begin with the following lemma.

\textbf{Lemma 2.} Let \( \Sigma \) be an \( n \)-character alphabet. Let \( \sigma \) be a random string of length \( q' \leq \sqrt{n} \), where each character of \( \sigma \) is chosen uniformly at random from \( \Sigma \). The probability that there are \( r \) or more instances of reappearing characters in \( \sigma \) is at most \( \sum_{b=r}^{q'-1} \left( \frac{e(q')^2}{nb} \right)^b \). In particular, if \( q' \leq \sqrt{n} \), \( n \geq \exp(2e) \), and \( r \geq 1 + c_1 \ln n \), then for any number \( c_1 \geq 1 \), the probability is at most \( n^{-c_1} \). With the same probability, the number of repeated characters is at most \( 2r \).
Proof: For a fixed integer \(1 \leq b \leq q' - 1\), we upper bound the number of strings with exactly \(b\) reappearing characters (where we are counting every instance of each reappearing character). There are at most \(\binom{q'}{b}\) combinations of reappearing characters. After fixing the combination of reappearing characters, we choose the values of the non-reappearing characters; there are at most \(n^{q' - b}\) choices. Then we choose the values of the reappearing characters. Note that there are at most \(q'\) choices for each reappearing character, so in total there are at most \((q')^b\) choices. Thus, the number of such strings is at most
\[
\binom{q'}{b} \cdot n^{q' - b} \cdot (q')^b \leq n^{q'} \left( \frac{eq'}{b} \right)^b \leq n^{q'} \cdot \left( \frac{e(q')^2}{nb} \right)^b.
\]
we conclude that the desired probability is at most
\[
\sum_{b=r}^{q'-1} \left( \frac{e(q')^2}{nb} \right)^b.
\]
When \(r \geq 1 + c_1 \ln n \geq 2e\) and \(q' \leq \sqrt{n}, \frac{e(q')^2}{n} < \frac{1}{2}\). Thus, the above summation is at most \(\sum_{b=1+c_1 \ln n}^{\infty} 2^{-b}\), which is at most \(n^{-c_1}\). In order for a character to be repeated, it must reappear; hence the number of repeated characters is at most twice the number of reappearing characters.

Now we define a string to be given by the names of the coordinates that are chosen to be updated. We refer to these coordinates as reappearing and repeated, corresponding to the usage in Lemma 2.

Corollary 3. Let \(b_0 = \lceil 1 + \ln 6 + c \ln n \rceil\). With high probability, i.e. with probability at least \(1 - \frac{1}{6}n^{-c}\), in the first phase there are at most \(b_0\) reappearing coordinates, and at most \(2b_0\) repeated coordinates.

Proof: By Lemma 2 (with \(q' = q/2\) and \(r = \lceil 1 + \ln 6 + c \ln n \rceil\)), with probability at least \(1 - \frac{1}{6}n^{-c}\), among the first \(q/2\) updates, the number of reappearing coordinate instances during this period is at most \(r = \lceil 1 + \ln 6 + c \ln n \rceil\). Note that the probability is strictly smaller than \(1 - \frac{1}{6}n^{-c}\) if \(r > 1 + \ln 6 + c \ln n \triangleq b_4\).

Let \(\mathcal{E}_0\) be the event specified in Corollary 3.

For a given phase \(\mathcal{P}\), let \(b_p\) be number of non-first updates to available coordinates during phase \(\mathcal{P}\). By the same argument as in Corollary 3 we obtain:

Corollary 4. With high probability, over all phase \(\mathcal{P}\) other than the first phase, \(b_p \leq 1 + \ln 6 + 2c \ln n\).

\(\mathcal{E}_4\) denotes the event that the bound on \(b_p\) in Corollary 4 holds for all Phases \(\mathcal{P}\) other than the first phase.

4.2 \(\mathcal{E}_1\) Occurs with High Probability

We set \(b_1 = \sqrt{3q(c \ln n + \ln 6)}\).

Lemma 5. With probability at least \(1 - \frac{1}{6}n^{-c}\), \(\mathcal{E}_1\) occurs.

Proof: Suppose \(s\) of the updates are not the first update to a coordinate. By a Chernoff bound, with probability at least \(1 - \frac{1}{6}n^{-c}\), the number of first updates to coordinates in \(J_1\) lies within the interval \(\left( \frac{q}{2} - \frac{s}{2} \right) \pm \frac{1}{2} \sqrt{3q(c \ln n + \ln 6)}\). Thus, with probability at least \(1 - \frac{1}{6}n^{-c}\), \(b_1\) is at most \(\sqrt{3q(c \ln n + \ln 6)}\).
4.3 A Biased Random Walk Showing $\varepsilon_3$ Occurs With High Probability

Next, we describe the rule for updating non-special coordinates after the first $\frac{t}{2}$ updates. We order such updates in chronological order (i.e., we order the updates by ignoring the updates to special coordinates), and we relabel the times of such updates, starting from $t = 1$. Recall that each non-special coordinate can assume values within small neighbourhoods around two values.

(A) At odd time steps, when the value of the chosen coordinate was in one neighbourhood before the update, the value is changed to a value in the other neighbourhood.

(B) At even time steps,

- if $m_p = m_N$, the value of the chosen coordinate is kept in the same neighbourhood.
- if $m_p > m_N$, if the chosen coordinate is an N-coordinate its value is kept in the same neighbourhood; otherwise, its value is changed to a value in the other neighbourhood.
- if $m_p < m_N$, we proceed as above but with the roles of P and N swapped.

The set-up has two targets in mind:

- First, we want the difference between $m_p$ and $m_N$ to remain small for a long time with high probability. Rule (B) is designed to achieve this.
- Second, we want that for a large $t$, among the moves $t + 1$ to $t + q/2$, at least $\lambda_2^q$ are left-moving, and another $\lambda_2^q$ are right-moving. As we will see, Rule (A) is designed to achieve this.

Let $X_t$ denote the value of $|m_p - m_N|$ at time $t$ (in the chronological ordering of updates to non-special coordinates). Initially, $X_0 = 0$. Observe that rules (A) and (B) imply:

- At odd time step $t$, if $X_{t-1} > 0$, then with probability at least $1/2$, $X_t = X_{t-1} - 1$, and otherwise, $X_t = X_{t-1} + 1$. If $X_{t-1} = 0$, then $X_t = 1$.
- At even time step $t$, if $X_{t-1} > 0$, then with probability at least $1/2$, $X_t = X_{t-1} - 1$, and otherwise $X_t = X_{t-1}$. If $X_{t-1} = 0$, then $X_t = 0$.

Since the above probabilities in the rule are not exactly $1/2$ analyzing it directly is difficult. Instead, we first analyze another random walk $\{Y_t\}$ in which the above probabilities are exactly $1/2$, and then use it to analyze $\{X_t\}$.

**Auxiliary Random Walk $\{Y_t\}$**. Let $p^t_c = \mathbb{P}[Y_{2t} = c]$. Note that $p^0_c = 1$, and for $c \geq 1$, $p^0_c = 0$. We have the following recurrence relations:

$$p^{2t+2}_0 = \frac{1}{2}p^{2t}_0 + \frac{1}{2}p^{2t}_1 + \frac{1}{4}p^{2t}_2$$

$$p^{2t+2}_1 = \frac{1}{2}p^{2t}_0 + \frac{1}{4}p^{2t}_1 + \frac{1}{4}p^{2t}_2 + \frac{1}{4}p^{2t}_3$$

$$p^{2t+2}_c = \frac{1}{4}p^{2t}_{c-1} + \frac{1}{4}p^{2t}_c + \frac{1}{4}p^{2t}_{c+1} + \frac{1}{4}p^{2t}_{c+2}, \quad c \geq 2.$$  

Then it is easy to prove by induction on $t$ that: for any $t \geq 0$, $p^{2t}_0 \geq p^{2t}_1$, and for any $c \geq 1$, $p^{2t}_c \geq 2 \cdot p^{2t}_{c+1}$. Thus, for any time $t \geq 0$ and $c \geq 1$, we have $p^{2t}_c \leq 1/2^c$, and consequently $\mathbb{P}[Y_{2t} > c] \leq 1/2^c$. Since $Y_{2t-1} \geq c+1$ implies that $Y_{2t} \geq c$, it follows that $\mathbb{P}[Y_{2t-1} \geq c+1] \leq 1/2^c$.  

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Back to Analyzing \( \{X_t\} \). Next, we derive similar tail bounds for \( \{X_t\} \) by using the auxiliary random walk. This is done by a “comparison technique”.

We consider two infinite sequences \( s^1 \) and \( s^2 \).

- Each term in \( s^1 \) is randomly and independently generated, with probability \( \frac{1}{4} \) of being \(+1\), with probability \( \frac{1}{4} \) of being \(0\), with probability \( \frac{1}{4} \) of being \(-1\), and with probability \( \frac{1}{4} \) being \(-2\).
- The generation of \( s^2 \) can be arbitrary; the only constraint is that every term must be non-positive.

Then we generate two sequences \( \{X_{2t}\}, \{Y_{2t}\} \) using the following rules. The first rule applies to both \( \{X_{2t}\} \) and \( \{Y_{2t}\} \); we use \( \{Z_{2t}\} \) to denote either of them.

- \( Z_{2t} = \max\{Z_{2(t-1)} + s^1_t, 0\} \).
- After applying the above rule to \( \{X_{2t}\} \), add \( s^2_t \) to \( X_{2t} \).

It is easy to check that the first rule is mimicking the random generation of \( \{Y_{2t}\} \). Observe that \( \{X_{2t}\} \) is actually a random walk biased towards the zero point. \( s^2 \) is used to mimic the marginal change due to the inexact probabilities when generating \( \{X_{2t}\} \); this is non-positive for sure.

We claim that for any fixed sequences \( s^1, s^2 \) and for all \( t, X_{2t} \leq Y_{2t} \) with certainty. First, note that this is true for \( t = 0 \). An easy induction then verifies the claim for all \( t \). For the first rule first computes \( Z_{2(t-1)} + s^1_t \), which maintains the relative order, as does taking a minimum with 0, as does adding the non-positive \( s^2_t \) to \( X_{2t} \).

Consequently, the tail bounds for \( \{Y_t\} \) also hold for \( \{X_t\} \). By a simple union bound, we have the following lemma:

**Lemma 6.** For any \( c \geq 1 \), with probability at least \( 1 - \frac{1}{6}n^{-c} \), for all \( t \leq n^c \), \( X_t \leq 1 + \log 6 + 2c \log n \iff b_3 \).

In other words, we have shown that Invariant \( \mathbb{E}_3 \) holds with probability at least \( 1 - \frac{1}{6}n^{-c} \) if \( b_3 = 1 + \log 6 + 2c \log n \). Let \( \mathcal{E}_3 \) be this event.

### 4.4 An Analogous Biased Random Walk Showing \( \mathcal{E}_2 \) Occurs With High Probability

We will bound \( \Delta = \sum_{j \in A \cap D} \delta_j \). Here, \( D \) is a subset of \( A \) and includes all available coordinates whose values are not the ideal values. Remember, at the end of the first phase, all coordinates that are not updated or updated just once will have value equal to one of \( \pm 1 \) or \( \pm (1 - \alpha) \); these values are the ideal values.

We want to keep \( |\Delta| \) small. Our approach is as follows.

After the first phase, if we update an available coordinate (a coordinate in set \( A \)), we will choose its new value to be in \( \pm 1 + \left[ \frac{2}{3} \nu, \nu \right] \) or in \( \pm (1 - \alpha) \pm \left[ \frac{2}{3} \nu, \nu \right] \). Therefore, once one coordinate \( i \) is in \( D \), it will remain in \( D \) forever.

Let \( D_+ \) denote the available coordinates in \( \pm 1 + \left[ \frac{2}{3} \nu, \nu \right] \) or \( \pm (1 - \alpha) + \left[ \frac{2}{3} \nu, \nu \right] \) and \( D_- \) denote the available coordinates in \( \pm 1 - \left[ \frac{2}{3} \nu, \nu \right] \) or \( \pm (1 - \alpha) - \left[ \frac{2}{3} \nu, \nu \right] \).

If \( \Delta \leq 0 \), then we make the following updates.

1. If the coordinate to be updated is in \( D_+ \), then we keep it in \( D_+ \);
2. if the coordinate to be updated is in $D_-$, then we move it to $D_+$;
3. if the coordinate to be updated is not in $D$ but in $A$, then we move it to $D_+$.

Let’s calculate the probability that each of the above updates occurs. If $\Delta \leq 0$, then $\sum_{i \in D_+} \delta_i + \sum_{i \in D_-} \delta_i \leq 0$. This implies $\frac{2}{3}|D_+| - |D_-| \leq 0$, and we obtain $|D_-| \geq \frac{2}{3}|D_+|$. Then, (1) happens with probability at most $\frac{3 |D_+|}{3 |A|}$; (2) happens with probability at least $\frac{2 |D_+|}{3 |A|}$; and (3) happens with probability $\frac{|A| - |D_+|}{|A|}$.

Next, we will see how these updates change the value $\Delta$.

- Update (1) will increase $\Delta$ by a value in $\left[\frac{2}{3} \nu, \frac{4}{3} \nu\right]$;
- update (2) will increase $\Delta$ by a value in $\left[\frac{4}{3} \nu, 2 \nu\right]$;
- update (3) will increase $\Delta$ by a value in $\left[\frac{2}{3} \nu, \nu\right]$.

The case that $\Delta \geq 0$ is similar.

As in the previous section, we also use discretization. We consider the following process. Suppose that $R$ is an integer valued random variable. WLOG, suppose that $R \geq 6$. Let $a^t$ be a series such that $0 \leq a^t \leq 1$ for any $t$. At time $t$, with probability $\frac{2}{3}a^t$, the value of $R$ is increased by 1; with probability $\frac{2}{3}a^t$, the value of $R$ is reduced by 4; and with probability $(1 - a^t)$, the value of $R$ is reduced by 2. We will show:

**Lemma 7.** $p_i^t \leq \left(\frac{2}{3}\right)^{i-6}$ for all $i > 6$ and all $t$.

**Proof.** The following recurrence specifies $p_i^t$:

$$p_{i+1}^t = \frac{2}{5} a^t p_{i+4}^t + \frac{3}{5} a^t p_{i-1}^t + (1 - a^t) p_{i+2}^t \quad \text{for } i > 6.$$  

Necessarily $p_0^t \leq 1$, verifying the claim for this case. Also, $p_i^t = 0$ for $i > 6$, verifying the claim in this case too. The claim for the remaining cases is verified by induction, for

$$p_{i+1}^t \leq \left(\frac{2}{3}\right)^{i-6} \left(\frac{32}{405} a^t + \frac{9}{10} a^t + \frac{4}{9} (1 - a^t)\right) \leq \left(\frac{2}{3}\right)^{i-6}.$$  

\qed

Let $q_i^t$ denote the probability that $R$ has value at least $i$ at time $t$. Clearly $q_i^t = \sum_{j \geq i} p_j^t \leq 3 \left(\frac{2}{3}\right)^{i-6}$.

Let $r_i^t$ be the probability that $\Delta/(\frac{2}{3} \nu)$ has value at least $i$. Clearly, $r_i^t \leq q_i^t$, as can be justified by means of an argument using two sequences, as in Section 4.3. We have shown:

**Lemma 8.** For any $c \geq 1$, with probability at least $1 - \frac{1}{6} n^{-c}$, for all $t \leq n^c$, $\Delta \leq \left[2 + \frac{\ln 18 + 2c \ln n}{3 \ln 3/2}\right] \nu$.

**Proof.** By a union bound, the probability that $\Delta/(\frac{2}{3} \nu) \geq 6 + i$, at any time during the first $n^c$ updates, is at most $3n^c \left(\frac{2}{3}\right)^i$ and this is bounded by $\frac{1}{6} n^{-c}$ if $\left(\frac{2}{3}\right)^i \leq \frac{1}{18} n^{-2c}$, which yields the claimed bound. \qed

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In other words, we have shown that Invariant 2 holds with probability at least \(1 - \frac{1}{6}n^{-c}\) if \(b_2 = 2 + \frac{\ln 18 + 2c \ln n}{3 \ln 3/2}\). Recall that \(\mathcal{E}_2\) is this event.

4.5 Completing the Construction

**Lemma 9.** Suppose that \(q \geq 3(1 + c \ln n + \ln 6)\) and \(n \geq 8\). Conditioned on events \(\mathcal{E}_0\) and \(\mathcal{E}_1\) (see Corollary 3 and Lemma 5), with probability \(1 - \frac{1}{6}n^{-2c}\), in each phase except the first, the number of coordinates in \(B \cup S\) that are updated, counting repetitions, is at most \(1 + \lg 6 + 2c \lg n \triangleq b_5\).

**Proof.** Consider some phase \(\mathcal{P}\) other than the first phase. Suppose that \(b\) updates are made to coordinates in \(B \cup S\) during phase \(\mathcal{P}\). This occurs with probability

\[
\left(\frac{q/2}{b}\right) \left(\frac{b_0 + b_1}{n}\right)^b \leq \left(\frac{eq}{2b}\right)^b \cdot \left(\frac{1 + \ln 5 + c \ln n + \sqrt{3q(c \ln n + \ln 6)}}{n^b}\right)^b
\]

\[
\leq \left(\frac{eq \cdot \frac{4}{3}q}{2bn}\right)^b = \left(\frac{2eq^2}{3bn}\right)^b \leq \left(\frac{e^\epsilon}{b}\right)^b \text{ as } q \leq \sqrt{n}.
\]

Taking a union bound over all \(b > 1 + \lg 6 + 2c \lg n\) gives a probability of at most \(\frac{1}{6}n^{-2c}\) as \(\epsilon < \frac{1}{2}\) if \(\lg n \geq 3\), i.e. if \(n \geq 8\).

We denote this event by \(\mathcal{E}_5\).

**Lemma 10.** In the first phase the number of left moving coordinates is \(\frac{1}{4}q - \frac{1}{2}(b_0 + b_1)\), as is the number of right moving coordinates.

Conditioning on events \(\mathcal{E}_5\), in each subsequent phase the number of left moving coordinates is at least \(\frac{1}{8}q - \frac{1}{2}\left[1 + \lg 6 + 2c \lg n\right] - 2b_3\).

**Proof.** In the first phase, by Corollary 3 at most \(b_0\) updates are reappearing, and by Lemma 4 at most \(b_1\) coordinates are put in \(B\). The remaining updates are left or right moving, with equal numbers of each. Thus there are at least \(\frac{1}{2}\left[\frac{1}{4}q - b_0 - b_1\right]\) left moving coordinates, as claimed.

For each remaining phase, by Lemma 9 at least \(q/2 - (1 + \lg 6 + 2c \lg n)\) updates are to available coordinates. By Corollary 4 at most \(\left[1 + \ln 6 + 2c \ln n\right]\) of these updates are reappearing in this phase. Of these coordinates, those updated at even times are either left- or right-moving. WLOG suppose that among these updates there are at least as many left-moving as right-moving ones. Then the number of left-moving updates is at least \(\frac{q}{8} - \frac{1}{2}\left[1 + \ln 6 + 2c \ln n\right]\). By Lemma 6 the number of right-moving updates in this phase is smaller by at most \(2b_3\), since at the start of the phase \(a_N - a_P \leq b_3\), and at the end of the phase \(a_N - a_P \geq -b_3\).

We set \(\lambda_1 q = \frac{1}{4}q - \frac{1}{2}(b_0 + b_1)\).

Recall (8). It suffices to have

\[
\frac{1}{8} \epsilon_\alpha q \geq \frac{3}{2} + 3\nu + \frac{1}{4} \epsilon_\alpha (b_0 + b_1) + \frac{1}{2} \epsilon_\alpha (b_0 + b_1) + \frac{3}{4} \epsilon_\alpha (b_0 + b_1).
\]

(12)
Next, we set $\lambda_2 q = \frac{1}{8} q - \frac{1}{2} \left[1 + \lg 6 + 2c \lg n\right] - 2b_3$.

Recall (11). It suffices to have

$$\frac{1}{16} q \epsilon \alpha \geq \frac{3}{2} + 3 \nu + \epsilon \left(\frac{1}{2} \alpha (b_0 + b_1 + b_3) + b_2 \nu\right) + \frac{1}{2} \epsilon \alpha \left[\frac{1}{2} \left[1 + \lg 6 + 2c \lg n\right] + 2b_3\right].$$

Clearly (13) subsumes (12).

We conclude that:

**Lemma 11.** If $q \geq 768b_0$ and $q \geq \frac{96}{\epsilon(1-\epsilon)} + 16b_0 + 56b_3 + 8b_2 + 8 \left[1 + \lg 6 + 2c \lg n\right] + 486$ then (12) and (13) hold.

**Proof.** The first condition ensures that $q \geq 16b_1 = 16\sqrt{3q(c \ln n + \ln 6)}$, which implies that $\frac{1}{32} q \epsilon \alpha \geq \frac{1}{2} \epsilon \alpha b_1$; this, together with the second condition, ensures that $\frac{1}{32} q \epsilon \alpha$ is also greater than or equal to the remaining terms in (13).

**Corollary 12.** It suffices that $q \geq \frac{96}{\epsilon(1-\epsilon)} + 2145 + 768c \ln n$.

**Proof.** As $\epsilon \leq \frac{1}{9}$, it is easy to see that the bound on $q$ implies the first condition in Lemma 11. To verify the second condition, we calculate as follows.

Recall that $b_0 = \left[1 + \ln 6 + c \ln n\right]$ (see Corollary 3), $b_2 = 2 + \frac{1}{3 \ln 3/2} (\ln 18 + 2c \ln n)$ (see Lemma 8), and $b_3 = 1 + \lg 6 + 2c \lg n$ (see the sentence after Lemma 6). Also, $\alpha = \frac{1}{2} (1 - \epsilon)$ and $\nu = \frac{3}{2} \ln 3/2 \alpha \leq \frac{1}{4} \alpha$. Thus the second condition is implied by

$$q \geq \frac{96}{\epsilon(1-\epsilon)} + 16b_0 + 56b_3 + 8b_2 + 216$$

$$q \geq \frac{96}{\epsilon(1-\epsilon)} + 16(2 + \ln 6 + c \ln n) + 56(1 + \lg 6 + 2c \lg n)$$

$$+ 8 \left(2 + \frac{1}{3 \ln 3/2} (\ln 18 + 2c \ln n)\right) + 216 \quad \text{suffices}$$

$$q \geq \frac{96}{\epsilon(1-\epsilon)} + 513 + 191c \ln n \quad \text{suffices.}$$

We have shown Theorem 11.

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