COINCIDENCES BETWEEN CALABI–YAU MANIFOLDS OF BERGLUND–HÜBSCH TYPE AND BATYREV POLYTOPES

A. A. Belavin,*† and M. Yu. Belakovskiy*

We consider the phenomenon of the complete coincidence of key properties of Calabi–Yau manifolds realized as hypersurfaces in two different weighted projective spaces. More precisely, the first manifold in such a pair is realized as a hypersurface in a weighted projective space, and the second is realized as a hypersurface in an orbifold of another weighted projective space. The two manifolds in each pair have the same Hodge numbers and the same geometry on the complex structure moduli space and are also associated with the same $\mathcal{N}=2$ gauged linear sigma model. We explain these coincidences using the correspondence between Calabi–Yau manifolds and the Batyrev reflexive polyhedra.

Keywords: superstring theory, compactification on Calabi–Yau manifold, mirror symmetry

DOI: 10.1134/S0040577920110045

1. Introduction

The set of Calabi–Yau (CY) manifolds that are used for compactification in superstring theory contains the set considered by Berglund and Hübsch [1], [2]. It comprises varieties defined in the weighted projective spaces $\mathbb{P}^4_{(k_1,k_2,k_3,k_4,k_5)}$ as the zero locus of quasihomogeneous polynomials $W(x_i)$ belonging to one of 16 types.

We consider cases where two manifolds constructed according to different polynomials of different Berglund–Hübsch types have completely coinciding key properties although the manifolds themselves are defined in different weighted projective spaces in the Berglund–Hübsch list. But we have not yet been able to show that the found manifolds are birationally equivalent.

We found two cases of the coincidences. In both of them, the first CY manifold is defined as a hypersurface in a weighted projective space, and the second is defined as an orbifold in another weighted projective space. In addition to the coincidence of the Hodge numbers, as we verified, in each of these two cases, the two CY manifolds have the same special Kähler geometry on the moduli space of complex structures. We also verified the mirror version of the JKLMR-conjecture [3] about the relation between the Kähler potential on the complex moduli space of the CY manifold and the corresponding partition function of $\mathcal{N}=2$ GLSM model on the two-dimensional sphere [4] in both cases.

*Landau Institute for Theoretical Physics, RAS, Chernogolovka, Moscow Oblast, Russia; Moscow Institute of Physics and Technology, Dolgoprudny, Moscow Oblast, Russia, e-mail: belavin@itp.ac.ru (corresponding author), belakovskiy.myu@phystech.edu.

†Institute for Information Transmission Problems, RAS, Moscow, Russia.

This research was supported by a grant from the Russian Science Foundation (Project No. 18-12-00439).

Prepared from an English manuscript submitted by the authors; for the Russian version, see Teoreticheskaya i Matematicheskaya Fizika, Vol. 205, No. 2, pp. 222–241, November, 2020. Received June 5, 2020. Revised June 25, 2020. Accepted June 26, 2020.
The coincidence of the GLSM models corresponding to the two CY manifolds in each of the two pairs shows that the mirror partners of each of these CY manifolds also coincide.

In Sec. 2, we briefly describe the method developed in [5]–[7] for calculating the Kähler potential on the complex structure moduli space. In Sec. 3, we use this method to calculate the Kähler potential for each CY manifold in each pair and find the abovementioned coincidences. In Sec. 4, we briefly describe the way to construct the GLSM model corresponding to a given CY manifold [8], [9]. Using this method, we find the models, and it turns out that the two manifolds in each pair yield the same model. In Sec. 5, we explain all the observed coincidences. For this, we use the correspondence between three-dimensional CY manifolds and four-dimensional reflexive polyhedra proposed by Batyrev [10]. We construct Batyrev polyhedra for each family following the approach proposed in [8] and find that the polyhedra in each pair indeed coincide.

2. Method for calculating the geometry on the complex structure moduli space

In this section, we briefly describe the technique for calculating [5]–[7] the Kähler potential of the complex structure moduli space of CY manifolds defined by the zero locus of a certain quasihomogeneous polynomial in a weighted projective space or its orbifold. This technique is based on the correspondence found in [11] between the cohomology ring of CY manifold defined as the zero locus of a polynomial \( W(x) \) of degree \( d = \sum_{i=1}^{5} k_i \) and the chiral ring \( R^Q \) defined by the same polynomial \( W(x) \).

It is known [12]–[14] that there exists a nonvanishing holomorphic \((3,0)\)-form \( \Omega \) on each CY manifold. The Kähler potential on the complex structure moduli space is defined as

\[
e^{-K^X_c} = \int_X \Omega \wedge \bar{\Omega}.
\]

We consider CY manifolds \( X \) defined as hypersurfaces in a weighted projective space

\[
P^4_{(k_1, \ldots, k_5)} = \{(x_1, \ldots, x_5) \mid (x_1, \ldots, x_5) \simeq (\lambda^{k_1} x_1, \ldots, \lambda^{k_5} x_5)\}.
\]

These hypersurfaces are defined by the vanishing of the sum of \( W_0(x) \) (one of the 16 types of Berglund–Hübsch polynomials) and the superposition of \( h=\dim H^{2,1} \) monomials corresponding to the deformations of the complex structure of the manifold \( X \):

\[
W(x) = W_0(x) + \sum_{s=1}^{h} \phi_s e_s = 0,
\]

\[
W(\lambda^{k_i} x_i) = \lambda^d W(x_i), \quad d = \sum_{i=1}^{5} k_i.
\]

The holomorphic \((3,0)\)-form in this case can be expressed explicitly in terms of the projective coordinates on \( \mathbb{P}^4_{(k_1, \ldots, k_5)} \):

\[
\Omega = x_5 \frac{dx_1 \wedge dx_2 \wedge dx_3}{\partial W(x)/\partial x_4}.
\]

Here, the form \( \Omega \) is restricted to the hypersurface defined by the zero locus of \( W \).

Let \( q_a \) be some basis in the middle homology group \( H_3(X, \mathbb{R}) \). Then we can rewrite the Kähler potential in terms of periods \( \omega_a \) over the cycles \( q_a \) and their intersection matrix \( C_{ab} = q_a \cap q_b \):

\[
e^{-K_c(X)} = \omega_a(\phi) C_{ab} \omega_b(\phi),
\]

\[
\omega_a(\phi) := \int_{q_a} \Omega.
\]
We now consider the chiral ring \( R^Q \). By definition, it is a subring of the Milnor ring \( R \) associated with the polynomial \( W_0 \),

\[
R = \frac{\mathbb{C}[x_1, \ldots, x_5]}{\langle \partial W_0(x) / \partial x_i \rangle}.
\]

(6)

The chiral ring \( R^Q \) contains those elements \( P(x_i) \) of the Milnor ring \( R \) that are invariant under the action of the so-called quantum symmetry group \( Q \):

\[
P(\omega^k x_i) = P(x_i), \quad \omega^d = 1.
\]

(7)

The ring \( R^Q \) is graded. It decomposes into a direct sum of four components whose elements are polynomials of degrees 0, \( d \), 2\( d \), and 3\( d \) in the weights of the projective coordinates \( x_i \). This grading corresponds to the Hodge structure on \( X \):

\[
R^Q = (R^Q)^0 \oplus (R^Q)^1 \oplus (R^Q)^2 \oplus (R^Q)^3,
\]

(8)

\[
H^1 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}.
\]

Let the set of monomials \( e_\mu \) be a basis in the chiral ring \( R^Q \). Then \( \eta_{\mu \nu} \) is an invariant pairing in this ring if it is defined as

\[
\eta_{\mu \nu} = \text{Res} e_\mu e_\nu \, d^5 x / \prod_{i=1}^5 \frac{\partial W_0}{\partial x_i}.
\]

(9)

We next introduce two differentials

\[
D_\pm = d \pm dW_0 \wedge
\]

(10)

and define two cohomology groups \( H^5_{D_\pm} \) corresponding to them. The groups \( H^5_{D_\pm} \) are isomorphic to \( R^Q \), and the set of 5-forms \( e_\mu \, d^5 x \) can be chosen as the basis in each of them.

We also define two relative homology groups \( H_5(C^5, \text{Re } W_0 \to \pm \infty) \) dual to \( H^5_{D_\pm} \) with respect to the pairing

\[
( e_\mu d^5 x, Q^\pm_a ) = \int_{Q^\pm_a} e_\mu e^{\mp W_0(x)} d^5 x.
\]

(11)

We assume that the cycles \( Q^\pm_a \) belong to the homology groups \( H_5(C^5, \text{Re } W_0 \to \pm \infty) \) with real coefficients and determine the bases in them.

The basic idea of the approach described in [5] is that the groups \( H_5, D_\pm \) and \( H_3(X, \mathbb{R}) \) are isomorphic. This isomorphism is established according to the following rule. The cycles \( q_a \in H_3(X, \mathbb{R}) \) and \( Q_a \in H_5(C^5, \text{Re } W_0 \to \pm \infty) \) correspond (are in a one-to-one correspondence) if and only if

\[
\int q_a \Omega = \int_{Q^+_a} e^{\mp W(x)} d^5 x.
\]

(12)

It hence follows that the intersection matrices of corresponding cycles in these two homology groups coincide:

\[
q_a \cap q_b = Q_a^+ \cap Q_b^- = C_{ab}.
\]

(13)

We now use an important relation obtained in [15] that connects the intersection matrix \( C_{ab} \) and the pairing \( \eta_{\mu \nu} \) in the chiral ring \( R^Q \):

\[
\eta_{\mu \nu} = \langle e_\mu, e_\nu \rangle = \int_{Q^+_a} e_\mu e^{-W_0} d^5 x C_{ab} \int_{Q^-_b} e_\nu e^{W_0} d^5 x.
\]

(14)
Introducing the matrix
\[ T_{a \mu}^\pm := \int_{Q_a^\pm} e_\mu e^\mp W_0 d^5 x, \] (15)
we can rewrite this relation as
\[ \eta_{\mu \nu} = T_{a \mu}^+ C_{ab} T_{b \nu}^- . \] (16)
Expressing the intersection matrix \( C_{ab} \) from this relation, we can rewrite the formula for the Kähler potential
\[ e^{-K_c(X)} = \omega_a(\phi) C_{ab} \omega_b(\phi) \] (17)
in the form
\[ e^{-K_c(X)} = \sigma_\mu^+ \eta_{\mu \lambda} M_{\lambda \nu} \sigma_\nu^-, \] (18)
where \( M := T^{-1}T, \sigma_\mu^\pm := T_{\mu a}^\pm \omega_a, \) and
\[ \omega_a(\phi) := \int_{q_a} \Omega = \int_{Q_a^+} e^\mp W(x) d^5 x. \]
The resulting formula for the Kähler potential is very useful for calculation. First, instead of the unknown intersection matrix \( C_{ab} \), it uses the pairing in the chiral ring \( R^Q \), which is easy to calculate. Second, if we introduce convenient basis cycles \( \Gamma_\mu^\pm \) in the relative homology groups \( H_5(\mathbb{C}^5, \text{Re} W_0 \to \pm \infty) \) dual to the chosen basis in \( H_5^{\pm} \) as
\[ \int_{\Gamma_\mu^\pm} e_\nu e^\mp W_0 d^5 x = \delta_{\mu \nu}, \] (19)
then the matrix \( T_{a \mu} \) becomes the transition matrix from the real cycles \( Q_a \) to the complex basis \( \Gamma_\mu \),
\[ \omega_a = T_{a \mu}^\pm \sigma_\mu^\pm . \] (20)
We note that the matrix \( M := T^{-1}T \) is independent of the choice of the real cycles \( Q_a \). Therefore, for calculations, we can use the basis of real cycles \( Q_a \) with which calculating the matrices \( T^\pm \) is easy. The Lefschetz thimbles are an example of such a choice \([7]\).

Third, the periods \( \sigma_\mu^\pm = \int_{\Gamma_\mu^\pm} e^\mp W(x) d^5 x \) defined as the integrals over the cycles \( \Gamma_\mu^\pm \) can be calculated rather easily. Namely, we can expand the integrand in \( \sigma_\mu^\pm = \int_{\Gamma_\mu^\pm} e^\pm W(x) d^5 x \) in the parameters \( \phi_a \):
\[ \sigma_\mu^\pm = \int_{\Gamma_\mu^\pm} e^\mp W(x) d^5 x = \sum_{m_1, \ldots, m_h} \prod_{s=1}^{h^2,1} \phi_{m_s}^{m_s} C_{\{m_1, \ldots, m_h\}}^\pm, \] (21)
\[ C_{\{m_1, \ldots, m_h\}}^\pm = \int_{\Gamma_\mu^\pm} e^\mp W_0(x) \prod_{s=1}^{h^2,1} e^{m_s} d^5 x. \]
Each term
\[ \int_{\Gamma_\mu^\pm} e^\mp W_0(x) \prod_{s=1}^{h^2,1} e^{m_s} d^5 x \]
in this formula depends only on the cohomology class of the monomial \( \prod_{s=1}^{h^2,1} e^{m_s} d^5 x \). Therefore, we can simplify the formula by reducing the degree of the monomial \( P \) using the relation
\[ P d^5 x = P' d^5 x + D\Psi, \] (22)
where \( \Psi \) is an appropriate 4-form. As a result of an inductive application of this procedure, we finally obtain an explicit expression for the periods \( \sigma_\mu^\pm \).
3. Coincidence of the special geometry on the moduli spaces of pairs of CY families

In this section, we consider two pairs of CY families. In each pair, the two families are defined differently, but after calculation and pairing, it turns out that the two families in each pair have the same Kähler potential on their complex structure moduli spaces.

3.1. The first pair of CY manifolds. We consider two CY families. They are defined by different types of Berglund–Hübsch polynomials in different projective spaces. Nevertheless, as shown by a direct calculation, the result for the special Kähler geometry is the same in both cases.

**Case 1a.** The CY family $X$ is defined as a hypersurface in $\mathbb{P}^4_{(23,55,34,53,34)}$ by

$$ W_0 + \phi_1 e_1 + \phi_2 e_2 = 0, $$

$$ W_0 = x_1^6 x_2 + x_2^3 x_3 + x_3^5 x_4 + x_4^3 x_5 + x_5^5 x_1. $$

The basis in the chiral ring $R^Q$ comprises six monomials:

$$ e_0 = 1, \quad e_1 = x_1 x_2 x_3 x_4 x_5, \quad e_2 = x_1^3 x_4^2 x_5, \quad e_3 = e_2^2, \quad e_4 = e_2 e_3, \quad e_5 = e_2^2 e_3. $$

The Hodge numbers of $X$ are $h^{2,1} = 2$ and $h^{1,1} = 95$. The monomials $e_1$ and $e_2$ correspond to deformations of the complex structure on $X$.

Following the approach described in Sec. 2, we can find an explicit expression for periods (21). Below, we note the fact that the two 5-forms $P_1 d^5 x$ and $P_2 d^5 x$ belong to the same cohomology class, i.e., that $P_1 d^5 x - P_2 d^5 x = D_2 \Psi$, as $P_1 d^5 x \sim P_2 d^5 x$.

After some calculations, we obtain the relation

$$ \prod_{j=1}^5 x_j^{a_j} d^5 x \sim (1 - B_{ij} (a_j + 1)) \prod_{j=1}^5 x_j^{a_j - M_{ij}} d^5 x, \quad i = 1, \ldots, 5, $$

where $W_0 := \sum_{i=1}^5 \prod_{j=1}^5 x_i^{M_{ij}}$ and $B_{ij} := (M)_{ij}^{-1}$. Using this relation, we can replace each monomial in (21) with a monomial of a lower degree:

$$ e_1^n e_2^m d^5 x \sim - \frac{1}{7} (n + 3m - 6) e_1^{n-1} e_2^{m-2} d^5 x, $$

$$ e_1^n e_2^m d^5 x \sim \frac{1}{7} (2n - m - 5) e_1^{n-3} e_2^{m+1} d^5 x. $$

Applying the two relations step by step, we can replace each monomial in (21) with one of the six monomials in the chiral ring basis as

$$ e_1^n e_2^m d^5 x \sim (-1)^m \frac{\Gamma^3((n + 3m + 1)/7)}{\Gamma^3(\mu/7)} \frac{\Gamma^2((2n - m + 2)/7)}{\Gamma^2(\nu/7)} e_{\mu} d^5 x, $$

$$ \mu = n + 3m - 6 \pmod{7}, \quad 1 \leq \mu \leq 6, $$

$$ \nu = 2\mu \pmod{7}, \quad 1 \leq \nu \leq 6. $$
Taking the definition of the cycles $\Gamma^+_{\mu}$ into account, for the periods over these cycles, we obtain the expression

$$\sigma_\mu(\phi_1, \phi_2) = \int_{\Gamma_\mu} e^{-W(x)}(x) d^5x = \sum_{n,m} \int_{\Gamma_\mu} e^{-W_0(x)} e^\nu \phi_1^m \phi_2^m \frac{\phi_1^n \phi_2^n}{n! m!} d^5x = \sum_{n,m \in \Sigma_\mu} (-1)^m \frac{\Gamma^3((n + 3m + 1)/7)}{\Gamma^3(\mu/7)} \frac{\Gamma^2((2n - m + 2)/7)}{\Gamma^2(\nu/7)} \phi_1^n \phi_2^m \frac{\phi_1^m \phi_2^m}{n! m!},$$

where

$$\Sigma_\mu = \{ n, m \in \mathbb{N}_0 \mid n + 3m - 6 = \mu \pmod 7, 0 \leq \mu \leq 6, \nu = \nu(\mu) = 2\mu \pmod 7, 0 \leq \nu \leq 6 \}. \quad (29)$$

To obtain an expression for the Kähler potential, we must now find the transition matrix $T^{\pm}_{\alpha \nu}$ from the cycles $\Gamma^\pm_{\mu}$ to some basis of real cycles $Q_d^\pm$. Before calculating, it is convenient to change the variables as $x_i = \prod_{j=1}^5 y_j^{7B_{ij}}$. Hence, in the new variables $y_i$, we have

$$W_0(x(y)) = \sum_{i=1}^5 y_i^7. \quad (30)$$

We now have a Fermat polynomial and can choose a basis of real cycles as described in [7], where the Lefschetz thimbles were used as real cycles. After the change of variables, we have

$$T_{\alpha \mu} = \int_{L_\alpha} e_{\alpha}(x(y)) e^{-W_0(x(y))} J(y) d^5 y, \quad (31)$$

where $J(y)$ is the Jacobian of the change of variables. Knowing $T_{\alpha \mu}$, we find the matrix $M = T^{-1}T$ and obtain

$$M_{\mu \nu} = \delta_{\mu, \nu} - \gamma^3 \left(\frac{\mu}{7}\right) \gamma^2 \left(\frac{\nu}{7}\right), \quad (32)$$

where $\delta(\mu) = 2\mu \pmod 7$, $1 \leq \delta \leq 6$.

As a result, we obtain an explicit expression for the Kähler potential on the complex structure moduli space:

$$e^{-K_{\phi}(\phi_1, \phi_2)} = \sum_{\mu=1}^6 \gamma^3 \left(\frac{\mu}{7}\right) \gamma^2 \left(\frac{\nu}{7}\right) |\sigma_\mu(\phi_1, \phi_2)|^2, \quad (33)$$

where $\nu(\mu) = 2\mu \pmod 7$, $1 \leq \nu \leq 6$.

**Case 1b.** The CY family is defined as a hypersurface in the orbifold of the weighted projective space $\mathbb{P}^4_{(3,2,7,2,7)/\mathbb{Z}_{21}}$ using the equation

$$W_0 + \phi_1^2 e_1 + \phi_2^2 e_2 = 0, \quad (34)$$

$$W_0 = \prod_{j=1}^5 y_j^{7B_{ij}}. \quad (35)$$

Here, the action of $\mathbb{Z}_{21}$ on this weighted projective space is given by the formula

$$(x_1, x_2, x_3, x_4, x_5) \rightarrow (\omega^{12} x_1, \omega^2 x_2, \omega^7 x_3, x_4, x_5),$$

where $\omega^{21} = 1$. 1444
The complete basis in the chiral ring is given by six monomials:
\[
e_0 = 1, \quad e_1 = x_1 x_2 x_3 x_4 x_5, \quad e_2 = x_1^3 x_2^3 x_4^3, \quad e_3 = e_2, \quad e_4 = e_2 e_3, \quad e_5 = e_2^2 e_3.
\] (36)

Its Hodge numbers are \(h^{2,1} = 2\) and \(h^{1,1} = 95\). The monomials \(e_1\) and \(e_2\) define the complete set of deformations of the complex structure.

The special Kähler geometry in this case was obtained in [16]. The result of that calculation is
\[
e^{-K_{\mu}(\phi_1, \phi_2)} = \sum_{\mu=1}^{6} \gamma^3 \left( \frac{\mu}{7} \right) \gamma^2 \left( \frac{\nu}{7} \right) |\sigma_\mu(\phi_1, \phi_2)|^2,
\] (37)

where \(\nu(\mu) = 2\mu \pmod{7}\), \(1 \leq \nu \leq 6\). As can be seen from this formula, the Kähler potential in this case completely coincides with that obtained in Case 1a. Therefore, the two differently defined families 1a and 1b have the same special Kähler geometry, which suggests the equivalence of these two families.

3.2. The second pair. We now consider the other pair of CY families.

Case 2a. The CY family is defined in \(\mathbb{P}^4_{(97,53,35,37,25)}\) using the equation
\[
W_0 + \phi_1 e_1 + \phi_2 e_2 = 0,
\]
\[
W_0 = x_1^2 x_2 + x_2^4 x_3 + x_3^6 x_4 + x_4^6 x_5 + x_5^6 x_1.
\] (38)

The basis in the chiral ring \(R^Q\) comprises six monomials:
\[
e_0 = 1, \quad e_1 = x_1 x_2 x_3 x_4 x_5, \quad e_2 = x_2^2 x_3^2 x_4^2, \quad e_3 = e_2, \quad e_4 = e_1 e_2, \quad e_5 = e_2^2 e_1.
\] (39)

The Hodge numbers in this case are \(h^{2,1} = 2\) and \(h^{1,1} = 122\). The monomials \(e_1\) and \(e_2\) define the complete set of deformations of the complex structure.

After calculations similar to those in Case 1a, we obtain the expression for the Kähler potential
\[
e^{-K_{\mu}(\phi_1, \phi_2)} = \sum_{\mu=1}^{6} \gamma^3 \left( \frac{\mu}{7} \right) \gamma^2 \left( \frac{\nu}{7} \right) |\sigma_\mu(\phi_1, \phi_2)|^2,
\] (40)

where \(\nu = 3\mu \pmod{7}\), \(1 \leq \nu \leq 6\).

Case 2b. The CY family is defined in \(\mathbb{P}^4_{(7,2,2,2,1)}/\mathbb{Z}_7^2\) using the equation
\[
W(x) = W_0 + \phi_1 e_1 + \phi_2 e_2 = 0,
\]
\[
W_0 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 x_1.
\] (41)

Here, the action of \(\mathbb{Z}_7^2\) on the weighted projective space \(\mathbb{P}^4_{(3,2,7,2,7)}\) is defined as
\[
(x_1, x_2, x_3, x_4, x_5) \to (x_1, \omega_1^{-1} x_2, \omega_1 x_3, \omega_2^{-1} x_4, \omega_2 x_5).
\] (42)

The complete basis in the chiral ring comprises six monomials:
\[
e_0 = 1, \quad e_1 = x_1 x_2 x_3 x_4 x_5, \quad e_2 = x_2^2 x_3^2 x_4^2 x_5^2, \quad e_3 = e_2, \quad e_4 = e_1 e_2, \quad e_5 = e_1 e_2^2.
\] (43)

1445
Although this case, in contrast to Cases 1a and 2a, does not relate to the “loop” type, calculating the special geometry here is almost the same as in those cases. Performing the same calculations as in the previous cases, we find expressions for the periods that completely coincide with the expressions in the previous cases:

\[
\sigma_\mu(\phi_1, \phi_2) = \sum_{n,m \in \Sigma_\mu} (-1)^n \frac{\Gamma((3n - m + 3)/7) \Gamma((n + 2m + 1)/7) \phi_1^n \phi_2^m}{\Gamma(\mu/7) \Gamma(\nu/7) n! m!}, \tag{44}
\]

\[
\Sigma_\mu = \{ n, m \in \mathbb{N}_0 \mid 3n - m - 4 = \mu \pmod{7}, \nu = -2\mu \pmod{7}, 0 \leq \nu \leq 6 \}, \tag{45}
\]

The final result for the Kähler potential on the complex moduli structure space is

\[
e^{-K_c(\phi_1, \phi_2)} = \sum_{\mu=1}^6 \gamma\left(\frac{\mu}{7}\right) \gamma^4\left(\frac{\nu}{7}\right) |\sigma_\mu(\phi_1, \phi_2)|^2, \tag{46}
\]

where \( \nu = 3\mu \pmod{7}, 1 \leq \nu \leq 6 \). As can be seen, it completely coincides with the result obtained in Case 2a, i.e., the two differently defined families 2a and 2b have the same special Kähler geometry. Therefore, as in the case of the first pair, we can expect the equivalence of these two families.

### 4. Coincidence of the GLSM models corresponding to the two CY families in each of the pairs

In this section following [8], [9], we describe the construction of the \( N=2 \) GLSM model corresponding to a CY manifold defined in some weighted projective space. Further, we find that the CY manifolds in Cases 1a and 1b and also in Cases 2a and 2b correspond to the same GLSM model, and we also verify the JKLMR conjecture [3].

The mirror version of the JKLMR conjecture [17]–[20] states that the partition function \( Z_Y \) of the GLSM model on the two-dimensional sphere, exactly calculated by the localization method in [21], [22], is related to the Kähler potential on the moduli space of complex structures for the CY manifold \( X \) that is the mirror of the vacuum manifold \( Y \) of the same model by

\[
Z_Y = e^{-K^X_Y}. \tag{47}
\]

To construct the GLSM model, we use the fan corresponding to the family \( Y \). As follows from [23], the ends of the vectors comprising the skeleton of the fan of the family \( Y \) correspond to the vertices of the Batyrev polyhedron for the family \( X \) that is mirror to \( Y \).

Therefore, starting with the polynomial \( W_X \) and its Batyrev polyhedron, we find the fan of the mirror family \( Y \) and use it to find the corresponding GLSM model with the vacuum manifold \( Y \), as was done in [8], [9], [24]. We construct the polyhedron corresponding to the family \( X \) determined by the polynomial \( W_X \) as follows. We rewrite the polynomial \( W_X = W_0 + \sum_{i=1}^h \phi_i e_a \) in the form

\[
W = \sum_{a=1}^5 \prod_{j=1}^5 x_i^{V_{aj}} + \sum_{a=6}^{h+5} \prod_{j=1}^5 \phi_{a} x_j^{V_{aj}}. \tag{48}
\]

Here, we introduce the \((h+5) \times 5\) matrix \( V_{ai} \), which is equivalent to the components of the \((h+5)\)-vectors \( V_a \). In fact, the vectors \( V_a \) are in the four-dimensional sublattice defined by the equation

\[
\sum_{i=1}^5 V_{ai} k_i = d. \tag{49}
\]
The convex hull of the points $V_a$ is a Batyrev four-dimensional reflexive polyhedron. The vertices of this polyhedron are the ends of the vectors of the skeleton of the fan for the mirror family $Y$ [25]. Because the fan of the family $Y$ consists of $5+h$ five-dimensional vectors, there exist $h$ linearly independent relations between them,

$$Q_{la}V_{ai} = 0, \quad l = 1, h, \quad a = 1, h + 5, \quad i = 1, 5. \quad (50)$$

We must find such a set $Q_{la}$ that is an integral basis in the space of linear relations between the vectors $V_{ai}$. As a result, we obtain $h$ sets of charges of $5+h$ chiral fields for the $h$ $U(1)$ gauge groups that are needed for determining the corresponding GLSM model.

### 4.1. Cases 1a and 1b

The verification of the JKLMR conjecture in Cases 1a and 1b easily reduces to a single computation. For family 1a, the vertices of the Batyrev polyhedron have the coordinates

$$v_1: (6, 1, 0, 0, 0), \quad v_2: (0, 3, 1, 0, 0), \quad v_3: (0, 0, 5, 1, 0), \quad v_4: (0, 0, 0, 3, 1), \quad v_5: (1, 0, 0, 0, 5), \quad v_6: (1, 1, 1, 1, 1), \quad v_7: (3, 0, 2, 0, 2). \quad (51)$$

For the family 1b, the coordinates of the vertices of the polyhedron are

$$v_1: (7, 0, 0, 0, 0), \quad v_2: (0, 7, 1, 0, 0), \quad v_3: (0, 0, 3, 0, 0), \quad v_4: (0, 0, 0, 7, 1), \quad v_5: (0, 0, 0, 3, 0), \quad v_6: (1, 1, 1, 1, 1), \quad v_7: (3, 3, 0, 3, 0). \quad (52)$$

Between vectors (51) and (52), there must exist two linearly independent relations with integer coefficients $Q_{1a}$ and $Q_{2a}$. We must choose the integer basis of the linear relations, i.e., choose two integer coefficients $Q_{1a}$ and $Q_{2a}$ of the linear relations such that the coefficients of any integer relation $Q'_a$ between the vectors $v_a$ can be represented as

$$Q'_a = m_1 Q_{1a} + m_2 Q_{2a}, \quad (53)$$

where $m_1, m_2 \in \mathbb{Z}$. We call $Q_{1a}$ and $Q_{2a}$ a $\mathbb{Z}$-basis in the space of the linear integer relations. It is easy to see that the $\mathbb{Z}$-basis of these relations can be chosen similarly in these two cases:

$$Q_{1a}v_a = 0, \quad l = 1, 2, \quad a = 1, 2, 3, 4, 5, \quad (54)$$

where

$$Q_{1a} = (1, 0, 1, 0, 1, -1, -2), \quad Q_{2a} = (0, 1, 0, 1, 0, -3, 1). \quad (55)$$

As a result, we obtain two sets of charges for the two $U(1)$ gauge groups of the corresponding GLSM model. We also choose the $R$-symmetry charges [21], [22], [24] as

$$q_1 = q_3 = q_5 = \frac{2}{7}, \quad q_2 = q_4 = \frac{4}{7}, \quad q_6 = q_7 = 0. \quad (56)$$

We can then write the partition function of the obtained sigma model as an expression depending on four real Fayet–Illiopoulos parameters [21], [22]:

$$Z(r_1, r_2, \theta_1, \theta_2) = \sum_{m_1, m_2 \in \mathbb{Z}} e^{-i\theta_1 m_1 - i\theta_2 m_2} \int \int \frac{d\tau_1}{2\pi i} \frac{d\tau_2}{2\pi i} e^{4\pi i \tau_1 + 4\pi i \tau_2} \times \prod_{a=1}^{7} \frac{\Gamma(q_a/2 + Q_{1a}(\tau_1 - m_1/2) + Q_{1a}(\tau_2 - m_2/2))}{\Gamma(1 - q_a/2 - Q_{1a}(\tau_1 + m_1/2) - Q_{1a}(\tau_2 + m_2/2))}. \quad (57)$$
Changing the variables now allows expressing the four real parameters with two complex variables,

\[ z_1 = \exp\left(-\frac{2\pi i}{7} r_1 - \frac{4\pi i}{7} r_2 - \frac{4\pi i}{7} \theta_1 - \frac{8\pi i}{7} \theta_2\right), \]

\[ z_2 = \exp\left(-\frac{6\pi i}{7} r_1 + \frac{2\pi i}{7} r_2 - \frac{12\pi i}{7} \theta_1 + \frac{4\pi i}{7} \theta_2\right) \]

(58)

or, more precisely,

\[ Z = \sum_{\mu} \sum_{(n,m), (\bar{n}, \bar{m}) \in \Sigma_{\mu}} z_1^{-n} z_2^{-m} z_1^{-\bar{n}} z_2^{-\bar{m}} \frac{\Gamma^3((1 + n + 3m)/7)}{\Gamma^3((1 + (1 + n + 3m)/7))} \times \]

\[ \times \frac{\Gamma^2(2 + 2n - m)/7}{\Gamma^2(1 - (2 + 2\bar{n} - \bar{m})/7)} \frac{(-1)^{n+m}}{n! m! \bar{n}! \bar{m}!} \times \]

\[ \times \sin^3\left(\frac{\pi}{7} (1 + n + 3\bar{m})\right) \sin^2\left(\frac{\pi}{7} (1 + 2\bar{n} - \bar{m})\right) \frac{\Gamma^3\left(\frac{1 + n + 3m}{7}\right)}{\Gamma^3\left(\frac{1 + 2 + 2\bar{n} - \bar{m}}{7}\right)} \times \]

\[ \times \Gamma^2\left(\frac{2 + 2n - m}{7}\right) \Gamma^3\left(\frac{1 + n + 3m}{7}\right) \Gamma^2\left(\frac{2 + 2\bar{n} - \bar{m}}{7}\right) \frac{\Gamma^3\left(\frac{2 + 2\bar{n} - \bar{m}}{7}\right)}{\Gamma^3\left(\frac{2 + 2n - m}{7}\right)} \]

(59)

We calculate this integral by the Cauchy theorem and obtain

\[ Z = \sum_{\mu} \sum_{(n,m), (\bar{n}, \bar{m}) \in \Sigma_{\mu}} z_1^{-n} z_2^{-m} z_1^{-\bar{n}} z_2^{-\bar{m}} \frac{\Gamma^3((1 + n + 3m)/7)}{\Gamma^3((1 + (1 + n + 3m)/7))} \times \]

\[ \times \frac{\Gamma^2(2 + 2n - m)/7}{\Gamma^2(1 - (2 + 2\bar{n} - \bar{m})/7)} \frac{(-1)^{n+m}}{n! m! \bar{n}! \bar{m}!} \times \]

\[ \times \sin^3\left(\frac{\pi}{7} (1 + n + 3\bar{m})\right) \sin^2\left(\frac{\pi}{7} (1 + 2\bar{n} - \bar{m})\right) \frac{\Gamma^3\left(\frac{1 + n + 3m}{7}\right)}{\Gamma^3\left(\frac{1 + 2 + 2\bar{n} - \bar{m}}{7}\right)} \times \]

\[ \times \Gamma^2\left(\frac{2 + 2n - m}{7}\right) \Gamma^3\left(\frac{1 + n + 3m}{7}\right) \Gamma^2\left(\frac{2 + 2\bar{n} - \bar{m}}{7}\right) \frac{\Gamma^3\left(\frac{2 + 2\bar{n} - \bar{m}}{7}\right)}{\Gamma^3\left(\frac{2 + 2n - m}{7}\right)} \]

\[ \times \sum_{(n,m), (\bar{n}, \bar{m}) \in \Sigma_{\mu}} \frac{\Gamma^3\left(\frac{1 + n + 3m}{7}\right)}{\Gamma^3\left(\frac{1 + 2 + 2\bar{n} - \bar{m}}{7}\right)} \frac{\Gamma^3\left(\frac{2 + 2n - m}{7}\right)}{\Gamma^3\left(\frac{2 + 2\bar{n} - \bar{m}}{7}\right)} \]

(60)

We can rewrite this expression as

\[ Z = \sum_{\mu} \gamma^3 \left(\frac{\nu}{7}\right) \gamma^2 \left(\frac{\mu}{7}\right) \sigma_{\mu} \left(-\frac{1}{z_1}, \frac{1}{z_2}\right)^2 \]

(61)

We hence obtain a change of variables relating the coordinates on the complex moduli structure space and the Fayet–Iliopoulos parameters

\[ \phi_1 = -\frac{1}{z_1}, \quad \phi_2 = \frac{1}{z_2}. \]

(62)

Therefore, the partition function for the GLSM model coincides with the expression for the Kähler potential \( e^{-K_{\mu}(\phi_1, \phi_2)} \) obtained above in Cases 1a and 1b.
4.2. Cases 2a and 2b. In the second pair, we again have two sets of vertices for the polyhedra:
seven vertices in Case 2a,

\[ v_1: (2, 1, 0, 0, 0), \quad v_2: (0, 4, 1, 0, 0), \quad v_3: (0, 0, 6, 1, 0), \]
\[ v_4: (0, 0, 0, 6, 1), \quad v_5: (1, 0, 0, 0, 6), \quad v_6: (1, 1, 1, 1, 1), \quad v_7: (0, 1, 2, 2, 2), \]

(63)

and seven vertices in Case 2b,

\[ v_1: (2, 0, 0, 0, 0), \quad v_2: (0, 7, 0, 0, 0), \quad v_3: (0, 0, 7, 0, 0), \]
\[ v_4: (0, 0, 7, 0), \quad v_5: (1, 0, 0, 7), \quad v_6: (1, 1, 1, 1), \quad v_7: (0, 2, 2, 2, 2). \]

(64)

Here, we again find the integer basis of the linear relations, which can be chosen the same in these two cases,

\[ Q_{1a}v_a = 0, \quad l = 1, 2, \quad a = 1, 2, 3, 4, 5, \]

(65)

where

\[ Q_{2a} = (1, 0, 0, 0, 0, -2, 1), \quad Q_{1a} = (0, 1, 1, 1, 1, -1, -3). \]

(66)

The \( R \)-symmetry charges are

\[ q_1 = q_2 = q_3 = q_4 = \frac{2}{7}, \quad q_5 = \frac{6}{7}, \quad q_6 = q_7 = 0. \]

(67)

After the change of variables

\[ z_2 = \exp \left( -\frac{4\pi}{7}r_2 + \frac{2\pi}{7}r_1 - \frac{8\pi i}{7}\theta_2 + \frac{4\pi i}{7}\theta_1 \right), \]
\[ z_1 = \exp \left( -\frac{2\pi}{7}r_2 - \frac{6\pi}{7}r_1 - \frac{4\pi i}{7}\theta_2 - \frac{12\pi i}{7}\theta_1 \right), \]

(68)

we obtain the partition function

\[ Z = \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \int \frac{d\tau_1}{2\pi i} \frac{d\tau_2}{2\pi i} \frac{dz_2}{z_2^{3(\tau_1-m_1/2)+(\tau_2-m_2/2)} z_1^{-3(\tau_1+m_1/2)+(\tau_2+m_2/2)} \times}
\times \frac{1}{z_2^{-(\tau_1-m_1/2)-2(\tau_2-m_2/2)} z_2^{-(\tau_1+m_1/2)-2(\tau_2+m_2/2)}} \frac{1}{\Gamma(1/7 + (\tau_1 - m_1/2)) \Gamma(6/7 - (\tau_1 - m_1/2)) \times}
\times \frac{1}{\Gamma(3/7 + (\tau_2 - m_2/2))} \frac{1}{\Gamma(-2(\tau_1 - m_1/2) + (\tau_2 - m_2/2))} \frac{1}{\Gamma(4/7 - (\tau_2 + m_2/2))} \frac{1}{\Gamma(1 + 2(\tau_1 + m_1/2) - (\tau_2 + m_2/2)) \times}
\times \frac{1}{\Gamma(1 + (\tau_1 + m_1/2) - 3(\tau_2 + m_2/2))}. \]

(69)

Integrating, we obtain the expression

\[ Z = \sum_{\mu=1}^{6} \gamma^4 \left( \frac{\mu}{7} \right) \gamma \left( \frac{\nu}{7} \right) \left| \sigma_{\mu} \left( \frac{1}{2}, \frac{1}{2} \right) \right|^2. \]

(70)

The change of variables

\[ \phi_1 = z_1^{-1}, \quad \phi_2 = -z_2^{-1} \]

(71)

relates the coordinates on the moduli space to the Fayet–Iliopoulos parameters, after which the expression for the Kähler potential \( e^{-K_0(\phi_1, \phi_2)} \) in both cases coincides with the partition function of the corresponding model.
5. Batyrev polyhedra and the coincidence of the CY families

In this section, we discuss the found coincidence of the CY properties in the first and the second pairs of CY families using the Batyrev approach [10]. We construct reflexive polyhedra corresponding to each CY family in the first pair and show that the families in Cases 1a and 1b define the same polyhedron. This also happens for the second pair.

The reflexive polyhedra are constructed following the scheme described in the preceding section. We know that the vectors \( v_a, a = 1, \ldots, b_{21} + 5 \), whose ends form the Batyrev polyhedron satisfy the relations

\[
\sum_{i=1}^{5} k_i v_{ai} = \sum_{i=1}^{5} k_i := d,
\]

i.e., they are in the same four-dimension hyperplane as the vector \( v_\rho \):

\[
v_\rho = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.
\]

After a shift of all vectors \( v_a \) to the vector \( v_\rho \), the polyhedron formed by the vectors \( \tilde{v}_a = v_a - v_\rho \) is in a hyperplane passing through the origin (we hereafter omit the tilde).

The reflexive polyhedron \( \Delta \) corresponding to a quasihomogeneous polynomial \( W_X \) by definition is the convex hull of the vectors composed of the degrees \( v_a \) of the monomials in \( W_X \). The vectors \( v_a \) can be written as linear combinations with integer coefficients of the vectors forming a basis of the four-dimensional lattice \( M \) [23]. The lattice \( M \) itself is defined by the equation \( \sum_{i=1}^{5} m_i k_i = 0 \), where \((m_1, m_2, m_3, m_4, m_5)\) are points of the five-dimensional lattice \( \mathbb{Z}^5 \) and \( k_i \) are weights of the weighted projective space.

If the CY family is defined as a hypersurface in a quotient of weighted projective space by some finite group \( H \), then the vectors \( v_a \) can be written as linear combinations with integer coefficients of the vectors forming a basis of the sublattice \( M/H \) of the lattice \( M \). Below, we show the difference between these two situations.

To verify that a set of vectors forms a basis in the lattice \( M \) that is a sublattice of \( \mathbb{Z}^5 \) defined by the equation

\[
\sum_{i=1}^{5} k_i m_i = 0
\]

(where the \( k_i \) are coprime), we use the following criterion.

Four linearly independent vectors \( u_1, u_2, u_3, u_4 \in M \) constitute a basis in \( M \) if and only if the volume \( Vol_{cell} \) of the cell spanning them is equal to the minimum possible. The volume \( Vol_{cell} \) is equal to \( |V| \), the length of the vector \( V \),

\[
|V| = \sqrt{V_1^2 + V_2^2 + V_3^2 + V_4^2 + V_5^2},
\]

whose components are defined as

\[
V_j = \epsilon_{j1} u_{11} + \epsilon_{j2} u_{12} + \epsilon_{j3} u_{13} + \epsilon_{j4} u_{14}.
\]

By definition, the vector \( V \) is parallel to the vector with the components \( k_i \). Because the vector \( k \) has integer irreducible components and therefore has the least length of all vectors parallel to it, this length
is equal to \( |k| = \sqrt{k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2} \). Hence and also from the requirement that the length of \( V \) is minimum, it follows that it is equal to \( \pm k \).

Hence, we obtain the formulation of the criterion: four vectors \( u_1, u_2, u_3, \) and \( u_4 \) constitute an integer basis in \( M \) if and only if the volume of the cell,

\[
\text{Vol}(k, u_1, \ldots, u_4) = \epsilon_{j_1} \epsilon_{j_2} \epsilon_{j_3} \epsilon_{j_4} k_{j_1} u_{j_1} u_{j_2} u_{j_3} u_{j_4},
\]

formed by the vector \( k \) with the components \( k_j \) and the vectors \( u_1, u_2, u_3, \) and \( u_4 \) satisfies the relation

\[
\text{Vol}(k, u_1, \ldots, u_4) = k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2. \tag{77}
\]

5.1. Cases 1a and 1b. We consider the first pair of CY families. The first of them is given by the equation

\[
W_1(x) = x_1^6 x_2 + x_2^3 x_3 + x_3^5 x_4 + x_4^3 x_5 + x_5^5 x_1 + \phi_1 x_1 x_2 x_3 x_4 x_5 + \phi_2 x_1^3 x_3^2 x_5^2 = 0
\]

in \( \mathbb{P}^4_{(23,55,28,53,34)} \). The second is given by the equation

\[
W_2(x) = x_1^7 + x_2^7 x_3 + x_3^7 x_4 + x_4^7 + x_5 + \psi_1 x_1 x_2 x_3 x_4 x_5 + \psi_2 x_1^3 x_3^2 x_4^3 = 0
\]

in \( \mathbb{P}^4_{(3,2,7,2,7)/\mathbb{Z}_{21}} \). The action of \( \mathbb{Z}_{21} \) is defined by the formula

\[
(x_1, x_2, x_3, x_4, x_5) \rightarrow (\omega^{12} x_1, \omega^2 x_2, \omega^7 x_3, x_4, x_5), \tag{78}
\]

where \( \omega^{21} = 1 \).

To construct Batyrev polyhedron corresponding to family 1a, we consider the lattice \( M \) given by the equation \( 23m_1 + 55m_2 + 28m_3 + 53m_4 + 34m_5 = 0 \) in \( \mathbb{Z}^5 \) with the coordinates \( m_i \). The vertices of the polyhedron in \( \mathbb{Z}^5 \) are the points

\[
\begin{align*}
  f_1 & : (5, 0, -1, -1, -1), &  f_2 & : (-1, 2, 0, -1, -1), &  f_3 & : (-1, -1, 4, 0, -1), \\
  f_4 & : (-1, -1, 2, 0), &  f_5 & : (0, -1, -1, -1, 4), &  f_6 & : (2, -1, 1, -1, 1).
\end{align*}
\]

As a basis, we take the vectors \( \{e_i\}_{i=1}^4 \) in \( M \)

\[
e_i = f_i, \quad i = 1, 2, 3, 4, \tag{80}
\]

or, more explicitly,

\[
e_1 : (5, 0, -1, -1, -1), \quad e_2 : (-1, 2, 0, -1, -1) \tag{81}
\]

\[
e_3 : (-1, -1, 4, 0, -1), \quad e_4 : (-1, -1, -1, 2, 0).
\]

These four vectors are indeed a basis in \( M \) because they satisfy the abovementioned criterion. The vertices of the polyhedron of family 1a in this basis have the coordinates

\[
\begin{align*}
  v_1 & : (1, 0, 0, 0), &  v_2 & : (0, 1, 0, 0), &  v_3 & : (0, 0, 1, 0), \\
  v_4 & : (0, 0, 0, 1), &  v_5 & : (-1, -2, -1, -2), &  v_6 & : (0, -1, 0, -1).
\end{align*}
\]

The convex hull of these six vertices \( \{v_i\}_{i=1}^6 \) determines the polyhedron of family 1a. This polyhedron is indeed reflexive because it has only one point in its interior (the origin) and the lattice points \( m_i \) in its facets satisfy the relations

\[
\sum_{i=1}^5 n_i m_i = 1, \tag{83}
\]

1451
where \( n_i \) is some vector with four coprime integer components.

We now construct the polyhedron for family 1b. In this case, the lattice \( N \) is given by the equation \( 3m_1 + 2m_2 + 7m_3 + 2m_4 + 7m_5 = 0 \) in \( \mathbb{Z}^5 \). The vertices of the polyhedron are the points

\[
\begin{align*}
  f_1' & : (6, -1, -1, -1, -1), \\
  f_2' & : (-1, 1, -1, -1, -1), \\
  f_3' & : (-1, 1, 1, 1, 1), \\
  f_4' & : (-1, 1, -1, -1, -1), \\
  f_5' & : (-1, 1, -1, 1, 1), \\
  f_6' & : (2, 2, -1, -1, -1).
\end{align*}
\]

If the manifold were defined as a hypersurface in the weighted projective space here, as in Case 1a, and not in its orbifold, then the construction would be complete at this stage. But in Case 1b, we must take into account that the manifold is defined in the orbifold; therefore, if we complete the construction at this stage, then it turns out that the obtained polyhedron is not reflexive. Although there is only one lattice point in its interior, its faces cannot be determined by equations of the form

\[
\sum_{i=1}^5 n_i m_i = 1,
\]

where \( n_i \) is a vector with four coprime integer components.

The reason that the construction did not lead to a reflexive polyhedron is the difference in the definitions of families 1a and 1b. The second family is defined in the quotient of the weighted projective space by the group \( \mathbb{Z}_{21} \). This fact must be taken into account in constructing the polyhedron. In fact, the polyhedron is defined not in the lattice \( N \) itself but in a sublattice. This sublattice \( N/\mathbb{Z}_{21} \) is defined by the equations

\[
\begin{align*}
  3m_1 + 2m_2 + 7m_3 + 2m_4 + 7m_5 &= 0, \\
  12m_1 + 2m_2 + 7m_3 &= 0 \pmod{21}.
\end{align*}
\]

Therefore, to construct the reflexive polyhedron of family 1b, we must choose a basis in the lattice \( N/\mathbb{Z}_{21} \). We verify that the set of four vectors \( \{f_1', f_2', f_3', f_4'\} \) is a basis in the sublattice \( N/\mathbb{Z}_{21} \).

First, each vector in the set \( \{f_1', f_2', f_3', f_4'\} \) satisfies the relation

\[
12(f_1')_i + 2(f_2')_i + 7(f_3')_i = 0 \pmod{21}.
\]

We must also verify that any vector of the sublattice \( N/\mathbb{Z}_{21} \) can be represented as the sum of vectors \( \{f_i'\}_{i=1}^4 \) with integer coefficients. For this, it suffices to show that the cell volume generated by four vectors \( \{f_1', f_2', f_3', f_4'\} \) from the sublattice \( N/\mathbb{Z}_{21} \) is exactly 21 times larger than the volume of the cell formed by the basis of the lattice \( N \) itself. This can be done as was shown above:

\[
\text{Vol}(k, e_1, e_2, e_3, e_4) = \det \begin{pmatrix}
  3 & 2 & 7 & 2 & 7 \\
  6 & -1 & -1 & -1 & -1 \\
  -1 & -1 & 2 & -1 & -1 \\
  -1 & 6 & 0 & 6 & 0 \\
  -1 & -1 & -1 & -1 & 2
\end{pmatrix} = 21 \cdot |k|^2 = 21 \cdot 115.
\]

This means that \( \{f_1', f_2', f_3', f_4'\} \) indeed forms a basis in the sublattice \( N/\mathbb{Z}_{21} \). In this basis, the vertices of the polyhedron have the coordinates

\[
\begin{align*}
  v_1' & : (1, 0, 0, 0), \\
  v_2' & : (0, 1, 0, 0), \\
  v_3' & : (0, 0, 1, 0), \\
  v_4' & : (0, 0, 0, 1), \\
  v_5' & : (-1, -2, -1, -2), \\
  v_6' & : (0, -1, 0, -1).
\end{align*}
\]

The polyhedra of families 1a and 1b indeed coincide.
5.2. Cases 2a and 2b. Here, we also consider two CY families. The first is given in $\mathbb{P}^4_{(97,53,35,37,25)}$ by the equation

$$W_1(x) = x_1^2 x_2 + x_2^4 x_3 + x_3^6 x_4 + x_4^5 x_5 + x_5^6 x_1 + \phi_1 x_1 x_2 x_3 x_4 x_5 + \phi_2 x_2 x_3^2 x_4^3 x_5^2 = 0.$$  

The second is given in $\mathbb{P}^4_{(3,2,7,2,7)}/\mathbb{Z}_7^2$ by the equation

$$W_2(x) = x_1^7 + x_2^7 x + x_3^7 + x_4^7 + x_5^7 x_1 + \psi_1 x_1 x_2 x_3 x_4 x_5 + \psi_2 x_2 x_3^2 x_4^3 x_5^2 = 0,$$

where the action of $\mathbb{Z}_7^2$ on the weighted projective space $\mathbb{P}^4_{(3,2,7,2,7)}$ is defined as

$$(x_1, x_2, x_3, x_4, x_5) \rightarrow (x_1, \omega_1^{-1} x_2, \omega_3 x_3, \omega_2^{-1} x_4, \omega_2 x_5).$$  

(90)

The polyhedron of family 2a is built like the polyhedron of family 1a:

$$f_1: (1, 0, -1, -1, -1), \quad f_2: (-1, 3, 0, -1, -1), \quad f_3: (-1, -1, 5, 0, -1), \quad f_4: (-1, -1, -1, 5, 0), \quad f_5: (0, -1, -1, -1, 5), \quad f_6: (-1, 0, 1, 1, 1).$$  

(91)

The set of four vectors $\{e_i\}_{i=1}^4$, $e_i = f_i$, $i = 1, \ldots, 4$ is a basis in the lattice $M = \{(m_1, \ldots, m_5) \mid 97m_1 + 53m_2 + 35m_3 + 37m_4 + 25m_5 = 0\}$. In this basis, the polyhedron of family 2a has the vertices

$$v_1: (1, 0, 0, 0), \quad v_2: (0, 1, 0, 0), \quad v_3: (0, 0, 1, 0), \quad v_4: (0, 0, 0, 1), \quad v_5: (-1, -1, -1, 3), \quad v_6: (0, 0, 0, -1).$$  

(92)

In Case 2b, the family is defined in the orbifold $\mathbb{P}^4_{(7,2,2,2,1)}/\mathbb{Z}_7^2$, and the $\mathbb{Z}_7 \times \mathbb{Z}_7$ action on the weighted projective space is given by the formula

$$(x_1, x_2, x_3, x_4, x_5) \rightarrow (x_1, \omega_1^{-1} x_2, \omega_3 x_3, \omega_2^{-1} x_4, \omega_2 x_5),$$  

$$\omega_1^7 = \omega_2^7 = 1.$$  

(93)

Similarly to Case 1b, we consider the lattice $N = \{m \in \mathbb{Z}^5 \mid 7m_1 + 2m_2 + 2m_3 + 2m_4 + m_5 = 0\}$. The vertices of the polyhedron are the points

$$f'_1: (1, -1, -1, -1, -1), \quad f'_2: (-1, 6, -1, -1, -1), \quad f'_3: (-1, -1, 6, -1, -1),$$  

$$f'_4: (-1, -1, 6, -1), \quad f'_5: (0, -1, -1, -1, 6), \quad f'_6: (-1, 1, 1, 1, 1).$$  

(94)

To construct the polyhedron, we must choose a basis in the sublattice $N/\mathbb{Z}_7^2$, which is defined by the equations

$$7m_1 + 2m_2 + 2m_3 + 2m_4 + m_5 = 0,$$

$$m_2 - m_3 = 0 \pmod{7}, \quad m_4 - m_5 = 0 \pmod{7}.$$  

(95)

We must now find a set of four vectors $\{e_i\}_{i=1}^4$ that generates the sublattice $N/\mathbb{Z}_7^2$.

Each vector in the set $\{f'_a\}_{a=1}^6$, $a = 1, \ldots, 6$, satisfies two relations:

$$(f'_a)_2 - (f'_a)_3 = 0 \pmod{7}, \quad (f'_a)_4 - (f'_a)_5 = 0 \pmod{7}.$$  

(96)
We now show that the volume of the cell generated by the four vectors \( \{f'_1, f'_2, f'_3, f'_4\} \) is exactly 49 times larger than the volume of the cell that forms a basis in the sublattice \( N \). This follows from the equation

\[
\text{Vol}(k; f'_1, f'_2, f'_3, f'_4) = \det \begin{pmatrix}
7 & 2 & 2 & 1 \\
1 & -1 & -1 & -1 \\
-1 & 6 & -1 & -1 \\
-1 & -1 & 6 & -1 \\
-1 & -1 & -1 & 6 \\
\end{pmatrix} = 7^2 \cdot |k|^2 = 49 \cdot 62.
\] (97)

This means that these four vectors are a basis of \( N/\mathbb{Z}_7 \). In this basis in \( N/\mathbb{Z}_7 \), the vertices of the polyhedron have the coordinates

\[
v'_1: (1, 0, 0, 0), \quad v'_2: (0, 1, 0, 0), \quad v'_3: (0, 0, 1, 0),
v'_4: (0, 0, 0, 1), \quad v'_5: (-1, -1, -1, -3), \quad v'_6: (0, 0, 0, -1).
\] (98)

Therefore, the polyhedra of the families in this pair also coincide.

6. Conclusion

We examined two cases of coincidence of the topology, the special geometries on moduli spaces, and the GLSM models for two differently defined CY manifolds. We found that the Batyrev polyhedra for such CY manifolds also coincide, which explains such a phenomenon in these cases. Most likely, this phenomenon is related to the fact that the mirror partners of the families in each pair are in the same weighted projective spaces. It would be interesting to understand how general this phenomenon of such coincidences between CY families is. It was recently shown in [26] that in some cases, the varieties defined by different Berglund–Hübsch polynomials in the same weighted projective spaces are birationally equivalent. Perhaps, the construction in [26] will allow generalizing the examples that we considered.

Acknowledgments. The authors are grateful to G. Koshevoy and A. Litvinov for the useful discussions.

Conflicts of interest. The authors declare no conflicts of interest.

REFERENCES

1. P. Berglund and T. Hübsch, “A generalized construction of mirror manifolds,” Nucl. Phys. B, 393, 377–391 (1993); arXiv:hep-th/9201014v1 (1992).
2. P. Berglund and T. Hübsch, “A generalized construction of Calabi–Yau models and mirror symmetry,” SciPost Phys., 4, 009 (2018); arXiv:1611.10300v3 [hep-th] (2016).
3. H. Jockers, V. Kumar, J. M. Lapan, D. R. Morrison, and M. Romo, “Two-sphere partition functions and Gromov–Witten invariants,” Commun. Math. Phys., 325, 1139–1170 (2014); arXiv:1208.6244v3 [hep-th] (2012).
4. E. Witten, “Phases of \( N=2 \) theories in two-dimensions,” Nucl. Phys. B, 403, 159–222 (1993); arXiv:hep-th/9301042v3 (1993).
5. K. Aleshkin and A. Belavin, “A new approach for computing the geometry of the moduli spaces for a Calabi–Yau manifold,” J. Phys. A: Math. Theor., 51, 055403 (2018); arXiv:1706.05342v4 [hep-th] (2017).
6. K. Aleshkin and A. Belavin, “Special geometry on the 101 dimensional moduli space of the quintic threefold,” JHEP, 1803, 018 (2018); arXiv:1710.11609v3 [hep-th] (2017).
7. K. Aleshkin and A. Belavin, “Exact computation of the special geometry for Calabi–Yau hypersurfaces of Fermat type,” JETP Lett., 108, 705–709 (2018); arXiv:1806.02772v2 [hep-th] (2018).
8. K. Aleshkin, A. Belavin, and A. Litvinov, “Two-sphere partition functions and Kahler potentials on CY moduli spaces,” JETP Lett., **108**, 710 (2018).
9. K. Aleshkin, A. Belavin, and A. Litvinov, “JKLMR conjecture and Batyrev construction,” J. Stat. Mech., **2019**, 034003 (2019); arXiv:1812.00478v3 [hep-th] (2018).
10. V. V. Batyrev, “Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties,” J. Alg. Geom., **3**, 493–535 (1994); arXiv:alg-geom/9310003v1 (1993).
11. W. Lerche, C. Vafa, and N. P. Warner, “Chiral rings in N=2 superconformal theories,” Nucl. Phys., **324**, 427–474 (1989).
12. P. Candelas and X. C. de la Ossa, “Moduli space of Calabi–Yau manifolds,” Nucl. Phys. B, **355**, 455–481 (1991).
13. P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes, “A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory,” Nucl. Phys. B, **359**, 21–74 (1991).
14. P. Berglund, P. Candelas, X. C. de la Ossa, A. Font, T. Hubsch, D. Jančić, and F. Quevedo, “Periods for Calabi–Yau and Landau–Ginzburg vacua,” Nucl. Phys. B, **419**, 352–403 (1994); arXiv:hep-th/9308005v2 (1993).
15. A. Chiodo, H. Iritani, and Y. Ruan, “Landau–Ginzburg/Calabi–Yau correspondence: Global mirror symmetry and Orlov equivalence,” Publ. IHES, **119**, 127–216 (2013); arXiv:1201.0813v3 [math.AG] (2012).
16. K. Aleshkin and A. Belavin, “Special geometry on the moduli space for the two-moduli non-Fermat Calabi–Yau,” Phys. Lett. B, **776**, 139–144 (2018); arXiv:1708.08362v2 [hep-th] (2017).
17. G. Bonelli, A. Sciarappa, A. Tanzini, and P. Vasko, “Vortex partition functions, wall crossing, and equivariant Gromov–Witten invariants,” Commun. Math. Phys., **333**, 717–760 (2015); arXiv:1307.5997v2 [hep-th] (2013).
18. J. Gomis and S. Lee, “Exact Kähler potential from gauge theory and mirror symmetry,” JHEP, **1304**, 019 (2013).
19. N. Doroud and J. Gomis, “Gauge theory dynamics and Kähler potential for Calabi–Yau complex moduli,” JHEP, **1312**, 099 (2013); arXiv:1309.2305v2 [hep-th] (2013).
20. E. Gerchkovitz, J. Gomis, and Z. Komargodski, “Sphere partition functions and the Zamolodchikov metric,” JHEP, **1411**, 001 (2014); arXiv:1405.7271v2 [hep-th] (2014).
21. F. Benini and S. Cremonesi, “Partition functions of $\mathcal{N}=(2,2)$ gauge theories on $S^2$ and vortices,” Commun. Math. Phys., **334**, 1483–1527 (2015); arXiv:1206.2356v3 [hep-th] (2012).
22. N. Doroud, J. Gomis, B. Le Floch, and S. Lee, “Exact results in $D=2$ supersymmetric gauge theories,” JHEP, **1305**, 093 (2013); arXiv:1206.2606v4 [hep-th] (2012).
23. K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, “Chap. 7: Toric geometry for string theory,” in: Mirror Symmetry (Clay Math. Monogr., Vol. 1), Amer. Math. Soc., Providence, R.I. (2003), pp. 101–142; arXiv:hep-th/000222v3 (2000).
24. A. A. Belavin and B. A. Eremin, “Partition functions of $\mathcal{N}=(2,2)$ supersymmetric sigma models and special geometry on the moduli spaces of Calabi–Yau manifolds,” Theor. Math. Phys., **201**, 1606–1613 (2019).
25. P. Candelas, X. C. de la Ossa, and S. Katz, “Mirror symmetry for Calabi–Yau hypersurfaces in weighted and extensions of Landau–Ginzburg theory,” Nucl. Phys. B, **450**, 267–290 (1995); arXiv:hep-th/9412117v1 (1994).
26. D. Favero and T. L. Kelly, “Derived categories of BHK mirrors,” Adv. Math., **352**, 943–980 (2019); arXiv:1602.05876v2 [math.AG] (2016).