The Ground State of the Sine-Gordon Soliton

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Abstract

At one loop, we provide an explicit formula for the ground state of the one-soliton sector in the Sine-Gordon theory. The state is given in the basis of eigenstates of the field operator, or equivalently as a Schrödinger wave functional. The formula readily generalizes to other solitons in other models and as an example we also provide the ground state of the kink in the (1+1)-dimensional $\phi^4$ double well.

1 Background Material

The prototypical strong-weak duality is that between the Sine-Gordon model and the massive Thirring model [1]. The central role in this duality is played by the Sine-Gordon soliton, which becomes the fundamental fermion in the massive Thirring model. The soliton is a solution of the classical equations of the motion of the classical Sine-Gordon theory. In the quantum theory, it is so far understood only in a singular limit of a tree-level approximation [2]. But what lies between these two limits? In this note, we will find the ground state of the one-soliton sector at one loop.

We will begin with a review of the Sine-Gordon soliton at one loop, treated in Ref. [3]. The Sine-Gordon theory describes a scalar field $\phi(x)$ in 1+1 dimensions. The Hamiltonian is

$$H = \int dx \mathcal{H}(x), \quad \mathcal{H}(x) = \frac{1}{2} : \pi(x) \pi(x) : + \frac{1}{2} : \partial_x \phi(x) \partial_x \phi(x) : - \frac{m^2}{\lambda} : \left( \cos(\sqrt{\lambda} \phi(x)) - 1 \right) :$$

(1.1)

where $m$ is a mass parameter, $\lambda$ is the coupling and $\pi$ is the conjugate momentum to $\phi$. The field and its conjugate can be expanded in oscillator modes

$$\phi(x) = \int \frac{dp}{2\pi} \phi_p e^{-ipx}, \quad \phi_p = \frac{1}{\sqrt{2\omega_p}} \left( a_p^\dagger + a_{-p} \right)$$

$$\pi(x) = \int \frac{dp}{2\pi} \pi_p e^{-ipx}, \quad \pi_p = i \sqrt{\frac{\omega_p}{2}} \left( a_p^\dagger - a_{-p} \right)$$

(1.2)

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where
\[ \omega_p = \sqrt{m^2 + p^2}, \quad [a_p, a_q^\dagger] = 2\pi\delta(p - q), \quad [\phi_p, \pi_q] = 2\pi i\delta(p + q). \] (1.3)
The normal ordering convention is defined with respect to \( a \) and \( a^\dagger \).

The classical equations of motion derived from (1.1) admit the soliton solution
\[ \phi_{cl}(x, t) = f(x), \quad f(x) = \frac{4}{\sqrt{\lambda}} \arctan e^{mx}. \] (1.4)

It will be convenient to define a new Hamiltonian \( H_K \) which is related to the original Hamiltonian by a similarity transformation
\[ H_K = D_f^{-1}H D_f, \quad D_f = \exp \left( -i \int dx f(x)\pi(x) \right) \] (1.5)
where \( D_f \) is the translation operator which shifts the field by the soliton solution. In classical field theory, \( H_K \) describes oscillations about the soliton configuration. As we have normal-ordered the Hamiltonian in (1.1), the theory is finite and regularization is not necessary. However, had we adapted a more general regulator, which would be necessary in a theory with fermions or more dimensions, then it is essential that the full, regulated \( H \) be used in Eq. (1.5).

Let \( |K\rangle \) be the soliton ground state and, motivated by [4, 5], define \( \mathcal{O} \) to be any operator such that
\[ |K\rangle = D_f\mathcal{O}|0\rangle. \] (1.6)
If \( E \) is the minimum soliton energy, then
\[ E|K\rangle = H|K\rangle = D_f H_K \mathcal{O}|0\rangle. \] (1.7)
Left multiplying by \( D_f^{-1} \) one finds
\[ H_K \mathcal{O}|0\rangle = E \mathcal{O}|0\rangle \] (1.8)
and so \( \mathcal{O}|0\rangle \) is the lowest eigenvector of \( H_K \).

It was shown in Ref. [6] that \( \mathcal{O} \) is equal to the identity plus quantum corrections and so (1.8) can be solved in perturbation theory. In this paper we will work at one-loop. Let \( Q_0 \) be the classical soliton energy. Then \( H_K = Q_0 + \int dx \mathcal{H}_{PT} \) is the free Pöschl-Teller Hamiltonian with Hamiltonian density
\[ \mathcal{H}_{PT} = \frac{\pi^2(x)}{2} + \frac{\partial_x \phi(x) \partial_x \phi(x)}{2} + \left( \frac{m^2}{2} - m^2 \text{sech}^2(mx) \right) \phi^2(x). \] (1.9)
whose classical equations of motion have constant frequency solutions \(g_k(x)\) parameterized by \(k\) and a bound state solution \(g_B(x)\) representing the soliton Goldstone mode

\[
g_k(x) = \frac{e^{-ikx}}{\sqrt{1 + m^2/k^2}} \left( 1 - im \frac{\tanh(mx)}{k} \right), \quad g_B(x) = \sqrt{\frac{m}{2}} \text{sech}(mx)
\]

with respective frequencies

\[
\omega_k = \sqrt{m^2 + k^2}, \quad \omega_B = 0.
\]

These have been normalized so that

\[
\int dx g_k^*(x) g_k(x) = 2\pi \delta(k_1 - k_2), \quad \int dx |g_B(x)|^2 = 1
\]

and they satisfy the reality conditions

\[
g_k^*(x) = g_{-k}(x), \quad g_B^*(x) = g_B(x).
\]

In the ground state sector it was convenient to decompose the field \(\phi(x)\) into plane waves (1.2) to obtain the Heisenberg operators \(a_p\). Similarly, in the one-soliton sector it is convenient to decompose \(\phi(x)\) into the constant frequency solutions (1.10)

\[
\phi(x) = \phi_0 g_B(x) + \int \frac{dk}{2\pi} \phi_k g_k(x), \quad \pi(x) = \pi_0 g_B(x) + \int \frac{dk}{2\pi} \pi_k g_k(x)
\]

\[
\phi_k = \frac{1}{\sqrt{2\omega_k}} \left( b_k^\dagger + b_{-k} \right), \quad \pi_k = i \frac{\sqrt{\omega_k}}{2} \left( b_k^\dagger - b_{-k} \right).
\]

Using the completeness relations (1.12) these can be inverted

\[
\phi_0 = \int dx \phi(x) g_B^*(x), \quad \pi_0 = \int dx \pi(x) g_B^*(x), \quad \phi_k = \int dx \phi(x) g_k^*(x)
\]

\[
\pi_k = \int dx \pi(x) g_k^*(x), \quad b_k^\dagger = \sqrt{\frac{\omega_k}{2\omega_k}} \phi_k - \frac{i}{\sqrt{2\omega_k}} \pi_k, \quad b_{-k} = \sqrt{\frac{\omega_k}{2\omega_k}} \phi_k + \frac{i}{\sqrt{2\omega_k}} \pi_k
\]

which fix the commutation relations

\[
[\phi_p, \pi_q] = 2\pi i \delta(p + q), \quad [\phi_0, \pi_0] = i, \quad [b_{k_1}, b_{k_2}^\dagger] = 2\pi \delta(k_1 - k_2).
\]

After a tedious calculation one finds [6]

\[
H_K = Q_1 + \int \frac{dk}{2\pi} \frac{\omega_k b_k^\dagger b_k + \pi_0^2}{2}
\]

where \(Q_1\) is the one-loop soliton energy. Thus the lowest eigenstate (1.8) solves

\[
b_k \mathcal{O} |0\rangle = \pi_0 \mathcal{O} |0\rangle = 0.
\]

So finishes our review of Ref. [6].
2 The Sine-Gordon Soliton State

We will work in the basis of states provided by the eigenvectors $|F\rangle$ of the field operator $\phi(x)$. Here $F$ is a real-valued function of the space coordinate $x$. The basis states are defined by the eigenvalue equation

$$\phi(x)|F\rangle = F(x)|F\rangle.$$  

(2.1)

Any state $|\Psi\rangle$ can be expanded in terms of the basis states $F$

$$|\Psi\rangle = \int DF\Psi[F]|F\rangle$$  

(2.2)

where $DF$ is a measure on the space of functions $F$ and $\Psi[F]$ is the complex-valued Schrodinger wave functional $\Psi$\textsuperscript{[7, 8]} evaluated on the function $F$. This is the analogue of a wave function in quantum mechanics

$$|\psi\rangle = \int dx\psi(x)|x\rangle.$$  

(2.3)

Just as in quantum mechanics, states can be equivalently described by Dirac kets $|\Psi\rangle$ or their matrix elements, the wave functionals $\Psi[F]$ and in fact both of these objects satisfy the same operator equations. Thus for simplicity we will work with the wave functionals and not the kets.

Before solving Eq. (1.18) for the soliton state, let us remind the reader of the solution of

$$a_p|0\rangle = 0$$  

(2.4)

which describes the ground state of the free massive scalar theory. Working in terms of wave functionals, this is

$$a_p\Psi_0 = 0.$$  

(2.5)

Inverting (1.2)

$$a_{-p} = \sqrt{\frac{\omega_p}{2}}\phi_p + \frac{i}{\sqrt{2\omega_p}}\pi_p.$$  

(2.6)

Let us try

$$\Psi_0 = \exp\left(\frac{i}{2}\int \frac{dq}{2\pi}\phi_q\omega_q\phi_{-q}\right).$$  

(2.7)

Using the commutator in (1.3) one finds

$$\pi_p\Psi_0 = \int \frac{dq}{2\pi}(2\pi i)\delta(p + q)\left(\frac{1}{2}\right)2\omega_q\phi_{-q}\Psi_0 = i\omega_p\phi_p\Psi_0$$  

(2.8)

and so

$$a_{-p}\Psi_0 = \sqrt{\frac{\omega_p}{2}}\phi_p\Psi_0 + \frac{i}{\sqrt{2\omega_p}}\pi_p\Psi_0 = 0$$  

(2.9)
establishing that $\Psi_0$ is the ground state wave functional of the vacuum sector.

Similarly, for the soliton wave functional corresponding to $\mathcal{O}|0\rangle$ try

$$\Psi_\mathcal{O} = \exp \left( -\frac{1}{2} \int \frac{dk}{2\pi} \phi_k \omega_k \phi_{-k} \right). \quad (2.10)$$

As $\Psi_\mathcal{O}$ is independent of $\phi_0$ and $\pi_0$ commutes with $\phi_k$, trivially

$$\pi_0 \Psi_\mathcal{O}. \quad (2.11)$$

In other words, this state is translation-invariant and so it has zero momentum. Following the argument above

$$\pi_k \Psi_\mathcal{O} = i\omega_k \phi_k \Psi_\mathcal{O} \quad (2.12)$$

and so $\Psi_\mathcal{O}$ is annihilated by $b_{-k}$ and solves (1.18). Therefore we have shown that

$$\mathcal{O}|0\rangle = \int \mathcal{D}F \Psi_\mathcal{O}[F]|F\rangle. \quad (2.13)$$

According to the definition (1.6), to obtain the kink state, we need to left multiply this result by

$$\mathcal{D}_f = \exp \left( -i\pi_0 f_B - i \int \frac{dk}{2\pi} \pi_k f_k \right), \quad f_B = \int dx f(x)g_B(x), \quad f_k = \int dx f(x)g_k(x). \quad (2.14)$$

Using the fundamental property of the translation operator

$$[\mathcal{D}_f, \phi(x)] = -f(x)\phi(x) \quad (2.15)$$

and the completeness relations (1.12) one finds the action of $\mathcal{D}_f$ on $\phi_0$ and $\phi_k$

$$[\mathcal{D}_f, \phi_0] = -f_B \mathcal{D}_f, \quad [\mathcal{D}_f, \phi_k] = -f_{-k} \mathcal{D}_f. \quad (2.16)$$

Thus we can compute the action of $\mathcal{D}_f$ on any function of $\phi_0$ and the set of $\phi_k$, it simply translates each argument. The soliton ground state is then

$$\Psi_K = \mathcal{D}_f \Psi_\mathcal{O} = \exp \left( -\frac{1}{2} \int \frac{dk}{2\pi} \left( \phi_k - f_{-k} \right) \omega_k \left( \phi_{-k} - f_k \right) \right). \quad (2.17)$$

This is our main result.
3 The Double Well Kink State

We expect the above construction to apply to stationary classical solutions in a range of quantum field theories. In this section we will show how trivially it is extended to the $\phi^4$ double well in 1+1 dimensions, first treated at one loop in Ref. [9]. This theory is described by the Hamiltonian

$$H = \int dx \left[ \frac{1}{2} : \pi(x) \pi(x) : + \frac{1}{2} : \partial_x \phi(x) \partial_x \phi(x) : + \frac{\lambda}{4} : \phi^2(x) (\phi(x) - 2v)^2 : \right]$$

(3.1)

and has a classical kink solution

$$\phi_{cl}(x,t) = f(x), \quad f(x) = \frac{m}{\sqrt{2\lambda}} \left( 1 + \tanh \left( \frac{mx}{2} \right) \right).$$

(3.2)

Again we define a kink Hamiltonian by the similarity transform (1.5), where $f(x)$ is now given by Eq. (3.2). At one loop the corresponding Hamiltonian density is

$$H_{PT} = \frac{\pi^2(x)}{2} + \frac{\partial_x \phi(x) \partial_x \phi(x)}{2} + \left( 2\beta^2 - 3\beta^2 \operatorname{sech}^2 (\beta x) \right) : \phi^2(x) :$$

(3.3)

where $\beta = m/2$. Like (1.9) in the case of the Sine-Gordon theory, this is a reflectionless Pöschl-Teller Hamiltonian. However, now it is at level 2 instead of level 1, due to the factor of 3.

The difference in level in the potential means that in addition to the Goldstone mode $g_B(x)$, $H_K$ has an additional, odd, classical bound state $g_{BO}(x)$. Overall the eigenfunctions are

$$g_k(x) = \frac{e^{-ikx}}{\sqrt{(1 + \beta^2/k^2)(1 + 4\beta^2/k^2)}} \left( 1 + \frac{\beta^2}{k^2} \left( 1 - 3\tanh^2(\beta x) \right) - 3i\frac{\beta}{k}\tanh(\beta x) \right)$$

$$g_B(x) = \frac{\sqrt{3\beta}}{2} \operatorname{sech}^2(\beta x), \quad g_{BO} = -i\frac{\sqrt{3\beta}}{2} \tanh(\beta x)\operatorname{sech}(\beta x).$$

(3.4)

The frequencies, completeness relations and reality conditions are as in the Sine-Gordon case except now we also have

$$\omega_{BO} = \beta\sqrt{3}, \quad \int dx |g_{BO}(x)|^2 = 1, \quad g_{BO}^*(x) = -g_{BO}(x).$$

(3.5)

Let us decompose the field and its conjugate

$$\phi(x) = \phi_0 g_B(x) + \phi_{BO} g_{BO}(x) + \int \frac{dk}{2\pi} \phi_k g_k(x)$$

$$\pi(x) = \pi_0 g_B(x) + \pi_{BO} g_{BO}(x) + \int \frac{dk}{2\pi} \pi_k g_k(x)$$

$$\phi_{BO} = \frac{1}{\sqrt{2\omega_{BO}}} \left( b_{BO}^\dagger - b_{BO} \right), \quad \pi_{BO} = i\frac{\sqrt{\omega_{BO}}}{2} \left( b_{BO}^\dagger + b_{BO} \right).$$

(3.6)
Using the completeness relations \(1.12\) these can be inverted

\[
\phi_{BO} = \int dx \phi(x) g^*_BO(x), \quad \pi_{BO} = \int dx \pi(x) g^*_BO(x)
\]

\[
b'^{\dagger}_{BO} = \sqrt{\omega_{BO}/2} \phi_{BO} - \frac{i}{\sqrt{2\omega_k}} \pi_{BO}, \quad b_{BO} = -\sqrt{\omega_{BO}/2} \phi_k - \frac{i}{\sqrt{2\omega_{BO}}} \pi_{BO}
\]  

(3.7)

which fix the commutation relations

\[
[\phi_{BO}, \pi_{BO}] = -i, \quad [\phi^*_{BO}, \pi_{BO}] = i, \quad [b_{BO}, b'^{\dagger}_{BO}] = 1.
\]  

(3.8)

Note that \(\phi_{BO}\) and \(\pi_{BO}\) are anti-Hermitian

\[
\phi^*_{BO} = -\phi_{BO}, \quad \pi^*_{BO} = -\pi_{BO}
\]  

(3.9)

which is the reason for the wrong sign in Eq. (3.8). We adopt a star instead of a dagger for Hermitian conjugation of \(\phi_{BO}\) for later convenience, because the field \(\phi\) in a Schrodinger wave functional is interpreted as a function and not an operator. However strictly speaking a dagger should be used at this step, as \(\phi_{BO}\) is an operator, to be replaced by a star only at the end of the calculation when we write the wave functional. The interpretation of \(\phi\) as a function, instead of an operator, in the wave functional is possible because \(\pi\) does not appear, and so \(\phi\) commutes with everything. In quantum mechanics the analogous statement is that \(x\) is an operator, but when the wave function is presented \(x\) may be interpreted as simply a coordinate.

After some calculation one finds \([10\]

\[
H_K = Q_1 + \int \frac{dk}{2\pi} \omega_k b'^{\dagger}_k b_k + \omega_{BO} b'^{\dagger}_{BO} b_{BO} + \frac{\pi^2_0}{2}.
\]  

(3.10)

Thus the lowest eigenstate solves

\[
b_k \mathcal{O}|0\rangle = b_{BO} \mathcal{O}|0\rangle = \pi_0 \mathcal{O}|0\rangle = 0.
\]  

(3.11)

This is identical to the Sine-Gordon case except for the \(b_{BO}\) condition, which states that the quantum harmonic oscillator corresponding to this oscillation mode of the kink is in its ground state. To solve this condition note that

\[
\pi_{BO} \exp \left( -\frac{1}{2} \phi_{BO} \omega_{BO} \phi^*_{BO} \right) = i \phi_{BO} \omega_{BO} \exp \left( -\frac{1}{2} \phi_{BO} \omega_{BO} \phi^*_{BO} \right)
\]  

(3.12)

and so

\[
b_{BO} \exp \left( -\frac{1}{2} \phi_{BO} \omega_{BO} \phi^*_{BO} \right) = 0.
\]  

(3.13)
Thus the Schrodinger wave functional corresponding to $O|0\rangle$ is

$$\Psi_O = \exp \left( -\frac{1}{2} \phi_{BO} \omega_{BO} \phi^*_{BO} - \frac{1}{2} \int \frac{dk}{2\pi} \phi_k \omega_k \phi^*_k \right). \quad (3.14)$$

As in the Sine-Gordon case, to obtain the kink ground state we need to multiply by

$$D_f = \exp \left( -i \pi_0 f_k - i \pi_{BO} f_{BO} - i \int \frac{dk}{2\pi} \pi_k f_k \right), \quad f_{BO} = \int dx f(x) g_{BO}(x). \quad (3.15)$$

The wrong-sign canonical commutation relations in $(3.8)$ lead to

$$[D_f, \phi_{BO}] = f_{BO} D_f, \quad [D_f, \phi^*_{BO}] = f^*_{BO} D_f = -f_{BO} D_f$$

and so the kink ground state Schrodinger wave functional is

$$\Psi_K = D_f \Psi_O = \exp \left( -\frac{1}{2} (\phi_{BO} + f_{BO}) \omega_{BO} (\phi^*_{BO} + f^*_{BO}) - \frac{1}{2} \int \frac{dk}{2\pi} (\phi_k - f_{-k}) \omega_k (\phi^*_k - f^*_{-k}) \right). \quad (3.17)$$

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