Recursive enumerability and elementary frame definability in predicate modal logic

Mikhail Rybakov† and Dmitry Shkatov‡

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Abstract

We investigate the relationship between recursive enumerability and elementary frame definability in first-order predicate modal logic. On the one hand, it is well-known that every first-order predicate modal logic complete with respect to an elementary class of Kripke frames, i.e., a class of frames definable by a classical first-order formula, is recursively enumerable. On the other, numerous examples are known of predicate modal logics, based on “natural” propositional modal logics with essentially second-order Kripke semantics, that are either not recursively enumerable or Kripke incomplete. This raises the question of whether every Kripke complete, recursively enumerable predicate modal logic can be characterized by an elementary class of Kripke frames. We answer this question in the negative, by constructing a normal predicate modal logic which is Kripke complete, recursively enumerable, but not complete with respect to an elementary class of frames. We also present an example of a normal predicate modal logic that is recursively enumerable, Kripke complete, and not complete with respect to an elementary class of rooted frames, but is complete with respect to an elementary class of frames that are not rooted.

1 Introduction

It has been observed (see, e.g., [5]) that first-order predicate modal logics built on top of propositional modal logics with essentially second-order Kripke semantics usually exhibit some undesirable properties. Indeed, for all “natural” propositional modal logics with essentially second-order Kripke semantics known from the literature—such as GL (Gödel-Löb), Grz (Grzegorczyk), as well as their “linear” counterparts GL.3 and Grz.3; the logic

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†Institute for Information Transmission Problems, Russian Academy of Sciences, and Higher School of Economics, Moscow, Russia, m.rybakov@mail.ru

‡School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa, shkatov@gmail.com
of finite Kripke frames; propositional dynamic logics; epistemic logics with the common knowledge operator; and branching-time temporal logics $\mathbf{CTL}$ and $\mathbf{CTL}^*$—the sets of predicate modal formulas valid on their frames are not recursively enumerable [6, 8, 9, 10], while the logics obtained by adding to their representations as Hilbert-style calculi, in cases where there exists one, the axioms and inference rules of the classical first-order logic are Kripke incomplete [4, 6, 8, 9, 10]. Thus, it would appear that, in predicate modal logic, completeness with respect to Kripke semantics with essentially second-order conditions and recursive enumerability might be incompatible. This, in particular, raises the question of whether for Kripke complete predicate modal logics recursive enumerability and completeness with respect to an elementary class of frames coincide.\footnote{A similar question can be posed for $\Delta$-elementary classes of frames, i.e., classes of frames defined by sets of first-order formulas; this is not the notion we consider in the present paper.}

A partial answer was given in [5], where it was shown that recursive enumerability and completeness with respect to an elementary class of frames do not coincide for Kripke complete quasi-normal predicate modal logics; furthermore, it was assumed in [5], without an explicit mention, that logics under consideration were exclusively those determined by rooted frames, i.e., frames generated by a single world. The question of whether recursive enumerability implies completeness with respect to an elementary class of frames for Kripke complete normal predicate modal logics has, however, remained open.

In the present paper, we answer that question in the negative, by exhibiting an example of a normal predicate modal logic that is Kripke complete, recursively enumerable, but not complete with respect to an elementary class of Kripke frames. We also show that the assumption of all frames for a logic being rooted may play a crucial role when answering this question: we construct a normal predicate logic that enjoys the required properties if we meta-logically restrict our attention to rooted frames, but ceases to enjoy them once all frames are taken into consideration. This might be of independent interest, as for most purposes in modal logic, it suffices to only consider rooted frames.

The paper is structured as follows. In Section 2, we introduce the necessary preliminaries. In Section 3, we present an example of a normal predicate modal logic that is Kripke complete, recursively enumerable, but not complete with respect to an elementary class of rooted frames; we also show that this logic is complete with respect to an elementary class containing frames that are not required to be rooted. Then, in Section 4, we present an example of a normal predicate modal logic that is Kripke complete, recursively enumerable, but not complete with respect to an elementary class of any frames. We conclude in Section 5.

2 Preliminaries

A first-order predicate modal language contains countably many individual variables; countably many predicate letters of every arity, including zero; Boolean connectives $\neg$ and $\land$; the unary modal connective $\Box$; and the quantifier $\forall$. Atomic formulas, formulas, as well as the symbols $\lor$, $\rightarrow$, $\exists$, and $\Diamond$ are defined in the usual way. We assume that $\lor$ and $\land$ bind stronger than $\rightarrow$. 
Given a formula $\varphi$, we denote by $md(\varphi)$ the modal depth of $\varphi$, which is defined inductively, as follows:

\[
\begin{align*}
md(\alpha) &= 0, \text{ where } \alpha \text{ is an atomic formula;} \\
md(\neg \varphi_1) &= \min\{md(\varphi_1), md(\varphi_2)\}; \\
md(\varphi_1 \land \varphi_2) &= \max\{md(\varphi_1), md(\varphi_2)\}; \\
md(\forall y \varphi_1) &= md(\varphi_1); \\
md(\Box \varphi_1) &= md(\varphi_1) + 1.
\end{align*}
\]

We also inductively define, for every $n \in \mathbb{N}$ and every formula $\varphi$, the formulas $\Box^n \varphi$ and $\Diamond^n \varphi$, as follows: $\Box^0 \varphi = \varphi$, $\Box^{n+1} \varphi = \Box \Box^n \varphi$; $\Diamond^0 \varphi = \varphi$, $\Diamond^{n+1} \varphi = \Diamond \Diamond^n \varphi$.

A normal predicate modal logic is a set $L$ of formulas containing the validities of the classical first-order logic as well as the formulas of the form $\Box(\varphi \to \psi) \to (\Box \varphi \to \Box \psi)$, and closed under predicate substitution, modus ponens, generalization (if $\varphi \in L$, then $\forall y \varphi \in L$), and necessitation (if $\varphi \in L$, then $\Box \varphi \in L$).

Modal formulas can be interpreted using Kripke semantics. A Kripke frame is a tuple $\mathfrak{F} = (W, R)$, where $W$ is a non-empty set of worlds and $R$ is a binary accessibility relation on $W$. A predicate Kripke frame is a tuple $\mathfrak{F}_D = (W, R, D)$, where $(W, R)$ is a frame and $D$ is a function from $W$ into a set of non-empty subsets of some set, the domain of $\mathfrak{F}_D$, satisfying the condition that $wRw'$ implies $D(w) \subseteq D(w')$. We call the set $D(w)$ the domain of $w$. If a predicate frame satisfies the condition that $wRw'$ implies $D(w) = D(w')$, we refer to it as a predicate frame with constant domains.

Note that a Kripke frame can be considered, in an obvious way, as a model for the classical first-order language in the signature $\{R, =\}$.

A frame $\mathfrak{F} = (W, R)$ is rooted if there exists $w_0 \in W$ such that $w_0R^*w$ holds for every $w \in W$, where $R^*$ is the reflexive and transitive closure of $R$. Given a frame $(W, R)$ and $w, w' \in W$, we say that $w'$ is accessible from $w$ or that $w$ sees $w'$ if $wRw'$ holds. We say that $w \in W$ is a dead end if $wRv$ does not hold for any $v \in W$. If $\mathfrak{C}$ is a class of frames and $\mathfrak{F} \in \mathfrak{C}$, we refer to $\mathfrak{F}$ as a $\mathfrak{C}$-frame.

A Kripke model is a tuple $\mathfrak{M} = (W, R, D, I)$, where $(W, R, D)$ is a predicate Kripke frame and $I$ is a function assigning to a world $w \in W$ and an $n$-ary predicate letter $P$ an $n$-ary relation $I(w, P)$ on $D(w)$. We refer to $I$ as an interpretation of predicate letters with respect to worlds in $W$.

We say that a model $(W, R, D, I)$ is based on the predicate frame $(W, R, D)$ and the frame $(W, R)$; similarly, we say that a predicate frame $(W, R, D)$ is based on the frame $(W, R)$. We also say that $(W, R, D)$ is the underlying predicate frame for a model $(W, R, D, I)$.

An assignment in a model is a function $g$ associating with every individual variable $y$ an element $g(y)$ of the domain of the underlying predicate frame.

The truth of a formula $\varphi$ at a world $w$ of a model $\mathfrak{M}$ under an assignment $g$ is defined inductively, as follows:

- $\mathfrak{M}, w \models^g P(y_1, \ldots, y_n)$ if $(g(y_1), \ldots, g(y_n)) \in I(w, P)$;
- $\mathfrak{M}, w \models^g \neg \varphi_1$ if $\mathfrak{M}, w \not\models^g \varphi_1$;
that $\varphi$. We say that $\varphi$ is true at a world $w$ of a model $M$ and write $M, w \models \varphi$ if $M, w \models \varphi$ holds for every $g$ assigning to free variables of $\varphi$ elements of $D(w)$. We say that $\varphi$ is valid at a world $w$ of a frame $\mathcal{F}$ if $M, w \models \varphi$ holds for every model $M$ based on $\mathcal{F}$. We say that $\varphi$ is true in $M$ and write $M \models \varphi$ if $M, w \models \varphi$ holds for every world $w$ of $M$. We say that $\varphi$ is valid on a predicate frame $\mathcal{F}_D$ and write $\mathcal{F}_D \models \varphi$ if $\varphi$ is true in every model based on $\mathcal{F}_D$. We say that $\varphi$ is valid on a frame $\mathcal{F}$ and write $\mathcal{F} \models \varphi$ if $\varphi$ is valid on every predicate frame ($\mathcal{F}_D$). We say that a set of formulas $\Gamma$ is valid on a frame $\mathcal{F}$ and write $\mathcal{F} \models \Gamma$ if $\mathcal{F} \models \varphi$ holds for every $\varphi \in \Gamma$. Finally, we say that a set of formulas is valid on a class of frames if it is valid on every frame from the class.

Let $M = \langle W, R, D, I \rangle$ be a model, $w \in W$, and $a_1, \ldots, a_n \in D(w)$. Let $\varphi(y_1, \ldots, y_n)$ be a formula whose free variables are among $y_1, \ldots, y_n$. We write $M, w \models \varphi[a_1, \ldots, a_n]$ to mean that $M, w \models \varphi(y_1, \ldots, y_n)$, where $g(y_1) = a_1, \ldots, g(y_n) = a_n$.

Given a class of frames $\mathcal{C}$, the set of predicate modal formulas valid on every $\mathcal{C}$-frame is denoted by $L(\mathcal{C})$; this set is a normal predicate modal logic.

A normal predicate modal logic $L$ is sound and complete with respect to a class of frames $\mathcal{C}$ if $L = L(\mathcal{C})$; in this case, for brevity, we also say that $L$ is complete with respect to $\mathcal{C}$. A logic is Kripke complete if it is sound and complete with respect to some class of frames.

A class $\mathcal{C}$ of frames is elementary if there exists a closed classical first-order formula $\Phi$ in the signature $\{R, =\}$ such that $\mathcal{F} \in \mathcal{C}$ if, and only if, $\Phi$ is true in $\mathcal{F}$ considered as a classical model, in which case we write $\mathcal{F} \models \Phi$.

The following proposition is well-known (see [7], [2, Proposition 3.12.8]).

**Proposition 2.1** Let $\mathcal{C}$ be an elementary class of frames. Then, $L(\mathcal{C})$ is recursively enumerable.

In the strict sense, the converse of Proposition 2.1 is not true, as some recursively enumerable logics are not complete with respect to any class of frames whatsoever (i.e., are Kripke incomplete), and thus, are not complete with respect to any elementary class; some examples have been mentioned in the Introduction (Section 1). A more interesting question is whether every recursively enumerable Kripke complete logic is a logic of an elementary class of frames.

The main contribution of this paper is in showing, which we do in Section 4, that this is not so for normal predicate logics—namely, we exhibit an example of a recursively enumerable Kripke complete normal predicate logic that is not complete with respect to an elementary class of frames. Prior to that, however, we consider, in the next section, a similar question for rooted frames.
3 An example over rooted frames

In this section, we exhibit the normal predicate modal logic $L_0$ which is recursively enumerable, Kripke complete, but not complete with respect to an elementary class of rooted frames. The restriction to rooted frames is sufficient for most purposes in modal logic; we show, however, later on in this section that in the context of the present inquiry this restriction does matter—while $L_0$ is not complete with respect to an elementary class of rooted frames, it is complete with respect to an elementary class of frames that are not required to be rooted. In the next section, we discard the restriction to rooted frames; the ideas introduced in this section are, however, reused in that context.

Let $F_n$, for $n \geq 1$, be the frame $\langle W_n, R_n \rangle$, where $W_n = \{w_1, \ldots, w_n, w^*\}$ and $R_n = \{(w_i, w_{i+1}) : 1 \leq i < n\} \cup \{(w_1, w^*)\}$; the frame $F_n$ is depicted in Figure 1.

Denote the set of all such frames by $\mathfrak{C}^*$, and let $\mathfrak{C}_0 = \{F_{2n} \in \mathfrak{C}^* : n \geq 1\}$; finally, let $L_0 = L(\mathfrak{C}_0)$.

The logic $L_0$ is Kripke complete by definition. We next show that $L_0$ is recursively enumerable and that $L_0$ is not complete with respect to an elementary class of rooted frames.

To show that $L_0$ is recursively enumerable, we effectively embed it into the classical first-order logic with equality $QCl_\omega$, whose set of theorems is known to be recursively enumerable.

Let $R$ and $D$ be binary, and $W$ unary, predicate letters not occurring in $\varphi$; intuitively, $W(x)$ means that $x$ is a world, $D(x, y)$ means that $y$ is an element of the domain of $x$, and $R(x, y)$ means that $y$ is accessible from $x$.

Let $ST_x(\varphi)$ be the standard translation (see [7], [2, Section 3.12]) of the predicate modal formula $\varphi$ into the language of $QCl_\omega$, defined as follows:

$$
\begin{align*}
ST_x(P(y_1, \ldots, y_m)) &= P'(y_1, \ldots, y_m, x); \\
ST_x(\varphi_1 \land \varphi_2) &= ST_x(\varphi_1) \land ST_x(\varphi_2); \\
ST_x(\neg \varphi_1) &= \neg ST_x(\varphi_1); \\
ST_x(\Box \varphi_1) &= \forall y (W(y) \land R(x, y) \rightarrow ST_y(\varphi_1)); \\
ST_x(\forall y \varphi_1) &= \forall y (\neg W(y) \land D(x, y) \rightarrow ST_x(\varphi_1)),
\end{align*}
$$

where the arity of $P'$ is one greater than the arity of $P$; the letter $P'$ is distinct from the letter $Q'$ if, and only if, $P$ is distinct from $Q$; and all the newly introduced individual variables are distinct from the previously used ones. Intuitively, the variable $x$ in $ST_x(\varphi)$ stands for the world at which $\varphi$ is being evaluated.
Let

\[ M = \exists x W(x) \land \forall x [W(x) \to \exists y D(x, y)] \land \forall x \forall y \forall z [W(x) \land W(y) \land \neg W(z) \land R(x, y) \land D(x, z) \to D(y, z)]. \]

The formula \( M \) describes general properties of predicate Kripke frames; it says that the set of worlds is non-empty, that the domain of every world is non-empty, and that, if world \( y \) is accessible from world \( x \), the domain of \( x \) is included in the domain of \( y \).

Let \( F_n \) be a classical first-order formula in the signature \( \{R, =\} \) describing, for a fixed number \( n \geq 2 \), the disjoint union \( \mathcal{F}_n^w \) of all the frames \( \mathcal{F}_m \in \mathcal{C}_0 \) such that \( m \leq n \). Since \( \mathcal{F}_n^w \) is finite, the formula \( F_n \) can be effectively constructed,—all we need to do is say which worlds exist in \( \mathcal{F}_n^w \), that those worlds are pairwise distinct, that there are no other worlds in \( \mathcal{F}_n^w \), and describe which worlds are, and which are not, related by the accessibility relation.

Next, for an arbitrary classical first-order formula \( \Phi \) in the signature \( \{R, =\} \), inductively define the formula \( \Phi^* \), as follows:

\[
\begin{align*}
(x = y)^* &= (x = y); \\
(R(x, y))^* &= R(x, y); \\
(\Phi_1 \land \Phi_2)^* &= \Phi_1^* \land \Phi_2^*; \\
(\neg \Phi_1)^* &= \neg \Phi_1^*; \\
(\forall x \Phi_1)^* &= \forall x (W(x) \to \Phi_1^*).
\end{align*}
\]

Lastly, given a predicate modal formula \( \varphi \), define

\[ \widehat{\varphi} = M \land F^*_{md(\varphi) + 3} \to \forall x (W(x) \to ST_x(\varphi)). \]

**Lemma 3.1** For every closed predicate modal formula \( \varphi \), the following holds: \( \varphi \in L_0 \) if, and only if, \( \widehat{\varphi} \in \text{QCL}_\omega \).

**Proof.** For the left-to-right, suppose that \( \widehat{\varphi} \notin \text{QCL}_\omega \), i.e., \( \widehat{\varphi} \) fails in some classical first-order model \( \mathfrak{A} \). We build a Kripke model, based on a \( \mathcal{C}_0 \)-frame, refuting \( \varphi \). Since \( \mathfrak{A} \models M \) and \( \mathfrak{A} \models F^*_{md(\varphi) + 3} \), we can construct from \( \mathfrak{A} \) a predicate Kripke frame \( \mathcal{F}_D \) based on a frame isomorphic to \( \mathcal{F}^w_{md(\varphi) + 3} \). Since \( \mathfrak{A} \not\models \forall x (W(x) \to ST_x(\varphi)) \), for some assignment \( g \), both \( \mathfrak{A} \models g W(x) \) and \( \mathfrak{A} \not\models g ST_x(\varphi) \) hold. Recall that the standard translation has the property (see, e.g., [2, Lemma 3.12.2]) that \( \mathfrak{A} \models g ST_x(\varphi) \) if, and only if, \( \mathfrak{M}, v \models \varphi \), where \( \mathfrak{A} \) and \( \mathfrak{M} \) are, respectively, classical and Kripke models agreeing on the interpretation of the predicate letters of \( \varphi \) and where \( v \) is the world corresponding to the element \( g(x) \) of \( \mathfrak{A} \). Therefore, we conclude that \( \mathfrak{M}, w \not\models \varphi \), where \( \mathfrak{M} \) is a model based on \( \mathcal{F}_D \) and \( w \) is the world corresponding, under the isomorphism, to \( g(x) \). Since \( \mathfrak{M} \) is based on a frame isomorphic to the frame \( \mathcal{F}^w_{md(\varphi) + 3} \), which is a disjoint union of \( \mathcal{C}_0 \)-frames, there exists a model \( \mathfrak{M}' \) based on a \( \mathcal{C}_0 \)-frame such that \( \mathfrak{M}', w \not\models \varphi \). Thus, \( \varphi \notin L_0 \).

For the converse, suppose that \( \varphi \notin L_0 \). Then, \( \mathfrak{M}, \widehat{\varphi} \not\models \varphi \), for some model \( \mathfrak{M} \) based on a \( \mathcal{C}_0 \)-frame and some world \( \widehat{w} \) in \( \mathfrak{M} \). We show that, then, \( \varphi \) fails in a model based on a \( \mathcal{C}_0 \)-frame \( \mathcal{F}_m \), where \( m \leq md(\varphi) + 3 \).
If \( \hat{w} = w_1 \), i.e., \( \hat{w} \) is the root of the frame underlying \( \mathcal{M} \), we define the sought model as follows. Consider the chain \( w_1 R w_2 R \ldots \) of worlds of \( \mathcal{M} \); if it contains more than \( s \) worlds, where

\[
s = \begin{cases} 
md(\varphi) + 1, & \text{if } md(\varphi) \text{ is odd}, \\
md(\varphi) + 2, & \text{if } md(\varphi) \text{ is even},
\end{cases}
\]

cut it off at \( w_s \) to obtain the chain \( w_1 R w_2 R \ldots R w_s \) together with \( w^* \) such that \( w_1 R w^* \). Since \( s \leq md(\varphi) + 3 \), we obtain that \( \mathcal{M}', w_1 \not\models \varphi \), for a model \( \mathcal{M}' \) based on one of the disjunct frames from \( \mathcal{F}_{md(\varphi) + 3} \).

If, on the other hand, \( \hat{w} \) is not the root of the frame underlying \( \mathcal{M} \), i.e., \( \hat{w} = w_k \), for some \( k \) \( \geq 1 \) (notice that the case \( \hat{w} = w^* \) is identical to \( k \) being the maximal index of a world in \( \mathcal{M} \)), we define the sought model as follows. First, if the chain \( w_k R w_{k+1} R \ldots \) contains more than \( md(\varphi) + 1 \) worlds, we cut it off at \( w_{k + md(\varphi)} \). Second, if \( md(\varphi) \) is even, we replace the chain \( w_1 R \ldots R w_{k-1} \) together with \( w^* \) accessible from \( w_1 \) by worlds \( w_0 \) and \( w^* \) with the accessibilities \( w_0 R w_k \) and \( w_0 R w^* \); if, on the other hand, \( md(\varphi) \) is odd, we put in an additional world between \( w_0 \) and \( w_k \), so that the length of the resultant chain is even. In either case, the length of the chain of worlds in \( \mathcal{M}' \) does not exceed \( md(\varphi) + 3 \). Hence, \( \mathcal{M}', w_k \not\models \varphi \), for a model \( \mathcal{M}' \) based on one of the disjunct frames from \( \mathcal{F}_{md(\varphi) + 3} \).

Therefore, due to the aforementioned property of the standard translation, there exists a classical first-order model \( \mathcal{A} \) such that \( \mathcal{A} \not\models M \wedge F^*_{md(\varphi) + 3} \rightarrow \forall x (W(x) \rightarrow ST_x(\varphi)) \). Thus, we conclude that \( \hat{\varphi} \not\in \text{QCl}_m \), as required.

\[\Box\]

**Lemma 3.2** Logic \( L_0 \) is recursively enumerable.

**Proof.** Immediate from Lemma 3.1.

It remains to show that \( L_0 \) is not complete with respect to an elementary class of rooted frames.

Given a positive integer \( n \), let

\[\alpha_n = \Diamond \Box \perp \wedge \Diamond^n \Box \perp.\]

It is easy to check that the formula \( \alpha_n \) is valid at worlds that see a dead end in one step and also in \( n \) steps.

**Lemma 3.3** Let \( n \) be a positive integer. Then, \( \neg \alpha_n \in L_0 \) if, and only if, \( n \) is even.

**Proof.** Suppose \( n \) is even. Let \( \mathcal{F}_m \in \mathcal{C}_0 \) and let \( w \) be a world in \( \mathcal{F}_m \). Assume that \( \mathcal{F}_m, w \models \Diamond \Box \perp \). Then, either \( w = w_1 \) or \( w = w_{m-1} \) (notice that \( m \geq 2 \), since \( \mathcal{F}_m \in \mathcal{C}_0 \)). Neither \( w_1 \) nor \( w_{m-1} \), however, can see a dead end in an even number of steps. Indeed, \( w_{m-1} \) can only see a dead end in one step. As for \( w_1 \), it can see two dead ends—one of them in one step, the other in \( m - 1 \) steps; but \( m - 1 \) is odd, since \( \mathcal{F}_m \in \mathcal{C}_0 \) implies that \( m \) is even. Therefore, \( \mathcal{F}_m, w \models \neg \alpha_n \) holds for every \( w \) in \( \mathcal{F}_m \). Thus, \( \neg \alpha_n \in L_0 \).
Suppose $n$ is odd. Then, $F_{n+1}, w_1 \models \alpha_n$ and, hence, $F_{n+1}, w_1 \not\models \neg \alpha_n$. Since $F_{n+1} \in C_0$, we conclude that $\neg \alpha_n \notin L_0$. \hfill \Box

Now, let
\[ alt_2 = \Box p_1 \lor \Box (p_1 \rightarrow p_2) \lor \Box (p_1 \land p_2 \rightarrow p_3), \]
where $p_1$, $p_2$, and $p_3$ are pairwise distinct propositional variables, i.e., 0-ary predicate letters. The formula $alt_2$ is valid at worlds that see at most two worlds (see, e.g., [1, Proposition 3.45]). Hence, $alt_2 \in L_0$.

**Lemma 3.4** Logic $L_0$ is not complete with respect to an elementary class of rooted frames.

**Proof.** Assume otherwise, i.e., let $L_0$ be complete with respect to an elementary class $C$ of rooted frames.

First, we prove that, for every $n \geq 3$,
\[ F_n \in C \iff n \text{ is even}. \]

Suppose $n$ is odd. Then, due to Lemma 3.3, $\neg \alpha_{n-1} \in L_0$. Since $F_n, w_1 \not\models \neg \alpha_{n-1}$, we obtain $F_n \notin C$.

Suppose $n$ is even. Then, due to Lemma 3.3, $\neg \alpha_{n-1} \notin L_0$. Thus, there exists a rooted frame $F' \in C$ and a world $w$ such that $F', w \models \alpha_{n-1}$. We show that, up to isomorphism, $F' = F_n$.

Let $\zeta = \Box p \rightarrow \Box \bot$. It is easy to check that $\zeta$ is valid at a world $w$ if, and only if, $w$ sees at most one world. Since only the roots of $C_0$-frames see more than one world, $\Box \zeta, \neg \zeta \rightarrow \Box \bot \in L_0$. Since $alt_2, \Box \zeta, \neg \zeta \rightarrow \Box \bot \in L_0$, if $F \in C$, then $F$ has a branching degree of at most two, no world in $F$ seen from another world sees more than one world (i.e., only the root of $F$ may see two worlds), and if the root of $F$ sees two worlds, one of them is a dead end. Since $F', w \models \alpha_{n-1}$, the world $w$ is the root of $F'$, and $w$ sees dead ends in one and $n-1$ steps. Therefore, as claimed, $F'$ is isomorphic to $F_n$.

Now, to obtain a contradiction, it remains to notice that classical first-order formulas cannot distinguish even and odd linear orders (see, e.g., [3, Corollary 3.12]). \hfill \Box

We, thus, obtain the following:

**Theorem 3.5** There exists a normal predicate modal logic which is Kripke complete, recursively enumerable, and not complete with respect to an elementary class of rooted frames.

**Proof.** Take $L_0$ as such a logic. \hfill \Box

We next show that the assumption of only considering rooted frames is essential to the example presented above.

**Proposition 3.6** There exists an elementary class $C^*_0$ of frames which are not rooted such that $L_0 = L(C^*_0)$. 

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**Proof.** We construct $\mathcal{C}_0^*$ so that it contains frames that resemble those in $\mathcal{C}_0$, but where every world $w_{2k}$ is marked off by an additional world $w_{2k}^*$, which sees $w_{2k}$. This enables us to tell apart evenly and oddly numbered elements of the chain of worlds of a frame. Then, we make sure that a chain of a frame in $\mathcal{C}_0^*$ does not end with an oddly numbered world. Notice that $\mathcal{C}_0^*$ might contain frames that do not look exactly like $\mathcal{C}_0$-frames with “additional” worlds $w_{2k}^*$; in particular, it might contain infinite frames; this is, however, immaterial to our argument.

We now describe $\mathcal{C}_0^*$-frames with classical first-order formulas (also, see Figure 2).

First, we say that $\mathcal{C}_0^*$-frames are irreflexive and do not contain transitive chains with more than two elements:

$$
\Phi_1 = \forall x \neg R(x, x) \land \forall x \forall y \forall z (R(x, y) \land R(y, z) \rightarrow \neg R(x, z)).
$$

Second, we describe the “bottom part” of a $\mathcal{C}_0^*$-frame, which looks as follows: the bottom-most world $w_1$ sees two worlds, $w^*$ and $w_2$, which is also seen from $w_2^*$, all these worlds being distinct (that $w_2$ is only seen from $w_1$ and $w_2^*$ follows from the formula $\Phi_3$ below):

$$
\Phi_2(w_1) = \exists w^* \exists w_2 \exists w_2^* [\forall x (R(w_1, x) \leftrightarrow x = w^* \lor x = w_2) \\
\land R(w_2^*, w_2) \land \neg \exists x R(x, w_1) \land \neg \exists x R(x, w_2^*) \\
\land \forall x (R(x, w^*) \rightarrow x = w_1) \land \neg \exists x R(w^*, x) \land w_1 \neq w_2^*].
$$

Third, we say that the bottom-most world $w_1$ is the only one that can see more than one world and that no world is seen from more than two worlds:

$$
\Phi_3(w_1) = \forall x (x \neq w_1 \rightarrow \neg \exists y \exists z (y \neq z \land R(x, y) \land R(x, z))) \land \\
\forall w \neg \exists x \exists y \exists z (x \neq y \land y \neq z \land x \neq z \land R(x, w) \land R(y, w) \land R(z, w)).
$$

Lastly, we describe the repetitive procedure of extending an $\mathcal{C}_0^*$-frame by appending to the topmost world $w$ two worlds $y$ and $x$, distinct from $w$ and each other, such that $wRyRx$ holds, as well as a world $z$ that lies off the main chain and sees $x$ (intuitively, $z$ marks off $x$ as the topmost world):

$$
\Phi_4 = \forall w [\exists y \exists z (y \neq z \land R(y, w) \land R(z, w)) \rightarrow \\
\neg \exists x R(w, x) \lor \\
\exists x \exists y \exists z (R(w, y) \land R(y, x) \land R(z, x) \land y \neq z \land x \neq w \land \\
\forall u (R(u, y) \rightarrow u = w) \land \neg \exists u R(u, z))].
$$
The procedure of extending a $C_0$-frame described by $\Phi_4$ can be carried out arbitrarily, including infinitely, many times.

Then, the class $C_0^*$ is defined to contain the frames satisfying the following classical first-order formula:

$$\exists w_1 (\Phi_1 \land \Phi_2(w_1) \land \Phi_3(w_1) \land \Phi_4).$$

We claim that $L(C_0^*) = L(C_0)$.

To see that $L(C_0) \subseteq L(C_0^*)$, let $\mathcal{F}, w \not\models \varphi$, for some $\mathcal{F} \in C_0^*$ and some $w$ in $\mathcal{F}$. Let $\mathcal{F}_w$ be a subframe of $\mathcal{F}$ generated by $w$.

If $w \neq w_1$, since the branching factor of $\mathcal{F}$ is two and only the bottom-most world $w_1$ of $\mathcal{F}$ can see two worlds, $\mathcal{F}_w$ is a chain of worlds. If we take the initial segment $\mathcal{F}_w$ of this chain of length at most $\text{md}(\varphi) + 1$, then clearly, $\mathcal{F}_w$ is isomorphic to a generated subframe of some frame in $C_0$ and, moreover, $\mathcal{F}_w, w \not\models \varphi$. Hence, in this case, $\varphi \notin L(C_0)$.

If, on the other hand, $w = w_1$, the world $w$ sees a dead end $w^*$ and also sees a chain $w_1 R w_2 R \ldots$ of worlds that is built in repetitive stages, starting from $w_1$ and $w_2$ and being extended at every stage by exactly two worlds; hence, the chain $w_1 R w_2 R \ldots$ is either infinite or contains an even number of worlds. In either case, we can find a frame $\mathcal{F} \in C_0$ such that $\mathcal{F}, w_1 \not\models \varphi$; thus, if $w = w_1$, we also obtain $\varphi \notin L(C_0)$.

To see that $L(C_0^*) \subseteq L(C_0)$, notice that, given $\mathcal{F}_n \in C_0$, where $\mathcal{F}_n = \langle W_n, R_n \rangle$, if we make every world $w_{2k} \in W_n$ accessible from an additional world $w^*_{2k}$, then the resulting frame $\mathcal{F}_n^*$ is in $C_0^*$. Since, for every $w \in W_n$, the subframe of $\mathcal{F}_n$ generated by $w$ coincides with the subframe of $\mathcal{F}_n^*$ generated by $w$, we obtain that $\varphi \notin L(C_0)$ implies $\varphi \notin L(C_0^*)$, as required.

Since $L_0 = L(C_0)$, we conclude that $L_1 = L(C_0^*)$.

Thus, the assumption of all frames under consideration being rooted is not insignificant in the context of the present enquiry. We do, however, show in the next section that, if we discard the restriction to rooted frames, the answer to the main question considered in this paper remains negative.

4 An example over arbitrary frames

In this section, we exhibit the normal predicate modal logic $L_1$ that is recursively enumerable, Kripke complete, but not complete with respect to an elementary class of any frames.

Given $n \geq 2$, let $\mathcal{G}_n$ be the frame $\langle W_n, R_n \rangle$, where $W_n = \{w_1, w_2, \ldots, w_n, w^*\}$ and $R_n = \{\langle w_i, w_{i+1} \rangle : 1 \leq i < n \} \cup \{\langle w_n, w_1 \rangle, \langle w_1, w^* \rangle\}$. In other words, $\mathcal{G}_n$ is a ring made up of $n$ worlds, such that one world of the ring sees a dead end; the frame $\mathcal{G}_n$ is depicted in Figure 3. Denote the set of all such frames by $C'$, and let $C_1 = \{\mathcal{G}_n \in C' : n \text{ is even}\}$; finally, let $L_1 = L(C_1)$.

To show that $L_1$ is recursively enumerable, we effectively embed it into the classical first-order logic with equality $\mathsf{QCL}_\omega$. The embedding is very similar to the one defined in the preceding section for $L_0$. The only difference is in the description, using a classical first-
order formula, of a particular frame that fails a modal formula in case the corresponding classical formula is not valid.

Let $G_n$ be a classical first-order formula in the signature $\{R, =\}$ that, for a fixed number $n \geq 2$, describes the disjoint union $G_n^\oplus$ of all the frames $G_m$ in $\mathfrak{C}_1$ such that $m \leq n$. Since $G_n^\oplus$ is finite, $G_n$ can be effectively constructed.

Let the formula $M$, as well as the translations $\cdot^*$ and $ST_x(\cdot)$, be defined as previously. Given a predicate modal formula $\varphi$, define

$$\bar{\varphi} = M \land G_{md(\varphi)+3}^* \rightarrow \forall x (W(x) \rightarrow ST_x(\varphi)).$$

**Lemma 4.1** For every closed predicate modal formula $\varphi$, the following holds: $\varphi \in L_1$ if, and only if, $\bar{\varphi} \in Q\text{Cl}_\mathfrak{C}_1$.

**Proof.** The left-to-right implication is argued as in Lemma 3.1.

For the right-to-left, assume that $\not\models L_1$, i.e., $\not\models M, w^* \not\models \varphi$, for some model $\mathfrak{M}$ based on a $\mathfrak{C}_1$-frame, say $\mathfrak{G}_n$, and some world $\bar{w}$ in $\mathfrak{M}$. We show that, then, $\varphi$ fails in a model based on a $\mathfrak{C}_1$-frame $\mathfrak{G}_m$, where $m \leq md(\varphi) + 3$.

If $\bar{w} = w^*$, then clearly $\not\models \mathfrak{M}, w^* \not\models \varphi$ holds for a model $\mathfrak{M}'$ based on the frame $\mathfrak{G}_2$.

Assume, on the other hand, that $\bar{w} = w_k$, for some $k \in \{1, \ldots, n\}$. If $n \leq md(\varphi) + 3$, then we are done. Otherwise, let $\mathfrak{G}'$ be either $\mathfrak{G}_{md(\varphi)+2}$ or $\mathfrak{G}_{md(\varphi)+3}$, whichever belongs to $\mathfrak{C}_1$ (one of them, clearly, does). It is easy to see that $\mathfrak{G}'$ contains a world $w'$ such that the subframe of $\mathfrak{G}'$ made up of worlds reachable from $w'$ in at most $md(\varphi)$ steps is isomorphic to the subframe of $\mathfrak{G}_n$ made up of worlds reachable from $\bar{w}$ in at most $md(\varphi)$ steps. Hence, $\not\models \mathfrak{M}', w' \not\models \varphi$ holds for a model $\mathfrak{M}'$ based on $\mathfrak{G}'$.

Therefore, due to the property of the standard translation mentioned in the proof of Lemma 3.1, there exists a classical first-order model $\mathfrak{A}$ such that $\not\models M \land G_{md(\varphi)+3}^* \rightarrow \forall x (W(x) \rightarrow ST_x(\varphi))$. Thus, $\bar{\varphi} \not\in Q\text{Cl}_\mathfrak{C}_1$. \hfill $\square$
Lemma 4.2 Logic $L_1$ is recursively enumerable.

Proof. Immediate from Lemma 4.1. □

It remains to show that $L_1$ is not complete with respect to an elementary class of frames. To that end, we define the following formulas:

$$\beta_n = \Diamond \Box \perp \land \Diamond^n \Box \perp \land \bigwedge_{k=1}^{n-1} \neg \Diamond^k \Diamond \Box \perp;$$

$$\gamma = \Diamond \Box \perp \lor (\Diamond p \rightarrow \Box p);$$

$$\delta^k_n = \Diamond^k \beta_n \land p \rightarrow \Box^n p;$$

$$\varepsilon_n = \beta_n \land p \rightarrow \Box^n (\beta_n \land p),$$

where $p$ is a propositional variable.

Lemma 4.3 The formula $\beta_n$ is valid at a world $w$ if, and only if, $w$ can see a dead end, can see in $n$ steps a world that sees a dead end, and cannot see in any number of steps a world that sees a dead end.

The formula $\gamma$ is valid on a frame $\mathfrak{F}$ if, and only if, only those worlds of $\mathfrak{F}$ that see a dead end can see more than one world.

The formula $\delta^k_n$ is valid on a frame $\mathfrak{F}$ if, and only if, every world in $\mathfrak{F}$ that can see in $k$ steps a world $w$ such that $w \models \beta_n$ can see itself in $n$ steps.

The formula $\varepsilon_n$ is valid on a frame $\mathfrak{F}$ if, and only if, every world $w$ of $\mathfrak{F}$ such that $w \models \beta_n$ cannot see in $n$ steps a world $w' \neq w$.

Proof. We only remark on $\varepsilon_n$, leaving the rest to the reader.

Notice that the truth status of $\beta_n$ at a world does not depend on the interpretation. If $\mathfrak{F}$ contains a world $w$ such that $w \models \beta_n$ and $w$ can see in $n$ steps a word $w' \neq w$, then $M, w \not\models \varepsilon_n$ if $M$ is a model based on $\mathfrak{F}$ such that $M, v \models p$ if, and only if, $v = w$. □

Recall that the formula $alt_2$ is valid at worlds that see at most two worlds.

Lemma 4.4 The following formulas belong to $L_1$: $alt_2; \gamma; \delta^k_n$, for every $k,n$; $\varepsilon_n$, for every $n$.

Proof. It is straightforward to check that every $\mathfrak{C}_1$-frame satisfies the properties associated with the formulas listed in the statement of the lemma. □

Lemma 4.5 Let $n$ be a natural number such that $n \geq 1$. Then, $\neg \beta_n \in L_1$ if, and only if, $n$ is odd.

Proof. Suppose $n$ is odd. Let $\mathfrak{G}_m \in \mathfrak{C}_1$ and let $w$ be a world in $\mathfrak{G}_m$. Assume that $\mathfrak{G}_m, w \models \Diamond \Box \perp$. Due to Lemma 4.3, this is only possible if $w = w_1$. Since in no $\mathfrak{C}_1$-frame can $w_1$ see a dead end in an odd number of steps, $\mathfrak{G}_m, w \models \neg \beta_n$ holds for every $w$ in $\mathfrak{G}_m$; thus, $\neg \beta_n \in L_1$.

Suppose $n$ is even. Then, due to Lemma 4.3, $\mathfrak{G}_n, w_1 \models \beta_n$ and, hence, $\mathfrak{G}_n, w_1 \not\models \neg \beta_n$. Since $\mathfrak{G}_n \in \mathfrak{C}_1$, we conclude that $\neg \beta_n \notin L_1$. □
Lemma 4.6 Logic $L_1$ is not complete with respect to an elementary class of frames.

Proof. Assume otherwise, i.e., let $L_1$ be complete with respect to an elementary class $C$ of frames. Further assume that $C$ is defined by a classical first-order formula $\Phi$ with quantifier rank $n$.

Since $2^n$ is even, due to Lemma 4.5, $\neg \beta_{2^n} \notin L_1$. Therefore, $C$ contains a frame $\mathfrak{F} = \langle W, R \rangle$ and $w_1 \in W$ such that $\mathfrak{F}, w_1 \models \beta_{2^n}$. Thus, $\mathfrak{F}$ contains worlds $w_1, \ldots, w_{2^n+1}, w_1^*, w_{2^n+1}^*$ such that $w_1Rw_2 \ldots w_{2^n}Rw_{2^n+1}$, as well as $w_1Rw_1^*$ and $w_{2^n+1}Rw_{2^n+1}^*$. Since $\mathfrak{F} \models \gamma$, every $w_i \in \{w_1, \ldots, w_{2^n}\}$ can see only one world, which is, thus, $w_{i+1}$. As $\mathfrak{F} \models \varepsilon_{2^n}$, due to Lemma 4.3, $w_1 = w_{2^n+1}$. Since $\mathfrak{F} \models alt_2$, we obtain $w_{2^n+1} = w_1^*$ or $w_{2^n+1}^* = w_2$. Since $\mathfrak{F} \models \delta_{2^n}^k$, for every $k$, no world in $X = \{w_1, \ldots, w_2^n\}$ is seen from a world in $W \setminus X$. Indeed, suppose otherwise, i.e., let $vRw$, for some $v \in W \setminus X$ and some $w \in X$. Then, $vRk^kw_1$ holds for some $k \in \mathbb{N}$. Since $\mathfrak{F}, v \models \delta_{2^n}^k$, there is a path of length $2^n$ from $v$ to $v$. Notice that the said path cannot contain worlds from $X$; hence, $w$ is not part of this path, and thus $v$ can see at least two worlds neither of which is a dead end. But, as $\mathfrak{F}, v \models \gamma$, the world $v$ also sees a dead end, which contradicts the fact that $\mathfrak{F}, v \models alt_2$.

Thus, $\mathfrak{F}$ looks like a ring of $2^n$ worlds, where one world in the ring also sees a dead end, and where no world not in the ring can see a world in the ring. Now, consider a frame $\mathfrak{F}'$ that looks like $\mathfrak{F}$, except that its ring is made up of $2^n + 1$, rather than $2^n$, worlds. Since, due to Lemma 4.5, $\neg \beta_{2^n+1} \notin L_1$, we obtain $\mathfrak{F}' \notin C$. One can, however, use Ehrenfeucht–Fraïssé games (see, e.g., [3, Chapter 3]) to show that $\Phi$ cannot distinguish $\mathfrak{F}$ from $\mathfrak{F}'$, which gives us a contradiction. \[\]

Theorem 4.7 There exists a normal predicate modal logic which is Kripke complete, recursively enumerable, and not complete with respect to an elementary class of frames.

Proof. Take $L_1$ as such a logic. \[\]

5 Conclusion

The main result of the present paper is the construction of an example of a normal predicate modal logic that is Kripke complete, recursively enumerable, and not complete with respect to an elementary class of Kripke frames, which solves the problem left open in [5]. Notice that our example is a conservative extension of the classical first-order logic, as no restrictions on either the sizes of the domains of the worlds or the possible interpretations of predicate letters have been used in its construction. Also notice that a similar example can be constructed, along the same lines, of a logic of predicate frames with constant domains.

It remains unclear to us whether a similar example can be constructed in the extensions of the intuitionistic predicate logic. A step toward answering that question would be the construction of a normal predicate modal logic, satisfying the properties studied in this
paper, whose frames are reflexive and transitive, as is the accessibility relation in the extensions of the intuitionistic logic. While it is not difficult to modify the example presented in Section 4 to construct a logic, satisfying the same properties, of reflexive frames, it is not clear to us whether an example over reflexive and transitive—or, simply, transitive—frames exists.

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