CLASSIFICATION OF CERTAIN CELLULAR CLASSES OF
CHAIN COMPLEXES

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Abstract. Let \((R, m)\) be a local commutative ring. Suppose that \(m\) is principal and that \(m^2 = 0\). We give a complete description of the cellular lattice of perfect chain complexes of modules over this ring.

1. Introduction

An explicit classification of thick subcategories of finite spectra \([\text{HS}98]\) and \([\text{DH}95]\) has been an important achievement of homotopy theory. This stable classification has been used to give an explicit classification of unstable Bousfield classes of finite suspension spaces \([\text{Bou}96]\). In contrast, an analogous classification of cellular classes of finite (suspension) spaces is out of reach as that would lead to a classification of ideals in the stable homotopy groups of spheres.

Thick subcategories of compact objects in the derived category of a ring are also well understood \([\text{Nee}92]\). Recently Bousfield classes (or Acyclic classes) of chain complexes have been classified, see \([\text{Sta}]\) or \([\text{Kie}]\). However, as for spaces, the classification of cellular classes is more subtle. The aim of this paper is to give some examples of rings for which a classification can be obtained. For some properties of cellular classes of chain complexes over a Noetherian ring, see \([\text{Kie}]\).

Throughout this paper all chain complexes are non-negatively graded chain complexes of modules over some fixed commutative ring.

Definition 1.1. Fix a chain complex \(A\). We let \(\mathcal{C}(A)\) denote the smallest collection of chain complexes satisfying the following properties: 1. The collection \(\mathcal{C}(A)\) contains \(A\), 2. It is closed under arbitrary sums and weak equivalences (i.e. homology isomorphisms) 3. If \(0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0\) is an exact sequence with \(X, Y \in \mathcal{C}(A)\) then also \(Z \in \mathcal{C}(A)\). If \(X \in \mathcal{C}(A)\) then we write that \(X \gg A\) and say that \(X\) is \(A\)-cellular.

In the paper we describe explicitly the cellular relations between perfect chain complexes of modules over a local ring \((R, m)\) such that \(m\) is principal and \(m^2 = 0\).

For such a ring and numbers \(i, j \geq 0\), we let:

\[
(\Sigma^i E_j)_n = \begin{cases} R & \text{for } i \leq n \leq i + j \\ 0 & \text{otherwise} \end{cases}
\]

with the differentials given by multiplication by \((-1)^i r\), where \(r\) is some generator of the maximal ideal \(m\). The isomorphism type of \(\Sigma^i E_j\) does not depend on the choice of the generator \(r\).

We prove that:

Theorem 1.2. Let \((R, m)\) be a local ring such that \(m\) is principal and \(m^2 = 0\). Let \(A\) be a perfect chain complex that is not weakly equivalent to 0. Then there exists \((i, j)\) such that:

\[\mathcal{C}(A) = \mathcal{C}(\Sigma^i E_j)\]

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Moreover \( \Sigma^j E_i \gg \Sigma^j E_i \) if and only if either \( i' \geq i \) or \( i' = i \) and \( j' \geq j \).

The key element in the proof of the theorem is a classification of the perfect chain complexes. We show that any perfect chain complex \( A \) splits into a sum of a contractible chain complex and a sum of \( \Sigma^j E_j \)'s (see Lemma 5.2).

2. Notation

We let \( R \) denote some arbitrary commutative ring. By a \textit{chain complex} \( X \) we mean a non-negatively graded chain complex of \( R \)-modules. We use the homological grading, i.e. the differential of \( X \) lowers the degree. Recall that the category of chain complexes of \( R \)-modules \( \text{Ch}_{\geq 0}(R) \) is a model category \cite{DS95}. In this model category a \textit{weak equivalence} is a map that induces an isomorphism on homology. A \textit{cofibration} is an injective map such that the cokernel is projective in each degree. A \textit{fibration} is map which is surjective in all positive degrees. We let \( \sim \) denote a weak equivalence. A cofibrant chain complex \( X \) is a chain complex such that the canonical map \( 0 \to X \) is a cofibration, or explicitly, it is a chain complex of projective modules.

If \( f : X \to Y \) is any map of chain complexes then we can factor \( f = f'' f' \), where \( f' : X \to X' \) is a cofibration and \( f'' : X' \sim Y \) is a weak equivalence \cite{DS95}.

Hence, any map is a cofibration up to a weak equivalence.

A complex \( X \) is called \textit{perfect} if it is cofibrant and \( \oplus_i X_i \) is finitely generated.

We let \( \text{Hom} \) denote the \textit{hom-complex}. It is defined as follows: if \( X, Y \in \text{Ch}_{\geq 0}(R) \) then \( \text{Hom}(X, Y)_n = \prod_i \text{hom}(X_i, Y_{i+n}) \) for \( n > 0 \) and \( \text{Hom}(X, Y)_0 \) is the set of maps of chain complexes from \( X \) to \( Y \) with the induced \( R \)-module structure. The differential takes \( \{ f_i : X_i \to Y_{i+n} \}_{i} \in \text{Hom}(X, Y)_n \) to \( \{ \partial f_i + (-1)^n f_{i-1} \partial \}_{i} \).

If \( A \) is cofibrant, then \( \text{Hom}(A, \bullet) \) preserves weak equivalences and fibrations (a consequence of Brown’s lemma, see \cite{DS95}).

We let \( \Sigma^i \) denote the \textit{shift operator} i.e \( (\Sigma^i X)_j = X_{j-i} \) and \( \partial \Sigma^i X = (-1)^i \partial \). The \textit{cone} of a map \( f : X \to Y \) is a chain complex \( C(f) \) defined by: \( C(f)_n = Y_n \oplus X_{n-1} \).

The differential of \( C(f) \) maps \( (y, x) \in C(f)_n \) to \( (\partial^X (y) + f(x), -\partial^Y (x)) \). Note that there is a canonical map \( Y \to C(f) \) and the cokernel of this map is isomorphic to \( \Sigma^1 X \). If \( 0 \to X \to Y \to Z \to 0 \) is an exact sequence of chain complexes then there is a natural map from the cone of \( X \to Y \to Z \). From the induced long exact sequences of homology, this map is a weak equivalence.

A more detailed account of the theory of chain complexes can be found for instance in \cite{Ves93}.

Associated to any module \( M \) are the \textit{sphere complex}, \( S^n(M) \), and the \textit{disk complex}, \( D^n(M) \), defined by:

\[
(S^n(M))_i = \begin{cases} M & i = n \\ 0 & i \neq n \end{cases} \quad (D^n(M))_i = \begin{cases} M & i = n \text{ or } i = n - 1 \\ 0 & \text{otherwise} \end{cases}
\]

With the differential \( \partial_n = 1_R \) in \( D^n(M) \). For short we let \( S^n := S^n(R) \) and \( D^n := D^n(R) \).

3. Cellular relation

Recall that there is an alternative description of cellularity via a universal property (see \cite{Far90}):

\textbf{Proposition 3.1.} Let \( X \) and \( A \) be cofibrant chain complexes. Then \( X \) is \( A \)-cellular if and only if for all maps \( f \) such that \( \text{Hom}(A, f) \) is a weak equivalence, the map \( \text{Hom}(X, f) \) is also a weak equivalence.

The cellular relation is transitive, in other words if \( X \gg A \) and \( Y \gg X \) then \( Y \gg A \).
To determine whether a given chain complex $X$ belongs to $C(A)$ is in general a hard question. In Proposition 3.2 we give a workable criteria for detecting cellularity for a very particular choice of ring $R$. We now list certain properties of cellularity.

**Proposition 3.2.**  
(i) All chain complexes are $S^0$-cellular.
(ii) If $X$ is acyclic (i.e. all homology group vanish), then $X \gg A$ for any chain complex $A$.
(iii) The collection $C(A)$ is closed under retracts.
(iv) Suppose that $A$ is a cofibrant chain complex. If $X \gg A$ and $H_0(A) \neq 0$ then there is a set $I$ and a map $f: \oplus_I A \to X$ such that $H_0(f)$ is surjective.
(v) If $X \gg A$ then $\Sigma^n X \gg A$ for all $n \geq 0$.
(vi) $\Sigma^1 X \gg \Sigma^1 A \iff X \gg A$.

**Proof.** We shall only give an outline of the proof. Statement (i) follows from the isomorphism $\text{Hom}(S^0, X) \cong X$ and $\Sigma^1$. To prove the second statement note first that $0 \in C(A)$ since $\text{Hom}(0, Y) = 0$. If $X$ is acyclic then $0 \to X$ is a weak equivalence and since $C(A)$ is closed under weak equivalences $X \gg A$. A retract of an isomorphism is an isomorphism so (iii) is a consequence of (3.1).

We fix a cofibrant chain complex $A$ and let $D$ denote the collection of all chain complexes $X$ such that there is a set $I$ and a map $f: \oplus_I A \to X$ surjective on $H_0$. It is a standard result in homological algebra ([Wei94], p. 388) that if $f: X \to Y$ is a weak equivalence and $g: A \to Y$ is any map, then because $A$ is cofibrant, there is a map $h: A \to X$ such that $g$ and $fh$ are homotopic. As a consequence, the collection $D$ is closed under weak equivalences. It is also closed under sums. Finally if $0 \to X \to Y \to Z \to 0$ is an exact sequence such that $X, Y \in D$ then also $Z \in D$. By definition $C(A)$ was the smallest collection satisfying these properties, hence $C(A) \subset D$ and we have proved (iv).

Statements (v) and (vi) are direct consequences of (3.1). \hfill \Box

4. **Two out of three property**

We say that a collection of chain complexes $C$ satisfies the **two out of three property** if given any exact sequence $0 \to X \to Y \to Z \to 0$ such that two out of $X$, $Y$ and $Z$ belong to $C$ then so does the third. The collection $C(A)$ does not in general satisfy the two out of three property. For instance it follows from (3.2) that $C(S^1)$ equals the collection of all chain complexes $X$ such that $H_0X = 0$. There is an exact sequence $0 \to S^0 \to D^1 \to S^1 \to 0$ and $D^1$ is $S^1$-cellular, but $S^0$ is not.

Collections of possibly unbounded chain complexes of modules over a Noetherian ring, satisfying the two out of three property, that are closed under sums and weak equivalences have been classified by Neeman in [Nee92]. They are in 1-1 correspondence with arbitrary sets of prime ideals in $R$.

A collection $C$ of chain complexes is closed under extensions if given any exact sequence $0 \to X \to Y \to Z \to 0$ with $X, Z \in C$ then also $Y \in C$. The collection $C(A)$ is in general not closed under extensions. In analogy with cellularity we can now define a relation called **acyclicity**:

**Definition 4.1.** Fix a chain complex $A$. Let $\mathcal{A}(A)$ denote the smallest collection of chain complexes satisfying the following properties: 1. The collection $\mathcal{A}(A)$ contains $A$, 2. It is closed under arbitrary sums, 3. If in an exact sequence $0 \to X \to Y \to Z \to 0$, either $X$ and $Y$ or $X$ and $Z$ belong to $\mathcal{A}(A)$, then so does the third. If $X \in \mathcal{A}(A)$ then we write that $X > A$ and say that $X$ is $A$-acyclic.

The collection $\mathcal{A}(A)$ is in particular closed under extensions. By definition $C(A) \subset \mathcal{A}(A)$. This inclusion is in general strict:
Example 4.2. Recall from the introduction the definition of \( E_i := \Sigma^0 E_i \). Up to a weak equivalence, \( E_1 \) is an extension of \( E_3 \) by \( E_2 \): Let \( r \) denote some generator of the maximal ideal. Multiplication by \( r \) in degree 0 and the zero map in higher degrees defines a map \( f : E_2 \to E_1 \). The cone of \( f \) is isomorphic to \( E_3 \). This gives the sequence: \( E_2 \to E_1 \to E_3 \).

We later show (Theorem 6.3) that \( E_3 \gg E_2 \) (that is \( E_3 \) is \( E_2 \)-cellular) and that \( E_1 \) is not \( E_2 \)-cellular. However \( E_1 > E_2 \) since \( E_1 \) is an extension of \( E_3 \) by \( E_2 \) and \( E_3 \gg E_2 \).

The relation \( > \) between perfect chain complexes of modules over a Noetherian ring is well understood. In fact \( X > A \) if and only if for every \( n \) the following holds:

if \( p \subset R \) is a prime ideal such that \( X_n \otimes R_p \neq 0 \) then \( A_i \otimes R_p \neq 0 \) for some \( i \leq n \).

The proof of this result can be found in the papers [Sta] and [Kie].

The following Proposition establishes an important connection between cellularity and acyclicity. An analogous result for topological spaces was obtained by Dror-Farjoun in [Far96].

Proposition 4.3. Fix a short exact sequence \( 0 \to X \to Y \to Z \to 0 \). If \( X \gg A \) and \( Z > \Sigma^1 A \) then \( Y \gg A \).

Proof. See [Kie]. A proof of the topological analog can be found in [Far96]. □

5. A Key Lemma

For the rest of this paper we fix a local commutative ring \( R \) with a principal maximal ideal \( m \) such that \( m^2 = 0 \). We let \( k \) denote the residue field \( R/m \). We choose some generator \( r \) of \( m \). Note that in such a ring all non-unitary elements are of the form \( x = r'r \), for some unit \( r' \). Good examples to keep in mind are \( R = \mathbb{Z}/(p^2) \) (\( p \) prime) and \( R = k[X]/(X^2) \), for some field \( k \).

Definition 5.1. An injective map \( D^n \to X \) is called an embedded disk.

Recall from the introduction the special class of chain complexes:

\[
(E_j)_n = \begin{cases} 
R & \text{if } n \leq j \\
0 & \text{if } n > j 
\end{cases}
\]

With \( \partial_n : (E_j)_n \to (E_j)_{n-1} \) multiplication by \( r \). For instance, \( E_\infty \) is a projective resolution of \( k \) and \( E_1 = S^0 \).

Lemma 5.2. Let \( X \) be any perfect chain complex. Then there is a splitting of \( X \):

\[
X \cong P \oplus Q
\]

where \( P \) is acyclic and \( Q \) can be written as a finite sum

\[
Q \cong \oplus \Sigma E_j
\]

Proof. The proof is divided into several steps and takes up the rest of this section. Fix some perfect chain complex \( X \).

Step 1

We first split off the contractible part of \( X \).

Remark 5.3. \( R \) is injective as a module over itself. Hence \( D^n \) is an injective object, i.e. any embedded disk \( D^n \to X \) is split.

A consequence of this remark is that \( X \) will split into a direct sum

\[
X \cong P \oplus \tilde{X}
\]

where \( P \) is an acyclic complex (a finite sum of disks) and \( \tilde{X} \) has no embedded disks.

Step 2

We can now assume that \( X \) has no embedded disks. We also want to assume that \( X_0 \neq 0 \).
Remark 5.4. For a perfect complex $X$, to have no embedded disks is equivalent to \( \partial_n(X_n) \subset rX_{n-1} \) for all $n$. Such a complexes are also known as minimal.

Suppose that $H_0(X) = 0$. Then $\partial_1$ is surjective. From the remark we see that $X_0 = 0$. Hence $X = \Sigma^1Y$ for some perfect complex $Y$.

**Step 3**

By step 1 and 2 we can assume that $X$ is a perfect chain complex, containing no embedded disks and that $X_0 \neq 0$.

Since $H_0(X) \neq 0$ there is a surjection $X \to S^0(k)$ and because $X$ is cofibrant this map factors:

$$X \to E_\infty \xrightarrow{\sim} S^0(k)$$

where $E_\infty \xrightarrow{\sim} S^0(k)$ is an acyclic fibration (i.e. a fibration and a weak equivalence).

The chain complex $X$ is perfect, in particular $\oplus_i X_i$ is finitely presented, so the map $X \to E_\infty$ factors through some $E_n$. We note that the map $X \to E_n$ is surjective in degree 0. There is some smallest $n_0$ such that there is a map $f : X \to E_{n_0}$ surjective in degree 0. We claim that $f$ is surjective and has a section, so that $X \cong E_{n_0} \oplus \hat{X}$.

First we show that $f$ is surjective. Suppose that this is not the case. Then there is some $m \leq n_0$ such that im $f_m \subset \ker \partial_m$. We truncate $f$ at $m-1$ and get a map $\hat{f} : X \to E_{m-1}$ with $\hat{f}_j = f_j$ for $j \leq m-1$ and $\hat{f}_j = 0$ for $j \geq m$. Then $\hat{f}$ is surjective in degree 0, contradicting the minimality of $n_0$. Hence $f$ is surjective.

We first fix some perfect complex $X$, so that there is some $\hat{X}$.

**Step 2**

We truncate $X$ at $m$ and get a map $f : X \to E_{m-1}$.

By step 1 and 2 we can assume that $X$ is a perfect chain complex, containing no embedded disks and that $X_0 \neq 0$.

Since $H_0(X) \neq 0$ there is a surjection $X \to S^0(k)$ and because $X$ is cofibrant this map factors:

$$X \to E_\infty \xrightarrow{\sim} S^0(k)$$

where $E_\infty \xrightarrow{\sim} S^0(k)$ is an acyclic fibration (i.e. a fibration and a weak equivalence).

The chain complex $X$ is perfect, in particular $\oplus_i X_i$ is finitely presented, so the map $X \to E_\infty$ factors through some $E_n$. We note that the map $X \to E_n$ is surjective in degree 0. There is some smallest $n_0$ such that there is a map $f : X \to E_{n_0}$ surjective in degree 0. We claim that $f$ is surjective and has a section, so that $X \cong E_{n_0} \oplus \hat{X}$.

First we show that $f$ is surjective. Suppose that this is not the case. Then there is some $m \leq n_0$ such that im $f_m \subset \ker \partial_m$. We truncate $f$ at $m-1$ and get a map $\hat{f} : X \to E_{m-1}$ with $\hat{f}_j = f_j$ for $j \leq m-1$ and $\hat{f}_j = 0$ for $j \geq m$. Then $\hat{f}$ is surjective in degree 0, contradicting the minimality of $n_0$. Hence $f$ is surjective.

We first fix some perfect complex $X$, so that there is some $\hat{X}$.

**Step 2**

We truncate $X$ at $m$ and get a map $f : X \to E_{m-1}$.

By step 1 and 2 we can assume that $X$ is a perfect chain complex, containing no embedded disks and that $X_0 \neq 0$.

Since $H_0(X) \neq 0$ there is a surjection $X \to S^0(k)$ and because $X$ is cofibrant this map factors:

$$X \to E_\infty \xrightarrow{\sim} S^0(k)$$

where $E_\infty \xrightarrow{\sim} S^0(k)$ is an acyclic fibration (i.e. a fibration and a weak equivalence).

The chain complex $X$ is perfect, in particular $\oplus_i X_i$ is finitely presented, so the map $X \to E_\infty$ factors through some $E_n$. We note that the map $X \to E_n$ is surjective in degree 0. There is some smallest $n_0$ such that there is a map $f : X \to E_{n_0}$ surjective in degree 0. We claim that $f$ is surjective and has a section, so that $X \cong E_{n_0} \oplus \hat{X}$.

First we show that $f$ is surjective. Suppose that this is not the case. Then there is some $m \leq n_0$ such that im $f_m \subset \ker \partial_m$. We truncate $f$ at $m-1$ and get a map $\hat{f} : X \to E_{m-1}$ with $\hat{f}_j = f_j$ for $j \leq m-1$ and $\hat{f}_j = 0$ for $j \geq m$. Then $\hat{f}$ is surjective in degree 0, contradicting the minimality of $n_0$. Hence $f$ is surjective.

We fix a generator $e_j$ of $(E_{n_0})_j$ for each $j$. From the surjectivity of $f_{n_0}$ it follows that there is some $x_{n_0} \in X_{n_0}$ such that $f_{n_0}(x) = e_{n_0}$. By remark there is some $x_{n_0-1}$ such that $r x_{n_0-1} = \partial(x_{n_0})$. Moreover $f_{n_0-1}(x_{n_0-1}) = a_{n_0-1} e_{n_0-1}$, for some unit $a_{n_0-1}$. Inductively we obtain a sequence of elements $(x_0, \ldots, x_n)$ and $(a_0, \ldots, a_n)$. We define a map $s : E_{n_0} \to X$ by $s_j(e_j) = x_j$, $j \leq n_0$. This is well defined since $\partial s_j(e_j) = r x_{j-1} = s_{j-1}(\partial(e_{j-1}))$. By construction $(f \circ s)_j(e_j) = a_j e_j$ so that $f \circ s$ is an isomorphism.

This determines a splitting of $X$ into

$$X \cong E_{n_0} \oplus \hat{X}$$

We can repeat the above discussion with $\hat{X}$ instead of $X$. This yields the required splitting formula.

\[ \square \]

6. Statement of Results

Recall that $R$ is assumed to be a commutative local ring with a principal maximal ideal $m$ such that $m^2 = 0$. We are now in a position to give a complete description of the cellular lattice of perfect complexes.

First we look at the cellularity relations among $E_i$'s. We begin with an observation.

**Remark 6.1.** If $i > j$ then there is no map $f : E_i \to E_j$, such that $H_0(f)$ is non-zero.

This remark in combination with the following proposition is enough to classify the cellular relations between the $E_i$'s.

**Proposition 6.2.** Let $X$ be a perfect complex such that $H_0(X) \neq 0$. Then $Y$ is $X$—cellular if and only if there is a set $I$ and a map $f : \oplus_{i \in I} X \to Y$ such that $f$ induces an epimorphism on the $H_0$.

**Proof.** We first fix some perfect complex $X$ with $H_0(X) \neq 0$. Let $D(X)$ denote the class of all complexes $Y$ such that there is some $I$ and $f : \oplus_{i \in I} Y \to X$ with $H_0(f)$ surjective. The statement is that $D(X) = C(X)$.
By Proposition 5.2. To prove that \( D(X) \subset C(X) \) we first note that since \( X \) is cofibrant, \( X \otimes \bullet \) preserves cellularity: if \( A \mid B \) then \( A \otimes X \gg B \otimes X \). Here \( \otimes \) denotes the ordinary tensor product of chain complexes (see [Wei94]). Since \( X \otimes S^0 \cong X \) and \( S^0(k) \gg S^0 \) we can conclude that \( X \otimes S^0(k) \gg X \). We can always assume that there are no embedded disks in \( X \) and in this case \( H_0(X \otimes k) \cong (X \otimes k)_0 \).

By assumption \( H_0(X) \neq 0 \), so there is a retraction: \( S^0(k) \to X \otimes k \to S^0(k) \). It follows from Proposition 5.2 that \( S^0(k) \gg X \otimes k \gg X \).

Fix a map \( f : \oplus_{i \in I} X \to Y \) such that \( H_0(f) \) is surjective. We can assume that \( f : \oplus_{i \in I} X \to Y \) is a cofibration (see section 2). Let \( Z := Y / f(\oplus_{i \in I} X) \). We wish to show that \( Z \) is \( X \)-cellular. By Proposition 4.3 it is enough to show that \( Z > \Sigma^1 X \).

We assumed that \( H_0(f_0) \) is surjective, so \( H_0(Z) \cong 0 \). As a consequence \( Z \gg \Sigma^1 \).

There is an isomorphism of \( R \)-modules \( m \cong k \). The exact sequence \( 0 \to k \to R \to k \to 0 \) shows that \( S^0 > S^0(k) \) or equivalently that \( S^1 > S^1(k) \). In the paragraph above we showed that \( S^1(k) \gg \Sigma^1 X \). In all we have that \( Z > \Sigma^1 X \), i.e. \( Z > \Sigma^1 X \). This concludes the proof of the proposition.

Consider the set \( S \) of all pairs \( (i, j) \) of non-negative integers. We order this set by declaring \( (i, j) < (i', j') \) if either \( i < i' \) or \( i = i' \) and \( j < j' \). To each perfect complex \( X \) we assign the subset \( S_X \subseteq S \) consisting of all pairs \( (i, j) \) such that \( \Sigma^i E_{j} \) appears in the splitting of \( X \) as in lemma 5.2. Finally we let \( (i_X, j_X) \) denote the minimal element of \( S_X \). We have shown that:

**Theorem 6.3.** Let \((R, m)\) be a local ring such that \( m \) is principal and \( m^2 = 0 \). Let \( X \) be a perfect complex over \( R \). Then

\[
\Sigma^i E_{j'} \gg \Sigma^i E_{j}
\]

if and only if \( (i, j) \leq (i', j') \). Moreover:

\[
\mathcal{C}(X) = \mathcal{C}(\Sigma^i X E_{j_X})
\]

unless \( X \) is contractible, in which case \( \mathcal{C}(X) = \mathcal{C}(0) \).

**References**

[Bou96] A. K. Bousfield, *Unstable localization and periodicity*, Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guixols, 1994), Progr. Math., vol. 136, Birkhäuser, Basel, 1996, pp. 33–50.

[DH95] Ethan S. Devinatz and Michael J. Hopkins, The action of the Morava stabilizer group on the Lubin-Tate moduli space of lifts, Amer. J. Math. 117 (1995), no. 3, 669–710.

[DS95] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.

[Far96] Emmanuel Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996.

[HS98] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) 148 (1998), no. 1, 1–49.

[Kie] Jonas Kiessling, Properties of cellular classes, In preparation.

[Nee92] Amnon Neeman, The chromatic tower for \( D(R) \), Topology 31 (1992), no. 3, 519–532, With an appendix by Marcel Bökstedt.

[Sta] Don Stanley, Invariants of \( t \)-structures and classification of nullity classes, arXiv:math/0602252 v1.

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.