The inverse problem for representation functions of additive bases

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Abstract

Let $A$ be a set of integers. For every integer $n$, let $r_{A,2}(n)$ denote the number of representations of $n$ in the form $n = a_1 + a_2$, where $a_1, a_2 \in A$ and $a_1 \leq a_2$. The function $r_{A,2} : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ is the representation function of order 2 for $A$. The set $A$ is called an asymptotic basis of order 2 if $r_{A,2}^{-1}(0)$ is finite, that is, if every integer with at most a finite number of exceptions can be represented as the sum of two not necessarily distinct elements of $A$. It is proved that every function is a representation function, that is, if $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ is any function such that $f^{-1}(0)$ is finite, then there exists a set $A$ of integers such that $f(n) = r_{A,2}(n)$ for all $n \in \mathbb{Z}$. Moreover, the set $A$ can be constructed so that $\text{card}\{a \in A : |a| \leq x\} \gg x^{1/3}$.

1 Representation functions

Let $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ denote the positive integers, nonnegative integers, and integers, respectively. Let $A$ and $B$ be sets of integers. We define the sumset

$$A + B = \{a + b : a \in A \text{ and } b \in B\},$$

and, in particular,

$$2A = A + A = \{a_1 + a_2 : a_1, a_2 \in A\}$$
and
\[ A + b = A + \{b\} = \{a + b : a \in A\}. \]

The restricted sumsets are
\[ A \hat{+} B = \{a + b : a \in A, b \in B, \text{ and } a \neq b\} \]
and
\[ 2 \& A = A \hat{+} A = \{a_1 + a_2 : a_1, a_2 \in A \text{ and } a_1 \neq a_2\}. \]

Similarly, we define the difference set
\[ A - B = \{a - b : a \in A \text{ and } b \in B\} \]
and
\[ -A = \{0\} - A = \{-a : -a \in A\}. \]

We introduce the counting function
\[ A(y, x) = \sum_{a \in A, y \leq a \leq x} 1. \]

Thus, \(A(-x, x)\) counts the number of elements \(a \in A\) such that \(|a| \leq x\).

For functions \(f\) and \(g\), we write \(f \gg g\) if there exist numbers \(c_0\) and \(x_0\) such that \(|f(x)| \geq c_0|g(x)|\) for all \(x \geq x_0\), and \(f \ll g\) if \(|f(x)| \leq c_0|g(x)|\) for all \(x \geq x_0\).

In this paper we study representation functions of sets of integers. For any set \(A \subseteq \mathbb{Z}\), the representation function \(r_{A,2}(n)\) counts the number of ways to write \(n\) in the form \(n = a_1 + a_2\), where \(a_1, a_2 \in A\) and \(a_1 \leq a_2\). The set \(A\) is called an asymptotic basis of order 2 if all but finitely many integers can be represented as the sum of two not necessarily distinct elements of \(A\), or, equivalently, if the function
\[ r_{A,2} : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\} \]
satisfies
\[ \text{card}(r_{A,2}^{-1}(0)) < \infty. \]

Similarly, the restricted representation function \(r_{A,2}(n)\) counts the number of ways to write \(n\) in the form \(n = a_1 + a_2\), where \(a_1, a_2 \in A\) and \(a_1 < a_2\). The set \(A\) is called a restricted asymptotic basis of order 2 if all but finitely many integers can be represented as the sum of two distinct elements of \(A\).

Let
\[ f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\} \]
be any function such that
\[ \text{card}(f^{-1}(0)) < \infty. \]

The inverse problem for representation functions of order 2 is to find sets \(A\) such that \(r_{A,2}(n) = f(n)\) for all \(n \in \mathbb{Z}\). Nathanson [4] proved that every function \(f\)
satisfying (1) and (2) is the representation function of an asymptotic basis of order 2, and that such bases $A$ can be arbitrarily thin in the sense that the counting functions $A(-x, x)$ tend arbitrarily slowly to infinity. It remained an open problem to construct thick asymptotic bases of order 2 for the integers with a prescribed representation function.

In the special case of the function $f(n) = 1$ for all integers $n$, Nathanson \[6\] constructed a unique representation basis, that is, a set $A$ of integers with $r_{A,2}(n) = 1$ for all $n \in \mathbb{Z}$, with the additional property that $A(-x, x) \gg \log x$. He posed the problem of constructing a unique representation basis $A$ such that $A(-x, x) \gg x^\alpha$ for some $\alpha > 0$.

In this paper we prove that for every function $f$ satisfying (1) and (2) there exist uncountably many asymptotic bases $A$ of order 2 such that $r_{A,2}(n) = f(n)$ for all $n \in \mathbb{Z}$, and $A(-x, x) \gg x^{1/3}$. It is not known if there exists a real number $\delta > 0$ such that one can solve the inverse problem for arbitrary functions $f$ satisfying (1) and (2) with $A(-x, x) \gg x^{1/3+\delta}$.

2 The Erdős-Turán conjecture

The set $A$ of nonnegative integers is an asymptotic basis of order 2 for $\mathbb{N}_0$ if the sumset $2A$ contains all sufficiently large integers. If $A$ is a set of nonnegative integers, then

$$0 \leq r_{A,2}(n) < \infty$$

for every $n \in \mathbb{N}_0$. It is not true, however, that if

$$f : \mathbb{N}_0 \to \mathbb{N}_0$$

is a function with

$$\text{card} \ (f^{-1}(0)) < \infty,$$

then there must exist a set $A$ of nonnegative integers such that $r_{A,2}(n) = f(n)$ for all $n \in \mathbb{N}_0$. For example, Dirac \[1\] proved that the representation function of an asymptotic basis of order 2 cannot be eventually constant, and Erdős and Fuchs \[3\] proved that the mean value $\sum_{n \leq x} r_{A,2}(n)$ of an asymptotic basis of order 2 cannot converge too rapidly to $cx$ for any $c > 0$. A famous conjecture of Erdős and Turán \[2\] states that the representation function of an asymptotic basis of order 2 must be unbounded. This problem is only a special case of the general inverse problem for representation functions for bases for the nonnegative integers: Find necessary and sufficient conditions for a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying $\text{card} \ (f^{-1}(0)) < \infty$ to be the representation function of an asymptotic basis of order 2 for $\mathbb{N}_0$.

It is a remarkable recent discovery that the inverse problem for representation functions for the integers, and, more generally, for arbitrary countably infinite abelian groups and countably infinite abelian semigroups with a group component, is significantly easier than the inverse problem for representation functions for the nonnegative integers and for other countably infinite abelian semigroups (Nathanson \[5\]).
3 Construction of thick bases for the integers

Let \([x]\) denote the integer part of the real number \(x\).

**Lemma 1** Let \(f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}\) be a function such that \(f^{-1}(0)\) is finite. Let \(\Delta\) denote the cardinality of the set \(f^{-1}(0)\). Then there exists a sequence \(U = \{u_k\}_{k=1}^{\infty}\) of integers such that, for every \(n \in \mathbb{Z}\) and \(k \in \mathbb{N}\),
\[
f(n) = \text{card}\{k \geq 1 : u_k = n\}
\]
and
\[
|u_k| \leq \left\lfloor \frac{k + \Delta}{2} \right\rfloor.
\]

**Proof.** Every positive integer \(m\) can be written uniquely in the form
\[
m = s^2 + s + 1 + r,
\]
where \(s\) is a nonnegative integer and \(|r| \leq s\). We construct the sequence
\[
V = \{0, -1, 0, 1, -2, -1, 0, 1, 2, -3, -2, -1, 0, 1, 2, 3, \ldots\}
\]
\[
= \{v_m\}_{m=1}^{\infty},
\]
where
\[
v_{s^2+s+1+r} = r \quad \text{for } |r| \leq s.
\]

For every nonnegative integer \(k\), the first occurrence of \(-k\) in this sequence is \(v_{k^2+1} = -k\), and the first occurrence of \(k\) in this sequence is \(v_{(k+1)^2} = k\).

The sequence \(U\) will be the unique subsequence of \(V\) constructed as follows. Let \(n \in \mathbb{Z}\). If \(f(n) = \infty\), then \(U\) will contain the terms \(v_{s^2+s+1+n}\) for every \(s \geq |n|\). If \(f(n) = \ell < \infty\), then \(U\) will contain the \(\ell\) terms \(v_{s^2+s+1+n}\) for \(s = |n|, |n|+1, \ldots, |n|+\ell-1\) in the subsequence \(U\), but not the terms \(v_{s^2+s+1+n}\) for \(s \geq |n|+\ell\). Let \(m_1 < m_2 < m_3 < \cdots\) be the strictly increasing sequence of positive integers such that \(\{v_{m_k}\}_{k=1}^{\infty}\) is the resulting subsequence of \(V\). Let \(U = \{u_k\}_{k=1}^{\infty}\), where \(u_k = v_{m_k}\). Then
\[
f(n) = \text{card}\{k \geq 1 : u_k = n\}.
\]

Let \(\text{card}\,(f^{-1}(0)) = \Delta\). The sequence \(U\) also has the following property: If \(|u_k| = n\), then for every integer \(m \notin f^{-1}(0)\) with \(|m| < n\) there is a positive integer \(j < k\) with \(u_j = m\). It follows that
\[
\{0, 1, -1, 2, -2, \ldots, n-1, -(n-1)\} \setminus f^{-1}(0) \subseteq \{u_1, u_2, \ldots, u_{k-1}\},
\]
and so
\[
k - 1 \geq 2(n-1) + 1 - \Delta.
\]

This implies that
\[
|u_k| = n \leq \frac{k + \Delta}{2}.
\]
Since $u_k$ is an integer, we have

$$|u_k| \leq \left\lfloor \frac{k + \Delta}{2} \right\rfloor.$$  

This completes the proof. □

Lemma 1 is best possible in the sense that for every nonnegative integer $\Delta$ there is a function $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ with $\text{card} \left( f^{-1}(0) \right) = \Delta$ and a sequence $U = \{u_k\}_{k=1}^\infty$ of integers such that

$$|u_k| = \left\lfloor \frac{k + \Delta}{2} \right\rfloor \quad \text{for all } k \geq 1. \quad (3)$$  

For example, if $\Delta = 2\delta + 1$ is odd, define the function $f$ by

$$f(n) = \begin{cases} 
0 & \text{if } |n| \leq \delta \\
1 & \text{if } |n| \geq \delta + 1
\end{cases}$$  

and the sequence $U$ by

$$u_{2i-1} = \delta + i,$$

$$u_{2i} = -(\delta + i)$$

for all $i \geq 1$.

If $\Delta = 2\delta$ is even, define $f$ by

$$f(n) = \begin{cases} 
0 & \text{if } -\delta \leq n \leq \delta - 1 \\
1 & \text{if } n \geq \delta \text{ or } n \leq -\delta - 1
\end{cases}$$  

and the sequence $U$ by $u_1 = \delta$ and

$$u_{2i} = \delta + i,$$

$$u_{2i+1} = -(\delta + i)$$

for all $i \geq 1$. In both cases the sequence $U$ satisfies (3).

**Theorem 1** Let $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ be any function such that

$$\Delta = \text{card} (f^{-1}(0)) < \infty.$$  

Let

$$c = 8 + \left\lfloor \frac{\Delta + 1}{2} \right\rfloor.$$  

There exist uncountably many sets $A$ of integers such that

$$r_{A,2}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}$$  

and

$$A(-x, x) \geq \left( \frac{x}{c} \right)^{1/3}.$$  

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Proof. Let 
\[ \Delta = \text{card}(f^{-1}(0)). \]
By Lemma 1, there exists a sequence \( U = \{u_k\}_{k=1}^{\infty} \) of integers such that
\[ f(n) = \text{card}(\{ i \in \mathbb{N} : u_i = n \}) \quad \text{for all integers } n \]  
and
\[ |u_k| \leq \frac{k + \Delta}{2} \quad \text{for all } k \geq 1. \]
We shall construct a strictly increasing sequence \( \{i_k\}_{k=1}^{\infty} \) of positive integers and an increasing sequence \( \{A_k\}_{k=1}^{\infty} \) of finite sets of integers such that, for all positive integers \( k \),

(i) \[ |A_k| = 2k, \]
(ii) There exists a positive number \( c \) such that 
\[ A_k \subseteq [-ck^3, ck^3] \]
(iii) \[ r_{A_k, 2}(n) \leq f(n) \quad \text{for all } n \in \mathbb{Z}, \]
(iv) For \( j = 1, \ldots, k \),
\[ r_{A_k, 2}(u_j) \geq \text{card}(\{ i \leq i_k : u_i = u_j \}). \]
Let \( \{A_k\}_{k=1}^{\infty} \) be a sequence of finite sets satisfying (i)-(iv). We form the infinite set
\[ A = \bigcup_{k=1}^{\infty} A_k. \]
Let \( x \geq 8c \), and let \( k \) be the unique positive integer such that
\[ ck^3 \leq x < c(k+1)^3. \]
Conditions (i) and (ii) imply that
\[ A(-x, x) \geq |A_k| = 2k > 2 \left( \frac{x}{c} \right)^{1/3} - 2 \geq \left( \frac{x}{c} \right)^{1/3}. \]
Since
\[ f(n) = \lim_{k \to \infty} \text{card}(\{ i \leq i_k : u_i = n \}), \]
conditions (iii) and (iv) imply that
\[ r_{A, 2}(n) = \lim_{k \to \infty} r_{A_k, 2}(n) = f(n) \]
for all \( n \in \mathbb{Z} \).
We construct the sequence \( \{A_k\}_{k=1}^{\infty} \) as follows. Let \( i_1 = 1 \). The set \( A_1 \) will be of the form \( A_1 = \{a_1 + u_{i_1}, -a_1\} \), where the integer \( a_1 \) is chosen so that 
\[
2A_1 \cap f^{-1}(0) = \emptyset \text{ and } a_1 + u_{i_1} \neq -a_1.
\]
This is equivalent to requiring that 
\[
2a_1 \not\in (f^{-1}(0) - 2u_{i_1}) \cup (-f^{-1}(0)) \cup \{-u_{i_1}\}. \tag{6}
\]
This condition excludes at most \( 1 + 2\Delta \) integers, and so we have at least two choices for the number \( a_1 \) such that 
\[
|a_1 + u_{i_1}| \leq 1 + \Delta \text{ and } a_1 \text{ satisfies (6).}
\]
Since \( |u_{i_1}| = |u_1| \leq (1 + \Delta)/2 \) and 
\[
|a_1 + u_{i_1}| \leq |a_1| + |u_{i_1}| \leq \frac{3(1 + \Delta)}{2},
\]
it follows that \( A_1 \subseteq [-c, c] \) for any \( c \geq 3(1 + \Delta)/2 \), and the set \( A_1 \) satisfies conditions (i)–(iv).

Let \( k \geq 2 \) and suppose that we have constructed sets \( A_1, \ldots, A_{k-1} \) and integers \( i_1 < \cdots < i_{k-1} \) that satisfy conditions (i)–(iv). Let \( i_k > i_{k-1} \) be the least integer such that 
\[
r_{A_{k-1,2}}(u_{i_k}) < f(u_{i_k}).
\]
Since 
\[
i_k - 1 \leq \sum_{n \in \{u_{i_1}, u_{i_2}, \ldots, u_{i_{k-1}}\}} r_{A_{k-1,2}}(n)
\]
\[
\leq \sum_{n \in \mathbb{Z}} r_{A_{k-1,2}}(n)
\]
\[
= \binom{2k - 1}{2}
\]
\[
< 2k^2,
\]
it follows that 
\[
i_k \leq 2k^2.
\]
Also, (5) implies that 
\[
|u_{i_k}| \leq \frac{i_k + \Delta}{2} \leq k^2 + \frac{\Delta}{2}. \tag{7}
\]
We want to choose an integer \( a_k \) such that the set 
\[
A_k = A_{k-1} \cup \{a_k + u_{i_k}, -a_k\}
\]
satisfies (i)–(iv). We have \( |A_k| = 2k \) if 
\[
a_k + u_{i_k} \neq -a_k
\]
and 
\[
A_{k-1} \cap \{a_k + u_{i_k}, -a_k\} = \emptyset,
\]
or, equivalently, if
\[ a_k \not\in (-A_{k-1} \cup (A_{k-1} - u_{i_k}) \cup \{-u_{i_k}/2\}). \tag{8} \]
Thus, in order for \( A_{k-1} \cup \{a_k + u_{i_k}, -a_k\} \) to satisfy condition (i), we exclude at most \( 2|A_{k-1}| + 1 = 4k - 3 \) integers as possible choices for \( a_k \).

The set \( A_k \) will satisfy conditions (iii) and (iv) if
\[ 2A_k \cap f^{-1}(0) = \emptyset \]
and
\[ r_{A_k,2}(n) = \begin{cases} r_{A_{k-1},2}(n) & \text{for all } n \in 2A_{k-1} \setminus \{u_{i_k}\} \\ r_{A_{k-1},2}(n) + 1 & \text{for } n = u_{i_k} \\ 1 & \text{for all } n \in 2A \setminus (2A_{k-1} \cup \{u_{i_k}\}). \end{cases} \]
Since the sumset \( 2A_k \) decomposes into
\[ 2A_k = 2(\{A_{k-1} \cup \{a_k + u_{i_k}, -a_k\}\}) = 2A_{k-1} \cup (A_{k-1} + \{a_k + u_{i_k}, -a_k\}) \cup \{u_{i_k}, 2a_k + 2u_{i_k}, -2a_k\}, \]
it suffices that
\[ (A_{k-1} + \{a_k + u_{i_k}, -a_k\}) \cap 2A_{k-1} = \emptyset, \tag{9} \]
\[ (A_{k-1} + \{a_k + u_{i_k}, -a_k\}) \cap f^{-1}(0) = \emptyset, \tag{10} \]
\[ (A_{k-1} + a_k + u_{i_k}) \cap (A_{k-1} - a_k) = \emptyset, \tag{11} \]
\[ \{2a_k + 2u_{i_k}, -2a_k\} \cap 2A_{k-1} = \emptyset \tag{12} \]
\[ \{2a_k + 2u_{i_k}, -2a_k\} \cap f^{-1}(0) = \emptyset \tag{13} \]
\[ \{2a_k + 2u_{i_k}, -2a_k\} \cap (A_{k-1} + \{a_k + u_{i_k}, -a_k\}) = \emptyset. \tag{14} \]
Equation (9) implies that the integer \( a_k \) must be chosen so that it cannot be represented either in the form
\[ a_k = x_1 + x_2 - x_3 - u_{i_k} \]
or
\[ a_k = x_1 - x_2 - x_3, \]
where \( x_1, x_2, x_3 \in A_{k-1} \). Since \( \text{card}(A_{k-1}) = 2(k-1) \), it follows that the number of integers that cannot be chosen as the integer \( a_k \) because of equation (9) is at most \( 2(2(k-1))^3 = 16(k-1)^3 \).

Similarly, the numbers of integers excluded as possible choices for \( a_k \) because of equations (10), (11), (12), (13), and (14) are at most \( 4\Delta(k-1), 4(k-1)^2, 8(k-1)^2, 2\Delta, \) and \( 8(k-1) \), respectively, and so the number of integers that cannot be chosen as \( a_k \) is
\[ 16(k-1)^3 + 12(k-1)^2 + (4\Delta + 8)(k-1) + 2\Delta \]
\[ = 16k^3 - 36k^2 + (32 + 4\Delta)k - 2\Delta - 12 \]
\[ \leq (16 + \Delta)k^3 - 4k^2 - 32k(k-1) - 2\Delta - 12. \]
Let 
\[ c = 8 + \left\lceil \frac{\Delta + 1}{2} \right\rceil. \]
The number of integers \( a \) with 
\[ |a| \leq ck^3 - k^2 - \left\lceil \frac{\Delta + 1}{2} \right\rceil = \left(8 + \left\lceil \frac{\Delta + 1}{2} \right\rceil\right)k^3 - k^2 - \left\lceil \frac{\Delta + 1}{2} \right\rceil \]
is
\[ \left(16 + 2 \left\lceil \frac{\Delta + 1}{2} \right\rceil\right)k^3 - 2k^2 - 2 \left\lceil \frac{\Delta + 1}{2} \right\rceil + 1 \]
\[ \geq (16 + \Delta)k^3 - 2k^2 - \Delta. \]
If the integer \( a \) satisfies (15), then (7) implies that 
\[ |a + u_i| \leq |a| + |u_i| \leq ck^3. \]
It follows that there are at least two acceptable choices of the integer \( a_k \) such that the set \( A_k = A_{k-1} \cup \{a_k + u_{i_k}, -a_k\} \) satisfies conditions (i)–(iv). Since this is true at each step of the induction, there are uncountably many sequences \( \{A_k\}_{k=1}^\infty \) that satisfy conditions (i)–(iv). This completes the proof. \( \square \)

We can modify the proof of Theorem 1 to obtain the analogous result for the restricted representation function \( \hat{r}_{A,2}(n) \).

**Theorem 2** Let \( f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\} \) be any function such that 
\[ \text{card}(f^{-1}(0)) < \infty. \]
Then there exist uncountably many sets \( A \) of integers such that 
\[ \hat{r}_{A,2}(n) = f(n) \quad \text{for all } n \in \mathbb{Z} \]
and 
\[ A(-x, x) \gg x^{1/3}. \]

**4 Representation functions for bases of order \( h \)**

We can also prove similar results for the representation functions of asymptotic bases and restricted asymptotic bases of order \( h \) for all \( h \geq 2 \).

For any set \( A \subseteq \mathbb{Z} \), the **representation function** \( r_{A,h}(n) \) counts the number of ways to write \( n \) in the form \( n = a_1 + a_2 + \cdots + a_h \), where \( a_1, a_2, \ldots, a_h \in A \) and \( a_1 \leq a_2 \leq \cdots \leq a_h \). The set \( A \) is called an **asymptotic basis of order \( h \)** if all but finitely many integers can be represented as the sum of \( h \) not necessarily distinct elements of \( A \), or, equivalently, if the function 
\[ r_{A,h} : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\} \]
satisfies
\[ \text{card}(r^{-1}_{A,h}(0)) < \infty. \]

Similarly, the restricted representation function \( \hat{r}_{A,h}(n) \) counts the number of ways to write \( n \) as a sum of \( h \) pairwise distinct elements of \( A \). The set \( A \) is called a restricted asymptotic basis of order \( h \) if all but finitely many integers can be represented as the sum of \( h \) pairwise distinct elements of \( A \).

**Theorem 3** Let \( f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\} \) be any function such that
\[ \text{card}(f^{-1}(0)) < \infty. \]

There exist uncountably many sets \( A \) of integers such that
\[ r_{A,h}(n) = f(n) \quad \text{for all } n \in \mathbb{Z} \]
and
\[ A(-x, x) \gg x^{1/(2h-1)}, \]
and there exist uncountably many sets \( A \) of integers such that
\[ \hat{r}_{A,h}(n) = f(n) \quad \text{for all } n \in \mathbb{Z} \]
and
\[ A(-x, x) \gg x^{1/(2h-1)}. \]

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