Vacuum Energy Induced by an External Magnetic Field in a Curved Space

Yu. A. Sitenko
D. G. Rakityansky†
Bogolyubov Institute for Theoretical Physics
National Academy of Sciences of Ukraine

Abstract
The asymptotic expansion of the product of an operator raised to an arbitrary power and an exponential function of this operator is obtained. With the aid of this expansion, the density of vacuum energy induced by a static external magnetic field of an Abelian or non-Abelian nature is expressed in terms of the DeWitt-Seeley-Gilkey coefficients.

1 Introduction
It is well known that modern theoretical and mathematical physics widely employs the asymptotic expansion of the heat kernel,

$$<x|e^{-tA}|x> \sim \sum_{l=0}^{\infty} E_l(x|A)t^{-\frac{d}{2}+l},$$

where $A$ — is a positive definite elliptic second-order differential operator acting in a section of a fiber bundle over a manifold with compact Fiemann base of dimension $d$ and where summation is performed over nonnegative integral values of $l$. The coefficients of this asymptotic expansion (DeWitt-Seeley-Gilkey coefficients [1, 2, 3]) are endomorphisms of a fiber in $x$ and represent local covariant quantities that are constructed from the coefficient functions of the operator $A$, curvature of a fiber and its base, and covariant derivatives.

* e-mail: yusitenko@bitp.kiev.ua
† e-mail: radamir@quantum.bitp.kiev.ua
In this study, we will consider the diagonal matrix element

\[ <x|A^\alpha e^{-tA}|x>, \]

where \( \alpha \) is a real-valued parameter, and obtain its asymptotic expansion for \( t \to 0_+ \). The coefficients in the resulting expansions are expressed in terms of the DeWitt-Seeley-Gilkey coefficients. For a certain relation between \( \alpha \) and \( d \), this expansion involves not only powers of \( t \) but also terms that are logarithmic in \( t \).

As an application of the aforementioned result, we consider the problem of vacuum energy induced by a static external magnetic field of an Abelian or non-Abelian nature. This problem is studied in spaces of dimensions \( d \geq 2 \).

2 Asymptotic Expansion of diagonal Matrix element

We make use of the method based on the symbolic calculus of pseudodifferential operators \([4, 5]\) and developed in \([6]\). This method is advantageous in that it can be applied to a wide variety of problems and in that it is manifestly covariant under gauge and general coordinate transformations.

We begin by expressing the operator \( e^{-tA}(t > 0) \) in terms of the resolvent as

\[ e^{-tA} = \int_C \frac{id\lambda}{2\pi} e^{-t\lambda} (A - \lambda)^{-1}, \quad (2) \]

where the contour \( C \) in the complex \( \lambda \) plane circumvents counterclockwise the spectrum of the operator \( A \), lying on the positive real semiaxis. The relation that will serve our present purpose and which is analogous to (2) has the form

\[ A^\alpha e^{-tA} = \int_C \frac{id\lambda}{2\pi} \lambda^\alpha e^{-t\lambda} (A - \lambda)^{-1}. \quad (3) \]

If \( \alpha \neq n \), where \( n \in \mathbb{Z} \) (\( \mathbb{Z} \) is the set of integers), the point \( \lambda = 0 \) is the branch point of the integrand on the right-hand side of (3). Drawing a cut in the complex \( \lambda \) plain along the negative real semiaxis from the origin to \(-\infty\), we can arrange the contour \( C \) in such a way that it lies in the half-plane \( \text{Re}\lambda > 0 \). This can be done because the operator \( A \) is strictly positive definite; hence, the lower bound of its spectrum nonzero: the contour \( C \) itresects the real axis at the point lying in the interval between zero and the lower bound of the spectrum. It follows that the matrix element of the operator in (3) is represented as an integral of the resolvent symbol, in just the same way as the matrix element of the operator in (2) is.
For the purposes of the ensuing analysis, it is more convenient to introduce the spectral parameter $\lambda$ in an alternative way. A positive definite operator $A$ can be represented as

$$A = A_0 + m^2$$

where $m^2 > 0$ and $A_0$ is a nonnegative operator (the lower bound of its spectrum is zero, so that zero modes can exist). Apart from (1), there also exists an asymptotic expansion that is associated with the kernel, but which corresponds to the operator $A_0$,

$$<x|e^{-tA_0}|x> \sim \sum_{l=0}^{\infty} E_l(x|A_0)t^{-\frac{d}{2}+l},$$

Taking this into account, we can express the operators on the left-hand sides of the equations (2) and (3) in terms of the resolvent of the operator $A_0$. This yields

$$e^{-tA} = e^{-tm^2} \int_C \frac{id\lambda}{2\pi} e^{-t\lambda}(A_0 - \lambda)^{-1},$$

$$A^\alpha e^{-tA} = e^{-tm^2} \int_C \frac{id\lambda}{2\pi}(\lambda + m^2)^\alpha e^{-t\lambda}(A_0 - \lambda)^{-1},$$

where the cut in the complex $\lambda$ plane goes from the point $-m^2$ to $-\infty$, and the contour $C$ intersects the real axis at some point lying in the interval $(-m^2, 0)$. For the diagonal matrix elements of the operators in (6) and (7), we obtain

$$<x|e^{-tA}|x> = e^{-tm^2} \int \frac{d^d k}{(2\pi)^d \sqrt{g(x)}} \int_C \frac{id\lambda}{2\pi} e^{-t\lambda} \sigma^0(x, k; \lambda),$$

$$<x|A^\alpha e^{-tA}|x> = e^{-tm^2} \int \frac{d^d k}{(2\pi)^d \sqrt{g(x)}} \int_C \frac{id\lambda}{2\pi}(\lambda + m^2)^\alpha e^{-t\lambda} \sigma^0(x, k; \lambda),$$

where $\sigma^0$ is the symbol of the resolvent of the operator $A_0$. This symbol can be expanded in a series in the powers of homogeneity as

$$\sigma^0(x, k; \lambda) = \sum_{l=0}^{\infty} \sigma^0_l(x, k; \lambda),$$

where $l$ takes nonnegative integral values. The functions $\sigma^0_l$ satisfy recursion relations. Substituting expansion (10) into (8) and performing the required integrations, we arrive at expansion (11).

As a result, we obtain

$$<x|A^\alpha e^{-tA}|x> \sim \sum_{l=0}^{\infty} E_l(x|A) \frac{\Gamma(\alpha + \frac{d}{2} - l)}{\Gamma(\frac{d}{2} - l)} t^{-\alpha - \frac{d}{2} + l}$$
\[ + \sum_{l=0}^{\infty} \sum_{l'=-l}^{\infty} E_{l+l'}(x|A_0) \frac{\Gamma(l' - \alpha - \frac{d}{2})}{\Gamma(-\alpha - l)\Gamma(l + 1)} m^{2(\alpha + \frac{d}{2} - l')}(t)^l, \quad \alpha + \frac{d}{2} \neq n; \]

\[ < x|A^{n-\frac{d}{2}} e^{-tA}|x > \sim \sum_{l=0}^{n-1} E_l(x|A) \frac{\Gamma(n-l)}{\Gamma(\frac{d}{2} - l)} t^{-n+l} + \]

\[ + \sum_{l=\max(n,0)}^{\infty} E_l(x|A) \frac{\ln(tm^2)^{-1} - \gamma + \psi(l + 1 - n) - \psi(\frac{d}{2} - l)}{\Gamma(\frac{d}{2} - l)\Gamma(l + 1 - n)} (-t)^{l-n} + \]

\[ + \sum_{l=0}^{\infty} \sum_{l'=\max(n+1,-l)}^{\infty} E_{l+l'}(x|A_0) \frac{\Gamma(l' - n)}{\Gamma(\frac{d}{2} - n-l)\Gamma(l + 1)} m^{2(n-l')}(t)^l, \]

where \( n \in \mathbb{Z}; \) the notation \( \max(a,b) \) means that, of the two quantities \( a \) and \( b, \) that which takes the larger value must appear in the corresponding expression; \( \psi(z) = \frac{d}{dz} \ln \Gamma(z) \) is the digamma function; and \( \gamma = -\psi(1) \) is the Euler constant. The first (finite) sum in the (12) naturally vanishes for \( n \leq 0. \)

## 3 Determination of the renormalized density of induced vacuum energy

The results of the preceding section can be used to solve the problem of the vacuum energy induced by the curvature of a fiber, the strength of a static external magnetic field. To determine the vacuum-energy density, it is necessary to specify the regularization and renormalization procedures. We apply the standard procedure, according to which ultraviolet divergencies are regularized by introducing a cut-off factor.

Let us consider a static \((d+1)\)-dimensional space-time endowed with a metric specified by the relation

\[ ds^2 = -(dx^0)^2 + g_{\mu\nu}(x)dx^\mu dx^\nu, \quad \mu, \nu = 1, ..., d, \quad (13) \]

where \( x^0 \) is the time. We write Dirac equation in the form

\[ (-i\partial_0 + H_D)\psi = 0, \]

where

\[ H_D = -i\alpha^\mu(x)\nabla_\mu + \beta m \]

is the Dirac Hamiltonian. For a spinor field, the zeroth component of energy-momentum tensor has the form

\[ T_{00} = \frac{i}{2} [\psi^+(\partial_0 \psi) - (\partial_0 \psi^+)\psi]. \]

4
For the vacuum energy of the second-quantized field, we obtain the expression.

\[ \int d^d x \sqrt{g(x)} \varepsilon(x) = -\frac{1}{2} \sum_\omega |\omega|, \]  

(17)

where \( g = \det g_{\mu\nu} \), and \( \varepsilon \) is the vacuum-energy density. Expression (17) is not well-defined, because the sum on the right-hand side can diverge for \( \omega \to \pm\infty \) (so-called ultraviolet divergence). Supplementing the summand with a factor that decreases exponentially for \( \omega \to \pm\infty \), we define a regularized vacuum energy as

\[ \int d^d x \sqrt{g(x)} \varepsilon_{\text{reg}}(x) = -\frac{1}{2} \sum_\omega |\omega| e^{-t\omega^2}. \]  

(18)

Let us represent the Klein-Gordon equation in the form

\[ (\partial_0^2 + H_S)\phi = 0 \]  

(19)

where

\[ H_S = -\Delta + \xi R(x) + m^2, \]  

(20)

For a scalar field, the zeroth component of the canonical energy-momentum tensor is given by

\[ T_{00} = (\partial_0 \phi^*)(\partial_0 \phi) + (\nabla_\mu \phi^*)(\nabla^\mu \phi) + (\xi R + m^2)\phi^*\phi; \]  

(21)

For the vacuum energy of a second-quantized scalar field, we then obtain the formal expression

\[ \int d^d x \sqrt{g(x)} \varepsilon(x) = \sum_{\omega > 0} \omega \]  

(22)

Accordingly, the regularized expression has the form

\[ \int d^d x \sqrt{g(x)} \varepsilon_{\text{reg}}(x) = \sum_{\omega > 0} \omega e^{-t\omega^2}. \]  

(23)

Thus, the regularized vacuum-energy density can be represented as

\[ \varepsilon_{\text{reg}}(x) = -\frac{1}{2} \text{tr} < x | H_D | \exp(-tH_D^2) | x > \]  

(24)

for a spinor field and as

\[ \varepsilon_{\text{reg}}(x) = \text{tr} < x | \sqrt{H_S} \exp(-tH_S) | x > \]  

(25)

for a scalar field
By using expansions (11) and (12) at $\alpha = \frac{1}{2}$, we obtain

\[
\langle x | A^{1/2} e^{-tA} | x \rangle \sim \frac{\sqrt{\pi}}{2} E_0(x | A_0) t^{-3/2} -
\]

\[
-\frac{1}{\pi} \sum_{l=0}^{\infty} \sum_{l' = 2}^{\infty} E_{l+l'}(x | A_0) \frac{\Gamma(l' - 3/2) \Gamma(l + 3/2)}{\Gamma(l + 1)} m^{3 - 2l'} t^{l'}, \quad d = 2,
\]

\[
\langle x | A^{1/2} e^{-tA} | x \rangle \sim \frac{2}{\sqrt{\pi}} E_0(x | A_0) t^{-2} -
\]

\[
-\frac{1}{\sqrt{\pi}} [m^2 E_0(x | A_0) - E_1(x | A_0)] t^{-1} -
\]

\[
-\psi(l-1/2)](-m^2)^{l-\ell} t^{l-2} - \frac{1}{\pi} \sum_{l=0}^{\infty} \sum_{l' = 3}^{\infty} E_{l+l'}(x | A_0) \frac{\Gamma(l' - 2) \Gamma(l + 3/2)}{\Gamma(l + 1)} m^{4 - 2l'} t^{l'}, \quad d = 3,
\]

\[
\langle x | A^{1/2} e^{-tA} | x \rangle \sim \frac{3\sqrt{\pi}}{4} E_0(x | A_0) t^{-5/2} -
\]

\[
-\frac{\sqrt{\pi}}{2} [m^2 E_0(x | A_0) - E_1(x | A_0)] t^{-3/2} -
\]

\[
-\frac{1}{\pi} \sum_{l=0}^{\infty} \sum_{l' = 3}^{\infty} E_{l+l'}(x | A_0) \frac{\Gamma(l' - 5/2) \Gamma(l + 3/2)}{\Gamma(l + 1)} m^{5 - 2l'} t^{l'}, \quad d = 4.
\]

Introducing the notation

\[
H_{D,0} = H_D |_{m=0}, \quad H_{S,0} = H_S |_{m=0},
\]

we can see that, both in the case of

\[
A_0 = H_{D,0}^2,
\]

and in the case of

\[
A_0 = H_{S,0}
\]

the operator $A_0$ has the form

\[
A_0 = -\Delta + X(x),
\]

where

\[
X(x) = \frac{i}{4} [\alpha^\mu(x), \alpha^\nu(x)] - F_{\mu\nu}(x) + \frac{1}{4} R(x)
\]

in the former case and

\[
X(x) = \xi R(x)
\]

in the latter case; in (33), we introduce the notation

\[
F_{\mu\nu}(x) = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x) - i[V_\mu(x), V_\nu(x)].
\]
4 Resume

We have obtained the asymptotic expansions (11) and (12) for the diagonal matrix element of the operator product $A^\alpha e^{-tA}$ for $t \to 0^+$. At $\alpha = -\frac{d}{2} + n$, where $n \in \mathbb{Z}$, the relevant expansion involves not only powers of $t$ but also terms that are logarithmic in $t$.

We have also solved the problem of vacuum energy induced by a static magnetic external magnetic field in a curved space.

References

[1] De Witt B. Dynamical Theory of Groups and Fields. New York: Gordon and Breach, 1965.

[2] Seeley R. T./ Proc. Symp. Pure. Math. 1967. V.10. P.288.

[3] Gilkey P. B./ J. Diff. Geom. 1975. V.10. P.601.

[4] Shubin M. A. Pseudodifferential Operators and Spectral Theory, New York: Springer-Verlag, 1987.

[5] Widom H. A./ Bull. Sci. Math. 1980. V.104. P.19.

[6] Gusynin V. P./ Nucl. Phys. 1990. V.B333 P.296.