Dynamic Chromatic Number of Regular Graphs

Meysam Alishahi
Department of Mathematics
Shahrood University of Technology, Shahrood, Iran
meysam.alishahi@shahroodut.ac.ir

Abstract
A dynamic coloring of a graph $G$ is a proper coloring such that for every vertex $v \in V(G)$ of degree at least 2, the neighbors of $v$ receive at least 2 colors. It was conjectured [B. Montgomery. Dynamic coloring of graphs. PhD thesis, West Virginia University, 2001.] that if $G$ is a $k$-regular graph, then $\chi_2(G) - \chi(G) \leq 2$. In this paper, we prove that if $G$ is a $k$-regular graph with $\chi(G) \geq 4$, then $\chi_2(G) \leq \chi(G) + \alpha(G^2)$. It confirms the conjecture for all regular graph $G$ with diameter at most 2 and $\chi(G) \geq 4$. In fact, we show that for any $k$-regular graph $G$, $\chi_2(G) - \chi(G) \leq 6 \ln k + 2$. Also, we show that for any $n$ there exists a regular graph $G$ whose chromatic number is $n$ and $\chi_2(G) - \chi(G) \geq 1$. This result gives a negative answer to a conjecture of [A. Ahadi, S. Akbari, A. Dehghan, and M. Ghanbari. On the difference between chromatic number and dynamic chromatic number of graphs. Discrete Math., In press].

Key words: Dynamic chromatic number, 2-colorability of hypergraphs.

Subject classification: 05C.

1 Introduction

Let $H$ be a hypergraph. The vertex set and the hyperedge set of $H$ are mentioned as $V(H)$ and $E(H)$, respectively. The maximum degree and the minimum degree of $H$ are denoted by $\Delta(H)$ and $\delta(H)$, respectively. For an integer $l \geq 1$, denote by $[l]$, the set $\{1, 2, \ldots, l\}$. A proper $l$-coloring of a hypergraph $H$ is a function $c : V(H) \rightarrow [l]$ in which there is no monochromatic hyperedge in $H$. We say a hypergraph $H$ is $t$-colorable if there is a proper $t$-coloring of it. For a hypergraph $H$, the smallest integer $l$ so that $H$ is $l$-colorable is called the chromatic number of $H$ and denoted by $\chi(H)$. Note that a graph $G$ is a hypergraph such that the cardinality of each $e \in E(G)$ is 2.

A proper vertex $l$-coloring of a graph $G$ is called a dynamic $l$-coloring [14] if for every vertex $u$ of degree at least 2, there are at least two different colors appearing in the neighborhood of $v$. The smallest integer $l$ so that there is a dynamic $l$-coloring of $G$ is called the dynamic chromatic number of $G$ and denoted by $\chi_2(G)$. Obviously, $\chi(G) \leq \chi_2(G)$. Some properties of dynamic coloring were studied in [3, 6, 10, 11, 14, 13]. It was proved in [11] that for a connected graph $G$ if $\Delta \leq 3$, then $\chi_2(G) \leq 4$ unless $G = C_5$, in which case $\chi_2(C_5) = 5$ and if $\Delta \geq 4$, then $\chi_2(G) \leq \Delta + 1$. It was shown in [14] that the difference between chromatic number and dynamic chromatic number can be arbitrarily large. However, it was conjectured that for regular graphs the difference is at most 2.

Conjecture 1. [14] For any regular graph $G$, $\chi_2(G) - \chi(G) \leq 2$
Also, it was proved in [14] that if $G$ is a bipartite $k$-regular graph, $k \geq 3$ and $n < 2^k$, then $\chi_2(G) \leq 4$. This result was extended to all regular bipartite graphs in [6].

In a graph $G$, a set $T \subseteq V(G)$ is called a \emph{total dominating set} if $G$ if for every vertex $v \in V(G)$, there is at least one vertex $u \in T$ adjacent to $v$. The set $T \subseteq V(G)$ is called a \emph{double total dominating set} if $T$ and its complement $V(G) \setminus T$ are both total dominating [6]. Also, by $I(G)$ and $IM(G)$ we refer to the set of independent and maximal independent sets in $G$, respectively.

### 2 results

The 2-colorability of hypergraphs has been studied in the literature and has lots of applications in the other area of combinatorics.

**Theorem 1.** [12] Let $H$ be a hypergraph in which every hyperedge contains at least $k$ points and meets at most $d$ other hyperedges. If $e(d + 2) \leq 2^k$, then $H$ is 2-colorable.

Assume that $G$ is a graph. Let $T \subseteq V(G)$ and define a hypergraph $H_G(T)$ whose vertex set is $\bigcup_{v \in T} N(v)$ and its hyperedge set is defined as follows

$$E(H_G(T)) \overset{\text{def}}{=} \{N(v)|v \in T\}.$$ 

Clearly, for any $f \in E(H_G(T))$, $\delta(G) \leq |f| \leq \Delta(G)$ and $\Delta(H_G(T)) \leq \Delta(G)$. Therefore $f$ meets at most $\Delta(G)(\Delta(G) - 1)$ other hyperedges.

It was shown by Thomassen [15] that for any $k$-uniform and $k$-regular hypergraph $H$, if $k \geq 4$, then $H$ is 2-colorable. This result can be easily extended to all $k$-uniform hypergraphs with the maximum degree at most $k$ [6], i.e., any $k$-uniform hypergraph $H$ with $k \geq 4$ and the maximum degree at most $k$, is 2-colorable. By considering Theorem [1] if $e(\Delta^2(G) - \Delta(G) + 2) \leq 2^k(G)$ (in the $k$-regular case, $k \geq 4$), then $H_G(T)$ is 2-colorable.

Next lemma is proved in [4] and extended to circular coloring in [5].

**Lemma 1.** [4] Let $G$ be a connected graph, and let $c$ be a $\chi(G)$-coloring of $G$. Moreover, assume that $H$ is a nonempty subgraph of $G$. Then there exits a $\chi(G)$-coloring $f$ of $G$ such that:

a) if $v \in V(H)$, then $f(v) = c(v)$ and

b) for every vertex $v \in V(G) \setminus V(H)$ there is a path $v_0v_1\ldots v_m$ such that $v_0 = v$, $v_m$ is in $H$, and $f(v_{i+1}) - f(v_i) = 1 \pmod{\chi(G)}$ for $i = 0, 1, \ldots, m - 1$.

Assume that $G$ is a given graph. The graph $G^2$ is a graph with the vertex set $V(G)$ and two different vertices $u$ and $v$ are adjacent in $G^2$ if $d_G(u, v) \leq 2$, i.e., there is a walk with length at most two between $u$ and $v$ in $G$.

The connection between the independence number and the dynamic chromatic number of graphs has been studied in [1]. The first part of the next theorem improves a similar result in [1] and in the second part of it we present an upper bound for the dynamic chromatic number of graph $G$ in terms of chromatic number of $G$ and the independence number of $G^2$. 
Theorem 2.

1) For any graph $G$ with $\chi(G) \geq 4$, $\chi_2(G) \leq \chi(G) + \alpha(G)$.

2) If $G$ is a $k$-regular graph with $\chi(G) \geq 4$, then $\chi_2(G) \leq \chi(G) + \alpha(G^2)$.

Proof. Let $uv$ be an edge of $G$. Assume that $c$ is a $\chi(G)$-coloring of $G$ such that $c(u) = 1$ and $c(v) = 3$. Let $H$ be the subgraph induced on the edge $uv$ and $f$ be a coloring as in Lemma 1. According to Lemma 1, $f(u) = 1$ and $f(v) = 3$. We call a vertex $v \in V(G)$, a bad vertex if $\deg(v) \geq 2$ and all vertices in $N(v)$ have the same color in $f$. Let $S$ be the set of all bad vertices. We claim that $S$ is an independent set in $G$. Suppose therefore (reductio ad absurdum) that this is not the case. We consider four different cases.

1. $\{u, v\} \subseteq S$. Since $u$ and $v$ are both bad vertices and $uv \in E(G)$, all the vertices in $N(u)$ have the color 3 and all the vertices in $N(v)$ have the color 1. Note that $\chi(G) \geq 4$. Therefore the coloring $f$ does not satisfy Lemma 1 and it is a contradiction.

2. There exists a vertex $x \neq v$ such that $\{u, x\} \subseteq S$ and $ux \in E(G)$. Since $v \in N(u)$ and $u \in N(x)$ and both $u$ and $x$ are bad, all the vertices in $N(u)$ and $N(x)$ have the colors 3 and 1 in the coloring $f$, respectively. According to Lemma 1 there is a path $v_0v_1 \ldots v_m$ in $G$ such that $v_0 = x$, $v_m \in V(H)$, and $f(v_{i+1}) = f(v_i) = 1$ (mod $\chi(G)$) for $i = 0, 1, \ldots, m - 1$. But, this is not possible, because all the neighbors of $x$ have the color 1 and $x$ has the color 3.

3. There exists a vertex $y \neq u$ such that $\{v, y\} \subseteq S$ and $vy \in E(G)$. This case is the same as the previous case.

4. There are two vertices $x$ and $y$ in $S \setminus \{u, v\}$ such that $xy \in E(G)$. Note that according to Lemma 1 there should be at least two vertices $z \in N(x)$ and $z' \in N(y)$ such that $f(z) = f(x) + 1$ and $f(z') = f(y) + 1$ (mod $\chi$). Since $x$ and $y$ are both bad vertices, all the vertices in $N(x)$ have the color $f(y)$ and all the vertices in $N(y)$ have the color $f(x)$. Therefore $f(z) = f(y)$ and $f(z') = f(x)$ (mod $\chi(G)$), but this is not possible because $\chi(G) \geq 4$.

Now, we know $S$ is an independent set in $G$ and so $|S| \leq \alpha(G)$. For any vertex $w \in S$, choose a vertex $x(w) \in N(w)$ and put all these vertices in $S'$. Assume that $S' = \{x_1, x_2, \ldots, x_1\}$. Consider a coloring $f'$ such that $f$ and $f'$ are the same on $V(G) \setminus S'$ and for any $x_i \in S'$, $x_i$ is colored with $\chi(G) + i$. One can easily check that $f'$ is a dynamic coloring of $G$ used at most $\chi(G) + \alpha(G)$ colors.

To prove the second part, assume that $G$ is a $k$-regular graph with $\chi(G) \geq 4$. Consider the coloring $f$ and the set $S$ as in the previous part. Assume that $G^2[S]$, i.e., the induced subgraph of $G^2$ on the vertices in $S$, has the components $G^2_1, G^2_2, \ldots, G^2_n$. Note that two different vertices $x, y \in S$ are adjacent in $G^2$ if and only if $N_G(x) \cap N_G(y) \neq \emptyset$ (since $S$ is an independent set, $xy \notin E(G)$). Therefore for any $1 \leq i \leq n$, all the vertices in

$$N_i = \bigcup_{x \in V(G^2_i)} N_G(x)$$
have the same color in the coloring $f$. For any $1 \leq i \leq n$, let $H_i$ be a hypergraph with the vertex set $N_i$ and with the hyperedge set

$$E(H_i) = \{N(x) \mid x \in V(G^2_i)\}.$$ 

It is clear that $H_i$ is a $k$-uniform hypergraph with $\Delta(H_i) \leq k$. Since $\chi(G) \geq 4$, we have $k \geq 4$ or $G = K_4$. If $G = K_4$, then $\chi_2(G) = 4 \leq \chi(G) + \alpha(G^2)$ and there is nothing to prove. Now, we can assume that $k \geq 4$. According to the discussion after Theorem 1, $H_i$ is 2-colorable. For any $1 \leq i \leq n$, let $(X_1^i, X_2^i)$ be a 2-coloring of $H_i$. Define $f''$ to be a coloring of $G$ such that $f''$ and $f$ are the same on $V(G) \setminus \bigcup X_1^i$ and for each $1 \leq i \leq n$, $f''$ has the constant value $i + \chi(G)$ on the vertices of $X_1^i$. It is easy to see that $f''$ is a $(\chi(G) + n)$-dynamic coloring of $G$. Obviously, $n \leq \alpha(G^2)$ and the proof is completed. 

In the proof of the second part of Theorem 2, we need the 2-colorability of all $H_i$’s and if some assumptions cause this property, then the remain of proof still works. Consequently, in view of the discussion after Theorem 1, we have the next corollary.

**Corollary 1.** Let $G$ be a graph such that $\chi(G) \geq 4$ and $e(\Delta(G) - \Delta(G) + 1) \leq 2^\delta(G)$. Then $\chi_2(G) \leq \chi(G) + \alpha(G^2)$.

**Remark.** Note that in the proof of Theorem 2, it is shown that for any $k$-regular graph $G$ with $\chi(G) \geq 4$, $\chi_2(G) \leq \chi(G) + \max_{I \in \mathcal{I}(G)} \text{com}(G^2[I])$ where $G^2[I]$ is the number of connected components of $G^2[S]$ and $S$ is an independent set given in the proof of Theorem 2. Therefore for any graph $G$ with $\chi(G) \geq 4$ and $e(\Delta(G) - \Delta(G) + 1) \leq 2^\delta(G)$ (in $k$-regular case $k \geq 4$),

$$\chi_2(G) \leq \chi(G) + \max_{I \in \mathcal{I}(G)} \text{com}(G^2[I]).$$

It is shown in [2] that if $G$ is a strongly regular graph except $C_5$ and $K_{m,m}$, then $\chi_2(G) - \chi(G) \leq 1$. Note that for a graph $G$ with diameter 2, the graph $G^2$ is a complete graph and $\alpha(G^2) = 1$. Therefore the second part of Theorem 2 extends this result to a larger family of regular graphs. In fact, every strongly regular graph has diameter at most 2, but according to the second part of Theorem 2, if $G$ is a $k$-regular graph with diameter 2 and $\chi(G) \geq 4$, then $\chi_2(G) = \chi(G) + \alpha(G^2) = 1$. Moreover, by the previous corollary, if $G$ is a graph with diameter 2, $\chi(G) \geq 4$ and $e(\Delta(G) - \Delta(G) + 1) \leq 2^\delta(G)$, then $\chi_2(G) - \chi(G) \leq 1$. We restate this result in the next corollary.

**Corollary 2.** Let $G$ be a graph with diameter 2, $\chi(G) \geq 4$ and $e(\Delta(G) - \Delta(G) + 1) \leq 2^\delta(G)$ (in $k$-regular case $k \geq 4$). Then $\chi_2(G) - \chi(G) \leq 1$.

The proof of the second part of Theorem 2 strongly depended on the assumption that $\chi(G) \geq 4$. In fact the only bipartite regular graphs with diameter 2 are complete regular bipartite graphs whose chromatic number and dynamic chromatic number are 2 and 4, respectively. But $K_{m,m}^2$ is a complete graph and so $\chi(K_{m,m}) + \alpha(K_{m,m}) = 3 \leq \chi_2(K_{m,m})$. For the case of $\chi(G) = 3$, if we set $G = C_5$, then $C_5^2 = K_5$ and $\chi_2(C_5) = 5 > \chi(C_5) + \alpha(C_5^2)$. 

4
Note that in the proof of Theorem 2 we assumed that \( \chi(G) \geq 4 \) because we want to use Lemma 1 to obtain a coloring \( f \) such that all the bad vertices related to \( f \) form an independent set in \( G \). However, if one finds a \( t \)-coloring \( f \) of \( G \) such that the set of bad vertices related to \( f \), \( S \), is an independent set in \( G \), then \( \chi_2(G) \leq t + \alpha(G) \) and if \( G \) is a \( k \)-regular graph with \( k \geq 4 \), then \( \chi_2(G) \leq t + \text{com}(G^2[S]) \).

Now, let \( G \) be a graph such that \( e(\Delta^2(G) - \Delta(G) + 1) \leq 2^{\delta(G)} \) (in \( k \)-regular case, \( k \geq 4 \)) and let \( I \) be an arbitrary maximal independent set in \( G \). Consider an optimum \( t \)-coloring \( c \) of \( G \) such that \( I \) is a color class in this coloring \( (t \) is the least possible number). Define \( H \) to be a hypergraph with vertex set \( I \) and the hyperedge set \( E(H) = \{ N(v) | v \in V(G) \& N(v) \subseteq I \} \). Since \( e(\Delta^2(G) - \Delta(G) + 1) \leq 2^{\delta(G)} \) (in \( k \)-regular case, \( k \geq 4 \)), \( H \) is 2-colorable. Let \( (X, Y) \) be a 2-coloring of \( H \). Recolor the vertices in \( Y \) with a new color \( t + 1 \) to obtain a \( (t + 1) \)-coloring \( f \) of \( G \). It is readily seen that \( S \), the set of the bad vertices related to \( f \), is a subset of \( I \) and therefore it is an independent set. By the same argument as in the proof of the second part of Theorem 2 one can show that \( \chi_2(G) \leq t + 1 + \text{com}(G^2[S]) \). Now, note that \( t \leq \chi(G) + 1 \) and so \( \chi_2(G) \leq t + 1 + \text{com}(G^2[S]) \leq \chi(G) + 2 + \max \text{com}(G^2[P]) \).

Let \( IM(G) \) be the set of all maximal independent sets in \( G \). Since \( I \) is an arbitrary maximal independent set in \( G \),

\[
\chi_2(G) \leq \chi(G) + \min_{I \in IM(G)} \max_{P \subseteq I} \text{com}(G^2[P]) + 2.
\]

In Theorem 2 we have the assumption \( \chi(G) \geq 4 \) and for a graph \( G \) with \( \chi(G) < 4 \), we can not use this theorem. In view of the above discussion, if we consider \( c \) as a \( \chi(G) \)-coloring of \( G \) such that the color class \( V_1 \) (all the vertices with color 1) is a maximal independent set in \( G(t = \chi(G)) \), then \( \chi_2(G) \leq \chi(G) + \alpha(G) + 1 \) and also, we have the next corollary.

**Corollary 3.** Let \( G \) be a graph such that \( e(\Delta^2(G) - \Delta(G) + 1) \leq 2^{\delta(G)} \) (in \( k \)-regular case, \( k \geq 4 \)). Then \( \chi_2(G) \leq \chi(G) + \alpha(G^2) + 1 \).

Erdős and Lovász 8 proved a very powerful lemma, known as the Lovász Local Lemma.

**The Lovász Local Lemma.** Let \( A_1, A_2, \ldots, A_n \) be evens in an arbitrary probability space. Suppose that each event \( A_i \) is mutually independent of a set of all \( A_j \) but at most \( d \) of the other events and \( \text{Pr}(A_i) \leq p \) for all \( 1 \leq i \leq n \). If \( ep(d + 1) \leq 1 \) then \( \text{Pr}(\bigcap_{i=1}^{n} A_i) > 0 \).

It was proved in 6 that for any \( k \)-regular graph \( G \), the difference between dynamic chromatic number and chromatic number of \( G \) is at most \( 14.06 \ln k + 1 \). In the next theorem we shall improve this result.

**Theorem 3.** For any \( k \)-regular graph \( G \), \( \chi_2(G) - \chi(G) \leq 6 \ln k + 2 \).

**Proof.** It is proved in 6, that for any regular graph \( G \), \( \chi_2(G) \leq 2\chi(G) \). Therefore for any \( k \)-regular graph \( G \) with \( k \leq 3 \), \( \chi_2(G) \leq 6 \leq 6 \ln k + 2 \). Now, we can assume that \( k \geq 4 \). Let \( V(G) = \{ v_1, v_2, \ldots, v_n \} \). For any permutation (total ordering) \( \sigma \in S_{V(G)} \), set

\[
I_\sigma = \{ v \in V(G) | v \prec_\sigma u \text{ for all } u \in N(v) \}.
\]
It is readily seen that $I_u$ is an independent set of $G$. Assume that $U \subseteq V(G)$ consists of all vertices that are not lied in any triangle. Now, choose $l$ permutations $\sigma_1, \ldots, \sigma_l$, randomly and independently. For any $u \in U$, let $A_u$ be the event that there are not a vertex $v \in N(u)$ and $\sigma_i$ such that the vertex $v$ precedes all of its neighbors in the permutation $\sigma_i$, i.e., $N(u) \bigcap \bigcup_{i=1}^{l} I_{\sigma_i} = \emptyset$. Since $u$ dose not appear in any triangle, one can easily see that $\Pr(A_u) = (1 - \frac{1}{k})^{kl} \leq e^{-l}$. Note that $A_u$ is mutually independent of all events $A_v$ for which $d(u, v) > 3$. Consequently, $A_u$ is mutually independent of all but at most $k^3 - k^2 + k + 1$ events. In view of Lovász Local Lemma, if $e(k^3 - k^2 + k + 2)e^{-l} \leq 1$, then none of the events $A_u$ happens with positive probability. In other words, for $l = \lceil 3 \ln k + 1 \rceil$, there are permutations $\sigma_1, \ldots, \sigma_l$ such that $A_u$ does not happen for any $u \in U$. It means that for any vertex $u \in U$, there is an $i$ $(1 \leq i \leq l)$ such that $N(u) \cap I_{\sigma_i} \neq \emptyset$. Note that if we set $T = \bigcup_{i=1}^{l} I_{\sigma_i}$, then $G[T]$, the induced subgraph on $T$, has the chromatic number at most $l$. Assume that $c_1$ is a proper $\chi(G[T])$-coloring of $G[T]$. Now, let $H$ be a hypergraph with the vertex set $T$ and the hyperedge defined as follows

$$E(H) = \{N(v) \mid v \in V(G), N(v) \subseteq T\}.$$  

One can check that $H$ is 2-colorable. If $H$ is an empty hypergraph, there is noting to prove. Otherwise, since $H$ is a $k$-uniform hypergraph with maximum degree at most $k$ $(k \geq 4)$, $H$ is 2-colorable. Assume that $c_2$ is a 2-coloring of $H$. It is obvious that $c = (c_1, c_2)$ is a 2l-coloring of $G[T]$. Now, consider a $(\chi(G) + 2l)$-coloring $f$ for $G$ such that the restriction of $f$ on $T$ is the same as $c$. One can check that $f$ is a dynamic coloring of $G$. \hfill \blacksquare

It is proved in [6] that for any $c > 6$, there is a threshold $n(c)$ such that if $G$ is a $k$-regular graph with $k \geq n(c)$ then, $G$ has a total dominating set inducing a graph with maximum degree at most $2c \log k$ (for instance if we set $c = 7.03$ then $n(c) \leq 139$). Note that in the proof of previous theorem, it is proved that any triangle free $k$-regular graph $G$ has a total dominating set $T$ such that the induced subgraph $G[T]$ has the chromatic number at most $\lceil 3 \ln k + 1 \rceil$.

In the rest of the paper by $G \times H$ and $G \square H$, we refer to the Categorical product and Cartesian product of graphs $G$ and $H$, respectively. It is well-known that if $G$ is a graph with $\chi(G) > n$, then $G \times K_n$ is a uniquely $n$-colorable graph, see [9].

It was conjectured in [1] that for any regular graph $G$ with $\chi(G) \geq 4$, the chromatic number and the dynamic chromatic number are the same. Here, we present a counterexample for this conjecture.

**Proposition 1** For any integer $n > 1$, there are regular graphs with chromatic number $n$ whose dynamic chromatic number is more than $n$.

**Proof.** Assume that $G_1$ is a $d$-regular graph with $\chi(G_1) > n$ and $m = |V(G_1)|$. Set $G_2 = G_1 \square C_{(n-1)(d+2)+1}$ and $G' = G_2 \times K_n$. Note that $G'$ is a uniquely $n$-colorable graph with regularity $(n - 1)(d + 2)$. Consider the $n$-coloring $(V_1, V_2, \ldots, V_n)$ for
where for \(1 \leq i \leq n\), \(V_i = \{(g, i) \mid g \in V(G_2)\}\). It is obvious that \(|V_i| = m((n-1)(d + 2) + 1)\) is divisible by \((n-1)(d + 2) + 1\). Now, for each \(1 \leq i \leq n\), consider \((S_1^i, S_2^i, \ldots, S_m^i)\) as a partition of \(V_i\) such that \(|S_j^i| = (n-1)(d+2) + 1\). Now, for any \(1 \leq i \leq n\) and \(1 \leq j \leq m\), add a new vertex \(s_{ij}\) and join this vertex to all the vertices in \(S_j^i\) to construct the graph \(G\). Note that \(G\) is an \(((n-1)(d+2)+1)\)-regular graph with \(\chi(G) = n\). Now, we claim that \(\chi_2(G) > n\). To see this, assume that \(c\) is an \(n\)-dynamic coloring of \(G\). Since \(G'\) is uniquely \(n\)-colorable, the restriction of \(c\) to \(V(G)\) is \((V_1, V_2, \ldots, V_n)\). In the other words, all the vertices in \(V_1\) have the same color in \(c\). But, all the neighbors of \(s_1^1\) are in \(V_1\) and this means that \(c\) is not a dynamic coloring. \(\blacksquare\)

However, it can be interesting to find some regular graph \(G\) with \(\chi(G) \geq 4\) and \(\chi_2(G) - \chi(G) \geq 2\).

Also, as a generalization of Conjecture \(\[\square\]\) it was conjectured \(\[\square\]\) that for any graph \(G\), \(\chi_2(G) - \chi(G) \leq \left\lceil \frac{\Delta(G)}{\delta(G)} \right\rceil + 1\). Here, we give a negative answer to this conjecture. To see this, assume that \(G_1\) is a graph with \(\chi(G_1) \geq 3\) and \(n\) vertices such that \(n > 3\Delta(G_1) + 5\). For any 2-subset \(\{u, v\} \subseteq V(G_1)\), add a new vertex \(x_{uv}\) and join this vertex to the vertices \(u\) and \(v\). Let \(G\) be the resulting graph form \(G_1\) by using this construction. Note that \(\chi_2(G) \geq n\), \(\Delta(G) = \Delta(G_1) + n - 1\), \(\delta(G) = 2\) and \(\chi(G) = \chi(G_1)\). Therefore if the conjecture was true, then we would have

\[
n - \Delta(G_1) - 1 \leq n - \chi(G_1) \leq \chi_2(G) - \chi(G) \leq \left\lceil \frac{\Delta(G_1) + n - 1}{2} \right\rceil + 1.
\]

Note that it is not possible because \(n > 3\Delta(G_1) + 5\), and consequently, \(n - \Delta(G_1) - 1 > \left\lceil \frac{\Delta(G_1) + n - 1}{2} \right\rceil + 1\).

Acknowledgment
The author wishes to express his deep gratitude to Hossein Hajiabolhassan for drawing the author’s attention to two counterexamples (Proposition \(\[\square\]\) and the example after that).

References

[1] A. Ahadi, S. Akbari, A. Dehghan, and M. Ghanbari. On the difference between chromatic number and dynamic chromatic number of graphs. \textit{Discrete Math.}, In press.

[2] S. Akbari, M. Ghanbari, and S. Jahanbekam. On the dynamic coloring of strongly regular graphs. \textit{Ars Combin.}, to appear.

[3] S. Akbari, M. Ghanbari, and S. Jahanbekam. On the list dynamic coloring of graphs. \textit{Discrete Appl. Math.}, 157(14):3005–3007, 2009.

[4] S. Akbari, V. Liaghat, and A. Nikzad. Colorful paths in vertex coloring of graphs. \textit{Electron. J. Combin.}, 18(1):Research Paper 17, 2011.

[5] M. Alishahi, A. Taherkhani, and C. Thomassen. Rainbow paths with prescribed ends. \textit{Electronic Journal of Combinatorics}, 18(1), 2011.
[6] Meysam Alishahi. On the dynamic coloring of graphs. *Discrete Appl. Math.*, 159(2-3):152–156, 2011.

[7] Noga Alon and Joel H. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2008.

[8] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In A. Hajnal, R. Rado, and V. T. Sós, editors, *Infinite and finite sets*, volume 10 of CMSJB, pages 609–627, Keszthely, 1973, 1975. North-Holland.

[9] C.D. Godsil and G. Royle. *Algebraic graph theory*. Graduate texts in mathematics. Springer, 2001.

[10] Hong-Jian Lai, Jianliang Lin, Bruce Montgomery, Taozhi Shui, and Suohai Fan. Conditional colorings of graphs. *Discrete Math.*, 306(16):1997–2004, 2006.

[11] Hong-Jian Lai, Bruce Montgomery, and Hoifung Poon. Upper bounds of dynamic chromatic number. *Ars Combin.*, 68:193–201, 2003.

[12] Colin McDiarmid. Hypergraph colouring and the lovász local lemma. *Discrete Math.*, 167/168:481–486, 1997.

[13] Xianyong Meng, Lianying Miao, Bentang Su, and Rensuo Li. The dynamic coloring numbers of pseudo-Halin graphs. *Ars Combin.*, 79:3–9, 2006.

[14] B. Montgomery. *Dynamic coloring of graphs*. PhD thesis, West Virginia University, 2001.

[15] Carsten Thomassen. The even cycle problem for directed graphs. *J. Amer. Math. Soc.*, 5(2):217–229, 1992.