Two-loop QCD corrections for 2 → 2 parton scattering processes

Maria Elena Tejeda-Yeomans

C. N. Yang Institute for Theoretical Physics
State University of New York at Stony Brook, New York 11794-3840, USA

Abstract. A summary is presented of the most recent matrix elements for massless 2 → 2 scattering processes calculated at two loops in QCD perturbation theory together with a brief review on the calculational methods and techniques used.

INTRODUCTION

The description of high energy processes at hadron colliders together with the study of the environment these events are emerging into, will rely heavily on the predictive power of the theory of Quantum Chromo-Dynamics (QCD). It is well known that QCD studies enable a better understanding of the color flow dynamics, which in turn facilitates improved predictions on observables such as $p_T$ distributions and production rates. Also, QCD studies have a direct impact on theoretical predictions for signals and their backgrounds, which are a key element in the quest for precision physics at the highest energies. A plethora of data on multi-particle final states will soon become available from high-energy collider runs and the comparisons of jet observables with theoretical predictions will be vital. In fact, missing higher order QCD theoretical predictions for such observables may be large and important. Therefore a systematic approach to perform these complex calculations, which usually involve the analysis of multi-leg and/or multi-loop amplitudes, becomes imperative. In recent years, new techniques and integral manipulation methods have been designed to achieve outstanding analytical results in this area. It is the purpose of this talk to review briefly the latest results on two-loop QCD corrections for massless 2 → 2 parton scattering processes and some of the techniques used. This is not an exhaustive review, so more details can be found in the references provided and the ones therein.

There are many important reasons why one might consider a next-to-next to leading order (NNLO) calculation[1] and to mention a few,

Reduced scale dependence The theoretical prediction for any observable $\Gamma$ should be

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2 E-mail address: tejeda@insti.physics.sunysb.edu. Partially supported by the National Science Foundation grant PHY-0098527.
independent of the renormalisation scale $\mu$. But, if we perform a fixed order calculation a scale dependence is introduced. The change due to the variation of renormalisation scale is formally one order higher than the one at which the theoretical prediction is given. Often, when giving a NLO result, a variation of $\mu$ around some hard scale is used as a tool to give the uncertainty in uncalculated higher orders but, this is an estimate of their size. The theoretical prediction can be improved with a complete NNLO result.

Improved jet description If we consider higher order corrections we automatically improve the matching between theoretically and experimentally defined jets. At leading order we are modeling jets with single parton emission, but at higher orders the phase space available is extended so that at NNLO up to three partons can combine to form a jet. Corrections beyond LO involve a better description of soft gluon radiation within the jet and this may provide a more accurate picture of its shape and structure.

Reduced power corrections When comparisons of NLO predictions with experimental data are made, the need for power corrections (terms of $O(1/Q^n)$) arises. The structure of these corrections can be motivated theoretically but they are always fitted to experiment. One might expect that higher order corrections may play a role in reducing the size of these power corrections.

Partonic cross sections beyond NLO

The description of hard scattering processes at future hadron colliders requires the study of the factorized structure of a cross section for processes with quarks and gluons in the initial state. Up to power corrections, the inclusive factorized cross section can be written as

$$\sigma(P_1, P_2) \sim \sum_{ij} \int dx_1 dx_2 f_{i/1}(x_1, \mu_F^2) f_{j/2}(x_1, \mu_F^2) \hat{\sigma}_{ij}(p_1, p_2, \alpha_s(\mu^2), s/\mu^2, s/\mu_F^2)$$  \hspace{1cm} (1)$$

where $p_i = x_i P_i$ is the momenta of the partons that initiate the hard scattering and $f_{a/h}$ are the parton distribution functions (pdf’s). The non-perturbative effects are comprised in these distribution functions (and into fragmentation functions, in the exclusive case). $\hat{\sigma}$ is the hard scattering partonic cross section which describes the interaction of partons $\{i, j\}$ that arose from hadrons $\{1, 2\}$ and can be calculated perturbatively, so that at NNLO for 2-particle production it can be written as

$$\hat{\sigma}_{\text{2 jet}} \sim \int \left[ \langle |\mathcal{M}^{(0)}| \mathcal{M}^{(0)} \rangle \right]_4 d\Phi_4 + \int \left[ \langle |\mathcal{M}^{(0)}| \mathcal{M}^{(1)} \rangle + \langle |\mathcal{M}^{(1)}| \mathcal{M}^{(0)} \rangle \right]_3 d\Phi_3 + \int \left[ \langle |\mathcal{M}^{(1)}| \mathcal{M}^{(1)} \rangle + \langle |\mathcal{M}^{(0)}| \mathcal{M}^{(2)} \rangle + \langle |\mathcal{M}^{(2)}| \mathcal{M}^{(0)} \rangle \right]_2 d\Phi_2$$  \hspace{1cm} (2)$$

where $[[]]_n$ indicates the number of particles in the final state with $d\Phi_n$ the corresponding phase space and $\mathcal{M}^{(i)}$ the $i$-th order scattering amplitude. After renormalisation, each
of the integrals in Eq. (2) is ultra-violet (UV) finite but infra-red (IR) divergent which manifests itself as poles in $\varepsilon$ (we adopt $D = 4 - 2\varepsilon$ as a dimensional regulator). The integration over phase space and the cancellation of poles requires the study of the kinematical regions where the additional radiated particles become unresolved. Finally, pdf’s and their evolution are needed at an accuracy that matches that of the matrix element calculation and for NNLO some great developments have already taken place[5].

**TWO-LOOP INTEGRALS**

The number of diagrams involved in the complete two-loop and one-loop matrix elements needed for the NNLO contribution to inclusive jet cross sections and photoproduction at hadron colliders, are shown in Table [6]. The types of integrals arising in these matrix-element calculations include integrals with scalar numerators

$$\int \frac{d^Dk}{i\pi^D} \int \frac{d^D\ell}{i\pi^D} \frac{\mathcal{F}(k_i \cdot k_j, k_i \cdot p_j, p_i \cdot p_j)}{A_1^{\nu_1} \cdots A_n^{\nu_n}}$$

where $k_i = k, \ell$ are the loop momenta, $p_i$ are the external momenta and $\mathcal{F}$ is a scalar function. Here, the massless propagators are denoted by $A_j \sim (k_i \pm \varphi + i\varepsilon)^2$ where $\varphi$ can be any of the loop or external momenta. There are other terms that would typically look like the one shown in Eq. (3), except now the function $\mathcal{F} \to \mathcal{F}^{\mu
u\cdots}(k, \ell)$ is a tensor that can depend on the loop-momenta of the system and/or other tensors (such as the metric tensor $g^{\mu\nu}$). Furthermore, the denominator of an integral provides the momentum flow in the graph and gives a complete description of its skeleton or topology which can be planar or non-planar. Within these two categories we will identify sub-topologies or pinchings, depending on whether or not a particular subset of propagators is absent. As technical aids, we use a couple of auxiliary general diagrams shown in Fig. [1], where each of the propagators is labeled by an integer and carries a specific momentum that fulfills conservation of momenta throughout. This auxiliary representation allows us to manipulate hundreds of integrals with the minimum amount of information and without compromising the accuracy of the integral description. So to refer to a

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3 The singularities are guaranteed to cancel for sufficiently inclusive physical quantities[2].
4 See for example, analytical[3] and numerical[4] applications.
Master integrals needed for the two-loop massless evaluation (see for example [10]) are used, whereby the propagators of the loop integrand scales increase. Different techniques arose due to this situation, among others are expressed in terms of integrations for real parameters over a particular range. In by eliminating, increasing or decreasing the values of \( \epsilon \) as series in \( \epsilon = (4 - D)/2 \). Usually, Feynman or Schwinger parameterizations (see for example [11]) are used, whereby the propagators of the loop integrand are expressed in terms of integrations for real parameters over a particular range. In both of these approaches, the loop momenta integration can be done easily and the remaining integrations over the parameters are doable for reasonably sized topologies. Sometimes, using these parametric representations does not leave an integral that can be solved easily, even more so when the number of loops, external legs and kinematical scales increase. Different techniques arose due to this situation, among others

- The Mellin-Barnes method is based on the representation for a sum to some power,
as a contour integral over a complex variable and the integration is then performed on straight contour lines parallel to the imaginary axis. After closing a contour, the result is the sum of all enclosed residues that may be expressed as a hypergeometric series. This has been used to obtain vital results for the two-loop boxes[7, 11].

- The Negative Dimensions technique consists of rewriting the integral over the parameters by introducing new ones through a multinomial expansion. Many conditions have to be satisfied among the parameters, which leads to the restriction: \( D \) must be a negative integer. Some results for MI have been obtained using this method[8].

- Numerical strategies are used, where any loop integral is stripped analytically of its IR singularities so that the finite integrals can be performed numerically. These methods can also be used to tackle the phase space integrations in the calculation of observables[4].

- The analytic evaluation of MI can also be carried out without explicit integration over loop momenta by deriving differential equations for MI in internal propagator masses or in external momenta. The equations can be solved with appropriate boundary conditions. This approach has been widely used to evaluate MI, as was done recently for \( e^+ e^- \to 3 \text{ jets} \)[8].

### Loop integral reduction

This approach produces an environment in which complex topologies with high powers on the propagators can be reduced down to integrals that can be solved with the methods described in the previous section. This reduction can be achieved using systems of equations that stem from Integration by Parts identities (IBP)[12] and exploiting the Lorentz invariance[9] of Feynman integrals. Let us review briefly how these identities are generated and help reduce the complexity of an integral. Consider a general two-loop integral

\[
I^{D} [v_1, \ldots, v_n] = \int \frac{d^D k}{i\pi^2} \int \frac{d^D \ell}{i\pi^2} \frac{1}{A_1^{v_1} \cdots A_n^{v_n}}, \quad (4)
\]

The idea behind IBP is to generate relations between loop integrals through a total derivative with vanishing surface terms, expressed as the following identity

\[
\int \frac{d^D k}{i\pi^2} \int \frac{d^D \ell}{i\pi^2} \frac{\partial}{\partial k_i^{\mu}} \left[ \frac{\gamma^\mu}{A_1^{v_1} \cdots A_n^{v_n}} \right] \equiv 0, \quad (5)
\]

where \( k, \ell \) are the loop-momenta and \( \gamma^\mu \) can be any internal or external momenta involved in the loop integration. Executing the derivative on all possible choices for \( \gamma^\mu \) will generate a set of relations[7] between integrals with dot products in the numerator.

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5 For a graph with \( m \) loops and \( n \) independent external momenta, we can generate \( m(m + n) \) identities.
These dot products can be rewritten in terms of linear combinations of propagators, by means of relations such as

\[ 2(k + g) \cdot (k + h) = (k + g)^2 + (k + h)^2 - (g - h)^2. \]  

(6)

With this simple step we can rewrite all the contents of the numerator in terms of propagators that may or may not be part of the denominator. We then say that the numerator is reducible if we can cancel it through and irreducible otherwise. In most cases, we can exchange the problem of calculating the original integral, for the problem of calculating a set of simpler integrals. We can imagine studying all the IBP identities for the topologies involved in a particular matrix-element calculation and applying the reduction procedure iteratively. Then we can assemble an algorithm that takes any integral and expresses it in terms of a few MI. This is precisely what we will review next.

**MATRIX ELEMENT EVALUATION AND RESULTS**

The ideas exposed above and extensions to them have been used in the past few years to tackle two-loop calculations. The way these calculations are put together varies, for example the chart in Fig. 3 shows some basic steps we followed to evaluate the two-loop QCD matrix-elements for $2 \to 2$ scattering. Steps 1 through 3 are mainly related to the automatic generation of diagrams using QGRAF\[13\] and the tensor algebra manipulation using FORM\[14\]. By step 4, we have the matrix element written as a linear combination of many different scalar and tensor two-loop integrals. These integrals can be rewritten in terms of MI (step 6), using a reduction system that was generated using MAPLE (step 5). The final step is the input of $\epsilon$-expansions for each MI, so that we get

\[ \langle \mathcal{M}^{(0)}, \mathcal{M}^{(2)} \rangle \sim \sum_{n=0}^{4} f_n (\alpha_s, \mu, \{s, t, u\}) \epsilon^n, \]  

(7)

where $\{s, t, u\}$ are the usual Mandelstam variables. The functions $f_n$ depend on the color factors $\{N_F, C_F, C_A, T_R\}$ and logarithms of ratios of the kinematic variables. The finite piece $f_0$ can contain up to 6 polylogarithms (Li$_n$ with $n = \{2, 3, 4\}$) of ratios in the kinematical variables, and the Riemann zeta function $\zeta(n)$ (with $n = \{2, 3\}$).

Several checks have served to verify the results obtained with this approach such as comparisons with QED processes\[15\] and also the study of the analytic structure of QCD amplitudes in the limit of forward and backward scattering together with the confirmation of the two-loop gluon Regge trajectory\[16\]. The fact that the singular analytical structure in Eq. (7) can be obtained independently, is also a powerful check on the calculation\[6\]. Catani\[17\] proposed the structure of the $1/\epsilon^n \forall n = \{2, 3, 4\}$ poles together with the color uncorrelated structure of the $1/\epsilon$ pole. For a while, the origins of this proposal did not exist. However, in the work with Sterman\[18\], we describe how the factorization properties of loop amplitudes lead to the exponentiation of double and single poles at each order in perturbation theory. The poles can then be assembled in terms of universal functions associated with the external partons. This formalism
FIGURE 3. Basic steps to follow in the evaluation of matrix elements

provides a way to generate the complete pole structure for multi-loop amplitudes at two loops and beyond.

CONCLUSION AND OUTLOOK

As reviewed, the past few years have seen a breakthrough in multi-loop integration technology and outstanding progress in the calculation of two-loop matrix elements in QCD. Much work remains to be done to have NNLO Monte Carlo numerical estimates but, looking back at the results we now have, it seems they will be available soon for the first basic scattering processes. This will enable an improved description of high-energy QCD phenomena.

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