Limits with Signed Digit Streams

Franziskus Wiesnet∗

Department of Mathematics, University of Trento,
Via Sommarive, 14, 38123 Trento, Italy, European Union
Franziskus.Wiesnet@unitn.it

July 30, 2018

Abstract

We work with the signed digit representation of abstract real numbers, which roughly is the binary representation enriched by the additional digit -1. The main objective of this paper is an algorithm which takes a sequence of signed digit representations of reals and returns the signed digit representation of their limit, if the sequence converges. As a first application we use this algorithm together with Heron’s method to build up an algorithm which converts the signed digit representation of a non-negative real number into the signed digit representation of its square root.

Instead of writing the algorithms first and proving their correctness afterwards, we work the other way round, in the tradition of program extraction from proofs. In fact we first give constructive proofs, and from these proofs we then compute the extracted terms, which is the desired algorithm. The correctness of the extracted term follows directly by the Soundness Theorem of program extraction. In order to get the extracted term from some proofs which are often quite long, we use the proof assistant Minlog. However, to apply the extracted terms, the programming language Haskell is useful. Therefore after each proof we show a notation of the extracted term, which can be easily rewritten as a definition in Haskell.

Keywords: signed digit code, real number computation, coinductive definitions, corecursion, program extraction, realizability, convergence of reals, square root, Heron’s method

1 Introduction and Motivation

There are several ways to define constructive real numbers. One of the best-known methods is to define them as Cauchy sequences of rational numbers with a Cauchy modulus. In our case, rather than we are not interested in a specific definition of real numbers, we are interested in their signed digit representation (SD code). Therefore, every quantifier on reals is non-computational, i.e. we write ∀x̄ and ∃x̄. These decorations just mean that the bounded variable does not appear in the computational content of the proof. Logically one can ignore the decorations. By x and y we denote variables of reals. Instead of using the

∗Marie Skłodowska-Curie fellow of the Istituto Nazionale di Alta Matematica
concrete real computationally, we use their signed digit representation in the
extracted term.
Attention should be paid to the equality between reals numbers. By equality =
between two reals we mean the “real equality”, which is an equivalence relation
and compatible with the usual operators and relations on the reals. The specific
definition of the real equality depends on the definition of the real numbers.
Generally, the real equality in not the same as the Leibniz equality. For instance,
if one defines real numbers as Cauchy sequences with modulus, the real numbers
\((2^{-n})_{n \in \mathbb{N}}, id\) and \((0)_{n \in \mathbb{N}}, id\) are equal w.r.t. the real equality but they are
not Leibniz equal.
One of the first paper \[13\] where signed digits are used to represent real
numbers, was published by Edwin Wiedmer in 1980. The SD code of reals is
similar to the binary code of reals but in addition to the digits 0 and 1, the SD
code has the digit \(-1\), which we also denote by \(\overline{1}\). Since every real \(x\) can be
represented as \(x = k + x'\), where \(k\) is an integer and \(-1 \leq x' \leq 1\), we work on
the interval \([-1, 1]\).
A real number \(x\) in \([-1, 1]\) has a binary representation if it can be written as
\[x = s \sum_{i=1}^{\infty} a_i 2^{-i},\]
where \(s \in \{-1, 1\}\) and \(a_i \in \{0, 1\}\) for every \(i\).
If one reads one by one the binary representation of a concrete real number, in
each step the interval in which the real number is located, is halved. Thus from
the binary code one can determine the real number arbitrarily exactly. On the
\[\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
& -1 & . & 0 & + & +1 \\
\hline
-1 & -1 & . & 0 & + & +1 \\
\hline
-11 & -10 & -01 & -00 & +01 & +00 & +10 & +11 \\
\hline
-1.11 & -1.10 & -1.01 & -1.00 & +1.00 & +1.01 & +1.10 & +1.11 \\
\hline
\end{array}\]
Figure 1: Visualization of the Binary Code
other hand it is not always possible to compute the binary representation of
a given real number or even to compute the binary representation of \(\frac{x+y}{2}\) out
of the binary representations of \(x\) and \(y\). Hereby “compute” means to get an
algorithm which takes the binary streams of \(x\) and \(y\) and gives back the digits
of the binary stream \(\frac{x+y}{2}\) one by one in finitely many steps for each digit. That
is, to compute finitely many binary digits of \(\frac{x+y}{2}\) can only use finitely many
binary digits of \(x\) and \(y\). This is not possible due to the “gaps” in the binary
representation. They are illustrated in Figure 2 at \(0, \frac{1}{2}, -\frac{1}{2}, \frac{3}{4}, \text{ and so on. Form}
The first digit (i.e. + or −) of a real \(x\), one can decide \(0 \leq x\) or \(x \leq 0\), with in
general can not be done for constructive reals.
The signed digit code fills these gabs. For a real number \(x \in [-1, 1]\) it is given by
\[x = \sum_{i=0}^{\infty} d_i 2^{-i},\]
Figure 2: Visualization of the Signed Digit Code

where \( d_i \in \{1, 0, 1\} \) for every \( i \). As the illustration in Figure 2 shows, to compute the first signed digit of a real number \( x \in [-1, 1] \) one has to decide which of the cases \( x \leq 0, -\frac{1}{2} \leq 0 \leq \frac{1}{2} \) or \( 0 \leq x \) holds. That this is possible, follows from the Comparability Theorem, which says that for reals \( x, y \) and \( z \) with \( x < y \) one has \( z \leq y \lor x \leq y \).

Figure 2 also shows that the SD code of a real number except \(-1\) and \(1\) is not unique, whereas the binary code is “almost” everywhere unique.

A stream of signed digits is an infinite list \( d_1 d_2 d_3 \ldots \) of elements in \( S_d := \{1, 0, 1\} \).

We will not prove something about signed digit streams directly, but we use the coinductively define predicate \( \mathrm{coI} \), which is given in the next section. For a real number \( x \) a realiser of \( \mathrm{coI} x \) is a signed digit stream of \( x \). With the Soundness Theorem of program extraction proven in [8, 10, 14] the proofs of the corresponding theorems for the signed digit streams are obtained.

The idea to use coinductive algorithms to describe the operators on the reals goes back to Alberto Ciaffaglione and Pietro Di Gianantonio [4]. The idea to use coinductively defined predicates and the Soundness Theorem in this context is due to Ulrich Berger and Monika Seisenberger [2]. The notation and definitions in this paper are taken from Kenji Miyamoto and Helmut Schwichtenberg [7].

For computing the extracted terms and verifying the correctness of the proofs, we have used the proof assistant Minlog [6] in some cases. After each proof we state its computational content not in the notation of Minlog but in the notation of Haskell, since the runtime of the programs in Haskell is shorter, and one can define the terms in a more readable way.

We now proceed as follows:

In Section 2 we give a definition of \( \mathrm{coI} \) and state the two axioms \( \mathrm{coI}^- \) and \( \mathrm{coI}^+ \) for this predicate. The two lemmas \( \mathrm{I} \) and \( \mathrm{II} \) are often used in the proofs below.

The main result of this work is Theorem \( \mathrm{I} \) the Convergence Theorem, at the end of Section 3. An application of this theorem is shown in Section 4. Here we use Heron’s method together with the Convergence Theorem to get a signed digit representation of the square root of a non-negative real number from its signed digit representation. Section 5 is about potential extensions of this work.
2 Formalisation

In this section we use the theory of coinductively defined predicates given in [10, 14] to formalise the statement that a real $x$ has an SD representation.

**Definition 1.** We define $\text{co} I$ as the greatest fixpoint of

$$\Phi(X) := \left\{ x \mid \exists_{d, x'} \left( Sd \ d \wedge X x' \wedge |x| \leq 1 \wedge x = \frac{d + x'}{2} \right) \right\}$$

Here $\exists_{x} A_{x} := \mu_{X} (\forall_{x}^{\lt}(A_{x} \to X))$ is the existential quantifier where the quantified variable $x$ does not appear in the computational content.

In a short form we have $\text{co} I := \nu_{X} (\Phi(X))$. Therefore a realiser of $\text{co} I x$ has the type

$$\tau(\text{co} I) = \mu_{\tau}(\tau(\Phi(X)) \to \tau(X)) = \mu_{\xi}(\text{Sd} \to \xi \to \xi).$$

Here we have identified $\tau(\text{Sd}) = \mu_{\xi}(\xi, \xi, \xi)$ with $\text{Sd}$ itself. We define $\text{Str} := \tau(\text{co} I)$ and with $\text{C}$ we denote the only constructor of $\text{Str}$. Then in Haskell notation $\text{Str}$ is given by

$$\text{Str} := \text{C Sd Str}.$$  

From the definition of $\text{co} I$ as greatest fixpoint of $\Phi$ we get two axioms for $\text{co} I$.

$$\text{co} I^- : \text{co} I \subseteq \Phi(\text{co} I)$$

$$\text{co} I^+ : X \subseteq \Phi(\text{co} I \cup X) \to X \subseteq \text{co} I$$

where $X$ is an unary predicate variable. The first axiom $\text{co} I^-$ says that $\text{co} I$ is a fixpoint of $\Phi$ and for obvious reasons it is called the elimination rule. Expressed in elementary formulas it is given by

$$\text{co} I^- : \forall_{x}^{\lt} \cdot \text{co} I x \to \exists_{d, x'} \left( Sd \ d \wedge \text{co} I x' \wedge |x| \leq 1 \wedge x = \frac{d + x'}{2} \right).$$

The type of this axiom is $\tau(\text{co} I^-) = \text{str} \to \text{Sd} \times \text{str}$ and a realiser of it is given by the destructor $\mathcal{D}$ with computation rule

$$\mathcal{D} : \text{str} \to \text{Sd} \times \text{str},$$

$$\mathcal{D}(\text{C}dv) := (d, v).$$

The destructor takes a stream and returns a pair consisting of its first digit and its tail. With the projectors $\pi_{0}$ and $\pi_{1}$ one gets the first digit and the tail, respectively.

The second axiom $\text{co} I^+$ is called the introduction axiom of $\text{co} I$ and says that $\text{co} I$ is the greatest fixpoint in the a strong sense. We use the following long version:

$$\text{co} I^+ : \forall_{x}^{\lt} X x \to$$

$$\forall_{x}^{\lt} \left( X x \to \exists_{d, x'} \left( Sd \ d \wedge (\text{co} I \cup X) x' \wedge |x| \leq 1 \wedge x = \frac{d + x'}{2} \right) \right)$$

$$\to \text{co} I x.$$
The type of this axiom depends on the type of the predicate variable $X$:

$$\tau^{(\text{co}I^+)} = \tau(X) \rightarrow (\tau(X) \rightarrow \mathbf{Sd} \times (\mathbf{Str} + \tau(X))) \rightarrow \mathbf{Str}$$

A realiser of $\text{co}I^+$ is the corecursion operator $\text{co}R$. It is given by the computation rule

$$\text{co}R : \tau(X) \rightarrow (\tau(X) \rightarrow \mathbf{Sd} \times (\mathbf{Str} + \tau(X))) \rightarrow \mathbf{Str},$$

$$(\text{co}Rf := C(\pi_0(ft))[\text{id}, \lambda t'. \text{co}R t' f]) \pi_1(ft),$$

where $[F_0, F_1][\text{in}_iT := F_iT$ for $i \in \{0, 1\}$. Here $\text{in}_0$ and $\text{in}_1$ are the two constructors of the type sum $\mathbf{Str} + \tau(X)$. If $\pi_1(ft)$ has the form $\text{in}_0 v$, the corecursion stops and we have $C(\pi_0(ft))v$ as signed digit representation. If it has the form $\text{in}_1 t'$, the corecursion goes further with the new argument $t'$. In both cases we have obtained at least the first digit of the stream. With iteration of the corecursion, if necessary, we generate each digit of the stream one by one.

Now we prove two lemmas, which are often used in our proofs:

**Lemma 1.** The predicate $\text{co}I$ is compatible with the real equality, i.e.

$$\forall \text{nc } x, y. \text{co}I x \rightarrow x = y \rightarrow \text{co}I y.$$

**Proof.** We apply $\text{co}I^+$ to the predicate $P x := \exists y (\text{co}I y \land x = y)$:

$$\forall_{x, y} \exists_{x'} (\text{co}I y \land x = y) \rightarrow
\forall_{x'} \left( Px \rightarrow \exists_{d, x'} \left( \mathbf{Sd} d \land (\text{co}I \cup P) x' \land |x| \leq 1 \land x = \frac{d + x'}{2} \right) \right) \rightarrow \text{co}I x$$

As one sees, it is sufficient to prove the second premise. Therefore we show

$$\forall_{x'} \left( \exists_{y'} (\text{co}I y \land x = y) \rightarrow \exists_{d, x'} \left( \mathbf{Sd} d \land (\text{co}I \cup P) x' \land |x| \leq 1 \land x = \frac{d + x'}{2} \right) \right).$$

Let $x, y$ with $\text{co}I y$ and $x = y$ be given. From $\text{co}I y$ we get $e \in \mathbf{Sd}$ and $y' \in \text{co}I$ with $|y| \leq 1$ and $y = \frac{e + y'}{2}$. We now define $d := e$ and $x' := y'$ and our goal follows directly because of $x = y$.

In the following proofs this theorem is used tacitly. If one have a model of reals in which the real equality is the same as the Leibniz equality, one do not need this theorem. The extracted term of it is given by

$$\lambda u. \text{co}R u(\lambda v. (\pi_0(Dv), \text{in}_0(\pi_1(Dv))))\).$$

For stream of the form $\mathbf{cd}u$ this term is the identity function. Since we always deal only with such streams, we drop this term hereafter.

**Lemma 2.**

$$\forall_{x, d} \mathbf{Sd} d \rightarrow \text{co}I x \rightarrow \text{co}I \frac{d + x}{2}$$
Proof. We apply \( ^{co}I^+ \) to the predicate
\[
Px := \exists_{d,x'} \left( Sd \land ^{co}Ix' \land x = \frac{d + x'}{2} \right).
\]
This leads to the formula
\[
\forall_{x} \exists_{d,x'} \left( Sd \land ^{co}Ix' \land x = \frac{d + x'}{2} \right)
\]
\[
\rightarrow \exists_{d,x'} \left( Sd \land \left( ^{co}I \cup P \right) x' \land |x| \leq 1 \land x = \frac{d + x'}{2} \right).
\]
In order to prove the goal formula, it is sufficient to prove the second premise. Therefore the new goal formula is
\[
\forall_{x} \exists_{d,x'} \left( Sd \land \left( ^{co}I \cup P \right) x' \land |x| \leq 1 \land x = \frac{d + x'}{2} \right)
\]
\[
\rightarrow ^{co}Ix.
\]
Formally the extracted term of this proof is given by
\[
\lambda_{d} \lambda_{u} \cdot ^{co}R u \lambda_{v} \langle d, \text{in} 0 \rangle.
\]
If we use the computation rule of \( ^{co}R \) once, we get \( \lambda_{d} \lambda_{u} \cdot C u \), which is identified with \( C \) itself. Therefore the constructor \( C \) is actually the computational content of this lemma.

3 Convergence Theorem

The aim of this chapter is to prove the convergence theorem in the SD code. It says that the limit of each convergent sequence in \( ^{co}I \) is also in \( ^{co}I \). As extracted term we expect a function which takes a sequence of signed digit streams (i.e. a term of type \( \mathbb{N} \rightarrow \text{Str} \)) and its modulus of convergence and returns a signed digit stream.

We do the proof step by step and therefore we first prove a few lemmas:

Lemma 3.
\[
\forall_{x} \exists_{d,x'} \left( Sd \land ^{co}Ix' \land x = \frac{d + x'}{2} \right)
\]
\[
\rightarrow ^{co}Ix.
\]

Proof. Since both formulas are shown analogously we only show the first formula. Therefore we use the introduction axiom of \( ^{co}I \), which is given by
\[
\forall_{x} \exists_{d,x'} \left( Sd \land \left( ^{co}I \cup P \right) x' \land |x| \leq 1 \land x = \frac{d + x'}{2} \right)
\]
\[
\rightarrow ^{co}Ix.
\]
Lemma 4. With this lemma we can now prove the following lemma easily: 

\[ \exists x : (x < 0 \land y \leq 0 \land y + 1 = x). \] 

Hence it is sufficient to prove the second premise. Let \( x, y, \sigma \in \mathbb{I}, y \leq 0 \) and \( y + 1 = x \) be given. Our goal is 

\[ \exists x : \left( \sigma d : \sigma d \in (\sigma \sigma \bigcup P \sigma) \land |x| \leq 1 \land x = \frac{d + x'}{2} \right). \]

From \( \sigma \sigma \sigma \) we get \( c \) and \( y' \) with \( \sigma \sigma \sigma \sigma \), \( \sigma \sigma \sigma \sigma \), \( \sigma \leq 1 \) and \( y = \frac{c + y'}{2} \). Independent on \( d \) and \( x' \) always get \( |x| \leq 1 \) out of \( |y| \leq 1, y \leq 0 \) and \( x = y + 1 \). In order to prove the remaining part of the formula, we do case distinction on \( \sigma \sigma \sigma \sigma \) e:

If \( c = -1 \), we define \( d := 1 \) and \( x' := y' \). Here \( \sigma \sigma \sigma \) and \( \sigma \sigma \sigma \) follow directly and we also have 

\[ x = y + 1 = \frac{-1 + y'}{2} + 1 = \frac{1 + y'}{2} = \frac{d + x'}{2}. \]

If \( c = 0 \), we define \( d := 1 \) and \( x' := y' + 1 \). We directly have \( \sigma \sigma \sigma \sigma \). In this case we prove \( \sigma \sigma \sigma \). Hence we know \( \sigma \sigma \sigma \sigma \), \( y' \leq 0 \) and \( x' = y' + 1 \). \( \sigma \sigma \sigma \sigma \) and \( x' = y' + 1 \) are already given and \( y' \leq 0 \) follows directly from \( y \leq 0 \) and \( y = \frac{0 + y'}{2} \).

The last case is \( c = 1 \). Because of \( y \leq 0, y = \frac{1 + y'}{2} \), \( \sigma \sigma \sigma \sigma \). \( \sigma \) is only possible that \( y \) is equal to 0 and therefore \( x = 1 \). Hence we define \( d := 1 \) and \( x' := 0 \). Then \( \sigma \sigma \sigma \sigma \) and \( x = \frac{d + x'}{2} \) are obviously true and \( \sigma \sigma \sigma \) is true because 1 has the SD representation 111.. .

A realiser of the first formula in this lemma is a function \( f \), which takes a signed digit stream of a real number \( x \) and returns a signed digit stream of \( x + 1 \) if \( x \leq 0 \).

Using the formal definition of the extracted term we get for \( f \) the term 

\[ f = \lambda v. \sigma \sigma \sigma \sigma \left( \lambda v. [1, \in_0(\pi_1(\delta v)), 1, \in_1(\pi_1(\delta v)), 1, \in_0(\delta v)] \right), \]

where \( [F_2, F_0, F_1]|d := F_d \) for \( d \in \sigma \sigma \sigma \sigma \) and \( \tilde{I} \) is the infinite list with each entry equal to 1. Another way to characterise this function \( f \) is to give its computation rules:

\[ f(C\sigma v) := C\sigma v \]
\[ f(C\sigma v) := C\sigma(fv) \]
\[ f(C\sigma v) := [1, 1, \ldots] \]

Analogously as extracted term of the second statement of this lemma, we get a function \( g : \sigma \sigma \rightarrow \sigma \sigma \) which is characterised by the rules

\[ g(C\sigma v) := [T, T, \ldots] \]
\[ g(C\sigma v) := C\sigma(gv) \]
\[ g(C\sigma v) := C\sigma v. \]

It takes a signed digit stream of a real \( x \) and returns a signed digit stream of \( x - 1 \) if \( 0 \leq x \).

With this lemma we can now prove the following lemma easily:

**Lemma 4.**

\[ v_{x}^{nc} \sigma \sigma \rightarrow |x| \leq \frac{1}{2} \rightarrow \sigma \sigma \]
Proof. From $\co I x$ we get $d, x'$ with $Sd, \co I x', |x| \leq 1$ and $x = \frac{d + x'}{2}$. Case distinction on $Sd d$ gives three cases:

$d = 1$: Here $2x = 1 + x'$ and with $|x| \leq \frac{1}{2}$ it follows $x' \leq 0$. Therefore the first formula of Lemma 3 gives $\co I (2x)$.

$d = -1$: The proof in this case is done analogously but we need the second statement of Lemma 3.

$d = 0$: This leads to $2x = x'$ and $\co I x'$ is already given.

If we keep the definition of the functions $f$ and $g$ form the computational context of the previous lemma, the computational content $D : \text{Str} \to \text{Str}$ of this lemma is given by

$$D = \lambda u. [g(\pi_1(Du)), \pi_1(Du), f(\pi_1(Du))]|\pi_0(Du).$$

Again we give a more readable characterisation of $D$ by the computation rules

- $D(C1u) := gu$
- $D(C0u) := u$
- $D(C1u) := fu$.

Lemma 5.

$$\forall x. \co I x \rightarrow \co I \left(\frac{x}{2} \pm \frac{1}{4}\right)$$

Proof. $\co I x$ gives $x' \in \co I$ and $d \in Sd$ with $x = \frac{d + x'}{2}$. We show only $\co I \left(\frac{x}{2} + \frac{1}{4}\right)$ because the proof of $\co I \left(\frac{x}{2} - \frac{1}{4}\right)$ is done analogously. Case distinction on $Sd d$ leads to the following three cases:

- $d = 1$ gives $\frac{x}{2} + \frac{1}{4} = \frac{2 + x'}{4} = \frac{1 + x'}{2}$,
- $d = 0$ gives $\frac{x}{2} + \frac{1}{4} = \frac{1 + x'}{4} = \frac{1 + x'}{2}$ and
- $d = -1$ gives $\frac{x}{2} + \frac{1}{4} = \frac{x'}{4} = \frac{x'}{2}$.

In each case we get $\co I \left(\frac{x}{2} + \frac{1}{4}\right)$ by using Lemma 3 twice.

We denote the extracted term of the proven statement by $q^+$. From the proof and the fact, that the extracted term of Lemma 3 is given by $C$, one easily sees that $q^+$ has the following computation rules:

- $q^+(Tu) := 00u$
- $q^+(0u) := 01u$
- $q^+(1u) := 10u$

Analogously, the computational content $q^-$ of the statement $\forall x. \co I x \rightarrow \co I \left(\frac{x}{2} - \frac{1}{4}\right)$ is characterised by

- $q^-(Tu) := Tu$
- $q^-(0u) := 0Tu$
- $q^-(1u) := 00u$.

Before we prove the Convergence Theorem, we have to give a definition of convergence. In this definition the witness of convergence is included:
Definition 2. Let a real $x$, a sequence $f : \mathbb{N} \to \mathbb{R}$ of reals and a modulus $M : \mathbb{Z}^+ \to \mathbb{N}$ be given. We say $f$ converges to $x$ with modulus $M$ if
\[ \forall p \forall n \geq M(p) |f(n) - x| \leq 2^{-p}. \]

Theorem 1. Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence of reals in $\mathbb{R}$ which converges to a real $x$ with a modulus $M$, then also $x$ is in $\mathbb{R}^I$.

Expressed in a formula:
\[ \forall x \forall n \forall f (n) \wedge \forall p \forall n \geq M(p) |f(n) - x| \leq 2^{-p} \Rightarrow \mathbb{R}^I(x) \]

Proof. We show the equivalent formula
\[ \forall n \exists_d \exists_d \exists_d (\forall n \forall f(n) \wedge \forall p \forall n \geq M(p) |f(n) - x| \leq 2^{-p}) \Rightarrow \mathbb{R}^I(x). \]

The existence quantifier $\exists_d$ is an inductively defined predicate and given by $\exists_d A(x) := \mu x (\forall x (A(x) \rightarrow \exists_d A(x)))$. This means that in contrast to $\exists'$ the quantified variable is also part of the computational content.

To prove this formula we apply $\mathbb{R}^I$ to the predicate
\[ P x := \exists_d \exists_d \exists_d (\forall n \forall f(n) \wedge \forall p \forall n \geq M(p) |f(n) - x| \leq 2^{-p}) \]

and show the second premise, which is given by
\[ \forall n \exists_d \exists_d \exists_d (\forall n \forall f(n) \wedge \forall p \forall n \geq M(p) |f(n) - x| \leq 2^{-p}) \rightarrow \exists_d x' (d \wedge (\mathbb{R} \cup P) x' \wedge |x| \leq 1 \wedge x = \frac{d + x'}{2}). \]

Let $x, f, M, \forall n \forall f(n)$ and $\forall p \forall n \geq M(p) |f(n) - x| \leq 2^{-p}$ be given. We show
\[ \exists_d d \wedge \exists_d x' (\forall n \forall f(n) \wedge \forall p \forall n \geq N(p) |g(n) - x'| \leq 2^{-p}) \wedge |x| \leq 1 \wedge x = \frac{d + x'}{2}. \]

Regardless of the choice of $x$ and $x'$ we get $\forall n |f(n)| \leq 1$ from $\forall n \forall f(n)$ and with $\forall p \forall n \geq M(p) |f(n) - x| \leq 2^{-p}$ it follows $|x| \leq 1$. Therefore in each case we consider $|x| \leq 1$ as proven.

Specializing $\forall n \forall f(n)$ to $M(4)$ leads to $\mathbb{I}f(M(4))$ and triple application of $\mathbb{R}^I$ gives $d_1, d_2, d_3 \in \mathbb{S}d$ and $y' \in \mathbb{R}^I$ such that $f(M(4)) = \frac{d_1 + 2d_2 + d_3 + y'}{8}$ or in short notation
\[ f(M(4)) = d_1 d_2 d_3 y'. \]

Now we do case distinction on this representation of $f(M(4))$.

If $f(M(4))$ has one of the forms $11d_3 y', 10d_3 y', 11y', 111y' \text{ or } 010y'$, it follows that $\frac{1}{8} \leq f(M(4))$. In this case we choose
\[ d := 1 \text{ and } x' := 2x - 1, \]

then $\mathbb{S}d d$ and $x = \frac{d + x'}{2}$ follow directly. Furthermore we define
\[ g(n) := 2f(M(4) \vee n) - 1 \]

for all $n \in \mathbb{N}$, where $m \vee l := \max\{m, l\}$, and
\[ N(p) := M(p + 1) \]
for all $p \in \mathbb{Z}^+$. The formula $\forall_p \forall_{n \geq N(p)} |g(n) - x'| \leq 2^{-p}$ is a direct consequence of $\forall_p \forall_{n \geq M(p)} |f(n) - x| \leq 2^{-p}$ and it remains to show $\forall_n ^{co} I g(n)$. We calculate

$$g(n) = 2f(M(4) \lor n) - 1 = 4 \left( \frac{f(M(4) \lor n)}{2} - \frac{1}{4} \right).$$

Lemma 5 gives $^{co} I \left( \frac{f(M(4) \lor n)}{2} - \frac{1}{4} \right)$. Furthermore we have $f(M(4)) \geq \frac{1}{8}$ and therefore

$$f(M(4) \lor n) = (f(M(4) \lor n) - x) + (x - f(M(4))) + f(M(4)) \geq -\frac{1}{16} - \frac{1}{16} + \frac{1}{8} = 0.$$  

Thus $0 \leq f(M(4) \lor n) \leq 1$ implies $\left| \frac{f(M(4) \lor n)}{2} - \frac{1}{4} \right| \leq \frac{1}{4}$ and Lemma 4 applied twice gives $^{co} I g(n)$.

If $f(M(4))$ has one of the forms $\mathbb{T}d_3 y', \mathbb{T}0d_3 y', \mathbb{T}11 y', \mathbb{T}10 y' \lor 0\mathbb{T}y'$ or $0\mathbb{T}y'$, it follows $f(M(4)) \leq -\frac{1}{4}$. Here we define

$$d := -1, x' := 2x + 1, g := \lambda_n (2f(M(4) \lor n) + 1) \text{ and } N := \lambda_p M(p + 1).$$

The proof in this case is analogous to the proof of the first case.

It remains to consider the case that $f(M(4))$ has one of the forms $00d_3 y', \mathbb{T}11 y', \mathbb{T}10 y' \lor 0\mathbb{T}y'$, $0\mathbb{T}y'$, $0\mathbb{T}y'$ or $0\mathbb{T}y'$. Here we have $-\frac{1}{2} \leq f(M(4)) \leq \frac{1}{4}$ and we define

$$d := 0 \text{ and } x' := 2x.$$  

The formulas $^{co} I d$ and $x = \frac{d + x'}{2}$ are obvious. In order to prove

$$\exists_p \exists_d \exists x \left( \forall_n ^{co} I g(n) \land \forall_p \forall_{n \geq N(p)} |g(n) - x'| \leq 2^{-p} \right),$$

we define

$$g := \lambda_n 2f(M(4) \lor n) \text{ and } N := \lambda_p M(p + 1).$$

The second part of the conjunction follows from $\forall_p \forall_{n \geq M(p)} |f(n) - x| \leq 2^{-p}$, which is given. And because of

$$|f(M(4) \lor n)| \leq |f(M(4) \lor n) - x| + |x - f(M(4))| + |f(M(4))|$$

$$\leq \frac{1}{16} + \frac{1}{16} + \frac{1}{4} \leq \frac{1}{2}$$

for every $n$, we have $\forall_n ^{co} I g(n)$ by Lemma 4.

We denote the extracted term by $\text{Lim}$. It has the type

$$\text{Lim} : (\mathbb{Z}^+ \rightarrow \mathbb{N}) \rightarrow (\mathbb{N} \rightarrow \text{Str}) \rightarrow \text{Str}.$$  

It takes the modulus of convergence and the sequence of streams and returns the stream of the limit value. In order to give an at most readable characterisation of $\text{Lim}$, we define the following sets:

$$\mathbb{R} := \{11v, 10v, 1\mathbb{T}1v, 1\mathbb{T}0v, 011v, 010v \mid v \in \text{Str} \}$$

$$\mathbb{M} := \{00v, 1\mathbb{T}1v, 1\mathbb{T}0v, 0\mathbb{T}1v, 0\mathbb{T}0v \mid v \in \text{Str} \}$$

$$\mathbb{L} := \{\mathbb{T}v, \mathbb{T}0v, 1\mathbb{T}v, 1\mathbb{T}0v, 0\mathbb{T}v, 0\mathbb{T}0v \mid v \in \text{Str} \}$$

$$\mathbb{T}.$$
These three sets correspond to the three cases in the proof and therefore we have the following rule for \( \text{Lim} \):

\[
\text{Lim } M F :=
\begin{cases} 
1 (\text{Lim } \lambda_p M(p+1) \lambda_n (\text{DD} q^- F(M(4) \lor n))) & \text{if } F(M(4)) \in \mathbb{R} \\
0 (\text{Lim } \lambda_p M(p+1) \lambda_n (\text{D} F(M(4) \lor n))) & \text{if } F(M(4)) \in \mathbb{M} \\
T (\text{Lim } \lambda_p M(p+1) \lambda_n (\text{DD} q^+ F(M(4) \lor n))) & \text{if } F(M(4)) \in \mathbb{L}
\end{cases}
\]

Because of readability, we have omitted some brackets, for example \( \text{D} F(M(4) \lor n) \) shall be read as \( \text{D}(F(M(4) \lor n)) \). The functions \( \text{D}, q^+ \) and \( q^- \) are the computational content of the lemmas above and, as in the proof, \( \lor \) between natural numbers means their maximum.

One note that the definition of the new sequence is not unique. For reasons of efficiency one should be flexible with the choice of the new sequence, which is called \( g \) in the proof above. For example by choosing \( g \) one can replace \( M(4) \lor n \) by \( M(4) + n \). The efficiency depends on the concrete sequence. In the next section we define the Heron sequence and apply it to \( \text{Lim} \). In this case the definition with \( M(4) \lor n \) is most efficient.

Since the definition of this function is not so easy, we give an example for an implementation as a Haskell program. First the sets \( \mathbb{R} \) and \( \mathbb{L} \) are realised as Boolean functions:

```haskell
rAux :: Str -> Bool
rAux (SdR ::: SdR ::: v) = True
rAux (SdR ::: SdM ::: v) = True
rAux (SdR ::: SdL ::: SdR ::: v) = True
rAux (SdR ::: SdL ::: SdM ::: v) = True
rAux (SdM ::: SdR ::: SdR ::: v) = True
rAux (SdM ::: SdR ::: SdM ::: v) = True
rAux v = False

lAux :: Str -> Bool
lAux (SdL ::: SdL ::: v) = True
lAux (SdL ::: SdM ::: v) = True
lAux (SdL ::: SdR ::: SdL ::: v) = True
lAux (SdL ::: SdR ::: SdM ::: v) = True
lAux (SdM ::: SdL ::: SdL ::: v) = True
lAux (SdM ::: SdL ::: SdM ::: v) = True
lAux v = False
```

Then one can define \( \text{cCoILim} \) by case distinction:

```haskell
cCoILim :: (Int -> Int) -> (Int -> Str) -> Str
cCoILim m f
| rAux (f (m 4)) = SdR ::: (cCoILim n (funcR m f))
| lAux (f (m 4)) = SdL ::: (cCoILim n (funcL m f))
| otherwise = SdM ::: (cCoILim n (funcM m f))
where n = \p -> (m (p+1))
```

In this implementation the constructor \( \text{C} \) is denoted by \( ::: \) and written as an infix. The tree elements in \( \text{Sd} \) are given by \( \text{SdR}, \text{SdM} \) and \( \text{SdL} \) and shall be interpreted by 1, 0 and \( T \), respectively.
4 Square Root

Definition 3. We define $H : \mathbb{R} \rightarrow \mathbb{N} \rightarrow \mathbb{R}$ by the computation rules

$$H(x,0) := 1$$
$$H(x,n+1) := \frac{1}{2} \left( H(x,n) + \frac{x}{H(x,n)} \right).$$

For every non-negative $x$ the sequence $\lambda_n H(x,n) =: H(x) : \mathbb{N} \rightarrow \mathbb{R}$ is the sequence, we get from Heron’s method with start value 1.

Please note that $H$ is well-defined for non-negative $x$ since one can easily prove $H(x,n) \geq 2^{-n}$ by induction.

Lemma 6. For $x \in [0,1]$ we have that $H(x)$ converges to $\sqrt{x}$ with modulus $\iota : \mathbb{Z}^+ \rightarrow \mathbb{N}$ which is the inclusion from $\mathbb{Z}^+$ to $\mathbb{N}$. Furthermore $\forall_n \sqrt{x} \leq H(x,n)$ holds.

Proof. Let $x \in [0,1]$ be given. For each $n \in \mathbb{N}$ we define $\Delta(x,n) := H(x,n) - \sqrt{x}$.

A short calculation gives

$$\Delta(x,n+1) = \frac{1}{2} \left( \frac{H(x,n) + x}{H(x,n)} \right) - \sqrt{x} = \frac{(H(n,x))^2 - 2H(x,n)\sqrt{x} + x}{2H(x,n)}$$

$$= \frac{(\Delta(x,n))^2}{2H(x,n)}.$$

By induction on $n$ one easily gets $0 \leq H(x,n)$ and therefore $0 \leq \Delta(x,n+1)$. Because of $\Delta(x,0) = 1 - \sqrt{x} \geq 0$ we have $\forall_n \sqrt{x} \leq H(x,n)$.

Furthermore we calculate as follows:

$$\Delta(x,n+1) = \frac{(\Delta(x,n))^2}{2H(x,n)} = \frac{1}{2} \Delta(x,n) \Delta(x,n) = \frac{1}{2} \Delta(x,n) \left( 1 - \frac{\sqrt{x}}{H(x,n)} \right)$$

$$\leq \frac{1}{2} \Delta(x,n).$$

Therefore by induction we have $|H(x,n) - \sqrt{x}| = \Delta(x,n) \leq 2^{-n}$ and this implies $\forall_p \forall_n \geq p |H(x,n) - \sqrt{x}| \leq 2^{-p}$ i.e. $H(x)$ converges to $\sqrt{x}$ with modulus $\iota$. \hfill \Box

This lemma does not have any computational content, but it says that $\iota$ is a witness for the convergence of $H(x)$ to $\sqrt{x}$ from above. In some special cases we can even improve the modulus:

Lemma 7. If $x \in [\frac{1}{2},1]$ we can also choose $\poslog : \mathbb{Z}^+ \rightarrow \mathbb{N}$ as modulus of the converges from $H(x)$ to $\sqrt{x}$. For a positive integer $p$ we define $\poslog(p)$ as the least natural number $n$ with $p \leq 2^n$.

Proof. Let $x \in [\frac{1}{2},1]$ be given. From Lemma 6 we know $\forall_n \sqrt{x} \leq H(x,n)$ and therefore $\forall_n \frac{1}{2} \leq H(x,n)$. In the proof of Lemma 6 the formula

$$\Delta(x,n+1) = \frac{(\Delta(x,n))^2}{2H(x,n)}$$

12
is proven. In total it follows $\Delta(x, n + 1) \leq (\Delta(x, n))^2$. Because of $\frac{1}{2} \leq \sqrt{x}$ we get by induction

$$\Delta(x, n) \leq 2^{-2^n}$$

for each natural number $n$. Hence for given $p$ and $n \geq \operatorname{poslog}(p)$ we have $p \leq 2^n$ and therefore

$$|H(x, n) - \sqrt{x}| = \Delta(x, n) \leq 2^{-2^n} \leq 2^{-p}$$

One possibility, to implement the function $\operatorname{poslog}$, is by defining an auxiliary function $\operatorname{auxlog} : \mathbb{Z}^+ \to \mathbb{N} \to \mathbb{N}$ with the computation rules

$$\operatorname{auxlog} p n := \begin{cases} n & \text{if } p \leq 2^n \\ \operatorname{auxlog} p (n + 1) & \text{otherwise} \end{cases}$$

and then setting $\operatorname{poslog}(p) := \operatorname{auxlog} p 0$. We do not use this lemma and the function $\operatorname{poslog}$ in the main theorem but we use it for concrete calculations to reduce their duration.

**Lemma 8.** For all $x \in ^{co}I$ with $\frac{1}{16} \leq x$ we have $\forall^{co}I(H(x, n))$. Expressed as a formula this means

$$\forall x.^{co}I x \rightarrow \frac{1}{16} \leq x \rightarrow \forall^{co}I(H(x, n)).$$

**Proof.** We use the results of [11] and of Section 3.3 from [13]. In both scripts there are the following statements proven:

$$\forall_{x, y}.^{co}I x \rightarrow ^{co}I y \rightarrow ^{co}I \frac{x + y}{2} \quad (1)$$

$$\forall_{x, y}.^{co}I x \rightarrow ^{co}I y \rightarrow |x| \leq y \rightarrow \frac{1}{4} \leq y \rightarrow ^{co}I \frac{x}{y} \quad (2)$$

With those formulas the proof of this lemma is done by induction on $n$:

For $n = 0$ it is easy since $H(x, 0) = 1$.

For arbitrary $n$ we have $H(x, n + 1) = \frac{1}{2} \left( H(x, n) + \frac{x}{\operatorname{poslog}(x, n)} \right)$. By Lemma 6 we have $\sqrt{x} \leq H(n, x)$ and therefore $\sqrt{\frac{1}{16}} = \frac{1}{4} \leq H(n, x)$ and $x \leq \sqrt{x} \leq H(x, n)$. Additionally the induction hypothesis claims $^{co}I(H(x, n))$. With (2) we have $^{co}I \frac{x}{\operatorname{poslog}(x, n)}$ and with (1) we get $^{co}I(H(x, n + 1))$.

With $^{co}I\text{Av}$ and $^{co}I\text{Div}$ we denote the computational content of (1) and (2). Each of these terms takes two streams of reals and returns a stream of their average and their quotient, respectively.

As we have used induction over $n$ to prove the lemma, the extracted term $\text{Heron}$ is defined by recursion:

$$\text{Heron} v 0 := [1, 1, \ldots]$$

$$\text{Heron} v (n + 1) := ^{co}I\text{Av}(\text{Heron} v n)(^{co}I\text{Div} v (\text{Heron} v n))$$

This is actually Definition 3 in the notation of streams.
Theorem 2.

\( \forall^\text{nc} \cdot \co I x \rightarrow 0 \leq x \rightarrow \co I \sqrt{x} \)

Proof. We apply the introduction axiom of \( \co I^+ \) to the predicate

\[ P x := 3 y (\co I y \land 0 \leq y \land \sqrt{y} = x) : \]

\[ \forall^\text{nc} \cdot \exists y (\co I y \land 0 \leq y \land \sqrt{y} = x) \rightarrow P x \rightarrow \exists_{d, x'} (\text{Sd} d \land (\co I \cup P) x' \land |x| \leq 1 \land x = \frac{d + x'}{2}) \rightarrow \co I x \]

In order to show the goal formula, we show the second premise:

\[ \forall^\text{nc} \cdot \exists y (\co I y \land 0 \leq y \land \sqrt{y} = x) \rightarrow \exists_{d, x'} (\text{Sd} d \land (\co I \cup P) x' \land |x| \leq 1 \land x = \frac{d + x'}{2}) \]

Let \( x, y \) with \( \co I y \), \( 0 \leq x \) and \( \sqrt{y} = x \) be given. \( |x| \leq 1 \) follows directly from \( 0 \leq y \leq 1 \) and \( y = \sqrt{x} \). Form \( \co I y \) we get \( d_1, d_2, d_3 \in \text{Sd} \) and \( y' \in \co I \) with \( y = d_1 d_2 d_3 y' \). We do case distinction into three cases:

If \( y \) has one of the forms \( \text{Id}_2 d_3 y' \), \( 0 \text{Id}_1 y' \) or \( 00 y' \), it follows \( y \leq 0 \) and therefore \( x = \sqrt{y} = 0 \). Hence we define \( d := 0 \) and \( x' := 0 \) then \( \text{Sd} d, \co I x' \) and \( x = \frac{d + x'}{2} \)

are obvious.

If \( y \) has one of the forms \( 000 y' \), \( 001 y' \), \( 01 y' \) or \( 11 y' \), we can rewrite \( y = 00 e y' \) for an \( e \in \{0, 1\} \). Here we define \( d := 0 \) and \( x' := \sqrt{\frac{e + y'}{2}} \). Then \( \text{Sd} d \) and

\[ x = \sqrt{y} = \sqrt{\frac{e + y'}{2}} = \frac{\sqrt{\frac{e + y'}{2}}}{2} = \frac{d + x'}{2} \]

follow. Furthermore we have \( \co I \left( \frac{e + y'}{2} \right) \) from Lemma 2, we have \( 0 \leq e + y' \) since \( 0 \leq y = \frac{e + y'}{2} \) and we have \( \sqrt{\frac{e + y'}{2}} = x' \), therefore \( P x' \).

The remaining case is that \( y \) has one the forms \( 010 y' \), \( 011 y' \), \( 11 y' \), \( 110 y' \), \( 10 d_3 y' \) or \( 11 d_3 y' \). Here it follows \( \frac{1}{2} \leq y \). Therefore we have \( \forall_{n} \co I (H(y, n)) \) by Lemma 8. Furthermore from Lemma 6 we know that \( H(y) \) converges to \( \sqrt{y} \) with modulus \( \nu : \mathbb{Z}^+ \rightarrow \mathbb{N} \). Thus by Theorem 1 we have \( \co I x \). Applying \( x \) and \( \co I x \) to \( \co I^+ \)

leads to

\[ \exists_{d, x'} (\text{Sd} d \land \co I x' \land |x| \leq 1 \land x = \frac{d + x'}{2}) \]

which proves the goal formular in this case. \( \Box \)

With the definitions of \( \text{Lim} \) as the extracted term of Theorem 1 and \( \text{Heron} \) as the extracted term of Lemma 8 we have the following rules for the computational
content \( \texttt{sqrt} : \texttt{Str} \rightarrow \texttt{Str} \) of this theorem:

\[
\begin{align*}
\text{sqrt}(1u) & := [0,0,\ldots] \\
\text{sqrt}(01u) & := [0,0,\ldots] \\
\text{sqrt}(00u) & := 0 \text{ sqrt } u \\
\text{sqrt}(011u) & := 0 \text{ sqrt } 1u \\
\text{sqrt}(111u) & := 0 \text{ sqrt } 1u \\
\text{sqrt } u & := \text{Lim } \iota (\text{Heron } u)
\end{align*}
\]

The last rule shall only be applied if the other rules do not fit. This algorithm has an inefficient runtime, which comes from the recursive definition of \texttt{Heron}. In each step the function \texttt{cCoIDiv} is used twice and it has already a quadratic runtime. Therefore \texttt{Heron} has an at least exponential runtime and Haskell already takes quite a few minutes to compute even the first digit of \texttt{Heron} \([1,0,0,\ldots]\). 10. By using \texttt{poslog} from Lemma 7 instead of \(\iota\) we get a bit more digits of \(\sqrt{\frac{1}{2}}\). If we enter

\[
\text{cCoILim poslog (heron phalf)}
\]

in Haskell and wait about one minute, we get

\[
+1 +1 0 \ -1 +1 \ -1 +1 \ 0 0 0 0 +1 \ -1 +1 \ -1 0 \ 0 +1 +1 +1 +1 \ -1 +1 +1 +1 -1 +1 +1
\]

as output. These are the first 29 digits of \(\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}\). One can check that this result is indeed valid.

To improve the runtime it could be helpful to use another programming language and another data type but this is not the subject of this paper.

5 Outlook and Suggestions

It is possible to generalise the Heron sequence for roots of higher order. For a positive integer \(n\) and \(x \in [0,1]\) we define

\[
G \ n \ x \ 0 := 1 \\
G \ n \ x \ (k + 1) := \frac{1}{n} \left( (n - 1) (G \ n \ x \ k) + \frac{x}{(G \ n \ x \ k)^{n-1}} \right).
\]

This sequence comes from Newton’s method applied to the function \(y \mapsto y^n - x\) and one easily sees \(G \ 2 = H\). With this formula and the modulus from Newton’s method one could prove a general version of Theorem 2 like

\[
\forall x. \text{ coLe } \rightarrow 0 \leq x \rightarrow \text{coI } \sqrt{x}.
\]

A difference of this generalisation is that one has to take a look at the first \(n + 1\) digits of the radicand to compute the first digit of the root. Another problem which increase the duration for higher roots is, that in the definition of \(G\) one divides by \((G \ n \ x \ k)^{n-1}\). If \((G \ n \ x \ k)\) is small, \((G \ n \ x \ k)^{n-1}\) is even smaller for large \(n\) and the smaller the divisor is, the longer is the duration of the division.
An generalisation of this is given in the context of functions with can be defined as power series. The main examples here are the trigonometric functions \( \sin \) and \( \cos \). They are given by

\[
\cos(x) := \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}
\]

\[
\sin(x) := \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}
\]

From Chapter 8 Section 3 of \[9\] follows that for \( \mathcal{C}^\omega Lx \) i.e. \( x \in [-1, 1] \) one can choose \( \nu : \mathbb{Z}^+ \to \mathbb{N} \) as modulus of convergence for both sequences. To prove that every real in these sequences are in \( \mathcal{C}^\omega I \), one can use the formulas (1) and (2) and Lemma 4.

As an even larger generalisation one could ask the question: How can we use Theorem 1 to get an algorithm which takes a continuous function \( f \) from \([-1, 1]\) to \([-1, 1]\) and returns a computable function which takes a signed digit stream of a real \( x \) and returns a signed digit stream of \( f(x) \)? Constructive continuous functions are for example defined in \[9\].

Another direction in which one could extend this work is to replace the signed digit code by the Gray code. Similar to the signed digit code, the Gray code is also suitable to represent real numbers. In contrast to the signed digit code, the Gray code is unique and it has the property that a small change in the value of a real number effects only a small change in the Gray code of this real number. To implement Gray code, one needs simultaneously defined types and simultaneously coinductively defined predicates. In \[11\] there are the analogous statements of Lemma 3 and Lemma 4 and also analogous statements to the formulas (1) and (2) for Gray code proven. The main goal here would be to prove Theorem 1 in terms of Gray code.

Acknowledgements

At the end of this paper the author would like to thank Peter Schuster and Helmut Schwichtenberg. Peter Schuster has supported him in writing this paper. Helmut Schwichtenberg has introduced him to the signed digit code and has given some background information.

The author also thanks the Istituto Nazionale di Alta Matematica “F. Severi” (INdAM) for their financial support. The results presented in this paper were achieved during the authors Marie Skłodowska-Curie fellowship by this institute.

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