Chern characters and Hirzebruch-Riemann-Roch formula for matrix factorizations

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Abstract

We study the category of matrix factorizations for an isolated hypersurface singularity. We compute the canonical bilinear form on the Hochschild homology of this category. We find explicit expressions for the Chern character and the boundary-bulk maps and derive an analog of the Hirzebruch-Riemann-Roch formula for the Euler characteristic of the Hom-space between a pair of matrix factorizations. We also establish $G$-equivariant versions of these results.

Introduction

Let $w$ be an element of a commutative ring $R$. A matrix factorization of $w$ is a $\mathbb{Z}/2$-graded finitely generated projective $R$-module $E = E_0 \oplus E_1$ together with an odd endomorphism $\delta_E$ such that $\delta_E^2 = w \cdot \text{id}_E$. Matrix factorizations have been a classical tool in the study of hypersurface singularity algebras (see [8]). In the geometric context the category of matrix factorizations measures the failure of every coherent sheaf on the hypersurface $w = 0$ to have a finite locally free resolution (see [25]). Matrix factorizations also appear prominently in the work of Khovanov and Rozansky [17] on link homology. In the context of the present paper the most relevant interpretation is the one suggested Kontsevich: to view matrix factorizations as D-branes in topological Landau-Ginzburg models (see [12]).

This paper is motivated by the desire to understand the rich structure arising on the Hochschild homology of the category $\text{MF}(w)$ of matrix factorizations of an isolated hypersurface singularity $w = 0$, where $w(x_1, \ldots, x_n)$ is a formal power series. According to the ideology of the open-closed topological string theory, the Hochschild (co)homology of the category of D-branes gives the state space for the closed string sector (see [18],[14]). In the case of Landau-Ginzburg model associated with an isolated singularity this Hochschild homology has been computed (see (0.1) below). One of the goals of our paper is to derive explicit formulas for some of the natural structures on this space from the (dg-)categorical point of view. In fact, the Hochschild homology of the category $\text{MF}(w)$ (in the orbifold setting and under additional assumptions on $w$) has an even richer structure: it can be identified with the state space of a certain cohomological field theory (in the sense of [19]) constructed by Fan, Jarvis and Ruan in [9]. The results of the present paper will be used in the sequel to construct a purely algebraic version of the Fan-Jarvis-Ruan theory.
Note also that the category $\text{MF}(w)$ for an isolated singularity fits naturally into the framework of noncommutative geometry developed from the point of view of dg-categories or $A_\infty$-categories (see [15], [6]). As shown in [6] it provides an example of a smooth and proper noncommutative space (in the sense of [20]). The classical Hirzebruch-Riemann-Roch formula for coherent sheaves on smooth projective varieties was recently generalized by Shklyar [31] to such noncommutative spaces (see also [4] where similar ideas are developed in the classical case). He showed that the Hochschild homology $HH_*(\mathcal{C})$ of a smooth proper dg-category $\mathcal{C}$ is equipped with a canonical nondegenerate bilinear form $\langle \cdot, \cdot \rangle$. His categorical Hirzebruch-Riemann-Roch formula expresses the Euler characteristic of the Hom-spaces between two objects in the derived category of $\mathcal{C}$ in terms of Chern characters (aka Euler classes) taking values in $HH_* (\mathcal{C})$ and the form $\langle \cdot, \cdot \rangle$. In fact, there exists an even more general formula computing the traces of certain endomorphisms of the Hom-spaces between two objects (see [4, Thm. 16] and Theorem 1.3.1 below). In the case of a Calabi-Yau category $\mathcal{C}$ this generalized formula is equivalent to the Cardy condition for the corresponding open-closed 2d TQFT (see [4, Thm. 15]).

In concrete situations the difficulty shifts to calculating explicitly the Hochschild homology of the corresponding category along with the Chern character map and the canonical bilinear form. Our second motivation was to work out these ingredients in the case of the ($\mathbb{Z}/2$-graded) dg-category of matrix factorizations of an isolated hypersurface singularity, including the $G$-equivariant version. This leads to a concrete Hirzebruch-Riemann-Roch formula for matrix factorizations.

Before formulating our results let us recall that for an isolated singularity $w \in k[[x_1, \ldots, x_n]]$ the Hochschild homology of the ($\mathbb{Z}/2$-graded dg-)category $\text{MF}(w)$ of matrix factorizations of $w$ is known to be isomorphic to the Milnor ring of $w$ (up to a twist): $HH_*(\text{MF}(w)) \cong A_w \cdot dx[n]$, \hfill (0.1)

where $A_w = k[[x_1, \ldots, x_n]]/J_w$ with $J_w = (\partial_1 w, \ldots, \partial_n w)$ and $dx = dx_1 \wedge \ldots \wedge dx_n$ (see [6]). We will derive the following formula for the Chern character $\text{ch}(\bar{E}) \in HH_0(\text{MF}(w))$ of a matrix factorization $\bar{E} = (E, \delta_E)$:

$$\text{ch}(\bar{E}) = \text{str}_R (\partial_1 \delta_E \cdots \partial_n \delta_E) \cdot dx \mod J_w \cdot dx, \hfill (0.2)$$

where we view $\delta_E$ as a matrix with entries in $R = k[[x_1, \ldots, x_n]]$ (by choosing a basis of the free module $E$) and take partial derivatives $\partial_i = \partial/\partial x_i$ component-wise, and $\text{str}_R$ is the supertrace of a matrix with entries in $R$. More generally, there is a canonical “boundary-bulk” map

$$\tau^{\bar{E}} : \text{Hom}^*(E, \bar{E}) \rightarrow HH_*(\text{MF}(w))$$

such that $\text{ch}(\bar{E}) = \tau^{\bar{E}}(\text{id}_{\bar{E}})$ and the above formula generalizes to

$$\tau^{\bar{E}}(\alpha) = \text{str}_R (\partial_1 \delta_E \cdots \partial_n \delta_E \circ \alpha) \cdot dx \mod J_w \cdot dx \hfill (0.3)$$

where $\alpha$ is an endomorphism of $\bar{E}$.

We also show that the formula (0.2) leads to the identification of the canonical bilinear form on $HH_*(\text{MF}(w))$ with the form

$$\langle f \otimes dx, g \otimes dx \rangle = (-1)^{\ell(g)} \text{tr}(f \cdot g), \hfill (0.4)$$
where \( tr \) is the well known Frobenius trace on the Milnor ring given by the generalized residue:
\[
\text{tr}(f) = \text{Res} \left[ f(x) \cdot dx_1 \wedge \ldots \wedge dx_n \right]_{\partial_1 w, \ldots, \partial_n w}
\]
(cf. [11], [10, ch.V]). As a consequence we obtain an analog of the Hirzebruch-Riemann-Roch formula for the Euler characteristic of the \( \mathbb{Z}/2 \)-graded space \( \text{Hom}^*(\bar{E}, \bar{F}) \):
\[
\chi(\bar{E}, \bar{F}) = \langle \text{ch}(\bar{E}), \text{ch}(\bar{F}) \rangle.
\]
(0.5)

More generally, for \( \alpha \in \text{Hom}^*(\bar{E}, \bar{E}) \) and \( \beta \in \text{Hom}^*(\bar{F}, \bar{F}) \) we have
\[
\text{str}_k(m_{\alpha, \beta}) = (-1)^{|\alpha|} \cdot \langle \tau^E(\alpha), \tau^F(\beta) \rangle,
\]
(0.6)

where \( m_{\alpha, \beta} \) is the endomorphism of \( \text{Hom}^*(\bar{E}, \bar{F}) \) sending \( f \) to \( (-1)^{|\alpha|+|\beta|} \beta \circ f \circ \alpha \) (see Theorem 4.1.4). All terms in the right-hand sides of (0.5) and (0.6) can be explicitly expressed in terms of partial derivatives of \( \delta_E, \delta_F \) and \( w \) via (0.3) and (0.4). We will also establish \( G \)-equivariant versions of formulas (0.2), (0.3), (0.4), (0.5) and (0.6) (see Theorems 2.5.4 and 4.2.1).

In the case of a quasihomogeneous potential one can also consider \( \mathbb{Z} \)-graded versions of the categories of matrix factorizations (see e.g. [35], [26]). An analog of the Hirzebruch-Riemann-Roch formula for these categories follows from this formula for the category of \( G \)-equivariant matrix factorizations, where \( G \) is an appropriate cyclic group (see section 4.4).

Note that in some particular cases the Hirzebruch-Riemann-Roch formula for matrix factorizations was proved in [35]. Our formula (0.3) for the map \( \tau^E \) is almost identical to the formula for the boundary-bulk map in the corresponding Landau-Ginzburg model for open topological strings (see [13], [30]): we get an extra sign \((-1)^{n/2}\) (see Corollary 3.2.4). Similar expression also appears in the explicit version of the Serre duality for matrix factorizations worked out by Murfet in [24]. In the present paper Serre duality does not appear; this connection will be discussed elsewhere.

The paper is organized as follows. In section 1 we review some general formalism of Hochschild homology for dg-categories. We also present in 1.3 a generalized categorical version of the Hirzebruch-Riemann-Roch theorem (a version of the Cardy condition). In section 2 we collect useful (and mostly well known) results on matrix factorizations, including calculation of the Hochschild homology of the dg-category \( MF(w) \) by the method of Dyckerhoff [6] (and present a \( G \)-equivariant version of this calculation). Section 3 contains the main computation leading to the explicit formula for the Chern character. Finally, in section 4 this formula is used to calculate the canonical bilinear form on Hochschild homology (based on the results of [14]) and to derive a Hirzebruch-Riemann-Roch-formula. We also present in 4.3 the explicit calculation of the boundary-bulk map \( \tau^{kst} \) for the particular matrix factorization \( kst \) (the stabilization of the residue field).

Notations and conventions. We work over a fixed ground field \( k \) of characteristic zero. All the dg-categories considered in this paper are assumed to be \( k \)-linear. We denote the tensor
product of $k$-vector spaces simply by $V \otimes W$. For a $\mathbb{Z}/2$-graded vector space $V = V^0 \oplus V^1$ we denote by $J_V$ the grading operator that sends a homogeneous vector $v$ to $(-1)^{|v|} \cdot v$.

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1 Hochschild homology for smooth proper dg-categories

We work with Hochschild homology in the context of dg-categories following the framework developed by Toën in [32]. Nice surveys of the relevant facts can be found in the papers [16] and [33]. An extension of this techniques to the $\mathbb{Z}_2$-graded case is explained in [6]. For the rest of this section we will discuss only the usual dg-categories, leaving it for the reader to spell out the $\mathbb{Z}/2$-graded version (essentially, one has to replace systematically all $\mathbb{Z}$-graded complexes by $\mathbb{Z}/2$-graded ones).

1.1 Categorical trace and Hochschild homology

Recall that with a dg-category $\mathcal{C}$ one associates the derived category $D(\mathcal{C})$ by considering right $\mathcal{C}$-modules (i.e., dg-functors from $\mathcal{C}^{op}$ to the category of complexes over $k$) up to quasi-isomorphism. The perfect derived category $\text{Per}(\mathcal{C}) \subset D(\mathcal{C})$ is the full subcategory of compact objects in $D(\mathcal{C})$ (see [16]). Note that $\text{Per}(\mathcal{C})$ is a triangulated category, and when discussing morphisms we usually stay at the level of triangulated categories (i.e., we are not interested in their explicit cocycle representatives). However, one needs to work on the level of dg-categories to obtain the correct category of functors involved in the definition of Hochschild homology. There is also a dg-version of the derived category obtained by considering the localization in the homotopy category of dg-categories. We denote by $\text{Per}_{dg}(\mathcal{C})$ the corresponding perfect dg-subcategory of the dg-derived category.

From now on we will consider only dg-categories that are Morita equivalent to homologically smooth and proper dg-algebras (see [16, 4.7]). Such dg-categories can be characterized by the condition that $\text{Per}_{dg}(\mathcal{C})$ is saturated (see [34, sec. 2.2]). An important example of such a category is provided by the dg-derived category of coherent sheaves on a smooth projective variety. As we will see below in section 2.4, another example is given by the category of matrix factorizations of an isolated singularity.

For every dg-category $\mathcal{C}$ we denote by $\mathcal{C}^{op}$ the opposite dg-category with the same set of objects but with the composition $f \circ g$ replaced with $(-1)^{|f||g|} g \circ f$. For an object $E \in \mathcal{C}$ we denote the same object viewed as an object of $\mathcal{C}^{op}$ by $E^\vee$. Note that there is a natural equivalence

$$\text{Per}(\mathcal{C}^{op}) \simeq \text{Per}(\mathcal{C})^{op}$$

that corresponds to the standard duality of left and right perfect modules (cf. [31, (3.6)]). The assumption of saturatedness implies that every dg-functor $\text{Per}_{dg}(\mathcal{C}) \to \text{Per}_{dg} \mathcal{C}'$ is homotopy equivalent to the functor associated with a kernel in $\text{Per}_{dg}(\mathcal{C}^{op} \otimes \mathcal{C}')$ (see [34, sec.
2.2] and [33, sec. 5.4]). In particular, there is a “diagonal” object
\[ \Delta = \Delta_C \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C}) \]
representing the identity functor from \( \text{Per}_{dg}(\mathcal{C}) \) to itself.

The definition of Hochschild homology, convenient for us (see [33, sec. 5.2.3]), uses a natural functor
\[ \text{Tr}_C : \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C}) \to \text{Per}(k) : F^{\vee} \otimes E \mapsto \text{Hom}_{\mathcal{C}}(F, E). \tag{1.1} \]
The Hochschild homology of \( \mathcal{C} \) is defined as
\[ HH_*(\mathcal{C}) = \text{Tr}_C(\Delta). \]
It is not hard to check using this definition that Hochschild homology is invariant under Morita equivalences.

**Example 1.1.1.** If \( A \) is a smooth proper dg-algebra then the diagonal object \( \Delta \) corresponds to a perfect resolution of \( A \) viewed as a right module over \( A^e := A^{op} \otimes A \), and the trace functor \( \text{Per}(A^e) \to \text{Per}(k) \) is given by \( \otimes_{A^e}^L \cdot A \), so the above definition reduces to the standard \( HH_*(A) = A \otimes_{A^e}^L A \) (as proved in [6, sec. 5.3]).

**Remark 1.1.2.** When computing the action of \( \text{Tr}_C \) on morphisms some signs appear due to the standard sign convention. Namely, for a pair of morphisms \( e : E_1 \to E_2, f : F_2 \to F_1 \) the induced morphism
\[ \text{Tr}_C(F_1^{\vee} \otimes E_1) = \text{Hom}_{\mathcal{C}}(F_1, E_1) \to \text{Hom}_{\mathcal{C}}(F_2, E_2) = \text{Tr}_C(F_2^{\vee} \otimes E_2) \]
is given by
\[ x \mapsto (f^{\vee} \otimes e)(x) = (-1)^{|e||f|+|x||f|} \cdot e \circ x \circ f. \]

Note that under the natural equivalence \( \sigma : \mathcal{C} \otimes \mathcal{C}^{op} \cong \mathcal{C}^{op} \otimes \mathcal{C} \) we have isomorphisms
\[ \sigma(\Delta_{\mathcal{C}^{op}}) \cong \Delta_{\mathcal{C}}, \]
\[ \text{Tr}_C \circ \sigma \cong \text{Tr}_{\mathcal{C}^{op}}, \tag{1.2} \]
which induce an isomorphism
\[ \overset{\sigma}{\nu} : HH_*(\mathcal{C}^{op}) \cong HH_*(\mathcal{C}) \]

Another basic property of Hochschild homology is the K"unneth isomorphism
\[ HH_*(\mathcal{C} \otimes \mathcal{D}) \cong HH_*(\mathcal{C}) \otimes HH_*(\mathcal{D}) \tag{1.3} \]
Since we consider only dg-categories that are Morita equivalent to dg-algebras, this follows from the K"unneth isomorphism for Hochschild homology of dg-algebras proved in the same way as for associative algebras (see [31, sec. 2.4], [22, sec. 4.25]).

Below we will use the following property of the functors \( \text{Tr}_C \).
**Lemma 1.1.3.** For saturated dg-categories $\mathcal{C}, \mathcal{D}$ and objects $F \in \text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{D})$ and $G \in \text{Per}(\mathcal{D}^{\text{op}} \otimes \mathcal{C})$ there is a natural isomorphism
\[
\text{Tr}_{\mathcal{D}}(F \circ G) \simeq \text{Tr}_{\mathcal{C}}(G \circ F) \quad (1.4)
\]
in $\text{Per}(k)$.

**Proof.** Consider the tensor product $F \otimes G \in \text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{D} \otimes \mathcal{D}^{\text{op}} \otimes \mathcal{C}^{\text{op}})$. We have a natural functor
[\text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{D} \otimes \mathcal{D}^{\text{op}} \otimes \mathcal{C}^{\text{op}}) \to \text{Per}(k)]
induced by $\text{Tr}_{\mathcal{C}}$ and $\text{Tr}_{\mathcal{D}}$. The isomorphism (1.4) reflects two ways of evaluating this functor on $F \otimes G$: by either first applying $\text{Tr}_{\mathcal{C}}$ and then $\text{Tr}_{\mathcal{D}}$, or vice versa. \qed

### 1.2 Functoriality

The most important property of Hochschild homology is its functoriality. For a pair of dg-categories $\mathcal{C}, \mathcal{D}$ as above and a dg-functor $F : \text{Per}_{dg}(\mathcal{C}) \to \text{Per}_{dg}(\mathcal{D})$ we are going to define a natural map
\[
F_* : HH_*^{\mathcal{C}} \to HH_*^{\mathcal{D}}. \quad (1.5)
\]

By saturatedness, for every $D \in \mathcal{D}$ the functor
[\mathcal{C}^{\text{op}} \to \text{Per}_{dg}(k) : C \mapsto \text{Hom}_{\mathcal{D}}(F(C), D)]
is represented by an object $G(D) \in \text{Per}_{dg}(\mathcal{C})$. This gives a functor $G : \text{Per}_{dg}(\mathcal{D}) \to \text{Per}_{dg}(\mathcal{C})$, which is a right adjoint to $F$. Using saturatedness again, we can view the compositions $G \circ F$ and $F \circ G$ as objects in $\text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{C})$ and $\text{Per}(\mathcal{D}^{\text{op}} \otimes \mathcal{D})$, respectively. The map (1.5) is defined now as the composition
\[
\text{Tr}_{\mathcal{C}}(\Delta_{\mathcal{C}}) \to \text{Tr}_{\mathcal{C}}(G \circ F) \simeq \text{Tr}_{\mathcal{D}}(F \circ G) \to \text{Tr}_{\mathcal{D}}(\Delta_{\mathcal{D}})
\]
of maps induced by the canonical adjunction morphisms $\Delta_{\mathcal{C}} \to G \circ F$ and $F \circ G \to \Delta_{\mathcal{D}}$ and by the isomorphism of Lemma 1.1.3.

**Lemma 1.2.1.** For functors $F : \text{Per}_{dg}(\mathcal{C}) \to \text{Per}_{dg}(\mathcal{D})$ and $F' : \text{Per}_{dg}(\mathcal{B}) \to \text{Per}_{dg}(\mathcal{C})$ we have $(F \circ F')_* = F_* \circ F'_*$. **Proof.** Let $G : \text{Per}_{dg}(\mathcal{D}) \to \text{Per}_{dg}(\mathcal{C})$ and $G' : \text{Per}_{dg}(\mathcal{C}) \to \text{Per}_{dg}(\mathcal{B})$ be the corresponding right adjoint functors. The required equality follows from the commutative diagram

$$
\begin{array}{ccc}
\text{Tr}_{\mathcal{B}}(G'F') & \xrightarrow{\simeq} & \text{Tr}_{\mathcal{C}}(G'G'F') \\
\text{Tr}_{\mathcal{C}}(F'G') & \xrightarrow{\simeq} & \text{Tr}_{\mathcal{D}}(F'G'G'F') \\
\text{Tr}_{\mathcal{B}}(\Delta) & \xrightarrow{\simeq} & \text{Tr}_{\mathcal{C}}(G'G'F') \\
\text{Tr}_{\mathcal{D}}(F \circ G) & \xrightarrow{\simeq} & \text{Tr}_{\mathcal{D}}(F'G'G')
\end{array}
$$
since the composition of the upper sequence of 6 diagonal arrows is equal to $F \circ F'$, while the composition of the lower 4 arrows is $(F \circ F')_*$. 

**Remark 1.2.2.** There are two other ways to define functoriality for Hochschild homology: one using explicit complexes (see e.g. [31, sec. 2.3]) and another using Serre functors (as in [4]). It is possible to connect our construction to both—this will be discussed elsewhere.

**Definition 1.2.3.** The Chern character (aka Euler class) of an object $E \in \text{Per}(\mathcal{C})$ is the element

$$\text{ch}(E) = (1_E)_*(1) \in HH_0(\mathcal{C}),$$

where $1_E : \text{Per}_{dg}(k) \to \text{Per}_{dg}(\mathcal{C})$ sends $k$ to $E$.

Combining the functoriality for the (dg-version of the) functor $\text{Tr}_e : \mathcal{C}^{\text{op}} \otimes \mathcal{C} \to \text{Per}(k)$ with Künneth isomorphism (1.3) one gets a canonical pairing

$$\langle \cdot, \cdot \rangle : HH_*(\mathcal{C}^{\text{op}}) \otimes HH_*(\mathcal{C}) \to k$$

(cf. [31, sec. 1.2]). This leads in a straightforward way to an analog of the Hirzebruch-Riemann-Roch formula for the Euler characteristics of the Hom-spaces:

$$\chi(\text{Hom}_e(E, F)) = \langle \text{ch}(E^\vee), \text{ch}(F) \rangle$$

(see [31, (1.2)]).

The pairing (1.7) is nondegenerate under our assumptions on $\mathcal{C}$. The proof of this fact given in [31, Thm. 6.2] uses the following connection of the canonical pairing $\langle \cdot, \cdot \rangle \mathcal{C}$ with the diagonal object $\Delta \in \text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{C})$. Consider the Chern character $\text{ch}(\Delta) \in HH_*(\mathcal{C}^{\text{op}}) \otimes HH_*(\mathcal{C})$. Then one can check that

$$\langle \cdot, \cdot \rangle_{\mathcal{C}} \otimes \text{id}(x \otimes \text{ch}(\Delta)) = x$$

for all $x \in HH_*(\mathcal{C})$, where $\langle h, h' \rangle_{\mathcal{C}} = \langle h', h \rangle_{\mathcal{C}}$ by (1.2).

We will use the following way of computing ch. Let us observe that for every $E \in \text{Per}(\mathcal{C})$ there is a natural map

$$c_E : E^\vee \otimes E \to \Delta_e$$

in $\text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{C})$ that corresponds to the canonical map of functors $\text{Hom}_e(E, ?) \otimes E \to \text{Id}_e$.

**Lemma 1.2.4.** Consider the map

$$\tau^E : \text{Hom}_e(E, E) \simeq \text{Tr}_e(E^\vee \otimes E) \xrightarrow{\text{Tr}_e(c_E)} \text{Tr}_e(\Delta) = HH_*(\mathcal{C}).$$

Then $\text{ch}(E) = \tau^E(\text{id}_E)$.

**Proof.** This is obtained by unraveling the definitions, since the functor $\text{Hom}_e(E, ?)$ is right adjoint to $1_E : \text{Per}_{dg}(k) \to \text{Per}_{dg}(\mathcal{C})$. 

\end{proof}
1.3 Generalized abstract Hirzebruch-Riemann-Roch Theorem

It is known that a Calabi-Yau dg-category gives rise to an open-closed 2d TQFT (see [5]). One of the equations of the open-closed TQFT, the so called Cardy condition, can be viewed as a generalization of the Hirzebruch-Riemann-Roch formula. It was observed in [4, Thm. 15] that the Cardy condition can be stated without assuming the Calabi-Yau property (in [4] this condition is called the ”Baggy Cardy Condition”). Here we prove a categorical version of the Cardy condition for arbitrary dg-category \( C \) such that \( \text{Per}_{dg}(C) \) is saturated.

**Theorem 1.3.1.** For a pair of objects \( A, B \in \mathcal{C} \) and elements \( \alpha \in \text{Hom}_{\mathcal{C}}(A, A), \beta \in \text{Hom}_{\mathcal{C}}(B, B) \) we have

\[
\langle \tau^A(\alpha^\vee), \tau^B(\beta) \rangle = \text{str}(m_{\alpha,\beta}),
\]

where \( \alpha^\vee \in \text{Hom}_{\mathcal{C}^\vee}(A^\vee, A^\vee) \) is induced by \( \alpha \), and \( m_{\alpha,\beta} \) is the endomorphism

\[
m_{\alpha,\beta} : \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{C}}(A, B) : f \mapsto (-1)^{|\alpha| |\beta| + |\alpha| |f|} \cdot \beta \circ f \circ \alpha.
\]

Note that the Hirzebruch-Riemann-Roch formula (1.8) is obtained by taking \( \alpha = \text{id}_A \) and \( \beta = \text{id}_B \). The proof of (1.12) will be based on the following compatibility of the maps \( \tau^A \) with the functoriality of Hochschild homology.

**Lemma 1.3.2.** Let \( F : \text{Per}_{dg}(\mathcal{C}) \to \text{Per}_{dg}(\mathcal{D}) \) be a dg-functor. Then for an object \( A \in \text{Per}(\mathcal{C}) \) the following diagram is commutative

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(A, A) & \xrightarrow{F^\ast} & \text{Hom}_{\mathcal{D}}(F(A), F(A)) \\
\tau^A & & \tau^{F(A)} \\
\text{HH}_*(\mathcal{C}) & \xrightarrow{F^\ast} & \text{HH}_*(\mathcal{D})
\end{array}
\]

**Proof.** Recall that the definition of \( F^\ast \) uses the natural isomorphism \( \text{Tr}_\mathcal{C}(G \circ F) \simeq \text{Tr}_\mathcal{D}(F \circ G) \) (see Lemma 1.1.3), where \( G : \text{Per}_{dg}(\mathcal{D}) \to \text{Per}_{dg}(\mathcal{C}) \) is the right adjoint functor to \( F \). Let \( F_A : \text{Per}_{dg}(\mathcal{C}) \to \text{Per}_{dg}(\mathcal{D}) \) denote the dg-functor

\[
F_A(?) = \text{Hom}_{\mathcal{C}}(A, ?) \otimes F(A).
\]

We have a natural morphism of dg-functors \( \phi : F_A \to F \) induced by the action of \( F \) on morphisms

\[
F : \text{Hom}_{\mathcal{C}}(A, ?) \to \text{Hom}_{\mathcal{D}}(F(A), F(?)).
\]

Hence, there is a commutative diagram

\[
\begin{array}{ccc}
\text{Tr}_\mathcal{C}(G \circ F_A) & \simeq & \text{Tr}_\mathcal{D}(F_A \circ G) \\
\text{Tr}_\mathcal{C}(\text{id}_G \circ \phi) & & \text{Tr}_\mathcal{D}(\phi \circ \text{id}_G) \\
\text{Tr}_\mathcal{C}(G \circ F) & \simeq & \text{Tr}_\mathcal{D}(F \circ G)
\end{array}
\]
It is easy to see that we have the following commutative diagrams in \( \text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}) \) and \( \text{Per}(\mathcal{D}^{\text{op}} \otimes \mathcal{D}) \), respectively:

\[
\begin{array}{cc}
A^\vee \otimes A & \xrightarrow{\psi} G \circ F_A \\
\downarrow c_A & \downarrow \text{id}_G \circ \phi \\
\Delta_c & G \circ F \\
\end{array}
\]

\[
\begin{array}{cc}
F_A \circ G & \xrightarrow{\eta} F(A)^\vee \otimes F(A) \\
\downarrow \phi \circ \text{id}_G & \downarrow c_{F(A)} \\
F \circ G & \xrightarrow{\Delta_D} \\
\end{array}
\]

where \( \psi \) and \( \eta \) are induced by the adjunction:

\[
\psi : \text{Hom}_\mathcal{E}(A,?) \otimes A \rightarrow \text{Hom}_\mathcal{E}(A,?) \otimes GF(A) \simeq G \circ F_A(?),
\]

\[
\eta : F_A \circ G(?) \simeq \text{Hom}_\mathcal{E}(A,G(?)) \otimes F(A) \rightarrow \text{Hom}_\mathcal{D}(F(A),?) \otimes F(A).
\]

Now we apply \( \text{Tr}_\mathcal{E} \) and \( \text{Tr}_\mathcal{D} \) to these two diagrams and use (1.14) to get the result. \( \square \)

Proof of Theorem 1.3.1. We apply Lemma 1.3.2 to the functor

\[
\text{Tr}_\mathcal{E} : \text{Per}_{dg}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}) \rightarrow \text{Per}_{dg}(k),
\]

the object \( A^\vee \otimes B \in \mathcal{C}^{\text{op}} \otimes \mathcal{C} \), and an element \( \alpha^\vee \otimes \beta \in \text{Hom}_{\mathcal{C}^{\text{op}} \otimes \mathcal{C}}(A^\vee \otimes B, A^\vee \otimes B) \). We have

\[
\text{Tr}_\mathcal{E}(A^\vee \otimes B) = \text{Hom}_\mathcal{E}(A, B)
\]

and

\[
\text{Tr}_\mathcal{E}(\alpha^\vee \otimes \beta) = m_{\alpha,\beta}
\]

(the sign in (1.13) appears from the definition of \( \text{Tr} \), see Remark 1.1.2). It remains to use the fact that the maps \( \tau^A \) are compatible with the Künneth isomorphism:

\[
\tau^{A^\vee \otimes B} = \tau^{A^\vee} \otimes \tau^B.
\]

\( \square \)

2 Matrix factorizations

In this section we collect some facts about categories of matrix factorizations (see e.g. [6] and KhR for more information).
2.1 Categories of matrix factorizations

Let $R$ be a commutative algebra over a field $k$. We fix an element $w \in R$ which will be called the potential.

**Definition 2.1.1.** A matrix factorization of the potential $w$ over $R$ is a pair $(E, \delta_E) = (E^0 \xrightarrow{\delta_0} E^1)$, (2.1)

where

- $E = E^0 \oplus E^1$ is a $\mathbb{Z}/2$-graded finitely generated projective $R$-module, and
- $\delta_E \in \text{End}^1_R(E)$ is an odd (i.e. of degree $1 \in \mathbb{Z}/2$) endomorphism of $E$, such that $\delta_E^2 = w \cdot \text{id}_E$.

Even though $\delta_E^2 \neq 0$, we will still call $\delta_E$ a “differential”.

If $E^0$ and $E^1$ are free $R$-modules with chosen bases, the differential $\delta_E$ can be represented by a block matrix

$$D = \begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix},$$

(2.2)

such that the matrices $D^0$ and $D^1$ give a factorization of the potential:

$$D^0D^1 = D^1D^0 = w \cdot I.$$

To a potential $w \in R$ we associate a $\mathbb{Z}/2$-dg-category $\text{MF}(w) = \text{MF}(R, w)$ whose objects are matrix factorizations of $w$ over $R$. The morphisms from $\bar{E} = (E, \delta_E)$ to $\bar{F} = (F, \delta_F)$ are elements of the $\mathbb{Z}/2$-graded module of $R$-linear homomorphisms

$$\text{Hom}_w(\bar{E}, \bar{F}) := \text{Hom}_{\text{Mod}_R}(E, F) = \text{Hom}_{\mathbb{Z}/2-\text{Mod}_R}(E, F) \oplus \text{Hom}_{\mathbb{Z}/2-\text{Mod}_R}(E, F[1]).$$

The $\mathbb{Z}/2$-graded dg-structure on $\text{MF}(R, w)$ is given by the differential $d$ defined on $f \in \text{Hom}_w(\bar{E}, \bar{F})$ as

$$df = \delta_F \circ f - (-1)^{|f|} f \circ \delta_E.$$

(2.3)

For $\bar{E}, \bar{F} \in \text{MF}(w)$ we set

$$\text{Hom}_w(\bar{E}, \bar{F}) = H^*(\text{Hom}_w(\bar{E}, \bar{F}), d).$$

Let

$$\text{HMF}(R, w) = H^0\text{MF}(R, w)$$

denote the homotopy category associated with the dg-category $\text{MF}(R, w)$. By definition, morphisms in this category are chain maps up to homotopy, i.e. the spaces $\text{Hom}_w^0(\bar{E}, \bar{F})$. The homotopy category $\text{HMF}(R, w)$ of matrix factorizations is naturally triangulated (see
e.g. [25]) with the shift functor induced from the functor $T : \text{MF}(R, w) \to \text{MF}(R, w)$ given by

$$T(E, \delta_E) = (E[1], -\delta_E), \quad T(f) = f[1], \text{ for } f \in \mathcal{H}om_w(\bar{E}, \bar{F}).$$

The category $\text{HMF}(R, w)$ can be viewed as a full triangulated subcategory of the perfect derived category of matrix factorizations $\text{Per}(\text{MF}(R, w))$ and is isomorphic to it when $R$ is regular and complete by [6, Thm. 4.6] (see [27] for a more general statement).

For a pair of elements $a, b \in R$ denote by $\{a, b\}$ the matrix factorization of the potential $ab$ given by

$$\{a, b\} = (R \xrightarrow{a} R) \in \text{MF}(R, ab).$$

(2.4)

An easy computation shows that

$$\text{Hom}_0^w(\{a, b\}, \{a, b\}) \simeq R/(Ra + Rb),$$

$$\text{Hom}_1^w(\{a, b\}, \{a, b\}) \simeq \{(x, y) \in R \oplus R \mid ax = by\}/R \cdot (b, a),$$

(2.5)

provided $w$ is not a zero divisor in $R$. In particular, the identity map of $\{a, b\}$ is homotopic to zero (and so $\{a, b\}$ represents the zero object in $\text{HMF}(R, ab)$) if and only if the ideal $(a, b)$ coincides with $R$. Also, if $R$ is a unique factorization domain and $\gcd(a, b) = 1$ then $\text{Hom}_1^w(\{a, b\}, \{a, b\}) = 0$.

If $G$ is a finite group of automorphisms of $R$ which fixes the potential $w$, one defines the $G$-equivariant $\mathbb{Z}/2$-graded dg-category of matrix factorizations $\text{MF}_G(w)$ (and the corresponding homotopy category $\text{HMF}_G(w)$), by requiring that all modules and morphisms should be $G$-equivariant (see e.g., [28]). In other words, in (2.1) $E$ should be a $\mathbb{Z}/2$-graded finitely generated projective $R$-module equipped with a compatible $G$-action, and $\delta_E$ has to be $G$-equivariant. Morphisms between $G$-equivariant matrix factorizations $\bar{E}$ and $\bar{F}$ should also be compatible with the action of $G$, so

$$\mathcal{H}om_{\text{MF}_G(w)}(\bar{E}, \bar{F}) = \mathcal{H}om_w(\bar{E}, \bar{F})^G.$$  

(2.6)

### 2.2 Tensor product and duality

The tensor product $\bar{E} \otimes_R \bar{E'}$ of two matrix factorizations $\bar{E} = (E, \delta_E) \in \text{MF}(R, w)$ and $\bar{E'} = (E', \delta_{E'}) \in \text{MF}(R, w')$ is defined as the pair

$$(E \otimes_R E', \delta_E \otimes \text{id}_{E'} + J_E \otimes \delta_{E'}),$$

(2.7)

where $J_E = (-1)^{|E|}$ is the grading operator. It is straightforward to check that $(E, \delta_E) \otimes (E', \delta_{E'})$ is a matrix factorization of the potential $w + w'$.

Note that we have a natural commutativity isomorphism in $\text{MF}(w + w')$:

$$\bar{E} \otimes_R \bar{E'} \simeq \bar{E'} \otimes_R \bar{E} : e \otimes e' \mapsto (-1)^{|e||e'|} e' \otimes e.$$  

(2.8)

The following definition was introduced in [3] (see also [17]).
Definition 2.2.1. Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) be two \( n \)-tuples of elements of \( R \). The matrix factorization
\[
\{a, b\} := \{a_1, b_1\} \otimes_R \cdots \otimes_R \{a_n, b_n\}
\] (2.9)
of the potential
\[
w = a \cdot b = a_1 b_1 + \cdots + a_n b_n
\]
is called the Koszul factorization corresponding to the pair \((a, b)\).

More explicitly, \(\{a, b\}\) is isomorphic as a \(\mathbb{Z}/2\)-graded \(R\)-module to the Koszul complex
\[
K_\bullet = \left( \bigwedge^\bullet_R (R^n), \delta \right),
\] (2.10)
where the differential is given by
\[
\delta = (\sum_{j=1}^n a_j e_j) \wedge ? + \iota(\sum_{j=1}^n b_j e_j^*).
\]
Here \((e_j)\) is the standard basis of \(R^n\), \((e_j^*)\) is the dual basis of the dual \(R\)-module, \(\iota\) denotes the contraction operator. The \(\mathbb{Z}/2\)-grading is induced by the \(\mathbb{Z}\)-grading of \(K_\bullet\).

We also need a \(G\)-equivariant version of Koszul factorizations. Suppose \(G\) is a finite group acting on \(R\) and \(w \in R\) is \(G\)-invariant. Assume we have a finite-dimensional \(G\)-module \(V\) and a pair of \(G\)-invariant elements \(\phi \in V \otimes R\) and \(\psi \in V^* \otimes R\) such that
\[
\langle \psi, \phi \rangle = w.
\]
Then we get a structure of a \(G\)-equivariant matrix factorization of \(w\) on the corresponding Koszul complex
\[
K_{\bullet}(V) = \bigwedge^\bullet_R (V \otimes R),
\] (2.11)
where the differential is given by
\[
\delta = \phi \wedge ? + \iota(\psi).
\]
We denote this \(G\)-equivariant matrix factorization by \(\{\phi, \psi\}\). Choosing a basis \((e_i)\) in \(V\) we can write \(\phi = \sum_i e_i \otimes a_i\), \(\psi = \sum_i e_i^* \otimes b_i\), where \((e_i^*)\) is the dual basis in \(V^\ast\). Then after forgetting a \(G\)-action we obtain
\[
\{\phi, \psi\} = \{a, b\}.
\]

For a pair of potentials \(w \in R\) and \(w' \in R'\) we have a natural external tensor product functor
\[
\text{MF}(R, w) \otimes \text{MF}(R', w') \to \text{MF}(R \otimes R', w \oplus w'),
\] (2.12)
where \(w \oplus w' := w \otimes 1_{R'} + 1_R \otimes w' \in R \otimes R'\), sending a pair \(\vec{E} \in \text{MF}(R, w)\), \(\vec{E}' \in \text{MF}(R, w')\) to
\[
\vec{E} \otimes \vec{E}' = p_1^* E \otimes p_2^* E',
\]
where \(p_1^*\) and \(p_2^*\) are given by the extension of scalars via \(R \to R \otimes R'\) and \(R' \to R \otimes R'\), respectively.
On the other hand, we can use objects of \(MF(R \otimes R', (-w) \oplus w')\) as kernels of dg-functors from \(MF(R, w)\) to \(MF(R', w')\) (cf. [17, sec. 4]). For this we need to work in the larger categories \(MF^\infty(R, w)\) of matrix factorizations of not necessarily finitely generated \(R\)-modules. Then a kernel \(K \in MF^\infty(R \otimes R', (-w) \oplus w')\) represents the dg-functor

\[
\Phi_K : MF^\infty(R, w) \to MF^\infty(R', w') : \overline{E} \mapsto p_{2*}(p_1^* \overline{E} \otimes_R K),
\]

where \(p_{2*}\) is the restriction of scalars via \(R' \to R \otimes R'\). In the case when \(R\) and \(R'\) are complete, there is a version of this construction involving the completed tensor product \(\hat{R} \hat{\otimes}_k R\).

To a matrix factorization \(\overline{E} = (E^0 \xrightarrow{\delta_0} E^1)\) of the potential \(w \in R\) we associate the following dual matrix factorization of \(-w\)

\[
E^* = ((E^0)^* \xrightarrow{-\delta_0^*} (E^1)^*) \in MF(R, -w), \tag{2.13}
\]

where for a projective \(R\)-module \(P\) we set \(P^* = Hom_R(P, R)\). In other words, for \(e^* \in E^*\) and \(e \in E\) we have

\[
\langle \delta_{E^*}(e^*), e \rangle = (-1)^{|e^*|} \langle e^*, \delta_E(e) \rangle
\]

which is the usual sign rule for defining the adjoint operator in the \(\mathbb{Z}/2\)-graded context (since \(|\delta_E| = 1\)). The functor \(\overline{E} \mapsto \overline{E}^*\) gives an equivalence

\[
MF(R, w)^{op} \simeq MF(R, -w). \tag{2.14}
\]

For any \(E, F \in MF(R, w)\) the tensor product \(F \otimes_R E^*\) is a \(\mathbb{Z}/2\)-graded complex (since it is a matrix factorization of the potential \(w + (-w) = 0\)) and we have a natural isomorphism of complexes of \(R\)-modules

\[
F \otimes_R E^* \simeq Hom_w(E, F) \tag{2.15}
\]

(note that our choice of sign in the definition (2.13) is compatible with this isomorphism).

For \(\overline{E} \in MF(R, w), \overline{F} \in MF(R, w')\) we have an isomorphism

\[
(\overline{E} \otimes F)^* \simeq \overline{F}^* \otimes \overline{E}^*
\]

of matrix factorizations of \(-w - w'\), given by the natural pairing between the \(R\)-modules \(F^* \otimes_R \overline{E}^*\) and \(E \otimes_R F\):

\[
\langle f^* \otimes e^*, e \otimes f \rangle = e^*(e) \cdot f^*(f),
\]

where \(e \in E, f \in F, e^* \in E^*, f^* \in F^*\). Note that the double dual is

\[
(\overline{E}^*)^* = \overline{E} := (E, -\delta_E),
\]

so the natural isomorphism \(\overline{E} \rightarrow (\overline{E}^*)^*\) is given by the grading operator \(J_E\). This is compatible with the sign convention

\[
\langle e, e^* \rangle = (-1)^{|e|} \langle e^*, e \rangle
\]
for \( e \in E, e^* \in E^* \) (the pairing is nonzero only if \(|e| = |e^*|\)).

Thus, we see that for \( \bar{E}, \bar{F} \in \text{MF}(R, w) \) there is a natural pairing

\[
\text{Hom}_w(E, F) \otimes \text{Hom}_w(F, E) \to R : A \otimes B \mapsto \text{str}_R(A \circ B)
\]

inducing the perfect duality between these complexes, where \( \text{str}_R \) denotes the supertrace of an \( R \)-linear operator \( C \) (it is equal to \( \text{tr}_R(C_{00}) - \text{tr}_R(C_{11}) \), where \( C_{ii} : E_i \to E_i \) are the components of \( C \)). Indeed, if we use the identifications \( \text{Hom}_w(\bar{E}, \bar{F}) \simeq \bar{F} \otimes_R \bar{E}^* \) and \( \text{Hom}_w(\bar{F}, \bar{E}) \simeq \bar{E} \otimes_R \bar{F}^* \) then the above pairing corresponds to the standard pairing

\[
\langle f \otimes e^*, e \otimes f^* \rangle = (-1)^{|f|} \langle e^*, e \rangle \cdot \langle f^*, f \rangle.
\]

In the case when \( R \) is a regular local \( k \)-algebra of dimension \( n \), and \( w \) is an isolated singularity (see sec. 2.4 below), by Grothendieck duality, the above duality induces an isomorphism

\[
\text{Hom}_w(\bar{E}, \bar{F})^* \simeq \text{Hom}_w(\bar{F}, \bar{E}) \otimes_R \omega_{R/k}[n]
\]

which means that \( \bar{E} \mapsto \bar{E} \otimes_R \omega_{R/k}[n] \) is a Serre functor on \( \text{HMF}(R, w) \) (see [1], [2]). An explicit formula for the duality trace maps is proved in [24].

### 2.3 Matrix factorizations and modules over hypersurface singularities

From now on we assume that \( R \) is a regular local \( k \)-algebra with the maximal ideal \( m \) and the residue field \( R/m = k \). Given a minimal set of generators \( x_1, \ldots, x_n \) of \( m \), we denote the corresponding derivations of \( R \) by \( \partial_i \). We will be mostly interested in the case when \( R \) is the ring of formal power series \( k[[x_1, \ldots, x_n]] \).

Matrix factorizations of a potential \( w \in m \subset R \) naturally arise in the study of maximal Cohen-Macaulay modules over the hypersurface algebra \( S = R/\mathfrak{w} \). (A maximal Cohen-Macaulay module over a commutative Noetherian local ring is a module whose depth is equal to the Krull dimension of the ring.) If

\[
(E, \delta_E) = (E^0 \xrightarrow{\delta_0} E^1)
\]

is a matrix factorization of \( \mathfrak{w} \in R \) then, since \( \mathfrak{w} \cdot E \subset \text{im}(\delta_E) \), the \( R \)-module \( \text{coker}(\delta_1) \) is naturally an \( S \)-module. Eisenbud showed in [8] that \( \text{coker}(\delta_1) \) is a maximal Cohen-Macaulay module and that any maximal Cohen-Macaulay module over \( S \) can be obtained this way. Moreover, he proved that the functor \( \text{Coker} : \text{MF}(R, \mathfrak{w}) \to \text{Mod}_S \) induces an equivalence of categories

\[
\text{HMF}(R, \mathfrak{w}) \to \text{MCM}(S),
\]

where \( \text{MCM}(S) \) is the stable category of maximal Cohen-Macaulay \( S \)-modules (the quotient of the full subcategory of maximal Cohen-Macaulay \( S \)-modules \( \text{MCM}(S) \subset \text{Mod}_S \) modulo free \( S \)-modules).
Buchweis [2] extended this result to a characterization of the stabilized derived category of $S$,
\[
\mathcal{D}^b_{\text{per}}(S) = \mathcal{D}^b(S)/\mathcal{D}^b_{\text{per}}(S),
\]
where $\mathcal{D}^b(S)$ is the bounded derived category of all complexes of $S$-modules with finitely generated cohomology and $\mathcal{D}^b_{\text{per}}(S)$ is the full triangulated subcategory of $\mathcal{D}^b(S)$ of perfect complexes (i.e. complexes quasi-isomorphic to a bounded complex of free $S$-modules). He proved that the natural functor $\text{MCM}(S) \to \mathcal{D}^b(S)$ is an equivalence of categories which, combined with (2.17), induces an equivalence of triangulated categories
\[
\text{HMF}(R, w) \to \mathcal{D}^b(S) \tag{2.18}
\]
(see [25] for a generalization). Thus, to every finitely generated $S$-module $M$ there corresponds a matrix factorization $M^{\text{st}}$ in $\text{HMF}(R, w)$ such that its image under (2.18) is isomorphic to $M$ in $\mathcal{D}^b(S)$. Following Dyckerhoff [6], we call $M^{\text{st}}$ the stabilization of $M$. (This term reflects the fact proved in [8] that $M$ has a free resolution which is stably 2-periodic.)

For a large class of $S$-modules the stabilizations are provided by Koszul factorizations (2.9).

**Proposition 2.3.1** ([8], cf. also [6]). Let $I$ be an ideal in $R$ generated by a regular sequence $b = (b_1, \ldots, b_n)$, and let $a = (a_1, \ldots, a_n)$ be an $n$-tuple of elements of $R$ such that $w = a \cdot b$. Then the Koszul factorization $\{a, b\} \in \text{HMF}(R, w)$ gives the stabilization of the $S$-module $R/I$.

### 2.4 Generators and Hochschild homology for matrix factorizations

From now on we assume that $R$ is the ring of formal power series $k[[x_1, \ldots, x_n]]$.

**Definition 2.4.1.** An element $w \in \mathfrak{m}$ is called an isolated singularity if its Tyurina algebra $R/(w, \partial_1 w, \ldots, \partial_n w)$ has finite dimension over $k$.

It is well known (see [23, Prop. (1.2)]) that in this case the Milnor ring
\[
\mathcal{A}_w = R/(\partial_1 w, \ldots, \partial_n w)
\]
is also finite dimensional (this uses the assumption that $k$ has characteristic zero).

The following result was proved by Dyckerhoff (cf. [6], Thm. 3.1, Thm. 4.2, Cor. 5.4).

**Theorem 2.4.2.** Let $w \in R$ be an isolated singularity. Let us view $R/\mathfrak{m} \simeq k$ as a module over $S = R/w$, and let $k^{\text{st}} \in \text{MF}(R, w)$ be the stabilization of $k$. Consider the $\mathbb{Z}/2$-dg-category $\text{MF}^\infty(R, w)$ of matrix factorizations involving free $R$-modules of possibly infinite rank. Then $\text{MF}^\infty(R, w)$ is quasi-equivalent to the $\mathbb{Z}/2$-dg-derived category of $\text{MF}(R, w)$ and is compactly generated by $k^{\text{st}}$. The $\mathbb{Z}/2$-dg-category $\text{Per}_{\text{dg}}(\text{MF}(R, w))$ is saturated.

As a consequence of the above theorem, one gets a derived equivalence of the category $\text{MF}(R, w)$ with the $\mathbb{Z}/2$-dg-algebra
\[
A = \text{Hom}_{\text{dg}}^\bullet(k^{\text{st}}, k^{\text{st}}),
\]

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so the computation of $HH_*(\MF(R, w))$ reduces to that of $HH_*(A)$. However, technically it is more convenient to use the category of matrix factorizations for the doubled potential

$$\tilde{w} := w(y_1, \ldots, y_n) - w(x_1, \ldots, x_n),$$

(2.19)

which is an element of the ring

$$R^e := R \hat{\otimes}_k R = k[[x_1, \ldots, x_n, y_1, \ldots, y_n]].$$

Using Theorem 2.4.2 one can deduce that the functor (2.12) induces an equivalence of perfect derived categories provided both $w$ and $w'$ are isolated singularities. Since the completion at the maximal ideal also does not change the perfect derived category by Theorem 4.6 of [6], we get an equivalence

$$\HMF(R^e, \tilde{w}) \simeq \Per(A^{\op} \otimes A) \simeq \Per(\MF(R, w)^{\op} \otimes \MF(R, w)).$$

(2.20)

To proceed with the computation of the Hochschild homology of $\MF(R, w)$ for an isolated singularity $w$ we need to use an explicit kernel in $\MF^\infty(R, \tilde{w})$ representing the identity functor on $\MF^\infty(R, w)$. As shown in [6, Cor. 5.4] (see also [17, Prop. 23]), the stabilized diagonal $\Delta^{st} \in \MF(R^e, \tilde{w})$ is such a kernel. Explicitly, this is the Koszul matrix factorization (see (2.9), (2.10))

$$\Delta^{st} = \{\Delta_1 w, \ldots, \Delta_n w; y_1 - x_1, \ldots, y_n - x_n\} = (K^\Delta_\bullet, \delta_K),$$

(2.21)

associated with the decomposition

$$\tilde{w} = w(y) - w(x) = \sum_{j=1}^n \Delta_j w \cdot (y_j - x_j),$$

where

$$\Delta_j w = \frac{w(x_1, \ldots, x_{j-1}, y_j, y_{j+1}, \ldots, y_n) - w(x_1, \ldots, x_{j-1}, x_j, y_{j+1}, \ldots, y_n)}{y_j - x_j} \in R^e.$$ 

(2.22)

For later calculations it is important to note that the difference derivative $\Delta_j w$ does not depend on $y_1, \ldots, y_{j-1}$ and on $x_{j+1}, \ldots, x_n$ and that

$$\Delta_j w|_{y=x} = \partial_j w.$$ 

(2.23)

The isomorphism of the functor $\Phi_{\Delta^{st}}$ represented by the kernel $\Delta^{st}$ with the identity functor is derived from the fact that for $E \in \MF^\infty(R, w)$ there is a natural map

$$p_2^*(p_1^* E \otimes K^\Delta_\bullet) \to E$$

(2.24)

induced by the composition $K^\Delta_\bullet \to K^\Delta_0 \to K^\Delta_0|_{y=x}$. One can easily check that the functor

$$\Tr : \MF(R^e, \tilde{w}) \to \Per_d(k)$$
is isomorphic to the restriction to the diagonal: $\bar{E} \mapsto \bar{E} \otimes \mathbb{R}$. Hence, the complex calculating $HH_{\ast}(\text{MF}(R, w))$ is obtained by restricting the differential $\delta_K$ to the diagonal $y = x$. This leads to the usual Koszul differential for the regular sequence $\partial_1 w, \ldots, \partial_n w$, so one obtains (see [6], Thm 5.7)

$$HH_{\ast}(\text{MF}(R, w)) \simeq H(w) := A_w \otimes dx[n],$$

(2.25)

where $dx = dx_1 \wedge \ldots \wedge dx_n$. Tensoring with the top-degree forms is important here because it makes the isomorphism compatible with the action of the group of symmetries of $w$.

### 2.5 $G$-equivariant Hochschild homology

Here we present an equivariant version of the results of section 2.4. Let $G$ be a finite group acting on the algebra $R = k[[x_1, \ldots, x_n]]$ by automorphisms (identical on $k$), and let $w \in R$ be a $G$-invariant isolated singularity. Recall that we denote $S = R/(w)$. Let $R \# G$ (resp., $S \# G$) denote the twisted group ring of $G$ over $R$ (resp., $S$).

There is a $G$-equivariant version of the equivalence (2.18),

$$\text{HMF}_G(R, w) \simeq \mathcal{D}^b_G(S),$$

where on the right one takes the quotient of the bounded derived category of finitely generated $S \# G$-modules by the subcategory generated by $S \# G$-modules that are free over $S$ (see [28]). In particular, to every $G$-equivariant finitely generated $S$-module $M$ there corresponds naturally a $G$-equivariant matrix factorization $M^{st}$, the stabilization of $M$.

Let us describe explicitly the stabilization of $k = R/m$. Since we work in characteristic zero, the surjective map of $G$-modules

$$m \to m/m^2 =: V$$

admits a $G$-equivariant splitting $s : V \to m$. Let $\psi \in (V^* \otimes R)^G$ be the element corresponding to $s : V \to R$. The $G$-equivariant map

$$\langle ?, \psi \rangle : V \otimes R \to m$$

is surjective. Hence, we can find a $G$-invariant element $\phi \in V \otimes R$ such that $\langle \phi, \psi \rangle = w$. Then $\{\phi, \psi\}$ is a $G$-equivariant Koszul matrix factorization (see (2.11)), and

$$k^{st} = \{\phi, \psi\}.$$  (2.26)

It will be convenient for what follows to choose a basis $(e_1, \ldots, e_n)$ in the space $V = m/m^2$ and use the elements

$$x_i = s(e_i) \in m, \quad i = 1, \ldots, n,$$  (2.27)

as a new set of variables (recall that we work with formal power series). With respect to these coordinates $G$ acts on $R$ by linear transformation, which will be frequently used in the rest of the paper.

Let $\text{MF}_G^\infty(w)$ denote the $\mathbb{Z}/2$-dg-category of $G$-equivariant matrix factorizations of free $R$-modules of possibly infinite rank.
Theorem 2.5.1. The category \( MF^\infty_G(w) \) is quasi-equivalent to the \( \mathbb{Z}/2 \)-dg-derived category of \( MF_G(R, w) \) and is compactly generated by the \( G \)-equivariant matrix factorization \( k^{st} \otimes k[G] \).

Proof. For a matrix factorization \( \bar{E} \in MF^\infty_G(w) \) we have

\[
\text{Hom}_w^*(k^{st} \otimes k[G], \bar{E})^G \simeq \text{Hom}_w^*(k^{st}, \bar{E}).
\]

Since \( k^{st} \) is a generator of \( HMF^\infty_G(w) \), this immediately implies that \( k^{st} \otimes k[G] \) is a generator of the homotopy category \( HMF^\infty_G(w) \). Now the same argument as in the proof of Theorem 4.2 of [6] implies that \( HMF^\infty_G(w) \) is quasi-equivalent to the derived category of \( MF_G(w) \).

To obtain an analog of the diagonal factorization (2.21) we start with an element

\[
\psi_\Delta = \sum_{i=1}^n e_i^* \otimes (y_i - x_i) \in V^* \otimes R^e,
\]

and set

\[
\phi_\Delta = \sum_{i=1}^n e_i \otimes \Delta iw \in V \otimes R^e,
\]

so that \( \langle \phi_\Delta, \psi_\Delta \rangle = \bar{w} = w(y) - w(x) \). If we view \( R^e \) as a \( G \)-module via the diagonal action on \( (x_1, \ldots, x_n, y_1, \ldots, y_n) \) then \( \psi_\Delta \) will be \( G \)-invariant (by our choice of variables (2.27)). Replacing \( \phi_\Delta \) with

\[
\phi^G_\Delta = |G|^{-1} \sum_{g \in G} g \cdot \phi_\Delta =: \sum_{i=1}^n e_i \otimes \tilde{\Delta} iw
\]

we obtain a \( G \)-equivariant Koszul matrix factorization

\[
\Delta^st_G := \{ \phi^G_\Delta, \psi_\Delta \} = \{ \tilde{\Delta} iw; y_1 - x_1, \ldots, y_n - x_n \}
\]

(see (2.11)) which is isomorphic to \( \Delta^st \) after forgetting the \( G \)-equivariant structure. Recall that the map \( V \otimes R^e \to V \otimes R \) (restriction to the diagonal \( y = x \)) sends \( \phi_\Delta \) to the \( G \)-invariant element \( \sum_{i=1}^n e_i \otimes \partial_i w \) (see (2.23)). Hence, we still have

\[
\tilde{\Delta} iw|_{y=x} = \partial_i w.
\]

Now we define a \( G \times G \)-equivariant matrix factorization of \( \bar{w} \)

\[
\Delta^st_{G \times G} := \bigoplus_y (\text{id} \times g)^* \Delta^st_G.
\]

We use objects \( K \in MF^\infty_{G \times G}(R^e, \bar{w}) \) as kernels of dg-functors:

\[
\Phi^G_K : MF^\infty_G(R, w) \to MF^\infty_G(R, w) : \bar{E} \mapsto p_{2*}(p_1^* \bar{E} \otimes_{R^e} K)^{G \times 1}.
\]

Proposition 2.5.2. The \( G \times G \)-equivariant matrix factorization \( \Delta^st_{G \times G} \) of \( \bar{w} \) is a kernel for the identity functor on \( HMF^\infty_G(R, w) \). Together with Theorem (2.5.1) this implies that the \( \mathbb{Z}/2 \)-dg-category \( \text{Per}_{dg}(MF_G(R, w)) \) is saturated.
Proof. The proof is parallel to the non-equivariant case (see [6, Cor. 5.4]).

Now in order to calculate $HH_*(MF_G(R, w))$ we have to compute $Tr^G(\Delta^st_{G\times G})$, where

$$Tr^G : MF_G\times G(R^e, \tilde{w}) \to \text{Per}(k)$$

is the categorical trace functor (1.1). It follows from the description (2.6) of morphisms in $MF_G(R, w)$ as $G$-invariants that $Tr^G$ is given by the restriction to the diagonal $y = x$ followed by taking $G$-invariants. Thus, we need first to compute the restriction of each summand $(id \times g)^*\Delta^st_G$ to the diagonal. We use an obvious isomorphism

$$(id \times g)^*\Delta^st_G|_\Delta \simeq \Delta^st_G|_{\Gamma_g},$$

where

$$\Gamma_g = \{(x, y) \mid y = gx\} \subset \text{Spec}(R\hat{\otimes}_k R)$$

is the graph of the action of $g$ on $\text{Spec}(R)$.

Furthermore, for a given element $g \in G$ we can choose variables $(x_1, \ldots, x_n)$ in such a way that

$$g(x_1, \ldots, x_n) = (\ell_1, \ldots, \ell_k, x_{k+1}, \ldots, x_n),$$

where $\ell_1, \ldots, \ell_k$ are linear forms in $x_1, \ldots, x_k$, and $\text{Span}(x_{k+1}, \ldots, x_n)$ is exactly the subspace of $g$-invariants in $\text{Span}(x_1, \ldots, x_n)$, i.e., the forms $(\ell_1 - x_1, \ldots, \ell_k - x_k)$ are linearly independent. Let us consider the restriction of $w$ to the subspace of $g$-invariants:

$$w_g(x_{k+1}, \ldots, x_n) := w|_{x_1 = \ldots = x_k = 0}.$$

In other words, we take the image of $w$ under the projection $R \to R^g = R/(x_1, \ldots, x_k)$.

Lemma 2.5.3. (i) For $j = k + 1, \ldots, n$ we have

$$\tilde{\Delta}_j w|_{\Gamma_g \cap \Delta} = \partial_j w_g,$$

and the sequence $(\partial_{k+1} w_g, \ldots, \partial_n w_g)$ in $R^g$ is regular, i.e., $w_g \in R^g$ is an isolated singularity.

(ii) The cohomology of the complex $\Delta^st_G|_{\Gamma_g}$ maps isomorphically to

$$H(w_g) = A_{w_g} \cdot dx_{k+1} \wedge \ldots \wedge dx_n[n - k],$$

via the restriction to $R/(x_1, \ldots, x_k) \otimes e_{k+1} \wedge \ldots \wedge e_n$ and passing to the quotient modulo $(\partial_{k+1} w_g, \ldots, \partial_n w_g)$ (where $dx_j$ gets identified with $e_j$).

Proof. (i) The equation (2.35) follows by setting $x_1 = \ldots = x_k = 0$ in the identity $\tilde{\Delta}_j w|_{\Delta} = \partial_j w$ for $j > k$ (see (2.31)). To check that $w_g$ is an isolated singularity let us differentiate the equation $w(gx) = w(x)$ with respect to the variables $x_1, \ldots, x_k$ and substitute $x_1 = \ldots = x_k = 0$. Using linear independence of the forms $\ell_1 - x_1, \ldots, \ell_k - x_k$ we derive that

$$\partial_i w|_{x_1 = \ldots = x_k = 0} = 0 \text{ for } i = 1, \ldots, k.$$

It follows that any critical point of $w_g$ is also a critical point of $w$ which implies our claim.
(ii) We have $\Delta_{st}^G = A_\bullet \otimes B_\bullet$, where $A_\bullet$ (resp., $B_\bullet$) is the matrix factorization of $\tilde{\nu}_A = \sum_{i=1}^k (y_i - x_i) \Delta_i w$ (resp., $\tilde{\nu}_B = \sum_{j=k+1}^n (y_j - x_j) \Delta_j w$), associated with this decomposition. Since $\tilde{\nu}_|_{\Gamma_g} = w(gx) - w(x) = 0$ and $\tilde{\nu}_B|_{\Gamma_g} = 0$, it follows that $\tilde{\nu}_A|_{\Gamma_g} = 0$. Hence, we have a decomposition into the product of two complexes $\Delta_{st}^G|_{\Gamma_g} = A_\bullet|_{\Gamma_g} \otimes B_\bullet|_{\Gamma_g}$, where $A_\bullet|_{\Gamma_g}$ is the complex associated with the decomposition

$$0 = w_A|_{\Gamma_g} = \sum_{i=1}^k (\ell_i - x_i) \cdot \Delta_i w,$$

and $B_\bullet$ is isomorphic to the Koszul complex for the sequence $(\tilde{\Delta}_{k+1} w|_{\Gamma_g}, \ldots, \tilde{\Delta}_n w|_{\Gamma_g})$, shifted by $n - k$ (and twisted by $dx_{k+1} \wedge \ldots \wedge dx_n$). Since the forms $\ell_1 - x_1, \ldots, \ell_k - x_k$ are linearly independent, it follows that the cohomology of $A_\bullet$ is isomorphic to $k$ and is concentrated in the term $A_0$, so

$$H^*(\Delta_{st}^G|_{\Gamma_g}) \simeq H^*(B_\bullet|_{\Delta \cap \Gamma_g}).$$

By part (i), this reduces to the Koszul complex for the regular sequence $(\partial_{k+1} w_g, \ldots, \partial_n w_g)$.

The above computation leads to the following decomposition of the equivariant Hochschild homology.

**Theorem 2.5.4.** Let $w \in R$ be an isolated singularity, invariant under a finite group of automorphisms $G$. Then we have

$$HH_*(MF_G(R, w)) \simeq (\bigoplus_{g \in G} H(w_g))^G,$$

where an element $g \in G$ acts on $\bigoplus_{g \in G} H(w_g)$ by sending $H(w_h)$ to $H(w_{ghg^{-1}})$.

**Remark 2.5.5.** It is easy to see that the isomorphism (2.36) does not depend on a choice of a $G$-invariant element $\phi^G_{\Delta} \in V \otimes R^e$ such that

$$\tilde{\nu} = \langle \phi^G_{\Delta}, \psi_\Delta \rangle,$$

where $\psi_\Delta$ is given by (2.28), and

$$\phi^G_{\Delta}|_{y=x} = \sum_{i=1}^n c_i \otimes \partial_i w.$$

**Remark 2.5.6.** E. Segal in [30] obtains a similar answer for a slightly modified version of Hochschild homology of $MF_G(R, w)$ (he used a version of the standard complex, where direct sums are replaced by direct products). It is not clear apriori how to identify his version with Hochschild homology defined in the usual way.
3 Chern character and boundary-bulk maps

Let $\bar{E} = (E, \delta_E)$ be a matrix factorization of $w$. Our goal in this section is to compute the image of the Chern character $\text{ch}(\bar{E}) \in HH_*(\text{MF}(R, w))$ (see (1.6)) under the isomorphism $HH_*(\text{MF}(R, w)) \simeq H(w)$ (see (2.25)). More generally, we will compute explicitly the boundary-bulk map (1.11)

$$\tau^{\bar{E}} : \text{Hom}_w(\bar{E}, \bar{E}) \to HH_*(\text{MF}(R, w)) \simeq H(w).$$

3.1 Technical lemmas

Let $\Delta_{\text{st}}$ be the diagonal matrix factorization of $\tilde{\omega} = w(y) - w(x)$ (see (2.21)). We will reduce the problem of computing the map $\tau^{\bar{E}}$ to finding a special element in the $\mathbb{Z}/2$-graded complex of $R^e$-modules

$$L_\bullet := \Delta_{\text{st}} \otimes_{R^e} p_2^* E^* \otimes_{R^e} p_1^* E \simeq \text{Hom}_{\tilde{w}}(E^* \boxtimes E, \Delta_{\text{st}}),$$

where the last isomorphism is the combination of the isomorphism (2.15) and the duality (2.16). Recall that $\Delta_{\text{st}} = (K_\bullet, \delta_K)$, where

$$K_\bullet = \bigoplus_{i=0}^n K_i, \quad K_i = \bigwedge_i (\mathbb{R}^e)^n.$$  

By (2.25), $H(w)$ is isomorphic to the cohomology of the complex $\Delta_{\text{st}}|_{y=x} = (K_\bullet|_{y=x}, \delta_K|_{y=x})$ concentrated in the term $K_n|_{y=x}$. Let us denote by

$$\pi : K_n|_{y=x} \to H(w)$$

the corresponding projection.

**Lemma 3.1.1.** Let $D \in L_{\text{even}}$ be a closed element such that

$$D_0|_{y=x} = 1 \otimes \text{id}_E \in K_0|_{y=x} \otimes E^* \otimes_R E,$$

where $D = \sum_{j=0}^n D_j$ with $D_i \in K_i \otimes_{R^e} p_2^* E^* \otimes_{R^e} p_1^* E$. Then the boundary-bulk map on $\alpha \in \text{Hom}_w(\bar{E}, \bar{E})$ is given by

$$\tau^{\bar{E}}(\alpha) = \pi(\text{str}(D_n|_{y=x} \circ \alpha)).$$

In particular, for the Chern character of $\bar{E}$ we have

$$\text{ch}(\bar{E}) = \pi(\text{str}(D_n|_{y=x})).$$

**Proof.** Recall that the external tensor product functor induces an equivalence of $\text{Per} (\text{MF}(R, w)^{\text{op}} \otimes \text{MF}(R, w))$ with the homotopy category of matrix factorizations of $\tilde{w} \in R^e$, such that the identity functor on $\text{MF}(R, w)$ corresponds to the diagonal matrix factorization $\Delta_{\text{st}}$. Since
the complex $L_\bullet$ gives morphisms from $\bar{E}^* \otimes \bar{E}$ to $\Delta^{st}$ in $\text{MF}(R^e, \bar{w})$, we have to find an element $c_E^\vee \in H^0(L_\bullet)$ inducing the canonical map (1.10)

$$c_E : \bar{E}^* \otimes \bar{E} \to \Delta^{st} : \alpha \mapsto \langle c_E^\vee, \alpha \rangle$$

(3.2)

and then restrict it to the diagonal. By definition, the map $c_E$ is characterized by the commutative diagram in $\text{HMF}^\infty(R, w)$

$$\begin{align*}
\bar{F} \otimes_R \bar{E}^* \otimes_k \bar{E} & \longrightarrow \text{Hom}_w(\bar{E}, \bar{F}) \otimes_k \bar{E} \overset{\text{ev}}{\longrightarrow} \bar{F} \\
p_2^*(p^*_1(F \otimes_R \bar{E}^*) \otimes_R \bar{E}) & \overset{id \otimes c_E}{\longrightarrow} p_2^*(p^*_1(F) \otimes_R \Delta^{st}) \overset{\gamma}{\longrightarrow} \bar{F}
\end{align*}$$

for $F \in \text{MF}(R, w)$, where the map $\gamma$ is (2.24). Note that the map in the first row is given by

$$f \otimes e^* \otimes e \mapsto f \cdot \langle e^* \otimes e, \text{id}_\bar{E} \rangle,$$

where we view $\text{id}_\bar{E}$ as an element in

$$\bar{E}^* \otimes_R \bar{E} \simeq \text{Hom}_w(\bar{E}, \bar{E}).$$

Since $\gamma$ is induced by the projection $K_\bullet \to K_0 \to K_0|_{y=x} \simeq R$, we obtain that the $K_0$-component of $c_E^\vee$ projects to $1 \otimes \text{id}_E$ under the projection $K_0 = R^e \to R^e/\mathfrak{d}_\Delta = R$, where $\mathfrak{d}_\Delta$ is the ideal of the diagonal. We observe that the latter projection coincides with the natural map

$$K_0 \to \text{coker}(K_1 \overset{\delta}{\rightarrow} K_0) \simeq \text{coker}(\delta_K : K_{odd} \to K_{even}).$$

By the equivalence (2.17), an element in $\text{Hom}^0_w(\bar{E}^* \otimes \bar{E}, \Delta^{st})$ is determined by the induced map on cokernels. Hence, for a closed element $D \in L_\bullet$ such that $D_0|_{y=x} = 1 \otimes \text{id}_E$, the image of $D$ in $H^0(L_\bullet)$ is equal to $c_E$. Thus, we can take $c_E^\vee = D$ in (3.2). When computing the restriction to the diagonal we recall that $K_{\bullet|x=x}$ can be identified with the shifted Koszul complex for the regular sequence $\partial_1 w, \ldots, \partial_n w$ (up to a twist by $dx_1 \wedge \ldots \wedge dx_n$), so its cohomology is concentrated in the $K_n|_{y=x}$ term. Hence, it remains to take the $K_n|_{y=x}$-component of the induced map

$$H^*(\bar{E}^* \otimes \bar{E}) \to H^*(K_{\bullet|x=x}) : \alpha \mapsto \langle D|_{y=x}, \alpha \rangle = \text{str}(D|_{y=x} \circ \alpha).$$

We will need the following result about Koszul complexes.

**Lemma 3.1.2.** Let $A$ be a commutative ring, $R = A[t_1, \ldots, t_n]$, and let

$$K_{\bullet}(t_1, \ldots, t_n) = \bigwedge_R^\bullet(R^e, \delta)$$

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be the Koszul complex for the sequence \((t_1, \ldots, t_n)\). Here \(\delta = \iota(\sum t_j e_j^*)\), where \((e_1, \ldots, e_n)\) is the standard basis of \(R^n\), \((e_1^*, \ldots, e_n^*)\) is the dual basis of the module \((R^n)^*\). Consider the following \(A\)-submodule in \(K_\bullet\):

\[
C = C(t_1, \ldots, t_n) = \sum_{i_1 < \ldots < i_k, k \geq 1} R_{i_1} \cdot e_{i_1} \wedge \ldots \wedge e_{i_k},
\]

where \(R_{i} = A[t_i, t_{i+1}, \ldots, t_n] \subset R\). Then

\[
K_\bullet = \ker(\delta) \oplus C.
\]

**Proof.** Let us use induction in \(n\). For \(n = 1\) we have \(C = R \cdot e_1\) and \(\ker(\delta) = R\), so the statement is clear. Suppose the statement holds for \(n - 1\). We have

\[
K_\bullet(t_1, \ldots, t_n) = K_\bullet(t_1) \otimes_A K_\bullet(t_2, \ldots, t_n).
\]

(3.3)

Let \(\delta_1\) (resp., \(\delta_{2, \ldots, n}\)) denote the differential in \(K_\bullet(t_1)\) (resp., \(K_\bullet(t_2, \ldots, t_n)\)). Note that under the isomorphism (3.3) we have a direct sum decomposition

\[
C(t_1, \ldots, t_n) = (1 \otimes C(t_2, \ldots, t_n)) \oplus (C(t_1) \otimes_A K_\bullet(t_2, \ldots, t_n)).
\]

In other words, we can write every element of \(K_\bullet(t_1, \ldots, t_n)\) in the form

\[
x = \sum_{i \geq 0} (t_i^1 \otimes a_i + (t_i^1 \cdot e_1) \otimes b_i),
\]

(3.4)

where \(a, b \in K_\bullet(t_2, \ldots, t_n)\), and this element is in \(C(t_1, \ldots, t_n)\) if and only if \(a_i = 0\) for \(i > 0\) and \(a_0 \in C(t_2, \ldots, t_n)\). Note that for \(f, g \in A[t_1]\) and \(a, b \in K_\bullet(t_2, \ldots, t_n)\) we have

\[
\delta(f \otimes a + (g \cdot e_1) \otimes b) = f \otimes \delta_{2, \ldots, n}(a) + g t_1 \otimes b - (g \cdot e_1) \otimes \delta_{2, \ldots, n}(b).
\]

(3.5)

This implies that

\[
g t_1 \otimes b = (g \cdot e_1) \otimes \delta_{2, \ldots, n}(b) \mod(\ker(\delta)),
\]

Also, we have

\[
f \otimes \ker(\delta_{2, \ldots, n}) \in \ker(\delta).
\]

Thus, starting with an arbitrary element \(x \in K_\bullet(t_1, \ldots, t_n)\) and applying the induction assumption to its decomposition (3.4) we can write \(a_i = a'_i + c_i\), where \(a'_i \in \ker(\delta_{2, \ldots, n})\) and \(c_i \in C(t_2, \ldots, t_n)\). Then we have

\[
x = \sum_{i \geq 0} (t_i^1 \otimes c_i + (t_i^1 \cdot e_1) \otimes b_i) \mod(\ker(\delta)).
\]

Furthermore, for \(i > 0\) we have

\[
t_i^1 \otimes c_i = (t_i^{i-1} \cdot e_1) \otimes \delta_{2, \ldots, n}(c_i) \mod(\ker(\delta)),
\]

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which proves that \( K_\ast(t_1, \ldots, t_n) = \ker \delta + C(t_1, \ldots, t_n) \). On the other hand, if the element \( x \) is in \( C(t_1, \ldots, t_n) \), so that \( a_i = 0 \) for \( i > 0 \) and \( a_0 \in C(t_2, \ldots, t_n) \), then the equation \( \delta x = 0 \) would give by (3.5)

\[
1 \otimes \delta_{2,\ldots,n}(a_0) + \sum_{i \geq 0} t_i^{i+1} \otimes b_i - \sum_{i \geq 0} (t_i^i \cdot e_1) \otimes \delta_{2,\ldots,n}(b_i) = 0.
\]

Since the first components of the three summands lie in complementary \( A \)-submodules we derive that \( \delta_{2,\ldots,n}(a_0) = 0 \) and \( b_i = 0 \) for all \( i \). By induction assumption, this implies that \( a_0 = 0 \) as well.

\[ \square \]

### 3.2 Formula for the Chern character and the boundary-bulk map

We keep the notation of section 3.1. Now we are going to construct an element \( D \in L_{\text{even}} \) satisfying the conditions of Lemma 3.1.1. Consider the operators

\[
\delta_\Delta = \iota(\sum_{j=1}^{n} (y_j - x_j)e_j^*), \quad \delta_w = (\sum_{j=1}^{n} \Delta_j w \cdot e_j) \land ?
\]

on \( K_\ast \) so that \( \delta_K = \delta_\Delta + \delta_w \). Let us fix an isomorphism

\[
E \simeq U \otimes R, \tag{3.6}
\]

where \( U \) is a \( \mathbb{Z}_2 \)-graded vector space. Then we can view the differential \( \delta_E \) of \( \tilde{E} \) as an \( R \)-valued odd endomorphism of \( U \). The decomposition (3.6) induces an isomorphism

\[
p_2^* \tilde{E}^* \otimes \text{Re} ~ p_1^* \tilde{E} \simeq U^* \otimes \text{Re} \simeq \text{End}(U) \otimes \text{Re}
\]

of matrix factorizations of \(-\tilde{w} = w(x) - w(y)\), where the differential \( \tilde{\delta} \) on \( \text{End}(U) \otimes \text{Re} \) acts by

\[
\tilde{\delta}(M) = \delta_E(x) \circ M - (-1)^{|M|} M \circ \delta_E(y).
\]

The complex \( L_\ast \) (see (3.1)) can now be expressed as \( L_\ast = K_\ast \otimes \text{End}(U) \), and its differential is given by

\[
\delta_L = (\delta_\Delta + \delta_w) \otimes \text{id}_{\text{End}(U)} + J_K \otimes \tilde{\delta},
\]

where \( J_K \) is the grading operator on the \( \mathbb{Z}/2 \)-graded space \( K_\ast \). Thus, if we write

\[
D = \sum_{j=0}^{n} D_j \in L_{\text{even}}
\]

with \( D_j \in K_j \otimes \text{End}(U) \) then the condition that \( D \) is \( \delta_L \)-closed is equivalent to the system

\[
(\delta_\Delta \otimes \text{id})(D_{j+1}) + (\delta_w \otimes \text{id})(D_{j-1}) + (-1)^j (\text{id} \otimes \tilde{\delta})(D_j) = 0, \tag{3.7}
\]

for \( j = 0, \ldots, n \), where we set \( D_{-1} = D_{n+1} = 0 \). For brevity we will write \( \delta_\Delta \) (resp., \( \delta_w \), resp., \( \tilde{\delta} \)) instead of \( \delta_\Delta \otimes \text{id} \) (resp., \( \delta_w \otimes \text{id} \), resp., \( \text{id} \otimes \tilde{\delta} \)).
Lemma 3.2.1. There exists a solution $D$ of (3.7) with $D_0 = 1 \otimes \text{id}_U \in \mathbb{R}^e \otimes \text{End}(U)$. Furthermore, let us consider the decomposition

$$D_j = \sum_{i_1 < \ldots < i_j} e_{i_1} \wedge \ldots \wedge e_{i_j} \otimes D_j(i_1, \ldots, i_j),$$

where $D_j(i_1, \ldots, i_j) \in \mathbb{R}^e \otimes \text{End}(U)$. Then there exists a unique $D$ satisfying (3.7) such that the coefficient $D_j(i_1, \ldots, i_j)$ does not depend on $y_k$ with $k < i_1$, for all $j$, $i_1 < \ldots < i_j$.

Proof. We define $D_j$ inductively starting with $D_0 = 1 \otimes \text{id}_U$. If all $D_i$ for $i \leq j$ are already defined, then to find $D_{j+1}$ we have to solve

$$-\delta_\Delta(D_{j+1}) = \delta_w(D_{j-1}) + (-1)^j \tilde{\delta}(D_j)$$

obtained from (3.7). Suppose first that $j = 0$. Then the equation becomes

$$\delta_\Delta(D_1) = -1 \otimes \tilde{\delta}(\text{id}_U) = 1 \otimes (\delta_E(y) - \delta_E(x)).$$

Since the right-hand side is zero for $y = x$, such $D_1$ exists. Applying Lemma 3.1.2 to $A = k[x_1, \ldots, x_n]$ and $t_j = y_j - x_j$, we can find a unique $D_1$ satisfying (3.9) such that $D_1(i)$ does not depend on $y_j$ with $j < i$. Now, assume that $j > 0$. Then the argument is similar, but we have to check first that the right-hand side of (3.8) is $\delta_\Delta$-closed. Indeed, using the same equation for $j - 1$ we get

$$\delta_\Delta(\delta_w(D_{j-1}) + (-1)^j \tilde{\delta}(D_j)) = \delta_\Delta \delta_w(D_{j-1}) + (-1)^j \tilde{\delta} \delta_\Delta(D_j) = (\delta_\Delta \delta_w + \tilde{\delta}^2)(D_{j-1}) - (-1)^j \tilde{\delta} \delta_w(D_{j-2}).$$

Using the identity $\delta_\Delta \delta_w + \tilde{\delta}^2 = -\delta_w \delta_\Delta$ and applying (3.8) for $j - 2$ we can rewrite this as

$$\delta_w[-\delta_\Delta(D_{j-1}) - (-1)^j \tilde{\delta}(D_{j-2})] = (\delta_w)^2(D_{j-3}) = 0$$

as claimed. 

Lemma 3.2.2. The unique solution $D$ of (3.7) constructed in Lemma 3.2.1 satisfies

$$D_j(n-j+1, \ldots, n)|_{y_{n-j+1} = x_{n-j+1}} = D_{j-1}(n-j+2, \ldots, n) \circ \partial_{n-j+1} \delta_E(x_1, \ldots, x_{n-j+1}, y_{n-j+2}, \ldots, y_n)$$

for $j = 1, \ldots, n$.

Proof. We use induction in $j$. For $j = 1$ the equation (3.10) takes form

$$D_1(n)|_{y_n = x_n} = \partial_n \delta_E(x).$$

To prove this let us rewrite (3.9) as

$$\sum_{j=1}^{n} (y_j - x_j) \cdot D_1(j) = \delta_E(y) - \delta_E(x).$$

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Since $D_1(n)$ does not depend on $y_1, \ldots, y_{n-1}$, substituting $y_1 = x_1, \ldots, y_{n-1} = x_{n-1}$ into this equation gives
\[(y_n - x_n) \cdot D_1(n) = \delta_E(x_1, \ldots, x_{n-1}, y_n) - \delta_E(x_1, \ldots, x_n),\]
which immediately implies (3.11).

Similarly, for $j > 1$ substituting $y_i = x_i$ for $i = 1, \ldots, n-j$ into the equation (3.8) with $j$ replaced by $j - 1$ and comparing the coefficients of $e_{n-j+2} \wedge \ldots \wedge e_n$ we get
\[(y_{n-j+1} - x_{n-j+1})D_j(n - j + 1, \ldots, n) = \left[(-1)^j \Delta(D_{j-1}(n - j + 2, \ldots, n)) - \sum_{i=n-j+2}^j (-1)^{n-j+2-i} \Delta_i \wedge \delta(D_{j-2}(n - j + 2, \ldots, i, \ldots, n))\right] |y_1=x_1,\ldots,y_{n-j}=x_{n-j}. \quad (3.12)\]

Now recall that $\Delta_i \wedge \delta$ and $D_m(i, \ldots)$ do not depend on $y_{n-j+1}$ for $i \geq n - j + 2$. Therefore, in the right-hand side of (3.12) only $\tilde{\delta}$ (that involves $\delta_E(y)$) depends on $y_{n-j+1}$. Hence, after differentiating the above equation with respect to $y_{n-j+1}$ and restricting to $y_{n-j+1} = x_{n-j+1}$ we obtain (3.10).

**Theorem 3.2.3.** Let $w \in R = k[[x_1, \ldots, x_n]]$ be an isolated singularity, and let $\mathcal{J}_w = (\partial_1 \wedge \ldots \wedge \partial_n \wedge w)$. Then the boundary-bulk map on an endomorphism $\alpha \in \text{Hom}_w^*(\bar{E}, \bar{E})$ of a matrix factorization $\bar{E} = (E, \delta_E) \in MF(R, w)$ is equal to
\[\tau^E(\alpha) = \text{str}_R(\partial_n \delta_E \circ \ldots \circ \partial_1 \delta_E \circ \alpha) \cdot dx_1 \wedge \ldots \wedge dx_n \mod \mathcal{J}_w \cdot dx_1 \wedge \ldots \wedge dx_n, \quad (3.13)\]
where we view $\delta_E$ and $\alpha$ as matrices with values in $R$ after choosing a basis in a free $R$-module $E$.

In particular, for the Chern character of $\bar{E}$ we have
\[\text{ch}(\bar{E}) = \text{str}_R(\partial_n \delta_E \circ \ldots \circ \partial_1 \delta_E) \cdot dx_1 \wedge \ldots \wedge dx_n \mod \mathcal{J}_w \cdot dx_1 \wedge \ldots \wedge dx_n. \quad (3.14)\]

**Proof.** Restricting (3.10) to the diagonal $y = x$ and combining the resulting equations for $j = 1, \ldots, n$ we deduce that
\[D_n|_{y=x} = e_1 \wedge \ldots \wedge e_n \otimes \partial_n \delta_E \circ \partial_{n-1} \delta_E \circ \ldots \circ \partial_1 \delta_E.\]
Now the required formulas follow from Lemma 3.1.1. \qed

**Corollary 3.2.4.** With the notations of Theorem 3.2.3 the expression
\[\text{str}_R(\partial_n \delta_E \circ \ldots \circ \partial_1 \delta_E \circ \alpha) \cdot dx_1 \wedge \ldots \wedge dx_n \mod \mathcal{J}_w \cdot dx_1 \wedge \ldots \wedge dx_n\]
is invariant under permutations of indices $(1, \ldots, n)$. Hence, we have
\[\tau^E(\alpha) = (-1)^{\binom{n}{2}} \cdot \frac{1}{n!} \cdot \text{str}_R((d \delta_E)^{\wedge n} \circ \alpha) \mod \mathcal{J}_w \cdot dx_1 \wedge \ldots \wedge dx_n.\]
3.3 \(G\)-equivariant Chern character

Now, we are going to discuss a \(G\)-equivariant version of the results of the previous section keeping the notation and assumptions of section 2.5.

Let \(\bar{E} = (E, \delta_E)\) be a \(G\)-equivariant matrix factorization of \(w\). We denote by

\[ \text{ch}_G(\bar{E}) \in HH_*(MF_G(R, w)) \]

the Chern character of \(\bar{E}\) and by

\[ \tau_{\bar{E}}^G : \text{Hom}_w(\bar{E}, \bar{E})^G \to HH_*(MF_G(R, w)) \]

the boundary-bulk map (1.11). Our goal is to compute explicitly for every \(g \in G\) and \(\alpha \in \text{Hom}_w(\bar{E}, \bar{E})^G\) the component

\[ \tau^E(\alpha)_g \in H(w_g) \]

of \(\tau_{\bar{E}}^G(\alpha) \in HH_*(MF_G(R, w))\) with respect to the decomposition (2.36).

Lemma 3.3.1. Let \(E\) be an \(R\#G\)-module, free of finite rank as \(R\)-module. There exists an isomorphism \(E \simeq U \otimes R\) of \(R\#G\)-modules, where \(U\) is a representation of \(G\).

Proof. Let \(U = E/\mathfrak{m}E\). Since we work in characteristic zero, we can choose a \(G\)-equivariant splitting \(U \to E\) of the surjective map \(E \to E/\mathfrak{m}E = U\) of \(G\)-modules. The induced map \(U \otimes R \to E\) will be an isomorphism by the standard argument using Nakayama Lemma. \(\square\)

We choose an isomorphism \(E \simeq U \otimes R\) as in the above Lemma and proceed as in section 3.1 to consider the complex

\[ L^G_\bullet = \Delta^s_G \otimes R^e \text{End}(U) \]

which is now equipped with a \(G\)-action (diagonal on \(R^e\)). Note that Lemma 3.1.1 still holds with the complex \(L^G_\bullet\) replaced by \(L^G_\bullet\) (which amounts to replacing the difference derivatives \(\Delta_i w\) with their \(G\)-equivariant version \(\tilde{\Delta}_i w\) defined by (2.29)).

For an element \(g \in G\) let us denote by

\[ \Delta_g := \Delta \cap \Gamma_g \subset \text{Spec}(R^e) \]

the intersection of the diagonal with the graph of \(g\). If we choose variables \((x_1, \ldots, x_n)\) so that \(g\) acts by the linear transformation (2.34) and \(\text{Span}(x_{k+1}, \ldots, x_n)\) is exactly the subspace of \(g\)-invariants in \(\text{Span}(x_1, \ldots, x_n)\), then \(\Delta_g\) is given by equations

\[ x_1 = y_1 = \ldots = x_k = y_k = 0, \ x_{k+1} = y_{k+1}, \ldots, x_n = y_n. \]

Lemma 3.3.2. We have

\[ \tau^E(\alpha)_g = \text{str}(D|_{\Delta_g} \circ g \circ \alpha) \quad (3.15) \]

where \(D\) defined as in Lemma 3.1.1.
Proof. By Lemma 1.2.4, we have to compute the components of the canonical morphism

\[ c_E^G : \hat{E}^* \boxtimes \hat{E} \to \bigoplus_{g \in G} (\text{id} \times g)^* \Delta_G. \]

We claim that its component corresponding to \( g = 1 \) coincides with the non-equivariant map (3.2). Indeed, for a kernel \( K \in MF_{G \times G}(\bar{w}) \) composing the functor \( \Phi^G_G : MF_G(w) \to MF_G(w) \) with the forgetful functor \( F : MF_G(w) \to MF(w) \) corresponds to forgetting the action of the second factor in \( G \times G \) on \( K \). Note that \( F \circ c_E^G \) is equal to the composition

\[ F \circ \Phi^G_G \to \Phi^G_G \circ F \to F \]

of morphisms between functors \( MF_G(w) \to MF(w) \). Our claim follows from the fact that the middle functor is represented by \( \bigoplus_{g \in G} (g^* \hat{E})^* \boxtimes \hat{E} \in MF_{G \times 1}(\bar{w}) \), so that the map \( \alpha \) corresponds to the natural embedding of kernels

\[ \hat{E}^* \boxtimes \hat{E} \to \bigoplus_{g \in G} (g^* \hat{E})^* \boxtimes \hat{E} \]

in \( MF_{G \times 1}(\bar{w}) \).

Thus, via the isomorphism (2.33) the restriction of the \( g \)-component of \( c_E^G \) to the diagonal \( y = x \) gets identified with the restriction of the non-equivariant map \( c_E^G \) to the graph \( \Gamma_g \). Hence, the \( g \)-component of \( \tau^E \) is obtained by restricting \( c_E^G \) to \( \Gamma_g \) and using the isomorphism

\[ \hat{E}^* \otimes_R \hat{E} \cong \hat{E}^* \otimes \hat{E}|_{\Gamma_g} \]

induced by the action of \( g \) on \( E \). It remains to apply Lemma 2.5.3. \( \square \)

Now we are ready to prove the formula for the \( G \)-equivariant Chern character and the boundary-bulk map.

Theorem 3.3.3. Fix an element \( g \in G \). Choose variables \( (x_1, \ldots, x_n) \) so that \( g \) acts by the linear transformation (2.34) and \( \text{Span}(x_{k+1}, \ldots, x_n) \) is exactly the subspace of \( g \)-invariants in \( \text{Span}(x_1, \ldots, x_n) \). Then for any \( G \)-equivariant matrix factorization \( \hat{E} = (\hat{E}, \delta_E) \) we have

\[ \tau^E_G(\alpha)_g = \text{str}_{R^g}([\partial_{x_k} \delta_E \circ \ldots \circ \partial_{x_k} \delta_E \circ g \circ \alpha]|_{x_1 = \ldots = x_k = 0}) \cdot dx_{k+1} \wedge \ldots \wedge dx_n \mod \mathcal{J}_w, \quad (3.16) \]

where \( \alpha \in \text{Hom}_w(\hat{E}, \hat{E})^G \) and \( R^g = R/(x_1, \ldots, x_k) \).

In particular,

\[ \text{ch}_G(\hat{E})_g = \text{str}_{R^g}([\partial_{x_k} \delta_E \circ \ldots \circ \partial_{x_k} \delta_E \circ g]|_{x_1 = \ldots = x_k = 0}) \cdot dx_{k+1} \wedge \ldots \wedge dx_n \mod \mathcal{J}_w. \quad (3.17) \]

Proof. First, note that the projection \( HH_*(MF_G(R, w)) \to H(w_g) \) factors as the composition

\[ HH_*(MF_G(R, w)) \to HH_*(MF_{G_0}(R, w)) \to H(w_g), \]

where \( G_0 \subset G \) is the subgroup generated by \( g \). Furthermore, when computing the projection \( HH_*(MF_{G_0}(R, w)) \to H(w_g) \) we can use either \( \Delta^G_{G_0} \) or \( \Delta^G_{G_0} \) (by Remark 2.5.5 applied to \( G_0 \)-equivariant situation). Let us work with

\[ \Delta^G_{G_0} = \{ \tilde{\Delta}_1 w, \ldots, \tilde{\Delta}_n w; y_1 - x_1, \ldots, y_n - x_n \} \]

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Lemma 3.3.5. Under the identification $\alpha$ is the dual matrix factorization to $\bar{\alpha}$.

Proof. Applying (3.16) for $\bar{E}^*$ and the dual endomorphism $\alpha^*$ we get

$$\tau^{\bar{E}}(\alpha^*)_g = \text{str}(\partial_{n+1}^* \circ \cdots \circ \partial_{k+1}^* \circ (g^{-1})^* \circ \alpha^*) \cdot dx_{k+1} \wedge \ldots \wedge dx_n \mod J_w =$$

$$(-1)^{\binom{n+1}{2} + (n-k)|\alpha|} \text{str}(\partial_n^*(\alpha \circ g^{-1} \circ \partial_{k+1}^* \circ \cdots \circ \partial_{n}^*)) \cdot dx_{k+1} \wedge \ldots \wedge dx_n \mod J_w =$$

$$(-1)^{(n-k)|\alpha|} \text{str}(\alpha \circ g^{-1} \partial_{k+1}^* \circ \cdots \circ \partial_{n}^*) \cdot dx_{k+1} \wedge \ldots \wedge dx_n \mod J_w =$$

where we used the equalities $\text{str}(M^*) = \text{str}(M)$, $\text{str}(M_1 \circ M_2) = (-1)^{|M_1||M_2|} \text{str}(M_2 \circ M_1)$ and $\alpha \circ g^{-1} = g^{-1} \circ \alpha$. It remains to use Corollary 3.3.4 to see that this is equal to the right-hand side of (3.16). □

We have also an equivariant version of Corollary 3.2.4.

**Corollary 3.3.4.** With the notations of Theorem 3.3.3 the expression

$$\text{str}_{R^0}([\partial_n \delta_E \circ \cdots \circ \delta_{k+1} \delta_E \circ g \circ \alpha]|_{x_1=\ldots=x_k=0}) \cdot dx_{k+1} \wedge \ldots \wedge dx_n \mod J_w$$

is invariant under permutations of indices $(k+1, \ldots, n)$. Hence, we have

$$\tau^E(\alpha) = (-1)^{\binom{n}{2}} \cdot \frac{1}{(n-k)!} \cdot \text{str}_{R^0}([\partial_n \delta_E \circ \cdots \circ \delta_{k+1} \delta_E \circ g \circ \alpha]|_{x_1=\ldots=x_k=0}) \cdot dx_{k+1} \wedge \ldots \wedge dx_n \mod J_w \cdot dx_{k+1} \wedge \ldots \wedge dx_n.$$

We have the following relation between the maps $\tau^E$ and $\tau^{\bar{E}}$, where $\bar{E}^* \in \text{MF}_G(R, -w)$ is the dual matrix factorization to $\bar{E}$.

**Lemma 3.3.5.** Under the identification $H(-w_g) = H(w_{g-1})$ for any $\bar{E} \in \text{MF}_G(R, w)$ we have

$$\tau^{\bar{E}}(\alpha^*)_g = \tau^E(\alpha)_{g^{-1}},$$

where $\alpha \in \text{Hom}_w(\bar{E}, \bar{E})^G$. In particular,

$$\text{ch}_G(\bar{E}^*)_g = \text{ch}_G(\bar{E})_{g^{-1}}.$$

**Proof.** Applying (3.16) for $\bar{E}^*$ and the dual endomorphism $\alpha^*$ we get

$$\tau^{\bar{E}}(\alpha^*)_g = \text{str}(\partial_{n} \delta_{\bar{E}} \circ \cdots \circ \partial_{k+1} \delta_{\bar{E}} \circ (g^{-1})^* \circ \alpha^*) \cdot dx_{k+1} \wedge \ldots \wedge dx_n \mod J_w =$$

where $G_0$ acting on $R$ changes only variables $x_1, \ldots, x_k$, it follows that $\bar{\Delta}_j w = \Delta_j w$ for $j > k$. Now the above equation can be verified by the same argument as in Lemma 3.2.2. □

## 4 Hirzebruch-Riemann-Roch formula

In this section we work out the explicit form of the categorical Hirzebruch-Riemann-Roch formula (1.8) for the categories $\text{MF}(R, w)$ and $\text{MF}_G(R, w)$, where $w \in R = k[[x_1, \ldots, x_n]]$ is an isolated singularity, using our calculation of the Chern characters from the previous section.
### 4.1 Non-equivariant case

Recall that the Hochschild homology $\HH_*(MF(R, w))$ can be identified with the $\mathbb{Z}/2$-graded space $H(w) = \mathcal{A}_w \otimes dx[n]$, where $\mathcal{A}_w$ is the Milnor ring of $w$. By (1.9), the canonical bilinear form (1.7) on $\HH_*(MF(R, w)) = H(w) = H(-w) = \HH_*(MF(R, -w))$ (4.1) is equal to the inverse of the element $\ch(\Delta_{st}) \in \HH_*(MF(R^e, \bar{w})) \simeq \HH_*(MF(R, -w)) \otimes \HH_*(MF(R, w))$, where the diagonal matrix factorization $\Delta_{st} \in MF(R^e, \bar{w})$ is the kernel (2.21) representing the identity functor on $MF(R, w)$. We can calculate the Chern character of $\Delta_{st}$ using the general formula (3.14) (this computation is contained implicitly in [14, sec. 5.1]).

**Proposition 4.1.1.** We have

$$\ch(\Delta_{st}) = (-1)^{\binom{n}{2}} \cdot \det(\Delta_j(\partial_i w)) \in \mathcal{A}_w \otimes dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n.$$  

**Proof.** By (3.14), this reduces to equality (21) of [14, sec. 5.1]. □

**Proposition 4.1.2.** Let $f_1, \ldots, f_n \in R = k[[x_1, \ldots, x_n]]$ be such that $\mathcal{A} = R/(f_1, \ldots, f_n)$ is finite-dimensional. Then the element

$$\delta = \det(\Delta_j(f_i)) \in \mathcal{A} \otimes \mathcal{A}$$

is the inverse to the nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathcal{A}$ given by

$$(f, g) = \operatorname{tr}(f \cdot g),$$

where the trace on $\mathcal{A}$ is the Grothendieck residue

$$\operatorname{tr}(h) = \operatorname{Res} \left[ h(x) \cdot dx_1 \wedge \ldots \wedge dx_n \right]_{f_1, \ldots, f_n}.$$  

**Proof.** The proof is similar to that of the equality (17) of [14]. However, the proof given in [14] is analytic, since it involves a small perturbation of the potential $w$. Let us show how to modify their argument to make it purely algebraic. We will use a version of the algebraic deformation constructed in [21].

First of all, let us restate the assertion in a more explicit form. We have to check that

$$((\cdot, \cdot) \otimes \text{id})(f \otimes \delta) = f$$  

(4.4)
for every $f \in A$. It is known that the trace form $(\cdot, \cdot)$ is nondegenerate, so it is sufficient to check that both sides of (4.4) have the same pairing with an arbitrary element $g \in A$. Thus, we have to check the equality

$$tr_{x,y}(h(x,y)\delta(x,y)) = tr(h(x,x)),$$

(4.5)

where $tr_{x,y}$ is the residue trace (4.3) on $A \otimes A$ associated with the elements $f_1(x), \ldots, f_n(x), f_1(y), \ldots, f_n(y) \in k[[x_1, \ldots, x_n, y_1, \ldots, y_n]]$.

The statement depends only on finitely many coefficients of $f_i$, so we can assume that all $f_i$ are polynomials. We can also assume that the field $k$ is algebraically closed. For each $i = 1, \ldots, n$ let $(f_i)$ denote the hypersurface in the projective space $\mathbb{P}^n_k$ defined by $f_i$. Let us consider a generic one-parameter deformation $(f^t_i)$, so that the intersection

$$\Sigma^t = \cap_{i=1}^n (f^t_i) \subset \mathbb{P}^n_k$$

is transversal for $t \neq 0$. Let $\mathcal{C} \subset \mathbb{P}^n_{k[t]}$ be the union of all irreducible components of $\Sigma^t$ containing the origin $0 \in \Sigma^0 \subset \mathbb{P}^n_k$. Then the projection $\mathcal{C} \to \text{Spec}(k[t])$ is a flat finite map, ramified only over $t = 0$. Let

$$\text{Spec}(B) = \mathcal{C} \times_{\text{Spec}(k[t])} \text{Spec}(k[[t]])$$

be the formal neighborhood of the central fiber $\mathcal{C}_0$. Then $B$ is a product of local algebras over $k[[t]]$ supported at points of $\mathcal{C}_0$. Let $P \subset B$ be the maximal ideal corresponding to the origin $0 \in \mathcal{C}_0 \subset \Sigma^0$. Then the localization $B_P$ is a finite flat algebra over $k[[t]]$ such that

$$B_P/(t) \simeq k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_n).$$

Since $B_P \otimes_{k[[t]]} k((t))$ is étale over $k((t))$, passing to the algebraic closure $L$ of $k((t))$ we find that $B_L = B_P \otimes_{k[[t]]} L$ is a product of the local rings of the intersection $\Sigma^t$ at a finite number of $L$-points. Now we can apply the argument of the proof of Theorem 4.2 in [21] to show that the residue for $k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_n)$ is the specialization of the sum of the residues for the collection $(f^t_1, \ldots, f^t_n)$ at $L$-points of a transversal intersection (corresponding to maximal ideals of $B_L$).\(^1\)

At this point we can mimic the argument of [14] (see [14, (26)–(28)]). Let

$$\text{tr}^t : B_L/(f^t_1, \ldots, f^t_n) \to L$$

denote the trace (4.3) associated with the deformed polynomials $(f^t_1, \ldots, f^t_n)$. Since in this we have only points of transversal intersection, we have

$$\text{tr}^t(f) = \sum_{p \in \text{Max}(B_L)} \frac{f(p)}{\det(\partial_j f^t_j(p))},$$

(4.6)

\(^1\)The difference of our approach with [21] is that we work infinitesimally from the beginning. This allows us to ignore intersection points at infinity.
where \( \text{Max}(B_L) \) is the set of maximal ideals of \( B_L \). Note that \( B_L \) is a localization of \( L[x_1, \ldots, x_n] \), so each \( p \in \text{Max}(B_L) \) can be viewed as an element of \( L^n \). Similarly,

\[
\text{tr}^t_{x,y}(h(x,y)) = \sum_{p,q \in \text{Max}(B_L)} \frac{h(p,q)}{\det(\partial_i f^i_j(p)) \cdot \det(\partial_i f^i_j(q))}.
\]

The crucial observation is that \( \delta^t(p,q) = 0 \) for a pair of distinct points \( p, q \in \text{Max}(B_L) \).

Indeed, by definition of \( \Delta \) (see (4.2)), we have for each \( i \)

\[
\sum_j (p_j - q_i) \cdot \Delta_j(f^i_1)(p, q) = f^i_1(q) - f^i_1(p) = 0.
\]

Since the vector \( (q_1 - p_1, \ldots, q_n - p_n) \) is not zero, the matrix \( (\Delta_j(f^i_1)(p, q)) \) is degenerate. Now we are ready to check the analog of the equation (4.5) for \( \text{tr}^t \). We have

\[
\text{tr}^t_{x,y}(h(x,y)\delta^t(x,y)) = \sum_{p,q \in \text{Max}(B_L)} \frac{h(p,q)\delta^t(p,q)}{\det(\partial_i f^i_j(p)) \cdot \det(\partial_i f^i_j(q))} = \sum_{p \in \text{Max}(B_L)} \frac{h(p,p)\delta^t(p,p)}{\det(\partial_i f^i_j(p))^2}.
\]

But \( \Delta_j(f^i_1)(x,x) = \partial_j f^i_1(x) \), so \( \delta^t(p,p) = \det(\partial_i f^i_1(p)) \), so by (4.6) the obtained expression is equal to \( \text{tr}^t(h(x,x)) \).

**Corollary 4.1.3.** The canonical bilinear form (1.7) for \( \mathcal{C} = \text{MF}(R, w) \) after the identification (4.1) coincides with

\[
\langle f \otimes dx, g \otimes dx \rangle = (-1)^{(\frac{n}{2})} \text{tr}(f \cdot g), \quad (4.7)
\]

where

\[
\text{tr}(f) = \text{Res} \left[ f(x) \cdot dx_1 \wedge \ldots \wedge dx_n \right]_{\partial_1 w, \ldots, \partial_n w}
\]

**Proof.** Since the canonical bilinear form is equal to the inverse of the Chern character \( \text{ch}(\Delta^w) \in H(-w) \otimes H(w) = H(w) \otimes H(w) \), the assertion follows from Propositions 4.1.1 and 4.1.2 applied to \( f = \partial_i w \).

Now the general categorical Hirzebruch-Riemann-Roch formula (1.8) specializes to the following explicit version for matrix factorizations.

**Theorem 4.1.4.** (i) For \( \bar{E}, \bar{F} \in \text{MF}(R, w) \) we have

\[
\chi(\text{Hom}_w(\bar{E}, \bar{F})) = \dim \text{Hom}^0_w(\bar{E}, \bar{F}) - \dim \text{Hom}^1_w(\bar{E}, \bar{F}) = \langle \text{ch}(\bar{E}), \text{ch}(\bar{F}) \rangle, \quad (4.8)
\]

where \( \langle \cdot, \cdot \rangle \) is given by (4.7) and the Chern characters are given by (3.14).

(ii) More generally, for \( \alpha \in \text{Hom}_w(\bar{E}, \bar{E}) \) and \( \beta \in \text{Hom}_w(\bar{F}, \bar{F}) \) we have

\[
\text{str}_k(m_{\alpha, \beta}) = \langle \tau_{\bar{E}}(\alpha), \tau_{\bar{F}}(\beta) \rangle, \quad (4.9)
\]

where \( m_{\alpha, \beta} \) is the endomorphism of \( \text{Hom}_w(\bar{E}, \bar{F}) \) induced by composing with \( \alpha \) and \( \beta \) (see (1.13)).
Proof. This follows from (1.12) using Corollary 4.1.3 and Lemma 3.3.5 (with the trivial group $G$). (To prove (4.8) it is sufficient to use the categorical Hirzebruch-Riemann-Roch formula (1.8) instead of (1.12).)

Remark 4.1.5. If the number of variables $n$ is odd then $\chi(\text{Hom}_w(\bar{E}, \bar{F})) = 0$ for any matrix factorizations $\bar{E}$ and $\bar{F}$. Indeed, this follows from Theorem 4.1.4, since in this case $HH_0 = 0$ so all Chern characters vanish.

Remark 4.1.6. The Chern character of the stabilization $k^s \in \text{MF}(R, w)$ of the residue field $k = R/\mathfrak{m}$ (viewed as $R/(w)$-module), vanishes for any number of variables $n > 0$ (see Proposition 4.3.4 below).

Example 4.1.7. Let us illustrate the Theorem for $w = x^n$ (where $n \geq 2$) and the Koszul matrix factorizations $\bar{E}_i = \{x^i, x^{n-i}\}, i = 1, \ldots, n - 1$. Assume first that $i \geq n/2$. Then the space $\text{Hom}_w^1(\bar{E}_i, \bar{F}_i)$ is isomorphic to $k[x]/(x^{n-i}) \cdot \alpha_i$, where $\alpha_i$ is the odd endomorphism of $\bar{E}_i$ given by

$$\alpha_i(e_0) = x^{2i-n}e_1, \quad \alpha_i(e_1) = -e_0.$$ 

We have

$$\tau^{\bar{E}_i}(\alpha_i) = \text{str}(\delta' \circ \alpha) \mod(x^{n-1}) = nx^{i-1} \mod(x^{n-1}).$$

For $i \leq n/2$ we can get a similar description of the maps $\tau^{\bar{E}_i}$ using the relation $\bar{E}_i \simeq \bar{E}_{n-i}[1]$, where $[1]$ is the change of parity functor. Notice that $\tau^{\bar{E}_i}$ is injective for every $i$. Also, we observe that for $\alpha \in \text{Hom}_w^1(\bar{E}_i, \bar{F}_i)$ and $\beta \in \text{Hom}_w^1(\bar{E}_j, \bar{F}_j)$ we have

$$\tau^{\bar{E}_i}(\alpha) \cdot \tau^{\bar{E}_j}(\beta) = 0$$

in $\mathcal{A}_w$ provided $(i, j) \neq (n/2, n/2)$. On the other hand, one can easily check that in this case the operator $m_{\alpha, \beta}$ is nilpotent, so $\text{str}_k(m_{\alpha, \beta}) = 0$ in agreement with the formula (4.9). Now let us consider the case $i = j = n/2$ (assuming that $n$ is even). Then $\alpha_i^2 = -\text{id}$ and the operator $m_{\alpha_i, \alpha_i}$ acts as identity on $\text{Hom}_w^0(\bar{E}_i, \bar{F}_i) \simeq k[x]/(x^i)$ and as $-\text{id}$ on $\text{Hom}_w^1(\bar{E}_i, \bar{F}_i) \simeq k[x]/(x^i)$. It follows that

$$\text{str}_k(m_{\alpha_i, \alpha_i}) = 2i = n.$$

On the other hand,

$$\langle nx^{i-1}, nx^{i-1} \rangle = \text{Res} \frac{n^2x^{n-2}dx}{nx^{n-1}} = n,$$

which again agrees with (4.9).

Example 4.1.8. Consider the Koszul matrix factorization $\bar{E} = \{x; x^2 + y^2\}$ of the $D_4$-singularity $w = x^3 + xy^2$. The Milnor ring for $w$ is

$$\mathcal{A}_w = R/(3x^2 + y^2, 2xy).$$

The formula (2.5) shows that $\text{Hom}_w^0(\bar{E}, \bar{E}) \simeq R/(x, x^2 + y^2) = R/(x, y^2)$ is 2-dimensional, while $\text{Hom}_w^1(\bar{E}, \bar{E}) = 0$. Hence,

$$\chi(\bar{E}, \bar{E}) = 2.$$
On the other hand, we have
\[ \partial_y \delta_E = 2y \cdot \iota(e^*) \quad \text{and} \quad \partial_x \delta_E = e \wedge + 2x \cdot \iota(e^*), \]
where 1, e is the standard basis of \( \mathcal{E} \). Hence
\[ \text{ch}(\mathcal{E}) = \text{str}(\partial_y \delta_E \circ \partial_x \delta_E) \cdot dx \wedge dy = 2y \cdot dx \wedge dy \in \mathcal{A}_w \otimes dx \wedge dy. \]
Hence,
\[ \langle \text{ch}(\mathcal{E}), \text{ch}(\mathcal{E}) \rangle = - \text{tr}(4y^2) = \text{Res} \left[ -\frac{4y^2 \cdot dx \wedge dy}{(3x^2 + y^2), 2xy} \right]. \]
To compute this generalized residue we change the variables to \( u = \sqrt{3}x + y \) and \( v = \sqrt{3}x - y \), and observe that \( u^2 = (3x^2 + y^2) + \sqrt{3}(2xy) \) and \( v^2 = (3x^2 + y^2) - \sqrt{3}(2xy) \). Therefore, the above expression is equal to
\[ \text{Res} \left[ -\frac{4(u-v)^2}{u^2, v^2} \cdot du \wedge dv \right] = 2 \]
in agreement with the Hirzebruch-Riemann-Roch formula (4.8).

4.2 \( G \)-equivariant case

Let \( G \) be a finite group acting on \( R = k[[x_1, \ldots, x_n]] \), and let \( w \in R \) be a \( G \)-invariant isolated singularity.

In order to compute the canonical bilinear form on \( HH_*(\text{MF}_G(R, w)) \) we first calculate the \( G \times G \)-equivariant Chern character of the \( G \times G \)-equivariant stabilized diagonal (see section 2.5)
\[ \Delta_{G \times G}^{st} = \bigoplus_{g \in G} (\text{id} \times g)^* \Delta_G^{st}. \quad \text{(4.10)} \]
We can use the formula (3.17) for the component \( \text{ch}_{G \times G}(\Delta_{G \times G}^{st})_{(g_1, g_2)} \), where \( (g_1, g_2) \in G \times G \). Note that if \( g_1 \neq g_2 \) then the supertrace in the right-hand side of this formula vanishes because the corresponding operator has no diagonal entries. In the case \( g_1 = g_2 = g \) the element \( (g, g) \in G \times G \) preserves each summand in the decomposition (4.10) and acts on each summand by essentially the same operator. Choosing the variables \( (x_1, \ldots, x_n) \) in such a way that \( g \) acts by linear transformations preserving \( \text{Span}(x_1, \ldots, x_r) \), and \( \text{Span}(x_{r+1}, \ldots, x_n) \) is exactly the subspace of \( g \)-invariants in \( \text{Span}(x_1, \ldots, x_n) \), we obtain
\[ \text{ch}_{G \times G}(\Delta_{G \times G}^{st})_{(g, g)} = [G] \cdot \text{str}(\delta \circ \ldots \circ \delta \circ \ldots \circ \delta \circ g) \big|_{x_1 = \ldots = x_r = y_1 = \ldots = y_r = 0} \quad \text{(4.11)} \]
where the supertrace is computed on the free \( R^e \)-module \( \Lambda^* V \otimes R^e \) of the diagonal matrix factorization \( \Delta_G^{st} = (\Lambda^* V \otimes R^e, \delta) \) and \( V = m/m^2 \). The element \( g \in G \) acts by the automorphism of the exterior algebra induced by the action of \( g \) on \( V \). In particular, \( g(e_j) = e_j \) for \( j > r \), and \( g \) preserves the subalgebra generated by \( e_1, \ldots, e_r \).
As in section 3.2 we use the decomposition \( \delta = \delta_\Delta + \delta_w \), where
\[
\delta_\Delta = \iota (\sum_{j=1}^{n} e_j^* \otimes (y_j - x_j)) \quad \text{and} \quad \delta_w = \{ \sum_{j=1}^{n} e_j \otimes \Delta_j w \} \wedge ?
\]
and \( \Delta_j \) is defined by (2.29). Let us further split \( \delta^\Delta \) into two parts \( \delta_\Delta = \delta^{\leq r}_\Delta + \delta^{\geq r}_\Delta \), where
\[
\delta^{\geq r}_\Delta = \iota (\sum_{j=1}^{n} e_j^* \otimes (y_j - x_j)).
\]
For \( j > r \) we have \( \partial_y \delta_\Delta = \partial_y \delta^{\geq r}_\Delta \) and \( \partial_x \delta_\Delta = \partial_x \delta^{\geq r}_\Delta \). Hence, the operators \( \partial_y \delta \) and \( \partial_x \delta \), as well as \( g \), preserve the filtration
\[
F_p = \bigwedge^{\geq p} (\bigoplus_{i=1}^{r} k \cdot e_i) \otimes \bigwedge^\bullet (\bigoplus_{j=r+1}^{n} k \cdot e_j) \otimes R^e
\]
on \( \bigwedge^\bullet (V) \otimes R^e \). Hence, we can pass to the induced endomorphism of the associated graded space, which allows us to replace \( \delta = \delta_\Delta + \delta_w \) with \( \delta^{\geq r}_\Delta + \delta^{\geq r}_w \), where
\[
\delta^{\geq r}_w = (\sum_{j=r+1}^{n} e_j \otimes \Delta_j w) \wedge ?
\]
One easily checks that the restriction of \( \delta^{\geq r}_\Delta + \delta^{\geq r}_w \) to \( x_1 = \ldots = x_r = y_1 = \ldots = y_r = 0 \) coincides with the differential for the stabilized diagonal \( \Delta^s_{\text{st}} \) of the potential \( \mathbf{w}^g = \mathbf{w}|_{x_1=\ldots=x_r=0} \) (tensored with the identity on \( \bigwedge^\bullet (\bigoplus_{i=1}^{r} k \cdot e_i) \)). Thus (4.11) can be rewritten as
\[
|G| \cdot \text{ch}(\Delta^s_{\text{st}}) \cdot \det[\text{id} - g; V/V^g],
\]
where the determinant is equal to the supertrace of \( g \) acting on \( \bigwedge^\bullet (\bigoplus_{i=1}^{r} k \cdot e_i) \) by the well-known property of the characteristic polynomial.

This brings us to the following \( G \)-equivariant version of the formula for the canonical pairing on the Hochschild homology (cf. Corollary 4.1.3) and of the Hirzebruch-Riemann-Roch formula for matrix factorizations (cf. Theorem 4.1.4). Recall that we have an isomorphism (2.36)
\[
HH_*(MF_G(R, \mathbf{w})) \simeq (\bigoplus_{g \in G} H(\mathbf{w}_g))^G,
\]
where \( \mathbf{w}_g \) is the restriction of the potential \( \mathbf{w} \) to the subspace of \( g \)-invariants (we can assume that \( G \) acts by linear transformations, see section 2.5).

**Theorem 4.2.1.** (i) Let
\[
\langle \cdot, \cdot \rangle : HH_*(MF_G(R, -\mathbf{w})) \otimes HH_*(MF_G(R, \mathbf{w})) \to k
\]
be the canonical bilinear form (1.7) for \( \mathcal{C} = MF_G(R, \mathbf{w}) \). Then for
\[
(h_g)_{g \in G} \in HH_*(MF_G(R, -\mathbf{w})) = (\bigoplus_{g \in G} H(-\mathbf{w}_g))^G \quad \text{and} \quad (h'_g)_{g \in G} \in HH_*(MF_G(R, \mathbf{w})) = (\bigoplus_{g \in G} H(\mathbf{w}_g))^G
\]
and...
we have
\[ \langle (h_g), (h'_g) \rangle = |G|^{-1} \sum_{g \in G} c_g \cdot \langle h_g, h'_g \rangle_{w_g}, \tag{4.12} \]
where \( \langle \cdot, \cdot \rangle_{w_g} \) is the canonical pairing (4.7) for the potential \( w_g \) and
\[ c_g = \det [\text{id} - g; V/V^g]^{-1}, \]
where \( V = \mathfrak{m}/\mathfrak{m}^2 \) and \( V^g \subset V \) is the subspace of \( g \)-invariants.

(ii) For \( \bar{E}, \bar{F} \in \mathcal{M}_G(R, w) \) we have
\[ \chi(\text{Hom}_w(\bar{E}, \bar{F})^G) = |G|^{-1} \sum_{g \in G} c_g \cdot \langle \text{ch}_G(\bar{E})_{g^{-1}}, \text{ch}_G(\bar{F})_g \rangle_{w_g}, \tag{4.13} \]
where \( \text{ch}_G(\bar{E})_g \) is given by (3.17).

More generally, for \( \alpha \in \text{Hom}_w(\bar{E}, \bar{E})^G \) and \( \beta \in \text{Hom}_w(\bar{F}, \bar{F})^G \) we have
\[ \text{str}(m_{\alpha, \beta}) = |G|^{-1} \sum_{g \in G} c_g \cdot \langle \tau^{\bar{E}_{g^{-1}}}(\alpha), \tau^{\bar{F}_g}(\beta) \rangle_{w_g}, \]
where \( m_{\alpha, \beta} \) is the endomorphism of \( \text{Hom}_w(\bar{E}, \bar{F})^G \) given by (1.13).

Proof. (i) Since the canonical bilinear form \( \langle \cdot, \cdot \rangle \) is equal to the inverse of the tensor \( \text{ch}_{G \times G}(\Delta^G_{x \times G}) \) (see (1.9)), the assertion follows from Proposition 4.1.1 and the calculation preceding the Theorem.

(ii) This follows from the generalized Hirzebruch-Riemann-Roch formula (1.12), the equation (4.12) and Lemma 3.3.5.

Example 4.2.2. Assume that the ground field \( k \) contains an \( n \)-th primitive root of unity \( \zeta \). Consider the potential \( w = x^n \in R = k[[x]] \) with the group of symmetries \( G = \mathbb{Z}/n \), where an element \( [m] \in \mathbb{Z}/n \) acts by \( x \mapsto \zeta^m \cdot x \). For each \( i \in \mathbb{Z} \) let us denote by \( \rho_i \) the character of \( G \) given by
\[ \rho_i([m]) = \zeta^{mi}. \]

For \( i = 1, \ldots, n-1 \), let us define the \( G \)-equivariant matrix factorization \( \bar{E}_i \) of \( x^n \) by setting \( (E_i)^0 = \rho_i \otimes R, (E_i)^1 = R \), such that after forgetting the \( G \)-equivariant structure we get \( \bar{E}_i = \{ x^i, x^{n-i} \} \). Note that in this case \( H(w)^G = 0 \), while \( H(w_{[m]})^G = H(w_{[m]}) = k \) for \( [m] \neq [0] \). The formula (3.17) in this case reduces to
\[ \text{ch}_G(\bar{E}_{[m]}) = \text{str}([m]|_{x=0}) \]
for \( [m] \neq [0] \). Thus, we obtain for \( [m] \neq [0] \):
\[ \text{ch}_G(\rho_a \otimes \bar{E}_i)_{[m]} = \rho_{a+i}([m]) - \rho_a([m]) = \zeta^m(\zeta^{mi} - 1). \]

On the other hand, \( c_{[m]} = (1 - \zeta^m)^{-1} \). Thus, we obtain
\[ \chi(\bar{E}_i, \rho_a \otimes \bar{E}_i) = n^{-1} \sum_{m=1}^{n-1} \frac{(\zeta^{-mi} - 1)\zeta^m(\zeta^{mi} - 1)}{1 - \zeta^m}. \]
A straightforward calculation allows us to rewrite the right hand side as
\[ \sum_{j=0}^{i-1} \delta[j][i-j] - \sum_{j=1}^{i} \delta[j][i]. \]

This agrees with the fact that
\[ \text{Hom}^0_w(\bar{E}_i, \bar{E}_i) = \rho_0 \oplus \rho_1 \oplus \ldots \oplus \rho_{i-1} \]
and
\[ \text{Hom}^1_w(\bar{E}_i, \bar{E}_i) = \rho_{-1} \oplus \rho_{-2} \oplus \ldots \oplus \rho_{-i}. \]

### 4.3 Bulk-boundary map for the stabilization of the residue field

Here we will compute the Chern character and the bulk-boundary map (1.11) for the stabilization $k^{\text{st}}$ of the residue field $k = R/m$. Recall that if we present $w$ as
\[ w = x_1 w_1 + \ldots + x_n w_n \text{ for } w_i \in R, \] (4.14)
then $k^{\text{st}} \in \text{MF}(R, w)$ is the Koszul matrix factorization
\[ k^{\text{st}} \{ w_1, \ldots, w_n; x_1, \ldots, x_n \} = (\bigwedge^\bullet (V) \otimes R, \delta), \]
where $V = m/m^2$ and
\[ \delta = \left( \sum_i e_i \otimes w_i \right) \wedge + \iota \left( \sum_i e_i^* \otimes x_i \right) \] (4.15)
with $e_i = x_i \mod m^2 \in V$. In the case when $w$ is preserved by a finite group of automorphisms $G$, we can equip $k^{\text{st}}$ with a $G$-equivariant structure (see (2.26)). The key property of $k^{\text{st}}$ is that for any $\bar{E} \in \text{MF}(R, w)$ there is an isomorphism
\[ \text{Hom}^w_w(\bar{E}, k^{\text{st}}) \simeq (E|_0)^*, \] (4.16)
where $E|_0$ is the restriction of $E$ to the origin (see [6, Lem. 5.3]). In particular, we obtain an isomorphism of $\mathbb{Z}/2$-graded vector spaces
\[ H := \text{Hom}^w_w(k^{\text{st}}, k^{\text{st}}) \simeq \bigwedge^\bullet \left( \bigoplus_{i=1}^n k \cdot e_i^* \right). \] (4.17)

Let us determine the algebra structure on $H$.

**Proposition 4.3.1.** Assume that $w \in m^2$ and choose $w_{ij} \in R$ such that
\[ w_j = \sum_{i=1}^n x_i w_{ij} \text{ for } j = 1, \ldots, n. \] (4.18)

Then for each $j = 1, \ldots, n$, the element
\[ \alpha_j = -\left( \sum_i e_i \otimes w_{ij} \right) \wedge + \iota(e_j^*) \in \text{Hom}^1_w(k^{\text{st}}, k^{\text{st}}) \] (4.19)
is closed. The cohomology classes \([\alpha_i] \in H\) generate \(H\) as a \(k\)-algebra and satisfy the relations
\[
[\alpha_i] \cdot [\alpha_j] + [\alpha_j] \cdot [\alpha_i] = -w_{ij}(0) - w_{ji}(0). \tag{4.20}
\]
In other words, \(H\) is isomorphic to the Clifford algebra associated with the quadratic form given by the matrix \((e^*_i, e^*_j) = -w_{ij}(0) - w_{ji}(0)\).

**Proof.** Simple direct computations show that \(\delta \circ \alpha_j + \alpha_j \circ \delta = 0\) and
\[
\alpha_i \circ \alpha_j + \alpha_j \circ \alpha_i = -(w_{ij} + w_{ji}) \cdot \text{id}.
\]
This shows that \(\alpha_j\) is closed. To deduce (4.20) we combine this with the fact that \(f \cdot \text{id}\) is a coboundary for any \(f \in m\). Indeed, if \(f = x_1 f_1 + \ldots + x_n f_n\) then
\[
f \cdot \text{id} = [\delta^{st}, (\sum_i e_i \otimes f_i) \wedge].
\]
Now it is easy to check that (4.17) becomes an algebra isomorphism, where the right-hand side is equipped with the Clifford algebra structure. \(\square\)

**Corollary 4.3.2.** If \(w \in m^3\) then \(H\) is supercommutative and (4.17) is an isomorphism of algebras.

**Remark 4.3.3.** In fact, the algebra \(H\) is equipped with an \(A_\infty\)-structure. In the case when \(w \in m^3\), the potential \(w\) can be recovered from this \(A_\infty\)-structure up to a change of variables (see [7, Thm. 7.1]).

Now let us calculate the \(G\)-equivariant Chern character of the stabilization of the residue field \(k^{st}\) (where \(G\) is a finite group of symmetries of \(w\)).

**Proposition 4.3.4.** Let \(G\) be a finite group of symmetries of \(w \in R\) and let \(k^{st} \in \text{MF}_G(R, w)\) be the \(G\)-equivariant stabilization of \(k\) (see (2.26)). Then
\[
\text{ch}_G(k^{st})_g = \begin{cases} 
\det(\text{id} - g; V), & \text{if } V^g = 0, \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** We can choose \(w_i\) in (4.14) so that the differential (4.15) is \(G\)-equivariant (see 2.5). When \(V^g = 0\) the formula (3.17) gives
\[
\text{ch}_G(k^{st})_g = \text{str}(g; \wedge^\cdot (V)) = \det(\text{id} - g; V).
\]
Now assume that \(V^g \neq 0\) and let us choose variables \((x_1, \ldots, x_n)\) in such a way that the action of \(g\) is given by
\[
g(x_1, \ldots, x_n) = (\ell_1, \ldots, \ell_r, x_{r+1}, \ldots, x_n),
\]
where $\ell_1, \ldots, \ell_r$ are linear forms in $x_1, \ldots, x_r$, and Span$(x_{r+1}, \ldots, x_n)$ is exactly the subspace of $g$-invariants in Span$(x_1, \ldots, x_n)$. Note that by assumption $r < n$. By (3.17), we need to prove that in this case

$$\text{str}([\partial_n \delta \circ \cdots \circ \partial_1 \delta \circ g]|_{x_1=\ldots=x_r=0}) = 0.$$  

Consider the filtration $\cdots \supset F_p \supset F_{p+1} \supset \cdots$ on $\bigwedge^\bullet(V) \otimes R$ with

$$F_p = \bigwedge^\geq_p \left( \bigoplus_{i=1}^r k \cdot e_i \right) \otimes \bigwedge^\bullet \left( \bigoplus_{j=r+1}^n k \cdot e_j \right) \otimes R.$$  

By passing to the associated graded space, as in the computation of the supertrace in (4.11), we can assume that $r = 0$. Thus, the problem is reduced to the case when $g$ acts trivially, and we have to show that

$$\text{str}_R(\partial_n \delta \circ \cdots \circ \partial_1 \delta) = 0$$

for $n > 0$. Consider the composition

$$\partial_n \delta \circ \cdots \circ \partial_1 \delta = (\iota(e^*_n) + p_n \wedge ?) \circ \cdots \circ (\iota(e^*_1) + p_1 \wedge ?),$$

where

$$p_j = \sum_{i=1}^n e_i \otimes \partial_j w_i.$$  

(4.22)

After expanding the right-hand side of (4.21) only the terms which contain equal amounts of $\iota(e^*_i)$ and $p_j \wedge ?$ factors will contribute to the supertrace. Now the assertion follows from Lemma 4.3.5 below.

**Lemma 4.3.5.** Let $V$ be a $k$-vector space with the basis $e_1, \ldots, e_n$. Suppose that we have operators $A_1, \ldots, A_r$ on $K = \bigwedge^\bullet(V) \otimes R$, such that for each $i = 1, \ldots, r$, either $A_i = \iota(e^*_m)$ for some $m$, or $A_i = (\sum_{j=1}^n e_j \otimes f_j) \wedge ?$ for some $f_1, \ldots, f_n \in R$. Then

$$\text{str}_R(A_1 \circ \cdots \circ A_r) = 0$$

unless all the operators $\iota(e^*_1), \ldots, \iota(e^*_n)$ appear among $A_1, \ldots, A_r$.

**Proof.** Let $I \subset \{1, \ldots, n\}$ be the set of all $i$ such that $\iota(e^*_i)$ appears among $A_1, \ldots, A_r$. Consider the decomposition

$$V = V_I \oplus V'_I$$

where $V_I := \bigoplus_{i \in I} k \cdot e_i$ and $V'_I = \bigoplus_{j \notin I} k \cdot e_j$.

Then each operator $A_i$ preserves the filtration

$$\bigwedge^\geq w (V'_I) \otimes \bigwedge^\bullet (V_I) \otimes R.$$
on $\bigwedge^\bullet(V) \otimes R$. After passing to the associated graded space, $A_i$ induces an operator of the form $\text{id} \otimes \tilde{A}_i$, where $\tilde{A}_i$ acts on $\bigwedge^\bullet(V_I) \otimes R$. Thus, we obtain

$$\text{str}_R(A_1 \circ \ldots \circ A_r) = \text{str}_k(\text{id}; \bigwedge^\bullet V_I) \cdot \text{str}_R(\tilde{A}_1 \circ \ldots \circ \tilde{A}_r; \bigwedge^\bullet V_I \otimes R) = 0$$

provided $\dim V_I > 0$, i.e., $I$ is a proper subset of $\{1, \ldots, n\}$.

**Remark 4.3.6.** For every $G$-equivariant matrix factorization $\tilde{E} = (E, \delta)$ of $w$ and a representation $\rho$ of $G$ there is an isomorphism

$$\text{Hom}_w(\tilde{E}, \text{k}^\ast \otimes \rho)^G \simeq \text{Hom}(E|_0, \rho)^G,$$

where $E|_0$ is the restriction of $E$ to the origin (see [6, Lem. 5.3]). The Hirzebruch-Riemann-Roch formula (4.13) together with the formula (3.17) and the above Proposition give the following expression for the Euler characteristic of the left-hand side of (4.23):

$$\chi(\text{Hom}_w(\tilde{E}, \text{k}^\ast \otimes \rho)^G) = |G|^{-1} \cdot \sum_{g, V^g = 0} \text{ch}_G(\tilde{E}^g) \cdot \text{tr}(g; \rho) = |G|^{-1} \cdot \sum_{g, V^g = 0} \text{str}(g^{-1}; E^g|_0) \cdot \text{tr}(g; \rho).$$

This is compatible with the standard formula for the Euler characteristic of the right-hand side of (4.23) because, as we will show,

$$\text{str}_k(g; E|_0) = 0 \text{ when } V^g \neq 0.$$

Indeed, we can assume that $g$ acts by linear transformations. Furthermore, replacing $G$ by the cyclic subgroup generated by $g$, the matrix factorization $\tilde{E}$ by its restriction to the subspace of $g$-invariants and the potential $w$ by $w_g$, we can assume that $G$ acts trivially on $R$. Then we have a decomposition

$$\tilde{E} = \bigoplus_{i=1}^N \rho_i \otimes \tilde{E}_i,$$

where $\tilde{E}_i$ are (non-equivariant) matrix factorizations of $w_g$ and $\rho_i$ are representations of $G$. Since $w_g \neq 0$, the superdimension of each $\mathbb{Z}_2$-graded space $\tilde{E}_i|_0$ vanishes. Hence, we have

$$\text{str}_k(g; E|_0) = \sum_{i=1}^N \text{tr}(g; \rho_i) \cdot \text{str}_k(\text{id}; \tilde{E}_i|_0) = 0.$$

Now let us consider the non-equivariant situation. We are going to calculate the boundary-bulk map for $k^\ast$ assuming that $w \in \mathfrak{m}^2$. Recall that by Proposition 4.3.1, the algebra $\text{Hom}_w(k^\ast, k^\ast)$ can be identified with a certain Clifford algebra.

**Proposition 4.3.7.** Let $w \in \mathfrak{m}^2$ and elements $(w_i)$ and $(w_{ij})$ are chosen as in (4.14) and (4.18). Let $[\alpha_1], \ldots, [\alpha_n]$ be the generators of the algebra $\text{Hom}_w(k^\ast, k^\ast)$ given by (4.19). Then the boundary-bulk map for $k^\ast$ is given by

$$\tau^{k^\ast}([\alpha_i] \circ \ldots \circ [\alpha_{ir}]) = 0 \text{ for } r < n.$$
and
\[ \tau^{kst}([\alpha_1] \circ \ldots \circ [\alpha_n]) = \frac{\text{Hess}(w)}{\mu} \cdot d\mathbf{x} \mod \partial_w \cdot d\mathbf{x}, \]
where \( \text{Hess}(w) = \det(\partial_i \partial_j w) \) is the Hessian and \( \mu = \dim A_w \) is the Milnor number of \( w \).

**Proof.** Recall that
\[ \alpha_j = \iota(e^*_j) - s_j \bigwedge ?, \]
where
\[ s_j = \sum_{i=1}^n e_i \otimes w_{ij}. \]

Hence, the formula (3.13) gives in our case
\[ (-1)^\binom{r}{2} \cdot \tau^{kst}([\alpha_1] \circ \ldots \circ [\alpha_n]) = \sum_{I \subseteq \{1, \ldots, n\}} \text{str}_R(A_I(p_1, 1) \ldots A_I(p_n, 1) A_{I^c}(-s_1, 1) \ldots A_{I^c}(-s_n, n)) \cdot d\mathbf{x}, \]
where \( p_j \) is given by (4.22). By Lemma 4.3.5, this expression is zero for \( r < n \). If \( (i_1, \ldots, i_r) = (1, \ldots, n) \), we get
\[ (-1)^\binom{r}{2} \cdot \tau^{kst}([\alpha_1] \circ \ldots \circ [\alpha_n]) = \sum_{I \subseteq \{1, \ldots, n\}} \text{str}_R(A_I(p_1, 1) \ldots A_I(p_n, n) A_{I^c}(-s_1, 1) \ldots A_{I^c}(-s_n, n)) \cdot d\mathbf{x}, \]
where \( I^c \) denotes the complement of \( I \) and
\[ A_I(v, i) = \begin{cases} \iota(e^*_i), & i \in I, \\ v\wedge ?, & i \notin I. \end{cases} \]

Using Lemma 4.3.5 again we see that we can skew-permute the operators in the product under the supertrace in (4.24). Thus, exchanging \( A_I(p_i, i) \) with \( A_{I^c}(-s_i, i) \) for each \( i \in I \) produces the factor \((-1)^{|I|}\), and we get
\[ \text{str}_R(A_I(p_1, 1) \ldots A_I(p_n, n) A_{I^c}(-s_1, 1) \ldots A_{I^c}(-s_n, n)) = \text{str}_R((\iota(v(1, 1) \wedge \ldots \wedge v(I, n)\wedge ?) \circ \iota(e^*_1) \circ \ldots \circ \iota(e^*_n))) = (-1)^\binom{r}{2} \cdot \det(v(I, 1), \ldots, v(I, n)), \]
where
\[ v(I, i) = \begin{cases} s_i, & i \in I \\ p_i, & i \notin I. \end{cases} \]

Summing over all subsets \( I \) in \( \{1, \ldots, n\} \) we obtain
\[ \tau^{kst}([\alpha_1] \circ \ldots \circ [\alpha_n]) = \det(s_1 + p_1, \ldots, s_n + p_n) \cdot d\mathbf{x} = \det(\partial_j w + w_{ij}) \cdot d\mathbf{x} \mod \partial_w \cdot d\mathbf{x}. \]

Using (4.14) and (4.18) we get
\[ \partial_j w = \sum_{i=1}^n x_i (\partial_j w_i + w_{ij}). \]

Now the assertion follows from Lemma 4.3.8 below applied to \( f_i = \partial_i w \). \( \square \)
Lemma 4.3.8. Let \( f_1, \ldots, f_n \in \mathfrak{m} \subset R = k[[x_1, \ldots, x_n]] \) be a regular sequence. Choose \( f_{ij} \in R \) such that
\[
f_j = \sum_{i=1}^{n} x_i f_{ij}.
\] (4.25)
Then
\[
det(\partial_i f_j) = \mu \cdot det(f_{ij}) \mod (f_1, \ldots, f_n),
\]
where \( \mu = \dim_k(R/(f_1, \ldots, f_n)) \).

Proof. Recall that by the general residue theory (see [11, III.9]), for a regular sequence \( f_1, \ldots, f_n \in \mathfrak{m} \subset R \) the ring \( R/(f_1, \ldots, f_n) \) is Gorenstein with the socle generated by the Jacobian \( \det(\partial_i f_j) \). Furthermore, we have
\[
\text{Res} \left[ \left. \frac{\det(\partial_i f_j) \cdot dx_1 \wedge \ldots \wedge dx_n}{f_1, \ldots, f_n} \right] \right. = \mu. \tag{4.26}
\]
On the other hand, using the adjoint matrix to \( (f_{ij}) \) one can immediately check that \( \det(f_{ij})x_i \) belongs to \( (f_1, \ldots, f_n) \), so \( \det(f_{ij}) \) belongs to the socle of \( R/(f_1, \ldots, f_n) \). Also, using (4.25) and the transformation law for the residue we obtain that
\[
\text{Res} \left[ \left. \frac{\det(f_{ij}) \cdot dx_1 \wedge \ldots \wedge dx_n}{f_1, \ldots, f_n} \right] \right. = 1. \tag{4.27}
\]
Since the socle is one-dimensional, comparing (4.26) with (4.27) we obtain the required formula.

4.4 Graded matrix factorizations

Let \( L \) be a commutative group with a fixed element \( \ell \in L \) such that the quotient \( L/\langle \ell \rangle \) is finite. Assume that the ring \( R = k[[x_1, \ldots, x_n]] \) is \( L \)-graded in such a way that each \( x_i \) is homogeneous. An \( L \)-graded free \( R \)-module is a free \( R \)-module equipped with an \( L \)-grading such that the basis elements are homogeneous.

Definition 4.4.1. For a potential \( w \in R \), homogeneous of degree \( 2\ell \), an \( L \)-graded matrix factorization of \( w \) is a pair of finitely generated \( L \)-graded free \( R \)-modules \( (E, \delta_E) \) equipped with \( R \)-linear maps \( \delta_0 : E^0 \to E^1 \) and \( \delta_1 : E^1 \to E^0 \), homogeneous of degree \( \ell \), such that \( \delta_0 \delta_1 = w \cdot \text{id} \) and \( \delta_1 \delta_0 = w \cdot \text{id} \).

Equivalently, we can view an \( L \)-graded matrix factorization as a \( \mathbb{Z}/2 \times L \)-graded free \( R \)-module \( E = E^0 \oplus E^1 \), equipped with an endomorphism \( \delta \) of bidegree \( (1, \ell) \in \mathbb{Z}/2 \times L \), such that \( \delta^2 = w \cdot \text{id} \).

For a pair of \( L \)-graded matrix factorizations \( \bar{E} = (E, \delta_E) \) and \( \bar{F} = (F, \delta_F) \) we define a \( \mathbb{Z} \)-graded complex \( \text{Hom}_{w,L}(\bar{E}, \bar{F}) \) by setting
\[
\text{Hom}_{w,L}(\bar{E}, \bar{F})^i = \text{Hom}^i_{\text{gr-Mod}_R}(E, F(i \cdot \ell)),
\]

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where $\text{Hom}_{\text{gr-Mod}_R}^0$ (resp., $\text{Hom}_{\text{gr-Mod}_R}^1$) is the space of morphisms of $\mathbb{Z}/2 \times L$-graded $R$-modules of bidegree $(0,0)$ (resp., $(1,0)$). The differential $d$ on $\mathcal{H}om_{w,L}(\bar{E}, \bar{F})$ is given by the usual formula (2.3). Note that the requirement that $\delta_E$ and $\delta_F$ have bidegree $(1,\ell) \in \mathbb{Z}/2 \times L$ implies that $d$ has degree 1. In this way we get a dg-category of $L$-graded matrix factorizations.

Consider the finite commutative group $G = L/\langle 2 \ell \rangle$, and let $G^* = \text{Hom}(G, k^*)$ be its dual group (we assume that $k$ contains a primitive root of unity of order $|G|$). An $L$-grading on a vector space $V$ induces a natural action of $G^*$ on $V$, so that $\gamma \in G^*$ acts on $V_l$, where $l \in L$, by the scalar multiplication with $\gamma(l \mod(2\ell)) \in k^*$. In particular, we have an action of $G^*$ on $R$ by algebra automorphisms. This action preserves $w$, since $w$ has degree $2\ell$.

Now suppose we have an $L$-graded matrix factorization $\bar{E} = (E, \delta_E)$. Then the $L$-grading on $E$ induces a $G^*$-action, so $\bar{E}$ can be viewed as a $G^*$-equivariant matrix factorization of $w$. It is easy to see that for a pair of $L$-graded matrix factorizations $\bar{E}$ and $\bar{F}$ one has an equality of $\mathbb{Z}/2$-graded complexes

$$\mathcal{H}om_{w,L}(\bar{E}, \bar{F}) = \mathcal{H}om_w(\bar{E}, \bar{F})^{G^*}.$$ 

Thus, we can apply Theorem 4.2.1 to calculate

$$\chi(\bar{E}, \bar{F}) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(\mathcal{H}om_{w,L}(\bar{E}, \bar{F}))$$

for $L$-graded matrix factorizations $\bar{E}$ and $\bar{F}$.

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