DERIVATION OF THE AHARANOV-BOHM POTENTIAL IN THE GINZBURG-LANDAU MODEL

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ABSTRACT. Starting from a step magnetic field, we derive the Aharonov-Bohm magnetic potential within the Ginzburg-Landau model of superconductivity. We then study the transition from normal to superconducting solutions and obtain oscillations consistent with the Little-Parks effect. Our results provide an example where the transition between superconducting and normal solutions is not monotone, and present a new aspect of magnetic steps besides their favoring of the celebrated edge states.

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1. INTRODUCTION

1.1. The Ginzburg-Landau model. Ginzburg and Landau introduced a phenomenological model of the response of superconducting materials to applied magnetic fields. The behavior of the material is described via the critical configurations of the Ginzburg-Landau functional, defined as follows,

\[ E[\psi, A] = \int_{\Omega} \left( |(\nabla - iA)\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) \, dx + \int_{\Omega} |\text{curl} (A - \mathfrak{f})|^2 \, dx, \tag{1.1} \]

where

Date: September 12, 2019.
Mathematics Subject Classification (2010): 35B40, 35P15, 35Q56.
The variational space for this functional depends on the nature of the magnetic potential, which satisfies

\[ |\psi|^2 \] measures the density of the superconducting electrons;

\[ \text{curl } \mathbf{A} \] measures the induced magnetic field in the sample.

In the two dimensional case we denote by \( \text{curl } \mathbf{A} = \partial_1 A_2 - \partial_2 A_1 \). The parameter \( \kappa \) will be fixed throughout this paper. For this reason, we skip it from the notation. On the opposite we will consider the variation of the parameter \( h > 0 \), that we will introduce in order to display the intensity of the applied magnetic field as follows. We rescale the Ginzburg-Landau functional in \( \mathbb{R}^2 \) by writing \( \mathcal{A} = h \mathbf{A} \) and \( \mathcal{F} = h \mathbf{F} \). Hence, we arrive at the new functional

\[
\mathcal{E}_h(\psi, \mathbf{A}) = \int_{\Omega} \left( |(\nabla - ih \mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) \, dx + h^2 \int_{\Omega} |\text{curl } (\mathbf{A} - \mathbf{F})|^2 \, dx. \tag{1.2}
\]

The variational space for this functional depends on the nature of the magnetic potential \( \mathbf{F} \). In fact:

- If \( \mathbf{F} \in H^1(\Omega; \mathbb{R}^2) \), then \( \mathcal{E}_h(\psi, \mathbf{A}) \) is well defined for all \( (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \).
- If \( \mathbf{F} \in L^p_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) \cap L^2_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2) \) for some \( q \in (1, 2) \), then \( \mathcal{E}_h(\psi, \mathbf{A}) \) is well defined for all \( \psi \in H^1_{\text{hF}}(\Omega; \mathbb{C}) \) and \( \mathbf{A} \in H^1(\Omega; \mathbb{R}^2) + \mathbf{F} \), where \( \psi \in H^1_{\text{hF}}(\Omega; \mathbb{C}) \) means that \( (\nabla - ih \mathbf{F})\psi \in L^2(\Omega; \mathbb{C}^2) \) and \( \psi \in L^2(\Omega; \mathbb{C}) \) (see Sec. 2 below for the precise definition of this space).

Hereafter the spaces of real-valued functions, complex-valued functions, and real vector-valued functions are denoted by \( L^p(\Omega), L^p(\Omega; \mathbb{C}), L^p(\Omega; \mathbb{R}^2) \) respectively. However the norms in these spaces are denoted by the same notation \( \| \cdot \|_{L^p(\Omega)} \).

The case of a uniform applied magnetic field, \( \text{curl } \mathbf{F} = 1 \), has been extensively studied in the literature (see the two monographs [1] [23] and the references therein), particularly in the framework of critical magnetic fields associated with the various phase transitions in the Ginzburg-Landau model. Recently, the analysis of non-uniform applied magnetic fields matches with some interesting physical phenomena like the Little-Parks effect [19] and the presence of edge states that concentrate on curves [1] [4] [8] [21]. More precisely, non-uniform magnetic fields could produce defects of topological nature [8]. We would like to address this kind of behavior by proving that a large uniform magnetic field applied on a small region of the sample (magnetic step) produces an effective energy involving the Aharonov-Bohm potential (see Theorem 1.3 in this paper); the later energy shows oscillations in the spirit of the Little-Parks effect (see Corollary 1.5 in this paper). Our contribution displays a new example where normal/superconducting oscillations exist, and at the same, presents a new aspect of magnetic steps besides their celebrated feature of producing edge states.

In this paper, we work under the hypothesis:

- \( \Omega \) is open, bounded, simply connected domain and with a boundary of class \( C^2 \);
- \( 0 \in \Omega \).

We fix \( \varepsilon_0 > 0 \) so that \( D(0, \varepsilon_0) := \{ x \in \mathbb{R}^2, \, |x| < \varepsilon_0 \} \subset \Omega \).

1.2. Aharonov-Bohm potential. This is the vector field

\[
\mathbf{F}_{AB}(x) = \left( \frac{-x_2}{2\pi |x|^2}, \frac{x_1}{2\pi |x|^2} \right) \quad (x = (x_1, x_2) \in \mathbb{R}^2), \tag{1.3}
\]

which satisfies

\[
\mathbf{F}_{AB} \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \quad \forall p \in [1, 2) \quad \text{and} \quad \text{curl } \mathbf{F}_{AB} = \delta_0 \text{ in } \mathcal{D}'(\mathbb{R}^2). \tag{1.4}
\]
1.3. Magnetic steps.

Magnetic steps, and the ground state energy

\[ E_{AB}(h) = \inf \{ E_{AB}(\psi, A) : (\psi, A) \in \mathcal{H}_{AB} \}, \]

where \( E_{AB}(\psi, A) \) is defined by (1.2) for \( \mathbf{F} = \mathbf{F}_{AB} \), i.e.

\[ E_{AB}(\psi, A) = \int_{\Omega} \left( \left| (\nabla - i h A) \psi \right|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) \, dx + h^2 \int_{\Omega} |\text{curl} (A - \mathbf{F}_{AB})|^2 \, dx. \]  

Note that \( E_{AB}(h) > -\infty \) because

\[ E_{AB}(\psi, A) = \int_{\Omega} \left( \left| (\nabla - i h A) \psi \right|^2 + \frac{\kappa^2}{2} (1 - |\psi|^2)^2 + h^2 |\text{curl} (A - \mathbf{F}_{AB})|^2 \right) \, dx - \frac{\kappa^2}{2} |\Omega|. \]

1.4. Magnetic steps to Aharonov-Bohm. For all \( \varepsilon \in (0, \varepsilon_0) \), define the vector field

\[ \mathbf{F}_\varepsilon(x) = \begin{cases} \mathbf{F}_{AB}(x) & \text{if } |x| > \varepsilon, \\ \frac{1}{\pi \varepsilon^2} \mathbf{A}_0(x) & \text{if } |x| < \varepsilon, \end{cases} \]

where \( \mathbf{A}_0(x) := \frac{1}{2}(-x_2, x_1) \). Note that \( \mathbf{F}_\varepsilon \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) \) and generates the following magnetic field

\[ B_\varepsilon := \text{curl} \mathbf{F}_\varepsilon = \frac{1}{\pi \varepsilon^2} \mathbf{1}_{D(0,\varepsilon)}, \]

which is an example of a magnetic step.

One interesting feature of magnetic steps is their manifestation of quantum mechanical edge states, a celebrated phenomenon extensively studied for linear models \[22, 14, 5\]. For superconductors with large Ginzburg-Landau parameter, magnetic steps also produce edge states in a non-linear framework \[3\] and enjoy an interesting analogy with piece-wise smooth domains \[1\] at the onset of superconductivity.

1.4. From magnetic steps to Aharonov-Bohm. We show a new feature of magnetic steps related to the Aharonov-Bohm potential. The connection can be seen formally by comparing (1.4) and

\[ B_\varepsilon \to \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2) \quad \text{as } \varepsilon \to 0. \]

To make this formal comparison precise, we introduce the following space

\[ \mathcal{H} = H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2), \]

and the following ground state energy

\[ E_{\varepsilon}(h) = \inf \{ E_{\varepsilon}(\psi, A) : (\psi, A) \in \mathcal{H} \}, \]

where \( E_{\varepsilon}(\psi, A) \) is defined by (1.2) for \( \mathbf{F} = \mathbf{F}_{\varepsilon} \), i.e.

\[ E_{\varepsilon}(\psi, A) = \int_{\Omega} \left( \left| (\nabla - i h A) \psi \right|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) \, dx + h^2 \int_{\Omega} |\text{curl} (A - \mathbf{F}_{\varepsilon})|^2 \, dx. \]  

The point now is to compare the ground state energies \( E_{AB}(h) \) and \( E_{\varepsilon}(h) \) (see (1.13) and (1.6)).

**Theorem 1.1.** For all \( h > 0 \), it holds,

\[ \lim_{\varepsilon \to 0^+} E_{\varepsilon}(h) = E_{AB}(h). \]

Furthermore, if \( h \in 2\pi \mathbb{Z} \),

\[ E_{AB}(h) = -\frac{\kappa^2}{2} |\Omega|. \]
and the energy $E_{AB_h}(\psi, A)$ in (1.7) is minimized for
\[(\psi := e^{i\frac{r\theta}{2}}, A := F_{AB}),\]
where $(r, \theta)$ denote the polar coordinates in $\mathbb{R}^2$.

1.5. Transition to the normal state. Given $\kappa, h > 0$, a critical point $(\psi, A)_{\kappa, h}$ of (1.7) is said to be a normal solution (or trivial solution) if $\psi = 0$ everywhere in $\Omega$; if $\psi$ is not identically 0 on $\Omega$, the critical point is said to be a superconducting solution.

In generic situations, all critical points become normal solutions after sufficiently increasing the intensity of the applied magnetic field \cite{9, 20}. For the Aharonov-Bohm potential, we will prove that such transition does not occur. In fact, every critical point $(\psi, A)_{\kappa, h}$ displays an oscillatory behavior by transitioning back and forth from normal to superconducting solutions. Examples of this sort are rare in the literature and are usually observed in non-simply connected domains \cite{S, 12}. On the opposite, generically, one observes a monotone transition from normal to superconducting solutions \cite{9, 7} on simply-connected domains.

For all $h \geq 0$, we introduce the eigenvalue
\[
\lambda_{AB}(h, \Omega) = \inf \left\{ \int_\Omega |(\nabla - ihF_{AB})u|^2 \, dx : u \in H^1_{0, F_{AB}}(\Omega; \mathbb{C}), \int_\Omega |u|^2 \, dx = 1 \right\}.
\]
This is the eigenvalue of the Aharonov-Bohm operator, $-(\nabla - ihF_{AB})^2$, defined by the Friedrichs theorem starting from the closed quadratic form (see \cite{3})
\[
H^1_{1, F_{AB}}(\Omega; \mathbb{C}) \ni u \mapsto \int_\Omega |(\nabla - ihF_{AB})u|^2 \, dx.
\]
The Aharonov-Bohm operator is self-adjoint in $L^2(\Omega; \mathbb{C})$ and has a compact resolvent, hence the eigenvalue $\lambda_{AB}(h, \Omega)$ is in the discrete spectrum (see \cite{17}).

Notice that
\[
\lambda_{AB}(h, \Omega) < \kappa^2 \implies \text{every minimizer of } E_{AB_h} \text{ is a superconducting solution}.
\]
This follows by using the test configuration $(tu_h, F_{AB})$, with $t$ sufficiently small and $u_h$ an eigenfunction of $\lambda_{AB}(h)$, so that
\[
E_{AB}(h) \leq E_{AB_h}(tu_h, F_{AB}) < 0.
\]
The result in Theorem 1.2 below complements (1.15). Its statement involves the constant $C_*(\Omega)$ introduced below, whose definition is related to the following space
\[
H^1_{0, \Omega}(\Omega, \text{div}0) = \{ u \in H^1(\Omega; \mathbb{R}^2) : \text{div} u = 0 \text{ in } \Omega, \; \nu \cdot u = 0 \text{ on } \partial \Omega \},
\]
where $\nu$ is the unit outward normal vector on $\partial \Omega$.

We introduce the following constant
\[
C_*(\Omega) = \frac{|\Omega|}{\lambda^D(\Omega)} \left( 2 + |\Omega|^{1/2}m_*(\Omega) \right),
\]
where
\[
m_*(\Omega) = \inf_{a \in H^1_{0,\Omega}(\Omega, \text{div}0) \setminus \{0\}} \frac{||\text{curl} a||^2_{L^2(\Omega)}}{||a||^2_{L^2(\Omega)}} \quad \text{and} \quad \lambda^D(\Omega) = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{||\nabla u||^2_{L^2(\Omega)}}{||u||^2_{L^2(\Omega)}}.
\]
In light of the compact embedding $H^1(\Omega; \mathbb{R}^2) \hookrightarrow L^4(\Omega; \mathbb{R}^2)$ and the celebrated curl-div inequality (see \cite[Prop. D.2.1]{4}), $||a||_{H^1(\Omega)} \leq C ||\text{curl} a||_{L^2(\Omega)}$, holding in $H^1_{0,\Omega}(\Omega, \text{div}0)$, we see that $m_*(\Omega) > 0$. That $\lambda^D(\Omega) > 0$ is a consequence of the Poincaré inequality; this is also the principal eigenvalue of the Dirichlet Laplacian on $\Omega$.

Theorem 1.2. If $\kappa$ and $h$ satisfy
\[0 < \kappa^2 < (1 + C_*(\Omega))^{-1} \lambda_{AB}(h, \Omega),\]
then every critical point $(\psi, A)_{\kappa, h}$ of the functional $E_{AB_h}$ is a normal solution.
Remark 1.3. Based on the existing results in the case of a uniform applied magnetic field and large Ginzburg-Landau parameter \(20, 6\), one would expect that the result in Theorem 1.2 holds for \(\kappa^2 \leq \lambda_{AB}(h, \Omega)\). However, in our present situation, the result is valid without restriction on the asymptotic behavior of \((\kappa, h)\), and for this reason, the minimizing magnetic potential \(A\) is no more close to the applied potential \(F_{AB}\), so that the constant \(C_*(\Omega)\) can not be neglected in our estimates.

The behavior of the eigenvalue \(\lambda_{AB}(h, \Omega)\), displayed in Theorem 1.4 below, is reminiscent of the one in a domain with a single hole \([11]\).

Theorem 1.4. The function \(h \mapsto \lambda_{AB}(h, \Omega)\) is continuous, \(2\pi\)-periodic and satisfies
\[
0 = \lambda_{AB}(0, \Omega) < \lambda_{AB}(h) \leq \lambda_{AB}(\pi, \Omega), \quad \forall h \in (0, 2\pi).
\]

In the special case where \(\Omega\) is the disc \(D(0, R)\), Theorem 1.4 follows easily by decomposition into Fourier modes and separation of variables \([15, \text{Prop. 2.1}]\).

Combining the results in \([1.15]\), Theorems 1.2 and 1.4 we can display the oscillations in the critical points of the functional \(E_{AB}\).

Corollary 1.5. Assume that \(\kappa\) satisfies
\[
0 < \kappa^2 < (1 + C_*(\Omega))\lambda_{AB}(\pi, \Omega).
\]

Consider the sequence \((h_n = \pi n)_{n \geq 1}\). It holds the following.

1. If \(n\) is even, then every minimizer \((\psi, A)_{\kappa, h_n}\) of \(E_{AB_n}\) is a superconducting solution.
2. If \(n\) is odd, then every critical point \((\psi, A)_{\kappa, h_n}\) of \(E_{AB_n}\) is a normal solution.

In the disc case, we recover the result obtained in our previous work \([15]\). However the result in \([15]\) is valid on a disc under the weaker condition when \(\kappa^2 < c_2^2\lambda_{AB}(\pi, D(0, R))\), where \(c_2\) is a constant satisfying \(c_2^2 < 1 + C_*(D(0, R))\), see \([1.17]\).

2. Magnetic Sobolev space

2.1. Hypotheses. Throughout this section, we assume that

\(U \subset \mathbb{R}^2\) is bounded, simply connected and with a \(C^2\) boundary;
\(I = \{x_1, \cdots, x_N\} \subset U\);
\(U_N = U \setminus \{x_1, \cdots, x_N\}\);
\(f \in L^q(U; \mathbb{R}^2) \cap L^2_{\text{loc}}(U_N; \mathbb{R}^2)\) is a given vector field, with \(1 < q < 2\);
\(\exists \varepsilon_0 \in (0, 1), \forall i \in \{1, \cdots, N\}, D(x_i, \varepsilon_0) \subset U\).

For all \(\varepsilon \in (0, \varepsilon_0)\), we set \(I_\varepsilon = \bigcup_{i=1}^N \overline{D(x_i, \varepsilon)}\); clearly \(I_\varepsilon \subset U\).

2.2. Definition of the magnetic Sobolev space. If \(\psi \in L^2(U; \mathbb{C})\) and \(f \in L^2_{\text{loc}}(U; \mathbb{R}^2)\), then \(f\psi \in L^1_{\text{loc}}(U; \mathbb{C}^2)\) and can be viewed as a distribution, i.e. \(f\psi \in \mathcal{D}'(U; \mathbb{C}^2)\). This allows us to define the magnetic Sobolev space \(H^1_1(U; \mathbb{C})\) in a straightforward manner. For a function \(\psi \in L^2(U; \mathbb{C})\) to be in \(H^1_1(U; \mathbb{C})\), we simply demand that the distribution \(\nabla \psi - if\psi\) is a function belonging to \(L^2(U; \mathbb{C}^2)\).

As long as \(f \not\in L^2_{\text{loc}}(U; \mathbb{R}^2)\), we can not insure any more that \(f\psi \in \mathcal{D}'(U; \mathbb{C}^2)\), and the condition \(\nabla \psi - if\psi \in L^2(U; \mathbb{C}^2)\) will be meaningless, since we can not assign a distributional sense of \(\nabla \psi - if\psi\) in the whole domain \(U\).

However, working under the hypotheses in Sec. 2.1 we can define the magnetic Sobolev space \(H^1_1(U_N; \mathbb{C})\) since we know that \(f \in L^2_{\text{loc}}(U_N; \mathbb{R}^2)\), where \(U_N = U \setminus \{x_1, \cdots, x_N\}\). Note that we do not distinguish between the spaces \(L^2(U)\) and \(L^2(U_N)\), because the set \(I = \{x_1, \cdots, x_N\}\) is finite.

Let us examine more closely this particular situation. Pick \(\psi \in L^2(U; \mathbb{C})\); since \(f \in L^2_{\text{loc}}(U_N; \mathbb{R}^2)\), we can view \(f\psi\) as a distribution on \(U_N\), i.e. \(f\psi \in \mathcal{D}'(U_N; \mathbb{C}^2)\); the condition \((\nabla - if)\psi \in L^2(U; \mathbb{C}^2)\) then means
\[ \exists g \in L^2(U; \mathbb{C}^2), \forall \varphi \in C_c^\infty(U_N; \mathbb{C}), \int_U \psi (\partial_j + i f_j) \varphi \, dx = - \int_U g_j \varphi \, dx, \quad j = 1, 2. \quad (2.1) \]

The problem is that we cannot utilise a test function \( \varphi \in C_c^\infty(U; \mathbb{C}) \), since the integral of \( f \psi \varphi \) on \( U \) would not make sense.

That is the motivation for the following general definition of the magnetic Sobolev space.

**Definition 2.1.** Under the Hypotheses of Sec. 2.1, we define the corresponding magnetic Sobolev space on \( U \) as follows

\[ H_1^f(U; \mathbb{C}) = \{ \psi \in L^2(U; \mathbb{C}) : (2.1) \text{ holds} \}. \]

**Remark 2.2.** The characterization of Sobolev spaces by means of the notion of absolute continuity on lines yields the pleasant property that \( W^{1,q}(U) = W^{1,q}(U_N) \) for all \( q \in [1, +\infty) \) (see [10 Thm. 6.1.3]). Consequently, if \( f \in L^2_{\text{loc}}(U; \mathbb{R}^2) \), then Definition 2.1 coincides with the usual one, i.e. the following condition holds for \( \psi \in H_1^f(U; \mathbb{C}) \) (compare with (2.1)):

\[ \exists g \in L^2(U; \mathbb{C}^2), \forall \varphi \in C_c^\infty(U; \mathbb{C}), \int_U \psi (\partial_j + i f_j) \varphi \, dx = - \int_U g_j \varphi \, dx, \quad j = 1, 2. \quad (2.2) \]

Indeed, supposing that \( \psi \in L^2(U; \mathbb{C}) \) such that \( g = (\nabla - i f)\psi \) in \( \mathcal{D}'(U_N; \mathbb{C}^2) \), we see that \( \psi \in W^{1,1}(U_N \cap K; \mathbb{C}) = W^{1,1}(U \cap K; \mathbb{C}) \) for any compact set \( K \subset U \), and \( \nabla \psi = g + if \psi \) becomes a locally integrable function on \( U \), so \( (\nabla - i f)\psi = g \) in \( \mathcal{D}'(U; \mathbb{C}^2) \) too.

Fortunately, our hypotheses on \( f \) will allow us to view \( f \psi \) as a distribution on \( U \) whenever \( \psi \in H_1^f(U; \mathbb{C}) \), thereby overcoming the technical difficulties in Definition 2.1. This is due to the following lemma.

**Lemma 2.3.** Under the hypotheses in Sec. 2.1, \( H_1^f(U; \mathbb{C}) \subset L^p(U; \mathbb{C}) \), for all \( p \in [2, +\infty) \). Furthermore, for all \( \psi \in H_1^f(U; \mathbb{C}) \), we have

\[ f \psi \in L^1(U, \mathbb{C}^2), \quad |\psi| \in H^1(U) \quad \text{and} \quad \|\nabla |\psi|\|_{L^2(U)} \leq \| (\nabla - i f)\psi \|_{L^2(U)}. \]

**Remark 2.4.** In light of Lemma 2.3, we see that \( f \psi \in \mathcal{D}'(U; \mathbb{C}^2) \), and now, we can interpret the condition \( (\nabla - i f)\psi \in L^2(U; \mathbb{C}^2) \) as follows

\[ \exists g \in L^2(U), \quad (\nabla - i f)\psi = g \text{ in } \mathcal{D}'(U). \]

We then can define the magnetic Sobolev space \( H_1^f(U; \mathbb{C}) \) as follows

\[ H_1^f(U; \mathbb{C}) = \{ \psi \in L^2(U; \mathbb{C}) : (2.2) \text{ holds} \}. \quad (2.3) \]

**Proof of Lemma 2.3.**

Let \( \psi \in H_1^f(U; \mathbb{C}) \) and consider the distributional derivative, \( \mathbf{u} := \nabla |\psi| \), in \( \mathcal{D}'(U_N) \). We first check that \( \mathbf{u} \) is a measurable vector function. Indeed, we will prove that, in \( \mathcal{D}'(U_N) \),

\[ \mathbf{u} = 1_{\{\psi(x) \neq 0\}} \Re \frac{\overline{\psi(x)}}{|\psi(x)|} \nabla \psi(x). \quad (2.4) \]

Pick an arbitrary test function \( \varphi \in C_c^\infty(U_N) \). Assume that \( I = \{a_1, \cdots, a_N\}, \varepsilon_0 > 0 \) and \( \text{supp} \varphi \subset U \setminus I_{\varepsilon_0} \) (see Sec. 2.1). Since \( f \in L^2_{\text{loc}}(U \setminus I; \mathbb{R}^2) \), we know that \( \psi \in H^1(U \setminus I_{\varepsilon_0}; \mathbb{C}) \) and (2.4) holds in \( \mathcal{D}'(U \setminus I_{\varepsilon_0}) \), hence

\[ \int_U |\psi(x)| \nabla \varphi(x) \, dx = - \Re \int_U 1_{\{\psi(x) \neq 0\}} \frac{\overline{\psi(x)}}{|\psi(x)|} \nabla \psi(x) \varphi(x) \, dx. \]

Next, we check that the function \( \nabla |\psi| \) is in \( L^2(U) \). Let \( \varepsilon \in (0, \varepsilon_0) \). By the diamagnetic inequality [18 Thm. 7.21, pp. 193], for almost every \( x \in U \setminus I_\varepsilon \), \( |\nabla |\psi|| \leq ||(\nabla - i f)\psi|| \). Consequently,

\[ \int_{U_\varepsilon} |\nabla |\psi||^2 \, dx \leq ||(\nabla - i f)\psi||^2_{L^2(U)}, \]
and by monotone convergence,
\[ \|\nabla \psi\|_{L^2(U \setminus I)}^2 = \lim_{\varepsilon \to 0^+} \int_{U_\varepsilon} |\nabla \psi|^2 \, dx \leq \|\nabla - if\|_{L^2(U)}^2 < +\infty. \]

This proves (1). Noting that by Hölder’s inequality, 
\[ H_\text{embedding}, \quad \text{there exists} \quad \psi \in H^1(U \setminus I) \]
and by monotone convergence, 
\[ \text{we get that} \quad \|\psi\|_{L^p(U \setminus I)} \leq \|\psi\|_{L^p(U)} + \|\nabla \psi\|_{L^2(U)} < +\infty. \]

A useful variant of Lemma 2.3 is given below.

Lemma 2.5. For all \((\psi, a) \in H^1_0(U; \mathbb{C}) \times (H^1(U; \mathbb{R}^2) + f)\), it holds,
\[
\begin{align*}
(1) & \quad (\nabla - ia)\psi \in L^2(U; \mathbb{C}^2); \\
(2) & \quad \|\nabla \psi\|_{L^2(U)} \leq \|\nabla - if\|_{L^2(U)} + \|a\psi\|_{L^2(U)} < +\infty.
\end{align*}
\]

Proof. Let \(a = a - f\). We know that \(a \in H^1(U; \mathbb{R}^2) \hookrightarrow L^4(U; \mathbb{R}^2)\). Consequently, \(a \psi \in L^2(U; \mathbb{C})\), by Hölder’s inequality. Thus
\[ \|\psi\|_{L^2(U)} \leq \|\psi\|_{L^2(U)} + \|a\psi\|_{L^2(U)} < +\infty. \]

This proves (1). Noting that \(a \in L^2(U; \mathbb{R}^2)\), we see that (2) follows from Lemma 2.3.

### 2.3. Compactness in the magnetic Sobolev space.

In the next sections, we will work with minimizing sequences of the functional with Aharonov-Bohm potential. We describe here the procedure of extracting convergent sub-sequences.

We continue to work with hypotheses in Sec. 2.1 and under the additional assumption \(f \in L^\infty_{\text{loc}}(U \setminus I; \mathbb{R}^2)\). Recall the space \(H^1_{00}(U; \text{div}0)\), of divergence free vector fields, introduced earlier in [14,10].

Proposition 2.6. Let \(M > 0\). Assume that \((\psi_n, a_n)_{n \geq 1} \subset H^1_0(U; \mathbb{C}) \times H^1_{00}(U; \text{div}0)\) such that:
\[ \forall n \geq 1, \quad \|\nabla - i(a_n + f)\psi_n\|_{L^2(U)} + \|\psi_n\|_{L^4(U)} + \|\text{curl} a_n\|_{L^2(U)} \leq M. \]

The following holds
\[
\begin{align*}
(1) & \quad \text{The sequences } (|\psi_n|)_{n \geq 1} \text{ and } (a_n)_{n \geq 1} \text{ are bounded in } H^1(U) \text{ and in } H^1(U; \mathbb{R}^2) \text{ respectively;} \\
(2) & \quad \text{The sequence } \|\nabla - i(\psi_n)\|_{L^2(U)} \text{ is bounded in } L^2(U; \mathbb{C}^2); \\
(3) & \quad \text{For all } \varepsilon \in (0, \varepsilon_0), \text{ the sequence } (\psi_{n, \varepsilon})_{n \geq 1} \text{ is bounded in } H^1(U \setminus I; \mathbb{C}); \\
(4) & \quad \text{There exist } (\psi, a) \in H^1_0(U; \mathbb{C}) \times H^1_{\text{div}}(U; \mathbb{R}^2) \text{ and a subsequence } (\psi_{n_k}, a_{n_k})_{k \geq 1} \text{ such that} \\
& \quad \limsup_{k \to +\infty} \|\nabla - i(a_{n_k} + f)\psi_{n_k}\|_{L^2(U)} \leq \|\nabla - i(\psi_n)\|_{L^2(U)} \\
& \quad \limsup_{k \to +\infty} \|\psi_{n_k}\|_{L^p(U)} = \|\psi\|_{L^p(U)} \quad (p \in \{2, 4\}) \\
& \quad \liminf_{k \to +\infty} \|\text{curl} a_{n_k}\|_{L^2(U)} \geq \|\text{curl} a\|_{L^2(U)}. \\
\end{align*}
\]

Proof.

#### Step 1. Proof of (1)-(3).

By Proposition 2.3 the sequence \((|\psi_n|)_{n \geq 1}\) is bounded in \(H^1(U)\). By the curl-div inequality [7, Prop. D.2.1], there exists \(C > 0\) such that,
\[ \forall u \in H^1_{00}(U; \text{div}0), \quad \|u\|_{H^1(U)} \leq C\|\text{curl} u\|_{L^2(U)}. \]

This proves (1). By the Sobolev embedding \(H^1(U; \mathbb{R}^2) \hookrightarrow L^p(U; \mathbb{R}^2)\), we get that \((a_n)_{n \geq 1}\) is bounded in \(L^p(U; \mathbb{R}^2)\), for all \(p \geq 2\). By Hölder’s inequality,
\[ \|a_n\psi_n\|_{L^2(U)} \leq \|a_n\|_{L^4(U)}\|\psi_n\|_{L^4(U)}, \]
and we get that \((a_n\psi_n)_{n \geq 1}\) is bounded in \(L^2(U; \mathbb{C}^2)\). By the Minkowski inequality,
\[ \|(|\nabla - i(\psi_n)| \psi_n\|_{L^2(U)} \leq \|(|\nabla - i(a_n + f)| \psi_n\|_{L^2(U)} + \|a_n\psi_n\|_{L^2(U)}, \]

\[ \|(|\nabla - i(a_n + f)| \psi_n\|_{L^2(U)} \leq \|(|\nabla - i(a_n + f)| \psi_n\|_{L^2(U)} + \|a_n\psi_n\|_{L^2(U)} \leq C\|\text{curl} a\|_{L^2(U)}, \]

\[ \|\psi_n\|_{L^4(U)} \leq \|\psi\|_{L^4(U)} \]

\[ \|a_n\psi_n\|_{L^2(U)} \leq \|a_n\|_{L^4(U)}\|\psi_n\|_{L^4(U)} \leq C\|\text{curl} a\|_{L^2(U)}. \]
which proves (2). Since \( U \) is bounded, \( L^4(U; \mathbb{C}) \hookrightarrow L^2(U; \mathbb{C}) \), hence \( (\psi_n)_{n \geq 1} \) is bounded in \( L^2(U; \mathbb{C}) \). Furthermore, \( f \in L^\infty(U \setminus I_e; \mathbb{R}^2) \) and
\[
\| \nabla \psi_n \|_{L^2(U \setminus I_e)} \leq \| (\nabla - if) \psi_n \|_{L^2(U)} + \| f \psi_n \|_{L^2(U \setminus I_e)},
\]
which proves (3).

**Step 2. Extraction of the subsequence**

By a diagonal sequence argument, the Banach-Alaoglu theorem and the compactness of the embedding \( H^1(U) \hookrightarrow L^p(U), \ p \in [2, +\infty) \), we can extract a subsequence \( (\psi_{n_k}, a_{n_k})_{k \geq 1} \), functions \( \psi \in H^1_{\text{loc}}(U \setminus I; \mathbb{C}), \ \zeta \in H^1(U), a \in H^1_{n_0}(U, \text{div0}) \) and \( w \in L^2(U; \mathbb{C}^2) \) such that
\[
\psi_{n_k} \rightarrow \psi \ \text{in} \ H^1_{\text{loc}}(U \setminus I; \mathbb{C})
\]
\[
\psi_{n_k} \rightarrow \psi \ \text{in} \ L^p_{\text{loc}}(U \setminus I; \mathbb{C}) \quad (p \in [2, \infty))
\]
\[
|\psi_{n_k}| \rightarrow \zeta \ \text{in} \ L^p(U) \quad (p \in [2, \infty))
\]
\[
a_{n_k} \rightarrow a \ \text{in} \ H^1_{n_0}(U, \text{div0})
\]
\[
a_{n_k} \rightarrow a \ \text{in} \ L^p(U; \mathbb{R}^2) \quad (p \in [2, \infty))
\]
\[
(\nabla - if) \psi_{n_k} \rightarrow w \ \text{in} \ L^2(U; \mathbb{C}^2).
\]

**Step 3. \( \psi \in L^p(U; \mathbb{C}) \).**

For all \( \varepsilon \in (0, \varepsilon_0) \) and \( p \in [2, \infty) \), \( |\psi_{n_k}| \rightarrow |\psi| \ \text{in} \ L^p(U \setminus I_e) \), hence \( |\psi| = \zeta \ \text{in} \ L^p(U \setminus I_e) \). By monotone convergence,
\[
0 = \lim_{\varepsilon \rightarrow 0^+} \int_{U \setminus I_e} \left( |\psi| - \zeta \right)^p dx = \int_U \left( |\psi| - \zeta \right)^p dx,
\]
hence \( |\psi| = \zeta \) a.e. in \( U \). Since \( \zeta \in H^1(U) \hookrightarrow L^p(U) \) with \( p \in [2, \infty) \), we deduce that \( \psi \in L^p(U; \mathbb{C}) \) and consequently
\[
\lim_{k \rightarrow +\infty} \int_U |\psi_{n_k}|^p dx = \int_U |\psi|^p dx \quad (p \in [2, \infty)).
\]

**Step 4. Convergence in \( L^p(U; \mathbb{C}) \).**

We prove that \( \psi_{n_k} \rightarrow \psi \) in \( L^p(U) \) as follows. Fix \( \varepsilon \in (0, \varepsilon_0) \). By the Hölder and Minkowski inequalities,
\[
\int_{I_e} |\psi_{n_k} - \psi|^p dx \leq |I_e|^{1/2} \left( \int_U |\psi_{n_k} - \psi|^{2p} dx \right)^{1/2} \leq \tilde{M}|I_e|^{1/2},
\]
where \( \tilde{M} = \left( \sup_{k \geq 1} \| \psi_{n_k} \|_{L^{2p}(U)} + \| \psi \|_{L^{2p}(U)} \right)^{2p} < +\infty \); moreover, \( \psi_{n_k} \rightarrow \psi \) in \( L^p(U \setminus I_e; \mathbb{C}) \). With this in hand, we deduce that
\[
0 \leq \limsup_{k \rightarrow +\infty} \int_U |\psi_{n_k} - \psi|^p dx = \limsup_{k \rightarrow +\infty} \left( \int_{I_e} |\psi_{n_k} - \psi|^p dx + \int_{U \setminus I_e} |\psi_{n_k} - \psi|^p dx \right) \leq \tilde{M}|I_e|^{1/2}.
\]
Sending \( \varepsilon \) to 0, we get the desired convergence, \( \lim_{k \rightarrow +\infty} \int_U |\psi_{n_k} - \psi|^p dx = 0 \).

**Step 5. \( \psi \in H^1(I; \mathbb{C}) \).**
Since \( f \in L^q(U; \mathbb{R}^2) \) and \( \psi \in L^p(U; \mathbb{C}) \) for all \( q \in [1, 2] \) and \( p \in [2, +\infty) \), we get that \( \psi \) and \( f \psi \) are distributions on \( U \). Hence \( (\nabla - if)\psi \in \mathcal{D}'(U; \mathbb{C}^2) \).

By Step 4 above, we get that \( (\nabla - if)\psi_{n_k} \to (\nabla - if)\psi \) in \( \mathcal{D}'(U; \mathbb{C}^2) \). In light of Step 2 above, the weak convergence of \( (\nabla - if)\psi_{n_k} \) to \( w \) in \( L^2(U; \mathbb{C}^2) \) yields the convergence in \( \mathcal{D}'(U; \mathbb{C}^2) \), hence the identity \( (\nabla - if)\psi = w \) in \( \mathcal{D}'(U; \mathbb{C}^2) \). This proves that \( (\nabla - if)\psi \in L^2(U; \mathbb{C}^2) \), and since \( \psi \in L^2(U; \mathbb{C}) \), we eventually get that \( \psi \in H^1_{F}(U; \mathbb{C}) \).

**Step 6.** End of the proof of (4).

For all \( \varepsilon \in (0, \varepsilon_0) \),
\[
\int_U |(\nabla - i(a_{n_k} + f))\psi_{n_k}|^2 \, dx \geq \int_{U \setminus I_\varepsilon} |(\nabla - i(a_{n_k} + f))\psi_{n_k}|^2,
\]
and \( (\nabla - i(a_{n_k} + f))\psi_{n_k} \to (\nabla - i(a + f))\psi \) in \( L^2(U \setminus I_\varepsilon; \mathbb{C}^2) \) by Step 2 above; this yields
\[
\liminf_{k \to +\infty} \int_U |(\nabla - i(a_{n_k} + f))\psi_{n_k}|^2 \, dx \geq \lim_{\varepsilon \to 0^+} \int_{U \setminus I_\varepsilon} |(\nabla - i(a + f))\psi|^2 \, dx = \int_U |(\nabla - i(a + f))\psi|^2 \, dx.
\]

By Monotone convergence
\[
\liminf_{k \to +\infty} \int_U |(\nabla - i(a_{n_k} + f))\psi_{n_k}|^2 \, dx \geq \int_U |\text{curl } a_{n_k}|^2 \, dx \geq \int_U |\text{curl } a|^2 \, dx.
\]

Finally, \( \text{curl } a_{n_k} \to \text{curl } a \) in \( L^2(U) \), which yields that
\[
\liminf_{k \to +\infty} \int_U |\text{curl } a_{n_k}|^2 \, dx \geq \int_U |\text{curl } a|^2 \, dx.
\]

\[\square\]

### 3. Minimizers with Aharonov-Bohm potential

In this section, we study the existence of minimizers of the GL functional in \([17]\) along with some of their properties.

#### 3.1. Gauge invariance.

Using gauge invariance, we can restrict the minimization of the functional in \([17]\) to the space of divergence free vector fields; the advantage being that such vector fields enjoy pleasant regularity properties.

**Proposition 3.1.** For all \( h > 0 \),
\[
E_{AB}(h) = \inf \{ E_{AB,h}(\psi, a + F_{AB}) \ : \ (\psi, a) \in H^1_{F_{AB}}(\Omega; \mathbb{C}) \times H^1_{\text{div0}}(\Omega, \text{div0}) \},
\]

where
- \( E_{AB}(h) \) is introduced in \([10]\);
- \( E_{AB,h} \) is the functional introduced in \([17]\);
- \( H^1_{\text{div0}}(\Omega, \text{div0}) \) is the space introduced in \([14,6]\).

**Proof.** Let \( (\psi, a := a + F_{AB}) \in H^1_{F_{AB}}(\Omega; \mathbb{C}) \times (H^1(\Omega; \mathbb{R}^2) + F_{AB}) \). By \([7]\) Prop. D.1.1, there exists \( \varphi \in H^2(\Omega) \) such that \( \tilde{a} := a - \nabla \varphi \in H^1_{\text{div0}}(\Omega, \text{div0}) \). Setting \( \tilde{\psi} = e^{ih\varphi}\psi \), it is clear that \( (\tilde{\psi}, \tilde{a}) \in H^1_{F_{AB}}(\Omega; \mathbb{C}) \times H^1_{\text{div0}}(\Omega, \text{div0}) \) and \( E_{AB,h}(\psi, a + F_{AB}) = E_{AB,h}(\tilde{\psi}, \tilde{a} + F_{AB}) \). \[\square\]
3.2. Existence of minimizers. Next we establish the existence of minimizing configurations.

**Proposition 3.2.** For all \( h > 0 \), there exists \((\psi, a) \in H^1_{hFAB}(\Omega; \mathbb{C}) \times H^1_{00}(\Omega, \text{div}0)\) such that

\[
E_{AB_h}(\psi, a + F_{AB}) = E_{AB}(h).
\]

**Proof.** We use the standard method of the calculus of variations. We choose a minimizing sequence \((\psi_n, a_n)_{n \geq 1} \subset H^1_{hFAB}(\Omega; \mathbb{C}) \times H^1_{00}(\Omega, \text{div}0)\) such that

\[
\lim_{n \to +\infty} E_{AB_h}(\psi_n, a_n + F_{AB}) = E_{AB}(h).
\]  

(3.1)

By (1.3), there exists \( M > 0 \) such that,

\[
\| (\nabla - ih(a_n + F_{AB}))\psi_n \|_{L^2(\Omega)} + \| \psi_n \|_{L^4(\Omega)} + h\| \text{curl } a_n \|_{L^2(\Omega)} \leq M, \quad \forall n \geq 1.
\]

We can apply Proposition 3.2 with \((\psi_n, a_n) = h a_n, f = h F_{AB}\). We get a subsequence \((\psi_{n_k}, a_{n_k})_{k \geq 1}\) and a configuration \((\psi, a) \in H^1_{hFAB}(\Omega; \mathbb{C}) \times H^1_{00}(\Omega, \text{div}0)\) such that

\[
E_{AB}(h) = \liminf_{k \to \infty} E_{AB_h}(\psi_{n_k}, a_{n_k} + F_{AB}) \geq E_{AB_h}(\psi, a + F_{AB}) \geq E_{AB}(h),
\]

where the identity on the left hand side follows from (3.1), and the last inequality on the right hand side follows from the definition of \( E_{AB}(h) \). \( \square \)

**Definition 3.3.** Given \( h > 0 \) and \((\psi, a) \in H^1_{hFAB}(\Omega; \mathbb{C}) \times H^1_{00}(\Omega, \text{div}0)\), we will use the following terminology:

- \((\psi, a)_h\) is said to be a minimizing configuration of \( E_{AB_h} \) if \( E_{AB_h}(\psi, a + F_{AB}) = E_{AB}(h) \), where \( E_{AB_h} \) and \( E_{AB}(h) \) are introduced in (1.7) and (1.3) respectively;
- \((\psi, a)_h\) is said to be a critical configuration of \( E_{AB_h} \) if \( \frac{d}{d t} E_{AB_h}(\psi + t \varphi, a + F_{AB}) \bigg|_{t=0} = 0 \) and \( \frac{d}{d t} E_{AB_h}(\psi, a + F_{AB} + t b) \bigg|_{t=0} = 0 \), for all \((\varphi, b) \in H^1_{hFAB}(\Omega; \mathbb{C}) \times H^1_{00}(\Omega, \text{div}0)\).

**Remark 3.4.** Obviously, every minimizing configuration is a critical configuration. Furthermore, every critical configuration \((\psi, a)_h\) satisfies

\[
\begin{cases}
- (\nabla - ihA)^2 \psi = \kappa^2(1 - |\psi|^2)\psi & \text{in } \Omega, \\
- \nabla^\perp (\text{curl } a) = \frac{1}{h} \text{Im}(\overline{\psi}(\nabla - ihA)\psi) & \text{in } \Omega, \\
\nu \cdot (\nabla - ihA)\psi = 0 & \text{on } \partial \Omega, \\
\text{curl } a = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(3.2)

where \( A = a + F_{AB}, \nu \) is the outward unit normal vector on \( \partial \Omega \), and the operator \( \nabla^\perp = (-\partial_{x_2}, \partial_{x_1}) \) is the Hodge gradient.

3.3. A priori estimates.

**Proposition 3.5.** There exists \( C_0 > 0 \) such that, given a critical configuration \((\psi, a)_h\) \( \in H^1_{hFAB}(\Omega; \mathbb{C}) \times H^1_{00}(\Omega, \text{div}0) \) of \( E_{AB_h} \), the following holds:

1. \( a \in H^2(\Omega; \mathbb{R}^2) \);
2. \( (\psi, a) \in C^\infty(\overline{\Omega} \setminus \{0\}; \mathbb{C}) \times C^\infty(\overline{\Omega} \setminus \{0\}; \mathbb{R}^2) \);
3. \( \| (\nabla - ihA)\psi \|_{L^2(\Omega)} \leq \kappa \| \psi \|_{L^2(\Omega)} \) where \( A = a + F_{AB} \);
4. \( \| \psi \|_{L^\infty(\Omega)} \leq 1 \);
5. \( \| a \|_{H^2(\Omega)} \leq \frac{C_0}{h} \kappa \| \psi \|_{L^2(\Omega)}^2 \).
**Proof.** Since $a \in H^1_{0\partial}(\Omega, \text{div}0)$, the second equation in (3.2) yields that $\text{curl} a \in H^1(\Omega)$. By the curl-div estimate (see [7, Prop. D.2.1]), $a \in H^2(\Omega; \mathbb{R}^2)$ and
\[
\|a\|_{H^2(\Omega)} \leq C_\Omega \|\text{curl} a\|_{H^1(\Omega)},
\]where $C_\Omega > 0$ depends on $\Omega$ only. This proves (1).

That $(\psi, a)$ is smooth in $\Omega \setminus \{0\}$ follows by a bootstrapping argument (see [23, Prop. 3.6]).

By Proposition 2.5, inequality (3) and (4) to write the last inequality.

Finally, we prove (5). By the last equation in (3.2), curl $A$ is smooth in $\Omega$, hence, by the Poincaré inequality
\[
\|\text{curl} a\|_{H^1(\Omega)} \leq C'_\Omega \|\nabla (\text{curl} a)\|_{L^2(\Omega)},
\]where $C'_\Omega$ depends on $\Omega$ only. Using (3.3) and the second equation in (3.2), we get
\[
\|a\|_{H^2(\Omega)} \leq \frac{C_\Omega C'_\Omega}{\hbar} \|\text{Im}(\overline{\psi}\nabla - i h A)\psi\|_{L^2(\Omega)} \leq \frac{C_\Omega C'_\Omega}{\hbar} \|\psi\|_{L^2(\Omega)},
\]where we used (3) and (4) to write the last inequality. $\square$

### 3.4 The non-degenerate case.

Our next result is that for a minimizing configuration the order parameter is actually in the space $H^1(\Omega; \mathbb{C})$ not just in the magnetic Sobolev space $H^1_{\text{mag}}(\Omega; \mathbb{C})$, except for the degenerate case where $h \in 2\pi \mathbb{Z}$. This is related to a magnetic Hardy inequality [10] (see Lemma 3.7 below).

**Proposition 3.6.** Given $r_0, \kappa, h > 0$ such that $D(0, r_0) \subset \Omega$, there exists $C > 0$ such that every minimizing configuration $\psi, A$ of $E_{\text{AB}}$ satisfies:
\[
2\pi \alpha(h) \|F_{\text{AB}}\psi\|_{L^2(\Omega)} + \|\nabla - i h F_{\text{AB}}\psi\|_{L^2(\Omega)} \leq C,
\]where
\[
\alpha(h) = \inf_{n \in \mathbb{Z}} \left| n - \frac{h}{2\pi} \right|.
\]
In particular, for $h \notin 2\pi \mathbb{Z}$, $\psi \in H^1(\Omega; \mathbb{C})$.

The proof relies on the following one dimensional spectral analysis.

**Lemma 3.7.** For all $h > 0$,
\[
\inf_{u \in H^1_{\text{per}}(0, 2\pi) \setminus \{0\}} \int_0^{2\pi} \frac{d}{d\theta} u - i \frac{h}{2\pi} u \left| u \right|^2 d\theta = \alpha(h)^2,
\]where $\alpha(h)$ is introduced in Proposition 3.6 and
\[
H^1_{\text{per}}(0, 2\pi) = \{ u \in H^1(0, 2\pi) : u(0) = u(2\pi) \}.\]
Proposition 3.8. For all $u \in H^1_{\text{per}}(0, 2\pi)$ let $v = e^{-i\frac{\kappa}{2\pi} \theta} u$. Notice that $v(0) = e^{ih} v(2\pi)$ and

$$q(u) := \int_0^{2\pi} \left| \frac{d}{d\theta} u - i\frac{h}{2\pi} u \right|^2 d\theta = \int_0^{2\pi} \left| \frac{d}{d\theta} v \right|^2 d\theta.$$ 

So we are led to determine the spectrum of the operator $L = -\frac{d^2}{dx^2}$ with the boundary condition

$$v(0) = e^{ih} v(2\pi). \quad (3.6)$$

It is easy to check that $e^{i\kappa \theta}$ is an eigenfunction associated with the eigenvalue $\lambda := w^2$, $w \in \mathbb{R}$, and that the boundary condition in $(3.6)$ reads as $w + \frac{h}{2\pi} \in \mathbb{Z}$.

Proof of Proposition 3.6. Using (4)-(5) in Proposition 3.5, the Hölder inequality and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we write

$$\|a \psi\|_{L^2(\Omega)} \leq \|a\|_{L^4(\Omega)} \|\psi\|_{L^4(\Omega)} \leq \tilde{C}_{\kappa} \frac{1}{h},$$

for a constant $\tilde{C}$ independent from $(\kappa, h)$. Consequently, by the Minkowski inequality and (3) in Proposition 3.5, we get

$$\|\nabla - ihF_{AB}\psi\|_{L^2(\Omega)} \leq \kappa |\Omega|^{1/2} + \tilde{C}_{\kappa}. \quad (3.6)$$

Now we express $\|\nabla - ihF_{AB}\psi\|_{L^2(D(0, r_0))}$ in polar coordinates $(r, \theta)$ as follows

$$\|\nabla - ihF_{AB}\psi\|_{L^2(D(0, r_0))} = \int_0^{2\pi} \int_0^{r_0} \left( |\partial_r \psi|^2 + \frac{1}{r^2} \left| \partial_\theta - i\frac{h}{2\pi} \psi \right|^2 \right) r dr d\theta.$$ 

Using Lemma 3.4, we infer the following estimate,

$$\|\nabla - ihF_{AB}\psi\|^2_{L^2(D(0, r_0))} \geq \int_0^{2\pi} \int_0^{r_0} \left( \frac{\alpha(h)}{r^2} |\psi|^2 \right) r dr d\theta = (2\pi \alpha(h))^2 \int_{D(0, r_0)} |F_{AB}\psi|^2 \, dx. \quad (3.6)$$

3.5. The degenerate case. We determine the minimizers of the functional in $(1.7)$ in the degenerate case where $h \in 2\pi \mathbb{Z}$.

Proposition 3.8. Assume that $h = 2\pi n_0$ with $n_0 \in \mathbb{Z}$. Then,

$$E_{AB}(h) = -\frac{\kappa^2}{2} |\Omega|$$

and every minimizer $(\psi, A)$ has the form

$$\psi = e^{in_0 \theta} \quad \text{and} \quad A = F_{AB}$$

with $c \in \mathbb{C}$ satisfying $|c| = 1$.

Proof. The inequality $E_{AB}(h) \geq -\frac{\kappa^2}{2} |\Omega|$ follows from $(1.8)$. To obtain the reverse inequality, we write

$$E_{AB}(h) \leq E_{AB}(u, F_{AB})$$

with $u = e^{in_0 \theta}$. Using polar coordinates, we notice that

$$\|\nabla - ihF_{AB}u\|^2 = |\partial_\theta u|^2 + \frac{1}{r^2} \left( \partial_\theta - i\frac{h}{2\pi} \right)^2 u = 0.$$ 

This proves that $u \in H^1_{F_{AB}}(\Omega; \mathbb{C})$, $E_{AB}(u, F_{AB}) = -\frac{\kappa^2}{2} |\Omega|$ and $(u, F_{AB})$ is a minimizer.

Now, assume that $(\psi, a)$ is a minimizing configuration of $E_{AB}$, i.e. $E_{AB}(\psi, a + F_{AB}) = E_{AB}(h) = -\frac{\kappa^2}{2} |\Omega|$. Notice that $a = 0$, since $a \in H^1_{\partial \Omega}(\Omega, \text{div}0)$ and (see $(1.8)$)

$$-\frac{\kappa^2}{2} |\Omega| + h^2 \int_{\Omega} |\text{curl} a|^2 \, dx \leq E_{AB}(\psi, a + F_{AB}) = -\frac{\kappa^2}{2} |\Omega|.$$
The same argument yields that $|\psi| = 1$ and $(\nabla - ih\mathbf{F}_{AB})\psi = 0$, since

$$
-\frac{\kappa^2}{2}|\Omega| + \int_{\Omega} \left( |(\nabla - ih\mathbf{F}_{AB})\psi|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 \right) \, dx \leq \mathcal{E}_{AB}\epsilon(\psi; \mathbf{F}_{AB}) = -\frac{\kappa^2}{2}|\Omega|.
$$

By introducing the ansatz $\psi = cu$, we find

$$
0 = (\nabla - ih\mathbf{F}_{AB})\psi = c(\nabla - ih\mathbf{F}_{AB})u + u\nabla c = u\nabla c
$$

hence $c$ must be a constant. Finally, the condition $|\psi| = 1$ yields that $|c| = 1$. 

\[\square\]

4. Minimizers with a magnetic step

In this section we study the minimizers of the functional $\mathcal{E}_{h,\epsilon}$ introduced in (1.13). Since this is associated with the magnetic potential $\mathbf{F}_\epsilon \in H^1(\Omega; \mathbb{R}^2)$, the corresponding magnetic field $B_\epsilon = \text{curl} \mathbf{F}_\epsilon$ is in $L^2(\Omega)$. Hence, we can use the results in [7 Thm. 10.2.1]. In particular, for all $h > 0$ and $\epsilon \in (0, \epsilon_0)$, there exists a configuration $(\Psi, \mathbf{A})_{h,\epsilon} \in \mathcal{H}$ such that

$$
\mathcal{E}_{h,\epsilon}(\Psi, \mathbf{A}) = E_\epsilon(h),
$$

(4.1)

where $E_\epsilon(h)$ is introduced in (1.12). A configuration satisfying (4.1) is said to be a minimizer of $\mathcal{E}_{h,\epsilon}$. Similarly as we did in Proposition 3.1, we can use the gauge invariance to select a configuration

$$(\psi_\epsilon, \mathbf{a}_\epsilon)_h \in \mathcal{H}_0 := H^1(\Omega; \mathbb{C}) \times H^1(\Omega, \text{div}0)$$

(4.2)

such that $(\psi_\epsilon, \mathbf{A}_\epsilon := \mathbf{a}_\epsilon + \mathbf{F}_\epsilon)$ is a minimizer of $\mathcal{E}_{h,\epsilon}$. Such a configuration is said to be a minimizing configuration of $\mathcal{E}_{h,\epsilon}$. It satisfies the following properties (see [7 Prop. 10.3.1 & Lem. 10.3.2]):

$$
\|\psi_\epsilon\|_{L^\infty(\Omega)} \leq 1, \quad \|\nabla - ih\mathbf{A}_\epsilon\psi\|_{L^2(\Omega)} \leq \kappa \|\psi_\epsilon\|_{L^2(\Omega)}, \quad \|\text{curl} \mathbf{a}_\epsilon\|_{L^2(\Omega)} \leq \frac{\kappa^2}{h}\|\psi_\epsilon\|_{L^2(\Omega)}.
$$

(4.3)  (4.4)  (4.5)

Furthermore, $(\psi_\epsilon, \mathbf{a}_\epsilon)_h$ is a solution of

$$
\begin{cases}
-\frac{1}{h^2}(\nabla - ih\mathbf{A}_\epsilon)^2 \psi_\epsilon = \kappa^2(1 - |\psi_\epsilon|^2)\psi_\epsilon & \text{in } \Omega, \\
\nabla \cdot (\text{curl} \mathbf{a}_\epsilon) = \frac{1}{h} \text{Im}(\overline{\psi_\epsilon} \nabla - ih\mathbf{A}_\epsilon)\psi_\epsilon & \text{in } \Omega, \\
\nu \cdot (\nabla - ih\mathbf{A}_\epsilon)\psi_\epsilon = 0 & \text{on } \partial\Omega, \\
\text{curl} \mathbf{a}_\epsilon = 0 & \text{on } \partial\Omega.
\end{cases}
$$

(4.6)

Using (3.3) and the second equation in (4.6), we can prove that (see the proof of Prop. 3.3(5)):

$$
\|\mathbf{a}_\epsilon\|_{H^1(\Omega)} \leq \frac{\tilde{C}}{h}.
$$

(4.7)

where $\tilde{C}$ does not depend on $\epsilon$ and $h$.

In the sequel, we study the behavior of the minimizing configurations of $\mathcal{E}_{h,\epsilon}$ as $\epsilon$ approaches 0.

**Proposition 4.1.** Given $\omega \subset \Omega \setminus \{0\}$ and $h > 0$, there exist $\epsilon_0 \in (0, 1)$ and $C > 0$ such that, for all $\epsilon \in (0, \epsilon_0)$, every minimizing configuration $(\psi_\epsilon, \mathbf{a}_\epsilon)_h$ of $\mathcal{E}_{h,\epsilon}$ satisfies

$$
\|\psi_\epsilon\|_{H^2(\omega)} \leq C.
$$

**Proof.** For $\epsilon_0$ sufficiently small and $\epsilon \in (0, \epsilon_0]$, we have

$$
\omega \subset \Omega \setminus \overline{D(0, 2\epsilon_0)} \subset \Omega \setminus \overline{D(0, \epsilon)}.
$$

Hence $\mathbf{F}_\epsilon = \mathbf{F}_{AB}$ on $\tilde{\omega} := \Omega \setminus \overline{D(0, \epsilon_0)}$, it is smooth and the first equation in (1.13) reads as follows

$$
-\Delta \psi_\epsilon + 2ih(\mathbf{a}_\epsilon + \mathbf{F}_{AB}) \cdot \nabla \psi_\epsilon + h^2|\mathbf{a}_\epsilon + \mathbf{F}_{AB}|^2\psi_\epsilon = \kappa^2(1 - |\psi_\epsilon|^2)\psi_\epsilon \text{ in } \tilde{\omega},
$$

The same argument yields that $|\psi| = 1$ and $(\nabla - ih\mathbf{F}_{AB})\psi = 0$, since

$$
-\frac{\kappa^2}{2}|\Omega| + \int_{\Omega} \left( |(\nabla - ih\mathbf{F}_{AB})\psi|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 \right) \, dx \leq \mathcal{E}_{AB}\epsilon(\psi; \mathbf{F}_{AB}) = -\frac{\kappa^2}{2}|\Omega|.
$$

By introducing the ansatz $\psi = cu$, we find

$$
0 = (\nabla - ih\mathbf{F}_{AB})\psi = c(\nabla - ih\mathbf{F}_{AB})u + u\nabla c = u\nabla c
$$

hence $c$ must be a constant. Finally, the condition $|\psi| = 1$ yields that $|c| = 1$. 

\[\square\]
since $\text{div}(a_e) = \text{div}(F_{AB}) = 0$. Using (4.3), (4.4) and (4.7), we get
$$\|\Delta \psi_e \|_{L^2(\Omega)} \leq \hat{C}.$$ 

Note that, on $\partial \Omega$, the following boundary condition holds (which results from the third equation in (4.6) and the boundary condition on $a_e \in H^1_{n0}(\Omega, \text{div}0)$, hence $\nu \cdot a_e = 0$, see (4.16):
$$\nu \cdot \nabla \psi_e = ih(\nu \cdot F_{AB})\psi_e,$$

with $\nu \cdot F_{AB}$ a continuous function on $\partial \Omega$, by smoothness of the boundary. Consequently, we can apply the $L^2$-elliptic estimates on $\partial \Omega$, by smoothness of the boundary. Consequently, we can apply the $L^2$-elliptic estimates on $\partial \Omega$, by smoothness of the boundary. Consequently, we can apply the $L^2$-elliptic estimates on $\partial \Omega$, by smoothness of the boundary.

**Proposition 4.2.** Given $\alpha \in (0, 1)$, $h > 0$ and a sequence $(\varepsilon_n)_{n \geq 1} \subset \mathbb{R}_+$ which converges to $0$, there exist $(\psi_n, a_n) \in H^1_{FAB}(\Omega; \mathbb{C}) \times H^1_{n0}(\Omega, \text{div}0)$ and a subsequence
$$(\psi_{n\lambda}, a_{n\lambda}, \varepsilon_{n\lambda})_{n\lambda \in I} \subset H^1(\Omega; \mathbb{C}) \times H^1_{n0}(\Omega, \text{div}0) \times \mathbb{R}_+$$

such that

1. $(\psi_{n\lambda}, a_{n\lambda})_{n\lambda}$ is a minimizing configuration of $\mathcal{E}_{h, \varepsilon_n}$;
2. For every open set $\omega \subset \Omega \setminus \{0\}$, $\psi_{n\lambda} \to \psi_*$ in $H^1(\omega; \mathbb{C})$;
3. $a_{n\lambda} \to a_* \quad \text{in} \quad H^1(\Omega; \mathbb{R}^2)$;
4. $\psi_{n\lambda}, \psi_* \in C^0_{\text{loc}}(\Omega \setminus \{0\}; \mathbb{C})$ and $a_{n\lambda}, a_* \in C^{0, \alpha}(\Omega; \mathbb{R}^2)$;
5. $\psi_{n\lambda} \to \psi_*$ in $C^0_{\text{loc}}(\Omega \setminus \{0\}; \mathbb{C})$ and $a_{n\lambda} \to a_* \quad \text{in} \quad C^{0, \alpha}(\Omega; \mathbb{R}^2)$;
6. $\|\psi_*\|_{L^\infty(\Omega)} \leq 1$;
7. $\psi_n \to \psi_*$ in $L^p(\Omega; \mathbb{C})$, for all $p \in [2, +\infty)$;
8. $\liminf_{n \to \infty} \mathcal{E}_{h, \varepsilon_n}(\psi_n, a_n + F_{\varepsilon_n}) \geq \mathcal{E}_{ABh}(\psi_*, a_* + F_{AB})$,

where $\mathcal{E}_{ABh}$ is the functional introduced in (1.7).

**Proof.** Consider a sequence $(\varepsilon_n)_{n \geq 1} \subset \mathbb{R}_+$ such that $\lim_{n \to \infty} \varepsilon_n = 0$. For all $n \geq 1$, choose a minimizing configuration $(\psi_{n\lambda}, a_{n\lambda})_{n\lambda}$ of $\mathcal{E}_{h, \varepsilon_n}$.

By Proposition 1.1 for all $r \in (0, \varepsilon_0]$, there exists $C_r, N_0 > 0$ such that
$$\|\psi_{n\lambda}\|_{H^2(\Omega_\varepsilon)} \leq C_r, \quad \forall n \geq N_0,$$

where $\Omega_\varepsilon = \Omega \setminus \overline{D}(0, r)$. By a diagonal sequence argument, we can construct a function $\psi_* : \Omega \setminus \{0\} \to \mathbb{C}$ and extract a subsequence of $(\psi_{n\lambda})_{n\lambda \in I_0}$ which is weakly convergent to $\psi_*$ in every $H^2(\Omega_\varepsilon; \mathbb{C})$, $r \in (0, \varepsilon_0]$.

At the same time, (1.7) yields a function $a_* \in H^2(\Omega; \mathbb{R}^2)$ and a subsequence $(a_{n\lambda})_{n\lambda \in I \subset I_0}$ which converges weakly to $a_*$ in $H^2(\Omega; \mathbb{R}^2)$.

In light of the estimates in (1.3)-(1.4), we see that the sequence $((\nabla - ihF_{\varepsilon_n})\psi_n)$ is bounded in $L^2(\Omega; \mathbb{C}^2)$. So we can extract a subsequence $((\nabla - ihF_{\varepsilon_n})\psi_{n\lambda})_{n\lambda \in I \subset I_1}$ that is weakly convergent in $L^2(\Omega; \mathbb{C}^2)$, and denote its weak limit by $g$.

By compactness of the embedding $H^2(U) \hookrightarrow H^1(U)$ and $H^2(U) \hookrightarrow C^{0, \alpha}(\overline{U})$, we can extract a further subsequence, $(\psi_{n\lambda}, a_{n\lambda})_{n\lambda \in I \subset I_2}$, which converges to $(\psi_*, a_*)$ in $H^1(\Omega_\varepsilon; \mathbb{C}) \times H^1(\Omega_\varepsilon; \mathbb{R}^2)$ and in $C^{0, \alpha}(\Omega_\varepsilon; \mathbb{C}) \times C^{0, \alpha}(\Omega_\varepsilon; \mathbb{R}^2)$. This proves (2)-(5).

The estimate $\|\psi_n\|_{L^\infty(\Omega)} \leq 1$ follows from (1.3); actually, for every $x \in \Omega \setminus \{0\}$, we can find $r > 0$ such that $x \in \Omega_\varepsilon$, and consequently $\psi_{n\lambda}(x) \to \psi_*(x)$. This proves (6) and also that $\psi_* \in L^p(\Omega; \mathbb{C})$ for all $p \geq 1$.

We can prove that $\psi_n \to \psi_*$ in $L^p(\Omega; \mathbb{C})$ by repeating the argument used in the proof of Proposition 2.6 (Step 4). In particular, we now know that
$$\lim_{n \to +\infty} \int_{\Omega} |\psi|^p \, dx = \int_{\Omega} |\psi_*|^p \, dx. \tag{4.8}$$
Now we prove that \( \psi_\varepsilon \in H^1_{\text{HF}}(\Omega; \mathbb{C}) \). Note that \( \mathbf{F}_{\varepsilon_n} \to \mathbf{F}_{\text{AB}} \) in \( L^q(\Omega; \mathbb{R}^2) \) for all \( q \in [1, 2) \); moreover, by (1.3), we deduce that
\[
(\nabla - i\hbar \mathbf{F}_{\varepsilon_n})\psi_n - (\nabla - i\hbar \mathbf{F}_{\text{AB}})\psi_n = -i\hbar (\mathbf{F}_{\varepsilon_n} - \mathbf{F}_{\text{AB}})\psi_n \to 0 \quad \text{in} \quad L^q(\Omega; \mathbb{C}^2).
\]
Since \( g \) is the weak limit of \( (\nabla - i\mathbf{F}_{\varepsilon_n})\psi_n \) in \( L^2(\Omega; \mathbb{C}^2) \), we deduce the following convergence in the distributional sense,
\[
(\nabla - i\hbar \mathbf{F}_{\text{AB}})\psi_n \to g \quad \text{in} \quad \mathcal{D}'(\Omega \setminus \{0\}; \mathbb{C}^2).
\]
Pick an arbitrary test function \( \varphi \in C_c^\infty(\Omega \setminus \{0\}; \mathbb{C}) \). By Hölder’s inequality,
\[
|(\psi_n - \psi_\varepsilon, (\nabla - i\hbar \mathbf{F}_{\text{AB}})\varphi)| \leq ||\psi_n - \psi_\varepsilon||_{L^p(\Omega)}||(\nabla - i\hbar \mathbf{F}_{\text{AB}})\varphi||_{L^q(\Omega)}
\]
with \( \frac{1}{p} + \frac{1}{q} = 1, p \in (2, +\infty) \) and \( q \in (1, 2) \); this proves that \( (\nabla - i\hbar \mathbf{F}_{\text{AB}})\psi_n \to (\nabla - i\hbar \mathbf{F}_{\text{AB}})\psi_\varepsilon \) in \( \mathcal{D}'(\Omega \setminus \{0\}; \mathbb{C}^2) \); consequently, \( (\nabla - i\hbar \mathbf{F}_{\text{AB}})\psi_\varepsilon = g \in L^2(\Omega; \mathbb{C}^2) \), which proves (8).

So far we have proved the statements (1)-(8) of Proposition 4.2; it remains to prove the statement (9). For \( n \in I \) sufficiently large and \( r \) sufficiently small, \( \mathbf{F}_{\varepsilon_n} = \mathbf{F}_{\text{AB}} \) in \( \Omega_r \); hence
\[
\int_\Omega |(\nabla - i(\mathbf{a}_{\varepsilon_n} + \mathbf{F}_{\varepsilon_n}))\psi_{\varepsilon_n}|^2 \, dx \geq \int_{\Omega_r} |(\nabla - i(\mathbf{a}_\varepsilon + \mathbf{F}_{\text{AB}}))\psi_\varepsilon|^2 \, dx.
\]
As a consequence of (4.3) and (4.4), we get
\[
\lim_{n \to \infty} \int_{\Omega_r} |(\nabla - i(\mathbf{a}_{\varepsilon_n} + \mathbf{F}_{\text{AB}}))\psi_{\varepsilon_n}|^2 \, dx = \int_{\Omega_r} |(\nabla - i(\mathbf{a}_\varepsilon + \mathbf{F}_{\text{AB}}))\psi_\varepsilon|^2 \, dx.
\]
By monotone convergence, we get further
\[
\lim_{r \to 0^+} \int_{\Omega_r} |(\nabla - i(\mathbf{a}_\varepsilon + \mathbf{F}_{\text{AB}}))\psi_\varepsilon|^2 \, dx = \int_{\Omega} |(\nabla - i(\mathbf{a}_\varepsilon + \mathbf{F}_{\text{AB}}))\psi_\varepsilon|^2 \, dx.
\]
This argument yields that
\[
\liminf_{n \to +\infty} \int_\Omega |(\nabla - i(\mathbf{a}_{\varepsilon_n} + \mathbf{F}_{\varepsilon_n}))\psi_{\varepsilon_n}|^2 \, dx \geq \int_{\Omega} |(\nabla - i(\mathbf{a}_\varepsilon + \mathbf{F}_{\text{AB}}))\psi_\varepsilon|^2 \, dx. \tag{4.9}
\]
Collecting (4.8), (4.9) and the convergence established in (5), we finish the proof of (9).

5. PROOF OF THE MAIN THEOREMS

In this section, we prove our main Theorems 1.1, 1.2 and 1.4.

Proof of Theorem 1.1

Lower bound. It follows from Proposition 1.2 that
\[
\liminf_{\varepsilon \to 0^+} E_\varepsilon(h) \geq E_{AB,h}(\psi_\varepsilon, \mathbf{a}_\varepsilon + \mathbf{F}_{\text{AB}}) \geq E_{AB}(h). \tag{5.1}
\]

Upper bound.

The non-degenerate case. Assume that \( h \not\in 2\pi\mathbb{Z} \) and consider a minimizing configuration \( (\psi, \mathbf{a}) \) of \( E_{AB,h} \). By Proposition 3.6 \( \psi \in H^1(\Omega; \mathbb{C}) \) and \( \mathbf{F}_{\text{AB}}\psi \in L^2(\Omega; \mathbb{C}^2) \). By dominated convergence,
\[
\lim_{\varepsilon \to 0^+} \int_{D(0,\varepsilon)} |\nabla \psi - i\hbar \mathbf{a}\psi|^2 \, dx = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \int_{D(0,\varepsilon)} |\mathbf{F}_{\text{AB}}\psi|^2 \, dx = 0. \tag{5.2}
\]
For all \( \varepsilon \in (0, \varepsilon_0], |\mathbf{F}_\varepsilon| \leq |\mathbf{F}_{\text{AB}}| \) in \( D(0,\varepsilon) \), hence, by (1.2),
\[
\lim_{\varepsilon \to 0^+} \int_{D(0,\varepsilon)} |\mathbf{F}_\varepsilon\psi|^2 \, dx = 0. \tag{5.3}
\]
Furthermore, for all $\varepsilon \in (0, \varepsilon_0]$, 

$$E_{AB}(h) = E_{ABh}(\psi, a + F_{AB}) \geq E_{h, \varepsilon}(\psi, a + F_{\varepsilon}) - \int_{D(0, \varepsilon)} |(\nabla - i h (a + F_{\varepsilon}))\psi|^2 \, dx$$

$$\geq E_{\varepsilon}(h) - \int_{D(0, \varepsilon)} |(\nabla - i h (a + F_{\varepsilon}))\psi|^2 \, dx.$$ 

Using (5.2) and (5.3), we get that 

$$\lim_{\varepsilon \to 0^+} \int_{D(0, \varepsilon)} |(\nabla - i h (a + F_{\varepsilon}))\psi|^2 \, dx = 0$$

and consequently

$$E_{AB}(h) \geq \limsup_{\varepsilon \to 0^+} E_{\varepsilon}(h).$$  \hspace{1cm} (5.4) 

Combining this and (5.1), we get that $(\psi_*, a_*)$ is a minimizing configuration of $E_{ABh}$.

The degenerate case. It remains to prove the inequality (5.4) when $h = 2\pi n_0$ and $n_0 \in \mathbb{Z}$. We introduce the test function defined in polar coordinates as follows 

$$w_{\varepsilon} = \chi_{\varepsilon, p}(r)u(\theta)$$

where 

$$p \in (0, 1), \quad \chi_{\varepsilon, p}(r) = \begin{cases} \left(\frac{r}{\varepsilon}\right)^p & \text{if } 0 < r < \sqrt{\varepsilon}, \\ 1 & \text{if } r \geq \sqrt{\varepsilon} \end{cases} \quad \text{and} \quad u(\theta) = e^{i n_0 \theta}.$$ 

Clearly, $w_{\varepsilon} \in H^1(\Omega; \mathbb{C})$ and 

$$\lim_{\varepsilon \to 0^+} \int_{\Omega} |w_{\varepsilon}|^2 \, dx = \lim_{\varepsilon \to 0^+} \int_{\Omega} |w_{\varepsilon}|^4 \, dx = |\Omega|.$$ 

Knowing that $(\nabla - i F_{AB})u = 0$ (see the proof of Proposition 3.8), and that $F_{AB} = F_{\varepsilon}$ in $\Omega \setminus D(0, \varepsilon)$, we get

\[
\int_{D(0, \sqrt{\varepsilon})} |(\nabla - i F_{\varepsilon})w_{\varepsilon}|^2 \, dx \\
= 2\pi \int_0^{\sqrt{\varepsilon}} \int_0^\infty |\partial_r w_{\varepsilon}|^2 r \, dr + \int_0^{\sqrt{\varepsilon}} \frac{1}{r} |(\partial_\theta - i n_0) w_{\varepsilon}|^2 r \, dr + \int_0^{\sqrt{\varepsilon}} \frac{1}{r} \left| \frac{\partial_\theta - i n_0 \varepsilon^2}{\varepsilon^2} \right| w_{\varepsilon}^2 \, dr \\
= 2\pi \int_0^{\sqrt{\varepsilon}} \left| \chi_{\varepsilon, p}(r)^2 \right| r \, dr + 2\pi \int_0^{\sqrt{\varepsilon}} \int_0^{n_0^2 / \varepsilon^2} \left(1 - \frac{r^2}{\varepsilon^2}\right)^2 \left| \chi_{\varepsilon, p}(r)^2 \right| r \, dr \\
\leq \pi \left( p + \frac{n_0^2 \varepsilon^p}{p} \right). 
\]

Writing $E_{\varepsilon}(h) \leq E_{h, \varepsilon}(w, F_{\varepsilon})$ then taking the limit as $\varepsilon \to 0^+$, we infer from the foregoing considerations that 

$$\limsup_{\varepsilon \to 0^+} E_{\varepsilon}(h) \leq \pi p - \frac{\kappa^2}{2} |\Omega|.$$ 

Now we send $p$ to 0 and get

$$\limsup_{\varepsilon \to 0^+} E_{\varepsilon}(h) \leq - \frac{\kappa^2}{2} |\Omega|,$$

which yields the inequality in (5.4) in the case $h \in 2\pi \mathbb{Z}$, thanks in particular to Proposition 3.8.
End of the proof. Having proved that \( \lim_{\varepsilon \to 0^+} E_\varepsilon(h) = E_{AB}(h) \), the remaining statements in Theorem 1.1 follow from Proposition 3.8. \( \Box \)

Proof of Theorem 1.2

Let \((\psi, a)_{\kappa, h}\) be a critical configuration of \(\mathcal{E}_{AB, \kappa} \), i.e. a solution of (3.2). We denote by \( \| \cdot \|_{p} \) the usual norm in \( L^p(\Omega) \). Expanding the term \( \| (\nabla - ih(a + F_{AB}))\psi \|_2^2 \), we find

\[
\| (\nabla - ih(a + F_{AB}))\psi \|_2^2 = \| (\nabla - ihF_{AB})\psi \|_2^2 - h^2\| a\psi \|_2^2 - 2h\langle j, a \rangle ,
\]

where

\[
j = \text{Im}(\nabla - ih(a + F_{AB}))\psi = -h\nabla^*\text{curl} a
\]

by the second equation in (3.2). Consequently, by integration by parts,

\[
\langle j, a \rangle = -h\langle \nabla^*\text{curl} a, a \rangle = h\langle \text{curl} a, \text{curl} a \rangle = \| \text{curl} a \|_2^2 .
\]

Therefore, after introducing the energy

\[
\mathcal{E}_0(\psi, a + F_{AB}) = \| (\nabla - ih(a + F_{AB}))\psi \|_2^2 - h^2\| a\psi \|_2^2 - \kappa^2\| \psi \|_2^2 + \frac{\kappa^2}{2}\| \psi \|_4^4
\]

we get the useful identity

\[
\mathcal{E}_0(\psi, a + F_{AB}) = \mathcal{E}_0(\psi, F_{AB}) - h^2\| a\psi \|_2^2 - 2h^2\| \text{curl} a \|_2^2 .  \tag{5.5}
\]

By the first equation in (3.2), \( \mathcal{E}_0(\psi, a + F_{AB}) = -\frac{\kappa^2}{2}\| \psi \|_4^4 \leq 0 \), hence we infer from (5.5) the following estimate

\[
0 \geq (\lambda_{AB}(h) - \kappa^2)\| \psi \|_2^2 - h^2\| a\psi \|_2^2 - 2h^2\| \text{curl} a \|_2^2 , \tag{5.6}
\]

after applying the min-max principle and the Hölder inequality to estimate the terms \( \| (\nabla - ihF_{AB})\psi \|_2^2 \) and \( \| a\psi \|_2^2 \) respectively.

We estimate the term \( \| \text{curl} a \|_2^2 \) using the second equation in (3.2) as follows,

\[
h^2\| \nabla \text{curl} a \|_2^2 \leq \| \psi (\nabla - ih(a + F_{AB}))\psi \|_2 \leq \| \psi \|_2^2\| (\nabla - ih(a + F_{AB}))\psi \|_2^2 \leq |\Omega|\kappa^2\| \psi \|_2^2 ,
\]

after using the estimates (3)-(4) in Proposition 3.6.

Recall the constant \( C_*(\Omega) \) introduced in (1.17). We now infer from (5.6) the following inequality

\[
0 \geq (\lambda_{AB}(h) - \kappa^2 - C_*(\Omega)\kappa^2)\| \psi \|_2^2 .
\]

This yields that \( \| \psi \|_2^2 = 0 \) when \( \kappa \) and \( h \) satisfy the relation \( \kappa^2 < (1 + C_*(\Omega))^{-1}\lambda_{AB}(h) \). \( \Box \)

Proof of Theorem 1.4

Step 1.

Choose \( \varepsilon_0 \in (0, 1) \) such that \( \overline{D(0, \varepsilon_0)} \subset \Omega \). For all \( \varepsilon \in (0, \varepsilon_0) \), we introduce the auxiliary eigenvalue, \( \lambda_\varepsilon(h, \Omega) \), in the perforated domain \( \Omega_\varepsilon := \Omega \setminus \overline{D(0, \varepsilon)} \), defined as follows,

\[
\lambda_\varepsilon(h, \Omega) = \inf_{u \in H^2_0(\Omega_\varepsilon)} \frac{\| (\nabla - ihF_{AB})u \|_{L^2(\Omega_\varepsilon)}^2}{\| u \|_{L^2(\Omega_\varepsilon)}^2} . \tag{5.7}
\]

Note that the flux of \( hF_{AB} \) around the hole \( D(0, \varepsilon) \) is

\[
\Phi_0 := \frac{1}{2\pi} \int_{|x| = \varepsilon} F_{AB}(x) \cdot dx = \frac{h}{2\pi} .
\]

By Thm. 1.1, for \( \varepsilon \in (0, \varepsilon_0) \), \( h \geq 0 \) and \( k \in \mathbb{Z} \), it holds the following,

\[
\lambda_\varepsilon(0, \Omega) = 0 \leq \lambda_\varepsilon(h, \Omega) \leq \lambda_\varepsilon(\pi, \Omega) \text{ and } \lambda_\varepsilon(h + 2k\pi, \Omega) = \lambda_\varepsilon(h, \Omega) . \tag{5.8}
\]

Furthermore, if \( h \in 2\pi\mathbb{Z} \), the function \( u_0 = e^{\frac{ih}{\pi}x} \) is a zero mode for the operator \( -((\nabla - ihF_{AB})^2) \), hence

\[
\forall h \in 2\pi\mathbb{Z} , \quad \lambda_\varepsilon(h, \Omega) = 0 = \lambda_{AB}(h, \Omega) . \tag{5.9}
\]
We shall show that
\[
\lim_{\varepsilon \to 0_+} \lambda_{\varepsilon}(h, \Omega) = \lambda_{AB}(h, \Omega), \quad \forall h \geq 0.
\] (5.10)

Then the $2\pi$-periodicity of $\lambda_{AB}(h, \Omega)$ in $h$ follows from [X].

**Step 2.**
Denote by $u \in H^1_{hF_{AB}}(\Omega; \mathbb{C})$ a normalized ground state of $\lambda_{AB}(h, \Omega)$. By the min-max principle,
\[
\lambda_{AB}(h, \Omega) \geq \| (\nabla - ihF_{AB})u \|_{L^2(\Omega)}^2 \geq \lambda_{\varepsilon}(h, \Omega) \| u \|_{L^2(\Omega)}^2.
\]

Consequently,
\[
\lambda_{AB}(h, \Omega) \geq \lim_{\varepsilon \to 0_+} \lambda_{\varepsilon}(h, \Omega)
\] (5.11)
since, by dominated convergence,
\[
\lim_{\varepsilon \to 0_+} \| u \|_{L^2(\Omega)}^2 = \| u \|_{L^2(\Omega)}^2 = 1.
\]

**Step 3.**
Assume that $0 < r < \varepsilon_0$ and $h \notin 2\pi \mathbb{Z}$. For all $\varepsilon \in (0, r)$, denote by $u_{\varepsilon} \in H^1(\Omega_\varepsilon; \mathbb{C})$ a normalized ground state of $\lambda_{\varepsilon}(h, \Omega)$; the eigenvalue equation $-(\nabla - ihF_{AB})^2 u_{\varepsilon} = \lambda_{\varepsilon}(h, \Omega) u_{\varepsilon}$ yields that
\[
\| u_{\varepsilon} \|_{H^2(\Omega_\varepsilon)} \leq C_r
\]
for some constant $C_r$ independent from $\varepsilon$. By a diagonal sequence argument, we can extract a sequence $(u_{\varepsilon_n})_{n \geq 1}$ and a function $u_* : \Omega \setminus \{0\} \to \mathbb{C}$ such that $(u_{\varepsilon_n})_{n \geq 1}$ converges to $u_*$ in $H^1(\Omega)$ for $0 < r < \varepsilon_0$.

It then results the following two inequalities,
\[
\| u_* \|_{L^2(\Omega_r)}^2 = \lim_{n \to +\infty} \| u_{\varepsilon_n} \|_{L^2(\Omega_r)}^2 \leq 1,
\]
and
\[
\liminf_{\varepsilon \to 0_+} \lambda_{\varepsilon}(h, \Omega) \geq \liminf_{n \to +\infty} \lambda_{\varepsilon_n}(h, \Omega) \geq \liminf_{n \to +\infty} \| (\nabla - ihF_{AB}) u_{\varepsilon_n} \|_{L^2(\Omega_r)}^2 = \| (\nabla - ihF_{AB}) u_* \|_{L^2(\Omega_r)}^2.
\]

Sending $r$ to 0 and using monotone convergence, we deduce that $\| u_* \|_{L^2(\Omega)} \leq 1$, $u_* \in H^1_{hF_{AB}}(\Omega)$ and
\[
\liminf_{\varepsilon \to 0_+} \lambda_{\varepsilon}(h, \Omega) \geq \| (\nabla - ihF_{AB}) u_* \|_{L^2(\Omega)}^2 \geq \lambda_{AB}(h, \Omega) \| u_* \|_{L^2(\Omega)}^2.
\] (5.12)

Let us prove that
\[
\limsup_{n \to +\infty} \left( \frac{1}{r} \| u_{\varepsilon_n} \|_{L^2(D(0,r) \setminus D(0,\varepsilon_n))}^2 \right) < +\infty.
\] (5.13)
The inequality in (5.13) results immediately from (5.11) and Lemma 3.7 in fact
\[
\left( \frac{\alpha(h)}{2\pi r} \right)^2 \| u_{\varepsilon_n} \|_{L^2(D(0,r) \setminus D(0,\varepsilon_n))}^2 \leq \| (\nabla - ihF_{AB}) u_{\varepsilon_n} \|_{L^2(D(0,r) \setminus D(0,\varepsilon_n))}^2 \leq \lambda_{\varepsilon}(h).
\]
It now results from (5.13)
\[
\| u_{\varepsilon_n} \|_{L^2(\Omega_r)}^2 = 1 - \| u_{\varepsilon_n} \|_{L^2(D(0,r) \setminus D(0,\varepsilon_n))} \geq 1 - \tilde{C} r,
\]
where $\tilde{C} > 0$ is independent from $r$ and $\varepsilon_n$; consequently,
\[
\| u_* \|_{L^2(\Omega_r)} \geq \lim_{n \to +\infty} \| u_{\varepsilon_n} \|_{L^2(\Omega_r)} \geq 1 - \tilde{C} r.
\]

Sending $r$ to 0, we get by monotone convergence that $\| u_* \|_{L^2(\Omega)}^2 \geq 1$. Inserting this into (5.12), we get
\[
\lim_{\varepsilon \to 0_+} \lambda_{\varepsilon}(h, \Omega) \geq \lambda_{AB}(h, \Omega) \quad (h \notin 2\pi \mathbb{Z}).
\]

Collecting this inequality, (5.11) and (5.9), we get (5.10) for $h \notin 2\pi \mathbb{Z}$.

**Step 4.**
Let us prove that $\lambda_{AB}(h, \Omega) > 0$ for $h \in (0, 2\pi)$. Choose $r > 0$ so that $D(0, r) \subset \Omega$. Suppose that $\lambda_{AB}(h) = 0$. By the min-max principle and [15] Prop. 2.1

$$0 = \lambda_{AB}(h, \Omega) \geq \lambda_{AB}(h, D(0, r)) \int_{D(0, r)} |u|^2 \, dx \geq \frac{\alpha(h)}{4r^2} \int_{D(0, r)} |u|^2 \, dx,$$

with $\alpha(h) > 0$ for $h \in (0, 2\pi)$, see [15]. Hence, $u = 0$ on $D(0, r)$; by the min-max principle and the diamagnetic inequality,

$$0 = \lambda_{AB}(h, \Omega) \geq \lambda^{N,D}(\Omega \setminus D(0, r)) \int_{\Omega \setminus D(0, r)} |u|^2 \, dx,$$

where $\lambda^{N,D}(\Omega \setminus D(0, r)) > 0$ is the eigenvalue of the Laplace operator with Neumann condition on $\partial \Omega$ and Dirichlet condition on $\partial D(0, r)$. This proves that $u = 0$ on $\Omega \setminus D(0, r)$ too and contradicts the fact that $u$ is a normalized eigenfunction of $\lambda_{AB}(h, \Omega)$.

**Step 5.**

As mentioned above, that $\lambda_{AB}(h)$ is $2\pi$-periodic follows from [6.5] and [6.10]. By Step 4, $\lambda_{AB}(h, \Omega) = 0$ if $h \in 2\pi \mathbb{Z}$.

To finish the proof of Theorem 1.4, it remains to show that $\lambda_{AB}(h, \Omega)$ is continuous at every $h_0 \in (0, 2\pi)$. If $h_0 \in (0, 2\pi)$, the result is a simple application of the min-max principle, since, in some neighborhood $I_0 \subset (0, 2\pi)$ of $h_0$, the space $H^1_{F_{AB}}(\Omega; \mathbb{C})$ is simply $H^1(\Omega; \mathbb{C}) \cap L^2(\Omega; \mathbb{C}; |F_{AB}|^2 \, dx)$ (see Proposition 3.6 and Lemma 3.7). So we treat the case $h_0 = 0$; we would like to prove that

$$\lim_{\epsilon \to 0^+} \lambda_{AB}(h, \Omega) = 0. \tag{5.14}$$

We introduce the following quasi-mode

$$u_0(x) = \begin{cases} 1 & \text{if } |x| > \epsilon_0 \\ |x| & \text{if } |x| < \epsilon_0. \end{cases}$$

The min-max principle and a straightforward computation yield

$$0 < \lambda_{AB}(h) \leq \frac{\|(-i\hbar F_{AB})u_0\|^2_{L^2(\Omega)}}{\|u_0\|^2_{L^2(\Omega)}} \leq C_0 h^2,$$

for some constant $C_0$ independent of $h \in (0, 2\pi)$; sending $h$ to 0, we get (5.14). $\square$

**Acknowledgements.** This work was partially supported by NYU-ECNU JRI Seed Fund for Collaborative Research. A. Kachmar’s research is partially supported by the Lebanese University in the framework of the project “Analytical and Numerical Aspects of the Ginzburg-Landau Model”. X.B. Pan was partially supported by the National Natural Science Foundation of China grant no. 11671143, and 11431005.

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