Entropy stabilization and property-preserving limiters for discontinuous Galerkin discretizations of nonlinear hyperbolic equations

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Abstract

The methodology proposed in this paper bridges the gap between entropy stable and positivity-preserving discontinuous Galerkin (DG) methods for nonlinear hyperbolic problems. The entropy stability property and, optionally, preservation of local bounds for the cell averages are enforced using flux limiters based on entropy conditions and discrete maximum principles, respectively. Entropy production by the (limited) gradients of the piecewise-linear DG approximation is constrained using Rusanov-type entropy viscosity, as proposed by Abgrall in the context of nodal finite element approximations. We cast his algebraic entropy fix into a form suitable for arbitrary polynomial bases and, in particular, for modal DG approaches. The Taylor basis representation of the entropy stabilization term reveals that it penalizes the solution gradients in a manner similar to slope limiting and requires semi-implicit treatment to achieve the desired effect. The implicit Taylor basis version of the Rusanov entropy fix preserves the sparsity pattern of the element mass matrix. Hence, no linear systems need to be solved if the Taylor basis is orthogonal and an explicit treatment of the remaining terms is adopted. The optional application of a vertex-based slope limiter constrains the piecewise-linear DG solution to be bounded by local maxima and minima of the cell averages. The combination of entropy stabilization with flux and slope limiting leads to constrained approximations that possess all desired properties. Numerical studies of the new limiting techniques and entropy correction procedures are performed for two scalar two-dimensional test problems with nonlinear and nonconvex flux functions.

Keywords: hyperbolic conservation laws, entropy stability, invariant domain preservation, discontinuous Galerkin methods, flux correction, slope limiting

1. Introduction

In recent years, significant advances have been made in the analysis and design of property-preserving high-resolution finite element schemes for hyperbolic problems. The essential properties of a physics-compatible approximation include entropy stability and positivity preservation. Entropy...
stable discontinuous Galerkin (DG) methods [9, 34] are usually derived using entropy conservative numerical fluxes and additional dissipative terms (fluxes and/or element contributions) depending on the gradients of entropy variables. Tadmor’s seminal work [38] provides a general entropy stability criterion for the analysis and design of such schemes [39, 40]. In the case of a piecewise-linear or higher-order DG approximation, excess entropy production by the gradients of the conserved quantities must be balanced using nonlinear artificial diffusion operators and/or limiters. The entropy correction term proposed by Abgrall [2] and its generalizations presented in [3] penalize the deviations of entropy variables from their cell averages using Rusanov-type entropy viscosity. Artificial diffusion operators of this kind are also widely used to enforce local discrete maximum principles, preservation of invariant domains, and/or positivity preservation in low-order components of residual distribution methods [1, 19] and algebraic flux correction (AFC) schemes [15, 25, 27] for hyperbolic problems. The use of flux and slope limiters makes it possible to adjust the amounts of artificial diffusion or the gradients of the numerical solution in an adaptive manner. A variety of algebraic [4, 5, 16, 19] and geometric [7, 23] limiting techniques can be found in the literature on finite volume and DG methods for hyperbolic conservation laws. The most recent approaches are backed by theoretical proofs of positivity preservation for cell averages and/or solution values at certain control points [11, 14, 20, 32, 41, 42]. However, positivity-preserving high-resolution schemes may converge to wrong weak solutions if they are not entropy stable [15, 29]. Conversely, an entropy stable high-order method may exhibit excellent convergence behavior in smooth regions but produce undershoots and overshoots in shock regions.

An algebraic limiting framework that ensures both entropy stability and preservation of local bounds was introduced in [29] in the context of AFC schemes for continuous Galerkin methods. In the present paper, we constrain piecewise-linear ($P_1$) Taylor basis DG discretizations using similar flux correction tools in addition to slope limiting. The proposed flux limiter guarantees that the cell averages satisfy a semi-discrete entropy inequality and a local maximum principle. The rates of entropy production and dissipation inside mesh cells are balanced using built-in gradient penalization which corresponds to a diagonal Taylor basis form of Abgrall’s [2] correction term. In the process of Runge-Kutta time integration, we treat this term implicitly and apply a vertex-based version [23] of the Barth-Jespersen [7] slope limiter. The final DG-$P_1$ approximation stays in the range determined by the local maxima and minima of property-preserving cell averages. In Sections 2-4, we present the new correction tools and explain the underlying design principles. The numerical examples of Section 5 illustrate the implications of entropy stability and the capability of the proposed algorithms to enforce the desired properties. We close this paper with a summary of the results and possible extensions in Section 6.

2. Entropy stabilization of DG schemes

Let $u(x,t)$ be a scalar conserved quantity depending on the space location $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$ and time instant $t \geq 0$. Consider an initial value problem of the form

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+,$$

(1a)

$$u(\cdot,0) = u_0 \quad \text{in } \mathbb{R}^d,$$

(1b)
where \( f = (f_1, \ldots, f_d) \) is a possibly nonlinear flux function and \( u_0 : \mathbb{R}^d \to \mathcal{G} \) is an initial data belonging to a convex set \( \mathcal{G} \). The set \( \mathcal{G} \) is called an invariant set of problem (1a)–(1b) if the exact solution \( u \) stays in \( \mathcal{G} \) for all \( t > 0 \) [17]. A convex function \( \eta : \mathcal{G} \to \mathbb{R} \) is called an entropy and \( v = \eta' \) is called an entropy variable if there exists an entropy flux \( q : \mathcal{G} \to \mathbb{R}^d \) such that \( v(u)f'(u) = q'(u) \). A weak solution \( u \) of (1a) is called an entropy solution if the entropy inequality

\[
\frac{\partial \eta}{\partial t} + \nabla \cdot q(u) \leq 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+
\]  

holds for any entropy pair \((\eta, q)\). For any smooth weak solution, the conservation law

\[
\frac{\partial \eta}{\partial t} + \nabla \cdot q(u) = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+
\]  

can be derived from (1a) using multiplication by the entropy variable \( v \), the chain rule, and the definition of an entropy pair. Hence, entropy is conserved in smooth regions and dissipated at shocks.

A numerical scheme is called positivity-preserving [32, 41, 42] or, more formally, invariant domain preserving (IDP) [14, 17, 20, 25] if the solution of the (semi-)discrete problem is guaranteed to stay in an invariant set \( \mathcal{G} \). A well-designed discretization of (1a) should also be entropy stable, i.e., a (semi-)discrete version of the entropy inequality (2) should hold. The lack of entropy stability is a typical reason for convergence of numerical methods to nonphysical weak solutions.

Let us discretize (1a) in space using a piecewise-linear DG approximation \( u_h \) on a computational mesh \( T_h \) consisting of \( E_h \) elements. The restriction of \( u_h \) to element \( K_i, i = 1, \ldots, E_h \) is a linear polynomial \( u_{ih} \in \mathbb{P}_1(K_i) \) which can be expressed in terms of basis functions \( \varphi_{ik} \in \mathbb{P}_1(K_i) \) thus:

\[
u_{ih} = \sum_{k=0}^{d} u_{ik} \varphi_{ik}, \quad i = 1, \ldots, E_h.
\]  

In this work, we use the modal Taylor basis which is defined by [23, 24, 31]

\[
\varphi_{i0} \equiv 1, \quad \varphi_{ik}(x) = \frac{e_k \cdot (x - x_{i0})}{\Delta x_{ik}}, \quad k = 1, \ldots, d,
\]  

where \( e_k = (\delta_{kl})_{l=1}^d \) is a standard basis vector of the Euclidean space \( \mathbb{R}^d \), \( x_{i0} = \frac{1}{|K_i|} \int_{K_i} x \, dx \) is the centroid of element \( K_i \) and \( \Delta x_{ik} = \max_{x,y \in K_i} |e_k \cdot (x - y)| \) is a scaling factor which improves the condition number of the mass matrix. The Taylor degrees of freedom

\[
u_{i0} = \frac{1}{|K_i|} \int_{K_i} u_{ih}(x) \, dx, \quad u_{ik} = \Delta x_{ik} \frac{\partial u_{ih}}{\partial x_k} \bigg|_{K_i}, \quad k = 1, \ldots, d
\]  

represent the cell average and the constant partial derivatives of the linear Taylor polynomial \( u_{ih} \).

For simplicity, we assume that the whole boundary of \( K_i \) lies in the interior of the computational domain \( \Omega_h = \bigcup_{i=1}^{E_h} K_i \) or periodic boundary conditions are imposed. Let \( E_i \) denote the integer set containing the numbers of mesh cells that share a side (boundary point in 1D, edge in 2D, face in 3D)
\[ S_{ij} = \partial K_i \cap \partial K_j \] with \( K_i \). Substituting (4) into a weak form of (1a) and using the notation \( n_{ij} \) for the unit outward normal to \( S_{ij} \), we obtain \( N_h = (d + 1)E_h \) semi-discrete equations of the form

\[
\sum_{l=0}^{d} \frac{d}{dt} m_{i,kl} \int_{K_i} \varphi_{ik} \varphi_{il} \, dx = - \sum_{j \in E_i} \int_{S_{ij}} \varphi_{ik} H(u_{ih}, u_{jh}, n_{ij}) \, ds + \int_{K_i} \nabla \varphi_{ik} \cdot f(u_{ih}) \, dx,
\]

where \( H(u_L, u_R, n) \) is a Lipschitz-continuous numerical approximation to the flux \( f(u) \cdot n \) such that \( H(u, u, n) = f(u) \cdot n \) and \( H(u_L, u_R, n) + H(u_R, u_L, -n) = 0 \).

The system of equations (7) for the time-dependent Taylor degrees of freedom can be written as

\[
\sum_{l=0}^{d} m_{i,kl} \frac{d}{dt} u_{il} = - \sum_{j \in E_i} |S_{ij}| F_{ij,k} + \int_{K_i} \nabla \varphi_{ik} \cdot f(u_{ih}) \, dx, \quad i = 1, \ldots, E_h,
\]

\[
m_{i,kl} = \int_{K_i} \varphi_{ik} \varphi_{il} \, dx, \quad F_{ij,k} = \frac{1}{|S_{ij}|} \int_{S_{ij}} \varphi_{ik} H(u_{ih}, u_{jh}, n_{ij}) \, ds.
\]

Note that \( m_{i,00} = |K_i| \) and \( m_{i,0l} = 0 = m_{i,k0} \) for \( k, l \in \{1, \ldots, d\} \). If the Taylor basis is orthogonal, which is the case, e.g., on uniform rectangular meshes, then the mass matrix \( (m_{i,kl})_{k,l=0}^{d} \) is diagonal.

For a given convex entropy \( \eta(u) \), the corresponding entropy variable \( v = \eta'(u) \) can be approximated by the linear polynomial \( v_{ih} = \sum_{k=0}^{d} v_{ik} \varphi_{ik} \) with Taylor coefficients

\[
v_0 = v(u_0), \quad v_{ik} = \eta''(u_0) u_{ik}, \quad k = 1, \ldots, d.
\]

Let \( b_i : \mathbb{P}_1(K_i) \times \mathbb{P}_1(K_i) \to \mathbb{R}_0^+ \) be a symmetric positive definite bilinear form. Define

\[
D_i(v_{ih}, w_{ih}) = b_i(v_{ih} - v_0, w_{ih} - v_0), \quad v_{ih}, w_{ih} \in \mathbb{P}_1(K_i).
\]

The assumption of positive definiteness (coercivity) implies that \( D_i(v_{ih}, v_{ih}) = 0 \) for the constant function \( v_{ih} \equiv v_0 \) and \( D_i(v_{ih}, v_{ih}) > 0 \) for all \( v_{ih} \in \mathbb{P}_1(K_i) \setminus \{v_0\} \).

As noticed by Abgrall [2], a dissipative stabilization term of the form \( \nu_i D_i(\varphi_{ik}, v_{ih}) \) can be used to control the rate entropy production inside \( K_i \). Leaving the value of the stabilization parameter \( \nu_i \geq 0 \) unspecified for the time being, we consider the entropy-corrected DG approximation

\[
\sum_{l=0}^{d} m_{i,kl} \frac{d}{dt} u_{il} = - \sum_{j \in E_i} |S_{ij}| F_{ij,k} + \int_{K_i} \nabla \varphi_{ik} \cdot f(u_{ih}) \, dx - \nu_i D_i(\varphi_{ik}, v_{ih}).
\]

If \( \varphi_{ik} \) are defined by (5), then \( D_i(\varphi_{0i}, v_{ih}) = 0 \) and \( D_i(\varphi_{ik}, v_{ih}) = b_i(\varphi_{ik}, v_{ih} - v_0) \) for \( k = 1, \ldots, d \).

By the chain rule, we have \( \frac{\partial \eta(u_{ih})}{\partial u_{ih}} = \eta'(u_{ih}) \frac{\partial u_{ih}}{\partial t} \), where \( \eta'(u_{ih}) = \nu(u_{ih}) \). It follows that

\[
\frac{d}{dt} \int_{K_i} \eta(u_{ih}) \, dx = \int_{K_i} v(u_{ih}) \frac{\partial u_{ih}}{\partial t} \, dx = \int_{K_i} \left[ v_{ih} \frac{\partial u_{ih}}{\partial t} + (v(u_{ih}) - v_{ih}) \frac{\partial u_{ih}}{\partial t} \right] \, dx.
\]
Multiplying the semi-discrete conservation law (12) by the Taylor coefficients \( v_{ik} \) of \( v_i \), summing over \( k = 0, \ldots, d \), and substituting the result into (13), we obtain the evolution equation

\[
|K_i| \frac{d\eta_i}{dt} = P_i(v\sub{ih}, u\sub{ih}) - \nu_i D_i(v\sub{ih}, v\sub{ih}),
\]

where \( \eta_i = \frac{1}{|K_i|} \int_{K_i} \eta(u\sub{ih}(x)) \, dx \) is the average entropy in \( K_i \) and

\[
P_i(v\sub{ih}, u\sub{ih}) = -\sum_{j \in E_i} |S_{ij}| \left( \sum_{k=0}^{d} v_{ik} F_{ij,k} \right) + \int_{K_i} \left[ \nabla v\sub{ih} \cdot f(u\sub{ih}) + (v(u\sub{ih}) - v\sub{ih}) \frac{\partial u\sub{ih}}{\partial t} \right] \, dx.
\]

**Remark 1.** The contribution of \( (v(u\sub{ih}) - v\sub{ih}) \frac{\partial u\sub{ih}}{\partial t} \) vanishes for the square entropy \( \eta(u) = \frac{u^2}{2} \). In general, the coefficients of \( \frac{\partial u\sub{ih}}{\partial t} = \sum_{i=0}^{d} \frac{d\psi_{ih}}{dt} \) are obtained by solving system (7) for \( \frac{d\psi_{ih}}{dt} \).

Suppose that the numerical fluxes \( H(\cdot, \cdot, \cdot) \) and stabilization parameters \( \nu_i \geq 0 \) are chosen to satisfy

\[
P_i(v\sub{ih}, u\sub{ih}) + \sum_{j \in E_i} |S_{ij}| G_{ij} \leq \nu_i D_i(v\sub{ih}, v\sub{ih}), \quad i = 1, \ldots, E_h,
\]

where \( G_{ij} \) is a consistent approximation to the average value \( \frac{1}{|S_{ij}|} \int_{S_{ij}} q(u) \cdot n \, ds \) of the entropy flux \( q(u) \) associated with \( v(u) \). Then a solution \( u\sub{ih} \) of (12) satisfies the cell entropy inequality

\[
|K_i| \frac{d\eta_i}{dt} + \sum_{j \in E_i} |S_{ij}| G_{ij} \leq 0
\]

which implies entropy stability of the semi-discrete DG scheme. If \( v\sub{ih} \neq v\sub{i0} \) then \( D_i(v\sub{ih}, v\sub{ih}) > 0 \) and for any finite value of \( P_i(v\sub{ih}, u\sub{ih}) \) the validity of (16) can be readily enforced using

\[
\nu_i = \begin{cases} 
\max \left\{ 0, \frac{P_i(v\sub{ih}, u\sub{ih}) + \sum_{j \in E_i} |S_{ij}| G_{ij}}{D_i(v\sub{ih}, v\sub{ih})} \right\} & \text{if } D_i(v\sub{ih}, v\sub{ih}) > 0, \\
0 & \text{if } D_i(v\sub{ih}, v\sub{ih}) = 0.
\end{cases}
\]

(18)

To ensure the validity of (16) for \( v\sub{ih} = v\sub{i0} \) and \( D_i(v\sub{ih}, v\sub{ih}) = 0 \), we constrain the averaged fluxes \( F_{ij,0} = \frac{1}{|S_{ij}|} \int_{S_{ij}} H(u\sub{ih}, u\sub{i0}, n\sub{ij}) \, ds \) to satisfy the entropy stability condition (cf. [9, 29, 34, 39])

\[
(v\sub{j0} - v\sub{i0}) F_{ij,0} \leq (\psi(u\sub{j0}) - \psi(u\sub{i0})) \cdot n\sub{ij},
\]

where

\[
\psi(u) = v(u) f(u) - q(u).
\]

Condition (19) provides a useful tool for derivation of entropy stable schemes. The following theorem shows that it is, indeed, sufficient for (16) and (17) to hold if \( v\sub{ih} = v\sub{i0} \) and \( D_i(v\sub{ih}, v\sub{ih}) = 0 \).
The possibility of using flux limiting to enforce entropy inequalities in addition to LED constraints was methods, and other representatives of local extremum diminishing (LED) flux correction schemes [26].

Theorem 1 (Entropy stability criterion). The semi-discrete DG scheme (12) is entropy stable if the fluxes $F_{ij,0}$ satisfy condition (19) and the parameter $\nu_i$ is defined by (18).

Proof. For $D_i(v_{ih}, v_{ih}) > 0$, the validity of (16) and (17) follows from (18). If $D_i(v_{ih}, v_{ih}) = 0$, then $v_{ih} = v_{i0}$ and the entropy production term reduces to $P_i(v_{i0}, u_{ih}) = -\sum_{j \in E_i} |S_{ij}| v_{i0} F_{ij,0}$. In view of the assumption that the flux $F_{ij,0}$ satisfies condition (19), we have the estimate

$$v_{i0} F_{ij,0} = \frac{v_{i0} + v_{i0}}{2} F_{ij,0} - \frac{v_{i0} - v_{i0}}{2} F_{ij,0} \geq \frac{v_{i0} + v_{i0}}{2} F_{ij,0} - \frac{1}{2}(\psi(u_{i0}) - \psi(u_{i0})) \cdot n_{ij}.$$

Exploiting the fact that $\psi(u_{i0}) \cdot \sum_{j \in E_i} |S_{ij}| n_{ij} = 0$, we can now estimate $P_i$ as follows:

$$P_i(v_{i0}, u_{ih}) \leq -\sum_{j \in E_i} |S_{ij}| \left( \frac{v_{i0} + v_{i0}}{2} F_{ij,0} - \frac{1}{2}(\psi(u_{i0}) - \psi(u_{i0})) \cdot n_{ij} \right)$$

$$= -\sum_{j \in E_i} |S_{ij}| \left( \frac{v_{i0} + v_{i0}}{2} F_{ij,0} - \frac{1}{2}(\psi(u_{i0}) + \psi(u_{i0})) \cdot n_{ij} \right).$$

Thus, we have $P_i(v_{i0}, u_{ih}) + \sum_{j \in E_i} |S_{ij}| G_{ij} \leq 0 = \nu_i D(v_{i0}, v_{i0})$ for $G_{ij}$ defined by

$$G_{ij} = \frac{v_{i0} + v_{i0}}{2} F_{ij,0} - \frac{1}{2}(\psi(u_{i0}) + \psi(u_{i0})) \cdot n_{ij}.$$

This proves the validity of inequalities (16) and (17) in the case $v_{ih} = v_{i0}$. □

3. Flux limiting and entropy corrections

The above analysis provides general guidelines for the design of entropy stable DG-$P_1$ schemes. In this section, we present practical algorithms for calculating numerical fluxes that ensure not only entropy stability but also positivity preservation for the cell averages. We also define a stabilization operator $D_i(\cdot, v_{ih})$ which penalizes the gradients of $u_{ih}$ to satisfy condition (16). The methodology to be presented combines and generalizes the entropy correction tools developed in [2, 3] and [29].

Many nonlinear high-resolution schemes are based on the idea of blending a property-preserving low-order flux $F^L_{ij}$ and a high-order target flux $F^H_{ij}$. The former is supposed to satisfy inequality constraints that guarantee entropy stability and/or the validity of (local) maximum principles. Using an adaptively chosen correction factor $\alpha_{ij} \in [0, 1]$, the convex combination $F_{ij} = (1 - \alpha_{ij})F^L_{ij} + \alpha_{ij}F^H_{ij}$ can be constrained to satisfy them as well. If the flux $F^H_{ij}$ possesses the desired properties, then $\alpha_{ij} = 1$ is the optimal choice. Otherwise, the best inequality-constrained approximation corresponds to the largest value of $\alpha_{ij} \in [0, 1]$ for which the property constraints can be shown to hold. This design philosophy is common to flux-corrected transport (FCT) algorithms, total variation diminishing (TVD) methods, and other representatives of local extremum diminishing (LED) flux correction schemes [26]. The possibility of using flux limiting to enforce entropy inequalities in addition to LED constraints was
explored in [29] in the context of continuous Galerkin methods and Lagrange finite elements. Building on this work, we equip our Taylor DG-\(P_1\) scheme (12) with limited fluxes of the form

\[
F_{ij,k} = (1 - \alpha_{ij})F_{ij,k}^L + \alpha_{ij}F_{ij,k}^H, \quad k = 0, \ldots, d.
\]  

The general flux function of the local Lax-Friedrichs (LLF) method is defined by (cf. [9, 17])

\[
H_{LLF}(u_L, u_R, n) = \frac{f(u_R) + f(u_L)}{2} \cdot n - \lambda(u_L, u_R, n) \frac{u_R - u_L}{2},
\]  

where \(\lambda(u_L, u_R, n)\) is the maximal speed of wave propagation in the direction parallel to \(n\), i.e.,

\[
\lambda(u_L, u_R, n) = \max_{\omega \in [0,1]} |f'(\omega u_L + (1 - \omega)u_R) \cdot n|.
\]  

As shown by Chen and Shu [9], condition (19) is satisfied for the low-order LLF flux

\[
F_{ij,k}^L = \frac{1}{|S_{ij}|} \int_{S_{ij}} \varphi_{ik} H_{LLF}(u_{i0}, u_{j0}, n_{ij}) \, dx, \quad k = 0, \ldots, d.
\]  

The high-order LLF flux \(F_{ij,k}^H\) of the unconstrained DG-\(P_1\) approximation is given by

\[
F_{ij,k}^H = \frac{1}{|S_{ij}|} \int_{S_{ij}} \varphi_{ik} H_{LLF}(u_{ih}, u_{jh}, n_{ij}) \, dx, \quad k = 0, \ldots, d.
\]  

Let us first determine an entropy correction factor \(\alpha_{ij}^{ES}\) such that the limited LLF flux

\[
F_{ij,0} = (1 - \alpha_{ij})F_{ij,0}^L + \alpha_{ij}F_{ij,0}^H
\]  

satisfies (19) for any \(\alpha_{ij} \in [0, \alpha_{ij}^{ES}]\). Following the derivation of algebraic entropy fixes for continuous Galerkin methods in [29], we use \(\alpha_{ij}^{ES} \in \{\alpha_{ij}^{ES1}, \alpha_{ij}^{ES2}, \alpha_{ij}^{ES3}\}\), where \(\alpha_{ij}^{ES1} \leq \alpha_{ij}^{ES2} \leq \alpha_{ij}^{ES3}\) are correction factors corresponding to different definitions of the nonnegative bound \(Q_{ij}\) in the formula

\[
\alpha_{ij}^{ES} = \begin{cases} 
Q_{ij} / P_{ij} & \text{if } P_{ij} > Q_{ij}, \\
1 & \text{otherwise},
\end{cases} \quad P_{ij} = (v_{j0} - v_{i0})(F_{ij,0}^H - F_{ij,0}^L).
\]  

The least dissipative entropy stability preserving upper bound for the rate \(\alpha_{ij} P_{ij}\) of entropy increase in \(K_i\) due to the limited antidiffusive flux \(\alpha_{ij} (F_{ij,0}^H - F_{ij,0}^L)\) is given by

\[
Q_{ij}^{ES1} = (\psi(u_{j0}) - \psi(u_{i0})) \cdot n_{ij} - (v_{j0} - v_{i0})F_{ij,0}^L = (v_{j0} - v_{i0}) \frac{\lambda_{ij}}{2} (u_{j0} - u_{i0})
\]

\[
+ (\psi(u_{j0}) - \psi(u_{i0})) \cdot n_{ij} - (v_{j0} - v_{i0}) \frac{f(u_{j0}) + f(u_{i0})}{2} \cdot n_{ij}.
\]
Nonnegativity of $Q_{ij}^{ES1}$ follows from the fact that condition (19) holds for the low-order flux $F_{ij,0}$: It is easy to verify that $\alpha_{ij} P_{ij} \leq Q_{ij}^{ES1}$ is a sufficient condition for (26) to satisfy (19). In accordance with Tadmor’s [39, 40] comparison principle for entropy conservative and entropy stable schemes, definition (27) guarantees entropy stability for any $Q_{ij} \in [0, Q_{ij}^{ES1}]$. To prevent the limited flux (26) from producing more entropy than the centered flux $F_{ij,0} = \frac{f(u_{j0})+f(u_{i0})}{2} \cdot n_{ij}$, we use

$$Q_{ij}^{ES2} = \max \left\{ 0, (v_{j0} - v_{i0}) \frac{\lambda_{ij}}{2} (u_{j0} - u_{i0}) + \min \left\{ 0, Q_{ij}^{CD} \right\} \right\}, \quad (29)$$

where $\lambda_{ij} = \lambda(u_{i0}, u_{j0}, n_{ij})$ and $Q_{ij}^{CD} = (\psi(u_{j0}) - \psi(u_{i0})) \cdot n_{ij} - (v_{j0} - v_{i0}) F_{ij,0}^{CD}$. Definition

$$Q_{ij}^{ES3} = \max \left\{ 0, (v_{j0} - v_{i0}) \frac{\lambda_{ij}}{2} - v_{ij} \right\} (u_{j0} - u_{i0}) + \min \left\{ 0, Q_{ij}^{CD} \right\} \quad (30)$$

makes it possible to further increase the levels of entropy dissipation using (cf. [29])

$$\nu_{ij} = \begin{cases} \max \left\{ 0, \frac{f(u_{j0})+f(u_{i0})}{2} - f(u_{i0}+u_{j0}) \right\} n_{ij} & \text{if } v_{j0} \neq v_{i0}, \\ 0 & \text{if } v_{j0} \neq v_{i0}. \end{cases} \quad (31)$$

The numerical examples of Section 5 illustrate the ramifications of different choices of $Q_{ij}$.

It is also possible to find a correction factor $\alpha_{ij}^{BP}$ which ensures preservation of the local bounds

$$u_{i0}^{\min} = \min_{j \in E_i} u_{j0}, \quad u_{i0}^{\max} = \max_{j \in E_i} u_{j0}, \quad (32)$$

where $E_i^*$ is the set containing the numbers of all elements $E_j$ such that $E_i$ and $E_j$ have at least one common vertex. Note that $i = j$ is also an element of $E_i^*$. Adapting the monolithic convex limiting (MCL) strategy [25, 28, 29] to the DG setting, we impose the inequality constraints

$$u_{i0}^{\min} \leq \bar{u}_{ij,0} := \bar{u}_{ij,0} + \frac{\alpha_{ij} F_{ij,0}^A}{\lambda_{ij}} \leq u_{i0}^{\max}, \quad (33)$$

where $F_{ij,0}^A = F_{ij,0}^H - F_{ij,0}^L$ is the raw antidiffusive flux and $\bar{u}_{ij,0}$ are intermediate states such that

$$\min\{u_{i0}, u_{j0}\} \leq \bar{u}_{ij,0} := \frac{u_{j0} + u_{i0}}{2} - \frac{f(u_{j0}) - f(u_{i0})}{2 \lambda_{ij}} \cdot n_{ij} \leq \max\{u_{i0}, u_{j0}\}. \quad (34)$$

The validity of the MCL constraints (33) is guaranteed for $\alpha_{ij} \leq \alpha_{ij}^{BP}$, where (cf. [25, 28])

$$\alpha_{ij}^{BP} = \begin{cases} \min \left\{ 1, \frac{\lambda_{ij} \min\{u_{ij,0}^{\max} - \bar{u}_{ij,0} - u_{ij,0}^{\min}\}}{F_{ij,0}^A} \right\} & \text{if } F_{ij,0}^A > 0, \\ 1 & \text{if } F_{ij,0}^A = 0, \\ \min \left\{ 1, \frac{\lambda_{ij} \min\{\bar{u}_{ij,0} - u_{ij,0}^{\max}\}}{F_{ij,0}^A} \right\} & \text{if } F_{ij,0}^A < 0. \end{cases} \quad (35)$$
In the next section, we prove that the fully discrete version of the flux-limited DG scheme satisfies a local maximum principle if integration in time is performed using an SSP Runge-Kutta method.

In light of the above, the limited fluxes (21) should be defined using $\alpha_{ij} = \alpha_{ij}^{ES}$ to enforce entropy stability, $\alpha_{ij} = \alpha_{ij}^{BP}$ to ensure preservation of local bounds, and $\alpha_{ij} = \min\{\alpha_{ij}^{ES}, \alpha_{ij}^{BP}\}$ to provide both properties. To conclude the derivation of the semi-discrete property-preserving DG scheme, we need to choose a coercive bilinear form $b_i(\cdot, \cdot)$ for definition (11) of the entropy stabilization operator $D_i$. The bilinear form of the Rusanov-type dissipation term employed in [3] can be written as

$$b_i(v, w) = \int_{K_i} vw \, dx, \quad v, w \in L^2(K_i)$$

and induces

$$D_i(\varphi_{ik}, v_{ih}) = \int_{K_i} \varphi_{ik}(v_{ih} - v_{i0}) \, dx, \quad k = 0, \ldots, d,$$

where $v_{i0}$ is the cell average of $v_{ih}$. This representation of $D_i(\varphi_{ik}, v_{ih})$ is suitable not only for nodal (e.g., Lagrange, Bernstein or Gauss-Lobatto) finite element bases but also for the Taylor basis (5).

In essence, the addition of $\nu_i D_i(\varphi_{ik}, v_{ih})$ on the right-hand side of (12) penalizes the deviations of $u_{ih}(x) = u_{i0} + (x - x_{i0}) \cdot \nabla u_{ih}$ from $u_{i0} = u_{ih}(x_{i0})$. For the Taylor basis (5), we have

$$D_i(\varphi_{i0}, v_{ih}) = 0, \quad D_i(\varphi_{ik}, v_{ih}) = \sum_{l=1}^{d} m_{i,kl} v_{il}, \quad k = 1, \ldots, d,$$

where $m_{i,kl}$ are entries of the consistent Taylor mass matrix, as defined by (9). For an orthogonal Taylor basis, the mass matrix is diagonal and, therefore, definition (38) simplifies to

$$D_i(\varphi_{ik}, v_{ih}) = m_{i,kk} v_{ik}, \quad k = 1, \ldots, d.$$  

This simplified definition may be used in the case of a non-orthogonal Taylor basis as well. The associated coercive bilinear form is given by $\tilde{b}_i(v_{ih}, w_{ih}) = \sum_{k=0}^{d} \tilde{m}_{i,kk} v_{ik} w_{ik}$.

**Remark 2.** For a nodal basis $\{\hat{\varphi}_{ik}\}_{k=1}^{d+1}$, the $L^2$ scalar product $b_i(v_{ih}, w_{ih})$ can be approximated by $\tilde{b}_i(v_{ih}, w_{ih}) = \sum_{k=1}^{d+1} \hat{m}_{i,k} \hat{v}_{ik} \hat{w}_{ik}$, where $\hat{v}_{ik}$ are nodal values or Bernstein coefficients and $\hat{m}_{i,k}$ are diagonal entries of the lumped element mass matrix. This approach corresponds to inexact numerical integration for $b_i(v_{ih}, w_{ih})$ and replaces (37) with $D_i(\hat{\varphi}_{ik}, v_{ih}) = \hat{m}_{i,k} (\hat{v}_{ik} - v_{i0})$, cf. [1, 2].

**Remark 3.** Algebraic counterparts of the Rusanov dissipation operator (37) are often used to construct low-order components of nonlinear high-resolution finite element schemes [1, 15, 17, 19, 27]. Different authors write them in different forms and give them different names. Representation of $D_i(\cdot, \cdot)$ in the basis-independent form (37) was used in [27] in the context of algebraic flux correction (AFC) schemes for continuous finite elements. We refer the interested reader to [27] for an in-depth discussion of its properties and existing relationships to other forms of Rusanov dissipation.
4. Time discretization and slope limiting

Let $F_{ij,k}$ be the limited fluxes defined by (21) and $\nu_i$ the stabilization parameter defined by (18). Substituting the diagonal form (39) of $D_i(\phi_{ik}, v_{ih})$ into (12), we obtain the semi-discrete problem

$$m_{i,kk} \frac{du_{ik}}{dt} = - \sum_{j \in E_i} |S_{ij}| F_{ij,k} + \int_{K_i} \nabla \phi_{ik} \cdot f(u_{ih}) \, dx - (1 - \delta_{0k}) \left[ \nu_i m_{i,kk} v_{ik} + \sum_{l=1}^{d} m_{i,kl} \dot{u}_{il} \right],$$

(40)

where $\dot{u}_{i,l}$ is the time derivative of the Taylor degree of freedom $u_{i,l}$, as defined by (12), or a suitable approximation thereof. For simplicity, we set $\dot{u}_{i,l} := 0$ and consider the reduced system

$$|K_i| \frac{du_{i0}}{dt} = - \sum_{j \in E_i} |S_{ij}| F_{ij,0},$$

(41)

$$m_{i,kk} \left( \frac{du_{ik}}{dt} + \nu_i v_{ik} \right) = - \sum_{j \in E_i} |S_{ij}| F_{ij,k} + \int_{K_i} \nabla \phi_{ik} \cdot f(u_{ih}) \, dx, \quad k = 1, \ldots, d$$

(42)

which is equivalent to (40) in the case of an orthogonal Taylor basis and corresponds to a locally conservative lumped-mass approximation otherwise (see [24] for a discussion of mass lumping and appropriate treatment of time derivatives in Taylor basis DG methods).

We advance $u_{i0}$ in time using an explicit SSP Runge-Kutta method with at least two stages (for accuracy and stability reasons). Each stage has the structure of the forward Euler update

$$|K_i| \bar{u}_{i0} = |K_i| u_{i0} - \Delta t \sum_{j \in E_i} |S_{ij}| F_{ij,0},$$

(43)

where $\Delta t > 0$ is the time step and the numerical flux $F_{ij,0}$ is defined at the same old time level as $u_{i0}$.

The zero-order term $m_{i,kk} \nu_i v_{ik}$ on the left-hand side of equation (42) must be treated implicitly to avoid uncontrolled growth of the solution gradients (in the absence of limiting) and severe time step restrictions. Approximating $v_{ik} = \eta''(u_{i0}) u_{ik}$, $k = 1, \ldots, d$ by $\eta''(u_{i0}) \bar{u}_{ik}$, we update $u_{ik}$ as follows:

$$m_{i,kk} \left( 1 + \Delta t \nu_i \eta''(u_{i0}) \right) \bar{u}_{ik} = m_{i,kk} u_{ik} - \Delta t \left( \sum_{j \in E_i} |S_{ij}| F_{ij,k} - \int_{K_i} \nabla \phi_{ik} \cdot f(u_{ih}) \, dx \right).$$

(44)

Note that $\eta''(u_{i0}) \geq 0$ for any convex entropy $\eta(u)$. For the square entropy $\eta(u) = \frac{u^2}{2}$, we have $\eta''(u_{i0}) = 1$, and the coefficient of the left-hand side reduces to $m_{i,kk} + \Delta t \nu_i$. All quantities that appear on the right-hand side, as well as the value of $\nu_i \geq 0$ are defined using the old solution $u_{ih}$.

The semi-implicit treatment of $\nu_i v_{ik}$ corresponds to Patankar-type source term linearization [8, 33] for positivity-preserving schemes. It guarantees that the magnitude of $\nabla \bar{u}_{ih}$ decreases as the value of the entropy viscosity coefficient $\nu_i$ is increased. Hence, the addition of this term penalizes steep gradients in the same manner as it does at the semi-discrete level. In our experience, the explicit treatment of
\( v_{i,k} \) has a devastating effect on the entropy stability behavior of the fully discrete scheme. Indeed, it may fail to control the magnitude of \( \nabla \bar{u}_{ih} \) in the desired manner and is likely to increase it significantly in situations when the rate of entropy production and the value of \( \nu_i > 0 \) become very large.

Importantly, the use of \( \alpha_{ij} \leq \alpha_{ij}^{BP} \) in the formula for \( F_{ij,0} \) guarantees preservation of invariant domains and local bounds for the cell averages of the entropy-stabilized DG-\( P_0 \) approximation. The following adapted version of a theorem proved in [25] is presented here for the reader’s convenience.

**Theorem 2** (Preservation of local bounds [25]). Let \( u_0 \) be evolved using (43), where \( F_{ij,0} \) is defined by (26) with \( \alpha_{ij} \in [0, \alpha_{ij}^{BP}] \) and \( \alpha_{ij}^{BP} \) defined by (35). Choose the time step \( \Delta t \) to satisfy

\[
\Delta t \sum_{j \in E_i} |S_{ij}| \lambda_{ij} \leq |K_i|.
\]

Then the cell average \( \bar{u}_{i0} \) satisfies the local maximum principle

\[
u_{i0}^{\min} \leq \bar{u}_{i0} \leq \nu_{i0}^{\max},
\]

where \( \nu_{i0}^{\min} \) and \( \nu_{i0}^{\max} \) are the local bounds defined in (32).

**Proof.** The SSP Runge-Kutta stage (43) can be written in the bar state form (cf. [17, 25])

\[
\bar{u}_{i0} = u_{i0} + \frac{\Delta t}{|K_i|} \sum_{j \in E_i} |S_{ij}| \lambda_{ij} \bar{u}_{ij,0} - u_{i0} = \left( 1 - \frac{\Delta t}{|K_i|} \sum_{j \in E_i} |S_{ij}| \lambda_{ij} \right) u_{i0} + \frac{\Delta t}{|K_i|} \sum_{j \in E_i} |S_{ij}| \lambda_{ij} \bar{u}_{ij,0}^{\star},
\]

where \( \bar{u}_{ij,0}^{\star} \) are the bound-preserving flux-corrected bar states defined by (33). For time steps satisfying condition (45), the result \( \bar{u}_{i0} \) is a convex combination of \( u_{i0} \in [\nu_{i0}^{\min}, \nu_{i0}^{\max}] \) and \( \bar{u}_{ij,0}^{\star} \in [\nu_{i0}^{\min}, \nu_{i0}^{\max}] \), which proves the validity of estimate (46).

A local maximum principle can also be proved for the steady-state limit of (43), see Appendix of [25].

In addition to flux limiting, the slopes of the DG solution can be adjusted to ensure that the value of \( \bar{u}_{ih} \) at any point \( x \in K_i \) will be bounded by the local minimum \( \bar{u}_{ih}^{\min} \) and maximum \( \bar{u}_{ih}^{\max} \) of the property-preserving cell averages. Since the linear polynomial \( \bar{u}_{ih} \), whose Taylor coefficients \( u_{ik} \) are defined by (43) and (44), attains its maxima and minima at the vertices \( x_{i1}, \ldots, x_{iN} \) of \( K_i \), we constrain its gradients using a vertex-based version [23, 24] of the Barth-Jespersen slope limiter [7]. Let \( E_p \), \( p \in \{1, \ldots, N\} \) be a subset of \( E_i^\star \) containing the numbers of cells to which the vertex \( x_p \) belongs (including \( K_i \)). Multiplying \( u_{ik} \), \( k = 1, \ldots, d \) by a slope limiting factor \( \beta_i \in [0,1] \), we constrain

\[
\bar{u}_{ih}^\star(x) = \bar{u}_{i0} + \beta_i \sum_{k=1}^d \bar{u}_{ik} \varphi_{ik}(x) = \bar{u}_{i0} + \beta_i (x - x_{i0}) \cdot \nabla \bar{u}_{ih}, \quad x \in K_i
\]

to satisfy the inequality constraints

\[
\min_{j \in E_p} \bar{u}_{j0} =: \bar{u}_{j0}^{\min} \leq \bar{u}_{ih}^\star(x_{ip}) \leq \bar{u}_{ih}^\star := \max_{j \in E_p} \bar{u}_{j0}, \quad p = 1, \ldots, N.
\]
The vertex-based slope limiter employed in [23] accomplishes this task by using the correction factor

\[
\beta_i = \min_{1 \leq p \leq N} \begin{cases} 
1, & \frac{\bar{u}_{i,p}^{\max} - \bar{u}_i}{(x_{i,p} - x_{i,0}) \cdot \nabla \bar{u}_{ih}} \leq 0, \\
\min \left\{ 1, \frac{\bar{u}_{i,p}^{\min} - \bar{u}_i}{(x_{i,p} - x_{i,0}) \cdot \nabla \bar{u}_{ih}} \right\} & \text{if} \ (x_{i,p} - x_{i,0}) \cdot \nabla \bar{u}_{ih} > 0, \\
1 & \text{if} \ (x_{i,p} - x_{i,0}) \cdot \nabla \bar{u}_{ih} = 0, \\
\min \left\{ 1, \frac{\bar{u}_{i,p}^{\min} - \bar{u}_i}{(x_{i,p} - x_{i,0}) \cdot \nabla \bar{u}_{ih}} \right\} & \text{if} \ (x_{i,p} - x_{i,0}) \cdot \nabla \bar{u}_{ih} < 0.
\end{cases}
\] (50)

**Remark 4.** In most cases, the slope-limited version of a DG-\(P_1\) scheme produces nonoscillatory results even if the bound-preserving flux limiter is deactivated. However, the assertion of Theorem 2 is generally not true for the high-order LLF fluxes \(F_{ij,k}^H\) or the entropy-limited fluxes \(F_{ij,k}^{ij,0}\) defined by (26) with \(\alpha_{ij} = \alpha_{ij}^{ES}\). As noticed by Moe et al. [32], it is essential to use a flux limiter for cell averages in addition to slope limiting if positivity preservation is a must. Indeed, the cell averages are the main unknowns of the discrete problem. If they are constrained properly, slope limiting can be replaced with less aggressive accuracy-preserving gradient corrections. For example, a suitable smoothness indicator (cf. [21, 35]) can be used to increase the value of the gradient penalization coefficient \(\nu_i\) in troubled cells. Following the analysis of monolithic algebraic flux correction schemes [6, 30], the formula for \(\nu_i\) can be designed to provide Lipschitz continuity of \(\nu_i D_i\), which is an essential requirement for proving well-posedness of the nonlinear discrete problem and convergence to steady-state solutions. If necessary, the final output may be postprocessed using the vertex-based slope limiter to eliminate undershoots and overshoots (if any) in DG-\(P_1\) solutions to be visualized or used to calculate derived quantities.

**Remark 5.** The assumptions of Theorem 1 do not guarantee that the cell averages defined by (43) will satisfy a fully discrete entropy inequality. However, an explicit SSP Runge-Kutta time discretization of the low-order LLF scheme corresponding to (26) with \(\alpha_{ij} = 0\) is entropy stable for any entropy pair, as shown by Guermond and Popov [17] in the context of first-order continuous finite element approximations that exhibit the same structure [16]. Hence, the algorithm for calculating \(\alpha_{ij}\) can be modified to check and enforce inequality constraints that imply fully discrete entropy stability.

5. Numerical examples

In this section, we perform numerical experiments for two nonlinear scalar test problems in 2D. In the description of numerical results, the methods under investigation are labeled as follows:

- DG0: standard DG-\(P_0\) discretization using the low-order LLF fluxes \(F_{ij,k}^L\),
- DG1: standard DG-\(P_1\) discretization using the high-order LLF fluxes \(F_{ij,k}^H\),
- ESX: entropy stable DG scheme using (21) with \(\alpha_{ij} = \alpha_{ij}^{ESX}, X \in \{1, 2, 3\}\).

We append the letters F and S to abbreviations of methods that use bound-preserving flux and slope limiting, respectively. For example, ES1F is the DG scheme defined by (21) with \(\alpha_{ij} = \min \{\alpha_{ij}^{ES1}, \alpha_{ij}^{BP}\}\) and ES1FS is its slope-limited counterpart which adjusts the gradients using (48) and (50).
In all numerical examples, we use the square entropy $\eta(u) = \frac{u^2}{2}$ and the associated entropy variable $v(u) = u$. Computations are performed on uniform rectangular meshes, on which the Taylor basis is orthogonal. Numerical solutions are advanced in time using the explicit third-order three-stage SSP Runge-Kutta method \[12\] and time steps satisfying the CFL-like condition (45) of Theorem 2. For visualization purposes, we project DG solutions into the space of continuous bilinear elements.

5.1. KPP problem

The KPP problem \[17, 18, 22\] is a challenging nonlinear test for verification of entropy stability properties. We use this problem to test different components of the method that we propose. In this series of 2D experiments, we solve equation (1a) with the nonlinear and nonconvex flux function

$$f(u) = (\sin(u), \cos(u))$$

in the computational domain $\Omega_h = (-2, 2) \times (-2.5, 1.5)$ using the initial condition

$$u_0(x,y) = \begin{cases} \frac{14\pi}{4} & \text{if } \sqrt{x^2 + y^2} \leq 1, \\ \frac{\pi}{4} & \text{otherwise.} \end{cases}$$

The entropy flux corresponding to $\eta(u) = \frac{u^2}{2}$ is $q(u) = (u\sin(u) + \cos(u), u\cos(u) - \sin(u))$. A simple (but rather pessimistic) upper bound for the guaranteed maximum speed (GMS) is $\lambda = 1$. More accurate GMS estimates can be found in \[18\]. The exact solution exhibits a two-dimensional rotating wave structure, which is difficult to capture in numerical simulations using high-order methods. The main challenge of this test is to prevent possible convergence to wrong weak solutions.

Snapshots of numerical solutions at the final time $t = 1.0$ are shown in Figs. 1–3. To demonstrate the effect of entropy stabilization and limiting, we compare the results produced by standard and entropy stable DG schemes without and with activation of bound-preserving limiters (see Fig. 1). The DG0 solution is highly dissipative but provides a correct qualitative description of the rotating wave structure. The oscillatory DG1 solution exhibits not only large undershoots and overshoots but also an entropy-violating merger of two shocks. The application of the vertex-based slope limiter in the DG1S version has the same stabilizing effect as gradient penalization via the Rusanov dissipation term $\nu_i\eta''(u_{i0})u_{ik}$. However, the DG1 fluxes $F_{ij,k}$ violate condition (19) and the semi-discrete target scheme is not entropy stable. As a consequence of insufficient entropy stabilization, spurious distortions are observed in the contour lines of the slope-limited DG1S solution. Moreover, the levels of entropy dissipation are barely enough to keep the twisted shocks separated. This unsatisfactory state of affairs illustrates the need for entropy stabilization and confirms the findings of Guermond et al. \[17\] who noticed that preservation of invariant domains does not guarantee convergence to entropy solutions.

The entropy stable schemes ES1, ES2, ES3 preserve the thin gap between the two shocks even without slope limiting. As the levels of entropy viscosity are increased by decreasing the value of $Q_{ij}$ in formula (27), the distance between the shocks increases and violations of global bounds become less pronounced. All DG-$P_1$ solutions calculated using bound-preserving flux and/or slope limiters look alike. They are well-resolved and free of undershoots/overshoots. It is not unusual that the vertex-based slope limiter produces such solutions even if no flux limiting is performed to constrain the local
DG0, $u_h \in [0.785, 10.992]$

DG1, $u_h \in [-1.794, 14.741]$

DG1S, $u_h \in [0.785, 10.996]$

ES1, $u_h \in [-0.066, 12.316]$

ES2, $u_h \in [0.037, 12.241]$

ES3, $u_h \in [0.213, 11.504]$

ES1S, $u_h \in [0.785, 10.996]$

ES2S, $u_h \in [0.785, 10.996]$

ES3S, $u_h \in [0.785, 10.996]$

ES1FS, $u_h \in [0.785, 10.996]$

ES2FS, $u_h \in [0.785, 10.996]$

ES3FS, $u_h \in [0.785, 10.996]$

Figure 1: KPP problem, DG solutions at $t = 1.0$ calculated using $h = \frac{1}{128}$, $\Delta t = 10^{-3}$. 

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range of the cell averages \( \bar{u}_{j0} \) that define the bounds for (49). However, the flux-limited version is generally safer because the validity of local maximum principles is guaranteed by Theorem 2.

The snapshots shown in Fig. 2 visualize the corresponding diagrams of Fig. 1 as surface plots to better illustrate the capability of DG0, DG1S, and ES1S to capture the rotating wave structure on the mesh with spacing \( h = \frac{1}{128} \). The results obtained with the three methods on a finer mesh (\( h = \frac{1}{256} \)) are displayed in Fig. 3. The fine-mesh DG0 and ES1S solutions illustrate the correct shock behavior. The DG1S solutions indicate that the use of slope limiting has a strong stabilizing effect but may fail to prevent entropy-violating behavior that does not cause violations of local bounds in (49).

5.2. Buckley-Leverett equation

In the second numerical experiment, we consider the two-dimensional Buckley-Leverett equation [10, 29]. The nonconvex flux function of the nonlinear conservation law to be solved is

\[
f(u) = \frac{u^2}{u^2 + (1-u)^2}(1, 1 - 5(1-u)^2).
\]  (53)
The computational domain is $\Omega_h = (-1.5, 1.5)^2$. The piecewise-constant initial condition is given by

$$u_0(x, y) = \begin{cases} 
1 & \text{if } x^2 + y^2 < 0.5, \\
0 & \text{otherwise}.
\end{cases}$$ (54)

Similarly to the KPP problem, the solution exhibits a rotating wave structure. For entropy stabilization purposes, we use $\eta(u) = \frac{u^2}{2}$ and the corresponding entropy flux $q(u) = (q_x(u), q_y(u))$, where

$$q_x = \frac{1}{4} \left[ \frac{2(u - 1)}{2u^2 - 2u + 1} - \log(2u^2 - 2u + 1) \right],$$

$$q_y = \frac{1}{12} \left[ -20u^3 + 15u^2 - \frac{9u + 6}{2u^2 - 2u + 1} - 3 \log(2u^2 - 2u + 1) - 15 \tan^{-1}(1 - 2u) \right].$$ (56)

An upper bound for the fastest wave speed can be found in [10]. We overestimate it by using $\lambda_{ij} = 3.4$.

In Figure 4, we show the numerical results at the final time $t = 0.5$ obtained using a uniform mesh with spacing $h = \frac{1}{128}$. The qualitative behavior of the DG solutions is similar to that for the more challenging KPP problem. The DG0 approximation is bound-preserving w.r.t. $\mathcal{G} = [0, 1]$ but the levels of numerical diffusion are too high. The DG1 scheme produces an oscillatory solution and gives rise to strong violations of the global bounds. The entropy stable approximations ES1, ES2, ES3 exhibit smaller undershoots and overshoots. The application of the vertex-based slope limiter eliminates them completely. The results obtained with the bound-preserving flux limiter look similar and are not presented here.

6. Conclusions

The main result of the presented work is the development of a methodology that pieces together individual components of property-preserving DG schemes for nonlinear hyperbolic problems. We have shown that a carefully tuned combination of flux limiting, entropy stabilization, and slope limiting makes it possible to satisfy all relevant inequality constraints by blending a low-order LLF approximation and a high-order target. Although the proposed framework was introduced in the context of scalar nonlinear conservation laws, its extension to systems appears to be straightforward. In the case of $m > 1$ conserved quantities, a generalized version of the entropy condition (19) and preservation of invariant domains can be readily enforced using (21) with a scalar correction factor $\alpha_{ij}$. Additionally, the components of the high-order target flux $F_{ij,0}^H$ can be pre-constrained to satisfy local maximum principles for certain quantities of interest (see [25] for details). Last but not least, the time integration procedure can be redesigned to be not only SSP but also entropy stability preserving [36]. It is hoped that the findings of this paper will provide useful insights and tools for such research endeavors.

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Figure 4: Buckley-Leverett problem, DG solutions at $t = 0.5$ calculated using $h = 1/128$, $\Delta t = 10^{-3}$.

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