Global regular axially symmetric solutions to the Navier-Stokes equations

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Abstract. Global regular axially symmetric solutions to the Navier-Stokes equations is proved. The solution is such that velocity belongs to $W^{2,1}_2(\Omega \times \mathbb{R}_+)$ and gradient of pressure to $L^2_2(\Omega \times \mathbb{R}_+)$, where $\Omega$ is a finite axially symmetric cylinder in $\mathbb{R}^3$ and the slip boundary conditions are imposed on its boundary. First we prove the existence of local solutions in the mentioned spaces for time less or equal $T$. Having such solutions we are able to show that swirl $(rv_\varphi)$ is the Hölder continuous. This gives a possibility to show that $\|v'\|_{H^1(\Omega)}$, $v' = (v_r, v_z)$, is bounded independently on time near the axis of symmetry. Similar estimate for $v'$ is obtained in a neighborhood located in a positive distance of the axis of symmetry. Separately we show that $\|v_\varphi\|_{H^1(\Omega)}$ is bounded by a quantity independent on time. This implies that the local solution can be prolonged on intervals $(kT, (k + 1)T)$, $k \in \mathbb{N}$, $T > 0$. Employing the decay estimates appropriate for the Navier-Stokes equations we show that however the external force does not decrease with time the norm $\|v(t)\|_{H^1(\Omega)}$ does not increase.
1. Introduction

In this paper we prove the existence of global regular axially symmetric solutions with large swirl to the Navier-Stokes equations. We consider the motion of an incompressible fluid in an axially symmetric cylinder with the slip boundary conditions. We generalize the results from \([Z1, Z2, Z3]\), where the periodic cylinder with respect to the variable along its axis and the case without any external force are considered. The slip boundary conditions are necessary because the main step in a proof of global estimate is the energy type estimate for vorticity (see \([L1, Z4, Z5, Z6]\)). To get such estimate we need integration by parts so appropriate boundary conditions for vorticity are necessary.

In this paper we consider the axially symmetric solutions (see Definition 1.1 below) to the following problem

\[
\begin{align*}
    v_t + v \cdot \nabla v - \nu \Delta v + \nabla p &= f & \text{in } \Omega^T = \Omega \times (0, T), \\
    \text{div } v &= 0 & \text{in } \Omega^T, \\
    v \cdot \bar{n} &= 0 & \text{on } S^T = S \times (0, T), \\
    \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha &= 0, \quad \alpha = 1, 2, & \text{on } S^T, \\
    v|_{t=0} &= v_0 & \text{in } \Omega,
\end{align*}
\]

where \(\Omega\) is an axially symmetric cylinder with boundary \(S = S_1 \cup S_2\), \(x = (x_1, x_2, x_3)\) is the Cartesian system of coordinates in \(\mathbb{R}^3\) such that \(x_3\) is the axis of the cylinder \(\Omega\). By \(S_1\) we denote the part of the boundary of the cylinder parallel to the \(x_3\)-axis and \(S_2\) is perpendicular to it. Next, \(v = v(x, t) = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3\) is the velocity of the considered fluid, \(p = p(x, t) \in \mathbb{R}\) the pressure, \(f = f(x, t) = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3\) the external force field, \(\nu > 0\) is the constant viscosity coefficient, \(\mathbb{D}(v) = \nabla v + \nabla v^T\) the dilatation tensor which is the double symmetric part of \(\nabla v\), \(\bar{n}\) is the unit outward normal vector to \(S\) and \(\bar{\tau}_\alpha, \alpha = 1, 2,\) is a tangent one.

To examine axially symmetric solutions to (1.1) we introduce the cylindrical coordinates \(r, \varphi, z\) by the relations \(x_1 = r \cos \varphi, x_2 = r \sin \varphi, x_3 = z\). Moreover, we introduce the vectors \(\bar{e}_r = (\cos \varphi, \sin \varphi, 0), \bar{e}_\varphi = (-\sin \varphi, \cos \varphi, 0), \bar{e}_z = (0, 0, 1)\) connected with the cylindrical coordinates. Then, the cylindrical coordinates of \(v\) and \(f\) are defined by the relations

\[
\begin{align*}
    v_r &= v \cdot \bar{e}_r, \quad v_\varphi = v \cdot \bar{e}_\varphi, \quad v_z = v \cdot \bar{e}_z, \\
    f_r &= f \cdot \bar{e}_r, \quad f_\varphi = f \cdot \bar{e}_\varphi, \quad f_z = f \cdot \bar{e}_z,
\end{align*}
\]

where the dot denotes the scalar product in \(\mathbb{R}^3\). Finally, by \(u = v_\varphi r\) we denote a swirl.
To describe the domain $\Omega$ and its boundary in greater details we introduce the notation

$$\Omega = \{ x \in \mathbb{R}^3 : r < R, |z| < a \},$$
$$S_1 = \{ x \in \mathbb{R}^3 : r = R, |z| < a \},$$
$$S_2 = \{ x \in \mathbb{R}^3 : r < R, z \in \{-a, a\} \},$$

where $R$ and $a$ are given positive numbers.

**Definition 1.1.** By the axially symmetric solutions we mean such solutions to problem (1.1) that

$$v_{r,\varphi} = v_{\varphi,\varphi} = v_{z,\varphi} = p, \varphi = 0,$$
$$f_{r,\varphi} = f_{\varphi,\varphi} = f_{z,\varphi} = 0.$$  

In the cylindrical coordinates equations (1.1) for the axially symmetric solutions can be expressed in the form (see [LL, K])

$$v_{r,t} + v \cdot \nabla v_r - \frac{v_{\varphi}^2}{r} - \nu \Delta v_r + \nu \frac{v_r}{r^2} = -p_r + f_r,$$

$$v_{\varphi,t} + v \cdot \nabla v_\varphi + \frac{v_r}{r} v_\varphi - \nu \Delta v_\varphi + \nu \frac{v_\varphi}{r^2} = f_\varphi,$$

$$v_{z,t} + v \cdot \nabla v_z - \nu \Delta v_z = -p_z + f_z,$$

$$v_{r,r} + v_{z,z} = -\frac{v_r}{r},$$

where $v \cdot \nabla = v_r \partial_r + v_z \partial_z$, $\Delta u = \frac{1}{r}(ru_r)_r + u_{zz}$.

Expressing the boundary conditions (1.1) in the cylindrical coordinates yields (see [Z4, Ch. 4, Lemma 2.1])

$$v_r = 0, \quad v_z = 0, \quad v_{r, r} = \frac{1}{r} v_\varphi \quad \text{on} \quad S_1,$$
$$v_{r, z} = 0, \quad v_{\varphi, z} = 0, \quad v_z = 0 \quad \text{on} \quad S_2.$$

Finally, initial conditions assume the form

$$v_r|_{t=0} = v_r(0), \quad v_\varphi|_{t=0} = v_\varphi(0), \quad v_z|_{t=0} = v_z(0).$$

We prove the existence of global axially symmetric solutions to (1.1) such that $v \in W^{2,1}_2(\Omega \times \mathbb{R}^+)$, $\nabla p \in L^2(\Omega \times \mathbb{R}^+)$. The proof is divided into
the following steps. First we prove the existence of a local solution \( v \in W^{2,1}_2(\Omega \times (0, T_0)) \), \( \nabla p \in L_2(\Omega \times (0, T_0)) \), where \( T_0 \) is sufficiently small. To show such existence we need that \( v(0) \in H^1(\Omega) \) and \( f \in L_2(\Omega \times (0, T_0)) \). The existence of the local solutions is proved by the Leray-Schauder fixed point theorem, where the necessary a priori estimate is possible thanks to the restriction on \( T_0 \) (see Lemma 2.5). We are not able to extend the local solution on the interval \((T_0, 2T_0)\) without an additional estimate on \( \|v(T_0)\|_{H^1(\Omega)} \). However, to prove a global existence we have to show that

\[
(1.9) \quad \|v(kT_0)\|_{H^1(\Omega)} \leq \|v(0)\|_{H^1(\Omega)}, \quad k \in \mathbb{N}.
\]

To show (1.9) we separate cylinder \( \Omega \) into two parts: a cylinder of radius \( r_0 \), \( \Omega_{r_0} = \{x \in \Omega : r < 2r_0\} \) (a neighborhood of the axis of symmetry) and the domain \( \Omega_{\bar{r}_0} = \{x \in \Omega : r > r_0/2\} \) (a neighborhood located in a positive distance from the \( x_3 \)-axis). Since \( \Omega = \Omega_{r_0} \cup \Omega_{\bar{r}_0} \) we subordinate a partition of unity corresponding to this division. The most difficult part is to prove the estimate

\[
(1.10) \quad \|v'\|_{V^1_2(\Omega_{T_0}^T)} \leq a_1,
\]

where \( v' = (v_r, v_z) \), \( V^1_2(\Omega^T) \) is defined in Section 2, \( T \in \mathbb{R}_+ \) and \( a_1 \) is a constant depending on data. However, (1.10) is proved for the local solution the bound \( a_1 \) does not depend on \( T \). To prove (1.10) we need the Hölder continuity of swirl \( u \) with its vanishing on the axis of symmetry and the restriction

\[
(1.11) \quad \max_{t} \max_{\Omega_{r_0}} |u| \leq \frac{\sqrt{5}}{8} \nu.
\]

The results are proved in Section 3.

Hence (1.11) holds for \( r_0 \) sufficiently small and also for \( v \in L_{10}(\Omega^T) \), so for local solution. This needs step by step in time approach. In view of (1.11) we have (1.10) with \( a_1 = a_1(1/r_0) \), where \( a_1 \) is an increasing function. Estimate (1.10) is proved in a series of lemmas (see Lemmas 4.1, 4.2, 4.3, 4.5, 4.6).

The next step is to prove the estimate (see Lemmas 6.4, 6.5)

\[
(1.12) \quad \|v'\|_{V^1_2(\Omega_{T_0}^T)} \leq a_2, \quad T \in \mathbb{R}_+,
\]

where \( a_2 \) is a constant depending on data. Finally, by Lemma 7.4 we have

\[
(1.13) \quad \|v_\varphi(t)\|_{H^1(\Omega)} \leq a_3, \quad t \in \mathbb{R}_+,
\]

where \( \varphi(t) \) is a solution of the system (1.9) with initial data in \( H^1(\Omega) \).
where \( a_3 \) depends also on data.

However, estimates (4.10), (4.12), (4.13) are proved for the local solution quantities \( a_1, a_2, a_3 \) do not depend explicitly on time. The dependence on time is only by time integral norms of the external force.

Combining (1.10), (1.12) and (1.13) we derive the a priori estimate

\[
\|v(t)\|_{H^1(\Omega)} \leq a_4, \quad t \in \mathbb{R}_+,
\]

where \( a_4 \) depends on data but not on \( t \).

A priori estimate (1.14) becomes a real estimate for the local solution in the interval \((0, T_0)\). Since (1.14) is a global type estimate the local solution can be extended step by step on \( \mathbb{R}_+ \) (see the Main Theorem below).

**Main Theorem.** Let \( T > 0 \) be given. Let \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Assume that

1. \( v(0) \in H^1(\Omega), f \in L_2(\Omega \times (kT, (k + 1)T)) \cap L_\infty(\mathbb{R}_+; L_{6/5}(\Omega)) \) (see Lemmas 2.2, 2.3)
2. \( u_0 = rv_\phi(0) \in L_\infty(\Omega), g = rf_\phi \in L_\infty(\Omega \times \mathbb{R}_+) \) (see Lemma 3.1)
3. \( u_0 \in C^\alpha(\Omega), g \in C^{\alpha, \alpha/2}(\Omega \times (kT, (k + 1)T)), \alpha \in (0, 1/2] \) (see Lemma 3.2)
4. \( \chi = v_{r,z} - v_{z,r}, \frac{\chi(0)}{r} \in L_2(\Omega), F = f_{r,z} - f_{z,r}, \frac{F}{r} \in L_2(\Omega \times (kT, (k + 1)T)) \), \( \frac{f_r(0)}{r} \in L_4(\Omega), \frac{f_r(0)}{r} \in L_{20/11}(\Omega \times (kT, (k + 1)T)) \)
5. there exists \( r_0 > 0 \) such that

\[
\|u\|_{L_\infty(\Omega_{r_0} \times (kT, (k + 1)T))} \leq \sqrt{\frac{a}{3}} \nu, \quad a < 1
\]

where \( \Omega_{r_0} = \{x \in \Omega : r < 2r_0\} \).

Then there exists a solution to problem (1.1) such that \( v \in W^{2,1}_2(\Omega \times (kT, (k + 1)T)) \), \( \nabla p \in L_2(\Omega \times (kT, (k + 1)T)) \) and the estimate holds

\[
\|v\|_{W^{2,1}_2(\Omega \times (kT, (k + 1)T))} + \|\nabla p\|_{L_2(\Omega \times (kT, (k + 1)T))} \\
\leq c(B_0(1/r_0, d_4, d_5, d_6, d_7, d_8, d_9) + \|f\|_{L_2(\Omega \times (kT, (k + 1)T))} \\
+ \|v(kT)\|_{H^1(\Omega)})
\]

where (see Lemma 8.2)

\[
\|v(kT)\|_{H^1(\Omega)} \leq c(B_0 + X(0)e^{-\nu_0 kT}),
\]

\[
\chi^2(0) = \frac{1}{r^2} \left\| \frac{\chi(0)}{r} \right\|_{L_2(\Omega)}^2 + \left\| \frac{\chi(0)}{r} \right\|_{L_2(\Omega)}^2
\]
and $B_0$ is an increasing positive function of its arguments, where $d_4$ is introduced in (2.1), $d_5$ in (2.2), $d_6$ in (3.1), $d_7$ in (2.15), $d_8$ and $d_9$ in Lemma 8.2.

In the famous paper of Caffarelli, Kohn, Nirenberg [CKN] there is shown that the singular set for solutions to the Navier-Stokes equations might have at most an one-dimensional Hausdorff measure. Therefore it can be expected that for axially symmetric solutions such singularity might appear on the axis of symmetry. It is shown in this paper that $v_r$, $v_\varphi$, $u$ vanish on the axis of symmetry, so the considered solution behaves there very regularly. In [Z8] existence of global regular axially symmetric solutions with prescribed sufficiently small initial swirl in some neighborhood of the axis of symmetry is proved. Then the property is preserved. In this paper the property is shown without any smallness restrictions on the initial data. Otherwise the necessary a priori estimate can not be derived.

In this paper we generalize the result from [Z1–Z3] to the problem with an external force and the slip boundary conditions on whole boundary. The result is appropriate for examining stability of the axially symmetric solutions.

The generalization is not trivial because the external force has a strong influence on any solution to the Navier-Stokes equations. The external force has an opposite influence to the dissipation. Therefore a global existence can be proved if the dissipation prevails the influence of the external force. To escape restrictions that velocity and the external force vanish as time converges to infinity we prove existence step by step on each finite time interval $[kT, (k+1)T]$, $k \in \mathbb{N}_0$. This approach needs that the following inequality must be shown

\begin{equation}
\|v((k+1)T)\|_{H^1(\Omega)} \leq \|v(kT)\|_{H^1(\Omega)}, \quad k \in \mathbb{N}_0,
\end{equation}

which follows from the decay properties of the Navier-Stokes equations (see Sections 7 and 8). We have to emphasize that to derive (1.16) $T$ must be sufficiently large. Moreover, the approach needs that existence must be proved in each time interval $[kT, (k+1)T]$, $k \in \mathbb{N}_0$, separately. This follows from Lemma 2.5 and Theorem 8.1.

The Main Theorem says that however $v_r$, $v_\varphi$ vanish on the axis of symmetry $v_z$ remains large and is only restricted by estimate (1.15). Hence, considering an inflow-outflow conditions on $S_2$ seems that $v_z$ can be made as much as we want.

This paper is organized in the following way. In Section 2 there are formulated energy type estimates for solutions to problem (1.1) without showing
existence (see Lemmas 2.2, 2.3). Existence is very well presented in [CKN, L2, T]. However, in our case, we do not need existence of weak solutions because existence in each time interval \((kT, (k + 1)T)\) is proved by the Leray-Schauder fixed point theorem. For this we need only an appropriate a priori estimate. Moreover, in Section 2 there are formulated problems for \(u\) (see (2.23)) and \(\chi\) (see (2.25)). Finally local existence of solutions to (1.1) \((v \in W^{2,1}_2(\Omega \times (kT, (k + 1)T)), \nabla p \in L_2(\Omega \times (kT, (k + 1)T)), k \in \mathbb{N}_0)\) is proved in Lemma 2.5. In Section 3 boundedness and the Hölder continuity of swirl is proved (see Lemmas 3.1, 3.2). In Section 4 a priori estimate in a neighborhood of the axis of symmetry is proved. A crucial point of getting the estimate is restriction (1.11) (see (4.30)) which implies a different treatment near the axis of symmetry and far of it.

In Section 6 in view of estimate \(\|\chi/r\|_{V^2_0(\Omega_{r_0})} \leq A\) (see (6.1)) there is proved that

\[
(1.17) \quad \|v'(t)\|_{H^1(\Omega)} \leq cA,
\]

where \(A\) is defined by (6.2). Finally, in Section 7 the estimate is found

\[
(1.18) \quad \|v_\varphi(t)\|_{H^1(\Omega)} \leq \varphi(A_*),
\]

where \(A_*\) is introduced in (6.5).

Thanks to estimates (1.17) and (1.18) global existence of regular solutions such \(v \in W^{2,1}_2(\Omega \times (kT, (k + 1)T)), \nabla p \in L_2(\Omega \times (kT, (k + 1)T))\) is proved step by step in Section 8.

2. Notation and auxiliary results

By \(c\) we denote a generic constant which changes its value from line to line. A constant \(c_k\) with index \(k\) is defined by the first formula, where it appears. By \(\phi\) we denote the generic functions which changes its form from formula to formula and is always positive and increasing function. By \(c(\sigma)\) we denote a generic constant increasing with \(\sigma\).

We use also the notation

\[
\Omega_\varepsilon = \{x \in \Omega : \varepsilon < r\},
\]

\[
\Omega_\varepsilon \cap \text{supp } \zeta = \Omega_{\varepsilon, \zeta}, \quad \Omega_\varepsilon \cap \text{supp } \zeta_{,r} = \Omega_{\varepsilon, \zeta, r}, \quad \Omega \cap \text{supp } \zeta_{,r} = \Omega_{\zeta, r}.
\]

**Definition 2.1.** By \(V^k_2(\Omega^T), k \in \mathbb{N} \cup \{0\} \equiv \mathbb{N}_0\), we denote a space of functions with the finite norm

\[
\|u\|_{V^k_2(\Omega^T)} = \|u\|_{L_\infty(0,T;H^k(\Omega))} + \|\nabla u\|_{L_2(0,T;H^k(\Omega))},
\]
where $H^0(\Omega) = L_2(\Omega)$ and $H^k(\Omega) = W^k_2(\Omega)$ is a Sobolev space with the finite norm $\|u\|_{H^k(\Omega)} = (\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^2 dx)^{1/2}$, where $D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $\alpha_i \in \mathbb{N}_0$, $i = 1, 2, 3$.

**Lemma 2.2.** Assume that $v(0) \in L_2(\Omega)$, $f \in L_\infty(\mathbb{R}_+; L_{6/5}(\Omega))$, $g = f \cdot \eta$, $u = v \cdot \eta$, $\eta = (-x_2, x_1, 0)$, $u(0) \in L_\infty(\Omega)$. Assume that there exist positive constants $d_0$, $d_1$, $d_2$ such that

$$
\|v(0)\|_{L_2(\Omega)} \leq d_0, \quad \|f\|_{L_\infty(\mathbb{R}_+; L_{6/5}(\Omega))} \leq d_2,
$$

$$
\sup_{k \in \mathbb{N}_0} \sup_{t \in (kT, (k+1)T]} \left| \int_{kT}^t \int_{\Omega} g dx dt' + \int_{\Omega} u(kT) dx \right| \leq d_1.
$$

Assume that $\nu_* = \frac{\nu}{c_k}$, where $c_k$ is the constant from the Korn inequality (see (2.4)). Assume that $\nu_* = \nu_1 + \nu_2$, $\nu_i > 0$, $i = 1, 2$.

Assume that $T > 0$ is fixed and $k \in \mathbb{N}_0$. Then for the weak solutions to problem (1.1) we have the estimates

$$
\|v(t)\|_{L_2(\Omega)}^2 \leq \frac{d_3^2}{1 - e^{-\nu_1 T}} + e^{-\nu_1 T} d_0^2 \equiv d_4^2, \quad t \in \mathbb{R}_+,
$$

where $d_3 = \frac{1}{\nu_1} \left( \frac{1}{\nu_2} d_2^2 + 2vd_1^2 \right)$ and

$$
\|v\|_{V_0^2(\Omega \times (kT, t))}^2 \leq \frac{d_3^2}{\nu_2} e^{\nu_1 T} + \left( 1 + \frac{1}{\nu_2} \right) \left( \frac{2d_3^2}{1 - e^{-\nu_1 T}} + e^{-\nu_1 kT} d_0^2 \right)
$$

$$
\leq \left( 1 + \frac{1}{\nu_2} \right) \left[ \frac{2d_3^2}{1 - e^{-\nu_1 T}} e^{\nu_1 T} + d_0^2 \right] \equiv d_5^2,
$$

where $t \in (kT, (k+1)T)$, $k \in \mathbb{N}_0$.

**Proof.** Multiplying (1.1) by $v$, integrating over $\Omega$, using (1.1)$_2$ and the boundary conditions we obtain

$$
\frac{1}{2} \frac{d}{dt} \|v\|_{L_2(\Omega)}^2 + \nu E_\Omega(v) = \int_{\Omega} f \cdot v dx,
$$

where

$$
E_\Omega(v) = \|\mathcal{D}(v)\|_{L_2(\Omega)}^2.
$$

From [Z4, Ch. 4, Lemma 2.4] we have the Korn inequality

$$
\|v\|_{H_1(\Omega)}^2 \leq c_k \left( E_\Omega(v) + \left| \int_{\Omega} v \cdot \eta dx \right|^2 \right),
$$

\hspace{1cm} 8
where \( \eta = (-x_2, x_1, 0) \), \( v \cdot \eta = rv \varphi = u \).

Now we calculate the last term on the r.h.s. of (2.4). Multiplying (1.1) by \( \eta \), integrating over \( \Omega \) and using (1.1) we obtain

\[
\frac{d}{dt} \int_\Omega v \cdot \eta dx - \int_\Omega v_i v_j \nabla_i \eta_j dx + \int_\Omega T_{ij} \nabla_i \eta_j dx
\]

(2.5)

\[
= \int_\Omega f \cdot \eta dx,
\]

where \( T(v, p) = \{ T_{ij} \}_{i,j=1,2,3} \) is the stress tensor of the form

\[
T(v, p) = \nu \mathbb{D}(v) - p I,
\]

\( I \) is the unit matrix and the summation convention over the repeated indices is assumed. Since \( \nabla \eta \) is an antisymmetric tensor equation (2.5) implies

\[
\frac{d}{dt} \int_\Omega v \cdot \eta dx = \int_\Omega f \cdot \eta dx.
\]

(2.6)

Integrating (2.6) with respect to time from \( kT \) to \( t \in (kT, (k+1)T) \) yields

\[
\int_\Omega u(t) dx = \int_k^t \int_\Omega g dx dt' + \int_\Omega u(kT) dx.
\]

(2.7)

Using (2.4) and (2.7) in (2.3) implies

\[
\frac{1}{2} \frac{d}{dt} \| v \|_{L^2(\Omega)}^2 + \nu_* \| v \|_{H^1(\Omega)}^2 \leq \int_\Omega f \cdot vd x + \nu \int_k^t \int_\Omega g dx dt'
\]

(2.8)

\[
+ \int_\Omega u(kT) dx,' 
\]

where the inequality is considered in the time interval \( [kT, (k+1)T] \).

Applying the Hölder and the Young inequalities to the first term on the r.h.s. of (2.8) and multiplying the result by 2 we derive

\[
\frac{d}{dt} \| v \|_{L^2(\Omega)}^2 + \nu_* \| v \|_{H^1(\Omega)}^2 \leq \frac{1}{\nu_*} \| f \|_{L^6(\Omega)}^2 + 2\nu d_1^2
\]

(2.9)
Employing the decomposition $\nu_*=\nu_1+\nu_2$ inequality (2.9) takes the form

$$
\frac{d}{dt}(v\|^2_{L^2(\Omega)}e^{\nu_1 t}) + \nu_2\|v\|^2_{H^1(\Omega)}e^{\nu_1 t} \leq \left(\frac{1}{\nu_*}d_2^2 + 2\nu d_1^2\right)e^{\nu_1 t} \equiv \nu_1 d_3^2 e^{\nu_1 t}.
$$

(2.10)

Omitting the second term on the l.h.s. of (2.10) and integrating the result with respect to time from $kT$ to $t \in (kT, (k+1)T]$ yields

$$
\|v(t)\|^2_{L^2(\Omega)} \leq d_3^2 + \|v(kT)\|^2_{L^2(\Omega)}e^{-\nu_1(t-kT)}.
$$

By iteration we obtain

$$
\|v(kT)\|^2_{L^2(\Omega)} \leq \frac{d_3^2}{1-e^{-\nu_1 T}} + \|v(0)\|^2_{L^2(\Omega)}e^{-\nu_1 kT}.
$$

Hence for $t \in (kT, (k+1)T]$ we get

$$
\|v(t)\|^2_{L^2(\Omega)} \leq d_3^2 \frac{2-e^{-\nu_1 T}}{1-e^{-\nu_1 T}} + \|v(0)\|^2_{L^2(\Omega)}e^{-\nu_1 T}
$$

(2.11)

so (2.1) holds. Integrating (2.10) with respect to time from $kT$ to $t \in (kT, (k+1)T]$ we have

$$
\|v(t)\|^2_{L^2(\Omega)} + \nu_2e^{-\nu_1 t} \int_{kT}^{t} \|v(t')\|^2_{H^1(\Omega)}e^{\nu_1 t'}dt' \leq d_3^2 + \|v(kT)\|^2_{L^2(\Omega)}e^{-\nu_1(t-kT)}.
$$

(2.12)

Continuing, we obtain

$$
\|v(t)\|^2_{L^2(\Omega)} + \nu_2e^{-\nu_1(t-kT)} \int_{kT}^{t} \|v(t')\|^2_{H^1(\Omega)}dt' \leq d_3^2 + \|v(kT)\|^2_{L^2(\Omega)}e^{-\nu_1(t-kT)}.
$$

(2.13)

Finally, (2.13) implies

$$
\int_{kT}^{t} \|v(t')\|^2_{H^1(\Omega)}dt' \leq \frac{d_3^2}{\nu_2}e^{\nu_1 T} + \frac{1}{\nu_2}\left(\frac{d_3^2}{1-e^{-\nu_1 T}} + e^{-\nu_1 kT}d_0^2\right).
$$

(2.14)

Combining (2.11) and (2.14) gives (2.2). This concludes the proof.
Lemma 2.3. Let the assumptions of Lemma 2.2 hold. Let \( f \in L_2(\Omega \times (kT, t)), t \in (kT, (k+1)T] \). For weak solutions to problem (1.3)–(1.8) the following estimate holds

\[
\begin{align*}
\|v\|_{L_2(\Omega \times (kT, t))}^2 + \left\| \frac{v_r}{r} \right\|_{L_2(\Omega \times (kT, t))}^2 + \left\| \frac{v_\phi}{r} \right\|_{L_2(\Omega \times (kT, t))}^2 \\
\leq c(T + 1)d_4^2 + c\|f\|_{L_2(\Omega \times (kT, t))}^2 \equiv d_7^2,
\end{align*}
\]

where \( t \in (kT, (k+1)T] \), \( k \in \mathbb{N}_0 \).

Proof. Multiplying (1.3) by \( v_r \), integrating over \( \Omega \) and using boundary conditions (1.7) yields

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega v_r^2 dx - \int_\Omega \frac{v_r^2}{r} v_r dx + \nu \int_\Omega (v_{r,r}^2 + v_{r,z}^2) dx \\
+ \nu \int_\Omega \frac{v_r^2}{r^2} dx = - \int_\Omega p_r v_r dx + \int_\Omega f_r v_r dx.
\end{align*}
\]

Multiplying (1.4) by \( v_\phi \), integrating over \( \Omega \) and using boundary conditions (1.7) implies

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega v_\phi^2 dx + \int_\Omega \frac{v_r}{r} v_\phi^2 dx + \nu \int_\Omega (v_{\phi,r}^2 + v_{\phi,z}^2) dx - \nu \int_{-a}^a v_\phi^2 dz \\
+ \nu \int_\Omega \frac{v_\phi^2}{r^2} dx = \int_\Omega f_\phi v_\phi dx.
\end{align*}
\]

Multiplying (1.5) by \( v_z \), integrating over \( \Omega \) and using the boundary conditions (1.7) we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega v_z^2 dx + \nu \int_\Omega (v_{z,r}^2 + v_{z,z}^2) dx = - \int_\Omega p_z v_z dx + \int_\Omega f_z v_z dx.
\end{align*}
\]

Adding the above equations, using (1.6), the inequalities

\[
\|v_\phi\|_{L_2(S_1)}^2 \leq \varepsilon \|\nabla v_\phi\|^2_{L_2(\Omega)} + c(1/\varepsilon)\|v_\phi\|^2_{L_2(\Omega)},
\]

\[
\int_\Omega (v_r^2 + v_\phi^2) dx \leq R^2 \int_\Omega \left( \frac{v_r^2}{r^2} + \frac{v_\phi^2}{r^2} \right) dx,
\]

...
and the Poincare inequality for \( v_z \) we arrive to the inequality

\[
\frac{d}{dt} \int_{\Omega} v^2 dx + \nu \int_{\Omega} |\nabla v|^2 dx + \nu \int_{\Omega} \left( \frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2} \right) dx \\
\leq c \left( \int_{\Omega} v_\varphi^2 dx + \int_{\Omega} f^2 dx \right).
\]

(2.19)

Integrating (2.19) with respect to time from \( kT \) to \( t \) and using (2.1) we obtain

\[
\begin{align*}
\|v\|^2_{L^2(\Omega \times (kT, t))} + \nu \int_{kT}^{t} \int_{\Omega} \left( \frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2} \right) dx dt' \\
\leq cT d_4^2 + c\|f\|^2_{L^2(\Omega \times (kT, t))} + \|v(kT)\|^2_{L^2(\Omega)}.
\end{align*}
\]

(2.20)

Using again (2.1) we obtain (2.15). This concludes the proof.

Let us consider the Stokes problem

\[
\begin{align*}
v_t - \text{div} \, \mathbb{T}(v, p) &= f & \text{in} \; \Omega^T, \\
\text{div} v &= 0 & \text{in} \; \Omega^T, \\
v \cdot \vec{n} &= 0, \; \vec{n} \cdot \mathbb{D}(v) \cdot \vec{\tau}_\alpha &= 0, \; \alpha = 1, 2, & \text{on} \; S^T, \\
v|_{t=0} &= v_0 & \text{in} \; \Omega.
\end{align*}
\]

(2.21)

**Lemma 2.4.** (see [S, Z7, ZZ]) Assume that \( f \in L_s(\Omega^T) \), \( v_0 \in W^{2-2/s}_s(\Omega) \), \( s \in (1, \infty) \), \( S_1 \in C^2 \). Then there exists a solution to problem (2.21) such that \( v \in W^{2,1}_s(\Omega^T) \), \( \nabla p \in L_s(\Omega^T) \) and there exists a constant \( c_0 = c(\Omega, s) \) such that

\[
\|v\|_{W^{2,1}_s(\Omega^T)} + \|\nabla p\|_{L_s(\Omega^T)} \leq c(\Omega, s)(\|f\|_{L_s(\Omega^T)} + \|v_0\|_{W^{2-2/s}_s(\Omega)}).
\]

(2.22)

From (1.4) and (1.7) we obtain the following problem for swirl \( u = rv_\varphi \)

\[
\begin{align*}
u_t + v \cdot \nabla u - \nu \Delta u + 2\nu \frac{u_r}{r} &= rf_\varphi \equiv g & \text{in} \; \Omega^T, \\
u_r &= \frac{2}{r}u & \text{on} \; S_1^T, \\
|z &= 0 & \text{on} \; S_2^T, \\
u|_{t=0} &= u_0 & \text{in} \; \Omega,
\end{align*}
\]

(2.23)

where \( v \) is the divergence free vector.
Let us introduce the $\varphi$-component of vorticity by

\begin{equation}
\chi = v_{r,z} - v_{z,r}.
\end{equation}

In view of (1.3), (1.5) and (1.7), $\chi$ is a solution to the problem (see [Z4, Ch. 4, Sect. 3 (3.1) and Lemma 2.2 (2.13)])

\begin{equation}
\chi_t + v \cdot \nabla \chi - \frac{v_r}{r} \chi - \nu \left[ \left( r \left( \frac{\chi}{r} \right)_r \right)_r + \chi_{zz} + 2 \left( \frac{\chi}{r} \right)_r \right] = \frac{2v_\varphi v_{\varphi,z}}{r} + F
\end{equation}

in $\Omega^T$, $\chi|_{S} = 0$ on $S^T$, $\chi|_{t=0} = \chi_0$ in $\Omega$, where $F = f_{r,z} - f_{z,r}$. To prove global regular solutions to problem (1.1) we need a priori estimate with weights which are singular on the axis of symmetry (see [Z1]). Therefore it is convenient to consider instead of problem (1.1) the following approximated problem

\begin{equation}
\begin{align*}
v_t + v \cdot \nabla v - \nu \Delta v + \nabla p &= f & \text{in } \Omega^T, \\
\text{div } v &= 0 & \text{in } \Omega^T, \\
\bar{n} \cdot v = 0, \quad \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha &= 0, & \alpha = 1, 2, \quad \text{on } S_1^T \cup S_2^T \cup S_2^T, \\
v|_{t=0} &= v_0 & \text{in } \Omega^T,
\end{align*}
\end{equation}

where $\Omega_\varepsilon = \{x \in \Omega : r > \varepsilon\}$, $S_\varepsilon = \{x \in \mathbb{R}^3 : r = \varepsilon, \ |z| < a\}$, $S_2 = \{x \in S_2 : r > \varepsilon\}$. Setting $\varepsilon = 0$ we obtain problem (1.1).

**Lemma 2.5.** Assume that $v(0) \in H^1(\Omega_\varepsilon)$, $f \in L_2(\Omega_\varepsilon^T)$, $\varepsilon \geq 0$, $T > 0$. Assume that $T$ is so small that

\begin{equation}
\varphi(c_0, c_1, c_2) T^{1/2} d_4 \left[ \| f \|_{L_2(\Omega_\varepsilon^T)}^3 + \| v(0) \|_{H^1(\Omega_\varepsilon)}^3 + T^{3/2} d_4 + 1 \right] \leq 1/2,
\end{equation}

where $\varphi$ is some positive increasing function of its arguments and $d_4$ is introduced in (2.1). Then there exists a solution to problem (2.26) such that $v \in W^{2,1}(\Omega_\varepsilon^T)$, $\nabla p \in L_2(\Omega_\varepsilon^T)$ and

\begin{equation}
\| v \|_{W^{2,1}(\Omega_\varepsilon^T)} + \| \nabla p \|_{L_2(\Omega_\varepsilon^T)} \leq 8 c_0 \left( \| f \|_{L_2(\Omega_\varepsilon^T)} + \| v(0) \|_{H^1(\Omega_\varepsilon)} \right),
\end{equation}

where $c_0$ appears in (2.22), $c_1$ in (2.30) and $c_2$ in (2.31). The existence and the estimate hold also for $\varepsilon = 0$.

**Proof.** Since we are going to prove the existence of solutions to problem (2.26) by the Leray-Schauder fixed point theorem we restrict the proof to
show a priori estimate (2.28) only because other steps of it are clear. In
the proof we omit the index \( \varepsilon \) for simplicity.
Applying Lemma 2.4 with \( s = 2 \) to problem (2.26) yields
\[
\|v\|_{W^{2,1}_2(\Omega^T)} + \|\nabla p\|_{L^2(\Omega^T)} \leq c_0(\|v \cdot \nabla v\|_{L^2(\Omega^T)} + \|f\|_{L^2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}).
\]
(2.29)
Now we examine the first term on the r.h.s. of (2.29). We estimate it by
\[
\left(\int_0^T dt \int_\Omega |v \cdot \nabla v|^2 dx\right)^{1/2} \leq \left(\int_0^T \|v(t)\|_{L^\infty(\Omega)}^2 \|\nabla v(t)\|_{L^2(\Omega)}^2 dt\right)^{1/2}.
\]
\[
\leq \sup_t \|\nabla v(t)\|_{L^2(\Omega)} \left(\int_0^T \|v(t)\|_{L^\infty(\Omega)}^2 dt\right)^{1/2} \equiv I_1.
\]
From [BIN, Ch. 3, Sect. 15] we have the interpolation
\[
\|v\|_{L^\infty(\Omega)} \leq c_1 \|v_{xx}\|_{L^2(\Omega)}^{3/4} \|v\|_{L^2(\Omega)}^{1/4} + c_1 \|v\|_{L^2(\Omega)}.
\]
(2.30)
Employing (2.30) in \( I_1 \) yields
\[
I_1 \leq c_1 \sup_t \|\nabla v(t)\|_{L^2(\Omega)} (T^{1/8} \|v_{xx}\|_{L^2(\Omega)}^{3/4} \sup_t \|v(t)\|_{L^2(\Omega)}^{1/4}
+ T^{1/2} \sup_t \|v(t)\|_{L^2(\Omega)}) \equiv I_2.
\]
Using the estimate
\[
\sup_t \|\nabla v(t)\|_{L^2(\Omega)} \leq \sup_t \|v(t)\|_{H^1(\Omega)}
\leq c_2(\|v\|_{W^{2,1}_2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)})
\]
(2.31)
and the energy estimate (2.1) in \( I_2 \) we obtain from (2.29) the inequality
\[
\|v\|_{W^{2,1}_2(\Omega^T)} + \|\nabla p\|_{L^2(\Omega^T)} \leq c_0 c_1 c_2(\|v\|_{W^{2,1}_2(\Omega^T)}
+ \|v(0)\|_{H^1(\Omega)})(T^{1/8} \|v\|_{W^{2,1}_2(\Omega^T)}^{3/4} d_4^{1/4} + T^{1/2} d_4)
+ c_0 \|f\|_{L^2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}).
\]
(2.32)
Assuming that \( T \) is so small that
\[
c_0 c_1 c_2(T^{1/8} \|v\|_{W^{2,1}_2(\Omega^T)}^{3/4} d_4^{1/4} + T^{1/2} d_4) \leq \frac{1}{2}
\]
(2.33)
we derive from (2.32) the inequality
\[
\|v\|_{W^{2,1}_2(\Omega^T)} + \|\nabla p\|_{L^2(\Omega^T)} \leq 2c_0 c_1 c_2.
\]
(2.34)
\[
\cdot (T^{1/8} \|v\|_{W^{3/4}_2(\Omega^T)} d_4^{1/4} + T^{1/2} d_4) \|v(0)\|_{H^1(\Omega)} + 2c_0(\|f\|_{L^2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}).
\]
The condition (2.33) is not written in a final form because it contains an unknown norm \( \|v\|_{W^{2,1}_2(\Omega^T)} \). Applying the Young inequality to the first expression on the r.h.s. of (2.34) yields
\[
\|v\|_{W^{2,1}_2(\Omega^T)} + \|\nabla p\|_{L^2(\Omega^T)} \leq \frac{\varepsilon^{4/3}}{4/3} \|v\|_{W^{2,1}_2(\Omega^T)}
\]
\[
+ \frac{1}{4\varepsilon^4} (2c_0 c_1 c_2 T^{1/8} d_4 \|v(0)\|_{H^1(\Omega)})^4 + 2c_0 c_1 c_2 T^{1/2} d_4 \|v(0)\|_{H^1(\Omega)} + 2c_0(\|f\|_{L^2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}).
\]
Setting \( \frac{\varepsilon^{4/3}}{4/3} = \frac{1}{2} \) we obtain that \( \varepsilon = \left( \frac{2}{3} \right)^{3/4} \) so the above inequality yields
\[
\|v\|_{W^{2,1}_2(\Omega^T)} + \|\nabla p\|_{L^2(\Omega^T)} \leq 27(c_0 c_1 c_2)^4 T^{1/2} d_4 \|v(0)\|_{H^1(\Omega)}^4
\]
\[
+ 4c_0 c_1 c_2 T^{1/2} d_4 \|v(0)\|_{H^1(\Omega)} + 4c_0(\|f\|_{L^2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}).
\]
Assuming that \( T \) is so small that
\[
27(c_0 c_1 c_2)^4 T^{1/2} d_4 \|v(0)\|_{H^1(\Omega)}^3 + 4c_0 c_1 c_2 T^{1/2} d_4 \leq 4c_0
\]
we obtain that (2.35) implies (2.28). Using (2.28) we express condition (2.33) in the form
\[
8(c_0 c_1 c_2)^4 T^{1/2} d_4 [(8c_0)^3 (\|f\|_{L^2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)})^3
\]
\[
+ (T^{1/2} d_4)^3 \leq \frac{1}{2}
\]
There exists a function \( \varphi(c_0, c_1, c_2) \) such that (2.36) and (2.37) imply (2.27). This concludes the proof.

3. Regularity of swirl \( u \)

Let us recall that \( u \) is a solution to problem (2.23). To show \( L^\infty \) estimate of \( u \) we need some notation
\[
A_k(t) = \{ x \in \Omega : u(x, t) > k \}, \quad u^{(k)} = \max\{u - k, 0\}
\]
\[
\mu(k) = \int_0^T \text{meas} \frac{\varepsilon}{\mu} A_k(t) dt,
\]
where \( \frac{3}{q} + \frac{2}{r} = \frac{3}{2} \).
Lemma 3.1. Assume that \( u_0 \in L_{\infty}(\Omega), \ g \in L_{\infty}(\Omega^T) \). Then there exists a constant \( d_6 \) depending on \( \| u_0 \|_{L_{\infty}(\Omega)}, \| g \|_{L_{\infty}(\Omega^T)} \) such that

\[
(3.1) \quad \| u \|_{L_{\infty}(\Omega^T)} \leq d_6.
\]

Proof. Multiplying (2.23) by \( u^{(k)} \) and integrating the result over \( \Omega \) yields

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u^{(k)}|^2 dx &+ \nu \int_{\Omega} |\nabla u^{(k)}|^2 dx - \nu \int_{-a}^{a} |u^{(k)}|_{r=R}^2 dz \\
&+ 2\nu \int_{\Omega} u_r u^{(k)} dr dz = \int_{\Omega} gu^{(k)} dx.
\end{align*}
\]

The last term on the l.h.s. of (3.2) equals

\[
\nu \int_{\Omega} (|u^{(k)}|^2),_r dr dz = \nu \int_{-a}^{a} |u^{(k)}|_{r=R}^2 dz
\]

because \( u^{(k)}|_{r=0} = 0 \). Otherwise the condition \( \int_0^T \int_{\Omega} \frac{u^2}{r^2} dx dt < \infty \) (see (2.15)) implies a contradiction for \( u > k \). Hence, (3.2) takes the form

\[
(3.3) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u^{(k)}|^2 dx + \nu \int_{\Omega} |\nabla u^{(k)}|^2 dx = \int_{\Omega} gu^{(k)} dx.
\]

Integrating (3.3) with respect to time and using that \( k > \| u_0 \|_{L_{\infty}(\Omega)} \) we obtain

\[
\begin{align*}
\| u^{(k)} \|_{V_2^0(\Omega^T)}^2 &\leq c \int_{\Omega^T} |gu^{(k)}| dx dt \\
&\leq c \| g \|_{L_{\infty}(\Omega^T)} \| u^{(k)} \|_{L_{10/3}(\Omega^T)} (\mu(k))^{7/10}
\end{align*}
\]

In view of imbedding \( V_2^0(\Omega^T) \subset L_{10/3}(\Omega^T) \) we have

\[
(3.5) \quad \| u^{(k)} \|_{V_2^0(\Omega^T)} \leq c \| g \|_{L_{\infty}(\Omega^T)} |\mu(k)|^{\frac{7}{10}(1+c)},
\]

where \( \frac{7}{10} = \frac{3}{10}(1+c) \) so \( \frac{7}{3} = 1 + c, \ k = \frac{4}{3} \) and \( \mu(k) \) is calculated for \( r = q = \frac{10}{3} \). From [LSU, Ch. 2, Sect. 6, Theorem 6.1] we conclude the proof.

Repeating the considerations from [Z3] and [LSU, Ch. 2, Sect. 8] we have
Lemma 3.2. (Hölder continuity of swirl). Let $g \in C^{\alpha,\alpha/2}(\Omega^T)$, $u_0 \in C^\alpha(\Omega)$, $\alpha \in (0,1/2]$. Let $v \in L_{10}(\Omega^T)$. Then $u \in C^{\alpha,\alpha/2}(\Omega^T)$.

For more details see [B].

Lemma 3.3. Let the assumptions of Lemmas 3.1 and 2.3 hold. Then

$$
\|v_r\|_{L^4(\Omega^T)}^4 \leq d_6^2 d_7^2, \quad t \leq T,
$$

where $d_6$ is introduced by (3.1) and $d_7$ by (2.15).

4. A priori estimates for $\chi$ in a neighborhood of the $x_3$-axis

First we examine the elliptic problem

$$
v_{r,z} - v_{z,r} = \chi \quad \text{in } \Omega,
$$

$$
v_{r,r} + v_{z,z} + \frac{v_r}{r} = 0 \quad \text{in } \Omega,
$$

$$
v_r|_{S_1} = 0, \quad v_z|_{S_2} = 0.
$$

Expressing (4.1)$_2$ in the form

$$
(rv_r)_r + (rv_z)_z = 0
$$

we have existence of a potential $\psi = \psi(r,z,t)$ such that

$$
v_r = \frac{\psi_z}{r}, \quad v_z = -\frac{\psi_r}{r}.
$$

Then the boundary conditions (4.1)$_3$ are satisfied if

$$
\psi|_{S_1} = 0 \quad \text{so } \psi(R,z,t) = 0 \quad \text{and } \psi|_{S_2} = 0 \quad \text{so } \psi(r,-a,t) = \psi(r,a,t) = 0.
$$

Therefore, problem (4.1) takes the form

$$
\left(\frac{\psi_z}{r}\right)_z + \left(\frac{\psi_r}{r}\right)_r = \chi \quad \text{in } \Omega,
$$

$$
\psi|_S = 0 \quad \text{on } S.
$$

To obtain an a priori estimate for $v' = (v_r, v_z)$ near the $x_3$-axis we have to work with problems with singular coefficients on it. Therefore, we can work either with functions vanishing sufficiently fast near the axis of symmetry or examining approximate solutions described by (2.26).
shall restrict our considerations to the second case because it seems to be more precise. Then problem (2.25) for \( \chi \) takes the form

\begin{equation}
(4.6)
\chi_t + v \cdot \nabla \chi - \frac{v_r}{r} \chi - \nu \left[ \left( r \left( \frac{\chi}{r}, r, r \right), r \right) + \chi_{zz} + 2 \left( \frac{\chi}{r} \right)_{,rr}, r \right] = \frac{2v_\varphi v_{\varphi z}}{r} + F \quad \text{in} \quad \Omega^T_{\varepsilon},
\end{equation}

\begin{align*}
\chi &= 0 \quad \text{on} \quad S^T_{1, \epsilon} \cup S^T_{2, \epsilon}, \\
\chi|_{t=0} &= \chi_0 \quad \text{in} \quad \Omega_{\varepsilon},
\end{align*}

where \( \Omega_{\varepsilon}, S_{\varepsilon}, 2\varepsilon \) are defined below problem (2.26).

The condition \( \chi|_{s_{\varepsilon}} = 0 \) follows from \([Z4, \text{Lemma 2.2}]\). Hence we assume that

\begin{equation}
(4.7) \quad \chi = 0 \quad \text{for} \quad r \leq \varepsilon.
\end{equation}

Solutions of (2.26) and (4.6) should be labeled with index \( \varepsilon \) which we omit for simplicity. To examine a behavior of solutions in a neighborhood of the axis of symmetry we introduce a smooth cut-off function \( \zeta_1 = \zeta_1(r) \) such that \( \zeta_1(r) = 1 \) for \( r \leq r_0 \) and \( \zeta_1(r) = 0 \) for \( r \geq 2r_0 \), where \( 2r_0 < R \).

Let us introduce the notation

\begin{equation}
(4.8) \quad \tilde{\chi} = \chi \zeta_1^2, \quad \tilde{v}_\varphi = v_\varphi \zeta_1, \quad \tilde{v}' = v' \zeta_1^2, \quad \tilde{F} = F \zeta_1^2,
\end{equation}

\begin{align*}
\tilde{F}' &= f' \zeta_1^2, \quad \tilde{F}_{\varphi} = f_{\varphi} \zeta_1.
\end{align*}

Then \( \tilde{\chi} \) is a solution to the problem

\begin{equation}
(4.9) \quad \tilde{\chi}_t + v \cdot \nabla \tilde{\chi} - \frac{v_r}{r} \tilde{\chi} - \nu \left[ \left( r \left( \frac{\tilde{\chi}}{r}, r, r \right), r \right) + \tilde{\chi}_{zz} + 2 \left( \frac{\tilde{\chi}}{r} \right)_{,rr}, r \right] = \frac{2\tilde{v}_\varphi \tilde{v}_{\varphi z}}{r} + \tilde{F} \quad \text{in} \quad \Omega^T_{\varepsilon},
\end{equation}

\begin{align*}
\tilde{\chi}|_{r=2r_0} &= 0, \quad \tilde{\chi}|_{r=\varepsilon} = 0, \quad \tilde{\chi}|_{s_2} = 0, \\
\tilde{\chi}|_{t=0} &= \tilde{\chi}_0 \quad \text{in} \quad \Omega_{\varepsilon}.
\end{align*}

**Lemma 4.1.** Let \( \Omega_{\varepsilon} = \{ x \in \Omega : 0 < \varepsilon < r \} \). Assume that there exists a weak solution to problem (1.1) described by Lemma 2.2. Assume that \( \tilde{v}_{\varphi} \in L_4(\Omega^T_{\varepsilon}), \chi \in L_{2p}(\Omega \cap \text{supp } \zeta_{1,r} \times (0, T)) \), where \( \text{supp } \zeta_{1,r} = \{ x \in \Omega_{\varepsilon} : r_0 \leq r \leq 2r_0 \} \) and \( \frac{\tilde{F}}{r} \in L_2(\Omega^T_{\varepsilon}) \). Then for sufficiently smooth solutions to
problem (1.1) we have
\[
\left\| \frac{\dot{X}}{r} \right\|_{L^\infty(0, t; L_2(\Omega_\varepsilon))}^2 + \nu \left\| \nabla \frac{\dot{X}}{r} \right\|_{L_2(\Omega_\varepsilon^t)}^2
\]
\[
\leq \frac{2}{\nu} \left\| \frac{\tilde{\varphi}}{r} \right\|_{L^4(\Omega_\varepsilon^t)}^4 + c(1/r_0)^2 + c(1/r_0)d_5\|\chi\|_{L_2(\Omega_\varepsilon^t)}^2 \\
+ \left\| \frac{\dot{X}(0)}{r} \right\|_{L_2(\Omega_\varepsilon)}^2 + c \left\| \frac{\tilde{F}}{r} \right\|_{L_2(\Omega_\varepsilon^t)}^2, \quad t \leq T,
\]
where \( \varepsilon < r_0 \) and \( d_5 \) is introduced in (2.2).

**Proof.** Multiplying (4.9) by \( \frac{\dot{X}}{r} \) and integrating the result over \( \Omega_\varepsilon \) we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\| \frac{\dot{X}}{r} \right\|_{L_2(\Omega_\varepsilon)}^2 + \nu \left\| \nabla \frac{\dot{X}}{r} \right\|_{L_2(\Omega_\varepsilon)}^2
= \int_{\Omega_\varepsilon} \left[ \nu \cdot \nabla \zeta_1^2 \frac{\dot{X}}{r^2} - \nu (\chi \zeta_1, r) \frac{\dot{X}}{r^2} \\
- \nu r \left( \frac{\chi}{r} \right) \frac{\dot{X}}{r^2} \frac{\dot{X}}{r^2} - \frac{2}{\nu} \frac{\dot{X}}{r} \frac{\dot{X}}{r^2} \right] dx
\]
\[
+ 2 \int_{\Omega_\varepsilon} \frac{\tilde{\varphi} \dot{\varphi}}{r} \frac{\dot{\chi}}{r} dx + \int_{\Omega_\varepsilon} \frac{\tilde{F} \dot{X}}{r^2} dx.
\]

Now we estimate the particular terms on the r.h.s. of (4.11). The second term implies
\[
\int_{\Omega_\varepsilon} \left( \frac{\dot{\chi}}{r} \right) o_r \frac{\dot{X}}{r} dx = - \int_{\Omega_\varepsilon} \frac{\tilde{\varphi}^2}{r^2} \frac{\dot{X}}{r} dz dx
\]
\[
\leq \frac{\varepsilon_1}{2} \int_{\Omega_\varepsilon} \left( \frac{\dot{X}}{r} \right)^2 dx + \frac{1}{2\varepsilon_1} \int_{\Omega_\varepsilon} \left( \frac{\tilde{\varphi}}{r} \right)^4 dx.
\]
To estimate the first term on the r.h.s. of (4.11) we use properties of the cut-off function \( \zeta_1 = \zeta_1(r) \). The first expression under the square bracket is bounded by
\[
c(1/r_0) \int_{\Omega_\varepsilon, \zeta_1, r} |v_r| \chi^2 dx.
\]
The fourth term under the square bracket is estimated by
\[
c(1/r_0) \int_{\Omega_\varepsilon, \zeta_1, r} \chi^2 dx.
\]
The second term under the square bracket equals

\[-\nu \int_{\Omega_\varepsilon} \chi, r \xi_1^2, \xi_1^2 \frac{\chi}{r^2} dx - \nu \int_{\Omega_\varepsilon} \chi \xi_1^2, r \xi_1^2 \frac{\chi}{r^2} dx \equiv I_1.\]

Integrating by parts in the first integral in \( I_1 \) it takes the form

\[-\frac{\nu}{2} \int_{\Omega_\varepsilon} (\chi^2, r) \frac{1}{r^2} \xi_1^2, \xi_1^2 dx = \frac{\nu}{2} \int_{\Omega_\varepsilon} \chi^2 \left( \xi_1^2, r \xi_1^2 \frac{1}{r} \right) dr dz.\]

Hence

\[|I_1| \leq c(1/r_0) \int_{\Omega_\varepsilon, \xi_1, r} \chi^2 dx.\]

Finally, the third term under the square bracket yields

\[-\frac{\nu}{2} \int_{\Omega_\varepsilon} \left[ \left( \frac{\chi}{r} \right)^2 \right] r, \xi_1^2, \xi_1^2 dx = \frac{\nu}{2} \int_{\Omega_\varepsilon} \left( \frac{\chi}{r} \right)^2 \left( \xi_1^2, r \xi_1^2 r \right) dr dz \equiv I_2.\]

Then,

\[|I_2| \leq c(1/r_0) \int_{\Omega_\varepsilon, \xi_1, r} \chi^2 dx.\]

Applying the Cauchy inequalities and the Poincaré inequality to the last term on the r.h.s. of (4.11) we estimate it by

\[\frac{\varepsilon_1^2}{2} \int_{\Omega_\varepsilon} \left| \nabla \frac{\chi}{r} \right|^2 dx + \frac{1}{2\varepsilon_2} \int_{\Omega_\varepsilon} \left| \frac{\tilde{F}}{r} \right|^2 dx.\]

Using the above estimates in (4.11) and assuming that \( \varepsilon_1 = \varepsilon_2 = \frac{\nu}{2} \) we obtain from (4.11) the inequality

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left\| \frac{\chi}{r} \right\|_{L^2(\Omega_\varepsilon)}^2 + \nu \left\| \nabla \frac{\chi}{r} \right\|_{L^2(\Omega_\varepsilon)}^2 \\
\leq \frac{1}{\nu} \left\| \frac{\tilde{v}_2}{r} \right\|_{L^4(\Omega_\varepsilon)}^4 + c(1/r_0) \int_{\Omega_\varepsilon, \xi_1, r} |v_r| \chi^2 dx \\
+ c(1/r_0) \int_{\Omega_\varepsilon, \xi_1, r} \chi^2 dx + c \int_{\Omega_\varepsilon} \left| \frac{\tilde{F}}{r} \right|^2 dx.
\end{align*}
\]
Integrating (4.12) with respect to time and using estimate (2.2) to the third term on the r.h.s. of (4.12) we have

\[ \left\| \frac{\hat{x}}{r} \right\|_{L^\infty(0,T;L^2(\Omega_{\varepsilon}))}^2 + \nu \left\| \nabla \frac{\hat{x}}{r} \right\|_{L^2(\Omega_{\varepsilon}^t)}^2 \leq \frac{2}{\nu} \left\| \frac{\hat{v}_\varphi}{r} \right\|_{L^4(\Omega_{\varepsilon}^t)}^4 + c(1/r_0) \int_{\Omega_{\varepsilon}^t,\xi_1,r} |v_r| \hat{\chi}^2 dx dt' \]

(4.13)

\[ + c(1/r_0)d_5^2 + \left\| \frac{\hat{F}}{r} \right\|_{L^2(\Omega_{\varepsilon}^t)}^2 + \left\| \frac{\hat{x}(0)}{r} \right\|_{L^2(\Omega_{\varepsilon})}^2, \quad t \leq T. \]

Applying the H"older inequality and using estimate (2.2) in the second term on the r.h.s. of (4.13) we estimate it by

\[ c(1/r_0)\|v\|_{L^{10/3}(\Omega_{\varepsilon}^t)} \|\chi\|_{L^{20/7}(\Omega_{\varepsilon}^t,\xi_1,r)}^2 \]

\[ \leq c(1/r_0)d_5 \|\chi\|_{L^{20/7}(\Omega_{\varepsilon}^t,\xi_1,r)}. \]

Using the estimate in (4.13) we obtain (4.10). This concludes the proof.

A crucial step in the proof of an a priori estimate in a neighborhood of the axis of symmetry is an estimate for $v_\varphi$ (see [Z1]). Therefore we localize equation (1.4) with boundary and initial conditions included in (1.7) and (1.8). Hence we have the following problem

\[ \ddot{v}_\varphi + v \cdot \nabla \dot{v}_\varphi + \frac{v_r}{r} \dot{v}_\varphi - v \Delta \dot{v}_\varphi + \nu \frac{\dot{v}_\varphi}{r^2} = v \cdot \nabla \xi_1 v_\varphi \]

\[ - 2\nu \nabla v_\varphi \nabla \xi_1 - \nu v_\varphi \Delta \xi_1 + f_\varphi \quad \text{in} \quad \Omega_{\varepsilon}^T, \]

\[ \ddot{v}_\varphi|_{t=0} = \ddot{v}_\varphi(0), \]

\[ \dot{v}_\varphi|_{r=2r_0} = 0, \quad \ddot{v}_\varphi|_{r=\varepsilon} = 0, \quad \ddot{v}_\varphi|_{z,s} = 0. \]

**Lemma 4.2.** Assume that $\ddot{f}_\varphi/\sqrt{r} \in L^{20/11}(\Omega_{\varepsilon}^t)$, $\ddot{v}_\varphi(0)/\sqrt{r} \in L^4(\Omega_{\varepsilon})$, $\varepsilon > 0$. Assume that estimates (2.2) and (3.1) hold, $d_5$ appears in (2.2) and $d_6$ in (3.1). Assume that $v$ is a solution of (1.1). Then for solutions of (4.14) the inequality is valid

\[ \frac{1}{8} \int_{\Omega_{\varepsilon}} \frac{\ddot{v}_\varphi^4(t)}{r^2} dx + \frac{3}{8} \nu \int_{\Omega_{\varepsilon}^t} \left| \nabla \ddot{v}_\varphi \right|^2 dx dt' + \frac{3}{4} \nu \int_{\Omega_{\varepsilon}^t} \frac{\ddot{v}_\varphi^4}{r^4} dx dt' \]

(4.15)

\[ \leq \frac{3}{2} \int_{\Omega_{\varepsilon}^t} \frac{|v_r| \ddot{v}_\varphi^4}{r^2} dx dt' + c(1/r_0)d_5^2(1 + d_6)d_6^2 + c \left\| \frac{\ddot{f}_\varphi}{r^{1/2}} \right\|_{L^{20/11}(\Omega_{\varepsilon}^t)}^4 \]

\[ + \frac{1}{4} \int_{\Omega_{\varepsilon}} \ddot{v}_\varphi^4(0) dx, \quad t \leq T. \]
Proof. Multiplying (4.14) by $\frac{\tilde{v}_{\varphi}|\tilde{v}_{\varphi}|^2}{r^2}$ and integrating over $\Omega_\varepsilon$ we obtain

\begin{equation}
\int_{\Omega_\varepsilon} \partial_t \tilde{v}_{\varphi} \frac{\tilde{v}_{\varphi}|\tilde{v}_{\varphi}|^2}{r^2} \, dx + \int_{\Omega_\varepsilon} v \cdot \nabla \tilde{v}_{\varphi} \frac{\tilde{v}_{\varphi}|\tilde{v}_{\varphi}|^2}{r^2} \, dx + \int_{\Omega_\varepsilon} \frac{v_r \tilde{v}_{\varphi}^4}{r} \, dx \\
- \nu \int_{\Omega_\varepsilon} \Delta \tilde{v}_{\varphi} \frac{\tilde{v}_{\varphi}|\tilde{v}_{\varphi}|^2}{r^2} \, dx + \nu \int_{\Omega_\varepsilon} \frac{|\tilde{v}_{\varphi}|^4}{r^4} \, dx \\
= \int_{\Omega_\varepsilon} v \cdot \nabla \zeta_1 v \varphi \frac{\tilde{v}_{\varphi}|\tilde{v}_{\varphi}|^2}{r^2} \, dx \\
- \nu \int_{\Omega_\varepsilon} [2 \nabla v \varphi \nabla \zeta_1 + v \varphi \Delta \zeta_1] \frac{\tilde{v}_{\varphi}|\tilde{v}_{\varphi}|^2}{r^2} \, dx \\
+ \int_{\Omega_\varepsilon} \tilde{f}_\varphi \tilde{v}_{\varphi} \frac{|\tilde{v}_{\varphi}|^2}{r^2} \, dx.
\end{equation}

Now we examine the particular terms in (4.16). The first term on the l.h.s. of (4.16) equals

$$\frac{1}{4} \frac{d}{dt} \int_{\Omega_\varepsilon} \tilde{v}_{\varphi}^4 \, dx.$$ 

The second term on the l.h.s. of (4.16) assumes the form

$$\frac{1}{4} \int_{\Omega_\varepsilon} v \cdot \nabla (\tilde{v}_{\varphi}^4) \frac{1}{r^2} \, dx = \frac{1}{4} \int_{\Omega_\varepsilon} [v \cdot \nabla \tilde{v}_{\varphi}^4 - v \cdot \nabla \frac{1}{r^2} \tilde{v}_{\varphi}^4] \, dx = \frac{1}{2} \int_{\Omega_\varepsilon} \frac{v_r \tilde{v}_{\varphi}^4}{r} \, dx,$$

where the first integral vanishes after integration by parts because $\text{div} \, v = 0$ and $v \cdot \bar{n}|_{S_{2\varepsilon}} = 0$, $\tilde{v}_{\varphi} = 0$ on $r = 2r_0$ and $r = \varepsilon$. Hence, the sum of the second and the third terms on the l.h.s. of (4.16) equals

$$\frac{3}{2} \int_{\Omega_\varepsilon} \frac{v_r \tilde{v}_{\varphi}^4}{r} \, dx.$$

Integrating by parts and using the boundary conditions in the fourth term
on the l.h.s. of (4.16) yields

\[ I_1 = \nu \int_{\Omega} \nabla \tilde{v}_\varphi \cdot \nabla \left( \frac{\tilde{v}_\varphi^2}{r^2} \right) dx = 3\nu \int_{\Omega} \left| \nabla \tilde{v}_\varphi \right|^2 dx - \frac{\nu}{2} \int_{\Omega} \left( \frac{\tilde{v}_\varphi^4}{r^2} \right)_r dr dz \]

\[ = 3\nu \int_{\Omega} \left| \frac{\nabla \tilde{v}_\varphi}{r} \right|^2 dx - \frac{\nu}{2} \int_{\Omega} \frac{\tilde{v}_\varphi^4}{r^2} dr dz - \nu \int_{\Omega} \frac{\tilde{v}_\varphi^4}{r^4} dx \]

\[ = \frac{3}{4} \nu \int_{\Omega} \left[ \frac{\nabla \tilde{v}_\varphi^2}{r^2} + \frac{\tilde{v}_\varphi^2 \nabla r}{r^2} \right] dx - \frac{\nu}{2} \int_{\Omega} \frac{\tilde{v}_\varphi^4}{r^2} dr dz + \frac{3}{4} \nu \int_{\Omega} \left| \frac{\nabla \tilde{v}_\varphi}{r} \right|^2 dx \]

Using the above considerations in (4.16), employing that \( \tilde{v}_\varphi |_{r=\epsilon} = 0 \) we obtain after integration with respect to time the equality

\[ \frac{1}{4} \int_{\Omega} \frac{\tilde{v}_\varphi^4(t)}{r^2} dx + \frac{3}{4} \nu \int_{\Omega} \left| \frac{\nabla \tilde{v}_\varphi}{r} \right|^2 dx dt' + \frac{3}{4} \nu \int_{\Omega} \frac{\tilde{v}_\varphi^4}{r^4} dx dt' \]

\[ = - \frac{3}{2} \int_{\Omega} \frac{v_r \tilde{v}_\varphi^2}{r^2} dx dt' + \int_{\Omega} v \cdot \nabla \varphi_1 \tilde{v}_\varphi \frac{|\tilde{v}_\varphi|}{r^2} dx dt' \]

\[ + \nu \int_{\Omega} \left[ 2 \nabla v_r \nabla \varphi + v_r \Delta \varphi \right] \tilde{v}_\varphi \frac{|\tilde{v}_\varphi|}{r^2} dx dt' + \int_{\Omega} \tilde{f}_\varphi \tilde{v}_\varphi \frac{|\tilde{v}_\varphi|}{r^2} dx dt' \]

\[ + \frac{1}{4} \int_{\Omega} \tilde{v}_\varphi^4(0) dx. \]
The second term on the r.h.s. of (4.17) is estimated by
\[ c(1/r_0)\|rv_\varphi\|^3_{L_\infty(\Omega^t_\varepsilon)} \int_{\Omega^t_\varepsilon} v^2 dx dt' \leq c(1/r_0)d_5^2 d_6^3, \]
where \( d_5 \) appears in (2.2) and \( d_6 \) in (3.1). Similarly the third integral is bounded by
\[ c(1/r_0)\|rv_\varphi\|^2_{L_\infty(\Omega^t_\varepsilon)} \int_{\Omega^t_\varepsilon} (|\nabla v_\varphi|^2 + |v_\varphi|^2) dx dt' \leq c(1/r_0)d_5^2 d_6^2. \]

Applying the Hölder and the Young inequalities to the term with \( \tilde{f}_\varphi \) we estimate it by
\[ \int_{\Omega^t_\varepsilon} |\tilde{f}_\varphi| \frac{|\tilde{v}_\varphi|^3}{r^2} dx dt' = \int_{\Omega^t_\varepsilon} \frac{|\tilde{f}_\varphi|}{r^{1/2}} \frac{|\tilde{v}_\varphi|^3}{r^{3/2}} dx dt' \leq \frac{\varepsilon_1^{4/3}}{4/3} \left( \frac{\tilde{v}_\varphi^2}{r} \right)_{L_{10/3}(\Omega^t_\varepsilon)} + \frac{1}{4\varepsilon_1^4} \left( \frac{\tilde{f}_\varphi}{r^{1/2}} \right)_{L_{20/11}(\Omega^t_\varepsilon)}^4. \]

Employing the above estimates in (4.17) and choosing \( \varepsilon_1 \) sufficiently small we obtain (4.15). This concludes the proof.

To examine problem (4.5) it is convenient to introduce new quantities \( \eta \) and \( \vartheta \) by the relations
\[ \psi = \eta r^2, \quad \chi = \vartheta r. \]

Then problem (4.5) assumes the form
\[ \Delta \eta + \frac{2\eta_r}{r} = \vartheta \quad \text{in} \quad \Omega, \quad \eta|_{\partial S} = 0, \quad \vartheta = 0 \quad \text{for} \quad r \leq \varepsilon. \]

Introducing the new quantities
\[ \tilde{\eta} = \eta \zeta_1^2, \quad \tilde{\vartheta} = \vartheta \zeta_1^2 \]
we see that problem (4.19) takes the form
\[ \Delta \tilde{\eta} + \frac{2\tilde{\eta}_r}{r} = \tilde{\vartheta} + 2\nabla \eta \nabla \zeta_1^2 + \eta \Delta \zeta_1^2 + \frac{2}{r} \zeta_1^2 r \eta \equiv \tilde{\vartheta} + \tilde{\vartheta}_1 \equiv \vartheta_2, \]
\[ \tilde{\eta}|_{\partial S} = 0, \quad \tilde{\eta}|_{r=2r_0} = 0, \quad \vartheta_2 = 0 \quad \text{for} \quad r \leq \varepsilon. \]
Lemma 4.3. Assume that \( \vartheta, r \in L^2(\Omega^T) \) and \( v \) is a weak solution satisfying (2.2). Then

\[
(4.22) \quad \int_{\Omega^t} |\nabla \tilde{\eta},r|_z^2 dx dt' + 6 \int_{\Omega^t} \frac{\tilde{\eta}^2,r}{r^2} dx dt' \leq \int_{\Omega^t} \vartheta^2,rr dx dt' + c(1/r_0)d_5^2,
\]

where \( d_5 \) is introduced in (2.2).

Proof. Differentiating (4.21) with respect to \( r \) and \( z \) yields

\[
(4.23) \quad \Delta \tilde{\eta},r - \frac{3\tilde{\eta},r}{r} + 2\tilde{\eta},rr = \vartheta,rr.
\]

Multiplying (4.23) by \( \tilde{\eta},r \) and integrating over \( \Omega \) implies

\[
\int_{\Omega} \Delta \tilde{\eta},r \tilde{\eta},r dx - 3 \int_{\Omega} \frac{\tilde{\eta}^2,r}{r^2} dx + 2 \int_{\Omega} \tilde{\eta},rr \tilde{\eta},r dx dz = \int_{\Omega} \vartheta,rr \tilde{\eta},r dx.
\]

Integrating by parts in the first term on the l.h.s. yields

\[
(4.24) \quad \int_{S_2} \tilde{\eta} \cdot \nabla \tilde{\eta},r \tilde{\eta},r dS_2 - \int_{\Omega} |\nabla \tilde{\eta},r|_z^2 dx - 3 \int_{\Omega} \frac{\tilde{\eta}^2,r}{r^2} dx + \left( \frac{\tilde{\eta}^2,r}{r^2} \right)_r dr dz = \left( \frac{\vartheta,rr \tilde{\eta},r}{r} \right)_r dx - \int_{\Omega} \vartheta,rr \tilde{\eta},rzz dx.
\]

The first integral on the l.h.s. of (4.24) equals

\[
I_1 \equiv \int_{S_2} \tilde{\eta},rzz \tilde{\eta},r dS_2.
\]

Expressing (4.21) in the cylindrical coordinates yields

\[
(4.25) \quad \tilde{\eta},rr + \tilde{\eta},zz + 3\frac{\tilde{\eta},r}{r} = \tilde{\vartheta} + \vartheta_1.
\]

In view of (4.21) we have that \( \tilde{\eta}|_{S_2} = 0 \) also \( \tilde{\eta},r|_{S_2} = 0 \) and \( \tilde{\eta},rr|_{S_2} = 0 \). Therefore \( \vartheta_1|_{S_2} = 0 \).

Projecting (4.25) on \( S_2 \) and using that \( \tilde{\vartheta}|_{S_2} = 0 \) and \( \vartheta_1|_{S_2} = 0 \) we have that \( \tilde{\eta},zz|_{S_2} = 0 \) so also \( \tilde{\eta},zrz|_{S_2} = 0 \). Therefore, \( I_1 = 0 \).
Since $\vartheta_2|_{S_2} = 0$ so also $\vartheta_{2,r}|_{S_2} = 0$. Then the first term on the r.h.s. of (4.24) vanishes. Hence (4.24) takes the form

\[
\int_\Omega |\nabla \tilde{\eta}_{rz}|^2 dx + 3 \int_\Omega \frac{\tilde{\eta}_{rz}^2}{r^2} dx + \int_{-a}^a \tilde{\eta}_{rz}^2 |_{r=0} dz
\]

(4.26)

= \int_\Omega \vartheta_{2,r} \tilde{\eta}_{rrz} dx.

Applying the Cauchy inequality to the r.h.s. of (4.26) we estimate it by

\[
\frac{1}{2} \int_\Omega \tilde{\eta}_{rrz}^2 dx + \frac{1}{2} \int_\Omega \vartheta_{2,r}^2 dx.
\]

Using this in (4.26) implies

\[
\int_\Omega |\nabla \tilde{\eta}_{rz}|^2 dx + 6 \int_\Omega \frac{\tilde{\eta}_{rz}^2}{r^2} dx \leq \int_\Omega \vartheta_{2,r}^2 dx.
\]

(4.27)

To estimate the r.h.s. of (4.27) we examine

\[
\int_\Omega \vartheta_{1,r}^2 dx \leq c(1/r_0) \int_{\Omega, \xi_1,r} (\eta_{rr}^2 + \eta_{r}^2 + \eta^2) dx
\]

\[
\leq c(1/r_0) \int_{\Omega, \xi_1,r} (\psi_{rr}^2 + \psi_r^2 + \psi^2) dx \leq c(1/r_0) \int_\Omega (v_{z,r}^2 + v_z^2) dx,
\]

where we used the equality $\psi(r, z) = \int_{-R}^R \psi(r', r, z) dr'$. Therefore, (4.27) takes the form

\[
\int_\Omega |\nabla \tilde{\eta}_{rz}|^2 dx + 6 \int_\Omega \frac{\tilde{\eta}_{rz}^2}{r^2} dx \leq \int_\Omega \vartheta_{2,r}^2 dx
\]

(4.28)

+ c(1/r_0) \int_\Omega (v_{z,r}^2 + v_z^2) dx.

Integrating (4.28) with respect to time and using (2.2) yield (4.22). This concludes the proof.

**Corollary 4.4.** Since $\tilde{\eta}_{rz} = \frac{\tilde{\psi}_r}{r} = \tilde{\eta}_{rz} = \left(\frac{\tilde{\chi}_r}{r}\right)_r$ and $\tilde{\vartheta} = \tilde{\chi}_r$ then (4.22) takes the form

\[
\int_{\Omega_t} \left|\nabla \left(\frac{\tilde{\chi}_r}{r}\right)_r\right|^2 dx + 6 \int_{\Omega_t} \frac{1}{r^2} \left(\frac{\tilde{\chi}_r}{r}\right)_r^2 dx
\]

(4.29)

\[
\leq \int_{\Omega_t^s} \left(\frac{\tilde{\chi}}{r}\right)_r^2 dx + c(1/r_0)d_3^2.
\]
Lemma 4.5. Assume that $\bar{\chi}(0) \in L^2(\Omega_\varepsilon)$, $\bar{F} \in L^2(\Omega_\varepsilon^T)$, $\tilde{v}(0) \bar{v} \in L^2(\Omega_\varepsilon^T)$, and $\frac{\tilde{v}(0)}{\bar{v}} \in L^2(\Omega_\varepsilon^T)$. Assume that (2.2) and (3.1) hold, where $d_5$ and $d_6$ are introduced. Assume that (4.30)

$$
\|u\|_{L^\infty(\Omega_{\varepsilon}^T)} \leq \sqrt[4]{\frac{5}{8}} \nu,
$$

which can be satisfied because $u$ vanishes on the axis of symmetry, it is the Hölder continuous and $\text{supp} \, \zeta$ is sufficiently small. Then the following a priori inequality holds

(4.31) \begin{equation}
\left\| \frac{\tilde{v}}{r} \right\|_{L^2(\Omega_{\varepsilon}^T)}^2 \leq c(1/r_0) d_5 \|\chi\|_{L^{20/7}(\Omega_{\varepsilon}^T, \zeta_{1, r})}^2 + cA^2, \quad t \leq T,
\end{equation}

where

$$
A^2 = c(1/r_0) d_5^2 [1 + (1 + d_6) d_6]\|\tilde{v}(0)/r\|_{L^2(\Omega_\varepsilon^T)} + \left\| \frac{\tilde{F}}{r} \right\|_{L^2(\Omega_{\varepsilon}^T)}^2
$$

$$
+ \left\| \frac{f_\varphi}{\sqrt{r}} \right\|_{L^{20/11}(\Omega_{\varepsilon}^T)}^4 + \left\| \frac{\tilde{v}_\varphi(0)}{\sqrt{r}} \right\|_{L^4(\Omega_{\varepsilon}^T)}^4.
$$

Proof. From (4.10) and (4.29) we have

(4.32) \begin{equation}
\int_{\Omega_{\varepsilon}^T} \left\| \nabla \left( \frac{\tilde{v}}{r} \right) \right\|_{r}^2 \, dx \, dt + 6 \int_{\Omega_{\varepsilon}^T} \frac{1}{r^2} \left( \frac{\tilde{v}}{r} \right)^2 \, dx \, dt' 
\end{equation}

$$
\leq \frac{2}{\nu^2} \int_{\Omega_{\varepsilon}^T} \frac{\tilde{v}_\varphi^4}{r^4} \, dx \, dt' + c(1/r_0) d_5 \|\chi\|_{L^{20/7}(\Omega_{\varepsilon}^T, \zeta_{1, r})} + cA^2_1,
$$

where

$$
A^2_1 = c(1/r_0) d_5^2 + \left\| \frac{\tilde{v}(0)}{r} \right\|_{L^2(\Omega_\varepsilon^T)}^2 + \left\| \frac{\tilde{F}}{r} \right\|_{L^2(\Omega_{\varepsilon}^T)}^2.
$$

To estimate the first term on the r.h.s. of (4.32) we use (4.15) in the form

(4.34) \begin{equation}
\int_{\Omega_{\varepsilon}^T} \frac{\tilde{v}_\varphi^4}{r^4} \, dx \, dt' \leq \frac{2}{\nu} \int_{\Omega_{\varepsilon}^T} \frac{|v_\varphi| \tilde{v}_\varphi^4}{r^2} \, dx \, dt' + cA^2_2,
\end{equation}

where

$$
A^2_2 = c(1/r_0) d_5^2 (1 + d_6) d_6 + \left\| \frac{\tilde{f}_\varphi}{\sqrt{r}} \right\|_{L^{20/11}(\Omega_{\varepsilon}^T)}^4 + \left\| \frac{\tilde{v}_\varphi(0)}{\sqrt{r}} \right\|_{L^4(\Omega_{\varepsilon}^T)}^4.
$$
Now we examine the first term on the r.h.s. of (4.34) in the following way

$$\frac{2}{\nu} \int_{\Omega_t^z} \frac{\bar{\nu}_r}{r^3} \epsilon \frac{\bar{\nu}^2}{r^2} dxdt' = \frac{2}{\nu} \int_{\Omega_t^z} \frac{\bar{\nu}_r}{r^3} u^2 \frac{\bar{\nu}^2}{r^2} dxdt'$$

$$\leq \frac{2}{\nu} \left[ \epsilon \int_{\Omega_t^z} \frac{\bar{\nu}^4}{r^4} dxdt' + \frac{1}{2\epsilon} \|u\|_{L^4(\Omega_t^z)}^4 \int_{\Omega_t^z} \frac{\bar{\nu}^2}{r^6} dxdt' \right].$$

Setting $\epsilon = \frac{\nu}{2}$ in the above inequality we obtain from (4.34) the inequality

$$(4.36) \quad \int_{\Omega_t^z} \frac{\bar{\nu}^4}{r^4} dxdt' \leq \frac{4}{\nu^2} \|u\|_{L^4(\Omega_t^z)}^4 \int_{\Omega_t^z} \frac{\bar{\nu}^2}{r^6} dxdt' + cA_2^2.$$  

Using (4.36) in (4.32) and applying the Hardy inequality

$$(4.37) \quad \int_{\Omega_t^z} \frac{1}{r^4} \left( \frac{\bar{\nu}_r}{r} \right)^2 dxdt \leq \int_{\Omega_t^z} \frac{1}{r^2} \left( \frac{\bar{\nu}_r}{r} \right)^2 dxdt$$

we obtain

$$6 \int_{\Omega_t^z} \frac{1}{r^2} \left( \frac{\bar{\nu}_r}{r} \right)^2 dxdt' \leq \frac{8}{\nu^4} \|u\|_{L^4(\Omega_t^z)}^4 \int_{\Omega_t^z} \frac{1}{r^2} \left( \frac{\bar{\nu}_r}{r} \right)^2 dxdt'$$

$$+ c(1/r_0)d_5 \|\chi\|_{L^{20/7}(\Omega_{t_1,r}^z)}^2 + c(A_1^2 + A_2^2).$$  

Assuming that

$$6 - \frac{8}{\nu^4} \|u\|_{L^4(\Omega_t^z)}^4 \geq 1,$$

which can be expressed in the form

$$(4.39) \quad \|u\|_{L^4(\Omega_t^z)} \leq \sqrt[4]{\frac{5}{8}} \nu,$$

we obtain from (4.38) the inequality

$$(4.40) \quad \int_{\Omega_t^z} \frac{1}{r^2} \left( \frac{\bar{\nu}_r}{r} \right)^2 dxdt' \leq c(1/r_0)d_5 \|\chi\|_{L^{20/7}(\Omega_{t_1,r}^z)}^2 + c(A_1^2 + A_2^2).$$

Employing (4.39) and (4.40) in (4.36) implies

$$\int_{\Omega_t^z} \frac{\bar{\nu}^4}{r^4} dxdt' \leq \frac{4}{\nu^2} \frac{5}{8} \nu^4 [c(1/r_0)d_5 \|\chi\|_{L^{20/7}(\Omega_{t_1,r}^z)}^2]$$

$$+ c(A_1^2 + A_2^2)] + cA_2^2$$

$$\leq c(1/r_0)d_5 \|\chi\|_{L^{20/7}(\Omega_{t_1,r}^z)}^2 + c(A_1^2 + A_2^2).$$
In view of (4.41) inequality (4.10) takes the form
\begin{equation}
\left\| \frac{\chi}{r} \right\|_{L_\infty(0,T;L_2(\Omega_\varepsilon))}^2 + \left\| \nabla \frac{\chi}{r} \right\|_{L_2(\Omega_\varepsilon)}^2 \leq c(1/r_0)d_5 \left\| \chi \right\|_{L_2(\Omega_\varepsilon)}^2 + c(A_1^2 + A_2^2).
\end{equation}

This concludes the proof.

**Lemma 4.6.** Let the assumptions of Lemma 4.5 hold. Then
\begin{equation}
\left\| \frac{\chi}{r} \right\|_{L_2(\Omega_\varepsilon)} \leq cA,
\end{equation}
where $A$ is introduced in Lemma 4.5.

**Proof.** From (4.31) we have
\begin{equation}
\left\| \frac{\chi}{r} \right\|_{L_{10/3}(\Omega_\varepsilon^{\lambda/2})} \leq c_1 \left\| \frac{\chi}{r} \right\|_{L_{10/3}(\Omega_\varepsilon^{\lambda})} + cA.
\end{equation}

Let us introduce the sets
\[ \Omega_\varepsilon^{(\lambda)} = \{(r, z) \in \Omega_\varepsilon : 0 < \varepsilon < r \leq r_0 - \lambda, |z| < a\} \]
and connect with them a set of cut-off functions such that
\[ \zeta^{(\lambda)} = \begin{cases} 1 & \text{for } (r, z) \in \Omega_\varepsilon^{(\lambda)} \\ 0 & \text{for } (r, z) \in \Omega_\varepsilon \setminus \Omega_\varepsilon^{(\lambda/2)} \end{cases} \]
Let $\vartheta = \frac{\chi}{r}$. Then (4.44) can be expressed in the form
\begin{equation}
\|\vartheta\|_{L_{10/3}(\Omega_\varepsilon^{(\lambda)} \times (0,t))} \leq c_1 \|\vartheta\|_{L_{10/3}(\Omega_\varepsilon^{(\lambda/2)} \setminus \Omega_\varepsilon^{(\lambda)} \times (0,t))} + cA.
\end{equation}
From (4.45) we have
\begin{equation}
\int_{\Omega_\varepsilon^{(\lambda)} \times (0,t)} |\vartheta|^{10/3} dx dt' \leq c_1 c_1^{10/3} \int_{\Omega_\varepsilon^{(\lambda/2)} \setminus \Omega_\varepsilon^{(\lambda)} \times (0,t)} |\vartheta|^{10/3} dx dt' + c_2 c^{10/3} A^{10/3},
\end{equation}

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where \( c_2 = 2^{10/3-1} \). Adding
\[
c_2c_1^{10/3} \int_{\Omega^{(\lambda)} \times (0,t)} |\vartheta|^{10/3} dx dt'
\]
to both sides of (4.46) we obtain
\[
\int_{\Omega^{(\lambda)} \times (0,t)} |\vartheta|^{10/3} dx dt' \\
\leq \frac{c_2c_1^{10/3}}{1 + c_2c_1^{10/3}} \int_{\Omega^{(\lambda/2)} \times (0,t)} |\vartheta|^{10/3} dx dt' + \frac{c_2c_1^{10/3}}{1 + c_2c_1^{10/3}} A^{10/3}.
\]
Introducing the notation
\[
f(\lambda) = \int_{\Omega^{(\lambda)} \times (0,t)} |\vartheta|^{10/3} dx dt', \quad \mu = \frac{c_2c_1^{10/3}}{1 + c_2c_1^{10/3}} < 1
\]
\[
K = \frac{c_2c_1^{10/3}}{1 + c_2c_1^{10/3}} A,
\]
we obtain from (4.47) the inequality
\[
f(\lambda) \leq \mu f(\lambda/2) + K,
\]
which implies the estimate
\[
f(\lambda) \leq \sum_{j=0}^{\infty} \mu^j K = \frac{1}{1 - \mu} K.
\]
Therefore, Lemma 4.6 is proved.

5. Estimate for \( \chi \) in a neighborhood located in a positive distance from the axis of symmetry

Let \( \zeta_2 = \zeta_2(r) \) be a smooth cut-off function such that \( \zeta_2(r) = 0 \) for \( r \leq r_0 \) and \( \zeta_2(r) = 1 \) for \( r \geq 2r_0 \). Let \( \{\zeta_1(r), \zeta_2(r)\} \) compose a partition of unity in the radial direction. Let us introduce the notation
\[
\bar{\chi} = \chi \zeta_2^2, \quad \bar{v}_\varphi = v_\varphi \zeta_2, \quad \bar{v}' = v' \zeta_2^2, \quad \bar{F} = F \zeta_2^2, \quad \bar{f}' = f' \zeta_2^2,
\]
\[
\bar{f}_\varphi = f_\varphi \zeta_2.
\]
In view of (4.6) function $\bar{\chi}$ is a solution to the problem
\begin{align*}
\bar{\chi},_t + v \cdot \nabla \bar{\chi} - \frac{v_r}{r} \bar{\chi} - \nu \left[ \left( \frac{\bar{\chi}}{r} \right),_r + \bar{\chi},_{zz} + 2 \left( \frac{\bar{\chi}}{r} \right),_r \right] \\
= v \cdot \nabla \zeta^2 \bar{\chi} - \nu \left[ \left( \chi \zeta^2 \right),_r + r \left( \frac{\bar{\chi}}{r} \right),_r \zeta^2 + 2 \left( \frac{\bar{\chi}}{r} \zeta^2 \right),_r \right] \\
+ \frac{2 \bar{v}_\varphi \bar{v}_\varphi,z}{r} + \bar{F} \quad \text{in } \Omega^T, \\
\bar{\chi}|_{r=r_0} = 0, \quad \bar{\chi}|_{S_1 \cup S_2} = 0, \\
\bar{\chi}|_{t=0} = \bar{\chi}_0 \quad \text{in } \Omega.
\end{align*}

(5.2)

**Lemma 5.1.** Assume that $v$ is a weak solution to problem (1.1) satisfying assumptions of Lemma 2.1. Let the assumptions of Lemma 3.1 hold. Let $d_1, d_5, d_7, d_6$ be constants introduced by (2.1), (2.2), (2.15), (3.1), respectively. Let $\bar{F} \in L_2(0,T;L_{6/5}(\Omega)), \bar{\chi}(0) \in L_2(\Omega)$. Then solutions to (5.2) satisfy the estimate
\begin{align*}
\left\| \frac{\bar{\chi}}{r} \right\|_{V^0_2(\Omega^t)} &\leq c [c(1/r_0)d_1^2 + 1] A_0, \quad t \leq T,
\end{align*}

(5.3)

where
\begin{align*}
A_0^2 = c(1/r_0)(d_5^2 + d_6^2) + \| \bar{F} \|^2_{L_2(0,T;L_{6/5}(\Omega))} + \frac{1}{r_0^2} \| \bar{\chi}(0) \|^2_{L_2(\Omega)}
\end{align*}

(5.4)

and $r_0$ is introduced by the definition of the cut-off function $\zeta_2(r)$.

**Proof.** Multiplying (5.2) by $\bar{\chi}/r^2$, integrating over $\Omega$ and using the boundary conditions yields
\begin{align*}
\frac{d}{dt} \int_{\Omega} \left( \frac{\bar{\chi}}{r} \right)^2 dx + \nu \int_{\Omega} \left| \nabla \frac{\bar{\chi}}{r} \right|^2 dx &= \int_{\Omega} v \cdot \nabla \zeta^2 \frac{\bar{\chi}}{r^2} dx \\
&- \nu \int_{\Omega} \left[ \left( \chi \zeta^2 \right),_r + r \left( \frac{\bar{\chi}}{r} \right),_r \zeta^2 + 2 \left( \frac{\bar{\chi}}{r} \zeta^2 \right),_r \right] \frac{\bar{\chi}}{r^2} dx \\
&+ 2 \int_{\Omega} \frac{\bar{v}_\varphi \bar{v}_\varphi,z}{r} \frac{\bar{\chi}}{r} dx + \int_{\Omega} \frac{\bar{F}}{r} \frac{\bar{\chi}}{r} dx.
\end{align*}

(5.5)

Now we estimate the terms from the r.h.s. of (5.5). We estimate the first term by
\begin{align*}
\varepsilon_1 \left\| \frac{\bar{\chi}}{r} \right\|^2_{L_6(\Omega)} + c(1/r_0, 1/\varepsilon_1) \| v \chi \|^2_{L_{6/5}(\Omega_{\zeta_2,r})}.
\end{align*}
The first term under the square bracket of the second term yields
\[ \int_{\Omega} \chi_{r, \zeta_2}^2 \frac{1}{r^2} \zeta_2^2 \chi \, dr \, dz + \int_{\Omega} \chi^2 \frac{1}{r^2} (\zeta_2^2, r, \zeta_2^2) \, dx \equiv I_1, \]
where the first integral in \( I_1 \) equals
\[ -\frac{1}{2} \int_{\Omega} \chi^2 \left( \frac{1}{r} \zeta_2, \zeta_2^2 \right) dr \, dz. \]
Hence
\[ |I_1| \leq c(1/r_0) \int_{\Omega \zeta_2, r} \chi^2 \, dx. \]
Similarly, the integral with the second term under the square bracket implies
\[ \int_{\Omega} \left( \frac{\chi_{r, \zeta_2}^2}{r} \right) \chi_{r, \zeta_2}^2 \chi r \, dr \, dz = -\frac{1}{2} \int_{\Omega} \chi^2 \left( \frac{1}{r^2} (\zeta_2^2, r, \zeta_2^2) \right) dr \, dz \equiv I_2. \]
Then
\[ |I_2| \leq c(1/r_0) \int_{\Omega \zeta_2, r} \chi^2 \, dx. \]
Finally, the integral with the last term under the square bracket is bounded by
\[ c(1/r_0) \int_{\Omega \zeta_2, r} \chi^2 \, dx. \]
Summarizing, the second term on the r.h.s. of (5.5) is bounded by
\[ c(1/r_0) \int_{\Omega \zeta_2, r} \chi^2 \, dx. \]
The third term on the r.h.s. of (5.5) yields
\[
\left| \int_{\Omega} \frac{\hat{v}_r^2}{r^2} \left( \frac{\chi}{r} \right) \, dx \right| \leq \varepsilon_2 \int_{\Omega} \left( \frac{\chi}{r} \right)^2 \, dx + c(1/\varepsilon_2) \int_{\Omega \zeta_2} \frac{\hat{v}_r^4}{r^4} \, dx \\
\leq \varepsilon_2 \int_{\Omega} \left( \frac{\chi}{r} \right)^2 \, dx + c(1/\varepsilon_2, 1/r_0) \int_{\Omega} v_r^4 \, dx.
\]
Finally, the last term on the r.h.s. of (5.5) is bounded by

$$\varepsilon_3 \left\| \frac{\bar{X}}{r} \right\|_{L_6(\Omega)}^2 + c\left(1/\varepsilon_3\right) \left\| \frac{\bar{F}}{r} \right\|_{L_6/5(\Omega)}^2.$$  

Employing the above estimates in (5.5) and choosing $\varepsilon_1 - \varepsilon_3$ sufficiently small yield

$$\frac{d}{dt} \int_{\Omega} \left\| \frac{\bar{X}}{r} \right\|_{L_\infty(0,t;L_2(\Omega))}^2 + \nu \left\| \nabla \frac{\bar{X}}{r} \right\|_{L_2(\Omega^t)}^2 \leq c(1/r_0) \left\| v \right\|_{L_2(\Omega)}^2 \left\| \chi \right\|_{L_3(\Omega_{c_2,r})}^2,$$

$$+ c(1/r_0) \left\| \chi \right\|_{L_2(\Omega_{c_2,r})}^2 + c(1/r_0) \int_{\Omega} v_\phi^4 \, dx + c \left\| \frac{\bar{F}}{r} \right\|_{L_6/5(\Omega)}^2.$$  

Integrating (5.6) with respect to time and exploiting estimates (2.1) and (3.6) we obtain

$$\left\| \frac{\bar{X}}{r} \right\|_{L_\infty(0,t;L_2(\Omega))}^2 + \nu \left\| \nabla \frac{\bar{X}}{r} \right\|_{L_2(\Omega^t)}^2 \leq c(1/r_0) d_1^2 \left\| \chi \right\|_{L_2(0,t;L_3(\Omega_{c_2,r}))}^2 + c(1/r_0) (d_5^2 + d_6^2 + d_7^2),$$

$$+ c \left\| \frac{\bar{F}}{r} \right\|_{L_2(0,t;L_6/5(\Omega))}^2 + \left\| \frac{\bar{X}(0)}{r} \right\|_{L_2(\Omega)}^2,$$

where $t \leq T$, $d_1$ is introduced in (2.1), $d_5$ in (2.2), $d_6$ in (3.1) and $d_7$ in (2.15). To estimate the first term on the r.h.s. of (5.7) by data we express (5.7) in the form

$$\left\| \bar{X} \right\|_{L_3(\Omega^t)} \leq c_1 \left\| \chi \right\|_{L_3(\Omega_{c_2,r}^t)} + A_0,$$

where $c_1 = (1/r_0) d_1$ and $A_0$ is defined by (5.4).

To apply the local considerations (see LSU, Ch. 4, Sect. 10) we introduce the sets $\Omega^{(\lambda)} = \{(r, z) \in \Omega : r \geq r_0 + \lambda\}$ and corresponding cut-off functions $\zeta^{(\lambda)}(x)$ such that $\zeta^{(\lambda)}(x) = 1$ for $x \in \Omega^{(\lambda)}$ and $\zeta^{(\lambda)}(x) = 0$ for $x \in \Omega \setminus \Omega^{(\lambda/2)}$, so $|\nabla \zeta^{(\lambda)}| \leq \frac{\lambda}{\lambda}$. Moreover, we assume that $r_0 + \lambda = 2r_0$.

Then (5.8) can be expressed in the form

$$\left\| \chi \right\|_{L_3(\Omega^{(\lambda)} \times (0,t))} \leq c_1 \left\| \chi \right\|_{L_3(\Omega^{(\lambda/2)} \setminus \Omega^{(\lambda)} \times (0,t))} + A_0.$$  

Hence

$$\left\| \chi \right\|_{L_3(\Omega^{(\lambda)} \times (0,t))} \leq 4c_1^3 \left\| \chi \right\|_{L_3(\Omega^{(\lambda/2)} \setminus \Omega^{(\lambda)} \times (0,t))} + 4A_0^3.$$  

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By the filling-hole argument we have

\[ (5.11) \quad \int_{\Omega^{(\lambda)} \times (0,t)} |\chi|^3 dx dt' \leq \frac{4c_1^3}{4c_1^3 + 1} \int_{\Omega^{(\lambda/2)} \times (0,t)} |\chi|^3 dx dt' + \frac{4}{4c_1^3 + 1} A_0^3. \]

Introducing the notation

\[ f(\lambda) = \int_{\Omega^{(\lambda)} \times (0,t)} |\chi|^3 dx dt', \quad \mu = \frac{4c_1^3}{4c_1^3 + 1} < 1, \quad K = \frac{4}{4c_1^3 + 1} A_0^3, \]

we obtain from (5.11) the inequality

\[ f(\lambda) \leq \mu f(\lambda/2) + K \]

which implies the estimate

\[ f(\lambda) \leq \sum_{j=0}^{\infty} \mu^j K = \frac{1}{1 - \mu} K. \]

Employing the estimate in (5.7) we get (5.3). This concludes the proof.

**6. Estimate for \( v' \)**

From Lemmas 4.5 and 4.6 we have

\[ (6.1) \quad \left\| \frac{\bar{X}}{r} \right\|_{V_2^0(\Omega_t^r')} \leq cA, \quad t \leq T, \]

where

\[ A^2 = c(1/r_0)d_2^2[1 + (1 + d_6)d_7^2] + \left\| \frac{\bar{X}(0)}{r} \right\|_{L_2(\Omega_r)}^2 \]

\[ \quad + \left\| \frac{\tilde{F}}{r} \right\|_{L_2(\Omega_t^r')}^2 + \left\| \frac{\tilde{f}_\varphi}{\sqrt{r}} \right\|_{L_{20/11}(\Omega_t^r')}^4 + \left\| \frac{\tilde{v}_\varphi(0)}{\sqrt{r}} \right\|_{L_{4}(\Omega_r)}^4, \quad t \leq T. \]

Next, Lemma 5.1 implies

\[ (6.3) \quad \left\| \frac{\bar{X}}{r} \right\|_{V_2^0(\Omega_t^r')} \leq c[c(1/r_0)d_2^2 + 1] A_0, \quad t \leq T, \]
\[ A_0^2 = c(1/r_0)(d_5^2 + d_6^2 d_7^2) + \|\bar{F}\|_{L_2(0,t;L_6/5(\Omega))}^2 \]

(6.4)

\[ + \frac{1}{r_0^2} \| \bar{\chi}(0) \|_{L_2(\Omega)}^2, \quad t \leq T. \]

Inequalities (6.1) and (6.3) imply

(6.5)

\[ \left\| \frac{\chi}{r} \right\|_{V_0^0(\Omega_\varepsilon)} \leq cA + c[c(1/r_0)d_1 + 1]A_0 \equiv cA_*, \quad t \leq T. \]

Let us consider problem (4.5). It is convenient to introduce new quantities \( \eta \) and \( \vartheta \) by the relations

(6.6)

\[ \psi = \eta r^2, \quad \chi = \vartheta r. \]

Then problem (4.5) assumes the form

(6.7)

\[ \Delta \eta + \frac{2\eta_r}{r} = \vartheta \quad \text{in} \quad \Omega, \quad \eta|_S = 0, \quad \vartheta = 0 \quad \text{for} \quad r \leq \varepsilon. \]

Since (6.5) holds we have

**Lemma 6.1.** Assume that \( \vartheta \in H^1(\Omega) \). Then for a sufficiently smooth solution of (6.7) we have

\[
\int \Omega \left( \eta^2 + |\nabla \eta|^2 + |\nabla \eta_r|^2 + |\nabla \eta_z|^2 + |\nabla \eta_{rz}|^2 + |\nabla \eta_{zz}|^2 \right) dx \\
+ \int \Omega \left( \eta_r^2 r^2 + \eta_{rr}^2 r^2 \right) dx + \int_{-a}^{a} \left( \eta_r^2 r = R + \eta_{rr}^2 r = R \right) dz \\
+ \int_{-a}^{a} \left( \eta^2 \right|_{r=0} + \eta_r^2 \right|_{r=0} + \eta_z^2 \right|_{r=0} + \eta_{rz}^2 \right|_{r=0} + \eta_{zz}^2 \right|_{r=0} \right) dz \\
\leq c \int \Omega \left( \vartheta^2 + \vartheta_r^2 + \vartheta_z^2 \right) dx.
\]

**Proof.** Multiplying (6.7) by \( \eta \), integrating over \( \Omega \) and using the boundary conditions yields

\[- \int \Omega |\nabla \eta|^2 dx + 2 \int \Omega \eta_r \eta_r r dr dz = \int \Omega \eta \vartheta dx.\]
Applying the Cauchy and the Young inequalities to the r.h.s. of the above equality gives

$$\int_{\Omega} |\nabla \eta|^2 dx + \int_{-a}^{a} \eta^2|_{r=0} dz \leq \varepsilon \int_{\Omega} \eta^2 dx + c(1/\varepsilon) \int_{\Omega} \vartheta^2 dx.$$

In view of sufficiently small $\varepsilon$ and the Poincare inequality we get the estimate

$$\int_{\Omega} (\eta^2 + |\nabla \eta|^2) dx + \int_{-a}^{a} \eta^2|_{r=0} dz \leq c \int_{\Omega} \vartheta^2 dx.$$

Differentiating (6.7) with respect to $r$, multiplying the result by $\eta \cdot r$ and integrating over $\Omega$ yields

$$\int_{\Omega} \Delta \eta \cdot \eta_r dx - 3 \int_{\Omega} \frac{\eta_r^2}{r^2} dx + 2 \int_{\Omega} \frac{\eta_r \eta_r r}{r} dx = \int_{\Omega} \vartheta \eta_r dx,$$

where we used that (6.7) takes the form

$$\eta_{rr} + \eta_{zz} + 3 \frac{\eta_r}{r} = \vartheta, \quad \text{so} \quad \eta_{rrr} + \eta_{rzz} + 3 \frac{\eta_{rr}}{r} - 3 \frac{\eta_r}{r^2} = \vartheta_r.$$

Integrating by parts in (6.10) and using that $\eta_r|_{S_2} = 0$ we obtain

$$\int_{S_1} \bar{n} \cdot \nabla \eta \cdot \eta_r dS_1 = \int_{\Omega} |\nabla \eta|^2 dx - 3 \int_{\Omega} \frac{\eta_r^2}{r^2} dx + \int_{\Omega} (\eta_{rr})_r drdz,$$

$$\int_{\Omega} \vartheta \eta_r dx.$$

In view of (6.11) we have

$$\bar{n} \cdot \nabla \eta_r|_{S_1} = \eta_{rr}|_{S_1} = -3 \frac{\eta_r}{r}|_{S_1}.$$

Moreover, integrating by parts in the r.h.s. of (6.12) gives

$$\int_{\Omega} (\vartheta \eta_r)_r drdz - \int_{\Omega} \vartheta (\eta_{rr} r + \eta_r) drdz,$$

where the first integral vanishes because $\vartheta|_{r=R} = 0$ and $\vartheta|_{r=0} = 0$. 

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Therefore, (6.12) takes the form

\[
\int_{\Omega} |\nabla \eta, r|^2 dx + 3 \int_{\Omega} \frac{\eta_r^2}{r^2} dx + 3 \int_{-a}^{a} \eta_r^2 |r=R| dz - \int_{-a}^{a} \eta_r^2 |r=R| dz \\
+ \int_{-a}^{a} \eta_r^2 |r=0| dz \leq \varepsilon \left( \int_{\Omega} \eta_r^2 dx + \int_{\Omega} \frac{\eta_r^2}{r^2} dx \right) + c(1/\varepsilon) \int \vartheta^2 dx
\]

(6.13)

For sufficiently small \( \varepsilon \), (6.13) implies the estimate

\[
\frac{1}{2} \int_{\Omega} |\nabla \eta, r|^2 dx + \frac{5}{2} \int_{\Omega} \frac{\eta_r^2}{r^2} dx + 2 \int_{-a}^{a} \eta_r^2 |r=R| dz + \int_{-a}^{a} \eta_r^2 |r=0| dz \\
\leq c \int \vartheta^2 dx.
\]

(6.14)

Differentiating (6.7)\(_1\) with respect to \( z \), multiplying the result by \( \eta, z \) and integrating over \( \Omega \) gives

\[
\int_{\Omega} \Delta \eta, z \eta, z dx + 2 \int_{\Omega} \eta, rz \eta, z r dz = \int \vartheta, z \eta, z dx.
\]

Integrating by parts and using that \( \eta, zz |_S = 0, \eta, z |_S = 0, \vartheta |_S = 0 \) we obtain

\[
\int_{\Omega} |\nabla \eta, z|^2 dx - \int_{\Omega} (\eta, z)_r r dz = \int \vartheta \eta, z dz.
\]

Continuing, we have

\[
\int_{\Omega} |\nabla \eta, z|^2 dx + \int_{-a}^{a} \eta^2 r |r=0| dz \leq \varepsilon \int_{\Omega} \eta, zz dx + c(1/\varepsilon) \int \vartheta^2 dx.
\]

Hence, for sufficiently small \( \varepsilon \), we obtain

\[
(6.15) \quad \int_{\Omega} |\nabla \eta, z|^2 dx + \int_{-a}^{a} \eta^2 r |r=0| dz \leq c \int \vartheta^2 dx.
\]

Differentiating (6.7)\(_1\) with respect to \( r \) and \( z \) yields

\[
\Delta \eta, rz - \frac{3\eta, zr}{r} + 2 \frac{\eta, zrr}{r} = \vartheta, zr.
\]

(6.16)
Multiplying (6.16) by \( \eta_{rz} \) and integrating over \( \Omega \) implies
\[
\int_\Omega \Delta \eta_{rz} \eta_{rz} \, dx - 3 \int_\Omega \frac{\eta_{rz}^2}{r^2} \, dx + 2 \int_\Omega \eta_{zr} \eta_{zr} \, drdz 
\]
(6.17)
\[
= \int_\Omega \partial_{rz} \eta_{zr} \, dx.
\]
Integrating by parts in the first integral on the l.h.s. yields
\[
\int_S \bar{n} \cdot \nabla \eta_{rz} \eta_{rz} \, dS - \int_\Omega |\nabla \eta_{rz}|^2 \, dx - 3 \int_\Omega \frac{\eta_{rz}^2}{r^2} \, dx 
\]
(6.18)
\[
+ \int_\Omega (\eta_{rz})_r \, drdz = \int_\Omega (\partial_{,r} \eta_{rz})_r \, dx - \int_\Omega \partial_{,r} \eta_{zr} \, dx.
\]
The first integral on the r.h.s. vanishes because \( \partial_{,r}|S_2 = 0 \) and the first integral on the l.h.s. equals
\[
\int_{S_1} \eta_{zr} \eta_{zr} \, dS_1 + \int_{S_2} \eta_{zr} \eta_{zr} \, dS_2 \equiv I_1.
\]
Since \( \eta_{rr} = -3 \frac{\eta_{,r}}{r} \) on \( S_1 \), so \( \eta_{rrz} = -3 \frac{\eta_{rz}}{r} \) on \( S_1 \) also. Therefore, the first integral in \( I_1 \) takes the form
\[
-3 \int_{S_1} \frac{\eta_{rz}^2}{r} \, r \, dz = -3 \int_{-a}^a \eta_{rz}^2 |_{r=R} \, dz.
\]
Projecting (6.11) on \( S_2 \) gives \( \eta_{zr} |_{S_2} = 0 \), so also \( \eta_{zrr} |_{S_2} = 0 \). Therefore, the second term in \( I_1 \) vanishes.
In view of the above considerations, (6.18) takes the form
\[
\int_\Omega |\nabla \eta_{zr}|^2 \, dx + 3 \int_{-a}^a \eta_{rz}^2 |_{r=R} \, dz + 3 \int_\Omega \frac{\eta_{rz}^2}{r^2} \, dx 
\]
(6.19)
\[
- \int_\Omega (\eta_{rz})_r \, drdz = \int_\Omega \partial_{,r} \eta_{zr} \, dx.
\]
Performing integration by parts in the last term on the l.h.s. and applying the Cauchy and the Young inequalities to the r.h.s. we derive
\[
\int_\Omega |\nabla \eta_{zr}|^2 \, dx + 6 \int_\Omega \frac{\eta_{rz}^2}{r^2} \, dx + 4 \int_{-a}^a \eta_{rz}^2 |_{r=R} \, dz 
\]
(6.20)
\[
+ 2 \int_{-a}^a \eta_{rz}^2 |_{r=0} \, dz \leq \int_\Omega \partial_{,r}^2 \, dx.
\]
Differentiating (6.7) twice with respect to $z$, multiplying the result by $\eta_{zz}$ and integrating over $\Omega$ we arrive to

$$\int_\Omega \Delta \eta_{zz} \eta_{zz} \, dx + 2 \int_\Omega \eta_{zzr} \eta_{zz} \, dx = \int_\Omega \partial_z \eta_{zz} \, dx.$$  

Since $\eta_{zz}$ vanishes on $S_1 \cup S_2$ the above equality takes the form

$$\int_\Omega |\nabla \eta_{zz}|^2 \, dx + \int_{-a}^{a} \eta_{zz}^2 \, dz = - \int_\Omega (\vartheta_z \eta_{zz}) \, dz + \int_\Omega \partial_z \eta_{zz} \, dx.$$  

(6.21)

Since $\eta_{zz}|_{S_2} = 0$ the first integral on the r.h.s. of (6.21) vanishes. Applying the Cauchy and the Young inequalities to the second term on the r.h.s. of (6.21) we derive

$$\int_\Omega |\nabla \eta_{zz}|^2 \, dx + \int_{-a}^{a} \eta_{zz}^2 \, dz \leq c \int_\Omega \vartheta_z^2 \, dx.$$  

(6.22)

From (6.9), (6.14), (6.15), (6.20) and (6.22) we obtain (6.8). This concludes the proof.

Estimate (6.8) does not contain the norm $\|\nabla \eta_{rr}\|_{L^2(\Omega)}$ because to estimate it we need vanishing of $\eta_{rrr}|_{S_1}$. But the boundary conditions on $S_1$ do not imply it. Therefore we recall a smooth cut-off function $\zeta = \zeta_1(r)$ such that $\zeta_1(r) = 1$ for $r \leq r_0$ and $\zeta_1(r) = 0$ for $r \geq 2r_0$, $2r_0 < R$. Introducing the notation

$$\tilde{\eta} = \eta \zeta_2^2, \quad \tilde{\vartheta} = \vartheta \zeta_1^2$$

we see that $\tilde{\eta}$ is a solution to the problem

$$\Delta \tilde{\eta} + \frac{2 \tilde{\eta}_r}{r} = \tilde{\vartheta} + 2 \nabla \tilde{\eta} \nabla \zeta_1^2 + \eta \Delta \zeta_1^2 + \frac{\eta \zeta_1^2 \vartheta}{r} \equiv \tilde{\vartheta} \equiv \vartheta_1 \equiv \vartheta_2,$$

$$\tilde{\eta}|_{S_2} = 0.$$  

Lemma 6.2. Let $\tilde{\eta}$ be a solution to (6.24). Let $\vartheta, \vartheta \in L^2(\Omega), \vartheta' \in H^1(\Omega)$. Then the following estimate holds

$$\int_\Omega |\nabla \tilde{\eta}|^2 r^2 \, dx + \int_\Omega |\tilde{\eta}_{rr}|^2 \, dx + \int_\Omega |\tilde{\eta}_{rrr}|^2 \, dx + \int_{-a}^{a} |\tilde{\eta}_r|^2 \, dz$$  

$$\leq c \int_\Omega \vartheta^2 \, dx + c \|\vartheta'|^2_{H^1(\Omega)}.$$  

(6.25)

$$\xi_{2014} \xi_{2014}$$
Proof. Differentiating (6.24) twice with respect to \( r \), multiplying the result by \( r^2 \tilde{\eta} \) and integrating over \( \Omega \) yields

\[
\int_{\Omega} (\Delta \tilde{\eta})_{,rr} r^2 \tilde{\eta}_{,rr} dx + 2 \int_{\Omega} \left( \frac{\tilde{\eta}_r}{r} \right)_{,rr} r^2 \tilde{\eta}_{,rr} dx \]

(6.26)

\[
= \int_{\Omega} \vartheta_{2,rr} r^2 \tilde{\eta}_{,rr} dx.
\]

Since \( \Delta \tilde{\eta} = \tilde{\eta}_{,rr} + \tilde{\eta}_{,zz} + \frac{\tilde{\eta}_r}{r} \) we have

\[
(\Delta \tilde{\eta})_{,rr} = \tilde{\eta}_{,rrrr} + \tilde{\eta}_{,zrrr} + \frac{\tilde{\eta}_{,rr}}{r} - 2 \frac{\tilde{\eta}_{,rr}}{r^2} + 2 \frac{\tilde{\eta}_r}{r^3}
\]

Employing the expression in (6.26) implies

\[
\int_{\Omega} \left( \Delta \tilde{\eta}_{,rr} - \frac{2 \tilde{\eta}_{,rr}}{r^2} + \frac{2 \tilde{\eta}_r}{r^3} \right) r^2 \tilde{\eta}_{,rr} dx
\]

(6.27)

\[
+ 2 \int_{\Omega} \left( \frac{\tilde{\eta}_r}{r} \right)_{,rr} r^2 \tilde{\eta}_{,rr} dx = \int_{\Omega} \vartheta_{2,rr} r^2 \tilde{\eta}_{,rr} dx,
\]

where \( \tilde{\eta}_{,rr} |_{S_2} = 0 \) and \( \tilde{\eta} \) vanishes with all derivatives on \( S_1 \).

Integrating by parts in (6.27) and using the boundary conditions we arrive to the equality

\[
\int_{\Omega} \nabla \tilde{\eta}_{,rr} \nabla (r^2 \tilde{\eta}_{,rr}) dx + 6 \int_{\Omega} |\tilde{\eta}_{,rr}|^2 dx
\]

(6.28)

\[
- 6 \int_{\Omega} \tilde{\eta}_{,rr} \tilde{\eta}_r dr dz - 2 \int_{\Omega} \tilde{\eta}_{,rrr} \tilde{\eta}_{,rr} dx = \int_{\Omega} \vartheta_{2,rr} r^2 \tilde{\eta}_{,rr} dx.
\]

Continuing calculations in (6.28) gives

\[
\int_{\Omega} |\nabla \tilde{\eta}_{,rr}|^2 r^2 dx + 2 \int_{\Omega} \tilde{\eta}_{,rrr} \tilde{\eta}_{,rr} dx + 6 \int_{\Omega} |\tilde{\eta}_{,rr}|^2 dx
\]

\[
- 6 \int_{\Omega} \tilde{\eta}_{,rr} \tilde{\eta}_r dr dz - 2 \int_{\Omega} \tilde{\eta}_{,rrr} \tilde{\eta}_{,rr} dx = \int_{\Omega} \vartheta_{2,rr} r^2 \tilde{\eta}_{,rr} dx.
\]

Next, we have

\[
\int_{\Omega} |\nabla \tilde{\eta}_{,rr}|^2 r^2 dx + 6 \int_{\Omega} |\tilde{\eta}_{,rrr}|^2 dx - 3 \int_{\Omega} \partial_r |\tilde{\eta}_r|^2 dr dz
\]

\[
= \int_{\Omega} (\vartheta_{2,rr} r^2 \tilde{\eta}_{,rr})_r dx - \int_{\Omega} \vartheta_{2,r} (r^2 \tilde{\eta}_{,rr})_r dx.
\]
Since $\tilde{\eta}_r|_{r=R} = 0$, $\tilde{\eta}_{rr}|_{r=R} = 0$, $\vartheta|_{r=\varepsilon} = 0$ we obtain

\begin{equation}
\int_{\Omega} |\nabla \tilde{\eta}_{rr}|^2 r^2 dx + 6 \int_{\Omega} |\tilde{\eta}_{rr}|^2 dx + 3 \int_{-a}^{a} |\tilde{\eta}_r|^2 |_{r=0} dz
\end{equation}

(6.29)

\begin{equation}
= - \int_{\Omega} (\partial_{2,r} \tilde{\eta}_{rr} r^2 + 2 \partial_{2,r} \tilde{\eta}_{rr} r) dx.
\end{equation}

(6.30)

Applying the Cauchy and the Young inequalities to the r.h.s. of (6.29) implies

\begin{equation}
\int_{\Omega} |\nabla \tilde{\eta}_{rr}|^2 r^2 dx + 6 \int_{\Omega} |\tilde{\eta}_{rr}|^2 dx + 6 \int_{-a}^{a} |\tilde{\eta}_r|^2 |_{r=0} dz
\end{equation}

(6.30)

\begin{equation}
\leq c \int_{\Omega} \vartheta^2 dx \leq c \int_{\Omega} \vartheta^2 dx + c \int_{\Omega} \vartheta^2 dx.
\end{equation}

The second integral on the r.h.s. of (6.30) will be estimated in the following way

\begin{equation}
\int_{\Omega} \vartheta^2 dx = \int_{\Omega} \left( 2 \nabla \eta \nabla \zeta_1 + \eta \Delta \zeta_1 + \frac{\eta \zeta_1}{r} \right)^2 r^2 dx
\end{equation}

\begin{equation}
\leq c \int_{\Omega_{\zeta_1,r}} (\vartheta^2 + \vartheta^2 + \vartheta^2) dx \leq c \int_{\Omega_{\zeta_1,r}} (\psi^2 + \vartheta^2 + \psi^2) dx
\end{equation}

\begin{equation}
\leq c \int_{\Omega_{\zeta_1,r}} (v^2 + v^2) dx + c \int_{\Omega_{\zeta_1,r}} \psi^2 dx \equiv I.
\end{equation}

Since $\psi|_{S_2} = 0$ we have

\begin{equation}
\psi(r, z, t) = \int_{-a}^{z} \psi_{,z} dz
\end{equation}

and the second term in $I$ is estimated by

\begin{equation}
c \int_{\Omega} v^2 dx.
\end{equation}

Summarizing,

\begin{equation}
\int_{\Omega} \vartheta^2 dx \leq c \|v\|^2_{H^1(\Omega)}.
\end{equation}

(6.31)
From (6.30) and (6.31) we obtain (6.25). This concludes the proof.

To find an estimate for $\eta_{,rrr}$ in a neighborhood located in a positive distance from the axis of symmetry we introduce the notation

$$\bar{\eta} = \eta \zeta^2, \quad \bar{\vartheta} = \vartheta \zeta^2.$$  \hfill (6.32)

Then (6.11) takes the form

$$\bar{\eta}_{,rr} + \bar{\eta}_{,zz} + 3 \frac{\bar{\eta}}{r} = \bar{\vartheta} + 2 \eta_{,r} \zeta^2 + \eta_{,rrr} + 3 \frac{\eta_{,r}}{r}.$$  \hfill (6.33)

**Lemma 6.3.** Let $\vartheta \in H^1(\Omega)$. Then

$$\| \bar{\eta}_{,rrr} \|_{L^2(\Omega)} \leq c \int_{\Omega} (\vartheta^2 + \vartheta^2_{,r} + \vartheta^2_{,z}) \, dx.$$  \hfill (6.34)

**Proof.** Differentiating (6.33) with respect to $r$ yields

$$\bar{\eta}_{,rr} + \bar{\eta}_{,rzz} + 3 \frac{\bar{\eta}}{r} - 3 \frac{\vartheta}{r^2} = \bar{\vartheta} + \left( 2 \eta_{,r} \zeta^2 + \eta_{,rrr} + \frac{3 \eta_{,r}}{r} \right).$$  \hfill (6.34)

In view of (6.8) we obtain from (6.34) the inequality

$$\| \bar{\eta}_{,rrr} \|_{L^2(\Omega)}^2 \leq c \| \bar{\eta}_{,rzz} \|_{L^2(\Omega)}^2 + c(1/r_0) \| \bar{\eta}_{,rr} \|_{L^2(\Omega)}^2$$

$$+ c(1/r_0) \| \eta_{,r} \|_{L^2(\Omega)}^2 + c \| \bar{\vartheta}_{,r} \|_{L^2(\Omega)}^2 + c(1/r_0)(\| \eta_{,rr} \|_{L^2(\Omega)}^2)$$

$$+ \| \eta_{,r} \|_{L^2(\Omega_{\zeta_2})}^2 + \| \eta \|_{L^2(\Omega_{\zeta_2})}^2) \leq c(1/r_0) \int_{\Omega} (\vartheta^2 + \vartheta^2_{,r} + \vartheta^2_{,z}) \, dx.$$  \hfill (6.35)

This concludes the proof.

However, the norm on the l.h.s. of (6.5) is over $\Omega_\varepsilon$ we can pass to the limit $\varepsilon = 0$ because the r.h.s. of (6.5) is independent of $\varepsilon$.

**Lemma 6.4.** Assume that $\bar{x} = V^0(\Omega^T)$. Then

$$\left\| \frac{v_r}{r} \right\|_{V^0_2(\Omega^T)} \leq c \left\| \frac{\bar{x}}{r} \right\|_{V^0_2(\Omega^T)}.$$  \hfill (6.36)

**Proof.** From (6.8) we have

$$\int_{\Omega} (\eta^2_{,z} + \eta^2_{,zrr} + \eta^2_{,zzr} + \eta^2_{,zzz}) \, dx \leq c \left\| \frac{\bar{x}}{r} \right\|_{H^1(\Omega)}^2.$$  \hfill (6.37)
Using that
\[ \int_{\Omega} \eta_z^2 \, dx = \int_{\Omega} \left| \frac{v_r}{r} \right|^2 \, dx, \]
\[ \int_{\Omega} (|\eta_{zrr}|^2 + |\eta_{zzr}|^2 + |\eta_{zzz}|^2) \, dx \]
\[ = \int_{\Omega} \left[ \left( \frac{v_r}{r} \right)^2,_{rr} + \left( \frac{v_r}{r} \right)^2,_{zr} + \left( \frac{v_r}{r} \right)^2,_{zz} \right] \, dx, \]
we obtain from (6.37) the estimate
\[ (6.38) \quad \left\| \frac{v_r}{r} \right\|_{H^2(\Omega)}^2 \leq c \left\| \frac{\chi}{r} \right\|_{H^1(\Omega)}^2. \]
Integrating (6.38) with respect to time implies
\[ (6.39) \quad \left\| \frac{v_r}{r} \right\|_{L^2(0,T;H^2(\Omega))}^2 \leq c \left\| \frac{\chi}{r} \right\|_{L^2(0,T;H^1(\Omega))}^2. \]
From (6.15) we have
\[ (6.40) \quad \int_{\Omega} \left| \nabla \frac{v_r}{r} \right|^2 \, dx = \int_{\Omega} |\nabla \eta_z|^2 \, dx \leq c \left\| \frac{\chi}{r} \right\|_{L^2(\Omega)}^2 \]
and (6.9) yields
\[ \int_{\Omega} \left| \frac{v_r}{r} \right|^2 \, dx = \int_{\Omega} \left( \frac{\psi}{r^2} \right)^2,_{z} \, dx \leq \int_{\Omega} |\nabla \eta|^2 \, dx \leq c \left\| \frac{\chi}{r} \right\|_{L^2(\Omega)}^2. \]
The above two estimates yield
\[ (6.41) \quad \left\| \frac{v_r}{r} \right\|_{H^1(\Omega)}^2 \leq c \left\| \frac{\chi}{r} \right\|_{L^2(\Omega)}^2. \]
In view of the assumptions of the lemma estimate (6.41) implies
\[ (6.42) \quad \left\| \frac{v_r}{r} \right\|_{L^\infty(0,T;H^1(\Omega))}^2 \leq c \left\| \frac{\chi}{r} \right\|_{L^\infty(0,T;L^2(\Omega))}^2. \]
From (6.39) and (6.42) we derive (6.36). This concludes the proof.
Next we have
Lemma 6.5. Assume that \( \chi r \in V_2^0(\Omega^T) \). Then

\[
\|v'\|_{V_2^1(\Omega^T)} \leq c \left\| \frac{\chi}{r} \right\|_{V_2^0(\Omega^T)}.
\]

Proof. Since \( v_r = \frac{\psi}{r} = r\eta _z, v_z = -\frac{\psi}{r} = -r\eta _r - 2\eta \) we have

\[
\int_\Omega \left( |\nabla v_r|^2 + |\nabla v_z|^2 + |v_r|^2 + |v_z|^2 \right) dx \leq c \int_\Omega \left( |\eta|^2 + |\nabla \eta|^2 \right) dx
\]

\[
+ |\nabla \eta _r|^2 + |\nabla \eta _z|^2 dx \leq c \left\| \frac{\chi}{r} \right\|_{L_2(\Omega)}^2,
\]

where (6.8) was used. Similarly,

\[
\int_\Omega \left( |\nabla ^2(v_r)|^2 + |\nabla ^2(v_z)|^2 + |v_r|^2 + |v_z|^2 \right) dx \leq c \int_\Omega \left( |\nabla ^2(r\eta _z)|^2 + |\nabla ^2(r\eta _r)|^2 + |\nabla \eta|^2 + |\eta|^2 \right) dx
\]

\[
\leq c \left\| \frac{\chi}{r} \right\|_{H^1(\Omega)}^2,
\]

where Lemmas 6.1, 6.2, 6.3 were employed. Taking \( L_\infty \) norm with respect to time to (6.44) and \( L_2 \) norm with respect to time to (6.45) we obtain (6.43). This concludes the proof.

7. Estimate for the angular component of velocity

Let us consider the problem

\[
v_{\varphi,t} - \nu \Delta v_{\varphi} + v' \cdot \nabla v_{\varphi} + \frac{v_r}{r} v_{\varphi} + \nu \frac{v_{\varphi}}{r^2} = f_{\varphi} \quad \text{in } \Omega^T,
\]

\[
v_{\varphi,r} = \frac{1}{r} v_{\varphi} \quad \text{on } S_1^T,
\]

\[
v_{\varphi,z} = 0 \quad \text{on } S_2^T,
\]

\[
v_{\varphi}|_{t=0} = v_{\varphi}(0) \quad \text{in } \Omega.
\]

Lemma 7.1. Assume that \( v_{\varphi}(0) \in H^1_0(\Omega), v_{\varphi} \in L_{5/2}(0,T;W^1_{5/2}(\Omega)), f_{\varphi} \in L_2(\Omega^T) \). Assume that \( A \) defined by (6.2) and \( A_0 \) defined by (6.4) are
finite. Let $A_*$ be introduced in (6.5). Then

$$
\frac{1}{4} \|v_{\varphi,t}\|_2^2 + \frac{\nu}{2} \|v_{\varphi}(t)\|_{H^1_0}(\Omega) + \frac{\nu}{2} \|v_{\varphi}(R, t)\|_{L^2(-a,a)}^2
\leq \frac{\nu}{2} \|v_{\varphi}(0)\|_{H^1_0}(\Omega) + \nu d_0^2 + \varphi(A_*) \|v_{\varphi}\|_{L^2_{5/2}(0,t;W^{1/2}_0(\Omega))}^2
+ \|f_{\varphi}\|_{L^2(\Omega)}, \quad t \leq T,
$$

(7.2)

where

$$
\|u\|_{H^1_0(\Omega)} = \left( \int_\Omega (|\nabla u|^2 + \frac{u^2}{r^2}) \, dx \right)^{1/2}.
$$

**Proof.** Multiplying (7.1) by $v_{\varphi,t}$ and integrating the result over $\Omega$ yields

$$
\int_\Omega v_{\varphi,t}^2 \, dx + \frac{\nu}{2} \frac{d}{dt} \int_\Omega |\nabla v_{\varphi}|^2 \, dx + \frac{\nu}{2} \frac{d}{dt} \int_\Omega \frac{v_{\varphi}^2}{r^2} \, dx
\leq \frac{\nu}{2} \int_\Omega v_{\varphi}^2 \, dx
+ \frac{1}{2\varepsilon_1} \int_\Omega v_{\varphi,t}^2 \, dx
+ \frac{1}{2\varepsilon_2} \int_\Omega \nabla v_{\varphi}^2 \, dx
+ \frac{1}{2\varepsilon_3} \int_\Omega v_{\varphi,t}^2 \, dx
+ \frac{1}{2} \int_\Omega f_{\varphi}^2 \, dx.
$$

(7.3)

Setting $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{2}$ we get

$$
\frac{1}{4} \int_\Omega v_{\varphi,t}^2 \, dx + \frac{\nu}{2} \frac{d}{dt} \int_\Omega |\nabla v_{\varphi}|^2 \, dx + \frac{\nu}{2} \frac{d}{dt} \int_\Omega \frac{v_{\varphi}^2}{r^2} \, dx
\leq \int_\Omega \frac{v_{\varphi}^2}{r^2} \, dx
+ \int_\Omega f_{\varphi}^2 \, dx
$$

(7.4)

Integrating (7.4) with respect to time and using that

$$
\|v'\|_{L^{10}(\Omega^T)} + \left\| \frac{v'}{r} \right\|_{L^{10}(\Omega^T)} \leq \varphi(A_*),
$$

where Lemmas 6.4 and 6.5 are employed, we obtain from (7.4) the in-
equality

\[
\frac{1}{4} \int_{\Omega'} v_{\varphi, t'}^2 \, dx \, dt' + \frac{\nu}{2} \int_{\Omega} |\nabla v_{\varphi}(t)|^2 \, dx + \frac{\nu}{2} \int_{\Omega} \frac{v_{\varphi}^2(t)}{r^2} \, dx
\]

\[\quad + \frac{\nu}{2} \int_{-a}^{a} v_{\varphi}^2|_{r=R, t=0} \, dz \leq \frac{\nu}{2} \int_{\Omega} |\nabla v_{\varphi}(0)|^2 \, dx + \frac{\nu}{2} \int_{\Omega} \frac{v_{\varphi}^2(0)}{r^2} \, dx\]

\[\quad + \frac{\nu}{2} \int_{-a}^{a} v_{\varphi}^2|_{r=R, t} \, dz + \varphi(A_\ast)(\|\nabla v_{\varphi}\|_{L_5/2(\Omega')}^2 + \|v_{\varphi}\|_{L_5/2(\Omega')}^2)\]

\[\quad + \|f_{\varphi}\|_{L_2(\Omega')}^2, \quad t \leq T.\]

Using that the third integral on the r.h.s. of (7.5) is bounded by \( cd_6 \) (see (3.1)) we derive (7.2). This concludes the proof.

To estimate the coefficient near \( \varphi(A_\ast) \) on the r.h.s. of (7.2) we introduce the Green function to the linear part of problem (7.1). Let us denote it by \( G \). To obtain an estimate independent of time we consider problem (7.1) in the intervals \((0, T_0)\) and \((kT_0, (k + 1)T_0)\), where \( k \in \mathbb{N} \) and \( T_0 \) is a given positive number. For \( t \in (0, T_0) \) problem (7.1) can be expressed in the following integral form

\[
v_{\varphi}(x, t) = \int_{\Omega'} \nabla y_\alpha G(x - y, t - \tau)v_{\alpha}(y, \tau)v_{\varphi}(y, \tau) \, dy \, d\tau
\]

\[- \int_{\Omega'} G(x - y, t - \tau) \left( \frac{v_r(y, \tau)}{r} v_{\varphi}(y, \tau) + \nu \frac{v_{\varphi}(y, \tau)}{r^2} \right) \, dy \, d\tau\]

\[\quad + \int_{\Omega'} G(x - y, t)v_{\varphi}(y, 0) \, dy + \int_{\Omega'} G(x - z, t - \tau) \frac{1}{r} v_{\varphi}(z, \tau) \, dz \, d\tau\]

\[\quad + \int_{\Omega'} G(x - y, t - \tau)f_{\varphi}(y, \tau) \, dy \, d\tau, \quad t \in (0, T_0).\]

To examine problem (7.1) in the interval \((kT_0, (k + 1)T_0)\), \( k \in \mathbb{N} \), we introduce a smooth cut-off function \( \zeta = \zeta(t) \) such that \( \zeta(t) = 1 \) for \( t \in [kT_0, (k + 1)T_0] \) and \( \zeta(t) = 0 \) for \( t \notin [(k - 1)/2]T_0, (k + 3/2)T_0 \].

Multiplying (7.1) by \( \zeta \) and introducing the notation \( \tilde{v}_{\varphi} = v_{\varphi}\zeta, \tilde{f}_{\varphi} = f_{\varphi}\zeta \).
we obtain

\[
\tilde{v}_\varphi, t - \nu \Delta \tilde{v}_\varphi + v' \cdot \nabla \tilde{v}_\varphi + \frac{v_r}{r} \tilde{v}_\varphi = \tilde{f}_\varphi + v_\varphi \dot{\zeta} \quad \text{in} \quad \Omega \times ((k - 1)T_0, (k + 2)T_0),
\]

\[
\tilde{v}_\varphi, r = \frac{1}{r} \tilde{v}_\varphi \quad \text{on} \quad S_1 \times ((k - 1)T_0, (k + 2)T_0),
\]

\[
\tilde{v}_\varphi, z = 0 \quad \text{on} \quad S_2 \times ((k - 1)T_0, (k + 2)T_0),
\]

\[
\tilde{v}_\varphi \big|_{t=(k-1)T_0} = 0.
\]

(7.7)

Using the Green function we express (7.7) in the following integral equation

(7.8)

\[
\tilde{v}_\varphi(x, t) = \int_{(k-1)T_0}^{t} \int_{\Omega} \nabla_{y, \alpha} G(x - y, t - \tau)v'_{\alpha}(y, \tau)\tilde{v}_\varphi(y, \tau) dy d\tau
\]

\[- \int_{(k-1)T_0}^{t} \int_{\Omega} G(x - y, t - \tau) \left( \frac{v_r(y, \tau)}{r} \tilde{v}_\varphi(y, \tau) + \nu \frac{\tilde{v}_\varphi(y, \tau)}{r^2} \right) dy d\tau
\]

\[+ \int_{(k-1)T_0}^{t} \int_{S_1} G(x - z, t - \tau) \frac{1}{r} \tilde{v}_\varphi(z, \tau) dz d\tau
\]

\[+ \int_{(k-1)T_0}^{t} \int_{\Omega} G(x - y, t - \tau) (\tilde{f}_\varphi(y, \tau) + v_\varphi(y, \tau) \dot{\zeta}(\tau)) dy d\tau,
\]

where \( t \in ((k - 1)T_0, (k + 2)T_0) \).

**Lemma 7.2.** Assume that \( T_0 > 0 \) is a given positive number. Assume that \( v_\varphi(0) \in W^{1/5}_{5/2}(\Omega) \), \( \tilde{f}_\varphi \in L^{5/3}_{5/2}(\Omega \times ((k - 1)T_0, (k + 1)T_0)) \), \( \frac{v_\varphi}{r} \in L^{10/3}_{10/2}(\Omega \times ((k - 1)T_0, (k + 1)T_0)) \) for \( k \in \mathbb{N} \). Then the following estimates hold

\[
\|v_\varphi\|_{W^{1/2}_{5/2}(\Omega^t)} \leq c(T_0) \left[ \varphi(A_*) d_7 + T_0^{2/5} d_6 + \|v_\varphi(0)\|_{W^{1/5}_{5/2}(\Omega)} \right]
\]

(7.9)

\[
\left\| \frac{v_\varphi}{r} \right\|_{L^{10/3}_{10/2}(\Omega{T_0})}^2 + \|f_\varphi\|_{L^{5/3}_{5/2}(\Omega{T_0})}, \quad t \leq T_0,
\]

\[
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\]

\[\text{Z99 6–2–2014}\]
To estimate the third term on the r.h.s. of (7.2) we assume that

Similarly, the second term on the r.h.s. of (7.12) is bounded by

where we used (2.7), (2.15), (6.5) and (6.43).

Then (7.11) takes the form

(7.10)

and

(7.11)

(7.12)

To estimate the third term on the r.h.s. of (7.2) we assume that \( \sigma = 5/2 \). Then (7.11) takes the form

Now we estimate the particular terms from the r.h.s. of (7.12). Applying the Hölder inequality the first term on the r.h.s. is bounded by

where we used (2.7), (2.15), (6.5) and (6.43).

Similarly, the second term on the r.h.s. of (7.12) is bounded by

\[ \|v \|_{L^{3/2}(\Omega)} \leq c(T_0) \left[ \|v'\|_{L^{3/2}(\Omega)} + \|v\|_{L^{3/2}(\Omega)} \right], \quad t \leq T_0. \]
Assuming that \( v_\varphi \geq 1 \) (otherwise we have regularity of axially symmetric solutions) the third term on the r.h.s. of (7.12) is estimated by

\[
\left\| \frac{v_\varphi}{r} \right\|^{2}_{L^{10/3}(\Omega^{2})}.
\]

The fourth term is bounded by

\[
cT_0^{2/5} \| u \|_{L^{\infty}(\Omega^{2})} \leq cT_0^{2/5} d_6,
\]

where (3.1) is used. Summarizing the above estimates we obtain (7.9).

Now we apply the potential theory to (7.8). Then we obtain

\[
\| v_\varphi \|_{W^{1,1/2}_{5/2}(\Omega \times (kT_0,t))} \leq \| \tilde{v}_\varphi \|_{W^{1,1/2}_{5/2}(\Omega \times ((k-1)T_0,(k+1)T_0))}
\]

\[
\leq c(T_0) \left[ \| v'_\varphi \|_{L^{5/2}_{5/3}(\Omega \times ((k-1)T_0,(k+1)T_0))}
\right.
\]

\[
\leq + \left. \| v_r \tilde{v}_\varphi \|_{L^{5/2}_{5/3}(\Omega \times ((k-1)T_0,(k+1)T_0))}
\right.
\]

\[
\leq + \left. \| v_r \|_{L^{5/2}_{5/3}(\Omega \times ((k-1)T_0,(k+1)T_0))}
\right.
\]

\[
\leq + \left. \| \tilde{v}_\varphi \|_{L^{5/2}(S_1 \times ((k-1)T_0,(k+1)T_0))}
\right.
\]

\[
\leq + \left. \| \tilde{f}_\varphi + v_\varphi \zeta \|_{L^{5/3}(\Omega \times ((k-1)T_0,(k+1)T_0))}
\right]
\]

where \( t \in [kT_0,(k + 1)T_0] \). Repeating the considerations leading to (7.9) we obtain (7.10). This concludes the proof.

To estimate the term \( \| \frac{v_\varphi}{r} \|_{L^{10/3}(\Omega \times ((k-1)T_0,(k+1)T_0))} \) \( k \in \mathbb{N} \), from the r.h.s. of (7.9) and (7.10) we introduce the quantity \( w = \frac{v_\varphi}{r} \), which is a solution to the problem

\[
w_{,t} + v' \cdot \nabla w + 2 \frac{v_r}{r} w - \nu \Delta w - 2 \nu \frac{v}{r} w_{,r} = \frac{f_\varphi}{r} \equiv g_0 \quad \text{in} \quad \Omega \times \mathbb{R}_+,
\]

\[
w_{,r} = 0 \quad \text{on} \quad S_1 \times \mathbb{R}_+,
\]

\[
w_{,z} = 0 \quad \text{on} \quad S_2 \times \mathbb{R}_+,
\]

\[
w|_{t=0} = w(0) \quad \text{in} \quad \Omega.
\]

**Lemma 7.3.** Assume that \( A_* \) defined by (6.5) is finite. Let \( g_0 = \frac{f_\varphi}{r} \) and \( |||g_0|||_2 = \sup_{k \in \mathbb{N}_0} ||g_0||_{L^2((kT_0,(k+1)T_0);L^2(\Omega))} \) be finite, \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( T_0 > 0 \) be given. Let \( w(0) \in L^2(\Omega) \). Then

\[
\| w \|_{L^{10/3}(\Omega \times (kT_0,t))} \leq \varphi(T_0, d_T, A_*, |||g_0|||_2) + e^{-\nu_1 k T_0} || w(0) ||_{L^2(\Omega)},
\]
where \( \varphi \) is an increasing positive function, \( d_7 \) is defined by (2.15), \( 0 < \nu_* < \nu \) and \( t \in (kT_0; (k + 1)T_0] \).

**Proof.** Multiplying (7.14) by \( w|w|^{s-2} \) and integrating over \( \Omega \) we obtain

\[
\frac{1}{s} \frac{d}{dt} \int_{\Omega} |w|^s dx + \frac{4\nu(s-1)}{s^2} \int_{\Omega} |\nabla |w|^{s/2}|^2 dx \\
\leq 2 \int_{\Omega} \left| \frac{v_r}{r} \right| |w|^s dx + \frac{2\nu}{s} \int_{\Omega} \partial_r |w|^s drdz + \int_{\Omega} |g_0| |w|^{s-1} dx
\]

(7.16)

Performing integration in the second term on the r.h.s. of (7.16) yields

\[
\frac{1}{s} \frac{d}{dt} \int_{\Omega} |w|^s dx + \frac{4\nu(s-1)}{s^2} \int_{\Omega} |\nabla |w|^{s/2}|^2 dx \\
\leq 2 \int_{\Omega} \left| \frac{v_r}{r} \right| |w|^s dx + \frac{2\nu}{s} \int_{\Omega} |w(R, z, t)|^s dz \\
+ \|g_0\|_{L^s(\Omega)} \|w\|^{s-1}_{L^s(\Omega)}.
\]

(7.17)

In view of the interpolation inequality

\[
\int_{-a}^{a} |w(R, z, t)|^s dz \leq \varepsilon \int_{\Omega} |\nabla |w|^{s/2}|^2 dx + \frac{c_1}{\varepsilon} \int_{\Omega} |w|^s dx
\]

with \( \varepsilon = \frac{s-1}{s} \) we derive from (7.17) the inequality

\[
\frac{d}{dt} \int_{\Omega} |w|^s dx + \frac{2\nu(s-1)}{s} \int_{\Omega} |\nabla |w|^{s/2}|^2 dx \leq 2s \int_{\Omega} \left| \frac{v_r}{r} \right| |w|^s dx \\
+ \frac{c_1}{s-1} \int_{\Omega} |w|^s dx + s \|g_0\|_{L^s(\Omega)} \|w\|^{s-1}_{L^s(\Omega)}.
\]

(7.18)

We are going to consider (7.18) for \( s \in \left[ \frac{3}{2}, 4 \right] \). Then (7.18) takes the form

\[
\frac{d}{dt} \int_{\Omega} |w|^s dx + \frac{2\nu}{3} \int_{\Omega} |\nabla |w|^{s/2}|^2 dx \leq 8 \int_{\Omega} \left| \frac{v_r}{r} \right| |w|^s dx \\
+ 2c_1 \int_{\Omega} |w|^s dx + 4 \|g_0\|_{L^s(\Omega)} \|w\|^{s-1}_{L^s(\Omega)}.
\]

(7.19)
Introducing the quantity

\( \eta = |w|^{s/2} \)

inequality (7.19) assumes the form

\[
\frac{d}{dt} \int_{\Omega} |\eta|^2 dx + \frac{2}{3} \nu \int_{\Omega} |
abla \eta|^2 dx \leq 8 \int_{\Omega} \left| \frac{v_r}{r} \right| \eta^2 dx
\]

\( + 2c_1 \int_{\Omega} |\eta|^2 dx + 4\|\eta\|_{L^s(\Omega)}\|g_0\|_{L^s(\Omega)}. \)

Since \( \eta, r|_S = 0, \eta, z|_S = 0 \) the following Poincare inequality holds

\[
\int_{\Omega} \eta^2 dx \leq c_p \int_{\Omega} |\nabla \eta|^2 dx + c_p \int_{\Omega} |\eta| dx.
\]

Introducing the splitting \( \frac{2}{3} \nu = \nu_1 + \nu_2, \nu_i > 0, i = 1, 2 \), we obtain from (7.21) the inequality

\[
\frac{d}{dt} \int_{\Omega} \eta^2 dx + \nu_1 \int_{\Omega} \eta^2 dx + \nu_2 \int_{\Omega} |
abla \eta|^2 dx \leq \nu_1 \int_{\Omega} |\eta| dx
\]

\( + 8 \int_{\Omega} \left| \frac{v_r}{r} \right| \eta^2 dx + 2c_1 \int_{\Omega} \eta^2 dx + 4\|\eta\|_{L^s(\Omega)}\|g_0\|_{L^s(\Omega)}. \)

Let \( \nu_* = \nu_1 \). Then from (7.23) we have

\[
\frac{d}{dt} \left( \int_{\Omega} \eta^2 dx e^{\nu_* t} \right) + \nu_2 \int_{\Omega} |\nabla \eta|^2 dx e^{\nu_* t}
\]

\( \leq \left[ \nu_* \int_{\Omega} |\eta| dx + 2c_1 \int_{\Omega} \eta^2 dx + 8 \int_{\Omega} \left| \frac{v_r}{r} \right| \eta^2 dx + 4\|\eta\|_{L^s(\Omega)}\|g_0\|_{L^s(\Omega)} \right] e^{\nu_* t}. \)

Integrating (7.24) with respect to time from \( kT_0 \) to \( t \in (kT_0, (k + 1)T_0] \), \( k \in \mathbb{N} \cup \{0\} \equiv \mathbb{N}_0 \), yields

\[
\int_{\Omega} \eta^2(t) dx e^{\nu_* t} + \nu_2 \int_{kT_0}^{t} \int_{\Omega} |\nabla \eta(t')|^2 dx e^{\nu_* t'} dt'
\]

\( \leq \int_{kT_0}^{t} \left[ \nu_* \int_{\Omega} |\eta| dx + 2c_1 \int_{\Omega} \eta^2 dx + 8 \int_{\Omega} \left| \frac{v_r}{r} \right| \eta^2 dx \)

\( + 4\|\eta\|_{L^s(\Omega)}\|g_0\|_{L^s(\Omega)} \right] e^{\nu_* t'} dt' + \int_{\Omega} \eta^2(kT_0) dx e^{\nu_* kT_0}. \)
Continuing, we have

\[
\int_\Omega \eta^2(t)dx + \nu_2 e^{-\nu_* t} \int_{k T_0}^t \int_\Omega |\nabla \eta(t')|^2 dx e^{\nu_* t'} dt'
\]

(7.26)

\[
\leq e^{-\nu_* t} \int_{k T_0}^t \left[ \nu_* \int_\Omega |\eta| dx + 2c_1 \int_\Omega \eta^2 dx + 8 \int_\Omega \frac{|v_r|}{r} \eta^2 dx \right. \\
\left. + 4 \|\eta\|_{L^2_2(\Omega)} \|g_0\|_{L^s(\Omega)} \right] e^{\nu_* t'} dt + e^{-\nu_0 T_0} \int_\Omega \eta^2(k T_0) dx,
\]

t \in (k T_0, (k + 1) T_0].

First we obtain an estimate for \( \int_\Omega \eta^2(k T_0) dx \) for any \( k \in \mathbb{N}_0 \). For this purpose we omit the second term on the l.h.s. of (7.26). Therefore we consider the inequalities

\[
\int_\Omega \eta^2((k + 1) T_0) dx
\]

(7.27)

\[
\leq \int_{k T_0}^{(k+1)T_0} \left[ \nu_* \int_\Omega |\eta| dx + 2c_1 \int_\Omega \eta^2 dx + 8 \int_\Omega \frac{|v_r|}{r} \eta^2 dx \right. \\
\left. + 4 \|\eta\|_{L^2_2(\Omega)} \|g_0\|_{L^s(\Omega)} \right] dt + e^{-\nu_0 T_0} \int_\Omega \eta^2(k T_0) dx.
\]

Take \( s = \frac{9}{5} \). Then the first integral on the r.h.s. of (7.27) is estimated by

\[
c \int_{k T_0}^{(k+1)T_0} dt \int_\Omega (|w|^{9/10} + |w|^{9/5}) dx
\]

\[
+ c \left( \int_{k T_0}^{(k+1)T_0} \int_\Omega \left( \frac{|v_r|}{r} \right)^{10} dx dt \right)^{1/10} \left( \int_{k T_0}^{(k+1)T_0} \int_\Omega |w|^2 dx dt \right)^{9/10}
\]

\[
+ c T_0^{2/45} \left( \int_{k T_0}^{(k+1)T_0} \int_\Omega |w|^{9/5} dx dt \right)^{2/5} \left( \int_{k T_0}^{(k+1)T_0} \int_\Omega |g_0|^{9/5} dx dt \right)^{5/9} \equiv I_1.
\]

Applying the Hölder inequality and the energy estimate (2.15) with \( d_7 \) implies

\[
I_1 \leq \varphi(\|\Omega\|, A_0)[T_0^{11/20} d_7^{9/10} + T_0^{1/10} d_7^{9/5} + \varphi(A_*) d_7^{9/10}]
\]

\[
+ T_0^a d_7^b \|g_0\|_{L_{9/5}(\Omega \times (k T_0, (k+1) T_0))}
\]

\[
\equiv A_1(T_0, d_7, A_*, \sup_k \|g\|_{L_{9/5}(\Omega \times (k T_0, (k+1) T_0))}), \quad a > 0, \quad b > 0.
\]

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Introducing the notation

\[ X_1(t) = \int_{\Omega} |w(t)|^{9/5} dx \]  

(7.28)

we obtain from (7.27) the inequality

\[ X_1((k + 1)T_0) \leq A_1 + e^{-\nu_* T_0} X_1(kT_0). \]  

(7.29)

Hence, (7.29) implies

\[ X_1(kT_0) \leq \frac{A_1}{1 - e^{-\nu_* T_0}} + e^{-k\nu_* T_0} X_1(0). \]  

(7.30)

In view of (7.30) we can consider (7.26) in the any interval \((kT_0, (k + 1)T_0), \ k \in \mathbb{N}_0, \) for

\[ \eta_1 = |w|^{9/10}. \]  

(7.31)

Then we obtain the estimate

\[
\| \eta_1 \|^2_{L^2_\Omega((\Omega \times (kT_0, t))} \leq (1 + e^{\nu_* T_0}) \left[ A_1(T_0) + \frac{A_1(T_0)}{1 - e^{-\nu_* T_0}} \right. \\
+ e^{-k\nu_* T_0} \| \eta_1(0) \|^2_{L^2_\Omega} \left. \right], \quad t \in (kT_0, (k + 1)T_0]. \]  

(7.32)

Using that

\[
\int_{\Omega} \eta^2(0) dx = \int_{\Omega} |w(0)|^s dx
\]

and

\[
\| \eta \|^2_{L^2_\Omega(T^*)} = \| |w|^{s/2} \|^2_{L^2_\Omega(T^*)} \geq \| w \|^s_{L^2_\Omega(T^*)}
\]

we obtain from (7.32) in the case \(s = \frac{9}{5}\) the estimate

\[
\| w \|_{L^3_\Omega((\Omega \times (kT_0, t))} \leq (1 + e^{\nu_* T_0})^{5/9} \left[ \frac{2 - e^{-\nu_* T_0}}{1 - e^{-\nu_* T_0}} A_1(T_0) \right. \\
+ e^{-k\nu_* T_0} \| w(0) \|^{9/5}_{L^9_\Omega(T^*)} \left. \right]^{5/9} \equiv A_2(T_0), \quad t \in (kT_0, (k + 1)T_0],
\]

(7.33)

where \(A_2\) depends also on the same quantities as \(A_1.\)
Next we consider the case $s = 2$. Then the first term on the r.h.s. of (7.27) is estimated by
\[
c\left[ \int_{kT_0}^{(k+1)T_0} dt \int_\Omega (|w| + |w|^2) dx \right.
\]
\[
+ \left( \int_{kT_0}^{(k+1)T_0} \int_\Omega \left| \frac{v_r}{r} \right|^{10} dx dt \right)^{1/10}
\left( \int_{kT_0}^{(k+1)T_0} \int_\Omega |w|^2 dx dt \right)^{9/10}
\]
\[
+ \|g\|_{L_2(\Omega \times (kT_0,(k+1)T_0))} \|w\|_{L_2(\Omega \times (kT_0,(k+1)T_0))}
\]
\[
\leq c[T_0 d_7 + d_7^2 + \varphi(A)T_0^{7/30}A_2(T_0) + \|g_0\|_{L_2(\Omega \times (kT_0,(k+1)T_0))}d_7]
\equiv A_3(T_0, d_7, A_2; \sup_k \|g_0\|_{L_2(\Omega \times (kT_0,(k+1)T_0))}).
\]

Setting
\[
(7.34) \quad X_2(t) = \int_\Omega |w(t)|^2 dx
\]
we obtain from (7.27) the inequality
\[
(7.35) \quad X_2((k + 1)T_0) \leq A_3(T_0) + e^{-\nu_2T_0} X_2(kT_0).
\]

Hence
\[
(7.36) \quad X_2(kT_0) \leq \frac{A_3(T_0)}{1 - e^{-\nu_2T_0}} + e^{-\nu_2kT_0} X_2(0).
\]

Using the above estimate in (7.26) with $s = 2$ we have
\[
\|w\|_{L_{10/3}(\Omega \times (kT_0,t))}^2 \leq (1 + e^{\nu_2T_0}) \left( \frac{2 - e^{-\nu_2T_0}}{1 - e^{-\nu_2T_0}} A_3(T_0) + e^{-\nu_2kT_0} \|w(0)\|_{L_2(\Omega)}^2 \right), \quad t \in (kT_0, (k + 1)T_0].
\]

From (7.37) it follows (7.15). This concludes the proof.

To prolong in time a local solution to (1.1) we need a version o f Lemma 7.1 for solutions to problem (7.7). Therefore, we have

**Lemma 7.4.** Assume that $g = rf_\varphi \in L_\infty(\Omega \times \mathbb{R}_+)$, $u(0) \in L_\infty(\Omega)$, $v(0) \in L_2(\Omega)$, $f \in L_\infty(\mathbb{R}_+; L_{6/5}(\Omega))$, $\frac{v_\varphi(0)}{r} \in L_2(\Omega)$. Let $T_0 > 0$ be given. Let $|||f_\varphi|||_2 = \sup_k ||f_\varphi||_{L_2(\Omega \times (kT_0,(k+1)T_0))} < \infty$, $|||\frac{f_\varphi}{r}|||_{9/5} = $
sup\(k\left\| \frac{L}{r} \right\|_{L_{9/5}(\Omega \times (kT_0, (k+1)T_0))} < \infty\). Let \(A*\) (introduced by (6.5)) be finite. Then

\[
\int_{(k-1)T_0}^{kT_0} \int_{\Omega} \tilde{v}_{\varphi,t}^2 dx dt + \nu \int_{\Omega} \left( \left| \nabla \tilde{v}_{\varphi}(kT_0) \right|^2 + \left| \frac{\tilde{v}_{\varphi}(kT_0)}{r} \right|^2 \right) dx
\]

\[
\leq cd_6^2 + c\|f_{\varphi}\|^2_2 + \frac{c}{T_0^{3/2}}d_6 d_7 + \varphi \left( \left\| A*, T_0, d_6, d_7, \left\| f_{\varphi} \right\|_{9/5}, \left\| f_{\varphi} \right\|_2 \right)\]

\[
+ ce^{-\nu \cdot kT_0} \left\| \frac{v_{\varphi}(0)}{r} \right\|^2_{L_2(\Omega)} \equiv A_4^2.
\]

where \(d_6\) is introduced in (3.1), \(d_7\) in (2.15) and \(A*\) in (6.5).

**Proof.** Multiplying (7.7) by \(\tilde{v}_{\varphi,t}\), and integrating the result over \(\Omega\) yield

\[
\int_{\Omega} \tilde{v}_{\varphi,t}^2 dx - \nu \int_{\Omega} \Delta \tilde{v}_{\varphi} \tilde{v}_{\varphi,t} dx + \nu \int_{\Omega} \frac{\tilde{v}_{\varphi}}{r^2} \tilde{v}_{\varphi,t} dx
\]

\[
= \int_{\Omega} \tilde{f}_{\varphi} \tilde{v}_{\varphi,t} dx + \int_{\Omega} \nu \tilde{v}_{\varphi,t} \tilde{v}_{\xi} dx - \int_{\Omega} \frac{\nu \tilde{v}}{r} \tilde{v}_{\varphi,t} dx - \int_{\Omega} \nu' \cdot \nabla \tilde{v}_{\varphi} \tilde{v}_{\varphi,t} dx.
\]

Integration by parts implies

\[
- \int_{\Omega} \Delta \tilde{v}_{\varphi} \tilde{v}_{\varphi,t} dx = - \int_{-a}^{a} \tilde{v}_{\varphi} \tilde{v}_{\varphi,t}|_{r=R} dz + \int_{\Omega} \nabla \tilde{v}_{\varphi} \cdot \nabla \tilde{v}_{\varphi,t} dx,
\]

where the boundary conditions (7.7) were used.

Applying the Cauchy inequality to the r.h.s. terms of (7.39) we get

\[
\frac{1}{2} \int_{\Omega} \tilde{v}_{\varphi,t}^2 dx + \frac{\nu}{2} \int_{\Omega} \left| \nabla \tilde{v}_{\varphi} \right|^2 dx + \frac{\nu}{2} \int_{\Omega} \left| \frac{\tilde{v}_{\varphi}}{r} \right|^2 dx
\]

\[
- \frac{\nu}{2} \int_{-a}^{a} \tilde{v}_{\varphi}^2 \mid_{r=R} dx \leq c \int_{\Omega} \tilde{f}_{\varphi}^2 dx + c \int_{\Omega} \tilde{v}_{\varphi,t}^2 dx
\]

\[
+ c \int_{\Omega} v_{\varphi,t}^2 \left| \nabla \tilde{v}_{\varphi} \right|^2 dx + c \int_{\Omega} \frac{v_{\varphi,t}^2}{r^2} \tilde{v}_{\varphi}^2 dx.
\]

Integrating (7.40) with respect to time from \(t = (k-1)T_0\) to \(t = kT_0\) we
obtain

\[
\frac{1}{2} \int_{(k-1)T_0}^{kT_0} \int_{\Omega} \tilde{v}_{\varphi,t}^2 \, dx \, dt + \frac{\nu}{2} \int_{\Omega} |\nabla \tilde{v}_{\varphi}(kT_0)|^2 \, dx + \frac{\nu}{2} \int_{\Omega} \left( \frac{\tilde{v}_{\varphi}(kT_0)}{r} \right)^2 \, dx
\]

\[
\leq \frac{\nu}{2} \int_{-a}^{a} \tilde{v}_{\varphi}(kT_0) |_{r=R} \, dz + c \int_{(k-1)T_0}^{kT_0} \int_{\Omega} \tilde{f}_\varphi^2 \, dx \, dt
\]

\[
+ c \int_{(k-1)T_0}^{kT_0} \int_{\Omega} \tilde{v}_{\varphi,\zeta}^2 \, dx \, dt + c \int_{(k-1)T_0}^{kT_0} \int_{(k1)T_0} \left( v'^2 |\nabla \tilde{v}_{\varphi}|^2 + \frac{v^2 r^2}{r^2} \tilde{v}_{\varphi}^2 \right) \, dxdt.
\]

The first term in estimated by \( c d_6^2 \), the second by the norm \(|\| f_\varphi \|_2 = \sup_k \| f_\varphi \|_{L_2((k-1)T_0,kT_0;L_2(\Omega))} \), the third by \( \frac{c}{T_0^{3/2}} \| v_{\varphi} \|_{L_4(\Omega \times \mathbb{R}^+)} \leq \frac{c}{T_0^{3/2}} d_6 d_7 \) (see (3.6)) and finally the last is bounded by

\[
\varphi \left( A_*, T_0, d_6, d_7, \left\| \frac{f_\varphi}{r} \right\|_{9/5}, ||| f_\varphi \||_2 \right) + c e^{-\nu kT_0} \left\| \frac{v_{\varphi}(0)}{r} \right\|_{L_2(\Omega)},
\]

where (7.10) and (7.15) were used. Using the above estimates in (7.41) implies (7.38). This concludes the proof.

Having estimate (7.38) we are able to consider problem (7.1) in the time interval \((kT_0, (k+1)T_0)\). Then instead of (7.2) we obtain

\[
(7.42) \quad ||v_{\varphi,t}||_{L_2(kT_0,t;\Omega)}^2 + ||v_{\varphi}(t)||_{H_0^1(\Omega)}^2 \leq c A_4^2,
\]

where \( t \in (kT_0, (k+1)T_0) \) and \( A_4 \) is introduced in (7.38).

8. Global existence

To prove global existence of axially symmetric solutions to problem (1.1) we first show long time existence (see Theorem 8.1). By the long time existence we mean that there exists a time \( T_e < \infty \), without any restrictions on \( T_e \) from above, such that axially symmetric solutions to (1.1) satisfy

\[
(8.1) \quad v \in W_2^{2,1}(\Omega^T_e), \quad \nabla p \in L_2(\Omega^T_e).
\]

The existence of such time \( T_e \) is connected with the assumption that some spatial norms of the external force must be integrated in appropriate powers over time interval \((0, T_e)\). Therefore, we prove global existence, so the
existence on the infinite time interval \((0, \infty)\), by prolonging the long time solution step by step on intervals \(((k-1)T_e, kT_e), k \in \mathbb{N}\). In such a way we relax restrictions on the external force, because otherwise looking for solution (8.1) with \(T_e = \infty\), we have to assume that \(f\) vanishes sufficiently fast as time goes to infinity. In the step by step approach we prove existence of such solutions that

\[
(8.2) \quad v \in \bigcup_{k \in \mathbb{N}} W_2^{2,1}(\Omega \times ((k-1)T_e, kT_e)), \quad \nabla p \in \bigcup_{k \in \mathbb{N}} L_2(\Omega \times ((k-1)T_e, kT_e)).
\]

Looking for solutions (8.2) we have to repeat the long time existence (see Theorem 8.1) in each step, so we should prove that if there exists a constant \(\alpha\) depending on the data norms such that

\[
(8.3) \quad \|v(0)\|_{H^1(\Omega)} \leq \alpha
\]

then

\[
(8.4) \quad \|v(kT_e)\|_{H^1(\Omega)} \leq \alpha \quad \text{for any } k \in \mathbb{N}.
\]

The above assertion holds for \(v_\varphi\) (see Lemma 7.4 and formula (7.42)). To prove it for \(v' = (v_r, v_z)\) we need first some decay estimate for \(\chi\). This will be shown in Lemma 8.2.

To formulate Theorem 8.1, we recall necessary notation. Let the constants \(d_0, d_1, \ldots, d_7\) be introduced by the relations (see Lemma 2.1):

\[
\|v(0)\|_{L_2(\Omega)} \leq d_0, \quad \|f\|_{L_\infty(\mathbb{R}_+; L_6/5^5(\Omega))} \leq d_2, \\
g = f \cdot \eta, \quad u = v \cdot \eta, \quad \eta = (-x_2, x_1, 0), \\
\sup_{k \in \mathbb{N}_0} \sup_{t \in [kT, (k+1)T]} \left| \int_{kT}^{t} \int_{\Omega} g dx dt' + \int_{\Omega} u(kT) dx \right| \leq d_1, \\
d_3 \geq c(d_1 + d_2), \quad d_4 \geq c(T)(d_0 + d_3), \quad d_5 \geq c(T)(d_0 + d_3), \\
d_7 \geq c(T)(d_4 + \sup_k \|f\|_{L_2(\Omega \times (kT,(k+1)T))}) \quad \text{(see (2.15)).}
\]

\[
d_6 = d_6(\|u(0)\|_{L_\infty(\Omega)}, \|g\|_{L_\infty(\Omega \times \mathbb{R}_+)}) \quad \text{(some function)} \quad \text{(see (3.1)).}
\]

Let us recall the partition of unity \(\{\zeta_1(r), \zeta_2(r)\}\) on the interval \((0, R)\) with the properties: \(\zeta_1, \zeta_2\) are smooth functions such that \(\zeta_1(r) = 1\) for \(r \leq r_0\), \(\zeta_1(r) = 0\) for \(r \geq 2r_0\), \(\zeta_2(r) = 0\) for \(r \leq r_0\) and \(\zeta_2(r) = 1\) for \(r \geq 2r_0\). Moreover, we have the notation: \(\tilde{w} = w\zeta_1, \tilde{w} = w\zeta_2\) for any function \(w\).
Let us introduce the quantities

\[ A^2(t) = c(1/r_0)d_0^2[1 + (1 + d_0)d_0^2] + \left\| \frac{\tilde{x}(0)}{r} \right\|^2_{L^2(\Omega)} \]
\[ + \left\| \frac{v_0(0)}{\sqrt{r}} \right\|^4_{L^4(\Omega)} + \left\| \frac{\tilde{F}}{r} \right\|^2_{L^2(\Omega')} + \left\| \frac{f_\varphi}{\sqrt{r}} \right\|^2_{L^{20/11}(\Omega')}, \]

(8.5)  \[ A^2_0(t) = c(1/r_0)(d_0^2 + d_0^6d_7^2) + \left\| \tilde{F} \right\|^2_{L^2(0,T;L^6/5(\Omega))} + \frac{1}{r_0^2} \left\| \tilde{x}(0) \right\|^2_{L^2(\Omega)}, \]
\[ A^2_4(t) = c d_6^2 + \left\| \tilde{f}_\varphi \right\|^2_2 + \frac{c}{T^{3/2}}d_6d_7 + \varphi (A, T, d_6, d_7, \left\| \frac{f_\varphi}{r} \right\|_{9/5}, \left\| f_\varphi \right\|_2, \left\| f_\varphi \right\|_2), \]

where \( \left\| w \right\|_s = \sup_{k \in \mathbb{N}_0} \left\| w \right\|_{L^s((k-1)T, kT; L^s(\Omega))}, s \in (1, \infty) \) and \( A_* = A + [c(1/r_0)d_2^2 + 1]A_0 \) (see (6.5)).

**Theorem 8.1. (long time existence).** Let \( T_* > 0 \) be given. Assume that \( v(0) \in H^1(\Omega), f \in L^2(\Omega^{T_*}) \) and \( A_1(T_0), A_0(T_0), A_4(T_0) \) are finite. Then there exists axially symmetric solution to problem (1.1) such that (8.1) holds.

**Proof.** Assume that \( T_* \) is so small that (2.27) is satisfied for \( T = T_* \). Then Lemma 2.5 implies a local existence of solutions to problem (1.1) such that \( v \in W^{2,1}_2(\Omega^{T_*}), \nabla p \in L^2(\Omega^{T_*}) \). To apply Lemma 2.5 we need that \( v(0) \in H^1(\Omega) \) and \( f \in L^2(\Omega^{T_*}) \). To extend the existence of local solutions on the interval \((T_* + 2T_0) \) (or to apply Lemma 2.5 for interval \((T_* + 2T_0) \)) we have to obtain an estimate for \( \|v(T_*)\|_{H^1(\Omega)} \) such that

\[ \|v(T_*)\|_{H^1(\Omega)} \leq \|v(0)\|_{H^1(\Omega)}. \]

Moreover, we need that

\[ \|f\|_{L^2(\Omega \times (T_* + 2T_0))} \leq \|f\|_{L^2(\Omega^{T_*})}. \]

Estimate (8.7) holds by assumptions. Therefore, we have to show (8.6) only. Under assumptions (8.5) and from (6.5) we have the estimate

\[ \left\| \frac{\chi}{r} \right\|_{V^o_2(\Omega^{T_*})} \leq c(1/r_0, d_1)(A(T_0) + A_0(T_0)). \]

Then (6.43) (Lemma (6.5)) implies

\[ \|v'\|_{V^o_2(\Omega^{T_*})} \leq c \left\| \frac{\chi}{r} \right\|_{V^o_2(\Omega^{T_*})}. \]
Hence
\[(8.10) \quad \|v'(T_*)\|_{H^1(\Omega)} \leq c(1/r_0, d_1)(A(T_e) + A_0(T_e)).\]

Similarly, (7.42) yields
\[(8.11) \quad \|v_\varphi(T_*)\|_{H^1(\Omega)} \leq c A_4(T_e).\]

Let
\[(8.12) \quad \alpha = c(A(T_e) + A_0(T_e) + A_4(T_e)).\]

Since
\[(8.13) \quad \|v(0)\|_{H^1(\Omega)} \leq \alpha\]
we obtain from (8.10) and (8.11) that
\[(8.14) \quad \|v(T_*)\|_{H^1(\Omega)} \leq \alpha\]
and we can apply Lemma 2.5. Assuming that \(T_* = \frac{T_e}{m}\) we can repeat the procedure \(m\) times and prove the theorem.

We can prove the theorem in a different way. We have existence of weak solutions to problem (1.1) such that \(v \in V_2^0(T_e)\) and the estimate holds
\[(8.15) \quad \|v\|_{V_2^0(T_e)} \leq c(\|v(0)\|_{L_2(\Omega)} + \|f\|_{L_2(T_e)}).\]

We restrict an increasing of regularity of the weak solutions by getting an estimate guaranteeing regularity (8.1). The precise procedure is very complicated but it could be presented.

Under assumptions (8.5) and Lemma 6.5 we have the estimate
\[(8.16) \quad \|v'\|_{V_2^1(T_e)} \leq c(A(T_e) + A_0(T_e)).\]

From the proof of Lemma 3.7 in [Z9] we have
\[(8.17) \quad \|v'\|_{L_2(T_e)} \leq c\|v'\|_{V_2^1(T_e)}.\]

Let us consider the Stokes system
\[
\begin{align*}
v_t - \text{div} \mathbb{T}(v, p) &= -v' \cdot \nabla v + f, \\
\text{div} v &= 0 \\
v \cdot \bar{n}|_S &= 0, \quad \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha|_S = 0, \quad \alpha = 1, 2, \\
v|_{t=0} &= v(0),
\end{align*}
\]
where the r.h.s. of (8.18) is treated as given. In view of (8.15) and (8.17) the r.h.s. of (8.18) belongs to $L_{5/3}(\Omega_T)$ under the assumption that $f \in L_{5/3}(\Omega_T)$ also. The last statement is satisfied because $f \in L_2(\Omega_T)$. Hence, in view of [Z7, ZZ], there exists a solution to (8.18) such that $v \in W^{2,1}_{5/3}(\Omega_T)$, $\nabla p \in L_{5/3}(\Omega_T)$ and the estimate holds

(8.19) \[ \|v\|_{W^{2,1}_{5/3}(\Omega_T)} + \|\nabla p\|_{L_{5/3}(\Omega_T)} \leq c(\text{data}). \]

In view of the imbedding

(8.20) \[ \|\nabla v\|_{L_{5/2}(\Omega_T)} \leq c\|v\|_{W^{2,1}_{5/3}(\Omega_T)} \]

we obtain that the r.h.s. of (8.18) belongs to $L_2(\Omega_T)$. Applying again [Z7, ZZ] we get that $v \in W^{2,1}_2(\Omega_T)$, $\nabla p \in L_2(\Omega_T)$ and the estimate is valid

(8.21) \[ \|v\|_{W^{2,1}_2(\Omega_T)} + \|\nabla p\|_{L_2(\Omega_T)} \leq c(\text{data}). \]

This concludes the proof.

Next we shall show that if time existence $T_e$, appeared in Theorem 8.1, is sufficiently large then there exists a solution to (1.1) with properties described by (8.2). The solution behaves similarly in each time interval $(kT_e, (k+1)T_e)$, $k \in \mathbb{N}_0$. To prove this we have to show that $\|v(kT_e)\|_{H^1(\Omega)}$ is bounded by the same constant for all $k$. For this some decay estimates are needed.

**Lemma 8.2.** Let $T_e > 0$ be a sufficiently large given number. Let there exist a weak solution to problem (1.1). Let assumptions of Lemmas 2.2, 2.3, 3.1, 3.2 be satisfied. Let the constants $d_4$ (see (2.1)), $d_5$ (see (2.2)), $d_7$ (see (2.15)), $d_6$ (see (3.1)) be finite. Let $r_0 > 0$ be so small that

\[ \sup_t \|rv\|_{L_\infty(\Omega_{r_0})} \leq \sqrt{\frac{\alpha}{3}} \nu, \quad a < 1, \]

holds, where $\Omega_{r_0} = \{ x \in \Omega : r < r_0 \}$. Let

\[ X^2(t) = \frac{1}{\nu^2} \left( \left\| \frac{v}{r} \right\|_{L_2(\Omega)}^2 + \left\| \frac{\chi}{r} \right\|_{L_2(\Omega)}^2 \right). \]

Let $X(0)$ be finite. Let the quantities

\[
\begin{align*}
    d_8 &= \sup_{k \in \mathbb{N}_0} \| f_\varphi \|_{L_4(\Omega \times (kT_e, (k+1)T_e))}, \\
    d_9 &= \sup_{k \in \mathbb{N}_0} \left\| \frac{F}{T} \right\|_{L_2(\Omega \times (kT_e, (k+1)T_e))},
\end{align*}
\]

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be finite. Then there exists a positive increasing function

\[ B_0 = B_0(1/r_0, d_4, d_5, d_6, d_7, d_8, d_9) \]

such that

\[ X(kT_e) \leq B_0 + X(0)e^{-\nu_0kT_e}, \]

\[ \|v(t)\|_{H^1(\Omega)} \leq c(B_0 + X(kT_e)), \]

hold for any \( k \in \mathbb{N}_0 \) and \( t \in (kT_e, (k+1)T_e) \).

**Proof.** From (4.12) we have

\[ \frac{1}{2} \frac{d}{dt} \left\| \frac{\tilde{X}}{r} \right\|_{L^2(\Omega_\varepsilon)}^2 + \frac{\nu}{2} \left\| \frac{\nabla \tilde{X}}{r} \right\|_{L^2(\Omega_\varepsilon)}^2 \leq \frac{1}{\nu} \left\| \frac{\tilde{v}}{r} \right\|_{L^4(\Omega_\varepsilon)}^4 \]

\[ + c(1/r_0) \int_{\Omega_{\varepsilon,\zeta_1,r}} \chi^2 dx + c(1/r_0) \int_{\Omega_{\varepsilon,\zeta_1,r}} |v_r| \chi^2 dx + c \int_{\Omega_\varepsilon} \left\| \frac{\tilde{F}}{r} \right\|^2 dx. \]

Moreover, using that \( \tilde{v}_\phi \big|_{r=\varepsilon} = 0 \), (4.16) yields

\[ \frac{1}{4} \frac{d}{dt} \int_{\Omega_\varepsilon} \frac{\tilde{v}_\phi^4}{r^4} dx + 3 \int_{\Omega_\varepsilon} \left\| \frac{\nabla \tilde{v}_\phi}{r} \right\|^2 dx \]

\[ + 3 \int_{\Omega_\varepsilon} \frac{\tilde{v}_\phi^4}{r^4} dx = -\frac{3}{2} \int_{\Omega_\varepsilon} \frac{v_r}{r} \frac{\tilde{v}_\phi^4}{r^4} dx + \int_{\Omega_\varepsilon} v \cdot \nabla \zeta_1 v \frac{\tilde{v}_\phi^2}{r^2} dx \]

\[ - \int_{\Omega_\varepsilon} [2v \nabla v_\phi \nabla \zeta_1 + v_\phi \Delta \zeta_1] \frac{\tilde{v}_\phi^2}{r^2} dx + \int_{\Omega_\varepsilon} \tilde{f} \tilde{v}_\phi \frac{\tilde{v}_\phi^2}{r^2} dx. \]

Now we examine the particular terms from the r.h.s. of (8.24). We estimate the first term by

\[ \frac{3}{2} \left\| vr_\phi \right\|_{L^2(\Omega_\varepsilon)}^2 \int_{\Omega_\varepsilon} \left\| \frac{v_r}{r^3} \right\|^2 \frac{\tilde{v}_\phi^2}{r^2} dx, \]

the second by

\[ \frac{\varepsilon_1}{2} \int_{\Omega_\varepsilon} \frac{\tilde{v}_\phi^4}{r^4} dx + \frac{1}{2\varepsilon_1} c(1/r_0) \int_{\Omega_{\varepsilon,\zeta_1,r}} v^2 v_\phi^4 dx \equiv I_1, \]

where the second term in \( I_1 \) is bounded by

\[ \frac{c(1/r_0)}{2\varepsilon_1} \left\| vr_\phi \right\|_{L^2(\Omega_\varepsilon)}^4 \int_{\Omega_{\varepsilon,\zeta_1,r}} v^2 dx. \]
The third integral on the r.h.s. of (8.24) is divided into two parts. The first part equals

\[-2\nu \int_{\Omega} \nabla v \nabla \zeta \frac{\tilde{v}^2}{r^2} dx = -2\nu \int_{\Omega} \nabla v \nabla \zeta \frac{v^3}{r^2} \zeta^3 dx = -\frac{\nu}{2} \int_{\Omega} v^4 \nabla \left( \frac{\zeta^3}{r^2} \right) dx \equiv I_2.\]

Hence

\[|I_2| \leq c(1/r_0) \int_{\Omega} v^4 dx.\]

Similarly, the second part of the third term on the r.h.s. of (8.24) is estimated by

\[c(1/r_0) \int_{\Omega} v^4 dx.\]

Finally, the last term on the r.h.s. of (8.24) is bounded by

\[\frac{\varepsilon_2}{2} \int_{\Omega} \frac{\tilde{v}^4}{r^4} dx + \frac{1}{2\varepsilon_2} \int_{\Omega} \frac{\tilde{\varphi}^2}{r^2} \tilde{v}_r^2 dx.\]

Assuming \(\varepsilon_1 = \varepsilon_2 = \frac{\nu}{8}\) and using the above estimates in (8.24) imply

\[
\frac{1}{4} \frac{d}{dt} \int_{\Omega} \frac{\tilde{v}^4}{r^2} dx + \frac{3}{4} \nu \int_{\Omega} \left| \frac{\tilde{\varphi}^2}{r} \right|^2 dx + \frac{5}{8} \nu \int_{\Omega} \frac{\tilde{v}^4}{r^4} dx \leq \frac{3}{2} \|rv_\varphi\|_{L^\infty(\Omega)}^2 \int_{\Omega} \frac{|\tilde{v}_r|}{r^3} \frac{\tilde{\varphi}^2}{r^2} dx + c(1/r_0)\|rv_\varphi\|_{L^\infty(\Omega)} v^2 dx \\
+ c(1/r_0) \int_{\Omega} v^4 dx + \frac{4}{\nu} \int_{\Omega} \tilde{f}_\varphi^2 \tilde{v}_r^2 dx.
\]

(8.25)
Employing (4.29) in (8.23) gives

\[
\frac{1}{2} \frac{d}{dt} \left\| \tilde{\chi} \right\|_{L^2(\Omega_r)}^2 + \frac{\nu_1}{2} \left\| \nabla \tilde{\chi} \right\|_{L^2(\Omega_r)}^2 + \frac{\nu_2}{2} \left( \int_{\Omega_r} \left| \nabla \left( \frac{\tilde{v}_r}{r} \right) \right|^2 dx \right.
\]

\[
+ 3\nu_2 \int_{\Omega_r} \frac{1}{r^2} \left( \frac{\tilde{v}_r}{r} \right)^2 dx \leq \frac{1}{\nu} \int_{\Omega_r} \frac{\tilde{v}_\phi^2}{r^4} dx + c(1/r_0) \int_{\Omega_r,\xi_1,r} \chi^2 dx
\]

(8.26)

\[
+ c(1/r_0) \int_{\Omega_r,\xi_1,r} |v_r|^2 dx + c \int_{\Omega_r} \left| \frac{\tilde{F}}{r} \right|^2 dx
\]

\[
+ \frac{\nu_2}{2} c(1/r_0) \int_{\Omega_r} (v_{z,r}^2 + v_z^2) dx,
\]

where \( \nu = \nu_1 + \nu_2, \nu_i > 0, i = 1, 2. \) Using (3.1) and applying the Cauchy inequality to the first term on the r.h.s. of (8.25) we get

\[
\frac{1}{4} \frac{d}{dt} \int_{\Omega_r} \frac{\tilde{v}_\phi^4}{r^2} dx + \frac{3}{4} \nu \int_{\Omega_r} \left| \nabla \frac{\tilde{v}_\phi^2}{r} \right|^2 dx + \left( \frac{5}{8} \nu - \frac{\varepsilon}{2} \right) \int_{\Omega_r} \frac{\tilde{v}_\phi^4}{r^4} dx
\]

(8.27)

\[
\leq \frac{9}{8\varepsilon} \|rv_\phi\|_{L^\infty(\Omega_r)}^4 \int_{\Omega_r} \frac{\tilde{v}_r^4}{r^6} dx + c(1/r_0)d_6^4 \int_{\Omega_r,\xi_1,r} v^2 dx
\]

\[
+ c(1/r_0) \int_{\Omega_r,\xi_1,r} v^4 dx + c \int_{\Omega_r} \frac{\tilde{f}_\phi^4}{r^2} dx.
\]

Setting \( \varepsilon = \frac{\nu}{4} \) and using the Hardy inequality

\[
\int_{\Omega_r} \frac{\tilde{v}_r^2}{r^6} dx \leq \int_{\Omega_r} \frac{1}{r^2} \left( \frac{\tilde{v}_r}{r} \right)^2 dx
\]

in (8.27) gives

\[
\frac{1}{4} \frac{d}{dt} \int_{\Omega_r} \frac{\tilde{v}_\phi^4}{r^2} dx + \frac{3}{4} \nu \int_{\Omega_r} \left| \nabla \frac{\tilde{v}_\phi^2}{r} \right|^2 dx + \frac{\nu}{2} \int_{\Omega_r} \frac{\tilde{v}_\phi^4}{r^4} dx
\]

(8.28)

\[
\leq \frac{9}{2\nu} \|rv_\phi\|_{L^\infty(\Omega_r)}^4 \int_{\Omega_r} \frac{1}{r^2} \left( \frac{\tilde{v}_r}{r} \right)^2 dx + c(1/r_0)d_6^4 \int_{\Omega_r,\xi_1,r} v^2 dx
\]

\[
+ c(1/r_0) \int_{\Omega_r,\xi_1,r} v^4 dx + c \int_{\Omega_r} \frac{\tilde{f}_\phi^4}{r^2} dx.
\]
Multiplying (8.28) by $\frac{2}{\nu^2}$ and adding to (8.26) we obtain

\[
\frac{1}{2\nu^2} \frac{d}{dt} \int_{\Omega_\varepsilon} \frac{\tilde{\varphi}^4}{r^2} dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega_\varepsilon} \frac{\chi^2}{r^2} dx + \frac{3}{2\nu} \int_{\Omega_\varepsilon} \left| \nabla \frac{\tilde{\varphi}}{r} \right|^2 dx \\
+ \frac{\nu_1}{2} \int_{\Omega_\varepsilon} \left| \nabla \tilde{x} \right|^2 dx + 3\nu_2 \int_{\Omega_\varepsilon} \left( \frac{\tilde{v}_r}{r} \right)^2 dx \\
\leq \frac{9}{\nu^3} \|rv\varphi\|_{L^4_\infty(\Omega_\varepsilon)}^4 \int_{\Omega_\varepsilon} \frac{1}{r^2} \left( \frac{\tilde{v}_r}{r} \right)^2 dx + c(1/r_0) \int_{\Omega_\varepsilon} \chi^2 dx \\
+ c(1/r_0) \int_{\Omega_\varepsilon, \zeta_1, r} |v_r| \chi^2 dx + c(1/r_0) \int_{\Omega_\varepsilon} (v_{z,r}^2 + v_{z}^2) dx \\
+ c(1/r_0) d_0^2 \int_{\Omega_\varepsilon, \zeta_1, r} v^2 dx + c(1/r_0) \int_{\Omega_\varepsilon} v_\varphi^4 dx + c \int_{\Omega_\varepsilon} \left| \tilde{F}/r \right|^2 dx \\
+ c \int_{\Omega_\varepsilon} \tilde{f}_\varphi^4 dx.
\]

(8.29)

Setting $\nu_2 = a\nu$, $a < 1$ and assuming that

\[
(8.30) \quad \frac{9}{\nu^3} \|rv\varphi\|_{L^4_\infty(\Omega_\varepsilon)}^4 \leq 3a\nu \quad \text{so} \quad \|rv\varphi\|_{L^4_\infty(\Omega_\varepsilon)}^4 \leq \frac{a\nu^4}{3}.
\]

For $a = \frac{16}{27}$ the above condition takes the form

\[
\|rv\varphi\|_{L^4_\infty(\Omega_\varepsilon)} \leq \frac{2\nu}{3},
\]

we obtain from (8.29) the inequality

(8.31)

\[
\frac{d}{dt} \left( \frac{1}{\nu^2} \left\| \frac{\tilde{v}_\varphi}{r} \right\|_{L^2_2(\Omega_\varepsilon)}^2 + \left\| \frac{\tilde{x}}{r} \right\|_{L^2_2(\Omega_\varepsilon)}^2 \right) + \left( \frac{3}{\nu} \right) \left\| \nabla \frac{\tilde{\varphi}}{r} \right\|_{L^2_2(\Omega_\varepsilon)}^2 \\
+ \nu_1 \left\| \nabla \frac{\tilde{x}}{r} \right\|_{L^2_2(\Omega_\varepsilon)}^2 \right) \leq c(1/r_0) \int_{\Omega_\varepsilon} \chi^2 dx + c(1/r_0) \int_{\Omega_\varepsilon, \zeta_1, r} |v_r| \chi^2 dx \\
+ c(1/r_0) \int_{\Omega_\varepsilon} (v_{z,r}^2 + v_{z}^2) dx + c(1/r_0) d_0^2 \int_{\Omega_\varepsilon} v^2 dx + c(1/r_0) \int_{\Omega_\varepsilon} v_\varphi^4 dx \\
+ c \int_{\Omega_\varepsilon} \tilde{f}_\varphi^4 dx + c \int_{\Omega_\varepsilon} \left| \tilde{F}/r \right|^2 dx
\]

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Let us recall that $\nu_1 = (1 - a)\nu$. Let $b < 1 - a$.

Then the second expression on the l.h.s. takes the form

$$
\nu \left( \frac{3}{\nu^2} \left\| \frac{\tilde{v}_r}{r} \right\|_{L^2(\Omega_e)}^2 + (1 - a - b) \left\| \frac{\nabla \tilde{x}}{r} \right\|_{L^2(\Omega_e)}^2 \right) + b \nu \left\| \frac{\nabla \tilde{x}}{r} \right\|_{L^2(\Omega_e)}^2
$$

Let $c_p$ be the constant from the Poincare inequality.

Let $\nu_* = \min \left\{ \frac{3\nu}{c_p}, \frac{(1 - a - b)\nu}{c_p} \right\}$ and let

$$
X(t) = \frac{1}{\nu^2} \left\| \frac{\tilde{v}_r}{r} \right\|_{L^2(\Omega_e)}^2 + \left\| \frac{\tilde{x}}{r} \right\|_{L^2(\Omega_e)}^2.
$$

Then (8.31) takes the form

$$
\frac{d}{dt} X + \nu_* X + b \nu \left\| \frac{\nabla \tilde{x}}{r} \right\|_{L^2(\Omega_e)}^2 \leq c(1/r_0) \int_{\Omega_e \times \xi_1} |v_r| \chi^2 dx
$$

$$
+ c(1/r_0) \int_{\Omega_e} (\chi^2 + v_{x,r}^2 + v_{z}^2 + d_0^2 v^2) dx + c(1/r_0) \int_{\Omega_e} v_{1,\varphi}^4 dx
$$

$$
+ c \int_{\Omega_e} \tilde{f}_{\varphi}^4 dx + c \int_{\Omega_e} \left\| \frac{\tilde{F}}{r} \right\|^2 dx.
$$

Multiplying (8.33) by $e^{\nu_* t}$ and integrating with respect to time from $kT$ to $t \in (kT, (k + 1)T)$, $k \in \mathbb{N}_0$, yields

$$
X(t)e^{\nu_* t} + b \nu \int_{kT}^t \left\| \frac{\nabla \tilde{x}(t')}{r} \right\|_{L^2(\Omega_e)}^2 \ e^{\nu_* t'} dt'
$$

$$
\leq c(1/r_0) \int_{kT}^t \int_{\Omega_e \times \xi_1} |v_r| \chi^2 dx e^{\nu_* t'} dt' + c(1/r_0)d_0^2(1 + d_0^2)e^{\nu_* t}
$$

$$
+ c(1/r_0)d_0^2 d_0^2 e^{\nu_* t} + \left( \|f_\varphi\|_{L^4(\Omega \times (kT, (k + 1)T))} \right) e^{\nu_* t} + X(kT)e^{\nu_* kT}.
$$

Introducing the notation

$$
d_s = \sup_k \|f_\varphi\|_{L^4(\Omega \times (kT, (k + 1)T))}, \quad d_\delta = \sup_k \left\| \frac{\tilde{F}}{r} \right\|_{L^2(\Omega \times (kT, (k + 1)T))}
$$
and estimating the first term on the r.h.s. of (8.34) by

\[ c(1/r_0) \int_{kT_e}^{t} \| v_r \|_{L_2(\Omega_{\varepsilon, \zeta_1, \tau})} \| \chi \|_{L_4(\Omega_{\varepsilon, \zeta_1, \tau})}^2 e^{\nu_\ast t'} dt' \]

\[ \leq c(1/r_0) d_4 \int_{kT_e}^{t} \| \chi \|_{L_4(\Omega_{\varepsilon, \zeta_1, \tau})} e^{\nu_\ast t'} dt' \]

\[ \leq \varepsilon \int_{kT_e}^{t} \left\| \frac{\nabla X}{r} \right\|^2_{L_2(\Omega_{\varepsilon, \zeta_1, \tau})} e^{\nu_\ast t'} dt' + c(1/r_0, d_1, d_5) e^{\nu_\ast t} \]

we obtain

(8.35)

\[ X(t) e^{\nu_\ast t} + b \nu \int_{kT_e}^{t} \left\| \frac{\nabla X}{r} (t') \right\|^2_{L_2(\Omega_{\varepsilon})} e^{\nu_\ast t'} dt' \]

\[ \leq \varepsilon \int_{kT_e}^{t} \left\| \frac{\nabla X}{r} \right\|^2_{L_2(\Omega_{\varepsilon, \zeta_1, \tau})} e^{\nu_\ast t'} dt' + B(1/r_0, d_1, d_5, d_7, d_8, d_9, 1/\varepsilon) \cdot e^{\nu_\ast t} + X(kT) e^{\nu_\ast kT_e}, \]

where \( B \) is a positive increasing function of its arguments.

By the local iteration technique (see \([LSU, \text{Ch. 4, Sect. 10}]) we get

(8.36)

\[ X(t) e^{\nu_\ast t} + b \nu \int_{kT_e}^{t} \left\| \frac{\nabla X}{r} (t') \right\|^2_{L_2(\Omega_{\varepsilon})} e^{\nu_\ast t'} dt' \leq 2B e^{\nu_\ast t} + 2X(kT) e^{\nu_\ast kT} \]

To obtain an estimate for \( X(kT_e) \) we skip the second term on the l.h.s. of (8.36) and set \( t = (k + 1)T_e \). Then we get

(8.37)

\[ X((k + 1)T_e) \leq 2B + 2X(kT) e^{-\nu_\ast T_e}. \]

Let \( B_0 = 2B \) and \( 2e^{-\nu_\ast T_e} \leq 1 \). Denoting \( \nu_0 = \frac{\nu_\ast}{2} \) we obtain

(8.38)

\[ X((k + 1)T_e) \leq B_0 + X(kT_e) e^{-\nu_0 T_e}. \]

Hence

(8.39)

\[ X(kT_e) \leq \frac{B_0}{1 - e^{-\nu_0 T_e}} + X(0) e^{-\nu_0 kT_e}, \quad k \in \mathbb{N}_0. \]
In view of (8.39) we obtain from (8.31) after integration with respect to time from $kT_e$ to $t \in (kT_e, (k + 1)T_e]$ the estimate

$$\left\| \frac{v_r^2}{r} \right\|^2_{L^2_v(\Omega \times (kT_e, t))} + \left\| \frac{\chi}{r} \right\|^2_{L^2_v(\Omega \times (kT_e, t))} \leq B_0 + X^2(e^{\nu_e(t - kT_e)})$$

The inequality implies the estimate

$$\|v(t)\|_{H^1(\Omega)} \leq c(B_0 + \chi(kT_e)), \quad t \in (kT_e, (k + 1)T_e],$$

for any $k \in \mathbb{N}_0$. This concludes the proof.

**Proof of the Main Theorem.** Let $T = T_e$. In view of Theorem 8.1 and Lemma 8.2 there exists a local solution to problem (1.1) such that

$$v \in W^{2,1}_2(\Omega \times (kT, (k + 1)T)), \quad \nabla p \in L^2(\Omega \times (kT, (k + 1)T))$$

and estimate (1.15) is valid. This ends the proof.

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