Gödel-type universes and chronology protection in Hořava-Lifshitz gravity

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In the attempts toward a quantum gravity theory, general relativity faces a serious difficulty since it is non-renormalizable theory. Hořava-Lifshitz gravity offers a framework to circumvent this difficulty, by sacrificing the local Lorentz invariance at ultra-high energy scales in exchange of power-counting renormalizability. The Lorentz symmetry is expected to be recovered at low and medium energy scales. If gravitation is to be described by a Hořava-Lifshitz gravity theory there are a number of issues that ought to be reexamined in its context, including the question as to whether this gravity incorporates a chronology protection, or particularly if it allows Gödel-type solutions with violation of causality. We show that Hořava-Lifshitz gravity only allows hyperbolic Gödel-type space-times whose essential parameters $m$ and $\omega$ are in the chronology respecting intervals, excluding therefore any noncausal Gödel-type space-times in the hyperbolic class. There emerges from our results that the famous noncausal Gödel model is not allowed in Hořava-Lifshitz gravity. The question as to whether this quantum gravity theory permits hyperbolic Gödel-type solutions in the chronology preserving interval of the essential parameters is also examined. We show that Hořava-Lifshitz gravity not only excludes the noncausal Gödel universe, but also rules out any hyperbolic Gödel-type solutions for physically well-motivated perfect-fluid matter content.

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I. INTRODUCTION

Even though general relativity is a highly successful classical field theory of gravity, it faces a serious difficulty in the attempts toward a theory of quantum gravity since one cannot quantize it by using the canonical quantization or path integral formalism — there emerges that it is a non-renormalizable theory. Hořava-Lifshitz (HL) gravity [1] offers framework to circumvent this difficulty, by sacrificing the local Lorentz invariance at ultra-high energy scales (typically trans-Planckian) in exchange of power-counting renormalizability. The Lorentz symmetry is abandoned by invoking a different kind of scaling, called anisotropic or Lifshitz scaling [2], between space and time, and it is expected that it is recovered at low and medium (sub-Planckian) energy scales (long distance).

Since the publication of Hořava proposal in 2009 [1], a great deal of effort has gone into the study of several features of Hořava-Lifshitz gravity. One can roughly group the contributions to this issue into two broad families. In the first, one finds articles devoted to the several aspects of Hořava-Lifshitz gravity as a quantum field theory. This class includes, among other matters, the attempts toward a consistent quantization of the theory [3, 4] and the calculation of counter-terms. In the second family, one has a number of interesting cosmological implications of Hořava-Lifshitz gravity, and the exam of some important solutions of Einstein’s equations in the framework of Hořava-Lifshitz gravity. This includes, for example, Friedmann-Lemaître-Robertson-Walker models [7–9] and black-hole solutions [10], anisotropic scaling as a solution to the horizon problem and as a way of having scale-invariant cosmological perturbations without inflation [11], dark matter as an integration constant [12], and bounce solutions in the early universe [13]. For some further references on several cosmological implications of Hořava-Lifshitz gravity see Ref. [14] and references therein quoted on this issue.

Chronology and causality are central ingredients in the foundation of the special relativity theory — chronology is preserved and causality is respected. The space-times of the general relativity have locally the same causal structure of the flat space-time of special relativity since a local chronology protection is inherited from the very fact that the space-times of general relativity are locally Minkowskian. On nonlocal (global) scale, however, significant differences may arise since Einstein’s field equations do not provide nonlocal (topological) constraints on the space-times. Indeed, it has long been known that there are solutions to the general relativity field equations
that present causal anomalies in the form of closed timelike curves (see, for example, Refs. [17]). The renowned model found by Gödel [18] is a well-known example of a solution to Einstein’s equations that makes it apparent that general relativity permits solutions with closed timelike world lines, despite its local Lorentzian character that leads to the local validity of the causality principle. The Gödel model is a solution of Einstein’s equations with cosmological constant $\Lambda$ for dust of density $\rho$, but it can also be interpreted as perfect-fluid solution with equation of state $p = \rho$ without cosmological constant. Owing to its unexpected features, Gödel’s model has a Recognizable importance and has motivated an appreciable number of investigations on rotating Gödel-type models as well as on causal anomalies not only in the context of general relativity (see, e.g. Refs. [19]) but also in the framework of other theories of gravitation (see, for example, Refs. [21]).

The chronology protection conjecture introduced by Hawking [13] suggests that even though closed timelike curves are classically possible to be produced, quantum effects are likely to prevent such time travel. In this way, the laws of quantum physics would prevent closed timelike curves from appearing.

If gravitation can be described by Horava-Lifshitz gravity theory there are a number of matters that ought to be reexamined in its framework, including the question as to whether this quantum gravity theory permits Gödel-type solutions with violation of causality, somehow incorporating or not the chronology protection conjecture for this family of spacetimes. Our chief aim in this paper is to examine this question by investigating the possibility of Gödel-type universes along with the question of breakdown of causality in Horava-Lifshitz quantum gravity.

We show that Horava-Lifshitz gravity only allows hyperbolic Gödel-type space-times whose essential parameters $m^2$ and $\omega^2$ are in the chronology respecting intervals, excluding therefore the noncausal Gödel-type space-times in this class. Thus, the famous noncausal Gödel model is not allowed in context of Horava-Lifshitz gravity. The question as to whether this quantum gravity theory permits hyperbolic Gödel-type solutions in the surviving chronology preserving interval of the essential parameters is also examined. We show that Horava-Lifshitz gravity not only excludes the noncausal Gödel model, but also rules out any hyperbolic Gödel-type solutions for physically well-motivated perfect-fluid matter content, which is the matter source for the Gödel universe in general relativity.

## II. Gödel-Type Metrics

It is well known that Gödel solution to the general relativity field equations is a member of the following broad family of space-time-homogeneous (ST-homogeneous) Gödel-type geometries, whose form in cylindrical coordinates $[[r, \phi, z]]$ is given by [21]

$$ds^2 = -[dt + H(r)d\phi]^2 + D^2(r)d\phi^2 + dr^2 + dz^2,$$

where the functions $H(r)$ and $D(r)$ are such that

$$\frac{H'}{D} = 2\omega,$$

$$\frac{D''}{D} = m^2,$$

where the prime denotes derivative with respect to $r$, and the parameters $(m, \omega)$ are constants such that $\omega^2 > 0$ and $-\infty \leq m \leq \infty$.

The ST-homogeneous Gödel-type space-times can be grouped in the following classes:

i. Hyperbolic, in which $m^2 = \text{const} > 0$ and

$$H = \frac{4\omega}{m^2} \sinh^2 \left( \frac{mr}{2} \right), \quad D = \frac{1}{m} \sinh \left( \frac{mr}{2} \right);$$

ii. Trigonometric, where $m^2 = \text{const} \equiv -\mu^2 < 0$ and

$$H = \frac{4\omega}{\mu^2} \sin^2 \left( \frac{\mu r}{2} \right), \quad D = \frac{1}{\mu} \sin \left( \frac{\mu r}{2} \right);$$

iii. Linear, in which $m = 0$ and

$$H = \omega r^2, \quad D = r.$$

We recall that in the above three families the constant $\omega$ is the vorticity of matter source, and that all Gödel-type metrics in the above classes are characterized by the two essential parameters $\omega$ and $m$: identical pairs $(m^2, \omega^2)$ determine isometric Gödel-type space-times [21], [24], [25]. Moreover, Gödel solution is just a particular case of the hyperbolic ($m^2 > 0$) class with $m^2 = 2\omega^2$.

## III. Violation of Causality and Horava-Lifshitz Gravity

The causality features in Horava-Lifshitz gravity can be looked upon as having two interconnected physically significant ingredients, namely the gravity theory, which involves the matter source, and the space-time geometry.

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2 For a good pedagogical overview with a fair list of references on the chronology protection conjecture see Visser [16].

3 This extends the investigations on these issues carried out in the framework of general relativity and other classical theories of gravity (see, e.g., Refs. [21],[23]).
Regarding the latter, we begin by rewriting the Gödel-type line element (1) in the form
\[ ds^2 = -dt^2 - 2H(r) dt d\phi + dr^2 + G(r) d\phi^2 + dz^2, \tag{7} \]
where \( G(r) = D^2 - H^2 \). In this form it is easy to show that existence of closed timelike curves, which allows for violation of causality, depends upon the sign of the metric function \( G(r) \). Indeed, from Eq. (7) one has that the circles defined by \( t, z, r = \text{const} \) become closed timelike curves whenever \( G(r) < 0 \). Thus, the causality features of all ST-homogeneous Gödel-type space-times can be investigated by using essentially this inequality together with the basic variables and the field equations of Hořava-Lifshitz gravity.

Regarding the second ingredient in the causality problem, we recall that the basic quantities of Hořava-Lifshitz gravity.

From this equation together with Eq. (7) one has that the circles defined by \( t, z, r = \text{const} \) become closed timelike curves whenever \( G(r) < 0 \). Thus, the causality features of all ST-homogeneous Gödel-type space-times can be investigated by using essentially this inequality together with the basic variables and the field equations of Hořava-Lifshitz gravity.

Regarding the second ingredient in the causality problem, we recall that the basic quantities of Hořava-Lifshitz gravity are the lapse (real) function \( N(t) \), the shift vector field \( N^i(t, \vec{x}) \) and the 3-D metric \( g_{ij}(t, \vec{x}) \) with which we write the spacetime metric in the ADM form\(^4\)

\[ ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{8} \]

For the hyperbolic family of Gödel-type metrics, from Eq. (9) one finds that

\[ g(r) = \frac{4}{m^2} \sinh^2 \left( \frac{mr}{2} \right) \left( 1 - 4\omega^2 \sinh^2 \left( \frac{mr}{2} \right) + 1 \right), \tag{10} \]

and therefore for \( 0 < m^2 < 4\omega^2 \) there is a critical radius \( r_c \) defined by \( G(r) = 0 \), namely

\[ \sinh^2 \frac{mr_c}{2} = \left[ \frac{4\omega^2}{m^2} - 1 \right]^{-1}, \tag{11} \]

such that \( G(r) > 0 \) for \( r < r_c \) and \( G(r) < 0 \) for \( r > r_c \). Hence, the circles \( t, z, r = \text{const} \) in the circular band with \( r > r_c \) are closed timelike curves.\(^5\) However, on the one hand the Riemannian (positive definite) character of the spatial metric \( g_{ij} \) implies that \( g = \det(g_{ij}) = \sqrt{G(r)} > 0 \), on the other hand the lapse is also an imaginary function in the noncausal region \( G(r) < 0 \) defined by \( t, z, r = \text{const} \) and \( r > r_c \). Thus, for the hyperbolic class, the noncausal space-times are excluded in Hořava-Lifshitz gravity. Therefore, from this result one has that the famous Gödel model, for which \( m^2 = 2\omega^2 \), is not permitted in Hořava-Lifshitz gravity.\(^6\)

A similar analysis holds for the remaining classes of Gödel-type space-times. Indeed, for the trigonometric class whose metric functions are given by Eq. (11) one finds that

\[ G(r) = \frac{4}{\mu^2} \sin^2 \left( \frac{\mu r}{2} \right) \left( \mu^2 - (4\omega^2 + \mu^2) \sin^2 \left( \frac{\mu r}{2} \right) \right), \tag{12} \]

and therefore \( G(r) \) has an infinite sequence of zeros. Thus, there is an infinite sequence of alternating causal \( [G(r) > 0] \) and noncausal \( [G(r) < 0] \) regions in the section \( t, z, r = \text{const} \), without and with noncausal circles, depending on the value of \( r = \text{const} \) (see Appendix for detailed calculations). Thus, for example, if \( G(r) < 0 \) for a certain range of \( r \) \((r_1 < r < r_2, \text{say})\) noncausal Gödel’s circles exist, whereas for \( r \) in the next circular band \( r_2 < r < r_3 \) (say) for which \( G(r) > 0 \) no such noncausal circles exist. Nevertheless, since in context of Hořava-Lifshitz gravity the spatial metric \( g_{ij} \) is positive definite, the regions of the underlying Gödel-type manifolds in which the chronology is violated, i.e. \( t, z, r = \text{const} \) with \( G(r) < 0 \), are excluded for the trigonometric family of spacetimes. In these regions the lapse function again becomes an imaginary function.

Finally, for the linear family, from Eq. (11) one easily finds

\[ G(r) = r^2 - r^4 \omega^2 = -r^2 (r \omega - 1) (r \omega + 1). \tag{13} \]

Thus, there is a critical radius \([G(r) = 0]\) given by \( r_c = 1/\omega \), such that for any radius \( r > r_c \) the inequality \( G(r) < 0 \) holds, and then the circles defined by \( t, z, r = \text{const} \) are closed timelike curves. Here again the positive definite character of spatial metric and the fact the lapse is a real function cannot be imposed in the regions of Gödel-type manifolds that violate the chronology \([G(r) < 0]\). Thus, the noncausal region of the linear family is excluded in Hořava-Lifshitz gravity.

To summarize the above results, we have shown that Hořava-Lifshitz gravity can only be consistently formulated in the chronology preserving regions of Gödel-type manifolds, excluding therefore the noncausal regions of

\(^4\) Throughout this paper we use Greek letters to denote spacetime coordinate indices, which are lowered and raised, respectively, with \( g_{\mu \nu} \) and \( g^{\mu \nu} \), and vary from 0 to 3, whereas the spatial components 1 - 3 are denoted by Latin lower case letters which are lowered and raised with \( g_{ij} \) and \( g^{ij} \), respectively.

\(^5\) We note that the only Gödel-type metric without such noncausal circles comes about when \( m^2 = 4\omega^2 \) (see Ref. [21]). In this case, the critical radius \( r_c \) → ∞, and hence the violation of causality of Gödel type is avoided.

\(^6\) We note that in the parameter interval \( m^2 > 4\omega^2 \) one has \( G(r) > 0 \). Thus, \( m^2 > 4\omega^2 \) defines the causal parameter interval in Gödel-type class of spacetimes, which is permitted in Hořava-Lifshitz context.
underlying Gödel-type manifolds for all classes of ST-homogeneous Gödel-type spacetimes. This excludes any Gödel-type space-times with violation of causality. Particularly for the hyperbolic class it rules out the well-known Gödel metric for which \( m^2 = 2\omega^2 \). As a matter of fact, since we have not used so far the Hořava-Lifshitz field equations, this result holds for any theory whose formulation relies on the suitable behavior of the ADM variables of the Gödel-type space-times.

The fact that Hořava-Lifshitz gravity does not permit hyperbolic Gödel-type metrics whose essential parameters \( m \) and \( \omega \) define noncausal Gödel-type space-time geometries does not signify that space-times such as wormholes, which are seen generically to lead to the creation of time machines, cannot be found in Hořava-Lifshitz gravity [20]. Moreover, although the presence of a single closed timelike curve as, for example, the above Gödel’s circles \( (t, z, r = \text{const} > r_c) \), is an unequivocal manifestation of violation of the chronology protection conjecture, a space-time may admit noncausal closed curves other than these Gödel’s circles. This means that the exclusion of all noncausal hyperbolic Gödel space-times can only be seen as a tiny suggestion that the chronology is protected in Hořava-Lifshitz gravity in the sense that it is protected for this type of causal anomaly of Gödel-type space-times.

Given that the noncausal interval of the essential parameters of hyperbolic Gödel-type space-times are excluded in Hořava-Lifshitz context, a question arises as to whether this theory permits solutions of its field equations in the region where the chronology is respected \( \{ G(r) > 0 \} \) for physically well-motivated matter content. In the next section we shall examine this question for a perfect-fluid matter source.

IV. HOŘAVA-LIFSHITZ GRAVITY

A. Field equations

Here we briefly introduce the Hořava-Lifshitz gravity and present its field equations in the form that will be used in the next section. We begin by recalling that the dynamical variables of this theory are the lapse function \( N(t) \), the shift vector field \( N^i(t, \vec{x}) \) and the spatial metric \( g_{ij}(t, \vec{x}) \) with which we rewrite an arbitrary spacetime line element

\[
ds^2 = g_{00} dt^2 + 2 g_{0i} \, dx^i \, dt + g_{ij} \, dx^i \, dx^j
\]

in the ADM form given by Eq. (8). Thus, the ADM variables can be expressed in terms of the metric components \( g_{\mu\nu} \) as \( N_i = g_{0i} \) and \( N = (g_{ij} N^i N^j - g_{00})^{1/2} \).

The Lagrangian for the Hořava-Lifshitz gravity we consider in this paper is given by [8]

\[
L = \sqrt{g} N \left[ \frac{2}{\kappa^2} \left( K_{ij} K^{ij} - \lambda K^2 \right) - \frac{\kappa^2}{2\mu} C_{ij} C^{ij} + \kappa^2 \mu^2 R^{ij} R_{ij} - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} \right] + \frac{\kappa^2 \mu^2}{8(1 - 3\Lambda)} \left( 1 - 4\Lambda \right) R^2 + \Lambda R - 3\Lambda^2 + \mathcal{L}_m \right],
\]

where

\[
K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right)
\]

is the extrinsic curvature, overdot stands for derivative with respect to \( t \), \( K = g^{ij} K_{ij} \) is its trace, \( R_{ij} \) is the Ricci tensor for the metric \( g_{ij} \),

\[
C_{ij} = \frac{\kappa^2 \mu^2}{\sqrt{g}} \nabla_k (R^k_{ij} - \frac{1}{4} R \delta^k_{ij})
\]

is the Cotton tensor, and \( \mathcal{L}_m \) is the matter Lagrangian, which depends on the matter fields and on the ADM variables. In Eq. (15), \( \Lambda \) is a cosmological constant, \( \kappa^2 \) is a gravitational constant, and \( \lambda, \nu, \mu \) are coupling parameters of the theory. It should be noticed that if one keeps the spatial derivative only up to the second order, for \( \lambda = 1 \) the general relativity is recovered. We also note that the Lagrangian (15) involves terms with different values of the critical exponent \( z \). To recover general relativity the \( z = 1 \) terms are necessary, whereas \( z = 3 \) terms are needed for renormalizability.

The equations of motion can now be obtained through the variation of the action defined by the Lagrangian (15) with respect to the ADM dynamical variables. Indeed, (i) variation with respect to \( N \) and \( g_{00} \) are related and the result is given by [8]

\[
\frac{\delta S}{\delta g_{00}} = \left( \frac{\delta S}{\delta N} + \frac{\delta S_m}{\delta N} \right) \frac{\delta N}{\delta g_{00}} = G^{00} - T^{00} = 0,
\]

where \( \frac{\delta S_m}{\delta g_{00}} = -\frac{1}{2N} \), \( G^{\mu\nu} \) is the generalized Einstein tensor with

\[
G^{00} = \frac{1}{2N} \left[ -\alpha (K_{ij} K^{ij} - \lambda K^2) + \beta C_{ij} C^{ij} \right. + \sigma + \gamma \left( \frac{\epsilon^{ijk}}{\sqrt{g}} R_{il} \nabla_j R_{kl} + \zeta R_{ij} R^{ij} + \eta R^2 + \zeta R \right)
\]

and \( T^{\mu\nu} \) is the energy-momentum tensor of matter: (ii) variation with respect to \( N_i = g_{0i} \) furnishes [8]

\[
\frac{\delta S}{\delta N_i} = G^{0i} - T^{0i} = 2\alpha \nabla_k (K^{0i} - \lambda K^{0k}) - T^{0i} = 0
\]

(iii) variation with respect to \( g_{ij} \) provides [8]

\[
G_{ij} = T_{ij}.
\]
where

$$G_{ij} = G_{ij}^{(1)} + G_{ij}^{(2)} + G_{ij}^{(3)} + G_{ij}^{(4)} + G_{ij}^{(5)} + G_{ij}^{(6)} \quad (22)$$

and

$$G_{ij}^{(1)} = 2\alpha N K_{ik} K_{j}^{k} - \frac{\alpha N}{2} K_{kl} K_{ij}^{kl} + \alpha (K_{ik} N_{j})^{k} + \alpha (K_{jk} N_{i})^{k} - \alpha (K_{ij} N_{k})^{k} + (i \leftrightarrow j),$$

$$G_{ij}^{(2)} = -2\alpha \lambda N K_{i k} + \frac{\lambda N}{2} K^{2} g_{ij} - \frac{\lambda \alpha}{\sqrt{g}} g_{ik} g_{j} \frac{\partial}{\partial \xi} (\sqrt{g} g_{kl}^{ij})$$

$$- \alpha (K g_{ik} N_{j})^{k} - \alpha (K g_{jk} N_{i})^{k} + \alpha (K g_{ij} N_{k})^{k} + (i \leftrightarrow j),$$

$$G_{ij}^{(3)} = N \xi R_{ij} - \frac{N}{2} (\xi R + \sigma) g_{ij} - \xi N_{ij} + \xi \Box N g_{ij} + (i \leftrightarrow j),$$

$$G_{ij}^{(4)} = 2N \eta R R_{ij} - \frac{N}{2} \eta R^{2} g_{ij} + 2\eta \Box (N R) g_{ij} - 2\eta (N R)_{ij} + (i \leftrightarrow j),$$

$$G_{ij}^{(5)} = \Box (N (\xi R_{ki} + \frac{\gamma}{2} C_{ki}^{j})^{i k} + (N (\xi R_{kl} + \frac{\gamma}{2} C^{kl})); k g_{ij} + (i \leftrightarrow j),$$

$$G_{ij}^{(6)} = \frac{1}{2} \frac{e^{ijkl}}{\sqrt{g}} \left[ (Q_{m i} ; k j l) + (Q_{m i} ; k n j l) - (Q_{m i} ; k n j l) - (Q_{m i} ; k n j l) + (Q_{m i} ; k n j l) + (Q_{m i} ; k n j l) \right] + 2N \xi R_{ik} R^{k}$$

$$- \frac{N}{2} (\beta C_{ik} C^{kl} + \gamma R_{kl} C^{kl} + \xi R_{kl} R^{kl}) g_{ij} - \frac{1}{2} Q_{kl} C_{ik} C_{ij} + (i \leftrightarrow j). \quad (23)$$

In the above field equations we have defined

$$\alpha = \frac{2}{\kappa^{2}}, \quad \beta = -\frac{\kappa^{2}}{2w^{4}}, \quad \gamma = \frac{\kappa^{2} \mu}{2w^{2}}, \quad \zeta = -\frac{\kappa^{2} \mu^{2}}{8};$$

$$\eta = \frac{\kappa^{2} \mu^{2}(1 - 4\lambda)}{2 A (1 - 3\lambda)}, \quad \xi = \frac{\kappa^{2} \mu^{2} A}{8 (1 - 3\lambda)}, \quad \tau = \frac{1}{1 - 3\lambda};$$

$$\sigma = -\frac{3 \kappa^{2} \mu^{2} A^{2}}{8 (1 - 3\lambda)}, \quad Q_{ij} \equiv N (\gamma R_{ij} + 2\beta C_{ij}). \quad (24)$$

B. Perfect fluid as source

Given that Hořava-Lifshitz gravity rules out the chronology violating of hyperbolic Gödel-type spacetimes, the question as to whether this theory admits Gödel-type solutions in the chronology preserving interval of the essential parameters, $m^{2} > 4\omega^{2}$, naturally arises here. In this section we shall examine this question by considering the hyperbolic class ($m^{2} > 0$) of Gödel-type [Eq. (21)] in the framework of Hořava-Lifshitz gravity. This is the most important class of Gödel-type spacetime geometries as it contains the two most relevant Gödel-type solutions of Einstein’s equations, namely Gödel solution [18], in which $m^{2} = 2\omega^{2}$, and the only causal Gödel-type solution found in Ref. [21], in which $m^{2} = 4\omega^{2}$.

To simplify the calculation of the geometrical quantities of the hyperbolic class that are required for the Hořava-Lifshitz field equations, we begin by introducing new (Cartesian) coordinates $t', x, y, z'$ defined through the following coordinate transformation

$$\tan \left[ \phi/2 + (m^{2}/4\omega) (t' - t) \right] = e^{-m r} \tan(\phi/2), \quad (25)$$

$$e^{m x} = \cosh(m r) + \sinh(m r) \cos \phi, \quad (26)$$

$$m y e^{m x} = \sinh(m r) \sin \phi, \quad (27)$$

$$z' = z, \quad (28)$$

and rewrite the line element of the hyperbolic family given by Eq. (21) in the form

$$ds^{2} = -[dt' + (2\omega/m) e^{m x} dy']^{2} + e^{2m x} dy^{2} + dx^{2} + dz'^{2}, \quad (29)$$

where $-\infty < t', x, y, z' < +\infty$. In this coordinates the field equations for this class of Gödel-type metrics become much simpler. We emphasize, however, that the following results hold for to the whole hyperbolic ($m^{2} > 0$) class of Gödel-type metrics.

From Eq. (24) one has that the ADM variables are given by $N_{i} = (0, -(2\omega/m) e^{m x}, 0)$, $N = 1/v$, and $g_{ij} = \text{diag}(1, G(x), 1)$, where

$$G(x) = v^{2} e^{2m x} \quad \text{with} \quad v = \sqrt{1 - \left( \frac{2\omega}{m} \right)^{2}}. \quad (30)$$
A straightforward calculation shows that the spatial metric $g_{ij}$ gives rise to the following non-zero components of the Christoffel symbols $\Gamma^2_{12} = m$ and $\Gamma^1_{22} = -mv^2e_{m}^{x}$, from which one has the non-vanishing component of the Riemannian curvature $R_{1212} = -m^2v^2e_{m}^{x}$. Thus the corresponding components of the Ricci tensor and the curvature scalar are given, respectively, by $R_{11} = -m^2$ and $R_{22} = -m^2v^2e_{m}^{x}$, $R = -2m^2$. Now, by using this Ricci tensor and scalar, it is easy to show that related Cotton tensor (see Eq. (16)) is $K_{12} = -v\omega e_{m}^{x}$, which gives $K = g^{ij}K_{ij} = 0$. This completes the calculations of the geometrical quantities of hyperbolic Gödel-type geometries, which are needed to have Hořava-Lifshitz field equations.

The other important ingredient of Hořava-Lifshitz field equations is the matter source. Similarly to the Gödel solution in the general relativity framework \textsuperscript{18}, we consider in this work a perfect fluid of density $\rho$ and pressure $p$. Thus, we have

$$T^{\mu\nu} = (\rho + p) u^{\mu}u^{\nu} + p g^{\mu\nu}. \quad (31)$$

Without loss of generality from now on we choose units such that $\kappa^2 = 1$. Taking into account (31), the field equations (32), (39) and (41) reduce to the following set of algebraic equations:

$$-4m^4\tau v\zeta - 2\Lambda m^2\tau v\zeta + 3\Lambda^2\tau v\zeta + 2m^4v\zeta + 4p^2v^2 - 2\rho - 2p = 0, \quad (32)$$

$$2\omega(p v - 2m^2) = 0, \quad (33)$$

$$-4m^4\tau\zeta - 3\Lambda^2\tau\zeta + 2m^4\zeta - p v + 4\omega^2 = 0, \quad (34)$$

$$-4m^4\tau v\zeta - 3\Lambda^2\tau v\zeta + 2m^4v\zeta + \rho v^2 - 12\omega^2v - p - p = 0, \quad (35)$$

$$4m^4\tau\zeta - 2\Lambda m^2\tau\zeta - 3\Lambda^2\tau\zeta - 2m^4\zeta - p v - 4\omega^2 = 0, \quad (36)$$

written in terms of independent parameters $\rho, \Lambda, \tau, \zeta, \omega$, and $m^2$.

We recall that to find a Gödel-type solution to the above algebraic equations (32)–(36) amounts to determining a pair $(m^2, \omega^2)$ in the chronology preserving interval $m^2 > 4\omega^2$. In what follows we shall show that such a pair does not exist, ruling out therefore any Gödel-type solution for a perfect fluid matter source in the hyperbolic class. To this end, we first solve Eq. (33) for $p$ to have

$$p = \frac{4m^2}{v}, \quad (37)$$

and then substitute the result back into the remaining field equations (32), (34)–(36), to obtain that

$$4m^4\tau v\zeta - 2\Lambda m^2\tau v\zeta - 3\Lambda^2\tau v\zeta - 2m^4v\zeta + 4\omega^2v - 8m^2v + \frac{8m^2}{v} + 2\rho = 0, \quad (38)$$

$$4m^4\tau\zeta + 3\Lambda^2\tau\zeta - 2m^4\zeta - 4\omega^2 + 4m^2 = 0, \quad (39)$$

$$4m^4\tau v\zeta + 3\Lambda^2\tau v\zeta - 2m^4v\zeta - \rho v^2 + 12\omega^2v^2 + \rho v + 4m^2 = 0, \quad (40)$$

$$4m^4\tau\zeta - 2\Lambda m^2\tau\zeta - 3\Lambda^2\tau\zeta - 2m^4\zeta - 4\omega^2 - 4m^2 = 0. \quad (41)$$

Now, we solve (39) and (41) for $\tau$ and $\zeta$ and find

$$\tau = \frac{2m^6}{(\Lambda m^2 + 3\Lambda^2)^2\omega^2 + 4m^6 - \Lambda m^4}, \quad (42)$$

$$\zeta = -\frac{(2\Lambda m^2 + 6\Lambda^2)\omega^2 + 8m^6 - 2\Lambda m^4}{\Lambda m^6 + 3\Lambda^2 m^4}, \quad (43)$$

which can be substituted into (38) and (40), in order to write the remaining two equations in the form

$$\rho v - \frac{16\omega^4}{m^2} + 12\omega^2 + 2m^2 = 0, \quad (44)$$

$$\rho v - 16\omega^2 + 8m^2 = 0. \quad (45)$$

From Eq. (45) we have

$$\rho = \frac{8}{v}(2\omega^2 - m^2), \quad (46)$$

This equation together with Eq. (44) gives

$$(4\omega^2 - m^2)(2\omega^2 - 3m^2) = 0 \quad (47)$$

whose solutions are

$$m^2 = \frac{2}{3}\omega^2 \quad \text{and} \quad m^2 = \frac{1}{4}\omega^2, \quad (48)$$

which are both outside the chronology preserving interval $m^2 > 4\omega^2$, making apparent that there is no perfect-fluid Gödel-type solution to the Hořava-Lifshitz field equations in the chronology preserving region.\footnote{Note, in addition, that for the solutions (48) one has, respectively, $v = 0$ and $v = \sqrt{5}i$, which gives that $p$ and $\rho$ are either undefined or imaginary quantities. This reinforce the fact that these solutions are not permitted in the framework of Hořava-Lifshitz gravity.} Furthermore, this result holds regardless of the equation of state $p/\rho$.

To close this section, some words of clarification regarding the results of Ref. \textsuperscript{27} are in order. First, we note that rather than dealing with the whole hyperbolic family of Gödel-type space-times in this reference only the particular case of Gödel metric has been considered. Second, we emphasize that their whole calculations were made without noticing, for example, that lapse
$N$ is not well-defined for particular case of Gödel metric. Thus, Hořava-Lifshitz gravity was improperly used in the chronology violating region to define an energy-momentum tensor associated to the Gödel metric. Indeed, since $m^2 = 2\omega^2$ for Gödel metric, Eq. (30) makes clear that, e.g., the lapse $N = 1/v$, with $v$ given by Eq. (30), is an imaginary function. An important outcome of the above results is that the famous Gödel spacetime cannot be a solution of Hořava-Lifshitz gravity no matter how exotic is the source one takes [27], since some dynamical variables can only be consistently defined in the chronology preserving for the range $m^2 > 4\omega^2$ of Gödel-type classes of manifolds.

V. CONCLUDING REMARKS

Despite its great success as a classical theory of gravity, general relativity faces a crucial difficulty in the attempts toward a quantum theory of gravity in that it is non-renormalizable. Hence, general relativity is viewed as an effective theory that breaks down at some energy scale, beyond which it is unsuitable to describe the gravitational interaction. Hořava-Lifshitz gravity evades this difficulty by invoking an anisotropic scaling between space and time, which amounts to sacrificing the local Lorentz invariance at ultra-high energy scales in exchange of power-counting renormalizability. The Lorentz symmetry is expected to be recovered at low and medium energy scales (long distance).

Chronology and causality are central ingredients in the foundation of the special relativity theory. These properties are naturally inherited locally by general relativity theory, whose space-times are locally Minkowskian. The nonlocal question, however, is left open, and violation of causality can come about. Indeed, it has long been known that there are solutions of Einstein’s equations that exhibit closed time-like curves. The Gödel model is the best known example of a cosmological solution of Einstein’s equations in which causality is violated at a nonlocal scale. In 1992 Stephen Hawking suggested that even though closed timelike curves can arise in the framework of classical theories of gravitation, quantum effects are likely to prevent chronological pathologies. In this way, the laws of quantum physics would prevent closed timelike curves from appearing.

In this paper we proceeded further with the investigations on the potentialities, difficulties, and limitations of Hořava-Lifshitz gravity by investigating the possibility of Gödel-type solutions to its field equations along with the question of breakdown of causality in Hořava-Lifshitz quantum gravity. We have shown that Hořava-Lifshitz gravity only allows the chronology respecting interval of the essential parameters of hyperbolic Gödel-type spacetimes, excluding therefore the noncausal hyperbolic Gödel-type space-times. Thus, there emerges from our results that the well-known noncausal Gödel model is not permitted in context of Hořava-Lifshitz gravity regardless of matter source, since some ADM dynamical variables can only be consistently defined in the chronology preserving parameter interval $m^2 > 4\omega^2$ of hyperbolic Gödel-type space-time family. As a consequence, Gödel metric ($m^2 = 2\omega^2$) cannot be suitably used to define an energy-momentum tensor through Hořava-Lifshitz field equations. This illustrates concretely that the existence of a preferred foliation of space-time brings on a distinctive causal structure in the context of Hořava-Lifshitz gravity. Such a special causal structure puts the violation of causality of the general relativity theory into a new perspective. It should be noted that since we have not used the Hořava-Lifshitz field equations to derive this result, it holds for any theory whose formulation relies on the suitable behavior of the ADM variables of the Gödel-type space-times. However, the fact that these gravity theories do not permit noncausal Gödel-type whose essential parameters $m$ and $\omega$ define noncausal hyperbolic Gödel-type space-time geometries does not signify that space-times such a wormhole, which are seem generically to lead to the creation of time machines, cannot be found in Hořava-Lifshitz gravity [20]. This means that the exclusion of all noncausal hyperbolic Gödel space-times can only be seen as a tiny suggestion that the chronology is protected in Hořava-Lifshitz gravity. The question as to whether Hořava-Lifshitz gravity theory allows hyperbolic Gödel-type solutions in the chronology preserving region of the essential parameters was also examined. We have shown that Hořava-Lifshitz gravity not only excludes the noncausal Gödel model, but also rules out any Gödel-type solutions of the hyperbolic class for physically well-motivated perfect-fluid matter content, which can be taken as the matter source for the Gödel universe in the general relativity theory.

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sequence of alternating causal $[G(r) > 0]$ and noncausal $[G(r) < 0]$ regions in the section $t, z, r = \text{const}$, without and with noncausal circles, depending on the value of $r = \text{const}$. To this end, all we have to do is to determine the behavior of the function $G(r)$ given by equation (12).

We begin by noting that this function has an infinite sequence of zeros $G(r_n) = 0$ ($n = 0, 1, 2, \ldots$) determined by the equations

$$\sin\left(\frac{\mu r_n}{2}\right) = 0 \quad (A.1)$$

and

$$\sin\left(\frac{\mu r_n}{2}\right) = \pm \frac{\mu^{-1} n}{\sqrt{4 \omega^2 + \mu^2}}, \quad (A.2)$$

whose roots are given by

$$r_1^{(n)} = \frac{2 \pi n}{\mu}, \quad n = 0, 1, 2, \ldots \quad (A.3)$$

for Eq. (A.1), and

$$r_2^{(n)} = -\frac{2}{\mu} \frac{\arcsin\left(\frac{\mu}{\sqrt{4 \omega^2 + \mu^2}}\right) - \pi n}{\mu}, \quad n = 1, 2, \ldots, \quad (A.4)$$

$$r_3^{(n)} = \frac{2}{\mu} \arcsin\left(\frac{\mu}{\sqrt{4 \omega^2 + \mu^2}}\right) + \pi n, \quad n = 0, 1, 2, \ldots \quad (A.5)$$

for Eqs. (A.2).

From equations (A.3), (A.4) and (A.5) one has that although the values of the maxima, minima and zeros of $G(r)$ change for different values of the parameters $\mu$ and $\omega$, the general behavior of $G(r)$ (shape of the curve, number of maxima, minima and zeros) does not depend on the specific values of $\mu$ and $\omega$. Thus, for example, for $\mu = 2$ and $\omega = 1/2$ one has that $G(r) = -\frac{1}{2} \sin^2 r$ whose graph is shown in the Figure 1. Different values of the parameters $\mu$ and $\omega$ would give rise to a curve with the similar global pattern but with different values for the minima, maxima and zeros.

Finally, from the above results one obtains the sequence of alternating causal $[G(r) > 0]$ and noncausal $[G(r) < 0]$ regions. Indeed, $G(r) > 0$ for

$$R_1 = \left\{ r \mid r_1^{(0)} \leq r \leq r_3^{(0)} = 1.11 \right\}, \quad (A.6)$$

$$R_n = \left\{ r \mid r_2^{(n-1)} \leq r \leq r_3^{(n-1)} \right\}, \quad n = 2, 3, \ldots, \quad (A.7)$$

and $G(r) < 0$ otherwise.

Finally, it should be noticed that the analysis carried out in this appendix fulfill a minor gap left in Ref. [21] in the study of the trigonometric class of Gödel-type geometries.

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