QUBIT EXCHANGE INTERACTIONS FROM PERMUTATIONS
OF CLASSICAL BITS

Hans-Thomas Elze
Dipartimento di Fisica “Enrico Fermi”, Università di Pisa,
Largo Pontecorvo 3, I-56127 Pisa, Italia
elze@df.unipi.it

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In order to prepare for the introduction of dynamical many-body and, eventually, field
theoretical models, we show here that quantum mechanical exchange interactions in
a three-spin chain can emerge from the deterministic dynamics of three classical Ising
spins. States of the latter form an ontological basis, which will be discussed with ref-
erence to the ontology proposed in the Cellular Automaton Interpretation of Quantum
Mechanics by ’t Hooft. Our result illustrates a new Baker-Campbell-Hausdorff formula
with terminating series expansion.

Keywords: qubit; cellular automaton; Ising model; Baker-Campbell-Hausdorff formula;
ontological state; superposition principle; measurement problem; quantum mechanics

1. Introduction

In distinction to many studies of quantum cellular automata or quantum walks,
be it for their interest in the foundations of (quantum) physics or in (information
processing) applications, in this article we will not assume quantum mechanics as
an ingredient from the beginning. Instead we will further discuss circumstances
when quantum mechanical features can be found in the behaviour of certain kinds
of classical cellular automata.

We recall that ontological states have been proposed to underly quantum and,
a fortiori, classical states of physical objects according to the Cellular Automaton
Interpretation of Quantum Mechanics developed by ’t Hooft. [1]

There is ample motivation to reexamine the foundations of quantum theory in
the light of classical concepts – concerning, above all, determinism and existence
of ontological states of reality. Last not least, the Born rule and the infamous
collapse of quantum states in measurement processes can find a surprising and
intuitive explanation, if quantum states are regarded as mathematical objects. A
different lucid argument supporting such a view has been given, e.g., in a paper by
Rovelli. [2] The quantum states represent mathematically fictitious superpositions
of ontological (micro) states, while classical states are ontological (macro) states,
or probabilistic superpositions thereof, as appropriate for nature’s vast range of
different scales. [1]
While the unification of General Relativity and Quantum Mechanics (QM) has not yet been achieved, it is a widely held belief that it will require drastic modifications of the respective foundations. The Cellular Automaton Interpretation (CAI) presents a comprehensive attempt to unveil a simpler structure beneath QM.

We have recently studied so-called Hamiltonian cellular automata, comprising a large class of arguably very simple Cellular Automata (CA). In particular, we have considered multipartite systems composed of simple two-state subsystems, akin to classical Ising models. They are candidates for ontological models underlying interacting many-body QM or quantum fields.

It must be emphasized that this goes beyond standard quantum theory by deforming structural elements in a specific way, such that textbook theory is recovered by taking a suitable continuum limit. We refer to Refs. for further discussions, as well as to other related attempts.

It may be useful to distinguish here between going beyond, as indicated, and recent reconstructions from various alternative sets of axioms, without changing the contents of quantum theory, see, e.g., Refs. The latter are shedding light on QM by offering new options for experiments, for example, or by allowing to define and study precisely generalized (non-)quantum theories. However, so far, they do not seem to address the possibility that quantum mechanics itself may be based on phenomena beneath that warrant the development of an encompassing theory founded on new principles.

In the following Section 2., we recapitulate the heuristic distinction between ontological, classical, and quantum states that we employ throughout this paper.

We then present, in Section 3., our study of the particular example of three Ising spins evolving linearly by permutations on their 3-bit space of eight states. This simple system fulfills the requirements of an ontological model. Yet we show that its dynamics can be described conveniently by a typical quantum mechanical Hamiltonian, incorporating Heisenberg exchange interactions. An important aspect of the underlying permutations is that they avoid the formation of would-be-quantum superposition states. Surprisingly, this aspect seems very well hidden when the equivalent QM language is used.

We discuss these findings, which can be seen to follow from a new Baker-Campbell-Hausdorff formula, and indicate next steps in this program to build a deterministic classical ontology for QM. Last not least, it will be interesting to explore further how the violation of Bell’s inequality can be understood in such detailed model cases that it is not a prohibitive issue in general has been argued by ’t Hooft.

2. Matters of language – distinguishing ontological, classical and quantum states

To begin with, we emphasize that quantum states here are considered to form part of the mathematical language used and, thus, bear an epistemological character.
They are “templates” for the description of the reality beneath which, according to CAI, consists of ontological states following a deterministic rule of evolution.\footnote{While we do not consider a changing set of states here, this could be of interest when pondering the evolution of the Universe.}

**Definition 2.1.** Ontological States (\(\mathcal{OS}\))

are states that a closed physical system can be in.

The set of all such \(\mathcal{OS}\) may be very large, possibly infinite. For simplicity, we assume that it is denumerable.

The physical reality “out there” comprises no superpositions of \(\mathcal{OS}\). – To construct a theory based on this attribute of \(\mathcal{OS}\), which is essential, may seem wrong, in view of the overwhelming role played by superpositions in quantum theory. However, it must be stressed that quantum superpositions “happen” in the realm of the language used, *i.e.*, the mathematical formalism employed successfully to describe observed phenomena.

Consequently, the \(\mathcal{OS}\) can only evolve by permutations among themselves. Denoting \(\mathcal{OS}\) by \(|A\rangle\), \(|B\rangle\), \(|C\rangle\), \(|D\rangle\), \ldots, \) for example, such a dynamics could be simply represented by:

\[
|A\rangle \to |D\rangle \to |B\rangle \to \ldots .
\]

This kind of evolution is the only possible one, unless the set of states itself changes, *i.e.*, grows or shrinks.\footnote{Furthermore, the Born rule can be related to a counting procedure and conserved two-time function of CA, which generalizes the norm of \(\mathcal{QS}\).}

Formally, we may declare the \(\mathcal{OS}\) to form a fixed orthonormal set, fixed once for all, and define a Hilbert space \(\mathcal{H}\) with respect to this preferred basis. – Diagonal operators on this basis are beables and their eigenvalues characterize physical properties of the states, corresponding to the labels \(A, B, C, \ldots\) used above.

The association of the particular Hilbert space \(\mathcal{H}\) with the set of \(\mathcal{OS}\), then, leads to the following definition.

**Definition 2.2.** Quantum States (\(\mathcal{QS}\))

are superpositions of \(\mathcal{OS}\), formally introduced in \(\mathcal{H}\).

These are templates for doing physics with the help of mathematics. – The amplitudes that specify a \(\mathcal{QS}\) need to be interpreted, when describing experiments in terms of these states. By experience, interpreting amplitudes in terms of probabilities has been an extraordinarily useful invention. Thus, the *Born rule* is built in by definition,\footnote{While we do not consider a changing set of states here, this could be of interest when pondering the evolution of the Universe.} One might consider instead to abandon the proportionality between absolute values squared of complex amplitudes and probabilities, which is not forbidden by any element of quantum theory, however, would unnecessarily complicate the available mathematical tools.\footnote{Furthermore, the Born rule can be related to a counting procedure and conserved two-time function of CA, which generalizes the norm of \(\mathcal{QS}\).}

We anticipate that generally it will not be easy to relate unitary evolution of \(\mathcal{OS}\) by permutations, as in (1), to a more or less familiar looking Hamilton operator,
in particular in the presence of interactions.\textsuperscript{[19,10,11,24,25]} Which motivates the study of the model of Section 3.

In any case, we recognize a good part of the machinery of quantum theory already in place, including the powerful means of unitary transformations in Hilbert spaces, but with the new perspective furnished by the existence of $\mathcal{OS}$ of reality.

Finally, we define \textit{classical states} in relation to $\mathcal{OS}$. – Usually, they are thought to describe certain limiting situations of QM, especially in the presence of environment induced decoherence, \textit{i.e.}, when the object under study is part of a larger interacting system. They have formed the realm of classical physics before the advent of QM. – Within the CAI, however, classical states belong to deterministic macroscopic systems, including billiard balls, pointers of apparatus, planets, etc., \textit{and} are formed of ontological states that are not resolved individually.

\textbf{Definition 2.3.} Classical States ($\mathcal{CS}$) of a closed physical (macro)system are probabilistic distributions of its $\mathcal{OS}$.

Repeatedly performed experiments or any kind of \textit{approximately} repeating evolution of a sufficiently but never completely isolated component of the overall system, say, the Universe, pick up different initial conditions regarding the $\mathcal{OS}$. Therefore, a suitable \textit{classical apparatus} forming part of such situations must generally be expected to yield different pointer positions as outcomes.

Furthermore, then, the probability of a particular outcome directly reflects the probability of having a particular $\mathcal{OS}$ as initial condition. This conservation law, the \textit{Conservation of Ontology}, follows directly from the absence of superpositions “out there” and the evolution of $\mathcal{OS}$ by permutations among themselves.\textsuperscript{[11]} Since, using quantum superpositions of $\mathcal{OS}$ to describe the initial state approximately, as good as possible, we obtain for an evolving $\mathcal{QS} \, |Q\rangle$:

\begin{equation}
|Q\rangle := \alpha |A\rangle + \delta |D\rangle + \ldots , \quad |\alpha|^2 + |\delta|^2 + \ldots = 1 ,
\end{equation}

then,

\begin{equation}
|Q\rangle \longrightarrow \alpha |D\rangle + \delta |B\rangle + \ldots ,
\end{equation}

\textit{i.e.}, the amplitudes remain the initial ones, while the $\mathcal{OS}$ evolve by permutations, \textit{e.g.} as in \textsuperscript{[11]}.

Hence, the reduction or collapse to a $\delta$-peaked distribution of pointer positions, the core of the \textit{measurement problem}, is an apparent effect. It arises due to the intermediary use of \textit{quantum mechanical templates}, in particular superposition states such as $|Q\rangle$, when describing the evolution of what in reality are $\mathcal{OS}$ that differ in different runs of an experiment, either $|A\rangle$ or $|D\rangle$ or \ldots in the example of \textsuperscript{[2]}.

According to CAI, superpositions of $\mathcal{OS}$ do not exist “out there” and, therefore, \textit{no collapse} or reduction to one of their components can occur!\textsuperscript{[11]}

This does, of course, not imply that quantum mechanical superposition states are to be avoided. On the contrary, part of the motivation for CAI and its perspective is to better understand, why they are so extremely effective in describing experiments probing nature.
It is encouraging for attempts to formulate an ontological theory that QM *per se* does not need any stochastic or nonlinear reduction process, which could modify the collapse-free linear unitary evolution. Measurements, according to CAI, are simply *interactions* between the degrees of freedom belonging to an object and those belonging to an apparatus, altogether evolving through ontological states. While the elegance and simplicity of this point of view can hardly be denied, the construction of examples of interacting systems is not entirely straightforward.

3. Permutations of classical bits with a QM Hamiltonian

We considered a discrete dynamical theory before that deviates drastically from quantum theory, at first sight. With the help of Sampling Theory, however, it has been shown that members of this class of Hamiltonian CA are mapped one-to-one to continuum models of nonrelativistic QM, in which a deformation through a new time or length scale enters.

Yet all these models have been one-body models, *i.e.* with forces that are not dynamical but external, described by an external potential in the Hamiltonian. This has led to consider *multipartite systems* in this context, aiming to arrive at interacting discrete many-body or field models.

Presently, we continue in this line, however, refer specifically to an ontological model with \( OS \) that evolve by permutations, as discussed in Section 2.

3.1. Some properties of permutations

Let \( N \) objects, \( A_1, A_2, \ldots, A_N \) ("states"), be mapped in \( N \) steps onto one another, involving all states exactly once. This can be represented by unitary \( N \times N \) matrices with one off-diagonal arbitrary phase per column and row and zero elsewhere.

For example, consider:

\[
\hat{U}_3 := \begin{pmatrix} 0 & e^{-i\phi_1} & 0 \\ 0 & 0 & e^{-i\phi_2} \\ e^{-i\phi_3} & 0 & 0 \end{pmatrix}, \quad \hat{U}_3\hat{U}_3^\dagger = 1 .
\]  

(4)

This permutation matrix has the property:

\[
(\hat{U}_3)^3 = e^{-i(\phi_1+\phi_2+\phi_3)} 1 ,
\]  

(5)

which immediately yields its eigenvalues as three roots of 1, which lie on the unit circle in the complex plane, multiplied by an overall phase.

Similarly, one finds for unitary \( N \times N \) matrices that represent permutations:

\[
(\hat{U}_N)^N = \exp(-i \sum_{k=1}^{N} \phi_k) 1 .
\]  

(6)

Defining the *Hamiltonian* by:

\[
\hat{U}_N := e^{-i\hat{H}_N^T} ,
\]  

(7)
with $T$ introducing a time scale, its eigenvalues follow from Eq. (6). This results in the diagonal form of the Hamiltonian:

$$\hat{H}_{N, \text{diag}} = \text{diag} \left( \frac{1}{NT} \left( \sum_{k=1}^{N} \phi_k + 2\pi n \right) | n = 0, \ldots, N - 1 \right).$$

(8)

In passing we mention that these simple results form the basis of so-called cog-wheel models. These deterministic models show interesting quantum mechanical features, when considered in suitable limiting situations. For $N \to \infty$ and $T \to 0$, with $NT \equiv \omega^{-1}$ fixed, a quantum harmonic oscillator is obtained, while $\omega \to 0$ yields a free quantum particle.

As mentioned, difficulties are encountered, when one tries to introduce interactions among such simplest building blocks, which by themselves end up to represent quantum mechanical one-body systems.

### 3.2. Ising spins as ontological degrees of freedom

Somehow, besides “global” evolution of $\mathcal{OS}$ by permutations, we need to have some additional “local” or internal structure of the states, in order to construct more interesting dynamical models.

Classical two-state Ising spins (Boolean variables or bits) are suitable elementary objects to compose a multipartite CA. For simplicity, we consider only a chain of three coupled Ising spins “1,2,3”, at present.

Interactions among the three spins, leading to permutations among their $2^3$ possible $\mathcal{OS}$, will be generated by spin exchange, a permutation $\hat{P}_{ij} (\equiv \hat{P}_{ji})$ involving two spins, labeled $i,j = 1,2,3$, with the following properties:

$$\hat{P}_{ij} |s_i, s_j\rangle := |s_j, s_i\rangle, \quad \hat{P}_{ji} \hat{P}_{ij} = (\hat{P}_{ij})^2 = 1,$$

(9)

where the states of a single spin are $s_k = \pm 1$ or, graphically, $s_k = \uparrow, \downarrow$, for “spin up, spin down”, respectively; we use the ket notation $|s_i, s_j\rangle$ to indicate that the first spin has value $s_i$, the second value $s_j$, and similarly for all three spins.

If we identify the two states $s_k = \pm 1$ of an Ising spin with the eigenstates of the Pauli matrix $\hat{\sigma}^z$, $\psi_+ = (1,0)^t$ and $\psi_- (0,1)^t$, respectively, then the unitary operator $\hat{P}_{ij}$ can be expressed in terms of the Pauli spin-1/2 matrices:

$$\hat{P}_{ij} = \frac{1}{2} (\hat{\sigma}_i \cdot \hat{\sigma}_j + 1),$$

(10)

where $\hat{\sigma}$ denotes the vector formed by $\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z$. This hints at a relation with QM of qubits, to which we shall come back in the following.

Finally, an elementary example suffices to show:

$$[\hat{P}_{ij}, \hat{P}_{jk}] \neq 0, \quad \text{for } i \neq k,$$

(11)

where no summation over $j$ is implied.
3.3. Dynamics

Our next task is to define the unitary operator \( \hat{U} \) that evolves the state of the three classical spins under consideration in a discrete time step \( T \) and extract the corresponding Hamiltonian \( \hat{H} \), cf. Eqs. (7)-(8). A suitable choice may simply be:

\[
\hat{U} := \hat{P}_{12} \hat{P}_{23} =: \exp(-i\hat{H}T) ,
\]

acting sequentially on the indicated pairs of spins.

An important simplification arises, because the numbers of up and down spins are conserved when the interaction described by Eq. (12) acts on one of the \( \Omega S \). Therefore, we expect \( \hat{U} \) to have a block diagonal structure.

Let us order the eight \( \Omega S \) in the following way: \(|1\rangle := |\uparrow,\uparrow,\uparrow\rangle\), \(|2\rangle := |\uparrow,\uparrow,\downarrow\rangle\), \(|3\rangle := |\uparrow,\downarrow,\uparrow\rangle\), \(|4\rangle := |\downarrow,\uparrow,\uparrow\rangle\), \(|5\rangle := |\downarrow,\downarrow,\downarrow\rangle\), \(|6\rangle := |\downarrow,\uparrow,\downarrow\rangle\), \(|7\rangle := |\uparrow,\downarrow,\downarrow\rangle\), \(|8\rangle := |\downarrow,\downarrow,\uparrow\rangle\). Then, indeed, the update operator \( \hat{U} \) of Eq. (12) can be represented by the following block diagonal 8 \( \times \) 8 matrix:

\[
\hat{U} = \begin{pmatrix}
1 & \hat{U}_3 & \\
\hat{U}_3 & & \\
& & 1
\end{pmatrix},
\]

where \( \hat{U}_3 \) is the unitary 3 \( \times \) 3 matrix defined in (4), with \( \phi_1 = \phi_2 = \phi_3 = 0 \) henceforth; all other matrix elements not explicitly given are zero.

Similarly as before, cf. Eq. (8), since \( \langle U \rangle^3 = 1 \), we immediately obtain also the diagonal form of the Hamiltonian defined in Eqs. (12):

\[
\hat{H}_{\text{diag}} = \frac{2\pi}{3T} \cdot \text{diag} (0, 0, 1, 2, 0, 1, 2, 0) .
\]

We note the degeneracy of the eigenvalues.

Furthermore, due to the simple structure of \( \hat{U} \), it is straightforward to find its eigenstates. The eigenstates corresponding to two of the zero eigenvalues of \( \hat{H}_{\text{diag}} \) are simply the states \(|1\rangle \) and \(|8\rangle \) defined before Eq. (13), with all spins either up or down. Next, we construct the eigenstates of \( \hat{U}_3 \). It is convenient to introduce an auxiliary basis, \(|\alpha\rangle := (1, 0, 0)^t\), \(|\beta\rangle := (0, 1, 0)^t\), \(|\gamma\rangle := (0, 0, 1)^t\). In terms of these orthonormal vectors, one finds the normalized eigenvectors:

\[
|A\rangle = (|\alpha\rangle + |\beta\rangle + |\gamma\rangle)/\sqrt{3} ,
\]

\[
|B\rangle = (|\alpha\rangle + e^{-2\pi i/3}|\beta\rangle + e^{2\pi i/3}|\gamma\rangle)/\sqrt{3} ,
\]

\[
|C\rangle = (|\alpha\rangle + e^{2\pi i/3}|\beta\rangle + e^{-2\pi i/3}|\gamma\rangle)/\sqrt{3} .
\]

which obey \( \hat{U}_3|A\rangle = |A\rangle, \hat{U}_3|B\rangle = e^{-2\pi i/3}|B\rangle, \hat{U}_3|C\rangle = e^{4\pi i/3}|C\rangle \), with the correct eigenvalues of \( \hat{U}_3 \). The unitary diagonalizing matrix \( \hat{D} \), which maps \{\(|\alpha\rangle, |\beta\rangle, |\gamma\rangle\} \) to \{\(|A\rangle, |B\rangle, |C\rangle\} \), is given by:

\[
\hat{D} = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & e^{-2\pi i/3} & e^{2\pi i/3} \\
1 & e^{2\pi i/3} & e^{-2\pi i/3}
\end{pmatrix} .
\]
It serves us to map part of the diagonalized Hamiltonian, which corresponds to one of the $\hat{U}_3$ blocks, namely $\hat{H}_{3, \text{diag}} := \text{diag}(0, 1, 2) \cdot \frac{2\pi}{3T}$, back to the corresponding part $\hat{H}_{\text{aux}}$ of $\hat{H}$ which acts on the auxiliary basis $\{|\alpha\rangle, |\beta\rangle, |\gamma\rangle\}$:

$$\hat{H}_{\text{aux}} = \hat{D}^\dagger \hat{H}_{3, \text{diag}} \hat{D}$$

$$= \frac{2\pi}{3T} \begin{pmatrix} 1 & c & c^* \\ c^* & 1 & c \\ c & c^* & 1 \end{pmatrix}, \quad \text{with} \quad c := -\frac{1}{2} + \frac{i}{2\sqrt{3}}.$$

Note the permutation of entries between one row or column and the next. This Hamiltonian, of course, appeared also in the analysis of the $N = 3$ cogwheel model.\footnote{1}

The crucial step, in order to arrive at the Hamiltonian $\hat{H}$ of Eq. (12), is to identify the auxiliary states with corresponding OS, which were listed before Eq. (13), and to express $\hat{H}_{\text{aux}}$ in terms of operators acting on the three Ising spins “1,2,3”, of which the ontological states are composed. This works as follows.

Let us identify $|\alpha\rangle \equiv |\uparrow, \uparrow, \downarrow\rangle$, $|\beta\rangle \equiv |\uparrow, \down, \uparrow\rangle$, and $|\gamma\rangle \equiv |\down, \up, \up\rangle$; we recall that in the last case, e.g., the notation means spin “1” down, spins “2” and “3” up. Thus, with Eq. (17) the following identification is obtained, for example:

$$\hat{H}_{\text{aux}} |\beta\rangle = \frac{2\pi}{3T} (c|\alpha\rangle + |\beta\rangle + c^*|\gamma\rangle)$$

$$\equiv \frac{2\pi}{3T} (c\hat{P}_{23} + 1 + c^*\hat{P}_{12}) |\uparrow, \down, \up\rangle,$$  \hspace{1cm} (18)

using the spin exchange operators introduced in Eq. (9); similar identifications follow for the other two members of the auxiliary basis,

$$\hat{H}_{\text{aux}} |\alpha\rangle \equiv \frac{2\pi}{3T} (1 + c^*\hat{P}_{23} + c\hat{P}_{13}) |\uparrow, \up, \down\rangle,$$  \hspace{1cm} (19)

$$\hat{H}_{\text{aux}} |\gamma\rangle \equiv \frac{2\pi}{3T} (c^*\hat{P}_{13} + c\hat{P}_{12} + 1) |\down, \up, \up\rangle,$$  \hspace{1cm} (20)

and, correspondingly, for an auxiliary Hamiltonian related to the second $3 \times 3$ block entering $\hat{U}$ in Eq. (13).

Finally, we recall that the rows or columns of $\hat{H}_{\text{aux}}$ are simply related by cyclic permutations. For a generic OS, say $|x, y, z\rangle$, with $x, y, z = \uparrow, \down$, such permutations can be represented by:

$$\hat{P}_{13}\hat{P}_{23}|x, y, z\rangle = |y, z, x\rangle,$$

$$\hat{P}_{13}\hat{P}_{23}^2|x, y, z\rangle = |z, x, y\rangle,$$ \hspace{1cm} (21)

and $(\hat{P}_{13}\hat{P}_{23})^3 = 1$. This allows to write the $(8 \times 8$ matrix) Hamiltonian $\hat{H}$ of Eq. (12), which incorporates especially Eqs. (18)-(20) but acts on the eight OS, in a concise form:

$$\hat{H} = \frac{2\pi}{3T} (1 + c^*\hat{P}_{13}\hat{P}_{23} + c(\hat{P}_{13}\hat{P}_{23})^2)$$

$$= \frac{2\pi}{3T} (1 + c\hat{P}_{23}\hat{P}_{13} + c^*\hat{P}_{13}\hat{P}_{23}),$$ \hspace{1cm} (22)
which is evidently self-adjoint; note that \((\hat{P}_{13}\hat{P}_{23})^\dagger = \hat{P}_{23}\hat{P}_{13} = (\hat{P}_{13}\hat{P}_{23})^2\). Furthermore, it is noteworthy that all parts of \(\hat{H}\) commute with each other, since 
\[ [\hat{P}_{23}\hat{P}_{13}, \hat{P}_{13}\hat{P}_{23}] = 1 - 1 \]  
in particular.

The explicit form of the Hamiltonian, Eq. (22), presents our main result. In the following, we will discuss some of its implications.

3.4. A Baker-Campbell-Hausdorff formula with a qubit Hamiltonian for bits

Here, we first look at an interesting formal aspect of our result describing the dynamics by permutations of \(\mathcal{O}_\mathcal{S}\) by a Hamilton operator. Returning to Eq. (12), together with Eq. (9), we see that the two permutations composing the evolution operator \(\hat{U}\) can be exponentiated separately as follows:
\[
\hat{P}_{ij} = i \exp(-\frac{i\pi}{2}\hat{P}_{ij}),
\]  
using \((\hat{P}_{ij})^2 = 1\). However, since \([\hat{P}_{12}, \hat{P}_{23}] \neq 0\), we cannot evaluate the Hamiltonian \(\hat{H}\) by simply adding the exponents obtained with the help of Eq. (23). To put it differently, we would like to know, what is
\[
-i\hat{H}T = \log(\hat{P}_{12}\hat{P}_{23}) ,
\]
for two noncommuting permutations acting on three Ising spins (or bits).

This kind of algebraic problem with noncommuting operators is familiar from QM or Lie group theory. Let \(\exp(X) \exp(Y) = \exp(Z)\). Then, a formal solution for \(Z\) in terms of \(X, Y\) is provided by the Baker-Campbell-Hausdorff formula (BCH):
\[
Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) - \frac{1}{24}[Y, [X, [X, Y]]] + \ldots ,
\]  
i.e., a series expansion in terms of increasingly complicated iterated commutators. The coefficients of the series are known, yet it is generally difficult to ascertain its convergence. Several exceptional cases are known when the series terminates. In recent works by Visser et al. and by Matone, with references to earlier work, some interesting new classes of such finite solutions have been constructed. 36, 37

Instead, in the preceding section, rather elementary considerations have lead us to results which can be summarized by the new terminating BCH formula:
\[
i^2 \exp\left(-i\frac{\pi}{2}\hat{P}_{12}\right) \exp\left(-i\frac{\pi}{2}\hat{P}_{23}\right) = \exp\left(-i\frac{2\pi}{3}(1 + c\hat{P}_{23}\hat{P}_{13} + c^*\hat{P}_{13}\hat{P}_{23})\right) ,
\]  
using Eqs. (12), (22), (23), and with \(c\) from (17); the coefficients of \(\pi/2\) in this formula can be modified by adding integer multiples of \(2\pi\), without changing the result. – To obtain this result from the general BCH formula (24), does not seem impossible but rather complicated.

In passing we mention that the right-hand side of Eq. (25) can be factorized:
\[
\ldots = \exp\left(-i\frac{2\pi}{3}\right) \exp\left(-i\frac{2\pi}{3}c\hat{P}_{23}\hat{P}_{13}\right) \exp\left(-i\frac{2\pi}{3}c^*\hat{P}_{13}\hat{P}_{23}\right) ,
\]  
(26)
since all terms in the exponent commute.

It will be interesting to find out, how our derivation can be generalized, e.g., for more than three Ising spins (or bits).

Next, we recall the relation between exchange operations (permutations) and Pauli matrices, Eq. (10), \( \hat{P}_{ij} = (\hat{\sigma}_i \cdot \hat{\sigma}_j + 1)/2 \). If we now allow the Ising spins, on which the Hamiltonian \( \hat{H} \) of Eq. (22) acts, to be embedded into the larger Hilbert space of three qubits, then \( \hat{H} \) expressed in terms of the appropriate Pauli matrices can be considered as a genuine quantum mechanical operator.

This brings about an interesting situation: Equations like (12) together with (22), as well as Eq. (24), are only valid with their precisely determined numerical coefficients. Furthermore, despite the quantum mechanical appearance of the Hamiltonian, the operator \( \hat{U} \) describes the evolution of ontological states. In particular, no superpositions of OS are produced, as it should be according to the discussion in Section 2.

Then, let us envision a realistic model of some sufficiently isolated part of the Universe which works along these lines – and someone in search of such an ontological model. By necessarily limited experimental means this physicist will not be able to determine all relevant dimensionless (coupling or charge) constants precisely.

In analogy to what would happen in the case of our present toy model, a resulting approximate Hamiltonian, except if it is diagonal and rather uninteresting, will produce unphysical superpositions of ontological states!

This can be illustrated simply, e.g., by perturbing the right-hand side of Eq. (23):

\[
i \exp\left(-i\frac{\pi}{2} (1 + \epsilon) \hat{P}_{ij}\right) = \hat{P}_{ij} - i \frac{\pi}{2} \epsilon \cdot 1 + O(\epsilon^2), \quad 0 < \epsilon \ll 1.
\]

The resulting sum of terms unavoidably creates superposition states, which lie in a Hilbert space for qubits rather than concerning Ising spins (or bits). Similarly, any approximation to the Hamiltonian of Eq. (22) or perturbation of the BCH formula (25) likely will produce apparent QM effects.

We conclude that an only approximately known ontological Hamiltonian must lead to misinterpretation, namely that the system under study behaves quantum mechanically, due to the presence of superpositions of OS.

A surprise worth further exploration.

4. Conclusions

We discuss the notion of Ontological States (OS) in the context of a composite system consisting of three classical Ising spins (or bits) and its dynamics. The OS have been introduced as the basis on which models possibly underlying quantum mechanical ones must be built.\(^{11}\)

\(^{11}\)Symmetry arguments should help. While still little is known or can justifiably be assumed about symmetry principles that operate at the level of OS, ’t Hooft’s discussion of “Demands and Rules” that an ontological model of the Universe should obey presents steps in this direction.\(^{38}\)
Characteristic for OS – the states a physical system can be in – is that they evolve deterministically by permutations among themselves, since the familiar superposition states appearing in QM belong to the mathematical theory describing experimental findings, but are not considered to exist “out there”.

Single-component Cellular Automata that we have studied earlier allowed to reconstruct the dynamics of one-body models in QM with external potentials in terms of deterministic ones that are characterized by a finite discreteness scale.

One is led to ask, whether in this setting, generally, there is room for OS and their particular permutation dynamics when it comes to interacting multipartite systems, i.e. many-body systems or fields.

To progress towards such more interesting complex situations, we propose to begin with systems composed of two-state subsystems, such as Ising spins. In particular, we presently study a three-spin chain and its discrete dynamics generated by permutations among the $2^3 = 8$ possible OS.

We show that this opens new possibilities, since the unitary evolution of the system can be described by a Hamilton operator, which appears to be of quantum mechanical kind. It incorporates spin exchange interactions in a nontrivial way.

Nevertheless, this evolution law with the derived exact Hamiltonian does not produce superpositions of OS. This is implied by a new Baker-Campbell-Hausdorff formula with finitely many terms that we obtain. It describes the underlying permutations by the exponential of an equivalent Hamiltonian.

We discuss that any approximation of the Hamiltonian, an inaccuracy of the fixed coupling constants in particular, would unavoidably lead from OS to superposition states and, thus, generate, typical quantum mechanical behaviour, of qubits in the present case. Which naturally opens the way for some speculations.

It should be interesting to extend our study to multipartite systems, e.g., by dividing a given number of Ising spins in two sets and ask under what conditions, or rather approximations, one can observe entanglement between subsystems. We emphasize that presently the discrete configuration space of ontological degrees of freedom (Ising spins or bits) is strictly smaller than the corresponding continuous Hilbert space of qubits. As we discussed, the latter is opened up ‘by mistake’ – namely, when the effects of QM arise due to an inaccurate treatment of the ontological dynamics generated by permutations. This means, by approximating the exact BCH formula. We leave this important question for future work.

However, to distinguish the role of ontic vs. epistemic (QM) aspects in complex situations can be very relevant, for example, when the formalism of quantum theory is successfully applied to situations that traditionally fall outside of physics.

They are known, e.g., in the Heisenberg model of ferromagnetism.
ently be combined in one dynamical scheme deserves reconsideration in the light of the present discussion, see Ref. [42] with references to earlier work.

Next steps to be considered should include the extension to larger systems, in order to see whether the present results can be generalized in a straightforward way, with respect to dimensionality, but also incorporating the finite maximal signal velocity demanded by special relativity. Furthermore, the prerequisites for models with internal symmetries need to be understood.

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