Multiple DP-Coloring of Planar Graphs Without 3-Cycles and Normally Adjacent 4-Cycles

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Abstract
The concept of DP-coloring of a graph is a generalization of list coloring introduced by Dvořák and Postle (J. Combin. Theory Ser. B 129, 38–54, 2018). Multiple DP-coloring of graphs, as a generalization of multiple list coloring, was first studied by Bernshteyn, Kostochka and Zhu (J. Graph Theory 93, 203–221, 2020). This paper proves that planar graphs without 3-cycles and normally adjacent 4-cycles are $(7m, 2m)$-DP-colorable for every integer $m$. As a consequence, the strong fractional choice number of any planar graph without 3-cycles and normally adjacent 4-cycles is at most $7/2$.

Keywords  DP-coloring · Fractional coloring · Strong fractional choice number · Planar graph · Cycles

Mathematics Subject Classification 05C10 · 05C15

1 Introduction

A $b$-fold coloring of a graph $G$ is a mapping $\varphi$ which assigns to each vertex $v$ a set $\varphi(v)$ of $b$ colors so that adjacent vertices receive disjoint color sets. An $(a, b)$-coloring of $G$ is a $b$-fold coloring $\varphi$ of $G$ such that $\varphi(v) \subseteq \{1, 2, \ldots, a\}$ for each vertex $v$.

The fractional chromatic number of $G$ is

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\[
\chi_f(G) = \inf \left\{ \frac{a}{b} : G \text{ is } (a, b) \text{-colorable} \right\}.
\]

An \textit{a-list assignment} of \(G\) is a mapping \(L\) which assigns to each vertex \(v\) a set \(L(v)\) of \(a\) permissible colors. A \textit{b-fold L-coloring} of \(G\) is a \(b\)-fold coloring \(\varphi\) of \(G\) such that \(\varphi(v) \subseteq L(v)\) for each vertex \(v\). We say \(G\) is \((a, b)\)-\textit{choosable} if for any \(a\)-list assignment \(L\) of \(G\), there is a \(b\)-fold \(L\)-coloring of \(G\). The \textit{choice number} of \(G\) is

\[
ch(G) = \min \{ a : G \text{ is } (a, 1) \text{-choosable} \}.
\]

The \textit{fractional choice number} of \(G\) is

\[
ch_f(G) = \inf \{ r : G \text{ is } (a, b) \text{-choosable for some positive integers } a, b \text{ with } a/b = r \}.
\]

The \textit{strong fractional choice number} of \(G\) is

\[
ch^*_f(G) = \inf \{ r : G \text{ is } (a, b) \text{-choosable for all positive integers } a, b \text{ with } a/b \geq r \}.
\]

It was proved by Alon, Tuza and Voigt [1] that for any finite graph \(G\), \(\chi_f(G) = ch_f(G)\) and moreover the infimum in the definition of \(ch_f(G)\) is attained and hence can be replaced by minimum. So the fractional choice number \(ch_f(G)\) of a graph is not a new invariant. On the other hand, the concept of strong fractional choice number, introduced in [8], was intended to be a refinement of \(ch(G)\). It follows from the definition that \(ch^*_f(G) \geq ch(G) - 1\). However, it remains an open question whether \(ch^*_f(G) \leq ch(G)\).

For a family \(\mathcal{G}\) of graphs, let

\[
ch(G) = \max \{ ch(G) : G \in \mathcal{G} \}, ch_f(G) = \sup \{ ch_f(G) : G \in \mathcal{G} \},
\]

\[
ch^*_f(G) = \sup \{ ch^*_f(G) : G \in \mathcal{G} \}.
\]

We denote by \(\mathcal{P}\) the family of planar graphs, and by \(\mathcal{P}_\Delta\) the family of triangle free planar graphs. It is known that \(ch(\mathcal{P}) = 5\), \(ch(\mathcal{P}_\Delta) = 4\), \(ch_f(\mathcal{P}) = 4\) and \(ch_f(\mathcal{P}_\Delta) = 3\). It is easy to see that \(ch^*_f(\mathcal{P}) \leq 5\) and \(ch^*_f(\mathcal{P}_\Delta) \leq 4\), and these are the best known upper bounds for \(ch^*_f(\mathcal{P})\) and \(ch^*_f(\mathcal{P}_\Delta)\), respectively. The best known lower bounds for \(ch^*_f(\mathcal{P})\) and \(ch^*_f(\mathcal{P}_\Delta)\) are obtained in [6, 7] respectively:

\[
ch^*_f(\mathcal{P}) \geq 4 + 1/3, \quad ch^*_f(\mathcal{P}_\Delta) \geq 3 + \frac{1}{17}.
\]

It would be interesting to find better upper or lower bounds for \(ch^*_f(\mathcal{P})\) and \(ch^*_f(\mathcal{P}_\Delta)\).

In particular, the following questions remain open:

**Question 1** Is it true that every planar graph is \((9, 2)\)-choosable?

**Question 2** Is it true that every triangle free planar graph is \((7, 2)\)-choosable?

It follows from the Four Color Theorem that every planar graph is \((4m, m)\)-colorable for any positive integer \(m\). However, the problem of proving every planar graph
is (9, 2)-colorable without using the Four Color Theorem remained open for a long time, before it was done by Cranston and Rabern in 2018 [3]. As a weaker version of Question 1, it was proved by Han, Kierstead and Zhu [5] that every planar graph \( G \) is 1-defective (9, 2)-paintable (and hence 1-defective (9, 2)-choosable), where a 1-defective coloring is a coloring in which each vertex \( v \) has at most one neighbor colored the same color as \( v \).

This paper studies a variation of Question 2. We consider a more restrictive family of graphs: the family of planar graphs without 3-cycles and without normally adjacent 4-cycles, where two 4-cycles are said to be normally adjacent if they share exactly one edge. We prove a stronger conclusion for this family of graphs, i.e., all graphs in this family are \((7m, 2m)\)-DP-colorable for all positive integer \( m \).

Definition 1 Let \( G \) be a graph. A cover of \( G \) is a pair \( \mathcal{H} = (L, H) \), where \( H \) is a graph and \( L : V(G) \to \text{Pow}(V(H)) \) is a function, with the following properties:

- The sets \( \{L(u) : u \in V(G)\} \) form a partition of \( V(H) \).
- \( E(H) = \bigcup_{(u,v) \in E(G)} E_H(L(u), L(v)) \), where \( E_H(L(u), L(v)) \) is a matching (not necessarily perfect and possibly empty).

Note that for a cover \( (L, H) \) of \( G \) defined above, for each vertex \( v \) of \( G \), \( L(v) \) is an independent set in \( H \). This is different from the definition in [2], where \( L(v) \) induces a clique for each vertex \( v \) of \( G \).

We denote by \( \mathbb{N} \) the set of non-negative integers. For a set \( X \), denote by \( \mathbb{N}^X \) the set of mappings \( f : X \to \mathbb{N} \). For a graph \( G \), we write \( \mathbb{N}^G \) for \( \mathbb{N}^{V(G)} \).

For \( f, g \in \mathbb{N}^G \), we write \( g \leq f \) if \( g(v) \leq f(v) \) for each vertex \( v \) of \( G \), and let \( f + g \in \mathbb{N}^G \) be defined as \( (f + g)(v) = f(v) + g(v) \) for each vertex \( v \) of \( G \). If \( G' \) is a subgraph of \( G \), \( f \in \mathbb{N}^G \), \( g \in \mathbb{N}^{G'} \), we write \( g \leq f \) if \( g(v) \leq f(v) \) for each vertex \( v \) of \( G' \).

For \( f \in \mathbb{N}^G \), an \( f \)-cover of \( G \) is a cover \( (L, H) \) of \( G \) with \( |L(v)| = f(v) \) for each vertex \( v \).

Definition 2 Let \( G \) be a graph and let \( \mathcal{H} = (L, H) \) be a cover of \( G \). An \( \mathcal{H} \)-coloring of \( G \) is a mapping \( \varphi : V(G) \to V(H) \) such that \( \varphi(v) \in L(v) \) and \( \{\varphi(v) : v \in V(G)\} \) is an independent set of \( H \). If for every \( f \)-cover \( \mathcal{H} \) of \( G \), there is an \( \mathcal{H} \)-coloring of \( G \), then we say \( G \) is \( \mathcal{H} \)-\( f \)-colorable. We say \( G \) is \( \mathcal{H} \)-\( k \)-colorable if \( G \) is \( \mathcal{H} \)-\( f \)-colorable for the constant mapping \( f \) with \( f(v) = k \) for all \( v \). The \( \mathcal{H} \)-chromatic number of \( G \) is defined as

\[
\chi_{\mathcal{H}D}(G) = \min \{ k : G \text{ is } \mathcal{H} \text{-} k \text{-colorable} \}.
\]
We may think of $L(v)$ as the list of colors assigned to $v$. In an $\mathcal{H}$-coloring of $G$, instead of forbidding two adjacent vertices $u, v$ receiving a same color, we require that the color assigned to $u$ is not adjacent to the color assigned to $v$ in $H$. As the adjacency between $L(u)$ and $L(v)$ can be an arbitrary matching, one can impose more varieties of restrictions on the coloring than list coloring. Thus $ch(G) \leq \chi_{DP}(G)$ for each graph $G$, and it is known that the difference $\chi_{DP}(G) - ch(G)$ can be arbitrarily large.

Multiple DP-coloring of graphs was first studied in [2].

**Definition 3** Assume $\mathcal{H} = (L, H)$ is a cover of $G$ and $g \in \mathbb{N}^G$. An $(\mathcal{H}, g)$-coloring is a mapping $\varphi$ which assigns to each vertex $v$ of $G$ a subset $\varphi(v)$ of $L(v)$ such that $|\varphi(v)| = g(v)$ and $\bigcup_{v \in V(G)} \varphi(v)$ is an independent set in $H$. We say $G$ is $(\mathcal{H}, g)$-colorable if there exists an $(\mathcal{H}, g)$-coloring of $G$. We say graph $G$ is $(f, g)$-DP-colorable if for any $f$-cover $\mathcal{H}$ of $G$, $G$ is $(\mathcal{H}, g)$-colorable. If $f,g \in \mathbb{N}^G$ are constant maps with $g(v) = b$ and $f(v) = a$ for all $v \in V(G)$, then $(\mathcal{H}, g)$-colorable is called $(\mathcal{H}, b)$-colorable, and $(f, g)$-DP-colorable is called $(a, b)$-DP-colorable.

The definitions of a cover $\mathcal{H}$ of $G$ and an $(\mathcal{H}, g)$-coloring of $G$ are slightly different from the definitions in [2], but they are equivalent.

**Definition 4** The fractional DP-chromatic number, $\chi_{DP}^*(G)$, of $G$ is defined in [2] as

$$\chi_{DP}^*(G) = \inf \{ r : G \text{ is } (a, b) \text{-DP-colorable for some } a/b = r \}. $$

We define the strong fractional DP-chromatic number as

$$\chi_{DP}^{**}(G) = \inf \{ r : G \text{ is } (a, b) \text{-DP-colorable for every } a/b \geq r \}. $$

As $(a, b)$-DP-colorable implies $(a, b)$-choosable, for any graph $G$,

$$ch_f(G) \leq \chi_{DP}^*(G) \text{ and } ch_f^*(G) \leq \chi_{DP}^{**}(G). $$

It follows from the definition that

$$\chi_{DP}^*(G) \leq \chi_{DP}(G) \text{ and } \chi_{DP}^{**}(G) \geq \chi_{DP}(G) - 1. $$

It was proved in [2] that there are large girth graphs $G$ with $\chi(G) = d$ and $\chi_{DP}^*(G) \leq d/\log d$. As $\chi_{DP}(G) \geq ch(G) \geq \chi(G)$, the difference $\chi_{DP}^{**}(G) - \chi_{DP}^*(G)$ can be arbitrarily large.

The following is the main result of this paper.

**Theorem 1** Let $G$ be a triangle free planar graph without normally adjacent $C_4$. Then $G$ is $(7m, 2m)$-DP-colorable for every integer $m$.

**Corollary 1** If $G$ is a triangle free planar graph without normally adjacent $C_4$, then $ch_f^*(G) \leq \chi_{DP}^{**}(G) \leq 7/2$. 

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The following notations will be used in the remainder of this paper. Assume $G$ is a graph. A $k$-vertex ($k^+$-vertex, $k^-$-vertex, respectively) is a vertex of degree $k$ (at least $k$, at most $k$, respectively). A $k$-face, $k^-$-face or a $k^+$-face is a face of degree $k$, at most $k$ or at least $k$, respectively. The notions of $k$-neighbor, $k^+$-neighbor, $k^-$-neighbor are defined similarly. Two faces are intersecting (respectively, adjacent or normally adjacent) if they share at least one vertex (respectively, at least one edge or exactly one edge). For a face $F \in F(G)$, if the vertices on $F$ in a cyclic order are $v_1, v_2, \ldots, v_k$, then we write $F = [v_1v_2 \ldots v_k]$, and call $F$ a $(d(v_1), d(v_2), \ldots, d(v_k))$-face.

We use the following conventions in this paper:

1. For any $f$-cover $\mathcal{H} = (L, H)$ of a graph $G$, for any edge $e = uv$ of $G$ with $f(u) \leq f(v)$, we assume that the matching between $L(u)$ and $L(v)$ has $f(u)$ edges, and hence saturates $L(u)$, because adding edges to the matching only makes it more difficult to color the graph.
2. If the vertices of a graph $G$ is labelled as $v_1, v_2, \ldots, v_n$, then a mapping $f \in \mathbb{N}^G$ will be given as an integer sequence $(f(v_1), \ldots, f(v_n))$.
3. For an $f$-cover $\mathcal{H} = (L, H)$ of a graph $G$, an induced subgraph $H'$ of $H$ defines an $f'$-cover $\mathcal{H}' = (L', H')$ of $G$, where for each vertex $v$, $L'(v) = L(v) \cap V(H')$ and $f'(v) = |L'(v)|$.

2 Strongly Extendable Coloring of a Subset

Assume $G$ is a graph, $f, g \in \mathbb{N}^G$, $X$ is a subset of $V(G)$, $\mathcal{H} = (L, H)$ is an $f$-cover of $G$. By considering restriction of these mappings, we shall treat $\mathcal{H}$ as an $f$-cover of $G[X]$. Hence we can talk about $(\mathcal{H}, g)$-coloring of $G[X]$.

Assume $G$ is a graph and $X$ is a vertex cut-set. If $G_1, G_2$ are induced subgraphs of $G$ such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = X$, then we say $G_1, G_2$ are the components of $G$ separated by $X$.

In an inductive proof, if every $(\mathcal{H}, g)$-coloring of $X$ can be extended to an $(\mathcal{H}, g)$ -coloring of $G_2$, then we can first find an $(\mathcal{H}, g)$-coloring of $G_1$, and then extend it to $G_2$ to obtain an $(\mathcal{H}, g)$-coloring of the whole graph. In our proofs below, usually $G_2$ do not have the property that every $(\mathcal{H}, g)$-coloring of $G[X]$ can be extended to an $(\mathcal{H}, g)$-coloring of $G_2$. Nevertheless, every $(\mathcal{H}, g)$-coloring $\varphi$ of $G[X]$ satisfying the property that $\varphi(v) \geq h(v)$ for some pre-chosen subsets $h(v)$ can be extended to an $(\mathcal{H}, g)$-coloring of $G_2$. In many cases, this property is enough for the induction to be carried out. This technique is frequently used in the proofs below. We first give a precise definition of the desired property.

Assume $\varphi$ is an $(\mathcal{H}, g)$-coloring of $G[X]$ and $\varphi'$ is an $(\mathcal{H}, g)$-coloring of $G$. If $\varphi'(v) = \varphi(v)$ for each vertex $v \in X$, then we say $\varphi'$ is an extension of $\varphi$. We say $\varphi$ is $(\mathcal{H}, g)$-extendable if there exists an $(\mathcal{H}, g)$-coloring of $G$ which is an extension of $\varphi$ to $G$. 

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Assume $G$ is a graph, $f, h, h' \in \mathbb{N}^G$, $h \leq h' \leq f$. Let $\mathcal{H} = (L, H)$ be an $f$-cover of $G$. Assume $\varphi$ is an $(\mathcal{H}, h)$-coloring of $G$. An $h'$-augmentation of $\varphi$ is an $(\mathcal{H}, h')$-coloring $\varphi'$ of $G$ such that $\varphi(v) \subseteq \varphi'(v)$ for each vertex $v \in V(G)$.

Definition 6 Assume $G$ is a graph, $X$ is a subset of $V(G)$, $f, g, h \in \mathbb{N}^G$ and $h \leq g \leq f$. Assume $\mathcal{H} = (L, H)$ is an $f$-cover of $G$. An $(\mathcal{H}, h)$-coloring $\varphi$ of $G[X]$ is called strongly $(\mathcal{H}, g)$-extendable if

- $\varphi$ has a $g$-augmentation.
- every $g$-augmentation of $\varphi$ is $(\mathcal{H}, g)$-extendable.

We say $(f, h)$ is strongly $(f, g)$-extendable from $X$ to $G$, written as $(f, h)_X \leq (f, g)_G$,

if for any $f$-cover $\mathcal{H} = (L, H)$ of $G$, there exists a strongly $(\mathcal{H}, g)$-extendable $(\mathcal{H}, h)$-coloring of $G[X]$.

The following lemma illustrates how the concept of strongly reducible coloring of an induced subgraph can be used to prove the $(f, g)$-DP-colorability of a graph.

Lemma 1 Assume $G$ is a graph, $X$ is a cut-set of $G$ and $G_1, G_2$ are components of $G$ separated by $X$. Assume $f, g, h \in \mathbb{N}^G$ and $h \leq g \leq f$. Let $f', g' \in \mathbb{N}^G$ be defined as follows:

1. $f'(v) = f(v) - \sum_{u \in N_G(v) \cap X} h(u)$ for $v \in V(G_2)$, and $f'(v) = f(v)$ for $v \notin V(G_2)$.
2. $g'(v) = g(v) - h(v)$ for $v \in X$, and $g'(v) = g(v)$ for $v \notin X$.

If $(f, h)_X \leq (f, g)_{G_1}$ and $G_2$ is $(f', g')$-DP-colorable, then $G$ is $(f, g)$-DP-colorable.

Proof Let $\mathcal{H} = (L, H)$ be an $f$-cover of $G$. Since $(f, h)_X \leq (f, g)_{G_i}$, there exists an $(\mathcal{H}, h)$-coloring $\varphi$ of $G[X]$, such that any $g$-augmentation $\varphi'$ of $\varphi$ can be extended to an $(\mathcal{H}, g)$-coloring of $G_1$.

Let $H' = H - \cup_{v \in X} N_H[\varphi(v)]$. It is straightforward to verify that $\mathcal{H}' = (L', H')$ is an $f'$-cover of $G_2$. Since $G_2$ is $(f', g')$-DP-colorable, there exists an $(\mathcal{H}', g')$-coloring $\psi$ of $G_2$.

For $v \in X$, let $\psi'(v) = \psi(v) \cup \varphi(v)$. Then $\psi'$, as a coloring of $G[X]$, is a $g$-augmentation of $\varphi$, and hence can be extended to an $(\mathcal{H}, g)$-coloring of $G_1$, which we also denote by $\psi'$. Then $\psi''$ defined as

$$
\psi''(v) = \begin{cases} 
\psi'(v), & \text{if } v \in V(G_1), \\
\psi(v), & \text{if } v \notin V(G_1) 
\end{cases}
$$

is an $(\mathcal{H}, g)$-coloring of $G$. \qed
In the formula \((f, h)_X \preceq (f, g)_G\), if \(h\) or \(g\) is a constant function, then we replace it by a constant. For example, we write \((f, b)_X \preceq (f, a)_G\) for \((f, h)_X \preceq (f, g)_G\) where \(h(v) = b\) for \(v \in X\) and \(g(v) = a\) for \(v \in V(G)\).

Note that in the statement \((f, h)_X \preceq (f, g)_G\), the values of \(h(v)\) for \(v \notin X\) are irrelevant.

**Lemma 2** Assume \(G\) is a graph, \(X\) is a subset of \(V(G)\), \(f, g, h, h' \in \mathbb{N}_G\) and \(h \leq h' \leq g \leq f\). Then

\[(f, h)_X \preceq (f, g)_G \Rightarrow (f, h')_X \preceq (f, g)_G.\]

If \(X'\) is a subset of \(X\), then

\[(f, h)_X \preceq (f, g)_G \Rightarrow (f, h)_X' \preceq (f, g)_G.\]

**Proof** Assume \(\mathcal{H} = (L, H)\) is an \(f\)-cover of \(G\) and \(\varphi\) is a strongly \((\mathcal{H}, g)\)-extendable \((\mathcal{H}, h)\)-coloring of \(G[X]\). Since \(\varphi\) has a \(g\)-augmentation, there is an \(h'\)-augmentation \(\varphi'\) of \(\varphi\). As any \(g\)-augmentation of \(\varphi'\) extends to a \(g\)-augmentation of \(\varphi\), we conclude that every \(g\)-augmentation of \(\varphi'\) is \((\mathcal{H}, g)\)-extendable. Hence \((f, h')_X \preceq (f, g)_G\).

The second half of the lemma is proved similarly and is omitted. \(\square\)

Note that for any \(h \leq g \leq f \in \mathbb{N}_G\), \(X \subseteq V(G)\),

\[(f, h)_X \preceq (f, g)_G\]

implies that \(G\) is \((f, g)\)-DP-colorable.

**Lemma 3** Assume \(G\) is a graph, \(X\) is a cut-set of \(G\) and \(G_1, G_2\) are components of \(G\) separated by \(X\). Assume \(X_i \subseteq V(G_i)\), \(X \subseteq X_i\), \(f, g, h_1, h_2 \in \mathbb{N}_G\), and for \(i = 1, 2\), \(h_i(v) = 0\) for \(v \notin X_i\). If \(h_1 + h_2 \leq g\), then

\[(f, h_1)_X \preceq (f, g)_{G_1}\] and \[(f, h_2)_X \preceq (f, g)_{G_2} \Rightarrow (f, h_1 + h_2)_{X_i \cup X_2} \preceq (f, g)_G.\]

**Proof** Assume \(\mathcal{H} = (L, H)\) is an \(f\)-cover of \(G\) and for \(i = 1, 2\), \(\varphi_i\) is an \((\mathcal{H}, h_i)\)-coloring of \(G[X_i]\) which is strongly \((\mathcal{H}, g)\)-extendable to \(G_i\). Let \(\varphi'\) be the multiple coloring of \(G[X_1 \cup X_2]\) defined as follows:

\[\varphi'(v) = \begin{cases} \varphi_1(v) \cup \varphi_2(v), & \text{if } v \in X, \\ \varphi_i(v), & \text{if } v \in X_i \setminus X_{3-i}. \end{cases}\]

Note that \(|\varphi'(v)| \leq (h_1 + h_2)(v)\) for \(v \in X\). By arbitrarily adding some colors from \(L(v)\) to \(\varphi'(v)\) if needed, we may assume that \(|\varphi'(v)| = (h_1 + h_2)(v)\) for \(v \in X\). Then \(\varphi'\) is an \((\mathcal{H}, h_1 + h_2)\)-coloring of \(G[X_1 \cup X_2]\). For any \(g\)-augmentation of \(\varphi'\), its restriction to \(X_i\) is a \(g\)-augmentation of \(\varphi_i\), and hence can be extended to an \((\mathcal{H}, g)\)-coloring \(\varphi_i'\) of \(G_i\). Note that \(\varphi_1'\) and \(\varphi_2'\) agree on the intersection \(V(G_1) \cap V(G_2) = X\). Hence the union \(\varphi_1' \cup \varphi_2'\) is an \((\mathcal{H}, g)\)-coloring of \(G\). Therefore \((f, h_1 + h_2)_{X_i \cup X_2} \preceq (f, g)_G.\) \(\square\)
Lemma 4 Assume $G$ is a 3-path $v_1v_2v_3$, $X = \{v_1, v_3\}$, $f, g, h \in \mathbb{N}^G$, with $h = (p, 0, p) \leq g \leq f$. If

$$f(v_1) - f(v_2) + f(v_3) \geq p, f(v_2) \geq g(v_1) + g(v_2) + g(v_3) - p,$$

then

$$(f, h)_X \leq (f, g)_X.$$

Proof We prove the lemma by induction on $p$. If $p = 0$, then $f(v_2) \geq g(v_1) + g(v_2) + g(v_3)$ implies that any $(H, g)$-coloring of $X$ can be extended to an $(H, g)$-coloring of $G$.

Assume $p > 0$. Assume $H = (L, H)$ is an $f$-cover of $G$. We consider two cases.

Case 1 $f(v_1), f(v_3) \leq f(v_2)$.

By our convention, $|L(v_2) \cap N_H(L(v_1))| = |L(v_2)| = f(v_2)$ for $i = 1, 3$. Since $f(v_1) - f(v_2) + f(v_3) \geq p$, $|L(v_2) \cap N_H(L(v_1)) \cap N_H(L(v_3))| \geq p$.

Let $U$ be a $p$-subset of $L(v_2) \cap N_H(L(v_1)) \cap N_H(L(v_3))$, and for $i = 1, 3$, let

$$
\phi(v_i) = N_H(U) \cap L(v_i).
$$

Then $\phi$ is an $(H, h)$-coloring of $G[X]$.

If $\phi'$ is a $g$-augmentation of $\phi$, then

$$
|L(v_2) - (N_H(\phi'(v_1)) \cup \phi'(v_3))| \geq f(v_2) - p - (g(v_1) - p) - (g(v_3) - p) \geq g(v_2).
$$

We can extend $\phi'$ to an $(H, g)$-coloring of $G$ by letting $\phi'(v_2)$ be a $g(v_2)$-subset of $L(v_2) - (N_H(\phi'(v_1)) \cup \phi'(v_3))$. So $\phi'$ is $(H, g)$-extendable.

Case 2 $f(v_1) > f(v_2)$ or $f(v_3) > f(v_2)$.

By symmetry, we may assume that $f(v_1) - f(v_2) > 0$. Let

$$s = \min \{f(v_1) - f(v_2), p\}.$$

Then there exists an $s$-element set $S$ of $L(v_1)$ such that

$$S \cap N_H(L(v_2)) = \emptyset.$$

We modify the mappings $f$, $g$, $h$ to $f'$, $g'$, $h'$ as follows:

- $f'(v_i) = f(v_i) - s$ for $i = 1, 2, 3$.
- $h'(v_i) = h(v_i) - s$ and $g'(v_i) = g(v_i) - s$ for $i = 1, 3$, $g'(v_2) = g(v_2)$.

It is straightforward to verify that $f'$, $g'$, $h'$ satisfy the condition of the lemma. So by induction hypothesis, $(f', h')_X \leq (f', g')_G$.

Let $T$ be an arbitrary $s$-subset of $L(v_2)$, and let $T'$ be an $s$-subset of $L(v_2)$ which contains $N_H(T) \cap L(v_2)$. Let $H' = H - (S \cup T \cup T')$. Then $H' = (L', H')$ is an $f'$-cover of $G$. Let $\phi'$ be a strongly $(H', g')$-extendable $(H', h')$-coloring of $G[X]$. 

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Let
\[ \varphi(v_1) = \varphi'(v_1) \cup S, \varphi(v_3) = \varphi'(v_3) \cup T. \]

We shall show that \( \varphi \) is a strongly \((\mathcal{H}, g)\)-extendable \((\mathcal{H}, h)\)-coloring of \( G[X] \).

For any \( g \)-augmentation \( \psi \) of \( \varphi \),
\[ \psi'(v_1) = \psi(v_1) - S, \psi'(v_3) = \psi(v_3) - T \]

is a \( g' \)-augmentation of \( \varphi' \). Hence \( \psi' \) can be extended to an \((\mathcal{H}', g')\)-coloring \( \varphi^* \) of \( G \).

Then \( \varphi^* = \psi^* \) except that \( \varphi^*(v_1) = \psi(v_1) \cup S \) and \( \varphi^*(v_3) = \psi(v_3) \cup T \) is an \((\mathcal{H}, g)\)-coloring of \( G \) which is an extension of \( \psi \). \( \square \)

The following corollary follows from Lemma 4, and will be used frequently.

**Corollary 2** Assume \( G \) is a 3-path \( v_1v_2v_3 \).

1. If \( f = (3m, 4m, 3m) \), then \( (f, 2m)_{v_1,v_2} \leq (f, 2m)_G \).
2. If \( f = (3m, 5m, 3m) \), then \( (f, m)_{v_1,v_3} \leq (f, 2m)_G \).

**Lemma 5** Assume \( G \) is a graph and \( v_1v_2v_3 \) is an induced 3-path in \( G \), \( f, g \in \mathbb{N}^G \) and \( k \leq g(v_1), g(v_3) \) is a positive integer such that \( g \leq f \) and \( f(v_1) + f(v_3) - f(v_2) \geq k \).

Let \( f', g' \in \mathbb{N}^G \) be defined as follows:

1. \( f'(v_2) = f(v_2) - k, g'(v_i) = g(v_i) - k \) for \( i \in \{1, 3\} \).
2. For \( v \neq v_2 \), \( f'(v) = f(v) - k|N_G[v] \cap \{v_1, v_3\}| \), and for \( v \neq v_1, v_3 \), \( g'(v) = g(v) \).

If \( G \) is \((f', g')\)-DP-colorable, then \( G \) is \((f, g)\)-DP-colorable.

**Proof** Assume \( \mathcal{H} = (L, H) \) is an \( f \)-cover of \( G \).

If \( f(v_1), f(v_3) \leq f(v_2) \), then \( |L(v_2) \cap N_H(L(v_1)) \cap N_H(L(v_3))| \geq k \). Let \( U \) be a \( k \)-subset of \( L(v_2) \cap N_H(L(v_1)) \cap N_H(L(v_3)) \). Let \( \varphi(v_i) = N_H(U) \cap L(v_i) \) for \( i = 1, 3 \).

Let \( H' = H - \cup_{v \in \{v_1, v_3\}} N_H[\varphi(v)] \) and \( L'(v) = L(v) \cap V(H') \). It is straightforward to verify that \( \mathcal{H}' = (L', H') \) is an \( f' \)-cover of \( G \). Since \( G \) is \((f', g')\)-DP-colorable, there exists an \((\mathcal{H}', g')\)-coloring \( \varphi' \) of \( G \). Let
\[ \varphi^*(v) = \varphi'(v) \text{ for } v \neq v_1, v_3 \text{ and } \varphi^*(v) = \varphi'(v) \cup \varphi(v) \text{ for } v = v_1, v_3. \]

It is easy to check \( \varphi^* \) is an \((\mathcal{H}, g)\)-coloring of \( G \) and hence \( G \) is \((f, g)\)-DP-colorable.

Assume \( f(v_1) > f(v_2) \) or \( f(v_3) > f(v_2) \). By symmetry, we may assume that \( f(v_1) - f(v_2) > 0 \). Let
\[ s = \min \{f(v_1) - f(v_2), k\}. \]
Then there exists an s-element set $S$ of $L(v_1)$ such that $S \cap N_H(L(v_2)) = \emptyset$. Note if $s < k$, then every vertex in $L(v_1) - S$ is adjacent to a vertex in $L(v_2)$ by our convention. Let $T$ be a $k$-subset of $L(v_2)$ with $|N_H(T) \cap L(v_2)| = k$. Let $T'$ (maybe empty) be a $(k - s)$-subset of $L(v_1) - S$ such that $N_H(T') \cap L(v_2) \subseteq N_H(T) \cap L(v_2)$.

Let $H' = H - N_H[T \cup T' \cup S]$ and $L'(v) = L(v) \cap V(H')$. Then $H' = (L', H')$ is an $f'$-cover of $G$. Since $G$ is $(f', g')$-DP-colorable, there then exists an $(H', g')$-coloring $\varphi'$ of $G$. Let

$$\varphi(v) = \varphi'(v) \text{ for } v \neq v_1, v_3 \text{ and } \varphi(v_1) = \varphi'(v_1) \cup S \cup T' \text{ and } \varphi(v_3) = \varphi'(v_3) \cup T.$$

Then $\varphi$ is an $(H, g)$-coloring of $G$ and hence $G$ is $(f, g)$-DP-colorable.

\[\square\]

### 3 $(f, 2m)$-DP-Colorable Graphs

Assume $\mathcal{H} = (L, H)$ is a cover of $G$. In the proofs below, we treat an induced subgraph $H'$ of $H$ as a cover $\mathcal{H}' = (L', H')$ of $G$, where $L'(v) = L(v) \cap V(H')$ is given implicitly.

Assume $g, g' \in \mathbb{N}^G$. By a partial $(\mathcal{H}, g)$-coloring of $G$, we mean an $(\mathcal{H}, g')$-coloring $\varphi$ for some $g' \leq g$. For example, $\varphi(v) = \emptyset$ for all vertices $v$ of $G$ is a partial $(\mathcal{H}, g)$-coloring of $G$, which we call the trivial partial $(\mathcal{H}, g)$-coloring of $G$. In constructing an $(\mathcal{H}, g)$-coloring $\varphi$ of $G$, we may start with the trivial partial $(\mathcal{H}, g)$-coloring $\varphi$ of $G$, then add colors to $\varphi(v)$ step by step, and eventually obtain a $(\mathcal{H}, g)$-coloring of $G$. For convenience, we shall denote by $\varphi$ all the partial $(\mathcal{H}, g)$-colorings of $G$ produced in the process.

**Lemma 6** Assume $k \geq 1$, $G$ is a $k$-path $v_1v_2...v_k$, and $f \in \mathbb{N}^G$ such that

1. $f(v_1) = f(v_k) = 3m$ and $f(v_i) = 3m$ or $5m$ for $i \in \{2, 3, \ldots, k - 1\}$,
2. $f(v_i) + f(v_{i+1}) \geq 8m$ for $i \in \{1, 2, \ldots, k - 1\}$.

Then

$$(f, m)_{\{v_i, v_{k}\}} \leq (f, 2m)_G.$$

In particular, $G$ is $(f, 2m)$-DP-colorable.

**Proof** We prove this lemma by induction on $k$. If $k = 1$, then the lemma is obviously true. Assume $k \geq 2$ and the lemma holds for shorter paths. Since $f(v_1) + f(v_2) \geq 8m$ and $f(v_1) = f(v_k) = 3m$, we know that $k \geq 3$. If $k = 3$, then this is Corollary 2. Assume $k \geq 4$.

If $f(v_i) = 3m$ for some $3 \leq i \leq k - 2$, then let $G_1$ be the path $v_1 \ldots v_i$ and $G_2$ be the path $v_i \ldots v_k$. By induction hypothesis,

$$(f, m)_{\{v_1, v_i\}} \leq (f, 2m)_{G_1}, \text{ and } (f, m)_{\{v_i, v_k\}} \leq (f, 2m)_{G_2}.$$
By letting $X = \{v_1, v_i, v_k\}$ and $h(v_1) = h(v_k) = m$ and $h(v_i) = 2m$, it follows from Lemma 3 that $(f, h)_X \leq (f, 2m)_G$, which implies $(f, m)_{\{v_1, v_k\}} \leq (f, 2m)_G$ by Lemma 2.

Assume $f(v_i) = 5m$ for $i = 2, \ldots, k - 1$ and $k \geq 4$. In this case, we show a stronger result: for $h(v_1) = m$ and $h(v_k) = 0$, $(f, h)_{\{v_1, v_k\}} \leq (f, 2m)_G$.

Assume $\mathcal{H} = (L, H)$ is an $f$-cover of $G$. We need to show that there exists an $m$-subset $S$ of $L(v_1)$ such that for any $2m$-subset $S'$ of $L(v_1)$ containing $S$, and any $2m$-subset $T$ of $L(v_k)$, there exists an $(\mathcal{H}, 2m)$-coloring $\psi$ of $G$ such that $\psi(v_1) = S'$ and $\psi(v_k) = T$.

Let $\mathcal{H}'$ be the restriction of $\mathcal{H}$ to $G - v_k$, except that $L'(v_{k-1}) = L(v_{k-1}) - N_H(T)$. Let $f'$ be the restriction of $f$ to $G - v_k$, except that $f'(v_{k-1}) = 3m$. Then $\mathcal{H}'$ is an $f'$-cover of $G - v_k$. By induction hypothesis, $(f', m)_{\{v_{i}, v_{k-1}\}} \leq (f', 2m)_{G - v_k}$. Hence there exists an $m$-subset $S$ of $L(v_1)$ such that for any $2m$-subset $S'$ of $L(v_1)$ containing $S$, there exists an $(\mathcal{H}', 2m)$-coloring $\psi$ of $G - v_k$. Now $\psi$ extends to an $(\mathcal{H}, 2m)$-coloring $\psi'$ of $G$ with $\psi'(v_k) = T$. □

**Lemma 7** Assume $G$ is a cycle $v_1v_2\ldots v_kv_1$ such that $k \geq 4$,

1. $f(v_i) = 3m$ or $5m$ for $i \in \{1, 2, \ldots, k\}$,
2. $f(v_i) + f(v_{i+1}) \geq 8m$ for $i \in \{1, 2, \ldots, k\}$.

Then $G$ is $(f, 2m)$-DP-colorable.

**Proof** If there are two vertices $v_i$ and $v_j$ with $f(v_i) = f(v_j) = 3m$, then let $P_1 = v_{i}v_{i+1}\ldots v_{j}$ and $P_2 = v_{j}v_{j+1}\ldots v_{i}$ be the two paths of $G$ connecting $v_i$ and $v_j$. By Lemma 6,

$$(f, m)_{\{v_i, v_j\}} \leq (f, 2m)_{P_1}, \text{ and } (f, m)_{\{v_i, v_j\}} \leq (f, 2m)_{P_2}.$$ 

It follows from Lemma 3 that $(f, 2m)_{\{v_i, v_j\}} \leq (f, 2m)_G$. So $G$ is $(f, 2m)$-DP-colorable.

Otherwise, we may assume that $f(v_i) = 5m$ for $i = 2, 3, \ldots, k$. Let $f' = f$ except that $f'(v_1) = f'(v_3) = 3m$. Then $f'$ satisfies the condition of the lemma, and by the previous paragraph, $G$ is $(f', 2m)$-DP-colorable, which implies that $G$ is $(f, 2m)$-DP-colorable. □

**Lemma 8** Assume $G = K_{1,3}$ is star with $v_4$ be the center and $\{v_1, v_2, v_3\}$ be the three leaves. Then for $f = (3m, 3m, 3m, 5m)$, $G$ is $(f, 2m)$-DP-colorable.

**Proof** Apply Lemma 5 to $(f, g)$ and $(v_1, v_4, v_2)$, it suffices to show that $G$ is $(f_1, g_1)$-DP-colorable, where $f_1 = (2m, 2m, 3m, 4m), g_1 = (m, m, 2m, 2m)$.

Apply Lemma 5 to $(f_1, g_1)$ and $(v_2, v_4, v_3)$, it suffices to show that $G$ is $(f_2, g_2)$-DP-colorable, where $f_2 = (2m, m, 2m, 3m), g_2 = (m, 0, m, 2m)$. (Now $v_2$ needs no more colors and can be deleted. However, to keep the labeling of the vertices, we do not delete it).
Apply Lemma 5 to \((f_2, g_2)\) and \((v_1, v_4, v_3)\), it suffices to show that \(G\) is \((f_3, g_3)\)-DP-colorable, where \(f_3 = (m, m, m, 2m)\), \(g_3 = (0, 0, 0, 2m)\), and this is obviously true.

\[\tag*{\square}\]

**Lemma 9** Assume \(G = K_{1,4}\) is a star with center \(v_5\) and four leaves \(v_1, v_2, v_3, v_4\). Let \(f = (2m, 2m, 2m, 2m, 4m)\), \(g = (m, m, m, 2m)\). Then \(G\) is \((f, g)\)-DP-colorable.

**Proof** Assume \(\mathcal{H} = (L, H)\) is an \(f\)-cover of \(G\). We construct an \((\mathcal{H}, g)\)-coloring \(\varphi\) of \(G\) as follows:

Initially let \(\varphi(v) = \emptyset\) for all \(v \in V(G)\).

Assume \(|N_H(L(v_1)) \cap N_H(L(v_2)) \cap L(v_5)| = a\). Let \(k = \min\{a, m\}\), let \(S_1(v_5)\) be a \(k\)-subset of \(N_H(L(v_1)) \cap N_H(L(v_2)) \cap L(v_5)\).

For \(i = 1, 2\), add \(L(v_i) \cap N_H(S_1(v_5))\) to \(\varphi(v_i)\). Let

\[H_1 = H - N_H[\varphi(v_1) \cup \varphi(v_2)],\]

and \(H_1 = (L_1, H_1)\).

Let \(g_1(v_i) = g(v_i) - k\) for \(i = 1, 2\), and \(g_1(v_j) = g(v_j)\) for \(j \neq 1, 2\).

It suffices to show that there exists an \((\mathcal{H}_1, g_1)\)-coloring of \(G\). If \(k = m\), then \(g_1(v_i) = 0\) for \(i = 1, 2\). So we can delete \(v_1, v_2\). As \(|L_1(v_5)| = 3m\), it follows from Lemma 4 that there exists an \((\mathcal{H}_1, g_1)\)-coloring of \(G\).

Assume \(k = a < m\). Then \(N_H(L_1(v_1)) \cap N_H(L_1(v_2)) = \emptyset\). As \(|L_1(v_5)| = 4m - k\) and \(|L_1(v_3)| = |L_1(v_4)| = 2m\), we have

\[|L_1(v_5) \cap N_H(L_1(v_3)) \cap N_H(L_1(v_4))| \geq k.\]

Let \(S_2(v_5)\) be a \(k\)-subset of \(L_1(v_5) \cap N_H(L_1(v_3)) \cap N_H(L_1(v_4))\). For \(i = 3, 4\), add \(L_1(v_i) \cap N_H(S_2(v_5))\) to \(\varphi(v_i)\). Let

\[H_2 = H_1 - N_{H_1}[\varphi(v_3) \cup \varphi(v_4)],\]

and \(H_2 = (L_2, H_2)\).

Let \(g_2(v_i) = g_1(v_i) - k\) for \(i = 3, 4\), and \(g_2(v_j) = g_1(v_j)\) for \(j \neq 3, 4\). It suffices to show that there exists an \((\mathcal{H}_2, g_2)\)-coloring of \(G\).

As \(N_{H_2}(L_2(v_1)) \cap N_{H_2}(L_2(v_2)) = \emptyset\), we conclude that \(|N_{H_2}(L_2(v_1)) \cap N_{H_2}(L_2(v_2)) \cap N_{H_2}(L_2(v_3)) \cap N_{H_2}(L_2(v_4))| \geq m - k\), or \(|N_{H_2}(L_2(v_2)) \cap N_{H_2}(L_2(v_3)) \cap L_2(v_5)| \geq m - k\). By symmetry, we assume that

\[|N_{H_2}(L_2(v_1)) \cap N_{H_2}(L_2(v_2)) \cap L_2(v_3)| \geq m - k.\]

Let \(S_3(v_5)\) be an \((m - k)\)-subset of \(L_2(v_5) \cap N_{H_2}(L_2(v_1)) \cap N_{H_2}(L_2(v_3))\). For \(i = 1, 3\), add \(L_2(v_i) \cap N_{H_2}(S_3(v_5))\) to \(\varphi(v_i)\). Let

\[H_3 = H_2 - N_{H_2}[\varphi(v_1) \cup \varphi(v_3)],\]

and \(H_3 = (L_3, H_3)\).

Let \(g_3(v_i) = g_2(v_i) - (m - k)\) for \(i = 1, 3\), and \(g_3(v_j) = g_2(v_j)\) for \(j \neq 1, 3\). It suffices to show that there exists an \((\mathcal{H}_3, g_3)\)-coloring of \(G\).

Observe that \(g_3(v_1) = g_3(v_3) = 0\), and hence \(v_1, v_3\) can be deleted. The remaining graph is a 3-path. It is easy to verify that \(|L_3(v_5)| = 3m - k\) and
$|L_3(v_2)| = |L_3(v_4)| = 2m - k$, $g(v_5) = 2m$ and $g_3(v_2) = g_3(v_4) = m - k$. It follows from Lemma 4 that $G$ is $(\mathcal{H}_3, g_3)$-colorable. \hfill $\Box$

**Corollary 3** For the graph $G$ and $f \in \mathbb{N}^G$ shown in Fig. 1, $G$ is $(f, 2m)$-DP-colorable.

**Proof** Let $G_1$ be the 3-path induced by $\{v_1, v_6, v_2\}$. By Corollary 2, $(f, m)_{\{v_1, v_2\}} \preceq (f, 2m)_{G_1}$.

Apply Lemma 1 to the cut-set $X = \{v_1, v_2\}$, it suffices to show that $G' = G \setminus \{v_1, v_2, v_3, v_4, v_5\}$ is $(f', g')$-DP-colorable, where $f' = (2m, 2m, 3m, 3m, 5m)$ and $g' = (m, m, 2m, 2m, 2m)$.

Apply Lemma 5 to the 3-path $v_3v_5v_4$ with $k = m$, it suffices to show that $G'$ is $(f_1, g_1)$-DP-colorable, where $f_1 = (2m, 2m, 2m, 2m, 4m)$ and $g_1 = (m, m, m, m, m, 2m)$. This follows from Lemma 9. \hfill $\Box$

**Lemma 10** For the graph $G$ and $f \in \mathbb{N}^G$ shown in Fig. 2. Let $g = (2m, 2m, 2m, 2m, m)$. Then $G$ is $(f, g)$-DP-colorable.

**Proof** Apply Lemma 5 to the 3-path $v_4v_3v_5$ with $k = m$, it suffices to show that $G' = G \setminus \{v_1, v_2, v_3, v_4\}$ is $(f', g')$-DP-colorable, where $f' = (3m, 5m, 3m, 2m)$ and $g' = (2m, 2m, 2m, 2m, m)$.

Let $G_1$ be the 3-path $v_1v_2v_3$ and $G_2$ be the single edge $v_3v_4$. Apply Corollary 2 to $G_1$ and Lemma 1, it suffices to show that $G_2$ is $(2m, m)$-DP-colorable, which is obviously true. \hfill $\Box$
Corollary 4  For the graphs $G$ and $f \in \mathbb{N}^G$ shown in Fig. 3, $G$ is $(f, 2m)$-DP-colorable.

Proof  First we show the left graph in Fig. 3 is $(f, 2m)$-DP-colorable. Let $G_1$ be the 3-path induced by $\{v_5, v_6, v_7\}$. By Corollary 2, $(f, m)_{v_5,v_7} \preceq (f, 2m)_{G_1}$. Apply Lemma 1 to the cut-set $X = \{v_5\}$, it suffices to show that $G_1 = G_1\setminus \{v_1, v_2, v_3, v_4, v_5\}$ is $(f', g')$-DP-colorable, where $f' = (3m, 5m, 4m, 3m, 2m)$ and $g = (2m, 2m, 2m, 2m, 2m, m)$. This follows from Lemma 10.

Next we consider the right graph in Fig. 3. Assume $\mathcal{H} = (L, H)$ is an $f$-cover of $G$. We construct an $(\mathcal{H}, g)$-coloring $\varphi$ of $G$ as follows: Let $S_1(v_5)$ be an $m$-subset of $L(v_5) - N_{\mathcal{H}}(L(v_6))$, and add $S(v_5)$ to $\varphi(v_5)$. Choose a $2m$-subset from $L(v_7) - N_{\mathcal{H}}(S(v_5))$ and add it to $\varphi(v_7)$. It suffices to prove $G' = G[[v_1, v_2, v_3, v_4, v_5, v_6]]$ has an $(f', g')$-DP-coloring, where $f' = (3m, 5m, 4m, 3m, 2m, 3m)$ and $g' = (2m, 2m, 2m, 2m, 2m, 2m)$. By Lemma 10, $G' - v_6$ has an $(f', g')$-DP-coloring $\varphi'$. Choose a $2m$-subset of $L(v_6) - \varphi'(v_5)$ and add the $2m$-subset to $\varphi(v_6)$. Let $\varphi(v_i) = \varphi'(v_i)$ for $i = 1, 2, 3, 4$ and $\varphi(v_5) = \varphi'(v_5) \cup S(v_5)$. Thus $\varphi$ is an $(\mathcal{H}, g)$-coloring of $G$. □

Corollary 5  For the graphs $G$ and $f \in \mathbb{N}^G$ shown in Fig. 4, $G$ is $(f, 2m)$-DP-colorable.
Proof Assume $G$ is any of the two graphs in Fig. 4, and $\mathcal{H} = (L, H)$ is an $f$-cover of $G$. Let $H' = H - L(v_8) \cap N_H(L(v_4))$ and $\mathcal{H}' = (L', H')$. Let $e = v_4v_8$. Then it suffices to show that $G' = G - e$ is $(\mathcal{H}', 2m)$-colorable.

By Corollary 2, the subgraph $G'[v_8, v_9, v_{10}]$ has an $(\mathcal{H}', 2m)$-coloring $\varphi_1$. Let $H'' = H' - L'(v_5) \cap N_{H'}(\varphi_1(v_3))$. It remains to prove that $G'' = G[(v_1, v_2, v_3, v_4, v_5, v_6, v_7)]$ is $(\mathcal{H}'', 2m)$-coloring. For the graph $G$ on the left, $\mathcal{H}''$ is an $f$'-cover of $G''$, where $f' = (3m, 5m, 3m, 3m, 3m, 3m, 3m)$. For the graph $G$ on the right, $\mathcal{H}''$ is an $f$'-cover of $G''$, where $f' = (3m, 5m, 5m, 3m, 3m, 3m, 3m)$. Now the conclusion follows from Corollary 4.

4 Proof of Theorem 1

Let $G$ be a counterexample to Theorem 1 with minimum number of vertices.

It is trivial that $G$ is connected and has minimum degree at least 3. Let $\mathcal{H} = (L, H)$ be a 7m-cover of $G$ such that $G$ is not $(\mathcal{H}, 2m)$-colorable. By our assumption, $E_H(L(u), L(v))$ is a perfect matching whenever $uv \in E(G)$.

Definition 7 A configuration in $G$ is an induced subgraph $G'$ of $G$, where each vertex $v$ of $G'$ is labelled with its degree $d_G(v)$ in $G$. A configuration $G'$ is reducible if $G'$ is $(f_{G'}, 2m)$-DP-colorable, where $f_{G'} \in \mathbb{N}^{G'}$ is defined as $f_{G'}(v) = 7m - 2(d_G(v) - d_G(v)m)$ for $v \in V(G')$.

Lemma 11 $G$ contains no reducible configuration.

Proof Assume $G'$ is a reducible configuration in $G$. By minimality of $G$, $G - G'$ has an $(\mathcal{H}, 2m)$-coloring $\varphi$. For $v \in V(G')$, let

$$L'(v) = L(v) - \cup_{u \in V(G)-V(G')}N_H(\varphi(u))$$

and $H' = H[\cup_{v \in V(G')}L'(v)]$.

Then $\mathcal{H}' = (L', H')$ is an $f_{G'}$-cover of $G'$. As $G'$ is reducible, $G'$ has an $(\mathcal{H}', 2m)$-coloring $\varphi'$. Then $\varphi \cup \varphi'$ is an $(\mathcal{H}, 2m)$-coloring of $G$, a contradiction.

Corollary 6 The configurations in Fig. 5 are reducible.

Proof The reducibility of configurations (a), (b), (c) follows from Lemma 6, (d) follows from Lemma 8, (e) and (f) follows from Lemma 7.

Now we prove the reducibility of configurations (g). Let $G' = G[(v_1, v_2, v_3, v_4, v_5)]$. Let $f_{G'}(v) = 7m - 2(d_G(v) - d_G(v)m)$. Then $f_{G'}(v_i) = 5m$ for $i = 1, 2$ and $f_{G'}(v_j) = 5m$ for $j = 3, 4, 5$. Assume $\mathcal{H} = (L', H')$ is an $f_{G'}$-cover of $G'$. We color $v_5$ with a 2m-subset $\varphi(v_5)$ of $L'(v_5) - N_{H'}(L'(v_5))$. Let $\mathcal{H} = \mathcal{H} - L'(v_3) \cap N_{H'}(\varphi(v_3))$. It suffices to prove $G'' = G[(v_1, v_2, v_3, v_4)]$ has
an \((\mathcal{H} \simeq, 2m)\)-coloring. As \(\mathcal{H} \simeq\) is an \(f''\)-cover, where \(f'' = (3m, 3m, 3m, 5m)\), this follows from Lemma 8.

**Lemma 12** If two 4-faces intersect at a 4-vertex, then one of them contains at most one 3-vertex.

**Proof** Assume that \(F_1\) and \(F_2\) are 4-faces intersect at a 4-vertex \(v\), and each of \(F_1, F_2\) contains at least two 3-vertices. Then either \(v\) is adjacent to three 3-vertices and hence \(G\) contains reducible configuration (d), or \(G\) contains a \((3, 3, 4, 3, 3)\)-path, which is the reducible configuration (b).

We call a 4-face \(F\) **light** if \(F\) is a \((4, 4, 3, 3)\)-face, a \((4, 5, 3, 3)\)-face or a \((4, 3, 5, 3)\)-face. (Note that \(G\) contains no \((4, 3, 4, 3)\)-face by Corollary 6 (e)).

Assume \(v\) is a 4-vertex. We say \(v\) is

1. **strong** if it is not incident with any light 4-face.
2. **normal** if it is incident with a light 4-face and three \(5^+\)-faces.
3. **weak** if it is incident with a light 4-face and a 4-face with no 3-vertex.
4. **very weak** if it is incident with a light 4-face and a 4-face with a 3-vertex.

Let \(v\) be a weak or very weak 4-vertex. If \(v\) has a 3-neighbor \(u\) such that \(vu\) is shared by a light 4-face and a 5-face \(F\), then \(F\) is called a **special 5-face** of \(v\). Note that a 4-vertex \(v\) can have at most two special 5-faces. A 5-face incident to \(v\) is called a **non-special 5-face** of \(v\) if it is not a special 5-face of \(v\). Note that a non-special 5-face of \(v\) can be a special 5-face of some other vertex.

**Lemma 13** A \((4, 4, 4, 3)\)-face does not intersect a \((4, 4, 3, 3)\)-face at a 4-vertex.

**Proof** Assume that a \((4, 4, 3, 3)\)-face intersects a \((4, 4, 4, 3)\)-face at a 4-vertex \(v\). Thus one of the graphs in Fig. 6 is a subgraph of \(G\). Assume \(G'\) on the left of Fig. 6...
is a subgraph of $G$. Since $G$ is triangle free, contains no $(3, 3, 3)$-path and no normally adjacent 4-cycles, $G'$ is an induced subgraph of $G$. We shall prove that $G'$ is reducible.

Assume $H \cong (L', H')$ is an $f_{G'}$-cover of $G'$. Note that $f_{G'} = (3m, 3m, 3m, 5m, 7m, 5m)$. We color $v_7$ with a $2m$-subset $\varphi(v_7)$ of $L'(v_7) - N_{H'}(L'(v_3))$. Let $H' \cong H - L'(v_4) \cap N_{H'}(\varphi(v_7))$. It suffices to prove $G'' = G[\{v_1, v_2, v_3, v_4, v_5, v_6\}]$ has an $(H \cong, 2m)$-coloring. As $H''$ is an $f''$-cover of $G''$, where $f'' = (3m, 3m, 3m, 3m, 7m, 5m)$, the result follows from Corollary 3. Thus $G'$ is reducible, a contradiction.

Assume the graph on the right of Fig. 6 is a subgraph of $G$. Then $G' = G[\{v_1, v_2, v_3, v_4, v_5\}]$ is the reducible configuration $(g)$ in Corollary 6, a contradiction. □

**Lemma 14** A $(4, 4, 4, 3)$-face does not intersect a $(4, 3, 5, 3)$-face at a 4-vertex.

**Proof** Assume a $(4, 3, 5, 3)$-face $F_1$ intersect a $(4, 4, 4, 3)$-face $F_2$ at a 4-vertex. By Corollary 6(d), a 4-vertex has at most two 3-neighbors. Thus the 4-cycles are as shown in Fig. 7. But the induced subgraph $G' = G[\{v_1, v_2, v_3, v_4, v_5, v_6\}]$ is reducible by Corollary 3, a contradiction. □

**Lemma 15** A $(4^+, 4^+, 4^+, 3)$-face contains at most one very weak 4-vertex.

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**Fig. 6** $(4, 4, 4, 3)$-face intersects $(4, 4, 3, 3)$-face

**Fig. 7** $(4, 3, 5, 3)$-face intersects $(4, 4, 4, 3)$-face
**Proof** Assume that $F = (v_1, v_2, v_3, v_4)$ is a $(4^+, 4^+, 4^+, 3)$-face and contains at least two very weak 4-vertices.

If $v_1$ and $v_2$ are very weak 4-vertices, then since a 4-vertex has at most two 3-neighbors, the light faces incident with $v_1$ and $v_3$ are $(4, 4^+, 3, 3)$-faces. This implies that $G$ has a $(3, 3, 4, 3, 3, 3)$-path $P$. Since $G$ contains no $(3, 3, 3)$-path, the path $P$ is induced. Hence $P$ is the reducible configuration (c), a contradiction.

Thus we assume that $v_1, v_2$ are very weak 4-vertices. Using the fact that a 4-vertex has at most two 3-neighbors, we conclude that $G$ contains one of the graphs in Fig. 8 as an induced subgraph. But by Corollary 4, the subgraph $G[v_1, v_2, v_4, v_5, v_6, v_7, v_8]$ is reducible, a contradiction. □

**Lemma 16** Assume a $(4, 4, 4, 4)$-face $F$ contains a weak 4-vertex, which is incident with a $(4, 3, 5, 3)$-face. Then $F$ contains at most two weak 4-vertices.

**Proof** Assume $F$ has three weak 4-vertices and at least one vertex in $F$ is incident with a $(4, 3, 5, 3)$-face. Then $G$ contains one of the graphs in Fig. 9 as a subgraph.

Assume $Q$ is one of the graphs in Fig. 9 which is a subgraph of $G$. We shall derive a contradiction by showing $G$ has a reducible configuration.

We first prove that $Q$ is an induced subgraph of $G$. If $Q$ is one of the graphs in (c), (d), (e), then $Q$ is an induced subgraph, since $G$ is triangle free, has no normally adjacent 4-faces, and $G$ contains no $(3,3,3)$-path.

Assume $Q$ is the graph depicted in (a). It follows easily from Lemma 6 and Lemma 7 that $G$ contains at most one of the edges $v_1v_4, v_1v_6, v_3v_4, v_3v_6$. If $Q$ is not an induced subgraph, then $G$ contains exactly one of the edges $v_1v_4, v_1v_6, v_3v_4$ and $v_3v_6$. By symmetry, we assume $v_1v_4 \in E(G)$. Then $G[v_4, v_1, v_2, v_3, v_9, v_7, v_8]$ is isomorphic to the right graph in Fig. 4. Thus $G[v_4, v_1, v_2, v_3, v_9, v_7, v_8]$ is reducible by Corollary 4, a contradiction.

Assume $Q$ is graph depicted in (b). Since $G$ contains no $(3,3,3)$-path and no $(3,4,3,4,3,4)$-cycle, $G$ contains at most one of the edges $v_2v_8, v_2v_{10}, v_3v_8, v_3v_{10}$. If $Q$ is not an induced subgraph of $G$, then $G$ contains exactly one of these edges. By symmetry, we assume $v_2v_8 \in E(G)$. Then $G[v_8, v_2, v_4, v_3, v_5, v_6, v_7]$ is isomorphic to the left graph in Fig. 4, which is reducible by Corollary 4, a contradiction.
So $Q$ is an induced subgraph of $G$.

It remains to prove that $Q$ or an induced subgraph of $Q$ is a reducible configuration. If $Q$ is the graph as depicted in (c) or (d), then it follows from Corollary 5 that $Q$ is reducible.

For the remainder of the proof, we consider Cases (a),(b),(e), depending on whether $Q$ is the graph depicted in Fig. 9a, b, e.

**Case (a)** Assume $Q' = G[\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}]$ is the subgraph in Fig. 9a. We shall prove that $Q'$ is reducible. Let $\mathcal{H}' = (L', H')$ is an $f_{Q'}$-cover of $Q'$. We shall construct an $(\mathcal{H}', 2m)$-coloring $\varphi$ of $Q'$.

![Fig. 9 weak 4-vertices in (4, 4, 4, 4)-face](image)
Choose an $m$-subset $S(v_g)$ from $L'(v_g) - N_H'(L'(v_g)) - N_H(L'(v_g))$ and add it to $\varphi(v_g)$.

Let $H \supseteq H - N_H[S(v_g)]$. It suffices to prove $Q'$ has an $(H \supseteq, g)$-coloring, where $g(v_g) = m$ and $g(v_i) = 2m$ for $i \in \{1, 2, \ldots, 8\}$. By Corollary 2, $v_1v_2v_3$ has an $(H \supseteq, 2m)$-coloring $\varphi_1$. Similarly, $v_4v_5v_6$ has an $(H \supseteq, 2m)$-coloring $\varphi_2$. Add an $m$-subset of $L''(v_g) - N_H(\varphi_1(v_g) \cup \varphi_2(v_g))$ to $\varphi(v_g)$. For $i = 7, 8$, let $\varphi(v_i)$ be a $2m$-subset of $L(v_i) - N_H(\varphi(v_g))$. Let $\varphi(v_i) = \varphi_1(v_i)$ for $i = 1, 2, 3$. Let $\varphi(v_j) = \varphi_2(v_j)$ for $i = 4, 5, 6$. Then $\varphi$ is an $(H', 2m)$-coloring of $Q'$.

Case (b) In this case, we shall prove that $Q$ is reducible.

Let $\mathcal{H}' = (L', H')$ be an $f_Q$-cover of $Q$. Let $L''(v_g) = L'(v_g) - N_H'(L'(v_g))$, and $L''(v_i) = L'(v_i)$ for $i = 8, 10$. Then $H \supseteq = (L'', H'[\cup_{i=8}^{10} L''(v_i)])$ is an $(3m, 4m, 3m)$-cover of $G[v_8, v_9, v_{10}]$. By Corollary 2, the 3-path $v_8v_9v_{10}$ has an $(H \supseteq, 2m)$-coloring $\varphi_1$.

Let $Q' = Q - \{v_8, v_9, v_{10}\}$. For $v \in Q'$, let $L'''(v) = L'(v) - N_H'(\varphi_1(v_g))$ and $H''' = H'[\cup_{v \in Q'} L'''(v)]$. Then $H \supseteq = (L'''', H''''')$ is an $f'''$-cover of $Q'$, where $f''' = (3m, 3m, 3m, 7m, 3m, 3m)$. It follows from Corollary 4 that $Q'$ has an $(H \supseteq, 2m)$-coloring $\varphi_2$. Then $\varphi = \varphi_1 \cup \varphi_2$ is an $(H', 2m)$-coloring of $Q$.

Case (e) Let $Q' = G\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. We shall prove that $Q'$ is reducible. Let $H' = (L', H')$ be an $f_Q$-cover of $Q$. Let $Q_1 = G\{v_1, v_6, v_7, v_2, v_8, v_g\}$. By Lemma 4, $\langle f_2', m \rangle\{v_1, v_2\} \leq \langle f_2, g_2 \rangle\{v_1\}$. Apply Lemma 1 to $Q'$, it suffices to show that $Q_2 = G\{v_1, v_2, v_3, v_4, v_5\}$ is $(f_2', g_2')$-DP-colorable, where $f_2' = (2m, 2m, 3m, 3m, 5m)$, $g_2' = (m, m, 2m, 2m, 2m)$. Apply Lemma 5 to the 3-path $v_3v_4v_5$ with $k = m$, it suffices to show that $Q_2$ is $(f_2'', g_2'')$-DP-colorable, where $f_2'' = (2m, 2m, 2m, 2m, 4m)$ and $g_2'' = (m, m, m, 2m)$. This follows from Lemma 9.

We shall use discharging method to derive a contradiction. Set the initial charge $ch(v) = 2d(v) - 6$ for every vertex $v$, $ch(F) = d(F) - 6$ for every face $F$. By Euler formula,

$$\sum_{x \in V(G) \cup F(G)} ch(x) < 0.$$ 

Denote by $\omega(v \rightarrow F)$ the charge transferred from a vertex $v$ to an incident face $F$. Below are the discharging rules:

R1 Each strong 4-vertex sends $\frac{2}{3}$ to each incident 4-face and $\frac{1}{3}$ to each incident 5-face.
R2 Each normal 4-vertex sends 1 to the incident light 4-face and $\frac{1}{3}$ to each incident 5-face.
R3 If $v$ is a weak 4-vertex and $F$ is a 4-face or 5-face incident with $v$, then

\[ \omega(v \rightarrow F) = \begin{cases} 
\frac{2}{3} & \text{if } v \text{ is strong, } F \text{ is a 4-face}; \\
\frac{1}{3} & \text{if } v \text{ is strong, } F \text{ is a 5-face}; \\
1 & \text{if } v \text{ is normal, } F \text{ is a 4-face}; \\
\frac{1}{3} & \text{if } v \text{ is normal, } F \text{ is a 5-face}; \\
0 & \text{if } v \text{ is weak, } F \text{ is a 4-face}; \\
\frac{1}{3} & \text{if } v \text{ is weak, } F \text{ is a 5-face}.
\end{cases} \]
R4 Assume $v$ is a very weak 4-vertex and $F$ is a 4-face or 5-face incident with $v$.

- (i) If $v$ is incident with a $(4, 4, 4, 3)$-face, then

$$\omega(v \to F) = \begin{cases} 1, & \text{if } F \text{ is a light 4-face,} \\ \frac{1}{2}, & \text{if } F \text{ is a non-light 4-face and } v \text{ has at most one special 5-face,} \\ \frac{1}{3}, & \text{if } F \text{ is a special 5-face of } v; \text{or } F \text{ is a non-light 4-face and } v \text{ has two special 5-faces,} \\ \frac{1}{6}, & \text{if } F \text{ is a non-special 5-face of } v. \end{cases}$$

- (ii) Otherwise,

$$\omega(v \to F) = \begin{cases} 1, & \text{if } F \text{ is a light 4-face,} \\ \frac{2}{3}, & \text{if } F \text{ is a (4, 4, 4, 3)-face,} \\ \frac{1}{3}, & \text{if } F \text{ is a special 5-face of } v, \\ 0, & \text{if } F \text{ is a non-special 5-face of } v. \end{cases}$$

R5 Each 5-vertex sends 1 to each incident 4-face and sends $\frac{2}{3}$ to each incident 5-face.

R6 Each 6+-vertex sends $\frac{4}{3}$ to each incident 4-face and sends $\frac{2}{3}$ to each incident 5-face.

**Observation 1** If $v$ is a very weak 4-vertex incident with a 5-face $F$ and $w(v \to F) = 0$, then $v$ has a 5-neighbor in $F$.

**Proof** Since $v$ is very weak and $w(v \to F) = 0$, $v$ is incident with a light face and a $(4, 4, 4, 3)$-face. By Lemmas 13 and 14, the light face is a $(4, 5, 3, 3)$-face. Since $w(v \to F) = 0$, $F$ is not special, hence the neighbor of $v$ shared by $F$ and the light face is a 5-vertex. □

Let $ch^*$ denote the final charge after performing the discharging process. It suffices to show that the final charge of each vertex and each face is non-negative.

We first check the final charge of vertices in $G$. 

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Lemma 17 \( ch^*(v) \geq 0 \) for \( v \in V(G) \).

**Proof** If \( d(v) = 3 \), \( ch^*(v) = ch(v) = 0 \).

If \( v \) is a strong 4-vertex, then since \( v \) is incident with at most two 4-faces, by R1, \( ch^*(v) \geq ch(v) - 2 \times \frac{2}{3} - 2 \times \frac{1}{3} = 0 \).

If \( v \) is a normal 4-vertex, then by R2, \( ch^*(v) \geq ch(v) - 1 - 3 \times \frac{1}{3} = 0 \).

Assume \( v \) is a weak 4-vertex. If \( v \) has two special 5-faces, then by R3, \( ch^*(v) \geq ch(v) - 1 - 3 \times \frac{1}{3} = 0 \).

If \( v \) has at most one special 5-face, then \( ch^*(v) \geq ch(v) - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{6} = 0 \).

Assume that \( v \) is a very weak 4-vertex. If \( v \) is incident with a \((4, 4, 4, 3)\)-face, then by Lemmas 13 and 14, \( v \) is incident with a \((4, 5, 3, 3)\)-face. Thus there is at most one special 5-face of \( v \). By R4 (i), \( ch^*(v) \geq ch(v) - 1 - \frac{2}{3} - \frac{1}{3} = 0 \). Otherwise, by R4 (ii), \( ch^*(v) \geq ch(v) - 1 - 3 \times \frac{1}{3} = 0 \).

If \( d(v) = 5 \), then \( v \) is incident with at most two 4-faces and by R5, \( ch^*(v) \geq ch(v) - 2 \times 1 - 3 \times \frac{2}{3} = 0 \).

If \( d(v) = k \geq 6 \), then \( v \) is incident with at most \( \lfloor \frac{k}{2} \rfloor \) 4-faces. Thus by R6, \( ch^*(v) \geq ch(v) - \frac{4}{3} \times \lfloor \frac{k}{2} \rfloor - (k - \lfloor \frac{k}{2} \rfloor) \times \frac{2}{3} \geq 0 \). \( \square \)

Now we check the final charge of faces. If \( F \) is a 6+-face, no charge is discharged from or to \( F \). Thus \( ch^*(F) = ch(F) = d(F) - 6 \geq 0 \).

Lemma 18 If \( F \) is a 4-face, then \( ch^*(F) \geq 0 \).

**Proof** Assume \( F \) is a 4-face. By Corollary 6 (a), \( F \) contains at most two 3-vertices.

**Case 1** \( F \) contains two 3-vertices.

Assume \( F \) contains a 6+-vertex. If \( F \) contains a 4-vertex \( v \), then by Lemma 12, \( v \) is a strong 4-vertex. Hence \( F \) receives \( \frac{4}{3} \) from the 6+-vertex by R6 and at least \( \frac{2}{3} \) from the other 4+-vertex by R1, R5 and R6. So \( ch^*(F) \geq 0 \).

If \( F \) contains two 5-vertices, then \( F \) receives 1 from each incident 5-vertex by R5, and hence \( ch^*(F) \geq 0 \).

Otherwise, \( F \) is a light 4-face, and receives 1 from each incident 4+-vertex by R2-R5, and hence \( ch^*(F) \geq 0 \).

**Case 2** \( F \) contains one 3-vertex.

If \( F \) contains no very weak 4-vertex, then every 4+-vertex in \( F \) sends at least \( \frac{2}{3} \) to \( F \) by R1, R5 and R6. Thus \( ch^*(F) \geq ch(F) + 3 \times \frac{2}{3} = 0 \).

Assume that \( F \) contains a very weak 4-vertex. If \( F \) is a \((4, 4, 4, 3)\)-face, then \( ch^*(F) \geq ch(F) + 3 \times \frac{2}{3} = 0 \) by R1 and R4 (i). Assume that \( F \) is not a \((4, 4, 4, 3)\)-face. Then \( F \) contains a 5+-vertex. By Lemma 15, \( F \) contains at most one very weak 4-vertex.

Thus \( ch^*(F) \geq ch(F) + 1 + \frac{2}{3} + \frac{1}{3} = 0 \) by R1, R4 (ii) and R5.

**Case 3** \( F \) contains no 3-vertex.

Assume \( F \) is a \((4, 4, 4, 4)\)-face. If no vertex of \( F \) is incident with \((4, 3, 5, 3)\)-face, then each vertex \( v \) of \( F \) has at most one 3-neighbor and hence has at most one special 5-face. So \( ch^*(F) \geq ch(F) + 4 \times \frac{1}{2} = 0 \) by R3.
If $F$ has a vertex $v$ incident with a $(4, 3, 5, 3)$-face, then $F$ contains at most two weak vertices by Lemma 16. Thus $\text{ch}^*(F) \geq \text{ch}(F) + 2 \times \frac{2}{3} + 2 \times \frac{2}{3} = 0$ by R1 and R3.

Assume $F$ is a $(4^+, 4^+, 4^+, 5^+)$-face. Then $\text{ch}^*(F) \geq \text{ch}(F) + 1 + 3 \times \frac{1}{3} = 0$ by R3 and R5.

This completes the check for 4-faces. \hfill $\Box$

**Lemma 19** If $F$ is a 5-face, then $\text{ch}^*(F) \geq 0$.

**Proof** Assume $F = (v_1, v_2, v_3, v_4, v_5)$ is a 5-face, and for $i = 1, 2, 3, 4, 5$, let $F_i$ be the face sharing the edge $v_iv_{i+1}$ with $F$ (the indices are modulo 5).

By Corollary 6, either $F$ contains at least three $4^+$-vertices or $F$ contains two $4^+$-vertices and one of them is a $5^+$-vertex.

If $F$ contains no weak and no very weak $4$-vertex, or $F$ is a special 5-face of every weak and very weak 4-vertex, then $F$ receives at least $\frac{1}{6}$ from each incident 4-vertex and $\frac{2}{3}$ from each incident 5$^+$-vertex by R1-R5. Hence $\text{ch}^*(F) \geq \text{ch}(f) + \frac{1}{3} + \frac{2}{3} = 0$.

Assume $F$ contains a weak or a very weak 4-vertex, say $v_1$, and $F$ is a non-special 5-face of $v_1$.

**Case 1** $v_1$ is a weak 4-vertex.

By symmetry, we may assume that $F_2$ is a light 4-face. Then $F_1$ is a 4-face with no 3-vertex, and hence $v_2$ is a $4^+$-vertex.

If $F_2$ is a $(4, 5, 3, 3)$-face, then since $F$ is a non-special 5-face of $v_1$, $v_5$ is a 5-vertex. Then $w(v_5 \rightarrow F) = 2/3$ and $w(v_i \rightarrow F) \geq 1/6$ for $i = 1, 2$. So $\text{ch}^*(F) \geq \text{ch}(F) + \frac{2}{3} + \frac{1}{6} \times 2 = 0$.

Assume $F_2$ is a $(4, 4, 3, 3)$-face. Each of $v_1, v_5$ sends at least 1/6 to $F$. If $F$ contains a 5$^+$-vertex, then $\text{ch}^*(F) \geq \text{ch}(F) + \frac{2}{3} + \frac{1}{6} \times 2 = 0$. Assume $F$ contains no 5$^+$-vertex. So by Corollary 6, $v_2$ and $v_4$ are 4-vertices. By Lemma 12, none of $F_1$ and $F_4$ is a light 4-face.

Assume $v_3$ is a 3-vertex. If $v_3$ is a weak or very weak 4-vertex for $i = 2, 4$, then $F$ is a special 5-face of $v_j$. So $v_j$ sends 1/3 to $F$ by R3, R4. If $v_j$ is a strong or normal 4-vertex, then $v_j$ also sends 1/3 to $F$ by R1, R2. Hence $\text{ch}^*(F) \geq \text{ch}(F) + \frac{1}{6} \times 2 + \frac{1}{3} \times 2 = 0$.

Assume $v_3$ is a 4-vertex. Then $F$ is a $(4, 4, 4, 4, 4)$-face. By Observation 1, each 4-vertex sends at least 1/6 to $F$. As $F$ is adjacent to at most two light 4-faces, at least one of the 4-vertices sends 1/3 to $F$. Hence $\text{ch}^*(F) \geq \text{ch}(F) + \frac{1}{6} \times 4 + \frac{1}{3} = 0$.

**Case 2** $v_1$ is a very weak 4-vertex.

Assume $F_2$ is a light 4-face. Then $F_1$ is a 4-face containing one 3-vertex. Note that $F_5$ is not a $(4, 3, 5, 3)$-face, for otherwise, $F$ is a special 5-face of $v_1$.

Assume first that $F_1$ is a $(4, 4, 4, 3)$-face. By Lemma 13, $F_5$ is a $(4, 5, 3, 3)$-face. Hence $v_5$ is a 5-vertex. If $v_2$ is a very weak 4-vertex, then either $v_3$ is a $5^+$-vertex.
or $F$ is a special 5-face of $v_2$. In both cases $ch^*(F) \geq ch(F) + 1 = 0$. If $v_2$ is a normal or a strong 4-vertex, then $w(v_5 \to F) = 2/3$ and $w(v_2 \to F) = 1/3$. Hence $ch^*(F) \geq ch(F) + 1 = 0$.

Assume $v_2$ is a 3-vertex. Then $v_3$ is a 4*-vertex, for otherwise $G$ contains a $(3, 3, 4, 3, 3)$-path, which is reducible. Note that $v_3$ is not weak or very weak, for otherwise there is a reducible $(3, 3, 4, 3, 3, 3)$-path. So $w(v_3 \to F) \geq 1/3$ and $ch^*(F) \geq ch(F) + 2/3 + 1/3 = 0$.

Assume $F_1$ is not a $(4, 4, 4, 3)$-face. By Case (1), $F$ is a special 5-face of $v$ for every weak 4-vertex $v$. So each 4-vertex of $F$ sends at least 1/3 to $F$ and each 5*-vertex sends at least 2/3 to $F$. Hence $ch^*(F) \geq ch(F) + 1 = 0$.

This completes the proof of Theorem 1.

Availability of Data and Material Not applicable.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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