PARTLY-LOCAL DOMAIN-DEPENDENT ALMOST
COMPLEX STRUCTURES

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Abstract. We fill a gap pointed out by Nick Sheridan in the proof of
independence of genus zero Gromov-Witten invariants from the choice of
divisor in the Cieliebak-Mohnke perturbation scheme [1].

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1. Introduction

A convenient scheme to regularize moduli spaces of genus zero pseudoholo-
morphic maps was introduced by Cieliebak-Mohnke [1]. In this scheme one
chooses a Donaldson hypersurface to stabilize the domains and then a generic
domain-dependent almost complex structure to achieve regularity. The scheme
leads to a definition of genus zero Gromov-Witten invariants over the rationals
that counts some geometric objects, as opposed to the more abstract perturba-
tion schemes in the Kuranishi or polyfold methods. The proof of independence
of Gromov-Witten invariants from the choice of Donaldson hypersurfaces in
[1, 8.18] depends on the construction of a parametrized moduli space for the
following situation: Given a type Γ of stable marked curve let \( \mathcal{U}_\Gamma \to \overline{\mathcal{M}}_\Gamma \)
denote the universal curve over the compactified moduli space \( \overline{\mathcal{M}}_\Gamma \) of curves
of type Γ. Let \( J_\tau(X,\omega) \) denote the space of \( \omega \)-tamed almost complex struc-
tures on the given symplectic manifold \( (X,\omega) \) with rational symplectic class
\( [\omega] \in H^2(X,\omega) \). A domain-dependent almost complex structure is a map
\[
J_\Gamma : \mathcal{U}_\Gamma \to J_\tau(X,\omega).
\]
Associated to a coherent collection of sufficiently generic choices \( J = (J_\Gamma) \) is a
Gromov-Witten pseudocycle \( \overline{\mathcal{M}}_{0,n}(X,\beta) \subset X^n \) for each number of markings
\( n \) and each class \( \beta \in H_2(X) \).
Naturally one wishes to show that the resulting pseudocycle is independent, up to cobordism between pseudocycles, from the choice of Donaldson hypersurface. Suppose that \( V', V'' \subset X \) are two Donaldson hypersurfaces and \( J' = (J'_{\Gamma}) \), \( J'' = (J''_{\Gamma}) \) are two collections of domain dependent almost complex structures depending on the intersection points with \( V' \) resp. \( V'' \), depending on some combinatorial type \( \Gamma' \) resp. \( \Gamma'' \). Consider the pullback

\[
(f'')^* J'_{\Gamma'}, \ (f')^* J''_{\Gamma''} : \mathcal{U}_{\Gamma} \to \mathcal{J}(X, V', V'')
\]

to a common universal curve \( \mathcal{U}_{\Gamma} \) for some type \( \Gamma \) recording both sets of markings (so that if \( \Gamma' \) resp. \( \Gamma'' \) has \( n' \) resp. \( n'' \) leaves then \( \Gamma \) has \( n' + n'' \) leaves). One wishes to construct a homotopy between \( (f'')^* J'_{\Gamma'}, \ (f')^* J''_{\Gamma''} \) to construct a cobordism between the corresponding pseudocycles \( \overline{\mathcal{M}}_n(X, \beta) \) and \( \overline{\mathcal{M}}_{n'}(X, \beta) \). Unfortunately, as pointed out by Nick Sheridan, the pullbacks \( (f'')^* J'_{\Gamma'}, \ (f')^* J''_{\Gamma''} \) do not satisfy the locality condition used to show compactness. That is, the restriction of the almost complex structures \( (f'')^* J'_{\Gamma'}, \ (f')^* J''_{\Gamma''} \) to some irreducible component \( C_v \) of the domain curve \( C \) are not independent of markings on other components \( C_v' \neq C_v \), because collapsed components \( C_v \) may map to non-special points \( f'(C_v) = \{w\} \in f'(C) \) under the forgetful map \( f' \).

In this note we modify the definition of the locality on the collapsed components so that one may homotope between the two domain-dependent almost complex structures without losing compactness. Instead of directly homotoping between the given pull-backs, one first homotopes each pullback to an almost complex structure that is equal to a base almost complex structure near any special point.

2. Partly local perturbations

We introduce the following notation for stable maps with two types of markings. Let \( \Gamma \) be a combinatorial type of genus zero stable curve with \( n = n' + n'' \) markings. Let \( V', V'' \) be Donaldson hypersurfaces in the symplectic manifold \((X, \omega)\), that is, symplectic hypersurfaces representing large multiples \( k'\lbrack \omega \rbrack \) resp. \( k''\lbrack \omega \rbrack \) of the symplectic class \( \lbrack \omega \rbrack \in H^2(X, \mathbb{Q}) \). Suppose \( V' \) and \( V'' \) intersect transversely. Let \( \mathcal{J}(X, V', V'') \) be the space of \( \omega \)-tamed almost complex structures on \( X \) that make \( V' \) and \( V'' \) almost complex. Let \( \mathcal{J}^E(X, V', V'') \subset \mathcal{J}(X, V', V'') \) be some contractible subset of almost complex structures \( J : TX \to TX \) preserving \( TV' \) and \( TV'' \) taming the symplectic form \( \omega \) and so that any non-constant pseudoholomorphic \( J \)-holomorphic map \( u : C \to X \) with some given energy bound \( E(u) < E \) to \( X \) meets \( V', V'' \) each in at least three but finitely many distinct points \( u^{-1}(V'), u^{-1}(V'') \) in the domain.
\( C \) as in \([1, 8.18]\). Let 
\[ J_{V',V''} \in \bigcap_E J^E(X,V',V'') \]
be a base almost complex structure that satisfies these conditions without restriction on the energy of the map \( u : C \to X \).

The universal curve breaks into irreducible components corresponding to the vertices of the combinatorial type. Let \( U_\Gamma \to M_\Gamma \) be the closure of the universal curve of type \( \Gamma \). For each vertex \( v \in \text{Vert}(\Gamma) \) let \( \Gamma(v) \) denote the tree with the single vertex \( v \) and edges those of \( \Gamma \) meeting \( v \). Let \( U_{\Gamma,v} \subset U_\Gamma \) be the component corresponding to \( v \), obtained by pulling back \( U_{\Gamma(v)} \) so that \( U_\Gamma \) is obtained from the disjoint union of the curves \( U_{\Gamma,v} \to M_\Gamma \) by identifying at nodes.

Cieliebak-Mohnke \([1]\) requires that the almost complex structure is equal to the base almost complex structure near the nodes. This condition is not true for domain-dependent almost complex structures pulled back under forgetful maps, and so must be relaxed as follows. Recall that Knudsen’s (genus zero) universal curve \( U_\Gamma \) \([3]\) is a smooth projective variety, and in particular a complex manifold. A \textit{domain-dependent almost complex structure} for type \( \Gamma \) of stable genus zero curve is an almost complex structure 
\[ J_\Gamma : T(U_\Gamma \times X) \to T(U_\Gamma \times X) \]
that preserves the splitting of the tangent bundle \( T(U_\Gamma \times X) \) into factors \( T \times TX \) and that is equal to the standard complex structure on the tangent space to the projective variety \( U_\Gamma \), and gives rise to a map from \( U_\Gamma \) to \( J(X,V',V'') \) with the same notation \( J_\Gamma \). Let 
\[ J^E_\Gamma(X,V',V'') \subset \text{Map}(U_\Gamma,J^E(X,V',V'')) \]
denote the space of such maps taking values in \( J^E(X,V',V'') \). With this definition, the standard proof of Gromov convergence applies: Any sequence \( u_\nu : C_\nu \to X \) of \( J_\Gamma \)-holomorphic maps with energy \( E(u) < E \) may be viewed as a finite energy sequence of maps to \( U_\Gamma \times X \). Therefore it has a subsequence with a Gromov limit \( u : C \to X \) where the stabilization \( C^s \) of \( C \) is a fiber of \( U_\Gamma \) and \( u \) is pseudoholomorphic for the pull-back of the restriction of \( J_\Gamma \) to \( C^s \).

If we restrict to sequences of maps \( u_\nu : C_\nu \to X \) sending the markings to \( V' \) or \( V'' \) then in fact \( C^s \) is equal to \( C \), since non-constant components of \( u \) with fewer than three markings are impossible.

We distinguish components of the curve that are collapsed under forgetting the first or second group of markings. Let 
\[ f' : U_\Gamma \to U_{\Gamma'}, \quad f'' : U_\Gamma \to U_{\Gamma''} \]
denote the forgetful maps forgetting the first \( n' \) resp. last \( n'' \) markings and stabilizing. Call a component of \( C \) \( f'-\text{unstable} \) if it is collapsed by \( f' \), and
\(f'-\text{stable}\) otherwise, in which case it corresponds to a component of \(f'(C)\). \(f''\)-unstable components are defined similarly.

**Definition 2.1. (Local and partly local almost complex structures)**

(a) A domain-dependent almost complex structure

\[ J_\Gamma : \mathcal{U}_\Gamma \to \mathcal{J}(X, V', V'') \]

is *local* if and only if for each \(v \in \text{Vert}(\Gamma)\) the restriction \(J_\Gamma|_{\mathcal{U}_{\Gamma,v}}\) is local in the sense that \(J_\Gamma|_{\mathcal{U}_{\Gamma,v}}\) is pulled back from some map \(J_{\Gamma,v}\) defined on the universal curve \(\mathcal{U}_{\Gamma(v)}\) and equal to \(J_{V,V'}\) near any special point of \(\mathcal{U}_{\Gamma,v}\).

(b) A domain-dependent almost complex structure

\[ J_\Gamma : \mathcal{U}_\Gamma \to \mathcal{J}(X, V', V'') \]

is \(f'-\text{local}\) if and only if

(i) for each \(v \in \text{Vert}(\Gamma)\) such that \(\mathcal{U}_{\Gamma,v}\) is \(f'-\text{stable}\) (that is, has sufficiently many \(V''\) markings) then \(J_\Gamma|_{\mathcal{U}_{\Gamma,v}}\) is local in the sense that \(J_\Gamma|_{\mathcal{U}_{\Gamma,v}}\) is pulled back from some map \(J_{\Gamma,v}\) defined on the universal curve \(\mathcal{U}_{\Gamma(v)}\) and equal to \(J_{V,V'}\) near any point \(z \in C\) mapping to a special point \(f'(z)\) of \(f'(C)\), and

(ii) for each \(v \in \text{Vert}(\Gamma)\) such that \(\mathcal{U}_{\Gamma,v}\) is \(f'-\text{unstable}\) (that is, does not have sufficiently many \(V''\) markings) then \(J_\Gamma|_{\mathcal{U}_{\Gamma,v}}\) is constant on each fiber of \(\mathcal{U}_{\Gamma,v}\).

The definition of \(f''\)-local is similar.

**Remark 2.2.** Note that \(f'\)-pullbacks \((f')^*J_{\Gamma''}\) are \(f'-\text{local}\), and local almost complex structures are \(f'-\text{local}\). The condition that an almost complex structure be \(f'-\text{local}\) is weaker than the condition that it be pulled back under \(f'\), because the restriction \(J_\Gamma|_{\mathcal{U}_{\Gamma,v}}\) is allowed to depend on special points \(z \in C\) that are forgotten under \(f'\).

**Remark 2.3.** One can reformulate the \(f'-\text{local}\) condition as a pullback condition for a forgetful map that forgets almost the same markings as those forgotten by \(f'\). Let \(C\) be a curve of type \(\Gamma\). Let \(C^{\text{us}} \subset C\) be the locus collapsed by \(f'\). For each connected component \(C_i, i = 1,\ldots,k\) of \(C^{\text{us}}\) mapping to a marking of \(f'(C)\) choose \(j(i)\) so that \(z_{j(i)} \in C_i\). Let \(I^{\text{us}} \subset \{1,\ldots,n\}\) denote the set of indices \(j\) of markings \(z_j \in C^{\text{us}}\) with \(z_j \neq z_{j(i)}, \forall i\). Forgetting the markings with indices in \(I^{\text{us}}\) and collapsing defines a map \(f : C \to f(C)\) such that any collapsed component of \(C\) maps to a special point of \(f(C)\). Let \(\Gamma'\) denote the combinatorial type of \(f(C)\). Then \(J_\Gamma\) is \(f'-\text{local}\) if and only if \(J_\Gamma = f^*J_{\Gamma'}\) is pulled back from a local domain-dependent almost complex structure \(J_{\Gamma'} : \mathcal{U}_{\Gamma'} \to \mathcal{J}(X, V', V'')\). Indeed, the collapsed components under \(C \to f(C)\) are the same as those of \(f' : C \to f'(C)\) since adding a single
Lemma 2.5. The space of structures is contractible, it suffices to show the existence of an extension almost complex structures. Since the space of $f$ is contractible. Any irreducible component of $f(C)$ is isomorphic, as a stable marked curve, to an irreducible component of $C$ not collapsed under $f'$.

Remark 2.4. There also exist domain-dependent almost complex structures that are both $f'$ and $f''$-local. Indeed suppose that $C$ is a curve of type $\Gamma$, and $K \subset \{1, \ldots, n' + n''\}$ is the set of markings on components collapsed by $f'$ or $f''$. Forgetting the markings $z_k$, $k \in K$ defines a forgetful map $f^{ss} : C \to f^{ss}(C)$, where $f^{ss}(C)$ is of some (possibly empty) type $\Gamma^{ss}$. Let $J_{\Gamma^{ss}} : \overline{U}_{\Gamma^{ss}} \to J(X, V', V'')$ be a domain-dependent almost complex structure for type $\Gamma^{ss}$. Then $(f^{ss})^* J_{\Gamma^{ss}}$ is both $f'$ and $f''$-local (taking the constant structure $J_{\Gamma^{ss}}$ if $\Gamma^{ss}$ is empty.)

Lemma 2.5. The space of $f'$-local resp. $f''$-local resp. $f'$ and $f''$-local almost complex structures tamed by or compatible with the symplectic form $\omega$ is contractible. Any $f'$-local resp. $f''$-local resp. $f'$ and $f''$-local $J_{\Gamma} | _{\partial \overline{U}_{\Gamma}}$ defined on the boundary $\partial \overline{U}_{\Gamma} := \overline{U}_{\Gamma} | _{\partial \overline{M}_{\Gamma}}$ extends to a $f'$-local resp. $f''$-local resp. $f'$ and $f''$-local structure $J_{\Gamma}$ over an open neighborhood of the boundary $\partial \overline{U}_{\Gamma}$ in $\overline{U}_{\Gamma}$.

Proof. Contractibility follows from the contractibility of tamed or compatible almost complex structures. Since the space of $f'$-local tamed almost complex structures is contractible, it suffices to show the existence of an extension of $J_{\Gamma}$ near any stratum $U_{\Gamma_1} \subset \overline{U}_{\Gamma}$ and then patch together the extensions. Local domain-dependent almost complex structures $J_{\Gamma}$ extend by a gluing construction in which open balls $U_+, U_-$ around a node are replaced by a punctured ball $V \cong U_+^\circ \cong U_-^\circ$ on which the almost complex structure is equal to the base almost complex structure $J_{V', V''}$.

In the partly-local case recall from Remark 2.3 that $J_{\Gamma}$ is the pull-back of a local almost complex structure $J_{\Gamma \uparrow}$ near any particular fiber of the universal curve. Define an extension of $J_{\Gamma \uparrow}$ near curves of type $\Gamma_1$ by first extending $J_{\Gamma \uparrow}$ and then pulling back. In more detail, let $C$ be such a curve and let $C_1, \ldots, C_k$ denote the connected components of $C$ collapsed by $f'$ to a non-special point of $f'(C)$. Choose a marking $z_i \in C_i$ and let $\Gamma^s$ resp. $\Gamma^t$ denote the type obtained from $\Gamma$ resp. $\Gamma_1$ by forgetting all markings on $C_i$ except $z_i$, for each $i = 1, \ldots, k$. Consider the forgetful map $f : \overline{U}_{\Gamma} \to \overline{U}_{\Gamma \uparrow}$ that forgets all but the marking $z_i$ on $C_i$. As discussed in Remark 2.3 $J_{\Gamma \uparrow}$ is the pullback of a complex structure $J_{\Gamma_1 \uparrow} : \overline{U}_{\Gamma_1 \uparrow} \to J(X, V', V'')$. 


Since the complex structure $J_{1,1}$ is constant equal to the base almost complex structure $J_{V',V''}$ near the nodes (which must join non-collapsed components) $J_{1,1}$ naturally extends to a domain-dependent almost complex structure $J_{1,1}$ on a neighborhood $\mathcal{N}_{1,1}$ of $\mathcal{U}_{1,1}$ in $\mathcal{U}_{1,1}$ by taking $J_{1,1}$ to equal $J_{V',V''}$ near the nodes. Now take $J_{1} = f^{*}J_{1,1}$ to obtain an extension of $J_{1}$ from $\mathcal{U}_{1,1}$ to a neighborhood $f^{-1}(\mathcal{N}_{1,1})$. The proof for $f'$ local or $f'$ and $f''$-local structures is similar. \hfill $\square$

3. Transversality

We wish to inductively construct partly-local almost complex structures so that the moduli spaces of stable maps define pseudocycles. Recall that the combinatorial type of a stable map is obtained from the type of stable curve by decorating the vertices with homology classes; we also wish to record the intersection multiplicities with the Donaldson hypersurfaces. More precisely, a type of stable map $u$ from $C$ to $(X,V',V'')$ consists of a type $\Gamma$ the stable curve $C$ (the graph with vertices corresponding to components and edges corresponding to markings and nodes) with the labelling of vertices $v \in \text{Vert}(\Gamma)$ by homology class $d(v) = [u[C_v]] \in H_2(X)$, labelling of the semi-infinite edges $e$ by either $V'$ or $V''$,\footnote{To obtain evaluation maps one should allow additional edges, but here we ignore evaluation maps.} and by the intersection multiplicities $m'(e), m''(e)$ with $V'$ and $V''$ (possibly zero if the corresponding marking does not map to $V'$ or $V''$). A stable map is adapted of type $\Gamma$ if each connected component of $u^{-1}(V')$ resp. $u^{-1}(V'')$ contains at least one marking $z_e$ corresponding to an edge $e$ labelled $V'$ resp. $V''$, and each marking $z_e$ maps to $V'$ or $V''$ depending on its label. A stable map is adapted of type $\Gamma$ if

(a) each connected component of $u^{-1}(V')$ resp. $u^{-1}(V'')$ contains at least one marking $z_e$ corresponding to an edge $e$ with labelling $m'(e) \geq 1$ resp. $m''(e) \geq 1$, and

(b) if $m'(e) \geq 1$ resp. $m''(e) \geq 1$, then the marking $z_e$ is mapped to $V'$ resp. $V''$.

By forgetting the extra data and stabilization one can associate to each type of stable maps to a type of stable curves. In notation we do not distinguish the two notions of types. Given a type of stable map $\Gamma$ choose a domain-dependent almost complex structure $J_{\Gamma}$. Denote by $\mathcal{M}_{\Gamma}(X, J_{\Gamma})$ the moduli space of adapted $J_{\Gamma}$-holomorphic stable maps $u : C \to X$ of type $\Gamma$, such that for each $v \in \text{Vert}(\Gamma)$ with $d(v) \neq 0$, the image of $u_v$ is not contained in $V' \cup V''$, and for each semi-infinite edge $e$ attached to $v$, the local intersection number of $u_v$ with $V'$ resp. $V''$ at $z_e$ is equal to $m'(e)$ resp. $m''(e)$. The moduli space $\mathcal{M}_{\Gamma}(X, J_{\Gamma})$ is locally cut out by a smooth map of Banach manifolds: Given a local trivialization of the universal curve given by an subset $\mathcal{M}_{\Gamma}^0 \subset \mathcal{M}_{\Gamma}$.
and a trivialization $C \times \mathcal{M}_\Gamma \rightarrow \mathcal{U}_\Gamma = \mathcal{U}_\Gamma|_{\mathcal{M}_\Gamma}$, we consider the space of maps $\text{Map}(C, X)_{k,p}$ of Sobolev class $k, p$ for $p \geq 2$ satisfying the above constraints and $k$ sufficiently large to the space of 0,1-forms with values in $TX$ given by the Cauchy-Riemann operator $\overline{\partial}_J$, associated to $J_\Gamma$. The linearization of this operator is denoted $D_u$ (or $D_u, J_\Gamma$ to emphasize dependence on $J_\Gamma$) and the map $u$ is called regular if $D_u$ is surjective. We call a type $\Gamma$ of stable map $u : C \rightarrow X$ crowded if there is a maximal ghost subtree of the domain $C_1 \subset C$ with more than one marking $z_e \in C_1$ and uncrowded otherwise. It is not in general possible to achieve transversality for crowded types using the Cieliebak-Mohnke perturbation scheme.

**Definition 3.1.** We say a domain-dependent almost complex structure $J_\Gamma$ is regular for a type of map $\Gamma$ if

(a) if $\Gamma$ is uncrowded then every element of the moduli space $\mathcal{M}_\Gamma(X, J_\Gamma)$ of adapted $J_\Gamma$-holomorphic maps is regular; and

(b) If $\Gamma$ is crowded then there exists a regular $J_\Gamma^*$ for some uncrowded type $\Gamma^*$ obtained by forgetting all but one marking $z_e$ on each maximal ghost component for curves of type $\Gamma$ such that $J_\Gamma^*$ is equal to $J_\Gamma$ on all non-constant components, that is, all components of $\overline{U}_\Gamma$ on which the maps $u : C \rightarrow X$ in $\mathcal{M}_\Gamma(X, J_\Gamma)$ are non-constant.

Recall the construction by Floer [2, Lemma 5.1] of a subspace of smooth functions with a separable Banach space structure. Let $\xi = (\xi_\ell, \ell \in \mathbb{Z}_{\geq 0})$ be a sequence of constants converging to zero. Let $\mathcal{J}_\Gamma(X)_\xi$ denote the space of domain-dependent almost complex structures of finite Floer norm as in [2, Section 5]. In particular, $\mathcal{J}_\Gamma(X)_\xi$ allows variations with arbitrarily small support near any point.

**Proposition 3.2.** (a) For a regular domain-dependent almost complex structure $J_\Gamma$, the pull-back $(f^*)^*J_\Gamma^*$ is regular, and similarly for the pull-back $(f^*)^*J_\Gamma$ for regular $J_\Gamma$.

(b) Suppose that $J_\Gamma|_{\partial \overline{U}_\Gamma}$ is $f'$-local and is a regular domain-dependent almost complex structure defined on the boundary $\partial \overline{U}_\Gamma \rightarrow \partial \overline{M}_\Gamma$. The set of regular $f'$-local extensions is comeager, that is, is the intersection of countably many sets with dense interiors.

(c) Any parametrized-regular homotopy $J_{\Gamma,t}|_{\partial \overline{U}_\Gamma}$ between two regular $f'$-local domain-dependent almost complex structures $J_{\Gamma,0}, J_{\Gamma,1}$ on the boundary $\partial \overline{U}_\Gamma$ may be extended to a parametrized-regular one-parameter family of $f'$-local structures $J_{\Gamma,t}$ equal to $J_{\Gamma,t}$ over $\overline{U}_\Gamma$.

**Proof.** Item (a) is immediate from the definition, since any variation of $J_\Gamma$ induces a variation of $(f^*)^*J_\Gamma$. (b) is an application of Sard-Smale applied to a universal moduli space. We sketch the proof which is analogous to that in Cieliebak-Mohnke [1, Chapter 5]. By Lemma 2.5, $J_\Gamma|_{\partial \overline{U}_\Gamma}$ has an extension
over the interior. For transversality, first consider the case of an uncrowded
type $\Gamma$ of stable map. Choose open subsets $L_{\Gamma}, N_{\Gamma} \subset \overline{U}_{\Gamma}$ of the boundary
resp. markings and nodes, such that $L_{\Gamma}$ is union of fibers of $\overline{U}_{\Gamma}$ containing the
restriction $\overline{U}_{\Gamma}|\partial M_{\Gamma}$ and $N_{\Gamma}$ is sufficiently small so that the intersection of the complement of each fiber of $U_{\Gamma}$ not meeting $L_{\Gamma}$
is non-empty. Let $\hat{M}^{\text{univ}}_{\Gamma}(X)$ denote the universal moduli space consisting of
pairs $(u, J_{\Gamma})$, where $u : C \to X$ is a $J_{\Gamma}$-holomorphic map of some Sobolev class
$W^{k,p}, kp \geq 3, p \geq 2$ on each component (with $k$ sufficiently large so that the
given vanishing order at the Donaldson hypersurfaces $V', V''$ is well-defined).
Let $J_{\Gamma}^{E}(X, N_{\Gamma}, S_{\Gamma}) \subset J_{\Gamma}^{E}(X)$ denote the space of $J_{\Gamma} \in J_{\Gamma}^{E}(X)_{\xi}$ that are $f'$-local
domain-dependent almost complex structures that agree with $J_{\Gamma}|_{V', \Gamma''}$ on the
neighborhood $N_{\Gamma}$ of the nodes and markings $z \in \overline{U}_{\Gamma}$ that map to special points $f'(z) \in \overline{U}_{\Gamma}|_{\Gamma}$ as in Definition 2.1, and equal to the given extension in the
neighborhood $L_{\Gamma}$ of the boundary, and constant on the components required by $f'$-locality in Definition 2.1. By elliptic regularity, $M_{\Gamma}^{\text{univ}}(X)$ is independent of the choice of Sobolev exponents.

The universal moduli space is a smooth Banach manifold by an application of the
implicit function theorem for Banach manifolds. Let $U_{\Gamma}^{i} \to \hat{M}^{\text{univ}}_{\Gamma}, i = 1, \ldots, m$ be a collection of open subsets of the universal curve $U_{\Gamma} \to \hat{M}^{\text{univ}}_{\Gamma}$ on
which the universal curve is trivialized via diffeomorphisms $U_{\Gamma}^{i} \to \hat{M}^{\text{univ}}_{\Gamma} \times C$.
The space of pairs $(u : C \to X, J_{\Gamma})$ with $[C] \in \hat{M}^{\text{univ}}_{\Gamma}, u$ of type $\Gamma$ of class
$W^{k,p}$ on each component, and $J_{\Gamma} \in J_{\Gamma}^{E}(X, N_{\Gamma}, S_{\Gamma})$ is a smooth separable
Banach manifold. Since we assume that $J_{\Gamma}$ is regular on the boundary $\partial U_{\Gamma}$, an
argument using Gromov compactness shows that by choosing $L_{\Gamma}$ sufficiently
small we may assume that $\tilde{D}_{u, J_{\Gamma}}$ is surjective for $[C] \in L_{\Gamma}$, since regularity
is an open condition in the Gromov topology [4, Section 10.7]. Let $\tilde{D}_{u, J_{\Gamma}}$, the
linearization of $(u, J_{\Gamma}) \mapsto \partial J_{\Gamma} u$, and suppose that $\eta$ lies in the cokernel of
$\tilde{D}_{u, J_{\Gamma}}$. We have $D_{u}^{\star} \eta^{\star} = 0$ where $D_{u}$ is the usual linearized Cauchy-Riemann
operator [4, p. 258] for the map; in the case of vanishing constraints at the
Donaldson hypersurfaces see Cieliebak-Mohnke [1, Lemma 6.6]. By variation
of the almost complex structure $J_{\Gamma}$ and unique continuation, $\eta$ vanishes on any
component on which $u$ is non-constant. On the other hand, for any constant
component $u_{v} : C_{v} \to X$, the linearized Cauchy-Riemann operator $D_{u_{v}}$ on a
trivial bundle $u_{v}^{\star} TX$ is regular with kernel $\ker(D_{u_{v}})$ the space of constant maps
$\xi : C_{u} \to (u_{v})^{\star} TX$. It follows by a standard inductive argument that the same
holds true for a tree $C' = \cup_{v \in \nu} C_{v}, du_{v}|_{C'} = 0$ of constant pseudoholomorphic
spheres so the element $\eta$ vanishes on any component $C_{v} \subset C$ on which $u$ is
constant. It follows that $M_{\Gamma}^{\text{univ}, j}(X)$ is a smooth Banach manifold. For a
comeager subset $J_{\Gamma}^{\text{reg}}(X) \subset J_{\Gamma}(X)$ of partly almost complex structures in the
space above, the moduli spaces $M_{\Gamma}(X) = M_{\Gamma}(X)|_{M_{\Gamma}^{\text{univ}}}$ are transversally cut
out for each $i = 1, \ldots, m$. The transition maps between the local trivializations
\[ \mathcal{M}_f \cap \mathcal{M}_g \to \text{Aut}(C) \] induce smooth maps \[ \mathcal{M}_f(X)|_{\mathcal{M}_f \cap \mathcal{M}_g} \to \mathcal{M}_g(X)|_{\mathcal{M}_f \cap \mathcal{M}_g} \] making \( \mathcal{M}_f(X) \) into a smooth manifold.

Next consider a crowded type \( \Gamma \). Let \( f : \Gamma \to f(\Gamma) \) be a map forgetting all but one marking on each maximal ghost component \( C' \subset C \) and stabilizing; the multiplicities \( m'(e), m''(e) \) at any marking \( z_e \) is the sum of the multiplicities of markings in its preimage \( f^{-1}(z_e) \). Define \( J_{\Gamma} f \) as follows.

(a) If \( U_{f'}, u \sim U_{f''}, v \) let \( J_{\Gamma} f|_{U_{f'}, v} \) be equal to \( J_{f'} f|_{U_{f'}, v} \).

(b) Otherwise let \( J_{\Gamma} f : U_{f', v} \to J^{E}(X, V', V'') \) be constant equal to \( J_{V', V''} \).

The map \( J_{\Gamma} f \) is continuous because any non-collapsed ghost component \( C_v \subset C \) must connect at least two non-ghost components \( C_{v_1}, C_{v_2} \subset C \) and the connecting points of the non-ghost components \( f^i(C_{v_1}), f^i(C_{v_2}) \) is a node of the curve \( f^j(f(C)) \) of type \( f^j(\Gamma^f) \). For a comeager subset of \( J_{\Gamma} \) described above, the complex structures \( J_{\Gamma} f \) are also regular by the argument for uncrowded types. Item (c) is a parametrized version of (b).

**Corollary 3.3.** There exists a regular homotopy \( J_{\Gamma, t}, t \in [-1, 1] \) between \( (f'')^* J_{\Gamma'} \) and \( (f')^* J_{\Gamma''} \) in the space of maps \( U_{f'} \to J(X, V', V'') \) that are \( f' \)-local for \( t \in [-1, 0] \) and \( f'' \)-local for \( t \in [0, 1] \).

**Proof.** Let \( J = (J_{\Gamma}) \) be a collection of regular domain-dependent almost complex structures that are both \( f' \) and \( f'' \)-local, as in Remark 2.4. By part (b) above, for each type \( \Gamma \) there exists a regular homotopy from \( J_{\Gamma} \) to \( (f')^* J_{\Gamma''} \) resp. \( (f'')^* J_{\Gamma'} \) extending given homotopies on the boundary. The existence of a regular homotopy now follows by induction. \( \square \)

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