Yukawa Couplings involving Excited Twisted Sector States for $\mathbb{Z}_N$ and $\mathbb{Z}_M \times \mathbb{Z}_N$ Orbifolds.

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ABSTRACT

We study the Yukawa couplings among excited twist fields which might arise in the low-energy effective field theory obtained by compactifying the heterotic string on $\mathbb{Z}_N$ and $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds.
1. Introduction

Of all known conformal field theories, orbifold models [1,3] offer the best solution to the heterotic string equations of motion with regard to a phenomenologically promising string background. The two-dimensional conformal orbifold field theory is described by the Euclidean action

\[ S = \frac{1}{\pi} \int d^2z \left( \partial_z X^i \partial_{\bar{z}} \bar{X}^i + \partial_{\bar{z}} X^i \partial_z \bar{X}^i \right), \]

(1.1)

with \( X^i \) the three complexified coordinates of the orbifold target space \( \mathcal{O} \). This space can be described as a six dimensional Euclidean space \( \mathcal{E} \) with its points identified under the action of a space group \( \mathcal{S} \) whose elements consists of both discrete rotations and translations. These elements are represented by the pair \((\alpha, l)\), where \( \alpha \) constitutes the point group of the orbifold and \( l \) is a vector taking values on a lattice on which the point group acts as an automorphism. The generator of the point group, denoted by \( \theta \), must be a discrete subgroup of \( SU(3) \) in order to obtain an effective four dimensional theory with \( N = 1 \) supersymmetry. If one restricts attention to abelian groups, two possibilities arise: \( \mathbb{Z}_N \) and \( \mathbb{Z}_M \times \mathbb{Z}_N \). A \( \mathbb{Z}_N \) group is generated by a twist \( \theta \) with \( \theta^N = 1 \) and the point group is then given by \( P = \{ \alpha = \theta^n; n = 0, 1, ..., N - 1 \} \). The action of \( P \) on the complex coordinates \( X^i \) is defined by

\[ \theta X^i = e^{2\pi iv^i} X^i, \quad \sum_i v^i = 0. \]

(1.2)

Similarly \( \mathbb{Z}_M \times \mathbb{Z}_N \) is generated by two twists \( \theta \) and \( \omega \), with \( \theta^M = 1, \omega^N = 1 \) where \( N \) is a multiple of \( M \). Details of the study and classification of these orbifolds can be found in [1, 3, 5]. For a particular model, it is always possible to find at least one lattice on which the point group acts as an automorphism. In the case of \( \mathbb{Z}_M \times \mathbb{Z}_N \), both point groups should be realized as automorphisms on the same lattice.

In addition to the twists acting on the left-moving supersymmetric sector of the heterotic string, a gauge twisting group \( \mathcal{G} \) is also introduced. This group acts on
the right-moving gauge degrees of freedom which, in the bosonic representation, can be realized in terms of shifts \( V \) on the \( E_8 \times E_8 \) lattice. These shifts are restricted by the modular invariance of the orbifold partition function \([1,4]\). For standard embedding of the point group into \( \mathcal{G} \) one obtains models with left and right global world sheet N=2 supersymmetry – (2,2) orbifolds. A generic choice of embedding (with or without Wilson lines) results in models with only left-moving N=2 supersymmetry – (0,2) orbifolds. Models with Wilson lines break the gauge group to a lower rank giving semi-realistic four dimensional models \([23]\).

The identification of the point \( x \) of the orbifold with \( \alpha x + l \) implies that a twisted string in the \( \alpha \) twisted sector is closed due to the action of the element \( g = (\alpha, (1-\alpha)f) \), where \( f \) is a fixed point or torus under the action of the point group element \( \alpha \). More precisely, \( g \) represents a conjugation class \( hgh^{-1} \), \( h \in \mathcal{S} \) which for prime orbifolds takes the form \( (\alpha, (1-\alpha)(f+\lambda)) \), where \( \lambda \) is an arbitrary lattice vector. For every fixed point (or torus) \( f \) in a given twisted sector \( \alpha \) one associates a vacuum state created by the action of a twist field \( \sigma_{\alpha,f} \) on the \( SL(2,\mathbb{C}) \) Neveu-Schwarz (NS) invariant vacuum. These fields are analogous to the spin fields which transform an NS vacuum into a Ramond (R) vacuum. However, unlike spin fields, twist fields have no realizations in terms of free fields and are normally defined in terms of their operator product expansions with the complex string coordinates associated with the orbifold target space \([5]\).

In the various twisted sectors, massless states are created by vertex operators constructed from the heterotic string degrees of freedom and the twist fields \( \sigma_{\alpha} \). In general these states are not physical and one has to take a linear combination of the twist operators. Also, one could construct massless states with excitations. The associated vertex operators will then contain, in addition to the other string degrees of freedom, the so-called excited twist fields defined by the operator product expansions of ground state twist operators with \( \partial_z X^i \) and \( \partial_z \bar{X}^i \). The allowed number of oscillators present in the massless excited states in a given sector can
be determined by using the mass formula for right movers

\[ \frac{m^2}{8} = N_{osc} + h_{KM} + h_{\sigma_\alpha} - 1, \]  

(1.3)

where \( N_{osc} \) is the fractional oscillator number, \( h_{\sigma_\alpha} \) is the conformal dimension of the twist field \( \sigma_\alpha \) and \( h_{KM} \) is the contribution from the gauge part to the conformal dimension of the matter fields. In the general case, for a particular choice of embedding and Wilson lines, one can write down an expression for \( h_{KM} \) in terms of the Casimirs and levels of the relevant gauge Kac–Moody algebras involved.

A partial comparison of orbifold string compactified theories with low energy physics requires the determination of their Yukawa couplings. These Yukawa couplings could then be employed to make connections with phenomenological studies such as the determination of quark and lepton masses and the mixing angles. Because no momenta in the twisted directions are allowed, these Yukawa couplings split into two factors, one determined by the string degrees of freedom and the other is purely a twist correlator. Of particular importance is the exponential dependence of the twist couplings on moduli [5-6] because of its possible bearing on hierarchies [7].

So far, the discussion in the literature has been mostly limited to the couplings involving only twisted sector ground states, Although it has been pointed out how excited twist field correlations can be computed [5]. Also, an example of such a calculation has been given in [6], though using different approach to that of [5]. In this paper we shall extend the discussion to Yukawa couplings involving twisted sector excited states that might arise in \( \mathbb{Z}_N \) and \( \mathbb{Z}_M \times \mathbb{Z}_N \) orbifolds. We have already given a brief discussion for \( \mathbb{Z}_M \times \mathbb{Z}_N \) orbifolds elsewhere [17]. For phenomenological studies, massless particles with the conformal dimensions of the standard model particles will be relevant. Therefore, we focus on massless states in the twisted sectors with the \( SU(3) \times SU(2) \times U(1) \) quantum numbers of quarks, leptons and electroweak Higgses.
It is of particular importance to be able to include these excited states in view of the fact that string threshold loop corrections to gauge coupling constants [18,19], consistent with their low energy values, have so far always involved modular weights for quarks and leptons requiring the use of excited twisted sector states. The $\mathbb{Z}_N$ and $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds considered are those with the point group realised in terms of the Coxeter elements (product of Weyl reflections) of Lie algebra root lattices [10, 21]. This work is organized as follows. Section 2 contains a brief review of the basic features of the conformal field theory of the twist fields. Sections 3 and 4 deal with the classification and the calculation of the Yukawa couplings involving excited twisted fields in $\mathbb{Z}_N$ and $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds respectively.

2. The Conformal Field Theory of Orbifolds

As mentioned in the introduction, a different vacuum state belongs to each fixed point or torus in a given twisted sector. These vacuum states are created by twist fields $\sigma_\alpha$ defined in terms of the operator product expansions

$$
\partial_z X_\sigma_\alpha(w, \bar{w}) \sim (z - w)^{-(1 - \eta_\alpha)} \tau_\alpha(w, \bar{w}),
$$

$$
\partial_{\bar{z}} X_\sigma_\alpha(w, \bar{w}) \sim (z - w)^{-(1 - \eta_\alpha)} \tilde{\tau}_\alpha(w, \bar{w}),
$$

(2.1)

where $\alpha = e^{2i\pi \eta_\alpha}$ and $0 < \eta_\alpha < 1$. The index $i$ labelling the $i$-th complex plane of the 6-dimensional compact manifold has been suppressed in (2.1). Clearly, (2.1) also serves as a definition of four different types of the so-called excited twist fields*. The fields $\sigma_\alpha$ are primary conformal fields satisfying the operator product expansion

$$
T(z)\sigma_\alpha(w) \sim \frac{h_\sigma \sigma_\alpha(w)}{(z - w)^2} + \frac{\partial_w \sigma_\alpha(w)}{(z - w)} + ..., \quad (2.2)
$$

where $T(z)$ is the right moving stress energy tensor of the underlying conformal

* Note that we are using the definitions of the excited twist fields $\tau_\alpha$, $\tau'_\alpha$, $\tilde{\tau}_\alpha$ and $\tilde{\tau}'_\alpha$ used in reference [13], rather than that used in our previous paper [17].
field theory. A similar expression holds for the left moving stress energy tensor \( \bar{T}(\bar{z}) \). The conformal dimensions of the field \( \sigma_\alpha \) are given by \( (h_\sigma, \bar{h}_\sigma) = \left( \frac{1}{2} \eta_\alpha (1 - \eta_\alpha), \frac{1}{2} \eta_\alpha (1 - \eta_\alpha) \right) \). From (2.1) one can read off the various conformal dimensions of the excited twist fields. For example, the conformal dimensions of \( \tau_\alpha \) and \( \tau'_\alpha \) are \( (h_\sigma + \eta_\alpha, \bar{h}_\sigma) \) and \( (h_\sigma + 1 - \eta_\alpha, \bar{h}_\sigma) \) respectively.

Yukawa couplings involving twisted sectors ground states are given by three-point functions involving fermionic and bosonic degrees of freedom. However, the crucial dependence on the deformation parameters or moduli and the particular fixed points or tori of the internal space is entirely contained in bosonic twist fields correlation functions \([5, 6]\) of the type

\[
Z = \prod_{i=1}^{3} Z_i, \tag{2.3}
\]

with

\[
Z_i = < \sigma_{\alpha,f_1}^i(z_1, \bar{z}_1) \sigma_{\beta,f_2}^i(z_2, \bar{z}_2) \sigma_{\gamma,f_3}^i(z_3, \bar{z}_3)> . \tag{2.4}
\]

Here \( i \) is the index labelling the \( i \)-th complex plane, \( \alpha, \beta \) and \( \gamma \) are the point group elements for the three twisted sectors involved and \( f_1, f_2 \) and \( f_3 \) are the corresponding fixed points or tori. The allowed trilinear couplings (2.4) are those with conjugation classes whose product contains the identity. This condition gives the so-called point group and space group selection rules. More precisely, let \( (\alpha, l_1), (\beta, l_2) \) and \( (\gamma, l_3) \) be space group elements associated with the \( \alpha, \beta \) and \( \gamma \) twisted sectors, where

\[
l_1 = (I - \alpha)(f_\alpha + \lambda_1), \tag{2.5}
\]

\[
l_2 = (I - \beta)(f_\beta + \lambda_2), \tag{2.6}
\]

\[
l_3 = (I - \gamma)(f_\gamma + \lambda_3), \tag{2.7}
\]

and \( \lambda_i, i = 1, 2, 3 \), denotes an arbitrary lattice vector. Then, provided that the
point group selection rule

\[ \alpha \beta \gamma = I \quad (2.8) \]

is already satisfied, the full space group selection rule contains the additional condition

\[ l_1 + l_2 + l_3 \text{ contains } 0. \quad (2.9) \]

Also, the Yukawa couplings involving ground states obey a further selection rule coming from the factors \( e^{i a \cdot H} \) in the left-moving part of the vertices, where \( H \) are the bosonic fields describing the NSR left-moving fermions and \( a \) is an \( SO(10) \) weight. This is the conservation of the \( H \)-lattice momentum, which is analogous to the conservation of ordinary momenta \( k \) which comes from the factors \( e^{ik^\mu x^\mu} \), with \( x^\mu \) the four uncompactified string coordinates.

To solve the conformal field theory of the orbifold model it remains to calculate the three-point functions of all of its primary fields. The tools for calculating the correlation functions of ground twist fields (for the simplest choice of twist fields) in orbifold models were first developed in [5]. Twisted sector Yukawa coupling have been investigated for the orbifolds \( \mathbb{Z}_3 \) [5-8], \( \mathbb{Z}_7 \) [9], \( \mathbb{Z}_N \) [11-14] and \( \mathbb{Z}_M \times \mathbb{Z}_N \) [15-16].

Roughly speaking, the method of evaluating the correlation functions in orbifold models [5] is as follows. One splits the coordinate fields \( X^i \) into a classical and a quantum piece,

\[ X^i(z, \bar{z}) = X^i_{\text{cl}}(z, \bar{z}) + X^i_{\text{qu}}(z, \bar{z}), \quad (2.10) \]

where \( X^i_{\text{cl}} \) is a classical solution of the equation of motion and \( X^i_{\text{qu}} \) are quantum fluctuations around the instanton classical solutions. The fact that the action (1.1) is bilinear in the fields \( X^i \) allows the splitting of the path integral representation of a four-point function into two factors, one representing the instanton solution,
and the other describing the quantum fluctuations. This is expressed as

\[ Z(x, \bar{x}) = Z_{qu}(x, \bar{x}) \sum_{X_{cl}} e^{-S(X_{cl})}, \]  

(2.11)

where \( x \) is the location of one of the vertices on the world sheet not fixed by \( SL(2, C) \) invariance.

Transporting the field \( X^i \) around a closed path \( C \), encircling twist fields with net zero twist, the following global monodromy conditions are obtained,

\[ \Delta_C X^i_{qu} = \oint_C dz \partial_z X^i_{qu} + \oint_C d\bar{z} \partial_{\bar{z}} X^i_{qu} = 0, \]

\[ \Delta_C X^i_{cl} = v^i, \]  

(2.12)

where the complex vectors \( v^i \) belong to the lattice coset which is obtained by multiplying the space group elements corresponding to twist vertices encircled by the contour \( C \).

Using the techniques of conformal field theory [24] together with the local and global monodromy conditions (eqns (2.1) and (2.12)), the four point function involving four ground twist fields is exactly solved. The operator product coefficients, i.e., the Yukawa couplings, are then deduced via the appropriate factorization of the four point function.
3. Excited Twisted Sector Yukawa couplings for $\mathbb{Z}_M \times \mathbb{Z}_N$ Orbifolds

In this section the calculation of the Yukawa couplings involving excited twisted states in the case of $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds is considered. From the phenomenological point of view, only the conformal dimensions of the standard model particles are relevant. Therefore, massless states in the twisted sectors with the $SU(3) \times SU(2) \times U(1)$ quantum numbers of quarks, leptons and electroweak Higgses are considered. Then from (1.3), the following restrictions on the allowed excitations in a given twisted sector are obtained [20],

\begin{align}
N_{osc} &\leq 1 - h_{\sigma_{\alpha}} - \frac{3}{5}, \quad \text{for } Q, \ u_c \text{ and } e_c \tag{3.1} \\
N_{osc} &\leq 1 - h_{\sigma_{\alpha}} - \frac{2}{5}, \quad \text{for } L, \ d_c \text{ and } H,
\end{align}

(\text{where we have chosen the Kac-Moody levels, } 3/5 k_1 = k_2 = k_3 = 1.) The inequality in (3.1) reflects the generic occurrence in orbifold models, before spontaneous symmetry breaking, of extra $U(1)$ gauge fields to which the quarks and leptons couple [23]. The above arguments guarantee the masslessness of the states. However whether or not they are present in the massless spectrum of the theory is determined by the generalized GSO projection of the orbifold symmetries [1, 2, 3].

The non-zero Yukawa couplings involving ground states obey a point group selection rule and a selection rule for the $H$-lattice momentum associated with bosonized NSR fermion degrees of freedom. These selection rules have already been written down in [21]. Allowed Yukawa couplings involving excited twisted sectors can be obtained from those of ground states with the insertions of $\partial_z X$ and $\partial_z \bar{X}$ vertices. The number of such insertions is restricted by the discrete symmetries of the two dimensional sub-lattices of the six dimensional compact manifold [6, 22]. This means that assuming the discrete symmetry acting on the $i$-th complex plane is of order $N$, the correlation functions involving $(\partial_z X^i)^m (\partial_z \bar{X}^i)^n$ are allowed only if

\[ m - n = 0 \mod N. \]  

(3.2)
As an illustration, consider the case of \( \mathbb{Z}_2 \times \mathbb{Z}_6 \) with lattice \( SO(4) \times G_2^2 \). In the orthonormal basis, the twist’s action on the internal complex string degrees of freedom is given by

\[
\begin{align*}
\theta X^1 &= -X^1, & \omega X^1 &= X^1, \\
\theta X^2 &= X^2, & \omega X^2 &= e^{2\pi i/6}X^2, \\
\theta X^3 &= -X^3, & \omega X^3 &= e^{-2\pi i/6}X^3,
\end{align*}
\]

where \( \theta \) and \( \omega \) the generators of the point groups associated with \( \mathbb{Z}_2 \) and \( \mathbb{Z}_6 \) respectively. The twisted sectors in the theory are \( \omega, \omega^2, \omega^3, \omega^4, \theta, \theta\omega, \theta\omega^2, \theta\omega^3 \).

Here the three-point functions among twisted sectors allowed by \( SO(10) \) (\( H \)-lattice) momentum conservation and point group selection rules are given by

\[
\begin{align*}
\langle \sigma_\omega \sigma_{\theta\omega^2} \sigma_{\theta\omega^3} \rangle, & \quad \langle \sigma_{\omega^2}\sigma_{\theta\omega} \sigma_{\theta\omega^3} \rangle, \quad \langle \sigma_{\omega^3}\sigma_\theta \sigma_{\theta\omega^3} \rangle \\
\langle \sigma_{\omega^4}\sigma_\theta \sigma_{\theta\omega} \rangle, & \quad \langle \sigma_{\omega^5}\sigma_\theta \sigma_{\theta\omega} \rangle, \quad \langle \sigma_{\omega^6}\sigma_\theta \sigma_{\theta\omega} \rangle.
\end{align*}
\]

Using equation (3.1), the allowed excitations in various sectors can be determined and we list them as follows:

\[
\begin{align*}
\omega : & \quad \partial \bar{z}X^2, \partial \bar{z}X^2\partial \bar{z}X^2, \partial \bar{z}\bar{X}^3, \partial \bar{z}\bar{X}^3\partial \bar{z}\bar{X}^3, \partial \bar{z}X^2\partial \bar{z}\bar{X}^3, \\
\omega^2 : & \quad \partial \bar{z}X^2, \partial \bar{z}\bar{X}^3, \\
\omega^4 : & \quad \partial \bar{z}X^2, \partial \bar{z}X^3, \\
\omega^5 : & \quad \partial \bar{z}X^3, \partial \bar{z}X^3\partial \bar{z}X^3, \partial \bar{z}X^2, \partial \bar{z}X^2\partial \bar{z}X^2, \partial \bar{z}X^3\partial \bar{z}X^2, \\
\theta \omega : & \quad \partial \bar{z}X^2, \\
\theta \omega^2 : & \quad \partial \bar{z}X^3.
\end{align*}
\]

It can be easily shown that two possible non-trivial excited three-point functions can be obtained from the coupling \( \langle \sigma_{\omega^4}\sigma_\theta \sigma_{\theta\omega} \rangle \), namely \( \langle \tau_{\omega^4}\tau_{\theta\omega} \sigma_{\theta\omega} \rangle \) and \( \langle \tau_{\omega^4}\sigma_{\theta\omega}^2 \tau_{\theta\omega}^2 \rangle \). The rest of the allowed couplings are simply two point functions.
Following the same procedure as that outlined above, it can be easily demonstrated that no non-trivial three-point functions exist for the orbifolds \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( \mathbb{Z}_2 \times \mathbb{Z}_4 \), \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) and \( \mathbb{Z}_3 \times \mathbb{Z}_3 \). The possible three-point functions involving excited couplings of all other \( \mathbb{Z}_M \times \mathbb{Z}_N \) orbifolds are tabulated in the Appendix, using the notation \( T_{pq} \) to denote the \( \theta^p \omega^q \) twisted sector. The excited twist fields which are found to be relevant in these cases are \( \tau^i_\alpha \), \( \tilde{\tau}^i_\alpha \), \( \hat{\tau}^i_\alpha \) and \( \hat{\tau}^i_\alpha' \), where the last two are, respectively, the excited twist fields created by applying \( \partial_z X^i \partial_{\bar{z}} X^i \) and \( \partial_{\bar{z}} \hat{X}^i \partial_z \hat{X}^i \) on the \( \alpha \)-twisted vacuum. However it is found that all the couplings involving \( \hat{\tau}_\alpha \) and \( \hat{\tau}_\alpha' \) are merely two point functions. Thus we are interested in the computation of correlation functions of the form

\[
\left( Z^3_i \right)_{\text{excited}} = \langle \tau^i_\alpha(z_1, \bar{z}_1) \tau'^i_\beta(z_2, \bar{z}_2) \sigma^i_\gamma(z_3, \bar{z}_3) \rangle, \tag{3.6}
\]

up to permutations of \( \alpha, \beta \) and \( \gamma \). In (3.6) and subsequent equations we suppress the dependence of the various fields on the fixed points or tori and assume that the fields are located at those satisfying the space group selection rule.

Normalization of the excited twist fields is necessary in order to create normalized states. The twisted mode expansion for \( X^i \) and \( \hat{X}^i \) suggests that the normalization factors for the excited fields \( \tau^i_\alpha \) and \( \tau'^i_\alpha \) are \( 1/\sqrt{2\eta_\alpha} \) and \( 1/\sqrt{2(1-\eta_\alpha)} \) respectively. This normalization could also be determined from the explicit calculations of the two-point function

\[
\langle \tau^i_{\alpha-1}(z_1, \bar{z}_1) \tau'^i_{\alpha}(z_2, \bar{z}_2) \rangle. \tag{3.7}
\]

The evaluation of (3.7) follows from that of the Green function [5]

\[
g(z, w, z_i) = -\frac{1}{2} \frac{\langle \partial_z X^i \partial_w \hat{X}^i \sigma_{\alpha-1}(z_1, \bar{z}_1) \sigma_{\alpha}(z_2, \bar{z}_2) \rangle}{\langle \sigma_{\alpha-1}(z_1, \bar{z}_1) \sigma_{\alpha}(z_2, \bar{z}_2) \rangle}. \tag{3.8}
\]

\( g \) is completely fixed by the local properties of the string coordinates in the presence
of twists, encoded in the operator product expansions (2.1), and the fact that
\[ g(z, w, z_i) \sim \frac{1}{(z-w)^2} + \text{finite}, \quad z \to w. \] (3.9)

Explicitly
\[ g(z, w, z_i) \sim (z - z_1)^{-\eta_\alpha}, \quad z \to z_1, \]
\[ \sim (z - z_2)^{-(1-\eta_\alpha)}, \quad z \to z_2, \]
\[ \sim (w - z_1)^{-(1-\eta_\alpha)}, \quad w \to z_1, \]
\[ \sim (w - z_2)^{-\eta_\alpha}, \quad w \to z_2. \] (3.10)

Using (3.9) and (3.10) we conclude that
\[ g(z, w, z_i) = (z - z_1)^{-\eta_\alpha}(z - z_2)^{-(1-\eta_\alpha)}(w - z_1)^{-(1-\eta_\alpha)}(w - z_2)^{-\eta_\alpha} \]
\[ \frac{\eta_\alpha(z - z_1)(w - z_2) + (1 - \eta_\alpha)(z - z_2)(w - z_1)}{(z-w)^2}. \] (3.11)

Then (3.7) is simply
\[ \langle \tau^i_{\alpha^{-1}}(z_1, \bar{z}_1)\tau'^i_{\alpha}(z_2, \bar{z}_2) \rangle = -2\langle \sigma_{\alpha^{-1}}(z_1, \bar{z}_1)\sigma_{\alpha}(z_2, \bar{z}_2) \rangle \]
\[ = \lim_{\substack{z \to z_1 \\\ \\\ \ \\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\\ \\"
as being simply

\[
\langle \hat{\tau}_i^{-1}(z_1, \bar{z}_1)\hat{\tau}_i^\prime(z_2, \bar{z}_2) \rangle = 4\langle \sigma_{\alpha^{-1}}(z_1, \bar{z}_1)\sigma_{\alpha}(z_2, \bar{z}_2) \rangle
\]

\[
\lim_{z, z' \to z_1, w, w' \to z_2} (z - z_1)^{-\eta_\alpha}(z' - z_1)^{-\eta_\alpha}(w - \bar{z}_2)^{-\eta_\alpha}(w' - \bar{z}_2)^{-\eta_\alpha}g'(z, w, z', w', z_i)
\]

\[
= 8(1 - \eta_\alpha)^2(-1)^{2\eta_\alpha - 2}(z_1 - z_2)^{-4(1 - \eta_\alpha)}|z_1 - z_2|^{-4h_{a\alpha}}.
\]

This implies that the normalization factors for \(\hat{\tau}_i^\prime\) and \(\hat{\tau}_i^\alpha\) are \(1/2\sqrt{\eta_\alpha}\) and \(1/2\sqrt{2(1 - \eta_\alpha)}\) respectively.

Having normalized the relevant fields, we turn to the calculation of the three point-function (3.6). This correlator can be written, using the operator products expansions (2.1), in terms of the internal right-moving complex string coordinates fields as

\[
Z_3^{\text{excited}} = \lim_{z \to z_1, w \to z_2} (z - z_1)^{1-\eta_\alpha}(w - z_2)^{\eta_\beta} \langle \partial_z X \partial_w \bar{X} \sigma_{\alpha}(z_1, \bar{z}_1)\sigma_{\beta}(z_2, \bar{z}_2)\sigma_{(\alpha\beta)^{-1}}(z_3, \bar{z}_3) \rangle,
\]

where the index \(i\) labelling the \(i\)-th complex plane has been suppressed in this and subsequent equations. Separating \(X\) into a classical part and a quantum part as in (2.10), we obtain

\[
\langle \partial_z X \partial_w \bar{X} \sigma_{\alpha}(z_1, \bar{z}_1)\sigma_{\beta}(z_2, \bar{z}_2)\sigma_{(\alpha\beta)^{-1}}(z_3, \bar{z}_3) \rangle = \sum_{X_{cl}} e^{-S_{cl}} \langle \partial_z X_{qu} \partial_w \bar{X}_{qu} \rangle_{3-\text{twists}} + \sum_{X_{cl}} e^{-S_{cl}} \partial_z X_{cl} \partial_w \bar{X}_{cl} Z_3^{\text{qu}},
\]

where

\[
\langle \partial_z X_{qu} \partial_w \bar{X}_{qu} \rangle_{3-\text{twists}} = \int D X_{qu} e^{-S_{qu}} \partial_z X_{qu} \partial_w X_{qu} \sigma_{\alpha}(z_1, \bar{z}_1)\sigma_{\beta}(z_2, \bar{z}_2)\sigma_{(\alpha\beta)^{-1}}(z_3, \bar{z}_3),
\]

\[
Z_3^{\text{qu}} = \int D X_{qu} e^{-S_{qu}} \sigma_{\alpha}(z_1, \bar{z}_1)\sigma_{\beta}(z_2, \bar{z}_2)\sigma_{(\alpha\beta)^{-1}}(z_3, \bar{z}_3).
\]
The classical fields have derivatives of the form
\[
\partial_z X_{cl} = d(z - z_1)^{-\eta_0}(z - z_2)^{-\eta_3}(z - z_3)^{-\eta_0 - \eta_3}, \\
\partial_w X_{cl} = a(w - z_1)^{-\eta_0}(w - z_2)^{-\eta_3}(w - z_3)^{-\eta_0 - \eta_3},
\]
which are consistent with the operator product expansions (2.1). The constant \(a\) must be chosen to be zero for an acceptable classical solution because otherwise the classical action is divergent. Consequently, the second term in (3.17) vanishes, and the moduli dependence of \(Z_{\text{excited}}^3\), which is contained in \(\sum X_{cl} e^{-S_{cl}}\), is exactly the same as for the three-point function with unexcited twist fields. In order to determine the overall normalization of the three-point function, which depends on the twisted sectors involved, we consider the calculation of the four-point function,

\[
Z_{\text{excited}}^4 = \frac{1}{2\sqrt{\eta_3(1 - \eta_0)}} \lim_{z \to z_4} (z - z_4)^{1-\eta_3}(w - z_2)^{\eta_0} \langle \partial_z X \partial_w \bar{X} \sigma_{\alpha^{-1}}(z_1, \bar{z}_1) \sigma_\alpha(z_2, \bar{z}_2) \sigma_{\beta^{-1}}(z_3, \bar{z}_3) \sigma_\beta(z_4, \bar{z}_4) \rangle,
\]
where \(1/2\sqrt{\eta_3(1 - \eta_0)}\) is a normalization factor due to the fields \(\sigma_\beta\) and \(\sigma_{\beta'}\). Again with the aid of the operator product expansions (2.1) this can be written as

\[
Z_{\text{excited}}^4 = \frac{1}{2\sqrt{\eta_3(1 - \eta_0)}} \lim_{z \to z_4} (z - z_4)^{1-\eta_3}(w - z_2)^{\eta_0} \sum_{X_{cl}} e^{-S_{cl}} \langle \partial_z X_{qu} \partial_w \bar{X}_{qu} \rangle_{4\text{-twists}} + Z_{qu}^4,
\]
where

\[
\langle \partial_z X_{qu} \partial_w \bar{X}_{qu} \rangle_{4\text{-twists}} = \int \mathcal{D}X_{qu} e^{-S_{qu}} \partial_z X_{qu} \partial_w \bar{X}_{qu} \sigma_{\alpha^{-1}}(z_1, \bar{z}_1) \sigma_\alpha(z_2, \bar{z}_2) \sigma_{\beta^{-1}}(z_3, \bar{z}_3) \sigma_\beta(z_4, \bar{z}_4)
\]
and

\[
Z_{qu}^4 = \int \mathcal{D}X_{qu} e^{-S_{qu}} \sigma_{\alpha^{-1}}(z_1, \bar{z}_1) \sigma_\alpha(z_2, \bar{z}_2) \sigma_{\beta^{-1}}(z_3, \bar{z}_3) \sigma_\beta(z_4, \bar{z}_4).
\]
The first term in (3.22) can be evaluated from the Green function in the presence
of four twists,
\[ h(z, w, z_i) = -\frac{1}{2} \frac{\partial_z X_{qu} \partial_z \bar{X}_{qu} \sigma_{\alpha-1}(z_1, \bar{z}_1) \sigma_{\alpha}(z_2, \bar{z}_2) \sigma_{\beta-1}(z_3, \bar{z}_3) \sigma_{\beta}(z_4, \bar{z}_4)}{\sigma_{\alpha-1}(z_1, \bar{z}_1) \sigma_{\alpha}(z_2, \bar{z}_2) \sigma_{\beta-1}(z_3, \bar{z}_3) \sigma_{\beta}(z_4, \bar{z}_4)}. \] (3.25)

Using the operator product expansions (2.1), and the fact that
\[ h(z, w, z_i) \sim \frac{1}{(z - w)^2} \text{ finite}, \quad z \to w, \] (3.26)

(3.25) can be shown to take the form
\[ h(z, w, z_i) = (z - z_1)^{-\eta_{\alpha}} (z - z_2)^{-\eta_{\alpha-1}} (z - z_3)^{-\eta_{\beta}} (z - z_4)^{-\eta_{\beta-1}} \]
\[ (w - z_1)^{-\eta_{\alpha}} (w - z_2)^{-\eta_{\alpha}} (w - z_3)^{-\eta_{\beta}} (w - z_4)^{-\eta_{\beta}} \]
\[ \left( \frac{P(z, w)}{(z - w)^2} + A(z_i) \right), \]
where the coefficients of the polynomial
\[ P(z, w) = \sum_{i,j=0}^{2} a_{ij} z^i w^j, \] (3.28)
are determined using (3.26) and the freedom of defining the finite part \( A \). These are given by
\[ a_{00} = z_1 z_2 z_3 z_4, \]
\[ a_{01} = (k - 1) z_2 z_3 z_4 - k z_1 z_3 z_4 + (l - 1) z_1 z_2 z_4 - l z_1 z_2 z_3, \]
\[ a_{10} = - k z_2 z_3 z_4 + (k - 1) z_1 z_3 z_4 - l z_1 z_2 z_4 + (l - 1) z_1 z_2 z_3, \]
\[ a_{11} = z_1 z_4 + z_2 z_3 + z_1 z_2 + z_2 z_4, \]
\[ a_{20} = z_1 z_3 - \frac{k + l}{2} z_1 z_3 + \frac{l - k}{2} z_1 z_4 + \frac{k - l}{2} z_2 z_3 + \frac{k + l}{2} z_2 z_4, \] (3.29)
\[ a_{21} = z_2 z_4 + \frac{k + l}{2} z_1 z_3 + \frac{k - l}{2} z_1 z_4 - \frac{k - l}{2} z_2 z_3 - \frac{k + l}{2} z_2 z_4, \]
\[ a_{12} = - k z_1 + (k - 1) z_2 - l z_3 + (l - 1) z_4, \]
\[ a_{22} = (k - 1) z_1 - k z_2 + (l - 1) z_3 - l z_4, \]
\[ a_{22} = 1. \]

Unlike the Green function in the presence of two twists, \( h \) can not be entirely determined from the local monodromy conditions and one has to fix the function
\( A(z_i) \) using the global monodromy conditions (2.12). The determination of \( A \) can be found in ref [13]. Thus we can write

\[
\lim_{{z \to z_4, w \to z_2}} (z - z_4)^{1-\eta_\beta} (w - z_2)^{\eta_\alpha} h(z, w, z_i) =
(\eta_\alpha - \eta_\beta)^{\gamma_2} \left( (x - 1) \partial_x \log I \right),
\]

where we have used \( SL(2, C) \) invariance to set

\[
z_1 = 0, \quad z_2 = x, \quad z_3 = 1, \quad z_4 = \infty,
\]

\[
A(z_i) = -\frac{1}{z_\infty} A(x, \bar{x}) = -\frac{1}{z_\infty} \left( \frac{\eta_\alpha - \eta_\beta}{2} \right) \left( x + x(1 - x) \log I(x, \bar{x}) \right)
\]

and

\[
I(x, \bar{x}) = J_2 \tilde{G}_1(x) H_2(1 - x) + J_1 G_2(x) \tilde{H}_1(1 - \bar{x}),
\]

with

\[
J_1 = \frac{\Gamma(\eta_\alpha) \Gamma(1 - \eta_\beta)}{\Gamma(1 + \eta_\alpha - \eta_\beta)}, \quad J_2 = \frac{\Gamma(1 - \eta_\alpha) \Gamma(\eta_\beta)}{\Gamma(1 + \eta_\beta - \eta_\alpha)},
\]

\[
G_1(x) = F(\eta_\alpha, 1 - \eta_\beta; 1; x), \quad G_2(x) = F(1 - \eta_\alpha, \eta_\beta; 1; x)
\]

and

\[
H_1(x) = F(\eta_\alpha, 1 - \eta_\beta; 1 + \eta_\alpha - \eta_\beta; x), \quad H_2(x) = F(1 - \eta_\alpha, \eta_\beta; 1 + \eta_\beta - \eta_\alpha; x).
\]

Finally, the second term in (3.22) can be evaluated by writing, in agreement with the local monodromy behaviour,

\[
\partial_z X_{cl} = cz^{-\eta_\alpha} (z - x)^{\eta_\alpha - 1} (z - 1)^{-\eta_\beta} (z - z_\infty)^{\eta_\beta - 1}
\]

\[
\partial_w X_{cl} = dw^{\eta_\alpha - 1} (w - x)^{-\eta_\alpha} (w - 1)^{\eta_\beta - 1} (w - z_\infty)^{-\eta_\beta}
\]

The normalization factors \( c \) and \( d \) are then derived in terms of hypergeometric functions using global monodromy conditions. The expressions for \( c \) and \( d \) can be found in ref [13].
To factorize, we take \( x \to \infty \). In this limit, the leading term of \( Z_{\text{excited}}^4 \) should factorize into a product of the Yukawa coupling (3.16) and the Yukawa coupling of unexcited twisted matter fields. From the behaviour of the hypergeometric functions as \( x \to \infty \), the second term in (3.22) gives no contribution – this is consistent with the remarks made after eq (3.20). Therefore it remains to factorize the quantum part of the first term in (3.22) which will give the overall normalization factor (the quantum part) of the Yukawa coupling (3.16). In the limit \( x \to \infty \), the quantum piece of the first term in (3.22) should read

\[
\lim_{x,\bar{x} \to \infty} \langle \tau_{\beta}(\infty)\tau'_{\alpha}(x)\sigma_{(\alpha\beta)}^{-1}(0)\rangle \langle \sigma_{\alpha\beta}(\infty)\sigma^{-1}(1)\sigma^{-1}(0) \rangle
\]

(3.36)

Using the methods of conformal field theory, we write

\[
\langle \tau_{\beta}(z_1, \bar{z}_1)\tau_{\alpha}'(z_2, \bar{z}_2)\sigma_{(\alpha\beta)}^{-1}(z_3, \bar{z}_3) \rangle = (-1)^{-\eta_\beta}Y_{\tau_{\beta}\tau_{\alpha}\sigma_{(\alpha\beta)}^{-1}}^{\text{qu}}
\]

\[
(\bar{z}_1 - z_2)^{h_{\sigma_{(\alpha\beta)}^{-1}}} - h_{\tau_{\alpha}'} - h_{\tau_{\beta}} (z_2 - z_3)^{h_{\tau_{\beta}} - h_{\tau_{\alpha}'} - h_{\sigma_{(\alpha\beta)}^{-1}}^{-1}} (z_1 - z_3)^{h_{\tau_{\alpha}'} - h_{\tau_{\beta}} - h_{\sigma_{(\alpha\beta)}^{-1}}^{-1}}
\]

(3.37)

which implies that,

\[
\langle \tau_{\beta}(\infty)\tau_{\alpha}'(x)\sigma_{(\alpha\beta)}^{-1}(0) \rangle = (-1)^{-\eta_\beta}Y_{\tau_{\beta}\tau_{\alpha}\sigma_{(\alpha\beta)}^{-1}}^{\text{qu}}|x|^{2(h_{\sigma_{(\alpha\beta)}^{-1}} - h_{\sigma_{(\alpha\beta)}^{-1}}^{-1})} x^{\eta_\alpha + \eta_\beta - 1}.
\]

(3.38)

Then, inserting (3.38) in (3.36), we obtain

\[
\lim_{x,\bar{x} \to \infty} \langle Z_{\text{excited}}^4 \rangle_{\text{qu}} = \lim_{x,\bar{x} \to \infty} (-1)^{-\eta_\beta} |x|^{2(h_{\sigma_{(\alpha\beta)}^{-1}} - h_{\sigma_{(\alpha\beta)}^{-1}}^{-1})} x^{\eta_\alpha + \eta_\beta - 1} Y_{\tau_{\beta}\tau_{\alpha}\sigma_{(\alpha\beta)}^{-1}}^{\text{qu}} Y_{\sigma_{(\alpha\beta)}^{-1}}^{\text{qu}}
\]

(3.39)

where \( Y_{\sigma_{(\alpha\beta)}^{-1}}^{\text{qu}} \) denotes the quantum contribution to the full Yukawa coupling.

Using (3.30), we finally obtain

\[
\langle Z_{\text{excited}}^4 \rangle_{\text{qu}} = \frac{1}{\sqrt{\eta_\beta(1 - \eta_\alpha)}} (-1)^{-\eta_\beta} x^{\eta_\alpha}(x - 1)^{\eta_\beta - 1} \left((1 - x)\partial_x log I\right) Z_{\text{qu}}^4.
\]

(3.40)

The factorization of (3.40) is now straightforward. Using the formulae of ref [13]
for $Z_4^{\text{qu}}$ and the behaviour of the hypergeometric functions as $x \to \infty$, we get

$$\lim_{x, \bar{x} \to \infty} \left( Z_{\text{excited}}^{4, \text{qu}} \right) = (-1)^{-\eta_\beta} \lim_{x, \bar{x} \to \infty} |x|^{2(h_{\alpha \beta} - h_{\alpha \alpha} - h_{(\alpha \beta)}^{-1})} x^{\eta_\alpha + \eta_\beta - 1} \sqrt{\frac{(1 - \eta_\alpha)}{\eta_\beta}} \sqrt{\frac{2V_\Lambda}{\text{ctg}(\pi(1 - \eta_\alpha)) + \text{ctg}(\pi(1 - \eta_\beta))}} \frac{\Gamma(\eta_\alpha)\Gamma(\eta_\beta)}{\Gamma(\eta_\alpha + \eta_\beta - 1)} Y_{\sigma_\alpha \sigma_\beta \sigma_\alpha (\alpha \beta)^{-1}}^{\text{qu}}$$

(3.41)

for $\eta_\beta > 1 - \eta_\alpha$, where $\Lambda$ is the volume of the lattice unit cell, and

$$\lim_{x, \bar{x} \to \infty} \left( Z_{\text{excited}}^{4, \text{qu}} \right) = (-1)^{-\eta_\beta} \lim_{x, \bar{x} \to \infty} |x|^{2(h_{\alpha \beta} - h_{\alpha \alpha} - h_{(\alpha \beta)}^{-1})} x^{\eta_\alpha + \eta_\beta - 1} \sqrt{\frac{\eta_\beta}{(1 - \eta_\alpha)}} \sqrt{\frac{2V_\Lambda}{\text{ctg}(\pi\eta_\alpha) + \text{ctg}(\pi\eta_\beta)}} \frac{\Gamma(1 - \eta_\alpha)\Gamma(1 - \eta_\beta)}{\Gamma(1 - \eta_\alpha - \eta_\beta)} Y_{\sigma_\alpha \sigma_\beta \sigma_\alpha (\alpha \beta)^{-1}}^{\text{qu}}$$

(3.42)

for $\eta_\beta < 1 - \eta_\alpha$. From the equations (3.39), (3.41) and (3.42) one can read off the quantum part of the Yukawa coupling (3.16) as

$$Y_{\tau_\beta \tau_\alpha \sigma_\alpha \sigma_\beta (\alpha \beta)^{-1}}^{\text{qu}} = \sqrt{\frac{(1 - \eta_\alpha)}{\eta_\beta}} \sqrt{\frac{2V_\Lambda}{\text{ctg}(\pi(1 - \eta_\alpha)) + \text{ctg}(\pi(1 - \eta_\beta))}} \frac{\Gamma(\eta_\alpha)\Gamma(\eta_\beta)}{\Gamma(\eta_\alpha + \eta_\beta - 1)}$$

(3.43)

for $\eta_\beta > 1 - \eta_\alpha$, and

$$Y_{\tau_\beta \tau_\alpha \sigma_\alpha \sigma_\beta^{-1}}^{\text{qu}} = \sqrt{\frac{\eta_\beta}{(1 - \eta_\alpha)}} \sqrt{\frac{2V_\Lambda}{\text{ctg}(\pi\eta_\alpha) + \text{ctg}(\pi\eta_\beta)}} \frac{\Gamma(1 - \eta_\alpha)\Gamma(1 - \eta_\beta)}{\Gamma(1 - \eta_\alpha - \eta_\beta)}$$

(3.44)

for $\eta_\beta < 1 - \eta_\alpha$.

The classical part of $Y_{\tau_\beta \tau_\alpha \sigma_\alpha \sigma_\beta^{-1}}$, which is basically that for $Y_{\sigma_\alpha \sigma_\alpha \sigma_\alpha \sigma_\beta^{-1}}$, can be calculated by factorizing the classical part $\sum_{X_{cl}} e^{-S(X_{cl})}$ in (3.22) or alternatively it could be calculated directly using the same techniques as in the four-point function and the fact that only one of the fields $\partial_z X_{cl}$ and $\partial_z \bar{X}_{cl}$ has a non-zero solution. The classical part of all Yukawa couplings for $Z_M \times Z_N$ Coxeter orbifolds has been calculated in [15].
4. Excited Twisted Sector Yukawa couplings for $Z_N$ Orbifolds

In this section we consider the calculations of the possible Yukawa couplings involving excited twisted sectors in $Z_N$ orbifolds. Here it is found that no such couplings exist in the $Z_3$, $Z_4$, $Z_7$, $Z_6$-I and $Z_8$-I models for massless states with quark and lepton quantum numbers. In the rest of the $Z_N$ models, the allowed couplings found are among the excited twist fields of the type $\tau_\alpha$, $\tau'_\alpha$, $\hat{\tau}_\alpha$ and $\hat{\tau}'_\alpha$, where $\hat{\tau}_\alpha$ and $\hat{\tau}'_\alpha$ are the doubly excited twist fields mentioned before eqn. (3.6). In addition to the couplings appearing in $Z_N \times Z_M$, two new kinds are found. These are

$$\langle \hat{\tau}_\beta(z_1, \bar{z}_1) \hat{\tau}'_\alpha(z_2, \bar{z}_2) \sigma(\alpha \beta)^{-1}(z_3, \bar{z}_3) \rangle,$$

$$\langle \tau_\beta(z_1, \bar{z}_2) \tau_\alpha(z_2, \bar{z}_2) \hat{\tau}'(\alpha \beta)^{-1}(z_3, \bar{z}_3) \rangle.$$  \hspace{1cm} (4.1)

All possible Yukawa couplings among excited twist fields for $Z_N$ orbifolds are tabulated in the Appendix, using the notation $T_p$ to denote the $\theta^p$ twisted sector.

The three-point functions in (4.1) can be evaluated using the same procedure outlined in the previous section. The first one can be written as

$$\langle \hat{\tau}_\beta(z_1, \bar{z}_1) \hat{\tau}'_\alpha(z_2, \bar{z}_2) \sigma(\alpha \beta)^{-1}(z_3, \bar{z}_3) \rangle = Y_{\hat{\tau}_\beta \hat{\tau}'_\alpha \sigma(\alpha \beta)^{-1}} Y_{\tau_\beta \tau_\alpha \sigma(\alpha \beta)^{-1}},$$

where the moduli-dependent classical part contained in $Y_{\tau_\beta \tau_\alpha \sigma(\alpha \beta)^{-1}}$ is the same as for ground twist field three-point function. The quantum piece is obtainable from the knowledge of the four point function,

$$\mathcal{Z}^4_{\text{excited}} = \mathcal{N} \langle \sigma^{-1}(z_1) \hat{\tau}'(z_2) \sigma^{-1}(z_3) \hat{\tau}_\beta(z_4) \rangle,$$  \hspace{1cm} (4.3)

where $\mathcal{N} = 1/8(1 - \eta_\alpha)\eta_\beta$ is a normalization factor due to the fields $\hat{\tau}'_\alpha$ and $\hat{\tau}_\beta$. With the aid of the operator product expansions (2.1), the four point function (4.3)
can be expressed as

\[ Z_{\text{excited}}^4 = \mathcal{N} \lim_{z, z' \to z_4} (z - z_4)^{1-\eta_3} (z' - z_4)^{1-\eta_3} (w - z_2)^{\eta_\alpha} (w' - z_2)^{\eta_\alpha} \]

\[ = \mathcal{N} \lim_{z, z' \to z_4} (z - z_4)^{1-\eta_3} (z' - z_4)^{1-\eta_3} (w - z_2)^{\eta_\alpha} (w' - z_2)^{\eta_\alpha} \sum_{X_{\text{cl}}} e^{-S_{\text{cl}}} \]

\[ \left( \langle \partial_z X_{\text{qu}} \partial_{z'} X_{\text{qu}} \partial_w \bar{X}_{\text{qu}} \partial_{w'} \bar{X}_{\text{qu}} \rangle_{4-\text{twists}} + \partial_z X_{\text{cl}} \partial_{z'} X_{\text{cl}} \partial_w \bar{X}_{\text{cl}} \partial_{w'} \bar{X}_{\text{cl}} Z_{\text{qu}}^4 \right), \]

where the notation used is that of the previous section. In order to obtain the quantum piece of the Yukawa coupling (4.2), we need to factorize the quantum part of the first term in (4.4) as the second term makes zero contribution (following the discussion of the previous section). This term can be expressed as

\[ \left( Z_{\text{excited}}^4 \right)_{\text{qu}} = \lim_{z, z' \to z_4} \mathcal{N} (z - z_4)^{1-\eta_3} (z' - z_4)^{1-\eta_3} (w - z_2)^{\eta_\alpha} (w' - z_2)^{\eta_\alpha} \]

\[ \langle \partial_z X_{\text{qu}} \partial_{z'} X_{\text{qu}} \partial_w \bar{X}_{\text{qu}} \partial_{w'} \bar{X}_{\text{qu}} \rangle_{4-\text{twists}} \]

\[ = \lim_{z, z' \to z_4} 4\mathcal{N} (z - z_4)^{1-\eta_3} (z' - z_4)^{1-\eta_3} (w - z_2)^{\eta_\alpha} (w' - z_2)^{\eta_\alpha} \]

\[ H(z, w, z', w') \langle \sigma_{\alpha^{-1}} (z_1, \bar{z}_1) \sigma_{\alpha} (z_2, \bar{z}_2) \sigma_{\beta^{-1}} (z_3, \bar{z}_3) \sigma_{\beta} (z_4, \bar{z}_4) \rangle \]

where

\[ H(z, w, z', w') = \frac{1}{4} \frac{\langle \partial_z X_{\text{qu}} \partial_w \bar{X} \partial_{z'} X_{\text{qu}} \partial_{w'} \bar{X} \rangle_{4-\text{twists}}}{\langle \sigma_{\alpha^{-1}} (z_1, \bar{z}_1) \sigma_{\alpha} (z_2, \bar{z}_2) \sigma_{\beta^{-1}} (z_3, \bar{z}_3) \sigma_{\beta} (z_4, \bar{z}_4) \rangle}. \]

The correlator (4.6) is calculated using local and global monodromy conditions and is given by
Using the methods of conformal field theory, one can write

\[ H(z, w, z', w', z_i) = (z - z_1)^{-\eta_\alpha} (z - z_2)^{\eta_\beta} (z - z_3)^{-\eta_\beta} (z - z_4)^{-\eta_\beta} \]

\[ (z' - z_1)^{-\eta_\alpha} (z' - z_2)^{\eta_\beta} (z' - z_3)^{-\eta_\beta} (z' - z_4)^{-\eta_\beta} \]

\[ (w - z_1)^{\eta_\alpha} (w - z_2)^{-\eta_\alpha} (w - z_3)^{\eta_\beta} (w - z_4)^{-\eta_\beta} \]

\[ (w' - z_1)^{\eta_\alpha} (w' - z_2)^{-\eta_\alpha} (w' - z_3)^{\eta_\beta} (w' - z_4)^{-\eta_\beta} \] (4.7)

\[ \left[ \frac{P(z, w)}{(z - w)^2 + A(z_i)} \right] \left( \frac{P(z', w')}{(z' - w')^2 + A(z_i)} \right)^2 \]

\[ + \left( \frac{P(z, w')}{(z - w')^2 + A(z_i)} \right) \left( \frac{P(z', w)}{(z' - w)^2 + A(z_i)} \right)^2 \]

Then, performing the various limits in (4.5) and setting \( z_1 = 0, z_2 = x, z_3 = 1 \) and \( z_4 = \infty \), we obtain

\[ \left( Z_4^{\text{excited}} \right)_{\text{qu}} = \frac{1}{\eta_\beta(1 - \eta_\alpha)} (-1)^{-2\eta_\beta x^2 \eta_\alpha (x - 1)^2 \eta_\beta - 2} \left( (1 - x) \partial_x log I \right)^2 Z_4^{\text{qu}} \] (4.8)

To factorize into \( Y_{\tilde{\tau}_\beta \tilde{\tau}_\alpha \sigma_{(\alpha \beta)}^{-1}} \), we take the limit \( x \to \infty \), in (4.8). In this limit we have

\[ \lim_{x, x \to \infty} \left( Z_4^{\text{excited}} \right)_{\text{qu}} = \lim_{x, x \to \infty} \langle \tilde{\tau}_\beta(\infty) \tilde{\tau}_\alpha'(x) \sigma_{(\alpha \beta)}^{-1}(0) \rangle \langle \sigma_{\alpha \beta}(\infty) \sigma_{\beta - 1}(1) \sigma_{\alpha^{-1}}(0) \rangle \]

(4.9)

Using the methods of conformal field theory, one can write

\[ \langle \tilde{\tau}_\beta(z_1) \tilde{\tau}_\alpha'(z_2) \sigma_{(\alpha \beta)}^{-1}(z_3) \rangle = (-1)^{-2\eta_\beta} Y_{\tilde{\tau}_\beta \tilde{\tau}_\alpha \sigma_{(\alpha \beta)}^{-1}} \]

\[ (z_1 - z_2)^{h_{\sigma_{(\alpha \beta)}} - h_{\tau'}_{(\alpha \beta)} - h_{\tau_{(\alpha \beta)}}} (z_2 - z_3)^{h_{\tau_{(\alpha \beta)}} - h_{\tau'_{(\alpha \beta)}} - h_{\sigma_{(\alpha \beta)}}} (z_1 - z_3)^{h_{\tau'}_{(\alpha \beta)} - h_{\tau_{(\alpha \beta)}}} \]

(4.10)

from which we obtain

\[ \langle \tilde{\tau}_\beta(\infty) \tilde{\tau}_\alpha'(x) \sigma_{(\alpha \beta)}^{-1}(0) \rangle = (-1)^{-2\eta_\beta} Y_{\tilde{\tau}_\beta \tilde{\tau}_\alpha \sigma_{(\alpha \beta)}^{-1}} |x|^{2(h_{\sigma_{(\alpha \beta)}} - h_{\tau_{(\alpha \beta)}} - h_{\tau'_{(\alpha \beta)}} - 1)} x^{2\eta_\alpha + 2\eta_\beta - 2} \]

(4.11)

Then, substituting (4.11) in (4.9), we obtain

\[ \lim_{x, x \to \infty} \left( Z_4^{\text{excited}} \right)_{\text{qu}} = \lim_{x, x \to \infty} (-1)^{-2\eta_\beta} |x|^{2(h_{\sigma_{(\alpha \beta)}} - h_{\tau_{(\alpha \beta)}} - h_{\tau'_{(\alpha \beta)}} - 1)} x^{2\eta_\alpha + 2\eta_\beta - 2} Y_{\tilde{\tau}_\beta \tilde{\tau}_\alpha \sigma_{(\alpha \beta)}^{-1}} Y_{\sigma_{\alpha \beta} \sigma_{\beta^{-1}} \sigma_{(\alpha \beta)}^{-1}} \] (4.12)
From the asymptotic nature of the hypergeometric functions as \( x \to \infty \), we obtain from (4.8)

\[
\lim_{x, \bar{x} \to \infty} \left( Z_{\text{excited}}^4 \right)_{qu} = (-1)^{-2\eta_{\beta}} \lim_{x, \bar{x} \to \infty} |x|^{2(h_{\sigma_{\beta}} - h_{\sigma_{\alpha}} - h_{\sigma_{(\alpha\beta)}^{-1}})} x^{2\eta_{\alpha} + 2\eta_{\beta} - 2 \frac{(1 - \eta_{\alpha})}{\eta_{\beta}}}
\]

\[
\frac{2V_{\Lambda}}{\sqrt{\text{ctg}(\pi(1 - \eta_{\alpha})) + \text{ctg}(\pi(1 - \eta_{\beta}))}} \frac{\Gamma(\eta_{\alpha})\Gamma(\eta_{\beta})}{\Gamma(\eta_{\alpha} + \eta_{\beta} - 1)} Y_{\sigma_{\beta}\sigma_{\alpha}\sigma_{(\alpha\beta)}^{-1}}^{qu}
\]

(4.13)

for \( \eta_{\beta} > 1 - \eta_{\alpha} \), and

\[
\lim_{x, \bar{x} \to \infty} \left( Z_{\text{excited}}^4 \right)_{qu} = \frac{2V_{\Lambda}}{\sqrt{\text{ctg}(\pi(1 - \eta_{\alpha})) + \text{ctg}(\pi(1 - \eta_{\beta}))}} \frac{\Gamma(\eta_{\alpha})\Gamma(\eta_{\beta})}{\Gamma(\eta_{\alpha} + \eta_{\beta} - 1)} Y_{\sigma_{\beta}\sigma_{\alpha}\sigma_{(\alpha\beta)}^{-1}}^{qu}
\]

(4.14)

for \( \eta_{\beta} < 1 - \eta_{\alpha} \). From (4.12), (4.13) and (4.14), one can read off the quantum part of the Yukawa coupling (4.5)

\[
Y_{\tau_{\beta}\tau_{\alpha}^{\prime}\sigma_{(\alpha\beta)}^{-1}}^{qu} = \frac{(1 - \eta_{\alpha})}{\eta_{\beta}} \frac{2V_{\Lambda}}{\sqrt{\text{ctg}(\pi(1 - \eta_{\alpha})) + \text{ctg}(\pi(1 - \eta_{\beta}))}} \frac{\Gamma(\eta_{\alpha})\Gamma(\eta_{\beta})}{\Gamma(\eta_{\alpha} + \eta_{\beta} - 1)}
\]

(4.15)

for \( \eta_{\beta} > 1 - \eta_{\alpha} \), and

\[
Y_{\tau_{\beta}\tau_{\alpha}^{\prime}\sigma_{(\alpha\beta)}^{-1}}^{qu} = \frac{\eta_{\beta}}{(1 - \eta_{\alpha})} \frac{2V_{\Lambda}}{\sqrt{\text{ctg}(\pi(1 - \eta_{\alpha})) + \text{ctg}(\pi(1 - \eta_{\beta}))}} \frac{\Gamma(\eta_{\alpha})\Gamma(\eta_{\beta})}{\Gamma(\eta_{\alpha} + \eta_{\beta} - 1)}
\]

(4.16)

for \( \eta_{\beta} < 1 - \eta_{\alpha} \).

Finally the second Yukawa coupling in (4.1) can be expressed as

\[
\langle \tau_{\beta}(z_1, \bar{z}_1)\tau_{\alpha}(z_2, \bar{z}_2)\tau_{\alpha}^{\prime}(z_3, \bar{z}_3) \rangle = Y_{\tau_{\beta}\tau_{\alpha}\tau_{\alpha}^{\prime}(\alpha\beta)}^{qu} Y_{\tau_{\beta}\tau_{\alpha}\tau_{\alpha}^{\prime}(\alpha\beta)}^{cl}
\]

(4.17)

Here the moduli-dependent classical part contained in \( Y_{\tau_{\beta}\tau_{\alpha}\tau_{\alpha}^{\prime}(\alpha\beta)}^{cl} \) is the same as for ground state twist field three-point functions. The quantum piece \( Y_{\tau_{\beta}\tau_{\alpha}\tau_{\alpha}^{\prime}(\alpha\beta)}^{qu} \)
is obtainable through the knowledge of the four point function

\[
Z^{\text{excited}}_4 = \mathcal{M} \lim_{\begin{array}{c}
\scriptstyle z \to z_1 \\
\scriptstyle w \to w_2 \\
\scriptstyle z' \to z_3 \\
\scriptstyle w' \to w_4
\end{array}} (z - z_1)^{\eta_\alpha}(w - w_2)^{1 - \eta_\beta}(z' - z_3)^{\eta_\beta}(w' - w_4)^{\eta_\beta}
\]

\[
\langle \partial_z X_{qu} \partial_w X \partial_{z'} X_{qu} \partial_{w'} X \rangle_{4-\text{twists}}
\]

\[
= 4\mathcal{M} \lim_{\begin{array}{c}
\scriptstyle z \to z_1 \\
\scriptstyle w \to w_2 \\
\scriptstyle z' \to z_3 \\
\scriptstyle w' \to w_4
\end{array}} (z - z_1)^{\eta_\alpha}(w - w_2)^{1 - \eta_\beta}(z' - z_3)^{\eta_\beta}(w' - w_4)^{\eta_\beta}
\]

\[
H(z, w, z', w', z_i)(\sigma_{\alpha^{-1}}(z_1, \bar{z}_1)\sigma_{\alpha}(z_2, \bar{z}_2)\sigma_{\beta^{-1}}(z_3, \bar{z}_3)\sigma_{\beta}(z_4, \bar{z}_4))
\]

\[
(4.18)
\]

where \( \mathcal{M} = 1/4(1 - \eta_\alpha)(1 - \eta_\beta) \) is a normalization factor. By performing the various limits in (4.18) and setting \( z_1 = 0, z_2 = x, z_3 = 1 \) and \( z_4 = \infty \), we obtain

\[
Z_4^{\text{excited}} = 4\mathcal{M}(-1)^{-\eta_\alpha - 1}(x)^{2\eta_\alpha}(x - 1)^{\eta_\beta - 1}(1 - x)^{\eta_\alpha - 1}
\]

\[
\left( 1 - \eta_\alpha \cdot \frac{\eta_\beta}{2} \cdot \frac{\eta_\alpha - \eta_\beta}{2} + (x - 1)\partial_x \log I \right)^2
\]

\[
+ \left( \frac{1}{2} (1 - \eta_\alpha)(1 - \frac{2}{x}) + \frac{1}{2} (1 - \eta_\alpha) - \frac{\eta_\alpha - \eta_\beta}{2} + (x - 1)\partial_x \log I \right)
\]

\[
\left( \frac{1}{2} (1 - \eta_\beta)(1 - \frac{2}{x}) + \frac{1}{2} (1 - \eta_\alpha) - \frac{\eta_\alpha - \eta_\beta}{2} + (x - 1)\partial_x \log I \right)
\]

\[
(4.19)
\]

To factorize into \( Y_{\tau_\beta\tau_\alpha'\tau_\alpha^{-1}}^{qu} \), we take the limit \( x \to \infty \), in (4.19). In this limit, using the methods of conformal field theory leads to

\[
\lim_{x, \bar{x} \to \infty} Z_4^{\text{excited}} = \lim_{x, \bar{x} \to \infty} (x)^{3\eta_\alpha + \eta_\beta - 2}\bar{x}^{2(h^s_{\beta} - h_{\sigma_{\alpha}}} - h_{\sigma_{\alpha}} - 1) \left( Y_{\tau_\beta\tau_\alpha'\tau_\alpha^{-1}}^{qu} \right)^2.
\]

\[
(4.20)
\]

From the behaviour of the hypergeometric functions as \( x \to \infty \), we find from (4.19)

\[
\lim_{x, \bar{x} \to \infty} Z_4^{\text{excited}} = \lim_{x, \bar{x} \to \infty} (x)^{2(h^s_{\beta} - h_{\sigma_{\alpha}} - h_{\sigma_{\alpha}}^{-1})}\bar{x}^{3\eta_\alpha + \eta_\beta - 2}
\]

\[
\frac{(1 - \eta_\beta - \eta_\alpha)^2}{(1 - \eta_\alpha)(1 - \eta_\beta)} \frac{4V_\Lambda}{\text{ctg}(\pi \eta_\alpha) + \text{ctg}(\pi \eta_\beta)} \frac{\Gamma^2(1 - \eta_\alpha)}{\Gamma^2(1 - (\eta_\alpha + \eta_\beta))}
\]

\[
(4.21)
\]

for \( \eta_\beta < 1 - \eta_\alpha \), and vanishing otherwise.
Then equations (4.20) and (4.21) lead to

\[ Y_{\tau_\beta \tau_\alpha \tau_\nu (\alpha, \beta)}^{\text{qu}} = 2 \sqrt{ \frac{V_\Lambda}{\text{ctg}(\pi \eta_\alpha) + \text{ctg}(\pi \eta_\beta)} } \frac{\Gamma(1 - \eta_\alpha) \Gamma(1 - \eta_\beta)}{\Gamma(1 - (\eta_\alpha + \eta_\beta))} \frac{(1 - \eta_\beta - \eta_\alpha)}{\sqrt{(1 - \eta_\alpha)(1 - \eta_\beta)}} \]  

(4.22)

The classical part of the Yukawa couplings for certain Coxeter \( Z_N \) orbifolds has been calculated in [11].

In summary, we have calculated the Yukawa couplings involving excited twist fields in all \( Z_N \) and \( Z_M \times Z_N \) Coxeter orbifolds. We have limited ourselves to those of phenomenological interest, i.e., excited twist fields with gauge quantum numbers of quarks, leptons and Higgses of the standard model. However, it is found that only the quantum part of these couplings is modified relative to the Yukawa couplings of unexcited twist fields. The classical part coming from worldsheet instantons is the same as before. Therefore our analysis is valid for all \( Z_N \) and \( Z_M \times Z_N \) orbifolds and not only the Coxeter type. Our results may be of significance in obtaining the detailed pattern of quark and lepton masses.

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5. APPENDIX

Possible excited three-point functions \( (\mathcal{Z}_i^3)_{excited} \) for the \( \mathbb{Z}_N \) and \( \mathbb{Z}_M \times \mathbb{Z}_N \) orbifolds. The point group elements \( \alpha \) associated with the various fields are represented by the subscript \( \eta_\alpha \).

**Z\(_6\)-II**

| Yukawa couplings | Possible excited three-point functions |
|------------------|----------------------------------------|
| \( T_1T_1T_4 \) | \( \langle \tau_{1/6}^1 \sigma_{1/6}^1 \tau_{4/6}^1 \rangle; \langle \sigma_{1/6}^1 \tau_{1/6}^1 \tau_{4/6}^1 \rangle \) |

**Z\(_8\)-II**

| Yukawa couplings | Possible excited three-point functions |
|------------------|----------------------------------------|
| \( T_1T_1T_6 \) | \( \langle \tau_{1/8}^1 \sigma_{1/8}^1 \tau_{6/8}^1 \rangle; \langle \sigma_{1/8}^1 \tau_{1/8}^1 \tau_{6/8}^1 \rangle \) |

**Z\(_{12}\)-I**

| Yukawa couplings | Possible excited three-point functions |
|------------------|----------------------------------------|
| \( T_2T_3T_7 \) | \( \langle \tau_{2/12}^3 \sigma_{9/12}^3 \tau_{1/12}^3 \rangle; \langle \sigma_{2/12}^3 \tau_{9/12}^3 \tau_{1/12}^3 \rangle \) |
| \( T_1T_2T_9 \) | \( \langle \sigma_{1/12}^1 \tau_{2/12}^1 \tau_{9/12}^1 \rangle; \langle \tau_{1/12}^1 \sigma_{2/12}^1 \tau_{9/12}^1 \rangle \) |
### \( Z_{12} \text{-II} \)

| Yukawa couplings | Possible excited three-point functions |
|------------------|----------------------------------------|
| \( T_1T_3T_8 \) | \( \langle \hat{\tau}^1_{1/12} \sigma^1_{3/12} \tau^1_{8/12} \rangle, \langle \sigma^1_{1/12} \tau^1_{3/12} \tau^1_{8/12} \rangle \) |
| \( T_1T_1T_{10} \) | \( \langle \tau^1_{1/12} \sigma^1_{1/12} \tau^1_{10/12} \rangle, \langle \sigma^1_{1/12} \tau^1_{1/12} \tau^1_{10/12} \rangle, \langle \hat{\tau}^1_{1/12} \sigma^1_{1/12} \hat{\tau}^1_{10/12} \rangle, \langle \sigma^1_{1/12} \hat{\tau}^1_{1/12} \hat{\tau}^1_{10/12} \rangle \) |
| \( T_2T_5T_5 \) | \( \langle \hat{\tau}^2_{10/12} \tau^2_{1/12} \sigma^2_{1/12} \rangle, \langle \hat{\tau}^2_{10/12} \sigma^2_{1/12} \tau^2_{1/12} \rangle, \langle \hat{\tau}^2_{10/12} \sigma^2_{1/12} \hat{\tau}^2_{1/12} \rangle, \langle \hat{\tau}^2_{10/12} \tau^2_{1/12} \tau^2_{1/12} \rangle \) |
### \( \mathbb{Z}_3 \times \mathbb{Z}_6 \)

| Yukawa Coupling | Possible excited three-point functions |
|-----------------|----------------------------------------|
| \( T_{01}T_{14}T_{21} \) | \( \langle \sigma_1^{2/3} \tau_2^{2/3} \tau_1^{2/3} \rangle, \)  
| | \( \langle \tau_1^{2/3} \tau_2^{2/3} \rangle \) |
| \( T_{02}T_{13}T_{21} \) | \( \langle \tau_1^{3/2} \tau_2^{3/2} \sigma_1^{3/2} \rangle, \)  
| | \( \langle \tau_1^{3/2} \sigma_1^{3/2} \tau_2^{3/2} \rangle \) |
| \( T_{04}T_{11}T_{21} \) | \( \langle \tau_1^{3/2} \tau_2^{3/2} \sigma_1^{3/2} \rangle, \)  
| | \( \langle \tau_1^{3/2} \sigma_1^{3/2} \tau_2^{3/2} \rangle \) |
| \( T_{05}T_{10}T_{21} \) | \( \langle \sigma_1^{3/2} \tau_2^{3/2} \rangle \),  
| | \( \langle \sigma_1^{3/2} \sigma_1^{3/2} \tau_2^{3/2} \rangle \) |
| \( T_{10}T_{13}T_{13} \) | \( \langle \tau_1^{3/2} \tau_2^{3/2} \sigma_1^{3/2} \rangle, \)  
| | \( \langle \tau_1^{3/2} \sigma_1^{3/2} \tau_2^{3/2} \rangle \) |
| \( T_{11}T_{11}T_{14} \) | \( \langle \sigma_1^{2} \tau_2^{2} \rangle, \)  
| | \( \langle \sigma_1^{2} \sigma_1^{2} \tau_2^{2} \rangle \) |
### $\mathbb{Z}_2 \times \mathbb{Z}_6'$

| Yukawa Coupling | Possible excited three-point functions |
|-----------------|-----------------------------------------|
| $T_0 T_{11} T_{14}$ | $\langle \tau_{1/6}^1 \tau_{4/6}^1 \sigma_{1/6}^1 \rangle, \langle \tau_{1/6}^2 \sigma_{1/6}^2 \tau_{4/6}^2 \rangle, \langle \tau_{4/6}^3 \tau_{4/6}^3 \sigma_{1/6}^3 \rangle, \langle \sigma_{1/6}^2 \tau_{4/6}^2 \tau_{4/6}^2 \rangle, \langle \tau_{4/6}^3 \sigma_{1/6}^3 \tau_{4/6}^3 \sigma_{1/6}^3 \rangle$ |

### $\mathbb{Z}_2 \times \mathbb{Z}_6$

| Yukawa Coupling | Possible excited three-point functions |
|-----------------|-----------------------------------------|
| $T_0 T_{11} T_{11}$ | $\langle \tau_{4/6}^2 \tau_{4/6}^2 \sigma_{1/6}^2 \rangle, \langle \tau_{4/6}^2 \sigma_{1/6}^2 \tau_{4/6}^2 \rangle$ |

### $\mathbb{Z}_6 \times \mathbb{Z}_6$

| Coupling | Possible excited three-point functions |
|----------|-----------------------------------------|
| $T_0 T_{14} T_{51}$ | $\langle \tau_{1/6}^2 \tau_{4/6}^2 \sigma_{1/6}^2 \rangle, \langle \sigma_{1/6}^2 \tau_{4/6}^2 \tau_{1/6}^1 \rangle$ |
| $T_0 T_{24} T_{41}$ | $\langle \tau_{1/6}^2 \tau_{4/6}^2 \sigma_{1/6}^2 \rangle, \langle \sigma_{1/6}^2 \tau_{4/6}^2 \tau_{1/6}^1 \rangle$ |
| $T_0 T_{14} T_{50}$ | $\langle \tau_{4/6}^3 \tau_{4/6}^3 \sigma_{1/6}^3 \rangle, \langle \tau_{4/6}^3 \sigma_{1/6}^3 \tau_{4/6}^3 \rangle$ |
| $T_0 T_{23} T_{41}$ | $\langle \tau_{4/6}^3 \tau_{4/6}^3 \sigma_{1/6}^3 \rangle, \langle \tau_{4/6}^3 \sigma_{1/6}^3 \tau_{4/6}^3 \rangle$ |
| $T_0 T_{32} T_{32}$ | $\langle \tau_{4/6}^3 \tau_{4/6}^3 \sigma_{1/6}^3 \rangle, \langle \tau_{4/6}^3 \sigma_{1/6}^3 \tau_{4/6}^3 \rangle$ |
| $T_0 T_{11} T_{51}$ | $\langle \tau_{4/6}^2 \tau_{4/6}^2 \sigma_{1/6}^2 \rangle, \langle \tau_{4/6}^2 \sigma_{1/6}^2 \tau_{4/6}^2 \rangle$ |

continued on next page
| $T_{04}T_{21}T_{41}$ | $\langle \tau^2_4/6 \tau^2_1/6 \tau^2_1/6 \rangle, \langle \tau^2_4/6 \tau^2_1/6 \sigma^2_1/6 \rangle$ |
|----------------------|--------------------------------------------------------------------------|
| $T_{05}T_{11}T_{50}$ | $\langle \tau^3_5 \tau^2_4/6 \sigma^2_1/6 \rangle, \langle \sigma^2_1/6 \tau^2_4/6 \tau^2_1/6 \rangle$ |
| $T_{05}T_{20}T_{41}$ | $\langle \tau^3_1/6 \tau^3_4/6 \sigma^3_1/6 \rangle, \langle \sigma^3_1/6 \tau^3_4/6 \tau^3_1/6 \rangle$ |
| $T_{10}T_{14}T_{42}$ | $\langle \tau^1_1/6 \tau^1_1/6 \tau^1_1/6 \rangle, \langle \sigma^1_1/6 \tau^1_1/6 \tau^1_1/6 \rangle$ |
| $T_{10}T_{15}T_{41}$ | $\langle \tau^3_1/6 \tau^3_1/6 \tau^3_1/6 \rangle, \langle \sigma^3_1/6 \tau^3_1/6 \tau^3_1/6 \rangle$ |
| $T_{20}T_{23}T_{23}$ | $\langle \tau^3_4/6 \tau^3_1/6 \sigma^3_1/6 \rangle, \langle \tau^3_4/6 \sigma^3_1/6 \tau^3_1/6 \rangle$ |
| $T_{11}T_{13}T_{42}$ | $\langle \tau^1_1/6 \tau^1_1/6 \tau^1_1/6 \rangle, \langle \sigma^1_1/6 \tau^1_1/6 \tau^1_1/6 \rangle$ |
| $T_{11}T_{14}T_{41}$ | $\langle \tau^1_4/6 \sigma^1_1/6 \tau^1_4/6 \rangle, \langle \tau^2_4/6 \tau^2_4/6 \sigma^2_1/6 \rangle, \langle \sigma^1_1/6 \tau^1_4/6 \tau^1_4/6 \rangle, \langle \sigma^2_1/6 \tau^2_4/6 \tau^2_4/6 \rangle, \langle \tau^3_4/6 \sigma^3_1/6 \tau^3_4/6 \rangle, \langle \tau^3_4/6 \tau^3_4/6 \sigma^3_1/6 \rangle \langle \sigma^1_1/6 \tau^1_4/6 \tau^1_4/6 \rangle$ |
| $T_{11}T_{15}T_{40}$ | $\langle \tau^1_1/6 \sigma^1_1/6 \tau^1_4/6 \rangle, \langle \sigma^1_1/6 \tau^1_1/6 \tau^1_4/6 \rangle$ |
| $T_{11}T_{23}T_{32}$ | $\langle \tau^3_4/6 \tau^3_1/6 \sigma^3_1/6 \rangle, \langle \tau^3_4/6 \sigma^3_1/6 \tau^3_1/6 \rangle$ |
| $T_{11}T_{24}T_{31}$ | $\langle \tau^1_4/6 \sigma^1_1/6 \tau^1_4/6 \rangle, \langle \sigma^1_1/6 \tau^1_4/6 \tau^1_4/6 \rangle$ |
| $T_{12}T_{12}T_{42}$ | $\langle \tau^1_1/6 \sigma^1_1/6 \tau^1_4/6 \rangle, \langle \sigma^1_1/6 \tau^1_1/6 \tau^1_4/6 \rangle$ |
| $T_{12}T_{13}T_{41}$ | $\langle \tau^1_1/6 \sigma^1_1/6 \tau^1_4/6 \rangle, \langle \sigma^1_1/6 \tau^1_1/6 \tau^1_4/6 \rangle$ |
| $T_{12}T_{14}T_{40}$ | $\langle \tau^1_1/6 \sigma^1_1/6 \tau^1_4/6 \rangle, \langle \sigma^1_1/6 \tau^1_1/6 \tau^1_4/6 \rangle$ |
| $T_{13}T_{13}T_{40}$ | $\langle \tau^1_1/6 \sigma^1_1/6 \tau^1_4/6 \rangle, \langle \sigma^1_1/6 \tau^1_1/6 \tau^1_4/6 \rangle$ |
| $T_{14}T_{20}T_{32}$ | $\langle \tau^3_1/6 \tau^3_4/6 \sigma^3_1/6 \rangle, \langle \sigma^3_1/6 \tau^3_4/6 \tau^3_1/6 \rangle$ |
| $T_{14}T_{21}T_{31}$ | $\langle \tau^2_4/6 \sigma^2_1/6 \tau^2_1/6 \rangle, \langle \tau^2_4/6 \tau^2_1/6 \sigma^2_1/6 \rangle$ |
| $T_{21}T_{21}T_{24}$ | $\langle \tau^2_1/6 \sigma^2_1/6 \tau^2_4/6 \rangle, \langle \sigma^2_1/6 \tau^2_1/6 \tau^2_4/6 \rangle$ |
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