CARTAN DECOMPOSITION OF THE MOMENT MAP

PETER HEINZNER AND GERALD W. SCHWARZ

Abstract. We investigate a class of actions of real Lie groups on complex spaces. Using moment map techniques we establish the existence of a quotient and a version of Luna’s slice theorem as well as a version of the Hilbert-Mumford criterion. A global slice theorem is proved for proper actions. We give new proofs of results of Mostow on decompositions of groups and homogeneous spaces.

1. Introduction

Let Z be a complex space with a holomorphic action of the complex reductive group $U^C$, where $U^C$ is the complexification of the compact Lie group $U$. We assume that Z admits a smooth $U$-invariant Kähler structure and a $U$-equivariant moment mapping $\mu : Z \to u^*$, where $u$ is the Lie algebra of $U$ and $u^*$ its dual. We assume that $G \subset U^C$ is a closed subgroup such that the Cartan decomposition $U^C = U \exp(iu) \simeq U \times iu$ induces a Cartan decomposition $G = K \exp p \simeq K \times p$ where $K = U \cap G$ and $p \subset iu$ is an $(\Ad K)$-stable linear subspace. We have the subspace $ip \subset u$ and by restriction an induced “moment” mapping $\mu ip : Z \to (ip)^*$. We define $M_{ip}$ to be the set of zeroes of $\mu ip$, and we define $M$ to be the set of zeroes of $\mu$. For a given $\mu$ we have the set $S_{UC}(M) := \{ z \in Z ; U^C \cdot z \cap M \neq \emptyset \}$ of semistable points with respect to $\mu$ and the $U^C$-action on $Z$. We call $S_C(M_{ip}) := \{ z \in Z ; G \cdot z \cap M_{ip} \neq \emptyset \}$ the set of semistable points of $Z$ with respect to $\mu_{ip}$ and the $G$-action on $Z$.

The set $M$ plays an important role in determining the closed $U^C$-orbits in $S_{UC}(M)$ and its quotient under the $U^C$-action (see Theorems 10.1 and 10.6 below). We show that $M_{ip}$ is the right analogue of $M$ for the action of $G$, as follows.

Theorem 1.1. Let $Z$, $G$, $M_{ip}$, $M$ be as above and set $Z' := S_C(M_{ip})$.

(1) (Corollary 11.1) An orbit $G \cdot z$ is closed in $Z'$ if and only if $G \cdot z \cap M_{ip} \neq \emptyset$.

(2) (Theorem 11.3) There is a quotient space $Z'/G$ which parametrizes the closed $G$-orbits in $Z'$. The inclusion $M_{ip} \hookrightarrow Z'$ induces a homeomorphism $M_{ip}/K \simeq Z'/G$.

(3) (Corollary 11.18) Let $z \in Z'$. Then $G \cdot z \cap Z'$ contains a unique orbit of minimal dimension and this orbit is closed in $Z'$.

(4) If $Z = S_{UC}(M)$, then

(a) (Proposition 11.2) $Z = S_C(M_{ip})$.

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(b) (Corollary 15.7) Let \( z \in Z \) and suppose that \( Y \subset \overline{G \cdot z} \) is closed and \( G \)-stable. Then there is a Lie group homomorphism \( \lambda : \mathbb{R} \to G \) such that \( \lim_{t \to -\infty} \lambda(t) \cdot z \) exists and is a point in \( Y \). The image of \( \lambda \) consists of semisimple elements of \( G \).

Note that (1b) implies that \( Z'/\!\!/G \) is metrizable and locally compact. If \( Z' \) is open, then from (3) one can deduce that \( Z'/\!\!/G \) is locally homeomorphic to semianalytic sets (Corollary 14.21).

If \( Z \) is a Stein space then it admits a smooth strictly plurisubharmonic \( U \)-invariant exhaustion function \( \rho \). Associated with \( \rho \) is a \( U \)-invariant Kähler structure and a moment mapping \( \mu \). Then \( \mathcal{M} \) is the Kempf-Ness set (see [KeNe78, Sc88]). Moreover, for any such \( \mu \), the equality \( Z = S_{U^c}(\mathcal{M}) \) holds automatically. Another interesting example of equality is the case where \( Z \) is the set of semistable points (in the sense of geometric invariant theory) relative to a \( U^c \)-linearized ample line bundle of a projective variety \( Z_0 \). In this case there also exists a \( U \)-invariant Kähler structure on \( Z \) and a \( \mu \) such that \( Z = S_{U^c}(\mathcal{M}) \). Moreover, \( Z'_0 := S_G(\mathcal{M}_{ip}) \) is then a \( G \)-stable open subset of \( Z_0 \) which is not usually \( U^c \)-stable (Remark 14.22).

Of course, there is much earlier work on quotients and slice theorems for actions of complex reductive groups, and there is also earlier work for actions of real groups. In particular, in the latter case, there are the papers of Richardson-Slodowy [RiSl90] and Luna [Lu75]. Here one has a complex representation space \( V \) of \( U^c \) and real forms \( V_{\mathbb{R}} \) of \( V \) and \( G \) of \( U^c \). One considers the action of \( G \) on \( V_{\mathbb{R}} \). Here \( V_{\mathbb{R}} \) is a Lagrangian subspace of \( V \). In the case where \( G \) is a real form of \( U^c \) and \( Z \) is a Kähler manifold there are also results about the structure of the \( G \)-action on Lagrangian submanifolds \( X \) of \( Z \) using moment map techniques (see, e.g., [O'SSj00] and references therein). These cases are rather special. The \( \mu_{ip} \)-component of \( \mu \) on \( X \) is completely determined by \( \mu \). One establishes results concerning \( \mathcal{M} \) and the \( U^c \)-action on \( Z \) and then restricts to \( X \). This works because \( X \cap \mathcal{M} = X \cap \mathcal{M}_{ip} \) and because the map \( \mu_t : Z \to \kappa^* \) obtained by restricting \( \mu \) to \( \kappa \) is constant on \( X \).

The results presented here are much more general: the group actions are not necessarily algebraic, the group \( G \) is not necessarily a real form of \( U^c \) and we consider the action on \( Z \), not just on a real form of \( Z \).

Besides the results mentioned above, we also consider several topics pertaining to proper actions and compact isotropy groups. In particular we show the following.

**Theorem 1.2.** (Proposition 8.4, Remark 11.3) Assume that \( Z = S_{U^c}(\mathcal{M}) \). Let \( X \) be a \( G \)-stable closed subset of \( Z \) such that the \( G \)-action on \( X \) is proper. Then the natural map \( G \times^K (\mathcal{M}_{ip} \cap X) \to X \) is a homeomorphism and a real analytic isomorphism if \( X \) and \( \mu_{ip} \) are real analytic.

We have a similar decomposition for the subset \( \text{Comp}_{ip}(Z) \) of points \( z \in Z \) such that \( G_z \) is compact (Theorem 6.6). The results on proper actions are applied to obtain decompositions, due to Mostow, for groups and homogeneous spaces. The application relies on properties of a distinguished strictly plurisubharmonic exhaustion of \( U^c \) related to the Cartan decomposition. See section 9.

Several of our results rely upon the notion of \( \mu_{ip} \)-adapted sets. A \( \mu_{ip} \)-adapted subset of \( Z \) is a \( K \)-invariant subset \( A \) of \( Z \) such that for all \( z \in Z \) and \( \xi \in \mathfrak{p} \), the curve \( (\exp it\xi) \cdot z \) lies in \( A \) for a connected set \( J \) of \( t \in \mathbb{R} \). Moreover, we require that if \( t_+ := \sup J < \infty \), then \( \mu_{ip}(\exp(it_+\xi) \cdot z)(it\xi) > 0 \) and a similar negativity condition if \( t_- := \inf J > -\infty \). We are able to show that every \( K \)-orbit in \( \mathcal{M}_{ip} \) has a neighborhood basis of open \( \mu_{ip} \)-adapted sets in the case that \( Z = S_{U^c}(\mathcal{M}) \) (Theorem 13.7). The \( \mu_{ip} \)-adapted sets have very nice properties. For example, if \( A_1 \) and \( A_2 \) are \( \mu_{ip} \)-adapted, then \( G \cdot A_1 \cap G \cdot A_2 = G \cdot (A_1 \cap A_2) \).
In case $U$ is commutative, we can prove Theorem 13.1 without the hypothesis $Z = S_{U^C}(M)$. This allows us to establish the “separation property” which is used in the proof of most of the statements in Theorem 14. In addition, if $U$ is commutative, then $S_G(M_{ip})$ is open and $U^C$-stable in $Z$. In general $S_G(M_{ip})$ is not $U^C$-stable. It would be interesting to know if $S_G(M_{ip})$ is always open in $Z$. Examples indicate that it should be extremely interesting to clarify the interplay of the various geometric objects associated with $\mu$, $\mu_{ip}$ and $\mu_t$.

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2. $G$-fiber bundles and slices

Let $H$ be a closed subgroup of the Lie group $G$ and $S$ an $H$-space. The twisted product $G \times^H S$ is the quotient $(G \times S)/H$ with respect to the $H$-action $(h, (g, x)) \mapsto (gh^{-1}, h \cdot x)$. Since the $G$-action on $G \times S$ given by multiplication from the left on the first factor commutes with the $H$-action there is an induced $G$-action on $G \times^H S$. We use the notation $[g, x]$ for $H \cdot (g, x) \in G \times^H S$. Note that $G \times^H S$ is a $G$-fiber bundle over $G/H$ with fiber $S$ associated to the $H$-principal bundle $G \to G/H$. Let $X$ be a $G$-space and $H$ a closed subgroup of $G$. An $H$-stable subspace $S$ of $X$ is said to be a global $H$-slice if the natural map $G \times^H S \to X$, $[g, z] \mapsto g \cdot z$ is an isomorphism. If $x \in S$, $S$ is $G_x$-stable, $G \cdot S$ is open in $X$ and $G \times^G S \to G \cdot S$ is an isomorphism, then $S$ is called a geometric slice at $x$.

3. Kähler structures

In this paper a complex space $Z$ is always a reduced complex space with countable topology. If $G$ is a Lie group, then a complex $G$-space $Z$ is a complex space with a real analytic action $G \times Z \to Z$ which for fixed $g \in G$ is holomorphic. For a complex Lie group $G$ a holomorphic $G$-space is a complex $G$-space $Z$ such that the $G$-action $G \times Z \to Z$ is holomorphic.

A Kähler structure $\omega$ on $Z$ is an open covering $\{U_\alpha\}$ of $Z$ together with smooth strictly plurisubharmonic functions $\rho_\alpha : U_\alpha \to \mathbb{R}$ such that $h_{\alpha\beta} = \rho_\alpha - \rho_\beta$ is pluriharmonic on $U_\alpha \cap U_\beta$. Here strictly plurisubharmonic means strictly plurisubharmonic with respect to perturbations (see [HeHuLo94]) and a pluriharmonic function is, as usual, a function which is locally the real part of a holomorphic function. Note that for smooth $Z$ one obtains the usual definition of a Kähler manifold whose Kähler form is given locally by $\omega_\alpha = -dd^c \rho_\alpha = 2i\partial \bar{\partial} \rho_\alpha$. Here $\partial$ and $\bar{\partial}$ are the usual exterior differential operators and $dd^c \rho(v) = d\rho(Jv)$ for every $v \in T_Z Z$ and smooth function $\rho$, where $J$ denotes multiplication by $i = \sqrt{-1}$ on the tangent space $T_Z Z$. For smooth $Z$ we will not distinguish between $\omega = \{\rho_\alpha\}$ as defined above and the associated Kähler form $\omega$ given by $\omega|U_\alpha := -dd^c \rho_\alpha$.

For a complex $G$-space $Z$ one has a natural notion of a $G$-invariant Kähler structure $\omega$. A moment map on a complex $G$-space $Z$ with respect to such an invariant Kähler structure is a smooth $G$-equivariant map $\mu : Z \to g^*$ such that, for every $G$-stable complex submanifold $Y$ of $Z$ and $\xi \in g$, we have

$$d\mu(\xi) = \iota_{\xi,Z} \omega_Y.$$

Here $\omega_Y$ denotes the Kähler form induced on $Y$ and $\iota_{\xi,Z}$ denotes the vector field on $Z$ induced by $\xi$ and the $G$-action. The map $\mu^\xi : Z \to \mathbb{C}$ sends $z \in Z$ to $\mu(z)(\xi)$ and $\iota_{\xi,Z}$ denotes contraction with $\xi$. Our condition on $d\mu(\xi)$ is equivalent to requiring that $\text{grad}(\mu^\xi) = J\xi_Z$ where the gradient is with respect to the underlying Riemannian structure on $Y$. 

Let $G$ be a complex reductive group, let $Z$ be a holomorphic $G$-space and let $\mathcal{O}_Z$ denote the structure sheaf of $Z$. A complex space $Y$ together with a holomorphic map $\pi: Z \to Y$ is said to be an analytic Hilbert quotient of $Z$ with respect to the $G$-action if

- $\pi$ is a $G$-invariant locally Stein map and
- $\mathcal{O}_Y = \pi_*\mathcal{O}_Z^G$.

Here locally Stein means that there is an open covering of $Y$ by open Stein subspaces $Q_\alpha$ such that $\pi^{-1}(Q_\alpha)$ is a Stein subspace of $Z$ for all $\alpha$, and $\pi_*\mathcal{O}_Z^G$ denotes the sheaf $Q \to \mathcal{O}_Z(\pi^{-1}(Q))^G$, $Q$ open in $Y$. If the analytic Hilbert quotient of $Z$ with respect to $G$ exists it will be denoted by $Z//G$. If $H$ is a reductive algebraic subgroup of $G$ and $Z//G$ exists, then $Z//H$ exists (see, e.g., \cite{HeMiPo98}).

Assume now that the analytic Hilbert quotient $Z//G$ exists. Then it has the following properties.

- For every Stein subspace $A$ of $Z//G$ the inverse image $\pi^{-1}(A)$ is a Stein subspace of $Z$.
- For every closed analytic $G$-invariant subspace $X$ of $Z$ the image $\pi(X)$ is a closed analytic subspace of $Z//G$ and the restriction $\pi|X: X \to \pi(X)$ is an analytic Hilbert quotient of $X$.
- The quotient map $\pi$ maps disjoint closed $G$-stable subsets of $Z$ onto disjoint closed subsets of $Z//G$.

If $Z$ is a Stein space, then the analytic Hilbert quotient exists and has the properties above (see \cite{He91}). See \cite{HeMiPo98} for the general case.

4. THE CARTAN DECOMPOSITION

Let $U$ be a compact Lie group. Then $U$ has a natural real linear algebraic group structure, and we denote by $U^\mathbb{C}$ the corresponding complex linear algebraic group \cite{Ch46}. The group $U^\mathbb{C}$ is reductive and is the universal complexification of $U$ in the sense of \cite{Ho65}. On the Lie algebra level we have the Cartan decomposition

$$u^\mathbb{C} = u + iu$$

with a corresponding Cartan involution $\theta: u^\mathbb{C} \to u^\mathbb{C}$, $\xi + i\eta \mapsto \xi - i\eta$, $\xi, \eta \in u$. We also use $\theta$ to denote the corresponding anti-holomorphic involution on $U^\mathbb{C}$. The real analytic map

$$U \times iu \to U^\mathbb{C}, \quad (u, \xi) \mapsto u \exp \xi$$

is a diffeomorphism. Since $g_1(u \exp \xi)g_2^{-1} = g_1u g_2^{-1} \exp(\text{Ad}(g_2) \cdot \xi)$ for $(g_1, g_2) \in U \times U$, $\xi \in u$ and $u \in U$, the isomorphism $U \times iu \cong U^\mathbb{C}$ is a $U \times U$-equivariant diffeomorphism. We refer to the decomposition $U^\mathbb{C} = U \exp(iu)$ as the Cartan decomposition of $U^\mathbb{C}$, and we fix it for the remainder of this paper.

Let $G$ be a real Lie subgroup of $U^\mathbb{C}$. We say that $G$ is compatible with the Cartan decomposition of $U^\mathbb{C}$ if $G = K \exp p$ where $K$ is a Lie subgroup of $U$ and $p$ is a $K$-stable linear subspace of $iu$. Note that

$$K \times p \to G, \quad (k, \xi) \to k \exp \xi$$

is a $(K \times K)$-equivariant diffeomorphism. The closure $\bar{G}$ of $G$ in $U^\mathbb{C}$ is given by $K \exp p$. In particular, $G$ is a closed subgroup of $U^\mathbb{C}$ if and only if $K$ is compact. For a closed subgroup $G$ the complexification $K^\mathbb{C}$ of $K$ is a closed complex subgroup of $U^\mathbb{C}$ and

$$K \times i\mathfrak{k} \to K^\mathbb{C}, \quad (k, \xi) \to k \exp \xi$$

is a $(K \times K)$-equivariant diffeomorphism.
Example 4.1. a) For any compact subgroup $K$ of $U$, both $K$ and its complexification $G = K^c$ are compatible with the Cartan decomposition of $U^c$. In particular, $G = U^c$ is an example of a compatible subgroup.

b) For any $\xi \in iu$, the group $G = \exp(\mathbb{R}\xi)$ is compatible. More generally, if $a \subseteq iu$ is a Lie subalgebra of $u^c$, then it is commutative and $G = \exp(i\alpha + a)$ is a compatible subgroup of $U^c$. Note that $K = \exp(ia)$ need not be compact.

c) Let $\sigma$ be an antiholomorphic involution of $U^c$ which commutes with $\theta$. Let $G$ be a $\theta$-stable open subgroup of $(U^c)^\sigma$. Then $G$ is a compatible real form of $U^c$ and $u = \mathfrak{k} \oplus i\mathfrak{p}$.

Remark 4.2. Let $G$ be a compatible subgroup of $U^c$. Then the smallest complex subgroup of $U^c$ which contains $G$ is compatible and $\theta$-stable, hence is a reductive algebraic subgroup of $U^c$.

5. Moment map decomposition

Let $Z$ be a holomorphic $U^c$-space and $G = K\exp\mathfrak{p}$ a compatible subgroup of $U^c$. We assume that $Z$ is Kähler with $U$-invariant Kähler structure $\omega$ and that there is a $U$-equivariant moment mapping $\mu: Z \to u^\ast$. For any linear subspace $\mathfrak{m}$ of $\mathfrak{u}$ the inclusion gives by restriction a map $\mu_\mathfrak{m}: Z \to \mathfrak{m}^\ast$. Thus we have an equivariant moment mapping $\mu_\xi: Z \to \mathfrak{t}^\ast$ with respect to the $K$-action and a $K$-equivariant mapping $Z \to (i\mathfrak{p})^\ast$ which we denote by $\mu_{i\mathfrak{p}}$.

For $\beta \in \mathfrak{t}^\ast$ let $\mathcal{M}(\beta)$ denote $\mu^{-1}(\beta)$ and set $\mathcal{M} := \mathcal{M}(0)$. If $\beta \in (i\mathfrak{p})^\ast$ then we set $\mathcal{M}_{i\mathfrak{p}}(\beta) := \mu_{i\mathfrak{p}}^{-1}(\beta)$, $\mathcal{M}_{i\mathfrak{p}} = \mathcal{M}_{i\mathfrak{p}}(0)$ and similarly for the $\mathfrak{t}$ component of $\mu$. Then $\mathcal{M}_{i\mathfrak{p}}$ and $\mathcal{M}_{\mathfrak{t}}$ are $K$-stable and $\mathcal{M}$ is $U$-stable. When necessary for clarity, we will also use the notation $\mathcal{M}(Z), \mathcal{M}_{i\mathfrak{p}}(Z)$, etc. If $u = \mathfrak{t} + i\mathfrak{p}$, then $\mathcal{M} = \mathcal{M}_{i\mathfrak{p}} \cap \mathcal{M}_{\mathfrak{t}}$. Since the $U^c$-action is holomorphic, for every $\xi \in u^c$ the one-parameter group $(t, z) \mapsto (\exp it\xi) \cdot z, t \in \mathbb{R}, z \in Z$, has derivative the vector field $J_z \xi = (i\xi)_Z$.

For a linear subspace $\mathfrak{m}$ of $u^c$ and $z \in Z$ we set $\mathfrak{m} \cdot z := \{\xi_z(z); \xi \in \mathfrak{m}\} \subset T_zZ$ and $\mathfrak{m}_z$ will denote $\{\xi \in \mathfrak{m}; \xi_z(z) = 0\}$. If $z \in Z$ is a smooth point, then, in $T_zZ$, we use $^\perp\omega$ to denote perpendicularity with respect to $\omega$ and we use $^\perp$ to denote perpendicularity with respect to the underlying Riemannian structure. If $M \subset U^c$, then we set $M \cdot z := \{m \cdot z; m \in M\}$.

Lemma 5.1. Let $z \in Z$ be a smooth point. Then

1. $\ker d\mu_\mathfrak{m}(z) = (\mathfrak{p} \cdot z)^\perp = (i\mathfrak{p} \cdot z)^{+\omega}$ and
2. $(\mathfrak{g} \cdot z)^\perp = (\mathfrak{t} \cdot z)^\perp \cap \ker d\mu_{i\mathfrak{p}}(z)$.

Proof. This follows from the basic equation $d\mu^\xi = i_{\xi_z}\omega$ and from the fact that $\omega(J_\cdot, \cdot)$ is the Riemannian metric on $T_zZ$.

Lemma 5.2. Let $z \in Z$ be a smooth point and assume that $z \in \mathcal{M}_{i\mathfrak{p}}$. Then

1. $\mathfrak{g} \cdot z = \mathfrak{t} \cdot z \oplus \mathfrak{p} \cdot z$ where $\mathfrak{t} \cdot z \perp \mathfrak{p} \cdot z$.
2. If $G$ is a real form of $U^c$, then $\dim\mathbb{R} \mathfrak{t} \cdot z \leq \dim\mathbb{R} \mathfrak{g} \cdot z$.

Proof. Since $z \in \mathcal{M}_{i\mathfrak{p}}$ and $\mu_\mathfrak{p}$ is $K$-equivariant, we have $\mathfrak{t} \cdot z \subset \ker d\mu_\mathfrak{p}(z)$ and therefore $\mathfrak{t} \cdot z \perp \mathfrak{p} \cdot z$, giving (1). If $G$ is a real form of $U^c$, then $u \cdot z = \mathfrak{t} \cdot z + (i\mathfrak{p}) \cdot z$. Now $(i\mathfrak{p}) \cdot z = J(\mathfrak{p} \cdot z)$ and $\mathfrak{p} \cdot z$ have the same dimension, so (2) follows from (1).

Example 5.3. Let $G$ be a real form of $U^c$. Assume that $Z$ is compact and $U$-homogeneous, e.g., $Z$ is a flag manifold of $U^c$. If $z \in \mathcal{M}_{i\mathfrak{p}}$, then $\dim\mathbb{R} Z = \dim\mathbb{R} \mathfrak{u} \cdot z \leq \dim\mathbb{R} \mathfrak{g} \cdot z$, hence $G \cdot z$ is open in $Z$.

Lemma 5.4. For $z \in Z$ and $\xi \in \mathfrak{p}$ we have $\mu_\mathfrak{p}(\exp \xi \cdot z) = \mu_\mathfrak{p}(z)$ if and only if $\xi \in \mathfrak{p}_z$. In particular, $\exp \mathfrak{p} \cdot z \cap \mathcal{M}_{i\mathfrak{p}}(\beta) = \{z\}$ for all $z \in \mathcal{M}_{i\mathfrak{p}}(\beta)$. 
Proof. Assume that \( \mu_{ip}(\exp t\xi \cdot z) = \mu_{ip}(z) \). Set \( Y = U^C \cdot z \) and let \( \omega_Y \) denote the Kähler form induced on \( Y \). Let \( \alpha(t) = (\exp t\xi)(z) \) and \( \beta(t) = \mu^t(\alpha(t)) \). Then \( \beta(0) = \beta(1) \) and \( \beta'(t) = d\mu^t(\xi)(\alpha(t)) = \omega_Y(J\xi_Z, \xi_Z)(\alpha(t)) \geq 0 \) since \( \omega_Y(J\cdot, \cdot) \) is the underlying Riemannian metric of the Kähler metric on \( Y \). We must have \( \beta'(0) = 0 \) which implies that \( \xi_Z(z) = 0 \). So \( \xi \in \mathfrak{p}_z \) and \( (\exp \xi) \cdot z = z \). □

Lemma 5.5. Let \( z \in \mathcal{M}_{ip} \). Then

1. \( G \cdot z \cap \mathcal{M}_{ip} = K \cdot z \).
2. \( G_z = K_z \exp \mathfrak{p}_z \simeq K_z \times \mathfrak{p}_z \).

In particular, \( G_z \) is compatible with the Cartan decomposition of \( U^C \).

Proof. Let \( g = k \exp \xi \in G \) where \( k \in K \) and \( \xi \in \mathfrak{p} \), and suppose that \( gz \in \mathcal{M}_{ip} \). Then \( \mu_{ip}((k \exp \xi) \cdot z) = \mu_{ip}((\exp \xi) \cdot z) = \mu_{ip}(z) = 0 \). Applying Lemma 5.4 we obtain (1). For (2) just notice that \( (k \exp \xi) \cdot z = z \) implies that \( \xi \in \mathfrak{p}_z \) by the argument above, so that \( (k \exp \xi) \cdot z = kz = z \), and \( k \in K_z \). So \( G_z = K_z \exp \mathfrak{p}_z \). □

Remark 5.6. Applying Lemma 5.5 in the case \( G = U^C \) and \( z \in \mathcal{M} \) we obtain:

1. \( U^C \cdot z \cap \mathcal{M} = U \cdot z \).
2. \( (U^C)_z = U_z \exp(\mathfrak{i}u_z) \simeq U_z \times \mathfrak{i}u_z \).

6. Compact isotropy groups

Let \( G \) be a compatible Lie subgroup of \( U^C \). Let \( Z \) be a holomorphic \( U^C \)-space with \( U \)-invariant Kähler form \( \omega \) and \( U \)-equivariant moment mapping \( \mu : Z \to \mathfrak{u}^* \).

Proposition 6.1. Let \( z \in Z \) be a smooth point. If \( G_z \) is compact, then \( \mathfrak{p}_z = \{0\} \). Moreover, the following are equivalent.

1. \( \mathfrak{p}_z = \{0\} \).
2. \( d\mu_{ip}(z) \) maps \( \mathfrak{p} \cdot z \) isomorphically onto \( (\mathfrak{i}p)^* \).
3. \( d\mu_{ip}(z) \) is surjective, i.e., \( \mu_{ip} \) is a submersion at \( z \).

Proof. The Cartan decomposition of \( G \) implies that exp \( \mathfrak{p}_z \) is compact if and only if \( \mathfrak{p}_z = \{0\} \). Thus compactness of \( G_z \) implies that \( \mathfrak{p}_z = \{0\} \). Since \( \text{Ker}(d\mu_{ip}(z)) = (\mathfrak{p} \cdot z)^\perp \), we see that \( \mathfrak{p} \cdot z \) is mapped isomorphically onto \( \mathfrak{i}p^* \) if and only if \( \mathfrak{p}_z = \{0\} \) if and only if \( d\mu_{ip}(z) \) is surjective. □

Remark 6.2. Since \( U^C \) is a Stein manifold, the isotropy group \( (U^C)_z \) is compact if and only if it is finite. If \( (U^C)_z \) is finite then \( d\mu(z) \) maps \( \mathfrak{i}u \cdot z \) isomorphically onto \( \mathfrak{u}^* \). The converse is false. Just consider the standard actions of \( SU(2, \mathbb{C}) \subset SL(2, \mathbb{C}) \) on \( \mathbb{C}^2 \). Then the standard moment mapping has surjective differential on \( \mathbb{C}^2 \setminus \{0\} \), while \( SL(2, \mathbb{C}) \) has one dimensional isotropy groups.

Corollary 6.3. Let \( G \) be closed in \( U^C \) and \( Z \) smooth. Then the set of \( z \in \mathcal{M}_{ip} \) such that \( G_z \) is compact is open in \( \mathcal{M}_{ip} \).

Proof. For \( z \in \mathcal{M}_{ip} \) the isotropy group \( G_z \) is compact if and only if \( \mathfrak{p}_z = \{0\} \) (Lemma 5.5). Since \( Z \) is smooth, the set of \( z \in Z \) such that \( \mu_{ip} \) has maximal rank at \( z \) is open. Now the claim follows from Proposition 6.1. □

Corollary 6.4. Let \( X \) be a (real) smooth \( G \)-stable submanifold of \( Z \) and assume that \( G_z \) is compact for all \( x \in X \). Then \( \mathcal{M}_{ip}(\beta) \cap X \) is smooth for every \( \beta \in (\mathfrak{i}p)^* \). If \( x \in \mathcal{M}_{ip}(\beta) \cap X \), then \( T_x(\mathcal{M}_{ip}(\beta) \cap X) = \text{Ker}(d(\mu_{ip}|_X)_x) = (\mathfrak{p} \cdot x)^\perp \), the perpendicular being taken in the tangent space \( T_x(X) \).
Proof. Proposition 5.3 applied to $U^C \cdot x$ for $x \in X$ shows that $d(\mu_p|_x)[X]$ maps $p \cdot x$ isomorphically onto $(ip)^*$. Hence $M_{ip}(\beta) \cap X$ is smooth and $T_x(M_{ip}(\beta) \cap X) = \text{Ker} d(\mu_p|_x)[x] = (p \cdot x)^\perp$. □

Remark 6.5. Assume that $Z$ is smooth and that the $U^C$-action on $Z$ has compact (hence finite) isotropy groups. Then Corollary 5.3 says that $M(\beta)$ is a smooth submanifold of $Z$ with tangent space $T_x(M(\beta)) = \text{Ker} d\mu_x = (iuy \cdot z)^\perp$ at $z \in M(\beta)$.

Let $G$ be closed in $U^C$ and let $Z$ be smooth. Let $Z_G^r$ denote $\{z \in Z; \mu_p$ is a submersion at $z\}$. The set $Z_G^r$ is $K$-stable and open in $Z$ (Proposition 6.1). Let $\text{Comp}_{ip}(Z) := \{z \in Z; G \cdot z \cap M_{ip} \cap Z_G^r \neq \emptyset\}$. We have the following slice theorem.

Theorem 6.6. (1) The set $\text{Comp}_{ip}(Z)$ is $G$-stable and for every $z \in \text{Comp}_{ip}(Z)$ the isotropy group $G_z$ is compact. In particular, $\text{Comp}_{ip}(Z) \subset Z_G^r$.

(2) Let $S := M_{ip} \cap Z_G^r$. Then $S$ is a smooth closed $K$-submanifold of $Z_G^r$, $G \cdot S = \text{Comp}_{ip}(Z)$ is open in $Z$ and the natural map $G \times K S \to \text{Comp}_{ip}(Z)$ is an isomorphism of $G$-manifolds.

Proof. The set $S = Z_G^r \cap M_{ip} = (\mu_p|_{Z_G^r})^{-1}(0)$ is a closed submanifold of $Z_G^r$. The natural map $\Phi: G \times K S \to Z, [g, s] \mapsto g \cdot s$, has image $\text{Comp}_{ip}(Z)$. We first show that $\Phi$ is injective. Let $g_j \in G$ and $s_j \in S$ be such that $g_1 \cdot s_1 = g_2 \cdot s_2$ and set $g := g_2^{-1}g_1$. Then $g = k \exp \xi$ for some $k \in K$ and $\xi \in p$ and $s_2 = g \cdot s_1$. This implies that $\exp \xi \cdot s_1 = k^{-1} \cdot s_2 \in M_{ip}$ and therefore that $\exp \xi \cdot s_1 = s_1$. But $G_{s_1} = K_{s_1}$ and $p_{s_1} = \{0\}$ by assumption (see Lemma 5.3). Consequently, $\xi = 0$ and $g = k$. Hence $[g_1, s_1] = [g_2 k, s_1] = [g_2, s_2]$. This shows injectivity of $\Phi$.

In order to show that $\Phi$ is an open embedding it is sufficient to prove that $\Phi$ is a submersion. But for any $s \in S$ the tangent space $T_{[e, s]}(G \times K S)$ is mapped onto $g \cdot s + (p \cdot s)^\perp = p \cdot s + (p \cdot s)^\perp = T_x(Z)$. Hence $\Phi$ is a submersion at any point $[e, s]$. Since $\Phi$ is $G$-equivariant it is a submersion at any point of $G \times K S$. Thus $\Phi$ is an isomorphism onto $\text{Comp}_{ip}(Z)$. □

Remark 6.7. If $S_0$ is a differentiable slice at $s_0 \in S$ for the $K$-action on $S$, then $K \cdot S_0$ is a slice for the $G$-action on $Z$ at $s_0$, i.e., $G \cdot S_0$ is open in $Z$ and $G \times K S_0 \to G \cdot S_0$ is a diffeomorphism.

Remark 6.8. The $G$-action on $\text{Comp}_{ip}(Z)$ is proper.

7. Invariant plurisubharmonic functions

Let $Z$ be a holomorphic $U^C$-space. Assume that we are given a smooth $U$-invariant strictly plurisubharmonic function $\rho$ on $Z$. Set $\mu^\xi(z) = \frac{d}{d\xi}|_{\xi=0}\rho((\exp i\xi) \cdot z), z \in Z, \xi \in u$. Note that $\omega_Y = -dd^c(\rho|Y)$ is a $U$-invariant Kähler form on every smooth complex $U$-submanifold $Y$ of $Z$.

Lemma 7.1. Let $\rho$ and $\mu$ be as above. Then $\omega = \{\rho\}$ defines a $U$-invariant Kähler structure on $Z$ and $\mu$ is a moment mapping.

Proof. Let $\xi \in u$ and let $Y$ be a complex $U$-submanifold of $Z$. Then, since $\rho$ is $U$-invariant and since $U$ acts as complex analytic isomorphisms of $Z$, we have that $L_{\xi_x} d^c(\rho|Y) = 0$ where $L_{\xi_x}$ denotes Lie derivative with respect to $\xi_Z$. By Cartan’s formula, $L_{\xi_x} = d \circ \iota_{\xi_x} + \iota_{\xi_x} \circ d$, hence $d\mu^\xi = d\iota_{\xi_x}(d^c(\rho|Y)) = -\iota_{\xi_x} dd^c(\rho|Y) = \iota_{\xi_x} \omega_Y$. □

In the situation of Lemma 7.1 we will say that the moment map is defined by or associated with $\rho$.

Remark 7.2. Let $G = K \exp p \subset U^C$ be compatible, let $\mu$ be associated to $\rho$ and let $z \in Z$. Since $\rho(K \cdot z) = \rho(z), d\rho(z)(t \cdot z) = 0$ and $M_{ip} = \{z \in Z; \rho|G \cdot z$ has a critical point at $z\}$. 

Let $X$ be a topological space. A function $f: X \to \mathbb{R}$ is said to be an *exhaustion*, if for every $r \in \mathbb{R}$ the set $\{x \in X; f(x) \leq r\}$ is compact.

**Example 7.3.** Let $Z$ be a representation space of $U^C$. Let $\rho(z) = ||z||^2$ where $|| \cdot ||$ is the $U$-invariant norm on $Z$ coming from a $U$-invariant hermitian inner product. Then $\rho$ is a $U$-invariant strictly plurisubharmonic exhaustion of $Z$. Every fiber of the quotient $\pi: Z \to Z//U^C$ intersects $M$ in a single $U$-orbit $U \cdot z$, where $U^C \cdot z$ is closed. Moreover, the inclusion $M \to Z$ induces a homeomorphism $M/K \simeq Z//U^C$. See [KeNe78] and [Sc88].

**Example 7.4.** Let $U = SU(2, \mathbb{C})$ with its action on the complex binary forms $Z$ of degree 3. Then $U^C = SL(2, \mathbb{C})$. Let $\rho(z) = ||z||^2$ as above. The open set of orbits with finite $U^C$ isotropy group consists of the closed $U^C$-orbits with isotropy group isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and a non-closed orbit $U^C \cdot z_0$ where $(U^C)z_0 = \{e\}$. However, $U^C \cdot z_0$ does not intersect $M$, so thatComp$_{fp}(Z)$ is the open set of closed orbits with $(\mathbb{Z}/3\mathbb{Z})$-isotropy. This is unavoidable since the $U^C$-action on Comp$_{fp}(Z)$ is proper. If the action were proper on the whole open set of orbits with finite isotropy, then the slice theorem for proper actions (see [Pa73]) would show that there is an open set of orbits with trivial isotropy, which is not the case.

**Proposition 7.5.** Let $G \subset U^C$ be a compatible subgroup and let $\rho$ be a smooth $U$-invariant strictly plurisubharmonic function on $Z$. Let $z \in Z$ and suppose that

1. $\rho|G \cdot z$ is an exhaustion, i.e., the map $\bar{\rho}: G/G_z \to \mathbb{R}$, $gG_z \mapsto \rho(g \cdot z)$, is an exhaustion.

Then

2. $\rho|G \cdot z$ has a minimum value.
3. $\rho|G \cdot z$ has a critical point.
4. $G \cdot z \cap M_{fp} \neq \emptyset$.
5. $G \cdot z$ is closed.

**Proof.** Clearly we have that (1) implies (2) implies (3) implies (4). Let $(g_n \cdot z)$ be a sequence in $G \cdot z$ such that $\lim g_n \cdot z = z' \in Z$. Since $\{g \cdot z \in G \cdot z; \rho(g \cdot z) \leq \rho(z') + 1\}$ is compact, passing to a subsequence, we may assume that $\lim g_n \cdot z = g_0 \cdot z$ for some $g_0 \in G$. Then $g_0 \cdot z = z' \in G \cdot z$ and we have (5). \qed

**Remark 7.6.** We will show later that (3) or, equivalently, (4) imply that $\rho|G \cdot z$ is an exhaustion. Thus (1) through (4) are equivalent, and they all imply (5). Of course, if $\rho$ is an exhaustion, then (5) implies (1). All this was first observed by Azad and Loeb using results of Mostow. We will obtain Mostow’s results from general moment map properties and will then repeat the argument of Azad and Loeb (Lemma 9.6).

**Remark 7.7.** From Lemma 5.5 we see that the intersection in (4) is a single $K$-orbit consisting of the points where $\rho$ takes on its minimal value.

**Remark 7.8.** A smooth strictly plurisubharmonic $U$-invariant exhaustion function on $Z$ exists if and only if $Z$ is a Stein space.

**Example 7.9.** Let $\mathfrak{g}$ be a real semisimple Lie algebra and $G$ the corresponding adjoint group. Let $\mathfrak{g}^C = \mathfrak{g} \otimes \mathbb{C}$ be the complexification of $\mathfrak{g}$ with corresponding adjoint group $G^C$. Then $G^C$ has a Cartan involution $\theta$ defining a maximal compact subgroup $U$ of $G^C$, and $G$ is a real form of $G^C$ with compatible Cartan decomposition $G = K \exp \mathfrak{p}$. Let $\kappa$ denote the Killing form on $\mathfrak{g}^C$ and set $\rho(z) = -\kappa(z, \theta(z))$ for $z \in Z := \mathfrak{g}^C$. The function $\rho$ is a $U$-invariant strictly plurisubharmonic exhaustion function on $Z$. Let $\mu$ denote the associated moment mapping. A simple calculation
shows that \( M = \{ z \in Z ; [z, \theta(z)] = 0 \} \). Moreover, we have \( g \cap M = g \cap M_{ip} \) since \( g \) is a Lagrangian subspace of \( g^C \) with respect to \( \omega = -dd^c \rho \). In particular, as is well-known, the orbit \( G \cdot x \) in \( g \) is closed if and only if \( G^C \cdot x \) is closed in \( g^C \), and the latter is the case if and only if \( x \) is semisimple.

For \( x \in g \) we can write \( x = x_t + x_p \) where \( x_t \in \mathfrak{k} \) and \( x_p \in \mathfrak{p} \). We have \( M_{ip} \cap g = \{ x = x_t + x_p ; [x_t, x_p] = 0 \} \). Now consider the set \( X := \{ x \in g ; G_x \text{ is compact } \} \). Using the Jordan decomposition for \( x \in M_{ip} \cap X \subset g \subset g^C \) one sees that compactness of \( G_x \) forces \( x \) to be semisimple. Then \( [x_t, x_p] = 0 \) implies that \( x_p \in \mathfrak{g}_x \), hence \( x_p = 0 \) and \( x \in \mathfrak{k} \). This shows that \( X = G \cdot S \) with \( S := X \cap \mathfrak{k} \). Applying Theorem 6.6 we see that \( X \) is open in \( g \), that the natural map \( G \times K \rightarrow X \) is a real analytic isomorphism and hence that \( G \) acts properly on \( X \).

8. Proper actions

As in §6, we consider proper \( G \)-actions. However, we do not assume that the holomorphic \( U^C \)-space under consideration is smooth, but we do assume that the moment mapping is associated to a strictly plurisubharmonic exhaustion.

Let \( Z \) be a holomorphic \( U^C \)-space with a \( U \)-invariant Kähler structure and moment map \( \mu : Z \rightarrow u^* \). Let \( G = K \exp \mathfrak{p} \) be a compatible Lie subgroup of \( U^C \).

**Proposition 8.1.** Assume that \( \mu \) is associated with a strictly plurisubharmonic \( U \)-invariant exhaustion and let \( X \) be a \( G \)-stable closed subset of \( Z \). If \( G \times X \rightarrow X \) is a proper action, then the canonical map \( \Phi : G \times K (M_{ip} \cap X) \rightarrow X ; (g, x) \mapsto g \cdot x \), is a homeomorphism. If \( X \) is a real analytic submanifold and \( \mu \) is real analytic, then \( \Phi \) is a real analytic isomorphism.

**Proof.** Let \( x \in X \). Then \( G \cdot x \) is closed, and by Proposition 7.3 and Lemma 6.3 \( G \cdot x \cap M_{ip} = K \cdot x_0 \) for some \( x_0 \in M_{ip} \). Thus \( G \times K (M_{ip} \cap X) \rightarrow X \) is a bijection. In order to prove that \( \Phi \) is a homeomorphism it is sufficient to show that \( G \times (M_{ip} \cap X) \rightarrow X ; (g, x) \mapsto g \cdot x \) is a proper map. Let \( (g_\alpha, x_\alpha) \in G \times (M_{ip} \cap X) \) be a sequence such that \( g_\alpha \cdot x_\alpha \rightarrow y_0 \in X \). Since \( \rho(g \cdot x) \geq \rho(x) \) for \( x \in M_{ip} \) and \( g \in G \), we have \( \rho(x_\alpha) \leq \rho(g_\alpha \cdot x_\alpha) \leq r \) for some \( r > \rho(y_0) \). Since \( \rho \) is an exhaustion, passing to a subsequence, we may assume that \( x_\alpha \rightarrow x_0 \). Since the \( G \)-action is proper, passing to a subsequence, we may assume that \( g_\alpha \rightarrow g_0 \). Then \( y_0 = g_0 \cdot x_0 \) and \( \Phi \) is proper, hence a homeomorphism.

If \( X \) is a manifold and \( \mu \) is real analytic, then \( M_{ip} \cap X \) is smooth and \( \Phi \) has maximal rank (Proposition 6.1). Hence \( \Phi \) is an isomorphism of manifolds. \( \square \)

See Example 7.9 for the case of \( G \) acting on the points in its Lie algebra with compact isotropy group.

**Corollary 8.2.** If \( G \) acts properly on \( Z \), then \( G \times K M_{ip} \rightarrow Z \) is a \( G \)-equivariant homeomorphism which is smooth for smooth \( Z \).

Assume now that \( G \) is a linear semisimple real algebraic group. This implies that \( G \) is compatible with the Cartan decomposition of \( G^C = U^C = U \cdot \exp \mathfrak{u} \) where \( U \) is a maximal compact subgroup of \( G^C \). Under this assumption we can establish the real analytic version of a theorem of Abels [Ab75], [HeHuKu93].

**Theorem 8.3.** Let \( X \) be a real analytic manifold with a proper real analytic \( G \)-action where \( G \) is as above. Then there is a closed \( K \)-stable real analytic submanifold \( S \) of \( X \) such that the map \( G \times K S \rightarrow X \) is a \( G \)-equivariant real analytic isomorphism.

**Proof.** By [He83] there is a Stein \( G^C \)-manifold \( Z \) and a closed \( G \)-equivariant embedding \( \iota : X \rightarrow Z \). Now choose a \( U \)-invariant strictly plurisubharmonic real analytic exhaustion \( \rho : Z \rightarrow \mathbb{R} \). Then we can apply Proposition 8.1. \( \square \)
Remark 8.4. Let $G$ be an arbitrary closed subgroup of a semisimple Lie group $\hat{G}$ where $\hat{G}$ has finitely many components and maps injectively into its universal complexification. Assume that $G$ acts properly and real analytically on the real analytic manifold $X$. The map $x \mapsto [e, x]$ realizes $X$ as a $G$-stable closed subspace of $\hat{X} := \hat{G} \times^G X$. Since the $\hat{G}$-action on $\hat{X}$ is proper we can apply Theorem 8.3 to obtain a $G$-equivariant real analytic map $q: X \to \hat{G}/\hat{K}$ where $\hat{K}$ is a maximal compact subgroup of $\hat{G}$. Finding a global slice for the $G$-action on $X$ reduces to finding a global slice for the $G$-action on $\hat{G}/\hat{K}$.

9. Decompositions of homogeneous spaces

In this section we consider proper actions on $U^C$. We obtain Mostow decompositions of homogeneous spaces of real reductive groups (see [Mos55a, Mos55b]).

In the following we will identify $U^C$ with $U \times iu$ as a $(U \times U)$-space (see Section 4). Let $B$ be an $(\text{Ad} U)$-invariant inner product on $u$. Define $\rho: U^C \to \mathbb{R}$ by $\rho(u \exp(i\eta)) = \frac{1}{2}B(\eta, \eta)$ for $u \in U$, $\eta \in u$. Then $\rho$ is $(U \times U)$-invariant, and it is a strictly plurisubharmonic function on $U^C$ (see AzLo92 or HGHeu02).

Lemma 9.1. Let $\rho$ be as above. Then

1. $\rho$ is an exhaustion of $U^C$.
2. Let $\mu: U^C \to u^*$ be defined using $\rho$ and the right action of $U$ on $U^C$. Then $\mu^\xi(u \exp(i\eta)) = B(\xi, \eta)$ for $u \in U$, $\xi, \eta \in u$.

Proof. For 1) it is enough to show that $B$ restricted to $u$ is an exhaustion, which is obvious. The proof of part 2) is slightly more complicated. Let $\xi, \eta \in u$ and $u \in U$. Then

$$
\mu^\xi(u \exp(i\eta)) = \frac{d}{dt}|_{t=0}\rho(u \exp(i\eta) \exp(-it\xi)) = \frac{d}{dt}|_{t=0}\rho(\exp(i\eta) \exp(-it\xi)) = \mu^\xi(\exp(i\eta)).
$$

Write $u = u_0 \oplus u_1 \oplus \cdots \oplus u_r$, where each $u_j$ is irreducible and the decomposition is orthogonal with respect to $B$. We have $[u_j, u_k] = 0$ for $j \neq k$. Write $\xi = \sum_j \xi_j$ where $\xi_j \in u_j$, $j = 0, \ldots, r$, and similarly for $\eta$. Then $\mu^\xi(\exp(i\eta)) = \sum_j \mu^j(\exp(i\eta))$. Let $u_j(t) \in \exp(u_j)$ and $\alpha_j(t) \in u_j$, $t \in \mathbb{R}$, such that $\exp(i\eta_j) \exp(-it\xi_j) = u_j(t) \exp(i\alpha_j(t))$. This gives $\exp(i\eta) \exp(-it\xi) = u_j(t) \exp(i(\alpha(t) + \tilde{\eta}_j))$ where $\tilde{\eta}_j = \sum_k \eta_k$. Then

$$
\rho(\exp(i\eta) \exp(-it\xi_j)) = \frac{1}{2}B(\alpha_j(t), \alpha_j(t)) + \frac{1}{2}B(\tilde{\eta}_j, \tilde{\eta}_j),
$$

and $\mu^\xi(\exp(i\eta)) = \sum_j \mu^j(\exp(i\eta_j))$.

From the above it is enough to prove (2) under the hypothesis that $\xi, \eta \in u_0$. Choose an embedding $U \to U(n, \mathbb{C})$ so that the polar decomposition of $U^C$ is induced by that of $U(n, \mathbb{C})$. Note that the restriction of $B$ to $u_0$ is a positive constant times the trace form $X, Y \mapsto \text{tr}(iX, iY)$, $X, Y \in u_0 \subset u(n, \mathbb{C})$. Thus we may assume that $B$ is the trace form.

We have $\exp(i\eta) \exp(-it\xi) = u(t) \exp(i\alpha(t))$ where $u(t) \in \exp(u_0)$, $\alpha(t) \in u_0$ and

$$
\exp(2i\alpha(t)) = \exp(-i\xi) \exp(2i\eta) \exp(-i\xi) =: \beta(t).
$$

Here $\exp$ is now the usual exponential map on matrices. It suffices to establish (2) for $\xi$ and $\eta$ close to 0 since the functions involved are real analytic. So we can assume that $2i\alpha(t) = \log \beta(t)$ is given by the usual power series $\log(A) = (A - I) - \frac{1}{2}(A - I)^2 + \frac{1}{3}(A - I)^3 \ldots$ where $I$ is the $n \times n$ identity matrix and $A$ is near $I$. Then

$$
\mu^\xi(\exp(i\eta)) = -\frac{1}{2} \frac{d}{dt}|_{t=0} \text{tr}(\frac{1}{2} \log(\beta(t), \frac{1}{2} \log(\beta(t))) = -\frac{1}{4} \text{tr}(\frac{d}{dt}|_{t=0} \log(\beta(t), \log(\beta(t)))
$$
where \(\text{tr}(\frac{d}{dt} \log \beta(t), \log \beta(t))\) is a convergent sum of terms
\[
(-1)^{n+1} \frac{1}{n} \text{tr}[\beta'(t)(\beta(t) - I)^{n-1} + (\beta(t) - I)\beta'(t)(\beta(t) - I)^{n-2} + \cdots + (\beta(t) - I)^{n-1}\beta'(t), \log \beta(t)].
\]
Using the identities satisfied by trace and the fact that \(\beta(t)\) and \(\log \beta(t)\) commute we can permute the terms involving \(\beta'(t)\) and \(\beta(t) - I\) to obtain
\[
-\frac{1}{4} \text{tr}(\frac{d}{dt} \log \beta(t), \log \beta(t)) = -\frac{1}{4} \text{tr}(\beta'(t)(\beta(t)-I)^{-1}, \log \beta(t)).
\]
Evaluating at \(t = 0\) and using the identities satisfied by trace we obtain
\[
\mu^k(\exp(i\eta)) = -\frac{1}{4} \text{tr}([-i\xi \exp(2i\eta) + \exp(2i\eta)(-i\xi)] \exp(-2i\eta), 2i\eta) = \text{tr}(i\xi, i\eta) = B(\xi, \eta).
\]

Remark 9.2. If we identify \(u\) and \(u^*\) using the inner product \(B\), we get that \(\mu(u \exp(i\eta)) = \eta\) for all \(u \in U, \eta \in u\). Thus after identifying \(U^C\) with \(U \times iu \cong U \times u\) the moment map is given by projection on the second component! Also, if we view \(U^C \cong U \times u^*\) as the cotangent bundle of \(U\), then \(\mu\) coincides with the standard moment map on the cotangent bundle associated with the standard symplectic form. Moreover, the complex structure induced on the tangent bundle of \(U\) by \(U^C \cong U \times u\) is the so-called adapted complex structure with respect to the Riemannian metric on \(U\) defined by \(B\) (see [LeSz91] and [GuSt91]).

Let \(G = K\exp(p)\) and \(H = L\exp(q)\) be compatible closed subgroups of \(U^C\). Assume that \(H\) is a subgroup of \(G\) such that \(L \subseteq K\) and \(q \subseteq p\). We apply the results of the previous section to the free proper actions of \(U^C\) and its compatible subgroups on \(Z := U^C\) by multiplication on the right; \(g \cdot z = zg^{-1}\).

Theorem 9.3. Write \(p = q + q^\perp\) where \(q^\perp\) is the perpendicular of \(q\) in \(p\) relative to \(B\). Then the \((H \times K)\)-equivariant map \(H \times (K \times q^\perp) \to G, (h, (k, \xi)) \mapsto k \exp(\xi)h^{-1}\), induces an \((H \times K)\)-equivariant isomorphism
\[
H \times^L (K \times q^\perp) \cong G
\]
where \(l(h, k, \xi) = (hl^{-1}, kl^{-1}, \text{Ad}(l)(\xi))\), \(l \in L, h \in H, k \in K, \xi \in q^\perp\). In particular, we have an induced \(K\)-equivariant isomorphism
\[
K \times^L q^\perp \to G/H.
\]

Proof. This is an application of Proposition 8.1 where \(X = G, Z = U^C, H\) plays the role of \(G\) and \(\rho\) is as in Lemma 9.1 From Lemma 9.1 we get the equivariant identification of \(\mathcal{M}_{iq}\) with \(K \times q^\perp\).

Remark 9.4. Theorem 9.3 implies that, as sets, \(G = K \exp(q^\perp)H\). Applying the same reasoning to \(U^C\) and \(G\) we obtain that \(U^C = U \exp(i\mathfrak{t})G\).

Let \(M\) be a compact Lie subgroup of \(U\) and \(\mathfrak{m}\) its Lie algebra.

Corollary 9.5. Define \(m^\perp\) to be the perpendicular of \(m\) in \(u\) relative to \(B\) and let \(M\) act on \(U \times im^\perp\) by \(g \cdot (u, i\xi) = (ug^{-1}, i\text{Ad}(g)(\xi), g \in M, u \in U, \xi \in m^\perp\). Then the map \(M^C \times (U \times im^\perp) \to U^C, (h, u, i\xi) \mapsto u \exp(i\xi)h^{-1}\), induces an \((M^C \times U)\)-equivariant isomorphism
\[
M^C \times^M (U \times im^\perp) \cong U^C
\]
where \(M^C\) acts on \(U^C\) by right multiplication and \(M\) acts on \(M^C\) by right multiplication. In particular, we have an induced \(U\)-equivariant isomorphism
\[
U \times^M im^\perp \cong U^C/M^C.
\]
Let $Z$ be a holomorphic $\mathcal{U}^C$-space and $\rho: Z \to \mathbb{R}$ a $U$-invariant smooth strictly plurisubharmonic function with associated moment map $\mu: Z \to \mathfrak{u}^*$. Let $G = K \exp \mathfrak{p}$ be a closed compatible subgroup of $U^C$. The following was proved by Azad and Loeb ([AzLo99]) using Mostow’s results. For completeness, we reproduce their argument.

Lemma 9.6. The restriction $\rho|G \cdot z$ is an exhaustion if (and only if) $G \cdot z \cap \mathcal{M}_\mathfrak{ip} \neq \emptyset$.

Proof. Let $z_0 \in G \cdot z \cap \mathcal{M}_\mathfrak{ip}$. Then $G_{z_0} = K_{z_0} \exp(p_{z_0})$ and by Theorem 9.3 $(p_{z_0})^\perp \to Z$, $\xi \to \exp(\xi) \cdot z_0$, is an injective immersion. Let $\tilde{\rho}: (p_{z_0})^\perp \to \mathbb{R}$ be defined by $\tilde{\rho}(\xi) = \rho(\exp(\xi) \cdot z_0)$. Since $\mathbb{C} \to \mathbb{R}$, $w \mapsto \rho(\exp(w \xi) \cdot z_0)$ is strictly subharmonic for $\xi \neq 0$ and does not depend on the imaginary part of $w$ (since $\rho$ is $U$-invariant), it follows that $\tilde{\rho}|\mathbb{C} \xi$ is strictly convex for every $\xi \neq 0$. Hence $0 \in (p_{z_0})^\perp$ is the unique critical point of $\tilde{\rho}$. This all implies that $\tilde{\rho}$ is an exhaustion [AzLo93]. It follows that $\rho|G \cdot z$ is an exhaustion.

10. Quotients by complex groups

Before we continue our study of $G$-actions, we recall some results about actions of complex reductive groups. Let $Z$ be a holomorphic $\mathcal{U}^C$-space with $U$-invariant Kähler structure $\omega$ and moment map $\mu$. Set $S_{\mathcal{U}^C}(\mathcal{M}) = \{z \in Z; \mathcal{U}^C \cdot z \cap \mathcal{M} \neq \emptyset\}$, the set of semistable points of $Z$ with respect to $\mu$ and the $\mathcal{U}^C$-action. The following result can be found in [HeLo94].

Theorem 10.1. (1) The set $S_{\mathcal{U}^C}(\mathcal{M})$ is open and $\mathcal{U}^C$-invariant, and there is an analytic Hilbert quotient $\pi: S_{\mathcal{U}^C}(\mathcal{M}) \to S_{\mathcal{U}^C}(\mathcal{M})//\mathcal{U}^C$.

(2) Each fiber of $\pi$ contains a unique closed $\mathcal{U}^C$-orbit which is the unique orbit of minimal dimension in that fiber.

(3) For every $z \in \mathcal{M}$, the orbit $\mathcal{U}^C \cdot z$ is closed in $S_{\mathcal{U}^C}(\mathcal{M})$.

(4) The inclusion $\mathcal{M} \to S_{\mathcal{U}^C}(\mathcal{M})$ induces a homeomorphism $\mathcal{M}/\mathcal{U} \cong S_{\mathcal{U}^C}(\mathcal{M})//\mathcal{U}^C$.

Remark 10.2. Suppose that $\mu$ and $\omega$ are associated to a smooth strictly plurisubharmonic exhaustion $\rho$ and $z \in Z$. Then $\rho|\mathcal{U}^C \cdot z$ takes on a minimum value, hence $z \in S_{\mathcal{U}^C}(\mathcal{M})$. Thus $Z = S_{\mathcal{U}^C}(\mathcal{M})$.

Regarding the existence of strictly plurisubharmonic functions we have the following result from [HeLo94], [HeHuLo94] and [HeHu96].

Theorem 10.3. Let $Z$, $\omega$ and $\mu$ be as above, and let $z_0 \in \mathcal{M}$. Then there is a neighborhood $Q_0$ of $\pi(z_0) \in S_{\mathcal{U}^C}(\mathcal{M})//\mathcal{U}^C$ such that, on $\Omega_0 := \pi^{-1}(Q_0)$, $\mu$ and $\omega$ are associated to a strictly plurisubharmonic function $\rho$ such that

(1) $(\pi \times \rho): \Omega_0 \to Q_0 \times \mathbb{R}$ is proper and $\rho$ is bounded from below.

(2) $(\pi \times \rho)^{-1}(\pi(z_0), \rho(z_0)) = \mathcal{U} \cdot z_0$.

Remark 10.4. Property (1) is not stated in [HeHu96]. But an easy modification of the proof of the exhaustion theorem in [HeHu96] gives the claimed statement (see [HeHu94]).

Remark 10.5. Property that the analytic Hilbert quotient $\pi: Z \to Z//\mathcal{U}^C$ exists. Then there is an open cover $\{\Omega_\alpha\}$ of $Z//\mathcal{U}^C$ where each $\Omega_\alpha$ and $\pi^{-1}(\Omega_\alpha)$ are Stein. Moreover, there are strictly plurisubharmonic $U$-invariant exhaustions $\rho_\alpha$ on $\pi^{-1}(\Omega_\alpha)$. Thus, locally, we are in the case that $Z = S_{\mathcal{U}^C}(\mathcal{M})$.

Finally, from [HeLo94] we have the existence of slices.
Let \( z \in \mathcal{M} \). Then there is a locally closed \((U^C)_z\)-invariant subspace \( S \subset Z \), \( z \in S \), such that \( V := U^C \cdot S \) is open in \( Z \) and such that the natural \( U^C \)-invariant holomorphic map \( \varphi: U^C \times (U^C)_z \to V \), \([g,s] \mapsto g \cdot s\), is biholomorphic.

Remark 10.7. By construction of \( S \), we may assume that it admits a \((U^C)_z\)-equivariant closed embedding into an open \((U^C)_z\)-stable Stein submanifold \( \hat{S} \) of the \((U^C)_z\)-representation space \( T_zZ \). This implies, in particular, that \( U^C \times (U^C)_z \ S \) can be realized as a closed subspace of the Stein \( U^C \)-manifold \( U^C \times (U^C)_z \hat{S} \).

11. Semistable points with respect to compatible subgroups

Let \( Z \) be a holomorphic \( U^C \)-space with equivariant moment map \( \mu: Z \to u^* \) and let \( G = K \exp p \) be a compatible closed subgroup of \( U^C \). For a subset \( Y \) of \( Z \) and a subset \( H \) of \( U^C \) we define \( S_H(Y) := \{ z \in Z; H \cdot z \cap Y \neq \emptyset \} \). We call \( S_G(M_{ip}) \) the set of semistable points of \( Z \) with respect to \( \mu_{ip} \) and the \( G \)-action. We already met \( S_{U^C}(\mathcal{M}(Z)) \) in the previous section. When necessary for clarity, we will use notation such as \( S_{U^C}(\mathcal{M}(Z)) \) and \( S_G(M_{ip}(Z)) \).

Remark 11.1. If \( \beta \in u^* \) is a fixed point with respect to the co-adjoint \( U \)-action, then \( z \mapsto \mu(z) - \beta \) defines a shifted \( U \)-equivariant moment map \( \mu - \beta: Z \to u^* \). The set of semistable points with respect to this shifted moment map is \( S_{U^C}(\mathcal{M}(\beta)) \). If \( \beta \) annihilates \( ip \), then \( (\mu - \beta)_{ip} = \mu_{ip} \) and the set of semistable points with respect to \( G \) remains unchanged.

Proposition 11.2. Let \( G \) be a closed compatible subgroup of \( U^C \). If \( Z = S_{U^C}(\mathcal{M}) \), then

1. every \( G \)-orbit in \( Z \) contains a closed \( G \)-orbit in its closure,
2. \( Z = S_G(M_{ip}) \) and
3. the closed \( G \)-orbits are precisely the orbits intersecting \( M_{ip} \).

Proof. We may assume that \( Z \) coincides with a fiber of \( \pi: Z \to Z//U^C \). The moment map \( \mu \) is then associated to a strictly plurisubharmonic exhaustion \( \rho \). Now let \( z \in Z \). Since \( \rho|G \cdot z \) is an exhaustion, it has a minimum at a point \( z_0 \). Thus \( \mu_{ip}(z_0) = 0 \), so \( z \in S_G(M_{ip}) \) and we have (2).

For any \( z_0 \in M_{ip} \), Lemma 9.6 shows that \( \rho|G \cdot z_0 \) is an exhaustion, so \( G \cdot z_0 \) is closed, and we have (1) and (3).

Remark 11.3. In many instances, one can use Theorem 10.3 to replace the hypothesis of the existence of a strictly plurisubharmonic \( U \)-invariant exhaustion by the hypothesis that \( Z = S_{U^C}(\mathcal{M}) \). For example, this can be done in the case of Proposition 8.1 and Corollary 8.2.

Now we have an important definition.

Definition 11.4. Let \( A \subset Z \) be \( K \)-stable. For \( z \in Z \) and \( \xi \in u \) let \( I_A(z;\xi) := \{ t \in \mathbb{R}; (\exp it\xi) \cdot z \in A \} \). Then \( A \) is said to be \( \mu_{ip} \)-adapted if for every \( z \in Z \), \( \xi \in ip \) and every nonempty connected component \( C \) of \( I_A(z;\xi) \) the following holds.

i) If \( t_- = \inf C > -\infty \), then \( \mu_{ip}^C((\exp it_-\xi) \cdot z) < 0 \) and

ii) if \( t_+ = \sup C < +\infty \), then \( \mu_{ip}^C((\exp it_+\xi) \cdot z) > 0 \).

Remark 11.5. Conditions i) and ii) are valid for every \( z \in Z \) if and only if they are valid for every \( z \in A \).

Remark 11.6. Let \( A \) be \( \mu_{ip} \)-adapted. Let \( \xi \in ip \) and \( z \in Z \). If \( \xi \cdot z = 0 \), then \( I_A(z;\xi) = \emptyset \) or \( \mathbb{R} \). Assume that \( \xi \cdot z \neq 0 \) and that \( I_A(z;\xi) \neq \emptyset \). Clearly \( I_A(z;\xi) \) is not a point. As we saw in the proof of Lemma 8.3, the function \( \lambda: \mathbb{R} \to \mathbb{R}, \lambda(t) = \mu^\xi(\exp it\xi \cdot z) \), is strictly increasing. It follows that \( I_A(z;\xi) \) cannot contain two disjoint components, i.e., \( I_A(z;\xi) \) is connected.
Remark 11.7. If $G = U^C$, then $\mu_{ip} = \mu$ and we have the notion of $\mu$-adapted sets.

We leave the proof of the following to the reader.

Lemma 11.8. Arbitrary unions and finite intersections of $\mu_{ip}$-adapted sets are $\mu_{ip}$-adapted.

Here is an important tool for constructing $\mu$-adapted neighborhoods (c.f. \textit{He91}).

Proposition 11.9. Let $Z$ be a holomorphic $U^C$-space. Assume that $\mu: Z \to u^*$ is associated with a $U$-invariant strictly plurisubharmonic smooth function $\rho: Z \to \mathbb{R}$. For $r \in \mathbb{R}$ set $\Delta_r(\rho) := \{z \in Z; \rho(z) < r\}$. Then $\Delta_r(\rho)$ is $\mu$-adapted.

Proof. Let $\xi \in u$ and $z_0 \in \Delta_r(\rho)$. By definition we have

$$\mu^\xi(\exp(it\xi) \cdot z_0) = \frac{d}{dt} \rho(\exp(it\xi) \cdot z_0).$$

If $\xi Z(z_0) = 0$ we have nothing to prove, so assume that $\xi Z(z_0) \neq 0$. Then the function $t \mapsto \mu^\xi(\exp(it\xi) \cdot z_0)$ is strictly increasing and the boundary of $\Delta_r(\rho)$ is contained in $\{z \in Z; \rho(z) = r\}$. Since $t \mapsto \rho(\exp(it\xi) \cdot z_0)$ is a strictly convex function this implies that $\Delta_r(\rho)$ is $\mu$-adapted. \hfill \square

The following lemma shows the utility of the notion of $\mu_{ip}$-adaptedness.

Lemma 11.10. If $A_1, A_2 \subset Z$ are $\mu_{ip}$-adapted, then $G \cdot (A_1 \cap A_2) \supset A_1 \cap G \cdot A_2$. In particular,

$$G \cdot (A_1 \cap A_2) = G \cdot A_1 \cap G \cdot A_2.$$

Proof. Suppose that $x_1 = g \cdot x_2$ where $g \in G$ and $x_j \in A_j$, $j = 1, 2$. There is a $k \in K$ and $\xi \in iP$ such that $g = k \exp i\xi$. Replacing $x_1$ by $k^{-1}x_1$ we may assume that $g = \exp(i\xi)$. Consider the path $\alpha(t) = \exp(it\xi) \cdot x_2$ from $[0, 1]$ to $Z$. It is enough to show that $\alpha(t_0) \in A_1 \cap A_2$ for some $t_0 \in [0, 1]$. This is obvious if $x_1 \in A_2$ or if $x_2 \in A_1$, so we may assume that there are $t_1, t_2 \in [0, 1]$ where $\alpha$ enters $A_1$, resp. leaves $A_2$. From $\mu_{ip}$-adaptedness we get that $\mu^\xi(\alpha(t_1)) < 0$ and $\mu^\xi(\alpha(t_2)) > 0$. But $\mu^\xi$ is increasing on the image of $\alpha$, so $t_1 < t_2$ and $\alpha(t) \in A_1 \cap A_2$ for $t \in [t_1, t_2]$. \hfill \square

Remark 11.11. Let $\Omega$ be open and $\mu_{ip}$-adapted and let $A$ be a $K$-stable closed subset of $\Omega$. Assume that $A$ is locally $G$-stable, i.e, for all $a \in A$ and $\xi \in iP$ the set $I_A(a; \xi)$ is a neighborhood of $0 \in \mathbb{R}$. Since $A$ is closed in $\Omega$, $I_A(a; \xi)$ is closed in $I_\Omega(a; \xi)$. Now the condition that $A$ is locally $G$-stable implies that $I_A(a; \xi)$ is open. Thus $I_A(a; \xi) = I_\Omega(a; \xi)$ and consequently $A$ is $\mu_{ip}$-adapted. The equality $I_A(a; \xi) = I_\Omega(a; \xi)$ also implies that $A = G \cdot A \cap \Omega$. In particular, $G \cdot A$ is closed in $G \cdot \Omega$. If $A$ is real analytic in $\Omega$, then $G \cdot A$ is real analytic in $G \cdot \Omega$. Note that if $\Omega$ is $G$-stable then local stability of $A$ implies that $A$ is $G$-stable.

Corollary 11.12. Suppose that every $K$-orbit in $M_{ip}$ has a neighborhood basis of open $\mu_{ip}$-adapted subsets. Let $C_1, C_2 \subset M_{ip}$ be closed, disjoint and $K$-stable. Then there are $G$-invariant open neighborhoods $\Omega_i$ of $C_i$, $i = 1, 2$, such that $\Omega_1 \cap \Omega_2 = \emptyset$.

Proof. Let $W_1$ and $W_2$ be disjoint open $K$-invariant neighborhoods of $C_1$ and $C_2$, respectively. Then each point $z \in C_1$ has an open $\mu_{ip}$-adapted neighborhood $W_z \subset W_1$. Let $W'_z$ be the union of the $W_z$ for $z \in C_1$, and construct $W'_2$ similarly. Then $W'_1$ and $W'_2$ are disjoint and $\mu_{ip}$-adapted, hence the sets $\Omega_i := G \cdot W'_i$, $i = 1, 2$, are $G$-invariant disjoint open neighborhoods of $C_1$ and $C_2$. \hfill \square

It would be ideal if we could show that every $K$-orbit in $M_{ip}$ has a basis of open $\mu_{ip}$-adapted neighborhoods. We are able to do this in the case that $Z = S_{UC}(M)$ (Theorem 13.7). However, we can get many results with the weaker property established in Corollary 11.12. This will follow from the special case where $U$ is commutative, which we handle in the next section.
Definition 11.13. We say that the separation property holds if any two $K$-stable disjoint closed subsets of $\mathcal{M}_{ip}$ are contained in disjoint open $G$-stable sets.

Proposition 11.14. Suppose that the separation property holds and let $z \in Z$. Then $G \cdot z \cap \mathcal{M}_{ip} \neq \emptyset$ if and only if $G \cdot z \cap \mathcal{M}_{ip} = K \cdot z_0$ for some $z_0 \in G \cdot z \cap \mathcal{M}_{ip}$.

Proof. Assume that $G \cdot x_1, K \cdot x_2 \subset G \cdot z \cap \mathcal{M}_{ip}$ and that $K \cdot x_1 \neq K \cdot x_2$. Let $\Omega_j$ be $G$-invariant disjoint neighborhoods of $K \cdot x_j$, $j = 1, 2$. Then $G \cdot z \subset \Omega_1 \cap \Omega_2 = \emptyset$, a contradiction. □

Definition 11.15. Let $X$ be a topological $G$-space. We say that the topological Hilbert quotient of $X$ by $G$ exists if $\sim$ is an equivalence relation, where $x \sim y$ for $x, y \in X$ if and only if $G \cdot x \cap G \cdot y \neq \emptyset$. If this is the case, then we define the Hilbert quotient to be the set of equivalence classes, denoted $X/G$, with the quotient topology. If $Y$ is another topological $G$-space with topological Hilbert quotient $Y/G$ and if $\varphi : X \rightarrow Y$ is continuous and $G$-equivariant, then $\varphi/G$ will denote the induced continuous map from $X/G$ to $Y/G$.

Remark 11.16. Let $Z$ be a holomorphic $U^C$-space. If the analytic Hilbert quotient $\pi : Z \rightarrow Z/U^C$ exists, then it is also the topological Hilbert quotient since $\pi(x) = \pi(y)$ if and only if $U^C \cdot x$ and $U^C \cdot y$ contain the same closed $U^C$-orbit.

Corollary 11.17. Suppose that the separation property holds. Then the topological Hilbert quotient $S_G(\mathcal{M}_{ip})/G$ exists. We have $x \sim y$ in $S_G(\mathcal{M}_{ip})$ if and only if $K \cdot x_0 = K \cdot y_0$, where $K \cdot x_0 = G \cdot x \cap \mathcal{M}_{ip}$ and $K \cdot y_0 = G \cdot y \cap \mathcal{M}_{ip}$. The inclusion of $\mathcal{M}_{ip}$ into $S_G(\mathcal{M}_{ip})$ induces a continuous bijection of $\mathcal{M}_{ip}/K$ with $S_G(\mathcal{M}_{ip})/G$.

Proof. Suppose that $z \in G \cdot x \cap G \cdot y$, $z \in S_G(\mathcal{M}_{ip})$, and that $G \cdot x \cap \mathcal{M}_{ip} = K \cdot x_0$ and that $y_0$ and $z_0$ are defined similarly. Then by Proposition 11.14, we must have that $K \cdot x_0 = K \cdot z_0 = K \cdot y_0$. □

Remark 11.18. Assume the separation property. Then every fiber of the quotient map $\pi : S_G(\mathcal{M}_{ip}) \rightarrow S_G(\mathcal{M}_{ip})/G$ intersects $\mathcal{M}_{ip}$ in a unique $K$-orbit $K \cdot z_0$. The orbit $G \cdot z_0$ is closed in $S_G(\mathcal{M}_{ip})$ and is the unique orbit of lowest dimension in the corresponding fiber of $\pi$ (Corollaries 14.16 and 14.18). Conversely, any closed orbit in $S_G(\mathcal{M}_{ip})$ intersects $\mathcal{M}_{ip}$, by definition. Thus the quotient map $\pi$ gives a parametrization of the closed $G$-orbits in $S_G(\mathcal{M}_{ip})$.

Lemma 11.19. Assume the separation property. Then $\pi|\mathcal{M}_{ip} : \mathcal{M}_{ip} \rightarrow S_G(\mathcal{M}_{ip})/G$ is a closed mapping.

Proof. Let $C \subset \mathcal{M}_{ip}$ be closed. We may assume that $C$ is $K$-invariant. Set $Y := \pi^{-1}(\pi(C))$. Then we must show that $Y$ is closed in $S_G(\mathcal{M}_{ip})$, so let $y_n \in Y$ and suppose that $y_n \rightarrow y \in S_G(\mathcal{M}_{ip})$. Let $y_0 \in G \cdot y \cap \mathcal{M}_{ip}$. If $y_0 \notin C$, then we can find disjoint open $G$-stable neighborhoods $\Omega_1$ of $y_0$ and $\Omega_2$ of $C$. For large $n$ we must have that $y_n \in \Omega_1$ which implies that $G \cdot y_n$ lies in the complement of $\Omega_2$. But then we cannot have $y_n \in \pi^{-1}(\pi(C))$, a contradiction. Hence $y_0 \in C$, $Y$ is closed in $S_G(\mathcal{M}_{ip})$ and $\pi(C)$ is closed. □

Corollary 11.20. Assume the separation property. If $A$ is a $G$-stable closed subset $S_G(\mathcal{M}_{ip})$, then $\pi(A)$ is closed. If $A_1$ and $A_2$ are two closed $G$-stable subsets of $S_G(\mathcal{M}_{ip})$, then we have $\pi(A_1) \cap \pi(A_2) = \pi(A_1 \cap A_2)$.

Proof. We have that $\pi(A) = \pi(A \cap \mathcal{M}_{ip})$ is closed. Similarly, $\pi(A_1) \cap \pi(A_2) = \pi(A_1 \cap \mathcal{M}_{ip}) \cap \pi(A_2 \cap \mathcal{M}_{ip}) = \pi(A_1 \cap A_2)$. □

Corollary 11.21. Assume the separation property. Then the bijection $\mathcal{M}_{ip}/K \rightarrow S_G(\mathcal{M}_{ip})/G$ is a homeomorphism. In particular, $S_G(\mathcal{M}_{ip})/G$ is metrizable and locally compact.
Remark 11.22. Note that the quotient map $\pi$ is open at every point of $M_{ip}$ and therefore open at every point of $G \cdot M_{ip}$.

12. Commutative groups

In this section $U$ will denote a commutative compact Lie group. Let $Z$ be a holomorphic $U^c$-space with moment map $\mu : Z \to u^*$ and let $G = K \exp p$ be a compatible Lie subgroup of $U^c$. Let $Ann ip$ denote the annihilator of $ip$ in $u^*$.

Proposition 12.1.

$S_G(M_{ip}) = \bigcup_{\beta \in Ann ip} S_{U^c}(M(\beta))$.

Proof. Let $z \in S_{U^c}(M(\beta))$ where $\beta \in Ann ip$. On the fiber $Y$ through $z$ of the quotient $\pi_\beta : S_{U^c}(M(\beta)) \to S_{U^c}(M(\beta))/U^c$ the shifted moment map $\mu - \beta$ is associated with a strictly plurisubharmonic $U$-invariant smooth exhaustion $\rho : Y \to \mathbb{R}$. Since $\rho|G \cdot z$ attains a minimum value we obtain that $\overline{G \cdot z} \cap M_{ip} \neq \emptyset$. Consequently, $S_{U^c}(M(\beta)) \subset S_G(M_{ip})$.

If $z \in S_G(M_{ip})$, choose $z_0 \in M_{ip} \cap \overline{G \cdot z}$. Then $z_0 \in M(\beta_0)$ where $\beta_0 := \mu(z_0) \in Ann ip$. This gives $z_0 \in M(\beta_0) \cap \overline{G \cdot z} \subset M(\beta_0) \cap U^c \cdot z$, i.e., $z \in S_{U^c}(M(\beta_0))$.

Remark 12.2. If $u = ip$, then $S_G(M_{ip}) = S_{U^c}(\mathcal{M})$.

Corollary 12.3. We have that $S_G(M_{ip})$ is an open $U^c$-stable subset of $Z$.

Remark 12.4. We conjecture that, even for noncommutative groups, $S_G(M_{ip})$ is always open in $Z$.

In the following we will assume that $G = K \exp p$ is a closed compatible subgroup of $U^c$.

Proposition 12.5. Every $K$-orbit $K \cdot z_0 \subset M_{ip}$ has a neighborhood basis consisting of $\mu_{ip}$-adapted open sets.

Proof. Shifting by an element in $Ann ip$ we may assume that $z_0 \in \mathcal{M}$. Let $M$ denote $U_{z_0}$. Then by the holomorphic slice theorem (Theorem 10.6) we may assume that $Z = U^c \times^{MC} S$ for some $M^c$-space $S$, and we may assume that the moment map is associated to a strictly plurisubharmonic function $\rho$. Let $\pi : Z \to Z/U^c$ denote the quotient map and let $q : Z \to U^c/M^c \cong U^c \cdot z_0$ denote the bundle projection. We may assume that $\rho \times \pi$ is proper, that $\rho$ is bounded from below and that $(\rho \times \pi)^{-1}(\rho(z_0), \pi(z_0)) = U \cdot z_0$. Moreover, we have that $U \cdot z_0 \cap G \cdot z_0 = K \cdot z_0$. For suppose that $u \cdot z_0 = k \exp \xi \cdot z_0$ where $u \in U$, $k \in K$ and $\xi \in p$. Then $u^{-1}k \exp \xi \in (U^c)_{z_0} = U_{z_0} \exp ip_{z_0}$, so $u^{-1}k \in U_{z_0}$ and $u \cdot z_0 = u(u^{-1}k) \cdot z_0 = k \cdot z_0$.

Since $G$ is closed in $U^c$ and $z_0 \in \mathcal{M}$, the group $G/G_{z_0}$ acts properly and freely on $U^c \cdot z_0$. Thus the quotient $U^c \cdot z_0/G$ is a Hausdorff space. Let $\tilde{q} : Z \to (U^c \cdot z_0)/G$ denote $q$ composed with the quotient map $U^c \cdot z_0 \to (U^c \cdot z_0)/G$. The map $(\pi \times \rho \times \tilde{q}) : Z \to Z/U^c \times \mathbb{R} \times (U^c \cdot z_0)/G$ is proper and satisfies $(\pi \times \rho \times \tilde{q})^{-1}(\pi(z_0), \rho(z_0), \tilde{q}(z_0)) = U \cdot z_0 \cap G \cdot z_0 = K \cdot z_0$. This implies that the sets $\pi^{-1}(Q) \cap \rho^{-1}((r, \infty)) \cap \tilde{q}^{-1}(B)$ where $Q$ is an open neighborhood of $\pi(z_0)$, $r > \rho(z_0)$ and $B$ is an open neighborhood of $\tilde{q}(z_0)$ form a basis of open $K$-invariant neighborhoods of $K \cdot z_0$ in $Z$. Moreover, $\pi^{-1}(Q) \cap \rho^{-1}((r, \infty))$ is $\mu$-adapted, hence $\mu_{ip}$-adapted, and $\tilde{q}^{-1}(B)$ is $\mu_{ip}$-adapted since it is $G$-invariant. Therefore the intersection remains $\mu_{ip}$-adapted. \hfill $\Box$

Remark 12.6. This paper makes essential use of the holomorphic slice theorem and the existence of a potential $\rho$ only in the case that $U$ is commutative. One can give short proofs of the required results in this case, thus circumventing the use of the results in [HeLo94]. Then our results in this paper generalize those of [HeLo94] and are independent of the results there.
13. The Separation Property

In this section we establish the separation property. Let $U$ be a compact Lie group.

Proposition 13.1. Let $G \subset U^C$ be closed and compatible. Let $A$ be a maximal connected subgroup of $\exp p$ and let $a$ denote its Lie algebra. Then $a$ is Abelian, so that $A \simeq \mathbb{R}^l$ for some $l$. We have $\text{Ad}(K)a = p$ and $G = KAK$.

Proof. Since $[p, p] \subset \mathfrak{l}$, both $a$ and $A$ are commutative. Write $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_r$ where $\mathfrak{g}_s$ is semisimple and $\mathfrak{g}_r$ is the radical of $\mathfrak{g}$. Then $\mathfrak{g}_r$ and $\mathfrak{g}$ are $\theta$-stable. Let $H$ be the Zariski closure of $G$ in $U^C$ and let $M$ be the Zariski closure in $U^C$ of the Lie subgroup of $G$ corresponding to $\mathfrak{g}_r$. Then $H$ and $M$ are $\theta$-stable, so they are reductive. Since $\mathfrak{g}_r$ is solvable, so is $M$, so that $M$ is a torus. Since $\mathfrak{g}_r$ is $G$-stable, $M$ is normal in $H$. Thus $M$ lies in the center of $H^0$. Going back to $\mathfrak{g}$, we see that $\mathfrak{g}_r$ is the center of $\mathfrak{g}$ and that we can choose $\mathfrak{g}_s = [\mathfrak{g}, \mathfrak{g}]$. Then $\theta$ respects the Levi decomposition of $\mathfrak{g}$ and $p = (\mathfrak{p} \cap \mathfrak{g}_s) \oplus (\mathfrak{p} \cap \mathfrak{g}_r)$. Since $a$ is maximal Abelian, it contains $\mathfrak{p} \cap \mathfrak{g}_r$, so $a = a' \oplus (\mathfrak{g}_r \cap \mathfrak{p})$ where $a'$ is maximal commutative in $\mathfrak{p} \cap \mathfrak{g}_s$. Then $\text{Ad}(K)a' = p \cap \mathfrak{g}_s$ [Helg78, Ch. 5 §6], so that $\text{Ad}(K)(a) = p$. It follows that $G = KAK$. \qed

Example 13.2. The Zariski closure of $G$ can be much larger than $G$. For example, let $U^C = (\mathbb{C}^*)^2$, $U = S^1 \times S^1$ and $G = \mathbb{R}^*$ where $\mathfrak{g} \simeq \mathbb{R} \subset \mathbb{R}^2 \simeq i\mathbb{R}$ has irrational slope. Then the Zariski closure of $G$ in $U^C$ is $U^C$. The real algebraic closure of $G$ in $(\mathbb{R}^*)^2$ is $(\mathbb{R}^*)^2$.

Let $A$ and $a$ be as in Proposition 13.1. Then the results in section 12 show that every point in $\mathcal{M}_{ip} \subset \mathcal{M}_{ia}$ has a neighborhood basis of $\mu_{ia}$-adapted open sets.

The following lemma was established during discussions with H. Stötzl.

Lemma 13.3. Let $\Omega_1$ and $\Omega_2$ be $\mu_{ia}$-adapted with $\Omega_1 \cap \Omega_2 = \emptyset$, and suppose that $W_i$ is a $K$-invariant subset of $\Omega_i$, $i = 1, 2$. Then $G \cdot W_1 \cap G \cdot W_2 = \emptyset$.

Proof. It is enough to show that $W_1 \cap G \cdot W_2 = \emptyset$. So assume that we have $g = k(\exp \xi)k' \in G = KAK$ and $w_i \in W_i$ such that $w_1 = g \cdot w_2$. Then replacing $w_1$ by $k^{-1}w_1$ and $w_2$ by $k'w_2$, we can assume that $w_1 = (\exp \xi)w_2$. Then $w_2 \not\in \Omega_1$ and $(\exp \xi)w_2 \in \Omega_1$, so there is a $t_1$ where $(\exp t\xi)w_2$ first enters $\Omega_1$. Similarly, $w_2 \in \Omega_2$ while $(\exp t\xi)w_2 \not\in \Omega_2$, so that there is a last $t_2$ such that $(\exp t\xi)w_2 \in \Omega_2$. By $\mu_{ia}$-adaptedness, $t_1 < t_2$, so that $(\exp t\xi)w_2 \in \Omega_1 \cap \Omega_2 = \emptyset$ for $t \in [t_1, t_2]$, a contradiction. \qed

Corollary 13.4. Let $C_1$ and $C_2$ be disjoint closed $K$-stable subsets of $\mathcal{M}_{ip}$. Then there are $G$-stable disjoint open subsets $\Omega_i \supset C_i$, $i = 1, 2$. Hence the separation property holds.

Proof. There are disjoint open subsets containing the $C_i$, hence disjoint open $\mu_{ia}$-adapted subsets $W_i$ containing the $C_i$, $i = 1, 2$. These in turn contain $K$-stable open neighborhoods $W'_i$ of $C_i$. Set $\Omega_1 = G \cdot W'_1$ and $\Omega_2 = G \cdot W'_2$. Then the $\Omega_i$ have the required properties. \qed

Now that we have the separation property, we have all the results of §11. For completeness, we restate them here.

Theorem 13.5. Let $Z$ be a holomorphic $U^C$-space with $U$-invariant Kähler form and moment mapping $\mu$, and let $G$ be a closed compatible subgroup of $U^C$. Then

1. The topological Hilbert quotient $\pi$ of $\mathcal{S}_G(\mathcal{M}_{ip})$ by $G$ exists, and the inclusion $\mathcal{M}_{ip} \to \mathcal{S}_G(\mathcal{M}_{ip})$ induces a homeomorphism $\mathcal{M}_{ip}/K \tilde{\to} \mathcal{S}_G(\mathcal{M}_{ip})/G$. In particular, $\mathcal{S}_G(\mathcal{M}_{ip})/G$ is metrizable and locally compact.

2. If $A \subset \mathcal{S}_G(\mathcal{M}_{ip})$ is closed, then $\pi(A) = \pi(A \cap \mathcal{M}_{ip})$ is closed.
(3) If $A_1$ and $A_2$ are closed and $G$-stable subsets of $S_G(\mathcal{M}_p)$, then $\pi(A_1) \cap \pi(A_2) = \pi(A_1 \cap A_2)$.
(4) If $G$ is commutative, then $S_G(\mathcal{M}_p)$ is open in $Z$ and every $K$-orbit in $\mathcal{M}_p$ has a neighborhood basis of $\mu_{ip}$-adapted open sets.

In case $Z = S_{UC}(\mathcal{M})$ we can establish more than the separation property.

**Proposition 13.6.** Assume that the analytic Hilbert quotient $\pi: Z \to Z/\!/U^C$ exists and that we have a $U$-invariant strictly plurisubharmonic function $\rho$ on $Z$, bounded below, such that $\pi \times \rho$ is proper. Let $\mu$ be associated with $\rho$ and let $x_0 \in \mathcal{M}_p$. Then $K \cdot x_0$ has a basis of $\mu_{ip}$-adapted neighborhoods. The elements of the neighborhood basis can be taken to be of the form $\Delta_{r_n}(\rho) \cap V_n$ where $r_n > \rho(x_0) := r_0$, $\Delta_{r_n}(\rho) = \{ z \in Z; \rho(z) < r_n \}$ and $V_n$ is a $G$-invariant neighborhood of $x_0$.

**Proof.** Fix a $K$-neighborhood $W_0$ of $K \cdot x_0$. Let $W_0 \supset W_1 \supset W_2 \supset \cdots \supset K \cdot x_0$ be a basis of $K$-neighborhoods of $K \cdot x_0$ and let $r_1 > r_2 > \cdots > r_0$ be such that $\lim r_n = r_0$. We show that $\Delta_{r_n}(\rho) \cap G \cdot W_n \subset W_0$ for $n$ sufficiently large. By Proposition 11.9, the sets $\Delta_{r_n}(\rho)$ are $\mu$-adapted, hence the sets $\Delta_{r_n}(\rho) \cap G \cdot W_n$ are $\mu_{ip}$-adapted and we have the proposition.

Suppose that the proposition is false. Then we can assume that there are $x_n \in W_n$, $g_n \in G$ such that $g_n \cdot x_n \notin W_0$ while $\rho(g_n \cdot x_n) \leq r_n$. We may assume that $\lim x_n = x_0$. Since $\pi \times \rho$ is proper, we may assume that $\lim g_n \cdot x_n = y_0 \in Z$. Let $z_0 \in G \cdot y_0 \cap \mathcal{M}_p$. We claim that $K \cdot z_0 = K \cdot x_0$. If not, then by Corollary 13.4, we can find disjoint $G$-neighborhoods $\Omega_1$ and $\Omega_2$ of $z_0$ and $x_0$, respectively. But we must have that $y_0 \notin \Omega_1$, so that $g_n \cdot x_n \notin \Omega_1$ for $n$ large. Thus $x_n \in \Omega_2$ where $x_n \to x_0 \in \Omega_2$. This is a contradiction.

Now $G \cdot y_0 \cap \mathcal{M}_p$ consists of the minima of $\rho$ on the closed $G$-orbits in $G \cdot y_0$, and we have just shown that these minima consist of the $K$-orbit of $x_0$. Since $\rho(x_0) = r_0$ and $\rho(y_0) \leq r_0$ we must have that $K \cdot y_0 = K \cdot x_0 \subset W_0$. But $g_n \cdot x_n \notin W_0$ and $g_n \cdot x_n \to y_0$, a contradiction. \hfill $\Box$

**Theorem 13.7.** Suppose that $Z = S_{UC}(\mathcal{M})$. Then every point $z \in \mathcal{M}_p$ has a neighborhood basis of $\mu_{ip}$-adapted open subsets.

**Proof.** By Theorem 10.3 and the Proposition above, every point in $\mathcal{M}_p$ has a neighborhood basis of $\mu_{ip}$-adapted open subsets. \hfill $\Box$

### 14. Slices

In this section we assume that $Z$ is a holomorphic $U^C$-space with a $U$-invariant Kähler structure and moment mapping $\mu$. We show that we can find slices at points of $\mathcal{M}_p$. In particular, if the analytic Hilbert quotient $Z/\!/U^C$ exists (e.g., $Z$ is Stein), then there are slices at points on closed $G$-orbits (see Remark 14.27 below). First we give a sufficient condition that $K$-equivariant maps extend to be $G$-equivariant maps.

Let $G = K \exp \mathfrak{p}$ be a closed compatible subgroup of $U^C$. A $K$-stable subset $\Omega$ of $Z$ is said to be orbit convex with respect to $G$ if for every $\xi \in \mathfrak{p}$ and $z \in \Omega$ the set $I_\Omega(z; \xi) = \{ t \in \mathbb{R}; \exp t\xi \cdot z \in \Omega \}$ is connected. Note that a $\mu_{ip}$-adapted subset of $Z$ is orbit convex with respect to $G$.

Let $\Omega \subset Z$ be $K$-stable and open and let $X$ be a topological $G$-space. Let $\varphi: \Omega \to X$ be $K$-equivariant. We say that $\varphi$ is locally $G$-equivariant if for every $x \in \Omega$ there is a neighborhood $V_x$ of $e \in \exp(\mathfrak{p})$ with $V_x \cdot x \subset \Omega$ such that $\varphi(g \cdot x) = g \cdot \varphi(x)$ for $g \in V_x$.

**Proposition 14.1.** Let $A$ be a subgroup in $\exp \mathfrak{p}$ such that $G = KAK$. Let $\Omega \subset \Omega_1 \subset \Omega_2 \subset Z$ be open sets where $\Omega$ and $\Omega_2$ are $K$-invariant and $\Omega_1$ is orbit convex with respect to $A$. Let $X$ be a topological $G$-space and let $\varphi: \Omega_2 \to X$ be a locally $G$-equivariant continuous map. Then there is a unique $G$-equivariant continuous map $\Phi: G \cdot \Omega \to X$ such that $\Phi|\Omega = \varphi$. 

Proof. For $z \in \Omega$ and $g \in G$, we define $\Phi(g \cdot z)$ to be $g \cdot \varphi(z)$. This is clearly the desired mapping, as long as we can prove that it is well-defined. So let $z_1, z_2 \in \Omega$ and $g_1, g_2 \in G$ and suppose that $g_1 \cdot z_1 = g_2 \cdot z_2$. We need to show that $g_1 \cdot \varphi(z_1) = g_2 \cdot \varphi(z_2)$. Equivalently, we have to show that \( g \cdot \varphi(z_1) = \varphi(z_2) \) where $g = g_2^{-1} g_1$. Write $g = k (\exp \xi) k'$ where $k, k' \in K$ and $\xi \in \mathfrak{a}$. Then $z_1' := k' \cdot z_1$ and $\exp \xi \cdot z_1' = z_2' := k^{-1} \cdot z_2$ are in $\Omega \subset \Omega_1$, hence $I_{\Omega_1}(z_1'; -i\xi)$ contains the interval $[0, 1]$. It follows that $\varphi(\exp(\xi) \cdot z_1') = \exp(\xi) \cdot \varphi(z_1')$ and from $K$-equivariance of $\varphi$ we finally obtain that $g \cdot \varphi(z_1) = \varphi(g \cdot z_1) = \varphi(z_2)$.

Remark 14.2. Suppose that $X$ is an analytic $G$-space and that $\varphi$ is $C^k$, $k \geq 1$, or real analytic. Then $\Phi$ is $C^k$ or real analytic. If $X$ is a complex $G$-space and $\varphi$ is holomorphic, then $\Phi$ is holomorphic.

The following is an analogue in our setting of Luna’s fundamental lemma.

Lemma 14.3. Let $X$ be a topological $G$-space and let $x \in X$. Let $\varphi : X \to Z$ be a $G$-equivariant continuous mapping which maps $K \cdot x$ injectively into $G \cdot z$ where $z = \varphi(x)$. Assume that $\varphi$ is a local homeomorphism at $x$ and that $z \in \mathcal{M}_G$. Then there is an open $G$-stable neighborhood $W$ of $G \cdot x$ which is mapped homeomorphically by $\varphi$ onto the open $G$-neighborhood $\varphi(W)$ of $G \cdot z$.

Proof. By equivariance, $\varphi$ is a local homeomorphism along $K \cdot x$. It follows that there is a neighborhood $\Omega$ of $K \cdot x$ which is mapped homeomorphically onto its image $\varphi(\Omega) \subset Z$. Let $\Omega_1$ be a $K$-stable neighborhood of $x$ which is contained in $\Omega$. Then $\varphi(\Omega_1)$ is a locally $G$-equivariant homeomorphism. Let $A \subset \exp \mathfrak{p}$ be a subgroup such that $G = KAK$. Then by Proposition 14.29 and Lemma 11.8 we may find an orbit convex (with respect to $A$) neighborhood $V_1$ of $K \cdot z$ inside $V_2 := \varphi(\Omega_1)$. Let $V$ be a $K$-invariant neighborhood of $K \cdot z$ which is contained in $V_1$. Since $(\varphi(\Omega_1)^{-1} V_2$ is a locally $G$-equivariant mapping, there is a unique $G$-equivariant extension $\psi$ of $(\varphi(\Omega_1)^{-1} V$ to $G \cdot V$. The composition $\varphi \circ \psi$ is the identity on $W := G \cdot \psi(V) \subset G \cdot \Omega_1$.

Remark 14.4. Suppose that $X$ is a real analytic (resp. complex) $G$-space. If $\varphi$ is a local analytic diffeomorphism (resp. local biholomorphism), then $\varphi|W$ is an analytic diffeomorphism (resp. biholomorphism). If $\varphi$ is a local $C^k$ diffeomorphism, $k \geq 1$, then $\varphi|W$ is a $C^k$ diffeomorphism.

Remark 14.5. If $G = U^C$ and $\varphi$ is holomorphic, then Lemma 14.3 is a strict generalization of the usual fundamental lemma in the complex analytic category. Similarly, Theorem 14.10, Theorem 14.28, and Proposition 14.29 below give statements in the complex analytic category. Proposition 14.1 in this setting may even be new.

We give a proof of the following well-known result.

Lemma 14.6. Let $z \in Z$ be a $U^C$-fixed point. Then there is a $U^C$-stable neighborhood $W$ of $z$ and a $U^C$-equivariant biholomorphic map $\psi : W \to \tilde{Z}$ where $\tilde{Z} \subset T_z(Z)$ is a $U^C$-stable closed subspace of a $U^C$-stable open subset of $T_zZ$.

Proof. Let $\psi$ be a $U$-equivariant biholomorphic map of a $U$-neighborhood $W'$ of $z \in Z$ onto a locally closed $U$-stable subspace $Z'$ of $T_zZ$ where $\psi(z) = 0$. We may assume that $Z'$ is closed in the ball $B$ of radius 1 for some $U$-invariant norm on $T_zZ$. Then $B$ is $\mu'$-adapted for $\mu'$ the moment mapping associated to the square of the norm function on $T_zZ$. Moreover, $Z'$ is locally $U^C$-stable (see Remark 14.11). It follows that $\tilde{Z} := U^C \cdot Z'$ is a closed analytic $U^C$-subspace of the $U^C$-stable open subset $U^C \cdot B$ such that $\tilde{Z} \cap B = Z'$. Note that $\psi : W' \to \tilde{Z}$ is automatically locally $U^C$-equivariant. Changing $\mu$ by a constant, we may assume that $z \in \mathcal{M}$. Then $z$ has a neighborhood basis of $\mu$-adapted open sets (Theorem 15.7) and by Proposition 14.1 we may assume that $\psi$ is defined on a $U^C$-invariant neighborhood of $z$. Now apply Lemma 14.3.
We now need a result on complete reducibility of certain representations of $G$.

**Lemma 14.7.** Let $V$ be a complex representation space of $U^C$. Then $V$ is completely reducible as a real representation of $G$.

**Proof.** We may assume that $V \simeq \mathbb{C}^n$ has the usual hermitian inner product $\langle \cdot,\cdot \rangle$ and that $U$ is a subgroup of $U(n, \mathbb{C})$. The real part of $\langle \cdot,\cdot \rangle$ is the usual real inner product $\langle \cdot,\cdot \rangle$ on the underlying real vector space $\mathbb{R}^{2n}$. Then $K$ consists of orthogonal matrices and elements of $\exp(p)$ are real symmetric matrices. Let $W$ be a real $G$-submodule of $V$. Taking perpendicular relative to $\langle \cdot,\cdot \rangle$ we have the $K$-stable subspace $W^\perp$. Moreover, if $w \in W$, $w^\perp \in W^\perp$ and $g \in \exp(p)$, then $0 = (g \cdot w, w^\perp) = (w, g \cdot w^\perp)$, so that $W^\perp$ is also $\exp(p)$-stable. Thus $W^\perp$ is $G$-stable. □

**Remark 14.8.** Let $z \in Z$ and let $H$ be a compatible reductive subgroup of $U^C$ fixing $z$. Then locally and $H$-equivariantly we may consider $Z$ as an $H$-stable subset of $T_zZ$. The $U$-invariant Kähler structure on $Z$ extends locally to an $(H \cap U)$-invariant Kähler structure on $T_zZ$ [Nara62].

**Corollary 14.9.** Let $z \in Z$ such that $G_z$ is a compatible subgroup of $U^C$ (e.g., $z \in M_{ip}$). Then the representation of $G_z$ on $T_zZ$ is completely reducible. In particular, there is a direct sum decomposition $T_zZ = T_z(G \cdot z) \oplus N_z$ where $N_z$ is $G_z$-stable.

**Proof.** The Zariski closure of $G_z$ in $U^C$ is reductive (Remark 142), and lies in $(U^C)_z$. Now applying the argument of Lemma 147 we can choose $N_z = (g \cdot x)^\perp$ for the inner product on $T_zZ$ coming from the Kähler structure. □

**Theorem 14.10.** Assume that $Z$ is smooth at $z \in M_{ip}$. Then there is a geometric $G$-slice at $z$, i.e., there is a $G_z$-invariant locally closed real analytic submanifold $S$ of $Z$, $z \in S$, such that $G \cdot S$ is open in $Z$ and such that the natural map $G \times^{G_z} S \to G \cdot S$, $[g,s] \mapsto g \cdot s$, is a real analytic isomorphism.

**Proof.** Let $H$ denote the analytic (hence algebraic) Zariski closure of $G_z$ in $U^C$. Note that $z$ is an $H$-fixed point. By the holomorphic slice theorem for fixed points (Lemma 146) there is an $H$-stable open neighborhood $D \subset T_zZ$ and an $H$-equivariant open embedding $i: D \to Z$ with $i(0) = z$. From Corollary 149 we know that there is a $G_z$-complement $N_z$ to $T_z(G \cdot z)$ in $T_zZ$. Let $\hat{S} := i(D \cap N_z)$ and let $\varphi: G \times^{G_z} \hat{S} \to Z$ denote the natural map. Then $\varphi$ is $G$-equivariant, a local diffeomorphism at $[e,z]$ and an isomorphism on the $G$-orbit through $[e,z]$. Applying Lemma 143 we obtain a $G$-neighborhood $\Omega$ of $[e,z] \in G \times^{G_z} \hat{S}$ which is mapped isomorphically onto $G \cdot \varphi(\Omega)$. But $\Omega = G \times^{G_z} S$ where $S = \Omega \cap \hat{S}$. □

**Remark 14.11.** By Lemma 142 we have $(g \cdot z)^\perp = (t \cdot z)^\perp \cap \ker(d\mu_{ip}(z))$. Thus, on the tangent space level, the slice for the $G$-action is the intersection of $M_{ip}$ with a slice for the $K$-action.

The slice theorem for the case that $z$ is not a smooth point of $Z$ requires more work. If we knew that $Z$ was a $G$-subspace of a smooth $G$-space $\hat{Z}$, then the slice theorem for $\hat{Z}$ would imply the slice theorem for $Z$. But the only way we see to find a $\hat{Z}$ is to prove the slice theorem for $Z$!

Assume that $z \in M_{ip}$ and let $H$ denote the Zariski closure of $G_z$ in $U^C$. By Lemma 140 we have an $H$-equivariant embedding $\psi$ of an open neighborhood $W$ of $z$ onto a locally analytic subset $\hat{Z} \subset T_zZ$ where $\psi(z) = 0$. Write $T_zZ = T_z(U^C \cdot z) \oplus N$ where $N$ is $H$-stable. The set $\hat{\Sigma} := \hat{Z} \cap N$ is an $H$-stable locally analytic subset of $N$ and $\Sigma := \psi^{-1}(\hat{\Sigma})$ is an $H$-stable analytic subset of $W$.

We have an $\text{Ad} H$-stable decomposition $u^C = (u^C)_z \oplus m$. Note that $m \to U^C/(U^C)_z$, $m \mapsto \exp(m)(U^C)_z$, is biholomorphic on an open neighborhood of $0 \in m$. 


Lemma 14.12. The holomorphic map $\varphi^0: m \times \Sigma \to Z$, $(m, s) \mapsto \exp(m) \cdot s$, is $G$-equivariant and there is an open $H$-stable neighborhood $V$ of $(0, z) \in m \times \Sigma$ such that $\varphi^0$ maps $V$ biholomorphically onto an open subset of $Z$.

Proof. By construction, $d\varphi^0_{(0,z)}$ is injective. This implies that $\varphi^0$ maps an open neighborhood $V = \Omega^0 \times \Sigma^0$ of $(0, z)$ biholomorphically onto a closed analytic subset of a neighborhood $Z^0$ of $z \in Z$. Here $\Omega^0$ is a connected neighborhood of $0 \in m$ and $\Sigma^0$ is a neighborhood of $z \in \Sigma$. Let $Z^0_\beta$, $\beta \in B$, be the irreducible components of $Z^0$. We may assume that each $Z^0_\beta$ is irreducible at $z$. There is an open neighborhood $\Omega$ of $e \in U^C$ such that $\Omega \cdot z$ is a locally closed submanifold of $Z^0_\beta$ for each $\beta$. Set $\tilde{Z}^0 := \psi(Z^0)$ and $\tilde{Z}^0_\beta := \psi(Z^0_\beta)$, $\beta \in B$. For each $\beta$, $\psi(\Omega \cdot z) \subset \tilde{Z}^0_\beta$ is a smooth submanifold through $0$ which is transversal to $N$, so we have

$$\dim \Sigma^0_{\alpha, \beta} = \dim Z^0_\beta - \dim \Omega \cdot z = \dim Z^0_\beta - \dim m$$

for every irreducible component $\Sigma^0_{\alpha, \beta}$ of $\tilde{Z}^0_\beta \cap N$ which contains $0$. For each $Z^0_\beta$ choose an irreducible component $\Omega^0 \times \Sigma^0_{\alpha, \beta}$ of $V$ such that $z \in \Sigma^0_{\alpha, \beta} \subset Z^0_\beta$. Since $\dim(m \times \Sigma^0_{\alpha, \beta}) = \dim Z^0_\beta$ we have $\varphi(\Omega^0 \times \Sigma^0_{\alpha, \beta}) = Z^0_\beta$. Hence $\varphi^0$ is a surjective $\lambda$-stable neighborhood of $(0, z)$ onto a closed analytic subset of a neighborhood $\Omega^0 \times \Sigma^0$ of $(0, z)$ and therefore biholomorphic.

Remark 14.13. We have $T_z \Sigma = N \subset T_z Z$.

Since $G_z$ is a compatible subgroup of $U^C$, the action of $G_z$ on $u^C$ is completely reducible. Let $m' \subset m$ be an $(\text{Ad}G_z)$-stable complement to $(u^C)_z + g$ in $u^C$. We have the twisted product $G \times G_z(m' \times \Sigma)$ where $G_z$ has the product action on $m' \times \Sigma$. There is the $G$-equivariant map $\varphi: G \times G_z(m' \times \Sigma) \to Z$, $(g, m, \sigma) \mapsto g \exp(m) \cdot \sigma$.

Proposition 14.14. There is a $G_z$-invariant open neighborhood $S$ of $(0, z) \in m' \times \Sigma$ such that $\varphi: G \times G_z S \to Z$ is a $G$-equivariant open embedding.

Proof. By Lemma 14.3 and Remark 14.4 it is enough to prove that $\varphi$ is an isomorphism in a neighborhood of $[e, (0, z)]$. Let $m'' \subset m$ be an $(\text{Ad}G_z)$-stable complement to $m'$ in $m$. Note that $m'' \simeq g/g_z$. It is easy to show that

$$\varphi': m'' \oplus m' \times \Sigma \to Z, \ (m'', m', \sigma) \mapsto \exp(m'') \exp(m') \cdot \sigma$$

is a diffeomorphism in a neighborhood of $(0, z)$. Since $\varphi^0$ is biholomorphic, it is enough to show that $\sigma := (\varphi^0)^{-1} \circ \varphi'$: $m \times \Sigma \to m \times \Sigma$ is a diffeomorphism near $(0, z)$. Clearly $d\sigma_{(0,z)}$ is the identity. Now we just apply the lemma below.

Lemma 14.15. Let $X$ be a germ at $0 \in \mathbb{C}^n$ of a complex analytic set. Let $\sigma: X \to X$, $\sigma(0) = 0$, be a germ of a real analytic mapping such that $\sigma(X) \subset X$. Assume that $d\sigma_0: T_0X \to T_0X$ is an isomorphism. Then $\sigma(X) = X$ (as germs).

Proof. We may assume that $T_0X = \mathbb{C}^n$. Then we may assume that a representative of $\sigma$ extends to a real analytic diffeomorphism $\tau: W_1 \to W_2$ where $W_1$ and $W_2$ are neighborhoods of $0$. We may assume that $X$ is represented by closed analytic subsets $X_j$ of $W_j$, $j = 1, 2$, such that $\tau(X_1) \subset X_2$.

We first assume that the $X_j$ are irreducible. Then each $X_j \setminus \text{Sing} X_j$ is connected and smooth where $\text{Sing} X_j$ has real codimension at least two in $X_j$. There is certainly a smooth point $x_1 \in X_1$ such that $x_2 := \sigma(x_1)$ is a smooth point in $X_2$. Since $\tau$ is an isomorphism, $\sigma$ is open at $x_1$. Hence there is a neighborhood $V_2$ of $x_2$ in $X_2 \setminus \text{Sing} X_2$ such that $\tau^{-1}(V_2) \subset X_1$. By the identity principle for real analytic maps we have $\tau^{-1}(X_2) \subset X_1$. Hence $\sigma$ is onto.
In general, each $X_j \setminus \text{Sing } X_j$ is a finite union of $d_j$ connected components whose closures are the irreducible components of $X_j$, where $d_1 = d_2$. By the argument above, if $Y$ is an irreducible component of $X_1$, then $\sigma(Y)$ is an irreducible component of $X_2$. Since $\tau$ is invertible, the irreducible components of $X_1$ are mapped onto the irreducible components of $X_2$. Hence $\sigma(X_1) = X_2$.  

Proposition 14.14 has several important corollaries.

**Corollary 14.16.** Let $z \in \mathcal{M}_{ip}$. Then $G \cdot z$ is closed in $S_G(\mathcal{M}_{ip})$ and closed in a $G$-stable open subset of $Z$. In particular, a $G$-orbit in $S_G(\mathcal{M}_{ip})$ is closed if and only if it intersects $\mathcal{M}_{ip}$.

**Proof.** By the proposition, $G \cdot z$ is closed in $G \cdot S$. Let $y \in \overline{G \cdot z} \setminus G \cdot z$. If $y \in S_G(\mathcal{M}_{ip})$, then $G \cdot y \cap \mathcal{M}_{ip}$ must be $K \cdot z$ by Corollary 14.17 but this is impossible since $y$ lies in the complement of $G \cdot S$.

**Corollary 14.17.** For every $z \in \mathcal{M}_{ip}$, the saturation $S_G(\{z\})$ is closed in an open $G$-stable neighborhood of $z$.

**Proof.** We may $G_z$-equivariantly identify $S \subset m' \times \Sigma$ with its image in $m' \times N \subset T_Z$. Then $G_z$ is compatible with the unitary structure on $T_Z$ and we have a topological Hilbert quotient $\pi : T_Z \to T_Z/G_z$. The null cone $\tilde{N} := \pi^{-1}(\pi(0))$ consists of the $y \in T_Z$ such that the closure of $G_z \cdot y$ contains 0. Then $\tilde{N} \cap S$ is closed in $S$, and $S_G(\{z\}) \cap G \cdot S \simeq G \times G_z (\tilde{N} \cap S)$ is closed in $G \cdot S$.

Similarly, we have

**Corollary 14.18.** Let $y \in S_G(\mathcal{M}_{ip})$ and let $z \in \overline{G \cdot y} \cap \mathcal{M}_{ip}$. Then $G \cdot z$ is the unique orbit of minimal dimension in $\overline{G \cdot y} \cap S_G(\mathcal{M}_{ip})$.

**Corollary 14.19.** Let $z \in \mathcal{M}_{ip}$. Then there is a smooth real analytic $G$-space $\tilde{Z}$ and a $G$-equivariant embedding of a $G$-neighborhood of $z$ into $\tilde{Z}$.

**Proof.** We have an embedding of a $G$-neighborhood of $z$ into $G \times G_z (m' \times N)$.

**Corollary 14.20.** Let $z \in \mathcal{M}_{ip}$. Then there is a $G$-invariant neighborhood $W$ of $z$ such that the topological Hilbert quotient $W/G$ exists.

**Proof.** As before, the topological Hilbert quotient $\pi : T_Z \to T_Z/G_z$ exists. We may identify $S$ with its image in $T_Z$. Let $S'$ denote $S \cap (\pi(S) \setminus \pi(S \setminus S'))$. Then $0 \in S'$, $S'$ is $G_z$-stable and open in $S$, and $S' \subset S_G(\mathcal{M}_{ip} \cap S')$. It follows that the topological Hilbert quotient of $S'$ exists, hence so does the topological Hilbert quotient of $W := G \times G_z S'$.

**Corollary 14.21.** Assume that $Z' := S_G(\mathcal{M}_{ip})$ is open and let $z \in \mathcal{M}_{ip}$. Then a neighborhood of $G \cdot z \in Z'/G$ is homeomorphic to a semianalytic set.

**Proof.** We continue from the proof above. We may assume that $S \subset Z'$. Then $(G \times G_z S)/G$ parametrizes the closed $G$-orbits, so we need only show that the closed $G_z$-orbits in $S$ are locally parametrized by a semianalytic set. We may consider $S$ as a subset of $V := T_Z$. We have a unitary structure on $V$ for which the image of $G_z$ in $GL(V)$ is compatible. Let $p_1, \ldots, p_d$ be generators of $\mathbb{R}[V]^{K_z}$ and let $p = (p_1, \ldots, p_d) : V \to \mathbb{R}^d$. Then $V/K_z$ is homeomorphic to the closed semialgebraic subset $p(V)$. Moreover, $\mathcal{M}_{ip} \subset V$ is semialgebraic, hence $p(\mathcal{M}_{ip})$ is also semialgebraic and parametrizes the closed $G_z$-orbits in $V$. We can find a nonnegative real valued $K_z$-invariant real analytic function $f$ (near 0) whose zeroes define $S$ near 0. Then $f = p^*h$ where $h$ is analytic.
in a neighborhood of $0 \in \mathbb{R}^d$. The zeroes of $h$ on $p(M_{ip})$ give the germ of the image of $M_{ip} \cap S$, where $(M_{ip} \cap S)/K_z$ parametrizes the closed $G_z$-orbits in $S$. Thus a neighborhood of $0 \in S//G_z$ is homeomorphic to a semianalytic set.

Remark 14.22. Let $Z$ be a projective variety with a $U^C$-linearized very ample line bundle $L$. Let $V$ denote the dual of the $U^C$-module $\Gamma(Z,L)$. Then $Z$ embeds into $P(V)$ and $S_G(M_{ip})$ is the intersection of $Z$ with the image of $V \setminus N$ in $P(V)$, where $N$ is the null cone of the $G$-action on $V$. Hence $S_G(M_{ip})$ is open in $Z$.

We may reformulate Proposition 14.14 as follows.

**Theorem 14.23.** For every $z \in M_{ip}$ there is a locally closed real analytic $G_z$-stable subset $S$ of $Z$, $z \in S$, such that the natural map $G \times^{G_z} S \to Z$ is a real analytic $G$-isomorphism onto the open set $G \cdot S$.

Remark 14.24. Assume that $Z' := S_G(M_{ip})$ is open. Then we may assume that $G \cdot S \subset Z'$. Let $\pi: Z' \to Z'/G$ be the quotient mapping. Then the image $C$ of the complement of $G \cdot S$ in $Z'$ is closed and does not contain the closed orbit $G \cdot z$. Thus replacing $G \cdot S$ by the inverse image of the complement of $C$ we may arrange that $G \cdot S$ is $G$-saturated. It follows automatically that $S$ is $G_z$-saturated.

Example 14.25. This is a continuation of Example 7.9. Let $\pi: g \to g//G$ denote the quotient map. The slice theorem implies that $\pi^{-1}(\pi(x)) = G \cdot x$ for every $x \in X$. In particular, the orbit space $X/G \cong \pi(X)$ is Hausdorff and the quotient map is given by restricting $\pi$ to $X$.

We now can state some variants of Theorem 6.6 and Proposition 8.1. For a general holomorphic $U^C$-space $Z$, let $\text{Comp}_G(Z)$ denote $\{z \in Z; \text{ for all } w \in G \cdot z \text{ the isotropy group } G_w \text{ is compact}\}$.

**Proposition 14.26.** If $Z' := S_G(M_{ip})$ is open in $Z$, then $\text{Comp}_G(Z')$ is open in $Z'$ and the natural map $G \times^K (M_{ip} \cap \text{Comp}_G(Z')) \to \text{Comp}_G(Z')$ is a homeomorphism. In particular, the $G$-action on $\text{Comp}_G(Z')$ is proper.

**Proof.** Let $C$ denote $\text{Comp}_G(Z')$. It follows from the slice theorem that $C$ is open in $Z'$ and that every $G$-orbit in $C$ is closed. Hence $G \times^K (M_{ip} \cap C) \to C$ is one to one and onto. We have a homeomorphism $(M_{ip} \cap C)/K \to C//G$. Let $\pi: C \to C//G$ denote the quotient mapping. Suppose that $[g_n, z_n] \in G \times^K (M_{ip} \cap C)$ such that $g_n \cdot z_n$ converges in $C$. Then $\pi(z_n)$ is convergent, so we can assume that $z_n \to z \in M_{ip} \cap C$. If $S$ is a slice at $z$, then the action $G \times G \cdot S \to G \cdot S$ is proper. Hence we can assume that $g_n \to g \in G$. Thus $G \times^K (M_{ip} \cap C) \to C$ is a homeomorphism.

**Remark 14.27.** Suppose that the analytic Hilbert quotient $Z//U^C$ exists and let $G \cdot z$ be a closed orbit. Then we may assume that $Z = S_{U^C}(M)$ (see Remark 10.5), so that $G \cdot z$ intersects $M_{ip}$. Hence there is a slice at $z$.

From Remark 10.5 and Theorem 14.23 we obtain

**Proposition 14.28.** If the analytic Hilbert quotient $Z \to Z//U^C$ exists, then $\text{Comp}_G(Z)$ is open in $Z$ and the natural map $G \times^K (M_{ip} \cap \text{Comp}_G(Z)) \to \text{Comp}_G(Z)$ is a homeomorphism. In particular, the $G$-action on $\text{Comp}_G(Z)$ is proper.

Here is a criterion for an equivariant map to be a homeomorphism.

**Proposition 14.29.** Let $X$ be a Hausdorff topological $G$-space such that the topological Hilbert quotient $\pi: X \to X//G$ exists and such that $X//G$ parametrizes the closed $G$-orbits. Assume that $Z = S_G(M_{ip}(Z))$ and that we have a continuous mapping $\varphi: X \to Z$ and the following properties.
Then $\varphi$ is a homeomorphism.

Proof. Let $G \cdot z$ be a closed orbit in $Z$. By Corollary [14.16] we may assume that $z \in \mathcal{M}_{ip}(Z)$. Since $\varphi$ is a local homeomorphism, the fiber $\varphi^{-1}(z)$ is discrete and consequently each $G$-orbit in $\varphi^{-1}(G \cdot z) \cong G \times G_z \varphi^{-1}(z)$ is open in $\varphi^{-1}(G \cdot z)$. Thus each $G$-orbit in $\varphi^{-1}(G \cdot z)$ is closed in $\varphi^{-1}(G \cdot z)$ and also closed in $X$. But $\varphi/G$ is a bijection, so it follows that $G \cdot x = \varphi^{-1}(G \cdot z)$ for some $x \in \varphi^{-1}(z)$ and then we claim that $\varphi^{-1}(K \cdot z) = K \cdot x$. For suppose that $\varphi(k \cdot \exp(\xi) \cdot x) = k' \cdot z$ where $k, k' \in K$ and $\xi \in \mathfrak{p}$. Then $\exp(\xi) \cdot z \in K \cdot z$ which implies that $\xi_z$ vanishes at $z$. Since $(\varphi|G \cdot x): G \cdot x \to G \cdot z$ is automatically smooth and is a local diffeomorphism, it follows that $\xi_X$ vanishes at $x$, and we have the claim. Note that since $\varphi/G$ is a bijection, the closed $G$-orbits in $X$ are precisely the inverse images of the closed $G$-orbits in $Z$.

We now show that $\varphi/G$ is a homeomorphism. Let $x \in X$ such that $G \cdot x$ is closed, let $V$ be a neighborhood of $G \cdot x$ in $X/G$ and set $W := \pi_X^{-1}(V)$. Since $\varphi$ is open and $G \cdot \varphi(x)$ is closed, $\pi_Z(\varphi(W))$ contains a neighborhood of $G \cdot \varphi(x)$ (Remark [14.22]). Hence $\varphi/G$ is open at $G \cdot x$ and $\varphi/G$ is a homeomorphism.

Now let $\mathcal{M}_{ip}(X)'$ denote the inverse image of $\mathcal{M}_{ip}(Z)$ in $X$. Then $\mathcal{M}_{ip}(X)' \to \mathcal{M}_{ip}(Z)$ is an open mapping, hence so is the composition $\mathcal{M}_{ip}(X)' \to Z/G$ and the quotient mapping $\mathcal{M}_{ip}(X)' \to X/G$. It follows that the inclusion $\mathcal{M}_{ip}(X)' \to X$ induces a homeomorphism $\mathcal{M}_{ip}(X)'/K \to X/G$. Then $\varphi|\mathcal{M}_{ip}(X)'$ is proper and a local homeomorphism, hence a covering map. Now (3), (4) and (5) imply that the covering has one sheet, i.e., $\varphi|\mathcal{M}_{ip}(X)'$ is a homeomorphism onto $\mathcal{M}_{ip}(Z)$. Let $x \in \mathcal{M}_{ip}(X)'$. Since $\varphi|(K \cdot x): K \cdot x \to K \cdot \varphi(x)$ is an isomorphism, Lemma [14.3] shows that $\varphi$ is a homeomorphism on a $G$-neighborhood of $G \cdot x$. Since $\varphi/G$ is a homeomorphism, it follows that $\varphi$ is a homeomorphism. \hfill \qed

Remark 14.30. Via the homeomorphism $\varphi$ we can impose a complex $G$-space structure on $X$ such that $\varphi$ is biholomorphic.

15. The Hilbert Mumford Criterion

Let $G$ be a compatible subgroup of $U^\mathbb{C}$ and choose a closed subgroup $A \subset \exp \mathfrak{p}$ such that $G = KAK$. Let $X$ be a $G$-space such that the topological Hilbert quotient $\pi: X \to X//A$ exists and is regular. That is, $X//A$ is Hausdorff and if $\xi$ is a point of $X//A$ and $C$ is a closed subset not containing $\xi$, then there are disjoint neighborhoods of $\xi$ and $C$. We also assume that $\pi$ parametrizes the closed $A$-orbits and maps closed $A$-stable subsets to closed subsets of $X//A$.

We use an argument of Richardson to establish the main part of the Hilbert Mumford criterion.

Proposition 15.1. Let $X$ be as above and let $Y$ be a closed $G$-stable subset of $X$. Then

$$S_G(Y) = K \cdot S_A(Y).$$

Proof. Since $S_G(Y)$ is $K$-stable and $S_A(Y) \subset S_G(Y)$, we have $K \cdot S_A(Y) \subset S_G(Y)$. Now let $z \in S_G(Y)$. If $\pi(K \cdot z) \cap \pi(Y) = \emptyset$, then it follows from regularity of $X//A$ that there are open $\pi$-saturated disjoint neighborhoods $\Omega_2$ of $Y$ and $\Omega_1$ of $K \cdot z$. Let $\Omega_1'$ be a $K$-stable neighborhood of $K \cdot z$ in $\Omega_1$ and define $\Omega_2'$ similarly. Then $G \cdot \Omega_1' \cap G \cdot \Omega_2' = \emptyset$, so that $z \notin S_G(Y)$, which is a contradiction. Thus there is a $k \in K$ such that $A \cdot k \cdot z \cap Y \neq \emptyset$, i.e., $z \in K \cdot S_A(Y)$. \hfill \qed
Let $A$ be a commutative connected simply connected real Lie group with Lie algebra $\mathfrak{a}$. Then the exponential map gives an isomorphism of $\mathfrak{a}$ with $A$. We consider here only finite dimensional continuous (hence real analytic) representations of $A$. We say that a real $A$-module $W$ is completely 1-reducible if it is completely reducible with each irreducible component being of dimension 1. Then every isotropy group of $W$ is connected (and therefore simply connected) and for every $x \in W$, the $A_x$-module $W$ is completely 1-reducible.

**Remark 15.2.** Let $W$ be a real $A$-module. If $W$ is completely 1-reducible, then one can choose an isomorphism $W \simeq \mathbb{R}^n$ such that the image $A'$ of $A$ in $\text{GL}(n, \mathbb{R})$ consists of real positive diagonal matrices. Then $A' \subset \exp i\mathfrak{u}(n, \mathbb{C})$ is a compatible subgroup of $\text{GL}(n, \mathbb{C})$ with its usual Cartan decomposition. Conversely, given an isomorphism $W \simeq \mathbb{R}^n$ such that the image of $A$ in $\text{GL}(n, \mathbb{C})$ is compatible and lies in $\exp i\mathfrak{u}(n, \mathbb{C})$, then the image consists of symmetric real matrices, so that $W$ is completely 1-reducible.

**Example 15.3.** Let $W$ be completely 1-reducible with isotypic decomposition $W = \bigoplus \chi_j W_{\chi_j}$ where $\chi_0$ is the trivial character. Then $W_{\chi_0} = W^A$. We may assume that $W$ is a real subspace of a holomorphic $U^\mathbb{C}$-representation such that the $U$-representation is unitary with respect to some Hermitian inner product $\langle \cdot, \cdot \rangle$ with associated norm $\| \cdot \|$. We may assume that $A \subset \exp(i\mathfrak{u})$. Let $\mu_{i\mathfrak{a}}$ denote the restriction to $W$ of the $i\mathfrak{a}$-component of the moment map associated with the strictly plurisubharmonic $U$-invariant exhaustion function $\rho := \frac{1}{2} \| \cdot \|^2$. A simple calculation shows that

$$\mu^\xi(z) = i\langle \xi, z \rangle \quad \xi \in \mathfrak{u}, z \in W.$$

For $z = z_0 + \cdots + z_r \in \bigoplus \chi_j W_{\chi_j}$ we obtain

$$\mu_{i\mathfrak{a}}(z) = i(\|z_1\|^2 \chi_1 + \cdots + \|z_r\|^2 \chi_r).$$

Hence

$$C := \mu_{i\mathfrak{a}}(W) = \{i(a_1 \chi_1 + \cdots + a_r \chi_r); a_j \geq 0\}$$

is a closed convex additive cone in $i\mathfrak{a}^*$.

Let $\mathfrak{g}(A)$ denote the set of one-parameter subgroups $\tau: \mathbb{R} \to A$. In the following we identify $\mathfrak{g}(A)$ with the set $\text{Hom}(\mathbb{R}, \mathfrak{a})$ of $\mathbb{R}$ linear maps from $\mathbb{R}$ into $\mathfrak{a}$.

**Lemma 15.4.** Let $W$ be a completely 1-reducible $A$-module. Then

$$\mathcal{S}_A(\{0\}) = \bigcup_{\tau \in \mathfrak{g}(A)} \mathcal{S}_{\tau(\mathbb{R})}(\{0\}).$$

**Proof.** We have to show that $\mathcal{S}_A(\{0\}) \subset \bigcup_{\tau \in \mathfrak{g}(A)} \mathcal{S}_{\tau(\mathbb{R})}(\{0\})$. Let $z \in \mathcal{S}_A(\{0\})$ and replace $W$ by the smallest $A$-submodule which contains $z$. We have the isotypic decomposition $W = \bigoplus \chi_j W_{\chi_j}$ where the $\chi_j: \mathfrak{a} \to \mathbb{R}$, $j = 1, \ldots, r$, are linear functions determining the weight spaces $W_{\chi_j} := \{w \in W; \exp \chi_j w = \exp(\chi_j(\xi))w \text{ for all } \xi \in \mathfrak{a}\}$. Since $W$ is spanned by $A \cdot z$, no $\chi_j$ is identically zero, and we have hyperplanes $H_j := \{\xi \in \mathfrak{a}; \chi_j(\xi) = 0\}$.

Choose a connected component $\mathfrak{a}^+$ of $\mathfrak{a} \setminus (\bigcup_{j} H_j)$ such that $\chi_1 > 0$ on $\mathfrak{a}^+$. For a weight $\chi_j$ we write $\chi_j > 0$ if $\chi_j|\mathfrak{a}^+ > 0$, and we write $\chi_j < 0$ if $-\chi_j > 0$. We have a decomposition $W = W^- \oplus W^+$ where $W^- = \oplus_{\chi_j < 0} W_{\chi_j}$ and $W^+ = \oplus_{\chi_j > 0} W_{\chi_j}$. We claim that $W^- = \{0\}$. If this is not the case, then $C \cap -C \neq \{0\}$ where $C$ is the cone generated by the $\chi_j$. Consequently, there is a nonzero element $c$ in the intersection, and we can write

$$c = \sum_j s_j \chi_j = -\sum_j t_j \chi_j, \quad s_j, t_j \in \mathbb{R}^+.$$
For $w = w_1 + \cdots + w_r \in W_{x_1} + \cdots + W_{x_r}$ set $f(w) := \prod_{j} \|w_j\|^{a_j + b_j}$. Then $f$ is $A$-invariant and continuous with $f(z) \neq 0$ and $f(0) = 0$. This contradicts the fact that $0 \in \overline{A \cdot z}$. Thus $W = W^+$ and for every $\xi \in a^+$ we have $\lim_{t \to -\infty} \exp(\xi t) \cdot z = 0$. Hence $z \in \bigcup_{r \in \mathbb{Z}} S_r(\mathbb{R})(Y)$. \hfill $\Box$

**Corollary 15.5.** The saturation $S_A(\{0\})$ is a finite union of linear subspaces of $W$.

Now let $Z$ be a holomorphic $U^C$-space with $U$-invariant Kähler structure and moment map $\mu : Z \to u^\ast$. Let $G$ be a closed compatible subgroup of $U^C$ and $X$ a $G$-stable closed subset of $Z$.

**Theorem 15.6.** Let $a \subset p$ be a maximal commutative subalgebra and $A = \exp(a)$ the corresponding subgroup of $G$. Assume that the quotient $\pi : X \to X/A$ exists and is regular, that $X/A$ parametrizes the closed $A$-orbits and that $\pi$ maps closed $A$-stable sets to closed sets. Let $z \in X$ and let $Y \subset \overline{G \cdot z}$ be closed and $G$-stable. Then there is a $k \in K$, $y \in Y$ and a one parameter subgroup $\tau : \mathbb{R} \to A_y$ such that $\lim_{t \to -\infty} \tau(t)k \cdot z = y$.

**Proof.** By Proposition 15.4 we have a $k \in K$ and a $y \in Y$ such that $\overline{Ak \cdot z} \cap G \cdot y \neq \emptyset$. We may assume that the intersection contains $y$. Then we may replace $U^C$ by the Zariski closure of $A$ in $U^C$, i.e., we may assume that $U^C$ is commutative. It follows that we may change $\mu$ by a constant such that $y \in \mathcal{M}$. Hence there is an open $U^C$-stable neighborhood $Z_0$ of $y$ such that the analytic Hilbert quotient $\pi : Z_0 \to Z_0/A^C$ exists and such that $U^C \cdot y$ is closed in $Z_0$. Then $A \cdot y$ is closed in $U^C \cdot y$ and hence in $Z_0$. We have an $A$-slice at $y$ for the $A$-action on $Z_0$, hence for the $A$-action on $X$. Thus we may assume that $X = A \times A_y S$ where $S$ is a closed $A_y$-stable subset of an open $A_y$-saturated subset of $T_y Z$ relative to the quotient $T_y Z \to T_y Z/A_y$. Since $A_y \subset \exp i(u_y)^C$ is a compatible subgroup of $(U_y)^C$, the representation on $T_y Z$ is completely 1-reducible, and using Lemma 15.4 we find the desired one-parameter subgroup $\tau : \mathbb{R} \to A_y$. \hfill $\Box$

**Corollary 15.7.** Assume that the analytic Hilbert quotient $Z/A^C$ exists. Let $z \in Z$ and let $Y \subset \overline{G \cdot z}$ be closed and $G$-stable. Then there is a $k \in K$, $y \in Y$ and a one parameter subgroup $\tau : \mathbb{R} \to A_y$ such that $\lim_{t \to -\infty} \tau(t)k \cdot z = y$.

**Proof.** By the results in section 11 we may assume that $Z = \mathcal{S}_{U^C}(M)$. Then the quotient $\pi : Z \to Z/A$ exists and has the desired properties. Now apply Theorem 15.6. \hfill $\Box$

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Fakultät für Mathematik, Ruhr Universität Bochum, Universitätsstrasse 150, D - 44780 Bochum
E-mail address: heinzner@cplx.rub.de

Department of Mathematics, Brandeis University, PO Box 549110, Waltham, MA 02454-9110
E-mail address: schwarz@brandeis.edu