The Straight Blow Up Section Family

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Introduction

Let \( X, Y \) be schemes over an algebraically closed base field \( k \) and \( Z \) a closed subscheme of \( X \times Y \). The main purpose of this paper is to introduce the straight blow up section family of the projection \( X \times Y \to X \) along \( Z \), which roughly speaking is an object that combines the universal properties of the universal section family of \( X \times Y \to X \) (c.f. Definition 3.2) and of the blow up of \( X \times Y \) along \( Z \). More precisely, it is a \( Y \)-scheme \( \tilde{Y} \) such that the pullback \((\text{Id}_X \times b)^{-1}(Z)\) is an effective Cartier divisor of \( X \times \tilde{Y} \) satisfying a suitable universal property.

When \( X \) is the base field we retrieve the classic blow up, but in general wide new phenomena may appear. For example, the resulting morphism \((\text{Id}_X \times b): X \times \tilde{Y} \to X \times Y\) is not necessarily birational or even generically finite, see Section 5.

Under Noetherian and projective assumptions, Theorem 3.8 asserts that the straight blow up section family exists. The result is mainly based on the existence of two other objects which deserve a special attention by themselves. Their interest relays on their role in the construction of many moduli spaces, examples of such constructions are Theorem 4.4 and Proposition 4.5.

The first object follows from the study of morphisms \( W \to X \) for which the base change of \( Z \to X \times Y \) by \( W \to X \) is an isomorphism. Theorem 2.5 asserts that they form a representable functor \( \text{Iso}_{X/Y}^{Z} \) on \( W \) which is represented by a closed subscheme \( X' \) of \( X \). This is the scheme theoretical version of Chevalley’s theorem for projections \( X \times Y \to X \) and for the “codimension zero stratum” (c.f. [8, IV3 Chapitre IV, Théorème 13.1.3, p.189, and Corollaire 13.1.5, p.190]).

Let \( f: X \times Y \to T \) be a \( k \)-morphism. The second object follows form the study of morphisms \( W \to X \) for which the restriction of \( f \) to \( W \times Y \) is constant along the fibres of the projection \( W \times Y \to W \). Theorem 2.7 shows that they form a category with a final object which is a closed subscheme of \( X \). We call it the \( f \)-constfy closed subscheme of \( X \) (c.f. Definition 2.6).

Now we describe our purpose introducing the straight blow up section family. Fix a morphism \( \pi: S \to B \). The author’s paper [2], introduces a generalisation of clusters of points of a scheme \( X \) to the relative case, clusters of sections of \( \pi \) ([2, Definition 2.11, p.7]). There the author defines analogue schemes to Kleiman’s iterated blow ups ([11, §4. p.36]) naturally parametrising clusters of sections of \( \pi \) of length \( r \), the Universal \( r \)-relative cluster family \( \text{Cl}_r \) of \( \pi \) ([2, Definitions 2.13, 2.17 and 2.19, pp.8-10]).
Assuming $S$ quasiprojective and $B$ projective, first the existence of the scheme $Cl_r$ is proved non-constructively through the existence of suitable Hilbert schemes (c.f. [2, Theorem 2.24, p.11]. The $f$-constfy construction will allow us relax the assumptions on $B$ to be proper. Second, it is shown that a recursive construction of $Cl_{r+1}$ from $Cl_r$ is possible. Such construction is analogous to the one in [11] for the absolute case. More precisely, there is a stratification of $Cl_r \times Cl_{r-1} \times Cl_r$ such that every irreducible component of $Cl_{r+1}$ is either (a) birational to a stratum or (b) composed entirely of clusters whose $(r+1)$-th section is infinitely near to the $r$-th, see [2, §2] and [2, Corollary 3.10.2].

So, the kind (a) irreducible components of $Cl_{r+1}$ may emerge as the blow up of $Cl_r \times Cl_{r-1} \times Cl_r$ along a suitable sheaf of ideals. This blow up morphism is unique, and the straight blow up section family is our attempt to characterise it. Theorem 4.10 shows, for $r = 1$, that the straight blow section family of the projection $B \times (Cl_r \times Cl_{r-1} \times Cl_r) \to B$ along a suitable closed subscheme satisfies the desired properties. That is, it is the union, with its non-necessarily reduced structure, of the kind (a) irreducible components of $Cl_{r+1}$.

The new techniques developed in this paper allow us to describe, for the moment just set theoretically, where the kind (b) irreducible components of $Cl_{r+1}$ emerge from, which are the ones missing in straight blow up section family of $B \times (Cl_r \times Cl_{r-1} \times Cl_r) \to B$.

In short, in Section 1 we fix some notation and present basic fact about the main constructions that we will use. Theorem 1.13 formalises the idea that blowing up a locally Noetherian scheme along a locally principal subscheme consists into shaving off those associated point of the ambient scheme lying on the locally principal subscheme.

Section 2 contains the constructions of the closed subscheme of $X$ representing the functor $\text{Iso}_Z^X : X \to Y$ and the $f$-constfy closed subscheme of $X$.

Section 3 contains the construction of the straight blow up section family.

Section 4 presents a way to apply the new developed techniques for the construction of $Cl_{r+1}$ from $Cl_r \times Cl_{r-1} \times Cl_r$ for the case $r = 1$, which is the inductive step for the whole construction.

Finally Section 5 presents some particular examples of the straight blow section family illustrating the new phenomena (the resulting morphism is not necessarily birational) and to show some possible applications such as to systematise small resolutions.

1 Preliminaries

We work over an algebraically closed base field $k$, that is, by a scheme we mean a $k$-scheme, by a morphism a morphism of $k$-schemes and by a homomorphism a homomorphism of $k$-algebras.

Let $X, Y, Z$ be schemes. We denote the identity morphism as $\text{Id}_X : X \to X$. Given a point $x \in X$, we denote by $k(x)$ its residue field. Usually we denote a monomorphism from $Z$ to $Y$ by $Z \hookrightarrow Y$ (almost all of them will be open or closed embeddings).

For the sake of simplicity, we do not distinguish a closed or open subscheme from its corresponding unique closed or open embedding. Let $f : X \hookrightarrow Y$ be a morphism and $Z$ a closed or open subscheme of $Y$. We denote by $f^{-1}(Z)$ the respectively closed or open subscheme of $X$ corresponding to the pullback by $f$.
of $Z \hookrightarrow Y$. When $f$ is also a closed or open embedding and there is no lack of confusion we denote $f^{-1}(Z)$ by $X \cap Z$.

1.1 Category Theory. This part collects some not so common general results on category theory that we use. We direct the curious reader to the references.

Definition 1.1. Let $\mathcal{F}: C \to \text{Set}$ be a contravariant functor on a category $C$ with values in sets. The category of elements of $\mathcal{F}$, denoted $\text{el}(\mathcal{F})$, is the comma category $(\ast \downarrow \mathcal{F})$ where $\ast$ is the inclusion functor from the one-element set to $\text{Set}$ (c.f. [13, Chapter III]). Namely, the objects of $\text{el}(\mathcal{F})$ are couples $(C, \eta)$ with $C$ an object of $C$ and $\eta$ an element of $\mathcal{F}(C)$. A morphism of $\text{el}(\mathcal{F})$ from $(C, \eta)$ to $(C', \eta')$ is a morphism $f: C \to C'$ of $C$ such that $\mathcal{F}(f)(\eta') = \eta$.

Remark 1.1.1. Let $\mathcal{F}: C \to \text{Set}$ be a contravariant functor on a category $C$ with values in sets, $C$ an object of $C$ and $\eta$ an element of $\mathcal{F}(C)$. The couple $(C, \eta)$ represents $\mathcal{F}$ if and only if it is the final object of $\text{el}(\mathcal{F})$.

The following definition and its subsequent lemma are a standard criterion for the representability of functors. We use the formulation of [14, Tag 01JJ], for another equivalent formulation see [4, Theorem 8.9, p.209].

Definition 1.2. Let $\mathcal{F}: \text{Sch} \to \text{Set}$ be a contravariant functor on the category of schemes with values in sets.

1. The functor $\mathcal{F}$ satisfies the sheaf property for the Zariski topology if for every scheme $T$ and every open covering $T = \bigcup_{i \in I} U_i$, and for any collection of elements $\xi_i \in \mathcal{F}(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ there exists a unique element $\xi \in \mathcal{F}(T)$ such that $\xi_i = \xi|_{U_i}$ in $\mathcal{F}(U_i)$.

2. A subfunctor $\mathcal{H} \subseteq \mathcal{F}$ is a rule that associates to every scheme $T$ a subset $\mathcal{H}(T) \subseteq \mathcal{F}(T)$ such that the map $\mathcal{F}(f): \mathcal{F}(T) \to \mathcal{F}(T')$ maps $\mathcal{H}(T)$ into $\mathcal{H}(T')$ for all morphisms of schemes $f: T' \to T$.

3. Let $\mathcal{H} \subseteq \mathcal{F}$ be a subfunctor. The subfunctor $\mathcal{H} \subseteq \mathcal{F}$ is representable by open immersions if for all pairs $(T, \xi)$, where $T$ is a scheme and $\xi \in \mathcal{F}(T)$ there exists an open subscheme $U_\xi \subseteq T$ with the following property:

(\ast) A morphism $f: T' \to T$ factors through $U_\xi$ if and only if $f^*\xi \in \mathcal{H}(T')$.

4. Let $I$ be a set. For each $i \in I$ let $\mathcal{H}_i \subseteq \mathcal{F}$ be a subfunctor. The collection $(\mathcal{H}_i)_{i \in I}$ covers $\mathcal{F}$ if and only if for every $\xi \in \mathcal{F}(T)$ there exists an open covering $T = \bigcup U_i$ such that $\xi|_{U_i} \in \mathcal{H}_i(U_i)$.

Lemma 1.3. Let $\mathcal{F}$ be a contravariant functor on the category of schemes with values in the category of sets. Suppose that

1. $\mathcal{F}$ satisfies the sheaf property for the Zariski topology,

2. there exists a set $I$ and a collection of subfunctors $\mathcal{F}_i \subseteq \mathcal{F}$ such that

(a) each $\mathcal{F}_i$ is representable,

(b) each $\mathcal{F}_i \subseteq \mathcal{F}$ is representable by open immersions, and

(c) the collection $(\mathcal{F}_i)_{i \in I}$ covers $\mathcal{F}$.
Then \( \mathcal{F} \) is representable.

1.2 Scheme theoretic image. First we review some basic facts about the schematic image of a morphism while we also set the notation. From Lemma 1.7, a second part starts whose main result is Proposition 1.10. It asserts that, for a flat \( S \)-scheme \( X \) and a closed subscheme \( Z \subseteq X \) flat over \( S \), the schematic closure of \( X \setminus Z \) commutes with flat base changes.

**Definition 1.4.** Let \( f : X \to Y \) be a morphism and \( i : Z \hookrightarrow Y \) a closed embedding. The morphism \( f \) is said to be **majorized** by \( i \) if \( f \) factors as \( X \xrightarrow{g} Z \xrightarrow{i} Y \). Here \( g \) is unique because \( i \) is a monomorphism.

**Remark 1.4.1.** If we fix a scheme \( Y \) and a closed embedding \( i : Z \hookrightarrow Y \), for a morphism \( f : X \to Y \), the property of being majorized by \( i \) is local on the source. Given an open cover \( \{ U_j \} \) of \( X \), if every morphism \( f|_{U_j} : U_j \to Y \) is majorized by \( i \) via a morphism \( g_j : U_j \to Z \), then any two morphisms \( g_j, g_j' \) agree on their overlap since \( i \) is a monomorphism.

**Lemma 1.5.** Consider the following Cartesian diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & Y \\
\downarrow{f} & & \\
X & \xrightarrow{g} & Z
\end{array}
\]

where \( i \) (hence also \( j \)) is a closed embedding. The closed embedding \( i \) majorizes \( f \) if and only if \( j \) is an isomorphism.

**Proof.** When \( j \) is an isomorphism, \( i \) majorizes \( f \) via \( j^{-1} \). If there is a morphism \( g : X \to Z \) with \( f = i \circ g \), then \( j \circ (\text{Id}_X \times_Y g) = \text{Id}_Y \) by definition and then \( (\text{Id}_X \times_Y g) \circ j = \text{Id}_Z \) because \( j \) is a monomorphism. \( \square \)

**Definition 1.6.** Let \( f : X \to Y \) be a morphism of schemes. The **scheme theoretic image** (or schematic image) of \( f \) is a closed subscheme \( \text{Im}(f) \) of \( Y \) majorizing \( f \) that satisfies the following universal property. If a closed embedding \( Z \hookrightarrow Y \) majorizes \( f \), then it also majorizes \( \text{Im}(f) \hookrightarrow Y \). We also call a diagram \( X \to \text{Im}(f) \to Y \) a scheme theoretic image. Given an open subscheme \( U \) of \( X \) the schematic closure of \( U \) in \( X \) is the schematic image of the open embedding \( U \hookrightarrow X \).

In addition, given a point \( x \in X \), we denote by \( \overline{x} \) the schematic image of the natural morphism \( \text{Spec}(k(x)) \to X \).

**Remark 1.6.1.** It is a standard result (c.f. \cite[Proposition 10.30]{EGA4}, \cite[I Chapitre I, §9.5, p.176]{EGA8} or \cite[Tag 01R5]{tag}) that the schematic image of any morphism \( f \) exists, by abstract nonsense it is unique up to unique isomorphism and, if the morphism \( f \) is quasi-compact, then the closed subscheme \( \text{Im}(f) \) of \( Y \) is defined by the quasi-coherent \( \mathcal{O}_Y \)-ideal \( \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X) \).

The following is another standard result about schematic images (c.f. \cite[Lemma 14.6, p.424]{EGA4}, \cite[Tag 0811]{tag} or \cite[IV Chapitre IV, Proposition 2.3.2, p.14]{EGA5}).


**Lemma 1.7.** Let $S$ be a ground scheme and $S' \to S$ a flat morphism. Let $f: X \to Y$ be a quasi-compact morphism of $S$-schemes with $\overline{X}$ its schematic image. The schematic image of the base change $f': X' \to Y'$ of $f$ by $S' \to S$ is the Cartesian product $\overline{X} \times_S S'$.

**Proposition 1.8.** Let $X \to Y$ and $Z \to Y$ be morphisms with $\overline{X}$ and $\overline{Z}$ their respective schematic images. Consider the following Cartesian diagram.

$$
\begin{array}{ccc}
X \times_Y Z & \longrightarrow & \overline{X} \times_Y Z \\
\downarrow \quad \quad \quad \downarrow \\
X \times_Y \overline{Z} & \longrightarrow & \overline{X} \times_Y \overline{Z} \\
\downarrow \quad \quad \quad \downarrow \\
X & \longrightarrow & \overline{X}
\end{array}
$$

(1.1)

If the top row and the middle column are schematic images, then so is the diagonal $X \times_Y Z \to \overline{X} \times_Y Z \to Y$.

**Proof.** We just need to check that $\overline{X} \times_Y \overline{Z} \to Y$ satisfies the required universal property. Let $T \to Y$ be a closed embedding majorizing $X \times_Y Z \to Y$. Then the following diagram commutes.

$$
\begin{array}{ccc}
X \times_Y Z & \longrightarrow & Z \\
\downarrow \quad \quad \quad \downarrow \\
T & \longrightarrow & Y
\end{array}
$$

So, there is a morphism $X \times_Y Z \to T \times_Y Z$ whose composition with (the closed embedding) $T \times_Y Z \to Z$ is the projection and, by the universal property of the top row of (1.1), there is a closed embedding $\overline{X} \times_Y Z \to T \times_Y Z$ whose composition with $T \times_Y Z \to Z$ is just the projection $\overline{X} \times_Y Z \to Z$. Now, the following diagram commutes.

$$
\overline{X} \times_Y Z \quad \longrightarrow \quad T \times_Y Z
\begin{array}{ccc}
\quad \\
\downarrow \quad \quad \quad \downarrow \\
\overline{X} & \longrightarrow & T
\end{array}
$$

So again, there is a morphism $\overline{X} \times_Y Z \to \overline{X} \times_Y T$ whose composition with (the closed embedding) $\overline{X} \times_Y T \to \overline{X}$ is the projection and, now by the universal property of the middle column of (1.1), there is a closed embedding $\overline{X} \times_Y Z \to \overline{X} \times_Y T$ whose composition with $\overline{X} \times_Y T \to \overline{X}$ is the projection. Hence, the following diagram commutes.

$$
\begin{array}{ccc}
\overline{X} \times_Y T & \longrightarrow & T \\
\downarrow \quad \quad \quad \downarrow \\
\overline{X} & \longrightarrow & Y
\end{array}
$$

and $T \to Y$ majorizes $\overline{X} \times_Y Z \to Y$. \qed
**Proposition 1.9.** Let $S$ be a ground scheme. Let $X'$ and $Y'$ be flat $S$-schemes. Let $X 	o X'$ and $Y 	o Y'$ be morphisms of $S$-schemes with $\overline{X}$ and $\overline{Y}$ their respective schematic image. If $\overline{X}$ and $\overline{Y}$ are flat $S$-schemes, then the closed embedding $\overline{X} \times_S \overline{Y} \to X' \times_S Y'$ is the schematic image of $X \times_S Y \to X' \times_S Y'$.

**Proof.** Consider the following Cartesian diagram.

\[
\begin{array}{cccccc}
X \times_S Y & \to & \overline{X} \times_S \overline{Y} & \to & X' \times_S Y & \rightarrow & Y \\
\downarrow & & \downarrow & & \downarrow & & \\
X \times_S Y & \to & \overline{X} \times_S \overline{Y} & \to & X' \times_S Y & \rightarrow & Y \\
\downarrow & & \downarrow & & \downarrow & & \\
X \times_S Y' & \to & \overline{X} \times_S \overline{Y}' & \to & X' \times_S Y' & \rightarrow & Y' \\
\downarrow & & \downarrow & & \downarrow & & \\
X & \to & \overline{X} & \to & X' & \rightarrow & S
\end{array}
\]

By Lemma 1.7, we can apply Proposition 1.8 to the three times three upper left sub-diagram. \qed

**Proposition 1.10.** Let $S$ be a ground scheme. Let $X$ and $Y$ be flat $S$-schemes and $Z$ a closed subscheme of $X$ flat over $S$. Then

\[
(X \times_S Y) \setminus (Z \times_S Y) \cong (X \setminus Z) \times_S Y.
\]

**Proof.** The schemes $(X \setminus Z) \times_S Y$ and $Z \times_S Y$ are respectively an open and a closed subscheme of $X \times_S Y$. It is straightforward to see that their union is the whole $X \times_S Y$ and that their intersection

\[
(Z \times_S Y) \times_{(X \times_S Y)} ((X \setminus Z) \times_S Y)
\]

is empty. So,

\[
(X \times_S Y) \setminus (Z \times_S Y) \cong (X \setminus Z) \times_S Y
\]

and by Proposition 1.9, since $\overline{Y} = Y$,

\[
(X \setminus Z) \times_S Y \cong (X \setminus Z) \times_S Y.
\]

\[
1.3 \text{ Blow up along a locally principal subscheme.} \quad \text{We show that blowing up a locally Noetherian scheme } X \text{ along a locally principal subscheme } Z \text{ consists of shaving off those associated points of } X \text{ lying on } Z, \text{ Theorem 1.13.}
\]

This, joint with Proposition 1.10, yields that for every integral scheme $B$ the blow up of $B \times X$ along a locally principal subscheme is again the Cartesian product of $B$ with a scheme $\tilde{X}$, which furthermore is a closed subscheme of $X$, see Proposition 1.15.

Let $X$ be a scheme. We recall the definitions of a locally principal subscheme and effective Cartier divisor of $X$ (c.f. [10], Remark 6.17.1, p.145), [4, Définition 11.24, p.301], [14, Tag 01WQ] or [8, IV, Définition 21.1.6, p.257, and Paragraphe 21.2.12, p.262]).
A locally principal subscheme of $X$ is a closed subscheme whose sheaf of ideals is locally generated by a single element.

An effective Cartier divisor of $X$ is a closed subscheme whose sheaf of ideals is locally generated by a single regular element.

Let $f, g: X \to Y$ be two morphisms and $U$ a open subscheme of $X$. When $U$ is (topologically) dense in $X$, it does not follow from the equation $f|_U = g|_U$ that $f = g$, it just follows that $f|_{X_{red}} = g|_{X_{red}}$. That motivates the following definition.

**Definition 1.11.** Let $X$ be a scheme. An open subscheme $U$ of $X$ is scheme theoretically dense in $X$ if, for every open $V$ of $X$, the schematic closure of $U \cap V$ in $V$ is equal to $V$ (c.f. [14, Tag 01RB] or [8, IV, Chapitre IV, Définition 11.10.2, p.171]).

**Remark 1.11.1.** In general, there are schemes $X$ with schematically dense open subschemes which are not (topologically) dense (c.f. [14, Tag 01RC]). But when the ambient scheme $X$ is locally Noetherian every open immersion is quasicompact (c.f. [14, Tag 01OX] or [8, I, Chapitre I, Proposition 6.6.4, p.153]) and then schematically dense implies dense (c.f. [14, Tag 01RD] or [8, IV, Chapitre IV, Remarque 11.10.3 (iv), p.171]).

**Proposition 1.12.** Let $X$ be a scheme and $Z$ a closed subscheme of $X$. Let $i : U \hookrightarrow X$ be the open subscheme complement of $Z$ in $X$ and $b : \overline{U} \hookrightarrow X$ its schematic closure. If $Z$ is a locally principal subscheme of $X$, then the closed embedding $b : \overline{U} \hookrightarrow X$ is the blow up of $X$ along $Z$.

Note that if $Z$ is an effective Cartier divisor then $\overline{U} = X$ (c.f. [8, IV, Chapitre IV, Corollaire 3.1.9, p.38]).

**Proof.** The open embedding $U \hookrightarrow X$ is an affine morphism because locally it is given by $\text{Spec}(A_f) \hookrightarrow \text{Spec}(A)$ for some ring $A$ and $f \in A$. Therefore $U \hookrightarrow X$ is quasi-compact (c.f. [8, II, Chapitre II, §5.1]), the sheaf $\mathcal{K} = \ker(\Theta_X \to i_*\mathcal{O}_U)$ is quasi-coherent and, by Remark 1.6.1, it defines the closed embedding $b : \overline{U} \hookrightarrow X$.

Since the blow up, by its universal property, can be computed locally, we may and do assume $X$ affine, say $X \cong \text{Spec}(A)$ for some ring $A$, and $Z$ principal, say defined by $f \in A$. Then, the open subscheme $U$ of $X$ is just $D(f) \cong \text{Spec}(A_f)$, the (coherent) $\mathcal{O}_X$-ideal $\mathcal{K}$ is defined by the ideal $\mathfrak{a} = \ker(A \to A_f) \subseteq A$ and the closed embedding $b$ corresponds to the natural homomorphism $A \to A/\mathfrak{a}$.

When $f \in A$ is nilpotent, the subscheme $U$ of $X$ is the empty scheme. Moreover, for all $n \gg 0$, the $n$-th graded components of the Rees algebra of the ideal $(f)$ of $A$ are zero. Hence, the blow up of $X$ along $Z$ is also the empty scheme.

Assume $f \in A$ non-nilpotent. The ideal $\mathfrak{a} \subseteq A$ is $\bigcup_{n \in \mathbb{N}} (0 : f^n)$ (c.f. [1, Proposition 3.11.i, p.41]). So, the closed subscheme $b^{-1}(Z)$ of $\overline{U}$ is an effective Cartier divisor because it is defined by the class of $f$ in $A/\mathfrak{a}$ which is a non-zerodivisor. Let $g : W \to X$ be a morphism with $g^{-1}(Z)$ an effective Cartier divisor of $W$. Affine locally $g$ is given by homomorphisms $\varphi : A \to B$ with $\varphi(f) \in B$ a non-zerodivisor. Hence, $\mathfrak{a} \subseteq \ker(\varphi)$ and $g$ factors through $b$.

We have seen that the blow up of every scheme $X$ along any locally principal subscheme $Z$ of $X$ is just the schematic closure of the open complement of $Z$.  

When the scheme $X$ is locally Noetherian there are no pathological associated points, see [14, Tag 020I], and then, as Theorem 1.13 shows, we can understand much better which parts of $Z$ are shaved off on the blowing up procedure.

**Theorem 1.13.** Let $X$ be a locally Noetherian scheme and $Z$ a locally principal subscheme of $X$. Let $T_Z$ be the subset of $X$ union of the underlying sets of $\overline{x}$ for all $x \in \operatorname{Ass}(X) \cap Z$. Let $V$ be its complement in $X$. Then $V$ is an open subscheme of $X$ and its schematic closure $\overline{V} \hookrightarrow X$ is the blow up of $X$ along $Z$.

**Proof.** First of all, the subset $T_Z$ of $X$ is closed because its intersection with every Noetherian affine open subscheme of $X$ is a union of finitely many closed subsets (c.f. [14, Tag 05AF] or [8, IV 2 Chapitre IV, Proposition 3.1.6, p.37]). Hence $V$ is an open subscheme of $X$.

Let $U$ be the open complement of $Z$ and $\overline{U}$ its schematic closure. Since $T_Z$ is a closed subset of $Z$, $U$ is an open subscheme of $V$ and of $\overline{V}$. We show that $U \hookrightarrow \overline{V}$ is schematically dense, then the claim follows from Proposition 1.12 and Remark 1.11.1 (recall that $U$ and $\overline{V}$ are locally Noetherian).

By definition of $T$, $\operatorname{Ass}(X) \cap U = \operatorname{Ass}(X) \cap V$ and, by [8, IV 2 Chapitre IV, Proposition 3.1.13, p.39], $\operatorname{Ass}(\overline{V}) \subseteq \operatorname{Ass}(X) \cap V$. So, $\operatorname{Ass}(\overline{V}) \subseteq U$ and then $U$ is a schematically dense subscheme of $\overline{V}$ (c.f. [8, IV 3 Chapitre IV, Proposition 11.10.10, p.172]).

**Lemma 1.14.** Let $X$ and $Y$ be locally Noetherian schemes. Let $(x,y) \in \operatorname{Ass}(X) \times \operatorname{Ass}(Y)$. The ring $\kappa(x) \otimes_{\kappa} \kappa(y)$ is an integral domain and its unique associated prime corresponds to an associated point $\xi_{x,y}$ of $X \times Y$ via the natural monomorphism $\operatorname{Spec}(\kappa(x) \otimes_{\kappa} \kappa(y)) \hookrightarrow X \times Y$. Furthermore the assignment $(x,y) \mapsto \operatorname{Ass}(X) \times \operatorname{Ass}(Y)$ with $\xi_{x,y} \in \operatorname{Ass}(X \times Y)$ is a one-to-one correspondence.

**Proof.** For all $(x,y) \in \operatorname{Ass}(X) \times \operatorname{Ass}(Y)$ the ring $\kappa(x) \otimes_{\kappa} \kappa(y)$ is an integral domain (see [15, Chapter III, Corollary 1 of Theorem 40, p.198]). Then the claim follows from [8, IV 2 Chapter IV, Corollaire 3.3.7, p.45], see also [8, I Chapitre I, Définition 9.1.2, p.169].

**Proposition 1.15.** Let $X$ be a locally Noetherian scheme and $B$ an integral locally Noetherian scheme. Let $Z$ be a locally principal subscheme of $B \times X$. Then there is a closed subscheme $X$ of $X$ such that the closed embedding $B \times X \hookrightarrow B \times X$ is the blow up of $B \times X$ along $Z$.

**Proof.** By Theorem 1.13, the blow up of $B \times X$ along $Z$ is the schematic closure of the open complement of the closed subset $T_Z$.

By Lemma 1.14 and since $B$ has a unique associated point $b$, there is a one-to-one correspondence between $\operatorname{Ass}(B \times X)$ and $\operatorname{Ass}(X)$. Clearly, given $x \in \operatorname{Ass}(X)$ the schematic closure of $\xi_{b,x} \hookrightarrow B \times X$ is $B \times \overline{x}$. So, by [8, I Chapitre I, paragraphe 3.2.8, p.108]

$$T_Z = \bigcup_{x \in C} (B \times \overline{x}) = B \times \left( \bigcup_{x \in C} \overline{x} \right)$$

where $C$ is the subset of points $x \in \operatorname{Ass}(X)$ such that $\xi_{x,b} \in Z$. Finally, by Proposition 1.10,

$$(B \times X) \setminus T_Z = B \times \left( X \setminus \left( \bigcup_{x \in C} \overline{x} \right) \right).$$
In Proposition 1.15 we assume the scheme $B$ integral, that is $B$ irreducible and reduced. If one of this assumptions fails, then there could be locally principal subschemes $Z$ of $B$ which are not effective Cartier divisors, and the blow up of $B$ along one of such $Z$ would not be an isomorphism. Hence, Proposition 1.15 requires both assumptions.

2 Constant morphisms

Let $X,Y$ be locally Noetherian schemes, $Z$ a closed subscheme of $X \times Y$ and $f: X \times Y \to Z'$ a morphism.

First we study morphisms $W \to X$ for which the base change of $Z \to X \times Y$ by $W \to X$ is an isomorphism. Theorem 2.5 asserts that they form a representable functor $\text{Iso}^Z_{X/Y}$ on $W$ which is represented by a closed subscheme of $X$.

This allows us to study morphisms $W \to X$ for which the restriction of $f$ to $W \times Y$ is constant along the fibres of the projection $W \times Y \to W$. Theorem 2.7, an immediate consequence of Theorem 2.5, shows that they form a category with a final object which is a closed subscheme of $X$. We call it the $f$-constfy closed subscheme of $X$ (c.f. Definition 2.6).

**Definition 2.1.** Let $f: X \to Y$ and $p: X \to S$ be morphisms. Consider the following Cartesian diagram

$$
\begin{array}{ccc}
Z & \to & Y \\
\downarrow & & \downarrow \Delta_Y \\
X \times_S X & \xrightarrow{f \times s f} & Y \times Y
\end{array}
$$

where $\Delta_Y$ is the diagonal. We say that the morphism $f$ is constant along the fibres of $p$ if the morphism $Z \to X \times_S X$ is an isomorphism.

The standard (and maybe more intuitive) definition of a morphism $f: X \to Y$ being constant along the fibres of another morphism $p: X \to S$ is that the following diagram commutes,

$$
\begin{array}{ccc}
X \times_S X & \xrightarrow{q_1} & X \\
q_2 \downarrow & & \downarrow f \\
X & \xrightarrow{f} & Y
\end{array}
$$

where the morphisms $q_1, q_2$ are the two projections. It just says that the kernel or equaliser of the two morphisms $f \circ q_1, f \circ q_2$ is the whole scheme $X \times_S X$ which is equivalent to Definition 2.1 (c.f. [5, Définition 1.4.2, p.34 and Proposition 1.4.10, p.37]).

**Proposition 2.2.** Let $f: X \times Y \to Z$ be a morphism and $p: X \times Y \to X$ the projection. If $f$ is constant along the fibres of $p$, then there is a unique morphism $g: X \to Z$ such that $f = g \circ p$.

**Proof.** It is particular case of a bigger result due to Alexander Grothendieck. The original result is [6, B.1 Théorème 2. (190-19)]. But we are using [3, Théorème 2.55, p.34] which applies to a slightly more general class of morphisms. The result may also be found at [14, Tag 03O3].
The result says that the functor of points $h_Z$ of $Z$ is a sheaf in the fpqc topology. In this case, the morphism $p$ itself is a cover of $X$ by the fpqc topology (see [3, Definition 2.34, p.28]). Since there is just one element covering the whole scheme $X$, there is just one overlap, the scheme $(X \times Y) \times_X (X \times Y)$ (such overlap would be trivial by the Zariski topology, but here it is not). So, whenever $f: X \times Y \to Z$ agrees with itself on this overlap, it extends uniquely to a morphism $g: X \to Z$. But this condition is equivalent to $f$ being constant along the fibres of $p$. \hfill \Box

**Lemma 2.3.** Let $A$ and $B$ be $k$-algebras. Given an ideal $a$ of $A$, we denote $a^e$ its extension to $A \otimes_k B$. Then, for every family of ideals $\{a_\lambda\}_{\lambda \in \Lambda}$ of $A$, 

$$ \left( \bigcap_{\lambda \in \Lambda} a_\lambda \right)^e = \bigcap_{\lambda \in \Lambda} a_\lambda^e. $$

**Proof.** The left inclusion is clear.

Consider a $k$-base $\{e_l\}_{l \in L}$ of $B$. The tensor product $A \otimes_k B$ is a free $A$-module and the set $\{1 \otimes e_l\}$ is an $A$-base for it (c.f. [12, Proposition 2.3, p.609]). Given an ideal $a$ of $A$ and $x = \sum_l a_l (1 \otimes e_l) \in A \otimes_k B$, the element $x$ belongs to $a^e = a \otimes_k B$ if and only if $a_l \in a$ for all $l$. So, when $x \in \cap_\lambda a_\lambda^e$, every element $a_l$ belongs to $\cap_\lambda a_\lambda$ and hence $x \in (\cap_\lambda a_\lambda)^e$. \hfill \Box

**Definition 2.4.** Let $X$ and $Y$ be schemes and $Z$ a closed subscheme of $X \times Y$. We define $\text{Iso}^Z_{X/Y}: \text{Sch} \to \text{Set}$ as the contravariant functor that sends an scheme $W$ to the set 

$$ \text{Iso}^Z_{X/Y}(W) = \{ (W \to X) \in \text{Sch}_X \mid W \times_X Z \to W \times Y \text{ is an isomorphism} \}. $$

Since isomorphisms are stable by base change given $\alpha: W' \to W$ and $g \in \text{Iso}^Z_{X/Y}(W)$, $\alpha^*(g) = g \circ \alpha \in \text{Iso}^Z_{X/Y}(W')$. Hence the functor $\text{Iso}^Z_{X/Y}$ is well defined over morphisms. It sends $\alpha: W' \to W$ to $\alpha^*|_{\text{Iso}^Z_{X/Y}(W')}$.

Theorem 2.5 below asserts that the functor $\text{Iso}^Z_{X/Y}$ is representable for all schemes $X$ and $Y$ and for every closed subscheme $Z$ of $X \times Y$. The representing scheme will be the empty scheme in many cases but not always. For example, when $X$ is equal to the base field, the projection $X \times Y \to X$ is just the identity morphism and the representing scheme is $Z$ itself.

**Theorem 2.5.** Let $X$ and $Y$ be schemes and $Z$ a closed subscheme of $X \times Y$. Then the functor $\text{Iso}^Z_{X/Y}$ is representable by a closed subscheme $X'$ of $X$.

Recall that by Remark 1.1.1 there is a universal property corresponding to the functor $\text{Iso}^Z_{X/Y}$ and the result of Theorem 2.5 may be stated as follows. There is a closed subscheme $X'$ of $X$ such that the base change $X' \times_X Z \to X' \times Y$ of the closed embedding $Z \hookrightarrow X \times Y$ by $X' \hookrightarrow X$ is an isomorphism and $X'$ satisfies the following universal property. All morphisms $W \to X$ such that the closed embedding $W \times_X Z \hookrightarrow W \times Y$ is an isomorphism are majorized by $X' \hookrightarrow X$.

**Proof of Theorem 2.5.** Assume first that $X$ and $Y$ are affine schemes, say $X \cong \text{Spec}(A)$ and $Y \cong \text{Spec}(B)$ for some rings $A$ and $B$. Then $X \times Y \cong \text{Spec}(A \otimes_k B)$ and the scheme $Z$ is determined by an ideal $m$ of $A \otimes_k B$. Denote by $a$ the ideal
of $A$ intersection of all ideals whose extension to $A \otimes_k B$ contains $m$. In this case, we claim that $a$ is the ideal defining the desired closed subscheme $X'$ of $X$.

Clearly, the closed embeddings $\text{Spec}(A/a) \times Y \hookrightarrow X \times Y$ and $\text{Spec}(A/a) \times X \hookrightarrow Z$ correspond respectively to the natural homomorphisms $A \otimes_k B \rightarrow A \otimes_k B$ and $A \otimes_k B \rightarrow A \otimes_k B$. But, by Lemma 2.3, the extension of $a$ to $A \otimes_k B$ contains $m$, that is $m + (a \otimes_k B) = a \otimes_k B$. Hence the schemes $X' \times_X Z$ and $X' \times Y$ are isomorphic.

Now, consider a morphism $W \rightarrow X$ such that the closed embedding $W \times_X X/\mathbb{Z} \hookrightarrow W \times Y$ is an isomorphism. By Remark 1.4.1, we may and we do assume $W$ affine, say $W \cong \text{Spec}(C)$ for some ring $C$, and $W \rightarrow X$ corresponding to an homomorphism $\varphi: A \rightarrow C$. Consider the following diagram of pushforwards.

\[
\begin{array}{ccc}
A & \rightarrow & C' = A/\ker(\varphi) \\
A \otimes_k B & \rightarrow & C' \otimes_k B \cong (A \otimes_k B)/\ker(\varphi) \otimes_k B \\
D = (A \otimes_k B)/m & \rightarrow & C' \otimes_A D \cong (A \otimes_k B)/\ker(\varphi) \otimes_k B + m \\
& & \rightarrow C \otimes_A D
\end{array}
\]

The homomorphism $C \otimes_k B \rightarrow C \otimes_A D$ is an isomorphism by the assumption on $W \rightarrow X$. The homomorphism $C' \otimes_k B \rightarrow C \otimes_k B$ is injective since so is $C' \rightarrow C$. Then,

$$\ker(C' \otimes_k B \rightarrow C' \otimes_A D) \subseteq \ker(C' \otimes_k B \rightarrow C \otimes_A D) = 0.$$ 

Hence, $m \subseteq \ker(\varphi) \otimes_k B$, that is the kernel of $\varphi$ is an ideal of $A$ whose extension to $A \otimes_k B$ contains $m$. So, by definition of $a$, $a \subseteq \ker(\varphi)$ and $X' \hookrightarrow X$ majorizes $W \rightarrow X$.

Second, consider the case $X$ any scheme and $Y$ an affine scheme. Fix an affine open cover $\{U_i\}_{i \in I}$ of $X$ and denote by $Z_i \hookrightarrow U_i \times Y$ the base change of $Z \hookrightarrow X \times Y$ by the open embedding $U_i \hookrightarrow X$. This case follows from the fact that the functors $\text{Iso}^Z_{U_i/Y}$ form a collection of subfunctors of $\text{Iso}^Z_X/Y$ satisfying the conditions of Lemma 1.3 (the inclusion $\text{Iso}^Z_{U_i/Y} \hookrightarrow \text{Iso}^Z_X/Y$ is given by composition with $U_i \hookrightarrow X$).

By the previous case, all functors $\text{Iso}^Z_{U_i/Y}$ are representable and they clearly cover $\text{Iso}^Z_X/Y$. Let us check that the collection $\text{Iso}^Z_{U_i/Y}$ satisfies conditions 2a and 2b of Lemma 1.3. Let $W$ be a scheme, $\{W_j\}_{j \in J}$ an open cover of $W$ and $\{\xi_j: W_j \rightarrow X\}_j$ a collection of morphisms belonging to $\text{Iso}^Z_X/Y(W_j)$ such that for all $j, k$, $\xi_j|_{W_j \cap W_k} = \xi_k|_{W_k \cap W_j}$. Then, there is a unique morphism $\xi: W \rightarrow X$ such that $\xi|_{W_j} = \xi_j$. Furthermore, since being an isomorphism is local on the target, $\xi \in \text{Iso}^Z_X(Y(W))$. Hence, the functor $\text{Iso}^Z_X/Y$ satisfies the sheaf property for the Zariski topology.

Now, let us show that every $\text{Iso}^Z_{U_i/Y} \hookrightarrow \text{Iso}^Z_X/Y$ is representable by open immersions. Fix $i \in I$, a scheme $W$ and $(\xi: W \rightarrow X) \in \text{Iso}^Z_X/Y(W)$. Then the desired open subscheme of $W$ is $W \times_U U_i$. Let $f: W' \rightarrow W$ be a morphism. If $f$ factors through the open embedding $W \times_U U_i \hookrightarrow W$, then $f^*(\xi): W' \rightarrow X$
factors through $U_i \hookrightarrow X$. The base change of $Z_i \hookrightarrow U_i \times Y$ by $W' \twoheadrightarrow U_i$ is an isomorphism because $\xi \in Iso_{X/Y}^Z$ and it is the base change of $Z \hookrightarrow X \times Y$ first by $\xi$ and then by $f$. So, $(W' \twoheadrightarrow U_i) \in Iso_{U_i/Y}^Z(W')$. To see the converse just observe that a morphism $W \twoheadrightarrow X$ factorises through an open embedding $U_i \hookrightarrow X$ if and only if $U_i$ contains its image (c.f. [4, Exercise 3.25, p.91]).

Observe that the scheme representing $Iso_{X/Y}^Z$ is a closed subscheme of $X$ because it is obtained gluing closed subschemes of each $U_i$ and $\{U_i\}_{i \in I}$ is a cover of $X$.

Finally consider the general case. Fix a cover $\{V_j\}_{j \in J}$ of $Y$ by affine open subschemes and denote by $Z_j$ the pullback of $Z$ by $X \times V_j \twoheadrightarrow X \times Y$. The functor $Iso_{X/V_j}^Z$ is representable by a closed subscheme $X_j$ of $X$. Then we claim that the scheme theoretic intersection $X' = \bigcap_{j \in J} X_j$ is the desired closed subscheme of $X$. To show it we check that $X'$ satisfies the universal property related to the functor $Iso_{X/Y}^Z$.

First we check that the base change $Z_{X'} \hookrightarrow X' \times Y$ of $Z \hookrightarrow X \times Y$ by $X' \twoheadrightarrow X$ is an isomorphism. For every $V_j \hookrightarrow Y$, by definition of $X_j$, the base change $(Z_j)_{X'} \hookrightarrow X' \times V_j$ of $Z_j \hookrightarrow X \times V_j$ by $X_j \hookrightarrow X$ is an isomorphism. So when we change again the base, now by $X' \twoheadrightarrow X_j$, we get another isomorphism. That is, the morphism $(Z_j)_{X'} \hookrightarrow X' \times V_j$ is an isomorphism as well. But the morphism $(Z_j)_{X'} \hookrightarrow X' \times V_j$ is also the pullback of $Z_{X'} \hookrightarrow X' \times Y$ by (the open embedding) $X' \times V_j \hookrightarrow X' \times Y$. Hence, since being an isomorphism is local on the target and the collection $\{X' \times V_j\}_{j \in J}$ is an open cover of $X' \times Y$, the morphism $Z_{X'} \hookrightarrow X' \times Y$ is an isomorphism.

Finally, we check that $X' \twoheadrightarrow X$ majorizes any morphism $W \twoheadrightarrow X$ such that $Z_W \hookrightarrow W \times Y$ is an isomorphism. Given $V_j \hookrightarrow Y$, since $Z_W \hookrightarrow W \times Y$ is an isomorphism, so is its pullback $(Z_j)_W \hookrightarrow W \times V_j$ by $W \times V_j \twoheadrightarrow W \times Y$. Hence, by the universal property of $X_j$, the closed embedding $X_j \hookrightarrow X$ majorizes $W \twoheadrightarrow X$ via a morphism $f_j : W \twoheadrightarrow X_j$. Observe that $X'$ is just the Cartesian product of $\{X_j \twoheadrightarrow X\}_{j \in J}$, closed embeddings are affine morphisms (c.f. [8, II Chapitre II, Proposition 1.6.2, p.14]) and arbitrary Cartesian products of affine morphisms exists in the category of schemes (c.f. [14, Tag 0CNH]). So, by the universal property of the Cartesian product, the morphisms $f_j$ define a morphism $f : W \twoheadrightarrow X'$ whose composition with $X' \twoheadrightarrow X$ is $W \twoheadrightarrow X$.

**Remark 2.5.1.** Let $X$ and $Y$ be schemes and $Z$ a closed subscheme of $X \times Y$. The scheme representing $Iso_{X/Y}^Z$ is equal to $X$ if and only if the closed embedding $Z \hookrightarrow X \times Y$ is an isomorphism.

**Definition 2.6.** Let $f : X \times Y \twoheadrightarrow Z$ be a morphism and $X'$ a closed subscheme of $X$. We call $X'$ a $f$-constfy if the morphism $f|_{X' \times Y}$ is constant along the fibres of the projection $X' \times Y \twoheadrightarrow X'$ and $X'$ satisfies the following universal property. For every morphism $W \twoheadrightarrow X$ such that $f|_{W \times Y}$ is constant along the fibres of the projection $W \times Y \twoheadrightarrow W$, the closed embedding $X' \hookrightarrow X$ majorizes $W \twoheadrightarrow X$.

If a $f$-constfy closed subscheme exists, by abstract nonsense it is unique up to unique isomorphism.

**Theorem 2.7.** For every morphism $f : X \times Y \twoheadrightarrow Z$ with $Z$ separated over $k$, the $f$-constfy closed subscheme $X'$ of $X$ exists.
Proof. The scheme \((X \times Y) \times_X (X \times Y)\) is isomorphic to \(X \times Y \times Y\) and, via this identification, the morphism \((X \times Y) \times_X (X \times Y) \to X\) is just the projection \(X \times Y \times Y \to X\). Also, via this identification, we define \(\tilde{f}: X \times Y \times Y \to Z \times Z\) as the product \(f \times_X f\). Consider the following Cartesian diagram,

\[
\begin{array}{ccc}
W & \longrightarrow & Z \\
\downarrow \quad & & \downarrow \Delta_Z \\
X \times Y \times Y & \overset{\tilde{f}}{\longrightarrow} & Z \times Z
\end{array}
\]

where \(\Delta_Z\) is the diagonal. Since \(Z\) is separated the morphism \(W \to X \times Y \times Y\) is a closed embedding. Now, clearly the desired closed subscheme \(X'\) of \(X\) is the scheme representing the functor \(\text{Iso}_X^W/Z\times Y\), which exists by Theorem 2.5.

Remark 2.7.1. Let \(X, Y\) be schemes, \(Z\) a closed subscheme of \(X \times Y\) separated as a \(k\)-scheme and \(f: X \times Y \to W\) a morphism. By Theorem 2.7, the \(f\)-constfy closed subscheme \(X_f\) of \(X\) exists. By Theorem 2.5, the closed subscheme \(X'\) of \(X\) representing \(\text{Iso}_{X_f}^Z/X\times Y\) exists. Consider the following Cartesian diagram,

\[
\begin{array}{ccc}
W & \longrightarrow & Z \\
\downarrow \quad & & \downarrow \Delta_Z \\
X \times Y \times Y & \overset{\tilde{f}}{\longrightarrow} & Z \times Z
\end{array}
\]

where \(\Delta_Z\) is the diagonal. Since \(Z\) is separated the morphism \(W \to X \times Y \times Y\) is a closed embedding. Now, clearly the desired closed subscheme \(X'\) of \(X\) is the scheme representing the functor \(\text{Iso}_X^W/Z\times Y\), which exists by Theorem 2.5.

Remark 2.7.1. Let \(X, Y\) be schemes, \(Z\) a closed subscheme of \(X \times Y\) separated as a \(k\)-scheme and \(f: X \times Y \to W\) a morphism. By Theorem 2.7, the \(f\)-constfy closed subscheme \(X_f\) of \(X\) exists. By Theorem 2.5, the closed subscheme \(X'\) of \(X\) representing \(\text{Iso}_{X_f}^Z/X\times Y\) exists. Consider the following Cartesian diagram,

\[
\begin{array}{ccc}
W & \longrightarrow & Z \\
\downarrow \quad & & \downarrow \Delta_Z \\
X \times Y \times Y & \overset{\tilde{f}}{\longrightarrow} & Z \times Z
\end{array}
\]

where \(\Delta_Z\) is the diagonal. Since \(Z\) is separated the morphism \(W \to X \times Y \times Y\) is a closed embedding. Now, clearly the desired closed subscheme \(X'\) of \(X\) is the scheme representing the functor \(\text{Iso}_X^W/Z\times Y\), which exists by Theorem 2.5.

3 The Straight Blow Up Section Family

Let \(B, X\) be locally Noetherian schemes and \(Z\) a closed subscheme of \(B \times X\). In this section we prove our main result, Theorem 3.8, which asserts the existence of the straight blow up section family of the projection \(B \times X \to B\) along \(Z\) when \(X\) is piecewise quasiprojective and \(B\) is projective and integral. Roughly speaking, it is an object that combines the universal properties of the universal section family of \(B \times X \to B\) (c.f. Definition 3.2) and of the blow up of \(B \times X\) along \(Z\).

More precisely, it is a \(X\)-scheme \(\tilde{X} \xrightarrow{b} X\) such that the pullback \((\text{Id}_B \times b)^{-1}(Z)\) is an effective Cartier divisor of \(B \times \tilde{X}\) and it satisfies a universal property.

We split its construction into two steps. First we construct a slightly more general object, the blow up section family of a morphism \(Y \to B\) along a closed subscheme \(T\) of \(Y\) (c.f. Definition 3.4 and Theorem 3.7). It is a section family \((\tilde{Y}, b')\) of \(Y \to B\) (c.f. Definition 3.2) such that the pullback \((b')^{-1}(T)\) is an effective Cartier divisor of \(B \times \tilde{Y}\) and it satisfies a universal property. Then, when \(Y\) is a product \(B \times X\) and \(Y \to B\) is the projection over \(B\), the straight blow up section family just requires to the morphism \(b: B \times \tilde{Y} \to B \times X\) to be non-twisted. So, by Proposition 2.2, it is essentially the “constfy” closed subscheme \(\tilde{X}\) of \(\tilde{Y}\) for the morphism \(B \times \tilde{Y} \to B \times X \to X\) (c.f. Definition 2.6).

Definition 3.1. We call a scheme which is a finite or countable disjoint union of quasiprojective schemes a piecewise quasiprojective scheme.

Definition 3.2. Let \(X \to Y\) be a morphism. Let \(W\) be a scheme and \(\rho: Y \times W \to X\) a morphism. The couple \((W, \rho)\) is a section family of \(X \to Y\) when the composition \(Y \times W \xrightarrow{\rho} X \to Y\) is the projection.
Let \((\tilde{X}, \psi)\) be a section family of \(X \to Y\). The couple \((\tilde{X}, \psi)\) is a \textit{universal section family} of \(X \to Y\) if it satisfies the following universal property. For every section family \((W, \rho)\) of \(X \to Y\) there is a unique morphism \(g: W \to \tilde{X}\) such that the following diagram commutes (see [2, Definition 1.4, p.4] and [7, II, C, no2, pp.380,381, le foncteur “ensembles des sections”]).

\[
\begin{array}{ccc}
Y \times W & \xrightarrow{Id_\times \times g} & Y \times \tilde{X} \\
\downarrow{\rho} & & \downarrow{\psi} \\
Y \times X & \xrightarrow{\psi} & X
\end{array}
\] (3.1)

If a universal section family of the morphism \(X \to Y\) exists, by abstract nonsense it is unique up to unique isomorphism.

**Theorem 3.3** (Grothendieck). \(X \to Y\) be a morphism of locally Noetherian schemes. If \(Y\) is proper and \(X\) is piecewise quasiprojective, then the universal section family \((\tilde{X}, \psi)\) of \(X \to Y\) exists and the scheme \(\tilde{X}\) is locally Noetherian and piecewise quasiprojective.

**Proof.** Fix a finite or countable disjoint union of quasiprojective schemes \(\bigsqcup_{i \in I} X_i\) of \(X\). For every \(i \in I\), the universal section family \((\tilde{X}_i, \psi_i)\) of \(X_i \to Y\) exists and \(X_i\) is piecewise quasiprojective (c.f. [9, §4.c, pp.267,268]). So, the couple \(\tilde{X} = \bigsqcup_{i \in I} X_i\) and \(\psi = \bigsqcup_{i \in I} \psi_i \circ \theta\) is the universal section family of \(X \to Y\), where \(\theta\) is the isomorphism from \(Y \times \tilde{X}\) to \(\bigsqcup_{i \in I}(Y \times \tilde{X}_i)\) (c.f. [4, Exercise 4.2, p.115]).

For an alternative exposition to [9] see [3, Chapter 5, §5.6.2, pp.132,133].

**Definition 3.4.** Let \(X \to Y\) be a morphism and \(Z\) a closed subscheme of \(X\). A \textit{blow up section family} of \(Z\) along \(X \to Y\) (which we call the \textit{centre}) is a section family \((\tilde{X}, b)\) of \(X \to Y\) such that the pullback \(b^{-1}(Z)\) is an effective Cartier divisor of \(Y \times \tilde{X}\) and it satisfies the following universal property. For all section families \((W, \rho)\) such that the pullback of \(b^{-1}(Z)\) is an effective Cartier divisor of \(Y \times W\), there is a unique morphism \(g: W \to \tilde{X}\) such that the following diagram commutes.

\[
\begin{array}{ccc}
Y \times W & \xrightarrow{Id_\times \times g} & Y \times \tilde{X} \\
\downarrow{\rho} & & \downarrow{b} \\
Y \times X & \xrightarrow{b \circ \psi} & X
\end{array}
\]

If a blow up section family of a morphism \(X \to Y\) along a closed subscheme \(Z\) of \(X\) exists, by abstract nonsense it is unique up to unique isomorphism.

**Definition 3.5.** Let \(B, X\) be schemes and \(Z\) a closed subscheme of \(B \times X\). A \textit{straight blow up section family} of \(Z\) along \(B \times X \to B\) (which we call the \textit{centre}) is a \(X\)-scheme \(\tilde{X}\) such that the pullback \((Id_B \times b)^{-1}(Z)\) is an effective Cartier divisor of \(B \times \tilde{X}\) and it satisfies the following universal property. For every \(X\)-scheme \(W \xrightarrow{f} X\) such that the pullback \((Id_B \times f)^{-1}(Z)\) is an effective Cartier divisor of \(B \times W\), there is a unique morphism \(g: W \to \tilde{X}\) such that \(f = b \circ g\).
If a straight blow up section family of a projection $B \times X \to B$ along a closed subscheme $Z$ of $B \times X$ exists, by abstract nonsense it is unique up to unique isomorphism.

For a given morphism $X \to Y$, consider the category $\mathcal{C}$ of section families $(W, \rho)$ of $X \to Y$ where a morphism form $(W, \rho)$ to $(T, \tau)$ is a morphism $f: W \to T$ such that $\rho = \tau \circ (Id_Y \times f)$. Then, if it exists, the universal section family of $X \to Y$ is just the final object of this category.

Now, with the blow up section family, we just consider a full subcategory $D$ of $\mathcal{C}$ with the same morphisms but whose objects are section families $(W, \rho)$ of $X \to Y$ such that $\rho^{-1}(Z)$ is an effective Cartier divisor of $Y \times W$. Then, if it exists, the blow up section family of $X \to Y$ along $Z$ is just the final object of $D$.

With the straight blow up section family, the scheme $X$ must be $Y \times X'$ for some scheme $X'$ and $X \to Y$ must be the projection. Then, we are considering a full subcategory of $D$ with the same morphisms but whose objects are section family $(W, \rho)$ in $D$ such that there is a morphism $g: W \to X'$ with $\rho = Id_Y \times g$. So again, if it exists, the straight blow up section family of $Y \times X' \to Y$ along $Z$ is the final object of this new category.

**Lemma 3.6.** Consider the following diagram of schemes,

$$
\begin{array}{ccc}
B \times Y & \xrightarrow{\phi} & B \times \tilde{X} \\
\downarrow{Id_B \times f} & & \downarrow{Id_B \times i} \\
B \times X' & & \\
\end{array}
$$

where $i$ is a closed embedding. If the previous diagram commutes, then there is a unique morphism $g: Y \to \tilde{X}$ such that $\tilde{\phi} = Id_B \times g$.

**Proof.** It is straightforward to see that, since $i$ is a monomorphism, the morphism composition of $\tilde{\phi}$ with the projection $B \times \tilde{X} \to \tilde{X}$ is constant along the fibres of the projection $B \times Y \to Y$. By Proposition 2.2, the closed embedding $i$ majorizes $f$ through a morphism $g: Y \to \tilde{X}$ and, since $Id_Y \times i$ is a monomorphism, $\tilde{\phi} = Id_B \times g$.

**Theorem 3.7.** Let $B$ be a proper integral locally Noetherian scheme, $X$ a piecewise quasiprojective scheme and $Z$ a closed subscheme of $X$. Let $X \to B$ be a morphism. Then the blow up section family of $X \to B$ along $Z$ exists.

**Proof.** The blow up $b': \text{bl}(Z, X) \to X$ of $X$ along $Z$ is quasiprojective. By Theorem 3.3, the universal section family $(X', \psi)$ of the composition $\text{bl}(Z, X) \to X \to B$ exists and $X'$ is locally Noetherian and piecewise quasiprojective.

Since $B$ is integral, by Proposition 1.15, the blow up of $B \times X'$ along the locally principal subscheme $(b' \circ \psi)^{-1}(Z)$ is given by a closed subscheme $\tilde{X}$ of $X'$. Set $b: B \times \tilde{X} \to X$ as the composition $B \times \tilde{X} \to B \times X' \to \text{bl}(Z, X) \to X$.

We claim that $(\tilde{X}, b)$ is the blow up section family of $X \to B$ along $Z$, let us check that it satisfies the required universal property.

Let $(Y, \rho)$ be a section family of $X \to B$ such that $\rho^{-1}(Z)$ is an effective Cartier divisor of $B \times Y$. By the universal property of the blow up $b': \text{bl}(Z, X) \to X$, there is a unique morphism $\varphi: B \times Y \to \text{bl}(Z, X)$ such that
$b' \circ \varphi = \rho$. The couple $(Y, \varphi)$ is a section family of $bl(Z, X) \rightarrow B$, so by the universal property of $(X', \psi)$ there is a unique morphism $f : Y \rightarrow X'$ such that $\varphi = \psi \circ (Id_B \times f)$. Now, $(Id_B \times f)^{-1} \left( (b' \circ \psi)^{-1}(Z) \right)$ and $\rho^{-1}(Z)$ are the same effective Cartier divisor of $B \times Y$ and then by the universal property of the blow up $B \times \tilde{X} \rightarrow B \times X'$ there is a unique morphism $\tilde{\varphi} : B \times Y \rightarrow B \times \tilde{X}$ whose composition with (the closed embedding) $B \times \tilde{X} \rightarrow B \times X'$ is $Id_B \times f$. Finally, Lemma 3.6 asserts that the morphism $\tilde{\varphi}$ is the product of $Id_B$ with a morphism $Y \rightarrow \tilde{X}$.

Theorem 3.8. Let $B$ be a projective integral scheme, $X$ a piecewise quasiprojective scheme and $Z$ a closed subscheme of $B \times X$. Then the straight blow up section family $\tilde{X} \rightarrow X$ of the projection $B \times X \rightarrow B$ along $Z$ exists.

Proof. By Theorem 3.7, the blow up section family $(X', b')$ of $B \times X \rightarrow B$ along $Z$ exists.

The morphism $b' : B \times X' \rightarrow B \times X$ is not necessarily the product of a morphism $X' \rightarrow X$ with $Id_B$. By Proposition 2.2, such a morphism $X' \rightarrow X$ exists if and only if the morphism $f$ composition of $b'$ and $B \times X \rightarrow X$ is constant along the fibres of the projection $B \times X' \rightarrow X'$. By Theorem 2.7, the $f$-constify closed subscheme $X''$ of $X'$ exists and the restriction of $b'$ to $B \times X''$ is the product of a morphism $b'' : X'' \rightarrow X$ with $Id_B$.

Now, the pullback $(Id_B \times b'')^{-1}(Z)$ is a locally principal subscheme of $B \times X''$. By Proposition 1.15, there is a closed embedding $\iota : \tilde{X} \rightarrow X''$ such that $Id_B \times \iota$ is the blow up of $B \times X''$ along $(Id_B \times b'')^{-1}(Z)$. Now $b = (b'' \circ \iota) : \tilde{X} \rightarrow X$ is the straight blow up section family of $B \times X \rightarrow B$ along $Z$.

\[ \]  

4 Universal 2-relative clusters family

Fix a morphism $\pi : S \rightarrow B$ with $S$ piecewise quasiprojective and $B$ projective and integral. So, its universal section family exists and we denote it by $(X, \psi)$ (c.f. Theorem 3.3). Here we present the final goal of this paper, namely the construction of the universal $(r + 1)$-relative cluster section family $Cl_{r+1}$ of $\pi$ from $Cl_r \times Cl_{r-1} \times Cl_r$. We restrict to the case $r = 1$, that is $Cl_2$ from $X \times X$, which is the inductive step for the whole construction.

The following is a preliminary proposition. We leave its proof to the reader.

Proposition 4.1. Let $S_0$ be the scheme $S \times B$, $\pi_0 : S_0 \rightarrow B \times X$ the morphism $\pi \times Id_X$ and $p : B \times X \rightarrow B$ the projection. Let $\psi_0 : B \times X \times X \rightarrow S_0$ be the morphism $(\psi \times Id_X) \circ \iota$ where $\iota : B \times X \times X \rightarrow B \times X \times X$ is the automorphism that twists the second and third factors. Then the universal section family of $\pi_0$ is $(X, \psi_0)$ and the universal section family of $p \circ \psi_0$ is $(X \times X, \psi_0)$. Now, fix the following notation. The blow up $S_1$ of $S_0$ along the image $\Delta$ of the section $(\psi \times Id_X) : B \times X \rightarrow S \times X$ of $\pi_0$, the exceptional divisor $E$ in $S_1$ and the morphism $\pi_1 : S_1 \rightarrow B \times X$ composition of the blow up morphism $E$.
\( \mathcal{S}_1 \rightarrow \mathcal{S}_0 \) and \( \pi_0 \). In addition, denote by \( q_1 : \mathcal{S}_1 \rightarrow X \) the composition of the blow up morphism and the projection \( \mathcal{S}_0 \rightarrow X \). By [2, §2] and [2, Corollary 3.10.2], there is a stratification of \( X \times X \) such that every irreducible component of \( \mathcal{C}_2 \) is either (a) birational to a stratum or (b) composed entirely of clusters whose second section is infinitely near to the first. Theorem 4.10 below asserts that the straight blow up section family \((X', b)\) of the projection \( B \times X \times X \rightarrow B \) along \((\psi_0)^{-1}(\Delta)\) (c.f. Theorem 3.8) is the union, with its non-necessarily reduced structure, of the kind (a) irreducible components of \( \mathcal{C}_2 \).

To finish this section we show that the kind (b) irreducible components of \( \mathcal{C}_2 \), the ones missing in the straight blow up section family construction, may only emerge from the universal section family of \( E \rightarrow B \), see Theorem 4.12.

**Notation 4.2.** We denote the universal section family of \((p \circ \pi_1) : \mathcal{S}_1 \rightarrow B \) by \((X_1, \psi_1)\) (c.f. Theorem 3.3).

We denote by \( b' : B \times X' \rightarrow \mathcal{S}_1 \) the unique morphism whose composition with the blow up morphism \( \mathcal{S}_1 \rightarrow \mathcal{S}_0 \) is equal to \( \text{Id}_B \times b \).

**Definition 4.3.** A 2-relative cluster family of \( \pi \) is a section family \((W, \theta)\) of \((p \circ \pi_1) : \mathcal{S}_1 \rightarrow B \) such that the morphism \( q_1 \circ \theta \) is constant along the fibres of the projection \( B \times W \rightarrow W \).

A universal 2-relative cluster family of \( \pi \) is a 2-relative cluster family \((\mathcal{C}_2, \rho)\) of \( \pi \) that satisfies the following universal property. For every 2-relative cluster family \((W, \theta)\) of \( \pi \), there is a unique morphism \( f : W \rightarrow \mathcal{C}_2 \) such that \( \theta = \rho \circ (\text{Id}_B \times f) \).

Notice that Definition 4.3 is simpler than [2, Definition 2.19, p.10] because we use the existence of the universal section family of \( \pi \), but they are equivalent. When a universal 2-relative cluster family of \( \pi \) exists, by abstract nonsense it is unique up to unique isomorphism.

Recall that by Proposition 2.2 for every 2-relative cluster family \((W, \theta)\) of \( \pi \), the morphism \( q_1 \circ \theta : B \times W \rightarrow X \) is constant along the fibres of the projection \( B \times W \rightarrow W \) if and only if there is a morphism \( W \rightarrow X \) such that \( q_1 \circ \theta \) commutes with the composition \( B \times W \rightarrow W \rightarrow X \).

**Theorem 4.4.** The \((q_1 \circ \psi_1)-\text{constfy} \) closed subscheme \( X_1^c \) of \( X_1 \) exists and, setting \( \psi^c = \psi_1|_{B \times X_1} \), the couple \((X_1^c, \psi^c)\) is the universal 2-relative cluster family of \( \pi : \mathcal{S} \rightarrow B \).

**Proof.** It follows immediately from the universal properties of the universal section family \((X_1, \psi_1)\) of \((p \circ \pi_1) : \mathcal{S}_1 \rightarrow B \) and of the \((q_1 \circ \psi_1)-\text{constfy} \) closed subscheme \( X_1^c \) of \( X_1 \).

**Proposition 4.5.** Let \( X_E \) be the closed subscheme of \( X_1 \) representing the functor \( \text{Iso}_{\mathcal{S}_1/B}^{\psi_1^{-1}(E)} \) and set \( \psi_E = \psi_1|_{B \times X_E} \). Then, the couple \((X_E, \psi_E)\) is the universal section family of \( E \rightarrow B \).

**Proof.** Clearly the couple \((X_E, \psi_E)\) is a section family of \( E \rightarrow B \), let us check that it satisfies the required universal property.

Let \((Y, \rho)\) be a section family of \( E \rightarrow B \). By the universal property of \((X_1, \psi_1)\) there is a morphism \( f : Y \rightarrow X_1 \) such that \( \rho = \psi_1 \circ (\text{Id}_B \times f) \). Then, by the transitivity of the Cartesian product and Lemma 1.5, the base change of \( \psi_1^{-1}(E) \rightarrow B \times X_1 \) by \( f : Y \rightarrow X_1 \) is an isomorphism. So, \((f : Y \rightarrow X_1) \in \text{Iso}_{\mathcal{S}_1/B}^{\psi_1^{-1}(E)}(Y) \) and
there is a unique morphism $g: Y \to X_E$ whose composition with $X_E \hookrightarrow X_1$ is $f$. Now, using that $E \hookrightarrow S_1$ is a monomorphism, it is straightforward to check that $\rho = \psi_E \circ (\text{Id}_B \times g)$.

\[ \square \]

**Notation 4.6.** Let $(W, \theta)$ be a 2-relative cluster family of $\pi$. Let $E_W$ be the pullback of $E \hookrightarrow S_1$ by $\theta$ (which is a locally principal subscheme of $B \times W$). We denote by $W'$ the closed subscheme of $W$ for which the closed embedding $B \times W' \hookrightarrow B \times W$ is the blow up of $B \times W$ along $E_W$ (c.f. Proposition 1.15).

**Notation 4.7.** Let $(W, \theta)$ be a 2-relative cluster family of $\pi$. We denote by $W_E$ the closed subscheme of $W$ representing the functor $\text{Iso}^E_{W/B}$ (c.f. Theorem 2.5).

For the particular case $W = X^\circ_E$, we simplify the notation $(X^\circ_E)^\prime$ by $X^\circ_E$ and $(X^\circ_E)_{E}$ by $X^\circ_E$. In addition we denote respectively by $\psi^\circ_E$ and $\psi_E$ the restrictions of $\psi: B \times X^\circ_E \to S_1$ to $B \times X^\circ_E$ and $B \times X^\circ_E$.

**Remark 4.7.1.** The scheme $X^\circ_E$ is isomorphic to the $(q_1 \circ \psi_E: B \times X^\circ_E \to X)$-constly closed subscheme of $X_E$, see Remark 2.7.1.

**Proposition 4.8.** The couple $(X^\circ_E, \psi^\circ_E)$ satisfies the following universal property. For all 2-relative cluster family $(W, \theta)$ of $\pi$ such that the pullback $\theta^{-1}(E)$ is an effective Cartier divisor of $B \times W$, there is a unique morphism $f: W \to X^\circ_E$ with $\theta = \psi^\circ_E \circ (\text{Id}_B \times f)$.

**Proof.** Let $(W, \theta)$ be such a 2-relative cluster family of $\pi$. By the universal property of $(X^\circ_E, \psi^\circ_E)$ (c.f. Theorem 4.4), there is a unique morphism $f: W \to X^\circ_E$ with $\theta = \psi^\circ_E \circ (\text{Id}_B \times f)$. By the universal property of the blow up $\text{Id}_B \times i: B \times X^\circ_E \hookrightarrow B \times X^\circ_E$, there is a unique morphism $\varphi: B \times W \to B \times X^\circ_E$ with $\text{Id}_B \times f = \varphi \circ (\text{Id}_B \times i)$. Finally, Lemma 3.6 asserts that the morphism $\varphi$ is the product of $\text{Id}_B$ with a unique morphism $W \to X^\circ_E$.

**Proposition 4.9.** Let $(W, \theta)$ be a 2-relative cluster family of $\pi$. Then, there are unique morphisms $g': W' \to X'$ and $g_E: W_E \to X^\circ_E$ such that $\theta|_{B \times W'} = \psi^\circ_E \circ (\text{Id}_B \times g')$ and $\theta|_{B \times W_E} = \psi^\circ_E \circ (\text{Id}_B \times g_E)$.

**Proof.** Let denote by $i$ the closed embedding $W' \hookrightarrow W$. Since the composition of $\theta|_{B \times W'}: B \times W \to S_1$ with the blow up $S_1 \to S_0$ is a section family of $\pi_0: S_0 \to B$, by Proposition 4.1, there is a unique morphism $h: W \to X \times X$ such that the corresponding diagram (3.1) commutes.

Since $E_W \hookrightarrow B \times W$ is also the pullback of $\psi_0^{-1}(\Delta) \hookrightarrow B \times X \times X$ by $\text{Id}_B \times h$, the pullback of $\psi_0^{-1}(\Delta)$ by $\text{Id}_B \times (h \circ i)$ is an effective Cartier divisor of $B \times W'$ and then such unique morphism $g: W \to X'$ exists by the universal property of $(X', h)$.

The base change of $E_W \hookrightarrow B \times W$ by $W_E \hookrightarrow W$ is an isomorphism and, via its inverse, $W_E$ is a section family of $E \to B$. Hence, first by the universal property of $(X_E, \psi_E)$ and second by the universal property of the $(q_1 \circ \psi_E)$-constly closed subscheme $X^\circ_E$ of $X_E$ (c.f. Remark 4.7.1), such morphism $g_E: W_E \to X^\circ_E$ exists.

**Remark 4.9.1.** Let $(W, \theta)$ be a 2-relative cluster family of $\pi$. If $E_W \hookrightarrow B \times W$ is an effective Cartier divisor, then $W = W'$. 

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Theorem 4.10. The couple \((X', b')\) satisfies the same universal property of \((X_1^c, \psi_1^c)\) (c.f. Proposition 4.8).

Proof. If follows from Proposition 4.9 and Remark 4.9.1.

Corollary 4.10.1. The scheme \(X'\) is a closed subscheme of \(X_1^c\) and \(X_1\).

Proposition 4.11. Let \((W, \theta)\) be a 2-relative cluster family of \(\pi\) with \(W\) integral. Then the scheme \(W\) is equal to either \(W'\) or \(W_E\).

Proof. Since \(W\) and \(B\) are integral, the locally principal closed subscheme \(E_W\) of \(B \times W\) is either an effective Cartier divisor or isomorphic to \(B \times W\). So, the claim follows from Remark 4.9.1 or Remark 2.5.1.

Theorem 4.12. Let \((W, \theta)\) be a 2-relative cluster family of \(\pi\). The scheme \(W_{\text{red}}\) is a closed subscheme of \(X'\cup W_E\) of \(W'\) and \(W_E\). In particular, the underlying topological spaces of \(X'\cup X_E^c\) and \(X_1^c\) are homeomorphic.

Proof. By Proposition 4.11 every irreducible component of \(W\), with its reduced structure, is a closed subscheme of either \(W'\) or \(W_E\).

5 Examples

In this section, we recover two classic constructions, the classic blow up (see Proposition 5.1) and an example of a small resolution, both as particular cases of the straight blow up section family.

We also present an example showing that the straight blow up section family may also behave quite different from such classic constructions, namely, the dimension of the ambient scheme may decrease.

5.1 The classic blow up. The following proposition shows the classic blow up as a particular case of the straight blow up section family. We leave its proof to the reader.
Proposition 5.1. Let $Y'$ be a closed subscheme of $Y$ and $b: \tilde{Y} \to Y$ the blow up of $Y$ along $Y'$. Then, the straight blow up section family of $X \times Y' \to X \times Y \to X$ is $(\text{Id}_X \times b): X \times \tilde{Y} \to X \times Y$. In particular, when $X = \text{Spec}(k)$, the straight blow up section family agrees with the classic blow up.

5.2 The dimension may decrease. We show an example of the straight blow up section family where an irreducible ambient space breaks down into two irreducible components and the dimension of one of them decrease by one.

First let us fix the following construction of the quasiprojective variety $U_n$. Let $S$ denote the polynomial ring $k[u,v]$ and $S_n$ the $k$-vector space of degree $n$ forms in $S$. Given a $k$-vector space $V$ we denote by $PV$ its corresponding projective space. So, we define $U_n \subseteq \mathbb{P}(S_n \times S_n \times S_n)$ as the quasiprojective variety corresponding to triplets of forms with no common roots.

Consider $S = \mathbb{P}^1_{u,v} \times \mathbb{P}^2_{x,y,z}$ and $Z \subseteq S$ the graph of $[u : v] \in \mathbb{P}^1 \to [u : v : 0] \in \mathbb{P}^2$, that is $Z = V_+(z, ex - uy)$. Now, we will replicate the construction of the straight blow up section family in Theorem 3.8 for the projection $S \to \mathbb{P}^1$ along $Z$.

The blow up $\tilde{S}$ of $S$ along $Z$ may be given globally as the closed subvariety

$$V_+(ax + by + cz, au + bv)$$

of $\mathbb{P}^2_{a,b,c}$. Consider the disjoint union $X$ for all pair of integers $n,m$ of the closed subvarieties $X_{n,m}$ of $U_n \times U_m$ determined by the equations on the coefficients given by the identities of polynomials,

$$AP + BQ + CR \equiv 0 \quad (5.1)$$

$$Au + Bv \equiv 0 \quad (5.2)$$

where $[P : Q : R] \in U_n$ and $[A : B : C] \in U_m$. Observe that these polynomial identities determine $(n + m + 1) + (m + 2)$ equations on the coefficients.

Now, the morphism $\psi: \mathbb{P}^1 \times X \to \tilde{S}$ that maps $(t, F) \in \mathbb{P}^1 \times X$ to $(t, F(t)) \in \tilde{S}$ (where $F(t)$ denotes the evaluation) is well defined, and the couple $(X, \psi)$ is the universal section family of $\tilde{S} \to \mathbb{P}^1$.

Let $p: S \to \mathbb{P}^2$ be the composition of the blow up $\tilde{S} \to S$ and the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$. So, the $(p \circ \psi)$-constfied closed subvariety $X'$ of $X$ is the disjoint union of the $X_{0,m}$ for all integer $m$.

Now we show that the varieties $X_{0,m}$ are empty for all $m > 1$ and we describe the cases $m = 0, 1$.

First note that $U_0 \cong \mathbb{P}^2$, so let us denote its coordinates by $[p : q : r] \in U_0$.

Assume that $m > 0$ and restrict to the affine open chart $r \neq 0$. By (5.2) there are $A', B' \in S_{m-1}$ with $A = -vA', \ B = uB'$ and $A' = B'$. So, (5.1) becomes $(qu - pv)B' + rC = 0$ and there is $C' \in S_{m-1}$ with $rC = -(qu - pv)C'$ and $B' = C'$. Therefore $A' = B' = C'$ but they can not share a single root, that is either $A' = B' = C' = \alpha \in k$ and $m = 1$ or $X_{0,m}$ is empty. By a similar discussion, for all $m > 0$, there are no points in $X_{0,m}$ with $r = 0$. So, setting $L = V(r) \subseteq U_0$, the variety $X_{0,1} \subseteq U_0 \times U_1$ is the image of the closed embedding

$$(p,q) \in k^2 \cong U_0 \setminus L \overset{\sim}{\longrightarrow} \{ p : q : 1 \}, \ [-v : u : qu - pv] \in X_{0,1}.$$ 

Finally assume $m = 0$. By (5.2), $A = B = 0$. So we may assume $C = 1$ and (5.1) becomes $r = 0$. Then simply

$$X_{0,0} = L \times \{ [0 : 0 : 1] \} \cong \mathbb{P}^1_{p,q}.$$
Hence, $X' = \mathbb{A}^2 \sqcup \mathbb{P}^1$ is the stratification of $\mathbb{P}^2$ by the standard affine chart $\mathbb{P}^2 \setminus V(L)$ and $V(L)$. The morphism $\psi|_{\mathbb{P}^1 \times X'} : \mathbb{P}^1 \times X' \to \tilde{S}$ sends $[[u : v], x'] \in \mathbb{P}^1 \times X'$ to

\[
\begin{cases}
[[u : v], [p : q : 1], [-v : u : qu - pv]] & \text{if } x' = (p, q) \in \mathbb{A}^2 \\
[[u : v], [p : q : 0], [0 : 0 : 1]] & \text{if } x' = [p : q] \in \mathbb{P}^1
\end{cases}
\]

and its composition with the blow up morphism $\tilde{S} \to S$ is just the product of $\text{Id}_{\mathbb{P}^1}$ with the natural morphism $b : X' \to \mathbb{P}^1$. Now, the pullback of $Z = V(z, vx - uy)$ by $\text{Id}_{\mathbb{P}^1} \times b$ is the empty set on $\mathbb{P}^1 \times \mathbb{A}^2$ and the diagonal on $\mathbb{P}^1 \times \mathbb{P}^1$, both an effective Cartier divisor, so $(X', b)$ is the straight blow up section family of $S \to \mathbb{P}^1$ along $Z$.

5.3 Small resolution. We present an example where the straight blow up section family along a natural centre becomes a small resolution. It indicates the possibility that the straight blow up section family would offer a procedure to systematise small resolutions.

Similarly to §5.2, we define for every integer $n$ the quasiprojective variety $V_n \subseteq \mathbb{P}(S_n \times S_n)$ corresponding to pairs of degree $n$ forms with no common roots.

Consider the variety $\mathbb{A}^4$ parametrising matrices

\[
M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}
\]

and the closed subvariety $D \subseteq \mathbb{A}^4$ where the rank of $M$ is not maximal, or equivalently where the determinant of $M$ is zero. Consider the variety $S = \mathbb{P}^1_{u,v} \times D$ and its incidence subvariety

\[
Z = \{ ([\lambda], M) \in S : MX^t = 0 \}.
\]

It is a classic result that the projection $S \to D$ restricted to $Z$ is an small resolution of $D$. It turns out that the straight blow up section family of the projection $S \to \mathbb{P}^1$ along $Z$ is isomorphic to $Z$ and then again an small resolution of $D$. We replicate the construction of the straight blow up section family in Theorem 3.8 for the projection $S \to \mathbb{P}^1$ along $Z$.

The blow up $\tilde{S}$ of $S$ along Z may be given globally by the equations $xa - zb$ and $ya - wb$ in $S \times \mathbb{P}^1_{a,b}$.

Now, we describe the closed subvariety of the universal section family of $\tilde{S} \to \mathbb{P}^1$ corresponding to “constfy” by $\tilde{S} \to D$. Clearly, it is the disjoint union $X$ for all integers $n$ of the closed subvarieties $X_n$ of $D \times V_n$ determined by the equations on the coefficients given by the identities of polynomials,

\[
xA - zB \equiv 0 \\
yA - wB \equiv 0
\]

where $[A : B] \in V_n$. The resulting morphism $b' : X \to D$ is for each component $X_n$ the composition of the closed embedding $X_n \to D \times V_n$ and the projection $D \times V_n \to D$. 

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It is straightforward to see that given \((x, y, z, w), [A : B] \in X_n\) either the forms \(A, B\) are constants or \((x, y, z, w) = 0\). That is,

\[
X = X_0 \coprod \left( \coprod_{n \geq 1} \{0\} \times V_n \right)
\]

where \(X_0 \cong Z\). So, the pullback \((\text{Id}_{P^1} \times b')^{-1}(Z)\) is an effective Cartier divisor in \(P^1 \times X_0\) and the whole \(P^1 \times X_n\) for all \(n \geq 1\). Hence the blow up of \(P^1 \times X\) along the locally principal \((\text{Id}_{P^1} \times b')^{-1}(Z)\) is \(P^1 \times X_0\), and then the straight blow up section family of \(S \to P^1\) along \(Z\) is \(b'|_{X_0} : X_0 \to D\).

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