Adiabatic charge and spin transport in interacting quantum wires

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We study charge and spin transport through an interacting quantum wire, caused by backscattering off an effective impurity potential with a periodic time-dependence. The adiabatic regime of this pump for charge and spin is shown to depend on the presence of interactions in the wire. For the non-interacting case the transported quantities in a period are calculated and found to be adiabatic (independent of pumping frequency) for all frequencies $\Omega \ll v_F/\ell$, where $\ell$ is the range of the scattering potential, and $v_F$ is the Fermi velocity; this result is along the lines of adiabatic pumping in mesoscopic systems. On the other hand, we show that for a wire with repulsive electron-electron interactions the adiabatic regime is confined to $h\Omega \ll \omega_F$, with $\omega_F$ an energy scale set by the backscattering potential. Using symmetry and scaling properties of the quantum wire Hamiltonian we relate the charge $Q$ and spin $S$ transported through the wire in a period $2\pi/\Omega$ to an integral involving quasi-static backscattering conductances. We also show that the pumped charge $Q$ (or the spin $S$) is quantized in the adiabatic limit if the conductance of the system is zero at the stable fixed point of the renormalization-group (RG) transformations. The quanta transported are given by the winding number of a complex coupling $\Gamma(\Omega t)$ as it traverses a closed path $C$ enclosing the origin $\Gamma = 0$. No adiabatic charge or spin is transported when the RG fixed point corresponds to a perfectly conducting wire, or if the path $C$ does not enclose the point $\Gamma = 0$. By contrast, for a RG marginal conductance – which is the case for non-interacting electrons – the charge transported in a cycle is not quantized, rather it is proportional to the area enclosed by the path or circuit $C$, in agreement with previous studies of non-interacting mesoscopic systems. Finite size, temperature, and frequency effects on the transported quantities follow from the relation between adiabatic transport coefficients (backscattering conductances) and the quasi-static conductance of the system.

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I. INTRODUCTION

Quantum adiabatic charge transport through a system is achieved by a slow periodic modulation in time of a set of external potentials. This phenomenon has been investigated in one-dimensional non-interacting systems with gapped excitations (band insulators) [1,2], for which case the pumped charge $Q$ (charge transported through the system in a single period) was shown to be quantized. In the context of adiabatic transport through quantum dots, recent theoretical work on pumping [3,4] finds no such universal behavior for $Q$, which strongly depends on how the pump is operated. On the other hand, for gapless systems in one space dimension (1D Luttinger liquids), it is possible to pump weakly quantized charge and spin by the temporal variation of a spatially localized potential [5].

In our earlier work, Ref. [6], the weak quantization of adiabatically pumped charge or spin in a Luttinger liquid was argued to follow from the repulsive nature of electron-electron interactions. These arguments were given by solving exactly the special case of the spinless Luttinger model with interaction parameter $K = 1/2$ in the presence of an impurity with arbitrary time-dependence $\Gamma(\Omega t)$. Quantization of pumped charge was shown in the limit $\Omega \rightarrow 0$ with power law corrections $\Delta Q \sim (\Omega/\omega_T)^7$ determined by the barrier energy scale $\omega_T$. The quanta transported are given by the winding number of the complex backscattering amplitude $\Gamma$. Therefore, the pumped quantities are invariant under (small) continuous deformations of the path $C$, around the origin of the complex plane, traced by $\Gamma$.

In this paper, we prove the adiabatic quantization of the pumped charge and spin for gapless interacting systems using the properties of chiral symmetry and scale invariance. We derive a connection between the adiabatically pumped quantities and an integral involving the quasistatic (equilibrium) value of the conductance through the system. The quasistatic conductance is calculated using an instantaneous renormalized (because of bulk interactions) backscattering amplitude $\tilde{\Gamma}(t)$. For example, in its simplest form for a spinless system, the expression for the pumped charge is:

$$Q = \frac{1}{2\pi i} \int \frac{d\Gamma}{\Gamma} \left[1 - 2\pi G_F \right], \quad (1)$$

where $G_F$ is the equilibrium conductance of the one-dimensional wire with the instantaneous (renormalized) backscattering amplitude $\tilde{\Gamma}(t)$. Our derivation generically links quantization of adiabatic charge transport to insulating renormalization group (RG) fixed points of the interacting wire with an impurity. Thus, whenever the backscattering potential leads to insulating behavior ($G_F \rightarrow 0$) in response to an applied DC voltage, the pumped charge is perfectly quantized, albeit the time to transport such charge adiabatically is large (we discuss in this paper the frequency scales for adiabaticity). The quanta is given by the winding number (about the origin $\Gamma = 0$) of the backscattering amplitude $\Gamma$. For the particular case of non-interacting electrons, for which the
backscattering amplitude is marginal in the RG sense, the aforementioned quantization is lost, and we show that our result recovers the “area law” relation obtained by Brouwer.

We emphasize that the relation between the adiabatic charge and spin transported in a periodic cycle and the quasistatic conductance is rather general: it applies to systems with both short and long-range bulk interactions, and is derived here without taking recourse to bosonization. Within the bosonization approach, arguments have been put forth in Ref. that the quantization is a consequence of pinning the bosonic field at the impurity to a minimum of a single cosine potential when the backscattering potential is relevant; the displacement of the bottom of the potential after a complete cycle leads to the quantization condition. Here we show that the mechanism of pumping is based on controlling the spatially localized backscattering process that breaks a continuous global (chiral) symmetry of the bulk one-dimensional system, and point out the crucial role of interactions in the wire in determining the transported quantities.

The rest of this paper is organized as follows. In Section II we consider the case of charge pumping through a quantum wire of interacting spinless fermions with gapless excitations. We introduce the model in Section IIA and relate it to a time dependent single impurity model, with a (complex) coupling $\Gamma(t)$. In Section IIB we show how the temporal variation of the phase of $\Gamma$ can lead to the transport of charge $Q$ through an otherwise clean quantum wire. This is done by making use of the Ward identities which are derived in that section. In Section IIC we obtain a relation between the pumped charge in a cycle and the backscattering-current conductance of the wire with renormalized parameter. In Section IID we relate the pumped quanta to the usual conductance of the quantum wire. In Section III we consider electrons with spin and derive expressions for the spin ($\mathbf{S}$) and charge ($\mathbf{Q}$) transported per cycle by the spin and charge pumps. These expressions allow us to look at the limits of quantization and at the effect of finite temperature and finite size on the value of $Q$ and $S$, which is the content of Section IV. In Section V we solve the time-dependent impurity problem exactly for non-interacting electrons, and also present details of the calculations used in our earlier publication, Ref. for the spinless Luttinger model at the $K = 1/2$ point. We identify the adiabatic regime, and compare the results with the approach of Sections II and III. Finally, in Section VI we conclude with a summary of our main contributions.

II. TRANSPORT OF SPINLESS ELECTRONS BY PARAMETRIC PUMPING

A. Model Hamiltonian

An adequate low energy description of a system of 1D fermions is obtained by separating the Fermi field $\psi(x)$ into right (R) and left (L) moving chiral fields near the Fermi points $\pm k_F$

$$\psi(x, t) = e^{ik_F x} \psi_R(x, t) + e^{-ik_F x} \psi_L(x, t). \quad (2)$$

This turns a non-relativistic action into a relativistic Dirac fermion action in (1+1) space-time dimensions, with the Fermi velocity $v_F$ playing the role of the speed of light. Our notation assumes spinless fermions although, as we show later, it can be straightforwardly generalized to include spin. We consider such a system with a Hamiltonian $H$ that preserves the global chiral symmetry:

$$\Psi(x) \rightarrow e^{i\alpha\gamma} \Psi(x), \quad \tilde{\Psi}(x) \rightarrow \tilde{\Psi}(x)e^{i\alpha\gamma}, \quad (3)$$

where

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad \tilde{\Psi} = \psi_L^\dagger \gamma^0 = (\psi_L^\dagger, \psi_R^\dagger). \quad (4)$$

Here the $\gamma$-matrices are $\gamma^{0,1} = \sigma_x, \gamma^i = i\gamma^0\gamma^i = -\sigma_z$, with $\sigma_x, \gamma_z$ being the Pauli matrices; and we choose $v_F = \epsilon = \hbar = 1$. The chiral symmetry ensures that the axial charge $\int dx \, j = \int dx \, \tilde{\Psi}\gamma^0\gamma_5\Psi$ is conserved in the absence of an external electromagnetic field (“chiral anomaly”).

Adding a short range impurity potential $V(x) - \text{non-zero}$ only in a finite range of length $\ell \ll L$, the length of the wire – gives rise to an additional term in the Hamiltonian $H$:

$$H_{imp} = \int dx \, V(x)\psi_L^\dagger(x)\psi_L(x) + H.c. \quad (5)$$

Here the first term describes the backscattering of right movers into left movers and vice versa, while the second term describes forward scattering of both right and left movers with identical scattering amplitude and phase shift. The idea of parametric pumping in a quantum wire is to generate a current by varying parameters that control the “shape” of this scattering potential. Since the current (axial charge $j$) involves the difference in the number of right and left movers in the wire, the forward scattering term, which does not distinguish between the right and left movers, plays no role in generating this current. It is only the backscattering terms which can lead to a non-zero current when the potential $V(x)$ is suitably manipulated. Thus, for the purpose of determining the pumped current at low energies (when the continuum field theory description holds), we can write down an effective Hamiltonian which describes the most relevant (in the RG sense) backscattering processes. Thus the impurity contribution to $H$ can be written as a local $(x = 0)$
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change sign in a cycle. For example, then it amounts to varying both the phase function potential with out of phase periodic functions \( \phi \) and \( \pi \) phase is counting (in natural units, with \( \hbar = e = v_F = 1 \)) the coupling \( \Gamma \) has zero dimension.

If we vary the “shape” of the potential \( V(x) \) in time, then it amounts to varying both the phase \( \phi \) and amplitude of \( \Gamma \) in time. This can be understood qualitatively as follows (see also Fig. 1). Consider a time \( t = 0 \) in Fig. 1 when the potential profile is symmetric about \( x = 0 \), then the Fourier transform is real and the phase \( \phi = 0 \). At a later time, when the profile is anti-symmetric, the phase is \( \phi = \pi/2 \), and then, as the potential changes sign, \( \phi = \pi \) until the potential regains its original shape and the phase goes to \( \phi = 2\pi \). Thus, by changing the potential profile, one gets an effective time dependent delta function potential \( |\Gamma(t)|e^{i\phi(t)} \delta(x) \).

1. Double barriers: a realization of \( \Gamma(t) \)

In the pumping device proposed in Ref. 1, the changing \( V(x,t) \) of Fig. 1 is realized by oscillating two potential barriers placed some distance apart:

\[
V(x,t) = V_0^-(x)f_-(t) + V_0^+(x)f_+(t),
\]

with out of phase periodic functions \( f_+ \) and \( f_- \) which change sign in a cycle. For example, \( f_-(t) = \cos(\omega t) \) and \( f_+(t) = \cos(\omega t + \theta) \). We identify

\[
\Gamma(t) = \hat{V}_0^-(2k_F)f_-(t) + \hat{V}_0^+(2k_F)f_+(t),
\]

where

\[
\hat{V}_0^- = \int dx e^{i2kFx} V_0^-(x).
\]

If \( V_0^-(x) \) is asymmetrical about \( x = 0 \), then the Fourier transforms \( \hat{V}_0^-(x) \) are not real and as a result \( \Gamma(t) \) is complex. Furthermore, if the functions are different so that \( \text{Arg}[\Gamma_0^-] \neq \text{Arg}[\Gamma_0^+] \) then by varying the amplitudes and the signs of the complex quantities \( f_-(\pm)\hat{V}_0^- \) (accomplished by varying \( f_\pm \)) one can reach any point in the complex \( \Gamma \) plane. Thus, by choosing appropriate functional forms \( f_\pm(t) \), one can trace any desired path in the complex \( \Gamma \) plane (as in Fig. 2).

2. Special case: simple form for \( V(x,t) \)

We choose delta function forms for \( V_0^-(x) \) and \( V_0^+(x) \):

\[
V_0^-(x) = \gamma^- \delta(x + \ell/2), \quad V_0^+(x) = \gamma^+ \delta(x - \ell/2)
\]

to obtain:

\[
V(x,t) = \gamma^- \delta(x + \ell/2) \cos \omega t + \gamma^+ \delta(x - \ell/2) \cos(\omega t + \theta),
\]

which is of the form \( \Gamma(t) = \Gamma_+ e^{i\omega t} + \Gamma_- e^{-i\omega t} \), where \( \Gamma_\pm = V_0^+ e^{-ikF \ell} \pm V_0^- e^{ikF \ell} \). Written in this way, the time dependence of \( \Gamma \) can be distinguished from that obtained by applying a voltage – when only one of \( \Gamma_\pm \) is present, as in Ref. 4.

The phase \( \phi \) of the complex valued parameter \( \Gamma \) is time dependent whenever \( 2k_F \ell \neq n\pi \) and \( \theta \neq n\pi \) \((n = 0, \pm 1, \ldots)\). The first condition is the off-resonance condition, essential for backscattering to occur, while the second is essential for traversing a closed path in the \( \Gamma \) plane with a non-zero area. The rate of change of this phase, \( \dot{\phi} \), can be written as:

\[
\dot{\phi}(t) = \frac{\omega \gamma^- \gamma^+ \sin \theta \sin 2k_F \ell}{X^2(t)},
\]

where

\[
X^2(t) = \cos^2 k_F \ell \left[ \gamma^+ \cos(\omega t + \theta) + \gamma^- \cos(\omega t) \right]^2 + \sin^2 k_F \ell \left[ \gamma^+ \cos(\omega t + \theta) - \gamma^- \cos(\omega t) \right]^2.
\]

It is clear from this expression that \( \dot{\phi}(t) \) does not change sign with time, and thereby takes the complex quantity \( \Gamma \) around a contour, as shown in Fig. 2.
B. Pumping parameter for the charge pump

In the rest of this section we will show that the phase $\phi$ of the complex quantity $\Gamma$ is the pumping parameter. The total Hamiltonian is:

$$\mathcal{H} = H + H_\Gamma.$$  

The backscattering term breaks the continuous chiral symmetry, and thus gives rise to the backscattering current

$$J_\Gamma(t) = \frac{1}{2} \int dx \partial_t \tilde{j}(t, x)$$

$$= \frac{\Gamma}{2} \int dx [\tilde{j}(t, x), H_\Gamma] = iQ + \text{H.c.},$$

which measures the rate of change of total charge of the right movers in the system due to the backward scattering of the chiral fermions. It is important to note that we do not consider, in $\mathcal{H}$, the forward scattering term arising from the impurity potential. This is because forward scattering involves local fermion density and therefore conserves the chiral charge.

In order to look at the equilibrium expectation value of $\mathcal{H}$ we consider the functional

$$Z[\Gamma] = \langle \mathcal{F}(\infty, -\infty) \rangle,$$  

with a constant (other than the adiabatic switching on and off) source term $H_\Gamma$ that connects the vacuum state of $H$ at times $t = -\infty$ with the same state at $t = \infty$. As a consequence of the global chiral symmetry a change of variables, as in $\mathcal{H}$, with an infinitesimal $\alpha$ leaves the Hamiltonian $H$ invariant. The only change comes from the source term: $H_\Gamma \rightarrow H_\Gamma + \alpha J_\Gamma$. This implies:

$$0 = \delta Z[\Gamma] = -i\alpha \int_{-\infty}^{\infty} dt \langle J_\Gamma(t) \rangle_{\Gamma}$$

where $\langle \cdots \rangle_{\Gamma}$ denotes the expectation value in equilibrium with a non-zero $\Gamma$. Since an equilibrium expectation value should be time-independent, it follows that $\langle J_\Gamma(t) \rangle_{\Gamma} = 0$.

When $\Gamma \rightarrow \Gamma(t)$ the Hamiltonian $\mathcal{H} \rightarrow \mathcal{H}[\Gamma(t)]$, and the system is driven out of equilibrium. Then the instantaneous backscattering current can be non-zero. In order to calculate this current we introduce the generating functional for a closed time path (CTP) $\mathcal{Z}[\Gamma^+]$.

$$\mathcal{Z}[\Gamma^+, \Gamma_-] = \langle 0 | \mathcal{F}(\infty, -\infty, 0) S_{\Gamma^+}(\infty, -\infty) 0 \rangle |\Gamma(t) \rangle,$$

where $|0\rangle$ is the state at time $t = -\infty$ when the source $\Gamma(t)$ is adiabatically switched on. The $S$-matrix

$$S_{\Gamma^+}(\infty, -\infty) = T \exp \left\{ -i \int_{-\infty}^{\infty} dt \left\{ H + H_{\Gamma^+}(t) \right\} \right\}$$

evolves the state $|0\rangle$ forward in time with a source $\Gamma^+(t)$, while $S_{\Gamma^-}(\infty, -\infty)$ evolves this state backward in time with a source $\Gamma_-(t)$. Note that choosing $\Gamma_+ = \Gamma_-$ for all times makes $\mathcal{Z}[\Gamma] = 1$. We also note here that our choice of a single initial state is valid only for the zero temperature formalism. In general a non-zero temperature (or any other initial mixture of states) can be straightforwardly accommodated by including an additional $S$-matrix operator $S(\infty - i\beta, -\infty)$ and tracing over all initial states.

1. Ward-Takahashi Identities

Because of the global chiral symmetry of $H$, not all Green functions generated by the functional $\mathcal{Z}[\Gamma_\pm]$ are independent. General identities between Green functions of different orders of the local operator $Q$ can be derived by imposing the constraint that the generating functional, and any expectation value derived from it, is invariant under the global chiral transformation $\mathcal{H}$. This implies that:

$$0 = \frac{\delta}{i\delta \alpha} \langle Q(t) \rangle = \langle Q(t) \rangle - \int_{-\infty}^{t} dt' \langle [Q(t), J_\Gamma(t')] \rangle,$$

where the angular brackets denote expectation value of the Heisenberg operator at time $t$, and $J_\Gamma(t) = i\Gamma(t)Q + \text{H.c.}$. Note that there is no restriction on the time dependence of $\Gamma(t)$, so that these relations constitute non-equilibrium generalizations of the Ward-Takahashi (WT) identities derived in equilibrium field theory. In terms of operators defined earlier, the WT identities are:

$$\langle H_\Gamma(t) \rangle = -i \int_{-\infty}^{t} dt' \langle [J_\Gamma(t), J_\Gamma(t')] \rangle$$

$$\langle J_\Gamma(t) \rangle = i \int_{-\infty}^{t} dt' \langle [H_\Gamma(t), J_\Gamma(t')] \rangle.$$  

2. Pumped charge as a consequence of phase change of $\Gamma$

In order to calculate the pumped charge let us consider a time interval $[t_0, t_1]$, with $t_0 < t_1$, in which the pumping parameter can be written as $\Gamma(t) = \Gamma_0(t) + \delta \Gamma(t)$, where $\delta \Gamma(t)$ vanishes smoothly outside the interval. We can now write our Hamiltonian as $\mathcal{H}_0 + \delta \mathcal{H}_\Gamma$, where $\delta \mathcal{H}_\Gamma$ vanishes with the perturbation $\delta \Gamma(t)$ outside the chosen time interval. Then the current generated by backscattering can be written in the interaction picture with respect to the Hamiltonian $\mathcal{H}_0$:

$$I(t) = \langle S^\dagger(t, t_0) J_\Gamma(t) S(t, t_0) \rangle.$$  

where all operators are in the interaction picture with respect to $\mathcal{H}_0$, and $J_\Gamma(t) = i\Gamma(t)Q + \text{H.c.}$. The $S$-matrix in the interaction picture is:

$$S(t, t_0) = Te^{-i \int_{t_0}^{t} dt' \delta \mathcal{H}_\Gamma(t')}.$$
Here $T$ stands for time ordering, and $\delta H_{\Gamma}(t)$ is the time-dependent perturbation in the interaction representation.

As a consequence of global chiral invariance, when the pumping parameter $\Gamma(t)$ has a constant phase $\phi_0$ in the time interval $[t_i, t_f]$ (where the system is in an equilibrium state for times $t < t_i$) then the pumped charge in that interval

$$i \frac{\partial}{\partial \phi_0} Z[\Gamma^(+), \Gamma(-)] = \int_{t_i}^{t_f} dt \left\langle J_{\Gamma_0}(t) \right\rangle = 0, \quad (20)$$

Therefore, a phase change is the single important pumping parameter. Notice that in order to achieve a time-dependent phase we need to vary a minimum of two experimental parameters.

C. Charge pump: backscattering conductance

Consider now a change in the phase of $\Gamma(t)$ as a perturbation:

$$\Gamma(t) = |\Gamma(t)| e^{i\phi(t)},$$

$$\delta \phi(t) = \phi(t) - \phi(t_0), \quad \text{and}$$

$$\Gamma_0(t) = |\Gamma(t)| e^{i\phi(t_0)}.$$  \hspace{1cm} (21a)

Then, the perturbation parameter can be explicitly written as:

$$\delta \Gamma(t) = \Gamma_0(t) \left[ \cos \delta \phi(t) + i \sin \delta \phi(t) - 1 \right]. \quad (21b)$$

Therefore, for a small enough change $\delta \phi(t)$ in the interval $[t_0, t_1]$, we can evaluate (18) perturbatively to first order in $\delta \phi(t)$:

$$I(t) = \left\langle J_{\Gamma_0}(t) \right\rangle - \delta \phi(t) \left\langle H_{\Gamma_0}(t) \right\rangle$$

$$- i \int_{t_0}^{t} dt' \delta \phi(t') \left\langle [J_{\Gamma_0}(t), J_{\Gamma_0}(t')] \right\rangle, \quad (22a)$$

where all operators are in the Heisenberg representation with respect to the unperturbed Hamiltonian $H[\Gamma_0]$.

Using the WT identities (73)-(77) in (22) we obtain:

$$I(t) \simeq \left\langle J_{\Gamma_0}(t) \right\rangle + i \int_{t_0}^{t} dt' \delta \phi(t')$$

$$\times \int_{-\infty}^{t'} dt'' \left\langle [J_{\Gamma_0}(t), J_{\Gamma_0}(t'')] \right\rangle, \quad (22b)$$

The pumped charge in a small time interval $[t_0, t_1]$ can be written as:

$$\delta Q(t_1; t_0) = i \int_{t_0}^{t_1} dt \int_{t_0}^{t} dt' \delta \phi(t')$$

$$\times \int_{-\infty}^{t'} dt'' \left\langle [J_{\Gamma_0}(t), J_{\Gamma_0}(t'')] \right\rangle, \quad (23)$$

where the first terms contribution vanishes due to (20). From the above expression we see that the role of WT identities is to ensure that a non-zero pumped charge is a consequence of temporal variation in the phase of the backscattering amplitude. We note that in obtaining Eq. (23) we have explicitly shown that $\phi(t)$ is the pumping parameter.

The approach introduced above can be used to find the current at any other time by using the following scheme. We divide the entire path traversed by $\Gamma$ into a sequence of $N$ sub-intervals, as shown in Fig. 3, starting from the equilibrium state at time $t = t_0$ with a coupling $\Gamma(t_0) = \Gamma$, and returning to the same value at $t_N = T$. The length of the $n$-th interval $[t_n, t_{n+1}]$ is chosen such that (for $t \in [t_n, t_{n+1}]$) the phase change $\delta \phi(t) = \phi(t) - \phi(t_n)$ is small enough for the linear approximation to hold. The current $I(t)$ for $t \in [t_n, t_{n+1}]$ is evaluated by going to the interaction representation with respect to the Hamiltonian $H[\Gamma_n] = H + H_{\Gamma_n}$. Here $H_{\Gamma_n}$ has a time-dependent coupling $\Gamma_n(t)$ that has a constant phase in the interval $t \in [t_n, t_{n+1}]$.

We can therefore write the current as:

$$I(t) = \left\langle S(t, t_n) J_{\Gamma}(t) S(t, t_n) \right\rangle,$$

where

$$S(t, t_n) = T e^{-i \int_{t_n}^{t} dt' \delta H_{\Gamma}(t')}. \quad (24)$$

Here $\delta H_{\Gamma}(t)$ vanishes for $t \neq [t_n, t_{n+1}]$. Within this linear response approximation we can write the expression for the total charge pumped in a cycle as a sum of contributions arising from each of the intervals:

$$Q = \sum_{n=0}^{N} \int_{t_n}^{t_{n+1}} dt \ I(t) = \sum_{n=0}^{N} \delta Q(t_{n+1}; t_n) \quad (25)$$

Because the various time intervals differ only in the value of the parameters that characterize the initial time of each interval, it is sufficient to calculate the charge pumped in any one time interval. Contributions from all other intervals can be obtained by appropriate relabeling of these parameters. Thus, for example, the charge pumped in a time interval $[t_0, t_1]$ is obtained from (22b), after exchanging the order of $t$ and $t'$ integrations:

$$\delta Q(t_1) = \int_{t_0}^{t_1} dt' \delta \phi(t') \int_{t_0}^{t_1} dt \int_{-\infty}^{t} dt'' K^{(R)}_0(t; t'') \quad (26)$$

where the retarded backscattering current-current correlator for $t \in [t_0, t_1]$ is:

$$K^{(R)}_0(t; t') = i \theta(t - t') \left\langle [J_{\Gamma_0}(t), J_{\Gamma_0}(t')] \right\rangle. \quad (27)$$

The charge pumped in any other interval can now be found by substituting $\Gamma_0 \leftrightarrow \Gamma_n$, and $\delta \Gamma(t) \leftrightarrow \delta \Gamma_n(t)$. Therefore the total charge pumped in a cycle can be expressed as:

$$Q = \sum_{n=0}^{N} \int_{t_n}^{t_{n+1}} dt' \delta \phi(t') \int_{t_n}^{t_1} dt \int_{-\infty}^{t} dt'' K^{(R)}_n(t; t''). \quad (28)$$
The function multiplying \( \delta \phi(t') \) is to be thought of as a generalized non-equilibrium conductance \( \mathcal{G} \) for the backscattering current. We write the total charge pumped in a cycle as:

\[
Q = \sum_{n=0}^{N} \int_{t_{n-1}}^{t_{n}} dt' \frac{1}{2\pi} \delta \phi(t') G_n(t'),
\]

(29)

where \( F_n(t-t') \) has been introduced as a smooth cut-off function that vanishes as \( t \to t_n \). In general, the quantity \( G_n \) is dependent on the precise form of \( F_n \). However, when the contribution from the end points is negligible compared to the contribution from the rest of the time interval \( t \in (t_n, t_{n-1}) \), we can expect some interval-dependent behavior. We therefore introduce a simple form for the cut-off function, and write:

\[
G_n = 2\pi \int_{t'}^{t} F_n(t-t') dt \int_{-\infty}^{t'} dt'' \mathcal{K}^{(R)}_n(t; t''),
\]

(30)

where \( \omega_n \to 0^+ \) in the aforementioned limit. Next, we note that the retarded correlator \( \mathcal{K}^{(R)}_n \) is in general not time translation invariant, because of the time-dependent coupling \( \Gamma(t) \). The function \( \int_{-\infty}^{t'} dt'' \mathcal{K}^{(R)}_n(t; t'') \) has a maximum along \( t'' = t \). For non-interacting electrons, because of the singular nature of the fermionic Green function, the retarded correlator \( \mathcal{K}^{(R)}_n(t; t'') \) has a maximum near \( t'' \approx t + 1/\Lambda_0 \), where \( \Lambda_0 \) is the upper cut-off of the problem, \( \lambda_0 \leq E_F \) (see Section [X] for details). Also the width of this maximum \( \delta t^* \approx 1/\Lambda_0 \). As a result, for a frequency \( \Omega \ll \Lambda_0 \) of variation of \( \Gamma \), we can treat the time-dependent parameters \( \Gamma(t') = \Gamma(t) = \Gamma(t'') \), with an error of the order of \( \Omega/\Lambda_0 \ll 1 \). Keeping in mind the other energy scale in the problem (the inverse length scale of the barrier, \( \Lambda_0 \), which defines the validity of single impurity approximation), the adiabatic limit is \( \Omega \ll \min[\Lambda_0, \Lambda_0] \). In this limit we can replace the time-dependent Hamiltonian \( \mathcal{H}[\Gamma(t)] \) by the instantaneous (or static) Hamiltonian \( \mathcal{H}[\Gamma(t')] \), and treat the retarded current correlator as time translation invariant.

In the presence of electron-electron interactions in the bulk, the nature of singularity of the fermionic Green function is modified. This deformation of the Fermi liquid picture implies that a new energy scale enters into the problem, thereby altering the aforementioned adiabatic criterion. In order to identify the correct adiabatic limit, we define dimensionless integration variables:

\[
\bar{t} = \Lambda t; \quad \bar{t}' = \Lambda t''
\]

whereby the integrand acquires a multiplicative factor of \( \Lambda^{-2} \), and rewrite \( G_n \) as:

\[
G_n \approx 2\pi \int_{\bar{t}}^{\infty} e^{-\omega_n \bar{t}} d\bar{t} \int_{-\infty}^{\bar{t}'} d\bar{t}'' \Lambda^{-2} \mathcal{K}^{(R)}_n(\bar{t}, \bar{t}'; \bar{t}'').
\]

(32)

Because of interactions in the wire, the composite operator acquires anomalous dimensions. We write the scaling dimension \( \Delta \) for the composite operator \( Q \) as \( \Delta = 1 - a \), where \( a \) is the anomalous dimension, and invoke the scaling hypothesis (for the equilibrium problem scaling holds whenever \( \Gamma |^2 \ll 1 \) for the range \( 0 < \Lambda \leq \Lambda_0 \); in the non-equilibrium case it is further restricted to \( \Omega \lesssim \Lambda \leq \Lambda_0 \)). Since the actual dimension of \( \mathcal{K} \) is fixed, we obtain:

\[
\mathcal{K}^{(R)}_n(\bar{t}/\Lambda + \bar{t}', \bar{t}'') = \Lambda^2 (\Lambda_0/\Lambda)^2 \mathcal{K}^{(R)}_n(\bar{t} + \Lambda t'; \bar{t}' + \Lambda t').
\]

(33)

As a consequence, the dimensionless coupling constant \( \Gamma \to \Gamma \equiv (\Lambda_0/\Lambda)^2 \Gamma \), and its time-dependence takes the form:

\[
\Gamma(t) = \Gamma(\Omega t/\Lambda + \lambda t').
\]

We can therefore identify an energy scale \( \omega_T = \Lambda \left| \Gamma \right|^{1/a} \), dependent only on the bare coupling \( \Gamma \) and the upper cut-off \( \Lambda_0 \). Choosing \( \Omega \lesssim \Lambda \lesssim \Lambda_0 \) ensures that, for \( a > 0 \), the renormalized (dimensionless) coupling constant \( \Gamma \gg \Gamma \). The corresponding energy scale \( \omega_T \) defines a crossover scale that separates two qualitatively different responses to the time-dependence: (i) for \( \Omega \ll \omega_T \), the response is an adiabatic modification of the ground state with a renormalized coupling \( \Gamma \), and (ii) for \( \Omega \gg \omega_T \), the response is a sudden modification of the ground state with \( \Gamma = 0 \). In other words, the singular contribution from the retarded correlator \( \mathcal{K}^{(R)}_n(t) \) acquires a width of the order of \( 1/\omega_T \), as compared to \( 1/\Lambda_0 \) for the non-interacting case. For example, in Section [X] we show that, for the Luttinger model with interaction parameter \( K = 1/2 \), the singularity of \( \mathcal{K}^{(R)}_n(t) \) at \( t \sim 1/\Lambda_0 \) has a width \( \delta t^* \sim 1/(\Lambda_0 |^2) \). Thus, if \( \Lambda_0/\omega_T \ll 1 \), then we can treat \( \Gamma(t') \approx \Gamma(t) \approx \Gamma(t) \). Therefore, \( G_n \) can be approximated by:

\[
G_n(t') \approx 2\pi \int_{0}^{\infty} e^{-\omega_n t} dt \int_{-\infty}^{\bar{t}'(t')} d\bar{t}'' \mathcal{K}^{(R)}(\bar{t} - \bar{t}''),
\]

(34)

where \( \omega_n \to 0 \), because the intervals \( [t_n-1, t_n] \) have lengths \( \sim 1/\Lambda \gg 1/\omega_T \); \( \mathcal{K}^{(R)}(t - \bar{t}'') \) is the retarded backscattering current correlator with the renormalized coupling constant \( \Gamma(t') \), independent of the integration variables \( t, \bar{t}' \). It is important to note that the retarded correlator \( \mathcal{K} \) is time-translational invariant in the arguments \( t, \bar{t}' \). This transformation, from a current correlator without time translation invariance to a renormalized correlator with time translation invariance, is explicitly shown in Section [X] for the special cases when the scaling dimension of \( Q \) is \( 1 \), and \( 1/2 \). A consequence of this time translation invariance is that we can rewrite the time integrals to obtain:

\[
G_n \approx 2\pi \int_{0}^{\infty} d\tau \left[ 1 - e^{-\omega_n \tau} \right] \mathcal{K}^{(R)}(\tau)
\]

(35)

The integral involving \( \mathcal{K}^{(R)}_n(t') \) can be readily identified as the conductance for backscattering current at frequency \( i\omega_n \).

\[
G_{\Gamma(t')} = 2\pi \frac{\mathcal{K}^{(R)}(0 + i\omega_n) - \mathcal{K}^{(R)}(0)}{\omega_n}.
\]

(36)
Here $\tilde{K}^{(R)}_n(0 + i \omega_n)$ represents the Fourier transform of $\tilde{K}^{(R)}(t)$ at frequency $\omega_n$, analytically continued to imaginary frequency $i \omega_n$ (with $\omega_n \to 0^+$). We emphasize that the conductance depends on the renormalized coupling $\Gamma(t')$ which is time-dependent. As a consequence we can write the total charge pumped in a cycle as an integral over the path of this coupling constant:

$$Q = \int_{\phi(t')=\phi_0}^{\phi(t')=\phi_0+2\pi} dt' \frac{\hat{G}}{\Gamma(t')} \frac{\hat{G}}{\Gamma(t')}$$

$$= \frac{1}{2\pi i} \oint \frac{d\Gamma}{\Gamma} \hat{G}.$$  (37)

Thus the problem of evaluating $Q$, for $\Omega \ll \omega_T$, reduces to calculating the dependence of the backscattering conductance on the renormalized impurity couplings. In the next section we shall link this quantity to the more familiar conductance of the fermionic current.

### D. Pumped charge and dc conductance

We now proceed to recast the expression (37) in terms of the response function for fermionic currents $\vec{j}$ to a vector potential $A(t)$:

$$G = \frac{\delta \langle \vec{j}(x,t) \rangle}{\delta A(t')}.$$  (38)

Let us recall that in the case of linear response to a perturbative $A$ which vanishes outside a region of length $L$, the conductance is given by (see, for example, Ref [44])

$$G(\omega) = \frac{1}{L \omega} \int_{-L/2}^{L/2} dy \int_{-\infty}^{0} dt \left( 1 - e^{-i\omega t} \right) \langle \tilde{\vec{j}}(0,0), \tilde{\vec{j}}(y,t) \rangle.$$  (39)

In order to find an expression for pumped current in the low frequency ($\Omega \ll \omega_T$) limit, we begin by writing the fermionic current at any point $x$ in the wire at time $t \in [t_n, t_{n+1}]$, in a way similar to (18) of Section 11, albeit now with the effective action with renormalized couplings $\hat{\Gamma}$:

$$I(x,t) = \langle S^\dagger(t,t_n) \vec{j}(x,t) S(t,t_n) \rangle.$$  (40)

Expanding this perturbatively to first order in the phase change $\delta \phi(t)$ of the coupling $\Gamma(t)$ in the time interval $[t_n, t_{n+1}]$, and using the notations of Section 11 we get:

$$I(x,t) = \langle \vec{j}(x,t) \rangle$$

$$- i \int_{t_n}^{t_{n+1}} dt' \delta \phi(t') \langle [\vec{j}(x,t), J_{\hat{\Gamma}}^R(t',t)] \rangle.$$  (41)

We can now invoke Eq. (11) and do an integration by parts to write the current as:

$$I(x,t) = \langle \vec{j}(x,t) \rangle$$

$$- i \int_{t_n}^{t_{n+1}} dt' \delta \phi(t') \int_0^{t_{n+1}} dt' K_n^{(R)}(t,t')$$  (42)

where the retarded current correlator:

$$K_n^{(R)}(t,t') = i \int_{-1/\epsilon}^{1/\epsilon} dt \theta(t-t') \langle [\vec{j}(x,t), \vec{j}(y,t')] \rangle.$$  (43)

and the subscript $n$ reminds us that this depends on the time dependent coupling $\Gamma_n(t')$. In the slow pumping limit the contribution to the integral over $t$ from the end-point $t_{n+1}$ is negligible (or the integral is dominated by the singular contribution near $t \approx t'$), expression (43) can be rewritten as:

$$\delta Q(t_n) = - \int_{t_n}^{t_{n+1}} dt' \delta \phi(t') \int_0^{t_{n+1}} dt K_n^{(R)}(t,t')$$  (44a)

$$+ \int_{t_n}^{t_{n+1}} dt' \delta \phi(t') \int_0^{\infty} dt \left[ 1 - e^{-\epsilon t} \right] K_n^{(R)}(t,t').$$  (44b)

As discussed in Section 11C when $\Omega/\omega_T \ll 1$, we can obtain the effective Hamiltonian that evolves the operators $\vec{j}$. This effective Hamiltonian has static renormalized couplings $\Gamma_n(t')$, yielding a time-translation invariant correlator:

$$K_n^{(R)}(t,t') \to \tilde{K}_n^{(R)}(t-t').$$

In this limit the expression (44) above, can be identified as the equilibrium dc conductance $\lim_{\epsilon \to 0^+} G_{\hat{\Gamma}}(0 + i \epsilon)$:

$$G_{\hat{\Gamma}}(0 + i \epsilon) = - \frac{1}{2} \int_0^{\infty} dt \left[ 1 - e^{-\epsilon t} \right]$$

$$\times \int_{-1/|\epsilon|}^{1/|\epsilon|} dy i \langle [\vec{j}(0^-), \vec{j}(y,0)] \rangle\tilde{\Gamma}_n(t'),$$  (45)
where the subscript \( n \) reminds us of the renormalized coupling constant \( \Gamma_n \). On the other hand, expression (14a) can be simplified using the current continuity equation
\[
\partial_x \hat{j}(x', t) = \partial_t \hat{\rho}(x', t); \; x' \neq 0,
\]
inside the expectation value:
\[
\int_0^\infty dt \frac{1}{2t} \int_{-\infty}^{1/e} dy \left\langle \hat{j}(x, t), \hat{j}(y, t') \right\rangle
= \int_0^\infty dt \frac{1}{2t} \int_{-\infty}^{1/e} dy \int_{-\infty}^{x=0^{-}} dx' \left\langle \partial_x \hat{j}(x', t), \hat{j}(y, t') \right\rangle
= \frac{1}{2i} \int_{-\infty}^{1/e} dy \int_{-\infty}^{x=0^{-}} dx' \int_{t'}^{t} dt \left\langle \rho(x', t), \hat{j}(y, t') \right\rangle
= \frac{i}{2} \int_{-\infty}^{1/e} dy \int_{-\infty}^{x=0^{-}} dx' \left( \frac{i}{\pi} \right) \partial_x \delta(x' - y) = \frac{1}{2\pi}.
\]

Here, in the last step, we have used the equal time anomalous commutator of fermionic density and current,
\[
\left[ \rho(x, t'), \hat{j}(y, t') \right] = \frac{i}{\pi} \partial_x \delta(x - y).
\]

Putting all the terms together we get:
\[
\delta Q(t_n) = \int_{t_{n-1}}^{t_n} dt' \delta \phi(t') \left[ \frac{1}{2\pi} - G_{\Gamma}(0 + i\epsilon) \right]
\]

The prescription to obtain the total charge pumped in a cycle, \( Q \), is exactly the same as in Section III.C, divide the entire path of \( \Gamma(t) \) into appropriate intervals, and find the charge pumped in each interval in the linear response approximation. We thus find the charge pumped through \( x = 0 \) in the limit \( \Omega \ll \omega_r \):
\[
Q = \frac{1}{2\pi i} \oint \frac{d\Gamma}{\Gamma} \left[ 1 - \lim_{\epsilon \to 0} 2\pi G_{\Gamma}(0 + i\epsilon) \right].
\]

III. TRANSPORT OF CHARGE AND SPIN USING PARAMETRIC PUMPING

We now include spin to analyze both the spin pump and the charge pump of Ref. 7. We begin by considering a Hamiltonian \( H \) which conserves the \( z \) component of the total spin, and chirality in each of the two spin states (\( \uparrow, \downarrow \)). Thus the axial charges
\[
\int dx \hat{j}^{c, a}(x, t) = \int dx \left[ \hat{j}_\uparrow(x, t) + \hat{j}_\downarrow(x, t) \right]
\]
are conserved. Here \( \hat{j}^{c, a} \) are the fermion current operators for charge and spin currents. The spin (charge) pump operates by adding to \( H \) a time-dependent potential, locally breaking the chiral symmetry in the spin (charge) sector. For the spin pump we consider a localized magnetic field in the \( z \) direction, acting as an impurity potential \( V_{\uparrow, \downarrow}(x, t) \) that couples anti-symmetrically to up (\( \uparrow \)) and down (\( \downarrow \)) spin states. In addition we consider a localized scattering potential \( V_{\uparrow, \downarrow}(x, t) \), coupling symmetrically to the spin states. This is shown schematically in Figure 3, where we introduce the notation \( V_{\uparrow, \downarrow} \equiv \hat{V}_{\uparrow, \downarrow} \).

![FIG. 3: Scattering potentials for up (\( \uparrow \)) and down (\( \downarrow \)) spins whose variation in time drives the spin pump](image3)

The additional term in the Hamiltonian is:
\[
\int_{L} dx \left[ V_{\uparrow}(x) \right. \left. \psi_\uparrow(x) \psi_\uparrow(x) + \psi_\downarrow(x) \psi_\downarrow(x) \right]
\]
\[
+ \int_{L} dx \left[ V_{\downarrow}(x) \psi_\downarrow(x) \psi_\downarrow(x) - \psi_\uparrow(x) \psi_\uparrow(x) \right]
\]

As we pointed out before in Section III.A, it is the backward scattering of the right and left movers from the...
we consider the case when the potentials $V_{o,c}(x,t) = V_{o,c}(x)f_{o,c}(t)$ with real periodic functions $f_{o,c}(t)$. In particular, for harmonic pumping we can choose $f_{c}(t) = \cos(\Omega t + \theta)$ and $f_{o}(t) = \cos\Omega t$. The elastic backward scattering of spin up (down) electrons is then given by the complex quantities:

$$\Gamma_{\pm}(t) = \int dx \ e^{2ik_Fx} \left[ V_c(x)f_c(t) \pm V_o(x)f_o(t) \right]$$

which are the $2k_F$-Fourier transforms of the potentials that scatter up or down ($\pm$) spin particles. Thus, if $V_c(x)$ and $V_o(x)$ are different functions of position (both being symmetric with respect to $x = 0$), then the complex quantities $V_{c,o}(2k_F)$ have different arguments (cf. Section 17A), and by suitably varying the real functions $f_{c,o}(t)$ (making them change sign) we can trace any path in the complex plane which winds around the origin. Furthermore, the difference between $\Gamma_{+}$ and $\Gamma_{-}$ is that they wind in opposite directions on the complex plane, as shown in Fig. 17A.

We can denote the composite operators, which have couplings $\Gamma_{\pm}$, by $Q_{\pm}$. It is more convenient, however, to define operators which transform simply under rotations in spin space. We therefore write the backscattering term in the Hamiltonian (51) by defining linear combinations of $\Gamma_{\pm}$:

$$H_\Gamma = \sum_{\beta = o,e} \Gamma_{\beta} Q_{\beta} + \Gamma_{\beta}^* Q^\dagger_{\beta}, \quad (53)$$

where the $Q_{o,c}$ field is odd (even) under the unitary operator $R_y(\pi) = e^{-i\pi/2} R_y$, which does a $\pi$-rotation about the $y$-axis in the spin space, and $\Gamma_{c,o} = \frac{i}{2} [\Gamma_{+} \pm \Gamma_{-}]$. The backscattering current that arises from non-conservation of the axial charges $\int dx \ j^{\sigma}(x)$ is now:

$$j^{\sigma}_R(t) = -\frac{1}{2} \int dx \ \partial_t j^{\sigma}(x) \quad (54)$$

$$i\Gamma_{\sigma} = i\Gamma_{o} Q_{o,c} + i\Gamma_{c} Q_{o,c} + H.c.$$

WT identities, similar to Eq. (17a,17b), can now be derived for each of $Q_{o,c}$ fields and their corresponding backscattering currents. As before (Sections 17C and 17E), WT identities ensure that there is no current without a change in the phase of $\Gamma_{\pm}$.

### A. Spin pump

We now consider the currents generated by a small change in the phase of the couplings $\Gamma_{\pm} = \Gamma_{c} \pm \Gamma_{o}$, in a time interval $t \in [t_0,t_1]$:

$$I^s(x,t) = \int dt \ \delta \phi_\sigma(t') \ \left[ j^s_\sigma(x,t') + \sigma j_R^c(t') \right]$$

where, as in Section 17C, $\hat{t}_0(t)$ is the time dependent renormalized coupling with a constant phase for $t \in [t_0,t_1]$. Next, we invoke Eq. (53), albeit with renormalized couplings, and do an integration by parts to write the current as:

$$I^s(x,t) = \sum_{\sigma = \pm} \int_{t_0}^{t_1} dt' \ \delta \phi_\sigma(t') \ \left[ K^{(R)}_{0,s,c}(t, t') + \sigma K^{(R)}_{s,c}(t, t') \right]$$

where $\epsilon = 2/L < \Omega$. The pumped spin (charge) in this interval can now be written as:

$$\delta Q^s(x,t) = \int_{t_0}^{t_1} dt' \ \sum_{\sigma = \pm} \delta \phi_\sigma(t') \ \left[ K^{(R)}_{0,s,c}(t, t') + \sigma K^{(R)}_{s,c}(t, t') \right]$$

which can be recast as:

$$\delta Q^s(x,t) = \int_{t_0}^{t_1} dt' \ \sum_{\sigma = \pm} \delta \phi_\sigma(t') \ \left[ K^{(R)}_{0,s,c}(t, t') + \sigma K^{(R)}_{s,c}(t, t') \right]$$

where

$$G_{s,c}(0 + i\epsilon) = -\frac{1}{2} \int_{-1/\epsilon}^{1/\epsilon} dt \ \left[ 1 - e^{-\epsilon t} \right] \ \left[ \tilde{j}^s(0^-, t'), \tilde{j}^b(y, 0) \right].$$

The mixed conductance $G_{s,c} = 0$, from the following symmetry argument: The Hamiltonian (with the impurity) is invariant under a combination of spin rotation.
\(R_{\pi}(\pi)\) and local magnetic field inversion, whereas the conductance \(G^{\sigma,c}_{F}\) \(\rightarrow -G^{\sigma,c}_{F}\). We can now use a less loaded notation by labeling the spin (charge) conductance by a single superscript. Thus we obtain the following expression for total spin and charge transported through any point in the wire in a pumping cycle:

\[
Q^s = \sum_{\sigma=\pm} \frac{1}{2\pi i} \oint \frac{d\Gamma^\sigma}{\Gamma^\sigma} [1 - \pi G^s_{F}],
\]

\[
Q^e = \sum_{\sigma=\pm} \frac{1}{2\pi i} \oint \frac{d\Gamma^\sigma}{\Gamma^\sigma} [1 - \pi G^e_{F}].
\]

(61)

Here \(G^{\sigma,c}_{F}\) is the DC charge (spin) conductance with renormalized impurity coupling constants \(\tilde{\Gamma}_{\pm}\). For the spin pump the choice of time dependence for the parameters \(V_{o,c}(x,t) = V_{o,c}(x)f_{o,c}(t)\) (discussed earlier) implies that

\[
\oint \frac{d\Gamma^+}{\Gamma^+} = - \oint \frac{d\Gamma^-}{\Gamma^-} = 2\pi i.
\]

(62)

We can now write the pumped spin (in integer values of \(\hbar/2\)) as:

\[
Q^s = 2 - \sum_{\sigma=\pm} \frac{1}{2\pi i} \oint \frac{d\Gamma^\sigma}{\Gamma^\sigma} \pi G^s_{F},
\]

(63)

\[
Q^e = - \sum_{\sigma=\pm} \frac{1}{2\pi i} \oint \frac{d\Gamma^\sigma}{\Gamma^\sigma} \pi G^e_{F}.
\]

(64)

B. Charge pump

The charge pump operates by replacing the magnetic field \(V_{o,c}(x,t)\) by a charge (spin independent) scattering potential. The above analysis, and especially the expressions \(\tilde{\Gamma}_{\pm}\), can be straightforwardly adapted to this case by choosing \(\Gamma^+ (t) = \Gamma^- (t)\). We get the charge pumped (in units of \(e\)) in a cycle:

\[
Q^e = 2 - \frac{1}{2\pi i} \oint \frac{d\Gamma^+}{\Gamma^+} 2\pi G^e_{F},
\]

(65)

while no spin is pumped.

IV. DISCUSSION: PROBING THE FIXED POINTS OF THE ONE DIMENSIONAL IMPURITY PROBLEM

In this section we discuss the behavior of pumped charge and spin for both the spin and charge pumps discussed in earlier sections. We find that bulk interactions play a crucial role in determining what the pumped charge or spin will be in both the asymptotic limits of low (\(\Omega \ll \omega_{F}\)) and high (\(\Omega \gg \omega_{F}\)) frequency temporal variations, where the crossover energy scale \(\omega_{F}\) has been introduced earlier. By contrast, for non-interacting electrons there is no such distinction. As we discuss below, it is the non-Fermi liquid behavior peculiar to one-dimensional interacting fermion systems, which is responsible for this difference. As a result we find that the pumped charge or spin behavior at low frequency probing probes the fixed point behavior of the equilibrium impurity problem.

The low frequency behavior of pumped charge and spin can be obtained from the expressions \(\tilde{\Gamma}_{\pm}\), \(\Gamma_o\), \(\tilde{\Gamma}_{\pm}\), and \(\tilde{\Gamma}_{\pm}\), derived earlier. We first consider the case of non-interacting electrons.

A. Non-interacting electrons

In Section \(\text{IV}\) we had pointed out that the impurity coupling \(\Gamma\) has zero dimension. In the absence of interactions this is also the scaling dimension of the renormalized coupling \(\tilde{\Gamma}\), which is thereby marginal in the RG sense. This absence of scaling with a change of cut-off (in the notation of Section \(\text{II}\), the exponent \(a = 0\)) means that if the barrier \(V(x) \ll E_F\), one can evaluate the conductance (of both the fermionic current and the backscattering current) at any energy scale using perturbation theory. To lowest order in the coupling \(\Gamma\) (a more complete calculation is given in Section \(\text{V}\)), we proceed as follows.

1. Free electron Propagator and the Current Correlator

In order to calculate the retarded backscattering current-current correlator \(\mathcal{K}^{(R)}(t,t')\) perturbatively in the time-dependent coupling \(\Gamma\), we need the (local in space) free electron propagator for each of the left (right) moving fermions. With our normalization (a constant density of states per unit length \(\nu_0 = 1/\pi\), and a cut-off \(\Lambda_0 = E_F\) this is:

\[
D(t) = \frac{\nu_0}{it + \text{sgn}(t)/\Lambda_0}.
\]

(66)

The zeroth order term of \(\mathcal{K}^{(R)}(t,t')\) is then given by:

\[
\mathcal{K}^{(R)}(t,t') = |\Gamma(t)||\Gamma(t')| \frac{4(t-t')}{\Lambda_0^2} \left( t - t' \right)^2 + 1/\Lambda_0^2.\]

(67)

If the frequency of variation of \(|\Gamma(t')|\) is much smaller than \(1/\Lambda_0\), then we can approximate \(\Gamma(t')| \approx |\Gamma(t)|\) inside the integral \(\int_{-\infty}^{t'} dt'' \mathcal{K}^{(R)}(t,t'')\), as it is dominated by the contribution near \(t'' \approx t - 1/\Lambda_0\). Furthermore, we can write \(|\Gamma(t)| \approx |\Gamma(t')|\). These approximations can be performed at every order in the expansion of \(\mathcal{K}^{(R)}\), so that we obtain, from \(\int_{-\infty}^{t'} dt'' \mathcal{K}^{(R)}(t,t'')\), a DC backscattering conductance which depends only the instantaneous parameter \(|\Gamma(t')|\). For the spinless case we find the lowest order contribution to the DC backscattering conductance:

\[
G_{\Gamma} \approx \frac{1}{\pi} |\Gamma|^2,
\]
Using this expression in Eq. (67) we get a non-quantized charge

$$Q \simeq \frac{1}{i\pi} \oint \left[ \Gamma^* d\Gamma - \Gamma d\Gamma^* \right].$$

The integral over the path can be rewritten (using Stoke’s theorem) as an integral over the area of the contour:

$$Q \simeq \frac{1}{\pi} \int_A dA,$$

where the differential area element $dA$ is shown in Fig. 3. The charge pumped is thus dependent only on the area covered in a cycle on the $\Gamma$-plane, not on where the area is located in the $\Gamma$-plane.

B. Interacting electrons: Luttinger liquid behavior, spinless case

In the presence of interactions in the one-dimensional system the renormalized coupling $\tilde{\Gamma}$ acquires anomalous dimension $\alpha$, which is indicative of the absence of a quasi-particle pole in the (bulk) electronic Green function:

$$\tilde{\Delta}(k) \sim (|k| - k_F)^{-1+a};$$

$a = 0$ for non-interacting electrons where we recover the quasi-particle pole. As a result, the coupling $\tilde{\Gamma}(\Lambda)$ changes with the cut-off $\Lambda$. In the language of the RG, the coupling is relevant when $a > 0$ and irrelevant when $a < 0$:

$$-\frac{d\tilde{\Gamma}}{d\ln \Lambda} = a\tilde{\Gamma}.$$

It is well-known that the backscattering impurity coupling is always relevant when the electron-electron interactions are repulsive so that $\tilde{\Gamma}(\Lambda)$ increases as $\Lambda \rightarrow \Omega$, the frequency of pumping. As $\Omega \rightarrow 0$ the DC conductance at zero temperature vanishes. This implies that the charge pumped in a cycle in the asymptotic limit of slow pumping is:

$$Q = \frac{1}{2\pi i} \oint \frac{d\Gamma}{\Gamma} \left[ 1 - \lim_{\epsilon \rightarrow 0} 2\pi G_{\Gamma^\prime}(0 + i\epsilon) \right] \equiv \frac{1}{2\pi i} \oint \frac{d\Gamma}{\Gamma} = 1$$

1. Frequency corrections

Let us now turn to the question of low-frequency corrections to the quantization of $Q$. As $\Omega$ approaches $\omega_T$, our assumption of replacing the upper limit of integration over variable $t$ in expression (68), or (69), by a cut-off $\omega_n \rightarrow 0^+$, is no longer accurate. Deviations from this arise as the interval length $|t_n - t_{n-1}|$ shortens with increase of pumping frequency $\Omega$ (in order for the linear approximation to hold). As a result, the dependence of $G_{\Gamma^\prime}$ on the end-point of the integration region, i.e., $t_n$, in each of the summed quantities in (69) grows; likewise for $G_{\Gamma^\prime}$ in (69). This dependence can be taken into account very simply by choosing $\omega_n \equiv \epsilon \sim \Omega$. Then the resulting quantity is the conductance $G_{\Gamma^\prime}(0+ic)$, simply related, by analytic continuation, to the real frequency conductance $G_{\Gamma^\prime}(\epsilon)$ for $\epsilon > 0$. The functional form $G_{\Gamma^\prime}(\epsilon)$, as $\epsilon \rightarrow 0$ is governed by the effective impurity coupling $\tilde{\Gamma}(\Lambda)$ at the energy scale $\Lambda \sim \Omega$. To determine this behavior we have to know the finite ac-bias conductance of the interacting system with an impurity which requires recourse to a particular model. Exact solutions for the Luttinger model with a time-independent impurity coupling $\Gamma$ have been shown to have a scaling form for the conductance $G(\omega) \equiv G((\omega/T_B))$ with the impurity strength dependent energy scale $T_B \sim \Lambda_0|\Gamma|^{1/(1-K)}$, and the Luttinger parameter $K < 1$ for repulsive interactions. In the limit $\omega \ll T_B$ the conductance vanishes as $(\omega/T_B)^{2/K-2}$. This can be used to find the asymptotic frequency dependence of the pumped charge as follows. We replace the impurity energy scale $T_B$ with the instantaneous one $\omega_T(t')$, and the energy scale set by the applied voltage/frequency, $\omega$, by that set by the cut-off $\epsilon$. Thus we obtain:

$$G_{\Gamma^\prime}(t') \sim \left( \frac{\epsilon}{\omega_T(t')} \right)^{2/K-2}$$

The pumped charge is therefore:

$$Q \simeq 1 - \frac{1}{2\pi i} \oint \frac{d\Gamma}{\Gamma} \times \left[ C_1 \Omega^{2/K-2}|\Gamma|^{2/K} + C_2 \Omega^{4/K-4}|\Gamma|^{4/K} + \ldots \right]$$

$$= 1 - \epsilon q_1(\Omega / \omega_T)^{2/K-2} - \epsilon q_2(\Omega / \omega_T)^{4/K-4} + \ldots$$

where $C_1, C_2$ are cut-off dependent non-universal constants, whose ratios are universal numbers. However, in the expansion of pumped charge $Q$ the ratios of coefficients, for example $q_1/q_2$, as well as the quantity $\omega_T$, are dependent on the details of the path followed by the coupling $\Gamma$ and are consequently non-universal. We note though, that the pumped quantity (in this case, the charge $Q$) is bounded from above (below) by its value in the case of a constant $|\Gamma| = \max(|\Gamma(t)|)$, which also corresponds to a larger (smaller) value for the energy scale $\omega_T$. The above considerations give us the frequency-dependent correction to the quantized pumped charge $Q$ in a one-dimensional system of spinless (spin-polarized) interacting electrons.

When the electron-electron interactions are attractive, then it is known that the impurity coupling is irrelevant. As a result the renormalized impurity coupling at the energy scale $\Lambda \sim \Omega$ is smaller than the bare coupling at energy scale $\Lambda_0 > \Lambda$ and the DC conductance $G_{\Gamma^\prime}$ can be calculated perturbatively. The lowest order correction corresponds to setting $\tilde{\Gamma} = 0$ which gives zero pumped charge. For the frequency dependent corrections we turn to the Luttinger model with $K > 1$. 

Using the exact results and the scaling arguments used earlier for the $K < 1$ case, we obtain the conductance $2\pi G_p \simeq 1 - q_0'(\Omega/\omega_T)^{2K-2}$, and the charge pumped in a cycle vanishes with a power law: $Q \sim (\Omega/\omega_T)^{2K-2}$.

### C. Quantization and fractional charge

While the above analysis for spinless electrons does not directly apply to the case of a quantum wire – where both spin species of electrons are present – it correctly describes a charge pump operating in a quantum Hall system, where the left and right movers are the chiral edge excitations. For integer Hall systems the chiral excitations carry integer charge and correspond to the non-interacting case discussed earlier. On the other hand, for fractional quantum Hall systems whose edge excitations are described by Wen’s edge-state theory, the chiral excitations carry fractional charge $\nu e$ and correspond (in our description of the pumping behavior) to interacting electrons with $K \equiv \nu = 1/m < 1$, where $m$ is the odd-integer denominator which characterizes the bulk fractional quantum Hall state. One interesting question that emerges out of the quantization of adiabatically charge in units of the electron charge $e$ is how charge fractionalization in strongly correlated one-dimensional systems (including edge state tunneling in fractional quantum Hall – FQH – systems) is manifest through pumping. Consider, for example, a time-dependent backscattering potential in a constriction of a $\nu = 1/3$ FQH bar: it will lead to the adiabatic transport of an electron charge $e$ within a (slow) period $\tau = 2\pi/\Omega$, or a pumping current $I_p = \frac{\tau}{\pi} \Omega$. To attain the same current with a non-equilibrium voltage, one must have $I_p = \nu \frac{\tau}{\pi} V$ instead. So while the charge pumped in a cycle is $e$, the relationship between the pumping frequency $\Omega$ and the non-equilibrium voltage $V$ is $\Omega = \nu e V/h$. Both the finite bias and pumped currents derive from the phase of a time-dependent backscattering potential, and the phase changing rates in the two cases are related by the Josephson relation $\Omega = e^* V/h$, with $e^* = \nu e$. So even though the charged pumped in a cycle is integer, independent measurements of the pumping frequency and voltage leading to the same current yields the fractional charge relation between $\Omega$ and $V$. Simply put, the existence of fractional charge in the FQH is directly related to the quantization of the Hall conductance, and to the quantization of pumped charge across the edges.

### D. Electrons with spin: spin and charge pump.

The quantization of the pumped spin in a cycle, $Q^s$, depends on the low energy behavior of the DC spin and charge conductances $G^s_{\ell}^{(c)}$ with renormalized impurity couplings $\tilde{\Gamma}_{\alpha,e}$. The behavior of these DC conductances can be obtained in a manner similar to that discussed above in the context of spinless fermions. Assuming that interactions in the one-dimensional system respect $SU(2)$ symmetry in spin, it is known that the impurity backscattering coupling $\tilde{\Gamma}_{\alpha,e}$ is relevant whenever the interactions are repulsive. As a consequence both the spin and charge conductances vanish at low energies with power law corrections which can be determined by taking recourse to a particular interaction model with spin. In particular for the Luttinger model one needs to only modify the exponents of the renormalization factors in the expressions for $Q$ by replacing $1/K \rightarrow 1/K_{c}^{-1}$, and putting $K_{s} = 2$ for the spin isotropic point $\nu = 1$ in order to get the behavior of $Q^{s,c}$. Note that the non-interacting case is given by $K_{c} = 2$, $K_{s} = 2$, and repulsive interactions imply $K_{c} < 2$. For the spin pump in a wire described by the Luttinger model with $K_{s} + K_{c} < 2$, we obtain:

$$Q^s \simeq 1 - \sum_{n=1}^{\infty} q_{n}^s \left( \frac{\Omega}{\omega_T} \right)^{2n(\frac{1}{K_{c}} + \frac{1}{K_{s}})-1} + \ldots$$

(71)

$$Q^c \simeq \sum_{n=1}^{\infty} q_{n}^c \left( \frac{\Omega}{\omega_T} \right)^{2n(\frac{1}{K_{c}} + \frac{1}{K_{s}})-1} + \ldots$$

(72)

For the charge pump the pumped charge in a cycle is quantized just as the spin is quantized for the spin pump, albeit with different non-universal constants $\{q_{n}^s\}$, as in the case of spinless fermions.

The more general case of pumping in a wire described by the Luttinger model with $K_{s} \neq 2$ can also be studied using the general relations derived in Equations (64) and (65). All we need, to determine the pumped charge and spin in a cycle, is the behavior of the DC conductance with renormalized impurity couplings. When the charge (spin) DC conductance vanishes we have quantized charge (spin) pumped in a cycle. Allowing for more general values of $K_{c}$ and $K_{s}$ we can refer to the plots of Kane and Fisher Ref. for depicting the regions, in the $K_{c} - K_{s}$ plane, where the $2k_F$ backscattering is relevant and where it is irrelevant. In the former regions the spin pump will transport a quantized spin per cycle, and the charge pump will transport a quantized charge per cycle, whereas in the latter both will give vanishing pumped charge. In passing we would like to note that the analysis of parametric pumping presented thus far does not apply to the case when more than one backscattering operator is relevant. In the Luttinger model this happens when $K_{c} < 2/9$, $K_{s} = 2$, when both the $2k_F$ backscattering and $6k_F$ backscattering terms are relevant.

### E. Finite size and temperature effects

Real experimental scenarios involve finite length systems contacted by wide leads which are usually described by Fermi liquid theory. This brings the length of the quantum wire $L_W$, as another important scale in the problem. The issue of contacts in the presence of an impurity in a Luttinger liquid model has been dealt with extensively in the literature. Here we use the relevant
results. Whenever $\Omega \gg v_F/L_W$ the (backscattering) conductance $G_\Omega$ is determined by the properties of the wire alone, so that the above considerations for quantization and corrections from non-zero $\Omega$ hold. On the other hand, when $L_W \ll v_F/\Omega$, $G_\Omega$ is determined by the properties of the external leads. In the RG analysis discussed earlier, this corresponds to the infrared divergence of $\Gamma$ being cut off by the energy scale $\hbar v_F/L_W$. Consequently the charge (spin) $Q^{(s)}(\omega)$ is independent of the pumping frequency, and also of the length $L_W \ll v_F/\Omega$ just as for the non-interacting case. To see the finite size scaling one can imagine operating the pump in a closed geometry, for example, in a ring. Then the finite size effects $G_F$ are similar to those for the ac-conductance in Ref. \[4\].

The dependence of $Q$ on the DC conductance also implies that the finite temperature effects (when $T > \hbar\Omega$) are similar to that of Ref. \[4\]. Consequently the low temperature ($\Omega < T < \tilde{\omega}_T$) expansion is similar to \[7\] – when $T$ is constant, let us note here that unlike the case earlier, this corresponds to the infrared divergence of $\tilde{\Gamma}$ the wire is assumed to be in equilibrium. The current, which is determined by the properties of the external leads. In the RG analysis discussed \[5\], we remind the reader that we have set $\hbar = 1 = v_F$. The density of states per unit length for a single spin species is $\nu_0 = 1/(\pi\hbar v_F) \equiv 1/\pi$, and the constraint that the ground state contain a fixed number of particles relates the upper cut-off of the theory to the Fermi energy $E_F$. The uniform density of particles in the ground state of the wire is $k_F/\pi$. In the absence of any time-dependence of $\Gamma$ the wire is assumed to be in equilibrium. The current, which is given by the difference in density of the left and right movers, is therefore zero. We now ask for the density difference as a function of time at a point in the wire to the right of the impurity. For convenience, we choose this point to be in the immediate vicinity of the impurity. The time evolution of the fields is (obtained by using the Fermion anticommutation relations):

\[
\begin{align*}
-i\partial_t \psi_R(x) &= \left[H, \psi_R(x)\right] = i\delta(x)\Gamma^*(t)\psi_L(x), \quad (75a) \\
-i\partial_t \psi_L(x) &= \left[H, \psi_L(x)\right] \\
&= -i\partial_x \psi_L(x) - \delta(x)\Gamma(t)\psi_R(x), \quad (75b)
\end{align*}
\]

Since we are interested in the fields in the immediate vicinity of the impurity, we integrate these equations to obtain:

\[
\begin{align*}
\Delta \psi_R(t) &= -i\Gamma^*(t)\frac{1}{2}\psi_L(t), \quad (76a) \\
\Delta \psi_L(t) &= -i\Gamma(t)\frac{1}{2}\psi_R(t), \quad (76b)
\end{align*}
\]

where

\[
\begin{align*}
\Delta \psi_R &= \psi_R(x = 0^+) - \psi_R(x = 0^-), \\
\Delta \psi_L &= \psi_L(x = 0^-) - \psi_L(x = 0^+), \\
\psi_{R,L} &= \psi_{R,L}(x = 0^+) + \psi_{R,L}(x = 0^-).
\end{align*}
\]

The field $\psi_R(x, t)$ being a right-mover is a free field to the left of the impurity, not being influenced by the presence of the impurity. Likewise for the field $\psi_L(x, t)$ to the right of the impurity. From the relations \[7\] we can determine the outgoing fields:

\[
\begin{align*}
\psi_R(0^+, t) &= \frac{-2i\Gamma^*(t)}{1 + |\Gamma(t)|^2}\psi_L(0^+, t) \\
&+ \frac{1 - |\Gamma(t)|^2}{1 + |\Gamma(t)|^2}\psi_R(0^-, t), \quad (78a) \\
\psi_L(0^-, t) &= \frac{-2i\Gamma(t)}{1 + |\Gamma(t)|^2}\psi_R(0^-, t) \\
&+ \frac{1 - |\Gamma(t)|^2}{1 + |\Gamma(t)|^2}\psi_L(0^+, t). \quad (78b)
\end{align*}
\]

The current for free electrons is given by the current in the outgoing channel:

\[
\begin{align*}
I(t) &= \frac{1}{k_F} \left< \psi_R^\dagger(0^+, t) i\partial_t \psi_R(0^+, t) \right> \\
&- \frac{1}{k_F} \left< \psi_L^\dagger(0^-, t) i\partial_t \psi_L(0^-, t) \right> + H.c., \quad (79)
\end{align*}
\]
as the incoming channel is taken to have zero net current:  
\[ \langle \psi_R^+(0^+, t) i \partial_t \psi_L(0^+, t) \rangle - \langle \psi_R^+(0^-, t) i \partial_t \psi_R(0^-, t) \rangle = 0. \]

Note that use has been made of Eq. (77) away from the impurity, to obtain time derivatives in the expression for the current.

Using Eq. (78), and the incoming state normalization that  
\[ \langle \psi_R^+(0^-, t) \psi_R(0^-, t) \rangle = \langle \psi_R^+(0^+, t) \psi_L(0^+, t) \rangle \equiv \frac{k_F}{2\pi}, \]
we can write the current (at zero temperature) as:
\[ I(t) = \frac{1}{i\pi} \frac{\hat{\Gamma}(t)\Gamma^*(t) - \hat{\Gamma}^*(t)\Gamma(t)}{1 + |\Gamma(t)|^2} \]
(80)

The pumped charge (at zero temperature) can now be written as:
\[ Q = \int dt \frac{1}{i\pi} \frac{\hat{\Gamma}(t)\Gamma^*(t) - \hat{\Gamma}^*(t)\Gamma(t)}{(1 + |\Gamma(t)|^2)^2} \]
(81)
\[ = \frac{1}{2\pi \epsilon t} \oint \frac{d\Gamma}{\Gamma} \left[ \frac{2|\Gamma|^2}{(1 + |\Gamma|^2)^2} \right] + C.C. \]
(82)

The reflection probability $|\Gamma|^2$ determines the pumped charge generated by backscattering of electrons from the barrier. The chief feature of non-interacting electron gas is the energy independence of the reflection probability (for low-energies). As a consequence, we find that the pumped current is determined by the instantaneous backscattering amplitude.

1. Spin pump

Including spins in the above analysis, to analyze the spin pump, is straightforward: different spin species are backscattered by different amplitudes (see Section 11). Therefore, all we have to do to determine the pumped charge and spin is to put indices of different spin species on $\Gamma$ in the expression (82). The spin current is then given by:
\[ I_{\sigma}(t) = I_{\uparrow}(t) - I_{\downarrow}(t), \]
(83)
\[ I_{\uparrow,\downarrow}(t) = \frac{1}{i\pi} \frac{\hat{\Gamma}_\pm \pm(t)\Gamma^*(\pm(t) - \hat{\Gamma}_\pm \pm(t)\Gamma(t)}{(1 + |\Gamma_\pm(t)|^2)^2} \]
(84)

The condition for generating a pure spin current requires tuning the amplitudes $\Gamma_\pm$ such that $I_{\uparrow} = -I_{\downarrow}$, and is given by:
\[ \hat{\phi}_+(t) \frac{|\Gamma_+(t)|^2}{(1 + |\Gamma_+(t)|^2)^2} = -\hat{\phi}_-(t) \frac{|\Gamma_-(t)|^2}{(1 + |\Gamma_-(t)|^2)^2} \]

where $\hat{\phi}_\pm(t)$ is the phase of $\Gamma_\pm(t)$. This requires some fine-tuning of the scattering amplitudes. For example, in the case of delta function potentials $V_{\epsilon,\sigma}(x)$ of Section 11, adjusting the distance $\ell$ between the barrier and the magnetic field to be such that $2k_F\ell = \pi/2$ satisfies this condition.

B. Exact solution for pumped current at $K = 1/2$

We now consider the case with non-zero interactions ($g_\pm \neq 0$) in the Luttinger model for spinless electrons. With the bulk Hamiltonian $H = H_{LL}$ at $K = 1/2$, the backscattering Hamiltonian $H_\Gamma$ of Eq. (21) has a scaling dimension $\Delta = K = 1/2$ – same as that of a Fermi field operator. The low energy behavior of the theory with the impurity is therefore identical to that of a system of chiral Fermions (different from the original interacting fermions) with a Hamiltonian $H_{LL}$.

\[ H = H + H_\Gamma = \int dx \left\{ \psi^\dagger(x) \left[ -i\partial_x \right] \psi(x) \right\} \]
(85)
\[ + \frac{i}{\sqrt{2}} \left[ \lambda \hat{a} \psi(0) + \lambda^* \hat{a}^\dagger \psi(0) \right], \]

where $\psi$ is a chiral Dirac Fermion and $\hat{a}$ is a Majorana Fermion representing the impurity. The impurity potential $\lambda = (1/\sqrt{\pi\alpha})\Gamma$, where $\Lambda_0 \equiv 1/\pi \alpha \lesssim E_F$ is the high energy cut-off of the bulk Hamiltonian $H$, and we recognize the dimensionless coupling $\Gamma$ from the notations used earlier. In this section we use the mapping to free fermions to find the exact expression for the non-equilibrium current arising due to the pumping parameter $\Gamma \to \Gamma(t)$ acquiring a time dependence. This expression was first used in our earlier publication, Ref. 8.

Here we show the details of our calculations for the charge transported across the wire in a cycle. In the asymptotically slow limit of pumping (frequency $\Omega/E_F \to 0$) the pumped charge is shown to be determined by the dc conductance of the system. Furthermore, a general scaling formula for the pumped charge is also conjectured.

We begin by defining chiral Majorana fermions
\[ \eta_1(x) = \frac{\psi(x) + \psi^\dagger(x)}{\sqrt{2}}, \quad \eta_2(x) = \frac{\psi(x) - \psi^\dagger(x)}{i\sqrt{2}}, \]
\[ \{\eta_j(x), \eta_j(x')\} = \delta(x - x'); \quad \{\eta_j(x), \eta_k(x')\} = 0; \quad \{\eta_1(0), \hat{a}\} = 0; \quad \{\hat{a}, \hat{a}\} = 1, \]

and denote the real and imaginary parts of the complex scattering matrix $\lambda(t)$ as:
\[ \text{Re} \lambda = \lambda_1; \quad \text{Im} \lambda = -\lambda_2; \quad |\lambda| = \lambda_m. \]
(87)

Next, we write the equations of motion for these fields:
\[ -i\partial_t \eta_1(x) = [H, \eta_1(x)] = i\partial_x \eta_1(x) + i\lambda_1 \delta(x) \hat{a}, \]
(88a)
\[ -i\partial_t \eta_2(x) = [H, \eta_2(x)] = i\partial_x \eta_2(x) + i\lambda_2 \delta(x) \hat{a}, \]
(88b)
\[ i\partial_t \hat{a} = -[H, \hat{a}] = i\lambda_1 \eta_1(0) + i\lambda_2 \eta_2(0). \]
(88c)

Now we note that the Majorana fields are right-movers and therefore, the fields to the right of the impurity are dependent on the fields to its left. The latter are, however, independent of the impurity potential.

To find how the chiral Majorana fermion fields at $x = 0^+$ are related to those at $x = 0^-$, we integrate across the impurity, and using the notation
\[ \Delta \eta_j = \frac{i}{2} \left[ \eta_j(0^+) - \eta_j(0^-) \right], \]

...
From Eq. (89) we get the relation between the operators:

\[ -\Delta \eta_1 = \frac{1}{2} \lambda_1 \dot{\alpha}; \quad -\Delta \eta_2 = \frac{1}{2} \lambda_2 \dot{\alpha}; \]
\[ \partial_t \dot{\alpha} = \lambda_1 [\Delta \eta_1 + \eta_1 (0^-)] + \lambda_2 [\Delta \eta_2 + \eta_2 (0^-)]. \]  
(89)

We will now define scaled fields:

\[ \eta_1 \to \lambda_1^{-1} \eta_1, \quad \eta_2 \to \lambda_2^{-1} \eta_2, \]

and eliminate \( \dot{\alpha} \) between the equations in (89) to obtain:

\[ -\partial_t \Delta \eta_1 = \frac{1}{2} \left[ \lambda_1^2 (\Delta \eta_1 + \eta_1 (0^-)) + (1 \to 2) \right] \]
\[ = -\partial_t \Delta \eta_2. \]

Since the scaled fields satisfy \( \Delta \eta_1 = \Delta \eta_2 \), we choose an ansatz for relating the fields across the impurity:

\[ \Delta \eta_1 = \bar{M}_1 * \eta_1 + \bar{M}_2 * \eta_2 = \Delta \eta_2 \]

Here \( * \) denotes the convolution:

\[ \bar{M}_1 * \eta_1 = \int_{-\infty}^{t} dt \bar{M}_1 (t; t_1) \eta_1 (0^-, t_1). \]

From Eq. (89) we get the following differential equations for the \( \bar{M} \)'s:

\[ \partial_t \bar{M}_j (t; t_1) + \frac{1}{2} \left[ \lambda_1^2 + \lambda_2^2 \right] \bar{M}_j (t; t_1) \]
\[ + \frac{1}{2} \lambda_j^2 \delta (t - t_1) = 0; \quad \text{for } j = 1, 2 \]

These can be solved straightforwardly to reveal:

\[ \bar{M}_j (t; t_1) = -\frac{\lambda_j^2 (t_1)}{2} \theta (t - t_1) \]
\[ \times \exp \left\{ -\int_{t_1}^{t} dt' \frac{1}{2} \left[ \lambda_1^2 (t') + \lambda_2^2 (t') \right] \right\}. \]  
(93a)

We thus find the relation between the scaled fields on either sides of the impurity:

\[ \eta_j (0^+, t) = \eta_j (0^-, t) + \sum_{k=1}^{2} 2 \bar{M}_k * \eta_k \]  
(93b)

Reverting to original unscaled fields we define the kernels:

\[ M_j (t; t_1) = 2 \bar{M}_j (t; t_1) \lambda_j^{-1} (t_1), \]

and write the relations between Majorana fermions across \( x = 0 \):

\[ \eta_1 (0^+, t) = \eta_1 (0^-, t) + \lambda_1 (t) \left[ M_1 * \eta_1 + M_2 * \eta_2 \right], \]
\[ \eta_2 (0^+, t) = \eta_2 (0^-, t) + \lambda_2 (t) \left[ M_1 * \eta_1^* + M_2 * \eta_2^* \right]. \]  
(95a)

From the equations (93) above we get the relations in terms of the Dirac fermions:

\[ \psi (0^+, t) = \psi (0^-, t) + \lambda^* (t) \left[ M_1 [\psi + \psi^*] \right. \]
\[ \left. + i M_2 [\psi^* - \psi] \right]. \]  
(96)

The above equation allows us to calculate the current at time \( t \). The current is given by the difference in the electron density across the impurity

\[ I (t) = \left\langle \psi^+(0^+, t) \psi (0^+, t) - \psi^+(0^-, t) \psi (0^-, t) \right\rangle, \]

(97)

where the expectation value is taken in the Heisenberg state at a time in the remote past when the system is in equilibrium. Using:

\[ \left\langle \psi^+(0^-, t) \psi (0^-, t') \right\rangle = \left\langle \psi (0^-, t) \psi^+(0^-, t') \right\rangle \]
\[ = D (t - t') \sum_{E} n (E) e^{i E (t - t')}, \]

we find:

\[ I (t) = \int_{-\infty}^{t} dt' \dot{\phi} (t') \cos [\phi (t) - \phi (t')] \]
\[ \times \int_{-\infty}^{t'} dt'' K (t, t''), \]

(99)

where the retarded current correlator

\[ K (t, t'') = \lambda_m (t) \lambda_m (t'') \exp \left\{ \frac{1}{2} \int_{t}^{t''} dt' \lambda_m^2 (t') \right\} \]
\[ \times \int_{-\Lambda_0}^{\Lambda_0} \frac{dE}{2\pi} n (E) \sin E (t - t''). \]  
(100)

C. Pumped charge: low frequency asymptotics

Consider the pumping cycle in a time interval \([t_i, t_f]\), so that \( \dot{\phi} (t) = 0 \), \( t < t_i \). The expression (93) above can be integrated over time to find the charge pumped in the cycle:

\[ Q = \int_{t_i}^{t_f} dt' \dot{\phi} (t') \int_{t_i}^{t_f} dt \cos [\phi (t) - \phi (t')] \]
\[ \times \int_{-\infty}^{t'} dt'' K (t, t'') \]
\[ \equiv \int_{t_i}^{t_f} dt' \dot{\phi} (t') G (t'), \]

(101)

We now seek the asymptotic behavior of \( G \) as the ratio \( \Omega / \Lambda_0 \to 0 \), where \( \Omega \) is the pumping frequency and \( \Lambda_0 \lesssim E_F \) is the upper cut-off of the field theory (53).
Consider first the case of zero temperature. Then the energy integral can be performed exactly and we obtain:

\[ K^{(0)}(t, t'') = -\lambda_m(t)\lambda_m(t') \frac{1}{2\pi} \int_{t}^{t''} d\lambda_n(i) \left[ 1 - \cos \Lambda_0(t - t'') \right] \]

This function has a maximum at \( t - t'' \approx 1/\Lambda_0 \), with a width \( \delta t' \approx 1/\omega_T + 1/\Lambda_0 \), where \( \omega_T = \lambda_m(t'') \). As a result, the time \( t' \) has to be within \( 1/\Lambda_0 \) of this maximum, and we can write: \( \lambda_m(\Omega t') \approx \lambda_m(t') \). We note that the neglected term, \( \delta \lambda_m/\lambda_m \), contributes only to even orders to the current, \( i.e., (\delta \lambda_m/\lambda_m)^{2n} \sim (\Omega/\omega_T)^{2n} \), for \( n = 1, 2, \ldots \). It can therefore be accounted for by a suitable renormalization of the energy scale \( \omega_T \).

We therefore focus on the analytics of the lowest order corrections, and write:

\[ G(t') = \int_{0}^{\infty} dt \int_{-\infty}^{0} dt'' \lambda_n^2(t') e^{-\frac{1}{2} \lambda_n^2(t')(t - t'')} \]

Using the time-translation invariance of the integrand, we have:

\[ G(t') = \int_{-\Lambda_0}^{\Lambda_0} \frac{dE}{2\pi} n(E) \sin[E(t - t'')] \]

where \( G(\Omega \to 0) \) is the dc conductance of the \( K = 1/2 \) Luttinger liquid with an impurity of strength \( \omega_T^2 \). It has a low temperature \( (T < \omega_T) \) behavior \( G \sim (T/\lambda_n^2(t'))^2 \). This implies that the pumped charge:

\[ Q = \frac{1}{2\pi} \int dt' \dot{\phi}(t') \left[ 1 - 2\pi G(\Omega \to 0) \right] \]

at zero temperature, is \( Q = e \).

D. Crossover temperature and scaling behavior

In order to look at the high frequency pumping limit, we consider the time dependence of \( \phi \) and \( \lambda_m \) with a principle frequency \( \Omega \), so that we can write:

\[ \lambda_m(t) \equiv \lambda_m(\Omega t), \quad \phi(t) \equiv \phi(\Omega t). \]

Then it is straightforward to see that the expression for pumped charge \( Q \) can be written so as to make all the \( \Omega \) dependence explicit (upon rescaling all the times by a factor \( \Omega \)):

\[ Q = \int_{0}^{2\pi} dt' \dot{\phi}(t') \int_{0}^{2\pi} dt \cos[\phi(t) - \phi(t')]
\times \int_{-\infty}^{\infty} dt'' \lambda_m(t)(t'') e^{\frac{1}{2} \lambda_n^2(t_t)/\Omega}
\times \int \frac{dE}{2\pi} \tilde{n}(E) \sin E(t - t'') \]

From this expression, a large-\( \Omega/\omega_T \) expansion follows when the argument of the exponential is a small quantity, \( i.e., \)

\[ \int_{0}^{2\pi} dt_0 \lambda_n^2(t_0) \ll 1, \]

where \( \lambda_m(t) \) has its times rescaled according to \( \Omega \).

Expanding the exponential we can compare the resulting expansion with the low-frequency expansion of \( Q \), by substituting for \( K \to 1/K \). We thus find the coefficients \( \{q_n\} \) for all integer powers of \( 1/\Omega \) in this dual expansion. It is clear that unless \( \lambda_m \) is time-independent, the ratios of these coefficients are not universal numbers.

A particular form of time-dependence for the parameters \( \phi(t) \) and \( \lambda_m(t) \) based upon the simple model of Section 1A is:

\[ \lambda_m^2(t) = \lambda_0^2 [1 + \delta_0 \cos 2\Omega t], \quad 0 \leq \delta_0 < 1 \]

\[ \phi(t) = \tan^{-1} \left[ \frac{\sqrt{1 + \delta_0 \cos 2\Omega t - \sqrt{1 - \delta_0^2 \sin^2 \Omega t}}}{\sqrt{1 + \delta_0 \cos 2\Omega t + \sqrt{1 - \delta_0^2 \sin^2 \Omega t}}} \right] \]

For \( \delta_0 = 0 \) this gives a constant \( \phi \) and so we can compute the charge pumped for any frequency \( \Omega \) in terms of known functions. We find, after straightforward but tedious algebra:

\[ Q = \frac{1}{2\pi} \int dt' \dot{\phi}(t') \left[ 1 - 2\pi G(\Omega) \right], \]

where \( G(\Omega) \) is the conductance at a dc voltage \( \Omega \) for the \( K = 1/2 \) Luttinger model and at zero temperature is given by:

\[ G(\Omega) = \frac{1}{\Omega} - \frac{\lambda_0^2}{2\pi \Omega} \tan^{-1} \left[ \frac{\Omega}{\lambda_0} \right]. \]

In Fig. 1 the pumped charge in a cycle \( Q \) is plotted versus \( R = \lambda_0^2 \Omega^2 = \omega_T^2 \) for three values of the parameter \( \delta_0 = 0, 0.5, 0.7 \). It is clear from the plots that the low and high frequency asymptotic behavior is the same in all three cases. The uppermost curve is for \( \delta_0 = 0 \) – a constant \( \lambda_m \) and therefore a circular pumping cycle – and has the largest pumped charge \( Q \) per cycle for any particular value of \( R \). This decrease in pumped charge for a
given frequency is to be expected, as an oscillatory amplitude leads the response function, $G$, out of phase with the driving force $\dot{\phi}$, thereby decreasing the current. As $R$ increases from 0, deviations from $\delta_0 = 0$ curve are apparent near $R \sim 10$; at low frequency (large $R$) the curves for increasing $\delta_0$ ($\delta_0 = 0.5$ and $\delta_0 = 0.7$ in the figure) converge with the curve for $\delta_0 = 0$ at progressively greater values of $R$, indicating the decoherent (non-adiabatic) nature of deviations from the circular shape of the pumping cycle.

Thus, although there is a range of frequency where pumped charge is dependent on all the details of the time-dependent parameters in the problem, the asymptotic behavior is universal. A scaling formula for the pumped charge can therefore be written when $\Gamma(t) \equiv \Gamma(\Omega t)$:

$$Q = \tilde{Q}(\Omega/\omega_T, \delta \omega_T/\omega_T)$$

(112)

where $\omega_T = \frac{1}{\hbar} \int_0^{2\pi/\Omega} dt \lambda_m^2(t)$ is a cross-over energy scale in the time-dependent problem, and $\delta \omega_T$ represents a collection of other (time-independent) parameters contained in $\Gamma$ similar to the quantity $\delta_0$ in Eq. (108). The behavior of the scaling function is described by:

$$\tilde{Q}(0, x) = 1, \quad \tilde{Q}(x, y) \leq \tilde{Q}(x, 0).$$

Let us note that it has not been proven that the inequality will hold for a more general form for $\lambda(t)$.

VI. CONCLUSION

In conclusion, we have studied the adiabatic limit of a quantum pump in a quantum wire, by mapping it to a problem of a time-dependent backscatterer in a Luttinger liquid. We have shown that the properties of scale invariance and chiral symmetry of the Luttinger liquid imply a relationship between the pumped quantities and the dc conductance. The difference between a Fermi liquid and a Luttinger liquid behavior in a one-dimensional system is clearly brought out by this mechanism of charge and spin transport. It is found that for a Fermi liquid the adiabatic regime – for which the pumped quantities per cycle are independent of pumping frequency – extends up to pumping frequency $\Omega < E_F$, whereas for the Luttinger liquid this regime of pumping is limited to $\Omega \ll \omega_T \ll E_F$. The barrier energy scale $\omega_T$ is found to be a crossover energy scale between adiabatic and impulsive (sudden) response to the time-dependence. This distinction is absent in the Fermi liquid picture. Another way of interpreting these results is to associate adiabaticity, for this mechanism of pumping, with the scale invariant fixed points of a wire with an impurity. For the non-interacting case the beta function for the impurity coupling is zero, consequently any pumping frequency implies an adiabatic response. For the interacting Luttinger liquid the low-energy fixed point corresponds to an infinite barrier. Therefore, the adiabaticity criterion necessarily depends on the ratio $\Omega/\omega_T$.

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We would like to emphasize that a strictly linear spectrum is not a necessary requisite for determining the nature of the adiabatic limit. This is because the adiabatic limit for a 1D system with global chiral symmetry is determined by a set of relevant operators, and terms that contribute to non-linearity of the spectrum are irrelevant in the RG sense.

Note that this does not mean that the continuum field theory itself has a cut-off $v_F/\ell$, but only that at higher energies the potential structure gets resolved.

Besides the local backscattering operator $Q$ that we explicitly consider, there are other operators present in the bare Hamiltonian, that correspond to multi-particle backscattering processes. In the bare Hamiltonian each of these operators, $Q'$, has a different scaling dimension, $\Delta'$, and a different (dimensionless) bare coupling constant, $\Gamma'$, which involves $2nk_F$-Fourier-transform ($n > 1$) of the potential $V(x)$. As should be clear from the analysis presented here, it is only the most relevant operator that determines the conductance in the adiabatic limit.