Saturating Stable Matchings

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Abstract

I relate bipartite graph matchings to stable matchings. I prove a necessary and sufficient condition for the existence of a saturating stable matching, where every agent on one side is matched, for all possible preferences. I extend my analysis to perfect stable matchings, where every agent on both sides is matched.

1 Introduction

A bipartite graph is a graph $G$ with two disjoint vertex sets, $X$ and $Y$, and an edge set $E$, such that there is no edge connecting two vertices in the same set. A common goal in bipartite graphs is to connect these two sets in a matching, defined as a subset of $E$ such that no two edges share a vertex. If a vertex is the endpoint of an edge in $M$, we say it is matched; otherwise it is unmatched. Hall [1935] gives a necessary and sufficient condition for a bipartite graph to have an $X$-saturating matching, where every vertex $x \in X$ is matched.

When we imagine vertices as agents and allow them to have preferences over the other side, we have Gale and Shapley [1962]'s classic stable marriage problem. A matching is stable if there does not exist a vertex pair $(x, y) \in X \times Y$ which are not matched together but prefer each other to their partner (note that their partner may be no one - i.e. they are unmatched) under that matching; we call this a blocking pair.

Gale and Shapley [1962] prove there always exists a stable matching, giving a constructive proof by developing their famed deferred acceptance algorithm. Since their paper, the study of matchings has been taken on by

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economists in a rather different way than they are studied in graph theory, focusing on stability rather than combinatorics [Roth and Sotomayor, 1992].

So, we have two major classical theorems looking at matchings from two different perspectives. Hall [1935] gives a theorem for the existence of a saturating matching (not necessarily stable), while Gale and Shapley [1962] give a theorem for the existence of a stable matching (not necessarily saturating).

As matching applications proliferate, it may be desirable from a social perspective to have every agent matched (e.g., match everyone to a vaccine or match every medical student to a residency). Hence the question arises: can we find saturating stable matchings?

2 Main result

Consider a bipartite graph $G = (X + Y, E)$. The neighborhood $N(x)$ of a vertex $x$ is the set of vertices adjacent to, or sharing an edge with, $x$, and the neighborhood of a set of vertices is the union of each vertex’s neighborhood. The degree $\text{deg}(x)$ is the number of adjacent vertices, or $|N(x)|$.

Define $P$ to be a set of preference relations, the elements of which are each vertex’s strict preference relation over the vertices in its neighborhood (it’s acceptable partners). If $x_1$ prefers $y_1$ to $y_2$, we write $x_1 : y_1 \succ y_2$. Every vertex prefers an acceptable vertex to being unmatched, and prefers being unmatched to an unacceptable vertex. The set of all possible preference instances is $\mathcal{P}$. SM means stable matching below.

Define the following two conditions for some vertex $x$:

\begin{align*}
|N(N(x))| & \leq |N(x)| \quad (1) \\
\exists y \in N(x)(\text{deg}(y) = 1) \quad (2)
\end{align*}

Lemma 1. If some $x \in X$ satisfies condition (1) or (2) (or both), then it is matched in all SMs in all preference instances.

Proof. For contradiction’s sake, assume that this $x$ is unmatched in some SM. By assumption, this $x$ satisfies $|N(N(x))| \leq |N(x)|$ or $\exists y \in N(x)(\text{deg}(y) = 1)$, or both.

Case 1: $\exists y \in N(x)(\text{deg}(y) = 1)$ Observe that this $y$ is unmatched, as it only has one acceptable vertex, and that is $x$, which is unmatched. This $x$ and $y$ form a blocking pair so this matching is unstable, hence contradiction.
Case 2: \( \not \exists y \in N(x) \) so that \( \deg(y) = 1 \) \( x \) must satisfy \( |N(N(x))| \leq |N(x)| \).

Observe that \( x \) is only unmatched if all of the vertices in \( N(x) \) are already matched. Since vertices \( N(x) \) can only be matched to vertices in \( N(N(x)) \), this means that \( |N(N(x))| - 1 \) (the number of vertices in \( N(N(x)) \) excluding \( x \) itself) vertices are matched to \( |N(x)| \) vertices. By the pigeonhole principle, this is a contradiction.

This exhausts all the cases. \( \square \)

Lemma 2. If some \( x \in X \) does not satisfy condition (1) nor (2), then there exists some preference instance \( P \in \mathbb{P} \) under which it is unmatched in every SM.

Proof. Observe that this \( x \) is such that \( |N(N(x))| > |N(x)| \) and \( (\forall y \in N(x))(\deg(y) > 1) \).

By the assumption, there exists some \( x \in X \) that satisfies \( |N(N(x))| > |N(x)| \) and \( \forall y \in N(x)(\deg(y) > 1) \). It suffices to show there exists a \( P \) which leaves \( x \) unmatched.

Consider the following preference instance \( P \):

- \( \forall a \in N(x) : i \succ x \), where \( i \in N(N(x)) - x \)
- \( \forall b \in N(N(x)) - x : j \succ k \), where \( j \in N(x) \) and \( k \in N(b) - N(x) \)
- all other preference relations are allowed to vary

Roughly speaking, \( P \) states that all of \( x \)'s options prefer all their other acceptable vertices to \( x \) itself, and that all of \( x \)'s competitors prefer to be matched to a vertex in \( N(x) \) over any other vertex that they find acceptable.

Now we can show that \( x \) is unmatched in all SMs. Assume for contradiction’s sake that there is some SM \( M \) in which \( x \) is matched to some vertex in \( N(x) \). Observe that \( \forall v \in N(N(x)) - x \) are matched to a vertex in \( N(x) \). If some \( v \) wasn’t, then, due to the stated preferences, and the fact that all vertices in \( N(x) \) have \( \deg > 1 \) by the second part of the assumption, said \( v \) would form a blocking pair with some vertex in \( N(x) \).

However, this means that \( |N(N(x))| \) vertices (all of \( x \)'s competitors plus \( x \) itself) are matched to \( |N(x)| \) vertices. But, since \( |N(N(x))| > |N(x)| \), by the pigeonhole principle, this is a contradiction.

Therefore, \( x \) is unmatched in all SMs under \( P \). \( \square \)

\footnote{Note that the negation of \( \deg(y) = 1 \) is \( \deg(y) > 1 \), because it is in some vertex’s neighborhood, and so \( \deg(y) > 0 \).}
Now the main result.

**Theorem 1.** Every SM is X-saturating for all preference instances if and only if for all \(x \in X\), conditions (1) or (2) (or both) hold.

*Proof.* First, the if direction. For all \(x \in X\), conditions (1) or (2) or both hold. By Lemma [1] the desired statement holds.

Next, the only if direction. The contrapositive, where some \(x\) satisfies neither condition, holds by Lemma [2]. □

### 2.1 Equivalent statements

For the market designer, it is not necessarily important if *all* SMs are saturating, but if at least one exists. Further, in a real-world matching market that uses the Gale-Shapley algorithm, a very particular SM is yielded, which is either the X-optimal or Y-optimal SM (depending on the algorithm configuration\(^2\)), so the designer may be particularly interested in whether the outputted SM is saturating.

Interestingly, these are really all the same question, thanks to the following.

**Theorem (McVitie and Wilson [1970]).** “In a marriage problem of \(n\) men and \(k\) women if any person is unmarried in one stable marriage solution he or she will be unmarried in all the stable solutions.”

So, if a single SM is X-saturating (no one is “unmarried”), then any other SM is also X-saturating (including the X-optimal one, and the X-pessimal one), and indeed all of them.

**Lemma 3.** For a given preference instance \(P\), an arbitrary SM is X-saturating if and only if all SMs are X-saturating.

*Proof.* Follows from McVitie and Wilson [1970]. □

**Corollary 1.** For a given preference instance \(P\), let the set of all SMs be \(\mathbb{M}\). Then, for all preference instances, an arbitrary \(M \in \mathbb{M}\) is X-saturating if and only if for all \(x \in X\), at least one of conditions (1) and (2) hold.

*Proof.* Follows from Theorem [1] and Lemma [3]. □

\(^2\)The X-optimal SM is such that \(\forall x \in X\) prefers it to every other SM. The X-pessimal SM is such that \(\forall x \in X\) prefers every other SM to it. The X-pessimal SM is also the Y-optimal SM [Roth and Sotomayor, 1992].
Thus, the biconditional in Theorem 1 is the same for the existence of an X-saturating matching, the X-optimal SM being X-saturating, or for any arbitrary SM the market designer is interested in. This equivalence holds for subsequent results in this paper, per Lemma 3.

3 Applications

3.1 Demonstrative examples

Looking at Figure 1a, conditions (1) and (2) are violated for $x_2$, since $|N(N(x_2))| = 2 > |N(x_2)| = 1$, so X-saturating SMs do not exist for all preference instances.

In Figure 1b, which simply added one vertex to 1a, both conditions now hold for $x_2$.

Lastly, in Figure 1c, condition (1) holds for $x_1$ and $x_2$. While $x_3$ violates condition (1), it does satisfy (2), as $\deg(y_3) = 1$ and $y_3 \in N(x_3)$.

3.2 Perfect matchings

Gale and Shapley [1962] considered $|X| = |Y| = n$ and every $x \in X$ is acceptable to $y \in Y$ (and vice versa) (i.e. preferences are complete). In graph theory, this is a complete bipartite graph. Gale and Shapley [1962] say that no vertex is unmatched after the execution of their algorithm. In graph-theoretic terms, for all preference instances, the SM given by their algorithm is perfect, meaning that every vertex is matched.

This is actually implied by Theorem 1 as $\forall x \in X$ and $\forall y \in Y$ satisfy condition (1): $\forall x \in X$, $|N(x)| = n$ and $|N(N(x))| = n$ (due to being a complete bipartite graph) so condition (1) is fulfilled, and similarly $\forall y \in$
Y. Therefore, by Theorem 1, for all preference instances, every SM is $X$-saturating and $Y$-saturating, and hence perfect.

In fact, if our matching market is in one “piece”, then the only way to obtain a perfect matching if $|X| = |Y| = n$ is if preferences are complete. In Figure 2b, we can visually see that there are two different pieces - called components in graph theory. Even though preferences are incomplete (e.g., $x_3$ does not find $y_1$ acceptable), clearly a perfect SM will always exist.

I first consider graphs with only one component, like Figure 2a. A graph with only one component is called a connected graph, defined by a path existing between any two vertices [West, 1996].

**Theorem 2.** Given a connected bipartite graph $G = (X + Y, E)$ with $|X| = |Y| = n$, all SMs are perfect for all preference instances if and only if $G$ is a complete bipartite graph.

**Proof.** First, the if direction. If $G$ is a complete bipartite graph, then $\forall x \in X$ and $\forall y \in Y$ satisfy condition (1), as discussed above, so by Theorem 1 all SMs in all preference instances are $X$-saturating and $Y$-saturating, meaning perfect.

In the other direction, proceed by induction on $n$. For the base case $n = 1$, there is one vertex each in $X$ and $Y$, and they have an edge as $G$ is connected. Clearly, this is a complete bipartite graph.

Next, assume that for some $n = k$, if all SMs are perfect for all preference instances, then $G_k$ is a complete bipartite graph.

We wish to show that for $n = k + 1$, $G_{k+1}$ is also a complete bipartite graph. $G_{k+1}$ is formed by adding a vertex to each $X$ and $Y$, called $x$ and
respectively. Because $G_{k+1}$ is connected, at least one of $x$ or $y$ must be connected to a vertex other than $y$ or $x$ respectively. Without loss of generality, say it is $x$, which is connected to some not-$y$ vertex $v \in Y$.

Observe that $|N(v)| = k + 1$, and so $|N(N(x))| = k + 1$. $x$ must be matched in all SMs in all preference instances. By the contrapositive of Lemma 2 $x$ must satisfy $k + 1 \geq |N(x)| \geq |N(N(x))| = k + 1$, and so $|N(x)| = k + 1$, which means $x$ is connected to every vertex in $Y$.

By similar reasoning, $y$ is also connected to every vertex in $X$, and hence we have a complete bipartite graph.

Thus, by induction, the “only if” statement holds. \hfill \Box

I now extend Theorem 2 for the case of all graphs, not just connected ones. Looking at Figure 2b, the two components individually exhibit complete preferences over vertices in the same component. There is a special name for such components: these are called bicliques [West, 1996]. In a biclique, every vertex is connected to every vertex in the other set (a generalization of cliques to bipartite graphs). A complete bipartite graph is itself a biclique.

**Corollary 2.** Given a bipartite graph $G = (X + Y, E)$ with $|X| = |Y|$, all SMs are perfect for all preference instances if and only if every component of $G$ is a biclique.

*Proof.* Follows by applying Theorem 2 to each component of the graph. \hfill \Box

### 3.3 Matching with compatibility constraints

Maaz and Papanastasiou [2020] developed the matching with compatibility constraints problem by studying the Canadian medical residency match and point out that positions are designated as either for English speakers or French speakers; some students, being bilingual, can apply to either. Theorem 1 allows us to easily generalize their model to $n$ compatibility classes.

The vertex set $X$ is divided into $n$ possibly overlapping sets $X = A_1 \cup A_2 \cup A_3 ... \cup A_n$. The set $A_i - \cup_{j \neq i} A_j$ must be nonempty for all $i, j \in [1, n]$, meaning that there must be at least one vertex in each class that is not in any other class. The set $Y$ is partitioned into $n$ disjoint subsets $Y = B_1 \cup B_2 ... \cup B_n$. See Figure 3 for a schematic. A vertex can not find another vertex acceptable if they do not belong to the same class; otherwise they may, but not necessarily. In the special case of compatibility-wise complete
(or CW-complete) preferences that [Maaz and Papanastasiou 2020] study, every vertex finds every vertex in the same class acceptable.

**Theorem 3.** With $n$ compatibility classes, every SM is $X$-saturating in all instances of CW-complete preferences if and only if $|B_i| \geq |A_i|$ for all $1 \leq i \leq n$.

**Proof.** First, the if direction. Take an arbitrary $x \in X$. It belongs to one or more compatibility classes; let this list of classes be stored in the vector $q$. Then $|N(x)| = \sum_{i \in q} |B_i|$. Further, observe that $N(N(x))$ is the set of all vertices in $X$ that also belong to the same compatibility classes, including $x$ itself. Thus, $|N(N(x))| = \sum_{i \in q} |A_i|$. Because $|B_i| \geq |A_i|$ for all $1 \leq i \leq n$, condition (1) holds for this $x$, and indeed for all $x$. By Lemma 1, the result follows.

Next, the “only if” direction. Assume for contradiction’s sake that there exists a compatibility class $i$ such that $|B_i| < |A_i|$. There exists at least one vertex $x \in X$ that is in the $i$th compatibility class and not in any other class. Then, $|N(x)| = |B_i| < |A_i| = |N(N(x))|$, so it does not fulfill condition (1). And, it does not fulfill condition (2) either because it cannot be connected to a vertex $y$ with degree of 1, as $y$ would violate CW-complete preferences, unless $x$ is the only vertex, but that violates $|B_i| < |A_i|$. By Lemma 2, there exists a preference instance with a SM that is not $X$-saturating, which is a contradiction.

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3This is a generalization of [Maaz and Papanastasiou 2020]’s restriction that there must be a non-zero amount of students that speak only English and a non-zero amount that speak only French.
Figure 3: Matching with compatibility constraints with 2 and 3 classes. Lines indicate compatibility between every vertex in the two subsets touched by the line’s endpoints; under CW-complete preferences, lines also indicate acceptability.

(a) 2 compatibility classes

(b) 3 compatibility classes
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