A Model Independent Ultraviolet Cutoff for Theories with Charged Massive Higher Spin Fields

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Abstract

We argue that the theory of a massive higher spin field coupled to electromagnetism in flat space possesses an intrinsic, model independent, finite upper bound on its UV cutoff. By employing the Stückelberg formalism we do a systematic study to quantify the degree of singularity of the massless limit in the cases of spin 2, 3, 3/2, and 5/2. We then generalize the results for arbitrary spin to find an expression for the maximum cutoff of the theory as a function of the particle’s mass, spin, and electric charge. We also briefly explain the physical implications of the result and discuss how it could be sharpened by use of causality constraints.
1 Introduction

Powerful arguments exist that forbid massless particles from interacting with electromagnetism (EM) or gravity when their spin $s$ exceeds a certain maximum value $[1, 2, 3]$. This is $s = 1$ for EM, and $s = 2$ for gravity. Charged, massive particles instead do exist. Hadronic resonances and open-string charged states are but two examples. Even more obviously, any classical charged, spinning object can be decomposed into irreducible representations of the Poincaré group, and is thus mathematically the same as a particle. If common sense rebels against treating a charged macroscopic top in the same manner as we treat an electron, it is because we usually (and correctly) associate other properties to the objects we call elementary particles. Chief among them is that they interact as point-like objects up to distance scales parametrically larger than their Compton wavelength. A classical spinning top clearly does not satisfy this condition. Neither do high-spin hadronic resonances. They are quark bound states whose mass is always much larger than their inverse size: the former is well above $1.5 \text{ GeV}$, while the latter is $\mathcal{O}(\Lambda_{QCD}) \approx 400 \text{ MeV}$. Even seemingly true elementary particles as charged high-spin excitations of the open sting are not truly point-like. Their mass is always larger than the sting scale $M_S$, i.e. the same parameter that also sets the intrinsic non-locality scale of string theory.

Are these properties accidents? Can a high-spin massive particle be described by a local Lagrangian up to arbitrarily high energy scales? The answer to this question is no, because the no-go theorems on EM and gravitationally coupled massless particles cited above imply that the cutoff of that Lagrangian must vanish in the massless limit $m \rightarrow 0$.

To find the explicit parametric dependence of the cutoff on the mass and the relevant coupling constant is the much more difficult problem we study in this paper. This task requires additional assumptions about the interacting high-spin particle; of course, the more the assumptions, the stronger the result. We aim at obtaining model-independent, universal limits on the cutoff of the effective action describing high-spin particles. For this reason we will not get the strongest possible bound on the cutoff $\Lambda$, but rather an upper bound that no theory can beat. Let us first state the limit we will obtain, and later discuss its meaning and the assumptions we need for its derivation.

The largest cutoff of the local effective action describing a massive charged particle of spin larger than one, coupled to EM (or, more generally to a massless, Abelian vector) is

$$\Lambda_s \leq C m e^{-1/(2s-1)}, \quad C = \mathcal{O}(1) \text{ constant.}$$

(1)

This formula is valid in the limit $e \ll 1$. With $e \approx 0.3$, EM coupling is at the limit of the range of validity of Eq. $[1]$. Still, Eq. $[1]$ is not empty: it states that high-spin particles
admit a local description even for energies above their mass, when they must be treated as true dynamical degrees of freedom in the effective action.

As we already mentioned, the cutoff (1) is an upper bound. In fact, the true cutoff is lower in all known examples of UV complete theories containing high-spin, charged massive particles. The most physical example of this fact is QCD. The state of affairs is even clearer in open string theory, where one finds an infinite tower of massive charged particles with spin larger than one. The dynamics of any one of those particles in an external constant EM field can be studied independently of all others. It is described by a non-minimal yet local Lagrangian due to Argyres and Nappi [4]. The Lagrangian propagates five degrees of freedom inside the standard Lorentzian light cone [4]. It is thus exempt from the pathologies found years ago by Velo and Zwanziger [5] in their study of EM-coupled massive spin-3/2 and 2. The Argyres-Nappi Lagrangian [4] escapes those problems at the cost of being extremely non-minimal: it contains an infinite tower of “quadrupole” non-renormalizable couplings involving two derivatives of the spin-2 field \( h_{\mu\nu} \) and arbitrary powers of the EM field strength \( F_{\mu\nu} \). Somewhat symbolically their generic form is

\[
\mathcal{L} = \left( \alpha' e F \right)^n \partial_\rho \partial_\sigma h^{\rho\sigma}.
\]

The cutoff scale at which a Lagrangian containing non-renormalizable terms such as (2) breaks down is \( \alpha'^{-1/2} = M_S \). Validity of the open-string perturbation series requires a string coupling constant \( g_S = e^2 \ll 1 \); thus, the cutoff of the effective Argyres-Nappi action is much smaller than the “optimal” one.

The case of charged, massive spin-1 is worth special attention, because some of the technical procedures we employ in this paper do not work there, for reasons we shall explain later. Nevertheless, Eq. (1) still holds, since the cutoff of a Lagrangian containing only a charged massive spin-1 and the EM field is indeed \( \mathcal{O}(m/e) \). In this case, we also know explicit UV completions of such a theory. One such completion is to embed the spin-1 field as a vector of an \( SU(2) \) gauge group broken to \( U(1) \) by a Higgs field, which is also a vector of \( SU(2) \). Then, the bound (1) simply means that either an extra degree of freedom (the neutral Higgs) exists, with mass smaller than \( \Lambda_1 = m/e \), or the theory becomes strongly interacting and unitarizes at \( \Lambda_1 \).

To summarize, the cutoff (1) is model independent, but its very generality means that we have very little information about the UV completion at or above that energy scale. To understand the technique we will be using to derive Eq. (1), let us recall what we are looking for. First of all, we shall limit ourselves to finding an effective action that generates sensible low-energy scattering amplitudes between states with a finite number
of hard quanta. So, we will not attempt to solve the Velo-Zwanziger causality problem or any other pathology that may arise in external non-trivial backgrounds. To compute these S-matrix elements, we need an effective local Lagrangian for a charged massive particle of arbitrary spin. While many choices exist in the literature for free Lagrangians, our choice is one with as few auxiliary fields as possible, e.g. the Singh-Hagen Lagrangian [6]. All those Lagrangians enjoy restricted gauge invariances in the massless limit. These gauge invariances are broken by the interactions. This phenomenon is at the root of the no-go theorems on interacting massless particles [1, 2, 3]. It also makes it hard to find the UV cutoff of the interacting Lagrangians, since gauge invariances of the kinetic term change the canonical dimensions of the fields. The way out of this problem is well known: introduce “compensating” degrees of freedom (Stückelberg fields) which restore the gauge invariance, and fix this newly introduced gauge invariance to set all kinetic terms to a canonical form – i.e. a form that gives canonical dimensions to all fields. Once all fields have canonical dimensions (1 for Bosons and 3/2 for Fermions), then the interacting Lagrangian will acquire certain non-renormalizable interactions involving the Stückelberg fields. These will be weighted by coupling constants with negative mass dimensions; symbolically

\[ L = L_{\text{renormalizable}} + \sum_{n>0} (\Lambda_n)^{-n} O_{n+4}, \]

where \( O_{n+4} \) denotes operators of dimension \((n + 4)\). Some of these operators can be eliminated by field redefinitions or by adding non-minimal terms to the Lagrangian \( L \); the smallest \( \Lambda_n \) in the surviving terms defines the ultimate cutoff of our effective Lagrangian. The procedure to follow can be described more fully and more technically in a few steps, namely:

1. Write a (non-gauge invariant) massive Lagrangian with minimal number of auxiliary fields (e.g., à la Singh and Hagen).

2. Introduce Stückelberg fields and Stückelberg gauge symmetry. Any auxiliary field appearing in the Lagrangian in step 1 that is not a (gamma)trace of the high-spin field must be identified as a trace of a Stückelberg field by appropriate field redefinitions. For such a field one obtains a gauge invariance for free.

3. Complexify the fields, if required, and introduce interaction with a new gauge field (e.g., electromagnetism) by replacing ordinary derivatives with covariant ones.

4. Diagonalize all kinetic terms, i.e., get rid of kinetic mixing by field redefinitions.
and/or by covariant gauge fixing terms.

5. Look for the most divergent term(s) in the Lagrangian, in an appropriate limit of zero mass and zero coupling. These terms will involve fields that are zero (i.e. gauge) modes of the free kinetic operator before gauge fixing. One needs to take care of the non-commutativity of covariant derivatives and correctly interpret terms proportional to the equations of motion.

6. Try to remove non-renormalizable terms by adding non-minimal terms. This may not always be possible.

7. Find the cutoff of the effective field theory, and interpret the physics implied by the divergent term(s).

Steps 1 and 2 lead us to a gauge invariant description of massive high spin fields — the so-called Stückelberg formalism \[7, 8, 9, 10, 11, 12\]. Such gauge invariant description is convenient in that it allows one to introduce interactions simply by replacing ordinary derivatives with covariant ones (it is also useful for studying partially massless theories that appear in (A)dS space-time \[16\]). One can obtain the Stückelberg invariant Lagrangian by starting with the massless Lagrangian \[17\] in (4+1)D, and then Kaluza-Klein reducing it to (3+1)D \[18, 19\]. The higher dimensional gauge invariance gives rise to the Stückelberg symmetry in lower dimension. By suitably gauge fixing the resulting Lagrangian, one gets a (non-gauge invariant) massive Lagrangian with minimal number of auxiliary fields, as mentioned in Step 1, which is equivalent to the Singh-Hagen Lagrangian \[6\] up to field redefinitions. Then the Stückelberg trick in Step 2 is done with the help of carefully constructed gauge invariant tensors.

The procedure outlined above was carried out for spin-2 fields in \[20, 21\]; in Section 2, we review the main points of these references and give an improved derivation of the cutoff \(\Lambda_2 = me^{-1/3}\). Section 3 is devoted to studying EM interactions of a spin-3 particle, where for the first time we find a new complication, namely an extra auxiliary scalar field that cannot be set to zero by gauge transformations. Sections 4 and 5 apply the Stückelberg method to charged Fermions; first spin-3/2 Fermions, then spin-5/2. Section 6 generalizes the findings of all previous Sections to arbitrary spin. There the bound \(1\) is at last derived. Section 7 summarizes our findings, and briefly discusses a few additional topics related to the physics of interacting high spin fields. In particular, taking heed of the well known case of charged spin-1 particles, we will describe there some alternative

\[^{1}\text{There exist other kind of gauge invariant descriptions for massive higher spins, e.g. the BRST method [13], the frame-like formulation [14], and the quartet formulation [15].}\]
possibilities for the UV completion of a theory of higher spin particles. We will also suggest that causality constraints in external background fields may give stronger model independent bounds on the UV cutoff.

2 Massive Spin-2 Field Coupled to EM

The electromagnetic interaction of massive spin-2 field has been studied by various authors \[5, 8, 20, 21, 22, 23, 24, 25\]. Here we consider flat space-time background.

First we write down the Pauli-Fierz Lagrangian \[26\] with Stückelberg fields, complexify the fields, and then replace ordinary derivatives with covariant ones. Thus we obtain

\[
L = -|D_\mu \tilde{h}_{\nu\rho}|^2 + 2|D_\mu \tilde{h}^{\mu\nu}|^2 + |D_\mu \tilde{h}|^2 - [D_\mu \tilde{h}^{*\mu\nu}D_\nu \tilde{h} + \text{c.c.}] - m^2[\tilde{h}_{\mu\nu} \tilde{h}^{\mu\nu} - \tilde{h}^* \tilde{h}],
\]

with

\[
\tilde{h}_{\mu\nu} = h_{\mu\nu} + \frac{1}{m} D_\mu \left( B_\nu - \frac{1}{2m} D_\nu \phi \right) + \frac{1}{m} D_\nu \left( B_\mu - \frac{1}{2m} D_\mu \phi \right). \tag{5}
\]

Lagrangian (4) now enjoys a covariant Stückelberg symmetry:

\[
\delta h_{\mu\nu} = D_\mu \lambda_\nu + D_\nu \lambda_\mu, \tag{6}
\]

\[
\delta B_\mu = D_\mu \lambda - m \lambda_\mu, \tag{7}
\]

\[
\delta \phi = 2m \lambda. \tag{8}
\]

Next we diagonalize the kinetic operators to make sure that the propagators in the theory have good high energy behavior, i.e., that all propagators are proportional to \(1/p^2\) for momenta \(p^2 \gg m^2\). The field redefinition:

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \phi, \tag{9}
\]

eliminates the kinetic mixings between \(\phi\) and \(h_{\mu\nu}, h\), and also generates a kinetic term for \(\phi\) with the correct sign. After adding the gauge fixing terms:

\[
L_{gf1} = -2 |D_\nu h^{\mu\nu} - (1/2)D^\mu h + mB^\mu|^2, \tag{10}
\]

\[
L_{gf2} = -2 |D_\mu B^\mu + (m/2)(h - 3\phi)|^2, \tag{11}
\]

\[\text{The authors in Ref.} \[8, 23, 27\] \text{also considered a gauge invariant description to investigate consistent interactions of massive high-spin fields. In case of spin-2, say, they introduce Stückelberg fields only in the mass term. This procedure already breaks Stückelberg invariance at tree level. Our approach, instead, guarantees by construction that Stückelberg symmetry is kept intact by the minimal substitution.}\]
we have exhausted all gauge freedom to obtain diagonal kinetic terms. We are left with

\[ L = h^* \mu\nu (\Box - m^2) h_{\mu\nu} - \frac{1}{2} h^* (\Box - m^2) h + 2B^* \mu (\Box - m^2) B^\mu + \frac{3}{2} \phi^* (\Box - m^2) \phi \]

\[- \frac{1}{4} F^2_{\mu\nu} + L_{\text{int}}. \]  

(12)

Here \( L_{\text{int}} \) contains all interaction operators, which have canonical dimension 4 through 8. Among the non-renormalizable operators the most potentially dangerous ones, in the high energy limit \( m \to 0 \), are the ones with the highest dimensionality. Since in the decoupling limit \( e \to 0 \) one is left only with marginal and relevant operators, any non-renormalizable term must contain at least one power of \( e \). Parametrically in \( e \ll 1 \), for any given operator dimensionality, the \( \mathcal{O}(e) \)-terms are more dangerous than the others. Therefore we are more interested in terms linear in \( e \).

If there exists an \( \mathcal{O}(e) \)-term that is proportional to the equations of motion, it can be eliminated by a local field redefinition. But this introduces \( \mathcal{O}(e^2) \)-terms, which contain even higher dimensional operators. One needs to see if these \( \mathcal{O}(e^2) \)-terms be canceled by adding local functions of the high spin field. At \( \mathcal{O}(e) \) we have the following operators of dimensionality 8 and 7:

\[ L_8 = \frac{e}{m^4} \partial_\mu F^{\mu\nu} [(i/2) \partial_\rho \phi^* \partial^\rho \partial_\nu \phi + \text{c.c.}] \equiv \frac{e}{m^4} \partial_\mu F^{\mu\nu} J_\nu \]  

(13)

\[ L_7 = \frac{ie}{m^3} F^{\mu\nu} \{ 2 \partial_\mu B^*_{\rho} \partial^\rho \phi - \partial_\mu B^*_{\nu} \Box \phi \} + \text{c.c.} \]  

(14)

The dimension-8 operator can be removed by the field redefinition:

\[ A_\mu \to A_\mu - (e/m^4) J_\mu, \]  

(15)

which yields

\[ L = - \frac{1}{4} F^2_{\mu\nu} + \frac{e}{m^4} \partial_\mu F^{\mu\nu} J_\nu + \ldots, \to - \frac{1}{4} F^2_{\mu\nu} + \frac{e^2}{4m^8} (\partial_\mu J_\nu - \partial_\nu J_\mu)^2 + \ldots \]  

(16)

This by itself does not improve the degree of divergence. The field redefinition is helpful, only if we may cancel the \( \mathcal{O}(e^2) \)-term in Eq. (16) by adding some local functions of \( \hat{h}_{\mu\nu} \). Such functions may be present if, for example, there exist other interactions, that are linear in \( F_{\mu\nu} \) and mix the spin-2 field with other more massive degrees of freedom. By integrating out these additional degrees of freedom, one ends up with additional EM
interactions at $\mathcal{O}(e^2)$, that involve only the spin-2 field. Indeed the term

$$L_{\text{add}} = \frac{e^2}{4}(\tilde{h}_{\mu\rho}^* \tilde{h}_{\rho\nu} - \tilde{h}_{\nu\rho}^* \tilde{h}_{\mu\rho}^*)^2$$

(17)

eliminates the $\mathcal{O}(e^2)$-term in Eq. (16). In doing so, we introduce terms proportional to $e^2/m^7$, coming both from $L_{\text{add}}$, and from the shift of $A_\mu$ acting on the dimension-7 operator in (14). We have

$$L_{11} = \frac{e^2}{2m^7} \{ \partial^\mu \partial_\sigma \phi^* \partial^\nu \partial_\sigma \phi - (\mu \leftrightarrow \nu) \} \{ 2\partial_\mu B_\rho^* \partial^\rho \partial_\nu \phi - \partial_\mu B_\rho^* \Box \phi \} + \text{c.c.} \quad (18)$$

We want to eliminate this dangerous dimension-11 operator up to a total derivative by adding local counter-terms. In Appendix A using cohomological arguments it is shown that this is impossible. The key point is that any local function of (ungauged) $\tilde{h}_{\mu\nu}$ is manifestly invariant under the symmetry:

$$B_\mu \rightarrow B_\mu + b_\mu x_\nu,$$

$$\phi \rightarrow \phi + c + c_\mu x^\mu,$$

(19)  (20)

where $b_\mu, b_\mu, c,$ and $c_\mu$ are constants, with $b_\mu = -b_\mu$. On the other hand, the dimension-11 operator in (18) is invariant under (19, 20) only up to a nontrivial total derivative\footnote{Under (19, 20) the redefined $U(1)$ field transforms as $\delta A_\mu = (e/m^4)\delta J_\mu = \partial_\mu[(ie/2m^4)c_\rho(\partial^\rho \phi - \partial^\rho \phi^*)]$, which is a total derivative. Since $A_\mu$ appears only in $F_{\mu\nu}$, we can set $\delta A_\mu = 0.$}. As shown in Appendix A, this property is sufficient to guarantee that $L_{11}$ cannot be canceled up to a total derivative by local counter-terms. In Appendix B we present a physical explanation of this fact, by showing that no local function of $\tilde{h}_{\mu\nu}$ exists, that completely cancels the contribution of $L_{11}$ to scattering processes.

However, as we will see now, addition of a dipole term leaves us only with terms proportional to $e/m^3$, which is already an improvement over field redefinition plus addition of local term. Indeed, a dipole term $ie\alpha F_{\mu\nu} h_{\mu\nu}^* \tilde{h}_{\rho\sigma}^*$ gives

$$L^{(\text{dipole})}_8 = \frac{e}{m^4} \partial_\mu F_{\mu\nu}[-(i\alpha/2)\partial_\rho \phi^* \partial^\rho \partial_\nu \phi + \text{c.c.}] \quad (21)$$

$$L^{(\text{dipole})}_7 = -\frac{ie\alpha}{m^3} F^{\mu\nu} \partial_\rho (\mu B_\rho^* \partial^\sigma \partial_\nu \phi + \text{c.c.}] \quad (22)$$

If we choose $\alpha = 1$, in our non-minimal Lagrangian the dimension-8 operators at $\mathcal{O}(e)$...
cancel, and we are left only with dimension-7 operators:

\[ L_7^{(NM)} = \frac{ie}{2m^3} F^\mu_\nu \left\{ 2\partial_{\lfloor \rho} B^{*}_{\rho} B^{\nu} \partial_\sigma \phi - \partial_{\lfloor \rho} B^{*}_{\rho} \Box \phi \right\} + \text{c.c.} \]  

(23)

These operators contain pieces that are not proportional to any of the equations of motion. Therefore the degree of divergence cannot be improved further. In the scaling limit: \( m \to 0 \) and \( e \to 0 \), such that \( e/m^3 \) = constant, the non-minimal Lagrangian reduces to:

\[ L = L_{\text{kin}} + \frac{ie}{2m^3} F^\mu_\nu \left\{ 2\partial_{\lfloor \rho} B^{*}_{\rho} \partial_\sigma \phi - \partial_{\lfloor \rho} B^{*}_{\rho} \Box \phi \right\} + \text{c.c.} \]  

(24)

The theory has an intrinsic cutoff:

\[ \Lambda_2 = \frac{m}{e^{1/3}}. \]  

(25)

It is worth pointing out a couple of interesting facts. First, the above Lagrangian has acquired a \( U(1) \) gauge invariance for the vector Stückelberg field \( B_\mu \), i.e., only the field strength of \( B_\mu \) appears in the Lagrangian. This is because an appropriately chosen dipole term cancels not only the \( \mathcal{O}(e) \) dimension-8 operator, but also any \( \mathcal{O}(e) \) dimension-7 operators that are generated by a transformation \( \delta B_\mu = \partial_\mu \theta \). Second, the dimension-7 operators in (23) all cancel for constant \( F^\mu_\nu \). In fact, for this special case, there exists only one possible dimension-7 operator, and no dimension-8 operators at all. The former, of course, can be eliminated by appropriately choosing the dipole coefficient.

### 3 Electromagnetically Coupled Massive Spin-3 Field

The Lagrange formulation for spin-3 fields has been studied in [28]. Their geometric and gauge theoretic aspects have been discussed in [29]. Electromagnetic and gravitational interactions of massive spin-3 fields have respectively been considered in [30] and [31].

For our purpose, what we need is a Stückelberg Lagrangian for massive spin-3 field that can be readily coupled to a \( U(1) \) field or gravity, while maintaining at the same time the covariant version of the Stückelberg symmetry. It can be obtained by the procedure described below. Let us start with the Lagrangian for a massless spin-3 field [17] in (4+1)D:

\[ L = -\frac{1}{2} (\partial_\mu H_{MNP})^2 + \frac{3}{2} (\partial_\mu H^M_{MNP})^2 + \frac{3}{2} (\partial_\mu H_N)^2 + 3H_P \partial_\mu \partial_\nu H_{MNP} + \frac{3}{4} (\partial_\mu H^M)^2, \]  

(26)

---

\[ 4 \text{This is the true cutoff of the theory, while the “optimistic” one: } \Lambda = m/\sqrt{e}, \text{ mentioned in [20] is too optimistic.} \]
where $H_N = H^M_{MN}$. The above Lagrangian has the gauge symmetry:

$$\delta H_{MNP} = \partial_M \Lambda_{NP}, \quad \Lambda^M_M = 0. \quad (27)$$

As we will see, the tracelessness condition on the gauge parameter has important consequences. Now we do a Kaluza-Klein (KK) reduction by writing

$$H_{MNP}(x^\mu, x_5) = \left(\frac{m}{2\pi}\right)^{1/2} \frac{1}{\sqrt{2}} \left\{ h_{MNP}(x^\mu) e^{imx_5} + \text{c.c.} \right\}, \quad (28)$$

where we compactify the $x_5$-dimension on a circle of radius $1/m$. In (3+1)D this gives rise to a spin-3 field $h_{\mu\nu\rho}$, a spin-2 field $W_{\mu\nu} \equiv -ih_{\mu\nu5}$, a vector field $B_\mu \equiv -h_{\mu55}$, and a scalar $\phi \equiv ih_{555}$. We also write the gauge parameter $\Lambda_{MN}$ as

$$\Lambda_{MN}(x^\mu, x_5) = \left(\frac{m}{2\pi}\right)^{1/2} \frac{1}{\sqrt{2}} \left\{ \lambda_{MN}(x^\mu) e^{imx_5} + \text{c.c.} \right\}, \quad (29)$$

so that in (3+1)D we have three gauge parameters: $\lambda_{\mu\nu}, \lambda_\mu \equiv -i\lambda_{5\mu}$, and $\lambda \equiv -\lambda_{55}$. The higher dimensional gauge invariance $(27)$ translates itself in lower dimension into the St"uckelberg symmetry:

$$\delta h_{\mu\nu\rho} = \partial_\mu (\lambda_{\nu\rho}), \quad (30)$$
$$\delta W_{\mu\nu} = \partial_\mu (\lambda_\nu) + m\lambda_{\mu\nu}, \quad (31)$$
$$\delta B_\mu = \partial_\mu \lambda + 2m\lambda_\mu, \quad (32)$$
$$\delta \phi = 3m\lambda. \quad (33)$$

The tracelessness of the 5D gauge parameter gives rise to the following condition:

$$\lambda^\mu_\mu = \lambda. \quad (34)$$

We can gauge-fix the KK-reduced Lagrangian by setting $W^T_{\mu\nu} = 0, B_\mu = 0, \text{ and } \phi = 0$. Note that because of the constraint $(34)$ only the traceless part $W^T_{\mu\nu}$ of the spin-2 field $W_{\mu\nu}$ can be set to zero. This means that the gauge-fixed 4D Lagrangian, which describes a massive spin-3 field, unavoidably contains an auxiliary scalar field $W$ — the trace of the would be spin-2 St"uckelberg field $W_{\mu\nu}$. We get

$$L = -\frac{1}{2} (\partial_\sigma h_{\mu\nu\rho})^2 + \frac{3}{2} (\partial_\mu h_{\nu\rho})^2 + \frac{3}{4} (\partial_\mu h^\mu)^2 + \frac{3}{2} (\partial_\mu h_{\nu})^2 + 3h_\mu \partial_\mu \partial_\nu h_{\nu\rho}$$
$$- \frac{m^2}{2} (h_{\mu\nu\rho} - 3h_\mu^2) + \frac{9}{16} (\partial_\mu W)^2 + \frac{9}{4} m^2W^2 - \frac{3}{4} m \partial_\mu h^\mu W. \quad (35)$$
where \( h_\mu = h_\rho^{\rho \mu} \) is the trace of the spin-3 field. This is the Lagrangian for a massive spin-3 field with minimal number of auxiliary fields. After some field redefinitions, it is the same as the Singh-Hagen spin-3 Lagrangian [6, 11]. Considering the gauge conditions one finds that one can exactly reproduce the KK-reduced Lagrangian \( \text{before gauge-fixing} \) by the following field redefinitions:

\[
\begin{align*}
    h_{\mu \nu \rho} &\rightarrow \tilde{h}_{\mu \nu \rho} = h_{\mu \nu \rho} - \frac{1}{m} \partial_\mu W_\nu \rho + \frac{1}{2m^2} \partial_\mu (\partial_\nu B_\rho) - \frac{1}{6m^3} \partial_\mu (\partial_\nu \partial_\rho) \phi \\
    + \frac{1}{4m} \eta_{\mu \rho} \partial_\rho \left( W - \frac{3}{4} \phi + \frac{1}{m} \partial_\sigma B^\sigma + \frac{1}{3m^2} \square \phi \right),
\end{align*}
\]

\( (36) \)

\[
\begin{align*}
    W &\rightarrow \tilde{W} = W - \frac{1}{3} \phi - \frac{1}{m} \partial_\mu B^\mu + \frac{1}{3m^2} \square \phi.
\end{align*}
\]

\( (37) \)

While the Lagrangian \( (35) \) does not have any manifest Stückelberg invariance, after performing the field redefinitions \( (36, 37) \), the Stückelberg invariance is manifest in a trivial manner, because in fact the tensors \( \tilde{h}_{\mu \nu \rho} \) and \( \tilde{W} \) themselves are invariant under the Stückelberg transformations \( (30-33) \). The most important lesson here is that when we couple the theory to a \( U(1) \) field or gravity, the covariant counterparts of the tensors \( (36, 37) \) are still invariant under the covariant Stückelberg transformations. Therefore, for spin-3 we have been able to construct a consistent Lagrangian that can be readily coupled to a gauge field, while maintaining at the same time the covariant version of the Stückelberg symmetry.

Now, we couple the massive spin-3 field to electromagnetism by complexifying the fields in the Lagrangian \( (35) \), and replacing ordinary derivatives with covariant ones \( \partial_\mu \rightarrow D_\mu \):

\[
\begin{align*}
    L &= -(D_\alpha \tilde{h}_{\mu \rho})^2 + 3|D_\mu \tilde{h}^{\mu \rho}|^2 + \frac{3}{2} D_\mu \tilde{h}^\mu |^2 + 3|D_\mu \tilde{h}_\mu|^2 - 3(D_\mu \tilde{h}_\nu^* D_\rho \tilde{h}^{\mu \rho} + \text{c.c.}) \\
    &\quad - m^2 (\tilde{h}^{*}_{\mu \rho} \tilde{h}^{\mu \rho} - 3 \tilde{h}^{*}_\mu \tilde{h}_\mu) + \frac{9}{8} |D_\mu \tilde{W}|^2 + \frac{9}{2} m^2 \tilde{W}^* \tilde{W} - \frac{3}{4} m (D_\mu \tilde{h}^\mu \tilde{W}^* + \text{c.c.}).
\end{align*}
\]

\( (38) \)

Here the twiddled fields are given by the covariant version of the tensors \( (36, 37) \):

\[
\begin{align*}
    \tilde{h}_{\mu \nu \rho} &= h_{\mu \nu \rho} - \frac{1}{m} D_\mu W_{\nu \rho} + \frac{1}{2m^2} D_\mu (D_\nu B_\rho) - \frac{1}{6m^3} D_\mu (D_\nu D_\rho) \phi \\
    &\quad + \frac{1}{4m} \eta_{\mu \rho} D_\rho \left( W - \frac{1}{3} \phi - \frac{1}{m} D_\sigma B^\sigma + \frac{1}{3m^2} D_\sigma D^\sigma \phi \right),
\end{align*}
\]

\( (39) \)

\[
\begin{align*}
    \tilde{W} &= W - \frac{1}{3} \phi - \frac{1}{m} D_\mu B^\mu + \frac{1}{3m^2} D_\mu D^\mu \phi.
\end{align*}
\]

\( (40) \)

One can explicitly work out the various terms in the Lagrangian, keeping in mind the non-
commutativity of covariant derivatives: \([D_\mu, D_\nu] = \pm ieF_{\mu\nu}\). The Lagrangian becomes the sum of three pieces:

\[
L = L_{\text{free}} + L_{\text{int}} - \frac{1}{4} F_{\mu\nu}^2. 
\]

(41)

Here \(L_{\text{free}}\) is the free part of the Lagrangian; it consists of kinetic terms, mass terms, and certain dimension-3 operators, but no higher dimensional operators. \(L_{\text{int}}\) is the interaction Lagrangian, that consists of various terms, each one containing at least one power of \(m\) and possibly an inverse power of e, and possibly an inverse power of \(m (1/m^6 \text{ at most})\). The higher dimensional operators appearing here are exactly what we are interested in. However, we first diagonalize the kinetic terms, by performing field redefinitions and appropriate gauge fixing, in such a manner that the propagators in the theory have a good high energy behavior. We have

\[
L_{\text{free}} = \left\{ -|\partial_\sigma h_{\mu\rho\sigma}|^2 + 3|\partial_\mu h^{\mu\rho\rho}|^2 + (3/2)|\partial_\mu h^{\mu}\rho|^2 + 3|\partial_\mu h_{\nu}|^2 + 3(h^*_\mu \partial_\rho \partial_\nu h^{\mu\rho\nu} + \text{c.c.)} 
- m^2(h^{\mu\rho\rho} h^{\mu\rho\rho} - 3h^*_\mu h^{\mu\rho\rho} - 3[B^*_\rho (\partial_\sigma h^{\mu\rho\rho} - \Box h^{\mu\rho\rho} - (1/2) \partial^\rho \partial^\mu h_\mu) + \text{c.c.}]
+(9/2)|\partial_\rho B^\rho|^2 \right\} + 3\left\{ -|\partial_\rho W_{\mu\nu}|^2 + 2|\partial_\mu W^{\mu\nu}|^2 + |\partial_\mu W|^2 + (W^* \partial_\mu \partial_\nu W^{\mu\nu} + \text{c.c.})
- [\phi^*(\partial_\mu \partial_\nu W^{\mu\nu} - \Box W) + \text{c.c.}] + (2/3)|\partial_\mu \phi|^2 \right\} + \frac{3m^2}{2}\left\{ [2h^*_\mu \partial^\rho W^{\mu\rho} - 4h^*_\mu \partial_\nu W^{\mu\nu} - \partial^\mu h^*_\mu (W - \phi) - \partial^\mu B^*_\mu (3W - \phi)] + \text{c.c.} \right\} + \frac{m^2}{2}|3W - \phi|^2.
\]

(42)

We see that among the dimension-4 kinetic operators, the spin-1 field \(B_\mu\) mixes only with the spin-3 field, \(h_{\mu\nu\rho}\), or its trace, while the spin-0 field \(\phi\) mixes only with the spin-2 field, \(W_{\mu\nu}\), or its trace. Both kinds of mixing can be eliminated by standard field redefinitions:

\[
\begin{align*}
h_{\mu\nu\rho} & \rightarrow h_{\mu\nu\rho} + \frac{1}{D} \eta_{(\mu\nu} B_{\rho)}, \\
W_{\mu\nu} & \rightarrow W_{\mu\nu} + \frac{1}{D - 2} \eta_{\mu\nu} \phi.
\end{align*}
\]

(43)\hspace{1cm} (44)

where \(D = 4\) is the space-time dimensionality. The other mixing terms can also be removed by adding to the Lagrangian the following gauge-fixing terms:

\[
\begin{align*}
L_{gf1} &= -3 |D_\rho h^{\mu\rho\rho} - (1/2)(D^\mu h_{\nu} + D^\nu h_{\mu}) - mW^{\mu\nu} + (m/4) \eta^{\mu\nu} W|^2, \\
L_{gf2} &= -6 |D_\rho W^{\mu\nu} - (1/2)D^\mu W - (m/2)h^{\mu\nu} - (5m/4) B^\mu |^2, \\
L_{gf3} &= -(15/4) |D_\mu B^\mu - mW - 2m\phi|^2.
\end{align*}
\]

(45)\hspace{1cm} (46)\hspace{1cm} (47)

Note that in adding the term \(L_{gf1}\) we have used up the freedom of a traceless symmetric rank-2 gauge parameter. Similarly \(L_{gf2}\) and \(L_{gf3}\) were added at the cost of a vector and a scalar parameter respectively. This fully fixes all gauge invariances. Now we are left with
a Lagrangian where all the kinetic terms are diagonal:

\[
L = h^{\ast \mu \nu \rho}(\Box - m^2)h_{\mu \nu \rho} - \frac{3}{2}h^{\ast \mu}(\Box - m^2)h_{\mu} + 3W^{\ast \mu \nu}(\Box - m^2)W_{\mu \nu} + \frac{15}{4}B^{\ast \mu}(\Box - m^2)B_{\mu}
\]

\[
- \frac{3}{2}W^{\ast}(\Box - m^2)W + \frac{5}{2}\phi^{\ast}(\Box - m^2)\phi - \frac{1}{4}F_{\mu \nu}^2 + L_{\text{int}}. \tag{48}
\]

Here all the propagators have the same pole, which is necessary to cancel spurious poles in tree-level physical amplitudes.

Now, we turn our attention to the interaction terms, which may contain higher dimensional operators. Note that with all the kinetic terms diagonalized, we must assign standard (engineering) canonical dimensions to the higher-order operators in the interaction Lagrangian, so that we can interpret ours as an effective field theory valid up to some cutoff determined by the most divergent term in the \( m \to 0 \) limit. Schematically,

\[
L_{\text{int}} = \sum_{n=4}^{10} L_n. \tag{49}
\]

Operators with canonical dimension \( n \) are contained in \( L_n \); these are multiplied by a factor \( m^{4-n} \). The fact that in the \( e \to 0 \) limit no irrelevant operators exist implies that any higher dimensional operator is at least linear in \( e \) (and therefore in \( F_{\mu \nu} \)). For fixed \( e \), in the high energy limit \( m \to 0 \), the higher the \( n \), the more potentially dangerous the operator is. Notice that the gauge fixing terms can only generate a few harmless, power-counting renormalizable interactions, that are regular in the massless limit.

The most interesting \( \mathcal{O}(e) \) non-renormalizable operators can be computed in a clever way. To wit: in view of the powers of \( 1/m \) appearing in the various terms in the twiddled fields \( \{39, 40\} \), it is clear that the dimension-10 and -9 operators may arise only from the first line of Lagrangian \( \{38\} \), which is nothing but the gauged Lagrangian for a massless spin-3 field. We write this as:

\[
L_{\text{massless}} = \tilde{h}^{\ast \mu_1 \mu_2 \mu_3} T^{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \tilde{h}_{\nu_1 \nu_2 \nu_3}, \tag{50}
\]

where

\[
T^{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \equiv -3\eta^{\mu_1 \mu_2}\eta^{\mu_3 \nu_3} D_{\mu_1} D_{\nu_1} + 3\eta^{\mu_1 \nu_1}\eta^{\mu_2 \nu_2} D_{\mu_1} D_{\mu_2} + 3\eta^{\mu_1 \mu_2}\eta^{\mu_3 \nu_3} D_{\nu_1} D_{\nu_2}
\]

\[
- \frac{3}{2}\eta^{\mu_2 \mu_3}\eta^{\nu_2 \nu_3} D_{\mu_1} D_{\nu_1} + (\eta^{\mu_1 \nu_1}\eta^{\mu_2 \nu_2} - 3\eta^{\mu_1 \mu_2} \nu_1 \nu_2)\eta^{\mu_3 \nu_3} D_{\mu} D_{\rho}, \tag{51}
\]

with symmetrization assumed in \((\mu_1, \mu_2, \mu_3)\) and in \((\nu_1, \nu_2, \nu_3)\). Keeping in mind that
\[ D_\mu \equiv \partial_\mu \pm ieA_\mu, \] we can expand (50) in powers of \( e \):

\[ L_{\text{massless}} = \tilde{h}^{(0)}_{\mu_1 \mu_2 \mu_3} T_0^{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \tilde{h}^{(0)}_{\nu_1 \nu_2 \nu_3} + i e \tilde{h}^{(1)}_{\mu_1 \mu_2 \mu_3} T_1^{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \tilde{h}^{(0)}_{\nu_1 \nu_2 \nu_3} + \text{c.c.} + \mathcal{O}(e^2), \quad (52) \]

where

\[ T_0^{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \equiv -3 \eta^{\mu_2 \nu_1} \eta^{\mu_3 \nu_2} \partial_{\mu_1} \partial_{\nu_1} + 3 \eta^{\mu_1 \nu_2} \eta^{\mu_3 \nu_3} \partial_{\mu_2} + 3 \eta^{\mu_1 \mu_2} \eta^{\mu_3 \nu_3} \partial_{\nu_3} \partial_{\nu_2} - \frac{3}{2} \eta^{\mu_2 \mu_3} \eta^{\nu_3 \nu_3} \partial_{\mu_1} \partial_{\nu_1} + (\eta^{\mu_1 \nu_2} \eta^{\mu_2 \nu_3} - 3 \eta^{\mu_1 \mu_2} \eta^{\nu_1 \nu_3}) \eta^{\mu_3 \nu_3} \square, \quad (53) \]

\[ \tilde{h}^{(0)}_{\mu \nu \rho} = h_{\mu \nu \rho} - \frac{1}{12 m} \eta_{(\mu \nu \rho)} \phi - \frac{1}{m} \left\{ \partial_{\mu} \left( W_{\nu \rho} - \frac{1}{4} \eta_{\nu \rho} W \right) + \text{cyclic in } (\mu, \nu, \rho) \right\} + \frac{1}{m^2} \left\{ \partial_{\mu} \left( \frac{1}{2} \partial_{(\nu B_{\rho})} - \frac{1}{4} \eta_{\nu \rho} \partial_\sigma B_\sigma \right) + \text{cyclic in } (\mu, \nu, \rho) \right\} - \frac{1}{3 m^3} \left\{ \partial_{\mu} \left( \partial_\nu \partial_\rho \phi - \frac{1}{4} \eta_{\nu \rho} \square \phi \right) + \text{cyclic in } (\mu, \nu, \rho) \right\}, \quad (54) \]

\[ T_1^{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \equiv -3 \eta^{\mu_2 \nu_1} \eta^{\mu_3 \nu_3} (A_{\mu_1} \partial_{\nu_1} - \overline{\partial}_{\mu_1} A_{\nu_1}) + 3 \eta^{\mu_1 \nu_2} \eta^{\mu_3 \nu_3} (A_{\mu_2} \partial_{\nu_2} - \overline{\partial}_{\mu_2} A_{\nu_2}) + 3 \eta^{\mu_1 \mu_2} \eta^{\nu_1 \nu_3} (A_{\mu_3} \partial_{\nu_3} - \overline{\partial}_{\mu_3} A_{\nu_3}) + (\eta^{\mu_1 \nu_1} \eta^{\mu_3 \nu_3} - 3 \eta^{\mu_1 \mu_2} \eta^{\nu_1 \nu_3}) \eta^{\mu_3 \nu_3} (A_{\rho} \partial_{\rho} - \overline{\partial}_{\rho} A_{\rho}), \quad (55) \]

and \( i e \tilde{h}^{(1)}_{\mu_1 \mu_2 \mu_3} \) is comprised of the \( \mathcal{O}(e) \) terms in \( \tilde{h}^{(0)}_{\mu \nu \rho} \). Note that the quantities inside all the braces in (54) give vanishing contribution when \( T_0^{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \) (with proper symmetrization) acts on them, because they are just gauge shifts of the massless spin-3 field. Thus, in view of Eq. (54), only the second term in (52) can produce operators of dimensionality 10 and 9 at \( \mathcal{O}(e) \). If we are interested in finding such operators, we need to consider only the term \( \tilde{h}^{(0)}_{\mu_1 \mu_2 \mu_3} i e T_1^{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \tilde{h}^{(0)}_{\nu_1 \nu_2 \nu_3} \) in (52). Explicit computation gives:

\[ L_{10} = \frac{i e}{m^6} F^{\mu} \left[ -\partial_{\rho} \partial_\sigma \partial_{\mu} \phi^* \partial_\rho \partial_\sigma \partial_\nu \phi - \frac{7}{24} \partial_\mu \square \phi^* \partial_\nu \phi \right] + c.c. \quad (56) \]

\[ L_9 = \frac{i e}{m^5} F^{\mu} \left[ 3 \partial_{\rho} \partial_{\mu} B_{\sigma}^* \partial_\rho \partial_\sigma \partial_\nu \phi - \partial_\rho \partial_{\mu} B_{\sigma}^* \partial_\rho \square \phi + \frac{3}{4} \square B_{\rho}^* \partial_\rho \phi + \frac{1}{8} \partial_\mu \partial_\rho B_{\sigma}^* \partial_\nu \square \phi \right] + c.c. \quad (57) \]

The situation is analogous to the spin-2 case, in that the dimension-10 operator is proportional to the Maxwell equations, so that it can be removed by a field redefinition of the \( U(1) \) gauge field \( A_\mu \). This produces terms proportional to \( e^2/m^2 \), which can be
eliminated by adding of the following local counter-term to the Lagrangian:

\[ L_{\text{add}} = \frac{e^2}{4} \left[ (\tilde{h}^{\mu\rho}_{\nu\sigma} - \tilde{h}^{\nu\mu}_{\rho\sigma}) + \frac{5}{3} (\tilde{h}^{\mu\nu}_{\rho\sigma} - \tilde{h}^{\nu\mu}_{\rho\sigma}) \right]^2. \] (58)

In the process, we end up having \( \mathcal{O}(e^2) \) dimension-15 operators:

\[ L_{15} = \frac{e^2}{2m^{11}} \left\{ \partial^{\mu} \partial_{\alpha} \partial_{\beta} \phi^* \partial^{\nu} \partial^{\rho} \partial^{\sigma} \partial_{\alpha} \phi + \frac{7}{24} \partial^{\mu} \Box \phi^* \partial^{\nu} \Box \phi - (\mu \leftrightarrow \nu) \right\} \]
\[ \times \left\{ 2 \partial_{\mu} \partial_{(\mu} B_{\nu)}^{\rho} \partial^{\rho} \partial_{\nu} \phi - \partial_{\mu} \partial_{(\mu} B_{\nu)}^{\rho} \partial^{\rho} \Box \phi \right\} + \text{c.c.} \] (59)

Now any local function of the (ungauged) tensor \( \tilde{h}_{\mu\nu\rho} \) enjoys the symmetry:

\[ B_{\mu} \rightarrow B_{\mu} + b_{\mu} + b_{\mu\nu} x^{\nu} + \frac{1}{2} b_{\mu\nu\rho} x^{\nu} x^{\rho}, \] (60)
\[ \phi \rightarrow \phi + c + c_{\mu} x^{\mu} + \frac{1}{2} c_{\mu\nu} x^{\mu} x^{\nu}, \] (61)

where \( b_{\mu}, b_{\mu\nu}, b_{\mu\nu\rho}, c, c_{\mu}, \) and \( c_{\mu\nu} \) are constants. \( c_{\mu\nu} \) is symmetric traceless, and \( b_{\mu\nu\rho} \) satisfies \( 4 b_{(\mu\nu\rho)} = b^{\alpha}_{(\alpha\mu}(h_{\nu\rho)} \). However, the resulting dimension-15 operators (59) are invariant under (60, 61) only up to a nontrivial total derivative. This sets some cohomological obstruction, shown in Appendix A. In complete analogy with the spin-2 case described in Section 2, they prevent field redefinitions plus addition of local terms from canceling all terms proportional to \( e^2/m^{11} \).

On the other hand, addition of dipole terms gives us an improved divergence, where the most divergent terms are proportional to \( e/m^5 \). To see this, we notice that at the dipole level we have two possible terms that respect parity:

\[ L_{\text{dipole}} = ieF^{\mu\nu} [\beta_1 \tilde{h}^{\mu\rho}_{\nu\sigma} \tilde{h}^\rho_{\nu} + \beta_2 \tilde{h}^{\mu}_{\nu} \tilde{h}^\nu_{\mu}]. \] (62)

They produce the following \( \mathcal{O}(e) \) operators:

\[ L_{10}^{(\text{dipole})} = \frac{ie}{m^6} F^{\mu\nu} \left[ \beta_1 \partial_{\rho} \partial_{\sigma} \partial_{\mu} \phi^* \partial^{\rho} \partial^{\sigma} \partial_{\nu} \phi + \frac{1}{8} (2\beta_2 - \beta_1) \partial_{\mu} \Box \phi^* \partial_{\nu} \Box \phi \right]. \] (63)

\[ L_9^{(\text{dipole})} = \frac{ie}{m^5} F^{\mu\nu} \left[ \frac{1}{12} (\beta_1 - 6\beta_2) \Box B_{\mu}^{\rho} \partial_{\nu} \Box \phi + \frac{1}{24} (7\beta_1 - 6\beta_2) \partial_{\mu} \partial^{\rho} B_{\sigma}^{\rho} \partial_{\nu} \Box \phi \right] \]
\[ - \frac{ie\beta_1}{m^5} F^{\mu\nu} \partial_{(\mu} \partial_{\rho} B_{\sigma)}^{\rho} \partial^{\rho} \partial_{\nu} \phi + \text{c.c.} \] (64)

All the dimension-10 operators at \( \mathcal{O}(e) \) cancel in our non-minimal Lagrangian if we set \( \beta_1 = 1, \) and \( \beta_2 = 5/3. \) With this choice of dipole coefficients there are enormous cancela-
tions in dimension-9 operators as well, and we are simply left with:

\[ L_{9}^{(NM)} = \frac{ie}{2m^5} F^{\mu\nu} \left\{ 2\partial_\mu \partial_\nu B_\epsilon^* \partial^\rho \partial^\sigma \partial_\rho \phi - \partial_\mu \partial_\nu B_\epsilon^* \partial^\rho \Box \phi \right\} + \text{c.c.} \quad (65) \]

Since now the dimension-9 operators contain pieces not proportional to the equations of motion, we cannot improve the degree of divergence further. In the scaling limit: \( m \to 0 \) and \( e \to 0 \), such that \( e/m^5 = \text{constant} \), we have the non-minimal Lagrangian:

\[ L = L_{\text{kin}} + \frac{ie}{2m^5} F^{\mu\nu} \left\{ 2\partial_\mu \partial_\nu B_\epsilon^* \partial^\rho \partial^\sigma \partial_\rho \phi - \partial_\mu \partial_\nu B_\epsilon^* \partial^\rho \Box \phi \right\} + \text{c.c.} \quad (66) \]

The intrinsic cutoff for our theory is then given by

\[ \Lambda_3 = \frac{m}{e^{1/5}}. \quad (67) \]

We notice that, just as in the case of spin-2, here as well the Lagrangian has acquired a \( U(1) \) gauge invariance for the vector Stückelberg \( B_\mu \), i.e., only its field strength shows up in the Lagrangian \((66)\). The origin of this gauge invariance can be understood as follows. When we add the non-minimal (dipole) terms \((62)\) to the minimal Lagrangian to cancel all the \( \mathcal{O}(e) \) dimension-10 operators, we effectively change \( T_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \), given in \((55)\), to some \( T'_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \) such that

\[ 0 = \left\{ \partial_{\mu_1} \left( \partial_{\mu_2} \partial_{\mu_3} \phi^* - \frac{1}{4} \eta_{\mu_2 \mu_3} \Box \phi^* \right) + \text{cyclic in } (\mu_1, \mu_2, \mu_3) \right\} ieT'_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \]

\[ \times \left\{ \partial_{\nu_1} \left( \partial_{\nu_2} \partial_{\nu_3} \phi - \frac{1}{4} \eta_{\nu_2 \nu_3} \Box \phi \right) + \text{cyclic in } (\nu_1, \nu_2, \nu_3) \right\} \quad (68) \]

In the above if we redefine \( \phi \to \phi + \theta \) we must have

\[ 0 = \left\{ \partial_{\mu_1} \left( \partial_{\mu_2} \partial_{\mu_3} \theta^* - \frac{1}{4} \eta_{\mu_2 \mu_3} \Box \theta^* \right) + \text{cyclic in } (\mu_1, \mu_2, \mu_3) \right\} ieT'_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \]

\[ \times \left\{ \partial_{\nu_1} \left( \partial_{\nu_2} \partial_{\nu_3} \phi - \frac{1}{4} \eta_{\nu_2 \nu_3} \Box \phi \right) + \text{cyclic in } (\nu_1, \nu_2, \nu_3) \right\} + \text{c.c.} \quad (69) \]

Now if the vector Stückelberg \( B_\mu \) has a gauge shift \( \delta B_\mu = \partial_\mu \varphi \), it is clear from \((54)\) that the gauge shift will result in the following dimension-9 operator:

\[ L_{9}^{(\text{gauge})} \sim \left\{ \partial_{\mu_1} \left( \partial_{\mu_2} \partial_{\mu_3} \varphi^* - \frac{1}{4} \eta_{\mu_2 \mu_3} \Box \varphi^* \right) + \text{cyclic in } (\mu_1, \mu_2, \mu_3) \right\} ieT'_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \]

\[ \times \left\{ \partial_{\nu_1} \left( \partial_{\nu_2} \partial_{\nu_3} \phi - \frac{1}{4} \eta_{\nu_2 \nu_3} \Box \phi \right) + \text{cyclic in } (\nu_1, \nu_2, \nu_3) \right\} + \text{c.c.}, \quad (70) \]
which vanishes by virtue of (69), if we identify \( \theta = -3m\phi \). Therefore, after adding the suitable dipole terms to cancel the dimension-10 operators, the dimension-9 operators we are left with must have a \( U(1) \) gauge invariance for the vector Stückelberg \( B_\mu \). Similar arguments hold for arbitrary integer spin \( s \).

### 4 EM Coupling of Massive Rarita-Schwinger Field

The electromagnetic interaction of massive spin-3/2 is an old problem. It has been studied by several authors [32, 5, 33, 34, 35, 36, 37]. Here we start with the Lagrangian for a free massive complex Rarita-Schwinger field:

\[
L = -i\bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho - im\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu.
\]  

(71)

The mass term does not enjoy the gauge invariance of the massless part under the transformation: \( \psi_\mu \rightarrow \psi_\mu + \partial_\mu \epsilon \), where \( \epsilon \) is a fermionic gauge parameter. The gauge invariance is made manifest by introducing a spin-1/2 Stückelberg field \( \chi \). When the system is coupled to a \( U(1) \) gauge field the Stückelberg invariant Lagrangian reads

\[
L = -i \left( \bar{\psi}_\mu - \frac{1}{m} \bar{\chi} \bar{D}_\mu \right) \gamma^{\mu\nu\rho} D_\nu \left( \psi_\rho - \frac{1}{m} D_\rho \chi \right) - im \left( \bar{\psi}_\mu - \frac{1}{m} \bar{\chi} \bar{D}_\mu \right) \gamma^{\mu\nu} \left( \psi_\nu - \frac{1}{m} D_\nu \chi \right),
\]  

(72)

which has the manifest gauged Stückelberg symmetry:

\[
\delta \psi_\mu = D_\mu \epsilon, \quad \delta \chi = m\epsilon.
\]  

(73, 74)

Working out the Lagrangian (72) one arrives at

\[
L = -i \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - im\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu + i(\bar{\psi}_\mu \gamma^{\mu\nu} D_\nu \chi + \bar{\chi} \bar{D}_\mu \gamma^{\mu\nu} \psi_\nu) + \frac{e}{2m} F_{\mu\nu} \left[ \bar{\chi} \gamma^{\mu\rho} \psi_\rho - \bar{\psi}_\rho \gamma^{\mu\rho} \chi - \bar{\chi} \gamma^{\mu\nu} \chi - (1/m) \bar{\chi} \gamma^{\mu\rho} D_\rho \chi \right],
\]  

(75)

where we have used the non-commutativity of covariant derivatives: \( [D_\mu, D_\nu] = ieF_{\mu\nu} \).

The field redefinition:

\[
\psi_\mu \rightarrow \psi_\mu + \frac{1}{2} \gamma_\mu \chi,
\]  

(76)
eliminates the kinetic mixings, and produces a kinetic term for $\chi$. The free part of the 
Lagrangian now becomes

$$L_{\text{free}} = -i\bar{\psi}_\mu \gamma^{\mu\nu} \partial_\nu \psi_\rho - \frac{3}{2}i\bar{\chi} \partial_\nu \chi - i m \{ \bar{\psi}_\mu \gamma^{\mu} \psi_\nu - 3\bar{\chi} \chi + (3/2)(\bar{\psi}_\mu \gamma^\mu \chi - \bar{\chi} \gamma^\mu \psi_\mu) \}. \quad (77)$$

The $\chi$-propagator is already well-behaved in the $m \to 0$ limit. The spin-3/2 propagator 
also acquires a good high energy behavior if we add the gauge fixing term [38]:

$$L_{\text{gf}} = \frac{i}{2} \bar{\psi}_\mu \gamma^{\mu} \gamma^\rho \partial_\nu \psi_\rho. \quad (78)$$

Note that the interaction part of the Lagrangian (75) contains marginal and irrelevant 
operators. After the shift (76) is made, we find the following terms, singular in the 
massless limit:

$$L_{\text{div}} = \frac{e}{2m} F_{\mu\nu}(\bar{\chi} \gamma^{\mu\nu} \psi_\rho - \bar{\psi}_\rho \gamma^{\mu\nu} \chi + \bar{\chi} \gamma^{\mu\nu} \chi) - \frac{e}{2m^2} F_{\mu\nu}(\bar{\chi} \gamma^{\rho} \partial_\rho \chi) + O(e^2). \quad (79)$$

Now we make use of the identity:

$$\gamma^{\mu\rho} = \gamma^\mu \gamma^\nu \gamma^\rho - \gamma^{\mu\nu} \gamma^\rho + \gamma^{\mu\rho} \gamma^\nu - \gamma^{\nu\rho} \gamma^\mu, \quad (80)$$

to write the $O(e)$ dimension-6 operators as

$$L_6 = \frac{e}{2m^2} \partial_\mu F^{\mu\nu}(\bar{\chi} \gamma^{\nu} \chi) - \frac{e}{4m^2} F_{\mu\nu}(\bar{\chi} \gamma^{\mu\nu} \partial_\rho \chi - \bar{\chi} \partial_\rho \gamma^{\mu\nu} \chi). \quad (81)$$

The entire set is invariant, up to a total derivative, under $\chi \to \chi + \xi$, with $\xi = \text{constant}$, 
which is a symmetry of any local function of $(\psi_\mu - \partial_\mu \chi/m)$. We note that addition of 
any non-minimal term does not help us improve the degree of divergence, because any 
such operator is necessarily irrelevant. For example, even a dipole term:

$$L_{\text{dipole}} = \frac{e\alpha}{m} F^{\mu\nu} \left( \bar{\psi}_\mu - \frac{1}{m} \bar{\chi} \gamma^\nu \right) \left( \psi_\nu - \frac{1}{m} D_\nu \chi \right) \quad (82)$$

introduces new dimension-6 operators. Clearly, higher order zero-mass poles can only 
make it worse. This does not mean, though, that such operators should not exist. They 
do, e.g., in supergravity effective Lagrangians; they simply lower the EFT cutoff. On the 
other hand, since all the dimension-6 operators in (81) are proportional to the equations

\footnote{This is true before the shift (76) is made. However, the shift does not yield operators with the highest dimensionality.}
of motion, one can cancel them by appropriate field redefinitions of $A_\mu$ and $\chi$. Indeed

$$A_\mu \rightarrow A_\mu - \frac{e}{2m^2} \bar{\chi} \gamma_\mu \chi,$$  \hspace{1cm} (83)

$$\chi \rightarrow \chi + \frac{ie}{6m^2} F_{\mu\nu} \gamma^{\mu\nu} \chi,$$  \hspace{1cm} (84)

serve the purpose, as cancelations occur because of contributions coming from the kinetic terms. The degree of divergence, at this point, is not improved though, as now we end up having a bunch of $O(e^2)$ dimension-8 operators. We must eliminate all such operators by adding local functions of $(\psi_\mu - \partial_\mu \chi/m)$, if it is possible in the first place.

These operators are of two distinct types. One contains four $\chi$’s and two derivatives, e.g., \( \partial_\mu (\bar{\chi} \gamma_\nu \chi) \bar{\chi} \gamma_\mu \rho \partial_\rho \chi \); the other contains two $\chi$’s, two $F_{\mu\nu}$’s, and one derivative, e.g., \( F_{\mu\nu} F_{\rho\sigma} \bar{\chi} \gamma^{\mu\nu} \gamma^{\rho\sigma} \partial_\chi \). Let us consider the former kind. We need to add a 4-Fermi term of the spin-3/2 field, like \((e^2/m^2) \bar{\psi} \psi \bar{\psi} \psi\), to ever get four $\chi$’s. Now each spin-1/2 Stückelberg field $\chi$ comes with one derivative and one power of $1/m$, so that we obtain an $O(e^2/m^6)$-term with four $\chi$’s, but also with four derivatives instead of two. Thus the 4-Fermi terms cannot produce the dangerous operators we wanted to eliminate. We conclude that the degree of divergence cannot be improved by field redefinitions and addition of local terms.

Therefore, in the scaling limit: $m \rightarrow 0$ and $e \rightarrow 0$, such that $e/m^2 = \text{constant}$, we have the following Lagrangian:

$$L = L_{\text{kin}} - \frac{e}{2m^2} F_{\mu\nu} (\bar{\chi} \gamma^{\mu\nu} \partial_\rho \chi).$$  \hspace{1cm} (85)

It describes an effective field theory with a finite cutoff

$$\Lambda_{3/2} = \frac{m}{\sqrt{e}}.$$  \hspace{1cm} (86)

It is the spin-1/2 Stückelberg field $\chi$ that becomes strongly coupled at high energies. The above is the intrinsic cutoff of an electromagnetically interacting massive Rarita-Schwinger field in flat space-time background.

## 5 Massive Spin-5/2 Coupled to EM

The field theory of massive and massless spin-5/2 fields has been discussed in [39]. For the massive case, gravitational and EM interactions were studied in Refs. [40, 41] and [37].

To obtain a Stückelberg Lagrangian for massive spin-5/2 field that can be readily coupled to electromagnetism or gravity, while maintaining at the same time the covariant
version of the Stückelberg symmetry, we start with the Lagrangian for a massless, complex
spin-5/2 field \[17\] in (4+1)D.

\[
L = -i \left[ \bar{\Psi}_{MN} \Gamma^A \partial_A \Psi^{MN} + 2 \bar{\Psi}_{MN} \Gamma^N \Gamma^A \partial_A \Gamma_P \Psi^{PM} - \frac{1}{2} \bar{\Psi}^M \Gamma^A \partial_A \Psi^N \right.

\left. -2(\bar{\Psi}_{MN} \Gamma^N \partial_A \Psi^{AM} - \text{h.c.}) + (\bar{\Psi}^M \Gamma^A \partial_B \Psi_{AB} - \text{h.c.}) \right].
\]
(87)

where \(\Gamma^A\)'s are 5D Dirac matrices. The Lagrangian enjoys the gauge symmetry:

\[
\delta \Psi_{MN} = \partial_M \Lambda_N + \partial_N \Lambda_M, \quad \Gamma^M \Lambda_M = 0.
\]
(88)

Analogous to the spin-3 case, the condition on the gauge parameter, namely gamma-
tracelessness, has important consequences. We Kaluza-Klein reduce the Lagrangian by
writing

\[
\Psi_{MN}(x^\mu, x_5) = \sqrt{\frac{m}{2\pi}} e^{i(mx_5 + \frac{\pi}{4}\gamma_5)} \psi_{MN}(x^\mu),
\]
(89)

where the \(x_5\)-dimension is compactified on a circle of radius \(1/m\). Here we have incor-
porated a chiral rotation for convenience. In (3+1)D the KK-reduction gives rise to a
spin-5/2 field \(\psi_{\mu\nu}\), a spin-3/2 field \(\xi_\mu \equiv -i\psi_\mu 5\), and a spin-1/2 field \(\chi \equiv -\psi_{55}\). We also
write the gauge parameter \(\Lambda_M\) as

\[
\Lambda_M(x^\mu, x_5) = \sqrt{\frac{m}{2\pi}} e^{i(mx_5 + \frac{\pi}{4}\gamma_5)} \lambda_M(x^\mu).
\]
(90)

We have two gauge parameters in (3+1)D: \(\lambda_\mu\), and \(\lambda \equiv -i\lambda_5\). The 5D gauge invariance
\(\text{[88]}\) reduces in lower dimension to the Stückelberg symmetry:

\[
\delta \psi_{\mu\nu} = \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu,
\]
(91)

\[
\delta \xi_\mu = \partial_\mu \lambda + m \lambda_\mu,
\]
(92)

\[
\delta \chi = 2m \lambda.
\]
(93)

The gamma-tracelessness of the 5D gauge parameter gives the following condition:

\[
\gamma^\mu \lambda_\mu \equiv \chi = \lambda.
\]
(94)

We gauge fix the KK-reduced Lagrangian by setting \(\xi_\mu = (1/4)\gamma_\mu \xi\) and \(\chi = 0\). Note
that the constraint \(\text{[94]}\) enables us to set to zero only the gamma-traceless part of the
spin-3/2 field \(\xi_\mu\). The gauge-fixed 4D Lagrangian, which describes a massive spin-5/2
field, then unavoidably contains an auxiliary spin-1/2 field \(\xi\) — the gamma-trace of the
would be spin-3/2 Stückelberg field $\xi_\mu$. One obtains

$$ L = -i \left[ \bar{\psi}_{\mu \nu} \partial \psi^{\mu \nu} + 2 \bar{\psi}_{\mu \nu} \gamma^\nu \partial \psi^{\mu \rho \mu} - \frac{1}{2} \bar{\psi} \partial \bar{\psi} - (2 \bar{\psi}_{\mu \nu} \gamma^\nu \partial \psi^{\rho \mu} - \bar{\psi} \gamma^\mu \partial \psi_{\mu \nu} - \text{h.c.}) \right] $$

$$ + i m \left[ \bar{\psi}_{\mu \nu} \psi^{\mu \nu} + 2 \bar{\psi}_{\mu \nu} \gamma^\mu \psi^{\rho \nu} - \frac{1}{2} \bar{\psi} \psi \right] - \frac{i}{4} \left[ 2 \bar{\xi} \gamma_{\mu} \partial \psi^{\mu \nu} + \bar{\psi} \partial \bar{\xi} - \text{h.c.} \right] $$

$$ - i \bar{\xi} (\partial - m) \bar{\xi}, $$

(95)

where $\psi \equiv \psi^\mu_\mu$ is trace of the spin-5/2 field. This is the Lagrangian for a massive spin-5/2 field with minimal number of auxiliary fields. After some field redefinitions it is the same as the Singh-Hagen spin-5/2 Lagrangian [6]. The equivalence of the two Lagrangians was shown explicitly in [19].

By considering the gauge fixing conditions, we find that we can exactly reproduce the KK-reduced Lagrangian before gauge fixing from the above Lagrangian, if we make the following field redefinitions in the latter:

$$ \psi_{\mu \nu} \rightarrow \tilde{\psi}_{\mu \nu} = \psi_{\mu \nu} - \frac{1}{m} \partial (\mu \psi_{\nu} \chi) + \frac{1}{2m^2} \partial (\mu \partial \psi_{\nu}) \chi + \frac{1}{4m} \gamma (\mu \partial \psi_{\nu}) \left[ \bar{\xi} - \frac{1}{2} \chi - \frac{1}{2m} \partial \chi \right], $$

(96)

$$ \bar{\xi} \rightarrow \tilde{\bar{\xi}} = \bar{\xi} - \frac{1}{2} \chi - \frac{1}{2m} \partial \chi. $$

(97)

Note that all the higher dimensional operators cancel, as they should. With these substitutions, the Stückelberg invariance is trivially manifest, because in fact the tensors $\tilde{\psi}_{\mu \nu}$, and $\tilde{\bar{\xi}}$ themselves are invariant under the Stückelberg transformations (91-93). Most importantly, when the theory is gauged, the covariant counterparts of the tensors (96, 97) still preserve the covariant Stückelberg symmetry. Therefore, we have been able to construct a consistent massive spin-5/2 Lagrangian that can be readily coupled to a gauge field, while maintaining the covariant version of Stückelberg symmetry.

To couple the theory to electromagnetism we replace ordinary derivatives with covariant ones: $\partial_\mu \rightarrow D_\mu$, so that

$$ L = -i \left[ \bar{\psi}_{\mu \nu} D \psi^{\mu \nu} + 2 \bar{\psi}_{\mu \nu} \gamma^\nu D \psi^{\rho \mu} - \frac{1}{2} \bar{\psi} D \bar{\psi} - (2 \bar{\psi}_{\mu \nu} \gamma^\nu D \psi^{\rho \mu} - \bar{\psi} \gamma^\mu D \psi_{\mu \nu} - \text{h.c.}) \right] $$

$$ + i m \left[ \bar{\psi}_{\mu \nu} \psi^{\mu \nu} + 2 \bar{\psi}_{\mu \nu} \gamma^\mu \psi^{\rho \nu} - \frac{1}{2} \bar{\psi} \psi \right] - \frac{i}{4} \left[ 2 \bar{\xi} \gamma_{\mu} D \psi^{\mu \nu} + \bar{\psi} D \bar{\xi} - \text{h.c.} \right] $$

$$ - i \bar{\xi} (\partial - m) \bar{\xi} - \frac{1}{4} F^2_{\mu \nu}, $$

(98)
where the twiddled fields are the covariant counterparts of the tensors: 

\[ \tilde{\psi}_{\mu\nu} = \psi_{\mu\nu} - \frac{1}{m} D(\mu \xi_\nu) + \frac{1}{2m^2} D(\mu D_\nu) \chi + \frac{1}{4m} \gamma(\mu D_\nu) \left[ \tilde{\gamma} - \frac{1}{2} \chi - \frac{1}{2m} \not{D} \chi \right], \]  
\[ \tilde{\gamma} = \gamma - \frac{1}{2} \chi - \frac{1}{2m} \not{D} \chi. \]

The gauged Lagrangian symbolically looks like:

\[ L = L_{\text{free}} + L_{\text{int}} - \frac{1}{4} F_{\mu\nu}^2. \]

\( L_{\text{free}} \) is the free part that consists of kinetic terms, mass terms, and mixed terms, but no higher dimensional operators. \( L_{\text{int}} \) is the interaction Lagrangian, that consists of various terms, each one containing at least one power of \( e \), and having canonical dimensionality 4 through 8. By redefinitions of the fields \( \psi_{\mu\nu} \) and \( \xi_\mu \), one can get rid of some of the kinetic mixings. Furthermore we can add suitable gauge fixing terms (thereby exhausting all gauge freedoms) to the Lagrangian to make sure that the propagators in the theory have good high energy behavior. Among others, the above steps also produce a kinetic term for \( \chi \). We do not explicitly carry out these steps, because they are not important for the rest of the section. The important point is that no field redefinition of \( \chi \) is needed.

We are interested in finding \( \mathcal{O}(e) \)-terms containing operators of the highest possible dimensionality. Note that dimension-8 operators may only come from spin-5/2 kinetic terms, i.e., the first line of (98), which is also the gauged version of the massless spin-5/2 Lagrangian. We write:

\[ L_{\text{massless}} = -i \tilde{\psi}_{\mu\nu} T^{\mu\nu\rho\sigma\lambda} D_\rho \tilde{\psi}_{\sigma\lambda}, \]

where \( T^{\mu\nu\rho\sigma\lambda} \) is a constant tensor, symmetric under \( \mu \leftrightarrow \nu \) and \( \sigma \leftrightarrow \lambda \), given by

\[ T^{\mu\nu\rho\sigma\lambda} = \frac{1}{2} \eta^{\mu\nu}(\eta^{\rho\lambda} \gamma^\sigma - \eta^{\sigma\lambda} \gamma^\rho) + \frac{1}{2} (\eta^{\mu\rho} \gamma^{\nu\lambda} + \eta^{\nu\rho} \gamma^{\mu\lambda}) + \frac{1}{2} (\eta^{\sigma\lambda} \eta^{\rho\mu} - \eta^{\rho\sigma} \eta^{\mu\lambda}) \gamma^\mu 
+ \frac{1}{2} (\eta^{\sigma\lambda} \eta^{\rho\mu} - \eta^{\rho\sigma} \eta^{\mu\lambda}) \gamma^\nu 
+ \left[ \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} \gamma^\lambda + \frac{1}{2} (\eta^{\rho\sigma} \gamma^{\mu\nu} + \eta^{\mu\nu} \gamma^{\rho\sigma}) \gamma^\lambda \right]. \]

In writing the above we made use of the gamma-matrix identity (80). We notice that apart from the terms in the brackets, \( T^{\mu\nu\rho\sigma\lambda} \) is antisymmetric under \( \rho \leftrightarrow \sigma \). We will find this property useful shortly, as we explore what happens when \( T^{\mu\nu\rho\sigma\lambda} D_\rho \) acts on a gauged
Stückelberg shift of $\psi_{\sigma\lambda}$. Armed with the symmetry properties, it is easy to show that

$$T^{\mu\nu\rho\sigma\lambda}D_\rho D_{(\sigma\lambda)} = \partial \eta^\mu \partial \eta^\nu \gamma^\sigma + \frac{1}{2} (\eta^\mu \gamma^\nu \gamma^\rho + \eta^\nu \gamma^\rho \gamma^\mu) + \eta^\sigma \gamma^\lambda \gamma^\mu + \eta^\lambda \gamma^\mu \gamma^\nu + \eta^\sigma \gamma^\lambda \eta^\mu \gamma^\nu$$

\(\lambda\)

where we have used \([D_\mu, D_\nu] = ieF_{\mu\nu}\). In view of Eq. (99), we write

$$\lambda_{\mu} = -\frac{1}{m} \left[ \xi_{\mu} - \frac{1}{4} \gamma_{\mu} \chi + \frac{1}{8} \gamma_{\mu} \chi \right] + \frac{1}{2m^2} \left[ D_\mu \chi - \frac{1}{4} \gamma_{\mu} \chi D_\chi \right],$$

with \(\lambda = -\frac{1}{2m} \chi\).

Now that all possible dimension-8 operators coming from the Lagrangian (98) are contained only in \(\bar{\lambda}_{(\mu} D_{\nu)} T^{\mu\nu\rho\sigma\lambda} D_\rho D_{(\sigma\lambda)}\), where \(\lambda_{\mu}\) is given by (105), one can easily find them all. Note that the terms on the second line of Eq. (104) do not contribute the dimension-8 operators, because they can at most produce dimension-7 operators. Therefore at \(O(e)\) we find the following operators:

$$L_8 = -\frac{1}{4m^4} (2ieF_{\rho\sigma}) \left\{ \bar{\chi} \partial_\nu \partial_\mu \bar{\chi} - \frac{1}{4} \bar{\chi} \partial_\gamma \partial_\nu \partial_\mu \chi \right\} \times \left[ \eta^\mu \eta^\nu \eta^\rho \eta^\sigma + \frac{1}{2} (\eta^\mu \gamma^\nu \gamma^\rho + \eta^\nu \gamma^\rho \gamma^\mu) + \eta^\lambda \gamma^\lambda \eta^\mu \eta^\nu \right] \left\{ \partial \lambda \chi - \frac{1}{4} \gamma \lambda \partial \chi \right\}.$$

Notice that the double-derivative tensor appearing in the braces above is zero under contraction with \(\gamma^\nu\). One can simplify the above by making use of various identities and (anti)commutation relations involving gamma matrices and products thereof, Bianchi identity etc. to obtain:

$$L_8 = -\frac{e}{2m^4} F_{\mu\nu}(\bar{\chi} \partial_\sigma \gamma^{\mu\nu} \partial_\rho \partial_\sigma \chi) + \frac{3e}{16m^4} F_{\mu\nu}(\bar{\chi} \partial_\gamma \gamma^{\mu\nu} \partial_\rho \partial_\gamma \chi).$$

Each term here is invariant up to a total derivative under \(\chi \rightarrow \chi + \varepsilon \varepsilon \mu \xi^\mu\), where \(\varepsilon\) is a constant spinor, and \(\varepsilon_\mu\) a constant vector-spinor. In fact, the above local transformation is a symmetry of the highest dimensional operators built out of any local function of the high spin field. Addition of any non-minimal terms at this point does not improve the degree of divergence; they rather make it worse by introducing new dimension-8 operators (see remarks in section 4). However, by using the identity (80), one can render the operators...
in Eq. (107) proportional to the equations of motion:

\[ L_8 = \frac{e}{2m^4} \partial_\mu F^{\mu\nu} \left( \tilde{\chi} \partial_\sigma \gamma_\nu \partial_\sigma \chi - \frac{3}{8} \tilde{\chi} \partial_\sigma F^{\mu\nu} \partial_\sigma \phi_\chi \right) - \frac{e}{4m^4} \left( \tilde{\chi} \partial_\sigma F^{\sigma\rho} \partial_\rho \phi_\chi - \frac{3}{8} \tilde{\chi} \partial_\sigma F^{\sigma\rho} \partial_\rho \phi_\chi \right) \]

(108)

where \( F \equiv F^{\mu\nu}_\gamma \), and (...) stands for hermitian conjugate. We are in a situation analogous to the spin-3/2 case. Although by appropriate field redefinitions of \( A_\mu \) and \( \chi \) one may cancel all these terms, the \( \mathcal{O}(e^2) \) dimension-12 operators obtained in the process cannot be eliminated by adding local functions of the high spin field (one can see it just by considering terms containing four \( \chi \)'s). In other words, the degree of divergence cannot be improved by field redefinitions plus addition of local terms.

In the scaling limit: \( m \to 0 \) and \( e \to 0 \), such that \( e/m^4=\text{constant} \), our Lagrangian reduces to:

\[ L = L_{\text{kin}} - \frac{e}{2m^4} F^{\mu\nu}(\tilde{\chi} \partial_\sigma \gamma^{\mu\nu\rho} \partial_\rho \phi_\chi) + \frac{3e}{16m^4} F^{\mu\nu}(\tilde{\chi} \partial_\sigma \gamma^{\mu\nu\rho} \partial_\rho \phi_\chi). \]

(109)

Thus we have an effective field theory with an intrinsic finite cutoff

\[ \Lambda_{5/2} = \frac{m}{e^{1/4}}. \]

(110)

It is again the spin-1/2 Stückelberg field \( \chi \) that plays the principal role, by being the strongest interacting mode at high energies.

### 6 Intrinsic Cutoff for Arbitrary Spin

Having explicitly worked out the examples of spin 2, 3, 3/2, and 5/2, we see a generic pattern in the expression for the intrinsic cutoff of the theory as a function of the particle’s mass, spin, and electric charge. One is tempted to conjecture that for any spin-\( s \) particle of mass \( m \) and electric charge \( e \), the parametric dependence of the cutoff on \( m \) and \( e \) is

\[ \Lambda_s = \mathcal{O} \left( \frac{m}{e^{1/(2s-1)}} \right). \]

(111)

We will now show that the above is indeed the expression for the upper bound on the cutoff. To do so we consider the cases of integer spin and half-integer spin separately.
6.1 Integer Spin $s$:

One can as usual follow the procedure described in the introduction. We outline steps 1 through 4 only briefly, because the details are not very important for our final conclusion. One starts with the massless Lagrangian in (4+1)D \[17\], and then Kaluza-Klein reduce it to obtain in (3+1)D a Stückelberg invariant Lagrangian for a massive spin-$s$ field. The KK-reduced Lagrangian before gauge fixing contains symmetric tensor fields of rank (spin) 0 through $s$; the Stückelberg symmetry reads:

$$
\delta \phi_s = \partial \lambda_{s-1}
$$

$$
\delta \phi_{s-1} = \partial \lambda_{s-2} + m \lambda_{s-1}
$$

... 

$$
\delta \phi_{s-k} = \partial \lambda_{s-k-1} + km \lambda_{s-k}
$$

... 

$$
\delta \phi_0 = sm \lambda_0
$$

The tracelessness of the 5D gauge parameter gives rise to $(s-2)$ conditions on the 4D gauge parameters:

$$
\lambda'_{s-1} = \lambda_{s-3}
$$

$$
\lambda'_{s-2} = \lambda_{s-4}
$$

... 

$$
\lambda'_2 = \lambda_0
$$

where prime denotes trace w.r.t. the Minkowski metric. Thus, we have at our disposal $s$ symmetric, traceless, independent gauge parameters: \{\lambda_0, \lambda_1, \lambda'_2, ..., \lambda'_{s-1}\}. This implies that the gauge fixed 4D Lagrangian not only contains a spin-$s$ field $\phi_s$ and its trace $\phi'_s$ (spin $s-2$), but also $(s-2)$ auxiliary fields with spins 0 through $(s-3)$: \{\phi'_2, \phi'_3, ..., \phi'_{s-1}\}, identified as traces of Stückelberg fields. One can construct the following symmetric
tensors that are invariant under the St"uckelberg transformations.

\begin{align}
\hat{\phi}_s &= \sum_{k=0}^{s} \frac{(-1)^k}{k! m_k} \partial^k \phi_{s-k}, \\
\hat{\phi}'_n &= \phi'_n - (n - 2) \phi_{n-2} - \left(\frac{2/m}{s-n+1}\right) \partial \phi_{n-1} \\
&\quad + \sum_{k=2}^{n} \frac{(-1)^k (s-n)!}{(s-n+k)! m_k^k} \left[ m^2 a(s, n) + 2 \partial \cdot \partial \right] \partial^{k-2} \phi_{n-k},
\end{align}

where \( n = 2, 3, \ldots, (s - 1) \), and \( a(s, n) \equiv \{s(n-3) - (n-1)(n-4)\}(s-n+1) \). The construction of the \((s-2)\) lower rank invariant tensors (113) has been possible because of the \((s-2)\) gauge conditions. Simplest among them is the invariant scalar:

\begin{align}
\hat{\phi}'_2 &= \phi'_2 - \left(\frac{s-2}{s}\right) \phi_0 - \frac{2/m}{s-1} \left[ \partial \cdot \phi_1 - \frac{1}{s m} \Box \phi_0 \right].
\end{align}

Consider the following field redefinitions in the gauge fixed Lagrangian:

\begin{align}
\phi'_2 &\rightarrow \tilde{\phi}'_2 = \hat{\phi}'_2 \\
\phi'_3 &\rightarrow \tilde{\phi}'_3 = \hat{\phi}'_3 + \left(\frac{2}{s-2}\right) \frac{1}{4 m} \partial \hat{\phi}'_2 \\
&\quad \cdots \\
\phi_s &\rightarrow \tilde{\phi}_s = \hat{\phi}_s + \frac{1/m}{2(s-1)} \eta \partial \left[ \hat{\phi}'_{s-1} + \sum_{n=2}^{s-2} \frac{b(s)}{m^{n-1}} \partial^{n-1} \hat{\phi}'_{s-n} \right],
\end{align}

where \( b(s) \) is some rational function of \( s \). Once (115-117) are performed, the St"uckelberg invariant KK-reduced Lagrangian before gauge fixing is reproduced. One can understand this fact by considering the gauge fixing conditions. The function \( b(s) \) can be determined as follows. Note that \( \tilde{\phi}_s \) and \( \phi_s \) are related by the gauge transformation:

\begin{align}
\tilde{\phi}_s &= \phi_s + \partial \lambda_{s-1} = \phi_s + \partial \lambda^T_{s-1} + \frac{1}{4} \eta \partial \lambda_{s-3}.
\end{align}

We can compare the coefficients of the terms containing \((1/m)^s\) in (117) and (118). Our gauge fixing conditions are such that \( \lambda_{k-1} \) contains at most \((1/m)^k\). Thus we only need to consider the term \( \partial \lambda^T_{s-1} \) in (118). In view of (112), the tracelessness of \( \lambda^T_{s-1} \) gives

\begin{align}
\tilde{\phi}_s &= \phi_s + \frac{(-1)^s}{s! m^s} \left[ \partial^s \phi_0 - \frac{2(s-1)}{(s-1)(s-2)} \eta \partial^{s-2} \Box \phi_0 \right] + \cdots
\end{align}

\[25\]
By comparing the above with Eq. (117), we can find $b(s)$.

The virtue of the above procedure is that the Stückelberg invariance is left intact by the minimal substitution $\partial_\mu \rightarrow D_\mu$. The gauged Lagrangian with explicit Lorentz indices is given by

$$
L = -|D_\rho \tilde{\phi}_{\mu_1\mu_2...\mu_s}|^2 + s|D^{\mu_1} \tilde{\phi}_{\mu_1\mu_2...\mu_s}|^2 + \frac{s(s-1)}{2} |D^\rho \tilde{\phi}_{\mu_1\mu_3...\mu_s}|^2 \\
+ \frac{s(s-1)}{2} [D^{\mu_1} D^{\mu_2} \tilde{\phi}_{\mu_1\mu_2...\mu_s} \tilde{\phi}^{\nu_1\mu_3...\mu_s} + \text{c.c.}] + \frac{s(s-1)(s-2)}{4} |D^{\mu_3} \tilde{\phi}_{\mu_1\mu_3...\mu_s}|^2 \\
= \frac{1}{4} m^2 \left( |\tilde{\phi}_{\mu_1\mu_2...\mu_s}|^2 - \frac{s(s-1)}{2} |\tilde{\phi}_{\mu_1\mu_3...\mu_s}|^2 \right) + \ldots - \frac{1}{4} F_{\mu\nu}^2, 
$$

where the twiddled fields are given by the covariant counterparts of (115-117), and the ellipses $\ldots$ stand for terms involving lower spin (auxiliary) fields. Non-commutativity of covariant derivative will give rise to higher dimensional interaction operators, that contain powers of $e_i$ in the above Lagrangian. Symbolically:

$$
L = L_{\text{free}} + L_{\text{int}} - \frac{1}{4} F_{\mu\nu}^2.
$$

By doing appropriate field redefinitions and adding gauge fixing terms (thereby exhausting all gauge freedom) we can produce diagonal kinetic operators for all the fields, so that the propagators in the theory have good high energy behavior. It is important to note that the spin-1 and spin-0 Stückelberg fields do not require any field redefinition.

The interaction Lagrangian may contain operators up to dimension $(2s+4)$. Operators of mass dimension $(2s+4)$ and $(2s+3)$ may only come from the first five terms of the Lagrangian (120); these terms comprise the gauged Lagrangian for a massless spin-$s$ field. An analysis similar to that presented for spin-3 shows that

$$
L_{\text{int}} = \tilde{\phi}_{\mu_1\mu_2...\mu_s}^{(0)} i e T_1^{\mu_1\mu_2...\mu_s;\nu_1\nu_2...\nu_s} \tilde{\phi}_{\nu_1\nu_2...\nu_s}^{(0)} + L_{\leq (2s+2)} + \mathcal{O}(e^2),
$$

where $\tilde{\phi}_{\mu_1\mu_2...\mu_s}$ is given by (117) or (119), and

$$
T_1^{\mu_1\mu_2...\mu_s;\nu_1\nu_2...\nu_s} \equiv \left\{ \eta^{\mu_1\nu_1} \eta^{\mu_2\nu_2} - \frac{s(s-1)}{2} \eta^{\mu_1\mu_2} \eta^{\nu_1\nu_2} \right\} \eta^{\mu_3\nu_3} \ldots \eta^{\mu_s\nu_s} (A_{\rho} \partial^\rho - \tilde{\partial}_s A_{\mu}) \\
+ \frac{s(s-1)}{2} \eta^{\mu_3\nu_3} \ldots \eta^{\mu_s\nu_s} \eta^{\mu_1\mu_2} (A^{\nu_1} \partial^2 - \tilde{\partial}_{\nu_1} A^{\nu_2}) + (\mu_i \leftrightarrow \nu_i)] \\
- \frac{s(s-1)(s-2)}{4} \eta^{\mu_1\nu_4} \ldots \eta^{\mu_s\nu_s} \eta^{\mu_1\mu_3} \eta^{\mu_3\nu_3} (A^{\mu_1} \partial^2 - \tilde{\partial}_{\mu_1} A^{\nu_1}) \\
- s \eta^{\mu_2\nu_2} \ldots \eta^{\mu_s\nu_s} (A^{\mu_1} \partial^1 - \tilde{\partial}_{\mu_1} A^{\nu_1}),
$$
with symmetrization assumed in \((\mu_1, \mu_2, \ldots, \mu_s)\) and in \((\nu_1, \nu_2, \ldots, \nu_s)\). Given Eq. \((122)\), it is not difficult to compute the dimension-\((2s + 4)\) and \(-(2s + 3)\) operators at \(\mathcal{O}(e)\). Let us denote the spin-0 Stückelberg field as \(\phi_0 \equiv \phi\), and the spin-1 one as \(\phi_1 \equiv B_\mu\). We have

\[
L_{2s+4} = -\frac{ie}{m^{2s}} F^{\mu\nu} \partial_\mu \phi |_{\mu=1} \partial_{\mu s+1} \partial_\mu \phi^* \partial^{\mu_1} \partial^{\mu_{s+1}} \partial_\nu \phi \\
+ \frac{ie}{m^{2s}} c_1(s) F^{\mu\nu} \partial_\mu \phi |_{\mu=1} \partial_{\mu s+3} \partial_\mu \phi^* \partial^{\mu_1} \partial^{\mu_{s+3}} \partial_\nu \partial_\nu \phi + \text{c.c.} \\
L_{2s+3} = -\frac{ie}{m^{2s-1}} s F^{\mu\nu} \partial_\mu \phi |_{\mu=1} \partial_{\mu s+2} B_{\mu}^1 \partial^{\mu_1} \partial^{\mu_{s+2}} \partial_\nu \partial_\nu \phi \nonumber + \text{c.c.} \\
+ \frac{ie}{m^{2s-1}} c_2(s) F^{\mu\nu} \partial_\mu \phi |_{\mu=1} \partial_{\mu s+3} B_{\mu}^3 \partial^{\mu_1} \partial^{\mu_{s+3}} \partial_\nu \partial_\nu \phi + \text{c.c.} \\
+ \frac{ie}{m^{2s-1}} c_3(s) F^{\mu\nu} \partial_\mu \phi |_{\mu=1} \partial_{\mu s+3} B_{\mu}^3 \partial^{\mu_1} \partial^{\mu_{s+3}} \partial_\nu \partial_\nu \phi + \text{c.c.,} 
\]

where the \(c(s)\)'s are rational functions of \(s\), which are not important in the subsequent discussion. Here the dimension-\((2s + 4)\) operators are proportional to the Maxwell equations: \(L_{2s+4} = (e/m^{2s}) \partial_\mu F^{\mu\nu} J_\nu\), where \(J_\nu\) is given by

\[
J_\nu \equiv \frac{i}{2} \left[ \partial_\mu \phi |_{\mu=1} \partial_{\mu s+1} \phi^* \partial^{\mu_1} \partial^{\mu_{s+1}} \partial_\nu \phi - c_1(s) \partial_\mu \phi |_{\mu=1} \partial_{\mu s+3} \phi^* \partial^{\mu_1} \partial^{\mu_{s+3}} \partial_\nu \partial_\nu \phi \right] + \text{c.c.} 
\]

Therefore, they can be eliminated by a field redefinition of the \(U(1)\) gauge field. The resulting \(\mathcal{O}(e^2)\) dimension-\((4s + 4)\) operators can be canceled by addition of local counterterms of the spin-\(s\) field. This leaves us with terms proportional to \(e^2/m^{4s-1}\).

We expect here a situation similar to that encountered in the cases of spin-2 and spin-3. Any local function of the (ungauged) tensor \(\phi_s\) is fully invariant under the symmetry:

\[
B_\mu \rightarrow B_\mu + b_\mu + \sum_{k=1}^{s-1} \frac{1}{k!} b_{\mu,\nu_1 \cdots \nu_k} x^{\nu_1} \cdots x^{\nu_k}, \\
\phi \rightarrow \phi + c + \sum_{k=1}^{s-1} \frac{1}{k!} c_{\mu_1 \cdots \mu_k} x^{\mu_1} \cdots x^{\mu_k}, 
\]

where \(b_\mu\), \(b_{\mu,\nu_1 \cdots \nu_k}\), \(c\), and \(c_{\mu_1 \cdots \mu_k}\) are constants. \(c_{\mu_1 \cdots \mu_k}\) is symmetric traceless; \(b_{\mu,\nu_1 \cdots \nu_k}\), which is symmetric under \((\nu_i \leftrightarrow \nu_j)\) by definition, satisfies for \(k = (s - 1)\) the following:

\[b_{(\mu,\nu_1 \cdots \nu_{s-1})} = 0, \quad b^p_{(\mu_2 \cdots \nu_{s-1})} = 0.\]

\[\text{Given that the dimension-}(4s + 3)\text{ operators obey this symmetry only up to a nontrivial total derivative, the cohomological argument presented}\]

\[\text{This set of conditions is actually stronger than necessary. But as long as } b_{\mu,\nu_1 \cdots \nu_{s-1}} \neq 0, \text{ it does not matter for our purpose.}\]
in Appendix A should prevent us from improving the degree of divergence any further. No new feature needed to the computation of the cohomology arises above spin-3, except for a clutter of Lorentz indices. So, we will assume that the cohomological obstruction present in the case of spin-2 and 3 exists for all integer spins.

However, one can always add to the minimal Lagrangian a dipole term, which depends on two free parameters $\beta_1$ and $\beta_2$.

\[
L_{\text{dipole}} = ie F^{\mu\nu} \left[ \beta_1 \tilde{\phi}_{\mu\nu}^s \phi_{\mu}^{\nu \mu_2 \ldots \mu_s} + \beta_2 \phi_{\mu}^{\nu_2 \mu_4 \ldots \mu_s} \phi_{\nu_2 \mu_4 \ldots \mu_s} \right].
\] (129)

We get

\[
L_{2s+4}^{(\text{dipole})} = \frac{ie}{m^{2s}} \beta_1 F^{\mu\nu} \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \partial_{\mu} \phi^* \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \partial_{\nu} \phi
\]

\[
+ \frac{ie}{m^{2s}} f_1(\beta_1, \beta_2) F^{\mu\nu} \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \partial_{\mu} \phi^* \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \partial_{\nu} \phi + \text{c.c.}
\] (130)

\[
L_{2s+3}^{(\text{dipole})} = - \frac{ie}{m^{2s-1}} \beta_1 F^{\mu\nu} \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} B^*_{\rho} \partial_{\mu} \phi^* \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \partial_{\nu} \phi + \text{c.c.}
\]

\[
+ \frac{ie}{m^{2s-1}} f_2(\beta_1, \beta_2) F^{\mu\nu} \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \phi \partial_{\mu} \phi^* \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \partial_{\nu} \phi + \text{c.c.}
\]

\[
+ \frac{ie}{m^{2s-1}} f_3(\beta_1, \beta_2) F^{\mu\nu} \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \partial_{\mu} \phi \partial_{\mu} \phi^* \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \partial_{\nu} \phi + \text{c.c.}
\] (131)

where $f(\beta_1, \beta_2)$’s are linear functions of $\beta_1, \beta_2$. The $O(e)$ dimension-(2s + 4) operators can be canceled by choosing $\beta_1 = 1$, and $\beta_2$ such that $f_1(\beta_1, \beta_2) + c_1(s) = 0$. As discussed in Section 3, this will give rise to a gauge invariance for the vector St"uckelberg in the $O(e)$ dimension-(2s + 3) operators. Since all the operators that do not respect this gauge invariance cancel, we must have $f_2(\beta_1, \beta_2) + c_2(s) = 0$, and $f_3(\beta_1, \beta_2) + c_3(s) = 0$.

Now in the scaling limit: $m \to 0$ and $e \to 0$, such that $e/m^{2s-1} =$constant, the non-minimal Lagrangian reduces to

\[
L = L_{\text{kin}} + \frac{ie F^{\mu\nu}}{m^{2s-1}} \{ \partial_{\mu_1} \ldots \partial_{\mu_{s-2}} \partial_{\mu} B^*_{\rho} \partial_{\mu_1} \ldots \partial_{\mu_{s-2}} \partial_{\nu} \phi - \text{c.c.} \}
\]

\[
- \frac{ie F^{\mu\nu}}{2m^{2s-1}} \{ \partial_{\mu_1} \ldots \partial_{\mu_{s-2}} \partial_{\mu} B^*_{\nu} \partial_{\mu_1} \ldots \partial_{\mu_{s-2}} \partial_{\nu} \phi - \text{c.c.} \}
\] (132)

Since some of the dimension-(2s + 3) operators are not proportional to any of the equations of motion, we cannot improve the degree of divergence. Thus the theory has an intrinsic UV cutoff, not higher than

\[
\Lambda_s = \frac{m}{e^{1/(2s-1)}} .
\] (133)

\^[8]Terms involving $\tilde{F}$ are not useful in canceling the existing non-renormalizable operators, because the two have opposite parity.
6.2 Half-integer Spin \( s = n + 1/2 \):

We start with the Lagrangian for a massless field of arbitrary half-integer spin \( s = n + 1/2 \) in (4+1)D \([17]\). Kaluza-Klein reduction to (3+1)D gives a St"uckelberg invariant Lagrangian for a massive spin-\( s \) field. The Lagrangian contains symmetric tensor-spinor fields of rank 0 through \( n \) (spin \( 1/2 \) through \( s \)), and enjoys the St"uckelberg symmetry:

\[
\begin{align*}
\delta \psi_n &= \partial \lambda_{n-1} \\
\delta \psi_{n-1} &= \partial \lambda_{n-2} + m \lambda_{n-1} \\
&\vdots \\
\delta \psi_{n-k} &= \partial \lambda_{n-k-1} + km \lambda_{n-k} \\
&\vdots \\
\delta \psi_0 &= nm \lambda_0
\end{align*}
\]

The 4D gauge parameters are not all independent; they satisfy the following \((n-1)\) conditions, thanks to the gamma-tracelessness of the 5D gauge parameter:

\[
\begin{align*}
\lambda'_{n-1} &= \lambda_{n-2} \\
\lambda'_{n-2} &= \lambda_{n-3} \\
&\vdots \\
\lambda'_1 &= \lambda_0
\end{align*}
\]

where prime denotes gamma-trace. We therefore have the freedom of \( n \) symmetric gamma-traceless independent gauge parameters, namely \( \{\lambda_0, \lambda'_1, \lambda'_2, ..., \lambda'_{n-1}\} \). The gauge fixed Lagrangian contains the following fields — a spin-\( s \): \( \psi_n \), a spin-\((s-1)\): \( \psi'_n \), two spin-\((s-2)\): \( \psi''_n, \psi'_{n-1} \), two spin-\((s-3)\): \( \psi''_{n-1}, \psi'_{n-2}, ..., \), two spin-1/2: \( \psi''_2, \psi'_1 \). All the auxiliary fields have been identified as (gamma)traces of the St"uckelberg fields.

The following tensor-spinor is invariant under the St"uckelberg transformation:

\[
\hat{\psi}_n \equiv \sum_{k=0}^{n} \frac{(-1)^k}{k! m^k} \partial^k \psi_{n-k}.
\]

Because of the \((n-1)\) conditions on the gauge parameters, one is also able to construct \((n-1)\) additional invariant tensor-spinors: \( \hat{\psi}'_1, \hat{\psi}'_2, ..., \hat{\psi}'_{n-1} \). For example,

\[
\hat{\psi}'_1 \equiv \psi'_1 - \left( \frac{n-1}{n} \right) \psi_0 - \frac{1}{nm} \partial \psi_0.
\]

29
is Stückelberg invariant, because $\lambda'_1 = \lambda_0$. Out of the invariant spinors \{\tilde{\psi}'_1, \tilde{\psi}'_2, \ldots, \tilde{\psi}'_{n-1}, \tilde{\psi}'_n\} one can further construct another set of invariant spinors \{\psi'_1, \psi'_2, \ldots, \psi'_{n-1}, \psi'_n\}, such that when the latter set replaces its untwiddled counterpart in the gauge fixed Lagrangian, it exactly reproduces the Stückelberg invariant KK-reduced Lagrangian before gauge fixing. The twiddled fields are constructed by considering the gauge fixing conditions. Note that $\tilde{\psi}'_n$ and $\psi'_n$ are related by the gauge transformation:

$$\tilde{\psi}'_n = \psi'_n + \partial \lambda_{n-1} = \psi'_n + \partial \lambda_{n-1} + \frac{1}{4} \gamma \partial \lambda_{n-2}.$$ (136)

The gauge fixing conditions are such that the highest power of $1/m$ in $\lambda_{k-1}$ is $(1/m)^k$. Thus only the term $\partial \lambda_{n-1}^{\gamma T}$ in (136) can contain $(1/m)^n$. In view of (134) we have

$$\tilde{\psi}'_n = \psi'_n + \frac{(-1)^n}{n! m^n} \left[ \partial^a + b_1(n) \gamma \partial^{a-1} \partial + b_2(n) \eta \partial^{a-2} \Box + b_3(n) \gamma \partial^{a-3} \Box \partial + \ldots \right] \psi_0 + \ldots$$ (137)

where $b(n)$'s are rational functions of $n$ such that the quantity in the brackets is the derivative of a gamma-traceless quantity (with indices symmetrized).

The Stückelberg invariant gauged Lagrangian contains interaction terms proportional to $e/m^{2n} = e/m^{2s-1}$. Such terms come only from the kinetic pieces of the spin-$s$ field. Furthermore, since the free spin-$s$ kinetic operator acting on $\partial \lambda_{n-1}^{\gamma T}$ gives zero, we only need to take into account the terms that have $O(e)$ contribution from the kinetic operator, and $O(1)$ contribution from $\tilde{\psi}'_n$. In other words, we need to consider the following

$$L_{\text{int}} = \tilde{\psi}_0^{\gamma(0)} \left( \frac{\partial}{\partial T_{\mu_1 \mu_2 \ldots \mu_s; \nu_1 \nu_2 \ldots \nu_n}} \right) \tilde{\psi}_0^{\gamma(0)} = L_{\leq (2s+2)} + O(e^2),$$ (138)

where $\tilde{\psi}_0^{\gamma(0)}$ is given by Eq. (137), and

$$T_{\mu_1 \mu_2 \ldots \mu_s; \nu_1 \nu_2 \ldots \nu_n} = \frac{n - 1}{2} \eta^{\mu_1 \mu_2} \cdots \eta^{\mu_s \nu_n} A_{\rho} \left( \left( n/2 \right) \eta^{\mu_1 \mu_2} \eta^{\rho \nu_1} + \eta^{\mu_1 \nu_1} \eta^{\mu_2 \rho} + \eta^{\mu_2 \nu_1} \eta^{\mu_1 \rho} \right) \gamma^{\nu_2}$$

$$+ \frac{n - 2}{4} \eta^{\mu_1 \mu_2} \cdots \eta^{\mu_s \nu_n} A_{\rho} \left( \eta^{\mu_1 \nu_1} \eta^{\mu_2 \nu_2} - \eta^{\mu_1 \nu_2} \eta^{\mu_2 \nu_1} \gamma^{\mu_2} + \left( \mu_1 \leftrightarrow \mu_2 \right) \right)$$

$$- \frac{n - 2}{4} \eta^{\mu_1 \mu_2} \cdots \eta^{\mu_s \nu_n} A_{\rho} \left( \eta^{\mu_1 \nu_1} \gamma^{\mu_2 \rho} + \eta^{\mu_2 \nu_1} \gamma^{\mu_1 \rho} \right)$$

$$+ \frac{n(n - 1)}{4} \eta^{\mu_1 \mu_2} \cdots \eta^{\mu_s \nu_n} A_{\rho} \eta^{\mu_1 \mu_2} \left( \eta^{\mu_2 \nu_1} \gamma^{\mu_1 \rho} - \eta^{\mu_1 \nu_2} \gamma^{\mu_2 \rho} \right)$$

$$+ \frac{(n - 1)(n - 2)}{4} \eta^{\mu_1 \mu_2} \cdots \eta^{\mu_s \nu_n} A_{\rho} \eta^{\mu_1 \mu_2} \left( \eta^{\mu_2 \nu_1} \gamma^{\mu_1 \rho} + \eta^{\mu_1 \nu_2} \gamma^{\mu_2 \rho} \right),$$ (139)

with symmetrization assumed in $(\mu_1, \mu_2, \ldots, \mu_n)$ and in $(\nu_1, \nu_2, \ldots, \nu_n)$. Given Eq. (138), we
can easily compute the $O(e)$ dimension-$(2s + 3)$ operators. We have

$$L_{2s+3} = -\frac{e}{2m^{2n}} F_{\mu\nu}(\bar{\chi} \partial_{\mu_1} \cdots \partial_{\mu_{n-1}} \gamma^{\mu\nu\rho} \partial_\rho \partial_{\mu_1} \cdots \partial_{\mu_{n-1}} \chi) + (...)$$

where we denoted the spin-1/2 Stückelberg field as $\psi_0 \equiv \chi$, and the ellipses (...) stand for similar terms containing $\partial \chi$. The terms in (140) can be rendered proportional to the equations of motion, but the degree of divergence cannot be improved by field redefinitions followed by addition of local terms (similar to the cases of spin-3/2 and -5/2). Addition of non-minimal terms does not help as well. Therefore, in the scaling limit: $m \rightarrow 0$ and $e \rightarrow 0$, such that $e/m^{2s-1}=$constant, the Lagrangian becomes:

$$L = L_{\text{kin}} - \frac{e}{2m^{2n}} F_{\mu\nu}(\bar{\chi} \partial_{\mu_1} \cdots \partial_{\mu_{n-1}} \gamma^{\mu\nu\rho} \partial_\rho \partial_{\mu_1} \cdots \partial_{\mu_{n-1}} \chi) + (...)$$

The effective field theory has an intrinsic finite cutoff

$$\Lambda_s = \frac{m}{e^{1/(2s-1)}}.$$ (142)

### 7 Conclusion

In this paper we argued that massive, charged particles of spin $s \geq 3/2$ can be described by a local effective action at energy scales parametrically higher than their mass. This effective action breaks down at or below an energy $E = O(\Lambda_s)$, $\Lambda_s = me^{1/(2s-1)}$.

The cutoff $\Lambda_s$ can be improved neither by adding local non-minimal coupling terms nor by local field redefinitions. For the bosonic case, the latter result was proved using a cohomological argument valid for all spins, described in details in Appendix A. Its physical meaning is illustrated for the case of spin-2 particles in Appendix B. The meaning of the cutoff $\Lambda_s$ is only that some new physics must happen at a scale not higher than $\Lambda_s$. This new physics could result in a strong coupling unitarization, or in the existence of new interacting degrees of freedom, lighter than the cutoff. In the first case, the theory becomes, to all effects, non-local at a scale not higher than $\Lambda_s$, because the spin-s particle develops a form factor which implies a finite, nonzero charge radius. In the second case, one could integrate out the new light degrees of freedom, but the resulting action would be non-local already below the scale $\Lambda_s$. The very possibility of introducing non-local counter-terms into the action invalidates the cohomological argument of Appendix A. This is the technical reason why lighter degrees of freedom may be essential for a complete UV embedding of our effective actions.

Notice that the examples given in the introduction exhibit both new light degrees of
freedom and non-localities at energy scales parametrically smaller than $\Lambda_s$. High-spin hadronic resonances have a natural inverse size $\mathcal{O}(\Lambda_{QCD}) \ll m$, and also interact with other lighter, lower-spin resonances. The Argyres-Nappi action [4] describes a perturbative string state, with intrinsic inverse size $\mathcal{O}(M_s) \ll m$, which interacts with and can decay into many other lighter string states.

A third possibility, namely that the theory reaches a UV fixed point, is not really different from strong coupling unitarization. The point is that the intrinsic cutoff we found was due to the explicit presence of power counting, non-renormalizable operators in the effective action of the spin-s particle. To make these operators UV irrelevant, their anomalous dimensions must change (run) by factors of order one from their tree-level values within a small energy window ($m < E < \Lambda_s$). For instance, in the case of spin-2, the operator (23) must change its dimension from 7 at $E = m$ to 4 or less at $E = me^{-1/3}$! This cannot be achieved by the logarithmic running implied by a perturbative effective local action containing only a massive spin-2 field and the photon: a dramatic, non-perturbative re-arrangement of the degrees of freedom must take place below the scale $\Lambda_s$.

If a UV fixed point does exist, what would it look like? This is difficult to say; what is easy to predict is that it will not look like a weakly interacting spin-s particle. This is because a UV fixed point must necessarily describe a massless high-spin particle. But, starting at spin-5/2, massless particles are forbidden to interact with any lower-spin fields (including graviton) by powerful no-go theorems [42, 1, 3]. For spin-2, a weaker theorem [3] implies that the particle must be electrically neutral, while all non-minimal terms, except for possibly a dipole, are ruled out because they cannot have dimension four, as demanded by the conformal invariance of the UV fixed point.

It is instructive to compare our results with claims contained in a recent preprint [43], where it is argued that, in the case of massive, charged spin-2, a new choice of St"uckelberg fields may yield a higher cutoff than what we estimated in this paper. This is of course impossible. First of all, the action presented in [43] is incomplete, since it does not contain the EM kinetic term, whose gauge completion under its new St"uckelberg transformations contains non-renormalizable interactions, singular in the massless limit. More importantly, any gauge-invariant completion must reduce in the unitary gauge to the minimal EM-coupled action plus a particular choice of non-minimal terms. Otherwise, the action would not describe a spin-2 only, but also other physical degrees of freedom even in perturbation theory. Once in the unitary gauge, we can use our procedure to introduce our set of St"uckelberg fields and proceed to show that the cutoff is as in Eq. (1). The core of our

\[^{9}\text{That preprint appeared about a month later than our paper.}\]
paper is indeed proving that the cutoff (1) cannot be changed by any field redefinition or addition of non-minimal terms. Moreover, the action considered in [43] contains a vertex $DhFDh$, needed to give a correct counting of degrees of freedom. Although such a term is proportional to $e/m^2$, one cannot conclude that the model has a cutoff of $m/\sqrt{e}$. This is because in the unitary gauge the propagator is singular in the massless limit, so that one cannot simply read off the cutoff by looking at the most divergent term. In other words, higher dimensional operators have canonical dimensions only when the propagator has good high energy behavior (i.e. $\sim 1/p^2$). This can be achieved by introducing Stückelberg fields and performing an appropriate covariant gauge-fixing. Only after the gauge fixing can one correctly identify the canonical dimensions of the higher dimensional operators. In particular, the true cutoff of the action in [43] is $me^{-1/6}$, which is much lower than our upper bound of $me^{-1/3}$.

We remind again the reader that our aim was to find the maximum cutoff of an effective Lagrangian of a single spin-$s$ field interacting with EM. Our Lagrangian should be thought only as a generating functional of the S-matrix. We are after a bound valid in perturbation theory, and not after a Lagrangian free of pathologies, which would require more powerful methods than the one we used; i.e. methods that can fix terms quadratic and higher order in the EM field. We emphasize that our theoretical maximum may never be reached in a truly consistent theory. For instance, the spin-2 Argyres-Nappi Lagrangian [4] has a cutoff much lower than $me^{-1/3}$.

We would like to conclude with a comment on spin-1 and an observation on a possible new approach to the problem of finding the intrinsic cutoff.

**Spin-1**

The cutoff of a charged massive spin-1 particle is known to be $O(m/e)$. In this special case we also know how to improve the UV behavior of the theory: it suffices to add another degree of freedom, a neutral Higgs scalar lighter than $O(m/e)$. To show that the cutoff in the absence of the neutral Higgs is no higher than $O(m/e)$, we start by writing the Lagrangian of a complex, massive spin-1 field $W_\mu$:

$$L = -\frac{1}{2} |\partial_\mu W_\nu - \partial_\nu W_\mu|^2 - m^2 W_\mu^* W_\mu. \quad (143)$$

We make it gauge invariant by introducing a scalar (Stückelberg) field $\phi$ through the substitution $W_\mu = V_\mu - \partial_\mu \phi/m$. The minimal substitution $\partial_\mu \to D_\mu \equiv \partial_\mu + ieA_\mu$ preserves
the gauged Stückelberg symmetry:

\[
\delta V_\mu = D_\mu \lambda, \quad (144)
\]

\[
\delta \phi = m \lambda. \quad (145)
\]

One thus obtains the following non-renormalizable operators

\[
L_{\text{div}} = - \left[ \frac{ie}{2m} F^{\mu\nu} \phi^*(D_\mu V_\nu - D_\nu V_\mu) + \text{c.c.} \right] - \frac{e^2}{2m^2} F_{\mu\nu}^2 \phi^* \phi. \quad (146)
\]

The presence of non-renormalizable interactions implies the existence of a UV cutoff \( \Lambda \sim m/e \). The tri-linear operators can be canceled by adding to the Lagrangian the dipole term \( ieF^{\mu\nu}W_\mu^*W_\nu \). The resulting non-minimal Lagrangian still contains tri-linear non-renormalizable operators:

\[
L_{\text{div}}^{(\text{NM})} = - \left[ \frac{ie}{m} \partial_\mu F^{\mu\nu} V_\nu^* \phi + \text{c.c.} \right] - \frac{ie}{m^2} \partial_\mu F^{\mu\nu} \phi^* \partial_\nu \phi + \mathcal{O}(e^2). \quad (147)
\]

Notice that the last term implies a cutoff \( m/\sqrt{e} \), parametrically lower than \( m/e \)! This hardly seems like a progress, but since all non-renormalizable terms in Eq. (147) are proportional to the photon equations of motion, they can be canceled by the local field redefinition:

\[
A_\mu \rightarrow A_\mu + \left\{ \frac{ie}{m} \left( V_\mu^* \phi + \frac{1}{2m} \phi^* \partial_\mu \phi \right) + \text{c.c.} \right\}. \quad (148)
\]

This redefinition generates a host of dimension-6 and dimension-7 operators, of schematic form \( (e^2/m^2)O_6 \), \( (e^2/m^3)O_7 \), and a dimension-8 operator proportional to \( e^2/m^4 \). The latter comes form the redefinition of the Maxwell action. It is the most dangerous one, since it introduces a lower cutoff \( m/\sqrt{e} \); but it can be eliminated by adding appropriate local counter-terms. All remaining non-renormalizable terms are \( \mathcal{O}(e^2) \), so that even though they may not – indeed cannot – be canceled by adding local counter-terms, they may be – indeed are – canceled by the non-local counter-terms obtained by embedding the theory into a spontaneously broken gauge theory with, e.g., gauge group \( SU(2) \) broken to \( U(1) \), and integrating out the neutral Higgs degree of freedom.

Notice that in this case one obtains the same UV scale \( m/e \) from either the minimal Lagrangian or the non-minimal one. Indeed, unless we knew in advance that a UV completion existed, the exercise of adding the dipole term \( ieF^{\mu\nu}W_\mu^*W_\nu \) would have made the UV behavior of the theory worse, lowering the cutoff from \( m/e \) to \( me^{-2/3} \).
Future Directions

The explicit example of the Argyres-Nappi action [4] potentially holds for us one last lesson. Its true cutoff is $M_S$, much less than our “optimal” one, $\Lambda_2$. Even ignoring the fact that this action is embedded in string theory, it still contains a host of non-renormalizable operators proportional to powers of $e F_{\mu\nu}/M_S^2$; therefore, already the interactions involving only transverse modes become strong at the scale $M_S/\sqrt{e} \ll \Lambda_2$. On the other hand, these non-minimal terms are necessary to ensure that the theory propagates causally five degrees of freedom in constant external EM fields [4]. This observation opens up the possibility of obtaining more stringent bounds on the UV cutoff of any high-spin theory by requiring causality in external backgrounds. This is similar, at least in spirit, to causality bounds found for certain effective field theories in [44] and, in other theories and with different methods, in [45].

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Appendix A

In this Appendix we are going to use cohomological argumentsto show that for integer spin $s$ it is impossible to eliminate up to a total derivative the dangerous $O(e^2)$ dimension-(4s+3) operator by adding local counter-terms. Note that the former arises after we make a field redefinition of the $U(1)$ gauge field to cancel the dimension-(2s+4) operator, and then add a local term to eliminate the resulting $O(e^2)$ dimension-(4s+4) operator. We start with the simplest case of $s = 2$. Then we use similar arguments for spin-3, and finally generalize them for arbitrary spin $s$.

For spin-2, we want to study the cohomology defined by the coboundary operator:

$$
\delta = \int d^4x \left[ (c + c_\mu x^\mu) \frac{\delta}{\delta \phi(x)} + (b_\mu + b_{\mu\nu} x^\nu) \frac{\delta}{\delta B_\mu(x)} \right],
$$

(A.1)
on space-time integrals of local functionals of the basic fields $\phi(x)$, $B_\mu(x)$, and derivatives thereof, modulo total derivatives. That is, we want to find the space of functionals $P(\phi, B_\mu, \partial_\nu \phi, ...)$ which obey $\delta \int d^4 x P = 0$, and are not of the form $\int d^4 x P = \delta \int d^4 x P'$ for some other $P'(\phi, B_\mu, \partial_\nu \phi, ...)$. Here $c, c_\mu, b_\mu, b_{\mu\nu}$ are space-time constant anti-commuting
variables. We allow no explicit dependence on the coordinates $x^\mu$ in either $P$ or $P'$. The integrand $P$ is defined modulo a total derivative, of course. To take this fact into account, we define another coboundary operator: $d \equiv \epsilon^\mu \partial_\mu$, with $\epsilon^\mu$ an anti-commuting variable. Prior to defining a scalar product and a Hermitian conjugate for $d$, $\epsilon^\mu$ is just another name for the coordinate differential $dx^\mu$. With the help of $d$, the cohomology now reads

$$\delta P = dQ, \quad P \neq \delta P' + dQ'.$$

To study the cohomology it is extremely useful to define a Hilbert space structure for the space of $P$'s. It allows us to define Hermitian conjugates for $\delta$ and $d$.

A Hilbert space structure can be defined on local functionals of fields as follows [46]. We Taylor expand a field $A(x)$ around $x^\mu = 0$:

$$A(x) = \sum_{n=0}^\infty \frac{1}{n!} a_{\mu_1...\mu_n}^* x^{\mu_1} ... x^{\mu_n}. \quad (A.3)$$

Then we promote each $a_{\mu_1...\mu_n}^*$ to a canonical Bosonic creation operator; the corresponding canonical annihilation operator $a_{\mu_1...\mu_n}$ is its Hermitian conjugate. By definition, we have the commutation relations:

$$[a_{\mu_1...\mu_n}, a_{\nu_1...\nu_n}^*] = \sum \delta_{\mu_1^{p(1)}}^{\nu_1} ... \delta_{\mu_n^{p(n)}}^{\nu_n}, \quad (A.4)$$

where the sum extends to all permutations $p(1), ..., p(n)$ of $n$ indices. Likewise, Hermitian conjugates of $c, c_\mu, b_\mu, b_{\mu\nu}, \epsilon^\mu$ are defined by imposing canonical anti-commutation relations for all these variables. In particular:

$$\{\epsilon^\mu, \epsilon_\nu^*\} = \delta_\nu^\mu. \quad (A.5)$$

A local function of the field $A$ and its derivatives defines a vector in the Fock space, which is obtained by applying the creation operators $a_{\mu_1...\mu_n}^*$ to the vacuum state $|0\rangle$. We have $a_{\mu_1...\mu_n}|0\rangle = 0$ for all $a$'s. The correspondence is

$$P(A, \partial_\mu A, ....) \rightarrow P(a^*, a_{\mu}^*, ....)|0\rangle. \quad (A.6)$$
On the Fock space, the derivative operator \( \partial_{\mu} \) reduces to
\[
\partial_{\mu} = \sum_{n=0}^{\infty} \frac{1}{n!} a_{\mu_1 \ldots \mu_n}^{*} a_{\mu_1 \ldots \mu_n}.
\]  
(A.7)

Its Hermitian conjugate is therefore
\[
\partial_{\mu}^{*} = \sum_{n=0}^{\infty} \frac{1}{n!} a_{\mu_1 \ldots \mu_n}^{*} a_{\mu_1 \ldots \mu_n}.
\]  
(A.8)

Thanks to these definitions and to the anti-commutation relations (A.5), the Laplacian operator \( \Delta \equiv (d + d^{*})^{2} \) can be written as
\[
\Delta = \epsilon_{\mu}^{\nu} \partial_{\mu} \epsilon_{\nu}^{\nu} + \epsilon_{\nu}^{\mu} \partial_{\nu} \epsilon_{\mu}^{\mu} = \partial^{\mu} \partial_{\mu} + \epsilon_{\mu}^{\mu} N, \quad N \equiv \sum_{n=0}^{\infty} \frac{1}{n!} a_{\mu_1 \ldots \mu_n}^{*} a_{\mu_1 \ldots \mu_n}.
\]  
(A.9)

On a monomial in the field and its derivatives, the particle number operator \( N \) reduces to the degree of the monomial. Eq. (A.9) uses the identity \([46\): \([\partial_{\mu}^{*}, \partial_{\nu}] = \delta_{\nu}^{\mu} N\), 
(A.10)

which in turn can be easily derived using definitions (A.7, A.8), and the commutator
\[
[a_{\mu_1 \ldots \mu_n}^{*} a_{\mu_1 \ldots \mu_n}^{*}, a_{\nu_1 \ldots \nu_n}^{*} a_{\nu_1 \ldots \nu_n}^{*}] = \frac{1}{n!} \delta_{\nu}^{\mu} a_{\mu_1 \ldots \mu_n}^{*} a_{\mu_1 \ldots \mu_n} + \frac{1}{(n-1)!} a_{\mu_1 \ldots \mu_n-\nu}^{*} a_{\mu_1 \ldots \mu_n-\nu}^{*} - \frac{1}{n!} a_{\mu_1 \ldots \mu_n}^{*} a_{\mu_1 \ldots \mu_n}.
\]  
(A.11)

The identification \( \epsilon_{\mu} = dx_{\mu} \) implies that \( \epsilon_{\mu}^{*} \epsilon_{\mu} = 4 - D_F \), where \( D_F \) counts the degree of the differential form obtained by the replacement \( \epsilon_{\mu} \rightarrow dx_{\mu} \), \( a_{\mu_1 \ldots \mu_n} \rightarrow \partial_{\mu_1} \ldots \partial_{\mu_n} A \) in the Hilbert space vector \( P(\epsilon, a, a_{\mu}, \ldots)|0\rangle \).

Now the \( d \) cohomology is isomorphic to the space of harmonic vectors, which are solutions of the equation \( \Delta \psi = 0 \). Since \( \Delta = (4 - D_F)N + \) positive operator, the cohomology is trivial unless either \( D_F = 4 \) or \( N = 0 \). This fact is useful for our purpose.

Let us now apply this machinery to our problem, where the fields are \( \phi \) and \( B_{\mu} \). For spin-2, the dimension-11 operator we want to cancel by adding local counter-terms is
\[
L_{11} = \frac{e^2}{2m^2} \{ \partial^{\rho} \partial_{\sigma} \phi^{*} \partial^{\nu} \partial_{\sigma} \phi - (\mu \leftrightarrow \nu) \} \{ 2 \partial_{[\mu} B_{\rho]}^{*} \partial^{\nu} \partial_{\rho} \phi - \partial_{[\mu} B_{\rho]}^{*} \Box \phi \}.
\]  
(A.12)

\[^{10}\text{Opposite to [46], we use the standard physicists’ conventions for creation and annihilation operators.}\]
We can write it as a four-form with \( N = 4 \), denoted by \( O_{(4,4)} \):

\[
O_{(4,4)} = \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta} L_{11} e^\alpha \varepsilon^\beta \varepsilon^\gamma \varepsilon^\delta.
\]  

(A.13)

Its variation under the operator \( \delta \), defined in Eq. (A.1), is a nonzero total derivative: a three form with \( N = 3 \). By labeling all operators appearing in our cohomology equations as \( O_{(D_F,N)} \) we get the equation

\[
\delta O_{(4,4)} = dO_{(3,3)}.
\]  

(A.14)

By applying \( \delta \) to both sides of this equation and using the fact that the \( d \) cohomology is soluble on operators with \( N = 2, D_F = 3 \) we get

\[
\delta O_{(3,3)} = dO_{(2,2)}.
\]  

(A.15)

Repeating the same argument on \( O_{(2,2)} \) we get \( \delta O_{(2,2)} = dO_{(1,1)} \) and finally

\[
d\delta O_{(1,1)} = 0.
\]  

(A.16)

Now, \( \delta O_{(1,1)} \) is a one form with \( N = 0 \), thus it also obeys \( \partial^\mu \delta O_{(1,1)} = 0 \). So, unless it vanishes identically, it is a nontrivial element of the \( d \) cohomology. In particular,

\[
\delta O_{(1,1)} \neq dO_{(0,0)}.
\]  

(A.17)

In the spin-2 case, an explicit computation gives\(^{11}\)

\[
\delta O_{(1,1)} = \frac{6e^2}{m^2} \varepsilon^{\mu\nu\rho\sigma} \bar{c}_\alpha c^\alpha b_{\mu\nu} c_\rho \varepsilon^\sigma \neq 0.
\]  

(A.18)

Now, in order to cancel the dangerous dimension-11 operator \( O_{(4,4)} \), there must exist another four-form \( \tilde{O}_{(4,4)} \), of the same dimension, such that the sum of the two is a total derivative

\[
O_{(4,4)} + \tilde{O}_{(4,4)} = dO_{(3,4)}.
\]  

(A.19)

The operator \( \tilde{O}_{(4,4)} \) is built with the St"{u}ckelberg fields, so that by construction it is annihilated by the operator \( \delta \); this gives us the equation

\[
\delta O_{(4,4)} = d\tilde{O}_{(3,4)}.
\]  

(A.20)

\(^{11}\)To avoid confusion, let us point out that \( \bar{c}, \bar{b}_\mu \) etc. are the ghost fields associated with the complex conjugate fields \( \phi^* \) and \( B^*_\mu \), not the Hermitian conjugates of \( c, b_\mu \) etc!
Comparing Eqs. (A.20) and (A.14) we find that \( \delta O_{(3,4)} - O_{(3,3)} \) is closed. Since the \( d \) cohomology is trivial on three-forms with \( N = 3 \) we get

\[
O_{(3,3)} = \delta O_{(3,4)} + dO_{(2,3)}.
\]

(A.21)

By applying \( \delta \) to both sides of Eq. (A.21), comparing the result with Eq. (A.15), and using the triviality of the \( d \) cohomology on two-forms with \( N = 2 \), and then repeating the argument once more, on one forms with \( N = 1 \), we finally arrive at

\[
O_{(1,1)} = \delta O_{(1,2)} + dO_{(0,1)}.
\]

(A.22)

This immediately says that \( \delta O_{(1,1)} \) is \( d \)-exact, in contradiction with Eq. (A.17). The conclusion is that the dangerous dimension-11 operator cannot be eliminated, up to a total derivative, by adding local counter-terms.

The analysis for spin-3 is quite similar. In view of Eqs. (60, 61), the coboundary operator \( \delta \) is given by

\[
\delta = \int d^4 x \left[ \left( c + c_\mu x^\mu + \frac{1}{2} c_{\mu\nu} x^\mu x^\nu \right) \frac{\delta}{\delta \phi(x)} + \left( b_\mu + b_{\mu\nu} x^\nu + \frac{1}{2} b_{\mu\nu\rho} x^\nu x^\rho \right) \frac{\delta}{\delta B_\mu(x)} \right].
\]

(A.23)

The four-form under consideration is

\[
O_{(4,4)} = \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta} L_{15} \epsilon^\alpha \epsilon^\beta \epsilon^\gamma \epsilon^\delta,
\]

(A.24)

where \( L_{15} \) is the dimension-15 operator (59) we want to eliminate. Here again, we have a nonzero \( \delta O_{(1,1)} \):

\[
\delta O_{(1,1)} = \frac{6 \epsilon^2}{m^{11}} \varepsilon^{\mu\nu\rho\sigma} \bar{c}_{\alpha\beta} c^{\alpha\beta\delta} b_{\mu\lambda} \epsilon^\lambda \epsilon_\sigma \neq 0.
\]

(A.25)

This leads us into contradictions if we want to eliminate (A.24) by local counter-terms.

We expect a similar story for generic spin \( s \), where the operator \( \delta \) is defined in accordance with Eqs. (127, 128). After all, the only difference with the previous examples is a much larger number of Lorentz indices, which we expect to entail algebraic complications but no qualitatively new feature. Once again, we expect a nonzero \( \delta O_{(1,1)} \), which would make it impossible to eliminate the \( \mathcal{O}(\epsilon^2) \) dimension-(4s + 3) operator.
Appendix B

In this Appendix, we give a physical explanation of the fact that for spin-2 the dimension-11 operator (18) cannot be eliminated, up to a total derivative, by adding local counter-terms. We have

\[ L_{11} = \frac{e^2}{2m} \{ \partial^\mu \partial_\sigma \phi^* \partial^\nu \partial_\rho \phi - (\mu \leftrightarrow \nu) \} \{ 2\partial_{[\mu} B^*_{\rho]} \partial^\rho \partial_\sigma \phi - \partial_{[\mu} B^*_{\rho]} \Box \phi \}. \quad (B.1) \]

Let us consider an explicit scattering process involving two \( \phi \)'s, one \( \phi^* \), and one \( B^*_\mu \). In the process under consideration, the \( \phi \)'s and \( \phi^* \) are incoming and on-shell, which produce an off-shell \( B^*_\mu \) at rest. In the rest frame of \( B^*_\mu \), let the 4-momenta of the two \( \phi \)'s be

\[ k^\mu_1 = E(1, \cos \theta, \sin \theta, 0), \quad k^\mu_2 = E(1, \cos \theta, -\sin \theta, 0), \quad (B.2) \]

and that of the \( \phi^* \) be

\[ p^\mu = 2E(\cos \theta, -\cos \theta, 0, 0). \quad (B.3) \]

We have \( k^2_1 = k^2_2 = p^2 = 0 \), which are just on-shell conditions in the limit \( m \to 0 \). The following scalar products will be useful:

\[ k_1 \cdot k_2 = -2E^2 \sin^2 \theta, \quad (B.4) \]
\[ p \cdot k_1 = -2E^2 \cos \theta(1 + \cos \theta) = p \cdot k_2. \quad (B.5) \]

We define \( G_{\mu\nu} \equiv \partial_{[\mu} B^*_{\nu]} \) and \( H_{\mu\nu} \equiv \partial_{[\mu} B^*_{\nu]} \). The momentum space amplitude that comes from (B.1) is given by\(^\text{12}\)

\[ A = -G^{*\mu\nu}(k_1 \cdot k_2) \{(p \cdot k_1)p^\mu k^\nu_2 + (p \cdot k_2)p^\mu k^\nu_1\} \]
\[ = -16E^6 \sin^2 \theta \cos^2 \theta(1 + \cos \theta)^2 G^{*}_{01}. \quad (B.6) \]

We would like to cancel this amplitude by adding some local counter-terms in the Lagrangian. Let us add to the Lagrangian the most general set of counter-terms of appropriate dimension:

\[ L_{\text{c-t}} = e^2 \left[ \alpha \text{Tr}(\hat{h}^* \hat{h}^* \hat{h}^* \hat{h}^*) + \beta \text{Tr}(\hat{h}^* \hat{h}^* \hat{h}^* \hat{h}^*) \text{Tr}\hat{h} \right. \]
\[ + \gamma \text{Tr}(\hat{h}^* \hat{h}^* \hat{h}^* \hat{h}^*) \text{Tr}\hat{h}^* + \delta \text{Tr}(\hat{h}^* \hat{h}^*) \text{Tr}(\hat{h}^* \hat{h}^*) + \varepsilon \text{Tr}(\hat{h}^* \hat{h}^*) \text{Tr}(\hat{h}^* \hat{h}^*) \right], \quad (B.7) \]

\(^\text{12}\)In this Appendix we set \( e^2/m^7 = 1 \).
where \( \alpha, \beta, \gamma, \delta, \varepsilon \) are dimensionless constants. These counter-terms in turn give new dimension-11 operators, some of which contribute to the process under consideration. The contribution to the amplitude coming from (B.7) is

\[
A' = 2\beta H^*_{\mu\nu}(k_1 \cdot k_2)\{(p \cdot k_2)k_1^\mu p^\nu + (p \cdot k_1)k_2^\mu p^\nu\} + 2\delta H^*_{\mu\nu}\{(p \cdot k_1)^2k_2^\mu k_2^\nu + (p \cdot k_2)^2k_1^\mu k_1^\nu\} \\
+ \{4\alpha H^*_{\mu\nu}k_1^\mu k_2^\nu + 2\gamma^* H^*_{\mu\nu}(k_1 \cdot k_2)\}(p \cdot k_1)(p \cdot k_2) + 4\varepsilon H^*_{\mu\nu}(k_1 \cdot k_2)^2 p^\mu p^\nu. \tag{B.8}
\]

We want to have \( A + A' = 0 \), for any possible configuration of \( B^*_\mu \), and for any angle \( \theta \).

Let us choose the configuration: \( B_\mu = (B_0(t), 0, 0, 0) \), for which we have \( G^*_{\mu\nu} = 0 \), and \( H^*_{\mu\nu} = \text{diag}(2\hat{B}_0^*, 0, 0, 0) \). In this case, \( A \) vanishes, and \( A' \) reduces to

\[
A' = 32\hat{B}_0^* E^6 \cos^2 \theta [ (1 + \cos \theta)^2 \{ \alpha + \delta + \gamma^* \sin^2 \theta \} + 2 \sin^2 \theta \{ \beta (1 + \cos \theta) + \varepsilon \sin^2 \theta \} ] \tag{B.9}
\]

This must vanish for any \( \theta \), in particular for \( \theta = 0 \). This gives \( \delta = -\alpha \), so that we have

\[
A' = 32\hat{B}_0^* E^6 \cos^2 \theta \sin^2 \theta (1 + \cos \theta) [ \gamma^* (1 + \cos \theta) + 2\beta + 4\varepsilon (1 - \cos \theta) ] \tag{B.10}
\]

In order for this to vanish for generic \( \theta \), we must have \( \gamma^* = -\beta, \varepsilon = -\beta/4 \). All these conditions constrain the amplitude (B.8) for any generic configuration of \( B^*_\mu \) to

\[
A' = -2\beta H^*_{\mu\nu}(p \cdot k_1)^2(k_1 - k_2)^\mu(k_1 - k_2)^\nu - 2\beta H^*_{\mu\nu}(k_1 \cdot k_2)(p \cdot k_1)^2 \\
+ \beta H^*_{\mu\nu}(k_1 \cdot k_2)(p \cdot k_1)^2(2(k_1 + k_2)^\mu p^\nu - (k_1 \cdot k_2)p^\mu p^\nu) \tag{B.11}
\]

Now we choose another configuration: \( B_\mu = (0, B_1(t), 0, 0) \). Then all components of \( H_{\mu\nu}^* \), but \( H_{01}^* = H_{10}^* \), are zero, so that \( H_{\mu\nu}^* = 0 \). Given \( (k_1 - k_2)^\mu = 2E(0, 0, \sin \theta, 0) \), and \( (k_1 + k_2)^\mu = 2E(1, \cos \theta, 0, 0) \), one finds that \( A' \) vanishes identically. Thus the two amplitudes cannot cancel in general, as \( A \) does not vanish in this case:

\[
A = -16E^6 \sin^2 \theta \cos^2 \theta (1 + \cos \theta)^2 \hat{B}_1^* \neq 0. \tag{B.12}
\]

In conclusion, no set of local counter-terms can eliminate \( L_{11} \). This signals a bad UV behavior, since the amplitude goes like \( E^6 \), and therefore blows up at high energies.

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