1. Introduction

This paper should be considered a sequel to [Raf05]. We continue here to study the geometry of Teichmüller space using combinatorial properties of curves on surfaces. The main result is a formula for the Teichmüller distance between two points in Teichmüller space, in terms of the combinatorial information extracted from short curves of these two points. Let $S$ be a surface of finite type with negative Euler characteristic and let $\sigma_1$ and $\sigma_2$ be two points in the thick part of Teichmüller space $\mathcal{T}(S)$ of $S$. Let $\mu_1$ and $\mu_2$ be short markings on $\sigma_1$ and $\sigma_2$, respectively.

**Theorem 1.1.** There exists $k > 0$ such that

$$d_T(\sigma_1, \sigma_2) \asymp \sum_Y [d_Y(\mu_1, \mu_2)]_k + \sum_\alpha \log [d_\alpha(\mu_1, \mu_2)]_k.$$  

In the above theorem, the first sum is over all subsurfaces of $S$ that are not annuli and the second sum is over all simple closed curves on $S$; $d_Y(\mu_1, \mu_2)$ measures the relative complexity of the restrictions of $\mu_1$ and $\mu_2$ to a subsurface $Y$, and $d_\alpha(\mu_1, \mu_2)$ measures the relative twisting of $\mu_1$ and $\mu_2$ around a curve $\alpha$; the function $[x]_k$ is equal to zero when $x < k$ and is equal to $x$ when $x \geq k$, that is, we take into account only terms that are large enough; and the function log is a modified logarithm so that, for $x \in [0, 1]$, log $x = 0$. A general version of this theorem, where $\sigma_1$ and $\sigma_2$ are not necessarily in the thick part, is stated in §6 (Theorem 6.1).

Other recent results relate the geometry of Teichmüller space to combinatorial spaces. In [MM99] Masur and Minsky show that the electrified Teichmüller space is quasi-isometric to the complex of curves and therefore is also $\delta$–hyperbolic. Brock has shown [Bro03] that Teichmüller space equipped with the Weil-Petersson metric is quasi-isometric to the pants complex. Most recent developments in studying the Weil-Petersson metric have resulted from this analogy.

To drive our formula, we need to acquire an understanding of how the length and the twisting parameter of a curve change along a Teichmüller geodesic. [Raf05] provides a description of short curves. In this paper, we prove the following “convexity” property for the length of a curve along a
Teichmüller geodesic. Let $g : \mathbb{R} \to \mathcal{T}(S)$ be a geodesic in the Teichmüller space of $S$. For a curve $\alpha$ on $S$, denote the hyperbolic length of the geodesic representative of $\alpha$ at $g(t)$ by $l_t$.  

**Theorem 1.2.** Assume $\alpha$ is balanced at $t_\alpha$ and $s \geq t_\alpha$ (respectively, $s \leq t_\alpha$). Then, for any $t \geq s$ ($t \leq s$), we have

$$\frac{1}{l_s} \preceq \frac{1}{l_t}.$$  

We also give the following estimate for the twisting parameter along a Teichmüller geodesic. Let $\nu_+$ be the stable foliation of the geodesic $g$. The twisting parameter around a curve $\alpha$ at $g(t)$ is (roughly) the number of times that $\nu_+$ twists around $\alpha$ relative to a curve perpendicular to $\alpha$ in the hyperbolic metric of $g(t)$, and is denoted by $tw_t^+$.  

**Theorem 1.3.** There exists a constant $d_\alpha > 0$ such that

$$tw_t^+(\alpha) = \frac{d_\alpha e^{-2(t-t_\alpha)}}{e^{2(t-t_\alpha)} + e^{-2(t-t_\alpha)}} \pm O(1/l_t)$$

Some notation. To simplify our presentation, we avoid keeping track of constants that depend on the topology of the surface only. Instead, we use the following notation: When two functions $f$ and $g$ are equal up to additive constants, that is, when there exists a $C$ depending on the topology of $S$, such that $g(x) - C \leq f(x) \leq g(x) + C$, we write $f(x) \leftrightarrow g(x)$. Similarly, $f(x) \succ g(x)$ and $f(x) \ll g(x)$ mean that the inequalities are true up to an additive constant. When an inequality is true up to a multiplicative constant, we use symbols $\times$, $\succ$ and $\prec$; and, when it is true up to an additive constant and a multiplicative constant, we use symbols $\times$, $\prec$ and $\succ$. For example, $f(x) \asymp g(x)$ means that there are constants $c$ and $C$, depending on the topology of the surface only, such that

$$\frac{1}{c} g(x) - C \leq f(x) \leq c g(x) + C.$$  

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2. Preliminaries  

2.1. Curves and markings. By a curve in $S$ we mean a non-trivial, non-peripheral, simple closed curve in $S$. The free homotopy class of a curve $\alpha$ is denoted by $[\alpha]$. By an essential arc $\omega$ we mean a simple arc, with endpoints on the boundary of $S$, that cannot be pushed to the boundary of $S$. In case $S$ is not an annulus, $[\omega]$ represents the homotopy class of $\omega$ relative to the
boundary of $S$. When $S$ is an annulus, $[\omega]$ is defined to be the homotopy class of $\omega$ relative to the endpoints of $\omega$.

Define $\mathcal{C}(S)$ to be the set of all homotopy classes of curves and essential arcs on the surface $S$. To simplify notation, we often write $\alpha \in \mathcal{C}(S)$ instead of $[\alpha] \in \mathcal{C}(S)$. Define a distance on $\mathcal{C}(S)$ as follows: For $\alpha, \beta \in \mathcal{C}(S)$, define $d_S(\alpha, \beta)$ to be equal to one if $\alpha \neq \beta$ and if $\alpha$ and $\beta$ can be represented by disjoint curves or arcs. Let the metric on $\mathcal{C}(S)$ be the maximal metric having the above property, i.e., $d_S(\alpha, \beta) = n$ if $\alpha = \gamma_0, \gamma_1, \ldots, \gamma_n = \beta$ is the shortest sequence of curves or arcs on $S$ such that, for $i = 1, \ldots, n$, $\gamma_{i-1}$ is distance one from $\gamma_i$. (See [MM99].)

Let $\{\alpha_1, \ldots, \alpha_m\}$ be a pants decomposition of $S$. A marking on $S$ is a set $\mu = \{(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ such that the curve $\beta_i$ is disjoint from $\alpha_j$, for $i \neq j$, and intersects $\alpha_i$ once (twice) if the surface filled by $\alpha_i$ and $\beta_i$ is a once-punctured torus (four-times-punctured sphere). The $\alpha_i$ are called the base curves of $\mu$. For every $i$, $\beta_i$ is called the transverse curve to $\alpha_i$ in $\mu$. When the distinction between the base curves and the transverse curves is not important, we represent a marking as a set of curves $\{\beta_1, \ldots, \beta_n\}$ including all the base curves and the transverse curves. Denote the space of all markings on $S$ by $\mathcal{M}(S)$ (see [MM00].)

### 2.2. Subsurface intersection and subsurface distance

Let $\nu$ be a subset of $\mathcal{C}(S)$ (e.g., curves appearing in a marking) or a singular foliation on $S$, and let $Y$ be a subsurface of $S$. We define the projection of $\nu$ to the subsurface $Y$ as follows: Let

$$f : \tilde{S} \to S$$

be a regular covering of $S$ such that $f^*(\pi_1(S))$ is conjugate to $\pi_1(Y)$ (the $Y$–cover of $S$). Since $S$ admits a hyperbolic metric, $\tilde{S}$ has a well-defined boundary at infinity. Let $\tilde{\nu}$ be the lift of $\nu$ to $\tilde{S}$. Components of $\tilde{\nu}$ that are essential arcs or curves on $\tilde{S}$, if any, form a subset of $\mathcal{C}(\tilde{S})$. The surface $\tilde{S}$ is homeomorphic to $Y$. We call the corresponding subset of $\mathcal{C}(Y)$ the projection of $\nu$ to $Y$ and will denote it by $\nu_Y$. If there are no essential arcs or curves in $\tilde{\nu}$, $\nu_Y$ is the empty set; otherwise we say that $\nu$ intersects $Y$ essentially. This projection depends on the homotopy class of elements of $\nu$ only.

Let $\nu$ and $\nu'$ be subsets of $\mathcal{C}(S)$ or singular foliations on $S$ that intersect a subsurface $Y$ essentially. We define the $Y$–intersection ($Y$–distance) between $\nu$ and $\nu'$ to be the maximum geometric intersection number in $Y$ (maximum distance in $\mathcal{C}(Y)$) between the elements of projections $\nu_Y$ and $\nu'_Y$ and denote it by

$$i_Y(\nu, \nu') \quad \text{(respectively, } d_Y(\nu, \nu')).$$

If $Y$ is an annulus whose core is the curve $\alpha$, then we also denote $i_Y(\nu, \nu')$ and $d_Y(\nu, \nu')$ by $i_\alpha(\nu, \nu')$ and $d_\alpha(\nu, \nu')$, respectively. The following lemma is well known.

**Lemma 2.1.** Let $Y$, $\nu$ and $\nu'$ be as above.
(1) If $Y$ is not an annulus, then
$$d_Y(\nu, \nu') \propto \log i_Y(\nu, \nu').$$

(2) For a curve $\alpha$,
$$d_\alpha(\nu, \nu') \propto i_\alpha(\nu, \nu').$$

2.3. Quadratic differentials. Let $q$ be a meromorphic quadratic differential of area one on $S$. (See [GL00] for definition and details.) We assume that $q$ has a discrete set of finite critical points (i.e., critical points of $q$ are either zeroes or poles of order 1). Corresponding to $q$, there are two singular measured foliations called the horizontal and the vertical foliations, which we denote by $\nu_+$ and $\nu_-$. We call the singular Euclidean metric $|q|$ the $q$–metric on $S$. For a curve $\alpha$ in $S$, the $q$–geodesic representative of $\alpha$ exists and is unique except for the case where it is one of the continuous family of closed geodesics in a flat annulus, which we refer to as the flat annulus corresponding to $\alpha$. (Some difficulties arise when $q$ has poles of order 1. See [Raf05] for precise definitions and discussion.) We denote the $q$–length of $\alpha$ by $l_q(\alpha)$, the horizontal length of $\alpha$ by $h_q(\alpha)$ and the vertical length of $\alpha$ by $v_q(\alpha)$. We also denote the $q$–length, the horizontal length and the vertical length of the $q$–geodesic representative of $\alpha$, by $l_q([\alpha])$, $h_q([\alpha])$ and $v_q([\alpha])$, respectively. In general, for any metric $\tau$, $l_\tau(\alpha)$ represents the $\tau$–length of $\alpha$ and $l_\tau(\alpha)$ represents the $\tau$–length of the $\tau$-geodesic representative of $\alpha$.

2.4. Regular and primitive annuli in $q$. Let $Y$ be a subsurface of $S$ and $\gamma$ be a boundary component of $Y$.\footnote{We always assume that curves are piecewise smooth.} The curvature of $\gamma$ with respect to $Y$, $\kappa_Y(\gamma)$, is well defined as a measure with atoms at the corners. We choose the sign to be positive when the acceleration vector points into $Y$. If $\gamma$ is curved non-negatively (or non-positively) with respect to $Y$ at every point, we say it is monotonically curved with respect to $Y$. Let $A$ be an open annulus in $S$ with boundaries $\gamma_0$ and $\gamma_1$. Suppose both boundaries are monotonically curved with respect to $A$ and $\kappa_A(\gamma_0) \leq 0$. Further, suppose that the boundaries are equidistant from each other, and the interior of $A$ contains no zeroes. We call $A$ a primitive annulus and write $\kappa(A) = -\kappa_A(\gamma_0)$. If $\kappa(A) > 0$, we call $A$ expanding and say that $\gamma_0$ is the inner boundary and $\gamma_1$ is the outer boundary. When $\kappa(A) = 0$, $A$ is a flat annulus and is foliated by closed Euclidean geodesics homotopic to the boundaries. The following lemma is useful for computing the modulus of a primitive annulus.

Lemma 2.2 ([Raf05] Lemma 3.6). Let $A$ and $\gamma_0$ be as above, and let $d$ be the distance between the boundaries of $A$. Then
$$\kappa \operatorname{Mod}(A) \propto \log \left( \frac{d}{l_q(\gamma_0)} \right) \quad \text{if } \kappa(A) > 0$$
$$\operatorname{Mod}(A) l_q(\gamma_0) = d \quad \text{if } \kappa(A) = 0$$
Minsky has shown that every annulus of large modulus contains a primitive annulus with comparable modulus.

**Theorem 2.3** (Minsky [Min92, Theorem 4.6]). There exists an $\epsilon_0 > 0$ such that, for a curve $\alpha$ in $S$, if $l_\sigma([\alpha]) \leq \epsilon_0$, then there exists a primitive annulus $A$ such that

$$\frac{1}{l_\sigma([\alpha])} \asymp \text{Mod}(A).$$

Throughout this paper, $\epsilon_0$ is a fixed constant smaller than the Margulis constant, such that the above theorem and Theorem 2.4 are true.

### 2.5. Product regions in Teichmüller space

The Teichmüller space of $S$, $\mathcal{T}(S)$, is the space of conformal structures on $S$ up to isotopy. The Teichmüller distance between two points $\sigma_1$ and $\sigma_2$ is defined as

$$d_\mathcal{T}(\sigma_1, \sigma_2) = \frac{1}{2} K(\sigma_1, \sigma_2),$$

where $K(\sigma_1, \sigma_2)$ is the smallest quasi-conformal dilatation of a homeomorphism from $\sigma_1$ to $\sigma_2$. Let $\Gamma$ be a system of disjoint curves on $S$, and let $\text{Thin}_\epsilon(\Gamma)$ denote the set of all $\sigma \in \mathcal{T}(S)$ such that, for all $\gamma \in \Gamma$, the length of $\gamma$ in $\sigma$, $l_\sigma(\gamma)$, is less than or equal to $\epsilon$. Let $\mathcal{T}_\Gamma$ denote the product space $\mathcal{T}(S \setminus \Gamma) \times \prod_{\gamma \in \Gamma} \mathbb{H}_\gamma$,

where $S \setminus \Gamma$ is considered as a punctured space and each $\mathbb{H}_\gamma$ is a copy of the hyperbolic plane. Endow $\mathcal{T}_\Gamma$ with the sup metric. Minsky has shown, for small enough $\epsilon$, that $\text{Thin}_\epsilon(\Gamma)$ has a product structure.

**Theorem 2.4** (Minsky [Min96]). The Fenchel-Nielsen coordinates on $\mathcal{T}(S)$ give rise to a natural homeomorphism $\pi: \mathcal{T}(S) \to \mathcal{T}_\Gamma$. There exists an $\epsilon_0 > 0$ sufficiently small that this homeomorphism restricted to $\text{Thin}_{\epsilon_0}(\Gamma)$ distorts distances by a bounded additive amount.

Note that $\mathcal{T}(S \setminus \Gamma) = \coprod_Y \mathcal{T}(Y)$, where the product is over all connected components $Y$ of $S \setminus \Gamma$. Let $\pi_0$ denote the component of $\pi$ mapping to $\mathcal{T}(S \setminus \Gamma)$, let $\pi_Y$ denote the component mapping to $\mathcal{T}(Y)$, and, for $\gamma \in \Gamma$, let $\pi_\gamma$ denote the component mapping to $\mathbb{H}_\gamma$. For the rest of the paper, we fix $L_0 > 0$ such that, for a hyperbolic metric $\sigma$ on $S$, if $l_\sigma(\alpha) \geq \epsilon_0$, then there exists a curve $\beta$ intersecting $\alpha$ with $l_\sigma(\beta) \leq L_0$.

### 3. Behavior of a Geodesic in the Thin Part of Teichmüller Space

In this section, we prove Theorem 1.2, restated as Theorem 3.1, and study how the combinatorics of short markings changes along a Teichmüller geodesic. We show that, for every curve $\alpha$ in $S$, there exists a connected interval where $\alpha$ is “short” (Corollary 3.3), and the projections of the short markings to a subsurface can only change while all the boundaries of that subsurface...
are short (Proposition 3.7). This is an essential component of the proof of the main theorem.

3.1. Teichmüller geodesics. For $t \in \mathbb{R}$, let $q_t$ be the quadratic differential obtained from $q$ by scaling its horizontal foliation by a factor of $e^t$, and its vertical foliation by a factor of $e^{-t}$. Define $g(t)$ to be the conformal structure corresponding to $q_t$. Then $g \colon \mathbb{R} \to \mathcal{T}(S)$ is a geodesic in $\mathcal{T}(S)$ parametrized by arc length. For a curve $\alpha$ in $S$, the horizontal and vertical lengths of $\alpha$ vary with time as follows:

\[
(2) \quad h_{q_t}(\alpha) = h_q(\alpha) e^{-t} \quad \text{and} \quad v_{q_t}(\alpha) = v_q(\alpha) e^t.
\]

We say $\alpha$ is balanced, mostly horizontal or mostly vertical at time $t$ if, respectively, $v_t([\alpha]) = h_t([\alpha])$, $v_t([\alpha]) \leq h_t([\alpha])$ or $v_t([\alpha]) \geq h_t([\alpha])$.

3.2. Hyperbolic length along a geodesic. The behavior of the hyperbolic length of a curve along a Teichmüller geodesic is somewhat mysterious. For the Weil-Petersson metric on $\mathcal{T}(S)$, the hyperbolic length of a curve along a geodesic is a convex function of time. In the Teichmüller metric, the quadratic differential length of a curve is also convex. The following result is a weaker but analogous statement. It roughly states that a curve assumes its shortest length when it is balanced and the length is “non-decreasing” as one moves away in either direction. Let $\sigma_t$ denote the hyperbolic metric on $g(t)$.

**Theorem 3.1.** Let $g$ be a geodesic in $\mathcal{T}(S)$ and $\alpha$ be a curve in $S$. Assume $\alpha$ is balanced at $t_\alpha$ and $s \geq t_\alpha$ (respectively, $s \leq t_\alpha$). Then, for any $t \geq s$ ($t \leq s$), we have

\[
(3) \quad \frac{1}{l_{\sigma_s}([\alpha])} \succ \frac{1}{l_{\sigma_t}([\alpha])}.
\]

**Remark 3.2.** The above inequality has no content if both $l_{\sigma_s}([\alpha])$ and $l_{\sigma_t}([\alpha])$ are large, because both quantities are within the additive error. However, if $l_{\sigma_s}([\alpha])$ is large, then (3) implies that $l_{\sigma_t}([\alpha])$ is bounded below for all $t \geq s$.

**Proof.** Let $F_t$ be the flat annulus corresponding to $\alpha$ in $q_t$. The modulus of $F_t$ is maximum at $t_\alpha$, and, for $t \in \mathbb{R}$,

\[
(4) \quad \text{Mod}(F_t) \asymp \text{Mod}(F_{t_\alpha}) e^{-2|t-t_\alpha|}.
\]
Let $A_t$ be as in Theorem 2.3 for hyperbolic metric $\sigma_t$, quadratic differential $q_t$, and curve $\alpha$ (if $l_t(\alpha) \geq \epsilon_0$, there is nothing to prove). If $A_t$ is flat, then

\begin{align*}
(\text{Mas85}) & \quad \frac{1}{l_{\sigma_s}(\alpha)} \asymp \frac{1}{\text{Ext}_{\sigma_s}(\alpha)} \\
(\text{by definition of Ext}_{\sigma_s}(\alpha)) & \quad > \text{Mod}(F_s) \\
(\text{Equation (4)}) & \quad > \text{Mod}(F_t) \\
(A_t \subset F_t) & \quad \geq \text{Mod}(A_t) \\
(\text{Theorem 2.3}) & \quad \asymp \frac{1}{l_{\sigma_t}(\alpha)}.
\end{align*}

Assume $A_t$ is not flat. Let $d$ be the distance between the boundary components of $A_t$ and $l$ be the length of the inner boundary of $A_t$. Let $\beta$ be a curve intersecting $\alpha$ whose hyperbolic length at $s$ is less than $L$, for some $L$ such that $e^L \asymp \frac{1}{l_{\sigma_s}(\alpha)}$. Using the “collar lemma” (Theorem ??), we have

\begin{align*}
(5) & \quad \frac{1}{l_{\sigma_s}(\alpha)} > \log \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)},
\end{align*}

But $\alpha$ is mostly vertical at $s$; therefore, for $t \geq s$,

\begin{align*}
& \quad l_{q_t}(\alpha) \asymp l_{q_{t\alpha}}(\alpha) e^{t-s}.
\end{align*}

The quadratic differential length of any curve grows at most exponentially; that is, for $t \geq s$,

\begin{align*}
& \quad l_{q_t}(\beta) \asymp l_{q_{t\alpha}}(\beta) e^{t-s}.
\end{align*}

Therefore,

\begin{align*}
(6) & \quad \frac{l_{q_s}(\beta)}{l_{q_s}(\alpha)} \geq \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)}.
\end{align*}

We also have $l_{q_t}(\beta) \geq d$ ($\beta$ has to cross $A_t$) and $l_{q_s}(\alpha) \leq l$ ($\alpha$ and the inner boundary of $A_t$ are homotopic). Therefore,

\begin{align*}
(\text{Equation (5)}) & \quad \frac{1}{l_{\sigma_s}(\alpha)} > \log \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)} \\
(\text{Equation (4)}) & \quad \geq \log \frac{l_{q_t}(\beta)}{l_{q_t}(\alpha)} \\
(\text{Lemma 2.2}) & \quad \geq \log \frac{d}{l} \\
(\text{Theorem 2.3}) & \quad \asymp \frac{1}{l_{\sigma_t}(\alpha)}.
\end{align*}

$\square$
Corollary 3.3. There exists $\epsilon_1$ such that, for any geodesic in the Teichmüller space and any curve $\alpha$ in $S$, there exists a connected (perhaps empty) interval $I_{\alpha}$ such that

1. for $t \in I_{\alpha}$, $l_{\sigma_t}(\alpha) \leq \epsilon_0$, and
2. for $t \notin I_{\alpha}$, $l_{\sigma_t}(\alpha) \geq \epsilon_1$.

The intersection of connected intervals is a connected interval (or an empty set). Therefore, a similar statement is also true for subsurfaces.

Corollary 3.4. Let $\epsilon_0$, $\epsilon_1$, and $g$ be as above. For every subsurface $Y$, there exists a connected interval $I_Y$ such that

1. for $t \in I_Y$, the hyperbolic lengths of all boundary components of $Y$ at $\sigma_t$ are less than or equal to $\epsilon_0$, and
2. for $t \notin I_Y$, there exists a boundary component of $Y$ whose hyperbolic length at $\sigma_t$ is greater than or equal to $\epsilon_1$.

3.3. A lower bound for distance in the Teichmüller space. Our main theorem describes how the distance between two points in Teichmüller space can be estimated by measuring the combinatorial complexity of curves of bounded size. Here we show that, if two curves of bounded length in $\sigma_1$ and $\sigma_2$ intersect each other a large number of times, then $\sigma_1$ and $\sigma_2$ are far apart in $T(S)$.

First we recall some properties of the extremal length. Let $\text{Ext}_\sigma(\alpha)$ denote the extremal length of $\alpha$ in $\sigma$. Minsky has shown (see [Min93]) that, for curves $\alpha$ and $\beta$ in $S$, and $\sigma \in T(S)$,

\[
\text{Ext}_\sigma(\alpha) \text{ Ext}_\sigma(\beta) \geq i_S(\alpha, \beta)^2.
\]

Kerckhoff’s theorem (see [Ker80]) states that, for points $\sigma_1$ and $\sigma_2$ in $T(S)$,

\[
K(\sigma_1, \sigma_2) = \sup_{\alpha} \frac{\text{Ext}_{\sigma_1}(\alpha)}{\text{Ext}_{\sigma_2}(\alpha)}.
\]

where the sup is over all curves on $S$. We also know (see [Mas85]) that, if the hyperbolic length of $\alpha$ is short (say, $l_\sigma(\alpha) \leq L_0$), then

\[
l_\sigma(\alpha) \asymp \text{Ext}_\sigma(\alpha).
\]

Proposition 3.5. Assume, for some $\sigma_1, \sigma_2 \in T(S)$ and curves $\alpha$ and $\beta$ in $S$, that $l_{\sigma_1}(\alpha) \leq L_0$ and $l_{\sigma_2}(\beta) \leq L_0$. Then

\[
d_T(\sigma_1, \sigma_2) \asymp \log i_S(\alpha, \beta).
\]

Proof. We have:

(Equation 7) \[ i_S(\alpha, \beta)^2 \leq \text{Ext}_{\sigma_1}(\alpha) \text{ Ext}_{\sigma_1}(\beta) \]

(Equation 8) \[ \leq \text{Ext}_{\sigma_1}(\alpha) \text{ Ext}_{\sigma_2}(\beta) K(\sigma_1, \sigma_2) \]

(Equation 9) \[ \asymp L_0^2 K(\sigma_1, \sigma_2) \]

Note that $L_0$ is a fixed constant depending on $S$ only. By taking the logarithm of both sides, we obtain the desired inequality. \qed
3.4. Combinatorics of short markings along a Teichmüller geodesic. For \( t \in \mathbb{R} \), let \( \mu_t \) be the shortest marking in \( \sigma_t \), constructed as follows. Let \( \alpha_1 \) be the shortest curve in \( S \) and \( \alpha_2 \) be the shortest curve disjoint from \( \alpha_1 \), and so on, to form a pants decomposition of \( S \). Then, let the transverse curve \( \beta_i \) be the shortest curve intersecting \( \alpha_i \) and disjoint from \( \alpha_j, i \neq j \).

Proposition 3.7 states that the projection of these markings to a subsurface \( Y \) stays in a bounded neighborhood in \( C(Y) \) while the geodesic is outside of the thin part of \( T(S) \) corresponding to \( Y \). The proof makes an essential use of the following theorem.

**Theorem 3.6** ([Raf05, Theorem 5.5]). Let \( \alpha \) be a curve in \( S \), \( \beta \) be a transverse curve to \( \alpha \), and \( Y \) be a component of \( S \setminus \alpha \) (\( Y \) is allowed to be an annulus). Assume \( l_{\sigma_t}(|\beta|) \leq L \). We have:

(1) If \( \alpha \) is mostly vertical, then
\[
 i_Y(\beta, \nu_+) \prec D_L.
\]

(2) If \( \alpha \) is mostly horizontal, then
\[
 i_Y(\beta, \nu_-) \prec D_L.
\]

Here, \( D_L \) is a constant depending on \( L \), with \( \log D_L \simeq e^L \).

**Proposition 3.7.** If \([r, s] \cap I_Y = \emptyset \), then
\[
d_Y(\mu_r, \mu_s) = O(1).
\]

**Proof.** Let \( L_1 \) be such that every curve of length larger than \( \epsilon_1 \) in a hyperbolic surface with geodesic boundary has a transverse curve of length less than \( L_1 \). For \( t \in [r, s] \), there exists a boundary component \( \gamma_t \) of \( Y \) whose \( \sigma_t \)-length is larger than \( \epsilon_1 \). Therefore, the marking \( \mu_t \) contains a curve \( \alpha_t \) with \( l_{\sigma_t}(\alpha_t) \leq L_1 \) that intersects \( Y \) nontrivially. The projection of \( \mu_t \) to \( Y \) has bounded diameter. Therefore it is sufficient to prove \( d_Y(\alpha_r, \alpha_s) = O(1) \).

The curve \( \gamma_t \) is either mostly horizontal or mostly vertical at time \( t \). The set of times at which \( Y \) has a boundary component of length larger than or equal to \( \epsilon_1 \) which is mostly horizontal (or mostly vertical) is closed. Therefore, either

(1) \( \gamma_r \) and \( \gamma_s \) are both mostly horizontal or both mostly vertical, or
(2) for some \( t \in [r, s] \), there are two curves \( \gamma_t \) and \( \gamma'_t \) whose lengths at \( \sigma_t \) are larger than or equal to \( \epsilon_1 \), and one is mostly horizontal and the other is mostly vertical (possibly \( \gamma_t = \gamma'_t \) and \( \gamma_t \) is balanced).

Case 1: If \( \gamma_r \) and \( \gamma_s \) are mostly vertical, Theorem 3.6 implies that
\[
 i_Y(\alpha_r, \nu_+) \prec D_{L_1} \quad \text{and} \quad i_Y(\alpha_s, \nu_+) \prec D_{L_1}.
\]
Therefore, using Lemma 2.1,
\[
d_Y(\alpha_s, \nu_+) \prec \log i_Y(\alpha_s, \nu_+) \prec \log D_{L_1}.
\]

\[\text{2There may be finitely many such markings.}\]
Similarly, $d_Y(\alpha_r, \nu_+) = O(1)$. This implies that $d_Y(\alpha_r, \alpha_s) = O(1)$. The proof is similar if $\gamma_r$ and $\gamma_s$ are both mostly horizontal.

Case 2: Assume (without loss of generality) that $\gamma_t$ is mostly horizontal and $\gamma'_t$ is mostly vertical. Let $\alpha_t$ and $\alpha'_t$ be the corresponding transverse curves in $\mu_t$ of length less than $L_1$. By the above argument,

$$d_Y(\alpha_t, \nu^-) = O(1) \quad \text{and} \quad d_Y(\alpha'_t, \nu^+) = O(1).$$

But the extremal lengths of $\alpha_t$ and $\alpha'_t$ are bounded by a constant depending on $L_1$. Equation (7) implies that $i_\mu(\alpha_t, \alpha'_t) = O(1)$, and, by Lemma 2.1, $d_Y(\alpha_t, \alpha'_t) = O(1)$. Therefore,

$$d_Y(\nu^+, \nu^-) = O(1).$$

Again, as above, the projection of each of $\alpha_s$ and $\alpha_r$ to $Y$ is close to the projection of either $\nu_+$ or $\nu_-$ to $Y$. Thus, (10) and the triangle inequality for $d_Y$ imply that

$$d_Y(\alpha_r, \alpha_s) = O(1).$$

\[\square\]

Corollary 3.8. If $I_Y = [c, d] \subset [a, b]$, then

$$d_Y(\mu_a, \mu_b) \asymp d_Y(\mu_c, \mu_d).$$

4. Twisting in the Hyperbolic Metric vs. Twisting in the Quadratic Differential Metric

Let $\alpha$ be a curve in $S$. Having a metric in $S$ enables us to define a twisting parameter for curves that cross $\alpha$. This, roughly speaking, is the number of times that a given curve twists around $\alpha$ in comparison with an arc that is perpendicular to the geodesic representative of $\alpha$. In this section we define a twisting parameter for $\nu_+$ and $\nu_-$ using metrics given by $q$ and $\sigma$, and we study how these two quantities are related. We use this to prove Theorem 1.3 at the end of this section.

Let $\bar{S}$ be the annular cover of $S$ with respect to $\alpha$. Let $\bar{q}$, $\bar{\nu}_+$ and $\bar{\nu}_-$ be the lifts of $q$, $\nu_+$ and $\nu_-$ to $\bar{S}$, respectively, and $\bar{\beta}_q$ be a geodesic arc connecting the boundaries of $\bar{S}$ that is perpendicular (in $\bar{q}$) to the geodesic representative of the core of $\bar{S}$, $\bar{\alpha}$. We define the twisting parameter of $\nu_+$ around $\alpha$ in $q$ to be the maximum intersection number of a leaf of $\bar{\nu}_+$ and $\bar{\beta}_q$, and we denote it by $tw_q(\nu_+, \alpha)$. When it is clear what $\alpha$ is, we denote this by $tw_q$. The twisting parameter $tw_q$ of $\nu_-$ around $\alpha$ in $q$ is defined similarly. Note that the maximum intersection number is at least one, that is, $tw_q$ are positive integers.

Let $F$ be the flat annulus in $q$ corresponding to $\alpha$ and let $\beta_q$ be an arc connecting the boundaries of $F$ that is perpendicular to the boundaries of $F$. The intersection number of the lift of a leaf of $\nu_+$ with $\beta_q$ is (up to small additive error) equal to the intersection number of the restriction of this leaf to $F$ with $\beta_q$. Therefore, to compute $tw_q$, it is sufficient to understand the picture in $F$. Consider an isometric embedding of the universal cover of $F$
in \( \mathbb{R}^2 \) such that the leaves of horizontal foliations are parallel to the \( x \)-axis and the leaves of vertical foliations are parallel to the \( y \)-axis (see Fig. 1).

\[
\theta \quad \text{proj} \vec{W} \quad \vec{H}
\]

\[
\vec{W} \quad \vec{V} \quad \text{proj}_\vec{W} \vec{V}
\]

**Figure 1. The universal cover of \( F \)**

Let \( \vec{W} \) be the vector representing the translation that generates the deck translation group. Let \( \vec{H} \) be the lift of a leaf of \( \nu_+ \) passing through the origin and \( \vec{V} \) be the same for \( \nu_- \). From the above discussion, we have:

\[
tw_q(\nu_+, \alpha) \propto \frac{\|\text{Proj}_{\vec{W}} \vec{H}\|}{\|\vec{W}\|} \quad \text{and} \quad tw_q(\nu_-, \alpha) \propto \frac{\|\text{Proj}_{\vec{W}} \vec{V}\|}{\|\vec{W}\|}.
\]

Let \( \theta \) be the angle between \( \vec{W} \) and the \( x \)-axis. It is easy to see, using similar triangles, that

\[
\frac{\|\text{Proj}_{\vec{W}} \vec{H}\|}{\|\text{Proj}_{\vec{W}} \vec{V}\|} = \frac{\sin^2 \theta}{\cos^2 \theta}.
\]

We also have \( \frac{h_q(\alpha)}{v_q(\alpha)} = \frac{\sin \theta}{\cos \theta} \). Therefore,

\[
tw_q^+ \propto \frac{h_q(\alpha)^2}{v_q(\alpha)^2}.
\]

(11)

This is a very useful equation that allows us to compute the \( q \)-twisting parameter of horizontal and vertical foliations around \( \alpha \) along a Teichmüller geodesic (see equation (15)).
We define the twisting parameter for a hyperbolic metric as follows. Let $\beta_\sigma$ be the shortest transverse curve to $\alpha$ in the hyperbolic metric $\sigma$. Define

$$tw_\sigma^+ = i(\nu_+, \beta_\sigma) \quad \text{and} \quad tw_\sigma^- = i(\nu_-, \beta_\sigma).$$

We would like to prove a statement similar to equation (11) for $\sigma$-twisting parameters. However, giving good estimates for $tw_\sigma^\pm$ is difficult when $\alpha$ is very short. The errors in our estimates get larger as $l_\sigma(\alpha)$ gets smaller.

Let $\bar{\beta}_\sigma$ be the lift of $\beta_\sigma$ to $\bar{S}$ whose end points are in different boundary components of $\bar{S}$. Our strategy is to relate $q$- and $\sigma$-twisting parameters by providing an upper bound for $i(\bar{\beta}_q, \bar{\beta}_\sigma)$.

**Lemma 4.1.** If $i(\bar{\beta}_q, \bar{\beta}_\sigma) = n$, then

$$\text{Ext}_\sigma(\beta_\sigma) \lesssim n^2 l_\sigma(\alpha).$$

**Proof.** By definition of the extremal length, for any metric $\tau$ on $S$ in the conformal class of $\sigma$,

$$\text{Ext}_\sigma(\beta_\sigma) \geq \frac{l_\tau(\beta_\sigma)^2}{\text{area}(S)}.$$  

To find a lower bound for $\text{Ext}_\sigma(\beta_\sigma)$, we need to find an appropriate metric $\tau$. First we establish some notation. Let $A$ be the largest regular neighborhood of $F$ that is still an annulus. Denote the boundary components of $A$ by $\alpha_0$ and $\alpha_c$, where $c$ is the $q$–distance between the boundaries of $A$. For $t \in (0, c)$, let $\alpha_t$ be a curve in $A$ that is equidistant from a $q$–geodesic representative of $\alpha$ and whose $q$–distance from $\alpha_0$ is $t$. These curves give a foliation of $A$ into curves in the homotopy class of $\alpha$. There is a subinterval $[a, b]$ of $[0, c]$ such that, for $t \in [a, b]$, $\alpha_t$ is a $q$–geodesic representative of $\alpha$. This gives a division of $A$ into three pieces, the flat annulus $F$ containing all $\alpha_t$, $t \in [a, b]$, and two expanding annuli $A_1$ and $A_2$ on the sides. Theorem 2.3 implies that $\text{Mod}(A) \asymp \frac{1}{l_\sigma(\alpha)}$. Using Lemma 2.2 we have

$$\frac{1}{l_\sigma(\alpha)} \asymp \log \frac{a}{l_q([\alpha])} + \frac{(b-a)}{l_q([\alpha])} + \log \frac{(c-b)}{l_q([\alpha])}. $$

As $t$ changes in the interval $[b, c]$, the length of $\alpha_t$ increases. The rate of change is equal to the curvature of $\alpha_t$, which is bounded above and below by constants depending on the topology of $S$ only. A similar statement is true for $A_1$ as well. Therefore,

$$l_q(\alpha_t) \asymp \begin{cases} l_q([\alpha]) + (a-t) & \text{if} \quad t \in [0, a] \\ l_q([\alpha]) & \text{if} \quad t \in [a, b] \\ l_q([\alpha]) + (t-b) & \text{if} \quad t \in [b, c] \end{cases}. $$

Denote $l_q(\alpha_t)$ by $\lambda_t$.

Let $Z$ be the union of $A$: the $\lambda_0$–neighborhood, $N_0$, of $\alpha_0$; and the $\lambda_c$–neighborhood, $N_c$, of $\alpha_c$. Define the metric $\tau$ in $S$ in the conformal class of $q$ as follows: if $x$ lies on a curve $\alpha_t$ in $A$, then we scale the $q$–metric at $x$ by a factor of $\frac{1}{\lambda_t}$; if $x$ is outside of $A$ and in $N_0$, then we scale the $q$–metric at $x$ by a factor of $\frac{1}{\lambda_0}$; if $x$ is outside of $A$ and in $N_c$, then we scale the $q$–metric at
Let $\bar{A}$ be the lift of $A$ to $\bar{S}$ that is an annulus, and let $\bar{\alpha}_t$ be the lift of $\alpha_t$ that is in $\bar{A}$ (this is to ensure that $\bar{\alpha}_t$ is a closed curves not an infinite line).

Let $\bar{\omega}$ be a sub-arc of $\bar{\beta}_\sigma$ with end points in $\bar{\beta}_q$ that goes around $\bar{S}$ once, that is, if $\bar{\omega}'$ is the sub-arc of $\bar{\beta}_q$ connecting the end points of $\bar{\omega}$, then $\bar{\gamma} = \bar{\omega} \cup \bar{\omega}'$ is a curve in the homotopy class of the core of $\bar{S}$. Let $\gamma$ be the projection of $\bar{\gamma}$ to $S$. Then $\gamma$ is in the homotopy class of $\alpha$ and therefore must intersect $A$ (otherwise, $A$ would not be maximal). Hence, $\gamma$ must intersect $\bar{A}$. But $\bar{\beta}_q$ is perpendicular to $\bar{\alpha}_t$, and, once it exits $\bar{A}$, it never returns. Therefore, $\bar{\omega}$ must intersect $\bar{A}$ as well.

Let $\bar{\alpha}_s$ be an equidistant curve in $\bar{A}$ intersecting $\bar{\omega}$ that has the shortest $\bar{q}$–length. We claim that

$$l_q(\bar{\omega}) \geq l_q(\bar{\alpha}_t) = \lambda_t.$$  

Assume $s > b$. The curve $\bar{\alpha}_s$ divides $\bar{S}$ into two annuli. Let $B$ be the annulus that contains $\bar{\alpha}_c$. For $t \in [b, s)$, the $\bar{q}$–length of $\bar{\alpha}_t$ is less than the $\bar{q}$–length of $\bar{\alpha}_s$. By assumption $\bar{\alpha}_s$ is the shortest equidistant curve intersecting $\bar{\omega}$, therefore, $\bar{\omega} \subset B$.

The curvature of $\bar{\alpha}_t$ with respect to $B$ is non-positive at all points. Therefore, the closest-point projection from $B$ to $\bar{\alpha}_t$ is length-decreasing. But the end points of $\bar{\omega}$ project to the same point in $\bar{\alpha}_t$ (because $\bar{\beta}_q$ is perpendicular to $\bar{\alpha}_t$), and the projection covers $\bar{\alpha}_t$ completely. Therefore, $l_q(\bar{\omega}) \geq l_q(\bar{\alpha}_t)$ in this case.

A similar argument holds if $t < a$. If $t \in [a, b]$, then $\bar{\omega}$ could intersect $\bar{\alpha}_t$ transversally, but, in this case, $\bar{\alpha}_t$ is a $\bar{q}$–geodesic and the curvature of $\bar{\alpha}_t$ is non-positive with respect to both annuli in $\bar{S} \setminus \alpha_t$. Therefore, the claim is true in all cases.

Let $\omega$ be the projection of $\bar{\omega}$ to $S$. If $\omega$ exits $Z$, then its $\tau$–length is larger than the $\tau$–distance between $A$ and $\partial Z$, which is equal to 1. Otherwise, $\omega \subset Z$. Therefore, at each point in $\omega$, $\tau$ is obtained from $q$ by scaling by a factor
of at least $\frac{1}{\lambda_t}$. Therefore,
\[ l_\tau(\omega) \geq \frac{1}{\lambda_t} l_q(\omega) \geq 1. \]

There are $(n-1)$ arcs like $\bar{\omega}$, and they all project down to different sub-arcs of $\beta_\sigma$. Therefore,
\[ l_\tau(\beta_\sigma) \geq n. \]

This implies that
\[ \text{Ext}_\sigma(\beta_\sigma) \geq \frac{l_\tau(\beta_\sigma)^2}{\text{area}_\sigma S} \geq \frac{n^2}{1/l_\sigma(\alpha)} = n^2 l_\sigma(\alpha). \quad \square \]

**Corollary 4.2.** For $\bar{\beta}_\sigma$ and $\bar{\beta}_q$ as before, we have
\[ i(\bar{\beta}_\sigma, \bar{\beta}_q) \prec \frac{1}{l_\sigma(\alpha)}. \]

**Proof.** The curve $\beta_\sigma$ is the shortest (in $\sigma$) transverse curve to $\alpha$. Therefore,
\[ \text{Ext}_\sigma(\beta_\sigma) \prec \frac{1}{l_\sigma(\alpha)}. \]

Applying the previous theorem we get
\[ \frac{1}{l_\sigma(\alpha)} > n^2 l_\sigma(\alpha), \]
which, using Lemma 4.1, implies the corollary. \quad \square

The following theorem is an immediate consequence of the definitions of the twisting parameters and of Corollary 4.2.

**Theorem 4.3.** The two twisting parameters are the same up to an additive error comparable to $\frac{1}{l_\sigma(\alpha)}$. That is,
\[ \text{tw}^\pm_\sigma = \text{tw}^\pm_q \pm O\left(\frac{1}{l_\sigma(\alpha)}\right). \]

4.1. **The twisting parameter along a Teichmüller geodesic.** In this section, we give estimates for the twisting parameters of $\nu_\pm$ around a curve $\alpha$ in $\sigma_t$. Let $d = d_\alpha(\nu_+, \nu_-)$. If $\alpha$ is not very short in $\sigma_t$, say $l_{\sigma_t}(\alpha) \geq \epsilon_0$, then it has a transverse curve that is not longer than $L_0$. Theorem 3.6 implies
\[ \{ \begin{align*} 
&\text{if } \alpha \text{ is mostly horizontal, } \text{tw}^+_{\sigma_t} \prec d_\alpha, \text{ and } \text{tw}^-_{\sigma_t} \prec 0 \quad \text{and} \\
&\text{if } \alpha \text{ is mostly vertical, } \text{tw}^+_{\sigma_t} \prec 0, \text{ and } \text{tw}^-_{\sigma_t} \prec d_\alpha
\end{align*} \]

In general, we know that $\text{tw}^+_{\nu_t} + \text{tw}^-_{\nu_t} \prec d$. Assume $\alpha$ is balanced at $t_\alpha$. Using Equations (11) and (2), we get
\[ \frac{\text{tw}^+_{\nu_t}}{\text{tw}^-_{\nu_t}} = \frac{e^{2(t-t_\alpha)} h_{\nu_t}(\alpha)^2}{e^{-2(t-t_\alpha)} v_{\nu_t}(\alpha)^2}. \]

But $h_{\nu_t}(\alpha) = v_{\nu_t}(\alpha)$. Therefore,
\[ \text{tw}^+_{\nu_t} \prec \frac{d_\alpha e^{2(t-t_\alpha)}}{e^{2(t-t_\alpha)} + e^{-2(t-t_\alpha)}} \quad \text{and} \quad \text{tw}^-_{\nu_t} \prec \frac{d_\alpha e^{-2(t-t_\alpha)}}{e^{2(t-t_\alpha)} + e^{-2(t-t_\alpha)}}. \]
This and Theorem 4.3 prove Theorem 1.3. The following theorem is a different statement for the same basic fact.

**Proposition 4.4.** Let \( \sigma_t \in T(S) \) and \( \alpha \) be a curve in \( S \) with \( l_{\sigma_t}(\alpha) \leq \epsilon_0 \). Let \( \sigma_t' \) be the point in \( T(S) \) obtained from \( \sigma_t \) by twisting along \( \alpha \) such that

\[
tw_{\sigma_t}^+ = \frac{d - \alpha e^{-2(t-t_\alpha)}}{e^{2(t-t_\alpha)} + e^{-2(t-t_\alpha)}}.
\]

Then \( d_T(\sigma_t, \sigma_t') = O(1) \).

**Proof.** Consider \( \pi : T(S) \to T(S \setminus \alpha) \times \mathbb{H}_\alpha \). We know that \( \pi_0(\sigma_t) = \pi_0(\sigma_t') \) and

\[
d_{\mathbb{H}_\alpha}(\pi_0(\sigma_t), \pi_0(\sigma_t')) \asymp \log (l_{\sigma_t}(\alpha) (tw_{\sigma_t}^+ - tw_{\sigma_t'}^+)).
\]

Theorem 4.3 implies that the \( \sigma_t \)–twisting and the \( q_\epsilon \)–twisting parameters of \( \nu_+ \) are equal up to an additive error that is comparable with \( \frac{1}{l_{\sigma_t}(\alpha)} \). Therefore, the right-hand side of the above equation is uniformly bounded. We have

\[
d_T(\sigma_t, \sigma_t') \asymp d_{\mathbb{H}_\alpha}(\pi_0(\sigma_t), \pi_0(\sigma_t')) = O(1) \quad \square
\]

5. **Proof of the main theorem**

In this section we prove Theorem 1.1. In §5.1 we show how a lower bound for the Teichmüller distance between two points in \( T(S) \) can be obtain by the combinatorial complexity between their short markings. In §5.2 we give an upper bound for the distance between two points in the Teichmüller space by constructing a path in \( T(S) \) of length comparable with the estimate given in Theorem 1.1.

5.1. **Lower estimate.** Let \( g : [a, b] \to T(S) \) be the geodesic segment in the Teichmüller space connecting \( \sigma_a \) to \( \sigma_b \). Recall that \( \sigma_t \) is the hyperbolic metric of \( g(t) \), and \( \mu_t \) is the short-marking on \( S \) corresponding to \( \sigma_t \).

**Lemma 5.1.** Let \( Y \) be a subsurface that is not an annulus and \( I = I_Y \cap [a, b] \). Then

\[
|I| \geq d_Y(\mu_a, \mu_b).
\]

**Proof.** Let \( I = [c, d] \), \( \tau_c = \pi_Y(\sigma_c) \) and \( \tau_d = \pi_Y(\sigma_d) \) (see Theorem 2.4). Let \( \eta_c \) and \( \eta_d \) be the short-markings on \( Y \) corresponding to \( \tau_c \) and \( \tau_d \), respectively. In fact, \( \eta_c \subset \mu_c \) and \( \eta_d \subset \mu_d \). We have

(\text{Theorem 2.4}) \quad |I| \geq d_{T(Y)}(\tau_c, \tau_d),

(\text{Proposition 3.5}) \quad \geq \log i_Y(\eta_c, \eta_d)

(\text{Lemma 2.4}) \quad \geq d_Y(\eta_c, \eta_d).

But \( d_Y(\eta_c, \eta_d) \asymp d_Y(\mu_c, \mu_d) \) (because they have the same projections to \( Y \)). Also, by Proposition 5.7 we have

\[
d_Y(\mu_a, \mu_c) = O(1) \quad \text{and} \quad d_Y(\mu_d, \mu_b) = O(1).
\]

This proves the lemma. \( \square \)
A similar lemma is true when the subsurface is an annulus. The difference is that, in Lemma 5.1, there is no restriction on the lengths of the boundaries of $Y$; but, for the next lemma to be true, we have to assume that $\alpha$ is not very short in $\sigma_a$ and $\sigma_b$. The proofs are almost identical.

**Lemma 5.2.** Let $\alpha$ be a curve in $S$ such that $l_{\sigma_a}(\alpha) \geq \epsilon_0$ and $l_{\sigma_b}(\alpha) \geq \epsilon_0$, and let $I = I_\alpha \cap [a, b]$. Then

$$|I| > d_\alpha(\mu_a, \mu_b).$$

**Proof.** Since $\alpha$ is not short at either end, either $I_\alpha$ is disjoint from $[a, b]$ or it is a subset of $[a, b]$. If $I_\alpha \cap [a, b] = \emptyset$, then Proposition 3.7 implies the lemma. If $I_\alpha = [c, d] \subset [a, b]$, then, by Corollary 3.8,

$$d_\alpha(\mu_a, \mu_b) \asymp d_\alpha(\mu_c, \mu_d).$$

Let $\beta_c$ and $\beta_d$ be curves transverse to $\alpha$ in markings $\mu_c$ and $\mu_d$, respectively. We have

$$i(\beta_c, \beta_d) = d_\alpha(\mu_c, \mu_d).$$

As in the previous lemma, using Theorem 2.4 and Proposition 3.5, we have

$$|I_\alpha| \asymp \log i(\beta_c, \beta_d).$$

The combination of the last three equations proves the lemma. $\square$

The following proposition provides a lower bound for the Teichmüller distance between two points in the thick part of $T(S)$.

**Proposition 5.3.** Let $\sigma_1$, $\sigma_2$ be in the $\epsilon_0$–thick part of $T(S)$ and $\mu_1$ and $\mu_2$ be the short-markings in $\sigma_1$ and $\sigma_2$, respectively. There exists a $k_0 > 0$ such that

$$d_T(\sigma_1, \sigma_2) \succ \sum_Y [d_Y(\mu_1, \mu_2)]_{k_0} + \sum_\alpha [d_\alpha(\mu_1, \mu_2)]_{k_0}.$$

**Proof.** Let $g: [a, b] \to T(S)$ be the geodesic segment connecting $\sigma_1$ and $\sigma_2$. Since the end points are in the thick part of $T(S)$, for every subsurface $Y$, $I_Y$ either is disjoint from $[a, b]$ or is a subset of $[a, b]$. Let $k_0$ be a constant such that, if $d_Y(\mu, \eta) \geq k_0$, then $I_Y \subset [a, b]$ (see Proposition 3.5). For $t \in I_Y$, the length of each boundary component of $Y$ is less than $\epsilon_0$. Therefore, there exists a constant $C$, depending on the topology of $S$, such that the number of subsurfaces with this property at each given time is at most $C$. Therefore,

$$d_T(\sigma, \tau) \geq \frac{1}{C} \sum |I_Y|.$$

Lemmas 5.1 and 5.2 imply the desired inequality. $\square$
5.2. The upper estimate. In [MM00], Masur and Minsky show how to change one marking to another through elementary moves (described below) efficiently. Their estimate for the number of necessary elementary moves closely resembles the estimate in Theorem 1.1. We use this sequence of elementary moves to construct an efficient path connecting two points in $T(S)$.

There are two types of elementary moves that transform a marking $\mu = \{(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ to a new marking.

1. Twist: Replace $\beta_i$ by $\beta'_i$, where $\beta'_i$ is obtained from $\beta_i$ by a Dehn twist or a half twist around $\alpha_i$.

2. Flip: Replace the pair $(\alpha_i, \beta_i)$ with $(\beta_i, \alpha_i)$ and, for $j \neq i$, replace $\beta_j$ with a curve $\beta'_j$ that does not intersect $\beta_i$, which is now a base curve, in such a way that $d_{\alpha_j}(\beta_j, \beta'_j)$ is as small as possible (see [MM00] for details).

In the first move, a twist can be positive or negative. A half twist is possible when $\alpha_i$ and $\beta_i$ intersect twice. The following is a consequence of work done in [MM00] and [?].

**Proposition 5.4.** There exists a large enough $k$ such that: For markings $\mu$ and $\mu'$, there exists a sequence of markings

$$
\mu = \mu_1, \ldots, \mu_n = \mu',
$$

where $\mu_i$ and $\mu_{i+1}$ differ by an elementary move except, for each $\alpha$ where $d_{\alpha}(\mu, \mu') \geq k$, there is an index $i_\alpha$ so that

$$
\mu_{i_\alpha+1} = D^p_{\alpha} \mu_{i_\alpha}, \quad \text{and} \quad |p| \asymp d_{\alpha}(\mu, \mu').
$$

Furthermore,

$$
n \asymp \sum_{Y \subset S} \left[ d_Y(\mu, \mu') \right] k,
$$

where the sum is over all subsurfaces $Y$ that are not annuli.

**Proof.** We use the definitions and notation used in [MM00] and [?]. [MM00] 4.6 and 4.20 state that there exists a complete hierarchy $H$ whose initial marking is $\mu$ and whose terminal marking is $\mu'$. Any complete marking has a resolution ([MM00 5.4]), that is, there is a sequence of markings

$$
\mu = \eta_1, \ldots, \eta_N = \mu'
$$

where $\eta_i$ and $\eta_{i+1}$ differ by an elementary move. For $k$ large enough, if $d_{\alpha}(\mu, \mu') \geq k$, the collar of $\alpha$ appears as a domain in $H$ ([MM00 6.2]) exactly once ([?, 5.15]), and the length of the corresponding geodesic in $H$ is comparable to $d_{\alpha}(\mu, \mu')$ ([MM00 6.2]). That is, the number of twist moves around $\alpha$ used in the resolution is comparable to $d_{\alpha}(\mu, \mu')$. The number of the remaining elementary moves is comparable to the sum of the lengths.
of geodesics in $H$ whose domains are not annuli, which is comparable to ([LM00, Lemma 6.2 and Equation (6.4)])

$$\sum_Y \left[ d_Y(\mu, \mu') \right]_k.$$ 

Our goal is, for any $\alpha$ where $d_\alpha(\mu, \mu') \geq k$, to rearrange the elementary moves in the resolution so that all the twist moves around $\alpha$ are applied consecutively. Then we replace the sequence of consecutive twists around $\alpha$ with one large step, which is applying $D_p^\alpha$, for some $p \asymp d_\alpha(\mu, \mu')$. This will result in the sequence described in the statement of the theorem and has the desired length condition.

We know ([?, 5.16]) that for every curve $\alpha$, the set $J_\alpha$ of indices $i$ such that $\alpha$ is a base curve in $\eta_i$ is an interval in $\mathbb{Z}$. Observe that when $\alpha$ is a base curve of a marking, a twist move around $\alpha$ and a twist move around any other curve can be rearranged without any complication. The trouble with the flip moves is that the outcome is not unique. Therefore, after rearranging a flip move and a twist move, we have to make sure the outcomes of two flip moves differ by just a twist around $\alpha$. For example, assume $\eta_{i-1}$, $\eta_i$ and $\eta_{i+1}$ all contain $\alpha$ as a base curve, $\eta_i$ is obtained from $\eta_{i-1}$ by a flip move and $\eta_{i+1} = D_\alpha \eta_i$. Then, replacing $\eta_i$ with $\eta'_i = D_\alpha \eta_{i-1}$ in our sequence will result in a sequence that is still a resolution of $H$. Because $\eta_i$ is obtained from $\eta_{i-1}$ by applying a flip move, $D_\alpha \eta_i$ is also obtained from $D_\alpha \eta_{i-1}$ by a flip move ($D_\alpha$ is a homeomorphism). Therefore, we can rearrange the elementary moves in $J_\alpha$ so that all the twist moves around $\alpha$ are done consecutively. 

**Remark 5.5.** The constant $k$ can be chosen as large as necessary, and the constants involved in (17) depend on $k$ and the topology of $S$ (see Theorem 6.12 in [MM99]). Therefore, we can assume $k \geq k_0$, where $k_0$ is as chosen in Proposition 5.3.

For a marking $\mu$, let $\text{short}(\mu)$ be the set of points in $\mathcal{T}(S)$ where all curves in $\mu$ have hyperbolic length less than $L_0$ ($L_0$ as on page 5). This is a compact subset of $\mathcal{T}(S)$. We define $f(\mu, \mu')$ to be the maximum distance between an element in $\text{short}(\mu)$ and an element in $\text{short}(\mu')$.

**Lemma 5.6.** If $i = i_\alpha$, where $\alpha$ is a curve with $d_\alpha(\mu, \mu') \geq k$, then

$$f(\mu_i, \mu_{i+1}) \asymp \log d_\alpha(\mu, \mu').$$

Otherwise,

$$f(\mu_i, \mu_{i+1}) = O(1).$$

**Proof.** Since $\text{short}(\mu)$ is compact, it is enough to bound the minimum distance between $\text{short}(\mu_i)$ and $\text{short}(\mu_{i+1})$.

Assume $i = i_\alpha$, for $\alpha$ as above, and let $\sigma$ be a point in $\text{short}(\mu_i)$. Then, for some $|p| \asymp d_\alpha(\mu, \eta)$, $\tau = D_p^\alpha \sigma$ is a point in $\text{short}(\mu_{i+1})$. The lengths of $\alpha$ in $\sigma$ and $\tau$ are less than $L_0$, therefore, $\sigma$ and $\tau$ are bounded distance from points $\sigma'$ and $\tau' = D_p^\alpha(\sigma')$, where the lengths of $\alpha$ in $\sigma'$ and $\tau'$ are less
than $\epsilon_0$. Taking $\Gamma = \{\alpha\}$ and $\pi$ as in Theorem 2.4, the following holds: the distance between $\sigma'$ and $\tau'$ equals, up to additive error, the distance in $\mathbb{H}_2$ between $\pi_{\alpha}(\sigma')$ and $\pi_{\alpha}(\tau')$, which, up to multiplicative error, equals $\log |p|$. Therefore, the distance between $\sigma$ and $\tau$ is comparable to $\log |p|$.

Otherwise, $\mu_i$ and $\mu_{i+1}$ differ by an elementary move. Note that there are only finitely many such pairs of markings up to homeomorphism. Therefore, there exists a uniform upper bound for the minimum distance between $\text{short}(\mu_i)$ and $\text{short}(\mu_{i+1})$, depending on the topology of $S$ only. \hfill $\square$

**Proposition 5.7.** Let $\sigma_1$, $\sigma_2$ be in the $\epsilon_0$–thick part of $\mathcal{T}(S)$ and $\mu_1$ and $\mu_2$ be the short-markings in $\sigma_1$ and $\sigma_2$, respectively. Then

$$d_{\mathcal{T}}(\sigma_1, \sigma_2) < \sum_Y \left[ d_Y(\mu_1, \mu_2) \right]_k + \sum_{\alpha} \log \left[ d_\alpha(\mu_1, \mu_2) \right]_k.$$ 

**Proof.** Let $\mu_1 = \tilde{\mu}_1, \ldots, \tilde{\mu}_n = \mu_2$ be the path in $\mathcal{M}(S)$ described in Proposition 5.4. For each $i$, let $\sigma_i$ be a point in $\text{short}(\tilde{\mu}_i)$ and let $g_i$ be the geodesic segment connecting $\sigma_i$ to $\sigma_{i+1}$. The distance in $\mathcal{T}(S)$ between $\sigma_1$ and $\sigma_2$ is less than the sum of the lengths of the $g_i$. Lemma 5.6 states that the lengths of the $g_i$ are uniformly bounded except when $i = i_0$ and $d_\alpha(\mu_1, \mu_2) \geq k$, in which case the length of $g_i$ is comparable with $\log d_\alpha(\mu_1, \mu_2)$. Therefore,

$$d(\sigma, \tau) < n O(1) + \sum_{\alpha} \log \left[ d_\alpha(\mu_1, \mu_2) \right]_k.$$ 

Proposition 5.3 finishes the proof. \hfill $\square$

**Proof of Theorem 1.1.** Propositions 5.3 and 5.7 provide a lower estimate and an upper estimate for the distance between $\sigma$ and $\tau$. Since $k \geq k_0$ (see Remark 5.6), the estimate given in Proposition 5.7 is smaller than the one given in Proposition 5.3. Therefore $d_{\mathcal{T}}(\sigma_1, \sigma_2)$ is comparable to

$$\sum_Y \left[ d_Y(\mu_1, \mu_2) \right]_k + \sum_{\alpha} \log \left[ d_\alpha(\mu_1, \mu_2) \right]_k.$$ 

\hfill $\square$

6. **The general case**

In this section we give an estimate for the distance between two arbitrary points in the Teichmüller space. Let $\sigma_1$ and $\sigma_2$ be two points in $\mathcal{T}(S)$ and $g: [a, b] \to \mathcal{T}(S)$ be the geodesic arc connecting them. If $\sigma_1$ and $\sigma_2$ are not in the thick part of $\mathcal{T}(S)$, then the set of short curves in $\sigma_1$ and $\sigma_2$ does not contain enough information to allow us to estimate the distance between $\sigma_1$ and $\sigma_2$; we also need to know how short these curves are. Therefore, our estimate for the distance contains terms measuring the distance between $\sigma_1$ and $\sigma_2$ and the thick part of Teichmüller space. An additional complication arises from the case where a curve is short in both $\sigma_1$ and $\sigma_2$ and remains short along the geodesic. However, the basic idea behind both Theorem 1.1 and Theorem 6.1 is that efficient paths in the space of markings are closely related to geodesics in Teichmüller space.
Let $\epsilon_0$ be as before. Define $\Gamma$ to be the set of curves that are short in both $\sigma_1$ and $\sigma_2$, and, for $i = 1, 2$, define $\Gamma_i$ to be the set of curves that are short in $\sigma_i$ but not in $\sigma_{3-i}$. Let $\mu_1$ and $\mu_2$ be short-markings on $\sigma_1$ and $\sigma_2$, respectively.

**Theorem 6.1.** The distance in $T(S)$ between $\sigma_1$ and $\sigma_2$ is given by the following formula:

$$d_T(\sigma_1, \sigma_2) \asymp \sum_Y \left[ d_Y(\mu_1, \mu_2) \right]_k + \sum_{\alpha \notin \Gamma} \log \left[ d_\alpha(\mu_1, \mu_2) \right]_k +$$

$$+ \max_{\alpha \in \Gamma} d_{\mathcal{H}_\alpha}(\sigma_1, \sigma_2) + \max_{i = 1, 2} \log \frac{1}{l_{\sigma_i}(\alpha)}.$$  \hspace{1cm} (18)

**Proof.** Theorem 2.4 implies that

$$d_T(\sigma_1, \sigma_2) \asymp \max_{\alpha \in \Gamma} d_{\mathcal{H}_\alpha}(\sigma_1, \sigma_2) + d_T(S \setminus \Gamma)(\pi_0(\sigma_1), \pi_0(\sigma_2)).$$

This accounts for the third term on the right-hand side of Equation (18). Therefore, without loss of generality, we can assume $\Gamma = \emptyset$.

Let $\sigma'_1$ and $\sigma'_2$ be points in the thick part of the Teichmüller space that have the same short-markings as $\sigma_1$ and $\sigma_2$. We have:

$$d_T(\sigma_1, \sigma_2) \leq d_T(\sigma_1, \sigma'_1) + d_T(\sigma'_1, \sigma'_2) + d_T(\sigma'_2, \sigma_2),$$

The sum of the first two terms in (18) is comparable with $d_T(\sigma'_1, \sigma'_2)$. Also,

$$d_T(\sigma_1, \sigma'_1) \asymp \max_{\beta \in \Gamma_1} \log \frac{1}{l_{\sigma_1}(\beta)} \text{ and } d_T(\sigma'_1, \sigma'_2) \asymp \max_{\gamma \in \Gamma_2} \log \frac{1}{l_{\sigma_2}(\gamma)}.$$

Therefore, the right side of (18) is an upper bound for $d_T(\sigma_1, \sigma_2)$ (up to additive and multiplicative constants).

To show that the right side of (18) is also a lower bound for $d_T(\sigma_1, \sigma_2)$, we follow the same argument as in §5.4. However, we cannot use Lemma 5.2 when $\alpha$ is short in either $\sigma_1$ or $\sigma_2$ and using the previous argument we can conclude only that

$$d_T(\sigma_1, \sigma_2) \asymp \sum_Y \left[ d_Y(\mu_1, \mu_2) \right]_k + \sum_{\alpha \notin \Gamma_1 \cup \Gamma_2} \log \left[ d_\alpha(\mu_1, \mu_2) \right]_k.$$  \hspace{1cm} (19)

For every $\alpha \in \Gamma_1$, we have

$$d_T(\sigma_1, \sigma_2) = \log K(\sigma_1, \sigma_2) \geq \log \left| \frac{\text{Ext}_{\sigma_1}(\alpha)}{\text{Ext}_{\sigma_2}(\alpha)} \right| \asymp \log \frac{1}{l_{\sigma_1}(\alpha)}.$$

A similar statement is true for $\alpha \in \Gamma_2$. Hence

$$d_T(\sigma_1, \sigma_2) \asymp \max_{i = 1, 2} \log \frac{1}{l_{\sigma_i}(\alpha)}.$$  \hspace{1cm} (20)

It remains to show, for $\alpha \in \Gamma_1 \cup \Gamma_2$, that $d_T(\sigma_1, \sigma_2) \asymp \log d_\alpha(\mu_1, \mu_2)$. Let $\beta_1$ and $\beta_2$ be the transverse curves to $\alpha$ in $\mu_1$ and $\mu_2$. We know

$$|tw_{\sigma_1}^+ - tw_{\sigma_2}^+| = |i_\alpha(\nu_+, \beta_1) - i_\alpha(\nu_+, \beta_2)|^+ \asymp i_\alpha(\beta_1, \beta_2) = d_\alpha(\mu_1, \mu_2).$$
Therefore, it is sufficient to show that \( d_T(\sigma_1, \sigma_2) \succ \log |tw_{q_1}^+ - tw_{q_2}^+| \). Theorem 4.3 implies that
\[
|tw_{q_1}^+ - tw_{q_2}^+| \succ \left| \frac{1}{l_{\sigma_1}(\alpha)} + \frac{1}{l_{\sigma_2}(\alpha)} \right| \;
\]
therefore,
\[
|tw_{q_1}^+ - tw_{q_2}^+| + \frac{1}{l_{\sigma_1}(\alpha)} + \frac{1}{l_{\sigma_2}(\alpha)} \succ |tw_{\sigma_1}^+ - tw_{\sigma_2}^+|,
\]
and Equation (1) implies that the \( q_t \)-twisting parameter changes at most exponentially fast; hence,
\[
d_T(\sigma_1, \sigma_2) \succ \log |tw_{q_1}^+ - tw_{q_2}^+|.
\]
We also know that
\[
d_T(\sigma_1, \sigma_2) \succ \log \frac{1}{l_{\sigma_1}(\alpha)} \quad \text{and} \quad d_T(\sigma_1, \sigma_2) \succ \log \frac{1}{l_{\sigma_2}(\alpha)}.
\]
From the last three equations, we can conclude
\[
d_T(\sigma_1, \sigma_2) \succ \log |tw_{\sigma_1}^+ - tw_{\sigma_2}^+|.
\]
Therefore,
\[
\forall \alpha \in \Gamma_1 \cup \Gamma_2, \quad d_T(\sigma_1, \sigma_2) \succ \log d_\alpha(\mu_1, \mu_2).
\]
The combination of Equations (19), (20) and (21) provides the desired lower bound and finishes the proof. □

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