UNIPOTENT QUANTUM COORDINATE RING AND
PREFUNDAMENTAL REPRESENTATIONS FOR TYPES $A_n^{(1)}$ AND $D_n^{(1)}$

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Abstract. We give a new realization of the prefundamental representations $L_{\tilde{\alpha}, a}$ introduced by Hernandez and Jimbo, when the quantum loop algebra $U_q(p)$ is of types $A_{n+1}^{(1)}$ and $D_{n+1}^{(1)}$, and the $r$-th fundamental weight $\varpi_r$ for types $A_n$ and $D_n$ is minuscule. We define an action of the Borel subalgebra $U_q(b)$ on the unipotent quantum coordinate ring associated to the translation by $\tilde{\alpha}_r$, and show that it is isomorphic to $L_{\tilde{\alpha}, a}$. We then give a combinatorial realization of $L_{\tilde{\alpha}, a}$ in terms of the Lusztig data of the dual PBW vectors.

1. Introduction

Let $U_q(g)$ be the quantum loop algebra associated to an affine Kac-Moody algebra $g$ of untwisted type. Let $I = \{1, \ldots, n\}$ be the index set for the Dynkin diagram of the finite-dimensional simple subalgebra $\tilde{g}$ of $g$. Let $x_{i,r}^{\pm}, k_{i,s}^{\pm}, h_{i,s} (i \in I, r, s \in \mathbb{Z}, s \neq 0)$ denote the Drinfeld generators of $U_q(g)$ [19].

A finite-dimensional irreducible $U_q(g)$-module (of type I) is generated by an eigenvector $v$ with respect to the currents $(\psi_i(z))_{i \in I}$ (see (2.1)) such that $x_{i,r}^+ v = 0$ for all $i \in I, r \in \mathbb{Z}$. The eigenvalue $\Psi_i(z)$ of $v$ with respect to $\psi_i^+(z)$ is given by a rational functions in $z$ of the following form

$\Psi_i(z) = q_i^\deg (P_i) P_i(q_i^{-1} z) P_i(q_i z)^{-1},$

where $P_i(z)$ is a polynomial in $z$ such that $P_i(0) = 1$ and $q_i$ is a power of $q$ depending on $i$. The sequence of polynomials $P = (P_i(z))_{i \in I}$ parametrizes the finite-dimensional irreducible $U_q(g)$-modules of type I [8, 9].

Let $U_q(b)$ be the Borel subalgebra $b$ of $g$. In [17], Hernandez and Jimbo introduced a category $\mathcal{O}$ of $U_q(b)$-modules, which contains the finite-dimensional $U_q(g)$-modules. An irreducible module in $\mathcal{O}$ is in general infinite-dimensional, and it is generated by an eigenvector vector $v$ with respect to $(\psi_i^+(z))_{i \in I}$ such that $e_i v = 0$ for all $i \in I$ and the eigenvalue $\Psi_i(z)$ for $\psi_i^+(z)$ is given by an arbitrary rational function in $z$. Here $e_i$ is the Chevalley generator of the positive part of $U_q(g)$.

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For a non-zero $a \in \mathbb{C}(q)$ and $r \in I$, the prefundamental representations $L_{r,a}^\pm$ are the $U_q(b)$-modules corresponding to $(\Psi_i(z))_{i \in I}$ such that
$$
\Psi_r(z) = (1 - az)^{\pm 1},
$$
and $\Psi_i(z) = 1$ elsewhere. It is shown in [17] that $L_{r,1}^\pm$ is irreducible and $L_{r,1}^-$ is a limit of Kirillov-Reshetikhin (simply KR) modules $W_{s,q_i^{-2s+1}}^{(r)}$ $(s \geq 1)$, which gives a representation theoretic explanation on the fact [16, 32] that the normalized $q$-character of $W_{s,q_i^{-2s+1}}^{(r)}$ [12] has a well-defined limit as $s \to \infty$. This implies that $L_{r,a}^\pm$ are $U_q(b)$-modules in $\mathcal{O}$, and any irreducible module in $\mathcal{O}$ is a subquotient of a tensor product of $L_{r,a}^\pm$, hence parametrized by the $I$-tuples of rational functions $\Psi_i(z)$ for $i \in I$ up to 1-dimensional $U_q(b)$-modules with a trivial action of $e_i$.

The category $\mathcal{O}$ plays an important role in generalizing the Baxter’s relation [11], where the $q$-character of a finite-dimensional $U_q(g)$-module $V$ is realized as a relation in the Grothendieck ring of $\mathcal{O}$ including the class of $V$ tensored by a certain tensor product of $L_{r,a}^\pm$’s. Furthermore, it is proved in [18] that the Grothendieck ring of a certain monoidal subcategory of $\mathcal{O}$ given by a restriction on the zeros and poles of $\Psi_i(z)$ for all $i \in I$ has a structure of cluster algebra, where the class of prefundamental representations $L_{r,a}^\pm$ form an initial seed.

Despite the importance of $L_{r,a}^\pm$, a realization of $L_{r,a}^\pm$ does not seem to be known much in general except for special cases. The purpose of this paper is to give an explicit realization of prefundamental representations $L_{r,a}^\pm$ when $g = A_n^{(1)}$, $D_n^{(1)}$ and $r \in I$ is minuscule, that is, the corresponding fundamental representation of $\hat{g}$ is minuscule.

Let $\pi_r$ denote the $r$-th fundamental weight for $\hat{g}$. The translation $t_{\pi_r}$ by $\pi_r$ in the extended Weyl group of $g$ is given by $t_{\pi_r} = \tau(w_r)^{-1}$ for some $\tau$ and $w_r$, where $\tau$ is an automorphism of the Dynkin diagram and $w_r$ is the element in the Weyl group of $g$. We consider the unipotent quantum coordinate ring associated to $w_r$, which we denote by $U_q^-(w_r)$. It is a subalgebra of $U_q^-(\hat{g})$, the negative part of $U_q(\hat{g})$, and this subalgebra is of special importance since it can be viewed as a $q$-analog of the coordinate ring of the unipotent subgroup associated to $w_r$, and has a quantum cluster algebra structure [13]. As a quantum cluster algebra, it also has a monoidal categorification in terms of the representations of quiver Hecke algebras [22].

We show that there exist two $U_q(b)$-module structures $\rho_{r,a}^\pm$ on $U_q^-(w_r)$ such that
$$
(1.1) \quad U_q^-(w_r) \cong L_{r,\pm,ac_r}^\pm,
$$
where $c_r$ is a scalar given in [11,17]. Note that we may assume that $U_q^-(w_r)$ is a subalgebra of $U_q^-(\hat{g}) \subset U_q^-(\hat{g})$ in our case. The action of $e_i$ on $U_q^-(w_r)$ for $i \in I$ is given by the usual $q$-derivation on $U_q^-(g)$, while the action of $e_0$ on $U_q^-(w_r)$ is given as a right (resp. left) multiplication by the dual PBW vector $x_0$ corresponding to the maximal root (see [8,17] for the definition of $x_0$) by which $U_q^-(w_r)$ is isomorphic to $L_{r,\pm,ac_r}^\pm$ (resp. $L_{r,-,ac_r}^\pm$) as a $U_q(b)$-module. Here $e_0$ is the Chevalley generator corresponding to the vertex 0 in the affine Dynkin diagram of $g$. We remark there is another construction of prefundamental
representations $L_{-a}$ and $L_{n-a}$ for $A_n^{(1)}$ in terms of $q$-oscillator representations \[2, 3\]. Our work can be viewed as a generalization of it.

Furthermore, we give an explicit combinatorial realization of $L_{r,a}$ in terms of the Lusztig data of the dual PBW vectors. More precisely, we identify the dual PBW vectors of $U_q^{-}(w_r)$ with the matrices consisting of the Lusztig data of the dual PBW vectors in a natural way and describe the $U_q(b)$-actions explicitly under this identification.

One of the key ingredients in the proof of \[1.1\] and the above combinatorial realization is that the $q$-derivations on the dual PBW vectors of $U_q^{-}(w_r)$ are well-understood by the minuscule representation $V(\varpi_i)$ of $U_q(\hat{g})$. This enables us to give a simple description of $U_q(b)$-actions on $U_q^{-}(w_r)$, in particular, the action of the current $\psi_i^+(z)$ on a maximal vector $1 \in U_q^{-}(w_r)$, which plays a crucial role in finding its loop highest weight.

Our realization of $L_{r,a}$ is partly motivated by the results in \[20, 27\], where the affine $A_n^{(1)}$-crystal structure on $U_q^{-}(w_r)$ is studied in terms of the Lusztig data of the PBW vectors of type $A_n$. More precisely, it is proved in \[20, 27\] that the associated affine crystal denoted by $B^J$ ($J := I / \{r\}$) is isomorphic to the limit of the crystals of KR modules $W_s^{(c)}$ as $s \to \infty$ (see \[19\] for type $D_n^{(1)}$). In particular, the Kashiwara operator $\tilde{e}_0$ on $B^J$ is given simply by increasing the multiplicity of the maximal root vector by 1 in the Lusztig data. This motivated the construction of $L_{r,a}$ in \[1.1\], and now we may understand $B^J$ as the crystal of $L_{r,1}$.

The paper is organized as follows. In Section 2, we briefly review necessary background on quantum loop algebras and the category $\mathcal{O}$. In Section 3 we define a $U_q(b)$-module structure $\rho_{r,a}$ of $U_q^{-}(w_r)$, which belongs to the category $\mathcal{O}$ (Theorem 3.1). In Section 4 we compute the eigenvalues of $1 \in U_q^{-}(w_r)$ with respect to $(\psi_i^+(z))_{i \in I}$ and show that $U_q^{-}(w_r) \cong L_{r,a_{-c}}$ in case of $\rho_{r,a}$ (Theorem 4.17). Then we define $\rho_{r,a}$ in a similar way, and show that $U_q^{-}(w_r) \cong L_{r,-a_{-c}}$ in this case (Theorem 4.22). In Section 5 we give an explicit realization description of $L_{r,a}$ in terms of the Lusztig data of the dual PBW vectors (Theorem 5.5 for type $A_n^{(1)}$ and Theorem 5.8 for type $D_n^{(1)}$).

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2. Preliminaries

2.1. Notations. Let $\mathbb{Z}_+$ denote the set of non-negative integers. Let $A = (a_{ij})_{0 \leq i, j \leq n}$ be the generalized Cartan matrix of symmetric affine type, and let $\mathfrak{g}$ be the affine Kac-Moody algebra associated to $A$. Put $I = \{1, \ldots, n\}$. Let $\hat{A} = (a_{ij})_{i,j \in I}$ be the Cartan matrix of finite type, and let $\hat{\mathfrak{g}}$ denote the associated finite-dimensional simple Lie algebra.

Let $\{\alpha_i \mid 0 \leq i \leq n\}$ be the set of simple roots for $\mathfrak{g}$, and let $Q = \oplus_{0 \leq i \leq n} \mathbb{Z} \alpha_i$ be the root lattice. Put $Q_+ = \oplus_{0 \leq i \leq n} \mathbb{Z}^+ \alpha_i$. Let $\{\Lambda_i \mid 0 \leq i \leq n\}$ be the fundamental weights for $\mathfrak{g}$, and let $P = \oplus_{0 \leq i \leq n} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta$ be the weight lattice of $\mathfrak{g}$, where $\delta$ is the imaginary null root.
Let us take a nondegenerate symmetric bilinear form $\left( \cdot, \cdot \right)$ on $P$ so that $(\alpha_i, \alpha_j) \in 2\mathbb{Z}_{\geq 0}$.

Let $\{ \alpha_i | i \in I \}$ be the set of simple roots for $\mathfrak{g}$ and let $\hat{Q} = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ be the root lattice of $\mathfrak{g}$. Let $\{ \omega_i | i \in I \}$ be the fundamental weights for $\mathfrak{g}$ and let $\hat{P} = \bigoplus_{i \in I} \mathbb{Z}\omega_i$ be the weight lattice of $\mathfrak{g}$. Note that $\theta$ is equal to the maximal root of $\mathfrak{g}$ by regarding $\hat{P}$ as a sublattice of $P/\mathbb{Z}\theta$.

Let $W$ be the affine Weyl group of $\mathfrak{g}$, which is generated by the simple reflection $s_i$ for $0 \leq i \leq n$, and let $\hat{W}$ be the Weyl group of $\hat{\mathfrak{g}}$, which is the subgroup generated by $s_i$ for $i \in I$. Let $w_0$ be the longest element of $W$. Note that $W$ is isomorphic to the semidirect product $\hat{W} \ltimes \hat{P}$ under the identification $s_0 \mapsto (s_0, \theta)$ and $s_i \mapsto (s_i, 0)$.

Let $\hat{W} = W \ltimes \hat{P}$ be the extended affine Weyl group. Let $\mathcal{T}$ be the set of bijections $\tau : I \cup \{0\} \rightarrow I \cup \{0\}$ such that $a_{\tau(i)\tau(j)} = a_{ij}$ for all $0 \leq i, j \leq n$. It is known that each element $\tau \in \mathcal{T}$ induces a unique automorphism $\psi_\tau$ of $W$ such that $\psi_\tau(s_i) = s_\tau(i)$. Then $W$ is a normal subgroup of $\hat{W}$ such that $\hat{W} \approx \hat{W}/W$ and $\hat{W} \approx \mathcal{T} \ltimes W$, where the action of $\mathcal{T}$ on $W$ is given by $\psi_\tau$ (see [21] for more details). An expression for $w \in \hat{W}$ is called reduced if $w = \tau s_{i_1} \cdots s_{i_{\ell(w)}}$, where $\tau \in \mathcal{T}$ and $\ell(w)$ is minimal. We call such $\ell(w)$ the length of $w \in \hat{W}$.

For $\lambda \in P$, we denote by $t_\lambda$ the element $(1, \lambda) \in \hat{W}$, which is called the translation by $\lambda$.

Let $q$ be an indeterminate. We put

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}} \quad (m \in \mathbb{Z}_+), \quad [m]_q! = [m]_q[m - 1]_q \cdots [1]_q \quad (m \geq 1), \quad [0]_q! = 1,$$

$$\left[ \begin{array}{c} m \\ k \end{array} \right]_q = \frac{[m]_q[m - 1]_q \cdots [m - k + 1]_q}{[k]_q} \quad (0 \leq k \leq m).$$

If there is no confusion, then we often write $[m]$ and $\left[ \begin{array}{c} m \\ k \end{array} \right]_q$ instead of $[m]_q$ and $\left[ \begin{array}{c} m \\ k \end{array} \right]_q$ for simplicity, respectively.

2.2. Quantum loop algebra. Let $\mathbb{k} = \mathbb{C}(q)$ be the base field. The quantum loop algebra $U_q(\mathfrak{g})$ is the associative $\mathbb{k}$-algebra generated by $e_i$, $f_i$, and $k_i^{\pm 1}$ for $0 \leq i \leq n$, and $C^{\pm \frac{1}{2}}$ subject to the following relations:

$$C^{\pm \frac{1}{2}}$$ are central with $C^{\frac{1}{2}}C^{-\frac{1}{2}} = C^{-\frac{1}{2}}C^{\frac{1}{2}} = 1$,

$$k_i k_j = k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad \prod_{i=0}^{n} k_i^{\pm n_i} = (C^{\pm \frac{1}{2}})^2,$$

$$k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j,$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}},$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^m e_i^{(1-a_{ij} - m)} e_j e_i^{(m)} = 0, \quad \sum_{m=0}^{1-a_{ij}} (-1)^m f_i^{(1-a_{ij} - m)} f_j f_i^{(m)} = 0 \quad (i \neq j),$$

for $0 \leq i, j \leq n$, where $e_i^{(m)} = e_i^m/[m]_q!$ and $f_i^{(m)} = f_i^m/[m]_q!$ for $0 \leq i \leq n$ and $m \in \mathbb{Z}_+$. Here $a_0, a_1, \ldots, a_n$ are the numerical labels of the Dynkin diagram associated with $A$ in [21].
There is a Hopf algebra structure on $U_q(\mathfrak{g})$, where the comultiplication $\Delta$ and the antipode $S$ are given by

\[
\Delta(k_i) = k_i \otimes k_i, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i,
\]
\[
S(k_i) = k_i^{-1}, \quad S(e_i) = -k_i^{-1}e_i, \quad S(f_i) = -f_i k_i,
\]
for $0 \leq i \leq n$. It is well-known in [4] that as a Hopf algebra, $U_q(\mathfrak{g})$ is also isomorphic to the algebra generated by $x_{i,r}^\pm$ ($i \in I, r \in \mathbb{Z}$), $k_i^\pm$ ($i \in I$), $h_i,r$ ($i \in I, r \in \mathbb{Z}\setminus\{0\}$), and $C^\pm$ subject to the following relations:

- $C^\pm$ are central with $C^\pm C^{-\pm} = C^{-\pm} C^\pm = 1$,
- $k_i k_j = k_j k_i$, $k_i k_i^{-1} = k_i^{-1} k_i = 1$,
- $k_i h_{j,r} = h_{j,r} k_i$, $k_i x_{j,r}^\pm k_i^{-1} = q^{\pm a_{ij}} x_{j,r}^\pm$,
- $[h_{i,r}, h_{j,s}] = \delta_{r,-s} \frac{1}{r} [r a_{ij}] C^r - C^{-r}$,
- $[h_{i,r}, x_{j,s}^\pm] = \frac{1}{r^2} [r a_{ij}] C^{r} x_{j,s}^\pm$,
- $x_{i,r+1}^\pm x_{j,s}^\pm - q^{\pm a_{ij}} x_{j,s}^\pm x_{i,r+1}^\pm = q^{\pm a_{ij}} x_{i,r}^\pm x_{j,s+1}^\pm - x_{j,s+1}^\pm x_{i,r}^\pm$,
- \[ [x_{i,r}^+, x_{j,s}^-] = \frac{C^{(r-s)/2} \psi_{i,r,s}^+ - C^{-(r-s)/2} \psi_{i,r,s}}{q - q^{-1}}. \]

where $r_1, \ldots, r_m$ is any sequence of integers with $m = 1 - a_{ij}$, $\mathfrak{S}_m$ denotes the group of permutations on $m$ letters, and $\psi_{i,r}^\pm$ is the element determined by the following identity of formal power series in $z$;

\[
(2.1) \quad \sum_{r=0}^{\infty} \psi_{i,r}^\pm z^r = k_i^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{s=1}^{\infty} h_{i,s} z^s \right).
\]

2.3. **Category $\mathfrak{O}$**. Let $U_q(\mathfrak{b})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, k_i^{\pm 1}$ for $0 \leq i \leq n$ and $C^\pm$. Let $t$ be the subalgebra of $U_q(\mathfrak{b})$ generated by $k_i^{\pm 1}$ for $i \in I$, and let $t^* = (k^*)^I$ be the set of maps from $I$ to $k^\times$ which is a group under pointwise multiplication.

Let $U_q(\mathfrak{g})^\pm$ (resp. $U_q(\mathfrak{g})^0$) be the subalgebras of $U_q(\mathfrak{g})$ generated by $x_{i,r}^\pm$ for $i \in I$ and $r \in \mathbb{Z}$ (resp. $k_i^{\pm 1}$, $\psi_{i,r}^{\pm}$ for $i \in I$, $r > 0$ and $C^\pm$). Then we have a triangular decomposition

\[
U_q(\mathfrak{g}) \cong U_q(\mathfrak{g})^- \otimes U_q(\mathfrak{g})^0 \otimes U_q(\mathfrak{g})^+.
\]

If we put $U_q(\mathfrak{b})^+ = U_q(\mathfrak{g})^+ \cap U_q(\mathfrak{b})$ and $U_q(\mathfrak{b})^0 = U_q(\mathfrak{g})^0 \cap U_q(\mathfrak{b})$, then we have $U_q(\mathfrak{b})^+ = \langle x_{i,r}^+ \rangle_{i \in I, r \geq 0}$ and $U_q(\mathfrak{b})^0 = \langle \psi_{i,r}^{\pm}, k_i^{\pm 1}, C^\pm \rangle_{i \in I, r > 0}$.

**Remark 2.1.** Throughout this paper, we assume that $C^\pm$ acts trivially on a $U_q(\mathfrak{b})$-module.

Let $V$ be a $U_q(\mathfrak{b})$-module. For $\omega \in t^*$, we define the weight space of $V$ with weight $\omega$ by

\[
V_\omega = \{ v \in V | k_i v = \omega(i) v \ (i \in I) \}.
\]
We say that $V$ is $t$-diagonalizable if $V = \bigoplus_{\omega \in t} V_\omega$.

A series $\Psi = (\Psi_{i,m})_{i \in I, m \geq 0}$ of elements in $k$ such that $\Psi_{i,0} \neq 0$ for all $i \in I$ is called an $\ell$-weight. We often identify $\Psi = (\Psi_{i,m})_{m \geq 0}$ with $\Psi = (\Psi_i(z))_{i \in I}$, an $I$-tuple of formal power series, where

$$\Psi_i(z) = \sum_{m \geq 0} \Psi_{i,m} z^m.$$  

We denote by $t^*_i$ the set of $\ell$-weights. Since $\Psi_i(z)$ is invertible, $t^*_i$ is a group under multiplication. Let $\varpi : t^*_i \rightarrow t^*$ be the surjective morphism defined by $\varpi(\Psi)(i) = \Psi_{i,0}$ for $i \in I$.

For $\Psi \in t^*_i$, we define the $\ell$-weight space of $V$ with $\ell$-weight $\Psi$ by

$$V_\Psi = \left\{ v \in V \mid \text{for any } i \in I \text{ and } m \geq 0, \exists p_{i,m} \in \mathbb{Z}_+ \text{ such that } (\psi^+_{i,m} - \Psi_{i,m})^{p_{i,m}} v = 0 \right\}.$$  

For $\Psi \in t^*_i$, we say that $V$ is of highest $\ell$-weight $\Psi$ if there exists a non-zero vector $v \in V$ such that

(i) $V = U_q(b)v$,  
(ii) $e_i v = 0$ for all $i \in I$,  
(iii) $\psi^+_{i,m} v = \Psi_{i,m} v$ for $i \in I$ and $m \geq 0$.

A non-zero vector $v \in V$ is called a highest $\ell$-weight vector of the weight $\Psi$ if it satisfies the conditions (ii) and (iii). There exists a unique irreducible $U_q(b)$-module of highest $\ell$-weight $\Psi$, which we denote by $L(\Psi)$ [17 Proposition 3.4].

**Definition 2.2.** [17 Definition 3.7] For $r \in I$ and $a \in k^\times$, let $L_{r,a}^\pm$ be an irreducible $U_q(b)$-module of highest weight $\Psi$ given by

$$\Psi_i(z) = \begin{cases} (1 - az)^{\pm 1} & \text{if } i = r, \\ 1 & \text{if } i \neq r. \end{cases}$$  

The representations $L_{r,a}^\pm$ are called **fundamental representations**.

For $i \in I$, let $\pi_i \in t^*$ given by $\pi_i(j) = q^{a_{ij}}$ $(j \in I)$. We define a partial order $\leq$ on $t^*$ by $\omega' \leq \omega$ if and only if $\omega'/\omega^{-1}$ is a product of $\pi_i^{-1}$'s. For $\lambda \in t^*$, put $D(\lambda) = \{ \omega \in t^* \mid \omega \leq \lambda \}$.

**Definition 2.3.** [17 Definition 3.8] The category $\mathcal{O}$ consists of $U_q(b)$-modules $V$ such that

(i) $V$ is $t$-diagonalizable,  
(ii) $\dim V_\omega < \infty$ for all $\omega \in t^*$,  
(iii) there exist $\lambda_1, \ldots, \lambda_s \in t^*$ such that the weights of $V$ are in $\bigcup_{j=1}^s D(\lambda_j)$.

The category $\mathcal{O}$ is a tensor category and the simple objects are given as follows.

**Theorem 2.4.** [17 Theorem 3.11] For $\Psi \in t^*_i$, $L(\Psi)$ is in the category $\mathcal{O}$ if and only if $\Psi_i(z)$ is rational for all $i \in I$.

Note that for $\Psi, \Psi' \in t^*_i$, $L(\Psi \Psi')$ is a subquotient of $L(\Psi) \otimes L(\Psi')$. Hence any irreducible $U_q(b)$-module in $\mathcal{O}$ is a subquotient of a tensor product of fundamental representations and 1-dimensional representations.
3. Unipotent quantum coordinate ring

3.1. Canonical basis. Let $U_q^-(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $f_i$ for $0 \leq i \leq n$. Recall that $U_q^-(\mathfrak{g})$ has a root space decomposition, that is, $U_q^-(\mathfrak{g}) = \bigoplus_{\beta \in Q_+} U_q(\mathfrak{g})_{\beta}$, where $U_q(\mathfrak{g})_{\beta} = \{ x | k_i x k_i^{-1} = q^{\alpha_i, \beta} x \ (0 \leq i \leq n) \}$. Write $\text{wt}(x) = \beta$ for $x \in U_q(\mathfrak{g})_{\beta}$.

Let us recall the (dual) canonical basis of $U_q^-(\mathfrak{g})$ (see [23, 24] for more details). Given $0 \leq i \leq n$, there exists a unique $\mathbf{k}$-linear map $e'_i : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ such that $e'_i(1) = 0$, $e'_i(f_j) = \delta_{ij}$ for $0 \leq j \leq n$, and

$$
e'_i(xy) = e'_i(x)y + q^{\text{wt}(x), \alpha_i} xe'_i(y),$$

for homogeneous $x, y \in U_q^-(\mathfrak{g})$. Then there exists a unique non-degenerate symmetric $\mathbf{k}$-valued bilinear form $(\ , \ )$ on $U_q^-(\mathfrak{g})$ such that

$$(1, 1) = 1, \quad (f_i x, y) = (x, e'_i(y)),$$

for $0 \leq i \leq n$ and $x, y \in U_q^-(\mathfrak{g})$.

For any homogeneous $x \in U_q^-(\mathfrak{g})$, we have $x = \sum_{k \geq 0} f_i^{(k)} x_{ik}$, where $e'_i(x_k) = 0$ for $k \geq 0$. Then we define $\tilde{f}_i x = \sum_{k \geq 0} f_i^{(k+1)} x_{ik}$. Let $A_0$ denote the subring of $\mathbf{k}$ consisting of rational functions regular at $q = 0$. Let

$$L(x) = \sum_{r \geq 0, 0 \leq i_1, \ldots, i_r \leq n} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} 1,$$

$$B(x) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} 1 \ (\mod qL(x)) \mid r \geq 0, 0 \leq i_1, \ldots, i_r \leq n \} \setminus \{ 0 \} \subset L(x)/qL(x).$$

The pair $(L(x), B(x))$ is called the crystal base of $U_q^-(\mathfrak{g})$.

Let $A = \mathbb{Z}[q, q^{-1}]$, and let $U_q^-(\mathfrak{g})_A$ be the $A$-subalgebra of $U_q^-(\mathfrak{g})$ generated by $f_i^{(k)}$ for $0 \leq i \leq n$ and $k \in \mathbb{Z}_+$. Let $- : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ be the automorphism of $\mathbb{C}$-algebras given by $q \mapsto q^{-1}$ and $f_i \mapsto f_i$ for $0 \leq i \leq n$. Then $(L(x), L(x), U_q^-(\mathfrak{g})_A)$ is balanced, that is, the map

$$E : L(x) \cap \overline{L(x)} \cap U_q^-(\mathfrak{g})_A \longrightarrow L(x)/qL(x)$$

is a $\mathbb{C}$-linear isomorphism.

Let $G$ denote the inverse of the map $(3.3)$. Then

$$G(x) := \{ G(b) \mid b \in B(x) \}$$

is an $A$-basis of $U_q^-(\mathfrak{g})_A$, which is called the canonical basis or global crystal basis. Let

$$G^{\text{up}}(x) := \{ G^{\text{up}}(b) \mid b \in B(x) \}$$

be the dual basis of $G(x)$ with respect to the bilinear form $(3.2)$, that is, $(G^{\text{up}}(b), G(b')) = \delta_{bb'}$ for $b, b' \in B(x)$. We call $G^{\text{up}}(x)$ the dual canonical basis of $U_q^-(\mathfrak{g})^{\text{up}}_A := \{ x \in U_q^-(\mathfrak{g}) \mid (x, U_q^-(\mathfrak{g})_A) \in A \}$.

Let $U_q^-(\mathfrak{g})^e$ be the $\mathbf{k}$-algebra generated by $U_q(\mathfrak{g})$ and $q^{\pm d}$ where $q^d$ commutes with $k_i^{\pm 1}$ and satisfies $q^{d} q^{-d} = 1$, $q^{d} e_i q^{-d} = q^{\delta_{0d} d} e_i$, and $q^{d} f_i q^{-d} = q^{-\delta_{0d} e_i}$. Given a dominant integral weight $\Lambda$ for $\mathfrak{g}$, let $V(\Lambda)$ be the irreducible highest weight module over $U_q^-(\mathfrak{g})^e$. Let $B(\Lambda)$ and $G(\Lambda) = \{ G_\Lambda(b) \mid b \in B(\Lambda) \}$ denote the crystal and canonical basis of $V(\Lambda)$, respectively.
Let \( * \) be the \( k \)-algebra anti-automorphism of \( U_q^{-}(\mathfrak{g}) \) given by \((f_i)^* = f_i^\alpha \) for \( 0 \leqslant i \leqslant n \). Then 

\[
L(x)^* = L(x) \quad \text{and} \quad B(x)^* = B(x).
\]

We may regard \( B(\Lambda) \subset B(x) \) by 

\[
(3.4) \quad B(\Lambda) = \{ b \in B(x) \mid \varepsilon^*_x(b) \leqslant \langle \alpha_i, \alpha_i^\vee \rangle \quad (0 \leqslant i \leqslant n) \},
\]

where \( \alpha_i^\vee \) is the simple coroot and \( \varepsilon_x^*(b) = \max \{ k \mid b^* = f_k^x(b^*_y) \) for some \( b \in B(x) \). \}

We have \( G_\Lambda(b) = \pi_\Lambda(G(b)) \) for \( b \in B(\Lambda) \), where \( \pi_\Lambda : U_q^{-}(\mathfrak{g}) \rightarrow V(\Lambda) \) is the canonical projection.

Let \( G^{qp}(\Lambda) = \{ G^{qp}(b) \mid b \in B(\Lambda) \} \) be the dual basis of \( G(\Lambda) \) with respect to the bilinear form on \( V(\Lambda) \) in [21] (4.2.4), (4.2.5)]. Let 

\[
\iota_\Lambda : V(\Lambda)^\vee \rightarrow U_q^{-}(\mathfrak{g}),
\]

be the dual of \( \pi_\Lambda \), where \( V(\Lambda)^\vee \) and \( U_q^{-}(\mathfrak{g}) \) are the duals of \( V(\Lambda) \) and \( U_q^{-}(\mathfrak{g}) \) with respect to [21] (4.2.4), (4.2.5)] and [3.2], respectively. Then we have 

\[
(3.5) \quad \iota_\Lambda(G^{qp}_\Lambda(b)) = G^{qp}(b),
\]

for \( b \in B(\Lambda) \). Here we identify \( V(\Lambda)^\vee \) with \( V(\Lambda) \), and \( U_q^{-}(\mathfrak{g}) \) with \( U_q^{-}(\mathfrak{g}) \).

### 3.2. Unipotent quantum coordinate ring \( U_q^{-}(w) \)

Let us recall the unipotent quantum coordinate ring (cf. [25]).

For \( 0 \leqslant i \leqslant n \), let \( T_i \) be the \( k \)-algebra automorphism of \( U_q(\mathfrak{g}) \) [31] given by 

\[
T_i(k_j) = k_j k_i^\alpha, \quad T_i(e_i) = -f_i k_i, \quad T_i(e_j) = \sum_{r + s = -a_{i,j}} (-1)^r q^{-r} f_i^{(s)} e_j^{(r)} \quad (j \neq i),
\]

\[
T_i(f_i) = -k_i^{-1} e_i, \quad T_i(f_j) = \sum_{r + s = -a_{i,j}} (-1)^r q^s f_i^{(r)} f_j^{(s)} \quad (j \neq i),
\]

for \( 0 \leqslant j \leqslant n \). Note that \( T_i = T_{i,i}^\alpha \) in [31].

Let \( \Delta^\pm \) be the set of positive (resp. negative) roots of \( \mathfrak{g} \). For \( w \in W \), let \( \Delta^+(w) = \Delta^+ \cap w \Delta^- \). Suppose that the length of \( w \) is \( \ell \). Let \( R(w) = \{ \tilde{w} = (i_1, \ldots , i_\ell) \mid 0 \leqslant i_j \leqslant n \text{ and } w = s_{i_1} \cdots s_{i_\ell} \} \) be the set of reduced expressions of \( w \). A 2-braid move on \( \tilde{w} \in R(w) \) is defined by \( (i_1, i_2, \ldots , i_\ell) \) for \( i, j \in I \) such that \( |i - j| > 1 \). For \( \tilde{w} = (i_1, \ldots , i_\ell) \in R(w) \), we have \( \Delta^+(w) = \{ \beta_k \mid 1 \leqslant k \leqslant \ell \} \), where 

\[
(3.6) \quad \beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_k) \quad (1 \leqslant k \leqslant \ell).
\]

For \( 1 \leqslant k \leqslant \ell \) and \( c \in \mathbb{Z}_+ \), let \( F(c \beta_k) = T_{i_1} \cdots T_{i_{k-1}}(f_{i_k}^{(c)}) \in U_q^{-}(\mathfrak{g})_{-c \beta_k} \). For \( c = (c_1, \ldots , c_\ell) \in \mathbb{Z}_+^\ell \), put

\[
F(c, \tilde{w}) = F(c_1 \beta_1) \cdots F(c_\ell \beta_\ell).
\]

Then \( \{ F(c, \tilde{w}) \mid c \in \mathbb{Z}_+^\ell \} \subset U_q^{-}(\mathfrak{g})_A \). Furthermore, we have \( F(c, \tilde{w}) \in L(\mathfrak{x}) \) and \( F(c, \tilde{w}) \) belongs to \( B(\mathfrak{x}) \), say \( b(c, \tilde{w}) \) [36].

We assume that \( \Delta^+(w) \) is linearly ordered by \( \beta_1 < \cdots < \beta_\ell \). Then the following \( q \)-commutation relation for the root vectors \( F(c_i \beta_k) \)'s holds [24].

\[
(3.7) \quad F(c_j \beta_\ell) F(c_i \beta_1) - q^{-(c_i \beta_1, c_j \beta_\ell)} F(c_i \beta_1) F(c_j \beta_\ell) = \sum_{c'} f_{c'} F(c', \tilde{w}),
\]
for \(i < j\) and \(c_i, c_j \in \mathbb{Z}_+\), where the sum is over \(c' = (c'_i)\) such that \(c_i \beta_i + c_j \beta_j = \sum_{k \leq j} c'_k \beta_k\) with \(c'_i < c_i\), \(c'_j < c_j\) and \(f_{c'} \in \mathbf{k}\).

**Definition 3.1.** For \(w \in W\), we denote by \(U_q^-(w)\) the vector space over \(\mathbf{k}\) generated by \(\{F(c, \tilde{w}) \mid c \in \mathbb{Z}_+^w\}\).

Note that \(U_q^-(w)\) does not depend on the choice of \(\tilde{w} \in R(w)\), and the \(q\)-commutation relation \((3.7)\) implies that \(U_q^-(w)\) is the \(\mathbf{k}\)-subalgebra of \(U_q^-(\mathfrak{g})\) generated by \(\{F(\beta_k) \mid 1 \leq k \leq \ell\}\). Moreover, it is shown in \([25]\) Theorem 4.25 that \(U_q^-(w)\) is compatible with the dual canonical basis in the following sense

\[
U_q^-(w) \cap U_q^-(\mathfrak{g})^\text{up} = \bigoplus_{b \in B(w)} AG^\text{up}(b),
\]

where \(B(w) = \{b(c, \tilde{w}) \mid c \in \mathbb{Z}_+^w\}\). Let \(\{F^{\text{up}}(c, \tilde{w}) \mid c \in \mathbb{Z}_+^w\}\) be the dual basis of \(\{F(c, \tilde{w}) \mid c \in \mathbb{Z}_+^w\}\). Then the left-hand side of \((3.8)\) is also the \(\mathcal{A}\)-subalgebra generated by \(\{F^{\text{up}}(\beta_k) \mid 1 \leq k \leq \ell\}\). Note that the same formula \((3.7)\) also holds for the dual root vectors \(F^{\text{up}}(\beta_k)\) where \(f_{c'} \in \mathcal{A}\) (see \([25]\) Theorem 4.27).

The subalgebra \(U_q^-(w)\) is called the unipotent quantum coordinate ring since the quotient \(\mathbb{C} \otimes_{\mathcal{A}} U_q^-(w)\) with respect to the \(\mathcal{A}\)-lattice generated by \(F^{\text{up}}(c, \tilde{w})\) is isomorphic to the coordinate ring of the unipotent subgroup \(N(w)\) of the Kac-Moody group associated to \(w\) \([25]\) Theorem 4.44.

### 3.3. \(U_q(b)\)-action on \(U_q^-(\varpi_r)\)

From now on, we assume that \(\mathfrak{g}\) is of type \(A_n^{(1)}\) \((n \geq 1)\) or \(D_n^{(1)}\) \((n \geq 4)\) and \(r \in I\) denotes an index such that \(\varpi_r\) is minuscule or the fundamental representation of \(\mathfrak{g}\) with highest weight \(\varpi_r\) is minuscule, that is,

\[
\begin{cases}
  r \in \{1, \ldots, n\} & \text{for type } A_n, \\
  r \in \{1, n - 1, n\} & \text{for type } D_n.
\end{cases}
\]

**Lemma 3.2.** Let \(t_{\varpi_r}\) be the translation by \(\varpi_r\). Suppose that

\(t_{\varpi_r}^{-1} = w_r \tau\),

for some \(\tau \in \mathcal{I}\) and \(w_r \in W\) of length \(\ell\). Then \(w_r\) satisfies the following properties:

1. \(w_r\) is the maximal element in the set \(W_r \setminus \tilde{W}\) of minimal length coset representatives, where \(W_r\) is the subgroup of stabilizers of \(\varpi_r\) in \(\tilde{W}\),
2. a reduced expression of \(w_r\) is unique up to 2-braid moves,
3. \(\Delta^+(w_r) = \{\beta \in \Delta^+ \cap \tilde{Q} \mid (\varpi_r, \beta) = 1\}\),
4. if \(\beta_k (1 \leq k \leq \ell)\) is a positive root in \((3.6)\) with respect to a reduced expression of \(w_r\), then we have \(\beta_1 = \alpha_r\) and \(\beta_\ell = \theta\),
5. \(w_r^{-1} = w_{r^*}\), where \(r^*\) is determined by \(\alpha_{r^*} = -w_0(\alpha_r)\).

**Proof.** Let \(J = \Gamma\setminus\{r\}\) (resp. \(J^* = \Gamma\setminus\{r^*\}\)) and \(w_0\) (resp. \(w_0^*\)) be the longest element of the parabolic subgroup \(\tilde{W}_J\) (resp. \(\tilde{W}_{J^*}\)) of \(\tilde{W}\). The assertion (1) is well-known (cf. \([7]\)).
Thus we have

\[(3.10) \quad w_0 = w_0^j w_r = w_r w_0^* .\]

Since \( w_r^{-1} \) is \( \varpi_r \)-minuscule (see [37, (2.1)] for its definition), (2) follows from [37 Proposition 2.1] (see also [20, Remark 5.2]). By the definition of being \( \varpi_r \)-minuscule, we have

\[\Delta^+(w_r) \subset \{ \beta \in \Delta^+ \cap \hat{Q} \mid (\varpi_r, \beta) = 1 \} .\]

As |\( \Delta^+(w_r) \)| = \( \ell(w_r) = \ell(w_0^j w_0) = \ell(w_0) - \ell(w_0^j) \), one can check that

\[|\Delta^+(w_r)| = \# \{ \beta \in \Delta^+ \cap \hat{Q} \mid (\varpi_r, \beta) = 1 \} ,\]

which gives (3).

We take a reduced expression \( \hat{\omega}_r = s_i s_{i_2} \cdots s_{i_\ell} \) of \( w_r \). It is easy to see that \( w_0^j (\alpha_r) = \theta \). It follows from (1) and [14 Lemma 2.2.1] that \( i_1 = r \) and \( i_\ell = r^* \). Thus \( \beta_1 = \alpha_r \) and

\[\beta_\ell = -w_r(\alpha_r^*) = w_0^j w_0(-\alpha_r^*) = w_0^j (\alpha_r) = \theta .\]

Since \( w_0 \) and \( w_0^j \) are involutions, it follows from (3.10) that

\[w_r^* = w_0^j w_0 = w_r^{-1} ,\]

which implies (5).

\[\square\]

**Remark 3.3.** For \( i \leq j \), put

\[s_{(i, j)} = s_is_{i+1} \cdots s_j .\]

A reduced expression of \( w_r \) is given as follows:

\[w_r = \begin{cases} s_{(r, n)} s_{(r-1, n-1)} \cdots s_{(1, n-r+1)} & \text{for type } A_n^{(1)} \text{ with } r \in I , \\ s_{(1, n)} s_{(1, n-2)}^{-1} & \text{for type } D_n^{(1)} \text{ with } r = 1 , \\ \hat{s}_1 \hat{s}_2 \cdots \hat{s}_{n-1} & \text{for type } D_n^{(1)} \text{ with } r = n , \end{cases}\]

where \( \hat{s}_k \) is given by

\[\hat{s}_k = \begin{cases} s_n s_{(k, n-2)}^{-1} & \text{if } k \text{ is odd} , \\ s_{(k, n-1)}^{-1} & \text{if } k \text{ is even} , \\ s_n & \text{if } n \text{ is even and } k = n - 1 . \end{cases}\]

Note that for type \( D_n^{(1)} \), a reduced expression of \( w_{n-1} \) is obtained from the one of \( w_n \) by replacing \( s_n \) with \( s_{n-1} \).

By abuse of notation, we put

\[(3.11) \quad U_q^- (\varpi_r) := U_q^- (w_r) .\]

The following is the main result in this section.

**Theorem 3.4.** For a minuscule \( \varpi_r \), there is a \( U_q(\mathfrak{g}) \)-module structure on \( U_q^- (\varpi_r) \) such that \( U_q^- (\varpi_r) \) is in the category \( \mathcal{O} \).
The rest of this subsection is devoted to proving Theorem 3.4. To define a $U_q(b)$-action on $U^-_q(\mathfrak{g})$, we will consider the $k$-linear operators (3.20) on $U^-_q(\mathfrak{g})$, and then verify that the operators satisfy the defining relations of $U_q(b)$.

Let us choose $\tilde{w} = (i_1, \ldots, i_\ell) \in R(w_r)$ and let $\{ \beta_k | 1 \leq k \leq \ell \}$ be as in (3.6) with respect to $\tilde{w}$. By Lemma 3.2.2, $F(\beta_k)$ (and hence $F^{up}(\beta_k)$) is independent of the choice of $\tilde{w}$ up to permutations. For $1 \leq k \leq \ell$, put $1_{\beta_k} \in B(\infty)$ such that

$$1_{\beta_k} = F^{up}(\beta_k) \pmod{qL(\infty)} \in B(\infty).$$

Let $\text{wt} : B(\infty) \to \mathbb{P}$ denote the weight function. For example, $\text{wt}(1_{\beta_k}) = -\beta_k$.

**Remark 3.5.** Note that for $b \in B(\Lambda)$ and $m \in \mathbb{Z}_{>0}$, we have

$$\tilde{e}_i^m(b) = 0 \iff e_i^m G^{up}_\Lambda(b) = 0 \iff (e_i')^m G^{up}(b) = 0$$

(24 Lemma 5.1.1]). Also, for $b \in B(\Lambda)$ with $\varepsilon_i(b) = 1$, we have

$$G^{up}_\Lambda(\tilde{e}_i(b)) = e_i G^{up}_\Lambda(b),$$

which is equivalent to $G^{up}(\tilde{e}_i(b)) = e_i G^{up}(b)$ (cf. [25 Theorem 3.14]).

**Lemma 3.6.** Let $1 \leq k \leq \ell$ be given.

1. We have

$$F^{up}(\beta_k) \in \iota_{\Lambda_r}(G^{up}(\Lambda_r)),$$

where $\Lambda_r$ is the $r$-th fundamental weight for $\mathfrak{g}$. In particular, $1_{\beta_k} \in B(\Lambda_r)$ by regarding $B(\Lambda_r)$ as a subset of $B(\infty)$ (cf. (3.4), (3.5)).

2. For $i \in I$, we have

$$e_i'(F^{up}(\beta_k)) = \begin{cases} F^{up}(\beta_k - \alpha_i) & \text{if } (\alpha_i, \beta_k) = 1 + \delta_{i,r}, \\ 0 & \text{otherwise.} \end{cases}$$

Here we understand $F^{up}(0) = 1$.

**Proof.** (1) First, we have $F^{up}(\beta_k) \in G^{up}(\infty)$ for $1 \leq k \leq \ell$ by [25 Theorem 4.29]. So by (3.5), it remains to show that $1_{\beta_k} \in B(\Lambda_r)$, equivalently

$$\varepsilon_i^\#(1_{\beta_k}) = \delta_{i,r}.$$  

It is clear that $\varepsilon_i^\#(1_{\beta_k}) = 0$ since $\beta_k - \alpha_0 \notin Q_+$. Let $i \in I \setminus \{ r \}$ be given. By Lemma 3.2.1, we have

$$\ell(s_i s_i \ldots s_{i_{k-1}}) = \ell(s_i \ldots s_{i_{k-1}}) + 1,$$

for $1 \leq k \leq \ell$. Then we have $T_i F^{up}(\beta_k) \in U^-_q(\mathfrak{g})$ (cf. [13 Propositions 7.1 and 7.4]), and hence $\varepsilon_i^\#(1_{\beta_k}) = 0$ since $T_i^{-1}(U^-_q(\mathfrak{g})) \cap U^-_q(\mathfrak{g}) = \text{Ker}(e_i')^\#$ (see [86 (3.4.4)]). On the other hand, we have $\varepsilon_r^\#(1_{\beta_k}) \neq 0$ since $\text{wt}(1_{\beta_k}) \neq 0$. Thus we conclude from the weight consideration that $\varepsilon_r^\#(1_{\beta_k}) = 1$. This proves that $1_{\beta_k} \in B(\Lambda_r)$.
(2) Let $B_0(\Lambda_r)$ be the connected component of 1 in $B(\Lambda_r)$ generated by $\tilde{e}_i, \tilde{f}_i$ for $i \in I$. We denote by $W \cdot \varpi_r$ the orbit of $\varpi_r$ by the action of $W$. Note that $B_0(\Lambda_r)$ is the crystal of the fundamental representation $V(\varpi_r)$ of $U_q(\mathfrak{g})$ with highest weight $\varpi_r$. Thus, the map

\[
B_0(\Lambda_r) \longrightarrow W \cdot \varpi_r
\]

is bijective because $B_0(\Lambda_r)$ is the crystal of the minuscule representation $V(\varpi_r)$. By (1), we have

\[
\{ \varpi_r - \beta_k \mid k = 1, \ldots, \ell \} \subset W \cdot \varpi_r.
\]

By (3.15) and [1, Lemma 1.4], one can check that $\varpi_r - \beta_k + \alpha_i \in W \cdot \varpi_r \iff (\varpi_r - \beta_k, \alpha_i) = -1 \iff (\alpha_i, \beta_k) = 1 + \delta_{i,r}$.

This yields that

\[
\varepsilon_i(1_{\beta_k}) = \begin{cases} 
1 & \text{if } (\alpha_i, \beta_k) = 1 + \delta_{i,r}, \\
0 & \text{otherwise}.
\end{cases}
\]

Therefore we have the assertion by (3.12) and (3.13). \hfill \Box

**Remark 3.7.** One can also check (3.14) directly by the formula in [6, Theorem 3.7] or [35, Proposition 7.4]. For example, see [19, Theorem 3.12] in which the authors give a combinatorial description of $\varepsilon_n^{\theta}$ for type $D_n$ by using [6, Theorem 3.7].

Consider the following root vector with the weight $\theta$:

\[
x_0 := F_{\text{up}}(\beta_{\ell}) = F_{\text{up}}(\theta).
\]

Let $J_0 = \{ i \in I \mid (\alpha_i, \theta) = 0 \}$ and $J_1 = I \setminus J_0$. If $\mathfrak{g}$ is of type $A_n^{(1)}$, then we have

\[
J_1 = \begin{cases} 
\{ 1 \} & \text{if } n \geq 1, \\
\{ 1, n \} & \text{if } n \geq 2,
\end{cases} \quad (\theta, \alpha_i) = \begin{cases} 
2 & \text{if } n = 1 \text{ and } i \in J_1, \\
1 & \text{if } n \geq 2 \text{ and } i \in J_1.
\end{cases}
\]

If $\mathfrak{g}$ is of type $D_n^{(1)}$, then we have $J_1 = \{ 2 \}$ and $(\theta, \alpha_2) = 1$.

**Lemma 3.8.**

(1) For $i \in I$, we have

\[
\begin{cases} 
\varepsilon_i(x_0) = 0 & \text{if } i \in J_0, \\
(\varepsilon_i')^2(x_0) = 0 & \text{if } i \in J_1.
\end{cases}
\]

(2) For $i \in J_1$, we have

\[
x_0 \cdot \varepsilon_i(x_0) = q^{-(1-\delta_{1,n})}\varepsilon_i(x_0) \cdot x_0.
\]

**Proof.** Note that (1) and (2) are straightforward for type $A_1^{(1)}$, so we assume that $n \geq 2$ when $\mathfrak{g}$ is of type $A_n^{(1)}$.

(1) By Lemma 3.6, $1_{\theta} = 1_{\beta_{\ell}}$ belongs to $B(\Lambda_r)$. Let us consider two cases.
Case 1. \( r \in I \setminus \{1, n\} \) in type \( A_n^{(1)} \) or \( r \in \{1, n-1, n\} \) in type \( D_n^{(1)} \). In this case, by (3.10), one can check that

\[
(\alpha_i, \theta) = 1 + \delta_{i,r} \iff i \in J_1.
\]

Thus, we have

\[
\varepsilon_i(1) = \begin{cases} 
0 & \text{if } i \in J_0, \\
1 & \text{if } i \in J_1. 
\end{cases}
\]

For \( i \in J_0 \), we have \( \tilde{c}_i \tilde{1}_\theta = 0 \) in \( B(\Lambda_r) \), which is equivalent to \( \tilde{c}_i \tilde{1}_\theta = 0 \) in \( B(\infty) \). Then it is equivalent to \( e_i' \xi_0 = 0 \) (see (3.12)). For \( i \in J_1 \), we have \( \varepsilon_i(1) = 1 \), and

\[
e_i' \xi_0 = e_i' G^{\text{up}}(1) = G^{\text{up}}(\tilde{c}_i \tilde{1}_\theta),
\]

\[
(e_i')^2 \xi_0 = (e_i)^2 G^{\text{up}}(1) = G^{\text{up}}(\tilde{c}_i^2 \tilde{1}_\theta) = 0,
\]

(see (3.12) and (3.13)).

Case 2. \( r = 1 \) or \( n \) in type \( A_n^{(1)} \). We may assume that \( r = 1 \), since the proof of the case of \( r = n \) is almost identical. In this case, since

\[
(\alpha_i, \theta) = 1 + \delta_{i,1} \iff i = n,
\]

we have

\[
\varepsilon_i(1) = \begin{cases} 
0 & \text{if } i \neq n, \\
1 & \text{if } i = n. 
\end{cases}
\]

By the same argument as in Case 1, we conclude that \( e_i' \xi_0 = 0 \) for \( i \neq n \), and \( (e_i')^2 \xi_0 = 0 \). Note that it is obvious that \( (e_1')^2 \xi_0 = 0 \), that is, \( (e_1')^2 \xi_0 = 0 \) for \( i \in J_1 \).

Hence, we complete the proof of (1).

(2) Put

\[
\tilde{J}_1 = \begin{cases} 
J_1 \setminus \{1, n\} & \text{if } \mathfrak{g} \text{ is of type } A_n^{(1)}, \\
J_1 & \text{if } \mathfrak{g} \text{ is of type } D_n^{(1)}. 
\end{cases}
\]

By Lemma 3.6(2) and (3.18)-(3.19), we have \( \tilde{c}_i \tilde{1}_\theta = 1_{\theta - \alpha_i} \) for \( i \in \tilde{J}_1 \). Then one can check the required relation directly by using the formula (3.7). We remark that for the case of type \( A_n^{(1)} \) with \( r = 1 \) or \( n \), since \( e_{\delta_{1,r-1} + \delta_{n,r} - n}(\xi_0) = 0 \), we also obtain the required relation in this case.

Let \( a \in \mathbb{k}^\times \). For \( 0 \leq i \leq n \), let \( e_i \) and \( k_i \) be the \( \mathbb{k} \)-linear operators on \( U_q^{-}(\mathfrak{g}) \) given by

\[
e_i(u) = \begin{cases} 
\xi_i' (u) & \text{if } i \in I, \\
a q^{-\theta, \beta} u \cdot \xi_0 & \text{if } i = 0, 
\end{cases}
\]

\[
k_i(u) = \begin{cases} 
q^{\alpha_i, \beta} u & \text{if } i \in I, \\
q^{-\theta, \beta} u & \text{if } i = 0. 
\end{cases}
\]

for \( u \in U_q^{-}(\mathfrak{g}) \) (\( \beta \in -Q_+ \)).

**Example 3.9.** Let us consider the above \( \mathbb{k} \)-linear operators on \( U_q^{-}(\mathfrak{g}) \) for type \( A_1^{(1)} \). In this case, we get

\[
w = s_1, \quad \theta = \alpha_1, \quad \xi_0 = f_1.
\]
By (3.21), one can check that

$$e'_i (f_1^m) = q^{-m+1}[m]f_1^{m-1},$$  

(3.21)

where we understand $f_1^{-1} = 0$ and $f_1^0 = 1$ (cf. [23, Section 3.1]).

Put $S_{i,j} = e_i^3 e_j - [3]e_i^2 e_j e_i + [3]e_i e_j^2 - e_i e_j^3$. Here $[3] = q^{-2} + 1 + q^2$ by definition. We show that

$$S_{i,j}(u) = 0,$$

for $u \in U_q^{-}(\mathfrak{g})$. Put $s = (\theta, \beta)$ and $t = (\beta, \alpha_1)$.

**Case 1.** $(i,j) = (0,1)$. By definition (3.20), one can check that

$$e_0^3 e_1(u) = a^3 q^{-3s} e'_1(u) f_1^3,$$

$$e_0^2 e_1 e_0(u) = a^3 q^{-3s+2} e'_1(u) f_1^3 + a^3 q^{-3s+t+2} u f_1^2,$$

$$e_0 e_1 e_0^2(u) = a^3 q^{-3s+4} e'_1(u) f_1^3 + a^3 q^{-3s+t+2}(1 + q^2) u f_1^2,$$

$$e_1 e_0^3(u) = a^3 q^{-3s+6} e'_1(u) f_1^3 + a^3 q^{-3s+t+2}(1 + q^2 + q^4) u f_1^2.$$

Thus, we have $S_{0,1}(u) = 0$ in this case.

**Case 2.** $(i,j) = (1,0)$. Similarly, we have

$$e_0^3 e_1(u) = a q^{-s} e'_1(u) f_1 + a q^{-s+t}(1 + q^2 + q^4) e'_1(u),$$

$$e_0^2 e_1 e_0(u) = a q^{-s-2} e'_1(u) f_1 + a q^{-s+t}(1 + q^2) e'_1(u),$$

$$e_0 e_1 e_0^2(u) = a q^{-s-4} e'_1(u) f_1 + a q^{-s+t} e'_1(u),$$

$$e_1 e_0^3(u) = a q^{-s-6} e'_1(u) f_1.$$

Thus, we have $S_{1,0}(u) = 0$ in this case.

By **Case 1** and **Case 2**, we have seen that $e_0$ and $e_1$ satisfy the quantum Serre relation. On the other hand, it is straightforward to check that $e_i$ and $k_j$ satisfy the relation $k_j e_i = q^{\alpha_j} e_i k_j$ for $i,j \in \{0,1\}$. Hence the $k$-linear operators $e_i$ and $k_i$ ($i = 0,1$) give a representation of $U_q(\mathfrak{b})$ on $U_q^{-}(\mathfrak{g})$ in the case of $A_1^{(1)}$.

In general, we have the following theorem.

**Theorem 3.10.** The operators $e_i$ and $k_i$ for $0 \leq i \leq n$ satisfy the defining relations of $U_q(\mathfrak{b})$. This gives a representation $\rho_{r,a}^+ : U_q(\mathfrak{b}) \rightarrow \text{End}_k(U_q^{-}(\mathfrak{g}))$ given by $\rho_{r,a}^+(e_i) = e_i$ and $\rho_{r,a}^+(k_i) = k_i$ for $0 \leq i \leq n$.

**Proof.** We verify that $e_i$ and $k_i$ satisfy the defining relations of $U_q(\mathfrak{b})$. Note that for type $A_1^{(1)}$, it is done by Example 3.9. Let us consider the remaining cases.

First, we prove the relations for $e_i$ and $k_j$ for $0 \leq i,j \leq n$. Clearly, we have $k_i k_j = k_j k_i$ for $0 \leq i,j \leq n$ by definition. Let us consider the relations between $e_i$ and $k_j$ for $0 \leq i,j \leq n$. Since $\delta = \alpha_0 + \theta$, we have $(\alpha_0, \alpha_1) = (\delta - \theta, \alpha_i) = (\theta, \alpha_i)$. Also, one can check that $(\theta, \theta) = 2$. This implies that $k_j e_i = q^{\alpha_j} e_i k_j$ for $0 \leq i,j \leq n$. 


Next, we prove the quantum Serre relations of \( e_i \) \((0 \leq i \leq n)\). By definition, \( \{ e_i \mid i \in I \} \) satisfies the relations by \[23\] Lemma 3.4.2. So it remains to show the quantum Serre relations of \( e_i \) and \( e_0 \) for \( 0 \leq i \leq n \). Let us consider two cases.

**Case 1.** If \( i \in J_0 \), then by Lemma 3.8(1), for \( u \in U_q^-(\g)_{\beta} \), we have

\[
e_i e_0 (u) = e_i \left( a q^{-\langle \theta, \beta \rangle} u \cdot x_0 \right) = a q^{-\langle \theta, \beta \rangle} \left\{ e_i'(u) \cdot x_0 + q^{\langle \beta, \alpha_i \rangle} u \cdot e_i'(x_0) \right\} = a q^{-\langle \theta, \beta+\alpha_i \rangle} e_i(u) \cdot x_0 = e_0 e_i(u),
\]

Hence \( e_i e_0 = e_0 e_i \) holds.

**Case 2.** Suppose that \( i \in J_1 \). We claim that

\[
e_0 e_i^2 - (q + q^{-1}) e_i e_0 + e_i^2 e_0 = 0, \tag{3.22}
\]

\[
e_i e_0^2 - (q + q^{-1}) e_0 e_i + e_0^2 e_i = 0. \tag{3.23}
\]

Let \( u \in U_q^-(\g)_{\beta} \) be given. Put \( s = (\theta, \beta) \) and \( t = (\beta, \alpha_i) \).

First, we have by using Lemma 3.8

\[
e_0 e_i^2(u) = a q^{-s-2} \left( e_i' \right)^2(u) \cdot x_0,
\]

\[
e_i e_0 e_i(u) = a q^{-s-1} \left( e_i' \right)^2(u) \cdot x_0 + a q^{-s-t+1} e_i'(u) \cdot e_i'(x_0),
\]

\[
e_i^2 e_0(u) = a q^{-s} \left( e_i' \right)^2(u) \cdot x_0 + a q^{-s+t+1} e_i'(u) \cdot e_i'(x_0) + a q^{-s+t} e_i'(u) \cdot e_i'(x_0).
\]

This implies (3.22). Similarly, we have

\[
e_i e_0^2(u) = a^2 q^{-2s+2} e_i'(u) \cdot x_0^2 + a^2 q^{-2s+2} u \cdot e_i'(x_0) \cdot x_0 + a^2 q^{-2s+t} u \cdot e_i'(x_0) \cdot x_0,
\]

\[
e_0 e_i e_0(u) = a^2 q^{-2s+1} e_i'(u) \cdot x_0^2 + a^2 q^{-2s+t+1} u \cdot e_i'(x_0) \cdot x_0,
\]

\[
e_i^2 e_i(u) = a^2 q^{-2s} e_i'(u) \cdot x_0^2.
\]

This implies (3.23).

**Remark 3.11.** For type \( A_n^{(1)} \) with \( r = 1 \) or \( r = n \), since \( e_i'_{\delta_1, r-1+\delta_1, r-n}(x_0) = 0 \), for \( u \in U_q^-(\g)_{\beta} \), we have

\[
e_0 e_r(u) = a q^{-\langle \theta, \beta \rangle-1} e_r'(u) x_0 = q^{-1} e_r e_0(u),
\]

which also implies the relations (3.22) and (3.23).

Now, we are ready to prove Theorem 3.4.

**Proof of Theorem 3.4.** By definition, we have \( x_0 \in U_q^-(\varpi_r) \). Hence \( U_q^-(\varpi_r) \) is invariant under \( e_0 \). By Theorem 3.10 it is enough to verify that \( U_q^-(\varpi_r) \) is invariant under \( e_i' \) for \( i \in I \). Suppose that \( e_i'_{F^{\text{up}}}(\beta_k) \neq 0 \). Note that \( e_i(1_{\beta_k}) = 1 \) by (3.10). By Lemma 3.8 we have

\[
e_i'_{F^{\text{up}}}(\beta_k) = F^{\text{up}}(\beta_k - \alpha_i) \in U_q^-(\varpi_r).
\]

Since \( e_i' \) is the derivation on \( U_q^-(\g) \) and \( U_q^-(\varpi_r) \) is the subalgebra generated by \( F^{\text{up}}(\beta_k) \), \( U_q^-(\varpi_r) \) is invariant under \( e_i' \) by (3.24). Hence \( U_q^-(\varpi_r) \) is a \( U_q(b) \)-module.
4. Realization of Prefundamental Representations

4.1. Affine PBW vectors and Drinfeld generators, $\psi^+_i, k$. Let $2\rho$ be the sum of positive roots of $\hat{g}$. Take a reduced expression $t_{2\rho} = s_{i_1} \cdots s_{i_N} \in W$ and define a doubly infinite sequence

$$i_N \ldots i_0 = i_{n+1} = i_0, i_1, i_2, \ldots,$$

by setting $i_k = i_{k (\text{mod } N)}$ for $k \in \mathbb{Z}$. Then (4.1) gives a convex order $<$ (cf. [34, Definition 2.1]) on the set of positive roots of $\hat{g}$:

$$\beta_0 < \beta_1 < \ldots < \beta_n,$$

where $\beta_k$ is given by

$$\beta_k = \begin{cases} s_{i_0} s_{i_1} \cdots s_{i_{k-1}} (\alpha_k) & \text{if } k \leq 0, \\ s_{i_1} s_{i_2} \cdots s_{i_{k-1}} (\alpha_k) & \text{if } k > 0. \end{cases}$$

For $k \in \mathbb{Z}$, we define the root vector $E_{\beta_k}$ by

$$E_{\beta_k} = \begin{cases} T^{-1}_{i_0} T^{-1}_{i_1} \cdots T^{-1}_{i_k} (e_{i_k}) & \text{if } k \leq 0, \\ T_{i_1} T_{i_2} \cdots T_{i_{k-1}} (e_{i_k}) & \text{if } k > 0. \end{cases}$$

We define the root vector $F_{\beta_k}$ in the same way with $e_{i_k}$ replaced by $f_{i_k}$ in (4.3). Recall that $E_{\beta_k} \in U_q^+(\hat{g})$ and $F_{\beta_k} \in U_q^-(\hat{g})$ for $k \in \mathbb{Z}$ by [31, Proposition 40.1.3]. In particular, if $\beta_k = \alpha_i$ for $0 \leq i \leq n$, then $E_{\beta_k} = e_i$ and $F_{\beta_k} = f_i$ (cf. [34, Corollary 4.3]).

Example 4.1. Let us consider the case of type $A^{(1)}_3$. By Remark 3.3, we have

$$t_{\varpi_1} = \tau_1 (s_3 s_2 s_1),$$

$$t_{\varpi_2} = \tau_2 (s_2 s_1 s_3),$$

$$t_{\varpi_3} = \tau_3 (s_1 s_2 s_3),$$

where $\tau_i$ is the diagram automorphism by the correspondence $\alpha_k \mapsto \alpha_{k+i (\text{mod } 4)}$. Recall that $\rho = \sum_{i=1}^{3} \varpi_i$. Then the reduced expression $(i_1, \ldots, i_{20})$ of $t_{2\rho}$ associated to the following expression (cf. [5])

$$t_{2\rho} = t_{\varpi_1} t_{\varpi_2} t_{\varpi_3} t_{\varpi_1} t_{\varpi_2}$$

is given by

$$(i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}) = (0, 3, 2, 1, 0, 2, 1, 3, 0, 1),$$

$$(i_{11}, i_{12}, i_{13}, i_{14}, i_{15}, i_{16}, i_{17}, i_{18}, i_{19}, i_{20}) = (2, 1, 0, 3, 2, 0, 3, 1, 2, 3).$$
Consider the doubly infinite sequence (4.1) induced from the above reduced expression. With respect to the corresponding convex order $<$, the positive real roots of the form $\delta - \alpha_i \ (i = 1, 2, 3)$ occur at the positions of the above bold numbers with

$$\cdots < \delta - \alpha_3 < \cdots < \delta - \alpha_2 < \cdots < \delta - \alpha_1 < \cdots .$$

In particular, the root vectors $E_{\delta - \alpha_i} \ (i = 1, 2, 3)$ are given by

$$E_{\delta - \alpha_1} = e_0 - q^{-1}e_2 e_0 e_2 - q^{-1}e_2 e_0 e_3 + q^{-2}e_2 e_3 e_0,$$

$$E_{\delta - \alpha_2} = e_0 e_1 e_3 - q^{-1}e_1 e_0 e_3 - q^{-1}e_3 e_0 e_1 + q^{-2}e_3 e_1 e_0,$$

$$E_{\delta - \alpha_3} = e_0 e_1 e_2 - q^{-1}e_1 e_0 e_2 - q^{-2}e_1 e_0 e_3 + q^{-2}e_1 e_2 e_0. \quad (4.5)$$

One can observe that the above formula (4.5) of $E_{\delta - \alpha_i}$ does not depend on the order of multiplication in an expression of $t_{2p}$ by $t_{w_i}$ (such as (4.4)). For example, let us consider the following expression

$$t_{2p} = t_{w_2} t_{w_1} t_{w_3} t_{w_2} t_{w_1} t_{w_3},$$

and then the reduced expression $(j_1, \ldots, j_{20})$ of $t_{2p}$ is given by

$$(j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8, j_9, j_{10}) = (0, 3, 1, 0, 2, 1, 0, 3, 0, 1),$$

$$(j_{11}, j_{12}, j_{13}, j_{14}, j_{15}, j_{16}, j_{17}, j_{18}, j_{19}, j_{20}) = (2, 1, 3, 2, 0, 3, 2, 1, 2, 3).$$

Here the positive real roots of the form $\delta - \alpha_i \ (i = 1, 2, 3)$ occur at the positions of the above bold numbers. With respect to the corresponding convex ordering $<'$, we have

$$\cdots <' \delta - \alpha_3 <' \cdots <' \delta - \alpha_2 <' \cdots .$$

If we take the doubly infinite sequence (4.1) associated to $(j_1, \ldots, j_{20})$, then by a direct computation, one check that the formulas of $E_{\delta - \alpha_i}$ are equal to the ones in (4.5).

In general, we have the following.

**Lemma 4.2.** For $i \in I$ and $k > 0$, the root vectors $E_{k\delta + \alpha_i}$ and $F_{k\delta + \alpha_i}$ are independent of the choice of a reduced expression of $t_{2p}$.

**Proof.** Let $<$ be a convex order associated with $t_{2p}$ as above. One can check that $\delta < \beta$ if and only if $\beta = k\delta - \alpha$ for some $k \geq 1$ and a positive root $\alpha$ of $\mathfrak{g}$ (cf. [5, Lemma 1.1]), that is, the convex order $<$ has coarse type $w_0$ (see [34, Definition 2.9]). Then the assertion follows from [34, Corollary 4.6].

Let $o : I \to \{ \pm 1 \}$ be a map such that $o(i) = -o(j)$ whenever $a_{ij} < 0$. Recall that

$$\psi^+_i,k = o(i)^k (q - q^{-1}) C^+ \frac{2}{k_i} \left( E_{k\delta - \alpha_i} E_{\alpha_i} - q^{-2} E_{\alpha_i} E_{k\delta - \alpha_i} \right), \quad (4.6)$$

for $i \in I$ and $k > 0$ (see [5, Proposition 1.2]). We remark that the right-hand side of (4.6) is also independent of the choice of a reduced expression of $t_{2p}$ due to Lemma 4.2.

The following lemma together with (4.6) enables us to compute the action of $\psi^+_i,k$ for $i \in I$ and $k \in \mathbb{Z}_{>0}$ on a $U_q(\mathfrak{b})$-module (cf. Example 4.3 and 4.24).
Lemma 4.3. For \( i \in I \) and \( k \in \mathbb{Z}_{>0} \), we have
\[
E_{(k+1)\delta-\alpha_i} = -\frac{1}{q + q^{-1}} \left( E_{\delta-\alpha_i} E_{\alpha_i} E_{k\delta-\alpha_i} - q^{-2} E_{\alpha_i} E_{\delta-\alpha_i} E_{k\delta-\alpha_i} \right) \nonumber
\]
\[
- E_{k\delta-\alpha_i} E_{\delta-\alpha_i} E_{\alpha_i} + q^{-2} E_{k\delta-\alpha_i} E_{\alpha_i} E_{\delta-\alpha_i} \right). \nonumber
\]

Proof. Thanks to Lemma 4.2, we may use the following relation [5, Proposition 1.2] (cf. [1]): for \( i, j \in I \) and \( k, l \in \mathbb{Z}_+ \) with \( l > 0 \),
\[
\left[ E_{\delta, i}, E_{k\delta \pm \alpha_j} \right] = \pm \delta(i) \delta(j) \frac{1}{l} \left[ l_{ij} \right] q E_{(l+k)\delta \pm \alpha_i}, \tag{4.7} \nonumber
\]
where \( k > 0 \) for \( k \delta - \alpha_j \). Here \( E_{\delta, i} \in U_q(g) \) is defined by the functional equation
\[
\exp \left( q - q^{-1} \sum_{l=1}^{\infty} E_{\delta, l} u^l \right) = 1 + \sum_{k=1}^{\infty} (q - q^{-1}) \tilde{\psi}_{i,k} u^k, \tag{4.8} \nonumber
\]
where \( \tilde{\psi}_{i,k} = E_{k\delta-\alpha_i} E_{\alpha_i} - q^{-2} E_{\alpha_i} E_{k\delta-\alpha_i} \). In particular, by comparing the coefficients of \( u \) in (4.8), we get
\[
E_{\delta, i} = \tilde{\psi}_{i,1} = E_{\delta-\alpha_i} E_{\alpha_i} - q^{-2} E_{\alpha_i} E_{\delta-\alpha_i}, \tag{4.9} \nonumber
\]
On the other hand, if we put \( l = 1 \) and \( i = j \) in (4.7), then for \( k > 0 \), we have
\[
- [2] E_{(k+1)\delta-\alpha_i} = E_{\delta, i} E_{k\delta-\alpha_i} - E_{k\delta-\alpha_i} E_{\delta, i}, \tag{4.10} \nonumber
\]
which implies the required formula by (4.9) and (4.10). \( \square \)

4.2. Highest \( \ell \)-weight of \( U_q^-((\omega_r)_a) \). Let \( U_q^-((\omega_r)_a) \) denote the \( U_q(b) \)-module defined by (3.20) in Theorem 3.10. We denote by \( 1 \in U_q^-((\omega_r)_a) \) a non-zero vector of weight 0, which is equal to \( 1 \in U_q^-((\omega_r)) \) up to a scalar multiplication. In this subsection, we compute the \( \ell \)-weight of \( 1 \) in \( U_q^-((\omega_r)_a) \) explicitly. It is crucial to prove that the \( U_q(b) \)-module \( U_q^-((\omega_r)_a) \) is isomorphic to the prefundamental representation \( L^+_{r,1} \) for \( r \) in (3.9) up to a shift of spectral parameter.

4.2.1. Type \( A_n^{(1)} \). Let us first give explicit examples to compute the \( \ell \)-weight of \( 1 \). Then the rest of this subsection will be devoted to generalize the following examples to an arbitrary rank \( n \) and \( r \in I \).

Example 4.4. Let us consider the case of type \( A_n^{(1)} \). We continue to use the convention in Example 3.9. By (4.10), it suffice to compute the action of \( E_{k\delta-\alpha_i} \) on \( 1 \) in order to compute the eigenvalue of \( \psi_{i,k}^+ \) for \( 1 \). In this case, \( t_{2p} = s_0 s_1 \). Thus, we have
\[
E_{\delta-\alpha_i} = e_0 \quad \text{and} \quad E_{\delta-\alpha_i} 1 = a \cdot \chi_0 = a \cdot f. \nonumber
\]
By Lemma 4.3 the action of \( E_{2\delta-\alpha_i} \) on \( 1 \) is given by
\[
E_{2\delta-\alpha_i} 1 = \frac{1}{q + q^{-1}} \left[ a^2 (1 - (1 + q^{-2}) + q^{-2}) \cdot f = 0, \right. \nonumber
\]
which implies that \( E_{k\delta-\alpha_i} 1 = 0 \) for \( k \geq 2 \) by Lemma 4.3. Hence, we have
\[
\psi_{i,k}^+ 1 = \begin{cases} \frac{-o(1)(q-q^{-1})q^{-2}a}{f} & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases} \nonumber
\]
Finally, the $\ell$-weight $\Psi(z)$ of $1$ is given by $(1 - o(1))(q - q^{-1})q^{-2}az$.

**Example 4.5.** As another example, we consider the case of $A_3^{(1)}$ with $r = 2$. Let us recall Example 4.1.

**Step 1.** By Lemma 3.6 and (4.5),

$$E_{\delta-\alpha_2} 1 = q^{-2}e_3e_1e_0 1 = q^{-2}ae_3e_1(x_0) = q^{-2}a(f_2).$$

Also, we have

$$E_{\delta-\alpha_2}(f_2) = q^{-2}e_3e_1e_0(f_2) = q^{-2}ae_3e_1(f_2x_0) = a(f_2e_1e_1(x_0)) = a(f_2).$$

**Step 2.** By Lemma 4.3 and the above computation, we observe that

$$E_{2\delta-\alpha_2} 1 = \frac{1}{q + q^{-1}}q^2(q^{-4} - q^{-4}(1 + q^{-2}) + q^{-6}) f_2 = 0.$$

Then it follows from Lemma 4.3 that $E_{k\delta-\alpha_2} 1 = 0$ for all $k \geq 2$.

**Step 3.** By (4.6), the eigenvalue of $1$ under $\psi_{2,k}^+$ is given by

$$\psi_{2,k}^+ 1 = \begin{cases} -o(2)(q - q^{-1})q^{-4}a1 & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

For $i = 1, 3$, we have $E_{\delta-\alpha_i} 1 = 0$ by 4.5 and Lemma 3.6(2). By Lemma 4.3 we conclude $\Psi_1(z) = \Psi_3(z) = 1$. Finally, the $\ell$-weight $\Psi(z)$ of $1$ is

$$\Psi(z) = (1, 1 - o(2)(q - q^{-1})q^{-4}az, 1).$$

Now, let us consider the case of type $A_n^{(1)} (n \geq 2)$ with $r \in I$. For simplicity, we denote by $\pm q^{-2+}$ the subset of $k$ consisting of elements of the form $\pm q^{-m}$ for $m \in \mathbb{Z}_+$. $\Psi^{(1)}$ can be written as

$$(\pm q^{2})^{n-1}(e_1 \ldots e_{n-1}\tilde{e}_{n-r})(e_n \ldots e_{n+2-r} e_{n+1-r} + \sum_{j_1, \ldots, j_n} C_{j_1, \ldots, j_n}(q)e_{j_1} \ldots e_{j_n},$$

where the sum is over the sequences $(j_1, \ldots, j_n)$ such that $\sum_{k=1}^n \alpha_{j_k} = \delta - \alpha_0$ with $j_n \neq n + 1 - r$ and $C_{j_1, \ldots, j_n}(q) \in \pm q^{-2+}$.

**Proof.** By Lemma 4.2 we have

- $w_{\ell-1}^{-1} = w_{n+1-r}$.
- $x$ does not depend on the choice of reduced expressions of $w_{\ell-1}$.

Thus we use the reduced expression in Remark 3.3. Set $[a,b]_q := ab - q^{-1}ba$. Note that $T_1(e_j) = [e_i,e_j]_q$ if $a_{ij} = -1$. We shall use induction on rank $n$. Since it is obvious when $n = 1$, we assume that $n > 1$.

Suppose that $r = 1$. Since $w_1^{-1} = w_n = s_n s_{n-1} \ldots s_1$, we have

$$x = T_n \ldots T_4 T_3 ((e_1)_q) = T_n \ldots T_4 T_3 ((e_2,e_1)_q) = T_n \ldots T_4 ((e_3,e_2)_q,e_1)_q)$$

$$= \ldots ((e_n,e_{n-1})_q,e_{n-2})_q, \ldots e_1)_q,$$
which implies the assertion.

Suppose that \( r > 1 \). For a reduced expression \( w = s_{j_1} \cdots s_{j_k} \), we let

\[ w^- = s_{j_1} \cdots s_{j_{k-1}} \quad \text{and} \quad T_w = T_{j_1} \cdots T_{j_k}. \]

Let \( v = s_{(n-r-1)}s_{(n-r-2)} \cdots s_{(1)} \) and \( y = T_{-r}(e_r) \). Note that \( v \in \langle s_1, \ldots, s_{n-1} \rangle \) and \( x = T_{s_{(n+1-r,n)}} \). By the induction hypothesis, \( y \) can be written as

\[ (-q^{-1})^{n-2}(e_1 \cdots e_{n-r-2}e_{n-r-1})(e_n \cdots e_{n+3-r}e_{n+2-r})e_{n+1-r} + A, \]

where \( A \) is of the form \( \sum_{j_1, \ldots, j_{n-1}} C_{j_1, \ldots, j_{n-1}}(q)e_{j_1} \cdots e_{j_{n-1}} \) with \( j_{n-1} \neq n - r \). As

\[ T_{s_{(n+1-r,n)}} = T_{n+1-r}T_{n+2-r} \cdots T_n, \]

it follows from (4.11) that

\[ T_{s_{(n+1-r,n)}}(e_k) = \begin{cases} e_k & \text{if } k = 1, \ldots, n - r - 1, \\ [e_{n+1-r}, e_{n-r}]_q & \text{if } k = n - r, \\ e_{k+1} & \text{if } k = n + 1 - r, \ldots, n - 1. \end{cases} \]

Therefore, we have

\[ x = T_{s_{(n+1-r,n)}}(y) = (-q^{-1})^{n-2}(e_1 \cdots e_{n-r-2}e_{n-r-1})(e_n \cdots e_{n+3-r}e_{n+2-r})[e_{n+1-r}, e_{n-r}]_q + T_{s_{(n+1-r,n)}}(A) \]

\[ = (-q^{-1})^{n-1}(e_1 \cdots e_{n-r-2}e_{n-r-1})(e_n \cdots e_{n+3-r}e_{n+2-r})e_{n-r}e_{n+1-r} \]

\[ + (-q^{-1})^{n-2}(e_1 \cdots e_{n-r-2}e_{n-r-1})(e_n \cdots e_{n+3-r}e_{n+2-r})e_{n+1-r}e_{n-r} + T_{s_{(n+1-r,n)}}(A), \]

which implies the assertion since the last two terms satisfy the required property by (4.12). \( \Box \)

The following formula plays an important role in the computation of the \( \ell \)-weight of

\[ 1 \in U_q^{-1}(\varpi_r)_a. \]

Lemma 4.7. For \( r \in I \), we have

\[ E_{\delta - \alpha_r} = (-q^{-1})^{n-1}(e_{r+1} \cdots e_{n+1}e_n)(e_{r-1} \cdots e_2e_1)e_0 + \sum_{j_1, \ldots, j_n} C_{j_1, \ldots, j_n}(q)e_{j_1} \cdots e_{j_n}, \]

where the sum is over the sequences \( (j_1, \ldots, j_n) \) such that \( \sum_{k=1}^{n} \alpha_{j_k} = \delta - \alpha_r \) with \( j_n \neq 0 \) and \( C_{j_1, \ldots, j_n}(q) \in \pm q^{-\mathbb{Z}_+} \).

Proof. Thanks to Lemma 4.1.2 \( E_{\delta - \alpha_r} \) does not depend on the choice of a reduced expression of \( t_{2\rho} \). By Lemma 3.2, we can write

\[ t_{2\rho} = t_{\varpi_r}t_{2\rho - \varpi_r} = \tau_r w_{r}^{-1}t_{2\rho - \varpi_r}, \]

where \( \tau_r \) is the Dynkin diagram automorphism of \( A_n^{(1)} \) sending \( i \) to \( i + r \) \(( \text{mod } n + 1) \). Let \( s_i, s_{i+1} \cdots s_i \) be a reduced expression of \( w_{r}^{-1} \). Note that \( w_{r}^{-1} = w_{n+1-r} \) and \( i_{\xi} = r \) by Lemma 3.2 Since \( t_{\varpi_r}(\alpha_r) = \alpha_r - \delta \), we know that

\[ E_{\delta - \alpha_r} = \tau_r T_{i_1}T_{i_2} \cdots T_{i_{\xi-1}}(e_r). \]

Here we understand \( \tau_r \) as the automorphism of \( U_q(g) \) induced by the corresponding Dynkin diagram automorphism. Thus the assertion follows from Lemma 4.6. \( \Box \)
Corollary 4.8. The action of $\mathbf{E}_{\delta - \alpha_i}$ on $\mathbf{1} \in U_q^-(\varpi_r)a$ is given by

$$\mathbf{E}_{\delta - \alpha_i} \mathbf{1} = \begin{cases} (-q^{-1})^{n-1} a(f_r) & \text{if } i = r, \\ 0 & \text{if } i \neq r. \end{cases}$$

Moreover, we have $\mathbf{E}_{k\delta - \alpha_i} \mathbf{1} = 0$ for $i \in I$ and $k \geq 2$.

Proof. We use Lemma 4.6 and Lemma 4.7 to compute the action of $\mathbf{E}_{\delta - \alpha_i} \mathbf{1}$. For $i = r$, we have by Lemma 4.7

$$\mathbf{E}_{\delta - \alpha_i} \mathbf{1} = (-q^{-1})^{n-1} (e_{r+1} \ldots e_{n-1} e_n)(e_{r-1} \ldots e_2 e_1) e_0 \mathbf{1}. $$

By Lemma 3.6 and (3.20), it is enough to consider the transition of the weight of $\xi_0$ along the action $(e_{r+1} \ldots e_{n-1} e_n)(e_{r-1} \ldots e_2 e_1)$. Consequently, we have $\mathbf{E}_{\delta - \alpha_i} \mathbf{1} = (-q^{-1})^{n-1} a(f_r)$.

For $i \neq r$, since $-\alpha_i$ is not a weight of $U_q^-(\varpi_r)$, we conclude that $\mathbf{E}_{\delta - \alpha_i} \mathbf{1} = 0$ for $i \neq r$. \hfill \Box

Remark 4.9. In the proof of Corollary 4.8 for $i \neq r$, one can check $\mathbf{E}_{\delta - \alpha_i} \mathbf{1} = 0$ directly by using Lemma 4.7 since it holds for any $r \in I$.

Corollary 4.10. The action of $\mathbf{E}_{\delta - \alpha_r}$ on $f_r \in U_q^-(\varpi_r)a$ is given by

$$\mathbf{E}_{\delta - \alpha_r}(f_r) = (-q^{-1})^{n-1} q^2 a(f_r^2).$$

Proof. We use Lemma 4.7 to compute the action of $\mathbf{E}_{\delta - \alpha_r}$ on $f_r$. Since $e_i^r f_r = 0$ for $i \neq r$, the summation of $e_{j_1} \ldots e_{j_n}$ ($j_n \neq 0$) in the formula of $\mathbf{E}_{\delta - \alpha_r}$ in Lemma 4.7 is vanished when acting on $f_r$. Thus, we have

$$\mathbf{E}_{\delta - \alpha_r}(f_r) = (-q^{-1})^{n-1} (e_{r+1} \ldots e_{n-1} e_n)(e_{r-1} \ldots e_2 e_1) e_0 (f_r).$$

Then since the action of $e_i$ ($i \neq r$) is defined by the derivation (3.1), we have

$$e_i(f_r f_r^{\text{up}}(\beta)) = q^{-(\alpha_r, \alpha_i)} f_r (e_i' f_r^{\text{up}}(\beta)).$$

The assertion follows from Lemma 3.6. \hfill \Box

Proposition 4.11. The $l$-weight $\Psi = (\Psi_i(z))_{i \in I}$ of $\mathbf{1} \in U_q^-(\varpi_r)a$ is given by

$$\Psi_i(z) = \begin{cases} 1 - ac_r z & i = r, \\ 1 & i \neq r, \end{cases}$$

where $c_r = (-1)^{n+1} a(r)(q - q^{-1}) q^{-n-1}$.

Proof. By (4.6) and Corollary 4.8

$$\psi_{\delta - \alpha_r}^+ \mathbf{1} = \begin{cases} (-1)^{n+1} a(r)(q - q^{-1}) q^{-n-1} a \mathbf{1} & i = r, \\ 0 & i \neq r. \end{cases}$$

We claim that $\psi_{\delta - \alpha_r}^+ \mathbf{1} = 0$ for all $k \geq 2$. It suffice to show that $\mathbf{E}_{k\delta - \alpha_r} \mathbf{1} = 0$ for all $k \geq 2$ due to the relation (4.6). Then the proof is by induction on $k$. When $k = 2$, by Lemma 4.8...
we have
\[ E_{2\delta - \alpha_i} = \frac{-1}{q + q^{-1}} \left( E_{\delta - \alpha_i} E_{\delta - \alpha_i} - q^{-2} E_{\delta - \alpha_i} E_{\delta - \alpha_i} \right). \]

If \( i \neq r \), we have \( E_{2\delta - \alpha_i} 1 = 0 \) by Corollary 4.8. Suppose that \( i = r \). By the above formula, we have
\[
E_{2\delta - \alpha_i} 1 = \frac{-1}{q + q^{-1}} \left( E_{\delta - \alpha_i} E_{\delta - \alpha_i} - q^{-2} E_{\delta - \alpha_i} E_{\delta - \alpha_i} + q^{-2} E_{\delta - \alpha_i} E_{\delta - \alpha_i} \right) 1.
\]

By Corollaries 4.8 and 4.10
\[ E_{\delta - \alpha_i} E_{\delta - \alpha_i} 1 = q^{-2n+2} a^2 (f_r 1), \quad E_{\alpha_i} E_{\delta - \alpha_i} 1 = q^{-2n+4} (1 + q^{-2}) a^2 (f_r). \]

Putting them in (4.13), we conclude that \( E_{2\delta - \alpha_i} 1 = 0 \). For \( k > 2 \), the induction step follows directly from Lemma 4.13. This completes the proof. \( \square \)

4.2.2. Type \( D_n^{(1)} \). Let us compute the \( \ell \)-weight of \( 1 \in U_q^- (\varpi_r) \) in the case of type \( D_n^{(1)} \) with \( r \in \{ 1, n-1, n \} \) following the arguments in Section 3.2.

Lemma 4.12.

1. If \( r = 1 \), then
\[
E_{\delta - \alpha_1} = q^{-2n+4} (e_2 e_3 \ldots e_{n-2} e_{n-1}) (e_{n-1} e_{n-2} \ldots e_2) e_0 + \sum_{j_1, j_2, \ldots, j_{n-3}} C_{j_1, j_2, \ldots, j_{n-3}} (q) e_{j_1} e_{j_2} \ldots e_{j_{n-3}},
\]
where the sum is over the sequences \( (j_1, j_2, \ldots, j_{n-3}) \) such that \( \sum_{k=1}^{n-3} \alpha_{j_k} = \delta - \alpha_1 \) with \( j_{n-3} \neq 0 \) and \( C_{j_1, j_2, \ldots, j_{n-3}} (q) \in \pm q^{\pm (2n-3)} \).

2. If \( r = n \), then
\[
E_{\delta - \alpha_n} = q^{-2n+4} (e_{n-2} e_{n-3} \ldots e_2 e_1) (e_{n-1} e_{n-2} \ldots e_2) e_0 + \sum_{j_1, j_2, \ldots, j_{n-3}} C_{j_1, j_2, \ldots, j_{n-3}} (q) e_{j_1} e_{j_2} \ldots e_{j_{n-3}},
\]
where the sum is over the sequences \( (j_1, j_2, \ldots, j_{n-3}) \) such that \( \sum_{k=1}^{n-3} \alpha_{j_k} = \delta - \alpha_n \) with \( j_{n-3} \neq 0 \) and \( C_{j_1, j_2, \ldots, j_{n-3}} (q) \in \pm q^{\pm (2n-3)} \).

3. If \( r = n-1 \), then the formula of \( E_{\delta - \alpha_{n-1}} \) is obtained from the one of \( E_{\delta - \alpha_n} \) by replacing \( e_{n-1} \) with \( e_n \) in which the sum is over the sequences \( (j_1, j_2, \ldots, j_{n-3}) \) such that \( \sum_{k=1}^{n-3} \alpha_{j_k} = \delta - \alpha_{n-1} \) with \( j_{n-3} \neq 0 \) and \( C_{j_1, j_2, \ldots, j_{n-3}} (q) \in \pm q^{\pm (2n-3)} \).

Proof. Let \( \mathfrak{g}' \) be the simple Lie algebra of type \( D_{n-1} \) associated to \( I \setminus \{ 1 \} \) and let \( \varpi'_i \) be the \( i \)-th fundamental weight of \( \mathfrak{g}' \). Put \( t^{-1}_{\varpi'_i} \equiv w_r \tau \) for some \( \tau \in \mathfrak{T}' \). Note that the reduced
expression \( i \) of \( w_r^{-1} \) can be rewritten by

\[
(4.14) \quad i = \begin{cases} 
1 \cdot i' \cdot 1 & \text{if } r = 1, \\
(r, r - 2, \ldots, 1) \cdot i' & \text{if } r = n \text{ and } r \text{ is even,} \\
(r - 1, r - 2 \ldots, 1) \cdot i' & \text{if } r = n \text{ and } r \text{ is odd,}
\end{cases}
\]

where \( i' \) is the sequence obtained from the reduced expression of \( (w_r')^{-1} \) by shifting the index by 1 (cf. Example 4.14). Then the formulas in (1) and (2) follow from the same argument as in Lemma 4.6 and Lemma 4.7 by using the reduced expression of \( w_r \). By similar computation as in Example 4.13, one can check that \( \tau \) where \( \tau \) is the Dynkin diagram automorphism so that \( \alpha_0 \leftrightarrow \alpha_1 \) and \( \alpha_3 \leftrightarrow \alpha_4 \) (cf. [7]). By Lemma 4.2 and (4.15), it is easy to check that

\[
E_{\delta - \alpha_1} = T_0T_2T_3T_4T_2(e_0) = q^{-4}(e_2e_3)(e_2e_2)e_0 + \sum_{j_1, j_2, \ldots, j_5} C_{j_1, j_2, \ldots, j_5}(q)e_{j_1}e_{j_2} \ldots e_{j_5},
\]

where the sum is over the sequences \((j_1, j_2, \ldots, j_5)\) such that \( \sum_{k=1}^{5} a_{j_k} = \delta - \alpha_1 \) with \( j_5 \neq 0 \).

**Case 2.** \( r = 4 \). In this case, we have

\[
(4.16) \quad t_{\varpi_1} = \tau_4 s_4(s_2s_4)(s_1s_2s_4),
\]

where \( \tau_4 \) is the Dynkin diagram automorphism so that \( \alpha_0 \leftrightarrow \alpha_4 \) and \( \alpha_1 \leftrightarrow \alpha_3 \) (cf. [7]). By Lemma 4.2 and (4.16), it is easy to check that

\[
E_{\delta - \alpha_4} = T_0T_2T_3T_2(e_0) = q^{-4}(e_2e_1)(e_2e_2)e_0 + \sum_{j_1, j_2, \ldots, j_5} C_{j_1, j_2, \ldots, j_5}(q)e_{j_1}e_{j_2} \ldots e_{j_5},
\]

where the sum is over the sequences \((j_1, j_2, \ldots, j_5)\) such that \( \sum_{k=1}^{5} a_{j_k} = \delta - \alpha_4 \) with \( j_5 \neq 0 \).

**Example 4.14.** Let us consider the case of \( D_4^{(1)} \) with \( r = 5 \). In this case, we have

\[
t_{\varpi_5} = \tau_5 s_3(s_3s_5)(s_2s_3s_4)(s_1s_2s_3s_5),
\]

where \( \tau_5 \) is the Dynkin diagram automorphism so that \( \alpha_0 \leftrightarrow \alpha_5 \), \( \alpha_1 \leftrightarrow \alpha_4 \) and \( \alpha_2 \leftrightarrow \alpha_3 \). Note that the reduced expression of \( \tau_5^{-1}t_{\varpi_5} \) can be rewritten by

\[(s_4s_3s_2s_1)(s_5)(s_3s_4)(s_2s_3s_5).\]

By similar computation as in Example 4.13, one can check that

\[
T_5T_3T_4T_2T_3(e_5) = q^{-4}(e_3e_4)(e_2e_3)e_5 + \sum_{j_1, j_2, \ldots, j_5} C_{j_1, j_2, \ldots, j_5}(q)e_{j_1}e_{j_2} \ldots e_{j_5},
\]
where the sum is over the sequences \((j_1, j_2, \ldots, j_5)\) such that \(\sum_{k=1}^{5} a_{j_k} = \delta - \alpha_1 - \alpha_2\) with \(j_5 \neq 5\). Note that \(s_4 s_3 s_2 s_1\) sends \(\alpha_i\) into \(\alpha_{i-1}\) for \(i = 2, 3, 4\). By applying \(T_1 T_2 T_3 T_1\) to the above formula, we have

\[
T^{-1}_{w_5} s_5 (e_5) = q^{-6} (e_2 e_3 e_5) (e_1 e_2 e_3) (e_4) + \sum_{j_1, j_2, \ldots, j_7} C_{j_1, j_2, \ldots, j_7} (q) e_{j_1} e_{j_2} \cdots e_{j_7},
\]

where the sum is over the sequences \((j_1, j_2, \ldots, j_7)\) such that \(\sum_{k=1}^{7} a_{j_k} = \delta - \alpha_0\) with \(j_7 \neq 4\).

Finally, by applying \(\tau_5\), we have

\[
E_{\delta - \alpha_5} = q^{-6} (e_3 e_2 e_1) (e_4 e_3 e_2) e_0 + \sum_{j_1, \ldots, j_7} C_{j_1, j_2, \ldots, j_7} (q) e_{j_1} e_{j_2} \cdots e_{j_7},
\]

where the sum is over the sequences \((j_1, j_2, \ldots, j_7)\) such that \(\sum_{k=1}^{7} a_{j_k} = \delta - \alpha_5\) with \(j_7 \neq 0\).

Corollary 4.15.

(1) The action of \(E_{\delta - \alpha_r}\) on \(1 \in U_q^{-}(\varpi_r)\) is given as follows:

\[
E_{\delta - \alpha_r} 1 = \begin{cases} 
q^{-2n+4}(f_r) & \text{if } i = r, \\
0 & \text{if } i \neq r.
\end{cases}
\]

Moreover, we have \(E_{k \delta - \alpha_r} 1 = 0\) for \(i \in I\) and \(k \geq 2\).

(2) The action of \(E_{\delta - \alpha_r}\) on \(f_r \in U_q^{-}(\varpi_r)\) is given by

\[
E_{\delta - \alpha_r}(f_r) = q^{-2n+6}(f_r^2).
\]

Proof. The proofs of (1) and (2) are almost identical with the ones of Corollaries 4.8 and 4.10 respectively, by using Lemma 4.12.

Proposition 4.16. The \(\ell\)-weight \(\Psi = (\Psi_i(z))_{i \in I}\) of \(1 \in U_q^{-}(\varpi_r)\) is given by

\[
\Psi_i(z) = \begin{cases} 
1 - a c_r z & \text{if } i = r, \\
1 & \text{if } i \neq r,
\end{cases}
\]

where \(c_r = o(r)(q - q^{-1})q^{-2(n-1)}\).

Proof. By following the proof of Proposition 4.11 with Corollary 4.15

\[
E_{k \delta - \alpha_r} 1 = \begin{cases} 
q^{-2n+4}(f_r) & \text{if } i = r \text{ and } k = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Then the assertion follows from the relation (4.6). \(\square\)

4.3. Realization of \(L_{i,a}^+\). Now we are in a position to state the main result in this paper.

Theorem 4.17. For a minuscule \(\varpi_r\) and \(a \in k^\times\), we have

\[
U_q^{-}(\varpi_r) \cong L_{i,a}^+.
\]
as a $U_q(b)$-module, where $c_r$ is given by

\begin{equation}
(4.17) \quad c_r = \begin{cases} 
(-1)^{n+1} a(r)(q - q^{-1})q^{-(n+1)} & \text{for type } A_1^{(1)}, \\
a(r)(q - q^{-1})q^{-2(n-1)} & \text{for type } D_4^{(1)}. 
\end{cases}
\end{equation}

**Proof.** By definition (3.20), \textbf{1} is a weight vector and $e_i \mathbf{1} = 0$ for all $i \in I$. Let $M$ be the $U_q(b)$-submodule of $U_q^-(\varpi_r)_a$ generated by \textbf{1}. By Propositions 4.11 (see Example 4.13) for type $A_1^{(1)}$ and 4.16, $L_{r,a,c}^+$ is a maximal quotient of $M$. Note that the character of $U_q^-(\varpi_r)_a$ with respect to the action of $k_i$ for $i \in I$ is

\begin{equation}
(4.18) \quad \text{ch} U_q^-(\varpi_r)_a = \frac{1}{\prod_{\beta \in \Delta^+(\varpi_r)} (1 - e^{-\beta})},
\end{equation}

where $e^{-\beta}$ denotes an element of a basis of the group algebra of $\tilde{Q}$. On the other hand, the character of $L_{r,a,c}^+$ is also equal to (4.18) by [17] Theorem 7.5 and Lemma 7.6 (cf. [33]), which implies that

\[ L_{r,a,c}^+ \cong M \cong U_q^-(\varpi_r)_a. \]

This completes the proof. \qed

**Remark 4.18.**

(1) In Section 4.2 we have seen that

\[ E_{k^\delta - \alpha_i} \mathbf{1} = \begin{cases} 
q^M a(f_r) & \text{if } k = 1 \text{ and } i = r, \\
0 & \text{otherwise},
\end{cases} \]

where $M$ is given in Corollaries 4.8 and 4.15. Since the Drinfeld generator $x_{i,k}^-$ is given by

\[ x_{i,k}^- = o(i)^k C^{-k} k_i E_{k^\delta - \alpha_i}, \]

for $k > 0$ (e.g. see [5] Lemma 1.5), we see that the action of $x_{i,k}^-$ on \textbf{1} is 0 for $k \geq 2$. By Theorem 4.17, the description of the action of $x_{i,k}^-$ on \textbf{1} may be viewed as a special case of [17] Proposition 7.3 (cf. [11] Theorem 6.3).

(2) For an affine type, the limit construction of $L_{r,a,c}^+$ by the family of KR modules $W_{s,q_i^e}^{(r)}$ for $s \geq 1$ in [17] is valid when $n_r = 1$ due to the convergence of $0$-action (see [17] Proposition 7.4]), where $q_i = q^{(\alpha_i, \alpha_i)/2}$ and $n_r$ is the multiplicity of $\alpha_r$ in the maximal root of $\tilde{g}$. In particular, for types $ADE$, $n_r = 1$ if and only if $\varpi_r$ is minuscule.

### 4.4. Realization of $L_{r,a}^-$. Let us consider the realization of $L_{r,a}^-$. Let $a \in k^\times$. For $0 \leq i \leq n$, we define the $k$-linear operators $e_i$ and $k_i$ on $U_q^-(g)$ by

\begin{equation}
(4.19) \quad e_i(u) = \begin{cases} 
e_i(u) & \text{if } i \in I, \\
x_{0} u & \text{if } i = 0,
\end{cases} \quad k_i(u) = \begin{cases} 
q^{(\alpha_i, \beta)} u & \text{if } i \in I, \\
q^{-(\theta, \beta)} u & \text{if } i = 0,
\end{cases}
\end{equation}

for $u \in U_q^-(g)\beta$ ($\beta \in -Q_+$). Here $x_0$ is given in (3.17). The main difference from (3.20) is that the action of $e_0$ is given by the left multiplication of $x_0$ without $q^{-(\theta, \beta)}$. 
Example 4.19. Let us consider the $k$-linear operators $e_i$ and $k_i$ (4.19) in type $A_1^{(1)}$. We use the notation $S_{i,j}$ in Example 3.9.

First, we show that the $e_i$ and $k_j$ $(i,j \in \{0,1\})$ satisfy the defining relations of $U_q(b)$. In this example, we only verify $S_{0,1}(u) = 0$ for $u \in U_q^{-}(g)_\beta$. It is almost identical to show that $S_{0,1}(u) = 0$, and it is straightforward to verify the relations for $e_i$ and $k_j$. We leave the details to the reader.

By definition (4.19), we get
\[
\begin{align*}
& e_0^3 e_1(u) = a^3 f_1^3 e_1'(u), \\
& e_0^2 e_1 e_0(u) = a^3 f_1^2 u + a^3 q^{-2} f_1^3 e_1'(u), \\
& e_0 e_1 e_0^2(u) = a^3 (q^{-2} + 1) f_1^2 u + a^3 q^{-4} f_1^3 e_1'(u), \\
& e_1 e_0^3(u) = a^3 (q^{-4} + q^{-2} + 1) f_1^2 u + a^3 q^{-6} f_1^3 e_1'(u).
\end{align*}
\]

Thus, we have $S_{0,1}(u) = 0$.

Next, let us compute the highest $\ell$-weight of $1$ (cf. Example 4.4). By Lemma 4.3, one can check that
\[
E_{k-\alpha, 1} = (-1)^{k-1} q^{-2(k-1)}(q - q^{-1})^{k-1} a^k f,
\]
where $k > 0$. Then it follows from (4.0) that the eigenvalue of $\psi_{1,k}^+$ on $1$ is given by
\[
\psi_{1,k}^+ 1 = (-1)^k o(1)^k (q - q^{-1})^k q^{-2k} a^k 1.
\]
Hence, the $\ell$-weight $\Psi(z)$ of $1$ is given by
\[
\frac{1}{1 + o(1)(q - q^{-1})q^{-2}z}.
\]

We have seen that $U_q^{-}(g)$ is a $U_q(b)$-module under (4.19) and $L^{-}_{1,\alpha}$ is a subquotient of $U_q^{-}(g)$ up to a shift of spectral parameter.

In this subsection, we will show that for a minuscule $\varpi_r$, $U_q^{-}(\varpi_r)$ is a $U_q(b)$-module under the actions in (4.19) and it is isomorphic to $L^{-}_{r,1}$ up to a shift of spectral parameter.

Theorem 4.20. The operators $e_i$ and $k_i$ for $0 \leq i \leq n$ satisfy the defining relations of $U_q(b)$. This gives a representation $\rho^{-}_{r,a} : U_q(b) \rightarrow \text{End}_k(U_q^{-}(g))$ defined by $\rho^{-}_{r,a}(e_i) = e_i$ and $\rho^{-}_{r,a}(k_i) = k_i$ for $0 \leq i \leq n$.

Proof. The proof is almost identical with the one of Theorems 3.4 and 3.10 except for verifying quantum Serre relations of $e_0$ and $e_i$ for $i \in J_1$. Note that the case of type $A_1^{(1)}$ is done by Example 4.19 so we consider the remaining cases.

For $i \in J_1$ (see 5.14 below), we claim that
\[
\begin{align*}
(4.20) & \quad e_0 e_i^2 - (q + q^{-1}) e_i e_0 e_i + e_i^2 e_0 = 0, \\
(4.21) & \quad e_i e_0^2 - (q + q^{-1}) e_0 e_i e_0 + e_0^2 e_i = 0.
\end{align*}
\]
Let \( u \in U_q^- (g)_\beta \) be given. Put \( s = (\alpha, \beta) \) and \( t = (\theta, \beta) \). By Lemma 4.18 we have
\[
  e_0 e_1^2(u) = ax_0 e_1^2(u),
  e_1 e_0 e_1(u) = a e'_1(x_0) e'_1(u) + a q^{-1} x_0 e_1^2(u),
  e_1^2 e_0(u) = a(q + q^{-1}) e'_1(x_0) e'_1(u) + a q^{-2} x_0 e_1^2(u),
\]
which implies (4.20). Similarly, we have
\[
  e_1 e_0^2(u) = a^2 q^{-1}(q + q^{-1}) e'_1(x_0) x_0 u + a^2 q^{-2} x_0^2 e'_1(u),
  e_0 e_1 e_0(u) = a^2 q^{-1} e'_1(x_0) x_0 u + a^2 q^{-1} x_0^2 e'_1(u),
  e_0^2 e_1(u) = a^2 x_0^2 e'_1(u),
\]
which implies (4.21). \( \square \)

Remark 4.21. For type \( A_n^{(1)} \) with \( r = 1 \) or \( n \), one can check that for \( u \in U_q^- (g)_\beta \),
\[
e_r e_0(u) = q^{-1} e_0 e_r(u),
\]
which also implies the relations (4.20) and (4.21) (cf. Remark 3.11).

In Section 3.3 we have seen that \( U_q^- (\varpi_r) \) is invariant under \( e'_i \) for \( i \in I \). Note that \( U_q^- (\varpi_r) \)
is also invariant under \( e_0 \) in (4.19). By abuse of notation, we also denote by \( U_q^- (\varpi_r)_a \) the \( U_q (b) \)-module \( U_q^- (\varpi_r) \) defined by (4.19).

The second main result of this paper is as follows.

Theorem 4.22. For a minuscule \( \varpi_r \) and \( a \in k^\times \), we have
\[
U_q^- (\varpi_r)_a \cong L_{r,-ac},
\]
as a \( U_q (b) \)-module, where \( c_r \) is given as in (4.17).

Remark 4.23. It is shown in [17] that \( L_{r,a}^+ \) is a \( \sigma \)-twisted dual of \( L_{r,a}^- \) as a representation of the asymptotic algebra. Furthermore, the \( q \)-characters of \( L_{r,a}^- \) and \( L_{r,a}^+ \) are far from being the same (e.g. see [11] Remark 4.4) while the characters of \( L_{r,a}^- \) and \( L_{r,a}^+ \) with respect to \( k_i \) \((i \in I)\) are equal and independent of the choice of the parameter \( a \in k^\times \) [17] Theorem 6.4].

In this sense, Theorem 4.22 looks quite interesting. However, it is not clear to us yet how the \( U_q (b) \)-actions (4.20) and (4.19) on \( U_q^- (\varpi_r) \) result in these two different \( U_q (b) \)-module structures.

In the remainder of this section, we prove Theorem 4.22. First, we compute the action of \( E_{\varpi_k} \), \((i \in I, k > 0)\) on \( 1 \) as in Section 4.2 where \( 1 \) is the weight vector in \( U_q^- (\varpi_r)_a \) of weight 0. The computation gives the \( \ell \)-weight of \( 1 \) by (4.6), which enable us to apply the argument in the proof of Theorem 4.17.

We remark that the computation for the \( \ell \)-weight of \( 1 \) in this case is more involved than in Section 4.2 since there is no cancellation as in the previous case when computing the action of \( E_{\varpi_k} \) (see Example 4.19 for type \( A_1^{(1)} \)). Let us illustrate this in the case of type \( D_4^{(1)} \) with \( r = 4 \) before considering a general case.
Example 4.24. Let us recall Example 4.13. We claim that
\[ E_{k^3-\alpha,1} = \begin{cases} 
(-1)^{k-1}q^{-6k+2}(q - q^{-1})^{k-1}a^k(f_4) & \text{if } i = 4, \\
0 & \text{otherwise}. 
\end{cases} \]

We proceed by induction on $k$. Thanks to Lemma 3.6 and Lemma 4.3, it suffices to verify the case of $i = 4$ (cf. Example 4.5).

Step 1. By Lemma 3.6 and Example 4.13, we have
\[ E_{\delta-\alpha,1} = q^{-4}e_2e_1e_3e_01 = q^{-4}a(f_4), \]
\[ E_{\delta-\alpha}(f_4) = q^{-4}e_2e_1e_3e_0(f_4) = q^{-4}a(f_4^2). \]

Step 2. Suppose that the claim holds for $k > 1$. By Lemma 4.3 and the induction hypothesis together with Step 1, we have
\[ (q + q^{-1})E_{(k+1)^3-\alpha,1} = -C_k(q^2 - q^{-2})(f_4) = -C_k(q - q^{-1})(q + q^{-1})(f_4), \]
where $C_k = (-1)^{k-1}q^{-6(k+1)+2}(q - q^{-1})^{k-1}a^{k+1}$. Hence, the claim is proved.

By \(4.0\), we have
\[ \psi_{i,k}^+ = \begin{cases} 
(-1)^4o(4)q^{-6k}(q - q^{-1})^{k}a^k1 & \text{if } i = 4, \\
0 & \text{otherwise}, 
\end{cases} \]

for $k > 0$. Hence we obtain the $\ell$-weight $\Psi(z)$ of $1$ as follows:
\[ \Psi(z) = \left(1, 1, 1, \frac{1}{1 + ac_4z}\right), \]
where $c_4 = o(4)q^{-6}(q - q^{-1})$.

4.4.1. Type $A_n^{(1)}$. Let us consider the case of type $A_n^{(1)}$ ($n \geq 2$) with $r \in I$ (see Example 4.19 for type $A_1^{(1)}$).

Lemma 4.25. The action of $E_{\delta-\alpha,1}$ on $f_r \in U_q^{-}(\varpi_r)_{a}$ is given by
\[ E_{\delta-\alpha,1}(f_r) = \begin{cases} 
(-1)^{n+1}q^{-n}a(f_r^2) & \text{if } i = r, \\
0 & \text{if } i \neq r. 
\end{cases} \]

Proof. By Lemma 4.7 and 4.19, we have
\[ E_{\delta-\alpha,1}(f_r) = (-q^{-1})^{n+1}(e_{r+1} \ldots e_{n-1}e_n)(e_{r-1} \ldots e_{2}e_1)e_0(f_r) \\
= (-q^{-1})^{n+1}a(e_{r+1} \ldots e_{n-1}e_n)(e_{r-1} \ldots e_{2}e_1)(x_0f_r) \\
= (-q^{-1})^{n+1}a(e_{r+1} \ldots e_{n-1}e_n)(f_{n+1}a)(x_0f_r) \\
= (-q^{-1})^{n+1}a(f_r^2). \]

The case of $i \neq r$ follows from Lemma 3.6 (see also Remark 4.5) \(\square\).
Lemma 4.26. The action of $E_{k\delta - \alpha}$ on $1 \in U_q^-(\varpi_r)_a$ ($k \geq 1$) is given as follows:

$$E_{k\delta - \alpha}, 1 = \begin{cases} 
(-1)^{kn-1}q^{-k(n+1)+2}(q - q^{-1})^{k-1}a^k(f_r) & \text{if } i = r, \\
0 & \text{if } i \neq r.
\end{cases}$$

Proof. The proof is by induction on $k$. When $i \neq r$, by the same argument in the proof of Corollary 4.8 we have $E_{\delta - \alpha}, 1 = 0$, and then it follows from Lemma 4.3 that $E_{k\delta - \alpha}, 1 = 0$ for $k \geq 1$.

Let us consider the case of $i = r$. For $k = 1$, by Lemma 4.7

$$E_{\delta - \alpha}, 1 = (-1)^{n-1}(e_{r+1} \ldots e_{n-1}e_n)(e_{r-1} \ldots e_1)e_0 f_r$$

$$= (-1)^{n-1}a(e_{r+1} \ldots e_{n-1}e_n)(e_{r-1} \ldots e_1)(x_0)$$

$$= (-1)^{n-1}a(e_{r+1} \ldots e_{n-1}e_n)(F^u(p_r - e_{n+1}))$$

$$= (-1)^{n-1}a(f_r).$$

Hence we have

$$(4.22) \quad E_{\delta - \alpha}, 1 = (-1)^{n-1}q^{-n+1}a(f_r).$$

Assume that the formula of $E_{k\delta - \alpha}, 1$ holds for $k \geq 1$. Let us compute the followings to proceed by induction on $k$. By induction and (4.22), one can check that

$$E_{\delta - \alpha}, E_{\alpha}, E_{k\delta - \alpha}, 1 = C_k \cdot (f_r) = E_{k\delta - \alpha}, E_{\alpha}, E_{\delta - \alpha}, 1,$$

where $C_k = (-1)^{(k+1)n-2}q^{-(k+1)(n+1)+4}a^{k+1}(q - q^{-1})^{k-1}$. Also by induction and Lemma 4.25 one can check that

$$E_{\alpha}, E_{\delta - \alpha}, E_{k\delta - \alpha}, 1 = C_k \cdot (1 + q^{-2})(f_r).$$

By Lemma 4.3 and the above computation, we get

$$E_{(k+1)\delta - \alpha}, 1 = -C_kq^{-2}(q - q^{-1})(f_r).$$

Hence, we have the desired formula of $E_{(k+1)\delta - \alpha}, 1$. This completes the induction. □

4.4.2. Type $D_n^{(1)}$. Let us assume that $n \geq 4$ and $r \in \{1, n-1, n\}$.

Lemma 4.27. The action of $E_{\delta - \alpha}$ on $f_r \in U_q^-(\varpi_r)_a$ is given by

$$E_{\delta - \alpha}, (f_r) = \begin{cases} 
q^{-2n+4}a(f_r^2) & \text{if } i = r, \\
0 & \text{if } i \neq r.
\end{cases}$$

Proof. Using Lemma 4.12, the formula follows from the same argument as in the proof of Lemma 4.26. □

Lemma 4.28. The action of $E_{k\delta - \alpha}$ on $1 \in U_q^-(\varpi_r)_a$ ($k \geq 1$) is given as follows:

$$E_{k\delta - \alpha}, 1 = \begin{cases} 
(-1)^{k-1}q^{-k(n-1)+2}(q - q^{-1})^{k-1}a^k(f_r) & \text{if } i = r, \\
0 & \text{if } i \neq r.
\end{cases}$$
Proof. The proof is by induction on \( k \) as in Lemma 4.26. Then the necessary computation for induction is almost identical to the one in the proof of Lemma 4.26 by using Lemma 4.12 and Lemma 4.27. The details are left to the reader.

4.4.3. Proof of Theorem 4.23. The character of \( U_q^{-}(\varpi_r)_a \) is equal to (4.18) and it is also the one of \( L_{r,a}^{-} \) for any \( a \in k^\times \) (cf. [17] Theorem 6.4]). By (4.16), Lemma 4.26 and Lemma 4.28 we obtain the \( \ell \)-weight \( \Psi = (\Psi_i(z))_{i \in I} \) of \( U \) as follows:

\[
\Psi_i(z) = \begin{cases} 
\frac{1}{1 + ac_r z} & \text{if} \ i = r, \\
1 & \text{if} \ i \neq r.
\end{cases}
\]

Then we may apply the argument in the proof of Theorem 4.17 to conclude that \( U_q^{-}(\varpi_r)_a \cong L_{r,-ac_r}^{-} \) as a \( U_q(b) \)-module.

Remark 4.29. For types \( A_n^{(1)} \) (\( n \geq 1 \)) and \( D_n^{(1)} \) (\( n \geq 4 \)) with a minuscule \( \varpi_r \), we have proved that \( U_q^{-}(\varpi_r) \) has a \( U_q(b) \)-module structure by which it is isomorphic to \( L_{r,a}^{-} \) up to a spectral parameter shift. It is natural to ask whether our approach could be extended to the non-symmetric types or a non-minuscule \( \varpi_r \). For types \( B_n^{(1)}, C_n^{(1)}, D_n^{(1)} \) with a low rank \( n \) and exceptional types \( E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)} \) and \( G_2^{(1)} \) based on computer experiments, we observe that for any \( r \in I \), the ordinary character of \( U_q^{-}(\varpi_r) \) is given as follows:

\[
(4.23) \quad \frac{1}{\prod_{[\beta] \in \Delta^+(\varpi_r)} (1 - e^{-\beta})^{[\beta]_r}},
\]

where \([\beta]_r \in \mathbb{Z}_+\) is the multiplicity of \( \alpha_r \) in the sum \( \beta = \sum_{s \in I} [\beta]_s \alpha_s \). Indeed, the formula (4.23) is equal to the one of the limit of the normalized characters of KR modules \( W_{k, q_i}^{(r)} \) conjectured by Mukhin-Young [33] (cf. [17]), which is proved recently in [28] for all untwisted types except for the case of type \( E_8^{(1)} \) with \( r \notin \{4, 8\} \). Here \( q_i = q^{\langle \alpha_i, \alpha_i \rangle}/2 \). We expect that there exists a \( U_q(b) \)-module structure on \( U_q^{-}(\varpi_r) \) similar to (4.20) (resp. (4.19)) for all untwisted types and \( r \in I \), which is isomorphic to \( L_{r,a}^{+} \) (resp. \( L_{r,a}^{-} \)) up to a spectral parameter shift.

Remark 4.30. Recall that in [17], Hernandez and Jimbo construct a module over an asymptotic algebra of \( U_q(g) \) by taking limits on the actions of Drinfeld generators of KR modules, which gives the prefundamental representation \( L_{r,1}^{+} \) for any \( r \in I \). It is not yet clear to us how this limit construction is related to the realization of \( L_{r,a}^{+} \) in this paper, where the actions of \( e_i \) and \( k_i \) (\( i \in I \)) are defined more directly on the space \( U_q^{-}(\varpi_r) \), and it would be interesting to clarify the connection between these two realizations.

5. Combinatorial realization of \( L_{r,a}^{+} \)

In this section, as a byproduct of Theorem 4.17 we give a combinatorial realization of \( L_{r,a}^{+} \) in terms of the Lusztig data of the dual PBW vectors.

Remark 5.1. We may try to give a combinatorial realization of \( L_{r,a}^{-} \) from Theorem 4.22. However, it doesn’t seem to easy to describe the operator \( \mathfrak{e}_0 \) in (4.19) explicitly since the
commutation relations associated to $\chi_0$ may be complicated in general. For this reason, we consider the combinatorial realization of $L_{r,a}^+$ only.

5.1. Type $A_n^\dagger$. Let us identify the weight lattice $\hat{P}$ for $\hat{g}$ with the abelian group generated by $\epsilon_i$ for $1 \leq i \leq n+1$ subject to the relation $\epsilon_1 + \cdots + \epsilon_{n+1} = 0$ so that $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n$. Let $r \in I$ be given. In this case, we have $\Delta^+(w_r) = \{ \epsilon_i - \epsilon_j \mid 1 \leq i \leq r < j \leq n + 1 \}$.

Let
\[
\mathcal{C}(n,r) = \{ \mathbf{c} = (c_{i,j})_{1 \leq i \leq r < j \leq n+1} \mid c_{i,j} \in \mathbb{Z}_+ \},
\]
and let
\[
\mathcal{U}(n,r) = \bigoplus_{\mathbf{c} \in \mathcal{C}(n,r)} \mathbf{k}[\mathbf{c}]
\]
be the $\mathbf{k}$-vector space with basis $\{ [\mathbf{c}] \mid \mathbf{c} \in \mathcal{C}(n,r) \}$ parametrized by $\mathcal{C}(n,r)$. We may identify $\mathbf{c}$ with an $r \times (n - r + 1)$ matrix with non-negative integral entries, where the row index $i$ are decreasing from top to bottom. For example, when $n = 6$ and $r = 3$,
\[
\mathbf{c} = \begin{pmatrix} c_{3,4} & c_{3,5} & c_{3,6} & c_{3,7} \\ c_{2,4} & c_{2,5} & c_{2,6} & c_{2,7} \\ c_{1,4} & c_{1,5} & c_{1,6} & c_{1,7} \end{pmatrix} \in \mathcal{C}(6,3).
\]

By convention, if $\mathbf{c}$ is an $r \times (n - r + 1)$ matrix with integral entries but $\mathbf{c} \notin \mathcal{C}(n,r)$, then we set $[\mathbf{c}] = 0$ in $\mathcal{U}(n,r)$. For $1 \leq k \leq r < l \leq n + 1$, let $\mathbf{1}_{k,l} = (c_{i,j}^{k,l})$ such that $c_{i,j}^{k,l} = \delta_{ik}\delta_{jl}$.

**Remark 5.2.** Throughout this section, we use the notation $[m]_q$ ($m \in \mathbb{Z}_+$) to distinguish the notation $[\mathbf{c}]$ for $\mathbf{c} \in \mathcal{C}(n,r)$.

Fix $a \in \mathbf{k}^\times$. For $\mathbf{c} = (c_{i,j}) \in \mathcal{C}(n,r)$, we set
\[
\text{wt}(\mathbf{c}) = - \sum_{1 \leq i < r \leq n+1} c_{i,j}(\alpha_i + \cdots + \alpha_{j-1}) = - \sum_{1 \leq i < r < j \leq n+1} c_{i,j}(\epsilon_i - \epsilon_j),
\]
and define
\[
\begin{align*}
e_i[\mathbf{c}] &= \begin{cases} q^{\sum_{r < l \leq n+1} c_{r,l} + \sum_{r < l \leq n+1} c_{1,l}}[\mathbf{c} + 1_{1,n+1}] & \text{if } i = 0, \\
q^{\sum_{r < l \leq n+1} q^{\alpha_i,\alpha_{i-1}}(\mathbf{c})_{i,l}[\mathbf{c} + 1_{i+1,l} - 1_{i,l}]} & \text{if } 1 \leq i < r, \\
[\mathbf{c}]_{i,r+1}q[\mathbf{c} - 1_{r,r+1}] & \text{if } i = r, \\
\sum_{1 \leq k \leq r} q^{\beta_k,\epsilon_i}(\mathbf{c})_{k,i+1}[\mathbf{c} + 1_{k,i+1} - 1_{k,i+1}] & \text{if } r < i \leq n,
\end{cases}
\end{align*}
\]
(5.1)

\[
k_i[\mathbf{c}] = \begin{cases} q^{(\alpha_i, \text{wt}(\mathbf{c}))}[\mathbf{c}] & \text{if } i \in I, \\
q^{-\theta, \text{wt}(\mathbf{c})}[\mathbf{c}] & \text{if } i = 0,
\end{cases}
\]
(5.2)

where we understand $c_{k,l} = 0$ if $k > r$ or $l \leq r$ and
\[
a_{i,t}(\mathbf{c}) = \sum_{r < t \leq l} (c_{i+1,t} - c_{i,t}), \quad b_{k,t}(\mathbf{c}) = \sum_{k < t \leq r} (c_{t,i} - c_{t,i+1}).
\]
Lemma 5.3. Let $\beta = \epsilon_k - \epsilon_l$ given with $1 \leq k < l \leq n + 1$. For $i \in I$ and $c \geq 1$, we have
\[
e' \, F^\up(c \beta) = \begin{cases} [c]_q F^\up((c - 1)\beta) & \text{if } i = r \text{ and } (k, l) = (r, r + 1), \\ [c]_q F^\up(\beta - \alpha_i) F^\up((c - 1)\beta) & \text{if } i \neq r \text{ and } (k = i \text{ or } l = i + 1), \\ 0 & \text{otherwise,} \end{cases}
\]
where we understand $F^\up(0) = 1$.

Proof. We prove only the case when $i = r$ since the other cases are proved by similar computation as in (5.3) below and Lemma 3.6(2).

First, we consider the case of $\beta = \alpha_r$. We use induction on $c$. The initial step follows from Lemma 3.6(2). Suppose $c > 1$. By [25, Proposition 4.26], one can verify that
\[
e'(\alpha_r) = q^{-1} e' \, F^\up((c - 1)\alpha_r)) = q^{-1} (F^\up((c - 1)\alpha_r) + q^{-2} F^\up(\alpha_r) ([c - 1]_q F^\up((c - 2)\alpha_r)))
\]
Next, suppose that $\beta \neq \alpha_r$. If $c = 1$, then we have $e' \, F^\up(\beta) = 0$ by Lemma 3.6(2). Since $e' \, F^\up(\beta)$ is a derivation on $U_q(\mathfrak{g})$, we have $e' \, F^\up(c \beta) = 0$ by induction on $c$. $\square$

The following is a well-known identity which can be checked directly by using (3.7).

Lemma 5.4. For $t < s \leq r < i \leq n + 1$ and $c \geq 1$, we have
\[
F^\up(\epsilon_s - \epsilon_{i+1}) F^\up(\epsilon_t - \epsilon_i) = F^\up(\epsilon_t - \epsilon_i) F^\up(\epsilon_s - \epsilon_{i+1}),
\]
\[
F^\up(\epsilon_t - \epsilon_i)) F^\up(\epsilon_s - \epsilon_i) = q^{-c} F^\up(\epsilon_s - \epsilon_i) F^\up(c(\epsilon_t - \epsilon_i)). \quad \square
\]

Theorem 5.5. For $a \in \mathbb{k}^\times$ and $r \in I$, $\mathcal{U}(n, r)_a$ becomes a $U_q(\mathfrak{b})$-module with respect to (5.1) and (5.2). Furthermore, we have
\[
\mathcal{U}(n, r)_a \cong U_q^{-}(\varpi_r)_a \cong L^+_c(\varpi_r),
\]
where $c_r$ is given as in (4.11).

Proof. Let us take the reduced expression of $w_r$ in Remark 3.3 whose convex order $<$ on $\Delta^+(w_r)$ is given by
\[
\epsilon_i < \epsilon_j < \epsilon_k < \epsilon_l \text{ if and only if } (i > k) \text{ or } (i = k \text{ and } j < l),
\]
for $\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l \in \Delta^+(w_r)$. Let $\phi$ be the $\mathbb{k}$-linear isomorphism given by
\[
\phi : \mathcal{U}(n, r)_a \rightarrow U_q^{-}(\varpi_r)_a, \quad [c] \mapsto \prod_{i < j} F^\up(\epsilon_i, \epsilon_j - \epsilon_j).
\]
is the ordered product with respect to $\prec$.

We may take a reduced expression of $w_r$ such that the resulting convex order $\prec'$ on $\Delta^+(w_r)$ is given by

$$
\epsilon_i - \epsilon_j \prec' \epsilon_k - \epsilon_l \text{ if and only if } (j < l) \text{ or } (i > k \text{ and } j = l),
$$

for $\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l \in \Delta^+(w_r)$, and define a $k$-linear isomorphism $\phi'$ as in (5.4) with respect to $\prec'$.

We claim that $\phi = \phi'$. Let us consider the following $r \times (n - r + 1)$ matrix.

$$
\begin{pmatrix}
  r & r + 1 & \ldots & n \\
  r - 1 & r & \ldots & n - 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 2 & \ldots & r
\end{pmatrix}
$$

If we read the entries of the above matrix row by row (resp. column by column) from top to bottom and then from left to right in each row (resp. from left to right and then from top to bottom in each column), then the resulting sequence is associated to the reduced expression of $w_r$ whose convex order is $\prec (5.4)$ (resp. $\prec' (5.6)$).

One can check that these reduced expressions are equal up to 2-braid moves (Recall that a 2-braid move means $ij = ji$ for $i, j \in I$ such that $|i - j| > 1$). Hence, we have by Lemma 5.3.

$$
\prod_{\prec} F_{c_{i,j}}^{\operatorname{up}}(\epsilon_i - \epsilon_j) = \prod_{\prec'} F_{c_{i,j}}^{\operatorname{up}}(\epsilon_i - \epsilon_j),
$$

where $\epsilon_i - \epsilon_j \in \Delta^+(w_r)$. Hence, the claim is proved.

Now, we show that $k_i$ and $k_i$ on $U(n, r)_a$ coincide with $e_i$ and $k_i$ on $U_\phi^-(\varpi_r)$ under $\phi$. It is clear that $\phi(k_i([c])) = k_i \phi([c])$ for $0 \leq i \leq n$. We also have $\phi(e_0([c])] = e_0 \phi([c])]$ by definition and [25, Proposition 4.26(2)], and $\phi(e_i([c])] = e_i \phi([c])]$ by Lemma 5.3. So it remains to check that $\phi(e_i([c])] = e_i \phi([c])]$ for the case when $i \neq 0, r$. Let $[c] \in U(n, r)$ be given with $c = (c_{i,j}) \in \mathcal{C}(n, r)$.

Case 1. Suppose that $1 \leq i < r$. In this case, we use the $k$-linear isomorphism $\phi'$ associated to $\prec'(5.6)$. Let $y_0 = \phi'([c])]$ and write

$$
y_0 = x_1 \cdot F^{\operatorname{up}}(c_{i+1,r+1}(\epsilon_{i+1} - \epsilon_{r+1}))F^{\operatorname{up}}(c_{i,r+1}(\epsilon_i - \epsilon_{r+1})) \cdot y_1,
$$

where $x_1$ is the product of $F^{\operatorname{up}}(c_{u,v}(\epsilon_u - \epsilon_v))$ such that $\epsilon_u - \epsilon_v \prec' \epsilon_{i+1} - \epsilon_{r+1}$, and $y_1$ is the product of the other root vectors. Inductively, we write $y_k$ as follows:

$$
y_k = x_{k+1} \cdot F^{\operatorname{up}}(c_{i+1,r+k+1}(\epsilon_{i+1} - \epsilon_{r+k+1}))F^{\operatorname{up}}(c_{i,r+k+1}(\epsilon_i - \epsilon_{r+k+1})) \cdot y_{k+1},
$$

where $x_{k+1}$ is the product of $F^{\operatorname{up}}(c_{u,v}(\epsilon_u - \epsilon_v))$ such that $\epsilon_u - \epsilon_v \prec' \epsilon_{i+1} - \epsilon_{r+1}$, and $y_{k+1}$ is the product of the other root vectors in $y_k$. By Lemma 5.3 and [25, Proposition 4.26],
\[ \epsilon_i' y_0 = q^{c_{i+1} + 1} x_1 \cdot F^{\uparrow}(c_{i+1, r+1}(\epsilon_{i+1} - \epsilon_{r+1})) e_i' \left( F^{\downarrow}(c_{i, r+1}(\epsilon_i - \epsilon_{r+1})) \cdot y_1 \right) \]
\[ = q^{c_{i+1} + 1} x_1 \cdot F^{\uparrow}(c_{i+1, r+1}(\epsilon_{i+1} - \epsilon_{r+1})) \]
\[ \cdot \left( [c_{i, r+1}] q F^{\uparrow}(\epsilon_{i+1} - \epsilon_{r+1}) F^{\downarrow}(c_{i, r+1} - 1)(\epsilon_i - \epsilon_{r+1}) \cdot y_1 \right) \]
\[ + q^{-c_i, r+1} F^{\downarrow}(c_{i, r+1}(\epsilon_i - \epsilon_{r+1})) \cdot e_i'(y_1) \]
\[ = [c_{i, r+1}] q x_1 \cdot F^{\uparrow}((c_{i+1, r+1} + 1)(\epsilon_{i+1} - \epsilon_{r+1}))(c_{i, r+1} - 1)(\epsilon_i - \epsilon_{r+1}) \cdot y_1 \]
\[ + q^{c_{i+1, r+1}-c_{i}, r+1} x_1 \cdot F^{\uparrow}(c_{i+1, r+1}(\epsilon_{i+1} - \epsilon_{r+1})) F^{\downarrow}(c_{i, r+1}(\epsilon_i - \epsilon_{r+1})) \cdot e_i'(y_1) \]

Similarly, we have
\[ \epsilon_i' y_k = [c_{i, r+k+1}] q x_{k+1} \cdot F^{\uparrow}((c_{i+1, r+k+1} + 1)(\epsilon_{i+1} - \epsilon_{r+k+1})) \]
\[ \cdot F^{\downarrow}((c_{i, r+k+1} - 1)(\epsilon_i - \epsilon_{r+k+1})) \cdot y_{k+1} \]
\[ + q^{c_{i+1, r+k+1}-c_{i, r+k+1}} x_{k+1} \cdot F^{\uparrow}(c_{i+1, r+k+1}(\epsilon_{i+1} - \epsilon_{r+k+1})) \]
\[ \cdot F^{\downarrow}(c_{i, r+k+1}(\epsilon_i - \epsilon_{r+k+1})) \cdot e_i'(y_{k+1}) \]

Combining (5.7) and (5.8), we conclude that \( \phi'(\epsilon_i[c]) = \epsilon_i \phi'(\epsilon_i[c]) \) for \( 1 \leq i < r \).

**Case 2.** Suppose that \( r < i \leq n \). We may apply the same arguments as in **Case 1** by using \( \phi \) or the linear order \( \prec (5.4) \). We leave the details to the reader. \qed

**Example 5.6.** Let
\[ c = \begin{pmatrix} c_{3, 4} & c_{3, 5} & c_{3, 6} & c_{3, 7} \\ c_{2, 4} & c_{2, 5} & c_{2, 6} & c_{2, 7} \\ c_{1, 4} & c_{1, 5} & c_{1, 6} & c_{1, 7} \end{pmatrix} \in \mathcal{C}(6, 3). \]

In (5.1), each coefficient of \([c']\) appearing in the expansion of \( e_i[c] \) is determined by \( c_{k,l} \)‘s which are relevant to \( \alpha_i \). For example, when \( i = 0 \), it is determined by \( c_{k,l} \)'s except for the gray ones:
\[ \begin{pmatrix} c_{3, 4} & c_{3, 5} & c_{3, 6} & c_{3, 7} \\ c_{2, 4} & c_{2, 5} & c_{2, 6} & c_{2, 7} \\ c_{1, 4} & c_{1, 5} & c_{1, 6} & c_{1, 7} \end{pmatrix} \]

Here \( \theta, \epsilon_i - \epsilon_j \neq 0 \) for \( i = 1 \) or \( j = 7 \). Then we have
\[ e_0[c] = a q^{\sum_{i<j<k} c_{i,j} + \sum_{i<j<k} c_{i,j}} \begin{pmatrix} c_{3, 4} & c_{3, 5} & c_{3, 6} & c_{3, 7} \\ c_{2, 4} & c_{2, 5} & c_{2, 6} & c_{2, 7} \\ c_{1, 4} & c_{1, 5} & c_{1, 6} & c_{1, 7} \end{pmatrix}. \]

As another example, when \( i = 2 \), the coefficients appearing in \( e_i[c] \) are determined by \( c_{k,l} \)'s except for the gray ones:
\[ \begin{pmatrix} c_{3, 4} & c_{3, 5} & c_{3, 6} & c_{3, 7} \\ c_{2, 4} & c_{2, 5} & c_{2, 6} & c_{2, 7} \\ c_{1, 4} & c_{1, 5} & c_{1, 6} & c_{1, 7} \end{pmatrix}. \]
Then we have
\[ e_2 [c] = \left[ c_{2,4} \right]_q \left[ \begin{array}{cccc} c_{1,4}, +1 & c_{3,5} & c_{2,6} & c_{1,7} \\ c_{2,4}, -1 & c_{2,5} & c_{2,6} & c_{1,7} \\ c_{1,4} & c_{1,5} & c_{1,6} & c_{1,7} \end{array} \right] + q^{a_{2,4}} \left[ c_{2,5} \right]_q \left[ \begin{array}{cccc} c_{1,4} & c_{3,5} & c_{2,6} & c_{1,7} \\ c_{2,4} & c_{2,5} & c_{2,6} & c_{1,7} \\ c_{1,4} & c_{1,5} & c_{1,6} & c_{1,7} \end{array} \right] + q^{a_{2,5}} \left[ c_{2,6} \right]_q \left[ \begin{array}{cccc} c_{1,4} & c_{3,5} & c_{2,6} & c_{1,7} \\ c_{2,4} & c_{2,5} & c_{2,6} & c_{1,7} \\ c_{1,4} & c_{1,5} & c_{1,6} & c_{1,7} \end{array} \right] \]

where \( a_{2,4}(c) = \sum_{3 \leq 3 \leq 1} (c_{3,t} - c_{2,t}) \) and \( a_{2,3}(c) = 0 \) by definition.

5.2. Type \( D^{(1)}_n \). Let us consider the case of type \( D^{(1)}_n \). Here we assume that \( r = 1 \) or \( n \). Note that the case of \( r = n - 1 \) is almost identical to the case of \( r = n \). We assume that the weight lattice of \( \hat{\mathfrak{g}} \) is \( \hat{P} = \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i \), with a symmetric bilinear form \( (, ) \) such that \( (\epsilon_i, \epsilon_j) = \delta_{i,j} \) for \( 1 \leq i, j \leq n \), so that \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) (for \( 1 \leq i \leq n - 1 \)) and \( \alpha_n = \epsilon_{n-1} + \epsilon_n \).

Then we have
\[ \Delta^+(w_r) \begin{cases} \{ \epsilon_i - \epsilon_i | 1 < i \leq n \} \cup \{ \epsilon_1 + \epsilon_i | 1 < i \leq n \} & \text{if } r = 1, \\ \{ \epsilon_i + \epsilon_j | 1 \leq i < j \leq n \} & \text{if } r = n. \end{cases} \]

For each \( \beta \in \Delta^+(w_r) \), we will denote the multiplicity of \( \beta \) in a PBW vector by
\[ \left\{ \begin{array}{ll} c_{i,j} & \text{if } \beta = \epsilon_i - \epsilon_j \text{ for } 1 \leq i < j \leq n, \\ \overline{c}_{i,j} & \text{if } \beta = \epsilon_i + \epsilon_j \text{ for } 1 \leq i < j \leq n, \end{array} \right. \]

where we assume that \( \overline{f} < \overline{f} \) for \( 1 \leq i < j \leq n \).

Set
\[ \mathcal{C}(n, r) = \left\{ \left[ c \right] = \left( (c_{1,i})_{1 < i \leq n}, (c_{c_i})_{1 < i \leq n} \right) \left| \begin{array}{l} c_{1,i}, c_{c_i} \in \mathbb{Z}_+ \\ 1 \leq i \leq n \end{array} \right\} \right. \text{ if } r = 1, \\ \left. \left\{ \left[ c \right] = \left( (c_{c_i})_{1 < i < j \leq n} \right) \left| \begin{array}{l} c_{c_i} \in \mathbb{Z}_+ \\ 1 \leq i < j \leq n \end{array} \right\} \right. \text{ if } r = n, \right. \]

and let
\[ \mathcal{U}(n, r) = \bigoplus_{\left[ c \right] \in \mathcal{C}(n, r)} k[c] \]

be the \( k \)-space with basis \( \left\{ \left[ c \right] \mid c \in \mathcal{C}(n, r) \right\} \) parametrized by \( \mathcal{C}(n, r) \), where we assume that \( [c] = 0 \) if \( c \notin \mathcal{C}(n, r) \). We may identify \( c \) with a strictly upper triangular \( n \times n \)-matrix (resp. a pair of \( 1 \times (n - 1) \)-matrices) if \( r = n \) (resp. \( r = 1 \)). For example, when \( n = 4 \) and \( r = 4 \),
\[ c = \left( \begin{array}{cccc} c_{c_1, c_1} & c_{c_1, c_2} & c_{c_1, c_3} \\ c_{c_2, c_1} & c_{c_2, c_2} & c_{c_2, c_3} \\ c_{c_3, c_1} & c_{c_3, c_2} & c_{c_3, c_3} \end{array} \right). \]

Fix \( a \in k^\vee \). Let us define the weight function and action of \( \epsilon_i \) on \( \mathcal{U}(r) \) as follows:

**Case 1.** \( r = 1 \). Put \( 1_{k,i} = \left( \left( c_{k,i} \right), \left( d_{k,i} \right) \right) \) with \( c_{a,b} = \delta_{a,b} \). For \( c = \left( (c_{1,i}), (c_{c_i}) \right) \in \mathcal{C}(n, 1) \), we set
\[ \text{wt}(c) = - \sum_{1 \leq i \leq n} \left( c_{1,i}(\epsilon_1 - \epsilon_i) + c_{c_i}(\epsilon_1 + \epsilon_i) \right), \]
and define

$$e_i[c] = \begin{cases} a q^{\sum_{c \in \Xi} (e_{c+1} + e_{c+1}) + e_{c+1}} [c + 1_{\Xi}] & \text{if } i = 0, \\ [c_1, 2] g [c - 1_{\Xi}] & \text{if } i = 1, \\ [c_1, i + 1] [c + 1_{i, i} - 1_{i, i + 1}] + q^{c_{1, i} - c_{1, i}} [c_{1, i}] g [c + 1_{i+1} - 1_{i, i}] & \text{if } 1 < i \leq n - 1, \\ [c_{n, r}] g [c + 1_{1, n} - 1_{n, r}] + q^{c_{1, n} - c_{1, r}} [c_{n, r}] g [c + 1_{n} - 1_{n, r}] & \text{if } i = n. \end{cases}$$

**Case 2.** $r = n$. Put $1_{k, i} = \left( \frac{k}{i}, \frac{i}{j} \right)$ with $c_{a, b} = \delta_{a, b}$. For $c = (c_{\alpha, \beta}) \in \mathcal{C}(n, n)$, we set

$$\text{wt}(c) = - \sum_{1 \leq i < j \leq n} c_{\alpha, \beta}(c_i + c_j),$$

and define

$$e_i[c] = \begin{cases} a q^{\sum_{c \in \Xi} \sum_{c} e_{c+1}} [c + 1_{\Xi}] + \sum_{1 \leq k < l \leq n} q^{a_k, l} (c_{\alpha, \beta})_q \left( \delta_{k, i + 1} [c + 1_{i+1} - 1_{i, i + 1}] + \delta_{k, i} [c + 1_{i+1} - 1_{i, i + 1}] \right) & \text{if } i = 0, \\ \sum_{1 \leq k < l \leq n} a_k, l (c_{\alpha, \beta})_q \left( \delta_{k, i + 1} [c + 1_{i+1} - 1_{i, i + 1}] + \delta_{k, i} [c + 1_{i+1} - 1_{i, i + 1}] \right) & \text{if } 1 \leq i < n, \\ [c_{n, r}] g [c - 1_{n, r}] & \text{if } i = n, \end{cases}$$

where $a_k, l (c_{\alpha, \beta})$ is given by

$$a_k, l (c_{\alpha, \beta}) = \begin{cases} \sum_{l \leq p} (c_{\alpha, \beta} - c_{\alpha, \beta}) & \text{if } k = i, \\ \sum_{p = i + 1}^n (c_{\alpha, \beta} - c_{\alpha, \beta}) + \sum_{q < l} (c_{\alpha, \beta} - c_{\alpha, \beta}) & \text{if } l = i. \end{cases}$$

Finally, we define the action of $k_i$ on $\mathfrak{U}(n, r)$ by

$$k_i[c] = \begin{cases} q^{\text{wt}(c)} [c] & \text{if } i \in I, \\ q^{-\text{wt}(c)} [c] & \text{if } i = 0. \end{cases}$$

**Example 5.7.** Let us consider the case of type $D_r^{(1)}$ with $r = 1, 4$ and $i = 2$.

1. Let $r = 1$. For $c = ((c_{1, 2}, c_{1, 3}, c_{1, 4}), (c_{1, 2}, c_{1, 3}, c_{1, 4})) \in \mathfrak{U}(1, 4)$, the action of $e_2$ is given by

$$e_2[c] = [c_{1, 3}] [\left( (c_{1, 2} + 1, c_{1, 3} - 1, c_{1, 4}), (c_{1, 2}, c_{1, 3}, c_{1, 4}) \right)] + q^{c_{1, 2} - c_{1, 3}} [c_{1, 3}] [\left( (c_{1, 2} + 1, c_{1, 3} - 1, c_{1, 4}), (c_{1, 2}, c_{1, 3}, c_{1, 4}) \right)].$$

2. Let $r = 4$ and

$$c = \left( \begin{array}{ccc} c_{1, 2} & c_{1, 3} & c_{1, 4} \\ c_{1, 2} & c_{1, 3} & c_{1, 4} \\ c_{1, 2} & c_{1, 3} & c_{1, 4} \end{array} \right) \in \mathfrak{U}(4, 4).$$
In this case, the action of $e_2$ is determined by $\sigma_{\mathcal{T}}$'s except for the gray ones:

$$
\left(
\begin{array}{ccc}
\sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} \\
\sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} \\
\sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} \\
\end{array}
\right).
$$

We have

$$a_{2,4}^2(c) = 0, \quad a_{1,2}^2(c) = \sigma_{\mathcal{T}} - \sigma_{\mathcal{T}}.$$

Hence, the action $e_2[c]$ is given by

$$e_2[c] = [c_{\mathcal{T}}]\begin{pmatrix}
\sigma_{\mathcal{T}} + 1 & \sigma_{\mathcal{T}} - 1 & \sigma_{\mathcal{T}} \\
\sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} \\
\sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} \\
\end{pmatrix} + q^{-\mathcal{T} - \mathcal{T}}[c_{\mathcal{T}}]\begin{pmatrix}
\sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} + 1 \\
\sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} \\
\sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} & \sigma_{\mathcal{T}} \\
\end{pmatrix}.$$

We denote by $\mathcal{U}(n,r)_{\mathcal{a}}$ the space $\mathcal{U}(n,r)$ with the above actions of $e_i$ and $k_i$ associated with $a \in \mathbb{k}^\times$.

**Theorem 5.8.** For $a \in \mathbb{k}^\times$ and $r \in \{1, n\}$, $\mathcal{U}(n,r)_{\mathcal{a}}$ becomes a $U_q(\mathfrak{b})$-module with respect to (5.9), (5.10), and (5.11). Furthermore, we have

$$\mathcal{U}(n,r)_{\mathcal{a}} \cong U_q^-(\mathfrak{w}_r)_{\mathcal{a}} \cong L^+_{r,\mathcal{a},c_r},$$

where $c_r$ is given as in (5.17).

**Proof.** The proof is almost identical to the one of Theorem 5.5. Let us explain briefly the proof of Theorem 5.8. First, we define a $\mathbb{k}$-linear map from $\mathcal{U}(n,r)_{\mathcal{a}}$ to $U_q^-(\mathfrak{w}_r)_{\mathcal{a}}$ as in (5.5).

**Case 1.** $r = 1$. Let us take the reduced expression of $w_1$ in Remark 5.3. In this case, the convex order $<$ of $w_1$ is given by

$$\epsilon_i - \epsilon_i < \epsilon_i + \epsilon_j,$$

$$\epsilon_i - \epsilon_j < \epsilon_i - \epsilon_k \iff j < k,$$

$$\epsilon_i + \epsilon_j < \epsilon_i + \epsilon_k \iff j > k,$$

for $i,j,k \in I$. Then we define the $\mathbb{k}$-linear map $\phi_1$ from $\mathcal{U}(n,1)_{\mathcal{a}}$ to $U_q^-(\mathfrak{w}_1)_{\mathcal{a}}$ by using the above convex order $<$ as in (5.5).

**Case 2.** $r = n$. Let us take the reduced expression of $w_n$ in Remark 5.3. Then the convex order $<$ of $w_n$ is given by

$$\epsilon_i + \epsilon_j < \epsilon_k + \epsilon_l \iff (j > l) \text{ or } (j = l, i > k)$$

for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$. Also, we may take a reduced expression of $w_n$ so that the corresponding convex order denoted by $<'$ is given by

$$\epsilon_i + \epsilon_j <' \epsilon_k + \epsilon_l \iff (i > k) \text{ or } (i = k, j > l)$$

for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$. We define the $\mathbb{k}$-linear maps $\phi_n$ and $\phi'_n$ from $\mathcal{U}(n,n)_{\mathcal{a}}$ to $U_q^-(\mathfrak{w}_n)_{\mathcal{a}}$ by using the above convex orders (5.12) and (5.13) as in (5.5), respectively.
We show that $\phi_n = \phi'_n$ as in type $A_{kn1}$, Let us consider a upper triangular $(n-1) \times (n-1)$ matrix $D_n$ such that its diagonal is given by $(n, n-1, n, n-1, \cdots)$ and the remaining non-zero part of $D_n$ is of the form

$\begin{pmatrix} * & n-2 & n-3 & n-4 & \cdots & 1 \\ * & n-2 & n-3 & \vdots & \vdots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & n-2 & n-3 & \cdots & * \\ * & n-2 & \cdots & \cdots & \cdots & \cdots \\ \end{pmatrix}$

where *'s denote the diagonal entries. For example, for type $D_{5(1)}$, the matrix $D_5$ is given by

$\begin{pmatrix} 5 & 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 3 \\ 4 \\ \end{pmatrix}$

If we read the entries of $D_n$ row by row (resp. column by column) from top to bottom and then from left to right in each row (resp. from left to right and then from top to bottom in each column), then the resulting sequence is associated to the reduced expression of $w_n$ whose convex order is $< (5.12)$ (resp. $< (5.13)$). It is straightforward to check that these reduced expressions are equal up to 2-braid moves. Thus, by Lemma 5.4, we prove the claim.

Now, we apply the argument in the proof of Theorem 5.5 to the $k$-linear maps $\phi_r$ $(r = 1, n)$. In particular, we should remark that in the case of $r = n$, the computation to show that $\phi_n(e_k[c]) = e_k \phi_n(c)$ for $k \neq 0, n$ is more involved. Let us explain this case in more details.

Let $r = n$ and $N = n^2 - n$. For $c = (\sigma^{-1}_{i,j})_{1 \leq i < j \leq n} \in \mathbb{Z}_N^N$, put

$F(c) = \prod_{i} F^{up}(\sigma^{-1}_{i,j}(\epsilon_i + \epsilon_j)) \in U_q^-(\mathbb{A}_n)$.

Suppose that $k \in I \setminus \{n\}$. Then $F(c)$ is rewritten as follows:

1. For $k + 1 < j \leq n$, we define

$x_j = y_j \cdot \left( \prod_{i = k \text{ or } k + 1}^{i \neq j} F^{up}(\sigma^{-1}_{i,j}(\epsilon_i + \epsilon_j)) \right) \cdot z_j,$

where $y_j$ and $z_j$ are given by

$y_j = \prod_{i < k + 1} F^{up}(\sigma^{-1}_{i,j}(\epsilon_i + \epsilon_j)), \quad z_j = \prod_{i > k} F^{up}(\sigma^{-1}_{i,j}(\epsilon_i + \epsilon_j)).$

Here $y_j$ and $z_j$ are assumed to be 1, if they are not defined.
(2) For $1 \leq i \leq k + 1$, we define
\[ x_i' = y_i' \cdot \left( \prod_{j = k \text{ or } k + 1}^{\infty} F^{\up}(c_{j,1}(\epsilon_i + \epsilon_j)) \right), \]
where $y_i'$ is given by
\[ y_i' = \prod_{j \neq k, k + 1}^{\infty} F^{\up}(c_{j,1}(\epsilon_i + \epsilon_j)). \]

Here if $y_i'$ and the term in $x_i'$ except for $y_i'$ are not defined, then we assume that they are equal to 1.

From (1) and (2), we rewrite $F(c)$ by
\[ F(c) = (x_n \cdot x_{n-1} \cdot \ldots \cdot x_{k+2}) \cdot (x'_{k+1} x'_{k'} \cdot \ldots \cdot x'_{1}) \]
\[ = (x_n \cdot x_{n-1} \cdot \ldots \cdot x_{k+2}) \cdot (x''_{k+1} x''_{k'} \cdot \ldots \cdot x''_{1}), \]
where $x''_i$ is obtained from $x'_i$ by rewriting with respect to the convex order $\prec'$ in (5.13) or using the map $\phi'_n$. Then we have
\[ e_k F(c) = e'_k(F(c)) \]
\[ = e'_i(x_n \cdot x_{n-1} \cdot \ldots \cdot x_{k+2}) \cdot x''_{k+1} x''_{k'} \cdot \ldots \cdot x''_{1} \]
\[ + q^{\sum_{p=k+2}^{n}(c_{p,1}-c_{p,2})} (x_n \cdot x_{n-1} \cdot \ldots \cdot x_{k+2}) \cdot e'_i(x''_{k+1} x''_{k'} \cdot \ldots \cdot x''_{1}). \]

Finally, we compute the above terms containing $e'_i$ as in Case 1 in the proof of Theorem 5.5 and one can check that $\phi_n(e_k[c]) = e_k \phi_n([c])$ for $k \in I \setminus \{n\}$. We leave the details to the reader. □

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