Comment on: “Rashba coupling induced by Lorentz symmetry breaking effects”. Ann. Phys. (Berlin) 526, 187 (2013)

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Abstract

We analyze the results of a paper on “the arising of a Rashba-like coupling, a Zeeman-like term and a Darwin-like term induced by Lorentz symmetry breaking effects in the non-relativistic quantum dynamics of a spin-1/2 neutral particle interacting with external fields”. We show that the authors did not obtain the spectrum of the eigenvalue equation but only one eigenvalue for a specific relationship between model parameters. In particular, the existence of allowed cyclotron frequencies conjectured by the authors is a mere artifact of the truncation condition used to obtain exact solutions to the radial eigenvalue equation.

In a paper published in this journal Bakke and Belich study “the arising of a Rashba-like coupling, a Zeeman-like term and a Darwin-like term induced by Lorentz symmetry breaking effects in the non-relativistic quantum dynamics of a spin-1/2 neutral particle interacting with external fields”. They derive an eigenvalue equation for the radial coordinate and solve it exactly by means of the Frobenius method. This approach leads to a three-term recurrence relation

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that enables the authors to truncate the series and obtain eigenfunctions with polynomial factors. They claim to have obtained the bound-state eigenvalues and eigenfunctions of the model. Since the truncation condition requires that a model parameter depends on the quantum numbers they conclude that not all the cyclotron frequencies are allowed. In this Comment we analyze the effect of the truncation condition used by the authors on the physical conclusions that they derive in their paper.

It is not our purpose to discuss the validity of the model but the way in which the authors solve the eigenvalue equation. For this reason we do not show the main equations displayed in their paper and restrict ourselves to what we consider relevant. We focus in the eigenvalue equation

\[ R''_s + \frac{1}{\xi} R'_s - \frac{\delta^2_s}{\xi^2} R_s - \frac{\alpha}{(2ma_2)^{3/4}} \xi R_s - \xi^2 R_s - \frac{\tau_s}{(2ma_2)^{1/4} \xi} R_s + WR_s = 0, \]

\[ \delta^2_s = \gamma^2_s + 2ma_1, \quad \gamma_s = l + \frac{1}{2}(1 - s), \quad \tau_s = \frac{gb\lambda}{4m} \gamma_s + \frac{gb\lambda}{8m}, \quad \alpha = gb\lambda m, \]

\[ \zeta = 2m (E - V_0), \quad W = \frac{\zeta}{(2ma_2)^{1/2}}, \quad a_2 = m\omega^2 \]

where \( l = 0, \pm 1, \pm 2, \ldots \), \( s = \pm 1 \), \( m \) is a mass, \( E \) the energy, \( a_1, a_2, V_0 \) parameters of the model potential \( V(\rho) = a_1 \rho^{-2} + a_2 \rho^2 + V_0 \) and \( a, b, \) and \( \lambda \) are constants that appear in the interactions included in the model. The authors choose units such that \( \hbar = c = 1 \) although there are rigorous ways of deriving dimensionless equations, as well as the choice of natural units [2].

The authors’ eigenvalue equation (1) is a particular case of

\[ \hat{L}R = WR, \]

\[ \hat{L} = -\frac{d^2}{d\xi^2} - \frac{1}{\xi} \frac{d}{d\xi} + \frac{\gamma^2}{\xi^2} - \frac{a}{\xi} + b\xi + \xi^2, \]  

where \( \gamma, a \) and \( b \) are arbitrary real numbers that have nothing to do with the parameters in equation (1). Since the behaviour at origin is determined by the term \( \gamma \xi^{-2} \) and the behaviour at infinity by the harmonic term \( \xi^2 \) we conclude that there are bound states for all \( -\infty < a, b < \infty. \)
By means of the ansatz
\[ R(\xi) = \xi^{\gamma} e^{-\frac{b}{2} \xi^2} P(\xi), \quad P(\xi) = \sum_{j=0}^{\infty} c_j \xi^j, \]  
(3)
we derive a three-term recurrence relation for the coefficients \( c_j \):
\[ c_{j+2} = \frac{b (2|\gamma| + 2j + 3) - 2a}{2 (j + 2) (2|\gamma| + j + 2)} c_{j+1} + \frac{4 (2|\gamma| + 2j - W + 2) - b^2}{4 (j + 2) (2|\gamma| + j + 2)} c_j, \]
\[ j = -1, 0, 1, \ldots, c_{-1} = 0, \quad c_0 = 1. \]  
(4)

In order to obtain polynomial solutions the authors force the termination conditions
\[ W = W_{\gamma}^{(n)} = \frac{8 (|\gamma| + n + 1) - b^2}{4}, \quad c_{n+1} = 0, \quad n = 1, 2, \ldots. \]  
(5)
Clearly, under such conditions \( c_j = 0 \) for all \( j > n \) and \( P(\xi) \) reduces to a polynomial of degree \( n \). In this way, they obtain analytical expressions for the eigenvalues and the radial eigenfunctions \( R_{\gamma}^{(n)}(\xi) \). For the sake of clarity and generality we will use \( \gamma \) instead of \( l \) as an effective quantum number.

For example, when \( n = 1 \) we have
\[ W_{\gamma}^{(1)} = \frac{8 (|\gamma| + 2) - b^2}{4}, \quad a_{\gamma}^{(1,1)} = \frac{2b (|\gamma| + 1) - \sqrt{b^2 + 8 (2|\gamma| + 1)}}{2}, \]
\[ a_{\gamma}^{(1,2)} = \frac{2b (|\gamma| + 1) + \sqrt{b^2 + 8 (2|\gamma| + 1)}}{2}, \]  
(6)
or, alternatively,
\[ b_{\gamma}^{(1,1)} = \frac{2 \left[ 2a (|\gamma| + 1) - \sqrt{a^2 + 2 (2|\gamma| + 3) (2|\gamma| + 1)^2} \right]}{(2|\gamma| + 1) (2|\gamma| + 3)}, \]
\[ b_{\gamma}^{(1,2)} = \frac{2 \left[ 2a (|\gamma| + 1) + \sqrt{a^2 + 2 (2|\gamma| + 3) (2|\gamma| + 1)^2} \right]}{(2|\gamma| + 1) (2|\gamma| + 3)}. \]  
(7)

When \( n = 2 \) we obtain a cubic equation for either \( a \) or \( b \), for example,
\[ W_{\gamma}^{(2)} = \frac{8 (|\gamma| + 3) - b^2}{4}, \]
\[ 4a^3 - 6a^2b (2|\gamma| + 3) + a \left( b^2 (12\gamma^2 + 36|\gamma| + 23) - 16 (4|\gamma| + 3) \right) \]
\[ - b (2|\gamma| + 1) \left( b^2 (2|\gamma| + 3) (2|\gamma| + 5) - 16 (4|\gamma| + 7) \right) \]
\[ = 0, \]  
(8)
from which we obtain either \( a_\gamma^{(2)}(b) \) or \( b_\gamma^{(2)}(a) \); for example, \( a_\gamma^{(2,1)}(b) \), \( a_\gamma^{(2,2)}(b) \), \( a_\gamma^{(2,3)}(b) \). In the general case we will have \( n + 1 \) curves of the form \( a_\gamma^{(n,i)}(b) \), \( i = 1, 2, \ldots, n + 1 \), labelled in such a way that \( a_\gamma^{(n,i)}(b) < a_\gamma^{(n,i+1)}(b) \) and it can be proved that all the roots are real [4,5]. Notice that Bakke and Belich completely overlooked such multiplicity of roots.

It is obvious to anybody familiar with conditionally solvable (or quasi-solvable) quantum-mechanical models (see [4–7] and, in particular, the remarkable review [8] and references therein for more details) that the approach just described does not produce all the eigenvalues of the operator \( \hat{\mathcal{L}} \) for a given set of values of \( \gamma, a \) and \( b \) but only those states with a polynomial factor \( P(\xi) \). Each of the particular eigenvalues \( W_\gamma^{(n)} \), \( n = 1, 2, \ldots \) corresponds to a set of particular curves \( a_\gamma^{(n,i)}(b) \). On the other hand, if we solve the eigenvalue equation (2) in a proper way we obtain an infinite set of eigenvalues \( W_{\nu,\gamma}(a, b) \), \( \nu = 0, 1, 2, \ldots \) for each set of real values of \( a, b \) and \( \gamma \). The condition that determines these allowed values of \( W \) is that the corresponding radial eigenfunctions \( R(\xi) \) are square integrable

\[
\int_0^\infty |R(\xi)|^2 \xi \, d\xi < \infty. \tag{9}
\]

Notice that \( \nu \) is the actual radial quantum number (that labels the eigenvalues in increasing order of magnitude), whereas \( n \) is just a positive integer that labels some particular solutions with a polynomial factor \( P(\xi) \). In other words: \( n \) is a fictitious quantum number given by the truncation condition (5).

It should be obvious to everybody that the eigenvalue equation (2) supports bound states for all values of \( a \) and \( b \) and that the truncation condition (5) only yields some particular solutions. Besides, according to the Hellmann-Feynman theorem [3] the true eigenvalues \( W_{\nu,\gamma}(a, b) \) of equation (2) are decreasing functions of \( a \) and increasing functions of \( b \)

\[
\frac{\partial W}{\partial a} = -\left\langle \frac{1}{\xi} \right\rangle, \quad \frac{\partial W}{\partial b} = \left\langle \xi \right\rangle. \tag{10}
\]

Therefore, for a given value of \( b \) and sufficiently large values of \( a \) we expect negative values of \( W \) that the truncation condition fails to predict. It is not
difficult to prove, from straightforward scaling \[2\], that
\[
\lim_{a \to \infty} \frac{W_{\nu, \gamma}}{a^2} = -\frac{1}{(2\nu + 2|\gamma| + 1)^2}.
\] (11)

What is more, we can conjecture that the pairs \[a_{(n,i)}^{(n,i)}(b), W_{(n,i)}^{(n,i)}\], \(i = 1, 2, \ldots, n+1\) are points on the curves \(W_{\nu, \gamma}(a), \nu = 0, 1, \ldots, n\), respectively, for a given value of \(b\).

The eigenvalue equation \[2\] cannot be solved exactly in the general case. In order to obtain sufficiently accurate eigenvalues of the operator \(\hat{\mathcal{L}}\) we resort to the reliable Rayleigh-Ritz variational method that is well known to yield increasingly accurate upper bounds to all the eigenvalues \[9\] (and references therein). For simplicity we choose the basis set of non-orthogonal Gaussian functions \(u_j(\xi) = c^{\nu+j}_j e^{-\xi^2/2}, j = 0, 1, \ldots\) and test the accuracy of these results by means of the powerful Riccati-Padé method \[10\].

As a first example, we choose \(n = 2, \gamma = 0\) and \(b = 1\) so that \(W^{(2)}_0 = 5.75\) for the three models \(\left[ a_0^{(2,1)} = -1.940551663, b = 1 \right], \left[ a_0^{(2,2)} = 1.190016441, b = 1 \right]\) and \(\left[ a_0^{(2,3)} = 5.250535221, b = 1 \right]\). The first four eigenvalues for each of these models are

\[
\begin{align*}
d_0^{(2,1)} &\rightarrow \begin{cases} W_0,0 = 5.750000000 \\
W_1,0 = 9.894040660 \\
W_2,0 = 14.06831985 \\
W_3,0 = 18.24977457 \end{cases}, \\
d_0^{(2,2)} &\rightarrow \begin{cases} W_0,0 = -0.1664353619 \\
W_1,0 = 5.750000000 \\
W_2,0 = 10.52307155 \\
W_3,0 = 15.06421047 \end{cases}, \\
d_0^{(2,3)} &\rightarrow \begin{cases} W_0,0 = -27.32460313 \\
W_1,0 = -0.5108147276 \\
W_2,0 = 5.750000000 \\
W_3,0 = 10.90599171 \end{cases}.
\end{align*}
\]

We appreciate that the eigenvalue \(W_0^{(2)} = 5.75\) coming from the truncation con-
dition (5) is the lowest eigenvalue of the first model, the second lowest eigenvalue of the second model and the third lowest eigenvalue for the third model. The truncation condition misses all the other eigenvalues for each of those models and for this reason it cannot provide the spectrum of the physical model for any set of values of $\gamma$, $a$ and $b$ as suggested by Bakke and Belich.

In the results shown above we have chosen model parameters on the curves $a_0^{(2,i)}(b)$. In what follows we consider the case $a = 2$, $b = 1$ that does not belong to any of those curves. For this set of model parameters the first five eigenvalues are $W_{0,0} = -3.230518994$, $W_{1,0} = 4.510929109$, $W_{2,0} = 9.532275968$, $W_{3,0} = 14.19728140$ and $W_{4,0} = 18.70978427$. As said above: there are square-integrable solutions (actual bound states) for any set of real values of $a$, $b$ and $\gamma$. The obvious conclusion is that the dependence of the frequency $\omega$ on the quantum numbers $n$, $l$, $s$ ($\omega_{n,l,s}$) and the consequent allowed cyclotron frequencies conjectured by Bakke and Belich [1] are just artifacts of the truncation condition (5). Such claims are nonsensical from a physical point of view. To be clearer, since there are bound states for all $a$ and $b$ then there are bound states for all $\omega$.

Figure 1 shows some eigenvalues $W^{(n)}_0 (b = 1)$ given by the truncation condition (red points) and the lowest variational eigenvalues $W_{\nu,0}(a,1)$ (blue lines). We clearly appreciate that the truncation condition (5) yields only some particular points of the curves $W_{\nu,0}(a,1)$. Therefore, any conclusion drawn from $W^{(n)}_\nu$ is meaningless unless one is able to organize these eigenvalues properly [5–7]. Bakke and Belich [1] completely overlooked this fact. The reason is that these authors appear to believe that the only acceptable solutions to the eigenvalue equation are those with polynomial factors $P(\xi)$. The fact is that this kind of solutions already satisfy equation (9) but they are not the only ones. Notice that the variational method also yields the polynomial solutions as shown by the fact that the blue lines connect the red points in Figure 1. In order to make the meaning of the eigenvalues $W^{(n)}_\nu$ and the associated multiplicity of roots $i = 1, 2, \ldots, n + 1$ clearer, Figure 1 shows an horizontal line (green, dashed) at $W = W^{(8)}_0$ that intersects the curves $W_{\nu,0}(a,1)$ exactly at the red points. The
most important conclusion of present analysis is that the occurrence of allowed oscillator frequencies are fabricated by Bakke and Belich by picking out some isolated eigenvalues $W_{\gamma}^{(n)}$ for some particular curve $a_{\gamma}^{(n,i)}(b)$. Since there are eigenvalues $W_{\nu,\gamma}(a, b)$ for all real values of $a$ and $b$ then there are bound states for every positive value of $\omega$ in their equations (1).

Summarizing: The authors make two basic, conceptual errors. The first one is to believe that the only possible bound states are those with polynomial factors $P(\xi)$. We have shown above that there are square-integrable solutions for model parameters $a$ and $b$ outside the curves $a_{\gamma}^{(n,i)}(b)$ associated to these polynomials. The second error is the assumption that the spectrum of the problem is given by the truncation condition (5). It is clear that this equation only provides one energy eigenvalue for a particular set of model parameters given by the curves just mentioned. From these mistakes the authors conjecture the existence of allowed cyclotron frequencies. Here we have shown that such allowed cyclotron frequencies are fabricated by Bakke and Belich by means of the truncation method. Therefore, such conclusion is nonsensical from both mathematical and physical points of view. It is clear that there are bound states for all values of $\omega$ because the eigenvalues $W_{\nu,\gamma}(a, b)$ are continuous functions of both $a$ and $b$.

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Figure 1: Eigenvalues $W_0^{(n)}(a,1)$ from the truncation condition (red points) and $W_{r,0}(a)$ obtained by means of the variational method (blue lines)

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