The fixation probability and time for a doubly beneficial mutant

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Abstract

For a highly beneficial mutant $A$ entering a randomly reproducing population of constant size, we study the situation when a second beneficial mutant $B$ arises before $A$ has fixed. If the selection coefficient of $B$ is greater than the selection coefficient of $A$, and if $A$ and $B$ can recombine at some rate $\rho$, there is a chance that the double beneficial mutant $AB$ forms and eventually fixes. We give a convergence result for the fixation probability of $AB$ and its fixation time for large selection coefficients.

1 Introduction

The spread of a beneficial mutant in a constant size population is a well-studied model in mathematical population genetics. A first approximation of its fixation probability has already been established by Haldane (1927), and the theory of one-dimensional diffusions can be used in order to obtain properties of the fixation time (Kimura and Ohta, 1969); see also Etheridge et al. (2006) and Hermisson and Pennings (2005). Adding recombination with a neutral locus raises new questions about the genealogy at the neutral locus as well as an opportunity to detect selection from a population sample (see e.g. Barton, 1998; Kaplan et al., 1989; Maynard Smith and Haigh, 1974; Nielsen et al., 2005; Stephan et al., 1992).

However, population genetic models become much more complex if we assume that a second (different) beneficial mutant arises during the spread of the first. Due to recombination, there is a chance that the first and the second beneficial mutant recombine to form an even fitter type. While Otto and Barton (1997) and Barton (1998) have studied the case that the second allele is less beneficial than the first, we will concentrate on the opposite case. This is even more interesting since only a more beneficial second mutant has a chance to survive against an almost fixed first mutant and form the fittest recombinant type. This model of competing selective sweeps has been studied in a series of papers (Chevin et al., 2008; Cutbertson et al., 2012; Hartfield and Otto, 2011; Kirby and Stephan, 1996; Stephan, 1995; Yu and Etheridge, 2010), but the picture is not complete yet.

Basic questions are the fixation probability, fixation time and the pattern of this scenario in genetic data under the competing sweeps model. In the present

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paper, we are dealing with the first two questions using a diffusion model. This complements previous work by Yu and Etheridge (2010) and Cuthbertson et al. (2012), who use a Moran model and studied the fixation probability of the fittest recombinant type. We are able to extend their results in several respects: (i) the probability of fixation is given explicitly (in the limit of large selection coefficients); (ii) conditional on fixation, we obtain an approximation of the fixation time of the fittest recombinant type.

The main method we use is based on the ancestral selection graph (Krone and Neuhauser 1997; Neuhauser and Krone 1997), which describes possible ancestral lines in selective population genetic models. Recently, this graph has been used to study the process of fixation for selective sweeps (Greven et al., 2016; Pokalyuk and Pfaffelhuber, 2013). Since we are dealing with two recombining loci, we have to follow Griffiths and Marjoram (1997) and add recombination events to this graph in order to obtain the ancestral selection recombination graph. This approach is not new and is implicit in the ancestral influence graph of Donnelly and Kurtz (1999). Lessard and Kermany (2012) combined selection and recombination in one graph for a fixed population size and a Markov chain in discrete time and Mano and Innan (2008) analysed the evolution of duplicated genes under the influence of selection and recombination. However, the limit of large selection coefficients including recombination has not been studied using the ancestral selection graph before.

After introducing the diffusion model in Section 2, we also give our main results. In Section 3, we explain our main technique, the ancestral selection recombination graph (ASRG) and translate the event of fixation within the diffusion model to properties of the ASRG in terms of a Markov jump process. Before we come to the proofs of the main results, we give auxiliary results in Section 4. Then, in Section 5, we are ready to give the proofs of our Theorems.

2 Model and main result

2.1 Competing sweeps

We use the standard diffusion model from population genetics including selection and recombination (see e.g. Ethier and Kurtz 1993; Ewens 2004; Ohta and Kimura, 1969) with the four types

\[ 0 \equiv ab, \quad 1 \equiv Ab, \quad 2 \equiv aB, \quad 3 \equiv AB, \]

which have selection coefficients \( 0 = \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 \equiv \alpha \). The evolution of the frequencies of these types is governed by the solution of the system of SDEs

\[
\begin{align*}
    dX_0 &= \left( -X_0 \sum_j \alpha_j X_j + \rho(X_1X_2 - X_0X_3) \right) dt + \sum_{j \neq 0} \sqrt{X_0X_j} dW_{0j}, \\
    dX_1 &= \left( X_1 \left( \alpha_1 - \sum_j \alpha_j X_j \right) + \rho(X_0X_3 - X_1X_2) \right) dt + \sum_{j \neq 1} \sqrt{X_1X_j} dW_{1j}, \\
    dX_2 &= \left( X_2 \left( \alpha_2 - \sum_j \alpha_j X_j \right) + \rho(X_0X_3 - X_1X_2) \right) dt + \sum_{j \neq 2} \sqrt{X_2X_j} dW_{2j}, \\
    dX_3 &= \left( X_3 \left( \alpha_3 - \sum_j \alpha_j X_j \right) + \rho(X_1X_2 - X_0X_3) \right) dt + \sum_{j \neq 3} \sqrt{X_3X_j} dW_{3j},
\end{align*}
\]  

(2.1)

where \( (W_{kl})_{k,l} \) is a family of independent Brownian motions and \( W_{lk} = -W_{kl} \). Note that \( X_0 + X_1 + X_2 + X_3 = 1 \) for all times, if the initial state satisfies this relation. Here, \( X_i(t) \) denotes the frequency of type \( i \) at time \( t \). We will write
\( X = (X(t))_{t \geq 0} \) with \( X = (X_0, X_1, X_2, X_3) \) for a solution of \((2.1)\), whose existence and uniqueness follows from standard theory [Ethier and Kurtz 1993].

### 2.2 Main results

We now give our main results on the fixation probability and fixation time (conditioned on fixation). Their proofs are given in Section 5.

**Theorem 1** (Fixation probability of type \(3 \equiv AB\)). **Let** \( c > 0, 0 < \delta, \psi < 1 \) and \( P_{\delta, \psi} \) be the distribution of the solution \( X \) of \((2.1)\), started in \( \xi_{\delta, \psi} = (1 - \delta - \alpha \psi, \alpha \psi, \delta, 0) \). **Assume that**

\[
\frac{\alpha_1}{\alpha} \xrightarrow{\alpha \to \infty} c_i,
\]

**with** \( 0 < c_1 < c_2 < c_3 = 1 \).

1. **If** \( \psi < \frac{c_1}{c_2} \), the fixation probability of \(3 \equiv AB\) satisfies

\[
\lim_{\alpha \to \infty} \left( \frac{1}{\alpha} \right) \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}_{\delta, \psi} (X_3(\infty) = 1) = c_2 \left( 1 - \frac{1 - c_2}{1 - c_1} \right) \frac{2\psi(1 - c_2)(1 - c_1)}{(c_2 - c_1)^2}. \tag{2.2}
\]

2. **If** \( \frac{c_1}{c_2} < \psi < 1 \), the fixation probability of \( AB \) satisfies

\[
\lim_{\alpha \to \infty} \left( \frac{1}{\alpha} \right) \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}_{\delta, \psi} (X_3(\infty) = 1) = 0. \tag{2.3}
\]

**Theorem 2** (Fixation time of type \(3 \equiv AB\)). **Assume the same situation as in Theorem 1.** and let \( S := \inf \{ t : X_3(t) = 1 \} \) (with \( \inf \emptyset = \infty \)). **Then,** for all \( \varepsilon > 0 \)

\[
\lim_{\alpha \to \infty} \left( \frac{1}{\alpha} \right) \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}_{\delta, \psi} \left( \left| \frac{\alpha}{\log \alpha} S - \left( \frac{1 - \psi}{c_2 - c_1} + \frac{2}{1 - c_2} \right) \right| > \varepsilon \right| X_3(\infty) = 1 \) = 0.
\]

**Remark 2.1** (Additive selection). **For** additive selection, **we have** \( c_1 + c_2 = c_3 = 1 \), **and** \((2.2)\) **in this case turns into**

\[
\lim_{\alpha \to \infty} \left( \frac{1}{\alpha} \right) \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}_{\delta} (X_{AB}(\infty) = 1) = 2c_2 \left( 1 - \frac{c_1}{c_2} \right) \frac{2\psi(1 - \psi)c_1}{(c_2 - c_1)^2}.
\]

This limit result matches with the approximation result presented in (5) of [Harfield and Otto 2011] for a semi-deterministic model. The result on the fixation time from Theorem 2 translates for additive selection to

\[
\lim_{\alpha \to \infty} \left( \frac{1}{\alpha} \right) \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}_{\delta} \left( \left| \frac{\alpha}{\log \alpha} S - \left( \frac{2c_2 - (1 + \psi)c_1}{c_1(c_2 - c_1)} \right) \right| > \varepsilon \right| T_3 < \infty \) = 0.
\]

Since \( 2c_2 - (1 + \psi)c_1 > 2(c_2 - c_1) \), this fixation time is even longer than for a single beneficial allele with selection coefficient \( \alpha_1 \).

**Remark 2.2** (Interpretation for finite populations). The limit results can be used as an approximation for finite populations with initial value \( x_2 = \delta = 1/N \). **Writing** \( X^N \) **for frequencies in the finite model,** \( s_i := \alpha_i/N, i = 0, ..., 3 \), **and inserting this value in** \((2.2)\) **leads to the approximation**

\[
\mathbb{P}_{\delta} (X^N_3(\infty) = 1) \approx 2s_2 \left( 1 - \frac{1 - c_2}{1 - c_1} \right) \frac{2\psi(1 - c_2)(1 - c_1)}{(c_2 - c_1)^2}.
\]
In addition, the fixation time from Theorem 2 is approximately
\[
\left(\frac{1 - \psi}{c_2 - c_1} + \frac{2}{1 - c_2}\right) \log N \frac{s}{s}
\]
generations.

**Remark 2.3** (Comparison with results by Yu and Etheridge (2010) and Cuthbertson et al. (2012)). In Yu and Etheridge (2010) and Cuthbertson et al. (2012), a similar model was studied. More precisely, they study a finite Moran model, but use as their main scenario \(\psi < c_1/c_2\) and compute an approximation of the fixation probability. The main difference between their result and ours is that they formulate the fixation probability in terms of a solution of ODEs (see (2.8) in Cuthbertson et al. (2012)), while (2.2) is explicit. This difference comes from the phase where the recombinant type \(3 \equiv AB\) forms. Here, using the full finite Moran model seems to be more difficult than working with the Ancestral Selection Recombination Graph, which consists of only a subset of all possible events arising in the population.

### 2.3 Heuristics

All results in Theorems 1 and 2 can already be understood heuristically. We will make use of several intuitions:

1. If the best type in the population has a small frequency, it can be approximated by a supercritical branching process, i.e. a solution of
   \[
   dX = \alpha X dt + \sqrt{X} dW. \tag{2.4}
   \]
   Here, \(\alpha\) is the fitness advantage against the bulk of the population. Recall that the survival probability of the SDE (2.4), starting in \(X_0 = x\) is given by \(1 - e^{-2\alpha x}\). Hence, if \(x \sim \alpha^{-\psi}\) for some \(\psi \in [0, 1)\), the fixation probability approaches 1.
   The solution of (2.4) can be conditioned to survive. In this case, it reaches \(\varepsilon > 0\) by time approximately \(\log \alpha / \alpha\).

2. If a type is established in the population, which means that its survival probability is close to 1, but its frequency is still small, its frequency can be approximated by the logistic equation \(dX = \alpha X (1 - X) dt\) (where \(\alpha\) is the fitness advantage against the bulk). In particular, for small \(X\), the growth of \(X\) is exponential with rate \(\alpha\). Moreover, the time it takes to reach \(1 - \varepsilon\) starting in \(\varepsilon > 0\) is \(O(1/\alpha)\).

3. If a type in the population has small frequency, and its fitness disadvantage against the bulk is \(\alpha\), it can be approximated by a subcritical branching process, i.e. a solution of
   \[
   dX = -\alpha X dt + \sqrt{X} dW.
   \]
   When started in \(\varepsilon > 0\), it dies out (i.e. \(X\) hits 0) by time approximately \(\log \alpha / \alpha\).

...on the fixation probability

Since the initial configuration is \(X(0) = (1 - \delta - \alpha^{-\psi}, \alpha^{-\psi}, \delta, 0)\), most individuals are type \(0 \equiv ab\) at the beginning. The survival probability of type \(1 \equiv Ab\) is approximately \(1 - e^{-2\alpha_1 \alpha^{-\psi}} \approx 1\) (recall \(\psi < 1\)), so we have approximately \(dX_1 = \ldots\)
\( \alpha_1 X_1 dt \) until \( X_1 \) reaches some small \( \varepsilon > 0 \). By the exponential growth, \( X_1 = \alpha^{-\psi} e^{\alpha_1 t} \), so the hitting time of \( \varepsilon \) is not earlier than

\[
t_1 := (\psi \log \alpha + \log \varepsilon)/\alpha = \frac{\psi \log \alpha}{c_1}. \tag{2.5}
\]

Moreover, for small \( \delta \), type 2 \( aB \) has a chance of \( 1 - e^{-2\alpha \delta} \approx 2\alpha \delta \approx 2\alpha \delta c_2 \) of surviving (recall that type 2 has to survive only against type 0 since 1 is still in low frequency; note that \( \delta \) goes to 0 first in Theorem 1). If type 2 survives, it follows the SDE \( dX = \alpha_2 X_2 dt + \sqrt{X_2} dW \) (at least until \( X_1 \) becomes too large), conditioned on survival, so it hits \( \varepsilon > 0 \) not earlier than at time

\[
t' := \frac{\log \alpha_2}{\alpha_2} \approx \frac{\log \alpha}{c_2}. \tag{2.6}
\]

Comparing \( t_1 \) and \( t'_1 \), we have to distinguish two cases. We start with the simpler one, which leads to 2 in the Theorem: If \( \psi > \frac{c_2}{c_1} \), we find that \( t'_1 < t_1 \), i.e. \( X_2 \) hits \( \varepsilon \) before \( X_1 \), and shortly after \( t'_1 \), we have that \( \dot{X}_2 \approx 1 \), so \( X_1 \) becomes subcritical. In particular, \( X_1 \) cannot have any macroscopic frequency, and type 3 \( AB \) has no chance to form by recombination. This already explains (2.3). However, if \( \psi < \frac{c_2}{c_1} \) (see 1 in the Theorem), we have \( t_1 < t'_1 \), i.e. the frequency \( X_1 \) hits \( \varepsilon \) quicker than \( X_2 \). Shortly after \( t_1 \), \( X_1 \) reaches frequency 1. So, by this time we have \( X_0 \approx 0, X_1 \approx 1, X_2 \approx e^{(\alpha_2 \psi \log \alpha)/\alpha_2} \approx e^{\alpha \psi / c_2}, X_3 = 0 \). In particular, establishment of \( X_2 \) happened. Then, \( X \) can be approximated by \( X_0 = X_3 = 0, X_1 = 1 - X_2 \).

\[
dX_2 = (\alpha_2 - \alpha_1)X_1X_2 dt, \quad dX_1 = -dX_2.
\]

In words, \( X_2 \) grows logistically at speed \( (\alpha_2 - \alpha_1) \) and \( X_1 = 1 - X_2 \). The time it takes \( X_2 \) to reach some \( \varepsilon > 0 \) is thus

\[
t_2 = t_1 + \frac{1}{c_2 - c_1} \left( 1 - \frac{c_2 \psi}{c_1} \right) \log \alpha. \tag{2.7}
\]

Once \( X_2 \) reaches some \( \varepsilon > 0 \), a recombination between 1 \( \equiv Ab \) and 2 \( \equiv aB \) can occur. It is crucial to note that if a recombinant arises by some time \( t \), its chance to survive depends on \( X_1 \) and \( X_2 \). Denoting by \( S_t/N \) the chance that a recombinant at time \( t \) survives, the chance that a surviving recombinant arises is, for a population of size \( N \),

\[
1 - \exp \left( -\rho I \right) \quad \text{with} \quad I := \int X_1(t)X_2(t)S_t dt. \tag{2.8}
\]

Moreover, as can be computed from Theorem 1.8 and Proposition 4.9 (recall \( \alpha_3 \equiv \alpha \)),

\[
S_t = 2 \frac{(\alpha - \alpha_1)(\alpha - \alpha_2)}{(\alpha - \alpha_1)X_2(t) + (\alpha - \alpha_2)X_1(t)}. \tag{2.9}
\]

Plugging this into the last display shows that, using \( dX_2 = (\alpha_2 - \alpha_1)X_1X_2 dt \),

\[
I = \int X_1(t)X_2(t)S_t dt
= 2 \frac{(\alpha - \alpha_1)(\alpha - \alpha_2)}{\alpha_2 - \alpha_1} \int \left( (\alpha - \alpha_1)x + (\alpha - \alpha_2)(1 - x) \right)^{-1} dx
= 2 \frac{(\alpha - \alpha_1)(\alpha - \alpha_2)}{(\alpha_2 - \alpha_1)^2} \log \frac{\alpha - \alpha_1}{\alpha - \alpha_2}.
\]
Therefore,
\[ P_{E_3,0}(X_3(\infty) = 1) \approx 2\alpha \delta c_2 \left( 1 - e^{-\rho \tau} \right) \approx 2\alpha \delta c_2 \left( 1 - \frac{(1 - c_1)}{1 - c_2} \right) - 2\rho \frac{(1 - c_1)(1 - c_2)}{(1 - c_1)(1 - c_2)} \],
and we have shown (2.2) from Theorem 1 heuristically.

**Remark 2.4 (Bounds).** We note that some bounds on the fixation probability can be established heuristically as well. Using that
\[ \rho I \approx \frac{2(1 - c_2)(1 - c_1) \log \left( \frac{1 - c_1}{c_2 - c_1} \right)}{(c_2 - c_1)^2} \leq 2 \frac{(1 - c_2)(1 - c_1)(1 - c_1)}{(c_2 - c_1)^2} = \frac{2}{c_2 - c_1}, \]
we obtain
\[ 2\alpha \delta c_2 \left( 1 - \exp \left( - 2\rho \frac{1 - c_2}{c_2 - c_1} \right) \right) \leq P_{E_3,0}(X_3(\infty) = 1) \leq 2\alpha \delta c_2 \left( 1 - \exp \left( - 2\rho \frac{1 - c_1}{c_2 - c_1} \right) \right). \]
This also follows from (2.6), which holds since the recombinant has at least to survive against type 1 and at most against type 2. Plugging the upper bound into (2.7), we obtain again
\[ P_{E_3,0}(X_3(\infty) = 1) \approx 2\alpha \delta c_2 \left( 1 - \exp \left( - \rho \int X_1(t)X_2(t)S_1(t) dt \right) \right) \leq 2\alpha \delta c_2 \left( 1 - \exp \left( - 2\rho \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} \int dX_2 \right) \right) \approx 2\alpha \delta c_2 \left( 1 - \exp \left( - 2\rho \frac{1 - c_1}{c_2 - c_1} \right) \right) \]
and a similar relation holds for the lower bound.

**...on the fixation time**

For developing a heuristics on the fixation time (see Theorem 2), we rescale time by the factor \( \alpha / (\log \alpha) \) for the moment, leading to a time-scale \( d\tau = \frac{\alpha}{\log \alpha} dt \). We have already seen above (see (2.5)) that \( 1 \equiv Ab \) hits frequency \( \varepsilon \) by time
\[ \tau_1 \approx \frac{\psi}{c_1}, \]
and that type 3 - it if arises - arises (see (2.6)) by time
\[ \tau_2 = \tau_1 + \frac{1}{c_2 - c_1} \left( \frac{c_2 \psi}{c_1} \right). \]
Since the phase where type 2 outcompetes type 1 only takes \( O(1/\alpha) \), we find that at time \( \tau_2^+ \) that \( X_1(\tau_2^+) \approx \varepsilon, X_2(\tau_2^+) \approx 1 - \varepsilon \) for some small \( \varepsilon > 0 \). So, the successful type \( 3 \equiv AB \) has to compete against type \( 2 \equiv aB \), all other types being of small frequency. The fixation time of type \( 3 \equiv AB \) is thus given by the classical result for a beneficial allele, which is by time
\[ \tau_3 = \tau_2 + \frac{2}{1 - c_2}. \]
In total, we find that
\[ \tau_3 \approx \frac{\psi}{c_1} + \frac{c_1 - c_2 \psi}{c_1(c_2 - c_1)} + \frac{2}{1 - c_2} = \frac{1 - \psi}{c_2 - c_1} + \frac{2}{1 - c_2}. \]
Detecting selection from a population sample is a formidable task and has benefited from new methods in the last two decades. Such methods are based on polymorphism data, as reviewed e.g. by Stephan (2016). Frequently, detecting strong selection is based on the hitchhiking effect, i.e. the reduced neutral genetic diversity around a beneficial locus at or near the time of its fixation. In the case of competing sweeps, simulation results from Kim and Stephan (2003) and Chevin et al. (2008) report a reduced hitchhiking effect in the case of competing sweeps (relative to the scenario of a single beneficial allele rising to fixation). This indicates that the reduction in neutral genetic diversity, caused by the fast fixation process of the beneficial alleles, is weaker for competing sweeps. In addition, Chevin et al. (2008) report an increased number of intermediate frequency variants.

Although we will not contribute to a quantitative understanding of genetic patterns under competing sweeps in this manuscript, we will add some ideas how the above findings can be understood. For this, recall from the heuristics above that type 3 only arises during the time when type 2 takes over a population of mostly type-1-individuals. It is possible that several type 3 recombinants occurring during this time contribute to fixation of type 3. Since different recombinants will have different cross-over points between the A- and B-locus, the case of multiple recombinants leads to a haplotype structure between the two selective loci. Since recombinants arise at nearly the same time, they will rise in frequency similarly, leading to (i) an increase in intermediate frequency variants due to the haplotype structure and consequently (ii) a reduction in the hitchhiking effect.

3 The Ancestral Selection Recombination Graph

For computing moments of a diffusion such as (2.1), a genealogical picture can be extremely helpful (see e.g. Alkemper and Hutzenthaler, 2007; Etheridge et al., 2006; Mano, 2009; Pokalyuk and Pfaffelhuber, 2013). Since we study here a scenario with random reproduction, recombination and selection, all these forces have to be taken into account in the genealogical picture.

For populations which evolve under selection, a general graph construction called the ancestral selection graph (ASG), was introduced by Neuhauser and Krone. It contains all information about the ancestry of a population sample. The basic idea is best explained by means of the graphical representation of the Moran model. When the ancestry of a sample of lines is examined backwards in time, selective events occur at certain time points. As long as we do not know the allelic type of the individuals initiating this event, we cannot decide if the selection event has taken place or not. This requires information about the allelic types taking part in this event. To handle this difficulty both potential ancestors are traced back. In doing so, all possible ancestors of a considered sample are included in the ASG. Later on when the types in the past are determined we can decide about the real ancestor of the considered line.

Related arguments were used for a situation with recombination by Griffiths and Marjoram (1997), which introduced the ancestral recombination graph. When the ancestry of two loci is considered simultaneously, a recombination event between the two loci leads to the situation that the alleles originate from different individuals. Hence to handle the complete ancestry both ancestors must be traced back.

Both selection and recombination events (viewed backwards in time) lead to the necessity to split ancestral lines in order to identify the correct genealogy. In particular, it is possible to combine these two graphs. This link was done in a very general way by Donnelly and Kurtz (1999), who construct an ancestral in-
fluence graph using the lookdown process, a particle representation of \((\text{2.1})\); see also \cite{LessardKermany2012}. As a result, one obtains the ancestral selection recombination graph (ASRG). Before we start, we provide in the next subsection a computational tool for the fixation probability which is based on the Ancestral Selection Recombination Graph.

### 3.1 A Markov jump process and the fixation probability

For the proofs of Theorems \[1\] and \[2\] we will have to translate the ASRG into a Markov jump process and a duality relation with the SDE \((\text{2.1})\). The process \(\mathcal{L}\) as defined below arises as a time-reversed version of the Ancestral Selection Recombination Graph in equilibrium for large \(\alpha\); see the next Subsection for more details. Again, indices 0, 1, 2, 3 will correspond to the types in the SDE.

**Definition 3.1** (The Markov jump process \(\mathcal{L}\)). We define the Markov jump process \(\mathcal{L} = (\mathcal{L}(t))_{t \geq 0}\) with \(\mathcal{L}(t) = (L_0(t), L_1(t), L_2(t), L_3(t))\): Starting in \(\mathcal{L}(0)\) with \(L_0(0) = 1, L_1(0) = 0, L_1(0) \sim \text{Poi}(2\alpha^{-1/\psi}), L_0(0) \sim \text{Poi}(2\alpha(1 - \alpha^{-\psi}))\), the dynamics is as follows: If \(\mathcal{L}(t) = \ell := (\ell_0, \ell_1, \ell_2, \ell_3)\), jumps occur to (note that \(\ell_i = (\delta_{ij})_{j=0,1,2,3}\) and setting \(c_0 := 0\))

\[
\begin{align*}
\ell + e_i & \quad \text{at rate } \alpha \ell_i, \quad i = 0, \ldots, 3, \\
\ell - e_i & \quad \text{at rate } \left(\frac{\ell_i}{2}\right) + \frac{1}{2} \sum_{j \neq i} \ell_i \ell_j (1 - c_i + c_j), \quad i = 0, \ldots, 3, \\
\ell + e_i - e_1 - e_2 & \quad \text{at rate } \frac{1}{2} \ell_i \ell_2 \frac{\rho}{\alpha}, \quad i = 0, 3, \\
\ell + e_i - e_3 & \quad \text{at rate } \frac{1}{2} \ell_i \ell_3 \frac{\rho}{\alpha}, \quad i = 1, 2.
\end{align*}
\]

**Remark 3.2** (\(\mathcal{L}\) as a chemical reaction network). Actually, the Markov-jump process can be seen as a special form of chemical reaction network with mass-action dynamics. Precisely, this network is given through the equations

\[
\begin{align*}
A_i & \xrightarrow{\alpha} 2A_i, \quad i = 0, 1, 2, 3, \\
A_i + A_j & \xrightarrow{(1 - c_i + c_j)/2} A_j, \quad i, j = 0, 1, 2, 3, \\
A_0 + A_3 & \xrightarrow{\rho/(2\alpha)} A_i, \quad i = 1, 2, \\
A_1 + A_2 & \xrightarrow{\rho/(2\alpha)} A_i, \quad i = 0, 3,
\end{align*}
\]

where \(L_i\) is the number of molecules of species \(A_i, i = 0, \ldots, 3\). Note that the first reaction looks like binary branching dynamics for all species, the second is a special form of resampling, and the remaining equations come from recombination events within the ASRG.

The next Proposition is fundamental in the proof of Theorem \[1\] Its proof is found in Subsection \[3.4\].

**Proposition 3.3** (Eventual fixation and \(\mathcal{L}\)). Let \(\mathcal{L}\) be as in Definition \[3.1\], \(X\) as in Theorem \[1\] and \(S\) as in Theorem \[3\].

1. Then,

\[
\lim_{\alpha \to \infty} \frac{1}{2\alpha} \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}_{\mathcal{L}, \psi} (X_3(\infty) = 1) = \lim_{\alpha \to \infty} \mathbb{P}(L_j(\infty) = 0, j \neq 3). \tag{3.1}
\]

2. For \(\tau > 0\),

\[
\lim_{\alpha \to \infty} \frac{1}{2\alpha} \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}_{\mathcal{L}, \psi} \left(\frac{\alpha}{\log \alpha} S < \tau\right) = \lim_{\alpha \to \infty} \mathbb{P} \left(L_j \left(\tau \frac{\log \alpha}{\alpha}\right) = 0, j \neq 3\right). \tag{3.2}
\]
Figure 3.1: If a coalescing event (A), a selective branching event (B) or a recombination branching event (C) occurs by time $\beta$, we connect the lines within the ASRG according to the rules as given in Definition 3.4. In all cases, $U_g$ is uniformly distributed on $[0, 1]$, and updated upon any event for the affected lines. For branching event, labels in $\{1, 2, 3, a\}$ indicate which kind of event happens.

### 3.2 Construction of the ASRG and duality

We will define the ASRG first, and then obtain some basic results and the precise connection to $\mathcal{A}^{(2)}$. Then, we give the connection to the jump process from Definition 3.3.

In order to distinguish between the forwards time $t$ and the genealogical time, the backwards (genealogical) time $\beta$ is introduced; see Figure 3.1 for an illustration.

**Definition 3.4 (ASRG).** 1. For $k \in \mathbb{N}$, we define a particle model $\mathcal{A} = \mathcal{A}^k = (\mathcal{A}^k_\beta)_{\beta \geq 0}$, which takes values in $E := \bigcup_{k=1}^{\infty} E_k$, $E_k := \{A \subset [0, 1] : \#A = k\}$, the set of finite subsets of $[0, 1]$ and $\mathcal{A}^k_0 \in E_k$. Elements of $\mathcal{A}^k$ are called particles. We set $\mathcal{A}^k_0 = \{U_1, ..., U_k\} \in E_k$ for some (on $[0, 1]$) uniformly distributed and independent random variables $U_i, i = 1, ..., k$. The dynamics of $\mathcal{A}^k$ is as follows:

(a) Coalescence: Every (unordered) pair of particles is replaced at rate 1 by a particle with a label that is uniformly distributed on $[0, 1]$ and independent of everything else.

(b) Branching: Every particle is replaced at rate $\alpha + \rho$ by two particles, with labels that are uniformly distributed on $[0, 1]$ and independent of everything else.

Along the path of $\mathcal{A}^k$, we mark each branching event with a label in $\{1, 2, 3, a\}$,
namely

1 with probability \( \frac{\alpha_1}{\alpha + \rho} \)

2 with probability \( \frac{\alpha_2 - \alpha_1}{\alpha + \rho} \)

3 with probability \( \frac{\alpha_3 - \alpha_2}{\alpha + \rho} \)

\( a \) with probability \( \frac{\rho}{\alpha + \rho} \).

(3.3)

The branching events with a label in \( \{1, 2, 3\} \) are called selective (branching) events and the events with the letter \( a \) are called recombination (branching) events. For a selective branching event, the branching particle will be denoted the outgoing particle and we mark one new particle as the incoming, the other one as the continuing particle. For a recombination event, mark one new particle as the \( A/a \)-particle, the other one as the \( B/b \)-particle. We denote the particle system which includes all these marks by \( B^k \), which we refer to by the Ancestral Selection Recombination Graph or ASRG for short.

2. For the particle system \( B^k = (B^k)_\beta \geq 0 \), consider the line-counting process \( (K^k)_\beta \geq 0 \), which starts in \( K^k_0 = k \) and jumps from \( k \) to

\[
\begin{align*}
&k - 1 \text{ at rate } q_{k,k-1} := \binom{k}{2} \\
&k + 1 \text{ at rate } q_{k,k+1} := (\alpha + \rho)k.
\end{align*}
\]

(3.4)

3. Let \( x := (x_0, x_1, x_2, x_3) \in \Sigma_3 \), the three-dimensional simplex and \( \tau > 0 \). Based on the ASRG \( B^k \), we define random variables \( J_1(\tau), ..., J_k(\tau) \in \{0, 1, 2, 3\} \). (Recall that time for the ASRG is running backward.) First, color all particles in \( B^k_\tau \) independently with color \( i \) with probability \( x_i \), \( i = 0, 1, 2, 3 \). From here on, work forwards through the graph, such that types are inherited along coalescence events; see Figure 3.2(A). Upon a selective branching event, do the following: If the label of the selective branching event is \( i \), the incoming line is of type \( j \), and the continuing line of of type \( k \), the outgoing line is of type (see Figure 3.2(B))

\[
1_{j \geq i} + 1_{j < i} k.
\]

(3.5)

(This means that a type with a number higher than the mark is inherited along the incoming line, but if this is not the case, the type of the continuing line is inherited.) Upon a recombination branching event, do the following: the \( A/a \)-locus is inherited along the \( A/a \)-particle, the \( B/b \)-locus is inherited along the \( B/b \)-particle. Precisely, use Table 1 for the types; see also Figure 3.2(C). By this procedure, we obtain \( J_1(\tau), ..., J_k(\tau) \in \{0, 1, 2, 3\} \), which are the types of the \( k \) initial particles of \( B^k \) at time \( \beta = 0 \). We denote the distribution of \( (J_i(\tau))_{i \leq \tau} \) by \( P^x_\tau \).

Proposition 3.5 (Duality). Let \( X \) be as in (2.1), and \( J_1(\tau), ..., J_k(\tau) \) as in Definition 3.4.3 for \( \tau > 0 \). Then, the duality relation

\[
\mathbb{E}_x \left[ \prod_{i=1}^k X_{j_i}(\tau) \right] = \mathbb{P}_x \left[ J_1(\tau) = j_1, ..., J_k(\tau) = j_k \right], \quad j_1, ..., j_k \in \{0, 1, 2, 3\}
\]

(3.6)

holds.
Table 1: Lookup Table for recombination branching events. The \( A/a \)-locus is inherited along the \( A/a \)-particle, the \( B/b \)-locus along the \( B/b \)-particle.

\[
\begin{array}{ccc}
A/a\text{-particle} & B/b\text{-particle} & \text{outgoing particle} \\
0 \text{ or } 2 & 2 \text{ or } 3 & 2 \\
0 \text{ or } 2 & 0 \text{ or } 1 & 0 \\
1 \text{ or } 3 & 2 \text{ or } 3 & 3 \\
1 \text{ or } 3 & 0 \text{ or } 1 & 1 \\
\end{array}
\]

Figure 3.2: If a coalescing event (A), a selective branching event (B) or a recombination branching event (C) occurs by time \( \beta \), we connect the lines within the ASRG according to the rules as given in Definition 3.4. In all cases, \( U_g \) is uniformly distributed on \([0, 1]\), and updated upon any event for the affected lines. For branching event, labels in \( \{1, 2, 3, a\} \) indicate which kind of event happens.
Proof. As shown in [Donnelly and Kurtz (1999)], an ASRG (as a special case of their ancestral influence graph) is the graph of possible ancestors of an infinitely large population, whose evolution of frequencies is given by (2.1). Note that the left-hand side of (3.6) is the probability that, in a sample of size \( k \), taken at time \( \tau \) from the population which evolves according to (2.1), the \( i \)th pick is of type \( j_i \), \( i = 1, ..., k \). The right-hand side follows all possible ancestries of this sample back until time 0, and then decides forwards in time which types are inherited (i.e. which selective events take place and where the \( A/a \) and \( B/b \)-locus was inherited in a recombination event. The right hand side therefore gives the probability that the ASRG and the labeling is such that types \( j_1, ..., j_k \) appear in the sample. Since the types of the lines at time \( \beta = \tau \) in the ASRG are used according to the starting distribution \( \pi \) (see Definition 3.4), the result follows. \( \square \)

### 3.3 Fixation and the ASRG

For computing the fixation probability of type \( 3 \equiv AB \), we will require the equilibrium, denoted \( \Pi \), for the line-counting process \( (K_\beta^k)_{\beta \geq 0} \).

**Proposition 3.6** (The probability of fixation and the ASRG). Let \( \Pi \) be Poisson-distributed with parameter \( \alpha + \rho \), conditioned to be positive, i.e.

\[
P(\Pi = k) = \frac{e^{-2(\alpha+\rho)} 2^k (\alpha + \rho)^k}{k!}.
\]  

(3.7)

1. For all \( k \), we have \( K_\beta^k \xrightarrow{\beta \to \infty} \Pi \), i.e. \( \Pi \) is the unique equilibrium for the line-counting process. This equilibrium is reversible.

2. Denote by \( (K_\beta^1)_{0 \leq \beta \leq \tau} \) the line-counting process, started in \( \Pi \), and let \( J_1(\tau), ..., J_\Pi(\tau) \) be as in Definition 3.4. Then,

\[
P_x(\Sigma_i(\infty) = 1) = \lim_{\tau \to \infty} P_x(J_1(\tau) = \cdots = J_\Pi(\tau) = i), \quad i = 0, 1, 2, 3.
\]  

(3.8)

Proof. 1. The line-counting process \( (K_\beta^k)_{\beta \geq 0} \) is a birth-death process with birth rate \( (\alpha + \rho)k \) and death rate \( \binom{k}{2} \). The unique, reversible equilibrium of this process is given by (3.7); see e.g. [Greven et al. (2016)], Proposition 3.3.

2. Consider some \( \tau, \tau' \to \infty \) with \( \tau' - \tau \to \infty \). Then, if \( A^k \) starts at time \( \beta = 0 \) with \( k \) lines, it has (in the limit of large times) \( \Pi \) lines at time \( \beta = \tau' - \tau \). From here on, we use a restart argument and start the ASRG with \( \Pi \) lines at time \( \tau' - \tau \), which we take as the initial time \( \beta = 0 \) of the restarted ASRG. Since \( \tau \) is large, in the ASRG \( (A^k)_{0 \leq \beta \leq \tau} \) we find with high probability a time \( 0 \leq \beta' \leq \tau \) with \#\( A^k \) = 1. The type of this single particle is inherited to the whole ASRG when viewed forwards in time. Therefore, for large \( \tau \), the ASRG is monotype, and the type is inherited along the graph forwards in time, so fixation of type \( i \) occurs iff all \( \Pi \) lines of the ASRG at time 0 carry type \( i \). Together with Proposition 3.5 we have proven the result. \( \square \)

Since the equilibrium \( \Pi \) is reversible, there is actually another way to compute the fixation probability. For this, we require another Markov jump process.

**Definition 3.7** (The Markov jump process \( \tilde{\mathcal{L}} \)). Let \( x \in \Sigma_3 \) be given. We define the Markov jump process \( \tilde{\mathcal{L}} = (\tilde{L}(t))_{t \geq 0} \) with \( \tilde{L}(t) = (\tilde{L}_0(t), \tilde{L}_1(t), \tilde{L}_2(t), \tilde{L}_3(t)) \). For the initial state, take \( \tilde{L}_i(0) \sim \text{Poi}(2(\alpha + \rho)x_i), i = 0, 1, 2, 3 \), independently, and
condition on \( L_0(0) + L_1(0) + L_2(0) + L_3(0) \geq 0 \). The dynamics is as follows: If \( L(t) = \ell := (\ell_0, \ell_1, \ell_2, \ell_3) \), jumps occur to (note that \( \ell_i = (\delta_{ij})_{j=0,1,2,3} \))

\[
\ell + \xi_i \text{ at rate } (\alpha + \rho)\ell_i, \quad i = 0, 1, 2, 3,
\]

\[
\ell - \ell_i \text{ at rate } \left( \frac{\ell_i}{2} \right) + \frac{1}{2} \sum_{j \neq i} \ell_i \ell_j \frac{\alpha - \alpha_i + \alpha_j}{\alpha + \rho} + \frac{1}{2} \ell_i (\ell_1 + \ell_2) \frac{\rho}{\alpha + \rho}, \quad i = 0, 3
\]

\[
\ell - \ell_i \text{ at rate } \left( \frac{\ell_i}{2} \right) + \frac{1}{2} \sum_{j \neq i} \ell_i \ell_j \frac{\alpha - \alpha_i + \alpha_j}{\alpha + \rho} + \frac{1}{2} \ell_i (\ell_0 + \ell_3) \frac{\rho}{\alpha + \rho}, \quad i = 1, 2
\]

\[
\ell + \ell_i - \ell_1 - \ell_2 \text{ at rate } \frac{1}{2} \ell_1 \ell_2 \frac{\rho}{\alpha + \rho} \text{ for } i = 0, 3,
\]

\[
\ell + \ell_i - \ell_0 - \ell_3 \text{ at rate } \frac{1}{2} \ell_0 \ell_3 \frac{\rho}{\alpha + \rho} \text{ for } i = 1, 2.
\]

We denote the distribution of \( \tilde{L} \) by \( \mathbb{P}_{\tilde{L}} \).

**Proposition 3.8** (The probability of fixation and the reversed ASRG). Let \( \Pi \) be as in Proposition 3.6. Let \( J_1(\tau), \ldots, J_\Pi(\tau) \) as in Definition 3.4 and \( \tilde{L} \) as in Definition 3.7. Then,

\[
\mathbb{P}_{\tilde{L}}(J_1(\tau) = \cdots = J_\Pi(\tau) = i) = \mathbb{P}_{\tilde{L}}(\tilde{L}_j(\tau) = 0, j \neq i).
\]

In particular, combining with (3.8)

\[
\mathbb{P}_{\tilde{L}}(X_\Pi(\infty) = 1) = \mathbb{P}_{\tilde{L}}(\tilde{L}_j(\infty) = 0, j \neq i).
\]  

**Proof.** We only have to show the first identity. Since the equilibrium \( \Pi \) of the line-counting process \( (K_\beta)_{0 \leq \beta \leq \tau} \) is reversible, it is in distribution equivalent to construct the ASRG forwards in time and using the same rates for coalescence and branching. Hence, the (unlabelled) ASRG \( A^\Pi \) can as well be constructed forwards in time. Denote the time-reversed ASRG by \( A^\Pi \). In this time direction, colorings of particles can be performed at time \( \beta = \tau, t = 0 \) already when initializing the ASRG. The difference is that coalescence events for \( A^\Pi \) are branching events for \( A^\Pi \) and vice versa. In \( A^\Pi \), coalescence events obtain the labels in \( \{1, 2, 3, a\} \) with probabilities as in (3.3), and upon a coalescence event with label \( \{1, 2, 3\} \), call one line (picked at random from both lines) incoming and the other one continuing; upon a coalescence event with label \( a \), call one line the \( A/a \)-line, the other one the \( B/b \)-line. Then, (recall that types are known in \( A^\Pi \) since we have constructed the graph forwards in time) types in coalescence events are decided using either (3.1) or Table 1. The resulting transitions are given in the definition of the process \( \tilde{L} \). Just as an example, consider the transition \( \ell \to \ell - \varepsilon_1 \). This occurs if either two lines of type 1 coalesce (no matter which label the coalescence event has), or if a line of type 1 coalesces with a second line. If the second line is of type 3 (which occurs at total rate \( \ell_1 \ell_3 \)), the chance that the coalescence event has mark in \( \{1, 2, 3\} \) is \( \alpha / (\alpha + \rho) \). If the line of type 1 is the incoming line, the outgoing line is of type 3 if the mark is in \( \{2, 3\} \), which has probability \( (\alpha - \alpha_1) / (\alpha + \rho) \). If the line of type 1 is the continuing line and the line of type 3 is the incoming line, then the outgoing line is of type 3 in all cases. In total, the rate of a decrease of lines of type 1 by a coalescence with a line of type 3 is

\[
\frac{1}{2} \ell_1 \ell_3 \left( \frac{\alpha}{\alpha + \rho} + \frac{\alpha - \alpha_1}{\alpha + \rho} \right);
\]

see Definition 3.7. Similar arguments lead to all other terms in the definition of \( \tilde{L} \). Hence, the types in \( \tilde{L} \) at time \( t = \tau \) are in distribution the same as the types in \( J_1, \ldots, J_\Pi \). This implies the assertion. \( \Box \)
3.4 Proof of Proposition 3.3

From (3.9), we know how to compute fixation probabilities from \( \tilde{L} \). However, the process \( L \) used in Proposition 3.3 differs from \( \tilde{L} \) in two respects. First, its initial distribution fixes \( L_2(0) = 1 \) rather than using some \( x \in \Sigma_3 \). Second, the dynamics of \( L \) and \( \tilde{L} \) are only the same for \( \alpha \to \infty \). In order to compare the initial conditions, we need the following simple lemma.

**Lemma 3.9** (Total variation distance of \( \Pi, \Psi_n \)). Let \( \Pi = \Pi_{\alpha, \rho} \) be as in Proposition 3.6, \( \Psi_n - n \sim \text{Poi}(2\alpha) \) for some \( n \in \mathbb{Z} \). Then, for large \( \alpha \), the total variation distance \( d_{TV} \) obeys

\[
d_{TV}(\Pi, \Psi_n) = o(1).
\]

**Proof.** Clearly, \( d_{TV}(\Pi, \Psi_0) = 0 \) since \( \mathbb{P}(\Psi_0 = 0) = e^{-2\alpha} \). Hence, using a recursion and the triangle inequality, it suffices to prove that

\[
d_{TV}(\Pi, \Psi_{i+1}) = o(1), \quad i \in \mathbb{Z}.
\]

We compute

\[
d_{TV}(\Psi_i, \Psi_{i+1}) = e^{-2\alpha} \left( 1 + \sum_{k=1}^{\infty} \left| \frac{(2\alpha)^k}{k!} - \frac{(2\alpha)^k - 1}{(k-1)!} \right| \right)
\]

\[
= e^{-2\alpha} \sum_{k=0}^{\infty} \frac{(2\alpha)^k}{k!} \left| 1 - \frac{k}{2\alpha} \right| + o(1)
\]

\[
= \mathbb{E}\left[ \left| 1 - \frac{\Psi_0}{2\alpha} \right| \right] + o(1) \leq \mathbb{E}\left[ \left( 1 - \frac{\Psi_0}{2\alpha} \right)^2 \right]^{1/2} + o(1) = o(1).
\]

\( \square \)

**Proof of Proposition 3.3**. Combining with (3.9), we have to show that for \( x = (1 - \alpha^{-\psi} - \delta, \alpha^{-\psi}, \delta, 0) \) and \( L \) starting as in Definition 3.1

\[
\lim_{\alpha \to \infty} \lim_{\delta \to 0} \frac{1}{2\alpha \delta} \mathbb{P}_{\xi, \psi}(\tilde{L}_j(\infty) = 0, j \neq i) = \lim_{\alpha \to \infty} \mathbb{P}(L_j(\infty) = 0, j \neq i). \tag{3.10}
\]

We have to show that the total variation distance between \( \tilde{L} \) and \( L \) is small for large \( \alpha \) (and in the limit \( \delta \to 0 \)). Therefore, we have to compare both, the initial condition and the dynamics of the two processes.

**Initial condition:** In \( \tilde{L} \), we mark the \( \Pi \) lines in \( L_0 \) independently with probabilities \( 1 - \alpha^{-\psi} - \delta, \alpha^{-\psi}, \delta \) and 0 with types 0, 1, 2, 3, respectively. In \( L \), we mark the \( \Psi \) lines such that \( L_2(0) = 1 \) and all other lines are marked with 0, 1, 3 independently with probabilities \( 1 - \alpha^{-\psi}, \alpha^{-\psi} \) and 0, respectively.

As we see from Lemma 3.9, the total variation distance for the total number of lines if we use \( \Psi_0 \) instead of \( \Pi \) lines at time 0 is negligible for large \( \alpha \). In addition, in the limit \( \delta \to 0 \), a necessary condition for fixation to occur is that at least one particle is marked by \( 2 \equiv aB \). Therefore, in \( L_0 \), this has approximate probability

\[
\lim_{\alpha \to \infty} \lim_{\delta \to 0} \frac{1}{2\alpha \delta} \mathbb{P}_{\xi, \psi}(\tilde{L}_j(\infty) = 0, j \neq 3)
\]

\[
= \lim_{\alpha \to \infty} \lim_{\delta \to 0} \frac{1}{2\alpha \delta} \mathbb{P}_{\xi, \psi}(\tilde{L}_j(\infty) = 0, j \neq 3|\tilde{L}_2(0) > 0) \cdot \mathbb{P}_{\xi, \psi}(\tilde{L}_2(0) > 0)
\]

\[
= \lim_{\alpha \to \infty} \lim_{\delta \to 0} \frac{1}{2\alpha \delta} \mathbb{P}_{\xi, \psi}(\tilde{L}_j(\infty) = 0, j \neq 3|\tilde{L}_2(0) = 1) \cdot \frac{1 - e^{-2\alpha \delta}}{1 - e^{-2(\alpha + \rho)}}
\]

where we have used Lemma 3.9 in the last step.
Dynamics: Note that the dynamics of \( \mathcal{L} \) arises from the dynamics of \( \tilde{\mathcal{L}} \), if we only take the leading terms in the rates of all transitions into account. For example, \( \mathcal{L} \to \tilde{\mathcal{L}} - \varepsilon_1 \) takes place due to encounters of types 0 and 1 at a rate

\[
\frac{1}{2} \ell_1 \ell_0 \frac{\alpha - \alpha_1 + \rho}{\alpha + \rho} = \frac{1}{2} \ell_1 \ell_0 (1 - c_1)(1 + o(1))
\]
as \( \alpha \to \infty \). Since the event of fixation has a monotone dependence on the parameters \( c_1, c_2 \), we can bound fixation probabilities and times if we slightly change these parameters around the limits.

2. Here, note that (3.9) does not hold for finite times, so we have to adapt our reasoning. First, note that (3.10) still holds at finite times (since the initial conditions as well as the dynamics can be coupled as above). Now, we adopt the argument of the proof of Lemma 2.5 in [Pokalyuk and Pfaffelhuber 2013]. (However, note that in the present paper, we are not working with conditional probabilities, such that the correction as in Remark 3.17 of [Greven et al. 2010] does not apply here.)

By Propositions 3.5 and 3.8, we have that

\[
\mathbb{P}_x \left( \frac{\alpha}{\log \alpha} S < \tau \right) = \mathbb{E}_x \left[ X_3 \left( \frac{\tau}{\log \alpha} \right) \right] = \mathbb{P}_x \left( J_1 \left( \frac{\tau}{\log \alpha} \right) = J_2 \left( \frac{\tau}{\log \alpha} \right) = \ldots = 3 \right).
\]

Clearly, by monotonicity in the number of particles,

\[
\mathbb{P}_x \left( J_1 \left( \frac{\tau}{\log \alpha} \right) = J_2 \left( \frac{\tau}{\log \alpha} \right) = \ldots = 3 \right) \leq \mathbb{P}_x \left( J_1 \left( \frac{\tau}{\log \alpha} \right) = \ldots = J_\Pi \left( \frac{\tau}{\log \alpha} \right) = 3 \right)
\]

\[
= \mathbb{P}_x \left( \tilde{L}_j \left( \frac{\tau}{\log \alpha} \right) = 0, j \neq 3 \right),
\]

which shows ‘\( \leq \)’ in (3.2) (using (3.10) at time \( \tau \frac{\log \alpha}{\alpha} \)). For ‘\( \geq \)’, it suffices to show that for any \( \varepsilon > 0 \), uniformly in \( x \),

\[
\mathbb{P}_x \left( \frac{\alpha}{\log \alpha} S < \tau + \varepsilon \right) = \mathbb{P}_x \left( J_1 \left( \left( \tau + \varepsilon \right) \frac{\log \alpha}{\alpha} \right) = J_2 \left( \left( \tau + \varepsilon \right) \frac{\log \alpha}{\alpha} \right) = \ldots = 3 \right)
\]

\[
\geq \mathbb{P}_x \left( \tilde{L}_j \left( \frac{\tau + \varepsilon}{\log \alpha} \right) = 0, j \neq 3 \right) - \varepsilon
\]
as \( \alpha \to \infty \). Recall that in the second term, the random variables \( J_1, J_2, \ldots \) are types picked at time \( \beta = 0 \), (i.e. at \( t = \left( \tau + \varepsilon \right) \frac{\log \alpha}{\alpha} \)) from \( \mathcal{F}_0^\infty \). Now, for some bounded \( N \) with \( \mathbb{P}(N \leq \Pi) \geq 1 - \varepsilon \), define \( K_1 = K_1 \left( \left( \tau + \varepsilon \right) \frac{\log \alpha}{\alpha} \right), \ldots, K_N = K_N \left( \left( \tau + \varepsilon \right) \frac{\log \alpha}{\alpha} \right) \), which are types of specific lines at time \( \beta = \varepsilon \frac{\log \alpha}{\alpha} \) (i.e. \( t = \tau \frac{\log \alpha}{\alpha} \)) with the property

\[
\{ K_1 = \cdots = K_N = 3 \} \subseteq \{ J_1 \left( (\tau + \varepsilon) \frac{\log \alpha}{\alpha} \right) = J_2 \left( (\tau + \varepsilon) \frac{\log \alpha}{\alpha} \right) = \ldots = 3 \}
\]
on the event \( \{ N \leq \Pi \} \). These \( N \) lines exist with high probability for large \( \alpha \), since following incoming lines at selective, and both lines at recombination branching events along the ASRG between times \( \tau \frac{\log \alpha}{\alpha} \) and \( (\tau + \varepsilon) \frac{\log \alpha}{\alpha} \), we find that coloring these lines with type 3 always leads to the event \( J_1 \left( (\tau + \varepsilon) \frac{\log \alpha}{\alpha} \right) = J_2 \left( (\tau + \varepsilon) \frac{\log \alpha}{\alpha} \right) = \ldots = 3 \). Since coalescence and recombination branching events bring the infinitely many lines down to less than \( 2\alpha \) lines in a time \( \mathcal{O}(1/\alpha) \) (see e.g. Proposition 6.9 of [Depperschmidt et al. 2012]), we have found the specific \( N \) lines with types \( K_1, \cdots, K_N \). Then, from Proposition 3.8 and since
Corollary 4.3
(Exponential growth) Let \( \alpha \) with (individual) birth rate \( \lambda \) and death rate \( \mu \). If \( \lambda > \mu \), then for any \( t, \alpha \downarrow 0 \),
\[
\mathbb{P}(L(t) = 0) - \mu / \lambda \leq \mu e^{-(\lambda - \mu)t} / \lambda,
\]
\[
\mathbb{P}(1 \leq L(t) \leq K) \leq C_{\lambda, \mu} Ke^{-(\lambda - \mu)t} \quad \text{if} \quad K \leq e^{(\lambda - \mu)t} / 6,
\]
\[
\mathbb{P}(\sup_{s \leq t} L(s) \geq K) \leq C_{\lambda, \mu} e^{(\lambda - \mu)t} / K.
\]

Corollary 4.3 (Exponential growth). Let \( (L(t))_{t \geq 0} \) be a binary branching process with (individual) birth rate \( \alpha \) and death rate \( \mu \). If \( L(0) = 1 \), then for \( t > 0 \),
\[
Y(t) := \frac{\log L(t \log \alpha / \alpha)}{\log \alpha} \xrightarrow{a \to \infty} M,
\]
where \( \mathbb{P}(M = -\infty) = c, \mathbb{P}(M = (1 - c)t) = 1 - c \).
Proof. By continuity, we only need to show the result for death rate equal to \( \alpha \). We use Lemma 4.2 with \( \lambda = \alpha \) and \( \mu = \alpha \). We get, for any \( \delta > 0 \)
\[
\mathbb{P}(Y(t) = -\infty) = \mathbb{P}(L(t) = 0) = c + \mathcal{O}\left(\frac{\alpha e^{-\alpha(1-c)t\log(\alpha)}}{\alpha}\right) \overset{\alpha \to \infty}{\approx} c,
\]
\[
\mathbb{P}(-\infty < Y(t) \leq (1 - c - \delta)t) = \mathbb{P}(1 \leq L(t\log(\alpha)/\alpha) \leq e^{\alpha(1-c-\delta)t\log(\alpha)/\alpha})
\leq C_1 e^{\alpha(1-c-\delta)t\log(\alpha)/\alpha} e^{-\alpha(1-c)t\log(\alpha)/\alpha} = C_1 e^{-\delta t\log(\alpha)} \overset{\alpha \to \infty}{\approx} 0,
\]
\[
\mathbb{P}(Y(t) > (1 - c + \delta)t) = \mathbb{P}(L(t\log(\alpha)/\alpha) > e^{\alpha(1-c+\delta)t\log(\alpha)/\alpha})
\leq C_1 e^{\alpha(1-c)t\log(\alpha)/\alpha} e^{-\alpha(1-c+\delta)t\log(\alpha)/\alpha} = C_1 e^{-\delta t\log(\alpha)} \overset{\alpha \to \infty}{\approx} 0.
\]
Hence the result follows. \( \square \)

The following two lemmata are extensions of Lemma 4.6 in [Greven et al. 2010].

**Lemma 4.4** (Growth rate of binary branching process). Let \((L(t))_{t \geq 0}\) be a binary branching process with (individual) birth rate \( \alpha \) and death rate \( \alpha \) + \( o(\alpha) \) for \( c \geq 0 \). If \( L(0) = c' \alpha^\psi \) for some \( \psi, c' > 0 \) and \( Z(t) := \psi + (1-c)t \), then
\[
Y := \left(\frac{\log L(t\log(\alpha)/\alpha)}{\log(\alpha)}\right)_{t \geq 0} \overset{\alpha \to \infty}{\to} Z.
\]

**Proof.** Again, we only prove the result for death rate \( \alpha \). We have that \( L(t\log(\alpha)/\alpha) = \alpha^{Y(t)} \). Clearly, \( Y(0) \overset{\alpha \to \infty}{\approx} \psi \), and \( Y \) is a Markov process with generator, for \( f \in C^1(\mathbb{R}) \)
\[
Gf(y) = \alpha y \log(1 + \frac{\log(a) - \log(\alpha)}{\log(a)}) - f(y) + c\left(f(y) - f(y)\right)
= \alpha y \log(1 + \frac{\log(a) - \log(\alpha)}{\log(a)}) - f(y) + c\left(f(y) - f(y)\right)
\overset{\alpha \to \infty}{\approx} \alpha y \log(1 + \frac{1}{\alpha}) - f(y) + c\left(f(y) - f(y)\right)
\overset{\alpha \to \infty}{\approx} (1-c)f'(y).\]

By standard arguments (see e.g. Ethier and Kurtz 1986), the result follows. \( \square \)

**Lemma 4.5** (Hitting times of super-critical branching process). Let \( z, c' > 0, c \in (0,1), \epsilon \in (0,c/2), \) and \( \mathcal{L} = (L_t)_{t \geq 0} \) be a birth-death process with birth rate \( b_k = \alpha k \) and death rate \( d_k = \alpha(1-c+\epsilon)k \), started in \( L_0 = z \alpha^{-1-\epsilon} \). Moreover, let \( T_n \) be the first time when \( L_t = n \). Then,
\[
\mathbb{P}(\frac{\alpha}{\log(\alpha)} T_n > \frac{2\epsilon}{c} | L_0 = z \alpha^{-1-\epsilon}) \overset{\alpha \to \infty}{\to} 0.
\]

**Proof.** It suffices to treat the case \( d_k = \alpha(1-c+\epsilon) \), since other cases have an earlier hitting time of \( z \alpha \). Now, \( \mathcal{L} \) is a supercritical branching process and we let \( \mathcal{M} = (M_t)_{t \geq 0} \) be the process of immortal lines in \( \mathcal{L} \). Then, by classical theory, we find that each line in \( \mathcal{L} \) belongs to \( \mathcal{M} \) with probability \( c - \epsilon \). Moreover, for some \( z' > 0, M_0 = z' \alpha^{-1-\epsilon} \) and \( \mathcal{M} \) is a pure branching process with splitting rate \( (c-\epsilon) \alpha \). If \( S_{z\alpha} \) is the first time \( t \) when \( M_\alpha = z\alpha \), we find by Lemma 4.4 that
\[
\frac{\alpha}{\log(\alpha)} T_{\alpha} \overset{\alpha \to \infty}{\approx} \epsilon \frac{c}{c-\epsilon} < \frac{2\epsilon}{c},
\]
since \( \log(L(t\log(\alpha)/\alpha)) / (\log(\alpha)) \) grows (for large \( \alpha \)) linearly at speed \( c - \epsilon \). Using \( S_{z\alpha} \geq T_{z\alpha} \), we conclude by
\[
\mathbb{P}(\frac{\alpha}{\log(\alpha)} T_{z\alpha} > \frac{2\epsilon}{c} | L_0 = z \alpha^{-1-\epsilon} ) \leq \mathbb{P}(\frac{\alpha}{\log(\alpha)} S_{z\alpha} > \frac{2\epsilon}{c} | M_0 = z' \alpha^{-1-\epsilon}) \overset{\alpha \to \infty}{\to} 0.
\]
\( \square \)
Lemma 4.6 (Fast middle phase of local sweep). Let \( \varepsilon, z > 0 \) with \( 0 < z < 2(1 - \varepsilon) \), and \( \mathcal{L} = (L_t)_{t \geq 0} \) be a birth-death process, started in \( L_0 = z\alpha \),

\[
\text{birth rate } b_k = \alpha k \text{ and death rate } d_k = \left( \frac{k}{2} \right) + \frac{1}{2} k(2\alpha - k)c + o(\alpha)
\]

for some \( c \in [0, 1) \). Moreover, let \( T_n \) be the first time when \( L_t = n \). Then,

\[
T_{2\alpha(1-\varepsilon)} = O\left( \frac{1}{\alpha} \right)
\]
as \( \alpha \to \infty \). In particular, we find a sequence \( \varepsilon, z \downarrow 0 \) such that

\[
T_{2\alpha(1-\varepsilon)} - T_{2\alpha z} = o\left( \frac{\log \alpha}{\alpha} \right).
\]

Proof. We rescale state and space by setting \( V^\alpha := (V^\alpha_t)_{t \geq 0} := (L_{t/\alpha}/\alpha)_{t \geq 0} \) and obtain that \( V^\alpha \xrightarrow{\alpha \to \infty} V \) with \( V = (V_t)_{t \geq 0} \) solving

\[
\frac{dV}{dt} = V(1 - \frac{V}{2}) - V(1 - \frac{V}{2})c = V(1 - \frac{V}{2})(1 - c),
\]
and starting in \( V_0 = z \). Since this process hits \( 1 - \varepsilon \) by some time of order 1, we find that \( L \) hits \( 2\alpha(1 - \varepsilon) \) by some time \( O(1/\alpha) \).

Lemma 4.7 (Hitting times of sub-critical branching process). Let \( z, c', c > 0 \), \( \gamma \in [0, 1], \rho \in (\gamma, 1], \varepsilon \in (0, c^2/p \wedge c) \), and \( \mathcal{L} = (L_t)_{t \geq 0} \) be a birth-death process with

\[
\text{birth rate } b_k = \alpha k \text{ and death rate } d_k \text{ such that } |d_k - \alpha(1 + c)k| \leq \varepsilon \alpha k, \text{ started in } L_0 = z\alpha^p.
\]

Moreover, let \( T_n \) be the first time when \( L_t = n \). Then,

\[
\mathbb{P}\left( \frac{\alpha}{\log \alpha} T'_{c' \alpha} - \frac{p - \gamma}{c} > 4\varepsilon \bigg| L_0 = z\alpha^p \right) \xrightarrow{\alpha \to \infty} 0
\]

and

\[
\mathbb{P}\left( \frac{\alpha}{\log \alpha} T_0 - \frac{p}{c} > 2\varepsilon \bigg| L_0 = z\alpha^p \right) \xrightarrow{\alpha \to \infty} 0.
\]

Proof. It suffices to show \((4.2)\), since the hitting time of 0 is (by the Markov property) the independent sum of the hitting time of \( c'\alpha \gamma \) and the extinction time, if the process is started in \( c'\alpha \gamma \).

Define \( S_m^c \) be the extinction time of a branching process \( \mathcal{M} = (M_t)_{t \geq 0} \) with (individual) branching rate \( \alpha \) and death rate \( \alpha(1 + c) \), when started in \( M_0 = m \). Then, from classical theory (see e.g. \cite{Harris1963}, Chapter V (3.4))) it follows, that

\[
f(t) := \mathbb{P}(S_1^c > t) = \frac{c}{(1 + c)e^{t\alpha c} - 1},
\]

\[
g_m(t) := \mathbb{P}(S_m^c > t) = 1 - (1 - f(t))^m.
\]

Hence, for any \( \varepsilon > 0 \),

\[
\mathbb{P}\left( \frac{\alpha}{\log \alpha} S_{2\alpha^p}^c - \frac{p}{c} > \varepsilon \right) = \alpha S_{2\alpha^p} \left( \frac{\log \alpha}{\alpha} \left( \frac{p}{c} + \varepsilon \right) \right) \to 0,
\]

\[
\mathbb{P}\left( \frac{\alpha}{\log \alpha} S_{2\alpha^p}^c - \frac{p}{c} < -\varepsilon \right) = 1 - \alpha S_{2\alpha^p} \left( \frac{\log \alpha}{\alpha} \left( \frac{p}{c} - \varepsilon \right) \right) \to 0.
\]

(4.3)

Stochastically, \( S_{2\alpha^p}^{c+} \leq T_{2\alpha^p} \leq S_{2\alpha^p}^{c-} \) and hence, for \( \varepsilon \leq c^2/p \),

\[
\mathbb{P}\left( \frac{\alpha}{\log \alpha} T_{2\alpha^p} - \frac{p}{c} < -2\varepsilon \right) \leq \mathbb{P}\left( \frac{\alpha}{\log \alpha} S_{2\alpha^p}^{c+} - \frac{p}{c + \varepsilon} < -2\varepsilon + \frac{p\varepsilon/c}{c + \varepsilon} \right) \xrightarrow{\alpha \to \infty} 0
\]
as well as
\[ \mathbb{P}\left( \frac{\alpha}{\log \alpha} T_{z\alpha^p} - \frac{p}{c} > 2\varepsilon \right) \leq \mathbb{P}\left( \frac{\alpha}{\log \alpha} S_{z\alpha^p} - \frac{p}{c - \varepsilon} > 2\varepsilon - \frac{p\varepsilon/c}{c - \varepsilon} \right) \xrightarrow{a.s.} 0 \]
by (4.3) and we are done.

\[ \square \]

4.2 Results on non-homogeneous birth-death processes

Kendall (1948) studied time-inhomogeneous birth-death processes; see also Bailey (1990) and Allen (2013). We recall a result on the generating function of such a process. In order to be self-contained, we also give the proof.

**Theorem 4.8** (Time-inhomogeneous birth-death process). Let \( \mathcal{L} = (L_t)_{t \geq 0} \) be a birth-death process with time-inhomogeneous birth rates \( \lambda_k(t) = \lambda_k + \gamma_t \) and death rates \( \mu_k(t) = \mu_k \). If \( L_0 = \ell \), the generating function \( g_\ell(z) := \sum_{k=0}^{\infty} \mathbb{P}(L_t = k)z^k \) satisfies

\[
g_\ell(z) = \left( 1 - \left( \frac{\exp \left( \int_0^t (\mu - \lambda)ds \right)}{1 - z} + \int_0^t \lambda_r \exp \left( \int_0^r (\mu_u - \lambda_u)du \right) dr \right) \right) \cdot \exp \left( -\int_0^t \gamma_s \left( \frac{\exp \left( \int_0^t (\mu - \lambda)ds \right)}{1 - z} + \int_0^t \lambda_r \exp \left( \int_0^r (\mu_u - \lambda_u)du \right) dr \right) \right) ds \right)^{-1}.
\]

In particular, the probability that the process eventually goes extinct, starting with \( \ell = 0 \), is given by

\[
\mathbb{P}(L_t = 0) = \exp \left( -\int_0^t \gamma_s \left( 1 + \int_s^t \mu_r e^{\int_s^r (\mu_u - \lambda_u)du} dr \right) \right) ds \right)^{-1}.
\]

**Proof.** We write, using \( p_k(t) := \mathbb{P}(L_t = k) \),

\[
\frac{\partial g_\ell(z)}{\partial t} = \frac{\partial}{\partial t} \mathbb{E}[z^{L_t}] = (\lambda_t(k - 1) + \gamma_t)p_t(k - 1) + \mu_t(k + 1)p_{k+1}(t)
\]

\[
-((\lambda_t + \mu_t)k + \gamma_t)p_k(t))^z
\]

\[
= ( - \lambda_t z(1 - z) + \mu_t(1 - z)) \frac{\partial g_\ell(z)}{\partial z} - \gamma_t(1 - z)g_\ell(z).
\]

Therefore, if \( x \) solves the ODE (with \( s \in [0, \ell] \))

\[
\frac{d}{ds}x_s = -\lambda_{t-s}x_s(1 - x_s) + \mu_{t-s}(1 - x_s) \text{ with } x_0 = z,
\]

we find that

\[
\frac{d}{ds} \mathbb{E}\left[ x_s^{L_s} \exp \left( -\int_0^{t-s} \gamma_{t-u}(1 - x_u)du \right) \right] = \mathbb{E}\left[ \left( ( - \lambda_s x_{t-s}(1 - x_{t-s}) + \mu_s(1 - x_{t-s})) \frac{\partial}{\partial z} z^{L_s} \right)_{z=x_{t-s}} - \gamma_s(1 - x_{t-s})x_{t-s}^{L_s} \right.
\]

\[
- \frac{\partial}{\partial z} \left. z^{L_s} \right|_{z=x_{t-s}} ( - \lambda_s x_{t-s}(1 - x_{t-s}) + \mu_s(1 - x_{t-s}))
\]

\[
+ x_{t-s}^{L_s} \gamma_s(1 - x_{t-s}) \exp \left( -\int_0^{t-s} \gamma_{t-u}(1 - x_u)du \right) \right]
\]

\[
= 0.
\]
Thus, the solution of (4.6) is given by
\[ g_t(z) = g_0(x_t) \exp \left( - \int_0^t \gamma_{t-s} (1 - x_s) ds \right). \]

Since
\[ x_s = 1 - \left( \frac{\exp \left( \int_{t-s}^t (\mu_r - \lambda_r) dr \right)}{1 - z} \right) + \int_{t-s}^t \lambda_r \exp \left( \int_s^r (\mu_u - \lambda_u) du \right) dr \]
solves (4.7), we find (4.4). Then, (4.5) arises for \( z = 0 \), since
\[ e^{\int_{t-s}^t (\mu_r - \lambda_r) dr} + \int_{t-s}^t \lambda_r e^{\int_{t-s}^r (\mu_u - \lambda_u) du} dr \]
\[ = e^{\int_{t-s}^t (\mu_r - \lambda_r) dr} + \int_{t-s}^t \mu_r e^{\int_{t-s}^r (\mu_u - \lambda_u) du} dr - \int_{t-s}^t (\mu_r - \lambda_r) e^{\int_{t-s}^r (\mu_u - \lambda_u) du} dr \]
\[ = 1 + \int_{t-s}^t \mu_r e^{\int_{t-s}^r (\mu_u - \lambda_u) du} dr. \] (4.8)

**Proposition 4.9** (Survival of a non-homogeneous branching process with immigration). Let \( 0 < c_1 < c_2 < 1 \) and \( \rho > 0 \) and let \( y_t \) solve \( dy = (c_2 - c_1) y(1 - y) \) with \( y_0 = 1/2 \). Let \( \mathcal{L} = (L(t))_{t \in \mathbb{R}} \) be a binary non-homogeneous branching process with \( L_{-\infty} = 0 \), such that every individual gives birth at rate \( \lambda = 1 \), dies at rate
\[ \mu_t = c_1 (1 - y_t) + c_2 y_t = y_t (c_2 - c_1) + c_1, \]
and immigration rate \( \gamma_t = \rho y_t (1 - y_t) \). Then, the probability that the process survives is
\[ \mathbb{P}[L_\infty > 0] = 1 - \left( \frac{1 - c_1}{1 - c_2} \right)^{\frac{1 - c_1 (1 - c_2)}{(c_2 - c_1)^2}}. \]

**Proof.** We have to compute, using Theorem 4.8 (see also the equality (4.8)),
\[ -\log \mathbb{P}(L_\infty = 0) = \lim_{t \to \infty} \int_{-t}^t \rho y_s (1 - y_s) \]
\[ \cdot \left( \exp \left( - \int_s^t (1 - c_1 - y_u (c_2 - c_1)) du \right) \right) \]
\[ + \int_s^t \exp \left( - \int_s^r (1 - c_1 - y_u (c_2 - c_1)) du \right) dr \] \[ - \int_s^r (1 - c_1 - y_u (c_2 - c_1)) du = (1 - c_1) (r - s) - \int_{y_s}^{y_r} \frac{1}{1 - z} dz \]
\[ = (1 - c_1) (r - s) + \log \left( \frac{1 - y_r}{1 - y_s} \right) \]
Therefore, we will use the stopping times $L$

\[ L = \inf\{t \geq 0 : L_i(t) \geq k\}. \] (4.9)

**Lemma 4.10** (Type 3 does not occur before type 2 reaches $\varepsilon \alpha$). Let $L$ be as in Definition 3.1. Then, for any $\varepsilon, \alpha \to 0$, we find

\[ \lim_{\alpha \to 0} \mathbb{P}(T^2_{\varepsilon, \alpha} < T^2_k) = 1. \] (4.10)

**Proof.** Type-3-particles only arise by an event of rate $\ell_1 \ell_2 \rho/(2\alpha)$, so the time $T^3_k$ is the same if we ignore the decrease in type-2-particles due to this event. In other words, we consider the rates of in- and decrease of type-2-particles, given that $\ell_2 \leq \varepsilon \alpha$ and $\ell_0 + \ell_1 + \ell_2 \leq 2\alpha(1 + \delta)$ (see Lemma 4.1)

\[ r_2^+ := \alpha \ell_2, \]
\[ r_2^- := \ell_2 \frac{1}{2} \ell_1 (1 - c_2 + c_1) + \frac{1}{2} \ell_2 \ell_0 (1 - c_2) \]
\[ \leq \ell_2 (1 - c_2 + c_1)(\ell_0 + \ell_1 + \ell_2) + (c_2 - c_1) \varepsilon \alpha \to \alpha \ell_2. \]
Hence, using Lemma 4.1 and Lemma 4.11, and \( \delta > 0 \) such that \((1-c_2+c_1)(1+\delta) < 1\),

\[
\mathbb{P}(T_1^3 \geq T_2^2) = \mathbb{E}\left[ \exp\left( -\int_{T_2^2}^{T_1^3} \frac{\rho}{2\alpha} L_1(t)L_2(t)dt \right) \right] \\
\geq \exp\left( -\mathbb{E}\left[ \int_{T_2^2}^{T_1^3} \frac{\rho}{2\alpha} L_1(t)L_2(t)dt \right] \right) \\
\geq \exp\left( -\mathbb{E}\left[ \int_{T_2^2}^{T_1^3} \rho(1+\delta)L_2(t)dt \right] \right) \\
\geq \exp\left( -\frac{\rho(1+\delta)}{\alpha(1-(1-c_2+c_1)(1+\delta))^{\varepsilon\alpha}} \right) \overset{\alpha \to \infty}{\approx} 1.
\]

\[\square\]

**Lemma 4.11** (Occupation time of birth-death process). Let \((L(t))_{t \geq 0}\) be a birth-death-process with birth rate \( \lambda_i \geq a_i \) and death rate \( \mu_i \leq b_i \). If \( a > b \) and \( L(0) = k \), for \( T_\ell := \inf\{t : L(t) = \ell\} \) and \( \ell > k \),

\[
\mathbb{E}\left[ \int_{0}^{T_\ell} L(s)ds \right] \leq \frac{\ell - k}{a - b}.
\]

**Proof.** We know that

\[
\left( L(t) - L(0) - \int_{0}^{t} \lambda_{L(s)} - \mu_{L(s)}ds \right)_{t \geq 0}
\]

is a (local) martingale, hence,

\[
\left( L(t) - L(0) - \int_{0}^{t} (a - b)L(s)ds \right)_{t \geq 0}
\]

is a (local) sub-martingale. Using optional stopping,

\[
\mathbb{E}\left[ \int_{0}^{T_\ell} (a - b)L(s)ds \right] \leq \mathbb{E}[L(T_\ell) - L(0)] = (\ell - k)\mathbb{P}(T_\ell < \infty) - k \cdot \mathbb{P}(T_\ell = \infty) \\
\leq \ell - k.
\]

\[\square\]

**Proposition 4.12** (Rescaling \( L \) when type 2 takes over). Let \( L(t) \) from Definition 3.1 be started with

\[
L_0(0) = o(\alpha), \quad L_1(0) = 2(1 - \varepsilon)\alpha + o(\alpha), \quad L_2(0) = 2\varepsilon\alpha + o(\alpha), \quad L_3(0) = 0.
\]

Moreover, let

\[
V_i^\alpha(t) := \frac{L_i(t/\alpha)}{\alpha}, \quad i = 0, 1, 2, \quad V_3^\alpha(t) := L_3(t/\alpha)
\]

and \( V = (V_0, V_1, V_2, V_3) \) be a process with \( V_0 = 0 \), \( V_1(0) = 2(1 - \varepsilon) \), \( V_2(0) = 2\varepsilon \) and \( V_3(0) = 0 \), such that

\[
dV_1 = (c_2 - c_1)V_1(1 - V_1/2)dt, \quad V_2 = 2 - V_1,
\]

and \( V_3 \) is a time-dependent binary branching process with splitting rate 1, death rate \( \frac{c_2}{2}V_2 + \frac{c_1}{2}V_1 \), and immigration rate \( \frac{\rho}{2}V_1V_2 \). Then,

\[
(V_0^\alpha, V_1^\alpha, V_2^\alpha, V_3^\alpha) \overset{\alpha \to \infty}{\longrightarrow} (V_0, V_1, V_2, V_3).
\]

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We use the representation of the process \( \mathcal{L} \) as described in Remark 3.2. For such networks, limit results using a scaling parameter (\( \alpha \) in our case) have been established in Ball et al. (2006), Kang and Kurtz (2013), Pfaffelhuber and Popovic (2015) and others. The following is an application of Lemma 2.4 of Pfaffelhuber and Popovic (2015) (see also Theorem 4.1 of Kang and Kurtz (2013)). We use the representation of the process \( \mathcal{L} \) using time-change equations of the form (recall that \( c_0 = 0 \))

\[
V_0(0) = V_0(0) + \frac{1}{\alpha} Y_{b_0} \left( \int_0^t \alpha V_0(s) ds \right) - \frac{1}{\alpha} Y_{d_0} \left( \frac{\alpha}{2} \int_0^t V_0(s) (V_0(s) - \frac{1}{\alpha}) ds \right) - \sum_{j=1,2} \frac{1}{\alpha} Y_{d_j 0} \left( \frac{1}{2} \int_0^t \rho V_0(s) V_j(s) ds \right) - \frac{1}{\alpha} Y_{d_3 0} \left( \int_0^t V_0(s) V_3(s) ds \right) - \frac{1}{\alpha} (Y_{r 12 0} + Y_{r 12 3}) \left( \int_0^t \frac{\rho}{2} V_1(s) V_2(s) ds \right),
\]

\[
V_1(0) = V_1(0) + \frac{1}{\alpha} Y_{b_1} \left( \int_0^t \alpha V_1(s) ds \right) - \frac{1}{\alpha} Y_{d_1} \left( \frac{\alpha}{2} \int_0^t V_1(s) V_3(s) ds \right) - \frac{1}{\alpha} Y_{d_3 1} \left( (1 - c_1/2) \int_0^t V_1(s) V_3(s) ds \right) + \frac{1}{\alpha} (Y_{r 0 3 1} + Y_{r 0 3 2}) \left( \int_0^t \frac{\rho}{2} V_1(s) V_3(s) ds \right),
\]

\[
V_2(0) = V_2(0) + \frac{1}{\alpha} Y_{b_2} \left( \int_0^t \alpha V_2(s) ds \right) - \frac{1}{\alpha} Y_{d_2} \left( \frac{\alpha}{2} \int_0^t V_2(s) V_3(s) ds \right) - \frac{1}{\alpha} Y_{d_3 2} \left( (1 - c_2/2) \int_0^t V_2(s) V_3(s) ds \right) + \frac{1}{\alpha} (Y_{r 0 3 2} + Y_{r 0 3 3}) \left( \int_0^t \frac{\rho}{2} V_1(s) V_3(s) ds \right),
\]

\[
V_3(t) = V_3(0) + Y_{b_3} \left( \int_0^t V_3(s) ds \right) - Y_{d_3} \left( \frac{1}{\alpha} \int_0^t V_3(s) (V_3(s) - 1) ds \right) - Y_{d_{13}} \left( \frac{C_1}{2} \int_0^t V_3(s) V_1(s) ds \right) - Y_{d_{23}} \left( \frac{C_2}{2} \int_0^t V_3(s) V_3(s) ds \right) + Y_{r 12 3} \left( \int_0^t \frac{\rho}{2} V_1(s) V_2(s) ds \right) - (Y_{r 0 3 1} + Y_{r 0 3 2}) \left( \int_0^t \frac{\rho}{2} V_0(s) V_3(s) ds \right),
\]

where all \( Y \)'s are independent rate-one Poisson processes.

From Lemma 4.1 \( (V_0, V_1, V_2, \mathcal{L}) \) satisfy the compact containment condition. Therefore, since \( V_3 \) is bounded by a binary branching process with immigration at most \( \rho \), it satisfies the compact containment condition as well. Therefore, neglecting all terms of lower order according to Lemma 2.4 of Pfaffelhuber and Popovic.
(i.e. using that \( \frac{1}{\alpha} V(t) \to^\alpha 0 \)), we can approximate \( V^\alpha \) by \( V \), which satisfies

\[
V_0(t) = V_0(0) + \int_0^t V_0(s) ds - \frac{1}{2} \int_0^t (V_0(s))^2 ds - \sum_{j=1,2} \frac{1}{2} (1 + c_j) \int_0^t V_0(s) V_j(s) ds,
\]

\[
V_1(t) = V_1(0) + \int_0^t V_1(s) ds - \frac{1}{2} \int_0^t (V_1(s))^2 ds - \sum_{j=0,2} \frac{1}{2} (1 - c_1 + c_j) \int_0^t V_1(s) V_j(s) ds,
\]

\[
V_2(t) = V_2(0) + \int_0^t V_2(s) ds - \frac{1}{2} \int_0^t (V_2(s))^2 ds
\]

\[
- \sum_{j=0,1} \left( \frac{1}{2} (1 - c_2 + c_j) \int_0^t V_2(s) V_j(s) ds \right),
\]

\[
V_3(t) = V_3(0) + V_{03} \left( \int_0^t V_3(s) ds \right) - V_{213} \left( \frac{c_1}{2} \int_0^t V_3(s) V_1(s) ds \right)
\]

\[
- V_{233} \left( \frac{c_2}{2} \int_0^t V_2(s) V_3(s) ds \right) + Y_{r123} \left( \int_0^t \rho V_1(s) V_2(s) ds \right).
\]

Since \( V_0^\alpha(0) \to^\alpha 0 \), we see that \( V_0 = 0 \). Consequently, since \( V_0 + V_1 + V_2 = 2 \) by Lemma 4.1, we see that \( V_1 + V_2 = 2 \) and therefore \( (V_1, V_2) \) satisfy

\[
dV_1 = \left( V_1 - \frac{1}{2} V_1 (V_1 + V_2) - \frac{1}{2} (c_2 - c_1) V_1 V_2 \right) dt = -\frac{1}{2} (c_2 - c_1) V_1 (2 - V_1) dt,
\]

\[
dV_2 = -dV_1,
\]

and \( V_3 \) is a branching process with splitting rate 1, (individual) death rate \( \frac{\rho}{2} V_1 + \frac{\rho}{2} V_2 \) and immigration rate \( \frac{\rho}{2} V_1 V_2 \), as claimed. \( \square \)

5 Proof of Theorems 1 and 2

In the light of Proposition 3.3, we have to show fixation of type 3 in \( L \). In order to do this, we give a fundamental result (Proposition 5.1) in Subsection 5.1 Proofs of both theorems are given in Sections 5.2 and 5.3 respectively.

5.1 A fundamental result; Proofs of both Theorems

The following result is fundamental for the proofs of our main results. Its proof is given in the next subsection. For illustration, consult Figures 5.1 and 5.2.

Proposition 5.1 (Scenarios of fixation and \( L \)). Assume the same situation as in Theorem 1 and \( L \) as in Definition 3.1

1. If \( \psi < \frac{c_1}{c_2} \), let

\[
p = 1 - \left( \frac{1 - c_2}{1 - c_1} \right)^{2(1 - \phi_2)(1 - \phi_1) \psi^{2(1 - 2\psi)}},
\]

\[
\tau_1 = \frac{\psi}{c_1},
\]

\[
\tau_2 = \tau_1 + \frac{1}{c_2 - c_1} \left( 1 - \frac{c_2 \psi}{c_1} \right),
\]

\[
\tau_3 = \tau_2 + \frac{1}{1 - c_2},
\]

\[
\tau_4 = \tau_3 + \frac{1}{1 - c_2},
\]

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and $Q = (Q_2, Q_3)$ be a stochastic process as follows. Starting with $(0, -\infty)$, the process is with probability $1 - c_2$

$$Q_2(\tau) = \begin{cases} 0, & \tau = 0, \\ -\infty, & \tau > 0 \end{cases} \quad \text{and} \quad Q_3(\tau) = -\infty, \quad (5.2)$$

with probability $c_2(1 - p)$

$$Q_2(\tau) = \begin{cases} c_2 \tau, & \tau \leq \tau_1, \\ \frac{c_2}{c_1} \psi + (c_2 - c_1)(\tau - \tau_1), & \tau_1 < \tau < \tau_2, \\ 1, & \tau > \tau_2 \end{cases} \quad \text{and} \quad Q_3(\tau) = -\infty, \quad (5.3)$$

with probability $c_2 p$

$$(Q_2(\tau), Q_3(\tau)) = \begin{cases} (c_2 \tau, -\infty), & \tau \leq \tau_1, \\ \left(\frac{c_2}{c_1} \psi + (c_2 - c_1)(\tau - \tau_1), -\infty\right), & \tau_1 < \tau \leq \tau_2, \\ (1, (c_3 - c_2)(\tau - \tau_2)), & \tau_2 < \tau \leq \tau_3, \\ (1 - (1 - c_2)(\tau - \tau_3), 1), & \tau_3 < \tau \leq \tau_4, \\ (-\infty, 1), & \tau > \tau_4. \end{cases} \quad (5.4)$$

Then, (with $\Rightarrow$ denoting convergence of finite-dimensional distributions),

$$\left(\frac{\log L_2(\tau \log \alpha / \alpha)}{\log \alpha}, \log L_3(\tau \log \alpha / \alpha)\right) \xrightarrow{\alpha \to \infty} (Q_2(\tau), Q_3(\tau))_{\tau \geq 0, \tau \neq \tau_2}. \quad (5.5)$$

In the last case (5.4), we find that $L_j(\tau \log \alpha / \alpha) \xrightarrow{\alpha \to \infty} 0$ for $j = 0, 1, 2$ if and only if $\tau \geq \tau_4$.

2. If $\frac{c_2}{c_1} < \psi < 1$, let

$$\sigma_1 = \frac{1}{c_2},$$

$$\sigma_2 = \sigma_1 + \frac{1 - \psi + c_1/c_2}{c_2 - c_1}.$$ 

Let $R = (R_1, R_2, R_3)$ be a stochastic process as follows. Starting with $(1 - \psi, 0, -\infty)$, the process is with probability $1 - c_2$

$$R_1(\tau) = \begin{cases} 1 - \psi + c_1 \tau, & 0 \leq \tau \leq \tau_1, \\ 1, & \tau > \tau_1 \end{cases} \quad \text{and} \quad (R_2, R_3) = (Q_2, Q_3) \text{ from } (5.2), \quad (5.5)$$

with probability $c_2$

$$(R_1(\tau), R_2(\tau), R_3(\tau)) = \begin{cases} (1 - \psi + c_1 \tau, c_2 \tau, -\infty), & 0 \leq \tau \leq \sigma_1, \\ (1 - \psi + c_1 \sigma_1 - (c_2 - c_1)(\tau - \sigma_1), 1, -\infty), & \sigma_1 < \tau \leq \sigma_2, \\ (-\infty, 1, -\infty), & \tau > \sigma_2. \end{cases} \quad (5.6)$$
Figure 5.1: For the limit of the process \( (Q^i)_{i=1,2,3} \) as \( \alpha \to \infty \) in the case \( \psi < \frac{c_1}{c_2} \), there are three possibilities. (Note that convergence towards \( Q^1 \) is not claimed in Proposition 5.1, but is displayed here for completeness.) (A) Type 2 \( \equiv aB \) does not even establish, leading to quick fixation of \( 1 \equiv Ab \). No 3 \( \equiv AB \) is produced. This happens with probability \( 1 - c_2 \). (B) Type \( aB \) establishes, but no successful type \( AB \) is created during the spread of type \( aB \). This happens with probability \( c_2(1-p) \). (C) Type \( aB \) establishes, successful types \( AB \) are created and they spread through the whole population. This happens with probability \( c_2p \).
Figure 5.2: For the limit of the process $(R_i)_{i=1,2,3}$ as $\alpha \to \infty$ in the case $\frac{c_1}{c_2} < \psi < 1$, there are two possibilities. In both cases, type 3 does not occur, so fixation of this type has a probability converging to 0.
Then, \[
\left( \frac{\log L_i(\tau \log \alpha)}{\log \alpha} \right)_{i=1,2,3,\tau \geq 0} \quad \overset{\alpha \to \infty}{\Longrightarrow} \quad (R_i(\tau))_{i=1,2,3,\tau \geq 0}.
\]

In particular, \(L_3 \overset{\alpha \to \infty}{\Longrightarrow} 0\).

**Proof.** During the proof, we will make use of sequences \(\varepsilon_n \downarrow 0\). Their precise value will change from occurrence to occurrence.

We start with the initial phase. Let \(\ell_1, \ell_2 \leq \varepsilon_n \alpha\) for \(\varepsilon_n \overset{\alpha \to \infty}{\to} 0\). We bound the rates \(r_i^+, r_i^-, i = 1, 2\) of in- and decrease of \(L_1\) and \(L_2\) before \(T^1_{\varepsilon_n \alpha} \wedge T^2_{\varepsilon_n \alpha}\). During this time, we have that \(\ell_0 \alpha = 2\alpha + o(\alpha)\) by Lemma 4.1 and \(\ell_3 = 0\) by Lemma 4.10. Hence, \(r_i^+, r_i^-\) satisfy

\[
\begin{align*}
r_i^+ &:= \ell_1, \quad i = 1, 2, \\
r_1^- &= \frac{1}{2} \ell_1 \ell_0 (1-c_1) + \ell_1 \cdot o(\alpha) = \ell_1 \alpha (1-c_1 + o(1)), \\
r_2^- &= \frac{1}{2} \ell_2 \ell_0 (1-c_2) + \ell_2 \cdot o(\alpha) = \ell_2 \alpha (1-c_2 + o(1))
\end{align*}
\]

before \(T^1_{\varepsilon_n \alpha} \wedge T^2_{\varepsilon_n \alpha}\). So, \(L_1\) behaves like a birth-death process as in Lemma 4.4 starting in \(c^\alpha, r_1^-, i = 1, 2\) of in- and decrease of \(L_1\) and \(L_2\) before \(T^1_{\varepsilon_n \alpha} \wedge T^2_{\varepsilon_n \alpha}\). During this time, we have that \(\ell_0 \alpha = 2\alpha + o(\alpha)\) by Lemma 4.1 and \(\ell_3 = 0\) by Lemma 4.10. Hence, \(r_i^+, r_i^-\) satisfy

\[
\begin{align*}
r_i^+ &:= \ell_1, \quad i = 1, 2, \\
r_1^- &= \frac{1}{2} \ell_1 \ell_0 (1-c_1) + \ell_1 \cdot o(\alpha) = \ell_1 \alpha (1-c_1 + o(1)), \\
r_2^- &= \frac{1}{2} \ell_2 \ell_0 (1-c_2) + \ell_2 \cdot o(\alpha) = \ell_2 \alpha (1-c_2 + o(1))
\end{align*}
\]

before \(T^1_{\varepsilon_n \alpha} \wedge T^2_{\varepsilon_n \alpha}\). So, \(L_1\) behaves like a birth-death process as in Lemma 4.4 starting in \(c^\alpha, r_1^-, i = 1, 2\) of in- and decrease of \(L_1\) and \(L_2\) before \(T^1_{\varepsilon_n \alpha} \wedge T^2_{\varepsilon_n \alpha}\). During this time, we have that \(\ell_0 \alpha = 2\alpha + o(\alpha)\) by Lemma 4.1 and \(\ell_3 = 0\) by Lemma 4.10. Hence, \(r_i^+, r_i^-\) satisfy

\[
\begin{align*}
r_i^+ &:= \ell_1, \quad i = 1, 2, \\
r_1^- &= \frac{1}{2} \ell_1 \ell_0 (1-c_1) + \ell_1 \cdot o(\alpha) = \ell_1 \alpha (1-c_1 + o(1)), \\
r_2^- &= \frac{1}{2} \ell_2 \ell_0 (1-c_2) + \ell_2 \cdot o(\alpha) = \ell_2 \alpha (1-c_2 + o(1))
\end{align*}
\]

2. We can now proceed to show our result for \(\frac{\alpha}{c^\alpha} < \psi < 1\). With probability \(c_2\), in the initial phase, according to Corollary 4.3, \((\log L_2((\tau \log \alpha)/\alpha))/\log \alpha\) increases approximately linearly with speed \(c_2\). In this case, for any \(\varepsilon_n \downarrow 0\) and \(\tau_\alpha = \frac{1}{c_2} - \varepsilon_n\), we find that \((\log L_2((\tau_\alpha \log \alpha)/\alpha))/\log \alpha\) \(\overset{\alpha \to \infty}{\to} 1\) whereas from Lemma 4.4 and \((\log L_1((\tau_\alpha \log \alpha)/\alpha))/\log \alpha\) \(\overset{\alpha \to \infty}{\to} 1 - \psi + \frac{c_\alpha}{c_2} < 1\). From Lemma 4.6, we see that we can choose \(\varepsilon_n\) such that \(L_2\) hits \(2\alpha(1-\varepsilon_n)\) after some time of duration \(o((\log \alpha)/\alpha)\). After \(T^2_{\alpha(1-\varepsilon_n)}\), we have that \(L_0 = o(\alpha), L_1 = o(\alpha), L_2 = 2\alpha + o(\alpha)\) and \(L_1\) has rate of decrease

\[
\begin{align*}
r_1^- &= \ell_1 \alpha (1-c_1 + c_2 + o(1)).
\end{align*}
\]

So from here on, \((\log L_1((\tau \log \alpha)/\alpha))/\log \alpha\) decreases linearly with speed \(c_2-c_1\) due to Lemma 4.7 and hits \(-\infty\) at time approximately \(\sigma_2\). During this whole process, the expected number of particles of type 3 which are created is bounded for some small \(\delta > 0\) and some \(c > 0\) by

\[
\mathbb{E} \left[ \int_0^{\sigma_2} \frac{\rho \log \alpha}{\alpha} L_1(s)L_2(s)ds \right] \leq c \frac{\log \alpha}{\alpha^2} \rho \alpha^{1-\psi+c_1/c_2+\delta} \alpha \delta s = o(1),
\]

so \((\log L_3((\tau \log \alpha)/\alpha))/\log \alpha = -\infty\) with high probability for all \(\tau\). This shows all assertions of 2.

1. We have already seen that initially \((\log L_2((\tau \log \alpha)/\alpha))/\log \alpha\) increases approximately linearly with speed \(c_2\) with probability \(c_2\), and with probability \(1-c_2\), we have the situation from (5.2). In the sequel, we assume the linear increase,
which happens with probability $c_2$. Since $\psi < \frac{\alpha}{c_2}$, we find with Lemma 4.3 that $(\log L_1((\tau_1 \log \alpha)/\alpha)/\log \alpha) \xrightarrow{\alpha \to \infty} 1$ and $(\log L_2((\tau_1 \log \alpha)/\alpha)/\log \alpha) \xrightarrow{\alpha \to \infty} c_2$. By the fast middle phase of a sweep from Lemma 4.6 for some $\epsilon_\alpha \downarrow 0$, it is $L_1((\tau_1 + \epsilon_\alpha)(\log \alpha)/\alpha) = 2\alpha + o(\alpha)$. From this, we see that $L_2$ has rate of decrease

$$r_2^- = \ell_2 \alpha(1 - c_2 + c_1 + o(1)),$$

as long as $\ell_2 \leq \epsilon_\alpha \alpha$ for some $\epsilon_\alpha \downarrow 0$. From Lemma 4.7, we see that $\log L_2((\tau_1 + \tau)(\log \alpha)/\alpha)$ increases linearly at speed $c_2 - c_1$ until $L_2$ hits $\epsilon_\alpha \alpha$ for some $\epsilon_\alpha \downarrow 0$, which happens at some time $(\tau_2 + \epsilon_\alpha)(\log \alpha)/\alpha$. Since we know that $T_3 > T_2^2$ with high probability for any $\epsilon_\alpha \downarrow 0$ from Lemma 4.10, we see from Proposition 4.12 that for $\epsilon > 0$ small enough,

$$\left( L_1 \left( T_2 \alpha + \frac{t}{\alpha} \right), L_2 \left( T_2 \alpha + \frac{t}{\alpha} \right), L_3 \left( T_2 \alpha + \frac{t}{\alpha} \right) \right)_{t \geq 0}$$

converges towards a process $(V^1, V^2, V^3)$ with $V^1(0) = 2(1 - \epsilon)$, $V^2(0) = \epsilon$ and $V^3(0) = 0$ as in Proposition 4.12. From Proposition 4.9 we find that $V^3$ survives with probability $p$ and goes extinct with probability $1 - p$. Therefore, with probability $1 - p$, we have that $L_3((\tau_2 + \tau)(\log \alpha)/\alpha) = 0$ for any $\tau > 0$ (and therefore $\log L_3((\tau_2 + \tau)(\log \alpha)/\alpha) = -\infty$). With probability $p$, the process $L_3$ survives, so for any small $\delta > 0$, by time $(\tau_2 + \delta)(\log \alpha)/\alpha$, the process $L_3$ has death rate

$$r_3^- = \ell_3 \alpha(1 - c_2 + o(1)),$$

and therefore, $\log L_3((\tau_2 + \tau)(\log \alpha)/\alpha)$ increases by Lemma 4.4 approximately linearly at speed $1 - c_2$ until $T_3^3$ for some $\epsilon_\alpha \downarrow 0$, hence $T_3^\alpha = \tau_3 + o(1)$ with probability $p$. From here on, the argument follows along the same line as in 2: During a time of duration of order $o((\log \alpha)/\alpha)$, $L_3$ grows logarithically up to $2(1 - \epsilon_\alpha)\alpha$, and then $\log L_2((\tau_2 + \tau)(\log \alpha)/\alpha)$ decreases linearly at speed $1 - c_2$ until it reaches 0 and then jumps to $-\infty$ at time $\tau_4$.

### 5.2 Proof of Theorem 1

1. By Proposition 3.3, the assertion of the Theorem translates to

$$\lim_{\alpha \to \infty} \mathbb{P}(L_j(\infty) = 0, j \neq 3) = c_2 p$$

with $p$ from Proposition 5.1. From the latter proposition, we see that type 3 only fixes (within $L$ in the sense that eventually $L_j(t) = 0$ for $j = 0, 1, 2$) in the case of 5.4 with probability $c_2 p$, which shows the assertion.

2. Here, we have to show that

$$\lim_{\alpha \to \infty} \mathbb{P}(L_j(\infty) = 0, j \neq 3) = 0.$$

This follows from Proposition 5.1, since $L_3 \xrightarrow{\alpha \to \infty} 0$ in all cases.

### 5.3 Proof of Theorem 2

Recall $\tau_4$ from Proposition 5.1 and note that

$$\tau_4 = \frac{1 - \psi}{c_2 - c_1} + \frac{2}{1 - c_2}.$$
which is the limit in probability of \( \frac{\alpha}{\log \alpha} S \) from the Theorem. First, for \( \varepsilon > 0 \), using Proposition 3.3

\[
\lim_{\alpha \to \infty} \lim_{\delta \to \infty} \mathbb{P}_{\xi, \psi} \left( \frac{\alpha}{\log \alpha} S < \tau_4 + \varepsilon | S < \infty \right) = \lim_{\alpha \to \infty} \lim_{\delta \to \infty} \frac{1}{2\alpha \delta} \mathbb{P}_{\xi, \psi} \left( \frac{\alpha}{\log \alpha} S < \tau_4 + \varepsilon \right)
\]

\[
= \lim_{\alpha \to \infty} \frac{\mathbb{P}(L_j((\tau_4 + \varepsilon) \log \alpha) = 0, j \neq 3)}{\mathbb{P}(L_j(\infty) = 0, j \neq 3)} = 1
\]

because, from Proposition 5.1, we see that both the numerator and denominator equal \( c_2p \) in the limit \( \alpha \to \infty \). Moreover, using the same arguments,

\[
\lim_{\alpha \to \infty} \lim_{\delta \to \infty} \mathbb{P}_{\xi, \psi} \left( \frac{\alpha}{\log \alpha} S < \tau_4 - \varepsilon | S < \infty \right) = \lim_{\alpha \to \infty} \frac{\mathbb{P}(L_j((\tau_4 - \varepsilon) \log \alpha) = 0, j \neq 3)}{\mathbb{P}(L_j(\infty) = 0, j \neq 3)} = 0,
\]

because the numerator is 0 according to Proposition 5.1 since no scenario gives fixation of type 3 already by time \( \tau_4 - \varepsilon \).

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