ESTIMATES FOR EIGENVALUES OF $\mathcal{L}$ OPERATOR ON SELF-SHRINKERS*

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Abstract. In this paper we study eigenvalues of the closed eigenvalue problem of the differential operator $\mathcal{L}$, which is introduced by Colding and Minicozzi in [4], on an $n$-dimensional compact self-shrinker in $\mathbb{R}^{n+p}$. Estimates for eigenvalues of the differential operator $\mathcal{L}$ are obtained. Our estimates for eigenvalues of the differential operator $\mathcal{L}$ are sharp. As an application of our estimates for eigenvalues, we give an optimal upper bound for the first eigenvalue and a characterization of compact self-shrinkers in $\mathbb{R}^{n+p}$ with arbitrary co-dimension is given by the first eigenvalue. Furthermore, we also study the Dirichlet eigenvalue problem of the differential operator $\mathcal{L}$ on a bounded domain with a piecewise smooth boundary in an $n$-dimensional complete self-shrinker in $\mathbb{R}^{n+p}$. For Euclidean space $\mathbb{R}^n$, the differential operator $\mathcal{L}$ becomes the Ornstein-Uhlenbeck operator in stochastic analysis. Hence, we also give estimates for eigenvalues of the Ornstein-Uhlenbeck operator.

1. INTRODUCTION

Let $X : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion from an $n$-dimensional Riemannian manifold $M^n$ into a Euclidean space $\mathbb{R}^{n+p}$. One considers a smooth one-parameter family of immersions:

$$F(\cdot, t) : M^n \to \mathbb{R}^{n+p}$$

satisfying $F(\cdot, 0) = X(\cdot)$ and

$$\left(\frac{\partial F(p, t)}{\partial t}\right)^N = H(p, t), \quad (p, t) \in M \times [0, T),$$

where $H(p, t)$ denotes the mean curvature vector of submanifold $M_t = F(M^n, t)$ at point $F(p, t)$. The equation (1.1) is called the mean curvature flow equation. A submanifold $X : M^n \to \mathbb{R}^{n+p}$ is said to be a self-shrinker in $\mathbb{R}^{n+p}$ if it satisfies

$$H = -X^N,$$

where $X^N$ denotes the orthogonal projection into the normal bundle of $M^n$ (cf. Ecker-Huisken [10]). Self-shrinkers play an important role in the study of the mean curvature flow since they are not only solutions of the mean curvature flow equation, but they also describe all possible blow up at a given singularity of a mean curvature flow. Huisken [11] proved that the sphere of radius $\sqrt{n}$ is the only closed embedded...
self-shrinker hypersurfaces with non-zero mean curvature. For classifications of complete non-compact embedded self-shrinker hypersurfaces, Huisken [12] and Colding and Minicozzi [13] proved that an \( n \)-dimensional complete embedded self-shrinker hypersurface with non-negative mean curvature and polynomial volume growth in \( \mathbb{R}^{n+1} \) is a Riemannian product \( S^k \times \mathbb{R}^{n-k}, 0 \leq k < n \). Smoczyk [14] has obtained several results for complete self-shrinkers with higher co-dimensions.

For study of the rigidity problem for self-shrinkers, Le and Sesum [13] and Cao and Li [1] have classified \( n \)-dimensional complete embedded self-shrinkers in \( \mathbb{R}^{n+p} \) with polynomial volume growth if the squared norm \( |A|^2 \) of the second fundamental form satisfies \( |A|^2 \leq 1 \). For a further study, see Colding and Minicozzi [5, 6], Ding and Wang [7], Ding and Xin [8, 9], Wang [15] and so on.

In [4], Colding and Minicozzi introduced a differential operator \( \mathcal{L} \) and used it to study self-shrinkers. The differential operator \( \mathcal{L} \) is defined by

\[
\mathcal{L} f = \Delta f - \langle X, \nabla f \rangle
\]

for a smooth function \( f \), where \( \Delta \) and \( \nabla \) denote the Laplacian and the gradient operator on the self-shrinker, respectively and \( \langle \cdot, \cdot \rangle \) denotes the standard inner product of \( \mathbb{R}^{n+p} \). We should notice that the differential operator \( \mathcal{L} \) plays a very important role in studying of \( n \)-dimensional complete embedded self-shrinkers in \( \mathbb{R}^{n+p} \) with polynomial volume growth in order to guarantee integration by part holds as in [4].

The purpose of this paper is to study eigenvalues of the closed eigenvalue problem for the differential operator \( \mathcal{L} \) on compact self-shrinkers in \( \mathbb{R}^{n+p} \) and eigenvalues of the Dirichlet eigenvalue problem of the differential operator \( \mathcal{L} \) on a bounded domain with a piecewise smooth boundary in complete self-shrinkers in \( \mathbb{R}^{n+p} \). Since the differential operator \( \mathcal{L} \) is self-adjoint with respect to measure \( e^{-\frac{|X|^2}{2}} \, dv \), where \( dv \) is the volume element of \( M^n \) and \( |X|^2 = \langle X, X \rangle \), we know that the closed eigenvalue problem:

\[
\mathcal{L} u = -\lambda u \quad \text{on} \quad M^n
\]

for the differential operator \( \mathcal{L} \) on compact self-shrinkers in \( \mathbb{R}^{n+p} \) has a real and discrete spectrum:

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \rightarrow \infty,
\]

where each eigenvalue is repeated according to its multiplicity. We shall prove the following:

**Theorem 1.1.** Let \( M^n \) be an \( n \)-dimensional compact self-shrinker in \( \mathbb{R}^{n+p} \). Then, eigenvalues of the closed eigenvalue problem (1.4) satisfy

\[
\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{2n - \min_{M^n} |X|^2}{4}).
\]

**Remark 1.1.** The sphere \( S^n(\sqrt{n}) \) of radius \( \sqrt{n} \) is a compact self-shrinker in \( \mathbb{R}^{n+p} \). For \( S^n(\sqrt{n}) \) and for any \( k \), the inequality (1.5) for eigenvalues of the closed eigenvalue problem (1.4) becomes equality. Hence our results in theorem 1.1 are sharp.
As an application our theorem 1.1, we obtain an optimal upper bound for the first
eigenvalue \( \lambda_1 \) and the equality holds if and only if \( M^n \) is the sphere of radius \( \sqrt{n} \),
which characterizes compact self-shrinkers in \( \mathbb{R}^{n+p} \) with arbitrary co-dimension by
the first eigenvalue of the differential operator \( \mathfrak{L} \).

**Corollary 1.1.** Let \( M^n \) be an \( n \)-dimensional compact self-shrinker in \( \mathbb{R}^{n+p} \). Then,
the first eigenvalue \( \lambda_1 \) of the closed eigenvalue problem (1.4) satisfies

\[
\lambda_1 \leq 2 - \frac{\min_{M^n} |X|^2}{n}
\]

and the equality holds if and only if \( M^n \) is the sphere of radius \( \sqrt{n} \).

**Remark 1.2.** S. Y. Cheng \[3\] proved that if the Ricci curvature \( \text{Ric}(M) \) of an
\( n \)-dimensional compact Riemannian manifold satisfies \( \text{Ric}(M) \geq (n-1) \), then the first
eigenvalue \( \lambda_1 \) of the Laplacian satisfies

\[
\lambda_1 \leq \frac{n\pi^2}{d^2}
\]

and the equality holds if and only if \( M \) is the sphere \( S^n(1) \) of radius 1. Hence, our
corollary 1.1 can be seen as a version for compact self-shrinkers in \( \mathbb{R}^{n+p} \).

Furthermore, from the recursion formula of Cheng-Yang \[2\], we can obtain an upper
bound for eigenvalue \( \lambda_k \):

**Theorem 1.2.** Let \( M^n \) be an \( n \)-dimensional compact self-shrinker in \( \mathbb{R}^{n+p} \). Then,
eigenvalues of the closed eigenvalue problem (1.4) satisfy, for any \( k \geq 1 \),

\[
\lambda_k + \frac{2n - \min_{M^n} |X|^2}{4} \leq (1 + \frac{a(\min\{n, k - 1\})}{n}) \frac{(2n - \min_{M^n} |X|^2)}{4})^{k^{2/n}},
\]

where the bound of \( a(m) \) can be formulated as:

\[
\begin{align*}
a(0) &\leq 4, \\
a(1) &\leq 2.64, \\
a(m) &\leq 2.2 - 4\log(1 + \frac{1}{50}(m - 3)), \quad \text{for} \quad m \geq 2.
\end{align*}
\]

In particular, for \( n \geq 41 \) and \( k \geq 41 \), we have

\[
\lambda_k + \frac{2n - \min_{M^n} |X|^2}{4} \leq \left( \frac{2n - \min_{M^n} |X|^2}{4} \right)^{k^{2/n}}.
\]

2. Preliminaries

Suppose \( X : M^n \rightarrow \mathbb{R}^{n+p} \) is an isometric immersion from Riemannian manifold
\( M^n \) into the \((n+p)\)-dimensional Euclidean space \( \mathbb{R}^{n+p} \). Let \( \{E_A\}_{A=1}^{n+p} \) be the
standard basis of \( \mathbb{R}^{n+p} \). The position vector can be written by \( X = (x_1, x_2, \cdots, x_{n+p}) \).
We choose a local orthonormal frame field \( \{e_1, e_2, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\} \) and the
dual coframe field \( \{\omega_1, \omega_2, \cdots, \omega_n, \omega_{n+1}, \cdots, \omega_{n+p}\} \) along \( M^n \) of \( \mathbb{R}^{n+p} \) such that
\( \{e_1, e_2, \cdots, e_n\} \) is a local orthonormal basis on \( M^n \). Thus, we have

\[
\omega_\alpha = 0, \quad n + 1 \leq \alpha \leq n + p
\]
on $M^n$. From the Cartan’s lemma, we have
\[ \omega_{i\alpha} = \sum_{j=1}^{n} h_{ij}^{\alpha} \omega_j, \quad h_{ij}^{\alpha} = h_{ji}^{\alpha}. \]

The second fundamental form $h$ of $M^n$ and the mean curvature vector $H$ are defined, respectively, by
\[ h = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} h_{ij}^{\alpha} \omega_i \otimes \omega_j e_{\alpha}, \quad H = \sum_{\alpha=n+1}^{n+p} \sum_{i=1}^{n} h_{ii}^{\alpha} e_{\alpha}. \]

One considers the mean curvature flow for a submanifold $X : M^n \to \mathbb{R}^{n+p}$. Namely, we consider an one-parameter family of immersions:
\[ F(\cdot, t) : M^n \to \mathbb{R}^{n+p} \]
satisfying $F(\cdot, 0) = X(\cdot)$ and
\[ \frac{\partial F(p, t)}{\partial t} = H(p, t), \quad (p, t) \in M \times [0, T), \]
where $H(p, t)$ denotes the mean curvature vector of submanifold $M_t = F(M^n, t)$ at point $F(p, t)$. An important class of solutions to the mean curvature flow equation (2.1) are self-similar shrinkers, which profiles, self-shrinkers, satisfy
\[ H = -X^N, \]
which is a system of quasi-linear elliptic partial differential equations of the second order. Here $X^N$ denotes the orthogonal projection of $X$ into the normal bundle of $M^n$.

In [4], Colding and Minicozzi introduced a differential operator $\mathfrak{L}$ and used it to study self-shrinkers. The differential operator $\mathfrak{L}$ is defined by
\[ \mathfrak{L}f = \Delta f - \langle X, \nabla f \rangle \]
for a smooth function $f$, where $\Delta$ and $\nabla$ denote the Laplacian and the gradient operator on the self-shrinker, respectively. For a compact self-shrinker $M^n$ without boundary, we have
\[ \int_{M^n} v \mathfrak{L} u e^{-\frac{|X|^2}{2}} dv \]
\[ = \int_{M^n} v(\Delta u - \langle X, \nabla u \rangle) e^{-\frac{|X|^2}{2}} dv \]
\[ = \int_{M^n} v \text{div}(e^{-\frac{|X|^2}{2}} \nabla u) dv \]
\[ = \int_{M^n} u \mathfrak{L} v e^{-\frac{|X|^2}{2}} dv, \]
that is,
\[ \int_{M^n} v \mathfrak{L} u e^{-\frac{|X|^2}{2}} dv = \int_{M^n} u \mathfrak{L} v e^{-\frac{|X|^2}{2}} dv, \]
for any smooth functions $u, v$. Hence, the differential operator $\mathcal{L}$ is self-adjoint with respect to the measure $e^{-\frac{|X|^2}{2}} dv$. Therefore, we know that the closed eigenvalue problem:

\begin{equation}
(2.4) \quad \mathcal{L} u = -\lambda u \quad \text{on } M^n
\end{equation}

has a real and discrete spectrum:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \longrightarrow \infty.$$ 

Furthermore, we have

\begin{equation}
(2.5) \quad \mathcal{L} x_A = -x_A.
\end{equation}

In fact,

\begin{align*}
\mathcal{L} x_A &= \Delta \langle X, E_A \rangle - \langle X, \nabla x_A \rangle \\
&= \langle \Delta X, E_A \rangle - \langle X, E^T_A \rangle \\
&= \langle H, E_A \rangle - \langle X, E^T_A \rangle \\
&= -\langle X^N, E_A \rangle - \langle X, E^T_A \rangle = -x_A.
\end{align*}

Denote the induced metric by $g$ and define $\nabla u \cdot \nabla v = g(\nabla u, \nabla v)$ for functions $u, v$. We get, from (2.5),

\begin{equation}
(2.6) \quad \mathcal{L} |X|^2 = \sum_{A=1}^{n+p} (2x_A \mathcal{L} x_A + 2\nabla x_A \cdot \nabla x_A) = 2(n - |X|^2).
\end{equation}

Here we have used

$$\sum_{A=1}^{n+p} \nabla x_A \cdot \nabla x_A = n.$$

**Proposition 2.1.** For an $n$-dimensional compact self-shrinker $M^n$ without boundary in $\mathbb{R}^{n+p}$, we have

$$\min_{M^n} |X|^2 \leq n = \frac{\int_{M^n} |X|^2 e^{-\frac{|X|^2}{2}} dv}{\int_{M^n} e^{-\frac{|X|^2}{2}} dv} \leq \max_{M^n} |X^N|^2.$$

**Proof.** Since $\mathcal{L}$ is self-adjoint with respect to the measure $e^{-\frac{|X|^2}{2}} dv$, from (2.6), we have

$$n \int_{M^n} e^{-\frac{|X|^2}{2}} dv = \int_{M^n} |X|^2 e^{-\frac{|X|^2}{2}} dv \geq \min_{M^n} |X|^2 \int_{M^n} e^{-\frac{|X|^2}{2}} dv.$$

Furthermore, since

\begin{equation}
(2.7) \quad \Delta |X|^2 = 2(n + \langle X, H \rangle) = 2(n - |X^N|^2),
\end{equation}

we have

$$n \leq \max_{M^n} |X^N|^2.$$

It completes the proof of this proposition. \(\square\)
3. Universal estimates for eigenvalues

In this section, we give proof of the theorem 1.1. In order to prove our theorem 1.1, we need to construct trial functions. Thank to $\mathfrak{L}X = -X$. We can use coordinate functions of the position vector $X$ of the self-shrinker $M^n$ to construct trial functions.

Proof of Theorem 1.1. For an $n$-dimensional compact self-shrinker $M^n$ in $\mathbb{R}^{n+p}$, the closed eigenvalue problem:

\[(3.1)\quad \mathfrak{L}u = -\lambda u \quad \text{on } M^n\]

for the differential operator $\mathfrak{L}$ has a discrete spectrum. For any integer $j \geq 0$, let $u_j$ be an eigenfunction corresponding to the eigenvalue $\lambda_j$ such that

\[
\begin{aligned}
\mathfrak{L}u_j &= -\lambda_j u_j \quad \text{on } M^n \\
\int_{M^n} u_i u_j e^{-\frac{|X|^2}{2}} dv &= \delta_{ij}, \quad \text{for any } i, j.
\end{aligned}
\]

From the Rayleigh-Ritz inequality, we have

\[
\lambda_{k+1} \leq -\frac{\int_{M^n} \varphi \mathfrak{L} \varphi e^{-\frac{|X|^2}{2}} dv}{\int_{M^n} \varphi^2 e^{-\frac{|X|^2}{2}} dv},
\]

for any function $\varphi$ satisfies $\int_{M^n} \varphi u_j e^{-\frac{|X|^2}{2}} dv, 0 \leq j \leq k$. Since $X : M^n \to \mathbb{R}^{n+p}$ is a self-shrinker in $\mathbb{R}^{n+p}$, we have

\[
(3.4) \quad H = -X^N.
\]

Letting $x_A, A = 1, 2, \cdots, n + p$, denote components of the position vector $X$, we define, for $0 \leq i \leq k$,

\[
(3.5) \quad \varphi_i^A := x_A u_i - \sum_{j=0}^{k} a_{ij}^A u_j, \quad a_{ij}^A = \int_{M^n} x_A u_i u_j e^{-\frac{|X|^2}{2}} dv.
\]

By a simple calculation, we obtain

\[
(3.6) \quad \int_{M^n} u_j \varphi_i^A e^{-\frac{|X|^2}{2}} dv = 0, \quad i, j = 0, 1, \cdots, k.
\]

From the Rayleigh-Ritz inequality, we have

\[
(3.7) \quad \lambda_{k+1} \leq -\frac{\int_{M^n} \varphi_i^A \mathfrak{L} \varphi_i^A e^{-\frac{|X|^2}{2}} dv}{\int_{M^n} (\varphi_i^A)^2 e^{-\frac{|X|^2}{2}} dv}.
\]
Since
\[ \mathcal{L}\varphi^A_i = \Delta\varphi^A_i - \langle X, \nabla\varphi^A_i \rangle \]
\[ = \Delta(x_A u_i - \sum_{j=0}^{k} a_{ij}^A u_j) - \langle X, \nabla(x_A u_i - \sum_{j=0}^{k} a_{ij}^A u_j) \rangle \]
\[ = x_A \Delta u_i + u_i \Delta x_A + 2 \nabla x_A \cdot \nabla u_i - \langle X, x_A \nabla u_i + u_i \nabla x_A \rangle \]
\[ - \sum_{j=0}^{k} a_{ij}^A \Delta u_j + \langle X, \sum_{j=0}^{k} a_{ij}^A \nabla u_j \rangle \]
\[ = -\lambda_i x_A u_i + u_i \mathcal{L} x_A + 2 \nabla x_A \cdot \nabla u_i + \sum_{j=0}^{k} a_{ij}^A \lambda_j u_j, \]
we have, from (3.7) and (3.8),
\[ (\lambda_{k+1} - \lambda_i) ||\varphi^A_i||^2 \]
\[ \leq - \int_{M^n} \varphi_i^A (u_i \mathcal{L} x_A + 2 \nabla x_A \cdot \nabla u_i) e^{-\frac{|X|^2}{2}} \, dv := W_i^A, \]
where
\[ ||\varphi^A_i||^2 = \int_{M^n} (\varphi_i^A)^2 e^{-\frac{|X|^2}{2}} \, dv. \]
On the other hand, defining
\[ b_{ij}^A = - \int_{M^n} (u_j \mathcal{L} x_A + 2 \nabla x_A \cdot \nabla u_j) u_i e^{-\frac{|X|^2}{2}} \, dv \]
we obtain
\[ (3.10) \quad b_{ij}^A = (\lambda_i - \lambda_j) a_{ij}^A. \]
In fact,
\[ \lambda_i a_{ij}^A = \int_{M^n} \lambda_i u_i u_j x_A e^{-\frac{|X|^2}{2}} \, dv \]
\[ = - \int_{M^n} u_j x_A \mathcal{L} u_i e^{-\frac{|X|^2}{2}} \, dv \]
\[ = - \int_{M^n} u_i \mathcal{L}(u_j x_A) e^{-\frac{|X|^2}{2}} \, dv \]
\[ = - \int_{M^n} u_i (x_A \mathcal{L} u_j + u_j \mathcal{L} x_A + 2 \nabla x_A \cdot \nabla u_j) e^{-\frac{|X|^2}{2}} \, dv \]
\[ = \lambda_j a_{ij}^A + b_{ij}^A, \]
that is,
\[ b_{ij}^A = (\lambda_i - \lambda_j) a_{ij}^A. \]
Hence, we have
\[ (3.11) \quad b_{ij}^A = -b_{ji}^A. \]
From (3.6), (3.9) and the Cauchy-Schwarz inequality, we infer

\[ W_i^A = - \int_{M^n} \varphi_i^A (u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i) e^{-\frac{|X|^2}{2}} dv \]

(3.12)

\[ = - \int_{M^n} \varphi_i^A (u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i - \sum_{j=0}^{k} b_{ij} A u_j) e^{-\frac{|X|^2}{2}} dv \]

\[ \leq \| \varphi_i^A \| \| u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i - \sum_{j=0}^{k} b_{ij} A u_j \|. \]

Hence, we have, from (3.9) and (3.12),

\[ (\lambda_{k+1} - \lambda_i) (W_i^A)^2 \]

\[ = (\lambda_{k+1} - \lambda_i) \| \varphi_i^A \|^2 \| u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i - \sum_{j=0}^{k} b_{ij} A u_j \|^2 \]

\[ \leq W_i^A \| u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i - \sum_{j=0}^{k} b_{ij} A u_j \|^2. \]

Therefore, we obtain

\[ (\lambda_{k+1} - \lambda_i) (W_i^A)^2 \leq (\lambda_{k+1} - \lambda_i) \| u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i - \sum_{j=0}^{k} b_{ij} A u_j \|^2. \]

(3.13)

Summing on \( i \) from 0 to \( k \) for (3.13), we have

\[ \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) (W_i^A)^2 \leq \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) \| u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i - \sum_{j=0}^{k} b_{ij} A u_j \|^2. \]

(3.14)

By the definition of \( b_{ij} A \) and (3.10), we have

\[ \| u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i - \sum_{j=0}^{k} b_{ij} A u_j \|^2 \]

\[ = \| u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i \|^2 \]

\[ - 2 \sum_{j=0}^{k} b_{ij} \int_{M^n} (u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i) u_j e^{-\frac{|X|^2}{2}} dv + \sum_{j=0}^{k} (b_{ij} A)^2 \]

(3.15)

\[ = \| u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i \|^2 - \sum_{j=0}^{k} (b_{ij} A)^2 \]

\[ = \| u_i \xi x_A + 2 \nabla x_A \cdot \nabla u_i \|^2 - \sum_{j=0}^{k} (\lambda_i - \lambda_j)^2 (a_{ij})^2. \]
Furthermore, according to the definitions of $W^A_i$ and $\varphi^A_i$, we have from (3.10)

$$W^A_i = - \int_{M^n} \varphi^A_i (u_i \mathcal{L} x_A + 2 \nabla x_A \cdot \nabla u_i) e^{-|X|^2} dv$$

$$= - \int_{M^n} (x_A u_i - \sum_{j=0}^k a^A_{ij} u_j) (u_i \mathcal{L} x_A + 2 \nabla x_A \cdot \nabla u_i) e^{-|X|^2} dv$$

$$= - \int_{M^n} (x_A u_i^2 \mathcal{L} x_A + 2 x_A u_i \nabla x_A \cdot \nabla u_i) e^{-|X|^2} dv$$

$$+ \sum_{j=0}^k a^A_{ij} \int_{M^n} u_j (u_i \mathcal{L} x_A + 2 \nabla x_A \cdot \nabla u_i) e^{-|X|^2} dv$$

$$= - \int_{M^n} (x_A \mathcal{L} x_A - \frac{1}{2} \mathcal{L} (x_A)^2) u_i^2 e^{-|X|^2} dv + \sum_{j=0}^k a^A_{ij} b^A_{ij}$$

$$= \int_{M^n} \nabla x_A \cdot \nabla x_A u_i^2 e^{-|X|^2} dv + \sum_{j=0}^k (\lambda_i - \lambda_j)(a^A_{ij})^2. \tag{3.16}$$

Since

$$2 \sum_{i,j=0}^k (\lambda_{k+1} - \lambda_i)^2(\lambda_i - \lambda_j)(a^A_{ij})^2$$

$$\sum_{i,j=0}^k (\lambda_{k+1} - \lambda_i)^2(\lambda_i - \lambda_j)(a^A_{ij})^2 - \sum_{i,j=0}^k (\lambda_{k+1} - \lambda_j)^2(\lambda_i - \lambda_j)(a^A_{ij})^2$$

$$= - \sum_{i,j=0}^k (\lambda_{k+1} - \lambda_i + \lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j)^2(a^A_{ij})^2$$

$$= -2 \sum_{i,j=0}^k (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)^2(a^A_{ij})^2, \tag{3.17}$$

from (3.14), (3.15), (3.16) and (3.17), we obtain, for any $A$, $A = 1, 2, \ldots, n + p$,

$$\sum_{i=0}^k (\lambda_{k+1} - \lambda_i)^2 \int_{M^n} \nabla x_A \cdot \nabla x_A u_i^2 e^{-|X|^2} dv$$

$$\leq \sum_{i=0}^k (\lambda_{k+1} - \lambda_i) \|u_i \mathcal{L} x_A + 2 \nabla x_A \cdot \nabla u_i\|^2. \tag{3.18}$$

On the other hand, since

$$\mathcal{L} x_A = - x_A, \quad \sum_{A=1}^{n+p} (\nabla x_A \cdot \nabla u_i)^2 = \nabla u_i \cdot \nabla u_i,$$
we infer, from (2.6),
\[
\sum_{A=1}^{n+p} \left\| u_i \xi_A + 2 \nabla x_A \cdot \nabla u_i \right\|^2 \\
= \sum_{A=1}^{n+p} \int_{M^n} \left( u_i \xi_A + 2 \nabla x_A \cdot \nabla u_i \right)^2 e^{-\frac{|X|^2}{2}} dv \\
= \sum_{A=1}^{n+p} \int_{M^n} \left( u_i^2(x_A)^2 - 4 u_i x_a \nabla x_A \cdot \nabla u_i + 4(\nabla x_A \cdot \nabla u_i)^2 \right) e^{-\frac{|X|^2}{2}} dv \\
= \sum_{A=1}^{n+p} \int_{M^n} \left( u_i^2(x_A)^2 - \nabla(x_A)^2 \cdot \nabla u_i^2 \right) e^{-\frac{|X|^2}{2}} dv + 4 \int_{M^n} \nabla u_i \cdot \nabla u_i e^{-\frac{|X|^2}{2}} dv \\
= \int_{M^n} \left( \xi |X|^2 + |X|^2 \right) u_i^2 e^{-\frac{|X|^2}{2}} dv + 4 \lambda_i \\
= \int_{M^n} \left( 2n - |X|^2 \right) u_i^2 e^{-\frac{|X|^2}{2}} dv + 4 \lambda_i \\
\leq (2n - \min_{M^n} |X|^2) + 4 \lambda_i.
\]

Furthermore, because of
\[
\sum_{A=1}^{n+p} \nabla x_A \cdot \nabla x_A = n,
\]

taking summation on \(A\) from 1 to \(n + p\) for (3.18) and using (3.19) and (3.20), we get
\[
\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{2n - \min_{M^n} |X|^2}{4}).
\]

It finished the proof of the theorem 1.1.

\[\square\]

**Proof of Corollary 1.1.** Taking \(k = 0\) in the theorem 1.1, we have
\[
\lambda^2 \leq \lambda_1 \left( 2 - \frac{\min_{M^n} |X|^2}{n} \right),
\]

namely,
\[
\lambda_1 \leq \left( 2 - \frac{\min_{M^n} |X|^2}{n} \right)
\]

because of \(\lambda_1 > 0\). If
\[
\lambda_1 = \left( 2 - \frac{\min_{M^n} |X|^2}{n} \right)
\]

holds, we know that all of inequalities in the proof of theorem 1.1 becomes equality. Thus, for any \(A\), we have that \(\varphi^A_0 = (x_A - a^A_{00})u_0\) is an eigenfunction for the first eigenvalue \(\lambda_1\). Hence,
\[
\lambda_1 = \left( 2 - \frac{\min_{M^n} |X|^2}{n} \right) = 1
\]
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from (2.5) since $u_0$ is constant. Since (3.19) becomes equalities, we have

$$|X|^2 = \min_{M^n} |X|^2$$

according to $u_0 = \text{constant}$. Therefore, we obtain, from (3.21), $|X|^2 = n$, that is, $M^n$ is the sphere of the radius $\sqrt{n}$.

4. Upper bounds for eigenvalues

In order to prove the theorem 1.2, we need to obtain better estimates for lower order eigenvalues.

**Proposition 4.1.** Let $M^n$ be an $n$-dimensional compact self-shrinker in $\mathbb{R}^{n+p}$. Then, eigenvalues of the closed eigenvalue problem (1.4) satisfy

$$\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \leq (2n - \min_{M^n} |X|^2) + 4\lambda_1.$$

**Proof.** Let $u_j$ be an eigenfunction corresponding to the eigenvalue $\lambda_j$ such that

$$\begin{cases}
\mathcal{L}u_j = -\lambda_j u_j & \text{on } M^n \\
\int_{M^n} u_i u_j e^{-\frac{|X|^2}{2}} dv = \delta_{ij}, \text{ for any } i, j.
\end{cases}$$

We consider an $(n+p) \times (n+p)$-matrix $B = (b_{AB})$ defined by

$$b_{AB} = \int_{M^n} x_i x_j u_{A+1} u_{B+1} e^{-\frac{|X|^2}{2}} dv.$$

From the orthogonalization of Gram and Schmidt, there exist an upper triangle matrix $R = (R_{AB})$ and an orthogonal matrix $Q = (q_{AB})$ such that $R = QB$. Thus,

$$R_{AB} = \sum_{C=1}^{n+p} q_{AC} b_{CB} = \int_{M^n} \sum_{C=1}^{n+p} q_{AC} x_C u_{A+1} u_{B+1} = 0, \text{ for } 1 \leq B < A \leq n+p.$$

Defining $y_A = \sum_{C=1}^{n+p} q_{AC} x_C$, we have

$$\int_{M^n} y_A u_{A+1} u_{B+1} = \int_{M^n} \sum_{C=1}^{n+p} q_{AC} x_C u_{A+1} u_{B+1} = 0, \text{ for } 1 \leq B < A \leq n+p.$$

Therefore, the functions $\varphi_A$ defined by

$$\varphi_A = (y_A - a_A) u_1, \quad a_A = \int_{M^n} y_A u_1^2 e^{-\frac{|X|^2}{2}} dv, \quad \text{for } 1 \leq A \leq n+p$$

satisfy

$$\int_{M^n} \varphi_A u_{B+1} = 0, \quad \text{for } 0 \leq B < A \leq n+p.$$

If $\lambda_1 = 1$, then, $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1} = 1$ since $x_A, A = 1, 2, \cdots, n+p$, are eigenfunctions corresponding to the eigenvalue 1. In this case, the result in the proposition 4.1 is obvious.
If $\lambda_1 < 1$, we have

$$
-\int_{M^n} x_A u_1 e^{-\frac{|x|^2}{2}} dv = \int_{M^n} \mathcal{L} x_A u_1 e^{-\frac{|x|^2}{2}} dv \\
= \int_{M^n} x_A \mathcal{L} u_1 e^{-\frac{|x|^2}{2}} dv = -\lambda_1 \int_{M^n} x_A u_1 e^{-\frac{|x|^2}{2}} dv.
$$

Hence,

$$
\int_{M^n} x_A u_1 e^{-\frac{|x|^2}{2}} dv = 0,
$$

for any $A = 1, 2, \cdots, n + p$. Since the eigenfunction $u_0$ corresponding to the eigenvalue $\lambda_0 = 0$ is a constant, we have

$$
(4.4) \quad \int_{M^n} \varphi_A u_0 e^{-\frac{|x|^2}{2}} dv = \int_{M^n} (y_A - a_A) u_0 u_1 e^{-\frac{|x|^2}{2}} dv = u_0 \int_{M^n} y_A u_1 e^{-\frac{|x|^2}{2}} dv = 0.
$$

Therefore, $\varphi_A$ is a trial function. From the Rayleigh-Ritz inequality, we have, for $1 \leq A \leq n + p$,

$$
(4.5) \quad \lambda_{A+1} \leq \frac{- \int_{M^n} \varphi_A \mathcal{L} \varphi_A e^{-\frac{|x|^2}{2}} dv}{\int_{M^n} (\varphi_A)^2 e^{-\frac{|x|^2}{2}} dv}.
$$

From the definition of $\varphi_A$, we derive

$$
\mathcal{L} \varphi_A = \Delta \varphi_A - \langle X, \nabla \varphi_A \rangle \\
= \Delta \{(y_A - a_A) u_1\} - \langle X, \nabla \{(y_A - a_A) u_1\}\rangle \\
= y_A \mathcal{L} u_1 + u_1 \mathcal{L} y_A + 2 \nabla y_A \cdot \nabla u_1 - a_A \mathcal{L} u_1 \\
= -\lambda_1 y_A u_1 - u_1 y_A + 2 \nabla y_A \cdot \nabla u_1 + a_A \lambda_1 u_1.
$$

Therefore, (4.5) can be written as

$$
(4.6) \quad (\lambda_{A+1} - \lambda_1) \|\varphi_A\|^2 \leq \int_{M^n} (y_A u_1 - 2 \nabla y_A \cdot \nabla u_1) \varphi_A e^{-\frac{|x|^2}{2}} dv.
$$

From the Cauchy-Schwarz inequality, we obtain

$$
\left( \int_{M^n} (y_A u_1 - 2 \nabla y_A \cdot \nabla u_1) \varphi_A e^{-\frac{|x|^2}{2}} dv \right)^2 \leq \|\varphi_A\|^2 \|y_A u_1 - 2 \nabla y_A \cdot \nabla u_1\|^2.
$$

Multiplying the above inequality by $(\lambda_{A+1} - \lambda_1)$, we infer, from (4.6),

$$
(\lambda_{A+1} - \lambda_1) \left( \int_{M^n} (y_A u_1 - 2 \nabla y_A \cdot \nabla u_1) \varphi_A e^{-\frac{|x|^2}{2}} dv \right)^2 \leq (\lambda_{A+1} - \lambda_1) \|\varphi_A\|^2 \|y_A u_1 - 2 \nabla y_A \cdot \nabla u_1\|^2
$$

$$
\leq \left( \int_{M^n} (y_A u_1 - 2 \nabla y_A \cdot \nabla u_1) \varphi_A e^{-\frac{|x|^2}{2}} dv \right) \|y_A u_1 - 2 \nabla y_A \cdot \nabla u_1\|^2
$$

Hence, we derive

$$
(4.8) \quad (\lambda_{A+1} - \lambda_1) \int_{M^n} (y_A u_1 - 2 \nabla y_A \cdot \nabla u_1) \varphi_A e^{-\frac{|x|^2}{2}} dv \leq \|y_A u_1 - 2 \nabla y_A \cdot \nabla u_1\|^2
$$
For any point \( p \)
\[
\phi \quad (4.10)
\]
we infer
\[
\sum_{A=1}^{n+p} y_A^2 = \sum_{A=1}^{n+p} x_A^2 = |X|^2,
\]
we infer
\[
\sum_{A=1}^{n+p} \| y_A u_1 - 2 \nabla y_A \cdot \nabla u_1 \|^2
\]
\[
= \sum_{A=1}^{n+p} \int_{M^n} \left( y_A^2 u_1^2 - 4 y_A u_1 \nabla y_A \cdot \nabla u_1 + 4(\nabla y_A \cdot \nabla u_1)^2 \right) e^{-\frac{|X|^2}{2}} dv
\]
\[
= \int_{M^n} \left( |X|^2 u_1^2 - \nabla |X|^2 \cdot \nabla u_1^2 + 4 \nabla u_1 \cdot \nabla u_1 \right) e^{-\frac{|X|^2}{2}} dv
\]
\[
= \int_{M^n} \left( |X|^2 u_1^2 + |X|^2 u_1^2 + 4 \nabla u_1 \cdot \nabla u_1 \right) e^{-\frac{|X|^2}{2}} dv
\]
\[
= \int_{M^n} \left( 2n - |X|^2 u_1^2 \right) e^{-\frac{|X|^2}{2}} dv + 4 \lambda_1 \leq (2n - \min_{M^n} |X|^2) + 4 \lambda_1.
\]
On the other hand, from the definition of \( \varphi_A \), we have
\[
\int_{M^n} \left( y_A u_1 - 2 \nabla y_A \cdot \nabla u_1 \right) \varphi_A e^{-\frac{|X|^2}{2}} dv
\]
\[
= \int_{M^n} \left( y_A^2 u_1^2 - a_A y_A u_1^2 + 2a_A u_1 \nabla y_A \cdot \nabla u_1 - 2y_A u_1 \nabla y_A \cdot \nabla u_1 \right) e^{-\frac{|X|^2}{2}} dv
\]
\[
= \int_{M^n} \left( y_A^2 u_1^2 - a_A y_A u_1^2 - a_A \varphi y_A u_1^2 + \frac{1}{2} \varphi y_A^2 u_1^2 \right) e^{-\frac{|X|^2}{2}} dv
\]
\[
= \int_{M^n} \nabla y_A \cdot \nabla y_A u_1^2 e^{-\frac{|X|^2}{2}} dv.
\]
For any point \( p \), we choose a new coordinate system \( \bar{X} = (\bar{x}_1, \cdots, \bar{x}_{n+p}) \) of \( \mathbb{R}^{n+p} \) given by \( X = X(p) = \bar{X}O \) such that \( (\frac{\partial}{\partial \bar{x}_1})_p, \cdots, (\frac{\partial}{\partial \bar{x}_n})_p \) span \( T_p M^n \) and at \( p \),
\[
g\left( \frac{\partial}{\partial \bar{x}_i}, \frac{\partial}{\partial \bar{x}_j} \right) = \delta_{ij}, \text{ where } O = (o_{AB}) \in O(n+p) \text{ is an } (n+p) \times (n+p) \text{ orthogonal matrix.}
\]
\[
\nabla y_A \cdot \nabla y_A = g(\nabla y_A, \nabla y_A) = \sum_{B,C=1}^{n+p} q_{AB} q_{AC} g(\nabla x_B, \nabla x_C)
\]
\[
= \sum_{B,C=1}^{n+p} q_{AB} q_{AC} \left( \sum_{D=1}^{n+p} o_{DB} \nabla \bar{x}_D, \sum_{E=1}^{n+p} o_{EC} \nabla \bar{x}_E \right)
\]
\[ = \sum_{A=1}^{n+p} (\lambda_{A+1} - \lambda_1) \int_{M^n} \left( y_A u_1 - 2 \nabla y_A \cdot \nabla u_1 \right) \varphi_A e^{-\frac{|x|^2}{2}} dv \]

\[ = \sum_{A=1}^{n+p} (\lambda_{A+1} - \lambda_1) \int_{M^n} \nabla y_A \cdot \nabla y_A u_1^2 e^{-\frac{|x|^2}{2}} dv \]

\[ = \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_{M^n} \nabla y_j \cdot \nabla y_j u_1^2 e^{-\frac{|x|^2}{2}} dv \]

\[ + \sum_{A=n}^{n+p} (\lambda_{A+1} - \lambda_1) \int_{M^n} \nabla y_A \cdot \nabla y_A u_1^2 e^{-\frac{|x|^2}{2}} dv \]

\[ \geq \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_{M^n} \nabla y_j \cdot \nabla y_j u_1^2 e^{-\frac{|x|^2}{2}} dv \]

Thus, we obtain, from (4.10) and (4.11),

\[ \lambda_{A+1} - \lambda_1 \int_{M^n} \left( y_A u_1 - 2 \nabla y_A \cdot \nabla u_1 \right) \varphi_A e^{-\frac{|x|^2}{2}} dv \]

\[ = \sum_{A=1}^{n+p} (\lambda_{A+1} - \lambda_1) \int_{M^n} \nabla y_A \cdot \nabla y_A u_1^2 e^{-\frac{|x|^2}{2}} dv \]

\[ = \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_{M^n} \nabla y_j \cdot \nabla y_j u_1^2 e^{-\frac{|x|^2}{2}} dv \]

\[ + (\lambda_{n+1} - \lambda_1) \int_{M^n} (n - \sum_{j=1}^{n} \nabla y_j \cdot \nabla y_j) u_1^2 e^{-\frac{|x|^2}{2}} dv \]

\[ = \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_{M^n} \nabla y_j \cdot \nabla y_j u_1^2 e^{-\frac{|x|^2}{2}} dv \]

\[ + (\lambda_{n+1} - \lambda_1) \int_{M^n} \sum_{j=1}^{n} (1 - \nabla y_j \cdot \nabla y_j) u_1^2 e^{-\frac{|x|^2}{2}} dv \]

\[ \geq \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1). \]
According to (4.8), (4.9) and (4.12), we obtain

$$\sum_{j=1}^{n}(\lambda_{j+1} - \lambda_1) \leq (2n - \min_{M^n}|X|^2) + 4\lambda_1.$$ 

This completes the proof of the proposition 4.1. \(\square\)

**Proof of Theorem 1.2.** Let

$$\mu_{i+1} = \lambda_i + \frac{2n - \min_{M^n}|X|^2}{4} > 0,$$

for any \(i = 0, 1, 2, \ldots\), according to the proposition 2.1. Then, we obtain from (1.5)

(4.13) $$\sum_{i=1}^{k}(\mu_{k+1} - \mu_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\mu_{k+1} - \lambda_i)(\mu_i).$$

From the proposition 4.1 and the recursion formula of Cheng and Yang \[2\], the proof of the theorem 1.2 is completed. \(\square\)

5. **The Dirichlet eigenvalue problem**

For a bounded domain \(\Omega\) with a piecewise smooth boundary \(\partial\Omega\) in an \(n\)-dimensional complete self-shrinker in \(\mathbb{R}^{n+p}\), we consider the following Dirichlet eigenvalue problem of the differential operator \(\mathcal{L}\):

(5.1) $$\begin{cases} \mathcal{L}u = -\lambda u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

This eigenvalue problem has a real and discrete spectrum:

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty,$$

where each eigenvalue is repeated according to its multiplicity. We have following estimates for eigenvalues of the Dirichlet eigenvalue problem (5.1).

**Theorem 5.1.** Let \(\Omega\) be a bounded domain with a piecewise smooth boundary \(\partial\Omega\) in an \(n\)-dimensional complete self-shrinker \(M^n\) in \(\mathbb{R}^{n+p}\). Then, eigenvalues of the Dirichlet eigenvalue problem (5.1) satisfy

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{2n - \inf_{\Omega}|X|^2}{4}).$$

From the recursion formula of Cheng-Yang \[2\], we can give an upper bound for eigenvalue \(\lambda_{k+1}^\prime\):

**Theorem 5.2.** Let \(\Omega\) be a bounded domain with a piecewise smooth boundary \(\partial\Omega\) in an \(n\)-dimensional complete self-shrinker \(M^n\) in \(\mathbb{R}^{n+p}\). Then, eigenvalues of the Dirichlet eigenvalue problem (5.1) satisfy, for any \(k \geq 1\),

$$\lambda_{k+1} + \frac{2n - \inf_{\Omega}|X|^2}{4} \leq (1 + \frac{a(\min\{n, k - 1\})}{n})(\lambda_1 + \frac{2n - \inf_{\Omega}|X|^2}{4})^{k/2},$$

where \(a = \frac{n}{n-1}\).
where the bound of $a(m)$ can be formulated as:

$$
\begin{align*}
    a(0) &\leq 4, \\
    a(1) &\leq 2.64, \\
    a(m) &\leq 2.2 - 4\log(1 + \frac{1}{50}(m - 3)), \quad \text{for} \quad m \geq 2.
\end{align*}
$$

In particular, for $n \geq 41$ and $k \geq 41$, we have

$$
\lambda_{k+1} + \frac{2n - \inf_{\Omega} |X|^2}{4} \leq (\lambda_1 + \frac{2n - \inf_{\Omega} |X|^2}{4})k^{2/n}.
$$

**Remark 5.1.** For the Euclidean space $\mathbb{R}^n$, the differential operator $\mathcal{L}$ is called Ornstein-Uhlenbeck operator in stochastic analysis. Since the Euclidean space $\mathbb{R}^n$ is a complete self-shrinker in $\mathbb{R}^{n+1}$, our theorems also give estimates for eigenvalues of the Dirichlet eigenvalue problem of the Ornstein-Uhlenbeck operator.

**Proof of Theorem 5.1.** By making use of the same proof as in the proof of the theorem 1.1, we can prove the theorem 5.1 if one notices to count the number of eigenvalues from 1.

By making use of the same assertion as in the proposition 4.1, we have

**Proposition 5.1.** Let $\Omega$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$ in an $n$-dimensional complete self-shrinker $M^n$ in $\mathbb{R}^{n+p}$. Then, eigenvalues of the Dirichlet eigenvalue problem (1.6) satisfy

$$
\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \leq (2n - \inf_{\Omega} |X|^2) + 4\lambda_1.
$$

**Proof of Theorem 5.2.** By making use of the same proof as in the proof of the theorem 1.2, we can prove the theorem 5.2 if one notices to count the number of eigenvalues from 1.

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