Entanglement measures with asymptotic weak-monotonicity as lower (upper) bound for the entanglement of cost (distillation)

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We propose entanglement measures with asymptotic weak-monotonicity. We show that a normalized form of entanglement measures with the asymptotic weak-monotonicity are lower (upper) bound for the entanglement of cost (distillation).

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It is quantum entanglement that led to the controversy over Einstein-Podolsky-Rosen experiment and the non-local nature of quantum mechanics.

The entanglement is the key ingredient in quantum information processing. The speedup in quantum computation is obtained through the parallel quantum operations on massively superposed states which are in general entangled. In particular, in quantum communications such as the quantum teleportation the entanglement is valuable resource that should not be wasted, since entanglements can only be obtained by (costly) non-local operations. For better understanding and manipulation of entangled states, we need to classify them as well as possible. Quantification of the entanglement degree namely the measure of entanglement is, thus, the central issue in quantum information theory.

A pair of important measures of entanglement, that is, the entanglement of cost  and entanglement of distillation  have been introduced in a pioneering work by Bennett et al. Recently the relation between the entanglement of cost  and the entanglement of formation  has been clarified. That is, . (Here  denotes a mixed state.) The pair of measures of entanglement have been shown to be the limits for other entanglement measures satisfying certain conditions.

In other words, any entanglement measure satisfying the conditions is the lower (upper) bound of the entanglement of cost (entanglement of distillation ). One of the conditions for the entanglement measures is monotonicity that entanglement measures cannot be increased by any local quantum operations with classical communications (LOCCs).

On the other hand, it has been shown that there is irreversibility in the asymptotic manipulations of entanglement. That is, for a class of states  for which  except for a few cases. Thus it is not yet clear whether  is strictly greater than  in general.

In this paper, we consider a relaxed and thus conceptually simple monotonicity requirement, namely asymptotic weak-monotonicity. We show that certain reparameterized entanglement measures satisfying the relaxed monotonicity are the lower (upper) bound for the entanglement of cost  (entanglement of distillation ). These bounds are different from those in Ref. 3. It is notoriously difficult to calculate the asymptotic measures of entanglement  or even bounds to them as noted above. Thus it is worthwhile for us to give clues for the bounds to the asymptotic measures of entanglement, namely the entanglement measures with asymptotic weak-monotonicity.

Now let us review basic conditions for the entanglement measures. (a) Non-negativity: . (b) Vanishing on separable states: . In addition to these, we require monotonicity conditions. Let LOCCs transform an initial state  into a final state . The weak-monotonicity is that the entanglement degree does not increase with LOCCs. (c) The weak-monotonicity : . Let us consider the case where LOCCs transforms an initial state  into an ensemble . Here  is positive integer and  is the probability that will outcome. The strong-monotonicity says that the expectation value of the entanglement degree does not increase with LOCCs. (c) The strong monotonicity: . Let us introduce the asymptotic weak-monotonicity. Consider the asymptotic transformation. Let us consider transformation of an initial state  to a final state . Here  is not assumed to be a pure state. We say that a state  can asymptotically be transformed to a state  if we can achieve a transformation  at least approximately by LOCCs with a condition . More precise definition is the following.

Definition 1: We say that a state  can asymptotically be transformed to a state  with LOCCs, if the followings are satisfied. For any , , and , there exist an  such that for any  we can transform a state  with LOCCs to a state , with the fidelity .

\[
F(\xi, \rho_{\text{initial}}^{\otimes n_i}) \geq 1 - \epsilon
\]

and with the success probability , where . Here and the Uhlmann fidelity .

\[\|\|_{1}\]
Asymptotic weak-monotonicity condition (c.3) If an initial state $\rho_{\text{initial}}$ can be asymptotically transformed to a final state $\rho_{\text{final}}$ by LOCCs, then $E(\rho_{\text{initial}}) \geq E(\rho_{\text{final}})$. □

In the asymptotic weak-monotonicity (c.3), exact transformation is not required: Even though we cannot exactly transform an initial state to a desired final state, we require that $E(\rho_{\text{initial}}) \geq E(\rho_{\text{final}})$ as long as we can asymptotically achieve the transformation.

Let us discuss on the difference between the entanglement measure with asymptotic weak-monotonicity and that of strong-monotonicity $\Box$. Consider the case where entanglement measures with strong-monotonicity are transformed by a monotonic function $m$. That is, consider a re-parameterized function $E'(\rho) = m(E(\rho))$. Here we assume that the re-parameterized entanglement measure $E'(\rho)$ satisfy the two obvious conditions (a) and (b). The re-parameterized function $E'(\rho)$ is no longer the entanglement measure with strong-monotonicity in general: the entanglement measure involves with processes where the number of distinguishable states varies. However, it is easy to see that the re-parameterization does not preserve strong-monotonicity in general with respect to the processes where the number of the states varies. However, it is clear that the asymptotic weak-monotonicity is preserved upon the re-parameterization, since we are considering the one-dimensional orderings in this case.

Proposition 1: The asymptotic weak-monotonicity is preserved upon the re-parameterization. Namely, if $E(\rho)$ is an entanglement measure with asymptotic weak-monotonicity and $E'(\rho) = m(E(\rho))$ where $m$ is a monotonic function, then $E'(\rho)$ is also an entanglement measure with asymptotic weak-monotonicity. □

Proposition 2: All entanglement measures with asymptotic weak-monotonicity gives rise to the same orderings for pure states.

Proof: Let us assume two entanglement measures $E_i$ ($i = A, B$) with asymptotic weak-monotonicity give rise to different orderings for two pure states $\rho$ and $\rho'$. The fact that the order is reversed in dependence on entanglement measures obviously means that entanglement-degree of $\rho$ is less than that of $\rho'$ in one of the two measures and vice versa in the other. That is, we have either

$$E_A(\rho) > E_A(\rho') \quad \text{and} \quad E_B(\rho) < E_B(\rho'),$$

(1)

or

$$E_A(\rho) < E_A(\rho') \quad \text{and} \quad E_B(\rho) > E_B(\rho').$$

(2)

Due to the asymptotic weak-monotonicity condition (c.3) and Eqs. (1) and (2), the state $\rho$ can neither be asymptotically transformed to the state $\rho'$ nor $\rho'$ to $\rho$. That is, the two states $\rho$ and $\rho'$ are (asymptotically) incomparable. However, for the two pure states $\rho$ and $\rho'$, we can achieve the asymptotic transformation of either $\rho^{\otimes n} \approx (\rho')^{\otimes n}$ or $\rho^{\otimes n} \approx (\rho')^{\otimes n}$, by entanglement concentration and dilution. This is in contradiction with the former statement. □

The Proposition 1 says that we can generate numerous entanglement measures with asymptotic weak-monotonicity from a single one. On the other hand, the unique entanglement measure for pure states is also an entanglement measure with asymptotic weak-monotonicity for pure states. Here the unique measure for a pure state is the followings: $E_p(|\Psi\rangle\langle\Psi|) = S(Tr_B|\Psi\rangle\langle\Psi|)$, where $S(\rho) = Tr(-\rho \log_2 \rho)$ and $B$ denotes the latter one of the two parties Alice and Bob. Thus by the proposition 2 for pure states the ordering of the entanglement measures with asymptotic weak-monotonicity is the same as that of the unique measure of entanglement. Therefore, we can fix the freedom of entanglement measure $E$ with asymptotic weak-monotonicity involved with the Proposition 1 by the following condition.

Normalization condition: For any pure state $\rho$, $E(\rho)$ is the same as the unique entanglement measure $E_p(\rho)$ for pure state $|\Psi\rangle$.

Although the normalization condition is involved with entanglement-degrees only for pure states, it is also fixing those for mixed states since entanglement-degrees of pure states are continuously distributed. We denote the entanglement measure with asymptotic weak-monotonicity satisfying the normalization condition as $\tilde{E}$.

Proposition 3: The normalized entanglement measure with asymptotic weak-monotonicity $\tilde{E}(\rho)$ is the lower bound for the entanglement of cost $E_C(\rho)$.

Proof: Assume that there exists a state $\rho$ such that $E_C(\rho) < E(\rho)$. Then we choose a pure state such that $E_p(|\Psi\rangle\langle\Psi|) = a$. This means that we can asymptotically distill $ma$ numbers of the maximally entangled states $|\phi^+\rangle\langle\phi^+|^{\otimes ma}$ from the $m$ copies of the state $|\Psi\rangle\langle\Psi|$. (Here $|\phi^+\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle).$) Then some of the $ma$ numbers of the maximally entangled states can be asymptotically transformed to $\rho^{\otimes n}$, since $mE_C(\rho) < ma$. What we have done is the asymptotic transformation $|\Psi\rangle\langle\Psi| \Rightarrow \rho$. By the definition 1, we have $\tilde{E}(|\Psi\rangle\langle\Psi|) \geq \tilde{E}(\rho)$. (Note that the continuity condition is not additionally necessary here while it is in the case of Ref. $\Box$. The asymptotic weak-monotonicity is defined such that the continuity condition is not necessary.)

By the normalization condition, we have $\tilde{E}(|\Psi\rangle\langle\Psi|) = a$. Thus we have $\tilde{E}(|\Psi\rangle\langle\Psi|) < \tilde{E}(\rho)$, which contradicts above obtained inequality. □

We can get a proposition for the entanglement of distillation $E_D(\rho)$ in a similar way.

Proposition 4: The normalized entanglement measure with asymptotic weak-monotonicity $\tilde{E}(\rho)$ is the upper bound for the entanglement of distillation $E_D(\rho)$.

Proof: Assume that there exists a state $\rho$ such that
\[ E(\rho) < a < E_D(\rho) \]. However, we can asymptotically distill \( mE_D(\rho) \) numbers of the maximally entangled states from the state \( \rho^m \), by definition. Then we choose a pure state such that \( E_p(\Psi\rangle\langle\Psi|) = a \). Some of the \( mE_D(\rho) \) numbers of the maximally entangled states can be asymptotically transformed to \( (|\Psi\rangle\langle\Psi|)^{\otimes m} \), since \( mE_D(\rho) > ma \). Thus, we have done asymptotic transformation \( \rho \Rightarrow |\Psi\rangle\langle\Psi| \). By definition 1, we have \( E(\rho) \geq E(|\Psi\rangle\langle\Psi|) \). By the normalization condition, we have \( E(|\Psi\rangle\langle\Psi|) = a \). Thus we have \( E(\rho) < E(|\Psi\rangle\langle\Psi|) \), which contradicts the above obtained inequality. \( \square \)

The entanglement measures with strong-monotonicity satisfying certain additional conditions in Ref. [23] are bounds for the asymptotic measures, namely the entanglement of distillation \( E_{C} \) and entanglement of distillation \( E_{D} \). As we have shown, the entanglement measures with asymptotic weak-monotonicity satisfying an additional normalization condition are also bounds for the asymptotic measures. Due to the existence of the additional conditions, the two types of bounds can be different.

In conclusion, we have proposed entanglement measures with asymptotic weak-monotonicity. The orderings of the entanglement measures \( E(\rho) \) with asymptotic weak-monotonicity for the pure states is the same as that of the unique measure of entanglement \( E_p(\rho) \) for the states \( |\Psi\rangle \). This fact enabled us to re-parameterize entanglement measures with asymptotic weak-monotonicity \( E(\rho) \) such that the \( E(\rho) \) coincides with \( E_p(\rho) \) for all pure states. We have shown that normalized entanglement measures with asymptotic weak-monotonicity \( \tilde{E} \) are lower (upper) bound for the entanglement of cost \( E_C \) (entanglement of distillation \( E_D \)).

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