Relativistic mechanism of superconductivity

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Abstract

According to the theory of relativity, the relativistic Coulomb’s force between an electron pair is composed of two parts, the main part is repulsive, while the rest part can be attractive in certain situations. Thus the relativistic attraction of an electron pair provides an insight into the mechanism of superconductivity. In superconductor, there are, probably at least, two kinds of collective motions which can eliminate the repulsion between two electrons and let the attraction being dominant, the first is the combination of lattice and electron gas, accounting for traditional superconductivity; the second is the electron gas themselves, accounting for high $T_c$ superconductivity. In usual materials, there is a good balance between the repulsion and attraction of an electron pair, the electrons are regarded as free electrons so that Fermi gas theory plays very well. But in some materials, when the repulsion dominates electron pairs, the electron gas will has a behavior opposite to superconductivity. In the present paper the superconducting states are discussed in terms of relativistic quantum theory in details, some significant results are obtained including quantized magnetic flux, London equation, Meissner effect and Josephson effect.

1 Introduction

In BCS theory, it is believed that the mechanism responsible for the transition to superconductivity is a coupling between electrons via the positive ions of metallic lattice. The electron-lattice-electron interaction provides an attraction between electrons which can lead to a ground state separated from excited states by an energy gap. Whereas, in recent years the discovery of high $T_c$ superconductivity has offered a challenge for BCS theory, the first great difficult in the extension of BCS theory is to discover a nature of interaction responsible both for the traditional and high $T_c$ superconductivities.

In the present paper, we propose a mechanism for superconductivity which is based on the relativity theory.

2 Relativistic Coulomb’s force

This section is a theoretical preparation for the next section.

Consider a particle moving in a inertial system with 4-vector velocity $u$, it satisfies

$$u_{\mu}u_{\mu} = -c^2 \quad (1)$$

The above equation is valid so that any force can never change $u$ in its magnitude but can change $u$ in its direction. We therefore conclude that the relativistic Coulomb’s force on a particle always acts in the direction orthogonal to the 4-vector velocity of the particle in the 4-dimensional space-time, rather than along the line joining a couple of particles. Simply, any 4-vector force $f$ satisfy the following orthogonal relation

$$u_{\mu}f_{\mu} = u_{\mu}m\frac{du_{\mu}}{d\tau} = m\frac{d(u_{\mu}u_{\mu})}{d\tau} = 0 \quad (2)$$

Suppose there are two charged particle $q$ and $q'$ locating at positions $x$ and $x'$ in the Cartesian coordinate system $S$ and moving at 4-vector velocities $u$ and $u'$ respectively, as shown in Fig.1, where we use $X$ to denote $x - x'$. The Coulomb’s force $f$ acting on particle $q$ is perpendicular (orthogonal) to the velocity direction of $q$, as illustrated in Fig.1, like a centripetal force, the force $f$ should make an attempt to rotate itself about its path center, the center may locate at the front or back of the particle $q'$, so the force $f$ should lie in the plane of $u'$ and $X$, then

$$f = Au' + BX \quad (3)$$

Where $A$ and $B$ are unknown coefficients, the possibility of this expansion was discussed in details in the paper[2], in where the expansion is not an assumption. Using the relation $f \perp u$, we get

$$u \cdot f = A(u \cdot u') + B(u \cdot X) = 0 \quad (4)$$
we rewrite Eq. (3) as

\[ f = \frac{A}{u \cdot X}[(u \cdot X)u' - (u \cdot u')X] \]  

(5)

It follows from the direction of Eq. (3) that the unit vector of the Coulomb’s force direction is given by

\[ \hat{f} = \frac{1}{c^2 r}[(u \cdot X)u' - (u \cdot u')X] \]  

(6)

because

\[ \hat{f} = \frac{1}{c^2 r}[(u \cdot X)u' - (u \cdot u')X] \]

\[ = \frac{1}{c^2 r}[(u \cdot R)u' - (u \cdot u')R] \]

\[ = -[(\hat{u} \cdot \hat{R})\hat{u}' - (\hat{u} \cdot \hat{u}')\hat{R}] \]

\[ = -\hat{u}' \cos \alpha + \hat{R} \sinh \alpha \]  

(7)

\[ |\hat{f}| = 1 \]  

(8)

Where \( \alpha \) refers to the angle between \( u \) and \( R, R \perp u', r = |R|, \hat{u} = u/c, \hat{u}' = u'/ic, \hat{R} = R/r \). Suppose that the magnitude of the force \( f \) has the classical form

\[ |f| = k \frac{q q'}{r^2} \]  

(9)

Combination of Eq. (2) with (4), we obtain a modified Coulomb’s force

\[ f = \frac{kqq'}{c^2 r^3}[(u \cdot X)u' - (u \cdot u')X] \]

\[ = \frac{kqq'}{c^2 r^3}[(u \cdot R)u' - (u \cdot u')R] \]  

(10)

This force is in the form of Lorentz force for the two particles.

It is follows from Eq. (10) that the force can be rewritten in terms of 4-vector components as

\[ f_\mu = qF_{\mu\nu}u_\nu \]  

(11)

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]  

(12)

\[ A_\mu = \frac{kq' u'_\mu}{c^2 r} \]  

(13)

Where we have used the relations

\[ \partial_\mu \left( \frac{1}{r} \right) = -\frac{R_\mu}{r^3} \]  

(14)

2.1 Lorentz gauge condition

From Eq. (13), because of \( u' \perp R \), we have

\[ \partial_\mu A_\mu = \frac{kq' u'_\mu}{c^2 r} \partial_\mu \left( \frac{1}{r} \right) = -\frac{kq' u'_\mu}{c^2 r} \left( \frac{R_\mu}{r^3} \right) = 0 \]  

(15)

It is known as the Lorentz gauge condition.

\[ \text{Figure 1: The Coulomb’s force acting on } q \text{ is perpendicular to the 4-vector velocity } u \text{ of } q, \text{ and lies in the plane of } u' \text{ and } X \text{ with the retardation with respect to } q'. \]

2.2 Maxwell’s equations

To note that \( R \) has three degrees of freedom on the condition \( R \perp u' \), so we have

\[ \partial_\mu R_\mu = 3 \]  

(16)

\[ \partial_\mu \partial_\mu \left( \frac{1}{r} \right) = -4\pi \delta(R) \]  

(17)

From Eq. (12), we have

\[ \partial_\nu F_{\mu\nu} = \partial_\mu \partial_\nu A_\nu - \partial_\nu \partial_\nu A_\mu = -\partial_\nu \partial_\nu A_\mu \]

\[ = -\frac{kq' u'_\mu}{c^2 r} \partial_\nu \partial_\nu \left( \frac{1}{r} \right) = \frac{kq' u'_\mu}{c^2 r} 4\pi \delta(R) \]

\[ = \mu_0 J'_\nu \]  

(18)

where we define \( J'_\nu = q' u'_\nu \delta(R) \). From Eq. (12), by exchanging the indices and taking the summation of them, we have

\[ \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \]  

(19)

The Eq. (18) and (19) are known as the Maxwell’s equations. For continuous media, they are valid as well.

2.3 Lienard-Wiechert potential

From Maxwell’s equations, we know there is a retardation time for action to propagate between the two particles, let \( d \) denote the distance from \( q' \) to \( O \) in Fig. 1, the retardation effect is measured by

\[ r = c \Delta t = \frac{d}{ic} = \frac{\hat{u}' \cdot X}{ic} = \frac{u'_\nu (x'_\mu - x_\nu)}{c} \]  

(20)

Then
\[ A_\mu = \frac{kq' u_\mu}{c^2 v} = \frac{kq'}{c} u'_\mu (x'_\nu - x_\nu) \] (21)

Obviously, Eq.(21) is known as the Lienard-Wiechert potential for a moving particle.

3 Attraction between electron pair

In the preceding section we have devoted into electrodynamic subjects in details, the purpose is to establish full confidence in Eq.(10), re-given by

\[ f = \frac{kqq'}{c^2 r^3}[(u \cdot X)u' - (u \cdot u')X] = \frac{kqq'}{c^2 r^3}[(u \cdot R)u' - (u \cdot u')R] \] (22)

from which we will in this section discuss a mechanism for superconductivity.

Obviously, for an electron pair, the first term of Eq.(22) can give an attraction between the two electrons in certain situations, the second term represents a repulsion which contributes to classical Coulomb’s force. To note that this attraction of an electron pair requires no phonon exchange, the attraction is definitely distinguishable from that in Cooper pair of BCS theory.

In superconductors, there are, probably at least, two kinds of collective motions which can eliminate the repulsion between two electrons and let the attraction being dominant, one is the combination of lattice and electron gas, another is electron gas themselves.

In traditional superconductor, the transition temperature are fairly low, the electron gas and lattice must combine to depress the repulsion in electron pair, because the isotrope effect makes it clear that ions in the metal play an essential role in superconductivity. By this view point, we can arrive at the same consequences as BCS theory.

In high \(T_c\) superconductor, for certain situations, the lattice may play less-important role in eliminating the repulsion in electron pair, the repulsion may be removed by the electron gas itself.

In normal material without superconductivity, there must be in balance between the repulsion and attraction in an electron pair, so that the fermi gas theory plays good enough for explaining metallic properties, where the electrons can be regarded as free electrons.

In the other hand, in some material, it is possible that the repulsion of electron pair becomes dominant, thus the material will have opposite behavior with respect to the superconductivity at low temperatures.

4 Quantum wave equations

No doubt, we honestly believe that Pythagoras theorem is valid in every point in an inertial frame of reference, this requirement is an abstract constraint on motion behavior of particle in the space-time.

Consider a particle displacing \((dx_1, dx_2, dx_3)\) in time interval \(dt\) at speed \(v\), Pythagoras theorem is written in the form:

\[(dx_1)^2 + (dx_2)^2 + (dx_3)^2 = (vdt)^2 \] (23)

Using the above equation, we find

\[(dx_1)^2 + (dx_2)^2 + (dx_3)^2 - (c dt)^2\]
\[= (vdt)^2 - (c dt)^2\]
\[= (ict)^2[1 - (\frac{v}{c})^2] \] (24)

With the help of new notations \(x_4 = ict\) and \(d\tau = dt \sqrt{1 - \frac{v^2}{c^2}}\), Defining 4-vector velocity

\[ u_1 = \frac{dx_1/d\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{dx_1}{d\tau} \] (25)
\[ u_2 = \frac{dx_2/d\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{dx_2}{d\tau} \] (26)
\[ u_3 = \frac{dx_3/d\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{dx_3}{d\tau} \] (27)
\[ u_4 = \frac{ict}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{dx_4}{d\tau} \] (28)

Eq.(24) can be rewritten as

\[ u_1^2 + u_2^2 + u_3^2 + u_4^2 = u_\mu u_\mu = -c^2 \] (29)

Where Greek index \(\mu\) takes on 1,2,3 and 4. Multiplying the last equation with the rest mass \(m\) of the particle, we obtain

\[ \frac{d(mu_\mu u_\mu)}{d\tau} = 2mu \frac{du}{d\tau} + 2mu_4 \frac{du_4}{d\tau} = 0 \] (30)

Separating the last equation into two equations by using new notations \(f\) and \(f_4\), we have

\[ \frac{mdu}{d\tau} = f \] (31)
\[ \frac{mdu_4}{d\tau} = - \frac{u \cdot f}{u_4} = f_4 \] (32)

The above equations represents the well-known relativistic dynamics given by

\[ m \frac{du_\mu}{d\tau} = f_\mu \] (33)
You see, indeed, the relativistic mechanics can be derived from ancient Pythagoras theorem. For the details, please see the paper\cite{9}.

In an electromagnetic field, the dynamic equation [Eq.(8)] is valid at every point in the space-time, no matter whether there is an actual particle passing the point considered, this means that there is a 4-vector velocity \( u \) at the point regardless of whether there exists a particle, in other words, the 4-vector velocity is the geometric character of the point in the electromagnetic field, it reflects some requirement arisen from Pythagoras theorem. Every 4-vector velocity at every point forms a 4-vector velocity field \( u(x_1, x_2, x_3, x_4) \) in the space-time ( like the geometrization of gravitational field ). The right side of Eq.(8) is electromagnetic field, while the left side of Eq.(8) is 4-vector velocity field. According to Green’s formula ( or Stokes’s theorem ), the geometric character of the point in the electromagnetic field, this means that there is a 4-vector velocity at every point forms a 4-vector velocity field, at every point.

\[
\psi \in \Phi \quad \text{in mathematics, further set } \Phi = \frac{\psi}{\sqrt{g}},
\]

According to Green’s formula ( or Stokes’s theorem ), any 4-vector velocity at every point forms a 4-vector velocity field, at every point.

Because the variables \( \partial u, \partial A, \partial u, \partial v \) are independent from \( u, v \), a solution satisfying Eq.(29) is

\[
\psi = \frac{\psi}{\sqrt{g}}.
\]

By substituting \( \psi = \frac{\psi}{\sqrt{g}} \) into the above equation, we obtain a very important equation

\[
(mu + qA)\psi = -i\hbar \partial \psi
\]

where \( \psi \) representing wave nature may be a complex mathematical function, its physical meanings can be determined from experiments after the introduction of the Planck’s constant \( \hbar \).

Substituting the last equation into Eq.(29) and eliminating \( u, v \), under different approximations we can derive out Klein-Gordon wave equation, Dirac wave equation and Schrodinger wave equation, for the details of the derivations please see the paper\cite{4}.

From Eq.(29) and Eq.(38), we obtain a new quantum wave equation

\[
-m^2 c^2 \psi^2 = (-i\hbar \partial - qA)\psi(-i\hbar \partial - qA)\psi
\]

Or in Gaussian units it is written as

\[
-m^2 c^2 \psi^2 = (-i\hbar \partial - \frac{q}{e}A)\psi(-i\hbar \partial - \frac{q}{e}A)\psi
\]

Its precision is guaranteed by Pythagoras theorem. Thus, in the present paper, we expect to find out more new results beyond Dirac, Klein-Gordon or Schrodinger equations. In the following section, we will discuss spin in details which plays an important role in superconducting states.

5 Spin in atom

5.1 the electron in hydrogen atom

In this section, we use Gaussian units, and use \( m_e \) to denote the rest mass of electron. We limit ourself to hydrogen atom and its spin.

In a spherical polar coordinate system \( (r, \theta, \phi, ic\tilde{t}) \), the nucleus of hydrogen atom provides a spherically symmetric potential \( V(r) = e/r \) for the electron motion. The wave equation (40) for the hydrogen atom in energy eigenstate \( \psi(r, \theta, \phi)e^{ic\tilde{t}}/\hbar \) may be written in the spherical coordinates:

\[
\frac{m^2 c^2}{\hbar^2} \psi^2 = \left( \frac{\partial \psi}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right)^2 + \left( \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \right)^2
\]

By substituting \( \psi = R(r)X(\theta)\phi(\phi) \), we separate the above equation into

\[
\frac{\partial \phi}{\partial \phi}^2 + \kappa \phi^2 = 0 \quad (42)
\]

\[
\left( \frac{\partial X}{\partial \theta} \right)^2 + \left[ \lambda - \frac{\kappa}{\sin^2 \theta} \right] X^2 = 0 \quad (43)
\]

\[
\left( \frac{\partial R}{\partial r} \right)^2 + \left[ \frac{1}{\hbar^2 c^2}(-E + \frac{e^2}{r}) - \frac{m^2 c^2}{\hbar^2} - \frac{\lambda}{r^2} \right] R^2 = 0 \quad (44)
\]

The Eq.(12) can be solved immediately, with the requirement that \( \phi(\phi) \) must be a periodic function, we find that its solution is given by
\[ \phi = C_1 e^{\pm \sqrt{2} r}, \quad m = \pm \sqrt{k} = 0, \pm 1, \pm 2, \ldots \] (45)

where \( C_1 \) is an integral constant.

Factoring Eq. (43), we get its two branches

\[ \frac{\partial X}{\partial \theta} \pm iX \sqrt{\lambda - \frac{m^2}{\sin^2 \theta}} = 0 \] (46)

It is easy to find their solutions

\[ X(\theta) = C_2 e^{\pm i \int \sqrt{\lambda - \frac{m^2}{\sin^2 \theta}} d\theta} \] (47)

where \( C_2 \) is an integral constant. The requirement of periodic function for \( X \) demands

\[ \int_0^{2\pi} \sqrt{\lambda - \frac{m^2}{\sin^2 \theta}} d\theta = \pm 2\pi k \quad k = 0, 1, 2, \ldots \] (48)

Factoring Eq. (42), we get its two branches

\[ \frac{\partial R}{\partial r} \pm \frac{i}{\hbar c} \sqrt{(-E + \frac{e^2}{r})^2 - m_2^2 c^4 - \frac{\lambda \hbar^2 c^2}{r^2}} = 0 \] (49)

and their solutions

\[ R(r) = C_3 e^{\pm \int \sqrt{(-E + \frac{e^2}{r})^2 - m_2^2 c^4 - \frac{\lambda \hbar^2 c^2}{r^2}} dr} \] (50)

where \( C_3 \) is an integral constant. The requirement that the radical wave function forms a "standing wave" in the range from \( r = 0 \) to \( r = \infty \) demands

\[ \frac{1}{\hbar c} \int_0^\infty \sqrt{(-E + \frac{e^2}{r})^2 - m_2^2 c^4 - \frac{\lambda \hbar^2 c^2}{r^2}} dr = \pm \pi s \] (51)

\[ s = 0, 1, 2, \ldots \]

Evaluating the definite integrals of Eq. (48) and Eq. (51) are standard excises for contour integrals in complex space.

Consider a contour \( C_4 \) which is a unit circle around zero, as shown in Fig. (a), using \( z = e^{i \theta} \), we have

\[ I_1 = \int_0^{2\pi} \sqrt{\lambda - \frac{m^2}{\sin^2 \theta}} d\theta = \int_{C_4} \sqrt{\lambda + \frac{4m^2 z^2}{(z^2 - 1)^2}} \frac{dz}{iz} \]

\[ = \int_{C_4} \frac{\sqrt{\lambda(z^2 - 1)^2 + 4m^2 z^2} dz}{iz} \]

\[ = \int_{C_4} \frac{1}{z} \left( \frac{1/2}{z - 1} - \frac{1/2}{z + 1} \right) \sqrt{\lambda(z^2 - 1)^2 + 4m^2 z^2} \frac{dz}{iz} \]

\[ = \int_{C_4} \frac{1}{z} \sqrt{\lambda(z^2 - 1)^2 + 4m^2 z^2} \frac{dz}{iz} \]

\[ = \lim_{\eta \to 1} \int_{C_4} \frac{1}{2(z - \eta)} \sqrt{\lambda(z^2 - 1)^2 + 4m^2 z^2} \frac{dz}{iz} \]

\[ = \int_{C_4} \frac{\sqrt{\lambda + O(z^2)} dz}{iz} \]

\[ = \lim_{\eta \to 1} \int_{C_4} \frac{|m| + O(z^2 - 1) dz}{z - \eta} \]

\[ = \lim_{\xi \to 1} \int_{C_4} \frac{|m| + O(z^2 - 1) dz}{z + \xi} \]

\[ = \begin{cases} 2\pi \sqrt{\lambda} & |\eta| > 1, |\xi| > 1 \\ 2\pi(\sqrt{\lambda} - |m|) & |\eta| < 10|\xi| < 1 \\ 2\pi(\sqrt{\lambda} - 2|m|) & |\eta| < 1, |\xi| < 1 \end{cases} \] (52)

Where the integrand has the poles at \( z = 0 \) and \( z = \pm 1 \), and we have chosen \( \sqrt{(z^2 - 1)^2 - (z^2 - 1)} \). Comparing with Eq. (13), the right side of Eq. (13) is required to take plus sign, we obtain

\[ \sqrt{\lambda} = \begin{cases} k & |\eta| > 1, |\xi| > 1 \\ k + |m| & |\eta| < 10|\xi| < 1 \\ k + 2|m| & |\eta| < 1, |\xi| < 1 \end{cases} \] (53)

We rename the integer \( \lambda \) as \( j^2 \) for a convenient in the following, i.e. \( \lambda = j^2 \).

Consider a contour \( C_5 \) consisting of \( C_\gamma, L_-, C_\delta \) and \( L \) around zero in the plane as shown in Fig. (b), the radius of circle \( C_\gamma \) is large enough and the radius of circle \( C_\delta \) is small enough. The integrand of the following equation has no pole inside the contour \( C \), so that we have

\[ \int_C \sqrt{\frac{(-E + \frac{e^2}{z})^2 - m_2^2 c^4 - \frac{j \hbar^2 c^2}{z^2}}{z^2}} dz = \int_{C_\gamma} + \int_{L_-} + \int_{C_\delta} + \int_{L} = 0 \] (54)

The integrals in Eq. (53) are

\[ \int_{C_\gamma} = \int_{C_\gamma} \sqrt{\frac{(-E + \frac{e^2}{z})^2 - m_2^2 c^4 - \frac{j \hbar^2 c^2}{z^2}}{z^2}} dz \]

\[ = \int_{C_\gamma} \left[ \sqrt{E^2 - m_2^2 c^4} - \frac{Ez}{\sqrt{E^2 - m_2^2 c^4}} \frac{1}{z} + O(\frac{1}{z^2}) \right] dz \]

\[ = -i2\pi Ee^2 \sqrt{E^2 - m_2^2 c^4} \]

\[ \frac{1}{\sqrt{E^2 - m_2^2 c^4}} = \frac{2\pi E e^2}{\sqrt{m_2^2 c^4} - E^2} \] (55)

\[ \int_{C_\delta} = \int_{C_\delta} \sqrt{\frac{(-E + \frac{e^2}{z})^2 - m_2^2 c^4 - \frac{j \hbar^2 c^2}{z^2}}{z^2}} dz \]

\[ = \int_{C_\delta} \frac{\sqrt{(-Ez + e^2)^2 - m_2^2 c^4 - j \hbar^2 c^2}}{z} \]

\[ = \int_{C_\delta} \frac{\sqrt{(-Ez + e^2)^2 - m_2^2 c^4 - j \hbar^2 c^2}}{z} dz \]
Thus Eq.(51) becomes

\[ E = m_e c^2 \left[ 1 + \frac{\alpha^2}{(\sqrt{j^2 - \alpha^2 + s}^2)\right]^{\frac{1}{2}} \]

Because the integrand is a multiple-valued function, when the integral takes over the path \( L_- \) we have \( z = e^{2\pi i} \), thus

\[ \int_{L_-} = \int_{L_-}^\delta \sqrt{\frac{-2Ee^2}{z}} dz = \int_{L_-}^\delta \sqrt{\frac{-2Ee^2}{w}} dz = \int_{L_-}^\delta \sqrt{\frac{-2Ee^2}{pe^{i\beta/2}}} dz = -\int_{L(\gamma \to \delta)} \sqrt{\frac{-2Ee^2}{pe^{i\beta/2}}} dz = -\int_{\gamma}^{\delta} = \int_{L} \]

For a further manifestation, to define \( z - b = w = pe^{i\beta} \), where

\[ b = \frac{E^2 + e^4 - m_e^2 c^4 - j^2 \hbar^2 c^2}{2Ee^2} \]

We have

\[ \int_{L_-} = \int_{L_-}^\delta \sqrt{\frac{-2Ee^2}{z - b}} dz \]

The relation of \( z \) and \( w \) has shown in Fig.2(c), to note that \( w \) rotates around zero with \( z \).

\[ \int_{L} = \frac{1}{2}(\int_{L} + \int_{L_-}) = \frac{1}{2}(\int_{C_{\gamma}} + \int_{C_{\delta}}) \]

Thus Eq.(51) becomes

\[ \pm \pi s = \frac{1}{\hbar c} \int_0^\infty \sqrt{(-E + \frac{e^2}{z})^2 - m_e^2 c^4 - \frac{j^2 \hbar^2 c^2}{z^2} dr} \]

\[ = -\frac{\pi E \alpha}{\sqrt{m_e^2 c^4 - E^2}} + \pi \sqrt{j^2 - \alpha^2} \]

where \( \alpha = e^2 / \hbar c \) is known as the fine structure constant.

The left side of the last equation is required to take minus, then we obtain the positive energy levels given by

\[ \frac{m_e^2 c^4 \psi^2}{\hbar^2} = \left( \frac{\partial}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial}{\partial \theta} \right)^2 + \frac{1}{\hbar^2 c^2} (-E + \frac{e^2}{r})^2 \psi^2 \]

5.2 hydrogen atom in an uniform magnetic field

If we put the hydrogen atom into an external uniform magnetic field \( B \) which is along \( z \) axis with vector potential \( (A_r, A_\theta, A_z) = (0, 0, \frac{1}{r} \sin \theta B) \), then according to Eq.(10), the wave equation is given by

\[ \frac{m_e^2 c^4 \psi^2}{\hbar^2} = \left( \frac{\partial}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial}{\partial \theta} \right)^2 + \frac{1}{\hbar^2 c^2} (-E + \frac{e^2}{r})^2 \psi^2 \]
\[ +(\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \varphi} - \frac{er \sin \theta B}{i \hbar c})^2 \]  

By substituting \( \psi = R(r)X(\theta)\phi(\varphi) \), we separate the above equation into

\[ \frac{\partial \phi}{\partial \varphi} - \kappa \phi = 0 \]  

(64)

\( (\frac{\partial X}{\partial \theta})^2 + [(\frac{\kappa}{\sin \theta} - \frac{e \sin \theta r^2 B}{i \hbar c})^2 + \xi(r)]X^2 = 0 \)  

(65)

\( (\frac{\partial R}{\partial r})^2 + \left[ \frac{1}{\hbar^2 c^2}(-E + \frac{e^2}{r}) - \frac{m^2 e^2}{\hbar^2} - \frac{\xi(r)}{r^2} \right]R^2 = 0 \)  

(66)

Where we have used the unknown constant \( \kappa \) and function \( \xi(r) \) to connect these separated equations. Eq.(64) has the solution

\[ \phi = C_1 e^{im\varphi}, \quad \kappa = im, \quad m = 0, \pm 1, \pm 2, ... \]  

(67)

Expanding Eq.(64) and neglecting the term \( O(B^2) \), we have a term \( -\frac{me^2\mu}{\hbar c} \) in it, by moving this term into Eq.(66) through \( \xi(r) = \lambda + \frac{me^2\mu}{\hbar c} \), we obtain

\[ (\frac{\partial X}{\partial \theta})^2 + [\lambda - \frac{m^2}{\sin^2 \theta}]X^2 = 0 \]  

(68)

\[ (\frac{\partial R}{\partial r})^2 + \left[ \frac{1}{\hbar^2 c^2}(-E + \frac{e^2}{r}) - \frac{m^2 e^2}{\hbar^2} - \frac{\lambda}{r^2} \right]R^2 = 0 \]  

(69)

The above two equations are the same as Eq.(43) and Eq.(44), except for the additional constant term \( -\frac{meB}{\hbar c} \). After the similar calculation in the preceding section, we obtain the energy levels of hydrogen atom in the magnetic field given by

\[ E = \sqrt{m_e^2 c^4 + \text{mechB}} \left[ 1 + \frac{\alpha^2}{(\sqrt{j^2 - \alpha^2 + s})^2} \right]^{-\frac{1}{2}} \]  

(70)

### 5.3 spectroscopic notation

In the usual spectroscopic notation of quantum mechanics, four quantum numbers: \( n, l, m_l \) and \( m_s \) are used to specify the state of an electron in an atom. After the comparison, we get the relations between the usual notation and our notation.

\[ n = j + s, \quad s = 0, 1, ...; j = 1, 2, ..., \]  

(71)

\[ l = j - 1, \]  

(72)

\[ \max(m_l) = \max(m) - 1 \]  

(73)

We find that \( j \) takes over 1, 2, ..., \( n \); for a fixed \( j \) (or \( l \)), \( m \) takes over \(-l + 1, -l, ..., 0, ..., l, l + 1 \). In the present work, spin quantum number is absent.

### 5.4 Zeeman splitting

According to Eq.(41), for a fixed \((n, l)\), equivalent to \((n, j = l + 1)\), the energy level of hydrogen atom will split into \( 2l + 3 \) energy levels in magnetic field, given by

\[ E = (mc^2 + \text{mechB}) \left[ 1 + \frac{\alpha^2}{(\sqrt{j^2 - \alpha^2 + s})^2} \right]^{-\frac{1}{2}} + O(B^2) \]  

(74)

Considering \( m = -(l + 1), -l, ..., 0, ..., l, l + 1 \), this effect is equivalent to the usual Zeeman splitting in the usual quantum mechanics given by

\[ E = E_{nl} + \frac{(m \pm 1)eB}{2mc} \]  

(75)

But our work works on it without spin concept.

### 5.5 angular momentum, Stern-Gerlach experiment and spin

From Eq.(45), the angular momentum of the electron in hydrogen atom is given by

\[ \mathbf{J} = J_x \hat{\mathbf{e}}_x + J_z \hat{\mathbf{e}}_z = rm_eu_0 \hat{\mathbf{e}}_x + r \sin \theta m_u \hat{\mathbf{e}}_z \]  

\[ = \frac{1}{\psi} (-ih \frac{\partial \psi}{\partial \theta}) \hat{\mathbf{e}}_x + \frac{1}{\psi} (-ih \frac{\partial \psi}{\partial \varphi}) \hat{\mathbf{e}}_z \]  

\[ = \hbar \sqrt{\lambda - \frac{m^2}{\sin^2 \theta}} \hat{\mathbf{e}}_x + m \hbar \hat{\mathbf{e}}_z \]  

\[ = \hbar \sqrt{(k + |m|)^2 - \frac{m^2}{\sin^2 \theta}} \hat{\mathbf{e}}_x + m \hbar \hat{\mathbf{e}}_z \]  

(76)

According to Eq.(42), there should be three ground states \((j = 1)\).

\[ 1: k = 0, m = 1, J = i (\cot \theta) \hbar \hat{\mathbf{e}}_x + \hbar \hat{\mathbf{e}}_z \]  

(77)

\[ 2: k = 0, m = -1, J = i (\cot \theta) \hbar \hat{\mathbf{e}}_x - \hbar \hat{\mathbf{e}}_z \]  

(78)

\[ 3: k = 1, m = 0, J = \hbar \hat{\mathbf{e}}_x \]  

(79)

Why did the hydrogen atoms split into two branches in Stern-Gerlach experiment? To note that \( J_\varphi \to \infty \) when \( \theta \to 0, \pi \) for the ground states1 and 2, maybe \( J_\varphi \) locks the alignment of the electrons with external magnetic fields along z-axis, considering the directions of Lorentz forces acting on the currents as shown in Fig.4 schematically (despite it is imaginary), so that the angular momenta \( J_z = \pm \hbar \) of the electrons can not be subject to external
magnetic direction as the usual way. Conversely, the ground state 3 can rotate its orientation like a coil, disappeared within one of the two branches in Stern-Gerlach experiment. In the fact, maybe the ground state 3 has changed into the ground states 1 or 2 before it rotates to its usual destination because of its increasing or decreasing $J_z$. This explanation has drawn an image for spin.

![Figure 3](image)

**Figure 3**: The motion of the electron in hydrogen atom.

### 5.6 the shell of atom

In atom with many electrons, two electrons can form a pair with $\Delta J = \pm \hbar$, the reason of pairing arises probably from the alignment of their $J_x$ and $J_z$, the situation can imagined as an analogy to the combination of the ground states 1 and 2 of hydrogen atom.

For a fixed $l$ (or $j$), $m$ takes over $2l + 3$ values, thus the electrons with $2l + 3$ single-states can only form $2l + 1$ pair-states, as shown in Fig.4, the $l$-shell contains $2(2l+1)$ electrons.

**Figure 4**: Single-states merge into pair-states.

Although the single state of $m = 0$, like the ground state 3 of hydrogen atom, maybe unstable, it in a pair can be stable because it can align its $J_x$ with its counterpart electron.

Although the shell structure of atom based on our work is the same as the usual one, the present work predicts that hydrogen atom has $2l + 3$ degenerate states for a fixed $l$.

### 5.7 an open problem

In order to meet the fine structure of hydrogen atom, we have chosen plus sign for $k$ in Eq.(48) and minus sign for $s$ in Eq.(51). In other words, there are more states we have not discussed yet, which correspond to minus sign for $k$ in Eq.(48) or plus sign for $s$ in Eq.(51). Two situations must be considered; there probably exists an unknown constraint forbidding the states, or experiment has not payed attention to observe the states, this is an open question.

### 6 Superconducting states

Now we return to superconducting states. In the preceding section we have devoted into hydrogen atom and spin subjects in details, the purpose is to establish full confidence in Eq.(38), re-given by

$$ (mu_\mu + qA_\mu)\psi = -i\hbar\partial_\mu\psi $$

where $\theta$ is an integral constant, $x_0$ and $x$ are the initial point and final point of the integral with arbitrary path in the space-time.

Let us take the integration of Eq.(81) over a closed path $L$ in the space, at any instant $t (dx_4 = 0)$, the single-valued wave function of Eq.(81) requires

$$ \frac{1}{\hbar} \oint_\mathbf{L} (mu_\mu + qA_\mu) \cdot dl = 2\pi b, \quad b = 0, \pm 1, \pm 2, \ldots $$

Where $dl$ is an element of the integral path.

### 6.1 quantized magnetic flux in superconducting ring

Let us take integral path $L$ in superconducting ring, because the electronic current is zero in the interior, i.e. $u = 0$, thus the magnetic flux through $L$ is given from Eq.(82) by

$$ \phi = \oint_\mathbf{L} B \cdot d\sigma = \oint_\mathbf{L} A \cdot dl = \frac{2\pi b}{q} $$

Where $d\sigma$ is an element of area on the surface bounded by the integral path $L$. By experiment $q = 2e$, the charge of an electron pair, thus the flux through the ring is quantized in integral multiples of $\pi\hbar/e$. 

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8
6.2 London equation

Let \( n \) denote the density of electrons in superconductor, the electronic current is given by

\[
j = -neu
\]

Then Eq. (82) can be rewritten as

\[
\frac{-m}{ne\hbar} \int_L (j + \frac{ne^2}{m} A) \cdot dl = \frac{-m}{ne\hbar} \int_L (\nabla \times j + \frac{ne^2}{m} B) \cdot d\sigma = 2\pi s
\]

If the closed integral path contains no vortex (or fluxoid), at where we obtain

\[
\nabla \times j + \frac{ne^2}{m} B = 0
\]

It is the well known London equation.

6.3 Meissner effect

According to a Maxwell equation

\[
\nabla \times B = \mu_0 j
\]

from London equation we obtain

\[
\nabla^2 B = B/\lambda_L^2
\]

Where \( \lambda_L^2 = m/(\mu_0 ne^2) \) is a constant called as the London penetration depth. Near the surface of superconductor, the magnetic field along the depth direction \( z \) is given by

\[
B(z) = B(0) \exp(-z/\lambda_L)
\]

Thus for superconductor, the London equation leads to the Meissner effect.

6.4 Josephson superconductor tunneling

Before we discuss Josephson effect, we firstly discuss Aharonov-Bohm effect.

Aharonov-Bohm effect  Let us consider the modification of two slit experiment, as shown in Fig. 5. Between the two slits there is located a tiny solenoid \( S \), designed so that a magnetic field perpendicular to the plane of the figure can be produced in its interior. No magnetic field is allowed outside the solenoid, and the walls of the solenoid are such that no electron can penetrate to the interior. There are two paths \( l_1 \) and \( l_2 \) bypassing the solenoid from the electron gun to the screen, the wave function \( \psi \) is given by

\[
\psi = e^{i \oint_{x_0(l_2)} (mu_\mu + qA_\mu) dx_\mu} + e^{i \oint_{x_0(l_1)} (mu_\mu + qA_\mu) dx_\mu} (90)
\]

because \( l_1 \) and \( l_2 \) are equivalent for Eq.(81). The probability is given by

\[
W = \psi(x)\psi^*(x) = 2 + 2 \cos \left( \frac{P}{\hbar} (l_1 - l_2) + \frac{1}{h} \int_{x_0(l_1)} qA_\mu dx_\mu - \frac{1}{h} l_2 \int_{x_0(l_2)} qA_\mu dx_\mu \right)
\]

\[
= 2 + 2 \cos \left[ \frac{P}{h} (l_1 - l_2) + \frac{1}{h} \int_{x_0(l_1)} qA_\mu dx_\mu - \frac{1}{h} \int_{x_0(l_2)} qA_\mu dx_\mu \right]
\]

\[
= 2 + 2 \cos \left[ \frac{P}{h} (l_1 - l_2) + \frac{\phi}{h} \right]
\]

(91)

where \( \phi \) denotes the inverse path to the path \( l_2 \), \( \phi \) is the magnetic flux that passes through the surface between the paths \( l_1 \) and \( l_2 \), and it is just the flux inside the solenoid.

Two Slits

Screen

Electron Gun

Figure 5: A diffraction experiment with adding a solenoid.

Now, constructive (or destructive) interference occurs when

\[
\frac{P}{h} (l_1 - l_2) + \frac{\phi}{h} = 2\pi b \quad \text{(or)} \quad b + \frac{1}{2}
\]

(92)

where \( b \) is an integer. We know that this effect is just the Aharonov-Bohm effect which was shown experimentally in 1960.
Josephson effect  Consider an insulator film occupying the space of range \((0, \delta)\) in \(x\)-axis with an applied voltage \(V\), according to Eq.(81), set the origin at \(x = 0\), the wave function of electron pair in the region \(x > \delta\) can be calculated by taking the integral path \(l_1\) across the insulator, it gives

\[
\psi_1 = e^{\frac{i}{\hbar} \int_0^\delta p \, dx + \frac{i}{\hbar} \int_0^\delta A_4 \, dx + i\theta_1}
\]

the wave function of electron pair in the region \(x > \delta\) can also be calculated by taking the integral path \(l_2\) around the circuit without crossing the insulator, it gives

\[
\psi_2 = e^{\frac{i}{\hbar} \int_\delta^x p \, dx + i\theta_2}
\]

To note that the electrons have the same momentum \(p\) in the superconductors whereas have an imaginary momentum in the insulator, to denote \(\int_0^\delta p \, dx = i\hbar k\delta\), we obtain the wave function

\[
\psi = \psi_1 + \psi_2 = e^{-k\delta + \frac{i}{\hbar} \int_\delta^x p \, dx + i\theta_1 + \theta_2} + e^{\frac{i}{\hbar} \int_\delta^x p \, dx + i\theta_2}
\]

According to Eq.(80), the superconducting current in \(x > \delta\) region is given by

\[
j = nqu = \frac{nq}{m} p = -i \frac{q}{2m} \left( \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) = \frac{2e}{m} p e^{-k\delta} \sin \left[ \frac{2eVt}{\hbar} + i(\theta_1 - \theta_2) \right]
\]

The result is just the Josephson tunneling effect, including DC Josephson effect and AC Josephson effect.

7 General superconducting states

It is easy to find that the electrons in Aharonov-Bohm experiment can regarded as being in a superconducting state because of lacking lattice-scattering, the similar situations like electron two split experiment etc. give us the same inspiration. Even the electrons in an atom can also be regarded as in a superconducting state. The feeling becomes much strong when we recognize that the two electrons with up and down spins in an atom always form a pair in the sell of the atom. Under the guidance of the general superconducting states, we think that the electron pairs in atom must have the same mechanism with Cooper pairs in superconductor. The relativistic mechanism of superconductivity suggested in the present paper just satisfies this requirement.

8 Conclusion

According to the theory of relativity, the relativistic Coulomb’s force between an electron pair is composed of two parts, the main part is repulsive, while the rest part can be attractive in certain situations. The relativistic attraction of an electron pair provides a insight into the mechanism of superconductivity.

In superconductor, there are, probably at least, two kinds of collective motions which can eliminate the repulsion between two electrons and let the attraction being dominant, the first is the combination of lattice and electron gas, accounting for traditional superconductivity; the second is the electron gas themselves, accounting for high \(T_c\) superconductivity.

In usual materials, there is a good balance between the repulsion and attraction of an electron pair, the electrons are regarded as free electrons so that Fermi gas theory plays very well. But in some materials, when the repulsion dominates electron pairs, the electron gas will has a behavior opposite to superconductivity.

The superconducting states are discussed in terms of relativistic quantum theory in the present paper, some significant results are obtained, including quantized magnetic flux, London equation, Meissner effect and Josephson effect.

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