Abstract—The groupcast index coding problem is the most general version of the classical index coding problem, where any receiver can demand messages that are also demanded by other receivers. Any groupcast index coding problem is described by its fitting matrix which contains unknown entries along with 1's and 0's. The problem of finding an optimal scalar linear code is equivalent to completing this matrix with known entries such that the rank of the resulting matrix is minimized. Any row basis of such a completion gives an optimal scalar linear code.

An index coding problem is said to be a joint extension of a finite number of index coding problems, if the fitting matrices of these problems are disjoint submatrices of the fitting matrix of the jointly extended problem. In this paper, a class of joint extensions of any finite number of groupcast index coding problems is identified, where the relation between the fitting matrices of the sub-problems present in the fitting matrix of the jointly extended problem is defined by a base problem. A lower bound on the minrank (optimal scalar linear codelength) of the jointly extended problem is given in terms of those of the sub-problems. This lower bound also has a dependence on the base problem and is operationally useful in finding lower bounds of the jointly extended problems when the minranks of all the sub-problems are known. Then, we provide a code construction (not necessarily optimal) for a special class of the jointly extended problem based on any codes of the base problem and all the sub-problems. A set of necessary conditions for the optimality of the constructed codes is given based on these bounds.

I. INTRODUCTION

The index coding problem (ICP) introduced in [1] is a source coding problem with some side-information present at the receivers. The sender broadcasts coded messages leveraging the knowledge of the side-information present at all the receivers, in order to reduce the number of transmissions required for all the receivers to decode their demanded messages. This problem is also related to topological interference management problem in wireless networks [2]. It also has applications in satellite communications where some users want to exchange their messages using a satellite [3], and the retransmission phase of downlink networks [4] among many others. In general, the ICP is NP-Hard. Optimal codelengths and optimal codes were given for some special classes of the ICP [5], [9]. Many works address the single unicast ICP (SUICP) where each receiver demands a unique message [6], [7]. The most general class of the ICP which subsumes SUICP is the groupcast ICP where any receiver can demand messages which are also demanded by other receivers.

The groupcast ICP was first studied in [6] where upper and lower bounds on the optimal codelength were given for any groupcast problem in terms of the optimal codelengths of two related SUICPs. In [8], a directed bipartite graph representation was introduced for the groupcast problem and capacity region was found when only particular coding schemes were allowed. The groupcast problem was represented as a directed hypergraph and bounds on the optimal broadcast rate were given in [9]. In [10], optimal scalar codelengths were obtained for a class of the groupcast problem, where each message is demanded by at most two receivers. The results are obtained based on the optimality of linear coding schemes for a related SUICP.

Characterisation of the optimal codelengths of SUICPs in terms of those of its sub-problems has been carried out in many works [5], [11]-[15]. A lifting construction was presented in [11], where a special class of SUICPs were obtained from another class of SUICPs. The optimal scalar linear codelength of the larger derived SUICP has been shown to be equal to that of the smaller SUICP. Optimal vector linear codes for a class of SUICPs were constructed using optimal scalar linear codes of other basic SUICPs in [5]. Graph homomorphism between complements of the side-information digraphs of two given SUICPs was used to establish a relation between their optimal codelengths [12]. Some special classes of rank invariant extensions of any SUICP were presented in [13], where the extended problems have the same optimal linear codelength as that of the original SUICP, generalizing the results of [11]. The notion of rank invariant extensions was extended to a class of joint extensions of any finite number of SUICPs in [14]. Two-sender SUICPs with a sub-problem being a joint extension of two SUICPs were solved for optimal scalar linear codelengths using those of the component single-sender SUICPs (sub-problems) [14]. In [15], capacity region of SUICPs with side-information digraphs being generalized lexicographic products of side-information digraphs of the component SUICPs was characterized in terms of those of the component SUICPs.

In this paper, we identify a class of joint extensions of a finite number of groupcast ICPs, where the relation between the sub-problems in the jointly extended problem is defined by a base problem. Optimal scalar linear codelength and optimal codes of the jointly extended problem are given in terms of those of the sub-problems and the base problem for a special class of the jointly extended problem introduced in this paper. This result generalizes the class of joint extensions solved in [14]. When the base problem and all the sub-problems are restricted to SUICPs, the class of jointly extended groupcast problems identified in this paper reduces to the class of SUICPs with the side-information digraphs being generalized lexicographic products of the component side-information digraphs [15]. Viewing any jointly extended problem from a matrix-completion perspective, the class of jointly extended problems solved in this paper extends the notion of generalized lexicographic products where any number of sub-problems can be groupcast ICPs.
The key results of this paper are summarized as follows.

- A lower bound on the minrank (optimal scalar linear codelength) is given for the class of jointly extended problems identified in this paper, in terms of those of the sub-problems and the base problem.
- A code construction (not necessarily optimal) is presented for a sub-class of the class of jointly extended problems, based on codes of the base problem and all the sub-problems.
- A set of necessary conditions are provided for the constructed codes to be optimal.

The remainder of the paper is organized as follows. Section II introduces the problem setup and establishes the notations and definitions used in this paper. Section III contains the main results of the paper. Section IV concludes the paper with directions for future work.

II. PROBLEM FORMULATION AND DEFINITIONS

In this section, we establish the notations and definitions used in this paper and formulate a class of the groupcast index coding problem that can be seen as joint extensions of smaller groupcast problems.

Matrices and vectors are denoted by bold uppercase and bold lowercase letters respectively. For any positive integer \( m, [m] \triangleq \{1, \ldots, m\} \). \( \mathbb{F}_q \) denotes the finite field of order \( q \). \( \mathbb{F}_q^{m \times d} \) denotes the vector space of all \( n \times d \) matrices over \( \mathbb{F}_q \).

We first define the notion of disjoint submatrices of a given matrix, which is needed to define the class of jointly extended problems dealt in this paper, and provide related notations.

**Definition 1** (Disjoint submatrices of a given matrix). A matrix \( L \) obtained by deleting some of the rows and/or some of the columns of \( M \) is said to be a submatrix of \( M \). The notation \( L \prec M \) denotes that \( L \) is a submatrix of \( M \). The set containing the indices of rows of \( M \) present in \( L \) is denoted by \( \text{row}(L, M) \). Indices of columns and rows are assumed to start from \( 1 \). A set of submatrices of a given matrix are said to be disjoint, if no two of the submatrices have elements indexed by the same ordered pair in the given matrix.

The number of rows and columns of any matrix \( M \) are denoted by \( R(M) \) and \( C(M) \) respectively. The \((i, j)\)th entry of matrix \( M \) is denoted as \( M_{i,j} \). The notation \( \left[ M \right]_{i,j} \) denotes the \((i, j)\)th component block matrix of \( M \), where the component block matrices of \( M \) (or equivalently the partition of \( M \) into component block matrices) are predefined by the construction of \( M \) using the same. \( M_{ij} \) denotes the matrix formed by stacking the rows of \( M \) indexed by the elements in the set \( \mathcal{R} \) in the ascending order of indices such that the row with the least row index forms the first row of \( M_{ij} \). For any matrix \( M \) over \( \mathbb{F}_q \), the rank of \( M \) over \( \mathbb{F}_q \) is denoted by \( \text{rk}_q(M) \). \( \langle M \rangle \) denotes the row space of \( M \). The transpose of \( M \) is denoted by \( M^T \).

We now define an upper-triangular matrix which is frequently used in this paper.

**Definition 2** (Upper-triangular matrices). A permutation matrix \( P \) is a square matrix that has exactly one 1 in each row and each column and 0’s elsewhere. Any \( p \times p \) permutation matrix \( P \) represents a permutation of \( p \) elements. For a \( p \times p \) matrix \( M_x \) containing unknown elements denoted by \( x \) along with some known elements, \( PM_x \) denotes the matrix obtained by applying the permutation described by \( P \) on the rows of \( M_x \). Similarly, \( M_xP \) denotes the matrix obtained by applying the permutation described by \( P \) on the columns of \( M_x \). A \( p \times p \) square matrix \( M \) is said to be upper-triangular if there exists two \( p \times p \) permutation matrices \( P \) and \( Q \) such that \( PMQ \) is an upper-triangular matrix. A matrix constructed using block matrices is called block upper-triangular, if the matrix obtained by replacing each block matrix by a scalar is upper-triangular. The block matrices can also be rectangular matrices. A \( p \times p \) matrix \( M_x \) (with some unknown elements denoted by \( x \)) is said to be upper-triangular if there exists two \( p \times p \) permutation matrices \( P \) and \( Q \) such that \( PM_xQ \) is an upper-triangular matrix with all the diagonal entries being equal to 1. The set of all \( p \times p \) upper-triangular matrices containing entries from \( \mathbb{F}_q \) and possible unknowns is denoted by \( U^p \).

We now explain the groupcast index coding problem setup.

An instance of the groupcast index coding problem consists of a sender with \( m \) independent messages given by \( M = \{x_1, x_2, \ldots, x_m\} \), where \( x_i \in \mathbb{F}_q^{d \times 1} \), \( i \in [m] \), and \( d \geq 1 \). There are \( n \) receivers. The \( j \)th receiver knows \( K_j \subseteq M \) (also known as its side-information) and wants \( W_j \subseteq M \setminus K_j \), \( j \in [n] \). Each message is demanded by at least one receiver. Without loss of generality, throughout the paper we assume that \( |W_j| = 1 \), \( \forall j \in [n] \). For a receiver demanding more than one message, we replace it by as many new receivers as the number of messages demanded by the original receiver, with each new receiver demanding a unique message which was demanded by the original receiver and having the same side-information as that of the original receiver. Hence, we assume that the \( j \)th receiver wants \( x_{f(j)} \), \( j \in [n] \), where the mapping \( f : [n] \rightarrow [m] \) gives the index of the wanted message. Let \( \mathcal{K} = (K_1, K_2, \ldots, K_n) \). Hence, we can describe an instance of the groupcast ICP using the quadruple \( (m, n, \mathcal{K}, f) \). The transmission is through a noiseless broadcast channel which carries symbols from \( \mathbb{F}_q \).

An index code over \( \mathbb{F}_q \) for an instance of the groupcast ICP, described by \( (m, n, \mathcal{K}, f) \), is an encoding function \( E : \mathbb{F}_q^{md \times 1} \rightarrow \mathbb{F}_q^{p \times 1} \) such that there exists a decoding function \( D_j : \mathbb{F}_q^{(r[K_j]d) \times 1} \rightarrow \mathbb{F}_q^{x_1} \) at \( j \)th receiver \( \forall j \in [n] \), with \( x_{f(j)} = D_j(E(x), K_j) \) for any realizations of \( K_j \) and \( x = (x_1, \ldots, x_m)^T \). The sender transmits \( E(x) \) with codelength \( r \). The smallest possible value of \( r \) is called the optimal codelength of the problem. If the encoding function is linear, the index code is given by \( Gx \), where \( G \in \mathbb{F}_q^{r \times md} \) is called the encoding matrix. With the encoding function being linear, if \( d = 1 \), the code is said to be scalar linear, else it is said to be vector linear. In this paper, we consider only scalar linear codes. If \( n = m \), the index coding problem (ICP) is called single unicast ICP (SUICP). For an SUICP, without loss of generality, we assume that the \( j \)th receiver wants \( x_j, j \in [n] \).

Any groupcast ICP can be represented using a fitting matrix which was introduced in [16] and was defined again in [13].
to include the groupcast problem. It contains unknown entries denoted by $x$. Each row of the fitting matrix represents a receiver and each column represents a message.

**Definition 3 (Fitting Matrix, [13]):** An $n \times m$ matrix $F_x$ is called the fitting matrix of an ICP described by $(m, n, K, f)$, where the $(i, j)$th entry is given by

$$F_x[i,j] = \begin{cases} x & \text{if } x_j \in K_i, \\ 1 & \text{if } j = f(i), \\ 0 & \text{otherwise}. \end{cases}$$

$\forall i \in [n]$, and $j \in [m]$.

The minimum rank of $F_x$ obtained by replacing the $x$’s in $F_x$ with arbitrary values from $F_q$ is called the minrank of $F_x$ or that of the ICP described by $(m, n, K, f)$ over $F_q$. It has been shown in [17] that the optimal code length of any scalar linear code over $F_q$ is equal to the minrank of $F_x$ over $F_q$, denoted as $\text{mrk}_{q}(F_x)$. We say $F = F_x$ (complete $F_x$ or equivalently $F$ is a completion of $F_x$), if $F$ is obtained from $F_x$ by replacing all the unknown elements by arbitrary elements from the given field of interest.

The notion of joint extensions of any finite number of SUICPs was introduced in [14]. We extend the definition to include joint extensions of any finite number of groupcast ICPs.

**Definition 4 (Joint Extension):** Consider $l$ ICPs where the $i$th ICP $I_i$ is described using the fitting matrix $F_x^{(i)}$, $i \in [l]$. An ICP $I_E$ whose fitting matrix is given by $F_x^{E}$ is called a jointly extended ICP (or simply a joint extension of $l$ ICPs, extended using ICPs $I_1, ..., I_l$, if $F_x^{E}$ consists of all $F_x^{(i)}$, $i \in [l]$, as its disjoint submatrices. The $l$ ICPs are called as the component problems (or sub-problems) of the jointly extended problem.

In this paper, we study a special class of joint extensions of $m_B$ groupcast ICPs described as follows. Let the ICP $I_B$ described by the $n_B \times m_B$ fitting matrix $F_x^{B}$, be called the base problem. Let $l_B$ be the number of occurrences of 1 in the $j$th column of $F_x^{B}$, $j \in [m_B]$. The superscript and subscript “$B$” stands for the base problem. The $i$th component ICP $I_i$ is described by the $n_i \times m_i$ fitting matrix $F_x^{(i)}$, $i \in [l_B]$. Then, we have the joint extension $I_E$ of the $m_B$ component ICPs with respect to the base problem $I_B$, described by the $n_E \times m_E$ fitting matrix $F_x^{E}$ as given below in terms of its block matrices, where $n_E = \sum_{i \in [m_B]} n_i l_j$, and $m_E = \sum_{i \in [m_B]} m_i$.

$$[F_x^{E}]_{i,j} = \begin{cases} X & \text{if } F_x^{B}_{i,j} = x, \\ F_x^{(j)} & \text{if } F_x^{B}_{i,j} = 1, \\ 0 & \text{otherwise}. \end{cases}$$

$\forall i \in [n_B]$, and $j \in [m_B]$. That is, $F_x^{E}$ is obtained from $F_x^{B}$ by replacing the 1’s in its $j$th column by $F_x^{(j)}$, and replacing $x$’s and 0’s by $X$’s and 0’s of appropriate sizes respectively. The dependence of $I_E$ on $(I_i)_{i \in [m_B]}$ and $I_B$ is denoted as $I_E(I_B; (I_i)_{i \in [m_B]})$. Throughout this paper, whenever we refer to blocks (or block matrices) of $F_x^{E}$, we refer to the block matrices that are induced by $F_x^{B}$ as seen in the construction of $F_x^{E}$ from the fitting matrices of the component ICPs based on the fitting matrix of the base problem. The $i$th row of block matrices in $F_x^{E}$ refers to the matrix $([F_x^{E}]_{i,1}, [F_x^{E}]_{i,2}, \ldots, [F_x^{E}]_{i,m_B})$, $i \in [n_B]$.

**Remark 1.** In a recent work [15], generalized lexicographic product of a finite number of side-information digraphs was introduced. The class of joint extensions introduced in this paper reduces to the generalized lexicographic product, if the base ICP $I_B$ and all the ICPs $(I_i)_{i \in [m_B]}$ are SUICPs.

When the base problem and all the component problems are SUICPs, the side-information digraph $G_0$ in the generalized lexicographic product in [15] corresponds to the base problem $I_B$ stated in this paper.

We illustrate the construction of the extended problem using two running examples, given the base problem and the component problems, in terms of the respective fitting matrices.

**Example 1.** Consider $m_B = n_B = 3$. The base problem $I_B$ is described by the fitting matrix $F_x^{B}$. Let the component problems $(I_i)_{i \in [m_B]}$ be described by $(F_x^{(i)})_{i \in [m_B]}$ respectively.

$$F_x^{B} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & x \\ x & 0 & 1 \end{pmatrix}, \ F_x^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

$$F_x^{(2)} = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}, F_x^{(3)} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$  

Observe that $n_1 = m_1 = 4, n_2 = m_2 = 2, n_3 = m_3 = 3$, and $l_1 = l_2 = l_3 = 1$. All the problems involved in the construction of the extended problem are SUICPs. The extended problem $I_E(I_B; (I_i)_{i \in [m_B]})$ is described by $F_x^{E}$ with $n_E = m_E = 4 + 3 + 2 = 9$. The block matrices of $F_x^{E}$ are indicated by the partition shown in $F_x^{E}$.

$$F_x^{E} = \begin{pmatrix} 1 & 0 & 0 & x & x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & x & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 1 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & x & x & 0 & 0 \\ x & x & x & x & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 1 & x & x & 0 & 0 \\ x & x & x & x & 0 & 0 & x & 1 & 0 \\ x & x & x & x & 0 & 0 & x & 1 & 1 \end{pmatrix}.$$  

The following example illustrates the construction of an extended problem which is a groupcast problem, with the base problem also being a groupcast problem.

**Example 2.** Consider $m_B = 4, n_B = 5$. The base problem and the component problems are described by the fitting matrices given below respectively.

$$F_x^{B} = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 0 & x & 1 \\ x & 1 & 0 & 0 \\ 0 & 0 & x & 1 \\ 0 & 0 & 1 & x \end{pmatrix}, F_x^{(1)} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$  

$$F_x^{(2)} = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}, F_x^{(3)} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, F_x^{(4)} = \begin{pmatrix} 1 \end{pmatrix}.$$
Observe that \(n_1 = 4, m_1 = 3, n_2 = m_2 = 2, n_3 = m_3 = 3\), and \(n_4 = m_4 = 1\). Also, \(l_1 = l_2 = l_4 = 1\) and \(l_3 = 2\). Note that \(I_1\) is a groupcast problem. The extended problem \(\mathcal{IE}((I_x; (I_i)_{i \in [m_B]}))\) is described by \(\mathbf{F}^E_x\) with \(n_E = 4 + 2 + (2 \times 3) + 1 = 13\), and \(m_E = 3 + 2 + 3 + 1 = 9\).

\[
\mathbf{F}^E_x = \begin{bmatrix}
1 & x & 0 & x & x & 0 & 0 & 0 & 0 \\
0 & x & 1 & 0 & x & x & 0 & 0 & 0 \\
x & 1 & 0 & x & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x & x & 1 & 0 & x & 0 \\
0 & 0 & 0 & x & x & x & 1 & 0 & 0 \\
0 & 0 & 0 & x & 1 & x & x & 1 & 0 \\
0 & x & x & 1 & x & 0 & 0 & 0 & 0 \\
0 & x & x & x & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x & x & x & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The following notations are required for the construction of a larger index code from component index codes. Let \(C_1\) and \(C_2\) be two codewords of length \(l_1\) and \(l_2\) respectively. \(C_1 + C_2\) denotes the element-wise addition of \(C_1\) and \(C_2\) after zero-padding the shorter message to at least significant positions to match the length of the longer message. The resulting length of the codeword is \(\text{max}(l_1, l_2)\). For example, if \(C_1 = 1010\), and \(C_2 = 1100\), then \(C_1 + C_2 = 0110\). \([a : b]\) denotes the vector obtained by picking the element from position \(a\) to \(b\) with position \(b\), starting from the most significant position of the codeword \(C\), with \(a, b \in [l]\), \(l\) being the length of \(C\). For example \(C_1[2 : 4] = 010\).

The results presented in this paper hold for any finite field. But, we consider only \(q = 2\) (binary field) for simplicity.

### III. MAIN RESULTS

In this section, we first provide a lower bound on the minrank of the jointly extended problem introduced in the previous section, in terms of those of the component problems and the upper-triangular submatrices of the base problem. Then, we provide a code construction (not necessarily optimal) for a special class of the jointly extended problem, in terms of those of the component problems and the base problem. We then provide necessary conditions for the optimality of the code construction.

The following lemma provides a lower bound on the minrank of \(\mathcal{IE}((I_x; (I_i)_{i \in [m_B]}))\). The proof follows on similar lines as that of Lemma 4.2 in [7]. We provide the proof for completeness.

The set of all upper-triangular submatrices of \(\mathbf{F}^B_x\) is given by

\[
\mathcal{U}_B = \{\mathbf{M}_x : \mathbf{M}_x < \mathbf{F}^B_x, \mathbf{M}_x \in \mathcal{U}_B^{((M_x))}\}.
\]

**Lemma 1** (A lower bound). For a given jointly extended ICP \(\mathcal{IE}((I_x; (I_i)_{i \in [m_B]}))\) we have

\[
\text{mrk}_q(\mathbf{F}^E_x) \geq \max\{\sum_{s \in \text{col}(\mathbf{M}_x, \mathbf{F}^B_s)} \text{mrk}_q(\mathbf{F}^E_s) : \mathbf{M}_x \in \mathcal{U}_B\}.
\]

**Proof.** Consider the submatrix \(\mathbf{M}^E_x\) corresponding to an \(\mathbf{M}_x\) constructed using the block matrices of \(\mathbf{F}^E_x\) as follows. Let \((s_i, t_j)\) with \(i \in [\mathcal{R}(\mathbf{M}_x)]\) and \(j \in [\mathcal{C}(\mathbf{M}_x)]\) be an element of the cartesian product given by \(\text{row}(\mathbf{M}_x, \mathbf{F}^B_x) \times \text{col}(\mathbf{M}_x, \mathbf{F}^B_x)\) for any \(\mathbf{M}_x \in \mathcal{U}_B\). Then, the \((i, j)\th\) block matrix of \(\mathbf{M}^E_x\) is given by \([\mathbf{F}^E_x]_{s_i, t_j}\). From the construction of \(\mathbf{M}^E_x\) and the fact that \(\mathbf{M}_x\) is an upper-triangular matrix, we see that \(\mathbf{M}^E_x\) can be written as a block upper-triangular matrix \(\mathbf{U}^E_x\), by permuting the rows and/or columns of block matrices of \(\mathbf{M}^E_x\) using the same permutations that make \(\mathbf{M}_x\) an upper-triangular matrix with all its diagonal entries being 1. To prove the lemma, we find the minrank of \(\mathbf{M}^E_x\) as \(\mathbf{M}^E_x\) corresponds to a sub-problem of \(\mathbf{F}^E_x\). Note that the minrank of any subproblem is not greater than that of the original problem. Hence, we first provide an upper bound for \(\text{mrk}_q(\mathbf{M}^E_x)\) and then provide a matching lower bound.

With all matrices \(\mathbf{F}^E_{x_j}, t_j \in \text{col}(\mathbf{M}_x, \mathbf{F}^B_x), j \in [\mathcal{C}(\mathbf{M}_x)]\), now being the diagonal block matrices of \(\mathbf{U}^E_x\), if \(\mathbf{F}(t_j) \approx \mathbf{F}_{x_j}^{(t_j)}\), then the block diagonal matrix \(\mathbf{D}^E\) with its diagonal block matrices being \(\mathbf{F}(t_j)\) in some order (due to the permutations applied on the rows and/or columns of block matrices of \(\mathbf{M}^E_x\)), we see that \(\mathbf{D}^E \approx \mathbf{U}^E_x\). As \(\text{rk}_q(\mathbf{D}^E) = \sum_{t_j \in \text{col}(\mathbf{M}_x, \mathbf{F}^B_x)} \text{rk}_q(\mathbf{F}(t_j))\). Thus, we have

\[
\text{mrk}_q(\mathbf{M}^E_x) = \text{mrk}_q(\mathbf{U}^E_x) \leq \text{rk}_q(\mathbf{D}^E) = \sum_{t_j \in \text{col}(\mathbf{M}_x, \mathbf{F}^B_x)} \text{mrk}_q(\mathbf{F}(t_j)),
\]

where in the last equality, we take \(\mathbf{F}(t_j) \approx \mathbf{F}_{x_j}^{(t_j)}\) such that \(\text{rk}_q(\mathbf{F}(t_j)) = \text{mrk}_q(\mathbf{F}(t_j))\).

Now, we provide a matching lower bound. If \(\mathbf{U}^E_x \approx \mathbf{U}^E_x\), then \(\mathbf{U}^E_x\) must be a block upper-triangular matrix. Note that the diagonal block entries \([\mathbf{U}^E_x]_{j, j'} \approx [\mathbf{U}^E_x]_{j, j'}\), where \([\mathbf{U}^E_x]_{j, j'}\) is equal to \(\mathbf{F}(t_j)\) for some \(t_j \in \text{col}(\mathbf{M}_x, \mathbf{F}^B_x)\) (due to the permutations applied on the rows and/or columns of block matrices of \(\mathbf{M}^E_x\)). Thus, we have

\[
\text{rk}_q(\mathbf{F}(t_j)) \geq \sum_{\hat{j} \in \text{col}(\mathbf{M}_x)} \text{rk}_q([\mathbf{U}^E_x]_{\hat{j}, \hat{j}'}) \geq \sum_{t_j \in \text{col}(\mathbf{M}, \mathbf{F}^B)} \text{mrk}_q(\mathbf{F}(t_j)),
\]

which yields a matching lower bound by choosing \(\mathbf{F}(t)\) such that \(\text{rk}_q(\mathbf{F}(t)) = \text{mrk}_q(\mathbf{F}(t))\). This completes the proof.

**Remark 2.** This lower bound resembles the MAIS (Maximum Acyclic Induced Subgraph) bound introduced in [16], which is a lower bound on the minrank of the SUICP. However, the bound given in Lemma I need not be equal to the MAIS bound for \(\mathcal{IE}((I_x; (I_i)_{i \in [m_B]}))\). The submatrix of the fitting matrix of a SUICP corresponding to any maximum acyclic induced subgraph of the side-information digraph is upper-triangular (as the subgraph is acyclic). Hence, we get the MAIS bound for the SUICP.

**Remark 3.** The bound given in Lemma I also has an operational significance in finding a lower bound on the minrank.
of the jointly extended problem using the minranks of some of the component sub-problems and the set $U_F$ instead of directly computing lower bounds like the MAIS bound which is computation intensive.

We illustrate the application of Lemma 1 with two running examples. In the first example, all the problems involved are SUICPs.

**Example 3** (Example 1 continued). In Example 1 we see that there are six upper-triangulable submatrices of $F^B_x$, out of which considering all the $2 \times 2$ submatrices of $F^B_x$ are sufficient to find the lower bound given in the lemma, as shown below. Note that $mrk_q(F_x^{(1)}) = 3$, $mrk_q(F_x^{(2)}) = 1$, and $mrk_q(F_x^{(3)}) = 2$.

$$M_x^{(1)} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \text{col}(M_x^{(1)}, F^B_x) = \{1, 2\},$$

$$\text{row}(M_x^{(1)}, F^B_x) = \{1, 2\}, \sum_{s \in \text{col}(M_x^{(1)}, F^B_x)} mrk_{q}(F_x^{(s)}) = 4.$$ 

$$M_x^{(2)} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \text{col}(M_x^{(2)}, F^B_x) = \{2, 3\},$$

$$\text{row}(M_x^{(2)}, F^B_x) = \{2, 3\}, \sum_{s \in \text{col}(M_x^{(2)}, F^B_x)} mrk_{q}(F_x^{(s)}) = 3.$$ 

$$M_x^{(3)} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \text{col}(M_x^{(3)}, F^B_x) = \{1, 3\},$$

$$\text{row}(M_x^{(3)}, F^B_x) = \{1, 3\}, \sum_{s \in \text{col}(M_x^{(3)}, F^B_x)} mrk_{q}(F_x^{(s)}) = 5.$$ 

Hence, according to the lemma we have $mrk_q(F^E_x) \geq 5$.

In the following example, the base problem and a component problem are groupcast ICPs.

**Example 4** (Example 2 continued). In Example 2 it can be easily seen that there are no $4 \times 4$ upper-triangulable submatrices of $F^B_x$, since any combination of 4 rows consists of at least four rows 1, 3 or rows 1 and 5, 4, which if present in a $4 \times 4$ submatrix, the submatrix is not upper-triangulable. This is because the problem induced by rows (1 and 3) and rows (3 and 4) contain a cycle. Note that $mrk_q(F_x^{(1)}) = mrk_q(F_x^{(3)}) = 2$ and $mrk_q(F_x^{(2)}) = mrk_q(F_x^{(4)}) = 1$. Consider the submatrix given below which is upper-triangulable (There exist row and column permutations necessary for this upper-triangulable matrix). Then.

$$M_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}, \text{row}(M_x, F^B_x) = \{1, 2, 4\},$$

$$\text{col}(M_x, F^B_x) = \{1, 3, 4\}, \sum_{s \in \text{col}(M_x, F^B_x)} mrk_{q}(F_x^{(s)}) = 5.$$ 

Hence, according to the lemma we have $mrk_q(F^E_x) \geq 5$.

The following lemma provides a code construction (not necessarily optimal) for a particular class of the jointly extended ICP $I_E(I_B; (I_i)_{i \in [m_B]})$ using codes of the component problems $(I_i)_{i \in [m_B]}$ and a code of the base problem $I_B$.

**Lemma 2** (An upper bound). For a given jointly extended ICP $I_E(I_B; (I_i)_{i \in [m_B]})$, let $F^{(j)} \approx F_x^{(j)}$, $\forall j \in [m_B]$, such that $r_j = mrk_q(F^{(j)})$, where $r_j$ is not necessarily equal to $mrk_q(F^{(j)})$. If there exists $(i)$ an upper-triangulable matrix $M_x$ such that $M_x \prec F^B_x$ and $\{t_1, t_2, \cdots, t_{l(E(x))}\} = \text{col}(M_x, F^B_x)$, where $(t_i)_{i \in [m_B]}$ is a permutation of $[m_B]$ such that $r_{t_1} \geq r_{t_2} \geq \cdots \geq r_{t_{m_B}}$, and $r_{t_{m_B}} \geq r_i$ for $i \geq r_B = C(M_x)$, and $(ii)$ there exists an $F^{B}_C \approx F^{B}_x$ with $r_B = mrk_q(F^{B}_C)$, such that the rows of $F^{B}_x$ indexed by the numbers in $\text{row}(M_x, F^B_x)$ are independent, then there exists a scalar linear code of length $\sum_{j} r_j$.

**Proof.** We provide a construction of a scalar linear code with the stated code length. For an easier visualization of the code construction and to alleviate the need of more notations, we permute the rows of the fitting matrices and the completions (given in the statement of the lemma) of all the component problems and the base problem as stated in the following. We also permute the columns of the fitting matrix and the completion of the base problem. Note that the permutation applied on the columns and/or rows of any given fitting matrix (mentioned above) is the same as that applied on the columns and/or rows of the respective completion. Then, we provide a code construction for the jointly extended ICP $I_E(I_B; (I_i)_{i \in [m_B]})$, where the base problem $(I_x)$ and all the component problems $(I_i)_{i \in [m_B]}$ have fitting matrices obtained by the above mentioned permutations of rows and/or columns of the original fitting matrices. Note that there is no loss of generality in proving the lemma for the extended problem $I_x^E$ as the permutations mentioned above rename the messages (in the case of column permutations) and receivers (in the case of row permutations), which do not change the extended problem, the base problem and the component problems.

The rows of $F^{(j)}$ and $F^{(j)}_x$ are permuted with the same permutation such that the first $r_j$ rows of $F^{(j)}$ are independent and span $F^{(j)}$ (the row space of $F^{(j)}$), $\forall j \in [m_B]$. Note that such a permutation exists as $r_j = mrk_q(F^{(j)})$, $\forall j \in [m_B]$. Let $s_1, s_2, \cdots, s_{r_B} = \text{row}(M_x, F^B_x)$, where $(s_i)_{i \in [m_B]}$ is a permutation of $[m_B]$ such that $s_{r_1} \geq s_{r_2} \geq \cdots \geq s_{r_B}$, and $s_{r_B} \geq s_i$ for $i \geq r_B$, where $r_{s_i} = mrk_q(F^{(j)})$ such that $F^{(j)}_x$, $s_{i,j} = 1, j \in [m_B]$. The rows of $F^B$ and $F^B_x$ are permuted with the same permutation such that the rows of $F^B$ indexed by the elements in $\text{row}(M_x, F^B_x)$ are mapped to the first $r_B$ rows of $F^B$ such that the row indexed by $s_i$ is mapped to the row indexed by $i$, $i \in [r_B]$. The columns of $F^B$ and $F^B_x$ are also permuted with the same permutation such that the columns of $F^B$ indexed by the elements in $\text{col}(M_x, F^B_x)$ are mapped to the first $r_B$ columns of $F^B$. Now, consider the fitting matrix $F^E_x$ of the jointly extended ICP $I_E(I_B; (I_i)_{i \in [m_B]})$, with the fitting matrices of the base problem $F^{B}_C$ and the component problems $(F^{(j)}_x)_{i \in [m_B]}$ obtained after the above mentioned permutations. (Observe that we have not renamed the problems, fitting matrices, and their completions obtained after the permutation to have brevity in the notation. Due to the permutation, we now have $(t_i)_{i \in r_B}$ (defined in the statement of the theorem) mapped to $[r_B]$ in some order, which form the new $(t_i)_{i \in r_B}$. Hence, we
now have \( r_{E,B} \geq r_i, \forall i \geq r_B, i \in [m_B] \). We now provide a completion of \( \hat{F}^{E}_{x} \) and show that it is a valid completion. Then, we prove that the codelength obtained by such a completion is in the row space of the first \( r_j \) rows, which is in turn in the row space of \( \hat{F}^{E} \). This completes the proof. \[ \square \]

We illustrate the use of Lemma 2 using a running example.

**Example 5.** (Example 1 continued) Consider the completions \( F^{(i)} \) of \( F_{x}^{(i)} \), \( i \in [3] \) as given below with \( r_1 = r_{kq}(F^{(1)}) = 3 \), \( r_2 = r_{kq}(F^{(2)}) = 1 \), and \( r_3 = r_{kq}(F^{(3)}) = 3 \).

\[
\begin{align*}
F^{(1)} &= \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
\end{pmatrix}, \\
F^{(2)} &= \begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}, \\
F^{(3)} &= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \\
F^{(1)} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}, \\
F^{(2)} &= \begin{pmatrix}
1 & 1 \\
\end{pmatrix}.
\end{align*}
\]

Note that only the completions \( F^{(1)} \) and \( F^{(2)} \) correspond to optimal codes as \( r_1 \) and \( r_2 \) are equal to \( mr_{kq}(F^{(1)}) \) and \( mr_{kq}(F^{(2)}) \) respectively. Also, \( r_1 \geq r_3 \geq r_2 \). Hence, letting \( t_1 = 1, t_2 = 3, \) and \( t_3 = 2 \), and taking the third submatrix \( M^{(3)}_{x} \) of \( F_{x}^{(i)} \) given in Example 3 (given below for easy reference), we see that the condition (i) given in Lemma 2 is satisfied.

\[
M^{(3)}_{x} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
\end{pmatrix}, \\
\text{row}(M^{(3)}_{x}, F^{B}_{x}) = \{1, 3\}, \\
\text{col}(M^{(3)}_{x}, F^{B}_{x}) = \{1, 3\}.
\]

Note that by taking \( F^{B} \approx F^{E} \) as given below, condition (ii) given in Lemma 2 is also satisfied.

\[
F^{B} = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
\end{pmatrix}, \\
r_B = r_{kq}(F^{B}) = \text{mr}_{kq}(F^{B}) = 2.
\]

Now we complete the fitting matrix \( F_{x}^{E} \) as given in Lemma 2 as shown below. Note that double lines (in \( F^{E} \)) used for partitioning correspond to the block matrices of \( F^{E}_{x} \). The single lines correspond to the construction given in Lemma 2. The matrix \( \hat{F}^{E} \) is also shown below.

\[
\begin{align*}
\hat{F}^{(1)} &= \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}, \\
\hat{F}^{(2)} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}, \\
\hat{F}^{(3)} &= \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}.
\end{align*}
\]

Consider the message set given by \( \mathcal{M} = \{x_1 = x^{(1)}_1, x_2 = x^{(2)}_1, x_3 = x^{(3)}_1, x_4 = x^{(4)}_1, x_5 = x^{(1)}_2, x_6 = x^{(2)}_2, x_7 = x^{(3)}_2\} \).

Consider the message set given by \( \mathcal{M} = \{x_1 = x^{(1)}_1, x_2 = x^{(2)}_1, x_3 = x^{(3)}_1, x_4 = x^{(4)}_1, x_5 = x^{(1)}_2, x_6 = x^{(2)}_2, x_7 = x^{(3)}_2\} \).
The index code is given by $\hat{F}^E_x$, where $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)^T$. Hence, the code is given by $C_E = (x_1 + x_2 + x_5 + x_6, x_2 + x_3, x_3 + x_4, x_1 + x_2 + x_7, x_2 + x_3 + x_8, x_3 + x_4 + x_9)$. It can be easily verified that all receivers can decode their wanted messages using their side-information and $C_E$.

Note that the code given by $F^B$ for the base problem is $C_B = (x_1^{(B)} + x_2^{(B)}, x_3^{(B)} + x_4^{(B)})$, where the message set for the base problem is given by $M_B = \{x_1^{(B)}, x_2^{(B)}, x_3^{(B)}\}$. Considering the codes of the component problems given by $C_1 = (x_1^{(1)} + x_1^{(2)}, x_2^{(1)} + x_3^{(1)}, x_3^{(1)} + x_1^{(2)})$, $C_2 = (x_1^{(3)} + x_2^{(2)})$, and $C_3 = (x_1^{(3)}, x_2^{(3)}, x_3^{(3)})$, we see that the code $C_E$ can also be written as $(C_1 + C_2, C_1 + C_3)$. This shows the dependence of the code $C_E$ on those of the base problem and the component problems. In the code $C_B$, $x_i^{(B)}$ is replaced by $C_i$, for $i \in [m_B]$, to obtain $C_E$.

We now provide an example with a given groupcast ICP.

**Example 6.** Consider the groupcast ICP given by the fitting matrix shown below with $n_E = 14$ and $m_E = 11$.

$$F^E_x = \begin{pmatrix}
1 & 0 & 0 & 0 & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 1 & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 \\
x & x & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & x & x & x & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & x & x & x & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x & x & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

It can be easily identified that this problem is a jointly extended ICP introduced in this paper. The fitting matrices of the component problems and the base problem are given below.

$$F^B_x = \begin{pmatrix}
1 & x & 0 & 0 & 0 \\
0 & 1 & 0 & x \\
x & 1 & 0 & 0 \\
0 & 0 & 1 & x \\
0 & 0 & 1 & x \\
0 & 0 & 1 & x \\
0 & 0 & 1 & x \\
0 & 0 & 1 & x \\
0 & 0 & 1 & x \\
0 & 0 & 1 & x \\
0 & 0 & 1 & x \\
0 & 0 & 1 & x \\
0 & 0 & 1 & x
\end{pmatrix},$$

$$F^{(1)}_x = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
x & 0 & 1 \\
x & 0 & 1 \\
0 & 0 & 1 \\
x & 0 & 1 \\
x & 0 & 1 \\
x & 0 & 1 \\
x & 0 & 1 \\
x & 0 & 1 \\
x & 0 & 1 \\
x & 0 & 1 \\
x & 0 & 1
\end{pmatrix},$$

$$F^{(2)}_x = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
x & 0 & 1
\end{pmatrix},$$

$$F^{(3)}_x = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
x & 0 & 1 \\
0 & 0 & 1 \\
x & 0 & 1
\end{pmatrix},$$

$$F^{(4)}_x = \begin{pmatrix}
1 & x & 0 \\
0 & 0 & 1
\end{pmatrix}.$$

Note that $mrk_q(F^{(1)}_x) = 4$, $mrk_q(F^{(2)}_x) = 3$, $mrk_q(F^{(3)}_x) = mrk_q(F^{(4)}_x) = 2$, and $mrk_q(F^B_x) = 3$. As in Example 6, we find that there are no $4 \times 4$ upper-triangulable submatrices of $F^E_x$. By enumerating all possible $3 \times 3$ submatrices, we see that there are three $3 \times 3$ upper-triangulable submatrices as given below.

$$M^{(1)}_x = \begin{pmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},$$

$$\text{row}(M^{(1)}_x, F^B_x) = \{1, 2, 3\},$$

$$\text{col}(M^{(1)}_x, F^B_x) = \{1, 2, 3\},$$

$$\sum_{s \in \text{col}(M^{(1)}_x, F^B_x)} mrk_q(F^{(s)}) = 9.$$
Now, we state and prove the main result of this section, which establishes the minrank of a special class of jointly extended problems identified in this paper.

**Theorem 1.** For a given jointly extended ICP $\mathcal{I}_E(\mathcal{I}_B; \{\mathcal{I}_j\}) \subseteq [m_B]$, with $r_j = mrk_q(F_{x}^{(j)}, \forall j \in [m_B]$. Let $r_{t_1} \geq r_{t_2} \geq \cdots \geq r_{t_m},$ where $t_j, j \in [m_B]$. If there exists (i) an upper-triangulable matrix $M_x \prec F_x^{B}$ such that

$$\sum_{s \in \text{col}(M_x,F_x^{B})} mrk_q(F_{x}^{(s)}) = \max\{\sum_{s \in \text{col}(M_x,F_x^{B})} mrk_q(F_{x}^{(s)}) : M_x \in \mathcal{U}_B\},$$

where $\text{col}(M_x,F_x^{B}) = \{t_1, t_2, \ldots, t_{\text{col}(M_x)}\}$, and there exists (ii) an $F_x^{B} \approx F_x^{B}$ with $r_B = r_k_q(F_x^{(B)} = C(M_x)$, such that the rows of $F_x^{B}$ indexed by the numbers in $\text{row}(M_x,F_x^{B})$ are independent, then we have $mrk_q(F_x^{B}) = \sum r_{t_i}$.

**Proof.** The proof follows directly from Lemmas 1 and 2 which provide a lower bound and the matching upper bound respectively, with the conditions stated in the theorem. □

The optimality of the scalar linear code given in Example 6 follows from this theorem. We provide another example to illustrate the use of the theorem.

**Example 7.** Consider the groupcast ICP given by the fitting matrix shown below with $n_E = 15$ and $m_E = 14$. The fitting matrices of the component problems and the base problem are also identified given below.

$$F_x^{E} = \begin{pmatrix} 1 & x & x & x & x & x & x & x & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & x & x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$F_x^{B} = F_x^{(3)} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & x \\ x & 0 & 1 \end{pmatrix}, F_x^{(1)} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

$$F_x^{(2)} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \end{pmatrix}, F_x^{(4)} = \begin{pmatrix} 1 & x & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $mrk_q(F_x^{(3)}) = mrk_q(F_x^{(2)}) = mrk_q(F_x^{(3)}) = 2$, and $mrk_q(F_x^{(1)}) = 3$. Observe that there are no upper-triangulable matrices of size $3 \times 3$ in $F_x^{B}$. Consider the following upper-triangulable submatrix.

$$M_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, row(M_x,F_x^{B}) = \{3, 4\},$$

$$col(M_x,F_x^{B}) = \{3, 4\}, \sum_{s \in \text{col}(M_x,F_x^{B})} mrk_q(F_{x}^{(s)}) = 5.$$ Consider the completion of $F_x^{B}$ given below. Observe that the last two rows are independent and span $(F_x^{B})$. Note that this choice of $F_x^{B}$ and $M_x$ satisfy conditions (i) and (ii) given in the theorem.

$$F_x^{B} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$ We complete $F_x^{E}$ as given in Lemma 2. The encoding matrix $F_x^{E}$ obtained by this completion is also given below. Observe that the codelength is $3 + 2 = 5$ as stated by the theorem.

$$F_x^{E} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$
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