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UNIQUENESS IN AN INVERSE BOUNDARY PROBLEM FOR A MAGNETIC SCHRÖDINGER OPERATOR WITH A BOUNDED MAGNETIC POTENTIAL

KATSIARYNA KRUPCHYK AND GUNTHER UHLMANN

Abstract. We show that the knowledge of the set of the Cauchy data on the boundary of a bounded open set in $\mathbb{R}^n$, $n \geq 3$, for the magnetic Schrödinger operator with $L^\infty$ magnetic and electric potentials determines the magnetic field and electric potential inside the set uniquely. The proof is based on a Carleman estimate for the magnetic Schrödinger operator with a gain of two derivatives.

1. Introduction and statement of result

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set, and let $u \in C_0^\infty(\Omega)$. We consider the magnetic Schrödinger operator,

$$L_{A,q}(x,D)u(x) := \sum_{j=1}^n (D_j + A_j(x))^2 u(x) + q(x)u(x)$$

$$= -\Delta u(x) + A(x) \cdot Du(x) + D \cdot (A(x)u(x)) + ((A(x))^2 + q(x))u(x),$$

where $D = i^{-1}\nabla$, $A \in L^\infty(\Omega, \mathbb{C}^n)$ is the magnetic potential, and $q \in L^\infty(\Omega, \mathbb{C})$ is the electric potential. We have $Au \in L^\infty(\Omega, \mathbb{C}^n) \cap \mathcal{E}'(\Omega, \mathbb{C}^n)$, and therefore,

$$L_{A,q} : C_0^\infty(\Omega) \to H^{-1}(\mathbb{R}^n) \cap \mathcal{E}'(\Omega)$$

is a bounded operator. Here $\mathcal{E}'(\Omega) = \{v \in \mathcal{D}'(\Omega) : \text{supp}(v) \text{ is compact}\}$.

Let us now introduce the Cauchy data for an $H^1(\Omega)$ solution $u$ to the equation

$$L_{A,q}u = 0 \quad \text{in} \quad \Omega,$$

in the sense of distributions. First, following [1,14], we define the trace space of the space $H^1(\Omega)$ as the quotient space $H^1(\Omega)/H^1_0(\Omega)$. The associated trace map $T : H^1(\Omega) \to H^1(\Omega)/H^1_0(\Omega)$, $Tu = [u]$, is the quotient map. Here $H^1_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the $H^1(\Omega)$-topology.

Notice that if $\Omega$ has a Lipschitz boundary, then the space $H^1(\Omega)/H^1_0(\Omega)$ can be naturally identified with the Sobolev space $H^{1/2}(\partial \Omega)$. Indeed, in this case the kernel of the continuous surjective map $H^1(\Omega) \to H^{1/2}(\partial \Omega)$, $u \mapsto u|_{\partial \Omega}$ is precisely $H^1_0(\Omega)$, see [12, Theorems 3.37 and 3.40].
For \( u \in H^1(\Omega) \) satisfying (1.1), we can define \( N_{A,q}u \), formally given by 
\[
N_{A,q}u = (\partial_\nu u + i(A \cdot \nu)u)|_{\partial\Omega},
\]
as an element of the dual space \( (H^1(\Omega)/H^1_0(\Omega))' \) as follows. For 
[eq:Cauchy]
\[
[g] \in H^1(\Omega)/H^1_0(\Omega),
\]
we set 
\[
(N_{A,q}u, [g]) = \int_\Omega (\nabla u \cdot \nabla g + iA \cdot (u\nabla g - g\nabla u) + (A^2 + q)ug) \, dx.
\]
(1.2)
As \( u \) is a solution to (1.1), \( N_{A,q}u \) is a well-defined element of \( (H^1(\Omega)/H^1_0(\Omega))' \).

We define the set of the Cauchy data for solutions of the magnetic Schrödinger equation as follows,
\[
C_{A,q} := \{(Tu, N_{A,q}u) : u \in H^1(\Omega) \text{ and } L_{A,q}u = 0 \text{ in } \Omega\}.
\]
The inverse boundary value problem for the magnetic Schrödinger operator \( L_{A,q} \) is to determine \( A \) and \( q \) in \( \Omega \) from the set of the Cauchy data \( C_{A,q} \).

Similarly to [20], there is an obstruction to uniqueness in this problem given by the following gauge equivalence of the set of the Cauchy data: if \( \psi \in W^{1,\infty} \) in a neighborhood of \( \overline{\Omega} \) and \( \psi|_{\partial\Omega} = 0 \), then \( C_{A,q} = C_{A+\nabla\psi,q} \), see Lemma 3.1 below. Hence, the map \( A \mapsto A + \nabla\psi \) transforms the magnetic potential into a gauge equivalent one but preserves the induced magnetic field \( dA \), which is defined by
\[
dA = \sum_{1 \leq j < k \leq n} (\partial_{x_j} A_k - \partial_{x_k} A_j) dx_j \wedge dx_k,
\]
in the sense of distributions. Here \( A = (A_1, \ldots, A_n) \). In view of this and of the fact that the magnetic field is a physically observable quantity, one may hope to recover the magnetic field \( dA \) and the electric potential \( q \) in \( \Omega \) from the set of the Cauchy data \( C_{A,q} \).

As it has been shown by several authors, the knowledge of the set of the Cauchy data \( C_{A,q} \) for the magnetic Schrödinger operator \( L_{A,q} \) does determine the magnetic field \( dA \) and the electric potential \( q \) in \( \Omega \) uniquely, under certain regularity assumptions on \( A \) and \( q \). In [20], this result was established for magnetic potentials in \( W^{2,\infty} \), satisfying a smallness condition, and \( L^\infty \) electric potentials. In [13], the smallness condition was eliminated for smooth magnetic and electric potentials, and for compactly supported \( C^2 \) magnetic potentials and \( L^\infty \) electric potentials. The uniqueness results were subsequently extended to \( C^1 \) magnetic potentials in [22], to some less regular but small potentials in [14], and to Dini continuous magnetic potentials in [17].

The purpose of this paper is to extend the uniqueness result to the case of magnetic Schrödinger operators with magnetic potentials that are of class \( L^\infty \). Our main result is as follows.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n, n \geq 3 \), be a bounded open set, and let \( A_1, A_2 \in L^\infty(\Omega, \mathbb{C}^n) \) and \( q_1, q_2 \in L^\infty(\Omega, \mathbb{C}) \). If \( C_{A_1,q_1} = C_{A_2,q_2} \), then \( dA_1 = dA_2 \) and \( q_1 = q_2 \) in \( \Omega \).
Notice in particular that in Theorem 1.1 no regularity assumptions on the boundary of $\Omega$ are required.

The key ingredient in the proof of Theorem 1.1 is a construction of complex geometric optics solutions for the magnetic Schrödinger operator $L_{A,q}$ with $A \in L^\infty(\Omega, \mathbb{C}^n)$ and $q \in L^\infty(\Omega, \mathbb{C})$. When constructing such solutions, we shall first derive a Carleman estimate for the magnetic Schrödinger operator $L_{A,q}$, with a gain of two derivatives, which is based on the corresponding Carleman estimate for the Laplacian, obtained in [19]. Another crucial observation, which allows us to handle the case of $L^\infty$ magnetic potentials is that it is in fact sufficient to approximate the magnetic potential by a sequence of smooth vector fields, in the $L^2$ sense.

We would also like to mention that another important inverse boundary value problem, for which the issues of regularity have been studied extensively, is Calderón’s problem for the conductivity equation, see [4]. The unique identifiability of $C^2$ conductivities from boundary measurements was established in [21]. The regularity assumptions were relaxed to conductivities having $3/2 + \varepsilon$ derivatives in [2], and the uniqueness for conductivities having exactly $3/2$ derivatives was obtained in [15], see also [3]. In [8], uniqueness for conormal conductivities in $C^{1+\varepsilon}$ was shown. The recent work [9] proves a uniqueness result for Calderón’s problem with conductivities of class $C^1$ and with Lipschitz continuous conductivities, which are close to the identity in a suitable sense.

The paper is organized as follows. Section 2 contains the construction of complex geometric optics solutions for the magnetic Schrödinger operator with $L^\infty$ magnetic and electric potentials. The proof of Theorem 1.1 is then completed in Section 3.

2. Construction of complex geometric optics solutions

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set. Following [5, 11], we shall use the method of Carleman estimates to construct complex geometric optics solutions for the magnetic Schrödinger equation $L_{A,q}u = 0$ in $\Omega$, with $A \in L^\infty(\Omega, \mathbb{C}^n)$ and $q \in L^\infty(\Omega, \mathbb{C})$.

Let us start by recalling the Carleman estimate for the semiclassical Laplace operator $-h^2\Delta$ with a gain of two derivatives, established in [19], see also [11]. Here $h > 0$ is a small semiclassical parameter. Let $\tilde{\Omega}$ be an open set in $\mathbb{R}^n$ such that $\Omega \subset \subset \tilde{\Omega}$ and let $\varphi \in C^\infty(\tilde{\Omega}, \mathbb{R})$. Consider the conjugated operator

$$P_\varphi = e^{\frac{h^2}{2}}(-h^2\Delta)e^{-\frac{h^2}{2}},$$

with the semiclassical principal symbol

$$p_\varphi(x, \xi) = \xi^2 + 2i\nabla \varphi \cdot \xi - |\nabla \varphi|^2, \quad x \in \tilde{\Omega}, \quad \xi \in \mathbb{R}^n.$$
We have for \((x, \xi) \in \overline{\Omega} \times \mathbb{R}^n, |\xi| \geq C \gg 1, \) that \(|p_\varphi(x,\xi)| \sim |\xi|^2\) so that \(P_\varphi\) is elliptic at infinity, in the semiclassical sense. Following \([11]\), we say that \(\varphi\) is a limiting Carleman weight for \(-h^2\Delta\) in \(\tilde{\Omega}\), if \(\nabla \varphi \neq 0\) in \(\Omega\) and the Poisson bracket of \(\text{Re} p_\varphi\) and \(\text{Im} p_\varphi\) satisfies,

\[
\{\text{Re} p_\varphi, \text{Im} p_\varphi\}(x, \xi) = 0 \quad \text{when} \quad p_\varphi(x, \xi) = 0, \quad (x, \xi) \in \tilde{\Omega} \times \mathbb{R}^n.
\]

Examples of limiting Carleman weights are linear weights \(\varphi(x) = \alpha \cdot x, \alpha \in \mathbb{R}^n, |\alpha| = 1,\) and logarithmic weights \(\varphi(x) = \log |x - x_0|, \) with \(x_0 \not\in \tilde{\Omega}.\) In this paper we shall only use the linear weights.

Our starting point is the following result due to \([19]\).

**Proposition 2.1.** Let \(\varphi\) be a limiting Carleman weight for the semiclassical Laplacian on \(\tilde{\Omega},\) and let \(\varphi_\varepsilon = \varphi + \frac{h}{\varepsilon} \varphi^2\). Then for \(0 < h \ll \varepsilon \ll 1\) and \(s \in \mathbb{R},\) we have

\[
\frac{h}{\sqrt{\varepsilon}} \|u\|_{H^{s+2}\text{scl}(\mathbb{R}^n)} \leq C \|e^{\varphi_\varepsilon/h}(-h^2\Delta)e^{-\varphi_\varepsilon/h}u\|_{H^s\text{scl}(\mathbb{R}^n)}, \quad C > 0,
\]

for all \(u \in C^\infty_0(\Omega).\)

Here

\[
\|u\|_{H^s\text{scl}(\mathbb{R}^n)} = \|\langle hD \rangle^su\|_{L^2(\mathbb{R}^n)}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2},
\]

is the natural semiclassical norm in the Sobolev space \(H^s(\mathbb{R}^n),\) \(s \in \mathbb{R}.\)

Next we shall derive a Carleman estimate for the magnetic Schrödinger operator \(L_{A,q}\) with \(A \in L^\infty(\Omega, \mathbb{C}^n)\) and \(q \in L^\infty(\Omega, \mathbb{C}).\) To that end we shall use the estimate \((2.1)\) with \(s = -1,\) and with \(\varepsilon > 0\) being sufficiently small but fixed, i.e. independent of \(h.\) We have the following result.

**Proposition 2.2.** Let \(\varphi \in C^\infty(\tilde{\Omega}, \mathbb{R})\) be a limiting Carleman weight for the semiclassical Laplacian on \(\tilde{\Omega},\) and assume that \(A \in L^\infty(\Omega, \mathbb{C}^n)\) and \(q \in L^\infty(\Omega, \mathbb{C}).\) Then for \(0 < h \ll 1,\) we have

\[
h\|u\|_{H^1\text{scl}(\mathbb{R}^n)} \leq C \|e^{\varphi/h}(h^2L_{A,q})e^{-\varphi/h}u\|_{H^{-1}\text{scl}(\mathbb{R}^n)},
\]

for all \(u \in C^\infty_0(\Omega).\)

**Proof.** In order to prove the estimate \((2.2)\) it will be convenient to use the following characterization of the semiclassical norm in the Sobolev space \(H^{-1}(\mathbb{R}^n),\)

\[
\|v\|_{H^{-1}\text{scl}(\mathbb{R}^n)} = \sup_{0 \neq \psi \in C^\infty_0(\mathbb{R}^n)} \frac{|\langle v, \psi \rangle_{\mathbb{R}^n}|}{\|\psi\|_{H^{1}\text{scl}(\mathbb{R}^n)}},
\]

where \(\langle \cdot, \cdot \rangle_{\mathbb{R}^n}\) is the distribution duality on \(\mathbb{R}^n.\)
Let \( \varphi_\varepsilon = \varphi + \frac{h}{\varepsilon} \varphi^2 \) be the convexified weight with \( \varepsilon > 0 \) such that \( 0 < h \ll \varepsilon \ll 1 \), and let \( u \in C_0^\infty(\Omega) \). Then for all \( 0 \neq \psi \in C_0^\infty(\mathbb{R}^n) \), we have

\[
|\langle e^{\varphi_\varepsilon/h} h^2 A \cdot D(e^{-\varphi_\varepsilon/h} u), \psi \rangle_{\mathbb{R}^n}| \leq \int_{\mathbb{R}^n} |hA \cdot \left(-u \left(1 + \frac{h}{\varepsilon} \varphi \right) D\varphi + hDu\right)\psi| \, dx \leq O(h)\|u\|_{H^1_{\text{loc}}(\mathbb{R}^n)} \|\psi\|_{H^1_{\text{loc}}(\mathbb{R}^n)}.
\]

We also obtain that

\[
|\langle e^{\varphi_\varepsilon/h} h^2 D \cdot (Ae^{-\varphi_\varepsilon/h} u), \psi \rangle_{\mathbb{R}^n}| \leq \int_{\mathbb{R}^n} |h^2 Ae^{-\varphi_\varepsilon/h} u \cdot D(e^{\varphi_\varepsilon/h} \psi)| \, dx \leq O(h)\|u\|_{H^1_{\text{loc}}(\mathbb{R}^n)} \|\psi\|_{H^1_{\text{loc}}(\mathbb{R}^n)}.
\]

Hence, using (2.3), we get

\[
\| e^{\varphi_\varepsilon/h} h^2 A \cdot D(e^{-\varphi_\varepsilon/h} u) + e^{\varphi_\varepsilon/h} h^2 D \cdot (Ae^{-\varphi_\varepsilon/h} u) \|_{H^{-1}_{\text{loc}}(\mathbb{R}^n)} \leq O(h)\|u\|_{H^1_{\text{loc}}(\mathbb{R}^n)}. \tag{2.4}
\]

Notice that the implicit constant in (2.4) only depends on \( \|A\|_{L^\infty(\Omega)}, \|\varphi\|_{L^\infty(\Omega)} \) and \( \|D\varphi\|_{L^\infty(\Omega)} \). Now choosing \( \varepsilon > 0 \) sufficiently small but fixed, i.e. independent of \( h \), we conclude from the estimate (2.1) with \( s = -1 \) and the estimate (2.4) that for all \( h > 0 \) small enough,

\[
\| e^{\varphi_\varepsilon/h} (-h^2 \Delta) e^{-\varphi_\varepsilon/h} u + e^{\varphi_\varepsilon/h} h^2 A \cdot D(e^{-\varphi_\varepsilon/h} u) + e^{\varphi_\varepsilon/h} h^2 D \cdot (Ae^{-\varphi_\varepsilon/h} u) \|_{H^{-1}_{\text{loc}}(\mathbb{R}^n)} \geq \frac{h}{C}\|u\|_{H^1_{\text{loc}}(\mathbb{R}^n)}, \quad C > 0.
\]

Furthermore, the estimate

\[
\| h^2(A^2 + q)u \|_{H^1_{\text{loc}}(\mathbb{R}^n)} \leq O(h^2)\|u\|_{H^1_{\text{loc}}(\mathbb{R}^n)}
\]

and the estimate (2.5) imply that for all \( h > 0 \) small enough,

\[
\| e^{\varphi_\varepsilon/h} (h^2 L_{A,q}) e^{-\varphi_\varepsilon/h} u \|_{H^{-1}_{\text{loc}}(\mathbb{R}^n)} \geq \frac{h}{C}\|u\|_{H^1_{\text{loc}}(\mathbb{R}^n)}, \quad C > 0.
\]

Using that

\[
e^{-\varphi_\varepsilon/h} u = e^{-\varphi_\varepsilon/h} e^{-\varphi^2/(2\varepsilon)} u,
\]

we obtain (2.2). The proof is complete. \( \square \)

Let \( \varphi \in C^\infty(\tilde{\Omega}, \mathbb{R}) \) be a limiting Carleman weight for \(-h^2 \Delta\) and set \( L_\varphi = e^{\varphi/h} (h^2 L_{A,q}) e^{-\varphi/h} \). Then we have

\[
\langle L_\varphi u, \overline{v} \rangle_{\tilde{\Omega}} = \langle u, L_\varphi^* \overline{v} \rangle_{\Omega}, \quad u, v \in C_0^\infty(\Omega),
\]

where \( L_\varphi^* = e^{-\varphi/h} (h^2 L_{A,q}^*) e^{\varphi/h} \) is the formal adjoint of \( L_\varphi \) and \( \langle \cdot, \cdot \rangle_{\Omega} \) is the distribution duality on \( \Omega \). We have

\[
L_\varphi^* : C_0^\infty(\Omega) \to H^{-1}(\mathbb{R}^n) \cap E'(\Omega)
\]
is bounded, and the estimate (2.2) holds for $L^*_\varphi$, since $-\varphi$ is a limiting Carleman weight as well.

To construct complex geometric optics solutions for the magnetic Schrödinger operator we need to convert the Carleman estimate (2.2) for $L^*_\varphi$ into the following solvability result. The proof is essentially well-known, and is included here for the convenience of the reader. We shall write

$$
\|u\|_{H^1_{\text{scl}}(\Omega)}^2 = \|u\|^2_{L^2(\Omega)} + \|hDu\|^2_{L^2(\Omega)} \quad \text{and} \quad \|v\|_{H^{-1}_{\text{scl}}(\Omega)} = \sup_{0 \neq \psi \in C_0^\infty(\Omega)} \frac{|\langle v, \psi \rangle_{\Omega}|}{\|\psi\|_{H^1_{\text{scl}}(\Omega)}}.
$$

**Proposition 2.3.** Let $A \in L^\infty(\Omega, \mathbb{C}^n)$, $q \in L^\infty(\Omega, \mathbb{C})$, and let $\varphi$ be a limiting Carleman weight for the semiclassical Laplacian on $\tilde{\Omega}$. If $h > 0$ is small enough, then for any $v \in H^{-1}(\Omega)$, there is a solution $u \in H^1(\Omega)$ of the equation

$$
e^{\varphi/h}(h^2L^*_A q) e^{-\varphi/h} u = v \quad \text{in} \quad \Omega,
$$

which satisfies

$$
\|u\|_{H^1_{\text{scl}}(\Omega)} \leq \frac{C}{h} \|v\|_{H^{-1}_{\text{scl}}(\Omega)}.
$$

**Proof.** Let $v \in H^{-1}(\Omega)$ and let us consider the following complex linear functional,

$$
L : L^*_\varphi C_0^\infty(\Omega) \to \mathbb{C}, \quad L^*_\varphi w \mapsto \langle w, \overline{v} \rangle_{\Omega}.
$$

By the Carleman estimate (2.2) for $L^*_\varphi$, the map $L$ is well-defined. Let $w \in C_0^\infty(\Omega)$. Then we have

$$
|L(L^*_\varphi w)| = |\langle w, \overline{v} \rangle_{\Omega}| \leq \|w\|_{H^1_{\text{scl}}(\mathbb{R}^n)} \|v\|_{H^{-1}_{\text{scl}}(\Omega)} \leq \frac{C}{h} \|v\|_{H^{-1}_{\text{scl}}(\Omega)} \|L^*_\varphi w\|_{H^{-1}_{\text{scl}}(\mathbb{R}^n)}.
$$

By the Hahn-Banach theorem, we may extend $L$ to a linear continuous functional $\tilde{L}$ on $H^{-1}(\mathbb{R}^n)$, without increasing its norm. By the Riesz representation theorem, there exists $u \in H^1(\mathbb{R}^n)$ such that for all $\psi \in H^{-1}(\mathbb{R}^n)$,

$$
\tilde{L}(\psi) = \langle \psi, \overline{v} \rangle_{\mathbb{R}^n}, \quad \text{and} \quad \|u\|_{H^1_{\text{scl}}(\mathbb{R}^n)} \leq \frac{C}{h} \|v\|_{H^{-1}_{\text{scl}}(\Omega)}.
$$

Let us now show that $L^*_\varphi u = v$ in $\Omega$. To that end, let $w \in C_0^\infty(\Omega)$. Then

$$
\langle L^*_\varphi u, w \rangle_{\Omega} = \langle u, \overline{L^*_\varphi w} \rangle_{\mathbb{R}^n} = \overline{\langle L^*_\varphi w, \overline{v} \rangle_{\Omega}} = \langle v, \overline{w} \rangle_{\Omega} = \langle v, \overline{v} \rangle_{\Omega}.
$$

The proof is complete. \qed

Let $A \in L^\infty(\Omega, \mathbb{C}^n)$. We shall extend $A$ to $\mathbb{R}^n$ by defining it to be zero in $\mathbb{R}^n \setminus \Omega$, and denote this extension by the same letter. Then $A \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n, \mathbb{C}^n) \subset L^p(\mathbb{R}^n, \mathbb{C}^n)$, $1 \leq p \leq \infty$. 


Let $\Psi_\tau(x) = \tau^{-n} \Psi(x/\tau), \; \tau > 0$, be the usual mollifier with $\Psi \in C_0^\infty(\mathbb{R}^n), \; 0 \leq \Psi \leq 1$, and $\int \Psi \, dx = 1$. Then $A^\tau = A * \Psi_\tau \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$ and

$$\|A - A^\tau\|_{L^2(\mathbb{R}^n)} = o(1), \; \tau \to 0. \quad (2.6)$$

A direct computation shows that

$$\|\partial^\alpha A^\tau\|_{L^\infty(\mathbb{R}^n)} = O(\tau^{-|\alpha|}), \; \tau \to 0, \; \text{for all} \; \alpha, \; |\alpha| \geq 0. \quad (2.7)$$

We shall now construct complex geometric optics solutions for the magnetic Schrödinger equation

$$L_{A,q}u = 0 \; \text{in} \; \Omega, \quad (2.8)$$

with $A \in L^\infty(\Omega, \mathbb{C}^n)$ and $q \in L^\infty(\Omega, \mathbb{C})$, using the solvability result of Proposition 2.3 and the approximation (2.6). Complex geometric optics solutions are solutions of the form,

$$u(x, \zeta; h) = e^{x \cdot \zeta/h}(a(x, \zeta; h) + r(x, \zeta; h)), \quad (2.9)$$

where $\zeta \in \mathbb{C}^n, \, \zeta \cdot \zeta = 0, \, |\zeta| \sim 1, \, a$ is a smooth amplitude, $r$ is a correction term, and $h > 0$ is a small parameter.

It will be convenient to introduce the following bounded operator,

$$m_A : H^1(\Omega) \to H^{-1}(\Omega), \; m_A(u) = D \cdot (Au),$$

where the distribution $m_A(u)$ is given by

$$\langle m_A(u), v \rangle_\Omega = -\int_\Omega Au \cdot Dv \, dx, \; v \in C_0^\infty(\Omega).$$

Let us conjugate $h^2L_{A,q}$ by $e^{x \cdot \zeta/h}$. First, let us compute $e^{-x \cdot \zeta/h} \circ h^2m_A \circ e^{x \cdot \zeta/h}$. When $u \in H^1(\Omega)$ and $v \in C_0^\infty(\Omega)$, we get

$$\langle e^{-x \cdot \zeta/h} h^2m_A(e^{x \cdot \zeta/h}u), v \rangle_\Omega = -\int_\Omega h^2 A e^{x \cdot \zeta/h}u \cdot D(e^{-x \cdot \zeta/h}v) \, dx$$

$$= -\int_\Omega (h\zeta \cdot Au + h^2 Au \cdot Dv) \, dx,$$

and therefore,

$$e^{-x \cdot \zeta/h} \circ h^2m_A \circ e^{x \cdot \zeta/h} = -h\zeta \cdot A + h^2m_A.$$

Furthermore, we obtain that

$$e^{-x \cdot \zeta/h} \circ (-h^2 \Delta) \circ e^{x \cdot \zeta/h} = -h^2 \Delta - 2ih\zeta \cdot D,$$

$$e^{-x \cdot \zeta/h} \circ h^2(A \cdot D) \circ e^{x \cdot \zeta/h} = h^2A \cdot D - hi\zeta \cdot A.$$

Hence, we have

$$e^{-x \cdot \zeta/h} \circ h^2L_{A,q} \circ e^{x \cdot \zeta/h} = -h^2 \Delta - 2ih\zeta \cdot D + h^2 A \cdot D - 2hi\zeta \cdot A + h^2m_A + h^2(A^2 + q). \quad (2.10)$$
We shall consider \( \zeta \) depending slightly on \( h \), i.e. \( \zeta = \zeta_0 + \zeta_1 \) with \( \zeta_0 \) being independent of \( h \) and \( \zeta_1 = \mathcal{O}(h) \) as \( h \to 0 \). We also assume that \( |\text{Re} \zeta_0| = |\text{Im} \zeta_0| = 1 \). Then we write (2.10) as follows,
\[
e^{-x \cdot \zeta/h} \circ h^2 L_{A,q} \circ e^{x \cdot \zeta/h} = -h^2 \Delta - 2ih \zeta_0 \cdot D - 2ih \zeta_1 \cdot D + h^2 A \cdot D - 2hi \zeta_0 \cdot A^2 - 2hi \zeta_0 \cdot (A - A^2) - 2hi \zeta_1 \cdot A + h^2 m_A + h^2 (A^2 + q).
\]

In order that (2.9) be a solution of (2.8), we require that
\[
\zeta_0 \cdot Da + \zeta_0 \cdot A^2 a = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{2.11}
\]
and
\[
e^{-x \cdot \zeta/h} h^2 L_{A,q} e^{x \cdot \zeta/h} = -(-h^2 \Delta a + h^2 A \cdot Da + h^2 m_A(a) + h^2 (A^2 + q) a)
+ 2ih \zeta_1 \cdot Da + 2hi \zeta_0 \cdot (A - A^2) a + 2hi \zeta_1 \cdot A a =: g \quad \text{in} \quad \Omega. \tag{2.12}
\]

The equation (2.11) is the first transport equation and one looks for its solution in the form \( a = e^{\Phi^d} \), where \( \Phi^d \) solves the equation
\[
\zeta_0 \cdot \nabla \Phi^d + i \zeta_0 \cdot A^2 = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{2.13}
\]

As \( \zeta_0 \cdot \zeta_0 = 0 \) and \( |\text{Re} \zeta_0| = |\text{Im} \zeta_0| = 1 \), the operator \( N\zeta_0 := \zeta_0 \cdot \nabla \) is the \( \bar{\partial} \)–operator in suitable linear coordinates. Let us introduce an inverse operator defined by
\[
(N_{\zeta_0}^{-1} f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{f(x - y_1 \text{Re} \zeta_0 - y_2 \text{Im} \zeta_0)}{y_1 + iy_2} dy_1 dy_2, \quad f \in C_0(\mathbb{R}^n).
\]

We have the following result, see [17, Lemma 4.6].

**Lemma 2.4.** Let \( f \in W^{k,\infty}(\mathbb{R}^n) \), \( k \geq 0 \), with \( \text{supp} \,(f) \subset B(0,R) \). Then \( \Phi = N_{\zeta_0}^{-1} f \in W^{k,\infty}(\mathbb{R}^n) \) satisfies \( N_{\zeta_0} \Phi = f \) in \( \mathbb{R}^n \), and we have
\[
\| \Phi \|_{W^{k,\infty}(\mathbb{R}^n)} \leq C \| f \|_{W^{k,\infty}(\mathbb{R}^n)}, \tag{2.14}
\]
where \( C = C(R) \). If \( f \in C_0(\mathbb{R}^n) \), then \( \Phi \in C(\mathbb{R}^n) \).

Thanks to Lemma 2.4, the function \( \Phi^d(x,\zeta_0;\tau) := N_{\zeta_0}^{-1}(-i\zeta_0 \cdot A^2) \in C^\infty(\mathbb{R}^n) \) satisfies the equation (2.13). Furthermore, the estimates (2.7) and (2.14) imply that
\[
\| \partial^\alpha \Phi^d \|_{L^\infty(\mathbb{R}^n)} \leq C_{\alpha} \tau^{-|\alpha|}, \quad \text{for all} \quad \alpha, \quad |\alpha| \geq 0. \tag{2.15}
\]

Owing to [21, Lemma 3.1], we have the following result, where we use the norms
\[
\| f \|_{L^2_\delta(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (1 + |x|^2)^\delta |f(x)|^2 dx.
\]

**Lemma 2.5.** Let \( -1 < \delta < 0 \) and let \( f \in L^2_{\delta+1}(\mathbb{R}^n) \). Then there exists a constant \( C > 0 \), independent of \( \zeta_0 \), such that
\[
\| N_{\zeta_0}^{-1} f \|_{L^2_\delta(\mathbb{R}^n)} \leq C \| f \|_{L^2_{\delta+1}(\mathbb{R}^n)}.
\]
Setting $\Phi(\cdot, \zeta_0) := N_{\zeta_0}^{-1}(-i\zeta_0 \cdot A) \in L^\infty(\mathbb{R}^n)$, it follows from Lemma 2.3 and the estimate (2.6) that $\Phi^\tau(\cdot, \zeta_0; \tau)$ converges to $\Phi(\cdot, \zeta_0)$ in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $\tau \to 0$.

Let us turn now to the equation (2.12). First notice that the right hand side $g$ of (2.12) belongs to $H^{-1}(\Omega)$ and we would like to estimate $\|g\|_{H^{-1}_{\text{sc}}(\Omega)}$. To that end, let $0 \neq \psi \in C^\infty_0(\Omega)$. Then using (2.15) and the fact that $\zeta_1 = O(h)$, we get by the Cauchy–Schwarz inequality,

$$\begin{align*}
\|\langle h^2 \Delta a, \psi \rangle_\Omega \| &\leq O(h^2/\tau^2)\|\psi\|_{L^2(\Omega)} \leq O(h^2/\tau^2)\|\psi\|_{H^1_{\text{sc}}(\Omega)}, \\
\|\langle h^2 A \cdot Da, \psi \rangle_\Omega \| &\leq O(h^2/\tau)\|\psi\|_{H^1_{\text{sc}}(\Omega)}, \\
|\langle 2ih\zeta_1 \cdot Da, \psi \rangle_\Omega | &\leq O(h^2/\tau)|\psi|_{H^1_{\text{sc}}(\Omega)}, \\
|\langle 2hi\zeta_1 \cdot Aa, \psi \rangle_\Omega | &\leq O(h^2)|\psi|_{H^1_{\text{sc}}(\Omega)}.
\end{align*}$$

Using (2.6) and (2.15), we have

$$\begin{align*}
|\langle 2hi\zeta_0 \cdot (A - A^2)a, \psi \rangle_\Omega | &\leq O(h)\|a\|_{L^\infty(\mathbb{R}^n)}\|A - A^2\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} \\
&\leq O(h)\|a\|_{L^\infty(\mathbb{R}^n)}\|A - A^2\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}.
\end{align*}$$

With the help of (2.6), (2.7), and (2.15), we obtain that

$$\begin{align*}
|\langle h^2 m_A(a), \psi \rangle_\Omega | &\leq \int_\Omega h^2 A^2 \cdot D\psi dx + \int_\Omega h^2 (A - A^2) \cdot D\psi dx \\
&\leq \int_\Omega h^2 (D \cdot (A^2 a)) \psi dx + O(h)\|A - A^2\|_{L^2(\Omega)}\|hD\psi\|_{L^2(\Omega)} \\
&\leq (O(h^2/\tau) + O(h)\|a\|_{L^\infty(\mathbb{R}^n)}\|A - A^2\|_{L^2(\Omega)})\|\psi\|_{H^1_{\text{sc}}(\Omega)}.
\end{align*}$$

We also have $\|h^2(A^2 + q)a\|_{L^2(\Omega)} \leq O(h^2)$. Thus, from the above estimates, we conclude that

$$\|g\|_{H^{-1}_{\text{sc}}(\Omega)} \leq O(h^2/\tau^2) + O(h)\|a\|_{L^\infty(\mathbb{R}^n)}\|A - A^2\|_{L^2(\Omega)}.$$

Choosing now $\tau = h^\sigma$ with some $0 < \sigma < 1/2$, we get

$$\|g\|_{H^{-1}_{\text{sc}}(\Omega)} = o(h) \quad \text{as} \quad h \to 0. \quad (2.16)$$

Thanks to Proposition 2.3 and (2.16), for $h > 0$ small enough, there exists a solution $r \in H^{1}(\Omega)$ of (2.12) such that $\|r\|_{H^{1}_{\text{sc}}(\Omega)} = o(1)$ as $h \to 0$.

The discussion led in this section can be summarized in the following proposition.

**Proposition 2.6.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set. Let $A \in L^\infty(\Omega, \mathbb{C}^n)$, $q \in L^\infty(\Omega, \mathbb{C})$, and let $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$, $\zeta = \zeta_0 + \zeta_1$ with $\zeta_0$ being independent of $h > 0$, $|\Re \zeta_0| = |\Im \zeta_0| = 1$, and $\zeta_1 = O(h)$ as $h \to 0$. Then for all $h > 0$ small enough, there exists a solution $u(x, \zeta; h) \in H^{1}(\Omega)$ to the magnetic Schrödinger equation $L_{A,q}u = 0$ in $\Omega$, of the form

$$u(x, \zeta; h) = e^{x \cdot \zeta/h}(e^{\Phi^\tau(x, \zeta_0; h)} + r(x, \zeta; h)).$$
The function $\Phi^\pm(\cdot, \zeta_0; h) \in C^\infty(\mathbb{R}^n)$ satisfies $\|\partial^\alpha \Phi^\pm\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha h^{-\sigma|\alpha|}$, $0 < \sigma < 1/2$, for all $\alpha$, $|\alpha| \geq 0$, and $\Phi^\pm(\cdot, \zeta_0; h)$ converges to $\Phi(\cdot, \zeta_0) := N^{-1}_0(-i\zeta_0 \cdot A) \in L^\infty(\mathbb{R}^n)$ in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $h \to 0$. Here we have extended $A$ by zero to $\mathbb{R}^n \setminus \Omega$. The remainder $r$ is such that $\|r\|_{H^{-1}_c(\Omega)} = o(1)$ as $h \to 0$.

3. Proof of Theorem 1.1

Let us begin by recalling the following auxiliary, essentially well-known, result which shows that the set of the Cauchy data for the magnetic Schrödinger operator remains unchanged if the gradient of a function, vanishing along the boundary, is added to the magnetic potential, see [17, Lemma 4.1], [20].

Lemma 3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $A \in L^\infty(\Omega, \mathbb{C}^n)$, $q \in L^\infty(\Omega, \mathbb{C})$, and let $\psi \in W^{1,\infty}$ in a neighborhood of $\overline{\Omega}$. Then we have

$$e^{-i\psi} \circ L_{A, q} \circ e^{i\psi} = L_{A + \nabla \psi, q}. \quad (3.1)$$

If furthermore, $\psi|_{\partial \Omega} = 0$ then

$$C_{A, q} = C_{A + \nabla \psi, q}. \quad (3.2)$$

Proof. Let us notice first that the assumption that $\psi \in W^{1,\infty}$ in a neighborhood of $\overline{\Omega}$ implies that $\psi$ is Lipschitz continuous on $\overline{\Omega}$, so that $\psi|_{\partial \Omega}$ is well-defined pointwise.

Since (3.1) follows by a direct computation, only (3.2) has to be established. To that end, let $u \in H^1(\Omega)$ be a solution to $L_{A, q} u = 0$ in $\Omega$. Then $e^{-i\psi} u \in H^1(\Omega)$ satisfies $L_{A + \nabla \psi, q}(e^{-i\psi} u) = 0$ in $\Omega$. Let us show that $T(e^{-i\psi} u) = Tu$. In other words, we have to check that

$$u(e^{-i\psi} - 1) \in H^1_0(\Omega). \quad (3.3)$$

Since the function $e^{-i\psi} - 1$ is Lipschitz continuous on $\overline{\Omega}$ and vanishes along $\partial \Omega$, we have $|e^{-i\psi(x)} - 1| \leq C d(x)$ for any $x \in \Omega$ and some constant $C > 0$. Here $d(x)$ is the distance from $x$ to the boundary of $\Omega$. Then (3.3) follows from the following fact: if $v \in H^1(\Omega)$ and $v/d \in L^2(\Omega)$, then $v \in H^1_0(\Omega)$, see [6, Theorem 3.4, p. 223].

Let us now show that $N_{A + \nabla \psi, q}(e^{-i\psi} u) = N_{A, q} u$. To that end, first as above, one observes that for $g \in H^1(\Omega)$, we have $[g] = [e^{i\psi} g]$. Thus,

$$(N_{A + \nabla \psi, q}(e^{-i\psi} u), [g])_\Omega = (N_{A + \nabla \psi, q}(e^{-i\psi} u), [e^{i\psi} g])_\Omega = (N_{A, q}(u), [g])_\Omega,$$

for any $[g] \in H^1(\Omega)/H^1_0(\Omega)$, and therefore, $C_{A, q} \subset C_{A + \nabla \psi, q}$. The proof is complete.

The first step in the proof of Theorem 1.1 is the derivation of the following integral identity based on the fact that $C_{A_1, q_1} = C_{A_2, q_2}$, see also [17, Lemma 4.3].
Proposition 3.2. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set. Assume that $A_1, A_2 \in L^\infty(\Omega, \mathbb{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega, \mathbb{C})$. If $C_{A_1,q_1} = C_{A_2,q_2}$, then the following integral identity

$$\int_\Omega i(A_1 - A_2) \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) dx + \int_\Omega (A_1^2 - A_2^2 + q_1 - q_2) u_1 u_2 dx = 0 \quad (3.4)$$

holds for any $u_1, u_2 \in H^1(\Omega)$ satisfying $L_{A_1,q_1} u_1 = 0$ in $\Omega$ and $L_{A_2,q_2}^* u_2 = 0$ in $\Omega$, respectively.

Proof. Let $u_1, u_2 \in H^1(\Omega)$ be solutions to $L_{A_1,q_1} u_1 = 0$ in $\Omega$ and $L_{A_2,q_2}^* u_2 = 0$ in $\Omega$, respectively. Then the fact that $C_{A_1,q_1} = C_{A_2,q_2}$ implies that there is $v_2 \in H^1(\Omega)$ satisfying $L_{A_2,q_2} v_2 = 0$ in $\Omega$ such that

$$Tu_1 = Tv_2 \quad \text{and} \quad N_{A_1,q_1} u_1 = N_{A_2,q_2} v_2.$$

This together with \cite{12} shows that

$$\left( N_{A_1,q_1} u_1, \left[ v_2 \right] \right)_\Omega = \left( N_{A_2,q_2} v_2, \left[ u_1 \right] \right)_\Omega = \left( N_{A_2,q_2}^* u_2, \left[ v_2 \right] \right)_\Omega = \left( N_{A_2,q_2}^* u_2, \left[ u_1 \right] \right)_\Omega.$$

Then the integral identity (3.4) follows from the definition (1.2) of $N_{A_1,q_1} u_1$ and $N_{A_2,q_2}^* u_2$. The proof is complete. \qed

We shall use the integral identity (3.4) with $u_1$ and $u_2$ being complex geometric optics solutions for the magnetic Schrödinger equations in $\Omega$. To construct such solutions, let $\xi, \mu_1, \mu_2 \in \mathbb{R}^n$ be such that $|\mu_1| = |\mu_2| = 1$ and $\mu_1 \cdot \mu_2 = \mu_1 \cdot \xi = \mu_2 \cdot \xi = 0$. Similarly to \cite{20}, we set

$$\zeta_1 = \frac{i h \xi}{2} + \mu_1 + i \sqrt{1 - h^2 |\xi|^2 / 4} \mu_2, \quad \zeta_2 = -\frac{i h \xi}{2} - \mu_1 + i \sqrt{1 - h^2 |\xi|^2 / 4} \mu_2, \quad (3.5)$$

so that $\zeta_j \cdot \xi = 0$, $j = 1, 2$, and $(\zeta_1 + \zeta_2)/h = i \xi$. Here $h > 0$ is a small enough semiclassical parameter. Moreover, $\zeta_1 = \mu_1 + i \mu_2 + O(h)$ and $\zeta_2 = -\mu_1 + i \mu_2 + O(h)$ as $h \to 0$.

By Proposition 2.6, for all $h > 0$ small enough, there exists a solution $u_1(x, \zeta_1; h) \in H^1(\Omega)$ to the magnetic Schrödinger equation $L_{A_1,q_1} u_1 = 0$ in $\Omega$, of the form

$$u_1(x, \zeta_1; h) = e^{-\zeta_1/h} (e^{\Phi_1^2(x, \mu_1 + i \mu_2; h)} + r_1(x, \zeta_1; h)), \quad (3.6)$$

where $\Phi_1^2(\cdot, \mu_1 + i \mu_2; h) \in C^\infty(\mathbb{R}^n)$ satisfies the estimate

$$\|\partial^\alpha \Phi_1^2\|_{L^\infty(\mathbb{R}^n)} \leq C_{\alpha} h^{-|\alpha|/2}, \quad 0 < \sigma < 1/2, \quad (3.7)$$

for all $\alpha$, $|\alpha| \geq 0$. $\Phi_1^2(\cdot, \mu_1 + i \mu_2; h)$ converges to

$$\Phi_1(\cdot, \mu_1 + i \mu_2) := N_{\mu_1 + i \mu_2}^{-1} (-i(\mu_1 + i \mu_2) \cdot A_1) \in L^\infty(\mathbb{R}^n), \quad (3.8)$$

in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $h \to 0$, and

$$\|r_1\|_{H^1_{\text{ac}}(\Omega)} = o(1) \quad \text{as} \quad h \to 0. \quad (3.9)$$
Similarly, for all $h > 0$ small enough, there exists a solution $u_2(x, \zeta; h) \in H^1(\Omega)$ to the magnetic Schrödinger equation $L_{\overline{A}_2, \overline{\varphi}_2} u_2 = 0$ in $\Omega$, of the form

$$u_2(x, \zeta; h) = e^{x \cdot \zeta z/h} (e^{\phi_2^+(x, -\mu_1 + i \mu_2; h)} + r_2(x, \zeta; h)), \quad (3.10)$$

where $\phi_2^+(\cdot, -\mu_1 + i \mu_2; h) \in C^\infty(\mathbb{R}^n)$ satisfies the estimate

$$||\partial^\alpha \phi_2^+||_{L^\infty(\mathbb{R}^n)} \leq C_\alpha h^{-\sigma|\alpha|}, \quad 0 < \sigma < 1/2, \quad (3.11)$$

for all $\alpha$, $|\alpha| \geq 0$. Furthermore, $\phi_2^+(\cdot, -\mu_1 + i \mu_2; h)$ converges to

$$\phi_2(\cdot, -\mu_1 + i \mu_2) := N^{-1}_{\mu_1 + i \mu_2} (-i(-\mu_1 + i \mu_2) \cdot \overline{A}_2) \in L^\infty(\mathbb{R}^n) \quad (3.12)$$

in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $h \to 0$, and

$$||r_2||_{H^1_{\text{sc}}(\Omega)} = o(1) \quad \text{as} \quad h \to 0. \quad (3.13)$$

We shall next substitute $u_1$ and $u_2$, given by (3.6) and (3.10), into the integral identity (3.4), multiply it by $u$, and let $h \to 0$. We first compute

$$hu_1 \nabla u_2 = \zeta e^{ix \xi} (e^{\phi_1^+ + \phi_2} + e^{\phi_2^+} + r_1 e^{\phi_2^+} + r_1 \overline{r}_2) + h e^{ix \xi} (e^{\phi_1^+ \nabla e^{\phi_2^+} + e^{\phi_2^+ \nabla r_2} + r_1 \nabla e^{\phi_2^+} + r_1 \nabla r_2}).$$

Recall that $\zeta_2 = -\mu_1 - i \mu_2 + \mathcal{O}(h)$. We shall show that

$$(\mu_1 + i \mu_2) \cdot \int_{\Omega} (A_1 - A_2) e^{ix \xi} e^{\phi_1^+ + \phi_2} dx \to (\mu_1 + i \mu_2) \cdot \int_{\Omega} (A_1 - A_2) e^{ix \xi} e^{\phi_1^+ + \overline{\phi}_2} dx,$$

as $h \to 0$, where $\Phi_1$ and $\Phi_2$ are defined by (3.8) and (3.12), respectively. To that end, we have

$$\left| (\mu_1 + i \mu_2) \cdot \int_{\Omega} (A_1 - A_2) e^{ix \xi} (e^{\phi_1^+ + \phi_2} - e^{\phi_1^+ + \overline{\phi}_2}) dx \right| \leq C \left| e^{\phi_1^+ + \phi_2} - e^{\phi_1^+ + \overline{\phi}_2} \right|_{L^2(\Omega)} \leq C \left| \phi_1^+ + \phi_2 - \phi_1 - \overline{\phi}_2 \right|_{L^2(\Omega)} \to 0,$$

as $h \to 0$. Here we have used the inequality

$$|e^z - e^w| \leq |z - w| e^{\max(\text{Re } z, \text{Re } w)}, \quad z, w \in \mathbb{C}, \quad (3.14)$$

obtained by integration of $e^z$ from $z$ to $w$, and the fact that $\Phi_j, \overline{\Phi}_j \in L^\infty(\mathbb{R}^n)$, $j = 1, 2$, and $||\phi_2^+||_{L^\infty(\mathbb{R}^n)} \leq C$ uniformly in $h$.

Now using the estimates (3.7), (3.9), (3.11) and (3.13), we get

$$\left| \int_{\Omega} i(A_1 - A_2) \cdot \zeta e^{ix \xi} (e^{\phi_2^+} + r_1 e^{\phi_2^+} + r_1 \overline{r}_2) dx \right| \leq C ||A_1 - A_2||_{L^\infty} \left( ||e^{\phi_2^+}||_{L^2} ||\overline{\phi}_2||_{L^2} + ||r_1||_{L^2} ||e^{\phi_2^+}||_{L^2} + ||r_1||_{L^2} ||\overline{\phi}_2||_{L^2} \right) = o(1),$$

where $\Phi_j, \overline{\Phi}_j \in L^\infty(\mathbb{R}^n)$, $j = 1, 2$, and $||\phi_2^+||_{L^\infty(\mathbb{R}^n)} \leq C$ uniformly in $h$. 


as \( h \to 0 \). We also obtain that
\[
\left| \int_{\Omega} h i (A_1 - A_2) \cdot e^{i x \cdot \xi} (e^{\Phi_1} \nabla e^{\bar{\Phi}_2} + e^{\Phi_2} \nabla \bar{\Phi}_1 + r_1 \nabla e^{\bar{\Phi}_2} + r_1 \nabla \bar{\Phi}_1) dx \right|
\leq O(h)(h^{-\sigma} + h^{-1}o(1) + o(1)h^{-\sigma} + o(1)h^{-1}) = o(1),
\]
as \( h \to 0 \). Here \( 0 < \sigma < 1/2 \). Furthermore,
\[
\left| h \int_{\Omega} (A_1^2 - A_2^2 + q_1 - q_2) e^{i x \cdot \xi} (e^{\Phi_1} + \bar{\Phi}_2) + e^{\Phi_2} + r_1 e^{\bar{\Phi}_2} + r_1 \bar{\Phi}_1) dx \right| = O(h),
\]
as \( h \to 0 \). Hence, substituting \( u_1 \) and \( u_2 \), given by (3.6) and (3.10), into the integral identity (3.14), multiplying it by \( h \), and letting \( h \to 0 \), we get
\[
(\mu_1 + i \mu_2) \cdot \int_{\mathbb{R}^n} (A_1 - A_2) e^{i x \cdot \xi} e^{\Phi_1(x, \mu_1 + i \mu_2) + \bar{\Phi}_2(x, -\mu_1 + i \mu_2)} dx = 0, \tag{3.15}
\]
where
\[
\begin{align*}
\Phi_1 &= N_{\mu_1 + i \mu_2}^{-1} (-i(\mu_1 + i \mu_2) \cdot A_1) \in L^\infty(\mathbb{R}^n), \\
\Phi_2 &= N_{\mu_1 + i \mu_2}^{-1} (-i(\mu_1 + i \mu_2) \cdot A_2) \in L^\infty(\mathbb{R}^n).
\end{align*}
\]
Notice that the integration in (3.15) is extended to all of \( \mathbb{R}^n \), since \( A_1 = A_2 = 0 \) on \( \mathbb{R}^n \setminus \Omega \).

The next step is to remove the function \( e^{\Phi_1 + \bar{\Phi}_2} \) in the integral (3.15). First using the following properties of the Cauchy transform,
\[
N_{\zeta}^{-1} f = N_{-\zeta}^{-1} f, \quad N_{-\zeta}^{-1} f = -N_{\zeta}^{-1} f,
\]
we see that
\[
\Phi_1 + \bar{\Phi}_2 = N_{\mu_1 + i \mu_2}^{-1} (-i(\mu_1 + i \mu_2) \cdot (A_1 - A_2)). \tag{3.16}
\]
We have the following result.

**Proposition 3.3.** Let \( \xi, \mu_1, \mu_2 \in \mathbb{R}^n \), \( n \geq 3 \), be such that \( |\mu_1| = |\mu_2| = 1 \) and \( \mu_1 \cdot \mu_2 = \mu_1 \cdot \xi = \mu_2 \cdot \xi = 0 \). Let \( W \in (L^\infty \cap E')(\mathbb{R}^n, \mathbb{C}^n) \) and \( \phi = N_{\mu_1 + i \mu_2}^{-1} (-i(\mu_1 + i \mu_2) \cdot W) \). Then
\[
(\mu_1 + i \mu_2) \cdot \int_{\mathbb{R}^n} W(x) e^{i x \cdot \xi} e^{\phi(x)} dx = (\mu_1 + i \mu_2) \cdot \int_{\mathbb{R}^n} W(x) e^{i x \cdot \xi} dx. \tag{3.17}
\]

**Proof.** The statement of the proposition for \( W \in C_0(\mathbb{R}^n, \mathbb{C}^n) \) is due to [7], with similar ideas appearing in [20]. See also [18, Lemma 6.2]. For the completeness and convenience of the reader, we shall give a complete proof of the proposition here.

Assume first that \( W \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n) \). Then by Lemma 2.4 we have
\[
\phi = N_{\mu_1 + i \mu_2}^{-1} (-i(\mu_1 + i \mu_2) \cdot W) \in C^\infty(\mathbb{R}^n). \tag{3.18}
\]
We can always assume that \( \mu_1 = (1, 0, \ldots, 0) \) and \( \mu_2 = (0, 1, 0, \ldots, 0) \), so that \( \xi = (0, 0, \xi'') \in \mathbb{R}^{n-2} \), and therefore,
\[
(\partial x_1 + i \partial x_2) \phi = -i(\mu_1 + i \mu_2) \cdot W \quad \text{in} \quad \mathbb{R}^n.
\]
Hence, writing \( x = (x', x'') \), \( x' = (x_1, x_2) \), \( x'' \in \mathbb{R}^{n-2} \), we get
\[
(\mu_1 + i \mu_2) \cdot \int_{\mathbb{R}^n} W(x) e^{ix \cdot \xi} e^{\phi(x)} dx = i \int_{\mathbb{R}^n} e^{ix'' \cdot \xi''} e^{\phi(x)} (\partial x_1 + i \partial x_2) \phi(x) dx
\]
\[
= i \int_{\mathbb{R}^{n-2}} e^{ix'' \cdot \xi''} h(x'') dx'',
\]
where
\[
h(x'') = \int_{\mathbb{R}^2} (\partial x_1 + i \partial x_2) e^{\phi(x)} dx' = \lim_{R \to \infty} \int_{|x'| \leq R} (\partial x_1 + i \partial x_2) e^{\phi(x)} dx' = \lim_{R \to \infty} \int_{|x'| = R} e^{\phi(x)} (\nu_1 + i \nu_2) dS_R(x').
\]
Here \( \nu = (\nu_1, \nu_2) \) is the unit outer normal to the circle \( |x'| = R \), and we have used the Gauss theorem.

It follows from (3.18) that \( |\phi(x', x'')| = \mathcal{O}(1/|x'|) \) as \( |x'| \to \infty \). Hence, we have
\[
e^{\phi} = 1 + \phi + \mathcal{O}(|\phi|^2) = 1 + \phi + \mathcal{O}(|x'|^{-2}) \quad \text{as} \quad |x'| \to \infty.
\]
Since
\[
\int_{|x'| = R} (\nu_1 + i \nu_2) dS_R(x') = \int_{|x'| \leq R} (\partial x_1 + i \partial x_2)(1) dx' = 0,
\]
we obtain that
\[
h(x'') = \lim_{R \to \infty} \int_{|x'| = R} \phi(x) (\nu_1 + i \nu_2) dS_R(x') = \lim_{R \to \infty} \int_{|x'| \leq R} (\partial x_1 + i \partial x_2) \phi(x) dx'
\]
\[
= - \int_{\mathbb{R}^2} i(\mu_1 + i \mu_2) \cdot W(x) dx',
\]
which shows (3.17) for \( W \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n) \).

To prove (3.17) for \( W \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n, \mathbb{C}^n) \), consider the regularizations \( W_j = \chi_j * W \in C_0^\infty(\mathbb{R}^n) \). Here \( \chi_j(x) = j^n \chi(jx) \) is the usual mollifier with \( 0 \leq \chi \in C_0^\infty(\mathbb{R}^n) \) such that \( \int \chi dx = 1 \). Then \( W_j \to W \) in \( L^2(\mathbb{R}^n) \) as \( j \to \infty \) and
\[
\|W_j\|_{L^\infty(\mathbb{R}^n)} \leq \|W\|_{L^\infty(\mathbb{R}^n)} \|\chi_j\|_{L^1(\mathbb{R}^n)} = \|W\|_{L^\infty(\mathbb{R}^n)}, \quad j = 1, 2, \ldots
\]
Furthermore, there is a compact set \( K \subset \mathbb{R}^n \) such that \( \text{supp} (W_j), \text{supp} (W) \subset K, \ j = 1, 2, \ldots \)
We set \( \phi_j = N_{\mu + i\mu_2}^{-1}(-i(\mu_1 + i\mu_2) \cdot W_j) \in C^\infty(\mathbb{R}^n) \). Then by Lemma 2.5, we know that \( \phi_j \to \phi \) in \( L^2_{\text{loc}}(\mathbb{R}^n) \) as \( j \to \infty \). Lemma 2.4 together with the estimate (3.19) implies that
\[
\|\phi_j\|_{L^\infty(\mathbb{R}^n)} \leq C\|W_j\|_{L^\infty(\mathbb{R}^n)} \leq C\|W\|_{L^\infty(\mathbb{R}^n)}, \quad j = 1, 2, \ldots .
\] (3.20)

For \( j = 1, 2, \ldots \), we have
\[
(\mu_1 + i\mu_2) \cdot \int_K W_j(x)e^{ix\cdot\xi}e^{\phi_j(x)}dx = (\mu_1 + i\mu_2) \cdot \int_K W_j(x)e^{ix\cdot\xi}dx.
\] (3.21)

The fact that the integral in right hand side of (3.21) converges to the integral in the right hand side of (3.17) as \( j \to \infty \) follows from the estimate
\[
\left| (\mu_1 + i\mu_2) \cdot \int_K (W_j(x) - W(x))e^{ix\cdot\xi}dx \right| \leq C\|W_j - W\|_{L^2(K)} \to 0, \quad j \to \infty.
\]

In order to show that the integral in the left hand side of (3.21) converges to the integral in the left hand side of (3.17) as \( j \to \infty \), we establish that \( I_1 + I_2 \to 0 \) as \( j \to \infty \), where
\[
I_1 := (\mu_1 + i\mu_2) \cdot \int_K (W_j(x) - W(x))e^{ix\cdot\xi}e^{\phi_j(x)}dx,
\]
\[
I_2 := (\mu_1 + i\mu_2) \cdot \int_K W(x)e^{ix\cdot\xi}(e^{\phi_j(x)} - e^{\phi(x)})dx.
\]

Using (3.20), we have
\[
|I_1| \leq Ce^{\|\phi_j\|_{L^\infty(\mathbb{R}^n)}}\int_K |W_j(x) - W(x)|dx \leq C\|W_j - W\|_{L^2(K)} \to 0, \quad j \to \infty.
\]

Using (3.14) and (3.20), we get
\[
|I_2| \leq C\|W\|_{L^\infty(\mathbb{R}^n)}\|e^{\phi_j(x)} - e^{\phi(x)}\|_{L^2(K)} \leq C\|\phi_j - \phi\|_{L^2(K)} \to 0, \quad j \to \infty.
\]

Here we have also used that \( \phi_j \to \phi \) in \( L^2_{\text{loc}}(\mathbb{R}^n) \) as \( j \to \infty \). Hence, passing to the limit as \( j \to \infty \) in (3.21), we obtain the identity (3.17). The proof is complete.

By Proposition 3.3 we conclude from (3.15) and (3.16) that
\[
(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} (A_1(x) - A_2(x))e^{ix\cdot\xi}dx = 0.
\] (3.22)

It follows from (3.22) that \( \mu \cdot (\hat{A}_1(\xi) - \hat{A}_2(\xi)) = 0 \) whenever \( \mu, \xi \in \mathbb{R}^n \) are such that \( \mu \cdot \xi = 0 \). Here \( \hat{A}_j \) is the Fourier transform of \( A_j, \ j = 1, 2 \). Let \( \mu_{jk}(\xi) = \xi_j e_k - \xi_k e_j \) for \( j \neq k \), where \( e_1, \ldots, e_n \) is the standard basis of \( \mathbb{R}^n \). Then \( \mu_{jk}(\xi) \cdot \xi = 0 \), and therefore,
\[
\xi_j(\hat{A}_{1,k}(\xi) - \hat{A}_{2,k}(\xi)) - \xi_k(\hat{A}_{1,j}(\xi) - \hat{A}_{2,j}(\xi)) = 0.
\]
Proposition 3.4. which is due to [17, Lemma 4.2].

$$\nabla$$

We want to add $$\supp (\psi)$$ near infinity, we have from [10, Theorem 4.5.11] that $$\in \mathcal{D}$$ such that $$\partial (A_1 - A_2) = 0$$ in $$\mathbb{R}^n$$.

Our next goal is to show that $$q_1 = q_2$$ in $$\Omega$$. First, viewing $$A_1 - A_2$$ as a 1–current and using the Poincaré lemma for currents, we conclude that there is $$\psi \in \mathcal{D}'(\mathbb{R}^n)$$ such that $$d\psi = A_1 - A_2 \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n)$$ in $$\mathbb{R}^n$$, see [10]. It follows from [10, Theorem 4.5.11] that $$\psi$$ is continuous on $$\mathbb{R}^n$$, and since $$\psi$$ is constant near infinity, we have $$\psi \in L^\infty(\mathbb{R}^n)$$. Therefore, $$\psi \in W^{1,\infty}(\mathbb{R}^n)$$, and without loss of generality, we may assume that there is an open ball $$B$$ such that $$\Omega \subset B$$ and $$\supp (\psi) \subset B$$.

We want to add $$\nabla \psi$$ to the potential $$A_2$$ without changing the set of the Cauchy data for $$L_{A_2,q_2}$$ on the ball $$B$$. To that end, we shall need the following result, which is due to [17, Lemma 4.2].

**Proposition 3.4.** Let $$\Omega, \Omega' \subset \mathbb{R}^n$$ be bounded open sets such that $$\Omega \subset \subset \Omega'$$. Let $$A_1, A_2 \in L^\infty(\Omega', \mathbb{C}^n)$$, and $$q_1, q_2 \in L^\infty(\Omega', \mathbb{C})$$. Assume that

$$A_1 = A_2 \quad \text{and} \quad q_1 = q_2 \quad \text{in} \quad \Omega' \setminus \Omega. \quad (3.23)$$

If $$C_{A_1,q_1} = C_{A_2,q_2}$$ then $$C'_{A_1,q_1} = C'_{A_2,q_2}$$, where $$C'_{A_j,q_j}$$ is the set of the Cauchy data for $$L_{A_j,q_j}$$ in $$\Omega'$$, $$j = 1, 2$$.

**Proof.** Let $$u'_1 \in H^1(\Omega')$$ be a solution to $$L_{A_1,q_1}u'_1 = 0$$ in $$\Omega'$$ and let $$u_1 = u'_1|_\Omega \in H^1(\Omega)$$. As $$C_{A_1,q_1} = C_{A_2,q_2}$$, there exists $$u_2 \in H^1(\Omega)$$ satisfying $$L_{A_2,q_2}u_2 = 0$$ in $$\Omega$$ such that

$$Tu_2 = Tu_1 \quad \text{and} \quad N_{A_2,q_2}u_2 = N_{A_1,q_1}u_1 \quad \text{in} \quad \Omega.$$

In particular, $$\varphi := u_2 - u_1 \in H^1_0(\Omega) \subset H^1(\Omega')$$. We define

$$u'_2 = u'_1 + \varphi \in H^1(\Omega'),$$

so that $$u'_2 = u_2$$ on $$\Omega$$. It follows that $$Tu'_2 = Tu'_1$$ in $$\Omega'$$. Let us show now that $$L_{A_2,q_2}u'_2 = 0$$ in $$\Omega'$$. To that end, let $$\psi \in C^\infty_0(\Omega')$$, and write

$$\langle L_{A_2,q_2}u'_2, \psi \rangle_{\Omega'} = \int_{\Omega'} \left( (\nabla u'_1 + \nabla \varphi) \cdot \nabla \psi + A_2 \cdot (Du'_1 + D\varphi)\psi \right) dx$$

$$+ \int_{\Omega'} \left( -A_2(u'_1 + \varphi) \cdot D\psi + (A_2^2 + q_2)(u'_1 + \varphi)\psi \right) dx.$$

Using (3.23), we have

$$\langle L_{A_2,q_2}u'_2, \psi \rangle_{\Omega'} = \int_{\Omega} (\nabla u_2 \cdot \nabla \psi + A_2 \cdot (Du_2)\psi - A_2u_2 \cdot D\psi + (A_2^2 + q_2)u_2\psi) dx$$

$$+ \int_{\Omega' \setminus \Omega} (\nabla u'_1 \cdot \nabla \psi + A_1 \cdot (Du'_1)\psi - A_1u'_1 \cdot D\psi + (A_1^2 + q_1)u'_1\psi) dx$$

$$+ \int_{\Omega' \setminus \Omega} (\nabla \varphi \cdot \nabla \psi + A_1 \cdot (D\varphi)\psi - A_1\varphi \cdot D\psi + (A_1^2 + q_1)\varphi\psi) dx.$$
As \( \varphi \in H^1_0(\Omega) \), we get
\[
\int_{\Omega \setminus \Omega'} (\nabla \varphi \cdot \nabla \psi + A_1 \cdot (D\varphi)\psi - A_1 \varphi \cdot D\psi + (A_1^2 + q_1)\varphi \psi) dx = 0.
\]
This together with the fact \( N_{A_{2,q_2}} u_2 = N_{A_{1,q_1}} u_1 \) in \( \Omega \) implies that
\[
\langle L_{A_{2,q_2}} u_2', \psi \rangle_{\Omega'} = \langle L_{A_{1,q_1}} u_1', \psi \rangle_{\Omega'} = 0,
\]
which shows that \( L_{A_{2,q_2}} u_2' = 0 \) in \( \Omega' \).

Arguing similarly, we see that \( N_{A_{2,q_2}} u_2' = N_{A_{1,q_1}} u_1' \) in \( \Omega' \), which allows us to conclude that \( C'_{A_{1,q_1}} \subset C'_{A_{2,q_2}} \). The same argument in the other direction gives the claim. \( \square \)

Let us extend \( q_j, j = 1, 2 \), to the open ball \( B \) by defining \( q_j = 0 \) in \( B \setminus \Omega \). Then using Proposition 3.4, Lemma 3.1 and the fact that \( \psi|_{\partial B} = 0 \), we obtain that
\[
C'_{A_{1,q_1}} = C'_{A_{2,q_2}} = C'_{A_{2} + \nabla \psi, q_2} = C'_{A_{1,q_2}}.
\]
This implies the following integral identity,
\[
\int_B (q_1 - q_2) u_1 \overline{u_2} dx = 0,
\]
valid for any \( u_1, u_2 \in H^1(B) \) satisfying \( L_{A_{1,q_1}} u_1 = 0 \) in \( B \) and \( L_{A_{1,q_2}} u_2 = 0 \) in \( B \), respectively.

Let us choose \( u_1 \) and \( u_2 \) to be the complex geometric optics solutions in \( B \), given by (3.6) and (3.10), respectively. In this case, it follows from (3.16) that \( \Phi_1^\sharp(\cdot, \mu_1 + i\mu_2; h) \) and \( \Phi_2^\sharp(\cdot, -\mu_1 + i\mu_2; h) \) converges to zero in \( L^2_{\text{loc}}(\mathbb{R}^n) \) as \( h \to 0 \).

Plugging \( u_1 \) and \( u_2 \) into (3.24) gives
\[
\int_B (q_1 - q_2) e^{ix \cdot \xi} \Phi_1^\sharp + \Phi_2^\sharp dx = - \int_B (q_1 - q_2) e^{ix \cdot \xi} (e^{\Phi_1^\sharp} + r_1 e^{\Phi_2^\sharp} + r_1 r_2) dx.
\]
Letting \( h \to 0 \), and using (3.7), (3.9), (3.11), and (3.13), we get
\[
\int_B (q_1 - q_2) e^{ix \cdot \xi} dx = 0,
\]
and therefore, \( q_1 = q_2 \) in \( \Omega \). The proof of Theorem 1.1 is complete.
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