Bell-inequality in path-entangled single photon and purity test

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Different degrees of freedom of single photons have been entangled and used as a resource for various quantum technology applications. We present a scheme to perform Bell’s test and show the transition from violation of CHSH inequality to its validity when the purity of single photon state decreases below 70% visibility, \( P < 0.7 \). Our procedure allows a purity test on any single photon source and to study quantum correlations on systems driven by dynamics where single particle entanglement with position space is prominent.

I. INTRODUCTION

Single photons and entangled photon pairs have been a very useful resource for several experimental tests of fundamentals of quantum physics [1, 2]. Lately, they have been extensively used for quantum technology applications [3–7]. Among different approaches, spontaneous parametric down-conversion (SPDC) is one of the matured methods to generate such photons [8]. Experimental feasibility to control various degrees of freedom of photons such as polarization [2, 9, 10], frequency [11], time-bin [12] and path [13–15] independently and simultaneously in different combinations have enabled generation of high-dimensional entangled states [16]. These higher dimensional entangled systems have shown significant improvement in processing quantum information. In contrast to the standard notion of entanglement that has been associated with two or more quantum systems [17–19], single-particle states are also known to exhibit quantum nonlocality [20–23]. This was further probed by a series of theoretical [24–32] and experimental investigations [33–39] establishing the single-particle entanglement to be between the spatial modes, rather than between the quantum systems. Over time, single-photon entangled states have also been widely used as a resource for quantum information processing [40–44] and quantum computation [45, 46].

For more than four decades since the proposal for Bell’s test [17] and experimental violation of the Clauser, Horne, Shimony, and Holt (CHSH) inequality [47], innumerable experiments have been performed on various systems to establish quantum nonlocality, characterize entanglement between systems and between different degrees of freedom of a single particle. CHSH inequality violation with single photons entangled in momentum and polarization was shown using a Mach-Zehnder interferometer setting [48, 49]. Violation with photon entangled in polarization and orbital angular momentum [50] and using homodyne detection measurement scheme [51] has also been reported.

In this work we first present a theoretical scheme to show the violation of CHSH inequality in path-entangled single photon state using non-interferometric approach. The scheme involves the use of beam splitters to control the probability amplitude of presence and absence of photon along the two paths and to obtain the probabilities associated with the four basis states needed to calculate CHSH parameter. We also model decoherence in the form of depolarizing channel [52] and by introducing the presence of multi-photons in the source and show the transition from violation of CHSH inequality to its validity (thermal state) with decrease in purity of single photon state (visibility). We demonstrate this experimentally by employing a setup composing of a probabilistic single photon source from the SPDC using type-II Beta Barium Borate (BBO) crystal and two detector modules. Using combination of wave plates we have experimentally reduced the entanglement visibility and mimicked the reduction in purity of single photons (depolarizing channel) to show the transition from violation of CHSH inequality to validity with decrease in purity. Unlike the previous experimental reports of such results using interferometer setting for single photons [48, 49], our measurement procedure allows a sequence of measurements without the need for any interferometry approach which need more experimental resources. For any quantum communication and computation applications of single photons, purity of source is a key factor. A simple test for purity using CHSH inequality violation presented here will be a very useful resource and will also be useful to experimentally verify and study quantum correlations in systems where single particle dynamics are associated with spatial modes.

The paper is organized as follows. In Section II we present an analytical description of a path-entangled single photon in a beam splitter setup and the procedure
II. THEORETICAL DESCRIPTION

A. Bell’s-test for path-entangled single photon

Bell’s test : Entangled photon pairs in polarization degree of freedom is given by,

\[ |\Psi\rangle_{EP} = \frac{1}{\sqrt{2}} \left[ |H\rangle_1 |V\rangle_2 + |V\rangle_1 |H\rangle_2 \right], \]  

(1)

where \( |H\rangle \) and \( |V\rangle \) represent the polarization states and subscript represent the two photons. Standard Bell’s test procedure on such entangled pairs involve rotation of polarization angle by \( \theta \) and \( \delta \) on each photon (two associated Hilbert spaces) independently just before they reach spatially separated detectors. Rotation of polarization angle results in spanning over all four basis states of the two photons. Then a series of coincidence events of detection in the form of \( E(\theta, \delta) \) are recorded for different configurations of angles. Using those values, CHSH parameter

\[ S = |E(\theta, \delta) - E(\theta', \delta') + E(\theta', \delta) - E(\theta, \delta')| \]  

(2)

is calculated. If the value of \( S > 2 \), violation of CHSH inequality (\( S \leq 2 \)) is observed suggesting nonlocal effect, an entangled state. Theoretically,

\[ E(\theta, \delta) = P_{00} + P_{11} - P_{01} - P_{10} \]  

(3)

where \( P_{ij} \) are the probabilities of different basis states of the composite system. Experimentally they can be obtained from coincidence counts of occurrence in different combination of basis states.

Path-entangled single photon : Path-entangled single photon state is generated when \( m \) spatial modes share a single photon. Then the state in \( m \) spatial mode will exist in a Hilbert space \( H = \bigotimes_{i=1}^{m} H_i \), where \( H_i \) is the Hilbert space of the \( i^{th} \) spatial mode. Each \( H_i \) will be spanned by \( \{|0\rangle_i, |1\rangle_i\} \) representing the photon occupancy, absence and presence of photon, respectively in the mode.

For a single photon passing through a 50:50 beam split-

FIG. 1: Schematic representation of two input of the beam splitter to realize four and two output modes as basis states is presented in the inset. Its physical realization using combination of beam splitters is presented in the main schematic. The scheme comprises of beam splitters with different splitting ratio and four single photon counting module will allow us to control and span all four basis states and calculate CHSH parameter, Bell’s test on a path-entangled single photon system.

ter, the path-entangled state will be in the form,

\[ |\Psi\rangle_{1-2} = \frac{1}{\sqrt{2}} \left[ |1\rangle_1 |0\rangle_2 + |0\rangle_1 |1\rangle_2 \right], \]  

(4)

where the subscript 1 and 2 represent the two spatial modes or paths for the photon. To calculate CHSH parameter one has to span over all the basis states associated with the two spatial mode and photon occupancy. In two photon system described earlier using polarization state we can perform rotation on the polarization state of the photon and obtain non-zero probability of state \( |HH\rangle \) and \( |VV\rangle \). But in single photon occupancy description in two spatial mode, the states \( |0\rangle_1 |0\rangle_2 \) and \( |1\rangle_1 |1\rangle_2 \) will always have a zero probability amplitude. However, using single photon state along two different input modes all the four spatial modes of the beam splitter in occupancy representation with non-zero probability can be realized. The basis states in the Fock state representation can then be written as \(|n_1, n_2, n_3, n_4\rangle\) and for a single photon case \( n_1 + n_2 + n_3 + n_4 = 1 \). This can be experimentally replicated by adding two identical beam splitters along the two output modes of the first beam splitter as show in Fig. 1. In the insets of Fig. 1 we also show the schematic of the beam splitters’ spatial modes for single photon and combination of input states for realisation of four and two basis states, respectively. The generic output state in the photon occupancy representation will be,

\[ |\Psi\rangle_{1-4} = a|1000\rangle + b|0100\rangle + c|0010\rangle + d|0001\rangle. \]  

(5)

When any two output modes have non-zero probability, eliminating the redundant modes, the state will be identi-
path-entangled single photon state given in Eq. (4).

Alternatively, the four basis states with non-zero probability from the two input spatial modes can also be re-written by associating each spatial modes, \( s_1 \) and \( s_2 \) with the transmitted and reflected component of the photon using states \(|0\rangle_{s_1}, |1\rangle_{s_1}\), respectively. In this representation all four basis states \(|0\rangle_{s_1}|0\rangle_{s_2}, |0\rangle_{s_1}|1\rangle_{s_2}, |1\rangle_{s_1}|0\rangle_{s_2}, |1\rangle_{s_1}|1\rangle_{s_2}\) can be spanned by introducing single photon in superposition of two input modes of the beam splitter with variable \( \theta \) controlling the splitting ratio. In two spatial mode representation, the state with all four basis states will be in the form,

\[
|\Psi\rangle_{s_1-s_2} = a|0\rangle_{s_1}|0\rangle_{s_2} + b|0\rangle_{s_1}|1\rangle_{s_2} + c|1\rangle_{s_1}|0\rangle_{s_2} + d|1\rangle_{s_1}|1\rangle_{s_2}
\]

(6)

In Fig. 2 we show the combination of beam splitters to span over all four basis state in Hilbert space \( s_1 \otimes s_2 \) representation. First, the transmitted and reflected components from mode \( s_1 \) are spatially separated and it is further subjected to splitting using two beam splitters which represents spatial mode \( s_2 \).

**FIG. 2:** Schematic representation pf combination of beam splitter in two spatial mode representation which is identical to photon occupancy representation. Operators \( B(\theta) \) and \( B(\delta) \) on each mode will help in spanning over all four basis states and obtain probabilities to calculate CHSH parameter.

In Fig. 2, when second pair of beam splitters is absent, the state \(|\Psi\rangle_{s_1-s_2}\) can be written as

\[
|\Psi\rangle_{p-s} = \frac{1}{\sqrt{2}} \left[ |0\rangle_{s_1}|0\rangle_{s_2} + |1\rangle_{s_1}|1\rangle_{s_2} \right].
\]

(7)

Since we only have access to change the splitting components along paths in both the description, we can only measure the probability of finding photon along each path. This results in an identical effect on both the representation showing the equivalence, \(|\Psi\rangle_{1-4} \equiv |\Psi\rangle_{s_1-s_2} \). Therefore, for all practical purpose for path-entangled single photon in beam splitter setting,

\[
|\Psi\rangle_{1-2} \equiv |\Psi\rangle_{p-s}.
\]

(8)

**Bell’s test on path-entangled single photon:** To perform Bell’s test on photon in path-entangled state given by Eq. (4), we need to calculate \( E(\theta, \delta) \). The parameter \( \theta \) and \( \delta \) will control the amplitudes of presence and absence of photon in two basis states associated with each of the paths and this can be realised using the beam splitter in operational form,

\[
B(\theta) = \begin{bmatrix} \cos(\theta) & i \sin(\theta) \\ i \sin(\theta) & \cos(\theta) \end{bmatrix}
\]

along each path. By measuring the probabilities along the four outputs paths of the beam splitter, \( E(\theta, \delta) \) can be calculated. However, before calculating \( E(\theta, \delta) \) we will first deliberate the equivalence of beam splitter operation on states in photon occupancy representation and in two spatial mode representation. In photon occupancy representation, any change in amplitude of presence (absence) of photon in one path directly affects the absence (presence) of the photon in the other path. Therefore, any change on path 1 using \( B(\theta) \) will reflect on path 2 and any change on path 2 using \( B(\delta) \) will reflect on path 1. In a generic description we can write down the state with controllable probability of presence of photon in each path in the form,

\[
|\Psi\rangle_{1-4} = \begin{bmatrix} B(\theta) & 0 \\ 0 & B(\theta) \end{bmatrix} \begin{bmatrix} B(\delta) & 0 \\ 0 & B(\delta) \end{bmatrix} \left( \frac{1}{\sqrt{2}} \left( |1000\rangle + |0001\rangle \right) \right)
\]

\[
= \begin{bmatrix} B(\theta + \delta) & 0 \\ 0 & B(\theta + \delta) \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{bmatrix} \cos(\theta + \delta) & i \sin(\theta + \delta) \\ i \sin(\theta + \delta) & \cos(\theta + \delta) \end{bmatrix}
\]

(10)

In two spatial mode representation as described in Fig. 2 where states are represented using transmitted and reflected components along each path, the action of \( B(\theta) \) and \( B(\delta) \) will be in the form,

\[
|\Psi\rangle_{s_1-s_2} = \left( B(\theta) \otimes B(\delta) \right) |\Psi\rangle_{p-s}
\]

\[
= \left( B(\theta + \delta) \otimes I \right) |\Psi\rangle_{p-s} = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos(\theta + \delta) & i \sin(\theta + \delta) \\ i \sin(\theta + \delta) & \cos(\theta + \delta) \end{bmatrix}
\]

(11)

Since the probabilities associated with basis states in Eq. (10) and Eq. (11) are identical, equivalence between the two representation of single photon in the given beam splitter configuration can be established. Therefore, from the measurements outputs from the given setting, CHSH for path-entangled state can be calculated using spatial mode states. Here we would like to note that for a path-entangled photon along four output modes of single beam
splitter as shown in the inset of Fig. 1, we will not have freedom to choose two different splitting ratios from the same beam splitter, we will have to consider \( \theta = \delta \). However, in a generic setting using multiple beam splitters, we will have freedom to use both \( \theta \) and \( \delta \).

The expression to calculate CHSH parameter in photon occupancy representation will be same as Eq. (2). In spatial mode setting due to the combined effect of beam splitter operation on spatial mode, \( E(\theta, \delta) = E(\theta + \delta) \) and \( S \) will be,

\[
S = |E(\theta + \delta) - E(\theta + \delta')| + |E(\theta' + \delta) + E(\theta' + \delta')|.
\]

In Fig. 3 we show the CHSH parameter as function of \( \theta \) and \( \delta' \) when \( \delta = 0 \) and \( \theta' = (\theta + \delta') \). The scheme for generation of a path-entangled single photon and perform Bell’s test using polarization degree of freedom will be identical to Fig. 2. The splitting ratio is controlled using the polarization rotator (or half wave plate (HWP)) and PBS. The basis states described earlier in this section, \( \{|0\rangle_{s_1}, |1\rangle_{s_1}\} \) will be replaced with the basis states for polarization degree of freedom, \( |H\rangle \) and \( |V\rangle \). The rotation operation on polarization degree of freedom is given by

\[
R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.
\]

Rotating the state of photon \(|H\rangle\) by angle \( \theta = \pi/4 \) we obtain \( \frac{1}{\sqrt{2}} (|H\rangle + |V\rangle) \) and after passing it through PBS we will have a photon in path-entangled state in the form of \(|\Psi\rangle_{p-s}\) as given in Eq. (7). To perform Bell’s test we will choose two angle of rotations, \( \theta \) to act on the Hilbert space associated with \( s_1 \), photon polarization and \( \delta \) on the Hilbert space associated with spatial mode \( s_2 \). Since the action of operator on one spatial model will also be an action on the other spatial degree of freedom, the effective state will be,

\[
|\Psi\rangle_{s_1-s_2} = (R(\delta) \otimes \mathbb{I})(R(\theta) \otimes \mathbb{I})|\Psi\rangle_{p-s}
= \frac{1}{\sqrt{2}} \begin{bmatrix} \cos(\theta + \delta) \\ \sin(\theta + \delta) \\ -\sin(\theta + \delta) \\ \cos(\theta + \delta) \end{bmatrix}.
\]

The expression \( E(\theta, \delta) \) which is a composition of proba-
bility measurements in different basis state will be

\[
E(\theta, \delta) = P_{00} + P_{11} - P_{01} - P_{10} = \cos 2(\theta + \delta). \tag{16}
\]

CHSH parameter for different pairs of angles \((\theta, \delta)\) and \((\theta', \delta')\) will be,

\[
S(\theta, \delta, \theta', \delta') = |E(\theta, \delta) - E(\theta, \delta')| + |E(\theta', \delta) + E(\theta', \delta')|. \tag{17}
\]

In Fig. 4 the parameter \(S(\theta', \delta')\) obtained when \(\theta = 3\pi/4\) and \(\delta = 7\pi/8\) is presented. Violation of CHSH inequality for various combination of parameters and a maximum violation of \(S = 2\sqrt{2}\) is observed for some combination of parameters.

In addition to mimicking the path-entangled single photon, using the presence of polarization degree of freedom, a similar setup using the combination of beam splitter and PBS along with polarizer can be explicitly used to calculate CHSH parameter for polarization and path degree of freedom of single photon.

**B. Purity test on single photon state**

**Depolarization**: Transition of a pure quantum state to a maximally mixed state can be effectively modeled using quantum depolarizing channel. For the path-entangled single photon state, it is a linear combination of Eq. (7) in its density matrix form \(\rho_{p-s} = |\Psi\rangle_{p-s}\langle\Psi|\) and maximally mixed state,

\[
D_\mathcal{P}(\rho_{p-s}) = \mathcal{P}\rho_{p-s} + \frac{(1-\mathcal{P})}{2} \mathbb{I}. \tag{18}
\]

Here \(D_\mathcal{P}\) is a completely positive trace preserving map and \(\mathcal{P}\) is the probability or purity level in this case. By passing the state \(\rho_{p-s}' = D_\mathcal{P}(\rho_{p-s})\) through additional pair of beam splitters, probabilities associated with all four basis states, \(E(\theta, \delta)\) and CHSH parameter \(S\) can be calculated.

**Multi-photon noise**: One of the common cause for reduction in the purity of single photons is the probability of multi photon pair generation in some of the sources. When multi-photons accompany single photons, purity of single photons reduces and tends towards thermal state. When the photons source comprising of mixture of single photon and multi-photon states is passed through the beam splitter, the mixed input state can be written in the form,

\[
\rho_m = \mathcal{P}\left(\rho_{p-s}\right) + \frac{(1-\mathcal{P})}{3} (\rho_1 + \rho_2 + \rho_3). \tag{19}
\]

Here, \(\rho_{p-s}\) represents the single photons, \(\rho_1 = \langle 01 | + | 10 \rangle \langle 01 | + | 10 \rangle\) represents multi photon detection in both the detectors, \(\rho_2 = |01\rangle \langle 01|\) and \(\rho_3 = |10\rangle \langle 10|\) represents unresolved multi-photon detection in either of the detector, respectively. The value of \(\mathcal{P}\) purity level of single photons in the mixed input state. The path-entangled state for the mixed input state is further passed through a pair of beams splitter to calculate CHSH parameter associated with the input state.

In Fig. 5 the maximum value for CHSH parameter obtained from numerical calculation with increase in purity of single photon state is shown. Increase in the value of \(S\) and violation of CHSH inequality can be recorded for \(\mathcal{P} > 0.7\).

**III. EXPERIMENTAL METHOD**

In SPDC process photons are generated in pairs and presence of single photon can be gated by heralding the other photon. Since SPDC is probabilistic in nature, we will have time windows with single photons, no photons or multi-photons. Therefore, if we are not gating the presence of photon by heralding, the field striking the beam splitter is treated as a thermal source when one examines the full temporal behaviour. Instead of examining the full temporal behaviour, if we examine the field and photons detection in the smaller time windows we can resolve and record more single photons. Experimentally, with a choice of low pump power, small time window, better detector efficiency with low dead time and by filtering the pump power from entering detector, one can resolve and detect more single photons even when it is not heralded. For our experiment we have chosen un-heralded source with a low pump power such that the minimum difference between the two detection obtained

![FIG. 5: Maximum value of $S$ parameter with increase in purity ($\mathcal{P}$) of single photons when subjected to depolarizing noise and when the input state is a mixture of single and multi-photon states. For $\mathcal{P}$ close to 0.7 we see the transition from validity of CHSH inequality to violation of CHSH inequality.](image-url)
by averaging over many trial runs is slightly higher than the detector dead time. Keeping the average minimum difference as a time window we resolve a good number of single photons. Though we will have many time window detecting no photons and a few window detecting multi-photons, single photon and no photon time windows outnumber giving us a random distribution of single photons over time.

A. Experimental setup

To experimentally demonstrate path-entangled single photon, a single photon generated from the SPDC process is used as a source. As per the schematic presented in Fig. 2 we will need four single photon detector modules (SPCM) for Bell’s test. However, we will use only two detectors and measure the outcomes from only one of the basis of the path $|0\rangle_s$ as shown in Fig. 2. With a controlled initial state and series of measurements for different configurations we reproduce the full output expected from the four single photon counting module.

The schematic of an experimental setup is shown in Fig. 6. A 1-mm-long BBO nonlinear crystal cut for type-II phase-matching is pumped by a continuous-wave diode laser (Surelock, Coherent) at 405 nm with a pump power of 40 mW. A half-wave plate (HWP) is placed to control the pump polarization for optimal phase-matching. A plano-convex lens (f=200 mm) is used to tightly focus the laser into the crystal. Down-converted degenerate photons at 810 nm are collected using lens (f=30 mm) and then coupled into single-mode fibers (SMF). Only orthogonally polarized photon pairs (|H⟩, |V⟩) are separated using a PBS and then coupled into single-mode fibers (SMF). Only horizontally (|H⟩) polarized single photons with photon count of 4600 c/s at 40 mW are used as our source to generate path-entangled photons.

A polarization controller, consisting of two quarter-wave plates (QWP) with a HWP in between, is used to maintain the initial polarization state of single photons to |H⟩. In our setup, the polarization controller is also used as a depolarizing element to show the transition from validity of CHSH inequality to the violation of inequality with an increase in purity of quantum state of a single photon. To generate a path-entangled single photons and perform Bell’s test measurements, splitting of photons along different paths are controlled using a combination of HWP and PBS. The single photon counting measurements in two paths are simultaneously performed using two SPCM ID120 from ID Quantique. By changing the HWP angle from 0 to 360 degrees, we recorded the photon counts in both paths. To test the purity of the photon state, we introduce depolarization channel by changing the angles in the waveplates present in polarization compensator. When the incident angle is not linearly polarized along the symmetry axis of the wave plate, the temporal walk-off will be acquired between the two polarization states. Though the detectors are insensitive to such short temporal walk-offs, the temporal distinguishability acquired during the propagation fulfils the role of the environment in general decoherence models [52]. Thus, by controlling the angles of the wave plates in polarization compensator, depolarization was introduced to change the visibility of photon state from 100% to 30%. Using the number of photon counts in each SPCM for all angles of HWP (H2) we reconstruct the probability of finding photons in all four basis states of path-entangled single photon and calculate CHSH parameter.

B. Measurement procedure

In the experimental setup we have used combination of HWP and PBS to control the photon splitting along the paths and generate path-entangled state. The HWP operation can be written as

$$H(\kappa) = \begin{bmatrix} \cos(2\kappa) & \sin(2\kappa) \\ -\sin(2\kappa) & \cos(2\kappa) \end{bmatrix}$$

(20)

and when only probabilities of states are considered, $R(\theta) \equiv H(\kappa/2)$. To reproduce the probability output identical to the one obtained from Eq. (15), the HWP is set to rotate the state by $R(\theta + \delta)$ in different combinations and the combination that gives probabilities of all
four basis states are given below,

\[
P_{00}(\theta, \delta) = \frac{C_{D1}(\theta, \delta)}{C_{D1}(\theta, \delta) + C_{D1}(\theta, \delta_\perp) + C_{D2}(\theta, \delta) + C_{D2}(\theta, \delta_\perp)}
\]

\[
P_{10}(\theta, \delta) = \frac{C_{D2}(\theta, \delta)}{C_{D1}(\theta, \delta) + C_{D1}(\theta, \delta_\perp) + C_{D2}(\theta, \delta) + C_{D2}(\theta, \delta_\perp)}
\]

\[
P_{11}(\theta, \delta) = \frac{C_{D1}(\theta, \delta_\perp)}{C_{D1}(\theta, \delta) + C_{D1}(\theta, \delta_\perp) + C_{D2}(\theta, \delta) + C_{D2}(\theta, \delta_\perp)}
\]

\[
P_{01}(\theta, \delta) = \frac{C_{D2}(\theta, \delta_\perp)}{C_{D1}(\theta, \delta) + C_{D1}(\theta, \delta_\perp) + C_{D2}(\theta, \delta) + C_{D2}(\theta, \delta_\perp)}.
\]

(21)

\(C_{D1}(\theta, \delta)\) and \(C_{D2}(\theta, \delta)\) are the first set of number of photon counts in detector 1 and detector 2 when the rotation angles are \(\theta\) and \(\delta\). \(C_{D1}(\theta, \delta_\perp)\) and \(C_{D2}(\theta, \delta_\perp)\) are the second set of number of photon counts in detector 1 and detector 2 when the rotation are \(\theta_\perp\) and \(\delta_\perp\). From the probability values we calculate \(E(\theta, \delta)\) and use that to obtain CHSH parameter \(S\). Using the experimental data for number of photon counts detected per second averaged over 100 trials we calculate parameter \(S(\theta', \delta')\) when \(\theta\) and \(\delta\) are fixed at \(3\pi/4\) and \(7\pi/8\).

The output is shown in Fig. 7 and it is in agreement with the numerical simulation obtained for same parameters shown in Fig. 4. Using the data for full range of rotation of input state, we can reconstruct parameter \(S\) for any combination of \(\theta\), \(\delta\), \(\theta'\) and \(\delta'\). The maximum value we could obtain from the experimental data is, \(S = 2.82\). Due to simplicity of setup with minimum devices (optical components and detectors), the deviation from expected maximum value is very low. To calculate CHSH parameter for setup with different rotation, \(\delta_1\) and \(\delta_2\) along each path the first two probability values in Eq. (21) are calculated for \(\delta_1\) and second two probability values are calculated for \(\delta_2\).

\[\delta\]

**FIG. 8:** CHSH parameter \(S\) as a function of \(\theta'\) and \(\delta'\) when \(\theta\) and \(\delta\) are fixed at \(3\pi/4\) and \(7\pi/8\). The plot is generated from the experimental photon counts (a) when visibility is 30% and (b) when visibility is 80%. Violation of inequality \((S(\theta', \delta') > 2)\) for a range of values \(\theta'\) and \(\delta'\) can be seen when the visibility is 80% with maximum value of 2.35 and at 30% visibility, inequality is not violated.

\[\delta\]

**FIG. 9:** Transition towards violation of CHSH inequality with increase in visibility. Theoretical expectation and experimental values for the maximum value of \(S\) when \(\theta = 3\pi/4\) and \(\delta = 7\pi/4\).

C. Test for purity of single photon state

Depolarizing channel with controllable parameters on single photon can be effectively realized using birefrin-
In this work we have presented a theoretical framework to calculate CHSH parameter for path-entangled single photons in beam splitter setting and has been experimentally demonstrated. By experimentally mimicking the effect of depolarizing channel, we have validated the use of CHSH as a test purity of single photon state. The theoretical results and experimental results we have obtained are in good agreement with each other. Though our theoretical scheme proposes the use of four detector (SPCM) module, we have used a measurement procedure that needs only two SPCM. The maximum value of CHSH parameter we have obtained is $S = 2.82$ and we have observed the transition towards violation of CHSH inequality happening around purity value of $P = 0.7$ and above. In our procedure, polarization degree of freedom was only used to have control over the splitting of photon along different paths and was not explicitly used as one of degree of freedom to calculate CHSH parameter but same procedure will hold to calculate CHSH parameter for polarization and path degree of freedom. The simple non-interferometric scheme we have used for the demonstration makes it a practically efficient way to test purity of single photon state from any source. The measurement procedure we have adopted will be very useful to experimentally calculate quantum correlation in particle-path (spatial mode) entangled system. This work can lead to further investigations towards loophole free Bell’s inequality [53] in path entangled or spatial mode setting, towards violation of CHSH inequality and still debated nonlocality effect in path-entangled state through quantum steering approach [54] and in temporal order [55].

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