Statistical properties of Cherenkov and quasi-Cherenkov superradiance

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We consider the effects of shot noise and particle energy spread on statistical properties of Cherenkov and quasi-Cherenkov superradiance emitted by a relativistic electron beam. In the absence of energy spread, we have found the root-mean-square deviations of both peak radiated power and instability growth time as a function of the number of particles. It is shown that energy spread can lead to a sharp drop in the radiated power of Cherenkov and quasi-Cherenkov superradiance at high currents.

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I. INTRODUCTION

The numerous processes occur in the electron bunch moving in the medium with the refractive index more than unity [1–7]. Cherenkov instability is one of them. As was demonstrated in [1–4], the instability evolves in the electron bunch even in the absence of an external action. At the initial stage, the instability is accompanied by particle bunching and exponential growth of the radiated power. This exponential growth is then stopped due to nonlinearity, and the pulse of Cherenkov superradiance is formed.

The influence of different parameters on the peak power of Cherenkov superradiance was studied in detail in [5]. The authors of [5] also assumed that the particles having the same initial energy undergo phase pre-modulation at the radiation frequency. As a result of phase pre-modulation, the electromagnetic oscillations start with spontaneous coherent emission from all particles rather than with spontaneous emission from individual electrons of a bunch.

However, it seems interesting to know the behavior of Cherenkov superradiance without pre-modulation, in which case the generation starts as spontaneous emission of electromagnetic waves from individual particles. To give a correct description of Cherenkov superradiance in this case, we need to consider statistical fluctuations due to shot noise and energy spread of electrons inherent in particle ensembles [8–11]. It is well-known [12–15] that because of shot noise in single-pass FELs, the radiated power and the instability growth time become stochastic quantities whose root-mean-square deviations have the same order of magnitude as their average values.

This paper studies the statistical properties of Cherenkov and quasi-Cherenkov superradiance without phase pre-modulation when the emission of electromagnetic waves begins with spontaneous emission from individual particles. The peak radiated power and the instability growth time are taken as stochastic quantities for statistical analysis. The peak power is the main output characteristic of short-pulse sources of electromagnetic radiation and the instability growth time is the parameter defining the minimum particle passage time in the generator that is necessary for the superradiant instability to evolve.

The paper’s outline is as follows. First, we derive a system of equations describing the interaction of charged particles with the radiation field in the medium. Further comes a detailed consideration of statistical properties of Cherenkov [2, 3] and quasi-Cherenkov [7] superradiance in the presence of shot noise alone and then the energy spread of electrons is added. The energy spread of electrons will appear to be an important factor limiting the peak radiated power of Cherenkov superradiance.

II. INTERACTION BETWEEN CHARGED PARTICLES AND THE ELECTROMAGNETIC FIELD

Let us consider the electron bunch of length L. The bunch is directed by a strong longitudinal magnetic field. As a result, there is no transverse displacement of particles. The effective interaction between the bunch and the electromagnetic wave is provided by Cherenkov synchronism condition: the electron velocity v_0 is close to the phase velocity v_{ph}.
Within the beam region, the radiation field is assumed to have a form
\[ E = \text{Re}(E_0(x,t)e^{i\Omega(t-x/v_0)}), \]

where the frequency \( \Omega \) satisfies the equation
\[ v_{ph}(\Omega) = v_0, \]

and the slowly varying complex amplitude of the wave \( E_0 \) satisfies the conditions:
\[ \left| \frac{1}{\Omega E_0} \frac{\partial E_0}{\partial t} \right| \ll 1, \]
\[ \left| \frac{v_0}{\Omega E_0} \frac{\partial E_0}{\partial x} \right| \ll 1. \]

In this case, the excitation equation for \( E_0 \) can be written as follows \[10\]
\[ \frac{1}{v_{gr}} \frac{\partial E_0}{\partial t} + \frac{\partial E_0}{\partial \tilde{x}} = -\frac{\beta^2 K}{2} I_0, \]

where \( v_{gr} \) is the group velocity, \( K \) is the coupling impedance, \( \beta = \Omega/v_0 \), and

\[ I_0 = \frac{\beta}{\pi} \int_{x-\pi/\beta}^{x+\pi/\beta} I(x,t)e^{-i(\Omega t-\beta x)} dx \]
\[ = \frac{\beta q_e}{\pi} \sum_{\alpha} v_{\alpha} e^{-i(\Omega t-\beta x_{\alpha})} \]

is the slowly varying complex amplitude of the current \( I \).

Now, let us turn to the analysis of the electron motion. For this purpose, we assume that all charged particles located within the interval \([x - \pi/\beta, x + \pi/\beta]\) are affected by the same force determined by the amplitude \( E_0 \) at point \( x \). As a result, the equations of particle motion has the following form:

\[ \frac{d\gamma_{\alpha}}{dt} = \frac{q_e v_0}{mc^2} \text{Re} E_0 e^{i\theta_{\alpha}}, \]
\[ \frac{d\theta_{\alpha}}{dt} = \Omega \left( \frac{v_{0\alpha}}{v(\gamma_{\alpha})} - 1 \right) \approx \Omega \left( \frac{1}{\gamma_{\alpha}^2} - \frac{1}{\gamma_{0\alpha}^2} \right). \]

Here, \( q_e \) and \( m \) are the charge and mass of the electron respectively, \( \gamma_{\alpha} = 1/\sqrt{1 - v_{\alpha}^2/c^2} \) is the Lorenz factor, and \( \theta_{\alpha} \) is the particle phase in the electromagnetic wave.

Let us introduce the average current density:
\[ I_{av} = \frac{\beta}{2\pi} \sum_{\alpha} q_e v_{\alpha} \approx \frac{\beta}{2\pi} N_\Lambda q_e v_0, \]

where \( N_\Lambda \) is the number of particles in the interval \([x - \pi/\beta, x + \pi/\beta]\).

Using (7) and coordinate transformation
\[ x = \tilde{x} + v_0 t, \]

we rewrite equation (4) as follows
\[ \frac{1}{v_{gr}} \frac{\partial E_0}{\partial t} + \frac{v_{gr}}{v_{gr}} \frac{\partial E_0}{\partial \tilde{x}} = -\frac{\beta^2 K}{N_\Lambda} \sum_{\alpha} e^{-i\theta_{\alpha}}. \]
For $v_0 > v_{gr}$ and $\gamma_0 \gg 1$, the substitution of dimensionless quantities

$$
\tau = C\beta t \sqrt{v_0^2 v_{gr}},
$$

$$
C = \frac{eI_{av}K}{2\gamma_0^3 mc^2},
$$

$$
z = C\beta \sqrt{\frac{v_0^2 v_{gr}}{v_0 - v_{gr}}},
$$

$$
F = \frac{eE_0}{\gamma_0 mc^2 \beta C^2} \sqrt{\frac{v_0^2 v_{gr}}{v_{gr}}},
$$

$$
\nu = 2C\gamma_0^2 \sqrt{\frac{v_{gr}}{v_0}},
$$

$$
\xi = C\beta L \sqrt{\frac{v_0^2 v_{gr}}{v_0 - v_{gr}}}.\tag{10}
$$

into (6) and (9) yields to the set of equations

$$
\frac{\partial F}{\partial \tau} - \frac{\partial F}{\partial z} = -\frac{2}{N_\lambda} \sum_\alpha e^{-i\theta_\alpha},
$$

$$
\frac{d^2 \theta_\alpha}{d\tau^2} = -\left(1 + \nu \frac{d\theta_\alpha}{d\tau}\right) \frac{3/2}{\Re(Fe^{i\theta_\alpha})},\tag{11}
$$

which should be supplemented with boundary and initial conditions

$$
\dot{\theta}_\alpha = 0,
$$

$$
F(\xi, \tau) = 0,
$$

$$
\theta_\alpha = 2\pi r_\alpha,\tag{12}
$$

where $r_\alpha$ are random variables uniformly distributed in the interval $[0; 1)$. One of the main parameters of short-pulse sources is the conversion ratio equal to the peak radiated power to the electron flow power ratio. In dimensionless units, $\eta$ is given by the expression

$$
\eta = \frac{v_{gr}}{v_0} \frac{|F_{\text{peak}}|^2}{8},\tag{13}
$$

where $F_{\text{peak}}$ is the peak value of the dimensionless amplitude $F$. Further, we shall use the reduced conversion ratio

$$
P_0 = \frac{|F_{\text{peak}}|^2}{8} \bigg|_{z=0}\tag{14}
$$

instead of $\eta$. The quantity $P_0$ differs from $\eta$ by the numerical factor $\frac{v_{gr}}{v_0}$.

### III. SHOT NOISE

As follows from (11), the behavior of charged particles in the absence of the energy spread is determined by three controlling parameters: the bunch length $\xi$, the nonlinearity parameter $\nu$, and the number of particles $N_e$. Hence, to explore the statistical properties of Cherenkov superradiance, we need to solve the set of equations (11) for various values of the controlling parameters. Because the initial phases $\theta_\alpha(0)$ are randomly distributed, the numerical experiment with each triple of values of $\xi, \nu$, and $N_e$ must be repeated many times. This procedure will give information about statistical characteristics of Cherenkov superradiance, the most important of which are the reduced conversion ratio $P_0$, the instability growth time $T_0$, and their relative root-mean-square deviations $\delta_P$ and $\delta_T$.

In the numerical analysis of statistical fluctuations of Cherenkov superradiance in the presence of shot noise, instead of the number $N_e$ of real electrons, we took the number $N = 360\gamma \ll N_e$ of large electrons with initial phases equal to

$$
\theta_\alpha(0) = \frac{2\pi \alpha}{N} + \sqrt{\frac{12N}{N_e}} r_\alpha, \alpha = 1..N.\tag{15}
$$
where \( r_\alpha \) are random variables uniformly distributed over the interval \([0; 1)\). It has been shown in \([10]\) that this procedure, boosting the performance of the program, simulates the shot noise correctly. We selected the following values of the controlling parameters: \( n = N_e/\xi = 2.7 \cdot 10^4, 1.08 \cdot 10^5, \nu = 0—2, \) and \( \xi = 1—4 \). The numerical experiment with each \((N_e, \xi, \nu)\) triple was repeated 100 times.

Figures 1 and 2 show the results of computation from which we can draw two very important conclusions. First, the instability growth time and the peak radiated power, which is proportional to the reduced conversion ratio, are weakly dependent on the number of particles \( N_e \). In accordance with \([4]\), the peak power increases with the bunch length \( \xi \). Second, the root-mean-square deviations of \( P_0 \) and \( T_0 \) are inversely proportional to the square root of the number of particles: \( \delta P \approx 11/\sqrt{N_e} \) and \( \delta T \approx 6/\sqrt{N_e} \).

At present, the electron beams for generating Cherenkov superradiance are obtained at high-current accelerators with explosive emission cathodes. Due to explosive electron emission, charged particles leave the cathode in separate portions, called ectons. A typical current of each ecton is \( I_e \sim 10 \, \text{A} \). The total current \( I \) produced by an accelerator is several kiloamperes. As a result, in estimating the fluctuations we should use the number of ectons \( \sim I/I_e \) instead of \( N_e \). Let the total current \( I \) be 2.6 kA, then we have \( \delta P = 0.68 \) and \( \delta T = 0.37 \). Let us note that the root-mean-square...
deviations are of the same order of magnitude as their averages.

**IV. ENERGY SPREAD**

To take account of the electron energy spread, we assume the initial quantities \( \hat{\theta}_\alpha(0) \) to be Gaussian random variables whose averages equal zero and the root-mean-square deviations \( \sigma = \frac{C\Delta \gamma_\alpha}{\gamma_0} \sqrt{\nu gr} \) (\( \Delta \gamma_\alpha \) is the root-mean-square deviation of the Lorenz factor).

\[
\begin{align*}
\frac{\partial F_0}{\partial \tau} + \frac{\partial F_\tau}{\partial z} + i\chi F_\tau &= -\frac{2}{N\lambda} \sum_\alpha e^{-i\theta_\alpha}, \\
\frac{\partial F_\tau}{\partial \tau} - \frac{\partial F_0}{\partial z} + i\chi F_0 &= 0, \\
\frac{d^2 \theta_\alpha}{d\tau^2} &= -\left(1 + \nu \frac{d\theta_\alpha}{d\tau}\right)^{3/2} \text{Re}(F_0 e^{i\theta_\alpha}).
\end{align*}
\]

Here, \( z \) and \( \xi \) are defined as \( z = C\beta x \sqrt{\nu_0^2/\nu_{gr}} \) and \( \xi = C\beta L \sqrt{\nu_0^2/\nu_{gr}} \) instead of (10). The parameter \( \chi \) in (16) is proportional to the dielectric susceptibility \( \chi_\tau \) [5].
The equations (16) should be supplemented with boundary and initial conditions

\[\dot{\theta}_\alpha = 0,\]
\[F_0(0, \tau) = 0,\]
\[F_\tau(\Lambda, \tau) = 0,\]
\[\theta_\alpha = 2\pi r_\alpha,\]

where \(\Lambda\) is a crystal thickness and \(r_\alpha\) are random variables uniformly distributed in the interval \([0; 1)\). In dimensionless units, the reduced conversion ratios are given by the expressions

\[P_0 = \frac{\nu|F_{0\text{peak}}|^2}{8}|_{z=\Lambda},\]
\[P_\tau = \frac{\nu|F_{\tau\text{peak}}|^2}{8}|_{z=0}.\]

We shall assume that \(\nu = 1.0\) and \(\xi = 1.0\). For this case, the peak intensity of cooperative radiation emitted in forward and backward directions is investigated as a function of the crystal thickness \(\Lambda\). The peak radiation intensity \(P_0\) appeared to increase monotonically until saturation is achieved (Fig. 4). At saturation, fluctuations in the intensity of radiation undergo a sharp drop. The growth of parameter \(\chi\) results in decreasing \(P_0\) and increasing \(P_\tau\) (Fig. 5, 6). We would like to note the fluctuations of quasi-Cherenkov superradiance under dynamical diffraction conditions correlate well with the results obtained in the previous section for Cherenkov radiation.

**FIG. 4:** Quasi-Cherenkov radiation in forward direction [solid curve — \(\chi = 0.1\), dashed curve — \(\chi = 0.4\)].

**FIG. 5:** Quasi-Cherenkov radiation in backward direction [solid curve — \(\chi = 0.1\), dashed curve — \(\chi = 0.4\)].

**VI. CONCLUSION**

In this paper, we have studied the statistical properties of Cherenkov and quasi-Cherenkov superradiance. For the Cherenkov superradiance, it has been shown that the relative root-mean-square deviations of the radiated power and
the instability growth time are $\delta_P \approx 11/\sqrt{N_e}$ and $\delta_T \approx 6/\sqrt{N_e}$, respectively. The fluctuations of quasi-Cherenkov superradiance under dynamical diffraction conditions correlate well with the results obtained for the Cherenkov superradiance. While investigating the quasi-Cherenkov superradiance, we have restricted ourselves to the case when the roots of the dispersion equation don’t coincide. The latter case demands more detailed computer simulations. This will be done in subsequent papers.

At present, electron beams for generating Cherenkov superradiance are obtained in high-current accelerators through explosive electron emission. As a result, the electron flow is emitted in separate portions, called ectons. To estimate $\delta_P, \delta_T$, we should use the number of ectons $\sim I/I_e \sim 100 \div 1000$ instead of $N_e$. As a result, the root-mean-square deviations in peak radiated power and instability growth time are comparable to their averages.

The particle energy spread leads to a sharp decrease of the peak radiated power. The influence of the energy spread grows with growing electron current.

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