Isospectral deformation of the reduced quasi-classical self-dual Yang–Mills equation

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Abstract

We derive new four-dimensional partial differential equation with the isospectral Lax representation by shrinking the symmetry algebra of the reduced quasi-classical self-dual Yang–Mills equation. Then we find a recursion operator for the obtained equation and construct Bäcklund transformations between this equation and the reduced quasi-classical self-dual Yang–Mills equation as well as the four-dimensional Martínez Alonso–Shabat equation.

Keywords: integrable partial differential equation, Lax representation, symmetry algebra, extension of Lie algebra

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1. Introduction

Integrable partial differential equations (PDEs) play an important role in modern physics and mathematics. While there are many non-equivalent definitions of integrability (see discussion in [48, 31]), the most universal perspective seems to be one based on the construction of Lax representations, which is the starting point for a number of powerful techniques for studying PDEs such as inverse scattering transformations for solitonic equations, Bäcklund transformations, recursion operators, nonlocal symmetries and conservation laws, Darboux transformations, see [46, 47, 37, 11, 1, 30, 40, 3] and references therein. The systematic and convenient framework for dealing with
these nonlocal geometric structures is provided by the theory of differential coverings introduced by A.M. Vinogradov in [44] and then elaborated in [20, 21], see also [16, 45, 19].

From both theoretical and practical viewpoints, of the most significance are PDEs that admit isospectral Lax representations, in other words, differential coverings with non-removable (spectral) parameter. Such equations are quite rare, especially in the multi-dimensional case, i.e., when the number of independent variables is greater than 2. For examples of four-dimensional PDEs with isospectral Lax representations see [6, 4, 7, 10, 32, 36, 2, 42, 5] and references therein.

The challenging unsolved problem in the theory of integrable equations is to find conditions that are formulated in inherent terms of a PDE under study and ensure existence of a Lax representation. Recently, an approach to this problem has been proposed in [33, 34, 35], where we show that for some PDEs their Lax representations can be inferred from the second twisted cohomology group of the contact symmetry algebras. In particular, paper [35] contains examples exhibiting that for some multi-dimensional PDEs their isospectral Lax representations are related to the distinguished structure of the symmetry algebras. Namely, in these examples the symmetry algebra of the PDE is of the form \( a_\circ \ltimes (\mathbb{R}_n[h] \otimes a_\infty) \), where \( a_\circ \) is a finite-dimensional Lie algebra with nontrivial second twisted cohomology group, \( a_\infty \) is an infinite-dimensional Lie algebra, and \( \mathbb{R}_n[h] = \mathbb{R}[h]/(h^{n+1} = 0) \) is the (commutative associative unital) algebra of truncated polynomials of the formal variable \( h \). Moreover, in [35] we show that the series of extensions of the symmetry algebra generated by maps \( \mathbb{R}_n[h] \mapsto \mathbb{R}_{n+1}[h] \mapsto \mathbb{R}_{n+2}[h] \mapsto \ldots \) produces the integrable hierarchy associated with the PDE under the study. It is natural to ask what happens when one shrinks the symmetry algebra of this type via replacing \( \mathbb{R}_n[h] \) by \( \mathbb{R}_{n-1}[h] \). This idea was exploited in [22], where for some four-dimensional integrable Monge–Ampère equations of Hirota type we have derived a number of their "symmetric deformations", that is, equations whose symmetry algebras are constructed by omitting some graded components of the symmetry algebra of the initial equation. The obtained equations turn out to admit Lax representations of both isospectral and non-isospectral types.

In the present paper we combine the techniques of [22] and [33, 34, 35] and consider the reduced quasi-classical self-dual Yang–Mills equation

\[
uyz = utx + uy ux - u_x u_{xy},
\]  

(1)
which admits the isospectral Lax representation

\[
\begin{align*}
    s_t &= \lambda s_y - u_y s_x, \\
    s_z &= (\lambda - u_x) s_x,
\end{align*}
\]

derived in [6]. In [35] we show that this Lax representation can be inferred from the structure of the Lie algebra of contact symmetries of this equation. This algebra is isomorphic to the semi-direct product \( q_3 = q_\phi \ltimes q_{3,\infty} \) of the 6-dimensional Lie algebra \( q_\phi \) and the infinite-dimensional ideal \( q_{3,\infty} \), which is the tensor product \( \mathbb{R}_2[h] \otimes w[t, z] \) of the \((\text{associative commutative unital})\) algebra of truncated polynomials \( \mathbb{R}_2[h] = \mathbb{R}[h]/(h^3 = 0) \) and the Lie algebra \( w[t, z] = \langle t^iz^j \partial_z \mid i, j \in \mathbb{N}_0 \rangle \). The subalgebra \( q_\phi \) has one-dimensional second twisted cohomology group, and the non-trivial 2-cocycle generates an extension \( \hat{q}_3 \) for \( q_3 \). The Maurer–Cartan (mc) forms of this extension provide the Wahlquist–Estabrook form for the Lax representation (2).

In this paper we construct the PDE that is defined by the shrunk Lie algebra \( q_2 = q_\phi \ltimes (\mathbb{R}_2[h] \otimes w[t, z]) \). This Lie algebra admits the extension \( \hat{q}_2 \) generated by non-trivial twisted 2-cocycle of \( q_\phi \). We show that the mc forms of \( \hat{q}_2 \) provide Lax representations that define equation

\[
    u_{ty} = u_y u_{xz} - u_z u_{xy} + y u_{yz}.
\]

This equation, to the best of our knowledge, has not yet appeared in the literature. We show that one of the the Lax representations of (3) contains non-removable spectral parameter. This allowed us to find a recursion operator for symmetries of (3).

By the construction, equation (3) can be considered as an integrable deformation of equation (1). Notice that (3) is not invariant with respect to the translation \( y \mapsto y + \epsilon \), while (1) admits this transformation as a symmetry. Likewise, equation (3) can be considered as a deformation of the four-dimensional Martínez Alonso–Shabat equation [25, 26, 32]

\[
    u_{ty} = u_y u_{xz} - u_z u_{xy}.
\]

This equation is related to equation (1) by the Bäcklund transformation [22]. We construct a Bäcklund transformation between equations (3) and (1) and thus we show that equations (3) and (1) are related by a Bäcklund transformation. Notice that the contact symmetry algebras of equations (1), (3), and (1) are pairwise non-isomorphic, therefore these equations are pairwise non-equivalent with respect to the pseudogroup of contact transformations.

\[ \text{(3)} \]
2. Preliminaries

2.1. Symmetries and differential coverings

The presentation in this subsection closely follows [17, 18], see also [20, 21, 45]. Let \( \pi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), \( \pi: (x^1, \ldots, x^n, u^1, \ldots, u^m) \mapsto (x^1, \ldots, x^n) \), be a trivial bundle, and \( J^\infty(\pi) \) be the bundle of its jets of the infinite order. The local coordinates on \( J^\infty(\pi) \) are \((x_i, u^\alpha, u^\alpha_I)\), where \( I = (i_1, \ldots, i_n) \) are multi-indices, and for every local section \( f: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m \) of \( \pi \) the corresponding infinite jet \( j_\infty(f) \) is a section \( j_\infty(f): \mathbb{R}^n \to J^\infty(\pi) \) such that \( u^\alpha_I(j_\infty(f)) = \frac{\partial f^\alpha}{\partial x^I} = \frac{\partial}{\partial x^I} f^\alpha_i = \sum_{\#I \geq 0} m \sum_{\alpha=1} u^\alpha_{i+1k} \frac{\partial}{\partial u^\alpha I} \). We put \( u^\alpha = u^\alpha_{(0,\ldots,0)} \). Also, we will simplify notation in the following way, e.g., in the case of \( n = 4, m = 1 \): we denote \( x^1 = t, x^2 = x, x^3 = y, x^4 = z \) and \( u^1_{(i,j,k,l)} = u_{txxyyz...} \) with \( i \) times \( t \), \( j \) times \( x \), \( k \) times \( y \), and \( l \) times \( z \).

The vector fields

\[
D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I \geq 0} \sum_{\alpha=1}^m u^\alpha_{i+1k} \frac{\partial}{\partial u^\alpha I}, \quad k \in \{1, \ldots, n\},
\]

(\( i_1, \ldots, i_k, \ldots, i_n \) + 1\( k = (i_1, \ldots, i_k + 1, \ldots, i_n) \)), are called total derivatives. They commute everywhere on \( J^\infty(\pi) \).

The evolutionary vector field associated to an arbitrary vector-valued smooth function \( \varphi: J^\infty(\pi) \to \mathbb{R}^m \) is the vector field

\[
E_\varphi = \sum_{\#I \geq 0} \sum_{\alpha=1}^m D_I(\varphi^\alpha) \frac{\partial}{\partial u^\alpha I}
\]

with \( D_I = D_{(i_1, \ldots, i_n)} = D^i_{j_1} \circ \ldots \circ D^i_{j_n} \).

A system of PDEs \( F_r(x^i, u^\alpha_I) = 0 \) of the order \( s \geq 1 \) with \( \#I \leq s \), \( r \in \{1, \ldots, R\} \) for some \( R \geq 1 \), defines the submanifold \( \mathcal{E} = \{ (x^i, u^\alpha_I) \in J^\infty(\pi) \mid D_K(F_r(x^i, u^\alpha_I)) = 0, \#K \geq 0 \} \) in \( J^\infty(\pi) \).

A function \( \varphi: J^\infty(\pi) \to \mathbb{R}^m \) is called a (generator of an infinitesimal) symmetry of equation \( \mathcal{E} \) when \( E_\varphi(F) = 0 \) on \( \mathcal{E} \). The symmetry \( \varphi \) is a solution to the defining system

\[
\ell_\mathcal{E}(\varphi) = 0,
\]

where \( \ell_\mathcal{E} = \ell_F|_\mathcal{E} \) with the matrix differential operator

\[
\ell_F = \left( \sum_{\#I \geq 0} \frac{\partial F_r}{\partial u^\alpha I} D_I \right).
\]
The symmetry algebra $\text{Sym}(\mathcal{E})$ of equation $\mathcal{E}$ is the linear space of solutions to $\mathcal{E}$ endowed with the structure of a Lie algebra over $\mathbb{R}$ by the Jacobi bracket $\{\varphi, \psi\} = \mathbf{E}_\varphi(\psi) - \mathbf{E}_\psi(\varphi)$. The algebra of contact symmetries $\text{Sym}_0(\mathcal{E})$ is the Lie subalgebra of $\text{Sym}(\mathcal{E})$ defined as $\text{Sym}(\mathcal{E}) \cap C^\infty(J^1(\pi))$.

Consider $W = \mathbb{R}^\infty$ with coordinates $w^s$, $s \in \mathbb{N}_0$. A differential covering of $\mathcal{E}$ locally is a trivial bundle $\tau: J^\infty(\pi) \times W \to J^\infty(\pi)$ equipped with extended total derivatives

$$\tilde{D}_{x^k} = D_{x^k} + \sum_{s=0}^\infty T_s^k(x^i, u_1^\alpha, w^j) \frac{\partial}{\partial w^s}$$

such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$ for all $i \neq j$ whenever $(x^i, u_1^\alpha) \in \mathcal{E}$. Define the partial derivatives of $w^s$ by $w^s_{x^k} = \tilde{D}_{x^k}(w^s)$. This yields the system of covering equations

$$w^s_{x^k} = T_s^k(x^i, u_1^\alpha, w^j)$$

that is compatible whenever $(x^i, u_1^\alpha) \in \mathcal{E}$. Dually, the differential covering is defined by the Wahlquist–Estabrook forms

$$dw^s - \sum_{k=1}^m T_s^k(x^i, u_1^\alpha, w^j) \, dx^k$$

as follows: when $w^s$ and $u_1^\alpha$ are considered to be functions of $x^1, \ldots, x^n$, forms (7) are equal to zero whenever system (6) holds.

2.2. Twisted cohomology of Lie algebras

For a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, its representation $\rho: \mathfrak{g} \to \text{End}(V)$, and $k \geq 1$ let $C^k(\mathfrak{g}, V) = \text{Hom}(\Lambda^k(\mathfrak{g}), V)$ be the space of all $k$–linear skew-symmetric mappings from $\mathfrak{g}$ to $V$. Then the Chevalley–Eilenberg differential complex

$$V = C^0(\mathfrak{g}, V) \xrightarrow{d} C^1(\mathfrak{g}, V) \xrightarrow{d} \ldots \xrightarrow{d} C^k(\mathfrak{g}, V) \xrightarrow{d} C^{k+1}(\mathfrak{g}, V) \xrightarrow{d} \ldots$$

is generated by the differential $d: \theta \mapsto d\theta$ such that

$$d\theta(X_1, \ldots, X_{k+1}) = \sum_{q=1}^{k+1} (-1)^{q+1} \rho(X_q) (\theta(X_1, \ldots, \hat{X}_q, \ldots, X_{k+1}))$$

$$+ \sum_{1 \leq p < q \leq k+1} (-1)^{p+q+1} \theta([X_p, X_q], X_1, \ldots, \hat{X}_p, \ldots, \hat{X}_q, \ldots, X_{k+1}).$$
The cohomology groups of the complex \((C^*(g, V), d)\) are referred to as the cohomology groups of the Lie algebra \(g\) with coefficients in the representation \(\rho\). For the trivial representation \(\rho_0: g \to \mathbb{R}, \rho_0: X \mapsto 0\), the cohomology groups are denoted by \(H^*(g)\).

Consider a Lie algebra \(g\) over \(\mathbb{R}\) with non-trivial first cohomology group \(H^1(g)\) and take a closed 1-form \(\alpha\) on \(g\) such that \([\alpha] \neq 0\). Then for any \(c \in \mathbb{R}\) define new differential \(d_c: C^k(g, \mathbb{R}) \to C^{k+1}(g, \mathbb{R})\) by the formula

\[
d_c \theta = d\theta - c \alpha \wedge \theta.
\]

From \(d\alpha = 0\) it follows that \(d^2 = 0\). The cohomology groups of the complex

\[
C^1(g, \mathbb{R}) \xrightarrow{d_{c_0}} \ldots \xrightarrow{d_{c_0}} C^k(g, \mathbb{R}) \xrightarrow{d_{c_0}} C^{k+1}(g, \mathbb{R}) \xrightarrow{d_{c_0}} \ldots
\]

are referred to as the twisted cohomology groups \([38, 39]\) of \(g\) and denoted by \(H^*_{c_0}(g)\).

3. Lie algebra \(q_2\), its extension, and Lax representations for equation \((3)\)

As it was shown in \([35]\), the structure equations for the Lie algebra of contact symmetries of equation \((1)\) have the form

\[
\begin{cases}
d\alpha &= 0, \\
DB &= \nabla_1(B) \wedge B, \\
D\Gamma &= \alpha \wedge \Gamma + \nabla_1(\Gamma) \wedge B + \frac{1}{2} \nabla_1(B) \wedge \Gamma, \\
D\Theta &= \nabla_2(\Theta) \wedge \Theta + h_0 \nabla_0(\Theta) \wedge \left( \frac{1}{2} \nabla_1(B) + h_0 \nabla_1(\Gamma) - \alpha \right) \\
&\quad + \nabla_1(\Theta) \wedge (B + h_0 \Gamma),
\end{cases}
\]

where

\[
B = \beta_0 + h_1 \beta_1 + \frac{1}{2} h_2^2 \beta_2, \quad \Gamma = \gamma_0 + h_1 \gamma_1,
\]

\[
\Theta = \sum_{k=0}^{2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{h_0^k h_1^i h_2^j}{i!j!} \theta_{k,i,j},
\]

and \(\alpha, \beta_i, i \in \{0, 1, 2\}, \gamma_l, l \in \{0, 1\}, \theta_{k,i,j}, k \in \{0, 1, 2\}, i, j \in \mathbb{N}_0\), are the MC forms of the Lie algebra \(q_3\), while \(h_0, h_1, h_2\) are formal parameters such that \(dh_i = 0\) and \(h_0^k = 0\) for \(k > 2\).
Now we shrink the Lie algebra $\mathfrak{q}_3$ by imposing the condition $h_0^k = 0$ for $k > 1$. Thus we have

$$\Theta = \sum_{k=0}^{1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{h_0^k h_1^i h_2^j}{i! j!} \theta_{k,i,j}$$

instead of (9) and

$$\begin{cases}
  d\alpha &= 0, \\
  dB &= \nabla_1(B) \wedge B, \\
  d\Gamma &= \alpha \wedge \Gamma + \nabla_1(\Gamma) \wedge B + \frac{1}{2} \nabla_1(B) \wedge \Gamma, \\
  d\Theta &= \nabla_2(\Theta) \wedge \Theta + h_0 \nabla_0(\Theta) \wedge \left( \frac{1}{2} \nabla_1(B) - \alpha \right) + \nabla_1(\Theta) \wedge (B + h_0 \Gamma)
\end{cases}$$

instead of (8).

System (11) implies that $H^1(\mathfrak{q}_2) = \langle \alpha \rangle$,

$$H^2_{c\alpha}(\mathfrak{q}_3) = \left\{ \begin{array}{ll}
  \langle [\gamma_0 \wedge \gamma_1] \rangle, & c = 2, \\
  \{[0]\}, & c \neq 2,
\end{array} \right.$$ 

and $H^2_{2\alpha}(\mathfrak{q}_3) \subseteq H^2_{2\alpha}(\mathfrak{q}_2)$. Equation

$$d\sigma = 2\alpha \wedge \sigma + \gamma_0 \wedge \gamma_1$$

with unknown 1-form $\sigma$ is compatible with the structure equations (11). System (11), (12) defines the structure equations for the extension $\hat{\mathfrak{q}}_2$ of the Lie algebra $\mathfrak{q}_2$.

Frobenius’ theorem allows one to integrate equations (11), (12) step by step. In particular, we have

$$\alpha = \frac{dq}{q}, \quad \beta_0 = a_0^2 \, dt, \quad \beta_1 = 2 \frac{da_0}{a_0} + a_1 \, dt, \quad \beta_2 = \frac{1}{a_0^2} \left( da_1 + \frac{1}{2} a_1^2 \, dt \right),$$

$$\gamma_0 = a_0 \, q \left( dz + y \, dt \right), \quad \gamma_1 = \frac{q}{a_0} \left( dy + \frac{1}{2} a_1 \left( dz + y \, dt \right) \right),$$

$$\sigma = q^2 \left( dv - y \, dz - \frac{1}{2} y^2 \, dt \right), \quad \theta_{0,0,0} = b_0 \, dt + b_1 \, dx,$$

$$\theta_{1,0,0} = \frac{b_1}{a_0} \left( du + b_2 \, dt + b_3 \, dx + b_0 b_1^{-1} \, dz \right),$$
where \( t, x, y, z, u, q \neq 0, a_0 \neq 0, a_1, b_0 \neq 0, b_1 \neq 0, b_2, \) and \( b_3 \) are free parameters (‘constants of integration’). We do not need explicit expressions for the other MC forms in what follows.

Now we impose the condition for form \( \theta_{1,0,0} - \gamma_0 \) to be a multiple of the contact form \( du - u_t dt - u_x dx - u_y dy - u_z dz \) on the bundle of jets of sections of the bundle \( \pi: \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4 \), \( \pi: (t, x, y, z, u) \mapsto (t, x, y, z) \), that is, we require

\[
\theta_{1,0,0} - \gamma_0 = \frac{q b_1}{a_0} (du - u_t dt - u_x dx - u_y dy - u_z dz).
\]

To achieve this, we rename the integration parameters as \( b_0 = -u_z u_y^{-1} + \frac{1}{2} a_1, \)
\( b_1 = u_y^{-1}, \)
\( b_2 = -u_t + \frac{1}{2} a_1 y u_y, \)
and \( b_3 = -u_x. \)

Then we consider form

\[
\tau = \sigma + c_1 \beta_0 + c_2 \gamma_0 + c_3 \gamma_1 + c_4 \theta_{0,0,0}
\]

\[
= q^2 \left( dv + K dt + \frac{c_4}{q^2 u_y} dx + \frac{c_3}{a_0 q} dy + \frac{c_3 a_1 + 2 a_0 (c - 2 a_0 - q y)}{a_0 q} dz \right),
\]
where \( K = -2 c_1 a_0 u_x + 2 c_1 a_0^3 + 2 c - 2 a_0^2 y q + c - 3 a_1 q y + c_4 a_0 a_1 - a_0 q^2 y^2 \) and \( c_1, \ldots, c_4 \in \mathbb{R} \) are constants. Without loss of generality we put \( c_3 = c_4 = -1. \)

Then we rename the integration parameters as

\[
a_0 = -\frac{w}{u_y^{1/2} v_x^{1/2}}, \quad a_1 = -\frac{2 w (v_z + y - c_2 w)}{u_y v_x}, \quad q = \frac{1}{u_y^{1/2} v_x^{1/2}}
\]

and obtain

\[
\tau = u_y v_x \left( dv - (u_z v_x + (w + y) v_z - (c_2 + c_1) w^2 + y w + \frac{1}{2} y^2) dt 
- v_x dx - v_z dz + u_y v_x w^{-1} dy \right).
\]

This is the restriction of the multiple of the contact form \( dv - v_t dt - v_x dx - v_y dy - v_z dz \) from the bundle of jets of sections of the bundle \( \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^4, (t, x, y, z, u, v) \mapsto (t, x, y, z) \), to the submanifold defined by the over-determined system

\[
\begin{align*}
v_t &= u_z v_x + (w + y) v_z - (c_2 + c_1) w^2 + y w + \frac{1}{2} y^2, \\
v_y &= -u_y v_x w^{-1}.
\end{align*}
\]
The compatibility condition of this system reads \((v_t)_y = (v_y)_t\). This equation entails \(c_2 + c_1 = -\frac{1}{2}\) and the following system for \(w\):

\[
\begin{align*}
(w_t &= u_z w_x + (w + y) w_z + E w u_y^{-1}, \\
(w_y &= -u_y w_x w^{-1} + 1,
\end{align*}
\]

(14)

where \(E = u t_y - u_z u x_y + u_y u x_z - y u y z\). The compatibility condition \((w_t)_y = (w_y)_t\) of system (14) gives new over-determined system

\[
\begin{align*}
(E_x &= (u x_y + 1) u^{-1} E, \\
E_y &= u y y u^{-1} E,
\end{align*}
\]

and the condition \((E_x)_y = (E_y)_x\) yields \(E = 0\), that is, (3). Substituting this into (14) and renaming \(w = p - y\) gives the Lax representation

\[
\begin{align*}
p_t &= u_z p_x + p p_z, \\
p_y &= \frac{-u_y p_x}{p - y}
\end{align*}
\]

(15)

for equation (3). Notice that \(p = \lambda = \text{const}\) is a solution to (15). Substituting for \(w = \lambda - y\), \(c_1 + c_2 = -\frac{1}{2}\) and renaming \(v = r + \frac{1}{2} \lambda^2 t\) produces another Lax representation

\[
\begin{align*}
r_t &= u_z r_x + \lambda r_z, \\
r_y &= \frac{u_y r_x}{y - \lambda}
\end{align*}
\]

(16)

for (3).

**Remark 1.** Systems (15) and (16) are not contact equivalent, instead, they are related in the following sense, c.f. [41, 13]: suppose that function \(P(t, x, y, z, s)\) defines function \(R(t, x, y, z)\) implicitly by equation

\[P(t, x, y, z, R(t, x, y, z)) \equiv \lambda.\]

Then \(R\) is a solution to (16) iff \(P\) is a solution to (15). ⋄

**Remark 2.** Parameter \(\lambda\) in the Lax representation (16) is non-removable, that is, there is no change of variable \(r \mapsto \tilde{r} = f(t, x, y, z, r)\) that eliminates \(\lambda\). In accordance with [21, §§ 3.2, 3.4], [12, 8, 28, 9], to ensure this claim it is sufficient to notice that symmetry \(\phi = u_y - t u_z\) of equation (3) has
no lift to a symmetry of system (16), while for the associated vector field
\[ V = -\partial_y + t \partial_z \]
there holds
\[
e^{\lambda V} \left( dr - u_z r_x dt - r_x dx - \frac{uyr_x}{y} dy - r_z dz \right)
= dr - (u_z r_x + \lambda r_z) dt - r_x dx - \frac{uyr_x}{y - \lambda} dy - r_z dz.
\]

4. Recursion operator for symmetries

To construct a recursion operator for equation (3) we use the consider-
ations based on ideas of [43], cf. [14, 15, 28, 24, 29, 36, 22] also. We find
shadows for (3) in the covering defined by system (16). One of the shadows
is
\[ s = r_x - y. \]
Differentiating (16) by \( x \) and substituting for \( r_x = s^{-1} \) yields
another Lax representation
\[
\begin{align*}
  s_t &= u_z s_x + \lambda s_z - u_{xz} s, \\
  s_y &= \frac{uy s_x - u_{xy} s}{y - \lambda}
\end{align*}
\] (17)
for (3). Note that \( s \) is a solution to the linearization
\[
\tilde{D}_t \tilde{D}_y (\phi) = u_z \tilde{D}_x \tilde{D}_y (\phi) + u_{xy} \tilde{D}_z (\phi) - u_y \tilde{D}_x \tilde{D}_z (\phi) - u_{xz} \tilde{D}_y (\phi)
+ y \tilde{D}_y \tilde{D}_z (\phi)
\] (18)
of (3) with the extended total derivatives \( \tilde{D}_t, \ldots, \tilde{D}_z \) from the covering
defined by (16). Now we put
\[
s = \sum_{n=-\infty}^{\infty} \lambda^n s_n.
\] (19)
Since equation (18) is independent of \( \lambda \), each \( s_n \) is a solution to this equation
as well. Substitution for (19) into (17) yields
\[
\begin{align*}
  \tilde{D}_t (s_{n+1}) &= u_z \tilde{D}_x (s_{n+1}) + \tilde{D}_z (s_n) - u_{xz} s_{n+1}, \\
  y \tilde{D}_y (s_{n+1}) &= u_y \tilde{D}_x (s_{n+1}) + \tilde{D}_y (s_n) - u_{xy} s_{n+1}
\end{align*}
\]
for each \( n \in \mathbb{Z} \). Fixing \( n \) and renaming \( s_{n+1} = \psi, s_n = \phi \), we get the recursion operators \( \psi = \mathcal{R}(\phi) \) and \( \phi = \mathcal{R}^{-1}(\psi) \) defined by systems

\[
\begin{cases}
\widetilde{D}_t(\psi) &= u_z \widetilde{D}_x(\psi) - u_{xz} \psi + \widetilde{D}_z(\phi), \\
\widetilde{D}_y(\psi) &= \frac{1}{y} \left( u_y \widetilde{D}_x(\psi) - u_{xy} \psi + \widetilde{D}_y(\phi) \right)
\end{cases}
\]  

(20)

and

\[
\begin{cases}
\widetilde{D}_y(\phi) &= y \widetilde{D}_y(\psi) + u_y \widetilde{D}_x(\psi) - u_{xy} \psi, \\
\widetilde{D}_z(\phi) &= \widetilde{D}_t(\psi) + u_z \widetilde{D}_x(\psi) - u_{xy} \psi,
\end{cases}
\]  

(21)

respectively. Direct computations show that systems (20) and (21) are compatible iff \( \phi \) and \( \psi \) are shadows of symmetries of equation (3) in the covering (16).

5. Bäcklund transformations

Consider equation (3) written in variables \( \tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{u} \),

\[
\tilde{u}_{t\tilde{y}} = \tilde{u}_{\tilde{y}z} \tilde{u}_{\tilde{x}\tilde{z}} - \tilde{u}_{z} \tilde{u}_{t\tilde{y}} + \tilde{y} \tilde{u}_{t\tilde{y}z}, \quad (22)
\]

and make the following point transformation:

\[
\tilde{t} = t, \quad \tilde{x} = z, \quad \tilde{y} = q, \quad \tilde{z} = y, \quad \tilde{u} = -x. \quad (23)
\]

The second prolongation of this transformation maps equation (22) to equation

\[
q_{yz} = q_{tx} + \frac{q q_y - q_t}{q_x} q_{xx} + \frac{q_z - q q_x}{q_x} q_{xy}. \quad (24)
\]

This equation is related to the reduced quasi-classical self-dual Yang–Mills equation (1) by a Bäcklund transformation

\[
\begin{cases}
q_t &= q q_y - u_y q_x, \\
q_z &= (q - u_x) q_x,
\end{cases}
\]  

(25)

Indeed, compatibility condition \((q_t)_z = (q_z)_t\) of system (25) yields (1), while from (25) we have

\[
\begin{cases}
u_x &= q - q z q_x^{-1}, \\
u_y &= (q_t - q q_y) q_x^{-1},
\end{cases}
\]  

(26)
and then \((u_x)_y = (u_y)_x\) gives equation (24). Thus the superposition of transformations (23) and (25) provides the Bäcklund transformation between equations (3) and (1).

As we have shown in [23], equation (11) is related by a Bäcklund transformation to equation (4). Therefore equations (3) and (4) are related by a Bäcklund transformation as well. To write this transformation explicitly we substitute (26) into (2). The resulting system

\[
\begin{align*}
s_t &= \lambda s_y - \frac{q_t - q_y}{q_x} s_x, \\
s_z &= \left(\lambda - \frac{q q_x - q_z}{q_x}\right) s_x
\end{align*}
\]

(27)

defines a Bäcklund transformation between equation (24) and

\[
s_{yz} = s_{tx} + \frac{\lambda s_y - s_t}{s_x} s_{xx} + \frac{s_z - \lambda s_x}{s_x} s_{xy}.
\]

(28)

Then the point transformation

\[
t = \tilde{t}, \quad x = -\tilde{u}, \quad y = \tilde{z} - \lambda \tilde{t}, \quad z = \tilde{x}, \quad s = \tilde{y}
\]

(29)

maps equation (28) to equation (4) written as

\[
\dot{u} \hat{t} \hat{g} \hat{u} \hat{x} \hat{z} - \dot{u} \hat{z} \hat{u} \hat{x} \hat{g}.
\]

Thus the superposition of transformations (23), (27), and (29) maps equation (3) to (4).

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References

[1] M.J. Ablowitz, P.A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, Cambridge, 1991

[2] D. M. J. Calderbank, B. Kruglikov. Integrability via geometry: dispersionless differential equations in three and four dimensions. \[\text{arXiv:1612.02753}\]

[3] A. Coley, D. Levi, R. Milson, C. Rogers, P. Winternitz (eds). *Bäcklund and Darboux Transformations. The Geometry of Solitons*. CRM Proceedings and Lecture Notes, 28, AMS, Providence, 2001

[4] B. Doubrov, E.V. Ferapontov. On the integrability of symplectic Monge–Ampère equations. J. Geom. Phys., 60 (2010), 1604–1616

[5] B. Doubrov, E.V. Ferapontov, B. Kruglikov, V.S. Novikov. On a class of integrable systems of Monge–Ampère type. J. Math. Phys., 58 (2017), 063508

[6] Ferapontov E.V., Khusnutdinova K.R. Hydrodynamic reductions of multi-dimensional dispersionless PDEs: the test for integrability. J. Math. Phys., 45 (2004), 2365–2377

[7] E.V. Ferapontov, K.R. Khusnutdinova, C. Klein. On linear degeneracy of integrable quasilinear systems in higher dimensions. Lett. Math. Phys., 96 (2011), 5–35

[8] S. Igonin, J. Krasil’shchik. On one-parametric families of Bäcklund transformations. In: T. Morimoto, H. Sato, K. Yamaguchi (eds.), *Lie Groups, Geometric Structures and Differential Equations — One Hundred Years After Sophus Lie*. Advanced Studies in Pure Mathematics, 37, pp. 99–114. Math. Soc. Japan, Tokyo, 2002

[9] S. Igonin, P. Kersten, I. Krasil’shchik. On one-parametric families of Bäcklund transformations. Preprint \[\text{arXiv:nlin/0010040}\] (2000)

[10] S. Igonin, M. Marvan. On construction of symmetries and recursion operators from zero- curvature representations and the Darboux–Egoroff system. J. Geom. Phys., 85 (2014), 106–123
[11] B.G. Konopelchenko. *Nonlinear Integrable Equations. Lecture Notes in Physics, 270*, Springer, 1987

[12] I.S. Krasil’shchik. On one-parametric families of Bäcklund transformations. Preprint DIPS-1/2000, The Diffiety Institute, Pereslavl-Zalessky (2000)

[13] I. Krasil’shchik. Integrability in differential coverings. J. Geom. Phys. 87 (2015), 296–304

[14] I. S. Krasil’shchik, P. H. M. Kersten. Deformations and recursion operators for evolution equations. in *Geometry in Partial Differential Equations*, World Scientific, River Edge, 1994, pp. 114–154

[15] I. S. Krasil’shchik, P. H. M. Kersten. Graded differential operators and their deformations: A computational theory for recursion operators. Acta Appl. Math. 41 (1995), 167–191

[16] I.S. Krasil’shchik, V.V. Lychagin, A.M. Vinogradov. *Geometry of Jet Spaces and Nonlinear Differential Equations*. Gordon and Breach, N.Y., 1986

[17] J. Krasil’shchik, A. Verbovetsky. Geometry of jet spaces and integrable systems. J. Geom. Phys. 61 (2011), 1633–1674

[18] J. Krasil’shchik, A. Verbovetsky, R. Vitolo. A unified approach to computation of integrable structures. Acta Appl. Math. 120 (2012), 199–218

[19] J. Krasil’shchik, A. Verbovetsky, R. Vitolo. *The Symbolic Computation of Integrability Structures for Partial Differential Equations*. Springer 2017

[20] I.S. Krasil’shchik, A.M. Vinogradov. Nonlocal symmetries and the theory of coverings. Acta Appl. Math. 2 (1984), 79–86

[21] I.S. Krasil’shchik, A.M. Vinogradov. Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations. Acta Appl. Math. 15 (1989), 161–209

[22] B.S. Kruglikov, O.I. Morozov. Integrable dispersionless PDEs in 4D, their symmetry pseudogroups and deformations. Lett. Math. Phys. 105 (2015), 1703–1723
[23] B.S. Kruglikov, O.I. Morozov. A Bäcklund transformation between the four-dimensional Martínez Alonso – Shabat and Ferapontov–Khusnutdinova equations. Theor. Math. Phys. 188 (3) (2016), 1358–1360

[24] A.A. Malykh, Y. Nutku, M.B. Sheftel. Partner symmetries and non-invariant solutions of 4-dimensional heavenly equations. J. Phys. A 37 (2004), 7527–7546

[25] L. Martínez Alonso, A.B. Shabat. Energy-dependent potentials revisited: A universal hierarchy of hydrodynamic type. Phys. Lett. A 299 (2002), 359–365

[26] L. Martínez Alonso, A.B. Shabat. Hydrodynamic reductions and solutions of a universal hierarchy. Theor. Math. Phys. 140 (2004), 1073–1085

[27] M. Marvan. Another look on recursion operators. in: Differential Geometry and Applications. (Brno 28 August – 1 September 1995, J. Janyška, I. Kolář, and J. Slovák, eds.), Masaryk Univ., Brno (1996), pp. 393–402

[28] M. Marvan. On the horizontal gauge cohomology and nonremovability of the spectral parameter. Acta Appl. Math. 72 (2002), 51–65

[29] M. Marvan, A. Sergyeyev. Recursion operators for dispersionless integrable systems in any dimension. Inverse Problems 28 (2) (2012), 025011

[30] V.B. Matveev, M.A. Salle. Darboux Transformations and Solitons. Springer, 1991

[31] A. Mikhailov (ed.) Integrability. Lecture Notes in Physics 767, Springer: Berlin, 2009

[32] O.I. Morozov. The four-dimensional Martínez Alonso–Shabat equation: Differential coverings and recursion operators J. Geom. Phys. 85 (2014), 75 – 80

[33] O.I. Morozov. Deformed cohomologies of symmetry pseudo-groups and coverings of differential equations. J. Geom. Phys. 113 (2017), 215–225
[34] O.I. Morozov. Deformations of infinite-dimensional Lie algebras, exotic cohomology, and integrable nonlinear partial differential equations. J. Geom. Phys. 128 (2018), 20–31

[35] O.I. Morozov. Lax representations with non-removable parameters and integrable hierarchies of PDEs via exotic cohomology of symmetry algebras. J. Geom. Phys. 143 (2019), 150–163

[36] O.I. Morozov, A. Sergiyevev. The four-dimensional Martínez Alonso–Shabat equation: reductions and nonlocal symmetries. J. Geom. Phys. 85 (2014), 40–45

[37] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov. Theory of Solitons. Plenum Press, N.Y., 1984

[38] S.P. Novikov. On exotic De-Rham cohomology. Perturbation theory as a spectral sequence. arXiv:math-ph/0201019 2002

[39] S.P. Novikov. On metric-independent exotic homology. Proc. Steklov Inst. Math. 251 (2005), 206–212

[40] P.J. Olver. Applications of Lie Groups to Differential Equations. 2nd Edition, Springer, 1993

[41] M.V. Pavlov, J.-H. Chang, Y.-T. Chen. Integrability of the Manakov–Santini hierarchy. arXiv: 0910.2400

[42] M.V. Pavlov, N. Stoilov. Three dimensional reductions of four-dimensional quasilinear systems. J. Math. Phys. 58 (2017), 111510

[43] A. Sergiyevev. A simple construction of recursion operators for multidimensional dispersionless integrable systems. J. Math. Anal. Appl. 454 (2017), 468–480

[44] A.M. Vinogradov. Category of partial differential equations, in: Lecture Notes in Mathematics, 1108 (1984), Springer-Verlag, Berlin, 77–102

[45] A.M. Vinogradov, I.S. Krasil’shchik (eds.) Symmetries and Conservation Laws for Differential Equations of Mathematical Physics [in Russian], Moscow: Factorial, 2005; English transl. prev. ed.: I.S. Krasil’shchik,
A.M. Vinogradov (eds.) *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*. Transl. Math. Monogr., 182, Amer. Math. Soc., Providence, RI, 1999

[46] H.D. Wahlquist, F.B. Estabrook F.B. Prolongation structures of nonlinear evolution equations. J. Math. Phys., 16 (1975), 1–7

[47] V.E. Zakharov. Integrable systems in multidimensional spaces. Lect. Notes Phys., 153 (1982), 190–216

[48] V. E. Zakharov (ed.) *What is integrability?* Springer-Verlag, Berlin, 1991