We consider a dilute gas of granular material inside a box, kept in a stationary state by shaking. A wall separates the box into two identical compartments, save for a small hole at some finite height $h$. As the gas is cooled, a second order phase transition occurs, in which the particles preferentially occupy one side of the box. We develop a quantitative theory of this clustering phenomenon and find good agreement with numerical simulations.

One of the most outstanding features of a gas of granular material is its tendency to spontaneously form highly concentrated regions or clusters [1–3]. So even in its gaseous state it behaves fundamentally different from a molecular gas, which keeps its uniform density. Apart from throwing light on the nonequilibrium properties of a granular gas, understanding this clustering instability is of major technological importance. Imagine a flow of rocks down a chute: whenever a very dense region has formed due to the instability, the rocks may easily get entangled and the flow is stuck.

The clustering instability is caused by the distinguishing feature of a granular gas in contrast to a molecular gas, namely that a fraction of the kinetic energy is irretrievably lost upon a collision, and simply heats the particles. In this paper, we will concentrate on the simplest case of a dilute gas for which collisions are binary, so the rate of energy loss grows quadratically with the density. This means a dense region rapidly cools, increasing the density even more according to the equation of state. Goldhirsch and Zanetti [2] used a hydrodynamic description of a dilute gas or “rapid granular flow” [4–8] to show that this mechanism leads to a long-wavelength instability of a homogeneous assembly of inelastic particles. However, in the absence of driving, the collapse quickly becomes complete and the original assumptions break down. In the system described in this letter, a stationary state is maintained by external driving. It is thus simple enough to allow for an analytical description, while it keeps the central aspects of the clustering phenomenon.

The experiment, first described by Schlichting and Nordmeyer [9], consists of a box of base area $12cm^2$ and height $20cm$, mounted on a shaker, and filled with $N = 100$ plastic particles of radius $r = 1mm$ (see Fig. 1). The box is separated into two equal parts by a wall which has a narrow horizontal slit at a height $h = 2.3cm$. When the shaker is operating at full power, the amplitude is vibration is approximately $A = 0.3cm$ and the frequency $f = 50Hz$. Even if all particles are on one side initially, they immediately distribute equally to both sides. Lowering the frequency below a critical value of $30Hz$ the symmetry is spontaneously broken, and particles settle preferentially on one side. Evidently, the wall greatly enhances the clustering described above, since it prevents a direct exchange of particles which would break up the cluster. The grains thus act as Maxwell’s demon [10], who preferentially lets particles pass from left to right or vice versa. As a result, a more ordered state is formed in which most particles are on one side. The demon must then absorb entropy, a role which in our system is assumed by the sand grains. Still another interpretation would be that of a dissipative structure, which in the stationary state is maintained by a flux of entropy [1].
one-dimensional column of a vibrated granular assembly, from which we calculate the flux of particles leaving a compartment at the height of the hole $h$. Below we show that (4) has a single solution for strong driving, but two asymmetric solutions become stable as the driving is lowered.

The problem of a shaken granular gas in a gravitational field has recently been treated in a number of papers [12, 13]. Equations of motion for the number density $n(z)$, pressure $p(z)$, and granular temperature $T(z)$ can be found from conventional kinetic theory [14]. The position $z$ is measured from the bottom of the container and the granular temperature is defined as $T = < v^2 > /d$, where $d$ is the dimension of space and $v$ the velocity of a particle. This definition of temperature is customary for granular media, but differs from the usual molecular temperature, which is recovered by formally setting $k_B = m$. Now the stationary equations become in the dilute limit

\[ p = mnT, \quad \partial_z p = -mgn, \]
\[ \kappa \partial_z \left[ T^{1/2} \partial_z T \right] = (2\kappa/3) \partial_z^2 T^{3/2} = Dn^2T^{3/2}. \]  

(2)

The first equation is the usual equation of state, the second the force balance, where $-mg$ is the force on a single particle. The third equation is the balance of heat flux and dissipation due to inelastic collisions. As usual, the thermal conductivity $\kappa T^{1/2}$ is proportional to the average particle velocity. In the simplest possible model of hard and smooth particles the energy loss in one collision is proportional to $(1 - e^2)T$ on the average. The coefficient of restitution $e$ measures the reduction $v_n' = -ev_n$ in the normal velocity of the particles. Together with the number of collisions being proportional to $n^2 T^{1/2}$ this explains the form of the loss term on the right hand side of the third equation (3).

To allow for a better comparison with numerical simulations, we will consider the two-dimensional case of circular discs of radius $r$, for which the coefficients are found to be $\kappa = \pi^{-1/2} m/r$, $D = 4\pi^{1/2} mr(1 - e)$. Next, we have to supply boundary conditions. To minimize wall effects, which are not essential to our problem, all wall collisions are assumed to be elastic. The top is left open. For simplicity, the bottom of the container is taken to move in a sawtooth manner, such that a colliding particle always finds it to move upward with velocity $v_b = Af$. Finally, the amplitude $A$ of the vibration is assumed to be very small compared with the mean free path, so that the bottom is effectively stationary. In summary this means that the $z$-component of the velocity of a particle colliding with the bottom is changed according to $v_z' = 2v_b - v_z$. The approximations of (2) imply that the velocity is close to a Maxwell-Boltzmann distribution, which allows to calculate the rate of energy input per unit width (or unit area in 3D) to

\[ Q = mn \left[ v_0 T(0) + \frac{2}{\sqrt{\pi}} v_0^2 b T(0)^{1/2} \right]. \]

(3)

We will see below that the first term in (3) dominates, the second typically being smaller by a factor of 10 in our simulations. Thus the two boundary conditions for $z = 0$ become

\[ p(0) = -gmN, \quad Q = -\kappa T^{1/2} \partial_z T(0). \]

(4)

The first equation comes from integrating the force balance, and $N$ is the total number of particles per unit width of the box. The second equation balances the energy input with the heat flux out of the bottom according to (3).

![FIG. 2. Temperature and density profiles of $N = 320$ grains in a box of width $w = 1.6$. The particles are circular disks of radius $r = 0.01$ and coefficient of restitution $e = 0.95$. The velocity of the bottom is $v_b = 0.149$, acceleration of gravity $g$ and mass $m$ is normalized to one The full line is the result of a particle simulation, the dotted line is the present theory, and the dashed line is (3).](image)

Since (4) is of third order, another boundary condition is needed for a unique solution. The missing third condition is found by observing that the solution of (3) for large $z$ is of the form $T^{3/2}(z) = cz + b$, so $c$ must be zero to prevent $T$ from becoming negative or infinity. This is the desired third condition, and solutions are easily found
by shooting for a temperature profile which is asymptotically constant. The resulting temperature and density profiles are shown as dotted lines in Fig. 2 for a typical set of parameters and are compared with a numerical simulation (solid lines). The simulation uses an event-driven code for smooth and hard particles, with the normal velocity reduced by a factor of $e$ as described above. The agreement is quite good, but gets worse if the number of particles is reduced, and deteriorates even more in three dimensions. The reason is that even for the parameters of Fig. 3 the temperature changes significantly over the length of the mean free path. As a result, the distribution of the $v_z$-velocities of particles hitting the bottom deviates significantly from a Maxwell-Boltzmann distribution at $T(0)$.

The solution given in 13 assumes that the temperature is constant, which is asymptotically correct for very small inelasticity. The resulting profiles are the dashed lines in Fig. 2. Evidently, the present theory is more accurate, but the constant temperature solution has the great merit of simplicity and still contains the essentials. Namely, at constant temperature the density is an exponential and $T_{\infty}$ is found from a balance of the energy input and dissipation:

$$n(z) = \frac{gmN}{T_{\infty}}e^{-gz/T_{\infty}}, \quad T_{\infty} = \left[\frac{2v_b}{DmN}\right]^2. \quad (5)$$

Here we have neglected the second term of (3) in favor of the first, which is again consistent for $e \approx 1$.

Returning to the original stability problem, the flux through a hole of area $S$ is found to be $F = S\eta(h)\sqrt{T(h)/2\pi}$. Using this formula, and solutions of (2)-(4) to find the profiles, one can look for solutions of (1). Since the total number of particles $N$ is conserved, these must be sought subject to the constraint $\bar{N} = \bar{N}_e + \bar{N}_r$, where the overbar denotes the number of particles relative to the half-width of the container.

Figure 3 shows the solutions of (1) as a function of $h$ for $\bar{N} = 225$ as the solid line. The instability follows a typical pitchfork bifurcation, so the average of the asymmetry parameter $\epsilon = (\bar{N}_e - \bar{N})/N$ increases continuously when $h$ is raised above a critical value. The circles, which are the results of a numerical simulation with $N = 360$ particles averaged over time, show quite satisfactory agreement. Figure 4 illustrates the temporal behavior of the asymmetry parameter $\epsilon$. The symmetric state rapidly becomes unstable and the system fluctuates around its new equilibrium position. Since there are more collisions on the denser side the temperature is lowered, causing particles to sink to the bottom. Thus a non-symmetric state becomes possible because the flux $F$ is no longer a monotonously increasing function of the number of particles as it would be in equilibrium. Another hallmark of a nonequilibrium system is that the stationary state is described by a flux balance (1), while the temperature on either side of the hole is not equal, as it would be in thermal equilibrium.

![Figure 3](image3.png)

**Fig. 3.** The bifurcation of the asymmetry $\epsilon$ as function of the height $h$. The full line is the result of the present theory, circles are numerical simulations with $N = 360$ particles in a box of half-width 1.6, averaged over time. All other parameters are the same as in Fig. 2. The dotted line is the result of the simplified theory.

![Figure 4](image4.png)

**Fig. 4.** The temporal evolution of the asymmetry parameter $\epsilon$ corresponding to the data point $h = 0.9$ in Fig. 3, but with only $N = 180$ particles.

Further insight can be gained from the simplified solution (3). A straightforward calculation leads to

$$\frac{\partial \epsilon}{\partial t} = F_0N \left[ (\epsilon - 1/2)^2e^{-\mu(\epsilon-1/2)^2} - (\epsilon + 1/2)^2e^{-\mu(\epsilon+1/2)^2} \right] + \xi, \quad (6)$$

where $\xi$ is a noise term to be considered later and $F_0$ is a constant. The stationary solutions for small noise are determined by the vanishing of the angular brackets, which is controlled by a single parameter

$$\mu = 4\pi ghr^2(1-e)^2\bar{N}^2/v_b^2. \quad (7)$$

If $\mu > 4$, the equation $[\ldots] = 0$ has three roots, and just above the bifurcation the asymmetric solutions are described by $\epsilon = \pm \sqrt{3(\mu - 4)}/16$, which is included as
the dashed line in Fig. 3. While there is an offset between the simplified theory and the simulation, it does describe the form of the bifurcation fairly well using the single parameter \( \mu \). It is intuitively clear that by increasing the number of particles, the inelasticity, or the height of the wall separating the two compartments, phase separation is enhanced. Heating, as pointed out in the introduction, favors a symmetric state.

Finally, it is possible to include fluctuations in our description, which are important if the number of particles is just a few hundred as in Fig. 4. In particular this leads to a softening of the transition, since immediately below the transition the system can switch between the two possible states. Assuming that the particles passing through the hole are uncorrelated, the noise \( \xi \) is Gaussian white noise \[16\] on a coarse-grained time scale, and

\[
< \xi(t)\xi(t') > = F_0 \left[ (\epsilon - 1/2)e^{-\mu(\epsilon - 1/2)^2} + (\epsilon + 1/2)e^{-\mu(\epsilon + 1/2)^2} \right] \delta(t-t'). \tag{8}
\]

Note that the constant \( F_0 \) can be eliminated by rescaling time, so the strength of the noise is controlled by the total number of particles alone. By considering the fluctuations around the local minimum to Gaussian order, we find \[16\] \( \sqrt{< (\epsilon - <\epsilon >)^2 >} = [4N(\mu - 4)]^{-1/2} \). Adjusting this formula to the critical value of \( \mu_{\text{cr}} = 5.4 \) found from simulation this gives \( \sqrt{< (\epsilon - <\epsilon >)^2 >} = 0.045 \) in reasonable agreement with Fig. 4. Of course, equations \[3\]-\[8\] also allow for a more detailed analysis of transitions between the two asymmetric states and many more questions relating to the critical fluctuations near the transition. This will be considered in more detail in future publications.

In conclusion, hydrodynamic equations are a very valuable tool to describe a dilute granular gas. The greatest problem lies in the formulation of the boundary conditions. The clustering instability can be understood in terms of a very simple experiment, which we model in a static, one-dimensional description.

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