Optimal Change-point Testing for High-dimensional Linear Models with Temporal Dependence

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Abstract

This paper studies change-point testing for high-dimensional linear models, an important problem that is not well explored in the literature. Specifically, we propose a quadratic-form-based cumulative sum (CUSUM) test to inspect the stability of the regression coefficients in a high-dimensional linear model. The proposed test is able to control the type-I error at any desired level and is theoretically sound for temporally dependent observations. We establish the asymptotic distribution of the proposed test under both the null and alternative hypotheses. Furthermore, we show that our test is asymptotically powerful against multiple-change-point alternative hypotheses and achieves the optimal detection boundary for a wide class of high-dimensional linear models. Extensive numerical experiments and a real data application in macroeconomics are conducted to demonstrate the promising performance and practical utility of the proposed test. In addition, some new consistency results of the Lasso (Tibshirani, 1996) are established under the change-point setting with the presence of temporal dependence, which may be of independent interest.

Keywords: Change-point testing; High-dimensional regression; CUSUM statistics; Temporal dependence; Minimax optimality
1 Introduction

Over the last two decades, due to the emergence of “big data”, high-dimensional linear models have become a key component of modern statistical modeling and found ubiquitous applications in a wide range of scientific fields, including biology, neuroscience, climatology, finance, economics, cybersecurity, among others. There exists a vast statistical literature studying the methodological, computational and theoretical aspects of high-dimensional linear models. We refer to Bühlmann and van de Geer (2011) for an excellent book-length review.

On the other hand, one of the most commonly encountered issues for “big data” is heterogeneity (Wang and Samworth, 2018). For sequentially collected data, heterogeneity often manifests itself through non-stationarity, where the data-generating mechanism experiences structural breaks or change-points over time. To address such issues, numerous statistical testing procedures have been proposed to examine the existence of structural breaks.

Change-point testing is a classical problem in statistics and has been extensively studied in the low-dimensional setting, we refer the readers to Aue et al. (2009), Shao and Zhang (2010), Matteson and James (2014), Kirch et al. (2015) and Zhang and Lavitas (2018) (among many others) for some recent work and Perron (2006) and Aue and Horváth (2013) for comprehensive reviews. More recently, there is a surge of interest in change-point testing under the high-dimensional setting, see for example Horvath and Hušková (2012), Cho and Fryzlewicz (2015), Jirak (2015), Wang and Samworth (2018), Enikeeva and Harchaoui (2019), Wang et al. (2021c) and Chakraborty and Zhang (2021). However, these works mainly focus on testing the stability of mean vector or covariance matrices of high-dimensional times series, and we are not aware of any valid change-point testing procedure for high-dimensional linear models.

The existing change-point literature for high-dimensional linear models focuses on change-point estimation, where the main objective is to locate the unknown structural breaks. Lee et al. (2016, 2018) and Kaul et al. (2019) study change-point estimation with the knowledge of a single change-point, and Leonardi and Bühlmann (2016) and Wang et al. (2021b) propose procedures for localization of multiple change-points. However, all these works are essentially built upon penalized model selection procedures that cannot be easily adapted for valid hypothesis testing of structural instability. Moreover, all these works require temporal independence, which may not be realistic for real data applications, especially for financial and economics studies.

The change-point testing literature for linear models can only be found under the low-dimensional setting, see the sup $F$ test and its extensions in Andrews (1993) and Bai and Perron (1998). How-
ever, since the sup $F$ test is based on OLS, it is degenerate under the high-dimensional setting where the dimension of covariates $p$ is larger than the sample size $n$. In addition, even if $p$ is smaller than $n$, the sup $F$ test can incur large size distortion for the case where $p$ and $n$ are on comparable scales.

To fill the gap in the literature, in this paper, we propose a valid and optimal change-point testing procedure, named QF-CUSUM, for high-dimensional linear models. Our main contributions are three-fold. First, to our best knowledge, QF-CUSUM is the first valid change-point testing procedure for high-dimensional linear models that can control the type-I error at any given level. Second, through a well-designed randomization component, the proposed test is theoretically sound for temporally dependent observations, which is more appealing and realistic for real data applications. Third, we show that QF-CUSUM is optimal in the sense that it achieves the minimax lower bound of the detection boundary for a wide class of high-dimensional linear models. In addition, we establish some new consistency results of the Lasso (Tibshirani, 1996) under the change-point setting, which may be of independent interest.

The rest of the paper is organized as follows. Section 2 proposes the QF-CUSUM test and establishes its asymptotic distribution under both the null and alternative hypotheses. The minimax optimality of QF-CUSUM is further established as well. Section 3 extends the result to temporally dependent observations under the framework of $\beta$-mixing. In Section 4, we conduct extensive numerical experiments to examine the finite sample performance of QF-CUSUM under both null and alternative hypotheses. A real data application is further implemented to illustrate the practical value of the proposed method. Section 5 concludes.

## 2 A Quadratic Form based CUSUM Test

We first formally introduce the change-point testing problem for the high-dimensional linear model. Denote $y_i \in \mathbb{R}$ as the univariate response and $x_i \in \mathbb{R}^p$ as the $p$-dimensional covariate. Given sequentially observed data $\{(x_i, y_i)\}_{i=1}^n$, the high-dimensional linear model assumes

$$ y_i = x_i^T \beta_i^* + \epsilon_i, \text{ for } i = 1, 2, \cdots, n, $$

where $\beta_i^*$ is the regression coefficient and $\epsilon_i$ is the random noise. Under the null hypothesis, the regression coefficient $\beta_i^*$ stays unchanged and we have

$$ H_0 : \beta_1^* = \beta_2^* = \cdots = \beta_n^*. $$
Under the alternative hypothesis, there exists $K \geq 1$ unknown change-points $1 \leq \eta_1 < \eta_2 < \cdots < \eta_K < n$ such that

$$H_a : \beta_1^* = \cdots = \beta_{\eta_1}^* \neq \beta_{\eta_1+1}^* = \cdots = \beta_{\eta_2}^* \neq \cdots \neq \beta_{\eta_K}^* = \cdots = \beta_n^*.$$

We proceed by imposing some mild assumptions on the high-dimensional linear model. We first define the sub-Gaussian family, which is extensively used in the high-dimensional literature. Later in Section 3, we further extend the results to the sub-Weibull family, which includes sub-Gaussian as a special case. A random variable $x \in \mathbb{R}$ is sub-Gaussian with sub-Gaussian norm $\|x\|_{\psi_2}$ if

$$\|x\|_{\psi_2} := \sup_{d \geq 1} (\mathbb{E}|x|^d)^{1/d} d^{-1/2} < \infty.$$  

Similarly, a random vector $x \in \mathbb{R}^p$ is sub-Gaussian with sub-Gaussian norm $\|x\|_{\psi_2}$ if $\|x\|_{\psi_2} := \sup_{v \in S^{p-1}} \|v^\top x\|_{\psi_2} < \infty$, where $S^{p-1}$ is the unit sphere in $\mathbb{R}^p$.

**Assumption 1.** The observations $\{(x_i, y_i)\}_{i=1}^n$ follow model (1) with independently and identically distributed covariates $\{x_i\}_{i=1}^n$ and random noise $\{\epsilon_i\}_{i=1}^n$.

**a. (Sub-Gaussian)** The random covariate $x_i$ is a $p$-dimensional sub-Gaussian random vector with $\mathbb{E}(x_i) = 0$ and $\|x_i\|_{\psi_2} = K_X$. The random noise $\epsilon_i$ is a sub-Gaussian random variable independent of $x_i$ with $\mathbb{E}(\epsilon_i) = 0$, $\text{Var}(\epsilon_i) = \sigma_\epsilon^2$ and $\|\epsilon_i\|_{\psi_2} = K_\epsilon$. Here, $K_X$ and $K_\epsilon$ are absolute constants $< \infty$.

**b. (Eigenvalue)** Denote $\Sigma = \text{Cov}(x_i)$, there exist absolute constant $c_x$ and $C_x$ such that the minimal and maximal eigenvalues of $\Sigma$ satisfy $\lambda_{\min}(\Sigma) \geq c_x > 0$ and $\lambda_{\max}(\Sigma) \leq C_x < \infty$.

**c. (Sparsity)** For each $i = 1, \ldots, n$, there exists a support set $S_i \subseteq \{1, \ldots, p\}$ such that

$$\beta_{i,j}^* = 0 \quad \text{for all} \quad j \notin S_i.$$  

In addition, there exists a sparsity parameter $1 \leq s \leq p$ such that the cardinality of the support set satisfies $|S_i| \leq s$ for all $i = 1, 2, \cdots, n$.

**d. (Infill)** Under $H_a$, there exist a fixed set of (relative) change-points $0 < \eta_1^* < \eta_2^* < \cdots < \eta_K^* < 1$ such that $\eta_k = \lfloor n \eta_k^* \rfloor$ for $k = 1, \cdots, K$.

**e. (SNR)** There exists a sufficiently large absolute constant $C_{\text{snr}}$ such that $n \geq C_{\text{snr}} s \log(p)$.

Assumption 1(a) and (b) are standard assumptions in the high-dimensional literature. Assumption 1(c) imposes sparsity conditions on the regression coefficient. Denote $S = \bigcup_{k=1}^K S_{\eta_k+1}$. By Assumption 1(d), the number of change-points $K$ is finite, and thus $|S| \leq Ks$. Assumption 1(e) imposes the standard effective sample size condition used in the high-dimensional literature. We remark that Assumption 1(e) allows the sparsity $s$ and dimensionality $p$ to vary/grow with $n$. In particular, Assumption 1(e) allows both the sparse case where $s = O(1)$ and the dense case where
\(s = p^\alpha\) for some \(\alpha \in (0, 1]\), as long as \(n \geq C_{\text{snr}} s \log(p)\) holds.

Given a generic interval \(I \subseteq (0, n]\), denote its cardinality as \(|I|\). To handle the high-dimensional setting where \(p > |I|\), in this following, we work with the Lasso estimator such that

\[
\hat{\beta}_I := \arg \min_{\beta \in \mathbb{R}^p} \frac{1}{|I|} \sum_{i \in I} (y_i - x_i^\top \beta)^2 + \frac{\lambda}{\sqrt{|I|}} \|\beta\|_1,
\]

(2)

where \(\| \cdot \|_1\) is the \(l_1\)-norm and \(\lambda\) is the Lasso tuning parameter to be specified later. Define \(\beta^*_I = \frac{1}{|I|} \sum_{i \in I} \beta^*_i\), by Assumption 1(a), we have \(\beta^*_I\) is unique minimizer of the population squared loss function \(\mathbb{E}(\sum_{i \in I} (y_i - x_i^\top \beta)^2)\) and thus can be viewed as the population version of \(\hat{\beta}_I\). In addition, denote the sample covariance matrix estimated based on the interval \(I\) as \(\hat{\Sigma}_I := \frac{1}{|I|} \sum_{i \in I} x_i x_i^\top\).

Lemma 1 provides a uniform consistency result for the Lasso estimator \(\hat{\beta}_I\) defined in (2) w.r.t. its population quantity \(\beta^*_I\) across all intervals \(I \subseteq (0, n]\), which is of independent interest. Lemma 1 is used extensively in the technical proof to analyze the later proposed quadratic form.

**Lemma 1.** Let \(\zeta > 0\) be any fixed constant in \((0, 1)\). Suppose Assumption 1 holds. There exists \(\lambda = C_\lambda \sqrt{\log p}\) with some sufficiently large constant \(C_\lambda\), such that under both \(H_0\) and \(H_a\), we have with probability at least \(1 - n^{-5}\), for all \(I \subseteq (0, n]\) with \(|I| \geq \zeta n\), it holds that

\[
\|\hat{\beta}_I - \beta^*_I\|_2^2 \leq C_5 \frac{\log p}{n}, \quad \|\hat{\beta}_I - \beta^*_I\|_1 \leq C_5 \sqrt{\frac{\log p}{n}}, \quad \|((\hat{\beta}_I - \beta^*_I)s_I)^\top s_I\|_1 \leq 3 \|((\hat{\beta}_I - \beta^*_I)s_I^\top s_I\|_1.
\]

where \(C\) is an absolute constant and \(S_I^c = \{1, 2, \cdots, p\} \setminus S_I\) with \(S_I\) being the support set of \(\beta^*_I\).

### 2.1 A bias-corrected quadratic form

To ease presentation, in the following, we introduce the quadratic form based CUSUM (QF-CUSUM) using the single change-point alternative as a motivating example. In other words, we assume under \(H_a\), there is only one unknown change-point \(\eta\) such that

\[
\beta^*_i = \beta^{(1)}\text{ for } 1 \leq i \leq \eta \text{ and } \beta^*_i = \beta^{(2)}\text{ for } \eta + 1 \leq i \leq n.
\]

We remark that the presented method and theory apply to the *multiple* change-point alternative.

For two regression coefficients \(\beta^{(1)}\) and \(\beta^{(2)}\), it is natural to directly measure their difference via the \(l_2\)-norm \(\|\beta^{(1)} - \beta^{(2)}\|_2\). However, under the regression context, an alternative and more relevant quantity is the quadratic form \((\beta^{(1)} - \beta^{(2)})^\top \Sigma(\beta^{(1)} - \beta^{(2)})\), which equals to \(\text{Var}(x_i^\top (\beta^{(1)} - \beta^{(2)}))\) as \(\text{Cov}(x_i) = \Sigma\). Note that under Assumption 1(a), we have that

\[
C_x \|\beta^{(1)} - \beta^{(2)}\|_2^2 \leq (\beta^{(1)} - \beta^{(2)})^\top \Sigma(\beta^{(1)} - \beta^{(2)}) \leq C_x \|\beta^{(1)} - \beta^{(2)}\|_2^2.
\]

Thus, in terms of theoretical magnitude, \(\|\beta^{(1)} - \beta^{(2)}\|_2^2\) and \((\beta^{(1)} - \beta^{(2)})^\top \Sigma(\beta^{(1)} - \beta^{(2)})\) are the
same and both can capture the change in the regression coefficient.

However, compared to \( \| \beta^{(1)} - \beta^{(2)} \|_2^2 \), the quadratic form \((\beta^{(1)} - \beta^{(2)})^T \Sigma (\beta^{(1)} - \beta^{(2)})\) further incorporates the covariance structure \( \Sigma \) of \( x \) and thus can better reflect the difference between two regression models \( y = x^T \beta^{(1)} + \epsilon \) and \( y = x^T \beta^{(2)} + \epsilon \). Therefore \((\beta^{(1)} - \beta^{(2)})^T \Sigma (\beta^{(1)} - \beta^{(2)})\) is preferred for change-point testing. We remark that \((\beta^{(1)} - \beta^{(2)})^T \Sigma (\beta^{(1)} - \beta^{(2)})\) is closely related to the explained variance in the regression literature, see for example Cai and Guo (2020), where the explained variance is defined as \( \beta^T \Sigma \beta \) for a regression model \( y = x^T \beta + \epsilon \).

Given a potential change-point location \( t \), to estimate the quadratic form, a natural choice is the plug-in estimator. Specifically, based on the Lasso estimator (2), we can obtain \( \hat{\beta}_{(0,t]} \) from \( \{(x_i, y_i)\}_{i=1}^t \) and \( \hat{\beta}_{(t,n]} \) from \( \{(x_i, y_i)\}_{i=t+1}^n \). Denote \( \hat{\Delta}_t = \hat{\beta}_{(0,t]} - \hat{\beta}_{(t,n]} \), one possible plug-in estimator we can use is

\[
(\hat{\Delta}_t^T \hat{\Sigma}_{(0,t]} \hat{\Delta}_t + \hat{\Delta}_t^T \hat{\Sigma}_{(t,n]} \hat{\Delta}_t)/2,
\]

which intuitively estimates \( \Delta_t^*^T \Sigma \Delta_t^* \) with \( \Delta_t^* = \beta_{(0,t]}^* - \beta_{(t,n]}^* \).

However, as will be made clear in the proof, due to the use of Lasso, the plug-in estimator has an intrinsic bias term that makes it technically difficult to analyze its asymptotic distribution under the null and alternative hypothesis. In particular, we have

\[
\hat{\Delta}_t^T \hat{\Sigma}_{(0,t]} \hat{\Delta}_t - \Delta_t^*^T \Sigma \Delta_t^* = 2\hat{\Delta}_t^T \hat{\Sigma}_{(0,t]} (\hat{\beta}_{(0,t]} - \beta_{(0,t]}^*) - 2\hat{\Delta}_t^T \hat{\Sigma}_{(0,t]} (\hat{\beta}_{(t,n]} - \beta_{(t,n]}^*) + \Delta_t^*^T (\hat{\Sigma}_{(0,t]} - \Sigma) \Delta_t^* - (\hat{\Delta}_t - \Delta_t^*)^T \hat{\Sigma}_{(0,t]} (\hat{\Delta}_t - \Delta_t^*).
\]

Due to the bias of the Lasso estimator, the asymptotic behavior of the first two terms in the above equation is technically difficult to analyze.

Thus, we further introduce bias correction terms to the plug-in estimator. Specifically, the first two terms can be estimated by \( \frac{1}{t} \sum_{i=1}^t \hat{\Delta}_i^T x_i(x_i^T \hat{\beta}_{(0,t]} - y_i) \) and \( \frac{2}{n-t} \sum_{i=t+1}^n \hat{\Delta}_i^T x_i(x_i^T \hat{\beta}_{(t,n]} - y_i) \), where the key idea is to make use of \( \mathbb{E}(y_i|x_i) = x_i^T \beta^* \). The bias-corrected quadratic form of \( \hat{\Delta}_t^T \hat{\Sigma}_{(0,t]} \hat{\Delta}_t \) is thus defined as \( \hat{\Delta}_t^T \hat{\Sigma}_{(0,t]} \hat{\Delta}_t + \frac{1}{t} \sum_{i=1}^t \hat{\Delta}_i^T x_i(y_i - x_i^T \hat{\beta}_{(0,t]}) - \frac{2}{n-t} \sum_{i=t+1}^n \hat{\Delta}_i^T x_i(y_i - x_i^T \hat{\beta}_{(t,n]}). \) Based on the same arguments, we have that the bias-corrected quadratic form of \( \hat{\Delta}_t^T \hat{\Sigma}_{(t,n]} \hat{\Delta}_t \) can be defined as \( \hat{\Delta}_t^T \hat{\Sigma}_{(t,n]} \hat{\Delta}_t + \frac{1}{n-t} \sum_{i=t+1}^n \hat{\Delta}_i^T x_i(y_i - x_i^T \hat{\beta}_{(t,n]}). \)

Combine the above arguments and sum the two bias-corrected quadratic forms of \( \hat{\Delta}_t^T \hat{\Sigma}_{(0,t]} \hat{\Delta}_t \) and \( \hat{\Delta}_t^T \hat{\Sigma}_{(t,n]} \hat{\Delta}_t \), we estimate \( \Delta_t^*^T \Sigma \Delta_t^* \) using the bias-corrected quadratic form

\[
\frac{1}{2}(\hat{\Delta}_t^T \hat{\Sigma}_{(0,t]} \hat{\Delta}_t + \hat{\Delta}_t^T \hat{\Sigma}_{(t,n]} \hat{\Delta}_t) + \frac{1}{t} \sum_{i=1}^t \hat{\Delta}_i^T x_i(y_i - x_i^T \hat{\beta}_{(0,t]}) - \frac{2}{n-t} \sum_{i=t+1}^n \hat{\Delta}_i^T x_i(y_i - x_i^T \hat{\beta}_{(t,n]}),
\]

which later serves as the building block of the proposed QF-CUSUM test.
A Goodness of Fit viewpoint: The proposed QF in (3) may seem complicated, however, we show that it is indeed intuitive and interpretable by rewriting (3) as a goodness of fit (GoF) statistic. Specifically, given a time point \( t \), under \( H_0 \) with no change-point, we have that \( \hat{\beta}_t := (\hat{\beta}_{(0,t]} + \hat{\beta}_{(t,n]}^2) / 2 \) can serve as a valid estimator for the regression coefficient. Thus, to differentiate \( H_0 \) and \( H_a \), we can examine the magnitude of the following GoF statistic,

\[
\frac{1}{t} \left( \sum_{i=1}^{t} (y_i - x_i^\top \hat{\beta}_{(0,t]}^i)^2 - \sum_{i=1}^{t} (y_i - x_i^\top \hat{\beta}_t)^2 \right) + \frac{1}{n-t} \left( \sum_{i=t+1}^{n} (y_i - x_i^\top \hat{\beta}_{(t,n]}^i)^2 - \sum_{i=t+1}^{n} (y_i - x_i^\top \hat{\beta}_t)^2 \right),
\]

where we declare the existence of change-points once the GoF statistic is “too large” for some \( t = 1, \ldots, n \). Elementary algebra shows that the GoF statistic is nothing but (3)/2.

2.2 QF-CUSUM

Based on the bias-corrected quadratic form (3), in this section, we propose QF-CUSUM for change-point testing in high-dimensional linear models.

Using Lemma 1, it is intuitive to see that under \( H_0 \), since \( \Delta_t^* = 0 \) for all \( t \), we have that the bias-corrected QF (3) is of order \( O_p(s \log p/n) \), which is in general degenerate and does not follow any pivotal distribution. To overcome this issue and design a non-degenerate test under \( H_0 \), we adopt a randomization strategy to inject additional variance into the bias-corrected QF, see similar approaches used in Cai and Guo (2020) to construct confidence intervals for high-dimensional linear models. We refer to Lopes et al. (2011), Srivastava et al. (2016) and Li et al. (2020) for other types of randomized tests proposed in the high-dimensional literature.

Specifically, we generate a sequence of random variables \( \{\xi_i\}_{i=1}^{n} \) such that \( \xi_i \overset{i.i.d.}{\sim} N(0, \sigma_\xi^2) \), where the variance level \( \sigma_\xi^2 \) will be specified later in Assumption 2. We then define the randomized bias-corrected QF as

\[
S_n(t) = \frac{1}{t} (\hat{\Delta}_t^\top \hat{\Sigma}_{(0,t]} \hat{\Delta}_t + \hat{\Delta}_t^\top \hat{\Sigma}_{(t,n]} \hat{\Delta}_t) + \frac{1}{t} \sum_{i=1}^{t} (2\hat{\Delta}_t^\top x_i + \xi_i)(y_i - x_i^\top \hat{\beta}_{(0,t]}^i) - \frac{1}{n-t} \sum_{i=t+1}^{n} (2\hat{\Delta}_t^\top x_i + \xi_i)(y_i - x_i^\top \hat{\beta}_{(t,n]}^i),
\]

and further define the proposed QF-CUSUM statistic as

\[
T_n(t) = \frac{1}{\sigma_c \sigma_\xi \sqrt{n}} \sqrt{\frac{t(n-t)}{n}} S_n(t),
\]

where we combine \( S_n(t) \) with the classical CUSUM weight function. We reject the null hypothesis if \( T_n(t) \) is larger than a pre-specified threshold.

Before presenting the formal theoretical results, we first discuss some high-level intuition of the
QF-CUSUM test in \((5)\). Using Lemma 1 and some additional technical lemmas in the Appendix, we can show that under \(H_0\), \(T_n(t)\) can be approximately written as

\[
T_n(t) = \frac{1}{\sigma_\epsilon \sigma_\xi} \sqrt{\frac{t(n-t)}{n}} \left[ \frac{1}{t} \sum_{i=1}^{t} \epsilon_i \xi_i - \frac{1}{n-t} \sum_{i=t+1}^{n} \epsilon_i \xi_i \right] + O_p \left( \frac{s \log p}{\sqrt{n}} \right) + O_p \left( \frac{s \log p}{\sqrt{n} \sigma_\xi} \right).
\]

The first term takes the form of the classical CUSUM statistic and we later control the last two noise terms with additional conditions in Assumption 2.

**Remark 1:** We remark that the randomization strategy is devised to derive a pivotal and non-degenerate asymptotic distribution under \(H_0\). More importantly, as will be seen in Section 3, the randomized error \(\{\xi_i\}_{i=1}^{n}\) is deliberately designed to avoid the estimation of the long-run variance (LRV) under temporal dependence, which is highly challenging even in low-dimensional settings (Shao and Zhang, 2010). In the literature, another strategy to avoid a degenerate test under \(H_0\) is to combine the designed test with another pivotal (possibly less powerful) test statistic, see for example Fan et al. (2015). However, in our current setting of high-dimensional change-point testing, it is not obvious how to construct another pivotal test and such a strategy will likely involve the estimation of LRV under temporal dependence. We thus do not pursue this direction.

**Assumption 2.** We have that

\[
\frac{s \log p}{\sqrt{n}} \rightarrow 0 \quad \text{and} \quad \sigma_\xi = A_n \cdot \frac{s \log p}{\sqrt{n}}
\]

for some diverging sequence \(A_n \rightarrow \infty\).

Assumption 2 requires that \(s \log p = o(\sqrt{n})\), which is needed to control the noise term of \(T_n(t)\) as discussed above and thus facilitates the derivation of the limiting distribution. This is a standard assumption in the literature that concerns the limiting distribution of the high-dimensional linear models, see e.g. Javanmard and Montanari (2013), van de Geer et al. (2014), Zheng and Raskutti (2019) and Cai and Guo (2020).

Throughout the rest of the manuscript, we will assume without loss of generality that \(p \geq n^\alpha\) for some \(\alpha > 0\). This is a convenient assumption commonly used in the literature for high-dimensional linear models. We remark that for \(p = o(n^\alpha)\), e.g. \(p = O(1)\) or \(p = \log n\), all of our theoretical results continue to hold, up to a logarithmic factor of \(n\). Below in Theorem 1, we establish a process convergence result for \(T_n(t)\) under the null hypothesis \(H_0\).

**Theorem 1.** Let \(\zeta > 0\) be any fixed constant in \((0, 1/2)\). Suppose Assumption 1 and Assumption 2 hold, and \(\lambda = C_\lambda \sqrt{\log p}\) for some sufficiently large constant \(C_\lambda\). Under \(H_0\), we have that

\[
T_n(\lfloor nr \rfloor) \Rightarrow G(r), \quad \text{over} \quad r \in [\zeta, 1 - \zeta],
\]

where \(G(r)\) is a Gaussian distribution.
where $\mathcal{G}(r) = [B(r) - rB(1)]/\sqrt{r(1-r)}$ for $r \in (0, 1)$ and $B(\cdot)$ is the standard Brownian motion. Furthermore, we have

$$
\max_{t=[n\zeta], [n\zeta]+1, \ldots, [n(1-\zeta)]} \mathcal{T}_n(t) \xrightarrow{d} \sup_{r \in [\zeta, 1-\zeta]} \mathcal{G}(r).
$$

Note that Theorem 1 holds for any interval $[\zeta, 1-\eta]$ for any $\zeta > 0$, but does not hold for the interval $[0, 1]$. This is a well-known phenomenon for the CUSUM statistic, see for example Andrews (1993). In fact, we have that $\sup_{r \in [0,1]} \mathcal{G}(r) = \infty$ (Corollary 1 in Andrews (1993)), resulting in a trivial critical value $\infty$ for $\mathcal{T}_n(t)$. Thus, we require the interval $[\zeta, 1-\zeta]$ to be bounded away from zero and one.

For any fixed $\zeta \in (0, 1/2)$, $\sup_{r \in [\zeta, 1-\zeta]} \mathcal{G}(r)$ is a pivotal distribution. For a given type-I error rate $\alpha$, define $\mathcal{G}_\alpha(\zeta)$ such that $\mathbb{P}(\sup_{r \in [\zeta, 1-\zeta]} \mathcal{G}(r) \geq \mathcal{G}_\alpha(\zeta)) = \alpha$. We reject the null hypothesis if

$$
\max_{t=[n\zeta], [n\zeta]+1, \ldots, [n(1-\zeta)]} \mathcal{T}_n(t) > \mathcal{G}_\alpha(\zeta).
$$

By Theorem 1, this gives a test with asymptotically correct type I error rate.

We now provide the power result of QF-CUSUM under $H_a$. Denote the change size $\kappa_k := \|\beta_{nk+1}^* - \beta_{nk}^*\|_2$ for $k = 1, 2, \ldots, K$. Theorem 2 provides the detection boundary of QF-CUSUM.

**Theorem 2.** Let $\zeta > 0$ be any fixed constant in $(0, 1/2)$. Suppose Assumption 1 and Assumption 2 hold, and $\lambda = C_{\lambda} \sqrt{\log p}$ for some sufficiently large constant $C_{\lambda}$. Under $H_a$, suppose the maximum change size satisfies

$$
\max_{1 \leq k \leq K} \kappa_k^2 \geq B_n \frac{s \log p}{n} \tag{6}
$$

for some diverging sequence $B_n \to \infty$ and $B_n/A_n \to \infty$ as $n \to \infty$. We have that

$$
\mathbb{P}\left( \max_{t=[n\zeta], [n\zeta]+1, \ldots, [n(1-\zeta)]} \mathcal{T}_n(t) > \mathcal{G}_\alpha(\zeta) \right) \to 1 \quad \text{as} \quad n \to \infty.
$$

Theorem 2 states that QF-CUSUM can detect changes in the regression coefficient once the maximum change size is larger than $s \log p/n$ by any diverging order $B_n$. Note that our result holds for any fixed $\zeta \in (0, 1/2)$ without requiring that the true change-point with the maximum jump size is located between $[n\zeta]$ and $[n(1-\zeta)]$. We show in Section 2.3 that (6) is the optimal detection boundary that can be achieved by any valid test. We remark that Theorem 2 indicates that QF-CUSUM works under both single and multiple change-point alternatives and allows multiscale changes, i.e. different magnitude of jump sizes among $\{\kappa_1, \kappa_2, \ldots, \kappa_K\}$.

Thanks to the bias correction terms in QF-CUSUM, we can further establish an asymptotic distribution for $\mathcal{T}_n(t)$ under the alternative hypothesis with proper centering and scaling. Recall
we define $\Delta_i^* = \beta_{(0,i]}^* - \beta_{(i,n]}^*$ and further define
$$
\mu(t) = \Delta_i^T \Sigma \Delta_i^* \quad \text{and} \quad \sigma^2(t) = \sigma_i^2 \sigma_i^T + 4 \sigma_i^2 \Delta_i^* \Sigma \Delta_i^* + \frac{1}{4} \text{E} \left[ \Delta_i^T (x_i x_i^T - \Sigma) \Delta_i^* \right]^2 ,
$$
and
$$
\psi_L(t) = \sigma^2(t) + \frac{1}{t} \sum_{i=1}^t \sigma_i^2 (\beta_i^* - \beta_{(0,i]}^*)^T \Sigma (\beta_i^* - \beta_{(0,i]}^*) , \quad \psi_R(t) = \sigma^2(t) + \frac{1}{n - t} \sum_{i=t+1}^n \sigma_i^2 (\beta_i^* - \beta_{(i,n]}^*)^T \Sigma (\beta_i^* - \beta_{(i,n]}^*).
$$

Here, $\mu(t)$ is the mean and $\psi_L(t)$ and $\psi_R(t)$ measure the variance of the randomized bias-corrected quadratic form $\mathcal{S}_n(t)$. Recall that $T_n(t) = \frac{1}{\sigma_x \sigma_x} \sqrt{\frac{n}{n - t}} \mathcal{S}_n(t)$.

**Theorem 3.** Let $\zeta > 0$ be any fixed constant in $(0,1/2)$. Suppose Assumption 1 and Assumption 2 hold, and $\lambda = C_\lambda \sqrt{\log p}$ for some sufficiently large constant $C_\lambda$. Under $H_a$, we have that for any fixed $r \in [\zeta, 1 - \zeta]$ and $t = \lfloor rn \rfloor$, it holds that
$$
\left( \sqrt{\frac{\psi_L(t)}{t}} + \frac{\psi_R(t)}{n - t} \right)^{-1} (\mathcal{S}_n(t) - \mu(t)) \xrightarrow{d} N(0,1).
$$

Theorem 3 provides a pointwise convergence result for the asymptotic behavior of $\mathcal{S}_n(t)$. An extension to process convergence is possible if we make the additional assumptions on the magnitude of change sizes $\kappa_1, \kappa_2, \ldots, \kappa_K$ at the $K$ change-points. Theorem 3 is a more delicate version of the power result presented in Theorem 2 and can in fact be used to prove the result in Theorem 2. In addition, note that under $H_0$, we have that $\mu(t) \equiv 0$ and $\sigma(t) = \psi_L(t) = \psi_R(t) \equiv \sigma_i^2 \sigma_i^T$. Thus, the convergence result in Theorem 3 under $H_a$ reduces to the one in Theorem 1 under $H_0$.

### 2.3 Lower bound of the detection boundary

In this section, we further establish a matching lower bound for QF-CUSUM, which states that, if the maximum change size $\max_{1 \leq k \leq K} \kappa_k^2$ is too small, no test can consistently distinguish between $H_0$ and $H_a$. This together with the results in Theorem 2 shows that our test procedure is asymptotically optimal (up to an arbitrarily slowly diverging factor). The proof employs the convex version of Le Cam’s lemma (see, e.g., Yu (1996)) and constructs a sequence of mixtures of high-dimensional linear models with change-points that can not be reliably discriminated when the signal $\kappa_k$ is relatively small compared to the noise level.

**Theorem 4.** Suppose the observations $\{(x_i, y_i)\}_{i=1}^n$ are generated according to Assumption 1 and that $s \log p / \sqrt{n} \to 0$ and $s \leq p^\alpha$ for any $\alpha < 1/2$. In addition, assume that $\{x_i\}_{i=1}^n \overset{i.i.d.}{\sim} N(0, I_p)$ and $\{\epsilon_i\}_{i=1}^n \overset{i.i.d.}{\sim} N(0, \sigma_i^2)$. For $0 < \zeta < 1/2$, consider
$$
H_0 : \beta_1^* = \ldots = \beta_n^* \quad \text{and} \quad H_a(b) : H_a \text{ holds and } \max_{1 \leq k \leq K} \kappa_k^2 \geq b \frac{s \log(p)}{n}.
$$

10
where $b$ is some positive constant. Let $\psi$ be any test function mapping $\{(x_i, y_i)\}_{i=1}^n$ to $\{0, 1\}$. If $b$ is a sufficiently small constant, we have
\[
\liminf_{n, p \to \infty} \inf_{\psi \in \mathcal{P}_1(b)} \sup_{P \in \mathcal{P}_1(b)} \mathbb{E}_0(\psi) + \mathbb{E}_P(1 - \psi) \to 1,
\]
where $\mathcal{P}_1(b)$ denotes the class of distributions satisfying $H_a(b)$, $\mathbb{E}_0(\psi)$ gives the type-I error of $\psi$ under $H_0$ and $\mathbb{E}_P(1 - \psi)$ gives the power of $\psi$ when the observations are generated according to $P$.

3 QF-CUSUM under Temporal Dependence

In practice, temporal dependence is the norm rather than exception for sequentially observed data. Thus, in this section, we further study the behavior of the QF-CUSUM test $T_n(t)$ under the context of temporal dependence. Specifically, we follow the $\beta$-mixing framework in Wong et al. (2020) for high-dimensional linear models, which also allows for heavy-tailed observations.

We emphasize that the QF-CUSUM test $T_n(t)$ studied here is exactly the same as the one proposed in Section 2. In other words, our proposed change-point testing procedure applies to the case of both temporal independence and dependence without any modification. We begin our discussion by reviewing the concept of $\beta$-mixing and define the sub-Weibull family, which includes the sub-Gaussian family used in Section 2 as a special case.

Given two sigma fields $\mathcal{F}_1$ and $\mathcal{F}_2$, we define the $\beta$-mixing coefficient as
\[
\beta(\mathcal{F}_1, \mathcal{F}_2) = \sup_{t \in \mathbb{Z}} \left| \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^J \mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j) \right|,
\]
where the supremum is over all pairs of finite partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of the sample space $\Omega$ such that $A_i \in \mathcal{F}_1$, $B_j \in \mathcal{F}_2$ for all $i, j$. Thus, given a sequence of random vectors $\{Z_t\}_{t=-\infty}^\infty$, we can define its $\beta$-mixing coefficient at lag $l \in \mathbb{N}$ as
\[
\beta(l) = \sup_{t \in \mathbb{Z}} \beta(\sigma(\{Z_{-\infty:t}\}), \sigma(\{Z_{t+l:\infty}\})),
\]
where $\sigma(\cdot)$ denotes the generated sigma field.

We now introduce the sub-Weibull family. For any $\gamma \in (0, 2]$\footnote{The sub-Weibull family further incorporates the case of $\gamma > 2$. However, a $\gamma > 2$ implies a lighter tail than the sub-Gaussian family, which is not useful in practice. Thus, we only consider $\gamma \in (0, 2]$.}, a random variable $x \in \mathbb{R}$ is called sub-Weibull with parameter $\gamma$ if there exists a constant $K > 0$ such that
\[
(\mathbb{E}|x|^d)^{1/d} \leq K d^{1/\gamma} \text{ for all } d \geq \min\{1, \gamma\}.
\]
We further define the sub-Weibull($\gamma$) norm to be $\|x\|_{\psi_\gamma} := \sup_{d \geq 1} (\mathbb{E}|x|^d)^{1/d} d^{-1/\gamma}$. A random vector


$x \in \mathbb{R}^p$ is said to be a sub-Weibull($\gamma$) random vector if $\|x\|_{\psi,\gamma} := \sup_{v \in S^{p-1}} \|v^\top x\|_{\psi,\gamma} < \infty$, where $S^{p-1}$ is the unit sphere in $\mathbb{R}^p$. Note that sub-Weibull with $\gamma = 2$ reduces to the sub-Gaussian family and with $\gamma = 1$ reduces to the sub-Exponential family.

We now introduce Assumption 3, which can be seen as the extension of Assumption 1 to the temporal dependence setting and further allows for heavy-tailed observations.

**Assumption 3.** The observations $\{(x_i, y_i)\}_{i=1}^n$ follow model (1) with strictly stationary covariates $\{x_i\}_{i=1}^n$ and random noise $\{\epsilon_i\}_{i=1}^n$.

a. (Eigenvalue) Denote $\Sigma = \text{Cov}(x_i)$, there exists absolute constant $c_x$ and $C_x$ such that the minimal and maximal eigenvalues of $\Sigma$ satisfy $\Lambda_{\min}(\Sigma) \geq c_x > 0$ and $\Lambda_{\max}(\Sigma) \leq C_x < \infty$.

b. (Noise) The random noise $\epsilon_i$ is independent of $x_i$ and we have $\mathbb{E}(\epsilon_i) = 0$ and $\text{Var}(\epsilon_i) = \sigma^2_\epsilon > 0$.

c. (Mixing) The data $\{(x_i, \epsilon_i)\}_{i=1}^n$ is geometrically $\beta$-mixing; i.e., there exist positive constants $c$ and $\gamma_1$ such that the $\beta$-mixing coefficient of $\{(x_i, \epsilon_i)\}_{i=1}^n$ satisfies $\beta(l) \leq \exp(-cl^{\gamma_1})$ for all $l \in \mathbb{N}$.

d. (Sub-Weibull) The covariate $x_i$ and random noise $\epsilon_i$ follow sub-Weibull distributions with parameter $\gamma_2$; i.e., $\|x_i\|_{\psi,\gamma_2} \leq K_X$ and $\|\epsilon_i\|_{\psi,\gamma_2} \leq K_\epsilon$ for some absolute constants $K_X$ and $K_\epsilon$.

e. (Sparsity and Infill) Assumption 1(c) and Assumption 1(d) hold.

Given the two positive constants $\gamma_1$ and $\gamma_2$ in Assumption 3, we further define a key quantity $\gamma = \left(\frac{1}{\gamma_1} + \frac{2}{\gamma_2}\right)^{-1}$ and introduce Assumption 4, which extends Assumption 2 used in the temporal independence setting.

**Assumption 4.** Assumption 2 holds. Furthermore, we have that

$$\max\left\{\log^{1/\gamma}(p), (s \log p)^{\frac{2}{\gamma} - 1}\right\} \to 0.$$

As discussed in Wong et al. (2020), the parameter $\gamma$ measures the difficulty brought by the temporal dependence and heavy-tailed condition, where a smaller $\gamma$ corresponds to a more difficult problem and thus requires a stronger Assumption 4. Note that for sub-Gaussian random variables with temporal independence (i.e. the case studied in Section 2), we have $\gamma_1 = \infty$ and $\gamma_2 = 2$. Thus Assumption 4 essentially reduces to Assumption 2.

### 3.1 Theoretical guarantees

In the following, we further establish the theoretical guarantees of QF-CUSUM under the null and alternative hypothesis with Assumption 3 and Assumption 4. The general proof strategy is the same as in Section 2.2.
Specifically, we establish a Lasso consistency result similar to Lemma 1 but under the $\beta$-mixing case. We show that under $H_0$, as long as the tail is not too heavy and the temporal dependence is not too strong, $T_n(t)$ can again be approximately written as

$$T_n(t) = \frac{1}{\sigma_t \sigma_\xi} \sqrt{\frac{t(n-t)}{n}} \left[ \frac{1}{t} \sum_{i=1}^t \epsilon_i \xi_i - \frac{1}{n-t} \sum_{i=t+1}^n \epsilon_i \xi_i \right] + O_p \left( \frac{s \log p}{\sqrt{n}} \right) + O_p \left( \frac{s \log p}{\sqrt{n \sigma_\xi}} \right).$$

Define $\sigma^2_{\xi L} = \sum_{t=-\infty}^{\infty} \text{Cov}(\epsilon_0, \epsilon_t)$, which is the long-run variance (LRV) of $\{\epsilon_i\}_{i=1}^n$ due to temporal dependence. It is well known that $\lim_{t \to \infty} \text{Var}(\frac{1}{t} \sum_{i=1}^t \epsilon_i) = \sigma^2_{\xi L} > \sigma^2_\xi$ and the estimation of LRV can be challenging as it requires a user-specified bandwidth parameter (Shao and Zhang, 2010).

Fortunately, note that due to the i.i.d. nature of the randomized error $\{\xi_i\}_{i=1}^n$, $T_n(t)$ automatically avoids this difficulty. Specifically, the first term still converges to the Gaussian process defined in Theorem 1, as $\text{Var}(\frac{1}{t} \sum_{i=1}^t \epsilon_i) = \sigma^2_\xi \sigma^2_\xi / t$ and $\text{Var}(\frac{1}{n-t} \sum_{i=t+1}^n \epsilon_i) = \sigma^2_\xi \sigma^2_\xi / (n-t)$ do not depend on the LRV $\sigma^2_{\xi L}$ but only on the marginal variance $\sigma^2_\xi$. Theorem 5 provides the theoretical guarantees for QF-CUSUM under both the null and alternative hypotheses.

**Theorem 5.** Let $\zeta > 0$ be any fixed constant in $(0, 1/2)$. Suppose Assumption 3 and Assumption 4 hold, and $\lambda = C_\lambda \sqrt{\log p}$ for some sufficiently large constant $C_\lambda$. Under $H_0$, we have that

$$T_n([nr]) \Rightarrow \mathcal{G}(r), \text{ over } r \in [\zeta, 1 - \zeta], \text{ and } \max_{t = [nr], [nr]+1, \ldots, [n(1-\zeta)]} T_n(t) \overset{d}{\to} \sup_{r \in [\zeta, 1-\zeta]} \mathcal{G}(r),$$

where $\mathcal{G}(r)$ is the Gaussian process defined in Theorem 1. Under $H_\alpha$, suppose the maximum change size satisfies

$$\max_{1 \leq k \leq K} \kappa_k^2 \geq B_n \frac{s \log p}{n}$$

for some diverging sequence $B_n \to \infty$ and $B_n / A_n \to \infty$ as $n \to \infty$. We have that

$$\mathbb{P} \left( \max_{t = [nr], [nr]+1, \ldots, [n(1-\zeta)]} T_n(t) > \mathcal{G}_\alpha(\zeta) \right) \to 1 \text{ as } n \to \infty.$$  

We can further establish an asymptotic distribution for $T_n(t)$ under the alternative hypothesis with proper centering and scaling. However, due to temporal dependence, the scaling factor is more complicated as it now involves LRV of the random noise $\{\epsilon_i\}_{i=1}^n$ and covariate $\{x_i\}_{i=1}^n$. Define

$$\mu(t) = \Delta_t^T \Sigma \Delta_t$$

and

$$\sigma^2(t) = \sigma^2_\xi \sigma^2_\xi + 4 \text{Var} \left( \frac{1}{t} \sum_{i=1}^t x_i^T \Delta_t^T \epsilon_i \right) + \frac{1}{4} \text{Var} \left( \frac{1}{t} \sum_{i=1}^t \Delta_t^T (x_i x_i^T - \Sigma) \Delta_t^T \right),$$

and further define

$$\tilde{\psi}_L(t) = \sigma^2(t) + \frac{1}{t} \sum_{i=1}^t \sigma^2_\xi (\beta_i^* - \beta_{0,t})^T \Sigma (\beta_i^* - \beta_{0,t}), \quad \tilde{\psi}_R(t) = \sigma^2(t) + \frac{1}{n-t} \sum_{i=t+1}^n \sigma^2_\xi (\beta_i^* - \beta_{t,n})^T \Sigma (\beta_i^* - \beta_{t,n}).$$

Note that the last two terms of $\tilde{\sigma}^2(t)$ involve LRV and reduce to the last two terms of $\sigma^2(t)$ when
there is no temporal dependence. Theorem 6 provides the extension of Theorem 3 under the framework of $\beta$-mixing.

**Theorem 6.** Let $\zeta > 0$ be any fixed constant in $(0, 1/2)$. Suppose Assumption 3 and Assumption 4 hold, and $\lambda = C_\lambda \sqrt{\log p}$ for some sufficiently large constant $C_\lambda$. Under $H_a$, we have that for any fixed $r \in \lfloor \zeta, 1 - \zeta \rfloor$ and $t = \lfloor rn \rfloor$, it holds that

$$\left( \frac{\bar{\psi}_L(t)}{t} + \frac{\bar{\psi}_R(t)}{n - t} \right)^{-1} (S_n(t) - \mu(t)) \overset{d}{\to} N(0, 1).$$

### 4 Numerical Studies

In this section, we conduct extensive numerical experiments to investigate the performance of the proposed QF-CUSUM for change-point testing in high-dimensional linear models. Section 4.1 examines the finite-sample size of the test and Section 4.2 studies the power performance of the test under single and multiple change-points. Section 4.3 presents a real data application on macroeconomics to further illustrate the practical utility of the proposed test.

To our best knowledge, there is no existing change-point test designed for high-dimensional linear models available in the literature. Thus, we instead compare QF-CUSUM with three change-point estimation algorithms for high-dimensional linear models: BSA in Leonardi and Bühmann (2016), VPC in Wang et al. (2021b) and SGL in Zhang et al. (2015). We emphasize that the comparison is for reference purposes only, as QF-CUSUM focuses on testing while BSA, VPC and SGL are designed for estimation.

BSA estimates change-points via the minimization of an $l_0$-penalized information criteria, where the minimizer is searched via a heuristic procedure based on binary segmentation. VPC transforms change-point estimation for high-dimensional linear models to change-point detection in the mean of a univariate time series via a projection strategy. SGL recasts change-point estimation into a sparse group Lasso (Simon et al., 2013) problem. We remark that theoretical guarantees of the three algorithms all require temporal independence.

For all three algorithms, we keep the tuning parameters at the default setting recommended by the authors. We rule that BSA rejects $H_0$ if the set of estimated change-points is non-empty, and vice-versa. Same applies to VPC and SGL.

**Estimation of key quantities for QF-CUSUM:** To operationalize QF-CUSUM, we need to select the tuning parameter $\lambda$ of the Lasso estimation and further estimate the noise level $\sigma_\varepsilon$ and sparsity level $s$. To do so, we make the mild assumption that the first and last $\zeta$-proportion of
the observations, \((x_i, y_i)_{i=1}^{C_n}\) and \((x_i, y_i)_{i=(1-C)n}^{n}\), are stationary segments without change. As discussed before, this is a common assumption in the change-point testing literature and is also needed for the process convergence result in Theorem 1, see Andrews (1993) and references therein. For the choice of the trimming parameter \(C\), following the recommendation in Andrews (1993), we set \(C = 0.15\) throughout this section.

Given \((x_i, y_i)_{i=1}^{C_n}\), we conduct a standard 10-fold cross-validation to select tuning parameters \(\lambda_{pre}\). Based on \(\lambda_{pre}\), we estimate the sparsity level \(\hat{\lambda}_{pre}\) of \(\beta_{pre}\) in \((x_i, y_i)_{i=1}^{C_n}\) as the number of nonzero entries in \(\beta_{pre}\) and estimate the noise level via \(\hat{\sigma}_{pre} = \sqrt{\frac{\sum_{i=1}^{C_n} (y_i - x_i^T \hat{\beta}_{pre})^2}{(C_n - \hat{\lambda}_{pre})}}\). We estimate \(\hat{\lambda}_{post}\) and \(\hat{\sigma}_{post}\) for \((x_i, y_i)_{i=(1-C)n}^{n}\) in the same way. Finally, we set \(\lambda = (\lambda_{pre} + \lambda_{post})/2\), \(\hat{\lambda} = (\hat{\lambda}_{pre} + \hat{\lambda}_{post})/2\) and \(\hat{\sigma} = (\hat{\sigma}_{pre} + \hat{\sigma}_{post})/2\). Based on Assumption 2, we set the variance level of the random error as \(\sigma_{\xi} = \hat{\sigma} \log p / \sqrt{n} \times \log \log n\).

As can be seen from Assumption 2 and Theorem 1, for the validity of QF-CUSUM under \(H_0\), the theoretical choice for \(\sigma_{\xi}\) only requires the knowledge of the upper bound of \(s\), though an unnecessarily large upper bound will negatively impact the power of QF-CUSUM. For high-dimensional linear models, theoretical guarantees for the estimation of \(s\) have been well studied under mild conditions in the variable selection literature, see Meinshausen and Bühlmann (2006), Wainwright (2009), Fan and Lv (2010) and references therein. In addition, consistency for the estimation of the variance level \(\sigma_{\xi}^2\) has also been established in the high-dimensional literature, see for example Sun and Zhang (2012) and Guo et al. (2019).

### 4.1 Size performance

We simulate the data from the high-dimensional linear model

\[ y_i = x_i^T \beta^* + \epsilon_i, \quad i = 1, 2, \ldots, n, \]

where \(x_i \in \mathbb{R}^p\) denotes the \(p\)-dimensional covariates and \(\epsilon_i\) is the noise term with \(\sigma_{\xi}^2 = 1\).

We vary the sample size \(n\) across \{200, 400\} and the dimension \(p\) across \{100, 200, 400\}. For the covariance matrix \(\Sigma\) of \(x_i\), we consider two cases: Toeplitz \(\Sigma_{ij} = 0.6^{|i-j|}\) or compound symmetric \(\Sigma_{ij} = 0.3 \mathbb{I}(i \neq j) + \mathbb{I}(i = j)\). Denote \(s\) as the sparsity of \(\beta^*\), we vary \(s\) across \{5, 10\}. Given \(s\) and \(p\), define \(\beta^o(i) = i/s\) for \(i = 1, 2, \ldots, s\) and \(\beta^o(i) = 0\) for \(s < i \leq p\). We set \(\beta^* = 3 \beta^o / \sqrt{\beta^o \Sigma \beta^o}\) to keep the quadratic form at the same magnitude across different simulation settings.

As for the temporal dependence among \((x_i)_{i=1}^{n}\) and \((\epsilon_i)_{i=1}^{n}\), we consider three cases: independence (Ind.), auto-regressive (AR) dependence, and moving-average (MA) dependence. For AR dependence, we generate \((x_i)_{i=1}^{n}\) via \(x_i = 0.3 x_{i-1} + \sqrt{1 - 0.3^2} \epsilon_i\) with \(\epsilon_i \overset{i.i.d.}{\sim} N(0, \mathbb{I}_p)\) and
generate \( \{\epsilon_i\}_{i=1}^n \) via \( \epsilon_i = 0.3\epsilon_{i-1} + \sqrt{1 - 0.3^2}\epsilon'_i \) with \( \epsilon'_i \overset{i.i.d.}{\sim} N(0,1) \). For MA dependence, we generate \( \{x_i\}_{i=1}^n \) via \( x_i = (e_i + 0.4e_{i-1})/\sqrt{1 + 0.4^2} \) with \( e_i \overset{i.i.d.}{\sim} N(0,1) \) and generate \( \{\epsilon_i\}_{i=1}^n \) via \( \epsilon_i = (e'_i + 0.4e'_{i-1})/\sqrt{1 + 0.4^2} \) with \( e'_i \overset{i.i.d.}{\sim} N(0,1) \).

For each simulation setting with different \((n, p, s, \Sigma)\) and temporal dependence, we set the target size at 5% and repeat the experiments 500 times. Table 1 summarizes the empirical size of QF-CUSUM across different simulation settings. In general, the size of QF-CUSUM is reasonable and improves with the sample size \( n \). When the sample size is small \((n = 200)\), a large dimension \( p \) seems to cause an undersize of the test, and Toeplitz covariance exhibits more notable oversize than the CS covariance. However, these phenomena alleviate significantly when \( n \) increases, suggesting the validity of our asymptotic theory. The temporal dependence does inflate the empirical size, however, the inflation is not significant especially for \( n = 400 \). We note that Theorem 1 requires \( s \log(p)/\sqrt{n} \to 0 \) for the convergence of QF-CUSUM to the Gaussian process, which is not always the case for the simulation settings we consider. However, the empirical size achieved here is still reasonable, indicating the robustness of QF-CUSUM.

| \((n, p, s) / \Sigma\) | Ind. | AR | MA |
|------------------------|------|----|----|
| \((200,100,5)\) | T | CS | T | CS | T | CS |
| \((200,200,5)\) | 3.60 | 1.60 | 7.20 | 3.80 | 4.80 | 2.20 |
| \((200,400,5)\) | 2.80 | 1.40 | 4.80 | 2.40 | 3.60 | 2.20 |
| \((200,100,10)\) | 6.00 | 6.00 | 6.80 | 6.20 | 8.20 | 5.60 |
| \((200,200,10)\) | 2.20 | 2.60 | 4.20 | 4.00 | 5.00 | 2.80 |
| \((200,400,10)\) | 2.20 | 2.20 | 1.80 | 1.60 | 1.00 | 2.00 |
| \((400,100,5)\) | 6.40 | 4.20 | 5.40 | 5.80 | 7.20 | 4.60 |
| \((400,200,5)\) | 4.80 | 2.80 | 6.60 | 5.60 | 7.40 | 5.40 |
| \((400,400,5)\) | 4.80 | 4.40 | 5.80 | 4.80 | 7.00 | 4.80 |
| \((400,100,10)\) | 6.60 | 4.80 | 5.60 | 5.40 | 6.60 | 5.20 |
| \((400,200,10)\) | 4.00 | 3.00 | 5.40 | 5.20 | 7.40 | 6.00 |
| \((400,400,10)\) | 4.40 | 4.40 | 5.80 | 4.20 | 5.00 | 3.60 |

Table 1: Empirical size of QF-CUSUM averaged over 500 experiments under different simulation settings. T stands for Toeplitz and CS stands for compound symmetric.

Table 2 summarizes the empirical size of BSA, VPC and SGL. Interestingly, the three algorithms exhibit quite different patterns. BSA almost never rejects the null hypothesis with an empirical size close to 0, which is later shown to negatively impact its ability to detect small-scale changes. VPC also under-rejects under temporal independence, but suffers from false positives for the case of temporal dependence, especially for the MA setting, where the rejection rate can be as high as
15%. SGL suffers from over-rejection quite a bit across all simulation settings.

| (n, p, s) / Σ | BSA | VPC | SGL | BSA | VPC | SGL | BSA | VPC | SGL |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| T CS T CS T CS | T CS T CS T CS | T CS T CS T CS | T CS T CS T CS |
| (200,100,5)   | 0.0 | 0.0 | 1.2 | 0.0 | 70 | 78 | 0.0 | 0.0 | 11.8 | 8.2 | 71 | 76 | 0.0 | 0.0 | 12.6 | 8.2 | 72 | 74 |
| (200,200,5)   | 0.0 | 0.0 | 2.0 | 0.4 | 59 | 79 | 0.0 | 0.0 | 11.8 | 6.4 | 60 | 75 | 0.0 | 0.0 | 13.0 | 8.0 | 59 | 80 |
| (200,400,5)   | 0.0 | 0.0 | 1.6 | 0.4 | 75 | 76 | 0.0 | 0.0 | 9.4 | 6.2 | 75 | 74 | 0.0 | 0.0 | 12.2 | 6.0 | 79 | 79 |
| (200,100,10)  | 0.0 | 0.0 | 0.2 | 0.2 | 60 | 77 | 0.0 | 0.0 | 7.8 | 6.0 | 55 | 75 | 0.0 | 0.0 | 8.4 | 7.2 | 54 | 74 |
| (200,200,10)  | 0.0 | 0.0 | 0.4 | 0.0 | 60 | 77 | 0.0 | 0.0 | 8.4 | 5.2 | 66 | 75 | 0.0 | 0.0 | 8.6 | 5.6 | 62 | 74 |
| (200,400,10)  | 0.0 | 0.0 | 0.2 | 0.0 | 73 | 70 | 0.0 | 0.0 | 7.0 | 5.4 | 76 | 68 | 0.0 | 0.0 | 6.4 | 5.2 | 75 | 71 |
| (200,100,5)   | 0.2 | 0.4 | 2.4 | 0.0 | 92 | 91 | 0.0 | 0.2 | 13.4 | 7.2 | 94 | 92 | 0.2 | 0.4 | 15.2 | 11.4 | 93 | 91 |
| (200,200,5)   | 0.0 | 0.0 | 1.8 | 0.2 | 90 | 84 | 0.0 | 0.0 | 12.6 | 8.0 | 90 | 84 | 0.0 | 0.0 | 12.6 | 9.6 | 87 | 84 |
| (200,400,5)   | 0.0 | 0.0 | 1.0 | 0.0 | 81 | 82 | 0.0 | 0.0 | 12.8 | 6.2 | 81 | 81 | 0.0 | 0.0 | 15.2 | 8.2 | 82 | 85 |
| (200,100,10)  | 0.2 | 0.0 | 1.4 | 0.4 | 88 | 85 | 0.4 | 0.2 | 8.8 | 6.0 | 89 | 86 | 0.4 | 0.2 | 12.2 | 8.0 | 89 | 87 |
| (200,200,10)  | 0.0 | 0.0 | 0.2 | 0.0 | 87 | 73 | 0.0 | 0.0 | 8.2 | 5.4 | 88 | 72 | 0.0 | 0.0 | 12.8 | 5.6 | 83 | 76 |
| (200,400,10)  | 0.0 | 0.0 | 0.0 | 0.0 | 77 | 76 | 0.0 | 0.0 | 6.8 | 5.2 | 82 | 73 | 0.0 | 0.0 | 10.4 | 5.4 | 82 | 76 |

Table 2: Empirical size of BSA, VPC and SGL averaged over 500 experiments under different simulation settings. T stands for Toeplitz and CS stands for compound symmetric.

### 4.2 Power performance

In this section, we further investigate the power performance of QF-CUSUM under both single and multiple change-point settings. We further compare with BSA and VPC. We remove SGL from comparison as it is quite sensitive to false positives as shown in Section 4.1. For the single change-point scenario, we consider

\[ y_i = \begin{cases} 
  x_i^\top \beta^* + \epsilon_i, & \text{if } 1 \leq i \leq n/2 \\
  x_i^\top \beta^*(1 + \kappa) + \epsilon_i, & \text{if } n/2 + 1 \leq i \leq n,
\end{cases} \]

and for the multiple change-point scenario, we consider

\[ y_i = \begin{cases} 
  x_i^\top \beta^* + \epsilon_i, & \text{if } 1 \leq i \leq n/3 \\
  x_i^\top \beta^*(1 + \kappa) + \epsilon_i, & \text{if } n/3 + 1 \leq i \leq 2n/3, \\
  x_i^\top \beta^* + \epsilon_i, & \text{if } 2n/3 + 1 \leq i \leq n.
\end{cases} \]

We follow the same simulation setting as in Section 4.1. The only difference is that we vary \( \kappa \) to control the change size. For the same \( \kappa \), the change size across different simulation settings \((n, p, s, \Sigma)\) is the same as \( \beta^* \) is normalized (see Section 4.1 for details). We vary the change size \( \kappa^2 \) across \( \{0, 0.125, 0.25, 0.5, 0.75, 1\} \) to study the power performance. Note that we reduce to the no
change-point setting in Section 4.1 for $\kappa = 0$.

Figure 1 (left column) gives the size-adjusted power curves of QF-CUSUM under the single change-point case. To conserve space, we present the result with temporal independence. Appendix E further presents results under the AR and MA dependence, where the phenomenon observed is essentially the same, suggesting the robustness of QF-CUSUM to temporal dependence. In general, with the same signal $\kappa^2$, the power of QF-CUSUM decreases as the dimension $p$ and sparsity $s$ grow and increases as the sample size $n$ increases. In addition, changes under the CS covariance is more difficult to be detected than the ones under the Toeplitz covariance.

For comparison, Figure 1 (right column) gives the power curves of BSA and VPC under the same simulation setting. To conserve space, we only provide the result for $n = 200$ as the result for $n = 400$ is similar. Compared to QF-CUSUM, VPC provides similar (and for some cases slightly better) power performance under the single change-point setting. On the other hand, BSA seems to suffer from power loss when the change size is small and is notably sensitive to the dimension $p$.

Figure 2 (left column) gives the size-adjusted power curves of QF-CUSUM for the case of multiple change-points. The phenomenon is similar to the one in Figure 1. Compared to Figure 1, it can be seen that QF-CUSUM experiences power loss due to the presence of non-monotonic change. However, QF-CUSUM is still able to detect the structural breaks for strong signals, which can be seen as numerical evidence for Theorem 2.

For comparison, Figure 2 (right column) gives the power curves of BSA and VPC under the same simulation setting. Compared to QF-CUSUM, VPC provides similar but slightly worse power performance, especially when the change size is small. BSA again suffers from power loss and is notably sensitive to the dimension $p$.

Localization result: QF-CUSUM is proposed for change-point testing, i.e. to detect the existence of change-points. On the other hand, after the null hypothesis is rejected, it is possible to further use the test statistic to estimate the change-point location. Though this is not the focus of our paper, we give a brief numerical illustration here for the case of single change-point estimation. Specifically, once QF-CUSUM rejects $H_0$, we estimate the location of the change-point as

$$\tilde{\eta} = \max_{t = \lfloor n \zeta \rfloor, \lfloor n \zeta \rfloor + 1, \ldots, \lfloor n(1 - \zeta) \rfloor} \tilde{T}_n(t),$$

where $\tilde{T}_n(t)$ equals to $T_n(t)$ defined in (5) with $\xi_i \equiv 0$ for all $i$. We remove the randomized error $\xi$ for change-point localization as it is mainly introduced to ensure a pivotal and non-degenerate asymptotic distribution under $H_0$ but does not play a role under $H_0$.

For illustration, we focus on the simulation setting with temporal independence and Toeplitz
Figure 1: Power performance of QF-CUSUM (left column), BSA and VPC (right column) for different simulation settings under the single change-point case with temporal independence.
Figure 2: Power performance of QF-CUSUM (left column), BSA and VPC (right column) for different simulation settings under the multiple change-point case with temporal independence.
covariance matrix, and set $s = 5$. We vary the sample size $n$ across $\{200, 400\}$, the dimension $p$ across $\{100, 200, 400\}$ and vary the change size $\kappa^2$ across $\{0.25, 0.5, 1\}$. Figure 3 provides the kernel density estimation for $\hat{\eta}/n$ based on 500 experiments for each simulation setting. As can be seen, $\hat{\eta}$ centers around the true change-point and its accuracy improves as $n$ increases and is relatively robust to the dimension $p$.

### 4.3 Real data application

In this section, we apply QF-CUSUM to test if there is any structural break in the relationship between the monthly growth rate of the US industrial production (IP) index, an important indicator of macroeconomic activity, and 127 other macroeconomic variables from Federal Reserve Economic Database (FRED-MD)\(^2\) (McCracken and Ng, 2016). The empirical analysis in He et al. (2022) suggests that a high-dimensional linear model based on the 127 predictors are overall significant for forecasting IP. Here, we further test if there is any change in the high-dimensional linear model.

Specifically, our response variable is $y_t = \log(\text{IP}_t / \text{IP}_{t-1}) \times 100$, where IP$_t$ denotes the US industrial production index for the month $t$. In other words, $y_t$ measures the monthly growth rate of IP (in percentage scale). The high-dimensional covariate includes 127 macroeconomic variables recorded for the month $t - 1$. We transform the raw data, i.e. the response and each individual covariate, into stationary time series and remove outliers using the MATLAB codes provided on the FRED-MD website, see also McCracken and Ng (2016) for more details. We focus our analysis on the time period from June 2005 to March 2022, which includes $n = 200$ observations.

The implementation of QF-CUSUM follows the same setting as in Sections 4.1-4.2. The only difference is that we set the trimming parameter $\zeta = 24/200 = 0.12$ instead of 0.15 to make the choice more interpretable (i.e. 2 years). The result based on $\zeta = 0.15$ is essentially the same. Figure 4 (left) plots the QF-CUSUM statistic $T_n(t)$ computed over the sample, where the null hypothesis of no change-point is clearly rejected at the 5% nominal level. Figure 4 (right) further plots the statistic $\tilde{T}_n(t)$ defined in (7), which gives an estimated change-point at Oct. 2019, close to the beginning of the Covid-19 pandemic. Note that the two statistics $T_n(t)$ and $\tilde{T}_n(t)$ are similar to each other, indicating that the randomized error $\{\xi_i\}_{i=1}^n$ is rather negligible compared to the change size, which thus suggests the robustness of our finding.

We further implement BSA, VPC and SGL for comparison. BSA detects no change-point. VPC estimates two change-points, one at Jul. 2009 and one at May 2019. SGL estimates three change-points, one at Jul. 2009 and one at May 2019. SGL estimates three

\(^2\)The dataset is publicly available at https://research.stlouisfed.org/econ/mccracken/fred-databases.
Figure 3: Localization performance of QF-CUSUM for different simulation settings under the single change-point case with temporal independence and $\Sigma = T$ and $s = 5$. 
change-points, Oct. 2005, Jun. 2020 and Apr. 2021. These results provide additional evidence that the Covid-19 pandemic seems to alter the relationship between the US industrial production index and the macroeconomic predictors considered.

5 Conclusion

In this paper, we study the problem of change-point testing for high-dimensional linear models. We propose QF-CUSUM, a quadratic-form based CUSUM test to inspect the stability of the regression coefficients in a high-dimensional linear model. QF-CUSUM can control the asymptotic type-I error at any desired level and is theoretically sound for temporally dependent observations. Furthermore, QF-CUSUM achieves the optimal minimax lower bound of the detection boundary for a wide class of high-dimensional linear models. Through extensive numerical experiments and a real data application in macroeconomics, we demonstrate the effectiveness and utility of QF-CUSUM. The focus of this paper is change-point testing. As for future research, a potential direction is to further investigate the theoretical properties of the change-point estimator based on QF-CUSUM, such as its consistency and optimal convergence rate.
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25
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27
Appendix

A Main Results related to the Quadratic Form

Throughout the appendix, we assume that \( p \geq n^\alpha \) for some \( \alpha > 0 \). It follows that

\[
\log(pn) = O(\log(p)).
\]

This is a convenience assumption commonly used in the high-dimensional statistical literature.

We begin by introducing the following conditions.

**Condition 1.** Let \( \zeta \in (0, 1/2) \) be any given constant and that \( \{\xi_i\}_{i=1}^n \) i.i.d. \( N(0, \sigma^2) \) be a collection of user-generated random variables. Suppose that for all interval \( I = (0, t] \) or \( (t, n] \) such that \(|I| \geq \zeta n\), the following additional conditions hold.

- **a.** Let \( C_S := \{v \in \mathbb{R}^p : \|v_S\|_1 \leq 3\|v_S\|_1\} \). Suppose that

  \[
  \left| v^\top (\Sigma_I - \Sigma)v \right| \leq C \sqrt{\frac{n \log(pn)}{|I|}} \|v\|_2^2
  \]
  
  for all \( v \in C_S \).

- **b.** Suppose that

  \[
  \left| \frac{1}{|I|} \sum_{i \in I} \epsilon_i x_i x_i^\top \beta \right| \leq C \sqrt{\frac{\log(pn)}{|I|}} \|\beta\|_1
  \]
  
  for all \( \beta \in \mathbb{R}^p \).

- **c.** Let \( \{u_i\}_{i=1}^n \) be a collection of non-random vectors. Suppose that

  \[
  \left| \frac{1}{|I|} \sum_{i \in I} u_i^\top x_i x_i^\top \beta - \frac{1}{|I|} \sum_{i \in I} u_i^\top \Sigma \beta \right| \leq C \left( \max_{1 \leq i \leq n} \|u_i\|_2 \right) \sqrt{\frac{\log(pn)}{|I|}} \|\beta\|_1
  \]
  
  for all \( \beta \in \mathbb{R}^p \).

- **d.** Suppose that

  \[
  \left| \frac{1}{|I|} \sum_{i \in I} \xi_i x_i x_i^\top \beta \right| \leq C \sigma \sqrt{\frac{\log(pn)}{|I|}} \|\beta\|_1
  \]
  
  for all \( \beta \in \mathbb{R}^p \).
Condition 2. For any \( r \in (0, 1) \) let \( t = \lfloor rn \rfloor \) and 
\[
\mathcal{G}_n(r) = \frac{\sqrt{n}}{\sigma_\xi \sigma_\epsilon} \left\{ \frac{1}{t} \sum_{i=1}^{t} \epsilon_i \xi_i - \frac{1}{n-t} \sum_{i=t+1}^{n} \epsilon_i \xi_i \right\}.
\]
Suppose that for \( r \in (0, 1) \),
\[
\mathcal{G}_n(r) \overset{D}{\to} \mathcal{G}(r),
\]
where the convergence is in Skorokhod topology and \( \mathcal{G}(r) \) is a Gaussian Process on \( (0, 1) \) with covariance function
\[
\sigma(r, s) = \frac{1}{r(1-s)} \quad \text{when} \quad 0 < s \leq r < 1.
\]

In Appendix A.6, we show that Condition 1 and Condition 2 hold with high probability when the random quantities \( \{x_i, \epsilon_i\}_{i=1}^n \) satisfy some commonly used distribution assumptions in high-dimensional statistics.

A.1 Null distribution under \( H_0 \)

Proof of Theorem 1:

Proof. We justify all the technical results supporting Theorem 1 under Condition 1 and Condition 2. Let \( \mathcal{R}_1(t), \mathcal{R}_2(t), \mathcal{R}_3(t) \) be defined as Proposition 1 and \( \mathcal{R}'_1(t), \mathcal{R}'_2(t), \mathcal{R}'_3(t) \) be defined as Proposition 2.

So with \( t = \lfloor rn \rfloor \),
\[
\mathcal{T}_n := \frac{\sqrt{n}}{2\sigma_\epsilon \sigma_\xi} \left\{ (\hat{\beta}_{0, t} - \hat{\beta}_{t, n})^\top \hat{\Sigma}_{0, t} (\hat{\beta}_{0, t} - \hat{\beta}_{t, n}) + \mathcal{R}_1(t) + 2\mathcal{R}_2(t) - 2\mathcal{R}_3(t) \right.
\[
+ \left. ((\hat{\beta}_{0, t} - \hat{\beta}_{t, n})^\top \hat{\Sigma}_{t, n} (\hat{\beta}_{0, t} - \hat{\beta}_{t, n}) + \mathcal{R}'_1(t) + 2\mathcal{R}'_2(t) - 2\mathcal{R}'_3(t) \right\}.
\]

Observe that under \( H_0 \), \( \beta^*_{0, t} = \beta^*_{t, n} = \beta^*_1 \) for all \( t \in \{1, \ldots, n\} \).

So for all \( t \),
\[
(\beta^*_{0, t} - \beta^*_{t, n})^\top \Sigma (\beta^*_{0, t} - \beta^*_{t, n}) = 0
\]
\[
\mathcal{T}_{n, \xi}(t) = \frac{2}{t} \sum_{i=1}^{t} \epsilon_i \xi_i, \quad \mathcal{S}_n(t) = 0, \quad \mathcal{T}'_{n, \xi}(t) = \frac{2}{n-t} \sum_{i=t+1}^{n} \epsilon_i \xi_i, \quad \mathcal{S}'_n(t) = 0,
\]
where \( \mathcal{T}_{n, \xi}(t), \mathcal{S}_n(t) \) are defined in Proposition 1 and \( \mathcal{T}'_{n, \xi}(t), \mathcal{S}'_n(t) \) are defined in Proposition 2.
Consequently by Proposition 1, it holds that
\[
\frac{\sqrt{n}}{\sigma_e \sigma_\xi} \left\{ \left( \tilde{\beta}_{(0,t)} - \tilde{\beta}_{(t,n)} \right)^\top \Sigma_{(0,t)} \left( \tilde{\beta}_{(0,t)} - \tilde{\beta}_{(0,t)} \right) + R_1(t) + 2R_2(t) - 2R_3(t) - \frac{2}{t} \sum_{i=1}^t \epsilon_i \xi_i \right\} \leq \frac{\sqrt{n}}{\sigma_e \sigma_\xi} C_1 (1 + \sigma_\xi) \frac{\log(pm)}{n} = C_1 \frac{\log(pm)}{\sigma_e \sigma_\xi \sqrt{n}} + C_1 \frac{\log(pm)}{\sigma_e \sqrt{n}}
= o(1).
\]

By Proposition 2, it holds that
\[
\frac{\sqrt{n}}{\sigma_e \sigma_\xi} \left\{ \left( \tilde{\beta}_{(0,t)} - \tilde{\beta}_{(t,n)} \right)^\top \Sigma_{(t,n)} \left( \tilde{\beta}_{(0,t)} - \tilde{\beta}_{(t,n)} \right) + R_1'(t) + 2R_2'(t) - 2R_3'(t) + \frac{2}{n - t} \sum_{i=t+1}^n \epsilon_i \xi_i \right\} \leq \frac{\sqrt{n}}{\sigma_e \sigma_\xi} C_2 (1 + \sigma_\xi) \frac{\log(pm)}{n} = C_2 \frac{\log(pm)}{\sigma_e \sigma_\xi \sqrt{n}} + C_2 \frac{\log(pm)}{\sigma_e \sqrt{n}}
= o(1).
\]
The above two displays together with Condition 2 directly give the desired result. \( \square \)

A.2 Power analysis under \( H_1 \)

Proof of Theorem 2:

Proof. Suppose \( H_1(B_n) \) holds. In view of Lemma 19, there exists \( t' \in [\zeta n, (1 - \zeta)n] \) such that
\[
(\beta^*_{(0,t')} - \beta^*_{(t',n)})^\top \Sigma(\beta^*_{(0,t')} - \beta^*_{(t',n)}) := \kappa_1^2 \geq B_n \frac{\log(pm)}{n}. \tag{8}
\]
It suffices to show that for sufficiently large \( n \), with probability goes to 1,
\[
\frac{\sqrt{n}}{2\sigma_e \sigma_\xi} \left\{ \left( \tilde{\beta}_{(0,t')} - \tilde{\beta}_{(t',n)} \right)^\top \Sigma_{(0,t')} \left( \tilde{\beta}_{(0,t')} - \tilde{\beta}_{(t',n)} \right) + R_1(t') + 2R_2(t') - 2R_3(t') \right\} > G_0, \tag{9}
\]
\[
\frac{\sqrt{n}}{2\sigma_e \sigma_\xi} \left\{ \left( \tilde{\beta}_{(t',n)} - \tilde{\beta}_{(0,t')} \right)^\top \Sigma_{(t',n)} \left( \tilde{\beta}_{(t',n)} - \tilde{\beta}_{(0,t')} \right) + R_1'(t') + 2R_2'(t') - 2R_3'(t') \right\} > G_0. \tag{10}
\]
The justification of Equation (9) is provided as the justification of Equation (10) is the same by symmetry.

Step 1. Under the alternative \( H_1(B_n) \), note that by Proposition 1, for all \( t \in [\zeta n, (1 - \zeta)n] \),
\[
\left\{ \left( \tilde{\beta}_{(0,t)} - \tilde{\beta}_{(t,n)} \right)^\top \Sigma_{(0,t)} \left( \tilde{\beta}_{(0,t)} - \tilde{\beta}_{(t,n)} \right) + R_1(t) + 2R_2(t) - 2R_3(t) 
- (\beta^*_{(0,t)} - \beta^*_{(t,n)})^\top \Sigma(\beta^*_{(0,t)} - \beta^*_{(t,n)}) - (\Xi_{n, \xi}(t) + G_n(t)) \right\} \leq C_1 (1 + \sigma_\xi) \frac{\log(pm)}{n}.
\]
Then by Equation (8),
\[
(\beta_{(0,t')} - \hat{\beta}_{(t',n)})^\top \Sigma_{(0,t')} (\beta_{(0,t')} - \hat{\beta}_{(t',n)}) + R_1(t') + 2R_2(t') - 2R_3(t') 
\geq \kappa^2 - |\mathfrak{S}_{n,\xi}(t')| - |\mathfrak{S}_n(t')| - C_1(1 + \sigma_\xi)\frac{5 \log(pn)}{n}.
\]  
(11)

So it suffices to show that
\[
\kappa^2 - |\mathfrak{S}_{n,\xi}(t')| - |\mathfrak{S}_n(t')| - C_1(1 + \sigma_\xi)\frac{5 \log(pn)}{n} \geq \frac{2 \log p}{\sqrt{n}}.
\]  
(12)

**Step 2.** By definition of $\mathfrak{S}_{n,\xi}(t')$,
\[
\mathfrak{S}_{n,\xi}(t') = \frac{2}{t'} \sum_{i=1}^{t'} x_i^\top (\beta^*_i - \beta^*_i) \epsilon_i + \frac{2}{t'} \sum_{i=1}^{t'} x_i^\top (\beta^*_i - \beta^*_i) \xi_i + \frac{2}{t'} \sum_{i=1}^{t'} \xi_i \epsilon_i 
+ (\beta^*_{(0,t')} - \beta^*_{(t',n)})^\top (\hat{\Sigma}(0,t') - \Sigma)(\beta^*_i - \beta^*_i) + \frac{2}{t'} \sum_{i=1}^{t'} (\beta^*_i - \beta^*_i)^\top x_i x_i^\top (\beta^*_i - \beta^*_i).
\]

By Assumption 1, $\{x_i\}_{i=1}^n$ and $\{\epsilon_i\}_{i=1}^n$ are sub-Gaussian random variables. Let $D_n$ be some slowly diverging sequence to be specified. By standard sub-exponential tail bounds and the fact that $t' \in [\zeta n, (1 - \zeta)n]$, it holds that with probability goes to 1,
\[
|\frac{2}{t'} \sum_{i=1}^{t'} x_i^\top (\beta^*_i - \beta^*_i) \epsilon_i| \leq C_2 \sqrt{\frac{D_n}{n}} \sqrt{(\beta^*_{(0,t')} - \beta^*_{(t',n)})^\top \Sigma(\beta^*_{(0,t')} - \beta^*_{(t',n)})} = C_2 \sqrt{\frac{D_n}{n}},
\]
\[
(\beta^*_{(0,t')} - \beta^*_{(t',n)})^\top (\hat{\Sigma}(0,t') - \Sigma)(\beta^*_{(0,t')} - \beta^*_{(t',n)}) \leq C_3 \sqrt{\frac{D_n}{t'} \|\beta^*_{(0,t')} - \beta^*_{(t',n)}\|^2} \leq C_3 \sqrt{\frac{D_n}{n} \kappa^2},
\]
\[
|\frac{2}{t'} \sum_{i=1}^{t'} \xi_i \epsilon_i| \leq C_4 \sigma_\xi \sqrt{\frac{D_n}{n}} \quad \text{and} \quad |\frac{2}{t'} \sum_{i=1}^{t'} x_i^\top (\beta^*_i - \beta^*_i) \xi_i| \leq C_4 \sigma_\xi \sqrt{\frac{D_n}{n}},
\]

In addition, since
\[
\frac{1}{t'} \sum_{i=1}^{t'} (\beta^*_i - \beta^*_i) \Sigma (\beta^*_i - \beta^*_i) = 0,
\]

31
by standard sub-exponential tail bounds, it holds that
\[
\left| \frac{1}{t'} \sum_{i=1}^{t'} (\beta^*_{[0,t']} - \beta^*_{[t',n]}) x_i x_i^\top (\beta^*_i - \beta^*_{[0,t']}) \right| = \frac{1}{t'} \sum_{i=1}^{t'} (\beta^*_{[0,t']} - \beta^*_{[t',n]}) x_i x_i^\top (\beta^*_i - \beta^*_{[0,t']}) \leq C_5 \sqrt{D_n \frac{\kappa}{n}} \max_{1 \leq i \leq n} \| \beta^*_i - \beta^*_{[0,t']} \|_2 \leq C_5' \sqrt{D_n \frac{\kappa}{n}},
\]
where
\[
\sum_{i=1}^{t'} (\beta^*_{[0,t']} - \beta^*_{[t',n]}) \Sigma (\beta^*_i - \beta^*_{[0,t']}) = 0
\]
is used in the equality. So
\[
|\mathcal{S}_n(t')| \leq C_6 \sqrt{\frac{D_n}{n}} + C_6' \sqrt{\frac{D_n}{n}} \kappa^2 + C_6'' \kappa \sqrt{\frac{D_n}{n}}.
\]
By definition of \( \mathcal{S}_n(t') \),
\[
\frac{1}{2} \mathcal{S}_n(t') = \frac{1}{n - t'} \sum_{i=t'+1}^{n} x_i^\top (\beta^*_{[0,t']} - \beta^*_{[t',n]}) \epsilon_i + \frac{1}{n - t'} \sum_{i=t'+1}^{n} (\beta^*_{[t',n]} - \beta^*_{[0,t']}) x_i x_i^\top (\beta^*_i - \beta^*_{[t',n]}).
\]
Note that by standard sub-exponential tail bounds, it holds that with probability goes to 1,
\[
\left| \frac{1}{n - t'} \sum_{i=t'+1}^{n} x_i^\top (\beta^*_{[0,t']} - \beta^*_{[t',n]}) \epsilon_i \right| \leq \sqrt{\frac{D_n}{n - t'} \kappa}
\]
and
\[
\left| \frac{1}{n - t'} \sum_{i=t'+1}^{n} (\beta^*_{[0,t']} - \beta^*_{[t',n]}) x_i x_i^\top (\beta^*_i - \beta^*_{[t',n]}) \right| = \frac{1}{n - t'} \sum_{i=t'+1}^{n} (\beta^*_{[0,t']} - \beta^*_{[t',n]}) x_i x_i^\top (\beta^*_i - \beta^*_{[t',n]}) - (\beta^*_{[0,t']} - \beta^*_{[t',n]}) \Sigma (\beta^*_i - \beta^*_{[t',n]}) \leq C_7 \sqrt{\frac{D_n}{n - t'} \kappa} \max_{1 \leq i \leq n} \| \beta^*_i - \beta^*_{[t',n]} \|_2 \leq C_7' \sqrt{\frac{D_n}{n \kappa}}.
\]
So with probability goes to 1,
\[
|\mathcal{S}_n(t')| \leq C_8 \sqrt{\frac{D_n}{n \kappa}}.
\]

**Step 3.** Suppose \( D_n \) is some slowly divergent sequence such that \( D_n = o(\log(n)) \) and \( \sqrt{D_n} \leq
\(B_n/A_n\), where we note that by assumption, \(B_n/A_n \to \infty\). So

\[
(11) \geq \kappa^2 - C_9 \left( \sqrt{\frac{D_n}{n}} \kappa + \sqrt{\frac{D_n}{n}} \kappa^2 + \frac{\sqrt{D_n} \sigma \xi}{\sqrt{n}} + (1 + \sigma \xi) \frac{\log(pn)}{n} \right) \geq \frac{1}{2} \kappa^2,
\]

where the last inequality holds because \(\sigma \xi = A_n \frac{\log(pn)}{\sqrt{n}}\), \(\kappa^2 \geq B_n \frac{\log(pn)}{n}\). Since \(\kappa^2 \geq B_n \frac{\log(pn)}{n}\) and \(\sigma \xi = A_n \frac{\log(pn)}{\sqrt{n}}\), for sufficiently large \(B_n\),

\[
(11) \geq \frac{1}{2} \kappa^2 > 2\sigma \xi \frac{\log(pn)}{\sqrt{n}} = 2\sigma \xi \frac{\log(pn)}{\sqrt{n}}.
\]

This above inequality directly implies that the power of test goes 1 under \(H_1(B_n)\).

\[\square\]

### A.3 Asymptotic distribution under \(H_1\)

**Proof of Theorem 3:**

**Proof.** Let \(r \in [\zeta, 1 - \zeta]\) and \(t = \lfloor rn \rfloor\). Without loss of generality, assume that \(\sigma = 1\). Let \(\Xi_{n, \xi}(t)\) and \(\Theta_{n}(t)\) be defined as in Proposition 1, and \(\Xi'_{n, \xi}(t)\) and \(\Theta'(t)\) be defined as in Proposition 2. Denote

\[
\Xi_{n, \xi}(t) + \Theta_{n}(t) + \Xi'_{n, \xi}(t) + \Theta'(t) = \mathcal{L}(r) + \mathcal{R}(r),
\]

where

\[
\mathcal{L}(r) = \frac{4}{t} \sum_{i=1}^{t} x_i^T (\beta_{[0,t]} - \beta_{[t,n]}^*) \xi_i + \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta_i - \beta_{[0,t]}^*) \xi_i + \frac{2}{t} \sum_{i=1}^{t} \xi_i \xi_i,
\]

\[
+ (\beta_{[0,t]}^* - \beta_{[t,n]}^*)^T (\hat{\Sigma}_{(0,t)} - \Sigma)(\beta_{[0,t]}^* - \beta_{[t,n]}^*);\]

\[
\mathcal{R}(r) = \frac{4}{n - t} \sum_{i=t+1}^{n} x_i^T (\beta_{[t,n]} - \beta_{[0,t]}^*) \xi_i + \frac{2}{n - t} \sum_{i=t+1}^{n} x_i^T (\beta_i - \beta_{[t,n]}^*) \xi_i - \frac{2}{n - t} \sum_{i=t+1}^{n} \xi_i \xi_i.
\]

Note that by assumption,

\[
\psi_L(r) \geq 4\sigma^2 \xi. \quad (13)
\]

Similarly, we have that

\[
\psi_R(r) \geq 4\sigma^2 \xi. \quad (14)
\]
Step 1. We have that with probability goes to 1,
\[
\left( \frac{\psi_L(r)}{nr} + \frac{\psi_R(r)}{n(1-r)} \right)^{-1} \left| \mathcal{S}_n(r) - 2\mu(r) - \mathcal{L}(r) - \mathcal{R}(r) \right|
\]
\[
= \left( \frac{\psi_L(r)}{r} + \frac{\psi_R(r)}{n-r} \right)^{-1} \left| \mathcal{S}_n(r) - 2\mu(r) - \mathcal{T}_n \xi(t) - \mathcal{S}_n(t) - \mathcal{T}_n^\prime \xi(t) - \mathcal{S}_n^\prime(t) \right|
\]
\[
\leq C(1 + \sigma_\xi \frac{s \log(pn)}{n}) \left( \frac{\psi_L(r)}{nr} + \frac{\psi_R(r)}{n(1-r)} \right)^{-1}
\]
\[
\leq C'(1 + \sigma_\xi \frac{s \log(pn)}{n}) \left( \sqrt{\frac{\sigma_\xi^2}{nr} + \frac{\sigma_\xi^2}{n(1-r)}} \right)^{-1}
\]
\[
\leq C''(1 + \sigma_\xi \frac{s \log(pn)}{\sqrt{n}}) \left( \frac{\sigma_\xi^2}{r} + \frac{\sigma_\xi^2}{1-r} \right)^{-1}
\]
\[
\leq C''(1 + \sigma_\xi \frac{s \log(pn)}{\sigma_\xi \sqrt{n}}),
\]
where the first inequality follows from Proposition 1 and Proposition 2, the second inequality follows from Equation (13) and Equation (14). By assumption, \( \frac{s \log(pn)}{\sqrt{n}} = o(1) \), \( \sigma_\xi = A_n \frac{s \log(pn)}{\sqrt{n}} \) for some diverging sequence \( A_n \). So \( \frac{(1+\sigma_\xi) s \log(pn)}{\sigma_\xi \sqrt{n}} = o(1) \).

Consequently it suffices to show that
\[
\sqrt{\frac{nr}{\psi_L(r)}} \mathcal{L}(r) \quad \text{and} \quad \sqrt{\frac{n(1-r)}{\psi_R(r)}} \mathcal{R}(r)
\]
both converge to \( N(0,1) \). For brevity, only the analysis for \( \mathcal{L}(r) \) is shown. The analysis of \( \mathcal{R}(r) \) is similar and will be omitted.

Step 2. Let \( r \in [\zeta, 1-\eta] \) and \( t = [rn] \). For \( 1 \leq i \leq t \), let
\[
l_1(i) = 4x_i^T (\beta_{[0,t]}^* - \beta_{[t,n]}^*) \xi_i + 2\xi_i \xi_i + (\beta_{[0,t]}^* - \beta_{[t,n]}^*)^T (x_i x_i^T - \Sigma)(\beta_{[0,t]}^* - \beta_{[t,n]}^*),
\]
\[
l_2(i) = 2x_i^T (\beta_i^* - \beta_{[0,t]}^*) \xi_i.
\]
Since
\[
\mathcal{L}(r) = \frac{1}{t} \sum_{i=1}^{t} l_1(i) + \frac{1}{t} \sum_{i=1}^{t} l_2(i),
\]
it suffices to show that \( \frac{1}{\sqrt{t}} \sum_{i=1}^{t} l_1(i) \) and \( \frac{1}{\sqrt{t}} \sum_{i=1}^{t} l_2(i) \) are asymptotically normal. Since \( \{ l_1(i) \}_{i=1}^{n} \) is a collection of i.i.d. random variables with second finite moments,
\[
\frac{1}{\sqrt{t \text{Var}(l_1(i))}} \sum_{i=1}^{t} l_1(i) \rightarrow N(0,1).
\]
Let \( \{\eta_q^*\}_{q=1}^Q = \{\eta_k^*\}_{k=1}^K \cap (0, r] \), where \( Q = 0 \) indicates that \( (0, r] \) contains no change points. Denote \( \eta_k = [n \eta_k^*] \) and

\[
J_1 = (0, \eta_1], \; J_2 = (\eta_1, \eta_2] \; \ldots \; J_Q = (\eta_Q, \eta_Q], \; J_{Q+1} = (\eta_Q, t].
\]

Observe that for \( i \in J_q \)

\[
\Var(l_2(i)) = 4\sigma^2(\beta_{nq}^* - \beta_{0,q}^*)^\top \Sigma(\beta_{nq}^* - \beta_{0,q}^*)
\]

and that

\[
\mathbb{E}(l_2(i)^4) = 16\mathbb{E}(\xi_i^4)\mathbb{E}\left\{x_i^\top(\beta_i^* - \beta_{0,i}^*)^4\right\} \leq C_3\sigma^4\xi_i^4\left\{ (\beta_{nq}^* - \beta_{0,q}^*)^\top \Sigma(\beta_{nq}^* - \beta_{0,q}^*) \right\}^2
\]

where the second inequality holds because \( \xi_i \) and \( x_i^\top(\beta_i^* - \beta_{0,i}^*) \) are both centered sub-Gaussian random variables and so there exists \( C_2 > 0 \) such that \( \sqrt{\mathbb{E}(\xi_i^4)} \leq C_2\mathbb{E}(\xi_i^2) = C_2\sigma^2 \) and

\[
\sqrt{\mathbb{E}\left\{x_i^\top(\beta_i^* - \beta_{0,i}^*)^4\right\}} \leq C_2\mathbb{E}\left\{x_i^\top(\beta_i^* - \beta_{0,i}^*)^2\right\} = C_2(\beta_{nq}^* - \beta_{0,q}^*)^\top \Sigma(\beta_{nq}^* - \beta_{0,q}^*).
\]

Denote \( \delta_q = \eta_q^* - \eta_{q-1}^* \) for \( 1 \leq q \leq Q \) and \( \delta_{Q+1} = r - \eta_Q^* \). We have

\[
\sum_{i=1}^t \frac{\mathbb{E}(l_2(i)^4)}{\left\{ \sum_{i=1}^t \Var(l_2(i)) \right\}^2} \leq \sum_{q=1}^{Q+1} \delta_q n C_3 \sigma^4 \xi_i^4 \left\{ (\beta_{nq}^* - \beta_{0,q}^*)^\top \Sigma(\beta_{nq}^* - \beta_{0,q}^*) \right\}^2
\]

\[
= C_3 \sum_{q=1}^{Q+1} \delta_q n \left\{ (\beta_{nq}^* - \beta_{0,q}^*)^\top \Sigma(\beta_{nq}^* - \beta_{0,q}^*) \right\}^2
\]

\[
\leq C_3 \sum_{q=1}^{Q+1} \delta_q n \left\{ (\beta_{nq}^* - \beta_{0,q}^*)^\top \Sigma(\beta_{nq}^* - \beta_{0,q}^*) \right\}^2
\]

\[
\leq C_3 \sum_{q=1}^{Q+1} \frac{\delta_q n}{\delta^2 n^2},
\]

where the first inequality follows from Equation (15) and Equation (16). Thus

\[
\lim_{n \to \infty} \frac{\sum_{i=1}^t \mathbb{E}(l_2(i)^4)}{\left\{ \sum_{i=1}^t \Var(l_2(i)) \right\}^2} \to 0
\]

and by Lyapunov’s Central Limit Theorem,

\[
\frac{\sum_{i=1}^t l_2(i)}{\sqrt{\sum_{i=1}^t \Var(l_2(i))}} \to N(0, 1)
\]
as desired. 

A.4 Additional Technical Results

**Proposition 1.** Suppose \( \{\xi_i\}_{i=1}^{n} \overset{i.i.d.}{\sim} N(0, \sigma^2) \). Suppose Assumption 1 and Condition 1 hold. Let

\[
\mathcal{R}_1(t) = 2 \sum_{i=1}^{t} (y_i - x_i^T \hat{\beta}_{(0,t)})(x_i^T (\hat{\beta}_{(0,t)} - \hat{\beta}_{(t,n)}) + \xi_i),
\]

\[
\mathcal{R}_2(t) = \frac{1}{n-t} \sum_{i=t+1}^{n} x_i^T \hat{\beta}_{(t,n)} (y_i - x_i^T \hat{\beta}_{(t,n)}),
\]

\[
\mathcal{R}_3(t) = \frac{1}{n-t} \sum_{i=t+1}^{n} x_i^T \hat{\beta}_{(0,t)} (y_i - x_i^T \hat{\beta}_{(t,n)}).
\]

Denote

\[
\mathcal{T}_{n,\xi}(t) = 2 \sum_{i=1}^{t} x_i^T (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \epsilon_i + 2 \sum_{i=1}^{t} x_i^T (\beta_{i}^* - \beta_{(0,t)}^*) \xi_i + 2 \sum_{i=1}^{t} \xi_i \epsilon_i
\]

\[
+ (\beta_{(0,t)}^* - \beta_{(t,n)}^*)^T (\hat{\Sigma}_{(0,t)} - \Sigma)(\beta_{(0,t)}^* - \beta_{(t,n)}^*) + 2 \sum_{i=1}^{t} x_i^T (\beta_{i}^* - \beta_{(0,t)}^*) x_i^T (\beta_{(0,t)}^* - \beta_{(t,n)}^*)
\]

\[
\mathcal{S}_n(t) = \frac{2}{n-t} \sum_{i=t+1}^{n} x_i^T (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \epsilon_i + \frac{2}{n-t} \sum_{i=t+1}^{n} (\beta_{(0,t)}^* - \beta_{(t,n)}^*) x_i x_i^T (\beta_{i}^* - \beta_{(0,t)}^*).
\]

For all \( t \in [\xi_n, (1-\xi)n] \), it holds that

\[
\left| (\beta_{(0,t)}^* - \beta_{(t,n)}^*)^T \hat{\Sigma}_{(0,t)}(\beta_{(0,t)}^* - \beta_{(t,n)}^*) + \mathcal{R}_1(t) + 2\mathcal{R}_2(t) - 2\mathcal{R}_3(t) - (\beta_{(0,t)}^* - \beta_{(t,n)}^*)^T \Sigma(\beta_{(0,t)}^* - \beta_{(t,n)}^*) - \{ \mathcal{T}_{n,\xi}(t) + \mathcal{S}_n(t) \} \right| 
\]

\[
\leq C(1 + \sigma_\xi) \frac{\varepsilon \log(pn)}{n}.
\]

**Proof.** In view of Theorem 7, it suffices to bound \( |\mathcal{R}_1(t) - \mathcal{R}_4(t)| \). Observe that

\[
\mathcal{R}_1(t) - \mathcal{R}_4(t) = 2 \sum_{i=1}^{t} (y_i - x_i^T \hat{\beta}_{(0,t)}) \xi_i = 2 \sum_{i=1}^{t} \epsilon_i \xi_i + 2 \sum_{i=1}^{t} x_i^T (\beta_{i}^* - \beta_{(0,t)}^*) + (\beta_{(0,t)}^* - \beta_{(0,t)}^*) \xi_i
\]

Note that by Condition 1d and Lemma 1,

\[
\left| \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta_{(0,t)}^* - \beta_{(0,t)}^*) \xi_i \right| \leq C_1 \sigma_\xi \sqrt{\frac{\log(p)}{t}} \|\beta_{(0,t)}^* - \beta_{(0,t)}^*\|_1 \leq C_2 \sigma_\xi \frac{\varepsilon \log(pn)}{n}.
\]

\[ \blacksquare \]
Proposition 2. Suppose \( \{\xi_i\}_{i=1}^n \overset{i.i.d.}{\sim} N(0, \sigma^2_\xi) \). Suppose Assumption 1 and Condition 1 hold. Let
\[
\mathcal{R}_1(t) = \frac{2}{n - t} \sum_{i=t+1}^{n} \left( y_i - x_i^T \hat{\beta}_{(t,n)} \right) \left( x_i^T (\hat{\beta}_{(t,n)} - \hat{\beta}_{(0,t)}) - \xi_i \right),
\]
\[
\mathcal{R}_2(t) = \frac{1}{t} \sum_{i=1}^{t} x_i^T \hat{\beta}_{(0,t)} \left( y_i - x_i^T \hat{\beta}_{(0,t)} \right),
\]
\[
\mathcal{R}_3(t) = \frac{1}{t} \sum_{i=1}^{t} x_i^T \hat{\beta}_{(t,n)} \left( y_i - x_i^T \hat{\beta}_{(0,t)} \right).
\]
Denote
\[
\mathcal{T}_{n,\xi}(t) = \frac{2}{n - t} \sum_{i=t+1}^{n} x_i^T (\beta^*_{(t,n)} - \beta^*_{(0,t)}) \xi_i + \frac{2}{n - t} \sum_{i=t+1}^{n} x_i^T (\beta^*_{(t,n)} - \beta^*_{(0,t)}) \xi_i - \frac{2}{n - t} \sum_{i=t+1}^{n} \xi_i \epsilon_i
\]
\[
+ (\beta^*_{(t,n)} - \beta^*_{(0,t)})^T (\hat{\Sigma}_{(t,n)} - \Sigma) (\beta^*_{(t,n)} - \beta^*_{(0,t)}) + \frac{2}{n - t} \sum_{i=t+1}^{n} x_i^T (\beta^*_{(t,n)} - \beta^*_{(0,t)}) x_i^T (\beta^*_{(t,n)} - \beta^*_{(0,t)}),
\]
\[
\mathcal{G}_n(t) = \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta^*_{(0,t)} - \beta^*_{(t,n)}) \epsilon_i + \frac{2}{t} \sum_{i=1}^{t} (\beta^*_{(0,t)} - \beta^*_{(t,n)}) x_i x_i^T (\beta^*_{(0,t)} - \beta^*_{(t,n)}).
\]
It holds that
\[
\left| (\hat{\beta}_{(t,n)} - \hat{\beta}_{(0,t)})^T \hat{\Sigma}_{(t,n)} (\hat{\beta}_{(t,n)} - \hat{\beta}_{(0,t)}) + \mathcal{R}_1(t) + 2\mathcal{R}_2(t) - 2\mathcal{R}_3(t) \right|
\leq C(1 + \sigma_\xi) \frac{s \log(pn)}{n}.
\]
Proof. Observe that Proposition 2 is the symmetric counterpart of Proposition 1. The proof of Proposition 2 is identical to Proposition 1 and is therefore omitted. \(\square\)

Proofs Related to Proposition 1 and Proposition 2

Throughout this subsection, we assume that \( \zeta \in (0,1/2) \) and that Condition 1 and Condition 2 hold. The justification of these conditions will be postponed to Appendix A.6

Lemma 2. For \( t \in \{1, \ldots, n\} \), denote
\[
\mathcal{R}_2(t) = \frac{1}{n - t} \sum_{i=t+1}^{n} x_i^T \hat{\beta}_{(t,n)} \left( y_i - x_i^T \hat{\beta}_{(t,n)} \right).
\]
If $n - t \geq \zeta n$, then it holds that

\[
\begin{align*}
&\left| (\hat{\beta}_{(t,n)}^\top \Sigma \hat{\beta}_{(t,n)} + 2R_2(t)) - \beta_{(t,n)}^* \Sigma \beta_{(t,n)}^* + \left\{ -2\beta_{(t,n)}^* (\Sigma - \hat{\Sigma}_{(t,n)})(\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*) \\
&\quad - \frac{2}{n - t} \sum_{i=t+1}^n \epsilon_i x_i^\top \beta_{(t,n)} - \frac{2}{n - t} \sum_{i=t+1}^n \beta_{(t,n)}^* x_i x_i^\top (\beta_{i}^* - \beta_{(t,n)}^*) \right\} \right| \leq O \left( \frac{s \log(pn)}{n} \right)
\end{align*}
\]

Proof. Note that

\[
\begin{align*}
&\hat{\beta}_{(t,n)}^\top \Sigma \hat{\beta}_{(t,n)} + 2R_2(t) - \beta_{(t,n)}^* \Sigma \beta_{(t,n)}^*

&= \frac{2}{n - t} \sum_{i=t+1}^n x_i^\top \hat{\beta}_{(t,n)} \left( \epsilon_i + x_i^\top (\beta_{i}^* - \beta_{(t,n)}^*) \right) \\
&+ 2\beta_{(t,n)}^* \Sigma (\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*) - 2\hat{\beta}_{(t,n)}^* \hat{\Sigma}_{(t,n)}(\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*) \\
&- (\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*)^\top \Sigma (\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*)
\end{align*}
\]

Step 1. For Equation (17), note that

\[
\left| \frac{2}{n - t} \sum_{i=t+1}^n x_i^\top \hat{\beta}_{(t,n)} \epsilon_i - \frac{2}{n - t} \sum_{i=t+1}^n x_i^\top \beta_{(t,n)}^* \epsilon_i \right|
\]

\[
= \left| \frac{2}{n - t} \sum_{i=t+1}^n x_i^\top (\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*) \epsilon_i \right|
\]

\[
\leq C_1 \sqrt{\frac{\log(pn)}{n - t}} \|\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*\|_1 \leq C_1 \frac{s \log(pn)}{n - t},
\]

where the first inequality follows from Condition 1b and the second inequality follows from Lemma 1. In addition,

\[
\left| \frac{2}{n - t} \sum_{i=t+1}^n (\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*)^\top x_i x_i^\top (\beta_{i}^* - \beta_{(t,n)}^*) - \frac{2}{n - t} \sum_{i=t+1}^n (\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*)^\top \Sigma (\beta_{i}^* - \beta_{(t,n)}^*) \right|
\]

\[
\leq \max_{t+1 \leq i \leq n} \|\beta_{i}^* - \beta_{(t,n)}^*\|_2 \sqrt{\frac{\log(pn)}{n - t}} \|\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*\|_1
\]

\[
\leq C_2 \frac{s \log(pn)}{n - t},
\]

where the first inequality follows from Condition 1c and the second inequality follows from Lemma 1 and Lemma 17. Note that

\[
\sum_{i=t+1}^n (\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*)^\top \Sigma (\beta_{i}^* - \beta_{(t,n)}^*) = (\hat{\beta}_{(t,n)} - \beta_{(t,n)}^*)^\top \Sigma \sum_{i=t+1}^n (\beta_{i}^* - \beta_{(t,n)}^*) = 0.
\]
Consequently,
\[
\left| (17) - \frac{2}{n-t} \sum_{i=t+1}^{n} \epsilon_i x_i^\top \beta^*_{(t,n)} - \frac{2}{n-t} \sum_{i=t+1}^{n} \beta^*_i x_i^\top x_i^\top (\beta^*_i - \beta^*_{(t,n)}) \right| \leq C_3 \frac{s \log(pn)}{n-t}.
\]

**Step 2.** By Lemma 1,
\[
\| (\hat{\beta}(t,n) - \beta^*_{(t,n)}) S \|^2_1 \leq 3 \| (\hat{\beta}(t,n) - \beta^*_{(t,n)}) S \|_1. \tag{20}
\]
So
\[
\frac{1}{2} \cdot (18) - \beta^*_{(t,n)}^\top (\Sigma - \hat{\Sigma}_{(t,n)}) (\hat{\beta}(t,n) - \beta^*_{(t,n)})
\]
\[
= (\hat{\beta}(t,n) - \beta^*_{(t,n)})^\top (\Sigma - \hat{\Sigma}_{(t,n)}) (\hat{\beta}(t,n) - \beta^*_{(t,n)})
\]
\[
\leq C_4 3 \frac{s \log(pn)}{n-t} \| \hat{\beta}(t,n) - \beta^*_{(t,n)} \|^2_2 \leq C_4 \frac{s \log(pn)}{n-t}.
\]
where the first inequality follows from Condition 1a and Equation (20), and the second inequality follows from Lemma 1 and the fact that
\[
\frac{s \log(pn)}{n-t} \leq \frac{s \log(pn)}{\zeta n} \leq \frac{1}{C_{snr} \zeta} < \infty.
\]

**Step 3.** Lemma 1 implies that
\[
(\hat{\beta}(t,n) - \beta^*_{(t,n)})^\top \Sigma (\hat{\beta}(t,n) - \beta^*_{(t,n)}) \leq C_2 \| \hat{\beta}(t,n) - \beta^*_{(t,n)} \|^2 \leq C_5 \frac{s \log(pn)}{n-t}.
\]

The desired result follows from putting all the previous steps together.

**Lemma 3.** For \( t \in \{1, \ldots, n\} \), denote
\[
\mathcal{R}_3(t) = \frac{1}{n-t} \sum_{i=t+1}^{n} x_i^\top \hat{\beta}_{(0,t)} (y_i - x_i^\top \hat{\beta}_{(t,n)})
\]
If \( \min \{t, n - t\} \geq \zeta n \), then with probability goes to 1, it holds that
\[
\left| \beta^*_{(0,t)}^\top \Sigma (\beta^*_{(t,n)} - \hat{\beta}_{(t,n)}) - \mathcal{R}_3(t) \right| \leq \left\{ \frac{1}{n-t} \sum_{i=t+1}^{n} x_i^\top \beta^*_{(0,t)} \epsilon_i + \frac{1}{n-t} \sum_{i=t+1}^{n} \beta^*_{(0,t)}^\top x_i^\top x_i^\top (\beta^*_i - \beta^*_{(t,n)}) - \beta^*_i (\Sigma - \hat{\Sigma}_{(t,n)}) (\beta^*_{(t,n)} - \hat{\beta}_{(t,n)}) \right\} \leq C_6 \frac{s \log(pn)}{n}.
\]

39
Proof. Note that

\[
\beta_{(0,t)}^* \Sigma (\beta_{(t,n)}^* - \hat{\beta}_{(t,n)}) - R_3(t) \\
= \beta_{(0,t)}^* \Sigma (\beta_{(t,n)}^* - \hat{\beta}_{(t,n)}) - \hat{\beta}_{(0,t)}^T \hat{\Sigma}_{(t,n)} (\beta_{(t,n)}^* - \hat{\beta}_{(t,n)}) \\
- \frac{1}{n-t} \sum_{i=t+1}^{n} x_i^T \hat{\beta}_{(0,t)} \epsilon_i - \frac{1}{n-t} \sum_{i=t+1}^{n} \hat{\beta}_{(0,t)}^T x_i \beta_i^* (\beta^*_i - \hat{\beta}_{(t,n)}^*)
\] (21)

Step 1. Observe that

\[
(21) = (\beta_{(0,t)}^* - \hat{\beta}_{(0,t)})^T \Sigma (\beta_{(t,n)}^* - \hat{\beta}_{(t,n)}) + \hat{\beta}_{(0,t)}^T (\Sigma - \hat{\Sigma}_{(t,n)}) (\beta_{(t,n)}^* - \hat{\beta}_{(t,n)}) \\
+ (\beta_{(0,t)}^* - \hat{\beta}_{(0,t)})^T (\Sigma - \hat{\Sigma}_{(t,n)}) (\beta_{(t,n)}^* - \hat{\beta}_{(t,n)}) \\
+ \beta_{(0,t)}^T (\Sigma - \hat{\Sigma}_{(t,n)}) (\beta_{(t,n)}^* - \hat{\beta}_{(t,n)}).
\]  

It holds that

\[
(\beta_{(0,t)}^* - \hat{\beta}_{(0,t)})^T \Sigma (\beta_{(t,n)}^* - \hat{\beta}_{(t,n)}) \leq C_t \| \beta_{(0,t)}^* - \hat{\beta}_{(0,t)} \|_2 \| \beta_{(t,n)}^* - \hat{\beta}_{(t,n)} \|_2 \leq C_1 \frac{s \log (pn)}{\sqrt{t (n-t)}} \leq C_1^* \frac{s \log (pn)}{n},
\]

where the second inequality follows from Lemma 1. In addition,

\[
(\beta_{(0,t)}^* - \hat{\beta}_{(0,t)})^T (\Sigma - \hat{\Sigma}_{(t,n)}) (\beta_{(t,n)}^* - \hat{\beta}_{(t,n)}) \\
\leq C_2 \sqrt{\frac{\log (pn)}{n}} \| \beta_{(0,t)}^* - \hat{\beta}_{(0,t)} \|_2 \| \beta_{(t,n)}^* - \hat{\beta}_{(t,n)} \|_1
\] (23)

where the first inequality follows from Condition 1c and the fact that the two collections of data \( \{x_i, \epsilon_i\}_{i=1}^t \) and \( \{x_i, \epsilon_i\}_{i=t+1}^n \) are independent, and the second inequality follows from Lemma 1. So

\[
\left| (21) - \beta_{(0,t)}^* (\Sigma - \hat{\Sigma}_{(t,n)}) (\beta_{(t,n)}^* - \hat{\beta}_{(t,n)}) \right| \leq C_3 \frac{s \log (pn)}{n}.
\]

Step 2. For Equation (22), note that

\[
\left| \frac{1}{n-t} \sum_{i=t+1}^{n} x_i^T \hat{\beta}_{(0,t)} \epsilon_i - \frac{1}{n-t} \sum_{i=t+1}^{n} x_i^T \beta_{(0,t)}^* \epsilon_i \right| \\
= \left| \frac{1}{n-t} \sum_{i=t+1}^{n} x_i^T (\hat{\beta}_{(0,t)} - \beta_{(0,t)}^*) \epsilon_i \right| \\
\leq \sqrt{\frac{\log (pn)}{n-t}} \| \hat{\beta}_{(0,t)} - \beta_{(0,t)}^* \|_1 \leq C_4 \frac{s \log (pn)}{\sqrt{t (n-t)}}.
\]

40
In addition, 
\[
\left| \frac{1}{n-t} \sum_{i=t+1}^{n} (\hat{\beta}_{(0,t)} - \beta_{(0,t)}^*)^\top x_i x_i^\top (\beta_{i}^* - \beta_{(t,n)}^*) - \frac{1}{n-t} \sum_{i=t+1}^{n} (\hat{\beta}_{(0,t)} - \beta_{(0,t)}^*)^\top \Sigma (\beta_{i}^* - \beta_{(t,n)}^*) \right| 
\]
\[
\leq \sqrt{\max_{1 \leq i \leq n} \| \beta_{i}^* - \beta_{(t,n)}^* \|_2^2 C} \sqrt{\frac{\log(pn)}{(n-t)} \| \hat{\beta}_{(0,t)} - \beta_{(0,t)}^* \|_1} \leq C s \frac{\log(pn)}{\sqrt{t(n-t)}},
\]
where the first inequality follows from Condition 1c and the second inequality follows from Lemma 1.

Note that
\[
\frac{1}{n-t} \sum_{i=t+1}^{n} (\hat{\beta}_{(0,t)} - \beta_{(0,t)}^*)^\top \Sigma (\beta_{i}^* - \beta_{(t,n)}^*) = (\hat{\beta}_{(0,t)} - \beta_{(0,t)}^*)^\top \Sigma \frac{1}{n-t} \sum_{i=t+1}^{n} (\beta_{i}^* - \beta_{(t,n)}^*) = 0.
\]
So
\[
\left| (22) + \frac{1}{n-t} \sum_{i=t+1}^{n} x_i^\top \beta_{(0,t)}^* e_i + \frac{1}{n-t} \sum_{i=t+1}^{n} \beta_{(0,t)}^* x_i^\top (\beta_{i}^* - \beta_{(t,n)}^*) \right| \leq C s \frac{\log(pn)}{n}.
\]

\[
\textbf{Lemma 4.} \text{ For } t \in \{1, \ldots, n\}, \text{ let }
\]
\[
R_4(t) = 2 \frac{t}{t} \sum_{i=1}^{t} \left( y_i - x_i^\top \hat{\beta}_{(0,t)} \right) x_i^\top (\hat{\beta}_{(0,t)} - \beta_{(t,n)}).
\]
If \(\min\{t, n - t\} \geq \zeta n\), then
\[
\left| (\hat{\beta}_{(0,t)} - \beta_{(t,n)})^\top \hat{\Sigma}_{(0,t)} (\hat{\beta}_{(0,t)} - \beta_{(t,n)}) + R_4(t) - (\beta_{(0,t)}^* - \beta_{(t,n)}^*)^\top \Sigma (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \right|
\]
\[
\leq \left\{ - \frac{2}{t} \sum_{i=1}^{t} x_i^\top (\beta_{(0,t)}^* - \beta_{(t,n)}^*) e_i - (\beta_{(0,t)}^* - \beta_{(t,n)}^*)^\top (\hat{\Sigma}_{(0,t)} - \Sigma) (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \right\} \leq C s \frac{\log(pn)}{n}.
\]

\[
\textbf{Proof.} \text{ Note that }
\]
\[
(\hat{\beta}_{(0,t)} - \beta_{(t,n)})^\top \hat{\Sigma}_{(0,t)} (\hat{\beta}_{(0,t)} - \beta_{(t,n)}) - (\beta_{(0,t)}^* - \beta_{(t,n)}^*)^\top \Sigma (\beta_{(0,t)}^* - \beta_{(t,n)}^*)
\]
\[
= 2 (\hat{\beta}_{(0,t)} - \beta_{(t,n)})^\top \hat{\Sigma}_{(0,t)} (\hat{\beta}_{(0,t)} - \beta_{(0,t)}^*)
\]
\[
+ (\beta_{(0,t)}^* - \beta_{(t,n)}^*)^\top (\hat{\Sigma}_{(0,t)} - \Sigma) (\beta_{(0,t)}^* - \beta_{(t,n)}^*)
\]
\[
- (\hat{\beta}_{(0,t)} - \beta_{(0,t)}^*)^\top \hat{\Sigma}_{(0,t)} (\hat{\beta}_{(0,t)} - \beta_{(0,t)}^*)
\]
\[
(24)
\]
\[
(25)
\]
\[
(26)
\]

41
Step 1. Note that

\[(24) + R_4(t) = \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta_{[0,t]} - \tilde{\beta}_{[t,n]}) \epsilon_i + \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta^*_i - \beta_{[0,t]}) x_i^T (\beta_{[0,t]} - \tilde{\beta}_{[t,n]}).\]

It holds that

\[
\left| \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta_{[0,t]} - \tilde{\beta}_{[t,n]}) \epsilon_i - \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta^*_i - \beta_{[0,t]}) x_i^T (\beta^*_i - \beta_{[0,t]}) \right|
\leq C_1 \sqrt{\frac{\log(pn)}{t}} \left( \|\beta_{[0,t]} - \beta^*_{[0,t]}\|_1 + \|\tilde{\beta}_{[t,n]} - \beta^*_{[t,n]}\|_1 \right)
\leq C'_1 \frac{s \log(pn)}{n},
\]

where the first inequality follows from Condition 1b, and the second inequality follows from Lemma 1. In addition,

\[
\left| \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta^*_i - \beta^*_{[0,t]}) x_i^T (\tilde{\beta}_{[0,t]} - \tilde{\beta}_{[t,n]}) - \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta^*_i - \beta^*_{[0,t]}) x_i^T (\beta^*_i - \beta_{[0,t]}) \right|
\leq \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta^*_i - \beta^*_{[0,t]}) x_i^T (\tilde{\beta}_{[0,t]} - \beta_{[0,t]}) + \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta^*_i - \beta^*_{[0,t]}) x_i^T (\tilde{\beta}_{[n]} - \beta^*_{[t,n]})
\]

\[
= \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta^*_i - \beta^*_{[0,t]}) x_i^T (\tilde{\beta}_{[0,t]} - \beta^*_{[0,t]}) - \frac{2}{t} \sum_{i=1}^{t} (\beta^*_i - \beta^*_{[0,t]})^T \Sigma (\beta^*_{[0,t]} - \beta^*_{[0,t]})
+ \frac{2}{t} \sum_{i=1}^{t} (\beta^*_i - \beta^*_{[0,t]})^T \Sigma (\tilde{\beta}_{[t,n]} - \beta^*_{[t,n]})
\]

\[
\leq C_2 \max_{1 \leq i \leq t} \|\beta^*_i - \beta^*_{[0,t]}\|_2 \sqrt{\frac{\log(pn)}{t}} \left( \|\tilde{\beta}_{[0,t]} - \beta^*_{[0,t]}\|_1 + \|\tilde{\beta}_{[t,n]} - \beta^*_{[t,n]}\|_1 \right)
\leq C'_2 \frac{s \log(pn)}{n},
\]

where the equality follows from the observation that

\[
\sum_{i=1}^{t} (\beta^*_i - \beta^*_{[0,t]})^T \Sigma (\beta_{[0,t]} - \beta^*_{[0,t]}) = 0, \quad \sum_{i=1}^{t} (\beta^*_i - \beta^*_{[0,t]})^T \Sigma (\tilde{\beta}_{[t,n]} - \beta^*_{[t,n]}) = 0,
\]

the second to last inequality follows from Condition 1c, and the last inequality follows from Lemma 1.

Step 2. It holds that

\[(25) - (\beta_{[0,t]} - \beta^*_{[t,n]})^T (\tilde{\Sigma}_{[0,t]} - \Sigma)(\beta^*_{[0,t]} - \beta^*_{[t,n]})
= 2(\beta^*_{[t,n]} - \beta^*_{[0,t]})^T (\tilde{\Sigma}_{[0,t]} - \Sigma)(\tilde{\beta}_{[0,t]} - \beta^*_{[t,n]})
+ (\beta_{[t,n]} - \beta^*_{[t,n]})^T (\tilde{\Sigma}_{[0,t]} - \Sigma)(\tilde{\beta}_{[0,t]} - \beta^*_{[t,n]}),
\]

42
Note that
\[
(\beta^*_t, n) - (\beta^*_0, t)^T (\hat{\Sigma}(0, t) - \Sigma)(\hat{\beta}(t, n) - \beta^*_t, n) \\
\leq C_3 \|\beta^*_t, n - \beta^*_0, t\| \frac{2 \log(pl)}{t} \|\beta^*_t, n - \beta^*_0, t\|_1 \\
\leq C'_3 s \log(pm) \frac{p}{n}
\]
where the first inequality follows from Condition 1c and the second inequality follows from Lemma 1. In addition,
\[
(\hat{\beta}(t, n) - \beta^*_t, n)^T (\hat{\Sigma}(0, t) - \Sigma)(\hat{\beta}(t, n) - \beta^*_t, n) \leq C_4 \|\hat{\beta}(t, n) - \beta^*_t, n\|^2_2 \leq C'_4 s \frac{\log(pm)}{n},
\]
where the first inequality follows from Condition 1a and the second inequality follows from Lemma 1.
So
\[
\left|(25) - (\beta(0, t) - \beta^*_0, t)^T (\hat{\Sigma}(0, t) - \Sigma)(\beta^*_0, t) - \beta^*_0, t)\right| \leq \frac{C_5 s \log(pm)}{n}.
\]

**Step 3.** Similarly
\[
(26) \leq \left| (\hat{\beta}(0, t) - \beta^*_0, t)^T \{\hat{\Sigma}(0, t) - \Sigma\} (\hat{\beta}(0, t) - \beta^*_0, t) \right| + \left| (\hat{\beta}(0, t) - \beta^*_0, t)^T \Sigma (\hat{\beta}(0, t) - \beta^*_0, t) \right| \\
\leq C_5 \sqrt{\frac{s \log(pm)}{n}} \|\hat{\beta}(0, t) - \beta^*_0, t\|^2_2 + C_2 \|\hat{\beta}(0, t) - \beta^*_0, t\|^2_2 \\
\leq C_5 s \log(pm) \frac{p}{n},
\]
where the first inequality follows from Condition 1a and the second inequality follows from Lemma 1.

**Theorem 7.** Suppose \(\min\{t, n - t\} \geq \zeta n\). Let
\[
R_4(t) = \frac{2}{t} \sum_{i=1}^{t} (y_i - x_i^T \hat{\beta}(0, t)) x_i^T (\hat{\beta}(0, t) - \hat{\beta}(t, n)).
\]
Denote
\[
\mathcal{S}_n(t) = \frac{2}{n-t} \sum_{i=t+1}^{n} x_i^T (\beta^*_t, n) - \beta^*_0, t) \epsilon_i + (\beta^*_0, t) - \beta^*_0, t)^T (\hat{\Sigma}(0, t) - \Sigma)(\beta^*_0, t) - \beta^*_0, t) \\
+ \frac{2}{n-t} \sum_{i=t+1}^{n} x_i^T (\beta^*_t, n) - \beta^*_0, t) x^T (\beta^*_0, t) - \beta^*_0, t),
\]
\[
\mathcal{G}_n(t) = \frac{2}{n-t} \sum_{i=t+1}^{n} x_i^T (\beta^*_t, n) - \beta^*_0, t) \epsilon_i + \frac{2}{n-t} \sum_{i=t+1}^{n} (\beta^*_t, n) - \beta^*_0, t) x_i x_i^T (\beta^*_t, n) - \beta^*_0, t)
\]
\[
\mathcal{G}_n(t) = \frac{2}{n-t} \sum_{i=t+1}^{n} x_i^T (\beta^*_t, n) - \beta^*_0, t) \epsilon_i + \frac{2}{n-t} \sum_{i=t+1}^{n} (\beta^*_t, n) - \beta^*_0, t) x_i x_i^T (\beta^*_t, n) - \beta^*_0, t)
\]
\[
\mathcal{G}_n(t) = \frac{2}{n-t} \sum_{i=t+1}^{n} x_i^T (\beta^*_t, n) - \beta^*_0, t) \epsilon_i + \frac{2}{n-t} \sum_{i=t+1}^{n} (\beta^*_t, n) - \beta^*_0, t) x_i x_i^T (\beta^*_t, n) - \beta^*_0, t)
\]
It holds that
\[
\begin{aligned}
&\left(\hat{\beta}_{(0,t)} - \hat{\beta}_{(t,n)}\right)^{\top}\hat{\Sigma}_{(0,t)}(\hat{\beta}_{(0,t)} - \hat{\beta}_{(t,n)}) - (\hat{\beta}_{(0,t)} - \hat{\beta}_{(t,n)})^{\top}\Sigma(\hat{\beta}_{(0,t)} - \hat{\beta}_{(t,n)}) \\
&+ R_4(t) + 2R_2(t) - 2R_3(t) - \left\{ \mathcal{I}_n(t) + \mathcal{G}_n(t) \right\} \leq C_{1} \frac{s \log(pn)}{n}
\end{aligned}
\]

\textbf{Proof.} Observe that
\[
\begin{aligned}
&(\hat{\beta}_{(0,t)} - \hat{\beta}_{(t,n)})^{\top}\hat{\Sigma}_{(0,t)}(\hat{\beta}_{(0,t)} - \hat{\beta}_{(t,n)}) + R_4(t) + 2R_2(t) = 2R_3(t) - (\beta_{(0,t)} - \beta_{(t,n)})^{\top}\Sigma(\beta_{(0,t)} - \beta_{(t,n)}) \\
&+ (\beta_{(0,t)} - \beta_{(t,n)})^{\top}\Sigma(\beta_{(0,t)} - \beta_{(t,n)}) + (\beta_{(0,t)} - \beta_{(t,n)})^{\top}\Sigma(\beta_{(0,t)} - \beta_{(t,n)}) + 2R_2(t) - 2R_3(t).
\end{aligned}
\]

\textbf{Step 1.} Note that
\[
(28) = 2\beta_{(0,t)}^{\top}\Sigma(\beta_{(t,n)} - \hat{\beta}_{(t,n)}) - 2R_3 + \hat{\beta}_{(t,n)}^{\top}\Sigma\hat{\beta}_{(t,n)} - \beta_{(t,n)}^{\top}\Sigma\beta_{(t,n)}^{\top} + 2R_2.
\]

By Lemma 2 and Lemma 3, it follows that
\[
\begin{aligned}
&\left|\left(28\right) + \left\{ - 2\beta_{(t,n)}^{\top}\left(\Sigma - \hat{\Sigma}_{(t,n)}\right)(\hat{\beta}_{(t,n)} - \beta_{(t,n)}) - \frac{2}{n-t} \sum_{i=t+1}^{n} \epsilon_i x_i^{\top} \beta_{(t,n)} - \frac{2}{n-t} \sum_{i=t+1}^{n} \beta_{(0,t)}^{\top} X_i x_i^{\top} (\beta_{(0,t)}^{\top} - \beta_{(t,n)}^{\top}) \right\} \right| \\
&+ 2\left\{ \frac{1}{n-t} \sum_{i=t+1}^{n} x_i^{\top} \beta_{(0,t)}^{\top} \epsilon_i + \frac{1}{n-t} \sum_{i=t+1}^{n} \beta_{(0,t)}^{\top} x_i x_i^{\top} (\beta_{(0,t)}^{\top} - \beta_{(t,n)}^{\top}) - \beta_{(0,t)}^{\top}(\Sigma - \hat{\Sigma}_{(t,n)})(\beta_{(0,t)}^{\top} - \beta_{(t,n)}^{\top}) \right\} \right\} \right| \\
&\leq C_1 \frac{s \log(pn)}{n}.
\end{aligned}
\]

Note that
\[
- \beta_{(t,n)}^{\top}\left(\Sigma - \hat{\Sigma}_{(t,n)}\right)(\hat{\beta}_{(t,n)} - \beta_{(t,n)}) = (\beta_{(0,t)} - \beta_{(t,n)})^{\top}\left(\Sigma - \hat{\Sigma}_{(t,n)}\right)(\beta_{(0,t)}^{\top} - \hat{\beta}_{(t,n)}^{\top})
\]
\[
= (\beta_{(0,t)} - \beta_{(t,n)}^{\top})^{\top}\left(\Sigma - \hat{\Sigma}_{(t,n)}\right)(\beta_{(t,n)} - \beta_{(t,n)}^{\top})
\]
\[
\leq C_2 \|\beta_{(0,t)} - \beta_{(t,n)}^{\top}\|_2 \sqrt{\frac{\log(p)}{n-t}} \|\beta_{(t,n)} - \beta_{(t,n)}^{\top}\|_1
\]
\[
\leq C_2' \frac{s \log(pn)}{\sqrt{(n-t)t}} \leq C_2'' \frac{s \log(pn)}{n},
\]

where the first inequality follows from Condition 1c and the second inequality follows from Lemma 1.

So
\[
\left|\left(28\right) + \left\{ \frac{2}{n-t} \sum_{i=t+1}^{n} \epsilon_i x_i^{\top} (\beta_{(t,n)}^{\top} - \beta_{(0,t)}^{\top}) + \frac{2}{n-t} \sum_{i=t+1}^{n} (\beta_{(t,n)}^{\top} - \beta_{(0,t)}^{\top}) x_i x_i^{\top} (\beta_{(0,t)}^{\top} - \beta_{(t,n)}^{\top}) \right\} \right| \leq C_3 \frac{s \log(pn)}{n}.
\]
Step 2. By Lemma 4,

$$\left| (27) - \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \epsilon_i - (\beta_{(0,t)}^* - \beta_{(t,n)}^*)^T (\hat{\Sigma}_{(0,t)} - \Sigma) (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \right| - \frac{2}{t} \sum_{i=1}^{t} x_i^T (\beta_i^* - \beta_{(0,t)}^*) x_i^T (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \leq C_4 \frac{s \log(pn)}{n}.$$

The desired result follows from the previous two steps.

A.5 Lasso Consistency

Proof of Lemma 1:

Proof. It follows from the definition of $\hat{\beta}_I$ that

$$\frac{1}{|I|} \sum_{i \in I} (y_i - x_i^T \hat{\beta}_I^2) + \frac{\lambda}{\sqrt{|I|}} \| \hat{\beta}_I \|_1 \leq \frac{1}{|I|} \sum_{i \in I} (y_i - x_i^T \beta_I^*)^2 + \frac{\lambda}{\sqrt{|I|}} \| \beta_I^* \|_1.$$

This gives

$$\frac{1}{|I|} \sum_{i \in I} \{x_i^T (\hat{\beta}_I - \beta_I^*) \}^2 + \frac{2}{|I|} \sum_{i \in I} (y_i - x_i^T \beta_I^*) x_i^T (\beta_I^* - \hat{\beta}_I) + \frac{\lambda}{\sqrt{|I|}} \| \hat{\beta}_I \|_1 \leq \frac{\lambda}{\sqrt{|I|}} \| \beta_I^* \|_1,$$

and therefore

$$\frac{1}{|I|} \sum_{i \in I} \{x_i^T (\hat{\beta}_I - \beta_I^*) \}^2 + \frac{\lambda}{\sqrt{|I|}} \| \hat{\beta}_I \|_1 \leq 2 \sum_{i \in I} \epsilon_i x_i^T (\hat{\beta}_I - \beta_I^*) + 2(\hat{\beta}_I - \beta_I^*)^T \frac{1}{|I|} \sum_{i \in I} x_i x_i^T (\beta_I^* - \beta_I^*) + \frac{\lambda}{\sqrt{|I|}} \| \beta_I^* \|_1. \quad (29)$$

To bound $\| \sum_{i \in I} x_i x_i^T (\beta_I^* - \beta_i^*) \|_\infty$, note that for any $j \in \{1, \ldots, p\}$, the $j$-th entry of $\sum_{i \in I} x_i x_i^T (\beta_I^* - \beta_i^*)$ satisfies

$$\mathbb{E} \left\{ \sum_{i \in I} x_{ij} x_i^T (\beta_I^* - \beta_i^*) \right\} = \sum_{i \in I} \mathbb{E} \left\{ x_{ij} x_i^T \{ \beta_I^* - \beta_i^* \} \right\} = \mathbb{E} \{ x_{1j} x_1^T \} \sum_{i \in I} \{ \beta_I^* - \beta_i^* \} = 0.$$

So $\mathbb{E} \{ \sum_{i \in I} x_i x_i^T (\beta_I^* - \beta_i^*) \} = 0 \in \mathbb{R}^p$. By Condition 1c,

$$\left| (\beta_i^* - \beta_I^*)^T \frac{1}{|I|} \sum_{i \in I} x_i x_i^T (\hat{\beta}_I - \beta_I^*) \right| \leq C_1 \left( \max_{1 \leq i \leq n} \| \beta_i^* - \beta_I^* \|_2 \right) \sqrt{\frac{\log(pn)}{|I|}} \| \hat{\beta}_I - \beta_I^* \|_1$$

$$\leq C_2 \sqrt{\frac{\log(pn)}{|I|}} \| \hat{\beta}_I - \beta_I^* \|_1$$

$$\leq \frac{\lambda}{8 \sqrt{|I|}} \| \hat{\beta}_I - \beta_I^* \|_1$$

where the second inequality follows from Lemma 17 and the last inequality follows from $\lambda = $
$C\lambda\sqrt{\log(pn)}$ with sufficiently large constant $C$. In addition by Condition 1b,

$$\frac{2}{|I|} \sum_{i \in I} \epsilon_i x_i^\top (\hat{\beta}_I - \beta^*_I) \leq C \sqrt{\frac{\log(pn)}{|I|}} \|\hat{\beta}_I - \beta^*_I\|_1 \leq \frac{\lambda}{8\sqrt{|I|}} \|\hat{\beta}_I - \beta^*_I\|_1.$$  

So (29) gives

$$\frac{1}{|I|} \sum_{i \in I} \{x_i^\top (\hat{\beta}_I - \beta^*_I)^2 + \frac{\lambda}{\sqrt{|I|}} \|\hat{\beta}_I\|_1 \leq \frac{\lambda}{4\sqrt{|I|}} \|\hat{\beta}_I - \beta^*_I\|_1 + \frac{\lambda}{\sqrt{|I|}} \|\beta^*_I\|_1.$$  

Let $\Theta = \hat{\beta}_I - \beta^*_I$. The above inequality implies

$$\frac{1}{|I|} \sum_{i \in I} \left( x_i^\top \Theta \right)^2 + \frac{\lambda}{2 \sqrt{|I|}} \|\hat{\beta}_I\|_1 \leq \frac{3\lambda}{2 \sqrt{|I|}} \|\hat{\beta}_I - \beta^*_I\|_1,$$  

which also implies that

$$\frac{\lambda}{2} \|\Theta_{S^c}\|_1 = \frac{\lambda}{2} \|\hat{\beta}_{I^c}\|_1 \leq \frac{3\lambda}{2} \|\hat{\beta}_I - \beta^*_I\|_1 = \frac{3\lambda}{2} \|\Theta_S\|_1.$$  

The above inequality and Condition 1a imply that

$$\frac{1}{|I|} \sum_{i \in I} \left( x_i^\top \Theta \right)^2 = \Theta^\top \Sigma \Theta \geq \Theta^\top \Sigma \Theta - C_3 \frac{s \log(pn)}{|I|} \|\Theta\|^2_2 \geq \frac{c_4}{2} \|\Theta\|^2_2,$$  

where the last inequality follows from Assumption 1e. Therefore Equation (30) gives

$$c' \|\Theta\|^2_2 + \frac{\lambda}{2 \sqrt{|I|}} \|\hat{\beta}_I - \beta^*_I\|_1 \leq \frac{3\lambda}{2 \sqrt{|I|}} \|\Theta_S\|_1 \leq \frac{3\lambda\sqrt{s}}{2 \sqrt{|I|}} \|\Theta\|_2$$  

and so

$$\|\Theta\|_2 \leq \frac{C\lambda\sqrt{s}}{\sqrt{|I|}}.$$  

This and the last inequality of Equation (31) also implies that $\|\Theta_S\|_1 \leq C\lambda\sqrt{s}$, since $\|\Theta_{S^c}\|_1 \leq 3\|\Theta_S\|_1$, it also holds that

$$\|\Theta\|_1 = \|\Theta_S\|_1 + \|\Theta_{S^c}\|_1 \leq 4\|\Theta_S\|_1 \leq \frac{4C\lambda\sqrt{s}}{\sqrt{|I|}}.$$  

A.6 Distributions Satisfying Condition 1 and Condition 2

In this subsection, we show that Condition 1 and Condition 2 hold with high probability when \( \{x_i, \epsilon_i\}_{i=1}^n \) are sampled from independent sub-Gaussian distributions. We begin by noting that Condition 2 is a well known functional CLT result. See e.g., Theorem A.4 of Francq and Zakoian (2019) and references therein. Throughout the section, we write

$$z \sim SG(\sigma_z^2)$$
if \( z \) is a sub-Gaussian random variable such that \( \|z\|_{\psi_2} \leq \sigma_z \).

**Theorem 8.** Suppose Assumption 1 holds. Denote \( E(x_i x_i^\top) = \Sigma \) and \( \mathcal{C}_S := \{ v : \mathbb{R}^p : \|v_{S^c}\|_1 \leq 3\|v_S\|_1 \} \). Then there exists constants \( c \) and \( C \) such that for all \( \eta \leq 1 \),

\[
\mathbb{P} \left( \sup_{v \in \mathcal{C}_S, \|v\|_2 = 1} \left| v^\top (\hat{\Sigma} - \Sigma) v \right| \geq C\eta \Lambda_{\max}(\Sigma) \right) \leq 2 \exp\left(-cn\eta^2 + 2s \log(p)\right). \tag{32}
\]

**Proof.** This is a well known restricted eigenvalue property for sub-Gaussian design. The proof can be found in Zhou (2009) or Loh and Wainwright (2011).

The following corollary, being a direct consequence of Theorem 8, justifies Condition 1a.

**Corollary 1.** Denote \( \mathcal{C}_S := \{ v : \mathbb{R}^p : \|v_{S^c}\|_1 \leq 3\|v_S\|_1 \} \). Under Assumption 1, with probability at least \( 1 - n^{-5} \), it holds that

\[
\left| v^\top (\hat{\Sigma}_I - \Sigma) v \right| \leq C \sqrt{s \log(pn)} \|v\|_2^2
\]

for all \( v \in \mathcal{C}_S \) and all \( I \subset (0, n] \) such that \( |I| \geq \zeta n \).

**Proof.** For any \( I \subset (0, n] \) such that \( |I| \geq \zeta n \), by Theorem 8, it holds that

\[
\mathbb{P} \left( \sup_{v \in \mathcal{C}_S, \|v\|_2 = 1} \left| v^\top (\hat{\Sigma}_I - \Sigma) v \right| \geq C\eta \Lambda_{\max}(\Sigma) \right) \leq 2 \exp\left(-c\zeta n\eta^2 + 2s \log(p)\right).
\]

Let \( \eta = C_1 \sqrt{s \log(pn)|I|/|I|} \) for sufficiently large constant \( C_1 \). With probability at least \( (pn)^{-7} \),

\[
\sup_{v \in \mathcal{C}_S, \|v\|_2 = 1} \left| v^\top (\hat{\Sigma}_I - \Sigma) v \right| \geq C_2 \sqrt{s \log(pn)|I|/|I|}.
\]

Since there are at most \( n^2 \) many different choice of \( I \), the desire result follows from a union bound argument.

The following lemma justifies Condition 1b and c. Note that Condition 1d follows from the same argument as that of Condition 1b.

**Lemma 5.** Suppose \( n \geq Cs \log(p) \) for sufficiently large \( C \) and that Assumption 1 holds. Then with probability at least \( 1 - n^{-5} \), it holds that

\[
\left| \frac{1}{|I|} \sum_{i \in I} \epsilon_ix_i^\top \beta \right| \leq C \sqrt{\frac{\log(pn)}{|I|} \|\beta\|_1} \tag{33}
\]
for all $\beta \in \mathbb{R}^p$ and all $\mathcal{I} \subset (0, n]$ such that $|\mathcal{I}| \geq \zeta n$.

Let $\{u_i\}_{i=1}^n \subset \mathbb{R}^p$ be a collection of deterministic vectors. Then it holds that with probability at least $1 - n^{-5}$

$$
\left| \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T x_i x_i^T \beta - \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T \Sigma \beta \right| \leq C \left( \max_{1 \leq i \leq n} ||u_i||_2 \right) \sqrt{\frac{\log(pn)}{|\mathcal{I}|} ||\beta||_1}
$$  \hspace{1cm} (34)

for all $\beta \in \mathbb{R}^p$ and all $\mathcal{I} \subset (0, n]$ such that $|\mathcal{I}| \geq \zeta n$.

**Proof.** The first bound is a well-known inequality. The proof of the first bound is also similar and simpler than the second one. For conciseness, the proof of the first bound is omitted.

For Equation (34), it suffices to show that

$$
P\left\{ \left| \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T x_i x_i^T \beta - \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T \Sigma \beta \right| \geq C \left( \max_{1 \leq i \leq n} ||u_i||_2 \right) \sqrt{\frac{\log(pn)}{|\mathcal{I}|} ||\beta||_1} \right\} \leq n^{-7}
$$

for any $\mathcal{I}$ such that $|\mathcal{I}| \geq \zeta n$. Note that

$$
\left| \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T x_i x_i^T \beta - \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T \Sigma \beta \right| = \left| \left( \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T x_i x_i^T - \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T \Sigma \right) \beta \right|
$$

$$
\leq \max_{1 \leq j \leq p} \left| \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T x_i \Sigma_{i,j} - \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T \Sigma_{i,j} \right| \|\beta\|_1.
$$

Note that $\mathbb{E}(u_i^T x_i x_{i,j}) = u_i^T \Sigma_{i,j}$ and in addition,

$$
u_i^T x_i \sim SG(C_x^2 ||u_i||_2^2) \text{ and } x_{i,j} \sim SG(C_x^2).
$$

So $u_i^T x_i x_{i,j}$ is a sub-exponential random variable such that

$$
u_i^T x_i x_{i,j} \sim SE(C_x^4 ||u_i||_2^2).
$$

As a result, for $\gamma < 1$ and every $j$,

$$
P\left( \left| \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T x_i x_{i,j} - u_i^T \Sigma_{i,j} \right| \geq \gamma \sqrt{\max_{1 \leq i \leq n} C_x^4 ||u_i||_2^2} \right) \leq \exp(-c \gamma^2 |\mathcal{I}|).
$$

By union bound,

$$
P\left( \sup_{1 \leq j \leq p} \left| \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T x_i x_{i,j} - \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} u_i^T \Sigma_{i,j} \right| \geq \gamma C_x^2 \left( \max_{1 \leq i \leq n} ||u_i||_2 \right) \right) \leq p \exp(-c \gamma^2 n).
$$

48
The desired result follows from taking $\gamma = C_\gamma \sqrt{\frac{\log(pn)}{n}}$ for sufficiently large $C_\gamma$ and an union bound over all possible interval $I$ such that $|I| \geq \zeta n$. 

\section{\textit{\beta}-mixing Time Series}

\subsection{The Null Distribution under \textit{\beta}-mixing}

Proof of Theorem 5 (under $H_0$):

Proof. We follow the same strategy used in the proof of Theorem 1. Consider the following conditions:

Condition 3.

c. There exists an absolute constant $C$ such that given a fixed $u \in \mathbb{R}^p$, we have that

$$\left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^T x_i x_i^T \beta - u^T \Sigma \beta \right\} \right| \leq C \|u\|_2 \sqrt{\frac{\log(pn)}{|I|}} \|\beta\|_1 \quad \text{for all } \beta \in \mathbb{R}^p.$$

e. Suppose that

$$\left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^T x_i x_i^T \beta - u^T \Sigma \beta \right\} \right| \leq C \sqrt{\frac{s \log(pn)}{|I|}} \|\beta\|_1$$

for all $u \in \mathbb{R}^p$ such that $\|u\|_2 \leq 1$ and $u \in \mathcal{C}(S)$ and all $\beta \in \mathbb{R}^p$.

To justify the limiting distribution under the null hypothesis $H_0$, it suffices to justify Condition 1 a, b, d, Condition 2 and Condition 3 under the new \textit{\beta}-mixing assumption.

Observe that Condition 3c is a less general version of Condition 1c. However, under the null hypothesis

$$H_0: \beta_1^* = \cdots = \beta_n^*,$$

all we need is Condition 3c, as the general version in Condition 1c is used for power analysis and alternative distribution.

We also note that for the independent setting, in the proof of Lemma 3, we verify Equation (23) using the fact that $\hat{\beta}_{(0,t]} - \beta_{(0,t]}$ is independent of the data $\{x_i, \epsilon_i\}_{i=t+1}^n$. In the presence of temporal
dependence, we need to justify Equation (23) using Condition 3e.

In view of the proof of Theorem 1, the desired result follows immediately from Lemma 10, Lemma 11, Lemma 12 and Lemma 13 in Appendix B.4.

For illustration purposes, we show Equation (23) as follows. Suppose all the good events hold in Lemma 14. Then

\[
(\hat{\beta}_{(0,t]} - \beta_{(0,t]}^*)^T (\Sigma - \hat{\Sigma}_{(t,n]}) (\beta_{(t,n]}^* - \hat{\beta}_{(t,n]}) \leq C_2 \frac{1}{\sqrt{n}} \|\hat{\beta}_{(0,t]} - \beta_{(0,t]}^*\|_2 \|\beta_{(t,n]}^* - \hat{\beta}_{(t,n]}\|_1
\]

\[
\leq C'_2 \frac{1}{\sqrt{n}} \sqrt{C_3 \log(\sqrt{n})} \sqrt{C'_3 \log(\sqrt{n})}
\]

\[
= C_4 g^2 \left( \frac{1}{n} \right)^{3/2} \leq C'_4 \frac{1}{n},
\]

where the first inequality follows from Condition 3e, the second inequality follows from Lemma 14, and the last inequality follows from the assumption that \( s \sqrt{\frac{\log(\sqrt{n})}{n}} = o(1) \).}

\[\square\]

B.2 Power Analysis under \( \beta \)-mixing

Proof of Theorem 5 (under \( H_a \)):

Proof. The proof is analogous to the proof of Theorem 2. Suppose \( H_a \) holds.

Step 1. Denote

\[\kappa = \max_{1 \leq k \leq K} \kappa_k.\]

It follows from Lemma 19 that there exists \( \eta \), being a change point of \( \{\beta_i^*\}_{i=1}^n \) such that

\[(\beta_{[0,\eta]}^* - \beta_{(\eta,n]}^*)^T \Sigma (\beta_{[0,\eta]}^* - \beta_{(\eta,n]}^*) \geq c_1 \kappa.\]

Therefore, to justify the test \( \psi \) has power going to 1 under \( H_a \), it suffices to show that for sufficiently large \( n \),

\[
\frac{\sqrt{n}}{2\sigma_x \sigma_{\xi}} \left\{ (\hat{\beta}_{[0,\eta]} - \hat{\beta}_{(\eta,n]} \right)^T \hat{\Sigma}_{[0,\eta]} (\hat{\beta}_{[0,\eta]} - \hat{\beta}_{(\eta,n]}) + R_1(\eta) + 2R_2(\eta) - 2R_3(\eta) \right\} > G_\alpha, \tag{35}
\]

\[
\frac{\sqrt{n}}{2\sigma_x \sigma_{\xi}} \left\{ (\hat{\beta}_{[\eta,n]} - \hat{\beta}_{(0,\eta]})^T \hat{\Sigma}_{[\eta,n]} (\hat{\beta}_{[\eta,n]} - \hat{\beta}_{(0,\eta]}) + R'_1(\eta) + 2R'_2(\eta) - 2R'_3(\eta) \right\} > G_\alpha. \tag{36}
\]

The justification of Equation (35) is provided as the justification of Equation (36) is the same by
Step 2. In view of Theorem 2, we need to justify that under \( H_a \), Condition 1 a, b, c, d, and Condition 3 continue to hold in the intervals \( (0, \eta] \) and \( (\eta, n] \). For illustration purpose, in Corollary 2 we show that Condition 1 a and b holds in the interval \( (0, \eta] \). The justification of other conditions follow from similar arguments and is omitted for brevity.

Step 3. Using Condition 1 a, b, c, d, and Condition 3 in the intervals \( (0, \eta] \) and \( (\eta, n] \), it follows from the same argument as in Proposition 1,

\[
\begin{align*}
& \left| (\hat{\beta}_{(0, \eta]} - \hat{\beta}_{(\eta, n]} )^\top \Sigma (\hat{\beta}_{(0, \eta]} - \hat{\beta}_{(\eta, n]} ) + R_1(\eta) + 2R_2(\eta) - 2R_3(\eta) \\
& - (\beta^*_{(0, \eta]} - \beta^*_{(\eta, n]} )^\top \Sigma (\beta^*_{(0, \eta]} - \beta^*_{(\eta, n]} ) - \{ \Theta_{n, \xi}(\eta) + \Theta_n(\eta) \} \right| \leq C_1(1 + \sigma_\xi)^{s \log(p_n)} n.
\end{align*}
\]

Here

\[
\begin{align*}
\Theta_{n, \xi}(\eta) = & \frac{2}{\eta} \sum_{i=1}^\eta x_i^\top (\beta^*_{(0, \eta]} - \beta^*_{(\eta, n]} ) \xi_i + \frac{2}{\eta} \sum_{i=1}^\eta x_i^\top (\beta^*_{(0, \eta]} - \beta^*_{(\eta, n]} ) \xi_i \\
 & + (\beta^*_{(0, \eta]} - \beta^*_{(\eta, n]} )^\top (\Sigma (\beta^*_{(0, \eta]} - \Sigma(\beta^*_{(\eta, n]} - \beta^*_{(\eta, n]} ) + \frac{2}{\eta} \sum_{i=1}^\eta (\beta^*_{(0, \eta]} - \beta^*_{(\eta, n]} )^\top x_i x_i^\top (\beta^*_{(0, \eta]} - \beta^*_{(\eta, n]} ),
\end{align*}
\]

\[
\Theta_n(\eta) = \frac{2}{n - \eta} \sum_{i=\eta+1}^n x_i^\top (\beta^*_{(0, \eta]} - \beta^*_{(\eta, n]} ) \xi_i + \frac{2}{n - \eta} \sum_{i=\eta+1}^n (\beta^*_{(0, \eta]} - \beta^*_{(\eta, n]} ) x_i x_i^\top (\beta^*_{(0, \eta]} - \beta^*_{(\eta, n]} ). \tag{37}
\]

Then under \( H_a \),

\[
\begin{align*}
(\hat{\beta}_{(0, \eta]} - \hat{\beta}_{(\eta, n]} )^\top \Sigma (\hat{\beta}_{(0, \eta]} - \hat{\beta}_{(\eta, n]} ) + R_1(\eta) + 2R_2(\eta) - 2R_3(\eta) \\
\geq \kappa^2 - |\Theta_{n, \xi}(\eta)| - |\Theta_n(\eta)| - C_1(1 + \sigma_\xi)^{s \log(p_n)} n.
\end{align*}
\]

So it suffices to show that

\[
\kappa^2 - |\Theta_{n, \xi}(\eta)| - |\Theta_n(\eta)| - C_1(1 + \sigma_\xi)^{s \log(p_n)} n \geq \frac{2G_n \sigma_\xi \sigma_\xi}{\sqrt{n}}. \tag{39}
\]

The rest of the proof follows from the the same argument of Theorem 2 and the union bound argument (similar as that in Corollary 2).

For illustration purposes, in Lemma 6, we demonstrate how to handle the term \(|\Theta_n(\eta)|\) in Equation (39). More precisely, we show that there exists a constant \( C_2 \) such that with probability goes
to 1,
\[ |\mathcal{S}_n(\eta)| \leq C_2 \sqrt{\frac{D_n}{n}}, \]
where \(D_n\) is any slowly diverging sequence.

Lemma 6. Suppose all the assumptions in Theorem 5 hold. Let \(\mathcal{S}_n(\eta)\) be defined as in Equation (37). Then with probability goes to 1,
\[ |\mathcal{S}_n(\eta)| \leq C_2 \sqrt{\frac{D_n}{n}}, \]
where \(\kappa = \max_{1 \leq k \leq K} \kappa_k\) and \(D_n\) is any slowly diverging sequence.

Proof. Step 1. We show that
\[ \frac{1}{n-\eta} \sum_{i=\eta+1}^{n} x_i^T (\beta^*_{(0,\eta]} - \beta^*_{(\eta,n)}) \epsilon_i = O\left(\sqrt{\frac{D_n}{n}}\right) \]
with probability goes to 1. Let
\[ z_i = x_i^T (\beta^*_{(0,\eta]} - \beta^*_{(\eta,n)}) \epsilon_i. \]
As a result, \(z_i\) is strictly stationary. Since \(z_i\) is measurable to \(\sigma(x_i, \epsilon_i)\), the process \(\{z_i\}_{i=1}^{\eta}\) is \(\beta\)-mixing with the same mixing coefficient as \(\{x_i, \epsilon_i\}_{i=1}^{\eta}\). In addition,
\[ \|z_i\|_{\psi_{\gamma_1/2}} = \|x_i^T (\beta^*_{(0,\eta]} - \beta^*_{(\eta,n)}) \epsilon_i\|_{\psi_{\gamma_1/2}} \]
\[ \leq \frac{c_1}{\kappa} \|x_i^T (\beta^*_{(0,\eta]} - \beta^*_{(\eta,n)}) \epsilon_i\|_{\psi_{\gamma_1}} \]
\[ \leq \frac{c_1}{\kappa} K \|\beta^*_{(0,\eta]} - \beta^*_{(\eta,n)}\|_2 K \epsilon = c_1' K \epsilon, \]
where \(c_1\) is a constant only depending on \(\gamma_1\) and the last equality follows from Lemma 18. Therefore by Theorem 11,
\[ \mathbb{P}\left\{ \frac{1}{n-\eta} \sum_{i=\eta+1}^{n} z_i \geq \delta \right\} \leq n \exp\left(\frac{-c_2(\delta \eta)^\gamma}{\delta c_3 \eta^2} \right) + \exp(-c_3 \delta^2 \eta), \]
where \(\gamma < 1\) is defined in Assumption 4. Let \(\delta = C_3 \sqrt{\frac{D_n}{n}}\) for sufficiently large \(C_3\). Since \(n-\eta \geq \zeta n\), this leads to
\[ \mathbb{P}\left\{ \frac{1}{n-\eta} \sum_{i=\eta+1}^{n} x_i^T (\beta^*_{(0,\eta]} - \beta^*_{(\eta,n)}) \epsilon_i \geq C_3 \kappa \frac{D_n}{n} \right\} = o(1), \]
as desired.
Step 2. In this step, we show that
\[ \mathbb{P}\left\{ \left| \frac{1}{n - \eta} \sum_{i=\eta+1}^{n} (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) x_i x_i^\top (\beta_i^* - \beta_{(\eta,n]}^*) \right| \geq C_3 \kappa \sqrt{\frac{D_n}{n}} \right\} = o(1) \]

Let \( \eta = \eta_q \) for some \( q \in \{1, \ldots, K\} \). Denote \( J_k = (\eta_{k-1}, \eta_k] \). Then
\[ (\eta, n] = \bigcup_{k=q+1}^{K+1} J_k. \]

Observe that
\[ \frac{1}{n - \eta} \sum_{i=\eta+1}^{n} (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) x_i x_i^\top (\beta_i^* - \beta_{(\eta,n]}^*) \]
\[ = \frac{1}{n - \eta} \sum_{i=\eta+1}^{n} (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) x_i x_i^\top (\beta_i^* - \beta_{(\eta,n]}^*) - (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) \Sigma (\beta_i^* - \beta_{(\eta,n]}^*) \]
\[ \leq \sum_{k=q+1}^{K+1} \frac{1}{n - \eta} \sum_{i \in J_k} (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) x_i x_i^\top (\beta_i^* - \beta_{(\eta,n]}^*) - (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) \Sigma (\beta_i^* - \beta_{(\eta,n]}^*) \]
\[ = \sum_{k=q+1}^{K+1} \left| \frac{1}{n - \eta} \sum_{i \in J_k} (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) x_i x_i^\top (\beta_i^* - \beta_{(\eta,n]}^*) - (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) \Sigma (\beta_i^* - \beta_{(\eta,n]}^*) \right|. \quad (40) \]

where we use the fact that \( \beta_i^* \) is unchanged in each of the interval \( J_k \). For each interval \( J_k \), the time series
\[ w_i := (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) x_i x_i^\top (\beta_i^* - \beta_{(\eta,n]}^*) \]
is strictly stationary beta-mixing. In addition, by similar calculations as in the previous step, it follows that
\[ \frac{1}{K} \| w_i \|_{\psi_1/2} = \frac{1}{K} \| (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) x_i x_i^\top (\beta_i^* - \beta_{(\eta,n]}^*) \|_{\psi_1/2} \]
\[ \leq \frac{c_1}{K} \| (\beta_{(0,\eta]}^* - \beta_{(\eta,n]}^*) X_i \|_{\psi_1} \| x_i^\top (\beta_i^* - \beta_{(\eta,n]}^*) \|_{\psi_1} \]
\[ \leq \frac{c_1}{K} K_X^2 \| \beta_{(0,\eta]}^* - \beta_{(\eta,n]}^* \|_2 \| \beta_i^* - \beta_{(\eta,n]}^* \|_2 = c_2 K_X^2. \]

Therefore by Theorem 11,
\[ \mathbb{P}\left\{ \frac{1}{\kappa | J_k |} \left| \sum_{i \in J_k} w_i - E(w_i) \right| \geq \delta \right\} \leq n \exp \left( -c_2 (\delta n)^\gamma \right) + \exp(-c_3 \delta^2 n), \]

where \( \gamma < 1 \) is defined in Assumption 4. Let \( \delta = C_3 \sqrt{\frac{D_n}{n}} \) for sufficiently large \( C_3 \). Since
\[ n - \eta \asymp n \times | J_k |, \]

53
this leads to
\[
P\left\{ \frac{1}{n - \eta} \sum_{i = \eta}^{n} (\beta_{i,0}^* - \beta_{i,n}^*) x_i x_i^T (\beta_{i,J_k}^* - \beta_{i,n}^*) - (\beta_{0,0}^* - \beta_{0,n}^*) \Sigma (\beta_{i,J_k}^* - \beta_{i,n}^*) \right\} \geq C_\delta \kappa \sqrt{\frac{D_n}{n}} = o(1).
\]
By a union bound argument and Equation (40), it holds that
\[
P\left\{ \frac{1}{n - \eta} \sum_{i = \eta + 1}^{n} (\beta_{i,0}^* - \beta_{i,n}^*) x_i x_i^T (\beta_{i}^* - \beta_{i,n}^*) \right\} \geq C_\delta (K + 1) \kappa \sqrt{\frac{D_n}{n}} = o(1)
\]
as desired.
\[
\square
\]

B.3 Alternative Distribution under $\beta$-mixing

Consider the alternative hypothesis
\[
H_a: \text{there exists at least one change point in } \{\beta_i^*\}_{i = 1}^n.
\]
Let $r \in [\zeta, 1 - \zeta]$ and $t = \lfloor rn \rfloor$. Denote
\[
\mathcal{G}(r) := (\hat{\beta}_{0,t} - \hat{\beta}_{t,n})^T \Sigma (\beta_{0,t}^* - \beta_{t,n}^*) + \mathcal{R}_1(t) + 2\mathcal{R}_2(t) - 2\mathcal{R}_3(t)
\]
\[
+ (\hat{\beta}_{0,t} - \hat{\beta}_{t,n})^T \Sigma (\beta_{0,t}^* - \beta_{t,n}^*) + \mathcal{R}_1'(t) + 2\mathcal{R}_2'(t) - 2\mathcal{R}_3'(t),
\]
where $\mathcal{R}_1(t), \mathcal{R}_2(t), \mathcal{R}_3(t)$ are defined as in Proposition 1 and $\mathcal{R}_1'(t), \mathcal{R}_2'(t), \mathcal{R}_3'(t)$ are defined as in Proposition 2. Furthermore, denote
\[
\mu(r) = \lim_{n \to \infty} (\beta_{0,t}^* - \beta_{t,n}^*)^T \Sigma (\beta_{t,n}^* - \beta_{t,n}^*)
\]
\[
\phi_L(r) = \lim_{n \to \infty} \left\{ 16\sigma^2_{L,1} + 4 \sum_{i = 1}^{t} \sigma^2_{L,1} (\beta_{i}^* - \beta_{(0,t)}^*)^T \Sigma (\beta_{i}^* - \beta_{(0,t)}^*) + 4\sigma^2_{L,1} \sigma^2_{L,2} \right\}, \quad \text{and}
\]
\[
\phi_R(r) = \lim_{n \to \infty} \left\{ 16\sigma^2_{R,1} + 4 \sum_{i = t + 1}^{n} \sigma^2_{R,1} (\beta_{i}^* - \beta_{(t,n)}^*)^T \Sigma (\beta_{i}^* - \beta_{(t,n)}^*) + 4\sigma^2_{R,1} \sigma^2_{R,2} \right\},
\]
where
\[
\sigma^2_{L,1} = \lim_{n \to \infty} \text{Var} \left\{ \frac{1}{\sqrt{t}} \sum_{i = 1}^{t} x_i x_i^T (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \epsilon_i \right\},
\]
\[
\sigma^2_{R,1} = \lim_{n \to \infty} \text{Var} \left\{ \frac{1}{\sqrt{n - t}} \sum_{i = t + 1}^{n} x_i x_i^T (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \epsilon_i \right\},
\]
\[
\sigma^2_{L,2} = \lim_{n \to \infty} \text{Var} \left\{ \frac{1}{\sqrt{t}} \sum_{i = 1}^{t} (\beta_{(0,t)}^* - \beta_{(t,n)}^*)^T (\Sigma_{(0,t)} - \Sigma) (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \right\},
\]
\[
\sigma^2_{R,2} = \lim_{n \to \infty} \text{Var} \left\{ \frac{1}{\sqrt{n - t}} \sum_{i = t + 1}^{n} (\beta_{(0,t)}^* - \beta_{(t,n)}^*)^T (\Sigma_{(t,n)} - \Sigma) (\beta_{(0,t)}^* - \beta_{(t,n)}^*) \right\},
\]
\[
54
\]
Note that by Lemma 9, all of $\sigma^2_{L,1}, \sigma^2_{R,1}, \sigma^2_{L,2}, \sigma^2_{R,2}$ are finite.

**Theorem 9.** Let $\zeta > 0$ be any fixed constant in $(0,1/2)$. Suppose Assumption 1 and Assumption 3 hold, and $\lambda = C_\lambda \sqrt{\log p}$ for some sufficiently large constant $C_\lambda$. Suppose in addition that $
{i=1}{n} N(0, \sigma^2_t)$ with $\sigma_t = A_n \frac{s \log (pn)}{n}$ for some sequence $A_n \to \infty$. Suppose that
\[
\frac{s \log (pn)}{\sqrt{n}} = o(1) \quad \text{and} \quad \max\{\log^{1/\gamma}(p), (s \log (np))^{\frac{2}{\gamma} - 1}\} = o(1),
\]
where $\gamma = \left(\frac{1}{n} + \frac{2}{c_2}\right)^{-1}$ and that $\gamma < 1$. Under $H_a$, for any fixed $r \in [\zeta, 1 - \zeta]$ and $t = [rn]$, it holds that
\[
\left(\frac{\sqrt{\Phi_L(r)}}{n_r} + \frac{\Phi_R(r)}{n(1 - r)}\right)^{-1} (\mathcal{G}_n(r) - 2\mu(r)) \to N(0, 1).
\]

**Proof of Theorem 9.** Let $r \in [\zeta, 1 - \zeta]$ and $t = [rn]$. Without loss of generality, assume that $\sigma_t = 1$. Following the same argument as Theorem 2, Proposition 1 and Proposition 2 continue to hold under Assumption 3. Let $\mathcal{X}_{n,\xi}(t)$ and $\mathcal{G}_n(t)$ be defined as in Proposition 1, and $\mathcal{X}'_{n,\xi}(t)$ and $\mathcal{G}'_n(t)$ be defined as in Proposition 2. Denote
\[
\mathcal{X}_{n,\xi}(t) + \mathcal{G}_n(t) + \mathcal{X}'_{n,\xi}(t) + \mathcal{G}'_n(t) = \mathcal{M}(r),
\]
where
\[
\mathcal{M}(r) = 4 \sum_{i=1}^{t} x_i^T (\hat{\beta}_{0,t} - \beta_{0,t}^*) \xi_i + \frac{4}{n - t} \sum_{i=t+1}^{n} x_i^T (\beta_{t,t}^* - \beta_{0,t}^*) \xi_i
\]
\[
+ \frac{2}{n} \sum_{i=1}^{t} x_i^T (\beta_{i,t}^* - \beta_{0,t}^*) \xi_i + \frac{2}{n - t} \sum_{i=t+1}^{n} x_i^T (\beta_{i,t}^* - \beta_{0,t}^*) \xi_i
\]
\[
+ \frac{2}{n} \sum_{i=1}^{t} \xi_i \xi_i - \frac{2}{n - t} \sum_{i=t+1}^{n} \xi_i \xi_i
\]
\[
+ (\beta_{0,t}^* - \beta_{t,t}^*)^T (\Sigma_{0,t} - \Sigma)(\beta_{0,t}^* - \beta_{t,t}^*) + (\beta_{t,t}^* - \beta_{0,t}^*)^T (\Sigma_{t,t} - \Sigma)(\beta_{t,t}^* - \beta_{0,t}^*).
\]

**Step 1.** Note that by assumption,
\[
\Phi_L(r) \geq 4\sigma^2_t \quad \text{and} \quad \Phi_R(r) \geq 4\sigma^2_t.
\]
We have that with probability goes to 1,

\[
\left( \frac{\Phi_L(r)}{nr} + \frac{\Phi_R(r)}{n(1-r)} \right)^{-1} \left| \mathcal{S}(r) - 2\mu(r) - M(r) \right|
\]

\[
\leq C(1 + \sigma_\xi) \frac{\text{s log}(pn)}{n} \left( \frac{\sqrt{\Phi_L(r)} + \sqrt{\Phi_R(r)}}{n} \right)^{-1}
\]

\[
\leq C'(1 + \sigma_\xi) \frac{\text{s log}(pn)}{n} \left( \frac{\sqrt{\sigma_\xi^2 r}}{nr} + \frac{\sqrt{\sigma_\xi^2 (1-r)}}{n(1-r)} \right)^{-1}
\]

\[
\leq C'(1 + \sigma_\xi) \frac{\text{s log}(pn)}{\sqrt{n}} \left( \frac{\sqrt{\sigma_\xi^2 r}}{r} + \frac{\sqrt{\sigma_\xi^2 (1-r)}}{(1-r)} \right)^{-1}
\]

\[
\leq C''(1 + \sigma_\xi) \frac{\text{s log}(pn)}{\sigma_\xi \sqrt{n}}
\]

where the first inequality follows from Proposition 1 and Proposition 2, the second inequality follows from Equation (49). By assumption, \( \frac{\text{s log}(pn)}{\sqrt{n}} = o(1) \), \( \sigma_\xi = A_n \frac{\text{s log}(pn)}{\sqrt{n}} \) for some diverging sequence \( A_n \). So \( \frac{(1+\sigma_\xi)\text{s log}(pn)}{\sigma_\xi \sqrt{n}} = o(1) \).

Consequently it suffices to show that

\[
\left( \frac{\Phi_L(r)}{nr} + \frac{\Phi_R(r)}{n(1-r)} \right)^{-1} M(r)
\]

converges to \( N(0,1) \).

**Step 2.** Since \( \{X_i\}_{i=1}^n, \{\epsilon_i\}_{i=1}^n \) and \( \{\xi_i\}_{i=1}^n \) are independent, so (45), (46), (47) and (48) are pairwise uncorrelated. Therefore, it suffices to show that each of (45)\~(48) are converging to normal random variables, as this would imply that (45)\~(48) are asymptotically independent. For brevity, the justifications of (45) and (46) are provided only, as the justification of (47) and (48) are either similar or simpler.

**Step 3.** For (45), note that each \( x_i^T (\beta_{0,t}^* - \beta_{t,n}^*) \epsilon_i \) is sub-Weibull(2\( \gamma_2 \)) \( \beta \)-mixing time series with mixing coefficient \( \exp(-cn^2) \). So by Theorem 10,

\[
\frac{1}{\sqrt{L_1 \sigma_{L,1}}} \sum_{i=1}^t x_i^T (\beta_{0,t}^* - \beta_{t,n}^*) \epsilon_i \rightarrow N(0,1) \quad \text{and} \quad \frac{1}{\sqrt{(n-t)\sigma_{R,1}}} \sum_{i=t+1}^n x_i^T (\beta_{t,n}^* - \beta_{0,t}^*) \epsilon_i \rightarrow N(0,1),
\]

where \( \sigma_{L,1}^2 \) are defined in Equation (41) and \( \sigma_{L,2}^2 \) are defined in Equation (42).
By Lemma 8, \( \frac{1}{\sqrt{t}} \sum_{i=1}^{t} x_i^T (\beta^*_{(0,t)} - \beta^*_{(t,n)}) \epsilon_i \) and \( \frac{1}{\sqrt{n-t}} \sum_{i=t+1}^{n} x_i^T (\beta^*_{(t,n)} - \beta^*_{(0,t)}) \epsilon_i \) are asymptotically independent, it follows that
\[
\frac{1}{\sqrt{t}} \sum_{i=1}^{t} x_i^T (\beta^*_{(0,t)} - \beta^*_{(t,n)}) \epsilon_i + \frac{1}{\sqrt{n-t}} \sum_{i=t+1}^{n} x_i^T (\beta^*_{(t,n)} - \beta^*_{(0,t)}) \epsilon_i \rightarrow N(0, \sigma^2_L + \sigma^2_L).
\]

**Step 4.** For (46), let \( \{ \eta^*_k \}_{q=1}^{Q} = \{ \eta^*_k \}_{k=1}^{K} \cap (0, r] \), where \( Q = 0 \) indicates that \((0, r] \) contains no change points. Denote \( \eta_k = |n \eta^*_k| \) and
\[
\mathcal{J}_1 = (0, \eta_1], \mathcal{J}_2 = (\eta_1, \eta_2] \ldots \mathcal{J}_Q = (\eta_Q, \eta_{Q+1}], \mathcal{J}_{Q+1} = (\eta_Q, t].
\]
Note that \( x_i^T (\beta^*_{i} - \beta^*_{(0,t)} \xi_i \) and \( X_j^T (\beta^*_{j} - \beta^*_{(0,t)} \xi_j \) are uncorrelated because \( \{ \xi_i \}_{i=1}^{n} \) are i.i.d. By Theorem 10, for each \( q \in \{1, \ldots, Q+1 \} \)
\[
\frac{1}{\sqrt{|\mathcal{J}_q| \nu_q}} \sum_{i \in \mathcal{J}_q} x_i^T (\beta^*_{i} - \beta^*_{(0,t)}) \xi_i \rightarrow N(0, 1).
\]
Here
\[
\nu^2_q = \lim_{n \rightarrow \infty} \text{Var} \left\{ \frac{1}{\sqrt{|\mathcal{J}_q| \nu_q}} \sum_{i \in \mathcal{J}_q} x_i^T (\beta^*_{i} - \beta^*_{(0,t)}) \xi_i \right\} = \lim_{n \rightarrow \infty} \frac{4}{|\mathcal{J}_q|} \sum_{i \in \mathcal{J}_q} \sigma^2_i (\beta^*_{i} - \beta^*_{(0,t)})^T \Sigma (\beta^*_{i} - \beta^*_{(0,t)}),
\]
where the second equality follows from the fact that \( \{ x_i^T (\beta^*_{i} - \beta^*_{(0,t)}) \xi_i \}_{i=1}^{n} \) are pairwise uncorrelated.

In addition, by Lemma 8, the collection
\[
\left\{ \frac{1}{\sqrt{|\mathcal{J}_q| \nu_q}} \sum_{i \in \mathcal{J}_q} x_i^T (\beta^*_{i} - \beta^*_{(0,t)}) \xi_i \right\}_{q=1}^{Q+1}
\]
are asymptotically independent. Since \( \bigcup_{q=1}^{Q+1} \mathcal{J}_q = (1, t] \), it follows that
\[
\frac{1}{\sqrt{t \nu_L}} \sum_{i=1}^{t} x_i^T (\beta^*_{i} - \beta^*_{(0,t)}) \xi_i \rightarrow N(0, 1), \tag{50}
\]
where \( \nu^2_L = \frac{1}{t} \sum_{i=1}^{t} \sigma^2_i (\beta^*_{i} - \beta^*_{(0,t)})^T \Sigma (\beta^*_{i} - \beta^*_{(0,t)}). \) Similarly
\[
\frac{1}{\sqrt{n-t \nu_R}} \sum_{i=t+1}^{n} x_i^T (\beta^*_{i} - \beta^*_{(0,t)}) \xi_i \rightarrow N(0, 1), \tag{51}
\]
where \( \nu^2_R = \frac{1}{n-t} \sum_{i=t+1}^{n} \sigma^2_i (\beta^*_{i} - \beta^*_{(0,t)})^T \Sigma (\beta^*_{i} - \beta^*_{(0,t)}). \) Since by Lemma 8, (50) and (51) are asym-
totically independent, it follows that
\[
\frac{1}{\sqrt{t}} \sum_{i=1}^{t} x_i^T (\beta_i^* - \beta_{(0,t)}) \xi_i + \frac{1}{\sqrt{n-t}} \sum_{i=1}^{n-t} x_i^T (\beta_i^* - \beta_{(0,t)}) \xi_i \to N(0, \nu_L^2 + \nu_R^2),
\]
as desired. 

\[\square\]

### B.3.1 Additional Technical Results

Throughout Appendix B.3.1, assume that \( \{z_i\}_{i=1}^{n} \) is a sequence of time series satisfying the following additional conditions.

**Assumption 5.**

a. Suppose that the time series \( \{z_i\}_{i=1}^{n} \) is strictly stationary and geometrically \( \beta \)-mixing; i.e., there exist constants \( c \) and \( \gamma_1 \) such that the \( \beta \)-coefficients of \( \{z_i\}_{i=1}^{n} \) satisfy
\[
\beta(n) \leq \exp(-cn^\gamma_1) \quad \text{for all} \quad n \in \mathbb{N}.
\]

b. Each \( z_i \) follows a sub-Weibull \( \gamma_2 \) distribution with \( \|z_i\|_{\psi_{\gamma_2}} \leq K \epsilon \) for \( i \in \{1, \ldots, n\} \).

c. It holds that \( \mathbb{E}(z_i) = 0 \) for all \( i \in \{1, \ldots, n\} \).

**Lemma 7.** Suppose \( \{z_i\}_{i=1}^{n} \) satisfies Assumption 5. Then
\[
\text{Var} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i \right) \leq \sum_{l=-\infty}^{\infty} |\text{Cov}(z_1, z_{l+1})| < \infty
\]

**Proof.** This is a well known property for \( \beta \)-mixing time series. Since \( z_i \) is sub-Weibull, all the moments of \( z_i \) exist and are finite. Since the \( \beta \)-coefficient of \( \{z_i\}_{i=1}^{n} \) decay exponentially, it is faster than any polynomial decay. The desired result follows from Corollary A.2 of Francq and Zakoian (2019).

\[\square\]

**Theorem 10 (\( \beta \)-mixing CLT).** Suppose \( \{z_i\}_{i=1}^{n} \) satisfies Assumption 5. Then
\[
\frac{1}{\sqrt{n \sigma_z^2}} \sum_{i=1}^{n} z_i \to N(0, 1),
\]
where \( \sigma_z^2 = \text{Var} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i \right) \).
Proof. This is a well known property for $\beta$-mixing time series. The desired result follows from Theorem A.4 of Francq and Zakoian (2019).

Lemma 8. Let $a \in (0, 1)$. Then

$$\lim_{n \to \infty} \text{Cov} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor an \rfloor} z_i, \frac{1}{\sqrt{n}} \sum_{j=\lfloor an \rfloor + 1}^{n} z_j \right) = 0.$$ 

Therefore $\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor an \rfloor} z_i$ and $\frac{1}{\sqrt{n}} \sum_{j=\lfloor an \rfloor + 1}^{n} z_j$ are asymptotically independent.

Proof. Denote $b_{ij} = \text{Cov}(z_i, z_j)$ and

$$\sigma_{LV}^2 = \sum_{k=-\infty}^{\infty} |\text{Cov}(z_1, z_k)|.$$ 

Then

$$\text{Cov} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor an \rfloor} z_i, \frac{1}{\sqrt{n}} \sum_{j=\lfloor an \rfloor + 1}^{n} z_j \right) = \frac{1}{n} \sum_{i=1}^{\lfloor an \rfloor} \sum_{j=\lfloor an \rfloor + 1}^{n} \text{Cov}(z_i, z_j) + \frac{1}{n} \sum_{i=\lfloor an \rfloor + 1}^{\lfloor an \rfloor - M} \sum_{j=\lfloor an \rfloor + 1}^{n} \text{Cov}(z_i, z_j).$$

Note that for $i \in \{1, \ldots, \lfloor an \rfloor - M\}$

$$\sum_{j=\lfloor an \rfloor + 1}^{n} |\text{Cov}(z_i, z_j)| \leq \sum_{k=M}^{\infty} |\text{Cov}(z_1, z_k)|.$$ 

So for sufficiently large $M$ and $i \in \{1, \ldots, \lfloor an \rfloor - M\}$,

$$\sum_{j=\lfloor an \rfloor + 1}^{n} |\text{Cov}(z_i, z_j)| \leq \epsilon.$$

Therefore

$$\frac{1}{n} \sum_{i=1}^{\lfloor an \rfloor - M} \sum_{j=\lfloor an \rfloor + 1}^{n} \text{Cov}(z_i, z_j) \leq \frac{1}{n} \sum_{i=1}^{\lfloor an \rfloor - M} \epsilon \leq a \epsilon.$$ 

In addition,

$$\frac{1}{n} \sum_{i=\lfloor an \rfloor - M + 1}^{\lfloor an \rfloor} \sum_{j=\lfloor an \rfloor + 1}^{n} \text{Cov}(z_i, z_j) \leq \frac{1}{n} \sum_{i=\lfloor an \rfloor - M + 1}^{\lfloor an \rfloor} \sum_{j=-\infty}^{\infty} |\text{Cov}(z_i, z_j)| \leq \frac{1}{n} \sum_{i=\lfloor an \rfloor - M}^{\lfloor an \rfloor} \sigma_{LV}^2 \leq \frac{M}{n} \sigma_{LV}^2.$$ 

So

$$\frac{1}{n} \sum_{i=1}^{\lfloor an \rfloor} \sum_{j=\lfloor an \rfloor + 1}^{n} |\text{Cov}(z_i, z_j)| \leq a \epsilon + \frac{M}{n} \sigma_{LV}^2.$$
Therefore
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\lfloor an \rfloor} \sum_{j=\lfloor an \rfloor+1}^{n} |\text{Cov}(z_i, z_j)| \leq a\epsilon.
\]

Since \( \epsilon \) is arbitrary, the desired result follows. \( \square \)

**Lemma 9.** The long-run variances \( \sigma^2_{L,1}, \sigma^2_{R,1}, \sigma^2_{L,2}, \sigma^2_{R,2} \) defined in (41)\textendash (44) are all finite.

**Proof.** The argument for each long-run variance is the same. For brevity, only \( \sigma^2_{L,1} \) is shown in detailed. Observe that \( \{x_i^\top (\beta_i^\ast - \beta_{(t,n)}^\ast)\epsilon_i\}_{i=1}^n \) are sub-Weibull(2γ2) \( \beta \)-mixing time series with mixing coefficient \( \exp(-cn^{\gamma_1}) \). The desired result follows immediately from Lemma 7. \( \square \)

**B.4 Technical results for Theorem 5**

**Lemma 10.** Let \( \mathcal{I} \subset (0, n] \) be any generic interval such that \( |\mathcal{I}| \geq \zeta n \). Under the same assumptions as in Theorem 5, then it holds that for sufficiently large constant \( C \),
\[
\mathbb{P}\left\{ |v^\top (\hat{\Sigma}_{\mathcal{I}} - \Sigma)| \geq C \sqrt{\frac{s \log(pn)}{|\mathcal{I}|} \|v\|_2^2} \forall v \in \mathcal{C}_S \right\} \leq 2 \exp \left\{ -c_1 \log(pn) \right\}, \quad \text{and} \quad (52)
\]
\[
\mathbb{P}\left\{ \left| \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} \epsilon_i x_i^\top \beta \right| \geq C \sqrt{\frac{s \log(pn)}{|\mathcal{I}|} \|eta\|_1} \forall \beta \in \mathbb{R}^p \right\} \leq 2np \exp \left\{ -c_2 n^{\gamma} \right\}, \quad (53)
\]

where \( c_1, c_2 \geq 3 \) are positive constants.

**Proof.** The deviation bound in Equation (52) is a straight forward adaption of Proposition 8 in Lasso Guarantees for \( \beta \)-mixing heavy-tailed time series by Wong et al. (2020), and the deviation bound in Equation (53) is Proposition 7 Wong et al. (2020). \( \square \)

**Corollary 2.** Let \( \{x_i, y_i\}_{i=1}^n \) be a collection of regression data satisfying Assumption 1 and Assumption 3. Suppose that
\[
\frac{s \log(pn)}{\sqrt{n}} = o(1), \quad \frac{s \log(pn)}{\sigma \sqrt{n}} = o(1) \quad \text{and} \quad \frac{\max\{\log^{1/\gamma}(p), (s \log(np))^{2/\gamma - 1}\}}{n} = o(1),
\]
where \( \gamma = \left( \frac{1}{\gamma_1} + \frac{2}{\gamma_2} \right)^{-1} \) and that \( \gamma < 1 \). Let \( \eta_q \) be a change point of \( \{\beta_i^n\}_{i=1}^n \) and let \( \mathcal{I}_q = (0, \eta_q] \).

Then it holds that for sufficiently large constants \( C, C' \)
\[
\mathbb{P}\left\{ |v^\top (\hat{\Sigma}_{\mathcal{I}_q} - \Sigma)| \geq C \sqrt{\frac{s \log(pn)}{|\mathcal{I}_q|} \|v\|_2^2} \forall v \in \mathcal{C}_S \right\} \leq C' \exp \left\{ -c_1 \log(pn) \right\}, \quad \text{and} \quad (54)
\]
\[
\mathbb{P}\left\{ \left| \frac{1}{|\mathcal{I}_q|} \sum_{i \in \mathcal{I}_q} \epsilon_i x_i^\top \beta \right| \geq C \sqrt{\frac{s \log(pn)}{|\mathcal{I}_q|} \|eta\|_1} \forall \beta \in \mathbb{R}^p \right\} \leq C' np \exp \left\{ -c_2 n^{\gamma} \right\}, \quad (55)
\]
where $c_1, c_2 \geq 3$ are positive constants.

Proof. The proof of Equation (54) follows from a straightforward union bound argument together with Equation (52) in Lemma 10. More precisely, let $\mathcal{I}_q = (\eta_{k-1}, \eta_k]$. Then

$$\mathcal{I}_q = \bigcup_{k=1}^{q} \mathcal{J}_k.$$ 

Note that $\beta_i^*$ is unchanged within each subinterval $\mathcal{J}_k$ and thus $\{x_i, y_i\}_{i=1}^{n}$ is strictly stationary within each $\mathcal{J}_k$. Since $|\mathcal{J}_k| \approx n$, it follows that for any $k$,

$$\mathbb{P}\left\{ |v^\top (\hat{\Sigma}_{\mathcal{J}_k} - \Sigma) v| \geq C \sqrt{\frac{\log(pn)}{|\mathcal{J}_k|}} \|v\|_2 \forall \ v \in C_S \right\} \leq 2 \exp \left\{ -c_1 \log(pn) \right\}.$$

Denote the good events $E_k := \left\{ |v^\top (\hat{\Sigma}_{\mathcal{J}_k} - \Sigma) v| \leq C \sqrt{\frac{\log(pn)}{|\mathcal{J}_k|}} \|v\|_2 \forall \ v \in C_S \right\}$. Then

$$\mathbb{P}\left( \bigcap_{k=1}^{K+1} E_k \right) \geq 1 - (K+1)2 \exp \left\{ -c_1 \log(pn) \right\}.$$ 

Under $\bigcap_{k=1}^{K+1} E_k$, it follows that for any $v \in C_S$,

$$v^\top (\hat{\Sigma}_{\mathcal{I}_q} - \Sigma) v = \frac{1}{|\mathcal{I}_q|} \sum_{i \in \mathcal{I}_q} \left\{ (x_i^\top v)^2 - v^\top \Sigma v \right\} = \frac{1}{|\mathcal{I}_q|} \sum_{k=1}^{q} \sum_{i \in \mathcal{J}_k} \left\{ (x_i^\top v)^2 - v^\top \Sigma v \right\} \leq \frac{1}{|\mathcal{I}_q|} \sum_{k=1}^{q} C \sqrt{|\mathcal{J}_k| \log(pn)} \|v\|_2 \leq qC \sqrt{\frac{\log(pn)}{|\mathcal{I}_q|}} \|v\|_2 \leq KC \sqrt{\frac{\log(pn)}{|\mathcal{I}_q|}} \|v\|_2,$$

where the first inequality follows from $\bigcap_{k=1}^{K+1} E_k$ and the second inequality follows from $|\mathcal{J}_k| \leq |\mathcal{I}_q|$. This gives Equation (54).

The argument of Equation (55) follows from the same argument of Equation (54) and is therefore omitted.

Lemma 11. Let $\zeta \in (0, 1/2)$ and $u \in \mathbb{R}^p$ be any deterministic vector. Under the same assumptions as in Theorem 5, it holds that

$$\mathbb{P}\left\{ \left| \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} \left\{ u^\top x_i x_i^\top \beta - u^\top \Sigma \beta \right\} \right| \geq C \|u\|_2 \sqrt{\frac{\log(pn)}{|\mathcal{I}|}} \|\beta\|_1 \forall \ \beta \in \mathbb{R}^p, \ |\mathcal{I}| \geq \zeta n \right\} \leq C' \exp \left( -c(n \log(pn))^{\gamma/2} \right) + \exp \left( -c' \log(pn) \right),$$

where $C, C', c, c'$ are all absolute constants and that $c, c' \geq 3$. 

\[61\]
Proof. Without loss of generality, assume \( \|u\|_2 = 1 \). For any given \( j \in \{1, \ldots, p\} \), let

\[
\begin{align*}
  z_i &= u^\top x_i x_{i,j} - u^\top \Sigma(,j) = u^\top x_i x_{i,j} - \mathbb{E}(u^\top x_i x_{i,j}).
\end{align*}
\]

As a result, \( z_i \) is strictly stationary. Since \( z_i \) is measurable to \( \sigma(x_i) \), the process \( \{z_i\}_{i=1}^n \) is \( \beta \)-mixing with the same mixing coefficient as \( \{x_i\}_{i=1}^n \). In addition,

\[
\|z_i\|_{\psi_{\gamma_2}/2} = \|u^\top x_i x_{i,j}\|_{\psi_{\gamma_2}/2} \leq c_1 \|u^\top x_i\|_{\psi_{\gamma_2}} \|x_{i,j}\|_{\psi_{\gamma_2}} \leq c_1 K_\chi^2,
\]

where \( c_1 \) is a constant only depending on \( \gamma_2 \). Therefore by Theorem 11,

\[
P\left\{ \frac{1}{|I|} \sum_{i \in I} z_i \geq \delta \right\} \leq C_2 n \exp \left( -c_2 (\delta n)^\gamma \right) + \exp(-c_3 \delta^2 n),
\]

(56)

where \( \gamma < 1 \) is defined in Assumption 4. Consequently, taking an union bound overall coordinates, it holds that

\[
P\left\{ \max_{1 \leq j \leq p} \left| \frac{1}{|I|} \sum_{i \in I} \{ u^\top x_i x_{i,j} - u^\top \Sigma(,j) \} \right| \geq C_1 \delta \forall \|u\|_2 = 1 \right\}
\]

\[
\leq C_2 np \exp \left( -c_2 (\delta n)^\gamma \right) + p \exp(-c_3 \delta^2 n)
\]

Taking a union bound over all \( I \) such that \( |I| \geq \zeta n \), it holds that

\[
P\left\{ \max_{1 \leq j \leq p} \left| \frac{1}{|I|} \sum_{i \in I} \{ u^\top x_i x_{i,j} - u^\top \Sigma(,j) \} \right| \geq \delta \text{ for all } I \subset (0, n) \text{ such that } |I| \geq \zeta n \right\}
\]

\[
\leq C_2 n^3 p \exp \left( -c_2 (\delta n)^\gamma \right) + n^2 p \exp(-c_3 \delta^2 n).
\]

Taking \( \delta = C_1 \sqrt{\frac{\log (pn)}{|I|}} \), it follows that

\[
P\left\{ \max_{1 \leq j \leq p} \left| \frac{1}{|I|} \sum_{i \in I} \{ u^\top x_i x_{i,j} - u^\top \Sigma(,j) \} \right| \geq C_1 \sqrt{\frac{\log (pn)}{|I|}} \text{ for all } I \subset (0, n) \text{ such that } |I| \geq \zeta n \right\}
\]

\[
\leq C_2 \exp \left( -c_2' (n \log (pn))^{\gamma/2} \right) + \exp \left( -c_3' \log (pn) \right),
\]

where \( \frac{\log^{\frac{3}{2}} (np)}{n^2} = o(1) \) is used in the last inequality. The desired result follows from the observation that for all \( \beta \in \mathbb{R}^p \),

\[
\left| \frac{1}{|I|} \sum_{i \in I} u^\top x_i x_i^\top \beta - \frac{1}{|I|} \sum_{i \in I} u^\top \Sigma \beta \right| \leq \max_{1 \leq j \leq p} \left| \frac{1}{|I|} \sum_{i \in I} \{ u^\top x_i x_i(j) - u^\top \Sigma(,j) \} \right| \|\beta\|_1.
\]

\[\square\]

Lemma 12. Let \( \zeta \in (0, 1/2) \) and \( u \in \mathbb{R}^p \) be any deterministic vector. Under the same assumptions
as in Theorem 5, it holds that
\[
P\left\{ \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_i^\top \beta - u^\top \Sigma \beta \right\} \right| \geq C \sqrt{\frac{s \log(pn)}{|I|}} \| \beta \|_1 \right. \forall \beta \in \mathbb{R}^p, \forall |I| \geq \zeta n, \forall \|u\|_2 = 1, u \in C_S \right\} \leq C \exp\left( -c(s \log(pn) n)^{\gamma/2} \right) + C' \exp(-c' s \log(pn)),
\]
where \(C, C', c, c'\) are all absolute constants and that \(c, c' \geq 3\).

**Proof.** Step 1. By Equation (56), it holds that for any \(u \in \mathbb{R}^p\) such that \(\|u\|_2 = 1\),
\[
P\left\{ \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_i^\top - u^\top \Sigma(., j) \right\} \right| \geq \delta \right\} \leq C_2 n \exp\left( -c_2(\delta n)^{\gamma} \right) + \exp(-c_3 n^2),
\]
where \(\gamma = \left( \frac{1}{\gamma_1} + \frac{2}{\gamma_2} \right)^{-1}\) as defined in Assumption 4. Denote
\[
K(s) = \{ u \in \mathbb{R}^p : \|u\|_2 = 1, \|u\|_0 \leq s \}.
\]
By Lemma F1 in the appendix of BasuMichailidis(2015),
\[
C_S \cap \{ u \in \mathbb{R}^p : \|u\|_2 = 1 \} \subset 5c\{\text{conv}\{K(s)\}\},
\]
where \(c\{\text{conv}\{K(s)\}\}\) is the closure of convex hull of \(K(s)\). So
\[
P\left\{ \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_i^\top - u^\top \Sigma(., j) \right\} \right| \geq \delta \right\} \leq \sup_{u \in 5c\{\text{conv}\{K(s)\}\}} \left\{ \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_i^\top - u^\top \Sigma(., j) \right\} \right| \geq \delta \right\}
\]
By Lemma 16, with \(V = \frac{1}{|I|} \sum_{i \in I} x_i x_i^\top (j) - \Sigma(., j)\), it holds that
\[
P\left\{ \sup_{u \in 5c\{\text{conv}\{K(s)\}\}} \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_i^\top - u^\top \Sigma(., j) \right\} \right| \geq \delta \right\} \leq \sup_{u \in 5K(s)} \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_i^\top - u^\top \Sigma(., j) \right\} \right| \geq \delta \right\}.
\]
By Lemma 15, it holds that
\[
P\left\{ \sup_{u \in K(s)} \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_i^\top - u^\top \Sigma(., j) \right\} \right| \geq \delta \right\} \leq C_2 n p^5 \exp\left( -c_2(\delta n)^{\gamma} \right) + p^5 \exp(-c_3 n^2).
\]
As a result
\[
P\left\{ \sup_{u \in 5K(s)} \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_i^\top - u^\top \Sigma(., j) \right\} \right| \geq 5\delta \right\} \leq C_2 n p^5 \exp\left( -c_2(\delta n)^{\gamma} \right) + p^5 \exp(-c_3 \delta^2 n).
\]
Combining the calculations in this step, it holds that
\[
P \left\{ \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_{i,j} - u^\top \Sigma(j, j) \right\} \right| \geq \delta \quad \forall \|u\|_2 = 1, u \in \mathcal{C}_S \right\} \leq C_2 np^3 5^5 \exp \left( -c_2(\delta n)^5 \right) + p^5 5^5 \exp \left( -c_3 \delta^2 n \right).
\] (57)

**Step 2.** Since
\[
\left| \frac{1}{|I|} \sum_{i \in I} u^\top x_i x_{i} \beta - \frac{1}{|I|} \sum_{i \in I} u^\top \Sigma \beta \right| \leq \max_{1 \leq j \leq p} \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_{i,j} - u^\top \Sigma(j, j) \right\} \right| \|\beta\|_1,
\]
it suffices to bound
\[
P \left\{ \max_{1 \leq j \leq p} \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_{i,j} - u^\top \Sigma(j, j) \right\} \right| \geq C_2 \sqrt{\frac{\log(pn)}{|I|}} \forall|I| \geq \zeta n, \forall \|u\|_2 = 1, u \in \mathcal{C}_S \right\}.
\]
Since there are at most \(n^2\) many intervals \(I\) such that \(|I| \geq \zeta n\), by Equation (57),
\[
P \left\{ \max_{1 \leq j \leq p} \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_{i,j} - u^\top \Sigma(j, j) \right\} \right| \geq \delta \forall|I| \geq \zeta n, \forall \|u\|_2 = 1, u \in \mathcal{C}_S \right\} \leq C_2 n^3 p^{s+1} 5^5 \exp \left( -c_2(\delta n)^5 \right) + n^2 p^{s+1} 5^5 \exp \left( -c_3 \delta^2 n \right).
\]

Taking \(\delta = C_3 \sqrt{\frac{s \log(pn)}{n}}\) for sufficiently large constant \(C_3\) and using the assumption that \(\frac{s \log(pn)}{n} \leq o(1)\), it holds that
\[
P \left\{ \sup_{u \in K(\zeta)} \left| \frac{1}{|I|} \sum_{i \in I} \left\{ u^\top x_i x_{i}^{\top} (j) - u^\top \Sigma(j, j) \right\} \right| \geq 5C_3 \sqrt{\frac{s \log(pn)}{n}} \forall \|u\|_2 = 1, u \in \mathcal{C}_S, |I| \geq \zeta n \right\} \leq C'_2 \exp \left( -c'_2(s \log(pn)n)^{\gamma/2} \right) + C'_3 \exp \left( -c'_3 s \log(pn) \right).
\]

\[\square\]

**Lemma 13.** Suppose \(\{\xi_i\}_{i=1}^n\) is a collection of i.i.d. Gaussian random variables \(N(0, \sigma^2_\xi)\) and \(\{\epsilon_i\}_{i=1}^n\) satisfies Assumption 3 with \(\text{Var}(\epsilon_i) = \sigma^2_\epsilon\). For any \(r \in (0, 1)\), let \(t = \lfloor rn \rfloor\) and
\[
\mathcal{G}_n(r) = \frac{\sqrt{n}}{\sigma_\xi \sigma_\epsilon} \left\{ \frac{1}{t} \sum_{i=1}^t \epsilon_i \xi_i - \frac{1}{n-t} \sum_{i=t+1}^n \epsilon_i \xi_i \right\}.
\]
Then
\[
\mathcal{G}_n(r) \overset{D}{\to} \mathcal{G}(r),
\]
where the convergence is in Skorokhod topology and \(\mathcal{G}(r)\) is a Gaussian Process on \(r \in (0, 1)\), with
covariance function
\[ \sigma(r, s) = \frac{1}{r(1-s)} \quad \text{when} \quad 0 < s \leq r < 1. \]

**Proof.** This is a direct consequence of Theorem 10 and the observation that
\[ \text{Cov}(\epsilon_i \xi_i, \epsilon_j \xi_j) = \mathbb{E}(\epsilon_i \xi_i \epsilon_j \xi_j) = \mathbb{E}(\epsilon_i \epsilon_j) \mathbb{E}(\xi_i) \mathbb{E}(\xi_j) = 0. \]

\[ \square \]

**Lemma 14.** Suppose all the assumptions in Theorem 5 holds and that
\[ \beta_1^* = \ldots = \beta_n^* = \beta^*. \]
Let \( \zeta \in (0, 1/2) \) be a given constant. For any generic interval \( I \), let \( \hat{\beta}_I \) be defined as in Equation (2). If \( \lambda = C_\lambda \sqrt{\log(pn)} \) for some sufficiently large constant \( C_\lambda \), then it holds that
\[ \mathbb{P}\left\{ \| \hat{\beta}_I - \beta^* \|_2 \leq \frac{Cs \log(pn)}{n} \forall I \subset (0, n], |I| \geq \zeta n \right\} \geq 1 - (pn)^{-3}; \]
\[ \mathbb{P}\left\{ \| \hat{\beta}_I - \beta^* \|_1 \leq C \sqrt{\frac{\log(pn)}{n}} \forall I \subset (0, n], |I| \geq \zeta n \right\} \geq 1 - (pn)^{-3}; \]
\[ \mathbb{P}\left\{ \| (\hat{\beta}_I - \beta^*)_{S^c} \|_1 \leq 3\| (\hat{\beta}_I - \beta^*)_S \|_1 \forall I \subset (0, n], |I| \geq \zeta n \right\} \geq 1 - (pn)^{-3}. \]

**Proof.** Note that the number of possible interval \( I \) is bounded by \( n^2 \). As a result, the proof is a simple adaption of Corollary 9 of Wong et al. (2020) and together with a union bound over all possible intervals \( I \).

\[ \square \]

**B.5 Additional technical results**

**Theorem 11.** Let \( \{x_j\}_{j=1}^n \) be a strictly stationary sequence of zero mean random variables that are subweibull(\( \gamma_2 \)) with subweibull constant \( K \). Suppose the \( \beta \)-mixing coefficients of \( \{x_j\}_{j=1}^n \) satisfy \( \beta(m) \leq \exp(-cm^{\gamma_1}) \). Let \( \gamma \) be given by
\[ \frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{2}{\gamma_2} \]
and suppose that \( \gamma < 1 \). Then for \( n \geq 4 \) and any \( \delta > 1/n \),
\[ \mathbb{P}\left\{ \frac{1}{n} \sum_{i=1}^n x_i \geq \delta \right\} \leq n \exp \left\{ - \frac{(\delta n)^\gamma}{K^\gamma C_1} \right\} + \exp \left\{ - \frac{\delta^2 n}{K^2 C_2} \right\}, \]
where the constants \( C_1, C_2 \) depend only on \( \gamma_1, \gamma_2 \) and \( c \).

**Proof.** This is Lemma 13 of WongLiTewari (2020).

\[ \square \]
Lemma 15. Let $V \in \mathbb{R}^p$ be any random vector. Suppose for all $u \in \mathbb{R}^p$ such that $\|u\|_2 = 1$, it holds that

$$\mathbb{P}\left\{ |u^\top V| \geq \delta \right\} \leq g(\delta).$$

Then

$$\mathbb{P}\left\{ \sup_{\|u\|_2 = 1, \|u\|_0 \leq s} |u^\top V| \geq \delta \right\} \leq p^s 5^s g(\delta/2).$$

Proof. This is a standard covering lemma. Choose $S \subset \{1, \ldots, p\}$ with $|S| = s$. Define

$$\Omega_S := \{u \in \mathbb{R}^p : \|u\|_2 = 1, \text{ support}(u) \subset S\}.$$

Then we have

$$\{u \in \mathbb{R}^p : \|u\|_2 = 1, \|u\|_0 \leq s\} = \bigcup_{|S| = s} \Omega_S.$$

Step 1. Let $S$ be given such that $|S| \leq s$. Let $\{w_i\}_{i=1}^M$ be a 1/2-net of $\Omega_S$. That is for any $u \in \Omega_S$, there exists $w_j$ such that $\|w_j - u\|_2 \leq 1/2$. By standard covering result, $\{w_i\}_{i=1}^M$ can be chosen so that $M \leq 5^s$.

Denote

$$\alpha = \sup_{u \in \Omega_S} |u^\top V|.$$

Since support($w_j - u$) $\subset S$ and $\|w_j - u\|_2 \leq 1/2$,

$$|(w_j - u)^\top V| \leq \alpha/2.$$

So we have for any $u \in \Omega_S$

$$|u^\top V| \leq \max_{1 \leq i \leq M} |w_i^\top V| + |(w_j - u)^\top V| \leq \max_{1 \leq i \leq M} |w_i^\top V| + \frac{\alpha}{2}.$$

This implies that

$$\alpha \leq \max_{1 \leq i \leq M} |w_i^\top V| + \frac{\alpha}{2}$$

or simply

$$\alpha \leq 2 \max_{1 \leq i \leq M} |w_i^\top V|.$$

As a result

$$\mathbb{P}\left\{ \sup_{u \in \Omega_S} |u^\top V| \geq \delta \right\} \leq \mathbb{P}\left\{ \max_{1 \leq i \leq M} |w_i^\top V| \geq \delta/2 \right\} \leq 5^s g(\delta/2).$$

Step 2. Since there are at most $\binom{p}{s} \leq p^s$ number of possible choices of $S$ such that $|S| \leq s$, it holds
Lemma 16. Denote
\[ \mathcal{K}(\delta) = \{ u \in \mathbb{R}^p : \|u\|_2 = 1, \|u\|_0 \leq s \} . \]

For any vector \( V \in \mathbb{R}^p \), it holds that
\[
\sup_{u \in \text{cl}(\text{conv}(\mathcal{K}(\delta)))} |u^\top V| \leq \sup_{u \in \mathcal{K}(\delta)} |u^\top V|.
\]

Proof. By continuity of the linear form, it suffices to show that
\[
\sup_{u \in \text{conv}(\mathcal{K}(\delta))} |u^\top V| \leq \sup_{u \in \mathcal{K}(\delta)} |u^\top V|.
\]

Let \( v \in \text{conv}(\mathcal{K}(\delta)) \). Then \( v = \sum_{l=1}^L \alpha_l v_l \) where \( L \geq 1 \), \( v_l \in \mathcal{K}(\delta) \) and \( 0 \leq \alpha_l \leq 1 \) such that \( \sum_{l=1}^n \alpha_l = 1 \). Then
\[
|v^\top V| \leq \sum_{l=1}^L \alpha_l |v_l^\top V| \leq \sup_{u \in \mathcal{K}(\delta)} |u^\top V|,
\]
which directly gives the desired result. \( \square \)

C Lower Bounds for Detection Boundary

Proof of Theorem 4:

Proof. To establish the lower bound, it suffices to consider the special case for \( K = 1 \) as the \( K > 1 \) will only result in better constants. Throughout the argument, it is assumed that \( x_i \overset{i.i.d.}{\sim} N_p(0, I_p) \) and \( \epsilon_i \overset{i.i.d.}{\sim} N(0, 1) \).

Step 1. Denote
\[ \mathcal{C} = \{ \beta \in \mathbb{R}^p : \beta(i) \in \{ \kappa / \sqrt{s}, 0, -\kappa / \sqrt{s} \}, \|\beta\|_0 = s \} . \]

Let \( \tilde{P}_{\beta} \) denote the joint distribution of \( \{y_i, x_i\}_{i=1}^n \) generated according to Assumption 1, where
\[
\beta^*_t = \begin{cases} 
0 & \text{when } 1 \leq t \leq n/2; \\
\beta & \text{when } (n/2 + 1) \leq t \leq n.
\end{cases}
\]
So under $\widetilde{P}_\beta$, there is only one change point $n/2$ in the time series and
\[ \|\beta^*_n - \beta^*_{n/2+1}\|^2 = \|\beta\|^2 = \kappa^2. \]
where $\Sigma = I_p$ is used. Let
\[ \kappa^2 = b \frac{s \log(p)}{n} \]
for sufficiently small constant $b$. So by assumption, $\kappa^2 < 1/8$.

Let $\tilde{P}_0$ denote the distribution of data generated according to Assumption 1 with $\beta^*_i = 0$ for all $i$. Let
\[ \widetilde{P}_1 = \frac{1}{|C|} \sum_{\beta \in C} \widetilde{P}_\beta, \]
where $|C|$ denotes the cardinality of $C$. Denote $E_\beta$ to be the expectation under the distribution of $\widetilde{P}_\beta$. Note that
\[
\inf_\psi \sup_{P \in P_1(b)} E_0(\psi) + E_P(1 - \psi) \geq \inf_\psi \sup_\beta E_0(\psi) + E_\beta(1 - \psi) \geq \inf_\psi E_0(\psi) + \frac{1}{|C|} \sum_{\beta \in C} E_\beta(1 - \psi) \geq 1 - \frac{1}{2} \|\widetilde{P}_0 - \widetilde{P}_1\|_1,
\]
where the last inequality is due to LeCam’s Lemma. Let $P^{n/2}_0$ denote the joint distribution of $\{y_i, x_i\}_{i=1}^{n/2}$ with $\beta^*_i = 0$ for all $1 \leq i \leq n/2$. In addition, let $P^{n/2}_\beta$ denote the joint distribution of $\{y_i, x_i\}_{i=1}^{n/2}$ with $\beta^*_i = \beta$ for all $1 \leq i \leq n/2$ and let
\[ P^{n/2}_1 = \frac{1}{|C|} \sum_{\beta \in C} P^{n/2}_\beta. \]
Since under $\tilde{P}_0$ or $\tilde{P}_1$, the distribution of $\{y_i, X_i\}_{i=1}^{n/2}$ are the same, straightforward calculations show that
\[ \|\tilde{P}_0 - \tilde{P}_1\|_1 = \|P^{n/2}_0 - P^{n/2}_1\|_1. \]
In addition, by Le Cam’s lemma,
\[
\inf_\psi \sup_{P \in P_1(b)} E_0(\psi) + E_P(1 - \psi) \geq 1 - \|P^{n/2}_0 - P^{n/2}_1\|_{TV}\]
where
\[ \|P_{0}^{n/2} - P_{1}^{n/2}\|_{TV} \leq \frac{1}{2} \sqrt{E_{P_{0}^{n/2}} \left( \frac{dP_{1}^{n/2}}{dP_{0}^{n/2}} \right)^2 - 1}. \]

The relationship between different norms of distributions can be found in Tsybakov (2009).

So it suffices to show
\[ E_{P_{0}^{n/2}} \left( \frac{dP_{1}^{n/2}}{dP_{0}^{n/2}} \right)^2 = 1 + o(1). \]  

(58)

**Step 2.** Note that \( \frac{dP_{\beta}}{dP_{0}} = \exp \left\{ y x^\top \beta - \frac{1}{2} (x^\top \beta)^2 \right\} \), where \( x \sim N(0, I_{p}) \). So
\[ E_{P_{0}} \left( \frac{dP_{\beta}}{dP_{0}} \frac{dP_{\beta'}}{dP_{0}} \right) = E_{X}(\exp\{\beta^\top xx^\top \beta'\}). \]

Straightforward calculations show that
\[ E_{P_{0}^{n/2}} \left( \frac{dP_{1}^{n/2}}{dP_{0}^{n/2}} \right)^2 = E_{\beta, \beta' \sim \mathcal{C}} \left\{ \left[ E_{X} \left( \exp\{\beta^\top xx^\top \beta'\} \right) \right]^{n/2} \right\} 
\leq E_{\beta, \beta' \sim \mathcal{C}} \left\{ \left( 1 - \left[ 2 \beta^\top \beta' + \kappa^4 - (\beta^\top \beta')^2 \right] \right)^{-n/4} \right\}, \]

where \( \beta \sim \mathcal{C} \) indicates that \( \beta \) is selected from \( \mathcal{C} \) uniformly at random. Since \( \log(1 - x)^{-1} \leq x + x^2 \), and
\[ 2\beta^\top \beta' + \kappa^4 \leq 2\kappa^2 + \kappa^4 \leq 2 \cdot \frac{1}{8} + \frac{1}{64} < 1/2, \]

69
it holds that

\[
\mathbb{E}_{\beta,\beta' \sim C}\left\{ \left( 1 - \left[ 2\beta^\top \beta' + \kappa^4 \right] \right)^{-n/4} \right\} \\
= \mathbb{E}_{\beta,\beta' \sim C}\left\{ \exp \left( \frac{n}{4} \log \frac{1}{1 - 2\beta^\top \beta' - \kappa^4} \right) \right\} \\
\leq \mathbb{E}_{\beta,\beta' \sim C}\left\{ \exp \left( \frac{n}{4} \left( 2\beta^\top \beta' + \kappa^4 + \left[ 2\beta^\top \beta' + \kappa^4 \right]^2 \right) \right) \right\} \\
\leq \mathbb{E}_{\beta,\beta' \sim C}\left\{ \exp \left( \frac{n}{4} \left( 4\beta^\top \beta' + 2\kappa^4 \right) \right) \right\} \\
= \exp \left( \frac{nk^4}{2} \right) \mathbb{E}_{\beta,\beta' \sim C}\left\{ \exp \left( n\beta^\top \beta' \right) \right\},
\]

where \( 4\beta^\top \beta' + 2\kappa^4 < 1 \) is used in the last inequality. Let \( J \) denote the support of \( \beta \), \( J' \) denote the support of \( \beta' \) and

\[
U =: |\{ j \in J \cap J' : \beta_j = \beta'_j \}|.
\]

As a result

\[
\beta^\top \beta' = 2\frac{\kappa^2}{s} U - \frac{\kappa^2}{s} |J \cap J'|.
\]

So

\[
\mathbb{E}_{\beta,\beta' \sim C}\left\{ \exp \left( n\beta^\top \beta' \right) \right\} = \mathbb{E}_{J \cap J'}\left\{ \exp \left[ n \left( 2\frac{\kappa^2}{s} U - \frac{\kappa^2}{s} |J \cap J'| \right) \right] \right\}.
\]

Conditioning on \( |J \cap J'|, U \sim Bin(|J \cap J'|, 1/2) \). So

\[
\mathbb{E}_{J \cap J'}\left\{ \exp \left[ n \left( 2\frac{\kappa^2}{s} U - \frac{\kappa^2}{s} |J \cap J'| \right) \right] \right\} \\
= \mathbb{E}_{J \cap J'} \mathbb{E}_{U \mid J \cap J'}\left\{ \exp \left[ n \left( 2\frac{\kappa^2}{s} U - \frac{\kappa^2}{s} |J \cap J'| \right) \right] \right\} \\
= \mathbb{E}_{J \cap J'}\left\{ \left( 1 - \frac{1}{2} + \frac{1}{2} e^{2n\frac{\kappa^2}{s}} \right)^{|J \cap J'|} \exp \left( - n \frac{\kappa^2}{s} |J \cap J'| \right) \right\} \\
= \mathbb{E}_{J \cap J'}\left\{ \cosh \left( \frac{nk^2}{s} \right)^{|J \cap J'|} \right\} \\
\leq \mathbb{E}_{J \cap J'}\left\{ \exp \left( \frac{nk^2}{s} |J \cap J'| \right) \right\},
\]
where \( \cosh(x) \leq \exp(x) \) for all \( x \) is used in the last inequality. Putting this together, it holds that

\[
\mathbb{E}_{P_{0}^{n/2}} \left( \frac{dP_{1}^{n/2}}{dP_{0}^{n/2}} \right)^{2} \leq \exp \left( \frac{n\kappa^{4}}{2} \right) \mathbb{E}_{\mathcal{J} \cap \mathcal{J}'} \left\{ \exp \left( \frac{n\kappa^{2}}{s} |\mathcal{J} \cap \mathcal{J}'| \right) \right\}
\]

(59)

**Step 3.** Let \( H(p, s, s) \) denote the hypergeometric distribution counting the number of black balls in \( s \) draws from an urn containing \( s \) black balls out of \( p \) balls. Then by Equation (59),

\[
\mathbb{E}_{P_{0}^{n/2}} \left( \frac{dP_{1}^{n/2}}{dP_{0}^{n/2}} \right)^{2} \leq \exp \left( \frac{n\kappa^{4}}{2} \right) \mathbb{E} \left\{ \exp \left( \frac{n\kappa^{2}}{s} H(p, s, s) \right) \right\}
\]

By Lemma 3 in the supplement of Arias-Castro et al. (2011), \( H(p, s, s) \) is stochastically smaller than \( Bin(s, s/(p - s)) \). So if \( B \sim Bin(s, s/(p - s)) \), then

\[
\mathbb{E}_{P_{0}^{n/2}} \left( \frac{dP_{1}^{n/2}}{dP_{0}^{n/2}} \right)^{2} \leq \exp \left( \frac{n\kappa^{4}}{2} \right) \mathbb{E} \left\{ \exp \left( \frac{n\kappa^{2}}{s} B \right) \right\}
\]

\[
= \exp \left( \frac{n\kappa^{4}}{2} \right) \left\{ 1 - \frac{s}{p - s} + \frac{s}{p - s} \exp \left( \frac{n\kappa^{2}}{s} \right) \right\}^{s}
\]

\[
\leq \exp \left( \frac{n\kappa^{4}}{2} \right) \left\{ 1 + \frac{s}{p - s} \exp \left( \frac{n\kappa^{2}}{s} \right) \right\}^{s}
\]

\[
\leq \exp \left( \frac{n\kappa^{4}}{2} + \frac{s^{2}}{p - s} \exp \left( \frac{n\kappa^{2}}{s} \right) \right).
\]

Since \( s \log(p) = o(\sqrt{n}) \), \( s \leq p^{\alpha} \) for some \( \alpha < 1/2 \), and \( \kappa^{2} = b \frac{s \log(p)}{n} \) for sufficiently small constant \( b \), \( \lim_{n,p \to \infty} \frac{n\kappa^{4}}{2} = 0 \) and \( \lim_{n,p \to \infty} \frac{s^{2}}{p - s} \exp \left( \frac{n\kappa^{2}}{s} \right) = 0 \). This directly implies that Equation (58) holds.
D Additional Technical Results

Lemma 17. Suppose Assumption 1 holds. Let $\mathcal{I} \subset [1, n]$. Denote $\kappa = \max_{k \in \{1, \ldots, K\}} \kappa_k$, where $\{\kappa_k\}_{k=1}^K$ are defined in Assumption 1. Then

$$\|\beta^*_{\mathcal{I}} - \beta_i\|_2 \leq C\kappa \leq CC\kappa,$$

for some absolute constant $C$ independent of $n$.

Proof. It suffices to consider $\mathcal{I} = [1, n]$ and $\beta_i = \beta_1$ as the general case is similar. Denote $\Delta_k = \eta_{k+1} - \eta_k$.

Observe that

$$\|\beta^*_{[1,n]} - \beta^*_1\|_2 = \left\| \frac{1}{n} \sum_{i=1}^n \beta^*_i - \beta^*_1 \right\|_2 = \left\| \frac{1}{n} \sum_{k=0}^{K} \Delta_k \beta^*_{\eta_k+1} - \frac{1}{n} \sum_{k=0}^{K} \Delta_k \beta^*_1 \right\|_2 \leq \frac{1}{n} \sum_{k=0}^{K} \|\Delta_k (\beta^*_{\eta_k+1} - \beta^*_1)\|_2 \leq \frac{1}{n} \sum_{k=0}^{K} \Delta_k (K+1) \kappa \leq (K+1) \kappa.$$  

By Assumption 1, both $\kappa$ and $K$ bounded above. \hfill \Box

Lemma 18. Let $t \in \mathcal{I} = (s, e] \subset [1, n]$. Denote $\kappa_{\max} = \max_{k \in \{1, \ldots, K\}} \kappa_k$, where $\{\kappa_k\}_{k=1}^K$ are defined in Assumption 1. Then

$$\sup_{0 < s < t < e \leq n} \|\beta^*_{[s,t]} - \beta^*_{[t,e]}\|_2 \leq C\kappa \leq CC\kappa,$$

for some absolute constant $C$ independent of $n$.

Proof. It suffices to consider $(s, e) = (0, n]$, as the general case is similar. Denote $\Delta_k = \eta_{k+1} - \eta_k$. 

72
Suppose that $\eta_q < t \leq \eta_{q+1}$. Observe that

$$
\|\beta^*_i - \beta^*_{(t,n)}\|_2^2
= \left\| \frac{1}{t} \sum_{i=1}^t \beta^*_i - \frac{1}{n-t} \sum_{i=t+1}^n \beta^*_i \right\|_2^2
= \left\| \frac{1}{t} \left( \sum_{k=0}^{q-1} \Delta_k \beta^*_{\eta_k+1} + (t-\eta_q) \beta^*_{\eta_q+1} \right) - \frac{1}{n-t} \left( \sum_{k=q+1}^K \Delta_k \beta^*_{\eta_k+1} + (\eta_{q+1} - t) \beta^*_{\eta_{q+1}} \right) \right\|_2^2
= \left\| \frac{1}{t} \left( \sum_{k=0}^{q-1} \Delta_k (\beta^*_{\eta_k+1} - \beta^*_{\eta_q+1}) \right) + \beta^*_{\eta_q+1} - \frac{1}{n-t} \left( \sum_{k=q+1}^K \Delta_k (\beta^*_{\eta_k+1} - \beta^*_{\eta_{q+1}}) \right) - \beta^*_{\eta_{q+1}} \right\|_2^2
\leq \frac{1}{t} \sum_{k=0}^{q-1} \Delta_k K \kappa + \frac{1}{n-t} \sum_{k=q+1}^K \Delta_k K \kappa \leq 2K \kappa.
$$

\[\square\]

**Lemma 19.** Let $0 < \zeta < 1/2$ be any constant sufficiently close to 0. Denote

$$
\kappa = \max_{k \in \{1, \ldots, K\}} \kappa_k,
$$

where $\{\kappa_k\}_{k=1}^K$ are defined in Assumption 1. Suppose $\{\beta^*_i\}_{i=1}^n$ is a collection of vectors satisfying Assumption 1 with $K > 0$. Then it holds that

$$
\max_{\zeta n \leq t \leq (1-\zeta)n} (\beta^*_{(0,t]} - \beta^*_{(t,n]} )^\top \Sigma (\beta^*_{(0,t]} - \beta^*_{(t,n]} ) \geq c \kappa^2
$$

for some sufficiently small constant $c$. In addition, suppose that $\zeta$ is sufficiently small such that

$$
\zeta \leq \eta^*_1 < \ldots < \eta^*_K \leq 1 - \zeta.
$$

Then there exists $q \in \{1, \ldots, K\}$ such that

$$
(\beta^*_{(0,\eta_q]} - \beta^*_{(\eta_q,n]})^\top \Sigma (\beta^*_{(0,\eta_q]} - \beta^*_{(\eta_q,n]}) \geq c' \kappa^2,
$$

where $\eta_q = n \eta^*_q$ and $c'$ is some sufficiently small constant.

**Proof.** Let

$$
\alpha_i = \Sigma^{1/2} \beta^*_i \quad \text{for all} \quad 1 \leq i \leq n.
$$

Then the vector CUSUM statistics of $\{\alpha_i\}_{i=1}^n$ is

$$
\tilde{\alpha}(t) := \sqrt{\frac{t(n-t)}{n}} \left( \frac{1}{t} \sum_{i=1}^t \alpha_i - \frac{1}{n-t} \sum_{i=t+1}^n \alpha_i \right).
$$

73
The time series $\{\alpha_i\}_{i=1}^n$ is a collection of piecewise constant vectors such that
\[
\alpha_i = \alpha_{i'} \quad \text{for all} \quad \eta_k + 1 \leq i \leq i' \leq \eta_{k+1}
\]
and that the jump sizes at the change points satisfy
\[
\|\alpha_{\eta_{k+1}} - \alpha_{\eta_k}\|_2^2 = (\beta_{\eta_{k+1}}^* - \beta_{\eta_k}^*)^\top \Sigma (\beta_{\eta_{k+1}}^* - \beta_{\eta_k}^*) \geq c_x \|\beta_{\eta_{k+1}}^* - \beta_{\eta_k}^*\|_2^2 = c_x \kappa_k^2,
\]
where $\kappa_k$ is defined in Assumption 1. By Lemma S.12 of Wang et al. (2021a), for any $k \in \{1, \ldots, K\}$,
\[
\max_{\zeta_n \leq t \leq (1-\zeta)n} \|\tilde{\alpha}(t)\|_2^2 \geq \frac{\Delta^2}{48n} c_x \kappa_k^2 \geq c_1 \kappa_k^2 n,
\]
where
\[
\Delta := \min_{1 \leq k \leq K} (\eta_{k+1} - \eta_k) \geq c_2 n
\]
is used in the last inequality. Therefore
\[
\max_{\zeta_n \leq t \leq (1-\zeta)n} \|\tilde{\alpha}(t)\|_2^2 \geq c_1 \kappa_k^2 n.
\]
The desired result follows by observing that
\[
\|\tilde{\alpha}(t)\|_2^2 = \frac{t(n-t)}{n} (\beta_{(0,t]}^* - \beta_{(t,n]}^*)^\top \Sigma (\beta_{(0,t]}^* - \beta_{(t,n]}^*)
\]
and so
\[
\max_{\zeta_n \leq t \leq (1-\zeta)n} (\beta_{(0,t]}^* - \beta_{(t,n]}^*)^\top \Sigma (\beta_{(0,t]}^* - \beta_{(t,n]}^*) = \max_{\zeta_n \leq t \leq (1-\zeta)n} \frac{n}{t(n-t)} \|\tilde{\alpha}(t)\|_2^2 
\geq \max_{\zeta_n \leq t \leq (1-\zeta)n} \frac{n}{t(n-t)} c_1 \kappa_k^2 n \geq c_1 \kappa^2.
\]
For Equation (60), note that by Proposition S.1 of Wang et al. (2021a), $\|\tilde{\alpha}(t)\|_2^2$ as a function of $t$ is maximized at the change points. Say $\eta_q$ is the maximizer. Then
\[
\|\tilde{\alpha}(\eta_q)\|_2^2 = \max_{\zeta_n \leq t \leq (1-\zeta)n} \|\tilde{\alpha}(t)\|_2^2 \geq c_1 \kappa_k^2 n.
\]
Therefore
\[
(\beta_{(0,\eta_q]}^* - \beta_{(\eta_q,n]}^*)^\top \Sigma (\beta_{(0,\eta_q]}^* - \beta_{(\eta_q,n]}^*) = \frac{n}{\eta_q(n-\eta_q)} \|\tilde{\alpha}(\eta_q)\|_2^2 \geq \frac{n}{\eta_q(n-\eta_q)} c_1 \kappa^2 n \geq c' \kappa^2,
\]
where the last inequality follows that by Assumption 1, $\eta_q \asymp n$ and $n - \eta_q \asymp n$.

E Additional Numerical Results
Figure 5: Power performance of QF-CUSUM under the single change-point case (left column) and multiple change-point case (right column) with AR temporal dependence.
Figure 6: Power performance of QF-CUSUM under the single change-point case (left column) and multiple change-point case (right column) with MA temporal dependence.