Note About Weyl Invariant Hořava-Lifshitz Gravity

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Abstract: We construct Hořava-Lifshitz gravities that are invariant under anisotropic Weyl scaling. This construction is based on an extension of the group of symmetries of healthy extended Hořava-Lifshitz gravity and RFDiff invariant Hořava-Lifshitz gravity. We find their Hamiltonian formulation and determine their constraint structure.

Keywords: Hořava-Lifshitz gravity
1. Introduction and Summary

In 2009 Petr Hořava formulated new proposal of quantum theory of gravity (now known as Hořava-Lifshitz gravity (HL gravity) that is power counting renormalizable [1, 2, 3] that is also expected that it reduces do General Relativity in the infrared (IR) limit. The HL gravity is based on an idea that the Lorentz symmetry is restored in IR limit of given theory while it is absent in its high energy regime. For that reason Hořava considered systems whose scaling at short distances exhibits a strong anisotropy between space and time,

\[ x' = lx, \quad t' = l^2 t. \]  

(1.1)

In \((D + 1)\) dimensional space-time in order to have power counting renormalizable theory requires that \(z \geq D\). It turns out however that the symmetry group of given theory is reduced from the full diffeomorphism invariance of General Relativity to the foliation preserving diffeomorphism

\[ x'^i = x^i + \zeta^i(t, x), \quad t' = t + f(t). \]  

(1.2)

Due to the fact that the diffeomorphism is restricted (1.2) one more degree of freedom appears that is a spin–0 graviton. It turns out that the existence of this mode could be dangerous since it has to decouple in the IR regime, in order to be consistent with observations, for detailed discussion, see [1] and [39]. More precisely, for further discussion it is necessary to stress that there exists two main versions of the HL gravity: projectable, where the lapse function \(N\) depends on \(N(t)\) only. This presumption has a fundamental consequence for the formulation of the theory since there is no local form of the Hamiltonian constraint but only the global one. The fact that this is the theory where the local Hamiltonian constraint is absent implies an existence of an additional scalar mode.

\[ ^1\text{For review and extensive list of references, see [1, 2, 3, 4].} \]
The second version of HL gravity is the version where the projectability condition is not imposed so that \( N = N(x,t) \). This form of HL gravity was extensively studied in [13, 14, 15, 16, 19, 20, 21, 22, 23, 24, 37]. It was shown in [16] that this version of HL gravity could really be an interesting candidate for the quantum theory of gravity without ghosts and without strong coupling problem despite its unusual Hamiltonian structure [19, 20].

Recently Hořava and Malby-Thompson in [26] proposed very interesting way how to eliminate the spin-0 graviton in the context of the projectable version of HL gravity. Their construction is based on an extension of the foliation preserving diffeomorphism in such a way that the theory is invariant under additional local \( U(1) \) symmetry. The resulting theory is known as non-relativistic covariant HL gravity [3]. It was shown in [26, 27] that the presence of this new symmetry implies that the spin-0 graviton becomes non-propagating and the spectrum of the linear fluctuations around the background solution coincides with the fluctuation spectrum of General Relativity.

We would like to stress that it is possible to formulate the modification of HL gravity that contains the correct number of physical degrees of freedom without introducing additional fields [34]. This model is based on the formulation of the HL gravity with reduced symmetry group known as restricted-foliation-preserving Diff \( (RFDiff) \) HL gravity [16, 25]. This is the theory that is invariant under following symmetries

\[
t' = t + \delta t , \quad \delta t = \text{const} , \quad x'^i = x^i + \zeta^i(x,t) .
\]

The characteristic property of given theory is the absence of the Hamiltonian constraint [25] either global or local. The construction presented in [34] was based on an extension of RFDiff HL action by an additional term that is function of scalar curvature and it is multiplied by Lagrange multiplier. It turned out that the number of physical degrees of freedom coincides with the physical number of degrees freedom of General Relativity while the theory possesses all pleasant properties of HL gravity.

In this paper we present yet another version of HL gravity that is now invariant under anisotropic Weyl scaling [1, 2] for general form of the potential term and for general \( \lambda \). Since lapse transforms non-trivially under Weyl scaling it seems to be natural to consider HL gravity without the projectability condition imposed. For that reason it makes sense to discuss invariance of the action under Weyl scaling in case of the healthy extended HL gravity or in the theory where the lapse function is absent as for example RFDiff HL gravity.

It is well known that the HL gravity that obeys the detailed balance condition is invariant under Weyl rescaling for special case \( \lambda = 1/3 \) for HL gravity in \( D = 3 \) dimensions. However it was quickly realized that HL gravity with condition of detailed balance should be generalized to the more general form of the potential given as the linear combination of spatial metric, Ricci tensors and its covariant derivatives. As a result the anisotropic

\[ \text{For another proposal of renormalizable theory of gravity, see [1, 2].} \]

\[ \text{This theory was also studied in [28, 29, 30, 31, 32].} \]

\[ \text{For another analysis of HL gravities with anisotropic Weyl invariance, see [38].} \]
Weyl invariance is lost even in the case of $\lambda = 1/3$. Then it turns out that the only way how to define Weyl invariant HL gravity for general $\lambda$ is to build it from the connection that is manifestly invariant under the Weyl transformation. To do this we introduce additional scalar field that transforms under Weyl scaling in such a way that it compensates the transformation of the connection. Clearly this new scalar field has the form of the Stückelberg field \[35\]. In fact, Weyl scaling symmetry can be fixed by setting this scalar field to be constant leading to the recovery of the original theory.

As we argued above we can introduce HL gravity invariant under Weyl scaling for two versions of HL gravity: The healthy extended HL gravity with spatial dependent lapse function and in case of RFDiff invariant HL gravity with lapse function absent. In the first case the requirement of the invariance of the action under Weyl scaling only introduces an additional degree of freedom to the theory. We expect that this new degree of freedom does not modify physical content of the theory due to the fact that it is accompanied by additional gauge symmetry. On the other hand the more interesting case occurs in case of RFDiff invariant HL gravity. We show that the resulting theory can be modified in the similar way as in case of the non-relativistic covariant HL gravity \[26\]. Explicitly, we introduce an additional term to the action that breaks the manifest Weyl symmetry of the theory. After this modification the action becomes invariant under Weyl scaling on condition that the parameter of transformation is covariantly constant. Then in order to restore the invariance for any value of this parameter we introduce the gauge field that transforms in an appropriate way under Weyl scaling, following similar procedure performed in \[26\]. Further, the Hamiltonian analysis of given theory shows that it has the same number of physical degrees of freedom as in case of General Relativity. On the other hand we should stress that the elimination of the scalar mode that is present in the original version of HL gravity is due to the fact that the newly introduced gauge field acts as Lagrange multiplier whose existence leads to the additional constraint in the theory. However we mean that it is sometimes useful to extend theory by additional symmetry. In particular, we can hope that the presence of an additional symmetry could help to find the relation between IR limit of HL gravity and General Relativity even if this important problem will not be studied in this paper. In particular when we formulate the RFDiff HL gravity invariant under Weyl scaling we have to introduce additional scalar field $N$ that can be interpreted as the lapse. Note also that the lapse $N$ and spatial metric components $g_{ij}$ transform non-trivially under Weyl scaling which is different from the case of the $\alpha-$ symmetry in covariant non-relativistic HL gravity. We also mean that Weyl invariant RFDiff HL gravity could be also considered as a new example of the power counting renormalizable theory of gravity with correct number of degrees of freedom.

On the other hand problems and open questions that are well known from the formulation of the non-relativistic covariant HL gravity hold in our case as well. For example, it is not completely clear how the IR limit of this theory is related to the General Relativity. In order to properly address this issue we should carefully analyze how the Weyl invariant RFDiff HL gravity couples to matter. In particular, it would be interesting to formulate the action for the probe in the context of Weyl invariant RFDiff HL gravity and study its dynamics. We should also stress one important point that is common for non-relativistic
covariant HL gravity, Weyl invariant RFDiff HL gravity and HL gravity with Lagrange multiplier which is the presence of the second class constraints with complicated structure so that we are not able to solve them in the full generality. Generally it is well known that is very difficult task to analyze theories with the second class constraints and then perform its quantum mechanical generalization. One way how to proceed is to implement the abelian conversion of the second class constraints so that the resulting theory can be formulated as the theory with the first class constraints. Then we can certainly apply the powerful BRST quantization of such a theory at least in principle. On the other hand the fact that the Poisson brackets between the second class constraints depend on the phase space variables implies that the resulting Hamiltonian will contain infinite number of terms so that it seems that given procedure is meaningless in the case of HL gravity.

Let us outline the content of given paper. In next section (2) we formulate healthy extended HL gravity that is invariant under anisotropic Weyl scaling. In section (3) we perform its Hamiltonian analysis. In section (4) we introduce RFDiff HL gravity invariant under Weyl scaling and we perform its Hamiltonian analysis in section (5).

2. Weyl Invariant Healthy Extended HL Gravity

In this section we extend the healthy extended HL gravity so that the resulting theory is invariant under anisotropic Weyl scaling for general value of parameter $\lambda$ and for general form of the potential term. As usual we begin our presentation with the introduction of the basis notation. Let us consider $D + 1$ dimensional manifold $\mathcal{M}$ with the coordinates $x^\mu, \mu = 0, \ldots, D$ and where $x^\mu = (t, \mathbf{x})$, $\mathbf{x} = (x^1, \ldots, x^D)$. We presume that this space-time is endowed with the metric $\hat{g}_{\mu\nu}(x^\rho)$ with signature $(-, +, \ldots, +)$. Suppose that $\mathcal{M}$ can be foliated by a family of space-like surfaces $\Sigma_t$ defined by $t = x^0$. Let $g_{ij}, i, j = 1, \ldots, D$ denotes the metric on $\Sigma_t$ with inverse $g^{ij}$ so that $g_{ij}g^{jk} = \delta^k_i$. We further introduce the operator $\nabla_i$ that is covariant derivative defined with the metric $g_{ij}$. We introduce the future-pointing unit normal vector $n^\mu$ to the surface $\Sigma_t$. In ADM variables we have $n^0 = \sqrt{-\hat{g}^{00}}, n^i = -\hat{g}^{0i}/\sqrt{-\hat{g}^{00}}$. We also define the lapse function $N = 1/\sqrt{-\hat{g}^{00}}$ and the shift function $N^i = -\hat{g}^{0i}/\hat{g}^{00}$. In terms of these variables we write the components of the metric $\hat{g}_{\mu\nu}$ as

\[
\begin{align*}
\hat{g}_{00} &= -N^2 + N_i g^{ij} N_j, \quad \hat{g}_{0i} = N_i, \quad \hat{g}_{ij} = g_{ij}, \\
\hat{g}^{00} &= -\frac{1}{N^2}, \quad \hat{g}^{0i} = \frac{N^i}{N^2}, \quad \hat{g}^{ij} = g^{ij} - \frac{N^i N^j}{N^2}.
\end{align*}
\]  

We further define the extrinsic curvature

\[
K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i)
\]  

and the generalized de Witt metric $G^{ijkl}$ in the form

\[
G^{ijkl} = \frac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl},
\]
where $\lambda$ is real constant. In this notation the HL action has the form
\begin{equation}
S = \frac{1}{\kappa^2} \int dt d^D x \sqrt{g} N [K_{ij} G^{ijkl} K_{kl} - \mathcal{V}(g)] ,
\end{equation}
where $\mathcal{V}(g)$ is a general function of $g_{ij}$ and its covariant derivative. The action (2.4) is invariant under foliation preserving diffeomorphism
\begin{equation}
t' - t = f(t) , \quad x'^i - x^i = \xi^i(t, x) .
\end{equation}
Our goal is to formulate HL gravity that is invariant under anisotropic Weyl transformation defined as
\begin{equation}
N' = \Omega^z N , \quad N'_i = \Omega^2 N_i , \quad g_{ij} = \Omega^2 g_{ij} .
\end{equation}
We construct this modification of HL gravity in the following way. It is easy to show that the component of the spatial connection
\begin{equation}
\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})
\end{equation}
transforms under (2.6) as
\begin{equation}
\Gamma'^k_{ij} = \Gamma^k_{ij} + \frac{1}{\Omega} (\delta^k_i \partial_j \Omega + \delta^k_j \partial_i \Omega - g^{kl} \partial_l \Omega g_{ij}) .
\end{equation}
Then we find that the extrinsic curvature $K_{ij}$ transforms under (2.4) as
\begin{equation}
K'_{ij} = \Omega^{2-z} K_{ij} + \Omega^{1-z} \nabla_n \Omega g_{ij} ,
\end{equation}
where we defined
\begin{equation}
\nabla_n X = \frac{1}{N} (\partial_t X - N^i \partial_i X) .
\end{equation}
It is important to stress that the potential term in HL gravity contains terms that are constructed from covariant derivatives and Ricci tensors of higher order. Then in order to formulate HL gravity that is invariant under (2.6) for general form of the potential it is natural to introduce the connection is manifestly invariant under (2.6). For that reason we introduce scalar field $\varphi(t, x)$ that transforms under anisotropic Weyl scaling as
\begin{equation}
\varphi'(t, x) = \Omega(t, x) \varphi(t, x)
\end{equation}
and define the connection with bar
\begin{equation}
\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} - \frac{1}{\varphi} (\delta^k_i \partial_j \varphi + \delta^k_j \partial_i \varphi - g^{kl} \partial_l \varphi g_{ij}) .
\end{equation}
Then with the help of (2.8) and (2.11) we easily find that $\bar{\Gamma}^k_{ij}$ is invariant under (2.6). As a result $D-$dimensional Riemann tensor constructed from connection with bar is invariant under (2.6) which also implies that corresponding Ricci tensor and scalar curvature transform as
\begin{equation}
\bar{R}'_{ij} = \bar{R}_{ij} , \quad \bar{R}' = \Omega^{-2} \bar{R} .
\end{equation}
Let us now discuss following specific form of the potential $V$ \[10\]

\[ V(g) = \zeta^2 g_0 + g_1 R + \frac{1}{\zeta^2} (g_2 R^2 + g_3 R_{ij} R^{ij}) + \]

\[ + \frac{1}{\zeta^2} (g_4 R^3 + g_5 R_{ij} R^{ij} + g_6 R^k_j R^l_i) + \]

\[ + \frac{1}{\zeta^2} [g_7 R \nabla^2 R + g_8 (\nabla_i R_{jk})(\nabla^i R^{jk})], \]

(2.14)

where the coupling constants $g_s$, $(s = 0, 1, 2, \ldots, 8)$ are dimensionless. The relativistic limit in the IR requires $g_1 = -1$ and $\zeta^2 = (16\pi G)^{-2}$. Then in order to find the potential term that is invariant under (2.6) we replace $R_{ij}$ with $\bar{R}_{ij}$ and multiply all expressions with appropriate powers of $\phi$ so that

\[ \bar{V}(\phi, g) = \zeta^2 g_0 + g_1 \phi^2 \bar{R} + \frac{1}{\zeta^2} (g_2 \phi^2 \bar{R}^2 + g_3 \bar{R}_{ij} \bar{R}^{ij}) + \]

\[ + \frac{1}{\zeta^2} \phi^4 (g_4 \phi^2 \bar{R}^3 + g_5 \phi^2 \bar{R}_{ij} \bar{R}^{ij} + g_6 \phi^2 \bar{R}^k_j \bar{R}^l_i) + \]

\[ + \frac{1}{\zeta^2} [g_7 \phi^2 \bar{R} \nabla^2 \phi + g_8 (\nabla_i \bar{R}_{jk})(\nabla^i \bar{R}^{jk})]. \]

(2.15)

Let us now construct the kinetic term that is invariant under Weyl scaling. To do this we introduce extrinsic curvature with bar defined as

\[ \bar{K}_{ij} = K_{ij} - \frac{1}{\phi} \nabla_n \phi g_{ij} \]

(2.16)

that transforms under (2.6) as

\[ \bar{K}_{ij}' = \Omega^{2-z} \bar{K}_{ij} \]

(2.17)

using (2.3) and the fact that

\[ \nabla_n \phi' = \Omega^{1-z} \nabla_n \phi + \phi \Omega^{-z} \nabla_n \Omega. \]

(2.18)

Then with the help of the fact that under Weyl scaling the generalized de Witt metric transforms as

\[ G'_{ijkl} = \Omega^4 G_{ijkl}, \quad G'^{ijkl} = \frac{1}{\Omega^4} G^{ijkl} \]

(2.19)

we find

\[ \bar{K}_{ij}' G'^{ijkl} \bar{K}_{kl}' = \Omega^{-2z} \bar{K}_{ij} G^{ijkl} \bar{K}_{kl}. \]

(2.20)

Collecting these results we propose following HL action that is invariant under anisotropic Weyl transformation

\[ S = \frac{1}{\kappa^2} \int dt d^D x \sqrt{\bar{g}} N \phi^{-(z+D)} [\phi^{2z} \bar{K}_{ij} G^{ijkl} \bar{K}_{kl} - \bar{V}(g, \phi)]. \]

(2.21)
Let us now discuss the spatial dependence of the lapse function $N$. Since the theory is invariant under anisotropic Weyl transformation with parameter that depends on space and time coordinates it is natural to presume that $N$ depends on $x$ and $t$ as well. Then the only way how to find consistent HL theory where the lapse is spatial dependent is to consider its healthy extended form that contains $D$–dimensional vector $a_i$ defined as

$$a_i = \frac{\partial_i N}{N}.$$  

(2.22)

The healthy extended HL gravity contains terms that depend on $a_i$ and that are invariant under spatial diffeomorphism. We denote this contribution by introducing an additional potential $\mathcal{V}_a(a, g, R)$. Note also that $a_i$ transforms under (2.6) as

$$a'_i(x, t) = a_i(x, t) + \frac{z}{\Omega} \partial_i \Omega(x, t).$$  

(2.23)

We see that in order to find the healthy extended HL gravity that is invariant under (2.6) we have to replace $a_i$ with $a_i \rightarrow \bar{a}_i \equiv a_i - \frac{z}{\Omega} \partial_i \Omega$ and include appropriate factors of $\varphi$ in corresponding expressions. In summary we propose following healthy extended HL gravity action that is invariant under anisotropic Weyl transformation

$$S = \frac{1}{\kappa^2} \int dtd^Dx \sqrt{g} N \varphi^{-(z+D)} \left[ \varphi^{2z} \bar{K}_{ij} \bar{G}^{ijkl} - \hat{V}(g, \varphi) - \hat{V}_a(\bar{a}, g, \bar{R}, \varphi) \right].$$  

(2.24)

3. Hamiltonian Analysis of Weyl Invariant Healthy Extended HL Gravity

In this section we perform the Hamiltonian analysis of the action (2.24). As a result of this analysis we will be able to determine the number of physical degrees of freedom.

As the first step we determine the momenta conjugate to $N, N^i, g_{ij}$ and $\varphi$ from (2.24)

$$p_N(x) = \frac{\delta S}{\delta \partial_t N(x)} \approx 0, \quad p^i(x) = \frac{\delta S}{\delta \partial_t N^i(x)} \approx 0,$$

$$\pi^{ij}(x) = \frac{\delta S}{\delta \partial_t g_{ij}(x)} = \frac{\delta S}{\delta \partial_t \varphi(x)} = \frac{\delta S}{\delta \partial_t \varphi(x)} = \frac{1}{\kappa^2} \sqrt{g} \varphi^{-(z+D)} \varphi^{2z} \bar{G}^{ijkl} \bar{K}_{kl},$$

$$p_{\varphi}(x) = \frac{2}{\kappa^2} \sqrt{g} \varphi^{-(z+D)} \varphi^{2z-1} g_{ij} \bar{G}^{ijkl} \bar{K}_{kl}.$$

(3.1)

As usual $p_N, p^i$ are primary constraints of the theory. Further, using relations given above we find an additional primary constraint

$$\mathcal{D} \equiv 2g_{ij} \pi^{ji} + p_{\varphi} \varphi \approx 0$$

(3.2)

and the bare Hamiltonian

$$H = \int d^Dx \left( N \mathcal{H}_T + N^i \mathcal{H}^i \right),$$

$$\mathcal{H}^i = p_{\varphi} g^{ij} \nabla_j \varphi - 2 \nabla_j \pi^{ji},$$

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\[ \mathcal{H}_T = \frac{\kappa^2}{\sqrt{g}} \varphi^{(z+D)} \frac{1}{\varphi^2} \varphi^{ij} G_{ijkl} \pi^{kl} + \frac{1}{\kappa^2} \sqrt{g} \varphi^{-(z+D)} (\tilde{V}(g, \varphi) + \bar{V}_a(\bar{a}, g, \bar{R}, \varphi)) \equiv \mathcal{H}_T + \frac{1}{\kappa^2} \sqrt{g} \varphi^{-(z+D)} \bar{V}_a(\bar{a}, g, \bar{R}, \varphi) \],

(3.3)

where \( \mathcal{H}_T \) coincides with the Hamiltonian constraint of the HL gravity without its healthy extension. Following the standard formalism of constraint system we introduce the extended Hamiltonian in the form

\[ H_T = \int d^D x \left( N (\mathcal{H}_T + \frac{1}{\kappa^2} \sqrt{g} \varphi^{-(z+D)} \bar{V}_a) + N_i \mathcal{H}_i + \lambda_N p_N + \lambda p^i + \lambda^D D \right) . 

(3.4)

Now we proceed to the analysis of the stability of the primary constraints. As usual the preservation of the primary constraints \( p_i(x) \approx 0 \) imply the secondary constraints

\[ \mathcal{H}_i(x) \approx 0 . 

(3.5)\]

It is convenient to introduce the following slightly modified smeared form of this constraint

\[ T_S(\xi) = \int d^D x (\xi^i(x) \mathcal{H}_i(x) + \xi^i(x) \partial_i N(x) p_N(x)) . 

(3.6)\]

Note that the additional term in \( T_S \) is proportional to the primary constraint \( p_N(x) \approx 0 \). Now we analyze the requirement of the preservation of the primary constraint \( \Theta_1(x) \equiv p_N(x) \approx 0 \) during the time evolution of the system. Explicitly, the time evolution of this constraint is governed by following equation

\[ \partial_t \Theta_1(x) = \{ \Theta_1(x), H_T \} = -\mathcal{H}_T(x) - \frac{1}{\kappa^2} \sqrt{g} \varphi^{-(z+D)} \bar{V}_a + \frac{1}{N} \partial_i \left( N \sqrt{g} \varphi^{-(z+D)} \frac{\delta \bar{V}_a}{\delta \bar{a}_i} \right)(x) \equiv -\Theta_2(x) \approx 0 , 

(3.7)\]

where \( \Theta_2 \) is the secondary constraint. Now we analyze the condition of the stability of the primary constraint \( D \). It turns out to be convenient to introduce the smeared form of the constraint \( D \) defined as

\[ D(\omega) = \int d^D x \omega(x)(D(x) + z N(x) p_N(x)) , 

(3.8)\]

that has following Poisson bracket

\[ \{ D(\omega), g_{ij} \} = -2 \omega g_{ij} , \quad \{ D(\omega), g^{ij} \} = 2 \omega g^{ij} , \]
\[ \{ D(\omega), p^{ij} \} = 2 \omega \pi^{ij} , \quad \{ D(\omega), \varphi \} = -\omega \varphi , \]
\[ \{ D(\omega), p_\varphi \} = \omega p_\varphi , \quad \{ D(\omega), N \} = -z \omega N , \]
\[ \{ D(\omega), a_i \} = -z \partial_i \omega , \quad \{ D(\omega), \bar{a}_i \} = 0 . 

(3.9)\]
Then we find

\[
\{ \mathbf{D}(\omega), \bar{\Gamma}_{ij} \} = 0
\]

(3.10)

and consequently

\[
\{ \mathbf{D}(\omega), \bar{R}_{ij}^{k} \} = 0, \quad \{ \mathbf{D}(\omega), \bar{R}_{ij} \} = 0, \quad \{ \mathbf{D}(\omega), \bar{R} \} = -2\omega \bar{R}.
\]

(3.11)

Using these results it is easy to see that

\[
\{ \mathbf{D}(\omega), \bar{V} \} = 0, \quad \{ \mathbf{D}(\omega), \bar{V}_a \} = 0, \quad \{ \mathbf{D}(\omega), \mathcal{H}_T \} = z\omega \mathcal{H}_T.
\]

(3.12)

In order to analyze the Poisson bracket including \( T_S(\xi) \) we need following Poisson brackets

\[
\{ T_S(\xi), g_{ij} \} = -\xi^k \partial_k g_{ij} - \partial_i \xi^k g_{kj} - g_{ik} \partial_j \xi^k,
\]

\[
\{ T_S(\xi), p_{ij} \} = -\partial_k p_{ij} \xi^k - p_{ij} \partial_k \xi^k + \partial_k \xi^i p_{kj} + p_{ik} \partial^i \xi^j,
\]

\[
\{ T_S(\xi), a_i \} = -\xi^j \partial_j a_i - \partial_i \xi^j a_j.
\]

(3.13)

Then we easily find

\[
\{ T_S(\xi), \mathcal{H}_T \} = -\xi^k \partial_k \mathcal{H}_T - \mathcal{H}_T \partial_k \xi^k,
\]

\[
\{ T_S(\xi), \nu_a(g) \} = -\partial_i \nu^i.
\]

\[
\{ T_S(\xi), D \} = -\partial_k D \xi^k - D \partial_k \xi^k.
\]

(3.14)

The last Poisson bracket has following smeared form

\[
\{ \mathbf{D}(\omega), T_S(\xi) \} = \mathbf{D}(-\partial_k \omega \xi^k).
\]

(3.15)

This result together with the last equation in (3.12) implies that the constraint \( \mathbf{D} \) is preserved during the time evolution of the system.

Including the secondary constraint \( \Theta_2 \) into the definition of the Hamiltonian we find that the total Hamiltonian takes the form

\[
H = \int d^Dx \left( N(\bar{\mathcal{H}}_T + \frac{1}{\kappa^2} \sqrt{g} \varphi^{-(z+D)}\bar{\nu}_a) + \lambda_i \nu^i + v^a \Theta_\alpha \right) + T_S(N^i) + \mathbf{D}(\lambda^D),
\]

(3.16)

where \( v^a \) are Lagrange multipliers related to the constraints \( \Theta_\alpha \).

As the next step we have to check the stability of the secondary constraints \( \Theta_2(x) \approx 0 \) , \( T_S(x) \approx 0 \). Using (3.13) and (3.14) we find

\[
\{ T_S(\xi), \Theta_\alpha(x) \} = -\partial_k \Theta_\alpha(x) \xi^k(x) - \Theta_\alpha(x) \partial_k \xi^k(x).
\]

(3.17)
With the help of this result and also (3.15) we find that the constraint \( T_S(\xi) \approx 0 \) is preserved during the time evolution of the system. Now we proceed to the analysis of the stability of the constraints \( \Theta_{1,2} \). First of all it is easy to see that

\[
\{ D(\omega), \Theta_1 \} = \omega \Theta_1 , \quad \{ D(\omega), \Theta_2 \} = z \omega \Theta_2 .
\]  
(3.18)

Then with the help of (3.17) we find that the Poisson brackets of \( \Theta' \)'s with \( D \) and with \( T_S \) vanish on the constraints surface. To proceed further we calculate following Poisson bracket

\[
\{ \Theta_1(x), \Theta_2(y) \} \equiv \triangle_{12}(x,y) = -\frac{1}{N} \partial_y \left( \sqrt{g} \frac{\delta^2 V}{\delta a_i(y) \delta a_j(y)} (\tilde{a}_j(y) \delta(x-y) - \partial_y \delta(x-y)) \right) .
\]  
(3.19)

Collecting these results we find that the time evolution of the constraint \( \Theta_1(x) \) is equal to

\[
\partial_t \Theta_1(x) = \{ \Theta_1(x), H_T \} \approx \int d^Dy v_2 \triangle_{12}(x,y) .
\]  
(3.20)

Clearly \( \partial_t \Theta_1 \approx 0 \) for \( v_2 = 0 \). In the same way we determine the time evolution of the constraint \( \Theta_2(x) \approx 0 \)

\[
\partial_t \Theta_2(x) = \{ \Theta_2(x), H_T \} \approx \int d^Dy \left( N \left\{ \Theta_2(x), \tilde{H}_T(y) + \frac{1}{k^2} \varphi^{-(z+D)} \sqrt{g} \tilde{V}_a(y) \right\} - v_1 \triangle_{12}(y,x) \right) = 0
\]  
(3.21)

using \( v_2 = 0 \). From (3.21) we see that the requirement that \( \partial_t \Theta_2(x) = 0 \) fixes the value of the Lagrange multiplier \( v_1 \). Equivalently, \( \Theta' \)'s are the second class constraints that according to standard analysis have to vanish strongly. As the result of this analysis we find following extended Hamiltonian

\[
H_T = H + T_S(N^i) + D(\lambda^D) + \int d^Dy \lambda_i p^i ,
\]  
(3.22)

where

\[
H = \int d^Dx N \left( \tilde{H}_T + \frac{1}{k^2} \varphi^{-(z+D)} \sqrt{g} \tilde{V}_a \right) .
\]  
(3.23)

As we argued above the constraints \( \Theta' \)'s are the second class constraints that should be solved for the canonical pair \( p_N, N \), at least in principle. At the same time it is necessary to replace the Poisson brackets between phase space variables \( (g_{ij}, \pi^{ij}) \) with the Dirac brackets

\[
\{ F(g, p), G(g, p) \}_D = \{ F(g, p), G(g, p) \} - \int d^Dx d^Dy \{ F(q, p), \Theta_\alpha(x) \} \triangle^{\alpha \beta}(x,y) \{ \Theta_\beta(y), G(p, q) \} ,
\]  
(3.24)
where $\Delta^{\alpha\beta}(x,y)$ is inverse of $\Delta_{\alpha\beta}(x,y)$ in a sense

$$
\int d^Dz \Delta_{\alpha\beta}(x,z) \Delta^{\beta\gamma}(z,y) = \delta^\gamma_\alpha \delta(x - y) \ .
$$

(3.25)

However due to the fact that the Poisson brackets between $g_{ij}, p^{ij}$ and $\Theta_1$ vanish we find that the Dirac brackets between canonical variables $g_{ij}, p^{ij}$ coincide with the Poisson brackets. Even if we are not able to solve the constraint $\Theta_2$ explicitly we can still determine the number of physical degrees of freedom knowing the constraint structure of the theory [36]. We have $N_{p.s.d.f.} = D(D + 1) + 2D + 4$ phase space variables $g_{ij}, \pi^{ij}, N, p_N, N_i, p^i, \varphi, p_\varphi$. On the other hand we have $N_{f.c.c.} = 2D + 1$ first class constraints $p^i \approx 0$, $\mathcal{H}_i \approx 0$, $D \approx 0$ and $N_{s.c.c.} = 2$ the second class constraints $\Theta_{1,2} \approx 0$. Then the number of physical degrees of freedom is

$$
N_{p.d.f.} = N_{p.s.d.f.} - 2N_{f.c.c.} - N_{s.c.c.} = (D^2 - D - 2) + 2 \ ,
$$

(3.26)

where the expression in parenthesis corresponds to the number of physical degrees of freedom of general relativity and where the factor 2 corresponds to the additional scalar mode whose presence is the general property of the healthy extended HL gravity. On the other hand the non-relativistic covariant HL gravity proposal completely eliminates this mode. In the next section we formulate the Weyl invariant RFDiff HL gravity where this scalar mode is also eliminated.

### 4. Weyl Invariant RFDiff HL Gravity

The RFDiff invariant HL gravity was introduced in [16] and further studied in [25, 32, 34]. This is the version of the HL gravity that is invariant under restricted foliation preserving diffeomorphism

$$
t' = t + \delta t \ , \quad \delta t = \text{const} \ , \quad x'^i = x^i + \zeta^i(t, x) \ .
$$

(4.1)

The simplest form of RFDiff invariant Hořava-Lifshitz gravity takes the form [25]

$$
S = \frac{1}{\kappa^2} \int dtd^Dx \sqrt{\tilde{g}}(\hat{K}_{ij}\tilde{G}^{ijkl}\hat{K}_{kl} - \mathcal{V}(g)) \ ,
$$

(4.2)

where we introduced modified extrinsic curvature

$$
\hat{K}_{ij} = \frac{1}{2}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i)
$$

(4.3)

that differs from the standard extrinsic curvature by absence of the lapse $N$. In fact, the absence of the lapse is the general property of RFDiff invariant HL gravity even if it is possible to consider more general case [16] where however $N$ transforms as scalar under (4.1). Now in order to formulate RFDiff HL gravity that is invariant under anisotropic Weyl transformation we proceed in the slightly different way than in section (2). Explicitly, we replace $g_{ij}$ and $N_i$ with corresponding variables with bar defined as

$$
\bar{g}_{ij} = \frac{1}{N^{2/z}} g_{ij} \ , \quad \bar{N}_i = \frac{1}{N^{2/z}} N_i \ .
$$

(4.4)
where $N$ transforms as scalar under (4.1). Clearly $\bar{g}$ and $\bar{N}_i$ are invariant under transformations

$$N'(x, t) = \Omega^2(x, t)N(x, t), \quad g'_{ij}(x, t) = \Omega^2(x, t)g_{ij}(x, t), \quad N'_i(x, t) = \Omega^2 N_i(x, t).$$

(4.5)

As a result any theory constructed from $\bar{g}$ and $\bar{N}_i$ is invariant under the transformation (4.5). On the other hand it is useful to find relation between quantities with and without bar. Explicitly

$$\bar{\Gamma}^k_{ij} = \frac{1}{2} \bar{g}^{kl}(\partial_i \bar{g}_{lj} + \partial_j \bar{g}_{li} - \partial_l \bar{g}_{ij}) = \Gamma^k_{ij} = \frac{1}{2} \left( a_i \delta_j^k + a_j \delta_i^k - g^{kl} a_l g_{ij} \right), \quad a_i = \frac{\partial_i N}{N}.$$

(4.6)

and

$$\bar{K}_{ij} = \frac{1}{2} \left( \partial_t \bar{g}_{ij} - \nabla_i \bar{N}_j - \nabla_j \bar{N}_i \right) = \frac{1}{N^2/z} \left( \bar{K}_{ij} - \frac{1}{z} \nabla_n N g_{ij} \right),$$

(4.7)

where again $\nabla_n X = \frac{1}{N}(\partial_t X - N^i \partial_i X)$. Then it is easy to see that RFDiff HL gravity defined by the action

$$S = \frac{1}{\kappa^2} \int dt d^D x \sqrt{\bar{g}} \bar{G}^{ijkl} \bar{K}_{kl} - V(\bar{g}, \bar{R}) =$$

$$= \frac{1}{\kappa^2} \int dt d^D x \sqrt{\bar{g}} N^{-D/z} \left[ (\bar{K}_{ij} - \frac{1}{z} \nabla_n N g_{ij}) \bar{G}^{ijkl} (\bar{K}_{kl} - \frac{1}{z} \nabla_n N g_{kl}) - V(\bar{g}, \bar{R}) \right]$$

(4.8)

is invariant under (4.5). Apparently we could expect this result since $N$ cannot change the physical content of the theory as follows from the way we introduced it into the action. At this place we would like to stress the analogy with the similar situation in non-relativistic covariant HL gravity. We argued in [32, 33] that the field $\nu$ that is present in the covariant non-relativistic HL gravity should be interpreted as Stückelberg field. Note that we could give the same interpretation to the field $N$ introduced above.

The idea how to make the mode $N$ non-trivial is to add to the action an additional term that breaks the manifest Weyl symmetry of the action. To do this we proceed in the similar way as in case of non-relativistic covariant HL gravity and consider following term

$$S_A = \frac{1}{\kappa^2} \int dt d^D x \sqrt{\bar{g}} \bar{G}(\bar{R}) \nabla_n N,$$

(4.9)

where $\bar{G}$ is the scalar function that depends on $\bar{R}$ only. Note that under Weyl transformation $S_A$ transforms as

$$S'_A = \frac{1}{\kappa^2} \int d^D x \sqrt{\bar{g}} \bar{G}(\bar{R}) \left( \nabla_n N + \frac{z}{\Omega} (\partial_i \Omega - N^i \partial_i \Omega) \right).$$

(4.10)
In other words the theory is not invariant under general local Weyl transformation but only under transformation with covariantly constant parameter:

$$\partial_t \Omega - N^i \partial_i \Omega = 0 .$$  \hspace{1cm} (4.11)

We see that this condition coincides with the restriction of the parameter of the $\alpha-$transformation in case of non-relativistic HL gravity [26]. Then following this paper we make the theory invariant for general $\Omega$ when we introduce the gauge field $A$ into the action $S_A$ so that

$$S_A = \frac{1}{\kappa^2} \int d^D x \sqrt{\bar{g}} (\bar{G}(\bar{R}) - \nabla_n N - A)$$  \hspace{1cm} (4.12)

and demand that $A$ transforms under Weyl scaling as

$$A'(t, x) = A(t, x) + \frac{z}{\Omega(t, x)} (\partial_t \Omega(t, x) - N^i \partial_i \Omega(t, x)) .$$  \hspace{1cm} (4.13)

Then we see that (4.12) is invariant under Weyl transformation for general values of the parameter $\Omega$. The presence of the field $A$ has crucial impact on the physical content of given theory as will be seen from its Hamiltonian analysis.

5. Hamiltonian Formalism for Weyl Invariant RFDiff HL gravity

For reader’s convenience we again write Weyl invariant RFDiff HL gravity action

$$S = \frac{1}{\kappa^2} \int d^D x dt \sqrt{\bar{g}} N^{-D/z} \left( (\hat{K}_{ij} - \frac{1}{z} \nabla_n N g_{ij}) \bar{G}^{ijkl} (\hat{K}_{kl} - \frac{1}{z} \nabla_n N g_{kl}) - V(\bar{g}, \bar{R}) + \bar{G}(\bar{R})(\nabla_n N - A) \right) .$$  \hspace{1cm} (5.1)

As the first step we determine momenta from (5.1)

$$\pi^{ij} = \frac{\delta S}{\delta \partial_t g_{ij}} = \frac{1}{\kappa^2} \sqrt{\bar{g}} N^{-D/z} \bar{G}^{ijkl} (\hat{K}_{kl} - \frac{1}{z} \nabla_n N g_{kl}) , \hspace{1cm} p^i = \frac{\delta S}{\delta \partial_t N_i} \approx 0 ,$$

$$p_N = \frac{\delta S}{\delta \partial_t N} = \frac{2}{\kappa^2} \sqrt{\bar{g}} N^{-D/z} \frac{1}{z} N g_{ij} \bar{G}^{ijkl} (\hat{K}_{kl} - \frac{1}{z} \nabla_n N g_{kl}) +$$

$$+ \frac{1}{\kappa^2} \sqrt{\bar{g}} N^{-D/z-1} \bar{G}(\bar{R}) , \hspace{1cm} p_A \approx 0 .$$  \hspace{1cm} (5.2)

We see that this theory possesses $D + 2$ primary constraints:

$$p^i \approx 0 , \hspace{1cm} p_A \approx 0 ,$$

$$\mathcal{D} \equiv z p_N N + 2 \pi^{ij} g_{ji} - \frac{1}{\kappa^2} \sqrt{\bar{g}} N^{-D/z} \bar{G}(\bar{R}) \equiv \mathcal{D}_0 - \frac{1}{\kappa^2} \sqrt{\bar{g}} N^{-D/z} \bar{G}(\bar{R}) \approx 0 .$$  \hspace{1cm} (5.3)
Following standard procedure we find Hamiltonian
\[
H_T = \int d^Dx (\mathcal{H}_T + N_i \mathcal{H}^i + v_A p_A + v_i p^i + v^D D),
\]
\[
\mathcal{H}^i = p_N g^{ij} \nabla_j N - 2 \nabla_k \pi^{ki},
\]
\[
\mathcal{H}_T = \frac{\kappa^2}{\sqrt{g}} N^{D/2} \pi^{ij} G_{ijkl} \pi^{kl} + \frac{1}{\kappa^2} \sqrt{g} N^{D/2} \mathcal{V}(g, \phi) + \frac{1}{\kappa^2} \sqrt{g} N^{D/2} \mathcal{G}(\bar{R}) A,
\]
where \( v_A, v_i, v^D \) are Lagrange multipliers corresponding to the primary constraints (5.3).

Now we proceed to the analysis of the preservation of these constraints. In case of \( p_A, p^i \) we find
\[
\partial_t p_A = \{ p_A, H_T \} = -\frac{1}{\kappa^2} \sqrt{g} N^{D/2} \mathcal{G} \equiv \Phi_1 \approx 0,
\]
\[
\partial_t p^i = \{ p^i, H_T \} = -\mathcal{H}^i \approx 0,
\]
where \( \Phi_1 \approx 0 \) and \( \mathcal{H}^i \approx 0 \) are secondary constraints. More intricate is the analysis of the preservation of the constraint \( D \). We start with the observation that the Poisson bracket between \( D \) and \( T_S(\xi) \) is proportional to \( D \) and hence it vanishes on the constraint surface. Further we show that the Poisson bracket between \( D_0 \) and \( \mathcal{H}_T \) vanishes. To begin with we determine the Poisson brackets between \( D_0 \) and all phase space variables
\[
\{ D_0(x), \pi^{ij}(y) \} = 2\pi^{ij}(x)\delta(x-y), \quad \{ D_0(x), g_{ij}(y) \} = -2g_{ij}(x)\delta(x-y),
\]
\[
\{ D_0(x), N(y) \} = -2N(x)\delta(x-y), \quad \{ D_0(x), p_N(y) \} = 2p_N(x)\delta(x-y).
\]
Then it is easy to see that
\[
\{ D_0(x), \bar{g}_{ij}(y) \} = 0
\]
that implies
\[
\{ D_0(x), \bar{R}(y) \} = 0, \quad \{ D_0(x), \sqrt{g(y)} \} = -D \sqrt{g(x)} \delta(x-y).
\]
Using this result together with
\[
\left\{ D_0(x), \frac{N^{D/2}}{\sqrt{g}} \pi^{ij} G_{ijkl} \pi^{kl}(y) \right\} = 0
\]
we find
\[
\{ D_0(x), \mathcal{H}_T(y) \} = 0.
\]
Now we proceed to the analysis of the time evolution of the second term in $D$ that is proportional to $G(\bar{R})$. We use the fact that
\[\{G(\bar{R})(x), \pi^{ij}(y)\} = \frac{dG}{d\bar{R}}(x) \frac{1}{N^{2/z}(y)} \cdot \]

Then with the help of formulas
\[\frac{\delta \bar{R}(x)}{\delta g_{ij}(y)} = -\bar{R}^{ij}(x)\delta(x - y) + \bar{\nabla}^i \bar{\nabla}^j \delta(x - y) - \tilde{g}^{ij}(x) \bar{\nabla}_k \tilde{\nabla}^k \delta(x - y),\]
we determine following Poisson bracket
\[\{G(\bar{R})(x), \pi^{ij}(y)\} = \frac{dG}{d\bar{R}}(x) \frac{1}{N^{2/z}(y)} \times \]
\[\times (-\bar{R}^{ij}(x)\delta(x - y) + \bar{\nabla}^i \bar{\nabla}^j \delta(x - y) - \tilde{g}^{ij}(x) \bar{\nabla}_k \tilde{\nabla}^k \delta(x - y)) \cdot \]

Collecting all these results together we determine the time evolution of the constraint $D$
\[\partial_t D = \{D, H_T\} \approx 2 \frac{dG}{d\bar{R}}(x) \Phi^{II}_D \approx 0 \]

where
\[\Phi^{II}_D = -\bar{R}^{ij} \bar{\nabla}_i N^{2/z} \pi^{kl} + \bar{\nabla}^i \bar{\nabla}^j \bar{\nabla}^k \bar{\nabla}^{i} + \frac{1 - \lambda}{\lambda D - 1} \bar{\nabla}^i \bar{\nabla}^i \pi \cdot \]

Note also that using definition of $D$ and $\Phi_1$ we can introduce $D_0$ as an independent constraint
\[D_0 = D - \Phi_1 \approx 0 \cdot \]

In summary, we have following set of constraints
\[p^i \approx 0, \quad p_A \approx 0, \quad H^i \approx 0, \quad D_0 \approx 0, \quad \Phi_1 \approx 0, \quad \Phi^{II}_D \approx 0 \cdot \]

Now we proceed to the analysis of the stability of all constraints taking into account an existence of the secondary constraints. This is simple task in case of $p^i$ and $p_A$ that are trivially preserved and form the first class constraints of the theory. In the same way we can show that $T_S(N)$ is generator of the spatial diffeomorphism and it is preserved as well. Let us now analyze briefly the constraint $D_0$. Since $\{D_0(x), \bar{g}_{ij}(y)\} = 0$ and $\{D_0(x), N^{2/z} \pi^{ij}(y)\} = 0$ we see that Poisson brackets between $D_0$ and $\Phi_1 \approx 0, \Phi^{II}_D \approx 0$ vanish as well. In other words $D_0$ is the first class constraint and its smeared form is the generator of the scaling transformation.
Finally we have to determine the time evolution of the constraint $\Phi_1$ and $\Phi_D^{II}$. To do this we firstly show that

$$\{ \Phi_1(x), \Phi_D^{II}(y) \} \approx \triangle(R, R_{ij}, x, y) + \frac{(1 - \lambda)(1 - D)}{D\lambda - 1} \nabla_i \nabla^i \nabla_j \nabla^j \delta(x - y),$$

(5.18)

where $\triangle$ is complicated expression containing covariant derivatives of delta function which is also proportional to $R_{ij}$ and $\bar{R}$. On the other hand the last expression is proportional to the ordinary function and it vanishes for $\lambda = 1$. We see that the Poisson bracket between $\Phi_1$ and $\Phi_D^{II}$ is non-zero and depends on the phase space variables. It is important to stress that for $\lambda \neq 1$ this Poisson bracket is non-zero over all phase space. An exceptional situation occurs for $\lambda = 1$ when we find that this Poisson bracket vanishes for the subspace of the phase space where $R_{ij} = 0$. We can analyze this situation as in [32, 34] with the result that the theory on the subspace $R_{ij} = 0$ is effectively topological.

Excluding this exceptional case we see that the Poisson bracket between $\Phi_1$ and $\Phi_D^{II}$ is non-zero so that they are the second class constraints. As a result we can easily determine the time evolution of the constrains $\Phi_1$ and $\Phi_D^{II}$. In case of $\Phi_1$ we find

$$\partial_t \Phi_1 = \{ \Phi_1, H_T \} \approx 2 \frac{dG}{dR} \Phi_D^{II} + \int d^Dx v_{II}^D \{ \Phi_1, \Phi_D^{II}(x) \} \approx$$

$$\approx \int d^Dx v_{II}^D \{ \Phi_1, \Phi_D^{II}(x) \} = 0$$

(5.19)

that implies $v_{II}^D = 0$. On the other hand the time evolution of the constraint $\Phi_D^{II}$ is equal to

$$\partial_t \Phi_D^{II} = \{ \Phi_D^{II}, H_T \} \approx \int d^Dx (\{ \Phi_D^{II}, H_T(x) \} +$$

$$+ v_1 \{ \Phi_D^{II}, \Phi_1(x) \} + v_D^{II} \{ \Phi_D^{II}, \Phi_D^{II}(x) \}) = 0.$$  

(5.20)

Now due to the fact that $v_{II}^D = 0$ we see that this equation determines $v_1$ as a functional of canonical variables, at least in principle. Say differently, we see that it is not necessary to impose additional constraints on the systems and hence we can stop here.

In summary, the constraint structure of given theory is the same as the constraint structure of non-relativistic covariant HL gravity studied in [32]. This should not be surprising when we note that these two theories differ in the form of the St"{u}ckelberg fields while the form of the additional term is the same. Explicitly, $\Phi_1$ and $\Phi_D^{II}$ are the second class constraints that, according to the standard analysis have to vanish strongly and hence they allow us to express two phase space variables as functions of remaining physical phase space variables, at least in principle. Even if we cannot solve these constraints explicitly in general case we can still determine the number of physical degrees of freedom. To do this note that there are $D(D + 1)$ gravity phase space variables $g_{ij}, \pi^{ij}, 2D$ variables $N_i, p^i$, 2 variables $A, p_A$ and 2 variables $N, p_N$. In summary the total number of degrees of freedom
is $N_{D.o.f} = D^2 + 3D + 4$. On the other hand we have $D$ first class constraints $\mathcal{H}^i \approx 0$, $D$ first class constraints $p^i \approx 0$, 2 first class constraints $\mathcal{D}_0 \approx 0, p_A \approx 0$ and two second class constraints $\Phi_1, \Phi_2$. Then we have $N_{f.c.c} = 2D + 2$ first class constraints and $N_{s.c.c.} = 2$ second class constraints. Then the number of physical degrees of freedom is

$$N_{D.o.f} - 2N_{f.c.c} - N_{s.c.c.} = D^2 - D - 2$$

that exactly corresponds to the number of the phase space physical degrees of freedom of $D + 1$ dimensional gravity.

As we argued previously in [32, 34] it is difficult to make further analysis of these second class constraints. For example, the symplectic structure defined by corresponding Dirac brackets is very complicated. Secondly, it is also subtle point to perform quantization of the theory with the second class constraints. In some situations it is useful to perform so named Abelian conversion [40]. In this process we extend given theory so that it becomes theory with the first class constraints only. Even if this procedure is possible in principle it is again very difficult to apply it in our case due to the complicated form of the Poisson bracket between the second class constraints. For these reasons the proper understanding Weyl invariant HL gravity is lacking.

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