A study of energy concentration and drain in incompressible fluids

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Abstract
In this paper, we examine two opposite scenarios of energy behaviour for solutions of the Euler equation. We show that if \( u \) is a regular solution on a time interval \([0, T)\) and if \( u \in L^r L^\infty\) for some \( r \geq \frac{2}{N} + 1 \), where \( N \) is the dimension of the fluid, then the energy at the time \( T \) cannot concentrate on a set of Hausdorff dimension smaller than \( N - \frac{2}{r-1} \). The same holds for solutions of the three-dimensional Navier-Stokes equation in the range \( 5/3 < r < 7/4 \).

Oppositely, if the energy vanishes on a subregion of a fluid domain, it must vanish faster than \((T - t)^{1-\delta}\), for any \( \delta > 0 \). The results are applied to find new exclusions of locally self-similar blow-up in cases not covered previously in the literature.

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1. Introduction

We consider evolution of an incompressible \( N \)-dimensional ideal fluid governed by the Euler equations

\[
\begin{align*}
    u_t + u \cdot \nabla u + \nabla p &= 0, \\
    \nabla \cdot u &= 0, \\
    u(t=0) &= u_0.
\end{align*}
\]

Here \( u \) is the velocity field, \( p \) internal pressure, and we assume the fluid domain is \( \mathbb{R}^N \), \( N \geq 3 \). For any initial condition \( u_0 \in H^{N/2 + \epsilon} \) there exists a unique local-in-time solution \( u \in C([0, T); H^{N/2 + \epsilon}) \) with the associated pressure given by

\[
p(t, x) = -\frac{|u(t, x)|^2}{N} + \text{P.V.} \int_{\mathbb{R}^N} K_{ij}(x - y)u_i(y)u_j(y) \, dy,
\]
where $K_{ij}(y) = \frac{N\delta_{ij} - \delta_{ij}|y|^2}{N\omega_N|y|^{N-2}}$, and $\omega_N = 2\pi^{N/2}(\Gamma(N/2))^{-1}$ is the volume of the unit ball in $\mathbb{R}^N$. A recent trend in the global regularity problem for (1) is to rule out model scenarios of blow-up that arise in numerical simulations. The model of particular relevance to this note is the locally self-similar blow-up given by

$$
\begin{align*}
  u(x, t) &= \frac{1}{(T-t)^{\frac{N}{2}}} v \left( \frac{x-x_0}{(T-t)^{\frac{N}{2}}} \right), \\
  p(x, t) &= \frac{1}{(T-t)^{\frac{N}{2}}} q \left( \frac{x-x_0}{(T-t)^{\frac{N}{2}}} \right),
\end{align*}
$$

for $|x-x_0| < \rho_0$, $t < T$ and $\alpha > -1$ (focusing case). These solutions emerge, for instance, in vortex line models of Kida’s high-symmetry flows (see Pelz and others [1, 8, 9, 10]), although previously self-similar blow-up has been observed as well, [2, 7]. In a recent joint effort with D. Chae [3] (see also [4, 6, 11]) solutions of the form (3) have been ruled out under additional integrability condition, $v \in L^p(\mathbb{R}^N) \cap C^1_{\text{loc}}(\mathbb{R}^N)$, $p \geq 3$, in the range $-1 < \alpha \leq \frac{N}{p}$ and $\alpha > \frac{N}{2}$. In the energy conservative scaling $\alpha = \frac{N}{2}$, self-similar solutions are excluded provided $v \in L^2$ and the power bounds $\frac{1}{|\cdot|^{2\alpha+\frac{N}{p}}} \lesssim |v(y)| \lesssim |y|^{1-\alpha}$ hold at infinity. The range $\frac{N}{p} < \alpha < \frac{N}{2}$, or just $\alpha < \frac{N}{2}$ if no $L^p$-condition is assumed, remains open at the moment. We remark that finiteness of the total energy of $u$ requires $v$ to satisfy the energy growth bound $\int_{|y| \leq L} |v(y)|^2 \, dy \lesssim L^{N-2\alpha}$, instead of $v \in L^2$. It follows from the arguments of [3], that a slightly better bound

$$
\int_{|y| \leq L} |v(y)|^2 \, dy \lesssim L^{N-2\alpha} o(1),
$$

along with $v \in L^p$, $p \geq 3$, implies $v = 0$.

The main motivation of this present work is to understand the general energetics of the Euler system that lies behind the results of [3], and consequently to exclude new cases of self-similar blow-up in the range $\alpha \leq \frac{N}{2}$. To illustrate the thrust of what follows let us consider a self-similar solution with $\alpha = \frac{N}{2}$. As $t \to T$, the energy density $|u(x, t)|^2$ tends weakly to the Dirac mass at $x_0$. We see two different types of anomaly: energy concentration on the ‘small’ set $\{x_0\}$, and energy drain elsewhere. One can study these phenomena for general solutions of (1) by introducing an energy measure at time $T$. Suppose that $u$ is a smooth solution of (1) on the interval $[0, T]$. Then the local energy equality

$$
\int_{\Omega} |u(t, x)|^2 \sigma(x) \, dx = \int_{\Omega} |u_0(x)|^2 \sigma(x) \, dx + \int_0^T \int_{\Omega} (|u|^2 + 2p) u \cdot \nabla \sigma \, dx \, dt,
$$

holds for all $0 < t < T$, and $\sigma \in C^\infty_0(\mathbb{R}^N)$. If, in addition, $u \in L^3(0, T); L^3(\Omega))$ on some subdomain $\Omega \subseteq \mathbb{R}^N$, then (5) guarantees that the limit of the right hand side exists as $t \to T$ for all $\sigma \in C^\infty_0(\Omega)$, and hence, so does the limit on the left-hand side. It therefore defines a non-negative measure on $\Omega$, which we call the energy measure and denote by $\mathcal{E}_T$. If the solution does not loose smoothness at time $T$, then trivially,

$$
d\mathcal{E}_T(x) = |u(T, x)|^2 \, dx.
$$

Thus, deviation from (6) can be viewed as a measure of severity of the blow-up. One way to quantify this deviation is to consider how low the Hausdorff dimension of a set of positive $d\mathcal{E}_T$-measure can be

$$
d_T = \inf \{d \geq 0 : \exists S \subseteq \Omega, \dim_H(S) \leq d, \mathcal{E}_T(S) > 0\}.
$$

In theorem 2.1 we will prove that the size of $d_T$ can be controlled from below by the growth of $L^\infty$-norm at the time of blow-up: if $u \in L^r L^\infty(\Omega)$, for some $r \geq \frac{2}{N} + 1$, then $d_T \geq N - \frac{2}{r-1}$.
and $E_T$ has no atoms if $r = \frac{2}{N} + 1$ (a more general statement is given in lemma 2.2). In the case of a self-similar solution, this translates into the following statement: if $v \in L_\infty$ in (3), then the energy at time $T$ cannot concentrate on sets of dimension smaller than $N - 2\alpha$. Unfortunately, our technique does not rule out concentration to a point under milder condition $\|u(\cdot)\|_\infty \in L^{\frac{2}{N}+1}_{weak}$ which is the kind of condition that appears in the case $\alpha = \frac{2}{N}$. Instead, we will examine this case, as well as $\alpha \leq N/2$, from the point of view of the opposite phenomenon—the energy drain.

In theorem 3.1 we interpret energy drain as a merger of two solutions, given and the trivial one. As a consequence of theorem 3.1, if $u \in L^r L^\infty$, for some $r > 1$, and $\|u(t)\|_2 \to 0$ on a domain $\Omega$ at time $T$, then the following improvement occurs $\|u(t)\|_2 \leq C_\delta (T - t)^{1-\delta}$, for all $\delta > 0$. This can be interpreted as curbing the influence of pressure on local uniqueness. Let us assume from now on that $u$ is a solution to (1) with smooth initial condition $u_0$ on $\Omega$. As a consequence of theorem 3.1, if $\alpha \in \mathbb{N} \setminus \{1\}$, then the improved rate in self-similar variables translates into the bound $\int_{L^r L^\infty} |v|^2 \, dy \lesssim L^{-N/4-\delta}$. This rate of decay of energy is strong enough to put $v$ automatically in all $L^p$ spaces for $\frac{N}{N+1} < p < N + 4 + \delta$. Obviously, the lower bound $|v| \geq |y|^{-N/4-\delta}$, even in a sector, is inconsistent with these implications. We thus obtain a more robust exclusion condition. In the case $\alpha < N/2$, condition (4) ensures energy drain in the entire region of self-similarity $|x - x_0| < \rho_0$. The improved rate gives the bound $\int_{|y| \leq L} |v|^2 \, dy \lesssim L^{-N/4-\delta}$, for all $\delta > 0$. This implies $v = 0$ in the range $\frac{N}{N+2} < \alpha < \frac{N}{2}$, without any $L^p$-condition as previously considered in [3]. The full list of exclusions based on energy drain is stated in corollary 3.2.

Adaptations of the general results above to the Navier–Stokes system is given in section 4. We show that the estimates on the linear term in many cases are subordinate to those obtained for the nonlinear term. However we believe that a more meaningful use of the parabolic nature of the equation may improve the results.

2. Energy concentration

Let us assume from now on that $u$ is a solution to (1) with smooth initial condition $u_0$ on the interval $[0, T)$ with the natural pressure given by (2). Note that $p$ is the unique pressure recovered from the momentum equation under the assumption that $p \to 0$ as infinity. We do not make any regularity assumption on $u$ at time $T$.

**Theorem 2.1.** Suppose $u \in L^r ([0, T); L^\infty (\Omega))$ for some $r \geq \frac{2}{N} + 1$. Then $d_T \geq N - \frac{2}{r-1}$. Moreover, if $r = \frac{2}{N} + 1$, then $E_T$ has no atoms.

Notice that $u \in L^r L^\infty$ and energy conservation $u \in L^\infty L^2$ already implies $u \in L^\infty L^3$, which is necessary to define the energy measure as in the introduction.

2.1. Case $r = \frac{2}{N} + 1$

The proof in this case is relatively short. Suppose, on the contrary, that $E_T ([x_0]) = \varepsilon_0 > 0$. Let us fix a small $\rho > 0$ so that $B_{4\rho}(x_0) \subset \Omega$. Let us fix a $C^\infty$-function $0 \leq \sigma \leq 1$ with $\sigma (x) = 1$ on $B_{1/2}(0)$ and $\sigma (x) = 0$ on $\mathbb{R}^N \setminus B_1 (0)$. Let $\sigma_\rho (x) = \sigma ((x - x_0)/\rho)$. From the energy equality (5) we obtain for all $t < T$

$$\varepsilon_0 \leq \int_\Omega |u(t, x)|^2 \sigma_\rho (x) \, dx + \int_t^T \int_\Omega ((|u|^2 + 2p)u \cdot \nabla \sigma_\rho) \, dx \, d\tau \leq C \int_{|x - x_0| \leq \rho} |u(t, x)|^2 \, dx + \frac{C}{\rho} \int_t^T \int_{|x - x_0| \leq \rho} (|u|^3 + |u||p|) \, dx \, d\tau.$$
Note that \( L^{\frac{3}{2} + 1} \cap L^\infty L^2 \subset L^3 L^{\frac{3N}{N-3}} \). Thus,
\[
\frac{1}{\rho} \int_{[t-x_0] \leq \rho} (|u|^3 + |u|p) \, dx \, dr \leq \|u\|_{L^3([t, T]; L^{\frac{3N}{N-3}})}^3.
\]
So, letting \( \rho \to 0 \) first and then \( t \to T \), we arrive at the contradictory statement \( \epsilon_0 = 0 \).

Further analysis reveals that if energy concentration is to occur it has to happen over a family of shrinking balls.

**Lemma 2.2.** Suppose \( u \in L^4([0, T); L^\infty(\Omega)) \). Suppose that for any \( A > 0 \),
\[
\liminf_{t \to T} \int_{[x-x_0] \leq A} |u(t, x)|^2 \, dx = 0.
\]
Then \( \mathcal{E}_T \) has no atom at \([x_0]\).

One can easily check that \( u \in L^{\frac{3}{2} + 1} L^\infty \), or a weak-type condition \( \|u(t)\|_\infty \leq o(T - t)(T - t)^{\frac{3}{2}} \) implies (8).

**Proof.** Let \( E = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx \) denote the total (time-independent) energy of the solution. The cubic term in the last integral has a straightforward estimate
\[
\int_{[x-x_0] \leq \rho} |u|^3 \, dx \leq E \|u(\tau)\|_\infty.
\]
We split the pressure as follows \( p = p_{loc} + p_1 + p_2 \), where \( p_{loc} \) is the local term of (2), and does not require attention, while
\[
p_1(x) = \int_{|y-x_0| < 2\rho} \quad p_2(x) = \int_{|y-x_0| \geq 2\rho}.
\]
We have, by the Calderon–Zygmund boundedness,
\[
\int_{[x-x_0] \leq \rho} |u|p_1 \, dx \leq \left( \int_{[x-x_0] \leq \rho} |u|^2 \, dx \right)^{1/2} \left( \int_{[x-x_0] \leq 2\rho} |u|^4 \, dx \right)^{1/2} \leq E \|u(\tau)\|_\infty.
\]
as to the \( p_2 \) term we have
\[
\int_{[x-x_0] \leq \rho} |u|p_2 \, dx \leq C \|u(\tau)\|_\infty \rho^N \sup_{[x-x_0] \leq \rho} \int_{|y-x_0| \geq 2\rho} \frac{1}{|x-y|^N} |u(y)|^2 \, dy \leq CE \|u(\tau)\|_\infty.
\]
Returning to the energy inequality, we obtain
\[
\epsilon_0 \leq C \int_{[x-x_0] \leq \rho} |u(t, x)|^2 \, dx + \frac{CE}{\rho} \int_t^T \|u(\tau)\|_\infty \, dr
\]
for any fixed \( A > 0 \), let \( \rho = A \int_t^T \|u(\tau)\|_\infty \, dr \). Letting \( t \to T \), the above implies \( \epsilon_0 \lesssim A^{\frac{-1}{2}} \), a contradiction.

2.2. Case \( r > \frac{2}{N} + 1 \)

Let us fix an \( 0 < L < 1 \), the subdomain \( \Omega_L = \{ x \in \Omega : \text{dist}(x, \delta \Omega) > L \} \), and without loss of generality assume that \( S \subset \Omega_L \). Let us fix a small \( \delta > 0 \) and large \( M \in \mathbb{N} \) such that
\[
M\delta > N - \frac{2}{r} - 1.
\]
The main technical ingredient is the following lemma.
**Lemma 2.3.** There exists a constant $C = C(\delta, L, u) > 0$ and $\rho_0 = \rho_0(\delta, L, u) > 0$ such that for every $x_0 \in \Omega_L$ and $\rho < \rho_0$ one has

$$\sup_{T-\rho < t \leq T} \int_{|x-x_0| \leq \rho} |u(t, x)|^2 \, dx \leq C\rho^{-N-2}. \quad (11)$$

The theorem follows immediately from lemma 2.3. Indeed, suppose that $\dim_H(S) = d < N - \frac{2}{\beta_1}$. Let $\delta > 0$ be so small that $d < N - \frac{2}{\beta_1} - \delta$. Then for every $\varepsilon > 0$ there exists a cover of $S$ by open balls $B_{\rho_i}(x_i)$, $\rho_i < \rho_0$, $x_i \in \Omega_L$, with

$$\sum_{i=1}^{\infty} \rho_i^{-N-\delta} < \varepsilon.$$

Then

$$\mathcal{E}_T(S) \leq \sum_{i=1}^{\infty} \int_{\Omega} \sigma_{2\rho_i} (x-x_i) \, d\mathcal{E}_T(x) = \sum_{i=1}^{\infty} \lim_{t \to T} \int_{\Omega} \sigma_{2\rho_i} (x-x_i) |u(t, x)|^2 \, dx$$

$$\leq C \sum_{i=1}^{\infty} \rho_i^{-N-\delta} \leq C\varepsilon.$$

**Proof of lemma 2.3.** To simplify our notation, let us assume that $x_0 = 0$, $T = 0$, and the time $t > 0$ is reversed. Let us introduce some notation first. Let us fix a positive time $t_0 > 0$, and $\rho < \rho_0$ where $\rho_0$ is small, but fixed, satisfying

$$\log_2(L/\rho_0) > 3M + 4. \quad (12)$$

The time $t_0$ will be determined later and will depend on $\rho$. Let $\sigma \in C_{0}^{\infty}(\mathbb{R}^N)$ with $\sigma = 1$ on $|x| < 1/2$ and $\sigma = 0$ on $|x| > 1$. Denote

$$E(t, \rho) = \int |u(x, t)|^2 \sigma(x/\rho) \, dx$$

$$f(t) = \|u(t)\|_{L^\infty(\Omega)}$$

$$F(t) = \int_{t_0}^{t} f(\tau) \, d\tau.$$

Note that $\int_{|x| \leq \rho} |u(x, t)|^2 \, dx \leq E(t, 2\rho)$. Our goal will be to establish uniform bounds on the energy $E(t, \rho)$ for all $t < t_0$ and $\rho < \rho_0$. In all computations below $\lesssim$ will denote an inequality that holds up to a constant independent of $\rho$ or $t$.

Let $\rho < \rho_0$ be fixed, and let $K = \lfloor \log_2(L/\rho) \rfloor - 1$. From the above, $K = 3M \geq 3$. Notice that the ball $\{|x| < 2^K \rho\}$ is still inside the domain $\Omega$. From the energy equality (5) we find

$$E(t, \rho) \leq E(t_0, \rho) + \frac{C}{\rho} \int_{t_0}^{t} \int_{|x| \leq \rho} (|u|^3 + |u||p|) \, dx \, d\tau. \quad (13)$$

Using introduced notation, we have

$$\int_{t_0}^{t} \int_{|x| \leq \rho} |u|^3 \, dx \, d\tau \leq \int_{t_0}^{t} f(\tau) E(\tau, 2\rho) \, d\tau.$$

As to the pressure term we split similar to the previous $p = p_{\text{loc}} + p_1 + p_2$, where

$$p_1(x) = \int_{|y| < 2\rho} ; \quad p_2(x) = \int_{|y| \geq 2\rho}.$$

The term with $p_1$ is estimated through the Calderon–Zygmund inequality as before, and gives

$$\int_{t_0}^{t} \int_{|x| \leq \rho} |u||p_1| \, dx \, d\tau \lesssim \int_{t_0}^{t} f(\tau) E(\tau, 4\rho) \, d\tau.$$
For $p_2$ we obtain for all $|x| \leq \rho$,

$$p_2(\tau, x) \leq \frac{1}{\rho^N} \sum_{k=3}^{K-3M} \frac{1}{2^N k} E(\tau, 2^k \rho) + \frac{2^{3N} M}{L^N} \|u\|_2.$$  

Thus,

$$\int_{|x| \leq \rho} |u| p_2 \, dx \, d\tau \lesssim \int_{t}^{b} f(\tau) \sum_{k=3}^{K-3M} \frac{1}{2^N k} E(\tau, 2^k \rho) \, d\tau + \rho^N F(t).$$  

Putting all the estimates together we obtain

$$E(t, \rho) \lesssim E(t_0, \rho) + \frac{1}{\rho} \sum_{k=1}^{K-3M} \frac{1}{2^N k} f(t) E(t_0, 2^k \rho) \, dt_2$$  

$$\lesssim E(t_0, \rho) + \frac{1}{\rho^N} \sum_{k=1}^{K-3M} \frac{1}{2^N k} f(t) E(t_0, 2^k \rho) \, dt_2$$  

$$+ \rho^{N-2} \sum_{k=1}^{K-3M} \frac{1}{2^N k} \int_{t_1}^{t_0} f(t_1) \int_{t_1}^{t_2} f(t_2) \, dt_2 \, d\tau$$  

$$+ \rho^{N-2} \sum_{k=1}^{K-3M} \frac{1}{2^N k} \int_{t_1}^{t_0} f(t_1) \int_{t_1}^{t_2} f(t_2) \, dt_2 \, d\tau$$  

Estimating all the energies at time $t_0$ trivially we obtain

$$E(t, \rho) \lesssim \rho^N f^2(t_0) + K \rho^{N-1} f^2(t_0) F(t) + \frac{1}{2} F^2(t) \rho$$  

$$\lesssim \rho^N f^2(t_0) K \left(1 + F(t)/\rho + \frac{1}{2!} (F(t)/\rho)^2\right) + \rho^3 \left(\frac{1}{2!} (F(t)/\rho)^2 + \frac{1}{3!} (F(t)/\rho)^3\right)$$  

$$\times \int_{t_1}^{t_0} f(t_1) \int_{t_1}^{t_2} f(t_2) \, dt_2 \, d\tau$$  

On the $M$th step we obtain

$$E(t, \rho) \lesssim K \rho^N f^2(t_0) \left(1 + \cdots + \frac{1}{(M-1)!} (F(t)/\rho)^{(M-1)}\right)$$  

$$+ \rho^N \left(\frac{1}{2!} (F(t)/\rho)^2 + \cdots + \frac{1}{M!} (F(t)/\rho)^M\right)$$  

$$+ \frac{1}{\rho^N} \sum_{k=3}^{K-3M} \frac{1}{2^N k} f(t) E(t_0, 2^k \rho) \, dt_2 \, d\tau.$$

Notice that the index $k$ must be allowed to reach 3 for the above step to be possible. With our choice of $K$, we therefore can make $M$ iterations of (14). Let $k_1 = k$ and $t_1 = \tau$. From (14) we obtain

$$E(t_1, 2^k \rho) \lesssim E(t_0, 2^k \rho) + \frac{1}{\rho} \sum_{k=1}^{K-3M} \frac{1}{2^N k} f(t_2) E(t_2, 2^k \rho) \, dt_2$$

$$+ 2^{(N-1)k_1} \rho^2 F(t_1).$$

Plugging back into (14) we obtain

$$E(t, \rho) \lesssim E(t_0, \rho) + \frac{1}{\rho} \sum_{k=1}^{K-3M} \frac{1}{2^N k} E(t_0, 2^k \rho)$$

$$+ \rho^{N-2} \sum_{k=1}^{K-3M} \frac{1}{2^N k} \int_{t_1}^{t_0} f(t) \int_{t_1}^{t_2} f(t_2) \, dt_2 \, d\tau$$

$$+ \rho^{N-2} \sum_{k=1}^{K-3M} \frac{1}{2^N k} \int_{t_1}^{t_0} f(t) \int_{t_1}^{t_2} f(t_2) \, dt_2 \, d\tau.$$
where the ‘sum\_M’ involves an M-tuple integral similar to the above. Replacing the energies in that integral trivially by \|u\|_2, we obtain

\[ \frac{1}{\rho^M} \text{sum}_M \lesssim (F(t)/\rho)^M. \]

We thus arrive at the following estimate for all 0 < t < t_0

\[ E(t, \rho) \lesssim K \rho^N f^2(t_0) \exp (F(t)/\rho) + \rho^N \exp (F(t)/\rho) + (F(t)/\rho)^M. \]

By Hölder, \( F(t) \lesssim t r^{-1/2} \|u\|_{L^r L^\infty(\Omega)} \). Suppose we can choose \( t_0 \sim \rho^{r-1+\delta} \) such that \( f^r(t_0) \lesssim 1/t_0 \).

Then, in view of (10), and \( K \lesssim |\log_2 \rho| \lesssim \rho^\delta \), the bound above would give the desired inequality (11). To find such \( t_0 \) we recall that \( f \in L^r([0,T)) \). Then starting from some \( t' > 0 \) and for all \( t < t' \) there exists a \( t_0 \in [t/2, t] \) such that \( f^r(t_0) \lesssim 1/t_0 \). Indeed, otherwise the integral of \( f \) would diverge logarithmically. In addition, \( t' \) depends only on \( \|f\|_r \). Therefore by further reducing the size of \( \rho \) to satisfy \( \rho^{r-1+\delta} \lesssim t' \) we obtain the desired conclusion. □

Applying theorem 2.1 to the case of self-similar solutions (3) with \( v \in L^\infty \) we immediately obtain the following conclusion.

**Corollary 2.4.** Suppose \( v \in L^\infty \), \( 0 < \alpha < \frac{N}{2} \). Then in the region where the solution \( u \) is given by (3) the energy does not concentrate on sets of dimension smaller than \( N - 2\alpha \).

### 3. Local merger and self-similar solutions

**Theorem 3.1.** Suppose \( u_1 \) and \( u_2 \) are two classical solutions to (1) on a time interval \([0,T)\), and \( \nabla u_1, u_2 \in L^r([0,T); L^\infty(\Omega)) \) or \( \nabla u_1 \in L^r([0,T); L^\infty(\Omega)) \) for some \( r > 1 \). Suppose further that

\[ \|u_1(t) - u_2(t)\|_{L^2(\Omega)} \to 0, \quad \text{as} \ t \to T. \] (15)

Then for every compactly embedded subdomain \( \Omega' \subset \Omega \) and every \( \delta > 0 \) there exists \( C > 0 \) such that

\[ \|u_1(t) - u_2(t)\|_{L^2(\Omega')} \leq C(T-t)^{1-\delta}, \quad \text{as} \ t \to T. \] (16)

**Proof.** Let us assume for definiteness that \( \nabla u_1, u_2 \in L^r([0,T); L^\infty(\Omega)) \). Let \( w = u_1 - u_2 \). Then \( w \) satisfies

\[ w_t = -w \cdot \nabla u_1 - u_2 \cdot \nabla w - \nabla q, \] (17)

where \( q \) is the associated pressure recovered via a relationship similar to (2). By the standard covering argument it suffices to show that for every \( x_0 \in \Omega \) there exists a \( \rho_0 > 0 \) such that

\[ \int_{|x-x_0| \leq \rho_0} |w(t,x)|^2 \, dx \lesssim C(T-t)^{2-\delta}. \] (18)

Let us assume for notational convenience that \( x_0 = 0 \in \Omega \). Let us denote

\[ E(t, \rho) = \int_{|x| \leq \rho} |w(t,x)|^2 \, dx. \]

The result will follow from the following energy estimate

\[ E(t, \rho) \lesssim \int_t^T f(\tau) E(\tau, 4\rho) \, d\tau + \int_t^T \sqrt{E(\tau, 4\rho)} \, d\tau. \] (19)
where \( f = \|u_1\|_\infty + \|\nabla u_1\|_\infty + \|u_2\|_\infty \). To see that, first, let us note the following interpolation inequality
\[
\|u_1\|_\infty \leq C_\alpha \|u_1\|_2^{\frac{2}{\alpha}} (\|u_1\|_\infty + \|\nabla u_1\|_\infty)^{\frac{\alpha}{\alpha}}.
\]
Since the energy of \( u_1 \) is bounded, we obtain \( \|u_1\|_\infty^{\frac{2}{\alpha}} \leq \max\{C_{\Omega,E} \|\nabla u_1\|_\infty\} \), and hence \( f \in L' \). Let us now consider
\[
\alpha_0 = \sup\{\alpha < 2 : \exists \rho > 0, \exists C > 0 : E(t, \rho) \leq C(T - t)^\alpha \text{ as } t \to T\}.
\]
From (19) it follows that \( \alpha_0 \geq 1 - 1/r \), and (18) is equivalent to \( \alpha_0 \geq 2 \). Assume that \( \alpha_0 < 2 \). Then let \( \delta > 0 \) be small, and \( \alpha > \alpha_0 - \delta \). Let \( \rho > 0 \) be as in the definition. Then, from (19),
\[
E(t, \rho/4) \leq \int_t^T f(\tau) E(\tau, \rho) \, d\tau + \int_t^T \sqrt{E(\tau, \rho)} \, d\tau
= (T - t)^{1/2 + \alpha_0 - \delta} + (T - t)^{\frac{\alpha}{\alpha_0}}.
\]
The power now is strictly grater than \( \alpha_0 \), provided \( \delta \) is small enough, which is a contradiction.

We now turn to proving (19). From the energy equality, for all \( t < T - \varepsilon < T \),
\[
E(t, \rho) \leq E(T - \varepsilon, \rho) + \frac{1}{\rho} \int_t^{T - \varepsilon} \int_{|x| \leq 2 \rho} \left( |u_2| |w|^2 + |w| |q| \right) \, dx \, dt
+ \int_t^{T - \varepsilon} \int_{|x| \leq 2 \rho} \|\nabla u_1\| |w|^2 \, dx \, dt.
\]
Letting \( \varepsilon \to 0 \) we obtain
\[
E(t, \rho) \leq C_\rho \int_t^T \int_{|x| \leq 2 \rho} \left( |u_2| |w|^2 + |w| |q| + \|\nabla u_1\| |w|^2 \right) \, dx \, dt.
\]
We have
\[
\int_{|x| \leq 2 \rho} (|u_2| + |\nabla u_1|) |w|^2 \, dx \leq (\|u_2(\tau)\|_\infty + \|\nabla u_1(\tau)\|_\infty) E(\tau, 2 \rho).
\]
The local part of the pressure adds another term \( \|u_1(\tau)\|_\infty E(\tau, 2 \rho) \). The nonlocal part is split as before,
\[
q_1(x) = \int_{|y| \leq 4 \rho} K_{ij}(x - y) (w_i u_1^{(1)} + w_i u_2^{(2)})(y) \, dy,
\]
\[
q_2(x) = \int_{|y| > 4 \rho} K_{ij}(x - y) (w_i u_1^{(1)} + w_i u_2^{(2)})(y) \, dy.
\]
We have
\[
\int_{|x| \leq 2 \rho} |w| |q_1| \, dx \leq \left( \int_{|x| \leq 2 \rho} |w|^2 \, dx \right)^{1/2} \left( \int_{|x| \leq 4 \rho} |w|^2 (|u_1| + |u_2|)^2 \, dx \right)^{1/2}
\leq E(\tau, 4 \rho) (\|u_1\|_\infty + \|u_2\|_\infty),
\]
while simply using that \( |q_2| \leq C_\rho \|u_1\|_2 \|u_2\|_2 \) we have
\[
\int_{|x| \leq 2 \rho} |w| |q_2| \, dx \leq C_\rho E(\tau, 2 \rho).
\]
Incorporating the above estimates into (22) we obtain
\[
E(t, \rho) \leq \int_t^T (\|u_1\|_\infty + \|\nabla u_1\|_\infty + \|u_2\|_\infty) E(\tau, 4 \rho) \, d\tau + \int_t^T \sqrt{E(\tau, 4 \rho)} \, d\tau,
\]
which is the desired inequality. \( \square \)
Assuming \( u_2 = 0 \) in the above theorem we can find several new exclusion criteria for self-similar blow-up. First let us assume that \( v \in C_{1\infty}^{1} \) and \( |v(y)| \lesssim |y|^{1-\delta} \), as \( y \to \infty \), for some \( 0 < \delta < 1 \), and \( \alpha > 0 \). This puts the solution \( v \) in \( L^{r}L^{\infty} \) for some \( r > 1 \) in the region of self-similarity. In the case \( \alpha = N/2 \), the natural energy assumption \( v \in L^{2} \) implies that \( \|u(t)\|_{2} \to 0 \) in any annulus \( 0 < \rho_1 < |x-x_0| < \rho_0 \). Thus, the solution merges with the trivial zero solution, and hence, theorem 3.1 implies

\[
\int_{L<|y|<2L} |v|^{2} \, dy \lesssim \frac{1}{L^{N+2-\delta}},
\]

for all \( \delta' > 0 \). Now, by Hölder for any \( p < 2 \) we have

\[
\int_{L<|y|<2L} |v(y)|^{p} \, dy \lesssim L^{N-\frac{2\delta}{p}} \left( \int_{L<|y|<2L} |v(y)|^{2} \, dy \right)^{p/2} < L^{N-(N+1)p+\delta'}. \]

This implies that \( v \in L^{p} \), for all \( N/4 < p < 2 \). Furthermore, if \( p > 2 \), then trivially,

\[
\int_{L<|y|<2L} |v(y)|^{p} \, dy \lesssim \frac{L^{(p-2)(1-\delta)}}{L^{N+2-\delta}},
\]

which implies \( v \in L^{p} \), if \( p \leq 4 + N \). In summary, \( v \in L^{2} \) implies

\[
v \in \bigcap_{\frac{4}{N+1} < p \leq 4+N} L^{p}(\mathbb{R}^{N}).
\]

Let us note that under the assumption \( |v(y)| \gtrsim \frac{1}{|y|^{\frac{N}{2}+\delta}} \) these solutions have been already excluded in [3]. So, the implications above are somewhat more general.

In the case \( \alpha < N/2 \), the natural energy bound on \( v \), coming from the boundedness of the global energy, is

\[
\int_{|y| \leq L} |v|^{2} \, dy \lesssim L^{N-2\alpha},
\]

while the energy drain condition becomes

\[
\int_{|y| \leq L} |v|^{2} \, dy \lesssim L^{N-2\alpha}o(1), \quad \text{as } L \to \infty.
\]

Notice that since \( N - 2\alpha > 0 \) this is equivalent to a similar condition over dyadic shells. Again, according to theorem 3.1 the improved energy bound becomes

\[
\int_{|y| \leq L} |v|^{2} \, dy \lesssim L^{N-2\alpha+\delta'},
\]

for any \( \delta' > 0 \). This excludes solutions in the range \( \alpha > \frac{N-2}{4} \), while in the range \( 0 < \alpha \leq \frac{N-2}{4} \) is inconsistent with the bound from below \( |v(y)| \gtrsim \frac{1}{|y|^\beta} \) for any \( \beta < 2\alpha + 1 \). We summarize our observations in the following corollary.

**Corollary 3.2.** Suppose \( v, q \) is a self-similar solution to (1) in the form (3). Suppose \( |v| \lesssim |y|^{1-\delta} \). Then \( v = 0 \) or nonexistent in any of the following cases

(i) \( \alpha = \frac{N}{2}, \ v \in L^{2} \setminus \bigcap_{\frac{N}{4} \leq p \leq N+2} L^{p}; \)
(ii) \( \frac{N-2}{4} < \alpha < \frac{N}{2}, \ \int_{|y| \leq L} |v|^{2} \, dy \leq L^{N-2\alpha}o(1); \)
(iii) \( 0 < \alpha \leq \frac{N-2}{4}, \ \int_{|y| \leq L} |v|^{2} \, dy \leq L^{N-2\alpha}o(1) \) and \( \frac{1}{|y|^\beta} \lesssim |v(y)|, \) for some \( \beta < 2\alpha + 1 \).
Corollary (3.2) can be used to exclude solutions with explicit asymptotic bounds, which are sufficient for (i), (ii) and (iii), respectively:

(i') \( \alpha = \frac{N}{2}, \ v \in L^2, \) and
\[
\frac{1}{|y|^{N+\delta}} \lesssim |v(y)| \lesssim |y|^{1-\delta},
\]
(see also [3]).

(ii') \( \frac{N-2}{4} < \alpha < \frac{N}{2}, \) and
\[
|v(y)| \lesssim o(1) |y|^\alpha.
\]

(iii') \( 0 < \alpha \leq \frac{N-2}{4}, \) and
\[
\frac{1}{|y|^{2\alpha+1-\delta}} \lesssim |v(y)| \lesssim o(1) |y|^\alpha.
\]

4. Adaptations to the Navier–Stokes system

Most of the results of the previous sections carry over to the viscous case too. Here we assume \( N = 3. \) Even though some of them may not be optimal for this case, we will show that the contribution of the linear term in most cases is of lower order, and thus is subordinate to the contribution of the nonlinear term. We start with the concentration results. Let us assume that \((u, p)\) is a solution to the Navier–Stokes equation
\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u, \\
\nabla \cdot u &= 0
\end{align*}
\]
(23)
with smooth initial data \( u_0. \) Suppose that \([0, T)\) is an interval of regularity of \( u. \) It was shown in [5] that the condition \( u \in L^1 L^\infty \) holds automatically on \([0, T).\)

**Corollary 4.1.** Let \( u \) be a regular solution to the Navier–Stokes equation on an interval \([0, T).\) Suppose (8) holds. Then \( \mathcal{E}_T \) has no atoms. In particular, \( \mathcal{E}_T \) has no atoms if \( u \in L^{5/3}([0, T); L^\infty(\Omega)).\)

**Proof.** Two additional terms that appear on the right-hand side of the energy equality are
\[
-\nu \int_0^T \int_\Omega |\nabla u|^2 \sigma_\rho \, dx \, dt + \frac{\nu}{2} \int_0^T \int_\Omega |u|^2 \Delta \sigma_\rho \, dx \, dt.
\]
(24)

The first term has a negative sign, so it can be dropped. For the second term we have the estimate
\[
\int_0^T \int_\Omega |u|^2 \Delta \sigma_\rho \, dx \, dt \leq \frac{1}{\rho^3} \int_0^T \int_\Omega |u|^2 \, dx \, dt \leq \int_0^T \left( \int_\Omega |u|^6 \, dx \right)^{1/3} \, dt
\]
\[
\leq \int_0^T \int_\Omega |\nabla u|^2 \, dx \, dt \to 0,
\]
as \( t \to T. \)

A minor modification makes it possible to extend the conclusion of theorem 2.1 to larger values of \( r. \) Let us note however that if \( r \geq 2, \) then \( u \) satisfies the Prodi–Serrin regularity condition, and hence, theorem 2.1 holds trivially.
Corollary 4.2. Suppose \( u \in L^r([0, T); L^\infty) \), \( 5/3 < r < 7/4 \), and \( u \) is a regular solution to the Navier–Stokes equation on the interval \([0, T)\). Then

\[
d_T \geq \frac{3r - 5}{r - 1}.
\]

Proof. Let us keep the original forward direction of time \( t \to T \). The initial energy inequality (14) should be replaced with

\[
E(t, \rho) \lesssim E(T - t_0, \rho) + \frac{1}{\rho} \sum_{k=1}^{K-2M} \frac{1}{2^{Nk}} \int_{T-t_0}^{t_f} f(\tau) E(\tau, 2^k \rho) \, d\tau
\]

\[
- v \int_{T-t_0}^{t_f} \int_{\Omega} |\nabla u|^2 \sigma_{\rho} \, dx \, d\tau + \frac{v}{2} \int_{T-t_0}^{t_f} \int_{\Omega} |u|^2 \Delta \sigma_{\rho} \, dx \, d\tau,
\]

for all \( T - t_0 < t < T \). The negative viscous term on the right hand side can be dropped as before, while the other viscous term can be estimated by

\[
\frac{v}{2} \int_{T-t_0}^{t_f} \int_{\Omega} |u|^2 \Delta \sigma_{\rho} \, dx \, d\tau \lesssim \frac{t_0}{\rho^2}.
\]

Subsequent iterations of this term result in the sum

\[
\frac{t_0}{\rho^2} (1 + \frac{F(t)}{\rho} + \cdots + \frac{1}{M!} (\frac{F(t)}{\rho}^M),
\]

which, given the choice of \( t_0 \), is comparable to \( \rho^{\frac{5}{1-r}} \). In order for this term to be less than the required \( \rho^{\frac{3}{r}} \), the exponent \( r \) has to satisfy \( r < \frac{7}{4} \). □

Clearly from the proof, if \( r \geq \frac{7}{4} \), then the dimension becomes \( d < \frac{2d - r}{2} \). However, it is somewhat unnatural that it becomes smaller, hence more singular, as \( r \) approaches its regularity threshold value \( r = 2 \).

The local merger results of section 3 carry over backward in time. This is because the basic energy inequality (21) has time direction reversed. Thus, assuming that the merger time is 0, and for \( t > 0 \), the extra viscous term that appears on the right hand side is

\[
\frac{v}{2} \int_{T-t_0}^{t_f} \int_{\Omega} |w|^2 \Delta \sigma_{\rho} \, dx \, d\tau \leq \frac{C}{\rho^r} \int_{0}^{t_f} E(\tau, \rho) \, d\tau.
\]

This term further appears in (19), and does not effect the rest of the proof.

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