Comment on hep-lat/9901005 v1-v3 by
W. Bietenholz

Ivan Horváth

Department of Physics, University of Virginia
Charlottesville, Virginia 22903, U.S.A.

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Abstract

I comment on the above paper(s) discussing the issue of non-ultralocality for Ginsparg-Wilson fermionic actions. The purpose of this note is to point out that the new claim in “v3” (Feb 24, 2000), alleging the proof of non-ultralocality for “all Ginsparg-Wilson fermions”, might not be substantiated. The remarkable evolution of versions of this paper is put in context with the literature existing at the time of their appearance.

1 The Setting

One of the interesting issues in lattice field theory is understanding the structural properties of Ginsparg-Wilson (GW) fermionic actions, i.e. the actions for which the chirally nonsymmetric part of the massless propagator \( R \equiv (D^{-1})_N \neq 0 \) is local \(^1\). Perhaps the most basic problem is clarifying under which conditions the GW property is (in)compatible with ultralocality of the action. This question can be usefully discussed already at the free level, because non-ultralocality of a free action implies non-ultralocality in any gauge-invariant interacting theory based on it.

It turns out that a lot can be said if one requires the underlying free theory to respect the symmetries of the hypercubic lattice structure \([1,2]\), i.e. translations and transformations of the hypercubic group. In fact, this is naturally the most important case since ideally, one prefers to work with the action respecting all crucial fundamental symmetries, i.e. hypercubic symmetries, gauge invariance, and chiral symmetry (in this case Ginsparg-Wilson-Lüscher (GWL) symmetry). It is worth noting that attempts to clarify this issue in the context where symmetry under hypercubic group is abandoned and only lattice translations are kept, were not very successful so far.

Current insight into the question of (in)compatibility of hypercubic symmetries with ultralocality of GW actions is based on two suggestions:

*ih3p@virginia.edu

\(^1\)For definitions, see Ref. [2]
**Part 1:** Studying the consequences of the GW property for ultralocal lattice Dirac operators restricted on lines corresponding to periodic directions in the Brillouin zone. This was introduced in Ref. [1] and fully developed in Ref. [2].

**Part 2:** Studying the consequences of the GW property at the origin of the Brillouin zone for two-dimensional restrictions of ultralocal lattice Dirac operators. This was introduced in Ref. [3] with the suggestion that corresponding analytic properties at the origin have powerful global consequences for the operator in ultralocal case, and may lead to the necessity of fermion doubling.

## 2 The Literature

It is useful to outline the evolution of the above issues in the literature chronologically. For clarity, I will refer to various versions of paper hep-lat/9901005 [4-6] as “v1-v3”.

1. In Ref. [1], the idea of Part 1 was introduced and necessary steps were performed to prove that canonical GW operators, i.e. operators satisfying \( \{D, \gamma_5\} = D\gamma_5D \), or \( R \equiv (D^{-1})_N = \frac{1}{2}I \), cannot be ultralocal. On the first page of that Letter it is also explicitly stated that the proof can be extended to all ultralocal \( R \), trivial in Dirac space. This and other simple generalizations were explicitly deferred to Ref. [2] for reasons of space.

2. The paper “v1” adopts the approach of Part 1, and the claim is made in the abstract of extending the proof of Ref. [1] to a “...much larger class of Ginsparg-Wilson fermions...”. While it is quite unclear from the paper what this larger class is, it is explicitly stated that the alleged proof applies to all cases for which \( R^{-1} \) is ultralocal, where \( R \) is a “Dirac scalar” i.e. trivial in spinor space (Ref. [4], pages 1,2). This would be an interesting new result but, unfortunately, it was not substantiated.

3. In paper “v2” the above claim of “v1” is changed, and it is stated that the proof rather applies to all cases for which \( R \) is ultralocal and trivial in spinor space (Ref. [5], pages 1,2). This is a result put forward in Ref. [1].

While “v2” uses the ingredients of Part 1, the satisfactory discussion of its merits as a proof (and the merits of its assumptions) would be rather involved. Some improvements were put forward later in “v3”, and this latest version will be discussed in Sec. 3 of this Comment.

4. Paper [2] describes in detail the consequences of the approach of Part 1, as indicated in Ref. [1]. The considerations on periodic directions of the Brillouin zone are shown to be sufficient to prove that infinitesimal GWL symmetry transformations must be non-ultralocal for arbitrary GW action in the presence of hypercubic symmetries (“weak non-ultralocality”). On the basis of weak non-ultralocality it is then proved that GW operators for which \( R \equiv (D^{-1})_N \) is ultralocal, can not be ultralocal. This contains the result announced in Ref. [1], and generalizes it further by showing that triviality in spinor space is not crucial.

5. The approach of Part 2 is proposed in Ref. [3] 2. It is pointed out that in the presence of hypercubic symmetries, the GW condition for ultralocal actions translates into

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2Ref. [3] is a contribution of the author to the proceedings of the conference “Lattice Fermions and the Structure of the Vacuum”, Dubna, Russia, Oct 5-9 1999. The list of participants at this conference includes the author of [4-6].
analytic properties of certain rational functions after suitable change of variables, and that two-dimensional restrictions of operators on the Brillouin zone already capture the required analytic structure. The crucial observation is that analyticity at the origin implies factorization properties of involved polynomials, which strongly constraints the global behaviour of the corresponding action, and may imply fermion doubling. This connection is encapsulated in the hypothesis (Ref. [3], page 6), reflecting the conjectured property of such factorizations. The hypothesis was proposed as a key to the problem of “strong non-ultralocality”, i.e. non-ultralocality of all doubler-free GW actions in the presence of hypercubic symmetries, as formulated in Ref. [2]. There is no resolution of the hypothesis to date.

(6) The version “v3” appears (Feb 24, 2000) with two major changes compared to “v2”:

(a) Parts of formalism and the main claim of “v2” are upgraded to the most general result of Ref. [2], i.e. discussion involves actions with \( R \equiv (D^{-1})_N \) ultralocal (not just trivial in spinor space). This now forms the STEP 1 of the paper.

(b) The approach Part 2 of Ref. [3] is adopted in the completely new part STEP 2. The argument is presented in such a way that STEP 1 and STEP 2 together are claimed to imply the non-ultralocality for “all Ginsparg-Wilson fermions” [6].

3 Discussion of “v3”

The purpose of this note is to point out that one can raise several objections to the arguments of “v3”, casting doubts about the result claimed in the paper. Some of these objections are described below.

(a) One of the starting points of “v3” is the suggestion that Eq. (2) represents the general ansatz for the restriction of arbitrary lattice Dirac operator in \( d \) dimensions to the two-dimensional momentum plane through \( D \rightarrow D(p_1, p_2, 0, \ldots, 0) \). This is supposed to be true if “We assume Hermiticity, discrete translation invariance, as well as invariance under reflections and exchange of the axes.” [6]3. If true, the statement of this nature should perhaps be proved. If not true, the additional possible terms (such as one proportional to \( \gamma_1\gamma_2 \), which is compatible with hypercubic symmetries) must be included and the proof should proceed with the presence of such terms.

(b) The GW property is encoded in the analyticity properties of Clifford components of \((D^{-1})_N\). “v3” uses the non-invertible change of variables (two to one) on the Brillouin zone \( c_\mu = 1 - \cos p_\mu \) (see Eq. (14)), and implicitly assumes that the required analyticity properties are inherited in new variables (see case (b) on page 6 of “v3”). In the proof one would expect the required analyticity properties to be defined, as well as careful justification that the above change of variables preserves them.

(c) Eq. (18) of “v3” introduces the following polynomial decomposition for symmetric polynomial \( K(c_1, c_2) \) (the fact that \( K \) must be symmetric has not been justified)

\[
K(c_1, c_2) = c_1 X(c_1, c_2) + c_2 X(c_2, c_1)
\]

3By “Hermiticity”, the author of “v3” perhaps means \( \gamma_5 \)-Hermiticity. However, there doesn’t appear to be a good reason or necessity to assume either of these (none is assumed Refs [1-3]).

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Since $K(0,0) = 0$, the required polynomial $X$ exists, but this representation is not unique. There are infinitely many $X$ that represent the same $K$, for example, $X \rightarrow X + c_2(c_1 - c_2)$. How then is $X$ fixed?

(d) The crucial part of the argument presented in STEP 2 of “v3” is the identification expressed by Eq. (22), which is supposed to follow from Eqs. (16,17) and (21). Unless there are hidden assumptions and arguments, there doesn’t appear to be any reason why this identification should hold. If some symmetric polynomial $P(c_1,c_2)$ can be written in terms of another polynomial $Q(c_1,c_2)$ as

$$P(c_1,c_2) = c_1(2-c_1)Q(c_1,c_2) + c_2(2-c_2)Q(c_2,c_1)$$

then for similar reasons to those discussed in item (c) above, there are infinitely many polynomials $Q$ that can be used for this decomposition. Since there is no uniqueness, how does the Eq. (22) follow? If a conclusion of such nature is possibly justifiable, then the proof would seem to require an explicit argument to that effect.

It should be emphasized in closing that the above objections are not aimed at excessive improvements of rigor in “v3”. The aim is to point out that there appear to be serious holes in the arguments, raising the worry that the claim of “v3” might simply not be justified at all. The problem of “strong non-ultralocality” of GW fermions is an important issue and it would be inherently useful to resolve it cleanly (by either giving a proof or a counterexample). Hopefully, the remarks in this Comment can contribute to eventually achieving that goal.

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References

[1] I. Horváth, Phys. Rev. Lett. 81 (1998) 4063.

[2] I. Horváth, Phys. Rev. D60 (1999), 034510.

[3] I. Horváth, hep-lat/9912030.

[4] W. Bietenholz, hep-lat/9901005 v1.

[5] W. Bietenholz, hep-lat/9901005 v2.

[6] W. Bietenholz, hep-lat/9901005 v3.

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4It also appears that the generic case $n_1 = n_2 = 0$ of “v3” should be discussed separately, because the polynomial structure is different.