Non-commutative deformation of Chern-Simons theory

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Abstract
The problem of the consistent definition of gauge theories living on the non-commutative (NC) spaces with a non-constant NC parameter $\Theta(x)$ is discussed. Working in the $L_\infty$ formalism we specify the undeformed theory, 3d abelian Chern-Simons, by setting the initial $\ell_1$ brackets. The deformation is introduced by assigning the star commutator to the $\ell_2$ bracket. For this initial set up we construct the corresponding $L_\infty$ structure which defines both the NC deformation of the abelian gauge transformations and the field equations covariant under these transformations. To compensate the violation of the Leibniz rule one needs the higher brackets which are proportional to the derivatives of $\Theta$. Proceeding in the slowly varying field approximation when the star commutator is approximated by the Poisson bracket we derive the recurrence relations for the definition of these brackets for arbitrary $\Theta$. For the particular case of $su(2)$-like NC space we obtain an explicit all orders formulas for both NC gauge transformations and NC deformation of Chern-Simons equation which is non-Lagrangian.
1 Introduction

In the standard approach to the definition of the gauge theory one needs the notion of the covariant derivative, \( D_a = \partial_a - i A_a \), as a generalization of the partial derivative \( \partial_a \) which transforms covariantly, \( D_a \rightarrow e^{i f(x)} D_a \), under the gauge transformations \( \delta f A_a = \delta A_a \). This notion is based on the Leibniz rule. The non-commutativity is a fundamental feature of the space-time which manifests itself at the very short distances. It can be introduced in the theory through the star product,

\[
\langle f \star g \rangle = f \cdot g + \frac{i}{2} \Theta^{ab}(x) \partial_a f \partial_b g + \ldots ,
\]

where \( \Theta^{ab}(x) \) is the non-commutativity parameter depending on the specific physical model. In some cases, like the open string dynamics in the constant \( B \)-field \([1]\), the non-commutativity parameter can be constant, however in general it is a function of coordinates. The coordinate dependence of \( \Theta \), in general, leads to the violation of the Leibniz rule,

\[
\partial_c (\langle f \star g \rangle) = (\partial_c f) \star g + f \star (\partial_c g) + \frac{i}{2} \partial_c \Theta^{ab}(x) \partial_a f \partial_b g + \ldots ,
\]
and makes impossible to follow the standard path for the formulation of NC
gauge theory. Let us note that in some particular cases, like the NC gauge theory
on D-branes in non-geometric backgrounds \[2\] the type of non-commutativity is
compatible with the Leibniz rule, so the standard reasoning can be used for the
definition of the NC field strength. At that, because of the non-geometry one has
to shift the field strength tensor by a closed two-form on the D-brane worldvolume
to construct the NC Yang-Mills action.

The problem with the violation of the Leibniz rule can be taken under control
if, e.g., instead of the partial derivative \(\partial_a\) one takes the inner one defined through
the star commutator, \(D_a = i[\cdot, x_a]_{\ast}\), like it was done in the approach of covariant
coordinates \[3\]. This however may lead to the problem with the correct commu-
tative limit. Another possibility discussed in the literature consists in using the
deformed Leibniz rule constructed with the help of the twist element of the Hopf
algebra \[4, 5\]. Here we mention that the twist element is known for the very few
examples of NC spaces \[6\].

In the recent work \[7\] in collaboration with Ralph Blumenhagen, Ilka Brunner
and Dieter Lüst we have formulated the \(L_\infty\)-bootstrap approach to the construc-
tion of non-commutative gauge theories. On the one hand, in the physical litera-
ture \(L_\infty\) structures were introduced for description of gauge theories \[8\], see also
\[9, 10\] for more details and recent references. On the classical level it contains all
necessary information about the theory including the gauge symmetry, the field
equations and the Noether identities. On the other hand, \(L_\infty\) algebras or the
strong homotopy Lie algebras \[11, 12\] is a natural framework for dealing with the
deformation since the Jacobi identities are required to hold only up to the total
derivative or the higher coherent homotopy. We note in particular that the proof
of the key result in deformation quantization, the Formality Theorem, is based
on the concept of \(L_\infty\) algebras \[13\].

The main idea of the \(L_\infty\) bootstrap approach consists in two steps. The
first one is to represent the original undeformed gauge theory, like the Chern-
Simons or the Yang-Mills, as well as the deformation introduced through the star
commutator as a part of a new \(L_\infty\) algebra by specifying the initial brackets \(\ell_1,\)
\(\ell_2,\) etc. Then solving the \(L_\infty\) relations (the higher Jacobi identities), \(\mathcal{J}_n = 0,\)
one determines the missing brackets \(\ell_n\) and completes the \(L_\infty\) algebra which
governs the NC deformation of the gauge transformations and the equations
of motion. In \[7\] we found the expressions for the gauge transformations and
the field equations up to the order \(O(\Theta^2)\) in the non-commutativity parameter.
However the calculations were extremely involved and it was not clear whether
the procedure can be extended to the higher or potentially all orders in \(\Theta\).

The purpose of the current work is to develop the ideas proposed in \[7\] in part
of the existence of the solution for the \(L_\infty\) bootstrap program, its construction
and the explicit examples. The key observation we made is that in each given order
\(n\) the consistency condition for the \(L_\infty\) bootstrap equations, \(\mathcal{J}_n = 0,\) is satisfied
as a consequence of the previously solved relations, \(\mathcal{J}_m = 0,\) with \(m \leq n\). We use
it to express the brackets \( \ell_n \) in terms of those which have been already found \( \ell_m \), \( m \leq n \). Aiming to provide explicit calculations we work in the slowly varying field approximation when the higher derivative terms in the star commutator are discarded and it is approximated by the Poisson bracket, so we set,

\[
\ell_2(f, g) = -\{f, g\} = -\Theta^{ab}(x) \partial_a f \partial_b g.
\]  

(1.3)

The construction of the algebra \( L_{\text{gauge}}^{\infty} \) describing the NC deformation of the abelian gauge transformations was previously discussed in the proceedings of the Durham Symposium on Higher Structures in M-Theory \[14\]. We provide it in the Section 3 for the completeness. The essentially new results regarding the derivation of the algebra \( L_{\text{full}}^{\infty} \) which also includes the equations of motion are contained in the Sections 4 and 5. In the Section 4 we discuss the NC deformation of the 3d abelian Chern-Simons theory for arbitrary \( \Theta \). The specific case of the rotation invariant NC space is analyzed in the Section 5. In this case the coordinates satisfy the \( su(2) \) algebra and thus, \( \Theta^{ab}(x) = 2 \theta \varepsilon^{abc} x_c \), with \( \theta \) being the small parameter. An important algebraic relations involving the Levi-Civita tensor \( \varepsilon^{abc} \) and arbitrary vector \( A^e \) are given in the Appendix. These relations allowed us to find an explicit all order expressions for the NC gauge transformations satisfying the relation,

\[
[\delta_f, \delta_g]A_a = \delta_{\{f,g\}}A_a,
\]

(1.4)

and the field equations, \( \mathcal{F}^a = 0 \), covariant under these gauge transformations, i.e., \( \delta_f \mathcal{F}^a = \{\mathcal{F}^a, f\} \), and reproducing in the commutative limit, \( \theta \to 0 \), the standard Chern-Simons equations, \( \varepsilon^{abc} \partial_b A_c = 0 \).

## 2 Basic facts from \( L_{\infty} \)-algebras

For the convenience of the reader in this Section we will briefly review the basic facts form the theory of \( L_{\infty} \)-algebras and its relation to the gauge theories. We start with a formal definition. In fact, \( L_{\infty} \)-algebras are generalized Lie algebras where one has not only a two-bracket, that is the commutator, but more general multilinear \( n \)-brackets with \( n \) inputs

\[
\ell_n : X^\otimes n \to X
\]

\[
x_1, \ldots, x_n \mapsto \ell_n(x_1, \ldots, x_n),
\]

(2.1)

defined on a graded vector space \( X = \bigoplus_m X_m \), where \( m \in \mathbb{Z} \), denotes the grading of the corresponding subspace. Each element \( x \in X \), has its own degree, meaning that if \( \text{deg}(x) = p \), this element belongs to the subspace \( X_p \). The concept of the degree is essential for the definition of the products \( \ell_n \). First, because these brackets are graded anti-symmetric according to,

\[
\ell_n(\ldots,x_1,x_2,\ldots) = (-1)^{1+\text{deg}(x_1)\text{deg}(x_2)} \ell_n(\ldots,x_2,x_1,\ldots).
\]

(2.2)
And second, because the result \( \ell_n(x_1, \ldots, x_n) \in X_p \), with
\[
p = \deg(\ell_n(x_1, \ldots, x_n)) = n - 2 + \sum_{i=1}^{n} \deg(x_i).
\]
(2.3)

The set of higher brackets \( \ell_n \) define an \( \mathbb{L}_\infty \) algebra, if they satisfy the infinitely many relations
\[
J_n(x_1, \ldots, x_n) := \sum_{i+j=n+1} (-1)^{(i-1)} \sum_{\sigma} (-1)^{\sigma} \chi(\sigma; x)
\]
\[
\ell_j(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0.
\]
(2.4)

The permutations are restricted to the ones with
\[
\sigma(1) < \cdots < \sigma(i), \quad \sigma(i+1) < \cdots < \sigma(n),
\]
(2.5)
and the sign \( \chi(\sigma; x) = \pm 1 \) can be read off from (2.2). The first relations \( J_n \) with \( n = 1, 2, 3, \ldots \) can be schematically written as
\[
J_1 = \ell_1 \ell_1, \quad J_2 = \ell_1 \ell_2 - \ell_2 \ell_1, \quad J_3 = \ell_1 \ell_3 + \ell_2 \ell_2 + \ell_3 \ell_1,
\]
\[
J_4 = \ell_1 \ell_4 - \ell_2 \ell_3 + \ell_3 \ell_2 - \ell_4 \ell_1,
\]
(2.6)

from which one can deduce the scheme for the higher Jacobi identities \( J_n \). More precisely, denoting \((-1)^{x_i} = (-1)^{\deg(x_i)}\), the first two \( \mathbb{L}_\infty \) relations read
\[
\ell_1(\ell_1(x)) = 0
\]
\[
\ell_1(\ell_2(x_1, x_2)) = \ell_2(\ell_1(x_1), x_2) + (-1)^{x_1} \ell_2(x_1, \ell_1(x_2)),
\]
(2.7)

which means that that \( \ell_1 \) is a nilpotent derivation with respect to the bracket \( \ell_2 \), and that in particular the Leibniz rule is satisfied. The full relation \( J_3 \) reads
\[
0 = \ell_1(\ell_3(x_1, x_2, x_3))
\]
\[
+ \ell_2(\ell_2(x_1, x_2), x_3) + (-1)^{(x_2+x_3)x_1} \ell_2(\ell_2(x_2, x_3), x_1)
\]
\[
+ (-1)^{(x_1+x_2)x_3} \ell_2(\ell_2(x_3, x_1), x_2)
\]
\[
+ \ell_3(\ell_1(x_1), x_2, x_3) + (-1)^{x_1} \ell_3(x_1, \ell_1(x_2), x_3) + (-1)^{x_1+x_2} \ell_3(x_1, x_2, \ell_1(x_3))
\]
(2.8)
and means that the Jacobi identity for the $\ell_2$ bracket holds up to $\ell_1$ exact terms. For the future needs we will also provide here the complete form of the $\mathcal{J}_4$ relation,

$$0 = \ell_1\left(\ell_4(x_1, x_2, x_3, x_4)\right)$$  \hspace{1cm} (2.9)

\begin{align*}
&-\ell_2\left(\ell_3(x_1, x_2, x_3), x_4\right) + (-1)^{x_3x_3}\ell_2\left(\ell_3(x_1, x_2, x_4), x_3\right) \\
&+(-1)^{(1+x_1)x_2}\ell_2\left(x_2, \ell_3(x_3, x_1), x_2\right) - (-1)^{x_1}\ell_2(x_1, \ell_3(x_2, x_3, x_4)) \\
&+\ell_3\left(\ell_2(x_1, x_2), x_3, x_4\right) + (-1)^{1+x_2x_3}\ell_3\left(\ell_2(x_1, x_3), x_2, x_4\right) \\
&+(-1)^{x_4(x_2+x_3)}\ell_3\left(\ell_2(x_1, x_4), x_2, x_3\right) - \ell_3(x_1, \ell_2(x_2, x_3), x_4) \\
&-\ell_4\left(\ell_1(x_1), x_2, x_3, x_4\right) - (-1)^{x_1}\ell_4(x_1, \ell_1(x_2), x_3, x_4) \\
&-(-1)^{x_1+x_2}\ell_4(x_1, x_2, \ell(x_3), x_4) - (-1)^{x_1+x_2+x_3}\ell_4(x_1, x_2, x_3, \ell(x_4)).
\end{align*}

The framework of $L_\infty$ algebras is quite flexible and it has been suggested that every classical perturbative gauge theory (derived from string theory), including its dynamics, is organized by an underlying $L_\infty$ structure. To see this, let us assume that the field theory has a standard type gauge structure, meaning that the variations of the fields can be organized unambiguously into a sum of terms each of a definite power in the fields. First we choose only two non-trivial vector spaces as

$$X_0 \quad X_{-1}$$

$$f \quad A_a,$$

(2.10)

where physically $X_0$ corresponds to the space of gauge parameters or functions $f$, and $X_{-1}$ contains the gauge fields $A_a$. Note that in this case $\ell_1(f) \in X_{-1}$ and can be non-zero, while $\ell_1(A) \in X_{-2}$, which is empty by now, i.e., $\ell_1(A) = 0$, by the construction. In this case, the only allowed non-trivial higher bracket are the ones with one gauge parameter $\ell_{n+1}(f, A^n) \in X_{-1}$, and two gauge parameters $\ell_{n+2}(f, g, A^n) \in X_0$. The graded symmetry in this case means,

$$\ell_n(\ldots, f, g, \ldots) = (-1)^{1+\deg(f)\cdot\deg(g)}\ell_n(\ldots, g, f, \ldots) = -\ell_n(\ldots, g, f, \ldots),$$

$$\ell_n(\ldots, f, A, \ldots) = -\ell_n(\ldots, A, f, \ldots),$$

$$\ell_n(\ldots, A, B, \ldots) = \ell_n(\ldots, B, A, \ldots).$$

The non-trivial $L_\infty$ relations are

$$\mathcal{J}_{n+2}(f, g, A^n) = 0 \quad \text{and} \quad \mathcal{J}_{n+3}(f, g, h, A^n) = 0,$$

with $\mathcal{J}_{n+2}(f, g, A^n) \in X_{-1}$, and $\mathcal{J}_{n+3}(f, g, h, A^n) \in X_0.$
The gauge variations are defined in terms of the brackets $\ell_{n+1}(f, A^n) \in X_{-1}$ as follows,

$$
\delta_f A = \sum_{n \geq 0} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_{n+1}(f, A, \ldots, A)_{n \text{ times}}
$$

$$
= \ell_1(f) + \ell_2(f, A) - \frac{1}{2} \ell_3(f, A, A) + \ldots.
$$

It was shown in [9, 15, 16], that the $L_\infty$ relations with two gauge parameters, $\mathcal{J}_{n+2}(f, g, A^n) = 0$, imply the off-shell closure of the symmetry variations

$$
[\delta_f, \delta_g] A = \delta_{-C(f, g, A)} A,
$$

where

$$
C(f, g, A) = \sum_{n \geq 0} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_{n+2}(f, g, A, \ldots, A)_{n \text{ times}}.
$$

Here we stress that the closure relation allows for a field dependent gauge parameter. The Jacobi identity for gauge variations

$$
\sum_{\text{cycl}} [\delta_f, [\delta_g, \delta_h]] \equiv 0,
$$

is equivalent to the $L_\infty$ relations with three gauge parameters $\mathcal{J}_{n+3}(f, g, h, A^n) = 0$. Thus, we see that the action of gauge symmetries on the fundamental fields is governed by an $L_\text{gauge}^\infty$ algebra.

We stress that in principle, the $L_\infty$ algebra may have an infinite number of the brackets $\ell_n$, which however, are not arbitrary, since should satisfy $L_\infty$ relations (2.4). As it was already mentioned in the introduction the idea of the $L_\infty$ bootstrap approach consists in representing the original undeformed gauge theory together with a deformation as a part of a new $L_\infty$ structure by setting initial brackets and solving $L_\infty$ relations to determine the algebra $L_\infty^{new}$, which corresponds to the consistent deformation of the original theory.

### 3 Non-commutative deformation of the abelian gauge transformations

To define the undeformed model, the abelian gauge algebra, we set the bracket $\ell_1(f) = \partial_a f$. The non-commutative deformation is introduced through the star commutator of functions which, from the consideration of anti-symmetry, should be assigned to the bracket $\ell_2(f, g) = i[f, g]_\star$. Just for the simplicity let us consider the limit of slowly varying, but not necessarily small gauge fields, i.e., we discard
the higher derivatives terms in the star commutator and take, \( \ell_2(f, g) = -\{f, g\} \), as a (quasi)-Poisson bracket defined in (1.3). This is a “self-consistent” approximation of non-commutativity since the main algebraic properties of the model are preserved. If we work with the NC deformations induced by the associative star product, the star commutator satisfies the Jacobi identity, so as the corresponding Poisson bracket.

Having non-vanishing brackets \( \ell_1(f) \) and \( \ell_2(f, g) \), one has to check the \( L_\infty \) relation, \( J_2(f, g) = 0 \), involving yet undetermined bracket \( \ell_2(f, A) \). It means that now the identity, \( J_2(f, g) = 0 \), becomes an equation on \( \ell_2(f, A) \). Solving this equation one may proceed to the next \( L_\infty \) relation, \( J_3(f, g, h) = 0 \), and define the next bracket \( \ell_3(f, g, A) \), etc. The procedure should be continued till no new bracket can be determined and all \( L_\infty \) relations are satisfied. Let us see how it works on practice.

### 3.1 Leading order contribution

The relation \( J_2(f, g) = 0 \), reads,

\[
\ell_1(\ell_2(f, g)) = -\left\{ \frac{\partial}{\partial_a} f, g \right\} - \left\{ f, \frac{\partial}{\partial_a} g \right\} - (\partial_i \Theta^{ij}) \partial_i f \partial_j g = \ell_2(\ell_1(f), g) + \ell_2(f, \ell_1(g)),
\]

from which one finds

\[
\ell_2(f, A) = -\{f, A_a\} - \frac{1}{2}(\partial_i \Theta^{ij}) \partial_i f A_j. \tag{3.2}
\]

Note that the solution is not unique, one may also set, e.g.,

\[
\ell'_2(f, A) = \ell_2(f, A) + s^i_a(x) \partial_i f A_j, \tag{3.3}
\]

with \( s^i_a(x) = s^a_i(x) \). By the definition of \( L_\infty \), \( \ell'_2(A, f) := -\ell'_2(f, A) \). The symmetry of \( s^i_a(x) \) implies that this choice of the bracket \( \ell'_2(f, A) \) also satisfies the equation (3.1). However, the symmetric part \( s^i_a(x) \partial_i f A_j \) can be always “gauged away” by \( L_\infty \)-quasi-isomorphism, physically equivalent to Seiberg-Witten map [1], see [17] for more details.

### 3.2 Next to the leading order

Then we have to define the bracket \( \ell_3(f, g, A) \) from the identity \( J_3(f, g, h) = 0 \), which reads,

\[
0 = \ell_2(\ell_2(f, g), h) + \ell_2(\ell_2(g, h), f) + \ell_2(\ell_2(h, f), g) + \ell_3(\ell_1(f), g, h) + \ell_3(f, \ell_1(g), h) + \ell_3(f, g, \ell_1(h)).
\]

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For simplicity, we replace it with

\[ \ell_2(\ell_2(f, g), h) + \ell_2(\ell_2(g, h), f) + \ell_2(\ell_2(h, f), g) = -\Pi^{ijk} \partial_i f \partial_j g \partial_k h. \]  

(3.5)

For associative non-commutative deformations we may just set, \( \ell_3(A, f, g) = 0 \), while in the non-associative case one needs non-vanishing bracket \( \ell_3(A, f, g) \) to satisfy it. We define

\[ \ell_3(A, f, g) = \frac{1}{3} \Pi^{ijk} A_i \partial_j f \partial_k g. \]  

(3.6)

The next step is the crucial for the whole construction. We have to analyze the relation \( \mathcal{J}_3(f, g, A) = 0 \), given by

\[ 0 = \ell_2(\ell_2(A, f), g) + \ell_2(\ell_2(f, g), A) + \ell_2(\ell_2(g, A), f) + \ell_1(\ell_3(A, f, g)) - \ell_3(A, \ell_1(f), g) - \ell_3(A, f, \ell_1(g)). \]  

(3.7)

For simplicity, we replace it with \( \mathcal{J}_3(g, h, \ell_1(f)) = 0 \), written in the form

\[ \ell_3(\ell_1(f), \ell_1(g), h) - \ell_3(\ell_1(f), \ell_1(h), g) = G(f, g, h), \]  

(3.8)

\[ G(f, g, h) := \ell_1(\ell_3(\ell_1(f), g, h)). \]

We will follow the logic of [18] for the solution of the above algebraic equation. By construction, the equation (3.8) is antisymmetric with respect to the permutation of \( g \) and \( h \). The graded symmetry of the \( \ell_3 \) bracket, \( \ell_3(\ell_1(f), \ell_1(g), h) = \ell_3(\ell_1(g), \ell_1(f), h) \), implies the identity on the l.h.s. of (3.8):

\[ \ell_3(\ell_1(f), \ell_1(g), h) - \ell_3(\ell_1(f), \ell_1(h), g) + \ell_3(\ell_1(h), \ell_1(f), g) - \ell_3(\ell_1(h), \ell_1(g), f) + \ell_3(\ell_1(g), \ell_1(h), f) - \ell_3(\ell_1(g), \ell_1(f), h) = 0. \]

(3.9)

Which in turn requires the graded cyclicity of r.h.s. of the eq. (3.8),

\[ G(f, g, h) + G(h, f, g) + G(g, h, f) = 0. \]  

(3.9)

The latter is nothing but the consistency condition for the eq. (3.8).

It is remarkable that the consistency condition (3.9) follows from the previously satisfied \( L_\infty \) relations, namely \( \mathcal{J}_2(f, g) = 0 \), and \( \mathcal{J}_3(g, h, f) = 0 \). Indeed, taking the definition of \( G(f, g, h) \), one writes

\[ G(f, g, h) + G(h, f, g) + G(g, h, f) = 0. \]  

(3.9)
Using $\mathcal{J}_2(f, g) = 0$, we may push $\ell_1$ out of the brackets and rewrite it as

$$
\ell_1 \left[ \ell_2(f, g, h) + \ell_2(g, h, f) + \ell_2(h, f, g) \right] =
\ell_1 \left[ \mathcal{J}_3(f, g, h) \right] \equiv 0.
$$

Which means that the consistency condition [3.9] holds true as a consequence of the previously satisfied $L_\infty$ relations. Taking into account [3.9] one may easily check that the following expression (symmetrization in $f$ and $g$ of the r.h.s. of the eq. (3.8)):

$$
\ell_3(\ell_1(f), \ell_1(g), h) = -\frac{1}{6} \left( G(f, g, h) + G(g, f, h) \right),
$$

has required graded symmetry and solves $\mathcal{J}_3(g, h, \ell_1(f)) = 0$.

Setting

$$
\ell_3(A, B, f) = \ell_3(\ell_1(f), \ell_1(g), h) |_{\ell_1(f) = A; \ell_1(g) = B},
$$

one gets,

$$
\ell_3(A, B, f) = -\frac{1}{6} \left( G_a{}^{ijk} + G_a{}^{jik} \right) A_i B_j \partial_k f
$$

$$
+ \frac{1}{6} \Pi^{ijk}(\partial_i A_k B_j \partial_k f - A_i \partial_j B_k \partial_k f) - \frac{1}{2} \Pi^{ijk}(\partial_i A_k B_j \partial_k f - A_i \partial_j B_k \partial_k f),
$$

with

$$
G_a{}^{ijk} = \frac{1}{3} \partial_a \Pi^{ijk} - \Theta^i m \partial_m \Theta^j k - \frac{1}{2} \partial_a \Theta^j m \partial_m \Theta^k i - \frac{1}{2} \partial_a \Theta^k m \partial_m \Theta^i j.
$$

At this point we would like to stress two main observations:

- The consistency condition (graded cyclicity) [3.9] holds true as a consequence of $L_\infty$ construction.
- Even in the associative case one needs higher brackets to compensate the violation of the standard Leibniz rule.

### 3.3 Higher relations

Once the brackets $\ell_3(f, g, A)$ and $\ell_3(f, A, B)$ are determined we may proceed to the next $L_\infty$ relation and find the brackets with four entries, $\ell_4$. First we analyze $\mathcal{J}_4(f, g, h, A) = 0$, which we rewrite in the form $\mathcal{J}_4(f, g, h, \ell_1(k)) = 0$. Taking into account [2.9] we write it explicitly as:

$$
\ell_4(\ell_1(f), g, h, \ell_1(k)) + \ell_4(f, \ell_1(g), h, \ell_1(k)) + \ell_4(f, g, \ell_1(h), \ell_1(k)) (3.13)
= F(f, g, h, k),
$$
with
\[ F(f, g, h, k) = \ell_2(\ell_4(f, g, \ell_1(k)), h) + \ell_2(g, \ell_3(f, h, \ell_1(k))) \\
- \ell_2(f, \ell_3(g, h, \ell_1(k))) + \ell_3(\ell_2(f, g), h, \ell_1(k)) - \ell_3(\ell_2(f, h), g, \ell_1(k)) \\
+ \ell_3(\ell_2(f, \ell_1(k)), g, h) - \ell_3(f, \ell_2(g, h, \ell_1(k))) + \ell_3(f, \ell_2(g, \ell_1(k)), h) \\
+ \ell_3(g, \ell_2(h, \ell_1(k))) . \]

The explicit form is given by
\[ F(f, g, h, k) = F_{ijkl} \partial_i f \partial_j g \partial_k h \partial_l k , \quad (3.14) \]
where
\[ 3F_{ijkl} = \Theta^{km} \partial_m \Pi^{ijl} + \Theta^{jm} \partial_m \Pi^{kil} + \Theta^{im} \partial_m \Pi^{jkl} \quad (3.15) \]
\[ \Pi^{kml} \partial_m \Theta^{ijl} + \Pi^{jml} \partial_m \Theta^{kili} + \Pi^{iml} \partial_m \Theta^{jkl} \]
\[ \frac{1}{2} \Pi^{ijm} \partial_m \Theta^{kli} + \frac{1}{2} \Pi^{jkl} \partial_m \Theta^{iml} + \frac{1}{2} \Pi^{km} \partial_m \Theta^{jil} . \]

Solution of the algebraic equations of the type (3.13) was given in [19]. By the construction \( F(f, g, h, k) \) is antisymmetric in first three arguments and the graded symmetry of \( \ell_4(\ell_1(f), g, h, \ell_1(k)) \) implies the graded cyclicity (consistency condition) for \( F(f, g, h, k) \), which now reads:
\[ F(f, g, h, k) - F(k, f, g, h) + F(h, k, f, g) - F(g, h, k, f) = 0 . \quad (3.16) \]

Again, the consistency condition (3.16) holds true as a consequence of the previous L\( \infty \) relations, graded symmetry and multilinearity of the brackets \( \ell_n \).

As previously the solution of (3.13) is constructed by taking the corresponding symmetrization of the r.h.s.:
\[ \ell_4(\ell_1(f), g, h, \ell_1(k)) = \frac{1}{8} (F(f, g, h, k) + F(k, g, h, f)) . \]

Then, setting
\[ \ell_4(A, g, h, B) = \ell_4(\ell_1(f), g, h, \ell_1(k))|_{\ell_1(f)=A; \ell_1(g)=B} \]
we conclude that
\[ \ell_4(A, g, h, B) = \left[ \frac{1}{16} \Pi^{jlm} \partial_m \Theta^{kij} + \frac{1}{16} \Pi^{jkm} \partial_m \Theta^{lij} - \frac{1}{16} \Pi^{ilm} \partial_m \Theta^{kjl} - \frac{1}{16} \Pi^{jkm} \partial_m \Theta^{lij} + \frac{1}{24} \Theta^{km} \partial_m \Pi^{ijl} - \frac{1}{24} \Theta^{lm} \partial_m \Pi^{ijk} \right] \partial_i g \partial_j f A_k B_l . \quad (3.17) \]

To complete the picture in this order let us also consider the L\( \infty \) relation:
\[ \mathcal{J}_4(f, g, A, B) = 0 , \]
which we replace with \( \mathcal{J}_4(f, g, \ell_1(h), \ell_1(k)) = 0 \), and write in the form of the equation:
\[ \ell_4(\ell_1(f), g, \ell_1(h), \ell_1(k)) - \ell_4(f, \ell_1(g), \ell_1(h), \ell_1(k)) = G(f, g, h, k) , \quad (3.18) \]
where

\[
G(f, g, h, k) = \ell_1(\ell_4(f, g, \ell_1(h), \ell_1(k)) \\
- \ell_2(\ell_3(f, g, \ell_1(h)), \ell_1(k)) - \ell_2(\ell_3(f, g, \ell_1(k)), \ell_1(h)) \\
+ \ell_2(\ell_3(f, \ell_1(h), \ell_1(k))) - \ell_2(\ell_3(g, \ell_1(h), \ell_1(k))) \\
- \ell_3(\ell_2(f, \ell_1(h)), g, \ell_1(k)) - \ell_3(\ell_2(\ell_1(k)), g, \ell_1(h)) \\
- \ell_3(\ell_2(g, \ell_1(h)), \ell_1(k)) - \ell_3(\ell_2(g, \ell_1(k)), \ell_1(h)) \\
+ \ell_3(\ell_2(f, g), \ell_1(h), \ell_1(k)) .
\]

By the construction, \(G(f, g, h, k)\) is antisymmetric in first two and symmetric in last two arguments, and as a consequence of the previous \(L_\infty\) relations it satisfies the graded cyclicity relation:

\[
G(f, g, h, k) + G(g, h, f, k) + G(h, f, g, k) = 0 .
\quad (3.19)
\]

Taking into account (3.19) one may check that the symmetrization in the last three arguments of the r.h.s. of the eq. (3.18),

\[
\ell_4(\ell_1(g), \ell_1(h), \ell_1(k)) = \frac{1}{12} (G(f, g, h, k) + G(f, h, k, g) + G(f, k, g, h)) ,
\quad (3.20)
\]

has the required graded symmetry and satisfies the equation in question.

### 3.4 Recurrence relations

For the higher relations, \(J_{n+2}(g, h, A^n) = 0\), we proceed in the similar way. First we substitute them by the equations \(J_{n+2}(g, h, \ell_1(f)^n) = 0\), which can be represented in the form

\[
\ell_{n+2}(\ell_1(f)^n, \ell_1(g), h) - \ell_{n+2}(\ell_1(f)^m, \ell_1(h), g) = G(f_1, \ldots, f_n, g, h) ,
\quad (3.21)
\]

where the right hand side, \(G(f_1, \ldots, f_n, g, h)\), is defined in terms of the previously defined brackets \(\ell_{m+2}(\ell_1(f)^m, \ell_1(g), h)\), with \(m < n\). It is symmetric in the first \(n\) arguments and antisymmetric in the last two by the construction. The graded symmetry of \(\ell_{n+2}(\ell_1(f)^n, \ell_1(g), h)\) implies the non-trivial consistency condition (since \(G(f_1, \ldots, f_n, g, h)\) is symmetric in first \(n\) arguments, one needs to check the cyclicity relation with respect to the permutation of the last three slots),

\[
G(f_1, \ldots, f_n, g, h) + G(f_1, \ldots, f_{n-1}, g, h, f_n) + G(f_1, \ldots, f_{n-1}, h, f_n, g) = 0 ,
\quad (3.22)
\]

which follows from the previous \(L_\infty\) relations and can be proved by induction.
Following [18] the solution of the equation (3.21) can be constructed taking the symmetrization of the r.h.s. in the first $n + 1$ arguments, i.e.,

$$
\ell_{n+2}(\ell_1(f)^n, \ell_1(g), h) = -\frac{1}{(n+1)(n+2)} \left( G(f_1, \ldots, f_n, g, h) \right) (3.23)
$$

$$
+ G(f_2, \ldots, f_n, g, f_1, h) + \cdots + G(f_n, \ldots, f_{n-1}, h).
$$

And finally we obtain the expression for $\ell_{n+2}(f, A^{n+1})$, substituting in the above expression all $\ell_1(f)$ with the corresponding fields $A$.

The identities with three gauge parameters $J_{n+3}(f, g, h, A^n) = 0$, $n > 1$, are substituted by the relations $J_{n+3}(f, g, h, \ell_1(k)^n) = 0$, written in the form:

$$
\ell_{n+3}(\ell_1(f), g, h, \ell_1(k)^n) + \ell_{n+3}(f, \ell_1(g), h, \ell_1(k)^n)
$$

$$
+ \ell_{n+3}(f, g, \ell_1(h), \ell_1(k)^n) = F(f, g, h, k_1, \ldots, k_n) .
$$

The r.h.s. $F(f, g, h, k_1, \ldots, k_n)$ is antisymmetric in first three arguments and symmetric in last $n$ arguments, and also should satisfy the graded cyclicity relation,

$$
F(f, g, h, k_1, \ldots, k_n) - F(k_1, f, g, h, k_2, \ldots, k_n)
$$

$$
+F(h, k_1, f, g, k_2, \ldots, k_n) - F(g, h, k_1, f, k_2, \ldots, k_n) = 0 ,
$$

which as before follows from the previous $L_\infty$ relations, graded symmetry and multi-linearity of the brackets $\ell_n$. The solution of (3.24) is constructed by taking the corresponding symmetrization of the r.h.s:

$$
\ell_{n+3}(f, g, h, \ell_1(k)^n) = -\frac{1}{n(n+2)} \left( F(f, g, h, k_1, \ldots, k_n) \right) (3.26)
$$

$$
+ F(f, g, k_1, \ldots, k_n, h) + \cdots + F(f, g, k_n, h, k_1, \ldots, k_{n-1}) .
$$

Again the expression for $\ell_{n+3}(f, g, A^{n+1})$ is obtained from (3.26) substituting all $\ell_1(f)$ by the fields $A$.

### 4 Non-commutative field dynamics and $L_\infty$ structure

It is remarkable that the dynamics of the theory, i.e. the equations of motion, are also expected to fit into an extended $L_{\infty}^{\text{full}}$ algebra. For this purpose one extends the vector space to $X_0 \oplus X_{-1} \oplus X_{-2}$

$$
X_0 \quad X_{-1} \quad X_{-2}
$$

$$
f \quad A_a \quad E_a
$$

(4.1)
where $X_{-2}$ also contains the equations of motion, i.e. $\mathcal{F} \in X_{-2}$. Now more higher brackets, namely $\ell_n(A^n) \in X_{-2}$, $\ell_{n+2}(f, E, A^n) \in X_{-2}$, and $\ell_{n+3}(f, g, E, A^n) \in X_{-1}$, can be non-trivial and should satisfy the following identities

$$J_{n+1}(f, A^n) = 0 \quad \text{and} \quad J_{n+2}(f, E, A^n) = 0. \quad (4.2)$$

The higher brackets $\ell_n(A^n)$ are special since they define the equation of motion, $\mathcal{F} = 0$, where

$$\mathcal{F} := \sum_{n \geq 1} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_n(A^n) = \ell_1(A) - \frac{1}{2} \ell_2(A^2) - \frac{1}{3!} \ell_3(A^3) + \ldots. \quad (4.3)$$

Now the $L_\infty$ structure admits that the closure condition (2.12) is only satisfied on-shell, i.e. there can be terms $\ell_{n+3}(f, g, \mathcal{F}, A^n) \in X_{-1}$ on the right hand side. The gauge variation of $\mathcal{F}$ reads

$$\delta_f \mathcal{F} = \ell_2(f, \mathcal{F}) + \ell_3(f, \mathcal{F}, A) - \frac{1}{2} \ell_4(f, \mathcal{F}, A^2) + \ldots \quad (4.4)$$

reflecting that, as opposed to the gauge field $A$, it transforms covariantly.

In this Section we discuss the consistent deformation of the field dynamics, i.e., the construction of $L_\text{full}_\infty$ algebra in the bootstrap approach. First we will make some general statements regarding the consistency condition and the solution of the equations (4.2). Then we will work out the non-commutative deformation of the abelian Chern-Simons theory. In this case we write the initial brackets as

$$\ell_1(f) = \partial_a f, \quad \ell_1(A) = \varepsilon^{ab} \partial_a A_b, \quad \ell_2(f, g) = -\{f, g\}. \quad (4.5)$$

The brackets $\ell_{n+1}(f, A^n)$ and $\ell_{n+2}(f, g, A^n)$ defining the pure gauge algebra $L_{\text{gauge}}_\infty$ were determined in the Section 3. The rest brackets $\ell_n(A^n)$, $\ell_{n+2}(f, E, A^n)$, and $\ell_{n+3}(f, g, E, A^n)$, should be found from the identities (4.2).

### 4.1 Leading order contribution

The first non-trivial $L_\infty$ relation is

$$J_2(f, A) := \ell_1(\ell_2(f, A)) - \ell_2(\ell_1(f), A) - \ell_2(f, \ell_1(A)) = 0, \quad (4.6)$$

which we rewrite as

$$\ell_2(\ell_1(f), A) + \ell_2(f, \ell_1(A)) = \ell_1(\ell_2(f, A)). \quad (4.7)$$

The r.h.s. is given and can be calculated as

$$\ell_1(\ell_2(f, A)) = -\varepsilon^{ab} \{\partial_a f, A_b\} \varepsilon - \{f, \varepsilon^{ab} \partial_a A_b\} \varepsilon - \varepsilon^{ab} \partial_a \Theta^{ij} \partial_i f \partial_j A_b - \frac{1}{2} \varepsilon^{ab} \partial_b \Theta^{ij} \partial_i \partial_a f A_j - \frac{1}{2} \varepsilon^{ab} \partial_b \Theta^{ij} \partial_i \partial_a A_j, \quad (4.8)$$
while the two brackets in the l.h.s. should be determined. The bracket \( \ell_2(f, E) \) should be antisymmetric with respect to the permutation of its arguments, so we identify

\[
\ell_2(f, \ell_1(A)) = -\{f, \epsilon^{ab}_c \partial_a A_b\}, \quad \text{thus} \quad \ell_2(f, E) = -\{f, E_a\}. \tag{4.9}
\]

The rest of the eq. (4.7) can be written in the form

\[
\ell_2(\ell_1(f), A) = P_{aijk}^1 \partial_i f \partial_j A_k + Q_{aijk}^1 A_i \partial_j \partial_k f + R_{aijkl}^1 \partial_i A_j \partial_k \partial_l f, \tag{4.10}
\]

where the coefficient functions \( P_{aijk}^1 \), \( Q_{aijk}^1 \) and \( R_{aijkl}^1 \) are given by

\[
\begin{align*}
P_{aijk}^1 &= \epsilon_{akm} \partial_m \Theta^{ij} - \frac{1}{2} \epsilon^{ajm} \partial_m \Theta^{ik}, \\
Q_{aijk}^1 &= \frac{1}{2} \epsilon^{ajm} \partial_m \Theta^{ki}, \\
R_{aijkl}^1 &= -\epsilon^{ajk} \Theta^{il}.
\end{align*} \tag{4.11}
\]

The solution of the equation (4.10) will be constructed following the logic of the previous section. There is a non-trivial consistency condition coming from the graded symmetry of the bracket \( \ell_2 \), which is satisfied as a consequence of the previously solved \( L_\infty \) relations. The relation \( J_2(f, \ell_1(g)) = 0 \), can be written as

\[
\ell_2(\ell_1(f), \ell_1(g)) = \ell_1(\ell_2(f, \ell_1(g))). \tag{4.12}
\]

The graded symmetry of \( \ell_2 \) bracket,

\[
\ell_2(\ell_1(f), \ell_1(g)) = \ell_2(\ell_1(g), \ell_1(f)), \tag{4.13}
\]

implies the consistency condition on the right hand side of (4.12),

\[
\begin{align*}
\ell_2(\ell_1(f), \ell_1(g)) &= \ell_2(\ell_1(g), \ell_1(f)), \\
\ell_1(\ell_2(f, \ell_1(g))) &= \ell_1(\ell_2(g, \ell_1(f))) = 0. \tag{4.14}
\end{align*}
\]

The later however is automatically satisfied due to JI, \( J_2(f, g) = 0 \), since

\[
\ell_1(\ell_2(f, \ell_1(g))) - \ell_1(\ell_2(g, \ell_1(f))) = 0. \tag{4.15}
\]

In the specific case of the deformation of Chern-Simons theory, i.e., eq. (4.8) the relation (4.15) implies

\[
\begin{align*}
P_{aijk}^1 \partial_i f \partial_j \partial_k g &+ Q_{aijk}^1 \partial_i g \partial_j \partial_k f + R_{aijkl}^1 \partial_i \partial_j \partial_k \partial_l f = P_{aijk}^1 \partial_i g \partial_j \partial_k f + Q_{aijk}^1 \partial_i f \partial_j \partial_k g + R_{aijkl}^1 \partial_i \partial_j f \partial_k \partial_l g, \tag{4.16}
\end{align*}
\]
which in turn yields the following relations between the coefficients $P_{1}^{aijk}, Q_{1}^{aijk}$ and $R_{1}^{aijkl}$:

$$Q_{1}^{aijk} = P_{1}^{aijk}, \quad \text{and} \quad R_{1}^{aijkl} = R_{1}^{a(kl)(ij)}. \quad (4.17)$$

We stress that these relations can be checked explicitly taking into account (4.11), however they follow from the construction of $L_{\infty}$ algebra. Using (4.17) the original equation (4.10) becomes

$$\ell_{2}(\ell_{1}(f), A) = P_{1}^{aijk} \left( \partial_{i} f \partial_{j} A_{k} + A_{i} \partial_{j} \partial_{k} f \right) + R_{1}^{aijkl} \partial_{i} A_{j} \partial_{k} \partial_{l} f, \quad (4.18)$$

implying the solution

$$\ell_{2}(B, A) = P_{1}^{aijk} \left( B_{i} \partial_{j} A_{k} + A_{i} \partial_{j} B_{k} \right) + R_{1}^{aijkl} \partial_{i} A_{j} \partial_{k} B_{l}. \quad (4.19)$$

The explicit form of $\ell_{2}(A, B)$ is given by

$$\ell_{2}(A, B) = - \epsilon^{ab}_{c} \{ A_{a}, B_{b} \}_{c} - \epsilon^{ab}_{c} \partial_{a} \Theta^{ij}(A_{i} \partial_{j} B_{b} + B_{i} \partial_{j} A_{b})$$

$$+ \frac{1}{2} \epsilon^{ab}_{c} \partial_{a} \Theta^{ij}(A_{i} \partial_{b} B_{j} + B_{i} \partial_{b} A_{j}), \quad (4.20)$$

which is in the perfect agreement with our previous result [7].

### 4.2 Next to the leading order

At this order there appear higher brackets $\ell_{3}$. The expressions for $\ell_{3}(A,f,g)$ and $\ell_{3}(A,B,f)$ were found in Sect. 3.1. Taking into account that now $X_{-2}$ is non trivial, one may also have non-vanishing $\ell_{3}(E,f,g) \in X_{-1}$, $\ell_{3}(E,A,f) \in X_{-2}$ and $\ell_{3}(A,B,C) \in X_{-2}$.

Let us start with $\ell_{3}(E,f,g)$. Such a term contributes to the closure condition $J_{3}(f,g,A) = 0$, which are however satisfied without it. Therefore, we can set $\ell_{3}(E,f,g) = 0$. Next we consider $J_{3}(E,f,g) = 0$, i.e.,

$$0 = \ell_{2}(\ell_{2}(E,f),g) + \ell_{2}(\ell_{2}(g,E),f) + \ell_{2}(\ell_{2}(f,g),E)$$

$$+ \ell_{3}(E,\ell_{1}(f),g) + \ell_{3}(E,f,\ell_{1}(g)) \quad (4.21)$$

from which one derives

$$\ell_{3}(E,A,f) = \frac{1}{2} \Pi^{ijk} \partial_{i} E_{a} A_{j} \partial_{k} f. \quad (4.22)$$

Finally, to determine $\ell_{3}(A,B,C)$, we consider $J(A,B,f)$ and write is as

$$\ell_{3}(A,B,\ell_{1}(f)) = r_{3}(A,B,f),$$

$$r_{3}(A,B,f) = - \ell_{1}(\ell_{3}(A,B,f)) - \ell_{3}(\ell_{1}(A),B,f) + \ell_{3}(A,\ell_{1}(B),f)$$

$$- \ell_{2}(\ell_{2}(A,B),f) - \ell_{2}(\ell_{2}(f,A),B) + \ell_{2}(\ell_{2}(B,f),A). \quad (4.23)$$
By the construction the r.h.s., \( r_3(A, B, f) \), is symmetric with respect to the permutation of \( A \) and \( B \). Before discussing the specific form of the r.h.s. for the deformation of the Chern-Simons theory let us prove the general formula:

\[
r_3(A, \ell_1(g), f) = r_3(A, \ell_1(f), g),
\]

which is implied by the graded symmetry of \( \ell_3 \) bracket,

\[
\ell_3(A, \ell_1(g), \ell_1(f)) = \ell_3(A, \ell_1(f), \ell_1(g)).
\]

First we write

\[
r_3(A, \ell_1(g), f) - r_3(A, \ell_1(f), g) =
-\ell_1 (\ell_3(A, \ell_1(g), f)) - \ell_1 (\ell_3(A, \ell_1(f), g))
-\ell_3(\ell_1(A), \ell_1(g), f) + \ell_3(\ell_1(A), \ell_1(f), g)
-\ell_2(\ell_2(A, \ell_1(g)), f) + \ell_2(\ell_2(A, \ell_1(f)), g)
-\ell_2(\ell_2(f, A), \ell_1(g)) + \ell_2(\ell_2(g, A), \ell_1(f))
+\ell_2(\ell_2(\ell_1(g), f), A) - \ell_2(\ell_2(\ell_1(f), g), A).
\]

Using the graded symmetry and the previously satisfied \( L_\infty \) relations, \( \mathcal{J}_2(f, g) = 0 \), and, \( \mathcal{J}_2(A, f) = 0 \), the r.h.s. of the above relation becomes

\[
-\ell_1 (\ell_3(A, \ell_1(g), f)) - \ell_1 (\ell_3(A, \ell_1(f), g))
-\ell_3(\ell_1(A), \ell_1(g), f) + \ell_3(\ell_1(A), \ell_1(f), g)
-\ell_2(\ell_2(A, \ell_1(g)), f) + \ell_2(\ell_2(A, \ell_1(f)), g)
+\ell_1(\ell_2(\ell_2(f, A), g)) - \ell_1(\ell_2(\ell_2(g, A), f))
+\ell_1(\ell_2(\ell_2(\ell_1(g), f), A)) - \ell_2(\ell_2(\ell_1(f), g), A),
\]

which in turn can be rearranged as a combination of two other previously satisfied \( L_\infty \) relations

\[
\ell_1 (\mathcal{J}_3(g, f, A)) - \mathcal{J}_3(\ell_1(A), g, f) \equiv 0.
\]

Now let us discuss the solution of the eq. (4.23) for the non-commutative deformation of CS theory. The calculation of the r.h.s. is quite involved, but straightforward. We represent it as

\[
\ell_3(A, B, \ell_1(f)) = P^{ijkl}_2 (A_i \partial_j B_k \partial_l f + B_i \partial_j A_k \partial_l f) +
Q^{ijkl}_2 (A_i B_j \partial_k \partial_l f + B_i A_j \partial_k \partial_l f) +
R^{ijkl}_2 (\partial_i f \partial_j A_k \partial_l B_m + \partial_i f \partial_j B_k \partial_l A_m) +
S^{ijkl}_2 (A_i \partial_j B_k \partial_l \partial_m f + B_i \partial_j A_k \partial_l \partial_m f),
\]

(4.29)
where
\[ P_{2}^{aijkl} = \varepsilon^{obj} \left( \frac{1}{2} \Theta^{lm} \partial_b \partial_m \Theta^{kl} + \frac{1}{6} \Theta^{km} \partial_b \partial_m \Theta^{il} + \frac{1}{6} \Theta^{im} \partial_b \partial_m \Theta^{kl} + \right. \]
\[ \left. \frac{1}{6} \partial_b \Theta^{km} \partial_m \Theta^{il} - \frac{1}{3} \partial_b \Theta^{im} \partial_m \Theta^{kl} \right) + \varepsilon^{abk} \left( \Theta^{lm} \partial_b \partial_m \Theta^{ij} - \frac{1}{2} \Theta^{jm} \partial_m \partial_m \Theta^{il} + \right. \]
\[ \left. \partial_b \Theta^{im} \partial_m \Theta^{il} - \frac{1}{2} \partial_b \Theta^{im} \partial_m \Theta^{il} \right) \quad (4.30) \]
\[ Q_{2}^{aijkl} = \varepsilon^{abl} \left( \frac{1}{6} \Theta^{im} \partial_b \partial_m \Theta^{jk} + \frac{1}{3} \partial_b \Theta^{im} \partial_m \Theta^{ik} \right) \]
\[ R_{2}^{aijklm} = \frac{1}{6} \varepsilon^{ijklm} + \frac{1}{2} \varepsilon^{abkm} \Pi^{ijl} + \frac{1}{2} \varepsilon^{abkl} \Pi^{ijm} + \frac{1}{2} \varepsilon^{ablm} \Theta^{ijl} \partial_b \Theta^{ilm}, \]
\[ S_{2}^{aijklm} = \frac{1}{6} \varepsilon^{ijklm} + \frac{1}{2} \varepsilon^{abklm} \Pi^{ijm} + \frac{1}{2} \varepsilon^{abkm} \Theta^{ijl} \partial_b \Theta^{ilm} + \frac{1}{2} \varepsilon^{ablm} \Theta^{ijl} \partial_b \Theta^{ilm} . \]

The relation (4.24) becomes
\[ P_{2}^{aijkl} A_i \partial_j \partial_k g \partial_l f + P_{2}^{aijkl} \partial_i g \partial_j A_k \partial_l f + \]
\[ Q_{2}^{aijkl} A_i \partial_l g \partial_j \partial_k f + Q_{2}^{aijkl} A_i \partial_j g \partial_j \partial_k f + \]
\[ R_{2}^{aijklm} \partial_l f \partial_j A_k \partial_j \partial_m g + R_{2}^{aijklm} \partial_l f \partial_j \partial_k g \partial_l A_m + \]
\[ S_{2}^{aijklm} A_i \partial_j \partial_k g \partial_l \partial_m f + S_{2}^{aijklm} \partial_i g \partial_j A_k \partial_l \partial_m f = \]
\[ P_{2}^{aijkl} A_i \partial_j \partial_k f \partial_l g + P_{2}^{aijkl} \partial_i f \partial_j A_k \partial_l g + \]
\[ Q_{2}^{aijkl} A_i \partial_l f \partial_j \partial_k g + Q_{2}^{aijkl} A_i \partial_f \partial_j \partial_k g + \]
\[ R_{2}^{aijklm} \partial_i g \partial_j A_k \partial_j \partial_m f + R_{2}^{aijklm} \partial_i g \partial_j \partial_j f \partial_k A_m + \]
\[ S_{2}^{aijklm} A_i \partial_j \partial_k f \partial_l \partial_m g + S_{2}^{aijklm} \partial_i f \partial_j A_k \partial_l \partial_m g . \]

Thus we obtain the following relations on the coefficient functions
\[ P_{2}^{aijkl} = P_{2}^{aijkl}, \]
\[ P_{2}^{aijkl} = Q_{2}^{aijkl} + Q_{2}^{aijkl}, \]
\[ S_{2}^{aijklm} = R_{2}^{aijklm} + R_{2}^{aijklm} . \]

We stress that the above relations are not manifest from the explicit form of the coefficient functions \( P_{2}^{aijkl}, Q_{2}^{aijkl}, R_{2}^{aijklm} \) and \( S_{2}^{aijklm} \) given by (4.30) correspondingly. They follow from the \( L_\infty \) relations, \( J_3(g, f, A) = 0, J_3(E, g, f) = 0, \) etc., which were also used to obtain the eq. (4.24). The situation here is absolutely the same as in the previous Section for the construction of \( L_\infty \) gauge-algebra.
The solution of the \( L_\infty \) relations in each given order \( n \) imply the non-trivial consistency conditions, which in turn are satisfied due to the previously solved lower order \( L_\infty \) relations.

The following expression

\[
\ell_3(A, B, C) = \frac{1}{2} P_{ijkl}^m ( A_i \partial_j B_k C_l + C_i \partial_j A_k B_l + B_i \partial_j C_k A_l + \\
C_i \partial_j B_k A_l + B_i \partial_j A_k C_l + A_i \partial_j C_k B_l ) + \\
P_{ijkl}^m ( A_i \partial_j B_k \partial_l C_m + C_i \partial_j A_k \partial_l B_m + B_i \partial_j C_k \partial_l A_m + \\
C_i \partial_j B_k \partial_l A_m + B_i \partial_j A_k \partial_l B_m + A_i \partial_j C_k \partial_l B_m ) ,
\]

by construction is symmetric in all arguments and due to the relations (4.32) satisfies the equation (4.29). Rewriting the first two lines of (4.33) in the more compact form,

\[
\frac{1}{2} \left( P_{ijkl}^a + P_{ijkl}^a \right) ( A_i \partial_j B_k C_l + C_i \partial_j A_k B_l + B_i \partial_j C_k A_l ) ,
\]

the final answer is given by

\[
\ell_3(A, B, C) = -\varepsilon^{ab} \left( \frac{1}{3} \Theta^{km} \partial_k \partial_m \Theta^{ij} - \frac{1}{6} \partial_b \Theta^{km} \partial_m \Theta^{ij} + (j \leftrightarrow k) \right ) \\
(\partial_a A_i B_j C_k + A_j \partial_a B_i C_k + A_k B_j \partial_a C_i) + \\
-\varepsilon^{ab} \left( \frac{1}{2} \Theta^{km} \partial_k \partial_m \Theta^{ij} + \frac{1}{2} \partial_b \Theta^{km} \partial_m \Theta^{ij} + (j \leftrightarrow k) \right ) \\
(\partial_i A_j B_k C_l + A_j \partial_i B_k C_l + A_k B_j \partial_i C_l) + \\
-\varepsilon^{ab} \left( \frac{1}{2} \partial_a \Theta^{ij} \partial_b \Theta^{kl} \right ) \\
(\partial_i A_k - \partial_k A_i) B_j C_l + A_j (\partial_i B_k - \partial_k B_i) C_l + A_l B_j (\partial_i C_k - \partial_k C_i) + \\
+ \frac{1}{2} \left( \varepsilon^{jil} \Pi^{km} + \varepsilon^{km} \Pi^{jil} - \varepsilon^{kli} \Pi^{jlm} + \varepsilon^{bk} \Theta^{jl} \partial_b \Theta^{im} \right ) \\
( A_i \partial_j B_k \partial_l C_m + C_i \partial_j A_k \partial_l B_m + B_i \partial_j C_k \partial_l A_m + \\
C_i \partial_j B_k \partial_l A_m + B_i \partial_j C_k \partial_l B_m + A_i \partial_j C_k \partial_l B_m ) .
\]

It is written in this form to match with [7].

4.3 Higher order relations

In the associative case the Jacobi identities of the type \( \mathcal{J}_n(A^{n-3}E, f, g) = 0 \), are satisfied automatically and we set \( \ell_n(A^{n-2}, E, f) = 0 \). The missing brackets
\( \ell_n(A^n) \) should be determined from the \( L_\infty \) relations \( J_n(f, A^{n-1}) = 0 \), which can be schematically represented as

\[
\ell_n \left( A^{n-1}, \ell_1(f) \right) = r_n \left( A^{n-1}, f \right),
\]

where the r.h.s. \( r_n \left( A^{n-1}, f \right) \) written in terms of the lower order brackets \( \ell_m, m < n \), by the construction is symmetric in first \( n - 1 \) arguments. By the induction one may prove the following relation

\[
r_n \left( A^{n-2}, \ell_1(g), f \right) = r_n \left( A^{n-2}, \ell_1(f), g \right).
\]

(4.37)

This relation is general, the specific form of undeformed theory, i.e., \( \ell_1(A) \) was not used to prove it.

Before writing the eq. (4.36) for the Chern-Simons case let us first make some observations regarding the equations (4.10) and (4.29) describing the first and second order deformations of the CS theory correspondingly. In both cases the r.h.s. does not contain higher derivatives of fields \( A \) (i.e., second derivatives, \( \partial \partial A \), third derivatives, etc.). The later is in agreement with the slowly varying field approximation. At that, the order of the first derivative terms \( (\partial A) \) is at the maximum second, e.g., \( R_{ijklm} \partial_i f \partial_j A_k \partial_l A_m \), the terms of the form \( (\partial A)^3 \), etc., do not appear. The same form of the r.h.s. remains in the third order deformation which we do not write here explicitly. For the deformation of Chern-Simons theory in the \( n \)-th order we conjecture the following form of the r.h.s.:

\[
\ell_n \left( A^{n-1}, \ell_1(f) \right) = P_{ijkl}^n(A) \partial_j A_k \partial_l f + Q_{ajk}^n(A) \partial_j A_k \partial_l f + R_{ijklm}^n(A) \partial_i f \partial_j A_k \partial_l A_m + S_{ijklm}^n(A) \partial_j A_k \partial_l \partial_m f,
\]

(4.38)

where the coefficient functions \( P_{ijkl}^n(A) \) and \( S_{ijklm}^n(A) \) are monomials of the degree \( n - 1 \), \( Q_{ajk}^n(A) \) is the monomial of the degree \( n \) and \( R_{ijklm}^n(A) \) is the monomial of the degree \( n - 2 \).

The relation (4.37) implies

\[
\frac{\delta P_{ijkl}^n}{\delta A_m} = \frac{\delta P_{ijkl}^n}{\delta A_l},
\]

\[
\frac{\delta Q_{ajk}^n}{\delta A_l} = P_{ajkl}^n,
\]

\[
\frac{\delta R_{ijklm}^n}{\delta A_p} = \frac{\delta P_{ijklm}^n}{\delta A_i},
\]

\[
\frac{\delta S_{ijklm}^n}{\delta A_i} = R_{ijklm}^n(A) + R_{ijklm}^{ai(lm)jk}.
\]

(4.39)

Using these relations one may show that

\[
\ell_n (A^n) = \frac{1}{n} P_{ijkl}^n A_l \partial_j A_k + \frac{1}{n - 1} R_{ijklm}^n A_l \partial_j A_k \partial_l A_m,
\]

(4.40)

solves the equation (4.38).
5 Lie-algebra like deformation

The main goal of this Section is to do some explicit calculations to illustrate the proposed ideas. We will work with the most simple and at the same time non-trivial situation taking the non-commutativity parameter $\Theta$ to be linear function of the coordinates and satisfying the Jacobi identity. Physically it corresponds, for example, to the $Q$-flux backgrounds in open string theory [20]. In this case (associative deformations) all higher brackets with two gauge parameters vanish, $\ell_{n+2}(f, g, A^n) = 0$, for $n > 0$, so

$$[\delta_f, \delta_g]A = \delta_{[f,g]}A. \quad (5.1)$$

For non-associative deformations induced by the quasi-Poisson structures the non-vanishing brackets of the type $\ell_{n+2}(f, g, A^n)$ are required to compensate the violation of the associativity.

5.1 NC $su(2)$-like deformation

We choose the non-commutativity parameter $\Theta^{ij}(x) = 2 \theta \varepsilon^{ijk} x^k$, which correspond to the rotationally invariant 3d NC space [21, 22, 23, 24, 25]. For the brevity of the calculations we will suppress the small parameter $\theta$ in this and the following subsections. However, we will restore $\theta$ in the Subsection 5.3 where we provide the summary of the main findings of this Section. The corresponding Poisson bracket is

$$\{f, g\} = 2 \varepsilon^{ijk} x^k \partial_i f \partial_j g. \quad (5.2)$$

For the first two brackets with one gauge parameter one finds,

$$\ell_2(f, A) = \{A_a, f\} + \varepsilon^{abc} A_b \partial_c f$$
$$\ell_3(f, A, A) = -\frac{2}{3} \left( \partial_a f A^2 - \partial_b f A^b A_a \right), \quad (5.3)$$

with $A^2 = A_b A^b$. Then, using the recurrence relations (5.64) we observe that the brackets $\ell_{n+3}(f, A^n)$ with the odd $n$ vanish, while for even $n$ they have the structure

$$\ell_{n+3}(f, A^n) = \left( \partial_a f A^2 - \partial_b f A^b A_a \right) \chi_n(A^2), \quad (5.4)$$

for some monomial function $\chi_n(A^2)$. The combination of (5.3) and (5.4) in (2.11) results in the following ansatz for the gauge variation:

$$\delta_f A_a = \partial_a f + \{A_a, f\} + \varepsilon^{abc} A_b \partial_c f + \left( \partial_a f A^2 - \partial_b f A^b A_a \right) \chi(A^2), \quad (5.5)$$

where the function $\chi(A^2)$ should be determined from the closure condition (5.1).
Let us write
\begin{align}
\delta_f (\delta_g A_a) - \delta_g (\delta_f A_a) - \delta_{f,g} A_a = \\
\{\delta_f A_a, g\} + \epsilon^{abc} \delta_f A_b \partial_c g + (2 \partial_a g A_b \delta_f A^b - \partial_b g \delta_f A^b a_a - \partial_b g A^b \delta_f A_a) \chi (A^2) \\
+ (\partial_a g A^2 - \partial_b g A^b A_a) \chi' (A^2) 2A_c \delta_f A^c \\
- \{\delta_g A_a, f\} - \epsilon^{abc} \delta_g A_b \partial_c f - (2 \partial_a f \delta_g A^b - \partial_b f \delta_g A^b a_a - \partial_b f A^b \delta_g A_a) \chi (A^2) \\
- (\partial_a f A^2 - \partial_b f A^b A_a) \chi' (A^2) 2A_c \delta_g A^c \\
- \partial_a \{f, g\} - \{A_a, \{f, g\}\} - \epsilon^{abc} A_b \partial_c \{f, g\} - (\partial_a \{f, g\} A^2 - \partial_b \{f, g\} A^b A_a) \chi (A^2).
\end{align}

After tedious but straightforward calculations we can rewrite the r.h.s. of (5.6) as

\begin{align}
[\partial_a g \partial_b f A^b - \partial_a f \partial_b g A^b] (1 + 3 \chi (A^2) + A^2 \chi' (A^2) + 2 A^2 \chi' (A^2)).
\end{align}

That is, requiring that
\begin{align}
2 t \chi' (t) + 1 + 3 \chi (t) + t \chi^2 (t) = 0, \quad \chi (0) = - \frac{1}{3},
\end{align}

we will obtain zero in the r.h.s. of (5.6). The solution of (5.8) is
\begin{align}
\chi (t) = \frac{1}{t} \left( \sqrt{t} \cot \sqrt{t} - 1 \right).
\end{align}

Thus, we have obtained in (5.5), (5.42) an explicit form of the non-commutative \textit{su}(2)-like deformation of the abelian Chern-Simons equations of motion up to the order \( O (\Theta^3) \) is given by:

\begin{align}
f^a := \epsilon^{abc} \partial_b A_c + \frac{1}{2} \epsilon^{abc} \{A_b, A_c\} + 2 A_b \partial_a A_b - A_a \partial_b A_b - A_b \partial_b A_a \quad \text{(5.10)}
\end{align}

\begin{align*}
- \frac{8}{3} \epsilon^{abc} A^2 \partial_b A_c + \frac{2}{3} \epsilon^{abm} A_m A^c \partial_b A_c - 2 \epsilon^{acm} A_m A^b \partial_b A_c \\
+ 2 \epsilon^{bcm} A_m A^a \partial_b A_c - \{A^2, A_a\} + O (A^4) = 0.
\end{align*}

Let us emphasise that the contribution of the order \( O (A^n) \) to the e.o.m. corresponds to the order \( O (\Theta^{n-1}) \) of the bi-vector \( \Theta \). So, the correct commutative limit here is evident.
The $L_\infty$ relations, $\mathcal{J}_4(E, f, g, h) = 0$, and $\mathcal{J}_4(E, A, f, g) = 0$, are satisfied automatically and we may set, $\ell_4(E, A, B, f) = 0$. The same can be shown for higher brackets of the form $\ell_{n+2}(E, A^n, f)$. Thus, we conclude that the gauge variation of the field equation (1.4) in case of the the associative deformation should obey the equation

$$\delta_f \mathcal{F} = \ell_2(f, \mathcal{F}) = \{\mathcal{F}, f\}.$$  \hfill (5.11)

The gauge variation of the field equation is proportional to the field equation itself, i.e., it is gauge invariant on-shell. Taking into account the form of the lower order brackets $\ell_n(A^n)$ we are looking the solution of the equation (5.11) in the form

$$\mathcal{F}^a = P^{abc}(A) \partial_b A_c + R^{abc}(A) \{A_b, A_c\},$$ \hfill (5.12)

with the initial condition

$$P^{abc}(0) = \varepsilon^{abc} \text{ and } R^{abc}(0) = \frac{1}{2} \varepsilon^{abc}.$$ \hfill (5.13)

Substituting (5.12) in (5.11) and introducing the notation

$$\delta_f = \delta_f + \{\cdot, f\},$$ \hfill (5.14)

one obtains in the l.h.s.:

$$\delta_f \mathcal{F}_a = (\delta_f P^{abc}) \partial_b A_c + P^{abc} \partial_b (\delta_f A_c) + 2 P^{amc} \varepsilon^{bem} \partial_b A_c \partial_e f$$
$$+ P^{abc} \{A_c, \partial_b f\} + \{P^{abc} \partial_b A_c, f\}$$
$$+ (\delta_f R^{abc}) \{A_b, A_c\} + \{R^{abc}, f\} \{A_b, A_c\}$$
$$+ 2 R^{abc} \{\delta_f A_b, A_c\} + R^{abc} \{\{A_b, f\}, A_c\} + R^{abc} \{A_b, \{A_c, f\}\}.$$ \hfill (5.15)

While the r.h.s. of (5.11) is just given by

$$\{P^{abc} \partial_b A_c + R^{abc} \{A_b, A_c\}, f\}.$$ \hfill (5.16)

Taking into account that due to Jacobi identity,

$$R^{abc} (\{A_b, \{A_c, f\}\} + \{A_c, \{f, A_b\}\} + \{f, \{A_b, A_c\}\}) \equiv 0,$$ \hfill (5.17)

the eq. (5.11) becomes

$$(\delta_f P^{abc}) \partial_b A_c + P^{abc} \partial_b (\delta_f A_c) + 2 P^{amc} \varepsilon^{bem} \partial_b A_c \partial_e f$$
$$+ P^{abc} \{A_c, \partial_b f\} + (\delta_f R^{abc}) \{A_b, A_c\} + 2 R^{abc} \{\delta_f A_b, A_c\} = 0.$$ \hfill (5.18)

We set separately

$$(\delta_f P^{abc}) \partial_b A_c + P^{abc} \partial_b (\delta_f A_c) + 2 P^{amc} \varepsilon^{bem} \partial_b A_c \partial_e f = 0,$$ \hfill (5.19)

and

$$P^{abc} \{A_c, \partial_b f\} + (\delta_f R^{abc}) \{A_b, A_c\} + 2 R^{abc} \{\delta_f A_b, A_c\} = 0.$$ \hfill (5.20)
Definition of the $P$-term

Let us first discuss the eq. (5.19). Taking into account the explicit form of $\tilde{\delta}_f$ the eq. (5.19) reads

$$\frac{\delta P_{abc}}{\delta A_e} (1 + A^2 \chi) + \frac{\delta P_{abc}}{\delta A_m} (\varepsilon^{mne} A_n - A^m A^e \chi) + P^{abm} \varepsilon^{cem} + 2 P^{amc} \varepsilon^{bem} + 2 P^{abe} A^c \left( \chi + A^2 \chi' \right) - P^{abc} A^e \chi - P^{abm} A_m \delta^{ce} \chi - 2 P^{abm} A_m A^c A^e \chi \right] \partial_b A_e \partial_c f + [P^{abc} (1 + A^2 \chi (A^2)) + P^{abm} \varepsilon^{mne} A_n - P^{abm} A_n A^e \chi (A^2)] \partial_b \partial_c f = 0.$$  

From which we obtain two separate conditions

$$\frac{\delta P_{abc}}{\delta A_e} (1 + A^2 \chi) + \frac{\delta P_{abc}}{\delta A_m} (\varepsilon^{mne} A_n - A^m A^e \chi) + P^{abm} \varepsilon^{cem} + 2 P^{amc} \varepsilon^{bem} + 2 P^{abe} A^c \left( \chi + A^2 \chi' \right) - P^{abc} A^e \chi - P^{abm} A_m \delta^{ce} \chi - 2 P^{abm} A_m A^c A^e \chi' = 0,$$  

and

$$(P^{ab} + P^{acb}) (1 + A^2 \chi) - P^{abm} \varepsilon^{cmn} A_n - P^{acm} \varepsilon^{bnm} A_n - P^{abm} A_n A^c \chi - P^{acn} A^b \chi = 0.$$  

Again the lower order brackets $\ell_n (A^n)$ indicate the anzatz

$$P^{abc} (A) = \varepsilon^{abc} F (A^2) + \varepsilon^{abm} A_m A^e G (A^2) + \varepsilon^{acm} A_m A^b H (A^2) + \varepsilon^{bcm} A_m A^a J (A^2) + A^a A^b A^c K (A^2) + A^a \delta^{bc} L (A^2) + A^b \delta^{ce} M (A^2) + A^c \delta^{ab} N (A^2).$$  

The equation (5.23) implies the following relations on the coefficient functions:

$$G + H (1 + A^2 \chi) - \chi F - M = 0,$$
$$K - \chi (L + M) - J - H = 0,$$
$$L (1 + A^2 \chi) + F + A^2 J = 0,$$
$$M (1 + A^2 \chi) + N - F + A^2 H = 0.$$  

Our strategy is to substitute (5.24) in (5.22) and collect the coefficients at the different powers of fields $A$, modulo the $A^2$. Starting with a quartic in $A$ contribution, $A^a A^b A^c A^e$, then cubic in $A$ structures, like $\varepsilon^{abm} A_m A^e A^e$, etc. up to the zero order in $A$ terms like $\delta^{ab} \delta^{ce}$. Equating to zero these coefficients we will obtain the system of differential equations on the coefficient functions $F, \ldots, N$.

At that, the key observation here is that not all these power in $A$ structures are independent. There are algebraic relations involving the Levi-Civita tensors
and vector fields $A^e$ described in the appendix. Using them we will reduce the number of different structures and thus the number of the equations on $F$, $G$, etc. These relations guarantee that the resulting system of differential equations is not overfull. The equation (5.22) does have the solution.

We start writing quartic in $A$ term in the l.h.s. of (5.22):

$$A^a A^b A^c A^e \left[ 2 K' - 2 \chi K - 2 \chi' (L + M + A^2 K') + 2 A^2 \chi' K' \right] .$$

(5.26)

The cubic in the field $A$ contribution is given by

$$\varepsilon^{abm} A_m A^e A^e \left[ 2 G' - \chi G + 2 A^2 \chi' G \right] +$$

$$\varepsilon^{acm} A_m A^b A^e \left[ 2 H' - 3 \chi H \right] + \varepsilon^{bcm} A_m A^a A^e \left[ 2 J' - 3 \chi J \right] +$$

$$\varepsilon^{ace} A^b A^c \left[ 2 \left( \chi + A^2 \chi' \right) H - K \right] +$$

$$\varepsilon^{abe} A^c \left[ 2 A^2 K - 2 A^2 \left( \chi + A^2 \chi' \right) H \right] .$$

(5.27)

At this point for the first time we make use the algebraic relation from the appendix to reduce the number of structures. Namely employing the identity

$$\varepsilon^{acm} A_m A^b - \varepsilon^{bcm} A_m A^a = -\varepsilon^{abe} A^2 + \varepsilon^{abm} A_m A^e ,$$

(5.28)

and setting, $J = -H$, one rewrites (5.27) as

$$\varepsilon^{abm} A_m A^e A^e \left[ 2 G' - \chi G + 2 A^2 \chi' G + 2 \left( \chi + A^2 \chi' \right) H - K \right] +$$

$$\varepsilon^{acm} A_m A^b A^e \left[ 2 H' - 3 \chi H \right] + \varepsilon^{bcm} A_m A^a A^e \left[ 2 J' - 3 \chi J \right] +$$

$$\varepsilon^{abe} A^c \left[ 2 A^2 K - 2 A^2 \left( \chi + A^2 \chi' \right) H \right] .$$

(5.29)

We stress that now it appeared the linear in $A$ contribution coming from the cubic ones.

We continue with the quadratic in the fields $A$ terms in the l.h.s. of (5.22),

$$\delta^{ae} A^b A^c \left[ (1 + A^2 \chi) K + G + 2(\chi + A^2 \chi') M \right] +$$

$$\delta^{bc} A^a A^e \left[ (1 + A^2 \chi) K + G + 2(\chi + A^2 \chi') L \right] +$$

$$\delta^{ce} A^a A^b \left[ K - J - H - \chi (L + M) \right] +$$

$$\delta^{bc} A^a A^e \left[ 2 L' - 2 \chi L + J \right] + \delta^{ae} A^b A^e \left[ 2 M' - 2 \chi M - H \right] +$$

$$\delta^{ab} A^e A^c \left[ 2 N' - 2 G \right] + \varepsilon^{acm} A_m \varepsilon^{ben} A_n H - \varepsilon^{acm} A_m \varepsilon^{ben} A_n J .$$

(5.30)

Using the identity (7.5) from the appendix which we write here for the convenience of the reader,

$$\varepsilon^{acm} A_m \varepsilon^{ben} A_n = \left( \delta^{ab} \delta^{ce} - \delta^{ae} \delta^{bc} \right) A^2$$

$$+ \delta^{bc} A^a A^e - \delta^{ce} A^a A^b - \delta^{ab} A^c A^e + \delta^{ae} A^b A^c ,$$

(5.31)

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we rewrite (5.30) as
\[
\delta^{ab} \delta^{ce} A^2 [H - J] - \delta^{ae} \delta^{bc} A^2 H + \delta^{ae} \delta^{be} A^2 J +
\]
\[
\delta^{ae} A^b A^c \left[ (1 + A^2 \chi) K + G + 2(\chi + A^2 \chi') M + H \right] +
\]
\[
\delta^{be} A^a A^c \left[ (1 + A^2 \chi) K + G + 2(\chi + A^2 \chi') L - J \right] +
\]
\[
\delta^{ce} A^a A^b \left[ K - J - H - \chi(L + M) - H + J \right] +
\]
\[
\delta^{bc} A^a A^e \left[ 2 L' - 2 \chi L + J + H \right] + \delta^{ac} A^b A^e \left[ 2 M' - 2 \chi M - H - J \right] +
\]
\[
\delta^{ab} A^c A^e \left[ 2 N' - 2 G - H + J \right].
\] (5.32)

At this point it is convenient to invert the order. First we will analyze the zero order in \( A \) contributions in the equation (5.22) and only then the linear in the fields \( A \) terms. Taking into account the first line of (5.32) the zero order in \( A \) terms in the l.h.s. of (5.22) are given by
\[
\delta^{ac} \delta^{be} \left[ F + (1 + A^2 \chi) M + J \right] + \delta^{ae} \delta^{bc} \left[ F + (1 + A^2 \chi) L - H \right] +
\]
\[
\delta^{ab} \delta^{ce} \left[ N - 2 F + H - J \right].
\] (5.33)

The significant simplification occurs if we set
\[
H = -J = 0.
\] (5.34)

In order to the equation (5.22) be satisfied the coefficients at the different structures in the l.h.s. should be equal to zero. Thus from (5.33) we get
\[
L = M = -\frac{F}{1 + A^2 \chi}, \quad \text{and} \quad N = 2F.
\] (5.35)

Equating to zero the coefficient at \( \delta^{ce} A^a A^b \) in (5.32) one finds,
\[
K = \chi(L + M).
\] (5.36)

Now let us return to the linear in \( A \) contributions to the left hand side of the equation (5.22). Taking into account (5.29) it can be written as
\[
\epsilon^{abc} A^e \left[ 2 F' - \chi F \right] +
\]
\[
\epsilon^{abe} A^c \left[ (1 + A^2 \chi) G + 2 N + 2(\chi + A^2 \chi') F \right. +
\]
\[
+ A^2 K - 2 A^2 \left( \chi + A^2 \chi' \right) H] +
\]
\[
\epsilon^{ace} A^b \left[ (1 + A^2 \chi) H + M \right] + \epsilon^{bce} A^a \left[ (1 + A^2 \chi) J - L \right] +
\]
\[
\epsilon^{abm} A_m \delta^{ce} \left[ G - \chi F \right] + \epsilon^{acm} A_m \left( 1 + A^2 \chi \right) H - \epsilon^{aem} A_m \delta^{bc} L +
\]
\[
\epsilon^{bem} A_m \delta^{ae} \left( 1 + A^2 \chi \right) J + \epsilon^{bcm} A_m \delta^{ac} M.
\] (5.37)

Here we remind that because of the algebraic identities from the appendix not all structures in the above expression are independent. Now using these identities
and previously defend coefficients we will reduce the number of terms in (5.37). First, using (7.4) and (5.35) we get rid of the terms,

$$-\epsilon^{acm} A_m \delta^{bc} L + \epsilon^{bem} A_m \delta^{ac} M,$$

substituting them with,

$$\epsilon^{abe} A^c L - \epsilon^{abm} A_m \delta^{ce} L.$$

Then we utilize the identity (7.1) to convey the terms,

$$\epsilon^{ace} A^b M - \epsilon^{bce} A^a L,$$

through the

$$-\epsilon^{abc} A^e L + \epsilon^{abe} A^c L.$$

We use that, $H = -J = 0$, from (5.34), and also notice that due to (5.35) and (5.36) the coefficients $K$, $L$ and $F$ satisfy the relation,

$$2 L + A^2 K = -2 F.$$

We conclude that the linear in $A$ contribution to the l.h.s. of the equation (5.22) given initially by (5.37) becomes,

$$\epsilon^{abc} A^e \left[ 2 F' - \chi F - L \right] +$$

$$\epsilon^{abe} A^c \left[ (1 + A^2 \chi) G + 2 F + 2 (\chi + A^2 \chi') F \right] +$$

$$\epsilon^{abm} A_m \delta^{ce} \left[ G - \chi F - L \right].$$

Again we set to zero the coefficients in (5.38) and obtain the relations

$$2 F' = \chi F + L,$$  \hspace{1cm} (5.39)

$$(1 + A^2 \chi) G + 2 F + 2 (\chi + A^2 \chi') F = 0,$$  \hspace{1cm} (5.40)

and

$$G = \chi F + L,$$  \hspace{1cm} (5.41)

The solution of the equation (5.39) with the initial condition, $F(0) = 1$, is

$$F(t) = \frac{\sin \sqrt{t} \cos \sqrt{t}}{\sqrt{t}}.$$  \hspace{1cm} (5.42)

The relation (5.41) defines the function $G$ in terms of previously found ones $\chi$ and $F$. The equation (5.40) is satisfied as a consequence of the relation (5.41).
and the differential equation (5.8). The same happens, for example, with the equation,

$$L' - \chi L = 0,$$  

resulting from the quadratic contribution (5.32). To show (5.43) one needs (5.8), (5.35) and (5.39). The careful check shows that the rest of the coefficients also vanishes. We stress that in order to the eq. (5.22) hold the function \(\chi(t)\) cannot be arbitrary, but necessarily the one which guarantees the condition (5.1), i.e., \([\delta_f, \delta_g] A = \delta_{(f,g)} A\).

**Definition of \(R\)-term**

Now let us discuss the eq. (5.20) and define the \(R\)-term in (5.12). Like in the case of the equation (5.19) the equation (5.20) is equivalent to two separate conditions,

$$-P_{abc} + 2 (1 + A^2 \chi) R^{abc} - 2 R_{acm} \varepsilon^{mbc} A_n + 2 R^{acm} A_m A^b \chi = 0,$$  

and

$$-P^{abc} + 2 (1 + A^2 \chi) R_{abc} - 2 R^{acm} \varepsilon_{mbc} A_n + 2 R^{acm} A_m A^b \chi = 0.$$  

Because of the contraction with the Poisson bracket, \(R^{abc} \{A_b, A_c\}\), the coefficient function \(R^{abc}(A)\) should be antisymmetric in \(b\) and \(c\). So we write,

$$R^{abc}(A) = \varepsilon^{abc} S (A^2) + (\varepsilon_{abm} A_m A^c - \varepsilon_{acm} A_m A^b) T (A^2) + \varepsilon_{bcm} A_m A^a U (A^2) + (\delta^{ab} A^c - \delta^{ac} A^b) V (A^2).$$  

Since the coefficient function \(P^{abc}(A)\) is already known, from the eq. (5.45) one finds,

$$S = V = \frac{F}{2 (1 + A^2 \chi)}, \quad T = \frac{\chi F}{2 (1 + A^2 \chi)} \quad \text{and} \quad U = 0.$$  

then the equation (5.44) can be checked explicitly.

**Comparison to the lower order brackets**

As a consistency check let us calculate the first order contributions to the equations of motion. Since

$$L = M = -\frac{\sin^2 \sqrt{\frac{t}{I}}}{t}, \quad S = V = \frac{\sin^2 \sqrt{\frac{t}{2I}}}{2t},$$  

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one finds, \( L(0) = M(0) = -1 \). Then \( N(0) = 2F(0) = 2 \), and \( S(0) = 1/2 \), so the first order contribution is given by

\[
2A_b \partial_a A_b - A_a \partial_b A_b - A_b \partial_b A_a + \frac{1}{2} \varepsilon^{abc} \{ A_b, A_c \}, \tag{5.49}
\]

which is in the perfect agreement with (5.10). Now,

\[
F'(0) = -\frac{2}{3}, \quad G(0) = -\frac{4}{3}, \quad V(0) = \frac{1}{2}, \tag{5.50}
\]

which results in

\[
-\frac{2}{3} \varepsilon^{abc} A^2 \partial_b A_c - \frac{4}{3} \varepsilon^{abm} A_m A^c \partial_b A_c + \{ A_a A^2 \}. \tag{5.51}
\]

The term with the Poisson bracket is exactly the same as in (5.10), but the coefficients at the first two terms are different. However, adding to the (5.51) the algebraic identity (7.2) from the Appendix multiplied by the factor \(-2\),

\[
-2 \varepsilon^{abc} A^2 + 2 \varepsilon^{bcm} A_m A^a - 2 \varepsilon^{acm} A_m A^b + 2 \varepsilon^{abm} A_m A^c \equiv 0,
\]

we arrive exactly to the equation (5.10).

**Action principle**

It was proposed in \([9]\) that to define an action principle for these equations of motion one needs an inner product

\[
\langle \, , \rangle : X_{-1} \otimes X_{-2} \to \mathbb{R} \tag{5.52}
\]

satisfying the cyclicity property

\[
\langle A_0, \ell_n(A_1, \ldots, A_n) \rangle = \langle A_1, \ell_n(A_0, \ldots, A_n) \rangle \tag{5.53}
\]

for all \( A_i \in X_{-1} \). Then, the equations of motion follow from the action

\[
S = \sum_{n \geq 1} \frac{1}{(n+1)!} (-1)^{\frac{n(n-1)}{2}} \langle A, \ell_n(A^n) \rangle
\]

\[
= \frac{1}{2} \langle A, \ell_1(A) \rangle - \frac{1}{3!} \langle A, \ell_2(A^2) \rangle - \frac{1}{4!} \langle A, \ell_3(A^3) \rangle + \ldots . \tag{5.54}
\]

For the field theoretical models on the NC \( su(2) \)-like space such a product coincides with the canonical Weyl-Moyal case, see \([26]\), i.e.:

\[
\langle A, E \rangle = \int d^3x \, A_a E^a. \tag{5.55}
\]
Taking this into account we observe that the term,
\[ \varepsilon^{abm} A_m A^c G (A^2) \partial_b A_c, \]  
(5.56)
in the equations of motion (5.12) simply cannot be reproduced from the variation of such an action, since
\[ A_a \varepsilon^{abm} A_m A^c G (A^2) \partial_b A_c \equiv 0. \]  
(5.57)
In principle, using the identity (7.2) one may read off the expression (5.56) in terms of the other contributions to the equations of motion of the form
\[ \varepsilon^{abc} A^2 G \partial_c A, -\varepsilon^{bcm} A_m A^a G \partial_b A_c \text{ and } \varepsilon^{acm} A_m A^b G \partial_b A_c. \]
However, since
\[ A_a \varepsilon^{acm} A_m A^b G (A^2) \partial_b A_c \equiv 0, \]  
(5.58)
it does not solve the problem with the Lagrangian description.

On the other hand, there is a known result about the rigidity of the Chern-Simons action [27], meaning essentially that up to the field redefinition any consistent deformation of the Chern-Simons action is proportional to the trivial one. Thus the absence of the action principle for the equation (5.12) means that possibly we obtained here some non-trivial deformation of the Chern-Simons theory.

As it was already mentioned in the introduction on the classical level \( L_\infty \) algebra encodes all necessary information about the gauge theory, see [10] for more details. In the Sections 2 and 4 we described how the gauge symmetry and the equations of motion fit into the \( L_\infty \) structure. The Noether identities are contained in the additional space \( X_{-3} \) which we didn’t take into account in this research. While the existence of the action principle appears as an additional restriction on the field theoretical model. The example of the NC deformation of Chern-Simons theory shows that the model can be non-Lagrangian and admit the description within the \( L_\infty \) formalism. In this sense the formalism of \( L_\infty \) structures is broader then the action principle or the Batalin-Vilkovisky formalism.

### 5.3 Summary of the results

Let us summarise the main results of the Section 5. Consider the three dimensional space endowed with the Poisson bracket,
\[ \{ x^i, x^j \} = 2 \theta \varepsilon^{ijk} x_k, \]  
(5.59)
which corresponds in the slowly varying field approximation to the 3d rotation invariant NC space. At this point we restore the small parameter \( \theta \) in the Poisson bracket (5.59).

In the subsection 5.1 we have shown that the gauge transformation of the gauge field \( A_a \) given by
\[ \delta f A_a = \partial_a f + \{ A_a, f \} + \theta \varepsilon^{abc} A_b \partial_c f + \theta^2 \left( \partial_a f A^2 - \partial_b f A^b A_a \right) \chi \left( \theta^2 A^2 \right), \]  
(5.60)
where
\[
\chi(t) = \frac{1}{t} \left( \sqrt{t} \cot \sqrt{t} - 1 \right), \quad \chi(0) = -\frac{1}{3}; \tag{5.61}
\]
close the algebra
\[
[\delta_f, \delta_g] A_a = \delta_{\{f,g\}} A_a. \tag{5.62}
\]
In the commutative limit, \( \theta \to 0 \), the transformations (5.60) become an ordinary abelian gauge transformations, \( \delta_f A_a = \partial_a f \). That is why we call (5.60) as a non-commutative deformation of the abelian gauge transformation.

In the subsection 5.2 the problem of the consistent non-commutative deformation of the 3d abelian Chern-Somins equations, \( \varepsilon^{abc} \partial_b A_c = 0 \), was addressed. To construct this deformation we solve the equation
\[
\delta_f F^a = \{ F^a, f \}, \tag{5.63}
\]
meaning that the field equation, \( F^a = 0 \), should transform covariantly (it is gauge invariant on-shell) under the gauge transformation (5.60). The solution is given by the following expression,
\[
F^a := P^{abc} (A) \partial_b A_c + R^{abc} (A) \{ A_b, A_c \} = 0, \tag{5.64}
\]
where
\[
P^{abc} (A) = \varepsilon^{abc} F \left( \theta^2 A^2 \right) + \theta^2 \varepsilon^{abm} A_m A^c G \left( \theta^2 A^2 \right) + \theta^3 A^a A^b A^c K \left( \theta^2 A^2 \right) + \theta A^a \delta^{bc} L \left( \theta^2 A^2 \right) + \theta A^b \delta^{ac} M \left( \theta^2 A^2 \right) + \theta A^c \delta^{ab} N \left( \theta^2 A^2 \right), \tag{5.65}
\]
and
\[
R^{abc} (A) = \varepsilon^{abc} S \left( \theta^2 A^2 \right) + \theta^2 \left( \varepsilon^{abm} A_m A^c - \varepsilon^{acm} A_m A^b \right) T \left( \theta^2 A^2 \right) + \theta \left( \delta^{ab} A^c - \delta^{ac} A^b \right) V \left( \theta^2 A^2 \right), \tag{5.66}
\]
and the coefficient functions are determined as
\[
F(t) = \frac{N(t)}{2} = \frac{\sin \sqrt{t} \cos \sqrt{t}}{\sqrt{t}},
\]
\[
G(t) = \frac{2 \sqrt{t} \cos 2 \sqrt{t} - \sin 2 \sqrt{t}}{2 t \sqrt{t}} ,
\]
\[
K(t) = -4 T(t) = -\frac{2 \sin \sqrt{t}}{t^2} \left( \sqrt{t} \cos \sqrt{t} - \sin \sqrt{t} \right),
\]
\[
L(t) = M(t) = -2 S(t) = -2 V(t) = -\frac{\sin^2 \sqrt{t}}{t}. \tag{5.67}
\]
Taking into account that
\begin{align*}
F(0) = 1, \quad G(0) = -\frac{4}{3}, \quad K(0) = \frac{2}{3}, \quad \text{and} \quad L(0) = -1,
\end{align*}
one finds
\begin{align*}
\lim_{\theta \to 0} P^{abc}(A) = \varepsilon^{abc}, \quad \text{and} \quad \lim_{\theta \to 0} R^{abc}(A) = \frac{1}{2} \varepsilon^{abc}.
\end{align*}
(5.69)
The later guaranties that in the commutative limit the equation (5.64) reproduces
the ordinary Chern-Simons equations,
\begin{align*}
\lim_{\theta \to 0} F^a = \varepsilon^{abc} \partial^b A_c.
\end{align*}
(5.70)
The equations (5.64) are non-Lagrangian. The further physical properties and
applications will be discussed elsewhere.

6 Conclusions

To construct the L_{\infty} structure with given initial terms one has to solve the L_{\infty}
relations, \( J_n = 0 \). The key observation we made in this work is that in each
given order \( n \) the consistency condition of the equation, \( J_n = 0 \), is satisfied as a
consequence of the previously solved L_{\infty} relations, \( J_m = 0, m \leq n \). Using this
observation we were able to derive the recurrence relations for the construction
of the L_{\infty} algebra describing the NC deformation of the abelian Chern-Simons
theory in the slowly varying field approximation. Using these recurrence relations
we made a conjecture regarding the form of the NC su(2)-like deformation of the
gauge transformations and the corresponding field equations. The functional
coefficients in the proposed anzatz were fixed from the closure conditions of the
gauge algebra and the requirement of the gauge covariance of the equations of
motion correspondingly.

We conclude that the problem formulated in the introduction regarding the
existence of the solution to the L_{\infty} bootstrap programe has the positive answer.
Moreover we were able to find an explicit exemple of such a solution. Thus we
can see that L_{\infty} algebra is not only the correct mathematical framework to deal
with the deformations but also is a powerful tool for the construction of these
deformations.

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7 Appendix: Important algebraic relations

Since we are in 3d, for any vector $A^e$ one may check that,

$$\varepsilon^{abc}A^c - \varepsilon^{bce}A^a + \varepsilon^{cea}A^b - \varepsilon^{eab}A^c \equiv 0.$$  \hspace{1cm} (7.1)

Contracting the above identity with $A_e$ we arrive at,

$$\varepsilon^{abc}A^2 - \varepsilon^{bcm}A_m A^a + \varepsilon^{acm}A_m A^b - \varepsilon^{abm}A_m A^c \equiv 0.$$  \hspace{1cm} (7.2)

Taking the derivative of (7.2) with respect to $A_e$ one finds,

$$2\varepsilon^{abc}A^e - \varepsilon^{abe}A_c - \varepsilon^{abm}A_m \delta^{ce} + \varepsilon^{ace}A^b + \varepsilon^{acm}A_m \delta^{be} - \varepsilon^{bce}A_a - \varepsilon^{bcm}A_m \delta^{ae} = 0.$$  \hspace{1cm} (7.3)

Now using (7.1) in (7.3) we end up with

$$\varepsilon^{abc}A^e - \varepsilon^{abm}A_m \delta^{ce} + \varepsilon^{acm}A_m \delta^{be} - \varepsilon^{bcm}A_m \delta^{ae} \equiv 0.$$  \hspace{1cm} (7.4)

One more identity we need is

$$\varepsilon^{acm}A_m \varepsilon^{ben}A_n = \left(\delta^{ab} \delta^{ce} - \delta^{ae} \delta^{bc}\right) A^2 + \delta^{bc}A^a A^c - \delta^{ce}A^a A^b - \delta^{ab}A^c A^e + \delta^{ae}A^b A^c.$$  \hspace{1cm} (7.5)

It can be obtained from (7.2) contracting it with $\varepsilon^{cen}$ and then renaming the indices.

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