METACYCLIC ACTIONS ON SURFACES

KASHYAP RAJEEVSARATHY AND APEKSHA SANGHI

Abstract. Let Mod(S_g) be the mapping class group of the closed orientable surface S_g of genus g ≥ 2. In this paper, we derive necessary and sufficient conditions under which two torsion elements in Mod(S_g) will have conjugates that generate a finite metacyclic subgroup of Mod(S_g). This yields a complete solution to the problem of liftability of periodic mapping classes under finite cyclic covers. As applications of the main result, we show that 4g is a realizable upper bound on the order of a non-split metacyclic action on S_g and this bound is realized by the action of a dicyclic group. Moreover, we give a complete characterization of the dicyclic subgroups of Mod(S_g) up to a certain equivalence that we will call weak conjugacy. Furthermore, we show that every periodic mapping class in a non-split metacyclic subgroup of Mod(S_g) is reducible. We provide necessary and sufficient conditions under which a non-split metacyclic action on S_g factors via a split metacyclic action. Finally, we provide a complete classification of the weak conjugacy classes of the finite non-split metacyclic subgroups of Mod(S_{10}) and Mod(S_{11}).

1. Introduction

Let S_g be the closed orientable surface of genus g ≥ 0, Homeo^+(S_g) be the group of orientation-preserving homeomorphisms of S_g, and let Mod(S_g) be the mapping class group of S_g. Given periodic elements F, G ∈ Mod(S_g) such that ⟨F, G⟩ is finite, a pair of conjugates F', G' (of F, G resp.) may (or may not) generate a subgroup isomorphic to ⟨F, G⟩ (see [4]). A natural question that arises in this context is whether one can derive equivalent conditions under which ⟨F', G'⟩ ∼= ⟨F, G⟩. In this paper, we answer this question in the affirmative for the case when ⟨F, G⟩ is a metacyclic group [6] by deriving elementary number theoretic conditions for the same. Moreover, considering the fact that every cyclic subgroup of Mod(S_g) that lifts under a finite cyclic cover lifts to metacyclic group (see [10]), our main result can also be viewed as a solution to the problem of liftability of periodic mapping classes under such covers up to conjugacy. Our main result (see Theorem 3.6) is a complete generalization of the main results in [3] and [4], where the analogous problem for split metacyclic groups was considered. The proof of this result applies the theory of group actions on surfaces [7,9] and Thurston’s orbifold theory [16, Chapter 13].

A finite metacyclic group M(u, n, r, k) of order u · n, amalgam r and twist factor k admits a presentation of the form

⟨F, G | G^n = F^r, F^n = 1, G^{-1}FG = F^k⟩.

2020 Mathematics Subject Classification. Primary 57K20, Secondary 57M60.

Key words and phrases. surface; mapping class; finite order maps; metacyclic subgroups.
A metacyclic group $\mathcal{M}(u,n,r,k)$ is said to be split if $\mathcal{M}(u,n,r,k) \cong \mathbb{Z}_{m'} \times \mathbb{Z}_{n'}$ for some positive integers $m', n'$. Consider a finite metacyclic subgroup $H = \langle F, G \rangle$ of $\text{Mod}(S_g)$ as above. In view of the Nielsen realization theorem \cite{8} \cite{12}, the $H$ acts on $S_g$ yielding a quotient orbifold $\mathcal{O}_H = S_g/H$, and $F$ has a Nielsen representative $F \in \text{Homeo}^+(S_g)$ of the same order. In a recent paper \cite{1}, a symplectic criterion has been derived for the liftability of mapping classes under regular cyclic covers. Moreover, it is well-known \cite{18} that for a $G \in \text{Mod}(S_g)$ of finite order, $|G| \leq 4g + 2$. These results motivate the following immediate corollary of our main result (from Section 3).

**Corollary 1.** Let $H = \langle F, G \rangle$ be a metacyclic subgroup of $\text{Mod}(S_{n(g−1)+1})$, where $|F| = n$ and $\langle F \rangle$ acts freely on $S_{n(g−1)+1}$ inducing a regular cover $S_{n(g−1)+1} \to S_g$.

(i) If $\mathcal{O}_H$ has genus zero, then $H$ is a split metacyclic group.

(ii) If $n$ is prime and $(4g + 2) | (n − 1)$, then there exists a $G \in \text{Mod}(S_g)$ of order $4g + 2$ that lifts to $G$.

In Section \cite{4} we provide several applications of our main theorem. A metacyclic group of the form $\mathcal{M}(2, 2n, n, −1)$ is called a dicyclic group denoted by $\text{Dic}_n$. As a first application, we derive a realizable bound on $|H|$ when $H < \text{Mod}(S_g)$ is a non-split metacyclic subgroup.

**Proposition 1.** Let $H < \text{Mod}(S_g)$ be a non-split metacyclic subgroup. Then $|H| \leq 4g$ and this bound is realized when $H \cong \text{Dic}_g$.

It may be noted here that a bound on the order of an arbitrary metacyclic subgroup of $\text{Mod}(S_g)$ was derived in \cite{15}. As an immediate application of Proposition \cite{4} we obtain the following.

**Corollary 2.** Let $H < \text{Mod}(S_g)$ be a finite metacyclic subgroup.

(i) If $H$ is non-split, then any $F \in H$ is a reducible mapping class.

(ii) If $H$ contains an irreducible mapping class $F$, then $H$ is split. Furthermore, if $H = \langle F, G \rangle$ and $\langle F \rangle \triangleleft H$ with $|F| = n$, then either $H \cong \mathbb{Z}_n \times \mathbb{Z}_2$ or $H \cong \mathbb{Z}_n \times \mathbb{Z}_3$.

Furthermore, taking motivation from Proposition \cite{1} we give a combinatorial classification of dicyclic subgroups of $\text{Mod}(S_g)$ (see Proposition \cite{4,8}).

Since a non-split metacyclic group is a quotient of a split metacyclic group with a cyclic subgroup, a natural question is when would a non-split metacyclic action on $S_g$ factor through a split metacyclic action. In this connection, we provide equivalent conditions under which a non-split metacyclic action on $S_g$ would lift under a regular cyclic cover to a split metacyclic action (see Proposition \cite{4,9}). By applying these conditions, we obtain the following corollary.

**Corollary 3.** The actions on $S_g$ of the metacyclic groups $\text{Dic}_n$, $\text{Dic}_n \times \mathbb{Z}_m$, and $\text{Dic}_n \times \mathbb{Z}_m \times \mathbb{Z}_p$, where $n$ is even integer and $m, p$ are odd integers with $\gcd(p, n) = 1$, factor via split metacyclic actions.

The geometric realizations of several split metacyclic actions on $S_g$ were described in \cite{4}. But the realizations of non-split metacyclic group actions on $S_g$ are far more challenging as these groups are not realizable as isometry groups of $\mathbb{R}^3$. However, Corollary \cite{3} (see also Proposition \cite{4,9}) enables
us to realize the lifts of certain metacyclic actions under suitably chosen regular cyclic covers. In Section 5, we describe nontrivial geometric realizations of some finite split metacyclic actions that are lifts of non-split metacyclic actions on $S_{10}$ and $S_{11}$ under regular cyclic covers. Finally, in Section 6, we provide classifications of the finite non-split metacyclic subgroups of $\text{Mod}(S_{10})$ and $\text{Mod}(S_{11})$ up to a certain equivalence that we call weak conjugacy (see Section 3) which is central to the theory developed in this paper.

2. Preliminaries

In this section, we introduce some basic notions from Thurston’s orbifold theory [16, Chapter 13] and the theory of group actions on surfaces [7, 9] that are crucial to the theory we develop in this paper.

2.1. Fuchsian groups. Let $\text{Homeo}^+(S_g)$ denote the group of orientation-preserving homeomorphisms of $S_g$, and let $H < \text{Homeo}^+(S_g)$ be a finite group. A faithful and properly discontinuous $H$-action on $S_g$ induces a branched covering $S_g \to O_H := S_g/H$ with $\ell$ cone points $x_1, \ldots, x_\ell$ on the quotient orbifold $O_H \approx S_{g_0}$ (which we will call the corresponding orbifold) of orders $n_1, \ldots, n_\ell$, respectively. Then the orbifold fundamental group $\pi_1^\text{orb}(O_H)$ of $O_H$ has a presentation given by

$$\langle \alpha_1, \beta_1, \ldots, \alpha_{g_0}, \beta_{g_0}, \xi_1, \ldots, \xi_\ell \mid \xi_1^{n_1}, \ldots, \xi_\ell^{n_\ell}, \prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i] \rangle.$$  

In classical parlance, $\pi_1^\text{orb}(O_H)$ is also known as a Fuchsian group [16, 9] with signature

$$\Gamma(O_H) := (g_0; n_1, \ldots, n_\ell),$$

and the relation $\prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i]$ appearing in its presentation in [11] is called the long relation. From Thurston’s orbifold theory [16, Chapter 13], we obtain exact sequence

$$1 \to \pi_1(S_g) \to \pi_1^\text{orb}(O_H) \xrightarrow{\phi_H} H \to 1,$$

where $\phi_H$ is called the surface kernel epimorphism. In this setting, we will require the following result due to Harvey [5], which gives a characterization of finite group action on surfaces.

**Theorem 2.1.** A finite group $H$ acts faithfully on $S_g$ with $\Gamma(O_H) = (g_0; n_1, \ldots, n_\ell)$ if and only if it satisfies the following two conditions:

(i) $\frac{2g - 2}{|H|} = 2g_0 - 2 + \sum_{i=1}^{\ell} \left( 1 - \frac{1}{n_i} \right)$, and

(ii) there exists a surjective homomorphism $\phi_H : \pi_1^\text{orb}(O_H) \to H$ that preserves the orders of all torsion elements of $\pi_1^\text{orb}(O_H)$.
2.2. Cyclic actions on surfaces. For \( g \geq 1 \), let \( F \in \text{Mod}(S_g) \) be of order \( n \). By the Nielsen-Kerckhoff theorem \([8, 12]\), \( F \) is represented by a standard representative \( F \in \text{Homeo}^+(S_g) \) of the same order. We will refer to both \( F \) and the group it generates, interchangeably, as a \( \mathbb{Z}_n \)-action on \( S_g \).

Each cone point \( x_i \in O(F) \) lifts to an orbit of size \( \frac{n}{n_i} \) on \( S_g \), and the local rotation induced by \( F \) around the points in each orbit is given by \( 2\pi c_i - 1 \) mod \( n_i \), where \( \gcd(c_i, n_i) = 1 \) and \( c_i c_i^{-1} \equiv 1 \) (mod \( n_i \)). Further, by virtue of Theorem 2.1, there exists an exact sequence:

\[
1 \to \pi_1(S_g) \to \pi_{\text{orb}}^1(O(F)) \xrightarrow{\phi(F)} \langle F \rangle \to 1,
\]

where \( \phi(F)(\xi_i) = F^{(\frac{n}{n_i})c_i} \), for \( 1 \leq i \leq \ell \). We will now introduce a tuple of integers that encodes the conjugacy class of a \( \mathbb{Z}_n \)-action on \( S_g \).

**Definition 2.2.** A cyclic data set of degree \( n \) is a tuple

\[
D = (n, g_0, d; (c_1, n_1), \ldots, (c_\ell, n_\ell)),
\]

where \( n \geq 2 \), \( g_0 \geq 0 \), and \( 0 \leq d \leq n - 1 \) are integers, and each \( c_i \in \mathbb{Z}_{n_i}^* \) such that:

(i) \( d > 0 \) if and only if \( \ell = 0 \) and \( \gcd(d, n) = 1 \), whenever \( d > 0 \),
(ii) each \( n_i \mid n \),
(iii) \( \lcm(n_1, \ldots, n_\ell) = N \), for \( 1 \leq i \leq \ell \), where \( N = n \) if \( g_0 = 0 \), and
(iv) \( \sum_{j=1}^{\ell} \frac{n}{n_j} c_j \equiv 0 \) (mod \( n \)).

The number \( g \) determined by the Riemann-Hurwitz equation

\[
\frac{2 - 2g}{n} = 2 - 2g_0 + \sum_{j=1}^{\ell} \left( \frac{1}{n_j} - 1 \right)
\]

is called the genus of the data set, denoted by \( g(D) \).

Note that quantity \( d \) (in Definition 2.2) will be non-zero if and only if \( D \) represents a free rotation of \( S_g \) by \( 2\pi d/n \), in which case, \( D \) will take the form \((n, g_0, d;)\). We will not include \( d \) in the notation of a data set, whenever \( d = 0 \).

By the Nielsen-Kerckhoff theorem, the quotient map \( \text{Homeo}^+(S_g) \to \text{Mod}(S_g) \) induces a correspondence between the conjugacy classes of finite-order maps in \( \text{Homeo}^+(S_g) \) and the conjugacy classes of finite-order mapping classes in \( \text{Mod}(S_g) \). This leads us to the following lemma primarily due to Nielsen \([11]\) which allows us to use data sets to describe the conjugacy classes of cyclic actions on \( S_g \) (see also \([14, \text{Theorem 3.8}] \) and \([9]\)).

**Lemma 2.3.** For \( g \geq 1 \) and \( n \geq 2 \), data sets of degree \( n \) and genus \( g \) correspond to conjugacy classes of \( \mathbb{Z}_n \)-actions on \( S_g \).

We will denote the data set corresponding to the conjugacy class of a periodic mapping class \( F \) by \( D_F \). For compactness of notation, we also write a data set \( D \) (as in Definition 2.2) as

\[
D = (n, g_0, d; ((d_1, m_1), \alpha_1), \ldots, ((d_\ell, m_\ell), \alpha_\ell)),
\]
where \((d_i, m_i)\) are the distinct pairs in the multiset \(S = \{(c_1, n_1), \ldots, (c_\ell, n_\ell)\}\), and the \(a_i\) denote the multiplicity of the pair \((d_i, m_i)\) in the multiset \(S\). Further, we note that every cone point \([x] \in \mathcal{O}_F\) corresponds to a unique pair in the multiset \(S\) appearing in \(D_F\), which we denote by \(P_x := (c_x, n_x)\).

Given \(u \in \mathbb{Z}_m^\times\) and \(\mathcal{G} \in H < \text{Homeo}^+(S_g)\) be of order \(m\), let \(\mathcal{F}_\mathcal{G}(u, m)\) denote the set of fixed points of \(\mathcal{G}\) with induced rotation angle \(2\pi u/m\). Let \(C_H(\mathcal{G})\) be the centralizer of \(\mathcal{G} \in H\) and \(\sim\) denote the conjugation relation between any two elements in \(H\). We conclude this subsection by stating the following result from the theory of Riemann surfaces [2], which we will use in the proof of our main theorem.

**Lemma 2.4.** Let \(H < \text{Homeo}^+(S_g)\) with \(\Gamma(\mathcal{O}_H) = (g_0; n_1, \ldots, n_\ell)\), and let \(\mathcal{G} \in H\) be of order \(m\). Then for \(u \in \mathbb{Z}_m^\times\), we have

\[
|\mathcal{F}_\mathcal{G}(u, m)| = |C_H(\mathcal{G})| \cdot \frac{1}{\mathcal{F}_\mathcal{G}(u, m)}
\]

2.3. **Metacyclic actions on surfaces.** Given integers \(u, n \geq 2\), \(r \mid n\) and \(k \in \mathbb{Z}_n^\times\) such that \(k^n \equiv 1 \pmod{n}\), a finite metacyclic action of order \(un\) (written as \(u \cdot n\)) on \(S_g\) is a tuple \((H, (\mathcal{G}, F))\), where \(H < \text{Homeo}^+(S_g)\), and

\[
H = \langle F, \mathcal{G} \mid F^r = \mathcal{G}^n, F^n = 1, \mathcal{G}^{-1}F\mathcal{G} = F^k \rangle \cong \mathcal{M}(u, n, r, k),
\]

Metacyclic groups have been completely classified in [3]. We will call the multiplicative class \(k\) the **twist factor** and \(r\) be the **amalgam** of the metacyclic action \((H, (\mathcal{G}, F))\). In the presentation above, if we further assume that \(r = n\), then \(\mathcal{M}(u, n, r, k) \cong \mathbb{Z}_n \times_k \mathbb{Z}_u\), which is also known as a **split metacyclic group**. Since split metacyclic actions on surfaces have been analyzed in [4], we will focus our attention on the finite non-split metacyclic subgroups of \(\text{Mod}(S_g)\).

Since \(\langle F \rangle < H\), \(\mathcal{G}\) would induce a \(\mathcal{G} \in \text{Homeo}^+(\mathcal{O}_F)\) (see [17]) that restricts to an order-preserving bijection on the set of cone points in \(\mathcal{O}_F\). We will call \(\mathcal{G}\), the **induced automorphism** on \(\mathcal{O}_F\) by \(\mathcal{G}\), and we formalize this notion in the following definition.

**Definition 2.5.** Let \(H < \text{Homeo}^+(S_g)\) be a finite cyclic group with \(|H| = n\). We say a \(\mathcal{G} \in \text{Homeo}^+(\mathcal{O}_H)\) is an **automorphism of** \(\mathcal{O}_H\) if for \([x], [y] \in \mathcal{O}_H\), \(k \in \mathbb{Z}_n^\times\) and \(\mathcal{G}([x]) = [y]\), we have:

(i) \(n_x = n_y\), and
(ii) \(c_x = kc_y\).

We denote the group of automorphisms of \(\mathcal{O}_H\) by \(\text{Aut}_k(\mathcal{O}_H)\).

We will require the following technical lemma that gives some basic properties of the induced automorphism.

**Lemma 2.6.** Let \(\mathcal{G}, F \in \text{Homeo}^+(S_g)\) be maps of orders \(m, n\), respectively, such that \(\mathcal{G}^{-1}F\mathcal{G} = F^k\), and let \(H = \langle F \rangle\). Then:

(i) \(\mathcal{G}\) induces a \(\bar{\mathcal{G}} \in \text{Aut}_k(\mathcal{O}_H)\) such that

\[
\mathcal{O}_H/\langle \mathcal{G} \rangle = S_g/\langle F, \mathcal{G} \rangle,
\]

(ii) \(|\bar{\mathcal{G}}|\) divides \(|\mathcal{G}|\), and
(iii) $|\mathcal{G}| < m$ if and only if $F^r = G^u$, for some $0 < r < n$ and $0 < u < m$.

3. Main theorem

In this section, we establish the main result of the paper by deriving equivalent conditions under which torsion elements $F, G \in \text{Mod}(S_g)$ can have conjugates $F', G' \in \text{Mod}(S_g)$ such that $(F', G') \cong \mathcal{M}(u, |F'|, r, k)$. We begin by introducing a weaker notion of conjugacy that arises very naturally in this context.

**Definition 3.1.** Two finite metacyclic actions $(H_1, (G_1, F_1))$ and $(H_2, (G_2, F_2))$ of order $u \cdot n$, amalgam $r$ and twist factor $k$ are said to be weakly conjugate if there exists an isomorphism, $\psi : \pi_1^{\text{orb}}(\overline{O}_{H_1}) \cong \pi_1^{\text{orb}}(\overline{O}_{H_2})$ and an isomorphism $\chi : H_1 \to H_2$ such that the following conditions hold.

(i) $\chi((G_1, F_1)) = (G_2, F_2)$.

(ii) For $i = 1, 2$, let $\phi_{H_i} : \pi_1^{\text{orb}}(\overline{O}_{H_i}) \to H_i$ be surface kernel epimorphisms. Then $(\chi \circ \phi_{H_1})(g) = (\phi_{H_2} \circ \psi)(g)$, whenever $g \in \pi_1^{\text{orb}}(\overline{O}_{H_1})$ is of finite order.

(iii) The pair $(G_1, F_1)$ is conjugate (component-wise) to the pair $(G_2, F_2)$ in Homeo$^+(S_g)$.

The notion of weak conjugacy defines an equivalence relation on metacyclic actions on $S_g$ and the equivalence classes thus obtained will be called weak conjugacy classes.

**Remark 3.2.** In view of the Nielsen-Kerckhoff theorem, the notion of weak conjugacy (from Definition 3.1) naturally extends to an analogous notion in $\text{Mod}(S_g)$ via the natural association

$$(\langle F, G \rangle, (G, F)) \leftrightarrow (\langle F, G \rangle, (G, F)).$$

For brevity, we will introduce the following notation.

**Definition 3.3.** Let $F, G \in \text{Mod}(S_g)$ be of finite order with $|F| = n$. Then for some $k \in \mathbb{Z}^\times_n \setminus \{1\}$, we say (in symbols) that $[F, G]_{u, r, k}$ is 1 if there exists conjugates $F', G'$ (of $F, G$ resp.) such that $(F', G') \cong \mathcal{M}(u, n, r, k)$.

We will now define an abstract tuple of integers that will encode the weak conjugacy class of a finite metacyclic action on $S_g$.

**Definition 3.4.** A metacyclic data set of degree $u-n$, twist factor $k$, amalgam $r$ and genus $g \geq 2$ is a tuple

$$\mathcal{D} = ((u \cdot n, r, k), g_0, [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \ldots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]),$$

where $u, n \geq 2$, the $n_{ij}$ are positive integers for $1 \leq i \leq \ell$, $1 \leq j \leq 2$, the $c_{ij} \in \mathbb{Z}_{n_{ij}}$, $r \mid n$ and $k \in \mathbb{Z}^\times_n$ such that $k^n \equiv 1 \pmod{n}$ and there exists a $w \in \mathbb{Z}$, satisfying the following conditions.

(i) \[
\frac{2g - 2}{un} = 2g_0 - 2 + \sum_{i=1}^\ell \left(1 - \frac{1}{n_{ii}}\right).
\]

(ii) (a) For each $i, j$, $n_{ij} \mid \frac{un}{r} = m$, $n_{ij} \mid n$, either $\text{gcd}(c_{ij}, n_{ij}) = 1$ or $c_{ij} = 0$, and $c_{ij} = 0$ if and only if $n_{ij} = 1$.

(b) For each $i$, $n_i = s_i$, where $s_i$ is the least positive integer satisfying the following conditions for some $t_i \in \mathbb{N}$:
(i) \( c_{i_1 \frac{m_1}{n_1}} s_i \equiv \ell_i u \pmod{m} \).

(ii) \( c_{i_2 \frac{n_2}{n_1}} (k^{c_{i_1 \frac{m_1}{n_1}} (s_i - 1)} + \cdots + k^{c_{i_1 \frac{m_1}{n_1}}} + 1) \equiv -t_i r \pmod{n} \).

(iii) \( \sum_{i=1}^\ell c_{i_1 \frac{m}{n_{i_1}}} \equiv wu \pmod{m} \).

(iv) Defining \( A := \sum_{i=1}^\ell c_{i_2 \frac{n}{n_2}} \prod_{s=i+1}^{\ell v} k^{c_{i_1 \frac{m}{m_2}}} \) and \( d := \gcd(n, k - 1) \), we have

\[
A \equiv \begin{cases} -wr \pmod{n}, & \text{if } g_0 = 0, \\ d\theta - wr \pmod{n}, & \text{for } \theta \in \mathbb{Z}_n, \text{ if } g_0 \geq 1. \end{cases}
\]

(v) If \( g_0 = 0 \), there exists \( (p_1, \ldots, p_{\ell v}), (q_1, \ldots, q_{\ell v}) \in \mathbb{Z}^{\ell v}, v \in \mathbb{N}, \) and \( a, b \in \mathbb{Z} \) such that the following conditions hold.

(a) \( \sum_{i'=1}^{\ell v} p_{i' c_1 \frac{m}{n_{i_1}}} \equiv 1 + au \pmod{m} \) and

\[
\sum_{i'=1}^{\ell v} c_{i_2 \frac{n}{n_2}} \left( \sum_{s=1}^{p_{i' c_1 \frac{m}{n_{i_1}}}} (p v - s) \right) \left( \prod_{i'=i'+1}^{\ell v} k^{p_{i' c_1 \frac{m}{n_{i_1}}}} \right) \equiv -ar \pmod{n},
\]

(b) \( \sum_{i'=1}^{\ell v} q_{i' c_1 \frac{m}{n_{i_1}}} \equiv bu \pmod{m} \) and

\[
\sum_{i'=1}^{\ell v} c_{i_2 \frac{n}{n_2}} \left( \sum_{s=1}^{q_{i' c_1 \frac{m}{n_{i_1}}} (q v - s)} \right) \left( \prod_{i'=i'+1}^{\ell v} k^{q_{i' c_1 \frac{m}{n_{i_1}}}} \right) \equiv 1 - br \pmod{n},
\]

where

\[
i \equiv \begin{cases} \ell' \pmod{\ell}, & \text{if } \ell' \not\equiv 0 \pmod{\ell}, \\ \ell, & \text{otherwise}. \end{cases}
\]

(vi) If \( g_0 = 1 \), there exists \( (p_1, \ldots, p_{\ell v}), (q_1, \ldots, q_{\ell v}) \in \mathbb{Z}^{\ell v}, m', n', a, b \in \mathbb{Z}, \) and \( v \in \mathbb{N} \) such that \( m' \mid m \) and \( n' \mid n \), satisfying the following conditions.

(a) \( \sum_{i'=1}^{\ell v} p_{i' c_1 \frac{m}{n_{i_1}}} \equiv m' + au \pmod{m} \) and

\[
\sum_{i'=1}^{\ell v} c_{i_2 \frac{n}{n_2}} \left( \sum_{s=1}^{p_{i' c_1 \frac{m}{n_{i_1}}}} (p v - s) \right) \left( \prod_{i'=i'+1}^{\ell v} k^{p_{i' c_1 \frac{m}{n_{i_1}}}} \right) \equiv -ar \pmod{n},
\]

(b) \( \sum_{i'=1}^{\ell v} q_{i' c_1 \frac{m}{n_{i_1}}} \equiv bu \pmod{m} \) and

\[
\sum_{i'=1}^{\ell v} c_{i_2 \frac{n}{n_2}} \left( \sum_{s=1}^{q_{i' c_1 \frac{m}{n_{i_1}}} (q v - s)} \right) \left( \prod_{i'=i'+1}^{\ell v} k^{q_{i' c_1 \frac{m}{n_{i_1}}}} \right) \equiv n' - br \pmod{n},
\]

where

\[
i \equiv \begin{cases} \ell' \pmod{\ell}, & \text{if } \ell' \not\equiv 0 \pmod{\ell}, \\ \ell, & \text{otherwise}, \end{cases}
\]
respectively. But by carefully applying Lemma 2.4, we see that
\[
\alpha \mapsto \beta \quad \text{gives an epimorphism.}
\]
Moreover, by carefully choosing the images of the torsion elements. Let \( D \) be a metacyclic data set of degree \( u \cdot n \) with twist factor \( k \), amalgam \( r \), and genus \( g \) such that \( r | n \) and \( k \in \mathbb{Z}_n^\times \), the metacyclic data sets of degree \( u \cdot n \) with twist factor \( k \), amalgam \( r \), and genus \( g \) correspond to the weak conjugacy classes of \( M(u, n, r, k) \)-actions on \( S_g \).

Proof. Let \( D \) be a metacyclic data set of degree \( u \cdot n \) with twist factor \( k \), amalgam \( r \) and genus \( g \) (as in Definition 3.4 above). We need to show that \( D \) represents the weak conjugacy class of a \( M(u, n, r, k) \)-action on \( S_g \) represented by \( (H, (\mathcal{G}, \mathcal{F})) \), where \( H = \langle \mathcal{F}, \mathcal{G} \rangle \). To see this, we first show the existence of an epimorphism \( \phi_H : \pi_1^{orb}(O_H) \to H \) that preserves the orders of the torsion elements. Let \( H \) and \( \pi_1^{orb}(O_H) \) have presentations given by
\[
\langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \xi_1, \cdots, \xi_l | \xi_1^n = \cdots = \xi_l^n = \prod_{j=1}^l \xi_j^{g_0} \prod_{i=1}^l [\alpha_i, \beta_i] = 1 \rangle,
\]
respectively.

We consider the map
\[
\xi_i \mapsto \phi_H(\xi_i) = G^{c_1 \cdot \frac{m}{n_1} \cdot \frac{n}{n_2} f^{c_2 \cdot \frac{n}{n_2}}} \quad \text{for } 1 \leq i \leq \ell,
\]
where \( m := \frac{un}{l} \). Then condition (ii) of Definition 3.4 would imply that \( \phi_H \)

is a map which is order-preserving on torsion elements. For clarity, we break the argument for the surjectivity of \( \phi_H \) into the following three cases.

Case 1: \( g_0 = 0 \). Then it follows from conditions (iii)-(iv) that \( \phi_H \) satisfies the long relation \( \prod_{i=1}^l \xi_i = 1 \). Moreover, the surjectivity of \( \phi_H \) follows from condition (v).

Case 2: \( g_0 \geq 2 \). In this case, \( \pi_1^{orb}(O_H) \) has additional hyperbolic generators (viewing them as isometries of the hyperbolic plane), namely the \( \alpha_i \) and the \( \beta_i \). Extending \( \phi_H \) by mapping \( \alpha_1 \mapsto \mathcal{G}, \beta_1 \mapsto \mathcal{F} \) yields an epimorphism. Moreover, by carefully choosing the images of the \( \alpha_i \) and the \( \beta_i \), for \( i \geq 2 \), conditions (iii)-(iv) would ensure that the long relation \( \prod_{j=1}^l \xi_j \prod_{i=1}^l [\alpha_i, \beta_i] = 1 \) is satisfied.

Case 3: \( g_0 = 1 \). In this case, \( \pi_1^{orb}(O_H) \) would have two additional hyperbolic generators, namely the \( \alpha_1 \) and the \( \beta_1 \). We extend \( \phi_H \) by defining \( \alpha_1 \mapsto \mathcal{G}^a \) and \( \beta_1 \mapsto \mathcal{F}^\beta \). We then apply conditions (iii),(iv) and (vi) to obtain the desired epimorphism.

It remains to be shown that \( D \) determines \( \mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_g) \) up to conjugacy. But by carefully applying Lemma 2.4, we see that
\[
D_F = (n, g_1; \langle (v_{ij}^{-1}, t_i) \rangle, \frac{t_i [\mathcal{G}^u (v_{ij}, t_i)]}{n} : v_{ij} \in \mathbb{Z}_n^\times, t_i | n),
\]
Moreover, condition (ii) follows from the fact that \( \phi \) satisfies condition (i) of Definition 3.4. Furthermore, conditions (iii)-(iv) follow from the long exact sequence:

\[
D_G = (m, g_2; ((u_{ij}^{-1}, m), m_i | \frac{m}{m_i})(u_{ij}, m_i)) : u_{ij} \in \mathbb{Z}_{m_i}, m_i | m),
\]

and \( g_1 \) is determined by Riemann-Hurwitz equation, and

\[
|\mathfrak{U}_{F}^n|(v_{ij}, t_i) = |\mathfrak{F}_{F}^n|(v_{ij}, t_i) - \sum_{t_i \in \mathbb{N}} \sum_{t_j \neq t_i, v_{ij} \equiv v_{i'j'} (\mod t_i, t_j | t_j)} |\mathfrak{U}_{F}^{n'}|(v_{i'j'}, t_{i'})|
\]

and \( g_2 \) is determined by Riemann-Hurwitz equation.

Conversely, consider the weak conjugacy class of \( \mathcal{M}(u, n, r, k) \)-actions on \( S_g \) represented by \( (H, (G, F)) \), where \( H = \langle F, G \rangle \). By Theorem 2.1, there exists a surjective homomorphism

\[
\phi_H : \pi_1^{orb}(O_H) \to H : \xi_i \mapsto H \mathbb{Z}_{m_i} F^{c_1} \mathbb{Z}_{m_1} F^{c_2} \mathbb{Z}_{m_2}, \text{ for } 1 \leq i \leq \ell,
\]

which is order-preserving on the torsion elements. This yields a metacyclic data set \( D \) of degree \( u \cdot n \) with twist factor \( k \), amalgam \( r \) and genus \( g \) as in Definition 3.4. By Theorem 2.1, \( D \) satisfies condition (i) of Definition 3.4. Moreover, condition (ii) follows from the fact that \( \phi_H \) is order-preserving on torsion elements. Furthermore, conditions (iii)-(iv) follow from the long relation satisfied by \( \pi_i^{orb}(O_H) \), and condition (v)-(vi) are implied by the surjectivity of \( \phi_H \). Thus, we obtain the metacyclic data set \( D \) of degree \( u \cdot n \) with twist factor \( k \), amalgam \( r \) and genus \( g \), and our assertion follows.

We denote the data sets \( D_F \) and \( D_G \) (representing the cyclic factors of \( H \)) derived from the metacyclic data set \( D \) appearing in the proof of Proposition 3.5 by \( D_1 \) and \( D_2 \), respectively. Thus, our main theorem will now follow from Proposition 3.5.

**Theorem 3.6 (Main theorem).** Let \( F, G \in \text{Mod}(S_g) \) be of orders \( n, m \), respectively. Then \( [F, G]_{u, r, k} = 1 \) if and only if there exists a metacyclic data set \( D \) of degree \( u \cdot n \), twist factor \( k \), amalgam \( r \), and genus \( g \) such that \( D_1 = D_F \) and \( D_2 = D_G \).

### 3.1. Liftability of torsion under finite cyclic covers

From the viewpoint of liftability, a metacyclic group \( \langle F, G \rangle \) acts on \( S_g \) if and only if there exists \( \mathcal{G} \in \text{Aut}_b(O_F) \) that lifts under the branched cover \( S_g \to O_F \) to \( G \). From Birman-Hilden theory [10], this is equivalent to requiring the existence of a short exact sequence:

\[
1 \to \langle F \rangle \to \langle F, G \rangle \to \langle \mathcal{G} \rangle \to 1.
\]

Let \( S_{h,b} \) denote the closed oriented surface of genus \( h \) with \( b \) marked points. Given a branched cover \( p : S_g \to S_{h,b} \), let \( \text{LMod}_p(S_{h,b}) \) (resp. \( \text{SMod}_p(S_g) \)) denote the liftable (resp. symmetric) mapping class group of \( p \). Our main theorem can now be equivalently stated as follows.
Theorem 3.7 (Main theorem-Alternative version). Let $p : S_g \to S_{h,b}$ be an $n$-sheeted cover with deck transformation group $\langle F \rangle \cong \mathbb{Z}_n$. Then $G \in \text{LMod}_p(S_{h,b})$ lifts to a $G \in \text{SMod}_p(S_g)$ if and only if there exists a metacyclic data set $D$ of degree $u \cdot n$, twist factor $k$, amalgam $r$, and genus $g$ such that $D_1 = D_F$ and $D_2 = D_G$.

Thus, the main theorem provides necessary and sufficient conditions under which periodic elements of mapping class groups will lift under finite cyclic covers.

Remark 3.8. Given a metacyclic data set $D = ((u \cdot n, r, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \ldots, [(c_{\ell_1}, n_{\ell_1}), (c_{\ell_2}, n_{\ell_2}), n_\ell])$, encoding the weak conjugacy class represented by $\langle F \rangle$, of degree $u \cdot n$, twist factor $k$, and genus $n(g - 1) + 1$. Following the notation in the proof of Proposition 3.6, let $\phi_H(\xi_i) = G^\gamma_i \mathcal{F}^{\delta_i}$, where $\gamma_i = c_{1i} m/n_{1i}, \delta_i = c_{2i} n/n_{2i}$, and $\xi_i \in \pi_1^\text{orb}(O_H)$ is the generator enclosing the cone point of order $n_i$. Now, as $\mathcal{F}$ generates a free action on $S_{n(g - 1)+1}$, we have that $\langle G^n \mathcal{F}^{\delta_i} \rangle \cap \langle F \rangle = \langle \text{id} \rangle$. Hence, it follows that $\langle G^n \mathcal{F}^{\delta_i} \rangle$ is a split metacyclic group for all $i$, and consequently, $\langle G^n, F \rangle$ is a split metacyclic group for all $i$.

Now, we claim that $\langle G^n_1, \ldots, G^n_{\ell}, F \rangle$ is a split metacyclic group. We establish this claim by inducting on $\ell$. From the preceding argument, the statement holds for $\ell = 1$. For $\ell = 2$, we have to show that $\langle G^n_1, G^n_2, F \rangle$ is a split metacyclic group. We can write $\langle G^n_1, G^n_2, F \rangle = \langle G', F \rangle$, where $\langle G' \rangle = \langle G^{\gcd(\gamma_1, \gamma_2)} \rangle = \langle G^n_1, G^n_2 \rangle$. Suppose we assume on the contrary that $\langle G'^a \rangle = F^b$, for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_n$ with $b \neq 0$. Then $\langle (G')^a \rangle \subseteq \langle G^n_1, G^n_2 \rangle$, and so we have that $\langle (G')^a \rangle \subseteq \langle G^n_1 \rangle$ or $\langle G^n_2 \rangle$, for some $t$ such that $(G')^a \neq 1$. Hence, it follows that $G^n_{\ell_1} = (G')^a = F^{bt}$ or $G^n_{\ell_2} = (G')^a = F^{bt}$, for some $t_1, t_2 \in \mathbb{Z}$, which contradicts the fact that $\langle G^n, F \rangle$ is a split metacyclic group. Therefore, our claim holds true for $\ell = 2$.

Suppose we assume that our claim holds for $\ell - 1$. By similar arguments (as above), we have that $\langle G^n_1, \ldots, G^n_{\ell}, F \rangle = \langle G'^a, F \rangle$, where $\langle G'^a \rangle$ is split metacyclic group. So, it immediately follows from the case $\ell = 2$ that our
claim holds for $\ell$. Since $\phi_H$ is surjective, we have $(G^\alpha F^{\delta_1}, \ldots, G^\alpha F^{\delta_r}) = H$, and hence it follows that $H$ is a split metacyclic group.

As mentioned earlier, the bound on the order of a periodic mapping class $G \in \text{Mod}(S_g)$ is $4g + 2$ (see [13]) which is realized by the action $D_G = (4g + 2, 0; (1, 2), (1, 2g + 1), (2g - 1, 4g + 2))$. This inspires the following corollary.

**Corollary 3.10.** Let $p : S_{n(g-1)+1} \to S_g$ be a finite $n$-sheeted regular cover with deck transformation group $(\mathcal{F}) \cong \mathbb{Z}_n$. If $n$ is prime and $(4g + 2) \mid (n - 1)$, then there exists a $G \in \text{LMod}_p(S_g)$ with $|G| = 4g + 2$.

**Proof.** Since $(4g + 2) \mid (n - 1)$, there exists a $k \in \mathbb{Z}_n^*$ such that $|k| = 4g + 2$. Let $G \in \text{SMod}_p(S_{n(g-1)+1})$ be a lift of $\mathcal{G}$. Now from Proposition 3.6 it can be easily seen that the metacyclic data set $D = ((n \cdot 4g + 2, n, k), 0; [(1, 2), (1, n), 2], [(1, 2g + 1), (n - k^2, n), 2g + 1], [(2g - 1, 4g + 2), (0, 1), 4g + 2])$ represents the weak conjugacy class of $\langle (\mathcal{F}, \mathcal{G}), \mathcal{G}, \mathcal{F} \rangle$ with $D_1 = D_F = (n, g, 1); D_2 = D_G = (4g + 2, \frac{(n - 1)(g - 1)}{4g + 2}; (1, 2), (1, 2g + 1), (2g - 1, 4g + 2))$ and $D_G = (4g + 2, 0; (1, 2), (1, 2g + 1), (2g - 1, 4g + 2))$. Hence, our assertion follows. □

4. Applications

4.1. Bound on the order of a non-split metacyclic action. In this subsection, we derive a realizable bound for the order of a non-split metacyclic subgroup of $\text{Mod}(S_g)$. We will need the following technical lemma concerning metacyclic groups.

**Lemma 4.1.** Let $H = \langle F, G \mid G^n = F^r, F^n = 1, G^{-1}FG = F^k \rangle$. Suppose that there exists $x, y \in H$ such that $H = \langle x, y \rangle$ and at least one of $x$ or $y$ is of prime order. Then $H$ is a split metacyclic group.

**Proof.** Let $x = G^{\alpha_1}F^{\beta_1}$ and $y = G^{\alpha_2}F^{\beta_2}$ and let us assume without loss of generality that $x$ is of prime order $p$. Suppose we assume on the contrary that $H$ is a non-split metacyclic group. Then either $G^{\alpha_1}F^{\beta_1} = F^\alpha$, for some $\alpha$, or $G^{\alpha_1}F^{\beta_1} = (G')^\beta$, for some $\beta$, where $G'$ is chosen so that $H = \langle F, G' \rangle$. This implies that $H = \langle G_1^{F_{\beta_1}}, G_1^{F_{\beta_2}} \rangle = \langle F^\alpha, G_1^{F_{\beta_1}} \rangle$. Since $F^\alpha$ is of prime order and $\langle F^\alpha \rangle \trianglelefteq H$, $H$ must be split metacyclic group, which contradicts our assumption. A similar argument works for the case when $G^{\alpha_1}F^{\beta_1} = (G')^\beta$. Hence, it follows that $H$ is a split metacyclic group. □

We call $\text{Dic}_n := \mathcal{M}(2, 2n, n, -1)$ the dicyclic group of order $4n$. We will now derive a realizable bound on the order of a finite non-split metacyclic subgroup of $\text{Mod}(S_g)$.

**Proposition 4.2.** Suppose that $H < \text{Mod}(S_g)$ is a finite non-split metacyclic group. Then $|H| \leq 4g$ and this bound is realized when $g$ is even and $H \cong \text{Dic}_g$.

**Proof.** We will show that if $H < \text{Mod}(S_g)$ such that $|H| > 4g$, then $H$ cannot be a non-split metacyclic group. If $\Gamma(\mathcal{G}_H) = (g_0; n_1, n_2, \ldots, n_\ell)$,
Corollary 4.4. Corollary 4.3 further yields the following.

A periodic mapping class is at least 2.

Our assertion now follows from the fact that the order of any irreducible metacyclic group.

Proof. Since \( |H| > 4g \), we have

\[
2g - 2 + \sum_{i=1}^{\ell} \left( 1 - \frac{1}{n_i} \right) = \frac{2g - 2}{|H|} < \frac{2g - 2}{4g} = \frac{g - 1}{2g} < \frac{1}{2}.
\]

from which it follows that \( g_0 = 0 \) and \( \ell = 3 \) or 4.

From Lemma 4.1 if \( g_0 = 0 \), \( \ell = 3 \), and there is a cone point of prime order, then \( H \) cannot be a non-split metacyclic group. So, by (4), when \( H \) is a non-split metacyclic group with \( |H| > 4g \), the possible signatures for \( \pi^b_0(O_H) \) are \( (0; 2, 2, 3, 3), (0; 2, 2, 3, 4), (0; 2, 2, 3, 5), (0; 2, 2, 2, n), (0; 4, 4, n), (0; 4, 6, 6), (0; 4, 6, 8), (0; 4, 6, 9), \) and \( (0; 4, 6, 10) \), where \( n < 2g \). We will now show that none of these signatures will arise from a non-split metacyclic action.

Assume that \( H \) is a metacyclic group. Then \( H = \langle F, G \rangle \), where \( F \in \text{Homeo}^+(S_g) \) with \( O_F \approx S_h \) and \( \bar{G} \in \text{Aut}_k(O_F) \). From Remark 3.3 we have that \( \Gamma(O_G) = \langle 0; m_1, m_2, \ldots, m_\ell \rangle \), where \( m_i \mid n_i \). First, we will consider the case when \( \ell = 4 \). If \( h \neq 0 \), from Proposition 2.3 we get \( \Gamma(O_G) \) equals either \( (0; 2, 2, 2, 2) \) or \( (0; 2, 2, 2, 2) \). Thus, \( F \) either generates a free action or \( \Gamma(O_F) = (1; \frac{n}{2}) \). But, by Proposition 2.3 and Corollary 3.9 we can see that neither of these possibilities occur when \( H \) is a non-split metacyclic group. If \( h = 0 \), then \( \Gamma(O_G) = \langle 0; u, u \rangle \), where \( u \in \{2,3\} \). Again, from Theorem 3.3 we see that \( \Gamma(O_F) \) equals one of \( (0; 2, 2, 3, 3), (0; 3, 3, 3, 3), (0; 3, 3, 4, 4), (0; 2, 2, 2, 3, 3), (0; 3, 3, 5, 5), (0; 2, 2, 2, 2, n), \) or \( (0; 2, 2, 2, 2, n/2) \). Hence, either \( H \) is a split metacyclic group or \( |H| \leq 4g \).

This completes our argument for \( \ell = 4 \).

Now, for \( \ell = 3 \), by similar arguments as above, we can conclude that \( \Gamma(O_F) \) is \( (0; n, n, n, n) \) or \( (0; 4, 4, 4, 4, 4) \). By Lemma 2.6 and Theorem 3.6 we see that \( |\bar{G}| = \bar{G} \), and so \( H \) is a split metacyclic group.

For the realization of the bound, when \( H \cong \text{Dic}_g \) and \( g \) is even, we can see that data set

\[
D = \{(2; 2g, 0, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (1, 2g), 4], [(0, 1), (2g-1, 2g), 2g]\}
\]

represents the weak conjugacy class of \( (H, (G, F)) \).

An immediate consequence of Proposition 4.2 is the following.

Corollary 4.3. Suppose that \( H < \text{Mod}(S_g) \) is a finite non-split metacyclic group. Then there exists no irreducible periodic mapping class in \( H \).

Proof. Since \( |H| \leq 4g \) and \( H \) is non-split, we have \( |F| \leq 2g \) for any \( F \in H \). Our assertion now follows from the fact that the order of any irreducible periodic mapping class is at least \( 2g + 1 \).

Corollary 4.4 further yields the following.

Corollary 4.4. Suppose that \( H = \langle F, G \rangle < \text{Mod}(S_g) \) is a finite non-split metacyclic group.
(i) If \( g = 2 \), then \( |F| \leq 2g \cdot |G| \leq 2g \), and \( |\bar{G}| \leq g \). Moreover, these bounds are realized when \( H \cong Q_8 \).

(ii) If \( g > 2 \), then \( |F| \leq 2g \), \( |G| \leq 2g - 2 \), and \( |\bar{G}| \leq g - 1 \). Moreover, the bound on \( |F| \) is realized when \( H \cong \text{Dic}_g \), where \( g \) is even, while the bounds on \( |G| \) and \( |\bar{G}| \) are realized when \( H \cong Q_8 \times \mathbb{Z}_{\frac{g-1}{2}} \), where \( g \equiv 3 \pmod{4} \).

**Proof.** From Corollary 4.3, we have that \( |F|, |G| \leq 2g \). Also, as \( |\bar{G}| < |G| \), from Lemma 2.6, we have \( |\bar{G}| \leq g \). Hence, the assertion in (i) follows immediately from Proposition 4.2.

Furthermore, by Proposition 4.2, the bound \( 4g \) on \( |H| \) is realized when \( H \cong \text{Dic}_g \) and \( g \) is even. It is apparent that \( |F| = 2g \) in \( H \), which realizes the required bound in the first part of (ii). Moreover, from the proof of Proposition 4.2, it is apparent that \( \text{Dic}_g \) will not realize the bounds for \( |G| \) and \( |\bar{G}| \). However, it can be easily seen that the bounds on \( |G| \) and \( |\bar{G}| \) are realized when \( H \cong Q_8 \times \mathbb{Z}_{\frac{g-1}{2}} \) with the weak conjugacy class \( (H,(G,F)) \) represented by the metacyclic data set

\[
((g-1) \cdot 4, 2, -1), 1; [(0,1),(1,2),2]).
\]

\[\square\]

We conclude this subsection with the following direct consequence of Corollary 4.3.

**Corollary 4.5.** Let \( p : S_g \to S_{0,3} \) be a finite \( n \)-sheeted cover with deck transformation group \( \langle F \rangle \cong \mathbb{Z}_n \). If \( \bar{G} \in \text{LMod}_p(S_{0,3}) \) lifts to a \( G \in \text{SMod}_p(S_g) \). Then \( H = \langle F, G \rangle \) is a split metacyclic group. Furthermore, either \( H \cong \mathbb{Z}_n \times_k \mathbb{Z}_2 \) or \( H \cong \mathbb{Z}_n \times_k \mathbb{Z}_3 \).

### 4.2. Dicyclic subgroups of \( \text{Mod}(S_g) \)

In Proposition 4.2, we saw the significance of dicyclic groups as bound-realizing metacyclic subgroups of \( \text{Mod}(S_g) \). This motivates a separate analysis of dicyclic actions, which is precisely what we undertake in this subsection. We recall that a dicyclic group of order \( 4n \) is given by \( \text{Dic}_n := \mathcal{M}(2,2n,n,-1) \). We will call a metacyclic data set of degree \( 2 \cdot 2n \), amalgam \( n \) and twist factor \(-1\), a dicyclic data set. Note that a dicyclic group is a non-split metacyclic group if and only if \( n \) is even. Thus, throughout this subsection, \( n \) will be assumed to be even. The following is an immediate consequence of Proposition 3.5.

**Corollary 4.6.** For \( g \geq 2 \) and \( n \geq 3 \), dicyclic data sets of degree \( 2 \cdot 2n \) and genus \( g \) correspond to the weak conjugacy classes of \( \text{Dic}_n \)-actions on \( S_g \).

**Remark 4.7.** Let \( H = \text{Dic}_n = \langle F, G \rangle < \text{Mod}(S_g) \). Then \( \bar{G} \) cannot fix a regular point in orbifold \( S_g/\langle F \rangle \). To see this, suppose we assume on the contrary that \( \bar{G}([x]) = [x] \), where \([x]\) is a regular point in \( \mathcal{O}(F) \). Then \( \text{Stab}_{(G,F)}(x) = \langle G^2 \rangle = \langle F^n \rangle \), which implies that \([x]\) is an order 2 cone point in \( \mathcal{O}(F) \), thereby yielding a contradiction.

The following proposition provides an alternative characterization of a \( \text{Dic}_n \)-action on \( S_g \).
Proposition 4.8. Let $F \in \text{Mod}(S_g)$ be of order $2n$. Then there exists a $G \in \text{Mod}(S_g)$ of order 4 such that $\langle F, G \rangle \cong \text{Dic}_n$ if and only if $DF$ has the form

(*) $(2n, g_0, d; (c_1, n_1), (-c_1, n_1), \ldots, (c_s, n_s), (-c_s, n_s))$

satisfying the following conditions.

(i) When $g_0$ is even, there exists an $i$ such that $(c_i, n_i) = (-c_i, n_i) = (1, 2)$.

(ii) When $g_0$ is odd, at least one of the following statements hold true.

(a) There exists $i,j$ with $i \neq j$ such that $(\pm c_i, n_i) = (\pm c_j, n_j) = (1, 2)$.

(b) $g_0 \geq 3$ and $\sum_{i=1}^{s} c_i \frac{2n}{n_i} \equiv 2a \pmod{2n}$ for some $a \in \mathbb{Z}$.

(c) $g_0 = 1$ and $\sum_{i=1}^{s} c_i \frac{2n}{n_i} \equiv 2 \pmod{2n}$.

(d) $g_0 = 1$, $\sum_{i=1}^{s} c_i \frac{2n}{n_i} \equiv 2a \pmod{2n}$ for some $a \in \mathbb{Z}$, and $\text{lcm}(n_1, \ldots, n_s) = 2n$.

Proof. Suppose that $DF$ has the form (*). Then $\mathcal{O}(\mathcal{F})$ is an orbifold of genus $g_0$ with $2s$ cone points $[x_1], [y_1], \ldots, [x_s], [y_s]$, where $\mathcal{P}_{x_i} = (c_i, n_i)$ and $\mathcal{P}_{y_i} = (-c_i, n_i)$, for $1 \leq i \leq s$. If $DF$ satisfies condition (i), then we may assume without loss of generality that $(c_1, n_1) = (-c_1, n_1) = (1, 2)$. Then up to conjugacy, let $\mathcal{G} \in \text{Aut}_k(\mathcal{O}(\mathcal{F}))$ be an involution such that $\mathcal{G}([x_i]) = [y_i]$, for $2 \leq i \leq s$, $\mathcal{G}([x_1]) = [x_1]$, and $\mathcal{G}([y_1]) = [y_1]$. To prove our assertion, it would suffice to show the existence of an involution $\mathcal{G} \in \text{Homeo}^+(S_g)$ that induces $\mathcal{G}$. This amounts to showing that there exists a metacyclic data set $\mathcal{D}$ of degree $2 \cdot 2n$ with amalgam $n$ and twist factor $-1$ encoding the weak conjugacy class $(H, (\mathcal{G}, \mathcal{F}))$ so that $D_{\mathcal{G}}$ has degree 4.

Consider the tuple

$$\mathcal{D} = ((2 \cdot 2n, n, -1), g_0/2; [(1, 4), (0, 1), 4], [(3, 4), (c', n'), 4], [(0, 1), (c_2, n_2), n_2], \ldots, [(0, 1), (c_s, n_s), n_s])$$

where $c' \frac{2n}{n} \equiv -\sum_{i=2}^{s} c_i \frac{2n}{n_i} \pmod{2n}$. It follows immediately that $\mathcal{D}$ satisfies conditions (i)-(iv) of Definition 3.4. By taking $v = 1$, we may choose $(p_1, \ldots, p_{s+1}) = (1, 0, \ldots, 0)$ to conclude that $\mathcal{D}$ also satisfies either condition (v)(a) or (vi)(a), based on the choice of $g_0$. If $g_0 = 0$, we have that $\text{lcm}(n_1, \ldots, n_s) = 2n$, from which condition (v)(b) follows. If $g_0 \neq 0$, then by carefully defining $\phi_H$ (as in Proposition 3.5) on the hyperbolic elements of $\pi_1^{\text{orb}}(\mathcal{O}_H)$, our claim is true. Thus, it follows that $\mathcal{D}$ is a metacyclic data set.

If $DF$ satisfies condition (ii)(a), then by a similar argument as above, we obtain the metacyclic data set

$$\mathcal{D} = ((2 \cdot 2n, n, -1), (g_0 + 1)/2; [(1, 4), (0, 1), 4], [(1, 4), (0, 1), 4], [(1, 4), (1, 2n), 4], [(1, 4), (c'', n''), 4], [(0, 1), (c_3, n_3), n_3], \ldots, [(0, 1), (c_s, n_s), n_s])$$

where $c'' \frac{2n}{n} \equiv 1 - \sum_{i=3}^{s} c_i \frac{2n}{n_i} \pmod{2n}$. Suppose that $DF$ satisfies conditions (ii)(b)-(d). Then again by an analogous argument as above, we obtain the
metacyclic data set
\[ D = ((2 \cdot 2n, n, -1), (g_0 + 1)/2; [(0, 1), (c_1, n_1), \ldots, [(0, 1), (c_s, n_s), n_s]). \]
Further, a direct application of Theorem 3.6 would show that \( D \) indeed encodes the weak conjugacy represented by \( (H, (G, F)) \), as desired.

The converse follows immediately from Remark 3.2, Remark 4.7 and Proposition 3.5. \( \square \)

4.3. Liftability of non-split metacyclic actions under regular cyclic covers. Considering the fact that every non-split metacyclic group is a quotient of a split metacyclic group, a natural question arises is when a given metacyclic action on \( S_g \) factor via a split metacyclic action. In other words, when does a metacyclic action on \( S_g \) lift under a regular cover to a split metacyclic action. In the following proposition (which follows directly from Theorem 3.6), we provide an equivalent condition for the liftability of a metacyclic action under a regular cyclic cover.

**Proposition 4.9.** Let \( p_v : S_{v(g-1)+1} \to S_g \) be a regular cyclic cover, and let \( H = (F, G) < \text{Mod}(S_g) \) be a finite non-split metacyclic group such that \( H \cong M(u, n, r, k) \) and the weak conjugacy class \( (H, (G, F)) \) encoded by the data set
\[ D = ((u \cdot n, r, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), \ldots, [(c_{1t}, n_{1t}), (c_{2}, n_{2}), n_2]). \]
Then \( H \) lifts under \( p_v \) to a split metacyclic group \( \tilde{H} = \langle \tilde{F}, \tilde{G} \rangle < \text{Mod}(S_{v(g-1)+1}) \) such that \( \tilde{H} \cong M(\nu u, n, r, k) \cong Z_n \times_k Z_{\nu u} \) if and only if

(i) \( \nu = n/r \) and
(ii) the weak conjugacy class \( (\tilde{H}, (\tilde{G}, \tilde{F})) \) is encoded by the data set
\[ \tilde{D} = ((m \cdot n, n, k), g_0; [(c'_{11}, n'_{11}), (c'_{12}, n'_{12}), \ldots, [(c'_{1t}, n'_{1t}), (c'_{2}, n'_{2}), n_2]), \]
where \( m = u, c'_{11} \equiv c_{11} + a_i n \pmod{m} \) and \( c'_{2} \equiv c_{2} - a_i r \pmod{n}, \) for some \( a_i \in Z. \)

An immediate consequence of Proposition 4.9 is the following.

**Corollary 4.10.** The actions on \( S_g \) of the metacyclic groups \( \text{Dic}_n, \text{Dic}_n \times Z_m, \) and \( \text{Dic}_n \times Z_m \times Z_p, \) where \( n \) is even and \( m, p \) are odd with \( \gcd(p, n) = 1, \) factor via split metacyclic actions.

Proposition 4.9 and Corollary 4.10 motivate the following conjecture.

**Conjecture 4.11.** Every non-split metacyclic action on \( S_g \) lifts under a suitably chosen finite regular cyclic cover to a split metacyclic action.

5. Geometric realizations of the lifts of non-split metacyclic actions

In this section, we use Corollary 4.10 to provide explicit geometric realizations of the lifts of some non-split metacyclic actions on \( S_{10} \) and \( S_{11}. \) These realizations implicitly assume the theory developed in [4] [13]. The associated weak conjugacy classes of these actions are represented by the metacyclic data sets listed in Tables 12 in Section 6.
Figure 1. The realization of a $H = \mathbb{Z}_{12} \rtimes_{-1} \mathbb{Z}_{4}$-action on $S_{21}$ which is the lift of a $\text{Dic}_6$-action on $S_{11}$ under the regular cyclic cover $p_2$. Here, $H = \langle \mathcal{F}, \mathcal{G} \rangle$, where $D_G = (4, 6; ((1, 6), 2), ((5, 6), 2))$ and $D_F = (12, 1; ((1, 6), 2), ((5, 6), 2))$. Note that the $\mathcal{G}$ maps each orbit of the $\langle \mathcal{F} \rangle$-action of size 2 with local rotation angle $2\pi/6$ to an orbit with local rotation angle $10\pi/6$ (and vice versa).

Figure 2. The realization of a $H = \mathbb{Z}_{20} \rtimes_{-1} \mathbb{Z}_{4}$-action on $S_{19}$ which is the lift of a $\text{Dic}_{10}$-action on $S_{10}$ under the regular cyclic cover $p_2$. Here, $H = \langle \mathcal{F}, \mathcal{G} \rangle$, where $D_G = (4, 0; ((1, 4), 2), ((1, 2), 19))$ and $D_F = (20, 0; ((1, 20), 2), ((19, 20), 2))$. Note that the four fixed points of $\mathcal{F}$ (marked in red) form an orbit of size 4 under the $\langle \mathcal{G} \rangle$-action where each fixed point with local rotation $2\pi/20$ is mapped to fixed point with local rotation $38\pi/20$ (and vice versa). The point marked in blue are distinct size 2 orbits of the $\langle \mathcal{G} \rangle$-action, while the points marked in black are the fixed points of $\mathcal{G}$. 
Classes of the non-split metacyclic subgroups of $\text{Mod}(S)$

Definition 6.1. Two metacyclic data sets $(n, \mathcal{G})$ and $(n', \mathcal{G}')$ are said to be equivalent if each fixed point with local rotation $2\pi/4$ is mapped to fixed point with local rotation $6\pi/4$ (and vice versa). The remaining fixed (and orbit) points of the $\langle \mathcal{G}' \rangle$-action are marked in blue.

6. Classification of the weak conjugacy classes in $\text{Mod}(S_{10})$ and $\text{Mod}(S_{11})$

In this section, we will use Theorem 5.3 to classify the weak conjugacy classes of the non-split metacyclic subgroups of $\text{Mod}(S_{10})$ and $\text{Mod}(S_{11})$. For brevity, we will further assume the following equivalence of the metacyclic data sets (i.e. the weak conjugacy classes).

Definition 6.1. Two metacyclic data sets

$$
\mathcal{D} = ((u \cdot n, r, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \ldots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_{\ell}])
$$

and

$$
\mathcal{D}' = ((u \cdot n, r, k), g_0; [(c'_{11}, n'_{11}), (c'_{12}, n'_{12}), n'_1], \ldots, [(c'_{\ell 1}, n'_{\ell 1}), (c'_{\ell 2}, n'_{\ell 2}), n'_{\ell}])
$$

are said to be equivalent if for each tuple $[(c_{j1}, n_{j1}), (c_{j2}, n_{j2}), n_j]$ satisfying the following conditions:

(i) $n'_j = n_j$

(ii) $c'_{j1} \equiv c_{j1} + au \pmod{m}$, where $m = u^2$, and

(iii) $c'_{j2} \equiv c_{j2} + b(m - 1) \pmod{n}$ for some $a_i, b_i, a \in \mathbb{Z}$.

Note that equivalent data sets $\mathcal{D}$ and $\mathcal{D}'$ as in Definition 6.1 satisfy $\mathcal{D}' = \mathcal{D}_i$, for $i = 1, 2$. We will now provide a classification of the weak conjugacy classes of finite non-split metacyclic subgroups of $\text{Mod}(S_{10})$ and $\text{Mod}(S_{11})$ (up to this equivalence) in Tables 1 and 2 respectively.
### Table 1. The weak conjugacy classes of finite non-split metacyclic subgroups of Mod$(S_{10})$. (*The suffix refers to the multiplicity of the tuple in the non-split metacyclic data set.)

| Group      | Weak conjugacy classes in Mod$(S_{10})$                                                                 | Cyclic factors $[D/G : D_F]$ |
|------------|---------------------------------------------------------------------------------------------------------|-------------------------------|
| $M(2, 4, 2, -1)$ | $[(8, 4, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 1; (1, 4), (3, 4), (1, 2), 6)]; (4, 0; (0, 1), (3, 4), (1, 2), 2)]$ |
|            | $[(2, 4, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 0; (0, 1), (3, 4), (1, 2), 2)]; (4, 1; (1, 4), (3, 4), (1, 2), 6)]$ |
| $M(2, 8, 4, -1)$ | $[(8, 4, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 0; (0, 1), (3, 4), (1, 2), 2)]; (4, 1; (1, 4), (3, 4), (1, 2), 6)]$ |
|            | $[(2, 8, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 0; (0, 1), (3, 4), (1, 2), 2)]; (4, 1; (1, 4), (3, 4), (1, 2), 6)]$ |
| $M(2, 12, 6, 7)$ | $[(8, 4, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 0; (0, 1), (3, 4), (1, 2), 2)]; (4, 1; (1, 4), (3, 4), (1, 2), 6)]$ |
| $M(2, 20, 10, -1)$ | $[(8, 4, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 0; (0, 1), (3, 4), (1, 2), 2)]; (4, 1; (1, 4), (3, 4), (1, 2), 6)]$ |

### Table 2. The weak conjugacy classes of finite non-split metacyclic subgroups of Mod$(S_{11})$ other than quaternions. (*The suffix refers to the multiplicity of the tuple in the non-split metacyclic data set.)

| Group      | Weak conjugacy classes in Mod$(S_{11})$                                                                 | Cyclic factors $[D/G : D_F]$ |
|------------|---------------------------------------------------------------------------------------------------------|-------------------------------|
| $M(2, 12, 6, -1)$ | $[(4, 8, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 8; (1, 2), 2)]; (12, 1; (1, 6), (5, 6)]$ |
| $M(4, 8, -1)$ | $[(4, 8, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 8; (1, 2), 2)]; (12, 1; (1, 6), (5, 6)]$ |
| $M(4, 8, -1)$ | $[(4, 8, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 8; (1, 2), 2)]; (12, 1; (1, 6), (5, 6)]$ |
| $M(4, 8, -1)$ | $[(4, 8, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 8; (1, 2), 2)]; (12, 1; (1, 6), (5, 6)]$ |
| $M(2, 20, 10, -1)$ | $[(4, 8, -1):0; [(0, 1), (1, 2), 2], [(1, 4), (0, 1), 4], [(5, 1), (3, 4), 4], [(1, 4), (3, 4), 4]]$ | $[(4, 8; (1, 2), 2)]; (12, 1; (1, 6), (5, 6)]$ |
Acknowledgements

The second author was partly supported by a UGC-JRF fellowship. The authors would also like to thank Dr. Neeraj Kumar Dhanwani for some helpful discussions.

References

[1] Nikita Agarwal, Soumya Dey, Neeraj K Dhanwani, and Kashyap Rajeevsarathy. Liftable mapping class groups of regular cyclic covers. arXiv preprint arXiv:1911.05682, to appear in the Houston Journal of Mathematics, 2021.
[2] Thomas Breuer. Characters and automorphism groups of compact Riemann surfaces, volume 280 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2000.
[3] Neeraj K. Dhanwani and Kashyap Rajeevsarathy. Commuting conjugates of finite-order mapping classes. Geom. Dedicata, 209:69–93, 2020.
[4] Neeraj K Dhanwani, Kashyap Rajeevsarathy, and Apeksha Sanghi. Split metacyclic actions on surfaces. arXiv preprint arXiv:2007.08279, to appear in the New York Journal of Mathematics, 2022.
[5] W. J. Harvey. Cyclic groups of automorphisms of a compact Riemann surface. Quart. J. Math. Oxford Ser. (2), 17:86–97, 1966.
[6] C. E. Hempel. Metacyclic groups. Comm. Algebra, 28(8):3865–3897, 2000.
[7] Svetlana Katok. Fuchsian groups. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.
[8] Steven P. Kerckhoff. The Nielsen realization problem. Ann. of Math. (2), 117(2):235–265, 1983.
[9] Alexander Murray Macbeath and HC Wilkie. Discontinuous groups and birational transformations: [Summer School], Queen's College Dundee, University of St. Andrews, [Department of Math.], Queen's College, 1961.
[10] Dan Margalit and Rebecca R. Winarski. Braids groups and mapping class groups: the Birman-Hilden theory. Bull. Lond. Math. Soc., 53(3):643–659, 2021.
[11] Jakob Nielsen. Die Struktur periodischer Transformationen von Flächen, volume 15. Levin & Munksgaard, 1937.
[12] Jakob Nielsen. Abbildungsklassen endlicher Ordnung. Acta Math., 75:23–115, 1943.
[13] Shiv Parsad, Kashyap Rajeevsarathy, and Bidyut Sanki. Geometric realizations of cyclic actions on surfaces. J. Topol. Anal., 11(4):929–964, 2019.
[14] Kashyap Rajeevsarathy and Prahlad Vaidyanathan. Roots of Dehn twists about multicurves. Glasg. Math. J., 60(3):555–583, 2018.
[15] Andreas Schweizer. Metacyclic groups as automorphism groups of compact Riemann surfaces. Geom. Dedicata, 190:185–197, 2017.
[16] William P. Thurston. The Geometry and Topology of Three-Manifolds, notes available at: http://www.msri.org/communications/books/gt3m/PDF.
[17] Thomas W. Tucker. Finite groups acting on surfaces and the genus of a group. J. Combin. Theory Ser. B, 34(1):82–98, 1983.
[18] A Wiman. Ueber die hyperelliptischen curven und diejenigen vom geschlechte p= 3, welche eindeutigen transformationen in sich zulassen” and “ueber die algebraischen curven von den geschlechtern p= 4, 5 und 6 welche eindeutigen transformationen in sich besitzen”. Svenska Vetenskaps-Akademiens Handlingar, Stockholm, 96, 1895.
