Basis for scalar curvature invariants in three dimensions

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Abstract

$\mathbb{L}$-non-degenerate spaces are spacetimes that can be characterized uniquely by their scalar curvature invariants. The ultimate goal of the current work is to construct a basis for the scalar polynomial curvature invariants in three-dimensional Lorentzian spacetimes. In particular, we seek a minimal set of algebraically independent scalar curvature invariants formed by the contraction of the Riemann tensor and its covariant derivatives up to fifth order of differentiation. We use the computer software invar to calculate an overdetermined basis of scalar curvature invariants in three dimensions. We also discuss the equivalence method and the Karlhede algorithm for computing Cartan invariants in three dimensions.

Keywords: scalar curvature invariants, three-dimensional Lorentzian spacetimes, equivalence method

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1. Introduction

Scalar curvature invariants are scalars constructed from the Riemann tensor and its covariant derivatives. Scalar curvature invariants can be used to study the inequivalence of metrics and curvature singularities [1]. In the case of the Lorentzian $\mathbb{L}$-non-degenerate spaces, they can be used to determine the equivalence of metrics [2]. Scalar curvature invariants have been utilized in the study of universal spacetimes [3], and in VSI and CSI spacetimes [4–6]. In particular, scalar curvature invariants have been studied due to their potential use in general relativity [1]. Primarily, the scalar curvature invariants formed from the Riemann tensor $R_{abcd}$ only (contractions involving products of the undifferentiated Riemann tensor only, the so-called algebraic invariants) have been investigated in four-dimensional (4D) Lorentzian spacetimes.

Scalar curvature invariants are of primary importance in $\mathbb{L}$-non-degenerate spacetimes [2, 4, 7]. Indeed, these spacetimes can be completely characterized by their scalar curvature...
invariants. This leads to the natural problem of attempting to find a basis for the scalar curvature invariants formed from the Riemann tensor (up to some order of covariant differentiation). Much work has gone into the problem of constructing a basis in the 4D Lorentzian algebraic case. In this case, one can form 14 functionally independent scalar curvature invariants. The smallest set that contains a maximal set of algebraically independent scalars consists of 17 polynomials [8]. As reviewed in the introduction of [9], a number of proposed independent sets were given by Narlikar and Karmarkar [10], Geheniau and Debever [11], Petrov [12], and others. All of these sets were shown to be deficient for various reasons. A set of algebraic invariants was presented by Carminati and McLenaghan [9], consisting of 16 curvature invariants, that contains invariants of the lowest possible degree and contains a minimal set for any Petrov type and for any specific choice of Ricci tensor type in the perfect fluid and Einstein–Maxwell cases. In general, the expressions relating invariants to the basis members of an independent set can be very complicated, and can be singular in certain algebraic cases.

Much less work has gone into studying curvature invariants formed using covariant derivatives of the Riemann tensor (differential invariants) [5, 13–16]. Our ultimate goal is to provide a basis of all such scalar curvature invariants. This is primarily a mathematical question. However, it also has applications in mathematical physics.

There are differing notions of what is meant by a basis. For example, the number $N(n, p)$ of algebraically independent quantities formed from the first $p$ derivatives of the Riemann tensor in dimension $n$ can be determined (see equation (3) below). These $N(n, p)$ components correspond to the independent components of the Riemann tensor and its covariant derivatives. Hence, in principle, a basis could consist of these $N(n, p)$ scalars because all other curvature invariants are functions of these scalars. However, the expressions relating these invariants can be very complicated, involving roots of high order, and they may therefore be singular in certain algebraic cases. For actual classification using invariants, a different type of basis is needed. The common solution to this problem is to seek a basis of scalars such that all other scalars are polynomials in this basis. This is the approach that has been widely used in the study of scalar curvature invariants in 4D Lorentzian signature.

Rather than searching for an independent basis, we may search for a complete basis \{${I_1, I_2, \ldots, I_n}$\}, meaning that any scalar curvature invariant can be expressed as a polynomial of the elements of \{${I_1, I_2, \ldots, I_n}$\} but no member of \{${I_1, I_2, \ldots, I_n}$\} can be expressed in terms of the others. For example, the smallest known complete set in the 4D algebraic Lorentzian case consists of 38 scalars [17].

The main motivation for studying this problem in three dimensions (3D) is mathematical, although there are some possible applications to mathematical physics [18, 19]. In particular, it may be applied to the initial value problem in general relativity; in this case a unique solution is given by a three-metric and 3D extrinsic curvature specified on a Cauchy hypersurface. Hence, an important step is to characterize the hypersurface using scalar curvature invariants.

The primary motivation for studying scalar curvature invariants comes from the $I$-non-degenerate theorem.

### 1.1. $I$-non-degenerate spacetimes

In [2] the class of 4D Lorentzian manifolds that can be completely characterized by the scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives was determined. The notion of an $I$-non-degenerate spacetime metric was introduced, which implies that the spacetime metric is locally determined by its curvature invariants. It was proven that a spacetime metric is either $I$-non-degenerate or a degenerate
Kundt metric. Therefore, a metric that is not characterized by its curvature invariants must be of Kundt form. This theorem is also true in 3D, and it is also likely true in arbitrary dimensions; in [7], a number of results that generalize these results to higher dimensions were introduced, and their consequences and potential physical applications were discussed.

Let us describe this theorem in a little more detail. A (one-parameter) metric deformation \( \tilde{\gamma}, \tau \in [0, \epsilon] \) of a spacetime \( (\mathcal{M}, g) \) is a family of smooth metrics on \( \mathcal{M} \) such that (i) \( \tilde{\gamma} \) is continuous in \( \tau \), (ii) \( \tilde{\gamma}_0 = g \), (iii) \( \tilde{\gamma}_\tau \) for \( \tau > 0 \) is not diffeomorphic to \( g \). Given a spacetime \( (\mathcal{M}, g) \) with a set of invariants \( I \), then if there does not exist a metric deformation of \( g \) with the same set of invariants as \( g \), then we will call the set of invariants non-degenerate. The spacetime metric \( g \) will be called \( I \)-non-degenerate. In [2], it was proven that a spacetime metric is either \( I \)-non-degenerate or the metric is a Kundt metric; that is, a Lorentzian metric that is \( I \)-non-degenerate is locally characterized uniquely by its scalar curvature invariants.

2. Bounds on number of covariant derivatives

We are interested in constructing a basis for the scalar curvature invariants up to some order of covariant differentiation in general Lorentzian spacetimes (although in this paper we are specifically interested in 3D). In other words, we seek a minimal (or overdetermined) set of algebraically independent scalar curvature invariants formed by the contraction of the Riemann tensor and its covariant derivatives up to some order of differentiation \( p \). Hence, we must determine the bound on \( p \). Obviously, we seek the minimum value of \( p \) such that all geometrical information of the spacetime can be obtained.

The bound \( q \equiv p + 1 \) is used in the Cartan–Karlhede algorithm for determining the equivalence of spacetimes [20]. The bound on the algorithm (by considering the ‘worst-case scenario’ for the number of steps required for the algorithm to finish) is

\[
q = N_0 + n + 1,
\]

where \( N_0 \) is the dimension of the isotropy (automorphism) group of the Riemann tensor and \( n \) is the dimension of the spacetime. At the final step \( q \) of the Cartan–Karlhede algorithm, new classification functions can be introduced. Therefore, in the classification we need the Riemann tensor and its first \( q \) covariant derivatives.

2.1. Bounds in 4D

We first recall some results in 4D. Cartan’s argument gives a bound of \( n(n + 1)/2 = 10 \) [1]. It is, however, impossible in the 4D Lorentzian case to have \( p > 7 \) (except for the constant curvature case). To see this suppose that \( p > 7 \). In this case, in the Karlhede algorithm we must find at most ten functions on the ten dimensional frame bundle (six Lorentz group parameters and four independent functions of spacetime). Now, since we need eight or nine steps to terminate the algorithm, then at most two of the parameters were already fixed at the beginning. Then the undetermined part of the Lorentz group would be of dimension at least four, but there is no choice of Weyl and Ricci curvature allowing this (i.e., invariant under a 4D subgroup of the Lorentz group). Then the curvature would be invariant under the whole Lorentz group, which implies that the spacetime is of constant curvature [1]. Furthermore, in 4D we know that if \( p \geq 6 \) then the spacetime is degenerate Kundt [7]. Therefore, for \( I \)-non-degenerate spacetimes, we have \( p \leq 5 \) (\( q \leq 6 \)). We note that \( N(4, 6) = 2094 \) (see equation (3) below).
It is conjectured that for $I$-non-degenerate spacetimes all Cartan invariants are determined (up to possible discrete complex transformations) by scalar polynomial curvature invariants (as in the Riemannian case) [2].

In 4D Lorentzian signature, a complete basis at zeroth order has 38 objects and contains Riemann scalars up to degree eleven [17]. In [15], Invar was used to determine 25 of the 38 objects in this basis. In this basis there are 24 syzygies between the invariants. The syzygies are polynomials, but they are not yet known. If Invar were able to be run to an arbitrary degree they would in principal be found, but this is not computationally feasible. Recently, using techniques from graph theory, Carminati and Lim claim to be able to reproduce Sneddon’s basis [8].

2.2. Bounds in 3D

The question of the bound for $p$ in 3D was addressed by Sousa, Fonseca, and Romero in [21]. In 3D spacetimes, the Weyl tensor vanishes, and the canonical frame of the Karlhede algorithm [20] is aligned with principal directions of the Ricci tensor rather than the Weyl tensor. All spacetimes have $N_0 \leq 1$, hence we have $q = p + 1 \leq 5$ (the same as the 3D Riemannian case, which has no two dimensional (2D) isotropy group). See table 1 for (the dimension of the) the isotropy groups of spacetimes of different Segre types of the Ricci tensor [21], and we also show the related Ricci type [5, 22].

It is common to discuss the Segre types of the traceless Ricci tensor [1] whence the case \{(1, 11)\} with full 3D Lorentz subgroup corresponding to spacetimes of constant curvature is of Ricci type O. There is a 2D isotropy group of the Lorentz group spanned by a boost and a null rotation (either of these together with a spatial rotation gives rise to the full 3D Lorentz group). The 2D subgroups arise as special cases of Segre types \{(1, 1), 1\} and \{(2, 1)\} in table 1 with one dimensional (1D) isotropy group (with an additional isotropy). However, these resulting spacetimes must be degenerate Kundt (i.e., not $I$-non-degenerate), and are not considered further here. Suppose that the Karlhede algorithm takes five steps to complete (i.e., $p = 4$). Then at most two of the parameters on the fibre bundle were fixed at the beginning. Then the undetermined part of the Lorentz group has $N_0 \geq 1$, and we see that this argument does not improve the bound in 3D as it does in the 4D case.

It was shown in [23] that this bound is sharp (i.e., $q = 5$ is attained); that is, a 3D Lorentzian manifold exists for which the fifth covariant derivative of the Riemann tensor is required to classify the spacetime invariantly. If the isotropy group is 1D, then $q$ can be 5. However, in the (general) cases (see the table) that do not have a one dimensional isotropy group (and are not degenerate Kundt), then $q \leq 4$. Thus, in these cases the classification is simpler. We note that $N(3, 4) = 147$ and $N(3, 5) = 228$. We might be able to improve upon $q = 4$ on a case by case basis. Indeed, there is a sense in which in general $q = 1$; note that $N(3, 1) = 18$.

2.3. Cartan invariants in 3D

As noted earlier, it has been conjectured that for $I$-non-degenerate spacetimes all Cartan invariants are determined by scalar polynomial curvature invariants [2]. Hence the equivalence method suggests an approach to determine a basis of all scalar curvature invariants.

In particular, the question of the maximal order of covariant derivative required for the invariant classification of a pseudo-Riemannian manifold, as discussed above [21, 23], is relevant in determining the worst case scenarios for implementing the equivalence algorithm.
This then provides an upper bound on the number of covariant derivatives needed for a basis of all scalar curvature invariants.

The equivalence problem in 3D was first considered in [21], and the bound $q = 5$ was determined, and using methods of two-component real spinors the isotropy groups and the canonical forms of algebraic classifications of the Ricci and Cotton–York spinors were given. The results were applied to the equivalence of Godel-type spacetimes in 3D [21] (also see [24]). A further, more comprehensive, example will be presented later.

As an illustration, let us consider the 3D (P-type I) Gödel-like metric as given in [21]

$$\phi = - \phi_1^2 + \phi_2^2 + \phi_3^2,$$

The Cartan invariants consist of three zeroth order invariants, one first order invariant (essentially the spin coefficient $\tau$), and two frame derivatives of the zeroth order invariants [24]:

$$I_0, I_1, I_2, I_0', I_1', I_2'$$

and a prime denotes differentiation with respect to $r$.

The three zeroth order polynomial scalar curvature invariants expressed as polynomials in the Cartan invariants are [24]

$$R^a_\alpha = -2I_1 + \frac{1}{2}I_0^2$$

$$R^{ab}R_{ab} = 2I_1^2 - 2I_1I_0^2 - \frac{1}{2}(I_0')^2 + \frac{3}{4}I_0^4$$

$$R^{abc}R_{abc} = -2I_1^3 + \frac{3}{4}I_1(I_0')^2 + 3I_1^2I_0^2 - \frac{3}{2}I_1I_0^3 + \frac{1}{8}I_0^6$$

The first order scalar invariants, expressed as polynomials in the Cartan invariants, can be written down explicitly

$$R^{\alpha \beta}_{\gamma \delta} = (-I_0I_0' + 2I_1')^2,$$

and where $R^{cde}R_{bcde}$ and $R^{bcde}R_{bcde}$ are more complicated explicit polynomial functions of $\{I_0, I_1, I_2, I_0', I_1', I_2'\}$ [24]. Thus the 3D Gödel spacetimes are characterized by these six zeroth and first order scalar curvature invariants.

| Segre-type | Ricci-type | Isotropy group | Dimension of isotropy group |
|------------|------------|----------------|-----------------------------|
| $[1, 11]$  | $I$        | none           | 0                           |
| $[1, 11]$  | $I$        | $SO(2)$       | 1                           |
| $[1, 11]$  | $D$        | $SO(1, 1)$    | 1                           |
| $[1, 11]$  | $\lambda_1 \neq 0$ | $D$           | $SO(2, 1)$                 |
| $\lambda_1 = 0$ | $O$            |                |                             |
| $[11]z$    | $I$        | none           | 0                           |
| $[21]$     | $II$       | none           | 0                           |
| $[21]$     | $\lambda_1 \neq 0$ | $II$           | $null$-rotations            |
| $\lambda_1 = 0$ | $N$            |                |                             |
| $[3]$      | $\lambda_1 \neq 0$ | $II$           | none                        |
| $\lambda_1 = 0$ | $III$        |                |                             |

Table 1. Ricci-type to Segre-type conversion scheme.
This example is intended to illustrate the applicability of the algebraic relations of the Cartan invariants at each order to the simplification and study of polynomial scalar curvature invariants. In 3D these curvature invariants are expressed in terms of the Ricci components; while in the appropriate frame these components become Cartan invariants. By choosing this frame basis, it is hoped that the algebraic relationships arising from the Bianchi identities, Ricci identities, frame derivatives, or additional requirements will reduce the number of Ricci components involved in the curvature invariants.

3. Basis of scalar curvature invariants in 3D

3.1. Algebraically independent scalars

The number of algebraically independent scalars constructible from the Riemann tensor and its covariant derivatives up to order $p$ is given by

$$N(n, p) = \frac{n[n + 1][n + p + 2]!}{2n!(p + 2)!} - \frac{(n + p + 3)!}{(n - 1)!(p + 3)!} + n,$$

(3)

with $p \geq 0$, except for $N(2, 2) = 1$.

In 3D, we note that $N(3, 0) = 3, N(3, 1) = 18, N(3, 2) = 45, N(3, 3) = 87, N(3, 4) = 147$, and $N(3, 5) = 228$. We remark that the case of Segre type with no isotropy group (types $(1, 11), (1\tilde{1}), (121)$, and $(3)$) thus have $N(3, 3) = 87$ algebraically independent curvature invariants. Similarly, those with isotropy groups of dimension 1, (types $(1, (11)), ((1, 1)1),$ and $(21)$) have $N(3, 4) = 147$ algebraically independent curvature invariants. We also note that for all values of $p$, $N(n, p)$ as given by equation (3) is equal to the number of independent components up to the $p$th derivative of the Riemann tensor given in [21] minus three (we can deduct three because of the coordinate conditions that can be imposed).

3.2. Computing the independent scalar curvature invariants

As can be seen above, the number of algebraically independent curvature invariants necessary is too large to compute by hand. We have therefore utilized the software package Invar [15, 16] running on top of Maple.

Relations among Riemann invariants are a result of symmetries of the Riemann tensor and its covariant derivatives. There exist five such symmetries that are taken into account by Invar [16].

1. Permutation symmetries
$$R_{baed} = -R_{abcd}, \quad R_{cdea} = R_{acde}.$$ 

2. Cyclic symmetry
$$R_{a[bcd]} = 0.$$ 

3. Bianchi identity
$$R_{ab[cd,e]} = 0.$$ 

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4. Commutation of derivatives. Given any tensor $T^{a_1 \ldots a_n b_1 \ldots b_m}$ we have the identity

$$V_d V_e T^{a_1 \ldots a_n b_1 \ldots b_m} = \sum_{k=1}^{n} R^{cde}_{\ell k} T^{a_1 \ldots e a_n b_1 \ldots b_m} = \sum_{k=1}^{m} R^{cde}_{\ell k} T^{a_1 \ldots a_n b_1 \ldots \ell e b_m}.$$ 

5. Dimensionally dependent identities (also known as Lovelock-type identities). Antisymmetrization in $n + 1$ indices gives zero (where $n$ is the dimension of the manifold).

Using the commands provided by Invar, we were able to compute a basis of scalar curvature invariants in 3D. Let us explicitly present an overdetermined basis of invariants up to $p = 1$ using the cases $[0], [0, 0], [0, 0, 0], [1, 1], [1, 1, 1, 1]$.

We have included 29 terms involving the first covariant derivative. We recall that there are 18 independent components of $R_{abc}$. However, as discussed earlier, a basis such that all scalar invariants are given as polynomials in the basis contains more terms. In table 2, we see the number of independent scalar curvature invariants obtained per ‘case’. A case in this context refers to the number of Riemann tensors (and their orders of differentiation) contracted to form scalar curvature invariants.

We note that the total number of curvature invariants returned by Invar up to the fourth derivative of the Riemann tensor is actually greater than the number provided by the formula (3).

4. Alternative bases

4.1. Components

From equation (3) and the values of the formula given below equation (3), we see that there are three independent components constructible from $R_{abc}$, 18 - 3 = 15 independent components constructible from $R_{abc}$, etc. This means that we can construct a basis consisting of

1 We refer to the case of invariants formed by contracting, for example, one undifferentiated Riemann tensor, two once-differentiated Riemann tensors, and one twice-differentiated Riemann tensor, as $[0, 1, 1, 2]$. To give another example, invariants obtained by contracting two undifferentiated Riemann tensors would be denoted by $[0, 0]$. 

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three zeroth order scalar curvature invariants, 15 invariants constructed from $R_{abc}$ only, and so on. So a possible basis is

\[ p = 0: R, R_a^b R_b^a, R_a^b R_b^c R_c^a, \]

\[ p = 1: R_{abc} R_{abc}, R_{abc} R_{def} R_{def}, \ldots, \]

Table 2. Number of algebraically independent scalar invariants ($N$) per ‘case’ and total per derivative ($\bar{N}$). Also see text, and the website [25].

| Case   | $N$ | $\bar{N}$ |
|--------|-----|-----------|
| [0]    | 1   |           |
| [0, 0] | 1   |           |
| [0, 0, 0] | 1 |           |
| [0, 0, 0, 0] | 0 | 3         |
| [0, 0, 0, 0, 0] |   |           |
| [0, 0, 1, 1] | 15 |           |
| [0, 0, 0, 1, 1] | 21 |           |
| [1, 1] | 3   |           |
| [0, 1, 1] | 18  |           |
| [1, 1, 1, 1] | 26  | 83        |
| [0, 2] | 2   |           |
| [0, 0, 2] | 4   |           |
| [0, 0, 0, 2] | 4   |           |
| [0, 0, 0, 0, 2] | 4   |           |
| [0, 0, 2, 2] | 37  |           |
| [1, 1, 2] | 28  |           |
| [0, 1, 1, 2] | 118 |           |
| [2]    | 1   |           |
| [2, 2] | 6   |           |
| [0, 2, 2] | 14  |           |
| [2, 2, 2] | 30  | 249       |
| [0, 1, 3] | 18  |           |
| [0, 0, 1, 3] | 50  |           |
| [1, 3] | 4   |           |
| [1, 2, 3] | 96  |           |
| [3, 3] | 8   |           |
| [0, 3, 3] | 25  | 199       |
| [0, 4] | 2   |           |
| [0, 0, 4] | 5   |           |
| [0, 0, 0, 4] | 7   |           |
| [1, 1, 4] | 34  |           |
| [2, 4] | 8   |           |
| [0, 2, 4] | 34  |           |
| [4]    | 1   |           |
| [4, 4] | 15  | 106       |
and other independent contractions over indices. We note that such a basis, with higher order (multiple contractions), are extremely difficult to compute. We could also use table 2 to construct such a basis. For example, the cases \([1, 1]\) and \([1, 1, 1, 1]\) provide 29 scalar invariants constructed from \(R_{abc}\) (alone), the cases \([2], [2, 2], [2, 2, 2]\) provide 37 scalar invariants constructed from \(R_{abcd}\), etc.

4.2. FKWC basis in 3D

Field theoretic calculations on curved spacetimes are non-trivial due to the systematic occurrence, in the expressions involved, of Riemann polynomials. These polynomials are formed from the Riemann tensor by covariant differentiation, multiplication and contraction. The results of these calculations are complicated because of the non-uniqueness of their final forms, since the symmetries of the Riemann tensor as well as the Bianchi identities can not be used in a uniform manner and monomials formed from the Riemann tensor may be linearly dependent in non-trivial ways. In [13], Fulling, King, Wybourne and Cummings (FKWC) systematically expanded the Riemann polynomials encountered in calculations on standard bases constructed from group theoretical considerations. They displayed such bases for scalar Riemann polynomials of order eight or less in the derivatives of the metric tensor and for tensorial Riemann polynomials of order six or less. We adopt the FKWC-notations \(\mathcal{R}^{sq}_{r}\) to denote, respectively, the space of Riemann polynomials of rank \(r\) (number of free indices), order \(s\) (number of differentiations of the metric tensor) and degree \(q\) (number of factors \(\nabla R\) ...), etc.

4.2.1. Riemann polynomials of rank 0 (scalars). In 3D, the most general expression for a scalar of order six or less in derivatives of the metric tensor is obtained by expanding it in the FKWC-basis for Riemann polynomials of rank 0 and order 6 or less as follows [13].

The sub-basis for Riemann polynomials of rank 0 and order 2 consists of a single element: \(R[R_{4, 1}^{0}]\).

The sub-basis for Riemann polynomials of rank 0 and order 4 has three elements: \(R, R^{2}, R_{pq}^{*} R_{pq}^{*}\).

The sub-basis for Riemann polynomials of order 6 and rank 0 in 3D consists of the 10 following elements [13]: \(R, R^{2}, R_{pq} R_{pq}^{*}, R_{pq}^{*} R_{pq}^{*}, [R_{4, 1}^{0}], R_{pq} R_{pq}^{*}, R_{pq}^{*} R_{pq}^{*}, R_{pq}^{*} R_{pq}^{*}, R_{pq} R_{pq}^{*}, R_{pq}^{*} R_{pq}^{*}, R_{pq}^{*} R_{pq}^{*}, R_{pq}^{*} R_{pq}^{*}, R_{pq}^{*} R_{pq}^{*}\), and so on. These scalar polynomials are not algebraically independent (e.g., \(R^{2}\) is not independent of \(R, RR_{pq} R_{pq}^{*}\) is not independent of \(R\) and \(R_{pq} R_{pq}^{*}\), etc).

5. Example: the Karlhede algorithm

The maximal order of covariant derivative required for the invariant classification of a 3D Lorentzian pseudo-Riemannian manifold \((q \leq 5 [23])\) is relevant in determining the worst case scenarios for implementing the equivalence algorithm. It may be possible to provide a minimal basis on a case by case basis; some cases may have a smaller \(q\) and will then have fewer algebraically independent scalar invariants. For example, we could consider Segre type \([(1, 1), 1]\) (Ricci type I or \(P\)-type I) with no isotropy group.
In the special case of the (general) \( P \)-type I 3D spacetime with invariant count \((3, 3)\), we have that \( q = 1 \). At zeroth order there are three independent scalar invariants (after using the Lorentz freedom). There are 18 scalar invariants involving the first covariant derivative, of which only 15 are algebraically independent due to the Bianchi identities [23]. Therefore, in order to classify these spacetimes we only need (after choosing which three terms in the Bianchi identities that are algebraically dependent), \( 3 + 15 = 18 \) Cartan invariants.

Let us apply the Karlhede equivalence algorithm to the \( P \)-type I 3D spacetimes, in order to identify the Cartan invariants that are algebraically independent. We then list some of the simplest scalar curvature invariants and try to relate them to the algebraically independent Cartan invariants. It is argued that, in this general case, we may use the algebraic independence of the Cartan invariants to determine the minimal number of algebraically independent scalar curvature invariants.

5.1. 3D spaces of \( P \)-type I

Following the notation of [23]; at zeroth order, we fix all frame freedom by ensuring \( \Psi_0 = \Psi_2 \) and \( \Psi_1 = \Psi_3 \), so that \( \dim H_0 = 0 \)—this ensures that the coframe is invariant; that is, for any diffeomorphism \( \Phi \) we have that \( \Phi^* e_\alpha = \tilde e_\alpha \), where \( e_\alpha \) is the frame basis. At zeroth order, there are three non-zero linearly independent components of the Ricci tensor, namely \( R_0, \Psi_0 \) and \( \Psi_2 \). Hence, at zeroth order there may be at most three functionally independent invariants arising.

Let us assume that all three invariants are functionally independent. Then, in order to complete the algorithm we must set \( q = 1 \) and compute the covariant derivative of the Ricci tensor. Instead of working with the complicated expressions of a rank 3 tensor \( R_{abc} \); we use the fact that the frame derivatives of the zeroth order invariants are already invariants (because the coframe is an invariant coframe) and simplify the components to produce a set of scalars consisting of frame derivatives and spin-coefficients

\[
D\Psi_0, \Delta\Psi_0, \delta\Psi_0, D\Psi_2, \Delta\Psi_2, \delta\Psi_2, DR, \Delta R, \delta R, \alpha, \epsilon, \lambda, \kappa, \pi, \gamma, \tau, \sigma, \nu
\]

Of these 18 quantities, only 15 are algebraically independent due to the Bianchi identities [23].

This completes the algorithm for the \( P \)-type I 3D spacetimes with invariant count \((3,3)\); in order to classify these spacetimes we need only choose which three terms in the Bianchi identities are algebraically dependent and list the remaining \( 3 + 15 = 18 \) Cartan invariants.

5.2. Cartan invariants

Let us consider the minimal basis of algebraically independent scalar curvature invariants. By completing the Cartan equivalence algorithm we produce two notable subsets of the Cartan invariants by choosing three functionally independent invariants \( I_i \), \( i \in [1, 4] \); we may write the remaining Cartan invariants:

- \( H_1 \), \( l \in [1, m] \): collection of all invariants that are functionally dependent (f.d.) on \( I_l \).
- \( G_r \), \( l' \in [1, n] \), \( n \leq m \): collection of all invariants that are algebraically dependent (a. d.) on \( I_l \).

Treating these as sets we may produce the set of all invariants that cannot be written as algebraic expressions of the three invariants \( I_i \):

- \( F_{0 I_0} \), \( I_0 \in [1, m-n] \), with \( \{ F_{0 I_0} \} = \{ H_1 \} \setminus \{ G_r \} \).
Potentially this set may contain invariants that are still algebraically dependent on each other, despite being algebraically independent on \( I_i \).

To eliminate those elements of \( \{ F_{0, I_0} \} \) that are algebraically dependent, we repeat the following steps after denoting \( I_i = I_i^{(0)} \).

1. Set \( q = 0 \).
2. Write \( \{ F_{0, I_0} \} \) as functions of \( I_i^{(0)} \); i.e., \( \{ F_{0, I_0} \} = \{ G_{I_0}(I_i^{(0)}) \} \). Choose four functionally independent invariants from this set and denote them as \( I_i^{(1)} \).
3. Eliminate those invariants which may be expressed algebraically in terms of \( I_i^{(1)} \), and denote the subset of invariants that are f.d. but not a.d. on \( I_i^{(1)} \) as \( F \{ I_0 \}(1) \).
4. If \( F \{ I_0 \}(1) \) has less than three elements, stop the algorithm. Otherwise set \( q = 1 \) and move to the next step for \( q > 0 \).
5. Write \( \{ F_{q, I_0} \} \) as functions of \( I_i^{(q-1)} \); i.e., \( \{ F_{q, I_0} \} = \{ G_{I_0}(I_i^{(q-1)}) \} \). Choose three functionally independent invariants from this set and denote them as \( I_i^{(q)} \).
6. Eliminate those invariants which may be expressed algebraically in terms of \( I_i^{(q)} \), and denote the subset of invariants that are f.d. but not a.d. on \( I_i^{(q)} \) as \( F_{q, I_0} \).
7. If \( F_{q, I_0} \) has less than three elements, stop the algorithm. Otherwise set \( q = q + 1 \) and repeat steps 5–7 until such a \( q \) is found.
8. Given \( q_0 \) such that \( F_{q_0, I_0} \) has less than three elements, we may express these as functions of \( I_i^{(q_0)} \) and determine whether they are algebraically dependent on \( I_i^{(q_0)} \) to produce the smaller subset \( F_{q_0, I_0} \) containing at most two invariants which are f.d on \( I_i^{(q_0)} \) but not a.d.

By applying this procedure we produce a list of Cartan invariants that are algebraically independent

\[
F_{\text{sol}}, = \bigcup_{j=0}^{q_0} I_i^{(j)} \bigcup \tilde{F}_{q_0, I_0}.
\]

The cardinality of this set determines the number of algebraically independent polynomial scalar curvature invariants. Perron’s theorem [26] allows us to reduce the number of algebraically independent scalar curvature invariants to the cardinality of \( F_{\text{sol}}, \) as any scalar curvature invariant may be written in terms of these quantities; that is, as a polynomial in \( \text{dim}(F_{\text{sol}},) \)-variables. Of course this result only applies when there are more polynomials \( f_i \) than variables \( x_j, j \in [1, n] \).

If we wish to determine the smallest number of scalar curvature invariants \( \{ S_i \} \) \( i \in [1, N] \) such that any other scalar curvature invariant may be written as a polynomial in terms of this set \( \{ S_i \} \), Perron’s theorem is no longer helpful. As an alternative we conjecture that it is possible to exploit the relationship between Cartan invariants and the scalar curvature polynomial invariants in \( I \)-non-degenerate spacetimes, either by directly expressing the Cartan invariants as polynomials of the scalar curvature invariants [27] or indirectly by identifying algebraically independent polynomials by their dependence on Cartan invariants.

### 5.3. Scalar invariants at zeroth and first order

As an illustration, we consider one of the ‘best’-case scenarios for 3D \( I \)-non-degenerate spacetimes, the \( P \)-type I spacetimes [23] where the three zeroth-order Cartan invariants are functionally independent. It should be emphasized that this is not a proof, but rather an argument towards the feasibility of such an approach.
At zeroth order, the three algebraically independent scalar curvature invariants consist of the Ricci scalar $R = R^a_a$ and
\[ S_{ab} S^{ab} = S^2 = 3 \Psi_2^2 + \Psi_0^2, \quad \text{and} \quad S_{ab} S^{ac} S^{bd} = -\Psi_3^2 + \Psi_2^2 \Psi_0^2, \]
where $S_{ab}$ is the trace-free Ricci tensor.

At first order, we may compute the following scalar curvature invariants of lowest degree, $R^1$, $R^2$, $R^3$ and $R^4$, respectively
\[
\frac{1}{4} R_{abc} R^{abc : c} = 12 \alpha \Psi_0 \kappa \Psi_2 + 12 \nu \Psi_2 \alpha \Psi_0 + 16 \nu \Psi_0^2 \varepsilon - 2 \alpha \Psi_0^2 \pi - 18 \nu \Psi_0^2 \kappa
\]
\[
- 12 \sigma \Psi_0 \Psi_2^2 + 4 \sigma \Psi_0^2 \lambda + 36 \lambda \Psi_0^2 \sigma - 12 \lambda \Psi_0 \Psi_2^2 - 2 \kappa \Psi_0^2 \nu
\]
\[
- 18 \alpha \Psi_0^2 \tau - 3 D(\Psi_2) \Delta(\Psi_0) - 4(\Psi_0) \Delta(\Psi_2)
\]
\[
- 16 \alpha \Psi_0^2 \nu + \frac{\delta(\Psi_0)^2}{2} + 3 \delta(\Psi_2)^2
\]
\[
\frac{1}{2} R_{abc} R^{abc : b} = -2 \alpha \Psi_0 \delta(\Psi_0) - 8 \alpha \Psi_0^2 \alpha - 6 \delta(\Psi_2) \alpha \Psi_0^2 - D(\Psi_2) \Delta(\Psi_2)
\]
\[
- \Delta(\Psi_0) D(\Psi_0) - D(\Psi_2) D(\Psi_0) + 9 \tau \Psi_0^2 \nu + \nu \Psi_0^2 + \kappa \Psi_0^2
\]
\[
- \Delta(\Psi_2) \Delta(\Psi_0) + 4 \delta(\Psi_2)^2 - 18 \nu \Psi_0^2 \kappa - 12 \nu \Psi_2 \Psi_0 - 2 \delta(\Psi_0)^2 \nu
\]
\[
- 12 \Delta(\Psi_0) \Delta(\Psi_2) - 4 \Psi_0 D(\Psi_0) + 4 \Delta(\Psi_0) \varepsilon \Psi_2
\]
\[
- 8 \alpha \Psi_0^2 \tau + 2 \delta(\Psi_0) \varepsilon \Psi_0 - 6 \delta(\Psi_2) \nu \Psi_0 + 6 \alpha \Psi_0 \delta(\Psi_0)
\]
\[
- 4 D(\Psi_2) \varepsilon \Psi_0 + 2 \delta(\Psi_2) \kappa \Psi_0 - 4 D(\Psi_2) \sigma \Psi_0
\]
\[
+ 12 \Delta(\Psi_2) \varepsilon \Psi_0 + 6 \tau \Psi_0 \kappa \Psi_2 - 2 \tau \Psi_0^2 \pi + 16 \Psi_0^2 \varepsilon
\]
\[
+ 24 \alpha \Psi_0 \nu \Psi_2^2 + 24 \alpha \Psi_2 \nu \Psi_0 - 6 \alpha \Psi_2 \nu \Psi_0 + 9 \tau \Psi_2^2
\]
\[
- 6 \alpha \Psi_0 \nu \Psi_2 + 6 \Psi_0 \alpha \Psi_0^2
\]
\[
R^{a b c} R_{b c} = 4 D(\Psi_2) \varepsilon \Psi_0 + 6 \alpha \Psi_0 \nu \Psi_2 + 6 \delta(\Psi_2) \alpha \Psi_0 + 6 \delta(\Psi_2) \nu \Psi_2
\]
\[
+ 2 \delta(\Psi_0) \kappa \Psi_0 + 4 \delta(\Psi_2) \sigma \Psi_0 - 6 \alpha \Psi_0 \nu \Psi_2 + 2 \delta(\Psi_2) \nu \Psi_0 + 4 \delta(\Psi_0) \kappa \Psi_0
\]
\[
- 6 \alpha \Psi_0 \nu \Psi_2 + 2 \delta(\Psi_2) \nu \Psi_0 + 4 \delta(\Psi_0) \kappa \Psi_0 + 4 \delta(\Psi_2) \sigma \Psi_0
\]
\[
+ 2 \delta(\Psi_0) \kappa \Psi_0 + 4 \delta(\Psi_2) \sigma \Psi_0 - 6 \alpha \Psi_0 \nu \Psi_2 + 2 \delta(\Psi_2) \nu \Psi_0 + 4 \delta(\Psi_0) \kappa \Psi_0
\]
\[
These invariants lack the following first order Cartan invariants in each
\[
R_{a ; b} R^{a} R_{b} = D(\Psi_2) D(\Psi_0) D(\Psi_0) - 2 \delta(\Psi_2) \kappa \Psi_0 + 6 \delta(\Psi_2) \kappa \Psi_0
\]
\[
+ 2 \delta(\Psi_0) \kappa \Psi_0 + 4 \delta(\Psi_2) \sigma \Psi_0 - 6 \alpha \Psi_0 \nu \Psi_2 + 2 \delta(\Psi_2) \nu \Psi_0 + 4 \delta(\Psi_0) \kappa \Psi_0
\]
\[
+ 2 \delta(\Psi_0) \kappa \Psi_0 + 4 \delta(\Psi_2) \sigma \Psi_0 - 6 \alpha \Psi_0 \nu \Psi_2 + 2 \delta(\Psi_2) \nu \Psi_0 + 4 \delta(\Psi_0) \kappa \Psi_0
\]
By choosing three non-zero quantities in the Bianchi identities to be algebraically dependent we may remove three more terms from these three invariants; however, such a
choice will depend on the spacetime and the non-vanishing of the Cartan invariants involved in the Bianchi identities.

It is clear that the first invariant will be algebraically independent from the remaining three invariants as the first two lack all mention of the frame derivatives of $R$, and the third lacks the quadratic expressions in terms of the frame derivatives of $R$. Similarly $R^4$ will be algebraically independent to $R^2$ and $R^3$. Thus by inspection we have found four scalar curvature invariants that cannot be written as polynomials of each other. Continuing in this manner, it is conjectured that this process can be continued for the list provided from the Invar package of the algebraically independent scalar curvature invariants up to first order, and is believed to be the first 18 scalar curvature invariants listed in section 3.2

6. Discussion

In this paper we have discussed a basis for scalar polynomial curvature invariants in 3D Lorentzian spacetimes. Results from previous work on curvature invariants were reviewed. In particular, we discussed the class of spaces that can be characterized uniquely by their scalar curvature invariants, the $I$-non-degenerate spacetimes [2]. Careful consideration was given to the bound $p$ on the order of the covariant derivatives of the Riemann tensor that must be included in the contractions that form the scalar polynomial curvature invariants. The Karlhede bound is a well-studied problem in 4D, and some results were reviewed. Using the results from [21], we presented the bound $q = p + 1 \leq 5$ in 3D. After establishing this bound, the computer software Invar was used to calculate an overdetermined basis of scalar curvature invariants in 3D. Alternative bases, and particularly the so-called FKWC basis, were discussed in 3D spacetimes. In addition, we discussed the equivalence method (using the Karlhede algorithm) for computing the Cartan invariants, which can be used to determine a basis of all scalar curvature invariants in $I$-non-degenerate spacetimes.

Future work might include considering 3D spaces of different algebraic types on a case by case basis. It may be possible to lower the bounds on $p$ in some cases, as has been the case in 4D by considering spaces of different Petrov types [1]. In addition, we could use the equivalence method approach to determine a basis of all scalar curvature invariants in other particular 3D spacetimes [24]. Finally, the determination of a basis for scalar polynomial curvature invariants in 5D would also be an interesting and useful problem.

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