Computational analysis of unsteady and steady magnetohydrodynamic radiating nanofluid flows past a slippery stretching sheet immersed in a permeable medium

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Abstract. This paper discusses unsteady/unsteady radiating magnetohydrodynamic (MHD) nanofluid flow over a slippery stretching sheet. Introducing similarity variables reduced the Partial Differential Equations (PDEs) to a new set of PDEs in which the solution was a function of two independent variables. For time integration, first-order explicit Euler method was performed and spatial derivatives were approximated by the finite differences. The steady flow solution was compared by the built-in 	extit{bvp4c} solver in MATLAB. The flow regime was controlled by a number of thermophysical parameters, namely thermal Grashof number (Gr), Lewis number (Le), Eckert number (Ec), Brownian motion (Nb), thermophoresis (Nt), heat source or sink (S), Prandtl number (Pr), magnetic field (M), and Darcy number (Da). The findings were evaluated by graphs and tables for velocity, temperature, and concentration profiles as well as the skin friction coefficient, the local Nusselt number, and the local Sherwood number. The results converged in the grid convergence test. In the unsteady flow, the temperature of the nanofluid was higher near the surface without the thermophoresis parameter (Nt) and significantly decreased in the presence of Nt. Moreover, concentration boundary layer thickness decreased with an increase in the Darcy number (Da).

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1. Introduction

Today, more than ever, cooling of devices is one of the most pressing needs in many industrial technologies because of their ever-increasing heat generation rates at both micro (e.g., computer chips) and macro (e.g., car engines) levels. An abundance of cases exist in which the operational cost of the end product and mechanical work is highly dependent on the cooling rate of a process. For instance, the process of metal extraction from ores necessitates accurate calculation of heat transfer rate \cite{1}. Other examples of the application of heat transfer rate can be found in electronic devices, vehicle cooling, heat exchangers, and nuclear reactors \cite{2}.
Thermal conductivity of fluids can be increased by adding nanoparticles 100 nm in size to the base fluid (which is usually water). A colloidal suspension containing a mixture of nanoparticles and water is generally identified as a nanofluid, a term coined by Choi and Eastman [3]. Nanofluids have gained a prime importance in all walls of life, e.g., vehicle coolants, brake fluids, domestic refrigerators, etc. [4-8]. Some of their usages are in electronics (heat transfer), agriculture (energy harvesting systems), medicine (anti-infection therapy, hyperthermia), etc. [9-11]. A very comprehensive review of nanofluids has been provided in [8]. Buongiorno [12] studied various slip mechanisms between nanoparticles and the base fluid. Seventy slip mechanisms, namely inertia, Brownian diffusion, thermophoresis, diffusiophoresis, magnus effect, fluid drainage, and gravity, went under their investigation and Brownian diffusion and thermophoresis were found as the important slip mechanisms in the absence of turbulent effects. They also observed enhancement in heat transfer under convective conditions. In another study, Das et al. [13] examined boundary layer flow of nanofluid over a stretching sheet in the presence of thermal radiation with unsteady stream condition. They made enquiries into the effects of Brownian motion and thermophoresis. An interesting study of time-dependent tangent hyperbolic nanofluid flow over a wedge has been presented in [14]. In this study, the shooting technique was adopted for two types of wedges, namely static and stretching and an enhanced temperature profile was seen against different physical properties. A revised approach to the solar energy aspect of time-dependent magnetohydrodynamic (MHD) cross nanofluid is discussed in [15]. The problem was treated numerically and the solution was achieved by applying the shooting technique. It was indicated that the magnetic parameter influenced the wall shear stress. For further information on this topic, the reader is referred to [16-18].

Many researchers have recently paid attention to the study flow and heat transfer in electrically conducting continuous fluid under the influence of an applied magnetic field considering the fact that it provides a substantially wide range of applications to aerodynamics and many engineering problems such as MHD generators, MHD pumps, plasma, jet printer, nuclear reactors, and liquid metals. Rossow [19] was probably the first researcher who studied the hydrodynamic behavior of the boundary layer on a semi-infinite flat plate in the presence of a uniform magnetic field. Thermal radiation effect on MHD flow was discussed in [20]. Also, slip MHD flow was discussed in [21].

Makinde et al. [22] examined the variable viscosity effect of nanofluid on a radially stretching surface with radiative heat. Nield and Kuznetsov [23] presented convective boundary layer nanofluid flow in a porous medium. Hayat et al. [24] discussed the simultaneous effect of internal heat generation in Jeffery fluid on a nonlinear stretching surface with variable thickness. Analysis of unsteady flow is also important in aerodynamics, nuclear plants, and space vehicles. Makinde et al. [25] considered multiple factors to present a finite difference solution for unsteady flow over a slippery stretching sheet in a porous medium. Malik et al. [26] discussed boundary layer flow of Casson nanofluid over a vertically exponentially stretching cylinder. Joshi et al. [27] found a dual solution for MHD 3D nanofluid flow over a permeable stretching and shrinking sheet surface with velocity slip and thermal radiation. Reddy et al. [28] discussed variable fluid properties of slip flow of copper (Cu) based nanofluid over a stretching sheet with convective boundary conditions. Hakeem et al. [29] discussed the partial slip effect of flow over a porous sheet considering thermal radiation and wall mass transfer. Cortell [30] discussed the heat transfer in a porous medium with internal heat generation and absorption. Cai et al. [31] considered the unsteady convective flow over a vertically stretching surface. For some more information on partial slip, cf. [32-36].

Numerical methods have become widespread procedures to solve coupled differential equations. Instances of such methods are Finite Difference Method (FDM), Finite Volume Method (FVM), Finite Element Method (FEM), spectral methods, variational iteration method, shooting methods, bvp4c, etc. The variables in FDM, FVM, and FEM are defined on nodes, cells, and elements, respectively. Owing to its simplicity and easy implementation, the FDM is preferred to FVM and FEM in solving Partial Differential Equations (PDEs) for fluid flow [37,38]. Sheikholeslam [39] presented the Control Volume Finite Element Method (CVFEM) to solve nanofluid flow inside a porous medium considering Brownian motion. Recently, Reza-E-Rabbi et al. [40] applied explicit finite difference scheme to an unsteady chemically reacting fluid flow over a stretching sheet with Brownian and thermophoresis effects. Makinde et al. [25] also employed the explicit FDM for an unsteady flow. For the system of coupled Ordinary Differential Equations (ODEs), the superior choice among numerical methods is the recently introduced highly accurate collocation solver bvp4c written in MATLAB [41].

The general structure of the paper is as follows. Section 2 gives the definition of the physical problem in mathematical terms. Section 3 provides discretization of PDEs for both the spatial and temporal variables. Section 4 presents a steady flow analysis. The findings are discussed in Section 5. Conclusions are drawn in Section 6 at the end.
2. Mathematical formulation

We assume two-dimensional, unsteady, incompressible, laminar, MHD boundary layer flow of an electrically conducting nanofluid over a slippery stretching sheet submerged in a porous medium. The geometry of the problem is shown in Figure 1, in which the $x$-axis is taken along the sheet and $y$-axis is normal to it.

Under the above-mentioned assumptions, the governing equations consisting of continuity, momentum, energy, and concentration are construed in the following form using the Buongiorno model [12,25]:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}
\]

\[
u_{xy} + u_{xy} + \nu_y = \frac{-\sigma B_0^2 u^2}{\rho} - \frac{\nu}{K} \beta (T - T_\infty), \tag{2}
\]

\[
T_x + u T_x + v T_y = \frac{1}{\rho C_p} \frac{\partial}{\partial y} (k T_y) + \frac{\nu}{C_p} (u_y)^2
+ \frac{\sigma B_0^2 u^2}{\rho C_p} + \frac{\nu u^2}{C_p K} - \frac{1}{\rho C_p} Q_T(T - T_\infty)
+ \frac{\sigma B_0^2 u^2}{\rho C_p} \frac{D_B}{T_\infty} (T_y)^2, \tag{3}
\]

\[
C_t + u C_x + v C_y = D_B (C_{yy}) + \frac{\sigma B_0^2 u^2}{\rho C_p} \frac{D_T}{T_\infty} (T_y), \tag{4}
\]

where $(u, v)$ are the velocity components in the $x$ and $y$ directions, $\mu$ is the coefficient of viscosity, $\rho$ is the density of the fluid, $\sigma$ is electrical conductivity of the fluid, $T$ is fluid temperature, $K$ is permeability of the porous medium, $\beta$ is the thermal expansion coefficient, $k$ is thermal conductivity, $C_p$ is the specific heat capacity, $q_r$ is radiative heat flux, $Q$ is the heat source coefficient, $C$ is concentration, $\tau = (\rho C)_p/(\rho C)_f$ with $(\rho C)_p$ and $(\rho C)_f$ respectively being heat capacities of nanofluid and base fluid respectively, $C_p$ is the specific heat at constant pressure, $D_B$ and $D_T$ are Brownian and thermophoretic diffusion coefficients, $T_\infty$ is the ambient fluid temperature, and $C_\infty$ is the ambient fluid concentration.

2.1. Boundary conditions

The incremental boundary conditions are given as:

\[
\begin{align*}
    u(x, y, 0) &= 0, & v(x, y, 0) &= 0, & T(x, y, 0) &= T_\infty, \\
    C(x, y, 0) &= C_\infty, \\
    u(x, 0, t) &= U_w + \frac{u_L}{L} y, & v(x, 0, t) &= 0, & T(x, 0, t) &= T_w, \quad C(x, 0, t) &= C_w, \tag{5}
\end{align*}
\]

\[
u \rightarrow 0, \quad T \rightarrow T_\infty, \quad C \rightarrow C_\infty \text{ as } y \rightarrow \infty,
\]

where $U_w = ax$ is the stretching velocity, $T_w = T_\infty + bx$ is the variable temperature of sheet with $T_\infty$ being a free stream constant, and $C_w = C_\infty + bx$ is the variable concentration of nanofluid on the sheet with $C_\infty$ being a constant.

2.2. Solution method

The following similarity parameters are introduced to get the non-dimensionalized form of the momentum, energy, and concentration equations as well as the boundary conditions:

\[
\eta = \sqrt{\frac{a}{\nu}}, \quad \psi(\eta) = \sqrt{\alpha \nu x} f(\eta), \quad \tau = at,
\]

\[
\begin{align*}
    u &= \alpha x \frac{\partial f}{\partial \eta}, \quad v = -\sqrt{\alpha \nu} f(\eta), \\
    \theta(\eta) &= \frac{T - T_\infty}{T_w - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty},
\end{align*}
\]

where $\eta$ is the similarity variable; $\tau$ is the non-dimensional time; and $\psi(\eta)$, $\theta(\eta)$, and $\phi(\eta)$ are dimensionless stream, temperature, and concentration functions, respectively. By employing the above similarity variables, one can reduce Eqs. (2)–(4) into the following nonlinear differential equations

\[
\begin{align*}
    \frac{\partial^2 f}{\partial \eta^2} + f \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta} \left( \frac{\partial f}{\partial \eta} \right)^2 + \left( M + \frac{1}{Da} \right) \frac{\partial f}{\partial \eta} = 0, \tag{7}
\end{align*}
\]

\[
\begin{align*}
    &\text{Pr} \frac{\partial \theta}{\partial \tau} + Pr \left( \theta \frac{\partial f}{\partial \eta} - f \frac{\partial \theta}{\partial \eta} \right) - \left( 1 + N_l \right) \frac{\partial \theta}{\partial \eta} \left( \frac{\partial f}{\partial \eta} \right)^2 \\
    &- Ec \text{Pr} \left( \frac{\partial^2 f}{\partial \eta^2} \right)^2 - Ec \text{Pr} \left( M + \frac{1}{Da} \right) \left( \frac{\partial f}{\partial \eta} \right)^2 \\
    &- Pr \theta \phi - Pr N_l \phi \frac{\partial \theta}{\partial \eta} - N_l \phi \text{Pr} \left( \frac{\partial \theta}{\partial \eta} \right)^2 = 0. \tag{8}
\end{align*}
\]

Figure 1. Flow pattern of a slippery stretching sheet.
\[
\frac{\partial \phi}{\partial \tau} + \text{Le} \left( \frac{\partial f}{\partial \eta} - f \frac{\partial \phi}{\partial \eta} \right) + \frac{\partial^2 \phi}{\partial \eta^2} - \frac{Nt}{Nb} \frac{\partial \theta}{\partial \eta^2} = 0, \tag{9}
\]

where \( M, \text{Pr}, Nt, Nb, \text{Da}, \text{Gr}, \text{Ec}, \text{S}, \) and \( \text{Le} \) are the magnetic parameter, Prandtl number, thermophoresis parameter, Brownian parameter, Darcy number, thermal Grashof number, Eckert number, heat source, and Lewis number, respectively. These parameters are defined as [25]:

\[
\text{Pr} = \frac{\mu C_p}{k}, \quad M = \frac{\sigma b^2}{\rho a}, \quad Nt = \frac{\tau D_T (T_w - T_\infty)}{\nu T_\infty},
\]

\[
Nb = \frac{\tau D_B (C_w - C_\infty)}{\nu}, \quad \text{Le} = \frac{\nu D_B}{\beta g b}, \quad \text{Gr} = \frac{\beta g b}{a^2},
\]

\[
\text{Da} = \frac{K a}{\nu}, \quad \text{Ec} = \frac{a n_w}{b C_p}, \quad S = \frac{Q}{\rho C_p}.
\]

After transformation, the boundary conditions (5) and (6) take the following forms:

\[
\frac{\partial f}{\partial \eta}(\eta, 0) = 0, \quad f(\eta, 0) = 0, \quad \theta(\eta, 0) = 0,
\]

\[
\phi(\eta, 0) = 0, \quad \theta(\eta, 0) = 0, \quad \phi(\eta, 0) = 0, \quad \theta(\eta, 0) = 0.
\]

The skin friction coefficient \( C_f \) is given by [25]:

\[
C_f = \frac{u_w}{\rho w^2}, \quad \text{where} \quad u_w = \frac{-\partial u}{\partial \eta}.
\]

The local Nusselt number \( Nu_x \) is [25]:

\[
Nu_x = \frac{x j_w}{k(T_w - T_\infty)}, \quad \text{where} \quad j_w = -D \frac{\partial C}{\partial y} |_{y=0}.
\]

After using the similarity transformations, Eqs. (13), (14), and (15) become:

\[
\sqrt{Re_x} C_f = \frac{\partial f}{\partial \eta}(0, \tau),
\]

\[
\frac{Nu_x}{\sqrt{Re_x}} = -(1 + N r) \frac{\partial \theta}{\partial \eta}(0, \tau),
\]

\[
\frac{Sh_x}{\sqrt{Re_x}} = -\frac{\partial \phi}{\partial \eta}(0, \tau).
\]

Here, \( Re_x = U x / \nu \) is a local Reynolds number.

### 3. Unsteady flow analysis

#### 3.1. Spatial and temporal discretization

We assume a domain \( [0, \eta_{\infty}] \times [0, \tau_{end}] \) and \( N + 1 \) grid points in \( \eta \)-direction with equidistant grid spacing \( \Delta \eta = \frac{\eta_{\infty}}{N} \) and take \( \tau_{end} \) as the end time with time step \( \Delta \tau = \frac{\tau_{end} - \tau_0}{n} \). The coordinates of the grid points \((i, n)\) are \((\eta_i, \tau_n)\) where \( \eta_i = i \Delta \eta, i = 0, 1, \ldots, N \) and \( \tau_n = n \Delta \tau, n = 0, 1, 2, \ldots \). The finite difference stencil is shown in Figure 2. For discretization in \( \eta \)-direction we apply backward finite difference approximation to the first-order derivatives and central difference approximations to the second-order derivative. For time integration, we use explicit Euler method. For convenience, in order to implement FDM in MATLAB, we reduce the order of the momentum equation to convert it into a second-order PDE. The order of unsteady temperature and concentration equations is not reduced since it is already determined in the second order. All the finite difference approximations of all derivatives are given below. Now, we explain the procedure for reducing the order of the unsteady momentum equation. Defining \( \frac{\partial f}{\partial \eta} = F \), Eqs. (7), (8), and (9), take the following form:

\[
\frac{\partial F}{\partial \tau} - f \frac{\partial F}{\partial \eta} - \frac{\partial^2 F}{\partial \eta^2} + (F)^2 + (M + \frac{1}{Da}) F
\]

\[
- \text{Gr} \theta = 0,
\]

\[
\text{Pr} \frac{\partial \theta}{\partial \tau} + \text{Pr} \left( \theta F - f \frac{\partial \theta}{\partial \eta} \right) - (1 + N r) \frac{\partial^2 \theta}{\partial \eta^2} + \text{Ec} \text{Pr} \left( \frac{\partial^2 F}{\partial \eta^2} \right)^2 - \text{Ec} \text{Pr} \left( M + \frac{1}{Da} \right) (F)^2
\]

![Figure 2. Finite difference grid.](image-url)
\[-\Pr S\theta - \Pr N b \frac{\partial \phi}{\partial \eta} \frac{\partial \phi}{\partial \eta} - Nt Pr \left( \frac{\partial \theta}{\partial \eta} \right)^2 = 0, \quad (17)\]

\[
L \frac{\partial \phi}{\partial \tau} + Le \left( \phi F - f \frac{\partial \phi}{\partial \eta} \right) - \frac{\partial^2 \phi}{\partial \eta^2} - \frac{Nt}{N b} \frac{\partial^2 \theta}{\partial \eta^2} = 0. \quad (18)
\]

The order of associated boundary conditions (10), (11), and (12) is reduced as:

\[
F(\eta, 0) = 0, \quad f(\eta, 0) = 0, \quad \theta(\eta, 0) = 0, \quad \phi(\eta, 0) = 0. \quad (19)
\]

\[
F(0, \tau) = 1 + \frac{\partial F}{\partial \eta}(0, \tau), \quad f(0, \tau) = 0, \quad \theta(0, \tau) = 1, \quad \phi(0, \tau) = 1, \quad (20)
\]

\[
F(\infty, \tau) = 0, \quad \theta(\infty, \tau) = 0, \quad \phi(\infty, \tau) = 0. \quad (21)
\]

To solve the above system of equations along with boundary conditions, we approximate the derivatives in the PDEs by linear combinations of function values at the grid points using the FDM. All the finite difference approximations for all orders are given by:

\[
\left( \frac{\partial F}{\partial \eta} \right)_{(i,n)} = \frac{F^n_{i+1} - F^n_i}{\Delta \eta},
\]

\[
\left( \frac{\partial^2 F}{\partial \eta^2} \right)_{(i,n)} = \frac{F^n_{i+1} - 2F^n_i + F^n_{i-1}}{(\Delta \eta)^2},
\]

\[
\left( \frac{\partial F}{\partial \tau} \right)_{(i,n)} = \frac{F^n_{i+1} - F^n_i}{\Delta \tau},
\]

\[
\left( \frac{\partial \theta}{\partial \eta} \right)_{(i,n)} = \frac{\theta^n_{i+1} - \theta^n_i}{\Delta \eta},
\]

\[
\left( \frac{\partial^2 \theta}{\partial \eta^2} \right)_{(i,n)} = \frac{\theta^n_{i+1} - 2\theta^n_i + \theta^n_{i-1}}{(\Delta \eta)^2},
\]

\[
\left( \frac{\partial \phi}{\partial \eta} \right)_{(i,n)} = \frac{\phi^n_{i+1} - \phi^n_i}{\Delta \eta},
\]

\[
\left( \frac{\partial^2 \phi}{\partial \eta^2} \right)_{(i,n)} = \frac{\phi^n_{i+1} - 2\phi^n_i + \phi^n_{i-1}}{(\Delta \eta)^2},
\]

\[
\left( \frac{\partial \phi}{\partial \tau} \right)_{(i,n)} = \frac{\phi^n_{i+1} - \phi^n_i}{\Delta \tau},
\]

where the superscripts ‘n’ and ‘n + 1’ denote the solution at the nth and (n + 1)th time levels. Also, i, i − 1, and i + 1 represent current, previous, and next locations of the solution. Inserting all these approximations into Eqs. (16), (17), and (18), we get:

\[
\frac{F^{n+1}_i - F^n_i}{\Delta \tau} - \frac{F^n_{i+1} - 2F^n_i + F^n_{i-1}}{(\Delta \eta)^2} = 0.
\]

\[
-\frac{f^n_{i+1} - F^n_i}{\Delta \eta} + (F^n)^2 + M + \frac{1}{Da} \frac{F^n_{i+1} - F^n_i}{\Delta \eta} - \text{Gr} \theta^n_i = 0,
\]

\[
\text{Pr} \frac{\theta^n_{i+1} - \theta^n_i}{\Delta \tau} + \text{Pr} \theta^n_i \phi^n_i - \text{Pr} f^n_i \left( \frac{\theta^n_{i+1} - \theta^n_i}{\Delta \eta} \right) \quad \text{notag}
\]

\[
-\left( 1 + Nr \right) \frac{\theta^n_{i+1} - 2\theta^n_i + \theta^n_{i-1}}{(\Delta \eta)^2} - \text{EPr} \left( \frac{F^n_{i+1} - F^n_i}{\Delta \eta} \right)^2 - \text{EPr} \left( M + \frac{1}{Da} \right) \left( F^n_i \right)^2 - \text{Pr} S \theta^n_i
\]

\[
-\text{Pr} N b \left( \frac{\theta^n_{i+1} - \theta^n_i}{\Delta \eta} \right) \left( \frac{\phi^n_{i+1} - \phi^n_i}{\Delta \eta} \right) - \frac{\phi^n_{i+1} - 2\phi^n_i + \phi^n_{i-1}}{(\Delta \eta)^2} - \text{Nt} \frac{\theta^n_{i+1} - 2\theta^n_i + \theta^n_{i-1}}{N b} (\Delta \eta)^2 = 0.
\]

The initial and boundary conditions involved are:

\[
F^n_0 = 0, \quad f^n_0 = 0, \quad \theta^n_0 = 0, \quad \phi^n_0 = 0.
\]

\[
F^n_0 = 1 + \frac{F^n_1 - F^n_0}{(\Delta \eta)}, \quad f^n_0 = 0, \quad \theta^n_0 = 1, \quad \phi^n_0 = 1, \quad F^n_{\eta_{-\infty}} = 0, \quad \theta^n_{\eta_{-\infty}} = 0, \quad \phi^n_{\eta_{-\infty}} = 0.
\]

The discretization of the skin friction coefficient, the local Nusselt number, and the local Sherwood number for the unsteady flow is performed through:

\[
\text{Skin friction coefficient}
\]

\[
C_f = \frac{\tau_w}{\rho u_w^2}, \quad \text{where} \quad u_w = \frac{\partial u}{\partial y}.
\]
\[
\sqrt{Re_x} C_f = \frac{\partial^2 f(0, \tau)}{\partial \eta^2} = \frac{\partial F(0, \tau)}{\partial \eta},
\]

\[
\sqrt{Re_x} C_f = \frac{F_0^\eta - F_0^n}{(\Delta \eta)}.
\]

**Local Nusselt number:**

\[\text{Nu}_x = \frac{xq_w}{k(T_w - T_\infty)}, \text{ where :} \]

\[q_w = -k \left( 1 + \frac{16\sigma^2 T_\infty^4}{3k^2} \right) \Omega \frac{\partial T}{\partial y}.\]

\[
\frac{\text{Nu}_x}{\sqrt{Re_x}} = -(1 + Nr) \frac{\partial \theta(0, \tau)}{\partial \eta}.
\]

\[
\frac{\text{Nu}_x}{\sqrt{Re_x}} = -(1 + Nr) \frac{\theta_0^n - \theta_0^0}{\Delta \eta}.
\]

**Local Sherwood number:**

\[\text{Sh}_x = \frac{xj_w}{k(C_w - C_\infty)}, \text{ where :} \]

\[j_w = -D \frac{\partial C}{\partial y}|_{y=0}.
\]

\[
\frac{\text{Sh}_x}{\sqrt{Re_x}} = -\frac{\partial \phi(0, \tau)}{\partial \eta}.
\]

\[
\frac{\text{Sh}_x}{\sqrt{Re_x}} = -\phi_0^n - \phi_0^0.
\]

**3.1.1. Grid convergence analysis of unsteady flow**

Because of the lack of data on unsteady flow for comparison, we tried to evaluate the convergence of our results by the grid convergence check of the underlying FDM. In Figures 3–6, we display grid convergence for velocity, temperature gradient, concentration gradient, and the skin friction coefficient. It is notable that almost 150 nodal points are enough to get convergent results.

**4. Steady flow analysis**

For the steady flow, the governing problem given in Eqs. (16)–(18) reduces into:

\[
f'' + f f'' - (f')^2 - \left( M + \frac{1}{Da} \right) f + Gr\theta = 0, \quad (22)
\]

\[
Pr[(\theta f' - f\theta') - Ec(f')^2 - Ec\left( M + \frac{1}{Da} \right)(f')^2 - S\theta - Nt\theta' - Nt(\theta')^2] - (1 + Nr)\theta'' = 0. \quad (23)
\]
Figure 6. Grid convergence for the skin friction coefficient.

\[ \text{Le}(\phi f' - f' \phi') - \phi'' - \frac{Nt}{Nb} \theta'' = 0, \quad (24) \]

with:

\[ f'(0) = 1 + \lambda f'(0), \quad f(0) = 0, \quad \theta(0) = 1, \quad \phi(0) = 1, \quad (25) \]

\[ f'(\infty) = 0, \quad \theta(\infty) = 0, \quad \phi(\infty) = 0. \quad (26) \]

These equations have been solved by the built-in solver bvp4c in MATLAB.

4.1. Numerical procedure

To implement bvp4c for computing solutions to Eqs. (22)–(26), we first define \( y_1 = f, \quad y_2 = f', \quad y_3 = f'', \quad y_4 = \theta, \quad y_5 = \theta', \quad y_6 = \phi, \quad \) and \( y_7 = \phi'. \) Then the first-order system is written as:

\[ \begin{align*}
  y_1 &= y_2, \\
  y_2 &= y_3, \\
  y_3 &= -y_1 y_3 + y_2^2 + \left( M + \frac{1}{Da} \right) y_2 - G \rho y_1, \\
  y_4 &= -y_1 y_4 - \lambda y_1 y_5 - \lambda y_3 y_5 - Nt y_5^2, \\
  y_5 &= -y_1 y_5 - y_3 y_5 - Nt y_5^2, \\
  y_6 &= y_4, \\
  y_7 &= y_5.
\end{align*} \]

\[ y_6 = \frac{Pr}{1 + Nt} \left[ (y_4 y_5 - y_1 y_5) - Ec y_5^2 - Ec \left( M + \frac{1}{Da} \right) \right] \]

\[ y_7 = -S y_4 - Nt y_5 y_7 - Nt y_5^2 \]

\[ y_6' = y_7, \quad y_7' = \phi'' = \text{Le}(y_6 y_7 - y_1 y_7) - \frac{Nt}{Nb} y_7, \quad (27) \]

Similarly, the boundary conditions used for implementation in MATLAB are:

\[ y_6(2) = 1 + \lambda y_6(3), \quad y_6(1) = 0, \]

\[ y_6(4) = 1, \quad y_6(6) = 1, \quad y_{inj}(2) = 0, \]

\[ \theta_{inj}(4) = 0, \quad \phi_{inj}(6) = 0. \quad (28) \]

The physical parameters in this study are the skin friction coefficient, the local Nusselt number, and the local Sherwood number, which can readily be obtained from Eqs. (13), (14), and (15), respectively, i.e.:

The skin friction coefficient:

\[ \sqrt{Re} C_f = -f''(0), \]

The local Nusselt number:

\[ \frac{Nu_x}{\sqrt{Re} x} = -(1 + Nt) \theta'(0), \]

The local Sherwood number:

\[ \frac{Sh_x}{\sqrt{Re} x} = -\phi'(0). \]

5. Results and discussion

The result in Table 1 show the excellent agreement with the literature for the skin friction coefficient in steady flow. In Table 2, the results for the skin friction coefficient indicate that its value increases with increase in the values of Pr and M. The local Nusselt number surges with the values of Pr, Gr, and Da. The local Sherwood number experiences an upward trend with Gr, Rd, Da, Nb, Ec, and Le.

Table 1. Comparing \(-f''(0)\) with results in the literature for steady flow analysis.

| \( M \) | Maboed and Das [42] | Maboed and Shateyi [43] | Present result (bvp4c) |
|--------|-----------------|-------------------|-------------------|
| 0      | 1.0000008       | 1.00000084       | 1.0000            |
| 1      | 1.4142125       | 1.4142356        | 1.4142            |
| 5      | 2.4494897       | 2.44948974       | 2.4495            |
| 10     | 3.3166247       | 3.31662419       | 3.3166            |
| 50     | 7.1414284       | 7.14142843       | 7.1414            |
| 100    | 10.048875       | 10.0488756       | 10.0490            |
| 500    | 22.3830293      | 22.3830293       | 22.3830            |
| 1000   | 31.6385841      | 31.638584        | 31.6386            |
Table 2. Values of $-f''(0)$, $-\theta'(0)$, and $-\phi'(0)$ with different values of Pr, M, Gr, λ, Nr, Da, Ec, Nb, Nt, and Le.

|        | Pr | M  | Gr | λ  | Nr | Da | Ec | Nb | Nt | Le | $-f''(0)$ | $-\theta'(0)$ | $-\phi'(0)$ |
|--------|----|----|----|----|----|----|----|----|----|----|------------|----------------|-------------|
| 0.72   | 0.5| 0.5| 0.1| 0.2| 0.2| 0.2| 0.1| 0.2| 1.5|    | 1.8624     | 0.2488        | 0.6630       |
| 1      |    |    |    |    |    |    |    |    |    | 2  | 1.8658     | 0.2912        | 0.5930       |
| 2      |    |    |    |    |    |    |    |    |    |    | 1.8746     | 0.3909        | 0.4330       |
| 0.72   | 0   | 0.5| 0.1| 0.2| 0.2| 0.2| 0.1| 0.2| 1.5|    | 1.7921     | 0.2689        | 0.6642       |
| 0.5    |    |    |    |    |    |    |    |    |    | 1  | 1.8624     | 0.2488        | 0.6630       |
| 0.72   | 0   | 0   | 0.1| 0.2| 0.2| 0.2| 0.1| 0.2| 1.5|    | 2.0063     | 0.1526        | 0.5549       |
| 1      |    |    |    |    |    |    |    |    |    | 2  | 1.7252     | 0.2962        | 0.7296       |
| 2      |    |    |    |    |    |    |    |    |    |    | 1.4600     | 0.3514        | 0.8555       |
| 0.72   | 0.5 | 0.5| 0   | 0.2| 0.2| 0.2| 0.1| 0.2| 1.5|    | 2.3691     | 0.2161        | 0.8439       |
| 0.3    |    |    |    |    |    |    |    |    |    | 1  | 1.3129     | 0.2645        | 0.4913       |
| 0.5    |    |    |    |    |    |    |    |    |    |    | 1.0167     | 0.2637        | 0.4115       |
| 0.72   | 0.5 | 0.5| 0.1 | 0.4| 0.2| 0.2| 0.1| 0.2| 1.5|    | 1.8607     | 0.2263        | 0.6974       |
| 0.6    |    |    |    |    |    |    |    |    |    |    | 1.8594     | 0.2088        | 0.7236       |
| 0.8    |    |    |    |    |    |    |    |    |    |    | 1.8583     | 0.1946        | 0.7442       |
| 0.72   | 0.5 | 0.5| 0.1 | 0.2| 0.5| 0.2| 0.1| 0.2| 1.5|    | 1.3564     | 0.4059        | 0.6770       |
| 0.7    |    |    |    |    |    |    |    |    |    |    | 1.1191     | 0.4889        | 0.6877       |
| 0.5    |    |    |    |    |    |    |    |    |    |    | 1.0264     | 0.5227        | 0.6949       |
| 0.72   | 0.5 | 0.5| 0.1 | 0.2| 0.2| 0.2| 0.1| 0.2| 1.5|    | 1.8658     | 0.4114        | 0.3639       |
| 0.3    |    |    |    |    |    |    |    |    |    |    | 1.8607     | 0.1673        | 0.8117       |
| 0.5    |    |    |    |    |    |    |    |    |    |    | 1.8573     | 0.0339        | 1.1093       |
| 0.72   | 0.5 | 0.5| 0.1 | 0.2| 0.2| 0.2| 0.1| 0.2| 1.5|    | 1.8621     | 0.2295        | 0.7858       |
| 0.5    |    |    |    |    |    |    |    |    |    |    | 1.8609     | 0.2131        | 0.8603       |
| 0.7    |    |    |    |    |    |    |    |    |    |    | 1.8602     | 0.1969        | 0.8749       |
| 0.72   | 0.5 | 0.5| 0.1 | 0.2| 0.2| 0.2| 0.1| 0.2| 1.5|    | 1.8627     | 0.2532        | 0.7738       |
| 0.2    |    |    |    |    |    |    |    |    |    |    | 1.8624     | 0.2488        | 0.6630       |
| 0.4    |    |    |    |    |    |    |    |    |    |    | 1.8620     | 0.2405        | 0.4601       |
| 0.72   | 0.5 | 0.5| 0.1 | 0.2| 0.2| 0.2| 0.2| 0.2| 0.7 |    | 1.8626     | 0.2536        | 0.3017       |
| 1      |    |    |    |    |    |    |    |    |    |    | 1.8625     | 0.2510        | 0.3944       |
| 1.5    |    |    |    |    |    |    |    |    |    |    | 1.8624     | 0.2488        | 0.6630       |

In Figures 7 and 8 an upsurge tendency is observed in the velocity profile with the unsteady parameter $\tau$ under the influence of thermophoresis and magnetic parameters.

Figures 9 and 10 also illustrate the effects of thermophoresis and magnetic parameters with rising value of the unsteady parameter $\tau$ for the temperature profile. In the absence of $Nt$, the temperature is higher and thermal boundary layer thickness increases. When $Nt = 2$, the temperature is reduced significantly, as indicated in Figure 9, due to the migration of molecules. Figure 10 demonstrates an increase in the temperature near the boundary for $M = 0$ and $M = 1$. When the magnetic parameter is non-zero, velocity is reduced and hence, the temperature rises in the fluid flow.
Figure 7. Computational velocity profile against various values of $Nt$.

Figure 8. Computational velocity profile against various values of $M$.

Figure 9. Computational temperature against various values of $Nt$.

Figure 10. Computational temperature against various values of $M$.

6. Conclusions

This study presented a numerical solution for unsteady and steady magnetohydrodynamic (MHD) nanofluid flow above a slippery stretching sheet immersed in a porous medium. The key findings are:

- In the steady flow, the skin friction coefficient rises with $Pr$ and $M$, while it declines with all other parameters. The local Nusselt number experiences an increasing trend with $Pr$, $Gr$, and $Da$. The local Sherwood number rises with $Gr$, $Rd$, $Da$, $Ec$, $Nb$, and $Le$;

- In the unsteady case, the convergence of the numerical discourse was observed in the graphs and
it was demonstrated that approximately 150 grid points sufficed for the numerical solution:

- In the unsteady case, with an increase in nondimensional time, the thermophoresis and magnetic parameters had the same effect, i.e. the momentum boundary layer thickness increase;
- For unsteady flow, the nanofluid concentration profile is strongly dependent on the Grashof number and it decreases with increasing Da;
- In the unsteady case, the required stability of concentration profile is met possible by fine tuning of the values of the thermophoresis and Brownian motion parameters;
- The effect of unsteadiness was also visible in temperature profile. A significant increase in thermal boundary layer was found with $Nt$ and $M$.

**Nomenclature**

- $(u,v)$ Velocity components
- $\mu$ Coefficient of viscosity
- $\rho$ Density of fluid
- $\sigma$ Electrical conductivity of fluid
- $T$ Fluid temperature
- $K$ Porous medium permeability
- $\beta$ Thermal expansion coefficient
- $k$ Thermal conductivity
- $C_p$ Specific heat capacity
- $q_r$ Radiative heat flux
- $Q$ Heat source coefficient
- $C$ Concentration
- $\tau$ Time
- $(\rho C)_p$ Heat capacities of nanofluid
- $(\rho C)_f$ Heat capacities of base fluid
- $C_p$ Specific heat at constant pressure
- $D_B$ Brownian coefficient
- $D_T$ Thermophoretic diffusion coefficient
- $T_\infty$ Ambient fluid temperature
- $C_\infty$ Ambient fluid concentration
- $Pr$ Prandtl number
- $Gr$ Grashof number
- $Le$ Lewis number
- $Nt$ Thermophoresis number
- $Nb$ Brownian motion parameter
\( \lambda \)  
Slip parameter

\( S \)  
Heat source parameter

\( Nr \)  
Thermal radiation parameter

\( Ec \)  
Eckert number

\( Da \)  
Darcy number

\( M \)  
Magnetic parameter

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