GAUSS q-ED FROM HEINE CUBED

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Abstract. We consider $q$-analytic derivations of the $q$-Gauss summation formula for a $2\phi_1$ that respect the symmetry in its upper parameters.

Among the many original results in the pioneering paper [1847] of Heine is his $q$-series analogue for the Gauss summation of a hypergeometric series. Recall that the $q$-series analogue $2\phi_1$ of the hypergeometric series $\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c-1)}$ is defined by

$$2\phi_1 \left( \begin{array}{c} a \\ b \\ c \end{array} ; q \right) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n} \frac{z^n}{(q;q)_n}.$$ 

Here and in what follows, it is assumed that $|q| < 1$ and that the lower parameter $c$ is not equal to $q^{-N}$ for any integer $N \geq 0$. In these terms, the $q$-Gauss summation formula of Heine asserts that if $\frac{a}{b} \leq q < \frac{c}{ab}$ then

$$2\phi_1 \left( \begin{array}{c} a \\ b \\ c \\ ab \end{array} ; q \right) = \frac{(c/a;q)\infty (c/b;q)\infty}{(c;q)\infty (c/ab;q)\infty}.$$

This appears as Formula 80 on page 307 of [1847]; naturally, it is there expressed after the fashion of the time.

The traditional proof of the $q$-Gauss summation formula derives it as a consequence of the fundamental Heine transformation for $2\phi_1$: this asserts that if $z$ and $b$ lie (along with $q$) in the open unit disc then

$$2\phi_1 \left( \begin{array}{c} a \\ b \\ c \\ ab \end{array} ; q \right) = \frac{(c/a;q)\infty (c/b;q)\infty}{(c;q)\infty (c/ab;q)\infty}.$$ 

This transformation of Heine is central to [1847]: in fact, it appears on page 306 as Formula 79 in the form

$$\frac{(c;q)\infty}{(b;q)\infty} 2\phi_1 \left( \begin{array}{c} a \\ b \\ c \\ ab \end{array} ; q \right) = \frac{(a/c;q)\infty (b/c;q)\infty}{(z;q)\infty (c;q)\infty} 2\phi_1 \left( \begin{array}{c} c/b \\ a \\ z \\ ab \end{array} ; q \right).$$

Heine establishes the $q$-Gauss summation formula by evaluating his transformation formula at $z = c/ab$ and invoking the $q$-binomial series, the latter of which appears as Formula 74 on page 303 of [1847].

This original proof by Heine is one of the few known proofs of the $q$-Gauss summation formula. Ramanujan sketched an approach based on the $q$-binomial series in one of the manuscripts included in [1988] along with his ‘Lost Notebook’. Bailey [1935] offered a less elementary approach based on the $q$-Dougall identity of Jackson for an $8\phi_7$. None of these $q$-analytic derivations of the $q$-Gauss summation formula respects the symmetry in its upper parameters $a$ and $b$: this claim is at once clear in the case of Ramanujan’s approach; in the case of Bailey’s approach, the validity of the claim becomes evident upon closer inspection. Our aim in this short note is to consider $q$-analytic derivations of the $q$-Gauss summation formula that fully respect its symmetry. Section 1 reviews the Heine transformation formula and its group of symmetries; Section 2 addresses the task of developing a symmetric derivation of the $q$-Gauss summation formula.
We begin with a mild notational simplification. As \( _2\phi_1 \) is the only \( q \)-hypergeometric series that we shall consider, we drop ‘2’ and ‘1’ as subscripts, transferring the base \( q \) to one of the vacaries. Thus, we define

\[
\phi_q \left( \frac{a}{c} ; \frac{b}{c} ; \frac{c}{b} ; \frac{z}{b} \right) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} z^n
\]

assuming throughout that \( q \) lies in the open unit disc and that \( cq^N \neq 1 \) for each integer \( N \geq 0 \).

With this understanding, the fundamental transformation of Heine [1847] asserts that if also \( b \) and \( z \) lie in the open unit disc then

\[
\phi_q \left( \frac{a}{c} ; \frac{b}{c} ; \frac{c}{b} ; \frac{z}{b} \right) = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty} \phi_q \left( \frac{c/b}{az} ; \frac{z}{b} \right).
\]

This is the way in which the transformation is presented according to custom. Our purposes will be better served by exploiting the symmetry of \( \phi_q \) in its upper parameters: thus, we shall cast the Heine transformation as

\[
\phi_q \left( \frac{a}{c} ; \frac{b}{c} ; \frac{c}{b} ; \frac{z}{b} \right) = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty} \phi_q \left( \frac{z}{az} ; \frac{c/b}{az} \right).
\]

Of course, the case \( b = 0 \) calls for separate treatment, which we leave to the reader.

For organizational reasons, we introduce the parameter-variable space

\[
S = \left\{ \left( \frac{a}{c} ; \frac{b}{c} ; \frac{z}{b} \right) : a, b, c, z \in \mathbb{C} \setminus \{0\} \right\}.
\]

On \( S \) we define the operator

\[
U : S \to S : \left( \frac{a}{c} ; \frac{b}{c} ; \frac{z}{b} \right) \mapsto \left( \frac{b}{a} ; \frac{c}{b} ; \frac{z}{c} \right)
\]

that interchanges the upper parameters. Symmetry of \( \phi_q \) in its upper parameters now reads

\[
\phi_q \circ U = \phi_q.
\]

On \( S \) we also introduce the operator

\[
H : S \to S : \left( \frac{a}{c} ; \frac{b}{c} ; \frac{z}{b} \right) \mapsto \left( \frac{z}{az} ; \frac{c/b}{az} \right)
\]

along with the ‘automorphy factor’

\[
h : S \to \mathbb{C} : \left( \frac{a}{c} ; \frac{b}{c} ; \frac{z}{b} \right) \mapsto \frac{(c;q)_\infty (az;q)_\infty}{(b;q)_\infty (az;q)_\infty}.
\]

The fundamental Heine transformation now asserts that if

\[
s = \left( \frac{a}{c} ; \frac{b}{c} ; \frac{z}{b} \right) \in S
\]

then

\[
\phi_q(Hs) = h(s)\phi_q(s)
\]

where it is assumed that \( b \) and \( z \) lie in the open unit disc. Note that the composite \( UH \) corresponds to the customary way of presenting the Heine transformation.
Theorem 1. The operators $U$ and $UH$ have period two, while $H$ has period six.

Proof. That $U$ and $UH$ have period two is a matter of simple verification. The verification that $H$ has period six is likewise simple; for future reference, we record the iterates

$$H^2\left(\begin{array}{c} a \\ b \\ c \end{array}; z \right) = \left(\begin{array}{c} b \\ abz/c \\ c/b \end{array}; c/b \right)$$

$$H^3\left(\begin{array}{c} a \\ b \\ c \end{array}; z \right) = \left(\begin{array}{c} c/b \\ c/a; abz/c \end{array}; abz/c \right)$$

and note that $H^3$ is an involution. □

It follows at once that the group generated by $U$ and the Heine operator $H$ is dihedral of order twelve, the involution $H^3$ generating its centre. The value of this symmetry group was noted by Rogers, who exploited it in his study [1893] of the Heine transformation.

The iterates of the Heine operator have readily calculable effects on $\phi_q$. If $s = \left(\begin{array}{c} a \\ b \\ c \end{array}; z \right) \in \mathbb{S}$ then

$$\phi_q(H^2s) = \phi_q(HHs) = h(Hs)\phi_q(Hs) = h(Hs)h(s)\phi_q(s)$$

where

$$h(s) = \frac{(c; q)_\infty(z; q)_\infty}{(b; q)_\infty(z; q)_\infty}$$

and

$$h(Hs) = \frac{(az; q)_\infty(b; q)_\infty}{(c/b; q)_\infty(bz; q)_\infty}$$

so that

$$\phi_q\left(\begin{array}{c} b \\ abz/c \\ c/b \end{array}; c/b \right) = \frac{(c; q)_\infty(z; q)_\infty}{(c/b; q)_\infty(bz; q)_\infty} \phi_q\left(\begin{array}{c} a \\ b \\ c \end{array}; z \right)$$

in view of the formula for $H^2$ recorded in the proof of Theorem 1. Here $c/b$ and $z$ lie in the open unit disc, the intermediate requirement $|b| < 1$ being removed by analytic continuation. The effect of the cube $H^3$ we record as follows.

Theorem 2. If $z$ and $abz/c$ lie in the open unit disc then

$$(z; q)_\infty \phi_q\left(\begin{array}{c} a \\ b \\ c \end{array}; z \right) = (abz/c; q)_\infty \phi_q\left(\begin{array}{c} c/b \\ c/a \\ abz/c \end{array}; abz/c \right).$$

Proof. Argue essentially as above, using the formulæ recorded in the proof of Theorem 1. Again, analytic continuation authorizes the lifting of catalytic intermediate requirements. □

The cube of the Heine transformation is sometimes called the $q$-Euler transformation. Note that it is symmetric in the upper parameters $a$ and $b$; in this, it agrees with the $q$-Gauss summation formula.
GAUSS SYMMETRICALLY

We now turn to the \( q \)-Gauss summation formula, aiming at a derivation that fully respects its symmetry in the upper parameters.

For purposes of comparison we first recall the traditional derivation, presented in [1847]. This derivation rests on the \( q \)-binomial series, according to which

\[
\sum_{n=0}^{\infty} \frac{(u; q)_n}{(q; q)_n} z^n = \frac{(u z; q)_\infty}{(z; q)_\infty}
\]

if \( z \) (along with \( q \) as usual) lies in the open unit disc. In the Heine transformation

\[
\phi_q \left( \frac{a}{c}; \frac{b}{c}; \frac{z}{c} \right) = \frac{(b; q)_\infty}{(c; q)_\infty} \phi_q \left( \frac{c}{a}; \frac{c/b}{c}; \frac{b}{c} \right)
\]

put \( z = c/ab \): the right-hand side then becomes

\[
\frac{(b; q)_\infty}{(c; q)_\infty} \phi_q \left( \frac{c}{a}; \frac{c/b}{c}; \frac{b}{c} \right)
\]

whence follows the \( q \)-Gauss summation formula

\[
\phi_q \left( \frac{a}{c}; \frac{b}{c}; \frac{c}{ab} \right) = \frac{(c/a; q)_\infty}{(b; q)_\infty} (c/b; q)_\infty.
\]

Alternatively, the square of the Heine transformation offers a more interesting derivation. In

\[
\phi_q \left( \frac{a}{c}; \frac{b}{c}; \frac{1}{c} \right) = \frac{(c/b; q)_\infty}{(c; q)_\infty} \phi_q \left( \frac{b}{cz}; \frac{c/b}{c}; \frac{1}{c} \right)
\]

again put \( z = c/ab \): the \( q \)-Gauss summation formula follows at once, because

\[
\phi_q \left( \frac{b}{c/a}; \frac{1}{c/b} \right) = 1
\]

by virtue of the fact that \((1; q)_n\) is the Kronecker delta \( \delta_{0n} \).

Of course, neither of these derivations treats the upper parameters even-handedly. The cube of the Heine transformation shows more promise, as it is symmetric in the upper parameters. In order to apply ‘Heine cubed’ it is convenient to record the following elementary fact, in the statement of which we continue to suppress the underlying assumption \(|q| < 1\).

**Theorem 3.**

\[
\lim_{z \to 1} (z; q)_\infty \phi_q \left( \frac{a}{c}; \frac{b}{c}; \frac{z}{c} \right) = \frac{(a; q)_\infty}{(c; q)_\infty} (b; q)_\infty.
\]

**Proof.** Simply apply the Abel limit theorem to

\[
(z; q)_\infty \phi_q \left( \frac{a}{c}; \frac{b}{c}; \frac{z}{c} \right) = (z q; q)_\infty \cdot (1 - z) \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} z^n.
\]

Here, the limit may instead be taken within a Stolz sector, as usual. Of course, this theorem hints at the notion of Abel summability for possibly divergent \( q \)-series.

After this preparation, the \( q \)-Gauss summation formula may be derived as follows.
Theorem 4. If $|q| < 1$ and $|c| < |ab|$ then

$$\phi_q \left( \begin{array}{ccc} a & b & c \\ c & ab \end{array} \right) = \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty}.$$ 

Proof. We start from the $q$-Euler transformation of Theorem 2 in the equivalent form

$$(cz/ab; q)_\infty \phi_q \left( \begin{array}{ccc} a & b & c \\ c & ab \end{array} \right) = (z; q)_\infty \phi_q \left( \begin{array}{ccc} c/b & c/a & z \\ c & b & ab \end{array} \right),$$

where $q$, $z$ and $cz/ab$ lie in the open unit disc. Now let $z \uparrow 1$: the left-hand side plainly has limit

$$(c/ab; q)_\infty \phi_q \left( \begin{array}{ccc} a & b & c \\ c & ab \end{array} \right)$$

while the right-hand side has limit

$$\frac{(c/b; q)_\infty (c/a; q)_\infty}{(c; q)_\infty}$$

on account of Theorem 3.

□

A purist will rightly object that the ‘symmetry’ of this approach to the $q$-Gauss summation formula is undermined somewhat by the circumstance that, whereas the $q$-Euler transformation is symmetric in $a$ and $b$, the Heine transformation from which it is derived as the cube is not. This objection can be countered by an alternative derivation of the $q$-Euler transformation that handles the upper parameters $a$ and $b$ on equal terms, avoiding use of the Heine transformation. Such a derivation is made possible by the $q$-Pfaff-Saalschütz identity. In his eight-page eighth chapter, Bailey [1935] deduces the $q$-Euler transformation and the $q$-Gauss summation formula from the $q$-Pfaff-Saalschütz identity. However, his derivation of the $q$-Pfaff-Saalschütz identity itself is perhaps not quite elementary, being based on the $q$-Dougall identity of Jackson for an $8 \phi_7$; more to the point, his proof of the $q$-Dougall identity involves a breaking of symmetry, in that one of the upper parameters (there called $b$ and $c$) is singled out for special handling. The attractive recent introduction to $q$-analysis [2020] by Johnson places us within reach of thoroughly symmetric $q$-analytic derivations of the $q$-Gauss summation formula. In Section 5.7 Johnson passes from the $q$-Euler transformation to the $q$-Pfaff-Saalschütz identity by a careful comparison of coefficients. On the one hand, reconstitution effects a passage in the opposite direction, yielding a symmetric derivation of the $q$-Euler transformation from the $q$-Pfaff-Saalschütz identity; on the other hand, the $q$-Pfaff-Saalschütz identity is a finite version of the $q$-Gauss summation formula. Last but not least, the final exercise in Section 5.7 establishes the $q$-Pfaff-Saalschütz identity from first principles, while simultaneously treating the upper parameters symmetrically.

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