SEMIGROUPS OF $sl_3(\mathbb{C})$ TENSOR PRODUCT INVARIANTS

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Abstract. We compute presentations for a family of semigroup algebras related to the problem of decomposing $sl_3(\mathbb{C})$ tensor products. Along the way we find new toric degenerations of the Grassmannian variety $Gr_3(\mathbb{C}^n)$ which $T$-invariant for $T \subset GL_n(\mathbb{C})$ the diagonal torus.

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1. Introduction

A central problem in combinatorial representation theory is to compute the dimension of the space of invariant vectors

$$(1) 
H(\vec{\lambda}) = (V(\lambda_1) \otimes \ldots \otimes V(\lambda_n))^G
$$

in a tensor product of irreducible representations of a reductive group $G$, or equivalently a reductive Lie algebra $\text{Lie}(G) = \mathfrak{g}$. This problem is related to many interesting results each with their own combinatorial flavor, such as the Littlewood-Richardson rule, Kostant’s multiplicity formula, Steinberg’s formula, Littelman’s path model, the quiver methods of Derksen and Weyman [DW], and the crystal bases of Kashiwara and Lusztig [Lu]. Interest in formulas for the dimensions of these spaces also stems from their numerous appearances in geometry, the Schubert calculus in the cohomology of flag varieties, and the Hilbert functions of classical configuration spaces from invariant theory to name two. See Kumar’s survey paper [K] for an account of many of these results.

It is an especially beautiful feature of reductive Lie groups that this dimension can always be found by counting the lattice points in a convex polytope which depends on the weights $\lambda_1, \ldots, \lambda_n$ and some extra combinatorial information, namely a trivalent tree $T$ with $n$ leaves labelled $\{1, \ldots, n\}$ see [BZ1], [BZ2], [GP], [KTW], [M]. Let $\Delta$ be a Weyl chamber of $\mathfrak{g}$, the following is proved in [M].

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Theorem 1.1. For each trivalent tree $T$ with $n$ labelled leaves, there is a cone $Q_T(\mathfrak{g})$, along with a linear map $\pi_T : Q_T(\mathfrak{g}) \rightarrow \Delta^n$, such that the number of lattice points in $\pi_T^{-1}(\lambda) = Q_T(\lambda)$ is equal to the dimension of $H(\lambda)$.

Many such cones can be constructed, but we will gloss over this point as only one type, see section 2. We find a combinatorial description of the defining relations of the semigroup $Q_T(sl_3(\mathbb{C}))$ for trees of arbitrary size and topology.

Theorem 1.2. The semigroup $Q_T(sl_3(\mathbb{C}))$ is generated by lattice points $w$ such that $\pi_T(w)$ are all fundamental weights of $sl_3(\mathbb{C})$. These generators are always subject to at most quadratic and cubic relations.

A rather beautiful feature of the semigroup $Q_T(sl_2(\mathbb{C}))$ is that its generators can be put in bijection with the paths in $T$ that connect distinct leaves, see \cite{SpSt}. We find a similar result holds for each $Q_T(sl_3(\mathbb{C}))$. We say a subtree $T' \subset T$ is proper if every leaf of $T'$ is a leaf of $T$.

Theorem 1.3. There are exactly 2 generators of $Q_T(sl_3(\mathbb{C}))$ for each proper $T' \subset T$. Each semigroup $Q_T(sl_3(\mathbb{C}))$ is minimally generated by $2(2^n - n - 1)$ elements.

We also find a combinatorial description of the defining relations of $Q_T(sl_3(\mathbb{C}))$.

Now for a few more words about the relation of the spaces $H(\lambda)$ to invariant theory. For general reductive $G$, the spaces $H(\lambda)$ serve as the global sections of an ample line bundle $L^\lambda$ on a configuration space.

\begin{equation}
M^\lambda = G \setminus [G/P_1 \times \ldots \times G/P_n].
\end{equation}

Here $P_i$ is the parabolic subgroup of $G$ associated to $\lambda_i$, $G/P_i$ is the flag variety, and the quotient is a Mumford Geometric Invariant Theory quotient. When $G = SL_m(\mathbb{C})$, and each $\lambda_i$ is rank 1, ie a multiple of the first fundamental weight $\omega_1$, the space $M^\lambda$ is a space of weighted configurations of $n$ points on the projective space $\mathbb{P}^m$.

Determining the structure of the projective coordinate ring $R^\lambda$ of this space is a classical problem in invariant theory, see for example \cite{HMSV} and \cite{D}, 11.2.

In \cite{M}, the first author constructed toric degenerations of $R^\lambda$ to sub-algebras of $\mathbb{C}[Q_T(\mathfrak{g})]$, for all $\lambda$. We hope that by understanding the basic structure of $Q_T(sl_3(\mathbb{C}))$, the necessary combinatorics to resolve this question can be brought into reach for $sl_3(\mathbb{C})$. A similar program was successful in resolving the problem for points on $\mathbb{P}^1$ in part by understanding the cone $Q_T(sl_2(\mathbb{C}))$, see \cite{HMSV}.

We now recall the first fundamental theorem of invariant theory for $sl_3(\mathbb{C})$, stated in a slightly peculiar way.
Theorem 1.4 (First Fundamental Theorem of Invariant Theory). The projective coordinate ring $P_{3,n}$ of the Plücker embedding of the Grassmannian variety $Gr_3(C^n)$ can be identified with the $sl_3(C)$ invariant vectors in the algebra $C[x_1,x_2,x_3]^\otimes n$.

In terms of representation theory, this says that the Plücker algebra $P_{3,n}$ is isomorphic to the sum $\bigoplus_{\lambda \in \mathbb{Z}^n_{\geq 0}} (V(\omega_1) \otimes \ldots \otimes V(\omega_1))^{sl_3(C)}$. Results of [M] imply that this algebra flatly deforms to the semigroup algebra $C[P_T]$ where $P_T \subset Q_T(sl_3(C))$ is the subsemigroup of lattice points $w$ such that $\pi(w) \in \Delta^w_{sl_3(C)}$ is always a tuple of multiples of the first fundamental weight $\omega_1$. In this sense, the semigroup algebras $C[P_T]$ are an analogue of the toric degenerations of $Gr_2(C^n)$ constructed by Sturmfels and Speyer in [SpSt].

\[(3) \quad \pi(w) = (r_1 \omega_1, \ldots, r_n \omega_1)\]

We also determine the structure of this semigroup. We state our answer in terms of proper subtrees of $T$, as above.

Theorem 1.5. The minimal generators of the semigroup $P_T$ are in bijection with proper subtrees $T' \subset T$ which have the property that for any two leaves $\ell_1, \ell_2 \in T'$, the number of trinodes on the unique path between $\ell_1$ and $\ell_2$ is odd. The relations among these generators are quadratic and cubically generated.

We will show that a degeneration constructed by Sturmfels and Miller, [M] of $Gr_3(C^n)$, to a toric variety corresponding to a semigroup of Gelfand-Tsetlin patterns is realized by $P_T \cap T_0$, where $T_0$ is a special tree.

2. Description of $Q_T(sl_m(C))$

In this section we describe the affine semigroups $Q_T(sl_m(C))$. Elements of these semigroups are built out of triangular diagrams called Berenstein-Zelevinsky triangles.

Definition 2.1 (Berenstein-Zelevinsky triangles). Let $BZ_3(m)$ be the semigroup of non-negative integer weightings of the vertices of the version of the diagram below with $m-1$ small triangles to an edge, which satisfy the condition that any pair of pairs $A$ and $B$ and $E, D$ on the opposite side of a hexagon in the middle of the diagram satisfies $A + B = E + D$.

We orient the edges of the triangle counter-clockwise, and label the entries along the three edges $(a_1, \ldots, a_{2(m-1)})$, $(b_1, \ldots, b_{2(m-1)})$, $(c_1, \ldots, c_{2(m-1)})$. We define the boundary weights $\lambda_i$ as follows.

$\lambda_1 = (a_1 + a_2)\omega_1 + \ldots + (a_{2m-3} + a_{2m-2})\omega_{m-1}$

$\lambda_2 = (b_1 + b_2)\omega_1 + \ldots + (b_{2m-3} + b_{2m-2})\omega_{m-1}$

$\lambda_3 = (c_1 + c_2)\omega_1 + \ldots + (c_{2m-3} + c_{2m-2})\omega_{m-1}$

Here $\omega_i = (1, \ldots, 1, 0, \ldots, 0)$ is the $i$-th fundamental weight of $sl_m(C)$.

We will utilize a dual description of this semigroup given by Knutson, Tao, and Woodward in terms of weighted graphs on the BZ triangle diagram called
honeycombs. These objects also appear in the work Gleizer and Postnikov on tensor product multiplicities, \cite{GP}.

A honeycomb is constructed from the BZ triangle by replacing each entry $a$ on a hexagon with an edge weighted $a$ connected to the center of the hexagon. A weight $b$ on a corner of the triangle is left to represent a vertex weighted by $b$. The graphs coming from this construction can be characterized as those graphs that are "balanced" at each vertex dual to the interior of a hexagon. This means that the sum of the vectors pointing to the corners of the hexagon, scaled by the weights, is 0. Two of these graphs are "added" by taking their union and counting multiplicities.

Elements of $Q_T(sl_m(\mathbb{C}))$ are now constructed by pasting copies of the this triangle together over the structure of $T$.

**Definition 2.2.** The semigroup $Q_T(sl_m(\mathbb{C}))$ is the set of weightings of the diagram obtained from the tree $T$ such that
Figure 4.

(1) the weighting of each triangle in $Q_T(sl_m(C))$ is a BZ triangle,

(2) for any pair of adjacent triangles meeting along common edges, with counterclockwise oriented weights $(a_1, \ldots, a_{2m-2}), (A_1, \ldots, A_{2m-2})$, we have $a_{2i-1} + a_{2i} = A_{2(m-i)-1} + A_{2(m-i)}$.

We dualize this definition to form honeycomb graphs corresponding to elements of $Q_T(sl_m(C))$. Each weighting in $Q_3(sl_m(C))$ is dualized to a honeycomb, and honeycombs which are compatible by condition (2) above can be glued along the "stitching" added in between the BZ triangles.

3. Generators and relations for $Q_T(sl_3(C))$

In this section we describe a presentation of $Q_T(sl_3(C))$ for all $T$, in particular we prove Theorem 1.2.

3.1. Generators. First we describe the algebraic structure of $Q_3(sl_3(C))$. The triangles in Figure 5 are $X, Y$ in the top left, $P_{12}, P_{23}, P_{31}$, in the bottom left, and $P_{21}, P_{32}, P_{13}$ in the bottom right. The triangles $X$ and $Y$ each represent the determinant form in their respective triple tensor products of fundamental representations, and $P_{ij}$ represents the unique invariant in $V(\omega_1) \otimes V(\omega_2)$ in the positions $i, j$ respectively.

Proposition 3.1. The semigroup $Q_3(sl_3(C))$ is generated by $X, Y, \text{ and } P_{ij}$ for $i, j \in \{1, 2, 3\}$ distinct, subject to the relation $XY = P_{12}P_{23}P_{31}$. 

Proof. This is easily established with a software package which handles polyhedral cones, such as 4ti2 [4ti2].

This result says that $Q_3(sl_3(C))$ is generated by triangles with only fundamental weights along their edges. This does not hold as $m$ gets large, as shown by the
triangle in Figure 7. As the triangle gets larger, the diagram admits increasingly chaotic graphs which cannot be decomposed.

Indeed generation results like Theorem 1.2 cannot hold at larger scale either. The quilt depicted in Figure 6 shows that this cannot be the case when $m > 3$. Notice that the bottom BZ triangle is a generator and the top is composite, yet no factorization of the top can be extended to the bottom.

Figure 5.

We extend this result to $BZ_T(3)$ for any tree $T$. The first part of Theorem 1.2 follows from the next lemma.

**Lemma 3.2.** Let $S$ be a semigroup with a map $\partial : S \to \mathbb{Z}^2_{\geq 1}$, and let $\pi_1 : Q_3(sl_3(\mathbb{C})) \to \mathbb{Z}^2_{\geq 0}$ be the projection onto one of the edges. The fiber product semigroup $S \times_{\mathbb{Z}^2_{\geq 0}} Q_3(sl_3(\mathbb{C}))$ defined by these maps is generated by the elements $(s, w)$ such that $s$ is a generator of $S$.

**Proof.** Let $(x, z)$ be any element in the fiber product $S \times_{\mathbb{Z}^2_{\geq 0}} Q_3(sl_3(\mathbb{C}))$. Let $x = s_1 + \ldots + s_k$ be a decomposition of $x \in S$, and $z = w_1 + \ldots + w_n$ be a decomposition of $z$ into generators of $Q_3(sl_3(\mathbb{C}))$. The image vectors $\partial(x_1), \ldots, \partial(x_n) \in \mathbb{Z}^2_{\geq 0}$ and $\pi_1(w_1), \ldots, \pi_1(w_n)$ must satisfy
(4) \[ \sum \partial(x_i) = \sum \pi_1(w_j) \]

with each \( \pi_1(w_j) \in \{(0,1), (1,0)\} \). Since each element of \( \mathbb{Z}_2^2 \) uniquely decomposes into a sum of the generators \((0,1), (1,0)\), we can group the elements \( w_j \) into disjoint sets such that \( \partial(x_i) = \sum \pi_1(w_j(i)) \). But then it follows that \( (x_i, \sum w_j(i)) \in S_{\mathbb{Z}_2^2} Q_3(sl_3(\mathbb{C})) \).

Figure 7.

An induction argument now shows that \( Q_T(sl_3(\mathbb{C})) \) is generated by quilts which have the property that the weighting obtained by any restriction to an internal triangle is either 0 or a generator of \( Q_3(sl_3(\mathbb{C})) \). We define the support of a generator \( w \in Q_T(sl_3(\mathbb{C})) \) to be the set of edges in \( T \) where \( w \) is non-trivial. If the support of \( w \) has disjoint connected components, it follows easily that \( w \) can be factored.

Furthermore, if \( w \) has connected support, the corresponding subtree of \( T \) must be a proper, as defined in the introduction. This follows from an inspection of the available building blocks of \( w \) coming from \( Q_3(sl_3(\mathbb{C})) \). The same observation shows that these elements must be minimal.

The number of proper subtrees of \( T \) is exactly \( 2^n - n - 1 \), as this is the number of subsets of the set of leaves, modulo singletons and the empty set. For every \( T' \subset T \) proper there are precisely two ways to give \( T' \) a complete weighting. This is because the weighting on any edge in a complete weighting determines by induction the weightings on all other edges. It follows that there are \( 2(2^n - n - 1) \) minimal generators for each \( Q_T(sl_3(\mathbb{C})) \), this proves Theorem 1.3.

3.2. Relations. We now describe a generating set for the ideal \( I_T(sl_3(\mathbb{C})) \) that vanishes on these generators in \( \mathbb{C}[Q_T(sl_3(\mathbb{C}))] \).

Definition 3.3. For a vertex \( v \in V(T) \) we let \( T_1, T_2, T_3 \) be the connected components of the forest obtained by cutting along the edges connected to \( v \).

For any weighting or graph \( w \in Q_T(sl_3(\mathbb{C})) \) we obtain elements \( w_3 \in Q_3(sl_3(\mathbb{C})) \), and \( w_{T_i} \in Q_{T_i}(sl_3(\mathbb{C})) \) \( i = 1, 2, 3 \) by restriction. Note that if \( [w_1 \ldots w_k] - [v_1 \ldots v_j] \in I_T(sl_3(\mathbb{C})) \) then the restrictions of this word to \( Q_3(sl_3(\mathbb{C})) \) and the \( Q_{T_i}(sl_3(\mathbb{C})) \) will
be in $I_3(sl_3(\mathbb{C}))$ and $I_{\mathcal{T}}(sl_3(\mathbb{C}))$ respectively. We will consider the following two types of relations.

![Diagram](image.png)

**Figure 8.**

Here $A, B, C, D$ are meant to represent weightings on the quilts on either side of the depicted edge. If $A$ and $B$ are both compatible with $C$ and $D$ along this edge, a swap can be made.

Similarly, the main relation of $Q_3(sl_3(\mathbb{C}))$ can be extended to any $Q_{\mathcal{T}}(sl_3(\mathbb{C}))$ at any vertex $v \in V(\mathcal{T})$ for any compatible $A, B, C, D, E, F, G, H, I$ for the appropriate quilts on $Q_{\mathcal{T}}(sl_3(\mathbb{C}))$, as depicted in Figure 10. We now complete the proof of Theorem 1.2.

**Proposition 3.4.** The two types of relations depicted in Figures 10 and 10 suffice to generate $I_{\mathcal{T}}(sl_3(\mathbb{C}))$.

**Proof.** We select a trinode connected to two leaves $T \subset \mathcal{T}$. We write $\mathcal{T} = T \cup T'$ for the complement of $T$. Suppose that $c^1 \ldots c^n = d^1 \ldots d^m$ is a relation among generators $c^1, d^1 \in Q_{\mathcal{T}}(sl_3(\mathbb{C}))$. We consider the restriction of the words $c^1_T \ldots c^n_T$; $d^1_T \ldots d^m_T$ and $c^1_{T'} \ldots c^n_{T'}$; $d^1_{T'} \ldots d^m_{T'}$, to $T$ and $T'$ respectively.

By induction, there is a sequence of modifications of the word $c^1_{T'} \ldots c^n_{T'}$, by the above relations which result in the word $d^1_{T'} \ldots d^m_{T'}$. Note that each step in this sequence does not change the list of values along the edge $e$ which separates $T$ from $T'$. As a consequence, each modification can be extended to a relation between weightings of $\mathcal{T}$ which are trivial over $T$. The result is a word $g^1_T \ldots g^k_T$ which differs from $d^1_T \ldots d^m_T$ only over the trinode $T$. 

![Diagram](image.png)

**Figure 9.**
Now, since the relation $W_1 W_2 W_3 = Y_1 Y_2$ does not change the list of values along the edge $e$, we may use it to transform $g^1_T \ldots g^k_T$ to $d^1_T \ldots d^m_T$. Furthermore, since the list of boundary values along $e$ does not change, each application of this relation can be lifted to relations on weightings of $T$ which are trivial over $T'$. The resulting word now differs from $d^1_T \ldots d^m_T$ only up to an application of the relation in Figure 9. □

4. Generators and relations for $P_T$

In this section we prove Theorem 1.5 and relate the a particular instance of our quilt semigroups to a semigroup of Gel’fand-Tsetlin patterns. Define $F : Q_T(sl_3(\mathbb{C})) \to \mathbb{R}$ to be the function obtained by first projecting to the boundary weightings in $(\mathbb{Z}_2^{\geq 0})^n$, then summing all of the second fundamental weight components. This function is non-negative on all of $Q_T(sl_3(\mathbb{C}))$. From our discussion in the introduction, the semigroup $P_T$ can be identified with the 0 set of $F$. It follows that $P_T$ is face of $Q_T(sl_3(\mathbb{C}))$. As a result, in order to prove Theorem 1.5, we need only find the generators of $Q_T(sl_3(\mathbb{C}))$ which lie in $P_T$.

**Lemma 4.1.** A minimal generator of $Q_T(sl_3(\mathbb{C}))$ is in $P_T$ if and only if its associated proper tree $T'$ has the property that the unique path between any two leaves $\ell_1, \ell_2 \in T'$ contains an odd number of trivalent vertices.

**Proof.** Any generator $w \in P_T$ can only the first fundamental weight as a boundary value. Assuming this is the value of $w$ at a leaf $\ell_1$, we see that each pair of trinodes encountered on the unique path to a second leaf $\ell_2$ switches the weighting, either from $(1,0)$ to $(0,1)$ or vice-versa. It follows that exactly an odd number of such trinodes can be encountered on any such path. □

We now look at a special case, $P_{T_0}$, where $T_0$ is the caterpillar tree, depicted in Figure 11. Note that all trinodes in $T_0$ are connected to some leaf. It follows that
the only generators in $P_{T_0}$ are those defined by the proper subtrees $T_{i,j,k}$, which contain one trinode, with three paths emanating to the leaves on the specified indices, for all $\{i,j,k\} \subset \{1, \ldots, n\}$.

These generators are then subject to the induced relations from $Q_T(sl_3(C))$. Since the generator $Y \in Q_3(sl_3(C))$ never appears at a trinode of a generator of $P_{T_0}$, the cubic relations we described above make no contribution. It follows that the ideal $I_{T_0}$ that vanishes on the generators of $P_{T_0}$ is purely quadratic and binomial.

For two generator $w_{i,j,k}$ and $w_{r,s,t}$ we have the following relations,

\begin{align}
(5) \quad w_{i,j,k}w_{r,s,t} &= w_{r,j,k}w_{i,s,t}, \quad i, r > j, s \\
(6) \quad w_{i,j,k}w_{r,s,t} &= w_{i,t,k}w_{r,s,j}, \quad s, j > t, k
\end{align}

In [MS], Theorem 14.16, Sturmfels and Miller construct a degeneration of the Plücker algebra $P_{3,m}$ to a semigroup $GT_n(\omega_3)$ (our notation) of Gel’fand-Tsetlin patterns for the dominant weight $\omega_3$ of $GL_n(C)$. These are weightings of the vertices of the diagram below by non-negative integers for some $\lambda \in \mathbb{Z}_{\geq 0}$ which satisfy the interlacing inequalities indicated by the arrows. Namely, whenever an arrow connects two entries $m \rightarrow f$, we must have $f \geq m$. 

![Diagram of Gel'fand-Tsetlin patterns](image)

**Figure 11. A caterpillar graph**

**Figure 12. A Gel'fand-Tsetlin pattern in $GT_6(\omega_3)$**
where \((i, j, k)\) group algebra \((\text{of indices } (i, j, k))\) as the generators above. However, if \(w\) is either of the binomial generators of \(\lor \sigma\), it is a simple exercise to check that it must automatically contain additional minimal generators. In this sense \(P_{I_0}\) is the simplest member of this family of semigroups. We also remark that it is possible to construct an explicit bijection \(P_{I_0} \cong GT_n(\omega_3)\), and this can also be done for the corresponding semigroups for all \(sl_m(\mathbb{C})\).

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