A NOTE ON THE PAPER
"FRACTIONAL ORDER PETTIS INTEGRAL EQUATIONS WITH MULTIPLE TIME DELAY IN BANACH SPACES"
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Abstract. On a recent paper Benchohra and Mostefai [2] presented some existence results for an integral equation of fractional order with multiple time delay in Banach spaces. In contrast to the classical case, when assumptions are expressed in terms of the strong topology, they considered another case, namely with the weak topology. It has some consequences for the proof. We present here some comments and corrections.

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1. Introduction

It is known, that in some cases it is natural to consider the problems for which the functions satisfies some assumptions with respect to the weak topology on the space $E$. Then we can try to find some “solutions” of the considered (fractional) problem, but we should be very careful when trying to prove such a kind of results.

When we study such a case we need first to define a solution. Likewise for the standard problems ($\alpha = (1,1)$) we have two natural possibilities: either to consider weakly continuous solutions or strongly continuous ones. In the first case we have weaker assumptions, but the solutions are in a bigger set. In both cases different topologies are investigated (weak topology on $C(J,E)$ and the topology of weak uniform convergence on $C_w(J,E)$). The
choice has some consequence in the proof and the authors of [2] studied the first case. Let me start with some comments and corrections.

**Definitions**

First of all, some definitions should be corrected or updated. The definition of Pettis integrability require a comment how to understand the double integral. Since \( J_a \times J_b \) is a measure space, then we use classical definition (cf. [5, 8], for instance). Unfortunately, the space of Pettis integrable functions cannot be normed as claimed in the paper, i.e.

\[
\|u\|_{L^1} = \int_0^a \int_0^b \|u(x, y)\| \, dx \, dy
\]

since such a class of functions need not to be strongly measurable and this integral does not exist, in general. The space \( P_1(J_a \times J_b, E) \) (a standard symbol) is normed by

\[
\|u\|_{P} = \sup_{\|\varphi\| \leq 1} \int_{J_a \times J_b} \varphi(u(x, y)) \, d\lambda(x, y),
\]

where \( \varphi \in E^* \) and \( \lambda \) stands for a Lebesgue measure on \( J_a \times J_b \). We should be very careful in studying this space, because it is normed but incomplete space (although it is barrelled space), so we will be unable to use any results valid only for Banach or Fréchet spaces.

We have similar problem with the definition of the fractional Pettis integral of order \( \alpha > 0 \). The proposed notion is a kind of iterated integrals. Since the Fubini theorem fails in general even for the classical Pettis integral (cf. [8]) we need to consider the definition based on the above “double” fractional-Pettis integral, but allowing us to correct the proof of the main result. All integrals for real-valued functions will be taken in the Lebesgue sense, but to stress the role of the Pettis-integrability we will such a kind of integrals by \((P)\int_A x(s) \, ds\).

First observe that for \((y, t) \in J_a \times J_b\) a map \( I : t \mapsto (y - t)^\alpha \cdot \chi_{[0,y]}(t) \) belongs to \( L^1(J_b) \). As a scalar integral \( \varphi[(P)\int_{[0,y]}(y - t)^\alpha^{-1}h_1(t) \, dt] \) is a convolution of two integrable maps \( I(t) \) and \( \varphi(h_1) \) for the Pettis-integrable functions \( h_1 : J_b \to E \) and by [11, Theorem 3.4] we get the existence of the following Pettis integrals \((P)\int_{[0,y]}(t - s)^\alpha h_1(s) \, ds \in E \) for any \( t \in J_a \). If we have ensured the Pettis integrability of the second iterated integral, for \( \alpha = (\alpha_1, \alpha_2), \alpha_i > 0, \ i = 1, 2 \), then

\[
(I^\alpha_t h)(x, y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}(P)\int_{[0,x]}(P)\int_{[0,y]}(x - s)^{\alpha_1^{-1}}(y - t)^{\alpha_2^{-1}}h(t, s) \, dt \, ds
\]
is the well-defined fractional order integral for a Pettis-integrable function $h$ (on $J_a \times J_b$). However, the above integral is not the Pettis fractional integral on $J_a \times J_b$. A detailed theory for such a kind of problems with different classes of Pettis integrable functions can be found in [13, Section 2] or in [4].

Note that the above definition cannot be easily related with fractional differential problems unless we consider the class of their pseudo-solutions (cf. [3]). The paper [2] stands for a (partial) extension for an earlier one [1] in the special case of equations instead of inclusions.

**Proof of Theorem 3.4 in [2].**

Let us first note, that the question how to understand a notion of a solution is missing in the paper [2]. The lack of a definition implies some mistakes in the original proof. Indefinite Pettis integrals are absolutely continuous ([11, Theorem 2.5]), so the values of $T$ are continuous with respect to each variable separately (iterated integrals). Unfortunately, it is not sufficient to have jointly continuous functions. To have a domain the space $C(J_a \times J_b, E)$ it is necessary to use (H2). Conversely, this assumption is not necessary to have well-defined operator $T$ on $C(J_a \times J_b, E)$, but to have the values in this space. As this assumptions guarantee the existence of the fractional “double” Pettis integral we need to use the Fubini theorem to have iterated Pettis integrals (the operator $T$). As claimed, it will be possible if we have an extra assumption (cf. [8, 3]):

$$(H0) \ E = X^*, \text{ where } X \text{ is a weakly compactly generated Banach space and it does not contain any isomorphic copy of } l_1.$$  

Alternatively, we need to assume both the existence of the “double” Pettis integral and iterated Pettis integrals (a problem even with measurability hypotheses). Now, the Definition 3.1. in [2] is justified. However, there is no “a.e. Pettis integrability” as claimed in this proof. We improved a part of the proof in which the authors tried to show that the operator $T : C(J_a \times J_b, E) \to C(J_a \times J_b, E)$ is well-defined (in particular some arguments for the existence of iterated Pettis integrals - it cannot be done like that, cf. [13]). Moreover, it is not clearly stated, that $G < \infty$ (or it is necessary to put an assumption that $g_i$’s ($i = 1, ..., m$) are continuous).

Now, we need to discuss all the steps of the proof.

**Step 1.** In this part we need to correct the estimation based on the norm $\|f(s, t, u(s, t))\|$. It cannot be place inside the integral sign, because
it is not integrable (is merely scalarly measurable). But if we put there $|\varphi f(s, t, u(s, t))|$ and the assumption (H3x) given below (scalar boundedness) we can get $\|T(u)(x, y)\| \leq R$. For $(x, y) \in J$ the proof should be based on the norm of $\|\Psi(x, y)\|$ and require that $\|\Psi(x, y)\| \leq R$ which implies such an additional correction for the definition of this number in the proof (it should be “sufficiently big”).

The proof of the Hölder continuity of $T(u)$ should be corrected in the same manner - a “hidden” estimation with the norm of $f$ can be replaced as indicated above. It is necessary to check carefully whether the integral is for vector-valued functions (Pettis iterated integrals) or for real-valued ones (Lebesgue integrals) and to correct a misprint with a sign in the fourth line of the estimation. We should note that the proof presented in [2] is only for the case $x_1 < x_2$ and $y_1 < y_2$. Fortunately, it can be relatively easy extended with some necessary changes in the definition of the set $Q$.

**Step 2.** To control weakly convergent sequences in $C(J_a \times J_b, E)$ we need to use Dobrakov’s theorem [6, Theorem 9] (see also [9]) and weak uniform convergence is not needed. It is claimed in the proof, that we have weak uniform convergence of $(f(x, y, u_n(x, y)))$, which is not true and then we have not weak uniform convergence of $(T(u_n))$. But the use of the Lebesgue dominated convergence theorem for the Pettis integral ([10, Theorem 8.2]) require only scalar dominant for a sequence and convergence of $(\varphi f(\cdot, \cdot, u(\cdot, \cdot)))$ in measure. In the considered case both conditions are satisfied (see the Assumption (H3x) below), so $T$ is, in fact, weakly-weakly sequentially continuous on $C(J_a \times J_b, E)$ (cf. also [3]).

**Step 3.** This part of the proof is based on two main theorems: Ambrosetti’s Lemma (derived from [9], in fact) and the Mönch fixed point theorem for the weak topology. There are many conditions expressed in terms of measures of (weak) noncompactness allowing us to use the last theorem. A very interesting discussion can be found in [7]. In the paper [2] a special case of Szufla’s condition (H5) is assumed, but the proof is related with it. To the best of our knowledge this condition cannot be applied in this case, but the proof will be unchanged when assuming another condition (i.e. (H4x) below) instead if it (see also [3]). Note that, in general, there is no property allowing us to put directly the measure of weak noncompactness into the integral sign (as in the original proof in [2]). It should be done as in [3, p. 273]. We have no place here to study other compactness-type conditions (see [7], for instance).
Assumptions

In view of the amendments to the proof we need to present a current set of Assumptions. The assumption (H0) is taken as above. Assumptions (H1) and (H2) we left unchanged, but instead of (H3) we need the following scalar boundedness condition:

\( (H3x) \) there exists a function \( p(x, y) \) integrable on \( J_a \times J_b \) such that for all \( \varphi \in E^* \) and almost all \( (x, y) \) we have

\[
|\varphi f(x, y, u)| \leq |p(x, y)||\varphi|.
\]

The exceptional null-set may be dependent on \( \varphi \). Of course, it implies a.e. boundedness of \( f \).

\( (H4x) \) \( \beta(f(J_a \times J_b \times X)) \leq h(\beta(X)) \) for each measurable subset \( X \) of \( E \), where \( h \) is a Kamke-type function (see [3]).

A final version of the main Theorem 3.4 from [2] can be stated as follows (a solution is on \( \tilde{J} \) not in \( J \)):

**Theorem 1.1.** Assume that conditions \( (H0), (H1), (H2), (H3x), (H4x) \) and \( (H5) \) hold true. If

\[
mG + \frac{\|p\|_\infty a_{\alpha_1} b_{\alpha_2}}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)},
\]

then the problem (1)-(2) has at least one solution on \( \tilde{J} \).

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