Many-body Systems Interacting via a Two-body Random Ensemble (I): Angular Momentum distribution in the ground states

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In this paper, we discuss the angular momentum distribution in the ground states of many-body systems interacting via a two-body random ensemble. Beginning with a few simple examples, a simple approach to predict \( P(I) \)'s, angular momenta \( I \) ground state (g.s.) probabilities, of a few solvable cases, such as fermions in a small single-\( j \) shell and \( d \) boson systems, is given. This method is generalized to predict \( P(I) \)'s of more complicated cases, such as even or odd number of fermions in a large single-\( j \) shell or a many-\( j \) shell, \( d \)-boson, \( sd \)-boson or \( sdg \)-boson systems, etc. By this method we are able to tell which interactions are essential to produce a sizable \( P(I) \) in a many-body system. The g.s. probability of maximum angular momentum \( J_{\text{max}} \) is discussed. An argument on the microscopic foundation of our approach, and certain matrix elements which are useful to understand the observed regularities, are also given or addressed in detail. The low seniority chain of 0 g.s. by using the same set of two-body interactions is confirmed but it is noted that contribution to the total 0 g.s. probability beyond this chain may be more important for even fermions in a single-\( j \) shell. Preliminary results by taking a displaced two-body random ensemble are presented for the \( I \) g.s. probabilities.

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1 INTRODUCTION

Many-body systems interacting via a scalar two-body random matrix elements are expected to have eigenstates which are (nearly) random superpositions of Slater determinants. Among all the quantum numbers, the particle number $n$ and the total angular momentum $I$ of the states are the only good quantum numbers. Regularities exhibited by finite many-body systems interacting via random matrix elements provide an excellent tool to study general features, which are independent of interactions, of many-body systems [1]. Therefore, robust regularities of many-body systems, if there are any, are both interesting and important.

Recently, Johnson, Bertsch, and Dean discovered [2] that the dominance of $0^+$ ground state (0 g.s.) of even fermion systems can be obtained by using a two-body random ensemble (TBRE). Further studies showed that quasi-ordered spectra can be obtained by using a two-body random ensemble [3, 4, 5]. The 0 g.s. dominance was soon confirmed in $sd$-boson systems [5, 6]. Therefore, this 0 g.s. dominance in even nucleon systems and boson systems is robust and insensitive to the detailed statistical properties of the random ensemble, suggesting that the features of pairing arise from a very large ensemble of two-body matrix elements and might be independent of the specific character of the force. An understanding of this 0 g.s. dominance is important, because this observation seems to be contrary to the traditional assumption which was taken previously. For example, in nuclear physics the 0 g.s. dominance in even-even nuclei is explained as a reflection of a strong pairing associated with a strong short-range attraction between identical nucleons. Very recently, interesting studies are performed to check whether the spectroscopy with random and/or displaced random ensembles can simulate the realistic systems [7, 8].

There have been a few efforts to understand this observation. In Ref. [9], it was indicated that there is a correspondence between a large distribution width of $0^+$ states and the 0 g.s. dominance. In Ref. [10], it was suggested that for a system of interacting bosons the probability that the ground state has a certain value of
the angular momentum is not really fixed by the full distribution of eigenvalues, but rather by that of the lowest one. In [10], Mulhall et al. discussed the 0 g.s. dominance fermions in a single-\(j\) shell by using geometric chaoticity and uniformly distributed random interactions. Kusnezov discussed the \(sp\)-boson systems by using random polynomials [11]. In Ref. [12, 13], the authors discussed the 0 g.s. dominance in \(sp\)- and \(sd\)-boson systems in terms of the mean field approach. In Ref. [14] Kaplan et al. studied the correlation between eigenvalues and spins of corresponding states in a few simple cases.

For fermions in a small single-\(j\) shell, it was recently shown [15] that the width of the energy distribution for each angular momentum \(I\) is not the key to understand the 0\(^+\) g.s. dominance, instead, the coefficients \(\alpha^J_{I\beta\beta'}\) (\(n\)-body matrix elements of two-body interaction \(A^J_\dagger \cdot A^J\)) with \(\beta = \beta'\) was suggested to provide a reasonable explanation of the distribution of angular momentum \(I\) g.s., where \(\alpha^J_{I\beta\beta'} = \langle n\beta I | A^J_\dagger \cdot A^J | n\beta' I \rangle\) with \(n\) being the particle number, \(\beta\) additional quantum number necessary to label the state, and \(I\) being the angular momentum of the state. The \(A^J_\dagger \cdot A^J\) will be defined later. It was assumed [15, 16] that the off-diagonal matrix elements \(\alpha^J_{I\beta\beta'} (\beta \neq \beta')\) are neglected as an approximation, which is based on an observation [15] that they are very small compared with diagonal matrix elements \(\alpha^J_{I\beta\beta}\) for \(I > 0\) states of even fermions in a small single-\(j\) shell, e.g., \(j = \frac{9}{2}, \frac{11}{2}\), if one uses a seniority conserved basis [17]. This method has a disadvantage that it is not applicable to more complicated cases, e.g., fermions in a large single-\(j\) shell, where the off-diagonal matrix elements become also important.

These studies are interesting and important, and have potentially impacted our understanding on the origin of one of the most characteristic features of nuclear spectra. All these approaches, however, address only simple or very specific (\(sp\) and \(sd\) bosons, or fermions in a small single-\(j\) shell) cases. None of these arguments could explain the regularities of angular momenta \(I\) g.s. probabilities such as those observed in [15]. It is therefore very desirable to construct a universal approach to understand the 0 g.s. dominance of even nucleon systems and \(sd\)-boson systems,
and meanwhile, to understand all the I g.s. probabilities (denoted as $P(I)$) of very different systems including both even and odd number of fermions in a single-$j$ or a many-$j$ shell. In this paper, we shall present an approach to predict the $P(I)$’s of all types of systems in a simple but universal procedure.

In this paper, we use $G_j$’s be a set of Gaussian-type random numbers with a width being 1 and an average being 0:

$$\rho(G_j) = \frac{1}{\sqrt{2\pi}} \exp(-G_j^2/2), \quad J = 0, 2, \ldots, 2j - 1,$$

(1)

where the $G_j$’s define the two-body matrix elements of fermions in a single-$j$ shell as follows:

$$H = \sum_J G_J A^J \cdot A^J \equiv \sum_J \sqrt{2J + 1} G_J \left( A^J \times \tilde{A}^J \right)^0,$$

$$A^J = \frac{1}{\sqrt{2}} \left( a_j^\dagger \times a_j^\dagger \right)^J, \quad \tilde{A}^J = -\frac{1}{\sqrt{2}} \left( \tilde{a}_j \times \tilde{a}_j \right)^J, \quad G_J = \langle j^2 J | V | j^2 J \rangle.$$

For fermions in a many-$j$ shell we use $G_J(j_1 j_2 j_3 j_4)$’s in stead of $G_j$’s. For pure $d$ boson systems, there are only 3 independent two-body matrix elements, parameterized by $c_0, c_2$ and $c_4$ which will be defined later. For $sd$-boson systems we shall also define the two-body hamiltonian separately. All the results, except a few cases which we shall note clearly, are obtained by taking the two-body matrix elements to be a two-body random ensemble (TBRE) described by Eq. (1).

This paper is organized as follows: In Sec. 2, We begin with a few simple but non-trivial cases---a $j = \frac{7}{2}$ shell with $n = 3$ and 4, and $d$-boson systems with the number of bosons running from 3 to 48. In these cases, eigenvalues are analytically given in terms of linear combinations of two-body matrix elements. These examples are very helpful to understand our simple approach proposed in this paper, and also interesting because they are analytically given. In Sections 3 and 4, we generalize our method and apply it to both even and odd numbers of fermions in a large single-$j$ or a two-$j$ shell, $sd$- and $sdg$-boson systems, where this approach continues to work reasonably well. For fermions in a single-$j$ shell it is found in this paper that the $G_2 = -1$ always produces $I = n$ ground state for even $n$ fermions and
\[ I = j - (n-1)/2 \] ground state for odd \( n \). In Sec. 5 we discuss the foundation of our simple approach, and suggest that the essential part of the \( I \) g.s. probability is related to interaction \( G_J \)'s which produce angular momentum \( I \) g.s. if \( G_J = -1 \) and others are zero. This disproves a popular idea that the 0 g.s. dominance of even fermions and bosons comes intrinsically from the two-body nature of the interactions and might be independent of the two-body interactions that used. In Sec. 6, we address the \( I_{\text{max}} \) g.s. probability. For fermions in a single-\( j \) shell. We present the eigenvalues of the \( I_{\text{max}} \) states with \( n = 3, 4, 5, 6 \). This picture can be generalized to explain a few features of the \( I_{\text{max}} \) g.s. probabilities of fermions in a two-\( j \) shell. In Sec. 7, We shall check the previous statements of “pairing” and seniority “chain” related to the 0 g.s. dominance for fermions in a single-\( j \) shell, where seniority is well defined. We confirm the finding of a low seniority chain suggested in \([2]\) but note that the contribution to the 0 g.s. probability beyond this chain may be more important. A summary of this work will be given in Sec. 8. In appendix A we present our preliminary results by using a displaced TBRE hamiltonian. In appendix B we give a simple algorism to calculate diagonal matrix elements of fermions in a single-\( j \) shell. In Appendix C, a few counter examples of the 0 g.s. dominance, and counter examples of \( I = j \) for odd fermions systems are also given.
2 A FEW SIMPLE SYSTEMS

Generally speaking, eigenvalues of a many-body system are not linear in terms of two-body matrix elements. In some simple cases such as fermions in a single-$j$ shell with $j \leq 7/2$ and pure $d$-boson systems, however, all eigenvalues are given in terms of linear combinations of two-body matrix elements. For these cases an understanding of the I g.s. probabilities using a concept of shift described in Ref. [15] is applicable: A state which has the largest and/or the smallest coefficients for a given $G_J$ in the eigenvalues is favored to be the ground state. These examples will provide a useful clue to obtain a universal approach.

2.1 $j = \frac{5}{2}$ and $\frac{7}{2}$ shells

Fermions in a single-$j$ ($j = \frac{5}{2}$ or $\frac{7}{2}$) shell are simple but non-trivial cases. An analytical relation between eigen-energies $E_I$ and two-body matrix elements $G_J$ are available.

The eigen-energies of states with 3 fermions in a $j = \frac{5}{2}$ shell are given by

$$E_{3/2} = \frac{15}{7} G_2 + \frac{6}{7} G_4,$$
$$E_{5/2} = \frac{2}{3} G_0 + \frac{5}{6} G_2 + \frac{3}{2} G_4,$$
$$E_{0/2} = \frac{9}{14} G_2 + \frac{33}{14} G_4. \quad (2)$$

The eigen-energies $E_I$ of states with 3 fermions in the $j = \frac{7}{2}$ shell are as follows:

$$E_{3/2} = \frac{9}{14} G_2 + \frac{33}{14} G_4 + 0 G_6,$$
$$E_{5/2} = \frac{11}{6} G_2 + \frac{5}{7} G_4 + \frac{65}{66} G_6,$$
$$E_{7/2} = \frac{3}{4} G_0 + \frac{5}{12} G_2 + \frac{2}{4} G_4 + \frac{13}{12} G_6,$$
$$E_{9/2} = \frac{13}{12} G_2 + \frac{150}{77} G_4 + \frac{49}{66} G_6,$$
$$E_{11/2} = \frac{5}{6} G_2 + \frac{13}{22} G_4 + \frac{53}{33} G_6,$$
$$E_{15/2} = 0 G_2 + \frac{15}{22} G_4 + \frac{51}{22} G_6. \quad (3)$$
For four fermions in a $j = \frac{7}{2}$ shell, each eigenstate is labeled using seniority number $v$ and angular momentum $I$. The eigen-energies $E_{I(v)}$ of states with 4 fermions are as follows:

$$E_{0(0)} = \frac{3}{2} G_0 + \frac{5}{6} G_2 + \frac{3}{2} G_4 + \frac{13}{6} G_6,$$

$$E_{2(2)} = \frac{1}{2} G_0 + \frac{11}{6} G_2 + \frac{3}{2} G_4 + \frac{13}{6} G_6,$$

$$E_{2(4)} = G_2 + \frac{42}{11} G_4 + \frac{13}{77} G_6,$$

$$E_{4(2)} = \frac{1}{2} G_0 + \frac{5}{6} G_2 + \frac{3}{2} G_4 + \frac{13}{6} G_6,$$

$$E_{4(4)} = \frac{7}{3} G_2 + \frac{1}{2} G_4 + \frac{8}{3} G_6,$$

$$E_{5(4)} = \frac{8}{7} G_2 + \frac{192}{77} G_4 + \frac{26}{11} G_6,$$

$$E_{6(2)} = \frac{1}{2} G_0 + \frac{5}{6} G_2 + \frac{3}{2} G_4 + \frac{19}{6} G_6,$$

$$E_{8(4)} = \frac{10}{21} G_2 + \frac{129}{77} G_4 + \frac{127}{33} G_6. \quad (4)$$

In Eqs. (2-4), bold (italic) font is used for coefficients which are the largest (smallest) among all the $I(v)$ states for a given $J$.

We rewrite Eqs. (2-4) as follows:

$$E_{I(v)} = \sum_J \alpha^J_{I(v)} G_J = \sum_J \alpha^J_k G_J, \quad (5)$$

where $k = I\beta\beta$ (= $I(v)$ in this subsection), and $\alpha^J_k$ satisfies a sum-rule \([17]\)

$$\sum_J \alpha^J_k = \frac{1}{2} n(n - 1). \quad (6)$$

A method to calculate the above $\alpha^J_k$ for 4 fermions in a single-$j$ shell is given in terms of 9-$j$ coefficients in appendix A, where explicit expressions are available for $v = 0$ states. The $\alpha^{J_{\text{max}}}_{I_{\text{max}}}$ of fermions in a single-$j$ shell with $n = 3$ to 6 is given in sec. 2.4.

By using a TBRE hamiltonian described by Eq. (1) and the eigen-energies given by Eqs. (2-4), it is easy to obtain the probability for each $I$ ground state ($I\text{ g.s.}$). On the other hand, one can predict the $I\text{ g.s.}$ probability without running a TBRE
hamiltonian. For example, the exact 0 g.s. probability of 4 fermions in the \( j = \frac{7}{2} \) shell is determined by the following integral:

\[
\int dG_0 \int dG_2 \int dG_4 \int dG_6 \int dE_0 \int_{E_0(0)} dE_2 \cdots \int_{E_0(0)} dE_8 \\
\delta \left( E_0(0) - \sum_J \alpha_{0(0)}^J G_J \right) \cdots \delta \left( E_{8(4)} - \sum_J \alpha_{8(4)}^J G_J \right) \rho(G_0) \rho(G_2) \rho(G_4) \rho(G_6). (7)
\]

The \( P(I) \)'s of fermions in a single-\( j \) shell with \( j = \frac{7}{2} \), or \( \frac{5}{2} \) are given in Tables I-III. The row “TBRE” corresponds to results obtained by using Eqs. (2-4) and 1000 sets of a TBRE hamiltonian. The row “pred1.” corresponds to the probabilities calculated by integrals such as Eq. (7) for the 0 g.s. probability of 4 fermions in a \( j = \frac{7}{2} \) shell. Using Eqs. (2-4) the distribution width, \( g_I(v_{\beta \gamma}) \), of each state, is equal to \( \sqrt{\sum_J (\alpha_{I(v)}^J)^2} \). These widths are listed in the last row of Tables I-III.

It is noticed that an understanding in terms of shifts works well: a state with one or more largest (or the smallest) \( \alpha^J_{I(v)} \) has a very large probability to be the ground state or the highest state. The \( P(I) \)'s of states without the largest and/or the \( \alpha^J_{I(v)} \) for a given \( J \) are very small. A schematic argument of such an observation was given in detail in [15, 16].

For 3 fermions in a \( j = \frac{5}{2} \) shell, all states have the largest \( \alpha^J_{I(v)} \). Therefore, the above argument by using shifts predicts that all \( P(I) \)'s are large, because all the 3 states have the largest \( \alpha^J_{I(v)} \) for a given \( J \). In Table I, all states have indeed large probabilities, which are obtained by running 1000 sets of a TBRE hamiltonian and shown in the row “TBRE”, be the ground. The state with \( I = \frac{5}{2} \) have only one \( \alpha^J_{I(v)} \) which is the largest among different \( I \) states, while the other two states with \( I = \frac{3}{2} \), \( \frac{9}{2} \) have both the largest and the smallest \( \alpha^J_{I(v)} \). Therefore, \( P(I = \frac{5}{2}) \) is smaller than \( P(I = \frac{5}{2}) \) and \( P(I = \frac{9}{2}) \).

For 3 fermions in a \( j = \frac{7}{2} \) shell, states with \( I = \frac{3}{2}, \frac{5}{2}, \frac{15}{2} \) have both the largest and smallest \( \alpha^J_{I(v)} \), and the state with \( I = \frac{7}{2} \) has the largest \( \alpha^J_{I(v)} \) (\( J = 0 \)). Therefore, the above argument predicts that the probabilities of \( I = \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \) and \( \frac{15}{2} \) g.s. are large. From Table II it is noticed that all these states have large probabilities to be the ground states, but, the state \( I = \frac{7}{2} \) state does not have a coefficient which is the
smallest among $\alpha_J^I$'s with a given $J$ but different $I$, indicating that the $I = \frac{7}{2}$ g.s. probability is a bit smaller than those of $I = \frac{3}{2}, \frac{5}{2}, \frac{15}{2}$. The $I = \frac{9}{2}$ and $\frac{11}{2}$ states do not have coefficients $\alpha_J^I$ which are the largest or the smallest among all the states. The predicted probabilities of these two $I$ g.s. are small. Note that it can be shown that the $I = \frac{11}{2}$ cannot be the ground state. All these features are confirmed by the $P(I)$'s, which are calculated by running 1000 sets of a TBRE Hamiltonian, in the column “TBRE”.

The situation is very similar in the case of 4 fermions in $j = \frac{7}{2}$ shell: in each of the four states ($I(v) = 0^+, 2^+(4), 4^+(4), 8^+$) which have large probabilities as the ground state (and the highest state), there are coefficients $\alpha_J^I(v)$ which are the largest and/or the smallest for different $I(v)$. Other states which do not have the largest and/or the smallest $\alpha_J^I$ coefficients have very small probabilities to be the ground state or the highest state. Refer to the column “TBRE” in Table III.

In Tables I-III, the $P(I)$'s obtained by running 1000 sets of a TBRE Hamiltonian, and those obtained by using integrals such as Eq. (7) for the 0 g.s. probability of 4 fermions for a $j = \frac{7}{2}$ shell, are well consistent with each other. Therefore, the $I$ g.s. probabilities $P(I)$ in the above cases are explained in terms of shifts, defined in Ref. [13].

However, the $P(I)$'s in Eq. (7) is not yet within reach of a simple procedure by hand, and one has to evaluate this integral numerically. It is then very interesting and important to find a simple method to evaluate the $I$ g.s. probabilities.

Let $N'_I$ be the number of both the smallest and largest $\alpha_J^I(v)$ with a fixed $J$ for a certain $I$, the $I$ g.s. probability is approximately given by $N'_I/(N_m)$, where $N_m = 2N - 1$, $N$ is the number of two-body matrix elements. The predicted $I$ g.s. probabilities by this method are also given in the row ”pred2.” of Tables I-III. A reasonable agreement is obtained, though there are small differences, compared with those obtained by running a TBRE Hamiltonian. Note that in the above examples, we use $N_m = 2N - 1$ because all $\alpha_J^I(v)(I \neq 0)$'s are 0 (there is no smallest $\alpha_J^I(v)$).
2.2 \textit{d}-boson systems

Similar to fermions in a small single-\textit{j} shell (\(j = \frac{5}{2}\) or \(j = \frac{7}{2}\)), the relation between the two-body matrix elements and the eigenvalues of \(d\)-boson systems is also linear.

The two-body hamiltonian of a \(d\)-boson system is given by

\[
H_d = \sum_{l} \frac{1}{2} \sqrt{2l + 1} c_l \left( \left(d^\dagger \times d^\dagger\right)^l \times \left(\tilde{d} \times \tilde{d}\right)^l \right)^0
\]

(8)

From Eq. (2.79) of Ref. [18], we have

\[
E = E_0 + \alpha' \frac{1}{2} n_d(n_d - 1) + \beta'[n_d(n_d + 3) - v(v + 3)] + \gamma'[I(I + 1) - 6n_d],
\]

(9)

where \(E_0\) contributes only to binding energies and not to excitation energies, \(n_d\) is the number of \(d\) bosons. Eq. (9) can be rewritten below:

\[
E(v, n_\Delta, I) = E_0'(n_d) - \beta' v(v + 3) + \gamma'I(I + 1).
\]

(10)

We cite Eq. (2.82) of Ref. [18]:

\[
\alpha' + 8\gamma' = c_4,
\]

\[
\alpha' - 6\gamma' = c_2,
\]

\[
\alpha' + 10\beta' - 12\gamma' = c_0,
\]

which may be rewritten below:

\[
\alpha' = \frac{1}{7}(4c_2 + 3c_4),
\]

\[
\beta' = \frac{1}{70}(7c_0 - 10c_2 + 3c_4),
\]

\[
\gamma' = \frac{1}{14}(-c_2 + c_4).
\]

(11)

Substituting these coefficients \(\beta', \gamma'\) into Eq. (10), and taking the two-body matrix elements \(c_0, c_2\) and \(c_4\) to be the TBRE defined in Eq. (1), one obtains that \textbf{only} \(I=0, 2\), and \(I_{\text{max}} (= 2n)\) have sizable \(I\) g.s. probabilities (other \(I\) g.s. probabilities are zero), which are shown in Fig. 1. We notice following regularities of \(P(I)\)'s vs. \(n_d\).
1. The $P(I_{\text{max}})$ is almost a constant (around 40%) for all $n_d$ ($\leq 4$);
2. The $P(0)$ and $P(2)$ are periodical, with a period $\delta(n_d)=6$.
3. All the $P(I_{\text{max}})$, $P(0)$ and $P(2)$ are near to 0, 20%, 40%, or 60%. The other $P(I)$’s are always zero.

Below we explain these observations. From Eq. (10) and Eq. (11), we have

\[
\begin{align*}
    &c_0 = 1, c_2 = c_4 = 0 : \quad E(v, n_\Delta, I) = E'_0(n_d) - \frac{1}{10}v(v+3); \\
    &c_2 = 1, c_0 = c_4 = 0 : \quad E(v, n_\Delta, I) = E'_0(n_d) + \frac{1}{7}v(v+3) - \frac{1}{14}I(I+1); \\
    &c_4 = 1, c_0 = c_2 = 0 : \quad E(v, n_\Delta, I) = E'_0(n_d) - \frac{3}{70}v(v+3) + \frac{1}{14}I(I+1),
\end{align*}
\]

where $E'_0(n_d)$ is a constant for all states. Based on Eq. (12), we obtain TABLE IV, which presents the angular momenta giving the largest (smallest) eigenvalues when $c_l = -1$ ($l = 0, 2, 4$) and other parameters are 0 for $d$ boson systems. In Table IV, $\kappa$ is a non-negative integer, and $n_d \geq 3$. These angular momenta appear periodically, originating from the reduction rule of $U(5) \rightarrow SO(3)$. From TABLE IV, one notices again that a certain $P(I)$ is large if one state with angular momentum $I$ has the largest and/or the smallest $\alpha^l_{l,\beta}$ (Eq. (9)) for a given $l$ ($l = 0, 2, 4$).

Note that when one searches for the smallest eigenvalue with $c_0 = -1$ and $c_2 = c_4 = 0$ in case A of Eq. (12), one finds that many $I$ states are degenerate at the lowest value. Therefore, again, we use $N_m = 3N - 1$ in predicting the $P(I)$’s by the formula $P(I) = N'_I/N_m$. The results are well consistent, without any exceptions, with those obtained by running a TBRE hamiltonian. Take $n = 4$ case as an example, $N'_{I=0} = 3$ and $N'_{I=I_{\text{max}}} = 2$. We predict that 0 g.s. probability is 60% and $I_{\text{max}} = 8$ g.s. probability is 40% while all other $I$ g.s. probabilities are zero. The 0 g.s. and $I_{\text{max}}$ g.s. probabilities given by diagonalizing a TBRE hamiltonian are 60.7% and 39.3%, respectively, and all other $I$ g.s. probabilities are zero. Note that our predicted $P(I)$s of $d$-boson systems are always consistent with those obtained by diagonalizing a TBRE hamiltonian in the examples that checked: $n_d=4$ to 48. In another sentence, the distribution of the $P(I)$’s can be explained satisfactorily by shifts produced by the largest and/or smallest $\alpha^l_{l,\beta}$. 11
3 FERMIONS IN A LARGE SINGLE-\(j\) SHELL

In this subsection we generalize the method proposed above, and study the \(P(I)\)'s of fermions in a single-\(j\) shell with particle number \(n=4, 5, 6, 7, 8\). The explanation of our approach will be addressed in Sec. 8.

The procedures of our approach to a general case are as follows. First, set one of the two-body matrix elements to be \(-1\) and all other interactions to be zero. Then one finds which angular momentum \(I\) gives the lowest eigenvalue among all the eigenvalues of the full shell model space. Suppose that the number of independent two-body matrix elements is \(N\), then the above procedure is iterated \(N\) times. Each time only one of the \(G_J\)'s is set to be \(-1\) while all the others are switched off. Next, among the \(N\) runs one counts how many times (denoted as \(N_I\)) of a certain angular momentum \(I\) gives the lowest eigenvalue among all the possible eigenvalues. Finally, the probability of \(I\) g.s. is given by \(N_I/N_m \times 100\%\). Below we use \(N_m = N\) (unless pointed out explicitly) although the largest eigenvalues are equivalent to the lowest eigenvalues. The reason is that these largest eigenvalues are usually (exactly or nearly) zero for many \(I\) matrices, especially for a many-\(j\) shell or a large \((j \geq 9/2)\) single-\(j\) shell. To have the “rule” as simple as possible, we shall use only the lowest eigenvalues with one of the \(G_J\)’s being set to be \(-1\) and others being switched off for fermion systems in a large single-\(j\) shell, a many-\(j\) shell, \(sd\)- and \(sdg\)-boson systems.

Fig. 2 presents a comparison between the predicted \(P(I)\)'s and those obtained by diagonalizing a TBRE hamiltonian of fermions in a \(j = \frac{9}{2}\) shell. We present two cases: \(n = 4\) (even) and \(n = 5\) (odd). The agreements are good. Note that such agreements do not deteriorate (or become even better) if one goes to cases of fermions in a larger single-\(j\) shell or a many-\(j\) shell where there are more two-body matrix elements.

Tables V-IX give the angular momenta \(I\) which produce the lowest eigenvalues for different two-body matrix elements and particle numbers, with \(G_J\) being \(-1\) and others being 0, and with \(n\) as much as possible. In Table VI, the number of \(N_0\) staggers with \(j\) at a period of \(\delta j=3\), which will certainly produce a staggering of the
0 g.s. probabilities with \( j \) for 4 fermion systems. Fig. 3 gives a comparison between the predicted \( P(0)'s \) (open squares) and those obtained by diagonalizing a TBRE Hamiltonian (solid squares) for \( n = 4 \) and 6. It could be seen that a good agreement is obtained for fermions in both a small single-\( j \) shell and a large single-\( j \) shell. The predicted 0 g.s. probabilities exhibit a similar staggering as those obtained by diagonalizing a TBRE Hamiltonian.

It is interesting to note that \( P(0)'s \) can also be fitted by empirical formulae. For example, \( P(0)'s \) can be predicted by

\[
\text{for } n = 4 : P(0) = \left[ \frac{(2j + 1)/6 + k}{j + \frac{1}{2}} \right] \times 100\%, \quad k = \begin{cases} 1 & \text{if } 2j = 3m \\ 0 & \text{if } 2j + 1 = 3m \\ -1 & \text{if } 2j - 1 = 3m \end{cases}
\]

\[
\text{for } n = 6 : P(0) = \left[ \frac{(2j)/3}{j - \frac{1}{2}} \right] \times 100\% ,
\]

where the \( "[\ ]" \) means to take the integer part. These empirical formulas are interesting because it presents a scenario without any calculations for very large-\( j \) cases where it would be too time-consuming to diagonalize a TBRE Hamiltonian.

A very interesting note of Tables V-IX is on the quadruple matrix elements \( G_2 \) term. It has been well known for a few decades, based on the seniority scheme, that the monopole pairing interaction always gives \( I = 0 \) ground state for even fermion systems in a single-\( j \) shell and \( I = j \) ground state for odd number of fermions in a single-\( j \) shell when \( G_0 \) is set to be -1 and others 0. However, little was known about the \( G_2 \) matrix elements in a single-\( j \) shell. The Tables V-IX show that the quadruple pairing interaction corresponding to \( G_2 \) always gives \( I = n \) ground state for even fermion systems and \( I = j - (n - 1)/2 \) ground state for an odd number of fermions when \( G_2 \) is set to be -1 and others 0. A study of this observation based on pair approximation is now in progress [19].
4 \textit{sd} and \textit{sdg} BOSONS, AND FERMIONS IN A MANY-\textit{j} SHELL

Although \textit{sd}- and \textit{sdg}-boson systems, and even and odd numbers of fermions in a many-\textit{j} shell are very different systems from the cases discussed above, our method is applicable. All features are explained similarly.

The hamiltonian of a \textit{sd}-boson system is as follows:

\[ H_{sd} = H_d + e_{sss} \frac{1}{2} (s^\dagger s^\dagger) (ss) + e_{sdddd} \left( \sqrt{\frac{1}{2}} (s^\dagger d^\dagger) (d\tilde{d})^2 + h.c. \right) + e_{sddd} \left( \sqrt{\frac{1}{2}} (s^\dagger s^\dagger) (d\tilde{d})^0 + h.c. \right) + e_{sdss} ((s^\dagger d^\dagger) \times (s\tilde{d}))^0, \]

where \( H_d \) is a two-body hamiltonian defined in Eq. (8).

TABLE X presents the angular momenta which give the lowest energies when one of the above parameters is set to be -1 and others 0. We predict, according to Table X, that only \( I = 0, 2, 2n \) g.s. probabilities are sizable, which is consistent with the previous observation \[5, 13\], that in \textit{sd}-boson systems interacting via a two-body random ensemble only \( I = 0, 2, \) and \( 2n \) (maximum) have large probabilities to be the ground state, the g.s. probabilities of other angular momenta are close to zero.

Fig. 4 shows a comparison of the predicted \( P(I) \)'s and those obtained by diagonalizing a TBRE hamiltonian of \textit{sd}-boson systems, with boson numbers ranging from 6 to 16. It is seen that a good agreement is obtained.

The case of fermions in a many-\textit{j} shell is the most complicated. Fig. 5 presents a detailed comparison of the predicted \( P(I) \)'s and those obtained by diagonalizing a TBRE hamiltonian, for fermions in a two-\textit{j} (\( j = \frac{7}{2}, \frac{5}{2} \)) shell with \( n=4 \) to 6. The predicted \( P(I) \)'s are reasonably consistent with those obtained by diagonalizing a TBRE hamiltonian.

For fermions in a many-\textit{j} shell, number of two-body matrix elements is usually large. In such cases, especially in odd-fermion systems, there are “quasi-degeneracy” problem in counting \( \mathcal{N}_T \): sometimes the lowest eigen-value is quite close to the
second lowest one when one uses $G_{j_1j_2j_3j_4} = -1$ and others 0. For such two-body matrix elements, one should actually introduce an additional “rule” in order to have a more reliable prediction. Namely, it is not appropriate to count $N_T$ in a simple procedure. In order to avoid confusions, however, we did not modify the way in counting $N_T$ of such cases throughout this paper. It is noted that the $I = \frac{7}{2}$ in Fig. 5b) and $I = \frac{3}{2}$ in Fig. 5d) belong to the case with “quasi-degeneracy”. Improvement of agreement between the predicted $P(I)$’s and those obtained by diagonalizing a TBRE hamiltonian can be achieved by appropriately considering the above “quasi-degeneracy”.

We have checked two-$j$ shells such as $(2j_1, 2j_2) = (5, 7), (5, 9), (11, 3), (11,5), (11,9)$ and $(13,9)$ with $n = 4, 5, 6$, $sd$-boson systems with $n$ up to 17, and $sdg$-boson systems with $n = 4, 5$, and 6, and all the agreements are reasonably good.
It is interesting and very important to know why our simple approach by numerical experiments can successfully produce g.s. probabilities which are in good agreement with those obtained by diagonalizing a TBRE hamiltonian. In this subsection, we shall provide a schematic explanation. A sound explanation may be much more sophisticated.

As mentioned above, the relation between the eigenvalues and the two-body matrix elements is usually not linear. However, eigenvalues are always linear in terms of two-body matrix elements in a “local” space (explained below). Namely, within the local space we may find linear relations between the eigenvalues and the two-body matrix elements. Therefore, instead of studying the effects of all the two-body matrix elements simultaneously, we dismantle the problem into \( N \) parts. In each part we focus on only one term of two-body matrix elements. As a schematic interpretation of our method, we take \( n \) fermions in a single-\( j \) shell as an example (It is easily recoginzed that this explanation is applicable to all complicated cases as well). Let us take a certain \( G_J = -1 \) and all \( G_{J'} = 0 \) \((J' \neq J)\), and diagonalize the two-body hamiltonian. Suppose that the eigenvalues are \( E_{I\beta}^J \), and their corresponding wavefunctions are

\[
\Phi(j^n, I\beta J) = \sum_{KK'\gamma} \langle j^{n-2}(K\gamma)j^2(K')|j^n I\beta J \rangle \Phi(j^{n-2}(K\gamma)) \times \Phi(j^2(K')).
\]  

Now we introduce a small perturbation by adding \( \{\epsilon G_J\} \). \( G_J = -1 \) and \( \{\epsilon G_{J'}\} \) define our \((J/2)\)-th local space of two-body matrix elements. Then the new eigenenergies are approximated by

\[
\left(E_{I\beta}^J\right)' = E_{I\beta}^J + \epsilon \frac{n(n-1)}{2} \sum_{KK'\gamma J'} \left[\langle j^{n-2}(K\gamma)j^2(K')|j^n I\beta J \rangle\right]^2 G_{J'}.
\]  

Namely, the \( E_{I\beta}^J \)'s are linear in terms of \( \{G_{J'}\} \) in the local space. For two-body matrix elements which are close to the above local space, the angular momentum
which gives the lowest eigenvalue among all $I'$s with $G_J = -1$ and others zero, continues to give the lowest eigenvalue. This means that the very large part of full space of a TBRE hamiltonian can be covered by the $N = j + \frac{1}{2}$ local subspaces defined above, especially one uses a TBRE hamiltonian which produce a large probability for small $|G_J'|s$. This is the phenomenology of our approach to predict the $I$ g.s. probabilities in this paper.

A further rationale can be seen from the following analysis. To exemplify briefly, let us take 4 fermions in a single-$j$ shell with $j = \frac{17}{2}$. In Fig. 6a) we set $G_{J_{\text{max}}} (J_{\text{max}} = 16) = -1$, and set all the other parameters are taken to be the TBRE but with a factor $\epsilon$ multiplied. We see that almost all cases of the g.s. belong to $I = I_{\text{max}}$ when $\epsilon$ is small (say, 0.4). If one uses $G_{J_{\text{max}}} (J_{\text{max}} = 16) = 1$, then the $P(I_{\text{max}}) \sim 0$, which means that the cases of the TBRE with $G_{J_{\text{max}}} < 0$ produce almost all the $I_{\text{max}}$ g.s. in a single-$j$ shell. In Fig. 6b) we present the results of the same system with $G_0$ being $-1$ and other $G_J$'s being the TBRE multiplied by $\epsilon$. It is seen similarly that the 0 g.s. is dominant for small $\epsilon$, and that if we switch off all the interactions which give the $I = 0$ lowest eigenvalue with a certain $G_J$ being $-1$ and others 0, then the 0 g.s. probabilities will be very small (such as 10%) or be close to zero ($\sim 2\%$). Therefore, by this method one readily find which interactions, not only monopole pairing, are important to favor the 0 g.s. dominance. Previously, Johnson et al. noticed that the 0 g.s. dominance is even independent of monopole pairing [4, 5, 6]. It was not known, however, whether a certain two-body matrix element is essential or partly responsible, and how to find which interactions are essential, in producing the 0 g.s. dominance for a given system.

A shortcoming of the above explanation is as follows: in our simple approach we set each $G_J = -1$ for each numerical experiment and find the angular momentum of the lowest state. In most cases we obtain degenerate lowest states if we set $G_J = 1$. Thus the local space of $\{G_J = 1 + \epsilon G_{J'} (J' \neq J)\}$ is not considered according to the above explanation. However, the good consistence of our predicted $I$ g.s. probabilities with those obtained by diagonalizing a TBRE hamiltonian seems
to indicate that the local space such as \( \{ G_J = 1 + \epsilon G_{J'} (J' \neq J) \} \) is considered via certain procedures, namely, the properties of local spaces defined by \( \{ G_J = -1 + \epsilon G_{J'} (J' \neq J) \} \) are, more or less, enough to represent the main features of the full space, suggesting that the total I.g.s. probabilities of \( G_J = -1 \) local spaces might be symmetric as a whole to those of the \( G_J = 1 \) local spaces.
6 The $I_{\text{max}}$ g.s. PROBABILITIES

For fermions in a single-$j$ shell, the highest angular momentum (denoted as $I_{\text{max}}$) state was found to have a sizable probability to be the g.s. \cite{10, 13}. This is explained by the observation that $N_{I_{\text{max}}} = 1$ always: One can easily notice that the eigenvalue of $I = I_{\text{max}}$ state is the lowest when $G_{J_{\text{max}}} = -1$ and other parameters are switched off. Because $N_{I_{\text{max}}} = 1$, the predicted $I_{\text{max}}$ g.s. probabilities of fermions in a single-$j$ shell are $\frac{1}{N} = \frac{1}{j+1/2} \times 100\%$, which are valid for all particle numbers (even or odd). It is predicted that the $I = I_{\text{max}}$ g.s. probabilities of fermions in a single-$j$ shell decrease gradually with $j$ and vanish at a large $j$ limit; they will not saturate at a sizable value with $j$.

Fig. 7a) shows the $I_{\text{max}}$ probabilities for different particle numbers in a single-$j$ shell. The agreement between $I_{\text{max}}$ g.s. probabilities obtained by diagonalizing a TBRE hamiltonian and those predicted by using a simple $\frac{1}{N} \times 100\%$ is good.

For $d$-boson systems, the $I_{\text{max}} = 2n$ g.s. probabilities for all $n$ are $\sim 40-42\%$. In Sec. 2.1.2, the predicted $P(I_{\text{max}})$'s are $N_{I_{\text{max}}}/5 = 40\%$, where $N_{I_{\text{max}}} = 2$.

For $sd$-boson systems it was found in Ref. \cite{3} that the $I_{\text{max}}$ g.s. probabilities are large, which can be actually explained in the same way. Among the two-body matrix elements, the interactions with $c_4 = -1$ and others being 0 produce the lowest eigenvalue for the $I_{\text{max}} = 2n$ state. The predicted $I = 2n$ g.s. probability is $1/N = 1/6 = 16.7\%$, which is independent of the boson number. This is consistent with that obtained by diagonalizing a TBRE hamiltonian ($\sim 15\%$). Note that the term $(s^\dagger d^\dagger)(sd)$ gives degenerate lowest eigenvalues for many $I$ states when $e_{sd\text{sd}}$ is set to be $-1$ and others are 0. Therefore, we use 6 (instead of 7) as the number of independent two-body matrix elements, $N$. The difference due to this minor modification is very small, though.

For $sdg$-boson systems, the predicted $I_{\text{max}} = 4ng.s$ probabilities is $1/N \sim 3.2\%$, where $N = 32$. The $I_{\text{max}}$ g.s. probabilities that we obtain by diagonalizing a TBRE hamiltonian are $3.3\%$, $4.2\%$, $3.3\%$ for $n = 4, 5, 6$, respectively.

The above argument of $P(I_{\text{max}})$'s can be generalized to more complicated cases,
such as fermions in a many-\(j\) shell, bosons with two or more different angular momenta (e.g., \(sdg\) bosons). Let us firstly take a two-\(j\) \((j_1, j_2)\) shell. Similar to the above argument for fermions in a single-\(j\) shell, it is predicted that the two angular momenta \(I'_{\text{max}} = I_{\text{max}}(j_1^n)\) and \(I_{\text{max}}(j_2^n)\) have g.s. probabilities which are around or larger than \(1/N \times 100\%\). Here \(I_{\text{max}}(j^n)\) refers to the maximum among all angular momenta of states constructed by \(j^n\) configurations. In another word, one can predict the lower limit of these \(I'_{\text{max}}\) g.s. probabilities. Second, for a boson system, e.g., a \(sdg\)-boson system, it is predicted that the \(I = I_{\text{max}}(d^n) = 2n\) g.s. probability is always larger than (or around) \(1/32 \times 100\% = 3.2\%\) (this probability obtained by diagonalizing a TBRE hamiltonian is \(\sim 12\%-13\%\)).

Fig. 7b) presents the \(I'_{\text{max}} = I_{\text{max}}(j_1^n), I_{\text{max}}(j_2^n)\) g.s. probabilities of fermions in a two-\(j\) shell, obtained by diagonalizing a TBRE hamiltonian. They are compared with the curve plotted using \(1/N\). It is noticed that the predicted lower limit of the \((I_{\text{max}})'\) g.s. probabilities works quite well. It is worthy to mention that other \(P(I)\)'s with \(I\) very near \((I_{\text{max}})'\) are almost zero (smaller than 1%) in these examples.

Now we study the eigenvalue of the \(I_{\text{max}}\) state by calculating the \(\alpha^{j}_{I_{\text{max}}}\) of fermions in a single-\(j\) shell. In doing so we present an argument of an observation that the \(\alpha^{j}_{I_{\text{max}}}\) is always lower than other eigenvalues of all other states while \(G_{j_{\text{max}}} = -1\) and others switched off. Equivalently, below we calculate \(\alpha^{j}_{I_{\text{max}}}\) by setting \(G_{j_{\text{max}}} = 1\) and others zero, which gives the \(\alpha^{j}_{I_{\text{max}}}\) the largest among \(\alpha^{j}_{I_{\text{max}}}\)'s of all \(I\)'s.

The calculation of \(\alpha^{j}_{I_{\text{max}}}\) is straightforward. By decoupling the two-body interaction operators and using analytical formulas of Clebsch-Gordon coefficients, one can obtain all \(\alpha^{j}_{I_{\text{max}}}\)'s. It is noticed easily that there are only positive contributions in this state and there are always cancellations in all the other states when \(G_{j_{\text{max}}} = 1\) and others \(G_{j}\)'s switched off. The reason is as below:

The wavefunction of the \(I_{\text{max}}\) state is known as

\[
|I_{\text{max}}M = I_{\text{max}}\rangle = |jm_1, jm_2, \cdots j_{m_n}\rangle = |jj, j(j-1), j(j-2), \cdots j(j+1-n)\rangle.
\]

All \(I \neq I_{\text{max}}\) states can be constructed by a successive orthogonalization with those
obtained by acting $J_-$ operator on $|I_{\text{max}} M\rangle$ state. It is easy to realize that both negative sign and positive sign appear in the wavefunctions of $I \neq I_{\text{max}}$ states. The coefficients in $|I_{\text{max}} M\rangle$ may be chosen to be positive for all $M$. The $\alpha_{J_{\text{max}}}^I$ is given by a summation of squares of all possible couplings in the wavefunction. Therefore, there is no cancellation in calculating $\alpha_{J_{\text{max}}}^I$. Cancellation appears if $I \neq I_{\text{max}}$.

Below we list some results for $n = 3$ to 6 fermions in a single-$j$ shell.

1). For $n = 3$:

\[
\alpha_{J_{\text{max}}}^I = 2 + \frac{2j}{2(4j - 3)},
\]

\[
\alpha_{J_{\text{max}}}^{-2} = \frac{3(2j - 2)}{2(4j - 3)},
\]

\[
\alpha_{J_{\text{max}}}^{-4} = \frac{5(2j - 3)(2j - 4)}{2(4j - 5)(4j - 7)},
\]

\[
\alpha_{J_{\text{max}}}^{-6} = \frac{35(2j - 4)(2j - 5)(2j - 6)}{8(4j - 7)(4j - 9)(4j - 11)}.
\]

2). For $n = 4$:

\[
\alpha_{J_{\text{max}}}^I = 3 + \frac{2j(10j - 11)}{2(4j - 3)(4j - 5)},
\]

\[
\alpha_{J_{\text{max}}}^{-2} = 2 - \frac{2(2j)^3 - 11(2j)^2 + 9(2j) + 15}{(4j - 3)(4j - 5)(4j - 7)},
\]

\[
\alpha_{J_{\text{max}}}^{-4} = \frac{5(2j - 3)(2j - 4)}{2(4j - 5)(4j - 7)},
\]

\[
\alpha_{J_{\text{max}}}^{-6} = \frac{35(2j - 4)(2j - 5)(2j - 6)}{8(4j - 7)(4j - 9)(4j - 11)}.
\]

3). For $n = 5$:

\[
\alpha_{J_{\text{max}}}^I = 4 + \frac{(2j)(8 \cdot 2j - 17)}{2(4j - 3)(4j - 5)} + \frac{5(2j)(2j - 1)(2j - 2)}{8(4j - 3)(4j - 5)(4j - 7)},
\]

\[
\alpha_{J_{\text{max}}}^{-6} = \frac{35(2j - 4)(2j - 5)(2j - 6)}{8(4j - 7)(4j - 9)(4j - 11)}.
\]

4). For $n = 6$:

\[
\alpha_{J_{\text{max}}}^I = 3 + \frac{2j - 3}{4j - 3} + \frac{148j^2 - 242j + 60}{4(4j - 3)(4j - 5)} + \frac{2j(2j - 1)(396j^2 - 1482j + 1356)}{8(4j - 3)(4j - 5)(4j - 7)(4j - 9)}.
\]
7 CORRELATION BETWEEN THE GROUND STATES OF FERMIONS IN A SINGLE-$j$ SHELL

It is interesting to see whether there is any correlation between different systems by the same two-body random interactions. For this purpose the cases of fermions in a single-$j$ shell is interesting because, as shown in Sec. 2.2, despite of the simplicity these cases exhibit most regularities of many-body systems interacting via a TBRE.

There are some evidences of correlation between ground states of systems with different particle number $n$ but the same set of random interactions. In this section we firstly report a correlation of $I$ g.s. probabilities for fermions in a single-$j$ shell, then examine another correlation discussed in [3], where this latter correlation was explained as reminiscence of the general seniority scheme [20].

The 0 g.s. probability of 4 fermions in a single-$j$ shell was found to fluctuate periodically with $\delta_j = 3$ [10, 15]. We have checked $n = 4$ up to $j = 33/2$, $n = 6$ up to $j = 27/2$ and $n = 8$ up to $j = 23/2$, which show that the 0 g.s. probabilities change synchronously [3]. The $I = j$ g.s. probability of 5 fermions in a single-$j$ shell exhibits a similar pattern [16]. In this paper we notice that this synchronous fluctuation appear at an interval $\delta_j = 3$ for 0 g.s. probability of systems with $n = 4$, 6, and 8 fermions and $I = j$ g.s. probability of odd $n(=5,7)$ fermions in a single-$j$ shell, but the origin of this correlation is not yet been available.

In a previous work [4] another correlation for fermions in a many-$j$ shell was reported: the pairing phenomenon seems to be favored simply as a consequence of the two-body nature of the interaction. The “pairing” here means that there is a large matrix element between the $S$ pair annihilation operator between the ground states of a $n$ fermion system to a $n-2$, $n-4$ · · · system. Below we examine this “pairing” correlation for fermions in a single-$j$ shell, where the seniority quantum number $v$ is well defined.

First we see the simplest case: 4 and 6 fermions in the $j = 11/2$ shell. The 0 g.s. probability for $n = 4$ and 6 is 41.2% and 66.4%, respectively. Among 1000 sets
of the TBRE hamiltonian, 364 sets give 0 g.s. for both \( n = 4 \) and 6 simultaneously. This means that a TBRE hamiltonian in which \( I = 0 \) is the ground state for \( n = 4 \) has a extremely large probability (around 90\%) produces \( I = 0 \) ground state for \( n = 6 \). The overlaps between the 0 g.s. of \( n = 6 \) and the g.s. of \( n = 4 \) coupled with an \( S \) pair for the same TBRE hamiltonian are in most cases around 0.8-0.9, while those between the 0 g.s. of \( n = 4 \) and that of \( S \) pair acting on the 0 g.s. of \( n = 6 \) are in most cases around 0.9-1.0. This strongly supports the finding in [3]: the \( S \) pair annihilation operator takes the 0 g.s. of \( n \) fermions to that of \( n - 2 \) fermions. It is noted that the expectation values of seniority for 0 g.s. is, more or less, randomly distributed from 0 to 4, in these cases. Namely, there seems no bias for very low seniority in the above calculation by using a TBRE hamiltonian.

By examining the 0 g.s. of 4 and 6 fermions in the \( j = 11/2 \) shell, one notices that the seniority \( v \)'s of these g.s. are quite close. For \( n = 6 \) there are strong seniority mixings between states with seniority \( v = 0 \) and 4, but no mixing between states with \( v = 6 \) and 0 (or 4). It is interesting to note that no TBRE hamiltonian of \( n = 6 \) for 0 g.s. with seniority 6 produces 0 g.s. of \( n = 4 \). It is noticed that the \( I = 0 \) state with \( v = 6 \) contributes \( \sim 24\% \) to the total 0 g.s. probability (66.4\%) for \( n = 6 \). Roughly speaking, in this small single-\( j \) shell most of the random two-body interactions which produce 0 g.s. with seniority \( v \sim 0-4 \) of \( n = 6 \) give 0 g.s. with a similar seniority for \( n = 4 \), and in these cases the picture given in [3] is appropriate.

As for the \( j = 13/2 \) shell where there are strong mixings between states with \( v = 6 \) and \( v = 0 \) (or 4), 0 g.s. probability is 22.3\% for \( n = 4 \) and 42.4\% for \( n = 6 \). Among 1000 runs we obtain 13\% sets of random interactions which give 0 g.s. for both \( n = 4 \) and \( n = 6 \). Among those cases which simultaneously produce 0 g.s. for \( n = 4 \) and 6, the overlaps between the 0 g.s. of \( n = 6 \) and the g.s. of \( n = 4 \) coupled with an \( S \) pair for the same TBRE hamiltonian are in most cases around 0.6-0.9, while those between the 0 g.s. of \( n = 4 \) and that of \( S \) pair acting on the 0 g.s. of \( n = 6 \) are in most cases around 0.8-1.0, indicating a similar picture described in [3].

Now we come to larger shells such as \( j = 15/2 \) to 23/2 with \( n = 4, \, 6, \) and 8.
We concentrate on the $j = 15/2$ shell which is enough to demonstrate our points of view. The 0 g.s. probability is 50.2%, 68.2% and 32.1%, for $n = 4, 6$ and 8, respectively. We have 31% (among the 1000 runs) of the TBRE parameters which produce 0 g.s. for all $n = 4, 6$ and 8, i.e., almost all those TBRE parameters which produce 0 g.s. for $n = 8$ produce 0 g.s. for $n = 4$ and 6. It is interesting to note that the expectation value of seniority $v$ of all 31% 0 g.s. of $n = 4, 6$ and 8 systems has a very small probability larger than 4, indicating a similar pattern given in [3].

On the other hand, it is noted that the fluctuation of 0 g.s. probabilities for $n = 4, 6$, and 8 fermions in the same single-$j$ shell can be large. For example, the 0 g.s. of 6 fermions in a $j=15/2$ shell is 68.2% while that of 8 fermions is 32.1%, which means that more than 50% of the 0 g.s. for $n = 6$ are not related to the chain in which the 0 g.s. of $n - 2$ fermions can be obtained by annihilating one $S$ pair on that of $n$ fermions. Fig. 8 presents a few cases of 4 or 6 fermions in a single-$j$ shell, where no bias of a low seniority is observed, indicating that the contribution to the total 0 g.s. probability beyond the a seniority chain described in [3] may be more important in realistic systems. For example, there are only 14% of the TBRE Hamiltonian parameters which produce 0 g.s. for both $n = 4$ and 6 fermions in a $j=25/2$, while the 0 g.s. probability of $n = 6$ is 57.1%.

Therefore, we conclude that the finding [3] of a chain of 0 g.s. which is linked by $S$ pair for fermions in a many-$j$ shell is also observed frequently for even fermions in a single-$j$ shell, and that this chain covers, however, only one part of the 0 g.s. and the contribution beyond this chain may be more important.

It is noted that no bias of low seniority in the 0 g.s. is observed in our calculations by using a TBRE Hamiltonian except that $n = 8$ fermions in a $j = 15/2$ shell, where no states with seniority 8 are observed in the $I = 0$ ground states, i.e., most of 0 g.s. have expectation value of $v$ from 0 to 4.
8 SUMMARY AND DISCUSSION

To summarize, we have presented in this paper our understanding of regularities of many-body systems interacting via a two-body random ensemble.

First, beginning with simple systems in which the relation between eigenvalues and two-body matrix elements is linear, we propose an integral to predict the $I$ g.s. probabilities, $P(I)$’s. The properties of $P(I)$’s are understood by the shifts, or the largest and/or smallest eigenvalues with only one of the two-body matrix elements switched on and others switched off. This argument is further developed to predict the $P(I)$’s by a simple formula $P(I) = N_I/N_m$, where $N_I$ is the number of a certain angular momentum $I$ gives the lowest eigenvalue among all the possible eigenvalues with one of the two-body matrix elements being $-1$ and others being zero, and $N_m$ is taken as the number of two-body matrix elements except for very few cases. This method, as we show by a variety of very different systems, is applicable to both (even or odd number of) fermion systems (with both a single-$j$ shell and a many-$j$ shell) and boson systems. The agreement between the predicted $I$ g.s. probabilities and those obtained by diagonalizing a TBRE hamiltonian is good. It is noted that this method predicts the 0 g.s. probability, and on the same footing it addresses other $I$ g.s. probabilities as well. Therefore, we provide in this paper a universal approach in studying the $I$ g.s. probabilities.

Next, we discuss the microscopic foundation of our simple approach by defining a local space defined by $G_J = -1 + \{\epsilon G_{J'}(J' \neq J)\}$. We show that the angular momentum $I$ which gives the lowest eigenvalue when $G_J = -1$ and others are zero continue to be the lowest if $\epsilon$ is small. One has the $I$ g.s. probability around 70-90% even $\epsilon$ is quite large (such as 0.5-1.0). This could be a naive explanation of the success of our approach, but a sound explanation is not yet available.

A discovery of this work is that we are able to tell (by numerical experiments) which interactions, not only the monopole pairing interaction, are essential in favoring 0 g.s. in both boson fermions systems. For instance, interactions with $J = 0, 6, 8, 12, 22$ give the 0 g.s. dominance for 4 fermions in a $j = \frac{31}{2}$ shell.
This disproves a popular idea that the 0 g.s. dominance essentially comes from the two-body nature of the interactions and might be independent of the form of the hamiltonian.

The simple $I_{\text{max}}$ g.s. probabilities of fermions in a single-$j$ shell are found, for the first time, to be determined by only the number of the two-body matrix elements and independent of particle number, and to follow a simply $1/N$ relation. This phenomenon is explained by the fact that $\mathcal{N}_{I_{\text{max}}} = 1$ for fermions in a single-$j$ shell. A generalization of this regularity to fermions in a many-$j$ shell and $(sd$ and $sdg)$ boson systems is shown to work well, too. Quite a few counter examples of the 0 g.s. dominance in both fermions in a single-$j$ shell or a multi-$j$ shell and bosons are found and explained.

One interesting note of the Tables V-IX is on quadruple pairing interaction for fermions in a single-$j$ shell. It is found that $I = n\ (j + 1 - n/2)$ give the lowest eigenvalue when $G_2 = -1$ and $G_{J\neq2} = 0$ for all even (odd) number of fermions in a single-$j$ shell that we have checked in this paper. It is not known whether this observation is always correct, and what the origin might be if it is.

In this work we also studied the seniority distribution for fermions in a single-$j$ shell where the seniority number is well defined. In a pioneering work $^3$ it was claimed that the (low) seniority chain is very important in the 0 g.s. dominance, namely, the 0 g.s. of $n$ fermions is related to 0 g.s. of $n-2, n-4$ and etc fermions via $S$ pairs approximately. We confirm this phenomenon in this work. But we also note that the 0 g.s. beyond this (approximate) seniority chain may be more important.

It has not yet understood at a more microscopic level that why the above $\mathcal{N}_0$ is large for even fermion systems, or even more specifically, why there is a staggering on the $\mathcal{N}_0$ for even fermions in a single-$j$ shell. Further consideration of this issue is warranted.

Finally, it is stressed that the $P(I)$’s discussed in this paper and the $\mathcal{P}(I)$’s $^2$ (the probabilities of energies averaged over all the states for a fixed angular momentum $I$ being the ground states) are different quantities. For even systems
the behavior of these two are accidentally similar. For odd-$A$ systems, however, the $\mathcal{P}(I)$’s, which were explained in terms of the randomness of two-body cfp’s, are very different from $P(I)$’s. This explicitly demonstrates that the $I$ g.s. probabilities (and 0 g.s. dominance) are not consequences of geometric chaoticity \cite{10}. We show in this work that the 0 g.s. dominance is actually related to two-body matrix elements which have specific features.

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APPENDIX A  Many-body systems interacting via a displaced TBRE

While a TBRE is symmetric to zero, an interesting question is that what the results are if one uses random interactions with only positive, or negative sign, or random interactions which are not symmetric to zero. This issue is both interesting and important because interactions in realistic systems, such as nuclei, atoms etc, are not symmetric to zero. Below we present preliminary results by using a displaced TBRE.

Let us firstly consider two arbitrary ensembles \{G'_J\} and \{G_J\}, which are related by a shift \( C \):

\[
G'_J = G_J + C, \quad J = 0, 2, \ldots, 2j - 1,
\]

(20)

where \( C \) is a constant.

For fermions in a single-\( j \) shell, the results using the ensemble \{G'_J\} are exactly the same as those obtained using \{G'_J\} except a shift \( \frac{n(n-1)}{2}C \) on the eigen-energy of the ground state. Therefore, a displacement of a TBRE is trivial in a single-\( j \) shell.

For fermions in a two-\( j \) shell, however, the role played by the displacement of the TBRE is very complicated. Below we mention only two extreme situations:

1. 4 fermions in a \((2j_1, 2j_2) = (11, 3)\) shell, by using a TBRE, a TBRE+5, and a TBRE-5, the 0 g.s. probabilities are 60.8%, 95.4%, and 100.0%, respectively. Here the 0 g.s. is more pronounced if the TBRE is displaced to either negative or positive.

2. 4 fermions in a \((2j_1, 2j_2) = (13, 9)\) shell, by using a TBRE, a TBRE+5, and a TBRE-5, the 0 g.s. probabilities are 44.8%, 2.7%, and 0.1%, respectively. Here the 0 g.s. is greatly quenched down (close to zero) if the TBRE is displaced to either negative or positive. One observes a similar situation for 4 fermions in a \((2j_1, 2j_2) = (7, 5)\) shell.

Concerning the effect of the shape of an ensemble, we consider the TBRE and an ensemble of uniformly changed random numbers between -1 and 1. The general features are quite similar. Slight differences appear if one multiply a factor to each \( G_J \), such as the RQE of [2]. There seems no essential difference between them, however.
APPENDIX B  Calculation of $\alpha^I_{\beta\beta}$ for 4 fermions in a single-$j$ shell

In this appendix, we calculate $\alpha^I_{\beta\beta}$ for 4 fermions in a single-$j$ shell by using $9j$ coefficients and coupled pair basis.

The normalized pair basis of a 4-fermion system is defined as follows:

$$\left|L_1 L_2 : I\right>_N = \frac{1}{\sqrt{N_{L_1 L_2 I}}} \left[A^{L_1 \dagger} \times A^{L_2 \dagger}\right]_I \left|0\right> = \frac{1}{\sqrt{N_{L_1 L_2 I}}} \left|L_1 L_2 : I\right>,$$  \hspace{1cm} (21)

where the subscript $N$ means that the state is normalized, and $N_{L_1 L_2 I}$ is the overlap $\langle L_1 L_2 : I|L_1 L_2 : I\rangle$, which is given as follows:

$$N_{L_1 L_2 I} = 1 + \delta_{L_1 L_2} - 4(2L_1 + 1)(2L_2 + 1).$$ \hspace{1cm} (22)

The matrix elements of two-body interaction within normalized pair basis are given in Eq. (A.26) of Ref. [22]:

$$N\langle L_1 L_2 : I|\sqrt{2J + 1} \left[A^{J \dagger} \times \tilde{A}^{J \dagger}\right]^0 |L_1 L_2 : I\rangle_N = \frac{1}{N_{L_1 L_2 I}} \sum_{R=\text{even}} (U_{L_1 L_2 J R})^2$$ \hspace{1cm} (23)

with

$$U_{L_1 L_2 J R} = \delta_{J L_1} \delta_{R L_2} + (-)^I \delta_{J L_2} \delta_{R L_1} - 4\hat{L}_1 \hat{L}_2 \hat{J} \hat{R} \left\{ \begin{array}{c} j \cr j \cr \end{array} \right. \left\{ \begin{array}{c} L_1 \cr L_2 \cr \end{array} \right\}. \hspace{1cm} (24)
$$

Here $\hat{L}$ is a short hand notation of $\sqrt{2L + 1}$.

The seniority $v = 0$ and 2 states are easy to construct:

$$\left\{ \begin{array}{c} |00 : 0\rangle_0 = |00 : 0\rangle_N, \hspace{0.5cm} v = 0, \\
|I0 : I\rangle_0 = |I0 : I\rangle_N, \hspace{0.5cm} v = 2 \end{array} \right. \hspace{1cm} (25)$$

The subscript 0 means the state is a seniority-conserved state. The seniority $v = 4$ state is given by

$$\left|L_1 L_2 : I\right>_0 = \frac{1}{\sqrt{1 - \alpha^2}} \left|L_1 L_2 : I\right>_N - \frac{\alpha}{\sqrt{1 - \alpha^2}} |I0 : I\rangle_N,$$ \hspace{1cm} (26)

where $L_1 \neq 0, L_2 \neq 0$, and $\alpha = N \langle I0 : I|L_1 L_2 : I\rangle_N$. Suppose there are more than one seniority 4 states, say, $|L_1' L_2' : I\rangle_0$ and $|L_1 L_2 : I\rangle_0$, one should orthonormalize them to have an orthonormalized basis. We note here that in this
case the matrix elements $0\langle L_1L_2 : I|\sqrt{2J+1} [A^J \times \tilde{A}^J]^0 |L_1L_2 : I\rangle_0$ and $0\langle L'_1L'_2 : I|\sqrt{2J'+1} [A^{J'} \times \tilde{A}^{J'}]^0 |L'_1L'_2 : I\rangle_0$ are not an invariant, their summation is an invariant.

The $\alpha^J_0$ of seniority 0 state is as follows:

$$0\langle 00 : 0|\sqrt{2J+1} [A^J \times \tilde{A}^J]^0 |00 : 0\rangle_0 = \begin{cases} 2j^2 - 1, & J = 0, \\ \frac{2j^2 - 1}{4j^2 + 1}, & J \neq 0. \end{cases}$$

(27)

The $9j$ coefficients are difficult to be further simplified unless one of them are 0. The $\alpha^J_{I_{\text{max}}}$ is derived in another way in Sect.III.4.

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APPENDIX C  A few interesting examples

The 0 g.s. dominance was reported to be robust in even fermion systems and sd-boson systems. Similarly, for fermions in a single-\(j\) shell it was believed that the \(I = j\) g.s. probability is the largest [10]. However, as shown in simple cases such as \(d\)-boson systems, these dominances are not always true. Here we list a few counter examples that we noticed in our work.

1. 4 fermions in a \(j = \frac{7}{2}\) shell, which was explained in terms of shifts in [15]; and 4 fermions in a \(j = \frac{13}{2}\) shell, which is understood in this paper by counting \(N_0\). In these two examples, \(P(2)\)'s are larger than \(P(0)\).

2. A two-\(j\) shell with \((2j_1, 2j_2) = (7, 5)\) and \(n = 4\), where the 0 g.s. probability is 21.5\% and that of 2 g.s. probability is 34.7\%;

3. two-\(j\) shells with \((2j_1, 2j_2) = (11, 7), (13, 9)\) and \(n = 5\), where \(j_1\) g.s. and \(j_2\) g.s. probabilities are small (\(\leq 20\%\)) while \(\frac{5}{2}\) g.s. probability is large (\(> 30\%\)), which means that an angular momentum which is not \(j_1\) or \(j_2\) may be favored to be the ground state for odd number of fermion systems;

4. \(d\)-bosons systems with \(n \neq 6\kappa\), while the 0 g.s. probability is less than 40\% (2 \% or \(\sim 36\%\)) while the \(I_{\text{max}}\) g.s. probability is \(\sim 42\%\). Especially, the cases with \(n = 6\kappa \pm 1\) (\(\kappa\) is a natural number) are extreme counter examples of the 0 g.s. dominance: the 0 g.s. probabilities are very close to zero periodically. All regularities of \(P(I)\)'s of \(d\)-boson systems are well understood in this paper.
References

[1] W. Satula, J. Dobaczewski, and W. Nazarewicz, Phys. Rev. Lett. 81, 3599; H. Hakkinen, J. Kolehmainen, M. Koskinen, P. O. Lipas, and M. Manninen, Phys. Rev. Lett. 78, 1034(1997).

[2] C. W. Johnson, G. F. Bertsch, D. J. Dean, Phys. Rev. Lett. 80, 2749(1998).

[3] C. W. Johnson, G. F. Bertsch, D. J. Dean, and I. Talmi, Phys. Rev. C 61, 014311(1999).

[4] C. W. Johnson, Rev. Mex. Fis. 45 S2, 25(1999).

[5] R. Bijker, A. Frank, Phys. Rev. Lett. 84, 420(2000); Phys. Rev. C 62, 14303(2000).

[6] D. Kusnezov, N. V. Zamfir, and R. F. Casten, Phys. Rev. Lett. 85, 1396(2000).

[7] M. Horoi, B. A. Brown, V. Zelevinsky, Phys. Rev. Lett. 87, 062501(2001).

[8] V. Velazquez, and A. P. Zuker, Phys. Rev. Lett. 88, 027502(2002); Y. M. Zhao, and A. Arima, to be published.

[9] R. Bijker, A. Frank, and S. Pittel, Phys. Rev. C60, 021302(1999).

[10] D. Mulhall, A. Volya, and V. Zelevinsky, Phys. Rev. Lett. 85, 4016(2000); Nucl. Phys. A682, 229c(2001); D. Mulhall, V. Zelevinsky, and A. Volya, nucl-th/0103069; V. Zelevinsky, D. Mulhall, and A. Volya, Yad. Fiz. 64, 579(2001).

[11] D. Kusnezov, Phys. Rev. Lett. 85, 3773(2000); ibid. 87, 029202 (2001); R. Bijker, and A. Frank, Phys. Rev. Lett. 87, 029201(2001).

[12] R. Bijker, and A. Frank, nucl-th/0201080, Phys. Rev. C, in press.

[13] R. Bijker, and A. Frank, nucl-th/0105027; nucl-th/0108054, Phys. Rev. C64, (R)061303(2001).
[14] Lev Kaplan, Thomas Papenbrock, and Calvin W. Johnson, Phys. Rev. **C63**, 014307(2001).

[15] Y.M. Zhao, and A. Arima, Phys. Rev. **C64**, (R)041301(2001).

[16] A. Arima, N. Yoshinaga, and Y.M. Zhao, Eur. J. Phys. A **13**, 105(2002); N. Yoshinaga, A. Arima, and Y.M. Zhao, J. Phys. G, in press.

[17] R. D. Lawson, *Theory of the Nuclear Shell Model* Clarendon, Oxford (1980).

[18] F. Iachello, and A. Arima, *the Interacting Boson Model*, Cambridge University Press (1987), P38.

[19] Y. M. Zhao, A. Arima, and N. Yoshinaga, to be published.

[20] I. Talmi, Nucl. Phys. **A172**, 1(1971).

[21] Y. M. Zhao, A. Arima, and N. Yoshinaga, to be published.

[22] N. Yoshinaga, T. Mizusaki, A. Arima, and Y. D. Devi, Prog. Theor. Phys. (supple) **125**, 65(1996).
TABLE I. The probability of each state to be the ground state and distribution width of each eigen-energy in the case of \( j = 5/2 \) shell with 3 fermions. All the states are labeled uniquely by their angular momenta \( I \). The probabilities of the row “TBRE” are obtained by 1000 runs of a TBRE hamiltonian, and those of “pred1.” are obtained by integrals such as Eq. (7) for \( 0^+ \) state of \( n = 4, j = \frac{7}{2} \) case. The row ”pred2.” are obtained by the new approach proposed in this paper. The distribution width, \( g_I \), of each eigen-energy, is listed in the last row.

| \( I \) | \( 3/2 \) | \( 5/2 \) | \( 9/2 \) |
|-------|-------|-------|-------|
| TBRE  | 40.1\% | 23.7\% | 36.2\% |
| pred1. | 41.82\% | 22.77\% | 36.37\% |
| pred2. | 40\% | 20\% | 40\% |
| \( g_I \) | 2.31 | 1.84 | 2.44 |

TABLE II. Same as Table I except that \( j = 7/2 \) shell.

| \( I \) | \( 3/2 \) | \( 5/2 \) | \( 7/2 \) | \( 9/2 \) | \( 11/2 \) | \( 15/2 \) |
|-------|-------|-------|-------|-------|-------|-------|
| TBRE  | 30.4\% | 29.7\% | 9.6\% | 4.4\% | 0.0\% | 25.9\% |
| pred1. | 31.14\% | 27.26\% | 9.70\% | 3.69\% | 0.00\% | 28.13\% |
| pred2. | 28.6\% | 28.6\% | 14.3\% | 0 | 0 | 28.6\% |
| \( g_I \) | 2.44 | 2.09 | 1.57 | 2.11 | 1.88 | 2.42 |

TABLE III. Same at Table II except that \( n = 4 \). All the eigenstates are uniquely labeled by \( I(v) \).

| \( I(v) \) | 0(0) | 2(2) | 2(4) | 4(2) | 4(4) | 5(4) | 6(2) | 8(4) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| TBRE  | 19.9\% | 1.2\% | 31.7\% | 0.0\% | 25.0\% | 0.0\% | 0.0\% | 22.2\% |
| pred1. | 18.19\% | 0.89\% | 33.25\% | 0.00\% | 22.96\% | 0.00\% | 0.02\% | 24.15\% |
| pred2. | 14.3\% | 0\% | 28.6\% | 0\% | 28.6\% | 0 | 0\% | 28.6\% |
| \( g_{I(v)} \) | 3.14 | 3.25 | 4.12 | 3.45 | 3.68 | 3.62 | 3.64 | 4.22 |
TABLE IV. The angular momenta which give the largest (smallest) eigenvalues when $c_1 = -1$ and other parameters are 0 for $d$ boson systems.

| $n$ | $c_0$ (min) | $c_2$ (min) | $c_2$ (max) | $c_4$ (min) | $c_4$ (max) |
|-----|-------------|-------------|-------------|-------------|-------------|
| 6$\kappa$ | 0           | 0           | $I_{\text{max}}$ | $I_{\text{max}}$ | 0           |
| 6$\kappa + 1$ | 2           | 2           | $I_{\text{max}}$ | $I_{\text{max}}$ | 2           |
| 6$\kappa + 2$ | 0           | 2           | $I_{\text{max}}$ | $I_{\text{max}}$ | 2           |
| 6$\kappa + 3$ | 2           | 0           | $I_{\text{max}}$ | $I_{\text{max}}$ | 0           |
| 6$\kappa + 4$ | 0           | 2           | $I_{\text{max}}$ | $I_{\text{max}}$ | 2           |
| 6$\kappa + 5$ | 2           | 2           | $I_{\text{max}}$ | $I_{\text{max}}$ | 2           |

TABLE V. The angular momenta which give the lowest eigenvalues when $G_J = -1$ and other parameters are 0 for a 3-nucleon system in a single-$j$ shell. Here we use a unit of $2I$, and M refers to $I_{\text{max}}$.

| $2j$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_{10}$ | $G_{12}$ | $G_{14}$ | $G_{16}$ | $G_{20}$ | $G_{22}$ | $G_{24}$ | $G_{26}$ | $G_{28}$ | $G_{30}$ |
|------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 5    | 2j   | 2j - 2 | M     |       |         |         |         |         |         |         |         |         |         |         |
| 7    | 2j   | 2j - 2 | 3     | M     |         |         |         |         |         |         |         |         |         |         |
| 9    | 2j   | 2j - 2 | 2j - 6 | 9     | M       |         |         |         |         |         |         |         |         |         |
| 11   | 2j   | 2j - 2 | 2j - 6 | 3     | 15      | M       |         |         |         |         |         |         |         |         |
| 13   | 2j   | 2j - 2 | 2j - 6 | 3     | 9       | 21      | M       |         |         |         |         |         |         |         |
| 15   | 2j   | 2j - 2 | 2j - 6 | 5     | 3       | 15      | 27      | M       |         |         |         |         |         |         |
| 17   | 2j   | 2j - 2 | 2j - 2 | 2j - 6 | 3     | 9       | 17      | 33      | M       |         |         |         |         |         |         |
| 19   | 2j   | 2j - 2 | 2j - 2 | 2j - 6 | 5     | 3       | 15      | 23      | M-6     | M       |         |         |         |         |         |
| 21   | 2j   | 2j - 2 | 2j - 2 | 2j - 6 | 11    | 3       | 9       | 17      | 29      | M-6     | M       |         |         |         |         |
| 23   | 2j   | 2j - 2 | 2j - 2 | 2j - 6 | 13    | 5       | 3       | 11      | 23      | 35      | M-6     | M       |         |         |         |
| 25   | 2j   | 2j - 2 | 2j - 2 | 2j - 6 | 15    | 11      | 3       | 9       | 17      | 29      | 41      | M-6     | M       |         |         |         |
| 27   | 2j   | 2j - 2 | 2j - 2 | 2j - 6 | 29    | 13      | 5       | 3       | 11      | 23      | 35      | 47      | M-6     | M       |         |         |         |
| 29   | 2j   | 2j - 2 | 2j - 2 | 2j - 2 | 2j - 6 | 15    | 11      | 3       | 9       | 17      | 29      | 41      | 53      | M-6     | M       |         |         |         |
| 31   | 2j   | 2j - 2 | 2j - 2 | 2j - 6 | 21    | 13      | 5       | 3       | 11      | 23      | 35      | 47      | 59      | M-6     | M       |         |         |         |
TABLE VI. The angular momenta which give the lowest eigenvalues when $G_J = -1$ and other parameters are 0 for 4 fermions in a single-$j$ shell.

| $2j$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ | $G_{10}$ | $G_{12}$ | $G_{14}$ | $G_{16}$ | $G_{18}$ | $G_{20}$ | $G_{22}$ | $G_{24}$ | $G_{26}$ | $G_{28}$ | $G_{30}$ |
|------|-------|-------|-------|-------|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|  7   |  0    |  4    |  2    |  8    |       |        |        |        |        |        |        |        |        |        |        |        |
|  9   |  0    |  4    |  0    |  0    | 12    |        |        |        |        |        |        |        |        |        |        |        |
| 11   |  0    |  4    |  0    |  4    |  8    | 16    |        |        |        |        |        |        |        |        |        |        |
| 13   |  0    |  4    |  0    |  2    |  2    | 12    | 20    |        |        |        |        |        |        |        |        |        |
| 15   |  0    |  4    |  0    |  2    |  0    |  0    | 16    | 24    |        |        |        |        |        |        |        |        |        |
| 17   |  0    |  4    |  6    |  0    |  4    |  2    |  0    |  0    | 20    | 28    |        |        |        |        |        |        |        |
| 19   |  0    |  4    |  8    |  0    |  2    |  8    |  2    | 16    | 24    | 32    |        |        |        |        |        |        |        |
| 21   |  0    |  4    |  8    |  0    |  2    |  0    |  0    |  0    | 20    | 28    | 36    |        |        |        |        |        |        |
| 23   |  0    |  4    |  8    |  0    |  2    |  0    |  10   |  2    |  0    | 24    | 32    | 40    |        |        |        |        |        |
| 25   |  0    |  4    |  8    |  0    |  2    |  4    |  8    |  10   |  6    |  0    | 28    | 36    | 44    |        |        |        |        |
| 27   |  0    |  4    |  8    |  0    |  2    |  4    |  2    |  0    |  0    |  4    | 20    | 32    | 40    | 48    |        |        |        |
| 29   |  0    |  4    |  8    |  0    |  0    |  2    |  6    |  8    | 12    |  8    |  0    | 24    | 36    | 44    | 52    |        |        |
| 31   |  0    |  4    |  8    |  0    |  0    |  2    |  0    |  0    | 14    | 16    |  6    |  0    | 32    | 40    | 48    | 56    |        |

TABLE VII. Same as TABLE V except that $n = 5$.

| $2j$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ | $G_{10}$ | $G_{12}$ | $G_{14}$ | $G_{16}$ | $G_{18}$ | $G_{20}$ |
|------|-------|-------|-------|-------|-------|--------|--------|--------|--------|--------|--------|
|  9   | 2$j$  | 2$j-4$|  9    |  3    |       | $I_{max}$|        |        |        |        |        |
| 11   | 2$j$  | 2$j-4$|  5    |  2$j$ |  5    | $I_{max}$|        |        |        |        |        |
| 13   | 2$j$  | 2$j-4$|  5    |  2$j$ |  7    |  5    | $I_{max}$|        |        |        |        |
| 15   | 2$j$  | 2$j-4$|  7    |  9    |  2$j$ |  7    |  31   | $I_{max}$|        |        |        |
| 17   | 2$j$  | 2$j-4$|  9    |  5    |  2$j$ | 11    |  9    |  41    | $I_{max}$|        |        |
| 19   | 2$j$  | 2$j-4$| 11    |  5    | 13    |  2$j$ | 13    |  5     |  51    | $I_{max}$|        |
| 21   | 2$j$  | 2$j-4$| 19    |  5    |  7    |  2$j$ | 15    |  2$j$  |  61    | $I_{max}$|        |
TABLE VIII. Same as TABLE V except that $n = 6$.

| $2j$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ | $G_{10}$ | $G_{12}$ | $G_{14}$ | $G_{16}$ | $G_{18}$ | $G_{20}$ | $I_{\text{max}}$ |
|------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|-----------|
| 11   | 0     | 6     | 4     | 0     | 0     | 0       | $I_{\text{max}}$ |
| 13   | 0     | 6     | 4     | 0     | 0     | 4       | 4       | $I_{\text{max}}$ |
| 15   | 0     | 6     | 0     | 6     | 0     | 0       | 0       | $I_{\text{max}}$ |
| 17   | 0     | 6     | 0     | 0     | 0     | 4       | 0       | 16      | $I_{\text{max}}$ |
| 19   | 0     | 6     | 0     | 6     | 0     | 0       | 0       | 4       | 22      | $I_{\text{max}}$ |
| 21   | 0     | 6     | 10    | 6     | 0     | 0       | 0       | 0       | 0       | 28      | $I_{\text{max}}$ |

TABLE IX. Same as TABLE V except that $n = 7$.

| $2j$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ | $G_{10}$ | $G_{12}$ | $G_{14}$ | $G_{16}$ | $G_{18}$ | $I_{\text{max}}$ |
|------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|-----------|
| 13   | $2j$  | $2j - 6$ | 11    | 1     | 11    | 5       | $I_{\text{max}}$ |
| 15   | $2j$  | $2j - 6$ | 13    | 3     | 3     | $2j$    | 9       | $I_{\text{max}}$ |
| 17   | $2j$  | $2j - 6$ | 7     | $2j$  | 2     | 13      | 9       | 23      | $I_{\text{max}}$ |

TABLE X. Same as TABLE V except that $sd$-boson systems. The matrix element corresponding to $e_{ssdd}$ are omitted because it always presents degenerate levels for many $I$ states.

| $n$  | $e_{ssss}$ | $e_{sddd}$ | $e_{ssdd}$ | $c_0$ | $c_2$ | $c_4$ | $I_{\text{max}}$ |
|------|------------|------------|------------|-------|-------|-------|-----------|
| 6    | 0          | 0          | 0          | 0     | 0     | 0     | $I_{\text{max}}$ |
| 7    | 0          | 0          | 0          | 2     | 2     | $I_{\text{max}}$ |
| 8    | 0          | 0          | 0          | 0     | 2     | $I_{\text{max}}$ |
| 9    | 0          | 0          | 0          | 2     | 0     | $I_{\text{max}}$ |
| 10   | 0          | 0          | 0          | 0     | 2     | $I_{\text{max}}$ |
| 11   | 0          | 0          | 0          | 2     | 2     | $I_{\text{max}}$ |
| 12   | 0          | 0          | 0          | 0     | 0     | 0     | $I_{\text{max}}$ |
| 13   | 0          | 0          | 0          | 2     | 2     | $I_{\text{max}}$ |
| 14   | 0          | 0          | 0          | 0     | 2     | $I_{\text{max}}$ |
| 15   | 0          | 0          | 0          | 2     | 0     | $I_{\text{max}}$ |
| 16   | 0          | 0          | 0          | 0     | 2     | $I_{\text{max}}$ |
**Figure captions:**

FIG. 1 The $I$ g.s. probabilities of $d$ bosons. The boson number $n$ runs from 4 to 44. Only states with $I = 0, 2,$ and $I_{\text{max}} = 2n$ are possible to be the ground. The 0 g.s., 2 g.s. and $I_{\text{max}} = 2n$ g.s. probabilities are very near to 0, 20%, 40% or 60%. The $P(0) \sim 0$ in $d$-boson systems with $n_d = 6\kappa \pm 1$. The predicted $P(I)$'s (open squares) are well consistent with those (solid squares) obtained by diagonalizing a TBRE hamiltonian. All regularities are explained by the reduction rule of $U(5) \rightarrow O(3)$.

FIG. 2 The $P(I)$'s of fermions in a single-$j$ shell. The solid squares are $P(I)$'s obtained by diagonalizing a TBRE hamiltonian and the open squares are $P(I)$'s predicted by the approach proposed in this paper. a) $j = \frac{9}{2}$ with 5 fermions; b) $j = \frac{9}{2}$ with 4 fermions. Good agreements are obtained for both even and odd-A cases.

FIG. 3 The $P(0)$'s of fermions in a single-$j$ shell. Solid squares are obtained from 1000 runs of a TBRE hamiltonian. The open squares are predicted $P(0)$'s in this paper. a) $n = 4$; b) $n = 6$. solid triangles in a) are obtained from an empirical formula, Eq. (15).

FIG. 4 The $P(0)$, $P(2)$ and $P(I_{\text{max}})$ of $sd$-boson systems. Solid symbols are $P(I)$’s obtained from 1000 runs of a TBRE hamiltonian. Open symbols are $P(I)$’s predicted in this paper. Only $I = 0, 2, I_{\text{max}}$ g.s. probabilities are included. All other $P(I)$’s obtained by diagonalizing a TBRE hamiltonian are close to zero, and the predicted $P(I)$’s are zero.

FIG. 5 Fermions in a two-$j$ shell with $(j_1, j_2) = (\frac{7}{2}, \frac{5}{2})$. $n = 4, 5, 6, 7$ in a), b), c) and d), respectively. Solid squares are obtained by 1000 runs of a TBRE hamiltonian and open squares are predicted by the method proposed in this paper.

FIG. 6 4 fermions in a $j = \frac{17}{2}$ shell. a) The $I_{\text{max}}$ g.s. probabilities with $G_{16} = \pm 1$ ($J_{\text{max}}=16$) and all other $G_J$ being a TBRE multiplied by $\epsilon$; b) The 0 g.s.
probabilities with $G_0 = \pm 1$ and all other $G_j$ being a TBRE multiplied by $\epsilon$. Refer to text for details.

FIG. 7 The $I_{\text{max}}$ g.s. probabilities. a) The $I_{\text{max}}$ g.s. probabilities of fermions in a single-$j$ shell, which are obtained by diagonalizing a TBRE hamiltonian (1000 runs). Solid squares are plotted by $\frac{1}{N} \times 100\%, N = j + \frac{1}{2}$. It is noted that the $I_{\text{max}}$ g.s. probabilities follow a $\frac{1}{N} \times 100\%$ relation well, and are independent of particle number, $n$. b) The $(I_{\text{max}}') = I_{\text{max}}(j_1^n)$ and $I_{\text{max}}(j_2^n)$ g.s. probabilities of fermions in two-$j$ shells, obtained by 1000 runs of a TBRE hamiltonian. The lower limit of the $(I_{\text{max}}')$ g.s. probabilities are predicted $\frac{1}{N} \times 100\%$ (solid squares). The $I'_{\text{max}}$ g.s. probabilities are reasonably consistent with the predictions.

FIG. 8 The seniority distribution in the angular momentum $I = 0$ ground states. No bias of low seniority is observed in these 4 and 6 fermions in a single-$j$ shell, which indicates that the contribution to the total 0 g.s. beyond a low seniority chain may be more important.
FIG. 1

(a) $P(l_{\text{max}})$

(b) $P(2)$

(c) $P(0)$
a) 

$P(I)$ vs. $I$ for 5 fermions in a $j=9/2$ shell.

- $\text{SM,}$ ---
- $\text{predicted,}$ --

b) 

$P(I)$ vs. $I$ for 4 fermions in a $j=9/2$ shell.

- $\text{SM,}$ ---
- $\text{predicted,}$ --

FIG. 2
FIG. 3

(a) $P(0)$ vs $j$ for $n=4$.

(b) $P(0)$ vs $j$ for $n=6$. 

- - - $0^\circ$, - - - pred., - - - empirical, $n=4$.

- - - TBRE, - - - Pred., $n=6$. 

Graphs showing $P(0)$ vs $j$ for different $n$ values.
**FIG. 4**

sd-boson systems

$P(I)$ vs. $n$

- ■ - 0g.s.(pred), ■ - 0g.s.(TBRE),
- ● - 2g.s.(pred), ● - 2g.s.(TBRE),
- ▲ - $I_{\text{max}}$g.s.(pred), ▲ - $I_{\text{max}}$g.s.(TBRE)
FIG. 5

a). $n=4$

b). $n=5$

c). $n=6$

d). $n=7$
The $G_0 = -1.0$, The $G_0 = 1.0$, The $\{G_{j\mu}\} \text{ are TBR } x\varepsilon$. Here $j=17/2$, $n=4$.

FIG. 6
FIG. 7

a) $I_{\text{max}}$ g.s. probabilities (in %)

b) $(I_{\text{max}}')$ g.s. probabilities (in %)

- $n=3$
- $n=4$
- $n=5$
- $n=6$
- $n=7$

$1/\sqrt{N}$

$j_1$, $j_2=11/2, 3/2$
$11/2, 5/2$
$11/2, 9/2$
$13/2, 9/2$
$9/2, 5/2$
$7/2, 5/2$
$11/2, 7/2$
FIG. 8

(a) $j=9/2$, $n=4$

(b) $j=11/2$, $n=4$

(c) $j=13/2$, $n=4$

(d) $j=15/2$, $n=4$

(e) $j=19/2$, $n=4$

(f) $j=19/2$, $n=6$