Observation of the dynamics of an ergodic quantum protocol in a photonic realization

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Abstract

Iterated quantum protocols with measurement-based selection lead to deterministic chaos for the evolving pure state representing an ensemble of qubits. Deterministic chaos for the pure quantum state may lead to ergodic evolution in the sense that initial states from any small area on the Bloch sphere will cover the whole sphere after a finite number of iterations. We realize two steps of an ergodic protocol in a photonic experiment, where initial qubit states are encoded in the polarization and path degrees of freedom of down-converted photons stemming from a parametric process. We numerically analyze the effect of noise on the time evolution and show that the protocol, described by a Lattès map, remains quasi-ergodic for any initial state if the initial noise is small. Tomographic reconstruction of the quantum states throughout the evolution is consistent with simulations and thus demonstrates ergodicity of the quantum dynamics.

1. Introduction

One of the central problems in the rapidly developing field of quantum information processing technology is the fundamental relation of noise, measurement and quantum dynamics. Recent announcements of quantum advantage are promising signposts on the way towards useful quantum computation [1]. The importance of the optical realization of quantum processors is stressed by the successful demonstration of bosons amplifying in an optical arrangement with a clear advantage over classical computers [2] and the proof-of-principle demonstration of quantum circuits on a programmable nanophotonic chip this year [3].

Measurement is an essential element in quantum computation and in non-classical quantum phenomena in general: it breaks the linear nature of the evolution associated with a closed system. The emerging nonlinearity in quantum dynamics is especially pronounced if we apply a minimal version of a universal quantum computer—consisting of an entangling two qubit gate (e.g. a controlled not gate) and a parameterized one qubit gate, followed by a measurement on one of the outputs and selection according to the result—repeatedly on an ensemble of equally prepared qubits. This iterated protocol features rich dynamics for pure initial quantum states, which sensitively depends on the parameter of the one qubit gate. For a certain parameter value, true chaos is present for all initial states [4], one can say the dynamics is ergodic. Ergodic behavior of a dynamical system is a key concept in physics since its introduction by Boltzmann [5–7]. In classical mechanics, the state of the system is represented by a point in a high dimensional phase space and its evolution follows a trajectory. The ergodic hypothesis states that, in some sense, the trajectory will reach everywhere in the allowed part of the phase space. This rather vague statement is then made precise in various ways in mathematical physics. A closely related concept is deterministic chaos [8, 9] referring to very high sensitivity to initial conditions often leading to trajectories reaching various parts of the phase space, even if they originate from a small neighborhood. The iterated nonlinear protocol can be called ergodic in the sense that if we fix a small area on the Bloch sphere...
representing the initial pure quantum state of the qubits, then after a finite number of steps the corresponding set of trajectories will cover the whole Bloch sphere. One may ask the question, what happens if in an experiment the preparation of the initial state is imperfect, i.e. the state of the ensemble can be described by a mixed state?

In this paper, we investigate the realistic behavior of an iterated ergodic protocol. We experimentally implement two steps of the protocol with linear optical elements. For 16 different pure initial states from various regions of the Bloch sphere we demonstrate that the dynamics follows the theoretically predicted behavior. Moreover, the agreement between the experimental results and the detailed analysis is very good when the tomographically reconstructed two-qubit mixed initial states are taken as inputs of the simulations. We have implemented numerical simulations which indicate that the ergodic property is maintained in general for reasonably small initial noise. Furthermore, our simulations with noise levels comparable to those of the actual experiment show that a few more iterations could already reveal the ergodicity of the protocol. Our results indicate that iterated complex rational polynomial maps can be directly realized [4] in a photonic scheme, with errors low enough to faithfully follow even the sensitive evolution of an ergodic protocol.

2. Ergodicity in iterated protocols

Let us consider an ensemble of identically prepared qubits. Applying an entangling quantum gate on pairs from the ensemble, followed by a projective measurement on one of the outputs, leads to a smaller ensemble of qubits if we keep the unmeasured qubits which belong to a certain, fixed measurement result. The quantum state of the resulting ensemble is, in general, a nonlinear transform of the initial state [10–15]. By iterating such a protocol, we arrive at an evolution of the quantum state which exhibits complex deterministic chaos [16–19]. The time-evolution of pure initial quantum states can be represented by a complex to complex map, which is a quadratic rational function. Let us parameterize the quantum state as \( |\psi\rangle = N(z) \left( |0\rangle + z|1\rangle \right) \) (where \( z \in \mathbb{C} \cup \{\infty\} \) with the normalization \( N(z) = 1/\sqrt{1 + |z|^2} \). One step of the iteration is then represented by a function \( f(z) \). We focus here on a specific protocol leading to the transformation represented by the function

\[
 f(z) = \frac{i + z^2}{1 + iz^2},
\]

The induced iterated dynamics leads to chaotic evolution for all initial states, since it is a Lattès map [4, 20]. Lattès maps play a distinguished role among dynamic maps, as they very effectively randomize the system’s state throughout the evolution. If we choose an arbitrary open neighborhood around any of the initial states, then we can always find a finite, critical number of iterations above which the image of the initial open set will cover the whole state space. This property is related to the fact that the Julia set [21] (loosely speaking the set of initial points for which the evolution is chaotic) equals to the set of all possible initial states, moreover all initial states exhibit exponential sensitivity [4]. These statements are based on rigorous proofs about the complex map \( f \), representing the dynamics [22].

3. Experimental setup

In our experiment, qubits are realized on the one hand by the polarization states of single photons \( |0\rangle_p = |H\rangle \) and \( |1\rangle_p = |V\rangle \), where \( H \) and \( V \) stand for horizontal and vertical polarization, respectively, and on the other hand by their spatial modes \( |0\rangle_s = |D\rangle \) and \( |1\rangle_s = |U\rangle \), where \( U \) denotes the upper path, \( D \) denotes the lower path) [23–25]. Single photons are generated via type-I spontaneous parametric down-conversion (SPDC) using a 0.5 mm-thick-\( \beta \)-barium-borate (BBO) crystal, pumped by a CW diode laser with a central wavelength 400.8 nm and 80 mW of power. An interferencer filter is used to restrict the single-photon bandwidth to 3 nm. Photons are then injected into the experimental setup shown in figure 1. Total coincidence counts are about 10,000 over a collection time of 10 s.

The single photons can be prepared in an arbitrary superposition of polarization states \( |\psi_0\rangle = N(z) \left( |H\rangle + z|V\rangle \right) \) after passing through a polarizing beam splitter (PBS), a quarter-wave plate (QWP) and a half-wave plate (HWP) with the corresponding setting angles. For the first iteration, the photons pass through a beam displacer (BD) which splits them into two different spatial modes \( (U \) and \( D \) depending on their polarizations. The HWP at 45\(^\circ\) in the upper mode, and the HWP at 0\(^\circ\) in the lower mode rotate the polarizations of the photons into \( |H\rangle \) in both of their spatial modes. Next, a QWP and a HWP with the same setting angles as those used for the polarization state preparation in the beginning plus a HWP at 45\(^\circ\) are applied so that the (unnormalized) states of the photons in their lower and upper spatial
expressed in polar coordinates, where \( u \) is the angle of both QWPs are 0 and that of the HWP is 22°. The second iteration of the protocol—unlike the first step—is realized by two-photon interference. Here, both qubits are represented by the respective polarization degree of freedom of the two photons. The quantum state transformation represented by the complex map of equation (1) in the first step of the protocol is finally attained by applying a sandwich-type set (QWP–HWP–QWP), where the setting angles of both QWPs are 0 and that of the HWP is 22.5°. After this element the state of the photon can be written as

\[
|\psi_1\rangle = N_{f(z)} \left[ |H\rangle + f(z)|V\rangle \right].
\] (2)

The second iteration of the protocol—unlike the first step—is realized by two-photon interference. The initial state of the photon is post-selected via a projection measurement on the other photon of the same pair. A state tomography is then performed on the final state.

4. Experimental results

In our experiment we chose 16 initial states exhibiting certain symmetries. One such symmetry is that after the second step all of them ends up on the same hemisphere. Another property of the chosen states is that after the second step all of them ends up on the same hemisphere. In order to characterize the quantum states of the actual physical qubits, we performed two-qubit tomography on the spatial and polarizational modes of the initially prepared single-photons, and single-qubit tomography on the photon polarization states after each iteration. The resulting state is then represented by a density operator

\[
\rho = \frac{1}{2} \begin{pmatrix}
1 + w & u - iv \\
1 + iv & 1 - w
\end{pmatrix},
\] (4)

where \( u, v, w \) are the usual Bloch sphere coordinates \( u = r \cos \phi \cos \theta \), \( v = r \sin \phi \cos \theta \), \( w = r \sin \theta \). The surface of the Bloch sphere contains the pure states, while the closer a state is to the center of the sphere, the more mixed it becomes; the very center of the sphere being the completely mixed state. Figure 2 shows the spherical coordinates of the states determined from the measured data. There is a fair agreement between the experimental results (red and blue points) and those predicted by iterating the map in equation (1) for the chosen pure states (black points), although the purity in the experiment is decreasing (as can be seen by the plots for the radial
Figure 2. Spherical coordinates of the 16 experimentally initialized states after the first iteration (left panel) and after the second iteration (right panel). Black crosses represent the theoretical results for ideal pure initial states, blue (red) symbols represent the experimentally measured states of photon 1 (2). Squares (dots) represent the first (second) member of the pairs of states which are indicated on the horizontal axis. Note that these pairs of states are expected to be transformed into the same final states (i.e. there is one theoretical value, represented by a single black cross). Note that θ and ϕ are measured in degrees. Error bars are not shown.

Note that after the second step the ϕ coordinate of all states lie in the prescribed 90° < ϕ < 270° hemisphere except for state no. 5 for which there is a small deviation ϕ = 271.8°.

5. Comparison of the experiment and the theoretical model

Let us compare the results of the actual experimental implementation and the idealized theoretical model. Noise is inevitably present when preparing a quantum state, thus we should assume a mixed initial state, represented by a density operator ρ0. In addition, there are other deviations from the ideal theoretical model. First, in the experiment, the initial states are encoded into two different degrees of freedom of two single photons generated by parametric downconversion. It is not easy to prepare fully identical states into these different degrees of freedom for both photons. Second, the spatial and polarizational states of a given photon may not be fully disentangled (the logarithmic negativities of the measured two-qubit (spatial + polarizational) initial states are shown in table 1). Note that the quality of the preparation of the initial states is indicated by the small values (in most cases below 0.1) of the logarithmic negativity \( E_N = \log_2 \| \rho_{TB}^{AB} \|_1 \), where \( \rho_{TB}^{AB} \) is the partial transpose of \( \rho_{AB}^{TB} \) with respect to the subsystem B.

In order to perform numerical simulations based on the experimental data, we have extended the model to take into account the possibility that in the case of the first iteration the input states are not product states (and can also be different for the two photons), while in the second iteration the inputs are all different (see appendix A for details). The results of the full simulations of the experiment are presented in table 1, where we compare the closeness of the simulated results with the actual measured data after the two iterations. The high fidelities (\( F(\rho_{\text{exp}}, \rho_{\text{the}}) = \text{Tr}^2(\sqrt{\sqrt{\rho_{\text{exp}}^\dagger \rho_{\text{the}}^\dagger \rho_{\text{exp}}^\dagger}}) > 0.93 \)) between theory and experiment confirm the consistency of the preparation, the transformation by the protocol and the tomographic reconstruction of the photonic quantum states. The Bures distance
Table 1. The logarithmic negativity ($E_N$) of the initial two-qubit (spatial + polarizational) states of photons 1 and 2. The fidelity ($F$) and Bures distance ($D_B$) of experimentally measured states after two iterations compared to the pure states obtained by the two-step iteration of the idealized pure-state protocol, and from the ones obtained by the simulation with the measured data as input. $E_N$, $F_{id}$, and $D_B$ are obtained from the experimental data.

| State | $E_N^{(1)}$ | $E_N^{(2)}$ | $F_{id}$ | $F_{sim}$ | $D_B^{id}$ | $D_B^{sim}$ |
|-------|-------------|-------------|---------|---------|-----------|-----------|
| 1     | 0.028       | 0.049       | 0.936   | 0.958   | 0.255     | 0.206     |
| 2     | 0.002       | 0.072       | 0.942   | 0.954   | 0.242     | 0.217     |
| 3     | 0.098       | 0.089       | 0.953   | 0.983   | 0.218     | 0.131     |
| 4     | 0.114       | 0.060       | 0.935   | 0.971   | 0.258     | 0.170     |
| 5     | 0.004       | 0.119       | 0.954   | 0.988   | 0.216     | 0.109     |
| 6     | 0.135       | 0.003       | 0.939   | 0.968   | 0.250     | 0.180     |
| 7     | 0.036       | 0.043       | 0.939   | 0.957   | 0.249     | 0.210     |
| 8     | 0.119       | 0.081       | 0.954   | 0.979   | 0.217     | 0.144     |
| 9     | 0.005       | 0.079       | 0.947   | 0.975   | 0.232     | 0.159     |
| 10    | 0.100       | 0.003       | 0.942   | 0.953   | 0.243     | 0.219     |
| 11    | 0.061       | 0.025       | 0.946   | 0.967   | 0.233     | 0.183     |
| 12    | 0.072       | 0.051       | 0.953   | 0.984   | 0.219     | 0.127     |
| 13    | 0.017       | 0.101       | 0.937   | 0.943   | 0.252     | 0.241     |
| 14    | 0.080       | 0.027       | 0.945   | 0.965   | 0.235     | 0.188     |
| 15    | 0.069       | 0.029       | 0.947   | 0.977   | 0.232     | 0.153     |
| 16    | 0.040       | 0.035       | 0.952   | 0.943   | 0.263     | 0.239     |

Figure 3. Monte Carlo simulation of 9 iterative steps of the map with random, noisy initial states ($N = 3 \times 10^4$) from a small neighborhood of the $z = 1$ state (spherical Bloch coordinates: $r = 0.99$, $\theta \in (90^\circ - \varepsilon, 90^\circ + \varepsilon)$, $\varphi \in (0^\circ - 2\varepsilon, 0^\circ + 2\varepsilon)$, with $\varepsilon = 0.57^\circ$). The minimum $r$ value during the iterations was $r_{min} = 0.48$.

$D_B(\rho, \sigma) = \sqrt{2(1 - \text{Tr}(\sqrt{\rho} \sigma \sqrt{\rho})))$ of experimentally measured states from the theoretically predicted states ($D_B^{id}$) as well as from simulated outputs based on measured input states ($D_B^{sim}$) are also shown as evidence of the successful implementation of the protocol.

6. Simulations with noisy initial states

The presented experimental setup realized two iterations of the dynamics and our results indicate that the evolution followed the theory rather faithfully. Realization of a few more steps could be enough to directly check ergodicity of the protocol, which is analytically proven for pure initial states. We examine here the evolution of mixed initial states with the help of simulations. On one hand, we know that for mixed initial states, the asymptotic evolution becomes simple, all mixed initial states tend to the completely mixed state, as it was shown by numerical simulations in [26]. Convergence to the completely mixed state is an asymptotic result, valid for a large number of iterations. On the other hand, one may expect that for a reasonably high purity of the initial state, the evolution approximately follows the pure-state evolution for some number of iterations before it gets closer to the completely mixed state.

It is, however, a non-trivial question, whether there exists an ergodic regime for mixed initial states as well. By ergodic we mean here that states from a small neighborhood of a slightly mixed initial state will approximately cover all regions of the Bloch sphere before they get very close to the fully mixed state. More precisely, let us fix an arbitrary initial direction $\theta_0$, $\varphi_0$ and a small neighborhood around them, and let us...
also fix an initial purity \((r = 1 - \epsilon\) with, \(\epsilon \ll 1\)) that is consistent with the experimentally measured initial purities presented in figure A1. Then we ask whether after a finite number of iterations this set of initial density matrices will cover all possible directions while their purity remains close to 1. We have performed Monte Carlo simulations by following the dynamics of initial states randomly chosen from the close neighborhood of a given state with a small fixed mixedness. Our numerical results indicate that states will practically cover all possible directions even if we take initial density matrices from the neighborhoods belonging to the repelling fixed point \(z = 1\) of the map \(f(z)\) of equation (1) (corresponding to \(\theta_0 = \pi/2,\ \varphi_0 = 0\)) where the decrease of the purity during the iteration is the fastest. We have found that for a reasonably small amount of initial noise, the simulated dynamics of such initial states will cover all possible spherical angles while their purity does not get close to zero as illustrated in figure 3. Thus we might call the dynamics quasi-ergodic.

7. Conclusions

The delicate interplay of deterministic chaos, sensitivity to the initial pure quantum states and the effect of classical noise shapes the dynamics of the presented iterative protocol. Our photonic experiment demonstrates two steps of a specially selected member of a family of nonlinear protocols, represented by a Lattes map. The performed experiment demonstrates the first few steps of this ergodic quantum dynamics on the Bloch sphere. Our numerical observation is that even though all mixed initial states eventually tend to the completely mixed state, for small initial noise the dynamics is ergodic within a good approximation for any initial state. Our results provide a proof of principle that ergodicity could be observable even in a noisy experimental setup, if one was able to perform just a few more steps of the presented scheme. Our work, being the first direct experimental realization of an iterated nonlinear quantum map paves the way for possible useful applications of these protocols, such as comparing a quantum state to a reference [27]. On the other hand, the increasing availability of quantum chips offers yet another way to implement this kind of protocols. In turn, the presented high sensitivity of this type of dynamics to noise makes it an ideal candidate to test noise levels in quantum computers. We note that after the submission of this manuscript, the realization of two/three steps of the same nonlinear protocol on various superconducting quantum computers was reported [28].

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Details of the measurement

In our experiment the 16 initial states that were chosen for preparation are summarized in table A1. The Bloch-sphere coordinates of the experimentally prepared states, determined from the partial trace of the two-qubit density matrices measured by two-qubit quantum state tomography, are shown in figure A1 together with the respective ideal pure states shown in table A1.

Appendix B. Details of the theoretical model

The transformations performed by the linear optical elements on the polarizational and spatial degrees of freedom of the photons in the experiment actually correspond to the following simplified scheme (originally presented in [4]): In the first step, the polarizational \((|H\rangle + z|V\rangle)\) and spatial \((|D\rangle + z|U\rangle)\) states of a given photon undergo a CNOT transformation where the polarization state is acting as the control for the CNOT. Then, only those amplitudes are kept, where the spatial part corresponds to \(|D\rangle\) (the spatial
Appendix C. One step of the protocol with nonidentical inputs

C.1. Nonseparable input

As a further refinement of the theoretical model we incorporated the possibility of having non-separable two-qubit inputs as well as separable but nonidentical inputs at the two steps of the protocol. We have used a combination of these theoretical schemes to simulate the two implemented steps with the measured initial data (two-qubit density matrices) as input. Here we present the relevant details. Let us assume that the input of the protocol is a general two-qubit mixed state $\rho$, which cannot be written as a product state

$$
\rho = \frac{1}{\rho_{11} + \rho_{14}} \begin{pmatrix} 
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{12} & \rho_{22} & \rho_{23} & \rho_{24} \\
\rho_{13} & \rho_{23} & \rho_{33} & \rho_{34} \\
\rho_{14} & \rho_{24} & \rho_{34} & \rho_{44}
\end{pmatrix}, \quad \rho_{11}, \rho_{22}, \rho_{33}, \rho_{44} \in \mathbb{R}^+, \quad \text{Tr}(\rho) = 1.
$$

(C.1)

Note that $\rho$ is written in the $\{|H\rangle, |D\rangle, |U\rangle, |V\rangle\}$ basis. After the effect of the CNOT transformation and the subsequent measurement of the second qubit in the state $|D\rangle$, the state of the first qubit becomes

$$
\rho' = \frac{1}{\rho_{11} + \rho_{44}} \begin{pmatrix} 
\rho_{11} & \rho_{14} \\
\rho_{14} & \rho_{44}
\end{pmatrix}.
$$

(C.2)

Then, the unitary $U_L$ transforms this into

$$
\hat{\rho} = U_L \rho' U_L^\dagger = \frac{1}{\Sigma A} \begin{pmatrix} A - iC & -iB + D \\
iB + D & A + iC
\end{pmatrix},
$$

(C.3)
Figure A1. Spherical coordinates of the 16 experimentally initialized states. Black crosses represent the theoretical results for ideal pure initial states, blue (red) symbols represent the experimentally measured states of photon 1 (2). Squares (dots) represent the spatial (polarization) qubit states of the photon represented by the same color. Note that $\theta$ and $\varphi$ are measured in degrees.

where

\[ A = \rho_{11} + \rho_{44} \]
\[ B = \rho_{11} - \rho_{44} \]
\[ C = \rho_{14} - \rho_{14}^* = 2 \text{Im}(\rho_{14}) \]
\[ D = \rho_{14} + \rho_{14}^* = 2 \text{Re}(\rho_{14}). \]  

(C.4)

C.2. Nonidentical inputs

Let us now assume that the two-qubit input state is a product of two non-identical mixed states $\rho^{(1)}$ and $\rho^{(2)}$:

\[
\rho = \rho^{(1)} \otimes \rho^{(2)} = \begin{pmatrix}
\rho_{11}^{(1)} \rho_{11}^{(2)} & \rho_{12}^{(1)} \rho_{12}^{(2)} & \rho_{11}^{(1)} \rho_{12}^{(2)} & \rho_{12}^{(1)} \rho_{12}^{(2)} \\
\rho_{12}^{(1)} \rho_{11}^{(2)} & \rho_{11}^{(1)} \rho_{12}^{(2)} & \rho_{12}^{(1)} \rho_{12}^{(2)} & \rho_{12}^{(1)} \rho_{12}^{(2)} \\
\rho_{11}^{(2)} \rho_{12}^{(1)} & \rho_{12}^{(2)} \rho_{11}^{(1)} & \rho_{12}^{(2)} \rho_{12}^{(1)} & \rho_{12}^{(2)} \rho_{12}^{(1)} \\
\rho_{12}^{(2)} \rho_{12}^{(1)} & \rho_{12}^{(2)} \rho_{12}^{(1)} & \rho_{12}^{(2)} \rho_{12}^{(1)} & \rho_{12}^{(2)} \rho_{12}^{(1)}
\end{pmatrix}. \]  

(C.5)
The above calculations hold for this case as well, the transformed density matrix can be written in the same form as in equation (C.3) with the parameters $A$, $B$, $C$, and $D$ now given as

$$A = \rho_{11}^{(1)} \rho_{11}^{(2)} + \rho_{12}^{(1)} \rho_{12}^{(2)},$$

$$B = \rho_{11}^{(1)} \rho_{11}^{(2)} - \rho_{12}^{(1)} \rho_{12}^{(2)},$$

$$C = \rho_{12}^{(1)} \rho_{12}^{(2)} - \left(\rho_{12}^{(1)} \rho_{12}^{(2)}\right)^*,$$

$$D = \rho_{12}^{(1)} \rho_{12}^{(2)} + \left(\rho_{12}^{(1)} \rho_{12}^{(2)}\right)^*.\quad (C.6)$$

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