A Qualitative Analysis to Simple Harmonic Motion

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Abstract

This paper proposes a qualitative analysis to the simple harmonic motion for students who are not mathematically well-prepared. It uses the variation in speed and acceleration to sketch the velocity-time curve. The curve appears to be sinusoidal, whose shape is largely determined by the fact that the gradient of a velocity-time graph is acceleration. This approach cannot determine the exact value of period and claim the curve to be sinusoidal, however, it complements rigorous mathematical analysis in that it is full of physical reasoning that the mathematical approaches lack of.

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Simple Harmonic Motion (henceforth SHM) is taught at various levels depending on students’ maturity in mathematics [1]. Without delving into much mathematics, this paper qualitatively analyzes SHM based on elementary knowledge in kinematics, Newton’s second law, Hooke’s law, and mechanical energy conservation, which are usually exposed to students before the introduction of SHM. The results from the analysis are used to sketch the velocity-time curve, and the shape of the curve is largely determined by the fact that the gradient of a velocity-time curve is acceleration. Even though the obtained curve cannot be claimed to be sinusoidal, however, whoever sees it would strongly suspect that it is. The value of this approach is that key features of the seemingly complicated motion can be revealed to students who are not mathematically well prepared. Moreover, students are exposed to intensive physical reasoning rather than dry mathematical manipulations even though they are mathematically prepared.

The qualitative approach in this paper and rigorous mathematical approaches, either referring to uniform circular motion or solving differential equation [2, 3], actually complement with each other. On the one hand, rigorous approaches give the exact value of the period and confirm that the velocity-time curve is indeed sinusoidal, which the qualitative approach is unable to provide. On the other hand, the qualitative approach is full of rich physics content that the mathematical approaches severely lack of. In particular, one of the two rigorous approaches, which refers to uniform circular motion, is criticized as nonphysical, arbitrary, artificial, imaginary, and even confusing to students [4–7].

I. PROBLEM AND BACKGROUND KNOWLEDGE

A mass $m$ is hooked with one end of a massless spring with stiffness $k$. The other end of the spring is fixed on a wall. The mass lies on a frictionless floor as illustrated in Fig. 1. The equilibrium position is chosen as origin of the $x$-axis, and it is denoted as $O$. The mass is pulled to the right to point $B$ by distance $A$ (usually $A$ is reserved as the notation for amplitude). Then it is released from rest. The mass moves back to equilibrium $O$. It won’t stop at $O$ but keeps moving to the left. The point when it is momentarily at rest before moving right back to $O$ is denoted as $C$. The task is to sketch the velocity-time curve.

Before introducing SHM, the following facts were made clear to students. They are the basis for the qualitative analysis and are listed in the order of the time students learned
Fact 1: The gradient of a velocity-time curve is acceleration.
Fact 2: Newton’s second law: the acceleration is proportional to the net force.
Fact 3: Hooke’s law: the elastic force is proportional to its deformation.
Fact 4: Mechanical energy is conserved: the sum of elastic potential energy and kinetic energy is equal to the initial elastic potential energy, that is, \( \frac{1}{2} k A^2 \).

II. QUALITATIVE ANALYSIS

With the above listed facts at hand, a few qualitative results are obtained for SHM. They are listed below followed by explanations leading to them. In Sec. III they will be used to sketch the velocity-time curve.

Result 1: The motion is periodic.
Result 2: The separation between equilibrium \( O \) and \( B \) is the same as that between \( O \) and \( C \), that is, \( |OC| = |BO| = A \).
Result 3: For two symmetric points with respect to the equilibrium \( O \), the mass passes them with the same speed.
Result 4: It takes the same amount of time to cover each of the four segments: $B \to O$, $O \to C$, $C \to O$, and $O \to B$. In other words, the time spent in each segment is $\frac{1}{4} T$.

Result 5: As the mass moves from $B$ to $O$, its speed increases at a decreasing rate, and the direction is negative.

Result 6: As the mass moves from $O$ to $C$, its speed decreases at an increasing rate, and the direction is still negative.

Result 7: As the mass moves from $C$ back to $O$, its speed increases at an increasing rate, and the direction is positive.

Result 8: As the mass moves from $O$ back to $B$, its speed decreases at an increasing rate, and the direction is positive.

As for Result 1, it is not hard to convince students that the motion is periodic, which enables us to focus curve sketching within only one period, even though its exact value is unknown with the qualitative analysis here.

Result 2 comes from Fact 4, that is, mechanical energy conservation. The kinetic energy is zero at both $B$ and $C$. Thereby, the elastic potential energy is the same, so is the extension for the spring. Result 3 also comes from Fact 4. Equal distance from the equilibrium means equal elastic potential energy. Thereby, the kinetic energy at these two points is the same, so is the speed. Result 3 is used to obtain Result 4.

It takes a while to explain Result 4. I here only show that it takes the same amount of time to move from $B$ to $O$ as that from $C$ to $O$. The rest is left as an exercise for the reader. According to Result 3, for any two points that are symmetric with respect to the equilibrium $O$, the mass travels with the same speed. The only difference is direction; the mass moves in opposite directions at the two symmetric points. However, the time to cover $BO$ or $CO$ does not depend on moving direction. Therefore, the time to cover $BO$ is the same as $CO$. Result 4 enables us to divide the period into 4 equal time intervals.

The analyses for Results 5-8 are similar, thereby, only the reasoning for Result 5 is given. As the mass moves from $B$ to $O$, its separation from the equilibrium $O$ decreases. According to Fact 4, its kinetic energy increases, so is its speed. On the other hand, according to Fact 3, the magnitude of the elastic force on it decreases, so is its magnitude of acceleration by Fact 2, or Newton’s second law.

Results 5-8 are important for curve sketching in Sec. III. They, together with Fact 1,
determine the shape of the velocity-time curve in crucial ways. To be specific, they determine
the monotonicity, concavity, and convexity of the curve.

III. VELOCITY-TIME CURVE

Before details of the curve are sketched, the following are listed as preparation without
explanation.

1. The curve is sketched within one period only by Result 1.
2. The period \( T \) is divided into 4 equal intervals by Result 4.
3. At \( t = 0, \frac{1}{2} T, \) and \( T \) the velocity is zero.
4. At \( t = \frac{1}{4} T, \frac{3}{4} T, \) the velocities are \(-v_m\) and \(+v_m,\) respectively, where \( v_m \) is the maximal
speed and \( v_m = A\sqrt{k/m}.\)
5. For \( t \in (0, \frac{1}{2} T), \) the velocity is negative; the curve is below the time axis.
6. For \( t \in (\frac{1}{2} T, T), \) the velocity is positive; the curve is above the time axis.

More details about the curve are discussed for each of the 4 time intervals. Please refer to
Fig. 2 when reading the following subsections.

A. For \( t \in [0, \frac{1}{4} T] \) or \( B \rightarrow O: \)

1. The mass moves to the left or the negative direction; the curve is below the time axis.
2. Its speed increases by Result 5; the curve becomes more and more negative.
3. The magnitude of acceleration decreases by Result 5; the tangent to the curve becomes
more and more shallow. Note the tangent at \( P \) is shallower than that at \( Q \) in Fig. 2.
4. At \( t = \frac{1}{4} T \) or at point \( O, \) the speed achieves its maximum \( v_m \) and its acceleration is
zero; the tangent to the curve is horizontal.

B. For \( t \in (\frac{1}{4} T, \frac{1}{2} T] \) or \( O \rightarrow C: \)

1. At \( t = \frac{1}{4} T \) or point \( O, \) the mass keeps moving to the left; the curve is still below the
time axis.
2. Its speed decreases by Result 6; the curve becomes less and less negative.
3. The magnitude of acceleration increases by Result 6; the tangent of the curve becomes steeper and steeper.

4. At \( t = \frac{1}{2} T \) or point \( C \), the speed is zero.

C. **For** \( t \in (\frac{1}{2} T, \frac{3}{4} T) \) or \( C \rightarrow O \):

1. At \( t = \frac{1}{2} T \) or point \( C \), the mass starts moving from rest to the right; the curve is above the time axis.
2. Its speed increases by Result 7; the curve becomes more and more positive.
3. The magnitude of acceleration decreases by Result 7; the tangent of the curve becomes more and more shallow.
4. At \( t = \frac{3}{4} T \) or point \( O \), the speed achieves its maximum \( v_m \) and the acceleration becomes zero; the tangent to the curve is horizontal.

D. **For** \( t \in (\frac{3}{4} T, T) \) or \( O \rightarrow B \):

1. At \( t = \frac{3}{4} T \) or point \( O \), the mass keeps moving to the right or in the positive direction; the curve is still above the time axis.
2. Its speed decreases by Result 8; the curve becomes less and less positive.
3. The magnitude of acceleration increases by Result 8; the tangent to the curve becomes steeper and steeper.
4. At $t = T$ or point $B$, the speed is zero.

IV. CONCLUSION AND DISCUSSION

This paper provides a qualitative analysis to SHM without either referring to circular motion or solving differential equation. The information contained in the obtained velocity-time curve is rich as far as the rather elementary knowledge, which the analysis is based upon, is concerned. In addition, students should always be encouraged to tackle seemingly unreachable new problems with whatever knowledge they have at hand. The displacement-time curve can similarly be sketched and it is left as an exercise.

With the experience in such analysis, students will not be surprised, feel comfortable, or even delighted when the sinusoidal solution is obtained from solving differential equation. It is a good opportunity for them to appreciate the interaction between physics and mathematics.

This approach may be introduced to students right after Newton’s second law and Hooke’s law are taught. It serves as an example to show how to graphically analyze a motion under varying force instead of a constant one. The author is surprised that no textbook employs this approach or at least includes it as an exercise. Moreover, he sincerely regards this qualitative approach as the proper way of teaching SHM to students.

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