Convergence and Optimization Results for a History-Dependent Variational Problem

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Abstract We consider a mixed variational problem in real Hilbert spaces, defined on the unbounded interval of time \([0, +\infty)\) and governed by a history-dependent operator. We state the unique solvability of the problem, which follows from a general existence and uniqueness result obtained in Sofonea and Matei (J. Glob. Optim. 61:591–614, 2015). Then, we state and prove a general convergence result. The proof is based on arguments of monotonicity, compactness, lower semicontinuity and Mosco convergence. Finally, we consider a general optimization problem for which we prove the existence of minimizers. The mathematical tools developed in this paper are useful in the analysis of a large class of nonlinear boundary value problems which, in a weak formulation, lead to history-dependent mixed variational problems. To provide an example, we illustrate our abstract results in the study of a frictional contact problem for viscoelastic materials with long memory.

Keywords History-dependent operator · Mixed variational problem · Lagrange multiplier · Mosco convergence · Pointwise convergence · Optimization problem · Viscoelastic material · Frictional contact

Mathematics Subject Classification (2000) 35M86 · 35M87 · 49J40 · 74M15 · 74M10

1 Introduction

Mixed variational problems involving Lagrange multipliers provide a useful framework in which a large number of problems with or without unilateral constraints can be cast and can be solved numerically. They are intensively used in Solid and Contact Mechanics as well as in many engineering applications. Existence and uniqueness results in the study of stationary mixed variational problems can be found in [4, 5, 7, 8, 14], for instance. Recently,
there is an interest in the study of time-dependent mixed variational problems involving a special case of operators, the so-called history-dependent operators. Such kind of operators arise in Solid and Contact Mechanics and describe memory effects in both the constitutive law and the interface boundary conditions. Reference in the field are [23–25]. The analysis of various mixed variational problems associated to mathematical models which describe the contact between a deformable body and a foundation can be found in [9–12, 15–17] and, more recently, in [1–3, 27].

The current paper represents a continuation of our previous paper [26]. There, we considered a new class of mixed variational problems with history-dependent operators and used arguments of saddle point and fixed point in order to prove its unique solvability. Here, we consider a special case of the history-dependent mixed variational problems considered in [26] for which we provide additional results. Everywhere in this paper we assume that $X$, $Y$ and $Z$ are real Hilbert spaces endowed with the inner products $(\cdot, \cdot)_X$, $(\cdot, \cdot)_Y$ and $(\cdot, \cdot)_Z$. The associated norms will be denoted by $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. Moreover, $0_X$ and $0_Y$ will represent the zero elements of the spaces $X$ and $Y$, and $X \times Y$ is their product space endowed with the canonical inner product. A typical element of $X \times Y$ will be denoted by $(u, \lambda)$. We also denote by $\mathbb{R}_+$ the set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$, and we use the notation $C(\mathbb{R}_+; X)$, $C(\mathbb{R}_+; Y)$ and $C(\mathbb{R}_+; Z)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in $X$, $Y$ and $Z$, respectively. Consider three operators $A : X \to X$, $S : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ and $\pi : X \to Z$, a form $b : X \times Y \to \mathbb{R}$, a set $\Lambda \subset Y$ and two functions $f : \mathbb{R}_+ \to Z$, $h : \mathbb{R}_+ \to X$. With these data we introduce the following problem.

**Problem 1** Find the functions $u : \mathbb{R}_+ \to X$ and $\lambda : \mathbb{R}_+ \to \Lambda$ such that

\[
(Au(t), v)_X + (Su(t), v)_X + b(v, \lambda(t)) = (f(t), \pi v)_Z \quad \forall v \in X, \quad (1.1)
\]

\[
b(u(t), \mu - \lambda(t)) \leq b(h(t), \mu - \lambda(t)) \quad \forall \mu \in \Lambda, \quad (1.2)
\]

for all $t \in \mathbb{R}_+$.

If $\Lambda \subset Y$ is unbounded, the unique solvability of Problem 1 is a direct consequence of the existence and uniqueness result obtained in [26]. However, in the present paper we remove this restriction, as explained in the next section.

The current paper has three main objectives. The first one is to provide a continuous dependence result for the solution to Problem 1 with respect to the data. The second one is to study an optimization problem related to this history-dependent mixed problem. Finally, the third objective is to illustrate how these abstract results can be used in the analysis of nonlinear boundary value problems which describe the contact of a deformable body with a foundation.

The rest of the paper is structured as follows. In Sect. 2 we present some preliminary material, including a general existence and uniqueness result for mixed problems of the form (1.1)–(1.2). In Sect. 3 we state and prove our main result, Theorem 3.2, which concerns the pointwise convergence of the solution to Problem 1 with respect to $A$, $S$, $b$, $\Lambda$, $f$ and $h$. Its proof is based on arguments of monotonicity, compactness, lower semicontinuity and Mosco convergence. Then, in Sect. 4 we apply this convergence result in the study of an optimization problem associated to Problem 1, for which we prove the existence of minimizers. Finally, in Sect. 5 we consider a mathematical model which describes the frictional contact of a viscoelastic body with an obstacle. In a variational formulation, this problem leads to a history-dependent mixed problem of the form (1.1)–(1.2). We illustrate how our
existence, uniqueness and convergence result can be applied in the analysis of this nonlinear problem.

2 Preliminary Results

In this section we present some preliminary results useful in the study of Problem 1 and, to this end, we consider the following assumptions.

\[ A : X \rightarrow X \text{ and, moreover:} \]
\[ \begin{aligned}
& (a) \text{ there exists } m_A > 0 \text{ such that } \\
& (Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall u, v \in X; \\
& (b) \text{ there exists } L_A > 0 \text{ such that } \\
& \|Au - Av\|_X \leq L_A \|u - v\|_X \quad \forall u, v \in X.
\end{aligned} \tag{2.1} \]

\[ S : C(\mathbb{R}^+; X) \rightarrow C(\mathbb{R}^+; X) \text{ and for each } m \in \mathbb{N} \]
\[ \begin{aligned}
& \text{there exists and } s_m \geq 0 \text{ such that } \\
& \|Su_1(t) - Su_2(t)\|_X \leq s_m \int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\
& \forall u_1, u_2 \in C(\mathbb{R}^+; X), \ t \in [0, m]. \tag{2.2}
\end{aligned} \]

\[ b : X \times Y \rightarrow \mathbb{R} \text{ is a bilinear form and, moreover:} \]
\[ \begin{aligned}
& (a) \text{ there exists } M_b > 0 \text{ such that } \\
& |b(v, \mu)| \leq M_b \|v\|_X \|\mu\|_Y \quad \forall v \in X, \ \mu \in Y; \\
& (b) \text{ there exists } \alpha_b > 0 \text{ such that } \\
& \inf_{\mu \in \Lambda, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha_b.
\end{aligned} \tag{2.3} \]

\[ \Lambda \text{ is a closed convex subset of } Y \text{ that contains } 0_Y. \tag{2.4} \]

Note that assumption (2.1) shows that the operator $A$ is strongly monotone and Lipschitz continuous. Moreover, following the terminology introduced in [23] and used in a large number of papers, assumption (2.2) shows that $S$ is a history-dependent operator. Finally, (2.3)(a) shows that the bilinear form $b$ is continuous and (2.3)(b) represents the well-known “inf-sup” condition.

We denote in what follows by $X \times \Lambda$ the product of $X$ and $\Lambda$ and we use the notation $C(\mathbb{R}^+; \Lambda)$, $C(\mathbb{R}^+; X \times \Lambda)$ for the set of functions defined on $\mathbb{R}^+$ with values in $\Lambda$ and $X \times \Lambda$, respectively. In addition, we recall the following existence and uniqueness results.

**Theorem 2.1** Assume (2.1), (2.3), (2.4). Then, given $g, k \in X$, there exists a unique pair $(u, \lambda) \in X \times \Lambda$ such that

\[ (Au, v)_X + b(v, \lambda) = (g, v)_X \quad \forall v \in X, \tag{2.5} \]
\[ b(u, \mu - \lambda) \leq b(k, \mu - \lambda) \quad \forall \mu \in \Lambda. \tag{2.6} \]
Theorem 2.2 Assume (2.1)–(2.4). Then, given $g, k \in C(\mathbb{R}_+, X)$, there exists a unique pair $(u, \lambda) \in C(\mathbb{R}_+; X \times \Lambda)$ such that

\[
(Au(t), v)_X + (Su(t), v)_X + b(v, \lambda(t)) = (g(t), v)_X \quad \forall v \in X,
\]

(2.7)

\[
b(u(t), \mu - \lambda(t)) \leq b(k(t), \mu - \lambda(t)) \quad \forall \mu \in \Lambda.
\]

(2.8)

Theorem 2.1 corresponds to Theorem 5.2 in [15] for $\Lambda$ unbounded and to Theorem 2.1 in [18] for $\Lambda$ bounded. Its proof is based on arguments of saddle points and the Banach fixed point principle. The proof of Theorem 2.2 can be carried out in several steps, based on Theorem 2.1 combined with a fixed point argument for history-dependent operators proved in [22]. Recall that Theorem 2.2 represents a particular case of Theorem 2.1 in [26], where the operator $S$ was assumed to depend on both the unknowns $u$ and $\lambda$, and $\Lambda$ was supposed to be unbounded. Nevertheless, by a slight modification of the proof of Theorem 2.1 in [26], it follows that Theorem 2.2 still remains valid if $\Lambda$ is bounded.

Consider now the following additional assumption.

\[
\pi : X \rightarrow Z \text{ is a linear continuous operator}
\]

(2.9)

which implies that there exists $c_0 > 0$ such that

\[
\|\pi v\|_Z \leq c_0 \|v\|_X \quad \forall v \in X.
\]

(2.10)

We complete Theorem 2.1 with the following existence, uniqueness and continuous dependence result.

Proposition 2.3 Assume (2.1), (2.3), (2.4), (2.9). Then, given $\eta, k \in X$ and $f \in Z$ there exists a unique pair $(u, \lambda) \in X \times \Lambda$ such that

\[
(Au, v)_X + (\eta, v)_X + b(v, \lambda) = (f, \pi v)_Z \quad \forall v \in X,
\]

(2.11)

\[
b(u, \mu - \lambda) \leq b(k, \mu - \lambda) \quad \forall \mu \in \Lambda.
\]

(2.12)

In addition, if $(u_1, \lambda_1)$ and $(u_2, \lambda_2)$ are the solutions of the problem (2.11)–(2.12) corresponding to the data $\eta_1, k_1 \in X, f_1 \in Z$ and $\eta_2, k_2 \in X, f_2 \in Z$, respectively, then there exists $d_0 > 0$ which depends only on $m_A, L_A, M_b, \alpha_b$ and $c_0$ such that

\[
\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \leq d_0(\|\eta_1 - \eta_2\|_X + \|f_1 - f_2\|_Z + \|k_1 - k_2\|_X).
\]

(2.13)

Proof We use assumption (2.9) and the Riesz representation theorem to define the element $g \in X$ by the equality

\[
(g, v)_X = (f, \pi v)_Z - (\eta, v)_X \quad \forall v \in X.
\]

The existence and uniqueness part in Proposition 2.3 is now a direct consequence of Theorem 2.1.

Denote in what follows by $(u_i, \lambda_i)$ the solution of the mixed problem (2.11)–(2.12) for the data $\eta = \eta_i, k = k_i \in X, f = f_i \in Z$, for $i = 1, 2$. Let $v \in X$. Then, using (2.11) it follows that

\[
(Au_1 - Au_2, v)_X + b(v, \lambda_1 - \lambda_2) = (f_1 - f_2, \pi v)_Z - (\eta_1 - \eta_2, v)_X
\]

(2.14)
and, using (2.1)(b), (2.9) we find that
\[ b(v, \lambda_1 - \lambda_2) \leq c_0 \| f_1 - f_2 \|_Z \| v \|_X + \| \eta_1 - \eta_2 \|_X \| v \|_X + L_A \| u_1 - u_2 \|_X \| v \|_X. \]

We now use (2.3)(b) and the previous inequality to obtain that
\[ \alpha_b \| \lambda_1 - \lambda_2 \|_Y \leq c_0 \| f_1 - f_2 \|_Z + \| \eta_1 - \eta_2 \|_X + L_A \| u_1 - u_2 \|_X. \quad (2.15) \]

On the other hand, (2.12) yields
\[ b(u_1 - u_2, \lambda_2 - \lambda_1) \leq b(k_1 - k_2, \lambda_2 - \lambda_1) \]
and, therefore, using condition (2.3)(a) we find that
\[ b(u_1 - u_2, \lambda_2 - \lambda_1) \leq M_b \| k_1 - k_2 \| X \| \lambda_1 - \lambda_2 \| Y. \quad (2.16) \]

We now take \( v = u_1 - u_2 \) in (2.14) and use (2.16) in the resulting inequality to deduce that
\( (Au_1 - Au_2, u_1 - u_2)_X \)
\[ \leq (f_1 - f_2, \pi u_1 - \pi u_2)_Z - (\eta_1 - \eta_2, u_1 - u_2)_X + M_b \| k_1 - k_2 \| X \| \lambda_1 - \lambda_2 \| Y. \]

Therefore, using the assumptions (2.1)(a), (2.9) and (2.10) it follows that
\[ M_A \| u_1 - u_2 \| X^2 \]
\[ \leq c_0 \| f_1 - f_2 \|_Z \| u_1 - u_2 \| X + \| \eta_1 - \eta_2 \|_X \| u_1 - u_2 \| X + M_b \| k_1 - k_2 \| X \| \lambda_1 - \lambda_2 \| Y. \quad (2.17) \]

We now use (2.17) and (2.15) together with the elementary inequalities
\[ ab \leq \frac{a^2}{2c} + \frac{cb^2}{2}, \quad (a + b + c)^2 \leq 3(a^2 + b^2 + c^2) \quad \forall a, b, c > 0 \]
to see that
\[ M_A \| u_1 - u_2 \| X^2 \leq \frac{c_0^2 \| f_1 - f_2 \|_Z^2}{2c_1} + c_1 \| u_1 - u_2 \| X^2 + \frac{\| \eta_1 - \eta_2 \|_X^2}{2c_1} + \frac{M_b^2 \| k_1 - k_2 \|_X^2}{2c_2} + \frac{c_2 \| \lambda_1 - \lambda_2 \| Y^2}{2}, \quad (2.18) \]
\[ \| \lambda_1 - \lambda_2 \| Y^2 \leq \frac{3}{\alpha_b} (c_0^2 \| f_1 - f_2 \|_Z^2 + \| \eta_1 - \eta_2 \|_X^2 + L_A^2 \| u_1 - u_2 \| X) \quad (2.19) \]

where \( c_1, c_2 \) are arbitrary positive constants. We now substitute (2.19) in (2.18) and choose
\( c_1 \) and \( c_2 \) such that
\[ M_A - c_1 - \frac{3c_2 L_A^2}{2\alpha_b} > 0. \]

In this way we deduce that there exists \( c_3 > 0 \), which depends only on \( M_A, L_A, M_b, \alpha_b \) and \( c_0 \) such that
\[ \| u_1 - u_2 \| X^2 \leq c_3 (\| f_1 - f_2 \|_Z^2 + \| \eta_1 - \eta_2 \|_X^2 + \| k_1 - k_2 \|_X) \quad (2.20) \]
Finally, using (2.19) and (2.20), after some algebra we obtain (2.13) with \( d_0 > 0 \) depending only on \( m_A, L_A, M_b, \alpha_b \) and \( c_0 \), which concludes the proof. \( \square \)

Next, we introduce the following assumptions on the data \( f \) and \( h \) of Problem 1.

\[
f \in C(\mathbb{R}_+; Z), \tag{2.21}
\]

\[
h \in C(\mathbb{R}_+; X). \tag{2.22}
\]

We have the following existence and uniqueness result.

**Proposition 2.4** Assume (2.1)–(2.4), (2.9), (2.21) and (2.22). Then Problem 1 has a unique solution \((u, \lambda)\). Moreover, the solution satisfies \((u, \lambda) \in C(\mathbb{R}_+; X \times \Lambda)\).

**Proof** We use assumption (2.9) and the Riesz representation theorem to define the function \( g : \mathbb{R}_+ \to X \) by equality

\[
(g(t), v)_X = (f(t), \pi v)_Z \quad \forall v \in X, \ t \in \mathbb{R}_+.
\]

Then, we use assumption (2.21) to see that \( g \in C(\mathbb{R}_+; X) \). Proposition 2.4 is now a direct consequence of Theorem 2.2. \( \square \)

We end this section by recalling the following version of the Weierstrass theorem.

**Theorem 2.5** Let \((X, \| \cdot \|_X)\) be a reflexive Banach space, \( K \) a nonempty weakly closed subset of \( X \) and \( J : X \to \mathbb{R} \) a weakly lower semicontinuous function. In addition, assume that either \( K \) is bounded or \( J \) is coercive, i.e., \( J(v) \to \infty \) as \( \|v\|_X \to \infty \). Then, there exists at least one element \( u \) such that

\[
u \in K, \quad J(u) \leq J(v) \quad \forall v \in K. \tag{2.23}
\]

The proof of Theorem 2.5 is based on standard arguments which can be found in many books and survey as, for instance, [13, 24].

### 3 A Convergence Result

The solution \((u, \lambda)\) obtained in Proposition 2.4 depends on \( A, S, b, \Lambda, f \) and \( h \). In this section we state and prove its convergence with respect to these data, which represents a crucial ingredient in the study of the optimization problem we shall consider in Sect. 4. Unless stated otherwise, all the sequences we introduce below are indexed upon \( n \in \mathbb{N} \) and all the limits, upper and lower limits are considered as \( n \to \infty \), even if we do not mention it explicitly. The symbols “\( \rightarrow \)” and “\( \rightharpoonup \)” denote the weak and the strong convergence in various spaces which will be specified. Nevertheless, for simplicity, we write \( g_n \to g \) for the convergence in \( \mathbb{R} \).

The functional framework is as follows. For each \( n \in \mathbb{N} \) we consider two operators \( A_n \) and \( S_n \), a form \( b_n \), a set \( A_n \) and two functions \( f_n \) and \( h_n \) which satisfy assumptions (2.1), (2.2), (2.3), (2.4), (2.21) and (2.22), respectively, with constants \( m_n, L_n, s^m, M_n, \alpha_n \). To avoid any confusion, when used with \( n \), we refer to these assumptions as assumptions (2.1)_n, (2.2)_n, (2.3)_n, (2.4)_n, (2.21)_n and (2.22)_n. Then, if condition (2.9) is satisfied, we deduce from Proposition 2.4 that for each \( n \in \mathbb{N} \) there exists a unique solution \((u_n, \lambda_n) \in C(\mathbb{R}_+; X \times \Lambda_n)\) for the following mixed variational problem.
Problem 2 Find the functions $u_n : \mathbb{R}_+ \to X$ and $\lambda_n : \mathbb{R}_+ \to \Lambda_n$ such that

\[
(A_n u_n(t), v)_X + (S_n u_n(t), v)_X + b_n(v, \lambda_n(t)) = (f_n(t), \pi v)_Z \quad \forall v \in X,
\]

(3.1)

\[
b_n(u_n(t), \mu - \lambda_n(t)) \leq b_n(h_n(t), \mu - \lambda_n(t)) \quad \forall \mu \in \Lambda_n
\]

(3.2)

for all $t \in \mathbb{R}_+$.

We now consider the following additional assumptions.

\[
\begin{cases}
\text{For each } n \in \mathbb{N} \text{ there exist } F_n \geq 0 \text{ and } \delta_n \geq 0 \text{ such that} \\
\quad (a) \|A_n v - Av\|_X \leq F_n(\|v\|_X + \delta_n) \quad \forall v \in X; \\
\quad (b) \lim_{n \to \infty} F_n = 0; \\
\quad (c) \text{the sequence } \{\delta_n\} \subset \mathbb{R} \text{ is bounded.}
\end{cases}
\]

(3.3)

There exist $m_0$, $L_0 > 0$ such that $m_n \geq m_0$, $L_n \leq L_0 \forall n \in \mathbb{N}$. (3.4)

\[
\begin{cases}
\text{For each } n, m \in \mathbb{N} \text{ there exist } F_n^m \geq 0 \text{ and } \delta_n^m \geq 0 \text{ such that} \\
\quad (a) \|S_n v(t) - Sv(t)\|_X \leq F_n^m(\max_{s \in [0,m]} \|v(s)\|_X + \delta_n^m) \\
\qquad \text{for all } v \in C(\mathbb{R}_+; X), \ t \in [0, m]; \\
\quad (b) \lim_{n \to \infty} F_n^m = 0, \ \forall m \in \mathbb{N}; \\
\quad (c) \text{the sequence } \{\delta_n^m\} \subset \mathbb{R} \text{ is bounded, } \forall m \in \mathbb{N}.
\end{cases}
\]

(3.5)

For each $m \in \mathbb{N}$ there exists $s_0^m > 0$ such that $s_n^m \leq s_0^m \forall n \in \mathbb{N}$. (3.6)

\[
\begin{cases}
\text{For all sequences } \{z_n\} \subset X, \ \{\mu_n\} \subset Y \text{ such that} \\
\quad z_n \to z \text{ in } X, \ \mu_n \to \mu \text{ in } Y, \text{ we have} \\
\quad \limsup b_n(w - z_n, \mu_n) \leq b(w - z, \mu) \quad \forall w \in X.
\end{cases}
\]

(3.7)

There exist $\alpha_0$, $M_0 > 0$ such that $\alpha_n \geq \alpha_0$, $M_n \leq M_0 \forall n \in \mathbb{N}$. (3.8)

\[
\begin{cases}
\text{For all sequences } \{v_n\} \subset X \text{ such that} \\
\quad v_n \to v \text{ in } X, \text{ we have } \pi v_n \to \pi v \in Y.
\end{cases}
\]

(3.9)

\[
\{A_n\} \text{ converges to } \Lambda \text{ in the sense of Mosco, i.e.:}
\]

\[
\begin{cases}
\text{(a) for each } \mu \in \Lambda \text{ there exists a sequence } \{\mu_n\} \text{ such that} \\
\quad \mu_n \in \Lambda_n \ \forall n \in \mathbb{N} \text{ and } \mu_n \to \mu \text{ in } Y; \\
\text{(b) for each sequence } \{\mu_n\} \text{ such that} \\
\quad \mu_n \in \Lambda_n \ \forall n \in \mathbb{N} \text{ and } \mu_n \to \mu \text{ in } Y, \text{ we have } \mu \in \Lambda.
\end{cases}
\]

(3.10)
\[
\begin{aligned}
\begin{cases}
(a) & f_n(t) \rightharpoonup f(t) \quad \text{in } Z \quad \text{as } n \to \infty, \quad \forall t \in \mathbb{R}_+; \\
(b) & \text{for each } m \in \mathbb{N} \text{ there exists } \omega_m > 0 \text{ such that } \\
& \| f_n(t) \|_Z \leq \omega_m \quad \forall t \in [0, m], \ n \in \mathbb{N}.
\end{cases}
\end{aligned}
\]

Note that assumption (3.9) shows that the linear operator \( \pi : X \to Y \) is completely continuous. Details on the convergence of sets in the sense of Mosco used in condition (3.10) can be found in [19]. Such a convergence was used in the recent papers [21, 29] in the study of convergence results for elliptic and history-dependent variational-hemivariational inequalities, respectively.

Remark 3.1 It follows from assumptions (2.1) and (3.4) that for each \( n \in \mathbb{N} \) the operator \( A_n \) satisfies condition (2.1) with the constants \( m_0 \) and \( L_0 \). On the other hand, assumptions (2.2) and (3.6) show that for each \( n \in \mathbb{N} \) the operator \( S_n \) satisfies condition (2.2) with the constant \( \xi^0_m \). Finally, assumptions (2.3) and (3.8) show that for each \( n \in \mathbb{N} \) the bilinear form \( b_n \) satisfies condition (2.3) with the constants \( \alpha_0 \) and \( M_0 \).

The main result of this section is the following.

**Theorem 3.2** Assume (2.1)–(2.4), (2.9), (2.21), (2.22) and, for each \( n \in \mathbb{N} \), assume (2.1)–(2.4), (2.21), (2.22). Moreover, assume (3.3)–(3.12) and denote by \( (u_n, \lambda_n) \) and \( (u, \lambda) \) the solutions of Problems 2 and 1, respectively. Then, for all \( t \in \mathbb{R}_+ \) the following convergences hold:

\[
\begin{aligned}
\quad & u_n(t) \to u(t) \quad \text{in } X, \quad (3.13) \\
\quad & \lambda_n(t) \rightharpoonup \lambda(t) \quad \text{in } Y. \quad (3.14)
\end{aligned}
\]

The proof of Theorem 3.2 will be carried out in several steps that we present in what follows. To this end, below in this section we assume that the hypotheses of Theorem 3.2 are satisfied and, for each \( n \in \mathbb{N} \), we consider the following auxiliary problems.

**Problem 3** Find \( u_0^0 \in X \) and \( \lambda_0^0 \in \Lambda_n \) such that

\[
\begin{aligned}
& (A_n u_0^0, v)_X + b_n(v, \lambda_0^0) = 0 \quad \forall v \in X, \\
& b_n(u_0^0, \mu - \lambda_0^0) \leq 0 \quad \forall \mu \in \Lambda_n.
\end{aligned}
\]

**Problem 4** Find the functions \( \tilde{u}_n : \mathbb{R}_+ \to X \) and \( \tilde{\lambda}_n : \mathbb{R}_+ \to \Lambda_n \) such that

\[
\begin{aligned}
& (A_n \tilde{u}_n(t), v)_X + (Su(t), v)_X + b_n(v, \tilde{\lambda}_n(t)) = (f_n(t), \pi v)_Z \quad \forall v \in X, \\
& b_n(\tilde{u}_n(t), \mu - \tilde{\lambda}_n(t)) \leq b_n(h_n(t), \mu - \tilde{\lambda}_n(t)) \quad \forall \mu \in \Lambda_n
\end{aligned}
\] for all \( t \in \mathbb{R}_+ \).
The first step is given by the following result.

**Lemma 3.3** For each \( n \in \mathbb{N} \), Problem 3 has a unique solution \((u_n^0, \lambda_n^0) \in X \times \Lambda_n\). Moreover, there exists \( a_0 > 0 \) such that

\[
\|u_n^0\|_X \leq a_0, \quad \|\lambda_n^0\|_Y \leq a_0 \quad \forall n \in \mathbb{N}.
\] (3.19)

**Proof** The existence and uniqueness part is a direct consequence of Theorem 2.1. Let \( n \in \mathbb{N} \). We use assumption (2.4) and test in (3.16) with \( \mu = 0_Y \) to obtain that \( b_n(u_n^0, \lambda_n^0) \geq 0 \).

We now take \( v = u_n^0 \) in (3.15) and use the previous inequality to see that

\[
(A_n u_n^0, u_n^0)_X \leq 0.
\]

Next, we write \( A_n u_n^0 = A_n u_n^0 - A_n 0_X + A_n 0_X \) and use assumption (2.1) to deduce that

\[
m_n \|u_n^0\|_X \leq \|A_n 0_X\|_X.
\] (3.20)

On the other hand, writing \( A_n 0_X = A_n 0_X - A 0_X + A 0_X \) and using inequality (3.3) yield

\[
\|A_n 0_X\|_X \leq F_n \delta_n + \|A 0_X\|_X.
\] (3.21)

We now combine inequalities (3.20) and (3.21), and use assumption (3.4) to see that

\[
\|u_n^0\|_X \leq \frac{1}{m_0} (F_n \delta_n + \|A 0_X\|_X).
\]

Finally, we use assumptions (3.3)(b), (c) to deduce that the sequence \( \{u_n^0\} \) is bounded in \( X \), i.e., there exists \( K_0 > 0 \) which does not depend on \( n \) such that

\[
\|u_n^0\|_X \leq K_0.
\] (3.22)

Next, we establish the boundedness of \( \{\lambda_n\} \) in \( Y \). To this end we use (3.15) to see that

\[
b_n(v, \lambda_n^0) = - (A_n u_n^0, v)_X \leq \|A_n u_n^0\|_X \|v\|_X
\]

for all \( v \in X \), which implies that

\[
\sup_{v \in X, v \neq 0_X} \frac{b_n(v, \lambda_n^0)}{\|v\|_X \|\lambda_n^0\|_Y} \leq \frac{1}{\|\lambda_n^0\|_Y} \|A_n u_n^0\|_X,
\]

if \( \lambda_n^0 \neq 0_Y \). Therefore,

\[
\inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b_n(v, \mu)}{\|v\|_X \|\mu\|_Y} \leq \frac{1}{\|\lambda_n^0\|_Y} \|A_n u_n^0\|_X,
\]

if \( \lambda_n^0 \neq 0_Y \). We now use assumption (2.3) on the bilinear form \( b_n \) together with the bound (3.8) to deduce that

\[
\|A_n u_n^0\|_X \leq \|\lambda_n^0\|_Y,
\] (3.23)
both when $\lambda_n^0 \neq 0_Y$ and $\lambda_n^0 = 0_Y$.

Next, we use assumptions (3.3)(a) and (2.1)(b) to see that
\[
\|A_n u_n\|_X \leq \|A_n u_n - A u_n\|_X + \|A u_n\|_X \\
\leq F_n(\|u_n\|_X + \delta_n) + \|A u_n - A 0_X\|_X + \|A 0_X\|_X \\
\leq F_n(\|u_n\|_X + \delta_n) + L_A \|u_n\|_X + \|A 0_X\|_X.
\]

Therefore, by assumptions (3.3)(b), (c) and the bound (3.22) we find that the sequence \{\lambda_n^0\} is bounded in $Y$, i.e., there exists $P_0 > 0$ which does not depend on $n$ such that
\[
\|\lambda_n^0\|_Y \leq P_0. \tag{3.24}
\]

Lemma 3.3 is now a direct consequence of the inequalities (3.22) and (3.24). $\square$

The second step is given by the following result.

**Lemma 3.4** For each $n \in \mathbb{N}$, Problem 4 has a unique solution $(\tilde{u}_n, \tilde{\lambda}_n) \in C(\mathbb{R}_+; X \times \Lambda_n)$. Moreover, for each $m \in \mathbb{N}$, there exists $\tilde{a}_m > 0$ such that
\[
\|\tilde{u}_n(t)\|_X \leq \tilde{a}_m, \quad \|\tilde{\lambda}_n(t)\|_Y \leq \tilde{a}_m \quad \forall t \in [0, m], n \in \mathbb{N}. \tag{3.25}
\]

**Proof** The existence and uniqueness part is a direct consequence of Proposition 2.4. Let $m, n \in \mathbb{N}$ and let $t \in [0, m]$. Note that both Problems 3 and 4 are problems of the form (2.11)–(2.12). Therefore, using Remark 3.1 it follows that we are in the position to use Proposition 2.3 to obtain the estimate
\[
\|\tilde{u}_n(t) - u_n\|_X + \|\tilde{\lambda}_n(t) - \lambda_n^0\|_Y \leq d_0(\|S u(t)\|_X + \|f_n(t)\|_Z + \|h_n(t)\|_X) \tag{3.26}
\]
where $d_0 > 0$ is a positive constant which does not depend on $n$. Denote
\[
\zeta_m = \max_{t \in [0, m]} \|S u(t)\|_X.
\]

Then, using inequality (3.26) and assumptions (3.11)(b) and (3.12)(b) we deduce that there exists $K_m > 0$ which does not depend on $n$ such that
\[
\|\tilde{u}_n(t) - u_n\|_X + \|\tilde{\lambda}_n(t) - \lambda_n^0\|_Y \leq K_m. \tag{3.27}
\]

We now write
\[
\|\tilde{u}_n(t)\|_X \leq \|\tilde{u}_n(t) - u_n^0\|_X + \|u_n^0\|_X,
\]
\[
\|\tilde{\lambda}_n(t)\|_Y \leq \|\tilde{\lambda}_n(t) - \lambda_n^0\|_Y + \|\lambda_n^0\|_Y
\]
then we use inequalities (3.27) and (3.19) to see that (3.25) holds with $\tilde{a}_m = K_m + a_0$. $\square$

The next step of the proof consists in the following convergence result.
Lemma 3.5 For each \( t \in \mathbb{R}_+ \), there exists a pair \((\widetilde{u}(t), \widetilde{\lambda}(t)) \in X \times Y\) and a subsequence of the sequence \((\tilde{u}_n, \tilde{\lambda}_n))\), still denoted by \((\tilde{u}_n, \tilde{\lambda}_n))\), such that

\[
\begin{align*}
\tilde{u}_n(t) &\rightharpoonup \tilde{u}(t) \quad \text{in } X, \\
\tilde{\lambda}_n(t) &\rightharpoonup \tilde{\lambda}(t) \quad \text{in } Y.
\end{align*}
\]

Moreover,

\[
\tilde{u}_n(t) \to \tilde{u}(t) \quad \text{in } X. \tag{3.30}
\]

Proof Let \( t \in \mathbb{R}_+ \) and let \( m \) be such that \( t \in [0, m] \). We use Lemma 3.4 and a standard compactness argument to see that there exists an element \( \tilde{u}(t) \in X \) and an element \( \tilde{\lambda}(t) \in Y \) such that (3.28) and (3.29) hold.

We now prove the strong convergence (3.30) and, to this end, we test with \( v = \tilde{u}_n(t) - \tilde{u}(t) \) in (3.17) to deduce that

\[
\begin{align*}
(A_n\tilde{u}_n(t), \tilde{u}_n(t) - \tilde{u}(t))_X + (Su(t), \tilde{u}_n(t) - \tilde{u}(t))_X + b_n(\tilde{u}_n(t) - \tilde{u}(t), \tilde{\lambda}(t))_Y \\
= (f_n(t), \pi\tilde{u}_n(t) - \pi\tilde{u}(t))_Z.
\end{align*}
\]

Therefore,

\[
\begin{align*}
(A_n\tilde{u}_n(t) - A_n\tilde{u}(t), \tilde{u}_n(t) - \tilde{u}(t))_X \\
= (A_n\tilde{u}(t), \tilde{u}(t) - \tilde{u}_n(t))_X + (Su(t), \tilde{u}(t) - \tilde{u}_n(t))_X + b_n(\tilde{u}(t) - \tilde{u}_n(t), \tilde{\lambda}(t))_Y \\
+ (f_n(t), \pi\tilde{u}_n(t) - \pi\tilde{u}(t))_Z
\end{align*}
\]

and, using Remark 3.1 we find that

\[
\begin{align*}
m_0\|\tilde{u}_n(t) - \tilde{u}(t)\|_X^2 \\
\leq (A_n\tilde{u}(t), \tilde{u}(t) - \tilde{u}_n(t))_X + (Su(t), \tilde{u}(t) - \tilde{u}_n(t))_X + b_n(\tilde{u}(t) - \tilde{u}_n(t), \tilde{\lambda}(t))_Y \\
+ (f_n(t), \pi\tilde{u}_n(t) - \pi\tilde{u}(t))_Z. \tag{3.31}
\end{align*}
\]

Next, using (3.3)(a) we find that

\[
\begin{align*}
(A_n\tilde{u}(t), \tilde{u}(t) - \tilde{u}_n(t))_X &= (A_n\tilde{u}(t) - A\tilde{u}(t), \tilde{u}(t) - \tilde{u}_n(t))_X + (A\tilde{u}(t), \tilde{u}(t) - \tilde{u}_n(t))_X \\
&\leq \|A_n\tilde{u}(t) - A\tilde{u}(t)\|_X \|\tilde{u}_n(t) - \tilde{u}(t)\|_X + (A\tilde{u}(t), \tilde{u}(t) - \tilde{u}_n(t))_X \\
&\leq F_n(\|\tilde{u}(t)\|_X + \delta_n) \|\tilde{u}_n(t) - \tilde{u}(t)\|_X + (A\tilde{u}(t), \tilde{u}(t) - \tilde{u}_n(t))_X.
\end{align*}
\]

We now pass to the upper limit in this inequality and use assumptions (3.3)(b), (c) and the convergence \( \tilde{u}_n(t) \rightharpoonup \tilde{u}(t) \) in \( X \) to see that

\[
\limsup (A_n\tilde{u}(t), \tilde{u}(t) - \tilde{u}_n(t))_X \leq 0. \tag{3.32}
\]

On the other hand, the convergence (3.28) implies that

\[
(Su(t), \tilde{u}(t) - \tilde{u}_n(t))_X \to 0. \tag{3.33}
\]
Moreover, taking \( z_n = \tilde{u}_n(t) \), \( \mu_n = \tilde{\lambda}_n(t) \) and \( w = \tilde{u}(t) \) in (3.7) yields

\[
\limsup b_n(\tilde{u}(t) - \tilde{u}_n(t), \tilde{\lambda}_n(t)) \leq 0. \tag{3.34}
\]

In addition, using assumptions (3.11)(a), (3.9) we find that

\[
\left( f_n(t), \pi \tilde{u}_n(t) - \pi \tilde{u}(t) \right)_Z \to 0. \tag{3.35}
\]

We now pass to the upper limit in the inequality (3.31) and use (3.32)–(3.35) to deduce that

\[
\limsup m_0 \left\| \tilde{u}_n(t) - \tilde{u}(t) \right\|^2_X \leq 0
\]

which shows that (3.30) holds and concludes the proof. \( \square \)

The next step completes the statement of Lemma 3.5 and it is as follows.

**Lemma 3.6** For each \( t \in \mathbb{R}_+ \), the following convergences hold:

\[
\tilde{u}_n(t) \to u(t) \text{ in } X, \tag{3.36}
\]

\[
\tilde{\lambda}_n(t) \to \lambda(t) \text{ in } Y. \tag{3.37}
\]

**Proof** Let \( t \in \mathbb{R}_+ \) and let \( m \in \mathbb{N} \) such that \( t \in [0, m] \). We first recall that for each \( n \in \mathbb{N} \) we have \( \tilde{\lambda}_n(t) \in \Lambda_n \). Moreover, using Lemma 3.5 it follows that passing to a subsequence, still denoted by \( \{\tilde{\lambda}_n(t)\} \), the convergence (3.29) holds. Therefore, assumption (3.10)(b) implies that

\[
\tilde{\lambda}(t) \in \Lambda. \tag{3.38}
\]

Next, we consider a subsequence of the sequence \( \{\tilde{u}_n, \tilde{\lambda}_n\} \), still denoted by \( \{\tilde{u}_n, \tilde{\lambda}_n\} \), such that (3.28) and (3.29) hold. Let \( n \in \mathbb{N} \) and \( v \in X \). We use assumptions (3.3)(a) and (2.1)(b) to see that

\[
\left\| A_n \tilde{u}_n(t) - A \tilde{u}(t) \right\|_X \leq \left\| A_n \tilde{u}_n(t) - A \tilde{u}_n(t) \right\|_X + \left\| A \tilde{u}_n(t) - A \tilde{u}(t) \right\|_X
\]

\[
\leq F_n \left( \left\| \tilde{u}_n(t) \right\|_X + \delta_n \right) + L_A \left\| \tilde{u}_n(t) - \tilde{u}(t) \right\|_X
\]

and, therefore, assumptions (3.3)(b), (c) and (3.30) imply that

\[
A_n \tilde{u}_n(t) \to A \tilde{u}(t) \text{ in } X. \tag{3.39}
\]

On the other hand, we write condition (3.7) with \( z_n = 0_X, \mu_n = \tilde{\lambda}_n(t), w = v \), then with \( z_n = v, \mu_n = \tilde{\lambda}_n(t) \) and \( w = 0_X \) to obtain

\[
\limsup b_n(v, \tilde{\lambda}_n(t)) \leq b(v, \tilde{\lambda}(t)) \quad \text{and} \quad b(v, \tilde{\lambda}(t)) \leq \liminf b_n(v, \tilde{\lambda}_n(t)),
\]

respectively. These inequalities show that

\[
b_n(v, \tilde{\lambda}_n(t)) \to b(v, \tilde{\lambda}(t)). \tag{3.40}
\]

Finally, note that the convergence (3.11)(a) implies that

\[
\left( f_n(t), \pi v \right)_Z \to \left( f(t), \pi v \right)_Z. \tag{3.41}
\]
Next, we pass to the limit in equality (3.17) and use the convergences (3.39)–(3.41) to see that
\[
(A\tilde{u}(t), v)_X + (Su(t), v)_X + b(v, \tilde{\lambda}(t)) = (f(t), \pi v)_Z .
\] (3.42)
Consider now an arbitrary element \( \mu \in \Lambda \). Using assumption (3.10)(a) we know that there exists a sequence \( \{\mu_n\} \) such that \( \mu_n \in \Lambda_n \) for each \( n \in \mathbb{N} \) and \( \mu_n \to \mu \) in \( Y \). This allows to use the inequality (3.18) to see that
\[
\liminf b_n(\tilde{u}_n(t) - h_n(t), \mu_n - \tilde{\lambda}_n(t)) \leq 0
\] (3.43)
on the other hand, writing condition (3.7) with \( w = 0_X, z_n = \tilde{u}_n(t) - h_n(t) \) and \( \mu_n - \tilde{\lambda}_n(t) \) instead of \( \mu_n \), we deduce that
\[
\limsup b_n(-\tilde{u}_n(t) + h_n(t), \mu_n - \tilde{\lambda}_n(t)) \leq b(-\tilde{u}(t) + h(t), \mu - \tilde{\lambda}(t))
\] or, equivalently,
\[
b(\tilde{u}(t) - h(t), \mu - \tilde{\lambda}(t)) \leq \liminf b_n(\tilde{u}_n(t) - h_n(t), \mu_n - \tilde{\lambda}_n(t)).
\] (3.44)
We combine inequalities (3.44) and (3.43) to find that
\[
b(\tilde{u}(t) - h(t), \mu - \tilde{\lambda}(t)) \leq 0.
\] (3.45)
Finally, we gather (3.38), (3.42) and (3.45) to conclude that the pair \((\tilde{u}(t), \tilde{\lambda}(t))\) satisfies (1.1)–(1.2). On the other hand, it follows from Proposition 2.3 that there exists a unique solution to the system (1.1)–(1.2), denoted \((u(t), \lambda(t))\). Therefore, we deduce that \( \tilde{u}(t) = u(t) \) and \( \tilde{\lambda}(t) = \lambda(t) \). Next, the proofs of Lemmas 3.4–3.5 combined with equalities \( \tilde{u}(t) = u(t) \) and \( \tilde{\lambda}(t) = \lambda(t) \) reveal the fact that the sequence \( \{(\tilde{u}_n(t), \tilde{\lambda}_n(t))\} \) is bounded in \( X \times Y \) and every subsequence of \( \{(\tilde{u}_n(t), \tilde{\lambda}_n(t))\} \) which converges weakly in \( X \times Y \) has the same limit \( (u(t), \lambda(t)) \). Therefore, by a standard argument we deduce that the whole sequence \( \{(\tilde{u}_n(t), \tilde{\lambda}_n(t))\} \) converges weakly in \( X \times Y \) to \((u(t), \lambda(t))\) or, equivalently, \( \tilde{u}_n(t) \rightharpoonup u(t) \) in \( X \) and \( \tilde{\lambda}_n(t) \rightharpoonup \lambda(t) \) in \( Y \). This shows that the weak convergence (3.37) holds. Moreover, repeating the arguments in Lemma 3.4 we deduce the strong convergence (3.36), which concludes the proof. □

We are now in a position to provide the proof of Theorem 3.2.

Proof Let \( n \in \mathbb{N}, t \in \mathbb{R}_+ \) and let \( m \in \mathbb{N} \) be such that \( t \in [0, m] \). A careful examination of Problems 2 and 4 reveals that, with an appropriate notation, these problems are governed by a system of the (2.11)–(2.12). Therefore, using Remark 3.1 we are allowed to apply Proposition 2.3 to obtain the estimate
\[
\|u_n(t) - \tilde{u}_n(t)\|_X + \|\lambda_n(t) - \tilde{\lambda}_n(t)\|_Y \leq d_0 \|S_n u_n(t) - Su(t)\|_X
\] (3.46)
where \( d_0 \) is a positive constant which depends only on \( m_0, L_0, \alpha_0, M_0 \) and \( c_0 \). Thus,
\[
\|u_n(t) - \tilde{u}_n(t)\|_X \leq d_0 \|S_n u_n(t) - Su(t)\|_X .
\] (3.47)
Let
\[ r_m = \max_{t \in [0,m]} \| u(t) \|_X. \] (3.48)

We write \( S_n u_n(t) - S u(t) = S_n u_n(t) - S_n u(t) + S_n u(t) - S u(t) \), then we use assumption (3.5) and Remark 3.1, again, to deduce that
\[ \| S_n u_n(t) - S u(t) \|_X \leq s_0^0 \int_0^t \| u_n(s) - u(s) \|_X ds + F_m^m (r_m + \delta_n^m). \] (3.49)

On the other hand, we have
\[ \| u_n(t) - u(t) \|_X \leq \| u_n(t) - \tilde{u}_n(t) \|_X + \| \tilde{u}_n(t) - u(t) \|_X. \] (3.50)

Therefore, combining (3.50), (3.47) and (3.49) we find that
\[ \| u_n(t) - u(t) \|_X \leq \| \tilde{u}_n(t) - u(t) \|_X + d_0 F_m^m (r_m + \delta_n^m) \]
and, using the Gronwall argument yields
\[ \| u_n(t) - u(t) \|_X \leq \| \tilde{u}_n(t) - u(t) \|_X + d_0 s_0^0 \int_0^t e^{d_0 s_0^0 (t-s)} (d_0 F_m^m (r_m + \delta_n^m) + \| \tilde{u}_n(s) - u(s) \|_X) ds. \] (3.51)

For all \( s \in [0, m] \) denote
\[ z_n^m(s) = d_0 F_m^m (r_m + \delta_n^m) + \| \tilde{u}_n(s) - u(s) \|_X. \] (3.52)

Then, (3.51) implies that
\[ \| u_n(t) - u(t) \|_X \leq z_n^m(t) + d_0 s_0^0 \int_0^t e^{d_0 s_0^0 (t-s)} z_n^m(s) ds. \] (3.53)

On the other hand, (3.25) and (3.48) imply that
\[ \| \tilde{u}_n(s) - u(s) \|_X \leq \| \tilde{u}_n(s) \|_X + \| u(s) \|_X \leq \tilde{a}_m + r_m \]
and, therefore, (3.52) yields
\[ |z_n^m(s)| \leq d_0 F_m^m (r_m + \delta_n^m) + \tilde{a}_m + r_m \quad \forall s \in [0, m]. \] (3.54)

We now use assumption (3.5)(b), (c) and Lemma 3.6 to see that the sequence of functions \( \{ z_n^m \} \) is bounded as \( n \to \infty \) and converges to zero, for all \( s \in [0, m] \), i.e.,

there exists \( Z_m > 0 \) such that \( |z_n^m(s)| \leq Z_m \quad \forall s \in [0, m], \quad n \in \mathbb{N}, \) (3.55)
\[ z_n^m(s) \to 0 \quad \text{as} \quad n \to \infty, \quad \forall s \in [0, m]. \] (3.56)
Therefore, we are in a position to use the Lebesgue theorem in order to see that
\[
\int_0^t e^{\int_0^s (t-s) \varepsilon_n^m(s) \, ds} \, ds \to 0,
\]
(3.57)
and, moreover,
\[
\varepsilon_n^m(t) \to 0 \quad \text{as} \; n \to \infty.
\]
(3.58)
We now use the convergences (3.57), (3.58) and inequality (3.53) to deduce that (3.13) holds.

Finally, note that (3.53), (3.54) and the convergence (3.13) allows us to use the Lebesgue theorem, again, in order to see that
\[
\int_0^t \| u_n(s) - u(s) \| \, ds \to 0
\]
and, in addition, assumption (3.5)(b), (c) yield
\[
F_n^m(r_m + \delta_n^m) \to 0.
\]
(3.59)
These two convergences combined with (3.49) imply that
\[
\| S_n u_n(t) - S u(t) \| \to 0
\]
and, using inequality (3.46) we deduce that
\[
\| \lambda_n(t) - \tilde{\lambda}_n(t) \| \to 0.
\]
(3.59)
We now write \( \lambda_n(t) - \lambda(t) = \lambda_n(t) - \tilde{\lambda}_n(t) + \tilde{\lambda}_n(t) - \lambda(t) \) then we use the strong convergence (3.59) and the weak convergence (3.37) to find that (3.14) holds, which concludes the proof.

Note that Theorem 3.2 states a pointwise convergence result, strongly for the first component of the solution \((u, \lambda)\) of Problem 1, and weakly for the second component. Considering appropriate assumptions on the data which guarantee a strong convergence result for the second component \(\lambda\) and/or a uniform convergence result for the solution \((u, \lambda)\) represents an open problem which, clearly, deserves to be studied in the future.

### 4 An Optimization Problem

In this section we apply Theorem 3.2 in the study of a general optimization problem associated to the history-dependent mixed variational problem (1.1)–(1.2). To this end we consider a reflexive Banach space \(W\) endowed with the norm \(\| \cdot \|_W\) and a nonempty subset \(U \subset W\).

For each \(p \in U\) we consider two operators \(A_p, S_p\), a form \(b_p\) and a set \(\Lambda_p\) which satisfy assumptions (2.1), (2.2), (2.3), (2.4), respectively, with constants \(m_p, L_p, s_p^m, M_p, \alpha_p\).
To avoid any confusion, when used with \(p\), we refer to these assumptions as assumptions (2.1)\(_p\), (2.2)\(_p\), (2.3)\(_p\), (2.4)\(_p\). Also, assume that the elements \(\tilde{f}_p\) and \(\tilde{h}_p\) are given and have the regularity.
\[ \tilde{f}_p \in Z, \]  
\[ \tilde{h}_p \in X. \]  

Let \( \theta \) and \( \zeta \) be two functions such that

\[ \theta \in C(\mathbb{R}_+; \mathbb{R}), \]  
\[ \zeta \in C(\mathbb{R}_+; \mathbb{R}) \]

and consider the functions \( f_p, h_p \) defined by

\[ f_p : \mathbb{R}_+ \to Z, \quad f_p(t) = \theta(t) \tilde{f}_p \quad \forall t \in \mathbb{R}_+, \]  
\[ h_p : \mathbb{R}_+ \to X, \quad h_p(t) = \zeta(t) \tilde{h}_p \quad \forall t \in \mathbb{R}_+. \]

Then, under the previous assumptions, if in addition the condition (2.9) is satisfied, we deduce from Proposition 2.4 that for each \( p \in U \) there exists a unique solution \((u_p, \lambda_p)\) \( \in C(\mathbb{R}_+; X \times \Lambda_p) \) to the following problem.

**Problem 5** Find \( u_p : \mathbb{R}_+ \to X \) and \( \lambda_p : \mathbb{R}_+ \to \Lambda_p \) such that

\[ \left( A_p u_p(t), v \right)_X + \left( S_p u_p(t), v \right)_X + b_p(v, \lambda_p(t)) = \left( f_p(t), \pi v \right)_Z \quad \forall v \in X, \]  
\[ b(u_p(t), \mu - \lambda_p(t)) \leq b_p(h_p(t), \mu - \lambda_p(t)) \quad \forall \mu \in \Lambda_p \]

for all \( t \in \mathbb{R}_+ \).

Consider a cost function \( L : X \times Y \times U \to \mathbb{R} \). We formulate the following optimization problem.

**Problem 6** Given \( t \in \mathbb{R}_+ \), find \( p^* \in U \) such that

\[ L(u_{p^*}(t), \lambda_{p^*}(t), p^*) = \min_{p \in U} L(u_p(t), \lambda_p(t), p). \]

To solve Problem 6 we consider the following assumptions.

\( U \) is a nonempty weakly closed subset of \( W \).  

For all sequences \( \{u_n\} \subset X, \{\lambda_n\} \subset Y \) and \( \{p_n\} \subset U \) such that

\[ u_n \rightharpoonup u \text{ in } X, \quad \lambda_n \rightharpoonup \lambda \text{ in } Y, \quad p_n \rightharpoonup p \text{ in } W, \]  
\[ \liminf \mathcal{L}(u_n, \lambda_n, p_n) \geq \mathcal{L}(u, \lambda, p). \]

There exists \( z : U \to \mathbb{R} \) such that

- (a) \( \mathcal{L}(u, \lambda, p) \geq z(p) \quad \forall u \in X, \lambda \in Y, p \in U \),  
- (b) \( \|p_n\|_W \to +\infty \implies z(p_n) \to \infty \).

\( U \) is a bounded subset of \( W \).
A typical example of function $L$ which satisfies conditions (4.11) and (4.12) is obtained by taking

$$L(u, \lambda, p) = g(u) + k(\lambda) + z(p) \quad \forall u \in X, \lambda \in Y, p \in U,$$

where $g : X \rightarrow \mathbb{R}_+$ is a lower semicontinuous function, $k : Y \rightarrow \mathbb{R}_+$ is a weakly lower semicontinuous function, and $z : U \rightarrow \mathbb{R}$ is a weakly lower semicontinuous coercive function, i.e., it satisfies condition (4.12)(b).

Our main result in this section is the following.

**Theorem 4.1** Assume (2.1)–(2.4) and (4.1)–(4.6) for each $p \in U$. Moreover, assume (2.9), (4.10), (4.11) and either (4.12) or (4.13). In addition, assume that for each sequence $\{p_n\} \subset U$ such that $p_n \rightharpoonup p$ in $W$, conditions (3.3)–(3.10) are satisfied with $A_n = A_{p_n}$, $A = A_p$, $m_n = m_{A_{p_n}}$, $L_n = L_{A_{p_n}}$, $S_n = S_{p_n}$, $S = S_p$, $s_m^n = s_{m_{p_n}}$, $s_m^p = s_m$, $b_{p_n} = b_n$, $\alpha_n = \alpha_{p_n}$, $\alpha = \alpha_p$, $A_n = A_{p_n}$, $A = A_p$ and

$$\tilde{f}_{p_n} \rightharpoonup \tilde{f}_p \text{ in } Z, \quad (4.14)$$

$$\tilde{h}_{p_n} \rightharpoonup \tilde{h}_p \text{ in } X. \quad (4.15)$$

Then, for each $t \in \mathbb{R}_+$, Problem 6 has at least one solution $p^*$.

**Proof** Let $t \in \mathbb{R}_+$ be fixed. We consider the function $J_t : U \rightarrow \mathbb{R}$ defined by

$$J_t(p) = L(u_{p}(t), \lambda_{p}(t), p) \quad \forall p \in U \quad (4.16)$$

together with the problem of finding $p^* \in U$ such that

$$J_t(p^*) = \min_{p \in U} J_t(p). \quad (4.17)$$

Assume that $\{p_n\} \subset U$ is such that $p_n \rightharpoonup p$ in $W$. Then, recall that (4.14) and (4.15) hold. Note that (4.1), (4.3), (4.5) and (4.14) show that the functions $f_n = f_{p_n}$ and $f = f_p$ satisfy condition (3.11). Moreover, (4.2), (4.4), (4.6) and (4.15) show that the functions $h_n = h_{p_n}$ and $h = h_p$ satisfy condition (3.12). Thus, since conditions (3.3)–(3.10), are satisfied (in the sense prescribed in the statement of the theorem), we are in a position to apply Theorem 3.2 in order to obtain that $u_{p_n}(t) \rightarrow u_p(t)$ in $X$ and $\lambda_{p_n}(t) \rightarrow \lambda_p(t)$ in $Y$. Therefore, using definition (4.16) and assumption (4.11) we deduce that

$$\lim inf J_t(p_n) = \lim inf L(u_{p_n}(t), \lambda_{p_n}(t), p_n) \geq L(u_p(t), \lambda_p(t), p) = J_t(p).$$

It follows from here that the function $J_t : U \rightarrow \mathbb{R}$ is weakly lower semicontinuous.

Assume now that (4.12) holds. Then, for any sequence $\{p_n\} \subset U$, we have

$$J_t(p_n) = L(u_{p_n}(t), \lambda_{p_n}(t), p_n) \geq z(p_n).$$

Therefore, if $\|p_n\|_W \rightarrow \infty$ we deduce that $J_t(p_n) \rightarrow \infty$ which shows that $J_t : U \rightarrow \mathbb{R}$ is coercive. Recall also the assumption (4.10) and the reflexivity of the space $W$. The existence of at least one solution to problem (4.17) is now a direct consequence of Theorem 2.5. On the other hand, if we assume that condition (4.13) is satisfied we are still in a position to apply Theorem 2.5. We deduce from here that, if either (4.12) or (4.13) holds, then there exists at least one solution $p^* \in U$ to the optimization problem (4.17). We now use the definition (4.16) to see that $p^*$ is a solution to Problem 6 which concludes the proof. \[ \square \]
5 A Viscoelastic Frictional Contact Problem

The abstract results in Sects. 3–4 are useful in the variational analysis of mathematical models which describe the equilibrium of deformable bodies in contact with an obstacle, the so-called foundation. In this section we illustrate their use in the study of a frictional contact model with linearly viscoelastic materials. For the description of additional models of contact as well as for details on the notation and preliminaries introduced below we refer the reader to [6, 20, 24, 25, 28, 30].

The physical setting is as follows. We consider a viscoelastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz continuous boundary $\Gamma$, divided into three measurable disjoint parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ such that $\text{meas}(\Gamma_1) > 0$. We assume that the body is fixed on $\Gamma_1$, is acted by given body forces and given surface tractions on $\Gamma_2$, and it is in frictional contact with an obstacle on $\Gamma_3$. The time interval of interest is $\mathbb{R}_+ = [0, +\infty)$, the contact is bilateral, that is, there is no separation between the body and the foundation, and it is associated to the Tresca friction law. Then, the equilibrium of the body in the physical setting above is described by the following boundary value problem.

**Problem 7** Find a displacement field $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a stress field $\sigma : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that

$$\sigma(t) = 2\beta \varepsilon(u(t)) + \eta \text{tr} \left( \varepsilon(u(t)) \right) \mathbf{I}_d + \int_0^t e^{-\omega(t-s)} \varepsilon(u(s)) \, ds \quad \text{in} \; \Omega, \quad (5.1)$$

$$\text{Div} \; \sigma(t) + f_0(t) = 0 \quad \text{in} \; \Omega, \quad (5.2)$$

$$u(t) = 0 \quad \text{on} \; \Gamma_1, \quad (5.3)$$

$$\sigma \nu(t) = f_2(t) \quad \text{on} \; \Gamma_2, \quad (5.4)$$

$$u_\nu(t) = 0, \quad \|\sigma_\tau(t)\| \leq g, \quad \sigma_\tau(t) = -g \frac{u_\tau(t)}{\|u_\tau(t)\|} \quad \text{if} \; u_\tau(t) \neq 0 \quad \text{on} \; \Gamma_3, \quad (5.5)$$

for all $t \in \mathbb{R}_+$.

Here and below in this section we do not mention the dependence of various functions with respect to the spatial variable $x \in \Omega \cup \Gamma$. Notation $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order $d$, and $\mathbf{I}_d$ stands for the unit tensor of $\mathbb{S}^d$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$u \cdot v = u_i v_i, \quad \|v\| = (v \cdot v)^{\frac{1}{2}} \quad \forall u, v \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}} \quad \forall \sigma, \tau \in \mathbb{S}^d,$$

and the zero element of these spaces will be denoted by $0$. Also, $v$ is the outward unit normal at $\Gamma$ and $u_\nu, u_\tau$ represent the normal and tangential components of $u$ on $\Gamma$ given by $u_\nu = u \cdot v$ and $u_\tau = u - u_\nu v$, respectively. Finally, $\sigma_\nu$ and $\sigma_\tau$ denote the normal and tangential stress on $\Gamma$, that is $\sigma_\nu = (\sigma \cdot v) \cdot v$ and $\sigma_\tau = \sigma v - \sigma_\nu v$.

We now provide a short description of the equations and boundary conditions in Problem 3. First, (5.1) represents the viscoelastic constitutive law of the material, in which $\beta$ and $\eta$ represent the Lamé coefficients, $\text{tr} \; \tau$ denotes the trace of the tensor $\tau$, $\omega$ is a relaxation coefficient and $\varepsilon(u)$ denotes the linearized strain field. Equation (5.2) is the equation
of equilibrium in which \( f_0 \) represents the density of the body forces and \( \text{Div} \) denotes the divergence operator. Conditions (5.3), (5.4) represent the displacement and traction boundary conditions, respectively, where \( f_2 \) denotes the density of given surface tractions which act on the part \( \Gamma_2 \) of the boundary. Finally, condition (5.5) represents the interface law on the contact surface. Equality \( u_\nu(t) = 0 \) shows that there is no separation between the body and the obstacle during the deformation process, i.e., the contact is bilateral. The rest of the condition in (5.5) represents the static version of the Tresca’s friction law, in which \( g \) denote the friction bound, assumed to be given.

In the study of the contact problem (5.1)–(5.5) we assume that the data satisfy the following conditions.

\[
\begin{align*}
\beta &\geq 0, \\
\eta &\geq 0, \\
\omega &\geq 0, \\
f_0 &\in C(\mathbb{R}_+; L^2(\Omega)^d), \\
f_2 &\in C(\mathbb{R}_+; L^2(\Gamma_2)^d), \\
g &\geq 0.
\end{align*}
\]

Everywhere below we use the standard notation for Sobolev and Lebesgue spaces associated to \( \Omega \) and \( \Gamma \) and we denote by \( \gamma : H^1(\Omega)^d \to L^2(\Gamma)^d \) the trace operator. For each element \( v \in H^1(\Omega)^d \) we use the notation \( v_\nu \) and \( v_\tau \) for the normal and tangential components of \( v \) on \( \Gamma \), that is, \( v_\nu = \gamma v \cdot \nu \) and \( v_\tau = \gamma v - v_\nu \nu \), respectively. Moreover, we use the notation \( \epsilon(v) \) for the associated linearized strain field, i.e.,

\[
\epsilon(v) = (\epsilon_{ij}(v)), \quad \epsilon_{ij}(v) = \frac{1}{2} (v_{i,j} + v_{j,i}),
\]

where an index that follows a comma denotes the partial derivative with respect to the corresponding component of \( x \), e.g., \( v_{i,j} = \frac{\partial v_i}{\partial x_j} \).

Next, for the displacement field we consider the space

\[
X = \{ v \in H^1(\Omega)^d : \gamma v = 0 \text{ on } \Gamma_1, \quad v_\nu = 0 \text{ on } \Gamma_3 \}.
\]

Since \( \text{meas}(\Gamma_1) > 0 \), it is well known that \( X \) is a real Hilbert space endowed with the canonical inner product

\[
(u, v)_X = \int_\Omega \epsilon(u) \cdot \epsilon(v) \, dx
\]

and the associated norm \( \| \cdot \|_X \). It follows from [15] that the space \( \gamma(X) \) is a closed subspace of the Hilbert space \( \gamma(H^1(\Omega)^d) \) and, therefore, it is a Hilbert space, too. Let \( Y \) be its dual (which, in turn, can be organized as a real Hilbert space) and denote by \( \langle \cdot, \cdot \rangle \) the duality pairing between \( Y \) and \( \gamma(X) \). Recall also that \( \gamma(X) \) is continuously embedded in \( L^2(\Gamma)^d \). Finally, we need the space \( Z = L^2(\Omega)^d \times L^2(\Gamma_2)^d \) equipped with the canonical inner product.

We now introduce the operators \( A : X \to X \), \( S : C(\mathbb{R}_+, X) \to C(\mathbb{R}_+, X) \) and \( \pi : X \to Z \), the form \( b : X \times Y \to \mathbb{R} \), the set \( \Lambda \subset Y \) and the function \( f : \mathbb{R}_+ \to Z \) defined by the
following equalities:

\[
(Au, v)_X = \int_{\Omega} \left( 2\beta \varepsilon(u) + \eta \operatorname{tr}(\varepsilon(u)) I_d \right) : \varepsilon(v) \, dx \quad \forall u, v \in X,
\]

\[
(Su(t), v)_X = \int_{\Omega} \left( \int_0^t e^{-\omega(t-s)} \varepsilon(u(s)) \, ds \right) : \varepsilon(v) \, dx \quad \forall u \in C(\mathbb{R}_+; X), \ v \in X,
\]

\[
b(v, \mu) = \langle \mu, \gamma v \rangle \quad \forall v \in X, \ \mu \in Y,
\]

\[
\pi v = (v, \gamma_2 v) \quad \forall v \in X,
\]

\[
\Lambda = \left\{ \mu \in Y : (\mu, \xi) \leq g \int_{\Gamma_3} \|\xi\|da \quad \forall \xi \in \gamma(X) \right\},
\]

\[
f(t) = (f_0(t), f_2(t)) \quad \forall t \in \mathbb{R}_+.
\]

Note that the definition of the operators \(A\) and \(S\) follows by using Riesz’s representation theorem. Moreover, here and below, \(\gamma_2 v \in L^2(\Gamma_2)^d\) denotes the restriction to \(\Gamma_2\) of the trace \(\gamma v \in L^2(\Gamma)^d\), for any \(v \in X\). In addition, the definitions (5.16) and (5.18) imply that

\[
(f(t), \pi v)_Z = \int_{\Omega} f_0(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot \gamma_2 v \, da \quad \forall v \in X, \ t \in \mathbb{R}_+.
\]

We now introduce a new variable, the Lagrange multiplier, denoted by \(\lambda\). It is related to the friction force \(\sigma_\tau\) on the contact zone \(\Gamma_3\) by equality

\[
\langle \lambda(t), \widetilde{v} \rangle = -\int_{\Gamma_3} \sigma_\tau(t) : \widetilde{v} \, da \quad \forall \widetilde{v} \in \gamma(X), \ t \in \mathbb{R}_+.
\]

A mixed variational formulation of Problem 7 can be easily obtained, based on equalities (5.19), (5.20) and integration by parts. It can be stated as follows.

**Problem 8** Find a displacement field \(u : \mathbb{R}_+ \to X\) and a Lagrange multiplier \(\lambda : \mathbb{R}_+ \to \Lambda\) such that

\[
(Au(t), v)_X + (Su(t), v)_X + b(v, \lambda(t)) = (f(t), \pi v)_Z \quad \forall v \in X,
\]

\[
b(u(t), \mu - \lambda(t)) \leq 0 \quad \forall \mu \in \Lambda
\]

for all \(t \in \mathbb{R}_+\).

The unique solvability of Problem 8 is given by the following existence and uniqueness result.

**Theorem 5.1** Assume (5.6)–(5.11). Then, Problem 8 has a unique solution \((u, \lambda) \in C(\mathbb{R}_+; X \times \Lambda)\).

**Proof** Let \(u, v, w \in X\). We use definition (5.13), inequality

\[
\operatorname{tr}(\tau) I_d : \tau = \left( \operatorname{tr}(\tau) \right)^2 \geq 0 \quad \forall \tau \in \mathbb{R}^d
\]

and assumption (5.7) to see that

\[
(Au - Av, u - v)_X \geq 2\beta \|u - v\|_X^2.
\]

\[\square\text{ Springer}\]
On the other hand, assumptions (5.6), (5.7) and inequality
\[ \| \text{tr}(\tau) I_d \| \leq d \| \tau \| \quad \forall \tau \in S^d \]
imply that
\[ (Au - Av, w)_X \leq (2\beta + d\eta) \| u - v \|_X \| w \|_X \]
and, therefore,
\[ \| Au - Av \|_X \leq (2\beta + d\eta) \| u - v \|_X. \quad (5.24) \]
Inequalities (5.23) and (5.24) show that the operator \( A \) defined by (5.13) satisfies condition (2.1) with \( m_A = 2\beta \) and \( L_A = 2\beta + d\eta \).

Let \( u, v \in C(\mathbb{R}_+; X) \) and \( w \in X \). We use definition (5.14) and the assumption (5.8) to deduce that
\[ (Su(t) - Sv(t), w)_X \leq \left( \int_0^t e^{-\omega(t-s)} \| u(s) - v(s) \|_X \, ds \right) \| w \|_X \]
\[ \leq \left( \int_0^t \| u(s) - v(s) \|_X \, ds \right) \| w \|_X \quad \forall t \in \mathbb{R}_+. \]
This proves that
\[ \| Su(t) - Sv(t) \|_X \leq \int_0^t \| u(s) - v(s) \|_X \, ds \quad \forall t \in \mathbb{R}_+ \]
which shows that the operator (5.14) satisfies condition (2.2) with \( s_m = 1 \), for each \( m \in \mathbb{N} \).

Next, we claim that the form \( b \) given by (5.15) satisfies condition (2.3). For the proof of this statement we refer the reader to [15], for instance. Moreover, it is obvious to see that the operator \( \pi \) defined by (5.16) satisfies condition (2.9). On the other hand, assumption (5.11) shows that the set \( \Lambda \) defined by (5.17) satisfies condition (2.4) and, finally, assumptions (5.9), (5.10) imply (2.21) for the element \( f \) given by (5.18). Recall also that condition (2.22) also holds, since \( h \) vanishes. Therefore, Theorem 5.1 is now a direct consequence of Proposition 2.4.

A pair \( (u, \lambda) \in C(\mathbb{R}_+; X \times \Lambda) \) which satisfies (5.21) and (5.22) for each \( t \in \mathbb{R}_+ \) is called a weak solution to Problem 7. We conclude from here that Theorem 5.1 provides sufficient conditions which guarantee the weak solvability of the contact problem (5.1)–(5.5).

We now study the continuous dependence of the solution to Problem 8 with respect to the data. To this end we assume that the density of body forces and traction are such that
\[ f_0(t) = \theta(t) \tilde{f}_0 \quad \forall t \in \mathbb{R}_+, \quad (5.25) \]
\[ f_2(t) = \zeta(t) \tilde{f}_2 \quad \forall t \in \mathbb{R}_+, \quad (5.26) \]
where the functions \( \theta \in C(\mathbb{R}_+; \mathbb{R}) \) and \( \zeta \in C(\mathbb{R}_+; \mathbb{R}) \) are given and, moreover,
\[ \tilde{f}_0 \in L^2(\Omega)^d, \quad (5.27) \]
\[ \tilde{f}_2 \in L^2(\Gamma_2)^d. \quad (5.28) \]
Note that in this case conditions (5.9) and (5.10) are satisfied. We also consider the product space $W = \mathbb{R}^3 \times L^2(\Omega)^d \times L^2(\Gamma_2)^d \times \mathbb{R}$ endowed with the canonical Hilbertian norm and let $U$ be the subset of $W$ defined by

$$U = \{ p = (\beta, \eta, \omega, \tilde{f}_0, \tilde{f}_2, g) \in W : \beta, \eta, \omega, g > \delta_0 \}$$

(5.29)

where $\delta_0 > 0$ is given. Moreover, for each $p = (\beta, \eta, \omega, f_0, f_2, g) \in U$ we denote by $(u_p, \lambda_p)$ the solution of Problem 8 obtained in Theorem 5.1. Then, we have the following convergence result.

**Theorem 5.2** For each sequence $\{p_n\} \subset U$ such that $p_n \rightharpoonup p$ in $W$, and for each $t \in \mathbb{R}_+$, the following convergences hold:

$$u_{p_n}(t) \to u_p(t) \quad \text{in } X,$$

(5.30)

$$\lambda_{p_n}(t) \rightharpoonup \lambda_p(t) \quad \text{in } Y.$$

(5.31)

**Proof** Let $\{p_n\} \subset U$ be a sequence of elements in $U$ such that $p_n \rightharpoonup p$ in $W$, and for each $t \in \mathbb{R}_+$, the following convergences hold:

$$u_{p_n}(t) \to u_p(t) \quad \text{in } X,$$

(5.30)

$$\lambda_{p_n}(t) \rightharpoonup \lambda_p(t) \quad \text{in } Y.$$

(5.31)

Moreover, for simplicity, denote $u_{p_n} = u_n$ and $\lambda_{p_n} = \lambda_n$. Then, it follows that $(u_n, \lambda_n) \in C(\mathbb{R}_+; \mathcal{X}) \times \mathcal{Y}$ and, in addition,}

$$b(u_n(t), \mu - \lambda_n(t)) \leq 0 \quad \forall \mu \in \Lambda_n.$$  

(5.37)

Assume now that

$$p_n = (\beta_n, \eta_n, \omega_n, \tilde{f}_0, \tilde{f}_2, g_n) \rightharpoonup p = (\beta, \eta, \omega, \tilde{f}_0, \tilde{f}_2, g) \quad \text{in } W,$$

which implies that

$$\beta_n \to \beta,$$

(5.38)

$$\eta_n \to \eta,$$

(5.39)

$$\omega_n \to \omega.$$  

(5.40)
\[ \bar{f}_{0n} \rightarrow \bar{f}_0 \text{ in } L^2(\Omega)^d, \]  
\[ \widetilde{f}_{02} \rightarrow \widetilde{f}_2 \text{ in } L^2(F_2)^d, \]  
\[ g_n \rightarrow g. \]  
(5.41)  
(5.42)  
(5.43)

Our aim in what follows is to apply Theorem 3.2 in the study of the mixed variational problems (5.36)–(5.37) and (5.21)–(5.22) and, to this end, in what follows we check the validity of conditions (3.3)–(3.12).

First, we use the convergences (5.38), (5.39), (5.40) to see that condition (3.3) is satisfied with

\[ F_n = 2|\beta_n - \beta| + d|\eta_n - \eta| \text{ and } \delta_n = 0. \]

Moreover, equalities \( m_n = 2\beta_n \) and \( L_n = 2\beta_n + d\eta_n \) show that the condition (3.4) holds, too.

Let \( n, m \in \mathbb{N}, t \in [0, m] \) and let \( \psi \in C(\mathbb{R}_+; V) \). Then, using (5.33) and (5.14) it is easy to see that

\[ \|S_n \psi(t) - S \psi(t)\|_X \leq \left( \int_0^t e^{-\omega_n(t-s)} - e^{-\omega(t-s)} \right) ds \left( \max_{s \in [0,m]} \|\psi(s)\|_X \right) \]

\[ \leq \int_0^t |e^{-\omega_n(t-s)} - e^{-\omega(t-s)}| ds \left( \max_{s \in [0,m]} \|\psi(s)\|_X \right). \]  
(5.44)

Moreover, using the mean value theorem we deduce that for all \( s \in [0, t] \) there exists \( \xi_n(s) \geq 0 \) such that following inequality holds,

\[ |e^{-\omega_n(t-s)} - e^{-\omega(t-s)}| \leq e^{-\xi_n(s)}(t-s)|\omega_n - \omega| \]

Using now the inequality

\[ e^{-\xi_n(s)}(t-s)|\omega_n - \omega| \leq m|\omega_n - \omega| \]

we deduce that

\[ |e^{-\omega_n(t-s)} - e^{-\omega(t-s)}| \leq m|\omega_n - \omega|. \]  
(5.45)

Finally, we combine (5.44) and (5.45) to deduce that condition (3.5) holds with \( F_n^m = m|\omega_n - \omega| \) and \( \delta_n^m = 0 \). Recall that, the proof of Theorem 5.1 reveals that \( s_n^m = 1 \) for all \( n, m \in \mathbb{N} \) and, therefore, condition (3.6) holds, too.

Next, conditions (3.7) and (3.8) are obviously satisfied since in our case \( b_n = b \) for each \( n \in \mathbb{N} \). On the other hand, the compactness of the embedding \( X \subset L^2(\Omega)^d \) combined with the compactness of the trace operator \( \gamma_2 : X \rightarrow L^2(F_2)^d \) shows that the operator (5.16) satisfies condition (3.9). Moreover, we note that \( \Lambda_n = \frac{\xi_n}{s} \Lambda \). Using now the convergence (5.43) it is easy to see that condition (3.10) holds. In addition, we note that the convergences (5.41) and (5.42), together with (5.25) and (5.26), imply (3.11) for \( f_n \) and \( f \) given by (5.35) and (5.18), respectively. Finally, note that, obviously, condition (3.12) is satisfied.

It follows from above that we are in position to use Theorem 3.2 in order to deduce that the convergences (5.30) and (5.31) hold for each \( t \in \mathbb{R}_+ \), which concludes the proof.

Besides the mathematical interest, the convergence results (5.30) and (5.31) are important from mechanical point of view since they provide the continuous dependence of the weak solution of Problem 7 with respect to the Lamé coefficients, the relaxation coefficient, the densities of the body forces and surface tractions, and the friction bound, at each time moment.
We now provide two examples of optimization problems associated to Problem 8 for which the abstract result in Theorem 4.1 holds. Everywhere below $U$ represents the set given by (5.29). The two problems we consider below have a common feature and can be casted in the following general form.

**Problem 9** Given $t \in \mathbb{R}_+$, find $p^* \in U$ such that

$$
\mathcal{L}(u_{p^*}(t), \lambda_{p^*}(t), p^*) = \min_{p \in U} \mathcal{L}(u_p(t), \lambda_p(t), p).
$$

Here $\mathcal{L} : X \times Y \times U \to \mathbb{R}$ is the cost functional. Both $U$ and $\mathcal{L}$ will change from example to example and, therefore, will be described below. We also recall that, given $p = (\beta, \eta, \omega, \tilde{f}_0, \tilde{f}_2, g) \in U$, $(u_p, \lambda_p)$ represents the solution of Problem 8 with the data $\beta$, $\eta$, $\omega$, $f_0$, $f_2$ and $g$ where $f_0$, $f_2$ are given by (5.25) and (5.26), respectively. Note that the existence of this solution is guaranteed by Theorem 5.1. Moreover, it follows from the proof of Theorem 5.2 that if $p_n \to p$ in $W$ then conditions (3.3)–(3.10) hold and, obviously, (3.11) and (3.12) hold, too. Therefore, the solvability of Problem 9 follows from Theorem 4.1, provided that conditions (4.10), (4.11) and either (4.12) or (4.13) are satisfied.

**Example 5.3** Let $\delta_1$, $\delta_2$, $M_0$, $M_2$ be positive constants such that $\delta_1 \leq \delta_2$, and consider a function $u_0 \in X$. Let $U$ and $\mathcal{L} : X \times Y \times U \to \mathbb{R}$ be defined by

$$
U = \{ p = (\beta, \eta, \omega, \tilde{f}_0, \tilde{f}_2, g) \in \tilde{U} : \beta, \eta, \omega, g \in [\delta_1, \delta_2],
\|f_0\|_{L^2(\Omega)^d} \leq M_0, \|f_2\|_{L^2(\Omega_2)^d} \leq M_2 \},
$$

$$
\mathcal{L}(u, \lambda, p) = \int_{f_3} \|u - u_0\|^2 \, da ~ \forall u \in X, \lambda \in \Lambda, p \in U.
$$

With this choice, the mechanical interpretation of Problem 9 is the following: given a contact process of the form (5.1)–(5.5), (5.25)–(5.26) and a time moment $t \in \mathbb{R}_+$, we are looking for a set of data $p = (\beta, \eta, \omega, f_0, f_2, g) \in U$ such that the corresponding displacement on the contact surface at $t$ is as close as possible to the “desired displacement” $u_0$. Note that in this case assumptions (4.10), (4.11) and (4.13) are satisfied. Therefore, Theorem 4.1 guarantees the existence of solutions to the corresponding optimization problem 9.

**Example 5.4** Let $u_0 \in X$ and $\lambda_0 \in Y$ be given, and let $c_1$, $c_2$, $c_3$ be strictly positive constants. Moreover, consider the set $U$ defined by (5.29) and the cost functional $\mathcal{L} : X \times Y \times U \to \mathbb{R}$ defined by

$$
\mathcal{L}(u, \lambda, p) = c_1\|u - u_0\|_X^2 + c_2\|\lambda - \lambda_0\|_Y^2 + c_3\|p\|_W^2
$$

$$
\forall u \in X, \lambda \in \Lambda, p \in U.
$$

With this choice, the mechanical interpretation of Problem 9 is the following: given a contact process of the form (5.1)–(5.5), (5.25)–(5.26) and a time moment $t \in \mathbb{R}_+$ we are looking for a set of data $p = (\beta, \eta, \omega, f_0, f_2, g) \in U$ such that the corresponding state of the body at $t$ is as close as possible to the “desired state” $(u_0, \lambda_0)$. Furthermore, this choice has to fulfill a minimum expenditure condition which is taken into account by the last term in the functional $\mathcal{L}$. In fact, a compromise policy between the two aims (“$u$ close to $u_0$”, “$\lambda$ close to $\lambda_0$” and “minimal data $p$”) has to be found and the relative importance of each criterion with respect to the other is expressed by the choice of the weight coefficients $c_1$, $c_2$. 

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and $c_3$. Note that in this case assumptions (4.10), (4.11) and (4.12) are satisfied. Therefore, Theorem 4.1 guarantees the existence of the solutions to the corresponding optimization problem 9.

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