Entropy and the cosmological constant: a spacetime-foam approach

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A simple model of spacetime foam, made by $N$ wormholes in a semiclassical approximation, is taken under examination. The Casimir-like energy of the quantum fluctuation of such a model and its probability of being realized are computed. Implications on the Bekenstein-Hawking entropy and the cosmological constant are considered.

1. Introduction

The problem of merging General Relativity with Quantum Field Theory is known as Quantum Gravity. One aspect of this merged theories is that at the Planck scale, spacetime could be subjected to topology and metric fluctuations. Such a fluctuating spacetime is known under the name of “spacetime foam” which can be taken as a model for the quantum gravitational vacuum. At this scale of lengths (or energies) quantum processes like black hole pair creation could become relevant. To establish if a foamy spacetime could be considered as a candidate for a Quantum Gravitational vacuum, we can examine the structure of the effective potential for such a spacetime. It has been shown that flat space is the classical minimum of the energy for General Relativity. However there are indications that flat space is not the true ground state when a temperature is introduced, at least for the Schwarzschild space in absence of matter fields. It is also argued that when gravity is coupled to $N$ conformally invariant scalar fields the evidence that the ground-state expectation value of the metric is flat space is false. Moreover it is also believed that in a foamy spacetime, general relativity can be renormalized when a density of virtual black holes is taken under consideration coupled to $N$ fermion fields in a $1/N$ expansion. With these examples at hand, we have led to consider the possibility of having a ground state different from flat space even at zero temperature. Units in which $\hbar = c = k = 1$ are used throughout the paper.

2. Escaping from Flat Space

If the effective potential (more precisely effective energy) is analyzed at one loop in a Schwarzschild background, we discover that there exists an imaginary contribution, namely flat space is unstable. What is the physical interpretation associated to this instability. We can begin by observing that the “simplest” quantum process approximating a foamy spacetime, in absence of matter fields, could be a black hole pair creation of the neutral type. One possibility of describing such a process is represented by the Schwarzschild-deSitter metric which asymptotically approaches the deSitter metric. Its degenerate or extreme version is best known as the Nariai metric. Here we have an external background, the cosmological constant $\Lambda_c$, which gives a nonzero probability of having a neutral black hole pair produced with its components accelerating away from each other. Nevertheless this process is believed to be highly suppressed, at least for $\Lambda_c \gg 1$ in Planck’s units. In any case,
metrics with a cosmological constant have different boundary conditions compared to flat space. The Schwarzschild metric is the only case available. Here the whole spacetime can be regarded as a black hole-anti-black hole pair formed up by a black hole with positive mass $M$ in the coordinate system of the observer and an anti black-hole with negative mass $-M$ in the system where the observer is not present. In this way we have an energy preserving mechanism, because flat space has zero energy and the pair has zero energy too. However, in this case we have not a cosmological force producing the pair: we have only pure gravitational fluctuations. The black hole-anti-black hole pair has also a relevant pictorial interpretation: the black hole with positive mass $M$ and the anti black-hole with negative mass $-M$ can be considered the components of a virtual dipole with zero total energy created by a large quantum gravitational fluctuation\[9\]. Note that this is the only physical process compatible with the energy conservation. The importance of having the energy behavior (asympotic) is related to the possibility of having a spontaneous transition from one spacetime to another one with the same boundary condition \[10\]. This transition is a decay from the false vacuum to the true one \[11,12\].

However, if we take account of a pair of neutral black holes living in different universes, there is no decay and more important no temperature is involved to change from flat to curved space. To see if this process is realizable we need to compute quantum corrections to the energy stored in the boundaries. These quantum corrections are pure gravitational vacuum excitations which can be measured by the Casimir energy, formally defined as

$$E_{\text{Casimir}}[\partial M] = E_0[\partial M] - E_0[0],$$

where $E_0$ is the zero-point energy and $\partial M$ is a boundary.

3. Building the foam

We begin to consider the following line element (Einstein-Rosen bridge) related to a single wormhole $ds^2 = -N^2(r)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$\(2\)

We wish to compute the Casimir-like energy

$$\Delta E(M) = E(M) - E(0) = \frac{\langle \Psi | H^{\text{Schw.}} - H^{\text{Flat}} | \Psi \rangle}{\langle \Psi | \Psi \rangle} + \frac{\langle \Psi | H_{a} | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

by perturbing the three-dimensional spatial metric $g_{ij} = \delta_{ij} + h_{ij}$. $\Delta E(M)$ is computed in a WKB approximation, by looking at the graviton sector (spin 2 or TT tensor) in a Schrödinger representation with trial wave functionals of the Gaussian form by means of a variational approach. The Spin-two operator is defined as

$$\left(\Delta_2\right)_j^a = -\triangle_j^a + 2R_j^a$$

where $\triangle$ is the Laplacian on a Schwarzschild background and $R_j^a$ is the mixed Ricci tensor whose components are:

$$R_j^a = \text{diag} \left\{ \frac{-2MG}{r^3}, \frac{MG}{r^3}, \frac{MG}{r^3} \right\}.$$

The total energy at one loop, i.e., the classical term plus the stable and unstable modes respectively, is

$$\Delta E_{\text{q.l.}} + \Delta E_s + \Delta E_u$$

where $\Delta E_{\text{q.l.}}$ is the quasilocal energy. For symmetric boundary conditions with respect to the bifurcation surface $S_0$ (such as this case $E_{\text{q.l.}} = E_+ - E_- = 0$). When the boundaries go to spatial infinity $E_\pm = M_{\text{ADM}}$. The Stable modes contribution is

$$\Delta E_s = -\frac{V}{32\pi^2} \left( \frac{3MG}{r^3} \right)^2 \ln \left( \frac{r_3^3\Lambda^2}{3MG} \right).$$

$\Lambda$ is a cut-off to keep under control the $\text{UV}$ divergence, we can think that $\Lambda \leq m_p$. For the unstable sector, there is only one eigenvalue in S-wave. This is in agreement with Coleman arguments on quantum tunneling: the presence of a unique negative eigenvalue in the second order perturbation is a signal of a passage from a
false vacuum to a true vacuum. The Rayleigh-Ritz method joined to a numerical integration technique gives \( E^2 = -1.17541/(MG)^2 \), to be compared with the value \( E^2 = -1.19/(MG)^2 \) of Ref.\[3\]. How to eliminate the instability?

We consider \( N_w \) coherent wormholes (i.e., noninteracting) in a semiclassical approximation and assume that there exists a covering of \( \Sigma \) such that \( \Sigma = \bigcup_{i=1}^{N_w} \Sigma_i \), with \( \Sigma_i \cap \Sigma_j = \emptyset \) when \( i \neq j \). Each \( \Sigma_i \) has the topology \( S^2 \times \mathbb{R}^1 \) with boundaries \( \partial \Sigma_i \perp \) with respect to each bifurcation surface. On each surface \( \Sigma_i \), quasilocal energy is zero because we assume that on each copy of the single wormhole there is symmetry with respect to each bifurcation surface. Thus the total energy for the collection is \( E^2 = N_wMG^2 \) and the total macroscopic energy functional is the product of \( N_w \) t.w.f.

\[
\Psi_{\text{tot}}^{\pm} = \Psi_1^{\pm} \otimes \Psi_2^{\pm} \otimes \cdots \Psi_{N_w}^{\pm} \tag{7}
\]

By repeating the same calculations done for the single wormhole for the \( N_w \) wormhole system, we obtain

a) The total Casimir energy (stable modes), at its minimum, is

\[ \Delta E_s(M) \sim -N_w^2 \frac{V}{64\pi^2} \frac{A^4}{c} \]

The minimum does not correspond to flat space \( \rightarrow \Delta E_s(M) \neq 0 \).

b) The initial boundary located at \( R_\pm \) will be reduced to \( R_\pm/N_w \).

c) Since the boundary is reduced there exists a critical radius \( \rho_c = 1.1134 \) such that \( \forall \rho \geq N_w \rho_c \), \( r_c \leq r \leq \rho \). \( \sigma(\Delta_2) = 0 \). This means that the system begins to be stable\[4,17\]. To be compared with the value \( \rho_c = 1.445 \) obtained by B. Allen in Ref.\[18\].

4. Area Spectrum, Entropy and the Cosmological constant

Bekenstein has proposed that a black hole does have an entropy proportional to the area of its horizon \( S_{bh} = \text{const} \times A_{hor} \[16\]. \) In natural units one finds that the proportionality constant is set to \( 1/4G = 1/4l_p^2 \), so that the entropy becomes \( S = A/4G = A/4l_p^2 \). Another proposal always made by Bekenstein is the quantization of the area for nonextremal black holes \( a_n = a_l^2(p(n + \eta)) \eta > -1 \quad n = 1, 2, \ldots \) The area is measured by the quantity

\[ A(S_0) = \int_{S_0} d^2x \sqrt{\sigma} \tag{8} \]

We would like to evaluate the mean value of the area

\[ A(S_0) = \langle \Psi_F | \hat{A} | \Psi_F \rangle = \langle \Psi_F | \int_{S_0} d^2x \sqrt{\sigma} | \Psi_F \rangle \tag{9} \]

computed on the foam state

\[ |\Psi_F\rangle = \Psi_1^{\perp} \otimes \Psi_2^{\perp} \otimes \cdots \Psi_{N_w}^{\perp}. \tag{10} \]

Consider \( \sigma_{ab} = \bar{\sigma}_{ab} + \delta \sigma_{ab} \) such that \( \int_{S_0} d^2x \sqrt{\sigma} = 4\pi \bar{r}^2 \) and \( \bar{r} \) is the radius of \( S_0 \)

\[ A(S_0) = \langle \Psi_F | \hat{A} | \Psi_F \rangle = 4\pi \bar{r}^2 \tag{11} \]

Suppose to consider the mean value of the area \( A \) computed on a given macroscopic fixed radius \( R \). On the basis of our foam model, we obtain

\[ A = \bigcup_{i=1}^{N} A_i, \quad \text{with} \quad A_i \cap A_j = \emptyset \quad \text{when} \quad i \neq j. \]

Thus

\[ A = 4\pi R^2 = \sum_{i=1}^{N} A_i = \sum_{i=1}^{N} 4\pi \bar{r}_i^2. \tag{12} \]

When \( \bar{r}_i \rightarrow l_p, A_i \rightarrow A_{l_p} \) and \[17\]

\[ A = N A_{l_p} = N 4\pi l_p^2 \Rightarrow S = \frac{A}{4G} = \frac{N 4\pi l_p^2}{4l_p^2} = N \pi. \]

Thus the macroscopic area is represented by \( N \) microscopic areas of the Planckian size. In this sense we will claim that the area is quantized. The first consequence is the mass quantization of the Schwarzschild black hole, namely

\[ S = 4\pi M^2 G = 4\pi M^2 l_p^2 = N \pi \Rightarrow M = \frac{\sqrt{N}}{2l_p}. \tag{13} \]
To be compared with Refs. [15–22]. A second consequence is that in de Sitter space, the cosmological constant is quantized in terms of $l_p$, i.e.,

$$S = \frac{3\pi}{l_p^2\Lambda_c} = \frac{A}{4l_p^2} = \frac{N4\pi l_p^2}{4l_p^2} = N\pi = \frac{3}{l_p^2N} = \Lambda_c.$$ (14)

It is possible to give an estimate of the total amount of Planckian wormholes needed to fill the space beginning from the Planck era $\left(\Lambda \sim (10^{16} - 10^{18} GeV)^2\right)$ up to the space in which we now live $\Lambda \leq (10^{-42} GeV)^2$.

$$\frac{1}{N}10^{38} GeV^2 = 10^{-84} GeV^2 \rightarrow N = 10^{122}, \quad (15)$$

in agreement with the observational data $\Lambda_c \lesssim 10^{-122}l_p^2$ coming from the Friedmann-Robertson-Walker cosmology constraining the cosmological constant [13].

5. Problems and Conclusions

There are several examples indicating that flat space cannot be taken as the ground state of a quantum theory of gravity. In particular beginning with the result that hot flat space is unstable with respect to a nucleation of a single black hole, we have discovered that flat space is unstable with respect to a neutral black hole pair creation. The instability generated by the pair (wormhole) is stabilized if a certain large number of wormholes is considered. Unfortunately, the model depends on a cutoff. Moreover, contributions coming from the decompositions of the metric at one loop and their interactions with the graviton sector are missing. Matter fields have been completely neglected. Nevertheless there are good signals of a spacetime foam: first of all the size of the energy fluctuations $\Delta E \propto 1/L^4 \propto \Lambda^4$. Area, Entropy and the Cosmological constant are quantized as a simple consequence of the covering property of our simple wormhole model of the foam.

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