Contradictory entropic joint uncertainty relations for complementary observables in two-level systems

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We show that different entropic measures of fluctuations lead to contradictory uncertainty relations for two complementary observables. We apply Tsallis and Rényi entropies to the joint distribution emerging from a noisy simultaneous measurement of both observables as well as to the product of their individual statistics, either intrinsic or of operational origin.

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I. INTRODUCTION

Historically, the joint uncertainty of pairs of observables has been mostly addressed in terms of the product of their variances. Nevertheless, there are situations where such formulation is not satisfactory enough [1, 2] and alternative approaches have been proposed, mainly in terms of diverse entropic measures [3–5]. More specifically in this work we will consider the Tsallis and Rényi entropies [6–13].

In previous works we have shown that entropic measures of fluctuations lead to contradicting intrinsic joint uncertainty relations for complementary observables [12, 13], among other surprising results [10, 14, 15]. More specifically, the maximum uncertainty states of some measures are the minimum uncertainty states of other measures, and vice versa. By intrinsic we mean that the entropies are computed with the exact statistics or each observable.

In this work we address an operational approach where the uncertainties are derived from the statistics of a practical schemes devised to measure both observables in the same system realization. Simultaneous measurements of complementary observables cannot be exact and some extra uncertainty is unavoidable introduced [16–18]. We study whether this extra uncertainty might affect the contradictions found in the intrinsic case. Besides, the simultaneous measurement provides a true joint probability distribution that enables an alternative assessment of joint uncertainty different from the one provided by the individual statistics, either intrinsic or of operational origin. The main conclusions we find is that the contradictions between different entropic measures persist, and new contradictions emerge between the entropy of the joint distribution and the entropies of the product of the individual distributions.

For simplicity we address this issue in the simplest quantum system described by a two-dimensional Hilbert space. This comprises very relevant practical situations such as the path-interference complementarity in two-beam interference. Finally, we address further tests that might be useful to understand these contradictions.

II. SYSTEM STATES

We consider two complementary observables represented by the Pauli matrices $\sigma_z$ and $\sigma_x$. In practical terms they may represent path and phase in two-beam interference, respectively. The system state is represented by the density matrix operator in the system space $\mathcal{H}_s$,

$$\rho = \frac{1}{2} (I + s \cdot \sigma) ,$$

where $I$ is the identity, $\sigma$ are the three Pauli matrices, and $s = \text{tr}(\rho \sigma)$ is a three-dimensional real vector with $|s| \leq 1$. For simplicity we will consider that $s$ lays in the $xz$ plane so $s_y = 0$. We will use the parametrization

$$s_x = s \sin \theta, \quad s_z = s \cos \theta,$$

where $s = |s|$. When $s = 1$ the state is pure $\rho = |\psi\rangle\langle\psi|$. With

$$|\psi\rangle = \cos \frac{\theta}{2} |+\rangle + \sin \frac{\theta}{2} |-\rangle,$$

where $|\pm\rangle$ are the eigenstates of $\sigma_z$ with eigenvalue $\pm 1$. Throughout we will consider the one-parameter families $S$ of pure states $s = 1$ in Eq. (2.3). The exact intrinsic statistics for the $\sigma_x$, $\sigma_z$ observables are

$$p_x = \frac{1}{2} (1 + xs_x), \quad p_z = \frac{1}{2} (1 + zs_z),$$

with $x = \pm 1$ and $z = \pm 1$.

There are two sets of states that compete to be either the minimum uncertainty states (as well as of maximum uncertainty within the set $S$) depending on the measure of fluctuations used [12, 13]. We will refer to them as extreme and intermediate states:

(i) The extreme states are the eigenstates of $\sigma_z$ or $\sigma_x$. These are pure states in Eq. (2.3) with $\theta = m \pi/2$ for integer $m$ (these are even multiples $\pi/4$ and $s_x = \pm 1, s_z = 0$ or $s_x = 0, s_z = \pm 1$). They present full certainty for one of the observables and complete uncertainty for the other one.

(ii) The intermediate states are the eigenstates of $\sigma_x \pm \sigma_z$. These are the pure states (2.3) with $\theta = (2m + 1) \pi/4$ for integer $m$ (this is the odd multiples of $\pi/4$ and $s_x = \pm s_z = \pm 1/\sqrt{2}$). They have essentially
the same statistics for both complementary observables so they might be considered as a finite-dimensional counterpart of the Glauber coherent states.

In Appendix B we point out that for the intermediate states there can be no true probability distribution underlying the operational joint distribution (3.3) to be found below. This allows us to say that the intermediate states are literally nonclassical states regarding the statistics of the observables $\sigma_x, \sigma_z$, while all the extreme states are fully classical in this respect.

III. JOINT MEASUREMENT AND STATISTICS

The simultaneous measurement of noncommuting observables requires involving auxiliary degrees of freedom, usually referred to as apparatus. In our case we consider an apparatus described by a two-dimensional Hilbert space $H_a$. The measurement to be performed in $H_a$ addresses the measurement of $\sigma_x$, while $\sigma_x$ is measured directly on the system space $H_s$. The system-apparatus coupling transferring information about $\sigma_x$ from the system to the apparatus is arranged via the following unitary transformation acting in $H_s \otimes H_a$

$$U = V_+|+\rangle\langle+| + V_-|\rangle\langle-|,$$  

where $V_\pm$ are unitary operators acting solely on $H_a$.

For simplicity the initial state of the apparatus $|a\rangle \in H_a$ is assumed pure so that the system-apparatus coupling leads to the following state (assumed pure) $|\rangle a\rangle \in H_s \otimes H_a$

$$U|\psi\rangle|a\rangle = \cos \frac{\theta}{2}|+\rangle|a_+\rangle + \sin \frac{\theta}{2}|\rangle|a_-\rangle,$$  

where $|a_\pm\rangle = V_\pm |a\rangle \in H_a$ are not orthogonal in general, with $\cos \delta = \langle a_+|a_-\rangle$ assumed to be a positive real number without loss of generality (see Fig. 2). The measurement in $H_a$ introducing minimum uncertainty is given by projection on the orthogonal vectors $|b_\pm\rangle$ (see Appendix A)

$$|b_\pm\rangle = \frac{1}{\cos \phi} \left( \cos \frac{\phi}{2}|a_+\rangle - \sin \frac{\phi}{2}|a_-\rangle \right),$$  

where $\phi = \pi/2 - \delta$.

The joint statistics for the simultaneous measurement of $\sigma_x$ in $H_s$ and $|b_\pm\rangle$ in $H_a$ is

$$\tilde{p}_{x,z} = \frac{1}{4} (1 + zs \sin \delta + xs \cos \delta),$$  

where $x = \pm 1$ represent the outcomes of the $\sigma_x$ measurement and $z = \pm 1$ the outcomes of the $|b_\pm\rangle$ measurement. The marginal statistics for both observables are

$$\tilde{p}_x = \frac{1}{2} (1 + xs \cos \delta), \quad \tilde{p}_z = \frac{1}{2} (1 + zs \sin \delta).$$  

When contrasted with the intrinsic statistics in Eq. (2.4) we get that the observation of $\sigma_x$ is exact for $\delta = 0$ while the observation of $\sigma_z$ is exact for $\phi = \delta = \pi/2$. For $\delta = \pi/4$ the extra uncertainty introduced by the unsharp character of the simultaneous observation is balanced between observables.

IV. UNCERTAINTY ASSESSMENTS

We will focus on the Tsallis $T_q$ and Rényi $R_q$ entropies

$$T_q(p_j) = \frac{1}{1 - q} \left( \sum_{j=1}^N p_j^q - 1 \right),$$  

and

$$R_q(p_j) = \frac{1}{1 - q} \ln \left( \sum_{j=1}^N p_j^q \right),$$  

where $q > 0$ is a real parameter and $p_j$ is the statistics of some observable with $N$ outcomes. In both cases minimum entropy $T_q = R_q = 0$ holds when all the probability is concentrated in a single outcome $p_j = \delta_{j,k}$ for any $k$,
while maximum entropy occurs when all the outcomes are equally probable \( p_j = 1/N \). Both include the Shannon entropy in the limit \( q \to 1 \). Moreover, for Gaussian-like statistics \( R_q \) is proportional to variance.

The operational entropies are always larger than the intrinsic ones \( T_q(\hat{p}_k) \geq T_q(p_k) \) and \( R_q(\hat{p}_k) \geq R_q(p_k) \) for \( k = x, z \) since comparing Eqs. (3.5) and (2.4) we can appreciate that the observation amounts to be a reduction of \( |s_k| \), and both entropies are monotonic functions of \( |s_k| \). Their only extreme holds at the uniform distribution \( p_j = 1/N \) which is an absolute maximum.

In order to assess the joint uncertainty of \( \sigma_x \) and \( \sigma_z \) using these entropies we will follow two strategies:

(i) On the other hand we will compute the Tsallis and Rényi entropies of the complete joint statistics \( T_q(\hat{p}_{x,z}) \) and \( R_q(\hat{p}_{x,z}) \).

(ii) On the other hand, for completeness we will consider also the entropies for the product of individual operational statistics \( T_q(\hat{p}_x \hat{p}_z) \) and \( R_q(\hat{p}_x \hat{p}_z) \). In this regard we recall that the Rényi entropy is additive,

\[
R_q(\hat{p}_x \hat{p}_z) = R_q(\hat{p}_x) + R_q(\hat{p}_z),
\]

while in general Tsallis is not, except for \( q \to 1 \), \( \hat{p}_x \hat{p}_z = T_q(\hat{p}_x) + T_q(\hat{p}_z) + (1-q)T_q(\hat{p}_x)T_q(\hat{p}_z) \).

Both strategies will be examined and compared with the entropy of the product of intrinsic uncertainties \( T_q(p_x p_z) \) and \( R_q(p_x p_z) \) for different choices of \( q \) as functions of \( \theta \) within the set \( S \) of states \( \{2,3\} \). For the sake of clarity and a simpler comparison we mostly focus on normalized quantities of the form:

\[
T_q \to \frac{T_q - T_{q,\min}}{T_{q,\max} - T_{q,\min}},
\]

and equivalently for \( R_q \), where \( T_{q,\max}, T_{q,\min} \) are the maximum and minimum of \( T_q \), respectively, when \( \theta \) is varied.

V. JOINT UNCERTAINTY FROM TSALLIS AND RÉNYI ENTROPIES

A. Joint uncertainty from Tsallis entropies

A numerical evaluation of the Tsallis entropies \( T_q(\hat{p}_{x,z}) \) shows that there are contradictions between different \( q \) values. This is illustrated in Fig. 3a where we have represented \( T_q(\hat{p}_{x,z}) \) for \( q = 1/2 \) and \( q = 2.5 \) as functions of \( \theta \). It can be appreciated that for \( q = 1/2 \) the minimum uncertainty states are the intermediate states \( \theta = \pi/4 \) while for \( q = 2.5 \) the minimum uncertainty states are the extreme states \( \theta = 0, \pi/2 \).

The contradictions between different \( q \) values also hold for the Tsallis entropies of the product statistics \( T_q(\hat{p}_x \hat{p}_z) \), as illustrated in Fig. 3b representing the cases \( q = 1/2 \) and \( q = 2.5 \) as functions of \( \theta \). It can be appreciated that for \( q = 1/2 \) the minimum uncertainty states are the extreme states \( \theta = 0, \pi/2 \) while for \( q = 2.5 \) the minimum uncertainty states are the intermediate states \( \theta = \pi/4 \). This result coincides with the conclusions derived from the intrinsic entropies \( T_q(p_x p_z) \) as shown in Fig. 3c. Moreover, the case \( q = 1/2 \) coincides with the conclusions of the product of variances, this is that the minimum uncertainty states are the extreme states.

In Fig. 4 we have represented the extreme-intermediate competition for minimum uncertainty by plotting the difference of their Tsallis entropies or dif-

FIG. 3: (a) Plot of the Tsallis entropies for the joint statistics \( T_q(\hat{p}_{x,z}) \) for \( q = 1/2 \) (solid line) and \( q = 2.5 \) (dashed line) as functions of \( \theta \). (b) Same as before but for the Tsallis entropies of the product of operational statistics \( T_q(\hat{p}_x \hat{p}_z) \). (c) Same as before but for the Tsallis entropies of the product of intrinsic statistics \( T_q(\hat{p}_x \hat{p}_z) \). All quantities have been properly normalized to lay between 0 and 1 as in Eq. (4.5).
different values of $q$

$$
\delta T_q(\tilde{p}_{x,z}) \propto T_q(\tilde{p}_{x,z})|_{\theta=\pi/4} - T_q(\tilde{p}_{x,z})|_{\theta=0}.
$$

(5.1)

When $\delta T > 0$ the extreme states are of minimum uncertainty while when $\delta T < 0$ the intermediate states are the minimum uncertainty states. We have also included the same quantity computed for the Tsallis entropies of the product of operational $\delta T_q(\tilde{p}_{x,z})$ and intrinsic $\delta T_q(p_{x,z})$ statistics.

We can appreciate that the contradictions between different $q$ values for $T_q(\tilde{p}_{x,z})$ take place between entropies with $q \in (2, 3)$ and $q \in (0, 2) \cup (3, \infty)$. For the intrinsic case $T_q(p_{x,z})$ contradictions arise between the $q$ values $q < 1.4313$ and $q > 1.4313$, while for $T_q(\tilde{p}_{x,z})$ they hold between $q < 1.3439$ and $q > 1.3439$. Thus, observation drives the critical $q$-values with respect to the intrinsic case. Moreover, there are two $q$ intervals $q \in (1.4313, 2) \cup (3, \infty)$ free of these contradictions.

These two figures clearly shows the double kind of contradictions we have found: (i) We have different conclusions for different $q$ values, both in the intrinsic and operational frameworks. (ii) For the same $q$ we have contradictions between the entropy of the joint distribution and the entropy of the product of individual distributions, irrespectively of whether this is intrinsic or operational.

**VI. FURTHER TESTS**

In this section we compute entropic-related quantities that might serve to test the meaning of the contradictions revealed in the preceding section.

**A. Difference of entropies**

The *bona fide* joint probability distribution given by the simultaneous measurement provides us a meaningful comparison of the entropies for the joint $\tilde{p}_{x,z}$ and product $\tilde{p}_{x}\tilde{p}_{z}$ statistics that may be helpful to understand the contradicting behaviors reported above by providing new information about these quantities.

For example we can compute the differences

$$
T_q(\tilde{p}_{x,z}) - T_q(\tilde{p}_{x,z}), \quad R_q(\tilde{p}_{x,z}) - R_q(\tilde{p}_{x,z}),
$$

(6.1)

where we have that

$$
\tilde{p}_{x,z} - \tilde{p}_{x}\tilde{p}_{z} = -\frac{1}{16} x^2 s^2 \sin(2\theta) \sin(2\delta).
$$

(6.2)

The difference of the statistics is maximum for the simultaneous measurement that treat forth observables symmetrically $\delta = \pi/4$ and for intermediate states $\theta = \pi/4$.

The Tsallis entropy differences are plotted in Fig. 5a showing that for certain $q$ values we have the paradoxical result that the entropy of the joint distribution $\tilde{p}_{x,z}$ is larger than the entropy of the product of the marginal distributions $\tilde{p}_{x}\tilde{p}_{z}$. In Fig. 5b we have plotted the difference of entropies for intermediate states revealing the $q$ values where the counterintuitive result $T_q(\tilde{p}_{x,z}) < T_q(\tilde{p}_{x}\tilde{p}_{z})$ holds, this is $q > 1.60$. Note that this counterintuitive result cannot account for the contradictions reported above since the corresponding $q$ regions are different.

Here again, with the Rényi entropies we get the same result with the same critical value $q = 1.60$. Differences between $R_q(\tilde{p}_{x,z})$ and $R_q(\tilde{p}_{x}\tilde{p}_{z})$ regarding minimum uncertainty states have been also found in Ref. [17].

**B. Mutual information**

For Shannon entropy there is an equivalence between mutual information and the entropy differences (6.1). For Tsallis and Rényi entropies such equivalence no longer holds in general. Thus it may be worth investigating whether mutual information displays any unexpected behavior. To this end we have examined the following definitions of Tsallis and Rényi mutual informations

$$
I_{T,q} = \frac{1}{q-1} \left[ \sum_{x,z} \frac{\tilde{p}_{x,z}^q}{(\tilde{p}_{x,z})^{q-1}} - 1 \right],
$$

$$
I_{R,q} = \frac{1}{q-1} \ln \left[ \sum_{x,z} \frac{\tilde{p}_{x,z}^q}{\tilde{p}_{x}\tilde{p}_{z}} \right].
$$

(6.3)

We have found no contradiction nor counterintuitive behavior since we have obtained always $I_{T,q} \geq 0$ and
The most frequent test is given by Fisher information. Nevertheless, and following the same spirit of Tsallis and Rényi entropies, several

two outcomes with probabilities \( p_+ = \cos^2(\eta - \theta/2) \) and \( p_- = \sin^2(\eta - \theta/2) \).

As a further test of practical origin we may consider the performance provided by the states in Eq. (2.3) in the vicinity of the identity. Consider-
ing the measurement of \( \eta \) statistics depending on the pa-
rameter \( \eta \) of the transformation, and the final expression is to be evaluated at \( \eta = 0 \). In our case there are only two outcomes with probabilities \( p_+ = \cos^2(\eta - \theta/2) \) and \( p_- = \sin^2(\eta - \theta/2) \).

C. Generalized Fisher information

FIG. 5: (a) Difference of Tsallis entropies \( T_q(\hat{p}_x, \hat{p}_z) - T_q(\hat{\rho}_{x,z}) \) as functions of \( \theta \) for \( q = 1.4 \) (solid line) and \( q = 2.5 \) (dashed line). (b) Same difference of Tsallis entropies as function of \( q \) for intermediate states \( \theta = \pi/4 \) showing counterintuitive behavior for \( q > 1.60 \).

\[ I_{T,q}(\text{intermediate}) \geq I_{T,q}(\text{extreme}) , \]
and equivalently for Rényi entropies.

\[ I_{T,q}(\text{intermediate}) \geq I_{T,q}(\text{extreme}) , \]

D. Extension to unbounded continuous Cartesian variables

For the sake of comparison we extend the analysis to a pair of complementary unbounded Cartesian variables, such as one-dimensional position and linear momentum of a particle, or the quadratures of a single-mode electromagnetic field. Let us focus on the quadrature case with a pair of quadrature operators \( X, Y \), satisfying the commutation relation \([X, Y] = 2i\). Furthermore, let us assume pure states with Gaussian intrinsic statistics

\[ p_k = \frac{1}{\sqrt{2\pi \Delta k}} \exp \left[ -\frac{k^2}{2(\Delta k)^2} \right] , \]  \( (6.5) \)

for \( k = x, y \) where \( \Delta k \) are the corresponding variances and we have assumed without loss of generality that \( \langle k \rangle = 0 \). We focus on the set \( S \) of pure, minimum uncertainty states for the product of variances \( \Delta x \Delta y = 1 \). This set includes the analog of the intermediate states when \( \Delta x = \Delta y = 1 \), which are the Glauber classical-like coherent states. The set \( S \) also includes the extreme states as the limits \( \Delta x \to 0 \) and \( \Delta y \to 0 \) (which are non-classical arbitrarily squeezed states tending to be eigenstates of a quadrature operator).

We consider a version of Tsallis entropy adapted to continuous distributions as

\[ T_q(p_k) = \frac{1}{1 - q} \left( \int_{-\infty}^{\infty} dk \, p_k^q - 1 \right) , \]  \( (6.6) \)

for \( k = x, y \), and similarly for the Rényi entropies. It is worth noting that in this case entropies are no longer positive definite.
In the first place we can note that within the set $S$ there are no contradictions between different $q$ measures for the Tsallis and Rényi entropies applied to the product of intrinsic statistics

$$T_q(p_x p_y) = \frac{1}{1-q} \left[ \frac{(2\pi \Delta x \Delta y)^{1-q}}{q} - 1 \right], \quad (6.7)$$

that take the same value for all the states with $\Delta x \Delta y = 1$. Similar conclusion is obtained for the Rényi entropies

$$R_q(p_x p_y) = \ln(\Delta x \Delta y) + \ln(2\pi) - \frac{1}{1-q} \ln q. \quad (6.8)$$

Regarding the simultaneous joint measurement of $X$ and $Y$ we resort to standard schemes [18] leading to a joint distribution

$$\hat{p}_{x,y} = \frac{\exp \left\{ \frac{-\Delta^2}{2[1+(\Delta x)^2]} - \frac{-\Delta^2}{2[1+(\Delta y)^2]} \right\}}{2\pi \sqrt{[1+(\Delta x)^2][1+(\Delta y)^2]}}. \quad (6.9)$$

Let us note that in this case $\hat{p}_{x,y} = \hat{p}_x \hat{p}_y$ so that there is no difference between the joint statistics and its product of marginals. The operational uncertainty is now

$$T_q(\hat{p}_{x,y}) = \frac{1}{1-q} \left\{ \frac{2\pi \sqrt{(1+\Delta x^2)(1+\Delta y^2)}}{q} \right\}^{1-q} - 1. \quad (6.10)$$

For all the $q$ values that we have considered we have found no contradictions between different $q$ values within the set of states $\Delta x \Delta y = 1$, so that the coherent states $\Delta x = \Delta y = 1$ are always the minimum uncertainty states. The same conclusion is obtained using Rényi entropies. Nevertheless this does not exclude that contradictions might be found for other state families.

Nevertheless, it might be worth noting that we have found contradictions between different $q$ values when considering the sum of Tsallis entropies for both observables $\sigma_x$ and $\sigma_z$, both intrinsic and operational, i. e.,

$$T_q(p_x) + T_q(p_y) = \frac{(2\pi)^{\frac{1}{1-q}}}{\sqrt{q}(1-q)} \left[ (\Delta x)^{1-q} + (\Delta y)^{1-q} - 2 \right], \quad (6.11)$$

and

$$T_q(\hat{p}_x) + T_q(\hat{p}_y) = \frac{(2\pi)^{\frac{1}{1-q}}}{\sqrt{q}(1-q)} \left[ 1 + (\Delta x)^2 \right]^{\frac{1}{1-q}} + \left[ 1 + (\Delta y)^2 \right]^{\frac{1}{1-q}} - 2. \quad (6.12)$$

We have plotted both in Fig 7 as a function of $\Delta x$ for different values of $q$ showing that the coherent states $\Delta x = 1$ can be either maximum or minimum. The transition from maximum to minimum is dictated by the $q$ value for which vanish the second derivative the left-hand sides of Eqs. (6.11) and (6.12) with respect to $\Delta x$ evaluated at $\Delta x = 1$. In the intrinsic case this holds for $q = 1$.

![Fig. 7](image_url)  

**FIG. 7:** (a) Plot of $T_q(p_x) + T_q(p_y)$ as a function of $\Delta x$ for $q = 1/2$ (solid line) and $q = 2$ (dashed line). (b) Plot of $T_q(\hat{p}_x) + T_q(\hat{p}_y)$ as a function of $\Delta x$ for $q = 2$ (solid line) and $q = 4$ (dashed line) and normalized to their value at $\Delta x = 1$.

this is for the Shannon entropy, while in the operational case it holds for $q = 3$. Despite we are aware of the lack of additivity of Tsallis entropies we find this result still striking. To some extent one might regard the difference between $T_q(p_j p_k)$ and $T_q(p_j) + T_q(p_k)$ as merely quantitative, but the qualitative difference pointed out by the above example might add valuable arguments to the discussion of whether $T_q(p_j)$ is a meaningful assessment of quantum fluctuations.

**VII. DISCUSSION**

We have presented several examples of application of Tsallis and Rényi entropies as potential measures of quantum uncertainty. To this end we are considering situations where quantum features are relevant, such as uncertainty relations and metrology. We hope that the contradicting cases presented in this work might proper grounds for the definition of suitable measures of quantum uncertainty beyond variance. Besides, they might provide different interpretations of the general idea of quantum uncertainty and provide different measures adapted to these new situations.

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Fig. 8: Illustration of the relation between the states $|b_{\pm}\rangle$ and $|a_{\pm}\rangle$.

Appendix A: Apparatus measurement introducing minimum uncertainty

Here we examine the optimum measuring strategy introducing a minimum additional noise in the joint measurement. On the one hand, the operational statistics $\tilde{p}_x$ does not depend on the measurement to be performed on the apparatus. This is because it corresponds to the measurement of $\sigma_x$ in the reduced state obtained after tracing over the apparatus degrees of freedom. On the other hand, the operational statistics $\tilde{p}_z$ can be described by projection on two orthogonal states $|b_{\pm}\rangle$ in the subspace spanned by $|a_{\pm}\rangle$ on the apparatus space $\mathcal{H}_a$, as illustrated in Fig. 8.

The objective is to look for the choice of angle $\varphi$ leading to minimum uncertainty. The observed $\tilde{p}_z$ and intrinsic $p_z$ probabilities are related in the form

$$\begin{pmatrix} \tilde{p}_+ \\ \tilde{p}_- \end{pmatrix} = M \begin{pmatrix} p_+ \\ p_- \end{pmatrix}, \quad M = \begin{pmatrix} \cos^2 \varphi & \cos^2 (\varphi + \delta) \\ \sin^2 \varphi & \sin^2 (\varphi + \delta) \end{pmatrix}. \quad (A1)$$

The condition of minimum uncertainty can be implemented as the minimum distance between $\tilde{p}_z$ and $p_z$, this is the minimum distance between $M$ and the identity matrix $I$. For example we may look for the minimum of

$$\text{tr} \left[ (M - I)^2 \right] = [\sin^2 \varphi + \cos^2 (\varphi + \delta)]^2, \quad (A2)$$

which is obtained for $2\varphi = \pi/2 - \delta$.

Appendix B: Nonclassicality after joint statistics

Since the measurement scheme is known it is possible to formally invert the process to obtain from the joint unsharp statistics $\tilde{p}_{x,z}$ in Eq. (3.3) a sharp one $p_{x,z}$ providing the correct marginals [10].

Regarding the intrinsic $p_k$ and operational $\tilde{p}_k$ statistics as two-dimensional column vectors, from Eq. (3.5) we have $\tilde{p}_k = M_k p_k$ with

$$M_k = \begin{pmatrix} \cos^2 \phi_k & \sin^2 \phi_k \\ \sin^2 \phi_k & \cos^2 \phi_k \end{pmatrix}, \quad (B1)$$

with $k = x, z$, being $\phi_x = \delta/2$, and $\phi_z = \pi/4 - \delta/2$.

The relation $\tilde{p}_k = M_k p_k$ can be inverted to obtain the intrinsic "true" or noiseless distribution in terms of the operational one $p_k = M_k^{-1} \tilde{p}_k$. Thus, regarding the operational joint distribution $p_{x,z}$ as a $2 \times 2$ matrix we may infer a "true" or "noiseless" joint distribution $p_{x,z}$ as

$$p_{x,z} = M_x^{-1} \tilde{p}_{x,z} M_z^{-1} = \frac{1}{4} (1 + z s_z + x s_x), \quad (B2)$$

where the superscript $t$ denotes matrix transposition. This distribution $p_{x,z}$ provides the true intrinsic distributions $p_x$ and $p_z$ as its proper marginals.

Quantum mechanics in this case is reflected in the fact that for some states $p_{x,z}$ takes negative values. These states can be termed nonclassical. Concerning our problem we note that the extreme states are classical $p_{x,z} \geq 0$ (say for example $p_{x,z} = (1 + z)/4$ for $s_z = 0$, and $s_z = 1$), while the intermediate states are all nonclassical $p_{x,z} < 0$, since for example for $s_x = s_z = 1/\sqrt{2}$ we get $p_{-1,-1} = (1 - \sqrt{2})/4 < 0$, and similarly for all the other intermediate states. Let us note that this is precisely the opposite behavior we find for the unbounded Cartesian variables, where the classical states are the intermediate states (the Glauber coherent states), while the nonclassical states are the extreme states (limit of arbitrarily squeezed states).

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