COLORING THE 600 CELL

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ABSTRACT. The 600 cell S has exactly 10 5-colorings. From these colorings we can construct the space of colorings $B(S)$. This complex has 1344 colorings, and is isomorphic to the space of 5 by 5 Latin Squares. These simplices split into 4 copies of a quotient of $S$ by an involution, and two copies of a space made up of even Latin Squares.

Introduction. The five-colorings of the 600 cell have a surprising structure. In order to define this structure, we recall [2] the definition of the space $B(X)$ of five-colorings of a 4-complex $X$:

Consider a five-coloring of $X$ to be a map from $X$ to the colors $\{1, 2, 3, 4, 5\}$. The vertices of $B(X)$ are all the distinct sets $f^{-1}(v)$, where $v$ is a color, and $f$ is a coloring. Every five-coloring $f$ determines a 4-simplex $\{f^{-1}(1), \ldots, f^{-1}(5)\}$, and all 4-simplices of $B(X)$ are of this form.

Since $B(X)$ is a 4-complex, we can once again compute the space $B^2(X)$ of its five-colorings. Recall the definition of the map $\varphi : X \to B^2(X)$:

If $p$ is a vertex of $X$, then $\varphi(p)$ consists of all vertices $f^{-1}(v)$ such that $p \in f^{-1}(v)$.

This is a well-defined simplicial map that sends 4-simplices to 4-simplices. $\varphi$ is sometimes an isomorphism, but usually is neither 1-1 nor onto.

The 600 cell. The 600 cell $S$ is the triangulation of the 3-sphere that has 120 vertices, 600 tetrahedra and the link of every vertex is an icosahedron. $S$ can be realized in $\mathbb{R}^4$ in such a way that all tetrahedra are regular [1].

$S$ has 10 five-colorings. It is very unusual to be able to determine all the five-colorings of a graph with 120 vertices. We are successful because there is an inductive way of constructing $S$, and five-colorings extend uniquely from one stage to the next. First, pick any vertex $v$, and fix a five-coloring $f$ of the star $S_0$ of $v$. $S_0$ is $v$ joined to an icosahedron. If $S_1$ is $S_0$ along with all tetrahedra that meet $S_0$, then $f$ has a unique extension to $S_1$. Next, if $S_2$ is $S_1$ along with all tetrahedra meeting $S_1$, then $f$ has a unique extension to $S_2$. Continuing, $f$ has a unique extension to successive shells, and $f$ extends uniquely to $S$. These assertions about unique extensions are the result of computer calculations.

Since the icosahedron has exactly 10 colorings, $S$ does as well.

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**B(S) has 25 vertices.** To find the vertices of B(S) we need an algebraic description of S and its colorings. S can be realized as the convex hull of the 120 points in \( \mathbb{R}^4 \) that lie in the quaternionic icosahedral group \( \mathcal{I} \). Let \( T \) be the subgroup of \( \mathcal{I} \) whose 24 vertices are the vertices of a 24 cell embedded in S. If \( p \) is an element of order 5 in \( \mathcal{I} \), then the five cosets \( Tp^k \) (\( k = 0,1,2,3,4 \)) are all disjoint. Similarly, the five cosets \( p^kT \) (\( k=0,1,2,3,4 \)) are all disjoint. Since each coset has 24 elements, \( 5 \cdot 24 = 120 \), and no two of the elements of \( T \) are adjacent in S, it follows that we have 2 colorings of S, namely \( \{ Tp^k, k = 1, \ldots , 5 \} \) and \( \{ p^kT, k = 1, \ldots , 5 \} \). We can construct 8 additional colorings by multiplying each of these colorings by \( p^1, p^2, p^3, \) or \( p^4 \) on the appropriate side.

Since we established computationally that there are exactly 10 colorings, it follows that these are all the colorings. Furthermore, we can identify the set of points of one color in these colorings with pairs \( (i, j) \) representing the double coset \( p^iTp^j \). A set of five pairs \( (i, j) \) is a coloring if they all have the same first coordinate or the same second, so B(S) is isomorphic to \( \Delta^4 \# \Delta^4 \), the Cartesian product of two 4-simplices [2]. This means that we can represent B(S) by the following diagram, where the vertices are the “.”’s, and the “.”’s in a row (or column) are the 4-simplices.

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**The 4-simplices of B^2(S) are Latin Squares.** A five coloring of B(S) is an assignment of one of the integers 1, 2, 3, 4, 5 to each “.” of the figure such that each row and each column has all distinct entries. Such labelings are exactly Latin Squares. It is easy to compute that there are 1344 5 by 5 Latin Squares, so B^2(S) has 1344 4-simplices.

**The vertices of B^2(S) are permutations.** A vertex of B^2(S) is the set of “.” that have the same label in a Latin Square. Such a set has one “.” in each row and in each column. These sets are permutations of \{1, 2, 3, 4, 5\}, so there are 120 vertices.

\[ B^3(S) = B(S) \]. In [2], we showed that \( S_5 \), the space of 5 by 5 Latin Squares, satisfies \( B^2(S_5) = S_5 \). Since we observed that \( B^2(S) = S_5 \), the result follows.

**The map \( \varphi : S \rightarrow B^2(S) \) is not 1-1.** Both S and T are fixed by the involution sending x to -x. Consequently, whenever a vertex v of S is in some 24-cell, -v is in the same 24-cell. It follows that \( \varphi(v) = \varphi(-v) \), and that \( \varphi \) is not 1-1 on S. Computationally (or by using the quaternionic representation), we find that -v is the only vertex with the same image as v, so \( \varphi \) is actually two to one on S.
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The quotient of $S$. $S$ has a fixed point free involution given by sending $x$ to $-x$, where we think of $S$ as given by quaternions. This quotient $P$ has 60 vertices, 300 cells, and is a triangulation of projective 3-space such that the link of every vertex is an icosahedron. We can consider $P$ as a higher dimensional analog of $K_5$ imbedded in the projective plane, since that triangulation is the quotient of the icosahedron by its involution. The dual of $P$ is an analog of the Peterson graph.

$B(S) = B(P)$. The existence of the covering map $S \longrightarrow P$ shows that each coloring of $P$ determines a coloring of $S$. The vertices of $B(S)$ are given by embedded 24-cells, and the 24-cells are fixed under the involution, so it follows that every vertex of $B(S)$ determines a unique vertex of $B(P)$. This gives the desired isomorphism.

The map $P \longrightarrow B^2(S)$ is 1-1. We saw above that the only distinct points $v, w$ satisfying $\varphi(v) = \varphi(w)$ were $v = -w$. This shows that the map from $P$ to $B^2(S)$ is 1-1. In other words,

There is a set of 300 5 by 5 Latin Squares whose structure is isomorphic to $P$.

Labeling vertices of $S$ with permutations. The vertices of $S_5$ can be identified with the permutations on $\{1, 2, 3, 4, 5\}$. Using the map $S \longrightarrow B^2(S) = S_5$, the vertices of $S$ can also be so identified. Computationally, we found that the 120 vertices determined 60 permutations, all the even ones. $v$ and $-v$ determine the same permutation.

$B^2(S)$ contains 4 disjoint copies of $P$. Since the vertices of $B^2(S)$ are permutations, there is a map given by $\sigma \longrightarrow \sigma^{-1}$ that is an automorphism $\eta$ of the simplicial structure. The two embeddings of $P$, $\varphi(P)$ and $\eta(\varphi(P))$, have no 4-simplices in common. If $\tau$ is any odd permutation, then $\tau \cdot \varphi(p)$ and $\tau \cdot \eta(\varphi(p))$ give two more copies of $P$, whose vertex set is the set of odd permutations.

The structure of the remainder. Once we have removed the 1200 4-simplices from the 1344 4-simplices of $S_5$ that are in the four copies of $P$, we are left with 144 4-simplices. These also have an amazing structure. They break into two isomorphic complexes with 60 vertices and 72 4-simplices, each isomorphic to the space of even permutations $Alt_5$. The vertices of this complex are all even permutations of $1, 2, 3, 4, 5$. Five permutations form a 4-simplex if it is possible to make a Latin Square with them so that all rows and columns are even permutations.

$B^2(S)$ is regular. Not only is it regular, but the link of a point is quite nice. If $I$ is the icosahedron,

$$\text{Link}(S_5, p) \approx B^2(I).$$
REFERENCES

[1] Patrick du Val, *Homographies, Quaternions, and Rotations*, Oxford University Press, 1964.

[2] Steve Fisk, *Coloring Theories*, Contemporary Mathematics, vol. 103, American Mathematical Society, 1989.