A HYBRID-MIXED FINITE ELEMENT METHOD FOR SINGLE-PHASE DARCY FLOW IN FRACTURED POROUS MEDIA

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Abstract. We present a hybrid-mixed finite element method for a novel hybrid-dimensional model of single-phase Darcy flow in a fractured porous media. In this model, the fracture is treated as an \((d - 1)\)-dimensional interface within the \(d\)-dimensional fractured porous domain, for \(d = 2, 3\). Two classes of fracture are distinguished based on the permeability magnitude ratio between the fracture and its surrounding medium: when the permeability in the fracture is (significantly) larger than in its surrounding medium, it is considered as a conductive fracture; when the permeability in the fracture is (significantly) smaller than in its surrounding medium, it is considered as a blocking fracture. The conductive fractures are treated using the classical hybrid-dimensional approach of the interface model where pressure is assumed to be continuous across the fracture interfaces, while the blocking fractures are treated using the recent Dirac-\(\delta\) function approach where normal component of Darcy velocity is assumed to be continuous across the interface. Due to the use of Dirac-\(\delta\) function approach for the blocking fractures, our numerical scheme allows for nonconforming meshes with respect to the blocking fractures. This is the major novelty of our model and numerical discretization. Moreover, our numerical scheme produces locally conservative velocity approximations and leads to a symmetric positive definite linear system involving pressure degrees of freedom on the mesh skeleton only. The performance of the proposed method is demonstrated by various benchmark test cases in both two- and three-dimensions. Numerical results indicate that the proposed scheme is highly competitive with existing methods in the literature.

1. Introduction

Numerical simulations of single- and multi-phase flows in porous media have many applications in contaminant transportation, oil recovery and underground radioactive waste deposit. Due to the highly conductive and blocking fractures in the porous media underground, it is still challenging to construct accurate numerical approximations \([1,2,3]\).

There are several commonly used mathematical models for simulating flows in porous media with conductive fractures, such as the dual porosity model \([4,5,6]\), single porosity model \([7]\), traditional discrete fracture model (DFM) \([8,9,10,11,12,13,14]\), embedded DFM (EDFM) \([15,16,17,18,19,20,21]\), the interface models \([22,23,24,25]\) and extended finite element DFM (XDFM) based on the interface models \([26,27,28,29,30]\), finite element method based on Lagrange multipliers \([31,32,33]\), etc. Among the above methods, the traditional DFM and the interface models have been intensively studied in the past decades.

The DFM is based on the principle of superposition. It uses a hybrid dimensional representation of the Darcy’s law, and treats the fractures as lower dimensional entries, with the thickness of the fracture as the dimensional homogeneity factor. The first DFM was introduced by Noorishad and Mehran \([8]\) in 1982 for single phase flows. Later, Baca et al. \([9]\) considered the heat and solute transport in fractured media. Subsequently, several significant numerical methods were applied to the DFM, such as the finite element methods \([10,11,12,13,14]\), vertex-centered finite volume methods \([34,35,36,37]\), cell-centered finite volume methods \([38,39,40,41,42]\), mixed finite

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element methods \[43, 44, 45, 46, 47, 48, 49, 50\], discontinuous Galerkin methods \[51\]. All the above works are limited on conforming meshes, i.e. the fractures are aligned with the interfaces of the background matrix cells. Therefore, it may suffer from low quality cells. Recently, Xu and Yang introduced the line Dirac-$\delta$ functions \[52\] to represent the conductive fractures and reinterpreted the DFM (RDFM) on nonconforming meshes. The basic idea is to superpose the conductivity of the fracture to that of the matrix. The main contribution in \[52\] is to explicitly represent the DFM introduced in \[12\] as a scalar partial differential equation. Therefore, with suitable numerical discretizations, such as the discontinuous Galerkin method, the RDFM can be applied to arbitrary meshes. To demonstrate that the RDFM is exactly the traditional DFM if the mesh is conforming, in \[52\] only finite element methods were considered. Therefore, local mass conservation was missing. Later, the enriched Galerkin and interior penalty discontinuous Galerkin methods were applied to RDFM in \[53\] and the contaminant transportation was also simulated.

Different from the traditional DFM, the interface model \[22, 23, 24, 25\] explicitly represent the fractures as interfaces of the porous media. Then the governing equation of the flow in the lower dimensional fracture was constructed. In the interface model, the matrix and fractures are considered as two systems, and the communication between them was given as the jump the normal velocity along the fractures. Therefore, different from RDFM, the interface model, though hanging nodes are allowable, cannot be applied to structured meshes and the fracture must be aligned with the interfaces of the meshes for the matrix. To fixed this limitation, the XDFM was proposed \[26, 27, 28, 29, 30\]. However, these methods may increase the degrees of freedom (DOFs) significantly, and can hardly be applied to fracture networks with high geometrical complexity \[54\].

As an alternative, the CutFEM \[55\] can be applied to non-conforming meshes. It couples the fluid flow in all lower dimensional manifolds. However, this method requires the fractures to cut the domain into completely disjoint subdomains, thus it is not applicable for media with complicated fractures.

Most of the above ideas work for problems with conductive fractures. However, if the media contains blocking fractures, most methods may not be suitable. To fix this gap, the projection-based EDFM (pEDFM) was introduced in \[18, 56\]. The effective flow area between adjacent matrix grids is computed as the difference between the original interface area and the projected area of the fracture segment. It will be zero if the fracture fully penetrates through the matrix cell. Olorode et al. \[57\] extended the pEDFM into three-dimensional compositional simulation of fractured reservoirs. However, the pEDFM still cannot describe the complex multiphase flow behavior in the matrix blocks within barrier fractures. Another approach is to follow the interface model introduced in \[58, 59, 60, 61\]. However, as demonstrated above, the interface model can only handle hanging nodes, and the fractures must align with the interfaces of the background mesh. Recently, Xu and Yang extended the RDFM \[52, 53\] to problems with blocking fractures in \[62\]. The basic idea is to apply Ohm’s law and superpose the resistance (the reciprocal of the permeability) of the blocking fracture to that of the matrix. Then a modified partial differential equation system was introduced and the local discontinuous Galerkin methods with suitable penalty were perfectly applied. If the problems contains only blocking fractures, the mixed finite element methods can easily be combined with RDFM.

In this paper, we combine the ideas in \[22\] and \[62\] to propose a novel model for single phase flows with both conductive and blocking fractures. In particular, the conductive fractures are modeled by using the interface model \[22\] where pressure continuity is enforced across the conductive fractures, and the blocking fractures are modelled as resistance terms involving Dirac-$\delta$ functions following the main idea in \[62\]. The separate treatment of conductive and blocking fractures, and the seamless combination of the conductive fracture interface model and the blocking fracture Dirac-$\delta$ function approach is the major novelty of our proposed model. We further discretize this new model using a hybrid-mixed finite element method, which produces locally conservative velocity approximations and leads to a symmetric positive definite linear system with globally coupled degrees of freedom.
(DOFs) only those of pressure on the mesh skeletons. Moreover, due to the use of Dirac-$\delta$ function approach for blocking fractures, the method does not require any mesh conformity with respect to the blocking fractures, which is the major novelty of our proposed scheme. We believe our approach is the simplest non-conforming mesh approach to blocking fractures that still yield locally conservative velocity approximations. We note that mesh conformity with respect to the conductive fractures is still required for our method, which is typical for interface models. We numerically demonstrate that our hybrid-mixed finite element scheme is highly competitive both in terms of computational efficiency and accuracy. We finally emphasize that the proposed hybrid-mixed formulation is different from the mixed method in [60] due to the use of different model for the interface conditions. We believe that our model is significantly simpler for complex fracture networks since we only use one matrix domain and one (codimension 1) conductive fracture domain throughout, while the mixed method formulation [60] needs to split the matrix and fracture domains into multiple disjoint sub-domains and require the modeling of codimension 1-3 fracture flows, which might be very tedious to perform for complex fracture networks.

The rest of the paper is organized as follows. In Section 2 we present the hybrid-dimensional model under consideration. We then formulate in Section 3 the hybrid-mixed finite element discretization of the model proposed in Section 2. Numerical results for various benchmark test cases are presented in Section 4. We conclude in Section 5.

2. The hybrid-dimensional model

2.1. Notation. We consider a bounded open domain $\Omega_m \subset \mathbb{R}^d$, $d = 2, 3$, which contains several $(d-1)$-dimensional conductive or blocking fractures. For simplicity, the fractures are assumed to be hyperplanes with smooth boundaries. We denote $\Omega_c$ as the $(d-1)$-dimensional open set containing all the conductive fractures, and $\Omega_b$ as the set containing all the blocking fractures. Assume the $(d-1)$-dimensional domain boundary $\partial \Omega_m = \Gamma_D \cup \Gamma_N$, with $\Gamma_D \cap \Gamma_N = \emptyset$. Furthermore, we denote the following sets of $(d-2)$-dimensional boundaries (intersections) associated with the set of conductive fractures $\Omega_c$:

- $\Gamma_{cc}$ is the set containing the intersections among conductive fractures.
- $\Gamma_{cb}$ is the set containing the intersections between 2 conductive and blocking fractures.
- $\Gamma_{cm}$ is the set containing the intersections between conductive fractures and domain boundary $\partial \Omega_m$, which is further split to $\Gamma_{cm} = \Gamma_{cm}^N \cup \Gamma_{cm}^D$, with $\Gamma_{cm}^N \in \Gamma_N$ and $\Gamma_{cm}^D \in \Gamma_D$.
- $\Gamma_{ci}$ is the boundary of $\Omega_c$ that does not intersect with the domain boundary $\partial \Omega_m$.

We set $\Gamma_c = \Gamma_{cc} \cup \Gamma_{cb} \cup \Gamma_{cm} \cup \Gamma_{ci}$ as the collections of all intersections of $\Omega_c$. An illustration of a typical hybrid-dimensional domain in two-dimensions is given in Figure 1.

We denote $\mathbf{n}_\Gamma$ as a uniquely oriented unit normal vector on a $(d-1)$-dimensional interface/boundary $\Gamma$, and denote $\eta_\Gamma$ as the in-plane unit (outer) normal vector on the $(d-2)$ dimensional boundary $\partial \Gamma$ of $\Gamma$, see Figure 2.

Let $\epsilon$ be the thickness of the fractures, which is assumed to be a small positive constant for simplicity. Let $K_m$ be the permeability tensor of the domain excluding the fractures $\Omega_m \setminus \{\Omega_c \cup \Omega_b\}$, $K_b \ll K_m$ be the (scalar) permeability in the normal direction of blocking fractures $\Omega_b$, and $K_c \gg K_m$ be the permeability tensor in the tangential direction of the conductive fractures $\Omega_c$.

2.2. The hybrid-dimensional flow model. The following hybrid-dimensional model is a combination of the conductive fracture treatment in [22] and blocking fracture treatment in [62]. In the bulk domain $\Omega_m \setminus \Omega_c$ excluding conductive fractures, we use the following barrier model:

\[
(K_m^{-1} + \frac{\epsilon}{K_b} \delta_{\Omega_b} n_{\Omega_b} \otimes n_{\Omega_b}) \mathbf{u} = -\nabla p, \quad \text{in} \quad \Omega_m \setminus \Omega_c, \tag{1a}
\]
\[
\nabla \cdot \mathbf{u} = f, \quad \text{in} \quad \Omega_m \setminus \Omega_c, \tag{1b}
\]
where $\mathbf{u}$ is the Darcy velocity, $p$ is the pressure, $f$ is the volume source term, $\delta_{\Omega_b}$ is the Dirac-$\delta$ function that takes values $\infty$ on the blocking fractures $\Omega_b$ and zero elsewhere, and $\mathbf{n}_{\Omega_b}$ is the unit normal vector on $\Omega_b$. Within the conductive fractures excluding intersections $\Omega_c \setminus \Gamma_c$, we use the following $(d - 1)$-dimensional Darcy’s law:

\begin{align}
(\epsilon K)^{-1} \mathbf{u}_c &= - \nabla_G p_c, \quad \text{in } \Omega_c \setminus \Gamma_c, \tag{1c} \\
\nabla_G \cdot \mathbf{u}_c &= [\mathbf{u}], \quad \text{in } \Omega_c \setminus \Gamma_c, \tag{1d}
\end{align}

where $\mathbf{u}_c$ is the (tangential) Darcy velocity in the conductive fractures, $p_c$ is the associated pressure, and the velocity jump $[\mathbf{u}] = (\mathbf{u}^+ - \mathbf{u}^-) \cdot \mathbf{n}_G$ represents the mass exchange between the conductive fractures and the surrounding media, where $\mathbf{u}^\pm(x) = \lim_{\tau \to 0^\pm} \mathbf{u}(x - \tau \mathbf{n}_G)$ for all $x \in \Omega_c$ is the bulk Darcy velocity evaluated on one side of the conductive fractures. Moreover, $\nabla_G$ and $\nabla_G \cdot$ are the usual surface gradient and surface divergence operators. The above hybrid-dimensional system
is closed with the following set of boundary/interface conditions:

\[ p = p_D, \quad \text{on } \Gamma_D, \quad (1e)\]
\[ \mathbf{u} \cdot \mathbf{n} = q_N, \quad \text{on } \Gamma_N, \quad (1f)\]
\[ p = p_c, \quad \text{on } \Omega_c, \quad (1g)\]
\[ [\mathbf{u}_c] = 0, \quad \text{on } \Gamma_{cc}, \quad (1h)\]
\[ p_c = p_D, \quad \text{on } \Gamma_{cm}, \quad (1i)\]
\[ \mathbf{u}_c \cdot \eta_\Gamma = 0, \quad \text{on } \Gamma_{cb} \cup \Gamma_{Ncm} \cup \Gamma_{ci}, \quad (1j)\]

where \([1g]\) ensures continuity of bulk pressure across conductive fractures, the no-flow boundary condition in \([1j]\) is imposed on the intersections \(\Gamma_{cb}, \Gamma_{Ncm}\), and \(\Gamma_{ci}\), and the jump term in \([1h]\) is

\[ [\mathbf{u}_c]_e := \sum_{\Gamma \subset \Omega_c \setminus \Gamma_E, \; e \in \Gamma} \mathbf{u}_c|_{\Gamma} \cdot \eta_\Gamma, \quad \forall e \in \Gamma_{cc}, \]

which represents mass conservation along intersections \(\Gamma_{cc}\). Note in particular that each conductive fracture containing the intersection \(e\) appears exactly twice in the above summation, and the in-plane normal velocity on the fracture is allowed to be discontinuous along the intersection \(e\). For example, the jump \([\mathbf{u}_c]|_H\) at node \(H\) in the configuration in Figure 1 is

\[ [\mathbf{u}_c]|_H := \sum_{\Gamma \in \{\{EH\},\{HK\},\{GH\},\{HJ\}\}} \mathbf{u}_c|_{\Gamma} \cdot \eta_\Gamma. \]

We note that in the above model \([1]\), the flow in the tangential direction in the blocking fractures is completely ignored as the permeability therein is much smaller than that of the surroundings, on the other hand, the flow in the normal direction is ignored on conductive fractures by the pressure continuity condition \([1g]\) since the permeability is much larger than that of the surroundings and the fluid has a tendency to flow along the tangential direction therein.

### 2.3. The hybrid-dimensional transport model.

We now consider a scalar quantity \(c\) that is transported through the porous medium subject to the velocity fields in the flow model \([1]\). Here \(c\) usually represents the concentration of a generic passive tracer. Similar to the flow treatment in the previous subsection, transport inside the blocking fractures is ignored. The concentrations \(c\) in the matrix and \(c_c\) in the conductive fractures are governed by the following advection equations, see e.g. \([63, 64, 25]\),

\[ \phi_m \frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u}c) = cf, \quad \text{in } \Omega_m \setminus \Omega_c \times (0,T), \quad (2a) \]
\[ \epsilon \phi_c \frac{\partial c_c}{\partial t} + \nabla_\Gamma \cdot (\mathbf{u}_c c_c) - [c\mathbf{u}] = 0, \quad \text{in } \Omega_c \times (0,T), \quad (2b) \]

with the following initial, interface, and boundary conditions

\[ c = c_c \quad \text{on } \Omega_c \times (0,T), \quad (2c)\]
\[ c = c_0 \quad \text{on } \Omega \times 0, \quad c_c = c_c,0 \quad \text{on } \Omega_c \times 0, \quad (2d)\]
\[ c = c_B \quad \text{on } \partial \Omega_{in} \times (0,T), \quad c_c = c_c,B \quad \text{on } \Gamma_{in} \times (0,T), \quad (2e)\]

where \(\{\phi_m, c_0, c_B, \partial \Omega_{in}\}\) and \(\{\phi_c, c_c,0, c_c,B, \Gamma_{in}\}\) represent the \{porosity, initial concentration, inflow concentration, and inflow boundary\} in the matrix and conductive fractures, respectively. Observe that concentration continuity \([2c]\) across the conductive fractures are enforced in the model \([2]\).
3. The hybrid-mixed finite element method

3.1. Preliminaries. Let $\mathcal{T}_h := \{K\}$ be a conforming simplicial triangulation of the domain $\Omega_m$. Let $\mathcal{E}_h$ be the collections of $(d-1)$-dimensional facets (edges for $d = 2$, faces for $d = 3$) of $\Omega_m$. Assume the mesh is fully fitted with respect to the conductive fractures, i.e., $\mathcal{T}_h^c := \Omega_c \cap \mathcal{E}_h$ is a $(d-1)$-dimensional simplicial triangulation of the domain $\Omega_c$. Here the mesh $\mathcal{T}_h$ is allowed to be un fitted with respect to the blocking fractures. Moreover, we denote $\mathcal{E}^c_h$ as the collection of $(d-2)$-dimensional facets of $\mathcal{T}_h^c$ (vertices for $d = 2$, edges for $d = 3$).

We use the lowest-order hybrid-mixed finite element methods to discretize the model (1). The following finite element spaces will be needed:

$$
V_h := \{ v \in [L^2(\mathcal{T}_h)]^d : v|_K \in RT_0(K), \quad \forall K \in \mathcal{T}_h \},
$$

$$
W_h := \{ w \in L^2(\mathcal{T}_h) : w|_K \in P_0(K), \quad \forall K \in \mathcal{T}_h \},
$$

$$
M_h := \{ \mu \in L^2(\mathcal{E}_h) : \mu|_F \in P_0(F), \quad \forall F \in \mathcal{E}_h \},
$$

$$
V_h^c := \{ v_c \in [L^2(\mathcal{T}_h^c)]^d : v|_F \in RT_0(F), \quad \forall F \in \mathcal{T}_h^c \},
$$

$$
M_h^c := \{ \mu \in L^2(\mathcal{E}_h^c) : \mu|_E \in P_0(E), \quad \forall E \in \mathcal{E}_h^c \},
$$

where $RT_0(S)$ is the Raviart-Thomas space of lowest order on a simplex $S$, and $P_0(S)$ is the space of constants.

We denote the following inner products:

$$
(\phi, \psi)_T := \sum_{K \in \mathcal{T}_h} \int_K \phi \psi \, dx, \quad \langle \phi, \psi \rangle_{\partial T} := \sum_{K \in \mathcal{T}_h} \int_{\partial K} \phi \psi \, ds,
$$

$$
(\phi, \psi)_F := \sum_{F \in \mathcal{T}_h^c} \int_F \phi \psi \, ds, \quad [\phi, \psi]_{\partial F} := \sum_{F \in \mathcal{T}_h^c} \int_{\partial F} \phi \psi \, dr,
$$

where $dx$ is for $d$-dimensional integration, $ds$ is for $(d-1)$-dimensional integration, and $dr$ is for $(d-2)$-dimensional integration. When $d = 2$, $\int_{\partial F} \phi \psi \, dr$ is simply the sum of point evaluations at the two end points of a line segment $F$.

3.2. The hybrid-mixed method for the flow model. The hybrid-mixed method for the hybrid-dimensional model (1) is given as follows: Find $(u_h, p_h, \tilde{p}_h, u_h^c, \tilde{u}_h^c) \in V_h \times W_h \times M_h \times V_h^c \times M_h^c$ with $\tilde{p}_h|_{\Gamma_D} = P_0(\Gamma_D)$ and $\tilde{u}_h^c|_{\Gamma_D} = P_0(\Gamma_D)$, where $P_0$ denotes the projection onto piecewise constants, such that

$$
(K_m^{-1} u_h, v_h)_{\mathcal{T}_h} + \int_{\Omega_h} \frac{\epsilon}{K_b} (u_h \cdot \nabla v_h) \, dx - \langle p_h, \nabla \cdot v_h \rangle_{\mathcal{T}_h} + \langle \tilde{p}_h, v_h \cdot n \rangle_{\partial \mathcal{T}_h} = 0, \quad (4a)
$$

$$
\langle \nabla \cdot u_h, q_h \rangle_{\mathcal{T}_h} - \langle f, q_h \rangle_{\mathcal{T}_h} = 0, \quad (4b)
$$

$$
-\langle u_h \cdot n, \tilde{q}_h \rangle_{\partial \mathcal{T}_h} + \langle \nabla \cdot u_h^c, \tilde{q}_h \rangle_{\mathcal{T}_h} + \int_{\Gamma_N} q_N \tilde{q}_h \, ds = 0, \quad (4c)
$$

$$
\langle (\epsilon K_c)^{-1} u_h^c, v_h^c \rangle_{\mathcal{T}_h} - \langle \tilde{p}_h, \nabla \cdot v_h^c \rangle_{\mathcal{T}_h} + \langle \tilde{p}_h^c, v_h^c \cdot \eta \rangle_{\partial \mathcal{T}_h} + \int_{\Gamma_{cb}} \alpha (\epsilon K_c)^{-1} (u_h^c \cdot \eta) (u_h^c \cdot \eta) \, dr = 0, \quad (4d)
$$

$$
-\langle u_h^c \cdot \eta, \tilde{q}_h \rangle_{\partial \mathcal{T}_c} = 0, \quad (4e)
$$

for all $(v_h, q_h, \tilde{q}_h, v_h^c, \tilde{v}_h^c) \in V_h \times W_h \times M_h \times V_h^c \times M_h^c$ with $\tilde{q}_h|_{\Gamma_D} = \tilde{v}_h^c|_{\Gamma_D} = 0$, where $\alpha > 0$ is a penalty parameter for the implementation of the no-flow boundary condition (1) on $\Gamma_{cb}$. In our numerical implementation, we take $\alpha = 10^6$.

We show that the scheme (4) is formally consistent with the hybrid-dimensional model (1):
(1) Equation (4a) is a discretization of the Darcy’s law \((1a)\) in the bulk using integration-by-parts and the following property of Dirac-\(\delta\) function:
\[
\int_{\Omega_m} \delta_{\Omega_h} \phi dx = \int_{\Omega_h} \phi \, ds.
\]

(2) Equation (4b) is the discretization of mass conservation \((1b)\) in the bulk.

(3) Equation (4c) simultaneously enforces (i) the continuity of normal velocity \(u_h \cdot n\) across interior element boundaries \(E_h \setminus (T_h^c \cup \Gamma_N)\), (ii) the boundary condition \((1d)\) on \(\Gamma_N\), and (iii) mass conservation \((1d)\) within the conductive fractures in \(T_h^c\).

(4) Equation (4d) is a discretization of the Darcy’s law \((1c)\) on the conductive fractures \(T_h^c\), where the pressure continuity condition \((1h)\) is also strongly enforced as \(\hat{\rho}_h\) both represents the bulk pressure on the element boundary \(E_h\) and the pressure within the conductive fracture \(T_h^c\). Moreover, the last term in (4d) is a penalty formulation of the no-flow boundary condition \((1f)\) on \(\Gamma_{cb}\). Note that \(\Gamma_{cb}\) is allowed to be not aligned with the facets of \(T_h^c\).

(5) Equation (4e) is a transmission condition that simultaneously enforces (i) continuity of in-plane normal velocity \(u_h^c \cdot \eta\) on interior facets \(E_h^c \setminus \{\Gamma_{cc} \cup \Gamma_{cm} \cup \Gamma_{cs}\}\), (ii) the mass conservation \((1i)\) on \(\Gamma_{cm}\), and (iii) the no-flow boundary condition \((1j)\) on \(\Gamma_{cs}\).

(6) The Dirichlet boundary condition \((1e)\) and \((1i)\) are imposed strongly through the corresponding degrees of freedom (DOFs) on \(\hat{\rho}_h\) and \(\hat{p}_h\), respectively.

The following result further shows that the scheme (4) is well-posed.

**Theorem 3.1.** Assume the measure of the Dirichlet boundary \(\Gamma_D\) is not empty, then the solution to the scheme (4) exists and is unique.

**Proof.** Since the equations in (4) leads to a square linear system, we only need to show uniqueness. Now we assume the source terms in (4) vanishes, i.e., \(f = p_D = g_N = 0\). Taking test function to be the same as trial functions in (4) and adding, we get
\[
(K^{-1} u_h, u_h)_{T_h} + \int_{\Omega_h} \frac{\epsilon}{K_h} (u_h \cdot n)^2 ds + \langle (\epsilon K_c)^{-1} u_h^c, u_h^c \rangle_{T_h^c} = 0.
\]

Hence, \(u_h = u_h^c = 0\). Since \(u_h = 0\), the inf-sup stability of the \(RT_0-P_0\) finite element pair implies that \(p_h = \hat{p}_h = C\) from (4a) where \(C\) is a constant. Since \(\Gamma_D\) is not empty and \(p_D = 0\), we get the constant \(C = 0\). Finally, restricting equation (4d) to a single element \(F \in T_h^c\) and using the fact that \(u_h^c = 0\) and \(\hat{p}_h = 0\), we get
\[
\int_{\partial F} \hat{p}_h v^c \cdot \eta \, ds = 0, \quad \forall v^c \in RT_0(F),
\]
which then implies that \(\hat{p}_h = 0\). This completes the proof. \(\square\)

3.3. **Static condensation and linear system solver.** The linear system (4) can be efficiently solved via static condensation, where the DOFs for \(u_h\), \(p_h\), and \(u_h^c\) can be locally eliminated, resulting in a coupled global linear system for the DOFs for \(\hat{\rho}_h\) and \(\hat{p}_h\), which is symmetric and positive definite. Efficient linear system solvers for the resulting condensed system is an interesting topic where one could design efficient decoupling algorithms or robust monolithic preconditioners. Here we simply use a sparse direct solver in the computation and postpone the detailed study of linear system solvers to our future work.

3.4. **Local pressure postprocessing.** We use the following well-known local (piecewise linear) pressure postprocessing to improve the accuracy of pressure approximation in the bulk: find
\[
p_h^* \in W_h^* := \{ w \in L^2(T_h) : w|_K \in P_1(K), \quad \forall K \in T_h\},
\]

where $P^1(K)$ is the space of linear polynomials on element $K$, such that
\[
\begin{align}
(\nabla p_h^*, \nabla q_h^*)_{\mathcal{T}_h} &= - (K^{-1} u_h, \nabla q_h^*)_{\mathcal{T}_h}, \\
(p_h^*, 1)_{\mathcal{T}_h} &= (p_h, 1)_{\mathcal{T}_h},
\end{align}
\]
for all $q_h^* \in W_h^*$.

3.5. The hybridized finite volume method for the transport model. We consider a standard cell-centered, first-order upwinding finite volume scheme for the transport model [2], coupled with the implicit Euler method for the temporal discretization. We hybridize the cell-centered finite volume scheme so that the coupled unknowns live on the mesh skeletons, which simplifies the definition of upwinding fluxes on the conductive fracture interactions (e.g. point $H$ in Figure 7). Hence we use piecewise constant spaces to approximate the matrix concentration $c_h \in W_h$ on the mesh $\mathcal{T}_h$, the matrix concentration $\widehat{c}_h \in M_h$ on the matrix mesh skeleton $\mathcal{E}_h$, and the fracture concentration $\widehat{c}_{c,h} \in M^c_h$ on the fracture mesh skeleton $\mathcal{E}^c_{h}$. The hybridized finite volume scheme with implicit Euler temporal discretization is given as follows: given data $(c_{h}^{n-1}, c_{h}^{n-1}) \in W_h \times M_h$ at time $t^{n-1}$, find $(c_{h}^{n}, \widehat{c}_{h}^{n}, \widehat{c}_{c,h}^{n}) \in W_h \times M_h \times M^c_h$ at time $t^n := t^{n-1} + \Delta t$ with $c_{h}^{n}|_{\partial \Omega_{in}} = P_0(c_{B}(t^n))$ and $\widehat{c}_{c,h}^{n}|_{\Gamma_{in}} = P_0(c_{B}(t^n))$ such that
\[
\begin{align}
\phi_{m} \frac{c_{h}^{n} - c_{h}^{n-1}}{\Delta t} + \langle u_h \cdot n c_{h}^{n}, d_h \rangle_{\mathcal{T}_h} &= (c_{h}^{n}, d_h)_{\mathcal{T}_h}, \\
- \langle u_h \cdot n \widehat{c}_{h}^{n}, d_h \rangle_{\mathcal{T}_h} + \epsilon \phi_{c} \frac{\widehat{c}_{h}^{n} - \widehat{c}_{h}^{n-1}}{\Delta t} + [u_{c} \cdot \eta \widehat{c}_{c,h}^{n}, d_h]_{\partial \mathcal{T}_h} &= 0,
\end{align}
\]
for all $(d_h, \widehat{d}_{h}, \widehat{d}_{c,h}) \in W_h \times M_h \times M^c_h$ with $\widehat{d}_{h}|_{\partial \Omega_{in}} = 0$ and $\widehat{d}_{c,h}|_{\Gamma_{in}} = 0$, where the upwinding fluxes are given as follows:
\[
\begin{align}
\widehat{c}_{h}^{n} \mid_{\partial K} &= \begin{cases} 
c_{h}^{n} & \text{if } u_h \cdot n_K > 0, \\
\widehat{c}_{h}^{n} & \text{if } u_h \cdot n_K \leq 0,
\end{cases} \\
\widehat{c}_{c,h}^{n} \mid_{\partial F} &= \begin{cases} 
c_{c,h}^{n} & \text{if } u_{c} \cdot \eta_F > 0, \\
\widehat{c}_{c,h}^{n} & \text{if } u_{c} \cdot \eta_F \leq 0.
\end{cases}
\end{align}
\]

3.6. Remarks on the mesh restrictions and comparison with existing methods. The proposed flow and transport solvers [1], [3] require the mesh to be fitted to the conductive fractures, while allowing for an unfitted treatment of the blocking fractures. While the derivation of numerical schemes that work on fully unfitted meshes is beyond the scope of this paper, here we propose a simple mesh postprocessing technique to convert a general unfitted background matrix mesh to an immersed mesh that is fitted to all the fractures. Similar immersing mesh techniques were used for interface problems [63, 66, 67, 68]. Below we illustrate the procedure of immersing a single fracture to an unfitted tetrahedral mesh in 3D:

(i) Represent the fracture geometry as the zero level set of a continuous piecewise linear function $\phi_h$ on the background mesh. Perturb $\phi_h$ slightly if necessary to avoid fracture pass through the background mesh nodes.

(ii) Loop over the background mesh edges, find the cut edges where $\phi_h$ has opposite sign on the two edge endpoints. For each cut edge, compute the coordinates of the cut vertex $v_c$ where $\phi_h(v_c) = 0$, and add $v_c$ to the mesh nodes.

(iii) Loop over the background mesh faces, find the cut faces which contains the cut vertices. Order the cut vertices based on their vertex label number. Loop over the cut vertices, for
each (sub-)face that contains the cut vertex, split the (sub-)face by 2 by connecting the cut vertex with the opposite (sub-)face node.

(iv) Loop over the background mesh elements, find the cut elements which contains the cut vertices. Order the cut vertices based on their vertex label number. Loop over the cut vertices, for each (sub-)element that contains the cut vertex, split the (sub-)element by 2 by connecting the cut vertex with the opposite two (sub-)element nodes that are not aligned with the cut edge.

The above recursive bisection procedure guarantees that the fracture lies on the boundary of the generated immersed mesh. The case with multiply intersecting fractures can be treated by recursion. Here we note that the generated immersed mesh is usually highly anisotropic since the background mesh is completely independent of the fracture configurations. Our numerical results in the next section suggest that the hybrid-mixed method (4) works well on these anisotropic immersed meshes. Typical 2D immersed meshes for complex fracture configurations are given in Figure 11 and Figure 14 below.

We now briefly compare our proposed fractured flow solver (4) with some existing schemes in [69], which were used to solve a series of 4 benchmark problems in 3D fractured porous media flow. Among the 17 schemes in [69, Table 1], 7 were shown to yield no significant deviations for all the tests, see [69, Figure 18], which include the multi-point flux approximation (UiB-MPFA), the lowest order mixed virtual element method (UiB-MVEM), and the lowest order Raviart-Thomas mixed finite element method (UiB-RT0) mainly developed by the research group in the University of Bergen [70, 71, 60], the MPFA scheme (USTUTT-MPFA) and the two-point flux approximation scheme (USTUTT-TPFA_Circ) developed by Flemisch et al. [72], the mimetic finite difference method (LANL-MFD) [73], and the hybrid finite volumes discontinuous hydraulic head method (UNICE_UNIGE-HFV_Disc) developed by Brenner et al. [74]. Among these 7 schemes, the first three schemes use a mixed dimensional interface model that require the modeling of co-dimension 1-3 fractured flows, where the mesh can be non-matching across subdomains, but needs to be geometrically conforming to the fractures. On the other hand, the last four schemes work on a mixed dimensional interface model where only fractured flow in co-dimension 1 were modeled, which require the mesh to be completely conforming to the fractures. All of these schemes yield a locally conservative velocity approximation. We further note that the two methods in [69] that allow for general nonconforming meshes, namely the Lagrange multiplier method [31, 32, 33] and the EDFM method [75], cannot handle blocking fractures and do not provide a locally conservative velocity approximation.

Numerical results of our proposed scheme (4) for the benchmark problems in [69] indicate that our results yield no significant deviations with the above mentioned 7 schemes, see details in the next section. Our scheme also produce a locally conservative velocity approximation, and the resulting linear system after static condensation is a symmetric positive definite (SPD) problem with global unknowns involve pressure DOFs on the mesh skeleton only. The number of the global unknowns of our scheme is roughly $N_F$, which is the total number of mesh faces, and the average nonzero entries per row in the system matrix is 7 (a pressure DOF on an interior tetrahedral face is connected to 6 neighboring face pressure DOF’s). Concerning the computational cost of our scheme, it is more expensive than the TPFA scheme (USTUTT-TPFA_Circ) which lead to an SPD system with roughly $N_C$ cell-wise pressure DOFs and about 5 nonzero entries per row in the system matrix, is slightly less expensive than the cell-based MPFA schemes (UiB-MPFA, USTUTT-MPFA), which lead to SPD systems with roughly $N_C$ cell-wise pressure DOFs and about 20-50 nonzero entries per row in the system matrix, and is significantly cheaper than the schemes UiB-MVEM, UiB-RT0, LANL-MFD, and UNICE_UNIGE-HFV_Disc, which lead to saddle point systems with total number of roughly $N_F$ velocity DOFs and $N_C$ pressure DOFs. Note that $N_F \approx 2N_C$. Hence, our proposed scheme is also highly competitive in terms of computational costs. Another distinctive
advantage of our scheme over these 7 schemes is that the mesh can be completely nonconforming to the blocking fractures.

4. Numerics

In this section, we present detailed numerical results for the proposed hybrid-mixed method for the four 2D benchmark test cases in [54] and the four 3D benchmark test cases in [69]. We name the method (4) as HM-DFM since it is a hybrid mixed method for a discrete fracture model. When plotting the pressure or hydraulic head distribution over line segments, we evaluate the second-order postprocessed solution in (5) for the proposed method. The focus of the numerical experiments is on the verification of the accuracy of our proposed flow model (1) and the associated method (4). Hence, we test the flow solver (4) for all the 8 benchmark cases. Meanwhile, we also test the accuracy of velocity approximation by feeding them to the transport problem (2), which is solved using the scheme (6) for three cases, namely Benchmark 2 in 2D, and Benchmark 5/6 in 3D. Furthermore, convergence study via mesh refinements was conducted for Benchmark 2 and Benchmark 6 below.

Our numerical simulations are performed using the open-source finite-element software NGSolve [76], https://ngsolve.org/. Jupyter notebooks for reproducing all numerical examples in this section can be found in the git repository https://github.com/gridfunction/fracturedPorousMedia. Visualization of meshes for the 3D benchmark examples and interactive contour plots of the pressure/hydraulic head can also be found therein.

4.1. Benchmark 1: Hydrocoin (2D). This example is originally a benchmark for heterogeneous groundwater flow presented in the international Hydrocoin project [77]. A slight modification for the geometry was made in [54, Section 4.1], and we follow the settings therein. In particular, the bulk domain is a polygon with vertices $A = (0, 150)$, $B = (400, 100)$, $C = (800, 150)$, $D = (1200, 100)$, $E = (1600, 150)$, $F = (1600, -1000)$, $G = (1500, -1000)$, $H = (1000, -1000)$, and $I = (0, -1000)$ measured in meters. There are two conductive fractures in the domain $\{BG\}$ and $\{DH\}$. The fracture $\{BG\}$ has thickness $\epsilon = 5\sqrt{2}m$ and the fracture $\{DH\}$ has thickness $\epsilon = 33/\sqrt{5}m$. The permeability (hydraulic conductivity) is $K_m = 10^{-8}$ m/s in the bulk and $K_c = 10^{-6}$ m/s in the fractures. Dirichlet boundary condition $p = \text{height}$ is imposed on the top boundary, and homogeneous Neumann boundary condition is imposed on the rest of the boundary. Here the unknown variable $p$ is termed as the piezometric head according to [77]. The quantity of interest is the distribution of the piezometric head $p$ along the horizontal line at a depth of 200m.

We apply the method (4) on a uniform triangular mesh with mesh size $h = 60$, see the left panel of Figure 3, which leads to 1,115 matrix elements and 44 fracture elements. On this mesh, the number of the globally coupled DOFs is 1,779, in which 1,691 DOFs are associated with the bulk hybrid variable $\hat{p}_h$, and 43 DOFs are associated with the fracture hybrid variable $\hat{p}_c$. In the right panel of Figure 3, we record the postprocessed piezometric head $\hat{p}_h$ in (5) along the line segment $z = -200m$, where $z$ is the horizontal direction, along with the reference data obtained from a mimetic finite difference method on a very fine mesh (with 889,233 DOFs). We observe that the results for the proposed method on such a coarse mesh already shows a good agreement with the reference data.

4.2. Benchmark 2: Regular Fracture Network (2D). This test case is originally from [6] and is modified by [54], which simulates a regular fracture network in a square porous media. The computational domain including the fracture network and boundary conditions is shown in Figure 4. The matrix permeability is set to $K_m = \mathbb{I}$, and fracture thickness is $\epsilon = 10^{-4}$. Two cases of fracture permeability was considered: (i) a highly conductive network with $K_c = 10^4 \mathbb{I}$, (ii) a blocking fracture with $K_b = 10^{-4}$. 
We apply the method [4] on a triangular mesh with 1,348 matrix elements and 91 fracture elements, see the left panel of Figure 5. For the blocking fracture case, we also present the result on a unfitted triangular mesh with 1,442 matrix elements.

For the conductive fracture case, the number of the globally coupled DOFs is 2,127, in which 2,041 DOFs are associated with the bulk hybrid variable $\hat{p}_h$, and 86 DOFs are associated with the fracture hybrid variable $\hat{p}_c$. The pressure distributions along two lines, one horizontal at $y = 0.7$ and one vertical at $x = 0.5$ are shown in Figure 6 along with the reference data obtained from a mimetic finite difference method on a very fine mesh (with 1,175,056 DOFs). Similar to the previous example, we observe that the results for the proposed method show a good agreement with the reference data.

For the blocking fracture case, the number of the globally coupled DOFs is 2,041 on the fitted mesh and is 2,188 on the unfitted mesh. The pressure distribution along the lines $(0,0.1) - (0.9,1.0)$
is shown in Figure 7. Again, we observe a very good agreement with reference data for the results on the fitted mesh. The result on the unfitted mesh case is slightly off due to mesh nonconformity, which is expected as it could not capture the pressure discontinuity across the blocking fractures.

![Fitted and unfitted meshes](image)

(a) a fitted mesh. (b) a unfitted mesh.

**Figure 5.** Benchmark 2: computational meshes. The fitted mesh on the left panel is used for both conductive and blocking fracture cases. The unfitted mesh on the right panel is used only for the blocking fracture case.

![Pressure distribution](image)

(a) Horizontal line at $y = 0.7$. (b) Vertical line at $x = 0.5$.

**Figure 6.** Benchmark 2 with conductive fractures: pressure distribution along two lines.

4.2.1. **Coupling with transport and convergence study with mesh refinements.** After the velocity fields are computed from the scheme (4), we feed them to the transport model (2), and solve it by using the hybrid finite volume scheme (6). We take the porosities $\phi_m = 0.1$, $\phi_c = 0.9$ in the model (2), with the initial concentrations $c_0 = c_{c,0} = 0$, and set the left boundary as the inflow boundary for the concentrations, with $c_B = c_{c,B} = 1$. The final time of simulation is $T = 0.1$. Convergence of our coupled scheme (4) and (6) is checked via a mesh refinement study, where the initial meshes are given in Figure 5, and three level of uniform mesh refinements are applied afterwards. The constant time step size is taken to be $\Delta t = 2^{-l} \times 5 \times 10^{-3}$, where $l$ is the mesh refinement level. Since there is no analytic solution to the problem, we provide a reference solution using the coupled scheme (4) and (6) on the fourth level refined fitted mesh (with about 345k elements) with a small time step size $\Delta t = 3.125 \times 10^{-5}$. Contour of matrix concentrations of the reference solution at time $t = 0.05$ and $t = 0.1$ are presented in Figure 8 where we clearly observe the conducting and blocking effects of the respective fractures. Moreover, we plot the computed matrix concentrations...
Finally, the $L^2$-errors in the matrix velocity and postprocessed matrix pressure, and the $L^2$-errors in the matrix concentration at final time $T = 0.1$ are recorded in Table 1 for the conductive fracture case, in Table 2 for the blocking fracture case on fitted meshes and in Table 3 for the blocking fracture case on unfitted meshes. From Table 1 for the conductive fracture case, we observe that the convergence rate in the velocity approximation is first order and that in the postprocessed pressure approximation is second order, which is consistent with the expected convergence behavior of the hybrid-mixed method for the equi-dimensional case [78, 79], and the convergence rate for the concentration is about $1/2$, which is also expected for the hybridized finite volume scheme due to the concentration discontinuities in the domain. Similar convergence behavior was observed in Table 2 for the blocking fracture case on fitted meshes. From Table 3 we observe $1/2$ order convergence for all three variables, where the degraded velocity and pressure convergence is due to nonconformity of the mesh with the fractures.

| mesh ref. lvl | $L^2$-err in $u_h$ rate | $L^2$-err in $p^*_h$ rate | $L^2$-err in $c_h(T)$ rate |
|---------------|-------------------------|--------------------------|---------------------------|
| 0             | 3.567e-02               | 3.786e-04                | 1.177e-01                 |
| 1             | 1.954e-02 0.87          | 1.061e-04 1.84           | 8.587e-02 0.45            |
| 2             | 1.029e-02 0.92          | 7.146e-06 2.00           | 5.883e-02 0.55            |
| 3             | 4.881e-03 1.08          | 2.863e-05 1.89           | 3.541e-02 0.73            |

Table 1. Benchmark 2 with conductive fractures (fitted mesh): history of convergence for the $L^2$-errors in $u_h$, $p^*_h$, and $c_h(T)$ along mesh refinements. Reference solution is obtained on the fourth level refined fitted mesh with a small time step size $\Delta t = 3.125 \times 10^{-5}$.

4.3. Benchmark 3: Complex Fracture Network (2D). This test case considers a small but complex fracture network that includes permeable and blocking fractures. The domain and boundary conditions are shown in Figure 10. The exact coordinates for the fracture positions are provided.
**Figure 8.** Benchmark 2: Matrix concentration at time $t = 0.05$ (left) and $t = 0.1$ (right). Top row: conductive fractures. Bottom row: blocking fractures. Color range: 0 (blue)–1 (red). Solution obtained on the fourth level refined mesh with a small time step size $\Delta t = 3.125 \times 10^{-5}$.

| mesh ref. lvl. | $L^2$-err in $u_h$ | rate | $L^2$-err in $p_h^*$ | rate | $L^2$-err in $c_h(T)$ | rate |
|----------------|---------------------|------|-----------------------|------|------------------------|------|
| 0              | 1.358e-02           | –    | 2.406e-04             | –    | 1.396e-01              | –    |
| 1              | 7.098e-03           | 0.94 | 6.402e-05             | 1.91 | 1.025e-01              | 0.44 |
| 2              | 3.607e-03           | 0.98 | 1.630e-05             | 1.97 | 7.149e-02              | 0.52 |
| 3              | 1.666e-03           | 1.11 | 3.566e-06             | 2.19 | 4.572e-02              | 0.64 |

**Table 2.** Benchmark 2 with blocking fractures (fitted mesh): history of convergence for the $L^2$-errors in $u_h$, $p_h^*$, and $c_h(T)$ along mesh refinements. Reference solution is obtained on the fourth level refined fitted mesh with a small time step size $\Delta t = 3.125 \times 10^{-5}$. 
Figure 9. Benchmark 2: Matrix concentration along the line $y = 0.7$ at time $t = 0.1$ for the solution on different meshes. LVL stands for the number of mesh refinement levels. Reference solution is obtained on the fourth level refined fitted mesh with a small time step size $\Delta t = 3.125 \times 10^{-5}$.

| mesh ref. lvl. | $L^2$-err in $u_h$ rate | $L^2$-err in $p_h^*$ rate | $L^2$-err in $c_h(T)$ rate |
|----------------|-------------------------|---------------------------|---------------------------|
| 0              | 7.611e-02               | 8.295e-02                 | 1.424e-01                 |
| 1              | 5.357e-02 0.51          | 5.890e-02 0.49            | 1.050e-01 0.44            |
| 2              | 3.991e-02 0.42          | 4.088e-02 0.53            | 7.418e-02 0.50            |
| 3              | 2.634e-02 0.60          | 2.899e-02 0.50            | 4.882e-02 0.60            |

Table 3. Benchmark 2 with blocking fractures (unfitted mesh): history of convergence for the $L^2$-errors in $u_h$, $p_h^*$, and $c_h(T)$ along mesh refinements. Reference solution is obtained on the fourth level refined fitted mesh with a small time step size $\Delta t = 3.125 \times 10^{-5}$.

in [54 Appendix C]. The fracture network contains ten straight immersed fractures. The fracture thickness is $\epsilon = 10^{-4}$ for all fractures, and permeability is $K_c = 10^4$ for all fractures except for
fractures 4 and 5 which are blocking fractures with $K_b = 10^{-4}$. Note that we are considering two subcases a) and b) with a pressure gradient which is predominantly vertical and horizontal respectively.

![Figure 10. Benchmark 3: computational domain and boundary conditions.](image)

We apply the method (4) on two sets of meshes: a triangular fitted mesh with 1,332 matrix elements and 88 fracture elements which was provided in the git repository [https://git.iws.uni-stuttgart.de/benchmarks/fracture-flow](https://git.iws.uni-stuttgart.de/benchmarks/fracture-flow) see left of Figure 11, and a triangular immersed fitted mesh with 1,370 matrix elements and 211 fracture elements obtained from a background unfitted mesh using the immersing mesh technique introduced in Section 3.6, see right of Figure 11. The globally coupled DOFs is 2,066 for the fitted mesh, and is 2,211 for the immersed mesh.

![Figure 11. Benchmark 3: computational meshes.](image)

The pressure distributions along the lines $(0,0.5)-(1.0,0.9)$ are shown in Figure 12. We observe
that the results on the two meshes are very close to each other, and they are in good agreements with the reference data obtained from a mimetic finite difference method on a very fine mesh with 1.8 million DOFs.

Figure 12. Benchmark 3: pressure distribution along line (0, 0.5)–(1, 0.9). HDG-DFM(a) is the numerical solution on the fitted mesh in Figure 11(a), HDG-DFM(b) is the numerical solution on the immersed fitted mesh in Figure 11(b).

4.4. Benchmark 4: a Realistic Case (2D). We consider a real set of fractures from an interpreted outcrop in the Sotra island, near Bergen in Norway. The size of the domain is 700 m × 600 m with uniform scalar permeability \( K_m = 10^{-14} m^2 \). The set of fractures is composed of 64 line segments, in which the permeability is \( K_c = 10^{-8} m^2 \). The fracture thickness is \( \epsilon = 10^{-2} m \). The exact coordinates for the fracture positions are provided in the above mentioned git repository. The domain along with boundary conditions is given in Figure 13. Similar to the previous example, we apply the method (4) on two set of conforming meshes: a fitted mesh consists of 10,807 matrix elements and 1,047 fracture elements provided in https://git.iws.uni-stuttgart.de/benchmarks/fracture-flow, see left of Figure 14, and an immersed fitted mesh consists of 5,473 matrix elements and 1,541 fracture elements obtained from a background unfitted mesh using the immersing mesh technique introduced in Section 3.6, see right of Figure 14. The number of the globally coupled DOFs is 17,253 for the fitted mesh (a), and 9,753 for the immersed mesh (b).

The pressure distribution along the two lines \( y = 500m \) and \( x = 625m \) are shown in Figure 15 along with the results for the mortar-DFM method with 25,258 DOFs from [54]. We observe that the three results are in good agreements with each other, with the HDG-DFM(b) using the least amount of DOFs.

4.5. Benchmark 5: Single Fracture (3D). This is the first benchmark case proposed in [69]. To be consistent with the notation in [69], the pressure and permeabilities are renamed as hydraulic head and hydraulic conductivities, respectively for this test case and the three examples following. Figure 16 illustrates the geometrical description. Here the domain \( \Omega \) is a cube-shaped region \((0m, 100m) \times (0m, 100m) \times (0m, 100m) \) which is crossed by a conductive planar fracture, \( \Omega_2 \), with a thickness of \( \epsilon = 10^{-2} m \). The matrix domain consists of subdomains \( \Omega_{3,1} \), above the fracture, and \( \Omega_{3,2} \) and \( \Omega_{3,3} \) below. The subdomain \( \Omega_{3,3} \) represents a heterogeneity within the rock matrix. The matrix conductivities are given in Figure 16 and the fracture conductivity is \( K_c = 0.1 \) so that...
Figure 13. Benchmark 4: Computational domain and boundary conditions.

Figure 14. Benchmark 4: computational meshes.

\( \varepsilon K_c = 10^{-3} \). Inflow into the system occurs through a narrow band defined by \( \{0m\} \times (0m, 100m) \times (90m, 100m) \). Similarly, the outlet is a narrow band defined by \( (0m, 100m) \times \{0m\} \times (0m, 10m) \). At the inlet and outlet bands, we impose the hydraulic head \( h_{in} = 4m \) and \( h_{out} = 1m \) respectively. The remaining parts of the boundary are assigned no-flow conditions. Following the setup in [69], we set \( c_B = 0.01m^{-3} \) at the inlet boundary for the transport problem. The matrix porosity \( \phi \) is taken to be 0.2 on \( \Omega_{3.1} \cup \Omega_{3.2} \) and 0.25 on \( \Omega_{3.3} \), and the fracture porosity \( \phi_c \) is taken to be 0.4. The final time of simulation is \( T = 10^9s \), and the time step size is \( \Delta t = 10^7s \).

We perform the method (4) and (6) on a coarse tetrahedral mesh with 10,232 matrix elements and 448 fracture elements and a fine tetrahedral mesh with 111,795 matrix elements and 1,758 fracture elements.
elements. The number of the globally coupled DOFs on the coarse mesh is 23,377, while that on the fine mesh is 235,619. The hydraulic head along the line \((0\text{m}, 100\text{m}, 100\text{m}) - (100\text{m}, 0\text{m}, 0\text{m})\) is shown in Figure 17 along with reference data and published spread provided in the git repository \url{https://git.iws.uni-stuttgart.de/benchmarks/fracture-flow-3d}. The reference data in Figure 17 is obtained from the USTUTT-MPFA method on a mesh with approximately 1 million matrix elements, while the shaded region depicts the area between the 10th and the 90th percentile of the published results in [69] on mesh refinement level 1 (left, \(\sim 10k\) cells) and refinement level 2 (right, \(\sim 100k\) cells). The match number results from evaluating at 100 evenly distributed evaluation points if the value for the HM-DFM method is between the respective lower and upper value. We observe that our result agrees with the reference values quite well, especially on the fine mesh.

Moreover, we plot the matrix concentration along the line \((0\text{m}, 100\text{m}, 100\text{m}) - (100\text{m}, 0\text{m}, 0\text{m})\) in Figure 18, and the fracture concentration along the line \((0\text{m}, 100\text{m}, 80\text{m}) - (100\text{m}, 0\text{m}, 20\text{m})\) at final time \(T = 10^9\text{s}\) in Figure 19, together with the published spread provided in the git repository, which depicts the area between the 10th and the 90th percentile of the published results in [69].
Figure 17. Benchmark 5: Hydraulic head in the matrix over the line
(0m, 100m, 100m)–(100m, 0m, 0m). Left: results on a coarse mesh with about 10k
cells. Right: results on a fine mesh with about 100k cells.

We observe that our results agree quite well with the provided data.

Figure 18. Benchmark 5: Hydraulic head in the matrix over the line
(0m, 100m, 100m)–(100m, 0m, 0m). Left: results on a coarse mesh with about 10k
cells. Right: results on a fine mesh with about 100k cells.

4.6. Benchmark 6: Regular Fracture Network (3D). This is the second benchmark case
proposed in [69], which is a 3D analog of Benchmark 2. The domain is given by the unit cube Ω =
(0m, 1m)^3 and contains 9 regularly oriented fractures, as illustrated in Figure 20. Dirichlet boundary
condition \( p = h = 1 \text{m} \) is imposed on the boundary \( \Gamma_D = \{(x, y, z) \in \partial \Omega : x, y, z > 0.875 \text{m}\} \),
Neumann boundary condition \( u \cdot n = -1 \text{m/s} \) is imposed on the boundary \( \partial \Omega_{\text{in}} = \{(x, y, z) \in \partial \Omega : x, y, z < 0.25 \text{m}\} \), and no-flow boundary condition is imposed on the remaining boundaries. The
heterogeneous matrix conductivity is illustrated in Figure 20 and the fracture conductivity is either
\[ K_c = 10^4 \text{m}^2, \] which represents a conductive fracture or \( K_b = 10^{-4} \text{m}^2 \) which represents a blocking fracture. The fracture thickness is \( \epsilon = 10^{-4} \text{m} \). For the transport equation, matrix porosity is taken to be \( \phi = 0.1 \), conductive fracture concentration is \( \phi_c = 0.9 \), and the inflow boundary condition \( c_B = 1 \text{m}^{-3} \) is set on the inlet boundary \( \partial \Omega_{in} \). Final time of the simulation is \( T = 0.25 \text{s} \).

**Figure 19.** Benchmark 5: Fracture concentration over the line \((0 \text{m}, 100 \text{m}, 80 \text{m})–(100 \text{m}, 0 \text{m}, 20 \text{m})\). Left: results on a coarse mesh with about \(10k\) cells. Right: results on a fine mesh with about \(100k\) cells.

We perform the method (4) on a coarse fitted tetrahedral mesh with 4,375 matrix elements and 944 fracture elements and a fine tetrahedral mesh with 36,336 matrix elements and 4,524 fracture elements. The number of the globally coupled DOFs on the coarse mesh is 13,373 for the conductive fracture case and 8,334 for the blocking fracture case (only DOFs for \( \hat{p}_h \) are global DOFs in this case), while that on the fine mesh is 94,738 for the conductive fracture case and 70,881 for the blocking fracture case. The hydraulic head along the diagonal line \((0 \text{m}, 0 \text{m}, 0 \text{m})–(1 \text{m}, 1 \text{m}, 1 \text{m})\) is shown in Figure 21 for the conductive fracture case and in Figure 22 for the blocking fracture case. We observe that our results agree with the reference values very well, which were obtained from the USTUTT-MPFA method on a mesh with approximately 1 million matrix elements. The small derivation of our result on the left panel of Figure 21 with the reference data is acceptable due to the use of a very coarse mesh.
We further performed a convergence study of the flow and transport solvers (4) and (6) via mesh refinements, and record the $L^2$-errors in matrix velocity and postprocessed pressure, and the $L^2$-errors in matrix concentration at final time $t = 0.25$ in Table 4 for the conductive fracture case and in Table 5 for the blocking fracture case, where the initial mesh is the coarse one with 4,375 tetrahedral elements. A total of three uniform mesh refinements was performed, and the solution on the third level mesh was used as the reference solution to calculate the associated errors. The time step size is taken to be $\Delta t = 2^{-l} \times 2.5 \times 10^{-3}$ s, where $l$ is the mesh refinement level. On the finest mesh, there are about 2.25 million tetrahedral elements and 4.5 million globally coupled DOFs. From both tables, we observe convergence of our schemes, and in particular the convergence...
rate for the velocity is approaching first order, that for the postprocessed pressure is approaching second order, and for the concentration is about first order.

| mesh ref. lvl. | $L^2$-err in $u_h$ rate | $L^2$-err in $p_h^*$ rate | $L^2$-err in $c_h(T)$ rate |
|---------------|------------------------|--------------------------|-----------------------------|
| 0             | 1.789e-01              | 1.456e-01                | 1.496e-01                   |
| 1             | 1.120e-01 0.68         | 5.886e-02 1.31           | 9.645e-02 0.63             |
| 2             | 6.181e-02 0.86         | 1.852e-02 1.67           | 5.102e-02 0.92             |

Table 4. Benchmark 6 with conductive fractures (fitted mesh): history of convergence for the $L^2$-errors in $u_h$, $p_h^*$, and $c_h(T)$ along mesh refinements. Reference solution is obtained on the third level refined fitted mesh with about 2.25 million matrix elements and time step size $\Delta t = 3.125 \times 10^{-4}$.

| mesh ref. lvl. | $L^2$-err in $u_h$ rate | $L^2$-err in $p_h^*$ rate | $L^2$-err in $c_h(T)$ rate |
|---------------|------------------------|--------------------------|-----------------------------|
| 0             | 1.791e-01              | 1.533e-01                | 1.288e-01                   |
| 1             | 1.118e-01 0.68         | 6.080e-02 1.33           | 8.139e-02 0.66             |
| 2             | 6.172e-02 0.86         | 1.891e-02 1.68           | 3.939e-02 1.05             |

Table 5. Benchmark 6 with blocking fractures (fitted mesh): history of convergence for the $L^2$-errors in $u_h$, $p_h^*$, and $c_h(T)$ along mesh refinements. Reference solution is obtained on the third level refined fitted mesh with about 2.25 million matrix elements and time step size $\Delta t = 3.125 \times 10^{-4}$.

Finally, in Figure 23 we plot slices of concentrations computed on the 3rd refined mesh at final time $t = 0.25$ along the five vertical planes $x = 0.1, x = 0.3, x = 0.5, x = 0.7$ and $x = 0.9$, and in Figure 24 we plot the evolution of mean concentration over time on the following three regions:

$$
\Omega_A := (0.5m, 1m) \times (0m, 0.5m) \times (0m, 0.5m),
\Omega_B := (0.5m, 0.75m) \times (0.5m, 0.75m) \times (0.75m, 1m),
\Omega_C := (0.75m, 1m) \times (0.75m, 1m) \times (0m, 0.75m).
$$

$$
\Omega_A := (0.5m, 1m) \times (0m, 0.5m) \times (0m, 0.5m),
\Omega_B := (0.5m, 0.75m) \times (0m, 0.75m) \times (0.75m, 1m),
\Omega_C := (0.75m, 1m) \times (0.75m, 1m) \times (0m, 0.75m).
$$

From the results in Figure 23 we clearly observe the different flow pattern for the conductive fracture case in the first row and the blocking fracture case in the second row. We further note that the mean concentrations reported in Figure 24 were presented in [69, Figure 10] (only) on the coarse mesh with about $4k$ matrix elements and a coarse time step size $\Delta t = 2.5 \times 10^{-3}s$. Our results on four set of meshes are close to each other and improve slightly as the mesh and time step size refines, and they are also qualitatively similar to the majority of the coarse-grid results in [69, Figure 10].

### 4.7. Benchmark 7: Network with Small Features (3D).

This is the third benchmark case proposed in [69], in which small geometric features exist that may cause trouble for conforming meshing strategies. The domain is the box $\Omega = (0m, 1m) \times (0m, 2.25m) \times (0m, 1m)$, containing 8 fractures; see Figure 25. Homogeneous Dirichlet boundary condition is imposed on the outlet boundary

$$
\partial \Omega_{out} := \{(x, y, z) : 0 < x < 1, y = 2.25, z < 1/3 or z > 2/3\},
$$
Figure 23. Benchmark 6: Matrix concentration at time $t = 0.25$ along the five vertical planes $x = 0.1, x = 0.3, x = 0.5, x = 0.7$ and $x = 0.9$. Top row: conductive fractures. Bottom row: blocking fractures. Color range: 0 (blue)– 1(red).

Figure 24. Benchmark 6: Mean matrix concentration over time on $\Omega_A$ (left), $\Omega_B$ (middle), and $\Omega_C$ (right). Top row: conductive fractures. Bottom row: blocking fractures. LVL stands for the number of mesh refinement levels.

inflow boundary condition $\mathbf{u} \cdot \mathbf{n} = -1 \text{m/s}$ is imposed on the inlet boundary $\partial \Omega_{in} := \{(x, y, z) : 0 < x < 1, y = 0, 1/3 < z < 2/3\}$,
Figure 25. Benchmark 7: Conceptual model and geometrical description of the domain.

and no-flow boundary condition is imposed on the remaining boundaries. The conductivity in the matrix is $K_m = 1 \text{m}^2$, and that in the fracture is $K_c = 10^4 \text{m}^2$. Fracture thickness is $\epsilon = 0.01 \text{m}$.

We perform the method (4) on a coarse tetrahedral mesh with 31,812 matrix elements and 3,961 fracture elements and a fine tetrahedral mesh with 147,702 matrix elements and 9,441 fracture elements. The number of the globally coupled DOFs on the coarse mesh is 83,022, while that on the fine mesh is 343,359. The hydraulic head along the line $(0.5 \text{m}, 1.1 \text{m}, 0 \text{m})$–$(0.5 \text{m}, 1.1 \text{m}, 1 \text{m})$ is shown in Figure 26, where the reference data is obtained with the USTUTT-MPFA scheme on a grid with approximately $10^6$ matrix cells. Here we observe a very good agreement with the reference data even on the coarse mesh.

Figure 26. Benchmark 7: Hydraulic head in the matrix over the line $(0.5 \text{m}, 1.1 \text{m}, 0 \text{m})$–$(0.5 \text{m}, 1.1 \text{m}, 1 \text{m})$. Left: results on a coarse mesh with about $32k$ cells. Right: results on a fine mesh with about $148k$ cells.

4.8. Benchmark 8: Field Case (3D). This is the last benchmark case proposed in [69]. The geometry is based on a postprocessed outcrop from the island of Algerøyna, outside Bergen, Norway, which contains 52 fracture. The simulation domain is the box $\Omega = (-500 \text{m}, 350 \text{m}) \times (100 \text{m}, 1500 \text{m}) \times (-100 \text{m}, 500 \text{m})$. The fracture geometry is depicted in Figure 27. Homogeneous
Dirichlet boundary condition is imposed on the outlet boundary
\[ \partial \Omega_{\text{out}} := \{500\} \times (100, 400) \times (-100, 100) \cup \{350\} \times (100, 400) \times (-100, 100), \]
uniform unit inflow \( u \cdot n = 1 \text{m/s} \) is imposed on the inlet boundary
\[ \partial \Omega_{\text{in}} := \{500\} \times (1200, 1500) \times (300, 500) \cup \{350\} \times (1200, 1500) \times (-200, 0) \cup \{350\} \times (1200, 1500) \times (300, 500). \]

Conductivity is \( K_m = 1 \text{m}^2 \) in the matrix, and \( K_c = 10^4 \text{m}^2 \) in the fracture. Fracture thickness is \( \epsilon = 10^{-2} \text{m} \).

**Figure 27.** Benchmark 8: Conceptual model and geometrical description of the domain.

We perform the method (4) on a tetrahedral mesh with 241,338 matrix elements and 47,154 fracture elements. The number of the globally coupled DOFs is 696,487.

The hydraulic head along the two diagonal lines \((-500m, 100m, -100m)\)–\((350m, 1500m, 500m)\) and \((350m, 100m, -100m)\)–\((-500m, 1500m, 500m)\) are shown in Figure 28 along with published results from [69]. Similar to Benchmark 4 in 2D, no reference data on refined meshes was provided for this problem due to its complexity. Comparing with the published results in Figure 28 we observe that our method still performs quite well.

**Figure 28.** Benchmark 8: Hydraulic head across the domain. (a): Profile from outlet \( \partial \Omega_{\text{out},0} \) towards the opposite corner. (b): Profile from outlet \( \partial \Omega_{\text{out},1} \) towards the opposite corner \( \partial \Omega_{\text{in}} \).
5. Conclusion

A novel hybrid-mixed method for single-phase flow in fractured porous media has been presented. Distinctive features of the scheme include local mass conservation, symmetric positive definite linear system, and allowing the computational mesh to be completely non-conforming to the blocking fractures.

Ample benchmark tests show the excellent performance of the proposed scheme, which is also highly competitive with existing work in the literature. Extension to the method to more complex fractured flow models and adaptation of the method to more general meshes consists of our on-going work. We will also investigate efficient preconditioning procedures for the associated linear system problem in the near future.

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