An Explicit Construction of Gauss-Jordan Elimination Matrix✩

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Abstract

A constructive approach to get the reduced row echelon form of a given matrix $A$ is presented. It has been shown that after the $k$th step of the Gauss-Jordan procedure, each entry $a_{ij}^k(i \neq j, j > k)$ in the new matrix $A^k$ can always be expressed as a ratio of two determinants whose entries are from the original matrix $A$. The new method also gives a more general generalization of Cramer’s rule than existing methods.

Key words: Gauss-Jordan Elimination, Cramer’s rule, Determinants

1. Introduction

Gauss-Jordan elimination is a variation of standard Gaussian elimination in which a matrix is brought to reduced row echelon form rather merely to triangular form. In contrast to standard Gaussian elimination, entries above and below the diagonal have to be annihilated in the process of Gauss-Jordan elimination. It has been shown that the Gauss-Jordan elimination is considerably less efficient than Gaussian elimination with backsubstitution when solving a system of linear equations. Despite its higher cost, Gauss-Jordan elimination can be preferred in some situations. For instance, it may be implemented on parallel computers when solving systems of linear equations [2]. In addition, it is well suited for computing the matrix inverse.

Applying Gauss-Jordan elimination to a given matrix $A$, we denote by $A^k$ the new matrix obtained after $k$th step of Gauss-Jordan elimination. In the

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present paper, we will show that each entry $a_{i,j}^k (i \neq j, j > k)$ in the matrix $A^k$ can always be expressed as a ratio of two determinants whose entries are from the original matrix $A$. In 2002, Gong et al. [1] first established a generalized Cramer’s rule, which can be applied to a problem in decentralized control systems. However, their method is restricted to deal with a class of particular systems of linear equations. In [5], Hugo Leiva has presented another generalization of Cramer’s rule, but the given formula is somewhat complicated. Different from the two methods mentioned above, our approach can also be used to directly construct one solution of $AX = b$. From this point of view, our method can give a generalized Cramer’s rule whose form is completely different from the existing results. We also hope that it is useful not only as a theoretical tool, but also as a practical calculation methods in the linear algebra community.

2. Main results

Lemma 2.1. [3] If $M$ is a square matrix and $a, b, c, d$ are scalars, then

$$
|M| \begin{vmatrix}
M & U & V \\
R & a & b \\
S & c & d \\
\end{vmatrix} = \begin{vmatrix}
M & U \\
R & a \\
M & U \\
S & c \\
\end{vmatrix} \begin{vmatrix}
M & V \\
R & b \\
M & V \\
S & d \\
\end{vmatrix}.
$$

Before presenting the main result, we first offer a recursive description of Bareiss’ standard fraction free Gaussian elimination [4].

$$a_{i,j}^{(k)} = \begin{vmatrix}
\begin{array}{ccc}
a_{11}^0 & \cdots & a_{1,k}^0 \\
\vdots & \ddots & \vdots \\
a_{k,1}^0 & \cdots & a_{k,k}^0 \\
a_{i1}^0 & \cdots & a_{ik}^0 \\
\end{array}
\end{vmatrix}, \quad i > k, j > k. \quad (1)
$$

$$a_{0,0}^{(-1)} = 1, a_{i,j}^{(0)} = a_{i,j}$$

$$a_{i,j}^{(k)} = \frac{a_{k,k}^{(k-1)} a_{i,j}^{(k-1)} - a_{i,k}^{(k-1)} a_{k,j}^{(k-1)}}{a_{k-1,k-1}^{(k-2)}}. $$

In what follows, in order to simplify the discussion, we also assume that the leading principal minors of a $n \times m$ matrix $A$ are nonzero.
Theorem 2.2. Let $A = (a_{ij})$ be a $n \times m$ matrix with entries from an arbitrary commutative ring and $A^k(0 \leq k \leq n)$ is defined as above. Bring $A$ to reduced row echelon form by Gauss-Jordan elimination. Then after the $k$th elimination step, each entry $a^k_{i,j}(i \neq j, j > k)$ in $A^k$ can be expressed as a ratio of two determinants whose entries are from the original matrix $A$.

Proof. Consider the following three cases:

1). Case 1: $i > k, j > k$. We shall show that

$$a^k_{i,j} = \frac{a^{(k)}_{i,j}}{a^{(k-1)}_{k,k}}, \quad (i > k, j > k).$$

By (1), it is easy to see that the conclusion is true. To see this, let us use induction on the elimination step $k$ as follows.

(i) When $k = 1$, it is clear that the equality (2) holds.

(ii) Now assume that the equality (2) is true for $k$. Then, when the elimination step is $k + 1$, we have

$$a^{k+1}_{i,j} = a^k_{i,j} - \frac{a^{(k)}_{i,k+1}a^k_{k,k}}{a^{(k-1)}_{k,k}} - \frac{a^{(k)}_{i,k+1}a^{(k-1)}_{i,k}}{a^{(k-1)}_{k,k}} = \frac{a^{(k)}_{i,j}}{a^{(k-1)}_{k,k}}.$$

This proves the equality (2).

2). Case 2: $i = k, j > k$. We shall claim that the below formula is true.

$$a^k_{i,j} = \frac{a^{(k-1)}_{i,j}}{a^{(k-1)}_{k,k}}.$$  

It is easy to prove this, since we want $a^{k-1}_{k,k} \leftarrow 1$, according to Gauss-Jordan elimination.

3). Case 3: $i < k, j > k$. First, Let us construct the following determinant:

$$a^{(k)}_{i,j} = \begin{vmatrix} a^0_{1,1} & \cdots & a^0_{1,i-1} & a^0_{1,i+1} & \cdots & a^0_{1,k} & a^0_{1,j} \\ a^0_{2,1} & \cdots & a^0_{2,i-1} & a^0_{2,i+1} & \cdots & a^0_{2,k} & a^0_{2,j} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a^0_{k,1} & \cdots & a^0_{k,i-1} & a^0_{k,i+1} & \cdots & a^0_{k,k} & a^0_{k,j} \end{vmatrix}_{k \times k}, \quad i < k, j > k \quad (4)$$

Next, we will claim that the following two recursion formulae hold.
Case 3-1. When $i \leq k - 2$, we have

$$a_{i,j}^{(k)} = -\frac{a_{i,k}^{(k-1)} a_{i,j}^{(k-1)} - a_{i,k}^{(k-1)} a_{k,j}^{(k-1)}}{a_{k-1,k}^{(k-2)}}, \quad i \leq k - 2, \ j > k. \quad (5)$$

Case 3-2. When $i = k - 1$, it follows that

$$a_{i,j}^{(k)} = \frac{a_{k,k}^{(k-2)} a_{i,j}^{(k-2)} - a_{i,k}^{(k-2)} a_{k,j}^{(k-2)}}{a_{k-2,k}^{(k-3)}}, \quad i = k - 1, \ j > k. \quad (6)$$

The proof of the equality (6) : Since the row index of each element in the right-hand side of (6) is bigger than its column index, the formula (1) is still available. By (1), we get

$$a_{k,k}^{(k-2)} = \begin{bmatrix} a_{11}^0 & \cdots & a_{1,k-2}^0 & a_{1,k}^0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k-2,1}^0 & \cdots & a_{k-2,k-2}^0 & a_{k-2,k}^0 \\ a_{k,1}^0 & \cdots & a_{k,k-2}^0 & a_{k,k}^0 \end{bmatrix}_{k \times k}, \quad a_{k-1,j}^{(k-2)} = \begin{bmatrix} a_{11}^0 & \cdots & a_{1,k-2}^0 & a_{1,j}^0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k-2,1}^0 & \cdots & a_{k-2,k-2}^0 & a_{k-2,j}^0 \\ a_{k,1}^0 & \cdots & a_{k,k-2}^0 & a_{k,j}^0 \end{bmatrix}_{k \times k}$$

$$a_{k-1,k}^{(k-2)} = \begin{bmatrix} a_{11}^0 & \cdots & a_{1,k-2}^0 & a_{1,k}^0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k-2,1}^0 & \cdots & a_{k-2,k-2}^0 & a_{k-2,k}^0 \\ a_{k,1}^0 & \cdots & a_{k,k-2}^0 & a_{k,k}^0 \end{bmatrix}_{k \times k}, \quad a_{k,j}^{(k-2)} = \begin{bmatrix} a_{11}^0 & \cdots & a_{1,k-2}^0 & a_{1,j}^0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k-2,1}^0 & \cdots & a_{k-2,k-2}^0 & a_{k-2,j}^0 \\ a_{k,1}^0 & \cdots & a_{k,k-2}^0 & a_{k,j}^0 \end{bmatrix}_{k \times k}$$

Partition the above determinants into 4 submatrices respectively, as follows:

$$M = \begin{pmatrix} a_{11}^0 & \cdots & a_{1,k-2}^0 \\ \vdots & \ddots & \vdots \\ a_{k-2,1}^0 & \cdots & a_{k-2,k-2}^0 \end{pmatrix}, \quad a = a_{k,k}^0, \quad b = a_{k,j}^0, \quad c = a_{k-1,k}^0, \quad d = a_{k-1,j}^0$$

$$U = (a_{1,k}^0, \ldots, a_{k,k}^0)^T, \quad V = (a_{1,j}^0, \ldots, a_{k,j}^0)^T, \quad R = (a_{k,1}^0, \ldots, a_{k,k}^0)^T, \quad S = (a_{k-1,1}^0, \ldots, a_{k-1,k-2}^0)^T.$$
The last equality can be guaranteed by (4).

A similar but somewhat more complicated method can be used to establish the proof of (5). According to (1) and (4), we have

\[
\begin{pmatrix}
  a_{11}^{(k-2)} & \cdots & a_{1,k-2}^{(k-2)} & a_{k-1}^{(k-2)} \\
  \vdots & \ddots & \vdots & \vdots \\
  a_{k-2,1}^{(k-2)} & \cdots & a_{k-2,k-2}^{(k-2)} & a_{k-2,k-1}^{(k-2)} \\
  a_{k-1,1}^{(k-2)} & \cdots & a_{k-1,k-2}^{(k-2)} & a_{k-1,k-1}^{(k-2)}
\end{pmatrix}
= a_{k-1,j}^{(k)}.
\]

Afterwards, expand \( a_{k-1,k-1}^{(k-2)} \) along the \( i \)th column, it follows that

\[
a_{k-1,k-1}^{(k-2)} = (-1)^{i+1} a_{1,j}^{0} M_{1} + \cdots + (-1)^{k-2+i} a_{k-2,i}^{0} M_{k-2} + (-1)^{k-1+i} a_{k-1,i}^{0} M_{k-1}
= \sum_{s=1}^{k-1} (-1)^{s+i} a_{s,i}^{0} M_{s}.
\]

Here, \( M_{s} \) is a \((k-2) \times (k-2)\) minor of \( a_{k-1,k-1}^{(k-2)} \).

\[
a_{k-1,k-1}^{(k-2)} a_{i,j}^{(k)} = - \sum_{s=1}^{k-1} \left[ (-1)^{s+i} a_{s,j}^{0} M_{s} a_{i,j}^{(k)} \right]
= - \sum_{s=1}^{k-1} \left[ (-1)^{s+i} a_{s,i}^{0} M_{s} (-1)^{k-1} a_{k-1,j}^{0} \right]
\]

\[
\begin{pmatrix}
  M_{s} & U_{s} & V_{s} \\
  R_{s} & a_{s} & b_{s} \\
  S_{s} & c_{s} & d_{s}
\end{pmatrix}
\] (7)
Let $\overline{M}_s$ be a square matrix whose determinant is $M_s$. Since the minor $M_s$ obtained by expanding $a^{(k-2)}_{k-1,k-1}$ along the $i$th column is exactly a minor of $a^{(k)}_{i,j}$, then one always can apply elementary row operations to $a^{(k)}_{i,j}$, such that the top left corner of $\overline{a}^{(k)}_{i,j}$ is exactly $\overline{M}_s$. Here, $\overline{a}^{(k)}_{i,j} = -a^{(k)}_{i,j}$.

According to Lemma 2.1, it follows that

\[
(7) = (-1)^{k+i} \sum_{s=1}^{k-1} (a^0_{s,i}) \begin{vmatrix}
\overline{M}_s & U_s \\
R_s & a_s \\
\overline{M}_s & V_s \\
R_s & b_s
\end{vmatrix}
\]

\[
= (-1)^{k+i} \sum_{s=1}^{k-1} (a^0_{s,i})(-1)^{k-s} \begin{vmatrix}
\overline{M}_s & U_s \\
\overline{M}_s & V_s \\
S_s & c_s \\
S_s & d_s
\end{vmatrix}
\]  (8)

Here, notice that

\[
|\overline{M}_s| = \begin{vmatrix}
a^0_{11} & \cdots & a^0_{1,i-1} & a^0_{1,i+1} & \cdots & a^0_{1,k-2} & a^0_{1,k-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a^0_{s-1,1} & \cdots & a^0_{s-1,i-1} & a^0_{s-1,i+1} & \cdots & a^0_{s-1,k-2} & a^0_{s-1,k-1} \\
a^0_{s+1,1} & \cdots & a^0_{s+1,i-1} & a^0_{s+1,i+1} & \cdots & a^0_{s+1,k-2} & a^0_{s+1,k-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a^0_{k-1,1} & \cdots & a^0_{k-1,i-1} & a^0_{k-1,i+1} & \cdots & a^0_{k-1,k-2} & a^0_{k-1,k-1}
\end{vmatrix}_{(k-2) \times (k-2)}
\]

$U_s = (a^0_{1,k}, \ldots, a^0_{s-1,k}, a^0_{s+1,k}, \ldots, a^0_{k-1,k})^T, V_s = (a^0_{1,j}, \ldots, a^0_{s-1,j}, a^0_{s+1,j}, \ldots, a^0_{k-1,j})^T, S_s = (a^0_{s,1}, \ldots, a^0_{s,i-1}, a^0_{s,i+1}, \ldots, a^0_{s,k-2}, a^0_{s,k-1})^T$

$a_s = a^0_{s,k}, b_s = a^0_{s,j}, C_s = a^0_{k,k}, d_s = a^0_{k,j}$.

Clearly,

\[
\begin{vmatrix}
\overline{M}_s & U_s \\
R_s & a_s
\end{vmatrix} = (-1)^{k-(s+1)}(-a^{(k-1)}_{i,k}) = (-1)^{k-s}a^{(k-1)}_{i,k},
\]

\[
\begin{vmatrix}
\overline{M}_s & V_s \\
R_s & b_s
\end{vmatrix} = (-1)^{k-(s+1)}(-a^{(k-1)}_{i,j}) = (-1)^{k-s}a^{(k-1)}_{i,j}.
\]

Let

\[
\begin{vmatrix}
\overline{M}_s & U_s \\
S_s & c_s
\end{vmatrix} = Q_s, \quad \begin{vmatrix}
\overline{M}_s & V_s \\
S_s & d_s
\end{vmatrix} = T_s
\]
Hence,
\[
(8) = \sum_{s=1}^{k-1} [(-1)^{i-s+1} a_{s,i}^{(k-1)} (a_{ij}^{(k-1)} Q_s - a_{i,k}^{(k-1)} T_s)].
\]

Additionally,
\[
-(a_{k,k}^{(k-1)} a_{i,j}^{(k-1)} - a_{i,k}^{(k-1)} a_{k,j}^{(k-1)}) = \\
\begin{vmatrix}
  a_{11}^0 & \cdots & a_{1,i-1}^0 & a_{1,i}^{(k-1)} & a_{1,i+1}^0 & \cdots & a_{1,k-1}^0 & a_{1,k}^0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_{k-1,1}^0 & \cdots & a_{k-1,i-1}^0 & a_{k-1,i}^{(k-1)} & a_{k-1,i+1}^0 & \cdots & a_{k-1,k-1}^0 & a_{k-1,k}^0 \\
  a_{k,1}^0 & \cdots & a_{k,i-1}^0 & a_{k,i}^{(k-1)} & a_{k,i+1}^0 & \cdots & a_{k,k-1}^0 & a_{k,k}^0 \\
  a_{11}^0 & \cdots & a_{1,i-1}^0 & a_{1,i}^{(k-1)} & a_{1,i+1}^0 & \cdots & a_{1,k-1}^0 & a_{1,k}^0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_{k-1,1}^0 & \cdots & a_{k-1,i-1}^0 & a_{k-1,i}^{(k-1)} & a_{k-1,i+1}^0 & \cdots & a_{k-1,k-1}^0 & a_{k-1,k}^0 \\
  a_{k,1}^0 & \cdots & a_{k,i-1}^0 & a_{k,i}^{(k-1)} & a_{k,i+1}^0 & \cdots & a_{k,k-1}^0 & a_{k,k}^0 \\
\end{vmatrix}
\]

And then, expand the above determinants along the ith column, we have
\[
-(a_{k,k}^{(k-1)} a_{i,j}^{(k-1)} - a_{i,k}^{(k-1)} a_{k,j}^{(k-1)}) = -\sum_{s=1}^{k} (a_{s,i}^0 (-1)^{i+s} a_{i,j}^{(k-1)} a_{i,k}^{(k-1)} B_s)
\]

Thereinto, 
\[
B_s = \begin{vmatrix}
  M_s & U_s \\
  S_s & C_s \\
\end{vmatrix} = Q_s, \quad A_s = \begin{vmatrix}
  M_s & V_s \\
  S_s & D_s \\
\end{vmatrix} = T_s,
\]
are two minors obtained by deleting the sth row, the ith column from the determinants 
\[
a_{k,k}^{(k-1)} a_{i,j}^{(k-1)}, a_{i,k}^{(k-1)} a_{k,j}^{(k-1)}
\]
respectively.

It is important to notice that when \(s = k\), we have \(A_k \equiv -a_{i,k}^{(k-1)}\) and \(B_k \equiv -a_{i,j}^{(k-1)}\). Therefore, when \(s = k\), we have
\[
\begin{vmatrix}
  a_{i,k}^{(k-1)} & a_{i,j}^{(k-1)} \\
  A_k & B_k \\
\end{vmatrix} \equiv 0.
\]

So, \((8) = (9)\). The equality \((5)\) holds clearly.

Now, we consider the third case: when \(i < k, j > k\), the below equality holds.
\[ a^k_{i,j} = \begin{cases} \frac{a^{(k)}_{i,j}}{a^{(k-1)}_{k,k}}, & i = k - 1, j > k, \\ \frac{(-1)^{k-1+i} a^{(k)}_{i,j}}{a^{(k-1)}_{k,k}}, & i \leq k - 2, j > k \end{cases} \]  \hfill (10)

Let us induce on \( k \) as follows.

(i). When \( k = 2, 3, 4, j > k \), it is very easy to verify that all the following equalities hold.

\[ a^2_{1,j} = \frac{a^{(2)}_{1,j}}{a^{(1)}_{2,2}}, \quad a^3_{1,j} = \frac{a^{(3)}_{1,j}}{a^{(2)}_{3,3}}, \quad a^3_{2,j} = \frac{a^{(3)}_{3,j}}{a^{(2)}_{3,3}}, \]
\[ a^4_{1,j} = \frac{a^{(4)}_{1,j}}{a^{(5)}_{4,4}}, \quad a^4_{2,j} = \frac{a^{(4)}_{3,j}}{a^{(5)}_{4,4}}, \quad a^4_{3,j} = \frac{a^{(4)}_{3,j}}{a^{(5)}_{4,4}}. \]

(ii). Suppose that when the elimination step is \( k \), (10) still holds. Then, when the elimination step is \( k + 1 \), since \( i < k + 1 \), thus there are two cases: \( i \leq k - 1 \) and \( i = k \).

(ii-1) When \( i \leq k - 2 \), we have

\[ a^{k+1}_{i,j} = \frac{a^k_{k+1,i+1} a^k_{i,j} - a^k_{i,k+1} a^k_{i+1,j}}{a^k_{k+1,i+1} a^k_{k+1,j} - a^k_{i,k+1} a^k_{i+1,j}} = (-1)^k \frac{a^k_{i,j}}{a^k_{k+1,i+1}}. \]

The last equality is guaranteed by (5).

(ii-2) When \( i = k - 1 \), we get

\[ a^{k+1}_{i,j} = \frac{a^k_{k+1,k+1} a^k_{k-1,j} - a^k_{k-1,k+1} a^k_{k+1,j}}{a^k_{k+1,k+1} a^k_{k-1,j} - a^k_{k-1,k+1} a^k_{k+1,j}} = \frac{(-1)^k a^k_{i,j}}{a^k_{k+1,k+1}}. \]
When $i = k$, it follows that

\[
\begin{align*}
  a_{k,j}^{k+1} &= \frac{a_{k+1,k+1}^{k+1} a_{k,j}^k - a_{k,k+1}^k a_{k,j}^{k+1}}{a_{k+1,k}^{(k+1)} a_{k,j}^{(k)} - a_{k,k+1}^{(k+1)} a_{k,j}^{(k)}} \\
  &= \frac{a_{k+1,k+1}^{(k+1)} a_{k,j}^{(k)} - a_{k,k+1}^{(k+1)} a_{k,j}^{(k)}}{a_{k+1,k}^{(k)} a_{k,j}^{(k)} - a_{k,k+1}^{(k)} a_{k,j}^{(k)}} \\
  &= \frac{a_{k+1,k}^{(k+1)} a_{k,j}^{(k-1)} - a_{k,k+1}^{(k+1)} a_{k,j}^{(k-1)}}{a_{k+1,k}^{(k)} a_{k,j}^{(k-1)} - a_{k,k+1}^{(k)} a_{k,j}^{(k-1)}} \\
  &= \frac{a_{k+1,k}^{(k-1)} a_{k,j}^{(k-1)} - a_{k,k+1}^{(k-1)} a_{k,j}^{(k-1)}}{a_{k+1,k}^{(k-1)} a_{k,j}^{(k-1)} - a_{k,k+1}^{(k-1)} a_{k,j}^{(k-1)}} \\
  &= \frac{a_{k,j}^{(k-1)}}{a_{k+1,k}^{(k-1)}},
\end{align*}
\]

Thus, the equality (10) holds. This completes the proof of Theorem 2.2.

According to the above results, we know that after $k$th Gauss-Jordan elimination step, each $a_{ij}^k (i \neq j, j > k)$ in $A^k$ can be represented as a ratio of two determinants, as follows:

\[
a_{i,j}^k = \left\{ \begin{array}{ll}
  a_{i,j}^{(k)} & i > k, j > k, \\
  a_{i,j}^{(k-1)} & i = k, j > k, \\
  a_{i,j}^{(k)} & i = k-1, j > k, \\
  a_{i,j}^{(k)} & i \leq k-2, j > k.
\end{array} \right.
\]

We believe that many results derived by Gauss-Jordan elimination may be directly reconstructed by (11). Clearly, by the above formula one can also easily construct one solution of $AX = b$. Thus, this method gives a generalized Cramer’s rule.

3. Conclusions

As far as we know, the presented approach has not been published. This new method due to its distinct features can be used in a wide range of scientific and engineering problems. For example, it provides a feasible method to solve a system of linear equations with parametric coefficients by polynomial interpolation technique \[8\]. In addition, this method can be further
developed to give an explicit expression for the elements of the solution of a constrained linear systems of equations [7]. Finally, it can also be applied to solve some integer programming problems.

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