Topological Casimir effect in Maxwell Electrodynamics on a Compact Manifold

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We study the Topological Casimir effect, in which extra vacuum energy emerges as a result of the topological features of the theory, rather than due to the conventional fluctuations of the physical propagating degrees of freedom. We compute the corresponding topological term in quantum Maxwell theory defined on a compact manifold. Numerically, the topological effect is much smaller than the conventional Casimir effect. However, we argue that the Topological Casimir Effect is highly sensitive to an external magnetic field, which may help to discriminate it from the conventional Casimir effect. It is quite amazing that the external magnetic field plays the role of the \( \theta \) state, similar to a \( \theta \) vacuum in QCD, or \( \theta = \pi \) in topological insulators.

I. INTRODUCTION. MOTIVATION.

The nature of the conventional Casimir energy is well-understood by now: the effect is due to the vacuum fluctuations of physical photons, which have slightly different propagating properties in the presence of boundaries in comparison with infinite Minkowski space. Essentially, the electromagnetic modes get modified as a result of nontrivial boundary conditions (BC). For the well-known example of parallel conducting plates, this tiny deviation leads to the well known expression for the Casimir energy

\[
E_C \equiv (E_{BC} - E_{\text{Minkowski}}) = -\frac{L^2 \pi^2}{720 a^3},
\]  

where \( a \) is the separation distance between the two plates of size \( L \). This extra energy gives rise to an attractive force per unit area (vacuum pressure) \[1\]

\[
P = -\frac{\partial (E_C/L^2)}{\partial a} = -\frac{\pi^2}{240 a^4}.
\]  

Today the Casimir force has been measured \[2\], confirming Casimir’s basic idea.

Since its original prediction, the Casimir effect has been studied for countless configurations with fields of various spins. The Casimir effect on nontrivial topological spaces has also been widely explored. In many such cases, a simple scalar field is considered and periodic or twisted boundary conditions are imposed to reflect the topological properties. The fluctuations of the physical field are quantized, for instance, by the periodicity of the space, yielding a Casimir energy similar to (1). See \[3\] for an overview.

However, in the case of gauge fields, we argue that it is important to not only account for the topology of the spacetime manifold, but also the relation between that and the gauge topology. Precisely this topology of the gauge group leads to the emergence of vacuum states that are physically identical but topologically inequivalent. These are known as winding states and are often overlooked in literature on the Casimir effect.

We will explicitly demonstrate in the present work that, for the Casimir effect formulated using a pure photon field on a spacetime manifold with toroidal topology, the non-trivial spatial and gauge topology together induce an additional vacuum pressure that has not been previously computed. Such an effect is purely topological in origin, resulting not from fluctuations of the physically propagating degrees of freedom as in the “conventional” Casimir effect, but rather from the tunnelling between different topological sectors. Mathematically, such phenomena are described by the fundamental group \( \pi_1[U(1)] \cong \mathbb{Z} \), where non-trivial mappings between the spacetime manifold and the gauge group assume the form of gauge transformations. Due to its topological nature, the extra contribution has some unique qualities, both theoretical and practical, that distinguish it from the conventional Casimir effect.

A simple way to get some feeling on the nature of these new topological contributions is to study Maxwell theory in two dimensions, which is essentially the Schwinger model without fermions. As is well known, Maxwell theory in two dimensions is empty, since there are no physical propagating degrees of freedom. Still, there are non-trivial topological sectors in the model which eventually lead to the emergence of the so-called \( \theta \) vacuum state. The construction of these topological sectors is sensitive to the size of the system. Therefore, it is not a surprise that the partition function will be also sensitive to the system’s size, including finite size along the compactified time direction \( \beta \), corresponding to a finite temperature \( T = 1/\beta \). We will elaborate on this example in great detail in section II, using the Hamiltonian as well as Euclidean path integral approaches to explain the nature of topological vacuum fluctuations. We also elaborate on the physical “reality” of these vacuum fluctuations in Appendix A where we compute some important observables, such as the topological susceptibility and entropy.

We also note here that the gauge-induced contributions to the Casimir energy on a compact manifold have been discussed previously in literature. Although no explicit computations were performed, it has been suggested in \[5\] that a sum over all gauge classes may be required to accommodate the non-trivial topological features of the theory. More recently, the computations for the electromagnetic field on general manifolds were explicitly done in \[4\], and papers referenced therein. Our goal here is to discuss some key elements of these new topological terms.
in a more physical and intuitive way, rather than through formal mathematics. Furthermore, we will discuss the relation between the $\theta$ states and the physical realization of these states by placing the system into a uniform external magnetic field. We speculate that a high sensitivity of the extra terms to the applied external magnetic field might be a key element which could allow one to measure these novel types of vacuum fluctuations in real experiments.

To conclude this introduction, we wish to comment on the title of this work and the term “Topological Casimir Effect” (TCE) which will be frequently used in the text below. In some literature this name can refer to the conventional Casimir effect on different topological spaces. However, in the current context, we will use it to strictly denote the additional topological contribution from tunnelling phenomena between the nontrivial topological sectors that make up the $\theta$-state, which is the true vacuum of the configuration. The effect is fundamentally different from that obtained by solely manipulating the spacetime topology and is unique to gauge fields. Exactly in this context, this term was introduced in [6] to emphasize that new extra contribution to the vacuum energy may emerge as a result of tunnelling events.

The structure of our presentation is as follows. In the next section, we review the relevant parts of the two dimensional Maxwell “empty” theory which does not have any physical propagating degrees of freedom, but does show nontrivial topological features. We study this system using the Euclidean path integral approach as well as the Hamiltonian formalism. In section III we generalize our construction to four dimensional Maxwell theory. Numerical estimates in this case suggest that TCE is generally much smaller than the conventional CE in normal circumstances. However, in section IV we advocate an idea that the effect is highly sensitive to a weak uniform external magnetic field. It is very similar to a construction of the so-called $\theta$ states in QCD. Finally, in section V which is our conclusion, we comment on some profound consequences the Topological Casimir Effect may have for cosmology. Furthermore, we advocate the idea that an experimental study of the Topological Casimir Effect in a laboratory might be considered as an investigation of the most intricate properties of the cosmological vacuum and the dark energy observed in our universe. In Appendix A we argue, using an “empty” two dimensional Maxwell model, that the topological vacuum fluctuations are very real and very physical and must be taken into consideration to satisfy some important consistency conditions such as the Ward Identities.

II. MAXWELL THEORY IN TWO DIMENSIONS

The 2d Maxwell model has been solved numerous times using very different techniques, see e.g. [7–9] for a review. We have nothing new to say here. Our goal is in fact quite different: we want to review this “empty” model by emphasizing some elements which will be crucial for our discussions of the Topological Casimir Effect in four dimensions.

A. Hamiltonian framework

We consider 2d Maxwell theory defined on the Euclidean torus $S^1 \times S^1$ with lengths $L$ and $\beta$ respectively. In the Hamiltonian framework we choose a $A_0 = 0$ gauge along with $\partial_1 A_1 = 0$. This implies that $A_1(t)$ is the only dynamical variable of the system with $E = A_1$. The Hamiltonian density, the Gauss law and the commutation relations are

$$\mathcal{H} = \frac{1}{2} E^2, \quad \partial_1 E|_{\text{phys.}} = 0, \quad [A_1(x), E(y)] = i\hbar \delta(x - y),$$

where $|\text{phys.}\rangle$ is the physical subspace. The Gauss law is satisfied only for the $x$-independent (zero) mode. Therefore, the problem is reduced to the quantum mechanical (QM) problem of a single zero mode living on a circle of circumference $L$. In other words, the configurations

$$A_1 \approx A_1 + \frac{2\pi n}{eL}, \quad n \in \mathbb{Z}$$

are gauge equivalent and must be identified. The fact that 2d Maxwell theory does not describe any physical propagating degrees of freedom is well known—it simply follows from the observation that the polarization of a photon must be perpendicular to its momentum. However, such a polarization can not live in the physical space as there is only one spatial dimension $x$, which is reserved for momentum. The presence of a single $x$-independent mode and the absence of all other $x$-dependent modes are manifestations of the “emptiness” of this theory.

The loop integral $\int dx A_1 = eA_1L$ plays the role of phase $\phi$ in the conventional QM problem for a particle on a circle with periodic boundary conditions. The commutation relation (3) then implies that the electric field $E$ is a constant in space and that it is quantized:

$$E = e n \quad n \in \mathbb{Z}.$$  

The hamiltonian $H = \mathcal{H}L$ and the corresponding eigenvalues $E_n$ for this system are well known and are given by

$$H = -\frac{1}{2L} \frac{d^2}{dA_1^2}, \quad E_n = \frac{1}{2} n^2 e^2 L.$$  

Consequently, the partition function for this system is

$$Z(\beta, L) = \sum_{n \in \mathbb{Z}} e^{-\beta E_n} = \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2} \beta L n^2 e^2}.$$  

The construction of the so-called $\theta$ states is also well known for this system [7]. The spectrum in this case is shifted as follows $E_n(\theta) = \frac{1}{2} (n + \frac{\theta}{\pi})^2 e^2 L$, such that the corresponding partition function now takes the form

$$Z(V, \theta) = \sum_{n \in \mathbb{Z}} e^{-\frac{e^2 L}{2} (n + \frac{\theta}{\pi})^2},$$  

where $V$ is the external magnetic field.
where \( V = \beta L \) is the two-volume of the system. Before we discuss the physical meaning of the obtained results in the context of our present work, we want to reproduce the same partition function for the same 2d Maxwell theory using the path integral approach. In this case, the interpretation of eq. (8) will be quite obvious and straightforward. Furthermore, it can easily be generalized to four dimensional Maxwell theory defined on a compact manifold.

### B. Euclidean Path Integral Approach

For path integral computations, we use a Wick rotation to describe the system in a Euclidean metric. Here the inverse temperature \( \beta = 1/T \) takes the role of an imaginary time component with periodic BC, such that we can consider a two dimensional Euclidean torus \( \beta \times L \). We follow [9] and introduce the classical “instantons” in order to describe the different topological sectors of the theory which are classified by the integer \( k \). The transitions between different topological \( k \)-sectors are described by these “instantons”, as given by the following configuration [9]:

\[
e E^{(k)} = \frac{2\pi k}{V}, \tag{9}
\]

where \( Q = \frac{2\pi}{L} E \) is the topological charge density and

\[
\int d^2 x \, Q(x) = \frac{e}{2\pi} \int d^2 x \, E(x) = k \tag{10}
\]

is the integer-valued topological charge in the 2d \( U(1) \) gauge theory, \( E(x) = \partial_0 A_1 - \partial_1 A_0 \) is the field strength\(^1\). The action of this classical configuration is

\[
\frac{1}{2} \int d^2 x E^2 = \frac{2\pi^2 k^2}{e^2 V}. \tag{11}
\]

This configuration corresponds to the topological charge \( k \) as defined by (10). The next step is to compute the partition function defined as follows

\[
Z(\theta) = \sum_{k \in \mathbb{Z}} \int \mathcal{D}A^{(k)} e^{-\frac{\beta}{2} \int d^2 x E^2 + i \frac{\pi}{2\beta} \int d^2 x E}. \tag{12}
\]

All integrals in this partition function are gaussian and can be easily evaluated using the technique developed in [9]. The result is

\[
Z(\beta, L, \theta) = \sqrt{\frac{2\pi}{e^2 V}} \sum_{k \in \mathbb{Z}} e^{-\frac{2\pi^2 k^2}{e^2 V} + ik \theta}, \tag{13}
\]

where the expression in the exponent represents the classical instanton configurations with action (11) and topological charge (10), while the factor in front is due to the fluctuations. The computation of this pre-exponent factor is reduced to a conventional quantum mechanical (QM) problem as the fluctuating field is in fact \( x \)-independent in the \( A_0 = 0 \) gauge, as mentioned in section II A. Therefore, the expression for the pre-exponent is

\[
\int \mathcal{D}A_0 A_1 e^{-\frac{\beta}{2} \int_0^L d\tau (\delta A_1)^2} \tag{14}
\]

A simple way to evaluate this path integral is to rescale the \( A_1 \) field according to its natural dimensionality \( A_1 \equiv a_1 (\frac{\pi}{\beta}) \) where the dimensionless variable \( 0 \leq a_1 \leq 1 \) fluctuates inside a unit interval according to (4). In terms of this rescaled field problem, (14) is reduced to a standard expression for a free particle with mass \( m \equiv L (\frac{\pi}{\beta})^2 \) such that

\[
\int \mathcal{D}a_1 A_1 e^{-\frac{\beta}{4} (\frac{\pi}{\beta})^2 \int_0^L d\tau (\delta A_1)^2} = \sqrt{\frac{m}{2\pi\beta}} = \sqrt{\frac{2\pi}{e^2 L^\beta}} \tag{15}
\]

which is precisely the pre-exponent factor in formula (13).

While expressions (8) and (13) look differently, they are actually identically the same, as the Poisson summation formula states:

\[
\sum_{n \in \mathbb{Z}} e^{-\frac{2\pi^2}{e^2 V} (n + \frac{\pi}{4\pi})^2} = \frac{2\pi}{e^2 V} \sum_{k \in \mathbb{Z}} e^{-\frac{2\pi^2 k^2}{e^2 V} + ik \theta}, \tag{16}
\]

see [10] with detailed discussions on the relation between Hamiltonian formalism and the path integral approach.

### C. Interpretation

The crucial observation for our present study is that this naively “empty” theory which has no physical propagating degrees of freedom, nevertheless shows some very nontrivial features of the ground state related to the topological properties of the theory. These properties are inherent features of the gauge theories and do not have counterparts in conventional scalar field theories. Rather these new properties are related to the presence of different topological sectors in the system, which we refer to as the “degeneracy” of the ground state, for short\(^2\). We interpret the nontrivial properties of the partition function

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1. One should not confuse the electric field from the Euclidean formulation (9) with an \( E \) field (5) computed in Minkowski space. In the former case it is an unphysical complex configuration saturating the path integral, while in the latter it is a “real” physical fluctuating electric field in the Hamiltonian formalism in Minkowski space.

2. Not to be confused with the conventional term “degeneracy”, when two or more physically distinct states are present in the system. In the context of this paper the “degeneracy” implies the existence of winding states \(|n\rangle\) constructed as follows: \( T|n\rangle = |n + 1\rangle \). In this formula the operator \( T \) is the large gauge transformation operator which commutes with the Hamiltonian \( [T, H] = 0 \). The physical vacuum state is unique and constructed as a superposition of \(|n\rangle\) states as follows \(|\theta\rangle = \sum |n\rangle \exp(in\theta)\langle n|\). In the path integral approach, the presence of \( n \) different sectors in the system is reflected by summation over \( k \in \mathbb{Z} \) in eq. (12,13).
(12,13) in this “empty” model as a result of tunnelling between these “degenerate” winding \(|n\) states. These tunnelling processes are happening all the time and the intensity of tunnelling is determined by the topological charge (10) and the size \(V\) of the compact manifold (11). A typical value of the topological charge \(k\) which saturates the series (13) in the large volume limit is very large, \(k \sim \sqrt{e^2 V} \gg 1\).

It is different from the conventional tunnelling in QM in that the tunnelling in our system corresponds to a transition between one and the same physical state, whereas that in QM describes a transition between physically distinct states. More specifically, the tunnelling in our case occurs between the winding \(|n\) states which are connected by large gauge transformations. Therefore, they correspond to one and the same physical state.

The key for our present work is the observation that the properties of these tunnelling processes are sensitive to the size of the system. In different words, the additional energy associated with these tunnelling processes is different for systems with different sizes and shapes. A direct manifestation of this sensitivity (when it is generalized to 4d case as we discuss below) is the emergence of the Topological Casimir Effect (TCE), when the vacuum energy and pressure depend on the size of the compact manifold on which the theory is defined.

Is this extra energy physical? Our ultimate answer is “yes”. We refer to Appendix A where we present some arguments suggesting that the extra energy related to the tunnelling processes in the “empty” theory can not be removed by any redefinition of observables. It must be present in the system for consistency of the theory. In particular, the Ward Identities can not be maintained without these tunnelling contributions, see Appendix A for details. Essentially, this extra contribution is precisely the source of violation of a commonly accepted (but generally wrong) receipt that the Casimir effect due to Maxwell photons could be obtained by multiplying the corresponding scalar expressions by a factor of two.

III. TOPOLOGICAL CASIMIR EFFECT IN QED IN FOUR DIMENSIONS

The topological structure of the gauge field in 2d can be easily generalized to higher dimensions. In four space-time dimensions, we can devise boundary conditions that give rise to very similar instanton-like configurations, with precisely the action (11) as found in 2d Maxwell theory on the torus. In this section we show that these topological degrees of freedom are completely decoupled from the propagating physical photons. Furthermore, the corresponding quantum fluctuations do not depend on the properties of the topological sectors (due to the linearity of the Maxwell equations), and can be treated in the conventional way. As a result we are able to focus on the new contributions and compare them to the conventional Casimir effect from literature. We shall see that

the topological Casimir effect is strongly suppressed on a Euclidean 4-torus where one of the spatial dimensions is much smaller than the others. In this case the well known formulae (1) and (2) are recovered. However, the main goal of this work is to study precisely those novel contributions which are sensitive to the system size.

A. Decoupling of the topological and conventional parts

To construct a theory defined on a Euclidean 4-torus, we consider a system with a box of sizes \(L_1 \times L_2 \times L_3 \times \beta\) in the respective directions. A torus is realized when we assume periodic boundary conditions on the physical fields in all directions, in which case we find a “degeneracy” of the vacuum state, just like in section II: by making a loop in the \(xy\)-plane, the \(A^\mu\) field can pick up a phase corresponding to a large gauge transformation. Working in Euclidean space and adopting the Lorentz gauge, it is simple to find a 4d generalization of the instanton potential from section II B that satisfies these boundary conditions. The 4d instanton potential is given by

\[
A^\mu_{\text{top}} = \left(0, -\frac{\pi k}{e L_1 L_2} x_2, \frac{\pi k}{e L_1 L_2} x_1, 0\right),
\]

where \(k\) is the winding number that labels the topological sector, and \(L_1, L_2\) are the dimensions of the plates in the \(x, y\)-directions respectively, which are assumed to be much larger than the distance between the plates \(L_3\). This classical configuration satisfies the periodic boundary conditions up to a large gauge transformation, and provides a topological magnetic flux in the \(z\)-direction:

\[
\mathcal{B}_{\text{top}} = \nabla \times \mathcal{A}_{\text{top}} = \left(0, 0, \frac{2\pi k}{e L_1 L_2}\right),
\]

in close analogy with the 2d case (9). The Euclidean action of the system becomes

\[
\frac{1}{2} \int d^4x \left\{ \mathcal{E}^2 + \left(\mathcal{B} + \mathcal{B}_{\text{top}}\right)^2 \right\},
\]

where the integration is over the Euclidean torus \(L_1 \times L_2 \times L_3 \times \beta\) and \(\mathcal{E}\) and \(\mathcal{B}\) are the dynamical quantum fluctuations of the gauge field. These terms were not present in the 2d model, but must here be taken into account due to the presence of real propagating physical photons. We find that the action can be easily split into the sum of a topological and a quantum part, because of the vanishing cross term

\[
\int d^4x \mathcal{B} \cdot \mathcal{B}_{\text{top}} = \frac{2\pi k}{e L_1 L_2} \int d^4x B_z = 0
\]

3 A Euclidean 4-torus in this case corresponds to a spatial 3-torus at finite temperature. This method, which corresponds to the Matsubara formalism, is a common way of calculating the Casimir effect at finite temperature.
Here the fact is used that the magnetic portion of quantum fluctuations in the \( z \)-direction, represented by \( B_z = \partial_x A_y - \partial_y A_x \), is a periodic function because \( \tilde{A} \) is periodic over the domain of integration. As a result, there is no coupling between the conventional quantum fluctuations described by photons with physical polarizations and the classical instanton potential (17), (18). Furthermore, the quantum fluctuations due to photons are not sensitive to the topological sector \( k \) of the theory, and therefore they decouple from the classical \( k \)-instanton contribution. Finally, the quantum fluctuations from photons must be computed in a box with size \( L_1 L_2 \) rather than in infinite space along \( x, y \) directions. The corresponding corrections, in principle, can be computed. Technically the computations would be quite tedious as they require the operation with Green’s functions defined on a finite manifold rather than in the infinite space. The computations can be performed for the trivial \( k = 0 \) topological sector as the corresponding corrections are independent of \( k \). These contributions are expected to produce some corrections \( \sim (1 + \frac{\alpha^2}{\beta^2}) \) to formula (2). However, we shall not elaborate on these terms in the present work. It is a part of the conventional partition function \( Z_0 \) computed for trivial topological sector \( k = 0 \).

The main lesson from the previous discussion is that the conventional quantum fluctuations are not sensitive to the topological sectors \( k \) as a result of linearity of the Maxwell equations. Therefore, they can be treated in completely separate ways, which greatly simplifies our analysis. For the partition function we can now write: 
\[ Z = Z_0 \times Z_{\text{top}} \]

The conventional part \( Z_0 \) is well-studied for toroidal BCs at finite temperatures [3]. It is \( k \)-independent, so it will not be elaborated here. In the rest of this section we will study the behaviour of \( Z_{\text{top}} \). We shall see that in the limit when \( L_1 L_2 \rightarrow \infty \) the partition function related to the topological effects yield \( Z_{\text{top}} = 1 \), and we recover the conventional Casimir effect (1), (2) which is computed from \( Z_0 \). However, we shall see that a number of novel and unusual features will emerge in the system when \( L_1, L_2 \) are large but remain finite, which is precisely the main subject of our studies.

B. Computing the topological pressure

The system of parallel plates is related to 2d Maxwell theory by dimensional reduction: taking a slice of the 4d system in the \( xy \)-plane will yield precisely the topological features of the 2d torus. Assuming that \( L_3 \) is much smaller than \( L_1 \) and \( L_2 \), the additional dimensions do not contribute toward \( Z_{\text{top}} \) as we noted above. Instead, the quantum corrections slightly modify \( Z_0 \) as they do not depend on the topological sectors \( k \), and can be factored out from \( Z_{\text{top}} \). With this set up, the classical action for configuration (18) takes the form
\[ \frac{1}{2} \int d^4x \tilde{B}_{\text{top}}^2 = \frac{2\pi^2 k^2 \beta L_3}{e^2 L_1 L_2} \] (21)
while the topological partition function becomes:
\[ Z_{\text{top}} = \sqrt{\frac{2\pi \beta L_3}{e^2 L_1 L_2}} \sum_{k \in \mathbb{Z}} e^{-\frac{2\pi^2 k^2 \beta L_3}{e^2 L_1 L_2}} \] (22)
where the 2d electric charge entering eqs. (11), (13) is expressed in terms of the 4d electric charge as follows
\[ e_{2d}^2 = \frac{e^2}{\beta L_3}, \quad e^2 \equiv \alpha. \] (23)

In this section we consider \( \theta = 0 \). More discussion on this matter is found in section IV.

One should note that the dimensional reduction which is employed here is not the most generic one. In fact, one can impose a non-trivial boundary condition on every slice in the 4d torus. However, the main goal of this work is not to classify the most generic BC, but to discuss the physical properties for the simplest possible case (18), i.e. when a nontrivial BC is imposed on a single slice, while keeping the trivial periodic BC for other slices.

With this objective in mind, it is useful to introduce the dimensionless parameter
\[ \tau \equiv 2\beta L_3 / e^2 L_1 L_2 \] (24)
such that the partition function \( Z_{\text{top}} \) can be written in a very simple form:
\[ Z_{\text{top}}(\tau) = \sqrt{\pi \tau} \sum_{k \in \mathbb{Z}} e^{-\pi^2 \tau k^2} = \sum_{n \in \mathbb{Z}} e^{-\frac{n^2}{\tau}}, \] (25)
where the Poisson summation formula (16) is used again. Our normalization of the partition function \( Z_{\text{top}} \) is such that in the limit \( L_1 L_2 = \infty \) the topological portion of the partition function \( Z_{\text{top}} = 1 \) so that we recover the conventional Casimir effect (1), (2) which is encoded in \( Z_0 \). The simplest way to check our normalization is to take the limit \( \tau \rightarrow 0 \) using the right hand side of eq. (25) when a single term with \( n = 0 \) contributes. It corresponds to very large instanton numbers \( k \sim \tau^{-1/2} \rightarrow \infty \) saturating the original series (22).

From \( Z_{\text{top}} \), we can calculate any thermodynamic property of the system, like the topological pressure between the plates
\[ P_{\text{top}} = \frac{1}{\beta L_1 L_2} \frac{\partial}{\partial \beta L_3} \ln Z_{\text{top}}. \] (26)

In the asymptotic limit where \( \tau \ll 1 \) one can use the dual representation of \( Z_{\text{top}} \) encoded by the Poisson re-summation formula (25) to find that
\[ P_{\text{top}} \approx \frac{e^2}{\beta^2 L_3} e^{-\frac{1}{\tau}}, \quad \tau \ll 1. \] (27)

\footnote{Note that the units are consistent. Indeed, in 4d, \( e^2 \sim \alpha \) is the dimensionless fine-structure constant. However, in 1+1 dimensions the QED coupling constant has units of \( (\text{length})^{-2} \).}
In this case, the topological pressure is exponentially suppressed, and when compared with the conventional Casimir pressure \( P \) it is clear that the topological effect is too small to measure experimentally\(^5\).

A few comments are in order. First of all, one can explicitly see that the original instanton formula (22) is consistent with our interpretation: that the additional energy and pressure due to the topological features of the system is the result of tunnelling events between different winding states, as discussed in section II C. Indeed, a non-analytical structure of eq. (22) with respect to coupling constant exp\((-1/\varepsilon^2\)) represents the typical behaviour for a tunnelling process. It is quite fortunate that the Poisson re-summation formula (25) allows us to analyze both regimes, at large as well as small \( \tau \). Secondly, even in this simple \( \tau \ll 1 \) case one can explicitly see that the sign of the effect is opposite to the conventional Casimir effect (2). This correction leads to repulsive rather than attractive forces. This “wrong” sign is a typical manifestation of the topological fluctuations, in contrast with the conventional vacuum fluctuations of photons with physical polarizations. See some additional comments on a “wrong sign” in Appendix A.

While \( \tau \ll 1 \) can be examined analytically, it is more interesting to study a system with \( \tau \approx 1 \) where this effect could be sufficiently large. We can satisfy the conditions \( L_1, L_2 \gg \beta, L_3 \), in order to use the dimensional reduction, and at the same time still achieve \( \tau \approx 1 \) because the small parameter \( \varepsilon^2 \) enters the denominator in eq. (24) in the definition for \( \tau \). In this case we have to resort to numerical approximations, since there is no closed form for the partition sum and pressure. In Figure 1, a numerical plot of \( P_{\text{top}} \) is shown for this regime. There is a large peak around \( \tau \approx 0.4 \) where the pressure, measured in units \( \frac{2}{L_1^2 L_2^2 \varepsilon^2} \), has an order of 1. The relative magnitude between the maximum topological pressure and the conventional Casimir pressure \( P \) using parallel ideal conductors at low temperature is thus approximately

\[
R_{\text{max}} \approx \frac{|P_{\text{top}}|}{|P|} \approx \frac{480 \beta L_3^4}{L_1^4 L_2^2 \varepsilon^2 \pi^2} \approx \frac{120 \beta}{\pi^2 \alpha} \cdot \frac{L_3^4}{L_1^4 L_2^2} \quad \text{(28)}
\]

This ratio (even at its maximum at \( \tau \approx 0.4 \)) is very small in a typical Casimir experiment setup with \( L_1, L_2 \gg L_3 \), in spite of the large numerical factor in front of formula (28). As we mentioned previously, the power-like corrections \( \sim L_3^2/L_1^2, L_3^2/L_2^2 \) are also expected to occur in \( \mathcal{Z}_0 \) resulting from the conventional vacuum fluctuations of physical photons. We expect these conventional corrections to be even smaller as they can not contain a parameterically enhanced factor \( 1/e^2 \) that is a unique feature of the topological vacuum fluctuations.

![Topological Casimir Pressure](image)

FIG. 1. The topological pressure on the 4d system of parallel plates as a function of \( \tau \equiv 2\beta L_3/e^2 L_1 L_2 \). Pressure is measured in units \( \frac{2}{L_1^2 L_2^2 \varepsilon^2} \).

To conclude this section, we find that there is a small, but very real, contribution to the Casimir effect that is purely due to topological features of the system. When QED is defined on a compact manifold such as a 4-torus, one needs to take into account the tunnelling processes which occur between the topologically inequivalent (but physically identical) winding states. These topological transitions are described in terms of integer magnetic fluxes (18). It is not surprising that the effect is exponentially small in normal circumstances (27). The effect remains very small (28) even at \( \tau \approx 1 \). Still, there is a hope to make it measurable by studying the Topological Casimir Effect (TCE) in the presence of some external magnetic field. We shall observe a high sensitivity of TCE to applied weak external magnetic field. This should be contrasted with conventional Casimir effect (1), (2) which can not be sensitive to external fields as vacuum photon fluctuations do not couple to external fields (since the Maxwell equations are linear). This topic is precisely the subject of the next section.

### IV. \( \theta \) Vacua and External Magnetic Fields

Now it is interesting to place our system into a region with a weak external magnetic field \( B_{z}^{\text{ext}} \) along the \( z \)-direction. The idea behind this construction is that the

\(^5\) The conventional Casimir pressure in eq. (2) does not account for the thermal correction, and is computed for a system with slightly different boundary conditions (metallic instead of periodic). However, the boundary conditions only change the pressure by a constant of order unity and the thermal correction is negligible for low temperatures, see [3]. Also note that even though \( \tau \ll 1 \), we are still in the low-temperature regime.
external magnetic field $B_{\text{ext}}$ will interfere with the integer topological flux (18) describing the tunnelling events. It is expected that such interference may skew the summation over the topological sectors, similar to the action of the so-called $\theta$ parameter (12), (13). As we shall demonstrate below, this is indeed what happens in our simple case considered in section III. In different words, we claim that by adding a constant magnetic field to the previous setup, an effective non-zero $\theta_{\text{eff}}$ parameter emerges in the system. The crucial point here is that we introduce this parameter which can be externally varied. By studying the corresponding responses to $\theta_{\text{eff}}$ variation, it gives us some hope that while the TCE is numerically very small (28), it is nevertheless very sensitive to a weak magnetic external field (in contrast with conventional Casimir effect (2)), and hopefully it can be eventually measured due to this sensitivity.

To construct a system as such, in addition to the topological flux through the xy-plane, we apply a real physical constant magnetic field $B_{\text{ext}} = \partial_z A_{\text{ext}}^y - \partial_y A_{\text{ext}}^z$ parallel to the z-direction (perpendicular to the xy-plane). The total $B_z$ field in the Euclidean metric is thus modified as follows:

$$B_z = B_{z}^2 + B_{z}^{\text{top}} + B_{z}^{\text{ext}}$$

where the total field decomposition consists of the same instanton potentials $A_{\mu}^{\text{top}}$ as in eq. (17), the external magnetic field potential $A_{\mu}^{\text{ext}}$ given above and the quantum fluctuations $A_{\mu}^{\text{pot}}$ around them.

The only difference from the previous construction is the additional external constant magnetic field. Note that the quantum fluctuations still decouple from the classical and external fields, similar to eq. (20), due to the periodicity of quantum fluctuations over the domain of integration,

$$\Delta S = (B_{z}^{\text{ext}} + \frac{2\pi k}{L_1 L_2 e}) \cdot \int d^4 x B_{z}^q = 0.$$  

The remaining part of the action is quadratic and thus path integration can be performed. The same calculation from the previous section follows and the partition function separates into a classical portion, which describes TCE, and a quantum portion that corresponds to the effect of photons in 4D, i.e. the well known Casimir effect. It is important that the conventional quadratic term represents the photon fluctuations over the domain of integration,

$$Z_{\text{top}}(\tau, \theta_{\text{eff}}) = \sqrt{\pi \tau} \sum_{k \in \mathbb{Z}} \exp \left[ -\pi^2 \tau \left( k + \frac{\theta_{\text{eff}}}{2\pi} \right)^2 \right]$$

where we introduced the effective theta parameter

$$\theta_{\text{eff}} = B_{z}^{\text{ext}} L_1 L_2 e$$

proportional to the external magnetic flux through the xy-plane in this particular system. It is clear from the partition sum (31) that a non-zero effective $\theta_{\text{eff}}$ skews the summation over topological sectors similar to the 2d example given by (12), (13). It is also clear that $\theta_{\text{eff}} = 2\pi m$ corresponds to integer flux $m$ through the xy-plane, which obviously can not modify the system, such that $Z_{\text{top}}(\tau, \theta_{\text{eff}})$ is $2\pi$ periodic in $\theta_{\text{eff}}$.

In what follows, we also need a “dual” representation for $Z_{\text{top}}(\tau, \theta_{\text{eff}})$ which is obtained by applying the Poisson re-summation formula (16)

$$Z_{\text{top}}(\tau, \theta_{\text{eff}}) = \sum_{n \in \mathbb{Z}} \exp \left[ -\frac{n^2}{\tau} + i \theta_{\text{eff}} n \right].$$

In representation (33), it is obvious that $\theta_{\text{eff}}$ being expressed in terms of the external magnetic field (32) can be thought of as a fundamental $\theta$ parameter. However, the corresponding “instanton charge” $n$ which normally enters with $\theta$ is not the same magnetic flux $k$ from our original construction (18) with classical action (21). Rather, it is some “dual” configuration with classical action $\sim 1$.

We note that $Z_{\text{top}}(\tau, \theta_{\text{eff}})$ is properly normalized in the limit $L_1, L_2 \to \infty$ which corresponds to $\tau \to 0$. In this case, only a single term with $n = 0$ in (33) survives, leading to the desired normalization $Z_{\text{top}}(\tau \to 0, \theta_{\text{eff}}) = 1$. Therefore, all conventional formulae for the Casimir effect determined by $Z_0$ are recovered in this limit as $Z$ is factorized $Z = Z_0 \times Z_{\text{top}}$ for our system.

Now we are in position to calculate the topological Casimir pressure contribution from free energy at finite temperature, by inserting the partition function into eq. (26). Like in section III B, the topological pressure has no closed form. For the limit where $\tau \ll 1$, we may obtain an asymptotic expansion. Using the dual representation (33) and keeping the leading order terms, we arrive at:

$$P_{\text{top}} \approx e^2 \frac{\cos(\theta_{\text{eff}})}{\beta^2 L_5^3} \exp(-1/\tau), \quad \tau \ll 1$$

which reduces to our previous formula (27) in the absence of the external magnetic field. As expected, the oscillatory effect with respect to $\theta_{\text{eff}}$ is present, but becomes exponentially suppressed because the tunnelling amplitudes naturally diminish in this limit.

In a more general case when $\tau \approx 1$, we have to use some numerical methods as asymptotic analysis is no longer sufficient. A numerical plot is shown in Figure 2, such that the variation with $\theta_{\text{eff}}$ is manifest. The pressure is clearly oscillatory with respect to $\theta_{\text{eff}}$ and its local extrema are attained at $n\pi$ where $n \in \mathbb{Z}$. Thus, by altering the magnetic flux, the topological Casimir pressure will also be modified accordingly. Additionally, at $\theta_{\text{eff}} = 0$ the pressure has a “wrong” sign, i.e. it is opposite to conventional Casimir effect (2), as we already discussed after eq.(27). This sign changes as a function of the external magnetic field, as can be seen in the plot. Such a variation can be interpreted as the result of interference between the external magnetic field and the
This page contains a discussion on the effects of the external magnetic field on the induced magnetic field in units $\frac{1}{L_1L_2}$. The induced magnetic field defined as (35) can be thought of as the magnetization of the system per unit volume, i.e. $\langle M \rangle = -(B_{ind})$, as the definition for $\langle M \rangle$ is identical to (35) up to a minus sign.

As before, the topological effects are exponentially suppressed at $\tau \ll 1$, as $Z_{top} \to 1$ with exponential accuracy at $\tau \ll 1$. The effect is much more pronounced in the range where $\tau \approx 1$, see Figure 3, where we plot the induced magnetic field in units $(L_1L_2)^{-1}$ as a function of $\theta_{eff}$. One should also remark here that the induced magnetic field defined as (35) can be thought of as the magnetization of the system per unit volume, i.e. $\langle M \rangle = -(B_{ind})$, as the definition for $\langle M \rangle$ is identical to (35) up to a minus sign.

Now we turn our attention to the magnetic susceptibility, which is similar to the topological susceptibility reviewed in Appendix A for 2d QED. This object is $\mathcal{P}$ and $\mathcal{CP}$ even and does not vanish at zero external field. The magnetic susceptibility measures the response of free energy to the introduction of a source term, which is represented in our case by $B^{ext} \sim \theta_{eff}$. To be more precise, we define $\chi_{mag}$ in a way which is similar to the topological susceptibility in Appendix A for 2d QED.

$\chi_{mag} = \int d^4x \langle B_z(x), B_z(0) \rangle = -\frac{1}{\beta V} \frac{\partial^2 \ln Z_{top}}{\partial B^{ext}_z}(36)$

where the integration is over the Euclidean torus $L_1 \times L_2 \times L_3 \times \beta$. With this definition, $\chi_{mag}$ is a dimensionless parameter, in contrast with 2d QED where $\chi_{E&M}$ has dimension $(mass)^2$, and in 4d QCD where $\chi_{QCD}$ has dimension $(mass)^4$. This is due to the fact that the topological density operator has dimension $(mass)^2$ in 2d QED and $(mass)^4$ in 4d QCD while our topological instanton (18) expressed in terms of the magnetic field has dimension $(mass)^2$. Nevertheless, we opted to keep definition (36) without inserting any additional dimensional parameters such as $L_1L_2$ into formula (36), to maintain the conventional definition in statistical mechanics.
where $\chi_{mag}$ is a dimensionless parameter (in units where $\hbar = c = 1$).

We can represent (36) in terms of dimensionless variables as follows,

$$\chi_{mag} = -\frac{2}{\tau} \frac{\partial^2 \ln Z_{top}(\tau, \theta_{eff})}{\partial \theta_{eff}^2}. \quad (37)$$

In the limit when $\tau \ll 1$, one can use analytical expression (33) to conclude that $\chi_{mag} \sim \exp(-\frac{1}{\tau})$ is strongly suppressed. It is consistent with our expectations that there should not be any magnetic correlations in the conventional Casimir experimental setups. However, for $\tau$ near the order of 1, the behaviour is quite nontrivial as shown in Figure 4. Note that in this case, the susceptibility is highly dependent on the external field and changes signs, which is extremely unusual for conventional systems. More specifically, $\chi_{mag}$ behaves like a weak diamagnetic for $\theta_{eff}$ as it should, and it does not vanish even at zero external magnetic field at $\theta_{eff} = 0$ due to topological fluctuations, similar to well studied cases of 2d QED and 4d QCD.

We note that this topological effect is quite distinct from the behaviour of the conventional Casimir effect. In the conventional quantization of electromagnetic fields in Minkowski space, there is no direct connection between the Casimir pressure and an external magnetic field. Although such coupling could be attained through fermionic interactions in higher order diagrams, it will be highly suppressed $\sim \alpha^2 B_\text{ext}^2 / m_e^4$ as the non-linear Euler-Heisenberg Effective Lagrangian would suggest. In fact the explicit computations of this effect have been done [11] and they fully agree with our order of magnitude estimates$^7$. Thus, in a pure Maxwell theory as we consider here, the photon-photon interactions should be trivial. However, in TCE, we see that the external magnetic field does have a non-trivial effect on physical quantities through its interactions with topological “instanton” fluctuations (18). To put this in more concrete terms, the numerical value for loop level corrections in the conventional Casimir effect is of order $10^{-20}$ even with a 1T external magnetic field. In contrast, the proposed correction to TCE in an external field is of order 1, as shown in Figure 2.

Therefore, the periodicity in all of the physical quantities with respect to the external magnetic field is a unique feature of TCE. It is not found in any of the typical Casimir results and can serve as a clear indicator to distinguish a topological effect from conventional Casimir effects.

V. CONCLUSION AND FUTURE DIRECTIONS

We have demonstrated the existence and properties of a new type of vacuum fluctuations in gauge fields, resulting from the summation over topological sectors. While most literature on the Casimir effect neglects these topological sectors, which are indeed absent in the topologically trivial Minkowski space, they need to be taken into account when the theory is formulated on a non-simply connected, compact manifold. Physics related to pure gauge configurations describing the topological sectors does not go away when one removes all unphysical degrees of freedom as a result of gauge fixing; instead, this physics reappears in a much more complicated form where the so-called Gribov ambiguities emerge [12]. See recent paper [4] and also some previous relevant discussions [13–15]. We opted to keep some gauge freedom in our analysis to study these topological sectors explicitly.

Now we can formulate the main results of the present paper. First of all, the physics behind the Topological Casimir Effect (TCE) is quite simple: there are tunnelling events in a theory formulated on a small compact manifold when the temperature is small but not identically zero. These transitions can be completely ignored for relatively large systems, but in general, these topological transitions interpolating between different winding states (which correspond to one and the same unique

$^6$ This holds for the present assumption where radiative corrections are not considered and the boundary conditions are not affected by the magnetic field. Note that in the case of the TCE it is the bulk vacuum that is sensitive to an external magnetic field.

$^7$ We are thankful to our anonymous referee for pointing out that the corresponding computations have in fact been explicitly performed in [11] for the case of conventional Casimir effect.

$^8$ A short historical remark is warranted here. The Gribov ambiguities [12] were originally discussed for non-abelian gauge theories in Minkowski space when one tries to completely remove all unphysical degrees of freedom in the Coulomb gauge. The corresponding Gribov ambiguities lead to strong infrared singularities in the consequent analysis. Another option is to deal explicitly with some unphysical degrees of freedom that effectively describe these topological sectors.
physical state) do occur. The amplitudes for these transitions depend on size and shape of the system. Therefore, it is not really a surprise that the vacuum energy associated with these tunnelling events depends on the size of the system, which ultimately implies an extra contribution to the Casimir pressure.

In general, the effect is numerically much smaller than the conventional Casimir effect with the ratio given by eq. (28). However, we argued that the effect is highly sensitive to small external magnetic fields which can serve as a clear indicator to distinguish TCE from conventional Casimir effects.

Our last comment is as follows. The TCE as we already mentioned in the introduction is a very generic phenomenon in gauge theories. It shows the algebraic sensitivity to the size of the system even when the theory has a mass gap. See [6, 16] and many references therein where TCE has been tested in various models, including QCD lattice computations. Our comment here is that the observed Dark Energy (DE) in the universe might be a direct manifestation of the TCE as argued in [6, 16] and references therein. The idea is based on two key elements. Firstly, the additional energy in Maxwell theory (defined on a compact manifold and discussed in this paper) is based on nontrivial topological properties formally expressed by the first homotopy group $\pi_1[U(1)] \sim \mathbb{Z}$. In four dimensions a similar structure emerges for non-abelian QCD where the third homotopy group is nontrivial $\pi_3[SU(3)] \sim \mathbb{Z}$. In this case one can argue that the system is algebraically sensitive to very large distances in spite of the fact that the theory has a mass gap. The second key element is based on the paradigm that the relevant definition of the energy which enters the Einstein equations is the difference $\Delta E \equiv (E - E_{\text{Mink}})$, similar to the Casimir effect (1), rather than the energy $E$ itself. In this case the difference between the two metrics (expanding universe with Hubble expansion rate $H$ and Minkowski space-time) as a result of TCE would lead to an estimate [6, 16]

$$\Delta E \sim L^{-1} A_{\text{QCD}}^3 \sim (10^{-3}\text{eV})^4,$$  \hspace{0.5cm} (38)

where $L$ is the visible size of the universe estimated as $L^{-1} \sim H \sim 10^{-33}\text{eV}$. Estimation (38) is amazingly close to the observed DE value today. In fact, a comprehensive phenomenological analysis of this model (the so-called "ghost dark energy" model) has been recently studied in a number of papers where comparisons have been made with the current observational data. (See references on observational papers in [6, 16].) The conclusion was that the model (38) is consistent with all presently available data, and we refer the reader to the original papers on analysis of the observational data.

Our comment relevant for the present study is that some very fascinating topological properties of the quantum vacuum may be, in principle, studied in a laboratory if the TCE in Maxwell theory, which is the main subject of the present work, can be experimentally measured. We conclude on this optimistic note.$^9$

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**Appendix A: Why topological vacuum fluctuations must be real and physically observable.**

The main subject of the present work is zero point (vacuum) fluctuations. There has always hung a shadow over this question as there have always been suspicions that those vacuum fluctuations are not really zero point fluctuations, but rather can be attributed to some other physics. See in particular the relatively recent paper [18] where it has been argued that the conventional Casimir effect can be computed without even mentioning such a notion as the "vacuum".

The main goal of this appendix is to argue, using a simple exactly solvable 2d model, that the topological vacuum fluctuations are very real and very physical. In other words, we want to present a few arguments suggesting that finite contributions resulting from topological features of the system cannot be removed by any means such as subtraction or redefinition of observables.

We start our study with the topological susceptibility $\chi$ defined as follows,

$$\chi \equiv \frac{\epsilon^2}{4\pi^2} \lim_{k \to 0} \int d^2 x \ e^{i k x} \langle TE(x)E(0) \rangle,$$ \hspace{0.5cm} (A1)

where $Q = \frac{e}{2\pi} E$ is the topological charge density and

$$\int d^2 x \ Q(x) = \frac{e}{2\pi} \int d^2 x \ E(x) = k$$ \hspace{0.5cm} (A2)

is the integer valued topological charge in the 2d $U(1)$ gauge theory, $E(x) = \partial_1 A_2 - \partial_2 A_1$ is the field strength. The $\chi$ measures response of the free energy to the introduction of a source term

$$L_\theta = i \theta \frac{e}{2\pi} \int d^2 x \ E(x).$$ \hspace{0.5cm} (A3)

The corresponding computations can be easily carried out as the partition function $Z(\theta)$ is known exactly, see

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$^9$ In fact, the idea to test some intriguing vacuum properties relevant for cosmology in a laboratory is not a very new idea. It has been advocated by G. Volovik for years, see recent review [17] and references therein.
section II. When differentiating the partition function twice with respect to $\theta$, we get a finite contribution in the infinite volume limit, $V \equiv \beta L \rightarrow \infty$, i.e.

$$\chi_{E&M} = -\frac{1}{\beta L} \frac{\partial^2 \ln Z(\theta)}{\partial \theta^2} \Bigr|_{\theta=0} = \frac{e^2}{4\pi^2}. \quad (A4)$$

Is contribution (A4) physical? This question immediately arises because we are dealing with “empty” Maxwell theory in two dimensions, where there are no physical propagating degrees of freedom in the system. Can we redefine the theory to remove all such terms from consideration once and for all? For example, one can use a prescription [19] which ignores the topological sectors and leads to a trivial partition function $Z = 1$, see eqs (4.1), (4.3) in [19]. Such a prescription would obviously be consistent with the conventional procedure which relates the Casimir effect for 4d Maxwell theory and 4d massless scalar field theory up to factor 2. However, such a prescription would not produce the contact term (A4) which must be present in the system for its consistency, as we shall argue below.

The question addressed above on “physical reality” of (A4) is not a purely academic question. If eq. (A4) is treated as a physical contribution, then the partition function $Z(\beta, L, \theta)$ which leads to (A4) is also physical. Therefore, the same partition function $Z(\beta, L, \theta)$ will also lead to an extra Casimir force $P \sim \partial \ln Z/\partial L$ which “in principle” is an observable quantity. In different words, if (A4) is physical, then there is an extra term in the Casimir energy which is not related to any asymptotic propagating degrees of freedom and is “in principle” observable. We present a few arguments below to advocate that (A4) is indeed physical.

Our argument goes as follows. We add a massless fermion field $\psi$ to the system to arrive at the well known 2d Schwinger model. The expression for the topological susceptibility in the 2d Schwinger QED model is known exactly [9, 16]

$$\chi_{QED} = \frac{e^2}{4\pi^2} \int d^2 x \left[ \delta^2(x) - \frac{e^2}{2\pi^2}K_0(\mu|x|) \right], \quad (A5)$$

where $\mu^2 = e^2/\pi$ is the mass of the single physical state in this model, and $K_0(\mu|x|)$ is the modified Bessel function of order 0, which is the Green’s function of this massive particle. The crucial observation here is as follows: any physical state contributes to $\chi_{QED}$ with negative sign

$$\chi_{\text{dispersive}} \sim \lim_{k \rightarrow 0} \sum_n \frac{\langle 0 | \frac{e^2}{2\pi^2} E^n | n \rangle \langle n | \frac{e^2}{2\pi^2} E | 0 \rangle}{-k^2 - m_n^2} < 0, \quad (A6)$$

In particular, the term proportional to $-K_0(\mu|x|)$ with the negative sign in eq. (A5) is the result of the only physical field of mass $\mu$. However, there is also a contact term $\frac{e^2}{4\pi^2}$ in eqs. (A5), (A4) which contributes to the topological susceptibility $\chi$ with the opposite sign, and which can not be identified according to (A6) with any contribution from any asymptotic state.

The first term $\frac{e^2}{4\pi^2}$ in this formula (A5) can be easily recognized as the expression for $\chi_{E&M}$ for 2d Maxwell theory (A4) which is originated from the topological sectors, and not related to propagating degrees of freedom.

This term has a fundamentally different, non-dispersive nature. In fact it is ultimately related to different topological sectors discussed in section II. This contact term must be present in the expression (A5) to satisfy the Ward Identity (WI) which states that $\chi_{QED}(m=0) = 0$, see [16] for the details. Without this contribution, it would be impossible to satisfy the Ward Identity because the physical propagating degrees of freedom can only contribute with sign ($-$) to the correlation function as eq. (A6) suggests, while WI requires $\chi = 0$ in the chiral limit $m = 0$. One can explicitly check that WI is indeed automatically satisfied only as a result of exact cancellation between conventional dispersive term with sign ($-$) and non-dispersive term (A4) with sign ($+$),

$$\chi = \frac{e^2}{4\pi^2} \int d^2 x \left[ \delta^2(x) - \frac{e^2}{2\pi^2}K_0(\mu|x|) \right] \quad (A7)$$

$$= \frac{e^2}{4\pi^2} \left[ 1 - \frac{e^2}{2\pi} \frac{1}{\mu^2} \right] = \frac{e^2}{4\pi^2} [1 - 1] = 0.$$

The lesson we learn from this simple exercise is that the contact term (A4) which is saturated by the topological sectors is physical, and it must be present in the system for its consistency.

The same contact term (A4, A5) can be also computed using the auxiliary ghost fields, the so-called Kogut-Susskind (KS) ghost, as has been originally done in ref. [20], see [16] for relevant discussions in the present context. This auxiliary ghost field effectively takes into account the presence of topological sectors which lead to (A4). The crucial element accounting for different topological sectors of the underlying theory, does not go away in KS-description. Rather, this information is now coded in terms of the unphysical ghost scalar field which provides the required “wrong” sign for contact term (A4, A5). The contact term in this framework is precisely represented by the ghost contribution replacing the standard procedure of summation over different topological sectors. At the same time, this unphysical ghost scalar field does not violate unitarity or any other important properties of the theory as consequence of Gupta-Bleuler-like condition on the physical Hilbert space, see [16] for the details in the given context.\footnote{10 It is important to emphasize that the KS ghost should not be confused with the conventional Fadeev-Popov ghost which is normally introduced into the theory to cancel out unphysical polarizations of the gauge fields. Instead, the KS ghost is introduced to account for the existence of topological sectors in the theory. A similar construction is also known for four dimensional non-abelian gauge theory where the corresponding color singlet field is called the Veneziano ghost, see [16] for references and details.}
Our second argument that the topological sectors must be taken into consideration is based on analysis of the entropy in the same 2d “empty” Maxwell theory. Before we formulate our argument, we want to make a short historical detour on the entropy studies in this “empty” model.

It has been claimed [21] that for spins zero and one-half fields, the one loop correction to the black hole entropy is equal to the entropy of entanglement, while for a spin one Maxwell field, the entropy has an extra term describing the contact interaction with the horizon. While the entropy is a positively defined entity, the Kabat contact term is negative [21]. Furthermore, this term being a total divergence can be represented as a surface term determined by the behaviour of the theory at arbitrarily large distances, i.e. it obviously has an infrared (IR) origin. More recently, it has been conjectured [16] that the Kabat contact term is originated from the same topological gauge sectors which saturate the topological susceptibility (A4). Indeed, both terms have “wrong” signs in comparison with what physical propagating degrees of freedom would produce, and both terms can be represented by surface integrals, see [16] for the details. Next step in this development was the computation of the entropy for the 2d Maxwell system, defined on a finite dimensional compact manifold with size $V = \beta L$, such that the IR physics can be properly treated [22], see also [23, 24] with related discussions. In this case the expression for the entropy can be easily computed from the partition function (8), (13), (16) discussed in section II, and is given by [22]

$$ S = \left( \ln Z + \frac{1}{2} \right) - \frac{1}{2} \left( \frac{4\pi^2}{c^2} \right) \cdot \chi_{E&M}, \quad (A8) $$

where $\chi_{E&M}$ is the topological susceptibility given by eq. (A4). One can explicitly see that the negative contribution is indeed present in the expression (A8) for the entropy. This term with the “wrong” sign in eq.(A8) is exactly proportional to the topological susceptibility (A1) in agreement with conjecture [16]. Furthermore, this term can be represented as a surface integral because $Q = \frac{\pi}{c} E$ entering (A1) is the topological charge density operator which is a total divergence. One should also emphasize that the entropy $S$ as well as its surface term $\sim \chi_{E&M}$ separately are gauge invariant observables. Also, while the term $\sim \chi_{E&M}$ can be represented as a surface integral, the entropy itself does not possesses such a surface representation. Finally, the entropy (A8) can be interpreted as the entanglement entropy because the only local observable is $E$, which is constant over space as shown in section II. It means that the measurements of $E$ will be perfectly correlated on the opposite sides of the system [22].

The crucial observation for the present paper is as follows. When the IR physics is properly treated, the entropy (A8) is obviously a positively defined function. Furthermore, as the theory under discussion is “empty” in the sense that it does not describe any physical propagating degrees of freedom in the bulk, one should expect that the entropy $S$ must vanish in the infinite volume limit $V \to \infty$. This expectation follows from the fact that the only dynamics in this system could be related to the so-called “edge states” which are localized at the boundary of the system but not in the bulk, similar to other topological field theories [8]. The only way this vanishing result could occur is the presence of a negative contribution which could cancel a conventional positive contribution present in (A8). In different words, the negative contribution in (A8) is a must in order to produce an anticipated vanishing entropy in the infinite volume limit $V \to \infty$.

As we discussed previously, this contribution with a “wrong” sign can not be identified with any physical propagating degrees of freedom. Rather it is related to the tunnelling processes between different topological sectors as discussed in section II. These discussions again support our claim that the topological sectors must be included into consideration for self-consistency of the theory. Therefore, the additional terms they produce leading to the Topological Casimir Effect should be considered as physically observable quantities. As the last comment of this Appendix: though the term (A4) with a “wrong” sign is a gauge invariant contribution, its explicit computation depends on a specific gauge-dependent technique being used. In particular, in the KS framework [20] this term is saturated exclusively by an unphysical ghost field, see explicit computations in [16]. Still, this term (A1) is physical as we argued above, and it can not be discarded on a sole basis that it is saturated by the artificial ghost.

The main lesson of this Appendix is that there are extra contributions to the vacuum energy due to non-trivial topological features of the gauge fields, which do not have counterparts in scalar field theory. Therefore, the standard receipt (that the contribution to the energy and pressure due to the physical Maxwell photons could be obtained directly from expressions for massless scalar field by multiplying the corresponding scalar expressions by factor two) does not represent a complete description of the ground state in the presence of the gauge fields.

In the four dimensional case the Veneziano ghost can not be confused with a Fadeev-Popov ghost as the Veneziano ghost being a singlet does not carry a color index, in contrast with Fadeev-Popov ghosts. The sole purpose of the Veneziano ghost is to saturate the contact term with the “wrong sign” in the topological susceptibility, similar to eq. (A4), (A5) in the 2d Schwinger model.
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