A SUP-HODGE BOUND FOR EXPONENTIAL SUMS

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Abstract. The $C$-function of $T$-adic exponential sums is studied. An explicit arithmetic bound is established for the Newton polygon of the $C$-function. This polygon lies above the Hodge polygon. It gives a sup-Hodge bound of the $C$-function of $p$-power order exponential sums.

1. Introduction

Let $p$ be a prime number, $\mathbb{F}_p = \mathbb{Z}/(p)$, $\mathbb{F}_p$ a fixed algebraic closure of $\mathbb{F}_p$, and $\mathbb{F}_{p^k}$ the subfield of $\mathbb{F}_p$ with $p^k$ elements.

Let $q > 1$ be a power of $p$, $W$ the ring scheme of Witt vectors, $\mathbb{Z}_q = W(\mathbb{F}_q)$, $\mathbb{Q}_q$ the fraction field of $\mathbb{Z}_q$, $\overline{\mathbb{Q}_p} = \lim_{\rightarrow k} \mathbb{Q}_p^k$, and $\mathbb{C}_p$ the $p$-adic completion of $\overline{\mathbb{Q}_p}$.

Let $\triangle \supseteq \{0\}$ be an integral convex polytope in $\mathbb{R}^n$, and $I$ the set of vertices of $\triangle$ different from the origin. Let

$$f(x) = \sum_{u \in \triangle} (a_u x^u, 0, 0, \ldots) \in W(\mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) \text{ with } \prod_{u \in I} a_u \neq 0,$$

where $x^u = x_1^{u_1} \cdots x_n^{u_n}$ if $u = (u_1, \cdots, u_n) \in \mathbb{Z}^n$.

Definition 1.1. For $k \in \mathbb{N}$, the sum

$$S_f(k, T) = \sum_{x \in (\mathbb{F}_{q^k}^\times)^n} (1 + T)^{\text{Tr}_{\mathbb{Z}_q^k/\mathbb{Z}_p}(f(x))} \in \mathbb{Z}_p[[T]]$$

is called a $T$-adic exponential sum. And the function

$$L_f(s, T) = \exp\left(\sum_{k=1}^{\infty} S_f(k, T) \frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_p[[T]][[s]],$$

as a power series in the single variable $s$ with coefficients in the $T$-adic complete field $\mathbb{Q}_p((T))$, is called an $L$-function of $T$-adic exponential sums.

Let $m$ be a positive integer, $\zeta_p^m$ a primitive $p^m$-th root of unity, and $\pi_m = \zeta_p^m - 1$. Then $S_f(k, \pi_m)$ is the exponential sum studied by Liu-Wei [LWe]. If $m = 1$, the exponential sum $S_f(k, \pi_m)$ was studied by Adolphson-Sperber [AS,AS2]. And, if $n = 1$, the exponential sum $S_f(k, \pi_m)$ was studied by Kumar-Helleseth-Calderbank [KHC] and Li [Li].

This research is supported by NSFC Grant No. 10671015.
Definition 1.2. The function
\[ C_f(s, T) = C_f(s, T; \mathbb{F}_q) = \exp\left(\sum_{k=1}^{\infty} -\left(q^k - 1\right)^{-n} S_f(k, T) \frac{s^k}{k}\right), \]
as a power series in the single variable \(s\) with coefficients in the \(T\)-adic complete field \(\mathbb{Q}_p((T))\), is called a \(C\)-function of \(T\)-adic exponential sums.

We have
\[ L_f(s, T) = \prod_{i=0}^{n-1} C_f(q^i s, T)^{-1} \frac{n-i+1}{n+i}, \]
and
\[ C_f(s, T) = \prod_{j=0}^{\infty} L_f(q^j s, T)^{-1} \frac{n-1}{n+j-1}. \]
So the \(C\)-function \(C_f(s, T)\) and the \(L\)-function \(L_f(s, T)\) determine each other. From the last identity, one sees that \(C_f(s, T) \in 1 + s\mathbb{Z}_p[[T]][[s]].\)

Let \(C(\triangle)\) be the cone generated by \(\triangle\), and \(M(\triangle) = M(\triangle) \cap \mathbb{Z}^n\). There is a degree function \(\text{deg}\) on \(C(\triangle)\) which is \(\mathbb{R}_{\geq 0}\)-linear and takes the values 1 on each co-dimension 1 face not containing 0. For \(a \notin C(\triangle)\), we define \(\text{deg}(a) = +\infty\).

Definition 1.3. A convex function on \([0, +\infty]\) which is linear between consecutive integers with initial value 0 is called the infinite Hodge polygon of \(\triangle\) if its slopes between consecutive integers are the numbers \(\text{deg}(a), a \in M(\triangle)\). We denote this polygon by \(H_\infty^{\triangle}\).

Liu-Wan [LWa] also proved the following.

Theorem 1.4 (Hodge bound). We have
\[ T - \text{adic NP of } C_f(s, T) \geq \text{ord}_p(q)(p - 1)H_\infty^{\triangle}, \]
where NP is the short for Newton polygon.

Denote by \([x]\) the least integer equal or greater than \(x\), and by \(\{x\}\) the fractional part of \(x\).

Definition 1.5. Let \(C \subseteq M(\triangle)\) be a finite subset. We define
\[ r_C = \max_{\beta} \left(\#\{a \in C \mid \{\text{deg}(pa)\}' \geq \beta\} - \#\{a \in C \mid \{\text{deg}(a)\}' \geq \beta\}\right). \]

Definition 1.6. Let \(a \subseteq M(\triangle)\). We define
\[ \varpi(a) = [p \text{deg}(a)] - [\text{deg}(a)] + r_{\{a \in M(\triangle) \mid \text{deg}(u) < \text{deg}(a)\} \cup \{a\}} - r_{\{a \in M(\triangle) \mid \text{deg}(u) < \text{deg}(a)\} \cup \{a\}}. \]

Definition 1.7. The arithmetic polygon \(p_\triangle\) of \(\triangle\) is a convex function on \([0, +\infty]\) which is linear between consecutive integers with initial value 0, and whose slopes between consecutive integers are the numbers \(\varpi_\triangle(a), a \in M(\triangle)\).
One can prove the following.

**Theorem 1.8.** We have then
\[ p_\triangle \geq (p - 1)\muH^\infty_\triangle. \]
Moreover, they coincide at the point \( n!\text{Vol}(\triangle) \).

Let \( D \) be the least positive integer such that \( \deg(M(\triangle)) \subseteq \frac{1}{D}\Z \). The main result of this paper is the following.

**Theorem 1.9.** If \( p > 3D \), then
\[ T - \text{adic NP of } C_f(s, T) \geq \text{ord}_p(q)p_\triangle. \]

From the above theorem we shall deduce the following.

**Theorem 1.10.** If \( p > 3D \), then, for \( t \in \C_p \) with \( 0 \neq |t|_p < 1 \), we have
\[ t - \text{adic NP of } C_f(s, t) \geq \text{ord}_p(q)p_\triangle. \]

2. **The \( T \)-adic Dwork Theory**

In this section we review the \( T \)-adic analogue of Dwork theory on exponential sums.

Let
\[ E(t) = \exp(\sum_{i=0}^{\infty} \frac{t^p^i}{i!}) = \sum_{i=0}^{+\infty} \lambda_i t^i \in 1 + t\Z_p[[t]] \]
be the \( p \)-adic Artin-Hasse exponential series. Define a new \( T \)-adic uniformizer \( \pi \) of \( \Q_p((T)) \) by the formula \( E(\pi) = 1 + T \). Let \( \pi^{1/D} \) be a fixed \( D \)-th root of \( \pi \). Let
\[ L = \{ \sum_{u \in M(\triangle)} c_u \pi^{\deg(u)}x^u : c_u \in \Q_q[[\pi^{1/D}]] \}. \]

Let \( a \mapsto \hat{a} \) be the Teichmüller lifting. One can show that the series
\[ E_f(x) := \prod_{a \neq 0} E(\pi\hat{a}x^u) \in L. \]

Note that the Galois group of \( \Q_q \) over \( \Q_p \) can act on \( L \) but keeping \( \pi^{1/D} \) as well as the variable \( x \) fixed. Let \( \sigma \) be the Frobenius element in the Galois group such that \( \sigma(\zeta) = \zeta^p \) if \( \zeta \) is a \( (q - 1) \)-th root of unity. Let \( \Psi_p \) be the operator on \( L \) defined by the formula
\[ \Psi_p(\sum_{i \in M(\triangle)} c_i x^i) = \sum_{i \in M(\triangle)} c_{pi} x^i. \]

Then \( \Psi := \sigma^{-1} \circ \Psi_p \circ E_f \) acts on the \( T \)-adic Banach module
\[ B = \{ \sum_{u \in M(\triangle)} c_u \pi^{\deg(u)}x^u \in L, \text{ ord}_T(c_u) \to +\infty \text{ if } \deg(u) \to +\infty \}. \]

We call it Dwork’s \( T \)-adic semi-linear operator because it is semi-linear over \( \Z_q[[\pi^{1/D}]] \).
Let \( b = \log_p q \). Then the \( b \)-iterate \( \Psi^b \) is linear over \( \mathbb{Z}_q[[\frac{1}{p^D}]] \), since

\[
\Psi^b = \Psi_p^b \circ \prod_{i=0}^{b-1} E^i_f(x^p^i).
\]

One can show that \( \Psi \) is completely continuous in the sense of Serre [Se]. So \( \det(1 - \Psi^b s \mid B/\mathbb{Z}_q[[\frac{1}{p^D}]] \) and \( \det(1 - \Psi s \mid B/\mathbb{Z}_p[[\frac{1}{p^D}]] \) are well-defined.

We now state the \( T \)-adic Dwork trace formula [LWa].

**Theorem 2.1.** We have

\[
C_f(s, T) = \det(1 - \Psi^b s \mid B/\mathbb{Z}_q[[\frac{1}{p^D}]]).
\]

**Lemma 2.2.** The Newton polygon of \( \det(1 - \Psi^b s^b \mid B/\mathbb{Z}_q[[\frac{1}{p^D}]] \) coincides with that of \( \det(1 - \Psi s \mid B/\mathbb{Z}_p[[\frac{1}{p^D}]] \).

**Proof.** Note that

\[
\det(1 - \Psi^b s \mid B/\mathbb{Z}_p[[\frac{1}{p^D}]] = \text{Norm}(\det(1 - \Psi^b s \mid B/\mathbb{Z}_q[[\frac{1}{p^D}]]),
\]

where Norm is the norm map from \( \mathbb{Z}_q[[\frac{1}{p^D}]] \) to \( \mathbb{Z}_p[[\frac{1}{p^D}]] \). The lemma now follows from the equality

\[
\prod_{\zeta_i=1} \det(1 - \Psi \zeta_i s \mid B/\mathbb{Z}_p[[\frac{1}{p^D}]] = \det(1 - \Psi^b s^b \mid B/\mathbb{Z}_p[[\frac{1}{p^D}]]).
\]

\[\square\]

Write

\[
\det(1 - \Psi s \mid B/\mathbb{Z}_p[[\frac{1}{p^D}]] = \sum_{i=0}^{+\infty} (-1)^i c_i s^i.
\]

**Theorem 2.3.** The \( T \)-adic Newton polygon of \( \det(1 - \Psi^b s \mid B/\mathbb{Z}_q[[\frac{1}{p^D}]] \) is the lower convex closure of the points

\[
(m, \text{ord}_T(c_{bm})), \ m = 0, 1, \ldots.
\]

**Proof.** By Lemma 2.2, the \( T \)-adic Newton polygon of \( \det(1 - \Psi^b s^b \mid B/\mathbb{Z}_q[[\frac{1}{p^D}]] \) is the lower convex closure of the points

\[
(i, \text{ord}_T(c_i)), \ i = 0, 1, \ldots.
\]

It is clear that \((i, \text{ord}_T(c_i))\) is not a vertex of that polygon if \( b \nmid i \). So that Newton polygon is the lower convex closure of the points

\[
(bm, \text{ord}_T(c_{bm})), \ m = 0, 1, \ldots.
\]

It follows that the \( T \)-adic Newton polygon of \( \det(1 - \Psi^b s \mid B/\mathbb{Z}_q[[\frac{1}{p^D}]] \) is the lower convex closure of the points

\[
(m, \text{ord}_T(c_{bm})), \ m = 0, 1, \ldots.
\]

\[\square\]
3. The arithmetic bound

In this section we prove the following.

**Theorem 3.1.** We have

\[ \text{ord}_T(c_{bm}) \geq p\Delta(m). \]

**Proof.** First, we choose a basis of \( B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\pi^{1/D}) \) over \( \mathbb{Q}_p(\pi^{1/D}) \) as follows. Fix a normal basis \( \xi_i, i \in \mathbb{Z}/(b) \) of \( \mathbb{F}_q \) over \( \mathbb{F}_p \). Let \( \xi_i \) be their Teichmüller lift of \( \bar{\xi}_i \). The system \( \xi_i, i \in \mathbb{Z}/(b) \) is a normal basis of \( \mathbb{Q}_q \) over \( \mathbb{Q}_p \). Then \( \{\xi_i x^u\}_{u \in M(\Delta), 1 \leq i \leq b} \) is a basis of \( B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\pi^{1/D}) \) over \( \mathbb{Q}_p(\pi^{1/D}) \).

Secondly, we write out the matrix of \( \Psi \) on \( B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\pi^{1/D}) \) with respect to the basis \( \{\xi_i x^u\}_{u \in M(\Delta), 1 \leq i \leq b} \). Write

\[ E_f(x) = \sum_{u \in M(\Delta)} \gamma_u x^u, \]

and

\[ \sigma^{-1}(\xi_j \gamma_{pu-w}) = \sum_{i=1}^b \gamma(\xi_i, (w,j)) \xi_i. \]

Then

\[ \Psi(\xi_j x^w) = \sum_{u \in M(\Delta)} \sigma^{-1}(\xi_j \gamma_u) \Psi_p(x^{u+w}) = \sum_{u \in M(\Delta)} \sigma^{-1}(\xi_j \gamma_{pu-w}) x^u = \sum_{u \in M(\Delta)} \sum_{i=1}^b \gamma(\xi_i, (w,j)) \xi_i x^u. \]

So \( \{\gamma(\xi_i, (w,j))\}_{u,w \in M(\Delta), 1 \leq i,j \leq b} \) is the matrix of \( \Psi \) on \( B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\pi^{1/D}) \) with respect to the basis \( \{\xi_i x^u\}_{u \in M(\Delta), 1 \leq i \leq b} \).

Thirdly, we claim that

\[ \text{ord}_T(\gamma(\xi_i, (w,j))) \geq \lceil \deg(pu-w) \rceil. \]

In fact, this follows from the equality

\[ \sigma^{-1}(\xi_j \gamma_{pu-w}) = \sum_{i=1}^b \gamma(\xi_i, (w,j)) \xi_i, \]

and the inequality \( \text{ord}_T(\gamma_u) \geq \lceil \deg(u) \rceil. \)

Finally, we show that

\[ \text{ord}_T(c_{bm}) \geq p\Delta(m). \]

Note that

\[ c_{bm} = \sum_A \det((\gamma(\xi_i, (w,j)))_{(u,i),(w,j) \in A}). \]
where \( A \) runs over all subsets of \( M(\triangle) \times \mathbb{Z}/(b) \) with cardinality \( bm \). So it suffices to show that

\[
\text{ord}_T(\det(\gamma_{(i,u),(j,\omega)})(i,u),(j,\omega)\in A)) \geq bp_\triangle(m).
\]

Note that

\[
\det(\gamma_{(i,u),(j,\omega)})(i,u),(j,\omega)\in A)) = \sum_{\tau \in S_A} \sum_{a \in A} \text{ord}_\pi(\gamma_{a,\tau(a)}),
\]

where \( S_A \) is the permutation group of \( A \). So it suffices to show that

\[
\sum_{a \in A} \text{ord}_\pi(\gamma_{a,\tau(a)}) \geq bp_\triangle(m), \quad \tau \in S_A.
\]

Since

\[
\text{ord}_T(\gamma_{(u,i),(w,j)}) \geq \lceil \deg(pu - w) \rceil,
\]

the theorem follows from the following.

**Theorem 3.2.** If \( p > 3D \), \( A \) is a subset of \( M(\triangle) \times \mathbb{Z}/(b) \) with cardinality \( bm \), and \( \tau \in S_A \), then

\[
\sum_{a \in A} \lceil \deg(p\nu(a) - \nu(\tau(a))) \rceil \geq bp_\triangle(m),
\]

where \( \nu(u,i) = u \).

**Proof.** We have

\[
\sum_{a \in A} \lceil \deg(p\nu(a) - \nu(\tau(a))) \rceil \geq \sum_{a \in A} \lceil \deg(p\nu(a)) - \nu(\tau(a)) \rceil
\]

\[
\geq \sum_{a \in A} \lceil \deg(p\nu(a)) \rceil - \lceil \deg(\nu(\tau(a))) \rceil + 1_{\{\deg(p\nu(a)) > \deg(\nu(\tau(a)))\}}
\]

\[
\geq \sum_{a \in A} \lceil \deg(p\nu(a)) \rceil - \lceil \deg(\nu(a)) \rceil + 1_{\{\deg(p\nu(a)) > \deg(\nu(\tau(a)))\}}.
\]

Choose a set \( B \) of cardinality \( |A| \) such that \( B \cap A \) is as big as possible under the condition that, for some \( \alpha \),

\[
\{a \in M(\triangle) \times \mathbb{Z}/(b) \mid \deg(\nu(a)) < \alpha\} \subseteq B \subseteq \{a \in M(\triangle) \times \mathbb{Z}/(b) \mid \deg(\nu(a)) \leq \alpha\}.
\]

Choose a permutation \( \tau_0 \) on \( B \cap A \) which agrees with \( \tau \) on \( (B \cap A) \cap \tau^{-1}(B \cap A) \). Extend it trivially to \( B \). We have

\[
\sum_{a \in A} \lceil \deg(p\nu(a)) \rceil - \lceil \deg(\nu(a)) \rceil \geq \sum_{a \in B} (\lceil \deg(p\nu(a)) \rceil - \lceil \deg(\nu(a)) \rceil) + 2\#(A \setminus B).
\]

We also have

\[
\sum_{a \in A} 1_{\{\deg(p\nu(a)) > \deg(\nu(\tau(a)))\}} \geq \sum_{a \in B} 1_{\{\deg(p\nu(a)) > \deg(\nu(\tau(a)))\}} - 2\#(A \setminus B).
\]

It follows that

\[
\sum_{a \in A} \lceil \deg(p\nu(a)) \rceil - \lceil \deg(\nu(a)) \rceil + 1_{\{\deg(p\nu(a)) > \deg(\nu(\tau(a)))\}}
\]
\[ \sum_{a \in B} \left\lfloor \deg(p\nu(a)) \right\rfloor - \left\lfloor \deg(\nu(a)) \right\rfloor + 1_{\{\deg(p\nu(a))' > \{\deg(\nu(a)_0)\}'\}} \]
\[ \geq \sum_{a \in B} \left( \left\lfloor \deg(p\nu(a)) \right\rfloor - \left\lfloor \deg(\nu(a)) \right\rfloor \right) + r_B, \]
where
\[ r_B = \max_{\beta} \{ \# \{ a \in B \mid \{\deg(p\nu(a))\}' \geq \beta \} - \# \{ a \in B \mid \{\deg(\nu(a))\}' \geq \beta \} \}. \]

Choose a set \( C \) of cardinality \( m \) such that for some \( \alpha \),
\[ \{ a \in M(\Delta) \mid \deg(\nu(a)) < \alpha \} \subseteq C \subseteq \{ a \in M(\Delta) \mid \deg(\nu(a)) \leq \alpha \}. \]
Recall that
\[ r_C = \max_{\beta} \{ \# \{ a \in C \mid \{\deg(pa)\}' \geq \beta \} - \# \{ a \in C \mid \{\deg(a)\}' \geq \beta \} \}. \]
It is easy to see that \( r_B = br_C \), and
\[ \sum_{a \in B} \left( \left\lfloor \deg(p\nu(a)) \right\rfloor - \left\lfloor \deg(\nu(a)) \right\rfloor \right) + r_B \]
\[ = b \sum_{a \in C} \left( \left\lfloor \deg(pa) \right\rfloor - \left\lfloor \deg(a) \right\rfloor \right) + br_C = bp_\Delta(m). \]
The theorem now follows. \( \square \)

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