Abstract

We present a manifestly $N = 2$ supersymmetric formulation of $N = 2$ super-$W_3$ algebra (its classical version) in terms of the spin 1 and spin 2 supercurrents. Two closely related types of the Feigin-Fuchs representation for these supercurrents are found: via two chiral spin $\frac{1}{2}$ superfields generating $N = 2$ extended $U(1)$ Kac-Moody algebras and via two free chiral spin 0 superfields. We also construct a one-parameter family of $N = 2$ super Boussinesq equations for which $N = 2$ super-$W_3$ provides the second hamiltonian structure.

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MIRAMARE-TRIESTE
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*Permanent address: Laboratory of Theoretical Physics, JINR, Dubna, Russian Federation.
†E-mail address: eivanov@ltp.jinr.dubna.su
‡E-mail address: krivonos@ltp.jinr.dubna.su
1 Introduction

For the last two years there has been considerable interest in supersymmetric extensions of Zamolodchikov’s $W_N$ algebras (see, e.g. [1]-[8]). Recently, $N = 2$ super-$W_3$ algebra has been constructed, both on the classical [2, 3] and full quantum [4] levels. It is generated by two $N = 2$ supermultiplets of currents, with the conformal spins $(1, \frac{3}{2}, \frac{3}{2}, 2)$ and $(2, \frac{5}{2}, \frac{5}{2}, 3)$, and exists at an arbitrary value of the central charge.

For setting up the conformal field theory associated with $N = 2$ super-$W_3$ and studying representations of this algebra it is of importance to know its free-field realizations. One more urgent problem which could have important implications in $N = 2$ super-$W_3$ gravity and the related matrix models consists in defining the general hamiltonian flow on $N = 2$ super-$W_3$ and finding out the generalized KdV-type hierarchy for which this algebra produces the second hamiltonian structure.

In the present letter we address both these problems. We give two field-theoretical realizations of $N = 2$ super-$W_3$: via spin $\frac{1}{2}$ and spin 0 chiral $N = 2$ supermultiplets. Both these realizations are of the Feigin-Fuchs type and place no restrictions on the central charge. We also construct the simplest nontrivial hamiltonian flow on $N = 2$ super-$W_3$ yielding a $N = 2$ superextension of the Boussinesq equation. This superextension turns out to involve a free parameter and is reducible in the bosonic sector to the Boussinesq equation only at the special value of this parameter.

We use the language of $N = 2$ superfields which radically simplifies computations and allows to present the final results in a manifestly $N = 2$ supersymmetric concise form. We restrict our study here to the classical version of $N = 2$ super-$W_3$ [3]. Extension to the full quantum $N = 2$ super-$W_3$ [4] seems to be more or less straightforward and will be presented elsewhere.

2 $N = 2$ super-$W_3$ algebra in terms of $N = 2$ supercurrents

In this Section we present a supercurrent formulation of the classical $N = 2$ super-$W_3$ algebra [3].

The basic currents of $N = 2$ super-$W_3$, in accordance with their spin content, can be naturally incorporated into the two $N = 2$ supercurrents $J(Z)$ and $T(Z)$ carrying the spins 1 and 2 [3]. The full set of the current OPE’s of ref.[3] can now be summarized as the three SOPE’s between these supercurrents.

The first SOPE states that the supercurrent $J(Z)$ generates the standard $N = 2$ superconformal algebra [3]-[12]:

$$J(Z_1)J(Z_2) = \frac{\theta_{12}}{Z_{12}^2} + \frac{\bar{\theta}_{12}\bar{D}J}{Z_{12}} - \frac{\theta_{12}DJ}{Z_{12}^2} + \frac{\theta_{12}\bar{\theta}_{12}J}{Z_{12}^2} + \frac{\theta_{12}\bar{\theta}_{12}\partial J}{Z_{12}}, \quad (2.1)$$

where

$$\theta_{12} = \theta_1 - \theta_2, \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2, \quad Z_{12} = z_1 - z_2 + \frac{1}{2}\left(\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1\right), \quad (2.2)$$

and $D, \bar{D}$ are the spinor covariant derivatives

$$D = \frac{\partial}{\partial \theta} - \frac{1}{2}\bar{\theta}\frac{\partial}{\partial \bar{z}}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial z}. \quad (2.3)$$

By $Z$ we denote the coordinates of $1D N = 2$ superspace, $Z = (z, \theta, \bar{\theta})$. 

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1
\{ \mathcal{D}, \mathcal{D} \} = -\frac{\partial}{\partial z} , \quad \{ \mathcal{D}, \mathcal{D} \} = \{ \mathcal{D}, \mathcal{D} \} = 0.

The next SOPE expresses the property that the spin 2 supercurrent $T(Z)$ can be chosen primary with respect to the $N = 2$ superconformal algebra:

$$
J(Z_1)T(Z_2) = \frac{\bar{\theta}_{12} \mathcal{D} T}{Z_{12}} - \frac{\theta_{12} \mathcal{D} T}{Z_{12}} + 2 \frac{\theta_{12} \bar{\theta}_{12} T}{Z_{12}^2} + \frac{\theta_{12} \bar{\theta}_{12} \partial T}{Z_{12}} .
$$

(2.4)

(it possesses zero $U(1)$ charge).

The last SOPE needed to close the algebra is that involving $T(Z_1)$, $T(Z_2)$. It is most complicated

$$
T(Z_1)T(Z_2) = -\frac{3}{2} c \frac{T(Z_2)}{Z_{12}^2} - 12 \frac{\theta_{12} \bar{\theta}_{12} J}{Z_{12}^3} + 12 \frac{\theta_{12} \mathcal{D} J}{Z_{12}^3} - 12 \frac{\bar{\theta}_{12} \mathcal{D} J}{Z_{12}^3} - 12 \frac{\theta_{12} \bar{\theta}_{12} \partial J}{Z_{12}^3}
$$

$$
+ \frac{5T(Z_2)}{Z_{12}^2} - 2 \left[ \mathcal{D}, \mathcal{D} \right] J + \frac{8}{c} J^2 \right) + \frac{\theta_{12} \mathcal{D} \left( 5T(Z_2) + \frac{8}{c} J^2 \right)}{Z_{12}^2}
$$

$$
- \frac{\bar{\theta}_{12} \mathcal{D} \left( 5T(Z_2) - \frac{8}{c} J^2 \right)}{Z_{12}^2} + \frac{\theta_{12} \bar{\theta}_{12} \left( \frac{3}{2} \left[ \mathcal{D}, \mathcal{D} \right] T - 6 \partial^2 J + U_3 \right)}{Z_{12}^2}
$$

$$
+ \frac{\theta_{12} \bar{\theta}_{12} \left( -2 \partial^3 J + \partial \left[ \mathcal{D}, \mathcal{D} \right] T + \frac{1}{2} \partial U_3 + \frac{1}{2} \partial \mathcal{D} \Psi \bar{z} + \frac{1}{2} \partial \mathcal{D} \bar{\Psi} \bar{z} - \frac{2}{c} \partial \left[ \mathcal{D}, \mathcal{D} \right] J^2 \right)}{Z_{12}^2}
$$

$$
+ \frac{\partial \left( 5T - 2 \left[ \mathcal{D}, \mathcal{D} \right] J + \frac{8}{c} J^2 \right)}{Z_{12}^2} .
$$

(2.5)

Here $\Psi \bar{z}(Z)$, $\bar{\Psi} \bar{z}(Z)$, $U_3(Z)$ are the composite supercurrents of the spins $\frac{7}{2}$, $\frac{7}{2}$, 3, respectively

$$
\Psi \bar{z} = \frac{8}{c} \partial \left( \mathcal{D} \mathcal{D} \right) J - \frac{72}{c} \mathcal{D} \mathcal{D} T + \frac{36}{c} \left[ \mathcal{D}, \mathcal{D} \right] J \mathcal{D} J + \frac{8}{c} \mathcal{D} \mathcal{D} T - \frac{128}{c^2} J^2 \mathcal{D} J + \frac{4}{c} \partial \mathcal{D} \mathcal{D} J
$$

$$
\bar{\Psi} \bar{z} = -\frac{8}{c} \partial \left( \mathcal{D} \mathcal{D} \right) J - \frac{72}{c} \mathcal{D} \mathcal{D} T + \frac{36}{c} \left[ \mathcal{D}, \mathcal{D} \right] J \mathcal{D} J + \frac{8}{c} \mathcal{D} \mathcal{D} T - \frac{128}{c^2} J^2 \mathcal{D} J - \frac{4}{c} \partial \mathcal{D} \mathcal{D} J
$$

$$
U_3 = \frac{56}{c} J \mathcal{T} - \frac{32}{c} J \left[ \mathcal{D}, \mathcal{D} \right] J + \frac{128}{c^2} J^3 + \frac{120}{c} \mathcal{D} J \mathcal{D} J .
$$

(2.6)

The complete correspondence with ref. 3 is achieved under the following definition of the component currents

$$
| J | = 4J_0 \quad , \quad | T | = T_0 + 4 \bar{T}_0 - \frac{128}{c} J_0^2
$$

$$
| \mathcal{D} J | = \bar{G}_0 \quad , \quad | \mathcal{D} T | = \frac{3}{4} \bar{T}_0 - \frac{64}{c} J_0 \bar{G}_0
$$

$$
| \mathcal{D} J | = -G_0 \quad , \quad | \mathcal{D} T | = -\frac{3}{4} \bar{T}_0 + \frac{64}{c} J_0 G_0
$$

(2.7)

$$
\frac{1}{2} \left[ \mathcal{D}, \mathcal{D} \right] J = T_0 + \bar{T}_0 \quad , \quad \frac{1}{2} \left[ \mathcal{D}, \mathcal{D} \right] T = \frac{3}{4} W_0 + \frac{32}{c} \left( T_0 + 4 \bar{T}_0 - \frac{128}{c} J_0^2 \right) J_0 + \frac{40}{c} G_0 \bar{G}_0 .
$$
The currents $J_0$, $G_0$, $G_0$, $T_0$, $T_0$, $U_0$, $U_0$, $W_0$ obey the OPE’s of $\mathbb{3}$ as a consequence of the SOPE’s (2.1), (2.4), (2.3).

It is worth noting that the spin content of our composite $N=2$ supercurrents $J^2$, $\Psi_{\frac{1}{2}}$, $\overline{\Psi}_{\frac{1}{2}}$, $U_3$ is larger than that of the set of composite currents figuring in the OPE’s of ref. $\mathbb{3}$: $$(2, \frac{5}{2}, \frac{5}{2}, 3; \frac{7}{2}, \frac{7}{2}, 4, 4, \frac{9}{2}, \frac{9}{2}, 3, \frac{7}{2}, \frac{7}{2}, 4) \quad \text{vs} \quad (3, \frac{7}{2}, \frac{7}{2}, 4, 4, \frac{9}{2}, \frac{9}{2}, 3, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}).$$

This differences stems from the fact that we are working in a manifestly $N=2$ supersymmetric superfield formalism, so the composites can appear only in groups forming $N=2$ supermultiplets. Of course, the missing composite currents are implicitly present also in the relations of ref. $\mathbb{3}$: they can be produced by action of the $N=2$ supersymmetry generators on the composites appearing explicitly.

Thus we have established the full $N=2$ superfields structure of $N=2$ super-$W_3$ algebra. Before closing this Section, let us indicate one more version of classical $N=2$ super-$W_3$ which is similar to the algebra called in ref. $\mathbb{13}$ “classical $W_3$” and follows from the $N=2$ super-$W_3$ defined above in the contraction limit $c = 0$. We call this contracted superalgebra $N=2$ super-$W_3^{cl}$.

In order to approach the limit $c \to 0$ in an unambiguous way, one needs beforehand to rescale the supercurrents as

$$J^{cl} = J, \quad T^{cl} = cT, \quad \Psi^{cl}_{\frac{1}{2}} = c^2 \Psi_{\frac{1}{2}}, \quad \overline{\Psi}^{cl}_{\frac{1}{2}} = c^2 \overline{\Psi}_{\frac{1}{2}}, \quad U_3^{cl} = c^3 U_3.$$  \hspace{1cm} (2.8)

Now it is straightforward to put $c = 0$ in SOPE’s (2.1), (2.4), (2.5) and to obtain the algebra $N=2$ super-$W_3^{cl}$:

$$J^{cl}(Z_1)J^{cl}(Z_2) = \frac{\theta_{12} D J^{cl}}{Z_{12}} - \theta_{12} D J^{cl} + \frac{\theta_{12} \theta_{12} J^{cl}}{Z_{12}^2} + \frac{\theta_{12} \theta_{12}}{Z_{12}},$$

$$J^{cl}(Z_1)T^{cl}(Z_2) = \frac{\theta_{12} D T^{cl}}{Z_{12}} - \theta_{12} D T^{cl} + \frac{2 \theta_{12} \theta_{12} T^{cl}}{Z_{12}^2} + \frac{\theta_{12} \theta_{12}}{Z_{12}},$$

$$T^{cl}(Z_1)T^{cl}(Z_2) = \frac{\theta_{12} \theta_{12} U_3^{cl}}{Z_{12}^2} + \frac{\theta_{12} \theta_{12} \overline{\Psi}^{cl}_{\frac{1}{2}}}{Z_{12}^2} + \frac{\theta_{12} \theta_{12} \overline{\Psi}^{cl}_{\frac{1}{2}}}{Z_{12}^2} (\overline{\Psi}^{cl}_{\frac{1}{2}} + \overline{\Psi}^{cl}_{\frac{1}{2}} + \overline{\Psi}^{cl}_{\frac{1}{2}}),$$  \hspace{1cm} (2.9)

where the composite supercurrents are now given by

$$\Psi^{cl}_{\frac{1}{2}} = -72 J^{cl} D J^{cl} + 8 J^{cl} D T^{cl} - 128 (J^{cl})^2 D J^{cl},$$

$$\overline{\Psi}^{cl}_{\frac{1}{2}} = -72 T^{cl} D J^{cl} + 8 J^{cl} D T^{cl} - 128 (J^{cl})^2 D J^{cl},$$

$$U_3^{cl} = 56 J^{cl} T^{cl} + 128 (J^{cl})^3.$$  \hspace{1cm} (2.10)

These relations are guaranteed to define a closed nonlinear algebra (with all the Jacobi identities satisfied) because they have been obtained from those of $N=2$ super-$W_3$ algebra via a contraction procedure.

### 3 Free superfield realizations of super-$W_3$

In this Section we construct two $N=2$ superfield realizations of $N=2$ super-$W_3$ algebra. They prove to be closely related to each other.
The first realization is via two \( N = 2 \) chiral spin \( \frac{1}{2} \) fermionic superfields \( \chi(Z) \), \( \xi(Z) \),

\[
\mathcal{D}\bar{\chi} = \overline{\mathcal{D}}\chi = 0 \quad , \quad \mathcal{D}\bar{\xi} = \overline{\mathcal{D}}\xi = 0 \quad ,
\]

with the two-point functions given by

\[
\langle \chi(Z_1)\bar{\chi}(Z_2) \rangle = \frac{1}{Z_{12}} + \frac{\theta_{12}\tilde{\theta}_{12}}{2Z_{12}^2} \quad , \quad \langle \xi(Z_1)\bar{\xi}(Z_2) \rangle = \frac{1}{Z_{12}} + \frac{\theta_{12}\tilde{\theta}_{12}}{2Z_{12}^2} .
\]

We have explicitly checked (this is a rather tedious labor despite the fact that we are using the condensed \( N = 2 \) superfield formalism) that the supercurrents

\[
J = -\chi\bar{\chi} - \xi\bar{\xi} + \sqrt{\frac{c}{8}} \left( \mathcal{D}\bar{\chi} - \mathcal{D}\chi \right)
\]

and

\[
T = \sqrt{\frac{c}{8}} \left( \partial\mathcal{D}\xi - \partial\overline{\mathcal{D}}\bar{\xi} \right) + 2\partial\chi\bar{\chi} + \partial\bar{\xi}\xi + \partial\bar{\xi}\chi - 2\partial\bar{\xi}\chi + \partial\chi\bar{\xi} - \partial\bar{\xi}\chi - \frac{40}{c} \bar{\xi}\chi\bar{\chi}
+ \mathcal{D}\xi \left( \mathcal{D}\chi + \overline{\mathcal{D}}\bar{\chi} + \mathcal{D}\xi - 3\overline{\mathcal{D}}\bar{\xi} \right) + \overline{\mathcal{D}}\bar{\xi} \left( -\mathcal{D}\chi - \overline{\mathcal{D}}\bar{\chi} + \mathcal{D}\xi \right)
+ \sqrt{\frac{c}{8}} \left[ \mathcal{D}\xi \left( 2\bar{\xi}\chi - 4\chi\bar{\xi} + \chi\bar{\chi} - \xi\bar{\xi} \right) + \mathcal{D}\chi \left( \xi\bar{\chi} - 2\bar{\xi}\chi - \xi\bar{\xi} \right) \right]
+ \sqrt{\frac{c}{8}} \left[ \overline{\mathcal{D}}\bar{\xi} \left( -4\bar{\xi}\chi + 2\bar{\xi}\chi + \chi\bar{\chi} - \xi\bar{\xi} \right) + \overline{\mathcal{D}}\bar{\chi} \left( 2\bar{\xi}\chi - \chi\bar{\chi} - \xi\bar{\xi} \right) \right]
\]

obey the defining SOPE’s \([2.1], [2.4], [2.5]\).

This realization naturally generalizes the one proposed in \([12]\) for \( N = 2 \) superconformal algebra (the corresponding supercurrent \( J(Z) \) is the \( \xi = 0 \) reduction of \([3.3]\)). The chiral superfields \( \chi(Z) \) and \( \xi(Z) \) are recognized as supercurrents generating \( N = 2 \) superextensions of two independent complex \( U(1) \) Kac-Moody algebras. Note the presence of the Feigin-Fuchs linear terms in \([3.3], [3.4]\). Just these terms ensure an arbitrary central charge in the present case (at the classical level) \([4]\).

In order to obtain one more field realization of \( N = 2 \) super-W\(_3\), one notices that the SOPEs \([3.2]\) can be reproduced starting from the following particular representation of the supercurrents \( \chi(Z), \xi(Z) \)

\[
\chi = -\overline{\mathcal{D}}\bar{U} \quad , \quad \bar{\chi} = \mathcal{D}U \quad , \quad \xi = -\overline{\mathcal{D}}\Phi \quad , \quad \bar{\xi} = \mathcal{D}\Phi ,
\]

where \( U(Z), \Phi(Z) \) are the spin 0 free chiral \( N = 2 \) superfields

\[
\mathcal{D}\Phi = \overline{\mathcal{D}}\Phi = 0 \quad , \quad \mathcal{D}\bar{U} = \overline{\mathcal{D}}U = 0
\]

\[
\langle U(Z_1)\bar{U}(Z_2) \rangle = \ln(Z_{12}) - \frac{\theta_{12}\tilde{\theta}_{12}}{2Z_{12}^2} \quad , \quad \langle \Phi(Z_1)\bar{\Phi}(Z_2) \rangle = \ln(Z_{12}) - \frac{\theta_{12}\tilde{\theta}_{12}}{2Z_{12}^2} .
\]

To get the free-superfield expressions for \( J(Z) \) and \( T(Z) \) one should replace the \( U(1) \) supercurrents \( \chi(Z), \xi(Z) \) in eqs. \([3.3], [3.4]\) by their particular representation \([3.3]\). For brevity, we quote the free-superfield form only for the \( N = 2 \) conformal supercurrent

\[
J = \overline{\mathcal{D}}\bar{U}\mathcal{D}U + \mathcal{D}\Phi\overline{\mathcal{D}}\Phi - \sqrt{\frac{c}{8}} \partial(U + \bar{U}) \quad .
\]

\(^2\)In the quantum case \( c \) is expected to be restricted to discrete series by the unitarity reasonings \([4]\).
This realization generalizes the free chiral superfield realization of $N = 2$ superconformal algebra given in [10]. We conjecture that it is closely related to the $N = 2$ supersymmetric Toda system associated with the superalgebra $sl(3|2)$.

Before ending this Section, it is worth saying a few words as to how we arrived at the above particular realizations of $N = 2$ super-$W_3$. These were prompted to us while we treated this superalgebra in the framework of the covariant reduction approach worked out earlier on the simpler examples of $W_2$ (Virasoro) and $W_3$ algebras [14, 15]. Without entering into details we only note that this approach allows to regard nonlinear (super)algebras of the $W_N$ type as special realizations of some associate linear higher-spin (super)algebras $W_\infty^N$. The appropriate (super)currents and scalar (super)fields (e.g., of the type considered above) naturally come out in the covariant reduction approach as the parameters of some coset (super)spaces connected with the infinite-dimensional (super)algebras just mentioned. Then the covariant relations between them of the type (3.3), (3.4), (3.8) arise as a result of imposing covariant constraints on the Cartan one-forms describing the geometry of these cosets. In more detail the applications of this geometric approach to $N = 2$ super-$W_3$ will be reported elsewhere [16]. Here we wish to stress that all the formulas and statements of the present paper are self-contained in their own right and do not require the reader to be familiar with the covariant reduction techniques.

4 N=2 super Boussinesq equation

In this Section we deduce $N = 2$ super Boussinesq equation and give the second hamiltonian structure for it.

It is well known that the bosonic Boussinesq equation has the second hamiltonian structure which is equivalent to the classical form of the $W_3$ algebra [17], namely

$$\dot{T} = [T, H] \quad , \quad \dot{W} = [W, H] \quad , \quad (4.1)$$

with

$$H = \int dz W \quad (4.2)$$

and the currents $T(z)$ and $W(z)$ obeying the OPE’s of the classical $W_3$ algebra (with an arbitrary central charge)\(^3\).

It is natural to define $N = 2$ super Boussinesq equation as the $N = 2$ superfield equation the second hamiltonian structure for which is induced by the $N = 2$ super-$W_3$ algebra (2.1), (2.4), (2.5). In other words, we consider the set of the evolution equations

$$\dot{\hat{T}} = [T, H] \quad , \quad \dot{\hat{J}} = [J, H] \quad (4.3)$$

where now hamiltonian $H$ is given by

$$H = \int dZ \left( T + \alpha J^2 \right) \quad . \quad (4.4)$$

We emphasize that the hamiltonian (4.4) is the most general one which can be constructed out of $J$ and $T$ under the natural assumptions that it respects $N = 2$ supersymmetry and has the

\(^3\)In a recent paper [8] it has been observed that the extended classical symmetry of this system is generated just by the spin 1 and spin 2 $N = 2$ supercurrents.

\(^4\)The commutators here and in the subsequent formulas are defined like in the quantum case as the radially ordered OPE’s (SOPE’s).
same dimension 2 as the bosonic Hamiltonian (4.2). Note the appearance of the free parameter \( \alpha \) in (4.4).

Now, using the previously established SOPE's (2.1), (2.4) and (2.5), we immediately find the explicit form of the sought \( N = 2 \) super Boussinesq equation:

\[
\begin{align*}
\dot{T} &= -2J''' + [\overline{D}, D]T' + \frac{88}{c} \partial (\overline{D} J D J) - \frac{28}{c} J' [\overline{D}, D] J - \frac{12}{c} J [\overline{D}, D] J' + \frac{256}{c^2} J^2 J' \\
&\quad + \left( \frac{40}{c} - 2\alpha \right) \overline{D} J D T + \left( \frac{40}{c} - 2\alpha \right) D J D T + \left( \frac{64}{c} + 4\alpha \right) J'T + \left( \frac{24}{c} + 2\alpha \right) JT'' \\
\dot{J} &= 2T' + \alpha \left( \frac{c}{4} [\overline{D}, D] J' + 4J J' \right) .
\end{align*}
\]

(4.5)

Note that, in contrast to the set (4.1) which can be equivalently rewritten as a single equation for the conformal stress tensor \( T \) (it is just what is usually called the Boussinesq equation), the set (4.5) cannot be reduced to one equation for the conformal supercurrent \( J \). Thus the \( N = 2 \) superextension of Boussinesq equation in general amounts to the system of coupled equations for 4 bosonic and 4 fermionic fields.

It is instructive to examine the bosonic subsector of (4.5), with all fermions omitted:

\[
\begin{align*}
\dot{\phi} &= 2v' + \frac{\alpha c}{4} u' + 4\alpha \phi \phi' \\
\dot{v} &= -2\phi''' + \omega' - \frac{28}{c} \phi' u - \frac{12}{c} \phi u' + \left( \frac{64}{c} + 4\alpha \right) \phi' v + \left( \frac{24}{c} + 2\alpha \right) \phi v' + \frac{256}{c^2} \phi^2 \phi' \\
\dot{u} &= 2\omega' + \frac{\alpha c}{4} \phi''' + 4\alpha \omega u' + 4\alpha \phi u' \\
\dot{\omega} &= -2u''' + v'' - \frac{128}{c} \phi' u' - \frac{60}{c} \phi \phi'' - \frac{12}{c} \phi \phi' u' - \frac{64}{c} \phi u' + \frac{512}{c^2} u \phi \phi' + \frac{256}{c^2} \phi^2 u' \\
&\quad + \left( 6\alpha + \frac{24}{c} \right) \omega \phi' + \left( 4\alpha + \frac{64}{c} \right) u' v + \left( 2\alpha + \frac{24}{c} \right) \phi \omega' \\
\end{align*}
\]

(4.6)

Here

\( J | = \phi \, , \, [\overline{D}, D] J | = u \, , \, T | = v \, , \, [\overline{D}, D] T | = \omega \) .

The set (4.6) contains the \( N = 0 \) Boussinesq equation in a very special manner. Namely, if we choose

\( \alpha = -\frac{4}{c} \) ,

(4.7)

then the equations (4.6) admit the following self-consistent truncation

\( \phi = 0 \, , \, u = 2v \).

(4.8)

In this case first of eqs. (4.6) is satisfied identically, while the second and third ones turn out to coincide and, together with the fourth equation, just give the Boussinesq equation

\[
\dot{\omega} = -3v''' - \frac{288}{c} \nu \nu' \, , \, \dot{v} = \omega' 
\]

(4.9)

Finally, we mention that the spinor and scalar \( N = 2 \) superfield realizations found for \( J(Z) \) and \( T(Z) \) in the previous Section give generalized super-Miura maps for the \( N = 2 \) Boussinesq

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\[5\] It would be interesting to compare eqs. (4.3) with another \( N = 2 \) extension of Boussinesq equation deduced in [6] within a generalized Lax representation (using \( N = 1 \) superfield formalism).
equation (4.3). The explicit form of the evolution equations for the Kac-Moody supercurrents $\xi$, $\chi$

\[ \dot{\xi} = [H, \xi], \quad \dot{\chi} = [H, \chi] \quad (4.10) \]
as well as for the scalar superfields $U$, $\Phi$ can be easily established expressing the hamiltonian in terms of these superfields and further employing the relations (3.2), (3.7). Eqs. (4.10) are related to eqs. (4.3) like the mKdV equation to the KdV one.

We postpone the analysis of the integrability properties of the $N = 2$ super Boussinesq equation (the existence of the Lax pair and infinite series of the conserved quantities, etc) to future publications.

5 Conclusion

To summarize, we have concisely rewritten classical $N = 2$ super-$W_3$ algebra of [3] in terms of two $N = 2$ supercurrents, found its Feigin-Fuchs type representations, in terms of two chiral $N = 2 U^c(1)$ Kac-Moody supercurrents and two free scalar chiral $N = 2$ superfields, and constructed a one-parameter family of $N = 2$ super Boussinesq equations the second hamiltonian structure for which is related to this superalgebra. We have also deduced a new classical nonlinear superalgebra $N = 2$ super-$W_3^{cl}$ by taking the contraction limit $c = 0$ in the defining relations of $N = 2$ super-$W_3$. In a forthcoming publication we will extend our consideration to the case of full quantum $N = 2$ super-$W_3$ algebra of ref. [4].

It is interesting to apply our manifestly $N = 2$ supersymmetric superfield formalism for constructing higher-$N$ superextensions of $W_3$. For instance, $N = 2$ superconformal algebra can be extended to the $N = 4 SU(2)$ one by adding a spin 1 chiral $N = 2$ supercurrent to $J(Z)$ (this additional supercurrent should possess the $U(1)$ charge +2, if one ascribes the charge +1 to the $N = 2$ superspace spinor coordinate $\theta$). To preserve the algebraic structure, we also have to add some extra $N = 2$ supercurrents to $T(Z)$ in order to complete the latter to an irreducible $N = 4$ supermultiplet (it can be primary or quasi-primary with respect to the $N = 4$ superconformal algebra) and then try to write a closed set of S OPE’s between all these supercurrents. The minimal way to enlarge $T(Z)$ so as to have still only one spin 3 current is as follows: we should add one extra real spin 1 $N = 2$ supercurrent and one complex spin $3/2$ $N = 2$ supercurrent (besides the chiral supercurrent extending $N = 2$ superconformal algebra to the $N = 4$ one). It is an open question whether these supercurrents can be forced to generate an $N = 4$ super-$W_3$ algebra.

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