3D two-color QCD at finite temperature and baryon density

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We study the phase diagram for two-color QCD in three-dimensional spacetime, as a function of temperature and baryon chemical potential, using the low-energy effective Lagrangian approach. We show one-loop renormalizability at zero temperature, and then use the one-loop effective Lagrangian at finite temperature and chemical potential to show that at low temperature there is a critical line separating the normal and diquark phase, with this critical line ending at a tricritical point. This phase structure is qualitatively similar to that found recently by Splittorff et al. for two-color QCD in four-dimensional spacetime, although the details are quite different, due to the different symmetries and the different loop and infrared properties of three-dimensional spacetime.

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I. INTRODUCTION

A systematic approach to the investigation of the symmetry breaking structure of fermion-gauge theories is to use low-energy effective Lagrangians \[1, 2\] to study the Goldstone modes corresponding to the spontaneously broken global symmetries \[3\]. Ideally, one would like to understand the phase diagram of fermion-gauge theories in the \((T, \mu)\) plane, where \(T\) is the temperature, and \(\mu\) is the (baryon number) chemical potential. However, it is not known how to analyze general fermion-gauge systems at finite baryon density, since the baryon number chemical potential makes the Euclidean Dirac operator non-Hermitian and the Boltzmann weight complex. This problem can be avoided, as advocated in \[4, 5, 6, 7, 8\], by considering “two-color QCD”, for which the fundamental representation of SU(2) is pseudoreal. From studies of 4D theories, it has long been appreciated that the SU(2) theory exhibits exotic types of spontaneous breakdown of global symmetry \[9\]. Quarks and charge-conjugated antiquarks are combined into an extended flavor multiplet, which is expected to break into its extended vector subgroup. This has the consequence that in lattice simulations the Boltzmann weight is real even at finite (baryon number) chemical potential \(\mu\). For an even number of flavors, the Boltzmann weight is also positive definite. Thus, analytic predictions can be quantitatively compared with Monte Carlo simulations in lattice gauge theory \[10, 11\], provided the lattice regularization respects the relevant flavor symmetry group. Such studies have recently been carried out \[12\] in 4D QCD with quarks in pseudoreal (and real) representations at finite \(\mu\).

In \[13\] we studied the \(T = 0\) case of 3D two-color QCD, and found that the phase structure predicted by the tree-level effective potential was qualitatively the same as the 4D case studied in \[3, 8, 14\], even though the details of the symmetry breaking in 3D are completely different from in 4D. This universality between 3D and 4D corresponds to the existence of two different breakings of the original global symmetry by a mass and a chemical potential term. In this paper we extend the 3D analysis of \[13\] to include nonzero temperature effects via the one-loop effective potential, as was done recently for the 4D system in \[15\]. The loop effects introduce another difference between 3D and 4D, as infrared physics is more significant in 3D. As a result, we find some similarities with the 4D case, but also some important differences.

In 3D, with an even number \(N_F\) of flavors of massless complex fermions, denoted by the \(N_F/2\) pairs \(\psi_f, \chi_f\), one can predict spontaneous flavor symmetry breaking patterns along similar lines to 4D QCD. In 3D the generic situation is that the continuous part of the global symmetry group is broken according to \[16, 17\]

\[
U(N_F) \to U(N_F/2) \times U(N_F/2)
\]

by the quark-antiquark condensate

\[
\sum_{f=1}^{N_F/2} \left( \langle \bar{\psi}_f \psi_f \rangle - \langle \bar{\chi}_f \chi_f \rangle \right)
\]

Evidence for such a symmetry breaking pattern has been observed in 3D lattice simulations \[18\] with gauge group SU(3). This pattern of flavor symmetry breaking can also be predicted for 3D QCD at large \(N_C\) using the Coleman–Witten argument \[19\]. The 3D symmetry breaking pattern in \[11\] is for quarks in a complex representation of the
gauge group, and is expected to apply to a U(1) theory and to SU($N_C$) theories with number of colors $N_C \geq 3$. For SU(2), with fundamental quarks, the symmetry breaking pattern is expected to be different again, due to the pseudoreality of the fundamental representation. The pseudoreality of the fundamental representation of SU(2) means that the naive U($N_F$) flavor symmetry is extended to USp($2N_F$), and the continuous part of this global symmetry group is predicted to break down in 3D as

$$\text{USp}(2N_F) \rightarrow \text{USp}(N_F) \times \text{USp}(N_F).$$

(3)

This is different from the generic 3D symmetry breaking pattern in [4], and also is different from 4D theories where the standard flavor symmetry breaking patterns are

$$U(N_F)_L \times U(N_F)_R \rightarrow SU(N_F)_V \ , \text{ for } SU(N_C) \geq 3$$

(4)

$$U(2N_F) \rightarrow \text{USp}(2N_F) \ , \text{ for } SU(2).$$

In 4D the U($N_F$) flavor symmetries are first broken to SU($N_F$), and then broken by the chiral condensate, with the net breakings as shown in [4]. Physically, the differences between the 3D and 4D cases reflect the differences between the anomalous discrete symmetries of parity and chirality in 3D and 4D, respectively.

In section II we review the global symmetries and their cosets that are relevant for the low energy effective Lagrangian description of the 3D two-color QCD system, with quarks in the fundamental representation. In section III we compute the one-loop effective action at finite temperature and finite baryon chemical potential. This result is used in section IV for a Landau-Ginzburg description of the phase diagram, using the rotation angle $\alpha$ which characterizes the tree-level vacuum state as an order parameter. Section V contains our conclusions and some discussion.

II. SYMMETRIES AND EFFECTIVE LAGRANGIAN

A. Enlarged flavor symmetry

The fermionic part of the microscopic Lagrangian for two-color QCD with $N_F = 2n$ flavors of quarks in Euclidean 3D space is

$$\mathcal{L}_f = \bar{\psi} D \psi + \bar{\chi} D \chi + m(\bar{\psi}\psi - \bar{\chi}\chi) - \mu(\bar{\psi}^\dagger \psi + \chi^\dagger \chi).$$

(5)

Here $\psi = \psi_f$, $\psi^\dagger = \psi^{\ast f}$, $\chi = \chi_f$, $\chi^\dagger = \chi^{\ast f}$ are independent two-component spinor fields, with the color index $i = 1, 2$ and flavor index $f = 1, \ldots, n$ being suppressed. Pauli matrices denoted as $\sigma_\nu$, with $\nu = 1, 2, 3$, are employed to represent the Euclidean Dirac matrix algebra, and those denoted as $\tau_\alpha$, with $\alpha = 1, 2, 3$, are employed to represent the gauge group algebra. We choose $x_3$ to be the Euclidean time direction and define $\tilde{\psi} = \psi^\dagger \sigma_3$, $\tilde{\chi} = \chi^\dagger \sigma_3$. The Dirac operator is $D = \sigma_\nu D_\nu$, and the covariant derivative is $D_\nu = \partial_\nu + i A_\nu$, and the gauge field $A_\nu = A_\nu^{\alpha} \tau_\alpha$ is Hermitian and $su(2)$ valued.

Due to the pseudoreality of the SU(2) Dirac operator, the kinetic part of the Lagrangian [4] is invariant under a symmetry group larger than the apparent U(2n) symmetry. One can put $\psi$, $\chi$, $\tilde{\psi}$, and $\tilde{\chi}$ into a single flavor $4n$-plet,

$$\Psi = \begin{bmatrix} \psi \\ \chi \\ \sigma_2 \tau_2 \tilde{\psi}^T \\ \sigma_2 \tau_2 \tilde{\chi}^T \end{bmatrix} ,$$

(6)

so that

$$\mathcal{L}_f = \frac{1}{2} \Psi^T \sigma_2 \tau_2 \tilde{\psi} \tilde{D} \Psi + \frac{m}{2} \Psi^T \sigma_2 \tau_2 \tilde{M} \Psi - \frac{\mu}{2} \Psi^T i \sigma_1 \tau_2 \tilde{C} \Psi$$

(7)

where the $4n \times 4n$ matrices $\tilde{I}$, $\tilde{M}$ and $\tilde{C}$ are

$$\tilde{I} = \begin{bmatrix} I_{2n} & \frac{-I_{2n}}{2n} \\ \frac{I_{2n}}{2n} & I_{2n} \end{bmatrix} , \quad \tilde{M} = \begin{bmatrix} \tilde{I}_{2n} & -I_{2} \\ I_{2} & \tilde{I}_{2n} \end{bmatrix} , \quad \tilde{C} = \begin{bmatrix} I_{2n} & \frac{-I_{2n}}{2n} \\ \frac{I_{2n}}{2n} & I_{2n} \end{bmatrix} .$$

(8)
The kinetic term in the microscopic Lagrangian (7) has extended flavor symmetry group being the unitary symplectic group USp(4n),

$$\Psi \rightarrow S \Psi, \quad S^T \hat{I} S = \hat{I}, \quad S^\dagger S = \mathbb{I}_{4n}. \quad (9)$$

The above extension of the flavor symmetry group is analogous to what happens in two-color QCD in 4D, where the conjugated right-handed spinor $\sigma_2 \tau_2 \psi_2^\dagger R$ transforms as the left-handed spinor $\psi_L$ does under gauge and Lorentz transformations, so that the chiral $SU(N_F)_L \times SU(N_F)_R$ symmetry is extended to $SU(2N_F)$.

In the 3D case, the flavor group USp(4n) is broken down to USp(2n) × USp(2n) explicitly by the mass term in (7), or spontaneously by the quark-antiquark condensate (2), if formed. In the latter case, the Goldstone manifold is thus given by the coset space USp(4n)/(USp(2n) × USp(2n)), which has $4n^2$ independent degrees of freedom, and can be parametrized as

$$\Sigma = S \Sigma_c S^T, \quad \Sigma_c = \hat{M}^\dagger, \quad (10)$$

where

$$S(x) = \exp \left( \frac{i \Pi(x)}{2F} \right), \quad \Pi(x) = \pi_a(x) X_a. \quad (11)$$

The fields $\pi_a$ are the Goldstone modes, and the $4n^2$ generators $X_a$ span the subspace $\text{usp}(4n) - (\text{usp}(2n) \oplus \text{usp}(2n))$. The choice of $\Sigma_c = \hat{M}^\dagger$ leads to the block representation of the generators $X_a$ as

$$\Pi = \frac{1}{2} \begin{bmatrix} \begin{bmatrix} P^\dagger & Q \\ Q^T & -P^* \end{bmatrix} & \\ \begin{bmatrix} P^* & -Q \\ Q^\dagger & P \end{bmatrix} \end{bmatrix}, \quad (12)$$

where $P$ and $Q$ are $n \times n$ complex matrices, each having $2n^2$ degrees of freedom. The chemical potential term in (7) explicitly breaks the extended flavor group USp(4n) down to its unextended U(2n) subgroup, while in the presence of both mass and chemical potential terms, the surviving global symmetry becomes $U(n) \times U(n)$, which is the intersection of USp(2n) × USp(2n) and U(2n).

### B. Low-energy effective Lagrangian

Coupling the fermionic Lagrangian (7) to the SU(2) gauge field gives the the microscopic Lagrangian

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr} F_{\mu \nu} F_{\mu \nu} + \mathcal{L}_f. \quad (13)$$

In the low energy ($\ll \Lambda_{\text{QCD}}$) regime, where fundamental particles are confined and Goldstone bosons dominate, we can define a low energy effective theory. The kinetic term of the effective Lagrangian describing the Goldstone modes $\Sigma$, parametrized as in (10) and (11), should be invariant under the action of the global USp(4n) group

$$\Sigma(x) \rightarrow s \Sigma(x) s^T, \quad s \in \text{USp}(4n) \quad (14)$$

with $s$ in the antisymmetric tensor representation. Invoking the standard symmetry principles of chiral perturbation theory, and the principle of local flavor symmetry, the effective Lagrangian is

$$L = \frac{F^2}{2} \text{tr} \nabla_\nu \Sigma \nabla_\mu \Sigma^\dagger - F^2 M^2 \text{tr} (\hat{M} \Sigma) \quad (15)$$

where the flavor-covariant derivatives are

$$\nabla_\nu \Sigma = \partial_\nu \Sigma - \mu (\Sigma B_\nu^\dagger + B_\nu^\dagger \Sigma) \quad (16)$$

with

$$B_\nu = \bar{B} \delta_{\nu,3}, \quad \bar{B} = \hat{C} I = \begin{bmatrix} \mathbb{I}_{2n} \\ -\mathbb{I}_{2n} \end{bmatrix}. \quad (17)$$
C. Phases at tree level

At zero temperature, the static part of the effective Lagrangian \( L_{\text{st}}(\Sigma) \), i.e. the effective potential, determines the vacuum alignment of the field \( \Sigma \). Introducing a dimensionless parameter \( \xi = 2\mu/M \) to represent the chemical potential, we find

\[
L_{\text{st}}(\Sigma) = F^2 M^2 \left( -\frac{\xi^2}{4} \text{tr}(\hat{M}\Sigma) - \frac{\xi^2}{4} \text{tr}((\hat{B}\Sigma\hat{B}\Sigma^\dagger) - n\xi^2) \right). \tag{18}
\]

The above two terms in \( \Sigma \) compete for the direction of the condensate which we denote by \( \bar{\Sigma} \). The condensate that gives the global minimum of the static tree level effective Lagrangian is

\[
\bar{\Sigma}_\alpha = M^\dagger \cos \alpha + J \sin \alpha, \tag{19}
\]

which is parametrized by \( \alpha \) where

\[
\cos \alpha = \min(1, \xi^{-2}). \tag{20}
\]

1. Normal phase

When \( \xi < 1 \), (i.e., \( \mu < \frac{M}{2} \)), the vacuum orientation of the condensate does not depend on \( \xi \) and is given by \( \bar{\Sigma} = \hat{M}^\dagger \). Expanding \( \Sigma \) around \( \hat{M}^\dagger \) using the Goldstone field defined in \[11\] and \[12\] we find

\[
L = L_{\text{st}}(\bar{\Sigma}_\alpha) + \frac{1}{2} \text{tr} \partial_\nu P^\dagger \partial_\nu P + \frac{M^2}{2} \text{tr} P^\dagger P + \frac{1}{2} \text{tr} \partial_\nu Q^\dagger \partial_\nu Q - 2\mu \text{tr} Q^\dagger Q + \left( \frac{M^2}{2} - 2\mu \right) \text{tr} Q^\dagger Q + \cdots. \tag{21}
\]

From this we can identify the excitations. The baryon charge \( b \) of the excitations are: \( b = 0 \) for \( P \) and \( P^\dagger \); \( b = 2 \) for \( Q \); and \( b = -2 \) for \( Q^\dagger \). The pole masses for these excitations are

\[
\begin{align*}
M & \text{ for } P \text{ and } P^\dagger, \\
M - 2\mu & \text{ for } Q, \\
M + 2\mu & \text{ for } Q^\dagger.
\end{align*} \tag{22}
\]

Each mode belongs to a representation of dimension \( n^2 \).

2. Diquark condensation phase

In the regime \( \xi > 1 \), (i.e., \( \mu > \frac{M}{2} \)), the vacuum condensate has \( U(n) \) degeneracy corresponding to \( n^2 \) true Goldstone modes. This change of massless modes indicates a second-order phase transition at \( \xi = 1 \). When \( \xi > 1 \) the configuration \[19\] begins to rotate away from \( \hat{M}^\dagger \) according to \( \cos \alpha = \xi^{-2} \). This rotation can be written as

\[
\bar{\Sigma}_\alpha = s_\alpha M^\dagger s^T_\alpha = s^2_\alpha M^\dagger, \quad s_\alpha = e^{i\frac{\pi}{4} X_2}, \tag{23}
\]

where \( X_2 \) is the generator that rotates \( M \) into \( J \). We parametrize the fluctuation around the vacuum \( \bar{\Sigma}_\alpha \) as

\[
\Sigma = s_\alpha S M^\dagger S^T s^T_\alpha, \tag{24}
\]

where \( S \) are generated by the unrotated generators \( X_\alpha \).

In this phase we expand around the rotated value of the condensate \( \bar{\Sigma}_\alpha \), as in \[24\]. To second order

\[
\begin{align*}
L &= L_{\text{st}}(\bar{\Sigma}_0) + \frac{1}{2} \text{tr} \partial_\nu P^\dagger_2 \partial_\nu P_S + \frac{1}{2} \text{tr} \partial_\nu P^\dagger_A \partial_\nu P_A + \frac{1}{2} \text{tr} (\partial_\nu Q^\dagger_2)^2 + \frac{1}{2} \text{tr} (\partial_\nu Q^\dagger_I)^2 \\
+ & i M \xi^{-1} \text{tr} (Q^\dagger_2 \partial_\nu Q_I - Q^\dagger_I \partial_\nu Q_2) + \frac{M^2}{2} \left[ \xi^2 \text{tr} P^\dagger_2 P_S^I + \xi^{-2} \text{tr} P^\dagger A P^I_A + (\xi^2 - \xi^{-2}) \text{tr} Q^I_2 \right] + \cdots. \tag{25}
\end{align*}
\]

Here \( P_{S,A} = (P \pm P^T)/2 \), and \( Q_{R,I} \) are the real and imaginary parts of the complex matrix \( Q \). The \( P_{S,A} \) fields have conventional dispersion relations with masses

\[
\begin{align*}
2\mu & \text{ for } P_S, \\
\frac{M^2}{2\mu} & \text{ for } P_A.
\end{align*} \tag{26}
\]
and lie in representations of dimension $n(n+1)$ and $n(n-1)$, respectively.

On the other hand, the dispersion laws for the $Q$ fields are unconventional due to the linear time derivative terms in (25), and are determined by the secular equation

$$\det \begin{bmatrix} [E^2 - p^2] & 2iEM\xi^{-1} \\ -2iEM\xi^{-1} [E^2 - p^2 - M^2(\xi^2 - \xi^{-2})] \end{bmatrix} = 0.$$ \hspace{1cm} (27)

Diagonalizing, we find one true Goldstone field, which we denote $\tilde{Q}$, and its orthogonal complement denoted $\tilde{Q}^\dagger$. These excitations have masses

$$\begin{align*}
0 & \text{ for } \tilde{Q} \\
2\mu\sqrt{1 + 3\left(\frac{M}{2\mu}\right)^4} & \text{ for } \tilde{Q}^\dagger
\end{align*}$$ \hspace{1cm} (28)

and each lies in a representation of dimension $n^2$. Note that fields with this unconventional form of dispersion also appear in the context of kaon condensation [21, 22], and gauge symmetry breaking via Bose condensation [23].

**III. ONE-LOOP FREE ENERGY AT FINITE DENSITY AND TEMPERATURE**

The bosonic low energy effective Lagrangian for 3D two-color QCD reads

$$L = F^2 \frac{2}{\pi} \text{tr} \nabla_\nu \Sigma(x) \nabla_\nu \Sigma(x)^\dagger - F^2 M^2 \text{tr} \tilde{M} \Sigma(x) + L_4.$$ \hspace{1cm} (29)

where the phenomenological constant $F^2$ has dimensions of mass in 3D, and $L_4$ is a collection of terms of order $O(\nabla, M)^4$. We take $M \geq 0$ without loss of generality.

**A. Normal phase**

Within the normal phase (the boundary of which will be determined in the next section), the Lagrangian reads

$$L = -4nF^2 M^2 + \frac{1}{2} P(x)^\dagger (-\partial^2 + M^2) P(x) + \frac{1}{2} Q(x)^\dagger (-\partial^2 - 4\mu\partial_3 + M^2 - 4\mu^2) Q(x) + L_4.$$ \hspace{1cm} (30)

Thus the one-loop free energy is given by

$$\Omega = -4nF^2 M^2 + n^2 \text{tr} \log(-\partial^2 + M^2) + n^2 \text{tr} \log(-\partial^2 - 4\mu\partial_3 + M^2 - 4\mu^2) + L_4[\Sigma = \tilde{M}]$$

$$= -4nF^2 M^2 + n^2 \int \frac{d^3p}{(2\pi)^3} \log(p^2 + M^2) + n^2 \int \frac{d^3p}{(2\pi)^3} \log(p^2 + 4i\mu p_3 + M^2 - 4\mu^2) + \Omega_4.$$ \hspace{1cm} (31)

Here $\Omega_4$ is a constant that is a combination of the contact term couplings. The sum-integral in the above is defined as

$$\sum\int d^d-1 p f(p_3, p) = 2\pi T \sum_{n=-\infty}^{\infty} \int d^d-1 p f(2n\pi T, p).$$ \hspace{1cm} (32)

The first sum-integral in (31) is standard

$$-\sum\int \frac{d^d p}{(2\pi)^d} \log(p^2 + M^2) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty dt e^{-tM^2} \sum_{N=-\infty}^\infty e^{-\frac{tM^2}{N^2}}$$

$$= \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} M^d + 4 \left(\frac{MT}{2\pi}\right)^{d/2} \sum_{N=1}^\infty N^{-d/2} K_{d/2} \left(\frac{NM}{T}\right)$$

$$= \frac{M^3}{6\pi} + \frac{T^2}{\pi} \left(M \text{Li}_2(e^{-\frac{M}{T}}) + T \text{Li}_3(e^{-\frac{M}{T}})\right).$$ \hspace{1cm} (33)
Note that the zero temperature part is finite in odd dimensional spacetime, and the finite temperature correction is expressed in terms of the polylogarithm function \[ \text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}. \] (34)

The second sum-integral in (31) is the conventional effective action for a bosonic field at finite chemical potential, \[ -\sum_{d=3} (2\pi)^d \frac{d^d p}{(2\pi)^d} \log ((p^2 + 2i\mu) + p^2 + \mu^2) \]
\[ = \frac{1}{(4\pi)^d/2} \int_0^\infty dt e^{-tM^2} \sum_{N=-\infty}^{\infty} e^{-\frac{N^2}{2t\pi}} \exp \left( \frac{2\mu N}{T} \right) \]
\[ = \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} M^d + 4 \left( \frac{MT}{2\pi} \right)^{d/2} \sum_{N=1}^{\infty} N^{-d/2} K_{d/2} \left( \frac{MN}{T} \right) \cosh \left( \frac{2\mu N}{T} \right) \] (35)
\[ \rightarrow \frac{M^3}{6\pi} + \frac{T^2}{2\pi} \left[ M \left( \text{Li}_2(e^{-\frac{M-2\mu}{T}}) + \text{Li}_2(e^{-\frac{M+2\mu}{T}}) \right) + T \left( \text{Li}_3(e^{-\frac{M-2\mu}{T}}) + \text{Li}_3(e^{-\frac{M+2\mu}{T}}) \right) \right] + \Omega_4. \] (36)

Once again, the zero T part is finite due to the odd dimensionality, and the finite T corrections are expressed in terms of polylog functions.

Bringing together the contributions (33) from the $P$ modes, and the contribution (36) from the $Q$ modes, the one-loop effective action in the normal phase, at finite $T$ and $\mu$, is
\[ \Omega = -4nF^2M^2 - n^2 \left[ \frac{M^3}{6\pi} + \frac{T^2}{\pi} \left( M \text{Li}_2(e^{-\frac{M}{T}}) + T \text{Li}_3(e^{-\frac{M}{T}}) \right) \right] \]
\[ -n^2 \left[ \frac{M^3}{6\pi} + \frac{T^2}{2\pi} \left[ M \left( \text{Li}_2(e^{-\frac{M+2\mu}{T}}) + \text{Li}_2(e^{-\frac{M-2\mu}{T}}) \right) + T \left( \text{Li}_3(e^{-\frac{M+2\mu}{T}}) + \text{Li}_3(e^{-\frac{M-2\mu}{T}}) \right) \right] \right] + \Omega_4. \] (37)

### B. Diquark condensation phase

We now compute the free energy, to 1-loop order, by expanding about the rotated vacuum
\[ \Sigma_\alpha = s_\alpha \bar{s}_{\alpha} T, \quad s_\alpha = \exp \left( \frac{i}{2} X_2 \right) \] (38)
which minimizes the tree level effective potential in the diquark condensation phase. We parameterize the fluctuations around $\Sigma_\alpha$ as
\[ \Sigma(x) = s_\alpha S(x) \bar{M}_T S(x)^T s^T_\alpha, \]
\[ S(x) = \exp \left( \frac{i\Pi(x)}{2F} \right), \quad \Pi(x) = \frac{1}{2} \begin{bmatrix} P & Q \\ Q^T & -P^T \end{bmatrix}. \] (39)

The quadratic Lagrangian reads
\[ L = -F^2\mu^2 tr (B \Sigma_\alpha B \Sigma_\alpha^T + \mu_4) - F^2M^2 tr \bar{M} \Sigma_\alpha \\
\quad + \frac{1}{2} P_S(x) (-\partial^2 + M_1^2 + \frac{1}{4} M_2^2) P_S(x) + \frac{1}{2} P_A(x) (-\partial^2 + M_1^2 + \frac{1}{4} M_2^2) P_A(x) \\
\quad + \frac{1}{2} Q(x) \begin{bmatrix} -\partial^2 + M_1^2 & M_3 \partial_3 \\ M_3 \partial_3 & -\partial^2 + M_2^2 \end{bmatrix} Q(x) + L_4. \] (40)

Here we have used the notations of \[ \text{14, 15}, \] expressing $M_1$, $M_2$ and $M_3$ in terms of the overall mass scale $M$, the baryon chemical potential $\mu$, and the rotation angle $\alpha$ which characterizes the rotated vacuum \[ \text{35} : \]
\[ M_1^2 = M^2 \cos \alpha - 4\mu^2 \cos 2\alpha, \]
\[ M_2^2 = M^2 \cos \alpha - 4\mu^2 \cos^2 \alpha, \]
\[ M_3^2 = 16\mu^2 \cos^2 \alpha. \] (41)
There are \( n(n+1) \) \( P_S \) modes, \( n(n-1) \) \( P_A \) modes, and \( (n^2 + n^2) \) \( Q \) modes. Integrating out these modes, one obtains the one-loop free energy

\[
\Omega = -F^2 \mu^2 \text{tr} (\hat{B} \Sigma_\alpha \hat{B} \Sigma_\alpha^T + \mathbb{1}_n) - F^2 M^2 \text{tr} \hat{M} \Sigma_\alpha \\
+ \frac{1}{2} n(n+1) \text{tr} \log(-\partial^2 + M^2_1 + \frac{1}{4} M^2_3) + \frac{1}{2} n(n-1) \text{tr} \log(-\partial^2 + M^2_2 + \frac{1}{4} M^2_3) \\
+ \frac{1}{2} n^2 \text{tr} \left[ -\partial^2 + M^2_1 \frac{M_3 \partial_3}{M_3 \partial_3} - \partial^2 + M^2_2 \right] + \Omega_4 \\
= -4nF^2 \left( \frac{M^2_1 + M^2_2 + M^2_3}{2} \right) \\
+ \frac{n(n+1)}{2} \int \frac{d^d p}{(2\pi)^d} \log(p^2 + M^2_1 + \frac{1}{4} M^2_3) + \frac{n(n-1)}{2} \int \frac{d^d p}{(2\pi)^d} \log(p^2 + M^2_2 + \frac{1}{4} M^2_3) \\
+ \frac{n^2}{2} \int \frac{d^d p}{(2\pi)^d} \log((p^2 + M^2_1)(p^2 + M^2_2) + p^2 M^2_3) + \Omega_4. 
\]

(42)

Here \( \Omega_4(\alpha) = L_4[\Sigma_\alpha] \) is given by substituting \( \Sigma(x) = \Sigma_\alpha \) into the \( O(\nabla, M)^4 \) terms of the effective Lagrangian.

The \( P_S \) and \( P_A \) mode contributions in (42) are precisely as for the \( P \) modes in the normal phase in (33), with \( \sqrt{M^2_1 + M^2_3} \) replaced by \( \sqrt{M^2_2 + M^2_3}/4 \) for the \( P_S \) modes, and by \( \sqrt{M^2_1 + M^2_3}/4 \) for the \( P_A \) modes. On the other hand, the final term in (42) is nontrivial because of the quartic nature of the operator. One approach is to factor this quartic as

\[
(p^2 + M^2_1)(p^2 + M^2_2) + p^2 M^2_3 = ((p_3 + i M y)^2 + p^2 + M^2 z^2)((p_3 - i M y)^2 + p^2 + M^2 z^2) - \left( \frac{M^2_1 - M^2_2}{2} \right)^2 
\]

(43)

where we have introduced the convenient dimensionless parameters \( y \) and \( z \), as in (44)

\[
y = \frac{M_3}{2 M}, \quad z = \frac{1}{M} \sqrt{\frac{M^2_1 + M^2_2}{2} + M^2_2/4}. 
\]

(44)

One can now expand the final term in (42) in terms of the difference

\[
M^2_1 - M^2_2 = 4 \mu^2 \sin^2 \alpha. 
\]

(45)

Such an expansion proves useful in the Landau-Ginzburg analysis for the region where the order parameter \( \alpha \) is small, as studied in the next Section.

Then the contribution of these \( Q \)-type modes is

\[
G_Q = -\int \frac{d^d p}{(2\pi)^d} \log ((p^2 + M^2_1)(p^2 + M^2_2) + p^2 M^2_3) \\
= -\int \frac{d^d p}{(2\pi)^d} \log (((p_3 + i M y)^2 + p^2 + M^2 z^2)((p_3 - i M y)^2 + p^2 + M^2 z^2)) \\
+ \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{M^2_1 - M^2_2}{2} \right)^{2k} \int \frac{d^d p}{(2\pi)^d} \frac{1}{((p_3 + i M y)^2 + p^2 + M^2 z^2)^k((p_3 - i M y)^2 + p^2 + M^2 z^2)^k}. 
\]

(46)

The logarithmic term in the second line of (46) factors in terms of conventional bosonic effective actions at finite chemical potential:

\[
G_Q^{(0)} = -\int \frac{d^d p}{(2\pi)^d} \log (((p_3 + i M y)^2 + p^2 + M^2 z^2)((p_3 - i M y)^2 + p^2 + M^2 z^2)) \\
= \frac{2}{(4\pi)^{d/2}} \int_0^{\infty} dt e^{-t(M z)^2} \sum_{N = -\infty}^{\infty} e^{-\frac{t^2}{4t^{d/2}+1}} \text{cosh} \left( \frac{M y N}{T} \right) \\
= \frac{2 \Gamma(-d/2)}{(4\pi)^{d/2}} (M z)^d + 8 \left( \frac{M z T}{2 \pi} \right)^{d/2} \sum_{N = 1}^{\infty} N^{-d/2} K_{d/2} \left( \frac{M z N}{T} \right) \text{cosh} \left( \frac{M y N}{T} \right) \\
\frac{d=3}{3 \pi} \left( M z ^3 + \frac{T^2}{\pi} \left[ M z \left( Li_2(e^{-M(z-y)/T}) + Li_2(e^{-M(z+y)/T}) \right) + T \left( Li_3(e^{-M(z-y)/T}) + Li_3(e^{-M(z+y)/T}) \right) \right] \right). 
\]

(47)
The vanishing of the coefficient of the term quadratic in shape of the effective potential can be used to deduce information about the phase structure of the system \[26\]. The effective action in the diquark phase, at finite temperature and chemical potential is

\[\Gamma = \frac{1}{2} \sum_{k} \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + M^2z^2)^2} (s(1-s))^{k-1} \int \frac{1}{0} ds (s(1-s))^{k-1} \int \frac{1}{T} e^{-tM^2(z^2-y^2)} T \sum_{n=-\infty}^{\infty} e^{-t[(2\pi nT)^2+(2s-1)^2M^2y(2\pi nT)]} \]

\[= \frac{1}{\Gamma(k)^2(4\pi)^d/2} \sum_{n=1}^{\infty} \left( \frac{N}{TM} \right)^{2k-2} \int_{0}^{1} du \frac{u(k-1) K_{\frac{1}{2}-2k} \left( \frac{MN}{T \sqrt{1-u}} \right)}{\sqrt{z-y^2u}} \cosh \left( \frac{MN}{T \sqrt{y(1-u)}} \right) + (T = 0 \text{ part}) \]

Motivated by the Landau-Ginzburg analysis of the next Section, we consider just the \(k = 1\) term, since higher terms are at least of order \(\alpha^8\). In \(d = 3\) the \(k = 1\) term in \[48\] reduces to a simple integral:

\[C_Q^{(1)} = \frac{1}{8\pi My} \left( \frac{1}{2} \log \frac{z+y}{z-y} + \int_{(z-y)/M}^{(z+y)/M} \frac{dx}{x(e^x-1)} \right) \]

Here the first term corresponds to the \(T = 0\) contribution, while the integral represents the finite \(T\) correction. Collecting together the contributions of the \(P_3\), \(P_4\) and \(Q\) modes, we obtain the following expression for the one-loop effective action in the diquark phase, at finite temperature and chemical potential

\[\Omega = -4nF^2M^2z^2 \]

\[-\frac{n(n+1)}{2} \left( \frac{M_1^4 + M_2^4/4}{6\pi} \right) + \frac{T^2}{\pi} \left( \frac{M_1^2 + M_2^2}{4} \right) \] \[= \frac{n(n-1)}{2} \left( \frac{M_1^4 + M_2^4/4}{6\pi} \right) + \frac{T^2}{\pi} \left( \frac{M_1^2 + M_2^2}{4} \right) \] \[-\frac{z^2}{3\pi} \right) + \frac{T^2}{\pi} \right) \left( \frac{M_1^2 + M_2^2}{4} \right) \] \[+ \frac{M_1^2 - M_2^2}{32\pi yM} \left[ \frac{1}{2} \log \frac{z+y}{z-y} + \int_{(z-y)/M}^{(z+y)/M} \frac{dx}{x(e^x-1)} \right] + O(M_1^2 - M_2^2)^4 + \Omega_4. \]

Note that the higher order corrections are at least of order \((M_1^2 - M_2^2)^4 \sim \mu^8 \sin^8 \alpha\).

\section{Landau-Ginzburg Model}

In the Landau-Ginzburg approach \[23\], we regard the rotation angle \(\alpha\) as an order parameter. For small \(\alpha\), the shape of the effective potential can be used to deduce information about the phase structure of the system \[23\]. The vanishing of the coefficient of the term quadratic in \(\alpha\) determines a critical line in the \(T\) and \(\mu\) plane.

\subsection{Landau-Ginzburg model at zero temperature}

It is important first to analyze the Landau-Ginzburg model at \(T = 0\) in order to understand the effect of one-loop renormalization. Since we expect the critical behavior to be near the tree-level \(T = 0\) critical point where \(\mu = M/2\),
we expand $\Omega$ in terms of the rotation angle $\alpha$ and also the deviation of chemical potential from $M/2$:

$$\bar{\mu} = \frac{\mu}{M} - \frac{1}{2}.$$  

(51)

Substituting the definitions of $M_1$, $M_2$, $M_3$, $z$ and $y$ into the $T = 0$ parts of the one-loop effective potential in (50) and expanding, one finds

$$\frac{\Omega_{T=0}}{-4nF^2M^2} = \text{cst.} + \frac{1}{32\pi} \frac{M}{F^2} \alpha^2 + \left(2 + \frac{2n + 1}{8\pi} \frac{M}{F^2}\right) \bar{\mu} \alpha^2 + O\left(\alpha^4 \log(\alpha^2 - 4\bar{\mu})\right).$$  

(52)

Note that the higher order terms in (50) are not relevant as they contribute terms at least of order $\alpha^8$. Also, we neglect the quartic contact terms $\Omega_4$. This is motivated by the 4D case [15] where these terms are negligibly small. The dimensionless combination $M/F^2$ in (52) is the loop expansion parameter. The vanishing of the coefficient of $\alpha^2$ leads to the critical point (at one-loop)

$$\bar{\mu} = -\frac{1}{64\pi} \frac{M}{F^2}.$$  

(53)

This can be rewritten as

$$\mu = \frac{1}{2} M \left(1 - \frac{1}{32\pi} \frac{M}{F^2}\right)$$  

(54)

which shows that the critical point is shifted by a finite one-loop correction from the tree-level $T = 0$ critical point $\mu = M/2$. Actually, this shift is just a mass renormalization since the one-loop renormalized pion mass $m_\pi$ is

$$m_\pi = M \left(1 + \frac{\Gamma(1 - d/2)}{8(4\pi)^{d/2}} \frac{M^{d-2}}{F^2}\right) \overset{d=3}{\to} M \left(1 - \frac{1}{32\pi} \frac{M}{F^2}\right)$$  

(55)

as computed in the Appendix. Thus, this result means that one-loop renormalization has the effect of shifting the $T = 0$ critical point from $\mu = M/2$ to $\mu = m_\pi/2$, where $m_\pi$ is the one-loop renormalized value of $M$.

### B. Landau-Ginzburg model at finite temperature

At finite temperature, the shape of the free energy as a function of the order parameter $\alpha$ changes as we move through the $T$ and $\mu$ plane. Given the explicit expression (50) it is straightforward to plot (using, e.g., Mathematica) the free energy for chosen values of $T$ and $\mu$. This gives some guidance as to the parameter region in which to search for a critical line. We find that the appropriate critical region of parameters is

$$0 < \mu - \frac{m_\pi}{2} \approx m_\pi g \ll T \ll m_\pi,$$  

(56)

where we have defined the natural dimensionless self coupling constant of pions by

$$g = \frac{M}{32\pi F^2}.$$  

(57)

In this region the temperature is far smaller than the mass of massive modes (the $P$’s and half of the $Q$’s), but is far larger than the mass of the almost Goldstone modes (the other half of the $Q$’s). We also have the intrinsic applicability limit of the low energy effective Lagrangian that other hadronic modes are not excited: i.e., $m_\pi \ll \Lambda_{\text{QCD}}$.

We expand the Landau-Ginzburg free energy (normalized by that at $\mu = T = 0$) up to the quartic order in $\alpha$,

$$\frac{\Omega}{-4nF^2M^2} = c_0 + c_2 \alpha^2 + c_4 \alpha^4 + O(\alpha^6).$$  

(58)

Using the expression (50), the coefficient of the quadratic term is given explicitly by

$$c_2 = -\frac{1}{2} + \frac{2\mu^2}{M^2} + \frac{1}{32\pi F^2 M^2} \left[-2nM^3 + 4(1 + 2n)\mu^2 M + 2(-nM^2 + 4(1 + n)\mu^2) T \log(1 - e^{-\frac{4M}{\mu}T})
+ (-M^2 + 4\mu M + 4\mu^2) nT \log(1 - e^{-\frac{M+2\mu}{\mu}T})
+ (-M^2 - 4\mu M + 4\mu^2) nT \log(1 - e^{-\frac{M+2\mu}{\mu}T})
- 4n\mu T^2 \left(\text{Li}_2(e^{-\frac{M+2\mu}{\mu}}) - \text{Li}_2(e^{-\frac{M+2\mu}{\mu}})\right)\right].$$  

(59)
Similarly, from (50) the coefficient of the quartic term is given explicitly by

\[
c_4 = \frac{1}{24} - \frac{\mu^2}{3M^2} + \frac{1}{768\pi F^2FM^2} \left[ 10nM^4 - 56(1+2n)\mu^2 + 48(2+3n)\frac{\mu^4}{M} \right. \\
+ \frac{3n(M^2 - 4\mu \mu^2 - 4\mu^2)^2}{(e^{\frac{M+2\mu}{M}} - 1)M} + \frac{3n(M^2 + 4\mu \mu^2 - 4\mu^2)^2}{(e^{\frac{M-2\mu}{M}} - 1)M} + \frac{6(nM^4 - 8(1+n)M^2\mu^2 + 32(1+n)\mu^4)}{(e^{\frac{M+2\mu}{M}} - 1)M} \\
+ 2n(M^2 - 4\mu \mu^2 - 40\mu^2)T log(1 - e^{-\frac{M+2\mu}{M}}) + 2n(M^2 + 4\mu \mu^2 - 40\mu^2)T log(1 - e^{-\frac{M-2\mu}{M}}) \\
+ 4(nM^2 - 16(1+n)\mu^2)T log(1 - e^{-\frac{M+2\mu}{M}}) + 8nM^2T \left( Li_2(e^{-\frac{M+2\mu}{M}}) - Li_2(e^{-\frac{M-2\mu}{M}}) \right) + 12n\mu^3 \int_{\frac{M+2\mu}{M}}^{\frac{M-2\mu}{M}} \frac{dx}{x} \coth \frac{x}{2} .
\]

The separation between the tree level part and the one-loop part is apparent in these expressions for \( c_2 \) and \( c_4 \). We stress that no approximation has been used so far, except that (58) is an expansion for small values of the order parameter \( \alpha \).

In the parameter region (59) we may obtain more compact approximate expressions for \( c_2 \) and \( c_4 \) by the use of the expansion formula of polylogarithms with positive integer indices,

\[
Li_n(e^{-\epsilon}) = \frac{\psi(n) + \gamma - \log(\epsilon)}{(n-1)!}(-\epsilon)^{n-1} + \sum_{k \geq 0, k \neq n-1} \frac{\zeta(n-k)}{k!}(-\epsilon)^k .
\]

and the formula

\[
\int_\epsilon^{\Lambda} \frac{dx}{x} \coth \frac{x}{2} = \frac{2}{\epsilon} + \log(\Lambda) + O(\epsilon^0) + O\left(\frac{e^{-\Lambda}}{\Lambda}\right) ,
\]

valid for \( 0 < \epsilon \ll 1 \ll \Lambda \). We define the dimensionless deviation parameter

\[
\delta = 1 - \frac{2\mu}{M} = -2\bar{\mu}
\]

which is of the same infinitesimal order as the coupling \( g \). Then we find the following simple approximate formulas for \( c_2 \) and \( c_4 \):

\[
c_2 \simeq -\delta + g \left( 1 - 2n \frac{T}{M} \log \frac{T}{M \delta} \right) ,
\]

\[
c_4 \simeq -\frac{1}{24} + g \frac{5n}{24M} \frac{T}{\delta} .
\]

In the Landau-Ginzburg approach, the critical line is characterized by the condition \( c_2 = 0 \), and the tricritical point is characterized by the conditions \( c_2 = c_4 = 0 \). Expressing dimensionful quantities in the unit of the renormalized pion mass \( M_\pi \), rather than the bare pion mass \( M \), the critical conditions given by the approximated coefficients (58) read

\[
\text{critical line: } \mu - \frac{M_\pi}{2} = ngT \log \frac{T}{\mu - 2\bar{\mu} + gm_\pi} ,
\]

\[
\text{tricritical point: } \mu_{\text{tri}} = \frac{M_\pi}{2} \left( 1 + \frac{g}{2 \log(5ng)} \right) , \quad T_{\text{tri}} = \frac{M_\pi}{n(5 - 2 \log(5ng))} .
\]

We have confirmed numerically that these expressions approximate very well the critical line and tricritical point derived numerically from the vanishing of the full expressions for \( c_2 \) and \( c_4 \) in (59), for the parameter region of concern. Accordingly our approximation is indeed self-consistent in that the critical region is within the region (59).

C. The critical line and a weakly interacting Bose gas

The critical line (60) has an interpretation in terms of a weakly interacting Bose gas, as in the 3+1 dimensional case studied in (14), where the diquark phase was argued to be a Bose condensate. The fact that the same structure
FIG. 1: Plot of the phase diagram for two flavors \((n = 1)\) and coupling \(g = 0.002\) which corresponds to \(M/F^2 = 0.2\). The solid line is the critical line in \(66\), and the x marks the tricritical point \(67\). The dotted line is an approximation based on expanding the effective potential to sixth order in \(\alpha\) and demanding it to have the form \(\Omega(\alpha) = c_0 \alpha^2 (\alpha^2 - \text{const.})^2 + \text{const.}\). Note that the scale has been magnified to show just the vicinity of the critical line.

appears in our \(2 + 1\) dimensional model is quite interesting, as Bose condensation is rather different in two spatial dimensions. So we show here explicitly how the usual Bose gas results are consistent with the Landau-Ginzburg effective Lagrangian approach of this paper.

Recall that for a free relativistic Bose gas in \(d\) spatial dimensions, the charge density is given by \(29\)

\[
\rho = \frac{\Gamma (d+1)}{\pi^{(d+1)/2}} T^d \left[ g_d \left( \frac{M}{T}, \frac{\mu}{M} \right) - g_d \left( \frac{M}{T}, -\frac{\mu}{M} \right) \right].
\]

The first term corresponds to the particles and the second to the antiparticles. For \(d = 2\) \(29\),

\[
g_2 \left( \frac{M}{T}, -\frac{\mu}{M} \right) = \text{Li}_2 \left( e^{(\mu - M)/T} \right) - \frac{M}{T} \log \left( 1 - e^{(\mu - M)/T} \right).
\]

In the \(2 + 1\) dimensional two-color QCD system discussed in this paper, the \(Q\) excitations have baryon charge 2, so we should replace the chemical potential in \(68,69\) by \(\mu \to 2\mu_B\), and the charge density in \(68\) by \(\rho \to n_B/2\). At low temperature the lightest modes, the \(Q\) modes, dominate. There are \(n^2\) such modes, where we recall that the number of flavors is \(N_f = 2n\). Furthermore, at low temperature only the particles contribute and so we expect

\[
n_B = 2n^2 T^2 \frac{g_2}{2\pi} \left( \frac{M}{T}, \frac{2\mu_B}{M} \right).
\]

In the Landau-Ginzburg approach discussed in the previous section, the critical line separating the normal and diquark phases is determined by the vanishing of the coefficient \(c_2\) of the term quadratic in the order parameter \(\alpha\) in the expansion of the free energy \(58\). At one-loop, the vanishing of \(c_2\) in \(59\) near the critical point \(\mu_B \approx m_\pi/2\) can be written as [here we use slightly less restrictive approximations than were used in obtaining \(60\)]

\[
\mu_B - \frac{m_\pi}{2} \approx \frac{nT^2}{32\pi F^2} \left[ \text{Li}_2 \left( e^{(2\mu_B - M)/T} \right) - \frac{M}{T} \log \left( 1 - e^{(2\mu_B - M)/T} \right) \right]
\]

\[
= \frac{nT^2}{32\pi F^2} g_2 \left( \frac{M}{T}, \frac{2\mu_B}{M} \right).
\]
Combining these two results (70) and (71) we see that
\[ n_B = 16N_f F^2 \left( \mu_B - \frac{m_\pi}{2} \right). \] (72)

In the effective Lagrangian approach the renormalized vacuum energy is [31]
\[ \mathcal{E}_{\text{vac}} = n_B \frac{m_\pi}{2} + \frac{n_B^2}{32F^2N_f} + \cdots. \] (73)
where the coefficients are fixed by the symmetries. Since the baryon number chemical potential is
\[ \mu_B = \frac{\partial \mathcal{E}_{\text{vac}}}{\partial n_B} \] (74)
we see that the relation [24] follows directly. This demonstrates the consistency of the Landau-Ginzburg effective Lagrangian approach with the standard Bose gas results, and shows that the critical line separating the normal and diquark phases describes a free Bose gas in two spatial dimensions.

V. CONCLUSION AND DISCUSSION

In this paper we have used the low energy effective field theory method at finite temperature and finite baryon density to investigate the phase structure of three dimensional parity invariant SU(2) QCD with fundamental quarks. We computed the one-loop effective potential at both \( T = 0 \) and finite \( T \), at nonvanishing baryon number chemical potential. At \( T = 0 \), the tree-level critical point \( \mu = M/2 \) between the normal and diquark phase receives a finite shift at one-loop, but this shift corresponds exactly to the mass renormalization, so that at one-loop the \( T = 0 \) critical point is \( \mu = m_\pi/2 \), where \( m_\pi \) is the one-loop renormalized pion mass. At nonzero temperature we found a simple expression [60] for the effective potential, suitable for a Landau-Ginzburg analysis in terms of the order parameter \( \alpha \) which describes the rotation of the tree-level vacuum away from the normal phase vacuum alignment. The subsequent Landau-Ginzburg analysis of this free energy revealed a critical line in the \( (\mu, T) \) plane, given approximately by [66]. This line separates the normal and diquark phases, and ends in a tricritical point whose location is given approximately by [67]. This general structure, with a critical line ending in a tricritical point, is qualitatively similar to that found in the 4D case [15], except that the \( T \log T \) temperature dependence for the critical line in [66] becomes a power law \( \sim T^{3/2} \) in the 4D case.

However, the details of the 3D phase diagram are much less clear than in the 4D case, due to the presence of infrared divergences, which become more pronounced at higher loops, where they will appear as powers. In a restricted region of the relevant coupling and parameters (such as \( g, T/M \) and \( \mu \)), the one-loop effective potential does describe the correct physics. The one-loop calculation indicates that some transition happens in the vicinity of the critical line [66], and the one-loop Landau-Ginzburg analysis suggests that this is a line of second-order phase transitions, as is the case in 4D [15]. However, in 3D there are infrared divergences as \( \mu \) tends to zero (\( \mu \to M/2 \)), which become more significant at higher loops. This suggests that a nonperturbative analysis may be necessary to identify the true nature of the phase transition. There is another reason to think that in the 3D case the critical line may not be one of second-order phase transitions: in two spatial dimensions, only truly massless modes Bose condense – massive modes, no matter how light, do not Bose condense [26]. If the \( Q \) modes are truly massless even at nonzero temperature, we would expect to be able to dimensionally reduce the system to two dimensions, in which case the Mermin-Wagner-Coleman theorem tells us that a continuous symmetry cannot be broken (in the absence of long-range interactions). What then is the fate of the diquark phase at nonzero \( T \)? Perhaps nonperturbative effects generate an exponentially small, \( \sim e^{-F^2/T} \) (rather than actually zero), mass for the modes (which at tree level are exactly massless), and the critical line is actually a cross-over between a normal and diquark phase, until the tricritical point is reached, after which it presumably becomes a line of first-order transitions. We mention that such a nonperturbative exponentially small mass has been observed in the thermodynamics of the large \( N O(N) \) sigma model in 3D [30]. This would mean that the standard perturbative effective potential approach to two-color QCD in 3D is good for describing the physics on some portion of the critical line where it is a noninteracting Bose gas, but away from this region the perturbative analysis is insufficient. It would be very interesting to investigate the relevant nonperturbative effects, perhaps from a large \( N_F \) analysis, or from a lattice study of 3D two-color QCD.

Acknowledgments

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APPENDIX A: ONE-LOOP RENORMALIZATION

In this Appendix we perform the one-loop renormalization of the pion mass and wave function. As the theory is renormalizable at each order of the chiral perturbation, renormalization constants are common for both normal and diquark phase. Thus we shall concentrate on the simpler case of the normal phase, and set $\Omega = 0$ as mentioned in section IV.A. We substitute $\phi^{(2)}$ at $\alpha = 0$ into the tree-level term of $\phi^{(1)}$ and expand in $P$ and $Q$ fields up to the quartic order. We write

$$P_{ab} = \phi^{(1)}_{ab} + i\phi^{(2)}_{ab}, \quad Q_{ab} = \phi^{(3)}_{ab} + i\phi^{(4)}_{ab}, \quad a, b = 1, \ldots, n.$$  \hspace{1cm} (A1)

Without loss of generality we choose to compute $\langle \phi^{(1)}(p)\phi^{(1)}(-p) \rangle$. The terms that contribute to this Green’s function are:

$$L_{\text{tree}} = \sum_{\alpha=1}^{4} \frac{M^2}{96F^2} \left[ 4(\phi^{(1)})^2 + 2(\phi^{(1)})^2 \left[ \sum_{\alpha=2}^{4} (\phi^{(1)})^2 + \sum_{\alpha=1}^{4} \sum_{\alpha=1}^{n} (\phi^{(1)})^2 + (\phi^{(1)})^2 \right] \right]$$

$$\quad - \frac{M^2}{96F^2} \left[ 4(\phi^{(1)})^2 + 2(\phi^{(1)})^2 \left[ \sum_{\alpha=1}^{n} (\phi^{(1)})^2 + \sum_{\alpha=1}^{n} \sum_{\alpha=1}^{n} (\phi^{(1)})^2 + (\phi^{(1)})^2 \right] \right].$$ \hspace{1cm} (A2)

There exist vertices of the form $\phi^{(1)}\partial_{\mu}\phi^{(1)}\phi^{(1)}\partial_{\nu}\phi^{(1)}$, $a \neq 1$, but they do not contribute to $\langle \phi^{(1)}(p)\phi^{(1)}) \rangle$.

Using these vertices, we have

$$\langle \phi^{(1)}(p)\phi^{(1)}(-p) \rangle = \frac{1}{(p^2 + M^2)^2} \left[ p^2 + M^2 + \frac{M^2}{96F^2} \cdot 1 \cdot 3 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2} + \frac{M^2}{48F^2} \cdot 1(3 + 8(n - 1)) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2} \right]$$

$$\quad - \frac{1}{96F^2} \cdot 1(12 + 8(n - 1)) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2} - \frac{1}{96F^2} \cdot 1(12 + 8(n - 1))(-p^2) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2} \right]$$

$$\quad = \frac{1}{(p^2 + M^2)^2} \left[ p^2 \left( 1 + \frac{2n + 1}{12F^2} \right) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2} \right] + M^2 \left( 1 + \frac{n - 1}{6F^2} \right) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2} \right]$$

$$\quad = \frac{1}{(p^2 + M^2)^2} \left[ p^2 \left( 1 + \frac{2n + 1}{12F^2} \right) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2} \right] + M^2 \left( 1 + \frac{n - 1}{6F^2} \right) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + M^2} \right].$$ \hspace{1cm} (A3)

We have discarded a term proportional to the divergent integral $\int \frac{d^4q}{(2\pi)^4}$ in the dimensional regularization.

The wave function renormalization should absorb the factor multiplying $p^2$,

$$\phi(p) = \phi(p)_{\text{ren}} \left( 1 + \frac{2n + 1}{24F^2} \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} M^{d - 2} \right).$$ \hspace{1cm} (A4)

Accordingly,

$$\langle \phi^{(1)}(p)_{\text{ren}}\phi^{(1)}(-p)_{\text{ren}} \rangle = \frac{1}{(p^2 + M^2)^2} \left[ p^2 + M^2 \left( 1 - \frac{1}{4F^2} \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} M^{d - 2} \right) \right].$$ \hspace{1cm} (A5)

At $p = 0$,

$$\langle \phi^{(1)}(0)_{\text{ren}}\phi^{(1)}/(0)_{\text{ren}} \rangle = \frac{1}{M^2} \left( 1 - \frac{\Gamma(1 - d/2)}{4(4\pi)^{d/2}} \frac{M^{d - 2}}{F^2} \right) \equiv \frac{1}{m^2_\pi}.$$ \hspace{1cm} (A6)

That is,

$$m_\pi = M \left( 1 + \frac{\Gamma(1 - d/2)}{8(4\pi)^{d/2}} \frac{M^{d - 2}}{F^2} \right).$$ \hspace{1cm} (A7)
Note that the $n$-dependent part of the coefficients of $p^2$ and $M^2$ in $A_3$ are identical, leading to the $n$-independent mass renormalization.

[1] J. Gasser and H. Leutwyler, “Chiral perturbation theory to one loop,” Annals Phys. 158, 142 (1984); “Chiral perturbation theory: Expansions in the mass of the strange quark,” Nucl. Phys. B 250, 465 (1985).

[2] S. Weinberg, “The Quantum Theory of Fields, Vol. 2: Modern Applications,” sect.19, Cambridge Univ. Press (Cambridge, UK, 1996).

[3] J. J. M. Verbaarschot and T. Wettig, “Random matrix theory and chiral symmetry in QCD,” Ann. Rev. Nucl. Part. Sci. 50, 343 (2000) arXiv:hep-ph/0003017.

[4] S. Hands, J. B. Kogut, M. P. Lombardo and S. E. Morrison, “Symmetries and spectrum of SU(2) lattice gauge theory at finite chemical potential,” Nucl. Phys. B 558, 327 (1999) arXiv:hep-lat/9902034.

[5] J. B. Kogut, M. A. Stephanov and D. Toublan, “On two-color QCD with baryon chemical potential,” Phys. Lett. B 464, 183 (1999) arXiv:hep-ph/9906346.

[6] J. B. Kogut, M. A. Stephanov, D. Toublan, J. J. M. Verbaarschot and A. Zhiltztsky, “QCD-like theories at finite baryon density,” Nucl. Phys. B 582, 477 (2000) arXiv:hep-ph/0001171.

[7] D. Toublan and J. J. M. Verbaarschot, “Dirac spectra and real QCD at nonzero chemical potential,” in Minneapolis 2002, Continuous advances in QCD, K.A. Olive, M.A. Shifman, M.B. Voloshin (Eds), (World Scientific, Singapore, 2002), p 265-290; arXiv:hep-th/0208021.

[8] J. J. M. Verbaarschot, “What really matters”, talk at the Channel Meeting, Plymouth, August 2002.

[9] M. E. Peskin, “The alignment of the vacuum in theories of technicolor,” Nucl. Phys. B 175, 197 (1980).

[10] E. Dagotto, F. Karsch and A. Moreo, “The strong coupling limit of SU(2) QCD at finite baryon density,” Phys. Lett. B 169, 421 (1986).

[11] C. Baillie, K. C. Bowler, P. E. Gibbs, I. M. Barbour and M. Rafaie, “The chiral condensate in SU(2) QCD at finite density,” Phys. Lett. B 197, 195 (1987).

[12] J. B. Kogut, D. Toublan and D. K. Sinclair, “The phase diagram of four flavor SU(2) lattice gauge theory at nonzero chemical potential and temperature,” Nucl. Phys. B 642, 181 (2002) arXiv:hep-lat/0205019; “SU(2) lattice gauge theory at nonzero chemical potential and temperature,” Talk at Lattice 2002, arXiv:hep-lat/0208076.

[13] G. V. Dunne and S. M. Nishigaki, “Two-color QCD in 3D at finite baryon density,” Nucl. Phys. B 654, 445 (2003) arXiv:hep-th/0210219.

[14] K. Splittorff, D. Toublan and J. J. M. Verbaarschot, “QCD with two colors at finite baryon density at next-to-leading order,” Nucl. Phys. B 620, 290 (2002) arXiv:hep-ph/0108040.

[15] K. Splittorff, D. Toublan and J. J. M. Verbaarschot, “Thermodynamics of chiral symmetry at low densities,” Nucl. Phys. B 639, 524 (2002) arXiv:hep-ph/0204076.

[16] R. D. Pisarski, “Chiral symmetry breaking in three-dimensional electrodynamics,” Phys. Rev. D 29, 2423 (1984).

[17] A. P. Polyakov, “Symmetry breaking patterns in (2+1)-dimensional gauge theories,” Phys. Rev. Lett. 60, 1920 (1988).

[18] P. H. Damgaard, U. M. Heller, A. Krasnitz and T. Madsen, “A quark-antiquark condensate in three-dimensional QCD,” Phys. Lett. B 440, 129 (1998) arXiv:hep-lat/9803012.

[19] S. R. Coleman and E. Witten, “Chiral symmetry breakdown In large N chromodynamics,” Phys. Rev. Lett. 45, 100 (1980).

[20] U. Magnea, “The orthogonal ensemble of random matrices and QCD in three dimensions,” Phys. Rev. D 63, 056005 (2000) arXiv:hep-th/9907096.

[21] T. Schafer, D. T. Son, M. A. Stephanov, D. Toublan and J. J. M. Verbaarschot, “Kaon condensation and Goldstone’s theorem,” Phys. Lett. B 522, 67 (2001) arXiv:hep-ph/0108210.

[22] V. A. Miransky and I. A. Shovkovy, “Spontaneous symmetry breaking with abnormal number of Nambu-Goldstone bosons and kaon condensate,” Phys. Rev. Lett. 88, 111601 (2002) arXiv:hep-ph/0108178.

[23] F. Sannino and K. Tuominen, “Spontaneous symmetry breaking in gauge theories via Bose-Einstein condensation,” arXiv:hep-ph/0303167.

[24] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, (Dover, 1998).

[25] E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics: Part 2, Theory of the Condensed State, (Pergamon Press, Oxford, 1980).

[26] K. Huang, Statistical Mechanics, (2nd Edition), John Wiley & Sons (New York, 1987).

[27] N. Goldenfeld, Lectures On Phase Transitions And The Renormalization Group, (Addison-Wesley, Reading, 1992).

[28] J. I. Kapusta, “Bose-Einstein Condensation, Spontaneous Symmetry Breaking, And Gauge Theories,” Phys. Rev. D 24, 426 (1981).

[29] H. E. Haber and H. A. Weldon, “Thermodynamics Of An Ultrarelativistic Bose Gas,” Phys. Rev. Lett. 46, 1497 (1981), “Finite Temperature Symmetry Breaking As Bose-Einstein Condensation,” Phys. Rev. D 25, 502 (1982).

[30] B. Rosenstein, B. J. Warr and S. H. Park, “Thermodynamics Of The O(N) Invariant Sigma Model In (2+1)-Dimensions,” Nucl. Phys. B 336, 435 (1990).

[31] Note that in (A3) we differ from $\Pi$ by a factor of 2 in the interaction term. This is because only half of the $Q$ modes contribute to the low $T$ Bose gas, and so only they should be selected out of the general $H$ field in this limit.