A COMPREHENSIVE COORDINATE SPACE RENORMALIZATION OF QUANTUM ELECTRODYNAMICS TO 2-LOOP ORDER

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ABSTRACT

We develop a coordinate space renormalization of massless Quantum Electrodynamics using the powerful method of differential renormalization. Bare one-loop amplitudes are finite at non-coincident external points, but do not accept a Fourier transform into momentum space. The method provides a systematic procedure to obtain one-loop renormalized amplitudes with finite Fourier transforms in strictly four dimensions without the appearance of integrals or the use of a regulator. Higher loops are solved similarly by renormalizing from the inner singularities outwards to the global one. We compute all 1- and 2-loop 1PI diagrams, run renormalization group equations on them and check Ward identities. The method furthermore allows us to discern a particular pattern of renormalization under which certain amplitudes are seen not to contain higher-loop leading logarithms. We finally present the computation of the chiral triangle showing that differential renormalization emerges as a natural scheme to tackle $\gamma_5$ problems.

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0. Introduction

All one-loop diagrams in Quantum Electrodynamics (QED) are products of propagators when written in coordinate space and, therefore, finite for separate external points. Apart from tadpoles, this is in fact true for any quantum field theory. This striking observation has been in the mind of the physics community since long ago though little practical use of it has been made. The need for renormalization comes through the fact that the product of propagators is not a distribution and, thus, has no Fourier transform. In crude terms, renormalization is a procedure to extend products of distributions into distributions.

Differential Renormalization (DR) [1] provides a prescription to implement this project which is in a sense minimal. The basic idea is to write an amplitude in position space and then express it as derivatives of a distribution less divergent at coincidence points. The typical computation of primitively divergent Feynman integrals is substituted for solving trivial differential equations. This prescription is complemented with the instruction of using the derivative in front of the distribution as acting to its left, which is standard in the theory of distributions.

We claim that differential renormalization is minimal because it never changes the value of a primitively divergent Feynman diagram away from its singularities, and neither does it modify the dimensionality of space-time or introduce a regulator. It is simple and addresses straightforwardly the concept of renormalization, rather than regularization. In spirit, it is close to BPHZ renormalization [2], in that both prescriptions deliver renormalized amplitudes graph by graph. In the absence of a renormalized lagrangian, DR needs a supplementary effort to prove unitarity. In [3], this problem has been analyzed for $\lambda\phi^4$-theory and perturbative unitarity proven to 3-loop order.

A number of theories have been used as a test for differential renormalization. In [4], the supersymmetric Wess-Zumino model was studied up to three loops with ease. Conformal invariance in QCD was exploited in [5] thanks to the position space nature of differential renormalization. The method also extends to massive theories [6] and to lower dimensional theories [7].

Other coordinate space regularization and renormalization methods have occasionally been used in the past. From Schwinger [8] to all the recent conformal field theory developments [9], the advantages of postponing loop integration to higher loops and the possibility of exploiting conformal invariance have led to numerous applications of coordinate space methods.
In this paper we present a comprehensive study of the differential renormalization of massless QED up to two loops. There are obvious reasons to undertake such a computation. QED is a physical theory with gauge symmetry. The consistency of the method passes a stringent test because of the existence of Ward identities (WIs) and potential problems hidden in overlapping divergences. It also faces the problem of the chiral anomaly and the correct treatment of $\gamma_5$.

Our results are in perfect agreement with other computational schemes (e.g., [8],[10],[11],[12],[13]) but are obtained in a simpler form. We devote Sections 1 and 2 to presenting the renormalized amplitudes up to two loops. Whereas several standard treatments of QED will perform the subtractions on mass-shell, here renormalized amplitudes are subtracted at an arbitrary and unphysical scale $M$, due to the fact that we are dealing with massless QED; this massless limit exists as a quantum field theory and provides substantial information about massive QED. We pay special attention to the way differential renormalization uses the basic WIs of the theory and how they are repeatedly checked in two-loop computations. Deeply rooted in the nature of DR lies the idea that each Feynman diagram has as many hidden scales as singular regions. Some of these scales are fixed by WIs. The rest are renormalization-scheme prescriptions. Keeping all this freedom manifest allows the method to “predict” that the renormalization group equations will display a particular organization. This “structured renormalization group”, discussed in Section 3, provides a deep insight into the absence of promotion of leading logs in some diagrams, which is seen as accidental in standard treatments. Finally, we review the calculation of the chiral anomaly in our regulator-free way in Section 5. We also compute the absence of infinite renormalizations (i.e., absence of ln’s) at two loops of the triangle amplitude. As differential renormalization never leaves four dimensions, it stands as a good candidate to work with $\gamma_5$-related observables. The anomaly is just one instance.

1. One-loop Renormalization

1.1 Conventions and definitions

In $d = 4$ Euclidean space, massless QED is described by the Lagrangian:

$$L = \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2d} (\partial_\mu A_\mu)^2 + \overline{\psi}(\partial + i e A)\psi,$$  \hspace{1cm} (1.1)
where $A_\mu(x)$ and $\psi(x)$ are the usual $U(1)$ gauge and Fermi fields, $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ is the gauge field strength, $a$ is a gauge fixing parameter, and $\gamma$-matrices satisfy the usual Clifford algebra, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

Loop diagrams are calculated with the following Feynman rules:

(i) the gauge field and massless fermion propagators are, respectively,

\[ G^{(0)}_{\mu\nu}(x - y; a) = \frac{1}{4\pi^2} \left[ \left( \frac{1}{2} - a \frac{\delta_{\mu\nu}}{(x - y)^2} \right) \delta^{(4)}(x - y) \right. \]

\[ S^{(0)}(x - y) = -\frac{1}{4\pi^2} \gamma_\mu \partial_\mu \frac{1}{(x - y)^2}, \]

(ii) to each vertex is associated a factor $ie\gamma_\mu$,

(iii) internal coordinates are integrated over,

(iv) closed fermion loops are multiplied by a factor $-1$, and

(v) diagrams have no symmetry factors.

We also use the convention that for derivative operators $O = \partial_\mu \partial_\nu$, etc., we take $O f(x - y)$ to mean $O(x)f(x - y)$, unless explicitly stated otherwise.

Renormalization of QED entails the renormalization of only three different vertex functions: the 2-point vacuum polarization, $\Pi_{\mu\nu}(x - y)$, the 2-point fermion self-energy, $\Sigma(x - y)$, and the 3-point vertex, $V_\mu(x - z, y - z)$. These will contribute to the effective action in the following way:

\[ \Gamma_{\text{eff}} = \int d^4x d^4y \left\{ \frac{1}{2} A_\mu(x) \left[ \left( \frac{1}{a} \partial_\mu \partial_\nu - \delta_{\mu\nu} \Box \right) \delta^{(4)}(x - y) - \Pi_{\mu\nu}(x - y) \right] A_\nu(y) \right. \]

\[ + \bar{\psi}(x) \left( \Box \delta^{(4)}(x - y) - \Sigma(x - y) \right) \psi(y) \}

\[ + \int d^4xd^4yd^4z \bar{\psi}(x) \left[ e\gamma_\mu \delta^{(4)}(x - y) \delta^{(4)}(x - z) + V_\mu(x - z, y - z) \right] \psi(y) A_\mu(z) + \cdots. \] (1.2)

As defined above, these renormalization parts will contain only 1PI loop contributions and $\Pi_{\mu\nu}$ and $\Sigma$ will, furthermore, lead to the following full photon and electron propagators:

\[ G_{\mu\nu} = G^{(0)}_{\mu\nu} + G^{(0)}_{\mu\rho} \cdot \Pi_{\rho\sigma} \cdot G^{(0)}_{\sigma\nu} + G^{(0)}_{\mu\rho} \cdot \Pi_{\rho\alpha} \cdot G^{(0)}_{\alpha\beta} \cdot \Pi_{\beta\nu} \cdot G^{(0)}_{\sigma\tau} + \cdots = (G^{(0)}_{\mu\nu}^{-1} - \Pi_{\mu\nu})^{-1} \]

\[ S = S^{(0)} + S^{(0)} \cdot \Sigma \cdot S^{(0)} + S^{(0)} \cdot \Sigma \cdot S^{(0)} \cdot \Sigma \cdot S^{(0)} + \cdots = (S^{(0)} - 1)^{-1}, \] (1.3)
where the dot indicates a convolution. To all loop orders, renormalized vertex parts satisfy renormalization group (RG) equations [14] and, as a consequence of gauge invariance, the following Ward identities:

$$\partial_\mu \Pi_{\mu\nu}(x-y) = 0,$$

(1.4)

$$\frac{\partial}{\partial z^\mu} V_\mu(x-z,y-z) = -ie[\delta^{(4)}(z-x) - \delta^{(4)}(z-y)]\Sigma(x-y).$$

(1.5)

The specific coefficients in the second Ward identity are uniquely fixed by the tree-level identity solved by $V_\mu^{(0)} = ie\gamma_\mu\delta^{(4)}(x-y)\delta^{(4)}(x-z)$ and $S^{(0)-1} = \phi\delta^{(4)}(x-y)$.

In what follows, we shall find the 1- and 2-loop corrections to $\Pi_{\mu\nu}, \Sigma$ and $V_\mu$. We will verify the above Ward identities, and the requirement that renormalization parts satisfy renormalization group equations will allow us to calculate the renormalization group functions of QED (beta-functions and anomalous dimensions).

### 1.2 Renormalization

The bare 1-loop vacuum polarization (Fig. 1a) reads:

$$\Pi^{(1)\text{bare}}_{\mu\nu}(x-y) = -\left(\frac{ie}{4\pi^2}\right)^2 \text{Tr} \left[\gamma_\mu \phi \frac{1}{(x-y)^2} \gamma_\nu \phi \frac{1}{(y-x)^2}\right]$$

$$= -\frac{\alpha}{3\pi^3} (\partial_\mu \partial_\nu - \delta_{\mu\nu} \Box) \frac{1}{(x-y)^4},$$

(1.6)

with $\alpha = e^2/4\pi$ the fine structure constant. Using the basic DR identity (cf. Appendix A for all the identities needed in renormalizing amplitudes up to two loops):

$$\frac{1}{x^4} = -\frac{1}{4\pi^2} \ln \frac{x^2 M^2}{x^2},$$

(1.7)

we find the renormalized value of the one-loop vacuum polarization:

$$\Pi^{(1)}_{\mu\nu}(x-y) = \frac{\alpha}{12\pi^3} (\partial_\mu \partial_\nu - \delta_{\mu\nu} \Box) \ln \frac{(x-y)^2 M^2_{\text{ren}}}{(x-y)^2},$$

(1.8)

with the proviso that the total derivatives above, as well as in all other renormalized amplitudes throughout, are to be understood as acting to the left. Here and below we adopt the convention of appending a subscript ($\Pi, \Sigma$ or $V$) to the renormalization scales appearing in the different renormalization parts, since these scales are a priori independent (but eventually related through Ward identities, as we shall see). This quantity gives us then the renormalized $AA$ 2-point function:

$$\Gamma^{AA}_{\mu\nu}(x-y) = \left(1 - \frac{1}{a}\right) \partial_\mu \partial_\nu - \delta_{\mu\nu} \Box) \delta^{(4)}(x-y) - \Pi^{(1)}_{\mu\nu}(x-y).$$

(1.9)
We now impose that it satisfy the usual RG equation:

\[
\left( M_{\Pi} \frac{\partial}{\partial M_{\Pi}} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_a(\alpha) \frac{\partial}{\partial a} - 2\gamma_A(\alpha) \right) \Gamma_{\mu\nu}^{AA}(x-y) = 0,
\]

(1.10)

where \( \beta(\alpha) \) is the QED \( \beta \)-function, \( \gamma_A(\alpha) \) is the anomalous dimension of the gauge field, and \( \gamma_a(\alpha) \) a \( \beta \)-function\" associated to the running of the gauge parameter \( a \). This is then easily seen to lead to the 1-loop values:

\[
\gamma_A(\alpha) = \frac{\alpha}{3\pi}
\]

and

\[
\gamma_a(\alpha) = \frac{2a\alpha}{3\pi}
\]

(1.11)

(\( \beta(\alpha) \) has dropped out above because it is of higher order in \( \alpha \)).

It is well-known that the above computation can be used in the framework of the background field method to find the beta-function of QED [15]. It turns out that the relation \( \beta(\alpha) = 2\alpha\gamma(\alpha) \) holds to all orders. Therefore, we can anticipate that the beta function at one-loop is

\[
\beta(\alpha) = \frac{2a^2}{3\pi}.
\]

(1.12)

This result will later be confirmed by an independent computation of the one-loop vertex.

The bare 1-loop fermion self-energy (Fig. 1b) is:

\[
\Sigma^{(1)\text{bare}}(x-y) = \left( \frac{e}{4\pi^2} \right)^2 \gamma_{\mu} \phi \frac{1}{(x-y)^2} \gamma_{\nu} \left[ \frac{1}{2} \frac{\delta_{\mu\nu}}{(x-y)^2} + \frac{1-a}{a} \frac{(x-y)_\mu(x-y)_\nu}{(x-y)^4} \right].
\]

(1.13)

The renormalization of this amplitude is straightforward, apart from the following important subtlety: if we use the same mass scale \( M_{\Sigma} \) in renormalizing the two parts coming from the two different pieces in the photon propagator, we will end up with a renormalized amplitude which is directly proportional to the gauge parameter \( a \); now, in Landau gauge \((a = 0)\), it would then vanish entirely, and it is easy to see this would be inconsistent with the Ward identity. It is in fact a known property of QED that, in Landau gauge, although the bare 1-loop fermion self-energy vanishes, the renormalized self-energy does not. In momentum space, it is equal to a finite constant times \( \phi \), and this is a reflection of the ambiguity inherent in the linearly divergent integral defining the amplitude. This turns out to be transparent in differential renormalization: in renormalizing the two pieces in question, no one tells us we should take the same mass scale \( M_{\Sigma} \), and so we do not. We renormalize the first piece (the
Feynman gauge part) with a scale $M_\Sigma$, and the second piece with a scale $M'_\Sigma$ related to the first one by $\ln M'_\Sigma = \ln M_\Sigma - \lambda$. This then leads to the following renormalized 1-loop fermion self-energy:

$$\Sigma^{(1)}(x - y) = \frac{\alpha}{16\pi^2} \partial \left[ \frac{a \ln(x - y)^2 M_\Sigma^2 + \lambda(1 - a)/2}{(x - y)^2} \right]$$

$$= \frac{a\alpha}{16\pi^2} \partial \left[ \frac{\ln(x - y)^2 M_\Sigma^2}{(x - y)^2} - \frac{\lambda(1 - a)}{8\pi} \partial \delta^{(4)}(x - y) \right].$$

(1.14)

As expected, in Landau gauge, all that survives is precisely the contact term mentioned above. We will see shortly that the Ward identity will not only determine a relation between the self-energy and vertex mass scales, but also a unique value for $\lambda$. The renormalized $\overline{\psi}\psi$ 2-point function,

$$\Gamma^{\overline{\psi}\psi}(x - y) = \partial \delta^{(4)}(x - y) - \Sigma^{(1)}(x - y),$$

(1.15)

then satisfies the RG equation:

$$\left( M_\Sigma \frac{\partial}{\partial M_\Sigma} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_\alpha(\alpha) \frac{\partial}{\partial a} - 2\gamma_\psi(\alpha) \right) \Gamma^{\overline{\psi}\psi}(x - y) = 0,$$

(1.16)

and this will lead to the following 1-loop fermion anomalous dimension:

$$\gamma_\psi(\alpha) = \frac{a\alpha}{4\pi}$$

(1.17)

(here, both $\beta$ and $\gamma_\alpha$ drop out).

Finally, we now renormalize the 1-loop vertex (Fig. 1c). Its bare expression is:

$$V^{(1)}_{\mu}(x - z, y - z) =$$

$$= \left( \frac{i e}{4\pi^2} \right)^3 \left( \gamma_\rho \gamma_\alpha \gamma_\mu \gamma_\nu \right) \partial_\alpha \frac{1}{(x - z)^2} \partial_\beta \frac{1}{(z - y)^2} \left[ \frac{1 + a}{2} \frac{\delta_\rho\sigma}{(x - y)^2} + (1 - a) \frac{(x - y)_\rho (x - y)_\sigma}{(x - y)^4} \right]$$

$$= \frac{-i}{32\pi^3} \left( \frac{\alpha}{\pi} \right)^{3/2} \left( \gamma_\rho \gamma_\alpha \gamma_\mu \gamma_\nu \right) \partial_\alpha \frac{1}{(x - z)^2} \partial_\beta \frac{1}{(z - y)^2} \left[ (a - 1) \partial_\rho \partial_\sigma + \delta_\rho\sigma \right] \ln(x - y)^2 \mu^2.$$  

(1.18)

Like for the fermion self-energy, the photon propagator in a generic gauge leads to two pieces to be renormalized, in principle with mass scales $M_V$ and $M'_V$. However, it turns out that this need not be done here, that is, the choice of the same $M_V$ throughout will not lead to any inconsistencies (still, one may choose different scales at will; this would simply correspond to different choices of scheme). We briefly sketch the procedure involved in renormalizing the expression above. In Feynman gauge ($a = 1$) it is most transparent [1]: there are three propagators $1/(x - x')^2$ forming a triangle, with derivatives acting on two of them. This has dimension $L^{-8}$ and therefore it is log divergent in the ultraviolet. One first integrates by parts until the two derivatives are acting on the same leg, say, $\partial_a \partial_b \frac{1}{(x - y)^2}$.
and then subtracts and adds a trace piece thus: \( (\partial_a \partial_b - \frac{1}{4} \delta_{ab} \Box \frac{1}{(x-y)^2}) = \frac{1}{4} \delta_{ab} \Box \frac{1}{(x-y)^2} \). The surface terms, with total derivatives acting outside, is power counting \( L^{-7} \) and thus finite, and the traceless combination of derivatives is also finite (due to tracelessness). The divergence has been isolated in the term \( \frac{1}{4} \delta_{ab} \Box \frac{1}{(x-y)^2} = -\pi^2 \delta_{ab} \delta^{(4)}(x-y) \), which turns out to be easily renormalizable through the DR identity, Eq.(1.7). For a generic gauge, the same principle of separating divergent terms into trace and traceless pieces applies, only in this case the \( \gamma \)-matrix structure in front complicates things a little: we integrate \( \partial_a \) and \( \partial_b \) above by parts onto the photon leg, and then look for the coefficient \( A \) to make the expression

\[
(\gamma_\rho \gamma_\mu \gamma_\nu \gamma_\sigma) (\partial_a \partial_b + A \delta_{ab} \Box) [(a-1) \partial_\rho \partial_\sigma + \delta_{\rho\sigma} \Box] \quad (1.19)
\]
a traceless (and thus finite) combination of derivatives. The appropriate value is \( A = -a/(a+3) \); adding and subtracting that trace piece, we are able to find the renormalized expression for the 1-loop vertex:

\[
V_\mu^{(1)}(x-z,y-z) = \frac{-i}{32\pi^3} \left( \frac{\alpha}{\pi} \right)^{3/2} \left\{ (\gamma_\rho \gamma_\sigma \gamma_\mu \gamma_\nu) \frac{\partial}{\partial x^a} \left[ \frac{1}{(x-z)^2} \partial_b \frac{1}{(z-y)^2} [(a-1) \partial_\rho \partial_\sigma + \delta_{\rho\sigma} \Box] \ln(x-y)^2 \mu^2 \right] + 4 [(a-1) \gamma_\mu \gamma_b - 2 \gamma_b \gamma_\mu] \frac{\partial}{\partial y^b} \left[ \frac{1}{(x-z)^2} \partial_\rho \frac{1}{(z-y)^2} \delta \rho\sigma (x-y)^2 \right] - \frac{16}{(x-z)^2 (z-y)^2} \gamma_\sigma \left[ \partial_a \partial_b - \frac{1}{4} \delta_{ab} \Box \right] \frac{1}{(x-y)^2} + 4\pi^2 a \gamma_\mu \Box \ln(x-z)^2 M_\nu^2 \delta^{(4)}(x-y) \right\}. \quad (1.20)
\]

We remark here on the fact that the renormalization piece above (containing \( \ln M_\nu^2 \)) is directly proportional to the gauge parameter \( a \), and this is also true of \( \Sigma^{(1)}(x-y) \) found above. This is a reflection of the well-known fact that, apart from vacuum polarization infinities, QED is 1-loop finite in Landau gauge.

The renormalized 3-point vertex

\[
\bar{\Gamma}_{\mu}^{A\psi}(x-z,y-z) = ie \gamma_\mu \delta^{(4)}(x-y) \psi(x-z) + V_\mu^{(1)}(x-z,y-z) \quad (1.21)
\]
satisfies the RG equation:

\[
\left( M_\nu \frac{\partial}{\partial M_\nu} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_\alpha(\alpha) \frac{\partial}{\partial \alpha} - 2 \gamma_\psi(\alpha) - \gamma_A(\alpha) \right) \Gamma_{\mu}^{A\psi}(x-z,y-z) = 0 \quad (1.22)
\]

with values found previously for \( \gamma_\psi \) and \( \gamma_A \), and

\[
\beta(\alpha) = \frac{2\alpha^2}{3\pi}. \quad (1.23)
\]
(and, again, $\gamma_a$ is of higher order than we are considering). This confirms the result from the background field method shown previously. Naturally, in the above equation the $a$-dependent pieces $(M \frac{\partial}{\partial M} \gamma)$ and the $a$-independent pieces $(\beta \frac{\partial}{\partial \alpha} \gamma_A)$ cancel separately.

We now consider the Ward identity for $V_\mu$ and $\Sigma$.

### 1.3 Ward Identity

For separate points $x \neq y \neq z$, Eq.(1.5) is trivially verified. The subtlety involved in a coordinate space Ward identity is, however, its validity at contact. In order to verify Eq.(1.5), then, we must be careful in particular not to lose any contact terms due to formal manipulations with delta functions, and the best way to guarantee a correct procedure is to integrate it over one of the external variables. This was done in [1] for Feynman gauge, and we use the same procedure here for an arbitrary gauge.

The integrated form of the Ward identity is:

$$
\int d^4y \frac{\partial}{\partial \gamma^\mu} V_\mu^{(1)}(x - z, y - z) = \ie \Sigma(x - z).
$$

Integrating the expression for the renormalized vertex, Eq.(1.20), over $y$, one finds:

$$
\int d^4y V_\mu^{(1)}(x - z, y - z) = -i \left( \frac{\alpha}{\pi} \right)^{3/2} \left( \partial_\mu \phi - \frac{1 + 2a}{4} \gamma_\mu \Box \right) \frac{1}{(x - z)^2} + \frac{a}{2} \gamma_\mu \ln \frac{(x - z)^2}{M^2} V(x - z).
$$

(1.24)

Acting now with the $z$ derivative:

$$
\int d^4y \frac{\partial}{\partial \gamma^\mu} V_\mu^{(1)}(x - z, y - z) = \frac{\ie^3}{64\pi^2} \Box \left( \frac{a \ln(x - z)^2 M_V^2 + (3/2 - a)}{(x - z)^2} \right).
$$

(1.25)

Now, we compare this with:

$$
\ie \Sigma^{(1)}(x - z) = \frac{\ie^3}{64\pi^2} \Box \left( \frac{a \ln(x - z)^2 M_V^2 + \lambda(1 - a)/2}{(x - z)^2} \right).
$$

(1.26)

Setting $a = 0$ gives $\lambda = 3$, and then, for any $a$, we also find:

$$
\ln \frac{M^2}{M_V^2} = \frac{1}{2}. \tag{1.27}
$$

We see then that a Ward identity relating two renormalized amplitudes will, in the context of differential renormalization, enforce a relation between the scales that renormalize the amplitudes. It is an important point we will analyze further that the converse is also true, namely, mass scales that renormalize amplitudes not related by a symmetry are not related, and this in turn also enforces constraints in the renormalization of these amplitudes. At two loops, in particular for the anomalous triangle, the use of the above mass relation will be crucial for the consistency of our calculations.
2. Two-loop Renormalization

At two loops, the 1PI diagrams renormalizing $\Pi_{\mu\nu}$, $\Sigma$ and $V_\mu$ are depicted in Fig. 2. Throughout this section, we will perform all calculations in Feynman gauge ($g = 1$); since $\gamma_a \frac{\partial}{\partial a}$ acting on any 2-loop diagram will be one order higher in $\alpha$, this will not affect the verification of RG equations.

2.1 Vacuum Polarization

We begin with the simplest of the two vacuum polarization diagrams (Fig. 2a), and from now on we will use translation invariance to set one external point to zero in all diagrams, for the sake of simplicity. Its bare expression is:

$$\Pi^{(2a)}_{\mu\nu}(x) = \left(\frac{ie}{4\pi^2}\right)^2 \int d^4u d^4v \ Tr \left( \gamma_{\mu} \frac{1}{(x-u)^2} \Sigma^{(1)}(u-v) \frac{1}{v^2} \gamma_{\nu} \frac{1}{x^2} \right). \quad (2.1)$$

Standard manipulations lead to:

$$\Pi^{(2a)\text{bare}}_{\mu\nu}(x) = \frac{e^4}{48\pi^6} \left[ \frac{1}{4} (\partial_{\mu} \partial_{\nu} - \delta_{\mu\nu}) \left( \ln \frac{x^2 M^2}{\Sigma} + \frac{5}{3} \ln \frac{x^2 M^2}{\Sigma} \right) - \delta_{\mu\nu} \ln \frac{x^2 M^2}{\Sigma} \right]. \quad (2.2)$$

We note that this diagram is not transverse by itself; the non-transverse piece will be cancelled when we consider the entire 2-loop vacuum polarization. The renormalization now proceeds with the straightforward use of the DR identities listed in Appendix A, and we find:

$$\Pi^{(2a)}_{\mu\nu}(x) = -\frac{1}{96\pi^2} \left(\frac{\alpha}{\pi}\right)^2 \left\{ (\partial_{\mu} \partial_{\nu} - \delta_{\mu\nu}) \left( \ln \frac{x^2 M^2}{\Sigma} + \frac{5}{3} \ln x^2 M^2 \right) - \delta_{\mu\nu} \ln \frac{x^2 M^2}{\Sigma} \right\}, \quad (2.3)$$

where $M$ is a new renormalization mass parameter appearing at two loops. The diagram with the fermion self-energy inserted on the lower leg of the loop will have the same value as this one. We note here that the $\ln M^2$ coming from the 1-loop self-energy subdivergence has been promoted to a $\ln^2 M^2$ at two loops: this is the self-consistency of the renormalization group at work. We will choose everywhere the same 2-loop mass scale $M$. In renormalization terms, this simply corresponds to some choice of renormalization scheme: because these appear as $\ln^2 M^2$, different values of $M$ will lead to amplitudes that differ by finite contact terms or, in other words, by finite renormalizations (and this will be true for the new $M$’s appearing at each loop order). Of course, once we set the (2-loop) $M$’s we like for the renormalization of $\Sigma$ and $V_\mu$, in particular, they will subsequently be related by the Ward identity (but this will not be of concern to us here, as we will not compute the 2-loop mass
relation). Throughout this section, then, we will only care to distinguish between $M_{\Sigma}, M_V$ or $M_H$
and other mass scales $M$ when the former appear as promoted ln’s, i.e., as $\ln^2 M_{\Sigma}$, etc..

The other vacuum polarization diagram, Fig. 2b, is the most difficult integral we have had to
perform. The bare expression is:

$$
\Pi_{\mu\nu}^{(2b)\text{bare}}(x-y) = \frac{(ie)^4}{(4\pi^2)^6} \int d^4u d^4v \text{Tr} \left( \gamma_{\mu} \partial_{\nu} \frac{1}{(x-u)^2 (y-v)^2} \gamma_{\nu} \partial_{\mu} \frac{1}{(v-y)^2 (u-x)^2} \right) \frac{1}{(u-v)^2}. 
$$

There are potential problems related to overlapping divergences in this amplitude, and so we must
examine how differential renormalization deals with them. There are two subdiagrams which contain
log singularities related to the regions $u \sim v \sim x$ and $u \sim v \sim 0$. We remove both divergences by
pulling out derivatives in $x$ and $y$ in a symmetric way as we did in the case of the one-loop vertex,
Eq.(1.20). More explicitly, the important intermediate step is

$$
\partial_u \frac{1}{(x-u)^2} \partial_v \frac{1}{(y-v)^2} \partial_u \frac{1}{(x-v)^2} \partial_v \frac{1}{(u-v)^2} = 
$$

$$
\left[ \frac{\partial}{\partial x^a} \frac{1}{(x-u)^2} \partial_v \frac{1}{(x-v)^2} \right] - \frac{1}{(x-u)^2} \left( \partial_u \partial_v - \frac{\delta_{cd}}{4} \right) \frac{1}{(x-v)^2} + \pi^2 \delta_{cd} \frac{\delta^4(x-v)}{(x-u)^2} \frac{1}{(u-v)^2} \quad (2.5)
$$

This manipulation makes it evident that each subdivergence is cured separately thanks to the fact that
in coordinate space the external points are kept apart (whereas, in momentum space, the momentum
integral would make the two singularities overlap). The rest of the computation, carried out with the
help of Feynman parameters, although somewhat lengthy, is conceptually simple since only a global
divergence needs further correction. The final renormalized value is:

$$
\Pi_{\mu\nu}^{(2b)}(x) = -\frac{1}{48\pi^2} \left( \frac{\alpha}{\pi} \right)^2 \left\{ - (\partial_{\mu} \partial_{\nu} - \delta_{\mu\nu} \Box) \left( \frac{\ln^2 x^2 M_V^2 + \frac{17}{4} \ln x^2 M^2}{x^2} \right) + \delta_{\mu\nu} \frac{\ln x^2 M^2}{x^2} \right\},
$$

Taking into account the mass relation (Ward identity), Eq.(1.28), the entire 2-loop renormalized
vacuum polarization reads:

$$
\Pi_{\mu\nu}^{(2)}(x) = 2\Pi_{\mu\nu}^{(2a)}(x) + \Pi_{\mu\nu}^{(2b)}(x) = \frac{1}{16\pi^2} \left( \frac{\alpha}{\pi} \right)^2 (\partial_{\mu} \partial_{\nu} - \delta_{\mu\nu} \Box) \frac{\ln x^2 M^2}{x^2}. \quad (2.7)
$$

This is automatically transverse, and furthermore the $\ln^2$ contributions have cancelled. This is an
important point which we will further elaborate in Section 3.
The 2-loop renormalized $AA$ 2-point function
\[
\Gamma^{AA}_{\mu\nu}(x) = \left(1 - \frac{1}{a}\partial_\mu \partial_\nu - \delta_{\mu\nu}\Box\right) \delta^{(4)}(x) - \Pi^{(1)}_{\mu\nu}(x) - \Pi^{(2)}_{\mu\nu}(x)
\]
will satisfy an RG equation, Eq.(1.10), with $M \partial_{\Sigma_{M}}$ substituted for $M \partial_{\Sigma_{M}} + M \partial_{\Sigma}$ (in general, an RG equation will include the sum of the derivatives w.r.t. all masses present in an amplitude). This will yield the 2-loop RG functions:
\[
\gamma_A(\alpha) = \frac{\alpha^3}{3\pi} + \frac{\alpha^2}{4\pi^2}
\]
and
\[
\gamma_a(\alpha) = -\frac{2a\alpha^3}{3\pi} - \frac{\alpha^2}{2\pi^2} = -2a\gamma_A(\alpha).
\]
As mentioned before, the same calculation in the background field method would have led us to the 2-loop beta function, which we will derive and present independently through the calculation of the 2-loop vertex in Sec. 2.3.

2.2 Fermion Self-Energy

Next, we consider 2-loop contributions to the fermion self-energy, shown in Figs. 2c-2e. In order to eventually verify RG equations and Ward identities, we also give the logarithmic mass derivative $M \partial_{M}$ acting on each one of the amplitudes. The first diagram is that of Fig. 2c:
\[
\Sigma^{(2c)\text{bare}}(x) = -\frac{(ie)^2}{(4\pi^2)^3} \int d^4u d^4v \frac{1}{x^2} \gamma_\mu \partial_1 \left(\frac{1}{(x-u)^2} \Sigma^{(1)}(u-v) \partial_1 \gamma_\mu\right). \tag{2.10}
\]
Again, integration by parts, properties of $\gamma$-matrices and DR identities lead to the renormalized value:
\[
\Sigma^{(2c)}(x) = -\frac{1}{128\pi^2} \left(\frac{\alpha}{\pi}\right)^2 \partial_1 \left(\ln x^2 M_S^2 + \ln x^2 M^2\right), \tag{2.11}
\]
where we have also used the identity
\[
\frac{1}{x^2} \partial_1 \left(\ln x^2 M_S^2\right) = \frac{1}{2} \partial_1 \left(\ln x^2 M_S^2 - 1/2\right). \tag{2.12}
\]
The mass derivative of this amplitude is:
\[
M \partial_{M} \Sigma^{(2c)} = -\frac{\alpha}{2\pi} \Sigma^{(1)} - \frac{\alpha^2}{16\pi^2} \Sigma^{(0)}. \tag{2.13}
\]
The next contribution is that of Fig. 2d:
\[
\Sigma^{(2d)\text{bare}}(x) = -\frac{(ie)^4}{(4\pi^2)^5} \int d^4u d^4v \frac{1}{u^2} \frac{1}{(x-v)^2} \gamma_\mu \partial_1 \frac{1}{(x-u)^2} \gamma_\nu \partial_1 \frac{1}{(u-v)^2} \gamma_\mu \partial_1 \gamma_\nu. \tag{2.14}
\]
This amplitude is solved by using vertex-like manipulations. It turns out that no new integrals are needed apart from those appearing in the two-loop vacuum polarization diagram 2b. We also make use of the following trick

\[ \Box \left( \frac{1}{(v-x)^2} \frac{1}{v^2} \right) = -4\pi^2 \left( \frac{\delta^{(4)}(v-x)}{v^2} + \frac{\delta^{(4)}(v)}{(v-x)^2} \right) + 2\partial_a \frac{1}{(v-x)^2} \partial_a \frac{1}{v^2}, \]

which simplifies part of the computation. The renormalized value we finally get is:

\[ \Sigma^{(2d)}(x) = \frac{1}{64\pi^2} \left( \frac{\alpha}{\pi} \right)^2 \phi \Box \frac{\ln^2 x^2 M^2_v}{x^2}, \quad (2.15) \]

and the mass derivative of this amplitude is:

\[ M \frac{\partial}{\partial M} \Sigma^{(2d)} = \frac{\alpha}{\pi} \Sigma^{(1)} - \frac{\alpha^2}{8\pi^2} \Sigma^{(0)}, \quad (2.16) \]

where we have used the 1-loop mass relation to express \( \Sigma^{(1)} \) with the mass scale \( M_\Sigma \) rather than \( M_V \).

We finally present the last of the 2-loop fermion self-energy diagrams (Fig. 2e):

\[ \Sigma^{(2e)}(x) = -\frac{(ie)^2}{(4\pi^2)^3} \gamma_\mu \frac{1}{x^2} \gamma_\nu \int d^4ud^4v \frac{1}{(x-v)^2} \Pi^{(1)}_{\mu\nu}(v-u) \frac{1}{u^2}. \quad (2.17) \]

The renormalization of this amplitude is straightforward, and we only point out that while there is in principle the possibility that the \( \ln M_\Pi \) in \( \Pi^{(1)}_{\mu\nu} \) would get promoted to a \( \ln^2 M_\Pi \), the \( \gamma \)-matrices in the amplitude, together with the transverse operator in \( \Pi^{(1)}_{\mu\nu} \), conspire to simply cancel all \( \ln^2 \)'s. This is again an instance of the feature we have seen previously in \( \Pi^{(2)}_{\mu\nu} \), namely, the apparently unexpected cancellation of some particular divergences. The final, renormalized result is:

\[ \Sigma^{(2e)}(x) = -\frac{1}{32\pi^2} \left( \frac{\alpha}{\pi} \right)^2 \phi \Box \frac{\ln x^2 M^2}{x^2}. \quad (2.18) \]

The mass derivative of this amplitude will be:

\[ M \frac{\partial}{\partial M} \Sigma^{(2e)} = -\frac{\alpha^2}{4\pi^2} \Sigma^{(0)}. \quad (2.19) \]

 Whereas we would generally expect the mass derivative of a 2-loop amplitude to generate 1-loop and tree-level amplitudes, we see this does not happen here. In Section 3, we will examine this more closely.
We now add all these amplitudes to find the 2-loop renormalized $\bar{\psi}\psi$ 2-point function:

$$\Gamma_{\bar{\psi}\psi}(x) = \frac{\alpha}{16\pi^2} \left( \frac{\alpha}{\pi} \right) \left( \frac{\ln x^2 M^2}{x^2} - \frac{1}{128\pi^2} \left( \frac{\alpha}{\pi} \right)^2 \left( \frac{\ln x^2 M^2}{x^2} - 7 \ln x^2 M^2 \right) \right).$$

(2.20)

We have the 1-loop correction in a generic gauge $a$, and the 2-loop terms in Feynman gauge. This is written thus because in RG equations we will need $\frac{\partial}{\partial a}$ on the 1-loop term but not on the 2-loop terms. The 2-loop fermion anomalous dimension given by the RG equations is (in Feynman gauge):

$$\gamma_{\psi}(\alpha) = \frac{\alpha}{4\pi} - \frac{3\alpha^2}{32\pi^2}.$$  

(2.21)

### 2.3 Vertex

We now turn to the computation of 2-loop corrections to the gauge coupling vertex, shown in Figs. 2f-l. Many of the integrals are extremely difficult to perform analytically, but fortunately they need not be done for the purposes of verifying RG equations and Ward identities (although we will not present the 2-loop relation between $M_V$ and $M_{\Sigma}$). The parts of these diagrams that do need to be computed fully are the renormalization pieces from both global and internal divergences, and those are reasonably simple to calculate. Because the expressions for the renormalized amplitudes are somewhat lengthy, we present them in Appendix B. Here, instead, we will explain briefly the different techniques we have used and subtleties involved as we go along. Again, for the purpose of later verifying RG equations and Ward identities, we also present here the result of the logarithmic mass derivative $M \frac{\partial}{\partial M}$ acting on each one of the renormalized amplitudes. We set $z = 0$ in all amplitudes.

The first diagram we consider is Fig. 2f:

$$V^{(2f)\text{bare}}_{\mu}(x,y) = -\frac{(ie)^3}{\left(4\pi^2\right)^4} \frac{1}{(x-y)^2} \int d^4u d^4v \gamma_{\mu} \phi \left( x \right) \frac{1}{u^2} \gamma_{\mu} \phi \left( u \right) \frac{1}{v^2} \Sigma^{(1)}(u-v) \phi \left( v \right).$$

(2.22)

The standard integrations by parts and separation into trace and traceless pieces which were used to renormalize the 1-loop vertex are used here as well, without any added complication. We also need to consider the diagram identical to the one above, but with the self-energy insertion on the other fermion leg, Fig. 2g. The procedure is identical to the one used above, and its contribution to the RG equations at two loops is also the same. From the renormalized expression given in Appendix B, we can calculate the mass derivative of this amplitude:

$$M \frac{\partial}{\partial M} \left[ V^{(2f)}_{\mu} + V^{(2g)}_{\mu} \right] = -\frac{\alpha}{\pi} V^{(1)}_{\mu} - \frac{3\alpha^2}{8\pi^2} V^{(0)}_{\mu}.$$  

(2.23)
Here and below, we are always taking $V^{(1)}_{\mu}$ in Feynman gauge, unless otherwise explicitly stated; we must also be careful to express the result in terms of $M_V$ and not $M_\Sigma$ (in the above, for instance, neglecting this would lead to a different coefficient for the contact piece $V^{(0)}_{\mu}$).

The second vertex diagram we compute, Fig. 2h, contains a vacuum polarization insertion:

$$V^{(2h)\text{bare}}_{\mu}(x, y) = \frac{(ie)^3}{(4\pi^2)^3} (\gamma_\rho \gamma_\sigma \gamma_\mu \gamma_\nu \gamma_\sigma) \partial_\sigma \frac{1}{x^2} \partial_\sigma \frac{1}{y^2} \int d^4ud^4v \frac{1}{(u-v)^2} \Pi^{(1)}_{\rho\sigma}(u-v) \frac{1}{(x-y)^2}. \quad (2.24)$$

To renormalize this, we integrate the derivatives $\partial_\sigma$ and $\partial_\tau$ by parts onto the photon leg. The transversality of $\Pi^{(1)}_{\mu\nu}$ and the gamma structure in front will arrange things so that the resulting combination of derivatives will automatically be traceless, thus obviating the need for further renormalization and (again) avoiding the promotion of $\ln M_\Pi$. This in turn leads to a peculiar form for the mass derivative:

$$M \frac{\partial}{\partial M} V^{(2h)}_{\mu} = -\frac{2\alpha}{3\pi} V^{(1)}_{\mu}(a = 0). \quad (2.25)$$

Now, we are left with the more difficult diagrams, which cannot be computed in closed form. Starting with the diagram of Fig. 2i, its bare value is:

$$V^{(2i)\text{bare}}_{\mu}(x, y) = \frac{(ie)^5}{(4\pi^2)^5} (\gamma_\rho \gamma_\sigma \gamma_\mu \gamma_\nu \gamma_\sigma) \times \frac{1}{(x-y)^2} \int d^4ud^4v \frac{1}{(u-v)^2} \partial_\sigma \frac{1}{u^2} \partial_\rho \frac{1}{v^2} \partial_\nu \frac{1}{(v-y)^2}. \quad (2.26)$$

We first of all renormalize the upper vertex (connecting points $z = 0$, $u$, and $v$), by integrating by parts in $z$ and separating the term integrated by parts into a trace and traceless piece (in the indices $b$ and $c$). The surface term is finite by power counting, and the trace piece easily renormalizes to a structure of the form

$$(\gamma_\rho \gamma_{\sigma} \gamma_\mu) \frac{1}{(x-y)^2} \partial_\rho \frac{1}{x^2} \partial_\sigma \frac{1}{y^2} \frac{\ln y^2 M_V^2}{y^2}, \quad (2.27)$$

which needs to be renormalized one more time (integration by parts and separation into trace and traceless pieces), giving a $\ln^2$ promotion through the use of a DR identity. This is easily done, and now we are finally left with the global divergence ($x \sim y \sim 0$) present in the traceless piece (in $bc$) we got after the first integration by parts. That term is power counting log divergent, and one might think the traceless combination of derivatives then makes it finite, but this is not so: the point is that there are other free indices in that expression ($a$ and $d$). The integration is indeed made finite by tracelessness in the $bc$ indices, because the $a$ and $d$ derivatives can be brought out of the integral, but there remains a global divergence as $x \sim y \sim 0$, because the expression is not traceless in all
indices. Attempting to separate the trace pieces from four free indices is hopeless because we cannot even do the integral in $u$ and $v$, and so we resort to a technique used to renormalize the nonplanar 3-loop 4-point diagram in $\lambda \phi^4$, valid for primitively divergent expressions in general. Details can be found in [1], and we do not give them here. The idea consists in writing the factor $\frac{1}{(x-y)^2}$ in front as

$$\frac{1}{(x-y)^2} = \frac{1}{(x-y)^4} = \frac{-(x-y)^2}{4} \ln \frac{(x-y)^2 M^2}{(x-y)^2}. \tag{2.28}$$

One can verify that the simple substitution above, as is, suffices to renormalize the global divergence we had (that is, that term will have a well-defined Fourier transform). The only subtlety in applying the mass derivative to the resulting renormalized amplitude lies in fact in this globally divergent piece. $M \frac{\partial}{\partial M}$ on the term above is proportional to $\delta^{(4)}(x-y)$ and, like in [1], one must verify that what multiplies this is a representation of $\delta^{(4)}(y-z)$ when $x \to y$. Adding up all contributions, the mass derivative then reads:

$$M \frac{\partial}{\partial M} V^{(2)}_{\mu} = \frac{\alpha}{2\pi} V^{(1)}_{\mu} + \frac{5\alpha^2}{16\pi^2} V^{(0)}_{\mu}. \tag{2.29}$$

The next diagram is that of Fig. 2j, with bare value:

$$V^{(2j)\text{bare}}_{\mu}(x,y) = -\frac{(ie)^5}{(4\pi^2)^6} \left( \gamma_{\rho} \gamma_{a} \gamma_{\mu} \gamma_{b} \gamma_{\nu} \gamma_{c} \gamma_{\rho} \gamma_{\nu} \right) \times \partial_{a} \frac{1}{x^2} \int d^4u d^4v \frac{1}{u^2} \frac{1}{(x-u)^2} \partial_{b} \frac{1}{v^2} \frac{1}{(u-v)^2} \frac{1}{(u-y)^2} \frac{1}{(v-y)^2}. \tag{2.30}$$

The first step in renormalizing this diagram is integration by parts of the $\partial_{b}$ and $\partial_{c}$ derivatives, and separation into trace and traceless parts. There will be two of each; the first trace piece is easily renormalizable with standard DR identities, and the second one involves the structure

$$\left( \gamma_{a} \gamma_{\mu} \right) \partial_{a} \frac{1}{x^2} \partial_{b} \frac{1}{y^2} K(x,y), \tag{2.31}$$

where

$$K(x,y) = \int d^4u \frac{1}{(x-u)^2(y-u)^2u^2}. \tag{2.32}$$

This is renormalized one more time by integrating the derivatives by parts onto the $K$-function and separating trace and traceless pieces. For the first time, we encounter renormalization structures that do not correspond to any lower loop diagrams, and because of consistency with RG equations, these must cancel in the only other diagram left to compute, Fig. 2l (the nonplanar diagram). We find that indeed they do. The only other divergent pieces remaining are the ones traceless in $cb$ and $cd$, which
can have a global divergence for the same reason as the equivalent term in the previous diagram: the presence of extra indices. These are primitively divergent, like in the previous diagram, and are solved in exactly the same way. Naturally, the diagram identical to this one, but with the vertex subdivergence on the opposite fermion leg, Fig. 2k, is solved in the same fashion, and leads to the same contribution to the 2-loop RG equations. We present the result of the mass derivative on these diagrams after the inclusion of the nonplanar diagram, Fig. 2l, precisely because there are structures in these three diagrams which do not correspond to any lower loop diagram, and which cancel when they are added. So, finally, the nonplanar diagram is:

\[ V^{(2\text{bare})}_\mu(x, y) = -\frac{(ie)^5}{(4\pi^2)^6} (\gamma_\nu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta \gamma_\mu) \times \int d^4u d^4v \frac{1}{(x-u)^2} \frac{1}{u^2} \frac{1}{(v-y)^2} \frac{1}{(v-y)^2} \frac{1}{(u-y)^2}. \] (2.33)

To renormalize this, we integrate \( \partial_c \) by parts around \( z = 0 \), and again separate trace and traceless pieces. Within the trace pieces, we will find the structure

\[ \gamma_\mu \left[ \frac{\partial}{\partial x^a} \left( \frac{1}{x^2 y^2} \frac{\partial}{\partial y^a} K(x, y) \right) + \frac{1}{x^2 y^2} \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^a} K(x, y) \right], \] (2.34)

and these will precisely cancel with structures found in the two previous diagrams. We can now present the mass derivative acting on the sum of these three diagrams:

\[ M \frac{\partial}{\partial M} [V^{(2j)}_\mu + V^{(2k)}_\mu + V^{(2l)}_\mu] = \frac{\alpha}{\pi} V^{(1)} \mu - \frac{\alpha^2}{8\pi^2} V^{(0)}_\mu. \] (2.35)

We are now ready to consider RG equations. These are easy to verify once we use the mass derivatives given above for the 2-loop vertices. The 2-loop \( \Gamma_\mu^{A\psi} \) vertex function,

\[ \Gamma_\mu^{A\psi}(x, y) = ie\gamma_\mu \delta^{(4)}(x-y)\delta^{(4)}(x) + V^{(1)}_\mu(x, y) + V^{(2j+2k+2l+2j+2k+2l)}_\mu(x, y), \] (2.36)

satisfies an RG equation, Eq.(1.22), confirming all the results given previously for different RG functions, and yielding also the 2-loop \( \beta \)-function:

\[ \beta(\alpha) = \frac{2\alpha^2}{3\pi} + \frac{\alpha^3}{2\pi^2}. \] (2.37)

This matches the value gotten by a background field calculation on the 2-loop vacuum polarization.

2.4 Ward identity
At two loops, the method employed previously to verify Ward identities, viz., integrating over an external variable, becomes computationally difficult and not very illuminating. For our purposes here, we shall consider instead the following simpler procedure: we apply the logarithmic mass derivative to both sides of the (2-loop) Ward identity, as we have seen above, this yields 1-loop and tree-level vertices and self-energies, whose Ward identities have already been verified, so that our problem is reduced to that of matching the coefficients of 1-loop and tree-level quantities on both sides of the identity. We have, on the one hand, gained much in simplicity but, of course, on the other hand, some information contained in the Ward identity will thus be lost, namely, the contact terms coming from the finite parts of vertices, since these latter have no mass scales and will vanish when the mass derivative is applied. Although in this way we cannot derive mass relations like we did at one loop, the match we find represents a highly nontrivial consistency check of our 2-loop computations.

Adding up all the contributions from Eqs.(2.13), (2.16) and (2.19), we find:

$$M \frac{\partial}{\partial M} \Sigma^{(2c+2d+2e)} = \frac{\alpha}{2\pi} \Sigma^{(1)} - \frac{7\alpha^2}{16\pi^2} \Sigma^{(0)}$$

(2.38)

and from Eqs.(2.23), (2.25), (2.29), (2.35),

$$M \frac{\partial}{\partial M} V^{(2f+2g+2h+2j+2k+2l)} = \frac{\alpha}{2\pi} V^{(1)} - \frac{3\alpha^2}{16\pi^2} V^{(0)} - \frac{2\alpha}{3\pi} V^{(1)}(a = 0).$$

(2.39)

Once we use the fact that, from Eq.(1.25)

$$\int d^4y \frac{\partial}{\partial z^\mu} V^{(1)}(x, y; a = 0) = \frac{3\alpha}{8\pi} \Sigma^{(0)}(x),$$

(2.40)

we immediately find the amplitudes we have calculated above do indeed satisfy the 2-loop Ward identity. It is also worthwhile noting, furthermore, that in fact we can divide the above 2-loop amplitudes into three sets that separately verify the Ward identity. They are:

$$\frac{\partial}{\partial z^\mu} V^{(2f+2g+2l)}(x, y) = -ie[\delta^{(4)}(x) - \delta^{(4)}(y)]\Sigma^{(2c)}(x - y),$$

(2.41)

$$\frac{\partial}{\partial z^\mu} V^{(2j+2k+2l)}(x, y) = -ie[\delta^{(4)}(x) - \delta^{(4)}(y)]\Sigma^{(2d)}(x - y),$$

(2.42)

$$\frac{\partial}{\partial z^\mu} V^{(2h)}(x, y) = -ie[\delta^{(4)}(x) - \delta^{(4)}(y)]\Sigma^{(2e)}(x - y).$$

(2.43)

The vertex diagrams in each one of these sets are generated by attaching an external photon line in every possible way to an internal fermion line of the corresponding self-energy diagram. This is a vestige of the fact that each bare vertex – individually – formally satisfies a Ward identity with the self-energy gotten by eliminating the external photon line from that vertex diagram.
3. Structured Renormalization Group

In this section we point out a certain structure exhibited by the renormalization of the different relevant vertex functions of QED. For that purpose, we gather here the results of the mass derivative of all 2-loop amplitudes.

Vacuum polarization:

\[ M \frac{\partial}{\partial M} [2\Pi_{\mu \nu}^{(2a)} + \Pi_{\mu \nu}^{(2b)}] = -\frac{\alpha^2}{2\pi^2} (\partial_\mu \partial_\nu - \delta_{\mu \nu} \square) \delta(x) \]  

Self-energy:

\[ M \frac{\partial}{\partial M} \Sigma^{(2c)} = -\frac{\alpha}{2\pi} \Sigma^{(1)} - \frac{\alpha^2}{16\pi^2} \Sigma^{(0)} \]  
\[ M \frac{\partial}{\partial M} \Sigma^{(2d)} = \frac{\alpha}{\pi} \Sigma^{(1)} - \frac{\alpha^2}{8\pi^2} \Sigma^{(0)} \]  
\[ M \frac{\partial}{\partial M} \Sigma^{(2e)} = -\frac{\alpha^2}{4\pi^2} \Sigma^{(0)} \]  

Vertex:

\[ M \frac{\partial}{\partial M} [V^{(2f)} + V^{(2g)}] = -\frac{\alpha}{\pi} V^{(1)} - \frac{3\alpha^2}{8\pi^2} V^{(0)} \]  
\[ M \frac{\partial}{\partial M} V^{(2h)} = -\frac{2\alpha}{3\pi} V^{(1)}(a = 0) \]  
\[ M \frac{\partial}{\partial M} V^{(2i)} = \frac{\alpha}{2\pi} V^{(1)} + \frac{5\alpha^2}{16\pi^2} V^{(0)} \]  
\[ M \frac{\partial}{\partial M} [V^{(2j)} + V^{(2k)} + V^{(2l)}] = \frac{\alpha}{\pi} V^{(1)} - \frac{\alpha^2}{8\pi^2} V^{(0)} \]

The feature the above equations clearly display, which has already been alluded to in Section 2, is the absence of promotion of 1-loop logarithms at two loops for (i) the 2-loop vacuum polarization, Eq.(3.1), (ii) the fermion self-energy with a vacuum polarization subdiagram, Eq.(3.4), and (iii) the vertex with a vacuum polarization subdiagram, Eq.(3.6). All the rest follow the expected pattern of log promotions. Given that the Ward identities provide a relation between \( M_\Sigma \) and \( M_V \), but not between \( M_\Pi \) and anything else, we might then understand the above as a manifest, converse statement to the Ward identity, namely, that the renormalization of the vacuum polarization runs entirely independently of the renormalization of the self-energy and vertex. Thus, for the 2-loop vacuum polarization, although both \( \Pi_{\mu \nu}^{(2a)} \) and \( \Pi_{\mu \nu}^{(2b)} \) contain promoted logs of \( M_\Sigma \) and \( M_V \) (cf. Eqs.(2.3) and (2.6)) coming from their respective subdivergences, in the entire 2-loop amplitude these promotions cancel. This is as it should be, since otherwise RG equations would imply a relation between the mass scales \( M_\Sigma \) (or \( M_V \)) and \( M_\Pi \), which cannot happen if the renormalization of
these amplitudes is to be independent. For the self-energy and vertex with vacuum polarization subdivergences, the same happens: no $\ln^2 M^2$ occur. The mass derivative of the vertex does give a 1-loop vertex, but it is in Landau gauge, and thus finite. For these amplitudes then, we can state that there are no genuine 2-loop divergences, and that is a direct consequence of the lack of a Ward identity relating the vacuum polarization to any other vertex functions.

We see then that the renormalization of these amplitudes has, so to speak, split into different sectors. A careful study of this structured renormalization pattern should furthermore allow us to make predictions about the renormalization at higher loops. In [11] (cf. p.423), for instance, the statement is made that all higher-loop vacuum polarization amplitudes with a single fermion loop (i.e., with no lower-loop vacuum polarization subdivergences) do not contain genuine higher loop divergences ($\ln^2$'s, or $1/\epsilon^2$'s in dimensional regularization, etc.). This feature, which may seem fortuitous in other renormalization methods, appears naturally in differential renormalization.

4. Chiral Anomaly

In this section, we review the computation of the anomalous triangle amplitude \( \langle j_\mu(x) j_\nu(y) j_5^\lambda(z) \rangle \) at one loop [1], and carry it on partially at two loops. Some of the main attractive features of differential renormalization are made manifest. We also compare it to another coordinate space, regulator-free computation of the anomaly due to K. Johnson [20] for the sake of completeness.

Since differential renormalization is strictly 4-dimensional and does not introduce any unphysical regulator fields, one is able to avoid the complications introduced by standard methods such as dimensional regularization (in necessitating ad hoc $\gamma_5$-prescriptions away from $d = 4$), and Pauli-Villars regularization. Furthermore, differential renormalization does not present the anomaly as coming from a symmetry that is broken by regularization artifacts, as is usually the case. Instead, we shall see that the bare triangle amplitude, without the $\gamma$-matrix trace factor in front, has singularities which bring in two renormalization scales and these, when the trace is included, lead to a finite amplitude with a continuous one-parameter shift freedom given by the quotient of these scales. It turns out that this freedom cannot accommodate simultaneously vector an axial conservation. Therefore, the triangle amplitude is overconstrained by the two Ward identities. In our scheme, both vector and axial symmetries appear manifestly on the same footing, and the shift in the anomaly from one to the other is transparent.
We begin with the 1-loop computation. The basic elements have already been spelled out in [1].

Here we simply sketch the key points. At one loop, the anomalous triangle is (Fig. 3):

\[ T_{\mu\nu\lambda}(x,y) = \langle j_\mu(x)j_\nu(y)j_5^\lambda(0) \rangle = 2 \text{Tr} [\gamma_5 \gamma_\lambda \gamma_\alpha \gamma_\nu \gamma_\mu \gamma_\rho] \frac{\partial_a}{y^2} \frac{\partial_b}{(x-y)^2} \frac{1}{x^2}, \]

where the factor of 2 reflects the inclusion of the Bose symmetrized diagram. The term

\[ t_{\alpha\beta\gamma}(x,y) = \partial_a \frac{1}{y^2} \partial_b \frac{1}{(x-y)^2} \partial_c \frac{1}{x^2} \]

is power counting \( L^{-9} \) and thus linearly divergent. This is renormalized in the same way we have treated triangles previously, with the difference that now two derivatives need to be taken out. This has already been done in [1], and we present the final result:

\[ t_{\alpha\beta\gamma}(x,y) = F_{\alpha\beta\gamma}(x,y) + S_{\alpha\beta\gamma}(x,y). \]

\( F_{\alpha\beta\gamma} \) is the finite part

\[
F_{\alpha\beta\gamma}(x,y) = \frac{\partial^2}{\partial x_\alpha \partial y_\beta} \left[ \frac{1}{x^2 y^2} \frac{\partial_c}{(x-y)^2} + \frac{\partial}{\partial x_\alpha} \left( \frac{1}{x^2 y^2} \left( \partial_b \partial_c - \frac{\delta_{bc}}{4} \right) \frac{1}{(x-y)^2} \right) \right] \\
- \frac{\partial}{\partial y_\beta} \left[ \frac{1}{x^2 y^2} \left( \partial_\alpha \partial_c - \frac{\delta_{\alpha c}}{4} \right) \frac{1}{(x-y)^2} \right] - \frac{1}{x^2 y^2} \left[ \partial_\alpha \partial_b \partial_c - \frac{1}{6} \left( \delta_{(ab} \partial_c) \delta \right) \frac{1}{(x-y)^2} \right]
\]

(4.2)

where (,) means unnormalized symmetrization in all indices, and \( S_{\alpha\beta\gamma} \) is the renormalization piece:

\[
S_{\alpha\beta\gamma}(x,y) = \frac{1}{4} \pi^2 \left\{ \left[ \delta_{bc} \frac{\partial}{\partial x_\alpha} - \delta_{\alpha c} \frac{\partial}{\partial y_\beta} \right] \delta(x-y) \frac{\ln M_1^2 x^2}{x^2} \\
- \frac{1}{3} \left[ \delta_{bc} \left( \frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial y_\beta} \right) + \delta_{\alpha c} \left( \frac{\partial}{\partial x_\beta} - \frac{\partial}{\partial y_\alpha} \right) + \delta_{ab} \left( \frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial y_\beta} \right) \right] \delta(x-y) \frac{\ln M_2^2 x^2}{x^2} \right\}.
\]

(4.3)

We note that, like for the 1-loop fermion self-energy, two different mass scales have been used to renormalize the different divergent trace pieces. Not only are we entitled to do that, but in fact one more time it will prove crucial that we do so.

At this point, we apply the traces outside to get \( T_{\mu\nu\lambda}(x,y) \):

\[ T_{\mu\nu\lambda}(x,y) = R_{\mu\nu\lambda}(x,y) + a_{\mu\nu\lambda}(x,y), \]

(4.4)

where \( R_{\mu\nu\lambda} \) comes from the finite part \( F_{\alpha\beta\gamma} \), and

\[
a_{\mu\nu\lambda}(x,y) = -16 \pi^4 \ln \frac{M_1}{M_2} \epsilon_{\mu\nu\lambda\rho} \left( \frac{\partial}{\partial x_\rho} - \frac{\partial}{\partial y_\rho} \right) \delta(x) \delta(y). \]

(4.5)
It is fairly simple to see that for \( x \neq y \neq 0 \), \( T_{\mu \nu \lambda} \) is conserved on all channels. However, just as in the vertex WI studied previously, the subtleties are of course in the contact terms. It is also worthwhile noting that while \( S_{abc} \) (and thus \( t_{abc} \)) indeed contains divergences, the precise \( \gamma \)-structure in front arranges things such as to give the final combination \( \ln \frac{M_1}{M_2} \) as the only renormalization mass dependence, implying that \( a_{\mu \nu \lambda} \) (and thus \( T_{\mu \nu \lambda} \)) is actually finite (because \( M \frac{\partial}{\partial M} \) on that vanishes). This leads us then to a very clear physical picture: we have a finite anomalous triangle, which however contains an ambiguity – in the choice of \( M \)s – coming from a power counting linearly divergent Feynman diagram.

By Fourier transforming into momentum space, one can verify the conservation laws satisfied by \( R_{\mu \nu \lambda} \) on all three channels. Given the form of \( a_{\mu \nu \lambda} \) found above, the vector and axial WIs on \( T_{\mu \nu \lambda} \) then read:

\[
\begin{align*}
\frac{\partial}{\partial x^\mu} T_{\mu \nu \lambda}(x, y) &= 8\pi^4 (1 + 2 \ln \frac{M_1}{M_2} \epsilon_{\nu \lambda \rho} \frac{\partial}{\partial y^\rho} \delta(x) \delta(y)) \\
\frac{\partial}{\partial y^\nu} T_{\mu \nu \lambda}(x, y) &= 8\pi^4 (1 + 2 \ln \frac{M_1}{M_2} \epsilon_{\mu \lambda \rho} \frac{\partial}{\partial x^\rho} \delta(x) \delta(y)) \\
- \left( \frac{\partial}{\partial x^\lambda} + \frac{\partial}{\partial y^\lambda} \right) T_{\mu \nu \lambda}(x, y) &= 16\pi^4 (1 - 2 \ln \frac{M_1}{M_2} \epsilon_{\mu \nu \lambda} \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial y^\rho} \delta(x) \delta(y)) .
\end{align*}
\]

This is the final and, as it were, a most “democratic” expression of the anomaly: we can tune \( \frac{M_1}{M_2} \) so that \( T_{\mu \nu \lambda} \) is conserved either on the vector channels or on the axial one, but never on both; we have been able to complete the calculation in a “scheme-free” fashion all the way to the end, without having to commit at any point to conservation on a particular channel.

We now proceed to the 2-loop calculations. The relevant diagrams are indicated in Fig. 4. Writing the amplitude as

\[
T_{\mu \nu \lambda}^{(2)}(x, y) = A_{\mu \nu \lambda}(x, y) + B_{\mu \nu \lambda}(x, y) ,
\]

where \( A_{\mu \nu \lambda} \) contains the contributions of the diagrams with fermion self-energy insertions, and \( B_{\mu \nu \lambda} \) contains the contributions of the diagrams with the vertex insertions, we can immediately write, as the result of a trivial computation:

\[
A_{\mu \nu \lambda}(x, y) = \frac{e^2}{32\pi^4} \text{Tr} \gamma_5 \gamma_\lambda \gamma_a \gamma_\nu \gamma_b \gamma_\mu \gamma_c \left( \frac{\partial_a}{y^2} \frac{\partial_b}{(y - x)^2} \frac{\partial_c}{x^2} \ln \frac{y^2 M_2^2}{x^2} + \frac{\partial_a}{y^2} \frac{\partial_b}{(y - x)^2} \frac{1}{x^2} + \frac{\partial_a}{y^2} \ln \frac{y}{y - x} \frac{1}{(y - x)^2} \frac{1}{x^2} \right) .
\]

To compute the diagrams related to vertex corrections we operate as follows. Let us take, for instance, the correction to the vertex at \( y \), which leads to the following integral:

\[
\int d^4 u d^4 v \frac{1}{(u - v)^2} \frac{\partial_a}{u^2} \frac{\partial_b}{(u - y)^2} \frac{1}{(v - y)^2} \frac{\partial_c}{(v - x)^2} \frac{1}{x^2} \]

(4.9)
Treating the vertex in the standard way, we integrate $\partial_b$ by parts over $y$, and separate a trace and a traceless piece, and a surface term. The last two terms do not produce logarithms. It is only the trace piece, which is very easy to compute, that brings in a log of the same kind as in Eq.(4.8). Putting together the log pieces of the three vertex correction diagrams we get

$$B_{\mu\nu\lambda}(x,y) = -\frac{e^2}{32\pi^4} \text{Tr} \left[ \gamma_5 \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta \right] \left( \partial_\alpha \frac{1}{y^2} \partial_\beta \frac{1}{(y-x)^2} \partial_\gamma \frac{\ln x^2 M_V^2}{x^2} + \partial_\alpha \frac{\ln y^2 M_V^2}{y^2} \partial_\beta \frac{1}{(y-x)^2} \partial_\gamma \frac{1}{x^2} + \partial_\alpha \frac{1}{y^2} \partial_\beta \frac{\ln (y-x)^2 M_V^2}{(y-x)^2} \partial_\gamma \frac{1}{x^2} \right) + \ldots ,$$

where the dots indicate the finite pieces not written explicitly here. Clearly, $A_{\mu\nu\lambda}$ and $B_{\mu\nu\lambda}$ cancel exactly except for the fact that the mass scale involved in $A_{\mu\nu\lambda}$ is $M_\Sigma$ whereas the one in $B_{\mu\nu\lambda}$ is $M_V$. The one-loop Ward identity shows that the mismatch is precisely proportional to the 1-loop amplitude, thus showing that the entire 2-loop amplitude is finite. The finite parts of $B_{\mu\nu\lambda}$ are extremely lengthy to calculate and we have not found a compact way to perform the computation.

In fact, the total amplitude is expected to vanish identically since, otherwise, as we now explain, the anomaly would get a finite renormalization.

QED is conformally invariant up to the photon propagator and the appearance of renormalization scales. Thus, the anomalous triangle is conformally covariant at one loop because it is finite and contains no photon lines. In any coordinate space treatment, this can be verified explicitly to one loop, whereas in momentum space calculations, this is obscured because plane waves transform easily under translations but not under conformal tranformations. This conformal property, when conjectured to all loops, is powerful enough to show the vanishing of all higher-loop triangle amplitudes, in an argument due to Baker and Johnson [16]. From the study of conformal transformations on functions of three variables, it turns out there is a unique nonlocal, conformal covariant, parity-odd, dimension 3, VVA tensor [19], and the 1-loop triangle has precisely this form. Now, from uniqueness, if the triangle amplitude is covariant to all orders, it means higher-loop contributions also have to have the form of the 1-loop triangle. But that would mean a renormalization of the very structure which gives rise to the chiral anomaly, and thus a renormalization of the anomaly itself, which is forbidden by the Adler-Bardeen theorem. Therefore, all higher-loop contributions to the basic triangle must vanish identically.

Let us finally mention a very nice coordinate space computation of the anomaly which makes no use of an ultraviolet regulator, due to Johnson [20]. His starting point is the conformally covariant
and manifestly finite form of the triangle gotten by acting out the $\gamma$-matrix trace in Eq.(4.1) from the beginning. Because there is, apart from this nonlocal structure, a unique contact term with precisely the same dimension and conformal properties [19], the triangle amplitude can have this local ambiguity, and its coefficient can be chosen to give conservation on either channel but not both. If we want conservation on the $x$ channel, for instance, we have:

$$\frac{\partial}{\partial x^\mu} \left[ T_{\mu\nu\lambda}(x,y) + a \epsilon^{\mu\nu\lambda\sigma} \left( \frac{\partial}{\partial x^\sigma} - \frac{\partial}{\partial y^\sigma} \right) \delta(x) \delta(y) \right] = 0 , \quad (4.11)$$

where we have shown explicitly the contact term in question, with coefficient $a$ to be determined. To find $a$, one then integrates $\partial_\mu T_{\mu\nu\lambda}$ in $x$ and $y$ against the "test" function $x^\alpha y^\beta$. This integral, remarkably, is entirely determined from the long-distance behavior of $T_{\mu\nu\lambda}$, and the only cutoff needed to perform it is an infrared one. Johnson also points out that this situation is reminiscent of the Poisson equation in classical electrostatics, where the coefficient of a contact term in a differential equation (the charge of a point particle) is also determined by the long-distance behavior of a field (Gauss' theorem).

5. Conclusion

In this paper, we have presented a detailed analysis of the differential renormalization of QED up to two loops. All computations are reasonably simple when compared to other approaches.

Our results can be summarized as follows. We have found explicit expressions for the 1- and 2-loop renormalized amplitudes of the vacuum polarization, self-energy of the fermion and vertex. Only some finite parts of the latter are left in the form of integrals. Ward Identities are verified and provide independent checks of the computations. Furthermore, the amplitudes obey renormalization group equations which yield the beta function and the various anomalous dimensions of all the basic fields in the theory. This renormalization group equations display a natural organization due to the freedom to use different renormalization subtractions for quantities not related by WIs. We have also presented the treatment of the anomalous triangle which, in the very spirit of differential renormalization, is regulator-free. It sides with other techniques which are based on the interplay between potential counterterms and symmetries rather than on explicit computations on a regulated theory.

One is rapidly enticed by the ease of computation of differential renormalization. It is our opinion that the method also deserves further consideration because of its natural treatment of chiral problems.
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Appendix A. Basic Differential Renormalization Identities

The following are the DR identities used to renormalize all of the amplitudes presented in the text:

\[ \frac{1}{x^4} = -\frac{1}{4} \ln \frac{x^2 M^2}{x^2} \]  
\[ \frac{1}{x^6} = -\frac{1}{32} \ln \frac{x^2 M^2}{x^2} \]  
\[ \frac{\ln x^2 M^2}{x^4} = -\frac{1}{8} \left( \frac{\ln^2(x^2 M^2) + 2 \ln x^2 M^2}{x^2} \right) \]  
\[ \frac{\ln x^2 M^2}{x^4} = -\frac{1}{8} \left( \frac{\ln^2(x^2 M^2) + 2 \ln x^2 M^2}{x^2} \right) \]

Appendix B. Two-loop renormalized vertices

We present here the final expressions for the 2-loop renormalized vertices. In the expressions below, a derivative w.r.t. \( z^\mu \) means \( \frac{\partial}{\partial z^\mu} = -\frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial y^\mu} \).

\[ V^{(2f)}_{\mu}(x, y) = \frac{i}{16\pi^3} \left( \frac{\alpha}{\pi} \right)^{5/2} (\gamma_\mu \gamma_\rho \gamma_\sigma) \left\{ \frac{\partial}{\partial z^\rho} \left[ \frac{\ln y^2 M^2}{y^2(x-y)^2} \frac{1}{x^2} \right] + \frac{\partial}{\partial z^\sigma} \left[ \frac{\ln y^2 M^2}{y^2(x-y)^2} \frac{1}{x^2} \right] \right\} \]  
\[ \frac{\ln y^2 M^2}{y^2(x-y)^2} \left( \partial_\mu \partial_\rho - \frac{\delta_{\rho \mu}}{4} \right) \frac{1}{x^2} + \frac{\pi^2}{8} \delta_{ab} \delta^{(4)}(x) \frac{\ln y^2 M_0^2}{y^2} \]  
\[ = \frac{i}{96\pi^3} \left( \frac{\alpha}{\pi} \right)^{5/2} \left\{ (\gamma_\rho \gamma_\alpha \gamma_\mu \gamma_\sigma) \left[ -\frac{\partial}{\partial x^\rho} \left( \frac{1}{x^2} \partial_\mu \frac{1}{y^2} (\partial_\rho \partial_\sigma - \delta_{\rho \sigma} \Box) L(x-y) \right) + \right. \frac{\partial}{\partial y^\rho} \left( \frac{1}{x^2 y^2} \gamma_\alpha (\partial_\rho \partial_\sigma - \delta_{\rho \sigma} \Box) L(x-y) \right) \right\} - \frac{16}{x^2 y^2} \gamma_\alpha \left( \partial_\mu \partial_\alpha - \frac{\delta_{\alpha \mu}}{4} \right) \ln \frac{(x-y)^2 M_0^2}{(x-y)^2} \]
where $L(x) = \ln x^2 \mu^2 - \frac{1}{2} \ln^2 x^2 M_H^2$.

\[
V_{\mu}^{(2)}(x, y) = \frac{i}{32\pi^2} \left( \frac{\alpha}{\pi} \right)^{5/2} \int d^4 u d^4 v \left\{ \left( \gamma_d \gamma_5 \gamma_\mu \gamma_\nu \gamma_\alpha \right) \frac{\partial}{\partial z^\nu} \left[ \frac{1}{(x-y)^2} \frac{1}{u^2} \frac{1}{v^2} \frac{1}{(x-u)^2} \frac{1}{(v-y)^2} \right] \right. \\
- \left( \gamma_d \gamma_5 \gamma_\alpha \right) \frac{(x-y)^2}{2} \ln (x-y)^2 M^2 \left( \frac{1}{(x-y)^2} \frac{1}{u^2} \right) - \frac{i}{16\pi^5} \left( \frac{\alpha}{\pi} \right)^{5/2} \left( \gamma_d \gamma_\mu \gamma_\alpha \right) \left\{ \frac{\partial}{\partial z^\nu} \left[ \frac{1}{(x-y)^2} \frac{\ln (x-y)^2 M^2}{y^2} \right] \right. \\
+ \frac{1}{(x-y)^2} \left( \frac{\partial \partial z^\nu}{\partial v} - \frac{\delta_{\nu \mu}}{4} \right) \frac{1}{y^2} \ln (x-y)^2 M^2 \right\} + \frac{i}{64\pi} \left( \frac{\alpha}{\pi} \right)^{5/2} \gamma_\mu \delta^{(4)}(x) \ln^2 y^2 M^2 + 2 \ln y^2 M^2.
\] (B.3)

\[
V_{\nu}^{(2)}(x, y) = \frac{i}{64\pi} \left( \frac{\alpha}{\pi} \right)^{5/2} \left( \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta \gamma_\rho \right) \int d^4 u d^4 v \left\{ \frac{\partial}{\partial z^\nu} \left[ \frac{1}{(x-y)^2} \frac{1}{u^2} \frac{1}{v^2} \frac{1}{(x-u)^2} \frac{1}{(v-y)^2} \right] \right. \\
- \frac{1}{x^2} \left( \frac{\partial \partial x^\nu}{\partial u} - \frac{\delta_{\nu \rho}}{4} \right) \frac{1}{y^2} \ln (x-y)^2 M^2 \right\} + \frac{i}{16\pi^5} \left( \frac{\alpha}{\pi} \right)^{5/2} \left( \gamma_d \gamma_\alpha \gamma_\mu \right) \left\{ \frac{\partial}{\partial z^\nu} \left[ \frac{1}{(x-y)^2} \frac{1}{u^2} \right] K(x, y) \right\} \\
+ \frac{\partial}{\partial y^\nu} \left[ \frac{1}{x^2} \frac{\partial}{\partial x^\nu} K(x, y) \right] - \frac{1}{x^2} \left( \frac{\partial \partial y^\nu}{\partial u} - \frac{\delta_{\nu \rho}}{4} \right) K(x, y) \right\} \\
- \frac{i}{32\pi^2} \left( \frac{\alpha}{\pi} \right)^{5/2} \gamma_\mu \left\{ \frac{1}{4 \sqrt{2}} \ln x^2 M^2 \frac{\ln y^2 M^2}{y^2} \right. \\
+ \frac{\partial}{\partial x^\nu} \left[ \frac{1}{(x-y)^2} \frac{\ln x^2 M^2}{y^2} \right] \left. \right\} + \pi^2 \delta^{(4)}(x-y) \left[ \frac{1}{x^2} \frac{\ln x^2 M^2}{y^2} \right] \right\}.
\] (B.4)

\[
V_{\mu}^{(2)}(x, y) = \frac{i}{128\pi^2} \left( \frac{\alpha}{\pi} \right)^{5/2} \left( \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta \gamma_\rho \right) \int d^4 u d^4 v \left\{ \frac{\partial}{\partial z^\nu} \left[ \frac{1}{(x-y)^2} \frac{1}{u^2} \frac{1}{v^2} \frac{1}{(x-u)^2} \frac{1}{(v-y)^2} \right] \right. \\
- \frac{i}{8\pi^7} \left( \frac{\alpha}{\pi} \right)^{5/2} \gamma_\beta \int d^4 u d^4 v \left[ \frac{1}{(x-y)^2} \frac{\partial \partial x^\rho}{\partial u} - \frac{1}{4 \sqrt{2}} \frac{\partial \partial x^\rho}{\partial v} \right] \frac{1}{(x-u)^2} \frac{1}{(v-y)^2} \\
+ \frac{i}{8\pi^7} \left( \frac{\alpha}{\pi} \right)^{5/2} \gamma_\mu \frac{\partial}{\partial x^\nu} \left[ \frac{1}{x^2 \sqrt{2}} \frac{\ln x^2 M^2}{y^2} \right] K(x, y) \right\} + \frac{i}{16\pi^5} \left( \frac{\alpha}{\pi} \right)^{5/2} \gamma_\mu \left\{ \frac{1}{4 \sqrt{2}} \ln x^2 M^2 \frac{\ln y^2 M^2}{y^2} \right. \\
- \frac{\partial}{\partial z^\nu} \left[ \frac{1}{(x-y)^2} \frac{\ln x^2 M^2}{y^2} \right] \left. \right\} + \pi^2 \delta^{(4)}(x-y) \left[ \frac{1}{x^2} \frac{\ln x^2 M^2}{y^2} \right] \right\}.
\] (B.5)
Figure captions:

Figure 1: 1-loop 1PI diagrams of QED.

Figure 2: 2-loop diagrams of QED.

Figure 3: 1-loop anomalous triangle diagram.

Figure 4: 2-loop contributions to anomalous ABJ amplitude.
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