ON ALMOST-EQUIDISTANT SETS

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Abstract. A finite set of points in $\mathbb{R}^d$ is called almost-equidistant if among any three distinct points in the set, some two are at unit distance. We proved that an almost-equidistant set in $\mathbb{R}^d$ has cardinality at most $5d^{13/9}$.

1. Introduction

A finite set of (unit) vectors in $\mathbb{R}^d$ is called almost orthogonal if among any three vectors of the set there is at least one orthogonal pair. Erdős asked [Ros91]: What is the largest cardinality $f_{\pi/2}(d)$ of an almost-orthogonal set in $\mathbb{R}^d$? Using an elegant linear algebraic argument Rosenfeld [Ros91] proved that $f_{\pi/2}(d) = 2d$. Clearly, we can take two sets of $d$ orthogonal vectors, and obtain an almost orthogonal set of size $2d$. Other nice proofs of Rosenfeld’s theorem were given in [Dea11, Theorem 3.5] and [Pud02, Theorem 6]. Moreover, Deaett exhibited a different example of almost orthogonal sets in $\mathbb{R}^d$ of size $2d$; see [Dea11, Theorem 4.11].

The following definition naturally generalizes the concept of almost orthogonal sets: A finite set $V$ of (unit) vectors in $\mathbb{R}^d$ is called almost $\alpha$-angular if among any three vectors of $V$, two of them form a fixed angle $\alpha$. In particular, an almost orthogonal set is an almost $\pi/2$-angular set. The following variation of Erdős’s problem has also been considered: What is the largest cardinality $f_{\alpha}(d)$ of an almost $\alpha$-angular set in $\mathbb{R}^d$, where $\alpha$ is a fixed angle? Bezdek and Lángi [BL99, Theorem 1] found that $f_{\alpha}(d) \leq 2d+2$ for any $\pi/2 < \alpha < \pi$ and $d \geq 2$. Moreover, this bound is tight for $\alpha = \alpha_d := 2 \arcsin(\sqrt{d+1}/(2d))$. Indeed, one can take $d+1$ unit vectors in $\mathbb{R}^d$ such that an angle between any two of them is $\alpha_d$; note that the ends of these vectors correspond to vertices of a regular $d$-simplex inscribed in the unit sphere with center at the origin. Therefore, the union of two such sets of vectors forms an almost $\alpha_d$-angular set of $2d+2$ vectors in $\mathbb{R}^d$. It is worth pointing out that the argument of Bezdek and Lángi is a natural modification of Rosenfeld’s approach.

The key idea of their proof is based on two facts: The first fact is that the matrix

$$V_\alpha := \langle v_i, v_j \rangle - \cos \alpha,$$

is positive semidefinite for $\pi/2 < \alpha < \pi$; the second fact is that the trace of the matrix $(V_\alpha - (1 - \cos \alpha)I_n)^3$ is equal to 0, where $I_n$ is the identity matrix of size $n$. For more details we refer the interested readers to Subsection 3.1. Since $\cos \alpha > 0$ for $0 < \alpha < \pi/2$, the matrix $V_\alpha$ can have one negative eigenvalue, and hence the argument of Bezdek and Lángi does not work for small $\alpha$. Unfortunately, we still do not know whether $f_{\alpha}(d) = O(d)$ holds for any $0 < \alpha < \pi/2$.

A finite set of $n$ points in $\mathbb{R}^d$ is called almost-equidistant if among any three points in the set, some two are at unit distance. The concept of an almost-equidistant set generalizes the notion of an almost $\alpha$-angular set. Indeed, if a set of unit vectors $\{v_1, \ldots, v_n\}$ is almost $\alpha$-angular then the set $\{kv_1, \ldots, kv_n\}$ is almost-equidistant, where $k = 1/(2 \sin(\alpha/2))$.

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The following question was posed by Zsolt Lángi: What is the largest cardinality $f_{ae}(d)$ of an almost-equidistant set in $\mathbb{R}^d$? Denote by $G = (V, E)$ the graph such that its vertices are points of an almost-equidistant set $V$ in $\mathbb{R}^d$ and edges are pairs of vertices that are at unit distance apart. Note that $G$ does not contain a clique of size $d + 2$ because it is impossible to embed a regular unit $(d + 1)$-simplex in $\mathbb{R}^d$. Furthermore, it does not contain an anticlique of size 3 because its vertex set is an almost-equidistant set. By the upper bound on the Ramsey number $R(3, d + 2)$ from the article [Gri83], we have $f_{ae}(d) \leq 2.4 \cdot (d+2)^2 / \log(d+2)$. Bezdek, Nadzódi and Visy [BNV03, Statement (2) in Theorem 4] proved that $f_{ae}(2) = 7$; notice that the famous Moser spindle has 7 vertices forming an almost-equidistant set in $\mathbb{R}^2$. Recently, Balko, Pór, Scheucher, Swanepoel and Valtr [BPSV17] showed that $f_{ae}(d) = O(d^{9/2})$. Also, they constructed an example of an almost-equidistant set in $\mathbb{R}^d$ with $2d + 4$ vertices for $d \geq 3$ and proved some upper and lower bounds for the largest cardinalities of almost-equidistant set in $\mathbb{R}^d$, $d \leq 9$. The main result of the present article is the following improvement.

**Theorem 1.** $f_{ae}(d) \leq 5d^{3/3}9$.

It is worth noting that more recently than this paper was submitted Kupavskii, Mustafa and Swanepoel [KMS17] proved $f_{ae}(d) = O(d^{3/3})$.

The article is organized in the following way. Section 2 presents some preliminaries. In particular, we discuss some properties of Euclidean distance matrices and unit distance graphs formed by vertices of almost-equidistant sets. In Section 3 we prove Theorem 1. In Section 4 we compare our and other approaches in papers, where some upper bounds on the largest size of an almost-equidistant set are proven. In Section 5 we look at some open problems related to almost-equidistant sets. Along the way, we obtain some facts (Lemma 2 and Lemma 7) that are useful in studying distance graphs in $\mathbb{R}^d$, for example, unit distance graphs and diameter graphs.

## 2. Preliminaries

### 2.1. Properties of Euclidean distance matrices

Suppose that $\{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ is an arbitrary set of distinct points, where $n \geq 2$. Let us consider the matrix

$$V := \|v_i - v_j\|^2.$$ 

**Lemma 2.** The matrix $V$ has exactly one positive eigenvalue.

**Proof.** Note that $V$ is not the zero matrix. As it is symmetric, it has a real eigenvalue different from 0. Since $\text{tr}(V) = 0$, the matrix $V$ has at least one positive eigenvalue. Let us prove that it has at most one positive eigenvalue. Clearly,

$$V = v^t j + j^t v - 2\langle v_i, v_j \rangle,$$

where $v := (\|v_1\|^2, \ldots, \|v_n\|^2)$ and $j := (1, \ldots, 1)$ are row vectors of size $n$, and $\langle v_i, v_j \rangle$ is the Gram matrix of the vectors $v_1, \ldots, v_n$.

It is easy to check that the eigenvalues of $v^t j + j^t v$ satisfy the equality

$$\det(v^t j + j^t v - \mu I_n) = (-1)^n \mu^{n-2} \left( \mu^2 - \left( \sum_{i=1}^n \|v_i\|^2 \right) \mu - \sum_{1 \leq i < j \leq n} (\|v_i\|^2 - \|v_j\|^2)^2 \right) = 0.$$ 

Thus the matrix $v^t j + j^t v$ has one positive eigenvalue, one negative eigenvalue and $n - 2$ eigenvalues equal to 0. Note that the Gram matrix $\langle v_i, v_j \rangle$ is positive semidefinite. In order to finish the proof of Lemma 2, we need to apply Weyl’s inequality [Wey12, Theorem 1] (or [Pra94, Theorem 34.2.1]).
Theorem 3 (Weyl’s inequality). Let $A$ and $B$ be Hermitian matrices of size $n$. Suppose that $\alpha_1 \leq \cdots \leq \alpha_n$ are eigenvalues of $A$, $\beta_1 \leq \cdots \leq \beta_n$ are eigenvalues of $B$, $\gamma_1 \leq \cdots \leq \gamma_n$ are eigenvalues of $A + B$. Then

$$\gamma_i \geq \alpha_j + \beta_{i-j+1} \text{ for } i \geq j \text{ and } \gamma_i \leq \alpha_j + \beta_{i-j+n} \text{ for } i \leq j.$$  

Remark. Actually, we will use only the inequality $\gamma_{n-1} \leq \alpha_{n-1} + \beta_n$.

By Weyl’s inequality, the second-largest eigenvalue of $V = (x^Tj + j^Tx) - 2\langle v_i, v_j \rangle$ is not positive, and so $V$ has at most one positive eigenvalue. □

Corollary 4. Let $U := V - J_n + I_n$, where $J_n$ is the matrix of ones of size $n \times n$. The matrix $U$ has at most one eigenvalue $> 1$ and at least $n - d - 2$ eigenvalues equal to 1.

Proof. It is enough to prove that the matrix $W := V - J_n$ has at most one positive eigenvalue and $\text{rank}(W) \leq d + 2$. Note that eigenvalues of $-J_n$ are $-n, 0, \ldots, 0$. By Weyl’s inequality, the second-largest eigenvalue of $W$ is not positive, and hence $W$ has at most one positive eigenvalue.

Let us prove that $\text{rank}(W) \leq d + 2$. Obviously,

$$W = v^Tj + j^Tv - 2\langle v_i, v_j \rangle - j^Tv = v^Tj + j^T(v - j) - 2\langle v_i, v_j \rangle.$$  

Therefore, we have

$$\text{rank}(W) = \text{rank}\left(v^Tj + j^T(v - j) - 2\langle v_i, v_j \rangle\right)$$

$$\leq \text{rank}(v^Tj) + \text{rank}(j^T(v - j)) = 1 + 1 = d + 2.$$  

The last inequality holds because $\text{rank}((v_i, v_j)) \leq d$ for the Gram matrix of vectors $v_1, \ldots, v_n$ in $\mathbb{R}^d$. □

2.2. Properties of almost-equidistant sets. Here and subsequently, we assume that $V$ is an almost-equidistant set in $\mathbb{R}^d$ and $U$ is the corresponding matrix for $V$; see Corollary 4. Let us prove the following useful lemma about $U$.

Lemma 5. $\text{tr}(U) = \text{tr}(U^2) = 0$.

Proof. Notice that $U$ with $ij$-entry $u_{ij}$ satisfies the following properties:

1. $u_{ii} = 0$ for all $i = 1, \ldots, n$;
2. $u_{ij}u_{jk}u_{ki} = 0$ for all triples $1 \leq i, j, k \leq n$.

By property (1), we have $\text{tr}(U) = 0$. Using property (2) we get $\text{tr}(U^2) = 0$. □

From now on, $G = (V, E)$ stands for the unit distance graph formed by the points in $V$; edges of $G$ are pairs of vertices that are at unit distance apart. We need the following simple lemma.

Lemma 6. A vertex in $G$ has at most $d + 1$ non-neighbors.

Proof. Assume to the contrary that there are $d + 2$ vertices in $G$ that are not adjacent to some vertex. Note that these vertices form a regular unit $(d + 1)$-simplex because $V$ is an almost-equidistant set, but it is impossible to embed a regular unit $(d + 1)$-simplex in $\mathbb{R}^d$, a contradiction. □

One of the key ingredients of the proof of Theorem 1 is the following lemma.

Lemma 7. Let $w_i$, $0 \leq i \leq k$, be vertices in $G$. Assume that

$$s = \sum_{i=1}^{k} (\|w_0 - w_i\|^2 - 1) \text{ and } |s| \geq \sqrt{k}.$$  

Then the number of vertices adjacent to all $w_i$, for $0 \leq i \leq k$ is at most $2d + 2$. 

Proof. Without loss of generality, assume that \( w_i, 1 \leq i \leq k \), are non-neighbors of \( w_0 \). Indeed, if among \( w_i, 1 \leq i \leq k \), there are vertices adjacent to \( w_0 \), then we delete them and show that the number of vertices adjacent to \( w_0 \) and its non-neighbors among \( w_i, 1 \leq i \leq k \), is at most \( 2d + 2 \).

Our proof of is based on the following theorem (see [DM94, Theorem 1]):

**Theorem 8.** Let \( X = \{x_1, \ldots, x_n\} \) and \( Y = \{y_1, \ldots, y_n\} \) be two point-sets in \( \mathbb{R}^d \). Then

\[
\sum_{1 \leq i, j \leq n} \|x_i - y_j\|^2 = \sum_{1 \leq i, j \leq n} \|x_i - x_j\|^2 + \sum_{1 \leq i, j \leq n} \|y_i - y_j\|^2 + n^2\|x - y\|^2,
\]

where \( x \) and \( y \) are barycenters of \( X \) and \( Y \), respectively, that is,

\[
x = (x_1 + \cdots + x_n)/n, \quad y = (y_1 + \cdots + y_n)/n.
\]

Note that \( w_1, \ldots, w_k \) form a regular unit \((k - 1)-\)simplex because \( V \) is an almost-equidistant set and \( \|w_0 - w_i\| \neq 1 \) for \( 1 \leq i \leq k \). We write \( S(v, r_0) \) for the sphere of radius \( r_0 \) with center \( v \). We claim that \( S := S(w_1, 1) \cap \cdots \cap S(w_k, 1) \subset S(o, r) \), where \( o \) is the center of \( w_1, \ldots, w_k \) and \( r := \sqrt{(k + 1)/(2k)} \). Indeed, let \( o' \) be the orthogonal projection of some point \( u \in S \) onto the affine hull of \( w_1, \ldots, w_k \). Since the triangles \( uu'w_i, 1 \leq i \leq k \), are right triangles with the common cathetus \( o'u \) and the congruent hypotenuses \( uu'w_i \), they are congruent, and consequently \( o' = o \). Using the Pythagorean theorem for the triangle \( uw_1 \) and the fact that the circumscribe radius of an unit \((k - 1)-\)simplex is \( \sqrt{(k - 1)/(2k)} \) we have \( (r')^2 = \|u - o\|^2 = (k + 1)/(2k) \).

Applying Theorem 8 for \( X = \{w_0, \ldots, w_0\} \) and \( Y = \{w_1, \ldots, w_k\} \) we obtain:

\[k(s + k) = \frac{k(k - 1)}{2} + k^2s^2, \quad \text{where } x := \|w_0 - o\|.
\]

Since \( |s| \geq \sqrt{k} \), we have

\[x^2 = \frac{s}{k} + \frac{k + 1}{2k} \geq \frac{1}{\sqrt{k}} + \frac{k + 1}{2k} = \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right)^2 \quad (1)
\]

or

\[x^2 = \frac{s}{k} + \frac{k + 1}{2k} \leq -\frac{1}{\sqrt{k}} + \frac{k + 1}{2k} = \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right)^2. \quad (2)
\]

Note that vertices in \( G \) adjacent to all \( w_i, 0 \leq i \leq k \), lie on the sphere \( w := S(w_0, 1) \cap S(o, r) \). Let us prove that its radius \( r' \) is at most \( 1/\sqrt{2} \). Suppose \( z \in w \). Then \( \|z - w_0\| = 1 \) and \( \|z - o\| = \sqrt{(k + 1)/(2k)} \). Note that the center of \( w \) should coincide with the orthogonal projection \( z' \) of \( z \) onto the line passing through \( o \) and \( w_0 \), and hence \( r' = \|z - z'\| \). Denote by \( \theta \) the angle \( z w_0 o \). Using the Law of Cosines we have

\[
\cos \theta = \frac{\|z - w_0\|^2 + \|w_0 - o\|^2 - \|z - o\|^2}{2\|z - w_0\|\|w_0 - o\|} = \frac{1 + x^2 - (k + 1)/2k}{2x} = \frac{k - 1}{4kx} + \frac{x}{2} =: g(x).
\]

It is easily seen that \( \sqrt{(k - 1)/(2k)} \) is the only point of local minimum of \( g(t) \) for \( t > 0 \). Note that

\[
\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2k}} < \sqrt{\frac{k - 1}{2k}} < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2k}}.
\]
Therefore, we have
\[
g(x) \geq \min \left\{ g\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2k}}\right), g\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2k}}\right)\right\} = \frac{1}{\sqrt{2}}
\]
because (1) and (2) hold. Thus \(\cos \theta \geq 1/\sqrt{2}\), and so \(r' = \|z - w_0\| \sin \theta \leq 1/\sqrt{2}\).

Suppose that \(V \cap w = \{u_1, \ldots, u_m\}\). Assume that \(\|u_i - u_j\| = 1\) for some \(1 \leq i, j \leq m\).
Since the radius of \(w\) is at most \(1/\sqrt{2}\), we have \(\angle u_i z' u_j = \alpha \geq \pi/2\), where \(\alpha\) is a fixed angle. Indeed, by the Law of Cosines we obtain that
\[
\cos \angle u_i z' u_j = \frac{\|u_i - z'\|^2 + \|u_j - z'\|^2 - \|u_i - u_j\|^2}{2\|u_i - z'\|\|u_j - z'\|} = \frac{2(r')^2 - 1}{2(r'^2)} \leq 0
\]
is a fixed number. Therefore, \(\{u_1 - z', \ldots, u_m - z'\}\) is an almost \(\alpha\)-angular set in \(\mathbb{R}^d\) (note that these vectors are not unit). By results of Rosenfeld [Ros91] and Bezdek–Lángi [BL99], we get \(m \leq 2d + 2\).

3. Proof of Theorem 1

Since \(f_{ae}(2) = 7\) (see [BNV03]), we can assume that \(d \geq 3\). We will consider two cases: The matrix \(U\) does not have eigenvalues \(> 1\) or has just one such eigenvalue.

The proof of the first case is an almost word-for-word repetition of the proof of Theorem 1 in [BL99]. The proof of the second case involves new ideas, in particular the Gershgorin circle theorem [Ger31] and Lemma 7.

3.1. Proof of the first case. Assume that the matrix \(U\) does not have eigenvalues \(> 1\). Denote by \(\lambda_1, \ldots, \lambda_k\) eigenvalues of \(U\) that are \(< 1\). By Corollary 4 and Lemma 5, we have \(k \leq d + 2\) and
\[
\sum_{i=1}^{k} (-\lambda_i) = n - k, \quad \sum_{i=1}^{k} (-\lambda_i)^3 = n - k.
\]
In order to finish the proof of the current case, we need the following lemma (see [BL99, Lemma 1]).

**Lemma 9.** Let \(x_1, \ldots, x_m\) be real numbers with the property that there exists \(y > 0\) such that \(x_i \geq -y\) for \(i = 1, \ldots, m\) and \(\sum_{i=1}^{m} x_i = (m + l)y\), where \(l \geq 0\). Then
\[
\sum_{i=1}^{m} x_i^3 \geq (m + 3l)y^3.
\]

Assume that \(n > 2k\). Introducing the notation \(l = n - 2k\) and \(y = 1\) we can rewrite the first equality in (3) as:
\[
\sum_{i=1}^{k} (-\lambda_i) = (k + l)y.
\]
Thus Lemma 9 implies that
\[
\sum_{i=1}^{k} (-\lambda_i)^3 \geq (k + 3l)y^3.
\]
Finally, according to the second equality of (3)
\[
\sum_{i=1}^{k} (-\lambda_i)^3 = (k + l)y^3,
\]
a contradiction. This completes the proof of this case.
3.2. **Proof of the second case.** Assume that the matrix $U$ has eigenvalue $\lambda > 1$. By Corollary 4, there is exactly one eigenvalue $> 1$ and there are at most $d + 1$ eigenvalues $< 1$. Denote them by $\lambda_1, \ldots, \lambda_k$. By Lemma 5, we have

$$\lambda + \sum_{i=1}^{k} \lambda_i + (n - k - 1) = 0, \quad \lambda^3 + \sum_{i=1}^{k} \lambda_i^3 + (n - k - 1) = 0.$$  

Without loss of generality, assume that $\lambda_1, \ldots, \lambda_l \leq 0$ and $\lambda_{l+1}, \ldots, \lambda_k > 0$, where $1 \leq l \leq k$. Therefore, we have

$$\sum_{i=1}^{l} (-\lambda_i) = \lambda + (n - k - 1) + \sum_{i=l+1}^{k} \lambda_i \geq n + \lambda - d - 2 \geq n - d - 1,$$

so

$$\lambda^3 = \sum_{i=1}^{l} (-\lambda_i)^3 + \sum_{i=l+1}^{k} (-\lambda_i)^3 - (n - k - 1) \geq \frac{(-\lambda_1 - \cdots - \lambda_l)^3}{l^2} - (k - l) - (n - k - 1) \geq \frac{(n - d - 1)^3}{(d + 1)^2} - n.$$  

Suppose, contrary to our claim, that $n = 5d^\beta \geq 4d^\beta + d + 1$ for some $\beta > 13/9$, thus

$$\lambda^3 \geq \frac{64d^{3\beta}}{2d^2} - 5d^\beta \geq 27d^{3\beta - 2},$$

and hence $\lambda \geq 3d^{3\beta - 2}/3$.

We need the Gershgorin circle theorem [Ger31] (or [Pra94, Problem 34.1]).

**Theorem 10 (Gershgorin circle theorem).** Every eigenvalue of a matrix $A = a_{ij}$ over $\mathbb{C}$ of size $n \times n$ belongs to one of the disks

$$\left\{ z \in \mathbb{C} : |a_{kk} - z| \leq \rho_k, \text{ where } \rho_k = \sum_{j \neq k} |a_{kj}| \right\} \text{ for } k = 1, \ldots, n.$$  

Using the Gershgorin circle theorem for the matrix $U$ we can assume

$$\sum_{j=2}^{n} \left| \|v_1 - v_j\|^2 - 1 \right| \geq \lambda \geq 3d^{3\beta - 2}/3.$$  

By Lemma 6, on the left-hand side there are at least $n - d - 2$ terms equal to 0. There is no loss of generality in assuming

$$\sum_{j=2}^{d+2} \left| \|v_1 - v_j\|^2 - 1 \right| \geq 3d^{3\beta - 2}/3.$$

Therefore, without loss of generality, we have

$$\sum_{j=2}^{2\left\lfloor d^{4/9} \right\rfloor + 1} \left| \left( \|v_1 - v_j\|^2 - 1 \right) \right| \geq 3d^{3\beta - 2}/3 \cdot \frac{2\left\lfloor d^{4/9} \right\rfloor}{d + 1} \geq 2d^{2/9}.$$  

The last inequality holds for $d \geq 3$. Thus we can certainly assume that

$$\sum_{j=2}^{\left\lfloor d^{4/9} \right\rfloor + 1} \left( \|v_1 - v_j\|^2 - 1 \right) \geq d^{2/9}.$$  

By Lemma 7, the number of vertices in $G$ that are adjacent to all $v_i$ for $1 \leq i \leq \left\lfloor d^{4/9} \right\rfloor + 1$ is at most $2d + 2$. By Lemma 6, the number of vertices that are not adjacent to at
least one of \( v_i, 1 \leq i \leq \lfloor d^{4/9} \rfloor + 1 \), is at most \((\lfloor d^{4/9} \rfloor + 1)(d + 1)\). Therefore, \( n \leq (2d + 2) + ((\lfloor d^{4/9} \rfloor + 1)(d + 1) + \lfloor d^{4/9} \rfloor + 1) < 5d^{13/9} \), a contradiction. \( \Box \)

Corollary 11. We have \( f_{\alpha}(d) \leq 5d^{13/9} \) for \( 0 < \alpha < \pi/2 \).

Proof. Suppose that \( \{v_1, \ldots, v_n\} \) is an almost \( \alpha \)-angular set of unit vectors in \( \mathbb{R}^d \). Clearly, the set \( \{kv_1, \ldots, kv_n\} \) is an almost-equidistant set of points in \( \mathbb{R}^d \), where \( k = 1/(2 \sin(\alpha/2)) \). Therefore, \( n \leq 5d^{13/9} \). \( \Box \)

4. Discussion

Now we will compare our proof with other proofs (see [BPS+17] and [KMS17]) of the upper bounds on the largest size of an almost-equidistant set.

The common idea is to estimate the sizes of two subsets \( T_1 \) and \( T_2 \) of an almost-equidistant set that are constructed using some third subset \( T \). In the present paper \( T \) is the union of a clique and a vertex that is not adjacent to the clique. Note that we use Rosenfeld’s method [Ros91] and the Gershgorin circle theorem to find \( T \). But the choice of \( T \) in other papers is quite simple: In [BPS+17] the subset \( T \) is any clique of size \( \lfloor d^{3/2} \rfloor \), and in [KMS17] the subset \( T \) is a clique of maximum cardinality.

In all papers the first subset \( T_1 \) contains vertices adjacent to all points of \( T \) (as in the present paper and in [BPS+17]) or to almost all points of \( T \) (as in [KMS17]). The second subset \( T_2 \) is just the complement of \( T_1 \). To bound the number of points in \( T_2 \), we apply the trivial bound as in [BPS+17], but the authors of [KMS17] used double counting. To estimate the size of \( T_1 \), the authors of [BPS+17] and [KMS17] applied some lemma\(^1\) that Daett [Dea11, Lemma 3.4] used to prove \( f_{\pi/2}(d) = 2d \). We bound the size of \( T_1 \) using a new tool (Lemma 7) that is based on results of Rosenfeld and Bezdek–Lángi.

So the key of differences of our approaches is that we try to follow Rosenfeld’s proof of the fact that almost orthogonal set in \( \mathbb{R}^d \) has at most \( 2d \) points, but the authors of other articles followed Daett’s proof.

5. Open problems

Unfortunately, we were not capable to prove the following natural conjecture.

Conjecture 12. \( f_{\text{ae}}(d) = O(d) \).

If the diameter of an almost-equidistant set \( V \) in \( \mathbb{R}^d \) is at least \( \sqrt{2} \), then we can easily prove that \( |V| \leq 4d + 6 \). Indeed, suppose \( \|v - u\| \geq \sqrt{2} \) for \( v, u \in V \). By Lemma 7, the number of points in \( V \) that are at unit distance from \( v \) and \( u \) is at most \( 2d + 2 \). By Lemma 6, the number of points in \( V \) that are not at unit distance from \( u \) or \( v \) is at most \( 2d + 2 \). Thus \( |V| \leq 4d + 6 \). This means that in order to prove Conjecture 12 we can assume that the diameter of an almost-equidistant set does not exceed \( \sqrt{2} \). Therefore, it would be natural to ask the following question.

Problem 13. A subset of \( \mathbb{R}^d \) is called an almost-equidistant diameter set if it is an almost-equidistant set in \( \mathbb{R}^d \) and has diameter 1. What is the largest cardinality of an almost-equidistant diameter set in \( \mathbb{R}^d \)?

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\(^1\)This lemma claims that rank \( A \geq \text{tr} A / \text{tr} A^2 \) for a Hermitian matrix \( A \).
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