Non–Abelian tensor hierarchy in (1,0) D=6 superspace

Igor A. Bandos *

* Department of Theoretical Physics, University of the Basque Country, UPV/EHU, P.O. Box 644, 48080 Bilbao, Spain and
† IKERBASQUE, Basque Foundation for Science, 48011, Bilbao, Spain

ABSTRACT: We present a set of constraints on superfield strengths of the non-Abelian p–form potentials in D=6 (1,0) superspace which reproduces, as their selfconsistency conditions, the equations of motion of the recently proposed (1,0) superconformal theory. These include the anti-self-duality conditions for the field strength of the non-Abelian 2-form potential, duality between field strengths of the non–Abelian vectors and 3-forms as well as of the non-Abelian four forms and scalar fields.

KEYWORDS: Supersymmetry, superspace, non–Abelian tensor fields, duality and self-duality, multiple p-branes.
1. Introduction

Recently, motivated by the search for a description of multiple M5-brane system [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16], the authors of [17, 18, 19] have constructed a class of new (1,0) superconformal models describing a hierarchy of non-Abelian scalar, vector and tensor fields and their supersymmetric partners. The action for the bosonic sectors of these theories have been constructed in a very recent [20] using the PST (Pasti–Sorokin-Tonin) approach [21].

In this paper we propose the set of constraints on the super-(p+1)-form field strengths of non-Abelian super-p-form potentials on (1,0) D=6 superspace which restrict the field content of these super-p-forms to the fields of the non-Abelian tensorial hierarchy of [17].
We show that these constraints reproduce the dynamical equations of the (1,0) superconformal model as their selfconsistency conditions. The set of these equations includes supersymmetrizations of the anti–self–duality condition for the 3-form field strength of non-Abelian 2-form (antisymmetric tensor) potential, as well as non-Abelian vectors—3-forms and scalars—4-forms duality relations.

Although the same equations were obtained in [17] from closure of the algebra of super-symmetry transformations on the spacetime fields, the super field formulation of tensorial hierarchy may be useful as it clarifies the structure of the theory. In particular, it provides a basis for the search [22] for supersymmetric generalization of the action of [20] and, hopefully, can provide an insight in looking for a (2,0) superconformal theory by superspace methods.

2. Non-Abelian $p$–forms in six dimensions

The (1,0) superconformal 6d field theories of [17, 18, 19] describe a hierarchy of non–Abelian scalar, vector and tensor fields $(Y_{ijr}, \phi^I, A_i^r, B_{I \mu}^r, C_{\mu \nu \rho \lambda}^r, A)$ and their supersymmetric partners. The upper indices $r, s, t = 1, \ldots, n_V$ enumerate vector multiplets. Generically, the gauge group is not semi-simple: it may have the structure of direct product and contain Abelian factors. The indices $I, J, K = 1, \ldots, n_T$ enumerate the tensor multiplets. The index $A$ is used to enumerate 4-forms, while the three forms are enumerated by the lower $r, s, t$ indices.

As in [24] we will use the differential form notation in which the bosonic field content can be described by spacetime differential $p$–forms (0-form corresponds to a scalar) as $(Y_{ijr}, \phi^I, A_i^r, B_{I \mu}^r, C_{\mu \nu \rho \lambda}^r, A)$. This is especially convenient for the description of tensorial hierarchy in superspace, because the differential form equations which do not involve the Hodge star operator can be manipulated without referring on what is the base manifold (supermanifold).

So let us define the generalized field strengths

\[ F^r = F_2^r = dA^r + \frac{1}{2} f_{rs}^t A^s \wedge A^t + g_{rI} B_{I2}^r, \]

\[ H_3^I = dB_{I2}^r + d_{st}^A A^s \wedge dA^t + \ldots + g^{Ir} C_{3r} , \]

\[ H_{4r} = dC_{3r} + \ldots + k_r^A C_{4A} \]

of the non-Abelian Yang-Mills, two form and three form potentials $A^r := A_1^r, B_{I2}^r$ and $C_{3r}$ by stating that they obey the Bianchi identities

\[ I_5^r := D F^r - g_{rI} H_3^I = 0 , \]

\[ I_4^I := D H_3^I - d_{st}^A F^s \wedge F^t - g^{Ir} H_{4r} = 0 , \]

\[ I_{5r} := D H_{4r} + 2 H_3^I \wedge F^s d_{sr} - k_r^A H_{5A} = 0 . \]

Here $f_{rs}^t = f_{[rs]}^t$, $d_{st}^A = d_{[st]}^A$ and $g_{rI}^I$ are constant and covariantly constant tensors. The list of their properties can be found in Appendix A as well as in the original paper [17]. The meaning of $f_{rs}^t$ is the structure constant of the gauge algebra, while $d_{st}^A$ defines the
nonlinear gauge field contribution to the 3–form field strength of the two form potential; \( g^r_I \), \( g^{Ir} \) and \( k^a_r \) are St"uckelberg couplings. We restrict ourselves by the case of tensorial hierarchy which allows for existence of an action, this is to say we assume the existence of the (not positively definite, Lorentz-type) metric \( \eta_{IJ} = \eta_{(IJ)} \), so that \( g^r_I = \eta_{IJ} g^{jr} \).

The knowledge of Bianchi identities \((2.4)-(2.6)\) is completely sufficient for the discussion below; we will not need the complete explicit expressions \((2.1)-(2.3)\) for the field strengths in terms of the non-Abelian p-form potentials (see \([17]\) and \([20]\) where the equations from \([17]\) are written in differential form notations). What we need is rather the explicit form of the covariant derivatives \( D \) which are used in \((2.4), (2.5), (2.6)\),

\[
\begin{align*}
D F^r := dF^r + F^t \wedge A^s d_I r^r, \\
D H_3^I := dH_3^I + H_3 J \wedge A_s d_I r^r J, \\
D H_4^r := dH_4^r - H_4 t \wedge A_s d_I r^r t.
\end{align*}
\]

In our notation the exterior derivative acts from the right, so that, \textit{e.g.}, \( d(F^r \wedge H_3^I) = F^r \wedge dH_3^I - dF^r \wedge H_3^I \).

Below we will consider the potential defined on the flat (1,0) D=6 superspace which we are going to describe now.

3. Tensor hierarchy in superspace

3.1 6d (1,0) superspace

The structure equations of flat 6D \( N = (1, 0) \) superspace \( \Sigma^{(6|8)} \) are

\[
\begin{align*}
dE^a &= -iE^{ai} \wedge E^b \gamma^a_{\alpha \beta} \epsilon_{ij}, \\
dE^{ai} &= 0.
\end{align*}
\]

Here \( E^a \) and \( E^{ai} \) denote 6 bosonic and 8 fermionic supervielbein 1–forms of \( \Sigma^{(6|8)} \), \( \epsilon_{ij} = -\epsilon_{ji} \) is normalized by \( \epsilon^{12} = 1 = -\epsilon_{12} \) and

\[
\gamma^a_{\alpha \beta} = -\gamma^a_{\beta \alpha} = \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} \tilde{\gamma}^{a \gamma \delta}
\]

are \( SO(1,5) \) Klebsh-Gordan coefficients (generalized Pauli matrices) which obey

\[
\begin{align*}
\gamma^a \gamma^b + \eta^{ab} \delta^\alpha_\beta &= 2\eta^{ab} \delta^\alpha_\beta, \\
\eta^{ab} &= \text{diag}(+,-,-,-,-), \\
\gamma^a \gamma^b &= -4\delta^a_\beta \gamma^\delta_\alpha, \\
\gamma^a \gamma^b &= -2\epsilon_{\alpha \beta \gamma \delta} \gamma^\delta_\alpha.
\end{align*}
\]

Notice that

\[
\begin{align*}
\gamma^{abc}_{\alpha \beta} &= \gamma^{abc}_{(\alpha \beta)} = \frac{1}{3!} \epsilon^{abcdef} \gamma_{def} \alpha \beta, \\
\gamma^{abc}_{\alpha \beta} &= \gamma^{abc}_{(\alpha \beta)} = -\frac{1}{3!} \epsilon^{abcdef} \gamma_{def} \alpha \beta
\end{align*}
\]

are self–dual and anti–self–dual, respectively, with respect to their antisymmetrized vector indices, and provide the complete basis for the symmetric 4 x 4 matrices with, respectively, two lower case and two upper-case 4-valued spinor indices \( \alpha, \beta, ... = 1, ..., 4 \). The other useful relations can be found in the Appendix B.
The structure equations can be easily solved by

\[ E^a = dx^a - i d\theta^\alpha \gamma^a_\alpha, \quad E^{ai} = d\theta^{ai}, \quad (3.5) \]

where \( Z^M = (x^m, \theta^{ai}) \) are local coordinates of \( \Sigma^{(6|8)} \) and \( \theta^\beta_i := \epsilon_{ij} \theta^{\beta j} \) so that \( d\theta^\alpha \gamma^a_\alpha = d\theta^{ai} \theta^{\beta j} \epsilon_{ij} \gamma^a_\alpha \beta \).

### 3.2 Constraints for the superspace field strengths

When our field strengths, and corresponding non-Abelian \( p \)-form potentials, are differential forms on superspace \( \Sigma^{(6|8)} \), they can be decomposed on the basis of wedge products of the supervielbein forms \( (3.5) \), \( E^A = (E^a, E^{ai}) := dZ^M E^A(Z) \),

\[ \mathcal{F}^r = \frac{1}{2} E^B \wedge E^A \mathcal{F}^r_{AB}(Z), \quad \mathcal{H}_r^I = \frac{1}{3!} E^C \wedge E^B \wedge E^A \mathcal{H}_{ABC}(Z), \quad \mathcal{H}_{4r} = \frac{1}{4!} E^D \wedge E^C \wedge E^B \wedge E^A \mathcal{H}_{ABCD} r(Z). \quad (3.6) \]

To restrict the huge field content of the generic super–\( p \)-form potentials to the fields of the \((1,0)\) superconformal theory of \([17] \), we impose the following set of constraints

\[ \mathcal{F}^r = iE^b \wedge E^{ai}(\gamma_b W^i_\alpha)_\alpha + \frac{1}{2} E^b \wedge E^a \mathcal{F}^r_{ab}, \quad (3.7) \]

\[ \mathcal{H}_3^I = \frac{i}{2} E^b \wedge E^{ai} \wedge E^{\beta j} \gamma_{\beta a} \epsilon_{ij} \Psi^I + \frac{i}{2} E^b \wedge E^a \wedge E^{ai}(\gamma_{ab} \Psi^I)_\alpha + \frac{1}{3!} E^c \wedge E^b \wedge E^a \mathcal{H}_{abc}^I, \quad (3.8) \]

\[ \mathcal{H}_{4r} = - \frac{i}{3!} E^c \wedge E^b \wedge E^a \wedge E^{ai}(\gamma_{\alpha a} \beta)_{\beta a} W^i_\alpha d_{Ia} \Phi^I + \frac{1}{4!} E^d \wedge E^c \wedge E^b \wedge E^a \mathcal{H}_{abcd} r. \quad (3.9) \]

Here \( W^i_\alpha \) and \( \Psi^I \) are fermionic spinorial superfields, while \( \Phi^I, \mathcal{F}^r_{ab} = \mathcal{F}^r_{ab}, \mathcal{H}^I_{abc} = \mathcal{H}^I_{[abc]} \) and \( \mathcal{H}_{abcd} = \mathcal{H}_{[abcd]} \) are bosonic scalar and antisymmetric tensor superfields which at this stage can be considered unrestricted. The leading components of these superfields will give rise to the physical fermionic fields of vector and tensor multiplets \( (\chi^\alpha r(x) = W^i_\alpha r_{\theta=0} \text{ and } \psi^I(x) = \Psi^I_{\theta=0}), \) to the scalar field of the tensor multiplet \( (\phi^I(x) \propto \Phi^I_{|\theta=0}), \) and to the field strengths of the vector gauge fields and of the higher form potentials of the tensorial hierarchy.

Actually, the above expressions for the superform field strengths collect the independent constraints together with some of their consequences. In particular, the true constraints on the supersymmetric Yang–Mills (SYM) field strength are \( \mathcal{F}^r_{ai} \beta_j = 0 \), which imply

\[ \{ D_{ai}, D_{bj} \} = 2i \epsilon_{ij} \gamma^a_\alpha \beta D_a. \quad (3.10) \]

Then the expression for the field strength 2-form reflecting this constraint is \( \mathcal{F}^r = iE^b \wedge E^{ai} \mathcal{F}^r_{ai} b + \frac{1}{2} E^b \wedge E^a \mathcal{F}^r_{ab} \). The fact that \( \mathcal{F}^r_{ai} b = i(\gamma_b W^r_\alpha)_\alpha \), as it is read from \( (3.7) \), follows as the solution of the selfconsistency conditions given by Bianchi identities.

As another example, the true constraints in Eq. \( (3.9) \) are \( \mathcal{H}^I_{\alpha i} \beta_i \mathcal{F}^r \tau = 0 \), while the expression for \( \mathcal{H}_ai \beta \mathcal{F}^r \), presented in the first term of Eq. \( (3.9) \), is obtained by studying their selfconsistency conditions given by the Bianchi identities\(^1\).

\(^1\) Actually, one can guess the possible structure of the first nonvanishing term in Eq. \( (3.9) \), write it with an arbitrary coefficient \( il \), and fix \( l = -\frac{1}{2} \) by studying the Bianchi identities.
3.3 Equations of motion from consistency of the superspace constraints

Studying the Bianchi identities (2.4) with the constraints (3.7) and (3.8) we find the structure of covariant derivatives of fermionic superfield of the SYM sector,

$$D_{\alpha i} W^r_j = \delta_{\alpha \beta} \left( Y_{ij} - \frac{1}{2} \epsilon_{ij} \Phi^r_I g_I^r \right) - \frac{1}{2} \epsilon_{ij} F^{ab r} (\gamma_{ab})_{\alpha \beta}, \quad (3.11)$$

as well as the relations

$$D_{\alpha i} F^r_{ab} = -2i \gamma_{[\alpha \beta} D_{\beta i} W^r_j + i \gamma_{ab} \beta \Psi^I \gamma_I g_I^r, \quad (3.12)$$

$$3 D_{[c} F^r_{ab]} = H^I_{abc} g_I^r. \quad (3.13)$$

Eq. (3.11) is equivalent to the following set of equations

$$D_{\alpha i} W^r_{\alpha i} = 4 \Phi^I g_I^r, \quad (3.14)$$

$$D_{\alpha i} W^r_j = \delta_{\alpha \beta} Y_{ij}, \quad (3.15)$$

$$F^r_{ab} = -\frac{1}{8} (\gamma_{ab})_{\alpha \beta} D_{\beta i} W^r_{\alpha i} \quad (3.16)$$

and actually these are obtained from the Bianchi identities. Notice that Eqs. (3.14) and (3.16) relate the superfields already present in (3.7) and (3.8) with higher components of the fermionic superfield $W^r_j$ which remains indefinite when studying the Bianchi identities.

Eq. (3.13) indicates that $F^r_{ab}$ superfield is a (generalized) field strength of a vector (super)field potential and Eq. (3.12) shows that all the higher components of $F^r_{ab}$ are present in fermionic superfields entering (3.7) and (3.8). Furthermore, using (3.12) in calculating the right hand sides (r.h.s.) of

$$\{D_{\alpha i}, D_{\beta j}\} W^r_{W_k} = 2i \epsilon_{ij} \gamma^r_{\alpha \beta} D_{\alpha} W^r_k = \delta_{\beta \gamma} \left( D_{\alpha i} Y_{jk} - \frac{1}{2} \epsilon_{jk} D_{\alpha i} \Phi^I g_I^r \right) + ((\alpha i) \leftrightarrow (\beta j)) - \frac{1}{2} \epsilon_{jk} D_{\alpha i} F^{ab r} (\gamma_{ab})_{\beta \gamma} + ((\alpha i) \leftrightarrow (\beta j)) \quad (3.17)$$

we find, after some algebraic manipulations,

$$i (\gamma^r_{\alpha} D_{\alpha} W^r_i)_\alpha = \frac{1}{3} D_{\alpha i} Y_{ij} + D_{\alpha i} \Phi^I g_I^r, \quad (3.18)$$

$$D_{\alpha i} Y_{jk} = 0, \quad (3.19)$$

$$D_{\alpha i} \Phi^I g_I^r = 2i \Psi^I \gamma_I g_I^r. \quad (3.20)$$

This set of equations indicates that Eq. (3.7) itself describes the off–shell constraints of the 6d (1,0) SYM model. Indeed, Eq. (3.18) implies that the l.h.s. of the Dirac equation for gaugino appears as a second component of the auxiliary superfield $Y_{ij}^r$, the leading component of which is the auxiliary field of the SYM supermultiplet.

Taking a look from the other side, Eqs. (3.18) and (3.19) can be collected in the following expression for the fermionic covariant derivative of the auxiliary superfield $Y_{ij}^r$

$$D_{\alpha i} Y_{jk} = 2i \epsilon_{ij} \left( \gamma^r_{\alpha} D_{\alpha} W^r_k - 2 \Psi^I \gamma_I g_I^r \right)_\alpha, \quad (3.21)$$
which shows that higher components of the auxiliary superfield are expressed through the leading components of already introduced basic superfields, i.e. through the fields of \((1,0)\) superconformal theory of \([17]\).

Passing to Binachi identities \((2.5)\), from its lowest dimensional \((\text{dim } 5/2)\) nontrivial component we find

\[
0 = \gamma_{\alpha i}^a \gamma_{\beta j}^b \gamma_{\gamma k}^c \left( \eta_{ab} D_{\alpha i} \Phi^I + 2i \gamma_{ab} \alpha^\beta \Psi^I_{\beta i} \right) + \text{cyclic permutation of } (\alpha, \beta, \gamma) .
\]

Its general solution is

\[
D_{\alpha i} \Phi^I = 2i \Psi^I_{\alpha i} .
\]  
(3.22)

The dim 3 component of the Bianchi identity \((2.5)\) gives the following set of equations for the fermionic derivative of the fermionic superfield \(\Psi^I_{\alpha i}\):

\[
\gamma_{\alpha i}^a \gamma_{\beta j}^b \frac{1}{8} \gamma_{\alpha i}^a D_{\beta i} \Psi^I_{\alpha i} = \mathcal{H}^I_{\alpha i} = \frac{1}{2} (\mathcal{H}^I_{\alpha i} - * \mathcal{H}^I_{\alpha i})
\]

and

\[
\mathcal{H}^I_{\alpha i} := \frac{1}{2} (\mathcal{H}^I_{\alpha i} + * \mathcal{H}^I_{\alpha i}) = -\frac{i}{4} d^i_{st} W^{si} \gamma_{\alpha i} W^t .
\]

(3.27)

Eq. \((3.24)\) is a generalization of the anti-self-duality conditions which includes fermionic contributions. Thus studying the Binachi identities we have obtained a dynamical equations from the superspace constraints \((3.8), (3.9)\). Hence, in distinction to \((3.7)\) these latter are \(on-shell\) constraints.

Contracting Eq. \((3.24)\) with \(\tilde{\gamma}^c c_2 c_3 \alpha \beta\), after some algebra we find from its irreducible parts

\[
\tilde{c} \alpha \beta D_{\alpha i} (\Psi^I_{\beta j}) = 2i W^s_i \gamma^c W^t_j d^l_{st} , \quad \tilde{c} \alpha \beta D_{\alpha i} (\Psi^I_{\beta j}) = 0 .
\]

(3.28)

These equations imply

\[
D_{\alpha i} (\Psi^I_{\beta j}) = -\frac{i}{2} \gamma_{\alpha i} \gamma_{\beta j} W^s_i \gamma_{\alpha i} W^t_j ,
\]

(3.29)

Furthermore, Eqs. \((3.24)\) and \((3.26)\) can be written in the form of \(D_{[\alpha i} \Psi^I_{\beta j]} = -\gamma_{\alpha i} \gamma_{\beta j} D_a \Phi^I\) and \(D_{[\alpha i} \Psi^I_{\beta j]} = -\frac{1}{6} \gamma_{\alpha i} \gamma_{\beta j} \mathcal{H}^- I\) so that

\[
D_{\alpha i} \Psi^I_{\beta j} = -\gamma_{\alpha i} \gamma_{\beta j} D_a \Phi^I - \frac{1}{6} \gamma_{\alpha i} \gamma_{\beta j} \mathcal{H}^- I
\]

(3.30)
Eqs. (3.30) and (3.29) together imply
\[ D_{\alpha i} \Psi^I_{\beta j} = \frac{1}{12} \epsilon_{ij} \gamma^{abc} H^{-I}_{abc} + \frac{1}{2} \epsilon_{ij} \gamma^{a}_{\alpha \beta} D_{\alpha} \Phi^I - \frac{i}{2} \gamma^{a}_{\alpha \beta} W^{s}_{i} \gamma_{a} W^{I}_{J} d^{I}_{s t}, \]  
(3.31)
where in the first term the superscript \(-\) can be actually omitted as far as only the anti-self dual part of the 3-rank field strength contributes to \(\gamma^{\alpha \beta}_{abc} H^{-}_{abc}\).

The dim 7/2 component of (2.5) determines the fermionic derivative of the 3-rank antisymmetric tensor superfield,
\[ D_{\alpha i} H^I_{abc} = 3i \gamma^{[abc} D_{\alpha} \Psi^I_{\beta i} + 6i F^s_{[ab}(\gamma_c) W^{I}_{d s t} \gamma_{a} W^{s}_{I} d^{I}_{s t} - i \gamma_{abc\alpha\beta} W^{\tilde{s}}_{I} \Phi^{J} d_{I} dr g^{J r}. \]  
(3.32)
Finally, the dim 4 component of (2.5) reads
\[ D_{[a} H^I_{bcd]} = \frac{3}{2} d^{I}_{s t} F^s_{[ab} F^{I}_{cd]} + \frac{1}{4} g^{I} H_{abcd r}. \]  
(3.33)
and implies that \(H^I_{abc}\) is a generalized field strength.

Now let us observe that (3.31) implies
\[ \{ D_{\alpha i}, D_{\beta j} \} \Psi^I_{\gamma k} = 2i \epsilon_{ij} \gamma^{a}_{\alpha \beta} D_{\alpha} \Psi^I_{\gamma k} = \]
\[ = \frac{1}{12} \epsilon_{jk} \gamma^{abc}_{\beta \gamma} D_{\alpha} H^{-I}_{abc} - i(\gamma^{a}_{\alpha \beta} D_{\alpha} W^{I}_{J} \gamma^{s}_{k} d^{I}_{k} r a + (\alpha i \leftrightarrow \beta j) + i \epsilon_{jk} \gamma^{a}_{\alpha \beta} D_{\alpha} \Psi^I_{\gamma i} + i \epsilon_{ij} \gamma^{a}_{\alpha \beta} d^{I}_{I} dr + (\alpha i \leftrightarrow \beta j) . \]  
(3.34)
To simplify the terms in the last line we have used the commutation relation
\[ [D_{\alpha i}, D_{\alpha}] \Phi^{I} = i(\gamma_{a} W^{I}_{i}) \alpha \Phi^{J} X_{r J} := 2i(\gamma_{a} W^{I}_{i}) \alpha \Phi^{J} (g^{J r} d^{I}_{r a} - g^{I} d_{r a} g^{J r}) . \]  
(3.35)
Substituting the expressions (3.32) and (3.31) into the fermionic derivatives of superfields in the r.h.s. of (3.34), after some algebra we obtain an equation one of the irreducible parts of which \((\times \epsilon_{ij} \varepsilon_{\alpha \beta} \gamma_{\delta})\) provides us with superfield generalization of Dirac equations for the fermions of tensorial multiplet,
\[ (\gamma^{a}_{\alpha \beta} \gamma_{k} \Phi^{I} \delta) = \frac{1}{2} (\gamma^{ab} W^{J}_{i} \alpha \gamma^{c} d^{I}_{r s} F^{s}_{ab} + \gamma^{s} \gamma_{i} W^{s} d^{I}_{r s} + \frac{1}{2} W_{i}^{s} d^{I}_{r s} \Phi^{J} (-g^{s} d_{r s} + 4 d_{r s} g^{I} s). \]  
(3.36)
The other irreducible parts of the above mentioned equation (symmetric in \((\alpha \beta)\) and \((i j)\)) are satisfied identically due to \(\gamma^{c} \gamma^{de I} \gamma_{ab} \gamma_{c} = -4 \gamma_{a} \gamma^{de I} \gamma_{b l}\), which one can easily prove using the gamma matrix algebra.

Now let us turn to the Bianchi identities (2.6). To this end we need the constraints for the 5-form \(H_{5 A}\) which we assume to be
\[ H_{5 A} = \frac{1}{4!} E^{d} \wedge E^{c} \wedge E^{b} \wedge E^{a} \wedge E^{\alpha i} H_{\alpha i a b c d} A + \frac{1}{5!} E^{c} \wedge E^{d} \wedge E^{c} \wedge E^{b} \wedge E^{a} H_{a b c d e} A . \]  
(3.37)
Then dim 7/2 and lower components of (2.6) are satisfied identically, \(^2\) while its dim 4 component, after some algebra, can be presented in the form

\[
0 = i\epsilon_{ij} \gamma^d_{\alpha\beta} \left( \mathcal{H}_{abcd} r - \frac{1}{2} \epsilon_{abcdef} F^{ef} s \Phi^I_{dIr} - \frac{i}{2} \epsilon_{abcdef} (W^{ks} \gamma^{ef} \Psi^I_k)_{dIr} \right) - i\gamma_{abcd\alpha\beta} (\Upsilon^s_{ij} \Phi^I_{dIr} - 2iW^s_{(i} \Psi^I_{j)}_{dIr} ) .
\]

(3.38)

Clearly, the first and the second line of Eq. (3.38) belong to different irreducible representations of \(SO(1,5)\) and thus vanish separately. Hence we have found the duality equation deformed by fermionic contribution,

\[
\mathcal{H}_{abcd} r - \frac{1}{2} \epsilon_{abcdef} F^{ef} s \Phi^I_{dIr} = \frac{i}{2} \epsilon_{abcdef} (W^{ks} \gamma^{ef} \Psi^I_k)_{dIr}
\]

(3.39)

and the superfield generalization of the equations for auxiliary scalar field of the SYM multiplet, also involving the fermionic superfields,

\[
\mathcal{E}_{ij r} := (\Upsilon^s_{ij} \Phi^I_{dIr} - 2iW^s_{(i} \Psi^I_{j)}_{dIr} ) = 0 .
\]

(3.40)

The next-to leading component of this auxiliary superfield equation gives the fermionic equation of motion. Indeed, after some algebra one finds that the symmetric in \((ijk)\) part of the equation \(D^a \Phi^I_{dIr} = 0\) is satisfied identically due to the first equation in (A.1), while the remaining irreducible part,

\[
\phi_{ij r} := \left( \gamma^a \mathcal{D}^a H^s_{W_i} \right) d_{Ir} + 2iW^s_{(i} \Psi^I_{j)}_{dIr} = 0 .
\]

(3.41)

The same equation can be obtained from (3.38) by multiplying it by \(\Phi^I_{dIr}\) and using Eqs. (3.40), (3.11) and (3.31) as well as the first equation in (A.1).

Dim 9/2 component of (2.6) gives the expression for the fermionic derivative of the 4th rank antisymmetric tensor superfield,

\[
\mathcal{D}_{(a} H_{bcde)} r = -12(\gamma^{abc} W^s_{i})_{a} r d_{Ir} - \frac{1}{2} (\gamma^{a} W^s_{i})_{a} \mathcal{D}_{a} \Phi^I d_{Ir} - \frac{i}{2} \epsilon_{abcdef} (W^{ks} \gamma^{ef} \Psi^I_k)_{dIr} + 4iD_{[a} (\Phi^I_{dIrs} W^s_{i} \gamma^{rs})_{dIrs} + \frac{i}{2} \epsilon_{abcdef} A_{kr}^A .
\]

(3.42)

One can state that Eq. (3.42) completely determines the fermionic derivative of the 4th rank antisymmetric tensor superfield \(\mathcal{H}_{abcd} r\) only if we express \(\mathcal{H}_{a_{1}abcdA} A_{kr}^A\) in terms of

\(^2\)Actually, the dim 7/2 component can be used to fix \(l = -1/6\) if we start with the constraint (3.8) with coefficient \(il\) for the first term, as it was discussed in the footnote 1.
already known superfields. To do this, it is convenient to study the Bianchi identities for
the 5-form field strength. According to \cite{17} they read
\[ I_{6A} := D\mathcal{H}_{5A} + c_{AIJ}\mathcal{H}_3^I \wedge \mathcal{H}_3^J + c_{A,B}\mathcal{F}^s \wedge \mathcal{H}_4^r + \ldots = 0 , \tag{3.44} \]
where \ldots denote the possible term which vanish when contracted with \( k_{A}^r \). The consistency
condition for the Bianchi identity (3.44), 'identity for identity' implies that
\[ DI_{6A} = 0 \iff DD\mathcal{H}_{5A} = -\mathcal{H}_{5B} \wedge \mathcal{F}^r X_{rA}^B \tag{3.45} \]
With our constraints the first nontrivial component equation in (3.44) has dimension 9/2.
It can be solved by
\[ \mathcal{H}_{ai\,abcd\,A} = i(\gamma_{abcd}\Psi_i^a)\Phi^I c_{AIJ} = \frac{i}{2}\epsilon_{abcdf}(\gamma^{ef}\Psi_i^f)\Phi^I c_{AIJ} . \tag{3.47} \]
Then the next, dim 4 component of (3.44) produces a supersymmetric generalization of
the duality relation between 4 form potential and scalar,
\[ \mathcal{H}_{abcd\,A} = \frac{1}{2}\epsilon_{abcdf} \left( c_{AIJ} \left( \Phi^{[I}_f \Phi^J_{f]} - 2i\Psi^{[I}_f \gamma^f_{I]} \Psi_i^J \right) - i c_{A,J[\{d\]}_{\{r} c_{s]}t]} \Phi^I W^{is} \gamma^f W_i^J \right) . \tag{3.48} \]
As we have already mentioned, (3.46) should be obeyed modulo terms which vanish
when contracted with \( k_{r}^A \). So, if we consider constraints (3.37) contracted with \( k_{r}^A \), their
consequence will contain the duality equation
\[ k_{r}^A \mathcal{H}_{abcd\,A} = \frac{1}{2}\epsilon_{abcdf} \left( \left( \Phi^{[I}_f \Phi^J_{f]} - 2i\Psi^{[I}_f \gamma^f_{I]} \Psi_i^J \right) k_{r}^A c_{AIJ} - i k_{r}^A c_{A,\{d\}_{\{r} \Phi^I W^{is} \gamma^f W_i^J \right) . \tag{3.49} \]
Notice that another \( SO(1, 6) \times SU(2) \) irreducible part of the dim 4 component of (3.44)
(the one \( \propto \epsilon_{ij} \gamma_{\alpha\beta}[a,(\ldots)_{bcd}] \) results in
\[ \left( \mathcal{H}_{bcd}^+ + \frac{i}{4}W^{k_s} \gamma_{bcd} W_k^t \left( d_{st} \right) c_{AIJ}\Phi^I \right) = 0 , \tag{3.50} \]
which is obeyed identically due to the supersymmetrized anti-self duality equation (3.27).
To obtain the first order bosonic equations (duality and anti-self-duality conditions)
derived from the closure of the supersymmetry algebra in \cite{17} (and, in their purely bosonic
limit, from the action in \cite{20}), we have to identify the leading components of the antisym-
metric tensor superfield with field strengths, \( \Upsilon_{ij|\theta=0} = \Upsilon_{ij}^r \) and \( \Phi^I|\theta=0 = 2\phi^I \), where the
only nontrivial coefficient appears.

The second order bosonic equations can be obtained from the (self)duality relations and
the purely bosonic higher dimensional components of the Bianchi identities. On the

\[ ^{3}\text{Actually the 7-form 'identity for identity' } J_{7A} = 0 \text{ reduces to } DI_{6A} = 0 \text{ when the lower form Bianchi}
\text{identities are satisfied. See Appendix C for its complete form.} \]
other hand, they can be obtained from the next-to-leading components of the superfield fermionic equations. As far as the fermionic equations, in their turn, can be obtained by acting on the duality equations by fermionic covariant derivatives, the coincidence of the second order bosonic equations obtained in this two ways provides an additional check of the consistency of our constraints or, in other words, of the equivalence of the superspace constraints and the spacetime component equations of (1,0) superconformal theory of [17]. Actually it is a superspace counterpart of checking the closure of supersymmetry algebra on the spacetime component equations, which was done in [17]. Below we perform a bit simplified version of this consistency check, following mainly the bosonic superfield contributions to the second order bosonic equations.

3.4 Second order bosonic equations and check of consistency of our superspace description of the (1,0) superconformal theory

To begin the final check of consistency of our superspace description, let us discuss how the fermionic equations can be obtained from the duality and anti-self-duality conditions.

3.4.1 Fermionic equations of motion from duality and anti-self-duality conditions

The tensor multiplet fermionic equation provided by the leading component of the superfield equation (3.36), which we denote by $E^{I}_{i} = 0$, appears also as the next-to leading component of the generalized anti–self–duality condition (3.27), which we denote by $T_{abc}^{+I} = 0$. Indeed, acting on this latter superfield equation by $D_{\alpha i}$, after some algebra one finds that the only nontrivial irreducible part of $D_{\alpha i}T_{abc}^{+I} = 0$ is $\tilde{\gamma}_{abc}^{I}D_{\alpha i}E^{I}_{i} = 0$, which coincides with Eq. (3.36). The other irreducible parts of $D_{\alpha i}T_{abc}^{+I} = 0$ are satisfied identically: after the use of Eq. (3.11) and of the Bianchi identity (3.32), which we denote by $E_{Iiabc}^{I} = 0$, one finds that $D_{\alpha i}T_{abc}^{+I} = i\tilde{\gamma}_{abcO\beta}E^{I}_{i}$. Similarly, the vector multiplet fermionic superfield equation (3.41), $E^{I}_{\beta i}r = 0$, can be obtained from the duality equation (3.39), $I_{c1...c4r}^{I} = 0$. This fact is expressed by the following form of the 5-form Bianchi identity with all but one bosonic indices (3.42):

$$I_{ai c1...c4 r} = D_{ai}I_{c1...c4 r} + \frac{1}{2}\epsilon_{abc1...c4}I_{ai}^{ab s} d_{Isr} - i\frac{1}{2}\epsilon_{abc1...c4}^{\gamma_{ab}}_{\alpha \beta} E_{Ii r} = 0.$$ (3.51)

Here $I_{ai ab s} = 0$ is the dim 5/2 component of the gauge field Bianchi identity, Eq. (3.12).

One can also check that no new equations appear when acting by fermionic covariant derivative on the 5-form–scalar duality $k^{A}_r I_{c1...c5 A} = 0$ (3.49) and using the dim 11/2 Bianchi identity

$$k^{A}_r I_{aic1...c5 A} := k^{A}_r D_{ai}H_{c1...c5 A} + \ldots = 0.$$ (3.52)

To resume, we have shown that the fermionic superfield equations (3.41) and (3.36) can be obtained (also) by acting by fermionic covariant derivatives on the duality and anti-self-duality conditions. No additional restrictions on the physical fields, beyond the equations of (1,0) superconformal theories of [17] are produced at this stage.
3.4.2 Second order equation for 3-form (super)field strength

The second order equation for the 3-form field strength $H_{abc}$ can be found calculating the r.h.s. of the identity $D_c H_{abc} = -\frac{1}{3!} \epsilon_{abcdef} D_c H_{def} + 2 D_c H_{abc} + 1$ with the use of the generalized anti-self-duality equation \( (3.27) \) and the tensorial Bianchi identities \( (3.33) \). In such a way one arrives at

$$E^{abI} := D_c H_{abcI} + \frac{1}{4} \epsilon_{abcdef} F_{cd} F_{ef} d_{rs} + \frac{1}{4} \epsilon_{abcdef} H_{cddefr} g^{rf} + i D_c W^{si}_i \gamma_{abc} W_{i d} d_{st} = 0 \quad (3.53)$$

Furthermore, using \( (3.39) \) we can present this equation in the form

$$E^{abI} := D_c H_{abcI} + \frac{1}{4} \epsilon_{abcdef} F_{cd} F_{ef} d_{rs} - F^{abI} \Phi d_{is} g^{rf} +$$

$$+ i D_c W^{si}_i \gamma_{abc} W_{i d} d_{st} + i W^{st}_i \gamma_{abc} g_{st} d_{st} = 0 \quad (3.54)$$

The second order bosonic equations can be also obtained from the next-to leading components of the superfield fermionic equations. The coincidence of the results of these two calculations provides an additional check of the equivalence of our superspace constraints and of the spacetime component formalism of \( [17] \): at this stage some extra restrictions on our fields, beyond the spacetime component equations for the fields of (1,0) superconformal theory, could appear if the constraints were too strong.

Acting by fermionic covariant derivative on the fermionic equations \( (3.36) \), which we denote by $E^{abI} = 0$, one finds that $D_{ab} E^{ab} = \frac{1}{4} \epsilon^{ab} \Phi^I + \frac{1}{2} \epsilon^{ab} \delta^{ab} - 2 i \epsilon^{ab} \gamma_{ij} g^{rf}$, where $E_{ij} = 0$ is the auxiliary superfield equation \( (3.40) \), $E^I = 0$ is the scalar superfield equation

$$E^I := \Box \Phi^I - J_{ab} F^{ab} d_{rs} - \frac{3}{2} \Phi^I g_r^s \Phi^s - \frac{1}{2} \Phi^I g_r^s - \frac{1}{2} \Phi^I g_r^s +$$

$$+ i D_a W^{si}_i \gamma_{abc} W_{i d} d_{st} + i W^{st}_i \gamma_{abc} (g_{is} d_{st} - g_{st} d_{st}) = 0 \quad (3.55)$$

and $\tilde{E}^{abI} = 0$ reads

$$\tilde{E}^{abI} := D_c H_{abcI} + \frac{1}{4} \epsilon_{abcdef} F_{cd} F_{ef} d_{rs} - \frac{1}{4} \epsilon_{abcdef} H_{cddefr} g^{rf} +$$

$$+ \text{fermions} = 0 \quad (3.56)$$

For simplicity, from now on we will mainly follow the bosonic superfield contributions to the second order bosonic (super)field equations.

On the first look, the tensorial equation \( (3.56) \), $\tilde{E}^{abI} = 0$, differs from \( (3.53) \), $E^{abI} = 0$, as far as the fourth term in the former is absent in the latter. However, a more close look permits to notice also the difference in the sign in front of the third terms, $\pm \frac{1}{4} \epsilon^{ab} H_{cddefr} g^{rf}$, and to appreciate that actually these equations coincide modulo the duality equation \( (3.33) \), $T_{abcd} = 0$. Resuming,

$$\tilde{E}^{abI} = E^{abI} - \frac{1}{12} \epsilon^{ab} T_{cddefr} g^{rf} \quad (3.57)$$

This relation between the forms of the second order equations obtained from the superfield fermionic equation and directly from the selfduality condition has an interesting consequence. As far as the superfield fermionic equation \( (3.36) \) itself can be obtained by acting by the fermionic covariant derivatives on \( (3.27) \) (see sec. 3.4.1), we can state that also the duality conditions \( (3.33) \) projected on $g^{rf}$, $T_{cddefr} g^{rf} = 0$, can be obtained from the generalized anti-self-duality superfield equation \( (3.27) \).
3.4.3 Second order equation for the gauge (super)field strength

Similarly, from the duality equation (3.39) we find
\[ \mathcal{E}_r^b := d_{irs} D_a (\Phi^I F^{ab}_s) + \frac{1}{3!} \epsilon^{bcdefg} F^a_{cd} H^{il}_{efg} d_{irs} - \frac{1}{5!} \epsilon^{bcdefg} H_{cdefg} A k^r_r + i D_a (W^{is} \gamma^{ab} \psi_i^J d_{irs}) = 0, \tag{3.58} \]
and then, using (3.48),
\[ \mathcal{E}_r^b = d_{irs} D_a (\Phi^I F^{ab}_s) + \frac{1}{3!} \epsilon^{bcdefg} F^a_{cd} H^{il}_{efg} d_{irs} - \frac{1}{2} k^r_r c_{AIJ} \Phi^{[I} D^b \Phi^{J]} + i k^r_r c_{AIJ} \psi_i^J \psi_i^J + \frac{i}{2} k^r_r c_{AIJ} u_{[d} t_{]J} \Phi^I W^{is} \gamma^J W^r_t + i D_a (W^{is} \gamma^{ab} \psi_i^J d_{irs}) = 0. \tag{3.59} \]

On the other hand, let us consider the equation \( D_{\beta j} E_{\alpha i}^r = 0 \) obtained by acting by the fermionic covariant derivatives on the fermionic superfield equation of motion (3.18), \( \mathcal{E}_{\alpha i}^r = 0 \). Using (3.11), (3.31) and (3.13) we find that the SU(2) tensorial part of this equation, \( D_{\beta j} E_{\alpha i}^r = 0 \), is satisfied identically, while the SU(2) singlet part, \( \epsilon^{ij} D_{\beta j} E_{\alpha i}^r = 0 \), gives rise to the self-dual part of the bosonic Bianchi identity (3.13) (from \( D_{(\beta} E_{\alpha) i}^r = 0 \)) and to
\[ D^a F^b_{ab} + \frac{1}{2} D_b \psi^I g^r_t + \frac{i}{2} W^{is} \gamma^J W^r_t + i \alpha^{\alpha} \beta \gamma D^a \beta \gamma d_{irs} = 0. \tag{3.60} \]

Although formally this looks like (the superfield generalization of) the interacting gauge field equation of motion it contains the term with auxiliary superfield \( \Upsilon^r_{ij} \). This reflects the off-shell nature of the SYM part of our constraints. In the interacting system the other constraints result in that \( \Upsilon^r_{ij} \) must be a solution of the algebraic equation (3.40). However, to use this equation, and thus to make Eq. (3.60) dynamical, we should multiply it by \( \Phi^I d_{irs} \). Then, using the Leibnitz rule to move the fermionic covariant derivatives in the last term of this equation, as well as Eqs. (3.40), (3.11), (3.31), (3.12), (3.22) and (3.18), after some algebraic manipulation we arrive at
\[ 0 = d_{irs} D^a (\Phi^I F^r_{ab}) + \frac{1}{3!} \epsilon^{bcdefg} F^c_{d} H^{efg} d_{irs} - \Phi^I D_b \psi^I g^r_t d_{irs} + \text{fermions}, \tag{3.61} \]

Notice that the above mentioned transformations have resulted in appearance of the second term \( \propto \epsilon F^r H^{-1} d_{irs} \), absent in (3.60), and in antisymmetrizing the indices of the product of invariant tensors in the coefficient for the scalar current, \( g^r_t d_{irs} \leftrightarrow g^r_t d_{irs} \). Now, the terms presented in (3.61) coincide with the ones in (3.59) because of the property \( g_r^t d_{irs} = \frac{1}{2} k^r_r c_{AIJ} \) (see (A.1)), and as far as, due to (3.27), \( H^efg = H^{-1} \epsilon^{efg} + \text{fermions} \).

3.4.4 Scalar (super)field equation and 5-form duality condition

The second order equation for scalar superfield (3.55) has been obtained above form the fermionic superfield equation of motion (3.39). On the other hand, let us consider the duality equation (3.49), which we denote by \( \mathcal{I}_{c_1 \ldots c_5} A_k^r = 0 \). Taking the covariant divergence of its Hodge dual, \( \frac{1}{\epsilon^{bc_1 \ldots c_5}} \mathcal{I}_{c_1 \ldots c_5} A_k^r = 0 \), and using the pure bosonic part of the Bianchi
identities \((3.44)\),

\[
\frac{1}{5!} \epsilon_{bc_1...c_5} D_6 \mathcal{H}_{c_1...c_5} A k^A_r = \frac{1}{3! \cdot 3!} \epsilon^{bcdefg} \mathcal{H}_{bed}^I \mathcal{H}_{efg}^J k^A_r c_{AIJ} + \frac{1}{2! \cdot 4!} k^A_r c_{AIJ} \epsilon^{abc_1...c_4} \mathcal{F}_{ab} s \mathcal{H}_{c_1...c_4} r,
\]

as well as Eqs. \((3.39)\) and \((3.27)\), we arrive at

\[
0 = \frac{1}{2} k^A_r c_{AIJ} \Phi^I \left( \Box \Phi^J - d_I^J \mathcal{F}_{ab} \mathcal{F}^{ab} t \right) + \text{fermions}. \tag{3.62}
\]

Notice that \(\epsilon^{bcdefg} \mathcal{H}_{bed}^I \mathcal{H}_{efg}^J = \epsilon^{bcdefg} \mathcal{H}_{bed}^I \mathcal{H}_{efg}^J = \text{fermions}\). This is the case due to Eq. \((3.27)\), which implies \(\mathcal{H}_{bed}^I = \mathcal{H}_{bed}^J + \text{fermions}\), and the identity \(\epsilon^{bcdefg} \mathcal{H}_{bed}^I \mathcal{H}_{efg}^J = -6 \mathcal{H}_{bed}^I \mathcal{H}_{efg}^J = 0\).

Substituting \((3.55)\) and following, for simplicity, only the contributions of the bosonic superfields, one finds that \((3.63)\) is equivalent to

\[
0 = -\frac{3}{2} k^A_r c_{AIJ} \Phi^J g_j^r \Phi^K g_k^s d_I^J + \frac{1}{2} k^A_r c_{AIJ} \Upsilon^r_{ij} \Upsilon^i j s d_I^J + \text{fermions}.
\]

Using the properties of the invariant tensors in \((A.1)\) one can find that this is indeed the case; namely, the first term in this equation vanishes identically, while the second is expressed through the fermionic bilinears with the use of auxiliary superfield equation \((3.40)\).

Thus we have obtained the second order bosonic equations by acting by bosonic derivatives on the duality conditions, and also, following mainly the bosonic superfield contributions, by acting by fermionic derivative on the fermionic superfield equations. The bosonic equations obtained on these two ways are equivalent; no additional restrictions on physical fields appears. This procedure is a superfield counterpart of searching for closure of supersymmetry algebra on the spacetime component equations performed in \([17]\).

### 3.5 Summary

Thus we have performed the complete investigation of the Bianchi identities \((2.4)\), \((2.5)\), \((2.6)\), \((3.44)\) with our superspace constraints \((3.7)\), \((3.8)\), \((3.9)\), \((3.37)\), \((3.47)\), have studied the consequences of this solution and found that our superspace constraints describe the \((1,0)\) superconformal theory of \([17]\).

All the physical fields of this \((1,0)\) superconformal theory appear as leading components of the fermionic and bosonic main superfields, \(W^i_{\alpha r}, \Psi_{\alpha i}, \Phi^I, \mathcal{F}_{ab}^r, \mathcal{H}_{abc}^I, \mathcal{H}_{abcd r}, \mathcal{H}_{abcde} A\), which enter the differential form representation of our constraints \((2.7)\), \((3.37)\), \((3.47)\). The auxiliary field of the SYM multiplet enters as leading component in the auxiliary superfield \(\Upsilon^r_{ij}\) which appears as one of the irreducible parts of the fermionic derivative of the \(W^i_{\alpha r}\).

The complete solution of the Bianchi identities, which we have described above, gives us the relations between main superfields, including the anti-self-duality and duality relations

\footnote{Actually, the meaning of the term 'main superfield' in the superspace literature is usually more restrictive, but the wider treatment of it in this section cannot lead to any confusion.}
algebraic auxiliary superfield equation (3.40), as well as the expressions for the covariant derivatives of the main superfields.

The next stage consisted in obtaining the consequence of our solution, which has been done by studying the results of the action of fermionic covariant derivatives on the above described relations between the main superfields and their fermionic derivatives. (Actually, we did this for all the relations but the superfield generalization of pure bosonic tensorial Bianchi identities which are dependent as we discuss in Appendix C). We have obtained the superfield generalization of the fermionic equations of motion (3.41) and (3.36) by acting on the duality and self-duality relations by fermionic covariant derivatives. The superfield generalization of the second order bosonic equations can be obtained by acting by the bosonic derivative on the duality and anti-self-duality equations and by acting by the fermionic covariant derivatives on the fermionic superfield equations. We have shown (sometimes for simplicity following the bosonic superfield contributions on one of two ways) that the purely bosonic equations obtained on these two ways coincide. This final check can be considered as a superfield counterpart of the closure of supersymmetry algebra on the equations.

To resume, our superspace constraints on the field strengths of the tensorial hierarchy (2.1)–(2.6) restrict the field content of the superfields to the fields of the (1,0) superconformal theory of [17], produce exactly the same equations of motion for the physical fields as were obtained from the closure of (1,0) supersymmetry algebra in [17], do not produce other restrictions on the physical fields of the (1,0) superconformal theory and, hence, are equivalent to the spacetime component equations of motion of this found in [17].

4. Conclusions

Thus we have shown that all the dynamical equations for the bosonic fields of the 6D (1,0) superconformal theories of [17] can be obtained from the superspace constraints (3.7), (3.8), (3.9) and (3.37). Instructively, these dynamical equations were obtained in the form of superfield duality and anti-self-duality conditions (3.27), (3.39) and (3.49).

We have shown that the superfield generalization of the fermionic equations can be obtained from these first order duality equations by acting by the fermionic covariant derivatives, while a suitable action by the bosonic covariant derivative produces the (superfield generalization of the) second order bosonic field equations. Following mainly the bosonic superfield contributions, we have also obtained the above mentioned second order bosonic equations by acting by fermionic derivatives on the fermionic superfield equations. This has been an additional consistency check, designed to convince the reader that our superspace constraints do not impose any additional condition on the fields of the (1,0) superconformal theory, but only the spacetime field equations of [17], has confirmed that our superfield formalism is equivalent to the spacetime component description of the 6D (1,0) superconformal theory developed in [17].

The superspace realization of the tensorial hierarchy of [17] developed in this paper clarifies the structure and provides a new look on the (1,0) superconformal theory of [17]. It can be useful in the search [22] for the supersymmetric generalization of the purely bosonic
action of [20] and hopefully, can provide an insight in the quest for a hypothetical (2,0) superconformal theory related to multiple M5–brane system. Probably to this end it will be useful to understand better the possible relation of our constraints with the 6d twistor approach of [23, 24].

Acknowledgments. The author is thankful to Dima Sorokin and Henning Samtleben for collaboration on related projects [20, 22] and encouraging comments. This work was supported in part by the research grant FPA2012-35043-C02-01 from the MEC of Spain, by the Basque Government Research Group Grant ITT559-10 and by the UPV/EHU under the program UFI 11/55. The hospitality and support of the Theoretical Department of CERN at very final stages of this work is greatly acknowledged.

A. Algebraic constraints on the constant tensors

The consistency conditions for the Bianchi identities of the tensorial hierarchy (2.4)–(2.6) require the tensors $f_{rst}^r$, $d_{rs}^I$, $g_{Ir}^r$, $k_{rA}^r$ to obey

$$d_{Ir(u}d_{vs)}^I = 0,$$

$$\left( d_{r(u}s}^J d_{v)}^I - d_{uv}^J d_{rs}^I + d_{Kr}^K d_{uv}^J \eta^{IJ} \right) g_J^s = f_{r(u}^s d_{v)}^I s,$$

$$3f_{[pq}^uf_{r]u}^s - g_J^s d_{u[p}^J d_{qr]}^u = 0,$$

$$X_{rs}^t \equiv d_{rs}^t g_t^s - f_{rs}^t = -k_{rA}^s c_A^s,$$

$$X_{rIJ} \equiv 4g_{[I}^J d_{rs]}^r = 2k_{rA}^s c_A^I c_A^J,$$

$$f_{rs}^t g_t^r - d_{rs}^t g_t^r I = 0,$$

$$g_K^r g_{[I}^J d_{jr]}^s = 0,$$

$$g_t^r g^I s = 0,$$

$$k_{rA}^s g_{Ir}^r = 0.$$  \hfill (A.1)

Notice also the relations which are valued at least when contracted with $k_{rA}^s$ matrix

$$X_{sAB} = c_{As}^I k_{B}^s,$$

$$c_{A(s}^I d_{I|p]}^t = c_{AIJ} d_{rs}^J,$$

$$c_{As}^I g_I^s = -2c_{AIJ} g^{Jt}.$$  \hfill (A.2)

B. Some useful identities for 6d gamma matrices

We use the metric of mostly minus signature $\eta^{ab} = diag(+, -, -, -, -, -)$. The $4 \times 4$ matrices $\gamma^{a}_{\alpha\beta}$ and $\tilde{\gamma}^{a\gamma\delta}$ obey

$$\gamma^{a}_{\alpha\beta} = -\gamma^{a}_{\beta\alpha} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \tilde{\gamma}^{a\gamma\delta},$$

$$\left( \gamma^{(a}_{\alpha} \gamma^{b)}_{\beta} \right) \gamma^{\delta}_{\alpha} = \eta^{ab} \delta^{\delta}_{\alpha},$$

$$\tilde{\gamma}^{a}_{\alpha\beta} \tilde{\gamma}^{\gamma\delta}_{\alpha} = -4 \delta^{[a}_{\alpha} \delta^{\gamma\delta}_{\beta]} ,$$

$$\gamma^{a}_{\alpha\beta} \gamma^{\delta}_{\alpha} = -2 \epsilon^{a}_{\alpha\beta\gamma\delta}.$$  \hfill (B.1)
\[ tr(\gamma_a \gamma^b) = 4 \delta_a^b, \quad tr(\gamma_{cd} \gamma^{ab}) = -8 \delta_c^a \delta^b_d, \]
\[ tr(\gamma^{abc} \gamma_{def}) = -24 \delta_{[a}^d \delta_{b]}^e \delta_{c]}^f - 4 \varepsilon_{abcdef}, \]
\[ \delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} = -\frac{1}{4} \gamma^{\alpha \beta} \gamma_{\alpha \beta}^{\delta}, \quad \delta^{(\gamma \delta)}_{\alpha \beta} = -\frac{1}{48} \gamma^{\alpha \beta} \gamma_{\alpha \beta}^{\delta}, \]
\[ \gamma^{\alpha \beta}_{\gamma} = -8 \delta_{\alpha}^\gamma \delta_{\beta}^\gamma + 2 \delta_{\alpha}^\gamma \delta_{\beta}^\delta, \quad \gamma^{\alpha \beta}_{\gamma} = -\varepsilon^{\alpha \beta \gamma \delta}, \]
\[ \gamma^{\alpha \beta \gamma}_{\delta} = 2 \epsilon^{\alpha \beta \gamma \delta} - 2 \gamma \gamma_{[\alpha \beta \gamma]}^{[\delta}, \quad \gamma^{\alpha \beta \gamma}_{\delta} = 2 \epsilon^{\alpha \beta \gamma \delta} - 3 \gamma \gamma_{[\alpha \beta \gamma]}^{[\delta]. \]

(B.2) \[ \delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} = -\frac{1}{4} \gamma^{\alpha \beta} \gamma_{\alpha \beta}^{\delta}, \quad \delta^{(\gamma \delta)}_{\alpha \beta} = -\frac{1}{48} \gamma^{\alpha \beta} \gamma_{\alpha \beta}^{\delta}, \]
\[ \gamma^{\alpha \beta}_{\gamma} = -8 \delta_{\alpha}^\gamma \delta_{\beta}^\gamma + 2 \delta_{\alpha}^\gamma \delta_{\beta}^\delta, \quad \gamma^{\alpha \beta}_{\gamma} = -\varepsilon^{\alpha \beta \gamma \delta}, \]
\[ \gamma^{\alpha \beta \gamma}_{\delta} = 2 \epsilon^{\alpha \beta \gamma \delta} - 2 \gamma \gamma_{[\alpha \beta \gamma]}^{[\delta}, \quad \gamma^{\alpha \beta \gamma}_{\delta} = 2 \epsilon^{\alpha \beta \gamma \delta} - 3 \gamma \gamma_{[\alpha \beta \gamma]}^{[\delta}. \]

(B.3) \[ BIs \text{ for } Ids \] of (1,0) superspace

(C.1) \[ J_4^I := DI_4^I + g_I^I = 0, \]
\[ J_3^I := DI_3^I + 2d_{st}^{i:j} , \]
\[ J_{5r} := DI_{5r} - 2I_4^I + 2H_4^I \wedge H_3^I - I_3^I d_{Isv} + k_r^A I_6^A = 0, \]
\[ k_r^A J_{7A} := k_r^A DI_6^A + k_r^A c_{A}^{uv} I_3^u \wedge H_4^I - 2k_r^A c_{AI} J_{H_3}^I \wedge I_4^I + k_r^A c_{A}^{uv} s_{I}^u \wedge I_5u = 0 \]
\[ I_q^A := (I_3^I, I_4^I, I_5r, k_r^A I_6^A) = 0. \]

(C.1) \[ J_4^I := DI_4^I + g_I^I = 0, \]
\[ J_3^I := DI_3^I + 2d_{st}^{i:j} , \]
\[ J_{5r} := DI_{5r} - 2I_4^I + 2H_4^I \wedge H_3^I - I_3^I d_{Isv} + k_r^A I_6^A = 0, \]
\[ k_r^A J_{7A} := k_r^A DI_6^A + k_r^A c_{A}^{uv} I_3^u \wedge H_4^I - 2k_r^A c_{AI} J_{H_3}^I \wedge I_4^I + k_r^A c_{A}^{uv} s_{I}^u \wedge I_5u = 0 \]
\[ I_q^A := (I_3^I, I_4^I, I_5r, k_r^A I_6^A) = 0. \]

The cancelations of the contributions which are not proportional to $I_q^A$ in the r.h.s.s of (C.1)–(C.4) occur due to the properties (A.3) of the constant tensor. Actually the relations (A.1) can be obtained by requiring this cancelation.

With our constraints a number of lower dimensional components of the BIs are satisfied due to the algebraic structure of the superfield strengths. Omitting these, we find the
following decomposition of the superspace Bianchi identities

\[
I_3^I = \frac{1}{2} E^b \wedge E^{\alpha i} \wedge E^{\beta j} I_{\alpha i \beta j b} + \frac{1}{2} E^c \wedge E^b \wedge E^{\alpha i} I_{\alpha i \beta j c} + \frac{1}{3!} E^c \wedge E^b \wedge E^{\alpha i} I_{\alpha i \beta j c}^r , \tag{C.6}
\]

\[
I_4^I = \frac{1}{3!} E^c \wedge E^{\alpha i} \wedge E^{\beta j} \wedge E^{\gamma k} I_{\alpha i \beta j \gamma k c} + \frac{1}{4} E^c \wedge E^b \wedge E^{\alpha i} \wedge E^{\beta j} I_{\alpha i \beta j b c} + \\
+ \frac{1}{3!} E^c \wedge E^b \wedge E^a \wedge E^{\alpha i} I_{\alpha i \beta j a c} + \frac{1}{4!} E^d \wedge E^c \wedge E^b \wedge E^{\alpha i} I_{\alpha i \beta j a c d} , \tag{C.7}
\]

\[
I_{5r} = \frac{1}{2 \cdot 3!} E^c \wedge E^b \wedge E^a \wedge E^{\alpha i} \wedge E^{\beta j} I_{\alpha i \beta j a b c r} + \frac{1}{4!} E^d \wedge E^b \wedge E^c \wedge E^a \wedge E^{\alpha i} I_{\alpha i \beta j a b c d r} + \\
+ \frac{1}{5!} E^d \wedge E^c \wedge E^b \wedge E^a \wedge E^{\alpha i} k_r^A I_{\alpha i \beta j a b c d r} , \tag{C.8}
\]

\[
k_r^A I_{6A} = \frac{1}{2 \cdot 4!} E^d \wedge E^c \wedge E^b \wedge E^a \wedge E^{\alpha i} \wedge E^{\beta j} k_r^A I_{\alpha i \beta j a b c d A} + \\
+ \frac{1}{5!} E^d \wedge E^c \wedge E^b \wedge E^a I_{\alpha i \beta j a b c d A} + \\
+ \frac{1}{6!} k_r I_{\alpha i \beta j a b c d e} , \tag{C.9}
\]

Generically, the first nontrivial component in the \( q \)-form \( BI I_{i}^{\lambda_q} \) has dimension \( q-1 \), i.e. carries all–but–two bosonic indices. The only exception is the nontrivial identity \( I_{\alpha i \beta j \gamma k} = 0 \) in \( \{C.7\} \) which results in identification \( \Psi_i^I = -i/2D_{\alpha i} \Phi \).

Substituting \( \{C.6\} \)–\( \{C.9\} \) into the \( Ids for Ids \) \( \{C.7\} \)–\( \{C.4\} \) one can study the independence of different components of different \( BIs \) and establish some Dragon–like theorems for the tensorial hierarchies in 6D superspace. We will not perform here such a complete study, but just mention a few particular results.

- For instance, one can establish that the \( SU(2) \) tensorial part of the dim 4 component of the \( BI \) \( \{4\} \) for the 4-form superfield strength, \( I_{\alpha i \beta j \gamma k a b c d r} = 0 \), is dependent on the lower dimensional \( BIs \), and that the independent parts of the \( SU(2) \) singlet \( \epsilon^{ij} I_{\alpha i \beta j \gamma k a b c d r} = 0 \), are the duality equation \( \{3.39\} \), \( I_{abcdefgh} = 0 \), and the auxiliary superfield equation \( \{3.40\} \), \( I_{ij} r = I_{(ij)} r = 0 \):

\[
I_{\alpha i \beta j \gamma k a b c d r} = -2i \epsilon_{ij} c_{\alpha i \beta j} I_{abcd r} - 2i \epsilon_{ij} \gamma_{abc} \alpha \beta I_{ij r} . \tag{C.10}
\]

- The independent parts of next-to-higher dimensional components of the \( BIs \), the ones with all–but–one bosonic indices, are defined by nontrivial solutions of the equation

\[
\gamma_{\alpha \beta}^{\lambda_a} \epsilon_{ij} \tilde{\gamma}_{\lambda_a \beta} I_{\gamma_k a b_1 \ldots b_{q-2}} = \text{cyclic}(\alpha i, \beta j \gamma k) = 0 , \tag{C.11}
\]

where

\[
\tilde{\gamma}_{\lambda_a} I_{\gamma_k a b_1 \ldots b_{q-2}} = \left( I_{\gamma_k a b}^I I_{\gamma_k a b_1 b_2}^I I_{\gamma_k a b_1 b_2 b_3}^I I_{\gamma_k a b_1 b_2 b_3 b_4}^I A k_r^A \right) , \tag{C.12}
\]

\[
I_{\gamma_k a b_1 b_2 b_3} r = I_{\gamma_k a b_1 b_2 b_3} r + \propto \eta_{[a} \Phi I_{\gamma_k b_{[2} b_3]} d_{I_{sr}} , \tag{C.13}
\]

\[
I_{\gamma_k a b_1 b_2 b_3 a} A k_r^A = I_{\gamma_k a b_1 b_2 b_3 a} A k_r^A + \propto k_r^A c_{AIJ} \Phi I_{\gamma_k b_{[2} b_4] a} I_{ij}^I \eta_{[a} I_{\gamma_k b_{2} b_4]} . \tag{C.14}
\]
It is easy to see that Eq. (C.11) is solved by
\[ \tilde{I}_{\gamma k a,b_1\ldots b_{q-2}}^{\Lambda q} \propto \gamma_{a\gamma\delta} I_{\delta k b_1\ldots b_{q-2}}^{\Lambda q} . \]
Hence the independent part of the next-to-higher dimensional components of the BIs are given by their gamma–traces,
\[ \mathcal{I}_{k b_1\ldots b_{q-2}}^{\Lambda q} = \tilde{\gamma}^{a\beta\gamma} I_{\gamma k a,b_1\ldots b_{q-2}}^{\Lambda q} = 0 . \]

- It is easy to check that the highest dimensional components of the differential form BIs, the ones with all bosonic indices, \( I_{b_1\ldots b_q}^{\Lambda q} = (I_{b_1\ldots b_3}^r, I_{b_1\ldots b_4}^I, I_{b_1\ldots b_5}^r, I_{b_1\ldots b_6}^A k_r^A) \) are dependent. They are satisfied identically due to the lower dimensional BIs (and their derivatives) and thus do not require a separate study of their consequences.

References

[1] E. Witten, “Some comments on string dynamics,” arXiv:hep-th/9507121 [hep-th].
[2] N. Lambert and C. Papageorgakis, “Nonabelian (2,0) Tensor Multiplets and 3-algebras,” JHEP 1008 (2010) 083, [arXiv:1007.2982 [hep-th]].
[3] N. Lambert, C. Papageorgakis, and M. Schmidt-Sommerfeld, “M5-Branes, D4-Branes and Quantum 5D super-Yang-Mills,” JHEP 1101 (2011) 083 [arXiv:1012.2882 [hep-th]].
[4] M. R. Douglas, “On D=5 super Yang-Mills theory and (2,0) theory,” JHEP 1102 (2011) 011 [arXiv:1012.2880 [hep-th]].
[5] H. Singh, “Super-Yang-Mills and M5-branes,” JHEP 1108 (2011) 136 [arXiv:1107.3408 [hep-th]].
[6] N. Lambert and P. Richmond, “(2,0) Supersymmetry and the Light-Cone Description of M5-branes,” JHEP 1202 (2012) 013 [arXiv:1109.6454 [hep-th]].
[7] N. Lambert, C. Papageorgakis, and M. Schmidt-Sommerfeld, “Deconstructing (2,0) Proposals,” Phys. Rev. D88 (2013) 026007 [arXiv:1212.3337 [hep-th]].
[8] P.-M. Ho, K.-W. Huang, and Y. Matsuo, “A Non-Abelian Self-Dual Gauge Theory in 5+1 Dimensions,” JHEP 1107 (2011) 021 [arXiv:1104.4040 [hep-th]].
[9] K.-W. Huang, “Non-Abelian Chiral 2-Form and M5-Branes,” arXiv:1206.3983 [hep-th].
[10] C.-S. Chu and S.-L. Ko, “Non-abelian Action for Multiple Five-Branes with Self-Dual Tensors,” JHEP 1205 (2012) 028 [arXiv:1203.4224 [hep-th]].
[11] C.-S. Chu, S.-L. Ko, and P. Vanichchapongjaroen, “Non-Abelian Self-Dual String Solutions,” JHEP 1209 (2012) 018 [arXiv:1207.1095 [hep-th]].
[12] C.-S. Chu and P. Vanichchapongjaroen, “Non-abelian Self-Dual String and M2-M5 Branes Intersection in Supergravity,” arXiv:1304.4322 [hep-th].
[13] F. Bonetti, T. W. Grimm, and S. Hohenegger, “A Kaluza-Klein inspired action for chiral p-forms and their anomalies,” arXiv:1206.1600 [hep-th].
[14] F. Bonetti, T. W. Grimm, and S. Hohenegger, “Non-Abelian Tensor Towers and (2,0) Superconformal Theories,” JHEP 1305 (2013) 129 [arXiv:1209.3017 [hep-th]].
[15] H. Singh, “The Yang-Mills and chiral fields in six dimensions,” JHEP 1302 (2013) 056 [arXiv:1211.3281 [hep-th]].
[16] H.-C. Kim and K. Lee, “Supersymmetric M5 Brane Theories on R x CP2,” arXiv:1210.0853 [hep-th].

[17] H. Samtleben, E. Sezgin, and R. Wimmer, “(1,0) superconformal models in six dimensions,” JHEP 1112 (2011) 062 [arXiv:1108.4060 [hep-th]].

[18] H. Samtleben, E. Sezgin, R. Wimmer, and L. Wulff, “New superconformal models in six dimensions: Gauge group and representation structure,” PoS CORFU2011 (2011) 071 [arXiv:1204.0542 [hep-th]].

[19] H. Samtleben, E. Sezgin, and R. Wimmer, “Six-dimensional superconformal couplings of non-abelian tensor and hypermultiplets,” [arXiv:1212.5199 [hep-th]]. ARXIV:1212.5199;

[20] I. Bandos, H. Samtleben and D. Sorokin, “Duality-symmetric actions for non-Abelian tensor fields,” Phys. Rev. D88, 025024 (2013) [12pp] [arXiv:1305.1304 [hep-th]].

[21] P. Pasti, D. P. Sorokin and M. Tonin, “On Lorentz invariant actions for chiral p forms,” Phys. Rev. D 55 (1997) 6292 [arXiv:hep-th/9611100].

[22] I. Bandos, H. Samtleben and D. Sorokin, work in progress.

[23] C. Saemann and M. Wolf, “Non-Abelian Tensor Multiplet Equations from Twistor Space,” arXiv:1205.3108 [hep-th]; “Six-Dimensional Superconformal Field Theories from Principal 3-Bundles over Twistor Space,” arXiv:1305.4870 [hep-th];

[24] S. Palmer and C. Saemann, “Six-Dimensional (1,0) Superconformal Models and Higher Gauge Theory,” arXiv:1308.2622 [hep-th].

[25] N. Dragon, “Torsion and Curvature in Extended Supergravity,” Z. Phys. C 2 (1979) 29.

[26] M. F. Sohnius, “Identities for Bianchi Identities,” in: Superspace and Supergravity: Proceedings of the Nuffield Workshop, Cambridge, June 16 - July 12, 1980. CUP, 1981, pp. 469-480 [Preprint ICTP/79-80/44]

[27] P. S. Howe and H. Nicolai, “Gauging N = 8 Supergravity in Superspace,” Phys. Lett. B 109 (1982) 269.