Adaptive Model Predictive Control for A Class of Constrained Linear Systems with Parametric Uncertainties*

Kunwu Zhang, Yang Shi

Department of Mechanical Engineering, University of Victoria, Victoria, B.C., Canada, V8W 3P6

Abstract

This paper investigates adaptive model predictive control (MPC) for a class of constrained linear systems with unknown model parameters. This is also posed as the dual control problem consisting of system identification and regulation. We first propose an online strategy for simultaneous unknown parameter identification and uncertainty set estimation based on the recursive least square technique. The designed strategy provides a contractive sequence of uncertain parameter sets, and the convergence of parameter estimates is achieved under certain conditions. Second, by integrating tube MPC with proposed estimation routine, the developed adaptive MPC provides a less conservative solution to handle multiplicative uncertainties. This is made possible by constructing the polytopic tube based on the consistently updated nominal system and uncertain parameter set. In addition, the proposed method is extended with reduced computational complexity by sacrificing some degrees of optimality. We theoretically show that both designed adaptive MPC algorithms are recursively feasible, and the perturbed closed-loop system is asymptotically stable under standard assumptions. Finally, numerical simulations and comparison are given to illustrate the proposed method.

Key words: Adaptive model predictive control; parameter identification; multiplicative uncertainties.

Model predictive control (MPC) has become one of the most successful methods for multivariable control systems since it provides an effective and efficient methodology to handle complex and constrained systems [12]. The main insight of MPC is to obtain a sequence of optimal control actions over the prediction horizon by solving an optimization problem. The prediction employed in MPC is conducted based on an explicit system model. Therefore having an accurate model is critical for obtaining the desirable performance. However, various categories of uncertainties, such as the measurement noise and the model mismatch, are common in practical control applications. Although standard MPC, where uncertainties are not considered in the optimal control problem, has proved to be inherently robust to arbitrarily small disturbances, its performance may become poor for many practical applications where the disturbances cannot be ignored, therefore robust MPC has attracted considerable attention in recent years [11,13,16]. However, if the uncertainties are constant or slowly changing, the performance of robust MPC is relatively conservative due to the fixed nominal model used in the optimal control. In addition, robust MPC requires the bounds of uncertainties to establish sufficient conditions, which are generally developed offline, for recursive feasibility and closed-loop stability. Therefore, the performance of robust MPC may be deteriorated for some practical control problems if the description on uncertainties is not accurate. To solve this problem, a general solution is to tune the parameters of robust MPC manually [21]. Alternatively, a promising solution is to adapt robust MPC parameters online with new available measurements, which is addressed by adaptive MPC.

In recent years, the problem of adaptive MPC has drawn increasing attention since it reduces conservatism of robust MPC by focusing on the dual control problem where the controller has a dual role: Regulation and identification [6]. Mayne and Michalska firstly proposed an adaptive MPC method in [17] for the input-constrained, nonlinear and uncertain systems, and the estimation convergence could be achieved by assuming that the MPC problem was recursively feasible. Later in [5], the persistent excitation (PE) condition was considered in the parameter estimation algorithm. Then a robust MPC algorithm was developed for linear systems represented in a controllable canonical form based on the comparison model with guaranteed recursive feasibility, and the

* This paper was not presented at any conference.

Email addresses: kunwu@uvic.ca (Kunwu Zhang), yshi@uvic.ca (Yang Shi).
closed-loop was system proved to be asymptotically stable. The authors in [1] considered the simultaneous estimation of the unknown parameter and error bound, and designed a min-max MPC scheme for the regulation problem.

A recursive least square (RLS) estimation incorporated with normal MPC was presented in [15], where an additional constraint related to the PE condition was imposed on the optimization problem to ensure the estimation convergence. But the PE constraint was formulated in a quadratic form resulting in a quadratically constrained quadratic program, hence the optimization problem became much more complicated. To simplify the PE constraint, a novel control law was proposed in [8] where the control signal consists of two parts for the purposes of state regulation and persistent excitation of input signals. Although the proposed adaptive MPC method guaranteed the convergence of parameter estimates, the system state could only be stabilized in a small region around the origin due to the presence of the PE constraint in the MPC optimization problem. A combination of the set-membership identification and homothetic tube MPC was proposed in [14], where a set-based prediction was considered based on the estimated state and designed a min-max MPC scheme for the regulation of uncertain parameter sets. Similarly, the new estimation routine to identify the parameters used in adaptive MPC are also updated based on the new estimation to reduce conservatism. Thanks to the contractive sequence of uncertainty sets, we theoretically show that the new adaptive MPC scheme is recursively feasible, and it also ensures asymptotic stability of the closed-loop system.

The remainder of this paper is organized as follows: Section 1 demonstrates the problem formulation. In Section 2, the estimation of the unknown parameter and the uncertainty set are discussed. Two adaptive MPC algorithms are presented in Sections 3 and 4, respectively, and related analysis of closed-loop properties is also given. Two numerical examples are given in Section 5, and finally Section 6 concludes this work.

1 Problem Formulation

1.1 Notation

Let \( \mathbb{R}, \mathbb{R}^n \) and \( \mathbb{R}^{m \times n} \) denote the sets of real numbers, column real vectors with \( n \) dimension and real matrices consisting of \( n \) columns and \( m \) rows, respectively. The notation \( \mathbb{N} \) denotes the set of non-negative integers, and \( \mathbb{N}_a^b \) \((b \geq a)\) is the finite set consisting of integers \( \{a, a+1, \cdots, b\} \). Given a vector \( x \in \mathbb{R}^n \), the Euclidean norm and infinity norm of \( x \), are denoted by \( ||x|| \) and \( ||x||_{\infty} \), respectively. We use \( ||x||_Q = x^TQx \) to denote the quadratic norm of \( x \) associated with the positive-definite matrix \( Q \). The Pontryagin difference of sets \( X \subseteq \mathbb{R}^n \) and \( Y \subseteq \mathbb{R}^n \) is denoted by \( X \ominus Y = \{z \in \mathbb{R}^n : z + y \in X; \forall y \in Y\} \), and the Minkowski sum is \( X + Y = \{x + y | x \in X, y \in Y\} \). We use \( I_n \) to denote an identity matrix of size \( n \).

1.2 Problem setup

Consider a discrete-time linear time-invariant (LTI) system with an unknown parameter \( \theta \in \mathbb{R}^{n_{\theta}} \)

\[
x_{k+1} = A(\theta)x_k + B(\theta)u_k,
\]

subject to a mixed constraint

\[
Fx_k + Gu_k \leq 1,
\]

where \( x_k \in \mathbb{R}^{n_x} \) and \( u_k \in \mathbb{R}^{n_u} \) are the state and input, respectively. The matrices \( A(\theta) \) and \( B(\theta) \) are the real
affine functions of \( \theta \), i.e., \( A(\theta) = A_0 + \sum_{i=1}^{n_0} A_i \theta_i, B(\theta) = B_0 + \sum_{i=1}^{n_0} B_i \theta_i \). \( \theta = [\theta_1, \theta_2, \ldots, \theta_{n_0}]^T \) is the vector of unknown parameters, which is uniquely identifiable. Let \( \theta^* \) denote the true value of \( \theta \), and it is assumed that \( \theta \) stays in an initially known set \( \Theta_0 = \{ \theta : \| \theta \| \leq r_0 \} \), where \( r_0 \) is the bound of uncertainty.

The objective of this work is to design a state feedback control law for the perturbed and constrained system in (1) while ensuring desirable closed-loop behaviors and robust constraint satisfaction by means of adaptive MPC. In particular, the control policy is characterized by the form

\[
\dot{x}_{k+1} = A_0 x_k + B_0 u_k + g(x_k, u_k) \hat{\theta}_{k+1} + K_c \hat{x}_k + K_c w_k (\hat{\theta}_k - \hat{\theta}_{k+1}),
\]

where \( \hat{x}_k \) is the state prediction from (4). This requirement can be satisfied by imposing an additional constraint on the system input, but introducing such a constraint may destabilize the system [8]. To solve this problem, we introduce the following filter \( w_k \) for the regressor \( g(x_k, u_k) \)

\[
w_{k+1} = g(x_k, u_k) - K_c w_k,
\]

then (1) can be rewritten as

\[
\dot{x}_{k+1} = A_0 x_k + B_0 u_k + g(x_k, u_k) \hat{\theta}_k + g(x_k, u_k) \hat{\theta}_k.
\]

In this section, we design an online parameter estimator to identify the unknown parameter with guaranteed non-increasing estimation error. In order to get an accurate description of uncertainty to design robust MPC parameters, an online strategy is provided to estimate the feasible solution set (FSS) of the unknown parameters, which is uniquely identifiable. Let \( 0 \) denote the true value of \( \theta_0 \). Let \( \hat{\theta}_0 \) be the initial estimate of \( \theta_0 \), based on (1) and (5), a state predictor at time \( k \) is designed as follows:

\[
\hat{x}_{k+1} = A_0 x_k + B_0 u_k + g(x_k, u_k) \hat{\theta}_{k+1} + K_c \hat{x}_k + K_c w_k (\hat{\theta}_k - \hat{\theta}_{k+1}),
\]

where \( \hat{x}_k = x_k - \hat{x}_k \) is the state prediction error, and \( K_c \) is the gain matrix. Then subtracting (1) from (6) yields

\[
\hat{x}_{k+1} = g(x_k, u_k) \hat{\theta}_{k+1} - K_c \hat{x}_k - K_c w_k (\hat{\theta}_k - \hat{\theta}_{k+1}).
\]

In order to establish an implicit regression model for \( \hat{\theta} \), we introduce an auxiliary variable \( \eta_k \) which is defined as follows

\[
\eta_k = \hat{x}_k - w_k \hat{\theta}_k.
\]

Then by substituting (5)-(7) into (8), one gets

\[
\eta_{k+1} = -K_c \eta_k.
\]

Based on this implicit regression model, we develop the following parameter estimator by using the standard RLS algorithm [9]

\[
\hat{\theta}_{k+1} = \hat{\theta}_k + \Gamma_{k+1}^{-1} w_k^T (\hat{x}_k - \eta_k),
\]

\[
\Gamma_{k+1} = \lambda \Gamma_k + w_k^T w_k,
\]

where \( \Gamma_0 = \sigma I_{n_0}; \sigma \) is the positive scalar, and \( \lambda \in (0, 1) \) is the forgetting factor. Then it follows from [9] that the estimation error is non-increasing the convergence of parameter estimates \( \hat{\theta}_k \) can be achieved if the sequence \( w_k \) is strongly persistently exciting [9].

Note that the convergence of the estimation error \( \hat{\theta}_k \), relies on the persistently exciting sequence of \( w_k \) by using the proposed estimation mechanism (10). It follows from (5) that the sequence \( w_k \) may satisfy the PE condition even the system is stable. Therefore, the proposed estimation mechanism (10) may provide a faster convergence than the estimator which relies only on the prediction error.

2.2 Uncertainty set estimation

Let \( V(\hat{\theta}_k) \) denote a Lyapunov function candidate for \( \hat{\theta}_k \) and \( V(\theta_k) = \hat{\theta}_k^T \Gamma_k \hat{\theta}_k \), then it follows from (10) that

\[
V(\hat{\theta}_{k+1}) - V(\hat{\theta}_k) \\
\leq \hat{\theta}_k^T (I - \Gamma_{k+1}^{-1} w_k^T w_k) \Gamma_{k+1} (I - \Gamma_{k+1}^{-1} w_k^T w_k) \hat{\theta}_k - \hat{\theta}_k^T \Gamma_k \hat{\theta}_k \\
\leq \hat{\theta}_k^T ((\lambda - 1) \Gamma_k) \hat{\theta}_k = (\lambda - 1) V(\hat{\theta}_k)
\]

Hence we have \( V(\hat{\theta}_k) \leq \lambda V(\hat{\theta}_{k-1}) \). Let \( V_{r_0} = \hat{\lambda} (\Gamma_0) r_0^2 \), and define the evolution of \( V_{r_k} \) as \( V_{r_{k+1}} = \lambda V_{r_k} \), then
one gets $V_{r_k} \geq V(\hat{\theta}_k)$ for all $k \geq 0$. Consequently, we can construct the following set for the uncertain parameters

$$\hat{\Theta}_k = \{ \theta \| \theta - \theta_k \|_{r_k} \leq V_{r_k} \}. \quad (11)$$

for all $k \geq 1$. Since the unknown parameters are uniquely identifiable, and stay in the a priori known set $\Theta_0$, the estimated uncertain parameter set must be the subset of $\Theta_0$. Therefore, the FSS of the unknown parameters estimated at each time instant is given by

$$\Theta_k = \Theta_{k-1} \cap \hat{\Theta}_k \quad (12)$$

where $\Theta_k$ denotes the uncertain parameter set for all $k \geq 0$. By choosing suitable $\hat{\Theta}_0, \Gamma_0$ and $V_{r_0}$, $\Theta_0$ can be equivalent to $\Theta_0$. In order to guarantee $\hat{\Theta}_{k+1} \in \Theta_k$, we apply an orthogonal projection onto the plane of the uncertainty set $\Theta_k$ to the parameter estimate, i.e.,

$$\hat{\theta}_{k+1} = Y(\hat{\theta}_k + \Gamma^{-1}_{k+1} w_{r_k}^T (\hat{x}_k - \eta_k), \Theta_k). \quad (13)$$

where $Y(\cdot, \cdot)$ denote the project operator. More details on the operator can be found in [6]. The following lemma shows the performance of the sequence of estimated uncertain parameter sets.

**Lemma 1** Let $\Theta_k$ and $\hat{\theta}_k$ denote the estimated parameter and uncertain parameter set updated by following (5)-(13) at time $k$, respectively. Suppose that $\theta^* \in \Theta_0$, then we have $\theta^* \in \Theta_k$ for all $k \geq 0$.

**Proof** Since $V(\hat{\theta}_{k+1}) \leq \lambda V(\hat{\theta}_k)$ and $V_{r_{k+1}} = \lambda V_{r_k}$, it follows from (11) that $\| \theta - \theta_k \|_{r_k} \leq V_{r_k}$ for all $\theta \in \Theta_{k+1}$, which implies that $\hat{\Theta}_{k+1} \subseteq \hat{\Theta}_k$. Following (12) yields $\Theta_k = \Theta_{0} \cap \Theta_{0} \cap \Theta_1 \cap \cdots \Theta_k$. Since $\theta^* \in \Theta_k$ and $\theta^* \in \Theta_0$ for all $k \geq 1$, then we have $\theta^* \in \Theta_k$ for all $k \geq 0$. \hfill \blacksquare

Generally, the terminal conditions and tightened sets employed in robust MPC are designed offline based on the initial uncertainty set. Hence having an accurate uncertainty set is crucial to obtain desirable closed-loop performance. It follows from (12) that a contractive sequence of uncertain parameter sets can be obtained. Therefore we can reduce the conservatism of robust MPC by updating the terminal conditions and tightened sets based on the new estimations obtained at each time instant, which is verified by the given numerical examples. In the following section, we will show a computationally tractable integration of tube MPC and the proposed estimator.

### 3 Adaptive Model Predictive Control - A Less Conservative Approach

In this section, we present a less conservative adaptive MPC (LC-AMPC) algorithm based on the tube MPC technique. The sufficient conditions for robust satisfaction of constraints (2) are presented, then we construct terminal conditions based on LMI computations, and summarize the LC-AMPC algorithm in Algorithm 1. Finally, this section concludes with the stability and feasibility analysis of the proposed method.

#### 3.1 Error tube and constraint satisfaction

Let $z_k$ denote the nominal system state at time $k$

$$z_{k+1} = A_{k+1} z_k + B_{k+1} u_k \quad (14)$$

where $A_{k+1} = A(\hat{\theta}_{k+1})$ and $B_{k+1} = B(\hat{\theta}_{k+1})$. Let $x_{l|k}$ denote the predicted system state $l$ steps ahead from time $k$; $N$ is the prediction horizon and $l \in \mathbb{N}_0^N$. Define the error state $e_{l|k}$ as $e_{l|k} = x_{l|k} - z_{l|k}$, then subtracting (1) from (14) results in

$$e_{l+1|k} = x_{l+1|k} - z_{l+1|k} = \phi^* e_{l|k} + \Delta \phi_{k+1} z_{l|k} + \Delta B_{k+1} v_{l|k}, \quad (15)$$

where $\phi^* = A(\theta^*) + B(\theta^*) K, \phi_{k+1} = A_{k+1} + B_{k+1} K$, $\Delta \phi_{k+1} = \phi^* - \phi_{k+1}$ and $\Delta B_{k+1} = B(\theta^*) - B_{k+1}$. Since $\Theta_k$ is compact and convex, we use a convex hull $Co(\hat{\theta}_k)$ to over approximate the estimated uncertain parameter set $\Theta_k$ where $j \in \mathbb{N}_0^{n_\phi}$ and $n_c$ is an integer denoting the number of points for the convex hull. Hence a set for the system pair $(A(\theta), B(\theta))$ at time $k$ can be approximated by a convex hull $Co(A_k^l, B_k^l)$ where $A_k^l = A(\hat{\theta}_k)$ and $B_k^l = B(\hat{\theta}_k)$. Similarly, let $\phi_k^l = A_k^l + B_k^l K$ then the convex hulls $Co(\phi_k^l)$ and $Co(B_k^l)$ that approximate the set of $\phi(\theta)$ and $B(\theta)$ can be found, respectively.

Inspired by the previous work [4], we construct a polytopic tube $S_{l|k} = \{ e_{l|k} | V e_{l|k} \leq \alpha_{l|k} \}$ for the error $e_{l|k}$ to handle multiplicative uncertainties, where $V$ is a matrix describing the shape of $S_{l|k}$ and $\alpha_k$ is the tube parameter. The following proposition shows a sufficient condition for the robust satisfaction of constraint (2).

**Proposition 2** Let $S_{l|k} = \{ e_{l|k} | V e_{l|k} \leq \alpha_{l|k} \}$. Suppose that $e_{l|k} \in S_{l|k}$, then $e_{l+1|k} \in S_{l+1|k}$. In addition, the constraint (2) is satisfied at each time instant if the fol-
ollowing conditions hold:

\[
1 \geq \begin{cases} 
H\alpha_{l|k} + (F + GK)z_{l|k} + Gv_{l|k}, l \in \mathbb{N}_0^{N-1} \\
H\alpha_{l|k} + (F + GK)z_{l|k}, l \in \mathbb{N}_0^\infty
\end{cases}
\]

\[
H^j_{k+1}\alpha_{l|k} + V(\Delta\phi_{k+1}^j z_{l|k} + \Delta B_{k+1}^j v_{l|k}) 
\leq \alpha_{l+1|k}, l \in \mathbb{N}_0^{N_0^\infty}, j \in \mathbb{N}_0^{n_c}
\]

(16a) \hspace{1cm} (16b)

where \(\Delta\phi_{k+1}^j = \phi_{k+1}^j - \phi_{k+1}^0\) and \(\Delta B_{k+1}^j = B_{k+1}^j - B_{k+1}^0\); \(H\) is a positive-semidefinite matrix such that \(HV = F + GK\); \(H^j_{k+1}\) is nonnegative and \(H^j_{k+1}V = V\phi_{k+1}^j\).

**Proof** Consider the uncertain input matrix \(B(\theta)\) in the system (1), this proof is completed by following the proof of Proposition 2 in [4].

Proposition 2 shows a sequence of tightened sets for the nominal system state. By introducing the tube parameters \(\alpha_{l|k}\) to the MPC optimization problem as extra decision variables, we can obtain the optimal tightened sets online.

### 3.2 Construction of terminal sets

Based on Proposition 2, we define the following dynamics of \(z_{l|k}\) and \(\alpha_{l|k}\) for \(l \in \mathbb{N}_0^{N_0^\infty}\) at time \(k\):

\[
\alpha_{l+1|k} = \max_{j \in \mathbb{N}_0^{n_c}} \{ H^j_{l+1}\alpha_{l|k} + V\Delta\phi_{l+1}^j z_{l|k} \},
\]

(17a)

\[
z_{l+1|k} = \phi_{l+1} z_{l|k},
\]

(17b)

where the maximization is taken for each element in the vector.

Let \(\varepsilon(P_{k+1}) = \{ x \in \mathbb{R}^n : x^T P_{k+1} x \leq 1 \}\) be a robustly positively invariant (RPI) set [2] for the system:

\[
x_{l+1|k} = (A(\theta) + B(\theta)K)x_{l|k}, \ \theta \in \Theta_{k+1}.
\]

then it is also invariant under the dynamics in (17b). Since using the ellipsoidal terminal set will make the optimization problem much more complicated, we employ a polytope to approximate the ellipsoid [19]. Here we use a set \(\varepsilon(P_k) = \{ x \in \mathbb{R}^n : V_k x \leq 1 \}\) to denote the polytopic super set of \(\varepsilon(P_k)\) in the following.

Let \(Z_k = \varepsilon(P_{k+1})\), then we have \(\phi_{k+1}^j Z_k \subseteq Z_k\) for all \(j \in \mathbb{N}_0^{n_c}\). Define \(Z_{l|k}\) as

\[
Z_{l|k} = (\phi_{k+1}^j)^l Z_k,
\]

(18)

based on Proposition 3 in [4], the following proposition is given to show the construction of the terminal set for the system in (17a) with respect to the RPI set \(\varepsilon(P)\).

**Proposition 3** Define

\[
\tilde{f}_{l|k} = \max_{z \in Z_{l|k}, j \in \mathbb{N}_0^{n_c}} \{ (F + GK)z \},
\]

\[
\tilde{c}_{l|k} = \max_{z \in Z_{l|k}, j \in \mathbb{N}_0^{n_c}} \{ V(\phi_{k+1} - \phi_{k+1})z \}.
\]

The set \(A_k = \{ \alpha ||\alpha||_\infty \leq \gamma_k, \alpha \geq 0 \}\) is invariant for the system in (17a) while the constraint \(H\alpha + (F + GK)z \leq 1\) is satisfied if the following condition holds

\[
\tilde{\gamma}_{l|k} \geq \gamma_k \geq \sum\limits_{l|k}
\]

(19)

where

\[
\sum\limits_{l|k} = \frac{\max_{j \in \mathbb{N}_0^{n_c}} ||\tilde{c}_{l|k}||_\infty}{1 - \max_{j \in \mathbb{N}_0^{n_c}} ||H_{l|k}^j||_\infty},
\]

\[
\tilde{\gamma}_{l|k} = \tilde{\gamma}_{l|k}^\infty.
\]

In addition, there exists a \(\gamma_k\) satisfying the condition (19) if \(l\) is sufficiently large.

**Proof** This proposition can be proved by following the proof of Proposition 3 in [4].

As shown in [4], the invariant set \(A_k\) for the system in (17a) is nonempty if \(||H_{l|k}^j||_\infty < 1\) for all \(k \geq 0\). This condition can be satisfied by choosing the appropriate \(V\) such that the set \(\{ x \mid V x \leq 1 \}\) is a contractive set of the set \(\varepsilon(P_1)\).

**Lemma 4** Consider the system in (17a). Suppose that the condition (19) will be satisfied after at least \(M_k\) steps, then we have \(M_k \geq M_{k+1}\) if the condition \(\Theta_{k+1} \subseteq \Theta_k\) holds.

**Proof** Based on the definition of \(\tilde{f}_{l|k}, \tilde{c}_{l|k}\) and \(H^j_{l|k}\), it can be derived that

\[
\tilde{\gamma}_{l+1|k} \geq \gamma_{l+1|k} \geq \sum\limits_{l+1|k}.
\]

(20)

Since the condition (19) holds for all \(l \geq M_k\), one gets \(\gamma_{M_k} \geq \sum_{M_k} \), hence \(\gamma_{M_k+1} \geq \sum_{M_{k+1}} \) by following (20). In addition, Equation (18) shows that \(Z_{l+1|k} \subseteq Z_{l|k}\), which results in \(\tilde{\gamma}_{l+1|k} \leq \tilde{\gamma}_{l|k}\) and \(\gamma_{l+1|k} \leq \gamma_{l|k}\). Therefore there must exist a set \(M_{k+1}\) such that

\[
M_{k+1} = \{ m \in \mathbb{N}_0^M : \gamma_m \geq M_{k+1} \}
\]

Based on the Proposition 3, it can be derived that

\[
M_{k+1} = \min_{m \in M_{k+1}} (M_k - m) \leq M_k.
\]
**Remark 5** From Proposition 3, it can be seen that extra $M_k$ is required to steer $\alpha_{i,k}$ into the terminal set $A_k$. Hence the prediction horizon is extended from $N$ to $N + M_k$ steps. Based on Lemmas 1 and 4, it can be derived that $M_k$ is inversely proportional to $k$. Hence the computational complexity of MPC optimization problem is non-increasing.

In order to construct the terminal set for the nominal state $z_{N|k}$, we have the following assumption:

**Assumption 6** Let $\varepsilon (P_k)$ and $\varepsilon (P_{k+1})$ denote the RPI set with respect to the uncertain parameter set $\Theta_k$ and $\Theta_{k+1}$, respectively. Then the following condition holds

$$\phi(\theta)x \in \varepsilon (P_{k+1}), \forall (x \times \theta) \in \varepsilon (P_k) \times \Theta_{k+1} \quad (21)$$

if $\Theta_{k+1} \subseteq \Theta_k$.

**Remark 7** A recent paper [19] shows how to determine the largest RPI set $\varepsilon (P_k)$ by using the linear matrix inequality (LMI) approach. Consider the similar methodology presented in [19], suitable matrices $P_k$ and $P_{k+1}$ that satisfy Assumption 6 can be found by imposing (21) as an additional constraint to the LMI problem. In addition, the condition (21) holds by choosing $P_{k+1}$ as $P_{k+1} = P_k$ since $\Theta_{k+1} \subseteq \Theta_k$. Hence there must exist a solution to $\varepsilon (P_{k+1})$ such that the condition (21) holds.

Based on Propositions 2 and 3, the terminal constraints for the systems in (17) are summarized as follows:

$$V_{k+1}z_{N|k} + D_{k+1}z_{N|k} \leq 1, \quad (22a)$$

$$0 \leq \alpha_{N+M|k} \leq \gamma_k 1, \quad (22b)$$

where $D_{k+1}$ is a positive semi-definite matrix satisfying $D_{k+1}V = V_{k+1}$ and $M_k$ is the horizon such that the condition (19) holds.

### 3.3 Construction of the cost function

Let $v_k = [v^T_{0|k}, v^T_{1|k}, v^T_{2|k}, \ldots, v^T_{N-1|k}]^T$. Define $E$ and $T$ as shift matrices such that $v_{0|k} = Ev_k$ and $v_k = Tv_{k+1}$, then the prediction of $z_{i|k}$ can be written as $\xi_{i+1|k} = \Psi_{k+1} \xi_i|k$, where $\xi_i|k = \begin{bmatrix} z_i|k \\ v_k \end{bmatrix}$, $\Psi_{k+1} = \begin{bmatrix} \phi_{k+1} & B_{k+1}E \\ 0 & T \end{bmatrix}$.

Similarly, the real system state $x$ can be predicted by using the following dynamics $\tilde{\xi}_{i+1|k} = \Phi^* \tilde{\xi}_i|k$, where

$$\tilde{\xi}_i|k = \begin{bmatrix} x_i|k \\ v_k \end{bmatrix}$$

and $\Phi^* = \begin{bmatrix} \phi^* & B^*E \\ 0 & T \end{bmatrix}$. In this work, we consider a cost function with a quadratic form $J_k = \sum_{i=0}^{\infty} (x_i^T_k Q x_i|k + u_i^T_k R u_i|k)$, where $Q$ and $R$ are two positive-definite matrices indicating the penalty matrices for the state and input, respectively. Note that the cost function $\bar{J}_k$ can be equivalently represented by $\bar{J}_k = \xi^T_{0|k} W^* \xi_{0|k}$ where $W^*$ is the solution of a Lyapunov equation

$$\Psi^* W^* (\Psi^*)^T - W^* + \tilde{Q} = 0 \quad (23)$$

with $\tilde{Q} = \begin{bmatrix} Q + K^T R K & K^T R E \\ E^T R K & E^T R E \end{bmatrix}$. Since $\phi^*$ is unknown, it is impossible to find the matrix $W^*$ exactly. Alternatively, we consider an over approximation of $J_k$ based on the uncertainty set updated at each time instant.

**Lemma 8** Define a new cost function $J_k$ as $J_k = \xi^T_{0|k} W_{k+1} \xi_{0|k}$, then $J_k \geq \bar{J}_k$ if the following condition

$$W_{k+1} \geq \begin{bmatrix} \phi(\theta) & B(\theta)E \\ 0 & T \end{bmatrix} W_{k+1} \begin{bmatrix} \phi(\theta) & B(\theta)E \\ 0 & T \end{bmatrix}^T + \tilde{Q}$$

holds for all $\theta \in \Theta_{k+1}$.

**Proof** From Lemma 1, we have $\theta^* \in \Theta_{k+1}$. Then following (24) yields $W_{k+1} \geq \Psi^* W_{k+1} (\Psi^*)^T + \tilde{Q}$. By substituting $\tilde{Q} = W^* - \Psi^* W^* (\Psi^*)^T$ into the above equation, we have $W_{k+1} - W^* \geq \Psi^* (W_{k+1} - W^*) (\Psi^*)^T \geq 0$. In addition, $J_k - \bar{J}_k = \xi^T_{0|k} W_{k+1} \xi_{0|k} - \xi^T_{0|k} W^* \xi_{0|k}$. Since $\xi_{0|k} = \xi_{0|k}$ and $W_{k+1} - W^* \geq 0$, it can conclude that $J_k \geq \bar{J}_k$ for all $\theta \in \Theta_{k+1}$.

For the sequence of weighting matrices $W_k$, we have the following assumption.

**Assumption 9** Let $W_{k+1}$ denote the weighting matrix at time $k$, if $\Theta_{k+1} \subseteq \Theta_k$, then the following condition holds for all $k \geq 0$

$$\xi^T_{0|k} W_{k+1} \xi_{0|k} \leq \xi^T_{0|k} W_k \xi_{0|k}. \quad (25)$$

**Remark 10** Since the sequence of sets $\Theta_k$ is contractive, Assumption 9 can be satisfied by choosing suitable matrices $W_k$ and $W_{k+1}$. Those matrices can be obtained by solving a sequence of appropriate LMIs. An example can be found in [10] for details.

### 3.4 Adaptive MPC algorithm with less conservatism

According to the developed terminal sets and cost function, the LC-AMPC algorithm is based on the following MPC optimization problem:

$$P_1 : \min_{v_k, \alpha_{i,k}} J_k = \xi^T_{0|k} W_{k+1} \xi_{0|k} \quad (2), (3), (14), (16a), (16b), (22a), (22b)$$

s.t. $z_{0|k} = x_k$
Algorithm 1 The Adaptive MPC algorithm

Input: At time $k$, given $x_k, \hat{x}_k, v_k, \Theta_k, P_k$ and $W_k$.
Output: Control input $u_k$.
1: if $\|x_k\| \geq \varepsilon_x$ or $V_{r_k} \geq \varepsilon_r$, then
2: Calculate the new estimation $\dot{\theta}_{k+1}$ and $\Theta_{k+1}$ by
using (5)-(13)
3: Calculate the matrices $P_{k+1}$ and $W_{k+1}$ such that
conditions (21) and (25) hold by using the LMI technique [3].
4: else
5: $\theta_{k+1} = \dot{\theta}_k$, $P_{k+1} = P_k$ and $W_{k+1} = W_k$.
6: end if
7: Update the constraints of the optimization problem
$P_1$ based on the new estimation.
8: Solve the optimization problem $P_1$ to obtain the optimal solution $v_\ast^k$.
9: Calculate the control input as $u_k = Kx_k + v_\ast^k$ and
then implement $u_k$ to the system.

At time instant $k$, we update the estimation of the unknown parameter and the uncertainty set based on new
measurements, then reformulate the optimization problem $P_1$ and recalculate the polytopic tubes parameters,
weighting matrix and terminal sets. It should be noted that those calculation procedures are not necessary if
the estimation of unknown parameters converges to the true value, or the estimation error is sufficiently small.
To reduce the redundant actions of the parameter estimation, we introduce a mechanism to check the estimation
error. Let $\varepsilon_x > 0$ and $\varepsilon_r > 0$ denote the tolerance of the estimation error and the error bound, then the adaptive MPC algorithm is summarized in Algorithm 1.

Theorem 11 Suppose that Assumptions 6 and 9 hold, and there is a feasible solution to the optimal control
problem $P_1$ when $k = 0$. Then $P_1$ is recursively feasible if $\Theta_k$ is updated by following proposed estimator.

Proof Since $\dot{\theta}_k$ is updated by following (5)-(13), from Lemma 1, we have $\Theta_{k+1} \subseteq \Theta_k$. Suppose that $P_1$ is feasible at time $k$. Let $v_\ast^k$ and $\{\alpha^i_{\ast} (\cdot)\}_{i \in \mathbb{N}_0^N}$ denote the optimal solution of the MPC problem at time $k$. Define a candidate input sequence at time $k+1$ as $v_{k+1} = \{v_1^s | v_2^s | \cdots | v_{N-1}^s | 0 \}$. Let $X_{\ast}^i_k$ denote the state tube such that $X_{\ast}^i_k = z_{\ast}^i_k \cap S_k$. The set $A_{k+1}^i$ is chosen as $A_{k+1}^i = A_k$.

Two cases are investigated to prove this theorem. Case (1): The estimation error is large enough hence $\theta_{k+2}$
and $\theta_{k+2}$ are updated by following (5)-(13).

• For $l \in \mathbb{N}_0^N$, we can formulate a candidate state tube at time $X_{0}^l | k+1$ such that $X_{\ast}^i_{k+1} = X_{\ast}^i_{k+1} | k$ based on $v_{k+1}$ and $S_k$. Then we have $\{z_{\ast}^i_{k+1} | S_{k+1}^i \}$ satisfying $z_{\ast}^i_{k+1} = S_{k+1}^i + S_{k+1}^i | k$. By Proposition 2, it verifies that the candidate sequence $\{z_{\ast}^i_{k+1} | \alpha^i_{\ast} (k+1) \} \in \mathbb{N}_0^N$ satisfies the constraints (2) and (16).
• It follows form (22a) that

$$
\begin{align*}
1 & \geq V_{k+1} z_{N+1} k + D_{k+1} \alpha_{\ast}^i (k+1) \\
& \geq V_{k+1} z_{N+1} k + D_{k+1} V e_{N} (k) \\
& \geq V_{k+1} z_{N+1} (k) + V e_{N} (k) \\
& = V_{k+1} x_{N} (k),
\end{align*}
$$

hence we have $X_{N} (k) \subseteq Z_{0} (k)$. As aforementioned, $X_{N} (k-1) \subseteq X_{N} (k)$, it results in $X_{N-1} (k+1) \subseteq Z_{0} (k)$.
Since $Z_{0} (k)$ is an RPI set, $v_{N} (k+1) = 0$ and $\theta_{k+2} \subseteq \theta_{k+1}$, it yields that $X_{N} (k-1) \subseteq \phi_{k+2} Z_{0} (k) \subseteq Z_{0} (k+1)$ by following Assumption 6. Hence we have $z_{N+1} (k) + V e_{N} (k+1) \subseteq Z_{0} (k+1)$ for all admissible $e_{N} (k+1)$. Since $\alpha_{\ast}^i (k+1)$ describes the upper bound of $e_{N} (k+1)$, there exists an $\alpha_{\ast}^i (k+1)$ such that the condition (22a) holds.

Case (2): $\theta_{k+2} = \dot{\theta}_{k+1}$ and $\theta_{k+2} = \dot{\theta}_{k+1}$. Since Assumption 6 holds and $\theta_{k+2} \subseteq \theta_{k+1}$ for this situation, we can prove the recursive feasibility by following the above procedure.

In summary, there is a feasible solution for the optimal control problem $P_1$ at time $k+1$ if it is feasible at time $k$.
Therefore $P_1$ is proved to be recursively feasible. ■

Theorem 12 Suppose that Assumptions 6 and 9 hold, then the system in (1) is asymptotically stable by applying
the adaptive MPC algorithm 1.

Proof Let $J_{k}^* \ast$ denote the optimal cost at time $k$ and $J_{k}^* \ast$ indicate the cost with respect to $z_{0} (k+1) + \tilde{v}_{k+1}$, it yields that $J_{k+1}^* - J_{k}^* \ast \leq J_{k+1} - J_{k}^* \ast$. In order to prove this theorem, the following two cases are investigated.

Case (1): $\theta_{k+2}$ and $\dot{\theta}_{k+2}$ are updated by following (5)-(13). By Lemma 8, we have

$$
\begin{align*}
\tilde{\xi}_{0}^T W_{k+1} + \tilde{\xi}_{0} (k+1) - J_{k}^* \ast \\
\geq \xi_{0}^T W_{k+1} \tilde{\xi}_{0} (k+1) - \xi_{0} (k) W_{k+1} \xi_{0} (k) \\
\leq \xi_{0}^T \tilde{\xi}_{0}^T W_{k+1} \Psi_{k+1} \xi_{0} (k) - \xi_{0} (k) W_{k+1} \xi_{0} (k) \\
\leq - \xi_{0}^T Q \xi_{0} (k)
\end{align*}
$$

7
According to the definition of $\bar{Q}$, we get $\xi_{0|k}^T \bar{Q} \xi_{0|k} = z_{0|k}^T Q z_{0|k} + u_{0|k}^T R u_{0|k}$. Since $Q$ is positive definite and $R$ is positive definite, it yields that $z_{0|k+1}^T W_{k+1} \xi_{0|k+1} - J^*_k < 0$. In addition, from Assumption 6, it can be derived that $J_{k+1} = z_{0|k+1}^T W_{k+1} \xi_{0|k+1} \leq \xi_{0|k+1}^T W_{k+1} \xi_{0|k+1}$, which indicates $J_{k+1} - J^*_k < 0$. In addition, $z_{0|k} = x_k$, then we have $x_k \to 0$ and $u_k \to 0$ as $k \to \infty$. The asymptotic stability is proved.

Case (2): $\Theta$ is asymptotically stable by using the proposed approach. In summary, the closed-loop system is asymptotically stable by using the proposed approach.

\[ \text{Remark 13} \quad \text{Note that Theorems 11 and 12 investigate the stability and feasibility of closed-loop system in (1). By following (5)-(13), the contractive sequence of uncertainties achieves, the proposed adaptive MPC method can ensure the recursive feasibility for all $\theta \in \Theta_k$. From Lemma 1, it can be derived that the following condition holds for all $\theta \in \Theta_k$ and $k \geq 1$} \]

\[ W_1 \geq \phi(\theta) B(\theta) E W_1 \phi(\theta) B(\theta) E^T + \bar{Q}. \quad (28) \]

Hence a fixed cost function can be formulated as $\hat{J}_k = \xi_{0|k}^T W_k \xi_{0|k}$.

\[ \text{4.3 Adaptive MPC algorithm with improved computational efficiency} \]

Similarly, the MPC optimization problem for the CE-AMPC algorithm is summarized as follows:

\[ \mathcal{P}_2 : \min_{v_k, \alpha_{\theta|k}} \hat{J}_k = \xi_{0|k}^T W_k \xi_{0|k} \]

s.t. $z_{0|k} = x_k$

\[ (2), (3), (14), (16a), (16b), (27a), (27b) \]

At time instant $k$, we estimate the parameter and error bound based on the new measurement, then reformulate the optimization problem $\mathcal{P}_2$. By solving this optimal control problem, the optimal solution $u_k^*$ can be obtained. The control input to be implemented is calculated as $u_k = K x_k + v_k^*$. At next time instant, we repeat the above procedure by using the new measurement to obtain the control input $u_{k+1}$.

**Theorem 14** Suppose that Assumptions 6 and 9 hold and the optimization problem $\mathcal{P}_2$ is feasible when $k = 0$, then $\mathcal{P}_2$ is recursively feasible for all $k > 0$.

**Proof** By replacing $P_k$ with $P_1$, this proof follows directly from the proof of Theorem 11.

\[ \text{4.2 Construction of the cost function} \]

Let $W_1$ be a weighting matrix which satisfies the condition (24) for all $\theta \in \Theta_1$. From Lemma 1, it can be derived that the following condition holds for all $\theta \in \Theta_k$ and $k \geq 1$

\[ W_1 \geq \phi(\theta) B(\theta) E W_1 \phi(\theta) B(\theta) E^T + \bar{Q}. \quad (28) \]

\[ \text{Remark 16} \quad \text{As shown in Theorems 14 and 15, the CE-AMPC method can also ensure the recursive feasibility, and the closed-loop system is asymptotically stable. Since the fixed cost function and terminal sets are used in CE-AMPC algorithm, it is relatively more conservative than LC-AMPC. But this algorithm generally requires the} \]
lower computational cost compared with LC-AMPC, it is more practical for some real-time control problems with strict requirements on computational efficiency. In addition, it follows from Lemma 4 that $M_k$ is non-increasing as $k$ increases. Hence, LC-AMPC requires shorter, at least equal, computing time at each time instant if the estimation error is sufficiently small.

5 Simulation Results

In this section, two numerical examples are presented to show the behavior of proposed adaptive MPC algorithms. The performance of parameter identification is demonstrated firstly, and then the proposed methods are compared with the robust MPC to verify our theoretical results.

**Example 1** Consider an LTI system of the form (1) from [8] with the scheduling parameters where

$$
A_0 = \begin{bmatrix} 0.42 & -0.28 \\ 0.02 & 0.6 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.6 & -0.4 \\ -0.6 & -0.85 \end{bmatrix},
$$

$$
B_0 = \begin{bmatrix} 0.3 & -0.4 \end{bmatrix}^T, \quad B_1 = \begin{bmatrix} 0.2 & -0.4 \end{bmatrix}^T.
$$

The system is subject to the constraints: \{ \| x \|_\infty \leq 17 \} and \{ \| u \|_\infty \leq 4 \}, then the appropriate constraint matrices $F$ and $G$ in (2) can be found. The initial knowledge on the uncertainty set is given by $\Theta_0 = \{ \theta \in \mathbb{R} | ||\theta|| \leq 0.5 \}$. The MPC prediction horizon is chosen as $N_t = 5$. The weighting matrices are set to $Q = I_2$ and $R = 10$. The prestabilizing feedback gain, minimum horizon and terminal region are derived as $K = [0.0780, 0.1668]$, $M_0 = 15$ and $\gamma_0 = 0.4795$, respectively. Other parameters used in Algorithm 1 are chosen as follows: $\epsilon_r = 0.01, \epsilon_s = 0.01, \lambda = 0.9$ and $\Gamma_0 = 0.15$.

In this simulation, the objective is to regulate the system state from an initial point $x_0 = [8, 8]^T$ to the origin. The real parameter $\theta^* = -0.2$ is given to evaluate the proposed parameter estimation algorithm. The PE-Tube MPC developed in [8] is introduced for comparison. Figure 1 shows the state trajectories obtained by using the proposed adaptive MPC and the PE-Tube MPC. The simulation results show that the proposed adaptive MPC algorithms can regulate the state to the origin, while the PE-Tube MPC can only stabilize the state within a small neighborhood of the origin. As shown in Fig. 2, the recursive feasibility is guaranteed by using the error tube (16). Figure 3 demonstrates the estimation convergence of proposed estimator, and the sequence of parameter estimates converges to the true value within a few steps. In summary, it can be seen that the proposed adaptive MPC methods can provide better regulation performance while ensuring the robust constraint satisfaction.

**Example 2** To demonstrate the improved performance of the proposed adaptive MPC methods, we compare them with the robust MPC [4]. Consider a LTI system with the scheduling parameters

$$
A_0 = \begin{bmatrix} 0.42 & -0.28 \\ 0.02 & 0.6 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.12 & -0.08 \\ -0.12 & -0.17 \end{bmatrix}, \quad A_2 = -A_1,
$$

$$
B_0 = \begin{bmatrix} 0.3 & -0.4 \end{bmatrix}^T, \quad B_1 = \begin{bmatrix} 0.04 & -0.08 \end{bmatrix}^T, \quad B_2 = -1.5B_1.
$$

The initial uncertainty set is given by $\Theta_0 = \{ \theta \in \mathbb{R}^2 | ||\theta|| \leq 1 \}$. The system is subject to input and state constraints \{ $x$\|$x$\|$_\infty$ \leq 17 $\} and \{ $u$\|$u$\|$_\infty$ \leq 4 $\}. The weighting matrices are chosen as $Q = \text{diag}(10, 0.001)$ and $R = 1$. Set the prediction horizon $N_t = 9$, then the prestabilizing feedback gain, minimum
The uncertain parameter sets estimated at time $k=0, 1, 2, 30$ are depicted in Fig. 6. It can be seen that the falsified parameters are removed by using the proposed estimator, and the uncertain parameter set is non-increasing.

6 Conclusion

In this paper, we have investigated adaptive MPC for constrained linear systems subject to multiplicative uncertainties. An online parameter estimator has been designed based on the RLS technique for simultaneous parameter identification and uncertain set estimation. By integrating the proposed estimator with tube MPC method, the multiplicative uncertainties have been handled with reduced conservatism, therefore giving rise to enhanced performance. Two stabilizing adaptive MPC algorithms, LC-AMPC and CE-AMPC, have been proposed to provide a tradeoff between conservatism and computational complexity. We have proven that the proposed adaptive MPC methods are recursively feasible, and the closed-loop systems are asymptotically stable. Numerical simulations and comparison studies have been given to demonstrate the effectiveness and advantages of the proposed adaptive MPC methods. Future research will be focused on dealing with both time-varying multiplicative and additive disturbances by using adaptive MPC.

References

[1] Veronica Adetola, Darryl DeHaan, and Martin Guay. Adaptive model predictive control for constrained nonlinear systems. Systems & Control Letters, 58(5):320–326, 2009.

[2] Franco Blanchini and Stefano Miani. Set-Theoretic Methods in Control. Springer, 2008.

[3] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkatakrishnan Balakrishnan. Linear Matrix Inequalities in System and Control Theory, volume 15. Society for Industrial Mathematics, Philadelphia, PA, USA, 1994.

[4] James Fleming, Basil Kouvaritakis, and Mark Cannon. Robust tube MPC for linear systems with multiplicative uncertainty. IEEE Transactions on Automatic Control, 60(4):1087–1092, 2015.

[5] Hiroaki Fukushima, Tae-Hyoung Kim, and Toshiharu Sugie. Adaptive model predictive control for a class of constrained linear systems based on the comparison model. Automatica, 43(2):301–308, 2007.

[6] Graham C Goodwin and Kwai Sang Sin. Adaptive Filtering Prediction and Control. Courier Corporation, 2014.

[7] Jurre Hanema, Mirea Lazar, and Roland Tóth. Stabilizing tube-based model predictive control: Terminal set and cost construction for LPV systems. Automatica, 85:137–144, 2017.
[8] Bernardo Hernandez and Paul Trodden. Persistently exciting tube MPC. In Proceedings of the 2016 American Control Conference (ACC2016), pages 948–953, Boston, MA, USA, July 6-8, 2016. IEEE.

[9] Richard M Johnstone and Brian DO Anderson. Exponential convergence of recursive least squares with exponential forgetting factor adaptive control. Systems & Control Letters, 2(2):69–76, 1982.

[10] Basil Kouvaritakis and Mark Cannon. Model Predictive Control: Classical, Robust and Stochastic. Springer, 2015.

[11] Huiping Li and Yang Shi. Robust distributed model predictive control of constrained continuous-time nonlinear systems: A robustness constraint approach. IEEE Transactions on Automatic Control, 59(6):1673–1678, 2014.

[12] Huiping Li and Yang Shi. Robust Receding Horizon Control for Networked and Distributed Nonlinear Systems, volume 83. Springer, 2016.

[13] Changxin Liu, Huiping Li, Jian Gao, and Demin Xu. Robust self-triggered minmax model predictive control for discrete-time nonlinear systems. Automatica, 89:333–339, 2018.

[14] Matthias Lorenzen, Frank Allgöwer, and Mark Cannon. Adaptive model predictive control with robust constraint satisfaction. In Proceedings of the 20th World Congress of the International Federation of Automatic Control (IFAC 2017), pages 3313–3318, Toulouse, France, July 9-14, 2017. Elsevier.

[15] Giancarlo Marafioti, Robert R Bitmead, and Morten Hovd. Persistently exciting model predictive control. International Journal of Adaptive Control and Signal Processing, 28(6):536–552, 2014.

[16] David Q. Mayne. Model predictive control: Recent developments and future promise. Automatica, 50(12):2967–2986, 2014.

[17] David Q Mayne and H Michalska. Adaptive receding horizon control for constrained nonlinear systems. In Proceedings of the 32nd IEEE Conference on Decision and Control (CDC1993), pages 1286–1291. IEEE, Dec. 1993.

[18] Anusha Nagabandi, Guangzhao Yang, Thomas Asmar, Gregory Kahn, Sergey Levine, and Ronald S Fearing. Neural network dynamics models for control of under-actuated legged millirobots. arXiv preprint arXiv:1711.05253, 2017.

[19] Hoaı̂-Nam Nguyen, Sorin Olaru, Per Olof Gutman, and Morten Hovd. Constrained control of uncertain, time-varying linear discrete-time systems subject to bounded disturbances. IEEE Transactions on Automatic Control, 60(3):831–836, 2015.

[20] Chris J Otafeow, Angela P Schoellig, Timothy D Barfoot, and Jack Collier. Learning-based nonlinear model predictive control to improve vision-based mobile robot path tracking. Journal of Field Robotics, 33(1):133–152, 2016.

[21] S Joe Qin and Thomas A Badgwell. A survey of industrial model predictive control technology. Control Engineering Practice, 11(7):733–764, 2003.

[22] Saša V Raković, Basil Kouvaritakis, Rolf Findeisen, and Mark Cannon. Homothetic tube model predictive control. Automatica, 48(8):1631–1638, 2012.