On Bilinear Invariant Differential Operators Acting on Tensor Fields on the Symplectic Manifold

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ON BILINEAR INVARIANT DIFFERENTIAL OPERATORS ACTING 
ON TENSOR FIELDS ON THE SYMPLECTIC MANIFOLD 

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Abstract. Let $M$ be an $n$-dimensional manifold, $V$ the space of a representation $\rho : GL(n) \rightarrow GL(V)$. Locally, let $T(V)$ be the space of sections of the tensor bundle with fiber $V$ over a sufficiently small open set $U \subset M$, in other words, $T(V)$ is the space of tensor fields of type $V$ on $M$ on which the group $\text{Diff}(M)$ of diffeomorphisms of $M$ naturally acts. For irreducible fibers with lowest weight the $\text{Diff}(M)$-invariant differential operators $D : T(V_1) \otimes T(V_2) \rightarrow T(V_3)$ are classified by the author elsewhere. Here the result is generalized to bilinear operators invariant with respect to the group $\text{Diff}_c(M)$ of symplectomorphisms of the symplectic manifold $(M, \omega)$. This is a preliminary report on the work in progress.

Among the new operators we mention a 2nd order one which determines an “algebra” structure on the space of metrics (symmetric forms) on $M$.

1. Let $\rho$ be a representation of the group $Sp(2m; \mathbb{R})$ in a $V_\rho$. A tensor field of type $\rho$ on a $2m$-dimensional symplectic manifold $M$ is an object $t$ defined in each local coordinate system $x$, in which the symplectic form is of the canonical form $\omega = \sum_{i \leq m} dx_i \wedge dx_{2m+1-i}$, by the vector $t(x) \in V_\rho$ so that the passage to other coordinates, $y$ (with the same property), is defined by the formula

$$t(y(x)) = \rho \left( \frac{\partial y(x)}{\partial x} \right) t(x).$$

Traditionally (see reviews [K1], [?]) the fibers of the tensor bundles were considered finite dimensional, but Leites showed recently [LKW] that on supermanifolds it is natural and fruitful to consider infinite dimensional fibers: this leads to semi-infinite cohomology of supermanifolds. Similar problem for symplectic manifolds was not studied yet.

The space of smooth tensor fields of type $\rho$ will be denoted by $T(\rho)$ or by $T(\lambda)$, where $\lambda = (\lambda_1, \ldots, \lambda_m)$ is the lowest (or, for finite dimensional representations, highest, for convenience) weight of the irreducible representation $\rho$.

In what follows the letters $\rho, \sigma, \tau$ will denote irreducible representations of $Sp(2m; \mathbb{R})$ and letters $\lambda, \mu, \nu$ their highest weights. (We should have considered lowest weight only, but in this preliminary report we stick to finite dimensional representations.)

**Examples of spaces of tensor fields**: 

a) $T(0) = C^\infty(M)$;

b) $T(1, 0, \ldots, 0) \cong \text{Vect} \cong \Omega^1$ is the space of vector fields or (which is the same on any symplectic manifold thanks to the nondegeneracy of $\omega$) the space of 1-forms on $M$;

c) $\prod_{r=many} T(1, \ldots, 1, 0, \ldots, 0)$ the space of primitive $r$-forms.

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On the space $T_e(\rho)$ of tensor fields of type $\rho$ with compact support, as indicated by the subscript, there is an invariant inner product

$$<\chi, \theta> = \int_M <\chi(x), \theta(x)> \omega_0^m,$$

where $<\cdot, \cdot>$ in the integrand is the $Sp(2m;\mathbb{R})$-invariant inner product on $V_\rho$.

A differential operator $B : T(p_1) \otimes T(p_2) \otimes \cdots \otimes T(p_n) \longrightarrow T(\tau)$ is called $n$-ary (unary, binary, etc. for $n = 1, 2$, respectively). Such an operator $B$ is called $Diff(M)$-invariant if it is uniquely expressed in all coordinate systems; it is $Diff_\omega(M)$-invariant if it is uniquely expressed in all coordinate systems in which the symplectic form is of the standard form.

2. The unary $Diff_\omega(M)$-invariant differential operators. All such operators are classified by A. N. Rudakov ([R1], [R2]):

- **0-th order**: the multiplication by a number;
- **1-st order**: the derivations of the primitive forms $d_+ : \prod T \longrightarrow \prod^{r+1}$ and $d_- : \prod^{r+1} \longrightarrow \prod^r$ ($0 \leq r \leq m - 1$). These operators are compositions of the exterior differential $d : \Omega^p \longrightarrow \Omega^{p+1}$ and the projection onto the space of primitive forms; recall that $\Omega^p = \prod^p \oplus \prod^{p-2} \oplus \cdots$ for $p \leq m$ and $\Omega^p \cong \Omega^{2m-p}$;
- **2-nd order**: $d_0 = d_+ \circ d_- : \prod T \longrightarrow \prod^1 (1 \leq r \leq m)$.

**Remark**. Rudakov’s theorem implies that other invariant operators that might spring to mind ($d_- \circ \omega d_-$, etc.) are multiples of the described ones.

3. The binary $Diff_\omega(M)$-invariant differential operators. Clearly, the operators $B^{*1} : T(\tau) \times T(p_2) \longrightarrow T(p_1)$ and $B^{*2} : T(p_1) \times T(\tau) \longrightarrow T(p_2)$ conjugate to (or 1- and 2-duals of) the invariant differential operator $B : T(p_1) \otimes T(p_2) \longrightarrow T(\tau)$ with respect to the inner product (IP) are also differential and invariant ones.

**0-th order operators** are obviously those of the form

$$Z(\chi, \theta) = pr(\chi(x) \otimes \theta(x)),$$

where $pr : V_\rho \otimes V_\sigma \longrightarrow V_\tau$ is the projection of the tensor product onto an irreducible component.

**1-st order operators:**

**Theorem**. Any bilinear 1-st order (with respect to all arguments) $Diff_\omega(M)$-invariant differential operator $B : T(\lambda) \otimes T(\mu) \longrightarrow T(\nu)$ is a linear combination of the following cases P1 – P8 (some of which host several distinct operators) and the operators obtained from them by 1- and 2-dualization and transposition of the arguments.

- **P1** $\lambda = (1, \ldots, 1, 0, \ldots, 0)$; weights $\mu$ and $\nu$ differ by a unit in $r$ places, $r \equiv p + 1 \pmod{2}$. For $r \leq p + 1$ there exist operators of the form $Z(d_+ \omega, \theta)$ and for $r \leq p - 1$ there exist operators of the form $Z(d_- \omega, \theta)$.

- **P2** The Lie derivative being restricted onto $Sp(2m;\mathbb{R})$-irreducible subspaces splits into several operators of the form $Z(d_\pm \omega, \theta)$ and an operator

$$L : Vect \times T(\rho) \longrightarrow T(\rho)$$

which cannot be reduced to operators of the form P1).

**Remark**. Observe that if $\xi \in \mathfrak{h}(M) \subset \mathfrak{vect}(M)$ is a Hamiltonian vector field, then, by identifying $\mathfrak{h}$ with $d\Omega^0$, we see that $d_+ \xi = d_- \xi = 0$ and in this case $L$ coincides with the Lie
derivative. Therefore, \( L \) determines a representation of the Lie algebra \( \mathfrak{h}(M) \) in the space \( T(\rho) \). It is not difficult to show that the invariance of \( B \) is equivalent to its \( \mathfrak{h}(M) \)-invariance:
\[
L(\xi, B(\chi, \theta)) = B(L(\xi, \chi), \theta) + B(\chi, L(\xi, \theta))
\]
for any \( \chi \in T(\rho), \theta \in T(\sigma), \xi \in \mathfrak{h}(M) \).

P3) \( S^k \text{vect} \times S^l \text{vect} \to S^{k+l-1} \text{vect} \) (clearly \( S^k \text{vect} \cong T(k,0,\ldots,0) \)) is the Poisson bracket (a.k.a. the symmetric Schouten’s concomitant) on (polynomial in momenta) functions on \( T^* M \).

P4) \( \lambda, \mu, \nu \) are of the form \((2,1,\ldots,1,0,\ldots,0)\) each, with \( p, q \) and \( r \) non-zero numerical marks, respectively, such that \( p+q+r \equiv 0 \mod 2, \quad |p-q| \leq r \leq p+q, \) and \( p+q+r \leq 2m+2 \).

If all inequalities are strict, then there exist \textbf{four distinct operators} defined on the spaces of such fields, otherwise there exist only \textbf{two distinct operators}. For \( p+q+r \leq 2m \) two of these four or two operators are obtained as restrictions of the \textit{Nijenhuis bracket}, or its conjugates, onto the subspaces
\[
T(2,1,\ldots,1,0,\ldots,0) \subset \Omega^p \otimes_{C^\infty(M)} \text{vect}.
\]

\textbf{Remark}. The remaining two operators (i.e., the ones which are not the restrictions of the Nijenhuis bracket) are new. I do not know anything about them except that they exist and the same applies to the following two cases P5) and P6).

P5) \( \lambda, \mu \) are of the same form as for P4), \( \nu = (3,1,1,\ldots,1,0,\ldots,0) \). There exists one operator for \( |p-q|+1 \leq r \leq p+q-1, \quad p+q+r \equiv 1 \mod 2, \quad p+q+r \leq 2m+1 \).

P6) \( \lambda, \mu \) are the same as in 4), \( \nu = (2,2,1,\ldots,1,0,\ldots,0) \) with \( r \) non-zero entries. The operator exists under the same conditions on \( p,q,r \) as for P5).

P7) \( \nu = (1,\ldots,1,0,\ldots,0) \); whereas \( \lambda, \mu \) and conditions on \( p,q,r \) are the same as in 5). In this case there exists a unique operator which is not reducible to operators of the form \( \partial Z/\partial x \). It is a restriction of the Nijenhuis bracket.

P8) \( \lambda = (2,0,\ldots,0) \); whereas \( \mu \) and \( \nu \) differ from each other by a unit at one place. There exists a unique such operator. Further on I’ll give arguments which enable one to express it, in principle, explicitly.

\textbf{4. 2nd order operators}. I could not \textit{classify} such operators so far. However, I was lucky to find one new invariant operator, denoted in the literature \( G_z \):
\[
G_z : T(2,0,\ldots,0) \times T(2,0,\ldots,0) \to T(2,0,\ldots,0).
\]

\textbf{Conjecture}. The operator \( G_z \) is a particular case of a more general operator:
\[
G_{z_{r,s}} : T(2,1,\ldots,1,0,\ldots,0) \times T(2,1,\ldots,1,0,\ldots,0) \to T(2,1,\ldots,1,0,\ldots,0).
\]

For \( m = 1 \) I got the explicit expression for the operator \( G_z \) in 1976. Let me reproduce it. In coordinates \( x, y \) we have \( \omega = dx \wedge dy \). Then
\[
G_z : a \cdot dx^2 + 2b \cdot dx \wedge dy + c \cdot dy^2, \quad a' \cdot dx^2 + 2b' \cdot dx \wedge dy + c' \cdot dy^2 \mapsto
\]
\[
\frac{\partial^2 g}{\partial x^2} dx^2 + 2 \frac{\partial^2 g}{\partial x \partial y} dx \wedge dy + \frac{\partial^2 g}{\partial y^2} dy^2 + (\{c,a'\} - \{a,c'\}) dx \wedge dy +
\]
\[
\left( \frac{\partial^2 a}{\partial y^2} - 2 \frac{\partial^2 b}{\partial x \partial y} + \frac{\partial^2 c}{\partial x^2} \right) (a' dx^2 + 2bdx \wedge dy + c' dy^2) + (\{a,b'\} - \{b,a'\}) dx^2 +
\]
\[
\left( \frac{\partial^2 a}{\partial y^2} - 2 \frac{\partial^2 b}{\partial x \partial y} + \frac{\partial^2 c}{\partial x^2} \right) (adx^2 + 2bdx \wedge dy + cdy^2) ((b,c') - \{c,b'\}) dy^2,
\]
where \( g = ac' - 2bb' + ca' \) and \( \{\cdot,\cdot\} \) is the Poisson bracket.
5. Sketch of the proof of Theorem. Set \( y_i = x_{2m+1-i} \) (1 \( \leq i \leq m \)), \( \partial_i = \frac{\partial}{\partial x_i}, \delta_i = \frac{\partial}{\partial y_i} \).

Denote the elements of the Lie algebra \( \mathfrak{sp}(2m; \mathbb{R}) \subset \mathfrak{h}(M) \) by
\[
e^i = y_i \partial_i, \quad e_{ii} = x_i \delta_i, \quad e^i_j = x_j \partial_i + y_i \delta_j.
\]

Then, clearly,
\[
e^i_j = e^i = y_i \partial_j + y_j \partial_i \quad \text{and} \quad e_{ij} = e_{ji} = x_i \delta_j + x_j \delta_i \quad \text{for} \quad i \neq j.
\]

Let \( I(\rho) \) be the space of differential operators from \( T(\rho) \) into \( C^\infty(M) \) with constant coefficients, i.e.,
\[
I(\rho) = \{ \sum P_i(\partial, \delta) u_i \mid u_i \in V_{\rho}^* \cong V_\rho \}.
\]

The grading in \( I(\rho) \) is induced by that in the space of polynomials \( P_1 \)'s, i.e., \( I(\rho)_0 \cong V_\rho \).

Define the pairing \( I(\rho) \times T(\rho) \rightarrow \mathbb{R} \) by the formula
\[
< Pu, x > = P(\langle u, \chi(x) \rangle)_{x=0}.
\]

On \( I(\rho) \), define the \( \mathfrak{h}(M) \)-action, dual to the action on \( T(\rho) \), via \( L \). Now, to describe the invariant operators it suffices to find all the \( \mathfrak{h}(M) \)-morphisms \( I(\tau) \rightarrow I(\rho) \otimes \mathbb{R} I(\sigma) \). It turns out that such a morphism is completely defined by the image of the highest vector \( v \in V_{\tau} = I(\tau)_0 \). Here we have fixed a Borel subalgebra \( \{ \sum_{i \leq j} a_{ij} x_i \partial_j \} \cap \mathfrak{sp}(2m; \mathbb{R}) \) so that
\[
w \in I(\rho) \otimes \mathbb{R} I(\sigma) \text{ can be the image of a highest weight singular vector if and only if}
\]
\[
e^i_{i+1} w = 0 \text{ for } 1 \leq i \leq m - 1 \text{ and } e^{m,m} w = 0 \quad \text{(conditions on } w \text{ to be a highest vector)}
\]
and
\[
(x_1^2 \delta_1) w = 0. \quad \text{(conditions of singularity of the vector)}
\]

The degree of \( w \in I(\rho) \otimes \mathbb{R} I(\sigma) \) is equal to the order of the corresponding differential operator. The general form of a vector of degree 1 is
\[
w = \sum_{i \leq m} \partial_i z^0_i + \delta_i t^0_i + \partial_i^\prime z^1_i + \delta_i^\prime t^1_i,
\]
where \( z^0_i, t^0_i \in V_{\rho} \otimes V_{\sigma}, \partial' (u \otimes v) = \partial u \otimes v, \partial''(u \otimes v) = u \otimes \partial v \). If \( w \) is a highest vector, then all vectors \( z, t \) are expressed in terms of \( z^0_1, z^1_1 \) which should satisfy
\[
e^i_{i+1} z^0_i = 0 \text{ for } 2 \leq i \leq m - 1, (e^1_2)^2 z^0_1 = 0, e^{m,m} z^1_1 = 0.
\]

The condition \( (x_1^2 \delta_1) w = 0 \) is equivalent to the equation
\[
e^i_{1,1} z^0_1 + e^{\prime\prime}_{1,1} z^1_1 = 0,
\]
where (double) prime means that the operator acts only on the first (second) multiple of the tensor product.

I have succeeded to define all the cases, where the above system possesses a solution in \( V_{\rho} \otimes V_{\sigma} \); though in certain cases I was not able to find the solution itself.

Here is an example of a successfully solved case (case 8):
\[
\lambda = (2,0,\ldots,0), \quad \nu = (\mu_1, \ldots, \mu_{k-1}, \mu_k + 1, \mu_{k+1}, \ldots);
\]
the case \( \nu_k = \mu_k - 1 \) is dual to this one. Let \( u_0 \in V_{\rho} \) be a highest vector, then
\[
u_0 \otimes v - \frac{1}{2} \sum_{2 \leq i \leq k} e^1_i u_0 \otimes e^1_i v \in V_{\rho} \otimes V_{\sigma}
\]
is a highest vector of weight \((\nu_1 + 1, \nu_2, \ldots, \nu_m)\). We conclude that

\[
\begin{align*}
z_0^1 &= u_0 \otimes e_{11}v - \sum_{2 \leq i \leq k} e_i^1 u_0 \otimes e_{11} e_i^1 v - \frac{1}{2} \sum_{2 \leq i < j \leq m} e_i^1 e_j^1 u_0 \otimes (e_{ij} + e_{ij}^1 + e_{1i} e_{1j}^1) v - \\
&\frac{1}{2} \sum_{2 \leq i \leq k} (e_i^1)^2 u_0 \otimes (e_{ii} + e_{ii}^1) v + \frac{1}{2} \sum_{2 \leq i \neq j \leq m} e_i^1 e_{1j} u_0 \otimes (e_i^j + e_i^j e_i^1) v + (\nu_i + e_i^1 e_i^1) v + \\
&\frac{1}{4} \sum_{2 \leq i \leq k} e_{ii} (e_i^1)^2 u_0 \otimes + \frac{\nu_i - 1}{2} \sum_{2 \leq i \leq k} e_{1i} e_i^1 u_0 \otimes e_i^1 v, \\
z_1^1 &= -e_{11} u_0 \otimes v + \sum_{2 \leq i \leq k} e_{1i} e_i^1 u_0 \otimes e_i^1 v.
\end{align*}
\]

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