HIGH-ORDER IMPLICIT TIME-MARCHING METHODS BASED ON GENERALIZED SUMMATION-BY-PARTS OPERATORS

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Abstract.
This article presents the application of the generalized summation-by-parts (GSBP) framework enabling nonuniform distributions of solution points that need not include boundary points to the construction of high-order fully-implicit time-marching methods. GSBP operators require significantly fewer solution points than classical finite-difference summation-by-parts (SBP) operators to achieve the same order of accuracy, leading to substantially more efficient time-marching methods. The theory of classical SBP time-marching methods is extended to time-marching methods which satisfy the generalized SBP definition. The properties of dual-consistent GSBP time-marching methods include: A and L-stability and superconvergence of linear functionals when integrated with the quadrature associated with the discretization. In addition those constructed with a diagonal norm are B-stable. Several model problems are simulated numerically to demonstrate the theoretical results of the article and to present an initial comparison of the efficiency of various classical SBP and GSBP time-marching methods.

Key words. Initial-Value Problems, Summation-by-Parts, Simultaneous-Approximation-Terms, Implicit Time-Marching Methods, Multistage methods, Superconvergence

AMS subject classifications.

1. Introduction. Many dynamical systems in science and engineering are modelled by stiff initial value problems (IVPs) either directly or through semi-discretization of partial differential equations. The numerical solution of stiff IVPs is limited by the stability of the time-marching method employed, motivating the use of unconditionally-stable implicit methods. The computational cost and complexity of such methods is offset by the freedom to choose much larger time steps and therefore fewer time steps, limited only by the desired accuracy of the solution. This also motivates the use of higher-order methods, as larger time steps can be used to obtain the same level of accuracy.

Recently, it was shown that high-order finite-difference (FD) operators satisfying the classical summation-by-parts (SBP) property [26], together with simultaneous approximation terms (SATs) [8, 9, 15, 16], can be used to construct high-order L-stable and B-stable implicit time-marching methods for application to stiff IVPs [29, 31]. In general, this approach leads to a global discretization of the problem in time: the solution at any point in time is fully coupled to all other solution points [31]. This motivates the use of a dual-consistent multiblock approach. The choice of a dual-consistent SAT coefficient decouples the solution within each block from the solution in subsequent blocks, enabling each block to be solved sequentially in time. The solution points within each block, however, remain fully coupled and must be solved for simultaneously [29]. The classical FD-SBP operators considered in [29, 31] are constructed with a uniform distribution of solution points which includes both boundary points of the time-domain.

The generalized framework presented in [12] enables the construction of high-order operators with significantly fewer solution points than the classical FD-SBP approach.
This is accomplished by enabling nonuniform distributions of solution points that need
not include the boundary points, thereby creating the potential for substantially more
efficient time-marching methods. What remains to be shown is that the accuracy and
stability theory of classical FD-SBP time-marching methods extends to those based
on the generalized framework.

The objective of this paper is to combine the ideas of [29, 31] and [12] to define a
new family of time-marching methods. This family includes some known methods as
well as some novel methods. The methods defined by this family have the following
properties: A and L-stability, and superconvergence of linear functionals. Moreover,
the subset of methods within the family based on diagonal norms is characterized
by B-stability. Finally, the generalized approach enables the construction of time-
marching methods that are substantially more efficient than the family of methods
associated with [29, 31].

The paper is organized as follows: Section 2 gives a brief review of the GSBP-SAT
approach presented in [12] within the context of IVPs. The accuracy and stability
of classical FD-SBP time-marching methods is extended to all GSBP time-marching
methods, in particular those which only satisfy the relaxed definition of the generalized
framework, in Sections 3 through 5. Numerical examples are presented in Section 6
to demonstrate various elements of the theory developed in the article. A summary
concludes the paper in Section 7.

2. The GSBP-SAT Approach. This article considers discrete approximations
of IVPs for nonlinear systems of ordinary differential equations (ODEs):

\[ \dot{Y} = F(Y, t), \quad Y(t_0) = Y_0, \quad \text{with} \quad t_0 \leq t \leq t_f, \]

where \( Y \in \mathbb{C}^M, \dot{Y} = \frac{dY}{dt}, F(Y, t) : \{\mathbb{C}^M, \mathbb{R}\} \rightarrow \mathbb{C}^M, \) and \( Y_0 \) is the vector of initial
data. Continuous functions are represented by script upper-case letters; for example
\( U(t) \in C^\infty[t_0, t_f] \) represents a function which is infinitely differentiable over \( t \in [t_0, t_f] \).
The restriction of such functions onto a distribution of solution points, \( t = [t_1, \ldots, t_n] \),
is represented by bold lower-case letters, for example \( u = [U(t_1), \ldots, U(t_n)]^T \) for the
function given above. Furthermore, vectors with the subscript \( d \), for example \( u_d \),
denote a numerical solution to a system of discrete equations. The average time scale
associated with the solution points \( t_i \) is defined as \( \Delta t_n = \frac{\Delta t_N}{n} \), where \( \Delta t_N = t_f - t_0 \)
is the size of the time domain, and \( n \) is the number of solution points in the operator.
Finally, the order of accuracy of a discrete linear operator is defined as the lowest
order of the truncation error at all solution points. Unless otherwise stated, the order
of accuracy is defined with respect to \( \Delta t_n \), for example order \( p \) implies \( O(\Delta t_n^p) \).

2.1. Generalized Summation-By-Parts Operators. The well-posedness of
an IVP requires the existence of a unique solution that is well-conditioned with respect
to the initial data. Assuming that a unique solution exists, the well-posedness of the
IVP can be proved through the use of the energy method in which a bound on the
norm of the solution, called an energy estimate, is derived with respect to the data
\[ [18, 19, 25]. \] To obtain the energy estimate, an inner product between the solution
and IVP is taken and the integrals are related to the initial data through the use
integration-by-parts (IBP).

This motivates the use of discrete operators that form a consistent analogy to
IBP such that the same process can be applied to prove the numerical stability of
the discretized IVP, provided a unique solution exists. The discrete analogue of IBP
is called summation-by-parts (SBP). In [12] a generalized definition of SBP (GSBP)
was introduced:

\[(2.2) \quad u^*HDv + u^*D^THv = u^*\tilde{E}v \approx \bar{U}V|_{t_0}^{t_f},\]

where \(u^*\) is the conjugate transpose of \(u\), and \(D\) is a linear first-derivative GSBP operator defined as:

**Definition 2.1. Generalized summation-by-parts operator [12]:**

A linear operator \(D\) is a GSBP approximation to the first derivative of order \(q\) on the distribution of solution points \(t = [t_1, \ldots, t_n]\), where all \(t_i\) are unique, if \(D\) satisfies:

\[(2.3) \quad Dt^j = jt^{j-1}, \quad j \in [0, q],\]

with \(q \geq 1\), where \(t^j = [t^j_1, \ldots, t^j_n]^T\) forms a monomial with the convention that \(t^{-1} = 0\), \(H\) is a symmetric positive definite (SPD) matrix which defines a discrete inner product and norm:

\[(2.4) \quad (u, v)_H = u^*Hv, \quad ||u||_H^2 = u^*Hu,\]

the product \(HD\) is given the symbol \(\Theta\), and \((\Theta + \Theta^T) = \tilde{E}\) such that

\[(2.5) \quad (t^i)^*\tilde{E}t^j = (i + j) \int_{t_0}^{t_f} \int_{t_0}^{t_f} \tilde{t}^{i+j-1} d\tilde{t} = t^{i+j} - t^{i+j}_0, \quad i, j, \in [0, r],\]

with \(r \geq q\).

In the classical SBP definition [26] \(\tilde{E} = \text{diag}[-1, 0, \ldots, 0, 1]\). Here \(u^*\tilde{E}v\) need only be a consistent and high-order approximation of \(\bar{U}V|_{t_0}^{t_f}\). This, however, does not preclude those operators which satisfy the classical SBP definition [7, 17, 26, 30, 34]. The existence of such operators and their relationship to a quadrature rule of order \(\tau\) is presented in [12]. Sharper bounds on some of these results are also presented in [3].

In this article we make a distinction between diagonal and nondiagonal-norm matrices. The latter will be referred to as dense norms following [12], but includes all nondiagonal norms whether they are strictly dense matrices or not. For example, classical FD-SBP operators with a full, or restricted-full norm are referred to as dense-norm operators.

**2.2. Simultaneous Approximation Terms for GSBP Operators.** Applying a GSBP approach to the IVP (2.1) necessitates a means to impose the initial data. A natural approach is to use simultaneous-approximation-terms (SATs) [29, 31] which weakly impose the initial data via penalty terms.

To apply the SAT technique requires an approximation of the solution at the individual boundaries of the time domain \(\tilde{y}_{t_0} \approx \mathcal{Y}(t_0)\) and \(\tilde{y}_{t_f} \approx \mathcal{Y}(t_f)\). The initial boundary is required to impose the initial data, and the final boundary is required in multiblock implementations to couple the solution between blocks. In the classical approach this is obtained from the solution values \(\tilde{y}_{d,0}\) and \(\tilde{y}_{d,n}\); however, in a generalized framework these values may be approximated by consistent and high-order projection of the solution values. The use of projection operators in the construction of SATs was first proposed in [1, 2, 13, 33], though not for operators which produce a consistent analogy to IBP. Here, a projection operator is defined by

\[(2.6) \quad \chi^T_{t_0}t^j = t^{0,j}_0 \quad \text{and} \quad \chi^T_{t_f}t^j = t^{f,j}_f, \quad j \in [0, q \leq r \leq n - 1].\]
Following [12], the fully-discrete form of (2.1) can then be written as

\[(D \otimes I_M)y_d = F_d - \sigma (H^{-1} \chi_{t_0} \otimes I_M)((\chi_{t_0}^T \otimes I_M)y_d - (I_n \otimes \gamma_0)),\]

with

\[y_d = \begin{bmatrix} y_{d,1} \\ \vdots \\ y_{d,n} \end{bmatrix}, \quad F_d = \begin{bmatrix} F(y_{d,1}, t_1) \\ \vdots \\ F(y_{d,n}, t_n) \end{bmatrix}, \quad y_{d,i} = \begin{bmatrix} y_{d,k,1} \\ \vdots \\ y_{d,k,M} \end{bmatrix},\]

where \(\sigma\) is the SAT penalty parameter. A convenient vector decomposition of \(\tilde{E} = \Theta + \Theta^T\) which relates the definition of the derivative operator and form of the SAT terms through the projection operators is [12]:

\[\tilde{E} = \Theta + \Theta^T = \chi_{t_f} \chi_{t_f}^T - \chi_{t_0} \chi_{t_0}^T.\]

This formulation of both \(\Theta\), i.e., that it have the property \(\Theta + \Theta^T = \chi_{t_f} \chi_{t_f}^T - \chi_{t_0} \chi_{t_0}^T\), and of the SAT terms is assumed for the rest of this article. The choice of SAT parameter for stability of time-marching methods based on GSBP operators of this form is addressed further in Section 4.

The following extension of Assumption 1 from [31] is required to guarantee a unique solution for inherently stable linear ODEs [29, 31].

**Assumption 2.2.** For \(\sigma < -\frac{1}{4}\), all eigenvalues of \(\Theta - \sigma \chi_{t_0} \chi_{t_0}^T\) have strictly positive real parts.

This Assumption is critical to the proof of L-stability and various accuracy theorems presented in later sections. A proof of the assumption for the classical second-order FD-SBP operator is presented in [31] with numerical demonstration of the assumption for higher-order diagonal-norm FD-SBP operators. Using a multiblock approach, in which all operators are identical and of fixed size, this can easily be verified numerically. This property has been verified for all GSBP operators presented in [12].

3. **Accuracy.** This section extends various formal accuracy results of FD-SBP time-marching methods to GSBP time-marching methods. The primary goal is to extend the theory of superconvergent linear functionals integrated with the quadrature associated with the SBP-SAT discretization. This was first shown for classical FD-SBP-SAT discretizations of initial boundary value problems in [21, 22] and extended for IVPs in [29]. The motivation is that in many numerical simulations, integrated quantities are often of more interest then the solution itself. In addition, the superconvergence of linear functionals implies that the solution projected to the final time-domain boundary of the operator is superconvergent. This is analogous to a Runge-Kutta method with lower stage-order and a higher-order solution update. Finally, the section ends with a brief discussion of the accuracy for problems with stiff source terms.

3.1. **Pointwise Accuracy.** This section introduces some useful definitions related to the error of the numerical solution, which will be used in future sections, and formalizes the pointwise accuracy of the solution. To begin, consider a scalar IVP which is linear with respect to the solution:

\[\gamma' = \lambda \gamma + G(t), \quad \gamma(t_0) = \gamma_0, \quad \text{with} \quad t_0 \leq t \leq t_f,\]
between interior and boundary components of the operator. The GSBP-SAT discretization of (3.1) is:

\[
Dy_{d} = \lambda y_{d} + g + \sigma H^{-1} \chi_{t_{0}} (\chi_{t_{0}}^{T} y_{d} - \gamma_{0}).
\]  

The truncation error is obtained by replacing the discrete solution in (3.2) with the continuous solution of (3.1) projected onto the distribution of solution points, \( y \), and rearranging:

\[
T_{e} = Dy - \lambda y - g - \sigma H^{-1} \chi_{t_{0}} (\chi_{t_{0}}^{T} y - \gamma_{0}) = Dy - \sigma H^{-1} \chi_{t_{0}} (\chi_{t_{0}}^{T} y - \gamma_{0}) - y'.
\]

This differs from the classical SBP result found in [29], \( T_{e} = Dy - y' \), since \( \chi_{t_{0}}^{T} y \) is not in general equal to \( \gamma_{0} \). Comparing (2.9) and Definition 2.1, the projection operator in (3.3) must be of order greater than or equal to \( q \). Thus, the truncation error is also by definition of order \( q \).

The pointwise accuracy of a GSBP time-marching method is defined by the lowest order in the error of the numerical solution at all points. This is found by taking the infinity norm of the error vector defined in terms of the truncation error (3.3):

\[
e = y - y_{d} = (\Theta - \sigma \chi_{t_{0}} \chi_{t_{0}}^{T} - \lambda H)^{-1} H T_{e}.
\]

To determine the pointwise accuracy first consider an extension of Assumption 2 and Corollary 1 from [29]:

**Assumption 3.1.** If \( \sigma < -\frac{1}{2} \) and \( \text{Re}(\lambda) \leq 0 \), then the individual elements of \( (\Theta - \sigma \chi_{t_{0}} \chi_{t_{0}}^{T} - \lambda H)^{-1} H \) are at most order unity.

**Corollary 3.2.** If Assumption 3.1 holds, \( \sigma < -\frac{1}{2} \) and \( \text{Re}(\lambda) \leq 0 \), then \( ||(\Theta - \sigma \chi_{t_{0}} \chi_{t_{0}}^{T} - \lambda H)^{-1} H||_{\infty} \leq O(1) \).

To justify the assumption, consider (3.1) with homogeneous initial condition and exact solution \( Y(t) = e^{-\lambda t} \int_{0}^{t} e^{\lambda \tau} G(\tau) d\tau \). The contribution of \( G(t) \) from each interval of size \( \Delta t_{0} \) to the exact solution is of order unity. Thus, we expect that the contribution of each element of \( g \) to the numerical solution \( y_{d} = (\Theta - \sigma \chi_{t_{0}} \chi_{t_{0}}^{T} - \lambda H)^{-1} H g \) be of order unity as well [29]. Finally, the pointwise accuracy of the discrete solution can be summarized with the following theorem:

**Theorem 3.3.** The pointwise solution accuracy \( ||e||_{\infty} \) of a GSBP-SAT discretization (3.2) of the IVP (3.1) using a GSBP operator of order \( q \) and a compatible SAT implementation is of order \( q \) provided Assumption 3.1 holds.

**Proof.** The proof is analogous to Proposition 5 in [29] without making a distinction between interior and boundary components of the operator. \( \square \)

This general result for GSBP time-marching methods is one order lower than Proposition 5 in [29] for classical FD-SBP time-marching methods. The reason is that there is not such a clear distinction between interior and boundary components of GSBP operators. However, the generalized framework enables the construction of high-order operators with significantly fewer solution points than classical FD-SBP operators, and can therefore be more efficient. In addition, Theorem 3.3 highlights the importance of Theorem 5.1 presented in Section 5, which shows that the pointwise accuracy of multiblock discretizations is one order higher.

**3.2. The Dual Problem.** This section presents a brief review of the dual problem, a key tool required for proving the superconvergence of linear functionals in Section 3.3. The derivation of the continuous dual problem

\[
-\Phi' = \lambda \Phi + \mathcal{K}(t), \quad \Phi(t_{f}) = \bar{a}, \quad \text{with} \quad t_{0} \leq t \leq t_{f}.
\]
and dual functional

\[ J(\Phi) = J(Y) = (\Phi, G) + \Phi |_{t_0}. \]

can be found in several FD-SBP references (e.g. [21, 29]). Here we derive the discrete dual problem and dual functional for GSBP time-marching methods which satisfy the generalized SBP definition.

To begin, consider a functional of the discrete primal problem (3.2):

\[ J_H(y_d) = (k, y_d)_H + \alpha \chi^T_{t_f} y_d. \]

Subtracting the inner product of the discrete primal problem (3.2) and a vector \( \phi_d \) leads to

\[ J_H(y_d) = (k, y_d)_H + \alpha \chi^T_{t_f} y_d - (\phi_d, D y_d - \lambda y_d - g - H^{-1} \sigma \chi_{t_0} (\chi^T_{t_0} y_d - Y_0))_H. \]

Making use of (2.9) and simplifying yields

\[ J_H(y_d) = (\phi_d, g)_H - \sigma \phi_d \chi_{t_0} Y_0 + (H^{-1} (\Theta + (\sigma + 1) \chi_{t_0} \chi^T_{t_0} + \bar{\lambda}) \phi_d - H^{-1} \chi_{t_f} (\chi^T_{t_f} \phi_d - \bar{\alpha}) + k, y_d)_H. \]

Rearranging, a discrete approximation to the dual functional (3.6) can be extracted:

\[ J_H(\phi_d) = (\phi_d, g)_H - \sigma \phi_d \chi_{t_0} Y_0. \]

This must be equal to the functional (3.7) of the discrete primal problem, which will be the case if:

\[ -D \phi_d = \bar{\lambda} \phi_d + k - (1 + \sigma) H^{-1} \chi_{t_0} \chi^T_{t_0} \phi_d - H^{-1} \chi_{t_f} (\chi^T_{t_f} \phi_d - \bar{\alpha}). \]

This condition (3.11) is also an approximation of the dual problem (3.5). Furthermore, (3.11) and (3.10) become consistent analogues of the continuous dual problem and functional when \( \sigma = -1 \). This is called dual-consistency [28]. The dual-consistent discrete dual problem and dual functional are:

\[ -D \phi_d = \bar{\lambda} \phi_d + k - H^{-1} \chi_{t_f} (\chi^T_{t_f} \phi_d - \bar{\alpha}), \]

and

\[ J_H(y_d) = (\phi_d, g)_H + \phi_d \chi_{t_0} Y_0 = J_H(\phi_d). \]

With these definitions, the error in the dual-consistent discrete dual problem (3.12) can be defined as:

\[ \tilde{e} = \phi - \phi_d = (-\Theta + \chi_{t_f} \chi^T_{t_f} - \bar{\lambda} H)^{-1} H \tilde{e}. \]

where the truncation error is:

\[ \tilde{e} = -D \phi - \bar{\lambda} \phi - k + H^{-1} \chi_{t_f} (\chi^T_{t_f} \phi - \bar{\alpha}) = -D \phi + H^{-1} \chi_{t_f} (\chi^T_{t_f} \phi - \bar{\alpha}) + \phi', \]

which is of order \( q \) by Definition 2.1 and (2.9).
3.3. Superconvergence. The objective of this section is to generalize the theory of superconvergent linear functionals presented for FD-SBP time-marching methods in [29] to GSBP time-marching methods. The classical theory applies to some GSBP operators which satisfy the classical SBP definition \((E = \text{diag}[-1,0, \ldots, 0,1])\), though in general it is not sufficient. Here, the order of bounds on superconvergence is sharpened for GSBP operators which satisfy the classical SBP definition extended for those which only satisfy the SBP definition of the generalized framework.

The accuracy with which GSBP norms can compute approximations of continuous L2 inner products, specifically inner products involving both the primal and dual problems, is critical. This motivates the following definition:

**Definition 3.4. Accuracy of a GSBP norm:** The order \(\rho\) of a norm \(H\) associated with a GSBP operator \(D = H^{-1}\Theta\) of order \(q\) is the order with which:

- \((y,z)_H\) approximates \((y,z)\); and
- \((\phi, Dy + H^{-1} \chi_{a_0}(\chi_{e_0}^T y - Y_h))_H\) approximates \((\Phi, Y')\),

where \(Y\) and \(\Phi\) is the solution to the continuous primal problem (3.1) and dual problem (3.5), and \(Y_h\) is the initial condition of the primal problem.

For diagonal norms associated with GSBP operators \(\rho = \min(2q + 1, \tau)\), and for dense norms associated with GSBP operators \(\rho = \min(2q + 1, s)\), where \(2[\frac{q}{2}] \leq s \leq \tau\). The derivation of these values, along with a more precise definition of \(s\) is presented in [3, 12]. Note that \(\rho\) is always greater than or equal to the order \(q\) of the GSBP operator itself.

To begin the investigation of superconvergent linear functionals, the dual problem is initially assumed to have homogeneous initial data, i.e. \(\alpha = 0\); then the theory is extended for the non-homogeneous case. The following Theorem presents a generalization of Proposition 7 for FD-SBP time-marching methods presented in [29] to GSBP time-marching methods:

**Theorem 3.5.** If \(y_d\) is the solution of a dual-consistent GSBP-SAT discretization (3.2) of the primal problem (3.1) with \(\text{Re}(\lambda) \leq 0\) using a GSBP operator and compatible SAT implementation of order \(q\) and associated with a norm of order \(\rho\), then the discrete functional \(J_H(y_d) = (k, y_d)_H\) approximates \(J(Y) = (K(t), Y)\) with order \(\min(2q + 1, \rho)\).

**Proof.** The proof follows analogously from Proposition 7 in [29]; however, there are a few subtleties that are important to address.

Following the presentation in [29], begin with the exact functional \(J(Y)\) and expand

\[
J(Y) = (k, y)_H + O(\Delta t^n) \\
= (k, y_d)_H + (k, y - y_d)_H + O(\Delta t^n).
\]

The first term is the discrete functional sought; therefore, what remains is to determine the accuracy to which \((k, y - y_d)_H \approx 0\). Augmenting \((k, y - y_d)_H\) with the inner product of the discrete dual solution and the error equation (3.4) rearranged to equal zero yields:

\[
(k, y - y_d)_H = (k, y - y_d)_H + (\phi_d, T_e)_H \\
= (\phi_d, H^{-1}(\Theta + \chi_{a_0} \chi_{e_0}^T - \lambda I)(y - y_d)) \\
= (k - (H^{-1}(\Theta + \chi_{a_0} \chi_{e_0}^T - \lambda I)\phi_d, y - y_d)_H + (\phi_d, T_e)_H.
\]

Consider the left-hand component of the first inner product in (3.17); using (2.9) and
simplifying yields

\[
\mathbf{k} - (H^{-1}(-\Theta + \tilde{E} + \chi_{t_0}T_0^{-1} - \tilde{\lambda}I)\phi_d
\]

(3.18)

\[
= \mathbf{k} - (H^{-1}(-\Theta + \chi_{t_1}T_1^{-1}) - \tilde{\lambda}I)\phi_d
\]

\[
= H^{-1}\Theta\phi_d + \tilde{\lambda}\phi_d + \mathbf{k} - H^{-1}\chi_{t_1}T_1^{-1}\phi_d,
\]

which is the discrete dual problem (3.12) with \(\alpha = 0\) and is therefore equal to zero. The second term in (3.17) can be expanded as

\[
(\phi_d, T_e)_H = (\phi_d, T_e)_H + (\phi, T_e)_H - (\phi, T_c)_H
\]

(3.19)

\[
= (\phi_d - \phi, T_e)_H + (\phi, Dy + H^{-1}\chi_{t_0}(\chi_{t_0}T_0^{-1}y - \gamma_0) - y')_H
\]

\[
= (\tilde{T_e}, T_c)_H + (\phi, Dy + H^{-1}\chi_{t_0}(\chi_{t_0}T_0^{-1}y - \gamma_0))_H - (\phi, y')_H.
\]

This is where the proof deviates from [29]. In the classical case, the second inner product has no contribution from the SATs, as \(\chi_{t_0}T_0^{-1}y = \gamma_0\). In the generalized case, this is not necessarily true; however, the accuracy of this term has been defined in Definition 3.4. In addition, the inner product of the truncation errors is no longer negligible and may become dominant if the order of the quadrature is significantly higher than the order of the GSBP operator. Thus, using Definition 3.4, this gives

\[
(\phi_d, T_e)_H = O(\Delta t_n^{\min(2q+1, \rho)}) + (\Phi, y') + O(\Delta t_n^\rho) - ((\Phi, y') + O(\Delta t_n^\rho))
\]

(3.20)

\[
= O(\Delta t_n^{\min(2q+1, \rho)}) + O(\Delta t_n^\rho) + O(\Delta t_n^\rho) = O(\Delta t_n^{\min(2q+1, \rho)}).
\]

Combining the results in (3.16) and (3.20), the discrete functional \(J_H(y_d) = (\mathbf{k}, y_d)_H\) approximates \(\mathcal{J}(\mathcal{Y}) = (K(t), \mathcal{Y})\) with order \(\min(2q + 1, \rho)\).

Given that the order of a diagonal norm \(\rho\) associated with a GSBP operator is at least the order \(q\) of the GSBP operator itself (See Definition 3.4), this is a very useful result. To be explicit, the order of linear functionals of the form \(\mathcal{J}(\mathcal{Y}) = (K(t), \mathcal{Y})\) for \(K(t) \in C^{\min(2q+1, \rho)}\) computed from dual-consistent GSBP time-marching methods is at least twice as high as the order of the time-marching method itself.

In contrast, dense norms associated with GSBP operators are not required to be significantly more accurate than the operator itself (See Definition 3.4). This often permits dense-norm GSBP operators to be of higher order relative to the accuracy of the norm than diagonal norm GSBP operators; however, this also means that the rate of superconvergence is generally not as significant with respect to the order of the time-marching method.

Next, consider the accuracy of the numerical solution at the final boundary of the time domain \(\chi_{t_f}T_0^{-1}y_d = \tilde{u}_f\). It was shown in Proposition 8 of [29], that for FD-SBP time-marching methods these values are superconvergent with the same rate as the linear functionals. The following lemma is introduced to simplify the extension of the proof to GSBP time-marching methods, where \(\mathbf{1} = [1, \ldots, 1]^T\):

**Lemma 3.6.** For a dual-consistent GSBP-SAT discretization, \(\chi_{t_f}^T(\Theta - \sigma\chi_{t_0}\chi_{t_0}^T)^{-1} = \mathbf{1}^T\).

**Proof.** Beginning with the assertion that \(\chi_{t_f}^T(\Theta - \sigma\chi_{t_0}\chi_{t_0}^T)^{-1} = \mathbf{1}^T\), multiplying through by \((\Theta + \chi_{t_0}\chi_{t_0}^T)\) and simplifying yields

\[
\chi_{t_f}^T = \mathbf{1}^T(\Theta + \chi_{t_0}\chi_{t_0}^T).
\]

(3.21)

Using the GSBP property, namely that \(\Theta + \Theta^T = \tilde{E}\), yields

\[
\chi_{t_f}^T = \mathbf{1}^T(-\Theta^T + \tilde{E} + \chi_{t_0}\chi_{t_0}^T),
\]

(3.22)
and using (2.9) leads to

\[(3.23)\]
\[
\chi_{\Gamma_f}^T = - (\Theta \mathbb{I})^T + \mathbb{I}^T \chi_{\Gamma_f} \chi_{\Gamma_f}.
\]

A dual-consistent GS BP operator satisfies the identities \(D_1 \mathbb{I} = \Theta \mathbb{I} = 0\) and \(\mathbb{I}^T \chi_{\Gamma_f} = 1\), eliminating the first term in the expression above and reducing the second term to \(\chi_{\Gamma_f}^T\). An identity is obtained, proving the lemma. \[\square\]

Using this result, the superconvergence of the solution projected to the final boundary of the time domain is now presented in the following Theorem:

**Theorem 3.7.** If \(y_d\) is the solution of a dual-consistent GS BP-SAT discretization (3.2) of the primal problem (3.1) with \(\text{Re}(\lambda) \leq 0\) using a GS BP operator and compatible SAT implementation of order \(q\) and associated with a norm of order \(\rho\), then the solution values projected to the boundary of the time domain \(\tilde{y}_{t_f} = \chi_{\Gamma_f}^T y_d\) approximates \(\gamma(t_f)\) with order \(\min(2q+1, \rho)\).

**Proof.** Consider the discrete primal problem (3.2) with dual-consistent SAT value \(\sigma = -1\). Rearranging, and left-multiplying by \(\chi_{\Gamma_f}^T\) gives

\[(3.24)\]
\[
\chi_{\Gamma_f}^T y_d = \tilde{y}_{t_f} = \chi_{\Gamma_f}^T (\Theta + \chi_{\Gamma_0}^T)^{-1} H (\lambda y_d + G + H^{-1} \chi_{\Gamma_0} Y_0).
\]

Using Lemma 3.6, yields

\[(3.25)\]
\[
\tilde{y}_{t_f} = (\mathbb{I}, \lambda y_d)_H + (\mathbb{I}, G)_H + \mathbb{I}^T \chi_{\Gamma_0} Y_0.
\]

Recall that \(\mathbb{I}^T \chi_{\Gamma_0} = 1\) for consistent GS BP operators. Furthermore, using Theorem 3.5 and Definition 3.4 and simplifying yields

\[(3.26)\]
\[
\tilde{y}_{t_f} = \left(1, \lambda \gamma\right) + \left(1, G(t)\right) + \gamma_0 + O(\Delta t_n^{\min(2q+1, \rho)})
\]  
\[
= \int_{t_0}^{t_f} [\lambda \gamma + G(t)] dt + \gamma_0 + O(\Delta t_n^{\min(2q+1, \rho)}).
\]

Substituting using the continuous primal problem (3.1) gives

\[(3.27)\]
\[
\tilde{y}_{t_f} = \int_{t_0}^{t_f} \gamma'(t) dt + \gamma_0 + O(\Delta t_n^{\min(2q+1, \rho)})
\]  
\[
= \gamma(t_f) + O(\Delta t_n^{\min(2q+1, \rho)}),
\]

thus completing the proof. \[\square\]

Hence superconvergence is obtained not only for linear functionals of the homogeneous dual problem, but also for the solution values projected to the boundary of the time domain. This further motivates the use of multiblock discretizations with minimum block sizes. Finally, combining Theorems 3.5 and 3.7, the accuracy of the general linear functional \(J_H(y_d) = (k, y_d)_H + \alpha \chi_{\Gamma_f}^T y_d\) follows immediately:

**Theorem 3.8.** If \(y_d\) is the solution of a dual-consistent GS BP-SAT discretization (3.2) of the primal problem (3.1) with \(\text{Re}(\lambda) \leq 0\) using a GS BP operator and compatible SAT implementation of order \(q\) and associated with a norm of order \(\rho\), then the discrete functional \(J_H(y_d) = (k, y_d)_H + \alpha \chi_{\Gamma_f}^T y_d\) approximates \(J(\gamma) = (K(t), \gamma) + \alpha \gamma(T)\) with order \(\min(2q+1, \rho)\), for \(K(t) \in C^\tau\).

**Proof.** The proof follows from Theorems 3.5 and 3.7. \[\square\]

These last two theorems are the primary results of this section: that linear functionals of the solution, as well as the solution projected to the end of the time domain, are superconvergent. The additional flexibility of the generalized framework enables the construction of smaller operators associated with more efficient quadrature rules than classical FD-SBP operators [12]. Therefore, not only are the time-marching methods more efficient, but the rate of superconvergence is also higher with respect to the number of solution points.
3.4. Stiff Source Terms. It is well known that implicit Runge-Kutta methods applied to problems with stiff source terms can suffer from what is called order reduction, where the convergence rate of the solution follows the accuracy of the internal stage approximations rather than the global order of the method. This is distinct from the case of stiff parasitic modes that might arise, for example, from semi-discrete approximations to PDEs and have negligible magnitude [27]. In that case, the accuracy of these modes is not important, only their numerical stability.

A well-known investigation of stiff source terms and order reduction was carried out by Prothero and Robinson [32], who proposed the following problem to study stiff source terms

\[ \frac{d\mathcal{Y}}{dt} = \lambda \mathcal{Y} + \Psi' - \lambda \Psi, \]

where \( \Psi(t) \) is the prescribed exact solution, and \( \lambda \leq 0 \) controls the stiffness of the source term. This is equivalent to the primal problem (3.1) with a stiff source term, and hence the stiffness comes from the homogeneous and particular solutions having different time scales. Numerical simulation [29, 31] indicates that similar order reduction is exhibited by classical FD-SBP-SAT discretizations, where the pointwise solution values can be considered analogous to the internal stage approximations of an Runge-Kutta method. This result is proved in Proposition 6 of [29], and can be extended for to all GSBP operators using an analogous proof. The extended Theorem is presented here without proof:

**Theorem 3.9.** The pointwise accuracy \( ||e||_\infty \) of a GSBP-SAT discretization (3.2) of the linear IVP (3.1) with a stiff source term \( \mathcal{G}(t) = \Psi' - \lambda \Psi \) using a GSBP operator and compatible SAT implementation of order \( q \) in the stiff limit is \( \frac{1}{q}O(q) \).

This result supersedes the superconvergence of linear functionals and the numerical solution at the final time domain boundary shown in Section 3.3. Therefore, in the case of stiff source terms the order of the GSBP operator becomes more important than the order of the associated norm. Dense-norm GSBP operators are less restricted by the order of the associated norm than diagonal-norm GSBP operators. Hence, dense-norm GSBP time-marching methods can be more efficient for this class of problems than diagonal-norm GSBP time-marching methods, when associated with norms of the same order. The only caveat is the loss of nonlinear stability for dense-norm GSBP time-marching methods shown in Section 4.2.

4. Stability. Thus far it has been shown that the extensions of the generalized framework enable the construction of more efficient time-marching methods than those based on classical FD-SBP theory. These methods are only desirable for stiff IVPs if they also maintain the same stability properties. In this section, several linear and nonlinear stability criteria are considered. The analysis of these criteria for time-marching methods based on classical FD-SBP-SAT discretizations was done in [29]. Here we extend the theory to GSBP time-marching methods.

4.1. Linear Stability. To begin the discussion of stability, consider the scalar linear ODE,

\[ \mathcal{Y}' = \lambda \mathcal{Y}, \quad \mathcal{Y}(t_0) = \mathcal{Y}_0, \quad \text{with} \quad t_0 \leq t \leq t_f, \]

where \( \mathcal{Y} \in \mathbb{C} \), and \( \lambda \) is a complex constant. It is well known that (4.1) is inherently stable for \( \text{Re}(\lambda) \leq 0 \) (e.g. [10, 27]). This can also be shown using the energy method [29].
In the discrete case, which for a GSBP time-marching method is given by

\[ Dy_d = \lambda y_d - H^{-1} \chi_{t_0} \left( \chi^T_{t_0} y_d - Y_0 \right), \]

unconditional stability of the numerical solution to the linear IVP (4.1) is defined as A-stability [10]. A numerical method applied to (4.1) is called A-stable if \( \text{Re}(\lambda) \leq 0 \) implies that

\[ |\tilde{y}_{t_f}| \leq |Y_0|, \]

where \( \tilde{y}_{t_f} = \mathcal{Y}(t_f) \) is the numerical solution at or projected to \( t_f \). All dual-consistent FD-SBP time-marching methods where shown to be A-stable in Proposition 1 of [29]. Using the ideas presented in [12], this proposition can be extended to include all GSBP time-marching methods, specifically those constructed with a general projection operator. We find it instructive to show the influence of the projection operators on a stability proof once, and leave the extension of similar proofs to the reader.

**Theorem 4.1.** All dual-consistent GSBP-SAT discretizations (4.2) of the linear IVP (4.1) with \( \lambda \leq 0 \) are A-stable.

**Proof.** The energy method is applied by left-multiplying (4.2) by \( y^* H \), equivalent to multiplying by the complex conjugate of the solution and integrating in time. Adding the conjugate transpose of the resulting equation, and simplifying leads to

\[ y^*_d (\Theta + \Theta^T) y_d = 2\text{Re}(\lambda)||y_d||^2_H + 2\sigma||\tilde{y}_{t_0}||^2 - \sigma \tilde{y}_{t_0} Y_0 - \sigma \tilde{y}_{t_0} \bar{f}_{t_0}. \]

Further simplifying using (2.9) gives

\[ ||\tilde{y}_{t_f}||^2 = 2\text{Re}(\lambda)||y_d||^2_H + (1 + 2\sigma)||\tilde{y}_{t_0}||^2 - \sigma (\tilde{y}_{t_0} Y_0 + \tilde{y}_{t_0} \bar{f}_{t_0}). \]

Finally, completing the square yields the discrete energy estimate

\[ ||\tilde{y}_{t_f}||^2 = 2\text{Re}(\lambda)||y_d||^2_H + (1 + 2\sigma)\left|\tilde{y}_{t_0} - \frac{\sigma}{1 + 2\sigma} Y_0\right|^2 - \frac{\sigma^2}{1 + 2\sigma} |Y_0|^2. \]

The numerical solution at time \( t_f \) will be bounded by the initial data, in the sense that \( ||\tilde{y}_{t_f}||^2 \leq L|Y_0|^2 \) with \( L \) a positive real number, provided that \( \text{Re}(\lambda) \leq 0 \) and \( \sigma < -\frac{1}{2} \). Furthermore, the numerical solution is guaranteed to be strictly less than the initial data if \( \sigma = -1 \), the dual-consistent SAT penalty value. This also renders the energy estimate optimally sharp

\[ ||\tilde{y}_{t_f}||^2 = 2\text{Re}(\lambda)||y_d||^2_H - ||\tilde{y}_{t_0} - Y_0||^2 + |Y_0|^2. \]

Given that \( \text{Re}(\lambda) \leq 0 \), the proof is complete

\[ ||\tilde{y}_{t_f}|| \leq |Y_0|. \]
Here, an alternate approach to [29] is given for the proof of L-stability, which is inspired by Proposition 3.8 in [20]. The proof is simplified by first introducing the following lemma:

**Lemma 4.2.** If Assumption 2.2 holds, then \((\Theta + \chi_{t_0} \chi_{t_0}^T)^{-1} \chi_{t_0} = \mathbb{I}\).

**Proof.** Beginning with the assertion that \((\Theta + \chi_{t_0} \chi_{t_0}^T)^{-1} \chi_{t_0} = \mathbb{I}\), multiplying through by \((\Theta + \chi_{t_0} \chi_{t_0}^T)\), and simplifying yields

\[
\chi_{t_0} = (\Theta + \chi_{t_0} \chi_{t_0}^T) \mathbb{I} = \Theta \mathbb{I} + \chi_{t_0} \chi_{t_0}^T \mathbb{I}.
\]

A consistent GSBP operator satisfies the identities \(D \mathbb{I} = \Theta \mathbb{I} = \mathbb{0}\) and \(\chi_{t_0}^T \mathbb{I} = 1\) eliminating the first term on right side of the expression above and reducing the second term to \(\chi_{t_0}\). This leads to an identity, proving the lemma. \(\square\)

The proof is now presented for the L-stability of dual-consistent time-marching methods based on GSBP operators:

**Theorem 4.3.** If Assumption 2.2 holds, then all dual-consistent GSBP-SAT discretizations (4.2) of the linear IVP (4.1) with \(\text{Re}(\lambda) \leq 0\) are L-stable.

**Proof.** The first requirement of L-stability, A-stability, follows from Theorem 4.1. For the additional requirement that \(|\tilde{y}_{t_f} - y_{t_f}| \to 0\) as \(|\lambda| \to \infty\), consider the GSBP-SAT discretization (4.2) of (4.1). Rearranging using Assumption 2.2 and Lemma 4.2, this yields

\[
y_d = (\Theta + \chi_{t_0} \chi_{t_0}^T)^{-1} H \lambda y_d + \mathbb{I} \mathbb{0}_0.
\]

which can also be used to obtain an expression for \(\tilde{y}_{t_f}\)

\[
\tilde{y}_{t_f} = (\Theta + \chi_{t_0} \chi_{t_0}^T)^{-1} H \lambda y_d + \mathbb{I} \mathbb{0}_0.
\]

Inserting (4.11) into (4.12), and recalling that \(\chi_{t_f}^T \mathbb{I} = 1\) for all GSBP operators yields

\[
\tilde{y}_{t_f} = (1 + \chi_{t_f}^T (\Theta + \chi_{t_0} \chi_{t_0}^T)^{-1} H [I - \lambda (\Theta + \chi_{t_0} \chi_{t_0}^T)^{-1} H]^{-1} \mathbb{I}) \mathbb{0}_0.
\]

Taking the limit as \(|\lambda| \to \infty\)

\[
\tilde{y}_{t_f} = (1 - \chi_{t_f}^T (\Theta + \chi_{t_0} \chi_{t_0}^T)^{-1} H [(\Theta + \chi_{t_0} \chi_{t_0}^T)^{-1} H]^{-1} \mathbb{I}) \mathbb{0}_0.
\]

and simplifying yields

\[
\tilde{y}_{t_f} = (1 - \chi_{t_f}^T \mathbb{I}) \mathbb{0}_0 = 0.
\]

This implies that \(|\tilde{y}_{t_f}| \to 0\) as \(|\lambda| \to \infty\) and completes the proof. \(\square\)

In addition to unconditional stability of inherently stable linear IVPs, this guarantees damping of parasitic modes. A further stability definition, called linear stability, was proposed in [29] for linear systems of ODEs

\[
\dot{Y} = AY, \quad Y(t_0) = Y_0, \quad \text{with} \quad t_0 \leq t \leq t_f
\]

where \(Y \in \mathbb{C}^M\), and \(A\) is an \(M \times M\) matrix. A numerical method applied to (4.16), for example obtained with a GSBP-SAT discretization,

\[
(D \otimes I_M)Y_d = (I_n \otimes A)Y_d - \sigma(H^{-1} \chi_{t_0} \otimes I_M)((\chi_{t_0}^T \otimes I_M)Y_d - (I_n \otimes Y_0)),
\]
is called linearly stable if $PA + A^TP$ non-negative definite implies that

$$|\tilde{y}_{t_f}| \leq |Y_0|,$$

where $P$ is an SPD matrix defining a discrete inner product and norm over $\mathbb{C}^M$:

$$\langle Y, Z \rangle_P = Y^*PZ, \quad ||Y||^2_P = Y^*PY.$$

The linear stability of all dual-consistent FD-SBP time-marching methods was presented in Proposition 2 of [29]. The extension of this Proposition for GSBP time-marching methods which only satisfy the relaxed SBP definition of the generalized framework follows analogously to Theorem 4.1. Therefore, the following Theorem is presented without proof:

**Theorem 4.4.** All dual-consistent GSBP-SAT discretizations (4.17) of the linear system of IVP (4.16) with $PA + A^TP$ non-negative definite are linearly stable.

In summary, all dual-consistent time-marching methods based on GSBP operators are unconditionally stable for linear problems, and furthermore provide damping of parasitic modes. These conditions are derived for linear problems, but are often sufficient for nonlinear problems as well.

### 4.2. Nonlinear Stability and Contractivity.

Next, consider a subset of the general IVP (2.1) which satisfy the one-sided Lipschitz condition [11]:

$$\text{Re}[\langle F(Y, t) - F(Z, t), Y - Z \rangle_P] \leq \nu ||Y - Z||^2, \quad \forall Y, Z \in \mathbb{C}^M, \text{ and } t \in \mathbb{R}$$

where $\nu \in \mathbb{R}$ is the one-sided Lipschitz constant, and $P$ is an SPD matrix defining a discrete inner product and norm as in (4.19).

An IVP (2.1) is said to be contractive if it satisfies the one-sided Lipschitz condition with $\nu \leq 0$. The significance of this condition is that the distance between any two solutions, $||Y(t) - Z(t)||$, does not increase with time [20].

In the discrete case, it is desirable for the numerical method to behave in a similar fashion for contractive problems. This motivates the following definition of B-stability [5]. A numerical method applied to the IVP (2.1) is called B-stable if the one-sided Lipschitz condition (4.20) with $\nu \leq 0$ implies that

$$||\tilde{y}_{t_f} - \tilde{z}_{t_f}||_P \leq ||Y_0 - Z_0||_P,$$

where $\tilde{y}_{t_f}$ and $\tilde{z}_{t_f}$ are the numerical solutions at time $t_f$ given initial data $Y_0$ and $Z_0$ respectively. In [29] it was shown that all dual-consistent diagonal-norm FD-SBP time-marching methods are B-stable; however, the result does not hold for those based on dense-norm FD-SBP operators. The proof for GSBP time-marching methods is analogously to Proposition 4 in [29] using the ideas presented in Theorem 4.1. The following Theorem is therefore presented without proof:

**Theorem 4.5.** All diagonal-norm dual-consistent GSBP-SAT discretizations (2.7) of the general IVP (2.1) which satisfy the Lipschitz condition (4.20) with $\nu \leq 0$ are B-stable.

This implies that time-marching methods based on GSBP operators are unconditionally stable for nonlinear problems which are contractive. A similar nonlinear
stability definition was presented in [6] for autonomous IVPs with monotonic functions. An extension of this idea for non-autonomous IVPs was introduced in [29], called energy stability. A numerical method is called energy stable if

\[
\text{Re}([\mathcal{Y}(t), \mathcal{F}(\mathcal{Y}, t)]) \geq 0, \quad \forall \mathcal{Y} \in \mathbb{C}^M, \quad \text{and} \quad t \in \mathbb{R}
\]

implies that

\[
\| \tilde{y}_{t_f} \|_P \leq \| \mathcal{Y}_0 \|_P.
\]

This property is held by dual-consistent time-marching methods based on classical diagonal-norm FD-SBP operators [29]. The proof for GSBP time-marching methods is analogous to Proposition 3 in [29] using the ideas of to Theorem 4.1. Therefore, the following Theorem is presented without proof:

**Theorem 4.6.** All diagonal-norm dual-consistent GSBP-SAT discretizations (2.7) of the general IVP (2.1) which satisfy (4.22) are energy stable, and hence monotonic.

In summary, in addition to the linear stability of all dual-consistent time-marching method based on GSBP operators, those associated with a diagonal norm satisfy the nonlinear stability criteria of energy stability and B-stability as well.

5. Multiblock approach. Thus far our analysis has assumed a global discretization of the problem in time. As discussed in the introduction, this can be prohibitively expensive and it may be advantageous to employ a multi-block implementation. This is analogous to repeated application of a one-step multistage method like a Runge-Kutta scheme. This approach is also necessitated for some GSBP time-marching methods, as the operators can have a fixed number of solution points (no repeating internal stencil). In this section, it will be shown that the stability theorems of Section 4 extend to the multi-block case. In particular, the conditions on the interface SAT penalty parameters of adjoining blocks are sought. This theory was originally developed in [9] for operators with satisfy the classical SBP definition with \( 
\tilde{E} = \text{diag}[-1, 0, \ldots, 0, 1] \). In this section, this theory is extended for all GSBP time-marching methods constructed from general projection operators. We will also investigate the change in pointwise order of accuracy when a multiblock approach is used in which the operators are identical and have a fixed number of solution points. In this case temporal refinement is obtained by adding additional blocks rather than additional internal approximations within pre-existing blocks.

For the discussion of stability, consider a two-block GSBP-SAT discretization of the linear IVP (4.1):

\[
\begin{align*}
\bar{H}^{-1} \begin{bmatrix}
\Theta_L & 0 \\
0 & \Theta_R
\end{bmatrix} \mathbf{y}_d &= \lambda \mathbf{y}_d + \bar{H}^{-1} \begin{bmatrix}
\sigma_L \mathbf{x}_{L} \mathbf{x}_{L}^T & -\sigma_L \mathbf{x}_{L} \mathbf{x}_{R}^T \\
-\sigma_R \mathbf{x}_{L} \mathbf{x}_{L}^T & \sigma_R \mathbf{x}_{R} \mathbf{x}_{R}^T
\end{bmatrix} \mathbf{y}_d \\
&+ \sigma H_L^{-1} \chi_{t_0} \left( \mathbf{T}_{t_0} \mathbf{y}_{d,L} - \mathbf{Y}_0 \right),
\end{align*}
\]

where the composite norm and solution are defined as

\[
\bar{H} = \begin{bmatrix}
H_L & 0 \\
0 & H_R
\end{bmatrix}, \quad \text{and} \quad \mathbf{y}_d = \begin{bmatrix}
\mathbf{y}_{d,L} \\
\mathbf{y}_{d,R}
\end{bmatrix},
\]

and the solutions in the left and right blocks are \( \mathbf{y}_{d,L} \) in \([t_0, \delta]\) and \( \mathbf{y}_{d,R} \) in \([\delta, t_f]\), respectively, with their interface at \( t_0 \leq \delta \leq t_f \). For a graphical depiction of this problem, see Figure 1. Applying the energy method, simplifying, and rearranging leads to
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\begin{align}
|\tilde{y}_{R,t_f}|^2 &= 2\text{Re}(\lambda)||Y_d||_{\mathcal{Y}}^2 + (1 + 2\sigma)|\tilde{y}_{L,t_0} - \frac{\sigma}{1+2\sigma}Y_0|^2 - \frac{\sigma^2}{1+2\sigma}|Y_0|^2 \\
&+ Y_d \begin{bmatrix}
(-1 + 2\sigma_L)\chi L \chi R^T & -(\sigma_L + \sigma_R)\chi L \chi R^T \\
-(\sigma_R + \sigma_L)\chi R \chi L^T & (1 + 2\sigma_R)\chi R \chi L^T
\end{bmatrix} Y_d,
\end{align}

which bounds the numerical solution at time $t_f$ by the initial data, in the sense that $|\tilde{y}_{R,t_f}|^2 \leq L|Y_0|^2$, with $L$ a positive real number, provided that $\text{Re}(\lambda) \leq 0$, $\sigma < -\frac{1}{2}$, and the matrix in the last term is negative semi-definite. This will occur when

$$\sigma_L = \sigma_R + 1, \quad \text{and} \quad \sigma_R \leq -\frac{1}{2}.$$  

This equivalent to the result obtained in [9] for the classical FD-SBP-SAT approach. Furthermore, if $\sigma_R$ is chosen to be $-1$, then the solution in the left block becomes independent of the solution in the right block. As a consequence, the block can be computed sequentially in time rather than simultaneously. This is the same result shown for multi-block FD-SBP time-marching methods in [29]. With the choice of $\sigma = \sigma_R = -1$ and $\sigma_L = 0$, the energy estimate becomes

$$|\tilde{y}_{R,t_f}|^2 = 2\text{Re}(\lambda)||Y_d||_{\mathcal{Y}}^2 - |\tilde{y}_{L,t_0} - Y_0|^2 - |\tilde{y}_{R,\delta} - \tilde{y}_{L,\delta}|^2 + |Y_0|^2,$$

and the method is A-stable:

$$|\tilde{y}_{R,t_f}|^2 \leq |Y_0|^2,$$

This procedure can also be applied to the other linear and nonlinear stability results of Section 4.

Next, we consider the accuracy of the dual-consistent multiblock approach for the primal problem. This divides the problem into two time scales: one associated with the number of blocks in the discretization, the other with the subdiscretization within the blocks. The division of time scales is exploited to show that the global pointwise accuracy of the dual-consistent multiblock approach is one order higher than shown for the single-block case in Theorem 3.3. Note that this does not supersede the expected order of convergence for problems with stiff source terms presented in Theorem 3.9. The theorem is now stated:

**Theorem 5.1.** The pointwise accuracy $||e||_{\infty}$ of a dual-consistent multiblock GSBP-SAT discretization (3.2) of the primal problem (3.1) with $\text{Re}(\lambda) \leq 0$ using identical GSBP operators of order $q$ and associated with a norm of order $\rho$, along with a compatible SAT implementation, is of order $\mathcal{O}(\Delta t_N^{\infty(q+1,\rho)})$ provided Assumption 3.1 holds.

**Proof.** Define two time scales: 1) the block time scale $\Delta t_N = \frac{t_f - t_0}{N}$, where $N$ is the number of blocks in the discretization; and 2) the nodal time scale $\Delta t_n = \frac{\Delta t_N}{n}$, where $n$ is the number of solution points in each block. This also relates the two time scales.
From Theorem 3.3 the pointwise accuracy of the solution within each block is of order $O(\Delta t_N^q)$. However, the contribution from each block is now of order $\Delta t_N$, rather than $t_f - t_0 = O(1)$. Thus the order of the the multiblock approximation is $O(\Delta t_N)O(\Delta t_N^q)$. Using the relationship between the two time scales yields the pointwise accuracy:

$$O(\Delta t_N)O(\Delta t_N^q) = O(\Delta t_N)O(\frac{1}{n^q} \Delta t_N^q) = O(\Delta t_N^{q+1}).$$

An implicit assumption in Theorem 3.3 is that the initial data is exact, or at least of order $q$. Theorem 3.7 states that for dual-consistent GSBP-SAT implementations, the solution projected to the boundary of a block, which becomes the initial data for the next block, is order $\rho$. Hence, the pointwise order of accuracy is limited to order $O(\Delta t_N^{\min(q+1,\rho)})$.

This theorem shows that using a multiblock approach, one is guaranteed an additional order of pointwise accuracy over the general case.

6. Numerical Examples. This section presents numerical solutions to three simple model problems with the purpose of demonstrating some of the theory presented in this article and providing an initial characterization of time-marching methods based on classical FD-SBP and GSBP operators for IVPs. A summary of the GSBP time-marching methods investigated is presented in Table 1 along with their associated properties and abbreviations used hereafter. This includes the classical FD-SBP operators which are a subset of GSBP operators. More details on the non-classical GSBP operators can be found in [12].

6.1. Prothero-Robinson problem. Here the Prothero-Robinson problem (3.28) is solved with exact solution $\Psi(t) = e^{-t}$, equivalent to the primal problem (3.1) with a source term $G(t) = -(1 + \lambda)e^{-t}$. The intent is to compare the convergence rates of the GSBP time-marching methods for problems with stiff and non-stiff source terms. The stiffness of the problem is controlled via the parameter $\lambda$. The choice of $\lambda = -2$ is selected for a non-stiff source term, and $\lambda = -1000$ for a stiff source term [29, 31]. All time-marching methods are implemented with dual-consistent SAT penalty values.

The primary results are convergence rates based on two error measures: 1) the
error in pointwise solution:

\[ e_{Y_i} = \|Y_d - y\|_{\tilde{H}} = \sqrt{(Y_d - y)^T \tilde{H} (Y_d - y)}, \]

where \( \tilde{H} \) is a composite norm consistent with the GSBP time-marching method; 2) the discrete L2 norm of the solution error at each block boundary, which in general is the projection of the pointwise solution values to the boundary:

\[ e_{Y_e} = \sqrt{\frac{\sum_{j=1}^{N} (\chi^T \chi_{d,j} - Y(j \Delta t N))^2}{N}}, \]

where \( N \) represents the number of blocks in the discretization. The former error measure is used to ascertain the pointwise accuracy of the solution, the latter the rate of superconvergence.

### 6.1.1. Accuracy of problems with a non-stiff source term \((\lambda = -2)\).

Table 2 summarizes the convergence rates of the Prothero-Robinson problem with a non-stiff source term. The results for time-marching methods based on classical FD-SBP operators are obtained with the minimum number of solution points per block. This has been shown to have similar accuracy to the global approach [29]. The solution error \( e_{Y_i} \) converges with a rate of \( q + 1 \) for both time-marching methods based on classical FD-SBP and GSBP operators, where \( q \) is the order of the operator. This is in line with Theorem 5.1 and what has been observed in [12, 29, 31].

The table highlights the relatively small number of solution points required by time-marching methods based on GSBP operators to achieve the same rate of convergence as those based on classical FD-SBP operators. This is seen even in the case of the time-marching methods based on diagonal-norm NC GSBP operators, a novel set of GSBP operators presented in [12], the accuracy of which is limited to \( q = \lfloor \frac{n}{2} \rfloor \leq n - 1 \). In all the other cases for both diagonal and dense-norm GSBP time-marching methods, the order of the scheme is limited to \( q = n - 1 \) independent of the order of the associated norm or quadrature.

If one considers only the solution error at the block boundaries \( e_{Y_e} \), the rate of convergence of GSBP time-marching methods is \( \min(2q + 1, \rho) \), where \( \rho \) is the order of the associated norm. This is in line with Theorem 3.8 for dual-consistent discretizations, which shows that the solution at the block boundaries is superconvergent, even when this solution is projected from the solution values.

The table also confirms the relatively small number of solution points required per block by GSBP time-marching methods to achieve the same rate of superconvergence of linear functionals as those based on classical FD-SBP operators. Furthermore, it shows the benefit of allowing the distribution of solution points to be nonuniform. For example, the rate of superconvergence, \( p_{Y_e} \), exhibited by the operators associated with Legendre-Gauss-Lobatto quadrature rules [7, 17] increases about twice as fast with the number of solution points as the operators associated with the Newton-Cotes quadrature rules [7, 12]. Also, there is a limit to the rate of superconvergence seen with the GSBP time-marching methods associated with Legendre-Gauss quadrature rules, a family of novel GSBP operators presented in [12]. Despite the associated quadrature having order \( 2q + 2 \), the order of convergence of the block boundary solutions is only \( 2q + 1 \) as per Theorem 3.8.

Figure 2 shows the relative magnitude of the solution error as a function of the number of degrees of freedom (DOFs), where the number of degrees of freedom is the
Table 2  
Prothero-Robinson Problem ($\lambda = -2$): Convergence rates, $p_{Yi}$ of the pointwise solution error, $e_{Yi}$, and $p_{Ye}$ of the solution error at block boundaries, $e_{Ye}$. The order of the SBP operator is given by $q$, $n$ is the number of solution points in each block, and $\tau$ is the order of the associated quadrature rule. The convergence rates were computed using a line of best fit.

| SBP | $q = 1$ | $q = 2$ | $q = 3$ | $q = 4$ |
|-----|---------|---------|---------|---------|
| Scheme | $n$ | $\tau$ | $p_{Yi}/p_{Ye}$ | $n$ | $\tau$ | $p_{Yi}/p_{Ye}$ | $n$ | $\tau$ | $p_{Yi}/p_{Ye}$ | $n$ | $\tau$ | $p_{Yi}/p_{Ye}$ |
| FD   | 2 | 2 | 1.9842 / 1.9699 | - | - | - | 8 | 4 | 3.9725 / 3.9927 | - | - | - |
| FDB  | 2 | 2 | 1.9842 / 1.9699 | - | - | - | 8 | 4 | 3.9725 / 3.9927 | - | - | - |
| NC   | 2 | 2 | 1.9878 / 1.9767 | 3 | 4 | 2.9916 / 3.9817 | 5 | 6 | 3.8428 / 3.9830 | 7 | 8 | 4.9552 / 7.8919 |
| NCD  | 2 | 2 | 1.9965 / 1.9856 | 3 | 4 | 2.9925 / 3.9733 | 4 | 6 | 3.9928 / 3.9973 | 5 | 6 | 4.9832 / 5.9958 |
| LGL  | 2 | 2 | 1.9878 / 1.9767 | 3 | 4 | 2.9916 / 3.9817 | 4 | 6 | 3.9761 / 3.9923 | 5 | 6 | 4.9589 / 7.8724 |
| LGID | 2 | 2 | 1.9965 / 1.9856 | 3 | 4 | 2.9925 / 3.9733 | 4 | 6 | 3.9816 / 3.9973 | 5 | 6 | 4.9589 / 7.8724 |
| LGR  | 2 | 3 | 1.9949 / 2.9901 | 3 | 5 | 2.9919 / 4.9825 | 4 | 7 | 3.9808 / 6.9472 | 5 | 9 | 4.9581 / 8.9119 |
| LG   | 2 | 4 | 1.9909 / 2.9892 | 3 | 6 | 2.9853 / 4.9784 | 4 | 8 | 3.9660 / 6.9676 | 5 | 10 | 4.9304 / 8.9109 |

Table 3  
Prothero-Robinson Problem ($\lambda = -1000$): Convergence rates, $p_{Yi}$ of the pointwise solution error, $e_{Yi}$, and $p_{Ye}$ of the solution error at block boundaries, $e_{Ye}$. The order of the SBP operator is given by $q$, $n$ is the number of solution points in each block, and $\tau$ is the order of the associated quadrature rule. The convergence rates were computed using a line of best fit.

| SBP | $q = 1$ | $q = 2$ | $q = 3$ | $q = 4$ |
|-----|---------|---------|---------|---------|
| Scheme | $n$ | $\tau$ | $p_{Yi}/p_{Ye}$ | $n$ | $\tau$ | $p_{Yi}/p_{Ye}$ | $n$ | $\tau$ | $p_{Yi}/p_{Ye}$ | $n$ | $\tau$ | $p_{Yi}/p_{Ye}$ |
| FD   | 2 | 2 | 1.0404 / 0.9202 | 8 | 4 | 2.0252 / 2.0617 | 12 | 6 | 3.0770 / 3.1449 | 16 | 8 | 4.0328 / 4.1979 |
| FDB  | 2 | 2 | 1.0404 / 0.9202 | - | - | - | 8 | 4 | 3.0627 / 3.1824 | - | - | - |
| NC   | 2 | 2 | 1.0181 / 0.9412 | 3 | 4 | 2.0003 / 1.9870 | 5 | 6 | 3.0497 / 3.0459 | 7 | 8 | 4.1544 / 4.2056 |
| NCD  | 2 | 2 | 1.0181 / 0.9412 | 3 | 4 | 2.0003 / 1.9870 | 4 | 6 | 3.0689 / 3.0312 | 5 | 6 | 4.0207 / 4.0899 |
| LGL  | 2 | 2 | 1.0181 / 0.9412 | 3 | 4 | 2.0003 / 1.9870 | 4 | 6 | 3.0433 / 3.0464 | 5 | 6 | 4.0129 / 4.1107 |
| LGLD | 2 | 2 | 1.0553 / 0.9773 | 3 | 4 | 2.0044 / 1.9941 | 4 | 6 | 3.0965 / 3.1039 | 5 | 6 | 4.0312 / 4.1599 |
| LGR  | 2 | 3 | 1.0229 / 1.9911 | 3 | 5 | 2.0701 / 3.0036 | 4 | 7 | 3.1314 / 4.1111 | 5 | 9 | 4.1725 / 5.1658 |
| LG   | 2 | 4 | 0.9941 / 1.9768 | 3 | 6 | 2.0576 / 2.9939 | 4 | 8 | 3.1192 / 4.1038 | 5 | 10 | 4.1657 / 5.1599 |
number of blocks $N$ multiplied by the number of solution points per block $n$. Results are presented for operators of order $q = 3$ with the understanding that the conclusions equally apply to operators of other orders, especially as the order increases. Figure 2(a) shows that the relative magnitude of the solution error produced by the various methods is very similar. The difference exists when only the solution error at block boundaries is considered, (see Figure 2(b)). In this case, the importance of the order of the associated norm is seen, as it dictates the maximum rate of superconvergence. In each case, the GSBP time-marching methods based on Legendre-Gauss-Radau and Legendre-Gauss quadrature rules have the lowest error with respect to DOFs. These schemes make use of all three extensions of the generalized framework. If one considers only operators whose distribution of solution points includes both time domain boundaries of the block, the diagonal-norm NC, LGL and FD schemes perform similarly. However, this does not take into account the larger number of solution points of the FD operators, which reduces their efficiency relative to the other methods (See Section 6.3). In fact, nearly all the GSBP time-marching methods presented in this figure have half the number of solution points of the one based on the dense-norm classical FD-SBP operator and a third the number of one based on the diagonal-norm classical FD-SBP operator.

6.1.2. Accuracy of problems with a stiff source term ($\lambda = -1000$). Table 3 shows a similar summary of convergence rates for the Prothero-Robinson problem with a stiff source term ($\lambda = -1000$). The convergence rates are taken from the region of the convergence plots where the mode associated with $\lambda = -1000$ is under-resolved. The analytical solution does not contain this mode, though due to discrete approximations it is found in the numerical solution. This is the source of the stiffness. As the number of solution points is increased, this under-resolved portion of solution begins to be adequately resolved, eliminating the stiffness. As predicted in Section 3.4, the convergence rate in the region where the stiff mode is under-resolved, of both the pointwise solution error and the error of the solution at block boundaries, drops to order $q$. The exceptions are GSBP time-marching methods associated with the Legendre-Gauss-Radau and Legendre-Gauss quadrature rules, which achieve a rate of convergence $q + 1$.

The table highlights the benefit of the time-marching methods based on dense-norm FDB and NCD operators over their diagonal-norm counterparts for problems with stiff source terms. The advantage of the dense-norm operators is that they achieve the same rate of convergence with fewer solution points in each block. This translates to reduced computational cost, and therefore increased efficiency. The trade-off in general is a loss of nonlinear stability.

Figure 2 also shows the relative magnitude of the solution error as a function of the number of DOFs for the Prothero-Robinson problem with a stiff source term ($\lambda = -1000$). As with the non-stiff source term, the magnitude of the solution error is nearly identical for the various discretizations of order $q$. The difference in the case of a stiff source term is that this trend is also exhibited in the convergence of the solution error at block boundaries. The only noticeable deviation is with the GSBP time-marching methods associated with Legendre-Gauss-Radau and Legendre-Gauss quadrature rules that have slightly larger error in the region of the convergence plots where the mode associated with $\lambda = -1000$ is under-resolved. However, the higher rate of convergence in this region means that the magnitude is quickly reduced with the increase of DOFs.
6.2. Van der Pol’s equation. Van der Pol’s equation is a second-order non-linear ODE

\[ y'' - \mu(1 - y^2)y' + y = 0, \]

which is solved as a first-order system:

\[
\begin{cases}
  y' = z \\
  z' = \mu(1 - y^2)z - y
\end{cases}
\]

where \( \mu \) is the stiffness parameter. The initial conditions are \( y = 2 \) and \( z = -0.655748310724991 \), and the time domain is \( t = [0, 0.5] \). The initial condition are chosen to give a smooth solution and are based on an expansion around \( \mu \). The aim is to evaluate the performance of GSBP time-marching methods for nonlinear problems. A relatively small value of 10 is chosen for \( \mu \), leading to a non-stiff problem. The error definitions used for the Prothero-Robinson problem are applied here to each component of the solution independently.

Table 4 summarizes the convergence of the GSBP time-marching methods for the nonlinear non-stiff van der Pol equation. A reduced set of results is presented, as they are very similar to the convergence rates for the Prothero-Robinson equation which is linear with respect to the solution. The significance of this, is that the superconvergence result derived in Section 3.3 appears to hold for general nonlinear problems as well. Indeed, if the operators are cast as Runge-Kutta methods, they satisfy the general order conditions for nonlinear problems up to the order predicted by Theorem 3.8. Proof of this is presented in [4].

6.3. Linear convection equation. The function of these simulations is to demonstrate the performance of the generalized SBP-SAT approach for systems of linear ODEs and to provide an initial comparison of relative efficiency. Here, the one-dimensional linear convection equation with unit wave speed,

\[
\frac{\partial U}{\partial t} = -\frac{\partial U}{\partial x},
\]

is solved in the time domain \( t = [0, 1] \) with the initial conditions \( U(t = 0, x) = e^{-\frac{1}{2}\left(\frac{x-\text{mean}}{\text{std}}\right)^2} \), and spatial domain \( x \in [0, 2] \) with inflow condition \( U(t, x = 0) = 0 \). The spatial derivative is discretized with a 100-block dual-consistent generalized SBP-SAT
discretization, where each block is a 10-node operator associated with Legendre-Gauss quadrature. This is chosen so that the spatial error is negligible with respect to the temporal error. This also leads to an IVP problem for a system of $M = 1000$ coupled linear ODEs of the form (2.1). The solutions were computed using MATLAB 2013a on an intel Core i7-3930K processor at 3.2GHz with 32GB of RAM.

The two error measures used for the SBP-SAT approach are: 1) the pointwise solution error:

\begin{equation}
\epsilon_{U,i} = \|\epsilon_s\|_{H},
\end{equation}

where $H$ is a composite norm consistent with the temporal SBP-SAT discretization,

\begin{equation}
\epsilon_{s,j,k} = \|u_{d,j,k} - u_{j,k}\|_{H_s},
\end{equation}

is the error with respect to the exact solution at each temporal solution point, $j$ is the block index, $k$ is the solution point index within a block, and $H_s$ is the composite norm associated with the generalized spatial GSBP-SAT discretization; and 2) the discrete $L_2$ norm of the solution error at each block boundary:

\begin{equation}
\epsilon_{Ue} = \sqrt{\frac{1}{N} \sum_{j=1}^{N} \epsilon_{s,j}^2},
\end{equation}

where $N$ represents the number of blocks in the discretization, and

\begin{equation}
\epsilon_{s,j} = \|\chi^T u_{d,j} - \mathcal{U}(t = j\Delta t_N, x)\|_{H_s}.
\end{equation}

Figure 3 shows the convergence of the pointwise solution error and the solution error at the block boundaries with respect to CPU time. This is shown for GSBP operators of order $q = 3$. As expected, the rate of convergence of the solution error is identical for the various methods. What stands out, is the relative computational cost associated with the number of solution points. The LG-based GSBP time-marching method is the most efficient; however, the other 4-point GSBP time-marching methods have similar efficiency. As the number of solution points increases, all the way up to the 8 and 12 point FD-SBP time-marching methods, the computational cost rises rapidly. This highlights the potential benefits of the generalized framework in the construction of efficient time-marching methods.

Considering the solution error at the block boundaries as a function of CPU time further amplifies the efficiency of the generalized approach. In this case, the order of the associated norm plays a large role in the rate of convergence. Even the dense norm NC based GSBP time-marching method associated with the lowest order norm, $\rho = 4$, is more efficient the FD-SBP time-marching methods in the range shown. With the generalized framework, the other GSBP time-marching methods were able to match or beat the rate of convergence of the FD-SBP time-marching methods, in addition to requiring significantly less CPU time. In summary, this simulation has shown the superior efficiency of the generalized approach over classical FD-SBP time-marching methods.

7. Conclusions. This article combines the generalized summation-by-parts framework originally presented in [12] and the work of [29, 31] on the construction of time-marching methods based on FD-SBP operators. The generalized framework enables
the construction of high-order time-marching methods with significantly fewer solution points per block than those based on classical FD-SBP operators. Given that the solution points within each block must be solved for simultaneously, this can greatly improve the efficiency of the approach.

GSBP time-marching methods are shown to maintain the same stability and accuracy properties as those based on classical FD-SBP operators. Specifically, all dual-consistent GSBP time-marching methods are shown to be linearly stable, provide damping of stiff modes, and those constructed with a diagonal norm are shown to be nonlinearly stable as well. These results are shown to hold for multi-block discretizations with the appropriate choice of SAT values. The theory of superconvergent linear functionals is also extended to the generalized framework. It is shown that if the solution is dual-consistent and sufficiently smooth, then linear functionals integrated with the quadrature associated with the GSBP time-marching method are guaranteed to converge with the minimum of: 1) the order of the norm; and 2) one plus twice the order of the GSBP operator itself. This theory also implies that the solution at the block boundaries is superconvergent, analogous to a Runge-Kutta method with low stage order and higher-order solution update. Finally, the pointwise accuracy of the multiblock approach is shown to be superconvergent to the order of the discretization plus one.

Numerical simulations are presented to demonstrate the theoretical results of the article. Superconvergence of the solution at block boundaries is observed for the Prothero-Robinson problem with a non-stiff source term and order-reduction to the order of the individual operators with a stiff source term. Numerical simulation of the non-stiff nonlinear van der Pol’s equation exhibit the same superconvergence as the linear Prothero-Robinson problem with non-stiff source term. This demonstrates that some of the accuracy results generated in this article for problems linear with respect to the solution can also hold for nonlinear problems. Finally, numerical simulation of the linear convection equation highlights the potential efficiency gains of the generalized framework when applied to the construction of time-marching methods.

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References.
[1] S. Abarbanel and A. Ditkowski, Multi-dimensional asymptotically stable 4th-order accurate schemes for the diffusion equation, tech. report, feb 1996.
[2] ———, Asymptotically Stable Fourth-order Accurate Schemes for the Diffusion Equation on Complex Shapes, J. Comput. Phys., 133 (1997), pp. 279–288.
[3] P. D. Boom, High-order implicit numerical methods for unsteady fluid simulation, PhD thesis, University of Toronto Institute for Aerospace Studies, 2015.
[4] P. D. Boom and D. W. Zingg, Runge-Kutta Characterization of the Generalized Summation-by-Parts Approach in Time, (2014). arXiv:1410.0202 [Math.NA].
[5] K. Burrage and J. C. Butcher, Stability Criteria for Implicit Runge-Kutta Methods, SIAM Journal on Numerical Analysis, 16 (1979), pp. pp. 46–57.
[6] ———, Non-linear Stability of a General Class of Differential Equation Methods, BIT Numerische Mathematik, 20 (1980), pp. 185–203.
[7] M. H. Carpenter and D. Gottlieb, Spectral methods on arbitrary grids, Journal of Computational Physics, 129 (1996), pp. 74–86.
[8] Mark H. Carpenter, David Gottlieb, and Saul S. Abarbanel, *Time-stable boundary conditions for finite-difference schemes solving hyperbolic systems: methodology and application to high-order compact schemes*, Journal of Computational Physics, 111 (1994), pp. 220–236.

[9] Mark H. Carpenter, Jan Nordström, and David Gottlieb, *A stable and conservative interface treatment of arbitrary spatial accuracy*, Journal of Computational Physics, 148 (1999), pp. 341–365.

[10] Gerhard Wanner, *Time-Marching Methods Based on GSBO Operators*, Springer, 2008.

[11] Christopher A. Kennedy and Mark H. Carpenter, *Additive Runge-Kutta Schemes for Convection-diffusion-reaction Equations*, Appl. Numer. Math., 44 (2003), pp. 139–181.

[12] Zdzislaw Jackiewicz, *Initial-Boundary Value Problems and the Navier-Stokes Equations*, vol. 47 of Classics in Applied Mathematics, SIAM, 2004.
pects of finite elements in partial differential equations, Academic Press, New York/London, 1974, ch. Finite element and finite difference methods for hyperbolic partial differential equations, pp. 195–212.

[27] Harvard Lomax, Thomas H. Pulliam, and David W. Zingg, Fundamentals of Computational Fluid Dynamics, Scientific Computation, Springer, 2001.

[28] J. Lu, An a posteriori Error Control Framework for Adaptive Precision Optimization using Discontinuous Galerkin Finite Element Method, PhD thesis, Massachusetts Institute of Technology, 2005.

[29] Tomas Lundquist and Jan Nordström, The SBP-SAT Technique for Initial Value Problems, Journal of Computational Physics, 270 (2014), pp. 86–104.

[30] Ken Mattsson, Martin Almquist, and Mark H. Carpenter, Optimal diagonal-norm SBP operators, Journal of Computational Physics, 264 (2014), pp. 91–111.

[31] Jan Nordström and Tomas Lundquist, Summation-by-parts in Time, Journal of Computational Physics, 251 (2013), pp. 487–499.

[32] A. Prothero and A. Robinson, On the stability and accuracy of one-step methods for solving stiff systems of ordinary differential equations, Mathematics of Computation, 28 (1974), pp. 145–162.

[33] Adam Reichert, Michel T. Heath, and Daniel J. Bodony, Energy stable numerical method for hyperbolic partial differential equations using overlapping domain decomposition, Journal of Computational Physics, 231 (2012), pp. 5243–5265.

[34] Bo Strand, Summation by parts for finite difference approximations for d/dx, Journal of Computational Physics, 110 (1994), pp. 47–67.
**Prothero-Robinson Problem:** Convergence of the pointwise solution error, $e_{yi}$, and the solution error at the block boundaries, $e_{Ye}$, with respect to DOFs. The order of the SBP operator is given by $q$, and the numerical suffix in the legend indicates the number of solution points in each block.
Fig. 3. **Linear Convection Equation**: Convergence of the pointwise solution error, $e_{yi}$, and the solution error at the block boundaries, $e_{ye}$ with respect to CPU time (s). The order of the SBP operator is given by $q$, and the numerical suffix in the legend indicates the number of solution points in each block.