AN INDIRECT BIE FREE OF DEGENERATE SCALES

JENG-TZONG CHEN∗,a,b,c,d,e

aDepartment of Harbor and River Engineering
National Taiwan Ocean University
Keelung, 20224, Taiwan

bDepartment of Mechanical and Mechatronic Engineering
National Taiwan Ocean University
Keelung, 20224, Taiwan

cDepartment of Civil Engineering
National Cheng-Kung University
Tainan, 70101, Taiwan

dBachelor Degree Program in Ocean Engineering and Technology
National Taiwan Ocean University
Keelung, 20224, Taiwan

eCenter of Excellence for Ocean Engineering
National Taiwan Ocean University
Keelung, 20224, Taiwan

SHING-KAI KAO, JENG-HONG KAO AND WEI-CHEN TAI
Department of Harbor and River Engineering
National Taiwan Ocean University
Keelung, 20224, Taiwan

Abstract. Thanks to the fundamental solution, both BIEs and BEM are effective approaches for solving boundary value problems. But it may result in rank deficiency of the influence matrix in some situations such as fictitious frequency, spurious eigenvalue and degenerate scale. First, the nonequivalence between direct and indirect method is analytically studied by using the degenerate kernel and examined by using the linear algebraic system. The influence of contaminated boundary density on the field response is also discussed. It’s well known that the CHIEF method and the Burton and Miller approach can solve the unique solution for exterior acoustics for any wave number. In this paper, we extend a similar idea to avoid the degenerate scale for the interior two-dimensional Laplace problem. One is the external source similar to the null-field BIE in the CHIEF method. The other is the Burton and Miller approach. Two analytical examples, circle and ellipse, were analytically studied. Numerical tests for general cases were also done. It is found that both two approaches can yield an unique solution for any size.

2020 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. degenerate scale, fundamental solution, indirect BIE, Burton and Miller approach, fictitious source point.

The financial support from the Ministry of Science and Technology, Taiwan under Grant No. MOST 106-2221-E-019-009-MY3 and No. MOST 107-2221-E-019 -003, and we also appreciate Mr. Kuen-Ting Lien to provide numerical results.

∗Corresponding author.
1. Introduction. Boundary integral equations (BIEs) and boundary element method (BEM) have been employed to solve radiation and scattering problems for many years. Thanks to the fundamental solution, there are stable approaches for the exterior acoustic wave [22]. However, fictitious frequency [1, 32] and spurious eigenvalue [7] appear in the exterior and interior acoustics, respectively. Burton and Miller [1] solved the problem by combining singular and hypersingular equations with an imaginary number. This technique was noted as the Burton and Miller approach. Mathematicians have paid attention on the mathematical foundations of BIEM/BEM [3]. Later, Schenck [32] proposed the combined Helmholtz interior integral equation formulation (CHIEF) method free of using the hypersingular integral. He collocated the point outside the domain as an auxiliary constraint to promote the rank of influence matrices. Chen et al. [4] extended the CHIEF method to combine the Helmholtz exterior integral equation formulation (CHEEF) method for overcoming the problem of spurious eigenvalues in the eigenproblem. Both problems stem from the rank deficiency of the influence matrix in the BIEM/BEM. Not only the Helmholtz equation but also the Laplace equation must overcome this rank-deficiency problem if the BEM/BIEM were employed.

Mathematical studies on the Laplace equation were investigated by mathematicians [2, 21, 30]. By employing the single-layer representation to solve the 2D Dirichlet problem, the unknown boundary density (constant base) cannot be determined due to the degenerate scale [20]. The representation may lose the constant term in the single-layer integral operator [14]. However, the degenerate scale does not occur in 3D problems since the fundamental solution is $1/r$ instead of $\ln r$. Adding a rigid body mode in the fundamental solution, $\ln r + \tau [15, 16]$, is the most direct way to avoid the degenerate scale. Nevertheless, this way just avoids the original degenerate scale but results in another degenerate scale. Lin and Wu [31] argued the conventional fundamental solution by using the dimensional analysis. The argument in the function, $\ln r, H_0(\kappa r), \cos \theta$ and $\sin \theta$ must be dimensionless. Otherwise, it disobeys the objectivity. This is why a degenerate scale occurs in the BEM/BIEM for the 2D Laplace equation. Based on the objectivity, Hu [25] proposed a necessary and sufficient boundary integral equation (BIE). From the viewpoint of physics, he introduced a constant in the BIE and derived a constraint using the objectivity. The objectivity means that the solution representation is independent of any observer system or any scaling change. This necessary and sufficient BIE is similar to the one presented by Fichera [23] as shown in Table 1. Chen et al. [17] employed the degenerate kernel function to analytically study the uniqueness solution of the BIE. The constraint is mathematically derived by them although it was called the objectiveness condition in the literature [25]. It’s also free of the degenerate scale, but this paper focused on the non-dimensional approach. In this paper, we discussed the nonequivalence of the direct and the indirect BEM/BIEMs by using the degenerate kernel and examined by using the linear algebraic system [33] in Section 2. The conventional regularization techniques for the non-unique solution in the indirect BIEM were mentioned in Section 3. We introduced the fictitious source method in Section 4. Regarding the general geometry, we followed the idea of using the scaling technique in Jaswon and Symm’s book [26]. Finally, we would use the BEM program to examine several cases. It was found that both analytical solutions and numerical results showed the validity of these methods.
2. Review of the conventional BIEM and their possible failure.

2.1. Conventional indirect BIEM. A degenerate scale results in non-uniqueness solutions when the BEM/BIEM is applied to solve a Laplace problem subject to the Dirichlet boundary condition. Here, we consider the Laplace problem as shown below:

\[ \nabla^2 u(x) = 0, \quad x \in D \]
\[ u(x) = \bar{u}(x), \quad x \in B \]  

(2.1)

where \( \nabla^2 \), \( D \) and \( B \) are the Laplace operator, the domain of interest and the boundary, respectively. The indirect BIE is shown below:

\[ 2\pi u(x) = \int_B U(x, s) \alpha(s) dB(s), \quad x \in D, \]  

(2.2)

where \( U(x, s) = \ln r \) is the fundamental solution, \( \alpha(s) \), is the unknown boundary density and \( r \) is the distance between a source point \( s \) and a field point \( x \). By setting the field point \( x = (\rho \cos \phi, \rho \sin \phi) \) and the source point \( s = (R \cos \theta, R \sin \theta) \) in the polar coordinates for the circular domain, the closed-form fundamental solution in Eq.(2.2) is expressed in the series form as follows:

\[ U(x, s) = \ln r = \begin{cases} 
\ln R - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\rho}{R} \right)^m \cos(m(\theta - \phi)), & R \geq \rho, \\
\ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{R}{\rho} \right)^m \cos(m(\theta - \phi)), & \rho \geq R.
\end{cases} \]  

(2.3)

The unknown boundary density \( \alpha(s) \) and the given boundary condition \( \bar{u}(s) \) are expanded in terms of Fourier series as

\[ \alpha(s) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta), \quad 0 \leq \theta \leq 2\pi \]  

(2.4)

and

\[ \bar{u}(x) = p_0 + \sum_{n=1}^{\infty} p_n \cos(n\phi) + \sum_{n=1}^{\infty} q_n \sin(n\phi), \quad 0 \leq \phi \leq 2\pi \]  

(2.5)

where \( a_0, a_n \) and \( b_n \) are unknown coefficients, \( p_0, p_n \) and \( q_n \) are given from the Dirichlet boundary condition. By substituting Eqs.(2.3) to (2.5) into Eq.(2.2), and using the orthogonal property of trigonometric functions for \( R = a \), the coefficient of the Fourier base is

\[ \begin{cases} 
2\pi a \ln a = p_0, \\
\frac{n}{\pi a} p_n, & n = 1, 2, 3 \cdots, \\
\frac{n}{\pi a} q_n, & n = 1, 2, 3 \cdots
\end{cases} \]  

(2.6)

where \( a \) is the radius of the circle for the domain. The response of field could be expressed by

\[ u(x) = \frac{1}{2\pi} \left( 2\pi a \ln a a_0 + \sum_{n=1}^{\infty} \left( \frac{\rho}{a} \right)^n p_n \cos(n\phi) + \sum_{n=1}^{\infty} \left( \frac{\rho}{a} \right)^n q_n \sin(n\phi) \right). \]  

(2.7)

Equation (2.6) indicates that the occurring mechanism of the degenerate scale is

\[ 2\pi a \ln a = 0. \]  

(2.8)
When \( a = 1 \), the value of \( a_0 \) cannot be determined. It proves that the degenerate scale of a circular domain is 1. The degenerate scale results in no solution or infinite solutions in the indirect BIE if the constant term of the Dirichlet boundary condition is nonzero or zero, respectively.

### 2.2. Conventional direct BIEM

The boundary integral formulation for the Laplace equation can be derived from Green’s third identity

\[
2\pi u(x) = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)t(s)dB(s), \quad x \in D, \tag{2.9}
\]

for the domain point, where \( t(s) = \frac{\partial u(s)}{\partial n_s} \), in which \( n_s \) denotes the unit outward normal vector at the source point \( s \), and the \( T(x,s) \) kernel function is defined by

\[
T(x,s) = \frac{\partial U(x,s)}{\partial n_s} \tag{2.10}
\]

where \( n_s \) denotes the unit outward normal vector at the source point \( s \). Equation (2.9) is also called the direct BIE. The null-field boundary integral equation is

\[
0 = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)t(s)dB(s), \quad x \in D^c, \tag{2.11}
\]

where \( D^c \) is the complementary domain. The unknown boundary density \( t(s) \) can be expanded by using the Fourier series,

\[
t(s) = g_0 + \sum_{n=1}^{\infty} g_n \cos(n\theta) + \sum_{n=1}^{\infty} h_n \sin(n\theta), \quad 0 \leq \theta \leq 2\pi. \tag{2.12}
\]

By substituting Eqs. (2.3), (2.5), (2.12) and the orthogonality conditions of trigonometric functions into the Eq.(2.11), we have

\[
2\pi a \ln a g_0 - \sum_{n=1}^{\infty} \frac{\pi a}{n} \cos(n\phi) g_n - \sum_{n=1}^{\infty} \frac{\pi a}{n} \sin(n\phi) h_n = - \sum_{n=1}^{\infty} \pi p_n \cos(n\phi) - \sum_{n=1}^{\infty} \pi q_n \sin(n\phi). \tag{2.13}
\]

By comparing with the coefficient of the Fourier base in Eq.(2.13), we have

\[
\begin{cases}
2\pi a \ln a g_0 = 0, \\
g_n = \frac{n}{a} p_n, \quad n = 1,2,3\cdots, \\
h_n = \frac{n}{a} q_n, \quad n = 1,2,3\cdots,
\end{cases} \tag{2.14}
\]

The response of field could be expressed by

\[
u(x) = \frac{1}{2\pi} \left( 2\pi a \ln a g_0 + p_0 + \sum_{n=1}^{\infty} \left( \frac{\rho}{a} \right)^n p_n \cos(n\phi) + \sum_{n=1}^{\infty} \left( \frac{\rho}{a} \right)^n q_n \sin(n\phi) \right) \tag{2.15}
\]

In Eq.(2.14), it is easily found that the value of \( g_0 \) is free if the size of the boundary is the degenerate scale, i.e. \( a = 1 \). However, the field response is unique in Eq. (2.15) due to the zero term of \( \ln a \) even though \( g_0 \) is free. The degenerate scale only results in the infinite solution in the direct BIE due to the operation of \( [T] \{ \tilde{u} \} \). According to Eqs. (2.7) and (2.15), it explains that the solution space of indirect and direct BIEs are not equivalent when the degenerate scale occurs. Similarly, the elliptical case can be analytically derived by the degenerate kernel in terms of the
elliptical coordinates. The closed-form fundamental solution in elliptical coordinates is expressed in the series form as follows:

\[
\begin{cases}
U^i(\xi_s, \eta_s; \xi_x, \eta_x) = \xi_s + \ln \frac{e}{2} \\
- \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_s} \cosh m\xi_x \cos mn_x \cos mn_s, \quad \xi_s \geq \xi_x, \quad (a) \\
U^e(\xi_s, \eta_s; \xi_x, \eta_x) = \xi_x + \ln \frac{e}{2} \\
- \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_x} \cosh m\xi_s \cos mn_x \cos mn_s \\
- \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_x} \sinh m\xi_s \sin mn_x \sin mn_s, \quad \xi_x \geq \xi_s, \quad (b)
\end{cases}
\]

The unknown boundary density of the indirect method, \(\alpha(s)\), the unknown boundary density of direct method \(t(s)\) and the given boundary condition \(\bar{u}(x)\) are expanded in terms of Fourier series as

\[
\alpha(s) = \frac{1}{J(\xi_s, \eta_s)} \left( a_0 + \sum_{n=1}^{\infty} a_n \cos(n\eta_s) + \sum_{n=1}^{\infty} b_n \sin(n\eta_s) \right), \quad (2.17)
\]

\[
\bar{u}(x) = p_0 + \sum_{n=1}^{\infty} p_n \cos(n\eta_x) + \sum_{n=1}^{\infty} q_n \sin(n\eta_x), \quad (2.18)
\]

and

\[
t(s) = \frac{1}{J(\xi_s, \eta_s)} \left( g_0 + \sum_{n=1}^{\infty} g_n \cos(n\eta_s) + \sum_{n=1}^{\infty} h_n \sin(n\eta_s) \right). \quad (2.19)
\]

We can follow the same step of deriving the circular case. When the size of the boundary is a degenerate scale, it may result in the constant term of the unknown boundary density which cannot be determined. The unknown constant term procures no solution or infinite solutions in the indirect BIE and infinite solutions in the direct BIE. Although the unknown boundary density of the direct BIE may perturb by a constant term, the field solution in Eq.(2.15) is still acceptable. The solution space of the indirect BIE depends on the given boundary condition. The field response of the indirect BIE is no solution in Eq.(2.7) since \(a_0\) cannot be determined.

2.3. Revisit of the Fredholm alternative theorem by using the fundamental theorem of linear algebra and the SVD. Here, the solution space for the indirect and direct BIEs is examined in the discrete system by using the Fredholm alternative theorem. In the discrete system, we have

\[
[U]\{\alpha\} = \{\bar{u}\}, \quad \text{the indirect BEM,} \quad (2.20)
\]

and

\[
[U]\{t\} = [T]\{\bar{u}\} = \{b\}, \quad \text{the direct BEM,} \quad (2.21)
\]
where \([U]\) and \([T]\) are the influence matrices, \(\alpha\), \(\bar{u}\) and \(t\) are the boundary densities. By using the singular value decomposition, \([U]\) and \([T]\) can be expressed by

\[
[U]_{(2N+1)\times(2N+1)} = [\Phi(U)] [\Sigma(U)] [\Psi(U)]^T, \tag{2.22}
\]

\[
[T]_{(2N+1)\times(2N+1)} = [\Phi(T)] [\Sigma(T)] [\Psi(T)]^T, \tag{2.23}
\]

where \(\sigma_i^{(U)}\) is the singular value, \(\phi_i^{(U)}\) and \(\psi_i^{(U)}\) are the left and the right singular vectors corresponding to \(\sigma_i^{(U)}\), respectively. The symbol in the parentheses denotes the corresponding matrix. The matrix \([T]\) is singular due to the rigid body mode while \([U]\) is singular only if the size of the boundary is a degenerate scale. The boundary data can be expressed as

\[
\{\alpha\} = \sum_{i=1}^{2N+1} \alpha_i \phi_i^{(U)}, \tag{2.24}
\]

\[
\{t\} = \sum_{i=1}^{2N+1} t_i \psi_i^{(U)}, \tag{2.25}
\]

\[
\{\bar{u}\} = \sum_{i=1}^{2N+1} u_i^{\text{ind}} \phi_i^{(U)} = \sum_{i=1}^{2N+1} u_i^{d} \psi_i^{(T)}, \tag{2.26}
\]

and

\[
\{b\} = \sum_{i=1}^{2N+1} u_i^{d} \sigma_i^{(T)} \phi_i^{(T)} = \sum_{i=1}^{2N+1} \beta_i \phi_i^{(U)}. \tag{2.27}
\]

When the degenerate scale occurs, i.e. \(\sigma_1^{(U)} = 0\), the range of \([U]\) is deficient as shown below:

\[
[U] \{\alpha\} = \sum_{i=1}^{2N+1} \alpha_i \sigma_i^{(U)} \phi_i^{(U)} \nonumber \]

\[
= \alpha_1 \cdot 0 \cdot \{\phi_1^{(U)}\} + \sum_{i=2}^{2N+1} \alpha_i \sigma_i^{(U)} \phi_i^{(U)} , \quad \text{for the indirect BEM}, \tag{2.28}
\]

and

\[
[U] \{t\} = \sum_{i=1}^{2N+1} \beta_i \sigma_i^{(U)} \phi_i^{(U)} \nonumber \]

\[
= \beta_1 \cdot 0 \cdot \{\phi_1^{(U)}\} + \sum_{i=2}^{2N+1} \beta_i \sigma_i^{(U)} \phi_i^{(U)} , \quad \text{for the direct BEM}. \tag{2.29}
\]

Following the Strang’s book[33], the null space of \([U]\) exists as shown in Fig. 1(a). According to the Fredholm alternative theorem, the indirect BEM results in

\[
\text{infinite solution, in case of} \quad \{\bar{u}\} \cdot \{\phi_1^{(U)}\} = u_i^{\text{ind}} = 0, \tag{2.30}
\]

or

\[
\text{no solution, in case of} \quad \{\bar{u}\} \cdot \{\phi_1^{(U)}\} = u_i^{\text{ind}} \neq 0, \tag{2.31}
\]
but the solution of the direct BEM yields
\[ \{ b \} \cdot \{ \phi^{(U)} \} = 0. \tag{2.32} \]
The main difference is that \([T]\) is introduced in Eq.(2.23) and its null space is not empty as shown in Fig. 1(b). Since \([T]\) is singular, i.e. \( \sigma_1^{(T)} = 0 \), it results in
\[ u_d^1 \alpha_1 \{ \phi^{(T)}_1 \} = \{ 0 \}. \tag{2.33} \]
Fortunately, the corresponding left null vector \( \{ \phi^{(T)}_1 \} \) is equal to \( \{ \phi^{(U)}_1 \} \). It is the reason why the direct BEM satisfies Eq.(2.26). In Fig. 1(a), if \( \{ \bar{u} \} \) is specified in the yellow range of dimension 1 as shown in Fig.1, \( \{ \alpha \} \) has no solution, while there are infinite solutions for \( \{ \alpha \} \) if \( \{ \bar{u} \} \) falls in the dimension 2N cyan range. For any specified \( \{ \bar{u} \} \) in Fig.1(b), the cyan range of \([T]\) mapping falls in the dimension 2N space. By combining Fig.1(a) and 1(b) in Fig. 1(c), we can find that it is impossible to have the case of no solution in the direct BEM, since \([T]\) \( \{ \bar{u} \} \) falls in the dimension 2N green range. This indicates that the non-equivalence of the solution space between the direct BEM and the indirect BEM is well explained.

3. Regularization techniques for the non-uniqueness solution in the indirect BIEM.

3.1. Adding a rigid body term. Since the constant term in the range of the integral operator cannot be represented, the most direct way is adding a rigid body mode, \( \tau \) in the fundamental solution. The fundamental solution is expressed as follows:
\[
U_r(x, s) = \ln |x-s| + \tau = \begin{cases} 
\ln R + \tau - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\rho}{R} \right)^m \cos (m (\theta - \phi)), & R \geq \rho, \\
\ln \rho + \tau - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{R}{\rho} \right)^m \cos (m (\theta - \phi)), & \rho \geq R.
\end{cases}
\tag{3.1}
\]
We used \( U_r(x, s) \) in place of \( U(x, s) \) in the Eq.(2.2). Similarly, the coefficient of the Fourier constant base is
\[ a_0 = \frac{1}{2\pi a (\ln a + \tau)^p}. \tag{3.2} \]
When \( \ln a + \tau = 0 \), the value of \( a_0 \) cannot be determined. The degenerate scale of a circular domain is shifted to \( a = e^{-\tau} \) instead of \( a = 1 \).

3.2. Burton and Miller approach (Mixed-potential method). Burton and Miller solved the problem by combining singular and hypersingular equations in the direct BIEs/BEM. In this paper, we focus on the indirect BIE and BEM. The similar idea is called the mixed-potential method as follows:
\[
\bar{u}(x) = \int_B \left( T(x, s) - ikU(x, s) \right) \alpha(s) dB(s). \tag{3.3}
\]
We applied the idea of Burton and Miller to solve the Laplace problem. For the Laplace equation, Eq. (3.3) is simply written as
\[
\bar{u}(x) = \int_B \left( T(x, s) - iU(x, s) \right) \alpha(s) dB(s). \tag{3.4}
\]
By substituting Eqs. (2.3), (2.4), (2.5) and (2.10) into Eq. (3.4), and employing the orthogonality conditions of trigonometric functions for $R = a$, the coefficient of the Fourier constant base is

$$a_0 = \frac{1}{2\pi} (1 - ia \ln a)^p.$$  

(3.5)

Since the term of $1 - ia \ln a$ is never zero, i.e. $0 - 0i$, the degenerate scale does not occur anymore.

4. A new approach using a fictitious source. Rank deficiency in the BIEM/BEM also appears in time-harmonic problems namely fictitious frequency and spurious eigenvalue. For the exterior acoustics, Schenck proposed a combined Helmholtz integral equation formulation (CHIEF) to avoid the fictitious frequency [32]. Following the same idea of CHIEF method, the CHEEF method was proposed to filter out the spurious eigenvalue of interior problems [2]. On the basis of the similar idea, the CHEEF method was also employed to deal with the degenerate scale [15].

The above approaches employed the null-field integral equation in the direct BEM. However, the null-field integral equation is not available in the indirect BIEM/BEM. Briefly speaking, the CHEEF method can be only applied to avoid the degenerate scale in the direct BIEM/BEM. Here, we introduced a fictitious source in the complement of the domain to modify the kernel function instead of using the constraint of the null-field equation. The modified fundamental solution is shown below:

$$U_m(x, s) = \ln \left| \frac{x - s}{x - s^*} \right| = \ln r - \ln r^*, \quad (4.1)$$

where we introduce the distance, $r^*$, between a fictitious source point, $s^*$, and the field point instead of the fixed characteristic length, in which $s^* \in D_c$. Both indirect and direct BIEM/BEMs can employ this approach to deal with the degenerate-scale problem.

4.1. Proof of two special cases - circle and ellipse.

$$U_m(x, s) = \ln \left| \frac{x - s}{x - s^*} \right| = \begin{cases} \ln R - \ln r^* - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{R}{\rho} \right)^m \cos (m(\theta - \phi)), & R \geq \rho, \\
\ln \rho - \ln r^* - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{R}{\rho} \right)^m \cos (m(\theta - \phi)), & R > \rho. \end{cases} \quad (4.2)$$

Because $s^* = (R^* \cos \theta^*, R^* \sin \theta^*) \in D_c$, $\ln r^*$ in Eq. (4.2) could be expressed as:

$$\ln r^* = \ln R^* - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{R^*}{\rho} \right)^m \cos (m(\theta^* - \phi)), \quad R^* > \rho. \quad (4.3)$$

By substituting Eqs. (2.4), (2.5), (4.2) and (4.3) into the Eq. (2.2) and the orthogonality conditions of trigonometric functions, we have

$$2\pi a \left( \ln a - \ln R^* + \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{a}{R^*} \right)^n \cos (n(\theta^* - \phi)) \right) a_0 - \sum_{n=1}^{\infty} \frac{\pi a}{n} \cos (n\phi) a_n$$

$$- \sum_{n=1}^{\infty} \frac{\pi a}{n} \sin (n\phi) b_n = p_0 + \sum_{n=1}^{\infty} p_n \cos (n\phi) + \sum_{n=1}^{\infty} q_n \sin (n\phi). \quad (4.4)$$
By comparing with the coefficient of the Fourier base in Eq. (4.4), we have

\[
\begin{align*}
    a_0 &= \frac{1}{2\pi a (\ln a - \ln R^*)} p_0, \\
    a_n &= \frac{n - \pi a}{\pi a} \left( p_n - \frac{2\pi a}{n} \left( \frac{a}{R^*} \right)^n \cos (n\theta^*) a_0 \right), \\
    b_n &= \frac{n - \pi a}{\pi a} \left( q_n - \frac{2\pi a}{n} \left( \frac{a}{R^*} \right)^n \sin (n\theta^*) a_0 \right).
\end{align*}
\]  

(4.5)

Since \( R^* > a \), i.e. \( \ln a - \ln R^* \neq 0 \), the degenerate scale never occurs. Similarly, the modified fundamental solution in the elliptical coordinates could be shown below:

\[
U_m(x, s) = \ln \frac{|x - s|}{|x - s^*|} \begin{cases} 
U^s(\xi_s, \eta_s; \xi_x, \eta_x) = \xi_s + \ln \frac{c}{2} - \ln r^* \\
- \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_s} \cosh m\pi \cos m\eta_s \cos m\eta_x \\
- \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_s} \sinh m\pi \sin m\eta_s \sin m\eta_x, \\
\xi_s \geq \xi_x, \quad (a) \\
U^l(\xi_s, \eta_s; \xi_x, \eta_x) = \xi_x + \ln \frac{c}{2} - \ln r^* \\
- \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_x} \cosh m\pi \cos m\eta_s \cos m\eta_x \\
- \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_x} \sinh m\pi \sin m\eta_s \sin m\eta_x, \\
\xi_x \geq \xi_s, \quad (b)
\end{cases}
\]

(4.6)

where the field point \( x = (c \cosh \xi_x \cos \eta_x, c \sinh \xi_x \sin \eta_x) \) and the source point \( s = (c \cosh \xi_s \cos \eta_s, c \sinh \xi_s \sin \eta_s) \) in the elliptical coordinates, \( c \) is the half distance between two foci. Because \( s^* = (c \cosh \xi^* \cos \eta^*, c \sinh \xi^* \sin \eta^*) \) \( D^* \), \( \ln r^* \) in Eq. (4.6) could be expressed as:

\[
\ln r^* = \xi^* + \ln \frac{c}{2} - \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi^*} \cosh (m\xi_0) \cos (m\eta_0) \cos (m\eta^*) \\
- \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi^*} \sinh (m\xi_0) \sin (m\eta_0) \sin (m\eta^*), \quad \xi^* > \xi_x.
\]  

(4.7)

Since the unknown boundary density \( \alpha(s) \) and the given boundary condition \( \tilde{u}(x) \) can be expanded in terms of Fourier series. Based on the orthogonality conditions of trigonometric functions and substituting Eqs. (2.17), (2.18), (4.6) and (4.7) into the Eq. (2.2), we have

\[
2\pi(\xi_0 - \xi^*) + \sum_{n=1}^{\infty} \frac{n}{2} e^{-n\xi^*} \cos (n\eta_0) \cos (n\eta^*) \\
+ \sum_{n=1}^{\infty} \frac{n}{2} e^{-n\xi^*} \sinh (n\xi_0) \sin (n\eta_0) a_0 \\
- \sum_{n=1}^{\infty} \frac{n}{2\pi} e^{-n\xi^*} \cos (n\eta_0) a_n \\
- \sum_{n=1}^{\infty} \frac{n}{2\pi} e^{-n\xi^*} \sinh (n\xi_0) b_n \\
=p_0 + \sum_{n=1}^{\infty} p_n \cos (n\eta_x) + \sum_{n=1}^{\infty} q_n \sin (n\eta_x).
\]  

(4.8)
By comparing with the coefficient of the Fourier bases in Eq. (4.8), we have:

\[
\begin{align*}
    a_0 &= \frac{1}{2\pi (\xi_0 - \xi^*)} p_0, \\
    a_n &= \frac{n}{2\pi e^{-n\xi_0} \cosh (n\xi_0)} \left( -p_n + \frac{4\pi n}{\pi} e^{-n\xi^*} \cosh (n\xi_0) \cos (n\eta^*) a_0 \right), \\
    b_n &= \frac{1}{2\pi e^{-n\xi_0} \sinh (n\xi_0)} \left( -q_n + \frac{4\pi n}{\pi} e^{-n\xi^*} \sinh (n\xi_0) \sin (n\eta^*) a_0 \right).
\end{align*}
\]  

(4.9)

Since \(\xi^* \neq \xi_0\), i.e. \(2\pi (\xi_0 - \xi^*) \neq 0\), the degenerate scale never occurs. In a circular domain or an elliptical domain, the unique solution can be obtained by introducing dimensionless quantity, \(r/r^*\). A fictitious source method can be applied to any geometric shape. If we have the degenerate kernel, the analytical solution can be obtained by using a fictitious source and the BIEM. For the general shape, we introduce an expansion ratio of scaling technique to analytically check the validity of the present method. Besides, the numerical solution can be also obtained by using a fictitious source and the BEM.

4.2. Proof of the general case. Although degenerate kernels play an important role in the theory of Fredholm equations and give a natural type approximation method, the degenerate kernel is not available for the general geometry. It’s why Golberg claimed “their use in practical problems seems to have taken a back seat to other methods such as collocation and quadrature” [24]. Thanks to the available degenerate kernel of circle and ellipse, we can theoretically prove the degenerate-scale free using the new approach in the above section. For the general geometry, we use the scaling technique (expansion ratio) to examine the modified kernel for the unique solution of the 2D Laplace problem for a general case. If \(B\) is an ordinary boundary, there exists a unique solution for \(\alpha(s)\) satisfying Eq.(2.2). For simplicity, we can assume a constant along the ordinary boundary. Then, Eq.(2.2) can be rewritten as

\[
\int_B U_m(x, s)\alpha(s)dB(s) = \int_B \ln r \\alpha(s) dB(s) - \int_B \ln r^* \\alpha(s) dB(s) \\
= \bar{u}(x) = \gamma, x \in B.
\]  

(4.10)

where \(\gamma\) is a constant for the boundary condition and it is not trivial. If the domain is expanded by \(p\) times, and the fictitious source point, \(s^*\) becomes \(ps^*\) in the complement of a domain. According to mapping properties, we have some conditions as follows:

\[
(x_1, x_2) \rightarrow (px_1, px_2) = p(x_1, x_2),
\]  

(4.11)

\[
dB(s) \rightarrow dB(ps) = pdB(s),
\]  

(4.12)

\[
U_m(x, s) \rightarrow U_m(px, ps) = \ln \frac{p|x - s|}{p|x - s^*|} = \ln \frac{|x - s|}{|x - s^*|} = U_m(x, s).
\]  

(4.13)

According to the conditions of Eq. (4.13) and Eq. (2.2), we have

\[
\bar{u}(px) = \int_{B_p} U_m(px, ps)\alpha(ps)dB(ps) = p \int_B U_m(x, s)\alpha(ps)dB(s), px \in B_p.
\]  

(4.14)

According to Eq. (4.14), there exists \(\alpha(ps) = \frac{1}{p}\alpha(s)\) satisfying the boundary condition, \(\bar{u}(px) = \gamma\) for the domain bounded by \(B_p\). Although the present method and the necessary and sufficient BIE [25] both satisfy the objectivity, they are different. The main differences are the constraint and added term as shown in Table
1. Degenerate scale was determined by several approaches, including degenerate kernel, complex variable, scaling technique, and the BEM. We have studied degenerate scales for different shapes by using the above four methods. All the results are summarized in Table 2 including references.

5. Numerical implementation. In order to understand the degenerate-scale mechanism and the above formulations, the discrete form of Eq.(2.2) is shown as:

\[
[U] \{\alpha\} = \{\bar{a}\},
\]

(5.1)

The influence matrix \([U]\) is singular at the degenerate scale. The influence matrix of the modified fundamental solution, \([U_e]\), for the circular or elliptical case is shown below:

\[
[U_e]_{N \times N} = [U]_{N \times N} - \begin{bmatrix}
\ell_1 \ln r_1^* & \ell_2 \ln r_1^* & \cdots & \ell_N \ln r_1^* \\
\ell_1 \ln r_2^* & \ell_2 \ln r_2^* & \cdots & \ell_N \ln r_2^* \\
\vdots & \vdots & \ddots & \vdots \\
\ell_1 \ln r_N^* & \ell_2 \ln r_N^* & \cdots & \ell_N \ln r_N^*
\end{bmatrix}_{N \times N},
\]

(5.2)

where \(\ell_i\) is the length of the \(i\)th element (\(i = 1, ..., N\)), \(N\) is the number of elements and \(r_j^*\) is the distance between the \(j\)th element and the fictitious source point. The main difference between the present method and other non-dimensional approaches is that \(r_j^*\) is replaced by a constant. In the numerical implementation, we adopted 50 constant elements in the indirect BEM. Circular, elliptical, triangular, square and half-circular domains are demonstrated to show their drop at the degenerate scale using previous methods. The minimum singular value of the influence matrix versus the characteristic scale of the domain is shown in Fig. 2. The two drop positions are the original degenerate scale, \(a_d\) and the shifted degenerate scale, \(a_r\). The position of degenerate scale is addressed in Table 3. All drop positions appear as theoretically predicted. If the adding part is a constant, the degenerate scale only shifts to another degenerate scale. Burton and Miller approach and the dimensionless quantity, \(r/r^*\), avoids the degenerate scale for any case. The comparison of the four kinds of kernel functions is shown in Table 4.

6. Conclusions. When the degenerate scale occurs, the solution spaces of the indirect and direct BIEM/BEM are not equivalent. It was analytically verified by using the degenerate kernel and was numerically examined by the linear algebraic system. Although the boundary data may contain the infinite solution, the field response is unique. In addition, the effects of regularization techniques would be examined. The proposed methods have overcome the degenerate scale problem in the 2D Laplace interior problem by using the dimensionless fundamental solutions. First, a similar idea of the CHEEF method for the direct BEM was extended to fictitious source in the complement of the domain for the indirect BEM. Burton and Miller approach was extended from the Helmholtz equation to the Laplace equation. The degenerate kernel was employed to verify the validity of dimensionless quantity, \(r/r^*\) for circular and elliptical cases. By using the scaling technique, the present method was also analytically examined in a general-shape case. Finally, the numerical experiments were also addressed by using the BEM. It is found that analytical solutions and numerical results of these two proposed methods both yield degenerate-scale free for any domain.
Table 1. Comparison of four BIEs

| Method                              | Representation of the solution                                                                 | Unknown coefficient of Fourier series, $\alpha$ | Degenerate scale |
|-------------------------------------|-----------------------------------------------------------------------------------------------|-----------------------------------------------|------------------|
| Conventional BIE                    | $u(x) = \int_\mathbb{S} U(x,s) \alpha(s) dB$                                                  | Fail, undetermined                            | Appear           |
| Necessary and sufficient BIE,       | $egin{cases} u(x) = \int_\mathbb{S} U(x,s) \alpha(s) dB + c \\ \int_\mathbb{S} \alpha(s) dB = 0 \end{cases}$ | $\alpha = 0$ and $c$ depends on the boundary condition | Disappear        |
| Fichera approach [25]              |                                                                                               |                                               |                  |
| Burton & Miller approach            | $u(x) - \int_\mathbb{S} \left( T(x,s) - iU(x,s) \right) \alpha_{\text{inc}}(s) dB$            | $\alpha_0$ is a complex value                  | Disappear        |
| Method of a fictitious source       | $\begin{aligned} u(x) &= \int_\mathbb{S} \left[ U(x,s) - U(x,s^*) \right] \alpha(s) dB \\ &= \int_\mathbb{S} U(x,s) \alpha(s) dB - U(x,s^*) \int_\mathbb{S} \alpha(s) dB \end{aligned}$ | $\alpha_0$ depends on the boundary condition | Disappear        |

Table 2. Study on the degenerate scale by using different approaches

| Method                              | Geometry shape                                                                 |
|-------------------------------------|--------------------------------------------------------------------------------|
| Degenerate kernel                   | ![Geometric shapes](image1.png)                                                 |
| Complex variable                    | ![Geometric shapes](image2.png)                                                 |
| Scaling technique                   | ![Geometric shapes](image3.png)                                                 |
| BEM                                 | ![Geometric shapes](image4.png)                                                 |
Table 3. Degenerate scales appearing in the conventional BEM, the approach of adding rigid body mode, the Burton and Miller approach and the method of introducing the fictitious source point.

| Shape | Conventional BEM | Adding rigid body term | Burton and Miller approach | Present method ($r/r^*$) |
|-------|------------------|------------------------|----------------------------|--------------------------|
|       |                  |                        |                            |                          |
|       | $a_0 = 1$        | $a_0 = 1.001$          | $\alpha = 0.3683$          | Disappear                | Disappear                |
|       |                  |                        |                            |                          |
|       | $a_0 = 1.5$      | $a_0 = 1.5015$         | $\alpha = 0.5520$          | Disappear                | Disappear                |
| $a = 3b$ |                  |                        |                            |                          |
|       | $a_0 = 2.37105$  | $a_0 = 2.3797$         | $\alpha = 0.8755$          | Disappear                | Disappear                |
|       |                  |                        |                            |                          |
|       | $a_0 = 1.69443$  | $a_0 = 1.6983$         | $\alpha = 0.6248$          | Disappear                | Disappear                |

#: The degenerate scale is moved to another one.

Table 4. Comparison of the four kernel functions.

| Method                        | Kernel function | Representation of the solution | Enriched range due to the adding part | Effect                        |
|-------------------------------|-----------------|--------------------------------|---------------------------------------|-------------------------------|
| Conventional BIE              | $U(x,s) - \ln|x - s| - \ln r$ | $u(x) = \int U(x,s)u(s)\mathrm{d}s(x)$ | N.A.                                  | Degenerate scale.             |
| Adding a rigid body mode, $r$ | $\ln r + r$, $r$ is a constant. | $u(x) = \int U(x,s)u(s)\mathrm{d}s(x) + \frac{1}{r^2} \int \nabla u(s)\mathrm{d}s(x)$ | A constant                     | Shifting to another degenerate scale. |
| Burton & Miller approach      | $U(x,s) - \frac{\partial U(x,s)}{\partial r} - \frac{\partial U(x,s)}{\partial n}$ | $u(x) = \int \left[U(x,s) - \frac{\partial U(x,s)}{\partial n}\right]u(s)\mathrm{d}s$ | A field depends on $r$ kernel | Free of degenerate scale.     |
| Method of a fictitious source | $\ln\frac{r^2}{r^2 - (s - x)\cdot (s - x)}$ | $u(x) = \int U(x,s)
abla u\cdot n\mathrm{d}s(x)$ | A field depends on $r^2$, $r$ | Free of degenerate scale.     |
Figure 1. Row space, column space, null space and range of $[U]$ and $[T]$ matrices in the indirect and direct BEMs when the degenerate scale occurs.
AN INDIRECT BIE FREE OF DEGENERATE SCALES

(a) A circle

(b) An ellipse

(c) A triangle

(d) A square

Figure 2. Numerical evidences for the degenerate scale of four cases in the BEM

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