THE REGULARITY OF A DEGENERATE GOURSAT PROBLEM FOR THE 2-D ISOTHERMAL EULER EQUATIONS

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(Communicated by Alain Miranville)

ABSTRACT. We study the regularity of solution and of sonic boundary to a degenerate Goursat problem originated from the two-dimensional Riemann problem of the compressible isothermal Euler equations. By using the ideas of characteristic decomposition and the bootstrap method, we show that the solution is uniformly $C^{1,\frac{1}{6}}$ up to the degenerate sonic boundary and that the sonic curve is $C^{1,\frac{1}{6}}$.

1. Introduction. We consider the two-dimensional isothermal compressible Euler equations

$$\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0, \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0,
\end{align*}$$

where $\rho$ is the density, $(u, v)$ is the velocity and $p$ is given by the isothermal pressure law

$$p(\rho) = \rho.$$

Comparing with the isentropic gas $p(\rho) = \rho^\gamma$ ($\gamma > 1$) studied in [20], the isothermal gas (2) has its own unique features and challenges. For example, the sonic speed is uniformly unit in the isothermal case which prevents us from getting some important relations in the isentropic case.

We are interested in the semi-hyperbolic patch problems originated from the two-dimensional Riemann problem for (1) (2). In the previous paper [7], Hu, Li and Sheng established the global existence of smooth solutions in a semi-hyperbolic region to (1) (2) by solving a degenerate Goursat problem which has a sonic curve as a boundary. In the current paper, we address the regularity of solutions near the sonic boundary and regularity of the sonic boundary.

2000 Mathematics Subject Classification. 35L65, 35L80, 35R35.

Key words and phrases. Compressible Euler equations, semi-hyperbolic patch, degenerate Goursat problem, sonic curve, characteristic decomposition.

The first author was supported by NSF of Zhejiang Province LY17A010019, NSFC 11301128, 11571088 and China Scholarship Council 201708330155.

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A semi-hyperbolic patch is a region in which a family of characteristics starts on a sonic curve and ends on either a sonic curve or a transonic shock wave [21]. This type of regions appears frequently in the two-dimensional Riemann problem for the compressible Euler equations in gas dynamics, see [12, 28, 5]. In addition, semi-hyperbolic patches may also occur in the transonic flow over an airfoil [3, 2], in rarefaction wave reflection along a compressive corner [17] and in Guderley shock reflection of the von Neumann triple point paradox [22, 23]. The study of semi-hyperbolic patch problems provides us important information about the parabolic degenerate curves. The results on the semi-hyperbolic patches make it possible to construct global solutions for general nonlinear mixed-type equations in future work. The analysis of the semi-hyperbolic patch problems was initiated by Song and Zheng [21] for the pressure-gradient system. In [16], Li and Zheng investigated the semi-hyperbolic patch problems of the two dimensional isentropic Euler equations. A different viewpoint to study this kind of problems for the pressure-gradient system was proposed by Song [19]. In [24], Wang and Zheng studied the regularity of the semi-hyperbolic patch problems for the pressure-gradient system. We also refer the reader to [6, 25, 26, 27] for the construction of classical sonic-supersonic solutions to the Euler system. The above works [7, 6, 21, 16, 19, 24, 25, 26, 27] on building smooth solutions are based on the idea of characteristic decomposition which is a powerful tool revealed in [4], see, e.g., [1, 9, 10, 11, 13, 14, 15, 18] for more applications.

In the present paper, we show that the smooth solution of the semi-hyperbolic patch problem of the isothermal compressible Euler equations (1) (2) is uniformly $C^{1,\frac{1}{6}}$ up to the degenerate sonic boundary and the sonic curve is $C^{1,\frac{1}{6}}$-continuous. In [20], the authors explored such regularity problems for the isentropic Euler system. They verified that there exists a global solution up to the sonic boundary and that the sonic boundary has $C^1$-regularity. The previous work [20] relies on the known result from [16] that $\partial^+ c + \partial^- c = 0$ on the sonic curve. However, for the current isothermal case (2), such a relation is not available. To overcome the difficulty, we introduce a novel set of change variables to show our regularity results without using any relations on the sonic curve, which is the main innovation of this article. Moreover, these change variables allow us to establish higher regularity of the solution and of the sonic boundary by using the bootstrap method inspired by [20]. The analysis in this paper was also applied to improve the regularity results for the isentropic Euler systems, see [8].

The rest of the paper is organized as follows. Section 2 is devoted to describing the problem and presenting the main results. In Section 3, we provide the characteristic decompositions of the inclination angles and show some property of solutions in the self-similar plane. In Section 4, we discuss the properties of solutions in a partial hodograph coordinates. Finally, we complete the proof of our main theorem in Section 5.

2. Set up of the problem and the main results. In this section, we set up the semi-hyperbolic patch problem of (1) (2) and then present the main results of this paper.

We first provide some preliminaries for (1) with (2). System (1) admits self-similar solutions. We introduce the self-similar variables $(\xi, \eta) = (x/t, y/t)$ and

...
transform (1)-(2) to
\[\begin{align*}
U(\ln \rho)_\xi + V(\ln \rho)_\eta + u_\xi + v_\eta &= 0, \\
Uu_\xi + Vu_\eta + (\ln \rho)_\xi &= 0, \\
Uv_\xi + Vv_\eta + (\ln \rho)_\eta &= 0, \\
\end{align*}\]
where \((U, V) = (u - \xi, v - \eta)\) is the pseudo-velocity. Assume the flow is ir-rotational with \(u_\eta = v_\xi\), then we can decouple a subsystem in terms of \((u, v)\) from (3)
\[\begin{align*}
(1 - U^2)u_\xi - UV(u_\eta + v_\xi) + (1 - V^2)v_\eta &= 0, \\
\end{align*}\]
The density can be restored from the momentum equations of (3) or from the Bernoulli law
\[\frac{U^2 + V^2}{2} + \ln \rho = -\phi, \quad \phi_\xi = U, \quad \phi_\eta = V,\]
where \(\phi\) is the pseudo-velocity potential.
It is easy to calculate that the positive/negative eigenvalues of (4) are
\[\Lambda_{\pm} = \frac{UV \pm \sqrt{U^2 + V^2 - 1}}{U^2 - 1},\]
from which we see that system (4) is of mixed-type: supersonic for \(U^2 + V^2 > 1\), subsonic for \(U^2 + V^2 < 1\). A point \((\xi, \eta)\) is called the sonic point if it satisfies
\[U^2(\xi, \eta) + V^2(\xi, \eta) = 1.\]
The set of points at which \(U^2 + V^2 = 1\) is called the sonic curve.
Now we give a description of the problem that is investigated in this paper. Suppose that \((\rho_1, 0, v_1)^T \in B\) and \((\rho_4, 0, 0)^T \in A\) are two constant states satisfying \(\rho_1 > \rho_4 > \rho_1 > v_1 > 0\). Let \(R_{14}(\eta)\) be a planar rarefaction wave of (1) (2) in the self-similar coordinates \((\xi, \eta)\), connecting these two constant states \((1)\) and \((4)\), defined by
\[R_{14} : \begin{cases}
\eta = v + 1, & (\eta_4 \leq \eta \leq \eta_1) \\
v = \ln \rho - \ln \rho_4, & (\rho_4 \leq \rho \leq \rho_1) \\
u = u_1 = u_4 = 0
\end{cases}\]
in the domain \(\xi > 0\), where \(\eta_i = v_i + 1, \ i = 1, 4\). Denote the point \((\xi, \eta) = (0, v_i + 1)\) by \(A_i\), which is a sonic point by (7). According to (6) and (8), the positive characteristic curve passing though \(A_i\) in \(R_{14}\) is
\[\Gamma_+^i : \eta = v_i + 1 + \ln(1 - \xi^2), \quad \xi \in [0, 1].\]
We set \(\xi_B = \sqrt{1 - e^{-v_1}} \in (0, 1)\) and denote the point \((\xi, \eta) = (\xi_B, 1)\) by \(B\) which is the intersection point of \(\Gamma_+^i\) with the bottom boundary of \(R_{14}\), see Fig. 1. From (8), the boundary data \((\rho, u, v) = (\bar{\rho}, \bar{u}, \bar{v})(\xi)\) on \(\overline{BA}\) are
\[\bar{B}A : \begin{cases}
\bar{\rho}(\xi) = \rho_4(1 - \xi^2)e^{v_1}, & \bar{u}(\xi) = 0, \\
\bar{v}(\xi) = v_1 + \ln(1 - \xi^2), & \xi \in [0, \xi_B].
\end{cases}\]
Let \(\xi_C \in (0, \xi_B)\) be a real number and \(\overline{BC} : \eta = \phi(\xi) \ (\xi_C \leq \xi \leq \xi_B)\) be a smooth curve. We assign the boundary data for \((\rho, u, v)\) on \(\overline{BC}\), \((\rho, u, v)(\xi, \phi(\xi)) = \)
Figure 1. The semi-hyperbolic patch.

\( (\hat{\rho}, \hat{u}, \hat{v})(\xi) \) such that

\[
\varphi(\xi) \in C^3, \quad \varphi''(\xi) > 0, \quad (\hat{\rho}, \hat{u}, \hat{v})(\xi) \in C^2, \quad (\hat{\rho}, \hat{u}, \hat{v})(\xi_B) = (\rho_4, 0, 0),
\]

\[
\hat{BC} : \quad \varphi' = \frac{(\hat{u} - \xi)(\hat{v} - \varphi) - \sqrt{(\hat{u} - \xi)^2 + (\hat{v} - \varphi)^2 - 1}}{(u - \xi)^{2} - 1}, \quad [(\hat{u} - \xi)^2 + (\hat{v} - \varphi)^2]_{\xi_C} = 1,
\]

\[
\hat{u}' + \frac{(u - \xi)(\hat{v} - \varphi) + \sqrt{(u - \xi)^2 + (\hat{v} - \varphi)^2 - 1}}{(u - \xi)^2 - 1} \hat{v}' = 0
\]

for any \( \xi \in [\xi_C, \xi_B] \). The last condition of (10) is the compatibility condition of boundary data on \( \hat{BC} \) with system (4). Moreover, by (6), the boundary conditions (10) imply that the smooth curve \( \hat{BC} \) is a smooth convex negative characteristic curve and the point \( C \) is a sonic point. The semi-hyperbolic patch problem is described as follows: assuming the boundary conditions (9) and (10) hold, we look for a smooth solution in curvilinear triangle \( \hat{ABC} \) and a simple wave solution in curvilinear triangle \( \hat{CBD} \), where \( \hat{AC} \) is sonic curve while \( \hat{CD} \) is an envelope, see Fig. 1.

The existence of smooth solutions to the semi-hyperbolic patch problem in the region \( \hat{ABC} \) was verified by Hu, Li and Sheng [7]. In this paper we are mainly concerned with the regularity of solutions near the sonic curve \( \hat{AC} \). The main conclusion of the paper can be stated as follows.

**Theorem 1.** Suppose that \( \arctan \varphi'(\xi_C) \in (-\frac{\pi}{2}, 0) \). Then there exists a global smooth solution \( (\rho, u, v) \in C^2 \) to the degenerate Goursat boundary value problem (3) (9)-(10) in the region \( \hat{ABC} \) with the sonic boundary \( \hat{AC} \). Moreover, the solution \( (\rho, u, v)(\xi, \eta) \) is uniformly \( C^{1, \frac{1}{2}} \) up to the sonic boundary \( \hat{AC} \) and the sonic curve \( \hat{AC} \) is \( C^{1, \frac{1}{2}} \)-continuous.

We need to overcome the major difficulty arising from the hyperbolic degeneracy of the Euler system at the sonic curves. The key is to prove that the term \( (R - S)/\cos \delta \) is uniformly bounded up to the sonic boundary \( \hat{AC} \), where the variables \( R, S \) and \( \delta \) are defined in Section 3. For the isentropic case studied in [20], the authors took the relation \( R - S = 0 \) on the sonic curve as a known result from
In this paper, such a relation is not available. In order to acquire our regularity theory, we introduce a partial hodograph transformation and a novel change variables \((\tilde{G}, \tilde{H})\), which play a crucial role in our analysis, see (50). This change variables allows us to prove the uniform boundedness of \((R - S)/\cos \delta\) by employing the bootstrap method. Finally, after returning to the self-similar variables, we establish the uniform regularity of smooth solutions of system (3) up to the sonic boundary \(\hat{AC}\).

3. Properties of solutions in the self-similar plane. In this section, we deliver the characteristic decompositions for the inclination angles of characteristics and then show an important property of solutions.

3.1. Characteristic decompositions in angle variables. We adopt the inclination angle variables of characteristics as the dependent variables, which was invented by Li and Zheng [14] and then was used successfully in many problems, see, e.g. [1, 6, 7, 10, 15, 18, 25, 27].

By standard manipulation, system (4) can be turned into the characteristic form
\[
\partial^\pm u + \Lambda_\pm \partial^\pm v = 0, \quad \partial^\pm = \partial_\xi + \Lambda_\pm \partial_\eta,
\]
(11)

Following [14] and [7], we introduce
\[
\tan \alpha = \Lambda_+, \quad \tan \beta = \Lambda_-,
\]
(12)

and denote
\[
\sigma = \frac{\alpha + \beta}{2}, \quad \delta = \frac{\alpha - \beta}{2}.
\]
(13)

Here \(\sigma\) is called the pseudo-streamline angle and \(\delta\) is called the pseudo-Mach angle. Then \((u, v)\) can be expressed in terms of \((\sigma, \delta)\)
\[
u = \xi - \frac{\cos \sigma}{\sin \delta}, \quad v = \eta - \frac{\sin \sigma}{\sin \delta},
\]
(14)

and the sonic curve reads now \(\{ (\xi, \eta) : \sin \delta (\xi, \eta) = 1 \}\).

Introducing the scaled characteristic fields
\[
\bar{\partial}^+ = \cos \alpha \partial_\xi + \sin \alpha \partial_\eta, \quad \bar{\partial}^- = \cos \beta \partial_\xi + \sin \beta \partial_\eta,
\]
(15)

then
\[
\partial_\xi = \frac{\cos \sigma \sin \delta (\bar{\partial}^+ + \bar{\partial}^-) - \sin \sigma \cos \delta (\bar{\partial}^+ - \bar{\partial}^-)}{\sin (2\delta)},
\]
\[
\partial_\eta = \frac{\sin \sigma \sin \delta (\bar{\partial}^+ + \bar{\partial}^-) + \cos \sigma \cos \delta (\bar{\partial}^+ - \bar{\partial}^-)}{\sin (2\delta)},
\]
(16)

Now, we obtain the characteristic form in terms of \((\sigma, \delta)\) from (11) and (14)
\[
\begin{cases}
\bar{\partial}^+ \sigma + \frac{\cos^2 \delta}{\sin^2 \delta} \bar{\partial}^+ \delta = -\cos (2\delta), \\
\bar{\partial}^- \sigma - \frac{\cos^2 \delta}{\sin^2 \delta} \bar{\partial}^- \delta = \cos (2\delta).
\end{cases}
\]
(17)

For later calculations, we present the equations for \((\alpha, \beta)\)
\[
\begin{cases}
\bar{\partial}^+ \alpha = \frac{-\cos (2\delta)}{\sin^2 \delta} \bar{\partial}^+ \delta - \cos (2\delta), \\
\bar{\partial}^- \alpha = \frac{1}{\sin^2 \delta} \bar{\partial}^- \delta + \cos (2\delta),
\end{cases}
\]
\[
\begin{cases}
\bar{\partial}^+ \beta = \frac{1}{\sin^2 \delta} \bar{\partial}^+ \delta - \cos (2\delta), \\
\bar{\partial}^- \beta = \frac{\cos (2\delta)}{\sin^2 \delta} \bar{\partial}^- \delta + \cos (2\delta),
\end{cases}
\]
(18)
and the characteristic decomposition of \((\alpha, \beta)\)

\[
\begin{align*}
\bar{\partial}^{-}\partial^{+}\alpha &= \left\{ \frac{\sin(2\delta)}{2} (3\tan^{2}\delta - 1) + \frac{1 - 4\sin^{2}\delta}{\sin(2\delta)\cos(2\delta)} \bar{\partial}^{-}\alpha - \frac{\partial^{+}\beta}{\sin(2\delta)} \right\} \partial^{+}\alpha, \\
\bar{\partial}^{+}\partial^{-}\beta &= \left\{ \frac{\sin(2\delta)}{2} (3\tan^{2}\delta - 1) - \frac{1 - 4\sin^{2}\delta}{\sin(2\delta)\cos(2\delta)} \partial^{+}\beta + \frac{\bar{\partial}^{-}\alpha}{\sin(2\delta)} \right\} \partial^{-}\beta.
\end{align*}
\]  

(19)

The detailed derivation of equations (17) and (19) can be found in \([7]\).

To deal with our problem, we further introduce two new variables

\[
R := -\tan(2\delta)\partial^{+}\alpha, \quad S := -\tan(2\delta)\partial^{-}\beta.
\]  

(20)

By a direct calculation, one has

\[
\bar{\partial}^{+}\partial^{-}\delta = \frac{\sin\delta}{2\cos\delta} R - \sin^{2}\delta, \quad \bar{\partial}^{-}\partial^{+}\delta = -\frac{\sin\delta}{2\cos\delta} S - \sin^{2}\delta.
\]  

(21)

Making use of (19), we can derive the characteristic decomposition of \((R, S)\)

\[
\begin{align*}
\bar{\partial}^{-}\partial^{+}R &= R \left\{ \frac{R - S}{\sin^{2}(2\delta)} - 2\sin(2\delta) \right\}, \\
\bar{\partial}^{-}\partial^{+}S &= -S \left\{ \frac{R - S}{\sin^{2}(2\delta)} - 2\sin(2\delta) \right\},
\end{align*}
\]  

(22)

which is considerably more concise than equation (19) and plays an important role in our analysis. Here we use the scaled characteristic directions \(\bar{\partial}^{-}\) and \(-\bar{\partial}^{+}\) since they are both pointing away from the sonic boundary \(\hat{AC}\).

3.2. Boundedness of \((R, S)\). In this subsection, we show the boundedness of \((R, S)\) defined in (20). We first present the properties of solutions of (17) established in [7].

**Proposition 3.1** ([7]). **Assuming the boundary conditions (9)-(10), the boundary data in terms of angle variables \((\alpha, \beta)\) satisfy**

\[
\begin{align*}
\frac{\pi}{2} &\leq \alpha \left|_{BA} \right. \leq \pi, \\
\frac{\pi}{2} &\leq \beta \left|_{BA} \right. = 0, \\
\frac{\pi}{2} &\leq \alpha \left|_{BC} \right. \leq \pi + \beta C, \\
\frac{\pi}{2} &\leq \beta C \left|_{BC} \right. \leq \pi + \beta C,
\end{align*}
\]  

(23)

and the smooth solution \((\alpha, \beta)(\xi, \eta)\) in the region \(ABC\) satisfies

\[
\alpha \geq \frac{\pi}{2}, \quad \beta \leq 0, \quad \frac{\pi}{4} \leq \delta \leq \frac{\pi}{2}, \quad \pm \bar{\partial}^{+}\alpha > 0, \quad \pm \bar{\partial}^{-}\beta < 0, \quad \pm \bar{\partial}^{+}\delta > 0.
\]  

(24)

**Furthermore, the \(\Gamma_{+}\)-characteristics are concave and \(\Gamma_{-}\)-characteristics are convex.**

Based on Proposition 3.1, we now establish the boundedness of \((R, S)\). For any point \((\xi, \eta)\) in the region \(ABC\), we draw the positive and negative characteristic curves up to the boundaries \(BC\) and \(BA\) at points \(B_{2}\) and \(B_{1}\), respectively, see Fig 2.

**Lemma 3.1.** **Assume conditions (23). For any point \((\xi, \eta)\) in the region \(ABC\), the functions \(R\) and \(S\) satisfy**

\[
0 < m e^{-2d} \leq R(\xi, \eta) \leq Me^{2d}, \quad 0 < m e^{-2d} \leq S(\xi, \eta) \leq Me^{2d},
\]  

(25)

where \(m = \min \{\min_{\hat{B}_{1}} R, \min_{\hat{B}_{2}} S\} > 0\), \(M = \max \{\max_{\hat{B}_{1}} R, \max_{\hat{B}_{2}} S\}\) and \(d\) is the diameter of the domain \(ABC\).
Proof. It is easy to see by the definitions of $(R, S)$ and (24) that $R > 0$ and $S > 0$ in the region $ABC$. Denote $F = (R - S)/\sin^2(2\delta) - 2\sin(2\delta)$. We next split the proof into two steps.

**Step I.** We show that $R$ and $S$ have a uniform upper bound in the whole region $ABC$ in (25). The proof is further divided into two cases.

**Case 1.** For any point $P \in ABC$, if $S \geq R$ entirely in the region $PB_1BB_2$, then by (24) we have $F \leq 0$ and

$$\bar{\partial} - \ln R = F \geq -2$$

on $\Gamma_1^-$. Integrating the above from $P$ to $Q_1$ and noticing the direction of $\bar{\partial}$ pointing toward $\hat{BC}$ yields

$$S|_{Q_1} < R|_{Q_1} \leq R|_{P_1} e^{2\left(\xi_{P_1} - \xi_{Q_1}\right)} = S|_{Q_1} e^{2\left(\xi_{Q_1} - \xi_{P_1}\right)}.$$  

If $R < S$ for some point on $\Gamma_1^-$, then there exists a neighborhood $N_1$ of $P_1$ so that $R > S$ holds for every point in $N_1 \cap \Gamma_1^-$ and $R = S$ at $Q_1 := \partial N_1 \cap \Gamma_1^-$. Hence we have

$$\bar{\partial} - \ln R \geq -2$$

on $N_1 \cap \Gamma_1^-$. It follows by integrating the above from $P_1$ to $Q_1$ that

$$S|_{P_1} < R|_{P_1} \leq R|_{Q_1} e^{2\left(\xi_{Q_1} - \xi_{P_1}\right)} = S|_{Q_1} e^{2\left(\xi_{Q_1} - \xi_{P_1}\right)}.$$  

**Case 2.** If there exists a point, say $P_1$, in $PB_1BB_2$, such that $R > S$ at $P_1$, then from the point $P_1$, we draw a negative characteristic curve, called $\Gamma_1^-$, up to the boundary $\hat{BB_2}$ at a point $P_0$, see Fig. 2. If $R \geq S$ always holds on $\Gamma_1^-$, then we have

$$F \geq -2 \sin(2\delta) \geq -2$$

on $\Gamma_1^-$. Integrating the above from $P_1$ to $P_0$ and noticing the direction of $\bar{\partial}$ gives

$$R|_{P_1} \leq S|_{P_1} \leq S|_{P_0} \leq \max_{\hat{BB_2}} S \leq M e^{2d}.$$  

If $R < S$ for some point on $\Gamma_1^-$, then there exists a neighborhood $N_1$ of $P_1$ so that $R > S$ holds for every point in $N_1 \cap \Gamma_1^-$ and $R = S$ at $Q_1 := \partial N_1 \cap \Gamma_1^-$. Hence we have

$$\bar{\partial} - \ln R \geq -2$$

on $N_1 \cap \Gamma_1^-$. It follows by integrating the above from $P_1$ to $Q_1$ that

$$S|_{P_1} < R|_{P_1} \leq R|_{Q_1} e^{2\left(\xi_{Q_1} - \xi_{P_1}\right)} = S|_{Q_1} e^{2\left(\xi_{Q_1} - \xi_{P_1}\right)}.$$  

(27)
From the point $Q_1$, we now draw a positive characteristic curve, called $\Gamma_1^+$, up to the boundary $BB_2$ at a point $Q_1$. We recall $F|_Q < 0$ and discuss it in the following two subcases.

**Subcase 2a.** If $F < 0$ always holds on $\Gamma_1^+$, then we find by the equation for $S$ that $-\tilde{\partial}^+ S \geq 0$ and

$$S|_{Q_1} \leq S|_{Q_1} \leq M,$$

which together with (27) finishes the proof.

**Subcase 2b.** If $F > 0$ holds for some point on $\Gamma_1^+$, then there exists a neighborhood $\mathbb{N}_2$ of $Q_1$ so that $F < 0$ holds for every point in $\mathbb{N}_2 \cap \Gamma_1^+$ and $F = 0$ at $P_2 := \partial\mathbb{N}_2 \cap \Gamma_1^+$. Moreover, we see that $-\tilde{\partial}^+ S \geq 0$ on $\mathbb{N}_2 \cap \Gamma_1^+$, and then

$$S|_{Q_1} \leq S|_{P_2},$$

which combined with (27) and the definition of $F$ leads to

$$S|_{P_2} < R|_{P_2} \leq S|_{P_2} e^{2(\xi_2,\xi_r)} = (R - 2\sin^2(\delta))|_{P_2} e^{2(\xi_2,\xi_r)} \leq R|_{P_2} e^{2(\xi_2,\xi_r)}. \quad (28)$$

Next, from $P_2$, we draw a negative characteristic curve, called $\Gamma_2^-$, up to the boundary $BB_1$ at a point $P_3$. There must hold

$$R|_{P_2} \leq R|_{P_2} e^{2(\xi_2,\xi_r)} \leq Me^{2(\xi_2,\xi_r)}, \quad (29)$$

if $R > S$ on $\Gamma_2^-$, or

$$R|_{P_3} \leq R|_{P_3} e^{2(\xi_2,\xi_r)}, \quad (30)$$

where $Q_2$ is a point on $\Gamma_2^-$ such that $R > S$ on $P_2 Q_2 \cap \Gamma_2^-$ and $R = S$ at $Q_2$. If (29) holds, then we combine it with (28) to obtain by using the property of positive characteristics

$$S|_{P_1} < R|_{P_1} \leq M e^{2(\xi_2,\xi_r)} \leq Me^{2d},$$

otherwise, we put (30) into (28) to find

$$S|_{P_1} < R|_{P_1} \leq R|_{Q_2} e^{2(\xi_2,\xi_r)} \leq R|_{Q_2} e^{2(\xi_2,\xi_r)} = S|_{Q_2} e^{2(\xi_2,\xi_r)},$$

and repeat the above processes to complete the proof.

**Step II.** We next show the lower bound of $R$ and $S$ in (25). The proof is similar to Step I. For any point $P(\xi, \eta) \in ABC$, if $S \geq R$ entirely on the whole region $PB_1 BB_2$, then $F \leq 0$ and one has

$$\tilde{\partial}^- R \leq 0 \quad \text{on} \quad PB_1 BB_2,$$

from which we find that

$$S|_P \geq R|_P \geq R|_{B_1} \geq Me^{-2d} > 0.$$

If $R > S$ holds on the whole segment $\overline{PB_2}$, it follows by the second equation of (22) that

$$-\tilde{\partial}^+ \ln S \leq 2\sin(2\delta) \leq 2.$$

Thus

$$R|_P \geq S|_P \geq S|_{B_2} e^{-2(\xi_2,\xi_r)} \geq Me^{-2d}.$$

Otherwise, we choose a point $P'$ on $\overline{PB_2}$ such that $R = S$ at the point $P'$ and $R > S$ on the segment $\overline{PP'}$. Then

$$R|_P \geq S|_P \geq S|_{P'} e^{-2(\xi_2,\xi_r)} = R|_{P'} e^{-2(\xi_2,\xi_r)}.$$
From the point $P'$, we draw the negative characteristic curve up to the boundary $\hat{BB}_1$ and repeat the processes as Subcases 2a and 2b in Step I to complete the proof.

\textbf{Remark 1.} Lemma 3.1 shows that the lower bound of $(R, S)(\xi, \eta)$ in (25) is independent of the distance from the point $(\xi, \eta)$ to the sonic boundary $\hat{AC}$. This fact will be used in the analysis of the paper.

4. Solutions in the partial hodograph coordinates. Let us first analyze what we need to obtain the regularity of the sonic boundary $\hat{AC}$. Since the sonic curve is the set of points at which $\sin \delta = 1$, we then consider the level curves $\ell^\varepsilon(\xi, \eta) = 1 - \sin \delta = \varepsilon$.

Here $\varepsilon$ is a small positive constant. According to (16), (18) and (21), it follows that

$$\ell^\varepsilon_\xi = \frac{\sin \beta \bar{\partial} + \sin \alpha \bar{\partial} - \sin \delta}{2 \sin \delta} = \frac{\sin \sigma}{4} (R + S) + \cos \sigma \left( \sin^2 \delta - \frac{\sin \delta}{4} \cdot \frac{R - S}{\cos \delta} \right),$$

and

$$\ell^\varepsilon_\eta = \frac{\cos \alpha \bar{\partial} - \cos \beta \bar{\partial} + \sin \delta}{2 \sin \delta} = \frac{\cos \sigma \cos \delta (\bar{\partial} - \bar{\partial} + \bar{\partial} - \bar{\partial}) - \sin \sigma \sin \delta (\bar{\partial} + \bar{\partial} + \bar{\partial} - \bar{\partial})}{2 \sin \delta} = \frac{-\cos \sigma}{4} (R + S) + \sin \sigma \left( \sin^2 \delta - \frac{\sin \delta}{4} \cdot \frac{R - S}{\cos \delta} \right).$$

Moreover, we have

$$\left(\ell^\varepsilon_\xi\right)^2 + \left(\ell^\varepsilon_\eta\right)^2 = \left(\frac{R + S}{16}\right)^2 + \left( \sin^2 \delta - \frac{\sin \delta}{4} \cdot \frac{R - S}{\cos \delta} \right)^2,$$

which along with (25) arrives

$$0 < \left(\frac{R + S}{16}\right)^2 \leq \left(\ell^\varepsilon_\xi\right)^2 + \left(\ell^\varepsilon_\eta\right)^2 \leq M^2 e^{4d} + \left(1 + \left|\frac{R - S}{\cos \delta}\right|\right)^2.$$  

It is obvious that, in order to obtain the regularity of $\hat{AC}$, we need to establish a uniform upper bound for the term $W := (R - S)/\cos \delta$.

4.1. The new equations. We introduce a transformation $(\xi, \eta) \to (z, t)$ by defining

$$t = \cos \delta(\xi, \eta), \quad z = \phi(\xi, \eta),$$

where $\phi$ is the pseudo-velocity potential defined in (5). Thanks to (21) and (25), the Jacobian of this transformation is

$$J := \frac{\partial(z, t)}{\partial(\xi, \eta)} = \sin(V \delta_\xi - U \delta_\eta) = \frac{\bar{\partial} + \bar{\partial} - \bar{\partial} - \bar{\partial}}{2 \sin \delta} = \frac{R + S}{4 \cos \delta} > 0,$$

in the whole region $ABC$.

In terms of this new coordinates $(t, z)$, one has

$$\bar{\partial} = -\left(\frac{1 - t^2}{2t} R - (1 - t^2) \sqrt{1 - t^2}\right) \partial_t - \frac{t}{\sqrt{1 - t^2}} \partial_z,$$
\[
\bar{\partial}^- = \left( \frac{1-t^2}{2t} S + (1-t^2)\sqrt{1-t^2} \right) \partial_t - \frac{t}{\sqrt{1-t^2}} \partial_z.
\]

Inserting the above into (22) yields a new closed system for \((R, S)\) under the coordinates \((t, z)\)

\[
\begin{align*}
R_t - \frac{2\bar{S}^2}{(S+2\sqrt{1-t^2})\sqrt{(1-t^2)^3}} R_z &= \frac{R}{(1-t^2)(S+2\sqrt{1-t^2})} \left\{ \frac{R-S}{2(1-t^2)} - 8t\sqrt{1-t^2} \right\}, \\
S_t + \frac{2\bar{R}^2}{(R-2\sqrt{1-t^2})\sqrt{(1-t^2)^3}} S_z &= \frac{S}{(1-t^2)(R-2\sqrt{1-t^2})} \left\{ \frac{S-R}{2(1-t^2)} + 8t\sqrt{1-t^2} \right\}.
\end{align*}
\]

Here we still use the notations \((R, S)(z, t)\) to denote \((R, S)(\xi(z, t), \eta(z, t))\). From (25), we set \(\bar{R} = 1/R\) and \(\bar{S} = 1/S\), which are uniformly bounded up to the sonic boundary. Then system (37) changes to

\[
\begin{align*}
\bar{R}_t - \frac{2\bar{S}^2}{(1+2\sqrt{1-t^2})\sqrt{(1-t^2)^3}} \bar{R}_z &= \frac{1}{(1-t^2)(1+2\sqrt{1-t^2})} \left\{ \frac{\bar{R}-\bar{S}}{2(1-t^2)} + 8t\bar{R}\bar{S}\sqrt{1-t^2} \right\}, \\
\bar{S}_t + \frac{2\bar{R}^2}{(1-2\sqrt{1-t^2})\sqrt{(1-t^2)^3}} \bar{S}_z &= \frac{1}{(1-t^2)(1-2\sqrt{1-t^2})} \left\{ \frac{\bar{S}-\bar{R}}{2(1-t^2)} - 8t\bar{R}\bar{S}\sqrt{1-t^2} \right\}.
\end{align*}
\]

Denote the eigenvalues of (38) as

\[\lambda_- = -\frac{2\bar{S}}{(1+2\sqrt{1-t^2})\sqrt{(1-t^2)^3}}, \quad \lambda_+ = \frac{2\bar{R}}{(1-2\sqrt{1-t^2})\sqrt{(1-t^2)^3}}\]

and

\[\partial_\pm = \partial_t + \lambda_\pm \partial_z, \quad \begin{align*} G &= \partial_+ \bar{R} - \partial_- \bar{R}, \\
H &= \partial_+ \bar{S} - \partial_- \bar{S}.
\end{align*}\]

It follows that

\[\frac{\lambda_+ - \lambda_-}{t^2} = \frac{2(\bar{R} + \bar{S})}{\sqrt{(1-t^2)^3}(1-2\sqrt{1-t^2}) \sqrt{1-t^2}}.
\]

and

\[\bar{R}_z = \frac{G}{\lambda_+ - \lambda_-}, \quad \bar{S}_z = \frac{H}{\lambda_+ - \lambda_-}.
\]

Making use of the following commutator relation [13]

\[\partial_- \partial_+ - \partial_+ \partial_- = \frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_+ - \lambda_-} (\partial_+ - \partial_-),\]

we get

\[\begin{align*}
\partial_- G &= \frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_+ - \lambda_-} G + (\partial_+ \partial_- \bar{R} - \partial_- \partial_+ \bar{R}), \\
\partial_+ H &= \frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_+ - \lambda_-} H + (\partial_+ \partial_+ \bar{S} - \partial_- \partial_+ \bar{S}).
\end{align*}
\]

By a tedious but straightforward calculation, we deduce

\[\frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_+ - \lambda_-} = \frac{2}{t} + f(z, t),\]

where

\[f(z, t) = \frac{3t}{1-t^2} + \frac{2(1-t^2)(\bar{R} - \bar{S} + 4t\bar{R}\bar{S}\sqrt{1-t^2})}{\sqrt{1-t^2}(1-2\sqrt{1-t^2}) \sqrt{1-t^2}}.
\]
Hence we obtain by (40)
\[ + \frac{16t^2\bar{R}\sqrt{(1-t^2)^3} + \bar{R} - \bar{S}}{\sqrt{(1-t^2)^3}(1-2t\bar{R}\sqrt{1-t^2})(1+2t\bar{S}\sqrt{1-t^2})}. \]
Furthermore, one arrives at
\[ (\partial_- \bar{R})_z = g_1(z,t)\bar{R}_z + g_2(z,t)\bar{S}_z, \quad (44) \]
where
\[ g_1(z,t) = \frac{1}{(1-t^2)(1+2t\bar{S}\sqrt{1-t^2})} \left( \frac{1}{2t(1-t^2)} + 8t\bar{S}\sqrt{1-t^2} \right), \]
\[ g_2(z,t) = -\frac{2t}{\sqrt{1-t^2}(1+2t\bar{S}\sqrt{1-t^2})^2} \left( \frac{-\bar{R} - \bar{S}}{2t(1-t^2)} + 8t\bar{R}\bar{S}\sqrt{1-t^2} \right) \]
\[ - \frac{1}{(1-t^2)(1+2t\bar{S}\sqrt{1-t^2})} \left( \frac{1}{2t(1-t^2)} - 8t\bar{R}\sqrt{1-t^2} \right), \]
and
\[ (\partial_+ \bar{S})_z = h_1(z,t)\bar{S}_z + h_2(z,t)\bar{R}_z, \quad (45) \]
where
\[ h_1(z,t) = \frac{1}{(1-t^2)(1-2t\bar{R}\sqrt{1-t^2})} \left( \frac{1}{2t(1-t^2)} - 8t\bar{R}\sqrt{1-t^2} \right), \]
\[ h_2(z,t) = -\frac{2t}{\sqrt{1-t^2}(1-2t\bar{R}\sqrt{1-t^2})^2} \left( \frac{-\bar{S} - \bar{R}}{2t(1-t^2)} - 8t\bar{R}\bar{S}\sqrt{1-t^2} \right) \]
\[ - \frac{1}{(1-t^2)(1-2t\bar{R}\sqrt{1-t^2})} \left( \frac{1}{2t(1-t^2)} + 8t\bar{S}\sqrt{1-t^2} \right). \]
Hence we obtain by (40)
\[ \partial_+\partial_- \bar{R} - \partial_-\partial_- \bar{R} = (\lambda_+ - \lambda_-)(\partial_- \bar{R})_z = g_1G + g_2H, \]
\[ \partial_+\partial_+ \bar{S} - \partial_-\partial_+ \bar{S} = (\lambda_+ - \lambda_-)(\partial_+ \bar{S})_z = h_1H + h_2G. \]
Putting the above and (43) into (41)-(42) gives
\[ \partial_- G = \left( \frac{2}{t} + f(z,t) + g_1(z,t) \right)G + g_2(z,t)H, \]
\[ \partial_+ H = \left( \frac{2}{t} + f(z,t) + h_1(z,t) \right)H + h_2(z,t)G, \]
that is,
\[ \partial_- G = \left( \frac{5}{2t} + \tilde{g}_1(z,t) \right)G + g_2(z,t)H, \quad (46) \]
\[ \partial_+ H = \left( \frac{5}{2t} + \tilde{h}_1(z,t) \right)H + h_2(z,t)G, \quad (47) \]
where
\[ \tilde{g}_1(z,t) = f(z,t) + \left( g_1(z,t) - \frac{1}{2t} \right) \]
\[ = f(z,t) + \frac{t(2-t^2)}{2(1-t^2)^2} - \frac{\bar{S}\sqrt{1-t^2}}{(1-t^2)^2(1+2t\bar{S}\sqrt{1-t^2})} + \frac{8t\bar{S}\sqrt{1-t^2}}{(1-t^2)(1+2t\bar{S}\sqrt{1-t^2})}, \]
\[ \tilde{h}_1(z, t) = f(z, t) + \left( h_1(z, t) - \frac{1}{2t} \right) \]

\[ = f(z, t) + \frac{t(2-t^2)}{2(1-t^2)^2} + \frac{\tilde{R}\sqrt{1-t^2}}{(1-t^2)^2(1+2t\tilde{S}\sqrt{1-t^2})} - \frac{8t\tilde{R}\sqrt{1-t^2}}{(1-t^2)(1-2t\tilde{R}\sqrt{1-t^2})}. \]

It is clear by Lemma 3.1 that the functions \( \tilde{g}_1 \) and \( \tilde{h}_1 \) are uniformly bounded near \( t = 0 \). Moreover, we denote

\[ \tilde{g}_2(z, t) = 2g_2(z, t) = -1 + \frac{2t\tilde{S}\sqrt{1-t^2}}{(1-t^2)^2(1+2t\tilde{S}\sqrt{1-t^2})} + \frac{16t^2\tilde{R}}{\sqrt{1-t^2}(1+2t\tilde{S}\sqrt{1-t^2})} \]

\[ - \frac{t^2(2-t^2)}{(1-t^2)^2} - \frac{2t}{\sqrt{1-t^2}(1+2t\tilde{S}\sqrt{1-t^2})^2} \left( \frac{\tilde{R} - \tilde{S}}{1-t^2} + 16t^2\tilde{R}\tilde{S}\sqrt{1-t^2} \right), \]

\[ \tilde{h}_2(z, t) = 2h_2(z, t) = -1 - \frac{2t\tilde{R}\sqrt{1-t^2}}{(1-t^2)^2(1-2t\tilde{R}\sqrt{1-t^2})} - \frac{16t^2\tilde{S}}{\sqrt{1-t^2}(1-2t\tilde{R}\sqrt{1-t^2})} \]

\[ - \frac{t^2(2-t^2)}{(1-t^2)^2} - \frac{2t}{\sqrt{1-t^2}(1-2t\tilde{R}\sqrt{1-t^2})^2} \left( \frac{\tilde{S} - \tilde{R}}{1-t^2} - 16t^2\tilde{R}\tilde{S}\sqrt{1-t^2} \right). \]

It is easily seen by Lemma 3.1 that \( \tilde{g}_2 \to -1 \) and \( \tilde{h}_2 \to -1 \) as \( t \to 0^+ \). We comment that here we only need the uniform boundedness of \( \tilde{R} \) and \( \tilde{S} \). Thus (46)-(47) can be rewritten as

\[ \partial_- G = \left( \frac{5}{2t} + \tilde{g}_1(z, t) \right) G + \tilde{g}_2(z, t) \frac{H}{2t}, \quad (48) \]

\[ \partial_+ H = \left( \frac{5}{2t} + \tilde{h}_1(z, t) \right) H + \tilde{h}_2(z, t) \frac{G}{2t}, \quad (49) \]

In addition, in order to obtain the explicit limit constructs of coefficients for equations, we recall (39) and (40) to introduce

\[ \tilde{G} = \frac{G}{t^2}, \quad \tilde{H} = \frac{H}{t^2}. \quad (50) \]

Then we finally have

\[ \partial_- \tilde{G} = \left( \frac{1}{2t} + \tilde{g}_1(z, t) \right) \tilde{G} + \tilde{g}_2(z, t) \frac{\tilde{H}}{2t}, \]

\[ \partial_+ \tilde{H} = \left( \frac{1}{2t} + \tilde{h}_1(z, t) \right) \tilde{H} + \tilde{h}_2(z, t) \frac{\tilde{G}}{2t}, \]

or

\[ \partial_- \left( t^{-\frac{1}{2}} \tilde{G} \right) = t^{-\frac{3}{2}} \left( t\tilde{g}_1 \tilde{G} + \frac{1}{2} \tilde{g}_2 \tilde{H} \right), \quad (51) \]

\[ \partial_+ \left( t^{-\frac{1}{2}} \tilde{H} \right) = t^{-\frac{3}{2}} \left( t\tilde{h}_1 \tilde{H} + \frac{1}{2} \tilde{h}_2 \tilde{G} \right). \quad (52) \]

4.2. Properties of solutions in partial hodograph plane. With the help of (51)-(52), we now analyze the properties of solutions near the line \( t = 0 \), i.e., the sonic boundary \( AC \).

Denote the region \( ABC \) in the self-similar \((\xi, \eta)\) plane in Fig. 1 by \( A'B'C' \) in the \((z, t)\) plane. Let \((\tilde{z}, 0)\) be any point on the degenerate line \( A'C' \) and let \( t_P \) be
a small positive number such that the point \( P(t_P, \bar{z}) \) stays in the domain \( A'B'C' \).

We use the notations \( z_+(P) \) and \( z_-(P) \) to represent the \( \lambda_+ \) and \( \lambda_- \)-characteristics passing through the point \( P \) and intersecting \( \hat{A'C'P} \) at \( P_1 \) and \( P_2 \), respectively. Since the functions \( \hat{g}_1, \hat{h}_1 \) are uniformly bounded near \( t = 0 \) and \( \hat{g}_2 \to -1, \hat{h}_2 \to -1 \) as \( t \to 0^+ \), then for any constant \( \nu \in (0, 1] \), we can choose \( t_P < 1 \) small enough such that

\[
|t\hat{g}_1(z, t)| \leq \frac{\nu}{4}, \quad |t\hat{h}_1(z, t)| \leq \frac{\nu}{4}, \quad |\hat{g}_2(z, t)| \leq 1 + \nu, \quad |\hat{h}_2(z, t)| \leq 1 + \nu
\]  

(53)

hold in the whole domain \( PP_1P_2 \). Denote \( \Omega_\nu(\bar{z}) \) the domain bounded by \( \hat{PP}_1 \), \( \hat{PP}_2 \) and the positive and negative characteristics starting from \((\bar{z}, 0)\). For any \((z, t) \in \Omega_\nu(\bar{z})\), we denote, respectively, \( a(z, t_a) \) and \( b(z, t_b) \) the intersection points of the \( \lambda_- \) and \( \lambda_+ \)-characteristics through \((z, t)\) with the boundaries \( \hat{PP}_1 \) and \( \hat{PP}_2 \), see Fig 3.

Let

\[
\hat{K} = \max \left\{ \max_{\Omega_\nu(\bar{z})} |\hat{G}(z_a, t_a)| + 1, \max_{\Omega_\nu(\bar{z})} |\hat{H}(z_b, t_b)| + 1 \right\},
\]

which is well-defined and uniformly bounded in the domain \( \Omega_\nu(\bar{z}) \). Now we are ready to prove the following key lemma.

**Figure 3.** The region of \( \Omega_\nu(\bar{z}) \).

**Lemma 4.1.** Let \((\bar{z}, 0)\) be any point on the degenerate line \( \hat{A'C'} \) and \( \nu \in (0, 1] \) be any constant, there exists a uniform positive constant \( \hat{K} \) such that there hold

\[
|t^\nu \hat{G}| \leq \hat{K}, \quad |t^\nu \hat{H}| \leq \hat{K}, \quad \forall (z, t) \in \Omega_\nu(\bar{z}).
\]  

(54)

**Proof.** We apply the bootstrap method used in [20] to prove the lemma. For a fixed \( \varepsilon \in (0, t_P) \), we denote

\[
\hat{\Omega}_\varepsilon := \{(z, t) : \varepsilon \leq t \leq t_P, z_-(P) \leq z \leq z_+(P)\} \cap \Omega_\nu(\bar{z}),
\]

and \( \hat{K}_\varepsilon = \max\{|t^\nu \hat{G}|, |t^\nu \hat{H}|\} \). If for any \( 0 < \varepsilon < t_P \), one has \( \hat{K}_\varepsilon \leq \hat{K} \), then (54) holds. Otherwise, we assume there exists \( \varepsilon_0 \in (0, t_P) \) such that \( \hat{K}_{\varepsilon_0} > \hat{K} \).

For \((z_{\varepsilon_0}, \varepsilon_0) \in \Omega_\nu(\bar{z})\), we use the notation \( z_\nu^+(t) (z_\nu^-(t)) \) to represent the negative (positive) characteristic curve passing through the point \((z_{\varepsilon_0}, \varepsilon_0)\) and intersecting
At point \( a(z_a, t_a) \) \( (b(z_b, t_b)) \). We integrate (51) along the negative characteristic from \( t \geq \varepsilon_0 \) to \( t_a \) and apply the definition of \( \hat{K} \) to obtain

\[
\left| \frac{G}{t^2} \right| = \left| \frac{G(z_a, t_a)}{t_a^2} + \int_{t}^{t_a} \frac{\tau \hat{g}_1 \hat{G} + \frac{1}{2} \hat{g}_2 \hat{H}}{t^{\frac{3}{2}}} (z_a(\tau), \tau) \, d\tau \right| \\
\leq \left| \frac{G(z_a, t_a)}{t_a^2} \right| + \int_{t}^{t_a} \frac{\frac{1}{4} |\hat{G}| + \frac{\varepsilon_0 \nu |\hat{H}|}{t^{\frac{3}{2}}} (z_a(\tau), \tau) \, d\tau \\
\leq \left| \frac{G(z_a, t_a)}{t_a^2} \right| + \frac{2 + 3\nu \varepsilon_0}{4} \hat{K}_{\varepsilon_0} \int_{t}^{t_a} \frac{1}{t^{\frac{3}{2} + \nu}} \, d\tau \\
= \left| \frac{G(z_a, t_a)}{t_a^2} \right| + \frac{2 + 3\nu \varepsilon_0}{4} \hat{K}_{\varepsilon_0} \frac{1}{\frac{3}{2} + \nu} \left( \frac{1}{t^{\frac{3}{2} + \nu}} - \frac{1}{t_a^{\frac{3}{2} + \nu}} \right) \\
< \frac{(2 + 3\nu) \varepsilon_0}{2 + 4\nu} t^{-\frac{3}{2} - \nu} < \hat{K}_{\varepsilon_0} t^{-\frac{3}{2} - \nu}.
\]

(55)

Hence, on the line segment \( \{ t = \varepsilon_0 \} \cap \hat{\Omega}_{\varepsilon_0} \), we have a strict inequality

\[
|\hat{G}|_{t=\varepsilon_0} < \hat{K}_{\varepsilon_0} \varepsilon_0^{-\nu}.
\]

(56)

A similar argument for the equation of \( H \) shows

\[
|\hat{H}|_{t=\varepsilon_0} < \hat{K}_{\varepsilon_0} \varepsilon_0^{-\nu}.
\]

(57)

By combining (56) and (57), we conclude that the maximum values of \( t^\nu \hat{G} \) and \( t^\nu \hat{H} \) in the domain \( \hat{\Omega}_{\varepsilon_0} \) can only happen on \( \varepsilon_0 < t \leq t_{p0} \). This assertion also holds in a larger domain \( \hat{\Omega}_{\varepsilon'} \), where \( \varepsilon' < \varepsilon_0 \). We repeat the above process and then can extend the domain larger and larger until the whole domain \( \hat{\Omega}_{\nu}(z) \). The proof of the lemma is completed.

Now, for any point \( (\tilde{z}, 0) \in \hat{\AC} \), let \( \mu \) be a small constant such that \( (\tilde{z} - \mu, \tilde{z} + \mu) \subset \hat{\AC} \). Denote \( Q_1^* \) \( (Q_2^*) \) the intersection point of the negative (positive) characteristic through \( Q_1 := (\tilde{z} - \mu, 0) \) \( (Q_2 := (\tilde{z} + \mu, 0)) \) with the boundary \( \hat{P} \). By using a similar argument showing Lemma 4.1, we can extend the result (54) in the region \( \hat{\Omega}_\nu(\tilde{z}_\nu) \) bounded by the boundaries \( \hat{P} \) \( Q_1^*, Q_1^* Q_1, Q_1^* Q_2, Q_2^*, Q_2^* Q_2^* \) and \( \hat{P} \).

**Lemma 4.2.** Let \( (\tilde{z}, 0) \) be any point on the degenerate line \( \hat{\AC} \) and \( \mu \) be a small positive constant such that \( (\tilde{z} - \mu, \tilde{z} + \mu) \subset \hat{\AC} \). Then, for any constant \( \nu \in (0, 1] \), there exists a positive constant \( \tilde{K} \) depending on the interval \( (\tilde{z} - \mu, \tilde{z} + \mu) \) such that there hold

\[
|t^\nu \hat{G}| \leq \tilde{K}, \quad |t^\nu \hat{H}| \leq \tilde{K}, \quad \forall (z, t) \in \hat{\Omega}_\nu(\tilde{z}_\nu).
\]

(58)

We next prove the uniform boundedness of \( \tilde{W} := \frac{\tilde{K} - \tilde{s}}{t} \).

**Lemma 4.3.** Under the assumptions of Theorem 1, the function \( \tilde{W}(z, t) \) is uniformly bounded up to the sonic boundary \( \hat{\AC} \).
Proof. It suffices to prove that the lemma holds near \(t = 0\). For any point \((\bar{z}, 0) \in \overline{AC'}\), we choose the domain \(PP_1P_2\) as before. By a direct calculation, one finds by \((38)\) that
\[
\bar{W}_i = I_1\bar{W} + I_2,
\]
where
\[
I_1 = \frac{t(2 - t^2)}{(1 - t^2)^2} + \frac{(\bar{R} - \bar{S})\sqrt{1 - t^2} + 4t(1 - t^2)\bar{R}\bar{S}}{(1 + 2t\bar{S}\sqrt{1 - t^2})(1 - 2t\bar{R}\sqrt{1 - t^2})},
\]
\[
I_2 = \frac{16RS(1 + t(S - R)\sqrt{1 - t^2})}{\sqrt{1 - t^2}(1 + 2t\bar{S}\sqrt{1 - t^2})(1 - 2t\bar{R}\sqrt{1 - t^2})} + \frac{\bar{S}(1 - 2t\bar{R}\sqrt{1 - t^2})}{R + S}(Gt^{\nu})t^{1-\nu} + \frac{\bar{R}(1 + 2t\bar{S}\sqrt{1 - t^2})}{R + S}(\bar{H}t^{\nu})t^{1-\nu}.
\]
It is obvious that \(I_1\) is uniformly bounded near \(t = 0\). Furthermore, we can choose \(\nu = 1/2\) in Lemma 4.2 to obtain that \(I_2\) is also uniformly bounded near \(t = 0\). Thus, the uniform boundedness of \(W\) follows directly from \((59)\).

Recalling the definitions of \(\bar{R}\) and \(\bar{S}\) and applying Lemma 3.1, we see that \(W = (R - S)/t\) is uniformly bounded in the whole domain \(A'B'C'\), that is, there exists a uniform constant \(K_1\) such that
\[
\left| \frac{R - S}{t}(z, t) \right| \leq K_1 \quad \forall (z, t) \in A'B'C'.
\]
Moreover, in view of \((40)\) and \((50)\), one arrives at
\[
R_z = -\frac{\sqrt{1 - t^2}G}{2R^2(R + S)}(1 - 2t\bar{R}\sqrt{1 - t^2})(1 + 2t\bar{S}\sqrt{1 - t^2}) - \bar{G},
\]
\[
S_z = -\frac{\sqrt{1 - t^2}H}{2S^2(R + S)}(1 - 2t\bar{R}\sqrt{1 - t^2})(1 + 2t\bar{S}\sqrt{1 - t^2}) - \bar{H}.
\]
According to \((60)\) and \((61)\), it is easy to check by the equations for \(R\) and \(S\) \((37)\) that the two derivatives \(R_z\) and \(S_z\) are uniformly bounded in the whole domain \(A'B'C'\) including the degenerate line \(t = 0\). Therefore, the two functions \(R\) and \(S\) approach a common value on the degenerate curve \(\overline{AC'}\) with at least a rate of \(\cos \delta\).

Finally, we show the following lemma.

**Lemma 4.4.** Under the assumptions of Theorem 1, the functions \(R(z, t)\), \(S(z, t)\) and \(W(z, t)\) are uniformly \(C^{1/2}\) continuous in the whole domain \(A'B'C'\), including the degenerate curve \(\overline{AC'}\).

**Proof.** Suppose first that \((z_1, 0)\) and \((z_2, 0)\) \((z_1 < z_2)\) are any two points on the degenerate curve \(\overline{AC'}\). Let \((z_m, t_m)\) be the intersection point of the positive characteristic \(z^+\) and negative characteristic \(z^-\) starting from \((z_1, 0)\) and \((z_2, 0)\) respectively. Recalling \((37)\), the characteristic curves \(z^\pm\) are defined by
\[
\left\{ \begin{array}{l}
\frac{d}{dt}z^+ = \frac{2}{(R - 2t\sqrt{1 - t^2})(1 - t^2)^3/2}t^2, \\
z^+_t = z_1,
\end{array} \right.
\]
\[
\left\{ \begin{array}{l}
\frac{d}{dt}z^- = -\frac{2}{(S + 2t\sqrt{1 - t^2})(1 - t^2)^3/2}t^2, \\
z^-_t = z_2,
\end{array} \right.
\]
from which and (25) one has
\[ K t_m \leq |z_m - z_i|^{\frac{1}{2}} \leq K t_m, \quad i = 1, 2 \]
(62)
for some positive constants \( K \) and \( \overline{K} \). Furthermore, it follows by (60) and system (37) that \( \partial_- R \) and \( \partial_+ S \) are uniform bounded. Thus we integrate \( \partial_+ S \) from \((z_1, 0)\) to \((z_m, t_m)\) and \( \partial_- R \) from \((z_2, 0)\) to \((z_m, t_m)\) to find
\[
|S(z_1, 0) - S(z_m, t_m)| \leq K_2 t_m, \quad |R(z_2, 0) - R(z_m, t_m)| \leq K_2 t_m
\]
for some uniform constant \( K_2 \). Hence we have
\[
|R(z_1, 0) - R(z_2, 0)| = |S(z_1, 0) - R(z_2, 0)| \\
\leq |S(z_1, 0) - S(z_m, t_m)| + |S(z_m, t_m) - R(z_m, t_m)| + |R(z_m, t_m) - R(z_2, 0)| \\
\leq (2K_2 + K_1)t_m \leq \frac{2K_2 + K_1}{K} |z_2 - z_1|^{\frac{1}{2}} = \frac{2K_2 + K_1}{K} |(z_2, 0) - (z_1, 0)|^{\frac{1}{2}}.
\]

We now consider any two points \((z_1, t_1)\) and \((z_2, t_2)\) \((z_1 \leq z_2, 0 \leq t_1 \leq t_2)\) in the region \( A'B'C' \). If \( z_1 = z_2 \), we recall the uniform boundedness of \( R_t \) to get
\[
|R(z_1, t_1) - R(z_2, t_2)| = |R(z_1, t_1) - R(z_1, t_2)| \leq \max_{A'B'C'} |R_t| \cdot |t_2 - t_1|.
\]
Next we assume \( z_1 < z_2 \) and divide the problem into two cases:

Case I. \( t_1 \geq (z_2 - z_1) \). In this case, we choose \( \nu = 1/2 \) in (58) and use (61) and the fact \( t_1 \geq (z_2 - z_1) \) to obtain
\[
|R(z_2, t_2) - R(z_1, t_1)| \leq |R(z_2, t_2) - R(z_2, t_1)| + |R(z_2, t_1) - R(z_1, t_1)| \\
\leq \max_{A'B'C'} |R_t| \cdot |t_2 - t_1| + |R_z| \cdot |z_2 - z_1| \\
\leq \max_{A'B'C'} |R_t| \cdot |t_2 - t_1| + K t_1^{\frac{1}{2}} \cdot |z_2 - z_1| \\
\leq \max_{A'B'C'} |R_t| \cdot |t_2 - t_1| + K \cdot |z_2 - z_1|^{\frac{1}{2}} \leq K |(z_2, t_2) - (z_1, t_1)|^{\frac{1}{2}}
\]
for some uniformly constant \( K \).

Case II. \( t_1 < (z_2 - z_1) \). In this case, we derive
\[
|R(z_2, t_2) - R(z_1, t_1)| \\
\leq |R(z_2, t_2) - R(z_2, t_1)| + |R(z_2, t_1) - R(z_2, 0)| \\
+ |R(z_2, 0) - R(z_1, 0)| + |R(z_1, 0) - R(z_1, t_1)| \\
\leq \max_{A'B'C'} |R_t| \cdot |t_2 - t_1| + \max_{A'B'C'} |R_t| \cdot t_1 + \frac{2K_2 + K_1}{K} |z_2 - z_1|^{\frac{1}{2}} + \max_{A'B'C'} |R_t| \cdot t_1 \\
\leq 2 \max_{A'B'C'} |R_t| \cdot |t_2 - t_1| + |z_2 - z_1| + \frac{2K_2 + K_1}{K} |z_2 - z_1|^{\frac{1}{2}} \leq K |(z_2, t_2) - (z_1, t_1)|^{\frac{1}{2}}
\]
for a uniform constant \( K \). In summary, we obtain that the function \( R(z, t) \) is uniformly \( C^4 \) continuous in the whole domain \( A'B'C' \). A similar argument leads to the uniform continuity of \( S(z, t) \).

To establish the uniform \( C^4 \)-continuity of \( W \), we derive the equations \( \partial_- W \) and \( \partial_+ W \) from (37) as follows:
\[
\partial_- W = \frac{2 - t^2}{1 - t^2} W + \frac{W(W - 2\sqrt{1 - t^2})}{2(1 - t^2)^2} \left( \frac{1}{S + 2t\sqrt{1 - t^2}} - \frac{1}{R - 2t\sqrt{1 - t^2}} \right) \\
- \frac{8}{\sqrt{1 - t^2}} \left( \frac{R}{S + 2t\sqrt{1 - t^2}} + \frac{S}{R - 2t\sqrt{1 - t^2}} \right) + \frac{4tS}{(R - 2t\sqrt{1 - t^2})\sqrt{(1 - t^2)^3}},
\]
and
\[
\partial_z W = \frac{(2 - t^2)W}{1 - t^2} - \frac{W(W - 2\sqrt{1 - t^2})}{2(1 - t^2)^2} \left( \frac{1}{S + 2t\sqrt{1 - t^2}} - \frac{1}{R - 2t\sqrt{1 - t^2}} \right)
\]
\[
- \frac{8}{\sqrt{1 - t^2}} \left( \frac{R}{S + 2t\sqrt{1 - t^2}} + \frac{S}{R - 2t\sqrt{1 - t^2}} \right) + \frac{4tRz}{(S + 2t\sqrt{1 - t^2})\sqrt{(1 - t^2)^3}}.
\]
which indicate by Lemma 4.3 and (61) that $|\partial_1 W|$ and $|\partial_z W|$ are uniform bounded. We reproduce the same argument as above for $R$ to end the proof of the lemma.

5. **Proof of the main theorem.** We now use the results of the previous section to complete the proof of Theorem 1, that is, we show that the solution $(p, u, v)(\xi, \eta)$ in the region $ABC$ is uniformly $C^{1-\frac{1}{4}}$ up to the sonic boundary $\overline{AC}$ and the sonic curve $\overline{AC}$ is $C^{1-\frac{1}{4}}$-continuous. The proof is divided into four steps.

**Step I.** We claim that the map $(\xi, \eta) \mapsto (z, t)$ defined in (34) is an one-to-one mapping. It suffices to show that, for any point $(z^*, t^*) \in A'B'C'$, there exists a unique point $(\xi^*, \eta^*) \in ABC$ such that $t^* = \cos(\delta(\xi^*, \eta^*))$, $z^* = \phi(\xi^*, \eta^*)$. We use the proof by contradiction argument to prove it. Assume $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$ are two distinct points in the domain $ABC$ such that $t^* = \cos(\delta(\xi_1, \eta_1)) = \cos(\delta(\xi_2, \eta_2))$ and $z^* = \phi(\xi_1, \eta_1) = \phi(\xi_2, \eta_2)$, which means $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$ are two points on a level curve $\ell^\delta(\xi, \eta) = 1 - \sin(\delta(\xi, \eta)) = \varepsilon \geq 0$. According to Lemma 4.3, equalities (31) and (32) are well defined in the whole domain $ABC$. Then using (5), (31) and (32), we compute
\[
\nabla \phi \cdot (\ell_{\xi}^\delta, -\ell_{\eta}^\delta) = \left( -\frac{\cos \sigma}{\sin \delta}, -\frac{\sin \sigma}{\sin \delta} \right) \cdot (\ell_{\eta}^\delta, -\ell_{\xi}^\delta)
\]
\[
= -\frac{\cos \sigma}{\sin \delta} \left\{ -\frac{\cos \sigma}{4}(R + S) + \sin \sigma \left( \sin^2 \delta - \frac{\sin \delta}{4} \cdot \frac{R - S}{\cos \delta} \right) \right\}
\]
\[
+ \frac{\sin \sigma}{\sin \delta} \left\{ \frac{\sin \sigma}{4}(R + S) + \cos \sigma \left( \sin^2 \delta - \frac{\sin \delta}{4} \cdot \frac{R - S}{\cos \delta} \right) \right\} = \frac{R + S}{4\sin \delta} > 0,
\]
which implies that, along each level curve $\ell^\delta(\xi, \eta) = \varepsilon$, the function $\phi$ is strictly monotonic contradicting with the assumption $\phi(\xi_1, \eta_1) = \phi(\xi_2, \eta_2)$.

**Step II.** We assert that the function $\delta(\xi, \eta)$ is uniformly $C^{\frac{1}{2}}$-continuous up to the sonic boundary $\overline{AC}$. Taking into account (16) and (21), we get
\[
\cos \delta \delta_{\xi} = \frac{\cos \sigma \sin \delta}{4}W - \cos \sigma \sin^2 \delta - \frac{\sin \sigma}{4}(R + S),
\]
\[
\cos \delta \delta_{\eta} = \frac{\sin \sigma \sin \delta}{4}W - \sin \sigma \sin^2 \delta + \frac{\cos \sigma}{4}(R + S),
\]
which together with Lemma 3.1 and Lemma 4.3 yields
\[
|\cos \delta \delta_{\xi}| \leq C, \quad |\cos \delta \delta_{\eta}| \leq C,
\]
where $C$ is a uniform positive constant. From (64), we obtain
\[
|\left( \frac{\pi}{2} - \delta \right)\delta_{\xi}| \leq \left| \frac{\pi}{2} - \delta \right| \left| \frac{\pi}{4} - \delta \right| C \leq \tilde{C} (1 - \delta), \quad |\left( \frac{\pi}{2} - \delta \right)\delta_{\eta}| \leq \left| \frac{\pi}{2} - \delta \right| \left| \frac{\pi}{4} - \delta \right| \tilde{C} \leq \tilde{C}
\]
for a uniform constant $\tilde{C}$, which means that the function $(\pi/2 - \delta)^2$ is uniformly Lipschitz continuous in terms of $(\xi, \eta)$, that is, for any two points $T_1 = (\xi_1, \eta_1)$ and

\[
\]
for some uniform constant $C$. Thus \( \bar{\psi}(\xi, \eta) \) is uniformly \( C^1 \)-continuous in the whole domain \( ABC \). In view of Lemma 4.4, we apply the above result to the functions \( R(z, t) \), \( S(z, t) \) and \( W(z, t) \) to find that the functions \( R(\xi, \eta), S(\xi, \eta) \) and \( W(\xi, \eta) \) are uniformly \( C^1 \)-continuous in terms of \( (\xi, \eta) \) in the whole domain \( ABC \).

Note that the uniform regularity of functions in the self-similar \( (\xi, \eta) \) plane is lower than in the partial hodograph plane, which is due to the degeneracy of the Euler system at the sonic curves.
Step IV. We now establish the uniform regularity of solution \((u, v, \rho)(\xi, \eta)\) in the whole region \(ABC\) and the regularity of sonic boundary \(\hat{AC}\). To do this, we first investigate the regularity of \(\sigma(\xi, \eta)\) defined in (13). Using again (16) and (21), we derive by (17)

\[
\begin{align*}
\sigma_\xi &= -\sin \sigma \sin \delta + \frac{\sin \sigma \cos \delta (R - S) + \cos \sigma \sin \delta (R + S)}{4 \sin^2 \delta}, \\
\sigma_\eta &= -\cos \sigma \sin \delta + \frac{\cos \sigma \cos \delta (R - S) + \sin \sigma \sin \delta (R + S)}{4 \sin^2 \delta}.
\end{align*}
\] (69)

From the above and Lemma 3.1, we see that \(\sigma(\xi, \eta)\) and consequently \(\cos \sigma(\xi, \eta)\) and \(\sin \sigma(\xi, \eta)\) are uniformly Lipschitz continuous in the whole domain \(ABC\). Moreover, employing \((R, S) \in C^{1, \frac{1}{6}}, \delta \in C^{1, \frac{1}{2}}\) and (69), we find that \(\sigma(\xi, \eta), \cos \sigma(\xi, \eta)\) and \(\sin \sigma(\xi, \eta)\) are uniformly \(C^{1, \frac{1}{6}}\) in the whole domain \(ABC\). From (63) and Step III, the function \(\sin \delta(\xi, \eta)\) is uniformly \(C^{1, \frac{1}{6}}\) in the whole domain \(ABC\).

Recalling the expressions \((u, v)(\xi, \eta)\) in (14) and the Bernoulli law (5), we obtain that \((u, v, \rho)(\xi, \eta)\) are uniformly \(C^{1, \frac{1}{6}}\) in the whole domain \(ABC\).

Now we prove the regularity of \(\hat{AC}\). By (31) and (32), we see that

\[
\begin{align*}
a(\xi, \eta) &= \left\{ \frac{\sin \sigma}{4} (R + S) + \cos \sigma \left( \sin^2 \delta - \frac{W \sin \delta}{4} \right) \right\}(\xi, \eta), \\
b(\xi, \eta) &= \left\{ -\frac{\cos \sigma}{4} (R + S) + \sin \sigma \left( \sin^2 \delta - \frac{W \sin \delta}{4} \right) \right\}(\xi, \eta),
\end{align*}
\]

which show \(a(\xi, \eta)\) and \(b(\xi, \eta)\) are uniformly \(C^{1, \frac{1}{2}}\)-continuous in the whole domain \(ABC\). Moreover, it is obvious by Lemma 3.1, Lemma 4.3 and the inequality (33) that there exist two uniform positive constants \(k\) and \(K\) such that

\[k \leq [a(\xi, \eta)]^2 + [b(\xi, \eta)]^2 \leq K.\]

Therefore the sonic boundary \(\hat{AC}\) is \(C^{1, \frac{1}{2}}\) continuous, and the proof of Theorem 1 is complete.

Acknowledgments. Y. B. Hu would like to thank the hospitality and the support of Department of Mathematics in University of Iowa during his visit in 2017-2018.

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