THE CLASSIFICATION PROBLEM FOR EXTENSIONS OF TORSION-FREE ABELIAN GROUPS, I

MARTINO LUPINI

Abstract. Let $C, A$ be countable abelian groups. In this paper we determine the complexity of classifying extensions $C$ by $A$, in the cases when $C$ is torsion-free and $A$ is a $p$-group, a torsion group with bounded primary components, or a free $R$-module for some subring $R \subseteq \mathbb{Q}$. Precisely, for such $C$ and $A$ we describe in terms of $C$ and $A$ the potential complexity class in the sense of Borel complexity theory of the equivalence relation $R_{\text{Ext}(C, A)}$ of isomorphism of extensions of $C$ by $A$. This complements a previous result by the same author, settling the case when $C$ is torsion and $A$ is arbitrary. We establish the main result within the framework of Borel-definable homological algebra, recently introduced in collaboration with Bergfalk and Panagiotopoulos. As a consequence of our main results, we will obtain that if $C$ is torsion-free and $A$ is either a free $R$-module or a torsion group with bounded components, then an extension of $C$ by $A$ splits if and only if it splits on all finite-rank subgroups of $C$. This is a purely algebraic statement obtained with methods from Borel-definable homological algebra.

1. Introduction

The study from the perspective of Borel complexity theory of the classification problem for extensions of two given countable abelian groups up to equivalence was initiated in [Lup22a]. The main contribution of that paper was to completely determine the potential Borel complexity of the relation of equivalence of extensions of $C$ by $A$, where $C$ is a given countable torsion abelian group, and $A$ is an arbitrary countable abelian group. In particular, the main result of [Lup22a] shows that such a relation can have arbitrarily high potential Borel complexity for suitable choices of abelian $p$-groups $C$ and $A$ for every prime number $p$.

In this paper, we initiate the study from the same viewpoint of the classification problem of extensions of a countable torsion-free abelian group $C$ by a countable abelian group $A$. Recall that an extension of $C$ by $A$ is a short exact sequence

$$0 \to A \to B \to C \to 0$$

in the abelian category of abelian groups. (In what follows, we will assume all the groups to be abelian.) Two such extensions

$$0 \to A \to B \to C \to 0$$

and

$$0 \to A \to B' \to C \to 0$$

are equivalent if there exists a group isomorphism $\eta : B \to B'$ making the following diagram commute:

$$\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & B'
\end{array} \quad \begin{array}{ccc}
& & C \\
\eta & & \\
& & \\
& & C
\end{array}$$

An extension of $C$ by $A$ splits if it is equivalent to the trivial extension

$$0 \to A \to A \oplus C \to C \to 0.$$
As it is customary in Borel complexity theory, we regard a classification problem as an equivalence relation on a Polish space. The complexity of two such classification problems \((X, E)\) and \((Y, F)\) is compared via the notion of Borel reducibility. A Borel reduction from \((X, E)\) to \((Y, F)\) is a Borel function \(X \to Y\) that induces an injective function between the quotient spaces \(X/E \to Y/F\). If there exists such a Borel reduction, then we say that \((X, E)\) is Borel reducible to \((Y, F)\), in symbols \(E \leq_B F\). We say that \(E\) and \(F\) are Borel bireducible if \(E \leq_B F\) and \(F \leq_B E\), in which case we write \(E \sim_B F\).

Several canonical classification problems are used as benchmarks to calibrate the complexity of other classification problems. These benchmarks include the relation \(=\) of equality of real numbers (where \(\mathbb{R}\) can be replaced with any other uncountable Polish space); the relation \(E\) to be:

- **trivial** if it has only one equivalence class;
- **smooth** if it is Borel reducible to \(=\);
- **essentially hyperfinite** if it is Borel reducible to \(E_0\);
- **essentially countable** if it is Borel reducible to \(E_{\infty}\);
- **classifiable by countable sets of reals** if it is Borel reducible to \(+\).

One has that \(= <_B E_0 <_B E_0^N <_B ^+=\). It is observed in [Lup22a, Section 3.4] that, adopting the notation above, one can regard extensions of \(C\) by \(A\) as an axiomatizable class of countable structures in a suitable countable language \(L_{C,A}\). The relation \(\mathcal{R}_{\text{Ext}}(C,A)\) of equivalence of extensions coincides with isomorphism as \(L_{C,A}\)-structures. This allows one to regard the relation of equivalence of extensions of \(C\) by \(A\) as an equivalence relation on a Polish space. Such a relation is Borel bireducible with the coset equivalence relation of the subgroup of coboundaries \(B\) of \((C,A)\).

Let \(E\) be an equivalence relation on a Polish space \(X\) associated with a continuous action of a non-Archimedean abelian Polish group. If \(E\) is classifiable by countable sets of reals and has uncountably many classes, then \(E\) is Borel bireducible to \(=\), \(E_0\), or \(E_0^N\); see Proposition 1.2. Thus, \(E_0\) and \(E_0^N\) are the only possible Borel complexity classes for the relation \(\mathcal{R}_{\text{Ext}}(C,A)\) for countable groups \(C\) with \(C\) torsion-free, whenever \(\mathcal{R}_{\text{Ext}}(C,A)\) is nontrivial and is classifiable by countable sets of reals. In [Lup22a], the pairs of countable groups \(C\) with \(C\) torsion for which \(\mathcal{R}_{\text{Ext}}(C,A)\) is reducible to \(=\), \(E_0\), and \(E_0^N\) are characterized. In this paper, we obtain an analogous result for countable torsion-free groups \(C\) and several countable groups \(A\). We denote by \(\mathbb{P}\) the set of prime numbers. For a set of prime numbers \(Q\), we let \(R_Q\) be the subring of \(\mathbb{Q}\) generated by \(\{1/q : q \in Q\}\). If \(C\) is a torsion-free group such that \(R(C) \subseteq R_Q\), then \(C\) can be regarded as an \(R_Q\)-module. For a torsion-free group \(A\), we let \(R(A) = R_{\pi(A)}\), where \(\pi(A)\) is the set of prime numbers \(q\) such that \(A\) is \(q\)-divisible. For a torsion group \(T\), we let \(T_q\) be the \(q\)-primary component of \(T\), and \(\tau(T)\) be the set of \(q\) in \(\mathbb{P}\) such that \(T_q\) is nonzero.

**Theorem A.** Let \(C, A\) be countable abelian groups, where \(C\) is torsion-free. We have that if \(\mathcal{R}_{\text{Ext}}(C,A)\) is nontrivial, then \(E_0\) is Borel reducible to \(\mathcal{R}_{\text{Ext}}(C,A)\).

1. Suppose that \(A\) is torsion-free.
   a. \(\mathcal{R}_{\text{Ext}}(C,A)\) is trivial if and only if \(C \otimes R(A)\) is a free \(R(A)\)-module;
   b. \(\mathcal{R}_{\text{Ext}}(C,A) \leq_B E_0\) if and only if \(C \otimes R(A)\) is the sum of a free \(R(A)\)-module and a finite-rank \(R(A)\)-module;
(c) If \( A \) is a free \( R(A) \)-module, then \( E_{\text{Ext}}(C,A) \leq_B E_0^\mathbb{N} \).

(2) Suppose that \( A \) is a reduced torsion group with bounded primary components.
(a) \( E_{\text{Ext}}(C,A) \) is trivial if and only if for every finite-rank subgroup \( C_0 \) of \( C \) there exists a set \( Q \subseteq \mathbb{P} \) such that \( Q \cap \tau(T) \) is finite and \( C_0 \otimes R_Q = \text{a free } R_Q \)-module;
(b) \( E_{\text{Ext}}(C,A) \leq_B E_0 \) if and only if there exists a finite-rank pure subgroup \( C_0 \) of \( C \) such that for every finite-rank pure subgroup \( C_1 \) of \( C \) containing \( C_0 \) there exists a set \( Q \subseteq \mathbb{P} \) such that \( Q \cap \tau(T) \) is finite and \( C_1/C_0 \otimes R_Q = \text{a free } R_Q \)-module;
(c) \( E_{\text{Ext}}(C,A) \leq_B E_0^\mathbb{N} \).

(3) Suppose that \( A \) is a reduced p-group for some \( p \in \mathbb{P} \). Let \( B \) be a tight p-basic subgroup of \( C \) (see Definition 7.1) and let \( r \) be the torsion-free rank of \( C/B \). Let also \( \beta \) be the least countable ordinal such that the \( \beta \)-th Ulm subgroup \( u_\beta(A) = A^{\beta} \) of \( A \) is bounded.
(a) \( E_{\text{Ext}}(C,A) \) is trivial if and only if \( E_{\text{Ext}}(C,A) \) is Borel reducible to \( E_0 \) if and only if \( r = 0 \) or \( \beta = 0 \);
(b) \( E_{\text{Ext}}(C,A) \sim_B E_0^\mathbb{N} \) if and only if \( r \neq 0 \) and \( \beta = 1 \).

For orbit equivalence relations associated with continuous actions of non-Archimedean abelian Polish groups, one can characterize Borel (bi)reducibility to \( E_0 \) or \( E_0^\mathbb{N} \) in terms of the notion of potential Borel complexity from [Lou94]. A Borel complexity class \( \Gamma \) is an assignment \( X \rightarrow \Gamma(X) \) from Polish spaces to collections of Borel subsets such that for every continuous function \( f : X \rightarrow Y \) between Polish spaces and for every \( A \in \Gamma(Y) \) one has that \( A \) is a Borel subset of \( Y \) and \( f^{-1}(A) \in \Gamma(X) \). We will be mostly concerned with the complexity class \( \Pi^0_\alpha \) and its dual \( \Sigma^0_\beta \) for \( \alpha < \omega_1 \) see [Kec95, Section 11.B] and the complexity class \( \Lambda_0 \) of differences of \( \Pi^0_\beta \) sets for \( \alpha < \omega_1 \); see [Kec95, Section 22.E] where it is denoted by \( D_3(\Pi^0_\beta) \). If \( \Gamma \) is one of such classes, then we say that \( \Gamma \) is the complexity class of \( A \) in \( X \) if \( A \in \Gamma(X) \) and \( X \setminus A \notin \Gamma(X) \).

**Definition 1.1.** An equivalence relation \( E \) on a Polish space \( X \) is potentially \( \Gamma \) if there exists an equivalence relation \( F \) on a Polish space \( Y \) such that \( F \in \Gamma(Y \times Y) \) and \( E \leq_B F \). The potential complexity class of \( E \) is \( \Gamma \) if \( E \) is potentially \( \Gamma \) and not potentially \( \bar{\Gamma} \).

One can reformulate Theorem A in terms of Definition 1.1 in view of the following result, which can be proved in the same way as [Lup22c, Proposition 5.6].

**Proposition 1.2.** Let \( E \) be the orbit equivalence relation associated with a continuous action of an abelian non-Archimedean Polish group.
(1) \( E \) is smooth if and only if \( E \) is potentially \( \Pi^0_1 \);
(2) \( E \) is essentially countable if and only if \( E \) is Borel reducible to \( E_0 \) if and only if \( E \) is potentially \( \Sigma^0_2 \), and \( E \) is Borel bireducible with \( E_0 \) if and only if \( \Sigma^0_2 \) is the complexity class of \( E \);
(3) \( E \) is classifiable by countable sets of reals if and only if \( E \) is Borel reducible to \( E_0^\mathbb{N} \) if and only if \( E \) is potentially \( \Pi^0_3 \), and \( E \) is Borel bireducible with \( E_0^\mathbb{N} \) if and only if \( \Pi^0_3 \) is the complexity class of \( E \).

In view of Proposition 1.2, the following can be seen as an extension of Theorem B to arbitrary countable ordinals.

**Theorem B.** Let \( C, A \) be countable abelian groups, where \( C \) is torsion-free and \( A \) is a reduced p-group for some \( p \in \mathbb{P} \). Let \( B \) be a tight \( p \)-basic subgroup of \( C \) (see Definition 7.1) and let \( r \) be the torsion-free rank of \( C/B \). Let also \( \beta \) be the least countable ordinal such that the \( \beta \)-th Ulm subgroup \( u_\beta(A) = A^{\beta} \) of \( A \) is bounded. If \( r = 0 \) or \( \beta = 0 \), then \( E_{\text{Ext}}(C,A) \) is trivial. If \( r \neq 0 \) and \( \beta \neq 0 \), then \( \Pi^0_{\beta+2} \) is the potential complexity class of \( E_{\text{Ext}}(C,A) \).

We will obtain these results in the context of Borel-definable homological algebra. This is the study of homological constructions and invariants such as Ext as taking value in the abelian category of abelian groups with a Polish cover. This is a category of formal quotients of abelian Polish groups by Polishable subgroups introduced by Bergfalk, Panagiotopoulos, and the author in [BLP20]. It was proved in [Lup22c] that the category of abelian groups with a Polish cover is the left heart of (the derived category of) the quasi-abelian category of abelian Polish groups.

An abelian group with a Polish cover is an abelian group \( G \) explicitly presented as a quotient \( \hat{G}/N \) of an abelian Polish group \( \hat{G} \) by a Polishable subgroup \( N \). A subgroup \( H \) of \( G \) is Polishable if it is of the form \( H/N \) for some Polishable subgroup \( \hat{H} \) of \( \hat{G} \). If \( G \) is a Borel complexity class, then we say that \( H \) is \( \Gamma \) in \( G \) if and only if \( H \in \Gamma(\hat{G}) \), and that \( \Gamma \) is the complexity class of \( H \) in \( G \) if and only if \( H \in \Gamma(\hat{G}) \). The sum \( G \oplus H \) of abelian
groups with a Polish cover \( G = \hat{G}/N \) and \( H = \hat{H}/M \) is the group with a Polish cover \((\hat{G} \oplus \hat{H})/(N \oplus M)\). A group homomorphism \( f : G \to H \) between groups with a Polish cover \( G = \hat{G}/N \) and \( H = \hat{H}/M \) is \textit{Borel-definable} if its graph
\[
\Gamma(f) = \{(x, y) \in G \oplus H : f(x) = y\}
\]
is a Polishable subgroup of \( G \oplus H \). This is equivalent to the assertion that \( f \) has a \textit{Borel lift}, which is a Borel function \( \tilde{f} : G \to \hat{H} \) such that \( f(x + N) = \tilde{f}(x) + M \) for every \( x \in \hat{G} \); see [Lup22c, Proposition 4.7]. Abelian groups with a Polish cover are the objects of an abelian category that has Borel-definable group homomorphisms as morphisms [Lup22c, Section 11]. The subobjects of an abelian group with a Polish cover are precisely its Polishable subgroups. Furthermore, a Borel-definable group homomorphism is monic if and only if it is injective, epic if and only if it is surjective, and an isomorphism if and only if it is bijective.

An abelian group with a non-Archimedean Polish cover is a group with a Polish cover \( G = \hat{G}/N \) where \( G \) is a non-Archimedean abelian Polish group and \( N \) is a non-Archimedean Polishable subgroup. A Polishable subgroup \( H = \hat{H}/N \) of \( G \) is non-Archimedean if \( \hat{H} \) is a non-Archimedean Polishable subgroup of \( G \). Abelian groups with a non-Archimedean Polish cover form an abelian subcategory of the category of abelian groups with a Polish cover.

Given countable abelian groups \( C, A \), the group \( \text{Ext}(C, A) \) parametrizing equivalence classes of extensions of \( C \) by \( A \) is the quotient of the non-Archimedean abelian Polish group \( \mathbb{Z}(C, A) \) by the non-Archimedean Polishable subgroup \( B(C, A) \). By definition, \( \text{Ext}(C, A) \) is a group with a non-Archimedean Polish cover. The relation \( \mathcal{R}_{\text{Ext}(C, A)} \) is Borel bireducible with the coset equivalence relation of \( B(C, A) \) within \( \mathbb{Z}(C, A) \). Thus, if \( \Gamma \) is one of the classes \( \Pi_0^0, \Sigma_0^0, \) \( D(\Pi_0^0) \) for \( \alpha < \omega_1 \), then by [Lup22c, Proposition 5.4] we have that \( \mathcal{R}_{\text{Ext}(C, A)} \) is potentiality \( \Gamma \) if and only if \( \{0\} \) is \( \Gamma \) in \( \text{Ext}(C, A) \), and \( \Gamma \) is the potential complexity class of \( \mathcal{R}_{\text{Ext}(C, A)} \) if and only if \( \Gamma \) is the complexity class of \( \{0\} \) in \( \text{Ext}(C, A) \).

In computing the complexity class of \( \{0\} \) in \( \text{Ext}(C, A) \) we will use the following important fact: Let \( G, H \) be abelian groups with a non-Archimedean Polish cover, and let \( H_0 \) be a non-Archimedean Polishable subgroup of \( H \). Suppose that \( f : G \to H \) is a Borel-definable group homomorphism, and \( \Gamma \) is the Borel complexity class \( \Pi_0^0, \Sigma_0^0, \) or \( D(\Pi_0^0) \) for \( \alpha < \omega_1 \). If \( H_0 \in \Gamma(H) \), then \( f^{-1}(H_0) \) is a Polishable subgroup of \( G \), and \( f^{-1}(H_0) \in \Gamma(G) \); see [Lup22c, Proposition 5.5].

Every abelian group with a Polish cover \( G \) is endowed with a canonical chain of subgroups \( s_\alpha(G) \) for \( \alpha < \omega_1 \), called \textit{Solecki subgroups}, such that \( s_\alpha(G) \) is the smallest Polishable \( \Pi_0^0 \)-subgroup of \( G \); see [Lup22c, Section 6].

Towards a proof of Theorem \( A \) and Theorem \( B \), we will compute the Solecki subgroups of \( \text{Ext}(C, A) \) of the given countable groups \( C, A \). Furthermore, we will characterize in Theorem 6.6 the first Solecki subgroup \( s_1(\text{Ext}(C, A)) \) of \( \text{Ext}(C, A) \) when \( C \) is torsion-free as the subgroup parametrizing extensions of \( C \) by \( A \) that are \textit{finite-rank-pure}. This is a strengthening of the notion of purity that we introduce in Section 5. We also prove therein that this notions defines an exact structure on the quasi-abelian category of torsion-free abelian groups. The assertion that \( \mathcal{R}_{\text{Ext}(C,A)} = \Pi_3^0 \) is equivalent to the assertion that \( s_1(\text{Ext}(C, A)) = 0 \), or that every finite-rank-pure extension of \( C \) by \( A \) splits. Thus, as a consequence of Theorem \( A \) and the characterization of the first Solecki subgroup from Theorem 6.6 one has the following.

\textbf{Theorem C.} Let \( C, A \) be countable abelian groups, where \( C \) is torsion-free. Suppose that \( A \) is a free \( \mathbb{R}(A) \)-module or a torsion group with bounded primary components. Then every finite-rank-pure extension of \( C \) by \( A \) splits.

This is a purely algebraic statement that is obtained by applying methods from Borel-definable homological algebra. Such a result showcases how the Borel-definable refinement of homological not only enriches with more structure the classical algebraic theory, but also gives new tools to establish purely algebraic results.

This paper is divided into eight sections, including this introduction. In Section 2 we recall some fundamental notions from category theory, including the notions of additive, exact, quasi-abelian, and abelian category. In Section 2 we briefly summarize the main results of [BLP20, Lup22c, Lup22b] concerning abelian groups with a Polish cover. The study of extensions of abelian groups and the group Ext in the context of groups with a Polish cover are introduced in Section 4. The notion of finite-rank-pure extension is introduced in Section 5, where it is also proved that finite-rank-pure extensions define an exact structure on the quasi-abelian category of torsion-free abelian groups. The complexity of the classification problem for extensions of torsion-free groups by torsion-free groups, which is the content of (1) of Theorem \( A \), is studied in Section 6. The characterization of the first Solecki subgroup in terms of finite-rank-pure extensions is also obtained in this section. Section 7 considers extensions of torsion-free
groups by $p$-groups, and Theorem B is established therein. Finally, Section 8 settles the case of extensions of torsion-free groups by torsion groups with bounded primary components, thereby proving (3) of Theorem A.

2. Preliminaries on exact and (quasi-)abelian categories

2.1. Additive categories. Recall that a preadditive category, also called an Ab-category, is a category $\mathcal{C}$ in which each hom-set $\text{Hom}_{\mathcal{C}}(A, B)$ for objects $A, B$ of $\mathcal{C}$ is an abelian group, in such a way that composition of morphisms is bilinear [ML98, Section I.8]. In a preadditive category, binary products and binary coproducts coincide, and are called biproducts. Furthermore, an object is initial if and only if it is terminal, in which case it is called a zero object; see [ML98, Section VIII.2] and [ML95, Section IX.1]. An additive category is a preadditive category that has a zero object, denoted by $0$, and such that every pair of objects $A, B$ has a biproduct, denoted by $A \oplus B$. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories is called additive if satisfies $F(f_0 + f_1) = F(f_0) + F(f_1)$ whenever $f_0, f_1 : A \to B$ are morphisms in $\mathcal{C}$. This is equivalent to the assertion that $F$ preserves biproducts of pairs of objects of $\mathcal{C}$; see [ML98, Section VIII.2]. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in an additive category is short-exact or a kernel-cokernel pair if $f$ is a kernel of $g$ and $g$ is a cokernel of $f$.

2.2. Exact categories. The notion of exact category has been introduced by Quillen in [Qui75, Section 2]; see also [Bü11, Definition 2.3.1] and [Bü10, Definition 2.1]. Let $\mathcal{A}$ be an additive category, and let $\mathcal{E}$ be a collection of short-exact sequences that is closed under isomorphism in the double arrow category $\mathcal{A}^{\to \to}$. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{E}$ is called pure short exact or a conflation, where $f$ is called an admissible monic or an inflation and $g$ is called an admissible epic or a deflation. The collection $\mathcal{E}$ is called an exact structure on $\mathcal{A}$ if it satisfies the following axioms:

1. Identities are inflations and deflations.
2. The composition of inflations is an inflation. The composition of deflations is a deflation.
3. The pushout of an inflation along an arbitrary morphism exists and yields an inflation. The pullback of a deflation along an arbitrary morphism exists and yields a deflation.

An exact category $(\mathcal{A}, \mathcal{E})$ is an additive category $\mathcal{A}$ endowed with an exact structure $\mathcal{E}$. A functor $F : (\mathcal{A}, \mathcal{E}) \to (\mathcal{A}', \mathcal{E}')$ between exact categories is exact or preserves exactness if it is additive and carries short sequences in $\mathcal{E}$ to short sequences in $\mathcal{E'}$.

2.3. Quasi-abelian categories. A quasi-abelian category [Bü10, Definition 4.1] (called almost abelian in [Rum01]) is an additive category such that:

1. every morphism has a kernel and a cokernel;
2. the class of kernels is stable under pushout along arbitrary morphisms, and the class of cokernels is stable under pullback along arbitrary morphisms;

see also [Sch99].

In a quasi-abelian category, one defines the image $\text{im}(f)$ of an arrow $f : A \to B$ to be the subobject $\ker(\text{coker}(f))$ of $B$, and the coinage $\text{coim}(f)$ to be the quotient $\text{coker}(\ker(f))$ of $B$. Then $f$ induces a unique arrow $\hat{f} : \text{coim}(f) \to \text{im}(f)$ such that $\text{im}(f) \circ \hat{f} \circ \text{coim}(f) = f$, which is both monic and epic [Bü10, Proposition 4.8]. By definition, the arrow $f$ is strict if $\hat{f}$ is an isomorphism [Sch99, Definition 1.1.1].

If $\mathcal{A}$ is a quasi-abelian category, and $\mathcal{B}$ is an additive (not necessarily full) subcategory of $\mathcal{A}$ that is closed under taking biproducts, kernels, and cokernels (and hence all finite limits and colimits), then $\mathcal{B}$ is also a quasi-abelian category [Rum01, Lemma 4].

Let $\mathcal{A}$ be a quasi-abelian category. Then $\mathcal{A}$ is an exact category when endowed with its maximal exact structure, consisting of all the kernel-cokernel pairs in $\mathcal{A}$ [Bü10, Proposition 4.4]; see also [Sch99, Remark 1.1.11].

An abelian category is a quasi-abelian category $\mathcal{M}$ such that every monic arrow is a kernel, and every epic arrow is a cokernel [ML98, Section VIII.3]; see also [ML95, Section IX.2]. This is equivalent to the assertion that every arrow in $\mathcal{M}$ is strict.
The category $\mathcal{G}$ of abelian groups is an abelian category, which we regard as an exact category with respect to its maximal exact structure. An object $I$ of an exact category $(A, \mathcal{E})$ is injective if the functor $\text{Hom}_A (-, I) : A^{\text{op}} \to \mathcal{G}$ is exact. Dually, an object $P$ of $(A, \mathcal{E})$ is projective if the functor $\text{Hom}_A (P, -) : A \to \mathcal{G}$ is exact.

3. Groups with a Polish cover and their Solecki subgroups

3.1. Groups with a Polish cover. Recall that we assume all the groups, including Polish groups, to be abelian. The following notion was introduced in [BLP20]. A group with a Polish cover is a group $G$ explicitly presented as a quotient $\hat{G}/N$ where $\hat{G}$ is a Polish group and $N$ is a Polishable subgroup of $\hat{G}$. This means that $N$ is a Borel subgroup of $\hat{G}$ such that there exists a Polish topology compatible with the induced Borel structure that turns $N$ into a Polish group. Equivalently, there exist a Polish group $L$ and a continuous (injective) homomorphism $L \to \hat{G}$ with image equal to $N$.

Suppose that $G = \hat{G}/N$ and $H = \hat{H}/M$ are groups with a Polish cover. A group homomorphism $\varphi : G \to H$ is Borel-definable if there exists a Borel lift for $\varphi$, namely a Borel function $\varphi : \hat{G} \to \hat{H}$ such that $\varphi(x + N) = \varphi(x) + M$ for $x \in G$. We regard groups with a Polish cover as objects of a category with Borel-definable group homomorphisms as morphisms. It is proved in [Lup22c, Section 11] that this is an abelian category, where the sum of two Borel-definable homomorphisms $\varphi, \psi : G \to H$ is defined by setting $(\varphi + \psi)(x) = \varphi(x) + \psi(x)$ for $x \in G$. By [Lup22c, Remark 3.7], a Borel-definable homomorphism between groups with a Polish cover is an isomorphism in the category of groups with a Polish cover if and only if it is a bijection.

We identify a Polish group $G$ with the group with a Polish cover $\hat{G}/N$ where $\hat{G} = G$ and $N = \{0\}$. By [Kec95, Theorem 9.10 and Theorem 12.17], this realizes the category of Polish groups and a continuous group homomorphisms as a full subcategory of the category of groups with a Polish cover. A group with a Polish cover $\hat{G}/N$ is Borel-definably isomorphic to a Polish group if and only if $N$ is a closed subgroup of $\hat{G}$.

Recall that a Polish group $G$ is non-Archimedean if it admits a countable basis of the identity consisting of open subgroups. In the abelian case, this is equivalent to the assertion that $G$ is pro-countable, namely it is the limit of a tower of countable groups. A group with a non-Archimedean Polish cover is a group with a Polish cover $G = \hat{G}/N$ where $\hat{G}$ and $N$ are non-Archimedean Polish groups. It is proved in [Lup22a, Section 2] that non-Archimedean groups with a Polish cover form an abelian subcategory of the abelian category of groups with a Polish cover.

An exact sequence in the category of groups with a Polish cover is simply a sequence of groups with a Polish cover that is exact in the category of groups and such that the group homomorphisms are Borel-definable subgroups. Furthermore, it is shown in [Lup22c, Section 8] that images and preimages of Polishable subgroups of groups with a Polish cover under Borel-definable group homomorphisms are Polishable subgroups.

3.2. Polishable subgroups. Suppose that $G = \hat{G}/N$ is a group with a Polish cover, and let $H = \hat{H}/N$ be a subgroup of $G$. Then we say that $H$ is:

- a Polishable subgroup of $G$ if $\hat{H}$ is a Polishable subgroup of $\hat{G}$;
- a non-Archimedean Polishable subgroup of $G$ if it is a Polishable subgroup of $G$ and furthermore $\hat{H}$ is a non-Archimedean Polish group.

We let $\overline{H}^G$ be the closure of $H$ in $G$ with respect to the quotient topology. It is proved in [Lup22c, Section 4] that images and preimages of Polishable subgroups of groups with a Polish cover under Borel-definable group homomorphisms are Polishable subgroups. Furthermore, it is shown in [Lup22c, Section 8] that images and preimages of non-Archimedean Polishable subgroups of non-Archimedean groups with a Polish cover under Borel-definable homomorphisms are non-Archimedean Polishable subgroups.

Recall that a complexity class $\Gamma$ is an assignment $X \mapsto \Gamma (X)$ where $X$ is a Polish space and $\Gamma (X)$ is a collection of subsets of $X$, such that $f^{-1}(A) \in \Gamma (X)$ for every continuous function $f : X \to Y$ between Polish spaces $X,Y$ and $A \in \Gamma (X)$. We say that a Polishable subgroup $H = \hat{H}/N$ of a group with a Polish cover $G = \hat{G}/N$ is $\Gamma$ in $G$ or belongs to $\Gamma (G)$ if $\hat{H} \in \Gamma (\hat{G})$. Furthermore, $H$ has complexity class $\Gamma$ in $G$ if $\hat{H} \in \Gamma (\hat{G})$ and $\hat{G} \setminus \hat{H} \notin \Gamma (\hat{G})$. Let $\Gamma$ be the complexity class $\Sigma_0^n$, $\Pi_0^n$, or $D(\Pi_0^n)$ for $\alpha < \omega_1$. It is proved in [Lup22b, Section 3] that if $\varphi : G \to H$ is a Borel-definable homomorphism between groups with a non-Archimedean Polish cover and $H_0$ is a non-Archimedean Polishable subgroup of $H$ such that $H_0 \in \Gamma (H)$, then $\varphi^{-1}(H_0) \in \Gamma (G)$. Furthermore, if $H = \hat{H}/M$ and $H_0 = \hat{H}_0/M$, then $H_0 \in \Gamma (H)$ if and only if the coset equivalence relation of $\hat{H}_0$ in $\hat{H}$ is potentially $\Gamma$. 

3.3. **Solecki subgroups.** Let $G$ be a group with a Polish cover. The first Solecki subgroup $s_1(G)$ of $G$ is the smallest $\Pi^0_3$ Polishable subgroup of $G$. It was shown by Solecki in [Sol99] that such a subgroup exists, and $s_1(G) \subseteq \{0\}^G$. One then defines the sequence $(s_\alpha(G))_{\alpha < \omega_1}$ of Solecki subgroups of $G$ by recursion on $\alpha$ by setting:

- $s_0(G) = \{0\}^G$;
- $s_{\alpha+1}(G) = s_1(s_\alpha(G))$;
- $s_\lambda(G) = \bigcap_{\alpha < \lambda} s_\alpha(G)$ for $\lambda$ limit.

It is proved in [Sol99] that there exists $\sigma < \omega_1$ such that $s_\sigma(G) = \{0\}$. The least such a countable ordinal is called the **Solecki rank** of $G$ in [Lup22c].

Proposition 2.28 of [Lup22a] shows that, for every $\alpha < \omega_1$, $s_\alpha(G)$ is the smallest $\Pi^0_{1+\alpha}$ Polishable subgroup of $G$. If $G$ is a non-Archimedean group with a Polish cover, then $s_\alpha(G)$ is a non-Archimedean Polishable subgroup of $G$ for every $\alpha < \omega_1$. The following characterization of the first Solecki subgroup can be easily obtained with the methods of [Sol99]; see [Lup22b].

**Lemma 3.1.** Suppose that $G = \hat{G}/N$ is a group with a Polish cover, and let $H = \hat{H}/N$ be a $\Pi^0_3$ Polishable subgroup of $G$. Suppose that:

1. $N$ is dense in $\hat{H}$;
2. for every open neighborhood $V$ of the identity in $N$, $\pi^G \cap \hat{H}$ contains an open neighborhood of the identity in $\hat{H}$.

Then $H = s_1(G)$.

By [Lup22b, Theorem 6.1] we have the following description of the complexity of $\{0\}$ in a group with a Polish cover.

**Theorem 3.2.** Suppose that $G$ is a group with a Polish cover. Let $\alpha = \lambda + n$ be the Solecki rank of $G$, where $\lambda < \omega_1$ is either zero or limit and $n < \omega$.

1. Suppose that $n = 0$. Then $\Pi^0_{1+\lambda}$ is the complexity class of $\{0\}$ in $G$;
2. Suppose that $n \geq 1$. Then:
   a. If $\{0\}$ is $\Pi^0_{1+n}$ and not $D(\Pi^0_{1+n+1})$ in $s_{\lambda+n-1}(G)$, then $\Pi^0_{1+\lambda+n+1}$ is the complexity class of $\{0\}$ in $G$;
   b. If $n \geq 2$ and $\{0\}$ is $D(\Pi^0_{1+n})$ in $s_{\lambda+n-1}(G)$, then $D(\Pi^0_{1+\lambda+n})$ is the complexity class of $\{0\}$ in $G$;
   c. If $n = 1$ and $\{0\}$ is $D(\Pi^0_{1+n})$ and not $\Sigma^0_2$ in $s_\lambda(G)$, then $D(\Pi^0_{1+\lambda+n+1})$ is the complexity class of $\{0\}$ in $G$;
   d. If $n = 1$ and $\{0\}$ is $\Sigma^0_2$ in $s_\lambda(G)$, then $\Sigma^0_{1+\lambda+1}$ is the complexity class of $\{0\}$ in $G$.

In particular, if $\{0\}$ is $\Pi^0_{1+n}$ and not $\Pi^0_{1+\beta}$ for $1 + \beta < 1 + \lambda + n$. If $\{0\}$ is a non-Archimedean Polishable subgroup of $G$, then the case (2c) is excluded.

4. **The group Ext of extensions**

4.1. **Extensions of groups.** Recall that we assume all the groups to be abelian. Suppose that $C$ is a countable group, and $A$ is a group with a non-Archimedean Polish cover. A (normalized 2-)cocycle on $C$ with values in $A$ is a function $c : C \times C \to A$ satisfying, for every $x, y, z \in C$:

- $c(x, y) = c(y, x)$;
- $c(x, 0) = 0$;
- $c(x, y + z) + c(y, z) = c(x, y) + c(x, y, z)$.

The collection $Z(C, A)$ of $A$-valued cocycles on $C$ is a closed subgroup of $A^{C \times C}$, and hence a group with a non-Archimedean Polish cover. An $A$-valued cocycle $c$ on $C$ is a **coboundary** if there exists a function $f : C \to A$ such that $c(x, y) = f(x) + f(y) - f(x + y)$. The collection $B(C, A)$ of coboundaries forms a non-Archimedean Polishable subgroup of $Z(C, A)$. We let $\text{Ext}(C, A)$ be the group with a non-Archimedean Polish cover $Z(C, A)/B(C, A)$. We also let $\text{Hom}(C, A)$ be the group of homomorphisms $C \to A$. This is a closed subgroup of $A^C$, and hence a group with a non-Archimedean Polish cover.

An extension $A \to X \to C$ of $C$ by $A$ defines a cocycle $c : (x, y) \mapsto t(x) + t(y) - t(x + y)$ on $C$ with values in $A$, where $t : C \to X$ is a right inverse for the quotient map $X \to C$, and $A$ is identified with a subgroup of $X$. This induces a bijective correspondence between equivalence classes of extensions of $C$ by $A$ and elements of $\text{Ext}(C, A)$. 


The trivial element of Ext \((C, A)\) corresponds to the equivalence class of extensions of \(C\) by \(A\) that split. Thus, the classification problem for extensions of \(C\) by \(A\) can be seen as the problem of classifying elements of \(\mathbb{Z}(C, A)\) up to the relation of belonging to the same \(B (C, A)\)-coset; see [Lup22a, Section 3.4].

If \(A \to X \xrightarrow{\Delta} C\) and \(A \to X' \xrightarrow{\Delta'} C\) are two extensions, then the sum of the corresponding equivalence classes in Ext \((C, A)\) is represented by their Baer sum. This is the extension \(A \to Y \xrightarrow{\Delta} C\) of \(A\) where

\[
Y = \{(x, x') \in X \oplus X' : p(x) = p(x')\}
\]

is the pullback of \(p\) and \(p'\), and \(A \to Y\) is the diagonal map.

### 4.2. Pure extensions

A group \(A\) is a countable group, and \(A\) is a group with a non-Archimedean Polish cover. An extension \(A \to X \xrightarrow{\pi} C\) of \(C\) by \(A\) is pure if, identifying \(A\) with its image inside \(X\), for every \(n \in \mathbb{N}\) one has that \(nX \cap A = nA\). This is equivalent to the assertion that for every subgroup \(X_0\) of \(X\) containing \(A\) such that \(X_0/A\) is finite, one has that the short exact sequence \(A \to X_0 \xrightarrow{\pi} C_0\) splits. (Asking that \(X_0/A\) is finitely-generated yields an equivalent definition.) In this case, we also say that \(A\) is a pure subgroup of \(X\), writing \(A \triangleleft X\), and that \(X \to C\) is a pure epimorphism [Fuc70, Section V.26]. Model-theoretically, this is equivalent to the assertion that \(A\) is positively existentially closed in \(X\) regarded as a structure in the language of groups; see [Fuc70, Theorem 28.5].

An \(A\)-valued cocycle \(c\) on \(C\) corresponds to a pure extension if and only if \(c|_{C_0 \times \mathbb{Z}}\) is a coboundary for every finite subgroup \(C_0\) of \(C\). The set \(B_w(C, A)\) of such cocycles is a closed subgroup of \(\text{Ext}(C, A)\). We let \(\text{PExt}(C, A)\) be the corresponding group with a non-Archimedean Polish cover \(B_w(C, A)/\mathbb{Z}(C, A)\). By [Lup22a, Lemma 3.2], if both \(C\) and \(A\) are countable, then \(B_w(C, A)\) is the closure of \(B(C, A)\) in \(\text{Ext}(C, A)\), and hence \(\text{PExt}(C, A) = s_\mathbb{Q}(\text{Ext}(C, A))\) is the closure of \(\{0\}\) in \(\text{Ext}(C, A)\). Clearly, when \(C\) is torsion-free, \(\text{Ext}(C, A) = \text{PExt}(C, A)\) and hence \(\{0\}\) is dense in \(\text{Ext}(C, A)\).

We also define \(\text{Ext}_w(C, A)\) to be the group with a non-Archimedean Polish cover

\[
\mathbb{Z}(C, A)/B_w(C, A) = \text{Ext}(C, A)/\text{PExt}(C, A),
\]

which is a Polish group when both \(A\) and \(C\) are countable. By definition, we have a Borel-definable short exact sequence

\[
0 \to \text{PExt}(C, A) \to \text{Ext}(C, A) \to \text{Ext}_w(C, A) \to 0.
\]

### 4.3. Exact sequences for Ext

Suppose that \(A \to B \to C\) is a short exact sequence of countable groups, \(A' \to B' \to C'\) is a Borel-definable short exact sequence of groups with a non-Archimedean Polish cover, \(G\) is a countable group, and \(G'\) is a group with a non-Archimedean Polish cover. Then we have two Borel-definable exact sequences

\[
0 \to \text{Hom}(C, G') \to \text{Hom}(B, G') \to \text{Hom}(A, G') \to \text{Ext}(C, G') \to \text{Ext}(B, G') \to \text{Ext}(A, G') \to 0
\]

and

\[
0 \to \text{Hom}(G, A') \to \text{Hom}(G, B') \to \text{Hom}(G, C') \to \text{Ext}(G, A') \to \text{Ext}(G, B') \to \text{Ext}(G, C') \to 0.
\]

Here, the Borel-definable homomorphism \(\text{Hom}(A, G') \to \text{Ext}(C, G')\) is defined by mapping \(\varphi \in \text{Hom}(A, G')\) to the cocycle

\[
(x, y) \mapsto (\varphi \circ t)(x) + (\varphi \circ t)(y) - (\varphi \circ t)(x + y),
\]

where we identify \(A\) as a subgroup of \(B\) and \(t : C \to B\) is a right inverse for the quotient map \(B \to C\). Similarly, the Borel-definable homomorphism \(\text{Hom}(G, C') \to \text{Ext}(G, A')\) is defined by mapping \(\psi \in \text{Hom}(G, C')\) to the cocycle

\[
(x, y) \mapsto (t' \circ \psi)(x) + (t' \circ \psi)(y) - (t' \circ \psi)(x + y),
\]

where \(t' : C' \to B'\) is a right inverse for the quotient map \(B' \to C'\).

If \(A \to B \to C\) and \(A' \to B' \to C'\) are pure short exact sequences, then the Borel-definable exact sequences above restrict to Borel-definable exact sequences

\[
0 \to \text{Hom}(C, G') \to \text{Hom}(B, G') \to \text{Hom}(A, G') \to \text{PExt}(C, G') \to \text{PExt}(B, G') \to \text{PExt}(A, G') \to 0
\]

and

\[
0 \to \text{Hom}(G, A') \to \text{Hom}(G, B') \to \text{Hom}(G, C') \to \text{PExt}(G, A') \to \text{PExt}(G, B') \to \text{PExt}(G, C') \to 0;
\]

see [Lup22a, Section 3.5].
Suppose now that $A$ is a group with a non-Archimedean Polish cover, $C$ is a countable group, and $C = F/R$ is a presentation of $C$ where $F$ is a countable free group and $R$ is a subgroup of $F$. As $\text{Ext}(F, A) = 0$, we have a Borel-definable isomorphism

$$\text{Ext}(C, A) \cong \frac{\text{Hom}(R, A)}{\text{Hom}(F|R, A)}$$

where $\text{Hom}(F|R, A)$ is the image of the restriction map

$$\text{Hom}(F, A) \rightarrow \text{Hom}(R, A).$$

This isomorphism restricts to a Borel-definable isomorphism

$$\text{PExt}(C, A) \cong \frac{\text{Hom}^f(R, A)}{\text{Hom}(F|R, A)}.$$ 

Here, $\text{Hom}^f(R, A) = \bigcap_E \text{Hom}(E|R, A)$ where $E$ ranges among the subgroups of $F$ containing $R$ such that $E/R$ is finite (or, finitely generated).

4.4. Ext and $\lim^1$. A tower of groups with a Polish cover is a sequence $(A^{(n)})_{n \in \omega}$ of groups with a Polish cover and Borel-definable homomorphisms $p^{(n,n+1)} : A^{(n+1)} \rightarrow A^{(n)}$. A morphism $(A^{(n)}) \rightarrow (B^{(n)})$ between towers is represented by a sequence $(m_k, \varphi(k))$ such that, for every $k \in \omega$, $m_k \in \omega$, $m_{k+1} > m_k$, $\varphi(k) : A^{(m_k)} \rightarrow B^{(k)}$ is a Borel-definable homomorphism, and $p^{(k,k+1)}(\varphi(k)) = \varphi(k)(p^{(m_k, m_{k+1}))}$. By definition, two such sequences $(m_k, \varphi(k))$ and $(m'_k, \varphi'(k))$ represent the same morphism $(A^{(n)}) \rightarrow (B^{(n)})$ if and only if there exists an increasing sequence $(\tilde{m}_k)$ in $\omega$ such that $\tilde{m}_k \geq \max \{m_k, m'_k\}$ and $\varphi(k)(p^{(m_k, \tilde{m}_k)}) = \varphi'(k)(p^{(m'_k, \tilde{m}_k)})$ for every $k \in \omega$. Towers of groups with a Polish cover are the object of a category $\mathcal{M}$, where morphisms are defined as above, and composition of morphisms and identity morphisms are defined in the obvious way. Defining the sum of morphisms pointwise endows $\mathcal{M}$ with the structure of an abelian category [KS06, Chapter 15]; see also [K So06, Proposition 4.5 in Section A.4].

Let $A = (A^{(n)})$ be a tower of groups with a Polish cover. Then one defines $\lim^1 A$ to be the group with a Polish cover $Z(A)/B(A)$, where:

- $Z(A)$ is the closed subgroup of $\prod_{n \leq m} A^{(n)}$ consisting of $(a_{n,m})$ satisfying
  $$a_{n_0,n_1} + p^{(n_1,n_2)}(a_{n_1,n_2}) = a_{n_0,n_2}$$
  for $n_0 \leq n_1 \leq n_2$;
- $B(A)$ is the Polishable subgroup of $Z(A)$ consisting of elements $(a_{n,m})$ for which there exists a sequence
  $$(b_n) \in \prod_n A^{(n)}$$
  such that $a_{n,m} = b_n - p^{(m,n)}(b_m)$ for every $n \leq m$.

This defines a functor $A \mapsto \lim^1 A$ from $\mathcal{M}$ to the category of groups with a Polish cover. Jensen’s theorem [Sch03, Theorem 6.1] allows one to express PExt in terms of $\lim^1$, as follows; see also [BLP20, Theorem 7.4].

**Theorem 4.1.** Suppose $C, A$ are countable groups, and $(C_n)_{n \in \omega}$ is an increasing sequence of finitely-generated subgroups of $C$ with union equal to $C$. Then $\text{PExt}(C, A)$ and $\lim^1 \text{Hom}(C_n, A)$ are Borel-definably isomorphic.

**Proof.** We can write $C = F/R$ where $F$ is a countable free group and $R \subseteq F$ is a subgroup. We have a Borel-definable isomorphism

$$\text{PExt}(C, A) \cong \frac{\text{Hom}^f(R, A)}{\text{Hom}(F|R, A)}.$$ 

For every $n \in \omega$, let $F_n \subseteq F$ be such that $F_n/R = C_n$. If $\varphi \in \text{Hom}^f(F, A)$, then for every $n \in \omega$ there exists $\psi_n \in \text{Hom}(F_n, A)$ such that $\psi_n|_R = \varphi$. One can choose $\psi_n$ in a Borel fashion from $\varphi$ by [Kec95, Theorem 12.17]. For $n \leq m$, $\psi_n - \psi_m|_{F_n}$ vanishes on $R$, and hence it induces a homomorphism $\psi_{n,m} : C_n \rightarrow A$. The sequence $(\psi_{n,m})_{n \leq m}$ determines an element of $\lim^1 \text{Hom}(C_n, A)$. The Borel function $\varphi \mapsto (\psi_{n,m})_{n \leq m}$ induces a Borel-definable homomorphism

$$\frac{\text{Hom}^f(R, A)}{\text{Hom}(F|R, A)} \rightarrow \lim^1 \text{Hom}(C_n, A),$$

which can be easily seen to be bijective. \qed
Given a tower of groups with a Polish cover $A = (A_n)_{n \in \omega}$, one can define the group with a Polish cover
\[
\lim A = \left\{ (a_n) \in \prod_{n \in \omega} A_n : \forall n \leq m, p^{(n,m)}(a_m) = a_n \right\}.
\]
This defines a functor from $\mathcal{M}$ to the category of groups with a Polish cover. Suppose that
\[
A \xrightarrow{f} B \xrightarrow{g} C
\]
is a short exact sequence in $\mathcal{M}$. This induces an exact sequence of groups with a Polish cover
\[
0 \to \lim A \to \lim B \to \lim C \xrightarrow{\delta} \lim^1 A \to \lim^1 B \to \lim^1 C \to 0;
\]
see [Mar00, Corollary 11.51]. The Borel-definable group homomorphism $\lim C \to \lim^1 A$ can be described as follows.
After replacing $A$, $B$, and $C$ with isomorphic towers, we can assume that $g$ is represented by the sequence of Borel-definable group homomorphisms $g_n : B_n \to C_n$ for $n \in \omega$, and $f$ is represented by the sequence of Borel-definable group homomorphisms $f_n : A_n \to B_n$ for $n \in \omega$, such that
\[
A_n \xrightarrow{f_n} B_n \xrightarrow{g} C
\]
is a Borel-definable short exact sequence for every $n \in \omega$; see [AM86, Proposition 3.3 in Section A.3]. Suppose that $(c_n)_{n \in \omega} \in \lim C$. Thus, for every $n \in \omega$, one can choose $b_n \in B_n$ such that $g_n(b_n) = c_n$. For $n \leq m$, there exists $a_{n,m} \in A$ such that $f_n(a_{n,m}) = b_n - p^{(n,m)}(b_m)$. The sequence $(a_{n,m})_{n \leq m}$ defines an element of $\lim^1 A$. The assignment $(c_n)_{n \in \omega} \mapsto (a_{n,m})_{n \leq m}$ defines the group homomorphism $\delta : \lim C \to \lim^1 A$, which is easily seen to be Borel-definable using [Kec95, Theorem 12.17].

4.5. Some lemmas about Ext. In this section, we recall and prove some lemmas about Ext to be used in the rest of the paper. The following two lemmas can be found in [Fuc70, Section 52]; see also [Lup22a].

**Lemma 4.2.** Fix a prime number $p$. Suppose that $A, C$ are countable groups such that $C$ is a $p$-group and $A$ is $p$-divisible. Then $\text{Ext}(C, A) = 0$.

**Lemma 4.3.** Suppose that $A, C$ are countable groups such that $A$ is torsion-free and $C$ is torsion. Let $D$ be the divisible hull of $A$. Then $\text{Ext}(C, A)$ is a Polish group isomorphic to $\text{Hom}(C, D/A)$.

Recall that the Prüfer $p$-group $\mathbb{Z}(p^\infty)$ is the quotient of $\mathbb{Z}[1/p]$ by $\mathbb{Z}$, where $\mathbb{Z}[1/p]$ is the subring of $\mathbb{Q}$ generated by $1/p$. The $p$-completion $L_p^\omega (B)$ of a countable group $B$ is the Polish group $\lim_{n \in \omega} (B/p^n B)$.

**Lemma 4.4.** Fix a prime number $p$, and let $T$ be the Prüfer $p$-group $\mathbb{Z}(p^\infty)$. Suppose that $B$ is a countable torsion-free group. Then $\text{Ext}(T, B)$ is a Polish group naturally isomorphic to $L_p^\omega (B)$. In particular, $\text{Ext}(T, B)$ is an uncountable Polish group whenever $B$ is nonzero.

**Proof.** By Lemma 4.3, we have that $\text{PExt}(T, B) = 0$. Notice that $T = \text{colim}_n T_n$ where $T_n = \mathbb{Z}/p^n \mathbb{Z}$ and $T_n \to T_{n+1}$ is induced by the map $\mathbb{Z} \to \mathbb{Z}$, $x \mapsto px$. Thus, we have natural continuous isomorphisms
\[
\text{Ext}(T, B) = \text{Ext}_w(T, B) \cong \lim_n \text{Ext}(T_n, B) \cong \lim_n \frac{B}{p^n B} = L_p^\omega (B).
\]
When $B$ is nonzero, we have that $B$ is unbounded, and hence $L_p^\omega (B)$ is an uncountable Polish group. \hfill \Box

The notion of $p$-basic subgroup can be found in [Fuc70, Section 32].

**Lemma 4.5.** Let $A$ be a countable torsion-free group, and let $T$ be the Prüfer $p$-group $\mathbb{Z}(p^\infty)$. Suppose that $B \subseteq A$ is a $p$-basic subgroup. Then the inclusion map $B \to A$ induces a Borel-definable isomorphism $\text{Ext}(T, B) \to \text{Ext}(T, A)$. In particular, $\text{Ext}(T, A)$ is an uncountable Polish group whenever $A$ is not $p$-divisible.

**Proof.** By Lemma 4.4, we have natural continuous isomorphisms $\text{Ext}(T, B) \cong L_p^\omega (B)$ and $\text{Ext}(T, A) \cong L_p^\omega (A)$. Thus, it suffices to verify that the inclusion $B \to A$ induces a continuous isomorphism $L_p^\omega (B) \to L_p^\omega (A)$. This follows from the fact that $B$ is dense in the $p$-adic topology of $A$ and the $p$-adic topology on $B$ is the induced topology by the $p$-adic topology of $A$; see [Fuc70, page 136]. Finally, notice that, when $A$ is not $p$-divisible, $B$ is nonzero. This, the conclusion follows from Lemma 4.4. \hfill \Box
Lemma 4.6. Fix a prime number $p$, and let $T$ be a $p$-group. Suppose that $B$ is a countable group that is not $p$-divisible. If $T$ is nonzero, then $\text{Ext}(T, B)$ is a nonzero Polish group. If $T$ is infinite and $B$ is torsion-free, then $\text{Ext}(T, B)$ is an uncountable Polish group.

Proof. Suppose that $T$ is nonzero. Thus, there exists an injective homomorphism $\mathbb{Z}/p\mathbb{Z} \to T$. This induces a surjective homomorphism

$$\text{Ext}(T, B) \to \text{Ext}((\mathbb{Z}/p\mathbb{Z}), B) \cong \frac{B}{pB} \neq 0.$$ 

This shows that $\text{Ext}(T, B)$ is nonzero.

Suppose now that $T$ is infinite and $B$ is torsion-free. If $T$ is not reduced, then $T$ has $\mathbb{Z}(p^\infty)$ as a direct summand, and hence $\text{Ext}(T, B)$ has $\text{Ext}(\mathbb{Z}(p^\infty), B)$ as a direct summand. By Lemma 4.5, $\text{Ext}(\mathbb{Z}(p^\infty), B)$ is uncountable. If $T$ is reduced, then there exists a surjective homomorphism

$$T \to (\mathbb{Z}/p\mathbb{Z})^{(\omega)}.$$ 

Since $B$ is torsion-free, $\text{Hom}(\text{Ker}(\pi), B) = 0$. Thus, $\pi$ induces an injective homomorphism

$$\left(\frac{B}{pB}\right)^{\omega} \cong \text{Ext}((\mathbb{Z}/p\mathbb{Z})^{(\omega)}), T) \to \text{Ext}(T, B).$$ 

Since $(\frac{B}{pB})^{\omega}$ is uncountable, this shows that $\text{Ext}(T, B)$ is uncountable. □

Lemma 4.7. Suppose that $L, T$ are countable torsion groups. For $p \in \mathbb{P}$, let $T_p, L_p$ be the $p$-component of $T, L$, respectively. Then $\text{Ext}(L, T)$ is Borel-definably isomorphic to $\prod_{p \in \mathbb{P}} \text{Ext}(L_p, T_p)$.

Proof. We have that

$$L \cong \bigoplus_{p \in \mathbb{P}} L_p$$

and hence

$$\text{Ext}(L, T) \cong \prod_{p \in \mathbb{P}} \text{Ext}(L_p, T).$$

For $p \in \mathbb{P}$, we have that $T = T_p \oplus T'$ where $T'$ is $p$-divisible. Thus,

$$\text{Ext}(L_p, T') = 0$$

by Lemma 4.2. Hence,

$$\text{Ext}(L_p, T) \cong \text{Ext}(L_p, T_p) \oplus \text{Ext}(L_p, T') \cong \text{Ext}(L_p, T_p).$$

Therefore,

$$\text{Ext}(L, T) \cong \prod_{p \in \mathbb{P}} \text{Ext}(L_p, T) \cong \prod_{p \in \mathbb{P}} \text{Ext}(L_p, T_p).$$

This concludes the proof. □

Lemma 4.8. Suppose that $A, L$ are countable groups, where $L$ is torsion. If $A_p$ is bounded for every $p \in \mathbb{P}$, then $\text{PExt}(L, A) = 0$, and $\text{Ext}(L, A)$ is a Polish group.

Proof. By [Lup22b, Lemma 4.6] we have a Borel-definable isomorphism

$$\text{PExt}(L, A) \cong \prod_{p \in \mathbb{P}} \text{PExt}(L_p, A_p).$$

For $p \in \mathbb{P}$, as $A_p$ is bounded, we have that

$$\text{PExt}(L_p, A_p) = 0$$

by [Lup22b, Lemma 4.21]. Thus,

$$\text{PExt}(L, A) = 0$$

and $\text{Ext}(L, A)$ is a Polish group. □
5. Finite-rank-pure extensions

5.1. Dependence in groups. In this section we recall some fundamental notions from abelian group theory; see [Fuc70, Section III.16]. A family \((x_i)_{i \in I}\) of elements of a group \(A\) is independent if, for every finite subset \(I_0 \subseteq I\) and \(m_i \in \mathbb{Z}\) for \(i \in I_0\),

\[
\sum_{i \in I_0} m_i x_i = 0
\]

implies that \(m_i x_i = 0\) for every \(i \in I_0\). This is equivalent to the assertion that \((x_i : i \in \omega)\) is the direct sum of the cyclic groups \((x_i)\) for \(i \in \omega\) [Fuc70, Lemma 16.1]. A Prüfer basis is a maximal independent set. A subgroup \(B \subseteq A\) is essential if and only if \(B \cap C \neq 0\) whenever \(C\) is a nonzero subgroup of \(A\). If \(B \subseteq A\) is essential, then every maximal independent family in \(B\) is also a maximal independent family in \(A\). One has that an independent family \((x_i)_{i \in I}\) in \(A\) is maximal if and only if the subgroup \(\langle x_i : i \in I \rangle\) of \(A\) generated by \(\{x_i : i \in I\}\) is essential [Fuc70, Lemma 16.2].

Suppose that \(A\) is a group and \(L \leq A\). An element \(g\) of \(A\) depends on \(L\) if there exists a dependence relation of the form

\[
0 \neq ng = n_1 a_1 + \cdots + n_k a_k
\]

where \(n, n_1, \ldots, n_k \in \mathbb{Z}\), \(a_1, \ldots, a_k \in L\), and \(ng \neq 0\) [Fuc70, Section III.16]. The set \((L)_{\mathsf{a}}\) of elements of \(A\) that depend on \(L\) is the smallest pure subgroup of \(A\) containing \(L\) [Fuc70, Section V.26].

If \(A\) is a group, then the rank \(r(A)\) of \(A\) is the size of any maximal independent family, while the torsion-free rank \(r_0(A)\) of \(A\) is the size of any maximal independent family consisting of elements of infinite order. Clearly, if \(A\) is a torsion-free group, then the rank and the torsion-free rank of \(A\) coincide.

Notice that if \(A, B\) are torsion-free groups and \(f : A \to B\) is a group homomorphism, then \(\ker (f)\) is a pure subgroup of \(A\), while \(f(A)\) is not necessarily a pure subgroup of \(B\). A short-exact sequence \(A \to B \to C\), where \(B\) is a torsion-free group, is pure if and only if \(C\) is torsion-free, in which case the rank of \(B\) is the sum of the ranks of \(A\) and \(C\); see [FS13, Lemma 2.1(iii)].

5.2. The category of torsion-free groups. We denote by \(\mathcal{G}\) the category of abelian groups. Let \(\mathcal{A}\) be the full subcategory of \(\mathcal{G}\) whose objects are torsion-free abelian groups. It follows from [Rum01, Theorem 2] that \(\mathcal{A}\) is a quasi-abelian category. An arrow \(f : A \to B\) in \(\mathcal{A}\) is monic if and only if it is injective, and epic if and only if \(B = \langle f(A) \rangle_{\mathsf{a}}\). Furthermore, an arrow \(f : A \to B\) is a kernel if and only if it is injective and \(f(A)\) is a pure subgroup of \(B\), and a cokernel if and only if it is onto.

If \(f : A \to B\) is an arrow in \(\mathcal{A}\), then its kernel in \(\mathcal{A}\) is the subobject \(\{a \in A : f(a) = 0\}\) of \(A\), and its image in \(\mathcal{A}\) is \(\langle f(A) \rangle_{\mathsf{a}} \subseteq B\). The cokernel of \(f\) is the quotient of \(B\) by \(\langle f(A) \rangle_{\mathsf{a}}\). The subobjects of a torsion-free group \(A\) in \(\mathcal{A}\) are precisely its pure subgroups.

Suppose that \(f_0 : B_0 \to A\) and \(f_1 : B_1 \to A\) are arrows in \(\mathcal{A}\). Their pullback is given by \(g_0 : C \to B_0\) and \(g_1 : C \to B_1\), where

\[
C = \{(b_0, b_1) \in B_0 \oplus B_1 : f_0(b_0) = f_1(b_1)\},
\]

\[
g_0(b_0, b_1) = b_0, \text{ and } g_1(b_0, b_1) = b_1.
\]

Suppose now that \(g_0 : C \to B_0\) and \(g_1 : C \to B_1\) are arrows in \(\mathcal{A}\). Their pushout is given by \(f_0 : B_0 \to A\) and \(f_1 : B_1 \to A\), where

\[
A = (B_0 \oplus B_1)/J
\]

\[
J = \{(-g_0(c), g_1(c)) : c \in C\}_{\mathsf{a}},
\]

\[
f_0(b_0) = (b_0, 0) + J \text{ and } f_1(b_1) = (0, b_1) + J.
\]

An arrow \(f : A \to B\) in \(\mathcal{A}\) is strict if and only if \(f(A)\) is a pure subgroup of \(B\). Since not every arrow in \(\mathcal{A}\) is strict, \(\mathcal{A}\) is not an abelian category.
5.3. Finite-rank-pure extensions. Recall that \( \mathcal{G} \) denotes the category of abelian groups, and \( A \) denotes the full subcategory of \( \mathcal{G} \) whose objects are torsion-free abelian groups. Let \( \mathcal{I} \) be the full subcategory of \( A \) consisting of finite-rank torsion-free groups. Following [Wal66], we say that a short exact sequence \( A \to B \to C \) of groups is \( \mathcal{I} \)-pure or finite-rank-pure if \( C \) is torsion-free and for every subgroup \( B' \) of \( B \) containing \( A \) such that \( B'/A \in \mathcal{I} \) one has that the short exact sequence \( A \to B' \to B'/A \) splits. Notice that this notion is considered in [Wal66] in the setting of abelian categories, while the category \( A \) is not abelian. So the statements of the main results of [Wal66] do not automatically apply in this context. If \( A \xrightarrow{i} B \to C \in \mathcal{I} \)-pure short exact sequence, then we also say that it is an \( \mathcal{I} \)-conflation, where \( f \) is an \( \mathcal{I} \)-inflation, and \( g \) is an \( \mathcal{I} \)-deflation. If \( A \subseteq B \) is a pure subgroup and the inclusion \( A \to B \) is \( \mathcal{I} \)-pure, then we say that \( A \) is an \( \mathcal{I} \)-pure subgroup of \( B \) and write \( A \lesssim_{\mathcal{I}} B \). The terminology is justified by the fact, which we proceed to show, that \( \mathcal{I} \)-pure short exact sequences of torsion-free groups form an exact structure on \( A \).

The same proofs as (iii) and (vi) in [Wal66, Theorem 2.1] give the following two lemmas.

**Lemma 5.1.** Suppose that \( A, B, C \) are groups, \( A \lesssim_{\mathcal{I}} B \), and \( B \lesssim_{\mathcal{I}} C \). Then \( A \lesssim_{\mathcal{I}} C \).

**Proof.** We have that \( C/A \) is torsion-free. Indeed, if \( x \in C \) and \( n \in \mathbb{N} \) is such that \( nx \in A \), then \( nx \in B \) and hence \( x \in B \) since \( B \lesssim_{\mathcal{I}} C \). Since \( A \lesssim_{\mathcal{I}} B \), this implies \( x \in A \).

Suppose that \( A \subseteq S \subseteq C \) with \( S/A \) finite-rank. As we have a surjective homomorphism \( S/A \to (S + A)/B \) and \((S + A)/B \subseteq C/B \) is torsion-free, we have that \( (S + A)/B \) has finite rank. Since \( B \) is \( \mathcal{I} \)-pure in \( C \), we have that the short exact sequence \( B \to S + A \to (S + A)/B \) splits, and \( B \) is a direct summand of \( S + B \). We can thus write \( S + B = B \oplus R \) for some subgroup \( R \) of \( S + A \). Let then \( i: S \to B \oplus R = S + B \) the inclusion map and \( p: B \oplus R \to B \) the canonical projection. Define \( E \subseteq B \) to be the image of \( S \) under \( p \circ i \). Since \( p \circ i: S \to E \) induces a surjective homomorphism \( S/A \to E/A \), and \( S/A \) has finite rank, we have that \( E/A \subseteq B/A \) is (torsion-free and) finite-rank. Thus, as \( A \) is \( \mathcal{I} \)-pure in \( B \), the short exact sequence \( A \to E \to E/A \) splits, and \( A \) is a direct summand of \( E \). We can thus write \( E = A \oplus T \) for some subgroup \( T \) of \( E \). We have that
\[
A \subseteq S \subseteq E \oplus R = A \oplus T \oplus R
\]
This implies that
\[
S = A \oplus ((T \oplus R) \cap S)
\]
and \( A \) is a direct summand of \( S \). This concludes the proof that \( A \) is \( \mathcal{I} \)-pure in \( C \).

**Lemma 5.2.** Suppose that \( A, B, C \) are groups, with \( A \subseteq B \subseteq C \). If \( A \lesssim_{\mathcal{I}} C \) and \( B/A \lesssim_{\mathcal{I}} C/A \), then \( B \lesssim_{\mathcal{I}} C \).

**Proof.** We have that
\[
(C/A)/(B/A) \cong C/B
\]
is torsion-free. Indeed, if \( x \in C \) and \( n \in \mathbb{Z} \) are such that \( nx \in B \) then \( nx + A \in B/A \subseteq C/A \) and hence \( x + A \in B/A \) and \( x \in B \).

Suppose that \( B \subseteq S \subseteq C \) with \( S/B \) finite-rank (and torsion-free). Then we have that \( (S/A)/(B/A) \cong S/B \) is finite-rank. Therefore, since \( B/A \lesssim_{\mathcal{I}} C/A \), the short-exact sequence \( B/A \to S/A \to (S/A)/(B/A) \) splits, and \( B/A \) is a direct summand of \( S/A \). We can write \( S/A = (B/A) \oplus (R/A) \) for some subgroup \( R \) of \( C \) containing \( A \). Since \( R/A \cong (S/A)/(B/A) \) is finite-rank, and \( A \lesssim_{\mathcal{I}} C \), we have that \( A \) is a direct summand of \( R \). So we can write \( R = A \oplus T \) for some subgroup \( T \) of \( C \). We thus have that
\[
S/A = B/A \oplus \frac{A \oplus T}{A}
\]
and hence \( S = B \oplus T \). This shows that \( B \) is \( \mathcal{I} \)-pure in \( C \).

**Lemma 5.3.** Suppose that \( p: B \to B/J \) is an \( \mathcal{I} \)-deflation and \( f: A \to B/J \) is an arrow in \( A \). Let \( g_0: C \to B \) and \( g_1: C \to A \) be their pullback in \( \mathcal{G} \). Then \( g_1 \) is an \( \mathcal{I} \)-deflation. Furthermore, if \( p \) is an arrow in \( A \), then \( g_0, g_1 \) are the pullback of \( p, f \) in \( A \).

**Proof.** Since \( p \) is an \( \mathcal{I} \)-deflation, we have that \( J \) is an \( \mathcal{I} \)-pure subgroup of \( B \). We have that
\[
C = \{(b,a) \in B \oplus A : f(a) = b + J\},
\]
\[
g_0(b,a) = b, \text{ and } g_1(b,a) = a.
\]
For $a \in A$, $f(a) = b + J$ for some $b \in B$, and hence $a = g_1(b, a)$ where $(b, a) \in C$. Thus, $g_1$ is onto. It remains to prove that $\ker(g_1)$ is an $\mathcal{J}$-pure subgroup of $C$.

Notice that

$$\ker(g_1) = J \oplus \{0\}.$$ 

Suppose that $\ker(g_1) \subseteq S \subseteq C$ and $S/\ker(g_1)$ has finite rank. Consider $L := g_1(S) \cap \ker(f) \subseteq A$ and observe that $\{0\} \oplus L \subseteq S$. Indeed if $a \in g_1(S) \cap \ker(f)$ then there exists $b \in B$ such that $(b, a) \in S$ and in particular

$$b + J = f(a) = J$$

and hence $b \in J$. Since $(b, 0) \in J \oplus \{0\} \subseteq S$ we must have $(0, a) = (b, a) - (b, 0) \in S$.

We have that $g_0(S) \subseteq B$ is such that $g_0(S)/J$ has finite rank, considering the epimorphism $S/\ker(g_1) \to g_0(S)/J$ induced by $g_0$. Thus, we have that $g_0(S) = J \oplus R$ for some finite-rank subgroup $R$ of $B_0$. Thus, we have that $S \subseteq (J \oplus \{0\}) \oplus (R \oplus \{0\}) \oplus (\{0\} \oplus g_1(S)) \subseteq B \oplus A$. This shows that $J \oplus \{0\}$ is a direct summand of $S$.

$\square$

**Lemma 5.4.** Suppose that $A, B, C, D$ are groups, $B \preceq \mathcal{J} A$, $i: B \to A$ is the corresponding $\mathcal{J}$-inflation (inclusion map), and $g: B \to D$ is a homomorphism. Let $f_0: D \to C$ and $f_1: A \to C$ be the pushout of $i$ and $g$ in $\mathcal{G}$. Then $f_0: D \to C$ is an $\mathcal{J}$-inflation. Furthermore:

- If $D$ is torsion-free, then $C$ is torsion-free;
- If $i$ and $g$ are arrows in $A$, then $f_0, f_1$ give their pushout in $A$.

**Proof.** We have that

$$C = \frac{D \oplus A}{J}$$

where

$$J = \{(-g(b), b) : b \in B\}.$$ 

Furthermore, $f_0: D \to C$, $d \mapsto (d, 0) + J$ and $f_1: A \to C$, $a \mapsto (0, a) + J$. Thus, $f_0(D) = (D \oplus B)/J \subseteq C$.

We now show that $f_0(D)$ is a pure subgroup of $C$. If $(d, a) + J \in C$ and $n \in \mathbb{N}$ are such that $n(d, a) + J \in f_0(D)$, then we have that $na \in B$ and hence $a \in B$ since $B \preceq \mathcal{J} A$. This implies that $(d, a) + J \in f_0(D)$.

We have that $f_0$ is injective. Indeed, if $d \in D$ is such that $f_0(d) = 0$, then $f_0(d) = 0$ for some $b \in B$. Hence, $b = 0$ and $d = -g(b) = 0$.

It remains to prove that $f_0(D)$ is $\mathcal{J}$-pure in $C$. Suppose that $f_0(D) \subseteq S \subseteq C$ and $S/f_0(D)$ is finite-rank. Then we have that $S = \hat{S}/J$ for some subgroup $\hat{S} \subseteq D \oplus A$. Then we have that

$$S/f_0(D) \cong \hat{S}/B$$

has finite rank, where

$$\hat{B} = D \oplus B \subseteq D \oplus A.$$ 

If $\pi: D \oplus A \to A$ is the second-coordinate projection, then $\pi$ induces an epimorphism

$$\hat{S}/\hat{D} \to \pi(\hat{S})/B \subseteq A/B.$$ 

Thus, we have that $\pi(\hat{S})/B$ has finite rank. Since $B$ is $\mathcal{J}$-pure in $A$, we have that $\pi(\hat{S}) = B \oplus R$ for some subgroup $R$ of $A$. Hence, we have that $\hat{S} \subseteq \hat{B} \oplus \hat{R}$ where $\hat{R} = D \oplus R$. Hence, we have that $\hat{S} = \hat{B} \oplus L$ for some subgroup $L$ of $D \oplus A$. Since

$$J \subseteq \hat{B} = D \oplus B$$

we have that

$$S = \hat{S}/J = \frac{\hat{B}}{J} \oplus L = f_0(D) \oplus L.$$ 

This concludes the proof that $f_0(D)$ is a direct summand of $S$, and $f_0(D) \preceq \mathcal{J} C$.

Suppose now that $D$ is torsion-free. We prove that $C$ is torsion-free. Indeed, if $(d, a) \in D \oplus A$ and $n \in \mathbb{N}$ are such that $(nd, na) \in J$ for some $b \in B$, then we have that $(nd, na) = (-g(b), b)$ for some $b \in B$. Since $B \preceq \mathcal{J} A$, this implies that $a \in B$ and hence $nd = -g(b) = -g(na) = -ng(a)$. Since $D$ is assumed to be torsion-free, this implies $d = -g(a)$ and hence $(d, a) \in J$.

If furthermore $i$ and $g$ are arrows in $A$, we have that $D$ and $A$ are torsion-free, and hence $D \oplus A$ is torsion-free and $C$ is torsion-free by the above. Thus, $f_0, f_1$ give the pushout of $i, g$ in $A$. $\square$
Proposition 5.5. The collection $E_3$ of 3-pure exact sequences of torsion-free groups forms an exact structure on the category $\mathcal{A}$ of torsion-free abelian groups.

Proof. It is clear that identity morphisms in $\mathcal{A}$ are 3-inflations and 3-deflations. By Lemma 5.1, the composition of 3-inflations is an 3-inflation. By Lemma 5.2, the composition of 3-deflations is an 3-deflation. By Lemma 5.4 the pushout of an 3-inflation by an arbitrary arrow in $\mathcal{A}$ is an 3-inflation. By Lemma 5.3, the pullback of an 3-deflation by an arbitrary arrow in $\mathcal{A}$ is an 3-deflation. □

Lemma 5.6. Suppose that $E : A \to X \xrightarrow{\beta} C$ is an 3-pure extension of $C$ by $A$. Let $\gamma : C' \to C$ be an arrow in $\mathcal{A}$ and $\alpha : A \to A'$ be an arrow in $\mathcal{G}$. Consider the extension $E\gamma : A \to Y \to C'$ defined as in [Fuc73, Section IX.50], obtained by letting

$$Y = \{(b, c') \in B \oplus C' : p(b) = \gamma(c')\}$$

be the pushout of $p$ and $\gamma$ and $A \to Y$ be the homomorphism $a \mapsto (a, 0)$. Consider also the extension $\alpha E : A' \to Z \to C$ defined as in [Fuc73, Section IX.50], obtained by letting

$$Z = \frac{A' \oplus X}{\{(\alpha(a), -a) : a \in A\}}$$

be the pushout of $\alpha$ and the inclusion $A \to X$, and $Z \to C$ be the homomorphism $[(a', x)] \mapsto p(x)$. Then $E\gamma$ and $\alpha E$ are 3-pure.

Proof. Since the pullback of an 3-deflation by an arbitrary arrow in $\mathcal{A}$ is an 3-deflation by Lemma 5.3, we have that $E\gamma$ is 3-pure. Similarly, since the pushout of an 3-inflation by an arbitrary arrow in $\mathcal{G}$ is an 3-inflation by Lemma 5.4, we have that $\alpha E$ is 3-pure. □

Corollary 5.7. The Baer sum of 3-pure extensions is 3-pure.

Proof. If $E, E'$ are 3-pure extensions of $C$ by $A$ and, respectively, of $C'$ by $A'$, then their direct sum $E \oplus E'$ is easily seen to be an 3-pure extension of $C \oplus C'$ by $A \oplus A'$. Adopting the notation from Lemma 5.6, we have that the Baer sum $E + E'$ of $E$ and $E'$ is $\nabla_C (E \oplus E') \Delta_A$, where $\Delta_A : A \to A \oplus A$, $a \mapsto (a, a)$ is the diagonal map and $\nabla_C : C \oplus C \to C$, $(x, y) \mapsto x - y$. It follows from Lemma 5.6 that $E + E'$ is 3-pure. □

For groups $C, A$, where $C$ is torsion-free, one defines $\text{PExt}^3 (C, A)$ to be the subgroup of $\text{Ext} (C, A)$ consisting of equivalence classes of 3-pure extensions of $C$ by $A$. Notice that $\text{PExt}^3 (C, A)$ is indeed a subgroup of $\text{Ext} (C, A)$ by Lemma 5.7. By Lemma 5.6 this defines a functor $\mathcal{A}^{op} \times \mathcal{G} \to \mathcal{G}$, which restricts to a bifunctor on $\mathcal{A}$ with values in $\mathcal{G}$ that is contravariant in the first variable and covariant in the second variable.

5.4. Finite-rank-projective groups. Let as above $\mathcal{I}$ be the class of finite-rank torsion-free groups.

Definition 5.8. A torsion-free group is finite-rank-decomposable if it is isomorphic to a direct sum of finite-rank torsion-free groups.

The proof of the following lemma is the same as [Fuc73, Theorem 87.1].

Lemma 5.9. A countable torsion-free group $A$ is finite-rank-decomposable if and only if every finite subset of $A$ is contained in a finite-rank direct summand of $A$.

Proof. Suppose that $A$ is a countable torsion-free group such that every finite subset $F$ of $A$ is contained in a finite-rank direct summand of $A$. Let $(a_n)$ be an enumeration of $A$. By assumption, there exists a finite-rank direct summand $A_1$ of $A$ containing $a_1$. Suppose that we have defined a chain $A_1 \subseteq \cdots \subseteq A_{n-1}$ of finite-rank direct summands of $A$ such that $\{a_1, \ldots, a_i\} \subseteq A_i$ for $i < n$. Then by assumption there exists a finite-rank direct summand $A_n$ of $A$ that contains $a_n$ and a maximal independent set in $A_{n-1}$. Since $A_n$ is a direct summand of $A$, this implies that $A_{n-1} \subseteq A_n$. This procedure defines an increasing sequence $(A_n)$ of finite-rank direct summands of $A$ with union equal to $A$. For every $n \in \omega$, one has that $A_{n+1} = A_n \oplus B_{n+1}$, where $B_{n+1}$ has finite rank. Setting $B_0 = A_0$, one has that $A = \bigoplus_{n \in \omega} B_n$, and hence $A$ is finite-rank-decomposable. As the converse implication is obvious, this concludes the proof. □
Proposition 5.10. Suppose that $C$ is a torsion-free group. Then there exist a finite-rank-decomposable torsion-free group $G$ and an $\mathcal{I}$-deflation $\pi : G \to C$. If $C$ is countable, then one can choose $G$ to be countable. If $C$ is an $R$-module for some subring $R \subseteq \mathbb{Q}$, then one can take $G$ to be an $R$-module.

Proof. Select injective homomorphisms $\phi_i : L_i \to C$ for $i \in I$, where $L_i$ is a finite-rank torsion-free abelian group, such that $\{\phi_i(L_i) : i \in I\}$ is the set of all finite-rank pure subgroups of $C$. Define then $G = \bigoplus_{i \in I} L_i$ and let $\phi : G \to C$ be the homomorphism induced by the family $(\phi_i)_{i \in I}$ via the universal property of the direct sum. We identify $L_i$ with a subgroup of $G$ for every $i \in I$. If $F \subseteq C$ is a finite-rank subgroup, then there exists $i \in I$ such that $F \subseteq \phi_i(L_i)$ and hence the homomorphism $F \to G, x \mapsto \phi_i^{-1}(x)$ witnesses that $\phi : G \to C$ is an $\mathcal{I}$-deflation.

Notice that if $B$ is a finite-rank pure subgroup of $C$ and $a_1, \ldots, a_n$ is a maximal independent set in $B$, then $B = \langle a_1, \ldots, a_n \rangle$. This shows that if $C$ is countable, then it has countably many finite-rank pure subgroups, and hence $G$ is countable. If $C$ is an $R$-module, then $L_i$ is an $R$-submodule of $C$ for every $i \in I$ and hence $G$ is an $R$-module. □

Recall that $\mathcal{A}$ denotes the category of torsion-free abelian groups, and $\mathcal{E}_\mathcal{I}$ is the collection of $\mathcal{I}$-pure short-exact sequences in $\mathcal{A}$. By Proposition 5.5, $\mathcal{E}_\mathcal{I}$ is an exact structure on $\mathcal{A}$. By definition, a torsion-free group $G$ is projective in $(\mathcal{A}, \mathcal{E}_\mathcal{I})$ if and only if for every $\mathcal{I}$-pure extension $B \to A \to C$ of torsion-free groups, one has that the canonical homomorphism $\text{Hom}(G, A) \to \text{Hom}(G, C)$ is surjective or, equivalently, the canonical homomorphism $\text{Ext}(G, B) \to \text{Ext}(G, A)$ is injective.

Lemma 5.11. Suppose that $G$ is a finite-rank-decomposable torsion-free group. Then for every $\mathcal{I}$-pure extension $B \to A \to C$, the canonical homomorphism $\text{Hom}(G, A) \to \text{Hom}(G, C)$ is surjective.

Proof. It suffices to consider the case when $G$ has finite rank. Suppose that $B \to A \to C$ is an $\mathcal{I}$-pure extension of a torsion-free group $C$ by a group $B$. Suppose that $\phi : G \to C$ is a homomorphism. Then $\phi(G) \subseteq C$ is a finite-rank subgroup of $C$. Thus, there exists a homomorphism $\rho : \phi(G) \to A$ such that $\pi \rho = \text{id}_{\phi(G)}$. Thus, $\phi := \rho \phi : G \to A$ is a homomorphism such that $\pi \phi = \phi$. This concludes the proof. □

The following proposition characterizes the torsion-free groups that are projective in $(\mathcal{A}, \mathcal{E}_\mathcal{I})$.

Proposition 5.12. Suppose that $C$ is a torsion-free group. The following assertions are equivalent:

1. For every $\mathcal{I}$-pure extension $B \to X \to A$, the canonical homomorphism $\text{Hom}(C, X) \to \text{Hom}(C, A)$ is surjective;
2. $\text{PExt}^3(C, A) = 0$ for every group $A$;
3. $C$ is projective in $(\mathcal{A}, \mathcal{E}_\mathcal{I})$;
4. $\text{PExt}^3(C, A) = 0$ for every torsion-free group $A$;
5. $C$ is a direct summand of a finite-rank-decomposable torsion-free group.

Proof. (1)$\Rightarrow$(2): If $A \to X \to C$ is an $\mathcal{I}$-pure extension of $C$ by $A$, then by assumption the identity map $C \to C$ lifts to a homomorphism $C \to X$. Thus, the extension $A \to X \to C$ splits. This shows that $\text{PExt}^3(C, A) = 0$. The same proof gives (3)$\Rightarrow$(4).

(1)$\Rightarrow$(3) and (2)$\Rightarrow$(4): Obvious.

(4)$\Rightarrow$(5): By Proposition 5.10 there exists an $\mathcal{I}$-pure extension $A \to G \to C$ of $C$ such that $G$ is a finite-rank-decomposable torsion-free group. By assumption, $\text{PExt}^3(C, A) = 0$. Thus, the short exact sequence $A \to G \to C$ splits and $C$ is a direct summand of $G$.

(5)$\Rightarrow$(1): By Lemma 5.11 every finite-rank-decomposable torsion-free group satisfies (1). Thus, the same holds for every direct summand of a finite-rank-decomposable torsion-free group. □

5.5. An exact sequence for $\text{PExt}^3$. The following proposition is the analogue of [Fuc70, Theorem 51.3 and Theorem 53.7] for $\text{PExt}^3$.

Proposition 5.13. Let $\mathcal{I}$ be the class of finite-rank torsion-free groups. Suppose that $A \to B \to C$ is an $\mathcal{I}$-pure short exact sequence of groups. Let also $G$ be a group. Then:

1. If $A$, $B$ (and, hence, $C$) are torsion-free, then the natural exact sequence $0 \to \text{Hom}(C, G) \to \text{Hom}(B, G) \to \text{Hom}(A, G) \to \text{Ext}(C, G) \to \text{Ext}(B, G) \to \text{Ext}(A, G)$
restricts to a natural exact sequence

$$0 \rightarrow \text{Hom}(C, G) \rightarrow \text{Hom}(B, G) \rightarrow \text{Hom}(A, G) \rightarrow \text{PExt}^3(C, G) \rightarrow \text{PExt}^3(B, G) \rightarrow \text{PExt}^3(A, G);$$

(2) If $G$ is torsion-free, then the natural exact sequence

$$0 \rightarrow \text{Hom}(G, A) \rightarrow \text{Hom}(G, B) \rightarrow \text{Hom}(G, C) \rightarrow \text{Ext}(G, A) \rightarrow \text{Ext}(G, B) \rightarrow \text{Ext}(G, C)$$

restricts to a natural exact sequence

$$0 \rightarrow \text{Hom}(G, A) \rightarrow \text{Hom}(G, B) \rightarrow \text{Hom}(G, C) \rightarrow \text{PExt}^3(G, A) \rightarrow \text{PExt}^3(G, B) \rightarrow \text{PExt}^3(G, C).$$

Proof. Suppose that $t : C \rightarrow B$ is a right inverse for the quotient map $\pi : B \rightarrow C$.

(1) First we show that the homomorphism $\text{Hom}(A, G) \rightarrow \text{Ext}(C, G)$ has image contained in $\text{PExt}^3(C, G)$. Suppose that $\varphi \in \text{Hom}(A, G)$, then $\varphi$ is mapped to the element of $\text{Ext}(C, G)$ represented by the cocycle $c : (x, y) \mapsto \varphi(t(x + y) - t(x) - t(y))$. If $F \subseteq C$ is a finite-rank subgroup then one can choose a right inverse $t : C \rightarrow B$ such that $t|_{F}$ is a homomorphism, and hence $c|_{F \times F} = 0$. This shows that $c$ defines an element of $\text{PExt}^3(C, G)$.

By functoriality of $\text{PExt}^3(-, G)$, the homomorphism $\text{Ext}(C, G) \rightarrow \text{Ext}(B, G)$ maps $\text{PExt}^3(C, G)$ to $\text{PExt}^3(B, G)$, and the homomorphism $\text{Ext}(B, G) \rightarrow \text{Ext}(A, G)$ maps $\text{PExt}^3(B, G)$ to $\text{PExt}^3(A, G)$.

We now prove that the kernel of $\text{PExt}^3(B, G) \rightarrow \text{PExt}^3(A, G)$ is contained in the image of $\text{PExt}^3(C, G) \rightarrow \text{PExt}^3(B, G)$. We have that such a kernel is the intersection of $\text{PExt}^3(B, G)$ with the image of $\text{Ext}(C, G) \rightarrow \text{Ext}(B, G)$. Let $c : B \times B \rightarrow G$ be a cocycle representing an element of this set. Then there exist a $G$-valued cocycle $\tilde{c}$ on $C$ and a function $f : B \rightarrow G$ such that

$$c(x, y) = f(x + y) - f(x) - f(y) + \tilde{c}(\pi(x), \pi(y))$$

for $x, y \in B$. In particular, we have that

$$c(x, y) = f(x + y) - f(x) - f(y)$$

for $x, y \in A$. Furthermore, we have that $c|_{L \times L}$ is a coboundary for every finite-rank subgroup $L$ of $B$. It remains to prove that $\tilde{c}$ represents an element of $\text{PExt}^3(C, G)$. Suppose that $L$ is a finite-rank subgroup of $C$. Then there exists a homomorphism $\rho : L \rightarrow B$ such that $\pi \rho(x) = x$ for every $x \in L$. The image $\rho(L) \subseteq B$ is a finite-rank subgroup of $B$. Thus, $c|_{\rho(L) \times \rho(L)}$ is a coboundary. Hence, there exists a function $g : \rho(L) \rightarrow G$ such that $c(x, y) = g(x) + g(y) - g(x + y)$ for $x, y \in \rho(L)$. Thus, for $x, y \in L$ we have that

$$\tilde{c}(x, y) = \tilde{c}(\pi \rho(x), \pi \rho(y)) = c(\rho(x), \rho(y)) - f(x + y) + f(x) + f(y) = g(\rho(x)) + g(\rho(y)) - g(\rho(x + y)) - f(x + y) + f(x) + f(y) = h(x) + h(y) - h(x + y)$$

where $h = g \circ \rho + f : L \rightarrow G$. This shows that $\tilde{c}|_{L \times L}$ is a coboundary.

(2) We begin with showing that the homomorphism $\text{Hom}(G, C) \rightarrow \text{Ext}(G, A)$ has image contained in $\text{PExt}^3(G, A)$. Suppose that $\varphi \in \text{Hom}(G, C)$ is a homomorphism. Then the image of $\varphi$ is the element of $\text{Ext}(G, A)$ represented by the cocycle on $G$ with coefficients in $A$ defined by $c(x, y) = (t \circ \varphi)(x + y) - (t \circ \varphi)(x) - (t \circ \varphi)(y)$. If $F \subseteq G$ is a finite-rank subgroup, then $\varphi|F$ is a finite-rank subgroup of $C$, whence one can choose the $t : C \rightarrow B$ of $\varphi$ such that $t|_{\varphi(F)}$ is a homomorphism. In this case, $c|_{F \times F} = 0$. This shows that $c$ represents an element of $\text{PExt}^3(G, A)$.

By functoriality of $\text{PExt}^3(G, -)$, the homomorphism $\text{Ext}(G, A) \rightarrow \text{Ext}(G, B)$ maps $\text{PExt}^3(G, A)$ to $\text{PExt}^3(G, B)$, and the homomorphism $\text{Ext}(G, B) \rightarrow \text{Ext}(G, A)$ maps $\text{PExt}^3(G, B)$ to $\text{PExt}^3(G, A)$.

It remains to prove that the kernel of $\text{PExt}^3(G, B) \rightarrow \text{PExt}^3(G, C)$ is equal to the image of $\text{PExt}^3(G, A) \rightarrow \text{PExt}^3(G, B)$. We have that the kernel of $\text{PExt}^3(G, B) \rightarrow \text{PExt}^3(G, C)$ is the intersection between $\text{PExt}^3(G, B)$ and the image of $\text{Ext}(G, A) \rightarrow \text{Ext}(G, B)$. An element of this set is represented by a cocycle $c : G^2 \rightarrow B$ such that there exist a function $f : G \rightarrow B$ and a cocycle $c_0 : G^2 \rightarrow A$ satisfying

$$c(x, y) = c_0(x, y) + f(x) + f(y) - f(x + y)$$
for \( x, y \in G \), where we identify \( A \) with a subgroup of \( B \), and \( c|_{L \times L} \) is a coboundary for every finite-rank subgroup \( L \) of \( G \). This clearly implies that \( c_0|_{L \times L} \) is a coboundary for every finite-rank subgroup \( L \) of \( G \), and hence \( c \) represents an element of \( \text{PExt}^3 (G, A) \), concluding the proof.

The same proof as Theorem 4.1 gives the following description of \( \text{PExt}^3 (C, A) \) in terms of \( \text{lim}^1 \).

**Theorem 5.14.** Suppose \( C, A \) are countable groups, and \( \{ C_n \}_{n \in \omega} \) is an increasing sequence of finite-rank subgroups of \( C \) with union equal to \( C \). Then \( \text{PExt}^3 (C, A) \) and \( \text{lim}^1 \text{Hom} (C_n, A) \) are Borel-definably isomorphic.

6. Extensions of torsion-free groups by torsion-free groups

In this section we study extensions of torsion-free groups by torsion-free groups. We will characterize the first Solecki subgroup of \( \text{Ext} (C, A) \) when \( C \) is torsion-free using finite-rank-pure extensions. Using this characterization, we will prove (1) of Theorem A.

6.1. Subrings of \( \mathbb{Q} \). Suppose that \( R \) is a subring of \( \mathbb{Q} \). Then we have that there exists a set \( S \) of prime numbers such that \( R \) is the localization \( R_S \) of \( \mathbb{Z} \) with respect to the multiplicative system generated by \( S \) [AM69, Chapter 3]. In particular, we have that \( R \) is a principal ideal domain (PID), being the localization of a PID [AM69, Proposition 3.11]. Recall that if \( R \) is a PID, \( M \) is a free \( R \)-module, and \( N \subseteq M \) is an \( R \)-submodule, then \( N \) is a free \( R \)-module and \( \text{rank}_R (N) \leq \text{rank}_R (M) \) [Rot09, Corollary 4.14]. Furthermore, torsion-free finitely-generated \( R \)-modules are free [Rot09, Corollary 4.16, Theorem 4.34].

Following [War72], for a torsion-free abelian group \( A \), we let \( \pi (A) \) be the set of prime numbers \( q \) such that \( A \) is \( q \)-divisible, and \( R (A) \) be the subring \( R_{\pi (A)} \) of \( \mathbb{Q} \) generated by \( \{ 1/p : p \in \pi (A) \} \). For a torsion group \( T \), we let \( \tau (T) \) be the set of prime numbers \( q \) such that the \( q \)-primary component of \( T \) is nonzero. Notice that a torsion-free group \( C \) such that \( \pi (A) \subseteq \pi (C) \) can be seen as an \( R (A) \)-module.

6.2. The reduction to \( R (A) \)-modules. Suppose that \( A, C \) are countable torsion-free groups. We observe in the next lemma that, in the study of \( \text{Ext} (C, A) \), one can assume without loss of generality that \( C \) is an \( R (A) \)-module; see [FS13, Lemma 2.6].

**Lemma 6.1.** Suppose that \( A, C \) are countable torsion-free groups. Then \( \text{Ext} (C \otimes R (A), A) \) and \( \text{Ext} (C, A) \) are Borel-definably isomorphic.

**Proof.** Consider the \( R (A) \)-module \( C \otimes R (A) \). Then \( C \subseteq C \otimes R (A) \) and the quotient group

\[
T := \frac{C \otimes R (A)}{C}
\]

is a torsion group such that

\[
\tau (T) \subseteq \pi (A).
\]

Thus, one can write

\[
T = \bigoplus_{p \in \pi (A)} T_p,
\]

where \( T_p \) is the \( p \)-primary component of \( T \). For \( p \in \pi (A) \), since \( T_p \) is a \( p \)-group, and \( A \) is \( p \)-divisible, we have that

\[
\text{Ext} (T_p, A) = 0
\]

by Lemma 4.2. Thus

\[
\text{Ext} (T, A) = \prod_{p \in \pi (A)} \text{Ext} (T_p, A) = 0.
\]

Considering the Borel-definable exact sequence associated with the short exact sequence \( C \to C \otimes R (A) \to T \), we have

\[
0 = \text{Ext} (T, A) \to \text{Ext} (C \otimes R (A), A) \to \text{Ext} (C, A) \to 0.
\]

Thus, \( \text{Ext} (C, A) \) and \( \text{Ext} (C \otimes R (A), A) \) are Borel-definably isomorphic. \( \square \)
6.3. Vanishing of Ext. Fix a countable torsion-free group $A$. We now recall the characterization of the countable torsion-free groups $C$ such that $\text{Ext}(C, A) = 0$ from [FS13]; see also [War72].

**Proposition 6.2.** Suppose that $A, C$ are countable torsion-free abelian groups. Then $\text{Ext}(C, A) = 0$ if and only if $C \otimes R(A)$ is a free $R(A)$-module.

**Proof.** By Lemma 6.1, after replacing $C$ with $C \otimes R(A)$, we can assume that $C$ is an $R(A)$-module.

Suppose that $C$ is a free $R(A)$-module. We claim that $\text{Ext}(C, A) = 0$. Since $C$ is a free $R(A)$-module, we can assume without loss of generality that $C = R(A)$. Notice that we have a short exact sequence $\mathbb{Z} \to R(A) \to T$ where $T$ is a torsion-free group with $\tau(T) \subseteq \pi(A)$. Thus, by Lemma 4.2, as in the proof of Lemma 6.1, we have that $\text{Ext}(T, A) = 0$. We also have that $\text{Ext}(\mathbb{Z}, A) = 0$ since $\mathbb{Z}$ is a free group. Thus, considering the exact sequence

$$0 \to \text{Ext}(T, A) \to \text{Ext}(R(A), A) \to \text{Ext}(\mathbb{Z}, A) \to 0,$$

we also have that $\text{Ext}(R(A), A) = 0$.

Conversely, suppose that $\text{Ext}(C, A) = 0$. We will show that $C$ is a free $R(A)$-module. By Pontryagin’s theorem, it suffices to prove that every finite-rank $R(A)$-submodule $C_0 \subseteq C$ is free; see [Fuc70, Theorem 19.1] for a proof in the case of $\mathbb{Z}$-modules.

If $C_0$ is an $R(A)$-submodule of $C$, then we have an epimorphism $\text{Ext}(C, A) \to \text{Ext}(C_0, A)$ and hence $\text{Ext}(C_0, A) = 0$. Hence, after replacing $C$ with $C_0$, we can assume that $C$ is a finite-rank $R(A)$-module. Let $E \subseteq C$ be a free $R(A)$-submodule of full rank. Thus, $T := C/E$ is a torsion group such that $\tau(T) \cap \pi(A) = \emptyset$. We claim that $T$ is finite. Write

$$T = \bigoplus_{p \in \tau(T)} T_p,$$

whence

$$\text{Ext}(T, A) = \prod_{p \in \tau(T)} \text{Ext}(T_p, A).$$

The short exact sequence $E \to C \to T$ induces an exact sequence

$$\text{Hom}(E, A) \to \text{Ext}(T, A) \to \text{Ext}(C, A) \to \text{Ext}(E, A) = 0.$$

By assumption, $\text{Ext}(C, A) = 0$. Since $E$ has finite rank, $\text{Hom}(E, A)$ is countable. Thus, $\text{Ext}(T, A)$ is countable. As $\text{Ext}(T_p, A) \neq 0$ for $p \in \tau(T)$ by Lemma 4.6, we have that $\tau(T)$ is finite. Furthermore, for every $p \in \tau(T)$, $\text{Ext}(T_p, A)$ is countable, and hence $T_p$ is finite by Lemma 4.6. Therefore, $T$ is finite. Hence, $C$ is a torsion-free finitely-generated $R(A)$-module, and hence a free $R(A)$-module. This concludes the proof. \qed

6.4. The finite-rank case. A similar proof as Proposition 6.2 allows one to compute the complexity of $\{0\}$ in $\text{Ext}(C, A)$ when $C$ has finite rank.

**Lemma 6.3.** Suppose that $A, C$ are countable torsion-free groups such that $C$ is finite-rank. If $\text{Ext}(C, A) \neq 0$, then $\Sigma^0_2$ is the complexity class of $\{0\}$ in $\text{Ext}(C, A)$.

**Proof.** Since $C$ is torsion-free, we have that $\text{Ext}(C, A) = \text{PExt}(C, A)$, and $\{0\}$ is dense in $\text{Ext}(C, A)$. Thus, it suffices to prove that $\{0\}$ is $\Sigma^0_2$ in $\text{Ext}(C, A)$. By Lemma 6.1, after replacing $C$ with $C \otimes R(A)$, we can assume that $C$ is an $R(A)$-module. Let $E \subseteq C$ be a free $R(A)$-submodule of full rank. Then $T := C/E$ is a torsion group. The short exact sequence $E \to C \to T$ induces a Borel-definable exact sequence

$$\text{Hom}(E, A) \to \text{Ext}(T, A) \to \text{Ext}(C, A) \to \text{Ext}(E, A) = 0,$$

where $\text{Ext}(T, A)$ is a Polish group by Lemma 4.3. Thus, $\text{Ext}(C, A)$ is definably isomorphic to

$$\text{Ran}(\text{Hom}(E, A) \to \text{Ext}(T, A)).$$

Since $E$ is has finite rank, $\text{Hom}(E, A)$ is countable, and hence $\text{Ran}(\text{Hom}(E, A) \to \text{Ext}(T, A))$ is $\Sigma^0_2$ in $\text{Ext}(T, A)$. Thus, $\{0\}$ is $\Sigma^0_2$ in $\text{Ext}(C, A)$.

Alternatively, one can let $D$ be the divisible hull of $A$, whence $D$ is a divisible torsion-free group, and $D/A$ is a divisible torsion group. Consider then the Borel-definable exact sequence

$$0 \to \text{Hom}(C, A) \to \text{Hom}(C, D) \to \text{Hom}(C, D/A) \to \text{Ext}(C, A) \to 0.$$
Thus, we have that $\text{Ext}(C, A)$ is definably isomorphic to

$$\frac{\text{Hom}(C, D/A)}{\text{Ran}(\text{Hom}(C, D) \to \text{Hom}(C, D/A))}.$$ 

Notice that $\text{Hom}(C, D)$ is countable since $C$ has finite-rank.

**Proposition 6.4.** Suppose that $A, C$ are countable torsion-free groups. The following assertions are equivalent:

1. $C \otimes R(A)$ is the sum of a free $R(A)$-module and a finite-rank $R(A)$-module;
2. $\{0\}$ is $\Sigma^0_2$ in $\text{Ext}(C, A)$.

**Proof.**

(1)⇒(2) By Lemma 6.1, we can assume without loss of generality that $C$ is an $R(A)$-module. Then we have that $C = C_0 \oplus C_1$ where $C_0$ is a free $R(A)$-module and $C_1$ is a finite-rank $R(A)$-module. Thus, we have Borel-definable isomorphisms

$$\text{Ext}(C, A) \cong \text{Ext}(C_0, A) \oplus \text{Ext}(C_1, A) \cong \text{Ext}(C_1, A)$$

by Proposition 6.2. The conclusion thus follows from Lemma 6.3.

(2)⇒(1) By Lemma 6.1, after replacing $C$ with $C \otimes R(A)$, we can assume that $C$ is an $R(A)$-module. Furthermore, we can assume without loss of generality that $C$ has infinite rank, otherwise there is nothing to prove. Let $(e_n)_{n \in \omega}$ be a maximal independent set in $C$, and let $E$ be the subgroup $\langle e_n : n \in \omega \rangle \subseteq C$.

For $n \in \omega$, let $E_n$ be the subgroup $\langle e_k : k \leq n \rangle \subseteq E$, and $E_{>n}$ be the subgroup $\langle e_k : k > n \rangle \subseteq E$. Let also $C_n$ be the pure subgroup of $C$ generated by $E_n$, and $C_{>n} := C/C_n$. Thus, for every $n \in \mathbb{N}$, we have a diagram with exact rows and columns:

$$\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
E_n & C_n & C_n/E_n \\
\downarrow & \downarrow & \downarrow \\
E & C & C/E \\
\downarrow & \downarrow & \downarrow \\
E_{>n} & C_{>n} & C_{>n}/E_{>n} \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}$$

This induces a diagram in the category of groups with a Polish cover with exact rows and columns:

$$\begin{array}{ccc}
0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
\text{Hom}(E_n, A) & \text{Ext}(C_n/E_n, A) & \text{Ext}(C_n, A) \\
\uparrow & \uparrow & \uparrow \\
\text{Hom}(E, A) & \text{Ext}(C/E, A) & \text{Ext}(C, A) \\
\sigma_n & \tau_n & \rho_n \\
\uparrow & \uparrow & \uparrow \\
\text{Hom}(E_{>n}, A) & \text{Ext}(C_{>n}/E_{>n}, A) & \text{Ext}(C_{>n}, A) \\
\uparrow & \uparrow & \uparrow \\
0 & 0 & 0
\end{array}$$

Notice that $\text{Ext}(C/E, A)$ is a Polish group by Lemma 4.3. Thus, as $\text{Hom}(E, A)$ is also a Polish group, the Borel-definable homomorphism $\delta : \text{Hom}(E, A) \to \text{Ext}(C/E, A)$ is a continuous homomorphism of Polish groups. Since by assumption $\{0\}$ is $\Sigma^0_2$ in $\text{Ext}(C, A)$, we have that $\text{Ran}(\delta)$ is $\Sigma^0_2$ in $\text{Ext}(C/E, A)$. Thus, we can find an increasing sequence $(K_\ell)_{\ell \in \omega}$ of closed subsets of the Polish group $\text{Ext}(C/E, A)$ such that

$$\text{Ran}(\delta) = \bigcup_{\ell \in \omega} K_\ell.$$
Thus, we have that

$$\text{Hom} (E, A) = \bigcup_{\ell \in \omega} \delta^{-1} (K_\ell)$$

where $\delta^{-1} (K_\ell)$ is closed in $\text{Hom} (E, A)$ for every $\ell \in \omega$. Thus, there exists $\ell \in \omega$ such that $\delta^{-1} (K_\ell)$ contains an open subset of $\text{Hom} (E, A)$. After replacing $K_\ell$ with a translate of $K_\ell$ inside of $\text{Ext} (C/E, A)$, we can assume that $\delta^{-1} (K_\ell)$ contains an open neighborhood of 0 in $\text{Hom} (E, A)$. Therefore, there exists $n \in \omega$ such that

$$\text{Ran} (\sigma_n) \subseteq \delta^{-1} (K_\ell).$$

This implies that the epimorphism $\pi_n$ is an isomorphism. Indeed, suppose that $x \in \ker (\pi_n)$. Then we have that $x = \delta_n (y)$ for some $y \in \text{Hom} (E_{>n}, A)$. Hence, $\tau_n (x) = \tau_n \delta_n (y) = \delta \tau_n (y) = 0$. Since $\tau_n$ is a monomorphism, this implies that $x = 0$. Thus, we have that $\text{Ext} (C_{>n}, A)$ is Borel-definably isomorphic to $\text{Ext} (C_{>n}/E_{>n}, A)$. As $C_{>n}/E_{>n}$ is a torsion group and $A$ is torsion-free, this implies that $\text{Ext} (C_{>n}/E_{>n}, A)$ is a Polish group by Lemma 4.3 and $\text{PExt} (C_{>n}/E_{>n}, A) = 0$. Thus, since $C_{>n}$ is torsion-free,

$$\text{Ext} (C_{>n}, A) = \text{PExt} (C_{>n}, A) \cong \text{PExt} (C_{>n}/E_{>n}, A) = 0.$$

Therefore, by Proposition 6.2, $C_{>n}$ is a free $R(A)$-module. Thus, the short exact sequence of $R(A)$-modules $C_n \rightarrow C \rightarrow C_{>n}$ splits, and $C = C_n \oplus C_{>n}$. As $C_n$ has finite rank, this concludes the proof.

6.5. The first Solecki subgroup and 3-pure extensions. Suppose that $C$ is a countable torsion-free group and $A$ is a countable group. Let $\mathcal{B}$ be the class of finite-rank torsion-free groups. Notice that, by definition, $\text{PExt}^3 (C, A)$ is the subgroup of $\text{Ext} (C, A)$ obtained as the intersection of the kernels of the Borel-definable homomorphisms $\text{Ext} (C, A) \rightarrow \text{Ext} (B, A)$ where $B$ varies among the finite-rank (pure) subgroups of $C$.

**Lemma 6.5.** Suppose that $C, A$ are countable groups, with $C$ torsion-free. Then $\{0\}$ is dense in $\text{PExt}^3 (C, A)$.

**Proof.** It suffices to prove that $\{0\}$ is dense in $\ker (\text{Ext} (C, A) \rightarrow \text{Ext} (B, A))$ for every finite-rank pure subgroup $B$ of $C$. Let $B$ be a finite-rank subgroup of $C$. Then we have a Borel-definable exact sequence

$$\text{Hom} (B, A) \rightarrow \text{Ext} (C/B, A) \rightarrow \text{Ext} (C, A) \rightarrow \text{Ext} (B, A) \rightarrow 0$$

Thus, we have a Borel-definable isomorphism

$$\ker (\text{Ext} (C, A) \rightarrow \text{Ext} (B, A)) \cong \frac{\text{Ext} (C/B, A)}{\text{ran} (\text{Hom} (B, A) \rightarrow \text{Ext} (C/B, A))}.$$

Thus, it suffices to prove that $\text{ran} (\text{Hom} (B, A) \rightarrow \text{Ext} (C/B, A))$ is dense in $\text{Ext} (C/B, A)$. Since $B$ is pure in $C$, $C/B$ is torsion-free and hence $\text{Ext} (C/B, A) = \text{PExt} (C/B, A)$. Thus, $\{0\}$ is dense in $\text{Ext} (C/B, A)$, and a fortiori $\text{ran} (\text{Hom} (B, A) \rightarrow \text{Ext} (C/B, A))$ is dense in $\text{Ext} (C/B, A)$.

**Theorem 6.6.** Suppose that $A, C$ are countable groups, with $C$ torsion-free. Let $\mathcal{B}$ be the class of finite-rank torsion-free groups. Then $\text{PExt}^3 (C, A)$ is equal to the first Solecki subgroup of $\text{Ext} (C, A)$.

**Proof.** By Proposition 8.3, $\{0\}$ is a $\Sigma^0_2$ subgroup of $\text{Ext} (B, A)$ for every finite-rank subgroup $B$ of $C$. Since $\text{PExt}^3 (C, A)$ is the intersection of the kernels of the Borel-definable homomorphisms $\text{Ext} (C, A) \rightarrow \text{Ext} (B, A)$ where $B$ varies among the finite-rank pure subgroups of $C$, it follows that $\text{PExt}^3 (C, A)$ is a $\Pi^0_3$ subgroup of $\text{Ext} (C, A)$.

By in Proposition 5.10 there exists an 3-pure short exact sequence $R \rightarrow F \rightarrow C$ such that $F$ is finite-rank-decomposable. Thus, we have a natural Borel-definable isomorphism

$$\text{Ext} (C, A) \cong \frac{\text{Hom} (R, A)}{\text{Hom} (F|R, A)},$$

where

$$\text{Hom} (F|R, A) = \text{ran} (\text{Hom} (F, A) \rightarrow \text{Hom} (R, A)).$$

For a subgroup $B$ of $C$, define $F_B = \{x \in F : x + R \in B\}$. Under such an isomorphism $\text{PExt}^3 (C, A)$ corresponds to

$$\frac{\text{Hom}^3 (R, A)}{\text{Hom} (F|R, A)}.$$
where $\text{Hom}^3(R, A)$ is the subgroup of homomorphisms $R \to A$ that have an extension $F_B \to A$ for every finite-rank (pure) subgroup $B$ of $C$.

By Lemma 3.1 and Lemma 6.5, it suffices to prove that if $V \subseteq \text{Hom}(F|R, A)$ is an open neighborhood of $0$ in $\text{Hom}(F|R, A)$, then the closure $\overline{V}^{\text{Hom}(R, A)}$ of $V$ inside $\text{Hom}(R, A)$ contains an open neighborhood of $0$ in $\text{Hom}^3(R, A)$. Suppose that $V \subseteq \text{Hom}(F|R, A)$ is an open neighborhood of $0$. Thus, there exist $x_0, \ldots, x_n \in F$ such that

$$\{\varphi_R : \varphi \in \text{Hom}(F, A), \forall i \leq n, \varphi(x_i) = 0\} \subseteq V.$$ 

Let $B$ be a finite-rank pure subgroup of $C$ such that $\{x_1, \ldots, x_n\} \subseteq F_B$. Thus, we have that

$$U = \left\{\varphi \in \text{Hom}^3(R, A) : \exists \hat{\varphi} \in \text{Hom}(F_B, A), \forall i \leq n, \hat{\varphi}(x_i) = 0, \varphi_R = \varphi\right\}$$

is an open neighborhood of $0$ in $\text{Hom}^3(R, A)$. We claim that $U \subseteq \overline{V}^{\text{Hom}(R, A)}$. Indeed, suppose that $\varphi_0 \in U$ and let $W$ be an open neighborhood of $\varphi_0$ in $\text{Hom}(R, A)$. Thus, there exist $y_0, \ldots, y_k \in R$ such that

$$\{\varphi \in \text{Hom}(R, A) : \forall i \leq \ell, \varphi(y_i) = \varphi_0(y_i)\} \subseteq W.$$ 

We need to prove that $V \cap W \neq \emptyset$. To this purpose, it suffices to show that there exists $\varphi \in \text{Hom}(F, A)$ such that $\varphi(y_i) = \varphi_0(y_i)$ for $i \leq \ell$ and $\varphi(x_i) = 0$ for $i \leq n$. Since $\varphi_0 \in U$, there exists $\hat{\varphi}_0 \in \text{Hom}(F_B, A)$ such that $\hat{\varphi}_0(x_i) = 0$ for $i \leq n$ and $\hat{\varphi}_0|_R = \varphi_0$.

Let $(z_k)_{k \in I}$ be a free $\mathbb{Z}$-basis of $F$. The elements $x_0, \ldots, x_n, y_0, \ldots, y_k$ of $F$ are contained in the subgroup $F_0$ of $F$ generated by $\{z_i : i \in I_0\}$ for some finite $I_0 \subseteq I$. Letting $(F_0, F_B)$ be the subgroup of $F$ generated by $F_0$ and $F_B$, we have that, since $B$ is pure in $C$,

$$\langle F_0, F_B \rangle / F_B \subseteq F/F_B \cong C/B$$

is finitely-generated and torsion-free, and hence free. Therefore, $\text{Ext}(\langle F_0, F_B \rangle / F_B, A) = 0$ and there exists $\psi \in \text{Hom}(\langle F_0, F_B \rangle, A)$ that extends $\hat{\varphi}_0$. One can then define $\varphi \in \text{Hom}(F, A)$ by setting

$$\varphi(z_i) = \begin{cases} \psi(z_i) & \text{if } i \in I_0, \\ 0 & \text{if } i \in I \setminus I_0. \end{cases}$$

This concludes the proof. \hfill $\square$

**Corollary 6.7.** Suppose that $A$ is a countable torsion-free group, and $C$ is a countable $R(A)$-module. Suppose that there exists an 3-pure short exact sequence $R \to F \to C$ where $F$ is a countable free $R(A)$-module. Then $\text{Ext}(C, A) = 0$.

**Proof.** Since $\text{Ext}(F, A) = \text{PExt}^3(F, A) = 0$, the 3-pure short exact sequence $R \to F \to C$ induces a Borel-definable homomorphism $\text{Hom}(R, A) \to \text{PExt}^3(C, A) \subseteq \text{Ext}(C, A)$ which in turn induces Borel-definable isomorphisms

$$\frac{\text{Hom}(R, A)}{\text{Hom}(F|R, A)} \cong \text{Ext}(C, A)$$

and

$$\frac{\text{Hom}(R, A)}{\text{Hom}(F|F, R, A)} \cong \text{PExt}^3(C, A).$$

Thus, $\text{Ext}(C, A) = \text{PExt}^3(C, A)$ and hence $\text{Ext}(C, A) = 0$. \hfill $\square$

### 6.6. The infinite-rank case

Suppose that $R \subseteq \mathbb{Q}$ is a subring. For an $R$-module $C$, we define the radical $K^R(C)$ to be the intersection of the kernels of homomorphisms $C \to R$. Notice that $K^R(C)$ is an $R$-submodule of $C$ such that $K^R(C/K^R(C)) = 0$. Thus, $K^R(-)$ is a radical in the sense of [Cha68, FOW82].

**Lemma 6.8.** Suppose that $C$ is a countable torsion-free $R$-module. Then $K^R(C)$ is a direct summand of $C$ and $C/K^R(C)$ is a free $R$-module. Thus, $K^R(K^R(C)) = K^R(C)$.

**Proof.** It suffices to prove that every finite-rank pure $R$-submodule of $C/K^R(C)$ is free. After replacing $C$ with $C/K^R(C)$ we can assume that $K^R(C) = 0$. Suppose that $P$ is a finite-rank pure $R$-submodule of $C$ and let $F$ be a full-rank free $R$-submodule of $P$. Thus, $P$ is the pure subgroup of $C$ generated by $F$. Since $K^R(C) = 0$, there is a homomorphism $f : C \to L$, where $L$ is a free $R$-module, such that $f |_F$ is injective. It follows that $f |_P$ is injective.
Indeed, if \( x \in P \) then there exists \( m \in \mathbb{N} \) such that \( mx \in F \). If \( f(x) = 0 \) then \( f(mx) = 0 \), and hence \( mx = 0 \) and hence \( x = 0 \) since \( C \) is torsion-free. As \( L \) is a free \( R \)-module, this implies that \( P \) is a free \( R \)-module. □

Suppose that \( C \) is countable torsion-free \( R \)-module. Define \( \Phi^R (C) := C/K^R (C) \). By Lemma 6.8, \( \Phi^R (C) \) is a free \( R \)-module such that \( C \cong K^R (C) \oplus \Phi^R (C) \).

**Proposition 6.9.** Suppose that \( A, C \) are countable torsion-free groups such that \( A \) is a free \( R \)-module for some subring \( R \subseteq \mathbb{Q} \). Then \( \text{PExt}^3 (C, A) = 0 \) and \( \{0\} \) is \( \Pi^0_3 \) in \( \text{Ext} (C, A) \).

**Proof.** As \( A \) is a free \( R \)-module, \( R = R(A) \) whenever \( A \) is nonzero. Thus by Lemma 6.8, Theorem 6.6, and Proposition 6.2, we can assume without loss of generality that \( C \) is an \( R \)-module such that \( C = K^R (C) \).

By Proposition 5.10 we have an \( \mathcal{I} \)-pure short exact sequence \( B \to F \overset{\pi}{\to} C \), where \( F \) is a finite-rank-decomposable countable torsion-free \( R \)-module. We can write \( F = \bigoplus_{i \in \omega} F_i \) where, for every \( i \in \omega \), \( F_i \) is a finite-rank torsion-free \( R \)-module. For \( i \in \omega \), we have that \( F_i = K^R (F_i) \oplus \Phi^R (F_i) \), \( K^R (F) = \bigoplus_{i \in \omega} K^R (F_i) \), and \( \Phi^R (F) = \bigoplus_{i \in \omega} \Phi^R (F_i) \).

We now prove that \( B = (\Phi^R (F) \cap B) \oplus (K^R (F) \cap B) \). Suppose that \( x \in B \). Thus, we can write

\[
x = x_0 + x'_0 + \cdots + x_n + x'_n
\]

where \( x_i \in K^R (F_i) \) and \( x'_i \in \Phi^R (F_i) \) for \( i \leq n \). If \( f \in \text{Hom} (C, R) \), then \( (f \circ \pi) |_{F_i} \in \text{Hom} (F_i, R) \), and hence \( f(\pi(x'_i)) = (f \circ \pi)(x'_i) = 0 \) for \( i \leq n \). As this holds for every \( f \in \text{Hom} (C, R) \) and \( \Phi^R (C) = 0 \), this implies that \( \pi(x'_i) = 0 \) for \( i \leq n \). Thus, \( x'_i \in B \) for \( i \leq n \), and hence \( x'_0 + \cdots + x'_n \in B \) and \( x_0 + \cdots + x_n \in B \). This concludes the proof that \( B = (\Phi^R (F) \cap B) \oplus (K^R (F) \cap B) \).

Thus, we have that

\[
C = F/B = \frac{\Phi^R (F)}{\Phi^R (F) \cap B} \oplus \frac{K^R (F)}{K^R (F) \cap B}.
\]

Notice that, as \( B \to F \to C \) is an \( \mathcal{I} \)-pure short exact sequence,

\[
\Phi^R (F) \cap B \to \Phi^R (F) \to \frac{\Phi^R (F)}{\Phi^R (F) \cap B}
\]

is an \( \mathcal{I} \)-pure short exact sequence, and

\[
K^R (F) \cap B \to K^R (F) \to \frac{K^R (F)}{K^R (F) \cap B}
\]

is an \( \mathcal{I} \)-pure short exact sequence. Since \( \Phi^R (F) \) is a free \( R \)-module, we have that

\[
\text{Ext} \left( \frac{\Phi^R (F)}{\Phi^R (F) \cap B}, A \right) = 0
\]

by Corollary 6.7. Since \( A \) is a free \( R \)-module, we have that \( K^R (A) = 0 \) and

\[
\text{Hom}(K^R (F), A) = \text{Hom}(K^R (F), K^R (A)) = 0.
\]

Thus, we have Borel-definable isomorphisms

\[
\text{PExt}^3 (C, A) \cong \text{PExt}^3 \left( \frac{K^R (F)}{K^R (F) \cap B}, A \right) \cong \frac{\text{Hom}(K^R (F) \cap B, A)}{\text{Ran}(\text{Hom}(K^R (F), A) \to \text{Hom}(K^R (F) \cap B, A))} = \text{Hom}(K^R (F) \cap B, A).
\]

Since \( \{0\} \) is dense in \( \text{PExt}^3 (C, A) \) by Lemma 6.5 and \( \text{Hom}(K^R (F) \cap B, A) \) is a Polish group by the above, we have that \( \text{PExt}^3 (C, A) = 0 \). Since \( \text{PExt}^3 (C, A) = \text{Hom}(\text{Ext}(C, A)) \) by Theorem 6.6, it follows that \( \{0\} \) is \( \Pi^0_3 \) in \( \text{Ext}(C, A) \). □

Part (1) of Theorem A is an immediate consequence of Proposition 6.9, Proposition 6.4, and Proposition 6.2.
7. Extensions of torsion-free groups by $p$-groups

In this section we compute the potential complexity class of the relation of isomorphisms of extensions of a given countable torsion-free group by a countable $p$-group, thus providing a proof of Theorem B and, in particular, (3) of Theorem A. Throughout the section, we adopt the following notation: we let $K$ be a countable torsion-free group, and $T$ be a countable $p$-group. One defines $pT = \{ px : x \in T \}$ and, recursively, $p^0 T = T$ and

$$p^\lambda T = \bigcap_{\alpha < \lambda} p (p^\alpha T)$$

for $\lambda < \omega_1$. One can easily prove by induction that the $\alpha$-th Ulm subgroup $T^\alpha$ of $T$ is equal to $p^{\omega\alpha}T$.

7.1. Tight $p$-basic subgroups of torsion-free groups. Suppose that $K$ is a countable torsion-free group, and that $A \subseteq K$ is a $p$-basic subgroup [Fuc70, Section 32]. Let $(e_i)_{i \in I}$ be a free basis for $A$, whence $(e_i)_{i \in I}$ is a maximal $p$-independent family in $K$ [Fuc70, Lemma 32.2]. For $x \in K$ and $n \in \mathbb{N}$ there exist $a_k^{(i)} \in \{0, 1, \ldots, p - 1\}$ for $k < n$ such that

$$x \equiv \sum_{i \in I} a_k^{(i)} p^k e_i \mod p^n K.$$

Furthermore, the $a_k^{(i)}$s are obtained as above for $n \leq n'$ agree for every $i \in I$ and $k < n$; see [BM67, Section 2.1].

Fix a prime number $p$. We denote by $\mathbb{Z}_p$ the ring of $p$-adic integers. If $I$ is a set, we let $\mathbb{Z}_p^I$ be the direct sum of copies of $\mathbb{Z}_p$ indexed by $I$, and $\mathbb{Z}_p^I$ be the product of copies of $\mathbb{Z}_p$ indexed by $I$. The function $\sigma : K \to \mathbb{Z}_p^I$ given by

$$x \mapsto \left( \sum_{k \in I} a_k^{(i)} \right)_{i \in I}$$

is a group monomorphism, which maps $A$ onto $\mathbb{Z}_p^{(I)}$ [BM67]. The following notion isolates the property given by (2) of [Mad69, Theorem 3.3].

**Definition 7.1.** Adopt the notations above. We say that $A$ is a **tight $p$-basic subgroup of $K$** if $\sigma (K)$ is contained in $\mathbb{Z}_p^{(I)}$.

The following lemma guarantees the existence of tight $p$-basic subgroups; see also [Mad69, Corollary 3.6].

**Lemma 7.2.** Suppose that $(K_n)_{n \in \omega}$ is an increasing sequence of finite rank $p$-pure subgroups of $K$ such that $K = \bigcup_{n \in \omega} K_n$. Then there exists a tight $p$-basic subgroup $A \subseteq K$ such that $A \cap K_n$ is a $p$-basic subgroup of $K_n$ for every $n \in \omega$.

**Proof.** Define, by recursion, $A_0$ to be a maximal $p$-independent set of $K_0$, and $A_{n+1}$ to be a maximal $p$-independent set of $K_{n+1}$ that extends $A_n$. Define then $A_n$ to be the $p$-basic subgroup of $K_n$ generated by $A_n$, and $A := \bigcup_{n} A_n$. Then we have that $A \cap K_n = A_n$ for every $n \in \omega$. Furthermore, $A$ is a $p$-basic subgroup of $K$ that is tight by the argument in the proof of [Mad69, Corollary 3.6].

By [Mad69, Theorem 3.3], if $A$ is a tight $p$-basic subgroup of $K$, and $T$ is a reduced $p$-group, then the map $\text{Hom} (K, T) \to \text{Hom} (A, T)$ induced by the inclusion map $A \to K$ is surjective.

7.2. Cotorsion completions of $p$-groups. Suppose that $T$ is a countable reduced $p$-group. Let $\mathbb{Z}[1/p]$ be the subgroup of $\mathbb{Q}$ consisting of elements of the form $ap^{-n}$ for $a \in \mathbb{Z}$ and $n \geq 0$. Let also $\mathbb{Z}(p^\infty)$ be the Prufer $p$-group $\mathbb{Z}[1/p]/\mathbb{Z}$. The cotorsion completion of $T$ is the group with a Polish cover

$$T^* := \text{Ext}(\mathbb{Z}(p^\infty), T).$$

Since $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{q \in \mathbb{P}} \mathbb{Z}(q^\infty)$, one also has that

$$\text{Ext}(\mathbb{Q}/\mathbb{Z}, T) \cong \prod_{q \in \mathbb{P}} \text{Ext} (\mathbb{Z}(q^\infty), T) \cong T^*,$$
where $\text{Ext}(\mathbb{Z}(q^\infty), T) = 0$ for $q \neq p$ by Lemma 4.2. We have that $T^*$ is a cotorsion group, namely $\text{Ext}(J, T^*) = 0$ for every torsion-free group $J$; see [Fuc70, Theorem 54.6]. More generally by [CE99, Chapter IV, Proposition 3.5], if $K$ is a group, then

$$\text{Ext}(K, T^*) = \text{Ext}(K, \text{Ext}(\mathbb{Z}(p^\infty), T)) \cong \text{Ext}(\text{Tor}(K, \mathbb{Z}(p^\infty)), T) \cong \text{Ext}(K_p, T)$$

where $K_p$ is the $p$-component of $K$ and $\text{Tor}(K, \mathbb{Z}(p^\infty)) \cong K_p$ is the torsion product; see [Fuc70, Section 62]. Furthermore, $T^*$ is reduced and hence $p$-reduced (being $p$-local) [Fuc70, Lemma 55.4].

The short exact sequence $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces a Borel-definable exact sequence

$$0 \to \text{Hom}(\mathbb{Z}, T) = T \to \text{Ext}(\mathbb{Q}/\mathbb{Z}, T) = T^* \to \text{Ext}(\mathbb{Q}, T) \to 0.$$

Identifying $T$ with its image inside $\text{Ext}(\mathbb{Q}/\mathbb{Z}, T)$, one has that $T$ is the torsion subgroup of $\text{Ext}(\mathbb{Q}/\mathbb{Z}, T)$, and $T^*/T$ is Borel-definably isomorphic to $\text{Ext}(\mathbb{Q}, T)$ [Fuc70, Lemma 55.1]. Since $\mathbb{Q}$ is torsion-free and divisible, one has that $\text{Ext}(\mathbb{Q}, T)$ is torsion-free and divisible as well. A similar argument, invoking the short exact sequence $\mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}(p^\infty)$, shows that $T^*/T$ is Borel-definably isomorphic to $\text{Ext}(\mathbb{Z}[1/p], T)$.

### 7.3. An exact sequence for extensions of $p$-groups

Suppose that $T$ is a reduced countable $p$-group. For $\alpha < \omega_1$ define

$$L_\alpha(T) := \lim_{\beta<\alpha} T/p^\beta T,$$

$$\kappa_\alpha : T \to L_\alpha(T)$$

to be the canonical homomorphism with kernel $T^\alpha$, and

$$E_\alpha(T) := L_\alpha(T)/\kappa_\alpha(T).$$

For $\alpha < \omega_1$, the short exact sequence $p^\alpha T \to T \to T/p^\alpha T$ induces a Borel-definable short exact sequence

$$0 \to \text{Ext}(\mathbb{Z}[1/p], p^\alpha T) \to \text{Ext}(\mathbb{Z}[1/p], T) \to \text{Ext}(\mathbb{Z}[1/p], T/p^\alpha T) \to 0.$$

Indeed, as $T$ is reduced, $T/p^\alpha T$ is ($p$-)reduced, and hence $\text{Hom}(\mathbb{Z}[1/p], T/p^\alpha T) = 0$. Observe also that $L_\alpha(T)$ is $p$-reduced, and hence $\text{Hom}(\mathbb{Z}[1/p], L_\alpha(T)) = 0$. In the following theorem, we identify $\text{Ext}(\mathbb{Z}[1/p], p^\beta T)$ for $\beta < \omega_1$ with a Polishable subgroup of $\text{Ext}(\mathbb{Z}[1/p], T)$.

**Theorem 7.3.** Suppose that $T$ is a reduced countable $p$-group, and $\alpha < \omega_1$ is a limit ordinal. Then there is a Borel-definable short exact sequence

$$0 \to \text{Ext}(\mathbb{Z}[1/p], T^\alpha) \to \bigcap_{\beta<\alpha} \text{Ext}(\mathbb{Z}[1/p], p^\beta T) \to \text{Hom}(\mathbb{Z}[1/p], E_\alpha(T)) \to 0$$

where $\iota$ is the inclusion map.

**Proof.** Fix an increasing sequence $(\alpha_n)_{n \in \omega}$ of countable ordinals with $\sup_n \alpha_n = \alpha$. Consider a short exact sequence $R \to F \to \mathbb{Z}[1/p]$ where $F$ is free. We identify $\text{Ext}(\mathbb{Z}[1/p], T)$ with

$$\frac{\text{Hom}(R, T)}{\text{Hom}(F|R, T)}$$

where

$$\text{Hom}(F|R, T) = \text{Ran}(\text{Hom}(F, T) \to \text{Hom}(R, T)).$$

We also identify $\text{Ext}(\mathbb{Z}[1/p], p^\beta T)$ for $\beta < \alpha$ with the Polishable subgroup

$$\frac{\text{Hom}(R, p^\beta T) + \text{Hom}(F|R, T)}{\text{Hom}(F|R, T)} \subseteq \frac{\text{Hom}(R, T)}{\text{Hom}(F|R, T)}.$$

Set

$$G := \bigcap_{\beta<\alpha} \left( \frac{\text{Hom}(R, p^\beta T) + \text{Hom}(F|R, T)}{\text{Hom}(F|R, T)} \right).$$

We construct a surjective Borel-definable homomorphism $G \to \text{Hom}(\mathbb{Z}[1/p], E_\alpha(T))$ as follows. Suppose that $h \in \text{Hom}(R, T)$ represents an element of $G$. Thus, one can define by recursion on $n \in \omega$ functions $h_n : R \to p^n T$ and $\varphi_n : F \to p^n T$ such that $h_0 = h$, and $\varphi_0|_R + \cdots + \varphi_n|_R + h_n = h$. Thus, we have that

$$\varphi_h := \sum_{n \in \omega} (\kappa_\alpha \circ \varphi_n) \in \text{Hom}(F, L_\alpha(T)).$$
defines a homomorphism such that \( \varphi_h|_R = \kappa_\alpha \circ h \). This implies that \( \varphi_h \) induces \( \overline{\varphi}_h \in \text{Hom}(\mathbb{Z}[1/p], E_\alpha(T)) \).

Suppose that \( \varphi_n, \varphi'_n : F \to p^{\alpha n} T \) for \( n \in \omega \) are obtained as above starting from \( h \). Set

\[
\varphi_h = \sum_{n \in \omega} (\kappa_\alpha \circ \varphi_n)
\]

and

\[
\varphi'_h = \sum_{n \in \omega} (\kappa_\alpha \circ \varphi_n).
\]

Since

\[
(\varphi_0 + \cdots + \varphi_n)|_R - (\varphi'_0 + \cdots + \varphi'_n)|_R \in \text{Hom}(F, p^{\alpha n} T)
\]

for every \( n \in \mathbb{N} \), we have that

\[
(\varphi_h - \varphi'_h)|_R = 0.
\]

Since \( \text{Hom}(\mathbb{Z}[1/p], L_\alpha(T)) = 0 \), this shows that \( \overline{\varphi}_h = \overline{\varphi}'_h \). Thus, \( \overline{\varphi}_h \) only depends on \( h \) and not from the choice of the functions \( \varphi_n \) from \( h \) as above. It is easily follows that the assignment \( h \mapsto \overline{\varphi}_h \) induces a group homomorphism \( \pi : G \to \text{Hom}(\mathbb{Z}[1/p], E_\alpha(T)) \). To show that \( \pi \) is Borel-definable, it suffices to prove that given \( h \) one can choose a sequence \( (\varphi_n)_{n \in \mathbb{N}} \) as above in a Borel fashion. To this purpose, notice that, for every \( n \in \omega \), the function

\[
\text{Hom}(R, p^{\alpha n} T) \times \text{Hom}(F, T) \to \text{Hom}(R, p^{\alpha n} T) + \text{Hom}(F|R, T)
\]

\[
(\rho, \varphi) \mapsto \rho + \varphi|_R
\]

is a surjective continuous homomorphism, and hence by [Kec95, Theorem 12.17] it has a Borel right inverse. Thus, for every \( n \in \mathbb{N} \), given \( h \in \text{Hom}(R, p^{\alpha n} T) + \text{Hom}(F|R, T) \) one can choose in a Borel fashion \( \rho \in \text{Hom}(R, p^{\alpha n} T) \) and \( \varphi \in \text{Hom}(F, T) \) such that \( h = \rho + \varphi|_R \). This concludes the proof that \( \pi : G \to \text{Hom}(\mathbb{Z}[1/p], E_\alpha(T)) \) is a Borel-definable homomorphism.

We now show that \( \pi \) is surjective. Suppose that \( \overline{\varphi} \in \text{Hom}(\mathbb{Z}[1/p], E_\alpha(T)) \). Thus, \( \overline{\varphi} \) is induced by a homomorphism \( \varphi : F \to L_\alpha(T) \). There exists \( h \in \text{Hom}(R, T) \) such that \( \kappa_\alpha \circ h = \varphi|_R \). We can write \( \varphi = \sum_{n \in \omega} (\kappa_\alpha \circ \varphi_n) \) where \( \varphi_n \in (\text{Hom}(F, p^{\alpha n} T)) \) for \( n \in \omega \). Therefore, for every \( n \in \omega \),

\[
h_{n+1} := (\varphi_0 + \cdots + \varphi_n)|_R - h \in (\text{Hom}(R, p^{\alpha n+1} T)).
\]

This shows that

\[
h \in \bigcap_{\beta < \alpha} \left( \text{Hom}(R, p^\beta T) + \text{Hom}(F|R, T) \right)
\]

and hence \( h \) represents an element \( h + \text{Hom}(F|R, T) \) of \( G \). Furthermore, the sequence \( (\varphi_n) \) is obtained from \( h \) as in the definition of \( \pi \), and hence

\[
\pi(h + \text{Hom}(F|R, T)) = \overline{\varphi}.
\]

This concludes the proof that \( \pi \) is surjective.

We now prove that the kernel of \( \pi \) is equal to

\[
\text{Ext}(\mathbb{Z}[1/p], p^{\alpha T}) = \frac{\text{Hom}(R, p^\alpha T) + \text{Hom}(F|R, T)}{\text{Hom}(F|R, T)}.
\]

If \( h \in \text{Hom}(R, p^{\alpha T}) \) then one can choose \( \varphi_n = 0 \) and \( h_n = h \) for every \( n \in \omega \) as in the definition of \( \pi \). Thus, \( \text{Ext}(\mathbb{Z}[1/p], p^{\alpha T}) \) is contained in \( \text{Ker}(\pi) \). Conversely, suppose that \( h \) represents an element of \( \text{Ker}(\pi) \), and let \( \varphi_n \) and \( h_n \) be as in the definition of \( \pi \). Thus, we have that

\[
\varphi_h = \sum_{n \in \omega} (\kappa_\alpha \circ \varphi_n) \in \text{Hom}(F, L_\alpha(T))
\]

induces the trivial homomorphism \( \mathbb{Z}[1/p] \to E_\alpha(T) \). Thus, there exists a function \( \rho : F \to T \) such that \( \varphi_h = \kappa_\alpha \circ \rho \).

After replacing \( h \) with \( h - \rho|_R \) we can assume that \( \rho = 0 \) and hence \( \varphi_h = 0 \). Fix \( x \in R \) and \( k \in \omega \). Then there exists \( n \geq k \) such that, \( (\varphi_0 + \cdots + \varphi_n)(x) \in p^{\alpha k} \). Since \( h_{n+1}(x) \in p^{\alpha n+1} T \subset p^{\alpha k}T \), we have that

\[
h(x) = (\varphi_0 + \cdots + \varphi_n)(x) + h_{n+1}(x) \in p^{\alpha k}T.
\]

As this holds for every \( x \in R \) and \( k \in \omega \), we have that \( h \in \text{Hom}(R, p^{\alpha T}) \) represents an element of \( \text{Ext}(\mathbb{Z}[1/p], p^{\alpha T}) \). This concludes the proof that \( \text{Ker}(\pi) = \text{Ext}(\mathbb{Z}[1/p], p^{\alpha T}) \).

\( \square \)
7.4. The complexity of $T$ in its cotorsion completion. In this section, we continue to assume that $T$ is a countable reduced $p$-group. We compute the complexity class of $T$ inside $T^* = \text{Ext}(\mathbb{Z}(p^\infty), T)$. In view of the Borel-definable isomorphism $T^*/T \cong \text{Ext}(\mathbb{Z}[1/p], T)$, this is equal to the complexity class of $\{0\}$ in $\text{Ext}(\mathbb{Z}[1/p], T)$. We will also describe the Solecki subgroups $s_\alpha (\text{Ext}(\mathbb{Z}[1/p], T))$ of $\text{Ext}(\mathbb{Z}[1/p], T)$ for $\alpha < \omega_1$.

**Lemma 7.4.** Suppose that $T$ is a countable reduced $p$-group of Ulm length 1. Then $\text{Ext}(\mathbb{Z}[1/p], T)$ is Borel-definably isomorphic to $\text{Hom}(\mathbb{Z}[1/p], E^\omega(T))$.

**Proof.** Consider the short exact sequence $T \to L_\omega(T) \to E^\omega(T)$. This induces a Borel-definable exact sequence

$$0 = \text{Hom}(\mathbb{Z}[1/p], L_\omega(T)) \to \text{Hom}(\mathbb{Z}[1/p], E^\omega(T)) \to \text{Ext}(\mathbb{Z}[1/p], T) \to \text{Ext}(\mathbb{Z}[1/p], L_\omega(T)) = 0,$$

where $\text{Hom}(\mathbb{Z}[1/p], L_\omega(T)) = 0$ since $L_\omega(T)$ is $p$-reduced, and $\text{Ext}(\mathbb{Z}[1/p], L_\omega(T)) = 0$ since $L_\omega(T)$ is algebraically compact, being complete in the $p$-adic topology [Fuc70, Section 39]. Thus, $\text{Ext}(\mathbb{Z}[1/p], T)$ is Borel-definably isomorphic to $\text{Hom}(\mathbb{Z}[1/p], E^\omega(T))$. \hfill $\square$

**Lemma 7.5.** Suppose that $T$ is countable reduced $p$-group. Let $\alpha < \omega_1$ be such that $p^\alpha T \neq 0$ for every $\beta < \omega\alpha$. Then $\Pi^{0}_{\alpha}$ is the complexity class of $\{0\}$ in $\text{Hom}(\mathbb{Z}[1/p], E^\omega_\alpha(T))$.

**Proof.** After replacing $T$ with $T/p^{\omega_\alpha}T$, we can assume that $T$ has Ulm length at most $\alpha$. In this case, $\kappa^{\omega_\alpha} : T \to L_\omega(T) = L_\omega(T)/T$. Since $T$ is countable, we have that $\{0\}$ is $\Sigma^0_3$ in $\text{Hom}(\mathbb{Z}[1/p], T)$, and hence $\{0\}$ is $\Pi^0_{\alpha}$ in $\text{Hom}(\mathbb{Z}[1/p], T)$. By [Lup22a, Lemma 4.11] we have an injective Borel-definable homomorphism $\text{Hom}(\mathbb{Z}[1/p], E^\omega_\alpha(T)) \to \text{Ext}(\mathbb{Z}(p^\infty), E^\omega_\alpha(T))$, and hence $\{0\}$ is $\Pi^0_{\alpha}$ in $\text{Hom}(\mathbb{Z}[1/p], E^\omega_\alpha(T))$.

The short exact sequence $T \to \mathbb{Z}[1/p] \to \mathbb{Z}(p^\infty)$ induces an injective Borel-definable homomorphism

$$\text{Hom}(\mathbb{Z}(p^\infty), E^\omega_\alpha(T)) \to \text{Hom}(\mathbb{Z}[1/p], E^\omega_\alpha(T)).$$

By [Lup22a, Lemma 4.14], $\{0\}$ is not $\Sigma^0_3$ in $\text{Hom}(\mathbb{Z}(p^\infty), E^\omega_\alpha(T))$. Therefore, $\{0\}$ is not $\Sigma^0_3$ in $\text{Hom}(\mathbb{Z}[1/p], E^\omega_\alpha(T))$. \hfill $\square$

**Lemma 7.6.** Suppose that $T$ is a countable reduced $p$-group. If $T$ is bounded, then $\text{Ext}(\mathbb{Z}[1/p], T) = 0$. If $T$ is unbounded, then $s_1(\text{Ext}(\mathbb{Z}[1/p], T)) = \text{Ext}(\mathbb{Z}[1/p], T^1)$, and $\Pi^0_{\alpha}$ is the complexity class of $s_1(\text{Ext}(\mathbb{Z}[1/p], T))$ in $\text{Ext}(\mathbb{Z}[1/p], T)$.

**Proof.** If $T$ is bounded, then in particular $T$ has Ulm length 1. Thus, by Lemma 7.4, $\text{Ext}(\mathbb{Z}[1/p], T)$ is isomorphic to $\text{Hom}(\mathbb{Z}[1/p], E^\omega(T))$. Since $T$ is bounded, $E^\omega(T) = 0$. Thus, $\text{Ext}(\mathbb{Z}[1/p], T) = 0$.

Suppose that $T$ is unbounded. We can write $\mathbb{Z}[1/p] = \mathbb{F}/R$, where $\mathbb{F}$ is the free group generated by $x_n$ for $n \in \omega$, and $R$ is the subgroup of $F$ generated by $px_{n+1} - x_n$ for $n \in \omega$.

Then we have Borel-definable isomorphisms

$$\text{Ext}(\mathbb{Z}[1/p], T)^1 \cong \frac{\text{Hom}(R, T)}{\text{Hom}(F|R, T)}$$

and

$$\text{Ext}(\mathbb{Z}[1/p], T) \cong \frac{\text{Hom}(R, T) + \text{Hom}(F|R, T)}{\text{Hom}(F|R, T)}.$$

Since $\mathbb{Z}[1/p]$ is torsion-free, $\text{Ext}(\mathbb{Z}[1/p], T^1) = \text{PExt}(\mathbb{Z}[1/p], T^1)$, and $\{0\}$ is dense in $\text{Ext}(\mathbb{Z}[1/p], T^1)$.

The short exact sequence $T^1 \to T \to T/T^1$ induces a Borel-definable short exact sequence

$$0 = \text{Hom}(\mathbb{Z}[1/p], T/T^1) \to \text{Ext}(\mathbb{Z}[1/p], T) \to \text{Ext}(\mathbb{Z}[1/p], T^1) \to \text{Ext}(\mathbb{Z}[1/p], T/T^1) \to 0.$$

Since $\text{Ext}(\mathbb{Z}[1/p], T/T^1)$ is Borel-definably isomorphic to $\text{Hom}(\mathbb{Z}[1/p], T/T^1)$ by Lemma 7.4, and $\Pi^0_{\alpha}$ is the complexity class of $\{0\}$ in $\text{Hom}(\mathbb{Z}[1/p], T/T^1)$ by Lemma 7.5, we have that $\Pi^0_{\alpha}$ is the complexity class of $\text{Ext}(\mathbb{Z}[1/p], T^1)$ in $\text{Ext}(\mathbb{Z}[1/p], T)$. Since $\mathbb{Z}[1/p]$ is torsion-free, $\text{Ext}(\mathbb{Z}[1/p], T/T^1) = \text{PExt}(\mathbb{Z}[1/p], T/T^1)$ and $\{0\}$ is dense in $\text{Ext}(\mathbb{Z}[1/p], T/T^1)$. Therefore, by Lemma 3.1, it suffices to prove that if $V$ is a neighborhood of 0 in $\text{Hom}(F|R, T)$, then $\overline{\text{Hom}(R,T)}$ contains an open neighborhood of 0 in $\text{Hom}(R, T^1) + \text{Hom}(F|R, T)$.

Let $V$ be a neighborhood of 0 in $\text{Hom}(F|R, T)$. We have that there exists $n_0 \in \omega$ such that

$$\{\eta_R : \eta \in \text{Hom}(F, T), \eta(x_n) = 0 \text{ for } n \leq n_0\} \subseteq V.$$
Define
\[ W := \{ \rho \in \text{Hom}(R, T^1) : \rho(px_{n+1} - x_n) = 0 \text{ for } n \leq n_0 \} . \]

It suffices to prove that \( W \subseteq \mathcal{R}_{\text{hom}(R, T)} \). Suppose that \( \rho_0 \in W \) and let \( N \) be an open neighborhood of \( \rho_0 \) in \( \text{Hom}(R, T) \). Thus, there exists \( n_1 > n_0 \) such that
\[ \{ \rho \in \text{Hom}(R, T) : \rho(px_{n+1} - x_n) = \rho_0(px_{n+1} - x_n) \text{ for } n < n_1 \} \subseteq N. \]

We need to prove that there exists \( \eta \in \text{Hom}(F, T) \) such that
\[ \eta(x_n) = 0 \]
for \( n \leq n_0 \) and
\[ p\eta(x_{n+1}) - \eta(x_n) = \eta(px_{n+1} - x_n) = \rho_0(px_{n+1} - x_n) \]
for \( n < n_1 \).

Define thus \( \eta(x_n) = 0 \) for \( n \leq n_0 \) and for \( n > n_1 \). Define recursively \( \eta(x_n) \in p^{\omega_1-n}T \) for \( n_0 < n \leq n_1 \) such that
\[ p\eta(x_n) = \rho_0(px_{n+1} - x_n) + \eta(x_{n_1}). \]

This is possible since \( \rho_0(px_{n+1} - x_{n+1}) \in T^1 \subseteq p^{\omega_1-n+1}T \) and \( \eta(x_{n_1}) \in p^{\omega_1-n+1}T \) by the inductive hypothesis if \( n - 1 > n_0 \), or by the fact that \( \eta(x_{n_0}) = 0 \) if \( n - 1 = n_0 \). This concludes the definition of \( \eta \) and the proof. \( \square \)

**Theorem 7.7.** Suppose that \( T \) is a countable reduced \( p \)-group. Let \( \lambda < \omega_1 \) be a limit ordinal or zero and \( n < \omega \). Then we have that:

1. If \( n < \omega \), then \( s_n(\text{Ext}(Z[1/p], T)) = \text{Ext}(Z[1/p], T^n) \) and \( \Pi_3^n \) is the complexity class of \( s_n(\text{Ext}(Z[1/p], T)) \) in \( s_n(\text{Ext}(Z[1/p], T)) \) if and only if \( T^{n+1} \) is unbounded, while \( s_n(\text{Ext}(Z[1/p], T)) = 0 \) if \( T^{n+1} \) is bounded;
2. If \( \lambda \) is a limit ordinal, then \( s_{\lambda+1}(\text{Ext}(Z[1/p], T)) = \text{Ext}(Z[1/p], T^\lambda) \) and \( \Pi_3^\lambda \) is the complexity class of \( s_{\lambda+1}(\text{Ext}(Z[1/p], T)) \) in \( s_{\lambda}(\text{Ext}(Z[1/p], T)) \) if and only if \( T \) has Ulm length at least \( \lambda \), while \( s_{\lambda}(\text{Ext}(Z[1/p], T)) = 0 \) if \( T \) has Ulm length less than \( \lambda \);
3. If \( \lambda \) is a limit ordinal and \( 2 \leq n < \omega \), then \( s_{\lambda+n}(\text{Ext}(Z[1/p], T)) = \text{Ext}(Z[1/p], T^{\lambda+n-1}) \) and \( \Pi_3^{\lambda+n} \) is the complexity class of \( s_{\lambda+n}(\text{Ext}(Z[1/p], T)) \) in \( s_{\lambda+n}(\text{Ext}(Z[1/p], T)) \) if and only if \( T^{\lambda+n-1} \) is unbounded, while \( s_{\lambda+n}(\text{Ext}(Z[1/p], T)) = 0 \) if \( T^{\lambda+n-1} \) is bounded;
4. If \( \lambda < \omega_1 \) is a limit ordinal, then \( s_{\lambda}(\text{Ext}(Z[1/p], T)) = \bigcap_{\beta < \lambda} \text{Ext}(Z[1/p], T^\beta) \).

**Proof.** (1) follows immediately from Lemma 7.6 by induction on \( n \). We now prove by induction that (2), (3), and (4) hold.

If \( \lambda \) is a limit ordinal and the conclusion holds for \( \beta < \lambda \), then
\[ s_{\lambda}(\text{Ext}(Z[1/p], T)) = \bigcap_{\beta < \lambda} \text{Ext}(Z[1/p], T^\beta) \]
by definition of Solecki subgroups and the inductive hypothesis.

Suppose now that \( \lambda \) is a limit ordinal, \( n < \omega \) and the conclusion holds for \( \beta \leq \lambda + n \). We now prove that it holds for \( \lambda + n + 1 \). By the inductive hypothesis we have that
\[ s_{\lambda+n}(\text{Ext}(Z[1/p], T)) = \text{Ext}(Z[1/p], T^{\lambda+n-1}) \].

Thus, by Lemma 7.6 applied to \( T^{\lambda+n-1} \) we have that
\[ s_{\lambda+n+1}(\text{Ext}(Z[1/p], T)) = s_1(\text{Ext}(Z[1/p], T^{\lambda+n-1})) = \text{Ext}(Z[1/p], T^{\lambda+n}) \].

Furthermore, if \( T^{\lambda+n-1} \) is unbounded, then \( \Pi_3^{\lambda+n} \) is the complexity class of \( s_{\lambda+n+1}(\text{Ext}(Z[1/p], T)) \) in \( s_{\lambda+n}(\text{Ext}(Z[1/p], T)) \). If \( T^{\lambda+n-1} \) is bounded, then \( s_{\lambda+n}(\text{Ext}(Z[1/p], T)) = \text{Ext}(Z[1/p], T^{\lambda+n-1}) = 0 \).

Suppose now that \( \lambda \) is a limit ordinal and the conclusion holds for \( \beta \leq \lambda \). We now prove that it holds for \( \lambda + 1 \). By the inductive hypothesis we have that
\[ s_{\lambda}(\text{Ext}(Z[1/p], T)) = \bigcap_{\beta < \lambda} \text{Ext}(Z[1/p], T^\beta) \].
By Theorem 7.3 we have a Borel-definable exact sequence
\[ 0 \to \text{Ext} (\mathbb{Z}[1/p], T^\lambda) \to s_\lambda (\text{Ext} (\mathbb{Z}[1/p], T)) \to \text{Hom} (\mathbb{Z}[1/p], E_{\omega \lambda} (T)) \to 0. \]
If \( T \) has Ulm length less than \( \lambda \), then \( T^\beta = 0 \) for some \( \beta < \lambda \) and hence \( E_{\omega \lambda} (T) = 0 \) and \( s_1 (\text{Ext} (\mathbb{Z}[1/p], T)) = 0 \). If \( T \) has Ulm length at least \( \lambda \), then \( \Phi^\lambda_3 \) is the complexity class of \( \{0\} \) in \( \text{Hom} (\mathbb{Z}[1/p], E_{\omega \lambda} (T)) \) by Lemma 7.5. Thus, \( \Phi^\lambda_3 \) is the complexity class of \( \text{Ext} (\mathbb{Z}[1/p], T^\lambda) \) in \( s_\lambda (\text{Ext} (\mathbb{Z}[1/p], T)) \). Furthermore, \( \text{Ext} (\mathbb{Z}[1/p], T^\lambda) = \text{PExt} (\mathbb{Z}[1/p], T^\lambda) \) and hence \( \{0\} \) is dense in \( \text{Ext} (\mathbb{Z}[1/p], T^\lambda) \). We now proceed as in the proof of Lemma 7.6, adopting the same notation. We identify \( \text{Ext} (\mathbb{Z}[1/p], T) \) with
\[ \frac{\text{Hom} (R, T)}{\text{Hom} (F|R, T)}. \]
Under this identification, for \( \beta \leq \lambda \), \( \text{Ext} (\mathbb{Z}[1/p], T^\beta) \) corresponds to
\[ \frac{\text{Hom} (R, T^\beta) + \text{Hom} (F|R, T)}{\text{Hom} (F|R, T)}, \]
Similarly, \( s_\lambda (\text{Ext} (\mathbb{Z}[1/p], T)) \) corresponds to
\[ \frac{\hat{H}}{\text{Hom} (F|R, T)}, \]
where
\[ \hat{H} = \bigcap_{\beta < \lambda} \left( \text{Hom} (R, T^\beta) + \text{Hom} (F|R, T) \right). \]
Thus, by Lemma 3.1, it suffices to prove that if \( V \) is a symmetric neighborhood of \( 0 \) in \( \text{Hom} (F|R, T) \), then the closure \( \overline{V}^H \) of \( V \) in \( \hat{H} \) contains an open neighborhood of \( 0 \) in \( \text{Hom} (R, T^\lambda) \). Fix a symmetric neighborhood \( V \) of \( 0 \) in \( \text{Hom} (F|R, T) \). Thus, there exists \( n_0 \in \omega \) such that
\[ \{ \eta|_R : \eta \in \text{Hom} (F, T), \eta (x_n) = 0 \text{ for } n \leq n_0 \} \subseteq V. \]
Define
\[ W := \left\{ \rho \in \text{Hom} (R, T^\lambda) : \rho (px_{n+1} - x_n) = 0 \text{ for } n \leq n_0 \right\}. \]
We claim that \( W \subseteq \overline{V}^H \). Indeed, fix \( \rho_0 \in W \) and let \( N \) be a symmetric neighborhood of \( 0 \) in \( \hat{H} \). Thus, there exist \( \beta \leq \lambda \) and \( n_1 > n_0 \) such that, for every \( \psi \in \text{Hom} (F, T) \) such that \( \psi (x_n) = 0 \) for \( n \leq n_1 \) and for every \( \varphi \in \text{Hom} (R, T^\beta) \cap \hat{H} \) such that \( \varphi (px_{n+1} - x_n) = 0 \) for \( n < n_1 \), one has that \( \psi|_R + \varphi \in W \). It remains to prove that \( (\rho_0 + N) \cap V \neq \emptyset \) or, equivalently, \( (\rho_0 + V) \cap N \neq \emptyset \). Thus, it suffices to prove that there exists \( \eta \in \text{Hom} (F, T^\beta) \) such that \( \eta (x_n) = 0 \) for \( n \leq n_0 \) and such that
\[ \rho_0 (px_{n+1} - x_n) = \eta (px_{n+1} - x_n) = \eta (x_{n+1}) - \eta (x_n) \]
for \( n < n_1 \). Define \( \eta \in \text{Hom} (F, T^\beta) \) by setting \( \eta (x_n) = 0 \) for \( n \leq n_0 \) and \( n > n_1 \). Define then recursively \( \eta (x_n) \) for \( n_0 < n \leq n_1 \) such that \( \eta (x_n) \in p^{n_1-n}T^\beta \) and \( \rho_0 (px_n - x_{n-1}) = \rho_0 (px_n - x_{n-1}) + \eta (x_{n-1}) \). This is possible since \( \rho_0 (px_n - x_{n-1}) \in T^\beta \subseteq T^{\beta +1} \subseteq p^{n_1-n}T^\beta \) and \( \eta (x_{n-1}) \in p^{n_1-n+1}T^\beta \) by the inductive hypothesis if \( n - 1 > n_0 \) or by the fact that \( \eta (x_{n_0}) = 0 \) if \( n - 1 = n_0 \). This concludes the proof.

**Corollary 7.8.** Suppose that \( T \) is an unbounded countable reduced \( p \)-group. Let \( \beta \) be the least countable ordinal such that \( T^\beta \) is bounded. Then \( \Phi^\beta_{\beta+2} \) is the complexity class of \( \{0\} \) in \( \text{Ext} (\mathbb{Z}[1/p], T) \). Furthermore, the Soleciki rank of \( \text{Ext} (\mathbb{Z}[1/p], T) \) is \( \beta \) if \( \beta \) is finite, and \( \beta +1 \) if \( \beta \) is infinite.

**Proof.** For clarity, we consider three cases.

1. Suppose that \( \beta \) is a limit ordinal. By definition we have that \( T^\beta \) is bounded, and \( T^\alpha \) is unbounded for \( \alpha < \beta \). By Theorem 7.7, \( s_{\beta+1} (\text{Ext} (\mathbb{Z}[1/p], T)) = \text{Ext} (\mathbb{Z}[1/p], T^\beta) = \{0\} \), and \( \Phi^\beta_3 \) is the complexity class of \( \{0\} \) in \( s_\beta (\text{Ext} (\mathbb{Z}[1/p], T)) \). Thus, the Soleciki rank of \( \text{Ext} (\mathbb{Z}[1/p], T) \) is \( \beta +1 \), and \( \Phi^\beta_{\beta+2} \) is the complexity class of \( \{0\} \) in \( \text{Ext} (\mathbb{Z}[1/p], T) \) by Theorem 3.2.

2. Suppose that \( \beta = n < \omega \). Since \( T \) is unbounded, \( n \geq 1 \). Furthermore, \( T^n \) is bounded, and \( T^{n-1} \) is unbounded. By Theorem 7.7, we have that \( s_n (\text{Ext} (\mathbb{Z}[1/p], T)) = \text{Ext} (\mathbb{Z}[1/p], T^n) = 0 \) and \( \Phi^\beta_3 \) is the complexity class of \( \{0\} \) in \( s_{n-1} (\text{Ext} (\mathbb{Z}[1/p], T)) \). Thus we have that the Soleciki rank of \( \text{Ext} (\mathbb{Z}[1/p], T) \) is \( n \), and \( \Phi^\beta_{n+2} \) is the complexity class of \( \{0\} \) in \( \text{Ext} (\mathbb{Z}[1/p], T) \) by Theorem 3.2.
(3) Suppose that $\beta = \lambda + n$ where $\lambda$ is a limit ordinal and $1 \leq n < \omega$. Then we have that $T^{\lambda+n}$ is bounded and $T^{\lambda+n+1}$ is unbounded. By Theorem 7.7 we have that $s_{\lambda+n+1}(\text{Ext}(\mathbb{Z}[1/p], T)) = \text{Ext}(\mathbb{Z}[1/p], T^{\lambda+n}) = 0$ and $\Pi^0_{\lambda+n+2}$ is the complexity class of $\{0\}$ in $s_{\lambda+n}(\text{Ext}(\mathbb{Z}[1/p], T))$. Thus the Solecki rank of $\text{Ext}(\mathbb{Z}[1/p], T)$ is $\lambda + n + 1$, and $\Pi^0_{\lambda+n+2}$ is the complexity class of $\{0\}$ in $\text{Ext}(\mathbb{Z}[1/p], T)$ by Theorem 3.2.

$\square$

7.5. Tight $p$-basic subgroups and Ext. In this section, we assume that $K$ is a countable torsion-free group and $T$ is a countable reduced $p$-group, and determine the complexity of $\{0\}$ in $\text{Ext}(K, T)$. The proofs of the following results follow [Mad69, Section 2]. Recall the notion of tight $p$-basic subgroup from Definition 7.1.

**Lemma 7.9.** Suppose that $K$ is a countable torsion-free group, $A$ is a tight $p$-basic subgroup of $K$, and $T$ is a countable reduced $p$-group. Then $\text{Ext}(K, T)$ is Borel-definably isomorphic to $\text{Hom}(K/A, T^*/T)$.

**Proof.** Since $A$ is a $p$-basic subgroup of $K$, $K/A$ is $p$-divisible and has trivial $p$-component. The short exact sequence $A \to K \to K/A$ induces a Borel-definable exact sequence

$$0 \to \text{Hom}(K, T) \to \text{Hom}(A, T) \to \text{Ext}(K/A, T) \to \text{Ext}(K, T) \to 0.$$ 

Since $A$ is tight $p$-basic subgroup of $K$, by [Mad69, Theorem 3.3] the map $\text{Hom}(K, T) \to \text{Hom}(A, T)$ is surjective. Thus, $\text{Ext}(K, T)$ is Borel-definably isomorphic to $\text{Ext}(K/A, T)$. 

The Borel-definable short exact sequence $T \to T^* \to T^*/T$ induces a Borel-definable exact sequence

$$\text{Hom}(K/A, T^*) \to \text{Hom}(K/A, T^*/T) \to \text{Ext}(K/A, T) \to \text{Ext}(K/A, T^*).$$

Since $K/A$ has trivial $p$-component, $\text{Ext}(K/A, T^*) = 0$. Furthermore, since $K/A$ is $p$-divisible and $T^*$ is $p$-reduced, $\text{Hom}(K/A, T^*) = 0$. Therefore, we have that $\text{Ext}(K/A, T)$ is Borel-definably isomorphic to $\text{Hom}(K/A, T^*/T)$. $\square$

**Theorem 7.10.** Suppose that $K$ is a countable torsion-free group, $A$ is a tight $p$-basic subgroup of $K$, and $T$ is a countable reduced $p$-group. Let $r$ be the torsion-free rank of $K/A$. Then $\text{Ext}(K, T)$ is definably isomorphic to $(T^*/T)^r \cong \text{Ext}(\mathbb{Z}[1/p], T)^r$.

**Proof.** By Lemma 7.9, $\text{Ext}(K, T)$ is definably isomorphic to $\text{Hom}(K/A, T^*/T)$. Suppose that $t(K/A)$ is the torsion subgroup of $K/A$, and set $J := (K/A)/t(K/A)$. The short exact sequence $t(K/A) \to K/A \to J$ induces a Borel-definable exact sequence

$$0 \to \text{Hom}(J, T^*/T) \to \text{Hom}(K/A, T^*/T) \to \text{Hom}(t(K/A), T^*/T).$$

Since $T^*/T$ is torsion-free, $\text{Hom}(t(K/A), T^*/T) = 0$. Thus, $\text{Hom}(K/A, T^*/T)$ is definably isomorphic to $\text{Hom}(J, T^*/T)$. Since $A$ is a $p$-basic subgroup of $K$, $K/A$ is $p$-divisible, and hence $J$ is $p$-divisible. Thus, by [FS13, Lemma 2.1], $r$ is the rank of $J$, and there is a short exact sequence $\mathbb{Z}[1/p]^{(r)} \to J \to L$ such that $L$ is torsion. This induces Borel-definable exact sequence

$$0 \to \text{Hom}(L, T^*/T) \to \text{Hom}(J, T^*/T) \to \text{Hom}(\mathbb{Z}[1/p]^{(r)}, T^*/T) \to \text{Ext}(L, T^*/T).$$

As $T^*$ is a cotorsion group and $L$ is torsion, $\text{Ext}(L, T^*) = 0$. The quotient map $T^* \to T^*/T$ induces a surjective homomorphism $\text{Ext}(L, T^*) \to \text{Ext}(L, T^*/T)$. Thus, $\text{Ext}(L, T^*/T) = 0$. Since $T^*/T$ is torsion-free, $\text{Hom}(L, T^*/T) = 0$. Thus, $\text{Hom}(J, T^*/T)$ is definably isomorphic to $\text{Hom}(\mathbb{Z}[1/p]^{(r)}, T^*/T)$. Since $T^*/T$ is torsion-free and divisible, we have Borel-definable isomorphisms

$$\text{Hom}(\mathbb{Z}[1/p]^{(r)}, T^*/T) \cong \text{Hom}(\mathbb{Z}[1/p], T^*/T)^r \cong (T^*/T)^r.$$

This concludes the proof. $\square$

As an immediate consequence of Theorem 7.10 and Corollary 7.8 we have the following result, which in particular implies Theorem B.

**Corollary 7.11.** Suppose that $K$ is a countable torsion-free group, $A$ is a tight $p$-basic subgroup of $K$, and $T$ is a countable reduced $p$-group. Let $\beta$ be the least countable ordinal such that $T^\beta$ is bounded, let $r$ be the torsion-free rank of $K/A$. If $r = 0$ or $\beta = 0$ then $\text{Ext}(K, T) = 0$. If $r \neq 0$ and $\beta \neq 0$, then:

- $\Pi^0_{\beta+2}$ is the complexity class of $\{0\}$ in $\text{Ext}(K, T)$;
- the Solecki rank of $\text{Ext}(K, T)$ is $\beta$ if $\beta$ is finite, and $\beta + 1$ if $\beta$ is infinite;
- if $\lambda < \omega_1$ is either zero or limit and $n < \omega$, then

$$s_{\lambda+n+1}(\text{Ext}(K, T)) = \text{Ext}(K, T^{1+\lambda+n}).$$
8. Extensions of torsion-free groups by torsion groups with bounded components

In this section, we compute the potential complexity class of the relation of isomorphisms of extensions of a countable torsion-free group by a countable torsion group with bounded components, giving a proof of (2) of Theorem A. We let adopt the following notation: \( K \) is a countable torsion-free group, and \( T \) is a countable torsion group, which we can assume without loss of generality to be reduced. For \( q \in \mathbb{P} \), we let \( T_q \) be the \( q \)-primary component of \( T \). We let \( \tau (T) \) be the set of primes \( q \in \mathbb{P} \) such that \( T_q \) is nonzero. If \( T, L \) are torsion groups, we say that \( T \) and \( L \) are orthogonal if \( \tau (L) \cap \tau (T) = \emptyset \). This is equivalent to the assertion that if \( A \) is a group that embeds in both \( T \) and \( L \), then \( A = 0 \). It follows from Lemma 4.7 and Lemma 4.6 that \( L \) and \( T \) are orthogonal if and only if \( \text{Ext} (L, T) = 0 \).

**Lemma 8.1.** Suppose that \( K \) is a countable torsion-free group. Let \( T \) be a countable reduced torsion group with bounded primary components. Suppose that
\[
T' = \bigoplus_{q \in \mathbb{P} \setminus Q} T_q
\]
for some finite subset \( Q \) of \( \mathbb{P} \). Then the inclusion map \( T' \to T \) induces a Borel-definable isomorphism \( \text{Ext} (K, T') \to \text{Ext} (K, T) \).

**Proof.** We have that the inclusion map \( T' \to T \) induces the first-coordinate inclusion
\[
\text{Ext} (K, T') \to \text{Ext} (K, T) \cong \text{Ext} (K, T') \oplus \text{Ext} (K, \prod_{q \in Q} T_q).
\]
This is an isomorphism since
\[
\text{Ext} (K, \prod_{q \in Q} T_q) \cong \prod_{q \in Q} \text{Ext} (K, T_q) = \{0\}
\]
by Corollary 7.11. \( \square \)

Recall that for a set \( Q \subseteq \mathbb{P} \) we let \( R_Q \) be the subring of \( Q \) generated by \( 1/q \) for \( q \in Q \).

**Lemma 8.2.** Suppose that \( K \) is a countable torsion-free group, and \( T \) is a countable reduced torsion group with bounded primary components. Suppose that \( Q \subseteq \mathbb{P} \) is such that \( Q \cap \tau (T) \) is finite. Then the inclusion \( K \to K \otimes R_Q \) induces a Borel-definable isomorphism \( \text{Ext} (K \otimes R_Q, T) \to \text{Ext} (K, T) \).

**Proof.** By Lemma 8.1, we can assume that \( Q \cap \tau (T) = \emptyset \). Consider the short exact sequence \( K \to K \otimes R_Q \to L \)
where \( L = (K \otimes R_Q)/K \) is a torsion group with \( \tau (L) \subseteq Q \). Since \( Q \cap \tau (T) = \emptyset \), we have that \( L \) and \( T \) are orthogonal and \( \text{Ext} (L, T) = 0 \) by Lemma 4.7. Thus, the surjective Borel-definable homomorphism \( \text{Ext} (K \otimes R_Q, L) \to \text{Ext} (K, L) \) is an isomorphism. \( \square \)

**Proposition 8.3.** Suppose that \( K \) is a finite-rank torsion-free group, and \( T \) is a countable torsion group with bounded primary components. Fix a free essential subgroup \( E \) of \( K \). Denote by \( Q_0 \) the set of \( q \in \mathbb{P} \) such that \( E \) is not \( q \)-pure in \( K \). Then \( \{0\} \) is \( \Sigma^0_2 \) in \( \text{Ext} (K, T) \). Furthermore, the following assertions are equivalent:
1. \( \text{Ext} (K, T) = 0 \);
2. the set \( Q_0 \cap \tau (T) \) is finite;
3. there is a set \( Q \subseteq \mathbb{P} \) such that \( Q \cap \tau (T) \) is finite and such that \( K \otimes R_Q \) is a free \( R_Q \)-module.

**Proof.** We have a Borel-definable isomorphism
\[
\text{Ext} (K, T) \cong \frac{\text{Ext} (K/E, T)}{\text{Ran} (\text{Hom} (E, T) \to \text{Ext} (K/E, T))}.
\]
Here, \( \text{Ext} (K/E, T) \) is a Polish group by Lemma 4.8. Since \( K \) has finite-rank, \( E \) has finite rank, and \( \text{Hom} (E, T) \) is countable. It follows that \( \text{Ran} (\text{Hom} (E, T) \to \text{Ext} (K/E, T)) \) is countable, and hence \( \{0\} \) is \( \Sigma^0_2 \) in \( \text{Ext} (K, T) \).

By Lemma 4.7, we have that
\[
\text{Ext} (K/E, T) \cong \prod_{q \in Q_0 \cap \tau (T)} \text{Ext} (Kq/Eq, T_q)
\]
where, for every $q \in Q_0 \cap \tau(T)$, $(K/E)_q$ is not $q$-divisible, $T_q$ is nonzero, and $\text{Ext}((K/E)_q,T_q)$ is a nonzero Polish group by Lemma 4.6.

(1) $\Rightarrow$ (2) Suppose that $Q_0 \cap \tau(T)$ is infinite. Thus, $\text{Ext}(K/E,T)$ is an uncountable Polish group, while

$$\text{Ran}(\text{Hom}(E,T) \to \text{Ext}(K/E,T))$$

is a countable subgroup. Thus, $\text{Ext}(K/E,T) \neq 0$.

(2) $\Rightarrow$ (3) Set $Q = Q_0$. Then $K \otimes R_Q = E \otimes R_Q$ is a free $R_Q$-module.

(3) $\Rightarrow$ (1) We have that $\text{Ext}(R_Q,T) \cong \text{Ext}(\mathbb{Z},T) = 0$ by Lemma 8.2. Thus, $\text{Ext}(A,T) = 0$ for every countable free $R_Q$-module. The conclusion thus follows from Lemma 8.2. $\square$

**Proposition 8.4.** Suppose that $K$ is a countable torsion-free group, and $T$ is a countable torsion group with bounded primary components. Let $(K_n)_{n \in \omega}$ be an increasing sequence of finite-rank pure subgroups of $K$ such that $K = \bigcup_{n \in \omega} K_n$. Then we have that the canonical map

$$\text{Ext}(K,T) \to \lim_n \text{Ext}(K_n,T)$$

is injective, and $\{0\}$ is $\Pi^0_3$ in $\text{Ext}(K,T)$.

**Proof.** The second assertion follows from the first one in view of Proposition 8.3. Let

$$\hat{T} = \prod_{q \in \mathbb{P}} T_q$$

be the $\mathbb{Z}$-adic completion of $T$; see [Fuc70, Section 7]. We have a Borel-definable exact sequences

$$0 \to \text{PExt}^3(K,T) \to \text{Ext}(K,T) \to \lim_n \text{Ext}(K_n,T) \to 0$$

where, by Theorem 5.14, $\text{PExt}^3(K,T)$ is Borel-definably isomorphic to $\lim^1 \text{Hom}(K_n,T)$. Thus, it suffices to prove that $\lim^1 \text{Hom}(K_n,T) = 0$.

As above, for every $q \in \mathbb{P}$, we have a Borel-definable exact sequence

$$0 \to \lim^1 \text{Hom}(K_n,T_q) \to \text{Ext}(K,T_q) \to \lim_n \text{Ext}(K_n,T_q) \to 0.$$

As $T_q$ is bounded, $\text{Ext}(K,T_q) = 0$ by Corollary 7.11. Thus, we

$$\lim^1 \text{Hom}(K_n,T) \cong \prod_{q \in \mathbb{P}} \lim^1 \text{Hom}(K_n,T_q) = 0.$$

The short exact sequence of towers of groups with a Polish cover

$$(\text{Hom}(K_n,T))_{n \in \omega} \to (\text{Hom}(K_n,T_n))_{n \in \omega} \to \frac{\text{Hom}(K_n,\hat{T})}{\text{Hom}(K_n,T_n)}_{n \in \omega}$$

induces a Borel-definable exact sequence of groups with a Polish cover

$$0 \to \text{Hom}(K,T) \to \text{Hom}(K,\hat{T}) \to \lim_n \frac{\text{Hom}(K_n,\hat{T})}{\text{Hom}(K_n,T_n)} \to \lim^1_n \text{Hom}(K_n,T) \to \lim^1_n \text{Hom}(K_n,\hat{T}) = 0.$$

It suffices to prove that the homomorphism

$$\text{Hom}(K,\hat{T}) \to \lim_n \frac{\text{Hom}(K_n,\hat{T})}{\text{Hom}(K_n,T_n)}$$

is surjective. Suppose that $(\varphi_n)_{n \in \omega}$ is an element of $\prod_n \text{Hom}(K_n,\hat{T})$ such that, for every $n \in \omega$,

$$\varphi_{n+1}|_{K_n} - \varphi_n \in \text{Hom}(K_n,T).$$

For every $n \in \omega$, we can write

$$\varphi_n = (\varphi_n^p)_{p \in \tau(T)} \in \text{Hom}(K_n,T) \subseteq \text{Hom}(K_n,\hat{T}) = \prod_{p \in \tau(T)} \text{Hom}(K_n,T_p).$$

Thus, we have that, for every $n \leq m < \omega$ and $x \in K_n$, the set of $p \in \tau(T)$ such that $\varphi_n^p(x) \neq \varphi_m^p(x)$ is finite. For $n \in \omega$ and $p \in \tau(T)$, since $\text{Ext}(K/K_n,T_p) = 0$ by Corollary 7.11, we have that the homomorphism
Fix an enumeration \((p_k)_{k \in \omega}\) of \(\tau(T)\) and an enumeration \((x_i)_{i \in \omega}\) of \(K\). One can define by recursion on \(\ell\) a strictly increasing sequence \((k_\ell)\) in \(\omega\) such that \(k_0 = 0\) and for every \(i \leq j \leq \ell\) and \(t \leq \ell\) such that \(x_t \in X_i\) and \(k \geq k_\ell\) one has that

\[
\varphi^p_k(x_i) = \varphi^p_j(x_i).
\]

For \(\ell \in \omega\) and \(k_\ell \leq k < k_{\ell+1}\) define

\[
\psi^p_k := \hat{\varphi}_k^p \in \text{Hom}(K, T_{p_k}).
\]

The family \((\psi^p)_{p \in \tau(T)}\) defines an element \(\psi\) of \(\text{Hom}(K, \hat{T})\). We claim that, for every \(n \in \omega\), \(\psi|_{K_n} - \varphi_n \in \text{Hom}(K_n, T)\).

Proposition 8.5. Suppose that \(K\) is a countable torsion-free group, and \(T\) is a countable torsion group with bounded primary components. We have that \(\{0\}\) is \(\Sigma^0_3\) in \(\text{Ext}(K, T)\) if and only if there exist a finite-rank pure subgroup \(K_0\) of \(K\) such that for every finite-rank pure subgroup \(K_1\) of \(K\) containing \(K_0\) there exists \(Q \subseteq \mathbb{P}\) such that \(Q \cap \tau(T)\) is finite and \(K_1/K_0 \otimes R_Q\) is a free \(R_Q\)-module.

Proof. Suppose that \(\{0\}\) is \(\Sigma^0_3\) in \(\text{Ext}(K, T)\). We can assume that \(K\) has infinite rank, as otherwise there is nothing to prove. Let \((e_n)_{n \in \omega}\) be a maximal independent set in \(K\), and let \(E\) be the subgroup \((e_n : n \in \omega)\) \(\subseteq K\).

For \(n \in \omega\), let \(E_n\) be the subgroup \((e_k : k \leq n)\) \(\subseteq E\), and \(E_{>n}\) be the subgroup \((e_k : k > n)\) \(\subseteq E\). Let also \(K_n\) be the pure subgroup of \(K\) generated by \(E_n\), and \(K_{>n} := K/K_n\). Thus, for every \(n \in \mathbb{N}\) we have a diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & E_n & K_n & K_n/E_n & 0 & K_n/K_n & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & E & K & K/E & 0 & K/K_K & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & E_{>n} & K_{>n} & K_{>n}/E_{>n} & 0 & K_{>n}/K_{>n} & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

This induces a diagram in the category of groups with a Polish cover:
Notice that $K/E$ is a torsion group and hence $\text{Ext}(K/E, T)$ is a Polish group by Lemma 4.8. Thus, the Borel-definable homomorphism $\delta : \text{Hom}(E, T) \to \text{Ext}(K/E, T)$ is a continuous homomorphism. Since by assumption $\{0\}$ is $\Sigma^0_2$ in $\text{Ext}(K, T)$, we have that $\text{Ran}(\delta)$ is $\Sigma^0_2$ in $\text{Ext}(K/E, T)$. Thus, we can find an increasing sequence $(C_\ell)_{\ell \in \omega}$ of closed subsets of the Polish group $\text{Ext}(K/E, T)$ such that

$$\text{Ran}(\delta) = \bigcup_{\ell \in \omega} C_\ell.$$ 

Thus, we have that

$$\text{Hom}(E, T) = \bigcup_{\ell \in \omega} \delta^{-1}(C_\ell)$$

where $\delta^{-1}(C_\ell)$ is closed in $\text{Hom}(E, T)$ for every $\ell \in \omega$. Thus, there exists $\ell \in \omega$ such that $\delta^{-1}(C_\ell)$ contains an open subset of $\text{Hom}(E, T)$. After replacing $C_\ell$ with a translate of $C_\ell$ inside of $\text{Ext}(K/E, A)$, we can assume that $\delta^{-1}(C_\ell)$ contains an open neighborhood of $0$ in $\text{Hom}(E, T)$. Therefore, there exists $n \in \omega$ such that

$$\text{Ran}(\sigma_n) \subseteq \delta^{-1}(C_\ell).$$

The same argument as in the proof of Proposition 6.4 implies that the epimorphism

$$\pi_n : \text{Ext}(K_{>n}/E_{>n}, T) \to \text{Ext}(K_{>n}, T)$$

is an isomorphism. Since $K_{>n}/E_{>n}$ is a torsion group, $\text{Ext}(K_{>n}/E_{>n}, T)$ is a Polish group by Lemma 4.8. Since $K_{>n}$ is torsion-free, we have that $\text{Ext}(K_{>n}, T) = \text{PExt}(K_{>n}, T)$ and $\{0\}$ is dense in $\text{Ext}(K_{>n}/K_{>n}, T) = 0$. Therefore, we must have $\text{Ext}(K_{>n}, T) = 0$.

Fix $m > n$. Thus, by the above $\text{Ext}(K_m/K_{>n}, T) = 0$. Hence, by Proposition 8.3 there exists $Q \subseteq \mathbb{P}$ such that $Q \cap \tau(T)$ is finite and $K_m/K_n \otimes R_Q = (K_m \otimes R_Q)/(K_n \otimes R_Q)$ is a free $R_Q$-module. As this holds for every $m > n$, the proof of the forward implication is concluded.

Conversely, suppose that there exist a finite-rank pure subgroup $K_0$ of $K$ such that for every finite-rank pure subgroup $K_1$ of $K$ containing $K_0$ there exists $Q \subseteq \mathbb{P}$ such that $Q \cap \tau(T)$ is finite and $K_1/K_0 \otimes R_Q$ is a free $R_Q$-module. Thus, we have that $\text{Ext}(A, T) = 0$ for every finite-rank pure subgroup $A$ of $K/K_0$. This implies that $\text{Ext}(K/K_0, T) = 0$ by Proposition 8.4. Thus, we have that the Borel-definable epimorphism $\text{Ext}(K, T) \to \text{Ext}(K_0, T)$ induced by the inclusion $K_0 \to K$ is an isomorphism. By Proposition Proposition 8.3, $\{0\}$ is $\Sigma^0_2$ in $\text{Ext}(K_0, T)$ since $K_0$ has finite-rank. Thus, $\{0\}$ is $\Sigma^0_2$ in $\text{Ext}(K, T)$. This concludes the proof.

Part (2) of Theorem A is a consequence of Proposition 8.4, Proposition 8.5, and Proposition 8.3.

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**School of Mathematics and Statistics, Victoria University of Wellington, PO Box 600, 6140 Wellington, New Zealand**  
*Email address: lupini@tutanota.com*