Optimal Detection For Sparse Mixtures*

T. Tony Cai† and Yihong Wu‡

Department of Statistics
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104, USA

May 5, 2014

Abstract

Detection of sparse signals arises in a wide range of modern scientific studies. The focus so far has been mainly on Gaussian mixture models. In this paper, we consider the detection problem under a general sparse mixture model and obtain an explicit expression for the detection boundary. It is shown that the fundamental limits of detection is governed by the behavior of the log-likelihood ratio evaluated at an appropriate quantile of the null distribution. We also establish the adaptive optimality of the higher criticism procedure across all sparse mixtures satisfying certain mild regularity conditions. In particular, the general results obtained in this paper recover and extend in a unified manner the previously known results on sparse detection far beyond the conventional Gaussian model and other exponential families.

Keywords: Hypothesis testing, high-dimensional statistics, sparse mixture, higher criticism, adaptive tests, total variation, Hellinger distance.

---

*The research was supported in part by NSF FRG Grant DMS-0854973.
†tcai@wharton.upenn.edu
‡yihongwu@wharton.upenn.edu
1 Introduction

Detection of sparse mixtures is an important problem that arises in many scientific applications such as signal processing [11], biostatistics [23], and astrophysics [8, 24], where the goal is to determine the existence of a signal which only appears in a small fraction of the noisy data. For example, topological defects and Doppler effects manifest themselves as non-Gaussian convolution component in the Cosmic Microwave Background (CMB) temperature fluctuations. Detection of non-Gaussian signatures are important to identify cosmological origins of many phenomena [24]. Another example is disease surveillance where it is critical to discover an outbreak when the infected population is small [25]. The detection problem is of significant interest also because it is closely connected to a number of other important problems including estimation, screening, large-scale multiple testing, and classification. See, for example, [6], [7], [12], [17], and [23].

1.1 Detection of sparse binary vectors

One of the earliest work on sparse mixture detection dates back to Dobrushin [11], who considered the following problem originating from multi-channel detection in radiolocation. Let Ray($\alpha$) denote the Rayleigh distribution with the density $\frac{2y}{\alpha} \exp\left(-\frac{y^2}{\alpha}\right), y \geq 0$. Let $\{Y_i\}_{i=1}^n$ be independently distributed according to Ray($\alpha_i$), representing the random voltages observed on the $n$ channels. In the absence of noise, $\alpha_i$’s are all equal to one, the nominal value; while in the presence of signal, exactly one of the $\alpha_i$’s becomes a known value $\alpha > 1$. Denoting the uniform distribution on $[n]$ by $U_n$, the goal is to test the following competing hypotheses

$$H_0^{(n)}: \alpha_i = 1, i \in [n], \quad \text{versus} \quad H_1^{(n)}: \alpha_i = 1 + (\alpha - 1)1_{\{i=J\}}, \quad J \sim U_n.$$  (1)

Since the signal only appears once out of the $n$ samples, in order for the signal to be distinguishable from noise, it is necessary for the amplitude $\alpha$ to grow with the sample size $n$ (in fact, at least logarithmically). By proving that the log-likelihood ratio converges to a stable distribution in the large-$n$ limit, Dobrushin [11] obtained sharp asymptotics of the smallest $\alpha$ in order to achieve the desired false alarm and miss detection probabilities. Similar results are obtained in the continuous-time Gaussian setting by Burnashev and Begmatov [5].

Subsequent important work include Ingster [20] and Donoho and Jin [12], which focused on detecting a sparse binary vector in the presence of Gaussian observation noise. The problem can be formulated as follows. Given a random sample $\{Y_1, ..., Y_n\}$, one wishes to test the hypotheses

$$H_0^{(n)}: Y_i \overset{i.i.d.}{\sim} \mathcal{N}(0, 1), i \in [n] \quad \text{versus} \quad H_1^{(n)}: Y_i \overset{i.i.d.}{\sim} (1 - \epsilon_n)\mathcal{N}(0, 1) + \epsilon_n\mathcal{N}(\mu_n, 1), i \in [n]$$  (2)

where the non-null proportion $\epsilon_n$ is calibrated according to

$$\epsilon_n = n^{-\beta}, \quad \frac{1}{2} < \beta < 1,$$  (3)

and the non-null effect $\mu_n$ grows with the sample size according to

$$\mu_n = \sqrt{2r \log n}, \quad r > 0.$$  (4)
Equivalently, one can write

\[ Y_i = X_i + Z_i \]  

(5)

where \( Z_i \sim \text{i.i.d.} \mathcal{N}(0, 1) \) is the observation noise. Under the null hypothesis, the mean vector \( X^n = (X_1, \ldots, X_n) \) is equal to zero; under the alternative, \( X^n \) is a non-zero sparse binary vector with \( X_i \sim \text{i.i.d.} (1 - \epsilon_n) \delta_0 + \epsilon_n \delta_{\mu_n} \), where \( \delta_a \) denotes the point mass at \( a \).

The detection boundary, which gives the smallest possible signal strength, \( r \), such that reliable detection is possible, is given by the following function in terms of the sparsity parameter \( \beta \):

\[
  r^*(\beta) = \begin{cases} 
  \beta - \frac{1}{2} & \frac{1}{2} < \beta \leq \frac{3}{4}, \\
  (1 - \sqrt{1 - \beta})^2 & \frac{3}{4} < \beta < 1.
\end{cases}
\]  

(6)

See Ingster [20] and Donoho and Jin [12]. Therefore, the hypotheses in (2) can be tested with vanishing probability of error if and only if the pair \((\beta, r)\) lies in the strict epigraph

\[
  \{ (\beta, r) : r > r^*(\beta) \},
\]  

(7)

which is called the detectable region. Furthermore, because the fraction of the non-zero mean is very small, most tests based on the empirical moments have no power in detection. Donoho and Jin [12] proposed an adaptive testing procedure based on Tukey’s higher criticism statistic and showed that it attains the optimal detection boundary (6) without requiring the knowledge of the unknown parameters \((\beta, r)\).

The above results have been generalized along various directions within the framework of two-component Gaussian mixtures. Jager and Wellner [22] proposed a family of goodness-of-fit tests based on the Rényi divergences [29, p. 554], including the higher criticism test as a special case, which achieve the optimal detection boundary adaptively. The detection boundary with correlated noise was established in [16] which also proposed a modified version of the higher criticism that achieves the corresponding optimal boundary. In a related setup, [4, 2, 3] considered detecting a signal with a known geometric shape in Gaussian noise. Minimax estimation of the non-null proportion \( \epsilon_n \) was studied in Cai, Jin and Low [7].

The setup of [20] and [12] specifically focuses on the two-point Gaussian mixtures. Although [20] and [12] provide insightful results for sparse signal detection, the setting is highly restrictive and idealized. In particular, it has the limitation that the signal strength must be a constant under the alternative, i.e., the mean vector \( X^n \) takes constant value \( \mu_n \) on its support. In many applications, the signal itself varies among the non-null portion of the samples. A natural question is the following: What is the detection boundary if \( \mu_n \) varies under the alternative, say with a distribution \( P_n \)? Motivated by these considerations, the following heteroscedastic Gaussian mixture model was considered in Cai, Jeng and Jin [6]:

\[
  H_0^{(n)} : Y_i \sim \text{i.i.d.} \mathcal{N}(0, 1) \quad \text{versus} \quad H_1^{(n)} : Y_i \sim \text{i.i.d.} (1 - \epsilon_n) \mathcal{N}(0, 1) + \epsilon_n \mathcal{N}(\mu_n, \sigma^2). 
\]  

(8)

In this case, [6, Theorems 2.1 and 2.2] showed that reliable detection is possible if and only if
where $r > r^*(\beta, \sigma^2)$ where $r^*(\beta, \sigma^2)$ is given by

$$r^*(\beta, \sigma^2) = \begin{cases} 
(2 - \sigma^2)(\beta - \frac{1}{2}) & \frac{1}{2} < \beta \leq 1 - \frac{\sigma^2}{4}, \sigma^2 < 2 \\
(1 - \sigma^2 \sqrt{1 - \beta})^2 & \text{otherwise}
\end{cases}\quad (9)$$

where $x_+ \triangleq \max(x, 0)$. It was also shown that the optimal detection boundary can be achieved by a double-sided version of the higher criticism test.

### 1.2 Detection of general sparse mixture

Although the setup in Cai, Jeng and Jin [6] is more general than that considered in [20] and [12], it is still restricted to the two-component Gaussian mixtures. In many applications such as the aforementioned multi-channel detection [11] and astrophysical problems [24], the sparse signal may not be binary and the distribution may not be Gaussian. In the present paper, we consider the problem of sparse mixture detection in a general framework where the distributions are not necessarily Gaussian and the non-null effects are not necessarily a binary vector. More specifically, given a random sample $Y^n = \{Y_1, ..., Y_n\}$, we wish to test the following hypotheses

$$H_0^{(n)} : Y_i \overset{i.i.d.}{\sim} Q_n \quad \text{versus} \quad H_1^{(n)} : Y_i \overset{i.i.d.}{\sim} (1 - \epsilon_n)Q_n + \epsilon_nG_n \quad (10)$$

where $Q_n$ is the null distribution and $G_n$ is a distribution modeling the statistical variations of the non-null effects. The non-null proportion $\epsilon_n \in (0, 1)$ is calibrated according to (3).

In this paper we obtain an explicit formula for the fundamental limit of the general testing problem (10) under mild technical conditions on the mixture. We also establish the adaptive optimality of the higher criticism procedure across all sparse mixtures satisfying certain mild regularity conditions. In particular, the general results obtained in this paper recover and extend all the previously known results mentioned earlier in a unified manner. The results also generalize the optimality and adaptivity of the higher criticism procedure far beyond the original equal-signal-strength Gaussian setup in [20, 12] and the heteroscedastic extension in [6]. In the most general case, it turns out that the detectability of the sparse mixture is governed by the behavior of the log-likelihood ratio evaluated at an appropriate quantile of the null distribution.

Although our general approach does not rely on the Gaussianity of the model, it is however instructive to begin by considering the special case of sparse normal mixture with $Q_n = \mathcal{N}(0, 1)$, i.e.,

$$\begin{cases} 
H_0^{(n)} : Y_i \overset{i.i.d.}{\sim} \mathcal{N}(0, 1) \\
H_1^{(n)} : Y_i \overset{i.i.d.}{\sim} (1 - \epsilon_n)\mathcal{N}(0, 1) + \epsilon_nG_n
\end{cases}\quad (11)$$

It is of special interest to consider the convolution model, where

$$G_n = P_n \ast \mathcal{N}(0, 1) \quad (12)$$

is a standard normal mixture and $\ast$ denotes the convolution of two distributions. In this case the hypotheses (11) can be equivalently expressed via the additive-noise model (5), where $X_i = 0$ under
the null and \( X^i \overset{\text{i.i.d.}}{\sim} (1 - \epsilon_n)\delta_0 + \epsilon_n P_n \) under the alternative. Based on the noisy observation \( Y^n \), the goal is to determine whether \( X^n \) is the zero vector or a sparse vector, whose support size is approximately \( n\epsilon_n \) and non-zero entries are distributed according to \( P_n \). Therefore, the distribution \( P_n \) represents the prior knowledge of the signal. The case of \( P_n \) being a point mass is treated in [20, 12]. The case of Rademacher \( P_n \) is covered in [21, Chapter 8]. The heteroscedastic case where \( P_n \) is Gaussian is considered in [6]. These results can be recovered by particularizing the general conclusion in the present paper.

Moreover, our results also shed light on what governs detectability in Gaussian noise when the signal does not necessarily have equal strength. For example, consider the classical setup (2) where the signal strength \( \mu_n \) is now a random variable. If we have \( \mu_n = \sqrt{2r \log n} X \) for some random variable \( X \), then the resulting detectable region is given by the Ingster-Donoho-Jin expression (20) scaled by the \( L_\infty \)-norm of \( X \). On the other hand, it is also possible that certain distributions of \( \mu_n \) induce different shapes of detectable region than Fig. 2. See Sections 3.1 and 5.2 for further discussions.

### 1.3 Organization

The rest of the paper is organized as follows. Section 2 states the setup, defines the fundamental limit of sparse mixture detection and reviews some previously known results. The main results of the paper are presented in Sections 3 and 4, where we provide an explicit characterization of the optimal detection boundary under mild technical conditions. Moreover, it is shown in Section 4 that the higher criticism test achieves the optimal performance adaptively. Section 5 particularizes the general result to various special cases to give explicit formulae of the fundamental limits. Discussions of generalizations and open problems are presented in Section 6. The main theorems are proven in Section 7, while the proofs of the technical lemmas are relegated to the appendices.

### 1.4 Notations

Throughout the paper, \( \Phi \) and \( \varphi \) denote the cumulative distribution function (CDF) and the density of the standard normal distribution respectively. Let \( \Phi = 1 - \Phi \). Let \( P^n \) denote the \( n \)-fold product measure of \( P \). We say \( P \) is absolutely continuous with respect to \( Q \), denoted by \( P \ll Q \), if \( P(A) = 0 \) for any measurable set \( A \) such that \( Q(A) = 0 \). We say \( P \) is singular with respect to \( Q \), denoted by \( P \perp Q \), if there exists a measurable \( A \) such that \( P(A) = 1 \) and \( Q(A) = 0 \). We denote \( a_n = o(b_n) \) if \( \limsup_{n \to \infty} \frac{|a_n|}{|b_n|} = 0 \), \( a_n = \omega(b_n) \) if \( b_n = o(a_n) \), \( a_n = O(b_n) \) if \( \limsup_{n \to \infty} \frac{|a_n|}{|b_n|} < \infty \) and \( a_n = \Omega(b_n) \) if \( b_n = O(a_n) \). These asymptotic notations extend naturally to probabilistic setups, denoted by \( o_P, \omega_P, \text{ etc.} \), where limits are in the sense of convergence in probability.

### 2 Fundamental limits and characterization

In this section we define the fundamental limits for testing the hypotheses (10) in terms of the sparsity parameter \( \beta \). An equivalent characterization in terms of the Hellinger distance is also
2.1 Fundamental limits of detection

It is easy to see that as the non-null proportion $\epsilon_n$ decreases, the signal is more sparse and the testing problem in (10) becomes more difficult. Recall that $\epsilon_n$ is given by (3) where $\beta \geq 0$ parametrizes the sparsity level. Thus, the question of detectability boils down to characterizing the smallest (resp. largest) $\beta$ such that the hypotheses in (10) can be distinguished with probability tending to one (resp. zero), when the sample size $n$ is large.

For testing between two probability measures $P$ and $Q$, denote the optimal sum of Type-I and Type-II error probabilities by

$$E(P, Q) \triangleq \inf_A \{P(A) + Q(A^c)\},$$

where the infimum is over all measurable sets $A$. By the Neyman-Pearson Lemma [27], $E(P, Q)$ is achieved by the likelihood ratio test: declare $P$ if and only if $\frac{dP}{dQ} \geq 1$. Moreover, $E(P, Q)$ can be expressed in terms of the total variation distance

$$TV(P, Q) \triangleq \sup_A |P(A) - Q(A)| = \frac{1}{2} \int |dP - dQ|$$

as

$$E(P, Q) = 1 - TV(P, Q).$$

For a fixed sequence $\{(Q_n, G_n)\}$, denote the total variation between the null and alternative by

$$V_n(\beta) \triangleq TV(Q_n^n, ((1 - n^{-\beta})Q_n + n^{-\beta}G_n)^n),$$

which takes values in the unit interval. In view of (15), the fundamental limits of testing the hypothesis (10) are defined as follows.

Definition 1.

$$\beta^* \triangleq \sup \{\beta \geq 0 : V_n(\beta) \to 1\},$$

$$\longrightarrow \beta^* \triangleq \inf \{\beta \geq 0 : V_n(\beta) \to 0\}.$$  

If $\longrightarrow \beta^* = \beta^*$, the common value is denoted by $\beta^*$.

As illustrated by Fig. 1, the operational meaning of $\longrightarrow \beta^*$ and $\beta^*$ are as follows: for any $\beta > \beta^*$, all sequences of tests have vanishing probability of success; for any $\beta < \beta^*$, there exists a sequence of tests with vanishing probability of error. In information-theoretic parlance, if $\longrightarrow \beta^* = \beta^* = \beta^*$, we say strong converse holds, in the sense that if $\beta > \beta^*$, all tests fail with probability tending to one; if $\beta < \beta^*$, there exists a sequence of tests with vanishing error probability.

Clearly, $\longrightarrow \beta^*$ and $\beta^*$ only depend on the sequence $\{(Q_n, G_n)\}$. The following lemma, proved in Appendix A, shows that it is always sufficient to restrict the range of $\beta$ to the unit interval.
Figure 1: Critical values of $\beta$ and regimes of (in)distinguishability of the hypotheses (11) in the large-$n$ limit.

Lemma 1.

$$0 \leq \beta^* \leq \bar{\beta}^* \leq 1.$$ (19)

In the Gaussian mixture model with $Q_n = \mathcal{N}(0, 1)$, if the sequence $\{G_n\}$ is parametrized by some parameter $r$, the fundamental limit $\beta^*$ in Definition 1 is a function of $r$, denoted by $\beta^*(r)$. For example, in the Ingster-Donoho-Jin setup (2) where $G_n = \mathcal{N}(\mu_n, 1)$, $\beta^*$, denoted by $\beta_{IDJ}^*$, can be obtained by inverting (6):

$$\beta_{IDJ}^*(r) = \begin{cases} \frac{1}{2} + r & 0 < r \leq \frac{1}{4} \\ 1 - (1 - \sqrt{r})^2 & r > \frac{1}{4} \end{cases}.$$ (20)

In terms of (20), the detectable region (7) is given by the strict hypograph $\{(r, \beta) : \beta < \beta^*(r)\}$. The function $\beta_{IDJ}^*$, plotted in Fig. 2, plays an important role in our later derivations. Similarly, for the heteroscedastic mixture (8), inverting (9) gives

$$\beta^*(r, \sigma^2) = \begin{cases} \frac{1}{2} + \frac{r - \sigma^2}{2 - \sigma^2} & 2\sqrt{r} + \sigma^2 \leq 2 \\ 1 - (1 - \sqrt{r})^2 & 2\sqrt{r} + \sigma^2 > 2 \end{cases}.$$ (21)

As shown in Section 5, all the above results can be obtained in a unified manner as a consequence of the general results in Section 3.

2.2 Equivalent characterization via the Hellinger distance

Closely related to the total variation distance is the Hellinger distance [26, Chapter 2]

$$H^2(P, Q) \triangleq \int (\sqrt{dP} - \sqrt{dQ})^2,$$

which takes values in the interval $[0, 2]$ and satisfies the following relationship:

$$\frac{1}{2} H^2(P, Q) \leq TV(P, Q) \leq H(P, Q) \sqrt{1 - \frac{H^2(P, Q)}{4}} \leq 1.$$ (22)

Therefore, the total variation distance converges to zero (resp. one) is equivalent to the squared Hellinger distance converges to zero (resp. two). We will be focusing on the Hellinger distance partly due to the fact that it tensorizes nicely under the product measures:

$$H^2(P^n, Q^n) = 2 - 2 \left(1 - \frac{H^2(P, Q)}{2}\right)^n.$$ (23)
Figure 2: Ingster-Donoho-Jin detection boundary (20) and the detectable region (below the curve).

Denote the Hellinger distance between the null and the alternative by

$$H_n^2(\beta) \triangleq H^2(Q_n, (1 - n^{-\beta})Q_n + n^{-\beta}G_n).$$

In view of (17) – (18) and (23), the fundamental limits $\overline{\beta}^*$ and $\underline{\beta}^*$ can be equivalently defined as follows in terms of the asymptotic squared Hellinger distance:

$$\overline{\beta}^* = \sup \left\{ \beta \geq 0 : H_n^2(\beta) = \omega(n^{-1}) \right\},$$

$$\underline{\beta}^* = \inf \left\{ \beta \geq 0 : H_n^2(\beta) = o(n^{-1}) \right\}.$$

3 Main results

In this section we characterize the detectable region explicitly by analyzing the exact asymptotics of the Hellinger distance induced by the sequence of distributions $\{(Q_n, G_n)\}$.

3.1 Characterization of $\beta^*$ for Gaussian mixtures

This subsection we focus on the case of sparse normal mixture with $Q_n = \mathcal{N}(0, 1)$ and $G_n$ absolutely continuous. We will argue in Section 3.3 that by performing the Lebesgue decomposition on $G_n$ if necessary, we can reduce the general problem to the absolutely continuous case.

We first note that the essential supremum of a measurable function $f$ with respect to a measure $\mu$ is defined as$$\operatorname{ess sup}_{x} f(x) \triangleq \inf \{ a \in \mathbb{R} : \mu(\{x : f(x) > a\}) = 0 \}.$$We omit mentioning $\mu$ if $\mu$ is the Lebesgue measure. Now we are ready to state the main result of this section.
Theorem 1. Let $Q_n = \mathcal{N}(0,1)$. Assume that $G_n$ has a density $g_n$ with respect to the Lebesgue measure. Denote the log-likelihood ratio by

$$\ell_n \triangleq \log \frac{g_n}{\varphi}.$$  \hfill (27)

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a measurable function and define

$$\beta^\ast = \frac{1}{2} + \max \{ \alpha(u) - u^2 + \frac{u^2}{2} \}.$$ \hfill (28)

1. If

$$\liminf_{n \to \infty} \frac{\ell_n(u \sqrt{2 \log n})}{\log n} \geq \alpha(u) \quad \text{uniformly in } u \in \mathbb{R},$$ \hfill (29)

then $\beta^\ast \geq \beta^\sharp$.

2. If

$$\limsup_{n \to \infty} \frac{\ell_n(u \sqrt{2 \log n})}{\log n} \leq \alpha(u) \quad \text{uniformly in } u \in \mathbb{R},$$ \hfill (30)

then $\beta^\ast \leq \beta^\sharp$.

Consequently, if the limits in (29) and (30) agree and $\alpha > 0$ on a set of positive Lebesgue measure, then $\beta^\ast = \beta^\sharp$.

Proof. Section 7.2.

Assuming the setup of Theorem 1, we ask the following question in the reverse direction: What kind of function $\alpha$ can arise in equations (29) and (30)? The following lemma (proved in Section 7.2) gives a necessary and sufficient condition for $\alpha$. However, in the special case of convolutional models, the function $\alpha$ needs to satisfy more stringent conditions, which we also discuss below.

Lemma 2. Suppose

$$\lim_{n \to \infty} \frac{\ell_n(u \sqrt{2 \log n})}{\log n} = \alpha(u),$$ \hfill (31)

holds uniformly in $u \in \mathbb{R}$ for some measurable function $\alpha : \mathbb{R} \to \mathbb{R}$. Then

$$\lim_{t \to \infty} \frac{1}{t} \log \int_{\mathbb{R}} \exp(t(\alpha(u) - u^2)) du = 0.$$ \hfill (32)

In particular, $\alpha(u) \leq u^2$ Lebesgue-a.e. Conversely, for all measurable $\alpha$ that satisfies (32), there exists a sequence of $\{G_n\}$, such that (31) holds.

Additionally, if the model is convolutional, i.e., $G_n = P_n * \mathcal{N}(0,1)$, then $\alpha$ is convex.

In many applications, we want to know how fast the optimal error probability decays if $\beta$ lies in the detectable region. The following result gives the precise asymptotics for the Hellinger distance, which also gives upper bounds on the total variation, in view of (22).
Theorem 2. Assume that (31) holds. For any $\beta \geq \frac{1}{2}$, the exponent of the Hellinger distance (24) is given by

$$\lim_{n \to \infty} \frac{\log H_n^2(\beta)}{\log n} = E(\beta),$$

where

$$E(\beta) = \underset{u \in \mathbb{R}}{\text{ess sup}} \{(2(\alpha(u) - \beta)) \wedge (\alpha(u) - \beta) - u^2\}$$

$$= \underset{u: \alpha(u) \leq \beta}{\text{ess sup}} \{2\alpha(u) - 2\beta - u^2\} \lor \underset{u: \alpha(u) > \beta}{\text{ess sup}} \{\alpha(u) - \beta - u^2\}$$

which satisfies $E(\beta) > -1$ (resp. $E(\beta) < -1$) if and only if $\beta < \beta^\#$ (resp. $\beta > \beta^\#$).

As an application of Theorem 1, the following result relates the fundamental limit $\beta^*$ of the convolutional models to the classical Ingster-Donoho-Jin detection boundary:

Corollary 1. Let $G_n = P_n * N(0,1)$. Assume that $P_n$ has a density $p_n$ which satisfies that

$$\lim_{n \to \infty} \frac{\log p_n(t\sqrt{2\log n})}{\log n} = -f(t)$$

uniformly in $t \in \mathbb{R}$ for some measurable $f : \mathbb{R} \to \mathbb{R}$. Then

$$\beta^* = \underset{t \in \mathbb{R}}{\text{ess sup}} \{\beta^*_{IDJ}(t^2) - f(t)\}$$

where $\beta^*_{IDJ}$ is the Ingster-Donoho-Jin detection boundary defined in (20).

It should be noted that the convolutional case of the normal mixture detection problem is briefly discussed in [6, Section 6.1], where inner and outer bounds on the detection boundary are given but do not meet. Here Corollary 1 completely settles this question. See Section 5 for more examples.

We conclude this subsection with a few remarks on Theorem 1.

Remark 1 (Extremal cases). Under the assumption that the function $\alpha > 0$ on a set of positive Lebesgue measure, the formula (28) shows that the fundamental limit $\beta^*$ lies in the very sparse regime ($\frac{1}{2} \leq \beta^* \leq 1$). We discuss the two extremal cases as follows:

1. Weak signal: Note that $\beta^* = \frac{1}{2}$ if and only if $\alpha(u) \leq u^2 - \frac{u^2 + 1}{2}$ almost everywhere. In this case the non-null effect is too weak to be detected for any $\beta > \frac{1}{2}$. One example is the zero-mean heteroscedastic case $G_n = N(0, \sigma^2)$ with $\sigma^2 \leq 2$. Then we have $\alpha(u) \leq \frac{u^2}{2}$.

2. Strong signal: Note that $\beta^* = 1$ if and only if there exists $u$, such that $|u| \geq 1$ and

$$\alpha(u) = u^2.$$
maximum. Indeed, to understand the implication of (38) more quantitatively, let us look at an even weaker condition: there exists \( u \) such that \( |u| \geq 1 \) and

\[
\limsup_{n \to \infty} \frac{1}{\log n} \log \frac{1}{\mathbb{P} \{ u^{-1}Y_n \geq \sqrt{2 \log n} \}} = 0,
\]

which, as shown in Appendix B, implies that \( \beta^* = 1 \).

**Remark 2.** In general \( \beta^* \) need not exist. Based on Theorem 1, it is easy to construct a Gaussian mixture where \( \beta^* \) and \( \beta^\sharp \) do not coincide. For example, let \( \alpha_0 \) and \( \alpha_1 \) be two measurable functions which satisfy Lemma 2 and give rise to different values of \( \beta^\sharp \) in (28), which we denote by \( \beta^\sharp_0 < \beta^\sharp_1 \). Then there exist sequences of distributions \( \{ G_n^{(0)} \} \) and \( \{ G_n^{(1)} \} \) which satisfy (31) for \( \alpha_0 \) and \( \alpha_1 \) respectively. Now define \( \{ G_n \} \) by \( G_{2k} = G_k^{(0)} \) and \( G_{2k+1} = G_k^{(1)} \). Then by Theorem 1, we have \( \beta^* = \beta^\sharp_0 < \beta^\sharp = \beta^\sharp_1 \).

### 3.2 Non-Gaussian mixtures

The detection boundary in [20, 12] is obtained by deriving the limiting distribution of the log-likelihood ratio which relies on the normality of the null hypothesis. In contrast, our approach is based on analyzing the sharp asymptotics of the Hellinger distance. This method enables us to generalize the result of Theorem 1 to sparse non-Gaussian mixtures (10), where we even allow the null distribution \( Q_n \) to vary with the sample size \( n \).

**Theorem 3.** Consider the hypothesis testing problem (10). Let \( G_n \ll Q_n \). Denote by \( F_n \) and \( z_n \) the CDF and the quantile function of \( G_n \), respectively, i.e.,

\[
z_n(p) = \inf \{ y \in \mathbb{R} : F_n(y) \geq p \}.
\]

If the log-likelihood ratio

\[
\ell_n = \log \frac{dG_n}{dQ_n}
\]

satisfies

\[
\lim_{n \to \infty} \sup_{s \geq (\log_2 n)^{-1}} \left| \frac{\ell_n(z_n(n^{-s})) \vee \ell_n(z_n(1-n^{-s}))}{\log n} - \gamma(s) \right| = 0
\]

as \( n \to \infty \) uniformly in \( s \in \mathbb{R}_+ \) for some measurable function \( \gamma : \mathbb{R}_+ \to \mathbb{R} \). If \( \gamma > 0 \) on a set of positive Lebesgue measure, then

\[
\beta^* = \frac{1}{2} + 0 \vee \text{ess sup}_{s \geq 0} \left\{ \gamma(s) - s + \frac{s \wedge 1}{2} \right\}.
\]

The function \( \gamma \) appearing in Theorem 3 satisfies the same condition as in Lemma 2. Comparing Theorem 3 with Theorem 1, we see that the uniform convergence condition (31) is naturally replaced.
by the uniform convergence of the log-likelihood ratio evaluated at the null quantile. Using the fact that \( \frac{z}{1+z^2} \leq \frac{\Phi(z)}{\varphi(z)} \leq \frac{1}{z} \) for all \( z > 0 \) [1, 7.1.13], which implies that

\[
\Phi(z) = \frac{\varphi(z)}{z} (1 + o(1))
\]

uniformly as \( z \to \infty \), we can recover Theorem 1 from Theorem 3 by setting \( \gamma(s) = \alpha(-\sqrt{s}) \vee \alpha(\sqrt{s}) \).

### 3.3 Decomposition of the alternative

The results in Theorem 1 and Theorem 3 are obtained under the assumption that the non-null effect \( G_n \) is absolutely continuous with respect to the null distribution \( Q_n \). Next we show that it does not lose generality to focus our attention on this case. Using the Hahn-Lebesgue decomposition [15, Theorem 1.6.3], we can write

\[
G_n = (1 - \kappa_n)G'_n + \kappa_n \nu_n
\]

for some \( \kappa_n \in [0, 1] \), where \( G'_n \ll Q_n \) and \( \nu_n \perp Q_n \). Put

\[
\epsilon'_n = \frac{\epsilon_n (1 - \kappa_n)}{1 - \epsilon_n \kappa_n} \quad \text{and} \quad Q'_n = (1 - \epsilon'_n)Q_n + \epsilon'_n G'_n
\]

which satisfies \( Q'_n \ll Q_n \). Then \( (1 - \epsilon_n)Q_n + \epsilon_n G_n = (1 - \epsilon_n \kappa_n)Q'_n + \epsilon_n \kappa_n \nu_n \). By Lemma 7,

\[
H^2(Q_n, (1 - \epsilon_n)Q_n + \epsilon_n G_n) = \Theta(\epsilon_n \kappa_n \vee H^2((1 - \epsilon')Q_n + \epsilon'_n G'_n))
\]

Therefore the asymptotic Hellinger distance of the original problem is completely determined by \( \epsilon_n \kappa_n \) and the square-Hellinger distance \( H^2((1 - \epsilon')Q_n + \epsilon'_n G'_n) \), which is also of a sparse mixture form, with \( (\epsilon_n, G_n) \) replaced by \( (\epsilon'_n, G'_n) \) given in (46). In particular, we note the following special cases:

1. If \( \epsilon_n \kappa_n = O(n^{-1}) \), then \( H^2(Q_n, (1 - \epsilon_n)Q_n + \epsilon_n G_n) = o(n^{-1}) \) (resp. \( \omega(n^{-1}) \)) if and only if \( H^2(Q_n, (1 - \epsilon'_n)Q_n + \epsilon'_n G'_n) = o(n^{-1}) \) (resp. \( \omega(n^{-1}) \)), which means that detectability of the original sparse mixture coincide with the new mixture.

2. If \( \epsilon_n \kappa_n = \omega(n^{-1}) \), then \( H^2(Q_n, (1 - \epsilon_n)Q_n + \epsilon_n G_n) = \omega(n^{-1}) \), which means that the original sparse mixture can be detected reliably. In fact, a trivial optimal test is to reject the null hypothesis if there exists one sample lying in the support of the singular component \( \nu_n \).

### 4 Adaptive optimality of Higher Criticism tests

As discussed in Section 2.1, the fundamental limit \( \beta^* \) of testing sparse normal mixtures (11) can be achieved by the likelihood ratio test. However, in general the likelihood ratio test requires the knowledge of the alternative distribution, which is typically not accessible in practice. To overcome this limitation, it is desirable to construct adaptive testing procedures to achieve the optimal performance simultaneously for a collection of alternatives. This problem is also known
as universal hypothesis testing. See, e.g., [19, 33, 32] and the references therein, for results on discrete alphabets. The basic idea of adaptive procedures usually involves comparing the empirical distribution of the data to the null distribution, which is assumed to be known.

For the problem of detecting sparse normal mixtures, it is especially relevant to construct adaptive procedures, since in practice the underlying sparsity level and the non-zero priors are usually unknown. Toward this end, Donoho and Jin [12] introduced an adaptive test based on Tukey’s higher criticism statistic. For the special case of (2), i.e., \( P_n = \delta \sqrt{2r \log n} \), it is shown that the higher criticism test achieves the optimal detection boundary (20) while being adaptive to the unknown non-null parameters \((\beta, r)\). Following the generalization by Jager and Wellner [22] via Rényi divergence, next we explain briefly the gist of the higher criticism test.

Given the data \( Y_1, \ldots, Y_n \), denote the empirical CDF by

\[
F_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1\{Y_i \leq t\},
\]

respectively. Similar to the Kolmogorov-Smirnov statistic [30, p. 91] which computes the \( L_\infty \)-distance (maximal absolute difference) between the empirical CDF and the null CDF, the higher criticism statistic is the maximal pointwise \( \chi^2 \)-divergence between the null and the empirical CDF. We first introduce a few auxiliary notations. Recall that the \( \chi^2 \)-divergence between two probability measures is defined as

\[
\chi^2(P \| Q) \triangleq \int \left( \frac{dP}{dQ} - 1 \right)^2 dQ.
\]

In particular, the binary \( \chi^2 \)-divergence function (i.e., the \( \chi^2 \)-divergence between Bernoulli distributions) is given by

\[
\chi^2(\text{Bern}(p) \| \text{Bern}(q)) = \frac{(p - q)^2}{q(1 - q)},
\]

where Bern\((p)\) denotes the Bernoulli distribution with bias \( p \). The higher criticism statistic is defined by

\[
HC_n \triangleq \sup_{t \in \mathbb{R}} \sqrt{n} \chi^2(\text{Bern}(F_n(t)) \| \text{Bern}(\Phi(t)))
\]

\[
= \sqrt{n} \sup_{t \in \mathbb{R}} \frac{|F_n(t) - \Phi(t)|}{\Phi(t)\Phi(t)}
\]

(48)

Based on the statistics (48), the higher criticism test declares \( H_1 \) if and only if

\[
HC_n > \sqrt{2(1 + \delta) \log \log n}
\]

(50)

where \( \delta > 0 \) is an arbitrary fixed constant.

The next result shows that the higher criticism test achieves the fundamental limit \( \beta^* \) characterized by Theorem 1 while being adaptive to all sequences of distributions \( \{G_n\} \) which satisfy the regularity condition (31). This result generalizes the adaptivity of the higher criticism procedure far beyond the original equal-signal-strength setup in [12] and the heteroscedastic extension in [6].

**Theorem 4.** Under the same assumption of Theorem 1, for any \( \beta > \beta^* \), the sum of Type-I and Type-II error of the higher criticism test (50) vanishes as \( n \to \infty \).
5 Examples

In this section we particularize the general result in Theorem 1 to several interesting special cases to obtain explicit detection boundaries.

5.1 Ingster-Donoho-Jin detection boundary

We derive the classical detection boundary (20) from Theorem 1 for the equal-signal-strength setup (2), which is a convolutional model with signal distribution

\[ P_n = \delta_{\mu_n} \]

and \( \mu_n \) in (4). The log-likelihood ratio is given by

\[ \ell_n(y) = \log \frac{\varphi(y - \mu_n)}{\varphi(y)} = -\frac{\mu_n^2}{2} + \mu_n y = -r \log n + \sqrt{2r \log n} y. \]

Plugging in \( y = u\sqrt{2 \log n} \), we have \( \ell_n(u\sqrt{2 \log n}) = -r \log n + 2u\sqrt{r} \log n \). Consequently, the condition (31) is fulfilled uniformly in \( u \in \mathbb{R} \) with

\[ \alpha(u) = 2u\sqrt{r} - r. \]

Straightforward calculation yields that

\[ \text{ess sup}_{u \in \mathbb{R}} \left\{ 2u\sqrt{r} - r - u^2 + \frac{u^2 \land 1}{2} \right\} = \begin{cases} r & 0 < r \leq \frac{1}{4} \\ \frac{1}{2} - (1 - \sqrt{r})^2 & r > \frac{1}{4} \end{cases} \]

Applying Theorem 1, we obtain the desired expression (20) for \( \beta^*(r) \).

As a variation of (51), the symmetrized version of (51)

\[ P_n = \frac{1}{2}(\delta_{\mu_n} + \delta_{-\mu_n}) \]

was considered in [21, Section 8.1.6], whose detection boundary is shown to be identical to (20). Indeed, for binary-valued signal distributed according to (54), we have

\[ \ell_n(u\sqrt{2 \log n}) = -\frac{\mu_n^2}{2} + \log \cosh(\mu_n u\sqrt{2 \log n}) \]

\[ = -r \log n + \log(n^{2u\sqrt{r}} + n^{-2u\sqrt{r}}) - \log 2 \]

which gives rise to

\[ \alpha(u) = 2|u|\sqrt{r} - r \]

Comparing (55) with (52) and (53), we conclude that the detection boundary (20) still applies.
5.2 Dilated signal distributions

Generalizing both the unary and binary signal distributions in Section 5.1, we consider \( P_n \) that is the distribution of the random variable

\[
X_n = \mu_n X
\]

where \( \mu_n > 0 \) is a sequence of positive numbers and \( X \) is distributed according to a fixed distribution \( P \), parameterizing the shape of the signal. In other words, \( P_n \) is the dilation of \( P \) by \( \mu_n \). We ask the following question: By choosing the sequence \( \mu_n \) and the random variable \( X \), is it possible to have detection boundaries which are shaped differently than the classical Ingster-Donoho-Jin detection boundary?

It turns out that for \( \mu_n = \sqrt{2 \log n} \), the answer to the above question is negative. As the next theorem shows, the detection boundary is given by that of the classical setup rescaled by the \( L_\infty \)-norm of \( X \). Note that (51) and (54) corresponds to \( P = \delta \sqrt{r} \) and \( P = \frac{1}{2} (\delta \sqrt{r} + \delta - \sqrt{r}) \), respectively.

**Corollary 2.** Consider the convolutional model \( G_n = P_n \ast N(0,1) \), where \( P_n \) is the distribution of \( \sqrt{2 \log n} X \). Then

\[
\beta^* = \beta_{IDJ}^*(\|X\|_{\infty}^2) = \begin{cases} 
\|X\|_{\infty}^2 + \frac{1}{2} & 0 < \|X\|_{\infty} \leq \frac{1}{2} \\
1 - (1 - \|X\|_{\infty})^2 & \|X\|_{\infty} > \frac{1}{2}.
\end{cases}
\]

**Proof.** Recall that \( \beta_{IDJ}^*(\cdot) \) denotes the Ingster-Donoho-Jin detection boundary defined in (20). Since the log-likelihood ratio is given by \( \ell_n(y) = \mathbb{E} \left[ \exp \left( -\frac{X_n^2}{2} + X_n y \right) \right] \), we have

\[
\ell_n(u \sqrt{2 \log n}) = \log \mathbb{E} \left[ n^{-X_n^2 + 2uX} \right] = \esssup_X \left\{ -X_n^2 + 2uX \right\} \log n (1 + o(1)),
\]

where we have applied Lemma 3 and the essential supremum in (58) is with respect to \( P \), the distribution of \( X \). Therefore \( \alpha(u) = \esssup_X \left\{ -X_n^2 + 2uX \right\} \). Applying Theorem 1 yields the existence of \( \beta^* \), given by

\[
\beta^* = \frac{1}{2} + \esssup_{u \in \mathbb{R}} \left\{ \esssup_X \left\{ -X_n^2 + 2uX \right\} - u^2 + \frac{u^2 \wedge 1}{2} \right\}
\]

\[
= \frac{1}{2} + \esssup_{u \in \mathbb{R}} \esssup_X \left\{ -X_n^2 + 2uX - u^2 + \frac{u^2 \wedge 1}{2} \right\}
\]

\[
= \esssup_X \beta_{IDJ}^*(X^2)
\]

\[
= \beta_{IDJ}^*(\|X\|_{\infty}^2),
\]

where (59) follows from the facts that \( \beta_{IDJ}^*(\cdot) \) is increasing and that \( \|X\|_{\infty} = \esssup |X| \). \( \square \)
Remark 3. Corollary 2 tightens the bounds given at the end of [6, Section 6.1] based on the interval containing the signal support. From (57) we see that the detection boundary coincides with the classical case with $\sqrt{r}$ replaced by $L_\infty$-norm of $X$. Therefore, as far as the detection boundary is concerned, only the support of $X$ matters and the detection problem is driven by the maximal signal strength. In particular, for $\|X\|_\infty \geq 1$ or non-compactly supported $X$, we obtain the degenerate case $\beta^* = 1$ (see also Remark 1 about the strong-signal regime). However, it is possible that the density of $X$ plays a role in finer asymptotics of the testing problem, e.g., the convergence rate of the error probability and the limiting distribution of the log-likelihood ratio at the detection boundary.

One of the consequences of Corollary 2 is the following: as long as $\mu_n = \sqrt{2 \log n}$, non-compactly supported $X$ results in the degenerate case of $\beta^* = 1$, since the signal is too strong to go undetected. However, this conclusion need not be true if $\mu_n$ behaves differently. We conclude this subsection by constructing a family of distributions of $X$ with unbounded support and an appropriately chosen sequence $\{\mu_n\}$, such that the detection boundary is non-degenerate: Let $X$ be distributed according to the following generalized Gaussian (Subbotin) distribution $P_\tau$ [31] with shape parameter $\tau > 0$, whose density is

$$p_\tau(x) = \frac{\tau}{2\Gamma(\tau)} \exp(-|x|^\tau).$$

(60)

Put $\mu_n = \sqrt{2r(\log n)^{\frac{1}{2}} - \frac{1}{2}}$. Then the density of $X_n$ is given by $v_n(x) = \frac{1}{\mu_n} p\left(\frac{x}{\mu_n}\right)$. Hence

$$v_n(t\sqrt{2\log n}) = \frac{\tau}{2\Gamma(\tau)\mu_n} n^{-\frac{1}{2}} |t|^\tau r^{-\frac{1}{2}},$$

which satisfies the condition (36) with $f(t) = |t|^\tau r^{-\frac{1}{2}}$. Applying Corollary 1, we obtain the detection boundary $\beta^*$ (a two-dimensional surface parametrized by $(r, \tau)$ shown in Fig. 3) as follows

$$\beta^* = \sup_{t \in \mathbb{R}} \left\{ \beta_{IDJ}^* t^2 - |t|^\tau r^{-\frac{1}{2}} \right\} = \sup_{z \geq 0} \{ \beta_{IDJ}^* (rz^2) - z^\tau \}$$

(61)

where (20) is the Ingster-Donoho-Jin detection boundary.

Equation (61) can be further simplified for the following special cases.

- $\tau = 1$ (Laplace): Plugging (20) into (61), straightforward computation yields

$$\beta^* = \frac{1}{2} \vee \left( 1 - \frac{1}{2\sqrt{r}} \right)^2 = \begin{cases} \left( 1 - \frac{1}{2\sqrt{r}} \right)^2 & r > \frac{3}{2} + \sqrt{2}, \\
\frac{1}{2} & r \leq \frac{3}{2} + \sqrt{2}. \end{cases}$$

- $\tau = 2$ (Gaussian): In this case we have $X \sim \mathcal{N}(0, \frac{1}{2})$ and $X_n \sim \mathcal{N}(0, r)$. This is a special case of the heteroscedastic case in [6], which will be discussed in detail in Section 5.3. Simplifying (61) we obtain

$$\beta^* = \frac{1}{2} \vee \frac{r}{1 + r},$$

which coincides with (67).
5.3 Heteroscedastic normal mixture

The heteroscedastic normal mixtures considered in (8) corresponds to

\[ G_n = \mathcal{N}(\mu_n, \sigma^2) \]

with \( \mu_n \) given in (4) and \( \sigma^2 \geq 0 \). In particular, if \( \sigma^2 \geq 1 \), \( G_n \) is given by the convolution \( G_n = \Phi \ast P_n \), where the Gaussian component \( P_n = \mathcal{N}(\mu_n, \sigma^2 - 1) \) models the variation in the signal amplitude.

For any \( u \in \mathbb{R} \),

\[
\ell_n(u\sqrt{2 \log n}) = \log \frac{\varphi\left(\frac{u\sqrt{2 \log n} - \mu_n}{\sigma}\right)}{\varphi(u\sqrt{2 \log n})} = \alpha(u) \log n,
\]

(62)

where

\[
\alpha(u) = u^2 - \frac{(u - \sqrt{r})^2}{\sigma^2}.
\]

Similar to the calculation in Section 5.1, we have\(^1\)

\[
\sup_{0 \leq s \leq 1} \left\{ \alpha(s) - \frac{s}{2} \right\} = \begin{cases} \frac{r}{2 - \sigma^2} & 2\sqrt{r} + \sigma^2 \leq 2 \\ \frac{1}{2} - \frac{(1 - \sqrt{r})^2}{\sigma^2} & 2\sqrt{r} + \sigma^2 > 2 \end{cases}
\]

(63)

and

\[
\sup_{s \geq 1} \{ \alpha(s) - s \} = -\frac{(1 - \sqrt{r})^2}{\sigma^2}.
\]

(64)

Note that \( \frac{r}{2 - \sigma^2} - \frac{(1 - \sqrt{r})^2}{\sigma^2} \geq \frac{(\sigma^2 + 2\sqrt{r} - 2)^2}{2\sigma^2(2 - \sigma^2)} \geq 0 \) if \( 2\sqrt{r} + \sigma^2 \leq 2 \). Assembling (63) – (64) and applying Theorem 1, we have

\[
\beta^*(r, \sigma^2) = \frac{1}{2} + \left( \frac{r}{2 - \sigma^2} \right) \vee \left( \frac{1}{2} - \frac{(1 - \sqrt{r})^2}{\sigma^2} \right)
\]

(65)

\[
= \begin{cases} \frac{1}{2} + \frac{r}{2 - \sigma^2} & 2\sqrt{r} + \sigma^2 \leq 2 \\ 1 - \frac{(1 - \sqrt{r})^2}{\sigma^2} & 2\sqrt{r} + \sigma^2 > 2. \end{cases}
\]

(66)

\(^1\)In the first case of (63) it is understood that \( \frac{r}{2} = 0 \).
Solving the equation $\beta^*(r, \sigma^2) = \beta$ in $r$ yields the equivalent detection boundary (9) in terms of $r$.

In the special case of $r = 0$, where the signal is distributed according to $P_n = \mathcal{N}(0, \tau^2)$, we have

$$\beta^*(0, 1 + \tau^2) = \frac{\tau^2 \lor 1}{1 + \tau^2 \lor 1}.$$ (67)

Therefore, as long as the signal variance exceeds that of the noise, reliable detection is possible in the very sparse regime $\beta > \frac{1}{2}$, even if the average signal strength does not tend to infinity.

5.4 Non-Gaussian mixtures

We consider the detection boundary of the following generalized Gaussian location mixture which was studied in [12, Section 5.2]:

$$H_0^{(n)}: Y_i \overset{i.i.d.}{\sim} P_r(\cdot) \text{ versus } H_1^{(n)}: Y_i \overset{i.i.d.}{\sim} (1 - \epsilon_n)(1 - \epsilon_n)P_r(\cdot) + \epsilon_nP_r(\cdot - \mu_n),$$ (68)

where $P_r$ is defined in (60), and $\mu_n = (r \log n)^{\frac{1}{2}}$. Since $z(1 - n^{-s}) = z(n^{-s}) = (s \log n)^{\frac{1}{2}}(1 + o(1))$ uniformly in $s$, (42) is fulfilled with $\gamma(s) = s - |s^{\frac{1}{2}} - r^{\frac{1}{2}}|$. Applying Theorem 3, we have

$$\beta^*(r) = \frac{1}{2} + 0 \lor \sup_{s \geq 0} \left(-|s^{\frac{1}{2}} - r^{\frac{1}{2}}| + \frac{s \land \frac{1}{2}}{2}\right) = \frac{1}{2} + 0 \lor \sup_{u \geq 0} \left(-|u - r^{\frac{1}{2}}| + \frac{u^{\frac{1}{2}} \land \frac{1}{2}}{2}\right).$$ (69)

It is easy to verify that (69) agrees with the results in [12, Theorem 5.1]. Similarly, the detection boundary for exponential-$\chi^2_m$ mixture in [12, Theorem 1.7] can also be derived from Theorem 3.

6 Discussions

We conclude the paper with a few discussions and open problems.

6.1 Moderately sparse regime $0 \leq \beta \leq \frac{1}{2}$

Our main results in Section 3 only concern the very sparse regime $\frac{1}{2} < \beta < 1$. This is because under the assumption in Theorem 1 that $\alpha > 0$ on a set of positive Lebesgue measure, we always have $\beta^* \geq \frac{1}{2}$. One of the major distinctions between the very sparse and moderately sparse regimes is the effect of symmetrization. To illustrate this point, consider the sparse normal mixture model (11). Given any $G_n$, replacing it by its symmetrized version $\tilde{G}_n(dx) \triangleq \frac{G_n(dx) + G_n(-dx)}{2}$ always increases the difficulty of testing. This follows from the inequality $H^2(\tilde{G}_n, \Phi) \leq H^2(G_n, \Phi)$, a consequence of the convexity of the squared Hellinger distance and the symmetry of $\Phi$. A natural question is: Does symmetrization always have an impact on the detection boundary? In
the very sparse regime, it turns out that under the regularity conditions imposed in Theorem 1, symmetrization does not affect the fundamental limit $\beta^*$, because both $G_n$ and $\tilde{G}_n$ give rise to the same function $\alpha$. It is unclear whether $\beta^*$ and $\beta^*$ remain unchanged if an arbitrary sequence $\{G_n\}$ is symmetrized. However, in the moderately sparse regime, an asymmetric non-null effect can be much more detectable than its symmetrized version. For instance, direct calculation (see for example [6, Section 2.2]) shows that $\beta^*(r) = \frac{1}{2} - r$ for $G_n = \delta_{n-r}$, but $\beta^*(r) = \frac{1}{2} - 2r$ for $G_n = \frac{1}{2}(\delta_{n-r} + \delta_{-n-r})$.

Moreover, unlike in the very sparse regime, moment-based tests can be powerful in the moderately sparse regime, which guarantee that $\beta^* \geq \frac{1}{2}$. For instance, in the above examples $G_n = \delta_{n-r}$ or $G_n = \frac{1}{2}(\delta_{n-r} + \delta_{-n-r})$, the detection boundary can be obtained by thresholding the sample mean or sample variance respectively. More sophisticated moment-based tests such as the excess kurtosis tests have been studied in the context of sparse mixtures [24]. It is unclear whether they are always optimal when $\beta < \frac{1}{2}$.

6.2 Adaptive optimality of higher criticism tests

While Theorem 4 establishes the adaptive optimality of the higher criticism test in the very sparse regime $\beta > \frac{1}{2}$, the optimality of the higher criticism test in the moderately sparse case $\beta < \frac{1}{2}$ remains an open question. Note that in the classical setup (2), it has been shown [6] that the higher criticism test achieves adaptive optimality for $\beta \in [0, \frac{1}{2}]$ and $\mu_n = n^{-r}$. In this case since $\mu_n = o(1)$, we have $\alpha \equiv 0$ and Theorem 1 thus does not apply. It is possible to obtain a counterpart of Theorem 1 and an analogous expression for $\beta^*$ for the moderately sparse regime if one assumes a similar uniform approximation property of the log-likelihood ratio, for example, $\ell_n(u, \sqrt{\log n}) = n^{-\alpha(u) + o(1)}$ for some function $\alpha$. Another interesting problem is to investigate the optimality of procedures introduced in [22] based on Rényi divergence under the same setup of Theorem 4.

7 Proofs

7.1 Auxiliary results

Laplace’s method (see, e.g., [13, Section 2.4]) is a technique for analyzing the asymptotics of integrals of the form $\int \exp(Mf) d\nu$ when $M$ is large. The proof of Theorem 1 uses the following first-order version of the Laplace’s method. Since we are only interested in the exponent (i.e., the leading term), we do not use saddle-point approximation in the usual Laplace’s method and impose no regularity conditions on the function $f$ except for the finiteness of the integral. Moreover, the exponent only depends on the essential supremum of $f$ with respect to $\nu$, which is invariant if $f$ is modified on a $\nu$-negligible set.

Lemma 3. Let $(X, \mathcal{F}, \nu)$ be a measure space. Let $F : X \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable. Assume that

$$\lim_{M \to \infty} \frac{\log F(x, M)}{M} = f(x)$$

(70)
holds uniformly in $x \in X$ for some measurable $f : X \to \mathbb{R}$. If $\int_X \exp(M_0 f) d\nu < \infty$ for some $M_0 > 0$, then
\begin{equation}
\lim_{M \to \infty} \frac{1}{M} \log \int_X F(x, M) d\nu = \esssup_{x \in X} f(x).
\end{equation}

Proof. First we deal with the case of $\esssup f = \infty$, which implies that $\nu(\{f > a\}) > 0$ for all $a > 0$. Moreover, by Chernoff bound, $\nu(\{f > a\}) < \exp(-M_0 a) \int \exp(M_0 f) d\nu < \infty$. By (70), for any $\epsilon > 0$, there exists $K > M_0$ such that
\begin{equation}
\exp(M(f(x) - \epsilon)) \leq F(x, M) \leq \exp(M(f(x) + \epsilon))
\end{equation}
for all $x \in X$ and $M \geq K$. Therefore, $\int F(x, M) d\nu \geq \exp(-M\epsilon) \int \exp(Mf) d\nu \geq \exp(M(a - \epsilon))\nu(\{f > a\})$ for any $M > 0$ and $a > 0$. Then $\liminf_{M \to \infty} \frac{1}{M} \log \int \exp(Mf) d\nu \geq a - \epsilon$. By the arbitrariness of $a$ and $\epsilon$, we have $\lim_{M \to \infty} \frac{1}{M} \log \int \exp(Mf) d\nu = \infty$.

Next we assume that $\esssup f < \infty$. By replacing $f$ with $f - \esssup f$, we can assume that $\esssup f = 0$ without loss of any generality. Then $f \leq 0$ $\nu$-a.e. Hence, by (72),
\begin{equation}
\int F(x, M) d\nu \leq \int \exp(M(f + \epsilon)) d\nu \leq \exp(M\epsilon) \int \exp(M_0 f) d\nu < \infty
\end{equation}
holds for all $M \geq K$. By the arbitrariness of $\epsilon$, we have
\begin{equation}
\limsup_{M \to \infty} \frac{1}{M} \log \int \exp(Mf) d\nu \leq 0.
\end{equation}

For the lower bound, note that, by the definition of $\esssup f = 0$, $\nu(\{f > -\delta\}) > 0$ for all $\delta > 0$. Therefore, by (72), we have
\begin{equation}
\int F(x, M) d\nu \geq \exp(-M\epsilon) \int \exp(Mf) d\nu \geq \exp(-M(\delta + \epsilon))\nu(\{f > -\delta\})
\end{equation}
for any $M > 0$ and $\delta > 0$. First sending $M \to \infty$ then $\delta \downarrow 0$ and $\epsilon \downarrow 0$, we have
\begin{equation}
\liminf_{M \to \infty} \frac{1}{M} \log \int \exp(Mf) d\nu \geq 0,
\end{equation}
completing the proof of (71).

The following lemma is useful for analyzing the asymptotics of Hellinger distance:

**Lemma 4.**
1. For any $b > 0$, the function $s \mapsto (\sqrt{1 + b(s-1)} - 1)^2$ is strictly convex on $\mathbb{R}_+$ and strictly decreasing and increasing on $[0, 1]$ and $[1, \infty)$, respectively.

2. For any $t \geq 0$,
\begin{equation}
(\sqrt{2} - 1)^2 t \land t^2 \leq (\sqrt{1 + t} - 1)^2 \leq t \land t^2.
\end{equation}

Proof. 1. Since $t \mapsto \sqrt{1 + t}$ is strictly concave, $s \mapsto (\sqrt{1 + b(s-1)} - 1)^2 = 2 + b(s-1) - 2\sqrt{b(s-1)}$ is strictly convex. Solving for the stationary point yields the minimum at $s = 1$. 

20
2. First we consider $t \geq 1$. Since $t \mapsto (\sqrt{1+t} - 1)^2 = t - 2\sqrt{1+t}$ is convex, $t \mapsto (\sqrt{1+t} - 1)^2$ is increasing. Consequently, we have $(\sqrt{2} - 1)^2 \leq (\sqrt{1+t} - 1)^2 \leq 1$ for all $t \in [1, \infty)$.

Next we consider $0 \leq t \leq 1$. By the concavity of $t \mapsto \sqrt{1+t}$, $t \mapsto \sqrt{1+t} - 1$ is decreasing. Hence $\sqrt{2} - 1 \leq \sqrt{1+t} - 1 \leq \frac{1}{2}$ for all $t \in [0, 1]$. Assembling the above two cases yields (73). \hfill \Box

The following lemmas are useful in proving Theorem 4:

**Lemma 5.** Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and $\mu$ be any measure on $\mathbb{R}$. The function $g$ defined by

$$g(s) = \operatorname{ess sup}_{q \geq s} f(q)$$

is decreasing and lower-semicontinuous, where the essential supremum is with respect to $\mu$.

**Proof.** The monotonicity is obvious. We only prove lower-semicontinuity, which, in particular, also implies right-continuity. Let $s_n \to s$. By definition of the essential supremum, for any $\delta$, we have

$$\mu\{q \geq s : f(q) > g(s) - \delta\} > 0.$$ By the dominated convergence theorem, $\mu\{q \geq s_n : f(q) > g(s) - \delta\} \to \mu\{q \geq s : f(q) > g(s) - \delta\}$. Hence there exists $N$ such that $\mu\{q \geq s_n : f(q) > g(s) - \delta\} > 0$ for all $n \geq N$, which implies that $g(s_n) \geq g(s) - \delta$ for all $n \geq N$. By the arbitrariness of $\delta$, we have

$$\lim_{n \to \infty} g(s_n) \geq g(s),$$

completing the proof of the lower semi-continuity. \hfill \Box

**Lemma 6.** Under the conditions of Theorem 1, for any $u \geq 0$,

$$\lim_{n \to \infty} \frac{\log((1 - F_n(u\sqrt{2\log n})) \wedge F_n(-u\sqrt{2\log n}))}{\log n} = v(u) \triangleq \operatorname{ess sup}_{q \geq u} \{\alpha(q) - q^2\}.$$

**Proof.** First assume that $u > 0$. Then

$$1 - F_n(u\sqrt{2\log n}) = \int_{y \geq u\sqrt{2\log n}} \exp(\ell_n(y))\phi(y)dy = \sqrt{\frac{\log n}{\pi}} \int_{q \geq u} \exp(\ell_n(q\sqrt{2\log n}))n^{-q^2}dq$$

where the last equality follows from Lemma 3. The proof for $u < 0$ is completely analogous. \hfill \Box

**7.2 Proofs in Section 3**

**Proof of Theorem 1.** Let $W \sim \mathcal{N}(0, 1)$. Put $\nu_n = (1 - n^{-\beta})\mathcal{N}(0, 1) + n^{-\beta}G_n$. Since $G_n \ll \Phi$ by assumption, we also have $\nu_n \ll \mathcal{N}(0, 1)$. Denote the likelihood ratio by $L_n = \frac{d\nu_n}{d\Phi} = \exp(\ell_n)$. Then

$$\frac{d\nu_n}{d\Phi} = 1 + n^{-\beta}(\exp(\ell_n) - 1). \quad (74)$$

(Direct part) Recall the notation $\beta^2$ defined in (28), which can be equivalently written as

$$\beta^2 = \frac{1}{2} + \operatorname{ess sup}_{u \in \mathbb{R}} \left\{\alpha_+(u) - u^2 + \frac{u^2 \wedge 1}{2}\right\}.$$
Assuming (29), we show that $\beta^* \geq \beta^2$ by lower bounding the Hellinger distance. To this end, fix an arbitrary $\delta > 0$. Let $\beta = \beta^2 - 2\delta$. Denote by $\lambda$ the Lebesgue measure on the real line. By definition of the essential supremum, $\lambda\{u : \alpha_\nu(u) - u^2 + \frac{u^2\lambda u^2}{2} \geq \beta + \delta - \frac{1}{2}\} > 0$. Since $-u^2 + \frac{u^2\lambda u^2}{2} \leq 0$ for all $u$ and $\beta + \delta - \frac{1}{2} \geq -\delta$, we must have $\lambda\{u : \alpha_\nu(u) - u^2 + \frac{u^2\lambda u^2}{2} \geq \beta + \delta - \frac{1}{2}, \alpha_\nu(u) \geq 0\} > 0$. Since, by assumption, $\lambda\{u : \alpha_\nu(u) > 0\} > 0$, there exists $0 < \epsilon \leq \frac{\delta}{2}$, such that

$$\lambda\left\{ u : \alpha_\nu(u) - u^2 + \frac{u^2\lambda u^2}{2} \geq \beta + \delta - \frac{1}{2}, \alpha_\nu(u) \geq 2\epsilon \right\} > 0. \tag{75}$$

By assumption (29), there exists $N_\epsilon \in \mathbb{N}$ such that

$$\ell_n(u \sqrt{2\log n}) \geq (\alpha_\nu(u) - \epsilon) \log n \tag{76}$$

holds for all $u \in \mathbb{R}$ and all $n \geq N_\epsilon$. From (75), we have either

$$\lambda\left\{ u : |u| \leq 1, \alpha_\nu(u) - \frac{u^2}{2} \geq \beta + \delta - \frac{1}{2}, \alpha_\nu(u) \geq 2\epsilon \right\} > 0 \tag{77}$$

or

$$\lambda\left\{ u : |u| \geq 1, \alpha_\nu(u) - u^2 \geq \beta + \delta - 1, \alpha_\nu(u) \geq 2\epsilon \right\} > 0. \tag{78}$$

Next we discuss these two cases separately:

**Case I:** Assume (77). Let

$$U = \frac{W}{\sqrt{2\log n}} \sim \mathcal{N}\left(0, \frac{1}{2\log n}\right). \tag{79}$$

The square Hellinger distance can be lower bounded as follows:

$$H_\nu^2(\beta) = H^2(P, \nu_\nu) = \int \left(\sqrt{\frac{d\nu_\nu}{dP}} - 1\right)^2 dP$$

$$= \mathbb{E}\left[ \left(\sqrt{1 + n^{-\beta}(\exp(\ell_n(U \sqrt{2\log n})) - 1) - 1}\right)^2 \right] \tag{80}$$

$$\geq \mathbb{E}\left[ \left(\sqrt{1 + n^{-\beta}(\exp(\ell_n(U \sqrt{2\log n})) - 1) - 1}\right)^2 1\{|U|\leq 1, \alpha(U) - \frac{U^2}{2} \geq \beta + \delta - \frac{1}{2}, \alpha(U) \geq 2\epsilon\} \right]$$

$$\geq \mathbb{E}\left[ \left(\sqrt{1 + n^{-\beta}(n^{\alpha(U)} - \epsilon) - 1} - 1\right)^2 1\{|U|\leq 1, \alpha(U) - \frac{U^2}{2} \geq \beta + \delta - \frac{1}{2}, \alpha(U) \geq 2\epsilon\} \right] \tag{81}$$

$$\geq \frac{(\sqrt{2} - 1)^2}{4} \mathbb{E}\left[ n^{(\alpha(U) - \epsilon - \beta)\wedge 2(\alpha(U) - \epsilon - \beta)} 1\{|U|\leq 1, \alpha(U) - \frac{U^2}{2} \geq \beta + \delta - \frac{1}{2}, \alpha(U) \geq 2\epsilon\} \right] \tag{82}$$

$$= \frac{(\sqrt{2} - 1)^2 \sqrt{\log n}}{4\sqrt{n}} \int n^{(\alpha(U) - \epsilon - \beta)\wedge 2(\alpha(U) - \epsilon - \beta) - u^2} 1\{|U|\leq 1, \alpha(U) - \frac{U^2}{2} \geq \beta + \delta - \frac{1}{2}, \alpha(U) \geq 2\epsilon\} du \tag{83}$$

$$\geq \frac{(\sqrt{2} - 1)^2 \sqrt{\log n}}{4\sqrt{n}} \lambda\left\{ |u| \leq 1, \alpha(U) - \frac{u^2}{2} \geq \beta + \delta - \frac{1}{2}, \alpha(U) \geq 2\epsilon \right\} n^{-1 + \frac{\epsilon}{2}} \tag{84}$$

where

- (80): By (74).
• (81): By Lemma 4.1 and (76).

• (82): Without loss of generality, we can assume that \( n' \geq 2 \). Then applying the lower bound in Lemma 4.2 yields the desired inequality.

• (83): We used the density of \( U \) defined in (79).

• (84): Given that \( |u| \leq 1 \) and \( \alpha(u) - u^2 \geq \beta + \delta - 1 + \frac{\delta}{2} \), we have both \( \alpha(u) - \epsilon - \beta - u^2 \geq -1 + \frac{\delta}{2} \) and \( 2\alpha(u) - 2\epsilon - 2\beta - u^2 \geq -1 + 2\delta - 2\epsilon \geq -1 + \delta \).

Case II: Now we assume (78). Following analogous steps as in the previous case, we have

\[
H_n^2(\beta) \geq \frac{(\sqrt{2} - 1)^2 \sqrt{\log n}}{4\sqrt{\pi}} \int n^{(\alpha(u) - \epsilon - \beta) \wedge 2(\alpha(u) - \epsilon - \beta) - u^2} 1_{\{|u| \geq 1, \alpha(u) - u^2 \geq \beta + \delta - 1, \alpha(u) \geq 2\epsilon\}} du \geq \frac{(\sqrt{2} - 1)^2 \sqrt{\log n}}{4\sqrt{\pi}} \{ |u| \geq 1, \alpha(u) - u^2 \geq \beta + \delta - 1, \alpha(u) \geq 2\epsilon\} n^{-1 + \frac{\delta}{2}} \tag{85}
\]

where (85) is due to the following: Since \(|u| \geq 1\) and \( \alpha(u) - u^2 \geq \beta + \delta - 1\), we have both \( \alpha(u) - \epsilon - \beta - u^2 \geq \delta - \epsilon - 1 \geq -1 + \frac{\delta}{2} \) and \( 2\alpha(u) - 2\epsilon - 2\beta - u^2 \geq \epsilon^2 - 2 + 2\delta - 2\epsilon \geq -1 + \delta \).

Combining (84) and (85) we conclude that \( H_n^2(\beta) = \omega(n^{-1}) \). By the arbitrariness of \( \delta > 0 \) and the alternative definition of \( \beta^* \) in (25), the proof of \( \beta^* \geq \beta^\sharp \) is completed.

(Converse part) Fix an arbitrary \( \delta > 0 \). Let

\[
\beta = \beta^\sharp + 2\delta. \tag{86}
\]

We upper bound the Hellinger integral as follows: First note that

\[
H_n^2(\beta) = \mathbb{E} \left[ \left( \sqrt{1 + n^{-\beta}(L_n - 1)} - 1 \right)^2 1_{\{L_n \geq 1\}} \right] + \mathbb{E} \left[ \left( \sqrt{1 + n^{-\beta}(L_n - 1)} - 1 \right)^2 1_{\{L_n \leq 1\}} \right]. \tag{87}
\]

Applying Lemma 4.1, we have

\[
\mathbb{E} \left[ \left( \sqrt{1 + n^{-\beta}(L_n - 1)} - 1 \right)^2 1_{\{L_n \leq 1\}} \right] \leq (\sqrt{1 - n^{-\beta} - 1})^2 \leq n^{-2\beta} = o(n^{-1}), \tag{88}
\]

since \( \beta > \beta^\sharp \geq \frac{1}{2} \) by (86). Consequently, the asymptotics of the Hellinger integral \( H_n^2(\beta) \) is dominated by the first term in (87), denoted by \( a_n \), which we analyze below using the Laplace method.

By (30), there exists \( N_\delta \in \mathbb{N} \) such that

\[
\ell_n(u \sqrt{2 \log n}) \leq (\alpha(u) + \delta) \log n \tag{89}
\]
holds for all \( u \in \mathbb{R} \) and all \( n \geq N_\delta \). Then

\[
 a_n \triangleq E \left[ \left( \sqrt{1 + n^{-\beta}(L_n - 1)} - 1 \right)^2 1_{\{L_n \geq 1\}} \right] 
\]

\[
 = E \left[ \left( \sqrt{1 + n^{-\beta}(\exp(\ell_n(U\sqrt{2\log n})) - 1)} - 1 \right)^2 1_{\{L_n \geq 1\}} \right] 
\]

\[
 \leq E \left[ \left( \sqrt{1 + n^{-\beta}(n^{\alpha(U)} + \delta) - 1} - 1 \right)^2 1_{\{\alpha(U) \geq -\delta\}} \right] 
\]

\[
 \leq E \left[ \left( \sqrt{1 + n^{\alpha(U)} + \delta - 1} - 1 \right)^2 \right] 
\]

\[
 \leq E \left[ n^{2(\alpha(U) + \delta - \beta)} \right] 
\]

\[
 = \sqrt{\frac{\log n}{\pi}} \int n^{2(\alpha(u) + \delta - \beta)} - u^2 \, du < \infty 
\]

where (90) and (91) are due to (89) and Lemma 4.2, respectively. Next we apply Lemma 3 to analyze the exponent of (92). First we verify the integrability condition:

\[
 \int n^{2(\alpha(u) + \delta - \beta)\wedge (\alpha(u) + \delta - \beta) - u^2} \, du \leq n^{\delta - \beta} \int n^{\alpha(u) - u^2} \, du < \infty 
\]

in view of (32). Applying (71) to (92), we have

\[
a_n \leq n^{\mathrm{ess \sup}_{u \in \mathbb{R}} \{ (2(\alpha(u) + \delta - \beta)\wedge (\alpha(u) + \delta - \beta) - u^2 \} + o(1). 
\]

By (86), \( \alpha(u) - u^2 + \frac{u^2 + 1}{2} \leq \beta - \frac{1}{2} - 2\delta \) holds a.e. Consequently, \( \alpha(u) - u^2 \leq \beta - 1 - 2\delta \) holds for almost every \( u \in (\infty, -1] \cup [1, \infty) \) and \( \alpha(u) - \frac{u^2}{2} \leq \beta - \frac{1}{2} - 2\delta \) holds for almost every \( u \in [-1, 1] \). These conditions immediately imply that

\[
 (2(\alpha(u) + \delta - \beta)) \wedge (\alpha(u) + \delta - \beta) - u^2 \leq -1 + \delta 
\]

holds a.e. Assembling (87) and (93), we conclude that \( H_n^2(\beta) = o(n^{-1}). \) By the arbitrariness of \( \delta > 0 \) and the alternative definition of \( \overline{\beta}^* \) in (26), the proof of \( \overline{\beta}^* \leq \beta^2 \) is completed.

Proof of Theorem 2. In view of the proof of Theorem 1, the desired (33) readily follows from combining (84), (85), (88) and (93).

Proof of Lemma 2. Put

\[
c(t) \triangleq \int \exp(t(\alpha(u) - u^2)) \, du. 
\]

(Necessity) Since \( c(t) \geq 0 \), it is sufficient to prove

\[
 \limsup_{t \to \infty} \frac{\log c(t)}{t} \leq 0. 
\]

Since \( \int g_n = 1 \), we have \( \int g_n(u\sqrt{\log n}) \, du = (\log n)^{-1/2} \). By assumption, \( g_n(u\sqrt{\log n}) = n^{\alpha(u) - u^2 + o(1)} \) uniformly in \( u \). Then for all \( \delta > 0 \), \( c(\log n) = \int n^{\alpha(u) - u^2} \, du \leq \frac{n^{\delta}}{\sqrt{\log n}} < \infty \) holds for sufficiently
large $n$. In particular, $c(\log n) \leq n^{o(1)}$. For general $t > 0$, let $n_1 = \lfloor \exp(t) \rfloor$, $n_2 = \lceil \exp(t + 1) \rceil$ and $t_i = \log n_i$, $i = 1, 2$. Put $p = \frac{t_2 - t_1}{t_2 - t_1}$, $q = \frac{t_2 - t_1}{t_2 - t_1}$, $a = \frac{t_1}{p}$ and $b = \frac{t_1}{q}$. Then $\frac{1}{p}, \frac{1}{q} \in [0, 1]$. Hölder’s inequality yields $c(t) = \int e^{c(a)(u - u^2)}c(b)(u - u^2)du \leq c(t_1)^{\frac{1}{p}}c(t_2)^{\frac{1}{q}} \leq c(\log n_1)c(\log n_2) \leq \exp(o(t))$, which gives the desired (96). It then follows from Lemma 3 that $\text{ess sup}_u \{\alpha(u) - u^2\} \leq 0$, i.e., $\alpha(u) \leq u^2$ a.e.

(Sufficiency) Let $\alpha$ be a measurable function satisfying (32). Let $G_n$ be a probability measure with the density

$$g_n(y) = \frac{1}{c(\log n)\sqrt{\log n}} \exp \left\{ \alpha \left( \frac{y}{\sqrt{2\log n}} \right) \log n - \frac{y^2}{2} \right\},$$

which is a legitimate density function in view of (95). Then the log-likelihood ratio satisfies

$$\ell_n(u\sqrt{\log n}) = \log \frac{\sqrt{\pi}}{c(\log n)\sqrt{\log n}} + \alpha(u),$$

which fulfills (31) uniformly.

For convolutional models, the convexity of $\alpha$ is inherited from the geometric properties of the log-likelihood ratio in the normal location model: Since $y \mapsto \log \frac{\pi \phi(y - x)}{\phi(y)}$ is convex for any random variable $X$ (see, e.g., [18, Property 3] and [14]), we have $\ell_n((1 - t)u + tv) \log n) \geq (1 - t)\ell_n(u\sqrt{2\log n}) + t\ell_n(v\sqrt{2\log n})$ for any $t \in [0, 1]$ and $u, v \in \mathbb{R}$. Dividing both sides by $\log n$ and sending $n \to \infty$, we have $\alpha((1 - t)u + tv) \geq (1 - t)\alpha(u) + t\alpha(v)$.

**Proof of Corollary 1.** Since $g_n = \varphi \ast p_n$, we have

$$g_n(u\sqrt{2\log n}) = \int_{\mathbb{R}} \varphi(u\sqrt{2\log n} - x)p_n(x)dx$$

$$= \sqrt{2\log n} \int_{\mathbb{R}} \varphi((u - t)\sqrt{2\log n})p_n(x\sqrt{2\log n})dx$$

$$= n^{o(1)} \int_{\mathbb{R}} n^{-(u - t)^2} - f(t) + o(1)dx$$

$$= n^{o(1)} \int_{\mathbb{R}} (u - t)^2 - f(t) + o(1)dx$$

where the last equality follows from Lemma 3. Plugging the above asymptotics into $\ell_n = \log \frac{\pi \varphi(y)}{\varphi(y)}$, we see that (31) is fulfilled uniformly in $u \in \mathbb{R}$ with $\alpha(u) = u^2 - \text{ess inf}_{z \in \mathbb{R}} \{(u - \sqrt{r}z)^2 + |z|^2\}$. Applying Theorem 1, we obtain

$$\beta^* = \frac{1}{2} + \text{ess sup}_{u \in \mathbb{R}} \text{ess sup}_{t \in \mathbb{R}} \left\{ -(u - t)^2 - f(t) + \frac{u^2 \wedge 1}{2} \right\}$$

$$= \frac{1}{2} + \text{ess sup}_{t \in \mathbb{R}} \left\{ -f(t) + \text{ess sup}_{u \in \mathbb{R}} \left\{ -(u - t)^2 + \frac{u^2 \wedge 1}{2} \right\} \right\}$$

$$= \sup_{t \in \mathbb{R}} \{\beta^*_\mathcal{IDJ}(t^2) - f(t)\}$$

where the last step follows from the (53).

**Proof of Theorem 3.** Let $W_n \sim Q_n$. Put $\nu_n = (1 - n^{-\beta})\Phi + n^{-\beta}G_n$. Since $G_n \ll Q_n$ by assumption, we also have $\nu_n \ll P$. Denote the likelihood ratio (Radon-Nikodym derivative) by $L_n = \frac{dG_n}{dQ_n} = \exp(\ell_n)$. Then

$$\frac{d\nu_n}{dQ_n} = 1 + n^{-\beta}(\exp(\ell_n) - 1).$$

(97)
Instead of introducing the random variable $U$ in (79) for the Gaussian case, we apply the quantile transformation to generate the distribution of $W_n$: Let $U$ be uniformly distributed on the unit interval. Then $S = \log \frac{1}{U}$ which is exponentially distributed. Putting $S_n = \frac{S}{\log n}$, we have

$$W_n^{(d)}(U) = z_n(1 - S_n) = z_n(1 - n^{-S_n}). \quad (98)$$

Set $r_n(s) = \ell_n \circ z_n(n^{-s})$ and $t_n(s) = \ell_n \circ z_n(1 - n^{-s})$, which satisfy

$$\sup_{s \geq \log n^2} |r_n(s) - r_0(s) \log n| \leq \delta \log n \quad (99)$$

$$\sup_{s \geq \log n^2} |t_n(s) - r_1(s) \log n| \leq \delta \log n \quad (100)$$

for all sufficiently large $n$. For the converse proof, we can write the square Hellinger distance as an expectation with respect to $S_n$:

$$H_n^2(\beta) = E \left[ \left( \sqrt{1 + n^{-\beta}(\exp(\ell_n(z_n(U))) - 1)} - 1 \right)^2 \mathbb{1}_{\{0 < U < \frac{1}{2}\}} \right]$$

$$+ E \left[ \left( \sqrt{1 + n^{-\beta}(\exp(\ell_n(z_n(1 - U))) - 1)} - 1 \right)^2 \mathbb{1}_{\{0 < U \leq \frac{1}{2}\}} \right].$$

Analogous to (88), by truncating the log-likelihood ratio at zero, we can show that the Hellinger distance is dominated by the following:

$$a_n = E \left[ \left( \sqrt{1 + n^{-\beta}(\exp(\ell_n(z_n(U))) - 1)} - 1 \right)^2 \mathbb{1}_{\{0 < U < \frac{1}{2}, r_n(S_n) \geq 0\}} \right]$$

$$+ E \left[ \left( \sqrt{1 + n^{-\beta}(\exp(\ell_n(z_n(1 - U))) - 1)} - 1 \right)^2 \mathbb{1}_{\{0 < U \leq \frac{1}{2}, t_n(S_n) \geq 0\}} \right]$$

$$= E \left[ \left( \sqrt{1 + n^{-\beta}(r_n(S_n)) - 1} - 1 \right)^2 \mathbb{1}_{\{S_n > \log n^2, r_n(S_n) \geq 0\}} \right]$$

$$+ E \left[ \left( \sqrt{1 + n^{-\beta}(t_n(S_n)) - 1} - 1 \right)^2 \mathbb{1}_{\{S_n \geq \log n^2, t_n(S_n) \geq 0\}} \right]$$

$$\leq E \left[ \left( \sqrt{1 + n^{-\beta}(n^{\alpha_0} + \delta - 1)} - 1 \right)^2 + \left( \sqrt{1 + n^{-\beta}(n^{\alpha_1} + \delta - 1)} - 1 \right)^2 \right]$$

$$\leq 2E \left[ n^{2(\alpha_0 \vee \alpha_1(U)) + \delta - \beta \wedge (\alpha_0 \vee \alpha_1(U)) + \delta - \beta} \right]$$

$$\leq n^{-1 - \delta} \quad (104)$$

where (101) follows from (97) – (98), (102) from (99) – (100) and (104) from (92) – (94). The direct part of the proof is completely analogous to that of Theorem 1 by lower bounding the integral in (101).
7.3 Proof of Theorem 4

Proof. Let \( U_i = \Phi(X_i) \), which is uniformly distributed on \([0, 1]\) under the null hypothesis. With a change of variable, we have

\[
HC_n = \sqrt{n} \sup_{t \in \mathbb{R}} \frac{|F_n(t) - \Phi(t)|}{\sqrt{\Phi(t)F(t)}} \quad \text{(105)}
\]

\[
= \sqrt{n} \sup_{0 < u < 1} \frac{|F_n(\Phi^{-1}(u)) - u|}{\sqrt{u(1 - u)}},
\]

which satisfies that \( \frac{HC_n}{\sqrt{2 \log \log n}} \xrightarrow{p} 1 \) [30, p. 604]. Therefore the Type-I error probability of the test (50) vanishes for any choice of \( \delta > 0 \). It remains to show that \( HC_n = \omega(\log \log n) \) under the alternative. To this end, fix \( 0 < s < 1 \) and put \( r_{n,s} = \Phi(\sqrt{2s \log n}) \) and \( \rho_{n,s} = (1 - n^{-\beta})\Phi(\sqrt{2s \log n}) + n^{-\beta}G_n(\sqrt{2s \log n}) \). By (105), we have

\[
HC_n \geq V_n(s) \triangleq \sqrt{n} \frac{|F_n(\sqrt{2s \log n}) - r_{n,s}|}{\sqrt{r_{n,s}(1 - r_{n,s})}} \quad \text{(107)}
\]

\[
= \frac{N_n(s) - nr_{n,s}}{\sqrt{nr_{n,s}(1 - r_{n,s})}},
\]

where \( N_n(s) \triangleq \sum_{i=1}^n 1\{x_i \geq \sqrt{2s \log n}\} \) is binomially distributed with sample size \( n \) and success probability \( \rho_{n,s} \). Therefore

\[
E[V_n(s)] = \sqrt{n} \frac{n \rho_{n,s} - r_{n,s}}{\sqrt{r_{n,s}(1 - r_{n,s})}} = n^{1/2 - \beta}G_n(\sqrt{2s \log n}) - r_{n,s},
\]

\[
\text{and}
\]

\[
\text{var}V_n(s) = \frac{\rho_{n,s}(1 - \rho_{n,s})}{r_{n,s}(1 - r_{n,s})}.
\]

By Chebyshev’s inequality,

\[
\mathbb{P}\left\{ V_n(s) \leq \frac{1}{2} E[V_n(s)] \right\} \leq \frac{4 \text{var}V_n(s)}{E[V_n(s)]^2} = \frac{4 \rho_{n,s}(1 - \rho_{n,s})}{n(\rho_{n,s} - r_{n,s})^2}.
\]

By Lemma 6,

\[
1 - G_n(\sqrt{2s \log n}) = n^{v(s) + o(1)},
\]

where \( v(s) = \text{ess sup}_{q \geq s} \{\alpha(q) - q\} \geq -s \). Plugging (111) into (109) and (110) yields

\[
E[V_n(s)] = n^{1/2 - \beta + v(s) + o(1)}
\]

and

\[
\mathbb{P}\left\{ V_n(s) \leq \frac{1}{2} E[V_n(s)] \right\} \leq n^{2\beta - s - 1 - 2v(s) + o(1)} + n^{\beta - 1 - v(s) + o(1)}.
\]

Suppose that \( \beta < \frac{1 + s}{2} + v(s) \). Then \( \mathbb{E}[V_n(s)] = \omega(\sqrt{\log \log n}) \). Moreover, we have \( 2\beta - s - 1 - 2v(s) < 0 \) and \( \beta - 1 - v(s) \leq \frac{s - 1}{2} < 0 \) since \( s < 1 \). Combining (107), (112) and (113), we obtain

\[
\mathbb{P}\{HC_n > \sqrt{(2 + \delta) \log \log n}\} = 1 - o(1),
\]
that is, the Type-II error probability also vanishes. Consequently, a sufficient condition for the higher criticism test to succeed is

\[
\beta < \sup_{0<s<1} \frac{1 + s}{2} + v(s) \tag{114}
\]

\[
= \text{ess sup}_{0<s<1} \frac{1 + s}{2} + v(s), \tag{115}
\]

where (115) follows from the following reasoning: By [28, Proposition 3.5], the supremum and the essential supremum (with respect to the Lebesgue measure) coincide for all lower semi-continuous functions. Indeed, \( v \) is lower semi-continuous by Lemma 5, and so is \( s \mapsto 1 + s^2 + v(s) \).

It remains to show that the right-hand side of (115) coincides with the expression of \( \beta^* \) in Theorem 1. Indeed, we have

\[
\text{ess sup}_{0 \leq s \leq 1} \{ s + 2v(s) \} = \text{ess sup}_{0 \leq s \leq 1} \left\{ s + 2 \sup_{q \geq s} \{ \alpha(q) - q \} \right\}
\]

\[
= \sup_{q \geq 0} \sup_{0 \leq s \leq 1} \{ 2\alpha(q) - 2q + s \}
\]

\[
= \sup_{q \geq 0} \{ 2\alpha(q) - 2q + q \wedge 1 \}
\]

Note that the second equality follows from interchanging the essential suprema: For any bi-measurable function \( (x, y) \mapsto f(x, y) \),

\[
\text{ess sup}_{x} \text{ess sup}_{y} f(x, y) = \text{ess sup}_{y} \text{ess sup}_{x} f(x, y) = \text{ess sup}_{x,y} f(x, y),
\]

where the last essential supremum is with respect to the product measure. Thus the proof of the theorem is completed. \( \square \)

### A Hellinger distances for mixtures

This appendix collects a few properties of total variation and Hellinger distances for mixture distributions.

**Lemma 7.** Let \( 0 \leq \epsilon \leq 1 \) and \( Q_1 \perp P \). Then

\[
H^2(P, (1 - \epsilon)Q_0 + \epsilon Q_1) = 2(1 - \sqrt{1 - \epsilon}) + \sqrt{1 - \epsilon} H^2(P, Q_0) \tag{116}
\]

which satisfies

\[
\frac{1}{4} \leq \frac{H^2(P, (1 - \epsilon)Q_0 + \epsilon Q_1)}{\epsilon \vee H^2(P, Q_0)} \leq 4 \tag{117}
\]

**Proof.** Since \( Q_1 \perp P \), there exists a measurable set \( E \) such that \( P(E) = 0 \) and \( Q_1(E) = 1 \). Then

\[
H^2(P, (1 - \epsilon)Q_0 + \epsilon Q_1) = 2 - 2 \int \sqrt{dP((1 - \epsilon)dQ_0 + \epsilon dQ_1)}
\]

\[
= 2 - 2\sqrt{1 - \epsilon} \int_{E^c} \sqrt{dP}dQ_0
\]

\[
= 2 - \sqrt{1 - \epsilon} (2 - H^2(P, Q_0)).
\]

The inequalities in (117) follow from (116) and the facts that \( \frac{\epsilon}{2} \leq \sqrt{1 - \epsilon} \leq \epsilon \) and \( 0 \leq H^2 \leq 2 \). \( \square \)
Lemma 8. For any probability measures \((P, Q)\), \(\epsilon \mapsto H^2(P, (1 - \epsilon)P + \epsilon Q)\) is decreasing on \([0, 1]\).

Proof. Fix \(0 \leq \epsilon < \epsilon' \leq 1\). Since \((1 - \epsilon)P + \epsilon Q = ((1 - \epsilon')P + \epsilon'Q)\frac{\epsilon}{\epsilon'} + \frac{\epsilon - \epsilon'}{\epsilon'}P\), the convexity of \(H^2(P, \cdot)\) yields
\[
H^2((1 - \epsilon)P + \epsilon Q, P) \leq \frac{\epsilon}{\epsilon'} H^2((1 - \epsilon')P + \epsilon'Q, P).
\]

We conclude this appendix by proving Lemma 1 presented in Section 2.1:

Proof. By Lemma 8, the function \(\beta \mapsto H^2_n(\beta)\) is decreasing, which, in view of the characterization (25) – (26), implies that \(\beta^* \leq \bar{\beta}^*\). Thus it only remains to establish the rightmost inequality in (19). To this end, we show that as soon as \(\beta\) exceeds 1, \(V_n(\beta)\) becomes \(o(1)\) regardless of the choice of \(\{G_n\}\): Fix \(\beta > 1\). Then
\[
V_n(\beta) = TV(\Phi^n, ((1 - n^{-\beta})\Phi + n^{-\beta}G_n)^n)
\leq TV(\delta^n_0, ((1 - n^{-\beta})\delta_0 + n^{-\beta}\delta_1)^n)
= 1 - (1 - n^{-\beta})^n
\leq n^{1 - \beta}
= o(1),
\]
where (118) follows from the data-processing inequality, which is satisfied for all \(f\)-divergences [9], in particular, the total variation: \(TV(P_Y, Q_Y) \leq TV(P_X, Q_X)\), where \(Q_Y|X = P_Y|X\) is any probability transition kernel.

Remark 4. While Lemma 8 is sufficient for our purpose in proving Lemma 1, it is unclear whether the monotonicity carries over to \(\epsilon \mapsto TV(P^n, ((1 - \epsilon)P + \epsilon Q)^n)\), since product measures do not form a convex set. It is however easy to see that \(\epsilon \mapsto TV(P^n, ((1 - \epsilon)P + \epsilon Q)^n)\) is decreasing, which follows from the proof of Lemma 8 with \(H^2\) replaced by \(TV\). It is also clear that \(\epsilon \mapsto H^2(P^n, ((1 - \epsilon)P + \epsilon Q)^n)\) is decreasing in view of (23).

B The implication of the condition (39)

In this appendix we show that (39) implies that \(\beta^* = 1\), i.e., for any \(\beta < 1\), the hypotheses in (11) can be tested reliably. Without loss of generality, we assume that \(u \geq 1\). Then
\[
\tau_n \triangleq G_n((\sqrt{2\log n}, \infty)) = n^{-o(1)},
\]
We show that the total variation distance between the product measures converge to one. Put \(A_n = (-\infty, \sqrt{2s\log n}]^n\). In view of the first inequality in (14), the total variation distance can be lower bounded as follows:
\[
V_n(\beta) \geq \Phi^n(A_n) - ((1 - n^{-\beta})\Phi + n^{-\beta}G_n)^n(A_n).
\]
Using (44), we have

\[ \Phi^n(A_n) = (1 - \Phi(\sqrt{2s \log n}))^n = 1 - \frac{n^{1-s}}{\sqrt{4\pi s \log n}}(1 + o(1)). \]

On the other hand,

\[ ((1 - n^{-\beta}) \Phi + n^{-\beta} G_n)^n(A_n) = (1 - (1 - n^{-\beta}) \Phi(\sqrt{2s \log n}) - n^{-\beta} \tau_n)^n \]
\[ = (1 - n^{-s} + o(1) - n^{-\beta - s} + o(1) - n^{-\beta + o(1)})^n \]
\[ = o(1) \]

where the last equality is due to \( 0 < \beta < 1 \leq s \). Therefore \( V_n(\beta) = 1 - o(1) \) for any \( \beta < 1 \), which proves that \( \beta^* = 1 \).

In fact, the above derivation also shows that the following maximum test achieves vanishing probability of error: declare \( H_1 \) if and only if \( \max_i |X_i| > |u|\sqrt{2 \log n} \). In general the maximum test is suboptimal. For example, in the classical setting (2) where \( G_n = \delta_{\mu_n} \), [12, Theorem 1.3] shows that the maximum test does not attain the Ingster-Donoho-Jin detection boundary for \( \beta \in \left[ \frac{1}{2}, \frac{3}{4} \right] \).

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Wiley-Interscience, New York, NY, 1984.

[2] E. Arias-Castro, E. J. Candès, H. Helgason, and O. Zeitouni. Searching for a trail of evidence in a maze. *Annals of Statistics*, 36(4):1726–1757, 2008.

[3] E. Arias-Castro, E. J. Candès, and Y. Plan. Global testing under sparse alternatives: ANOVA, multiple comparisons and the higher criticism. *Annals of Statistics*, 39(5):2533–2556, 2011.

[4] E. Arias-Castro, D. L. Donoho, and X. Huo. Near-optimal detection of geometric objects by fast multiscale methods. *IEEE Transactions on Information Theory*, 51(7):2402–2425, Jul. 2005.

[5] M. V. Burnashev and I. A. Begmatov. On a problem of detecting a signal that leads to stable distributions. *Theory of Probability & Its Applications*, 35(3):556–560, 1990.

[6] T. T. Cai, J. X. Jeng, and J. Jin. Optimal detection of heterogeneous and heteroscedastic mixtures. *Journal of the Royal Statistical Society. Series B (Methodological)*, 73(5):629 – 662, Nov. 2011.

[7] T. T. Cai, J. Jin, and M. G. Low. Estimation and confidence sets for sparse normal mixtures. *Annals of Statistics*, 35(6):2421–2449, 2007.

[8] L. Cayon, J. Jin, and A. Treaster. Higher criticism statistic: detecting and identifying non-gaussianity in the WMAP first-year data. *Monthly Notices of the Royal Astronomical Society*, 362(3):826–832, 2005.
[9] I. Csiszárr. Information-type measures of difference of probability distributions and indirect observations. *Studia Sci. Math. Hungar.*, 2:299–318, 1967.

[10] L. de Haan and A. Ferreira. *Extreme Value Theory: An Introduction*. Springer, New York, NY, 2006.

[11] R. L. Dobrushin. A statistical problem arising in the theory of detection of signals in the presence of noise in a multi-channel system and leading to stable distribution laws. *Theory of Probability & Its Applications*, 3(2):161–173, 1958.

[12] D. L. Donoho and J. Jin. Higher criticism for detecting sparse heterogeneous mixtures. *Annals of Statistics*, 32(3):962–994, 2004.

[13] A. Erdélyi. *Asymptotic expansions*. Number 3. Dover Publications, New York, NY, 1956.

[14] R. Esposito. On a Relation between Detection and Estimation in Decision Theory. *Information and Control*, 12:116–120, 1968.

[15] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC Press, Boca Raton, FL, 1992.

[16] P. Hall and J. Jin. Innovated higher criticism for detecting sparse signals in correlated noise. *Annals of Statistics*, 38(3):1686 – 1732, 2010.

[17] P. Hall, Y. Pittelkow, and M. Ghosh. Theoretical measures of relative performance of classifiers for high dimensional data with small sample sizes. *Journal of the Royal Statistical Society. Series B (Methodological)*, 70(1):159–173, 2008.

[18] C. Hatsell and L. Nolte. Some Geometric Properties of the Likelihood Ratio. *IEEE Transactions on Information Theory*, 17(5):616–618, 1971.

[19] W. Hoeffding. Asymptotically optimal tests for multinomial distributions. *Annals of Mathematical Statistics*, pages 369–401, 1965.

[20] Y. I. Ingster. On some problems of hypothesis testing leading to infinitely divisible distributions. *Mathematical Methods of Statistics*, 6(1):47 – 69, 1997.

[21] Y. I. Ingster and I. A. Suslina. *Nonparametric goodness-of-fit testing under Gaussian models*. Springer, New York, NY, 2003.

[22] L. Jager and J. A. Wellner. Goodness-of-fit tests via phi-divergences. *Annals of Statistics*, 35(5):2018–2053, 2007.

[23] X. J. Jeng, T. T. Cai, and H. Li. Optimal sparse segment identification with application in copy number variation analysis. *Journal of the American Statistical Association*, 105(491):1156–1166, 2010.
[24] J. Jin, J. L. Starck, D. L. Donoho, N. Aghanim, and O. Forni. Cosmological non-gaussian signature detection: comparing performance of different statistical tests. *EURASIP Journal on Applied Signal Processing*, 15:2470 – 2485, 2005.

[25] M. Kulldorff, R. Heffernan, J. Hartman, R. Assunção, and F. Mostashari. A space–time permutation scan statistic for disease outbreak detection. *PLoS Medicine*, 2(3):e59, 2005.

[26] L. Le Cam. *Asymptotic methods in statistical decision theory*. Springer-Verlag, New York, NY, 1986.

[27] J. Neyman and E. S. Pearson. On the problem of the most efficient tests of statistical hypotheses. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 231:289–337, 1933.

[28] H. X. Phu and A. Hoffmann. Essential supremum and supremum of summable functions. *Numerical functional analysis and optimization*, 17(1-2):161–180, 1996.

[29] A. Rényi. On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, pages 547–561, 1961.

[30] G. R. Shorack and J. A. Wellner. *Empirical processes with applications to statistics*. John Wiley & Sons, 1986.

[31] T. Taguchi. On a generalization of Gaussian distribution. *Annals of the Institute of Statistical Mathematics, Part A*, 30(1):211–242, 1978.

[32] J. Unnikrishnan, D. Huang, S. P. Meyn, A. Surana, and V. V. Veeravalli. Universal and composite hypothesis testing via mismatched divergence. *IEEE Transactions on Information Theory*, 57(3):1587–1603, 2011.

[33] O. Zeitouni, J. Ziv, and N. Merhav. When is the generalized likelihood ratio test optimal? *IEEE Transactions on Information Theory*, 38(5):1597–1602, 1992.