Orbital Magnetic Susceptibility of Disordered Mesoscopic Systems

Moshe Goldstein and Richard Berkovits

The Minerva Center, Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

In this paper we study the orbital weak-field susceptibility of two-dimensional diffusive mesoscopic systems. For the previously unstudied regime of temperatures lower than the mean level spacing we find unexpected strong temperature as well as statistical ensemble dependence of the average and typical susceptibilities. An explanation for these features is given in terms of the long tail of the zero-temperature susceptibility distribution, including the parametric form of the temperature dependence. For temperatures higher than the mean level spacing we calculate the difference between the true canonical ensemble and the effective grand-canonical ensemble. We also perform numerical simulations, which seem to generally confirm previous theoretical predictions for this regime of temperatures, although some difficulties arise. The important role of gauge-invariance, especially how it renders Random Matrix Theory inapplicable to the study of orbital susceptibility, is discussed. We conclude by considering interaction effects, giving a new interpretation to previous results as well as demonstrating the influence of in-plane magnetic field on the interaction-induced orbital response.

PACS numbers: 73.21.-b, 73.23.Ra, 75.75.+a

I. INTRODUCTION

Orbital magnetism is a purely quantum phenomenon, completely absent in classical physics, due to a theorem of Born and van-Leeuwen. It is therefore not surprising that the magnetic response of mesoscopic systems, in which the particles move phase-coherently across the sample, shows many peculiar features, and is generally much larger than the magnetic response of macroscopic systems.

Although this issue has been the subject of an intense scrutiny for almost two decades, several issues remain obscure. Since the problem was examined almost exclusively using diagrammatic perturbation expansion for diffusive systems and semiclassical analysis for ballistic ones, the behavior of the susceptibility at temperatures lower than the mean level spacing was not adequately understood. The consideration of the role of the different statistical ensembles is also incomplete, as we discuss below, even for temperatures higher than the mean level spacing. Moreover, although much numerical effort was concentrated on investigating the magnetic response of multiply-connected geometries (persistent current), no similar effort was directed into understanding diffusive singly-connected geometries. Our aim in this paper is to give some theoretical considerations for the above mentioned issues, and also to compare these and previous results with numerical simulations.

We will start with a survey of previous results. Here, as well as in our analytical and numerical calculations, we limit ourselves to the case of singly-connected two dimensional systems of spinless noninteracting electrons (except in section V), and to the linear response regime. We will always consider the susceptibility per unit area.

The orbital magnetic response of macroscopic systems is the celebrated Landau diamagnetic susceptibility\[\chi_L = -e^2/24\pi mc^2.\] This form is restricted to the linear response regime, \(\hbar\omega_c < T, \) where \(T\) is the temperature (we use throughout units where the Boltzmann constant is unity) and \(\omega_c = eH/mc\) is the cyclotron frequency (\(H\) denotes the applied magnetic field). The Landau susceptibility is temperature independent, up to temperatures of the order of the Fermi energy \(\epsilon_F,\) above which the above result is multiplied by \(\epsilon_F/T.\)

This macroscopic Landau susceptibility is quite small. In a semiclassical picture this is a consequence of the absence of flux-enclosing trajectories in infinite clean systems. However, in mesoscopic systems such trajectories do exist, due to scattering by either impurities or the sample boundaries. These two mechanisms tend to strongly enhance the susceptibility as described below. In the following we will assume that the dephasing caused by inelastic scattering (by other electrons, phonons, etc.) is negligible, i.e., \(L_\phi > L,\) where \(L_\phi\) is the phase coherence length, and \(L\) is the system’s size.

We will first consider disorder-induced susceptibility, which is dominant in the diffusive regime, \(k_F L < \ell < L,\) where \(k_F\) is the Fermi wave-vector and \(\ell\) is the elastic mean free path. The presence of disorder causes large sample specific fluctuations. Thus, the typical susceptibility, measured by the r.m.s. value \(\delta\chi,\) is much larger than the average susceptibility, where the average is over an ensemble of disorder realizations. For temperatures below the Thouless energy \(E_C = \hbar D/L^2\) (\(D = v_F\ell/2\) is the diffusion coefficient in two dimensions), the typical susceptibility is expected to vary as \(|\chi_L|/(k_F\ell)\ln(E_C/T))^{1/2}.\) The form of the logarithmic factor is correct in the very low field regime \(\Phi < (T/E_C)^{1/2}\Phi_0,\) where \(\Phi = HS\) is the total magnetic flux through the system (\(S\) is the system’s area), and \(\Phi_0 = hc/e\) is the flux quantum. These fluctuations may persist even in temperatures above the Thouless energy, where the temperature dependent factor changes to \((E_C/T)^{1/2},\) with an increased range of validity for linear response.

We now regard the disorder averaged susceptibility. Here we should distinguish between the Canonical Ensemble (CE) and the Grand Canonical Ensemble (GCE).
The GCE corresponds to a situation where the systems in the ensemble of disorder realizations can exchange particles with a heat bath with a fixed, system independent chemical potential. The CE corresponds to a situation where the systems are isolated, with the same fixed number of particles in each. Usually it was assumed that the CE can be described by an Equivalent Grand Canonical Ensemble (EGCE), in which each system can exchange particles with a different heat bath, with its chemical potential adjusted to the realization so that the thermally averaged number of particles in the system equals the given fixed particle number. Thus, the difference appears only after averaging over an ensemble of systems with different disorder realizations, but not for a single system. However, it was argued several times in the past that the CE and EGCE are really equivalent only at zero temperature or at high temperatures (higher than the mean level spacing $\Delta$ or even the Thouless energy $E_C$). For intermediate temperatures the level occupancy in the CE cannot be described by the Fermi-Dirac distribution.

It is predicted that in the GCE the ensemble averaged susceptibility will be equal to the Landau diamagnetic response. Disorder only modifies this result by a small contribution, of order $(k_F \ell)^{-1}$. On the other hand, in the EGCE there is an additional paramagnetic contribution, of order $|\chi_L|(k_F \ell)\Delta/T$. This contribution is much larger than the Landau susceptibility, and will dominate both the sign and the magnitude of the response up to temperatures of the order of the Thouless energy $E_C$. This paramagnetic part of the response is expected to be strongly field dependent even in the weak field regime $\Phi < \Phi_0$, and linear response is expected to apply only for very low fields, for which $\Phi < (T/E_C)^{1/2}\Phi_0$. As we will show in the appendix, moving to the pure CE adds another term, of order $|\chi_L|(k_F \ell)(\Delta/T)^2$, which is smaller than the difference between the GCE and the EGCE, but may still be larger than the Landau susceptibility. All this applies to the average susceptibility; the r.m.s. susceptibility is always assumed to be ensemble-independent, at least to the leading order.

In the past few years it was recognized that the disorder-induced susceptibility can be important even in the ballistic regime $\ell > L$, and will cease to exists only in the clean limit, $\ell > (k_F L)$, where the disorder causes only a small perturbation to the energy levels. However, the situation in this case is still unclear, and will not be discussed in the following.

It should be noted that all the quoted results in the diffusive regime were obtained using diagrammatic perturbation theory, and are therefore valid only for temperatures above the mean level spacing $\Delta$. It was conjectured, using different semi-quantitative arguments, that there is almost no temperature dependence for $T < \Delta$, and that in this regime the values of the average and typical susceptibility are given by the above expressions evaluated at $T \sim \Delta$.

In clean mesoscopic systems the susceptibility is enhanced by boundary effects. Here, the spatially limited electron motion results in oscillations in the susceptibility as a function of the electron density. The oscillation’s period is of order $k_F L$, where $k_F$ is the Fermi wavevector. These oscillations are accompanied by large paramagnetic peaks as a function of electron density. These peaks result from the occurrence of degenerate or nearly-degenerate levels, and are eliminated as the temperature $T$ increases above the mean level spacing $\Delta$. The oscillations, however, persist up to much higher temperatures, of the order of the inverse time of flight across the sample, $h v_F / L$, beyond which they decay exponentially with the temperature ($v_F$ is the Fermi velocity). In all the cases the susceptibility is much larger than the Landau value, and is of order $|\chi_L|(k_F L)^\alpha$, where the exponent $\alpha$ is of order unity, depending on whether the system is classically integrable or chaotic, and also on the specific geometry in the former case. Introducing disorder into the system makes these effects decay as $\exp(-L/\ell)$, i.e., they should disappear in the diffusive regime $\ell < L$.

Since in reality the system’s size and shape cannot be set accurately, one usually averages the results over a distribution of sizes around an average value. This size averaging smears out the oscillations, and reduces the size averaged GCE susceptibility to the Landau value for temperatures higher than the mean level spacing. The size averaged EGCE susceptibility, as well as the size averaged r.m.s. susceptibility remain enhanced by factors which are again powers of $k_F L$. In this paper we will use the clean case only for comparison with the diffusive case, so we will not perform size-averaging.

The paper is organized as follows: In section II we will first show that the source of difficulties in evaluating the response of diffusive systems in the range $T < \Delta$ is related to the problem of keeping the gauge-invariance of the results in the disorder averaging, and that this makes Random Matrix Theory (RMT) inapplicable for evaluating the susceptibility. This discussion will also serve to introduce notations and formulas for usage in subsequent calculations. In section III we will then examine our numerical results for both clean and diffusive systems in the various statistical ensembles. These will be seen to generally agree with the theoretical predictions for $T > \Delta$, although not in all the details. For $T < \Delta$ an unexpected strong temperature and statistical ensemble dependence of the typical susceptibility in the diffusive regime is observed, together with similar but weaker effects in the average susceptibility. This will be shown in section IV to stem from long tails in the susceptibility distribution. It will enable us to work out the parametric form of the r.m.s. susceptibility temperature dependence, as well as the reasons for its statistical ensemble dependence. Finally, we will consider interaction effects in section V.

We will give a new way of interpreting the reasons for interaction-induced susceptibility, as well as a numerical demonstration of its dependence on applied in-plane magnetic field.
II. CONSEQUENCES OF GAUGE INVARIANCE

We consider a system of spinless non-interacting electrons confined to a finite area and moving in a random potential of impurities and a magnetic field. The system is described by the following Hamiltonian:

\[
\hat{H} = \frac{1}{2m} \left( \hat{p} - \frac{e}{c} \mathbf{A} \right)^2 + V_{\text{conf}} + V_{\text{dis}},
\]

where \( \hat{p} \) is the momentum operator, \( \mathbf{A} \) is the vector-potential, \( V_{\text{conf}} \) is the confining potential and \( V_{\text{dis}} \) is the random impurity potential.

If the magnetic field is weak, its influence can be taken as a perturbation. Since the eigenstates in zero field can be chosen real and are non-degenerate (due to level repulsion), the first order correction to the energy (containing the term linear in the magnetic field in the Hamiltonian to first order in perturbation theory) identically vanish, i.e., there is no spontaneous magnetic moment.

The second order correction to the energy determines the susceptibility, which is composed of the familiar Larmor diamagnetic term (containing the term quadratic in the magnetic field in the Hamiltonian to first order in perturbation theory) and van-Vleck paramagnetic term (containing the term linear in the magnetic field in the Hamiltonian to second order in perturbation theory). At finite temperature they take the following form:

\[
\chi_{\text{Larmor}} = -\frac{1}{S H^2 mc^2} \sum_k f(\epsilon_k) \langle k | A^2 | k \rangle,
\]

\[
\chi_{\text{van-Vleck}} = -\frac{1}{S H^2} \frac{e}{2mc} \sum_{k \neq l} \frac{f(\epsilon_k) - f(\epsilon_l)}{\epsilon_k - \epsilon_l} \times |\langle k | \mathbf{A} \cdot \hat{p} + \hat{p} \cdot \mathbf{A} | l \rangle|^2,
\]

where \( |k\rangle, |l\rangle \) are the eigenstates of the Hamiltonian without magnetic field and \( f(\epsilon) \) is the average occupation of a level with energy \( \epsilon \), whose form depends on the required ensemble, and will be given at the end of this section.

It has been noted several times in the past \cite{11,12} that there is a large cancellation between these two contributions. We interpret this as a consequence of gauge invariance. If one replaces the vector potential \( \mathbf{A} \) by \( \mathbf{A} + \nabla \phi \), where \( \phi \) is some function of the coordinates, the observables of the system cannot change. When treating the magnetic field as a perturbation, this means that \( \phi \) must cancel between all the terms of the same order in the perturbation theory. Indeed, it can be shown that the changes of the Larmor and van-Vleck susceptibilities due to the gauge transformation identically cancel. This explains why these two seemingly separate contributions are strongly correlated and tend to cancel each other, leaving behind only the gauge invariant part of their sum.

In fact, the expression for the Larmor susceptibility can be brought into a form very similar to the expression for the van-Vleck susceptibility, thus demonstrating the deep connection between them. The resulting formula for the total susceptibility reads (in the family of gauges obeying \( \nabla \cdot \mathbf{A} = 0 \)):

\[
\chi = -\frac{1}{2S H^2} \left( \frac{e}{mc} \right)^2 \sum_{k \neq l} \frac{f(\epsilon_k) - f(\epsilon_l)}{\epsilon_k - \epsilon_l} \int_S \int_{S'} d\mathbf{r} d\mathbf{r}' \sum_{\alpha,\beta} (A_\alpha(\mathbf{r}) - A_\alpha(\mathbf{r}'))(A_\beta(\mathbf{r}) - A_\beta(\mathbf{r}')) \\
\times \psi_k(\mathbf{r}) \nabla_\alpha \psi_l(\mathbf{r}) \psi_l(\mathbf{r}') \nabla_\beta \psi_k(\mathbf{r}'),
\]

where the indices \( \alpha, \beta \) run over the two Cartesian coordinates and the integrations are over the system’s area. Since the wave functions in this expression are most strongly correlated when \( \mathbf{r} = \mathbf{r}' \), but the integrand vanishes there identically, we can understand why the resulting average susceptibility is much smaller than the Larmor or van-Vleck terms separately.

Another lesson we can learn from this is that when trying to calculate the disorder averaged susceptibility, one should check this does not destroy the gauge invariance of the results. For example, the proof of gauge invariance in our case relies on the relation \( \langle k | \hat{p} | l \rangle = i(\epsilon_k - \epsilon_l)/\hbar \langle k | l \rangle \). This implies a connection between matrix elements of the eigenstates and their eigenvalue difference which is usually absent in many averaging schemes, especially RMT.

This problem does not arise in diagrammatic calculations, where the average or r.m.s susceptibilities are expressed in terms of the energy shifts of the diffuson and Cooperon propagators, since the equations governing them are manifestly gauge invariant. Neither is there a problem in the semiclassical approach, where the influence of weak magnetic fields is taken only through the phase accumulated in closed orbits, which depends only
on the total flux through the orbits and is thus again
gauge invariant. However, for temperatures lower than
the mean level spacing those approaches are inapplicable.
In this regime RMT is usually used, but, as mentioned
above, it fails to give gauge invariant results.

To conclude this section we give the explicit form of the
mean level occupation \( f(\epsilon_n) \) of the \( n \)-th single particle
level. In the GCE or EGCE it is simply the Fermi-Dirac
distribution function,
\[
\hat{f}_{\text{GCE}}(\epsilon) = \frac{1}{e^{(\epsilon - \mu)/T} + 1},
\]
where the chemical potential \( \mu \) is system independent in
the GCE. In the EGCE it is system specific, determined
by the requirement that the thermally averaged particle
number in the EGCE, \( \sum_n f(\epsilon_n) \), will be equal to the
constant particle number in the sample. For the CE the oc-
cupation function can be represented by Darwin-Fowler
integrals, which are convenient for both analytical and
numerical calculations:
\[
f_{\text{CE}}(\epsilon) = \frac{\int f_{\text{GCE}}(\epsilon'; z) e^{-\Omega(z)/T} \, dz}{\int e^{-\Omega(z)/T} \, dz}^{\sum n} \tag{6}
\]
where
\[
\Omega(z) = -T \sum_n \ln \left(1 + ze^{-\epsilon_n/T}\right)
\]
is the grand canonical potential, \( z = \exp(\mu/T) \) is the fu-
gacity, and the integrations are taken around any closed
contour in the complex \( z \) plane encircling the origin. This
expression can be shown to reduce to the Fermi-Dirac
distribution function both for zero temperature and for
temperatures much higher than the mean level spacing,
where the integral can be evaluated using saddle-point
approximation (The saddle-point is the fugacity corre-
sponding to the sample-specific chemical potential in the
EGCE).

III. NUMERICAL SIMULATIONS

In this section we will present the results of our numerical
simulations for the susceptibility. For this purpose
we use the familiar Anderson tight-binding Hamiltonian:
\[
\hat{H} = \sum_s \epsilon_s \hat{n}_s - t \sum_{\langle s,s' \rangle} e^{i \phi_{s,s'}} \hat{a}_s^\dagger \hat{a}_{s'}
\]
where \( \hat{a}_s^\dagger, \hat{a}_s \) and \( \hat{n}_s \) denote electron creation, annihi-
lation and number operators, respectively, for a state on
site \( s \) of a square lattice. The first term is a random on-
site potential, where \( \epsilon_s \) is chosen randomly in the range
\([-W/2, W/2]\); the second is the hopping or kinetic term,
where the sum is over nearest-neighbor sites \( s \) and \( s' \), \( t \) is
the overlap integral, and the phase \( \phi_{s,s'} = e^{i \phi_{s,s'}} \\text{A} \cdot \text{d} \text{r} \)
gives the influence of the external magnetic field. The
Anderson model has the advantage of being discrete, but,
unlike numerical discretization of a continuous Hamilton-
ian, it has all the required properties of a Hamiltonian
(i.e., being Hermitian, gauge-invariant, etc.)

In the calculation we use expressions for the Larmor
and van-Vleck susceptibilities for the Anderson Hamil-
tonian. Since the magnetic field is assumed small, we
first expand the phase exponent containing the magnetic
field in Eq. (7) in series. As for the continuous Hamilton-
ian \( \hat{H} \), the linear term in the field taken to first order in
perturbation theory identically vanishes. We are left with
the second order corrections to the energy – the quadratic
term in the field taken to first order in perturbation the-
ory (Larmor susceptibility), and the linear term in the field
taken to second order in perturbation theory (van-
Vleck susceptibility). We thus get formulas analogous to
Eqs. (2), (3) for the susceptibility, which are used in all
the subsequent calculations. The level occupancies are
calculated in the required ensembles, using Eqs. (5), (6).
We note that even though the contour of integrations in
Eq. (6) can be evaluated in principle using any path in the
complex plane encircling the origin, to get sensible
results in numerical integration one should use a contour
passing through or near the sample specific saddle point
(the EGCE fugacity) in a direction where this point is a
maximum of the integrand. A suitable form of the con-
tour is thus a circle around the origin, and this choice
was used in our calculations.

The calculations were usually made (unless otherwise
specified) on a \( 17 \times 24 \) lattice (the sizes were chosen mu-
tually prime so that there’s no degeneracy even in the
clean limit, and our non-degenerate perturbation theory
is applicable). We have made calculations both on clean
(\( W = 0 \)) systems and disordered systems in the diffu-
sive regime, especially with the disorder values \( W = 2.0 \)
and \( W = 4.0 \). For the diffusive systems the results were
averaged over 2500 realizations of disorder, unless other-
wise specified. Dependence on the system’s size was
deduced from comparison to the results on \( 13 \times 19 \) and
\( 8 \times 13 \) lattices. The susceptibility is usually plotted as a
function of electron filling (i.e., the ratio of the number
of electrons and the number of lattice sites). To enable
comparison this is done not only in the CE or EGCE, but
also in the GCE, in which case we refer to the ensemble
and thermally averaged filling.

A. Macroscopic Clean Systems

For comparison with subsequent results for finite sys-
tems, we first show the orbital susceptibility of the tight-
binding Hamiltonian \( \hat{H} \) for infinite clean lattice. The
zero temperature susceptibility was calculated in Ref. \( 2 \)
and is given by the following expression :
\[
\chi(T = 0, \epsilon_F) = -\frac{a^2 e^2}{24 \hbar^2 c^2} \left[ (\epsilon_F^2 - 8\hbar^2) \rho(\epsilon_F) - E(\epsilon_F) \right],
\]
where

\[ \rho(\epsilon) = \frac{1}{2\pi^2t} K \left( \sqrt{1 - \left(\epsilon/4t\right)^2} \right) \]

is the density of states (\(K\) denotes the complete elliptic integral of the second kind), \(E(\epsilon_F)\) is the total energy of the filled states, and \(a\) is the lattice constant. At the band edges we get the value \(-a^2e^2t/12\pi\hbar^2c^2\), which is the Landau susceptibility with the appropriate effective mass. At nonzero temperatures the susceptibility can be evaluated from its zero temperature value using the connection

\[ \chi(T, \mu) = \int \chi(0, \epsilon)f'_{\text{GCE}}(T, \epsilon)d\epsilon, \]

where \(f'_{\text{GCE}}\) is the energy derivative of the Fermi-Dirac distribution function (of course, in this macroscopic limit the difference between statistical ensembles disappears).

The susceptibility is plotted in Fig. 1 as a function of the filling \(\nu = \int \rho(\epsilon)f_{\text{GCE}}(\epsilon)d\epsilon\). We can see that the susceptibility has a diamagnetic Landau value only at the band edges at zero temperature, and that its sign changes to paramagnetic near the band center. At half-filling it actually exhibits a logarithmic singularity at zero temperature. Those non-analytic behaviors at the band’s center and edges cause noticeable temperature dependence of the susceptibility there, even for temperatures much lower than the Fermi energy (the only energy scale in macroscopic clean systems), although in the rest of the band temperature has a significant influence only when it reaches the Fermi energy. This behavior is connected to the inclusion of the magnetic field only as a phase factor in the hopping amplitude. However, if we remember in the sequel to compare our results for finite, possibly disordered systems to this macroscopic limit, we would not have any trouble interpreting the results.

### B. Mesoscopic Clean Systems

We now turn to finite clean systems. The results for the susceptibility of a system on a 17 \(\times\) 24 lattice are shown in Fig. 2 in the EGCE (which is equivalent to the GCE in the absence of disorder), for various temperatures. The mean level spacing \(\Delta\) is approximately 0.02t. We can see here all the expected phenomena described in the introduction. At zero temperature, or finite temperatures much smaller than the mean level spacing, there are very large paramagnetic peaks, occurring where the level spacing is small. Their exact position and height strongly depend on the system’s area and the ratio of its length and width. Those peaks are smeared out when the temperature approaches the level spacing, leaving behind oscillations whose amplitude and

![FIG. 1: The orbital susceptibility of an infinite ordered lattice as a function of filling for various temperatures.](image1)

![FIG. 2: The orbital susceptibility of a finite clean lattice as a function of filling for various temperatures in the EGCE. Low temperature results are presented in the upper panel, high temperature results are presented in the lower panel.](image2)
FIG. 3: The orbital susceptibility of a finite ordered lattice as a function of filling for various temperatures in the CE. Low temperature results are presented in the upper panel, high temperature results are presented in the lower panel.

As regards the CE, the results for the same values of parameters are shown in Fig. 3. We can see that moving from the EGCE to the CE is qualitatively equivalent to lowering the temperature in the EGCE, as noted by various authors in the past.\cite{16,17} Hence, the peaks and oscillations are somewhat larger in the CE, and thus persist to higher temperatures. The difference disappears at both zero temperature and very high temperatures (of the order of the temperature where the mesoscopic effects in the EGCE disappear), but is observable for intermediate temperatures. However, even when non-negligible, this difference is smaller than the EGCE susceptibility.

C. Mesoscopic Disordered Systems, GCE

We now introduce disorder into the system. The results for the average susceptibility in the GCE are shown in Fig. 4. The GCE susceptibility is expected, as noted in the introduction, to be equal to the macroscopic susceptibility, shown in Fig. 1. Indeed, the general behavior of the GCE susceptibility resembles the macroscopic susceptibility (especially in its sign as a function of filling). This is immediately obvious regarding the case \( W = 4.0 \), and is also true for \( W = 2.0 \) if the oscillations as a function of filling are averaged out (These oscillations will be discussed below). In addition, the GCE susceptibility
(ignoring again the oscillating part for \( W = 2.0 \)) shows no temperature dependence, up to about \( T \approx 0.1t \). This is in accordance with the macroscopic behavior if we remember that there are no zero temperature singularities as a function of filling for finite samples, so only the high temperature macroscopic temperature dependence persists.

There are, however, two modifications compared to the macroscopic results. The first is that the susceptibility of the disordered mesoscopic systems is clearly smaller than the macroscopic value. It also decreases when disorder increases, although quite slowly. This can be interpreted as stemming from corrections to susceptibility (beyond the leading order macroscopic value), which are expected to vary as \((k_F \ell)^{-1}\). Since \((k_F \ell)^{-1}\) is not so small in our simulations, these corrections can be important and lead to a weak disorder dependence, and their sign is apparently negative.

A second and more pronounced difference between the GCE average susceptibility and its macroscopic value is the above mentioned filling-dependent oscillations for \( W = 2.0 \). The oscillations’ position is the same as the oscillations in the ordered systems shown above in Fig. 2, i.e., of periodicity varying as \( L^{-1} \). The amplitude of the oscillations, however, shows little dependence on the system’s size, but decreases strongly when disorder or temperature increase. This effect is due to the fact that even though for \( W = 2.0 \) the systems of the size considered are regarded diffusive, showing, for example, a Wigner level spacing distribution, the elastic mean free path \( \ell \) is only about a third of the size of the considered systems, so that ballistic effects can still be observed. In fact, the density of states still shows ballistic oscillations for \( W = 2.0 \), which are smeared out as disorder increases toward \( W = 4.0 \). Thus, the susceptibility at \( W = 2.0 \) shows a mixture of diffusive and ballistic behaviors. We note that size-averaging will smear out these oscillations.

Referring now to fluctuations in the susceptibility, the results for the r.m.s. susceptibility of the same system are shown in Fig. 4. We can clearly see that the r.m.s. susceptibility is much larger than the average. It also decreases strongly with increasing disorder, implying the expected \(|\chi_m|/(k_F \ell)\) dependence (This can be seen quantitatively by noting that from Fermi’s golden rule one can estimate \( \ell \) to vary as \( W^{-2} \), which is quite accurately the W dependence of the r.m.s susceptibility in our data). Since \( k_F \ell \sim g \), where \( g \) is the dimensionless conductance, the enhancement factor agrees well with the values of \( g \approx 20 \) and \( g \approx 3 \) for \( W = 2.0 \) and \( W = 4.0 \) respectively. (The values \( g \) were calculated from non-universal corrections to the inverse participation ratio, and agrees well with estimates based on Fermi’s golden rule.)

A surprising feature of the results is that the r.m.s. susceptibility depends strongly on temperature even for temperatures much smaller from the mean level spacing \( \Delta \) (which is, approximately, 0.022\( t \) and 0.025\( t \) for \( W = 2.0 \) and \( W = 4.0 \), respectively) This will be discussed later on. Moreover, the r.m.s. susceptibility becomes equal to the average susceptibility for a temperature of about 0.1\( t \approx 5\Delta \) for both \( W \) values. As mentioned in the introduction, significant fluctuations in the susceptibility (i.e., larger than the average value) are expected to persist even in temperatures higher than the Thouless energy. However, since the enhancement factor \( k_F \ell \) is not so large in our simulations as in real systems, absence of fluctuations in temperatures above the Thouless energy is quite reasonable. On the other hand, fluctuations below the Thouless energy should be significant. This implies interpreting the temperature 0.1\( t \approx 5\Delta \), for which the difference between the average and r.m.s. susceptibilities disappears, as the Thouless energy times a numerical factor (note that our estimate for \( g \) indicates a much higher Thouless energy for \( W = 2.0 \)).

However, although this temperature goes as \( L^{-2} \) as the Thouless energy should vary, it shows very little if any disorder dependence, so its identification with the Thouless energy is quite problematic. This may support the claim of some authors that in the logarithmic factor
FIG. 6: The disorder averaged orbital susceptibility of a finite disordered lattice as a function of filling for various temperatures in the EGCE, for both $W = 4$ (upper panel) and $W = 4.0$ (lower panel).

$\ln(E_C/T)$ in the r.m.s susceptibility, the Thouless energy should be replaced by some lower energy scale. Our results seem to suggest that this scale, which is disorder independent, is connected to the mean level spacing $\Delta$.

D. Mesoscopic Disordered Systems, EGCE

We now turn to discuss disordered systems in the EGCE. The results for the average susceptibility are shown in Fig. 6. We see again the oscillations in the $W = 2.0$ susceptibility. Their explanation is the same as in the GCE case discussed above, so we now ignore them and refer to the smooth part of the susceptibility.

As is expected, the difference between the average EGCE susceptibility and the GCE susceptibility is positive. It is also much larger than the GCE susceptibility except near the band edges. This is somewhat problematic since precisely at the band edges the macroscopic susceptibility equals the Landau value. It may be explained by the fact that those states are more localized than the other states, and thus their susceptibility is not so much enhanced.

The difference between the average EGCE and GCE susceptibilities is strongly disorder dependent, confirming the identification of the zero temperature enhancement factor as $k_F \ell$ (As for the r.m.s. GCE susceptibility discussed above, it has quite accurately a $W^{-2}$ dependence). Temperature dependence shows up for temperatures higher than for the r.m.s. GCE susceptibility discussed above, but much lower than the mean level spacing. In fact the difference between the EGCE and GCE practically disappears for $T = 0.02t \approx 0.01t \approx 0.4\Delta$ for $W = 2.0$ and $W = 4.0$ respectively. Although this temperature is much smaller than the Thouless energy (based on our estimate $g \approx 20$ and $g \approx 3$ for $W = 2.0$ and $W = 4.0$ respectively), its $L^{-2}$ size dependence as well as its disorder dependence indicate that it might be considered as connected with the Thouless energy reduced by a numerical factor. We may note, however, that the disorder dependence of this temperature is quite weaker than the corresponding dependence of the Thouless energy (the above mentioned $W^{-2}$ law). Combined with the quite low value of this temperature, its meaning is not completely clear.

As for the fluctuations in the susceptibility, The results for the r.m.s. susceptibility are shown in Fig. 6. As is usually conjectured, it behaves quite similarly to the GCE r.m.s. susceptibility, and all the discussion regarding the GCE applies also here. The only exception is the very low temperature regime, (below $0.01t \approx 0.5\Delta$), where the EGCE r.m.s. susceptibility is well above the GCE values, the difference being larger for $W = 2.0$. It is also much more “noisy”, as can be clearly seen in the $T = 0$ curves. We will discuss this feature in the next section.

E. Mesoscopic Disordered Systems, CE

To conclude this section, we move to the CE. The results for both the averaged and r.m.s. susceptibilities are very similar to the EGCE results, the difference being only few percents of the EGCE values and thus not shown. This is quite expectable for $T > \Delta$, since, as we show in the appendix, the difference between the CE and EGCE is parametrically smaller than the difference between the EGCE and the GCE.

We may note, however, that the difference between the CE and EGCE is more pronounced for the r.m.s susceptibility at very low temperatures (below $0.01t \approx 0.5\Delta$), where there is also a difference between the EGCE and the GCE r.m.s. susceptibilities. We believe those two phenomena has a similar origin, and will discuss it in the next section.
FIG. 7: The disorder averaged r.m.s. orbital susceptibility of a finite disordered lattice as a function of filling for various temperatures in the EGCE, for both $W = 2.0$ (upper panel) and $W = 4.0$ (lower panel).

IV. ZERO TEMPERATURE SUSCEPTIBILITY DISTRIBUTION

In this section we will examine the zero temperature susceptibility distribution, and use it to explain the strong dependence of the typical and average susceptibilities on both temperature and the statistical ensemble for temperatures lower than the mean level spacing.

The zero temperature susceptibility distribution in our systems is shown in Fig. 8 for $W = 2.0$ and $W = 4.0$ and for both the CE or EGCE and the GCE. In all the cases the results are for half-filling. The numerically obtained distributions are shown together with Gaussian fits. The tails of the distributions are shown in the insets, together with fits to functional forms which we derive below. We can see that all the distributions are peaked at (or near) zero susceptibility, and are somewhat asymmetric around their maximum. They are, however, fitted quite well by a Gaussian near the peaks. On the other hand, The tails are clearly non-Gaussian, decaying somewhat faster in the negative direction and much more slowly in the positive direction. We note that the susceptibility distributions for other fillings, where the average susceptibility is different, are similar in form to Fig. 8, but show the required tendency toward higher or lower values.
We now compare the differences between the several cases shown. We can see that the CE distribution is wider than the GCE distribution for both disorder values, and has a stronger tendency for positive values. In particular, the positive tail of the CE distribution is larger than the corresponding tail in the GCE. When comparing the two disorder values, note we have chosen a factor of 4 between the scales of the two graphs. This factor is both as the ratio of the widths of the Gaussian fits and a consequence of the expected \( W^{-2} \) dependence of the susceptibility moments. We can see that after this rescaling the results for \( W = 2.0 \) are similar to those for \( W = 4.0 \), although the distributions are somewhat wider in the former case.

In two recent papers, Serota has attempted addressing the problem finding the zero temperature average CE susceptibility and its fluctuations for diffusive systems. His discussion was based on the expressions for the Larmor and van-Vleck susceptibilities, Eq. (2) and Eq. (3), respectively. In order to take into account the above mentioned strong cancellation between the Larmor and van-Vleck terms, he conjectured that the only contribution surviving this cancellation is the largest van-Vleck term composed of a matrix element and an energy difference between the highest occupied level and the lowest unoccupied level, which we will denote by \(-\) and \(+\) respectively. At zero temperature one is thus left with the following expression for the zero temperature susceptibility:

\[
\chi = \frac{1}{SH^2} \left( \frac{e^2}{2mc} \right)^2 \frac{\langle Q^2 \rangle}{\epsilon_+ - \epsilon_-} \cdot \frac{\langle A \cdot \hat{p} + \hat{p} \cdot A \rangle}{\epsilon_+ - \epsilon_-}.
\]

Using an Gor’kov-Eliashberg like argument to evaluate the squared matrix element semiclassically, and using the Wigner distribution for the level spacing, Serota has given expressions for the average and r.m.s. susceptibilities, which vary as \( |\chi_L|/|k_F\ell| \) and \( |\chi_L|/|k_F\ell| \ln(E_C/\Delta)^{1/2} \). These expressions are similar to the corresponding expressions for temperatures higher than the mean level spacing \( \Delta \) which we quoted in the introduction, if the temperature \( T \) is replaced by \( \Delta \).

However, this approach is quite problematic in view of our previous discussion of the importance of the gauge invariance of the susceptibility. Eq. (4) is clearly not gauge invariant, since the Larmor–van-Vleck cancellation was taken into account a is too crude fashion. As a result, Eq. (4), can only give positive susceptibility values, whereas our numerically computed distribution, Fig. 5, exhibits also negative values.

In spite of this problem, the approximation of Eq. (4) is still useful for treating the tail of the susceptibility distribution. Our numerical simulations indicate that anomalously large positive susceptibility values are caused by small level separation between the highest occupied level and the lowest unoccupied level. In those cases, the term in the van-Vleck susceptibility involving these two levels is much larger than all the other terms. Furthermore, in this case it is also approximately gauge invariant, since a gauge transformation adds to the squared matrix element in Eq. (4) contributions smaller by a factor of \((\epsilon_+ - \epsilon_-)/\Delta\) or less, which are thus negligible for very close levels. Moreover, the average and r.m.s. of the squared matrix element are seen in the numerics to be almost independent of the energy separation for small separations, so that the form of the tail of the susceptibility distribution is dominated by the distribution of the energy denominator. This, in turn, can be simply obtained from Wigner’s distribution, according to which the probability for small level separation varies linearly with the separation. Thus, the susceptibility distribution varies as \( P(\chi) d\chi \sim \chi^{-3} d\chi \). This dependence is fitted to the CE distributions in the insets of Fig. 8 and agrees quite well with the numerical results.

As previously noted by Serota, this distribution has a first moment but no second or higher moments, i.e., the r.m.s. susceptibility diverges. Serota suggested regularizing this divergence by noting that in any given experimental realization a finite magnetic field is used, so that for very small energy separations the non-degenerate perturbation theory used to obtain Eq. (4) is inapplicable. A degenerate perturbation theory must be used instead, and this gives a finite magnetization.

All this is true, however, at zero temperature. At higher temperatures Eq. (4) should be multiplied by the factor \((f(\epsilon^-) - f(\epsilon_+))/2\), where \(f(\epsilon)\) is the mean level occupation. This factor tends to linearly zero for level separation smaller than the temperature, and thus cuts off the divergence caused by the denominator. For temperatures higher than \(h\omega_c\) this factor, and not a finite magnetic field, is dominant in keeping the susceptibility finite. In fact, using Serota’s estimate for the squared matrix element in Eq. (4), the r.m.s. susceptibility can be shown to vary as \( |\chi_L|/|k_F\ell| \ln(E_C/\Delta)^{1/2} \). This means that the logarithmic temperature dependence of the r.m.s. susceptibility does not saturate for temperatures smaller than the level spacing, but persists to much lower temperatures, and, in the limit of zero magnetic field, continues down to zero temperature, where the r.m.s. susceptibility diverges.

This may serve to explain the strong temperature dependence of the r.m.s. susceptibility seen in our numerical results. It also suggests that the saturation of this effect at very low temperatures (less than tenth of the mean level spacing) is only due to the finite ensemble used, whereas in reality the r.m.s. susceptibility diverges as the temperature becomes lower. To test this we compared the zero temperature CE r.m.s. susceptibility (which is identical to the EGCE susceptibility in this case) calculated numerically on ensembles of different numbers of disorder realizations. We observed, indeed, that the numerically calculated r.m.s. susceptibility grows with the ensemble size, indicating it is really divergent, and thus temperature dependent for all temperatures.

All the above refers to the CE or EGCE. Since the number of electrons is fixed, the probability of having a small separation between the lowest unoccupied level and
the highest occupied level is given by the Wigner distribution. In the GCE, however, there is a low probability for the fixed chemical potential to fall between two close levels, and this probability is simply proportional to the separation itself when the separation is small. The tail of the susceptibility distribution thus varies as $\chi^{-3}$ instead of $\chi^{-4}$ in the CE or EGCE. This dependence is fitted to the GCE distributions in the insets of Fig. 8 and agrees quite well with the numerical results. Thus, the zero temperature r.m.s. susceptibility is expected to be finite in the GCE (although higher moments are still divergent). Numerical simulations have indeed shown that increasing the ensemble size only helps smoothing the r.m.s. GCE susceptibility. This means that the r.m.s. GCE susceptibility really saturates as a function of temperature for low enough temperatures.

We thus arrive at a surprising conclusion: in contrast to what is usually taken for granted, the r.m.s. susceptibility is different in the CE (or EGCE) and in the GCE, diverging as a function of temperature when approaching absolute zero in the former case, while tending to a constant value in the former. Moreover, since the CE is qualitatively similar to the EGCE at a lower temperature, the diverging temperature dependence explains why we can see a noticeable difference between the r.m.s. susceptibilities of the CE and the EGCE at very low temperatures. (It is not very large, however, since the temperature dependence is only logarithmic.)

To conclude this section we may remark that since the main difference between the CE or EGCE and the GCE is connected with the events of small level separation at the Fermi level, we can understand why the average CE and EGCE susceptibilities show a temperature dependence even for temperatures below the mean level spacing. However, the average susceptibility is not divergent at zero temperature. This means that we cannot refer only to the tails of the susceptibility distribution, and the analysis becomes much more involved. In any case, however, we can conclude that the usual assumption of no temperature dependence when $T < \Delta$ cannot be taken for granted even when treating the average susceptibility.

\section*{V. INTERACTION EFFECTS}

It is generally accepted that interaction should have a strong effect on the magnetic susceptibility. The interaction contribution to the susceptibility is expected to be positive for a repulsive interaction and negative for an attractive one \cite{8,19,20}. For diffusive systems this effect is believed to be insensitive to the ensemble average chosen, and to have small fluctuations around its mean value. In these systems, for weak interaction and temperatures higher than the mean level spacing, this contribution varies as $\lambda |\chi_L| (k_F L)^{\alpha}$ for some geometry-dependent exponent $\alpha$ with a more complicated temperature-dependent factor. In the following, however, we will concentrate at zero temperature and repulsive interaction.

In order to avoid the heavy computation involved in exact diagonalization or Hartree-Fock calculations, we will restrict ourselves to weak short-range interactions, which can be treated perturbatively to first order. We thus take as our model the usual Hubbard on-site interaction (electron spin is restored in this section):

$$U = U_H \sum_s n_{s;\uparrow} n_{s;\downarrow},$$

where $U_H$ specifies the interaction strength, and the arrow subscripts denote spin direction. To first order in $U_H$ the change in the system’s energy at zero temperature is thus:

$$\Delta E = U_H \sum_s n_{\uparrow}(s) n_{\downarrow}(s),$$

where

$$n_\sigma(s) = \sum_{k=1}^{n_{\text{e}}} |\psi_{k;\sigma}(s)|^2$$

is the number of electrons with spin projection $\sigma = \uparrow, \downarrow$ at lattice site $s$. In this later expression $\psi_{k;\sigma}$ denotes the $k$th single particle eigenfunctions without interaction, with spin projection $\sigma$ (The wave functions are, of course, spin independent; the spin index is used only for clarity); and $n_\sigma$ is the total number of electrons with spin projection $\sigma$ in the system (which is conserved by the interaction). We will take these numbers as independent, as they may be set different experimentally using an in-plane magnetic field. This applies to the CE or EGCE; in the GCE the sum is over single-particle levels with energies lower than the Fermi energy (which can again depend on the spin direction).

The first-order interaction correction to the susceptibility is $\Delta \chi = -S^{-1} \partial^2 \Delta E / \partial H^2$. Since for infinite clean systems $n_\sigma(s)$ is position independent both with and without an applied magnetic, $\Delta E$ is field-independent and $\Delta \chi$ vanishes. This is in agreement with previous results \cite{20}, according to which interaction corrections to the susceptibility are of the third order in the coupling strength ($U_H$ in our case). On the other hand, for finite and/or disordered systems, $n_\sigma(s)$ is non-uniform. Moreover, an applied magnetic field breaks time-reversal symmetry and drives the electronic density distribution to be more uniform. This makes $\Delta E$ smaller, and thus $\Delta \chi$ is positive. This is again in accordance with previous results \cite{7,8,19,20}.

We now turn to numerical results. Since one can show that the change in the square of the absolute value of the wave-function amplitude at a given point due to the applied magnetic field is quadratic in the applied field,
we have

\[ \Delta \chi = - \frac{U_H}{S} \sum_s \left[ \frac{\partial^2 n_{\uparrow}(s)}{\partial H^2} n_{\downarrow}(s) + n_{\uparrow}(s) \frac{\partial^2 n_{\downarrow}(s)}{\partial H^2} \right]. \] (12)

We may thus interpret the first term in the brackets as the correction to the orbital susceptibility of the spin-up electrons due to their interaction with spin down electrons, and similarly for the second term. The first term was calculated numerically with our usual parameters, and the results are shown in Fig. 9 for the two values of disorder strength \((W = 2.0)\) and \((W = 4.0)\). Since we are using numerical differentiation, our results for \((W = 0)\), which seemed unreliable to us are not shown. For simplicity, we have taken the interaction strength as our unit of energy, i.e., \(U_H = t\).

As in the non-interacting case, for \(W = 2.0\) we can see ballistic boundary-induced oscillations. The non-oscillating part of the interaction correction shows the expected disorder dependence (varying as \(W^{-2}\) as \(t\) should) and is size independent (This latter result applies also to the oscillating part). An interesting feature is the strong peak in \(\Delta \chi\) when the filling of spin-up electrons equals the filling of spin-down electrons. This can be explained by treating the interaction in the Hartree-Fock manner, as an effective single-particle potential. Obviously, the interaction between two electrons in the same orbital single-particle level but different spins is stronger than in the case where the two electrons reside in two different orbital levels. Thus, the single-particle energies of spin-up electrons up to the \(n_{\downarrow}\)-th level are raised by a repulsive interaction more than the single particle energies above the \(n_{\downarrow}\)-th level. Now, the expression for the van-Vleck susceptibility, Eq. 11 contains at zero temperature only energy denominators between occupied and unoccupied levels, and is thus dominated by the lowest unoccupied levels and the highest occupied levels. When the filling of spin-up electrons differs enough from the filling of spin-down electrons, the energies of these lowest unoccupied levels and highest occupied levels are raised by approximately the same amount by the interaction, so their differences do not change very much and the corrections to the susceptibility are small. However, when the filling of spin-up electrons is approximately equal to the filling of spin-down electrons, the highest unoccupied levels are raised by the interaction more than the lowest unoccupied levels so that the energy differences become smaller and \(\Delta \chi\) is enhanced, as we have seen in the numerical results.

To conclude this section we note that our numerical results indicate that the r.m.s. value of \(\Delta \chi\) is much larger than its average, and that the average of \(\Delta \chi\) in the GCE is significantly lower than its CE (or EGCE) value, in contrast to what is usually expected. It seems that the distribution of \(\Delta \chi\) at zero temperature shares with the distribution of the non-interacting susceptibility the property of having a long tail with diverging moments, causing a strong temperature and statistical-ensemble dependence. This point, however, deserves further investigation.

VI. CONCLUSIONS

To conclude this paper, we summarize our main findings:

(i) For temperatures lower than the mean level spacing, we have shown how the requirement of gauge-invariance causes a strong cancellation between the Larmor and van-Vleck susceptibilities, and how it makes using usual averaging procedures, like RMT, inapplicable for our discussion. We have also seen that, contrary to what one usually expects, the r.m.s. susceptibility depends strongly both on temperature and statistical ensemble in this regime of low temperatures. This was explained in term of the long tail of the zero temperature suscepti-
bility distribution, resulting in a diverging r.m.s. zero

temperature susceptibility in the CE or EGCE but not

in the GCE, and causing a logarithmic temperature de-

pendence in the former case for all temperatures down
to zero. We believe this to be responsible also to the strong
temperature dependence of the average CE and EGCE
susceptibilities for this low temperature regime.

(ii) For temperatures higher than the mean level spac-
ing our numerical results generally agree with previous
calculations. They, together with analytical calculations
show that the difference between the CE and the EGCE,
which was not considered so far in connection with or-

tbital susceptibility, is negligible for diffusive systems, and
is not the leading order contribution in the clean limit.
However, the meaning of the temperature where the
EGCE susceptibility becomes equal to the average value, are not
clear. Especially in the latter case, identifying the tem-
perature with the Thouless energy is quite problematic.

(iii) Considering interaction between the electrons, we
have given a new interpretation of its first order con-

tribution. We have also demonstrated its dependence on
in-plane magnetic field, and explained this effect in terms
of a single-particle picture.

Acknowledgments

We would like to thank B. Shapiro for useful discus-
sions. Financial support from the Israel Science Founda-
tion is gratefully acknowledged.

APPENDIX: DIFFERENCE BETWEEN
ENSEMBLES IN THE MESOSCOPIC REGIME

In this appendix we apply the general formulation of
Kamenev and Gefen\cite{17} regarding the differences between
ensembles to our problem. They calculated $\delta F$, the

disorder-averaged difference between the canonical free
energy $F$ and the expression $\Omega + \mu N$, the Legendre trans-

formation of the grand canonical potential $\Omega$ calculated
for system-independent chemical potential $\mu$. The num-
ber of particles $N$ is thus sample-specific in the grand-
canonical case, and the chemical potential is set so that
the disorder and thermal averaged particle number will
be equal to the fixed particle number in the CE. Treating
the regime of temperatures higher than the mean level
spacing $\Delta$, they give the following expression, correct
to the zeroth order in the small parameter $T/\epsilon_F$:

$$\delta F = \frac{\pi T}{2} \sum_{w=0}^{\infty} \frac{1}{(w+1)!} \left( \frac{\Delta}{T} \right)^w \int_0^\infty dt \frac{t^2}{\sinh^2 \pi t} \tilde{K} \left( \frac{\Delta}{T} t \right),$$

(A.1)

where

$$\tilde{K}(t) = \frac{1}{\Delta} \int_{-\infty}^{\infty} d\omega e^{-i\omega t/\Delta} K(\omega)$$

is the Fourier transform of the two-level correlation func-
tion $K(\omega)$. The $w = 0$ term is the difference between
the EGCE and the GCE, while the $w = 1$ is the leading
order difference between the CE and the EGCE. Only
these terms will be considered in what follows.

For temperatures lower than the Thouless energy $E_C$,
the magnetic-field dependent part of $K(\omega)$ can be ap-
proximated by the zero-mode Cooperon contribution
(since we are treating temperatures higher than the mean
level spacing, perturbative analysis applies):

$$K(\omega) = \frac{\Delta^2}{2\pi^2} \left[ \epsilon_H^2 - \omega^2 \right] / (\epsilon_H^2 + \omega^2)^2.$$  

(A.2)

where $\epsilon_H \sim E_C (\Phi/\Phi_0)^2$ is the leading order shift in
the lowest Cooperon eigenmode due to the applied magnetic
field (valid for small magnetic fields, whose flux through
the sample $\Phi$ is smaller than the quantum flux $\Phi_0$).

Inserting this into Eq. (A.1) we get for the difference
between the EGCE and the GCE the following expres-
sion, previously derived by Oh et al.\cite{18}, for the magneti-
zation per unit area:

$$M_{EGCE} - M_{GCE} \sim \frac{\Delta}{\Phi_0} \int_{-\infty}^{\infty} dt \frac{t^2}{\sinh^2 (\pi t)} e^{-(\epsilon_H/T)t}.$$  

(A.3)

From this it follows that

$$M_{EGCE} - M_{GCE} \sim \frac{\Delta}{\Phi_0} \left\{ \begin{array}{ll}
E_{C} \Phi \\
\frac{T}{E_C} \Phi_0, \quad \Phi \ll \Phi_c;
\end{array} \right.$$  

(A.4)

where $\Phi_c = (T/E_C)^{1/2} \Phi_0$. The difference between
the CE and the EGCE is given by:

$$M_{CE} - M_{EGCE} \sim \left( \frac{\Delta}{T} \right)^2 \frac{E_{C} \Phi}{\Phi_0} \int_{-\infty}^{\infty} dt \frac{t^4}{\sinh^2 (\pi t)} e^{-(\epsilon_H/T)t}.$$  

(A.5)

From this it follows that

$$M_{CE} - M_{EGCE} \sim \frac{\Delta^2}{T \Phi_0} \left\{ \begin{array}{ll}
E_{C} \Phi \\
\left( \frac{T}{E_C} \right)^2 \Phi_0, \quad \Phi \ll \Phi_c;
\end{array} \right.$$  

(A.6)

In our parameter regime we can see the the difference
between the CE and the EGCE is parametrically smaller
than the difference between the EGCE and the GCE. In
fact, the former effect is of the same order of magnitude as
the contribution of higher order perturbative corrections
to $K(\omega)$ to the latter. This is in accordance with our
numerical results.
1 J.H. van Leeuwen, J. Phys. (Paris) 2, 361 (1921).
2 L.D. Landau and E.M. Lifshitz, Statistical Physics I (Pergamon Press, Oxford, 1985).
3 P. Skudlarski and G. Vignale, Phys. Rev. B 43, 5764 (1991).
4 J.M. van Ruitenbeek, D.A. van Leeuwen, Phys. Rev. Lett. 67, 640 (1991); J.M. van Ruitenbeek, Z. Phys. D 19, 247 (1991).
5 K. Richter, D. Ullmo, R.A. Jalabert, Phys. Rep. 276, 1 (1996); and references cited therein.
6 E. Gurevich and B. Shapiro, J. Phys. I 7, 807 (1997).
7 R.A. Jalabert, in the CXLIII Course New Directions in Quantum Chaos on the International School of Physics "Enrico Fermi", eds. G. Casati, I. Guarneri and U. Smilansky (Elsevier, Amsterdam, 1999); and references cited therein.
8 S. Oh, A.Yu. Zyuzin and R.A. Serota, Phys. Rev. B 44, 8858 (1991).
9 B.L. Altshuler, Y. Gefen, Y. Imry and G. Montambaux, Phys. Rev. B 47, 10335 (1993).
10 B. Shapiro, Waves in Random Media 9, 271 (1999).
11 R.A. Serota, Solid State Commun. 117, 99 (2001).
12 R.A. Serota, Physica E 15, 211 (2002).
13 Y. Gefen, D. Braun and G. Montambaux, Phys. Rev. Lett. 73, 154 (1994).
14 A. Altland and Y. Gefen, Phys. Rev. B 51, 10671 (1995).
15 E. McCann and K. Richter, Phys. Rev. B 59, 13026 (1999).
16 R. Denton, B. Mühschlegel and D.J. Scalapino, Phys. Rev. B 7, 3589 (1973).
17 A. Kamenev and Y. Gefen, Phys. Rev. B 56, 1025 (1997).
18 L.P. Gor’kov and G.M. Eliashberg, Sov. Phys. JETP 21, 940 (1965).
19 L.G. Aslamazov and A.I. Larkin, Sov. Phys. JETP 40, 321 (1975).
20 B.L. Altshuler, A.G. Aronov and A.Yu. Zyuzin, Sov. Phys. JETP 57, 889 (1983).
21 V.N. Prigodin and B.L. Altshuler, Phys. Rev. Lett. 80, 1944 (1998).