Logarithmic quantum wave equation with variable nonlinear coupling

Konstantin G. Zloshchastiev

Institute of Systems Science, Durban University of Technology, P.O. Box 1334, Durban 4000, South Africa

E-mail: http://bit.do/kgz

Abstract. We study a possibility of a dynamical behavior of the nonlinear coupling in the quantum wave equation of a logarithmic type. In order to construct a viable model, this coupling is associated with some kind of the quantum temperature conjugated to the Everett-Hirschman entropy. A combined quantum-mechanical and field-theoretical model is proposed, which leads to the logarithmic equation with a variable nonlinear coupling. It is expected that such models will shed light on a fundamental nature of the temperature – as a notion which can be derived from the basic quantum-mechanical principles preceding the classical notions of heat and entropy.

1. Introduction

Assuming three spatial dimensions (lower-dimensional cases can be studied by analogy), let us consider the logarithmic quantum wave equation, for historical reasons referred here as the logarithmic Schrödinger equation (LogSE):

\[ i\hbar \partial_t \Psi = \left( \hat{H} - b \ln(a^3|\Psi|^2) \right) \Psi \]

\[ = \left[ \frac{\hbar^2}{2m} \Delta + V_{\text{ext}} - b \ln(a^3|\Psi|^2) \right] \Psi, \]

where the coupling

\[ b = b(\vec{r}, t) \]

measures the strength of a nonlinear interaction, \( a \) is a constant parameter of dimensionality length required to make the argument of the logarithm dimensionless, and \( m \) is a particle’s mass (also assumed constant here), and \( V_{\text{ext}} = V_{\text{ext}}(\vec{r}, t) \) is an external potential. Here \( \Psi = \Psi(\vec{r}, t) \) is a wavefunction normalized to a number \( N \):

\[ \langle \Psi | \Psi \rangle \equiv \int \frac{|\Psi|^2 d^3\vec{r}}{\mathcal{V}} = N \geq 1, \]

and \( \Delta = \nabla \cdot \nabla \) is the Laplacian (in the relativistic version the derivative part would be replaced by a d’Alembertian), and \( \mathcal{V} \) is a volume occupied by our system.

The simplest case of Eq. (2),

\[ b(\vec{r}, t) = b_0 = \text{const}, \]

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was historically first to study [1, 2, 3, 4]. The corresponding models were proven to be very instrumental when dealing with extensions of quantum mechanics [3, 5], physics of quantum fields and particles [1, 6, 7, 8, 9, 10, 11], optics and transport or diffusion phenomena [12, 13, 14], nuclear physics [15, 16], theory of dissipative systems and quantum information [17, 18, 19, 20, 21, 22, 23], theory of quantum liquids and superfluidity [24, 25, 27, 28], and theory of physical vacuum and classical and quantum gravity [29, 30, 31].

Notwithstanding the success of the models with a constant coupling (4), few questions remained open. Firstly, what is a specific value of this coupling? In Refs. [3, 15, 16] it was shown that for a large class of conserved quantum systems the logarithmic term must be dominated by an external potential, therefore, the nonlinear coupling must be rather small. On the other hand, in theory of superfluid He-4 this term plays a crucial role [25]. Therefore, it seems that this coupling can take different values depending on physical conditions, i.e., it can vary from system to system. Essentially, it means that its value cannot be a fundamental constant but rather some dynamical function, whereas the relation (4) can be a leading-order approximation with a finite domain of applicability.

Secondly, if this coupling is a nontrivial notion then what would be its physical meaning? One possible idea is advocated in Refs. [3, 18, 24] to relate this coupling to some kind of a (non-thermal) temperature $T_\Psi$, which is conjugated to the Everett-Hirschman entropy [32, 33], the latter being defined as

$$S_\Psi = -\langle \Psi | \ln (a^3|\Psi|^2) |\Psi \rangle = -\int_V |\Psi|^2 \ln (a^3|\Psi|^2) d^3\vec{r},$$  

(5)

where the Boltzmann constant is assumed to be $k_B = 1$ from now on. Therefore, the logarithmic quantum wave equation can be viewed as a minimization condition not for the energy of a system

$$E = \langle \Psi | \hat{H} |\Psi \rangle,$$

but for the quantum “internal energy”, defined as

$$U_\Psi = \langle \Psi | \hat{H} |\Psi \rangle + T_\Psi S_\Psi,$$  

(6)

where

$$T_\Psi = \left( \frac{\partial U_\Psi}{\partial S_\Psi} \right)_V = b,$$  

(7)

some details can be found in Ref. [24]. In Eq. (6), the term $\langle \Psi | \hat{H} |\Psi \rangle$ comes from the system’s dynamics, while the term $T_\Psi S_\Psi$ is responsible for dissipative effects related to handling the information about a system

$$I_\Psi = -\log_2(a^3|\Psi|^2) = -\ln(a^3|\Psi|^2)/\ln 2.$$  

(8)

These entropy considerations result in uncertainty relations which can complement the Heisenberg one [3, 34, 35]. Therefore, in what follows we shall focus on an interpretation of the nonlinear coupling and logarithmic term in terms of $T_\Psi$ and Everett-Hirschman entropy.

Thirdly, while the Everett-Hirschman entropy’s consideration seem to be an important step towards understanding the dynamical behavior of temperature $T_\Psi$, which we refer as the Everett-Hirschman’s or simply quantum temperature from now on, remains an open question. If it is a dynamical quantity then what are the laws governing it?

These three questions will be a main subject of our study.
2. The model
Since we assume the nonlinear coupling to be a dynamical value, we can define it as a scalar function of a field \( \sigma(\vec{r}, t) \), i.e., \( b = b(\sigma) \) and \( T_\psi = T_\psi(\sigma) \). Due to gauge invariance of a field, the corresponding wave equation must depend not on a field itself but on its derivative, such as the gradient. Since the latter is a vector, whereas the coupling must be a scalar function, we assume

\[
b = b \left( \vec{n} \cdot \vec{\nabla} \sigma, \vec{\nabla} \sigma \cdot \vec{\nabla} \sigma, \ldots \right),
\]

where \( \vec{n} = \vec{r}/r = \vec{r}/\sqrt{\vec{r} \cdot \vec{r}} \) is a normal radius vector, and \( r = \sqrt{\vec{r} \cdot \vec{r}} \) is an absolute value of the radius vector.

With these assumptions in hands, we can define the simplest (“minimal”) variable-coupling model as

\[
i \hbar \partial_t \Psi = \left[ -\frac{\hbar^2}{2m} \Delta + V_{\text{ext}} - \vec{n} \cdot \vec{\nabla} \ln(a^3 |\Psi|^2) \right] \Psi,
\]

\[
\Delta \sigma = 4\pi \rho_\sigma,
\]

where \( \rho_\sigma = \rho_\sigma(\vec{r}, t) \) is the energy density of the field \( \sigma \). Furthermore, it is natural to assume that this field’s distribution is correlated with our system, therefore, we can impose

\[
\rho_\sigma = f(\rho),
\]

where \( f \) is a function which is being chosen in a specific model, and

\[
\rho = |\Psi|^2
\]

is a probability density of our system; in case of condensates \( N \gg 1 \), \( \rho \) would be an actual particle density of a condensate.

One can see that the resulting model (10) - (13) is not only nonlinear but also coupled, therefore some robust simplifications must be made in order to extract an essential physical information.

From now on we shall be interested in a case of a trapless system,

\[
V_{\text{ext}} = 0.
\]

In this case, we can set

\[
\sigma = \sigma(r),
\]

and rewrite Eqs. (10) and (11) in the form:

\[
i \hbar \partial_t \Psi + \frac{\hbar^2}{2m} \Delta \Psi + \frac{d\sigma}{dr} \ln(a^3 |\Psi|^2) \Psi = 0,
\]

\[
\Delta \sigma - 4\pi f(|\Psi|^2) = 0.
\]

Furthermore, we also assume our quantum-mechanical system to be localized

\[
\lim_{r \to \pm \infty} \rho_\sigma = \lim_{r \to \pm \infty} r^2 |\Psi|^2 = 0.
\]

Under these conditions, Eq. (17) can be written in a series form, and Eqs. (16) and (17) can be decoupled from each other [36].

Our model thus suggests that the nonlinear coupling is not a constant in general. If for some system, the \( \sigma \)-field’s charge \( Q_\sigma \), is nonzero then at \( r \to 0 \) the coupling must diverge polynomially,
whereas at large $r$ the coupling tends to a constant, so that one recovers Eq. (4). It is interesting to notice that the constant part of the coupling, $b_0$, is actually induced by a regular localized part of the $\sigma$-field. This explains why the solutions of the conventional logarithmic equation (4) are so instrumental in describing regular extended objects, such as Q-balls and extended particles [6, 7, 8, 9, 10, 11], and superfluid droplets [24, 25]. The extended equation’s properties give us some hope that this model will be also instrumental when dealing with singular or point-like objects.

3. Conclusion
We study a possibility of a dynamical behavior of the nonlinear coupling in the quantum wave equation of a logarithmic type. In order to construct a viable model, this coupling is associated with some kind of the quantum temperature conjugated to the Everett-Hirschman entropy. A combined quantum-mechanical and field-theoretical model is proposed, which leads to the logarithmic equation with a variable nonlinear coupling.

This coupling function approaches a constant at large distance from a system’s center, where our equation becomes the conventional logarithmic Schrödinger equation. At short distances, the minimally-coupled model’s behavior must differ from that of the conventional logarithmic Schrödinger equation. We also expect this model to be more applicable to the conservative quantum-mechanical systems.

In the full model, the coupling constant $b_0$ is no longer a built-in parameter of a theory, cf. Eq. (4), but one of the integration constants of evolution equations, such as Eqs. (16) and (17), therefore, its value can vary from system to system. Thus, the full model allows us to reduce a number of parameters of the theory and make it more self-consistent and self-sufficient.

Furthermore, in the models with variable nonlinear coupling, a nature of the quantum EH temperature (6) starts to become clearer. In this regard, it is also interesting to study a possible connection between this temperature and the conventional (thermal) one. The latter is still a notion which is well-defined at a classical level only (e.g., as measured by bringing a system into a thermal equilibrium with some calibrated system), but its quantum analogue’s nature is still obscure.

This problem manifests itself most strikingly when dealing with quantum gases and liquids such as Bose-Einstein condensates (BEC). Strictly speaking, one can not measure the temperature of a condensate without affecting it – for instance, the standard method consists of switching a trap off, using a laser upon the condensate’s atoms and measuring the scattered light to deduce their ‘temperature’ from experimental profiles of density and momentum distributions, assuming that the energy spectrum is also known. Whether such “statistical” temperature is a genuine (thermal) temperature of a condensate remains an open question from a rigorous approach’s viewpoint, especially bearing in mind that: (i) experimental condensates are energetically isolated, so no thermal equilibrium can be achieved with a calibrated system, (ii) the relations between temperature and the above-mentioned distributions and spectrum are derived from one or another theoretical model which must be thus presumed to be a priori valid for that particular condensate, and (iii) this detrapping measurement method is unlikely to be reliable for strongly-interacting quantum Bose liquids which tend to confine their atoms into a droplet in absence of trapping potentials [24, 25, 28].

Therefore, we can make a conjecture here that the origins of the conventional thermal temperature come from basic quantum-mechanical notions, such as those discussed in the introductory section above, whereas its current statistical or classical thermodynamical definitions are derivative concepts. In other words, the concept of temperature could be introduced without referring to classical concepts, such as the heat or average kinetic energy, but rather underlies them.

Finally, properties of this logarithmic equation with a variable nonlinear coupling and its
solutions will also be a subject of our future work. It is expected that they give us more insights regarding the nature of the Everett-Hirschman entropy and conjugate temperature.

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