Approximate Symmetry Analysis of Gardner Equation

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Abstract

In this paper, we apply the method of approximate transformation groups proposed by Baikov, Gazizov and Ibragimov [1, 2], to compute the first-order approximate symmetry for the Gardner equations with the small parameters. We compute the optimal system and analyze some invariant solutions of these types of equations. Particularly, general forms of approximately Galilean-invariant solutions have been computed.

Keywords: Gardner equation, Approximate symmetry, Optimal system, Approximate Galilean-invariant solutions

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1 Introduction

The Korteweg-de Vries (KdV) equation is a mathematical model to describe weakly nonlinear long waves. There are many different variations of this PDE, but its canonical form is,

\[ u_t - 6uu_x + u_{xxx} = 0. \]  

(1.1)

The KdV equation (1.1) owes its name to the famous paper of Diederik Korteweg and Hendrik de Vries, but much of the significant work on the KdV equation was initiated by the publication of several papers of Gardner et al. (1967-1974). The remarkable discovery of Gardner et al. (1967) that KdV equation was integrable through an inverse scattering transform marked the beginning of soliton theory.

In 1968 Miura [9, 10] introduced "Miura transformation",

\[ u = v^2 + v_x, \]  

(1.2)

to determine an infinite number of conservation laws. If we put \( v = \frac{1}{2} \epsilon^{-1} + \epsilon w \) which \( \epsilon \) is an arbitrary real parameter, then Miura transformation becomes

\[ u = \frac{1}{4} \epsilon^{-2} + w + \epsilon w_x + \epsilon^2 w^2. \]

But, since any arbitrary constant is a trivial solution of KdV equation, it may be removed by a Galilean transformation, so we just consider "Gardner transformation", means,

\[ u = w + \epsilon w_x + \epsilon^2 w^2, \]

substituting above transformation in KdV equation, observe that \( w \) satisfies in "Gardner equation",

\[ w_t - 6(w + \epsilon^2 w^2)w_x + w_{xxx} = 0, \]  

(1.3)

for all \( \epsilon \).

Obviously, if \( \epsilon = 0 \), Gardner equation becomes the KdV equation. Next, we observe that the for more detailed exposition and references refer to [3, 8].
These two nonlinear equations are integrable with inverse scattering method. But in this paper, we analyse them with a method which is introduce by Baikov, Gazizov and Ibragimov [1, 2]. This method which is known as "approximate symmetry" is a combination of Lie group theory and perturbations. There is a second method which is also known as "approximate symmetry" due to Fushchich and Shtelen [4] and later followed by Euler et al [5, 6]. For a comparison of these two methods, we refer the interested reader to the papers [12, 13].

2 Notations and Definitions

Before continuing we need to present some definitions and theorems of the book of Ibragimov and Kovalev [7].

If a function $f(x, \varepsilon)$ satisfies the condition

$$\lim_{\varepsilon \to 0} \frac{f(x, \varepsilon)}{\varepsilon^p} = 0,$$

it is written $f(x, \varepsilon) = o(\varepsilon^p)$ and $f$ is said to be of order less than $\varepsilon^p$.

If $f(x, \varepsilon) - g(x, \varepsilon) = o(\varepsilon^p)$, the functions $f$ and $g$ are said to be approximately equal (with an error $o(\varepsilon^p)$) and written as $f(x, \varepsilon) = g(x, \varepsilon) + o(\varepsilon^p)$, or, briefly $f \approx g$ when there is no ambiguity. The approximate equality defines an equivalence relation, and we join functions into equivalence classes by letting $f(x, \varepsilon)$ and $g(x, \varepsilon)$ to be members of the same class if and only if $f \approx g$.

Given a function $f(x, \varepsilon)$, let

$$f_0(x) + \varepsilon f_1(x) + \ldots + \varepsilon^p f_p(x),$$

be the approximating polynomial of degree $p$ in $\varepsilon$ obtained via the Taylor series expansion of $f(x, \varepsilon)$ in powers of $\varepsilon$ about $\varepsilon = 0$. Then any function $g \approx f$ (in particular, the function $f$ itself) has the form

$$g(x, \varepsilon) = f_0(x) + \varepsilon f_1(x) + \ldots + \varepsilon^p f_p(x) + o(\varepsilon^p).$$

Consequently the expression [2.1] is called a canonical representative of the equivalence class of functions containing $f$. Thus, the equivalence class of functions $g(x, \varepsilon) \approx f(x, \varepsilon)$ is determined by the ordered set of $p + 1$ functions $f_0(x), f_1(x), \ldots, f_p(x)$. In the theory of approximate transformation groups, one considers ordered sets of smooth vector-functions depending on $x$'s and a group parameter $a$:

$$f_0(x, a), f_1(x, a), \ldots, f_p(x, a),$$

with coordinates

$$f_0^i(x, a), f_1^i(x, a), \ldots, f_p^i(x, a), \quad i = 1, \ldots, n.$$

Let us define the one-parameter family $G$ of approximate transformations

$$\bar{x}^i \approx f_0^i(x, a) + \varepsilon f_1^i(x, a) + \ldots + \varepsilon^p f_p^i(x, a), \quad i = 1, \ldots, n \quad (2.2)$$

of points $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ into points $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n) \in \mathbb{R}^n$ as the class of invertible transformations

$$\bar{x} = f(x, a, \varepsilon), \quad (2.3)$$

with vector-functions $f = (f^1, \ldots, f^n)$ such that

$$f^i(x, a, \varepsilon) \approx f_0^i(x, a) + \varepsilon f_1^i(x, a) + \ldots + \varepsilon^p f_p^i(x, a), \quad i = 1, \ldots, n.$$

Here $a$ is a real parameter, and the following condition is imposed:

$$f(x, 0, \varepsilon) = x.$$

The set of transformations [2.2] is called a one-parameter approximate transformation group if

$$f(f(x, a, \varepsilon), b, \varepsilon) \approx f(x, a + b, \varepsilon)$$

for all transformations [2.3].
unlike the classical Lie group theory, $f$ does not necessarily denote the same function at each occurrence. It can be replaced by any function $g \approx f$.

Let $G$ be a one-parameter approximate transformation group:

$$
\tilde{z}^i \approx f(z, a, \varepsilon) \equiv f_0^i(z, a) + \varepsilon f_1^i(z, a), \quad i = 1, \ldots, N. \tag{2.4}
$$

An approximate equation

$$
F(z, \varepsilon) \equiv F_0(z) + \varepsilon F_1(z) \approx 0 \tag{2.5}
$$

is said to be approximately invariant with respect to $G$, or admits $G$ if

$$
F(\tilde{z}, \varepsilon) \approx F(f(z, a, \varepsilon), \varepsilon) = o(\varepsilon)
$$

whenever $z = (z^1, \ldots, z^N)$ satisfies Eq. (2.5).

If $z = (x, u, u_1, \ldots, u_k)$, then (2.5) becomes an approximate differential equation of order $k$, and $G$ is an approximate symmetry group of the differential equation.

**Theorem 2.1.** Equation (2.5) is approximately invariant under the approximate transformation group (2.4) with the generator

$$
X = X_0 + \varepsilon X_1 \equiv \xi'(z) \frac{\partial}{\partial z^i} + \varepsilon \xi_1(z) \frac{\partial}{\partial z^i}, \tag{2.6}
$$

if and only if

$$
\left[ X^{(k)} F(z, \varepsilon) \right]_{F \approx 0} = o(\varepsilon),
$$

or

$$
\left[ X_0^{(k)} F_0(z) + \varepsilon \left( X_1^{(k)} F_0(z) + X_0^{(k)} F_1(z) \right) \right]_{\varepsilon \approx 0} = o(\varepsilon). \tag{2.7}
$$

In which $k$ is order of equation, and $X^{(k)}$ is $k^{th}$ order prolongation of $X$. The operator satisfying Eq. (2.7) is called an infinitesimal approximate symmetry of, or an approximate operator admitted by Eq. (2.5). Accordingly, Eq. (2.7) is termed the determining equation for approximate symmetries.

**Remark 2.2.** The determining equation (2.7) can be written as follows:

$$
X_0^{(k)} F_0(z) = \lambda(z) F_0(z), \tag{2.8}
$$

$$
X_1^{(k)} F_0(z) + X_0^{(k)} F_1(z) = \lambda(z) F_1(z). \tag{2.9}
$$

The factor $\lambda(z)$ is determined by Eq. (2.8) and then substituted in Eq. (2.9). The latter equation must hold for all solutions of $F_0(z) = 0$.

Comparing Eq. (2.8) with the determining equation of exact symmetries, we obtain the following statement.

**Theorem 2.3.** If Eq. (2.5) admits an approximate transformation group with the generator $X = X_0 + \varepsilon X_1$, where $X_0 \neq 0$, then the operator

$$
X_0 = \xi_0 \frac{\partial}{\partial z^i}, \tag{2.10}
$$

is an exact symmetry of the equation

$$
F_0(z) = 0. \tag{2.11}
$$

**Remark 2.4.** It is manifest from Eqs. (2.8), (2.9) that if $X_0$ is an exact symmetry of Eq. (2.11), then $X = \varepsilon X_0$ is an approximate symmetry of Eq. (2.5).

**Definition 2.5.** Eqs. (2.11) and (2.5) are termed an unperturbed equation and a perturbed equation, respectively. Under the conditions of Theorem 2.3, the operator $X_0$ is called a stable symmetry of the unperturbed equation (2.11). The corresponding approximate symmetry generator $X = X_0 + \varepsilon X_1$ for the perturbed equation (2.10) is called a deformation of the infinitesimal symmetry $X_0$ of Eq. (2.11) caused by the perturbation $\varepsilon F_1(z)$. In particular, if the most general symmetry Lie algebra of Eq. (2.11) is stable, we say that the perturbed equation (2.5) inherits the symmetries of the unperturbed equation.
3 Approximate Symmetry of Gardner Equation

Referring to Gardner equation \([1.3]\), if we put \(\varepsilon = \epsilon^2\) for small real parameter \(\epsilon\), it becomes

\[
w_t - 6(w + \varepsilon w^2)w_x + w_{xxx} = 0. \tag{3.1}\]

Now, we can use Remark \([2.2]\) and Theorem \([2.3]\) to provide an infinitesimal method for calculating approximate symmetries \([2.6]\) for above differential equations with a small parameter.

3.1 Exact Symmetries

Let us consider the approximate group generators in the form

\[
X = X_0 + \varepsilon X_1 = (\alpha_0 + \varepsilon \alpha_1) \frac{\partial}{\partial x} + (\beta_0 + \varepsilon \beta_1) \frac{\partial}{\partial t} + (\eta_0 + \varepsilon \eta_1) \frac{\partial}{\partial w}, \tag{3.2}
\]

where \(\alpha_i, \beta_i\) and \(\eta_i\) for \(i = 0, 1\) are unknown functions of \(x, t\) and \(w\).

Solving the determining equation

\[
X_0^{(k)} F_0(z) \bigg|_{F_0(z)=0} = 0, \tag{3.3}
\]

for the exact symmetries \(X_0\) of the unperturbed Gardner equation, means KdV equation we obtain

\[
\alpha_0 = C_1 - 6C_3 t + C_4 x, \quad \beta_0 = C_2 + 3C_4 t, \quad \eta_0 = C_3 - 2C_4 w, \tag{3.4}
\]

where \(C_1, \ldots, C_4\) are arbitrary constants. Hence,

\[
X_0 = (C_1 - 6C_3 t + C_4 x) \frac{\partial}{\partial x} + (C_2 + 3C_4 t) \frac{\partial}{\partial t} + (C_3 - 2C_4 w) \frac{\partial}{\partial w}. \tag{3.5}
\]

Therefore, unperturbed Gardner equation, means KdV equation, admits the four-dimensional Lie algebra with the basis

\[
X_0^1 = \frac{\partial}{\partial x}, \quad X_0^2 = \frac{\partial}{\partial t}, \quad X_0^3 = 6t \frac{\partial}{\partial x} - \frac{\partial}{\partial w}, \quad X_0^4 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2w \frac{\partial}{\partial w}. \tag{3.6}
\]

3.2 Approximate Symmetries

First we need to determine the auxiliary function \(H\) by virtue of Eqs.\([2.8]\), \([2.9]\) and \([2.5]\), i.e., by the equation

\[
H = \frac{1}{\varepsilon} \left[ X_0^{(k)} (F_0(z) + \varepsilon F_1(z)) \right]_{F_0(z)+\varepsilon F_1(z)=0}.
\]

Substituting the expression \([3.5]\) of the generator \(X_0\) into above equation we obtain the auxiliary function

\[
H = 12 w w_x (C_4 w - C_3). \tag{3.7}
\]

Now, calculate the operators \(X_1\) by solving the inhomogeneous determining equation for deformations:

\[
X_1^{(k)} F_0(z) \bigg|_{F_0(z)} + H = 0.
\]

the above determining equation for this equation is written as

\[
X_1^{(3)} (w_t - 6w w_x + w_{xxx}) + 12 w w_x (C_4 w - C_3) = 0.
\]

Solving this determining equation yields that \(C_4 = 0\), and hence,

\[
\alpha_1 = C_5 + C_8 x - 6C_7 t, \quad \beta_1 = C_6 + 3C_8 t, \quad \eta_1 = C_7 - 2(C_3 + C_6) w.
\]

Then, we obtain the following approximate symmetries of the Gardner equation :

\[
\begin{align*}
v_1 &= \frac{\partial}{\partial x}, & v_2 &= \frac{\partial}{\partial t}, & v_3 &= 6t \frac{\partial}{\partial x} + (2\varepsilon w - 1) \frac{\partial}{\partial w}, \\
v_4 &= \varepsilon v_1, & v_5 &= \varepsilon v_2, & v_6 &= \varepsilon (6t \frac{\partial}{\partial x} - \frac{\partial}{\partial w}) = \varepsilon v_3, & v_7 &= \varepsilon (x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2w \frac{\partial}{\partial w}).
\end{align*} \tag{3.8}
\]
Because of $C_4 = 0$, the scaling operator

$$X_0^4 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2w \frac{\partial}{\partial w},$$

is not stable. Hence, the Gardner equation does not inherit the symmetries of the KdV equation.

In the first-order of precision, we have the following Commutator table, shows that the operators (3.8) span an seven-dimensional approximate Lie algebra, and hence generate an seven-parameter approximate transformations group.

|   | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ |
|---|------|------|------|------|------|------|------|
| $v_1$ | 0 | 0 | 0 | 0 | 0 | 0 | $v_4$ |
| $v_2$ | 0 | 0 | $6v_1$ | 0 | 0 | $6v_4$ | $3v_5$ |
| $v_3$ | 0 | $-6v_1$ | 0 | 0 | $-6v_4$ | 0 | $-2v_6$ |
| $v_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_5$ | 0 | 0 | $6v_4$ | 0 | 0 | 0 | 0 |
| $v_6$ | 0 | $-6v_4$ | 0 | 0 | 0 | 0 | 0 |
| $v_7$ | $-v_4$ | $-3v_5$ | 2$v_6$ | 0 | 0 | 0 | 0 |

It is worth noting that the seven-dimensional approximate Lie algebra $\mathfrak{g}$ is solvable and its finite sequence of ideals is as follows:

$$0 \subset \langle v_4 \rangle \subset \langle v_4, v_5 \rangle \subset \langle v_4, v_5, v_6 \rangle \subset \langle v_4, v_5, v_6, v_7 \rangle \subset \langle v_1, v_4, v_5, v_6, v_7 \rangle \subset \langle v_1, v_2, v_4, v_5, v_6, v_7 \rangle \subset \mathfrak{g}$$

### 4 Optimal System for Gardner Equation

In general, to each $s$-parameter subgroup $H$ of the full symmetry group $G$ of a system of differential equations in $p > s$ independent variables, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions to the system. We need an effective, systematic means of classifying these solutions, leading to an optimal system of group-invariant solutions from which every other such solution can be derived.

**Definition 4.1.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. An optimal system of $s$-parameter subgroups is a list of conjugacy inequivalent $s$-parameter subalgebras with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of $s$-parameter subalgebras forms an optimal system if every $s$-parameter subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\mathfrak{h} = Ad(g(h)), g \in G.$

**Theorem 4.2.** Let $H$ and $\tilde{H}$ be connected $s$-dimensional Lie subgroups of the Lie group $G$ with corresponding Lie subalgebras $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ of the Lie algebra $\mathfrak{g}$ of $G$. Then $H = gHg^{-1}$ are conjugate subgroups if and only if $\mathfrak{h} = Ad(g(\mathfrak{h}))$ are conjugate subalgebras. (Proposition 3.7 of 

By theorem 4.2, the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. To compute the adjoint representation, we use the Lie series

$$Ad(exp(\mu v_i))v_j = v_j - \mu[v_i, v_j] + \frac{\mu^2}{2} [v_i, [v_i, v_j]] - \cdots = exp(ad(-\mu v_i))v_j, \quad (4.1)$$

where $[v_i, v_j]$ is the commutator for the Lie algebra, $\mu$ is a parameter, and $i, j = 1, \ldots, 7$. In this manner, we construct the table with the $(i, j)$-th entry indicating $Ad(exp(\mu v_i)v_j)$. 


If an optimal system of one-dimensional Approximate Lie algebras of the Gardner equation is provided by

\[ V = v_1, \ v_4, \ v_2 + \alpha v_6, \ v_3 + \alpha v_2 + \beta v_5, \ v_5 + \alpha v_1, \ v_6 + \alpha v_1 + \beta v_5, \ v_7 + \alpha v_1 + \beta v_2 + \gamma v_3. \]

**Proof.** Consider the approximate symmetry algebra \( \mathfrak{g} \) of the Gardner equation, whose adjoint representation was determined in the table. Our task is to simplify as many of the coefficients \( a_i \) as possible through judicious applications of adjoint maps to \( V_1 \). So that \( V_1 \) is equivalent to \( V_1' \) under the adjoint representation.

Given a non-zero vector

\[ V_1 = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 + a_6 v_6 + a_7 v_7. \]

First suppose that \( a_7 \neq 0 \). Scaling \( V_1 \) if necessary, we can assume that \( a_7 = 1 \).

As for the 7th column of the table, we have:

\[ V_1' = Ad(exp((a_4 - 3a_6a_5) v_1)) \circ Ad(exp(\frac{a_5}{3} v_2)) \circ Ad(exp(\frac{-a_6}{2} v_3)) V_1 = (a_1 - 3a_2a_6 - 2a_3a_5) v_1 + a_2 v_2 + a_3 v_3 + v_7. \]

The remaining approximate one-dimensional subalgebras are spanned by vectors of the above form with \( a_7 = 0 \). If \( a_3 \neq 0 \), we have

\[ V_2 = a_1 v_1 + a_2 v_2 + v_3 + a_4 v_4 + a_5 v_5 + a_6 v_6. \]

Next we act on \( V_2 \) to cancel the coefficients of \( v_1, v_4, v_6 \) as follows:

\[ V_2' = Ad(exp(\frac{a_1}{6} v_2)) \circ Ad(exp(\frac{a_1 a_6 + 2a_4}{12} v_5)) \circ Ad(exp(\frac{a_6}{2} v_7)) V_2 = a_2 v_2 + v_3 + \frac{3a_2a_6 + 2a_5}{2} v_5. \]

If \( a_3, a_7 = 0 \) and \( a_2 \neq 0 \), the non-zero vector

\[ V_3 = a_1 v_1 + v_2 + a_4 v_4 + a_5 v_5 + a_6 v_6, \]

is equivalent to:

\[ V_3' = Ad(exp(\frac{-a_1}{6} v_3)) \circ Ad(exp(\frac{a_1 a_5 - 3a_4}{18} v_6)) \circ Ad(exp(\frac{-a_5}{3} v_7)) V_3 = v_2 + a_6 v_6. \]

If \( a_2, a_3, a_7 = 0 \) and \( a_6 \neq 0 \), we scale to make \( a_6 = 1 \). Then

\[ V_4 = a_1 v_1 + a_4 v_4 + a_5 v_5 + v_6, \]

is equivalent to \( V_4 \) under the adjoint representation:

\[ V_4' = Ad(exp(\frac{a_4}{6} v_2)) V_4 = a_1 v_1 + a_5 v_5 + v_6. \]
if \( a_2, a_3, a_6, a_7 = 0 \) and \( a_5 \neq 0 \), In the same way as before, the non-zero vector
\[
V_5 = a_1 v_1 + a_4 v_4 + v_5,
\]
can be simplified:
\[
V'_5 = Ad(exp(-a_4 v_7))V_5 = a_1 v_1 + v_5.
\]
if \( a_2, a_3, a_5, a_6, a_7 = 0 \) and \( a_1 \neq 0 \), we act on
\[
V_6 = v_1 + a_4 v_4,
\]
by \( Ad(exp(-a_4 v_7)) \), to cancel the coefficient of \( V_4 \), leading to
\[
V'_6 = Ad(exp(-a_4 v_7))V_6 = v_1.
\]
The last remaining case occurs when \( a_1, a_2, a_3, a_5, a_6, a_7 = 0 \) and \( a_4 \neq 0 \), for which our earlier simplifications were unnecessary. Because the only remaining vectors are the multiples of \( v_4 \), on which the adjoint representation acts trivially.

5 Approximately differential invariants for the Gardner equation

In this section we use two different methods to compute an approximately invariant solutions.

In the beginning of this section we compute an approximately invariant solution based on the \( X = v_1 + v_6 \).

The approximate invariants for \( X \) are determined by the equation:
\[
X(J) = \left( \frac{\partial}{\partial x} + \epsilon(6t \frac{\partial}{\partial x} - \frac{\partial}{\partial w}) \right) (J_0 + \epsilon J_1) = o(\epsilon).
\]
Equivalently:
\[
\frac{\partial}{\partial x}(J_0) = 0, \quad \frac{\partial}{\partial x}(J_1) + (6t \frac{\partial}{\partial x} - \frac{\partial}{\partial w})(J_0) = 0.
\]
The first equation has two functionally independent solutions \( J_0 = t \) and \( J_0 = w \).

The simplest solutions of the second equation are respectively, \( J_1 = 0 \) and \( J_1 = x \). Therefore we have two independent invariants \( w + \epsilon x = \varphi(t) \) and \( \varphi(t) = x \) respect to \( X = v_1 + v_6 \).

Letting \( w + \epsilon x = \varphi(t) \) we obtain the \( w = \varphi(t) - \epsilon x \) for the approximately invariant solutions.

Then, upon substituting in Gardner equation transform into the below equation:
\[
\varphi' + 6(\epsilon - \varphi + \epsilon(\varphi - \epsilon x)^2 = o(\epsilon).
\]
Therefore, in our approximation we have \( \varphi = \varphi' = 0 \). Thus, invariant solution to Gardner equation Corresponding to \( X \) is:
\[
w = -\epsilon x.
\]

In this manner, we compute functionally approximate invariants respect to the generators of lie algebra and optimal system, as shown in Table 3 below.

Where the unknown functions \( F, G, H \) are considered as follows:
\[
F(x, t, w) = xw( - 1 - 2\gamma + \frac{3\alpha}{6\gamma t + \alpha}) + x^2 \left( \frac{3\alpha \gamma}{2(6\gamma t + \alpha)^2} - \frac{\gamma^2 + \gamma}{6\gamma t + \alpha} \right),
\]
\[
G(x, t, w) = -\frac{8\gamma t^3 + 2\alpha t^2 - \beta tx}{\beta}, \quad H(x, t, w) = -\frac{\gamma(1 + 2\gamma)t^2}{2\beta} + 2(1 - \gamma)tw.
\]
As you can see, the constant $t,w$ and use it for finding an approximately invariant solution by looking for the invariant perturbation of the : 

Now, we apply a different technique to find approximate Galilean-invariant solution. The reason is that the dependent variables are not expanded in a perturbation series \[12\].

Therefore we use another technique to find approximate invariant solutions for the Gardner equation.

5.1 Approximate Galilean-invariant solution

Now, we apply a different technique to find Approximate Galilean-invariant solutions for the Gardner equation. We know that the general form of Galilean-invariant solutions to the unperturbed Gardner equation, means KdV equation, look as follows:

$$W := \frac{c-x}{6t}.$$ 

The function $W$ is invariant under the operator $6t \frac{\partial}{\partial x} - \frac{\partial}{\partial w}$.

we consider the approximate symmetry $X_\varepsilon = v_3 + k_1v_4 + k_2v_5 + k_3v_6 + k_4v_7$ of the perturbed Gardner equation. Thus we will take the approximate symmetry

$$X_\varepsilon = 6t \frac{\partial}{\partial x} - \frac{\partial}{\partial w} + \varepsilon \left( k_1 \frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial t} + k_3(6t \frac{\partial}{\partial x} - \frac{\partial}{\partial w}) + k_4(x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2w \frac{\partial}{\partial w}) \right) = X_0 + \varepsilon X_1,$$

and use it for finding an approximately invariant solution by looking for the invariant perturbation of the:

$$w = W + \varepsilon v(x,t).$$ (5.1)

The invariant equation for above equation is:

$$X_\varepsilon \left( w - W - \varepsilon v(x,t) \right) \bigg|_{E=1} = o(\varepsilon).$$ (5.2)

Note that $X_0(w - W)$ vanishes identically. Therefore, Eq. (5.2) becomes

$$X_1(w - W) - X_0(v) = 0,$$

Therefore, we obtain the following differential equation for $v(t,x)$

$$6tv_t = \frac{k_1 + k_4 c}{6t} + k_2 \left( \frac{c-x}{6t^2} \right) + \left( \frac{c-x}{3t} \right).$$

As you can see, the constant $k_3$ is removed. It is easy to integrate this equation in the ”natural” variables

$$z = W = \frac{c-x}{6t}, \quad y = t.$$
Then it becomes:

\[ v_z + \frac{k_1 + k_4 c + 6 k_2 z}{6y} + 2z = 0. \]

The integration yields

\[ v = -\frac{(k_1 z + k_4 cz + 3 k_2 z^2)}{6y} - z^2 + F(y). \]

Returning to the variables \(t, x\), we have

\[ v = -\frac{(k_1 + k_4 c)(c - x)}{36t^2} - k_2 \frac{(c - x)^2}{72t^3} - \left( \frac{c - x}{6t} \right)^2 + F(t). \]

Inserting this \(v\) in (5.1) and substituting in the perturbed Gardner equation we obtain \(t \frac{dF(t)}{dt} + F(t) = 0\), so \(F(t) = \frac{C}{t}\), where \(C\) is an arbitrary constant.

Thus, the approximate symmetry \(X_\varepsilon\) provides the following the approximately Galilean-invariant solutions:

\[ w = W + \varepsilon \left( -\frac{(k_1 + k_4 c)(c - x)}{36t^2} - k_2 \frac{(c - x)^2}{72t^3} - \left( \frac{c - x}{6t} \right)^2 + \frac{C}{t} \right). \]

Galilean-invariant solution, means \(W\), and Approximate Galilean-invariant solution, means \(w\), are displayed below for \(-3 \leq x \leq 3, 0.1 \leq t \leq 3, c = C = k_1 = k_2 = k_4 = 1\) and \(\varepsilon = 0.1\), respectively.

![Figure 1: Comparison of Galilean-invariant and Approximate Galilean-invariant solution](image)

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