HYPERELLIPTIC THREEFOLDS WITH GROUP $D_4$, THE DIHEDRAL GROUP OF ORDER 8

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Abstract. We give a simple construction for the hyperelliptic threefolds with group $D_4$.

Introduction

A Generalized Hyperelliptic Manifold is the quotient $X = T/G$ of a complex torus $T$ by the free action of a finite group $G$ which contains no translations. We say that we have a Generalized Hyperelliptic Variety if moreover the torus $T$ is projective, i.e., it is an Abelian variety $A$.

Recently D. Kotschick observed that the classification of Generalized Hyperelliptic Manifolds of complex dimension three was not complete, since the case where $G$ is the dihedral group $D_4$ of order 8 was excluded (by H. Lange in [La01]) but it does indeed occur. Indeed F.E.A. Johnson in the preprint [Jo18] showed that a construction due to Dekimpe, Halenda and Szczepański of a flat manifold $M$ of real dimension 6 with holonomy equal to $D_4$ (see [DHS08]) would give the desired Manifold (which is projective, as remarked by Kotschick being Kähler with second Betti number $= 2$).

We describe all such examples explicitly, following the method of Lange, which was based on the classification of automorphisms of complex tori of dimension 2 given by Fujiki in [Fu88].

The family we give is exactly the one obtained by taking all possible complex structures on the flat manifold $M$, and the upshot is that all these hyperelliptic complex manifolds $X$ are quotients of the product of three elliptic curves by a translation of order 2.

1. The example

Let $E, E'$ be any two elliptic curves,

$$E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau), \quad E' = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau').$$

Set

$$A' := E \times E \times E', \quad A := A'/\langle \omega \rangle, \quad \text{where } \omega := (1/2, 1/2, 0).$$

Theorem 1.1. The Abelian variety $A$ admits a fixed point free action of the dihedral group $D_4 := \langle r, s | r^4 = 1, s^2 = 1, (rs)^2 = 1 \rangle$.

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such that $D_4$ contains no translations.

Proof. Set, for $z := (z_1, z_2, z_3) \in A'$:

$$r(z_1, z_2, z_3) := (z_2, -z_1, z_3 + 1/4) = R(z_1, z_2, z_3) + (0, 0, 1/4)$$
$$s(z_1, z_2, z_3) := (z_2 + b_1, z_1 + b_2, -z_3) = S(z_1, z_2, z_3) + (b_1, b_2, 0),$$

where $b_1 := 1/2 + \tau/2$, $b_2 := \tau/2$.

**Step 1.** It is easy to verify that $r, R$ have order exactly 4 on $A'$, and that $R(\omega) = \omega$, so that $r$ descends to an automorphism of $A$, of order exactly 4. Moreover, any power $r^j$, $0 < j \leq 3$ acts freely on $A$, since the third coordinate of $r^j(z)$ equals $z_3 + j/4$.

**Step 2.** $s^2(z) = z + \omega$, since $b_1 + b_2 = 1/2$; moreover $S(\omega) = \omega$, hence $s$ descends to an automorphism of $A$ of order exactly 2.

**Step 3.** We have

$$rs(z) = (z_1 + b_2, -z_2 - b_1, -z_3 + 1/4),$$

hence

$$(rs)^2(z) = (z_1 + 2b_2, z_2, z_3) = z,$$

and we have an action of $D_4$ on $A$, since the respective orders of $r, s, rs$ are precisely 4, 2, 2.

**Step 4.** We claim that also the symmetries in $D_4$ act freely on $A$ and are not translations. Since there are exactly two conjugacy classes of symmetries, those of $s$ and $rs$, it suffices to observe that these two transformations are not translations: in the next step we show that they both act freely.

**Step 5.** It is rather immediate to see that $rs$ acts freely, since $rs(z) = z$ is equivalent to

$$(b_2, -2z_2 - b_1, -2z_3 + 1/4)$$

being a multiple of $\omega$ in $A'$. But this is absurd, since $2\omega = 0$, and $b_2 = \tau/2 \neq 0, 1/2$.

The condition that $s$ acts freely, since $s(z) = z$ is equivalent to

$$\gamma := (z_2 - z_1 + b_1, z_1 - z_2 + b_2, -2z_3)$$

being a multiple of $\omega$ in $A'$.

However the sum of the first two coordinates of multiples of $\omega$ is 0, and this equation is not satisfied by $\gamma$, since $b_1 + b_2 = 1/2 \neq 0$.

\[ \square \]

**Corollary 1.1.** The above family of hyperelliptic threefolds $X$ with group $D_4$ forms a complete two dimensional family. The Kähler manifolds with the same fundamental group as $X$ yield an open subspace of the Teichmüller space of $X$ parametrized by the two halfplanes containing $\tau, \tau'$ respectively.

**Proof.** Take the above family of varieties $X = A/G$, where $G = D_4$, and observe that, setting $U := \mathbb{C}^3$, $H^1(\Theta_A)^G = (U \otimes U^\vee)^G$ can be calculated as follows. We have $U = U_1 \oplus W$, where $U_1, W$ are real (self-conjugate) representations, $U_1$ is irreducible and $W$ is a character of $G/(r)$.

Hence $(U \otimes U^\vee)^G = (U \otimes U^\vee)^G = \text{End}(U_1 \oplus W)^G$ has dimension 2 by Schur’s lemma.
Following Theorem 1 of [CC17], and since, as we show below in proposition 1.3, there is only one possible Hodge-Type, we conclude that the open subspace of the Teichmüller space of $X$ corresponding to Kähler manifolds is irreducible and equal to the product of two halfplanes. 

□

Remark 1.2. Let \( R := \mathbb{R}_4 := \mathbb{Z}[x]/(x^2 + 1) \) be the 4th cyclotomic ring, also called the ring of Gaussian integers. We denote by $\sigma$ the Galois involution sending $z = a + x b \mapsto \sigma(z) := a - x b$. We define, according to Dekimpe et al. ([DHS08]) the following $\mathcal{R}$-module:

\[
L := \mathcal{R} \oplus \mathcal{R} \oplus (\mathbb{Z}e_5 \oplus \mathbb{Z}e_6) =: L_1 \oplus L_2 \oplus L_3,
\]

where the module $L_3$ is the trivial $\mathcal{R}$-module. The real torus $T := (L \otimes \mathbb{R})/L$ admits a free action of the dihedral group $D_4$, defined as follows:

\[
\begin{align*}
r(z_1, z_2, z_3) &= (xz_1 + x/2, xz_2 + 1/2, z_3 - 1/4e_5), \\
s_2(z_1, z_2, z_3) &= (x\sigma(z_1), \sigma(z_2) + 1/2, -z_3).
\end{align*}
\]

It is easy to see that the flat manifolds $A/D_4$ are the same as the flat manifolds $T/D_4$.

In fact, we define $\omega_1, \omega_2 \in \mathbb{C}^3$ as the vectors

\[
\omega_1 := (1/2, 1/2, 0), \omega_2 := (\tau, 0, 0).
\]

Then we set $L_1$ to be the free $\mathcal{R}$-module generated by $\omega_1$, $L_2$ the free $\mathcal{R}$-module generated by $\omega_2$, and $L_3$ the trivial $\mathcal{R}$-module generated by $(0, 0, 1)$ and $(0, 0, \tau')$.

We have then that $A$ is the quotient $\mathbb{C}^3/L$.

**Proposition 1.3.** The above $\mathcal{R}$-module $L$ admits a unique Hodge-Type. More precisely, the $D_4$-invariant complex structures form a 2-dimensional complex family, obtained choosing respective 1 dimensional subspaces $U(1) \subset V(1) = V_3$ and $U(i) \subset V(i)$, such that

\[
(**) \quad U(1) \oplus \overline{U(1)} = V(1), \quad U(i) \oplus \overline{sU(i)} = V(i),
\]

and defining $U(-i) := SU(i)$.

**Proof.**

Consider the complex vector space

\[
V := L \otimes \mathbb{C} = (L_1 \otimes \mathbb{C}) \oplus (L_2 \otimes \mathbb{C}) \oplus (L_3 \otimes \mathbb{C}),
\]

where we observe that each summand is stable by the action of $D_4$.

By looking at the eigenspace decomposition of $V$ with respect to the linear action of $r$, given by the diagonal matrix with entries $(x, x, 1)$ we can decompose:

\[
V = (V_1(i) \oplus V_1(-i)) \oplus (V_2(i) \oplus V_2(-i)) \oplus V_3.
\]

The second summand is the conjugate of the first, the fourth is the conjugate of the third. To get a free action of $D_4$ one must give a Hodge decomposition

\[
V = U \oplus \overline{U},
\]
where the holomorphic subspace $U$ must be invariant by $R$, and must split as the sum of three eigenspaces for $R$:

$$U = U(i) \oplus U(-i) \oplus U(1).$$

Let us look at $V_1(i)$: it is spanned by $(1 - ix, 0, 0)$ and the linear part of $s$ sends it to $-i(1 + ix)$, an element in $V_1(-i)$.

Similarly $S$ sends $V_2(i)$ to $V_2(-i)$, hence

$$SV(i) = V(-i) = \overline{V(i)}.$$

Since $S$ preserves the complex structure $U$, we obtain that

$$S(U(i)) = U(-i),$$

so all eigenspaces have dimension one.

We have already seen that it must hold $U(-i) = SU(i)$, from which follows that $S(U(i)) = U(-i)$.

The condition $V = U \oplus \overline{U}$ amounts then to the two properties (**). One can directly verify that the second holds true on some open set of the Grassmannian.

However, this also follows from the explicit description of our examples.

\[\square\]

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