Landau damping in cylindrical inhomogeneous plasmas

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Abstract. We study the Landau damping for electromagnetic waves in a inhomogeneous cylindrical plasma in the presence of a strong axial magnetic field. For this task we use the kinetic theory for plasmas. This kind of phenomena can not be studied using only macroscopic fluid models. The results of our work can be relevant to identify the behavior of different electromagnetic normal modes propagating through the system, in particular, the energy of the damped modes can be transferred to the plasma particles through resonant interactions and according to the mode polarization can be used in modern particle accelerators based in plasmas.

1. Introduction

In plasmas, which is a matter state where the charged particle dynamics is governed by the electromagnetic fields, there are two fundamental approaches to model the behavior of this kind of systems. One of these two approaches consider the plasma matter as a continuous fluid described by macroscopic variables such as density, current densities, pressure and temperature and uses macroscopic transport equations to find the temporal behavior of these variables. This temporal evolution have to be solved in a coupled way with the Maxwell equations giving the dynamics of the electromagnetic fields \cite{1}. The reason for this coupling is that the charged particles dynamics determines the electromagnetic fields behavior and vice-versa. The fluid approach allows the description and understanding of a vast of phenomena, however there are behaviors that a macroscopic fluid model can not describe. These special phenomena deals with those related to microscopic behaviors involving the detailed structure of the statistical distribution of particles. These phenomena are not included in a fluid approach since the macroscopic fluid variables are constructed as statistical averages of microscopic quantities \cite{1}.

On the other hand, plasma matter can be modeled from the point of view of statistical mechanics, which considers the evolution of statistical distribution functions associated to each plasma species. The natural variables of these distribution functions corresponds to microscopic particles velocity and positions. Thus, this approach offers a more detailed description of the plasma since by following the temporal behavior of the distribution functions phenomena such resonant interactions between particles and waves can be studied \cite{1, 2, 3}. One of the important phenomena related with interactions between plasma particles and waves corresponds to the Landau Damping what is a physical mechanism that occurs when low frequency waves interact...
with plasma particles and lose energy due to this interaction. This effect is explained by a resonant interaction between charged particles and the electric field of the wave, that occurs for particles with velocities approximately equal to the phase velocity of the wave. By the shape of the velocity distribution function of particles, it can occurs that particles with slightly less velocities respect to the wave phase velocity are accelerated gaining energy from the wave while particles with slightly large velocities respect to the the wave phase velocity are decelerated losing energy to the wave. When the slope of the velocity distribution is negative, there are more particles gaining energy from the wave than those losing energy to the wave which leads to wave damping [1]. The main task of this work is to find numerically the Landau Damping for a low frequency wave mode propagating in a particular bounded plasma system surrounded by a metallic cylinder such it occurs in laboratory plasmas [5, 14]. This kind of studies can be relevant to explain heating phenomena in plasmas due to the energy extraction from the waves or to explore possible plasma based particles accelerators mechanisms.

2. Kinetic model for the plasma

A complete description of the plasma dynamics from the kinetics point of view require the equations that describe the dynamics of the electromagnetic fields in the system (2), (3), (4), (5), which coupled with the equations (1) (one for each species) that describe the dynamics of the distribution function originate to the following system, known as Vlasov-Maxwell equations:

\[ \frac{\partial f_\alpha}{\partial t} + v \cdot \frac{\partial f_\alpha}{\partial x} + \frac{q_\alpha}{m_\alpha} (E + \mathbf{v} \times \mathbf{B}/c) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0, \tag{1} \]

\[ \nabla \cdot \mathbf{E} = 4\pi \sum_\alpha n_\alpha q_\alpha \int f_\alpha d\mathbf{v} + 4\pi \rho_{ext}, \tag{2} \]

\[ \nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = 4\pi \sum_\alpha n_\alpha q_\alpha \int \mathbf{v} f_\alpha d\mathbf{v} + \frac{4\pi}{c} \mathbf{J}_{ext}, \tag{3} \]

\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{4} \]

\[ \nabla \cdot \mathbf{B} = 0, \tag{5} \]

3. Dispersion relation

In order to study the dynamics of the electromagnetic fields, the Vlasov-Maxwell system of equations needs to be solved. For this task we use a Fourier-Bessel expansion for the spatial behavior of the quantities while the temporal behavior is dealt with Laplace transforms [5, 12, 15]. This procedure allows us to find the dielectric tensor of the system which describes the electromagnetic response of the plasma. In this work we study the wave propagation in a cylindrical radially inhomogeneous plasma which is confined to a plasma metallic cylinder by a strong magnetic field. Taking into account that this plasma is surrounded by conductor boundaries, the fields must accomplish the respective boundary conditions. This leads to a boundary value problem which originates an eigenvalue problem where the eigenvectors corresponds to the expansion coefficients for the fields while the eigenvalues represent the eigen wave frequencies. Thus, we obtain a family of normal electromagnetic modes propagating through the system. We are interested in the propagation of small perturbations around the plasma equilibrium state which we suppose that the particles distribution function for the velocities is described by a Maxwell-Boltzman distribution given by \( F_{\alpha 0}(v) \). We consider that the equilibrium plasma is homogeneous both in angular and axial direction but presents a radial inhomogeneity described by the radial function \( g_{\alpha 0}(r) \) which we selected as a function to reproduce experimental data [5]. Dealing with the linearized Vlasov-Maxwell equations around
the equilibrium state we obtain the dielectric tensor that is further used to construct the wave equation for the electric fields in the system. In particular, we analyze the transverse magnetic modes which are characterized by the axial component of the electrical field which we expand as a Fourier-Bessel series as [5]:

\[ E_{zm}^m (r) = \sum_{l=1}^{\infty} A^{(l)(n)}_m (k) \frac{\sqrt{2} J_m (P_{ml})}{a J_{m+1} (X_{ml})} \]

where \( J_m (X) \) are the Bessel functions of first kind and order \( m \), \( P_{ml} \) is the radial wavenumber coming from the boundary condition \( E_{zm}^m (a) = 0 \) where \( a \) is the cylinder radius. This implies \( J_m (P_{ml} a) = 0 \). Thus, \( P_{ml} a = X_{ml} \), where \( X_{ml} \) are the zeros of the equation \( J_m (X) = 0 \). Replacing the electric field Fourier Bessel expansion in the wave equation the coefficients comes from the solution of the system of equations given by:

\[- \left( P_{ml}^2 + k^2 \right) A^{(l)(n)}_m (k) + 2 I_n (k, \omega) \sum_{l'=1}^{\infty} C^{l'}_{e,m} A^{(l')(n)}_m (k) = 0, \]

where we defined the convolution coefficients as:

\[ C^{l'}_{e,m} = \frac{2}{a^2 J_{m+1} (X_{ml}) J_{m+1} (X_{ml'})} \int_0^{a} dr \, J_m (P_{ml}) J_m (P_{ml'}) \rho^0 (r), \]

and define the function

\[ I_n (k, \omega) = \frac{\omega^2 (0)}{k^2} \int_{-\infty}^{\infty} \frac{dF(v)}{dv} \left( \frac{v - \frac{\omega}{k}}{\xi^2} \right) dv. \]

A non trivial solution of equations (7) for the coefficients \( A^{(l)(n)}_m \) requires that the determinant of the matrix associated to the system of equations vanish. This leads to the dispersion relation for the system \( \omega(k) \). Thus, for each value of \( k \) we look for the eigenfrequencies \( \omega = \omega_{mn}(k) \) that satisfy the dispersion relation four our boundary problem where \( m \) and \( n \) are integers. However, plasma is a system of many degrees of freedom, thus, for each value of \( k \) there are in principle infinite number of eigenfrequencies. Therefore, it is impossible to find all these eigenfrequencies, thus, in a specific study it is necessary to focus in find a particular eigenfrequency for each value of \( k \). These eigenfrequencies must be obtained from the solution of equations (7). The matrix associated to this system of equations has elements depending \( k \) and \( \omega \) in an intricated way and also involve the calculation of the convolution coefficients defined above for inhomogeneous radial density profiles. This matrix multiplies the column vector defined by the set of coefficients \( A^{(l)(n)}_m (k) \) which are the corresponding eigenvectors and they define the axial component of the electric field according to equation (6).

3.1. Description of the numerical scheme

Equation (7) constitutes an eigenvalue problem where the associated matrix itself depends on the eigenvalues. Therefore, the solution of our problem requires the implementation of a numerical method to solve eigenvalue problems where the associated matrix is a function of the eigenvalues. In this direction [16] based on the traditional Newton iterative roots solver scheme developed an efficient numerical method to solve these kind of problems. This method developed uses projection operators in order to avoid slow convergence and requires in its initialization an approximate eigenvalue. In the application of this method it is important to emphasize that if the approximate eigenfrequency is chosen near to the correct value few iterations are required to achieve the convergence. For the specific details of the implementation of this method see [16]. Between other advantages of this modified scheme is that the matrix can be in general
defined at the complex frequencies plane as the case of our work and also the matrix can be nonsymmetric. The matrix dependence of the complex frequencies can be in general nonlinear. Also, this scheme avoid singularities of the matrix very near the eigenfrequencies where the matrix determinant vanish. The method also allows to find the eigenfrequencies and field expansion coefficients (eigenvectors) simultaneously [16].

3.2. Landau Damping
As it was explained in the introduction, one of the basic phenomena about the wave propagation in plasmas is the Landau Damping of low frequency waves which is obtained through the plasma kinetic theory. This effect is a consequence of wave propagation in plasmas whose equilibrium is described by a Maxwellian velocity distribution and produces an energy transference from waves to particles moving with close velocities to the wave phase velocity, causing an effective wave energy damping. For an harmonic wave with amplitude proportional to \( \exp(i\omega t) \), Landau Damping reflects its existence as a negative frequency imaginary part, i.e, \( \omega = \Re(\omega) + i\Im(\omega) \) with \( \Im(\omega) < 0 \). We used in our study the radial density profile for the plasma particles at equilibrium given by [5]

\[
g_{e0}(r) = \frac{1}{1 + e^{(2r-a)/\gamma}}
\]  

(10)

For this calculation we used \( \gamma = 0.1 \) and \( a = 16 \) cm.

3.3. Numerical result and its physical interpretation
At Figure 1 we show the negative imaginary part (Landau damping) of the frequency for the fundamental low frequency transverse magnetic mode \( TM_{m=0,n=1} \) obtained numerically. This figure shows that the Landau Damping of the fundamental low frequency mode exhibits two behaviors according to the ranges of normalized wavenumbers \( (ka) \). The first range where \( 0 \leq ka \leq 5 \) shows that the imaginary part of the normalized frequency is vanishing small, which means that waves in this range are not damped. This is explained from the physical point of view taking into account that in this range the phase velocity of the waves is large, and the probability of find electrons with large velocities is small since we are working in a linear theory where the velocity distribution is approximately Maxwellian. Therefore, in this range there is not exist the required velocity gradient of the distribution function required to have an effective Landau Damping (see [1]). On the other hand, as \( ka \) increases, the phase velocity of the waves decreases. Again, since the velocity distribution of electrons is close to a Maxwellian, the probability of find electrons in this smaller velocity range is higher, and also the velocity gradient of the velocity distribution is more appreciable. Therefore, in this range of higher \( ka \) values the imaginary part is negative reflecting the Landau Damping of the waves where the wave delivers energy to electrons.

4. Conclusions
Using the linearized Vlasov-Maxwell equations and expansions for the spatial behavior of the variables we solved numerically the dispersion relation for transverse magnetic modes propagating in a plasma bounded by a metallic waveguide in the presence of a strong axial magnetic field with a radial density profile. Our study allowed us to find the frequencies of these modes with a negative imaginary part which corresponds to Landau damped modes. We focused our analysis to the damping of the fundamental low frequency transverse magnetic mode finding that for small wavenumbers there is no damping of the waves while the damping is more effective as wavenumbers increases. These results are explained since the distribution function of the electrons is close to a Maxwellian velocity distribution.
Figure 1. Landau damping of the fundamental low frequency transverse magnetic mode. The wave frequency is normalized respect to the electronic plasma frequency evaluated at $r = 0$, i.e., $\omega_{pe}(0)$.

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