Relativistic Bohmian interpretation of quantum mechanics

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Abstract. I present a relativistic covariant version of the Bohmian interpretation of quantum mechanics and discuss the corresponding measurable predictions. The covariance is encoded in the fact that the nonlocal quantum potential transforms as a scalar, which is a consequence of the fact that the nonlocal wave function transforms as a scalar. The measurable predictions that can be obtained with the deterministic Bohmian interpretation cannot be obtained with the conventional interpretation simply because the conventional probabilistic interpretation does not work in the case of relativistic quantum mechanics.

INTRODUCTION

The Bohmian interpretation of quantum mechanics and quantum field theory [1, 2, 3, 4, 5] is a promising approach towards the solution of the problem of measurement in quantum theory. However, two objections on this interpretation are often posed. First, it is a nonlocal hidden variable theory, so it seems to be in contradiction with special theory of relativity. Second, it does not seem to lead to new measurable predictions, so its value seems to be more philosophical than scientific. Here I review some of my recent results originally presented in [6, 7] which show that the Bohmian interpretation of quantum mechanics can be formulated such that it is both nonlocal and compatible with special theory of relativity. Moreover, it turns out that such a formulation leads to new measurable predictions, allowing for an experimental verification of the relativistic Bohmian interpretation of quantum mechanics. Before discussing the relativistic case, I also present a short review of the nonrelativistic Bohmian interpretation.

NONRELATIVISTIC BOHMIAN INTERPRETATION

Quantum mechanics (QM) is described by the Schrödinger equation

\[
\left[ -\frac{\hbar^2 \nabla^2}{2m} + V \right] \psi = i \hbar \partial_t \psi. \tag{1}
\]

By writing the wave function in the polar form

\[
\psi(x,t) = R(x,t)e^{iS(x,t)/\hbar}, \tag{2}
\]
the complex Schrödinger can be written as a set of two real equations. One of them is the quantum Hamilton-Jacobi equation

$$\frac{(\nabla S)^2}{2m} + V + Q = -\partial_t S,$$

(3)

while the other one is the conservation equation

$$\partial_t R^2 + \nabla \left( R^2 \frac{\nabla S}{m} \right) = 0.$$  

(4)

Here the quantum potential $Q$ is defined as

$$Q \equiv -\frac{\hbar^2}{2m} \nabla^2 R.$$

(5)

The conservation equation implies that $|\psi|^2$ can be interpreted as a probability density.

The quantum Hamilton-Jacobi equation given above takes the same form as the classical Hamilton-Jacobi equation, except for the fact that the quantum Hamilton-Jacobi equation contains an additional $Q$-term. This suggests the Bohmian interpretation, according to which the particle has a deterministic trajectory $x(t)$ that satisfies

$$\frac{dx}{dt} = \frac{\nabla S}{m}.$$  

(6)

Eq. (6) is identical to an analogous equation in the classical Hamilton-Jacobi theory. Moreover, Eq. (6) combined with the quantum Hamilton-Jacobi equation implies the quantum Newton equation

$$m \frac{d^2 x}{dt^2} = -\nabla(V + Q),$$

(7)

which takes the same form as the classical Newton equation, except for an additional quantum force generated by the quantum potential.

Now consider a statistical ensemble of particle positions, with the probability density $\rho(x,t)$. The probability density $\rho$ satisfies the conservation equation (4) with $R^2 \to \rho$. If $\rho(x,t_0) = R^2(x,t_0)$ for some initial time $t_0$, then (6) together with the conservation equation provides that $\rho(x,t) = R^2(x,t)$ for all $t$. This provides a statistical but deterministic explanation of the rule that $|\psi|^2$ represents the probability density. According to the Bohmian interpretation, all QM uncertainties emerge from the ignorance of the actual initial particle position $x(t_0)$. In this interpretation, there is no need for a wave-function “collapse”.

But why $\rho(x,t_0) = R^2(x,t_0)$ at the initial time $t_0$? This is because such a distribution corresponds to the statistical equilibrium. There are two variants of this claim. Valentini explains this through a quantum H-theorem [8], which says that the system will eventually approach the equilibrium distribution even if the initial distribution was not the equilibrium one. Dürr, Goldstein, and Zanghì [9, 10] explain this by invoking a typicality argument.
From the above, it is clear that the Bohmian interpretation explains the results of measurements on probabilities for particle positions, without any theory on quantum measurements. But what about measurements of other variables, such as momentum or energy? For other variables, the agreement with the conventional QM rules can be obtained only by considering the theory of quantum measurements. To see this, assume that we measure a hermitian operator $\hat{A}$. The eigenstates $\psi_a(x)$ of this operator satisfy

$$\hat{A}\psi_a(x) = a\psi_a(x),$$

where $a$ are the eigenvalues. The quantum state can be expanded in terms of these eigenstates as

$$\psi(x,t) = \sum_a c_a(t)\psi_a(x).$$

Now let $y$ be the position corresponding to the measuring apparatus. According to the von Neumann measurement scheme, the interaction between the measured system and the measuring apparatus causes the entanglement of the form

$$\Psi(x,y,t) = \sum_a c_a(t)\psi_a(x)\chi_a(y).$$

In the Bohmian context, the crucial property of the measuring apparatus is that different $\chi_a(y)$ do not overlap:

$$\chi_a(y)\chi_{a'}(y) = 0 \text{ for } a \neq a'.$$

This implies that

$$|\Psi(x,y,t)|^2 = \sum_a |c_a(t)|^2|\psi_a(x)|^2|\chi_a(y)|^2,$$

where the mixed terms vanish. By averaging over $x$, we obtain

$$\rho(y,t) = \sum_a |c_a(t)|^2|\chi_a(y)|^2 = |\chi(y,t)|^2,$$

where

$$\chi(y,t) \equiv \sum_a c_a(t)\chi_a(y).$$

Now we apply the Bohmian interpretation to both $x$ and $y$. The functions $\chi_a(y)$ form nonoverlapping localized channels, so that a particle with the position $x(t)$ enters only one of the localized channels. From (13), one infers that the probability for $y$ to take a value from the support of $\chi_a(y)$ is equal to $|c_a(t)|^2$. In other words, the probability to measure the eigenvalue $a$ of the operator $\hat{A}$ is $|c_a(t)|^2$. We emphasize that even the position is measured in this way. Thus the theory of measurement presented above explains the effective wave-function collapse: The wave function remains a superposition, but the particle moves in the same way as if $\psi(x)$ collapsed to $\psi_a(x)$.

The generalization of the results above to the many-particle case is straightforward. From a many-particle wave function $\psi(x_1, \ldots, x_n,t)$ one calculates the corresponding many-particle quantum potential $Q(x_1, \ldots, x_n,t)$. Consequently, the quantum force on a particle with the trajectory $x_a(t)$, $a = 1, \ldots, n$, is equal to

$$F_a(x_1, \ldots, x_n,t) = -\nabla_a Q(x_1, \ldots, x_n,t).$$
The force on one particle depends on the instantaneous position of all other particles. This is a manifestation of the nonlocality of QM. In this way, the Bohmian interpretation is (the simplest!) nonlocal hidden variable interpretation of QM, fully consistent with the Bell theorem!

At the end of this section, let us briefly discuss the advantages and disadvantages of the Bohmian interpretation. The main advantage is the fact that the Bohmian interpretation is conceptually the most similar to classical mechanics, which makes it conceptually very clear and appealing. The main disadvantages (emphasized by those who are skeptical about the Bohmian interpretation) are the following ones:

1. The Bohmian interpretation is technically more complicated than the conventional interpretation.
2. This interpretation does not lead to new measurable predictions.\(^1\)
3. The Bohmian hidden variables are nonlocal (owing to the instantaneous action at a distance), which possibly violates special relativity.

In the rest of this paper, I will demonstrate that the second and third of these three problems can be solved.

**RELATIVISTIC BOHMIAN INTERPRETATION**

A relativistic quantum spinless particle satisfies the Klein-Gordon equation

\[
(\partial^{\mu} \partial_{\mu} + m^2) \psi(x) = 0, \quad (16)
\]

where \( x = (t, \mathbf{x}) \), the signature of the metric is \(+, -, -, -\), and we take \( \hbar = c = 1 \). The quantity \( |\psi|^2 \) is not conserved, so it cannot be interpreted as a probability density. The conserved current is

\[
j_\mu = i \psi^* \partial_\mu \psi. \quad (17)
\]

However, even for positive-frequency solutions, it is possible that \( j_0(x) < 0 \) at some regions of spacetime. Consequently, \( j_0 \) cannot be interpreted as a probability density either.

The standard resolution of this problem is second quantization (known also under the name quantum field theory), where \( \psi \) is not a wave function describing probabilities, but an observable, represented by a field operator \( \hat{\psi} \). However, there is a problem with this interpretation. If, at the fundamental level, \( \psi \) should not be interpreted as a wave function that determines probabilities of particle positions, then it is not clear why the probabilistic interpretation of \( \psi \) is in agreement with experiments for nonrelativistic particles.

\(^1\) Since the time-observable is not well defined in the conventional interpretation of QM, there are some indications that the Bohmian interpretation may lead to new measurable predictions on the time-variable. However, since the theory of measurements has not been used in the existing attempts to give the Bohmian predictions on the time-variable, the correctness of these attempts is dubious.
To solve this problem, I propose that both fields and particle positions are fundamental entities [6, 11]. The field operator $\hat{\phi}$ satisfies

$$(\partial^\mu \partial_\mu + m^2)\hat{\phi}(x) = 0.$$  \hspace{1cm} (18)

Here $\hat{\phi}$ is a hermitian (uncharged) field, which provides that negative densities are not related to a negative charge. An $n$-particle wave function is

$$\psi(x_1, \ldots, x_n) = (n!)^{-1/2}S_{\{x_a\}}\langle 0|\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|n\rangle,$$  \hspace{1cm} (19)

where $|n\rangle$ is a Lorentz-invariant $n$-particle state, $|0\rangle$ is the vacuum, and $S_{\{x_a\}}$ denotes the symmetrization needed because the field operators do not commute for nonequal times. Eq. (19) relates second quantization with first quantization, because the right-hand side contains quantities that refer to second quantization (quantum field theory), while the left-hand side represents a quantity related to first quantization. Note that the wave function depends not only on $n$ space positions, but also on $n$ times. In this way, space and time are treated on an equal footing, which provides the relativistic covariance.

One can also introduce $n$ particle currents (one for each $a$) as

$$j^\mu_a = i\psi^* \partial^\mu_a \psi.$$  \hspace{1cm} (20)

(For other physical aspects of particle currents and their difference with respect to charge currents, see [12, 13, 14].) The currents are conserved, i.e.,

$$\partial^\mu_a j_{a\mu} = 0.$$  \hspace{1cm} (21)

The wave function (19) satisfies the $n$-particle Klein-Gordon equation

$$\left(\sum_a \partial^\mu_a \partial_\mu_a + nm^2\right)\psi(x_1, \ldots, x_n) = 0.$$  \hspace{1cm} (22)

By writing

$$\psi = Re^{iS},$$  \hspace{1cm} (23)

one obtains the relativistic conservation equation

$$\sum_a \partial^\mu_a (R^2 \partial_\mu_a S) = 0,$$  \hspace{1cm} (24)

and the relativistic quantum Hamilton-Jacobi equation

$$-\frac{\sum_a (\partial^\mu_a S)(\partial_\mu_a S)}{2m} + \frac{nm}{2} + Q = 0,$$  \hspace{1cm} (25)

where the relativistic quantum potential is

$$Q = \frac{1}{2m} \sum_a \partial^\mu_a \partial_\mu_a R.$$  \hspace{1cm} (26)
It is crucial to note that the quantum potential $Q(x_1, \ldots, x_n)$ is nonlocal, but relativistic invariant!

Now we introduce the Bohmian interpretation. We postulate the Bohmian equation of motion

$$\frac{dx^\mu_a}{ds} = -\frac{1}{m} \partial^\mu_a S.$$ (27)

This is equivalent to

$$\frac{dx^\mu_a}{ds} = \frac{j^\mu_a}{2m\psi^*\psi}.$$ (28)

Consequently, the relativistic quantum Newton equation reads

$$m \frac{d^2x^\mu_a}{ds^2} = \partial^\mu_a Q.$$ (29)

The Bohmian equation of motion is covariant, which is incoded in the fact that $s$ is not a specific time coordinate, but an auxiliary scalar parameter. Indeed, $s$ can be eliminated from (28) by writing

$$\frac{dx^\mu_a}{dx^\mu_b} = \frac{j^\mu_a}{j^\mu_b}.$$ (30)

The trajectories are integral curves of the vector field $j^\mu_a$. In this way, the Bohmian interpretation is relativistic covariant (since there is no preferred time coordinate), but nonlocal (since $Q$ is nonlocal)!

The claim above that the Bohmian interpretation is covariant requires an additional explanation.\(^2\) Eq. (28) can be viewed as an equation that determines one trajectory in the $4n$-dimensional configuration space. As an integral curve, such a trajectory is uniquely determined by one “initial” position in the configuration space. However, the physical trajectories are $n$ trajectories in the 4-dimensional spacetime. The corresponding “initial” position corresponds to $n$ arbitrary points in spacetime that are proclaimed to have the same value of the parameter $s$. This arbitrariness corresponds to an arbitrary “preferred” synchronization among $n$ particles. However, the theory is still covariant in the sense that there is no a priori preferred synchronization in the equations of motion. Instead, the structure of the covariant equations of motion is such that they allow a large number of different solutions. Consequently, a choice of a “preferred” synchronization is only related to a choice of one of the solutions to the equations of motion.\(^3\) As known even from local classical theories, the property of covariance refers to the equations of motion, not to their solutions.

A problem with such a covariant Bohmian interpretation is the following [17]. In general, one does not know the initial probability distribution $\rho(x_1, \ldots, x_n)$, so it seems that the theory does not have a predictive power. However, in the next section, I demonstrate that, in some cases, there are measurable predictions, not equivalent to those of the conventional interpretation.

\(^2\) I am grateful to D. Dürr and S. Goldstein for the discussions on that issue.

\(^3\) There is also a possibility for choosing a preferred foliation of spacetime in a dynamical way, as, for example, in [15, 16].
ONE-PARTICLE CASE

In this section, I discuss some physical consequences of the relativistic Bohmian equations of motion presented in the preceding section. For simplicity, I consider the case of one particle.

Solutions of the Klein-Gordon equation are momentum eigenfunctions

$$\psi_p(x) = e^{-i p \cdot x} = e^{-i (\omega t - px)},$$  \hfill (31)

where

$$\omega = \pm \sqrt{p^2 + m^2}. \hfill (32)$$

The wave function

$$\psi(x) = \langle 0 | \hat{\phi}(x) | 1 \rangle \hfill (33)$$

contains only positive frequencies. Nevertheless, if the superposition contains different positive frequencies, $j^0$ may be negative at some regions of spacetime. The Bohmian equation of motion can be written as

$$\frac{dx^\mu}{ds} \propto j^\mu, \hfill (34)$$

where the scalar of proportionality is an arbitrary scalar function of $x$. As the trajectory in spacetime is an integral curve of the vector field $j^\mu(x)$, this trajectory does not depend on the choice of that scalar function.

The trajectories have several unusual properties. First, a particle moves backwards in time where $j^0 < 0$. Consequently, at a single time, particle may have two or more positions. In addition, at some points, particles move superluminally (i.e., faster than $c \equiv 1$). Nevertheless, this superluminal motion is consistent with relativity! The simplest way to understand this is to observe that the quantum Hamilton-Jacobi equation can be seen as an equation of the form $p^\mu p_\mu = m^2_{\text{eff}}$ where

$$m^2_{\text{eff}}(x) = m^2 + \frac{\partial^\mu \partial^\nu R(x)}{R(x)} \hfill (35)$$

is the effective squared mass. It is clear that the effective squared mass may be negative at some $x$, which corresponds to tachyons at these $x$.

It is important to emphasize that the unusual properties above are not in contradiction with observations, because the theory of quantum measurements implies that the observed velocities are never superluminal and the observed position is always a single position. For example, consider a measurement of velocity. The eigenfunction $\psi_p(x)$ is both the 4-momentum eigenstate and the 4-velocity eigenstate. Consequently, during a measurement we have an entangled state of the form

$$\Psi(x, y) = \int d^3 p \, c_p \psi_p(x) \chi_p(y). \hfill (36)$$

This implies that the measured velocity is always the velocity corresponding to one of the eigenfunctions $\psi_p(x)$, which is never superluminal. The case of measurement of the
position is similar. As in the nonrelativistic case, the functions $\chi_a(y)$ form nonoverlapping channels, so that a particle can enter only one of the localized channels. Consequently, when the position is measured, the particle cannot be found at two or more positions (channels) at the same time.

We also emphasize that superluminal velocities are consistent with causality. That is, there are no causal paradoxes, because the trajectories in spacetime are uniquely and self-consistently determined by the initial condition (see also [18]).

Now we are ready to discuss the nontrivial measurable predictions of the relativistic Bohmian interpretation. Assume that initially $j_0(x,t_0) \geq 0$. In this case, the initial probability density at $t_0$ is known:

$$\rho(x,t_0) = j_0(x,t_0). \quad (37)$$

Now assume that for $t > t_0$, $j_0 < 0$ at some regions. By explicitly calculating the trajectories for all possible initial conditions at $t_0$, one can calculate $\rho(x,t)$ at any $t$. We are interested in the measurable $\rho$, so we need to use the theory of quantum measurements. Assume that the interaction with the apparatus that measures particle positions starts at some particular time $t_1$. In this case, the particles cannot move backwards in time for $t \geq t_1$. Consequently, some of the trajectories in the region $t_0 < t < t_1$ cannot realize as physical trajectories [7]. As a result, for the measurable probability density at $t_1$ one finds [7]

$$\rho(x,t_1) = \begin{cases} j_0(x,t_1) & \text{on } \Sigma', \\ 0 & \text{on } \Sigma^+ \cup \Sigma^-. \end{cases} \quad (38)$$

where $\Sigma^\pm$ and $\Sigma'$ are defined as follows. $\Sigma^-$ is the set of all space points at $t_1$ at which $j_0 < 0$. $\Sigma^+$ is a similar set with $j_0 > 0$, such that any point on $\Sigma^+$ is connected to a point on $\Sigma^-$ via a trajectory in the region $t_0 < t < t_1$. $\Sigma'$ is the set of all other points with $j_0 \geq 0$ at $t_1$ that are not contained in $\Sigma^+$. (See also [7] for a pictorial representation of this.)

Eq. (38) says that, at $t_1$, the particle cannot be found at any point in the region at which $j_0 < 0$, as well as at those points with $j_0 > 0$ that are connected with the $j_0 < 0$ region by a trajectory. This result cannot be obtained without calculating the trajectories. Possible experimental verification of the prediction (38) would be the experimental proof that quantum particles really do have Bohmian trajectories!

**CONCLUSION AND OUTLOOK**

From our results, two important conclusions can be drawn. First, the Bohmian interpretation can be formulated such that it is both relativistic covariant and nonlocal. Second, for the case $j_0 < 0$, the conventional interpretation of relativistic quantum mechanics does not have any prediction on the distribution of particle positions. On the other hand, the Bohmian interpretation does!

Our results also suggest some prospects for the future experiments. I believe that we should start thinking how to prepare states with $j_0 < 0$ and how to measure the particle distribution for this case. Since there are no experimentally accessible elementary particles with spin zero, perhaps the experiments could be performed with photons,
which also satisfy a second-order differential equation similar to the Klein-Gordon equation, so that negative particle densities also take place. Even if the result will not confirm the Bohmian prediction considered in this work, the point is that there is no standard prediction on that issue. Consequently, the result will certainly tell us something new and fundamental about relativistic quantum mechanics.

ACKNOWLEDGMENTS

This work was supported by the Ministry of Science and Technology of the Republic of Croatia under Contract No. 0098002.

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