The inverse scattering method for cylindrical gravitational waves

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Abstract

The initial-value problem for cylindrical gravitational waves is studied through the development of the inverse scattering method scheme. The inverse scattering transform in this case can be viewed as a transformation of the Cauchy data to the data on the symmetry axis. The Riemann-Hilbert problem, which serves as inverse transformation, is formulated in two different ways. We consider Einstein-Rosen waves to illustrate the method.

1 Introduction

In this paper we study the cylindrical gravitational waves model with two polarization modes. The space-time of the model possesses two commuting one-parameter isometry groups of which one is isomorphic to $\mathbb{R}$ and the other to $SO(2)$, and is non-stationary solution of the vacuum Einstein equations. The generators of these groups are called Killing vector fields. In the case of one polarization (both Killing fields are hypersurface orthogonal) this model is also known as Einstein-Rosen waves. The theory of Einstein-Rosen waves is linear, however, still physically interesting and even in recent time provides us with new important results both in classical and quantum scope Ref. [1, 2] where one finds additional discussion and reference list. The polarized model is non-linear and hence far more difficult to study.
In two Killing reduction of vacuum or electro-vacuum general relativity Einstein equations belong to the class of integrable equations. By the other words they can be written as a compatibility condition of an auxiliary linear system. Although in this paper we use the linear system proposed in Ref. [3] it is worth mentioning that there exits essentially different approach Ref. [4].

The best tool to study integrable equations is the Inverse Scattering Method (ISM). There are many works where the ISM is applied for constructing exact solutions and there are a few papers where the physically relevant models are considered. The author knows only Ref. [5] where the model of colliding gravitational plane waves was investigated and Rev. [6] where the solutions with disconnected horizon were analyzed.

The main goal of the present paper is to develop a framework to study the cylindrical gravitational waves model.

2 Field equations and boundery conditions

The metric of the space-time with two commuting space-like Killing fields can always be chosen locally in the following form

$$ds^2 = ds^2_{I1} + Xd\phi^2 + 2Wd\phi dz + Vdz^2,$$  \hspace{1cm} (1)

where

$$ds^2_{I1} = \gamma_{ab}dx_adx_b$$

is the metric of an orthogonal to both Killing vectors two-surface. The variables $X, W, V$ depend only on the coordinates $x_a$ adapted to this surface.

Let

$$g = \begin{pmatrix} V & W \\ W & X \end{pmatrix}, \rho^2 = \det g.$$  

Then, the first group of Einstein equations can be written as

$$d*\rho dgg^{-1} = 0.$$  \hspace{1cm} (2)

Here $*$ is the Hodge operator with respect to the metric $ds^2_{I1}$. The function $\rho$ has a space-like gradient hence it can be used as a radial coordinate. Let the time coordinate be dual to space one, $dt = *d\rho$. Coordinate chart constructed
is often called the Weyl canonical coordinates. We employ it throughout the paper. In these coordinates the 2-metric $ds^2_{II}$ is given by

$$ds^2_{II} = f(\rho, t) \left( -dt^2 + d\rho^2 \right), \quad f(\rho, t) = \frac{X}{\rho^2} e^{-2\gamma}$$

(3)

The last formula is a definition of the function $\gamma$. Then the system (2) takes the form

$$-(\rho g, t g^{-1})_{,t} + (\rho g, \rho g^{-1})_{,\rho} = 0.$$  

(4)

We impose the following boundary conditions on $g$

$$g = \begin{pmatrix} V & \rho^2 \hat{W} \\ \rho^2 \hat{W} & \rho^2 \hat{X} \end{pmatrix}$$

(5)

where $V, \hat{W}$ and $\hat{X}$ are smooth functions not equal to zero and

$$V = 1 + J/\rho + O(1/\rho^2), \quad \hat{X} = 1 - J/\rho + O(1/\rho^2), \quad \hat{W} = O(1/\rho^3)$$

at $\rho = \infty$. Here $J$ is a constant independent on $t$.

An alternative formulation of Einstein equations (2) exists. For this, let us introduce matrix potential

$$dH = \rho \ast dgg^{-1}, \quad Y = H_{12}.$$  

(6)

Here $Y$ is the Ernst potential that is determined by the transition Killing vector field. The functions $Y$ and $V$ are independent dynamical variables which set a solution of (2) completely. Potentials $Y$ and $V$ satisfy the Ernst equation which we will not use in this paper.

Using (5) one can easily prove that

$$\rho g_{,\rho} g^{-1} = \begin{pmatrix} 0 & \partial_t Y \\ 0 & 2 \end{pmatrix}, \quad \rho g_{,\rho} g^{-1} = 0$$

(7)

at $\rho = 0$ and

$$\rho g_{,\rho} g^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} O(1/\rho) & O(1/\rho^3) \\ O(1/\rho) & O(1/\rho) \end{pmatrix}, \quad \rho g_{,\rho} g^{-1} = \begin{pmatrix} O(1/\rho) & O(1/\rho^3) \\ O(1/\rho) & O(1/\rho) \end{pmatrix},$$

(8)

(9)
at $\rho = \infty$.

The second group of Einstein equations allows one to determine the coefficient $\gamma$ from the matrix $g$. From (2) it is possible to show that $\partial_t \gamma = 0$ as $\rho = 0$. The solution is regular at the symmetry axis iff $\gamma = 0$ for $\rho = 0$. However, in this case $\gamma \to \gamma_0 \neq 0$ as $\rho \to \infty$ and the conical singularity appears at the space infinity. Its angle of deficit can be treated as the energy of the system [2]. It is interesting to note that the deficit angle of the conical singularity at the symmetry axis of the stationary solutions with disconnected horizon has the sense of the force between the black holes (see Ref.[3] and references therein). In the present paper, we restrict ourselves to the study of the system (4).

3 Auxiliary linear problem

System of equations (4) is the compatibility condition for the following pair of matrix linear differential equations:

$$D_1 \Psi = U(\omega, \rho, t) \Psi, \quad U(\omega, \rho, t) = \frac{\rho^2 g_{\rho \rho} g^{-1} + \omega \rho g_{t \rho} g^{-1}}{\rho^2 - \omega^2},$$  \hspace{1cm} (10)$$

$$D_2 \Psi = V(\omega, \rho, t) \Psi, \quad V(\omega, \rho, t) = \frac{\rho^2 g_{t t} g^{-1} + \omega \rho g_{\rho t} g^{-1}}{\rho^2 - \omega^2},$$  \hspace{1cm} (11)$$

Here $D_1$ and $D_2$ are the commuting differential operators

$$D_1 = \partial_\rho - \frac{2\omega \rho}{\omega^2 - \rho^2} \partial_\omega, \quad D_2 = \partial_t - \frac{2\omega^2}{\omega^2 - \rho^2} \partial_\omega,$$  \hspace{1cm} (12)$$

and $\omega$ is a complex parameter that does not depend on $\rho, t$. We also use the $U - V$-pair representation in which $\omega$ is a dependent parameter. To be more precise, let $\omega$ be a root of the equation

$$\omega^2 + 2(t - k)\omega + \rho^2 = 0$$  \hspace{1cm} (13)$$

where $k$ is an independent parameter. Determined through (13), $\omega$ satisfies

$$\partial_\rho \omega = \frac{2\omega \rho}{\rho^2 - \omega^2}, \quad \partial_t \omega = \frac{2\omega^2}{\rho^2 - \omega^2}.$$  \hspace{1cm} (14)$$
Passing from $\Psi(\omega)$ to $\Psi'(k) = \Psi(\omega(k))$, one obtains from \cite{lo,li} that
\begin{equation}
\partial_\rho \Psi' = U(\omega(k), \rho, t)\Psi', \; \partial_t \Psi' = V(\omega(k), \rho, t)\Psi'.
\end{equation}

We will denote the solution of (15) by $\Psi(k)$ missing the prime for brevity. It is worth mentioning that solving (15) with the fixed branch of the root one finds the solution of \cite{lo,li} only in the analyticity domain of $\omega(k)$.

We will follow the general scheme for investigating integrable equations Ref.\,[7] omitting many technical details.

Eq. (13) is invariant w. r. t. the transformation $\omega \to \rho^2/\omega$. Now suppose $\omega_{\pm}(k)$ are root’s branches such that $\text{Im}\omega_+ > 0$ and $\text{Im}\omega_- < 0$. Then the cuts of $\omega_{\pm}$ are half-lines $(-\infty, t - \rho]$ and $[t + \rho, \infty)$. Before proceed we mention some useful properties of $\omega_{\pm}$. Note that
\begin{equation}
\omega_+(k) = \frac{\rho^2}{\omega_-(k)}, \quad \bar{\omega}_+(\bar{k}) = \omega_-(k).
\end{equation}

As $\rho \to \infty$ the functions $\omega_{\pm}(k, \rho, t)$ satisfy uniformly in $k, t$ the following estimate
\begin{equation}
\omega_{\pm}(k, \rho, t) = \pm i\rho + (k - t) + O(1/\rho).
\end{equation}

Furthermore, introduce the monodromy matrixes $T_{\pm}(k, \rho, \rho')$. By definition, they are solutions to
\begin{equation}
\partial_\rho T_{\pm}(\rho, \rho') = U(\omega_{\pm}(k), \rho, t)T_{\pm}(\rho, \rho'), \; T_{\pm}(\rho, \rho) = I.
\end{equation}

Besides, zero caveture condition,
\begin{equation}
U_{,t} - V_{,\rho} + [U, V] = 0,
\end{equation}

reveals that
\begin{equation}
\partial_t T_{\pm}(\rho, \rho') = V(\omega_{\pm}(k), \rho)T_{\pm}(\rho, \rho') - T_{\pm}(\rho, \rho')V(\omega_{\pm}(k), \rho').
\end{equation}

For symmetric real matrix $g$ the system \cite{lo, li} remains invariant under the transformations
\begin{equation}
\Psi(\omega, \rho, t) \to g\tilde{\Psi}^{-1}(\frac{\rho^2}{\omega}, \rho, t), \; \Psi(\omega, \rho, t) \to \tilde{\Psi}(\bar{\omega}, \rho, t)
\end{equation}
where the tilde denotes transposition. Taking into account (16) one gets

\[ T_+(k, \rho, \rho') = g(\rho)\tilde{T}^{-1}_-(k, \rho, \rho')g^{-1}(\rho'), \quad T_+(k, \rho, \rho') = \tilde{T}_-(\bar{k}, \rho, \rho') \]  

(21)

Note that we regard the parameter \( k \) as a parameter on the complex plane, not on the Riemann surface of the root, and understand equations (18) as two different ones. Let us remark also that \( T_+(\omega, \rho, \rho') \) and \( T_-(\omega, \rho, \rho') \) are solutions of (10) as \( \text{Im} \omega \geq 0 \) and \( \text{Im} \omega \leq 0 \) respectively.

Let us determine the Jost functions,

\[ \Psi_\pm(k, \rho) = \lim_{\rho' \to \infty} T_\pm(k, \rho, \rho')e_\pm(k, \rho'), \quad e_\pm(k, \rho') = \begin{pmatrix} 1 & 0 \\ 0 & \omega_\pm(k, \rho') \end{pmatrix}. \]  

(22)

They are analytic in the \( k \)-plane with the cuts \((-\infty, t - \rho] \) and \([t + \rho, \infty)\). Moreover, from equation (19) we derive that they are solutions of compatible system of equations (15). Since \( \lim_{\rho' \to \infty} e_-(\rho)g^{-1}(\rho)e_+(\rho) = I \) we see also that

\[ \Psi_+(k) = g\tilde{\Psi}_-(k), \quad \Psi_+(k) = \bar{\Psi}_-(\bar{k}). \]  

(23)

It is worth mentioning that \( \det \Psi_\pm(k) = \omega_\pm(k) \).

Recall that functions \( \Psi_+(\omega) \) and \( \Psi_-(\omega) \) are analytic solutions of (10), (11) as \( \text{Im} \omega > 0 \) and as \( \text{Im} \omega < 0 \) respectively. Hence for real \( \omega \) there is a matrix \( T(k) \) such that

\[ \Psi_+(\omega) = \Psi_-(\omega)T(k), \quad k = t + \frac{\omega^2 + \rho^2}{2\omega}. \]  

(24)

Since \( \Psi_\pm(\omega) \) are coupled by transformation (20) and \( \det \Psi_\pm(\omega) = \omega, \ T(k) \) satisfies

\[ T(k) = \tilde{T}(k), \quad T(\bar{k}) = T^{-1}(k), \quad \det T(k) = 1, \]  

(25)

by the other words it is a symmetric unitary matrix. It is interesting to note that if initial data \((V - 1, X - 1, V', X')\) are of compact support then \( T - I \) is also of compact support. In general case we have \( T(k) = I + O(1/|k|) \) as \(|k| \to \infty\).

Let

\[ \chi_\pm(\omega) = \Psi_\pm(\omega)e^{-1}(\omega), \quad e(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}. \]  

(26)
It is possible to prove that $\chi_{\pm}$ are regular at $\omega = 0$ and $\chi_{\pm}(\infty) = I$. The last condition and symmetry (20) reveal that

$$g = \chi_{\pm}(0) \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}$$

(27)

Since $\chi_{-}(0) = \chi_{+}(0)$ from (24) we derive that $T_{12}(k) = T_{21}(k) = O(1/|k|^2)$ at $k = \infty$.

Boundary condition (7) yields

$$V(\omega)|_{\rho=0} = -\frac{1}{\omega} \begin{pmatrix} 0 & \partial t Y_0 \\ 0 & 2 \end{pmatrix}$$

Solving (11) one obtains

$$\Psi_{\pm}(\omega)|_{\rho=0} = \begin{pmatrix} 1 & -Y(t) \\ 0 & \omega \end{pmatrix} T_{\pm}(k), \quad T_{\pm}(k) = T_{\mp}(\bar{k})$$

(28)

Here $T_{\pm}$ is analytic in $k$ as $\text{Im} k > 0$. Inserting (28) in (24) we come to identities:

$$T_{-1}(k)T_{+}(k) = T(k) = T_{+}(k)T_{-1}(k)$$

(29)

It follows from normalization of $\chi_{\pm}$ at $\omega = \infty$ that

$$\lim_{\omega \to \infty} e(\omega)T_{\pm}(k)e^{-1}(\omega) = I$$

(30)

As a function of $\omega$, $T_{\pm}(k(\omega))$ is analytic in domains: $\text{Im}\omega > 0, |\omega| > \rho$ and $\text{Im}\omega < 0, |\omega| < \rho$. Then using (24) and (29) one can easily check that

$$\Phi_{-}(\omega) = \begin{cases} \Psi_{+}(\omega)T_{-1}(k) & \text{Im}\omega > 0, |\omega| > \rho \\ \Psi_{-}(\omega)T_{-1}(k) & \text{Im}\omega < 0, |\omega| > \rho \end{cases}$$

(31)

is analytic as $|\omega| > \rho$ while

$$\Phi_{\pm}(\omega) = \begin{cases} \Psi_{+}(\omega)\tilde{T}_{-}(k) & \text{Im}\omega > 0, |\omega| < \rho \\ \Psi_{-}(\omega)\tilde{T}_{+}(k) & \text{Im}\omega < 0, |\omega| < \rho \end{cases}$$

(32)

is analytic as $|\omega| < \rho$. On the circle $|\omega| = \rho$, $\Phi_{\pm}$ satisfy the conjugation condition,

$$\Phi_{+}(\omega) = \Phi_{-}(\omega)G(k), \quad G(k) = T_{+}(k)\tilde{T}_{-}(k) = T_{-}(k)\tilde{T}_{+}(k).$$

(33)
Set $\hat{\chi}_\pm = \Phi_\pm e^{-1}$. The matrices $\Phi_+$ and $\Phi_-$ are still coupled by transformation (20). Using it and the fact that $\hat{\chi}_-(\infty) = I$ which follows from (30), we come to

$$g = \hat{\chi}_+(0) \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}$$

(34)

Due to the boundary condition (5) the limit, $\lim_{\rho \to 0} \hat{\chi}_+(0)$, exists. On the other hand we have from (33) that

$$\lim_{\rho \to 0} \hat{\chi}_+(0) = \lim_{\omega \to 0} \left( \begin{array}{cc} 1 & -Y(t) \\ 0 & \omega \end{array} \right) G(k)e^{-1}(\omega).$$

The above limit exits if and only if $Y(t) = G_{12}(t)/G_{22}(t)$. Eventually,

$$\begin{pmatrix} V(t) & \hat{W}(t) \\ 0 & \hat{X}(t) \end{pmatrix} = \begin{pmatrix} \frac{G_{22}(t)}{G_{12}(t)} & \frac{2G_{12}(t)G_{22}(t) - 2G_{12}(t)\dot{b}_1G_{22}(t)}{G_{12}(t)G_{22}(t)} \\ 0 & \frac{G_{22}(t)}{G_{22}(t)} \end{pmatrix}$$

(35)

where unimodularity of $G(k)$ was used as well.

We summarize the results of this section as follow. For any solution of the initial-value problem posed in Section 2 there exits analytic in upper $k$-plane matrix, $T_+(k)$, that satisfies the conditions,

$$T_+ T_+^\dagger = T_+^\dagger T_+, \quad \text{Im} T_+ T_+^\dagger = 0$$

for real $k$. Here $\dagger$ is the Hermitian conjugate. It uniquely determines symmetric unitary matrix, $T(k)$, and symmetric real matrix, $G(k)$. Then the inverse problem is reduced to the Riemann-Hilbert problem (RHP) (24) or (33). In addition, there is one to one correspondence between $G(k)$ and the data on the symmetry axis, for example $Y(t)$ and $V(t)$ (35).

## 4 Einstein-Rosen waves

In this section we apply the results of the previous section to a space-time with diagonal metric ($W = 0$). Set $X = \rho^2 e^{-2\psi}$. Then the system (4) reduces to one linear equation, viz

$$-\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} = 0$$

(36)
The above equation is the compatibility condition of the following pair of scaler linear equations

\[
\partial_\mu F = \frac{2\rho^2 \partial_\mu \psi + 2\rho \omega_\pm \partial_t \psi}{\rho^2 - \omega_\pm^2} F, \quad \partial_t F = \frac{2\rho^2 \partial_t \psi + 2\rho \omega_\pm \partial_\mu \psi}{\rho^2 - \omega_\pm^2} F.
\]  

(37)

The solution of the system (13) is given by

\[
\Psi_\pm(k, \rho, t) = \left( \begin{array}{cc}
e^{\theta_\pm(k, \rho, t)} & 0 \\
0 & \omega_\pm e^{-\theta_\pm(k, \rho, t)} \end{array} \right)
\]

with

\[
\theta_\pm(k, \rho, t) = -\int_{\rho}^{\infty} d\rho' \left( \frac{2\rho'^2 \partial_\rho \psi}{\rho'^2 - \omega_\pm^2} + \frac{2\rho' \omega_\pm \partial_t \psi}{\rho'^2 - \omega_\pm^2} \right).
\]

The function \(k(\omega) = t + \frac{\omega^2 + \rho^2}{2\omega} \) maps the upper half-plane(lower half-plane) into \(k\)-plane with the cuts \((-\infty, t - \rho] \text{ and } [t + \rho, \infty)\). The function \(\omega_+(k, \rho')(\omega_-(k, \rho'))\) maps the \(k\)-plane with the cuts \((-\infty, t - \rho'] \text{ and } [t + \rho', \infty)\) into the upper half-plane(lower half-plane). Analyzing these mappings in the case when \(\rho \leq \rho'\) we conclude that

\[
\omega_+(k(\omega + i0), \rho') - \omega_-(k(\omega - i0), \rho') = 2i\sqrt{\rho'^2 - (k - t)^2}
\]

as \(|k - t| \leq \rho'\) and zero otherwise. Note that \(|k - t| \geq \rho\) for any real \(\omega\) and \(\omega_\pm(k, \rho') = k - t \pm i\sqrt{\rho'^2 - (k - t)^2}\) as \(\rho \leq |k - t| \leq \rho'\) and real \(k\). Therefore

\[
\theta_+(k(\omega + i0), \rho, t) - \theta_-(k(\omega - i0), \rho, t) = 2i\Delta(k)
\]

(38)

where

\[
\Delta(k) = -\int_{|k-t|}^{\infty} d\rho' \frac{(k - t)\partial_\rho \psi + \rho' \partial_t \psi}{\sqrt{\rho'^2 - (k - t)^2}}.
\]

The piece-wise analytic function \(\theta(\omega), \theta(\omega) = \theta_-(\omega)\) as \(\text{Im}\omega < 0\) and \(\theta(\omega) = \theta_+(\omega)\) as \(\text{Im}\omega > 0\), is a unique solution of the RHP (38) with \(\theta(\infty) = 0\). This solution can be written as

\[
\theta(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \frac{\Delta(k(\lambda))}{\lambda - \omega}.
\]

(39)
Then we can represent the solution of (36) as

\[ \psi(t, \rho) = \frac{1}{2} \theta(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \frac{\Delta(k(\lambda))}{\lambda}. \]  

(40)

Changing the variable in the above integral one has

\[ \psi(t, \rho) = \frac{1}{\pi} \int_{-\infty}^{t-\rho} dk \frac{\Delta(k)}{(k-t)^2 - \rho^2} + \frac{1}{\pi} \int_{t+\rho}^{\infty} dk \frac{\Delta(k)}{(k-t)^2 - \rho^2}. \]  

(41)

5 Conclusions

In this paper we outline the ISM scheme for the cylindrical symmetric waves model. However, to complete this scheme one needs to analyze the smoothness property of the matrices \( T(k) \) and \( G(k) \). We assume that for smooth initial data they are also smooth. The relation of the matrix \( G \) with the data on the symmetry axis allows to determine the behavior of the solution at the time infinities through the study of the asymptotic properties of \( G \). The matrix \( T \) defines the behavior of the solution at the null infinities and we assume that it is uniquely restored from the data on the null infinities. An important open question is the derivation of the formula for the angle of deficit at the space infinity in terms of the matrix \( T \) or \( G \). We plan to consider the problems mentioned in separate papers without promising to do it soon.

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