Strong Asymptotics of Planar Orthogonal Polynomials: Gaussian Weight Perturbed by Finite Number of Point Charges

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Abstract
We consider the orthogonal polynomial $p_n(z)$ with respect to the planar measure supported on the whole complex plane
$$e^{-N|z|^2} \prod_{j=1}^{\nu} |z - a_j|^{2c_j} \, dA(z)$$
where $dA$ is the Lebesgue measure of the plane, $N$ is a positive constant, $\{c_1, \ldots, c_{\nu}\}$ are nonzero real numbers greater than $-1$ and $\{a_1, \ldots, a_{\nu}\} \subset \mathbb{D} \setminus \{0\}$ are distinct points inside the unit disk. In the scaling limit when $n/N = 1$ and $n \to \infty$ we obtain the strong asymptotics of the polynomial $p_n(z)$. We show that the support of the roots converges to what we call the “multiple Szegő curve,” a certain connected curve having $\nu + 1$ components in its complement. We apply the nonlinear steepest descent method [9, 10] on the matrix Riemann-Hilbert problem of size $(\nu + 1) \times (\nu + 1)$ posed in [22]. © 2023 Wiley Periodicals, LLC.

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1 Introduction and Main Result

Let \( \{c_1, \ldots, c_\nu\} \) be a set of nonzero real numbers greater than \(-1\) and \( \{a_1, \ldots, a_\nu\} \) be a set of distinct points inside the unit disk. Let \( p_{n,N}(z) \) be the monic polynomial of degree \( n \) satisfying the orthogonality relation

\[
\int_{\mathbb{C}} p_{n,N}(z) \overline{p_{m,N}(z)} e^{-N|z|^2} \prod_{j=1}^{\nu} |z - a_j|^{2c_j} \, dA(z) = h_n \delta_{nm}, \quad n, m \geq 0.
\]

Here \( dA \) is the Lebesgue area measure on the complex plane, \( N \) is a positive constant, \( h_n \) is the positive norming constant and \( \delta_{nm} = 1 \) when \( n = m \) and \( \delta_{nm} = 0 \) when \( n \neq m \).

We consider the scaling limit where \( n \) and \( N \) both go to \( \infty \) while \( \lim_{n \to \infty} n/N = 1 \). We will set \( N = n \) without losing generality since the orthogonality gives the relation

\[
p_{n,N}(z; a_1, \ldots, a_\nu) = \left( \frac{n}{N} \right)^{n/2} p_{n,n}(\sqrt{N/n} z; \sqrt{N/n} a_1, \ldots, \sqrt{N/n} a_\nu).
\]

The asymptotic behavior of the orthogonal polynomials for a general planar measure given by \( \exp(-NQ(z)) \, dA(z) \) for a general external field \( Q : \mathbb{C} \to \mathbb{R} \) has been an open problem in relation to the normal matrix model, two dimensional Coulomb gas and Hele-Shaw problems [26, 27]. One motivation for this open problem is to obtain the limiting kernel, which is the ultimate object to understand the statistical properties of the Coulomb gas. Such limiting kernel can be studied when the asymptotic behavior of the orthogonal polynomials are given. Another motivation is to calculate the partition function, which is equivalent to the averaged characteristic polynomials in the Ginibre ensemble and has been of interest in multiplicative chaos [24].

The asymptotic behaviors are known only for special choices of \( Q \) [3–5, 7, 12, 13, 17–21, 23]. All these results are based on the planar orthogonal polynomial being also a complex orthogonal polynomial on certain contours, hence amenable to the existing methods of asymptotic analysis. For a general class of \( Q \) Hedenmalm and Wennman [14] have found the asymptotic behavior of the orthogonal polynomials “outside the droplet”. Here the droplet is a certain two-dimensional compact set in the plane that is the support of the corresponding equilibrium measure. This general result however does not identify the limiting support of the roots, because the roots are mostly found — except a finite number of them — inside the droplet as their results have reassured. The main goal of this paper is to find the strong asymptotics for the new class of planar orthogonal polynomials given in (1.1).

Notations. We set \( N = n \). Though \( N = n \) we will keep both \( N \) and \( n \), preserving their separate roles — \( N \) as a real-valued parameter and \( n \) as the integer-valued degree of polynomial — as much as possible. We define \( p_n(z) = p_{n,n}(z) \). We denote \( \mathbb{D} = \{z : |z| < 1\} \) and \( \sum c = \sum_{j=1}^{\nu} c_j \). We use both the bar and the superscript * for the complex conjugation, e.g., \( \bar{z} \) and \( z^* \).
We do not consider the case when (some) \( a_j \)'s are outside \( \mathbb{D} \). The reason is partly because we have been motivated by the results of [24] and [8], where the main question is the asymptotic behavior of the partition function of the Coulomb gas ensemble as the function of \( \{a_j\}_{j=1}^\nu \subset \mathbb{D} \) and \( \{c_j\}_{j=1}^\nu \). This problem will be studied in our subsequent publication based on the results of this paper. Another application of our results can be the universal behavior of the Coulomb gas in the vicinity of a point singularity (such as \( a_j \)) which has been studied in [1] using Ward’s equation.

**Asymptotics for \( \nu = 1 \):** When \( \nu = 1 \) the full asymptotic behavior has been found [21]; the roots of the polynomial converge towards the generalized Szegő curve that depends only on \( a_1 \) but not on \( c_1 \). See Figure 1.1. The limiting support of the roots is given by the simple closed curve (which is exactly the *Szegő curve* when \( a_1 = 1 \))

\[
\Gamma = \{ z \in \mathbb{D} : \log |z| - \text{Re}(\overline{a}_1 z) = \log |a_1| - |a_1|^2 \}.
\]

The curve divides the plane into the unbounded domain \( \Omega_0 \) and the bounded domain \( \Omega_1 \) such that \( \mathbb{C} = \Omega_0 \cup \Omega_1 \cup \Gamma \). The strong asymptotics of the polynomial \( p_n \) is given by

\[
(1.2) \quad p_n(z) = \begin{cases} 
\frac{z^{n+c_1}}{(z-a_1)^{c_1}} \left( 1 + \mathcal{O}\left( \frac{1}{N^{\infty}} \right) \right), & z \in \Omega_0, \\
-\frac{a_1(1-|a_1|^2)c_1-1}{N^{1-c_1}\Gamma(c_1)} \frac{e^{N(\overline{a}_1 z) + \log a_1 - |a_1|^2}}{z-a_1} \left( 1 + \mathcal{O}\left( \frac{1}{N} \right) \right), & z \in \Omega_1,
\end{cases}
\]

where \( \mathcal{O}(1/N^{\infty}) \) stands for \( \mathcal{O}(1/N^m) \) for an arbitrary \( m > 0 \). The error bounds are uniform in any compact subset of the corresponding domain. In \( \Omega_0 \) the branch is chosen such that \( z^{n+c_1}/(z-a_1)^{c_1} \sim z^n \) as \( |z| \to \infty \) with the branch cut \([0, a_1]\).

When \( z \) is near \( \Gamma \) but away from \( a_1 \) the strong asymptotics is given by the sum of the two asymptotic expressions given above, hence zeros of \( p_n(z) \) line up along \( \Gamma \) with the inter-distance of order \( \mathcal{O}(1/N) \).

When \( z \) is near \( a_1 \) we define the local zooming coordinate

\[
\zeta(z) = -N(\overline{a}_1 z - \log z + \log a_1 - |a_1|^2),
\]

which maps \([0, a_1]\) to the negative real axis. We have

\[
(1.3) \quad p_n(z) = \frac{z^{n+c_1}}{(z-a_1)^{c_1}} \frac{\zeta(z)^{c_1}}{e^{\zeta(z)}} \left( \frac{e^{\zeta(z)}}{\zeta(z)^{c_1}} - f_{c_1}(\zeta(z)) + \mathcal{O}\left( \frac{1}{N} \right) \right),
\]

where the multivalued function \( \zeta^c \) is defined with the principal branch and \( f_{c}(\zeta) \) is defined by the two conditions: \( f_{c}(\zeta) \to 0 \) as \( |\zeta| \to 0 \), and \( e^{\zeta}/\zeta^c - f_{c}(\zeta) \) is entire. See Appendix A for more details about \( f_{c}(\zeta) \). Zeros of the above entire function are shown in Figure 1.2.
Multiple Szegő curve: The goal of this paper is to generalize these results to the case of $\nu > 1$. We obtain that the roots of the polynomial converge towards what we call the multiple Szegő curve, a certain merger of $\nu$ number of the generalized Szegő curves. The multiple Szegő curve, that we will denote by $\Gamma$, is determined in terms of $\{a_1, \ldots, a_\nu\} \subset \mathbb{D}$ and it divides the plane into $\nu + 1$ domains, the unbounded domain $\Omega_0$, and the $\nu$ number of bounded domains: $\Omega_1, \ldots, \Omega_\nu$ such that $\mathbb{C} = \Gamma \cup \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_\nu$. See Figure 1.3 for an example when $\nu = 4$.

To define the multiple Szegő curve, let $L = (l_1, \ldots, l_\nu)$ be a set of real numbers. We define a continuous function $\Phi^L(z) : \mathbb{D} \to \mathbb{R}$ by

\begin{equation}
\Phi^L(z) = \max\{\log|z|, \text{Re}(a_1z) + l_1, \ldots, \text{Re}(a_\nu z) + l_\nu\}.
\end{equation}
We define the bounded domains that depend on $L$ by

$$\Omega_j = \text{Int}\{z \in \mathbb{D} | \Phi^L(z) = \text{Re}(\overline{a}_j z) + l_j\}, \quad j = 1, \ldots, \nu,$$

where $\text{Int}A$ stands for the interior of $A$, the largest open subset of $A$. We also define the unbounded region $\Omega_0$ by

$$\Omega_0 = \mathbb{D}^c \cup \text{Int}\{z \in \mathbb{D} | \Phi^L(z) = \log |z|\}.$$

Note that $\Omega_j$ for $j \neq 0$ can be empty in some cases.

**Theorem 1.1.** For a given $\{a_1, \ldots, a_\nu\}$ there exists the unique set of real numbers, $L = (l_1, \ldots, l_\nu)$, such that

$$a_j \in \partial \Omega_j \text{ for } j = 1, \ldots, \nu.$$

Given $\{a\text{’s}\}$ the above theorem uniquely determines $L$ and, in turn, $\Phi^L$ and $\Omega_j$’s. It allows us to define the following.

**Definition 1.2.** For a given $\{a_j\text{’s}\}$ we define the multiple Szegő curve $\Gamma$ by

$$\Gamma = \bigcup_{j=1}^\nu \partial \Omega_j,$$

where $\Omega_j$’s are defined by (1.5) in terms of the unique $L$ that is given by Theorem 1.1.

Later in Lemma 2.6 we will prove that $\Gamma$ is included in $\mathbb{D}$. 

---

**Figure 1.3.** The multiple Szegő curve with $\nu = 4$. Dotted line is the unit circle. The plane is divided into five domains by the curve; the bounded region adjacent to $a_1$ is $\Omega_1$, the bounded region adjacent to $a_3$ is $\Omega_3$, the region containing the origin is $\Omega_4$ and the last remaining bounded region is $\Omega_2$. 

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Theorem 1.1 says that $a_j \in \partial \Omega_j$. It means that $a_j$ is adjacent to another domain $\Omega_k$ for some $k \neq j$. In such case we define the following notation:

$$j \rightarrow k \iff a_j \in \partial \Omega_k \text{ and } j \neq k.$$ \hfill (1.8)

**Definition 1.3.** Define the chain of $a_j$ by the ordered subset $(k_s, k_{s-1}, \ldots, k_1) \subset \{1, \ldots, \nu\}$ such that $k_s = j$ and

$$k_s \rightarrow k_{s-1} \rightarrow \cdots \rightarrow k_1 \rightarrow 0.$$ \hfill (1.9)

For example, in Figure 1.3, the chain of $a_4$ is $(4, 2, 3)$ or, equivalently, $4 \rightarrow 2 \rightarrow 3 \rightarrow 0$.

**Remark 1.1.** In this paper we consider only generic cases when the multiple Szegő curve is smooth at every $a_j$. This means that the point $a_j$ is on the boundary of exactly two domains $\Omega_j$ and $\Omega_k$ for $k \neq j$, which implies that the chain of $a_j$ is unique. It is possible that $a_j$ belongs to the boundary of three or more domains. See Figure 1.4 (the left picture). Though we omit such cases for brevity our method still applies to such non-generic cases and it modifies only Theorem 1.6. It is also possible that $\Omega_j$ is empty for some $j$. See Figure 1.4 (the middle and the right pictures). In this case we expect new type of local behavior showing up near $a_j$.

When $j \rightarrow k$, the continuity of $\Phi^L(z)$ at $a_j \in \partial \Omega_j \cap \partial \Omega_k$,

$$\lim_{z \to a_j, z \in \Omega_j} \Phi^L(z) = \lim_{z \to a_j, z \in \Omega_k} \Phi^L(z),$$

gives

$$|a_j|^2 + l_j = \begin{cases} \text{Re}(\bar{a}_k a_j) + l_k & \text{if } k \neq 0, \\ \log |a_j| & \text{if } k = 0. \end{cases}$$ \hfill (1.10)
When $j \to k$ we define the complex numbers $\ell_j$’s such that

\[(1.11) \quad |a_j|^2 + \ell_j = \begin{cases} \bar{a}_k a_j + \ell_k & \text{if } k \neq 0, \\ \log a_j & \text{if } k = 0. \end{cases} \]

These relations uniquely determine all the $\ell_j$’s inductively for a given chain; for example, the chain in (1.9) gives $\ell_{k_1} = \log a_{k_1} - |a_{k_1}|^2$ and $\ell_{k_2} = -|a_{k_2}|^2 + \bar{a}_{k_1} a_{k_2} + \ell_{k_1}$ and so on. Solving the relations inductively for the chain in (1.9) we get

\[\ell_j = \log a_{k_1} - |a_{k_1}|^2 + \sum_{i=2}^{s} (\bar{a}_{k_{i-1}} a_{k_i} - |a_{k_i}|^2).\]

By (1.10) and (1.11), one can also observe that $\text{Re} \ell_j = l_j$.

**Branch cuts for non-integer $c_j$’s:** Whenever there is a non-integer exponent $c_j$ one must be careful about the branch of the multivalued function. For example, an expression like $z^{c_1}$ has infinitely many branches when $c_1$ is irrational. The precise definition of the branches of the multivalued functions are needed to state the main results.

First we define various branch cuts; See Figure 1.5.

\[(1.12) \quad \hat{\mathcal{B}} = \bigcup_{j=1}^{\nu} \hat{B}_j \text{ where } \hat{B}_j = \{a_j t : 0 \leq t \leq 1\},\]

\[(1.13) \quad \mathcal{B} = \bigcup_{j=1}^{\nu} B_j \text{ where } B_j = \{a_j t : t \geq 1\},\]

\[(1.14) \quad B[k] = \left( \bigcup_{j=1}^{\nu} B_{jk} \right) \cup B_k \text{ where } B_{jk} = \{a_j + (a_j - a_k)t : t \geq 0\}.\]

In all these branch cuts, we define the orientations of the branch cuts by the directions of increasing $t$.

For the sake of presentation we will assume that no three points from $\{0, a_1, \ldots, a_\nu\}$ are collinear, which implies that the branch cuts do not overlap with each other. Such assumption can be disposed of with a perturbation argument.

We now define the exact branches of the multivalued functions that appear in this paper.

(i) $(z - a_j)^{c_j}$ is analytic away from $B_j$. One may choose any branch for this function but one should stick to the choice throughout the paper.

(ii) For $j \neq k$ we define $[(z - a_j)^{c_j}]_{B[k]}$ to be analytic away from $B_{jk}$ and

\[(1.15) \quad [(z - a_j)^{c_j}]_{B[k]} = (z - a_j)^{c_j} \text{ when } z \in B_k \cup \hat{B}_k \text{ and } j \neq k.\]

We also define $[(z - a_j)^{c_j}]_{B[j]} = (z - a_j)^{c_j}$. 
FIGURE 1.5. Various branch cuts for $\nu = 4$. $B = \{B_1, B_2, B_3, B_4\}$ (black rays), $\hat{B} = \{\hat{B}_1, \hat{B}_2, \hat{B}_3, \hat{B}_4\}$ (red rays), and $B[1] = B_1 \cup \{B_{21}, B_{31}, B_{41}\}$ (blue rays). The branch cuts of $W(z)$ are $B$. The branch cuts of $W_1(z)$ are $B[1]$.

(iii) We define

$$W(z) = \prod_{j=1}^{\nu} (z - a_j)^{c_j}.$$  

(iv) We define

$$W_k(z) = \prod_{j=1}^{\nu} [(z - a_j)^{c_j}]_{B[k]}.$$  

It follows that $W_k(z) = W(z)$ when $z$ is in a neighborhood of $B_k \cup \hat{B}_k$. Note that $W_k(z)$ has the branch cut on $B[k]$.

(v) $z^{c_j}$ is analytic away from $B_j \cup \hat{B}_j$. We select the branch such that $(z - a_j)^{c_j}/z^{c_j} \rightarrow 1$ as $z$ goes to $\infty$ along $B_2$.

(vi) $[z^{c_j}]_{B[k]}$ has the branch cut on $B_{jk} \cup \hat{B}_j$. We select the branch such that

$$[z^{c_j}]_{B[k]} = z^{c_j} \text{ when } z \in B_k \cup \hat{B}_k.$$  

(vii) We define $z^{\sum c} = \prod_{j=1}^{\nu} z^{c_j}$. We use the shortened notation $\sum c = \sum_{j=1}^{\nu} c_j$.

(viii) We define $[z^{\sum c}]_{B[k]} = \prod_{j=1}^{\nu} [z^{c_j}]_{B[k]}$. 


When \(c_j\)'s are all integer-valued there is no ambiguity in the choice of branches. We have \(z^{c_j} = [z^{c_j}]_{B[k]}\) and \((z - a_j)^{c_j} = [(z - a_j)^{c_j}]_{B[k]}\) and therefore \(W_k(z) = W(z)\). Also the final results should be independent of the choice of the branch made in (i).

**Definition 1.4.** Let us define the phase factor \(\tilde{\eta}_{kj}\) by

\[
\tilde{\eta}_{kj} := \frac{[(z - a_j)^{c_j}]_{B[k]} W_j(z)}{(z - a_j)^{c_j} W_k(z)}, \quad z \in B_{jk}.
\]

Let \((k_s, k_{s-1}, \ldots, k_1)\) be the chain of \(a_j\). Then we define the constant \(\text{chain}(j)\) by

\[
\text{chain}(j) = \frac{1 + \sum_{i \neq k_1} c_i N \sum_{i=1}^s (c_{k_i} - 1)}{\Gamma(c_{k_1})(1 - |a_{k_1}|^2)^{1-c_{k_1}}}
\times \prod_{i=1}^{s-1} \frac{\tilde{\eta}_{k_i,k_{i+1}}(a_{k_{i+1}} - a_{k_i})^{c_{k_i}} |a_{k_i} - a_{k_{i+1}}|^{2(c_{k_{i+1}} - 1)}}{\Gamma(c_{k_{i+1}})(a_{k_i} - a_{k_{i+1}})^{c_{k_{i+1}}+1}}.
\]

Above \(z^{n+\sum c} = z^n \cdot z^{\sum c}\) and \((a_{k_i} - a_{k_{i+1}})^{c_{k_{i+1}}}\) that appears in \(\text{chain}[j]\) is the evaluation of \((z - a_{k_{i+1}})^{c_{k_{i+1}}}\) at \(z = a_{k_i}\).

**Strong asymptotics of \(p_n\):** We now state the main results. See Figure 1.4 for numerical support.

**Theorem 1.5.** Let \(\{a_1, \ldots, a_\nu\} \subset \mathbb{D} \setminus \{0\}\). In a generic case (see Remark 1.1 above), if \(z\) is away from \(\Gamma\), as \(N \to \infty\) such that \(n/N = 1\), the polynomial \(p_n\) satisfies

\[
p_n(z) = \begin{cases} 
\frac{z^{n+\sum c}}{W(z)} \left(1 + O\left(\frac{1}{N^\infty}\right)\right), & z \in \Omega_0, \\
-\exp \left[N(\bar{\alpha}_j z + \ell_j)\right] \frac{(z - a_j)^{c_j \text{chain}(j)}}{W_j(z)} \left(1 + O\left(\frac{1}{N}\right)\right), & z \in \Omega_j.
\end{cases}
\]

The error bounds are uniform over any compact subset in the corresponding regions.
When $z$ is near $\Gamma$ but away from $a_j$’s the strong asymptotics of $p_n$ is given by the sum of the two asymptotic expressions from the adjacent domains as below:

$$p_n(z) = \begin{cases} 
\frac{z^{n+\sum c}}{W(z)} \left( 1 + \mathcal{O} \left( \frac{1}{N^{\infty}} \right) \right) \\
- \exp \left[ N(\alpha_j z + \ell_j) \right] \frac{(z - a_j)^{c_j}}{W_j(z)} \frac{\text{chain}(j)}{z - a_j} \left( 1 + \mathcal{O} \left( \frac{1}{N} \right) \right), \\
\text{when } z \text{ near } \Gamma_{j0}, \\
- \exp \left[ N(\alpha_j z + \ell_j) \right] \frac{(z - a_j)^{c_j}}{W_j(z)} \frac{\text{chain}(j)}{z - a_j} \left( 1 + \mathcal{O} \left( \frac{1}{N} \right) \right) \\
- \exp \left[ N(\alpha_k z + \ell_k) \right] \frac{(z - a_k)^{c_k}}{W_k(z)} \frac{\text{chain}(k)}{z - a_k} \left( 1 + \mathcal{O} \left( \frac{1}{N} \right) \right), \\
\text{when } z \text{ near } \Gamma_{jk}. 
\end{cases}$$
The error bounds are uniform over any compact subset of $\Omega_0 \cup \Omega_j \cup \Gamma_j0 \setminus \{a's\}$ for the former and $\Omega_k \cup \Omega_j \cup \Gamma_jk \setminus \{a's\}$ for the latter. The error bound $O(1/N^\infty)$ stands for $O(1/N^m)$ for all $m > 0$.

In a neighborhood of $a_j \in \partial\Omega_k \cap \partial\Omega_j$, $k \neq j$, we define the local zooming coordinate $\zeta$ by

\begin{align}
\zeta(z) &= -N(\bar{a}_j z - \log z + \log a_j - |a_j|^2) & \text{when } k = 0, \\
\zeta(z) &= -N((\bar{a}_j - \bar{a}_k) z - (\bar{a}_j - \bar{a}_k) a_j) & \text{when } k \neq 0.
\end{align}

**Theorem 1.6.** In a generic case (see Remark 1.1 above), if $a_j \in \partial\Omega_j \cap \partial\Omega_k$, $k \neq j$, we have the following asymptotic behavior when $N \to \infty$ such that $n/N = 1$

\begin{equation}
 p_n(z) = A_k(z) \frac{\zeta c_j}{e^c} \left( \frac{e^c}{\zeta c_j} - f c_j(\zeta) + O\left(\frac{1}{N}\right) \right), \quad z \in D_{a_j},
\end{equation}

where $D_{a_j}$ is a sufficiently small disk centered at $a_j$ with a fixed radius and $A_k(z)$ is the leading asymptotics in $\Omega_k$ as written in Theorem 1.5, i.e.,

\begin{equation}
 A_k(z) = \begin{cases} 
 z^{n+\sum c} \frac{W(z)}{\zeta c_j}, & k = 0, \\
 -\exp\left[N(\bar{a}_k z + \ell_k)\right] \frac{(z - a_k)^{\ell_k}}{W_k(z)} \frac{\text{chain}(k)}{z - a_k}, & k \neq 0.
\end{cases}
\end{equation}

The error bound is uniform over $D_{a_j}$.

**Remark 1.2.** We did not pursue the case when one of the $a_j$’s is at the origin. We believe that such case yields the same results.

**Plan of the paper:** In Section 2, we describe the multiple Szegö curve. We prove Theorem 1.1 which essentially states that the multiple Szegö curve is uniquely given in terms of $\{a_j\}_{j=1}^\nu$. The basic idea is to determine the $\nu$ number of real-valued constants $l_j$’s such that the function $\Phi^L$ (1.4) has the discontinuities exactly at each $a_j$’s. From the asymptotics of $p_n$, one can see that $\Phi^L$ becomes the logarithmic potential of the limiting zeros, that is, $\lim_{n \to \infty} \frac{1}{n} \log |p_n(z)|$. In the proof of Theorem 1.1, we present an algorithm to determine $l_j$’s in finite steps, which we also use to generate the figures of multiple Szegö curves in this paper. This section can be read independently from the other sections, where we embark on the Riemann-Hilbert analysis.

The remaining parts of the paper are all devoted to Riemann-Hilbert analysis of the Riemann-Hilbert problem for $Y_n$ that has been obtained in [22]. We refer to that paper for the connection between the planar orthogonal polynomial and the multiple orthogonal polynomial, hence the Riemann-Hilbert problem.

In Section 3, we construct the $\nu \times \nu$ matrix function $\Psi(z)$ that will be used in the subsequent Riemann-Hilbert analysis. This matrix function will be used later when we define the matrix $T$ in Section 4.2 such that the matrix $T$ has the jump matrix in terms of elementary functions, a preparatory step for the Riemann-Hilbert
analysis. The construction of $\Psi$ becomes lengthy mostly to handle the non-integer values of $c$’s. We need a systematic placement of all the branch cuts, especially $B[k]$’s, such that the jumps along the branch cuts decay properly at the end. If $c$’s are all integer-valued one can skip this section and simply assume $\Psi(z) = \tilde{\Psi}(z)$ (3.6). The other parts, and the main idea of using Riemann-Hilbert problem, are all required for the integer case.

In Section 4 we apply the nonlinear steepest decent method [10] on the corresponding Riemann-Hilbert problem of size $(\nu+1) \times (\nu+1)$ and perform successive transformations, $Y \to \tilde{Y}$ (4.2), $\tilde{Y} \to T$ (4.7) and $T \to S$ (4.23). The technique is quite similar to the standard technique in $2 \times 2$ Riemann-Hilbert problem [9, 16]. We define the global parametrix $\Phi$ (4.34) that satisfies the approximate Riemann-Hilbert problem of $S$.

In Section 5 we construct the local parametrices near each $a_j$’s to match the global parametrix. The local parametrix has little difference from the $\nu = 1$ case [21]. As in the $\nu = 1$ case we also construct a rational function $R$ to improve the global parametrix $\Phi$ into $R\Phi$ such that to match the local parametrices better. This construction, called “partial Schlesinger transform” has been introduced in [6] and also used in [21].

In Section 6, based on the global and the local parametrices in the previous sections, we define $S^\infty$ (6.1), which is the strong asymptotics of $S$. By applying the small norm theorem to $S^\infty S^{-1}$ we prove Theorem 1.5 and Theorem 1.6.

In Appendix A we explain the special function, a certain truncation of the exponential function, that appears in the local parametrix.

## 2 Multiple Szegő curve

In this section we define the multiple Szegő curve that depends on the set of points:

$$\{a_1, \ldots, a_\nu\} \subset \mathbb{D}, \quad \nu \geq 2.$$  

Let $\Lambda$ be an $\nu$-dimensional vector with real entries

$$\Lambda = (\lambda_1, \ldots, \lambda_\nu),$$

and $\Phi^\Lambda$ be the continuous and piecewise smooth function given by

$$\Phi^\Lambda(z) = \max\{\log |z|, \Re(\overline{a}_1 z) + \lambda_1, \ldots, \Re(\overline{a}_\nu z) + \lambda_\nu\}. \tag{2.1}$$

Then we define the regions,

$$K_j^\Lambda = \{z \in \mathbb{D} | \Phi^\Lambda(z) = \Re(\overline{a}_j z) + \lambda_j\}, \quad j = 1, \ldots, \nu, \tag{2.2}$$

$$K_0^\Lambda = \{z \in \mathbb{D} | \Phi^\Lambda(z) = \log |z|\} \cup \mathbb{D}^c.$$  

One can visualize the function $\Phi^\Lambda$ as follows. The graphs of the $\nu + 1$ functions inside the max function in (2.1) give $\nu + 1$ surfaces among which the $\nu$ of them
are planes. The aerial view of these mutually intersecting surfaces selects the maximal function in each corresponding region defined in (2.2). As one increases or decreases $\lambda_j$ the corresponding surface moves up or down respectively, and the regions $K_j^\Lambda$ expands or shrinks as well.

**Theorem 2.1.** There exists a maximal vector $L = (l_1, \ldots, l_\nu)$ such that

$$a_j \notin \text{Int}(K_j^L) \text{ for all } j = 1, \ldots, \nu.$$  

By maximal vector we mean that, for any other vector $L' > L$, the property (2.3) does not hold for some $j$. We say $L' > L$ if the the vector $L' - L$ is a nonzero vector without any negative entry.

The property (2.3) defines a closed set in the parameter space of $\Lambda$, i.e., the set

$$S = \{ \Lambda : a_j \notin \text{Int}(K_j^\Lambda) \text{ for all } j = 1, \ldots, \nu \} \subset \mathbb{R}^\nu$$

is closed. The set $S$ is also nonempty. When $a_j \neq 0$ we can choose $\lambda_j$ to be sufficiently small such that $\log |a_j| > |a_j|^2 + \lambda_j$ which leads to $a_j \notin K_j^\Lambda$. Then $\lambda_k$ for $k \neq j$ can be chosen sufficiently small such that $\text{Re}(\overline{a}_k a_k) + \lambda_j > |a_k|^2 + \lambda_k$ hence $a_k \notin K_j^\Lambda$. In this way, we can find $\Lambda$ such that $a_j \notin K_j^L$ for all $j$’s.

**Lemma 2.2.** If $\Lambda \in S$ there exists some $j$ such that $\Phi^\Lambda(a_j) = \log |a_j|.$

The lemma gives the upper bound on $S$ since, if $\Lambda \in S$, $\text{Re}(\overline{a}_k a_j) + \lambda_k \leq \Phi^\Lambda(a_j) = \log |a_j|$ and, therefore, $\lambda_k \leq \log |a_j| - \text{Re}(\overline{a}_k a_j) \leq \log |a_j| + |a_j|.$

To prove the existence of a maximal vector let $l_1 = \sup_S \lambda_1 = \max_S \lambda_1$ where we use that $S$ is closed. Inductively we define

$$l_j = \sup\{ \lambda_j | \Lambda \in S, \lambda_1 = l_1, \ldots, \lambda_{j-1} = l_{j-1} \}.$$  

Then $L = (l_1, \ldots, l_\nu) \in S$ and it is a maximal vector. Hence Theorem 2.1 is proven with the following proof of Lemma 2.2.

**Proof.** (Proof of Lemma 2.2) Assuming otherwise, for $\Lambda \in S$ and for any $j$, there exists $k \neq j$ such that

$$\Phi^\Lambda(a_j) = \text{Re}(\overline{a}_k a_j) + \lambda_k.$$  

Since the condition (2.3) means that $|a_j|^2 + \lambda_j \leq \Phi^\Lambda(a_j)$ we get that

$$|a_j|^2 - \text{Re}(\overline{a}_k a_j) \leq \lambda_k - \lambda_j \text{ for some } k \neq j.$$  

Let us use the notation $j \sim k$ to represent the above inequality. Repeating the argument, there exists some $\ell \neq k$ such that $k \sim \ell$. Since the index set is finite, the chain of arrows (i.e. the chain of inequalities), $j \sim k \sim \ell \sim \ldots$, must eventually repeat some entry and form a closed loop. Without losing generality let the closed loop be $1 \sim 2 \sim \cdots \sim s \sim 1$. Adding up the corresponding inequalities, we get

$$|a_1|^2 - \text{Re}(\overline{a}_2 a_1) + |a_2|^2 - \text{Re}(\overline{a}_3 a_2) + \cdots + |a_s|^2 - \text{Re}(\overline{a}_1 a_s) \leq (\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_3) + \cdots (\lambda_s - \lambda_1) = 0.$$
The left hand side is the half the inner product, \((A - \hat{A}) \cdot (A - \hat{A})^*\) (with the complex conjugation denoted by *), where the vectors \(A\) and \(\hat{A}\) are given by

\[ A = (a_1, \ldots, a_s), \quad \hat{A} = (a_2, a_3, \ldots, a_s, a_1). \]

This leads to \(a_1 = a_2 = \cdots = a_s\), a contradiction. \(\square\)

If \(L\) be maximal in the sense of Theorem 2.1 and if \(a_j \notin K_j^L\) for some \(j\) then one can increase \(l_j\) slightly without breaking the condition (2.3). This shows that the maximal \(L\) occurs only if every \(a_j\) is in \(\partial K_j^L\). Then there are two possibilities.

\[
\begin{align*}
(2.5) \quad \Phi^L(a_j) &= |a_j|^2 + l_j = \log |a_j| \quad \text{if} \quad a_j \in \partial K_j^L, \\
(2.6) \quad \Phi^L(a_j) &= |a_j|^2 + l_j = \Re(\bar{a}_k a_j) + l_k \quad \text{if} \quad a_j \in \partial K_k^L \text{ for some } k \neq j.
\end{align*}
\]

According to the notation defined in (1.8) we note that the former case corresponds to \(j \to 0\) and the latter case corresponds to \(j \to k\).

**Lemma 2.3.** Given \(j \neq 0\), the chain of arrows, \(j \to k \to \ell \to \ldots\), eventually leads to \(\cdots \to 0\) without repeating any entry.

**Proof.** Idea of the proof is similar to that of Lemma 2.2. Given successive relations, \(j \to k \to \ell \to \ldots\), it is enough to show that the chain of arrows never visits any nonzero number twice. To prove this statement, assume that we have a loop \(j_1 \to j_2 \to \cdots \to j_s \to j_1\) with all \(j\)’s being nonzero. We get

\[
(|a_{j_1}|^2 - \Re(\bar{a}_{j_2} a_{j_1})) + (|a_{j_2}|^2 - \Re(\bar{a}_{j_3} a_{j_2})) + \cdots + (|a_{j_s}|^2 - \Re(\bar{a}_{j_1} a_{j_s}))
= (l_1 - l_2) + (l_2 - l_3) + \ldots + (l_s - l_1) = 0.
\]

By the argument after the equation (2.4), we obtain \(a_{j_1} = a_{j_2} = \cdots = a_{j_s}\), a contradiction. \(\square\)

**Definition 2.4.** From Lemma 2.3, for each \(a_j\), there exists chains \(j \to \cdots \to 0\). We define the level of \(a_j\) by the smallest number of arrows among all the chains that starts with \(j\). For example, the level of \(a_j\) is one if \(j \to 0\). Lemma 2.3 says that the level of \(a_j\) should be \(\leq \nu\).

**Remark 2.1.** For a generic choice of \(\{a_1, \ldots, a_m\} \subset \mathbb{D}\), an \(a_j\) is adjacent to exactly two regions, i.e., \(a_j \in K_j^L \cap K_j^L\) for exactly one \(k \neq j\) among \(0 \leq k \leq m\). It means that \(j \to k\). For a generic case there exists a unique chain \(j \to \cdots \to 0\) for each \(a_j\). For a non-generic case \(a_j\) can be at the boundary of three or more regions. To avoid too much technicality we do not consider such case. See Figure 1.4 for a non-generic case.

**Proof.** (Proof of Theorem 1.1) Since the existence part is proven in Theorem 2.1 we only prove the uniqueness. We claim that the following iterative steps finds \(L = (l_1, \ldots, l_\nu)\) in Theorem 1.1, hence \(L\) is unique.

**Algorithm to find \(L\).**
1. Set \( \lambda_j = \log |a_j| - |a_j|^2 \) for all \( j = 1, \ldots, \nu \). (If \( a_j = 0 \) then set \( \lambda_j = \log |a_k| - |a_k|^2 \) for some \( k \neq j \).)

2. Define \( \lambda_j = \Phi^\Lambda (a_j) - |a_j|^2 \) for all \( j \). Note that \( \tilde{\lambda}_j \geq \lambda_j \) since \( \Phi^\Lambda (a_j) \geq |a_j|^2 + \lambda_j \).

3. Redefine \( \lambda_j = \tilde{\lambda}_j \) and, accordingly, \( \Lambda \).

4. Repeat the above two steps \( \nu \) times.

5. Set \( L = \Lambda \).

If \( a_j \) is of level one, i.e. \( j \to 0 \), \( \lambda_j = l_j \) is obtained by the step 1. Since, for all \( j \), \( |a_j|^2 + l_j = \Phi^L (a_j) \geq \log |a_j| \), we get \( \Lambda \leq L \) after the step 1, i.e. \( \lambda_j \leq l_j \) for all \( j \). In the prospect of using induction, let us assume that \( \lambda_j = l_j \) for all the \( a_j \)'s up to the \( k \)-th level while \( \Lambda \leq L \). Let \( a_q \) be of level \( k + 1 \), i.e. \( q \to q' \to \ldots 0 \) where \( a_q' \) is of level \( k \). Since we already have \( \lambda_{q'} = l_{q'} \) by the assumption, the step 2 gives \( \tilde{\lambda}_q = \Phi^\Lambda (a_q) - |a_q|^2 \geq \text{Re}(\bar{a}_q a_q) + l_q - |a_q|^2 = l_q \) where the last equality is from \( q \to q' \). On the other hand, we have, for all \( j \), \( \lambda_j = \Phi^\Lambda (a_j) - |a_j|^2 \leq \Phi^L (a_j) - |a_j|^2 = l_j \) since \( \Lambda \leq L \). The last two sentences lead to \( \lambda_q = l_q \) and \( \Lambda \leq L \). The step 3 will then give \( \lambda_q = l_q \) and \( \Lambda \leq L \). This shows that the step 2 and 3 can be repeated inductively.

Since the largest level of \( a_j \) is \( \leq \nu \), the induction will give \( \Lambda = L \) after \( \nu \) iterations.

This ends the proof of Theorem 1.1. \( \square \)

Let us repeat the definition of \( \Gamma \) in (1.7) using that \( \Omega_j = \text{Int} K_j^L \) as can be seen from (1.5).

**Definition 2.5.** (Multiple Szegő curve) Given \( \{a_j\}_{j=1}^\nu \subset \mathbb{D} \), we define the multiple Szegő curve

\[
\Gamma = \bigcup_{j=1}^\nu \partial K_j^L ,
\]

where \( K_j^L \) are defined by (2.2) and \( L \) is the unique vector that is asserted in Theorem 1.1 and can be obtained explicitly by the iterative algorithm. We define the oriented arc,

\[
\Gamma_{jk} = K_j^L \cap K_k^L , \quad j \neq k , \quad \{j, k\} \subset \{0, 1, \ldots, \nu\} ,
\]

whose orientation is such that \( K_j^L \) sits to the left with respect to the traveller who follows the orientation along the arc, i.e., \( K_j^L \) is at the + side of \( \Gamma_{jk} \).

From the definition, the contours, \( \Gamma_{jk} \) and \( \Gamma_{kj} \), have the opposite orientations. Also, \( \Gamma_{jk} \) will be a straight line segment only if both indices are nonzero.

**Lemma 2.6.** For \( \{a_j\}_{j=1}^\nu \subset \mathbb{D} \), the corresponding multiple Szegő curve \( \Gamma \) is in \( \mathbb{D} \), i.e. \( \partial K_0 \subset \mathbb{D} \).
Proof. Let \( L \) be the unique vector in Theorem 1.1. We first show that

\[
\log |z| \leq \Phi^L(z) \leq \frac{1}{2}(|z|^2 - 1).
\]

The first inequality is trivial from the definition of \( \Phi^L \). Let us prove the latter inequality. Let \( 1 \rightarrow 2 \rightarrow 0 \) \((1.8)\) be a chain. Since \( a_2 \in K_0 \) we have \( \Phi^L(a_2) = \log |a_2| \leq \frac{1}{2}(|a_2|^2 - 1) \), i.e. the inequality holds at \( a_2 \). In fact, the inequality holds for all \( z \in \text{Int} K_2 \) because

\[
\frac{1}{2}(|z|^2 - 1) - \text{Re}(a_2 z) - l_2 = \frac{1}{2}(|z - a_2|^2 - |a_2|^2 - 1) - l_2
\]

has the global minimum at \( z = a_2 \). Since \( \Phi^L \) is continuous, the inequality holds at \( a_1 \in \partial K_2 \). By the same argument the inequality holds for all \( z \in \text{Int} K_1 \). By induction, the argument applies to any chain and, therefore, the inequality holds for all \( K_j, j = 1, \ldots, \nu \).

Now we assume that \( K_j \) intersects \( \partial \mathbb{D} \). Then there exists a point \( p \in K_j \cap \partial \mathbb{D} \). By the squeezing inequalities \((2.8)\) we have \( \Phi^L(p) = \frac{1}{2}(|p|^2 - 1) = 0 \). Since \( a_j \) is the unique minimum of \( \frac{1}{2}(|z|^2 - 1) - \Phi^L(z) \) in \( K_j \) by \((2.9)\) applied to \( a_j \), we have \( \frac{1}{2}(|a_j|^2 - 1) - \Phi^L(a_j) < \frac{1}{2}(|p|^2 - 1) - \Phi^L(p) = 0 \), a contradiction to the established inequality \((2.8)\). □

### 3 Dealing with non-integer \( \nu \)'s

We use Theorem 1 in \([22]\) which states that the polynomial \( p_n \) is a multiple orthogonal polynomial of type II \([2, 11, 15]\) and Theorem 2 in \([22]\) which relates such multiple orthogonal polynomial to a \((\nu + 1) \times (\nu + 1)\) Riemann-Hilbert problem. Let us recapture both theorems by Theorem 3.1 and Theorem 3.2 below.

Let us assume that \( a_j \)'s are all distinct for a simple presentation. Without losing generality we set

\[
0 \leq \arg a_1 < \cdots < \arg a_{\nu} < 2\pi
\]

and let \( \gamma \) be a simple closed curve given by \( \overrightarrow{a_1 a_2} \cup \overrightarrow{a_2 a_3} \cup \cdots \cup \overrightarrow{a_{\nu} a_1} \) where \( \overrightarrow{AB} \) stands for the line segment connecting \( A \) and \( B \). We assign the orientation to \( \gamma \) such that the curve encloses the origin in counterclockwise direction.

**Theorem 3.1.** Let \( \kappa = \lfloor n/\nu \rfloor \) and set \((n_1, \ldots, n_{\nu})\) by

\[
n_j = \begin{cases} 
\kappa + 1 & \text{if } j \leq n - \kappa \nu, \\
\kappa & \text{otherwise}.
\end{cases}
\]

For a fixed \( 1 \leq k \leq \nu \) the polynomial \( p_n(z) \) satisfies the orthogonality,

\[
\int_{\gamma} p_n(z) z^i \chi_j^{(1)}(z) dz = 0 \quad \text{for} \quad 0 \leq i \leq n_j - 1 \quad \text{and} \quad 1 \leq j \leq \nu,
\]
with respect to the $\nu$ multiple measures $\{\chi_1^{(1)} dz, \ldots, \chi_\nu^{(1)} dz\}$ given by

$$
\chi_j^{(k)}(z) = W(z) \int_{\gamma} \frac{dz}{2\pi i} \frac{W(s)}{W(\bar{s})} \prod_{i=1}^{\nu} (s - \bar{a}_i)^{\delta_{ij}} e^{-Nz_s} ds.
$$

Above the integration contour starts at $\bar{a}_k$ and extends to $\infty$ in the angular direction of $\arg \bar{z}$ while avoiding $\cup_j \{\bar{a}_j t : t \geq 1\}$. We will use an alternative but equivalent expression

$$(3.3) \quad \chi_j^{(k)}(z) = W(z) \left( \int_{\alpha_k} z^{x \times \infty} W(s) \prod_{i=1}^{\nu} (s - a_i)^{\delta_{ij}} e^{-Nz_s} ds \right)^*,$$

where the superscript $\ast$ stands for the complex conjugation, and the integration contour starts from $\alpha_k$ and escapes to $z \times \infty$, the infinity in the angular direction of $\arg z$, while avoiding $\mathbf{B}$ (1.13). We also note that the above definitions make sense only for $z \notin \mathbf{B}$.

**Theorem 3.2.** Let $\gamma$ and $\chi_j^{(k)}(z)$ be given above, then the Riemann-Hilbert problem

$$(3.4) \quad Y(z) \text{ has the continuous limit values from each side of } \gamma \text{ and is holomorphic in } \mathbb{C} \setminus \gamma,$$

$$
Y_+(z) = Y_-(z) = \begin{bmatrix}
1 & \chi_1^{(1)}(z) & \cdots & \chi_\nu^{(1)}(z) \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}, \quad z \in \gamma,
$$

$$
Y(z) = (I_{\nu+1} + \mathcal{O}(\frac{1}{z})) \text{ diag } (z^n, z^{-n_1}, \ldots, z^{-n_\nu}), \quad z \to \infty,
$$

has the unique solution given by

$$
Y(z) = \begin{bmatrix}
p_n(z) & \frac{1}{2\pi i} \int_{\gamma} \frac{p_n(w)\chi_1^{(1)}(w)}{w - z} dw & \cdots & \frac{1}{2\pi i} \int_{\gamma} \frac{p_n(w)\chi_\nu^{(1)}(w)}{w - z} dw \\
q_n^{-1}(1) & \frac{1}{2\pi i} \int_{\gamma} \frac{q_{n-1}^{(1)}(w)\chi_1^{(1)}(w)}{w - z} dw & \cdots & \frac{1}{2\pi i} \int_{\gamma} \frac{q_{n-1}^{(1)}(w)\chi_\nu^{(1)}(w)}{w - z} dw \\
\vdots & \vdots & \ddots & \vdots \\
q_n^{(\nu)} & \frac{1}{2\pi i} \int_{\gamma} \frac{q_{n-1}^{(\nu)}(w)\chi_1^{(1)}(w)}{w - z} dw & \cdots & \frac{1}{2\pi i} \int_{\gamma} \frac{q_{n-1}^{(\nu)}(w)\chi_\nu^{(1)}(w)}{w - z} dw
\end{bmatrix}
$$

where $q_n^{(j)}(z)$ is the polynomial of degree $n - 1$ satisfying the orthogonality condition:

$$
\int_{\gamma} q_n^{(j)}(z) z^m \chi_i^{(1)}(z) dz = 0, \quad 0 \leq m \leq n_i - 1 - \delta_{ij}, \quad 1 \leq i, j \leq \nu,
$$
and the leading coefficient of \( q_{n-1}^{(j)}(z) \) is given by
\[
\left( \frac{-1}{2\pi i} \int_{\gamma} q_{n-1}^{(j)}(w) w^{n_j - 1} \chi_j^{(1)}(w) \, dw \right)^{-1}.
\]

**Remark 3.1.** In all the Riemann-Hilbert problems that we subsequently write, we will state only the jump conditions and the boundary behaviors. We assume the analyticity of the solution away from the jump contours and the existence of the continuous boundary values, that we denote by the subscripts \(+\) and \(−\), alongside the contours.

The goal of the next four subsections is to define the \( \nu \times \nu \) matrix function \( \Psi(z) \) that we will use for the subsequent Riemann-Hilbert analysis. These sections are needed mostly to be able to handle non-integer values of \( c_j \)’s. We first define \( \tilde{\Psi} \) which has its branch cut on \( B \). The method of Riemann-Hilbert analysis is to construct, by a succession of transformations, the Riemann-Hilbert problem with a desirable jump condition. The jump on \( B \) that is originated from the non-integer \( c_j \)’s turns out not desirable – the jump matrix could increase exponentially in \( N \). The cure is to deform the jump conditions on \( B \) into the jump conditions on \( B[k] \)'s. This is exactly done in Section 3.3 by defining the transformation matrix \( V \) and \( \Psi(z) = V(z) \tilde{\Psi}(z) \).

This section, unfortunately, involves rather lengthy manipulation of intertwined branch cuts. If one is interested in only integer \( c_j \)'s then one can skip ahead to Section 4, simply by using \( \tilde{\Psi} \) (3.6) in the place of \( \Psi \).

**3.1 Construction of \( \tilde{\Psi} \) and \( \tilde{\Psi} \)**

Let us define a shorthand notation
\[
\eta_j = e^{-2\pi ic_j}, \quad j = 1, \ldots, \nu.
\]
It will be convenient to define piecewise analytic row vectors for \( j = 1, \ldots, \nu \) by
\[
\tilde{\psi}_j(z) = W(z)^{-1} \begin{bmatrix} \chi_1^{(j)}(z), \ldots, \chi_{\nu}^{(j)}(z) \end{bmatrix}, \quad z \notin B \cup \tilde{B}.
\]
Using the above definitions, let us define \( \nu \times \nu \) matrix:
\[
\tilde{\Psi}(z) = \begin{bmatrix} \tilde{\psi}_1(z) \\ \vdots \\ \tilde{\psi}_\nu(z) \end{bmatrix}.
\]

**Lemma 3.3.** The matrix \( \tilde{\Psi} \) satisfies a jump discontinuity,
\[
\tilde{\Psi}_+(z) = \tilde{J}_j \tilde{\Psi}_-(z), \quad z \in \tilde{B}_j \cup B_j,
\]
where

\[
\begin{bmatrix}
\eta_j^{-1} - 1 \\
I_{j-1} \\
\vdots \\
\eta_j^{-1} - 1 \\
0 \\
\vdots \\
\eta_j^{-1} - 1 \\
0 \\
\eta_j^{-1} - 1 \\
I_{\nu-j} \\
\eta_j^{-1} - 1
\end{bmatrix}, \quad \text{for } j = 1, \ldots, \nu.
\]

Let us remark about the notations that will be used throughout the paper. The subscript \(\pm\) will be used for the boundary values on the sides of the said contour. In (3.7) for example, \(\tilde{\Psi}_+(z)\) refers to the boundary value of \(\tilde{\Psi}(z)\) on the + side of \(\tilde{\mathbb{B}}_j \cup \mathbb{B}_j\), i.e., the left side from the point of view of the traveler that walks along the oriented contour. Note that the orientations of \(\tilde{\mathbb{B}}_j\) and \(\mathbb{B}_j\) are given by (1.12) and (1.13).

Here and below, we will express matrices in block form as in (3.8). We will use \(I_k\) for an identity matrix (block) of size \(k \times k\), and \(0\) for zero matrix (block) of appropriate size that is determined by the size of its neighboring blocks. For example the four \(0\)'s in \(\tilde{J}_j\) are of sizes \((j-1) \times (\nu-j)\), \(1 \times (j-1)\), \(1 \times (\nu-j)\), and \((\nu-j) \times (j-1)\), respectively.

**Proof.** (Proof of Lemma 3.3) From the jump conditions

\[
W_+(z) = \eta_j W_-(z), \quad z \in \mathbb{B}_j, \quad j = 1, \ldots, \nu,
\]

we obtain

\[
[\tilde{\psi}_j(z)]_+ = \eta_j^{-1} [\tilde{\psi}_j(z)]_- \quad z \in \mathbb{B}_j \cup \tilde{\mathbb{B}}_j.
\]

Using that \(\tilde{\psi}_j(z) - \tilde{\psi}_k(z)\) is analytic in the whole complex plane we get

\[
[\tilde{\psi}_k(z)]_+ = [\tilde{\psi}_k(z) - \tilde{\psi}_j(z)]_+ + [\tilde{\psi}_j(z)]_+ = [\tilde{\psi}_k(z) - \tilde{\psi}_j(z)]_- + \eta_j^{-1} [\tilde{\psi}_j(z)]_-
\]

\[
= [\tilde{\psi}_k(z)]_- + (\eta_j^{-1} - 1) [\tilde{\psi}_j(z)]_-, \quad z \in \mathbb{B}_j \cup \tilde{\mathbb{B}}_j.
\]

This proves the lemma. \(\square\)

We also define \(\tilde{\psi}_j\) to be analytic in \(\{z | \arg z \neq \arg a_j + \pi\}\) such that

\[
\tilde{\psi}_j(z) = \tilde{\psi}_j(z) \quad \text{when } \arg a_j < \arg z < \arg a_{j+1},
\]

where we use \(a_{\nu+1} = a_1\) when \(j = \nu\).
Using the above definitions, let us define $\nu \times \nu$ matrix:

\[
\tilde{\Psi}(z) = \begin{bmatrix}
\hat{\psi}_1(z) \\
\vdots \\
\hat{\psi}_\nu(z)
\end{bmatrix}.
\]

Next we find the linear transform from $\tilde{\Psi}(z)$ to $\tilde{\Psi}(z)$. To describe the transform, we need notations to handle the situation: given $z \in \mathbb{C}$ we want to refer to $\{a_j's\}$ by the order of $\{\arg(z/a_j)\}'s$, i.e., by the order of the angular distances from $z$ with respect to the origin.

Let

\[ I = \{1, \ldots, \nu\} \]

and we define

\[ s = s(z) = \#\{k \in I : 0 < \arg(z/a_k) \leq \pi\}. \]

We define $r(k) = r(k; z)$ and $l(k) = l(k; z)$ to be the renaming of the indices in $I$,

\[ I = \{r(1; z), \ldots, r(s; z)\} \cup \{l(1; z), \ldots, l(\nu - s; z)\}, \]

such that to satisfy

\[-\pi < \arg\frac{z}{a_{l(1)}} < \cdots < \arg\frac{z}{a_{l(\nu-s)}} < 0 < \arg\frac{z}{a_{r(s)}} < \cdots < \arg\frac{z}{a_{r(1)}} \leq \pi.\]

Above and below we sometimes write, for example, $r(1)$ and $l(2)$ instead of $l(1, z)$ and $l(2, z)$ when the second argument $z$ is clear from the context. The same will be true for $s$, i.e. $s$ instead of $s(z)$. See Figure 3.1.
Lemma 3.4. Let \( I(\cdot) = I(\cdot; z) \), \( r(\cdot) = r(\cdot; z) \) and \( s = s(z) \) be defined as above. We have

\[
\hat{\psi}_{r(k)}(z) = \hat{\psi}_{r(k)}(z) + \sum_{j=k+1}^{s} (1 - \eta_{r(j)}) \hat{\psi}_{r(j)}(z), \quad k = 1, \ldots, s,
\]

\[
\hat{\psi}_{l(k)}(z) = \eta_{l(k)} \hat{\psi}_{l(k)}(z) + \sum_{j=k+1}^{\nu - s} (\eta_{l(j)} - 1) \hat{\psi}_{l(j)}(z), \quad k = 1, \ldots, \nu - s,
\]

for \( z \notin \hat{B} \cup B \) and \( -z \notin \hat{B} \cup B \).

Proof. These expansions are obtained from the integral representations. For example, the integration contour for \( \tilde{\psi}_{r(k)}(z) \), which is from \( a_{r(k)} \) to \( z \times \infty \), is the sum of the contours around \( B_j \)'s as shown in Figure 3.2 for \( r(k) = 1 \). The red dashed contour can be expressed into the sum of blue contours enclosing \( \{B_1, B_2, \ldots, B_6\} \). The lemma follows since the contour enclosing \( B_j \) clockwise corresponds to \( (1 - \eta_j) \hat{\psi}_j \). Since \( \hat{\psi}_j \) has the branch cut on \( \{z | \arg z = \arg a_j + \pi \} \), \( \hat{\psi}_{r(k)} \) (resp., \( \hat{\psi}_{l(k)} \)) is analytic in the angular sector from the argument of \( B_{r(k)} \) (resp., \( B_{l(k)} \)) to \( \arg z \) in the counterclockwise (resp., clockwise) direction. \( \square \)

Alternatively one can also understand this relation between \( \tilde{\psi} \) and \( \hat{\psi} \) through the integral representations as explained in Figure 3.2.
Let us define the piecewise constant $\nu \times \nu$ matrix $\tilde{V}(z)$

$$
\tilde{V}(z) = \sum_{k=1}^{s} \left( e_{\tau(k),\tau(k)} + \sum_{j=k+1}^{s} (1 - \eta_{\tau(j)}) e_{\tau(k),\tau(j)} \right) + \sum_{k=1}^{\nu-s} \left( \eta_{\tau(k)} e_{\tau(k),\tau(k)} + \sum_{j=k+1}^{\nu-s} (\eta_{\tau(j)} - 1) e_{\tau(k),\tau(j)} \right)
$$

(3.11)

where $e_{ij}$ stands for the basis of $\nu \times \nu$ matrices whose only nonzero entry is 1 at the $(i,j)$th entry.

Using the above matrix $\tilde{V}(z)$ we have, by Lemma 3.4,

$$
\tilde{\Psi}(z) = \tilde{V}(z) \tilde{\Psi}(z).
$$

(3.12)

### 3.2 Construction of $\Psi$

We will define $\Psi(z)$ by

$$
\Psi(z) = V(z) \tilde{\Psi}(z),
$$

(3.13)

in terms of the piecewise constant $\nu \times \nu$ matrix $V(z)$ that we define below.

For $z \notin \tilde{\mathcal{B}} \cup \mathcal{B}$ and for $i \in \mathcal{I}$ consider the line segment $\overline{a_i z} = \{ a_i + t(z - a_i) | 0 \leq t \leq 1 \}$. We define

$$
q = q(i) = q(i; z) = \# \{ \text{intersections of } \overline{a_i z} \text{ with } \mathcal{B} \setminus \mathcal{B}_i \}.
$$

If $0 < \arg(z/a_i) \leq \pi$ then by simple geometry $\overline{a_i z}$ can intersect only $\mathcal{B}_{\tau(s)}$ but not $\mathcal{B}_{\ell(s)}$, and vice versa for $-\pi < \arg(z/a_i) \leq 0$. For example, $q(1; z) = 3$ in Figure 3.3. And by versus for $-\pi < \arg(z/a_i) \leq 0$.

Let us consider the case $i = \tau(k)$ for some $k$, then all the $q$ number of intersections occur with $\mathcal{B}_{\tau(p)}$ for $k < p \leq s$. Hence we can define $p_1(i; z), p_2(i; z), \ldots, p_q(i; z)$ such that $\mathcal{B}_{\tau(p_s)}$ intersects $\overline{a_i z}$ and satisfies

$$
k < p_1 < p_2 < \cdots < p_q \leq s \quad \text{where } i = \tau(k).
$$

Similarly, for $-\pi < \arg(z/a_i) \leq 0$, we define $p_1(i; z), p_2(i; z), \ldots, p_q(i; z)$ such that $\mathcal{B}_{\ell(p_s)}$ intersects $\overline{a_i z}$ and satisfies

$$
k < p_1 < p_2 < \cdots < p_q \leq \nu - s \quad \text{where } i = l(k).
$$

We note that $p_s$'s are determined by the two arguments, $i$ and $z$. Therefore we will write $p_s = p_s(i; z)$ or, if the second argument is identified from the context, simply $p_s(i)$. Similarly we write $q(i; z)$ or $q(i)$.
Now we can define the $\nu \times \nu$ matrix $V(z)$ for $z \notin \hat{B} \cup B$.

\begin{equation}
V(z) = I_\nu + \sum_{k=1}^{s} \sum_{i=1}^{q(\tau(k))} \left( \prod_{j=1}^{i-1} \eta_{\tau(p_j)} \right) \left( \eta_{\tau(p_i)} - 1 \right) e_{\tau(k),\tau(p_i)} \text{ (where } p_* = p_*(\tau(k); z))
+ \sum_{k=1}^{\nu-s} \sum_{i=1}^{q(l(k))} \left( \prod_{j=1}^{i-1} \eta_{l(p_j)}^{-1} \right) \left( \eta_{l(p_i)}^{-1} - 1 \right) e_{l(k),l(p_i)} \text{ (where } p_* = p_*(l(k); z)).
\end{equation}

One can see that $V(z)$ is a block matrix with $s \times s$ block and $(\nu - s) \times (\nu - s)$ block. And each block has triangular structure due to $p_* > k$, which leads to $\det V(z) = 1$.

We also note that when $z$ is near $B_j \cup \hat{B}_j$ the line segment $\overrightarrow{a_jz}$ does not intersect $B \setminus \{a_j\}$ and therefore $q(j; z) = 0$. This implies that the the $j$th row of $V(z)$ vanishes except $[V(z)]_{jj} = 1$, where $[M]_{jk}$ stands for the $(j,k)$th entry of the matrix $M$. Then it follows that the $j$th row of $V(z)^{-1}$ also vanishes except $[V(z)^{-1}]_{jj} = 1$.

Let us recall (1.14)

$$B[j] = \bigcup_{i \neq j} B_{ij} \cup B_j$$

where $B_{ij} = \{a_i + (a_i - a_j)t \mid 0 \leq t < \infty\}$, $1 \leq i, j \leq \nu$,
with the orientation given by the direction of increasing $t$. We recall (1.17) that $W_j(z)$ is the analytic continuation of $W(z)$ (1.16) such that $W_j(z) = W(z)$ in a neighborhood of $B_j$ and $W_j(z)$ is analytic away from $B[j]$.

**Lemma 3.5.** When $z \in B_{jk}$, we have

$$
(z - a_j)^{c_j} \bigg[ ((z - a_j)^{c_j})_{B[k]} \bigg]_+ = \begin{cases} 
1, & 0 < \arg(a_j/a_k) \leq \pi, \\
1/\eta_j, & -\pi < \arg(a_j/a_k) \leq 0,
\end{cases}
$$

where $+$ denotes the boundary value evaluated from the $+$ side of $B_{jk}$.

**Proof.** When $0 < \arg(a_j/a_k) \leq \pi$, by the definition of $[(z - a_j)^{c_j}]_{B[k]}$ in (1.15) we have

$$
[(z - a_j)^{c_j}]_{B[k]} = (z - a_j)^{c_j}, \quad \text{when } z \in B_k \cup \hat{B}_k.
$$

As $[(z - a_j)^{c_j}]_{B[k]}$ is analytic away from $B_{jk}$, it follows from Figure 3.4 that

$$
[(z - a_j)^{c_j}]_{B[k]} = (z - a_j)^{c_j}, \quad \text{when } z \text{ is in the (+) side of } B_{jk}.
$$

For $-\pi < \arg(a_j/a_k) \leq 0$ we have, by the similar argument,

$$
[(z - a_j)^{c_j}]_{B[k]} = \eta_j(z - a_j)^{c_j}, \quad \text{when } z \text{ is in the (+) side of } B_{jk}.
$$

\[ \square \]

**Lemma 3.6.** Let $z \in B_{jk}$ be sufficiently close to $a_j$ such that the line segment $\overrightarrow{a_jz}$ does not intersect $B \setminus \{a_j\}$. Let

$$
\mathbf{r}(\ast) = \mathbf{r}(\ast; z) \quad \text{and} \quad I(\ast) = I(\ast; z),
$$

with the orientation given by the direction of increasing $t$. We recall (1.17) that $W_j(z)$ is the analytic continuation of $W(z)$ (1.16) such that $W_j(z) = W(z)$ in a neighborhood of $B_j$ and $W_j(z)$ is analytic away from $B[j]$.\[ \square \]
and

\[ q = q(k; z), \quad p_i = p_i(k; z) \quad \text{for } i = 1, \ldots, q. \]

For \( 0 < \arg(z/a_k) \leq \pi \) we have

\[
\left[ \frac{W_k(z)}{W_j(z)} \right] = \prod_{i=1}^q \eta_{\tau(p_i)}^{-1} \quad \text{and} \quad \left[ \frac{W_k(z)}{W_j(z)} \right] = \eta_j \prod_{i=1}^q \eta_{\tau(p_i)}, \quad z \in B_{jk},
\]

and for \( -\pi < \arg(z/a_k) \leq 0 \) we have

\[
\left[ \frac{W_k(z)}{W_j(z)} \right] = \prod_{i=1}^q \eta_{\tau(p_i)} \quad \text{and} \quad \left[ \frac{W_k(z)}{W_j(z)} \right] = \eta_j^{-1} \prod_{i=1}^q \eta_{\tau(p_i)}, \quad z \in B_{jk},
\]

Proof. Let \( z \in B_{jk} \) be sufficiently close to \( a_j \) such that the line segment \( \overline{a_j z} \) does not intersect \( B \setminus \{a_j\} \). For such \( z \) we have

\[
\frac{W(z)}{W_j(z)} = 1
\]

because \( W_j(z) = W(z) \) when \( z \) is near \( B_j \).

For \( z \notin B[k] \) the line segment \( \overline{a_k z} \) does not intersect \( B[k] \). If \( 0 < \arg(z/a_k) \leq \pi \) the same line segment crosses \( B_{\tau(p_i;z)} \) for \( i = 1, \ldots, q \) where \( p_i = p_i(k; z) \) and \( q = q(k; z) \). We get

\[
\frac{W_k(z)}{W(z)} = \prod_{i=1}^q \eta_{\tau(p_i)}^{-1}.
\]

If \( -\pi < \arg(z/a_k) \leq 0 \) the similar consideration gives

\[
\frac{W_k(z)}{W(z)} = \prod_{i=1}^q \eta_{\tau(p_i)}.
\]

Now let \( z \) be very close to \( B_{jk} \). For \( z \) in the + side of \( B_{jk} \), we get \( \tau(p_q) = j \). And we get, for \( 0 < \arg(z/a_k) \leq \pi \),

\[
\frac{W_{k,-}(z)}{W(z)} = \prod_{i=1}^q \eta_{\tau(p_i)}^{-1} \quad \text{and} \quad \frac{W_{k,+}(z)}{W(z)} = \eta_j \prod_{i=1}^q \eta_{\tau(p_i)}^{-1}.
\]

For \( -\pi < \arg(z/a_k) \leq 0 \), the same consideration gives

\[
\frac{W_{k,+}(z)}{W(z)} = \prod_{i=1}^q \eta_{\tau(p_i)} \quad \text{and} \quad \frac{W_{k,-}(z)}{W(z)} = \eta_j^{-1} \prod_{i=1}^q \eta_{\tau(p_i)}.
\]

This proves the lemma when \( z \in B_{jk} \) is near \( a_j \). Since \( B_{jk} \) is a branch cut of \( W_k \) and since it does not intersect the branch cuts of \( W_j \), \( [W_k(z)/W_j(z)]_{\pm} \) is a constant function over the whole \( B_{jk} \). \( \square \)
Let us define the constant $\eta_{kj}$ by

\begin{equation}
\eta_{kj} \equiv \frac{W_j(z)}{W_{k,+}(z)}, \quad z \in B_{jk},
\end{equation}

where $+$ means the boundary value evaluated from the $+$ side of $B_{jk}$.

When $z \in D_{aj}$, by Lemma 3.5 and Lemma 3.6 with the definitions of $\tilde{\eta}_{kj}$ (1.19) and $\eta_{kj}$ (3.19), one can see that

\begin{equation}
\tilde{\eta}_{kj} = \begin{cases} 
\eta_{kj}, & 0 < \arg(a_j/a_k) \leq \pi, \\
\eta_j \eta_{kj}, & -\pi < \arg(a_j/a_k) \leq 0.
\end{cases}
\end{equation}

The jump condition of $\Psi$ is given by the following proposition.

**Proposition 3.7.** Let $\Psi(z)$ be defined in (3.13) and $W(z)$ be defined by

\begin{equation}
W(z) = \text{diag}(W_1(z), \ldots, W_\nu(z)),
\end{equation}

we have the following jump conditions of $\Psi(z)$

\begin{equation}
\begin{cases}
\Psi_+(z) = (I_\nu - (\eta_j - 1)\eta_{kj} e_{kj}) \Psi_-(z), & z \in B_{jk}, \\
[W(z)\Psi(z)]_+ = [W(z)\Psi(z)]_- & z \in B_j, \\
[W(z)\Psi(z)]_+ = \tilde{J}_j [W(z)\Psi(z)]_- & z \in \hat{B}_j.
\end{cases}
\end{equation}

**Proof.** Let us find the jump at $z \in B_{jk}$.

We first assume $0 < \arg(z/a_k) \leq \pi$. The line segment $\overrightarrow{a_k z}$ intersects the branch cuts $\{B_{r(p_i)}\}_{i=1}^q$, where $p_i = p_i(k; z)$ and $q = q(k; z)$. Among them will be $B_j$ and we will define $1 \leq s \leq q$ such that $r(p_s; z) = j$. From (3.18) it follows that

\begin{equation}
\eta_{kj} = \frac{W_j(z)}{W_{k,+}(z)} = \prod_{i=1}^{s-1} \eta_{r(p_i)}.
\end{equation}

Let $z_+$ be approaching $z$ from the $+$ the side of $B_{jk}$, and $z_-$ from the $-$ side of $B_{jk}$. Then $\overrightarrow{a_k z_+}$ intersects all of $\{B_{r(p_i)}\}_{i=1}^q$ whereas $\overrightarrow{a_k z_-}$ intersects all but $B_{r(p_s)} = B_j$. One can also see that the line segment $\overrightarrow{a_j z}$, which is a subset of $\overrightarrow{a_k z}$, intersects exactly $\{B_{p_i}\}_{i=s+1}^q$ away from $a_j$. 

From this observation we obtain the $j$th row and the $k$th row of the matrix $\mathbf{V}(z)$ (3.14).

\begin{align*}
\mathbf{V}_-(z) &= I_\nu + \sum_{i=1}^{q} \left( \prod_{\xi=1}^{i-1} \eta_{\tau(p_\xi)} \right) (\eta_{\tau(p_i)} - 1) \mathbf{e}_{k\tau(p_i)} \\
&\quad + \sum_{i=s+1}^{q} \left( \prod_{\xi=s+1}^{i-1} \eta_{\tau(p_\xi)} \right) (\eta_{\tau(p_i)} - 1) \mathbf{e}_{j\tau(p_i)} + \ldots,
\end{align*}

\begin{align*}
\mathbf{V}_+(z) &= I_\nu + \sum_{\substack{i=1 \atop i \neq s}}^{q} \left( \prod_{\xi=1}^{i-1} \eta_{\tau(p_\xi)} \right) (\eta_{\tau(p_i)} - 1) \mathbf{e}_{k\tau(p_i)} \\
&\quad + \sum_{i=s+1}^{q} \left( \prod_{\xi=s+1}^{i-1} \eta_{\tau(p_\xi)} \right) (\eta_{\tau(p_i)} - 1) \mathbf{e}_{j\tau(p_i)} + \ldots.
\end{align*}

The other rows does not change across $B_{jk}$. We claim

(3.23) \quad \mathbf{V}_+(z) = [I_\nu - \left( \prod_{i=1}^{s-1} \eta_{\tau(p_i)} \right) (\eta_j - 1) \mathbf{e}_{kj}] \mathbf{V}_-(z).

It is enough to check the jump on $k$th row. First we check the $(k, \tau(p_i))$th entry of (3.23) for $i > s$.

\begin{align*}
[V_-(z)]_{k,\tau(p_i)} &= \left( \prod_{\xi=1}^{i-1} \eta_{\tau(p_\xi)} \right) (\eta_{\tau(p_i)} - 1) \\
&\quad - \left( \prod_{i'=1}^{s-1} \eta_{\tau(p_{i'})} \right) (\eta_j - 1) \left( \prod_{\xi=s+1}^{i-1} \eta_{\tau(p_\xi)} \right) (\eta_{\tau(p_i)} - 1) \\
&= \left( \prod_{\xi=1}^{i-1} \eta_{\tau(p_\xi)} \right) (\eta_{\tau(p_i)} - 1) = [V_+(z)]_{k,\tau(p_i)},
\end{align*}

where we have used $\eta_j = \eta_{\tau(p_s)}$. For $i = s$, the $(k, \tau(p_s)) = (k, j)$th entry of (3.23) becomes

\[0 = [V_+(z)]_{kj} = [V_-(z)]_{kj} - \left( \prod_{\xi=1}^{i-1} \eta_{\tau(p_\xi)} \right) (\eta_{\tau(p_i)} - 1).
\]

A similar and simpler calculation gives the identity for $i < s$.

From Lemma 3.6 the jump relation (3.23) gives the jump relation in this lemma.

Let us repeat the similar proof for $-\pi < \text{arg}(z/a_k) \leq 0$. If $z_+$ approaches $z$ from the + side of $B_{jk}$, $\overrightarrow{a_k z_+}$ intersects the branch cuts $\{B_{i}(p_i)\}_{i=1}^{q}$. Whereas
approaching from the $-$ side, $\alpha_k z^+$ intersects $\{B_{l(p_i)}\}_{i=1}^q$ except $B_{l(p_s)} = B_j$. And we obtain the $j$th row and the $k$th row of the matrix $V(z)$ (3.14).

$$V_+(z) = I_\nu + \sum_{i=1}^q \left( \prod_{\xi=1}^{i-1} \eta_{\ell(p_\xi)}^{-1} \right) (\eta_{\ell(p_i)}^{-1} - 1)e_{k,i(p_i)}$$

$$+ \sum_{i=s+1}^{q} \left( \prod_{\xi=s+1}^{i-1} \eta_{\ell(p_\xi)}^{-1} \right) (\eta_{\ell(p_i)}^{-1} - 1)e_{j,i(p_i)} + \ldots,$$

$$V_-(z) = I_\nu + \sum_{i=1}^q \left( \prod_{\xi=1}^{i-1} \eta_{\ell(p_\xi)}^{-1} \right) (\eta_{\ell(p_i)}^{-1} - 1)e_{k,i(p_i)}$$

$$+ \sum_{i=s+1}^{q} \left( \prod_{\xi=s+1}^{i-1} \eta_{\ell(p_\xi)}^{-1} \right) (\eta_{\ell(p_i)}^{-1} - 1)e_{j,i(p_i)} + \ldots,$$

This gives the jump relation by

$$V_+(z) = \left[ I_\nu + \left( \prod_{i=1}^{s-1} \eta_{\ell(p_i)}^{-1} \right) (\eta_j^{-1} - 1)e_{kj} \right] V_-(z) = \left[ I_\nu + \eta_{kj}(1 - \eta_j)e_{kj} \right] V_-(z),$$

where the last equality is obtained by Lemma 3.6. This ends the proof of the jump on $B_{jk}$.

Now let us prove the jump on $z \in B_j$.

We claim that the jump relation on $B_j$ in Proposition 3.7 is equivalent to the following identity using Lemma 3.6.

(3.24) $$V_+(z)\tilde{J}_j = \begin{bmatrix} I_{j-1} & \eta_j^{-1} \\ \eta_j^{-1} & I_{\nu-j} \end{bmatrix} V_-(z).$$

It is clear that only the $j$th column of $V(z)$ can have discontinuity on $B_j$. So it is enough to look at the $(+, j)$th entries of the above claim. Since the $j$th row of $V(z)$ consists of zeros except 1 at $(j, j)$th entry, the $(j, j)$th entry holds the identity.

For $k \neq j$ we consider $0 < \arg(z/\alpha_k) \leq \pi$ and $-\pi < \arg(z/\alpha_k) \leq 0$ separately. For $0 < \arg(z/\alpha_k) \leq \pi$, $\alpha_k z^+$ intersects $B_j$ while $\alpha_k z^-$ does not intersect $B_j$. Let $q = q(k; z_+)$ and $p_i = p_i(k; z_+)$ such that $v(p_q) = j$. The $k$th row of $V_\pm(z)$ are given by

$$V_+(z) = e_{kk} + \sum_{i=1}^{q} \left( \prod_{\xi=1}^{i-1} \eta_{\ell(p_\xi)} \right) (\eta_{\ell(p_i)} - 1)e_{k,\tau(p_i)},$$

$$V_-(z) = e_{kk} + \sum_{i=1}^{q-1} \left( \prod_{\xi=1}^{i-1} \eta_{\ell(p_\xi)} \right) (\eta_{\ell(p_i)} - 1)e_{k,\tau(p_i)}.$$
The \((k, j)\)th entry of the left hand side of (3.24) becomes

\[
(\eta_j^{-1} - 1) \left[1 + \sum_{i=1}^{q} \left( \prod_{\xi=1}^{i-1} \eta_{\kappa(p_\xi)} \right) (\eta_{\kappa(p_i)} - 1) \right] + \left( \prod_{\xi=1}^{q-1} \eta_{\kappa(p_\xi)} \right) (\eta_{\kappa(p_q)} - 1)
\]

\[
= (\eta_j^{-1} - 1) \left[1 + \sum_{i=1}^{q} \left( \prod_{\xi=1}^{i-1} \eta_{\kappa(p_\xi)} \right) (\eta_{\kappa(p_i)} - 1) - \left( \prod_{\xi=1}^{q} \eta_{\kappa(p_\xi)} \right) \right] = 0,
\]

where the last equality is obtained by telescoping cancellation. This is exactly the \((k, j)\)th entry of \(V_-(z)\) because \(\overrightarrow{a_k z} \to\) does not intersect \(B_j\).

We repeat the same argument for \(-\pi < \arg(z/a_k) \leq 0\). The line segment \(\overrightarrow{a_k z} \to\) intersects \(B_j\) while \(\overrightarrow{a_k z} \to\) does not intersect \(B_j\). Let \(q = q(k; z_-)\) and \(p_i = p_i(k; z_-)\) such that \(l(p_q) = j\). We can write

\[
V_-(z) = e_{kk} + \sum_{i=1}^{q} \left( \prod_{\xi=1}^{i-1} \eta_{\kappa(p_\xi)}^{-1} \right) (\eta_{\kappa(p_i)}^{-1} - 1) e_{k,l(p_i)},
\]

\[
V_+(z) = e_{kk} + \sum_{i=1}^{q-1} \left( \prod_{\xi=1}^{i-1} \eta_{\kappa(p_\xi)}^{-1} \right) (\eta_{\kappa(p_i)}^{-1} - 1) e_{k,l(p_i)}.
\]

The \((k, j)\)th entry of the left hand side of (3.24) becomes

\[
(\eta_j^{-1} - 1) \left[1 + \sum_{i=1}^{q-1} \left( \prod_{\xi=1}^{i-1} \eta_{\kappa(p_\xi)}^{-1} \right) (\eta_{\kappa(p_i)}^{-1} - 1) \right] = (\eta_j^{-1} - 1) \left( \prod_{\xi=1}^{q-1} \eta_{\kappa(p_\xi)}^{-1} \right),
\]

which is exactly the \((k, j)\)th entry of \(V_-(z)\).

Since

\[
W_+(z)^{-1}W_-(z) = \sum_{i=1}^{\nu} \frac{W_{j,-}(z)}{W_{j,+}(z)} e_{ii} = \begin{bmatrix} I_{j-1} & \eta_j^{-1} \\ \eta_j^{-1} & I_{\nu-j} \end{bmatrix},
\]

the identity (3.24) proves the jump relation on \(B_j\).

Lastly, we claim that the jump relation on \(\bar{B}_j\) in Proposition 3.7 is equivalent to the following identity using Lemma 3.6.

\[
V_+(z)\bar{J}_jV_-(z)^{-1} = W(z)^{-1}\bar{J}_jW(z).
\]
We first evaluate the product of $V(z)$ with a matrix with only $j$th column being nonzero as follows:

$$V(z) \left( \sum_{i=1}^{\nu} e_{ij} \right)$$

$$= \sum_{k=1}^{s} e_{\tau(k),j} \left( 1 + \sum_{i=1}^{\nu - s} q(\tau(k);\xi) \left( \prod_{\xi=1}^{i-1} \eta_{\tau(p_{\xi})} \right) \left( \eta_{\tau(p_{i})} - 1 \right) \right) \quad \text{where } p_* = p_*(\tau(k); z)$$

$$+ \sum_{k=1}^{\nu - s} e_{l(k),j} \left( 1 + \sum_{i=1}^{\nu - s} q(l(k);\xi) \left( \prod_{\xi=1}^{i-1} \eta_{l(p_{\xi})}^{-1} \right) \left( \eta_{l(p_{i})}^{-1} - 1 \right) \right) \quad \text{where } p_* = p_*(l(k); z)$$

$$= \sum_{k=1}^{s} e_{\tau(k),j} \left( \prod_{\xi=1}^{\nu} \eta_{\tau(p_{\xi})} \right) + \sum_{k=1}^{\nu - s} e_{l(k),j} \left( \prod_{\xi=1}^{\nu} \eta_{l(p_{\xi})}^{-1} \right) \quad \text{with } p_* \text{ as above}$$

$$= \sum_{i=1}^{\nu} \frac{W(z)}{W_i(z)} e_{ij}.$$

From above we get

$$V(z) \left( \sum_{i=1}^{\nu} e_{ij} \right) V(z)^{-1} = \sum_{i=1}^{\nu} \left[ \frac{W(z)}{W_j(z)} \right] e_{ij} = \sum_{i=1}^{\nu} \frac{W_j(z)}{W_i(z)} e_{ij}, \quad z \in \hat{B}_j.$$

The identity (3.25) is proved because $\tilde{J}_j = I_\nu + (\eta_j^{-1} - 1) \sum_{i=1}^{\nu} e_{ij}$.

### 3.3 Large $z$ behavior of $\Psi(z)$

We will identify the asymptotic behavior of $\Psi(z)$ as $z$ goes to $\infty$.

**Lemma 3.8.** Let $\Psi(z)$ and $\tilde{\Psi}(z)$ be defined in (3.13) and (3.12) respectively, we have

$$\Psi(z) = V(z) \tilde{V}(z) \tilde{\Psi}(z)$$

and

(3.26)

$$V(z) \tilde{V}(z)$$

$$= I_\nu + \sum_{k=1}^{s} \sum_{j=k+1}^{s} \left( q(\tau(k);z) \right) \left( \prod_{i=1}^{\nu} \eta_{\tau(p_{i})} \right) \left( 1 - \eta_{\tau(j)} \right) \text{e}_{\tau(k),\tau(j)} \quad \text{where } p_* = p_*(\tau(k); z)$$

$$+ \sum_{k=1}^{\nu - s} \sum_{j=k+1}^{\nu - s} \left( q(l(k);z) \right) \left( \prod_{i=1}^{\nu} \eta_{l(p_{i})}^{-1} \right) \left( \eta_{l(j)} - 1 \right) \text{e}_{l(k),l(j)} \quad \text{where } p_* = p_*(l(k); z),$$

where $\mathcal{P}(k; z) = \{p_1(k; z), \ldots, p_q(k; z)\}$ and $q = q(k; z)$. 

---

**In summary,** we have evaluated the product of $V(z)$ with a matrix and identified the asymptotic behavior of $\Psi(z)$ as $z$ goes to infinity. The identity (3.25) is established, and we have derived (3.26) for the product $V(z) \tilde{V}(z)$. These results provide insights into the behavior of planar orthogonal polynomials in the large $z$ limit.
Proof. By the definitions of $V(z)$ and $\tilde{V}(z)$ in (3.14) and (3.11), we get the $(r(k), r(j))$th entry of $V(z)\tilde{V}(z)$ by the following calculation

\[
\begin{align*}
\left[ V(z)\tilde{V}(z) \right]_{(r(k), r(j))} &= \left[ V(z) \right]_{r(k)\text{th row}} \left[ \tilde{V}(z) \right]_{r(j)\text{th column}} \\
&= \left[ \left( e_{r(k), r(k)} + \sum_{i=1}^{q(r(k))} \left( \prod_{j=1}^{i-1} \eta_{r(p_j)} \right) (\eta_{r(p_i)} - 1) e_{r(k), r(p_i)} \right) \right]_{(r(k), r(j))} \\
&\times \left( e_{r(j), r(j)} + (1 - \eta_{r(j)}) \sum_{k=1}^{j-1} e_{r(k), r(j)} \right)_{(r(k), r(j))} \\
&= \begin{cases} 
0 & \text{if } j \in \mathcal{P}(r(k); z) \\
\left( 1 + \sum_{i=1}^{q(r(k))} \left( \prod_{s=1}^{i-1} \eta_{r(p_s)} \right) (\eta_{r(p_i)} - 1) \right) (1 - \eta_{r(j)}) & \text{if } j \notin \mathcal{P}(r(k); z)
\end{cases}
\end{align*}
\]
Similarly, we obtain the $(l(k), l(j))$th entry of $V(z)\tilde{V}(z)$

$$
\begin{align*}
    [V(z)\tilde{V}(z)]_{(l(k), l(j))} &= [V(z)]_{l(k)th \text{ row}} [\tilde{V}(z)]_{l(j)th \text{ column}} \\
    &= \left( e_{l(k), l(k)} + \sum_{i=1}^{q(l(k))} \left( \prod_{j=1}^{i-1} \eta_{l(p_j)}^{-1} \right) \left( \eta_{l(p_i)}^{-1} - 1 \right) e_{l(k), l(p_i)} \right) \\
    &\quad \times \left( \eta_{l(j)} e_{l(j), l(j)} + (\eta_{l(j)} - 1) \sum_{k=1}^{j-1} e_{l(k), l(j)} \right) \\
    &= \begin{cases} 
        \left( 1 + \sum_{i=1}^{q(l(k))} \left( \prod_{s=1}^{i-1} \eta_{l(p_s)}^{-1} \right) \left( \eta_{l(p_i)}^{-1} - 1 \right) \right) \left( \eta_{l(j)} - 1 \right) & \text{if } j \in \mathcal{P}(l(k); z) \\
        + \left( \prod_{s=1}^{q(l(k))} \eta_{l(p_i)}^{-1} \right) \left( \eta_{l(j)} - 1 \right) & \text{if } j \notin \mathcal{P}(l(k); z) \\
        0 & \text{if } j \in \mathcal{P}(l(k); z) \\
        \left( \prod_{i=1}^{q(l(k))} \eta_{l(p_i)}^{-1} \right) \left( \eta_{l(j)} - 1 \right) & \text{if } j \notin \mathcal{P}(l(k); z). 
    \end{cases}
\end{align*}
$$

\[ \square \]

**Proposition 3.9.** Let $K > 0$ be sufficiently large such that $B \cup B[j]$ do not intersect in $\{z : |z| > K\}$. For $|z| > K$, we define $\psi_j(z)$ to be the row vector whose $k$th entry is given by

\begin{equation}
    [\psi_j(z)]_k = \left( \int_{\gamma_j} W_j(s) \prod_{i=1}^{\nu} (s - a_i)^{-\delta_{ik}} e^{-Nz s} ds \right)^*,
\end{equation}

where the integration contour $\gamma_j = \{a_j + (z - a_j)t, \ t \geq 0\}$ is oriented in the direction of increasing $t$. Then we have

$$
[\Psi(z)]_{jk} = [\psi_j(z)]_k,
$$

where $\Psi(z)$ is defined at (3.13).

**Proof.** For $z$ as given in the proposition and $1 \leq j < s(z)$, we will show that

\begin{equation}
    [\Psi(z)]_{\tau(j)th \text{ row}} = \psi_{\tau(j)}(z).
\end{equation}
There exist $\mathcal{P}(\tau(j); z) = \{p_1(\tau(j); z), p_2(\tau(j); z), \ldots, p_q(\tau(j); z)\}$ and $q = q(\tau(j); z)$ such that exactly $\{B_{\tau(p_1)}, B_{\tau(p_2)}, \ldots, B_{\tau(p_q)}\}$ among $B \setminus \{B_{\tau(j)}\}$ intersects the line segment $\overrightarrow{a_{\tau(j)}z}$. Then we have

$$[\psi_{\tau(j)}(z)]_k = [\hat{\psi}_{\tau(j)}(z)]_k + \sum_{i > j, i \notin \mathcal{P}(\tau(j); z)}^{s(z)} \left( \oint_{B_{\tau(i)}, \tau(j)} W_{\tau(j)}(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_{ik}} e^{-Nz_s} ds \right)^*,$$

where the integration contour is enclosing $B_{\tau(i), \tau(j)}$ in the clockwise orientation. See Figure 3.5.

Let us assume that $\{B_{\tau(i), \tau(j)}\}_{j < i \leq s(z), i \notin \mathcal{P}(\tau(j); z)}$ and $\{B_{\tau(i)}\}_{j < i \leq s(z), i \notin \mathcal{P}(\tau(j); z)}$ are connected to $\infty$ by the same order in the following sense: when $i < i'$

$$0 < \arg a_{\tau(i')} - \arg a_{\tau(i)} < \pi, \quad 0 < \arg(a_{\tau(i')} - a_{\tau(j)}) - \arg(a_{\tau(i)} - a_{\tau(j)}) < \pi.$$

In such case, $\{B_{\tau(i), \tau(j)}\}_{j < i \leq s(z), i \notin \mathcal{P}(\tau(j); z)}$ can be smoothly deformed into $\{B_{\tau(i)}\}_{j < i \leq s(z), i \notin \mathcal{P}(\tau(j); z)}$ without changing the homotopy relation in the branch cuts of $W_{\tau(j)}(z)$. Let $W_{\tau(j)}(z)$ change into $\tilde{W}_{\tau(j)}(z)$ by this deformation, then we have

$$[\psi_{\tau(j)}(z)]_k = [\tilde{\psi}_{\tau(j)}(z)]_k + \sum_{i > j, i \notin \mathcal{P}(\tau(j); z)}^{s(z)} \left( \oint_{B_{\tau(i)}} \tilde{W}_{\tau(j)}(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_{ik}} e^{-Nz_s} ds \right)^*.$$

\[\text{FIGURE 3.5. The red dashed line is the integration contour of } \psi (3.27) \text{ and the blue dashed lines are the integration contours of } \hat{\psi}\text{'s (3.9).}\]
For a given $r(i)$ with $i \notin \mathcal{P}(r(j); z)$, we claim

\[
\left( \oint_{\tilde{B}_r(i)} \tilde{W}_{r(j)}(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_i k} e^{-N \pi s d} ds \right)^* = \left( \prod_{\xi=1}^{q(r(j))} \eta_{r(p \xi)} \right) \left( \oint_{\tilde{B}_r(i)} W(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_i k} e^{-N \pi s d} ds \right)^*.
\]

(3.30)

It follows from the fact that

\[
\left( \tilde{W}_{r(j)}(w) \right)^* = \left( \prod_{\xi=1}^{q(r(j)):z} \eta_{r(p \xi)} \right) \left( W(w) \right)^*, \quad \text{when } w \text{ is near } a_{r(i)}.
\]

This is proven because the line segment $\alpha_{r(j)} a_{r(i)}$ intersects exactly $\{ \tilde{B}_r(p \xi) \}_{\xi=1, p \xi < i}$ among the branch cuts of $W(z)$ and the same line segment does not intersect branch cuts of $\tilde{W}_{r(j)}(z)$. If there is $r(i')$ with $j < i' < i$ and $i' \notin \mathcal{P}(r(j); z)$ such that $\tilde{B}_{r(i')}$ intersects the line segment $\alpha_{r(i')} a_{r(i)}$, then we have

\[
0 < \arg \alpha_{r(i)} - \arg \alpha_{r(i')} < \pi, \quad 0 < \arg (\alpha_{r(i')} - \alpha_{r(j)}) - \arg (\alpha_{r(i)} - \alpha_{r(j)}) < \pi,
\]

which contradicts the assumption (3.29).

We also note that $\tilde{W}_{r(j)}(z) = W(z)$ when $z$ is near $B_{r(j)}$. As a consequence of the claim in (3.30), we have

\[
[\psi_{r(j)}(z)]_k = [\tilde{\psi}_{r(j)}(z)]_k
\]

(3.31)

\[
+ \sum_{i > j, \xi \notin \mathcal{P}(r(j); z)}^{s(z)} \left( \oint_{B_{r(i), r(j)}} W_{r(j)}(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_i k} e^{-N \pi s d} ds \right)^*
\]

\[
= [\tilde{\psi}_{r(j)}(z)]_k + \sum_{i > j, \xi \notin \mathcal{P}(r(j); z)}^{s(z)} \left( \oint_{B_{r(i, r)}} \tilde{W}_{r(j)}(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_i k} e^{-N \pi s d} ds \right)^*
\]

\[
= [\tilde{\psi}_{r(j)}(z)]_k
\]

(3.31)

\[
+ \sum_{i > j, \xi \notin \mathcal{P}(r(j); z)}^{s(z)} \left( \oint_{B_{r(i)}} W(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_i k} e^{-N \pi s d} ds \right)^* \prod_{\xi=1}^{q(r(j))} \eta_{r(p \xi)} (1 - \eta_{r(i)}) [\tilde{\psi}_{r(i)}(z)]_k.
\]
A similar calculation shows that
\[
[\Psi(z)]_{(j)\text{th row}} = \psi_{(j)}(z).
\]

We observe that both \(\tau(j)\)th row of \(V(z)\) (3.14) and the matrix \(\tilde{V}(z)\) (3.11) do not change under changing the locations of \(a_{\tau(i)}\)'s as long as \(\arg a_{\tau(i)}\)'s are all preserved for the corresponding \(i\)'s and \(\{B_{\tau(i)}\}\)'s do not intersect \(a_{\tau(j)}\). One can see that any given \(a_{\tau(i)}\)'s can be deformed in this way into the distribution described in (3.29). Under this deformation, both \(\tilde{\psi}\)'s (3.9) and \(\psi\)'s (3.27) can be analytically continued in the space of parameters \(\{\tilde{a}_{\tau(i)}\}_{i>j, i \notin \mathcal{P}}\). Similar argument for \(a_{\tilde{\tau}(s)}\) proves (3.31) for arbitrary \(a\)'s.

\[\quad\]

**Proposition 3.10.** For \(j\) and \(k\) are from \(\{1, \ldots, \nu\}\), \([\psi_j(z)]_k\) is analytic away from \(\mathcal{B} \cup \tilde{\mathcal{B}} \cup \mathcal{B}[j]\) and the strong asymptotic behavior of \([\psi_j(z)]_k\) is given by
\[
[\psi_j(z)]_k = \frac{C_{jk} e^{-Nz\pi_j}}{(Nz)^{c_j+n_j+1-\delta_{kj}}} \left(1 + \mathcal{O}\left(-\frac{1}{z}\right)\right), \quad z \to \infty,
\]
where
\[
C_{jk} = \lim_{\omega \to a_j} \frac{W_j(\omega)}{(\omega - a_j)^{c_j}} \prod_{i \neq j} (a_j - a_i)^{n_i - \delta_{ik}} \Gamma(c_j + n_j + 1 - \delta_{kj}).
\]
Also \(z^{n+\sum c} \psi_j(z)\) is bounded as \(z\) goes to the origin. Furthermore, \(W_j(z)\psi_j(z) - W_k(z)\psi_k(z)\) is bounded near the origin for \(j, k \in \{1, \ldots, \nu\}\).

**Proof.** By Proposition 3.7, we have \([\psi_j(z)]_k\) is analytic away from \(\mathcal{B} \cup \tilde{\mathcal{B}} \cup \mathcal{B}[j]\).

By Proposition 3.9, we get
\[
[\psi_j(z)]_k = \left(\int_{\gamma_j} W_j(s) \prod_{i=1}^\nu (s - a_i)^{n_i - \delta_{ik}} e^{-N\bar{\varpi} s} ds\right)^*,
\]
where the integration contour \(\gamma_j\) is described in (3.27). Changing the integration variable such that \(X = N(s - a_j)(\bar{\zeta} - \bar{\alpha}_j)\), or, equivalently, \(s = a_j + \frac{X}{N(\bar{\zeta} - \bar{\alpha}_j)}\), we obtain
\[
[\psi_j(z)]_k = \left(\int_0^\infty W_j\left(\frac{X}{N(\bar{\zeta} - \bar{\alpha}_j)} + a_j\right) \prod_{i=1}^\nu \left(\frac{X}{N(\bar{\zeta} - \bar{\alpha}_j)} + a_j - a_i\right)^{n_i - \delta_{ik}}\right.
\]
\[
\times e^{-N\bar{\varpi}\left(a_j + \frac{X}{N(\bar{\zeta} - \bar{\alpha}_j)}\right)} dX\bigg)^* 
\]
\[
= \left(\int_0^\infty \lim_{\omega \to a_j} \frac{W_j(\omega)}{(\omega - a_j)^{c_j}} \prod_{i \neq j} (a_j - a_i)^{n_i - \delta_{ik}} \left(\frac{X}{N\bar{\zeta}}\right)^{c_j+n_j-\delta_{kj}} e^{-N\bar{\varpi}a_j - \frac{X}{N\bar{\zeta}}} dX\right)^* 
\]
\[
\times \left(1 + \mathcal{O}\left(-\frac{1}{z}\right)\right).
\]
We have (3.32) by \( (a_j - a_i + \frac{X}{N(z - \eta_j)}) \sim (a_j - a_i) \) and \((z - a_j) = z (1 + \mathcal{O}(1/z))\) as \(z \to \infty\).

When \(z \to 0\), by the definition of \([\tilde{\psi}_j(z)]_k\) in (3.3), we have

\[
[\tilde{\psi}_j(z)]_k = \left( \int_{a_j}^{z \times \infty} W(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_{ik}} e^{-N\bar{z} s} ds \right)^* \\
= \left( \int_{a_j}^{0} W(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_{ik}} e^{-N\bar{z} s} ds \right)^* \\
+ \left( \int_{0}^{z \times \infty} W(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_{ik}} e^{-N\bar{z} s} ds \right)^*,
\]

where the 1st term in the 2nd equality is finite. For the 2nd term, we will change the integration variable such that \(X = N s \bar{z}\), we obtain

\[
\left( \int_{0}^{\infty} W(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_{ik}} e^{-N\bar{z} s} ds \right)^* \\
= \Gamma \left( \frac{n + \sum c}{N\bar{z}} \right) \prod_{i=1}^{\nu} \left( \frac{X}{N\bar{z}} - a_i \right)^{n_i - \delta_{ik}} e^{-X d X} \\
= \frac{\Gamma \left( n + \sum c \right)}{(N\bar{z})^{n+\sum c}} (1 + \mathcal{O}(z)),
\]

where we apply \(\frac{X}{N\bar{z}} - a_i = \frac{X}{N\bar{z}} (1 + \mathcal{O}(z))\), \(z \to 0\) to the last equality. Therefore, \(z^{n+\sum c}[\tilde{\psi}_j(z)]_k\) is bounded as \(z\) goes to the origin. It follows by \(\Psi(z) = V(z)\tilde{\Psi}(z)\) in (3.13) and \(V(z)\) (3.14) is piecewise constant function, \(z^{n+\sum c}\tilde{\psi}_j(z)\) is also bounded as \(z\) goes to the origin.

To prove \(W_j(z)\tilde{\psi}_j(z) - W_k(z)\tilde{\psi}_k(z)\) is bounded near the origin for \(j, k \in \{1, \ldots, \nu\}\), it is enough to show that \(\tilde{\psi}_j(z) - \tilde{\psi}_k(z)\) is bounded as \(z\) goes to the origin for \(j, k \in \{1, \ldots, \nu\}\). By the definition of \([\tilde{\psi}_j(z)]_\nu\) in (3.3), we have

\[
[\tilde{\psi}_j(z) - \tilde{\psi}_k(z)]_\nu = \left( \int_{a_j}^{a_k} W(s) \prod_{i=1}^{\nu} (s - a_i)^{n_i - \delta_{ik}} e^{-N\bar{z} s} ds \right)^*,
\]

which is the integral of an entire function over a compact set, this shows that \(\tilde{\psi}_j(z) - \tilde{\psi}_k(z)\) is bounded near the origin for \(j, k \in \{1, \ldots, \nu\}\). Expanding \(W_{\tau(j)}(z)\tilde{\psi}_{\tau(j)}(z)\) in terms of \(W(z)\tilde{\psi}\)’s by (3.14) and (3.13), we observe that the sum of the linear coefficient of the expansion is independent of \(\tau(j)\) and given by

\[
\frac{W_{\tau(j)}(z)}{W(z)} \left( \sum_{i=1}^{q(\tau(j))} \prod_{j=1}^{i-1} \eta_{\tau(p_j)}(\eta_{\tau(p_i)} - 1) \right) = \frac{W_{\tau(j)}(z)}{W(z)} \prod_{i=1}^{q(\tau(j))} \eta_{\tau(p_i)} = 1,
\]
where we apply Lemma 3.6 to the last identity. Similarly, we expand $W_{l(j)}(z)[\psi_{l(j)}(z)]_{i'}$ in terms of $W(z)\tilde{\psi}$’s, we have
\[
\frac{W_{l(j)}(z)}{W(z)} \left( \sum_{i=1}^{q(l(k))} \left( \prod_{j=1}^{i-1} \eta_{l(p_i)}^{-1} \right) \left( \eta_{l(p_i)}^{-1} - 1 \right) \right) = \frac{W_{l(j)}(z)}{W(z)} \prod_{i=1}^{q(l(j))} \eta_{l(p_i)}^{-1} = 1.
\]
Consequently, if one expands $[W_{j}(z)\tilde{\psi}_{j}(z) - W_{k}(z)\tilde{\psi}_{k}(z)]_{i'}$ in terms of $W(z)\tilde{\psi}$’s the sum of the linear coefficient is zero and, therefore, $[W_{j}(z)\tilde{\psi}_{j}(z) - W_{k}(z)\tilde{\psi}_{k}(z)]_{i'}$ can be expressed as the sum of pairwise difference of $W(z)\tilde{\psi}$’s. Since the difference of a pair, $[\tilde{\psi}_{j}(z) - \tilde{\psi}_{k}(z)]$, is entire for $j, k \in \{1, \ldots, \nu\}$, we have that $W_{j}(z)\tilde{\psi}_{j}(z) - W_{k}(z)\tilde{\psi}_{k}(z)$ is bounded near the origin for $j, k \in \{1, \ldots, \nu\}$. □

Let us define the $\nu \times \nu$ matrix functions $\Psi_0(z)$ and $\Psi_j(z)$ by,
\[
\Psi_0(z) := C \mathbf{E}(z) N_0(z) \Psi(z), \quad \Psi_j(z) := W(z)^{-1} C \mathbf{E}(z) N_j(z) W(z) \Psi(z), \quad j = 1, \ldots, \nu,
\]
where
\[
C = \text{diag} \left( \frac{N_{c_1+n_1} \cdots N_{c_{\nu}+n_{\nu}}}{C_{11} e^{N_{11}} \cdots C_{\nu\nu} e^{N_{\nu\nu}}} \right),
\]
\[
C_{jj} = \lim_{\nu \to a_j} \frac{W_j(\nu)}{\nu} \prod_{i \neq j} (\nu - a_i)^{n_i} \Gamma(c_j + n_j),
\]
\[
\mathbf{E}(z) = \text{diag}(E_1(z), \ldots, E_{\nu}(z)) \quad \text{where } E_j(z) = \exp \left[ N(a_j z + \ell_j) \right],
\]
\[
N_0(z) = \text{diag}(z^{c_1+n_1}, \ldots, z^{c_{\nu}+n_{\nu}}),
\]
\[
N_j(z) = \begin{bmatrix}
I_{j-1} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \ddots & I_{\nu-j}
\end{bmatrix} + [z^{n+\sum c}]_{B[j]}.
\]

We remind that $[z^{n+\sum c}]_{B[j]}$ has the branch cuts on $B[j] \cup \widehat{B}$, see the definition in (1.18).

We also note that
\[
\frac{z^{\sum c}}{W(z)} = \frac{[z^{\sum c}]_{B[j]}}{W_j(z)}.
\]

**Lemma 3.11.** For $z \notin B \cup B[j] \cup \widehat{B}$, $\{\Psi_j(z)\}, j \in \{0, 1, \ldots, \nu\}$ satisfy the following properties,
(i) $\Psi_j(z)$ for $j \neq 0$ is bounded near the origin;
(ii) $\Psi_0(z) = I_\nu + O(1/z)$, $z \to \infty$;
(iii) $\det \Psi_0(z) = 1$, $|\det \Psi_j(z)| = 1$, $j = 1, \ldots, \nu$;
(iv) $\Psi_j(z)$ is analytic away from $B \cup B[j] \cup \bar{B}$.

Proof. Analyticity of $\Psi_j(z)$ follows from Proposition 3.7. $\Psi_j(z)$, $j \neq 0$ is bounded near the origin is because of the statements, $z^n + \sum c_j \Psi_j(z)$ is bounded as $z$ goes to the origin and $W_j(z) \Psi_j(z) - W_k(z) \psi_k(z)$ is bounded near the origin for $j, k \in \{1, \ldots, \nu\}$, in Proposition 3.10. The strong asymptotics of $\Psi_0(z)$ is due to (3.32) in Proposition 3.10.

Finally we prove (iii). Since

$$\partial_z \tilde{\Psi}_j(z) = \left( \int_{a_j}^{z \to \infty} (-Ns) \prod_{i=1}^{\nu} (s - a_i)^{n_i + c_i - \delta_{ik} e^{-N s} ds } \right)^*$$

$$= - \left( \int_{a_j}^{z \to \infty} \prod_{i=1}^{\nu} (s - a_i)^{n_i + c_i e^{-N s} ds } \right)^* - N \bar{a}_k \tilde{\Psi}_j(z)$$

$$= - \sum_{i=1}^{\nu} (n_i + c_i) \tilde{\Psi}_j(z) - N \bar{a}_k \tilde{\Psi}_j(z),$$

we have

$$\partial_z \tilde{\Psi}(z) = -\tilde{\Psi}(z) \left( \begin{array}{ccccc}
1 + c_1 & \cdots & 1 + c_1 & \cdots & \cdots \\
1 & \cdots & \cdots & \cdots & \cdots \\
n_{\nu} + c_{\nu} & \cdots & 1 + c_\nu & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array} \right) + \left( \begin{array}{ccccc}
N \bar{a}_1 & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array} \right).$$

Therefore,

$$\partial_z \det \tilde{\Psi}(z) = \det \tilde{\Psi}(z) \left( -N \sum_{j=1}^{\nu} \bar{a}_j - \frac{n + \sum c}{N z} \right).$$

Solving the differential equation, we get

$$\det \tilde{\Psi}(z) = \frac{\text{Const}. e^{-N (\sum_{j=1}^{\nu} \tau_j) z}}{z^{n + \sum c}},$$

where the constant term is a constant function in a connected region of $\mathbb{C} \setminus \{B \cup \bar{B}\}$.

By the definition of $\Psi_0(z)$ with $\det \Psi(z) = 1$ and the asymptotics of $\Psi_0$ in (ii), we have $\det \Psi_0(z) = 1$. One compares $\Psi_0(z)$ and $\Psi_j(z)$, they must have the same determinant up to a phase factor. The phase factor is due to the branch cuts chosen for $z^{n + \sum c}$.

4 Transformations of Riemann-Hilbert problem

In the previous section we have constructed a $\nu \times \nu$ matrix $\Psi$ (3.13). Now we deal with the full $(\nu + 1) \times (\nu + 1)$ matrices and we adopt the following notations.
We will use the index from \( \{0, 1, \ldots, \nu\} \) to count the entries of the matrices such that, for \((\nu + 1) \times (\nu + 1)\) matrix \( M \), \([M]_{jk}\) refers to the entry in the \((j + 1)\)th row and the \((k + 1)\)th column. We prefer such numbering because our matrices are structured such that the 1st row and the 1st column play a distinct role than the other rows and columns. By this new convention, from now on, the \( j \)th row in any \((\nu + 1) \times (\nu + 1)\) matrix will refer to the \((j + 1)\)th row in the old convention.

We apply the method of nonlinear steepest descent analysis [10] and define successive transformations of \( Y \) into \( \tilde{Y} \) (4.2), \( T \) (4.7) and \( S \) (4.23). We will finish the section by defining the global parametrix \( \Phi \) (4.34).

### 4.1 \( \tilde{Y} \) transform

Let us redefine the Riemann-Hilbert problem for \( Y \) in Theorem 3.2, with the deformation of jump contours as described below.

**Riemann-Hilbert problem for \( Y \):**

\[
\begin{align*}
Y_+(z) &= Y_-(z) \left( \begin{array}{c}
1 & W(z) \tilde{\psi}_1(z) \\
0 & I_{\nu}
\end{array} \right), & z \in \bigcup_j \Gamma_{j0}, \\
Y_+(z) &= Y_-(z) \left( \begin{array}{c}
1 & W(z) \tilde{\psi}_1(z) \\
0 & I_{\nu}
\end{array} \right)^{-1} \left( \begin{array}{c}
1 & W(z) \tilde{\psi}_1(z) \\
0 & I_{\nu}
\end{array} \right)_+, & z \in \mathcal{B} \cap (\Omega_0)^c,
\end{align*}
\]

\[
Y(z) = \left( I_{\nu+1} + \mathcal{O} \left( \frac{1}{z} \right) \right) \begin{bmatrix}
z^n \\
z^{-n_1} \\
\vdots \\
z^{-n_\nu}
\end{bmatrix}, & z \to \infty,
\]

\[
Y(z) = \mathcal{O}(1), & z \to a_j.
\]

Here \( \tilde{\psi}_1 \) is defined at (3.5).

The above Riemann-Hilbert problem is by deforming the jump contour \( \gamma \) in Theorem 3.2, such that the resulting contour is along the boundary of the domain \((\text{clos } \Omega_0)^c \setminus \mathcal{B}) \). See Figure 4.1. When the contour goes around the branch cut \( \mathcal{B} \) (1.13) the jump matrix can be expressed as in the second equation by the product of the jumps that come from either sides of the branch cut. The subscripts \( \pm \) of the jump matrices on \( \mathcal{B} \cap (\Omega_0)^c \) stand for the boundary values evaluated from the \( \pm \) sides of \( \mathcal{B} \) respectively.

**Lemma 4.1.** \( \mathcal{B}_{kj} \) does not intersect \( \Omega_j \).

*Proof.* Since \( a_k \in \text{clos } \Omega_k \) we have \( \text{Re}(\overline{a}_k z) + l_k \geq \text{Re}(\overline{a}_j z) + l_j \) at \( z = a_k \). Since \( \text{Re}(\overline{a}_k z) - \text{Re}(\overline{a}_j z) \) increases as one moves from \( a_k \) to \( \infty \) along \( \mathcal{B}_{kj} \) (1.14) we have that \( \text{Re}(\overline{a}_k z) + l_k \geq \text{Re}(\overline{a}_j z) + l_j \) for \( z \in \mathcal{B}_{kj} \). \( \square \)
By Proposition 3.7 the \( j \)th row of \( \Psi \) (3.13) has nontrivial jump only on \( B_{\ast j} \), where \( \ast \) stands for all possible numbers from \( \{1, \ldots, \nu\} \). Therefore \( W_j \psi_j \) is analytic away from \( B_{\ast j} \) and \( B(1.12) \). Since \( B_{\ast j} \) does not intersect \( \Omega_j \) by the previous lemma \( W_j \psi_j \) is analytic on \( \Omega_j \setminus B \).

Then \( W_j(z)\tilde{\psi}_j(z) - W(z)\tilde{\psi}_1(z) \) is analytic in \( \Omega_j \setminus (B \cup \tilde{B}) \).

Let us define \( \tilde{Y}(z) \) by

\[
\begin{align*}
\tilde{Y}(z) &= Y(z), \quad z \in \Omega_0, \\
\tilde{Y}(z) &= Y(z) \left[ \begin{array}{c}
1 + \frac{W_j(z)\psi_j(z) - W(z)\tilde{\psi}_1(z)}{I_{\nu}} \\
0
\end{array} \right], \quad z \in \Omega_j \setminus (B \cup \tilde{B}),
\end{align*}
\]

where \( j = 1, \ldots, \nu \).

**Riemann-Hilbert problem for \( \tilde{Y} \):**

\[\begin{align*}
\tilde{Y}_+(z) &= \tilde{Y}_-(z) \left[ \begin{array}{c}
1 + \frac{W_k(z)\psi_k(z)}{I_{\nu}} \\
0
\end{array} \right], \quad z \in \Gamma_{k0}, \\
\tilde{Y}_+(z) &= \tilde{Y}_-(z) \left[ \begin{array}{c}
1 + \frac{W_k(z)\psi_k(z) - W(z)\tilde{\psi}_j(z)}{I_{\nu}} \\
0
\end{array} \right], \quad z \in \Gamma_{kj}, \\
\tilde{Y}_+(z) &= \tilde{Y}_-(z) \left[ \begin{array}{c}
1 + \frac{W_j(z)\psi_j(z) - W(z)\tilde{\psi}_1(z)}{I_{\nu}} \\
0
\end{array} \right], \quad z \in \tilde{B} \cap \Omega_j,
\end{align*}\]

\[
\tilde{Y}(z) = \left( I_{\nu+1} + \mathcal{O} \left( \frac{1}{z} \right) \right) \begin{bmatrix}
z^n \\
z^{-n_1} \\
\vdots \\
z^{-n_\nu}
\end{bmatrix}, \quad z \to \infty,
\]

\[
\tilde{Y}(z) = \mathcal{O}(1), \quad z \to a_j,
\]
where \( j \neq k \) and \( 1 \leq j, k \leq \nu \). One can check that the jump on \( B \) is absent because \( W_j \psi_j \) is analytic on \( B \) by Proposition 3.7.

Using the fact that \( \tilde{\psi}_j(z) - \tilde{\psi}_1(z) \) is analytic everywhere for any \( j \), the jump on \( \tilde{B} \cap \Omega_j \) can be written by

\[
\tilde{Y}_+(z) = \tilde{Y}_-(z) \left[ \begin{array}{c} 1 \\ 0 \\ \vdots \\ \frac{1}{I_{
u+j}} \end{array} \right] W_j(z) (\psi_{j+}(z) - \psi_{j-}(z)) - W(z) \left( \tilde{\psi}_{j+}(z) - \tilde{\psi}_{j-}(z) \right), \quad z \in \tilde{B} \cap \Omega_j.
\]

Let us define

\[
\tilde{Y}_0(z) = \begin{cases} \tilde{Y}(z), & z \in \Omega_0, \\ \tilde{Y}(z) \left[ \begin{array}{c} 1 \\ -W_k(z) \psi_k(z) \\ \vdots \\ I_{
u+j} \end{array} \right], & z \in \Omega_k \end{cases} \text{ for } 1 \leq k \leq \nu.
\]

By the jump condition of \( \tilde{Y} \) (4.3) the only jump of \( \tilde{Y}_0 \) is at \( \tilde{B} \).

### 4.2 \( T \) transform

Let us define

\[
G_0(z) = \text{diag} (z^{-n}, z^{n_1}, \ldots, z^{n_\nu}),
\]

\[
G_j(z) = \begin{bmatrix} E_j(z)^{-1} & I_{j-1} \\ I_{\nu-j} & E_j(z) \end{bmatrix} \quad \text{for } j \neq 0.
\]

Define \( T(z) \) by

\[
T(z) = \begin{bmatrix} 1 & 0 \\ 0 & C^{-1} \end{bmatrix} \tilde{Y}(z) \begin{bmatrix} 1 & 0 \\ 0 & \Psi_j(z)^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} G_j(z), \quad z \in \Omega_j,
\]

for \( j = 0, 1, \ldots, \nu \), where we use \( \Psi_0 \) and \( \Psi_j \) in (3.33).

We note that \( G_0 \) and \( G_j \) are made of the exponents of those functions that appeared in the definitions of the multiple Szegö curve, which corresponds to the support of the limiting roots of \( p_n \). The transform is to separate the leading exponential behavior of \( Y \) as \( n \to \infty \), and it corresponds to the so called \( g \)-function transform.

The jump of \( T \) on \( \Gamma_{j0} \) is given by

\[
T_-(z)^{-1}T_+(z) = G_0(z)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Psi_0(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} G_j(z)
\]

\[
\times \begin{bmatrix} 1 & 0 \\ 0 & \Psi_j(z)^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} G_j(z).
\]
Using the fact that \( \psi_j(z) \Psi(z)^{-1} \) is the \( \nu \) dimensional row vector \( e_j := (\ldots, 0, 1, 0, \ldots) \) with only nonvanishing entry being at the \( j \)th entry, the above jump of \( T \) on \( \Gamma_{j_0} \) is given by

(4.9)

\[
M_{j_0}(z) = \begin{bmatrix}
\frac{z^n}{E_j(z)} & 0 & \frac{W_j(z)}{z^\Sigma c} & 0 \\
0 & \frac{E_1(z)z^cW_j(z)}{z^{n+\Sigma c}} & \cdots & 0 \\
0 & 0 & \frac{E_{j-1}(z)z^{j-1}W_j(z)}{z^{n+\Sigma c}} & \cdots \\
0 & 0 & 0 & \frac{E_{j+1}(z)z^{j+1}W_j(z)}{z^{n+\Sigma c}} \\
0 & 0 & 0 & 0 \\
\frac{E_\nu(z)z^\nu W_j(z)}{z^{n+\Sigma c}} & \cdots & \frac{E_{\nu-1}(z)z^{\nu-1}W_j(z)}{z^{n+\Sigma c}} & \frac{E_\nu(z)z^\nu W_{\nu-1}(z)}{z^{n+\Sigma c}}
\end{bmatrix}
\]

The following decomposition will be useful.

(4.10) \( M_{j_0}(z) = M_0(z)^{-1}J_j(z)M_j(z), \quad j = 1, \ldots, \nu, \)

where

\[
M_0(z) = \begin{bmatrix}
\frac{1}{E_1(z)} & 0 & \cdots & 0 \\
\frac{E_1(z)}{W_1(z)} & \frac{1}{z^n} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\frac{E_{\nu}(z)}{W_\nu(z)} & \cdots & \frac{1}{z^{\nu-1+n}} & \frac{1}{z^{\nu-1+n}} \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

\[
M_j(z) = \begin{bmatrix}
\frac{1}{E_j(z)W_j(1)} & I_{j-1} & 0 & 0 \\
-\frac{E_{j-1}(z)}{E_j(z)W_{j-1}(z)} & \frac{1}{z^{n+\Sigma c}} & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
-\frac{E_{\nu}(z)}{E_j(z)W_{\nu}(1)} & \cdots & \frac{1}{z^{\nu-j+n}} & \frac{1}{z^{\nu-j+n}} \\
0 & 0 & \cdots & I_{\nu-j}
\end{bmatrix}
\]
and

\[
J_j(z) = \begin{pmatrix}
0 & 0 & W_j(z) & 0 \\
\vdots & \ddots & \vdots & \vdots \\
-\frac{z^{c_j}}{W_j(z)} & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots \\
0 & 0 & 0 & \cdots & z^{c_\nu}
\end{pmatrix}.
\]

We set that \(z^{c_j}\) has the branch cut on \(B_j\) and the branch cut of \(z^{\sum_c}\) comes from the factorization \(\prod_{j=1}^{\nu} z^{c_j}\). One can check that \(M_0\) has no branch cut on \(B\).

Let us define \(T_0(z)\) by

\[
(4.11) \quad T_0(z) = \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & \Psi_0(z)^{-1} & 0 & 0 \\
0 & 0 & G_0(z) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} G_0(z).
\]

We have

\[
T(z) = T_0(z), \quad z \in \Omega_0
\]

and for \(j \neq 0\) we have

\[
(4.12) \quad T(z) = T_0(z)M_{j0}(z), \quad z \in \Omega_j
\]

by the similar calculation as in the jump of \(T\) on \(\Gamma_{j0}\) (4.8).

Then the jump of \(T\) on \(\Gamma_{jk}\) is given in terms of the above definitions by

\[
(4.13) \quad T_0(z) - T_0(z) - 1 T_0(z) = M_{k0}(z) - 1 T_0(z)M_{j0}(z) = M_{k0}(z) - 1 J_k(z) - 1 J_j(z)M_j(z).
\]

The jump of \(T\) on \(B_{jk} \cap \Omega_0\) is given by

\[
(4.14) \quad T_0(z) - 1 T_0(z) = G_0(z) - 1 \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & \Psi_0(z)^{-1} & 0 & 0 \\
0 & 0 & G_0(z) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} G_0(z)
\]

\[
= \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & \Psi_0(z)^{-1} & 0 & 0 \\
0 & 0 & G_0(z) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} G_0(z)
\]

\[
D(z) (I_{\nu+1} + \eta_{kj}(\eta_j - 1)e_{kj}) D(z)^{-1},
\]

where

\[
D(z) = G_0(z) - 1 \begin{pmatrix}
1 & 0 & 0 \\
0 & E(z) & 0 \\
0 & 0 & N_0(z)
\end{pmatrix} = \text{diag}(z^n, E_1(z)z^{c_1}, \ldots, E_\nu(z)z^{c_\nu}).
\]

Lemma 4.2. We have

\[
(4.15) \quad M_{0,-}(z)T_0(z) - 1 T_0(z)M_{0,+}(z) - 1 = T_0(z)^{-1} T_0(z), \quad z \in B_{jk} \cap \Omega_0.
\]
Proof. On $B_{jk} W_k(z)$ (1.17) has nontrivial jump and we get
\begin{align}
D(z)^{-1}M_{0,-}(z)D(z)(I_{\nu+1} + \eta_{kj}(\eta_j - 1)e_{kj})D(z)^{-1}M_{0,+}(z)^{-1}D(z)
\end{align}
\begin{align}
= \begin{bmatrix}
- \frac{1}{W_{\nu}(z)} & 0 \\
\vdots & I_{\nu} \\
\frac{1}{W_{\nu}(z)} & 0
\end{bmatrix}^{-1}(I_{\nu+1} + \eta_{kj}(\eta_j - 1)e_{kj})
\begin{bmatrix}
- \frac{1}{W_{\nu}(z)} & 0 \\
\vdots & I_{\nu} \\
\frac{1}{W_{\nu}(z)} & 0
\end{bmatrix} + 
\end{align}
\begin{align}
= I_{\nu+1} + \eta_{kj}(\eta_j - 1)e_{kj} + \left(\frac{\eta_j - 1}{W_{k,+}(z)} + \frac{1}{W_{k,+}(z)} - \frac{1}{W_{k,-}(z)}\right)e_{k0}
\end{align}
\begin{align}
= I_{\nu+1} + \eta_{kj}(\eta_j - 1)e_{kj}.
\end{align}

The last equality is obtained by $W_{k,+}(z) = W_{k,-}(z)\eta_j$ for $z \in B_{jk}$. Conjugating by $D(z)$ and using (4.14) the proof is complete.\[\square\]

Since $T(z) = T_0(z)M_{i0}(z)$ for $z \in \Omega_i$ when $i \neq 0$ (4.12) the jump of $T$ on $B_{jk} \cap \Omega_i$ is given by
\begin{align}
T_-(z)^{-1}T_+(z) = (T_0(z)M_{i0}(z))^{-1}(T_0(z)M_{i0}(z))_+
\end{align}
\begin{align}
= [M_0(z)^{-1}J_i(z)M_i(z)]_1^{-1}T_0,-(z)^{-1}T_{0,+}(z)\left[M_0(z)^{-1}J_i(z)M_i(z)\right]_+
\end{align}
\begin{align}
= M_{i,-}(z)^{-1}J_{i,-}(z)^{-1}T_{0,-}(z)^{-1}T_{0,+}(z)J_{i,+}(z)M_{i,+}(z)
\end{align}
\begin{align}
= (D(z)^{-1}J_i(z)M_i(z))^{-1}(I_{\nu+1} + \eta_{kj}(\eta_j - 1)e_{kj})(D(z)^{-1}J_i(z)M_i(z))_+.
\end{align}

The 3rd equality is obtained by Lemma 4.2.

The jump of $T$ on $B_j \cap \Omega_0$ is given by
\begin{align}
T_-(z)^{-1}T_+(z) = G_0(z)^{-1}
\end{align}
\begin{align}
= I_{\nu+1}.
\end{align}

Here we have used that $\Psi_0(z)$ (3.33) does not jump on $B$. It can be seen by $\Psi_-(z)\Psi_+(z)^{-1} = W(z)^{-1}W(z)_+$ from Proposition 3.7 and that $N_0(z)$ (3.37) satisfies the same jump as $W(z)$ (3.21) on $B$.

Using again that $\Psi_0(z)$ does not jump on $B$ one can also see that $T_0$ (4.11) does not jump on $B$ since $\bar{Y}_0$ does not jump on $B$. The jump of $T$ on $B_j \cap \Omega_k$ for $k \neq 0$
is given by

\[(4.19)\]
\[
T_{-}(z)^{-1}T_{+}(z) = (T_{0}(z)M_{k0}(z))^{-1} (T_{0}(z)M_{k0}(z))_{+}
\]
\[
= M_{k0,-}(z)^{-1}M_{k0,0}(z)
\]
\[
= M_{k,-}(z)^{-1}J_{k,-}(z)^{-1}M_{0,-}(z)M_{0,+}(z)^{-1}J_{k,+}(z)M_{k,+}(z)
\]
\[
= M_{k,-}(z)^{-1}J_{k,-}(z)^{-1}J_{k,+}(z)M_{k,+}(z)
\]
\[
= M_{k,-}(z)^{-1} (I_{\nu+1} + (\eta_j - 1)e_{jj}) M_{k,+}(z).
\]

The 4th equality is obtained by the fact that $M_0$ has no jump on $B$.

**Lemma 4.3.** When $z \in \hat{B}_j$, we have

\[
N_{k,-}(z) J_j^{-1} N_{k,+}(z)^{-1} = \begin{cases} 
I_{\nu}, & k = j, \\
I_{\nu} + (\eta_j - 1)[z^{n+\sum c}] e_{jj}, & k \neq j.
\end{cases}
\]

Here $[z^{n+\sum c}]$ stands for the boundary value at the $-$ side of $\hat{B}_j$.

By Lemma 4.3, the jump of $T$ on $\hat{B}_j \cap \Omega_k$ for $k \neq 0$ is given by

\[
T_{-}(z)^{-1}T_{+}(z) = G_k(z)^{-1}
\]
\[
= G_k(z)^{-1} \left[ \begin{array}{cc}
1 & 0 \\
0 & C^{-1}
\end{array} \right]
\]
\[
+ G_k(z)^{-1} \left[ \begin{array}{cc}
1 & 0 \\
0 & W(z)^{-1} E(z) N_k(z)
\end{array} \right]^{-1}
\]
\[
\times \left[ \begin{array}{cc}
1 & 0 \\
0 & W(z) \Psi(z)
\end{array} \right]_+ G_k(z)
\]
\[
= G_k(z)^{-1} \left[ \begin{array}{cc}
1 & 0 \\
0 & W(z)^{-1} E(z) N_k(z)
\end{array} \right]^{-1}
\]
\[
\times \left[ \begin{array}{cc}
1 & 0 \\
0 & J_j
\end{array} \right]^{-1}
\]
\[
\times \left[ \begin{array}{cc}
1 & 0 \\
0 & N_k(z)^{-1} E(z)^{-1} W(z)
\end{array} \right]_+ G_k(z)
\]
\[
= G_k(z)^{-1} \left[ \begin{array}{cc}
1 & 0 \\
0 & W(z)^{-1} E(z) N_k(z)
\end{array} \right]^{-1}
\]
\[
\times \left[ \begin{array}{cc}
1 & 0 \\
0 & J_j
\end{array} \right]^{-1}
\]
\[
\times \left[ \begin{array}{cc}
1 & 0 \\
0 & E(z)^{-1} W(z)
\end{array} \right]_+ G_k(z), & k = j,
\]
\[
= G_k(z)^{-1} \left[ \begin{array}{cc}
1 & 0 \\
0 & W(z)^{-1} E(z)
\end{array} \right]^{-1}
\]
\[
\times (I_{\nu+1} + (\eta_j - 1)[z^{n+\sum c}] e_{jj})
\]
\[
\times \left[ \begin{array}{cc}
1 & 0 \\
0 & E(z)^{-1} W(z)
\end{array} \right]_+ G_k(z), & k \neq j,
\]

\[(4.20)\]
\[
= \begin{cases} 
I_{\nu+1}, & k = j, \\
I_{\nu+1} + (\eta_j - 1)[z^{n+\sum c}] W_j(z) E_j(z) W_k(z) e_{kj}, & k \neq j.
\end{cases}
\]
In the 1st equality, we have used the fact that $\tilde{Y}_-^{-1}\tilde{Y}_+$ (4.4) does not contribute to the jump because of the following computation. When $z \in \tilde{B}_j$ we have

$$
\begin{align*}
&[W_j(z)(\psi_{j,+}(z) - \psi_{j,-}(z)) - W(z)(\tilde{\psi}_{j,+}(z) - \tilde{\psi}_{j,-}(z))]\Psi_+(z)^{-1} \\
=& W_j(z)[\Psi_-(z)\Psi_+(z)^{-1} - I_\nu]_{j \text{th row}} \\
&- W(z)[V(z)^{-1}\Psi_-(z)\Psi_+(z)^{-1} - V(z)^{-1}]_{j \text{th row}} \\
=& W_j(z)[W(z)^{-1}\tilde{J}_j^{-1}W(z) - I_\nu]_{j \text{th row}} \\
&- W(z)[V(z)^{-1}W(z)^{-1}\tilde{J}_j^{-1}W(z) - V(z)^{-1}]_{j \text{th row}} \\
=& [\tilde{J}_j^{-1}W(z) - I_\nu]_{j \text{th row}} - W(z)[V(z)^{-1}V(z)\tilde{J}_j^{-1}V(z)^{-1} - V(z)^{-1}]_{j \text{th row}} \\
=& [\tilde{J}_j^{-1}W(z) - W(z)]_{j \text{th row}} - W(z)[(\tilde{J}_j^{-1} - I_\nu)V(z)^{-1}]_{j \text{th row}},
\end{align*}
$$

Since the $j$th row of $\tilde{J}_j^{-1} = \eta_j e_j$ where $e_j$ is the row basis vector whose only nonzero entry being 1 at the $j$th entry, the above becomes

$$
(4.21) \quad W_j(z)(\eta_j - 1)e_j - W(z)(\eta_j - 1)[V(z)^{-1}]_{j \text{th row}}
$$

which vanishes because $[V(z)^{-1}]_{j \text{th row}} = e_j$.

Collecting all the jumps that we obtained in (4.8),(4.13),(4.14),(4.17),(4.18),(4.19) and (4.20), we have the following Riemann-Hilbert problem.

**Riemann-Hilbert problem for $T$:**

$$
(4.22) \quad \begin{cases}
T_+(z) = T_-(z)M_0(z)^{-1}J_j(z)M_j(z), & \text{for } z \in \Gamma_{j0}, \\
T_+(z) = T_-(z)M_k(z)^{-1}J_k(z)M_j(z), & \text{for } z \in \Gamma_{jk}, \\
T_+(z) = T_+(z), & \text{for } z \in B_j \cap \Omega_0, \quad z \in \tilde{B}_j \cap \Omega_j, \\
T_+(z) = T_-(z)M_{k,-}(z)^{-1}(I_{\nu+1} + (\eta_j - 1)e_{jj})M_{k,+}(z), & \text{for } z \in B_j \cap \Omega_i, \quad i \neq j, \\
T_+(z) = T_-(z)\left(\eta_{\nu+1} + (\eta_j - 1)[z^{n+\sum c_j} - \frac{W_j(z)}{E_j(z)W_k(z)}e_{kj}]\right), & \text{for } z \in \tilde{B}_j \cap \Omega_k, \quad k \neq j, \\
T_+(z) = T_-(z)D(z)(I_{\nu+1} + \eta_k \eta_j - 1)e_{kj}D(z)^{-1}, & \text{for } z \in B_{jk} \cap \Omega_0, \\
T_+(z) = T_-(z)\left(D(z)^{-1}J_i(z)M_i(z)\right)^{-1} \times (I_{\nu+1} + \eta_k \eta_j - 1)e_{kj})\left(D(z)^{-1}J_i(z)M_i(z)\right), & \text{for } z \in B_{jk} \cap \Omega_i, \quad i \neq k, \\
T(z) = I_{\nu+1} + O(1/z), & \text{as } z \to \infty, \\
T(z) = O(1), & \text{as } z \to a_j,
\end{cases}
$$

where $j \neq k$ and $1 \leq j, k \leq \nu$. 

4.3 S transform: lens opening

Let us define $S(z)$ by

\[
S(z) = \begin{cases} 
T(z), & \text{when } z \in \Omega_0 \setminus U, \\
T(z)M_0(z)^{-1}, & \text{when } z \in \Omega_0 \cap U, \\
T(z)M_j(z)^{-1}, & \text{when } z \in \Omega_j \text{ for } j = 1, \ldots, \nu,
\end{cases}
\]

where $U \subset \mathbb{D}$ is a fixed neighborhood of $\bigcup_{j=1}^{\nu} \Gamma_{j0}$.

By the definition of $S$ in (4.23) the jump of $S$ on $\partial U \cap \Omega_0$ is given by

\[
S_-(z)^{-1}S_+(z) = M_0(z)^{-1}.
\]

By the jump of $T$ on $\Gamma_{j0}$ in (4.9) we have the jump of $S$ on $\Gamma_{j0}$ is given by

\[
S_-(z)^{-1}S_+(z) = M_0(z)M_{j0}(z)M_j(z)^{-1} = J_j(z).
\]

By the jump of $T$ on $\Gamma_{jk}$ in (4.13) we have the jump of $S$ on $\Gamma_{jk}$ is given by

\[
S_-(z)^{-1}S_+(z) = M_k(z)M_{k0}(z)^{-1}M_{j0}(z)M_j(z)^{-1} = J_k(z)^{-1}J_j(z).
\]

By Lemma 4.2 the jump of $S$ on $B_{jk} \cap \Omega_0$ is given by

\[
S_-(z)^{-1}S_+(z) = M_0(z)T_-(z)^{-1}T_+(z)M_0(z)^{-1} = T_-(z)^{-1}T_+(z) = D(z)\left(I_{\nu+1} + \eta_{kj}(\eta_j - 1)e_{kj}\right)D(z)^{-1}
\]

\[
= \left(I_{\nu+1} + \eta_{kj}(\eta_j - 1)\frac{E_k(z)}{E_j(z)}e_{kj}\right).
\]

By the jump of $T$ on $B_{jk} \cap \Omega_i$ in (4.17) we have the jump of $S$ on $B_{jk} \cap \Omega_i$ is given by

\[
S_-(z)^{-1}S_+(z) = M_{i-}(z)T_-(z)^{-1}T_+(z)M_{i+}(z)^{-1}
\]

\[
= (D(z)^{-1}J_i(z))^{-1}(I_{\nu+1} + \eta_{kj}(\eta_j - 1)e_{kj})(D(z)^{-1}J_i(z))_+
\]

\[
= \begin{cases} 
I_{\nu+1} + \eta_{kj}(\eta_j - 1)\frac{E_k(z)}{E_j(z)}e_{kj}, & i = j, \\
I_{\nu+1} + \eta_{kj}(\eta_j - 1)\frac{E_k(z)}{E_j(z)}e_{kj}, & i \neq j.
\end{cases}
\]

By the jump of $T$ in $B_j \cap \Omega_0$ in (4.18) the jump of $S$ on $B_j \cap \Omega_0$ is given by

\[
S_-(z)^{-1}S_+(z) = M_{0-}(z)T_-(z)^{-1}T_+(z)M_{0+}(z)^{-1} = M_{0-}(z)M_{0+}(z)^{-1} = I_{\nu+1}.
\]

By the jump of $T$ in $B_j \cap \Omega_k$ in (4.19) the jump of $S$ on $B_j \cap \Omega_k$ is given by

\[
S_-(z)^{-1}S_+(z) = M_{k-}(z)T_-(z)^{-1}T_+(z)M_{k+}(z)^{-1} = I_{\nu+1} + (\eta_j - 1)e_{jj}.
\]
By the jump of $T$ in $\tilde{\mathcal{B}}_j \cap \Omega_k$ in (4.20) the jump of $S$ on $\tilde{\mathcal{B}}_j \cap \Omega_k$ is given by

\[
S_-(z)^{-1}S_+(z) = M_{k,-}(z)T_-(z)^{-1}T_+(z)M_{k,+}(z)^{-1}
\]

\[
= \begin{cases} 
M_{k,-}(z)M_{k,+}(z)^{-1}, & k = j, \\
M_{k,-}(z) \left( I_{\nu+1} + (\eta_j - 1)\left[ z^n + \sum c \right] - \frac{W_j(z)}{E_j(z)W_k(z)}e_{kj} \right) M_{k,+}(z)^{-1}, & k \neq j,
\end{cases}
\]

\[
= \begin{cases} 
I_{\nu+1} - \frac{(\eta_j - 1)\left[ z^n + \sum c \right] - e_{j0}}{E_j(z)W_j(z)}e_{j0}, & k = j, \\
I_{\nu+1} + (\eta_j - 1)\frac{W_j(z)\left[ z^n + \sum c \right] - e_{kj}}{W_k(z)E_j(z)}e_{kj}, & k \neq j.
\end{cases}
\]

Collecting all the jumps of $S$ we have the following Riemann-Hilbert problem.

**Riemann-Hilbert problem for $S$:**

\[
S_+(z) = S_-(z)J_j(z), \quad z \in \Gamma_{j0},
\]

\[
S_+(z) = S_-(z)J_k(z)^{-1}J_j(z), \quad z \in \Gamma_{jk},
\]

\[
S_+(z) = S_-(z)M_0(z)^{-1}, \quad z \in \partial U \cap \Omega_0,
\]

\[
S_+(z) = S_-(z), \quad z \in \tilde{\mathcal{B}}_j \cap \Omega_0, \quad z \in \tilde{\mathcal{B}}_j \cap \Omega_j,
\]

\[
S_+(z) = S_-(z) \left( I_{\nu+1} + (\eta_j - 1)e_{jj} \right), \quad z \in \tilde{\mathcal{B}}_j \cap \Omega_k,
\]

\[
S_+(z) = S_-(z) \left( I_{\nu+1} - \frac{(\eta_j - 1)\left[ z^n + \sum c \right] - e_{j0}}{E_j(z)W_j(z)} \right), \quad z \in \tilde{\mathcal{B}}_j \cap \Omega_j,
\]

\[
S_+(z) = S_-(z) \left( I_{\nu+1} + \eta_{kj}(\eta_j - 1)\frac{E_k(z)z^{c_k}}{E_j(z)z^{c_j}}e_{kj} \right), \quad z \in \tilde{\mathcal{B}}_{jk} \cap \Omega_0,
\]

\[
S_+(z) = S_-(z) \left( I_{\nu+1} - \eta_{kj}(\eta_j - 1)\frac{E_k(z)z^{c_k}}{E_j(z)W_j(z)}e_{kj} \right), \quad z \in \tilde{\mathcal{B}}_{jk} \cap \Omega_j,
\]

\[
S_+(z) = S_-(z) \left( I_{\nu+1} + \eta_{kj}(\eta_j - 1)\frac{E_k(z)z^{c_k}}{E_j(z)} \right), \quad z \in \tilde{\mathcal{B}}_{jk} \cap \Omega_i, \quad i \neq j, k,
\]

\[
S(z) = I_{\nu+1} + O(1/z), \quad z \to \infty,
\]

\[
S(z)M_0(z) = O(1), \quad \text{as } z \to a_j \text{ in } \Omega_0,
\]

\[
S(z)M_j(z) = O(1), \quad \text{as } z \to a_k \text{ or } z \to a_j \text{ in } \Omega_j,
\]

where $j \neq k$ and $1 \leq j, k \leq \nu$.

We note that the jump conditions on $\tilde{\mathcal{B}}_j$ and on $\tilde{\mathcal{B}}_{jk}$ are all exponentially small as $N$ grows away from the points $\{a_1, \ldots, a_\nu\}$ because $z^n/E_j(z) = \exp(N \log z - \ldots)$.
\( \alpha_j z - \ell_j \)) is exponentially small on \( \hat{B}_j \) and \( E_k(z)/E_j(z) = \exp(N((\alpha_j - \alpha_j)z + \ell_j - \ell_k)) \) is exponentially small on \( B_{jk} \).

### 4.4 Global Parametrix

We set up the model Riemann-Hilbert problem of \( \Phi(z) \) from that of \( S \) by ignoring the jump matrices that are exponentially small as \( N \to \infty \).

\[
\begin{cases}
\Phi_+(z) = \Phi_-(z)J_j(z), & z \in \Gamma_{j0}, \\
\Phi_+(z) = \Phi_-(z)J_k(z)^{-1}J_j(z), & z \in \Gamma_{jk}, \\
\Phi_+(z) = \Phi_-(z)(I_{\nu+1} + (n_{ij} - 1)e_{jj}), & z \in B_j \cap \Omega_k, k \neq j, \\
\Phi(z) = I_{\nu+1} + O(1/z), & z \to \infty,
\end{cases}
\]

(4.33)

where \( 1 \leq j, k \leq \nu \).

A solution of the model Riemann-Hilbert problem is given by

\[
\Phi(z) = \begin{pmatrix}
\text{diag} \left( \sum_{c} \frac{z-a_1}{W(z)}, \ldots, \frac{z-a_{\nu}}{W(z)} \right), & z \in \Omega_0, \\
0 & \ddots & 0 & 0 \\
0 & & \ddots & 0 & 0 \\
-\frac{(z-a_j)^{c_j}}{W(z)^{c_j}} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & (z-a_{\nu})^{c_{\nu}}
\end{pmatrix}
\]

(4.34)

where \( j = 1, \ldots, \nu \). We assign the branch cut of \( (z - a_j)^{c_j} \) on \( B_j \) for each \( j = 1, \ldots, \nu \). One can check the jump on \( \Gamma_{j0} \) by (3.39). We note that \( z^{\sum c}/W(z) \) is analytic away from \( \hat{B} \).
We obtain the following jump relations.

\[(4.35) \quad [S(z)\Phi(z)^{-1}]_+ = [S(z)\Phi(z)^{-1}]_-(I_{\nu+1} + \frac{(\eta_j - 1)[z^{n+\sum c}]_e}{E_j(z)(z-a_j)^{c_j}}\mathbf{e}_{0j}), \quad z \in \tilde{B}_j \cap \Omega_j,\]

\[(4.36) \quad [S(z)\Phi(z)^{-1}]_+ = [S(z)\Phi(z)^{-1}]_-(I_{\nu+1} + \frac{W_j(z)}{W_k(z)} \frac{(\eta_j - 1)[z^{n+\sum c}]_e}{E_j(z)(z-a_j)^{c_j}}\mathbf{e}_{0j}), \quad z \in \tilde{B}_j \cap \Omega_k,\]

\[(4.37) \quad [S(z)\Phi(z)^{-1}]_+ = [S(z)\Phi(z)^{-1}]_-(I_{\nu+1} + \eta_{kj}(\eta_j - 1)\frac{E_k(z)(z-a_j)^{c_k}}{E_j(z)(z-a_j)^{c_j}}\mathbf{e}_{kj}), \quad z \in B_{jk},\]

\[(4.38) \quad [S(z)\Phi(z)^{-1}]_+ = [S(z)\Phi(z)^{-1}]_-(I_{\nu+1} + \sum_{i=1}^{\nu} \frac{E_i(z)(z-a_j)^{c_i}}{z^{n+\sum c}}\mathbf{e}_{i0}), \quad z \in \partial U \cap \Omega_0,\]

where \(j \neq k\).

5 Local Parametrices

Near \(a_j\)'s the jump matrices of \(\Phi (4.33)\) do not converge to the jump matrices of \(S (4.32)\). We therefore need the local parametrix around \(a_j\) that satisfies the exact jump condition of \(S\). In Section 5.1 and 5.2 we construct the local parametrices separately when \(a_j \in \Gamma_{j0}\) and when \(a_j \in \Gamma_{jk}\) for \(k \neq 0\). In Section 5.3 we construct a rational matrix function \(R\Phi\) such that the improved global parametrix, \(R\Phi\), matches the local parametrix better.

5.1 \(a_j \in \Gamma_{j0}\)

Let \(D_{a_j}\) be a disk neighborhood of \(a_j\) with a fixed radius \(r\) such that the map \(\zeta : D_{a_j} \to \mathbb{C}\) given below is univalent.

\[(5.1) \quad \zeta(z) = -N(\bar{a}_jz - \log z + \ell_j).\]

This is linearly approximated by

\[(5.2) \quad \zeta(z) = \frac{1}{a_j} - \frac{|a_j|^2}{a_j}N(z-a_j)(1 + \mathcal{O}(z-a_j)), \quad \text{when } z \to a_j.\]

Note that \(\zeta\) maps \(B_j (1.13)\) into the positive real axis and \(\tilde{B}_j (1.12)\) into the negative real axis.
Let us define the diagonal matrix function whose diagonal entries are nonvanishing and analytic at $a_j$ by

\begin{equation}
Q_j(z) = I_{\nu+1} + \left( \frac{\zeta(z)^{c_j/2}z^{c_j/2}}{(z - a_j)^{c_j/2}} - 1 \right) e_{00} + \left( \frac{(z - a_j)^{c_j/2}}{\zeta(z)^{c_j/2}z^{c_j/2}} - 1 \right) e_{jj}.
\end{equation}

We choose the branch cut of $\zeta^{c_j}$ at the negative real axis such that $Q_j$ is analytic in $\Gamma_{aj}$.

We define the matrix functions $U(a_j, z)$ and $F_j(\zeta(z))$ by

\begin{equation}
U(a_j, z) = I_{\nu+1} - \left( \sum_{i\neq 0,j} \frac{\tilde{\eta}_{kj} E_i(z)(z - a_i)^{c_i}}{E_j(z)(z - a_j)^{c_j}} B[i][\eta_{ij}] \right), \quad a_j \in \Gamma_{j0},
\end{equation}

\begin{equation}
F_j(\zeta(z)) = I_{\nu+1} + f_{c_j}(\zeta(z))e_{0j},
\end{equation}

where $f_{c}(\zeta)$ is defined at (A.1). See Appendix A for more detail about $f_{c}(\zeta)$.

Inside $\Gamma_{aj}$, from the definitions of $U(a_j, z)$ and $F_j(\zeta(z))$, one can see that

\begin{equation}
U_{+}(a_j, z) = U_{-}(a_j, z) \left( I_{\nu+1} + \left( \frac{\eta_{kj} E_k(z)(z - a_k)^{c_k}}{E_j(z)(z - a_j)^{c_j}} B[k][\eta_{kj}] \right) \right)_+^{-}, \quad z \in B_{jk},
\end{equation}

\begin{equation}
F_{j,+}(\zeta(z)) = F_{j,-}(\zeta(z)) \left( I_{\nu+1} + (\eta_j - 1) \frac{e^{\zeta(z)}}{[\zeta(z)]_{c_j}^+} e_{0j} \right), \quad z \in \tilde{B}_{j},
\end{equation}

where, in (5.6), $\pm$ means the boundary value evaluated from the + or − side of $B_{jk}$.

**Lemma 5.1.** $F_j(\zeta)Q_j(z)^{-1}U(a_j, z)\Phi(z)$ satisfies the exact Riemann-Hilbert problem of $S(z)$ in $\Gamma_{aj}$.

**Proof.** When $z \in \tilde{B}_{j} \cap \Gamma_{aj}$, using the identity $z^{n}/E_j(z) = e^{\zeta}$ from the definition of $\zeta$ in (5.1), the jump of $F_j(\zeta)Q_j(z)^{-1}U(a_j, z)$ is given by

\begin{equation}
(F_j(\zeta)Q_j(z)^{-1}U(a_j, z))^+ = \sum \left( \frac{e^{\zeta(z)}}{[\zeta(z)]_{c_j}^+} e_{0j} \right) Q_j(z)^{-1}U(a_j, z)
\end{equation}

\begin{equation}
= U(a_j, z)^{-1}Q_j(z) \left( I_{\nu+1} + (\eta_j - 1) \frac{e^{\zeta(z)}}{[\zeta(z)]_{c_j}^+} e_{0j} \right) \Phi(z) = U(a_j, z)^{-1}Q_j(z) \left( I_{\nu+1} + (\eta_j - 1) \frac{z^{n+\sum c_j}}{E_j(z)(z - a_j)^{c_j}} e_{0j} \right) U(a_j, z)
\end{equation}

\begin{equation}
= I_{\nu+1} + (\eta_j - 1) \frac{z^{n+\sum c_j}}{E_j(z)(z - a_j)^{c_j}} e_{0j},
\end{equation}
which matches the jump condition of \( S\Phi^{-1} \) on \( \hat{\mathbb{B}}_j \cap D_{a_j} \) in (4.35). In the 3rd equality we have used that \( [\zeta^{c_j} z^{c_j}]_- = [\zeta^{c_j} z^{c_j}]_+ \). We remark that \( Q_j \) was chosen in hindsight such that (5.8) holds while the jump of \( F_j \) at (5.7) is written only in terms of the local coordinate \( \zeta \).

When \( z \in B_{jk} \cap D_{a_j} \), we have
\[
(F_j(\zeta)Q_j(z)^{-1}U(a_j, z))_{-} = U_-(a_j, z)^{-1}U_+(a_j, z)
\]
\[
= I_{\nu+1} + \left( \frac{\eta_{kj} E_k(z) (z - a_k)^{c_k}}{E_j(z) [(z - a_j)^{c_j}]_{B[k]} } e_{kj} \right)_{-}
\]
\[
= I_{\nu+1} + (\eta_j - 1) \frac{\eta_{kj} E_k(z) (z - a_k)^{c_k}}{E_j(z) [(z - a_j)^{c_j}]_{B[k]} } e_{kj},
\]
which agrees with the jump of \( S\Phi^{-1} \) on \( B_{jk} \cap D_{a_j} \) in (4.37). In the 2nd equality \( \pm \) means the boundary value evaluated from the \( + \) or \( - \) side of \( B_{jk} \). The last equality is obtained by Lemma 3.5 and the relation (3.20).

Lastly, we need to show that \( F_j Q_j^{-1} U \Phi \) satisfies the boundedness of \( S \) in (4.32) as \( z \to a_j \).

When \( z \in D_{a_j} \cap \Omega_j \) we have
\[
Q_j(z) F_j(\zeta(z)) Q_j(z)^{-1} U(a_j, z) \Phi(z) M_j(z)
\]
\[
= \left( I_{\nu+1} + \frac{\zeta(z)^{c_j} z^{c_j}}{W_j(z)} \sum_{c_j} f_{c_j} (\zeta(z)) e_{00} \right) U(a_j, z) \Phi(z) M_j(z)
\]
\[
= \Phi(z) + \frac{\zeta(z)^{c_j} z^{c_j}}{W_j(z)} \left( \frac{z^n}{E_j(z) \zeta(z)^{c_j}} - f_{c_j} (\zeta(z)) \right) e_{00}
\]
\[
= \Phi(z) + \frac{\zeta(z)^{c_j} z^{c_j}}{W_j(z)} \left( \frac{e^{\zeta(z)}}{\zeta(z)^{c_j}} - f_{c_j} (\zeta(z)) \right) e_{00}.
\]
Since \( e^{\zeta}/\zeta^{c_j} - f_{c_j}(\zeta) \) is an entire function in \( \zeta \), the (0, 0)th entry is bounded. The boundedness of the other entries follow from the boundedness of the corresponding entries in \( \Phi(z) \). By a similar argument,
\[
Q_j(z) F_j(\zeta(z)) Q_j(z)^{-1} U(a_j, z) \Phi(z) M_0(z) = \mathcal{O}(1)
\]
as \( z \to a_j \) and \( z \in \Omega_0 \). This ends the proof of Lemma 5.1. \( \square \)

### 5.2 \( a_j \in \Gamma_{jk} \)

Similar to the above subsection we define
\[
\zeta(z) = -N((\overline{a}_j - a_k) z + \ell_j - \ell_k).
\]
This is linearly approximated by
\[
\zeta(z) = N(\sigma_k - a_j)(z - a_j)(1 + \mathcal{O}(z - a_j)), \quad \text{when } z \to a_j.
\]
Note that $\zeta$ maps $\Gamma_{jk}$ into the imaginary axis and $B_{jk}$ into the negative real axis.

Let us define the diagonal matrix function whose diagonal entries are nonvanishing and analytic at $a_j$ by
\begin{equation}
(5.12)
Q_j(z) = I_{\nu+1} + \left( \frac{[z - a_j]^{c_j/2}}{\zeta^{c_j/2} - a_k^{c_j/2}} \right) e_{jj} + \left( \frac{\zeta^{c_j/2} - a_k^{c_j/2}}{([z - a_j]^{c_j/2})B_{[k]} - 1} \right) e_{kk}.
\end{equation}

We require $Q_j$ being analytic in $D_{a_j}$ therefore we set that $\zeta^{c_j/2}$ has the branch cut on the negative real axis which results in $(z - a_j)^{c_j/2}$ having the branch cut on $B_{jk}$. The subscript at $([z - a_j]^{c_j/2})B_{[k]}$ refers to this fact.

We define the matrix functions $U(a_j, z)$ and $F_j(\zeta)$ by
\begin{equation}
(5.13) \quad U(a_j, z) := I_{\nu+1} + \frac{\eta' z^{n+c}}{E_j(z)(z - a_j)^{c_j}} \eta_0 \nu_j - \sum_{i \neq 0,j,k} \frac{E_i(z)(z - a_j)^{c_i}}{E_j(z)(z - a_j)^{c_j}B[i]} e_{ij}, \quad \alpha_j \in \Gamma_{jk},
\end{equation}
\begin{equation}
(5.14) \quad F_j(\zeta(z)) := I_{\nu+1} - \tilde{\eta}_{kj} f_{c_j}(\zeta(z)) e_{kj}.
\end{equation}

Above we define the constant $\eta' = 1$ if $\tilde{B}_j$ (1.12) sits in $\Omega_j$, while $\eta' = W_j(z) / W_k(z)|_{z \in \Omega_k \cap D_{a_j}}$ if $\tilde{B}_j$ sits in $\Omega_k$.

Inside $D_{a_j}$, from the definitions of $U(a_j, z)$ and $F_j(\zeta)$, one can see that
\begin{equation}
(5.15) \quad U_+(a_j, z) = U_-(a_j, z) \left( I_{\nu+1} + \left( \frac{z^{n+c}}{E_j(z)(z - a_j)^{c_j}} \eta_0 \right) \right)^+,
\end{equation}
\begin{equation}
(5.16) \quad U_+(a_j, z) = U_-(a_j, z) \left( I_{\nu+1} + \left( \frac{W_j(z)}{W_k(z)} \frac{z^{n+c}}{E_j(z)(z - a_j)^{c_j}} \eta_0 \right) \right)^+,
\end{equation}
\begin{equation}
(5.17) \quad U_+(a_j, z) = U_-(a_j, z) \left( I_{\nu+1} + \left( \frac{E_i(z)(z - a_j)^{c_i}}{E_j(z)(z - a_j)^{c_j}B[i]} \eta_0 \right) \right)^-,
\end{equation}
\begin{equation}
(5.18) \quad F_{j,+}(\zeta) = F_{j,-}(\zeta) \left( I_{\nu+1} + (\eta_j - 1) \frac{\tilde{\eta}_{kj} e^{\zeta(z)} - e_{kj}}{[\zeta(z)]^c} \right), \quad z \in B_{jk},
\end{equation}
where, in the above equations, $\pm$ means the boundary value evaluated from the $+$ or $-$ side of the corresponding contour.

**Lemma 5.2.** $F_j(\zeta)Q_j(z)^{-1}U(a_j, z)\Phi(z)$ satisfies the exact Riemann-Hilbert problem of $S(z)$ in $D_{a_j}$. 
Proof. On $B_{jk} \cap D_{a_j}$, using the identity $E_k(z)/E_j(z) = e^{^{c}}$ from the definition of $\zeta$ in (5.11), the jump of $F_j(\zeta)Q_j(z)^{-1}U(a_j, z)$ is given by

\[
(F_j(\zeta)Q_j(z)^{-1}U(a_j, z))_-^-(F_j(\zeta)Q_j(z)^{-1}U(a_j, z))_+^{-1} = U(a_j, z)^{-1}Q_j(z)F_{j,-}(\zeta)^{-1}F_{j,+}(\zeta)Q_j(z)^{-1}U(a_j, z)
\]

\[
= U(a_j, z)^{-1}Q_j(z) \left( I_{\nu+1} + (\eta_j - 1) \frac{\eta_{kj}e^{^{c}}}{\zeta_j^e} e_{kj} \right) Q_j(z)^{-1}U(a_j, z)
\]

\[
(5.19) = U(a_j, z)^{-1}Q_j(z) \left( I_{\nu+1} + \eta_{kj}(\eta_j - 1) \frac{[z - a_j]^{c_j} + e^{^{c}}}{[z - a_j]^{c_j} - e_{kj}} \right) Q_j(z)^{-1}U(a_j, z)
\]

\[
= U(a_j, z)^{-1} \left( I_{\nu+1} + \eta_{kj}(\eta_j - 1) \frac{E_k(z)(z - a_j)^{c_k}}{E_j(z)(z - a_j)^{c_j} e_{kj}} \right) U(a_j, z)
\]

\[
= I_{\nu+1} + \eta_{kj}(\eta_j - 1) \frac{E_k(z)(z - a_j)^{c_k}}{E_j(z)(z - a_j)^{c_j} e_{kj}}
\]

which matches the jump condition of $S\Phi^{-1}$ on $B_{jk} \cap D_{a_j}$ in (4.37). In the 3rd equality $[(z - a_j)^{c_j}]_+$ comes from the $+$ side of $[(z - a_j)^{c_j}]_B[k]$. The 3rd equality is obtained by Lemma 3.5 and the relation (3.20). We also note that the $-$ side of $\zeta_j^e$ corresponds to the $+$ side of $B_{jk}$ (1.14), hence $[\zeta_j^e]_-$ refers to the boundary values evaluated from the $+$ side of $B_{jk}$. We remark that $Q_j$ was chosen in hindsight such that (5.19) holds while the jump of $F_j$ at (5.18) is written only in terms of the local coordinate $\zeta$.

The other jump conditions of $S\Phi^{-1}$ in (4.35), (4.36) and (4.37) can be satisfied by the following calculations.

When $z \in \tilde{B}_j \cap D_{a_j} \cap \Omega_j$ and $a_j \in \Gamma_{jk}$ we have

\[
(F_j(\zeta)Q_j(z)^{-1}U(a_j, z))_-^-(F_j(\zeta)Q_j(z)^{-1}U(a_j, z))_+^{-1} = U_-(a_j, z)^{-1}U_+(a_j, z)
\]

\[
= I_{\nu+1} + \left( \frac{z^{n+\sum e}}{E_j(z)(z - a_j)^{c_j} e_{0j}} \right) \mid_+
\]

\[
= I_{\nu+1} + \frac{(\eta_j - 1) [z^{n+\sum e}]_ -}{E_j(z)(z - a_j)^{c_j} e_{0j}}
\]
When \( z \in B_j \cap D_{a_j} \cap \Omega_k \) and \( a_j \in \Gamma_{jk} \) we have
\[
(F_j(\zeta)Q_j(z)^{-1}U(a_j, z))^{-1} (F_j(\zeta)Q_j(z)^{-1}U(a_j, z))_+ = U_-(a_j, z)^{-1}U_+(a_j, z)
\]
\[
= I_{\nu+1} + \left( \frac{W_j(z)}{W_k(z)} \frac{z^{n+\sum c}}{E_j(z)(z-a_j)^c} e_{0j} \right)_+
\]
\[
= I_{\nu+1} + \left( \frac{W_j(z)}{W_k(z)} (\eta_j - 1) \left[ z^{n+\sum c} \right]_+ e_{0j} \right).
\]

When \( z \in B_j \cap D_{a_j} \) and \( a_j \in \Gamma_{jk} \) the calculation is similar to that in (5.9).
Lastly, we need to show that \( F_j Q_j^{-1} U \Phi \) satisfies the boundedness of \( S \) in (4.32) as \( z \to a_j \).

When \( z \in D_{a_j} \cap \Omega_j \) we have
\[
Q_j(z) F_j(\zeta(z))Q_j(z)^{-1}U(a_j, z) \Phi(z) M_j(z)
\]
\[
= \left( I_{\nu+1} - \frac{\zeta^{c_j}(z - a_k)^c \eta_{kj} f_{c_j}(\zeta(z)) e_{kj}}{[(z - a_j)^c]_B[k]} \right) U(a_j, z) \Phi(z) M_j(z)
\]
\[
= \Phi(z) + \left( \frac{\zeta^{c_j}(z - a_k)^c \eta_{kj} f_{c_j}(\zeta(z)) \zeta^{c_j}(z - a_k)^c}{W_j(z)[(z - a_j)^c]_B[k]} - \frac{E_k(z)(z - a_k)^c}{E_j(z)W_k(z)} \right) e_{j0}
\]
\[
= \Phi(z) + \frac{\zeta(z)^c (z - a_k)^c}{W_k(z)} \left( \frac{f_{c_j}(\zeta(z))}{E_j(z)\zeta^{c_j}(z)} \right) e_{j0}
\]
\[
= \Phi(z) + \frac{\zeta(z)^c (z - a_k)^c}{W_k(z)} \left( \frac{f_{c_j}(\zeta(z)) - \eta_{kj} \zeta^{c_j}(z)}{\zeta^{c_j}(z)} \right) e_{j0}.
\]

The 3rd equality is obtained by (1.19). Since \( f_{c_j}(\zeta) - e^{c_j} / \zeta^{c_j} \) is an entire function in \( \zeta \), the \((j, 0)\)th entry is bounded. The boundedness of the other entries are inherited from the boundedness of the corresponding entries in \( \Phi(z) \). By a similar argument, we also have \( Q_j(z) F_j(\zeta(z)) Q_j(z)^{-1} U(a_j, z) \Phi(z) M_k(z) = O(1) \) as \( z \to a_j, z \in \Omega_k \) and \( k \neq j \). This ends the proof of Lemma 5.2.

\[ \square \]

### 5.3 Construction of \( \mathcal{R} \) and \( H \)

To match the local parametrices obtained in Section 5.1 and 5.2 with the global parametrix obtained in Section 4.4 along \( \partial D_{a_j} \), we need to modify the global parametrix \( \Phi \) into \( \mathcal{R} \Phi \) with a rational function \( \mathcal{R} \) with poles at \( \{a_j\} \). This is called the “partial Schlesinger transform” [6], and it was used also for \( \nu = 1 \) [21].

From the asymptotic expansion of \( f_\nu(\zeta) \) in Appendix A the asymptotic expansion of \( F_j(\zeta) \) as \( \zeta \to \infty \) follows. We denote the coefficients in the expansion by \( \alpha_i(c_j) \) as below.
\[
F_j(\zeta) = \begin{cases} 
I_{\nu+1} + \sum_{i=1}^{\infty} \frac{\alpha_i(c_j)}{\zeta^i} e_{0j}, & \text{if } a_j \in \Gamma_{j0}, \\
I_{\nu+1} - \eta_{kj} \left( \sum_{i=1}^{\infty} \frac{\alpha_i(c_j)}{\zeta^i} \right) e_{kj}, & \text{if } a_j \in \Gamma_{jk}.
\end{cases}
\]

(5.20)

In fact \(\alpha_i(c_j)\) can be explicitly written by (A.2), see Appendix A.

Let \(F_j^{(m)}(\zeta)\) be the truncated asymptotic expansion of \(F_j(\zeta)\) given by

\[
F_j^{(m)}(\zeta) = \begin{cases} 
I_{\nu+1} + \sum_{i=1}^{m} \frac{\alpha_i(c_j)}{\zeta^i} e_{0j}, & \text{if } a_j \in \Gamma_{j0}, \\
I_{\nu+1} - \eta_{kj} \left( \sum_{i=1}^{m} \frac{\alpha_i(c_j)}{\zeta^i} \right) e_{kj}, & \text{if } a_j \in \Gamma_{jk}.
\end{cases}
\]

(5.21)

It follows that

\[
\hat{F}_j(\zeta) := F_j(\zeta) \left( F_j^{(m)}(\zeta) \right)^{-1} = I_{\nu+1} + \mathcal{O} \left( |\zeta|^{-m-1} \right) \quad \text{as } |\zeta| \to \infty.
\]

(5.22)

For a function \(f\) with pole singularity at \(z = \alpha\) let us define

\[
[f(z)]_{\text{sing}(a)} = \frac{1}{2\pi i} \oint_{a} \frac{f(s)}{z - s} ds,
\]

which represents the singular part of the Laurent expansion of \(f(z)\) at \(z = \alpha\). The integration contour circles around \(a\) such that \(a\) is the only singularity of the integrand inside the circle; especially \(z\) must be outside the circle.

**Lemma 5.3.** Let \(R(z)\) be a rational matrix function of size \(\nu + 1\) by \(\nu + 1\) whose \((p, q)\)th entry is given by \(r_{p, q}(z)\) and \(0 \leq p, q \leq \nu\), where

\[
\begin{align*}
\begin{cases}
\begin{align*}
r_{0,j}(z) &= \left[ \frac{\zeta(z)^{c_j}}{(z - a_j)^{c_j}} \sum_{i=1}^{m} \frac{\alpha_i(c_j)}{\zeta(z)^i} \right]_{\text{sing}(a_j)}, \\
r_{k,j}(z) &= -\eta_{kj} \left[ \frac{\zeta(z)^{c_j}}{(z - a_j)^{c_j}} |B[k]| \sum_{i=1}^{m} \frac{\alpha_i(c_j)}{\zeta(z)^i} \right]_{\text{sing}(a_j)}
\end{align*}
\end{cases}
\end{align*}
\]

(5.23)

For any \(p\) that belongs to the chain of \(a_j\), i.e. \(j \to k \to \ldots \to p \to \ldots\), we set \(r_{p, j}\) by the recursively applying the following relation:

\[
r_{p, j}(z) = [r_{p, k}(z)r_{k, j}(z)]_{\text{sing}(a_j)}.
\]

(5.24)

For all other entries we set

\[
r_{p, q}(z) = \delta_{pq},
\]

where \(\delta_{pq} = 1\) for \(p = q\) and \(\delta_{pq} = 0\) for \(p \neq q\). With the above definitions the matrix function \(H_j(z)\) given by

\[
H_j(z) := Q_j(z)^{-1} R(z) Q_j(z) \left( F_j^{(m)}(\zeta(z)) \right)^{-1}
\]

(5.25)
is holomorphic at \( a_j \). We note that \( r_{k,j}(z) \) has pole only at \( a_j \) and only for \( k \) that belongs to the chain of \( a_j \).

We have the following asymptotic behavior

\[
(5.26) \quad r_{0,j}(z) = \frac{\text{chain}(j)}{z - a_j} \left( 1 + O \left( \frac{1}{N} \right) \right)
\]

uniformly over a compact subset of \( \mathbb{C} \setminus D_{a_j} \), where the constant \( \text{chain}(j) \) is defined at (1.20).

For \( p \neq 0 \) and \( z \in \mathbb{C} \setminus D_{a_j} \), we have

\[
(5.27) \quad r_{p,j}(z) = O(N^C)
\]

for a fixed finite \( C \).

Proof. When \( a_j \in \Gamma_{j0} \), \( F_j^{(m)}(z) \) is given in (5.21). We have

\[
H_j(z) = \sum_{q \neq j} \left[ Q_j(z)^{-1} R(z) Q_j(z) \right]_{pq} e_{pq} + \left( \frac{z - a_j}{\zeta j z \sum c} \right) \left( \frac{r_{0,j}(z)}{r_{0,j}(z)} \right) e_{0j} + \left( \frac{r_{p,j}(z)}{r_{p,0}(z)} \right) e_{pj} - \sum_{p \notin \{0,j\}} \left( \frac{r_{j,0}(z)}{(z - a_j)^{c_j}} \sum \frac{\alpha_i(c_j)}{\zeta i} \right) e_{jj}.
\]

Let us discuss each term. In the 1st summation, since \( q \neq j \), \( r_{p,q}(z) \) in the summation are all holomorphic at \( a_j \) and it follows that the summation is holomorphic at \( a_j \). The 2nd term with \( e_{0j} \) becomes holomorphic exactly because of (5.23). The 3rd term with the summation vanishes because \( r_{p,j} = r_{p,0} = 0 \) for the corresponding \( p \)'s. The last term also vanishes because \( r_{j,0} = 0 \) by definition.
When $a_j \in \Gamma_{jk}$, we have

\begin{equation}
H_j(z) = \sum_{q \neq j}^{\nu} \left[ Q_j(z)^{-1} R(z) Q_j(z) \right]_{pq} e_{pq} \nonumber \\
+ \frac{\zeta c_j/2(z - a_k)^c}{[(z - a_j)^c_j/2] B[k]} \left[ [(z - a_j)^c_j] B[k] r_{0,j}(z) + r_{0,k}(z) \tilde{\eta}_{kj} \sum_{i=1}^{m} \alpha_i(c_j) \right] e_{0j} \nonumber \\
+ \left( \frac{\zeta c_j}{[z - a_j]^{c_j}} r_{j,k}(z) \tilde{\eta}_{kj} \sum_{i=1}^{m} \alpha_i(c_j) \right) e_{jj} \nonumber \\
+ \left( \frac{[(z - a_j)^c_j] B[k] r_{k,j}(z) + r_{0,k}(z) \tilde{\eta}_{kj} \sum_{i=1}^{m} \alpha_i(c_j) \right) e_{kj} \nonumber \\
+ \sum_{p \notin \{0,j,k\}} \left( \frac{[(z - a_j)^c_j] B[k] r_{p,j}(z) + \zeta c_j/2(z - a_k)^c/2 r_{p,k}(z) \tilde{\eta}_{kj} \sum_{i=1}^{m} \alpha_i(c_j \right) e_{pj} \right. 
\end{equation}

The 1st term with the summation is holomorphic by the similar argument as above. The term with $e_{jj}$ vanishes because $r_{j,k}(z) = 0$ for $j$ does not belong to the chain of $a_k$. The term with $e_{kj}$ is holomorphic at $a_j$ exactly by the definition (5.23). For the term with $e_{0j}$ to be holomorphic one obtains the following (recursive) relation:

\begin{equation}
R_{0,j}(z) = - \left[ R_{0,k}(z) \frac{\zeta c_j(z - a_k)^c \tilde{\eta}_{kj} \sum_{i=1}^{m} \alpha_i(c_j)}{[(z - a_j)^c_j] B[k]} \right] \text{sing}(a_j) \quad \text{when } a_j \in \Gamma_{jk}. 
\end{equation}

In the last term with the summation in (5.28) to be holomorphic we obtain

\begin{equation}
R_{p,j}(z) = - \left[ R_{p,k}(z) \frac{\zeta c_j(z - a_k)^c \tilde{\eta}_{kj} \sum_{i=1}^{m} \alpha_i(c_j)}{[(z - a_j)^c_j] B[k]} \right] \text{sing}(a_j) \quad \text{when } j \to k \to p \neq 0. 
\end{equation}

The above two relations combined with the definition of $r_{k,j}(z) \ (5.23)$, gives (5.24).

Now let us prove the asymptotic behaviors of $r_{0,j}(z)$ when $a_j \in \partial \Omega_0$. Using the linear approximation (5.2) we may write

\begin{equation}
r_{0,j}(z) = \frac{\zeta c_j \left( \sum_{i=1}^{m} \alpha_i(c_j) \right) \text{sing}(a_j)}{(z - a_j)^c_j \zeta(z)} + \frac{\zeta c_j \left( \sum_{i=2}^{m} \alpha_i(c_j) \right) \text{sing}(a_j)}{(z - a_j)^c_j \zeta(z)} 
\end{equation}

\begin{equation}
= \frac{N c_{j-1} \sum_{i \neq j}^{i} \alpha_i(c_j)}{\Gamma(c_j)(1 - (z - a_j)^{c_j})} \frac{1}{z - a_j} + \frac{1}{2 \pi i} \int_{a_j}^{s} \frac{\zeta(s)^{c_j} \left( \sum_{i=2}^{m} \alpha_i(c_j) \right) ds}{(s - a_j)^{c_j} \zeta(s) \ z - s} 
\end{equation}

\begin{equation}
= \frac{N c_{j-1} \sum_{i \neq j}^{i} \alpha_i(c_j)}{\Gamma(c_j)(1 - (z - a_j)^{c_j})} \frac{1}{z - a_j} + \left( 1 + O \left( \frac{1}{N} \right) \right), \quad z \notin D_{a_j}. 
\end{equation}
For the integration in the 2nd line we may choose the circular integration contour centered at $a_j$ with half the radius of $D_{a_j}$ such that $|\zeta(s)| > CN$ over the integration contour for some positive constant $C$. Then the integral is bounded by $O(N^{c_j-2})$ and we obtain the estimate.

Note that the coefficient of $1/(z - a_j)$ in the 1st term of the 2nd line in (5.30) comes from the evaluation of $\zeta(z)^{c_j} z^{\sum c_j / (z - a_j)^{c_j}}$ at $z = a_j$. We use that

$$\zeta(z)^{c_j} = N^{c_j} (1 - |a_j|^2)^{c_j} \frac{(z - a_j)^{c_j}}{z^{c_j}} (1 + O(z - a_j))$$

to determine the exact branch of the exponents.

By a similar consideration, we obtain

$$r_{k,j}(z) = -\frac{N^{c_j-1} \bar{\eta}_{kj}(a_j - a_k)^{c_k} |a_k - a_j|^{2c_j}}{\Gamma(c_j) \left[ (a_k - a_j)^{c_j} \right]_{B[k]} (\bar{a}_k - \bar{a}_j) z - a_j} \frac{1}{(z - a_j)^{c_j} (1 + O(z - a_j))}, \quad a_j \in \Gamma_{jk}, \ k \neq 0,$$

where we have used the following identity to evaluate the correct branch of the leading coefficient.

$$\zeta^{c_j} = N^{c_j} |a_k - a_j|^{2c_j} \frac{[(z - a_j)^{c_j} \left[ (a_k - a_j)^{c_j} \right]_{B[k]} (1 + O(z - a_j)).$$

Here $[(a_k - a_j)^{c_j} \left[ B[k] \right]$ means the evaluation of $[(z - a_j)^{c_j} \left[ B[k] \right]$ at $z = a_k$.

Finally we estimate the $r_{0,j}$ when $j \to k$ and $k \neq 0$ using the relation (5.29).

$$r_{0,j}(z) = -r_{0,k}(z) \frac{\zeta^{c_j} (z - a_k)^{c_k} \bar{\eta}_{kj} \alpha_1(c_j)}{[(z - a_j)^{c_j} \left[ B[k] \right]} \frac{\zeta^{c_j} (z - a_k)^{c_k} \bar{\eta}_{kj} \sum_{i=2}^{m} \alpha_i(c_j)}{\left[ (z - a_j)^{c_j} \right]_{B[k]} \left[ (a_k - a_j)^{c_j} \right]_{B[k]} (\bar{a}_k - \bar{a}_j) z - a_j} -\frac{1}{z - a_j} + O(\|r_{0,k}\|_\infty N^{c_j-2}),$$

where $\|r_{0,k}\|_\infty$ is the norm taken over $D_{a_j}$ and the error bound is uniformly over a compact subset in $\mathbb{C} \setminus D_{a_j}$. This allows us to define the constant $\text{chain}(j)$ by

$$N^{\mu_j} \lim_{N \to \infty} \frac{r_{0,j}(z)}{N^{\mu_j}} = \frac{\text{chain}(j)}{z - a_j}, \quad \text{for all } j = 1, \ldots, \nu,$$

where $\mu_j$ is fixed such that the limit is non-trivial. From the above two equations we obtain the recurrence relation

$$\mu_j = \mu_k + c_j - 1$$

and

$$\text{chain}(j) = -\frac{\text{chain}(k)}{a_j - a_k} \frac{N^{c_j-1} \bar{\eta}_{kj}(a_j - a_k)^{c_k} |a_k - a_j|^{2c_j} \Gamma(c_j) [(a_k - a_j)^{c_j}]_{B[k]} (\bar{a}_k - \bar{a}_j)}{\left[ (a_k - a_j)^{c_j} \right]_{B[k]} (\bar{a}_k - \bar{a}_j)}.$$
For a given chain $j = k_s \rightarrow k_{s-1} \rightarrow \cdots \rightarrow k_1 \rightarrow 0$ the above relation provides the recurrence relation that can be solved with the initial condition given by (5.30) as below.

$$
\mu_j = \sum_{i=1}^{s} (c_{k_i} - 1),
$$

(5.32)

$$
\text{chain}(j) = \frac{1 + \sum_{i \neq k_1} c_i N \sum_{i=1}^{s} (c_{k_i} - 1)}{\Gamma(c_{k_1})(1 - |a_{k_1}|^2)^{1 - c_{k_1}}}
\times \prod_{i=1}^{s-1} \frac{\tilde{\eta}_{k_i,k_i+1}(a_{k_{i+1}} - a_{k_i})^{c_{k_i}} |a_{k_i} - a_{k_{i+1}}|^2}{\Gamma(c_{k_i+1})[(a_{k_i} - a_{k_{i+1}})^{c_{k_i+1}} B[k_i]|a_{k_i} - a_{k_{i+1}}|^2].}
$$

Using the identity (1.15) about the branches of multivalued functions the above expression of $\text{chain}(j)$ becomes the original definition at (1.20). Note that $s$ is the level of $a_j$ as defined in Definition 2.4 and, if $s = 1$, the product part is one.

For other $r_{p,q}(z)$'s with $p \neq 0$ similar estimates as in (5.30) and (5.31) can be made and shown to be bounded by $O(N^C)$ with a finite $C$. □

From the above lemma we realize that all $r_{p,q}$'s grows (or decays) algebraically in $N$ away from $a_q$. Therefore we have

$$
\mathcal{R}(z) = O\left( N^C \right), \quad \text{when } z \in \partial D_{a_j}
$$

for some fixed finite $C$.

When $z \in \partial D_{a_j}$, by the definition of $H_j(z)$ in (5.25) and the boundedness of $\mathcal{R}(z)$, we have

$$
H_j(z) = O\left( N^{C'} \right),
$$

for some finite $C'$.

### 6 Strong asymptotics

Combining all the constructions of the (improved) global and the local parametrices we define $S^\infty(z)$ by

(6.1)

$$
S^\infty(z) := \begin{cases} 
\mathcal{R}(z)\Phi(z), & z \notin \cup_{j=1}^{\nu} D_{a_j}, \\
Q_j(z)H_j(z)F_j(\zeta(z))Q_j(z)^{-1}U(a_j,z)\Phi(z), & z \in D_{a_j}, \quad j = 1, \ldots, \nu.
\end{cases}
$$

This will be the strong asymptotics of $S$ (4.32) as $N \to \infty$ as we prove now.

#### 6.1 Error analysis

We define the error matrix by

(6.2)

$$
\mathcal{E}(z) := S^\infty(z)S(z)^{-1}.
$$
Lemma 6.1. Let $\mathcal{E}(z)$ be given above. Then $\mathcal{E}(z) = I_{\nu+1} + \mathcal{O} \left( 1/N^\infty \right)$ as $N \to \infty$ uniformly over a compact set of the corresponding region. Here the error bound $O(1/N^\infty)$ stands for $O(1/N^m)$ for an arbitrary $m > 0$.

Proof. When $z \in \partial D_{a_j}$, we have

$$\mathcal{E}_+(z)\mathcal{E}_-(z)^{-1} = Q_j(z)H_j(z)F_j(\zeta(z))Q_j(z)^{-1}U(a_j, z)R(z)^{-1}$$

$$= \mathcal{R}(z)Q_j(z) \left( F_j^m(\zeta(z)) \right)^{-1}F_j(\zeta(z))Q_j(z)^{-1}U(a_j, z)R(z)^{-1}$$

$$= \mathcal{R}(z)Q_j(z)\hat{F}_j(\zeta)Q_j(z)^{-1}(I_{\nu+1} + \mathcal{O}(e^{-CN})) R(z)^{-1}$$

$$= \mathcal{R}(z)(I_{\nu+1} + \mathcal{O}(N^C)) R(z)^{-1}$$

for some $C > 0$. The 2nd equality is obtained by (5.25). The 3rd equality is obtained by (5.22) and the fact that, since $E_j(z)$ is dominant near $a_j$, $\|U - I_{\nu+1}\|_{\infty}$ is exponentially small as $N$ grows on $\partial D_{a_j}$ from (5.4) and (5.13). By Lemma 5.3, $R(z)$ can be written as an upper triangular matrix by a reordering of $\{a_i\}_{i=1}^\nu$ such that the chain $j \to k$ always satisfies $j > k$, for instance by reordering $\{a_i\}'s$ by their levels. Then $R(z) - I_{\nu+1}$ is a nilpotent matrix and we have

$$\mathcal{R}(z)^{-1} = I_{\nu+1} + \sum_{i=1}^{\nu-1} (-1)^i (\mathcal{R}(z) - I_{\nu+1})^i = I_{\nu+1} + \mathcal{O}(N^C)$$

for some fixed finite $C$ that does not depend on $m$ (5.33). It means that, by increasing $m$, we can make $\|\mathcal{E}_+(z)\mathcal{E}_-(z)^{-1} - I_{\nu+1}\|_{\infty}$ on $\partial D_{a_j}$ as small as we want.

When $z$ is on all the jump contours of $S\Phi^{-1}$ and $z \notin \cup_{j=1}^\nu D_{a_j}$, we have

$$\mathcal{E}_+(z)\mathcal{E}_-(z)^{-1} = (S^\infty(z)S(z)^{-1})_+ (S^\infty(z)S(z)^{-1})_-^{-1}$$

$$= \mathcal{R}(z) (S(z)\Phi(z)^{-1})_+ (S(z)\Phi(z)^{-1})_-^{-1} \mathcal{R}(z)^{-1}. $$

By the jump conditions of $S\Phi^{-1}$ in (4.35),(4.36),(4.37) and (4.38), $\|\mathcal{R}(z)^{-1} \mathcal{R}(z)\|_{\infty}$ is exponentially small as $N$ grows we have

$$\mathcal{E}_+(z)\mathcal{E}_-(z)^{-1} = I_{\nu+1} + \mathcal{O}(e^{-CN}), \quad \text{when } z \notin \cup_{j=1}^\nu D_{a_j}$$

for some $C > 0$.

By Lemma 5.1 and Lemma 5.2, $S(z)\Phi(z)^{-1}$ and $S^\infty(z)\Phi(z)^{-1}$ have the same jump conditions in $D_{a_j}$. Therefore, $\mathcal{E}$ does not have any jump in $D_{a_j}$.

From the boundedness of $S$ in (4.32) and the definition of $S^\infty$ in (6.1), as $z \to a_j$ from $\Omega_i$, both $S(z)M_i(z) = \mathcal{O}(1)$ and $S^\infty(z)M_i(z) = \mathcal{O}(1)$ are bounded for $i \in \{0, \ldots, \nu\}$. Therefore $\mathcal{E} = S^\infty M_i(SM_i)^{-1}$ is bounded $z \to a_j$ in $\Omega_i$.

Since $\mathcal{E}_+(z)\mathcal{E}_-(z)^{-1} = I_{\nu+1} + \mathcal{O}(1/N^\infty)$ when $z \in \partial D_{a_j}$ and $j = 1, \ldots, \nu$ and the other jump conditions of $\mathcal{E}$ in (6.3) are exponentially small in $N$ away from $\partial D_{a_j}$, by the small norm theorem (e.g. Theorem 7.171 in [9], [10] or [25]), we obtain $\mathcal{E}(z) = I_{\nu+1} + \mathcal{O}(1/N^\infty)$. □
By the definition of $\mathcal{E}(z)$ in (6.2) and the above lemma, we have

\[(6.4) \quad S(z) = (I_{\nu+1} + O\left(1/N^{\infty}\right)) S^\infty(z).\]

### 6.2 Proof of Theorem 1.5

**Lemma 6.2.** On $z \not\in D_{a_j}$ we have

\[(6.5) \quad \left[\left(I_{\nu+1} + O\left(1/N^{\infty}\right)\right) \mathcal{R}(z)\right]_{1st \ row} = \left(1 + O\left(1/N^{\infty}\right)\right) [\mathcal{R}(z)]_{1st \ row}.\]

**Proof.** Recall that $O(1/N^{\infty})$ stands for $O(1/N^m)$ for an arbitrary $m > 0$. From Lemma 5.3 all the entries of $\mathcal{R}(z)$ is growing at most polynomially in $N$ and, therefore, we have $(I + O(1/N^{\infty})) \mathcal{R}(z) = \mathcal{R}(z) + O(1/N^{\infty})$. Furthermore, by (5.26) in Lemma 5.3, all the entries in the 1st row of $\mathcal{R}(z)$ are nonvanishing away from $\{a_j\}_{j=1}^\nu$. This leads to $[O(1/N^{\infty})]_{1st \ row} = (1+O(1/N^{\infty})) [\mathcal{R}(z)]_{1st \ row}$. \[\square\]

**Corollary 6.3.** On $z \in D_{a_j}$ we have

\[(6.6) \quad \left[\left(I_{\nu+1} + O\left(1/N^{\infty}\right)\right) Q_j(z) H_j(z)\right]_{1st \ row} = \left(1 + O\left(1/N^{\infty}\right)\right) [Q_j(z) H_j(z)]_{1st \ row}.\]

**Proof.**

LHS of (6.6) = \[
\left[\left(I_{\nu+1} + O\left(1/N^{\infty}\right)\right) \mathcal{R}(z) Q_j(z) \left(\mathcal{F}_j^{(m)}(\zeta(z))^{-1}\right)\right]_{1st \ row}
\]
= \[
\left[\left(I_{\nu+1} + O\left(1/N^{\infty}\right)\right) \mathcal{R}(z)\right]_{1st \ row} Q_j(z) \left(\mathcal{F}_j^{(m)}(\zeta(z))^{-1}\right)
\]
= \[
\left(1 + O\left(1/N^{\infty}\right)\right) [\mathcal{R}(z)]_{1st \ row} Q_j(z) \left(\mathcal{F}_j^{(m)}(\zeta(z))^{-1}\right)
\]
= RHS of (6.6),

where the 1st equality is obtained by the definition of $H_j$ in (5.25) and the 3rd equality is obtained by the above lemma. \[\square\]

In the following proofs, we will use the facts

\[(6.7) \quad [\Phi(z) M_j(z) G_j(z)^{-1}]_{1st \ column}

= \left(\begin{array}{c}
z^{m+c} - \frac{(z-a_1)^{c_1} E_1(z)}{W_1(z)}, \ldots, -\frac{(z-a_\nu)^{c_\nu} E_\nu(z)}{W_\nu(z)}
\end{array}\right)^T, \text{ when } z \in \Omega_j,

where $j = 0, 1, \ldots, \nu$.\]
\[ Y(z) = \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} S(z)G_0(z)^{-1} & 1 \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Psi_0(z) \end{bmatrix} \]

Here the error bound is uniformly over a compact subset of \( \Omega_0 \) therefore one can always choose \( U \), the neighbourhood of \( \bigcup_j \Gamma_{j0} \), small enough such that the compact subset in question sits in \( \Omega_0 \setminus U \).

For \( z \in \Omega_0 \setminus D_{a_j} \), we have

\[ p_n(z) = [Y(z)]_{11} = [(I_{\nu+1} + \mathcal{O}(1/N^{\infty})) S^{\infty}(z)G_0(z)^{-1}]_{11} \]

Using (4.2), (4.7), (4.23) and (6.4), when \( z \in \Omega_j \), we have

\[ Y(z) = \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} S(z)M_j(z)G_j(z)^{-1} & 1 \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Psi_j(z) \end{bmatrix} \]

\[ \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ \widetilde{W}(z) \psi_1(z) - W_j(z)\psi_j(z) \end{bmatrix} . \]

For \( z \in \Omega_j \setminus D_{a_j} \), we have

\[ p_n(z) = [Y(z)]_{11} = [(I_{\nu+1} + \mathcal{O}(1/N^{\infty})) S^{\infty}(z)M_j(z)G_j(z)^{-1}]_{11} \]

where \( j = k_s \to k_{s-1} \to \cdots \to k_1 \to 0 \). The 2nd equality is obtained by (6.9). The 3rd equality is obtained by (6.1). The 4th equality is obtained by Lemma 6.2.
and (6.7). The 5th equality is obtained by the fact that $E_j(z)$ is dominant in $\Omega_j$. The last equality is obtained by (5.26).

Using (4.2), (4.7), (4.23) and (6.4), when $z \in \Omega_0 \cap U$, we have

\[
Y(z) = \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} S(z)M_0(z)G_0(z)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Psi_0(z) \end{bmatrix} \\
= \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} \left( I_{\nu+1} + O\left( \frac{1}{N^{\infty}} \right) \right) S^{\infty}(z)M_0(z)G_0(z)^{-1} \times \begin{bmatrix} 1 & 0 \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Psi_0(z) \end{bmatrix}. 
\]

(6.10)

For $z \in \Omega_0 \cap U$ and near $\Gamma_{j0}$ we have

\[
p_n(z) = [Y(z)]_{11} = \left[ (I_{\nu+1} + O\left( 1/N^{\infty} \right)) S^{\infty}(z)M_0(z)G_0(z)^{-1} \right]_{11} \\
= \left[ (I_{\nu+1} + O\left( 1/N^{\infty} \right)) R(z) \right]_{\text{1st row}} \left[ \Phi(z)M_0(z)G_0(z)^{-1} \right]_{\text{1st column}} \\
= z^n \left( \sum \frac{(z-a_j)^c r_{0,j}(z) E_j(z)}{W_j(z)} - \sum_{i \neq j} \frac{(z-a_i)^c r_{0,i}(z) E_i(z)}{W_i(z)} \frac{z^n}{z^n} \right) \times \left( 1 + O\left( \frac{1}{N^{\infty}} \right) \right) \\
= z^{n+\sum c} W(z) \left( 1 + O\left( \frac{1}{N^{\infty}} \right) \right) - \frac{E_j(z)(z-a_j)^c}{W_j(z)} \frac{z^n}{z^n} \left. \right|_{z \to a_j} \left( 1 + O\left( \frac{1}{N} \right) \right). 
\]

The 2nd equality is obtained by (6.10). The 3rd equality is obtained by (6.1). The 4th equality is obtained by Lemma 6.2 and (6.7). The last equality is obtained by (5.26) and the fact that $z^n$ and $E_j(z)$ are the most dominant in the vicinity of $\Gamma_{j0}$. A similar calculation can be done for $z \in \Omega_j$ and near $\Gamma_{j0}$.

Similar to the case of $z \in \Omega_j \setminus D_{a_j}$, when $z \in \Omega_j$ and near $\Gamma_{jk}$ we have

\[
p_n(z) = E_j(z) \left( \frac{z^n+\sum c}{W_j(z)E_j(z)} - \sum_{i=1}^{\nu} \frac{r_{0,i}(z)(z-a_i)^c E_i(z)}{W_i(z)E_j(z)} \right) \left( 1 + O\left( \frac{1}{N^{\infty}} \right) \right) \\
= -\frac{r_{0,j}(z)E_j(z)(z-a_j)^c}{W_j(z)} \left( 1 + O\left( \frac{1}{N^{\infty}} \right) \right) \\
- \frac{r_{0,k}(z)E_k(z)(z-a_k)^c}{W_k(z)} \left( 1 + O\left( \frac{1}{N^{\infty}} \right) \right) \\
= -\frac{E_j(z)(z-a_j)^c \text{chain}(j)}{W_j(z)} \left( 1 + O\left( \frac{1}{N} \right) \right) \\
- \frac{E_k(z)(z-a_k)^c \text{chain}(k)}{W_k(z)} \left( 1 + O\left( \frac{1}{N} \right) \right). 
\]
The 2nd equality is obtained by the fact that \( E_j(z) \) and \( E_k(z) \) are dominant in the vicinity of \( \Gamma_{jk} \). The last equality is obtained by (5.26). A similar calculation can be done for \( z \in \Omega_k \) and near \( \Gamma_{jk} \).

This ends the proof of Theorem 1.5. \( \square \)

6.3 Proof of Theorem 1.6

Proof. (Proof of Theorem 1.6) For \( z \in D_{a_j} \cap \Omega_j \) and \( a_j \in \Gamma_{j0} \) we have

\[
p_n(z) = [Y(z)]_{11} = \left[ (I_{\nu+1} + O(1/N^\infty)) S^\infty(z) M_j(z) G_j(z)^{-1} \right]_{11}
\]

\[
= \left[ (I_{\nu+1} + O(1/N^\infty)) Q_j(z) H_j(z) \right]_{1st \ row}
\]

\[
\times \left[ F_j(\zeta(z)) Q_j(z)^{-1} U(a_j, z) \Phi(z) M_j(z) G_j(z)^{-1} \right]_{1st \ column}
\]

\[
= \left( I_{\nu+1} + O(1/N^\infty) \right) \left[ Q_j(z) H_j(z) \right]_{1st \ row}
\]

\[
\times F_j(\zeta(z)) Q_j(z)^{-1} U(a_j, z) \left[ \Phi(z) M_j(z) G_j(z)^{-1} \right]_{1st \ column}.
\]

The 2nd equality is obtained by (6.9). The 3rd equality is obtained by (6.1). The 4th equality is obtained by Lemma 6.2 and Corollary 6.3. Moreover, by the definition of \( H_j \) in (5.25), we have

\[
\left[ H_j(z) \right]_{1st \ row} = \sum_{q \neq j} \left[ Q_j(z)^{-1} R(z) Q_j(z) \right]_{0q} e_{0q}
\]

\[
+ \left( \frac{(z - a_j)^{c_j} r_{0,j}(z)}{\zeta c_j z \sum c} - \sum_{i=1}^m \frac{\alpha_i(c_j)}{\zeta^i} \right) e_{0j}.
\]

Above, by Lemma 5.3, \( r_{0,q} \) grows (or decays) algebraically in \( N \) away from \( a_q \) for \( q \neq j \). Let \( h_j(z) \) be defined by

\[
h_j(z) = \frac{(z - a_j)^{c_j}}{\zeta c_j z \sum c} r_{0,j}(z) - \sum_{i=1}^m \frac{\alpha_i(c_j)}{\zeta^i}
\]

\[
= \frac{(z - a_j)^{c_j}}{\zeta c_j z \sum c} \frac{N^{c_j - 1} a_j^{1 + \sum_{i \neq j} c_i}}{\Gamma(c_j)(1 - |a_j|^2)^{1-c_j}} \frac{1}{z - a_j} \left( 1 + O \left( \frac{1}{N} \right) \right) - \sum_{i=1}^m \frac{\alpha_i(c_j)}{\zeta^i}
\]

\[
= O \left( \frac{1}{N} \right), \quad z \in \partial D_{a_j},
\]
where we used (5.30) at the second equality. Since $h_j$ is holomorphic in $D_{a_j}$, the above bound holds in $D_{a_j}$. Therefore,

$$p_n(z) = \left( \frac{z^{n+\sum c}}{W_j(z)} - \frac{z^{\sum c} \zeta^{c_j} E_j(z)}{W_j(z)} \left( f_{c_j}(z) + h_j(z) \right) \right) \left( 1 + \mathcal{O}\left( \frac{1}{N^\infty} \right) \right)$$

$$= \frac{z^{n+\sum c}}{W_j(z)} \left( 1 + \mathcal{O}\left( \frac{1}{N^\infty} \right) \right) - \frac{z^{\sum c} \zeta^{c_j}}{W_j(z) e^\xi} \left( f_{c_j}(\zeta) + \mathcal{O}\left( \frac{1}{N} \right) \right) \cdot$$

The 1st equality is obtained by the facts that $r_{0,q}$ grows (or decays) algebraically in $N$ away from $a_q$ for $q \neq j$ and $E_j(z)$ is dominant in $\Omega_j$. The 2nd equality is obtained by the identity $z^n/E_j(z) = e^{\xi}$ from the definition of $\zeta$ in (5.1) and the estimate in (6.11). A similar calculation can be done for $z \in D_{a_j} \cap \Omega_0$ and $a_j \in \Gamma_{j0}$.

For $z \in D_{a_j} \cap \Omega_j$ and $a_j \in \Gamma_{jk}$ we have

$$p_n(z) = [Y(z)]_{11} = \left( [I_{\nu+1} + \mathcal{O}\left( 1/N^\infty \right) ]S^\infty(z)M_j(z)G_j(z)^{-1} \right)_{11}$$

$$= \left( [I_{\nu+1} + \mathcal{O}\left( 1/N^\infty \right) ]Q_j(z)H_j(z) \right)_{1 \text{st row}}$$

$$\times \left( F_j(\zeta(z))Q_j(z)^{-1}U(a_j, z)\Phi(z)M_j(z)G_j(z)^{-1} \right)_{1 \text{st column}}$$

$$= \left( [I_{\nu+1} + \mathcal{O}\left( 1/N^\infty \right) ][Q_j(z)H_j(z)]_{1 \text{st row}}$$

$$\times \left( F_j(\zeta(z))Q_j(z)^{-1}U(a_j, z)\left[ \Phi(z)M_j(z)G_j(z)^{-1} \right] \right)_{1 \text{st column}} \cdot$$

The 2nd equality is obtained by (6.9). The 3rd equality is obtained by (6.1). The 4th equality is obtained by Lemma 6.2 and Corollary 6.3. Moreover, by the definition of $H_j$ in (5.25) and the relation in (5.24), we have

$$\left[ H_j(z) \right]_{1 \text{st row}} = \sum_{q \neq j}^{\nu} [Q_j(z)^{-1}R(z)Q_j(z)]_{0q} e_{0q}$$

$$+ \frac{\zeta^{c_j}/2(z - a_k)^{c_k}/2 r_{0,k}(z)}{[(z - a_j)^{c_j/2}]_{B[k]}} \left( \frac{[z - a_j]^{c_j} B[k] r_{k,j}(z)}{\zeta^{c_j}(z - a_k)^{c_k}} + \tilde{\eta}_{kj} \sum_{i=1}^{m} \frac{\alpha_i(c_j)}{\zeta^i} \right) e_{0j}.$$
\[ p_n(z) = -\left( \frac{r_{0,k}(z)E_k(z)(z-a_k)^{c_k}}{W_k(z)} \right) \left( 1 + O\left( \frac{1}{N^\infty} \right) \right) \]
\[ - \left( \frac{r_{0,k}(z)E_j(z)\eta_{k,j} (-f_{c_j}(\zeta) + h_j(z)) \zeta^{c_j}(z-a_k)^{c_k}}{W_j(z)[(z-a_j)^{c_j}]B_{k,j}(z-a_j)^{-c_j}} \right) \left( 1 + O\left( \frac{1}{N^\infty} \right) \right) \]
\[ = -E_k(z)r_{0,k}(z)(z-a_k)^{c_k} \]
\[ \times \left( \frac{1}{W_k(z)} \left( 1 + O\left( \frac{1}{N^\infty} \right) \right) \right) - \frac{\zeta^{c_j} (f_{c_j}(\zeta) + O(1/N))}{W_k(z)e^{\zeta}} \]
\[ = -\frac{E_k(z)(z-a_k)^{c_k}}{z-a_k} \text{chain}(k) \frac{\zeta^{c_j}}{W_k(z)e^{\zeta}} \left( \frac{e^\zeta}{\zeta^{c_j}} - f_{c_j}(\zeta) + O\left( \frac{1}{N} \right) \right). \]

The 1st equality is obtained by the facts that \(r_{0,q}\) grows (or decays) algebraically in \(N\) away from \(a_q\) for \(q \neq j\) and \(E_j(z)\) is dominant in \(\Omega_j\). The 2nd equality is obtained by (1.19), the identity \(E_k(z)/E_j(z) = e^{\zeta}\) from the definition of \(\zeta\) in (5.11) and the estimate in (6.12). A similar calculation can be done for \(z \in D_{a_j} \cap \Omega_k\) and \(a_j \in \Gamma_{jk}\).

This ends the proof of Theorem 1.6. \qed

**Appendix: \(f_c(\zeta)\) and its properties**

Let us define \(f_c(\zeta)\) by the two conditions, \(f_c(\zeta) \to 0\) as \(|\zeta| \to 0\) and \(e^\zeta/\zeta^c - f_c(\zeta)\) is entire. The integral representation of \(f_c(\zeta)\) can be written by

\[ f_c(\zeta) = -\frac{1}{2\pi i} \int_{\mathcal{L}} \frac{e^s}{s^c(s-\zeta)} ds, \quad \zeta \notin \mathbb{R} \cup \{0\}. \]

The integration contour \(\mathcal{L}\) is enclosing the negative real axis counterclockwise from \(-\infty - i\epsilon\) to \(-\infty + i\epsilon\) for an infinitesimal \(\epsilon > 0\) such that \(\zeta\) is on the other side of \(\mathcal{L}\) from the negative real axis. When \(c\) is a positive integer \(\zeta^c f_c(\zeta)\) is exactly the first \(c\) terms in the Taylor expansion of \(\exp(\zeta)\). We take the principal branch for \(\zeta^c\).

As \(|\zeta| \to \infty\) we have the expansion

\[ f_c(\zeta) = \sum_{i=1}^{\infty} \frac{\alpha_i(c)}{\zeta^i} \quad \text{where} \quad \alpha_i(c) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{s^{i-1}e^s}{s^{c_j}} ds = \frac{\sin(c\pi)\Gamma(i-c)}{\pi(-1)^{i-1}}. \]

We also note that \(\alpha_1(c) = 1/\Gamma(c)\). When \(c\) is integer we notice that \(\alpha_i = 0\) when \(i > c\) and, therefore, \(f_c(\zeta)\) is written in terms of a finite truncation of the Taylor series of the exponential function.

Let us show that \(e^\zeta/\zeta^c - f_c(\zeta)\) is an entire function in \(\zeta\) as follows.

\[ \frac{e^\zeta}{\zeta^c} - f_c(\zeta) = \frac{1}{2\pi i} \int_{\zeta} \frac{e^s}{s^c(s-\zeta)} ds + \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{e^s}{s^c(s-\zeta)} ds, \]

where the first integration contour is the small circle around $\zeta$ directed counter-clockwise. The two integration contours can be deformed into a single contour that encloses the negative real axis and $\zeta$, hence the resulting integral has the analytic continuation onto $\zeta \in \mathbb{R}^- \cup \{0\}$.

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