Algebraic Geometry in Discrete Quantum Integrable Model and Baxter’s T-Q Relation

Shao-shiung Lin
Department of Mathematics, Taiwan University
Taipei, Taiwan
(email: lin@math.ntu.edu.tw)

Shi-shyr Roan
Institute of Mathematics, Academia Sinica
Taipei, Taiwan
(email: maroan@ccvax.sinica.edu.tw)

Abstract

We study the diagonalization problem of certain discrete quantum integrable models by the method of Baxter’s T-Q relation from the algebraic geometry aspect. Among those the Hofstadter type model (with the rational magnetic flux), discrete quantum pendulum and discrete sine-Gordon model are our main concern in this report. By the quantum inverse scattering method, the Baxter’s T-Q relation is formulated on the associated spectral curve, a high genus Riemann surface in general, arisen from the study of the spectrum problem of the system. In the case of degenerated spectral curve where the spectral variables lie on rational curves, we obtain the complete and explicit solution of the T-Q polynomial equation associated to the model, and the intimate relation between the Baxter’s T-Q relation and algebraic Bethe Ansatz is clearly revealed. The algebraic geometry of a general spectral curve attached to the model and certain qualitative properties of solutions of the Baxter’s T-Q relation are discussed incorporating the physical consideration.

PACS: 02.10.Rn, 03.65.Fd, 5.30, 75.10J.
Key words: Baxter T − Q relation, Hofstadter model, Discrete quantum pendulum, Discrete sine − Gordon model.

1Presented at the 10th Colloquium “Quantum groups and integrable systems”, Prague, 21-23 June, 2001.
2Supported in part by NSC grant 89-2115-M-001-037 of Taiwan.
1 Introduction

This report is a review based on our results in [10, 11], the relevant work on the subject can be found in the references therein, especially on the connection with [3, 5, 8]. We consider the diagonalization problem of certain discrete quantum integrable chains by the method of Baxter’s $T$-$Q$ relation [2] within the context of quantum inverse scattering method [4, 9]. The models we are going to discuss here are the Hofstadter type model (with the rational magnetic flux), discrete quantum pendulum and discrete (quantum) sine-Gordon model. Our approach to the eigenvalue problem of these Hamiltonians is from the algebraic geometry aspect by taking the algebraic structure of rational functions of the underlying spectral curve into account, then formulating the problem into the Baxter’s $T$-$Q$ relation [2]; a method often not only more general, but also mathematically tractable than the usual Bethe Ansatz technique. This is indeed the case when the spectral curve degenerates into rational curves for these models, where the complete solutions of Baxter’s $T$-$Q$ polynomial equation are obtained, and both the qualitative and quantitative features of eigenvalues and eigenfunctions have been studied in details incorporating the physical consideration. The method starts with the transfer matrix technique by encoding the Hamiltonian into the transfer matrix of a fixed finite size $L$, which arises from a Yang-Baxter solution for the $R$-matrix of XXZ model. The Baxter’s $T$-$Q$ relation is constructed from the action of transfer matrices on the Baxter vacuum state over the spectral curve. The Hofstadter type Hamiltonian is on $L = 3$, and the discrete quantum pendulum and sine-Gordon model are on $L = 4$, (see (I) (II) (III) of Sect 2). The mathematical treatment to solutions of these Baxter’s $T$-$Q$ polynomial equations takes explicit advantage of special features only presented in $L = 3, 4$. For a general spectral curve, it is a Riemann surface of a very high genus, however with a close link with elliptic curves for both cases. The qualitative analysis in the geometry of spectral curves has been made in respect to Baxter’s $T$-$Q$ relation. When the spectral curve is totally degenerated into rational curves, the Baxter’s $T$-$Q$ relation is descended to a polynomial equation for an arbitrary size $L$. In this situation, though the geometry of the spectral curve becomes trivial, the determination of the solutions inevitably requires the subtle analysis of Baxter vacuum state to extract the essential data for polynomials involved, also the necessary algebraic study of a certain “over-determined” system of $q$-difference equations for a root of unity $q^N = 1$, which has still been a difficult problem for an arbitrary finite size $L$. For $L = 3, 4$ in accordance with models concerning us here, a sound mathematical derivation of solutions of the Baxter’s $T$-$Q$ polynomial equation has been carried out in [10, 11]. The relationship between the Baxter’s $T$-$Q$ polynomial equation and algebraic Bethe Ansatz becomes clearer in this scheme.

Notations. In this article, $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ will denote the ring of integers, real, complex numbers respectively, $\mathbb{N} = \mathbb{Z}_{> 0}$, $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, and $i = \sqrt{-1}$. For the $N$-dimensional vector space $\mathbb{C}^N$, we will present a vector $v \in \mathbb{C}^N$ as a sequence of coordinates, $(v_k \mid k \in \mathbb{Z})$, with the $N$-periodic condition, $v_k = v_{k + N}$, i.e., $v = (v_k)_{k \in \mathbb{Z}}$. The standard basis of $\mathbb{C}^N$ will be denoted by $|k\rangle$, with the dual basis of $\mathbb{C}^N$ by $\langle k|$ for $k \in \mathbb{Z}_N$. For a positive integers $n$, we denote $\otimes \mathbb{C}^N$ the tensor product of $n$-copies of the vector space $\mathbb{C}^N$. We use the notation of $q$-shifted factorials,

$$(a; q)_0 = 1, \quad (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n \in \mathbb{N}.$$ 

2 Baxter’s $T$-$Q$ Relation on the Spectral Curve

Throughout this report, $N$ will always denote an odd positive integer with $M = \lfloor \frac{N}{2} \rfloor$,

$$N = 2M + 1, \quad M \geq 1.$$
and \( \omega \) is a primitive \( N \)-th root of unity, \( q := \omega^{\frac{1}{2}} \) with \( q^N = 1 \), i.e., \( q = \omega^{M+1} \).

Let \( Z, X \) be a pair of operators generating the Weyl algebra and \( Y := ZX \) with the Weyl commutation relation and the \( N \)-th power identity,

\[
ZX = \omega XZ, \quad XY = \omega^{-1} YX, \quad YZ = \omega^{-1} ZY; \quad Z^N = X^N = Y^N = I.
\]

The above relation is related to the Yang-Baxter solution for the study of sine-Gordon lattice model by using the Weyl operators

It satisfies the Yang-Baxter relation,

\[
[\hbar R, \hbar C]_{\omega} = [\hbar R, \hbar C]_{\omega, 1} = \omega^{c} \hbar C, \quad \hbar C = \hbar C^* = \hbar C^{-1}.
\]

Using \( X, Y, Z \) and the identity operator, there is a solution of the Yang-Baxter equation for a slightly modified \( R \)-matrix of the XXZ-model \([5]\) with the \( L \)-operator given by the following matrix of the operator-valued entries acting on the quantum space \( C^N \), depending on the parameter \( \hbar = [a : b : c : d] \) in the projective 3-space \( P^3 \),

\[
L_\hbar(x) = \begin{pmatrix} aY & xbX \\
xcZ & d \end{pmatrix}, \quad x \in C.
\]

It satisfies the Yang-Baxter relation,

\[
R(x/x')(L_\hbar(x) \otimes 1)(1 \otimes L_\hbar(x')) = (1 \otimes L_\hbar(x'))(L_\hbar(x) \otimes 1)R(x/x')
\]

where

\[
R(x) = \begin{pmatrix} x\omega - x^{-1} & 0 & 0 & 0 \\
0 & \omega(x - x^{-1}) & \omega - 1 & 0 \\
0 & \omega - 1 & x - x^{-1} & 0 \\
0 & 0 & x\omega - x^{-1} & xq - x^{-1} \end{pmatrix}.
\]

The above relation is related to the Yang-Baxter solution for the \( R \)-matrix of XXZ-model in \([6]\) on the study of sine-Gordon lattice model by using the Weyl operators \( U, V \): \( UV = q^{-1} VU \), \( U^N = V^N = 1 \),

\[
L_\hbar^*(x) = \begin{pmatrix} aqU & xbV^{-1} \\
xcV & dU^{-1} \end{pmatrix}, \quad R(x) = \begin{pmatrix} xq - x^{-1}q^{-1} & 0 & 0 & 0 \\
0 & x - x^{-1} & q - q^{-1} & 0 \\
0 & q - q^{-1} & x - x^{-1} & 0 \\
0 & 0 & 0 & xq - x^{-1}q^{-1} \end{pmatrix}.
\]

In fact, with the identification \( Z = VU, X = V^{-1}U \), or equivalently, \( U = q^{-\frac{1}{2}} Y^\frac{1}{2}, V = q^\frac{1}{2} ZY^{-\frac{1}{2}} \), the relation between \( L_\hbar(x) \) and \( L_\hbar^*(x) \) is given by \( L_\hbar^*(x) = L_\hbar(x)Y^{-\frac{1}{2}}q^\frac{1}{2} \). By the matrix-product on auxiliary spaces and tensor-product of quantum spaces, the \( L \)-operator for a finite size \( L \) with the period boundary condition and the parameter \( \hbar = (h_0, \ldots, h_{L-1}) \in (P^3)^L \), \( L_\hbar(x) = \otimes_{j=0}^{L-1} L_{\hbar_j}(x) \) again satisfies the Yang-Baxter equation, hence it gives rise to the commuting family of the transfer matrices

\[
T_{\hbar}(x) = \text{tr}_{aux}(L_\hbar^*(x)), \quad x \in C.
\]

The same conclusion holds for \( L_\hbar^*(x) \) and \( T_{\hbar}^*(x) \), and the following relation holds for these two families of transfer matrices:

\[
T_{\hbar}^*(x) = T_{\hbar}^*(x)D^{-\frac{1}{2}} \quad \text{where} \quad D := q^{-L} \otimes Y.
\]
The transfer matrix $T^\tilde{h}(x)$ acts on the quantum space $\otimes^n \mathbb{C}^N$, also its dual space $\otimes^n \mathbb{C}^{N^*}$. As $T^\tilde{h}(x)$ for $x \in \mathbb{C}$ form a commuting family of operators, all the operators $T^\tilde{h}(x)$ of $\otimes \mathbb{C}^{N^*}$ can be simultaneously diagonalized by common eigenvectors $\langle \varphi |$ with the eigenvalue $\Lambda(x)(= \Lambda_\varphi(x)) \in \mathbb{C}[x]$. The quest of spectrum of $T^\tilde{h}(x)$ (or $T^*_\tilde{h}(x)$) is the problem for our main concern in this work. It is not hard too see that $T^\tilde{h}(x)$ can be expressed as an even polynomial of $x$ with operator-coefficients:

$$T^\tilde{h}(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} T_{2j} x^{2j}, \quad [T_{2j}, T_{2k}] = 0$$

with $T_0 = (q^L \prod_{j=0}^{L-1} a_j) D + \prod_{j=0}^{L-1} d_j$. This implies that the eigenvalue $\Lambda(x)$ is an even polynomial with degree $\leq \lfloor \frac{n}{2} \rfloor$ and $\Lambda(0) = q^n \prod_{j=0}^{L-1} a_j + \prod_{j=0}^{L-1} d_j$ for $s \in \mathbb{Z}_N$. For $L = 3, 4$, the expressions of $T_{2j}$ are listed as follows:

$L = 3, \quad T_2 = c_0 b_1 d_2 Z \otimes X \otimes I + d_0 c_1 b_2 I \otimes Z \otimes X + b_0 d_1 c_2 X \otimes I \otimes Z + (a_j Y \leftrightarrow d_j, b_j X \leftrightarrow c_j Z)$;

$L = 4, \quad T_2 = d_0 d_1 d_2 d_3 1 \otimes Z \otimes X + b_0 d_1 d_2 c_3 X \otimes 1 \otimes Z + c_0 b_1 d_2 d_3 Z \otimes X \otimes 1 + d_0 c_1 b_2 d_3 1 \otimes Z \otimes X + d_0 c_1 a_2 b_3 1 \otimes Z \otimes Y \otimes X + c_0 a_1 b_2 d_3 Z \otimes Y \otimes X + 1 + (a_j Y \leftrightarrow d_j, b_j X \leftrightarrow c_j Z)$;

$$T_4 = b_0 c_1 b_2 c_3 D C^{-1} + c_0 b_1 c_2 b_3 C, \quad C := Z \otimes X \otimes Z \otimes X$$

A general scheme to study the diagonalization problem of $T^\tilde{h}(x)$ by a ”difference-like” equations over an algebraic curve via the Baxter vacuum state has been proposed in [5, 10], of which we are going to describe the method as follows.

In computing the spectra of $T^\tilde{h}(x)$, one can apply the gauge transform on $L_{h_j}(x)$ in a special form,

$$\tilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = A_j L_{h_j}(x) A_j^{-1} + A_j = \begin{pmatrix} 1 & \xi_j - 1 \\ 1 & \xi_j \end{pmatrix}, \quad 0 \leq j \leq L - 1,$$

with $A_L := A_0$. We have

$$\tilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix} F_{h_j}(x, \xi_j - 1, \xi_{j+1}) & -F_{h_j}(x, \xi_j - 1, \xi_{j+1} - 1) \\ F_{h_j}(x, \xi_j, \xi_{j+1}) & -F_{h_j}(x, \xi_j, \xi_{j+1} - 1) \end{pmatrix},$$

where

$$F_{h}(x, \xi, \xi') := \xi' a Y - x b X + \xi' \xi x c Z - \xi d .$$

Accordingly, for $\tilde{h} \in (P^3)^L$ and $\xi = (\xi_0, \ldots, \xi_{L-1}) \in (CP^1)^L$, the modified $L$-operator becomes

$$\tilde{L}_{\tilde{h}}(x, \tilde{\xi}) := \bigotimes_{j=0}^{L-1} L_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix} \tilde{L}_{\tilde{h};11}(x, \tilde{\xi}) & \tilde{L}_{\tilde{h};12}(x, \tilde{\xi}) \\ \tilde{L}_{\tilde{h};21}(x, \tilde{\xi}) & \tilde{L}_{\tilde{h};22}(x, \tilde{\xi}) \end{pmatrix}, \quad \xi_L := \xi_0 .$$

As the procedure of gauge transform keeps the same trace, we have $T^\tilde{h}(x) = \text{tr}_{aux}(\tilde{L}_{\tilde{h}}(x, \tilde{\xi}))$. For a given $\tilde{h}$, we shall consider the variable $(x, \tilde{\xi})$ only lying on the curve $C_{\tilde{h}}$ defined by the system of equations,

$$C_{\tilde{h}}: \quad \xi_j^N = (-1)^N \frac{\xi_j \xi_{j+1} + a_j x_j b_j c_j}{\xi_j \xi_{j+1} - d_j}, \quad j = 0, \ldots, L - 1 .$$

The Baxter vacuum state3 over $C_{\tilde{h}}$ is the family of vectors $|p\rangle \in \otimes^L \mathbb{C}^N$ with the form

$$|p\rangle := \langle p_0 \rangle \otimes \ldots \otimes \langle p_{L-1} \rangle \in \otimes^L \mathbb{C}^N, \quad p \in C_{\tilde{h}} ,$$

3The Baxter vacuum state here is phrased by the Baxter vector in [5, 10]
where \(|p_j\rangle\) is the vector in \(\mathbb{C}^N\) governed by the relation,

\[
\langle 0|p_j\rangle = 1, \quad \frac{\langle m|p_j\rangle}{\langle m-1|p_j\rangle} = \frac{\xi_{j+1} a_j \omega^m - x b_j}{-\xi_j (\xi_{j+1} x \omega^m - d_j)} \quad .
\]

The constraint of \((x, \xi)\) on the curve \(C_h\) ensures that the following properties hold for the Baxter vacuum state,

\[
\bar{L}_{h,11}(x, \xi)|p\rangle = |\tau_- p\rangle \Delta_-(p), \quad \bar{L}_{h,22}(x, \xi)|p\rangle = |\tau_+ p\rangle \Delta_+(p), \quad \bar{L}_{h,21}(x, \xi)|p\rangle = 0 ,
\]

where \(\Delta_{\pm}, \tau_{\pm}\) are (rational) functions and automorphisms of \(C_h\) defined by

\[
\Delta_{\pm}(x, \xi_0, \ldots, \xi_{L-1}) = \prod_{j=0}^{L-1} (d_j - x \xi_{j+1} c_j) ,
\]

\[
\Delta_{\pm}(x, \xi_0, \ldots, \xi_{L-1}) = \prod_{j=0}^{L-1} \xi_j (a_j d_j - x^2 b_j c_j) ,
\]

\[
\tau_{\pm}: (x, \xi_0, \ldots, \xi_{L-1}) \mapsto (q^\pm 1 x, q^{-1} \xi_0, \ldots, q^{-1} \xi_{L-1}) .
\]

This implies that under the action of the transfer matrix, the Baxter vacuum state is decoupled as the sum of those after the \(\tau_{\pm}\)-transformations:

\[
T_h^\dagger(x)|p\rangle = |\tau_- p\rangle \Delta_-(p) + |\tau_+ p\rangle \Delta_+(p) , \quad \text{for} \quad p \in C_h .
\]

For a common eigenvector \(\varphi \in \otimes^L \mathbb{C}^N\) of transfer matrices \(T_h(x)\), the eigenvalue \(\Lambda(x)\) is a polynomial of \(x\), i.e., \(\Lambda(x) \in \mathbb{C}[x]\). The function \(Q(p)\) on \(C_h\) defined by \(Q(p) := \langle \varphi | p \rangle\) for \(p \in C_h\) satisfies the following Baxter’s \(T\)-\(Q\) relation\(^4\) for \(T_h(x)\),

\[
\Lambda(x) Q(p) = \Delta_-(p) Q(\tau_-(p)) + \Delta_+(p) Q(\tau_+(p)) , \quad \text{for} \quad p \in C_h .
\]

The above equation is equivalent to the Baxter’s \(T\)-\(Q\) relation for \(T^\ast(x)\) with certain scaling modification of \(\Delta_{\pm}\), (see \(\square\)). The spectral curve \(C_h\) is a branched cover the rational curve, \(x\)-complex line. In \(\square\), \(\Lambda(x)\) is a polynomial of \(x\), and \(Q(p)\) is a rational function on the Riemann surface \(C_h\). For the solutions of the Baxter’s \(T\)-\(Q\) relation \(\square\) in the spectrum problem of the transfer matrix \(T_h(x)\), it requires the understanding of Baxter vacuum state so that one can extract the necessary properties of zeros and poles of \(Q(p)\), which appears to be a challenging problem in algebraic geometry.

In this article, we shall concern only the case \(L = 3, 4\) for the reason of their special feature connecting with certain Hamiltonian spin chains of physical interest. In these cases, the crux of the transfer matrix \(T(x)\) is the term \(T_2\), which can be converted to the following Hamiltonian. We won’t give the exact identification here, the details can be found in \(\square\) \(\square\).

(1) \(L = 3\). The term \(T_2\) is equivalent to the following Hofstadter type Hamiltonian proposed by Faddeev and Kaschaev \(\square\), ( for the identification using the convention in this report, see \(\square\)):

\[
H_{FK} = \mu (\alpha U + \alpha^{-1} U^{-1}) + \nu (\beta V + \beta^{-1} V^{-1}) + \rho (\gamma W + \gamma^{-1} W^{-1})
\]

where \(U, V, W\) are unitary operators satisfying the Weyl commutation relations and the \(N\)-th power identity property:

\[
UV = \omega VU, \quad VW = \omega WV, \quad WU = \omega UW; \quad U^N = V^N = W^N = 1 .
\]

\(^4\)The Baxter \(T\)-\(Q\) relation here was called the Bethe equation in \(\square\)
For a particular case of $H_{FK}$ with $\rho = 0$, the system becomes the Hofstadter Hamiltonian,\[ H_{Hof} = \mu(\alpha U + \alpha^{-1} U^{-1}) + \nu(\beta V + \beta^{-1} V^{-1}), \]a renowned Bloch electron system with a constant external magnetic field (and the rational magnetic flux in this situation).

For $L = 4$, the interesting physical systems arise from $T_2^*$ with the following constraints on parameters and their relations with the $C, D$ in $T_0, T_4$,

\[
\begin{align*}
    a_jd_j &= q^{-1}, \\
    b_jc_j &= \begin{cases} 
        -k^{-1}, & j = 0, 2, \\
        -k, & j = 1, 3, 
    \end{cases} \\
    C &= \frac{c_1}{k^2c_0c_2}.
\end{align*}
\]

With the operators $U_j$,

\[
    U_1 = Z \otimes X \otimes 1 \otimes 1, \quad U_2 = 1 \otimes Z \otimes X \otimes 1, \quad U_3 = 1 \otimes 1 \otimes Z \otimes X, \quad U_4 = X \otimes 1 \otimes 1 \otimes Z,
\]

one can discuss the following two systems, (for details, see [11]).

(II) Discrete quantum pendulum. This is the case (10) together with the further constraints,

\[
    D = 1; \quad \frac{d_0c_1d_3}{k_c^2} U_2 = \frac{k_c^2}{k_c^2} U_4^{-1} (=: Q_{n-1}), \quad \frac{c_1}{k_c^2d_2d_3} U_1^{-1} = \frac{k_c^2d_1e_2}{c_3} U_3 (=: Q_n).
\]

The coefficients of $T^*(x)$ in (3) now have the form,

\[
    T_0^* = T_4^* = 2, \quad -T_2^* = 2(Q_n + Q_{n-1} + Q_{n-1} + Q_{n-1}^{-1}) + k(qQ_nQ_{n-1} + q^{-1}Q_n^{-1}Q_{n-1}) + k^{-1}(qQ_nQ_{n-1} + q^{-1}Q_n^{-1}Q_{n-1}).
\]

The above $-T_2^*$ is the Hamiltonian of discrete quantum pendulum in [3], which is subject to the evolution equation,

\[
    Q_{n+1}Q_{n-1} = \left(\frac{k + qQ_n}{1 + qkQ_n}\right)^2, \quad Q_nQ_{n-1} = q^2Q_{n-1}Q_n.
\]

(III) Discrete sine-Gordon (SG) Hamiltonian. This is the situation under the condition (11) with one further identification,

\[
    \frac{c_1}{k_c^2d_2d_3} D_{1/2} U_2 = \frac{k_c^2}{k_c^2} D_{1/2} U_4^{-1}.
\]

We have

\[
    T_0^* = D_{1/2} + D_{-1/2}, \quad T_4^* = D_{-1/2} + D_{1/2},
\]

\[
    -T_2^* = \frac{k_c^2d_2d_3}{c_1} D_{1/2} U_1 + \frac{k_c^2d_2d_3}{c_1} D_{-1/2} U_2 + \frac{k_c^2d_2d_3}{c_3} D_{1/2} U_3 + \frac{k_c^2d_2d_3}{c_3} D_{-1/2} U_4 + \frac{c_1}{k_c^2d_2d_3} D_{1/2} U_1^{-1} + \frac{c_1}{k_c^2d_2d_3} D_{-1/2} U_1^{-1} + \frac{c_1}{k_c^2d_2d_3} D_{1/2} U_1^{-1} + \frac{c_1}{k_c^2d_2d_3} D_{-1/2} U_1^{-1}.
\]

The above $-T_2^*$ can be identified with the discrete quantum sine-Gordon integral in [3].
3 The T-Q Polynomial Equation for Rational Spectral Curves

In this section we consider the T-Q relation in the case when the spectral curve $C_R$ is totally degenerated into rational curves. In this situation, though geometric features of the spectral data become trivial, the quest for solutions of the Baxter’s T-Q relation has still raised certain subtle algebraic problems. We shall first summarize some basic facts of reducing the Baxter’s T-Q relation into a polynomial equation for an arbitrary size $L$, then feed the scheme into the case $L = 3, 4$ and state our results on the models (I) (II) (III) of Sect. 2, ( for the detailed account of the derivation, see [10, 11]).

By the rational degenerated spectral curve, we mean the coordinates $\xi_j^N$ in the equation (3) of $C_R$ are constants, independent of the variable $x$. Now the parameter $h_j$s and the variables $\xi_j$s are subject to the constraints:

$$\frac{b_j^N c_j^N}{a_j^N c_j^N} = \frac{a_j^N b_j^N}{c_j^N d_j^N+1}, \quad \xi_j^{2N} = \frac{a_j^N b_j^N}{c_j^N d_j^N} \text{ for } 0 \leq j \leq L - 1. \quad (11)$$

One special case is given by the relations, $a_j = q^{-1}d_j$, $b_j = q^{-1}c_j$, and $\xi_j^N = 1$ for $j = 0, \ldots, L - 1$ [3, 11]. For the condition (11), we define $r_j = \sqrt{\frac{b_j^{-1}d_j^{-1}}{a_j^{-1}c_j^{-1}}}$, for $j \in \mathbb{Z}_L$. Then $C_R$ contains the $\tau_{\pm}$-invariant curve $C$, which is sufficient for our purpose to study the Baxter’s T-Q equation,

$$C := \{(x, \xi_0, \ldots, \xi_{L-1}) \mid r_0^{-1}\xi_0 = \cdots = r_{L-1}^{-1}\xi_{L-1} = q^l, \ l \in \mathbb{Z}_N\}. \quad (3)$$

We shall identify $C$ with $\mathbb{P}^1 \times \mathbb{Z}_N$ via the correspondence:

$$C = \mathbb{P}^1 \times \mathbb{Z}_N, \quad (x, r_0q^l, \ldots, r_{L-1}q^l) \leftrightarrow (x, l).$$

The automorphisms $\tau_{\pm}$ on $C$ become $\tau_{\pm}(x, l) = (q^{\pm 1}x, l - 1)$, and the action (3) of $T(x)(:= T_R)$ on $|x, l\rangle$ now takes the form,

$$T(x)|x, l\rangle = |q^{-1}x, l - 1\rangle \Delta_-(x, l) + |qx, l - 1\rangle \Delta_+(x, l), \quad (12)$$

where $\Delta_{\pm}$ are functions of $x$:

$$\frac{\Delta_-(x, l)}{d_0 \cdots d_{L-1}} = \prod_{j=0}^{L-1} (1 - xq^j d_j^{-1}c_j r_j^{j+1}), \quad \frac{\Delta_+(x, l)}{d_0 \cdots d_{L-1}} = \prod_{j=0}^{L-1} \frac{1 - x^2 a_j^{-1}d_j^{-1}b_j c_j}{1 - xq^{-1}a_j^{-1}b_j r_j^{j+1}}. \quad (11)$$

With the substitutions,

$$(d_0 \cdots d_{L-1})^{-1}T(x) \mapsto T(x), \quad (d_0 \cdots d_{L-1})^{-1}\Delta_\pm(x, l) \mapsto \Delta_\pm(x, l),$$

the relation (12) still holds for the modified $\Delta_\pm$, now with the expression,

$$\Delta_-(x, l) = \prod_{j=0}^{L-1} (1 - x c_j^* q^{j+1}), \quad \Delta_+(x, l) = \prod_{j=0}^{L-1} \frac{1 - x^2 c_j^{*2}}{1 - x c_j^* q^{j+1}}$$

where $c_j^* = d_j^{-1}c_j r_j^{j+1}(= a_j^{-1}b_j r_j^{j+1})$. Furthermore, one can convert the expression (3) of the Baxter vacuum state over the elements of $C$ to the following component-expression of the Baxter’s vector $|x, l\rangle$:

$$\langle k | x, l \rangle = q^{|k|} \prod_{j=0}^{L-1} \frac{(x c_j^* q^{-l-2}; \omega^{-1})_{k_j}}{(x c_j^* q^{l+2}; \omega)_{k_j}} \cdot$$
Then into a single one. From now on, we shall always use the letter $m$ the corresponding Baxter’s vacuum state to extract the essential ingredients for the solutions of the analysis of the Baxter vacuum state to extract the essential ingredients for the solutions of the corresponding Baxter’s $T$-$Q$ relation. Indeed, by the Fourier transform method on the Baxter vacuum state of $C$, then applying certain “diagonalization” process of the transfer matrix on the modified Baxter vacuum state, one obtains the following results, (for the detailed derivation, see $[10, 11]$).

**Theorem 1** Let $f^e$, $f^o$ be the functions on $C$,

$$f^e(x, 2n) := \prod_{j=0}^{L-1} \frac{(xc_j^e; \omega^{-1})_{n+1}}{(xc_j^e; \omega)_{n+1}} , \quad f^o(x, 2n + 1) := \prod_{j=0}^{L-1} \frac{(xc_j^o q^{-1}; \omega^{-1})_{n+1}}{(xc_j^o q; \omega)_{n+1}}.$$  

For $x \in \mathbb{P}^1, l \in \mathbb{Z}_N$, we define the following vectors in $\otimes C^N$,

$$|x\rangle^e = \sum_{n=0}^{N-1} |x, 2n\rangle^e (x, 2n) \omega^{jn} , \quad |x\rangle^o = \sum_{n=0}^{N-1} |x, 2n + 1\rangle^o f^o(x, 2n + 1) \omega^{jn} ,$$

$$|x\rangle^+ = |x\rangle^e q^{-l} u(qx) + |x\rangle^o u(x) \quad \text{where} \quad u(x) := \prod_{j=0}^{L-1} (1 - x^N c_j^N)(xc_j^e q; q^2)_M.$$  

Then

(i) $|x\rangle^e u(qx) = |x\rangle^o q^l u(qx)$, or equivalently, $|x\rangle^+ = 2q^{-l}|x\rangle^e u(qx) = 2|x\rangle^o u(qx)$.

(ii) The $T(x)$-transform on $|x\rangle^+$, $l \in \mathbb{Z}_N$, is given by

$$q^{-l} T(x)|x\rangle^+ = |q^{-1}x\rangle^+ \Delta_-(x, -1) + |qx\rangle^+ \Delta_+(x, 0).$$

(iii) For a common eigenvector $\langle \varphi |$ of $T(x)$ with the eigenvalue $\Lambda(x)$, the function $Q^+_l(x) := \langle \varphi | x\rangle^+_l$ and $\Lambda(x)$ are polynomials with the properties:

$$\text{deg} Q^+_l(x) \leq (3M + 1)L, \quad \text{deg} \Lambda(x) \leq 2\left[\frac{L}{2}\right], \quad \Lambda(x) = \Lambda(-x), \quad \Lambda(0) = q^{2l} + 1,$$

and the following Baxter’s $T$-$Q$ equation holds:

$$q^{-l} \Lambda(x) Q^+_l(x) = \prod_{j=0}^{L-1} (1 - xc_j^e q^{-1}) Q^+_l(xq^{-1}) + \prod_{j=0}^{L-1} (1 + xc_j^e) Q^+_l(xq).$$

Furthermore for $0 \leq m \leq M$, $Q^+_m(x), Q^+_N(x) \in x^m \prod_{j=0}^{L-1} (1 - x^N c_j^N) \mathbb{C}[x]$.

\[ \square \]

By (iii) of the above theorem, the Baxter’s $T$-$Q$ equations for the sectors $m, N - m$ can be unified into a single one. From now on, we shall always use the letter $m$ to denote an integer between 0 and $M$,

$$0 \leq m \leq M.$$  

By introducing the polynomials $\Lambda_m(x), Q(x)$ via the relation,

$$(\Lambda_m(x), x^m \prod_{j=0}^{L-1} (1 - x^N c_j^N) Q(x)) = (q^{-m} \Lambda(x), Q^+_m(x)), \quad (q^m \Lambda(x), Q^+_N(x)).$$

8
the equations in Theorem 3 (iii) for \( l = m, N - m \), are equivalent to the following polynomial equation of \( Q(x), \Lambda_m(x) \):

\[
\Lambda_m(x)Q(x) = q^{-m} \prod_{j=0}^{L-1} (1 - xc_j^* q^{-1})Q(xq^{-1}) + q^m \prod_{j=0}^{L-1} (1 + xc_j^*)Q(xq),
\]

(13)

with the following constraints of \( Q(x) \) and \( \Lambda_m(x) \),

\[
\deg Q(x) \leq ML - m, \quad \deg \Lambda_m(x) \leq 2\left\lceil \frac{L}{2} \right\rceil, \quad \Lambda_m(x) = \Lambda_m(-x), \quad \Lambda_m(0) = q^m + q^{-m}.
\]

By (8), the above \( \Lambda_m(x) \) is indeed the eigenvalue of \( T^s(x) \), and (13) corresponds the Baxter’s T-Q equation for \( T^s(x) \) in the sectors \( m, N - m \).

We now consider the solutions of the equation (13) for \( L = 3, 4 \) for the models (I) (II) (III) in Sect. 2. We shall consider the \( c_j^* \)'s to be generic. Furthermore, for \( L=4 \), the parameter \( \tilde{h} \) will be confined in the regime,

\[
qa_j = d_j = 1, \quad -b_j = c_j = \begin{cases} \frac{k^{-1}}{2} & \text{for even } j, \\ \frac{k^1}{2} & \text{for odd } j, \end{cases}
\]

and the eigenvalue \( \Lambda_m(x) \) in (13) will be a reciprocal polynomial,

\[
\Lambda_m(x) = q^m + q^{-m} + \lambda x^2 + (q^m + q^{-m})x^4
\]

i.e., \( \Lambda_m(x) = \Lambda_m^\dagger(x) \). Here for \( P(x) \in \mathbb{C}[x], P^\dagger(x) \) is the polynomial defined by

\[
P^\dagger(x) = x^{\deg(P)}P(x^{-1}).
\]

The Baxter’s T-Q polynomial equation with a reciprocal \( \Lambda_m(x) \) will be called a symmetric one. Note that for the discrete quantum pendulum (II) and discrete SG Hamiltonian (III) in the above regime, we have \( C = 1 \) by (11), hence the reciprocal property of \( \Lambda_m(x) \) always hold. The following results were shown in Theorem 3 of [10] and Theorem 3 [11]:

**Theorem 2** (i) For \( L = 3 \), the Baxter’s T-Q polynomial equation (13) has \( N \) distinct eigenvalues \( \Lambda_m(x) \), each of which has one-dimensional eigenspace generated by a monic eigen-polynomial \( Q(x) \) of degree \( d = 3M - m \) with \( Q(0) \neq 0 \). In this case, the corresponding Hofstadter type Hamiltonian (8) has the parameters \( \alpha = \beta = \gamma = 1 \).

(ii) For \( L = 4 \) and the symmetric Baxter’s T-Q polynomial equation (13), there are \( N \) distinct eigenvalues \( \Lambda_m(x) \), each of which has one-dimensional eigenspace generated by a monic eigen-polynomial \( Q(x) \) of degree \( d = 4M - 2m \) with \( Q(0) \neq 0 \) and \( Q^\dagger(x) = \pm Q(x) \). Furthermore among these \( N \) eigen-polynomials, there are \( M + 1 \) of \( Q(x) \)’s with the type \( Q^\dagger(x) = Q(x) \), and the other \( M \) ones are of the type \( Q^\dagger(x) = -Q(x) \). In particular, the Baxter’s T-Q polynomial relation of the SG model is described by the sector \( C = 1 \) for all sectors \( m \), and the discrete quantum pendulum is the one with \( C = 1, m = 0 \).

\( \square \)

Now we discuss the relation between the T-Q polynomial equation (13) and the usual Bethe Ansatz formulation in the physical literature. In (13) when \( Q(0) \neq 0 \), which is the case in Theorem 2, one can write

\[
Q(x) = \prod_{i=1}^{d}(x - \frac{1}{z_i}), \quad z_i \in \mathbb{C}^*.
\]
Substituting $z_l^{-1}$ in (13), the $z_i$s satisfy the following Bethe Ansatz relations,

$$q^{L+2m+d} \prod_{j=0}^{L-1} \frac{z_l + c_j^i}{q z_l - c_j^i} = \prod_{i=1,i\neq k}^{d} \frac{z_i - q z_l}{q z_i - z_l}, \quad l = 1, \ldots, d. \quad (14)$$

For the Hofstadter type Hamiltonian (8), by Theorem 2 (i) the Bethe Ansatz (14) for the sector $m$ has the form

$$q^{m+\frac{d}{2}} \prod_{j=0}^{2} \frac{z_l + c_j^i}{q z_l - c_j^i} = \prod_{i=1,i\neq k}^{3M-m} \frac{z_i - q z_l}{q z_i - z_l}, \quad 1 \leq l \leq 3M - m.$$ 

The above relation for the sector $m = M$ is the one postulated in [3], where in some special case, one can reproduce the spectrum of Hofstadter Hamiltonian found in [12]. For the SG model and the discrete quantum pendulum with the parameter $k$, by Theorem 2 (ii), the Bethe Ansatz (14) now takes the form

$$\left( \frac{z_l^2 + 2icq^2 z_l - q}{q z_l^2 - 2icq^2 z_l - 1} \right)^2 = \prod_{i=1,i\neq l}^{4M-2m} \frac{z_i - q z_l}{q z_i - z_l}, \quad 1 \leq l \leq 4M - 2m,$$

where $c = \frac{1}{2}(k^\frac{1}{2} + k^\frac{-1}{2})$. The solutions of the above Bethe Ansatz represent the spectra for those model arising from the rational degenerated spectral curve.

4 The General Spectral Curve for Hofstadter Model, Discrete Quantum Pendulum and Discrete Sine-Gordon Model

We are going to discuss certain geometrical aspect of the Hofstadter, discrete quantum pendulum and SG models in a general spectral curve case. Now the curves $C_h$ [3] are Riemann surfaces with a very high genus. However, the values of $\xi_i^{N}$s for $C_h$ are determined only by the variables $\eta(=\xi_0^N), y(=x^N)$, which defines the curve

$$B_h : \quad C_h(y)\eta^2 + (A_h(y) - D_h(y))\eta - B_h(y) = 0,$$

where $A_h, B_h, C_h, D_h$ are polynomials of $y$ expressed by the relation,

$$\left( \begin{array}{cc} -A_h(y) & B_h(y) \\ C_h(y) & -D_h(y) \end{array} \right) := \prod_{j=0}^{L-1} \left( \begin{array}{cc} -a_j^N & b_j^N \\ yc_j^N & -d_j^N \end{array} \right).$$

The curve $C_h$ becomes a (branched) cover of $B_h$.

For the Hofstadter model [3], it is the case when $L = 3$ with $a_0 = d_0 = 0$, $b_0 = c_0 = 1$ and $h_1, h_2$ generic. For the discrete quantum pendulum and SG model, the constraints of the parameter $h$ are given those given in [10]. In all these cases, the curves $B_h$ form a family of elliptic curves, covered by the spectral family $C_h$. The geometrical feature of these covering families has been explored in cooperation with the Baxter’s $T$-$Q$ relation on $C_h$, but only on the qualitative aspect [10, 11]. Nevertheless, the quantitative study of the $T$-$Q$ relation, which is the core of the problem, has still remained a difficult task, due to the lack of proper understanding of the Baxter vacuum state over $C_h$. The prospect of employing the elliptic function theory related to the family $B_h$ to the Baxter $T$-$Q$ relation would be a challenging problem, which we leave to future work.
Further Remarks

We have made a short review of our recent results on Hofstadter type models, the quantum pendulum and the discrete sine-Gordon model by the transfer matrix technique within the frame work of quantum inverse scattering method. The diagonalization problem of Hamiltonians for these model relies on solving the Baxter’s $T$-$Q$ relation over a spectral curve. This relation arises from the transfer matrix (of a finite size $L$) on the Baxter’s vacuum state, to which we have conducted a systematic study from algebraic geometry aspect. The complexity of the spectral curve, consequently the solvability of its related $T$-$Q$ equation, depends on the length $L$ and the parameters appeared in the Hamiltonian. For the three models which we consider in this report, the transfer matrix has the size $L = 3, 4$, and the spectral curves with generic parameters are all related to elliptic curves. When the spectral curve is totally degenerated into rational curves, the Baxter’s $T$-$Q$ relation can be converted into a polynomial equation, in which case we have obtained a complete solutions of the diagonalization problem of these models. The rigorous mathematical derivation enables us to have a better understanding on both the qualitative and quantitative nature of the Baxter’s $T$-$Q$ polynomial equation, and Bethe Ansatz incorporating the relevant physical consideration. Indeed, our study in this context strongly suggests that some further mathematical feature of the Baxter’s $T$-$Q$ polynomial equation could possibly appear as a certain $q$-Strum-Liouville problem at roots of unity $q^N = 1$. Such program is now under investigation, the results of which we hope to report in near future.

References

[1] M.Ya. Azbel, Energy spectrum of a conduction electron in a magnetic field, Sov. Phys. JETP 19 (1964) 634-645.

[2] R. Baxter, Partition function of the eight-vertex lattice model, Ann. Phys. 70 (1972) 193-228; eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain, I, II, III, Ann. Phys. 76 (1973) 1-24, 25-47, 48-71.

[3] A. Bobenko, N. Kutz and U. Pinkall, The discrete quantum pendulum, Phys. Letts. A 177 (1993) 399-404.

[4] L. D. Faddeev, How algebraic Bethe Ansatz works for integrable models, eds. A. Connes, K. Gawedzki and J. Zinn-Justin, Quantum symmetries/ Symmetries quantiques, Proceedings of the Les Houches summer school, Session LXIV, Les Houches, France, August 1- September 8, 1995, North-Holland (1998) 149-219.

L. A. Takhtadzhian and L. D. Faddeev, The quantum method of the inverse problem and the Heisenberg XYZ model, Russ. Math. Surveys 34 (1979) 11-68.

[5] L. D. Faddeev and R. M. Kashaev, Generalized Bethe Ansatz equations for Hofstadter problem, Comm. Math. Phys. 155 (1995) 181-191, hep-th/9312133.

[6] D. R. Hofstadter, Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields, Phys. Rev. B 14 (1976) 2239-2249.

[7] A. G. Izergin and V. E. Korepin, Lattice versions of quantum field theory models in two dimensions, Nucl. Phys. B 205 (1982) 401- 413;

V. O. Tarasov, Irreducible monodromy matrices for $R$-matrix of the $XXZ$-model and lattice local quantum Hamiltonians, Theor. Math. Phys. 63 (1985) 175-196 (in Russian).
[8] J. Kellendonk, N. Kutz and R. Seiler, Spectra of quantum integrals, in *Discrete Integrable Geometry and Physics*, eds. A I Bobenko and R. Seiler, Oxford Lectures Series in Mathematics and Applications, No 16, 1999, 247-297.

[9] P. P. Kulish and E. K. Sklyanin, Quantum spectral transform method. Recent developments, eds. J. Hietarinta and C. Montonen, Lecture Notes in Physics 151 Springer (1982) 61-119.

[10] S. S. Lin and S. S. Roan, Algebraic geometry approach to the Bethe equation for Hofstadter type models, cond-mat/9912473.

[11] S. S. Lin and S. S. Roan, Baxter’s T-Q relation and Bethe Ansatz of discrete quantum pendulum and sine-Gordon model, hep-th/0105140.

[12] P. B. Wiegmann and A. V. Zabrodin, Bethe-ansatz approach to Hofstadter problem, quantum group and magnetic translations, Nucl. Phys. B 422 (1994) 495-514; Algebraization of difference eigenvalue equations related to $U_q(sl_2)$, Nucl. Phys. B 451 (1995) 699-724, cond-mat/9501129.