Chromatic Polynomials of Complements of Bipartite Graphs

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Abstract We define a biclique to be the complement of a bipartite graph, consisting of two cliques joined by a number of edges. In this paper we study algebraic aspects of the chromatic polynomials of these graphs. We derive a formula for the chromatic polynomial of an arbitrary biclique, and use this to give certain conditions under which two of the graphs have chromatic polynomials with the same splitting field. Finally, we use a subfamily of bicliques to prove the cubic case of the \( \alpha + n \) conjecture, by showing that for any cubic integer \( \alpha \), there is a natural number \( n \) such that \( \alpha + n \) is a chromatic root.

Keywords Chromatic polynomials · Chromatic roots · Algebraic integers · Bipartite complements

1 Introduction

If \( q \) is a positive integer, then a proper \( q \)-colouring of a graph \( G \) is a function from the vertices of \( G \) to a set of \( q \) colours, with the property that adjacent vertices receive different colours. The chromatic polynomial \( P_G(x) \) of \( G \) is the unique monic polynomial which, when evaluated at \( q \), gives the number of proper \( q \)-colourings of \( G \). A chromatic root of \( G \) is a zero of \( P_G(x) \). The chromatic polynomial has been the subject of much study; see [3] for a comprehensive introduction.

This paper is a study of the chromatic polynomials of a family of simple graphs we shall call bicliques. This family is easily described: each member is simply the

1 Note that we vary from the usual definition, in not insisting that every possible edge between cliques is present.

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complement of a bipartite graph, consisting of two cliques joined by a number of edges. When we need to be more specific, we shall refer to a biclique in which the two cliques are of size \( j \) and \( k \) as a \((j, k)\)-biclique. By convention, \( k \) will be greater than or equal to \( j \), and we shall refer to the edges between the two cliques as bridging edges.

The present study of bicliques was originally undertaken with the aim of extending the proof of Cameron’s \( \alpha + n \) conjecture, which suggests that for any algebraic integer \( \alpha \) there is a positive integer \( n \) such that \( \alpha + n \) is a chromatic root. Cameron and Morgan [1] proved the conjecture in the quadratic case, by showing that any quadratic integer is an integer shift of a chromatic root of a \((2, k)\)-biclique, but no further progress has subsequently been made.

In Sect. 2, we give a simple construction of the chromatic polynomial of an arbitrary biclique. We then use this construction in Sect. 3 to examine relations between bicliques having chromatic polynomials with the same splitting field. In Sect. 4 we justify the original motivation for studying bicliques, by applying them to prove the cubic case of the \( \alpha + n \) conjecture, and finally we remark on the potential suitability of bicliques for proving the general conjecture.

### 2 Chromatic Polynomials of Bicliques

A matching of a graph is a set of edges of that graph, no two of which are incident to the same vertex. When we refer to the size of a matching, we refer to the number of edges in that matching; an \( i \)-matching is a matching of size \( i \). Let \( m^i_G \) be the number of \( i \)-matchings of a graph \( G \); then the matching numbers of \( G \) are the elements of the sequence \((m^0_G, m^1_G, m^2_G, \ldots)\). We will write that two graphs are matching equivalent if they have the same matching numbers.

Now, for some positive integers \( j \) and \( k \), let \( G \) be a \((j, k)\)-biclique, and let \( \bar{G} \) be the complement of \( G \) (obtained by replacing edges of \( G \) with non-edges, and vice-versa). Then \( \bar{G} \) is a subgraph of the complete bipartite graph \( K_{j,k} \). We shall construct the chromatic polynomial of \( G \) by considering matchings of \( \bar{G} \).

Given some matching of \( G \), partition the vertices of \( G \) such that two vertices are contained in the same part if and only if the corresponding vertices of \( \bar{G} \) are joined by an element of the matching. Then, by assigning a different colour to each part of this partition, we obtain a proper colouring of \( G \). Conversely, any proper colouring of \( G \) corresponds to a partition induced by some matching of \( \bar{G} \). Thus we can compute the chromatic polynomial of \( G \) by counting \( x \)-colourings of partitions induced by matchings of \( \bar{G} \), as follows.

Let \( (x)_k \) denote the falling factorial \( x(x - 1) \cdots (x - k + 1) \). If each part receives a different colour, then there are \( (x)_{j+k-i} \) ways of assigning \( x \) colours to a partition induced by an \( i \)-matching of \( \bar{G} \) (as any such partition consists of \( j+k-i \) parts). Thus:

\[
P_G(x) = \sum_{M} (x)_{j+k-|M|},
\]

where the sum is over all possible matchings \( M \) of \( \bar{G} \). Note that this construction in fact gives us the unique decomposition of \( P_G(x) \) into a sum of chromatic polynomials.