TOPOLOGICAL CHARACTERISTIC FACTORS ALONG CUBES OF MINIMAL SYSTEMS

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Abstract. In this paper we study the topological characteristic factors along cubes of minimal systems. It is shown that up to proximal extensions the pro-nilfactors are the topological characteristic factors along cubes of minimal systems. In particular, for a distal minimal system, the maximal \((d - 1)\)-step pro-nilfactor is the topological cubic characteristic factor of order \(d\).

1. Introduction. This paper is motivated by the work of Glasner on topological characteristic factors in topological dynamics \([10]\) and the work of Host and Kra on the multiple ergodic averages \([16]\). In \([10]\), Glasner studied the topological characteristic factors along arithmetic progressions, and his work is the counterpart of Furstenberg’s work \([7]\) in topological dynamics. The present work is dedicated to the topological characteristic factors along cubes, which may be considered as the counterpart of \([16]\) in topological dynamics.

1.1. Characteristic factors in ergodic theory. The connection between ergodic theory and additive combinatorics was built in the 1970’s with Furstenberg’s beautiful proof of Szemerédi’s theorem via ergodic theory \([7]\). Furstenberg \([7]\) proved Szemerédi’s theorem via the following multiple recurrence theorem: let \(T\) be a measure preserving transformation on the probability space \((X, \mathcal{A}, \mu)\), then for every integer \(d \geq 1\) and \(A \in \mathcal{A}\) with positive measure,

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n} A \cap T^{-2n} A \cap \ldots \cap T^{-dn} A) > 0.
\]

So it is natural to ask about the convergence of these averages, or more generally about the convergence in \(L^2(X, \mu)\) of the multiple ergodic averages (or called non-conventional averages)

\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \ldots f_d(T^{dn} x),
\]

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where \( f_1, \ldots, f_d \in L^\infty(X, \mu) \). After nearly 30 years’ efforts of many researchers, this problem was finally solved in [16] (see [25] for an another proof).

In the study of multiple ergodic averages, the idea of characteristic factors plays a very important role. This idea was suggested by Furstenberg in [7], and the notion of “characteristic factors” was first introduced by Furstenberg and Weiss in [9].

**Definition 1.1.** [9] Let \((X, \mathcal{X}, \mu, T)\) be a measure preserving system and \((Y, \mathcal{Y}, \mu, T)\) be a factor of \(X\), and let \(d \geq 1\) be an integer. We say that \(Y\) is a \(L^2\) (resp. a.e.)-characteristic factor of \(X\) if for all \(f_1, \ldots, f_d \in L^\infty(X, \mu)\),

\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x) - \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}(f_1|\mathcal{Y})(T^n x) \mathbb{E}(f_2|\mathcal{Y})(T^{2n} x) \cdots \mathbb{E}(f_d|\mathcal{Y})(T^{dn} x) \to 0
\]

in \(L^2(X, \mu)\) (resp. almost everywhere).

Finding a characteristic factor for a scheme often gives a reduction of the problem of evaluating limit behavior of multiple ergodic averages to special systems. The structure theorem of [16, 25] states that for an ergodic system \((X, \mathcal{X}, \mu, T)\) if we want to understand the multiple ergodic averages

\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \cdots f_d(T^{dn} x),
\]

we can replace each function \(f_i\) by its conditional expectation on some \((d-1)\)-step pro-nilsystem (the 0-step system is a trivial system and the 1-step pro-nilsystem is the Kronecker’s one). Thus we can reduce the problem to the study of the same average in a nilsystem, i.e. reducing the average in an arbitrary system to a more tractable question.

In [16], tools like dynamical parallelepipeds, ergodic uniformity seminorms etc., were introduced and resulted useful to study the multiple averages and many others. One of main results of [16] is the following theorem of multiple ergodic averages along cubes.

**Theorem 1.2.** [16, Theorem 1.2] Let \((X, \mathcal{X}, \mu, T)\) be a measure preserving system, and \(d \geq 1\) be an integer. Then for functions \(f_\epsilon \in L^\infty(X, \mu)\), \(\epsilon \in \{0, 1\}^d, \epsilon \neq (0, \ldots, 0)\), the averages

\[
\prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{n \in [M_1, N_1) \times \cdots \times [M_d, N_d)} \prod_{\epsilon \neq (0, \ldots, 0)} f_\epsilon(T^{N_\epsilon} x)
\]

converge in \(L^2(X, \mu)\) as \(N_1 - M_1, N_2 - M_2, \ldots, N_d - M_d\) tend to \(+\infty\).

We may define the characteristic factor of (1) similarly as defined in Definition 1.1. To prove theorem above the authors in [16] showed that the \(d\)-dimensional average along cubes has the same characteristic factor as the average along arithmetic progressions of length \(d\), which is a \((d-1)\)-step pro-nilsystem. The main result of the paper is to give the topological counterpart of this fact, that is, to show that pro-nilfactors are the topological characteristic factors along cubes of minimal systems.
1.2. Topological characteristic factors along arithmetic progressions. The counterpart of characteristic factors in topological dynamics was first studied by Glasner in [10]. There, the author studied the characteristic factors for the transformation \( T \times T^2 \times \ldots \times T^d \) according to the following definition.

Definition 1.3. Let \((X, T)\) be a topological dynamical system and let \( \pi : (X, T) \to (Y, T) \) be a factor map. A subset \( L \) of \( X \) is called \( \pi \)-saturated if \( \{x \in L : \pi^{-1}(\pi(x)) \subseteq L\} = \pi L \), i.e. \( L = \pi^{-1}(\pi(L)) \).

Here is the definition of topological characteristic factors along arithmetic progressions:

Definition 1.4. [10] Let \((X, T)\) be a topological dynamical system and \( d \geq 1 \) be an integer. Let \( \pi : (X, T) \to (Y, T) \) be a factor map and \( \sigma_d = T \times T^2 \times \ldots \times T^d \). The topological dynamical system \((Y, T)\) is said to be a topological characteristic factor (along arithmetic progressions) of order \( d \) if there exists a dense \( G_\delta \) set \( X_0 \) of \( X \) such that for each \( x \in X_0 \) the orbit closure \( L_x = \overline{\{(x, \ldots, x), \sigma_d\}} \) is \( \pi \)-saturated. That is, \((x_1, x_2, \ldots, x_d) \in L_x \) iff \((x_1', x_2', \ldots, x_d') \in L_x\), where \( \pi(x_i) = \pi(x_i') \) for all \( i \).

In [10], it is shown that up to a canonically defined proximal extension, a characteristic family for \( T \times T^2 \times \ldots \times T^d \) is the family of canonical PI flows of class \( d-1 \). In particular, if \((X, T)\) is a distal minimal system, then its largest class \( d-1 \) distal factor is its topological characteristic factor of order \( d \). In particular, if \((X, T)\) is a weakly mixing system, then the trivial system is its topological characteristic factor. For more related results we refer the reader to [10].

A unsolved problem is:

Conjecture 1. If \((X, T)\) is a distal minimal topological dynamical system, then its maximal \((d-1)\)-step pro-nilfactor is its topological characteristic factor along arithmetic progressions of order \( d \).

1.3. Topological characteristic factors along cubes and main results of the paper. First we define topological characteristic factors along cubes. The transformation group related to (1) is the face group \( F^{[d]} \). Please refer to next section for the precise definition. Note that the group \( F^{[d]} \) acts on \( X^{2^d} \) and it acts on the first coordinate as an identity map.

Definition 1.5. Let \((X, T)\) be a topological dynamical system and \( d \geq 1 \) be an integer. Let \( \pi : (X, T) \to (Y, T) \) be a factor map. The topological dynamical system \((Y, T)\) is said to be a topological cubic characteristic factor of order \( d \) or topological characteristic factor along cubes of order \( d \) if there exists a dense \( G_\delta \) set \( X_0 \) of \( X \) such that for each \( x \in X_0 \) the set \( F_x = p_* \left( \overline{\{(x, \ldots, x), F^{[d]}\}} \right) \) is \( \pi^{2^d-1} = \pi \times \ldots \times \pi \) saturated, where \( p_* : X^{2^d} \to X^{2^d-1} \) is the projection onto the last \( 2^d - 1 \) coordinates. That is, for each \( x \in X_0 \),

\[
\overline{\{(x, \ldots, x), F^{[d]}\}} = \{x\} \times (\pi^{2^d-1})^{-1}(\pi^{2^d-1}F_x).
\]
One of main results of this paper is that up to proximal extensions the maximal $(d-1)$-step pro-nilfactor is the topological cubic characteristic factor of order $d$. To be precise, we will show the following theorem:

**Theorem 1.6.** Let $(X, T)$ be a minimal topological dynamical system and $d \geq 1$ be an integer. Let $\pi : (X, T) \to (\mathbb{Z}^{d-1}, T)$ be the factor map to the maximal $(d-1)$-step pro-nilfactor. Then there is a commutative diagram of homomorphisms of minimal topological dynamical systems

$$
\begin{array}{ccc}
X & \xrightarrow{\theta'} & X' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
\mathbb{Z}^{d-1} & \xrightarrow{\theta} & Y'
\end{array}
$$

such that $(Y', T)$ is the topological cubic characteristic factor of order $d$ of $(X', T)$, where $\theta, \theta'$ are proximal extensions.

When $X$ is distal or weakly mixing, the proximal extensions $\theta, \theta'$ in theorem are trivial (i.e. isomorphisms). That is:

**Corollary 1.**

1. Let $(X, T)$ be a minimal distal topological dynamical system and $d \geq 1$ be an integer. Then the maximal $(d-1)$-step pro-nilfactor is the topological cubic characteristic factor of order $d$.

2. Let $(X, T)$ be a minimal weakly mixing topological dynamical system and $d \geq 1$ be an integer. Then the trivial system is the topological cubic characteristic factor of order $d$.

We do not know whether we may remove proximal extensions in Theorem 1.6, so we have the following question:

**Question 1.** Let $(X, T)$ be a minimal topological dynamical system and $d \geq 1$ be an integer. Is the maximal $(d-1)$-step pro-nilfactor the topological cubic characteristic factor of order $d$?

2. **Preliminaries.** In the article, integers, nonnegative integers and natural numbers are denoted by $\mathbb{Z}, \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$ respectively. In the following subsections we give the basic background in topological dynamics necessary for the article.

2.1. **Topological dynamical systems.** By a topological dynamical system we mean a pair $(X, T)$ where $X$ is a compact metric space (with metric $\rho$) and $T : X \to X$ is a homeomorphism. For $n \geq 2$ we write $(X^n, T^{(n)})$ for the $n$-fold product system $(X \times \cdots \times X, T \times \cdots \times T)$. The diagonal of $X^n$ is

$$
\Delta_n(X) = \{(x, \ldots, x) \in X^n : x \in X\}.
$$

When $n = 2$ we write $\Delta_2(X) = \Delta(X)$. The orbit of $x \in X$ is given by $\mathcal{O}(x, T) = \{T^n x : n \in \mathbb{Z}\}$. For convenience, sometimes we denote the orbit closure of $x \in X$ under $T$ by $\overline{\mathcal{O}(x, T)}$ or $\overline{\mathcal{O}(x)}$, instead of $\overline{\mathcal{O}(x, T)}$.

A topological dynamical system $(X, T)$ is transitive if for any two nonempty open sets $U$ and $V$ there is $n \in \mathbb{Z}$ such that $U \cap T^{-n} V \neq \emptyset$. Equivalently, $(X, T)$ is transitive if and only if there exists $x \in X$ such that $\overline{\mathcal{O}(x, T)} = X$; such $x$ is called a transitive point. We say $(X, T)$ is weakly mixing if the product system $(X^2, T^{(2)})$ is transitive. A topological dynamical system $(X, T)$ is minimal if $\overline{\mathcal{O}(x, T)} = X$.
for every \( x \in X \). A point \( x \in X \) is minimal or almost periodic if the subsystem \((\overline{O(x,T)},T)\) is minimal.

A factor map \( \pi : X \to Y \) between the topological dynamical systems \((X,T)\) and \((Y,S)\) is a continuous onto map which intertwines the actions (i.e. \( \pi \circ T = S \circ \pi \)); we say that \((Y,S)\) is a factor of \((X,T)\) and that \((X,T)\) is an extension of \((Y,S)\).

In this paper we also make use of a more general definition of a topological system. That is, instead of just a single transformation \( T \), we consider commuting transformations \( T_1, \ldots, T_k \) of \( X \). We summarize some basic definitions and properties of systems in the classical setting of one transformation. Extensions to the general case are straightforward.

2.2. Cubes and faces. Cube and face groups were introduced by Host and Kra in [16]. We refer the reader to [16, 18] for more details.

Let \( X \) be a set, and let \( d \geq 1 \) be an integer. We view element in \( \{0,1\}^d \) as a sequence \( \epsilon = \epsilon_1 \ldots \epsilon_d \) of 0’s and 1’s. Let \( V_d = \{0,1\}^d \) and \( V_d^* = V_d \setminus \{0\} \), where \( 0 = 00 \ldots 0 \in \{0,1\}^d \).

If \( n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \) and \( \epsilon \in \{0,1\}^d \), we define
\[
 n \cdot \epsilon = \sum_{i=1}^{d} n_i \epsilon_i.
\]

We denote \( X^{2^d} \) by \( X^{[d]} \). A point \( x \in X^{[d]} \) can be written as
\[
x = (x_\epsilon : \epsilon \in \{0,1\}^d).
\]

A point \( x \in X^{[d]} \) can be decomposed as \( x = (x', x'') \) with \( x', x'' \in X^{[d-1]} \), where \( x' = (x_{\epsilon_0} : \epsilon \in \{0,1\}^{d-1}) \) and \( x'' = (x_{\epsilon_1} : \epsilon \in \{0,1\}^{d-1}) \). Hence \( x_0 \) is the first coordinate of \( x \). As examples, points in \( X^{[2]} \) are like
\[
(x_{00}, x_{10}, x_{01}, x_{11}).
\]

We can also isolate the first coordinate, writing \( X^{[d]}_1 = X^{2^d-1} \) and then writing a point \( x \in X^{[d]} \) as \( x = (x_0, x_*) \), where \( x_* = (x_\epsilon : \epsilon \neq 0) \in X^{[d]}_1 \).

For \( x \in X \), we write \( x^{[d]} = (x, x, \ldots, x) \in X^{[d]} \). The diagonal of \( X^{[d]} \) is \( \Delta^{[d]} = \{x^{[d]} : x \in X\} \).

2.3. Dynamical parallelepipeds.

Definition 2.1. Let \((X,T)\) be a topological dynamical system and let \( d \geq 1 \) be an integer. We define \( Q^{[d]}(X) \) to be the closure in \( X^{[d]} \) of elements of the form
\[
(T^{n_1} x, T^{n_2} x, \ldots, T^{n_d} x : \epsilon = \epsilon_1 \epsilon_2 \ldots \epsilon_d \in \{0,1\}^d),
\]
where \( n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \) and \( x \in X \). When there is no ambiguity, we write \( Q^{[d]} \) instead of \( Q^{[d]}(X) \). An element of \( Q^{[d]}(X) \) is called a (dynamical) parallelepiped of dimension \( d \).

As examples, \( Q^{[2]} \) is the closure in \( X^{[2]} = X^4 \) of the set
\[
\{(x, T^m x, T^n x, T^{m+n} x) : x \in X, m, n \in \mathbb{Z}\}
\]
and \( Q^{[3]} \) is the closure in \( X^{[3]} = X^8 \) of the set
\[
\{(x, T^m x, T^n x, T^{m+n} x, T^p x, T^{m+p} x, T^{n+p} x, T^{m+n+p} x) : x \in X, m, n, p \in \mathbb{Z}\}.
\]
Definition 2.2. Let $\phi : X \to Y$ and $d \geq 1$ be an integer. Define $\phi^{[d]} : X^{[d]} \to Y^{[d]}$ by $(\phi^{[d]}(x))_\epsilon = \phi x$, for every $x \in X^{[d]}$ and every $\epsilon \in \{0, 1\}^d$. Let $(X, T)$ be a topological dynamical system and $d \geq 1$ be an integer. The diagonal transformation of $X^{[d]}$ is the map $T^{[d]}$.

Definition 2.3. Face transformations are defined inductively as follows: Let $T^{[0]} = T$, $T^{[1]}_1 = \text{id} \times T$. If $\{T_j^{[d-1]}\}_{j=1}^{d-1}$ is defined already, then set

$$
T_j^{[d]} = T_j^{[d-1]} \times T_j^{[d-1]}, \quad j \in \{1, 2, \ldots, d - 1\},
$$

$$
T_j^{[d]} = \text{id}^{[d-1]} \times T^{[d-1]}.
$$

The face group of dimension $d$ is the group $F^{[d]}(X)$ of transformations of $X^{[d]}$ spanned by the face transformations. The cube group or parallelepiped group of dimension $d$ is the group $G^{[d]}(X)$ spanned by the diagonal transformation and the face transformations. We often write $F^{[d]}$ and $G^{[d]}$ instead of $F^{[d]}(X)$ and $G^{[d]}(X)$, respectively. For $G^{[d]}$ and $F^{[d]}$, we use similar notations to the ones used for $X^{[d]}$: namely, an element of either of these groups is written as $S = (S_\epsilon : \epsilon \in \{0, 1\}^d)$. In particular, $F^{[d]} = \{S \in G^{[d]} : S_0 = \text{id}\}$. Let $p_\tau : X^{[d]} \to X^{[d]}_\tau$ be the projection. Then all transformations of $G^{[d]}$ and $F^{[d]}$ factor through the projection $p_\tau$ and induce transformations of $X^{[d]}_\tau$. We denote the corresponding groups by $G^{[d]}_\tau$ and $F^{[d]}_\tau$ respectively.

For convenience, we denote the orbit closure of $x \in X^{[d]}$ under $F^{[d]}$ by $\overline{F^{[d]}(x)}$, instead of $\overline{O(x, F^{[d]})}$. It is easy to verify that $Q^{[d]}$ is the closure in $X^{[d]}$ of

$$
\{S_x^{[d]} : S \in F^{[d]}, x \in X\}.
$$

If $x$ is a transitive point of $X$, then $Q^{[d]}$ is the orbit closure of $x^{[d]}$ under the group $G^{[d]}$.

If $(X, T)$ is minimal, then for all $x \in X$, $(\overline{F^{[d]}(x^{[d]}), F^{[d]}})$ is minimal ([22, Theorem 3.1]), and $(Q^{[d]}, G^{[d]})$ is minimal ([19, Lemma 4.1] or [22, Corollary 4.9]).

2.4. Nilmanifolds and nilsystems. Let $G$ be a group. For $g, h \in G$ and $A, B \subset G$, we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator of $g$ and $h$ and $[A, B]$ for the subgroup spanned by $\{[a, b] : a \in A, b \in B\}$. The commutator subgroups $G_j, j \geq 1,$ are defined inductively by setting $G_1 = G$ and $G_{j+1} = [G_j, G]$. Let $d \geq 1$ be an integer. We say that $G$ is $d$-step nilpotent if $G_{d+1}$ is the trivial subgroup.

Let $G$ be a $d$-step nilpotent Lie group and $\Gamma$ be a discrete cocompact subgroup of $G$. The compact manifold $X = G/\Gamma$ is called a $d$-step nilmanifold. The group $G$ acts on $X$ by left translations and we write this action as $(g, x) \mapsto gx$. The Haar measure $\mu$ of $X$ is the unique probability measure on $X$ invariant under this action. Let $\tau \in G$ and $T$ be the transformation $x \mapsto \tau x$ of $X$. Then $(X, \mu, T)$ is called a $d$-step nilsystem. In the topological setting we omit the measure and just say that $(X, T)$ is a $d$-step nilsystem.

If $(X_i, T_i)_{i \in \mathbb{N}}$ are systems with $\text{diam}(X_i) \leq 1$ and $\pi_i : X_{i+1} \to X_i$ are factor maps, the inverse limit of the systems is defined to be the compact subset of $\prod_{i \in \mathbb{N}} X_i$ given by $\{(x_i)_{i \in \mathbb{N}} : \pi_i(x_{i+1}) = x_i\}$, and we denote it by $\lim (X_i, T_i)_{i \in \mathbb{N}}$. It is a compact metric space endowed with the distance $d((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) = \sum_{i \in \mathbb{N}} 1/2^j \rho_i(x_i, y_i)$, where $\rho_i$ is the metric in $X_i$. We note that the maps $T_i$ induce naturally a transformation $T$ on the inverse limit.

The following structure theorem characterizes inverse limits of nilsystems using dynamical parallelepipeds.
Theorem 2.4 (Host-Kra-Maass). [19, Theorem 1.2.] Assume that \((X,T)\) is a transitive topological dynamical system and let \(d \geq 2\) be an integer. The following properties are equivalent:

1. If \(x,y \in \mathbb{Q}^d\) have \(2^d-1\) coordinates in common, then \(x = y\).
2. If \(x,y \in X\) are such that \((x,y,\ldots,y) \in \mathbb{Q}^d\), then \(x = y\).
3. \(X\) is an inverse limit of \((d-1)\)-step minimal nilsystems.

A transitive system satisfying one of the equivalent properties above is called a system of order \((d-1)\) or a \((d-1)\)-step pro-nilsystem.

Further development of the theory of cubes in an abstract setting, calling these structures nilspaces, were given by Host and Kra [17], Antolin Camarena and Szegedy [2, 23], Gutman, Manners, and Varjú [13, 14, 15] and Candela [3, 4] etc.

2.5. Proximal, distal and regionally proximal relations. Let \((X,T)\) be a topological dynamical system. Fix \((x,y) \in X^2\). It is a proximal pair if \(\inf d(T^n x, n \in \mathbb{Z})\) \(T^n y) = 0\); it is a distal pair if it is not proximal. Denote by \(P(X,T)\) the set of proximal pairs of \((X,T)\). It is also called the proximal relation. A topological dynamical system \((X,T)\) is equicontinuous if for every \(\epsilon > 0\) there exists \(\delta > 0\) such that \(d(x,y) < \delta\) implies \(d(T^n x, T^n y) < \epsilon\) for every \(n \in \mathbb{Z}\). It is distal if \(P(X,T) = \Delta(X)\). Any equicontinuous system is distal.

Let \((X,T)\) be a minimal system. The regionally proximal relation \(RP(X,T)\) is defined as: \((x,y) \in RP(X,T)\) if there are sequences \(x_i, y_i \in X, n_i \in \mathbb{Z}\) such that \(x_i \to x, y_i \to y\) and \((T \times T)^n_i (x_i, y_i) \to (z,z), i \to \infty\), for some \(z \in X\). It is well known that \(RP(X,T)\) is an invariant closed equivalence relation and this relation defines the maximal equicontinuous factor \(X_{eq} = X/RP(X,T)\) of \((X,T)\) (for example see [24, Chapter V]).

2.6. Regionally proximal relation of order \(d\).

Definition 2.5. Let \((X,T)\) be a topological dynamical system and let \(d \geq 1\) be an integer. The points \(x,y \in X\) are said to be regionally proximal of order \(d\) if for any \(\delta > 0\), there exist \(x', y' \in X\) and a vector \(n = (n_1, \ldots, n_d) \in \mathbb{Z}^d\) such that \(\rho(x,x') < \delta, \rho(y,y') < \delta\), and

\[
\rho(T^{\epsilon n} x', T^{\epsilon n} y') < \delta \text{ for any } \epsilon \in \{0,1\}^d \setminus \{0\}.
\]

In other words, there exists \(S \in F^{[d]}\) such that \(\rho(S x', S y') < \delta\) for every \(\epsilon \in \{0,1\}^d \setminus \{0\}\). The set of regionally proximal pairs of order \(d\) is denoted by \(RP^{[d]}(X,T)\) (or by \(RP^{[d]}(X,T)\) in case of ambiguity), and is called the regionally proximal relation of order \(d\).

The notion of the regionally proximal relation of order \(d\) was introduced by Host, Kra and Maass in [19] \((d = 2\) in [20]). The regionally proximal relation of order \(d\) for the topological dynamical systems under more general group action was studied in [12].

It is easy to see that \(RP^{[d]}\) is a closed and invariant relation. Observe that ([19])

\[
P(X,T) \subset \ldots \subset RP^{[d+1]} \subset RP^{[d]} \subset \ldots \subset RP^{[2]} \subset RP^{[1]} = RP(X,T).
\]

The following theorems proved in [19] (for minimal distal systems) and in [22] (for general minimal systems) tell us conditions under which the pair \((x,y)\) belongs to \(RP^{[d]}\) and the relation between \(RP^{[d]}\) and \(d\)-step pro-nilsystems.
Theorem 2.6. [22] Let \((X,T)\) be a minimal topological dynamical system and let \(d \geq 1\) be an integer. Then
\begin{enumerate}
\item \((x,y) \in \text{RP}^d[X,T]\) if and only if \((x,y,\ldots,y) \in Q^{d+1}\) if and only if \((x,y,\ldots,y) \in F^{d+1}(x^{d+1})\).
\item \(\text{RP}^d\) is an equivalence relation.
\item \((X,T)\) is a system of order \(d\) if and only if \(\text{RP}^d[X,T] = \Delta_X\).
\end{enumerate}

Theorem 2.7. [22] Let \(\pi : (X,T) \to (Y,S)\) be a factor map between minimal topological dynamical systems and let \(d \geq 1\) be an integer. Then
\begin{enumerate}
\item \(\pi \times \pi([\text{RP}^d(X,T)]) = [\text{RP}^d(Y,S)]\).
\item \((Y,T)\) is a system of order \(d\) if and only if \(\text{RP}^d(X,T) \subset R_x\).
\end{enumerate}
In particular, the quotient of \((X,T)\) under \([\text{RP}^d(X,T)]\) is the maximal \(d\)-step pronil-factor of \(X\) (i.e. the maximal factor of order \(d\)).

Let \(Z_d = X/\text{RP}^d[X,T]\) and \(\pi_d : (X,T) \to (Z_d,T_d)\) be the factor map. The system \(Z_0\) is the trivial system and the system \(Z_1\) is the maximal equicontinuous factor \(X_{eq}\).

2.7. Some fundamental extensions. Let \((X,T)\) and \((Y,S)\) be topological dynamical systems and let \(\pi : X \to Y\) be a factor map. We say that \(\pi\) is an open extension if it is open as a map; and \(\pi\) is a semi-open extension if the image of every nonempty open set of \(X\) has nonempty interior. An important fact is that any factor map of minimal systems is semi-open (for example see [24, Chapter II, Subsection 9.17]).

An extension \(\pi\) is proximal if \(\pi(x_1) = \pi(x_2)\) implies \((x_1,x_2) \in \text{P}(X,T)\), and \(\pi\) is distal if \(\pi(x_1) = \pi(x_2)\) and \(x_1 \neq x_2\) implies \((x_1,x_2) \notin \text{P}(X,T)\). An extension \(\pi\) is almost one to one if there exists a dense \(G_4\) set \(X_0 \subset X\) such that \(\pi^{-1}(\{\pi(x)\}) = \{x\}\) for any \(x \in X_0\). It is easy to see that any almost one to one extension between minimal systems is proximal [24, Chapter VI].

An extension \(\pi\) between minimal systems is called a relatively incontractible (RIC) extension if it is open and for every \(n \geq 1\) the relation
\[
R_\pi^n = \{(x_1,\ldots,x_n) \in X^n : \pi(x_i) = \pi(x_j), \forall 1 \leq i \leq j \leq n\}
\]
has a dense set of minimal points. In particular, a RIC extension \(\pi\) is open, and hence \(\pi^{-1} : Y \to 2^X, y \mapsto \pi^{-1}(y)\) is continuous.

A distal extension between minimal systems is RIC. Every factor map between minimal systems can be lifted to a RIC extension by proximal extensions (see [6] or [24, Chapter VI]).

Theorem 2.8. Given a factor map \(\pi : X \to Y\) between minimal topological dynamical systems \((X,T)\) and \((Y,S)\) there exists a commutative diagram of factor maps (called RIC-diagram or EGS-diagram\(^1\))
\[
\begin{array}{ccc}
X & \xleftarrow{\theta} & X' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
Y & \xleftarrow{\theta} & Y'
\end{array}
\]
such that
\begin{enumerate}
\item \(\theta'\) and \(\theta\) are proximal extensions;
\end{enumerate}

\(^{1}\)EGS stands for Ellis, Glasner and Shapiro [6].
(b) \( \pi' \) is a RIC extension;
(c) \( X' \) is the unique minimal set in \( R_{\theta\theta} = \{(x,y) \in X \times Y' : \pi(x) = \theta(y)\} \) and \( \theta' \) and \( \pi' \) are the restrictions to \( X' \) of the projections of \( X \times Y' \) onto \( X \) and \( Y' \) respectively.

Note that when \( \pi \) is RIC, the proximal extensions \( \theta, \theta' \) in the theorem above are trivial (i.e. isomorphisms). In particular, when \( \pi \) is distal or \( Y \) is trivial, the proximal extensions \( \theta, \theta' \) are trivial.

3. Topological characteristic factors along cubes. Let \( (X,T) \) be a minimal topological dynamical system and \( d \geq 1 \) be an integer. By Theorem 2.7, \( (Z_d, T_d) = (X/\mathbb{RP}^d(X), T_d) \) is the maximal \( d \)-step pro-nilfactor of \( (X,T) \). For convenience, we also use the symbol \( T \) to denote the action on \( Z_d \), that is, \( (Z_d, T) \) is the maximal \( d \)-step pro-nilfactor of \( (X,T) \). Let \( \pi_d : (X,T) \to (Z_d, T) \) be the factor map.

In this section we will prove the main results of the paper. First we will show that modulo proximal extensions \( Q^d(X) \) is \( \pi_{d-1} \)-saturated. Then using this result we will prove that modulo proximal extensions the maximal \( (d-1) \)-step pro-nilfactor \( (Z_{d-1}, T) \) is the topological cubic characteristic factor of order \( d \) of \( (X,T) \).

3.1. Parallelepiped \( Q^d \). The following lemma generalizes Lemma 4.2 in [5], which gives a condition when a point \( x \in X^d \) belongs to \( Q^d(X) \).

**Lemma 3.1.** Let \( (X,T) \) be a minimal topological dynamical system and \( d \geq 1 \) be an integer. Let \( \pi : (X,T) \to (Z_{d-1}, T) \) be the factor map to the maximal \( (d-1) \)-step pro-nilfactor. If points \( x_1, x_2, \ldots, x_{2^d} \in X \) satisfy the following coditions:

1. \( x = (x_1, x_2, \ldots, x_{2^d}) \in R_\theta^d \), that is, \( \pi(x_1) = \pi(x_2) = \cdots = \pi(x_{2^d}) \);
2. \( \pi_1 : \overline{\mathcal{O}(x,T^d)} \to X \) is semi-open, where \( \pi_1 \) is the projection to the first coordinate,

then \( \{x_1, x_2, \ldots, x_{2^d}\}^d \subset Q^d(X) \). In particular, \( x = (x_1, x_2, \ldots, x_{2^d}) \in Q^d(X) \).

**Proof.** We first prove the following claim.

**Claim.** If \( (x_1, \alpha_*) \in \{x_1, x_2, \ldots, x_{2^d}\}^d \cap Q^d(X) \), then

\( (x_i, \alpha_*) \in Q^d(X) \) for any \( i \in \{1, 2, \ldots, 2^d\} \).

**Proof of Claim.** Fix an \( i_0 \in \{1, 2, \ldots, 2^d\} \), we will show that \( (x_{i_0}, \alpha_*) \in Q^d(X) \). Let \( U_1, U_2, \ldots, U_{2^d} \) be neighborhoods of \( x_1, x_2, \ldots, x_{2^d} \) respectively. Since \( \pi_1 \) is semi-open, we have that \( V_1 = \text{int} \pi_1((U_1 \times U_2 \cdots \times U_{2^d}) \cap \overline{\mathcal{O}(x,T^d)}) \neq \emptyset \).

It is obvious that \( V_1 \subset U_1 \). Set \( V_2 = U_2, \ldots, V_{2^d} = U_{2^d} \). We can write \( (x_1, \alpha_*) = (x_{s(e)})_{e \in \{0,1\}^d} \), where \( s : \{0,1\}^d \to \{1, 2, \ldots, 2^d\} \) is a function with \( s(0) = 1 \).

By the definition of \( V_1 \), it is easy to see that

\( V_1 \times V_2 \cdots \times V_{2^d} \cap \mathcal{O}(x,T^d) \neq \emptyset \).

So there exists \( n_0 \in \mathbb{Z} \) such that

\[ x_1 \in T^{-n_0}V_1, x_2 \in T^{-n_0}V_2, \ldots, x_{2^d} \in T^{-n_0}V_{2^d}, \]

thus \( (x_1, \alpha_*) = (x_{s(e)})_{e \in \{0,1\}^d} \in \prod_{e \in \{0,1\}^d} T^{-n_0}V_{s(e)} \).
From the hypothesis, \((x_1, \alpha_\ast) \in \mathbb{Q}^{[d]}(X)\), so there exists \(n \in \mathbb{Z}^d\) and \(x \in X\) such that
\[
(T^{m \cdot \epsilon} x)_{\epsilon \in \{0,1\}^d} \subset \prod_{\epsilon \in \{0,1\}^d} T^{-n_\epsilon} V_{s(\epsilon)}.
\]
It follows that
\[
\bigcap_{\epsilon \in \{0,1\}^d} T^{-n \cdot \epsilon} V_{s(\epsilon)} \neq \emptyset.
\]
Set \(W_1 = \bigcap_{\epsilon \in \{0,1\}^d} T^{-n \cdot \epsilon} V_{s(\epsilon)}\) and \(W_2 = U_2 \ldots , W_{2d} = U_{2d}^t\). Note that \(W_1 \subset V_1\).

Since \(W_1 \subset V_1\), by the definition of \(V_1\) we have that
\[
(W_1 \times W_2 \cdots \times W_{2d}) \cap O(x, T^{[d]}) \neq \emptyset.
\]
Thus there exists \(n_1 \in \mathbb{Z}\) such that
\[
x_1 \in T^{-n_1} W_1, x_2 \in T^{-n_1} W_2, \ldots , x_{2d} \in T^{-n_1} W_{2d}.
\]
Since \((x_1, x_{i_0}) \in R_\pi \subset \mathbb{RP}^{[d-1]}(X)\), by Theorem 2.6
\[
(x_{i_0}, (x_1^{[d]})) = (x_{i_0}, x_1, \ldots , x_1) \in \mathbb{Q}^{[d]}(X).
\]
Note that \((x_{i_0}, x_1, \ldots , x_1) \in T^{-n_1} W_{i_0} \times T^{-n_1} W_1 \cdots \times T^{-n_1} W_1\), so there exist some \(m \in \mathbb{Z}^d\) and \(x' \in X\) such that
\[
(T^{m \cdot \epsilon} x')_{\epsilon \in \{0,1\}^d} \subset T^{-n_1} W_{i_0} \times T^{-n_1} W_1 \cdots \times T^{-n_1} W_1.
\]
It follows that
\[
W_{i_0} \cap \bigcap_{\epsilon \in \{0,1\}^d \setminus \{0\}} T^{-m \cdot \epsilon} W_1 \neq \emptyset.
\]
Since \(W_1 = \bigcap_{\epsilon \in \{0,1\}^d} T^{-n \cdot \epsilon} V_{s(\epsilon)}\), we have that
\[
W_{i_0} \cap \bigcap_{\epsilon \in \{0,1\}^d \setminus \{0\}} T^{-m \cdot \epsilon} \bigcap_{\eta \in \{0,1\}^d} T^{-n \cdot \eta} V_{s(\eta)} \neq \emptyset.
\]
In particular, we have that
\[
W_{i_0} \cap \bigcap_{\epsilon \in \{0,1\}^d \setminus \{0\}} T^{-(m+n) \cdot \epsilon} V_{s(\epsilon)} \neq \emptyset.
\]
Since \(W_1 \subset V_1 \subset U_i, i \in \{1, 2, \ldots , 2^d\}\), it follows that
\[
U_{i_0} \cap \bigcap_{\epsilon \in \{0,1\}^d \setminus \{0\}} T^{-(m+n) \cdot \epsilon} V_{s(\epsilon)} \neq \emptyset.
\]
Note that \(U_i\) is arbitrary for each \(i \in \{1, 2, \ldots , 2^d\}\), by definition we have that \((x_{i_0}, \alpha_\ast) \in \mathbb{Q}^{[d]}(X)\). The proof of claim is complete.

Now we prove the lemma. Let \(y \in \{x_1, x_2, \ldots , x_{2^d}\}^{[d]}\) and \(l(y)\) denote the number of \(x_1\)'s appearing in \(y\). We prove the lemma by induction on \(l(y)\). If \(l(y) = 2^d\), then \(y = (x_1, x_1, \ldots , x_1) \in \mathbb{Q}^{[d]}(X)\).

Assume that \(y \in \mathbb{Q}^{[d]}(X)\) whenever \(l(y) = k \geq 1\). We show that if \(l(y) = k - 1\) then \(y = (y_0, x_\ast) \in \{0,1\}^d \subset \mathbb{Q}^{[d]}(X)\). Since \(l(y) = k-1 < 2^d\), there exists \(\epsilon_0 \in \{0,1\}^d\) such that \(y_{\epsilon_0} \neq x_1\). By using an Euclidean permutation, we may assume that \(\epsilon_0 = 0\). That is, \(y = (y_0, y_\ast)\) with \(y_0 \neq x_1\). Let \(z = (x_1, y_\ast)\). Then \(z \in \{x_1, x_2, \ldots , x_{2^d}\}^{[d]}\) and \(l(z) = k\). By the inductive assumption \(z = (x_1, y_\ast) \in \mathbb{Q}^{[d]}(X)\). By Claim, \(y = (y_0, y_\ast) \in \mathbb{Q}^{[d]}(X)\). The proof is complete.

By Lemma 3.1, we have the following corollary immediately.
Corollary 2. Let \((X,T)\) be a minimal topological dynamical system and \(d \geq 1\) be an integer. Let \(\pi : (X,T) \to (Z_{d-1},T)\) be the factor map to the maximal \((d-1)\)-step pro-nilfactor. If \(x \in R^d_\pi\) is a \(T^d\)-minimal point, then \(x \in Q^d(X)\).

Proof. Since \(x \in R^d_\pi\) is a \(T^d\)-minimal point, \(p_1 : \overline{O}(x,T^d) \to X\) is semi-open. The result follows from Lemma 3.1. \(\square\)

Now we have that if the factor map to the maximal \((d-1)\)-step pro-nilfactor is RIC, then \(Q^d(X)\) is \(\pi^d\)-saturated.

Proposition 1. Let \((X,T)\) be a minimal topological dynamical system and \(d \geq 1\) be an integer. Let \(\pi : (X,T) \to (Z_{d-1},T)\) be the factor map to the maximal \((d-1)\)-step pro-nilfactor. If \(\pi\) is RIC, then

\[
Q^d(X) = (\pi^d)^{-1}(Q^d(Z_{d-1})).
\]

Proof. First it is obvious that \((\pi^d)^{-1}(Q^d(Z_{d-1})) \supset Q^d(X)\). Now we show the other direction: \((\pi^d)^{-1}(Q^d(Z_{d-1})) \subset Q^d(X)\).

If \(x \in R^d_\pi\) is a \(T^d\)-minimal point, then by Corollary 2, \(x \in Q^d(X)\). Since \(\pi\) is RIC, the set of \(T^d\)-minimal points is dense in \(R^d_\pi\) and it follows that \(R^d_\pi = (\pi^d)^{-1}(\Delta_{Z_{d-1}}^d) \subset Q^d(X)\). Since \(\pi\) is RIC, \((\pi^d)^{-1}\) is continuous as mentioned in Subsection 2.7. Thus

\[
(\pi^d)^{-1}(Q^d(Z_{d-1})) = (\pi^d)^{-1}(\overline{O}(\Delta_{Z_{d-1}}^d,G^d))
\]

\[= \overline{O}((\pi^d)^{-1}(\Delta_{Z_{d-1}}^d),G^d) \subset Q^d(X).\]

The proof is complete. \(\square\)

We point out that we only use the fact that \(R_\pi \subset RP^{d-1}(X)\) in the proofs above. Since a distal extension is RIC, we have the following corollary.

Corollary 3. Let \((X,T)\) be a minimal distal topological dynamical system and \(d \geq 1\) be an integer. Let \(\pi : (X,T) \to (Z_{d-1},T)\) be the factor map to the maximal \((d-1)\)-step pro-nilfactor. Then

\[
Q^d(X) = (\pi^d)^{-1}(Q^d(Z_{d-1})).
\]

Remark 1. In Corollary 3, \(Z_{d-1}\) is the minimal factor such that the result holds. That is, if \(\pi' : X \to Y\) is a factor map such that \(Q^d(Y) = (\pi'^d)^{-1}(Q^d(Y))\), then \(Z_{d-1}\) is a factor of \(Y\).

To see this, we only need to show \(R_{\pi'} \subset RP^{d-1}(X)\). Now assume \((x_1,x_2) \in R_{\pi'}\), then \((\pi'^d)(x_1,x_2,\ldots,x_2) \in \Delta_{Z_{d-1}}^d \subset Q^d(Y)\), so \((x_1,x_2,\ldots,x_2) \in (\pi'^d)^{-1}(Q^d(Y)) = Q^d(X)\) and \((x_1,x_2) \in RP^{d-1}(X)\) by Theorem 2.6.

In the general case, we have that modulo proximal extensions \(Q^d(X)\) is \(\pi^d\)-saturated.

Theorem 3.2. Let \((X,T)\) be a minimal topological dynamical system and \(d \geq 1\) be an integer. Let \(\pi : (X,T) \to (Z_{d-1},T)\) be the factor map to the maximal \((d-1)\)-step
pro-nilfactor. Then there is a commutative diagram of homomorphisms of minimal topological dynamical systems

\[\begin{array}{ccc}
X & \xrightarrow{\theta'} & X' \\
\pi \downarrow & & \pi' \downarrow \\
Z_{d-1} & \xrightarrow{\theta} & Y'
\end{array}\]

such that \(Q^{[d]}(X') = (\pi'^{[d]})^{-1} Q^{[d]}(Y')\), where \(\theta, \theta'\) are proximal extensions.

**Proof.** By Proposition 2.8, \(\pi'\) is RIC. According to Proposition 1, we only need to prove \(R_{\pi'} \subset \text{RP}^{[d-1]}(X')\). Let \((x_1', x_2') \in R_{\pi'}\), then \(\pi'(x_1') = \pi'(x_2')\) and \(\theta\pi'(x_1') = \theta\pi'(x_2')\). Since the diagram is commutative, we have \(\pi\theta'(x_1') = \pi\theta'(x_2')\). It follows that \((\theta'(x_1'), \theta'(x_2')) \in R_{\pi} = \text{RP}^{[d-1]}(X)\).

By theorem 2.7, there exists \((x_1'', x_2'') \in \text{RP}^{[d-1]}(X')\) such that \((\theta'(x_1''), \theta'(x_2'')) = (\theta'(x_1'), \theta'(x_2'))\). Since \(\theta'\) is proximal, we have \((x_1', x_1''), (x_2', x_2'') \in R_{\theta'} \subset P(X') \subset \text{RP}^{[d-1]}(X')\).

Note that \(\text{RP}^{[d-1]}(X')\) is an equivalence relation (Theorem 2.6), we have \((x_1', x_1'') \in \text{RP}^{[d-1]}(X')\), so \(R_{\pi'} \subset \text{RP}^{[d-1]}(X')\). The proof is complete. \(\square\)

3.2. A counterexample. Let \(\pi : (X, T) \to (Z_{d-1}, T)\) be the factor map. We use the following classical system to show that without additional conditions, \(Q^{[d]}(X)\) may not be \(\pi^{[d]}\)-saturated.

**Example 1.** Sturmian system.

Let \(\alpha\) be an irrational number in the interval \((0, 1)\) and \(R_{\alpha}\) be the irrational rotation on the (complex) unit circle \(\mathbb{T}\) generated by \(e^{2\pi i \alpha}\). Set

\[A_0 = \{e^{2\pi i \theta} : 0 \leq \theta < (1 - \alpha)\}\] \[A_1 = \{e^{2\pi i \theta} : (1 - \alpha) \leq \theta < 1\}\,.

Consider \(z \in \mathbb{T}\) and define \(x \in \{0, 1\}^\mathbb{Z}\) by: for all \(n \in \mathbb{Z}\), \(x_n = i\) if and only if \(R_{\alpha}^n(z) \in A_1\). Let \(X \subset \{0, 1\}^\mathbb{Z}\) be the orbit closure of \(x\) under the shift map \(\sigma\) on \(\{0, 1\}^\mathbb{Z}\), i.e. for any \(y \in \{0, 1\}^\mathbb{Z}\), \((\sigma(y))_n = y_{n+1}\). This system is called Sturmian system. It is well known that \((X, \sigma)\) is a minimal almost one-to-one (hence proximal) extension of \((\mathbb{T}, R_{\alpha})\). Moreover, it is an asymptotic extension.

Let \(\pi : X \to \mathbb{T}\) be the former extension and consider \((x_1, x_2) \in R_{\pi} \setminus \Delta X\). Then \((x_1, x_2)\) is an asymptotic pair and thus \((x_1, x_2) \in \text{RP}^{[d]}\) for any integer \(d \geq 1\). It is showed in [5, Example 4.8] that it is not possible that for any \(d \in \mathbb{N}\), \(\{x_1, x_2\}^d \subset Q^{[d]}(X)\). Hence

\[Q^{[d]}(X) \neq (\pi'^{[d]})^{-1}(\pi'^{[d]}(Q^{[d]}(X)))\,.

That is, \(Q^{[d]}(X)\) is not \(\pi^{[d]}\)-saturated.

3.3. Topological characteristic factors along cubes. In this subsection we will use results developed above to show that up to proximal extensions the maximal \((d - 1)\)-step pro-nilfactor is the topological cubic characteristic factor of order \(d\).

First we need the following lemma, which can be proved by the method in [1] or [10, Section 4]. We set \(Q^{[d]}[x] = \{z \in Q^{[d]}(X) : z_0 = x\}\).
Lemma 3.3. Let \((X, T)\) be a minimal topological dynamical system and \(d \geq 1\) be an integer. There exists a dense \(G_δ\) set \(X_0 \subset X\) such that for each \(x \in X_0\) we have that
\[
Q^{[d]}[x] = \overline{F^{[d]}}(x^{[d]}).
\]

Proposition 2. Let \((X, T)\) be a minimal topological dynamical system and \(d \geq 1\) be an integer. Let \(\pi : (X, T) \to (Z_{d-1}, T)\) be the factor map to the maximal \((d-1)\)-step pro-nilfactor. If \(\pi\) is RIC, then \(Z_{d-1}\) is the topological cubic characteristic factor of order \(d\). That is, there exists a dense \(G_δ\) set \(X_0 \subset X\) such that for each \(x \in X_0\) with \(y = \pi(x)\) we have that
\[
\overline{F^{[d]}}(x^{[d]}) = \{x\} \times (\pi^*[d])^{-1} F^{[d]}(y^{[d]}).
\]

Proof. By Lemma 3.3, there exists a dense \(G_δ\) set \(X_0 \subset X\) such that for each \(x \in X_0\), \(Q^{[d]}[x] = \overline{F^{[d]}}(x^{[d]})\). Let \(x \in X_0\) and \(y = \pi(x)\). It is obvious that
\[
\overline{F^{[d]}}(x^{[d]}) \subset \{x\} \times (\pi^*[d])^{-1} F^{[d]}(y^{[d]}).
\]
Now we prove
\[
\overline{F^{[d]}}(x^{[d]}) \supset \{x\} \times (\pi^*[d])^{-1} F^{[d]}(y^{[d]}).
\]
Let \((x, z) \in \{x\} \times (\pi^*[d])^{-1} F^{[d]}(y^{[d]})\). Then
\[
(\pi(x), \pi^*[d](z)) = (y, \pi^*[d](z)) \in \overline{F^{[d]}}(y^{[d]}) \subset Q^{[d]}(Z_{d-1}).
\]
Hence \(\pi^{[d]}(x, z) \in Q^{[d]}(Z_{d-1})\). Since \(\pi\) is RIC, by Proposition 1,
\[
Q^{[d]}(x) = (\pi^{[d]})^{-1} \left(Q^{[d]}(Z_{d-1})\right).
\]
It follows that \((x, z) \in Q^{[d]}(X)\). Hence \((x, z) \in Q^{[d]}[x] = \overline{F^{[d]}}(x^{[d]}).\) The proof is complete. \(\square\)

The following theorem shows that up to proximal extensions the maximal \((d-1)\)-step pro-nilfactor is the topological cubic characteristic factor of order \(d\).

Theorem 3.4. Let \((X, T)\) be a minimal topological dynamical system and \(d \geq 1\) be an integer. Let \(\pi : (X, T) \to (Z_{d-1}, T)\) be the factor map to the maximal \((d-1)\)-step pro-nilfactor. Then there is a commutative diagram of homomorphisms of minimal topological dynamical systems
\[
\begin{array}{ccc}
X & \xleftarrow{\theta'} & X' \\
\pi \downarrow & & \downarrow \pi' \\
Z_{d-1} & \xleftarrow{\theta} & Y'
\end{array}
\]
such that \((Y', T)\) is the topological cubic characteristic factor of order \(d\) of \((X', T)\) and \(\theta, \theta'\) are proximal extensions.

Proof. It follows from Theorem 3.2 and Proposition 2. \(\square\)

When \(X\) is distal or weakly mixing, by the remark after Theorem 2.8 the proximal extensions \(\theta, \theta'\) in the theorem above are trivial (i.e. isomorphisms). We get

Corollary 4. 1. Let \((X, T)\) be a distal minimal topological dynamical system and \(d \geq 1\) be an integer. Then the maximal \((d-1)\)-step pro-nilfactor is the topological cubic characteristic factor of order \(d\).
2. Let \((X, T)\) be a weakly mixing minimal topological dynamical system and \(d \geq 1\) be an integer. Then the trivial system is the topological cubic characteristic factor of order \(d\).

Remark 2. In fact, for a weakly mixing minimal system we can say more. In [22], it is shown that for a weakly mixing minimal system \((X, T)\) and \(d \in \mathbb{N}\),

1. \((Q^d, G^d)\) is minimal and \(Q^d = X^d\).
2. For all \(x \in X\), \((F^d(x^d), F^d)\) is minimal and \(F^d(x^d) = \{x\} \times X^d = \{x\} \times X^{d-1}\).

### 3.4. Topological characteristic factors along cubes of distal systems

Corollary 4 says that for a distal minimal system the maximal \((d-1)\)-step pro-nilfactor is the topological cubic characteristic factor of order \(d\). In fact we can say more. In this subsection, we use a different approach to deal with distal systems. We will show the following result:

**Proposition 3.** Let \((X, T)\) be a minimal distal topological dynamical system and \(d \geq 1\) be an integer. Let \(\pi : (X, T) \to (\mathbb{Z}_{d-1}, T)\) be the factor map to the maximal \((d-1)\)-step pro-nilfactor. Then for each \(x \in X\) with \(y = \pi(x)\) we have that

\[
\overline{F^d(x^d)} = \{x\} \times (\pi_x^d)^{-1}(\overline{F^d(y^d)}).
\]

In particular, \(\mathbb{Z}_{d-1}\) is the topological cubic characteristic factor of order \(d\).

**Remark 3.** In the definition of the topological cubic characteristic factor of order \(d\), we require (3) holds for a dense \(G_\delta\) set. But for distal systems, Proposition 3 shows (3) holds for all \(x \in X\).

First we need some lemmas. By the proof of [22, Theorem 3.1] we can show the following lemma. Please see [11] for another proof.

**Lemma 3.5.** Let \((X, T)\) be a topological dynamical system and \(d \geq 1\) be an integer. If \(x \in X^d\) is an id \(T^d\)-minimal point, then \(x\) is an \(F^d\)-minimal point.

Recall that \(Q^d[x] = \{z \in Q^d(X) : z_0 = x\}\).

**Lemma 3.6.** [22, Proposition 5.2] Let \((X, T)\) be a minimal topological dynamical system and \(d \geq 1\) be an integer. If \(x \in Q^d[x]\), then \(x^d \in \overline{F^d(x)}\). Especially, \((\overline{F^d(x^d)}), F^d)\) is the unique \(F^d\)-minimal subset in \(Q^d[x]\).

**Lemma 3.7.** Let \((X, T)\) be a minimal topological dynamical system and \(d \geq 1\) be an integer. Let \(\pi : (X, T) \to (\mathbb{Z}_{d-1}, T)\) be the factor map to the maximal \((d-1)\)-step pro-nilfactor. Assume that \(x = (x_\epsilon)_{\epsilon \in \{0, 1\}} \in R^d_{\pi}\) is an id \(T^d\)-minimal point. Then \(x \in \overline{F^d(x_0^d)}\).

**Proof:** In fact we will show that if \(x = (x_\epsilon)_{\epsilon \in \{0, 1\}} \in R^d_{\pi}\) is a \(F^d\)-minimal point, then \(x \in \overline{F^d(x_0^d)}\). By Lemma 3.5, an id \(T^{d-1}\)-minimal point is \(F^{d-1}\)-minimal, Lemma 3.7 follows from the above result.

To simplify the notation, we write \(x = x_0\) in what follows. We show that if \(x = (x_\epsilon)_{\epsilon \in \{0, 1\}} \in R^d_{\pi}\) is a \(F^d\)-minimal point, then \(x \in \overline{F^d(x_0^d)}\). We proceed by induction on \(d\). When \(d = 1\), \(\mathbb{Z}_{d-1} = Z_0\) is the trivial system and \(F^1\) is spanned by \(T^1 = \text{id} \times T\). Assume that \(x = (x, x_1) \in R^2_{\pi} = X \times X\) is id \(T\)-minimal. Since \((X, T)\) is minimal, it is obvious that \(x \in \overline{F^1(x, x)}\).
To explain the idea of the proof, we show the case $d = 2$. The idea comes from the proof of [22, Theorem 6.4]. Assume that
\[ x = (x_{00}, x_{10}, x_{001}, x_{110}, x_{0011}) = (x, x_{01}, x_{001}, x_{111}) \]
is a $\mathcal{F}^2$-minimal point. Take \( \{n_k\} \subset \mathbb{Z} \) such that \( T^{n_k} x_{10} \to x, k \to \infty \) and assume that \( T^{n_k} x_{11} \to x_{11}^\prime, k \to \infty \). Then
\[ (id \times T \times id \times T)^{n_k} (x_{10}, x_{01}, x_{110}) \to (x, x_{01}, x_{110}^\prime), k \to \infty. \]
Now take \( \{m_k\} \subset \mathbb{Z} \) such that \( T^{m_k} x_{01} \to x, k \to \infty \) and assume that \( T^{m_k} x_{11} \to x_{11}^\prime, k \to \infty. \) Then
\[ (id \times id \times T \times T)^{m_k} (x_{01}, x_{110}, x_{110}^\prime) \to (x, x, x_{110}^\prime), k \to \infty. \]

Since id $\times T \times id \times T = T_1^2 \in \mathcal{F}^2$ and id $\times id \times T \times T = T_2^2 \in \mathcal{F}^2$, we have \( (x, x, x, x_{110}^\prime) \in \overline{\mathcal{F}^2}(x) \) and \( (x, x_{110}) \in R_\pi \). As \( \pi : X \to Z_1 \), we have that \( (x, x_{110}^\prime) \in \mathcal{R} \mathcal{P}^1(\pi) \). It follows that \( (x, x, x_{110}, x_{110}^\prime) \in \mathcal{Q}^2 \). As \( x \) is a $\mathcal{F}^2$-minimal point, so does \( (x, x, x_{110}, x_{110}^\prime) \). By Lemma 3.6, \( (\overline{\mathcal{F}^2}(x_{110}), \mathcal{F}^2) \) is the unique $\mathcal{F}^2$-minimal subset in \( \mathcal{Q}^2 \), so \( (x, x, x_{110}, x_{110}^\prime) \in \overline{\mathcal{F}^2}(x_{110}) \). Since \( x \) is a $\mathcal{F}^2$-minimal point and \( (x, x, x_{110}, x_{110}^\prime) \in \overline{\mathcal{F}^2}(x) \), we have that \( x \in \overline{\mathcal{F}^2}(x_{110}) \). We are done for the case \( d = 2 \).

Now we assume that the result holds for \( d - 1 \). Assume that \( x \in R^d_\pi \) is a $\mathcal{F}^d_{\pi}$-minimal point. We show that \( x \in \overline{\mathcal{F}^d}(x_{\pi}) \). First we have the following claim, whose proof is given later.

**Claim.** There is some \( x' \in X \) such that \( (x, x') \in R_\pi \) and
\[ (x_{\pi}^d, x') = (x, x, \ldots, x, x') \in \overline{\mathcal{F}^d}(x). \]

Assuming the claim, since \( R_\pi = \mathcal{R} \mathcal{P}^{d-1}(\pi) \), we have that \( (x_{\pi}^d, x') \in \mathcal{Q}^{d-1}[\pi] \). This can be seen using Euclidean permutations (or see [22, Lemma 6.2] for a proof). Since \( x \in R^d_\pi \) is a $\mathcal{F}^d_{\pi}$-minimal point, so does \( (x_{\pi}^d, x') \). By Lemma 3.6 \( (\overline{\mathcal{F}^d}(x_{\pi}^d), \mathcal{F}^d) \) is the unique $\mathcal{F}^d$-minimal subset in \( \mathcal{Q}^d \), and we have that \( (x_{\pi}^d, x') \in \overline{\mathcal{F}^d}(x_{\pi}^d) \). Since \( x \) is a $\mathcal{F}^d$-minimal point and \( (x_{\pi}^d, x') \in \overline{\mathcal{F}^d}(x) \), we have that \( x \in \overline{\mathcal{F}^d}(x_{\pi}^d) \).

Now we prove the Claim. We show the case \( d = 3 \), and general case is similar. Let
\[ x = (x_{000}, x_{100}, x_{010}, x_{110}, x_{001}, x_{101}, x_{011}, x_{111}). \]
Then \( x = x_{000} \). Since \( x \) is a $\mathcal{F}^3$-minimal point, the points in the lower dimensional faces are also minimal. For example, \( (x_{000} = x, x_{100}, x_{010}, x_{110}) \) is $\mathcal{F}^2$-minimal. Thus by the inductive hypothesis, \( (x, x_{100}, x_{010}, x_{110}) \in \overline{\mathcal{F}^2}(x_{100}) \). Hence there is some sequence \( F_k^1 \in \mathcal{F}^2 \) such that
\[ F_k^1 (x_{000}, x_{100}, x_{010}, x_{110}) \to x_{100} = (x, x, x), k \to \infty. \]
We may assume that
\[ F_k^1 (x_{001}, x_{101}, x_{011}, x_{111}) \to (x_{001}, x_{101}, x_{011}, x_{111}), k \to \infty. \]
Since \( F_k^1 \times F_k^1 \in \mathcal{F}^3 \), we have that
\[ (x, x, x, x_{001}, x_{101}, x_{011}, x_{111}) \in \overline{\mathcal{F}^3}(x) \].
Since $x$ is a $\mathcal{F}^3$-minimal point, $(x, x, x, x_{001}, x_{101}, x_{011}, x_{111})$ is $\mathcal{F}^3$-minimal and it follows that $(x, x, x_{001}, x_{101})$ is $\mathcal{F}^2$-minimal. Thus by the inductive hypothesis, $(x, x, x_{001}, x_{101}) \in \mathcal{F}^2(x^{[2]})$. Hence there is some sequence $F_k^2 \in \mathcal{F}^2$ such that

$$F_k^2(x, x, x_{001}, x_{101}) \to x^{[2]} = (x, x, x, x), \quad k \to \infty.$$ 

We may assume that

$$F_k^2(x, x, x_{011}, x_{111}) \to (x, x, x_{011}, x_{111}), \quad k \to \infty.$$ 

Let $F_k^2 = (F_k^{21}, F_k^{22})$, where $F_k^{21}$ and $F_k^{22}$ act on $X^2$. Then $(F_k^{21}, F_k^{21}, F_k^{22}, F_k^{22}) \in \mathcal{F}^3$, we have that

$$(x, x, x, x, x, x_{011}, x_{111}) \in \mathcal{F}^3(x).$$

Similarly we have $(x, x, x_{011})$ is $\mathcal{F}^2$-minimal. Again by the inductive hypothesis, there is some sequence $F_k^3 \in \mathcal{F}^2$ such that

$$F_k^3(x, x, x_{011}) \to x^{[2]} = (x, x, x), \quad k \to \infty.$$ 

We may assume that

$$F_k^3(x, x, x_{111}) \to (x, x, x), \quad k \to \infty.$$ 

Let $F_k^3 = (f_k^1, f_k^2, f_k^3, f_k^4)$. Then $(f_k^1, f_k^1, f_k^2, f_k^3, f_k^3, f_k^4) \in \mathcal{F}^3$, we have that

$$(x, x, x, x, x, x_{111}) \in \mathcal{F}^3(x).$$

It is easy to check that $(x, x^t) \in R_\pi$. The proof is complete. \hfill $\square$

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