We briefly review the significance of quantum corrections in the total energy of systems with voids: bubble nuclei, atomic clusters and the inhomogeneous phase of neutron stars.

It was suggested a long time ago that very large nuclei might not undergo a Coulomb explosion if they acquire a new topology, that of a bubble or a torus. When a void is formed, while the density and therefore the total volume is kept unchanged, the surface area of such a nucleus naturally increases and that leads to an increased surface energy and less binding. However, at the same time the average distance between protons increases as well and the total Coulomb energy then decreases. The balancing of these two types of energy and the fact that configurations with larger binding energy than the familiar compact geometries exists is the reason why bubble–like and torus–like nuclei could in principle be someday observed. It was realized however that shell effects play a crucial role in stabilizing these new shapes. During the last decade many experimentalists have tried to manufacture highly charged metallic clusters, but, again, Coulomb repulsion prevented their creation. The idea that objects with a different topology, in particular bubble–like charged metallic clusters could be a possible route to create highly charged metallic clusters was recently put forward, and again the stabilizing role of the shell corrections was noted as playing a decisive role.

There was an aspect of bubble systems, which for mysterious reasons never caught the attention of previous authors: Where should one position a bubble inside a nucleus? Symmetry considerations seem to suggest that a spherical

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bubble should be placed at the center of a spherical system. A closer look will show however that there is something more than mere symmetry and that Coulomb energy plays perhaps the most important role in stabilizing the bubble position. It is relatively straightforward to show that if one were to displace a bubble from the center of a nucleus the Coulomb energy would increase. When considering Coulomb effects, one can think of a bubble as being a charged object, having the same charge density as the rest of the matter, but of opposite sign. One can then easily evaluate the Coulomb force acting on a bubble. Inside a spherical uniformly charged object the electric field is radial and can be easily be evaluated using Gauss law:

\[ E(r) = \frac{4\pi \rho_0 r}{3}, \]  

(1)

where \( \rho_0 \) is the charge density. Thus the force acting on a bubble is simply the integral over the bubble “effective charge” times the electric field

\[ F_b = -\int_{\text{bubble}} d^3r \frac{4\pi \rho_0^2 r}{3} = -\frac{4\pi \rho_0^2 V_b}{3} \mathbf{R}, \]  

(2)

where \( V_b \) is the volume of the bubble and \( \mathbf{R} \) is the position vector of the bubble center, with respect to the nuclear center. When the suggestion was made to make cavities inside charged metallic clusters it became clear to us that the above argument is incomplete and there is no apparent physical candidate responsible for determining the optimal bubble position inside a homogeneous fermi system. As freshmen physics students know, there is no electric field inside a metal in the absence of electric currents. If in the case of nuclei one could invoke, either symmetry arguments (for some not totally clear reasons) or, better yet, the stabilizing role of Coulomb force, it was not obvious what made a bubble system stable in the case of a metal cluster. None of the “usual suspects” (volume, surface, curvature or Coulomb energies) seem to play any significant role and one might naturally expect that if there is something happening in a metal cluster, a similar mechanism should most likely be operative in a nucleus as well. (In metal clusters one has of course to deal with additional ionic degrees of freedom, however, many cluster properties are determined mostly by the electrons alone and the ions are merely spectators.) The solution to the above puzzle was rather simple, but at the same time to a large extent unexpected as well: the physics of a bubble is governed by pure quantum effects, known in nuclear physics as shell corrections and in quantum fielded theory as Casimir energy. Instead of presenting formulas and results of numerical calculations we shall limit ourselves here to a general discussion of some of the novel aspects of these systems and refer the interested readers to the available references.
When one mentally starts pushing a bubble around inside a finite fermi system one obviously excites such a system, if initially the bubble was in its optimal position and therefore the entire system in its ground state. Since the displacement of a bubble will affect many particles, bubble displacements are naturally collective excitations. Apart from collective pairing excitations, perhaps no other collective mode in a fermi system is purely quantum in nature. Since shell corrections effects scale with particle number as $\propto N^{1/6}$, see Ref. 7, and other collective modes involve some degree of surface deformation, and therefore their effects scale with particle number as $\propto N^{2/3}$, one can expect that bubble displacements would correspond to perhaps the softest collective modes possible. Our vast experience seem at this point to lend support to the idea that symmetry should play a major role in determining the optimal position of a bubble, since shell correction effects are largest for spherical systems. To some extent this is true, see Ref. 5, however with many provisos. Even if a system is “magic”, once one would displace a bubble off center significantly, the potential energy surface becomes rather flat. One would also expect that the amplitude of the shell corrections will become smaller when the bubble is significantly off center, since classically the motion of a particle in the corresponding single–particle potential is chaotic to a large degree and the single–particle spectrum is expected to have no large gaps. As our detailed numerical results show this expectation is hardly ever true. In all our numerical analyses so far we have used hard wall potentials (which thus partially explains the origin of the term quantum billiards in our title), for which there is significant evidence that they do reproduce the realistic spectra with sufficient accuracy for the purpose of computing the gross shell structure. A particular feature of the shell correction energy evaluated for hard wall potential, and which we do not expect to survive entirely in a selfconsistent calculation, is particularly interesting however, as it underlines a general trend. We have observed in Ref. 5, and later confirmed as a general feature in Refs. 10, that the amplitude of the shell correction energy increases as the bubble approaches the boundary of the system. This is particularly puzzling, since the closer the bubble is to the system boundary the classical motion is more chaotic and one would naturally expect then the shell energy to decrease, but not to increase. Part of the explanation is that a particular periodic orbit becomes prominent and leads to a significant “scarring” of the single–particle density of states. This is the orbit bouncing between the points of closest approach. The relative size of the bubble also plays a major role. If the fractional volume of the bubble is small, then the shell correction energy oscillates with a relatively small amplitude when compared with a bubble with a larger fractional volume.

Sidestepping the question of bubble stability and of the energy cost of
bubble formation, one can reasonably ask a number of quite relevant questions as well: “Why not have a system with two or more bubbles?” Neutron stars have been predicted a long time ago to have a locally inhomogeneous phase, often referred to as “the pasta phase” \(^\text{[11]}\). Due to the same type of interplay between the surface and Coulomb energies, at depths of about 0.5 km below the surface of a neutron star and at densities just below nuclear saturation density a new phase is favored, where spherical and rod–like nuclei embedded in a neutron gas, plates, cylinders and bubbles exist. Almost all previous analyses of this phase have been performed in the liquid drop or Thomas–Fermi approximations. It was determined that on the way inside a neutron star, while the average density is increasing, there is a well defined sequence of phases: nuclei → rods → plates → tubes → bubbles → uniform matter. The energy of each of these phases is significantly below the energy of the uniform phase at the same average density, irrespective of nuclear model used \(^\text{[11]}\). The energy differences between various phases even though are very small, of the order of keV’s per fm\(^3\), are apparently independent of the model for the nuclear forces used. The various models for nuclear forces can lead to significant variations in the values of the interface surface tension. Shell correction energy on the other hand is known to be of geometric origin essentially. Since in infinite matter the presence of various inhomogeneities does not lead to the formation of discrete levels, one might call the corresponding energy correction for neutron matter the Casimir energy \(^\text{[6]}\). The inhomogeneous phase of a neutron star is basically nothing else but a Sinai billiard, a model which is widely popular in classical and quantum chaos studies. In a first approximation one can treat various objects in the inhomogeneous phase as spherical, cylindrical or plate like voids in a neutron gas.

In order to better appreciate the nature of the problem we are addressing here, let us consider the following situation. Let us imagine that two spherical identical bubbles have been formed in an otherwise homogeneous neutron matter. For the sake of simplicity, we shall assume that the bubbles are completely hollow. We shall ignore for the time being the role of the Coulomb interactions, as their main contribution is to the smooth, liquid drop or Thomas–Fermi part of the total energy. Then one can ask the following apparently innocuous question: “What determines the most energetically favorable arrangement of the two bubbles?” According to a liquid drop model approach (completely neglecting for the moment the possible stabilizing role of the Coulomb forces) the energy of the system should be insensitive to the relative positioning of the two bubbles. Using Gutzwiller trace formula one can show that pure quantum
effects lead to an approximate interaction energy of the following form

\[
E_{\text{int}} = \frac{\hbar^2 k_F^2}{2m} \frac{R^2}{\pi a(a - 2R)} \left\{ \cos \left[ 2k_F(a - 2R) \right] \frac{- \sin \left[ 2k_F(a - 2R) \right]}{4k_F(a - 2R)^2} \right\}
\]

(3)

where \( R \) is the bubble radius, \( a \) is the distance between the bubble centers and \( k_F \) is the Fermi wave length, see Fig. 1. It came as surprise to us to find that two bubbles have a long range interaction. In condensed matter physics a similar type of interaction is known for about a half of a century, the interaction between two impurities in a fermi gas. The fact that this interaction oscillates suggest the intriguing possibility of forming di–bubble molecules with various radii. However, until one will determine the inertia of a di–bubble system it is not obvious whether such a molecule could indeed exist. It can be shown that in the case of three or more bubbles the interaction among them contains besides the expected pair–wise interaction we have just described, also genuine three–body, four–body and so forth interactions. The interaction Rel. (3) has its origin in the existence of the periodic orbit bouncing between the two bubbles. In the case of three or more bubbles there are distinct periodic orbits bouncing between three or more objects, which are the reason these genuine three and more body interactions arise.

Using semiclassical methods (Gutzwiller trace formula), we have analyzed the structure of the shell energy as a function of the density, filling factor, lattice distortions and temperature. The main lesson we have learned is that
the amplitude of the shell energy effects is comparable with the energy differences between various phases determined in simpler liquid drop type models. Our results suggest that the inhomogeneous phase has perhaps an extremely complicated structure, maybe even completely disordered, with several types of shapes present at the same time.

At higher densities in neutron stars one expects that quarks and mesons will lead to similar structured mixed phases. The formation of either quark–gluon droplets embedded in a hadron gas or of hadron droplets embedded in a quark–gluon plasma has been studied and predicted for almost a decade. One naturally expects that similar quantum corrections are relevant in these cases as well.

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