NP-completeness Proof: RBCDN Reduction Problem

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Suppose \{R_1, \ldots, R_k\} is the set of all possible regions \([?]\) of graph \(G\). Consider a \(k\)-dimensional vector \(C\) whose \(i\)-th entry, \(C[i]\), indicates the number of connected components in which \(G\) decomposes when all nodes in \(R_i\) fails. Then, region-based component decomposition number (RBCDN) of graph \(G\) with region \(R\) is defined as \(\alpha_R(G) = \max_{1 \leq i \leq k} C[i]\).

Suppose the RBCDN of \(G\) with region \(R\) is \(\alpha_R(G)\). If \(\alpha_R(G)\) is considered to be too high for the application and it requires RBCDN of the network not to exceed \(\alpha_R(G) - K\), for some integer \(K\). Assuming each additional link \(l_i\) that can be added to the network has a cost \(c(i)\) associated with it, find the least cost link augmentation to the network so that its RBCDN is reduced from \(\alpha_R(G)\) to \(\alpha_R(G) - K\). Formal description of the decision version of this problem is given below.

RBCDN Reduction Problem (RBCDN-RP)

INSTANCE: Given
(i) a graph \(G = (V, E)\) where \(V = \{v_1, \ldots, v_n\}\) and \(E = \{e_1, \ldots, e_m\}\) are the sets of nodes and links respectively,
(ii) the layout of \(G\) on a two dimensional plane \(LG = (P, L)\) where \(P = \{p_1, \ldots, p_n\}\) and \(L = \{l_1, \ldots, l_m\}\) are the sets of points and lines on the 2-dimensional plane,
(iii) region \(R\) defined to be a circular area of radius \(r\).
(iv) cost function \(c(e) \in \mathbb{Z}^+, \forall e \in \bar{E}\), where \(\bar{E}\) is complement of the link set \(E\).
(v) integers \(C\) and \(K\) (\(K \leq \alpha_R(G)\), where \(\alpha_R(G)\) is the RBCDN of \(G\)).

QUESTION: Is it possible to reduce the RBCDN of \(G\) by \(K\) by adding edges to \(G\) (from the set \(\bar{E}\)) so that the total cost of the added links is at most \(C\)?
NP-Completeness Proof of $RBCDN-RP$

We prove that $RBCDN-RP$ is NP-complete by a transformation from the Hamiltonian Cycle in Planar Graph Problem (HCPGP) which is known to be NP-complete \cite{?}. A Hamiltonian Cycle in an undirected graph $G = (V, E)$ is a simple cycle that includes all the nodes. A graph is a planar if it can be embedded in a plane by mapping each node to a unique point in the plane and each edge is a line connecting its endpoints, so that no two lines meet except at a common endpoint \cite{?}.

Hamiltonian Cycle in Planar Graph Problem (HCPGP)
INSTANCE: Given an undirected planar graph $G = (V, E)$.
QUESTION: Does $G$ contains a Hamiltonian Cycle?

**Theorem 1** $RBCDN$ is NP-complete.

*Proof:* It is easy to verify whether a set of additional edges of total cost $\leq C$ reduces the $RBCDN$ of graph $G$ with region $R$ from $\alpha_R(G)$ to $\alpha_R(G) - K$. Therefore $RBCDN-RP$ is in NP.

From an instance of the HCPGP (a planar graph $G = (V, E)$) we create an instance of the $RBCDN-RP$ (the layout $LG' = (P', L')$ of a graph $G' = (V', E')$) in the following way. First, we do a straight line embedding of the planar graph $G$ on a plane so that lines corresponding to links in $G$ do not intersect each other. Such an embedding can be carried out in polynomial time \cite{?}. We call this layout $LG'' = (P'', L'')$. We create another layout $LG' = (P', \emptyset)$, by setting $P' = P''$ and $L' = \emptyset$. The graph $G'$ corresponding to the layout $LG' = (P', L')$ is the instance of $RBCDN-RP$ created from the instance of HCPGP. We define a region $R$ to be circular area of sufficiently small radius $r$, such that if a region fails, it can only destroy (i) a single node with all links incident on it,
or (ii) a single link. Since the created instance of $RBCDN-RP$ has no links ($E' = L' = \emptyset$), the $RBCDN$ of $G'$ with region $R$ is $n$ where $n = |V'| = |P'|$.

We set the parameters $C$ and $K$ of the instance of the $RBCDN-RP$ to be equal to $n$ and $n - 1$ respectively. We assign costs to the links of $\bar{E}'$ in the following way. The cost of a link $c(e) = 1$, if $e \in (E \cap \bar{E}')$ and $c(e) = \infty$, if $e \in (\bar{E} \cap \bar{E}')$.

If the instance of the $HCPGP$ has a Hamiltonian Cycle, we can use the set of links that make up the cycle, to augment the link set $E'$ of the instance $G' = (V', E')$ of the $RBCDN-RP$. The augmented $G'$, $(G'_{aug})$, is now a simple cycle that involves all the nodes. With the given definition of region $R$ (a small circle of radius $r$), only one node can be destroyed when a region fails. Accordingly $RBCDN$ of $G'_{aug}$ is 1. It may be recalled that $RBCDN$ of $G'$ is $n$. Accordingly, augmentation of the link set of $G'$ reduced its $RBCDN$ by $n - 1$. Due to the specific cost assignment rule of the links, the total cost of link augmentation is $n$. Therefore, if the $HCPGP$ instance has a Hamiltonian Cycle, the $RBCDN$ of the instance of $RBCDN-RP$ can be reduced by $K$ with augmentation cost $\leq C$.

Suppose that it is possible to reduce the $RBCDN$ of the instance of $RBCDN-RP$ by $K$ with augmentation cost being at most $C$. This implies that the $RBCDN$ of $G'$ can be reduced from $n$ to 1 (as $K = n - 1$) when it is augmented with additional links with total cost at most $n$ (as $C = n$). In order for the $RBCDN$ of $G'_{aug}$ to be 1, the node connectivity of $G'_{aug}$ must be at least 2. A $n$ node graph that has the fewest number of links and yet is 2-connected, is a cycle that includes all the nodes. As $G'$ had no links, this implies at least $n$ links must have been added to create the augmented graph $G'_{aug}$. Given that the cost of a link $c(e) = 1$, if $e \in (E \cap \bar{E}')$ and $c(e) = \infty$, if $e \in (\bar{E} \cap \bar{E}')$, and total cost of link augmentation is at most $n$, it is clear that the links used in augmenting $G$ must be from the set $(E \cap \bar{E}')$. These links are part of the edge set of the instance of $HCPGP$. Accordingly, the instance of $HCPGP$ must have a Hamiltonian Cycle.