Second Zagreb indices of transformation graphs and total transformation graphs

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Abstract: In this paper, we obtained some new properties of Zagreb indices. We mainly give explicit formulas to the second Zagreb index of semitotal-line graph (or middle graph), semitotal-point graph and total transformation graphs $G^{xyz}$.

Keywords: Degree of vertices, Zagreb indices, transformation graphs.

MSC: 05C90, 05C35, 05C12.

1. Introduction

Let $G = (V, E)$ be a graph. We denote the number of vertices of $G$ by $n$ and the number of edges by $m$, i.e., $|V(G)| = n$ and $|E(G)| = m$. The degree of a vertex $v$, denoted by $d_G(v)$ is the number of edges incident to $v$. For undefined terminologies, we refer the reader to [1]. A graph invariant is any function on a graph that does not depend on a labeling of its vertices and are called topological indices. Hundreds of different invariants have been employed to date (with unequal success) in QSAR/QSPR studies. Among them two are Zagreb indices. Due to their chemical relevance, they have been subject of numerous papers in literature [2–5]. There two invariants are called the first Zagreb index and second Zagreb index [6–11] and are defined as:

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v),$$

respectively.

In fact, one can rewrite the first Zagreb index as:

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$

Noticing the contribution of nonadjacent vertex pairs when computing the weighted Winer polynomials of certain composite graphs, the authors in [6] defined the first Zagreb coindex and the second Zagreb coindex as:

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)] \quad \text{and} \quad \overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u) d_G(v),$$

respectively.

1.1. Transformation and total transformation graphs

Transformation graphs receives information from the original graph and converts source information into a new structure. If it is possible to figure out the given graph from the transformed graph in polynomial time, such operation may be used to survey miscellaneous structural properties of the original graph considering the transformation graphs. Therefore it fosters the research of transformation graphs and their structural properties [12].
It is interesting to see that

Also for a given graph

are adjacent if and only if they are adjacent or incident in

graph

For a positive integer

2. Results

the second Zagreb index of semitotal-point graph, semitotal-line graph and eight total transformation graphs.

Observation 2. For any nonempty graph G, we have

\[ \sum_{u,v \in E(G)} [d_G(u)^2 + d_G(v)^2] = \sum_{w \in V(G)} d_G(w)^3 = \xi_3(G). \]

Theorem 1. [6] Let G be any nontrivial graph of order n and size m. Then

\[ \overline{M_2}(G) = 2m^2 - M_2(G) - \frac{1}{2} M_1(G). \]

In the next theorem, the explicit formulas of first Zagreb index are given [19].
Theorem 2. [19] Let G be any nontrivial graph of order n and size m. Then

\[ M_1(T_1(G)) = M_1(G) + 2M_2(G) + \xi_3(G), \]
\[ M_1(T_2(G)) = 4(m + M_1(G)), \]
\[ M_1(G^{+++}) = 4M_1(G) + 2M_2(G) + \xi_3(G), \]
\[ M_1(G^{---}) = (m + n)((m + n)^2 + 6m - 2n + 1) + 8m + 2(m + n - 3)M_1(G) + 2M_2(G) + \xi_3(G), \]
\[ M_1(G^{+++}) = mn(m + n - 8) + 16m + 2(n - 4)M_1(G) + 2M_2(G) + \xi_3(G), \]
\[ M_1(G^{--}) = n(n - 1)^2 + m(m + 3)^2 - 2(m + 3)M_1(G) + 2M_2(G) + \xi_3(G), \]
\[ M_1(G^{+-}) = m(m + 3)^2 - 2(m + 1)M_1(G) + 2M_2(G) + \xi_3(G). \]

In the following Lemma, the order and size of transformation graphs are given [19].

Lemma 1. [19] Let G be a nontrivial graph of order n and size m. Then

\[ |V(T_1(G))| = m + n, \quad |E(T_1(G))| = m = \frac{1}{2}[2m + M_1(G)]. \]
\[ |V(T_2(G))| = m + n, \quad |E(T_2(G))| = 3m. \]
\[ |V(G^{+++})| = m + n, \quad |E(G^{+++})| = m = \frac{1}{2}[4m + M_1(G)]. \]
\[ |V(G^{---})| = m + n, \quad |E(G^{---})| = m = \frac{1}{2}[(m + n - 1)(m + n) - 4m - M_1(G)]. \]
\[ |V(G^{+-})| = m + n, \quad |E(G^{+-})| = m = \frac{1}{2}[2m(n - 2) + M_2(G)]. \]
\[ |V(G^{--})| = m + n, \quad |E(G^{--})| = m = \frac{1}{2}[m(m + n) + n(n + 1) - M_1(G)]. \]
\[ |V(G^{++})| = m + n, \quad |E(G^{++})| = m = \frac{1}{2}[m(m + 7) - M_1(G)]. \]
\[ |V(G^{+-})| = m + n, \quad |E(G^{+-})| = m = \frac{1}{2}[n(m + n - 1) + m(n - 8) + M_1(G)]. \]
\[ |V(G^{++})| = m + n, \quad |E(G^{++})| = m = \frac{1}{2}[n(n - 1) + M_1(G)]. \]
\[ |V(G^{+-})| = m + n, \quad |E(G^{+-})| = m = \frac{1}{2}[m(m + 2n - 1) - M_1(G)]. \]

In the next Lemma, the edge partition of transformation graphs in terms of \( E(G) \) and \( E(L(G)) \) are given.

Lemma 2. Let G be a nontrivial graph of order n and size m. Then

1. \( E(G^{+++}) = E(G) \cup E(L(G)) \cup 2E(G), \)
2. \( E(G^{---}) = E(G) \cup E(L(G)) \cup (n - 2) \cdot E(G), \)
3. \( E(G^{++}) = E(G) \cup E(L(G)) \cup (n - 2) \cdot E(G), \)
4. \( E(G^{--}) = E(G) \cup E(L(G)) \cup 2E(G), \)
5. \( E(G^{+-}) = E(G) \cup E(L(G)) \cup 2E(G), \)
6. \( E(G^{+-}) = E(G) \cup E(L(G)) \cup (n - 2) \cdot E(G), \)
7. \(E(G^{++}) = E(\overline{G}) \cup E(L(G)) \cup 2E(G)\),
8. \(E(G^{---}) = E(G) \cup E(L(G)) \cup (n - 2) \cdot E(G)\).

**Theorem 3.** Let \(G\) be a nontrivial graph of order \(n\) and size \(m\). Then \(M_2(T_1(G)) = 4(4m + M_1(G))\).

**Proof.** Note that for \(u \in V(T_1(G)) \cap V(G)\), \(d_{T_1(G)}(u) = 2d_G(u)\) and for \(u \in V(T_1(G)) \cap E(G)\), \(d_{T_1(G)}(u) = 2\). Therefore by Lemma 2,

\[
M_2(T_1(G)) = \sum_{u \in V(T_1(G))} (2d(u))^2 + 2 \sum_{u \in V(T_1(G)) \cap V(G)} 4d(u) = 4(4m + M_1(G)).
\]

as desired. \(\square\)

**Theorem 4.** Let \(G\) be a nontrivial graph of order \(n\) and size \(m\). Then \(M_2(T_2(G)) = 2M_1(G) + 4M_2(G) + \xi_3(G)\).

**Proof.** Suppose \(e = uv\) is a vertex in \(T_2(G)\). It can be easily seen that \(d_{T_2(G)}(e) = d_G(u) + d_G(v)\) and if \(u \in V(T_2(G)) \cap V(G)\), then \(d_{T_2(G)}(u) = d_G(u)\). Therefore by Lemma 2,

\[
M_2(T_2(G)) = \sum_{u \in V(T_2(G)) \cap V(G)} (d(u) + d(v))^2 + 2d(u) \sum_{u \in V(T_2(G)) \cap E(G)} (d(u) + d(v)) = 2M_1(G) + 4M_2(G) + \xi_3(G).
\]

This completes the proof. \(\square\)

**Theorem 5.** Let \(G\) be a nontrivial graph of order \(n\) and size \(m\). Then \(M_2(G^{++}) = 8M_1(G) + 6M_2(G) + \xi_3(G)\).

**Proof.** Note that \(E(G^{++}) = E(G) \cup E(L(G)) \cup 2E(G)\) and for \(u \in V(G^{++}) \cap V(G)\), \(d_{G^{++}}(u) = 2d_G(u)\) and for \(u \in V(G^{++}) \cap E(G)\), \(d_{G^{++}}(u) = d_G(u) + d_G(v)\). Therefore by Lemma 2,

\[
M_2(G^{++}) = \sum_{u \in V(G^{++}) \cap V(G)} 2d(u)^2 + \sum_{u \in V(G^{++}) \cap V(G)} (d(u) + d(v))^2 + 4d(u) \sum_{u,v \in E(G)} (d(u) + d(v))
\]

\[
= 4M_1(G) + 2M_2(G) + \xi_3(G) + 4M_1(G) + 4M_2(G)
\]

\[
= 8M_1(G) + 6M_2(G) + \xi_3(G).
\]

as asserted. \(\square\)

**Theorem 6.** Let \(G\) be a nontrivial graph of order \(n\) and size \(m\). Then \(M_2(G^{---}) = mn(m + n)^2 - 2m(m + n)(3n + 4) + m(n + 16) - 3(m + n - 1)M_1(G) + 2(n - 1)M_2(G) + \xi_3(G)\).

**Proof.** Note that \(E(G^{---}) = E(\overline{G}) \cup E(L(G)) \cup (n - 2) \cdot E(G)\), and for \(u \in V(G^{---}) \cap V(G)\), \(d_{G^{---}}(u) = m + n - 1 - 2d_G(u)\) and for \(u \in V(G^{---}) \cap E(G)\), \(d_{G^{---}}(u) = m + n - 1 - (d_G(u) + d_G(v))\). Therefore by Lemma 2,

\[
M_2(G^{---}) = \sum_{u \in V(G^{---}) \cap V(G)} (m + n - (1 + 2d_G(u))^2 + \sum_{u \in V(G^{---}) \cap E(G)} (m + n - (1 + (d_G(u) + d_G(v))))^2
\]

\[
+(n - 2)(m + n - (1 + 2d_G(u)) \sum_{u,v \in E(G)} (m + n - (1 + (d_G(u) + d_G(v))))
\]

\[
= [m(m + n)^2 + 2m(m + n) + 17m + 4M_1(G)] + [m(m + n)^2 + 2m(m + n) + 2m(m + n) + m
\]

\[
-2(m + n - 1)M_1(G) + 2M_2(G) + \xi_3(G)] + (n - 2) [m(m + n)^2 - 6m(m + n) + 5m
\]

\[
- (m + n + 3)M_1(G) + 2M_2(G)].
\]

This completes the proof. \(\square\)
In fully analogous manner we arrive also at:

**Theorem 7.** Let $G$ be a nontrivial graph of order $n$ and size $m$. Then

$$M_2(G^{++}) = m^3 + m(m - 4)(n + 1) - 2(m + 2) + [n(m + 1) - 2(m + 2)] M_1(G) + 2m_2(G) + \xi_3(G),$$

$$M_2(G^{+ -}) = m(m + 3)^2 + m(n - 1)[2m + n + 5] - 2(m + n + 2) M_1(G) + 2m_2(G) + \xi_3,$$

$$M_2(G^{++}) = m(m + 3)(m + 11) - 2(m + 3) M_1(G) - 2m_2(G) + \xi_3(G),$$

$$M_2(G^{+-}) = m[(m + n)^2 + (n - 4)^2] + m(n - 2)(n - 4)(n - 4m) - 10m(m + n) + 9m[(n - 2)^2 + mn - 11] M_1(G) + \xi_3(G),$$

$$M_2(G^{++}) = m(n - 1)^2 + 2(m - 1) M_1(G) + 2m_2(G) + \xi_3(G),$$

$$M_2(G^{+ -}) = m(m^2 + 1) + m(m + n)(m + n - 2) + m^2(n - 2)(m + n + 1) - (n(m + 2) - 2) M_1(G) + 2m_2(G) + \xi_3(G).$$

Applying Theorem 1, from the results of Theorems 3-7 and Lemma 1, we can deduce expressions for the second Zagreb coindex of the transformation graphs and total transformation graphs $G^{xy}$s. These are collected in the following:

**Corollary 8.** Let $G$ be a graph of order $n$ and size $m$. Then

$$M_2(T_1(G)) = 2m^2 - 16m + (M_1(G))^2 + 4(m - 1) M_1(G) - \frac{1}{2} [M_1(G) + 2m_2(G) + \xi_3(G)],$$

$$M_2(T_2(G)) = 18m^2 - 2m - 4 M_1(G) - 4m_2(G) - \xi_3,$$

$$M_2(G^{++}) = \frac{1}{2} [4m^2 + (M_1(G))^2 + 4(2m - 1) M_1(G) - 2m_2(G) - \xi_3(G)] - 8 M_1(G) - 6m_2(G) - \xi_3(G),$$

$$M_2(G^{--}) = \frac{1}{2} [(m + n - 1)(m + n) - (4m + M_1(G))]^2 - mn(m + n)^2 + 2m(m + n)(3n + 4) + m(n + 16)
+ 3(m + n) M_1(G) - 2(n - 1) M_2(G) - \xi_3(G) - \frac{1}{2} [(m + n)((m + n)^2 + 6m - 2n + 1]
77 + 8m + 2(m + n - 3) M_1(G) + 2m_2(G) + \xi_3(G)],$$

$$M_2(G^{+-}) = \frac{1}{2} [(2m(n - 2) + m_1(G))^2 - mn(m + n - 8) - 16m - 2(n - 4) M_1(G) - 2m_2(G) - \xi_3(G)]
- [m^3 + m(n - 4)(n + 1) - 2(m + 2)] + [n(m + 2) - 2(m + 4)] M_1(G) + 2m_2(G) + \xi_3,$$

$$M_2(G^{+ -}) = \frac{1}{2} [(m + n + n + 1) - M_1(G)]^2 - [n(n - 1)^2 + m(m + 3)^2 - 2(m + 3) M_1(G) + 2m_2(G)]
+ \xi_3(G)] - [m(m + 3)^2 + m(n + 1)(2m + n + 5) - 2(m + n + 2) M_1(G) + 2m_2(G) + \xi_3],$$

$$M_2(G^{+-}) = \frac{1}{2} [m(m + 7) - M_1(G)]^2 - [m(m + 3)^2 - 2(m + 1) M_1(G) + 2m_2(G) + \xi_3(G)]
- [m(m + 3)(m + 11) - 2(m + 3) M_1(G) - 2m_2(G) + \xi_3(G)],$$

$$M_2(G^{++}) = \frac{1}{2} [m(n + n - 1) + m(n - 8) + M_1(G)]^2 - [(m + n)(n + m) - 2(n + 4m)]
-m[(n - 4)^2 + 9] + 2(n - 2) M_1(G) + 2m_2(G) + \xi_3(G)] - [m((m + n)^2 + (n - 4)^2]
-m(n - 2)(n - 4)(n - 4m) - 10m(m + n) + 9m[(n - 2)^2 + mn - 11] M_1(G) + \xi_3(G).$$
\[ M_2(G^{-+}) = \frac{1}{2} \left[ n(n-1) + M_1(G)^2 - [n(n-1)^2 + 2M_2(G) + \xi_3(G)] - [m(n-1)^2 + 2(n-1)M_1(G) + 2M_2(G) + \xi_3(G)] \right], \]
\[ M_2(G^{+-}) = \frac{1}{2} \left[ m(m+2n-1) - M_1(G)^2 - [m((nm+1) + (m+n)(m+n-2)^2 - [m(m^2+1) + m(m+n)(m+n-2))] \right] - m^2(m+2)(m+n+1) - (m(m+n+2)(m+n+2))M_1(G) + 2M_2(G) + \xi_3(G). \]

**Proof.** The proof follows from the Theorems 1 and 3-7. \( \square \)

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