OPTIMAL SPINOR SELECTIVITY FOR QUATERNION BASS ORDERS

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Abstract. Let $A$ be a quaternion algebra over a number field $F$, and $\mathcal{O}$ be an $\mathcal{O}_F$-order of full rank in $A$. Let $K$ be a quadratic field extension of $F$ that embeds into $A$, and $B$ be an $\mathcal{O}_F$-order in $K$. Suppose that $\mathcal{O}$ is a Bass order that is well-behaved at all the dyadic primes of $F$. We provide a necessary and sufficient condition for $B$ to be optimally spinor selective for the genus of $\mathcal{O}$. This partially generalizes previous results on optimal (spinor) selectivity by C. Maclachlan [Optimal embeddings in quaternion algebras. J. Number Theory, 128(10):2852—2860, 2008] for Eichler orders of square-free levels, and independently by M. Arenas et al. [On optimal embeddings and trees. J. Number Theory, 193:91–117, 2018] and by J. Voight [Chapter 31, Quaternion algebras, volume 288 of Graduate Texts in Mathematics. Springer-Verlag, 2021] for Eichler orders of arbitrary levels.

1. Introduction

Let $F$ be a number field, and $\mathcal{O}_F$ be its ring of integers. Let $A$ be a quaternion $F$-algebra. Two orders $\mathcal{O}$ and $\mathcal{O}'$ in $A$ are said to be in the same genus if their $p$-adic completions $\mathcal{O}_p$ and $\mathcal{O}'_p$ are isomorphic at every finite prime $p$ of $F$. For example, given a fixed nonzero integral ideal $n \subseteq \mathcal{O}_F$ coprime to the reduced discriminant $\delta(A)$ of $A$, all Eichler orders (i.e. intersections of two maximal orders) of level $n$ in $A$ form a single genus. In particular, all maximal orders (which are just Eichler orders of level $n = \mathcal{O}_F$) form a single genus. Fix a genus $\mathcal{G}$ of orders in $A$. Let $K/F$ be a quadratic field extension that embeds into $A$, and $B$ be an order in $K$. The selectivity question for $\mathcal{G}$ and $B$ can be stated as follows:

**Question 1.1.** Whether and when can $B$ embed into every order in $\mathcal{G}$?

If $B$ embeds into some but not all members of $\mathcal{G}$, then we say that $B$ is selective for $\mathcal{G}$. In such a case, one further asks how to determine the members of $\mathcal{G}$ that do admit an embedding of $B$.

The selectivity question has several variants. Recall that an optimal embedding of $B$ into $\mathcal{O}$ is an embedding $\varphi : K \to A$ such that $\varphi(K) \cap \mathcal{O} = \varphi(B)$. If we substitute the word “embedding” by “optimal embedding” in Question 1.1, then we get the optimal selectivity question. Generally, the (optimal) selectivity question admits a satisfactory answer only if $A$ satisfies the Eichler condition, that is, $A$ is
split at an infinite place of $F$. Indeed, almost all literature \cite{[13,15,18,20]} on (optimal) selectivity assumes the Eichler condition. If the Eichler condition fails, then $F$ is necessarily a totally real field, and $A$ is ramified at all the infinite places of $F$. Such quaternion algebras are called $\textit{totally definite}$. For arbitrary quaternion algebras including the totally definite ones, we can formulate a general notion called $\textit{optimal spinor selectivity}$ as in Definition \ref{def:2.3}. If $A$ satisfies the Eichler condition, then “optimal spinor selectivity” reduces to “optimal selectivity” by Remark \ref{rem:2.2}. It was observed in \cite{[20]} that understanding optimal spinor selectivity plays a crucial role in computing certain class numbers associated to orders in totally definite quaternion algebras. For this reason, we focus on optimal spinor selectivity in this paper.

The study on selectivity questions was initiated by Chevalley \cite{[12]}. Modern research on this topic is heavily influenced by the work of Chinburg and Friedman \cite{[13]}, which gives a complete answer to Question \ref{ques:1.1} for the genus of maximal orders in $A$. Indeed, it was them who coined the term “selectivity”. Independently, Guo and Qin \cite{[15]} and Chan and Xu \cite{[10]} generalized the result to Eichler orders. Arenas-Carmona \cite{[5]} and Linowitz \cite{[18]} obtained selectivity theorems for more general classes of orders. More broadly, selectivity results in the context of $A$ being a central simple $F$-algebra have been obtained by Linowitz and Shemanske \cite{[19]} and Arenas-Carmona \cite{[2,3,4]}.

As for optimal selectivity, Maclachlan \cite{[20]} first obtained a theorem for Eichler orders of square-free levels. Independently, Arenas et al. \cite{[1]} and Voight \cite{[25]} Chapter 31] removed the square-free condition and obtained theorems for Eichler orders of arbitrary levels. Their results have been generalized by Chia-Fu Yu and the second named author to quaternion orders with nonzero Eichler invariants at all finite primes of $F$ in \cite{[26]} §2.2. See Definition \ref{def:2.4} for the notion of the Eichler invariant of a quaternion order at a prime $p$ of $F$. Since quaternion orders with nonzero Eichler invariants everywhere are Bass by \cite{[7]} Corollary 2.4 and Proposition 3.1], the next step is naturally to consider arbitrary quaternion Bass orders. In this paper, we study optimal spinor selectivity under the assumption that the genus $G$ consists of Bass orders well-behaved (to be made precise in \cite{[24]} §2.10) at the dyadic primes. The main result will be stated in Theorem \ref{thm:2.6}. See \cite{[14]} §37] for the general theory of Bass orders in finite dimensional $F$-algebras.

This paper is organized as follows. We introduce the preliminary notions and state our main theorem in §2. The definition and basic properties of quaternion Bass orders will be recalled in §3. The proof of the main theorem will be carried out in three steps in §4. We construct an interesting family of examples in §5.

\textbf{Notation.} Throughout this paper, $F$ is either a number field or a nonarchimedean local field of characteristic not equal to 2. We will always fix a quaternion $F$-algebra $A$, and write $\mathcal{O}$ for an $\mathcal{O}_F$-order in $A$. When $F$ is a number field, we write $\text{Ram}(A)$ for the finite set of places of $F$ that are ramified in $A$, and $\text{Ram}_\infty(A)$ (resp. $\text{Ram}_f(A)$) for the set of the infinite (resp. finite) ramified places. If $p$ is a finite prime of $F$ and $M$ is a finite dimensional $F$-vector space or a finite $\mathcal{O}_F$-module, we write $M_p$ for the $p$-adic completion of $M$. In particular, $F_p$ is the $p$-adic completion of $F$, whose $p$-adic discrete valuation is denoted by $\nu_p : F_p^\times \to \mathbb{Z}$. Let $\mathring{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$ be the profinite completion of $\mathbb{Z}$. If $X$ is a finitely generated $\mathbb{Z}$-module or a finite dimensional $\mathbb{Q}$-vector space, we set $\hat{X} = X \otimes_{\mathbb{Z}} \mathring{\mathbb{Z}}$. For example, $\hat{A}$ is the ring of finite adeles of $A$, and $\hat{\mathcal{O}} = \prod_p \mathcal{O}_p$.  


2. Basic notions and the main theorem

In this section, we introduce some preliminary notions and state our main result.

By definition, two orders $O$ and $O'$ in $A$ are said to be in the same genus if they are locally isomorphic everywhere, or equivalently, if there exists $x \in \hat{A}^\times$ such that $\bar{O}' = x\bar{O}x^{-1}$. The orders $O$ and $O'$ are said to be of the same type if they are isomorphic, or equivalently, if there exists $\alpha \in A^\times$ such that $O' = \alpha O\alpha^{-1}$. Let $[O] := \{\alpha O\alpha^{-1} \mid \alpha \in A^\times\}$ be the type of $O$, and $\text{Tp}(O)$ be the finite set of types of orders in the genus of $O$. We regard $\text{Tp}(O)$ as a pointed set with the base point $[O]$. If $\mathcal{G} := \mathcal{G}(O)$ denotes the genus of $O$, then we write $\text{Tp}(\mathcal{G})$ for the type set $\text{Tp}(O)$ with the base point omitted. The type set of maximal orders in $A$ is denoted by $\text{Tp}(A)$.

The quaternion algebra $A$ admits a canonical involution $\alpha \mapsto \bar{\alpha}$ such that $\text{Tr}(\alpha) = \alpha + \bar{\alpha}$ and $\text{Nr}(\alpha) = \alpha\bar{\alpha}$ are respectively the reduced trace and reduced norm of $\alpha \in A$. Given a set $X \subseteq \hat{A}$, we write $X^1$ for the subset of elements with reduced norm 1, that is,

\[ X^1 := \{x \in X \mid \text{Nr}(x) = 1\}. \]

**Definition 2.1** ([8 §1]). Two orders $O$ and $O'$ in $A$ are said to be in the same spinor genus if there exists $x \in A^\times \hat{A}^1$ such that $\bar{O}' = x\bar{O}x^{-1}$.

We write $O \sim O'$ if $O$ and $O'$ are in the same spinor genus. The spinor genus of $O$ is denoted by $[O]_{\text{sg}}$. For the genus $\mathcal{G} = \mathcal{G}(O)$, the set of spinor genera within $\mathcal{G}$ is denoted by $\text{SG}(\mathcal{G})$. We often write $\text{SG}(O)$ for the pointed set $\text{SG}(\mathcal{G})$ with the base point $[O]_{\text{sg}}$. By definition, there is a canonical projection of pointed sets

\[ \text{Tp}(O) \twoheadrightarrow \text{SG}(O), \quad [O'] \mapsto [O']_{\text{sg}}. \]

**Remark 2.2.** When $A$ satisfies the Eichler condition, Brzezinski [8 Proposition 1.1] shows that the above map is a bijection, that is, each spinor genus of orders consists of exactly one type.

Let $K/F$ be a quadratic field extension that embeds into $A$. In light of the Hasse-Brauer-Noether-Albert Theorem [22 Theorem 32.11] [24 Theorem III.3.8], this is to say that no place of $F$ which is ramified in $A$ splits in $K$. Given orders $B \subset K$ and $O \subset A$, we write $\text{Emb}(B, O)$ for the set of optimal embeddings of $B$ into $O$, that is

\[ \text{Emb}(B, O) := \{\varphi \in \text{Hom}_F(K, A) \mid \varphi(K) \cap O = \varphi(B)\}. \]

The unit group $O^\times$ acts from the right on $\text{Emb}(B, O)$ by conjugation: $\varphi \mapsto u^{-1}\varphi u$ for any $u \in O^\times$. Thanks to the Jordan-Zassenhaus Theorem [14 Theorem 24.1, p. 534], the number of orbits

\[ m(B, O, O^\times) := |\text{Emb}(B, O)/O^\times|, \]

is always finite (which holds true in the local case as well). According to [24 Corollary 30.4.8], there exists $O' \in \mathcal{G}$ such that $\text{Emb}(B, O') \neq \emptyset$ if and only if $\text{Emb}(B_p, O_p) \neq \emptyset$ for every finite prime $p$ of $F$. The latter condition depends only on the genus $\mathcal{G}$ and not on the choice of $O$. We define

\[ \Delta(B, O) = \begin{cases} 1 & \text{if } \exists O' \text{ such that } O' \sim O \text{ and } \text{Emb}(B, O') \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases} \]
Clearly, \( \Delta(B, \mathcal{O}) = 0 \) if there exists a finite prime \( p \) of \( F \) such that \( \text{Emb}(B_p, \mathcal{O}_p) = \emptyset \). The symbol \( \Delta(B, \mathcal{O}) \) is featured prominently in class number formulas studied in [26, Corollary 3.4 and Theorem 3.8].

**Definition 2.3.** We say \( B \) is optimally spinor selective for \( \mathcal{G} \) if \( \{ \mathcal{O} \in \mathcal{G} \mid \Delta(B, \mathcal{O}) = 1 \} \) is a nonempty proper subset of \( \mathcal{G} \), in which case a spinor genus \( [\mathcal{O}]_{\mathcal{SG}} \subseteq \mathcal{G} \) with \( \Delta(B, \mathcal{O}) = 1 \) is said to be optimally selected by \( B \).

From Remark 2.2 if \( A \) satisfies the Eichler condition, then each spinor genus consists of exactly one type of orders, and hence in this case there is no difference between “optimal spinor selectivity” here and “optimal selectivity” in [1], [20], [25].

To state our main theorem, we introduce some invariants of orders. Given a finite prime \( p \) of \( F \), we write \( \nu_p : F^\times \to \mathbb{Z} \) for the associated normalized \( p \)-adic discrete valuation. Let \( \delta(\mathcal{O}) \) be the reduced discriminant of \( \mathcal{O} \), and \( f(B) \) be the conductor of \( B \), i.e. the unique nonzero integral ideal of \( F \) such that \( B = O_F + f(B)O_K \). Put

\[
\nu_p(O) := \nu_p(\delta(\mathcal{O})), \quad \text{and} \quad t_p(B) := \nu_p(f(B)).
\]

Note that \( \nu_p(O) = 0 \) if and only if \( O_p \cong \text{Mat}_2(O_{F_p}) \). Similarly, \( \nu_p(O) = 1 \) if and only if one of the following is true:

- \( A \) is split at \( p \), and \( O_p \) is an Eichler order of level \( pO_{F_p} \);
- \( A \) is ramified at \( p \), and \( O_p \) is the unique maximal order of \( A_p \).

**Definition 2.4 ([7, Definition 1.8]).** Let \( \mathfrak{t}_p := O_F/p \) be the finite residue field of \( p \), and \( \mathfrak{t}_p' = \mathfrak{t}_p/p = pO_p \). Let \( E \) be the unique quadratic field extension. Then \( \text{Mat}_2(O_{F_p}) \), the quotient of \( O_p \) by its Jacobson radical \( J(O_p) \), falls into the following three cases:

\[
O_p/J(O_p) \cong \mathfrak{t}_p \times \mathfrak{t}_p, \quad \mathfrak{t}_p, \quad \text{or} \quad \mathfrak{t}_p',
\]

and the Eichler invariant \( e_p(O) \) of \( O \) at \( p \) is defined to be \( 1, 0, -1 \) accordingly. As a convention, if \( O_p \cong \text{Mat}_2(O_{F_p}) \), then its Eichler invariant is defined to be \( 2 \).

Similarly, let \( (K/p) \) be the Artin symbol of \( K \) at \( p \), which takes value \( 1, 0, -1 \) according to whether \( p \) is split, ramified or inert in the extension \( K/F \).

For example, if \( A \) is ramified at \( p \) and \( O_p \) is maximal, then \( e_p(O) = -1 \). It is shown in [7, Proposition 2.1] that \( e_p(O) = 1 \) if and only if \( O_p \) is a non-maximal Eichler order (particularly, \( A \) is split at \( p \)). From [7, Corollary 4.3], if \( e_p(O) = 0 \), then \( \nu_p(O) \geq 2 \) (see also the discussion above).

Central to the theory of spinor optimal selectivity is the class field \( \Sigma_{\mathcal{G}}/F \) associated to the genus \( \mathcal{G} \) and the map \( (O, O') \mapsto \rho(O, O') \in \text{Gal}(\Sigma_{\mathcal{G}}/F) \) on pair of orders \( O, O' \in \mathcal{G} \). These two notions have been stable for almost all variants of selectivity theory, cf. [18, §3] and [25, §31.1]. Following [24, §III.4], we write \( F_A^\times \) for the subgroup of \( F^\times \) consisting of all elements that are positive at each place in \( \text{Ram}_\infty(A) \). The Hasse-Schilling-Maass theorem [22, Theorem 33.15] implies that

\[
\text{Nr}(A^\times) = F_A^\times.
\]

Let \( \mathcal{N}(\hat{O}) \) be the normalizer of \( \hat{O} \) in \( \hat{A}^\times \). The pointed set \( \text{SG}(\mathcal{O}) \) of spinor genera in \( \mathcal{G} \) admits the following adelic description (cf. [8, Propositions 1.2 and 1.8])

\[
\text{SG}(\mathcal{O}) \simeq (A^\times \hat{A})/\hat{A}^\times/\mathcal{N}(\hat{O}) \xrightarrow{N} F_A^\times/\hat{F}^\times/\text{Nr}(\mathcal{N}(\hat{O})) = F_A^\times/\hat{F}^\times/\text{Nr}(\mathcal{N}(\hat{O})),
\]

where the two double coset spaces are canonically bijective via the reduced norm map. It follows that \( \text{SG}(\mathcal{O}) \) is naturally equipped with an abelian group structure,
with its distinguished point \( [\mathcal{O}]_{sg} \) as the identity element. Since \( \text{Nr}(\mathcal{N}(\hat{O})) \) is an open subgroup of \( \hat{F}^\times \) containing \((\hat{F}^\times)^2\), the group \( \text{SG}(\mathcal{O}) \) is a finite elementary 2-group \[18, \text{Proposition } 3.5\]. Clearly, the group \( \text{Nr}(\mathcal{N}(\hat{O})) \) depends only on the genus \( \mathcal{G} \) and not on the choice of \( \mathcal{O} \).

**Definition 2.5** ([2 §2], [18 §3]). The spinor genus field of \( \mathcal{G} \) is the abelian field extension \( \Sigma_{\mathcal{G}} / F \) corresponding to the open subgroup \( F_A^\times \text{Nr}(\mathcal{N}(\hat{O})) \subseteq \hat{F}^\times \) via the class field theory \[17\] Theorem X.5.

By the definition of \( \Sigma_{\mathcal{G}} \), there are isomorphisms:

\[
\text{SG}(\mathcal{O}) \simeq F_A^\times \backslash \hat{F}^\times / \text{Nr}(\mathcal{N}(\hat{O})) \simeq \text{Gal}(\Sigma_{\mathcal{G}} / F).
\]

Given another order \( \mathcal{O} \in \mathcal{G} \), we define \( \rho(\mathcal{O}, \mathcal{O}') \) to be the image of \([\mathcal{O}']_{sg} \in \text{SG}(\mathcal{O})\) in \( \text{Gal}(\Sigma_{\mathcal{G}} / F) \) under the above isomorphism. More canonically, we regard the base point free set \( \text{SG}(\mathcal{G}) \) as a principal homogeneous space over \( \text{Gal}(\Sigma_{\mathcal{G}} / F) \) via (2.9). Then \( \rho(\mathcal{O}, \mathcal{O}') \) is the unique element of \( \text{Gal}(\Sigma_{\mathcal{G}} / F) \) that sends \([\mathcal{O}]_{sg} \) to \([\mathcal{O}']_{sg}\).

Clearly, \( \rho(\mathcal{O}, \mathcal{O}') \) enjoys the following properties:

(a) \( \rho(\mathcal{O}, \mathcal{O}') = 1 \) if and only if \( \mathcal{O} \sim \mathcal{O}' \);
(b) \( \rho(\mathcal{O}, \mathcal{O}') = \rho(\mathcal{O}', \mathcal{O}) \);
(c) \( \rho(\mathcal{O}, \mathcal{O}'') = \rho(\mathcal{O}, \mathcal{O}')\rho(\mathcal{O}', \mathcal{O}'') \).

We will postpone the definition and basic properties of Bass orders to the next section. Taking that for granted, we are now ready to state the main theorem.

**Theorem 2.6.**

(I) Let \( \mathcal{G} \) be a genus of Bass orders in \( A \), and \( \mathcal{O} \) be a member of \( \mathcal{G} \). Assume that \( \mathcal{O} \) is well-behaved at every dyadic prime \( q \) of \( F \) in the following sense:

\[
n_q(\mathcal{O}) = 2 \text{ if } e_q(\mathcal{O}) = 0, \quad \forall q \mid (2O_F).
\]

Let \( K / F \) be a quadratic field extension that embeds into \( A \), and \( B \) be an order in \( K \). Suppose that \( \text{Emb}(B, \mathcal{O}_p) \neq \emptyset \) for every finite prime \( p \) so that \( B \) is optimally embeddable into some member of \( \mathcal{G} \). Then \( B \) is optimally spinor selective for \( \mathcal{G} \) if and only if \( K \subseteq \Sigma_{\mathcal{G}} \) and for every nondyadic prime \( p \) with \( e_p(\mathcal{O}) = 0 \) and \( (K/p) = 0 \), one of the following conditions holds

(i) \( n_p(\mathcal{O}) \geq 2i_p(B) + 3 \);
(ii) \( n_p(\mathcal{O}) = 2i_p(B) + 1 \), \( A \) is split at \( p \), and \( |\mathcal{O}_p| = 5 \);
(iii) \( n_p(\mathcal{O}) = 2i_p(B) + 1 \), \( A \) is ramified at \( p \), and \( |\mathcal{O}_p| = 3 \).

(II) If \( B \) is optimally spinor selective for \( \mathcal{G} \), then both of the following hold true:

(a) for any two \( O_F \)-orders \( \mathcal{O}, \mathcal{O}' \in \mathcal{G} \),

\[
\Delta(B, \mathcal{O}') = \rho(\mathcal{O}', \mathcal{O})|_K + \Delta(B, \mathcal{O}),
\]

where \( \rho(\mathcal{O}', \mathcal{O})|_K \in \text{Gal}(K / F) \) is the restriction of \( \rho(\mathcal{O}', \mathcal{O}) \in \text{Gal}(\Sigma_{\mathcal{G}} / F) \) to \( K \), and the summation is taken inside \( \mathbb{Z}/2\mathbb{Z} \) with the canonical identification \( \text{Gal}(K / F) \simeq \mathbb{Z}/2\mathbb{Z} \);

(b) exactly half of the spinor genera in \( \mathcal{G} \) are optimally selected by \( B \).

**Remark 2.7.** Suppose that \( \mathcal{O} \) is an order in \( A \) satisfying

\[
e_p(\mathcal{O}) \neq 0 \text{ for every finite prime } p \text{ of } F.
\]

Then \( \mathcal{O} \) is automatically Bass by \[7, \text{Corollary } 2.4 \text{ and Proposition } 3.1\]. Moreover, the conditions (2.10) and (i-iii) above are all vacuous for this case. Therefore, if \( \mathcal{O} \) satisfies condition (2.12), then \( B \) is optimally spinor selective for the genus \( \mathcal{G} \) if and
only if $K \subseteq \Sigma_{q}$. Since all Eichler orders satisfy \cite{2.12}, we recover partial cases of \cite[Theorem 1.1]{1} and \cite[Theorem 31.1.7]{25}. On the other hand, if condition \cite{2.12} is dropped, we can easily construct examples where $K \subseteq \Sigma_{q}$, but $\Delta(B, \mathcal{O}) = 1$ for every $\mathcal{O} \in \mathcal{G}$. See \S 5 for a family of examples.

Given a Bass order $\mathcal{O} \subset A$ and an order $B \subset K$, in order to apply Theorem \ref{theo2.6}, a priori, we need to know whether $\text{Emb}(B_{p}, \mathcal{O}_{p}) = \emptyset$ (equivalently, $m(B_{p}, \mathcal{O}_{p}, \mathcal{O}_{p}^{\ast}) = 0$) or not for every $p$. If $e_{p}(\mathcal{O}) \in \{1, 2\}$, then $A$ is split at $p$ and $\mathcal{O}_{p}$ is an Eichler order. In this case, the method for computing $m(B_{p}, \mathcal{O}_{p}, \mathcal{O}_{p}^{\ast})$ is well known and has historically been studied by Eichler, Hijikata and many others. See \cite[\S II.3 and \S 30.6]{24} and \cite[\S 3.5]{25} for some expositions. If $e_{p}(\mathcal{O}) \in \{-1, 0\}$, Brzeziński \cite{9} produced recursive formulas for $m(B_{p}, \mathcal{O}_{p}, \mathcal{O}_{p}^{\ast})$. For example, if $e_{p}(\mathcal{O}) = 0$ and $n_{p}(\mathcal{O}) = 2$, then $m(B_{p}, \mathcal{O}_{p}, \mathcal{O}_{p}^{\ast})$ can be read off directly from \cite[(3.14) and (3.17)]{9}. See also Corollary \ref{cor3.7} for an application of Brzeziński’s result in the case that $p$ is nondyadic, $e_{p}(\mathcal{O}) = 0$, and $n_{p}(\mathcal{O}) \geq 3$.

From \cite[Lemma 2.8]{25}, the condition $K \subseteq \Sigma_{q}$ can be characterized purely in terms of local conditions. For the reader’s convenience, we recall this lemma below.

**Lemma 2.8.** We have $K \subseteq \Sigma_{q}$ if and only if both of the following conditions hold:

(i) $K$ and $A$ are ramified at exactly the same (possibly empty) set of real places of $F$;

(ii) $\text{Nr}(\mathcal{N}(\mathcal{O}_{p})) \subseteq \text{Nr}(K_{p}^{\ast})$ for every finite prime $p$ of $F$.

Thanks to the explicit description of normalizers of local Bass orders \cite[Theorems 2.2 and 2.5]{9}, we have the following characterization of $\Sigma_{q}$.

**Proposition 2.9.** Let $\mathcal{G}$ and $\mathcal{O}$ be as in Theorem \ref{theo2.6}. Then $\Sigma_{q}/F$ is the maximal abelian extension of exponent 2 satisfying all of the following conditions:

1. $\Sigma_{q}$ is unramified at each of the following places:
   (1a) an infinite place of $F$ that is split in $A$,
   (1b) a finite prime $p$ with $e_{p}(\mathcal{O}) = 2$, i.e., $\mathcal{O}_{p} \simeq \text{Mat}_{2}(O_{F_{p}})$,
   (1c) a finite prime $p$ with $e_{p}(\mathcal{O}) = 1$ and $n_{p}(\mathcal{O}) \equiv 0 \pmod{2}$,
   (1d) a finite prime $p$ with $e_{p}(\mathcal{O}) = -1$ and $A_{p} \simeq \text{Mat}_{2}(F_{p})$;

2. $\Sigma_{q}$ splits completely at each of the following finite prime $p$ of $F$:
   (2a) $e_{p}(\mathcal{O}) = -1$ and $A$ is ramified at $p$,
   (2b) $e_{p}(\mathcal{O}) = 1$ and $n_{p}(\mathcal{O}) \equiv 1 \pmod{2}$,
   (2c) $e_{p}(\mathcal{O}) = 0$ and $n_{p}(\mathcal{O}) = 2$,
   (2d) $e_{p}(\mathcal{O}) = 0$, $n_{p}(\mathcal{O}) \geq 3$, $A$ is split at $p$, and $-1 \not\in t_{p}^{\times 2}$,
   (2e) $e_{p}(\mathcal{O}) = 0$, $n_{p}(\mathcal{O}) \geq 3$, $A$ is ramified at $p$, and $-1 \in t_{p}^{\times 2}$;

3. if $p$ is a finite nondyadic prime of $F$ with $e_{p}(\mathcal{O}) = 0$, $n_{p}(\mathcal{O}) \geq 3$, and
   - either $A$ is split at $p$ with $-1 \in t_{p}^{\times 2}$,
   - or $A$ is ramified at $p$ with $-1 \not\in t_{p}^{\times 2}$,
   then either $p$ splits completely in $\Sigma_{q}$, or $\Sigma_{q} \otimes_{F} F_{p}$ is a direct sum of copies of a quadratic extension $M_{p}/F_{p}$ whose ring of integers embeds into $\mathcal{O}_{p}$.

Such a quadratic extension $M_{p}/F_{p}$ is necessarily ramified by \cite[Proposition 1.12]{9}, and it is uniquely determined by $\mathcal{O}_{p}$ according to Lemma \ref{lem3.5}. From Proposition \ref{prop2.9}(2c) and assumption \cite{2.12} on $\mathcal{G}$, if $q$ is a dyadic prime with $e_{q}(\mathcal{O}) = 0$ (hence $n_{p}(\mathcal{O}) = 2$ by the assumption \cite{2.10}), then $\Sigma_{q}/F$ splits completely at $q$. This proposition will be proved at the end of \S 3.
Part (II) of Theorem 2.6 follows directly from [26, Theorem 2.11]. To prove the first part, we reduce it to local considerations as well. See §4.1 for the details. It is clear from the above discussion that we rely heavily on the fundamental work of Brzezinski [9]. In theory, it is possible apply his result to remove assumption (2.10) and to obtain an optimal spinor selectivity theorem for all Bass orders. However, our method is built upon explicit computations, which becomes too complicated at the dyadic primes. We leave such an endeavor to a more adventurous reader.

3. Quaternion Bass orders

In this section, we recall the definition and basic properties of quaternion Bass orders. Our main references are the work by Brzezinski [7,9] and by Chari et al. [11].

We keep the notation of previous sections, except that $F$ is allowed to be either a number field or a nonarchimedean local field of characteristic not equal to two. In the local case, a quadratic extension of $F$ means a quadratic semisimple $F$-algebra, that is, either $F \times F$ or a quadratic field extension of $F$. We denote the unique maximal ideal of $O_F$ by $p$, and its residue field by $k$. We drop the subscript $p$ and write $\nu, n, i, e$ for $\nu_p, n_p(O), i_p(B), e_p(O)$, respectively. See (2.6) and Definition 2.4.

Given an order $O$ in the quaternion $F$-algebra $A$, an overorder of $O$ is an order $O'$ in $A$ containing $O$. An overorder $O' \supseteq O$ is called a minimal overorder of $O$ if it is minimal with respect to inclusion among the orders properly containing $O$.

**Definition 3.1.** An order $O$ in $A$ is Gorenstein if its dual lattice $O^\vee := \{ x \in A \mid \text{Tr}(xO) \subseteq O_F \}$ is projective as a left (or right) $O$-module. It is called a Bass order if every overorder of $O$ (including $O$ itself) is Gorenstein.

As noted by Bass [6] himself, Gorenstein orders are ubiquitous. Being Gorenstein is a local property, that is, $O$ is Gorenstein if and only if $O_p$ is Gorenstein for every finite prime $p$ of $F$ (when $F$ is a number field). Consequently, being Bass is a local property as well. Bass orders enjoy many equivalent characterizations. We merely mention one of them that is mostly relevant to our current quest.

**Theorem 3.2.** An order $O \subset A$ is Bass if and only if it is basic, i.e. there exists a semisimple quadratic $F$-algebra $L$ whose ring of integers $O_L$ embeds into $O$.

This theorem is proved by Brzezinski [9] Proposition 1.11] in the local case, and by Chari et al. [11, Theorem 1.2] in the number field case. See [11, Corollary 1.3] for more characterizations of Bass orders.

For the rest of this section, we assume that $F$ is local unless specified otherwise. Recall that an order $O$ is hereditary if every $O$-lattice in a free $A$-module is $O$-projective [14, p. 76]. From [7, Proposition 1.2], $O$ is hereditary if and only if $n(O) \leq 1$. If $O$ is Bass but non-hereditary, then we have some further information on the quadratic $F$-algebra $L$ with $\text{Emb}(O_L, O) \neq \emptyset$ from [9, Proposition 1.12]:

- if $e(O) = 1$, then $L = F \times F$;
- if $e(O) = -1$, then $L/F$ is the unique quadratic unramified field extension;
- if $e(O) = 0$, then $L/F$ is a ramified field extension.

In fact, if $e(O) = 0$ and $n(O) = 2$, then $L/F$ can be any arbitrary quadratic ramified extension according to [9, (3.14)]. If $F$ is nondyadic, $e(O) = 0$ and $n(O) \geq 3$, then we prove in Lemma 3.5 that such an $L/F$ is uniquely determined by $O$. 
As mentioned in Remark 2.7, any order $\mathcal{O}$ with $e(\mathcal{O}) \neq 0$ is automatically Bass. If $e(\mathcal{O}) = 1$, then $\mathcal{O}$ is a non-maximal Eichler order, and it has exactly two minimal overorders. Suppose that $e(\mathcal{O}) \in \{-1, 0\}$ and $\mathcal{O}$ is Bass but non-hereditary. Then from [7, Proposition 1.12], $\mathcal{O}$ has a unique minimal overorder $\mathcal{M}(\mathcal{O})$, which is Bass by default. From [7, Propositions 3.1 and 4.1],

\begin{equation}
(3.1) \quad n(\mathcal{M}(\mathcal{O})) = \begin{cases} n(\mathcal{O}) - 2 & \text{if } e(\mathcal{O}) = -1; \\ n(\mathcal{O}) - 1 & \text{if } e(\mathcal{O}) = 0, \end{cases}
\end{equation}

and $e(\mathcal{M}(\mathcal{O})) = e(\mathcal{O})$ if $\mathcal{M}(\mathcal{O})$ is again non-hereditary. Thus starting from $\mathcal{M}(\mathcal{O}) := \mathcal{O}$, we define $\mathcal{M}^i(\mathcal{O}) := \mathcal{M}(\mathcal{M}^{i-1}(\mathcal{O}))$ recursively to obtain a unique chain of Bass orders terminating at a hereditary order $\mathcal{M}^{m}(\mathcal{O})$:

\begin{equation}
(3.2) \quad \mathcal{O} = \mathcal{M}^0(\mathcal{O}) \subset \mathcal{M}^1(\mathcal{O}) \subset \mathcal{M}^2(\mathcal{O}) \subset \cdots \subset \mathcal{M}^{m-1}(\mathcal{O}) \subset \mathcal{M}^m(\mathcal{O}),
\end{equation}

where each $\mathcal{M}^i(\mathcal{O})$ is a Bass non-hereditary order for $0 \leq i \leq m - 1$. Furthermore,

- $m = n(\mathcal{O}) - 1$ if $e(\mathcal{O}) = 0$; and
- $m = \lfloor n(\mathcal{O})/2 \rfloor$ if $e(\mathcal{O}) = -1$, where $x \mapsto \lfloor x \rfloor$ is the floor function.

The order $\mathcal{M}^m(\mathcal{O})$ is called the hereditary closure of $\mathcal{O}$ and will henceforth be denoted by $H(\mathcal{O})$. If $e(\mathcal{O}) = -1$, then $H(\mathcal{O})$ is always a maximal order by [7, Proposition 3.1]. Thus when $e(\mathcal{O}) = -1$, $n(\mathcal{O})$ is even if $A \simeq \text{Mat}_2(F)$, and $n(\mathcal{O})$ is odd if $A$ is ramified (i.e. $A$ is division).

Recall that $\nu : F^\times \to \mathbb{Z}$ denotes the discrete valuation of $F$. We say that an element $x \in A^\times$ is even (resp. odd) if $\nu(\text{Nr}(x))$ is even (resp. odd). Let $N^0(\mathcal{O})$ be the even normalizer group of $\mathcal{O}$, that is,

\begin{equation}
(3.3) \quad N^0(\mathcal{O}) := \{ x \in A^\times \mid x\mathcal{O}x^{-1} = \mathcal{O}, \text{ and } \nu(\text{Nr}(x)) \equiv 0 \pmod{2} \}.
\end{equation}

**Lemma 3.3.** If $e(\mathcal{O}) = -1$, then

\begin{equation}
(3.4) \quad \text{Nr}(N(\mathcal{O})) = \begin{cases} F^\times & \text{if } A \text{ is ramified,} \\ F^\times \mathcal{O}_E^2 & \text{if } A \text{ is split.} \end{cases}
\end{equation}

**Proof.** Let $E := E_{ur}$ be the unique unramified quadratic field extension of $F$. Since $e(\mathcal{O}) = -1$, there exists an embedding $\varphi : O_E \to \mathcal{O}$. Indeed, if $\mathcal{O}$ is non-hereditary, this follows from [9, Proposition 1.12] as discussed above. If $\mathcal{O}$ is hereditary, then $n(\mathcal{O}) = 1$, which implies that $A$ is division and $\mathcal{O}$ is the unique maximal order in $A$. Hence $\text{Emb}(O_E, \mathcal{O}) \neq \emptyset$ by [24, Corollary II.1.7]. For simplicity, let us identify $O_E$ with its image in $\mathcal{O}$ via $\varphi$. From [24, Proposition V.1], $N_{E/F}(O_E^\times) = O_F^\times$. It follows that

\begin{equation}
\text{Nr}(N(\mathcal{O})) \supseteq \text{Nr}(F^\times O_E^\times) = F^\times O_F^\times,
\end{equation}

which is an index 2-subgroup in $F^\times$. On the other hand, from [9, Theorem 2.2], $N(\mathcal{O}) = N^0(\mathcal{O})$ if and only if $n(\mathcal{O})$ is even (which happens if and only if $A$ is split as observed above). The lemma follows immediately. \qed

In the local case, Bass orders can be described explicitly according to [9, §1] (see also [7, Propositions 5.4 and 5.6]). For reasons to be explained in the proof of

\footnote{Caution: there is a minor typo in the line immediately below equation (1.5) in [9], instead of $0 \leq r - s \leq 1$, it should read $0 \leq s + r \leq 1$. Compare with [7, Proposition 5.4(c)]. We should emphasize that this bears no effect on the validity of results of [9], as most of the deduction relies on [9, (2.8)] instead.}
Theorem [2.5] (see §1.1), we focus exclusively on local Bass orders of Eichler invariant zero.

Let $\pi$ be a uniformizer of $F$. Pick an element $\varepsilon \in F$ in the following way:

- if $A$ is split (i.e. $A \simeq \text{Mat}_2(F)$), then put $\varepsilon = 0$;
- if $A$ is ramified, then choose $\varepsilon \in O_F^\times$ such that $1 - 4\varepsilon \in O_F^\times \cap O_F^{x_2}$ (i.e. $1 - 4\varepsilon$ is a unit but a non-square). The existence of such a unit is guaranteed by [21 63:4].

The assumption on $\varepsilon$ implies that

$$1 + \beta + \varepsilon\beta^2 \in O_F^\times, \quad \forall \beta \in O_F. \tag{3.5}$$

We can choose an $F$-basis $\{1, x_1, x_2, x_3\}$ of $A$ satisfying the following conditions:

$$x_1^2 = x_1 - \varepsilon, \quad x_2^2 = \pi, \quad x_2x_1 = (1 - x_1)x_2, \quad x_3 = x_1x_2. \tag{3.6}$$

Indeed, if $A$ is split, then we put as in [9 (2.7)]:

$$x_1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad x_2 := \begin{bmatrix} 0 & 1 \\ \pi & 0 \end{bmatrix}, \quad x_3 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3.7}$$

If $A$ is ramified, then from [21 63:3], $F(x_1)$ coincides with $E_{ur}$, the unique unramified quadratic field extension of $F$, and the existence of a basis satisfying (3.6) is guaranteed by [21 Corollary II.1.7].

Given a set $X$ in a finite dimensional $F$-vector space $V$, we write $(X)$ for the $O_F$-submodule of $V$ spanned by $X$. As remarked right after Definition [2.4] if $e(O) = 0$, then $n(O) \geq 2$. According to [9 (2.8)], after replacing $\pi$ by another suitable uniformizer if necessary, every Bass order $O$ with $e(O) = 0$ and $n(O) = n$ is isomorphic to

$$\langle 1, x_{\alpha\beta}, \pi^r x_1, \pi^s x_3 \rangle, \quad \text{where} \tag{3.8}$$

$$r + s = n - 1, \quad 0 \leq r - s \leq 1, \quad x_{\alpha\beta} = \alpha x_1 + x_2 + \beta x_3, \quad \alpha \in p, \ \beta \in O_F, \tag{3.9}$$

$$\text{and} \quad 1 + \beta \in O_F^\times \quad \text{if $A$ is split}. \tag{3.10}$$

There is no other restriction on $\beta$ when $A$ is ramified. From (3.9),

$$r = \left\lfloor \frac{n}{2} \right\rfloor \geq 1, \quad \text{and} \quad s = \left\lfloor \frac{n-1}{2} \right\rfloor \geq 0. \tag{3.11}$$

Let $O$ be as in [3.3]. Given $y \in A$, the discriminant of $y$ is defined to be

$$\Delta(y) := \text{Tr}(y)^2 - 4\text{Nr}(y).$$

If $y \in O$, we write $y$ uniquely as $a + bx_{\alpha\beta} + c\pi^r x_1 + d\pi^s x_3$ with $a, b, c, d \in O_F$. Direct calculation yields (see [9 (3.19), (3.20)])

$$\text{Tr}(y) = 2a + ba + c\pi^r, \tag{3.12}$$

$$\text{Nr}(y) = a^2 + a(ba + c\pi^r) + \varepsilon(ba + c\pi^r)^2$$

$$- \pi \left[ b^2 + b(b\beta + d\pi^s) + \varepsilon(b\beta + d\pi^s)^2 \right], \tag{3.13}$$

$$\Delta(y) = (ba + c\pi^r)^2(1 - 4\varepsilon)$$

$$+ 4\pi \left[ b^2 + b(b\beta + d\pi^s) + \varepsilon(b\beta + d\pi^s)^2 \right]. \tag{3.14}$$

In particular, we have

$$\text{Tr}(x_{\alpha\beta}) = \alpha, \quad \text{Nr}(x_{\alpha\beta}) = \alpha^2 \varepsilon - \pi(1 + \beta + \varepsilon\beta^2), \quad \Delta(x_{\alpha\beta}) = \alpha^2(1 - 4\varepsilon) + 4\pi(1 + \beta + \varepsilon\beta^2).$$
Clearly, $F(x_{\alpha\beta})$ is a ramified quadratic extension of $F$, and $x_{\alpha\beta}$ is a uniformizer of $F(x_{\alpha\beta})$. Particularly, the ring of integers of $F(x_{\alpha\beta})$ coincides with $\langle 1, x_{\alpha\beta} \rangle$. Since $r \geq 1$ by \((3.11)\), it is clear from \((3.13)\) that
\[(3.15) \quad y \in O_{F}^\times \quad \text{if and only if} \quad a \in O_{F}^\times.
\]

**Lemma 3.4.** If $e(O) = 0$ and $F$ is nondyadic, then
\[(3.16) \quad \text{Nr}(O^\times) = O_{F}^\times^2.
\]

**Proof.** For any $u \in O_{F}^\times$, we write $\tilde{u}$ for its canonical image in the residue field $\mathfrak{t} = O_F/\mathfrak{p}$. From Hensel’s lemma and the assumption that $\text{char}(\mathfrak{t}) \neq 2$, we have
\[(3.17) \quad O_{F}^\times^2 = \{ u \in O_{F}^\times \mid \tilde{u} \in \mathfrak{t}^\times^2 \}.
\]

The equality \((3.16)\) follows directly by combining \((3.13)\), \((3.15)\) and \((3.17)\). $\square$

When $F$ is nondyadic, it has exactly two ramified quadratic extensions up to isomorphism, namely $F(\sqrt{\pi})$ and $F(\sqrt{\pi\mathfrak{t}})$, where $u \in O_{F}^\times \setminus O_{F}^\times^2$.

**Lemma 3.5.** Suppose that $F$ is nondyadic, and $O \subset A$ is a Bass order with $e(O) = 0$ and $n(O) \geq 3$. Up to isomorphism, there exists a unique quadratic extension $M/F$ such that $O_M$ embeds into $O$.

**Proof.** Only the uniqueness of such an $M$ need to be proved. From \([3] \text{ Proposition 1.12}\), $M/F$ is necessarily ramified. Write $O$ as in \((3.8)\) and suppose that $M/F$ is a ramified quadratic extension such that there exists an embedding $\varphi : O_M \to O$. According to \([23] \text{ Proposition I.18}\), the characteristic polynomial over $F$ of the uniformizer $\pi_M \in M$ is an Eisenstein polynomial of degree 2, and $O_M = \langle 1, \pi_M \rangle$. If we put $y := \varphi(\pi_M)$, then $\text{Tr}(y) \in \mathfrak{p}$ and $\text{Nr}(y) \in O_{F}^\times \pi$. Write $y = a + b_x \alpha + c \pi x_1 + d \pi^2 x_3$ with $a, b, c, d \in O_F$ as before. Since $n(O) \geq 3$, we have $r \geq s \geq 1$ by \((3.11)\). From \((3.12)\) and \((3.14)\), the previous conditions on $\text{Tr}(y)$ and $\text{Nr}(y)$ is equivalent to
\[(3.18) \quad \pi | a \quad \text{and} \quad b \in O_{F}^\times.
\]

Without lose of generality, we can replace $\pi_M$ by $b^{-1} \pi_M$ and assume that $b = 1$. An easy calculation shows that $\Delta(y)/\Delta(x_{\alpha\beta}) \equiv 1 \pmod{\mathfrak{p}}$. It follows from \((3.17)\) that $\Delta(y) \equiv \Delta(x_{\alpha\beta})O_{F}^\times^2$. Therefore, $M \simeq F(x_{\alpha\beta})$, and hence it is uniquely determined up to isomorphism. $\square$

Let $O$ be an arbitrary Bass order of Eichler invariant 0. The normalizer group $\mathcal{N}(O)$ has been described effectively in \([3] \text{ Theorem 2.5}\]). First, suppose that $n(O) = 2$. Then $\mathcal{M}(O)$ is the hereditary closure of $O$, and
\[(3.19) \quad O = O_F + \mathfrak{J}(\mathcal{M}(O)),
\]
where $\mathfrak{J}(\mathcal{M}(O))$ denotes the Jacobson radical of $\mathcal{M}(O)$. It follows that $\mathcal{N}(O) = \mathcal{N}(\mathcal{M}(O))$. If $A$ is split, then $\mathcal{M}(O)$ is an Eichler order of level $\mathfrak{p}$; if $A$ is ramified, then $\mathcal{M}(O)$ is the unique maximal order. In both cases, the reduced norm of the normalizer group $\mathcal{N}(\mathcal{M}(O))$ coincides with $F^\times$. Therefore,
\[(3.20) \quad \text{Nr}(\mathcal{N}(O)) = F^\times \quad \text{if} \quad e(O) = 0 \quad \text{and} \quad n(O) = 2.
\]

Next, suppose that $n(O) \geq 3$. Write $O$ as in \((3.8)\) and put $M = F(x_{\alpha\beta})$. From \([3] \text{ Theorem 2.5}\], the even normalizer group $\mathcal{N}^0(O)$ is a subgroup of index 2 in $\mathcal{N}(O)$, and
\[(3.21) \quad \mathcal{N}(O) = \mathcal{N}^0(O) \sqcup \mathcal{N}^0(O)x_{\alpha\beta}.
\]
See [9] p. 177. In particular,
\begin{equation}
N^0(O) = \left( A_0 \gamma - O \right) M^x \equiv M^x.
\end{equation}

Since \([F^x : N_{M/F}(M^x)] = 2\) by local class field theory, we find that
\begin{equation}
\text{Nr}(N(O)) \text{ coincides with either } F^x \text{ or } N_{M/F}(M^x).
\end{equation}

For simplicity, we only write down \(N^0(O)\) under the assumption that \(F\) is nondenadic:
\begin{equation}
N^0(O) = F^x M(O)^x \left( F^x M(O)^x \sigma_0 \right)
\end{equation}
where \(\sigma_0 \in H(O)^x \setminus M_{n-2}(O)^x\). More explicitly, from [9] (2.10), we have
\begin{align*}
\sigma_0 &= -1 - \frac{A_0}{2} + A x_{a, \beta} + 2 x_1, \\
A &= \frac{-2(1 - 2) \pi (1 + \beta + \beta^2 \varepsilon)}{\alpha^2 (1 - 4 \varepsilon) + 4 \pi (1 + \beta + \beta^2 \varepsilon)} \in O_F.
\end{align*}

By direct calculation,
\begin{equation}
\text{Nr}(\sigma_0) = - \left( \frac{1 + \frac{A_0}{2}}{2} (1 - 4 \varepsilon) - \pi A^2 (1 + \beta + \beta^2 \varepsilon) \right) \in O_F^x.
\end{equation}

**Lemma 3.6.** Write \(O\) as in \((3.3)\) and put \(M = F(x_{a, \beta})\). Suppose that \(n(O) \geq 3\) and \(F\) is nondenadic. Then \(\text{Nr}(N(O)) = N_{M/F}(M^x)\) if and only if one of the following conditions holds:
\begin{enumerate}
\item \(A\) is split, and \(-1 \notin t^x\),
\item \(A\) is ramified, and \(-1 \notin t^x\).
\end{enumerate}

**Proof.** Since \(M/F\) is tamely ramified, \(N_{M/F}(O_M^x) = O_F^x\) according to [23] Corollary V.7. Combining (3.10), (3.21) and (3.24), we see that \(\text{Nr}(N(O)) = N_{M/F}(M^x)\) if and only if \(\text{Nr}(\sigma_0) \in O_F^x\). If \(A\) is split, then \(\varepsilon = 0\) by construction. It follows from (3.17) that \(\sigma_0 \in O_F^x\) if and only if \(-1 \notin F^x\) in this case. Next, suppose that \(A\) is ramified. Then \((1 - 4 \varepsilon) \notin O_F^x\) by construction. Thus \(\text{Nr}(\sigma_0) \in O_F^x\) if and only if \(-1 \notin F^x\) in this case. Lastly, since \(F\) is nondenadic, \(-1 \notin F^x\) if and only if \(-1 \notin t^x\).

**Proof of Proposition 2.12.** Suppose that \(F\) is a number field, and \(G\) is a genus of Bass orders in \(A\) satisfying (2.12). Let \(O\) be an arbitrary member of \(G\), and \(\Sigma\) be the spinor genus field of \(G\). The description of \(\Sigma\) at the infinite places of \(F\) or at a finite prime \(p\) with \(e_p(O) \in \{1, 2\}\) is well known. See [25] Proposition 31.2.1 for example. Thus we focus on the finite primes \(p\) with \(e_p(O) \in \{-1, 0\}\).

First, suppose that \(e_p(O) = -1\). If \(A_p\) is split, then \(\text{Nr}(N(O_p)) = F_p^x O_F^x\) by (3.4). Hence \(\Sigma/G\) is unramified at \(p\). This proves part (1d) of Proposition 2.12.

Similarly, if \(A_p\) is ramified, then \(\text{Nr}(N(O_p)) = F_p^x\), and hence \(\Sigma/G\) splits completely at \(p\). Part (2a) of the proposition follows.

Next, suppose that \(e_p(O) = 0\). If one of the following conditions holds:
\begin{itemize}
\item \(n_p(O) = 2\),
\item \(e_p(O) = 0, n_p(O) \geq 3, A\) is split at \(p\), and \(-1 \notin t^x\),
\item \(e_p(O) = 0, n_p(O) \geq 3, A\) is ramified at \(p\), and \(-1 \notin t^x\),
\end{itemize}
then \(\text{Nr}(N(O_p)) = F_p^x\) according to (3.20), (3.23) and Lemma 3.6. Note that the condition \(n_p(O) \geq 3\) implies that \(p\) is nondenadic by assumption (2.12). This proves
part (2c)–(2e) of Proposition 2.9. Part (3) of the proposition follows directly from Lemma 3.6. 

4. THE PROOF OF THE MAIN THEOREM

We carry out the proof of Theorem 2.6 in three steps.

4.1. Step (I): reduction to the local case. Let $F$ be a number field, and $\mathcal{G}$ be a genus of Bass orders in $A$ satisfying condition (2.12). Fix an order $\mathcal{O}$ in $\mathcal{G}$. Let $K/F$ be a quadratic field extension embeddable into $A$, and $B$ be an order in $K$. Suppose that for each finite prime $p$ of $F$, there exists an optimal embedding $\varphi_p : B \rightarrow \mathcal{O}$. Put

\begin{equation}
\mathcal{E}_p(\varphi_p, B_p, \mathcal{O}_p) := \{ g_p \in A_p^\times \mid \varphi_p(K_p) \cap g_p \mathcal{O}_p g_p^{-1} = \varphi_p(B_p) \}.
\end{equation}

If $\varphi_p, B_p$ and $\mathcal{O}_p$ are clear from the context, we simply write $\mathcal{E}_p$ for $\mathcal{E}_p(\varphi_p, B_p, \mathcal{O}_p)$. Clearly, $\mathcal{E}_p$ is invariant under left translation by $\varphi(K_p^\times)$ and right translation by $N(\mathcal{O}_p)$. The map $g_p \mapsto g_p^{-1} \varphi_p g_p$ induces a bijection

\begin{equation}
\varphi_p(K_p^\times) \setminus \mathcal{E}_p \simeq \text{Emb}(B_p, \mathcal{O}_p).
\end{equation}

Moreover,

\begin{equation}
\mathcal{E}_p \supseteq \varphi(K_p^\times) N(\mathcal{O}_p).
\end{equation}

It follows that $\text{Nr}(\mathcal{E}_p)$ is a finite subgroup of $F_p^\times$ of index at most 2, and $\text{Nr}(\mathcal{E}_p)$ does not depend on the choices of $\varphi_p \in \text{Emb}(B_p, \mathcal{O}_p)$. See [26, §2.2]. Let us define a finite set of primes of $F$ as follows

\begin{equation}
S := \{ p \mid c_p(\mathcal{O}) = 0 \quad \text{and} \quad (K/p) \neq 1 \}.
\end{equation}

From [26, Theorem 2.11], $B$ is optimally spinor selective for $\mathcal{G}$ if and only if

\begin{equation}
K \subseteq \Sigma_\mathcal{G}, \quad \text{and} \quad \text{N}_{K/F}(K_p^\times) = \text{Nr}(\mathcal{E}_p) \quad \text{for every} \quad p \in S.
\end{equation}

Assume that $K \subseteq \Sigma_\mathcal{G}$ from now on. From Proposition 2.9 if $p \in S$, then both of the following conditions hold:

\begin{enumerate}
\item[(†)] $p$ is nondyadic, $c_p(\mathcal{O}) = 0$, and $n_p(\mathcal{O}) \geq 3$,
\item[(‡)] $K_p$ is the unique ramified quadratic extension of $F_p$ such that $O_{K_p}$ embeds into $\mathcal{O}_p$, and $\text{Nr}(N(\mathcal{O}_p)) = N_{K_p/F_p}(K_p^\times)$. From Lemma 3.6 the latter condition is equivalent to the following:
\begin{itemize}
\item either $A_p$ is split with $-1 \in \mathbb{F}_p^{\times,2}$,
\item or $A_p$ is ramified with $-1 \not\in \mathbb{F}_p^{\times,2}$.
\end{itemize}
\end{enumerate}

Thus we may replace $S$ in (4.3) by

\begin{equation}
S := \{ p \mid \text{both conditions (†) and (‡) hold at } p \}.
\end{equation}

To prove Theorem 2.6 we need to show that for each $p \in S$ as above, $\text{Nr}(K_p^\times) = \text{Nr}(\mathcal{E}_p)$ if and only if one of the following conditions holds

\begin{enumerate}
\item[(i)] $n_p(\mathcal{O}) \geq 2i_p(B) + 3$,
\item[(ii)] $n_p(\mathcal{O}) = 2i_p(B) + 1$, $A$ is split at $p$, and $|\mathbb{F}_p| = 5$,
\item[(iii)] $n_p(\mathcal{O}) = 2i_p(B) + 1$, $A$ is ramified at $p$, and $|\mathbb{F}_p| = 3$.
\end{enumerate}

See [26] for the definitions of $n_p(\mathcal{O})$ and $i_p(B)$. If $A$ is split at $p$, then the assumption that $p$ is nondyadic and $-1 \in \mathbb{F}_p^{\times,2}$ already implies that $|\mathbb{F}_p| \geq 5$. Thus in (ii) or (iii), the value of $|\mathbb{F}_p|$ is precisely the minimal one in each respective case.
4.2. **Step (II): the recursion.** From now on, we work exclusively in the local case under the assumptions (i) and (ii) above. More explicitly, $F$ is assumed to be a nonarchimedean nondyadic local field with prime ideal $p$ and residue field $\mathfrak{p}$, and $O$ is a Bass order in $A$ with $e(O) = 0$ and $n(O) \geq 3$. Moreover, $K/F$ is the unique ramified quadratic extension such that $O_K$ embeds into $O$, and $\mathrm{Nr}(\mathcal{N}(O)) = N_{K/F}(K^\times)$. Let $B$ be an order in $K$ with $\mathrm{Emb}(B, O) \neq \emptyset$. We drop the subscript $p$ and write $\varphi, E$ for $\varphi_p, E_p$ etc.

**Lemma 4.2.1.** If $B = O_K$, then $\mathrm{Nr}(E) = N_{K/F}(K^\times)$.

*Proof.* From the proof of [9, Theorem 3.10, p. 180], the even normalizer group $\mathcal{N}(O)$ acts transitively from the right by conjugation on the set of (optimal) embeddings $\mathrm{Emb}(O_K, O)$. Hence we have

\[ E = \varphi(K^\times)\mathcal{N}(O). \tag{4.7} \]

Since $\mathrm{Nr}(\mathcal{N}(O)) = N_{K/F}(K^\times)$ by our assumption, the equality $\mathrm{Nr}(E) = N_{K/F}(K^\times)$ follows directly from (4.7). \qed

Now assume that $i(B) \geq 1$. For simplicity, let $\mathcal{M}(B)$ be the unique order in $K$ such that $i(\mathcal{M}(B)) = i(B) - 1$. A cornerstone of our proof of Theorem 2.6 is the following (slightly adjusted) lemma of Brzezinski [9, Lemma 3.18].

**Lemma 4.2.2.** Let $L/F$ be a semisimple quadratic extension, and $R \subset O_L$ be an order with $i(R) \geq 1$. Then every optimal embedding $R \to O$ extends to an optimal embedding $\mathcal{M}(R) \to \mathcal{M}^2(O)$, and every optimal embedding $\mathcal{M}(R) \to \mathcal{M}^2(O)$ whose image is not in $O$ restricts to an optimal embedding $R \to O$. Moreover, for each optimal embedding $\mathcal{M}(R) \to \mathcal{M}^2(O)$, its image is not in $O$ with the exception of $L = K$ and $\mathcal{M}(R) = O_K$ (i.e. $i(R) = 1$).

For the exceptional case, Brzezinski has “$L \supset F$ ramified” instead of “$L = K$”. However, since we assume that $F$ is nondyadic and $n(O) \geq 3$, if $L/F$ is ramified and $L \neq K$, then $\mathrm{Emb}(O_L, O) = \emptyset$ by Lemma 3.3.

Now fix $\varphi \in \mathrm{Emb}(B, O)$. Applying Lemma 4.2.2 for $L = K$ and $R = B$, we see that

\[ \varphi \in \mathrm{Emb}(\mathcal{M}(B), \mathcal{M}^2(O)) \quad \text{and} \quad E(\varphi, B, O) \subseteq E(\varphi, \mathcal{M}(B), \mathcal{M}^2(O)). \tag{4.8} \]

Moreover,

\[ E(\varphi, B, O) = E(\varphi, \mathcal{M}(B), \mathcal{M}^2(O)) \quad \text{if} \quad i(B) > 1. \tag{4.9} \]

Starting from a pair $(B, O)$ with $i(B) \geq 1$ and $n(O) \geq 3$, we apply (4.9) repeatedly until we arrive at a pair of orders $(\bar{B}, \bar{O})$ for which (4.9) no longer applies. In other words, the recursion halts after $k$ steps once we hit one of the following conditions:

\[ i(\bar{B}) = i(B) - k = 1 \quad \text{or} \quad n(\bar{O}) = n(O) - 2k < 3. \tag{4.10} \]

For simplicity, let us put $\bar{E} := E(\varphi, \bar{B}, \bar{O})$. By construction, $E = \bar{E}$, so

\[ \mathrm{Nr}(\bar{E}) = N_{K/F}(K^\times) \quad \text{if and only if} \quad \mathrm{Nr}(\bar{E}) = N_{K/F}(K^\times). \tag{4.11} \]

Thus we may replace $(B, O)$ by $(\bar{B}, \bar{O})$ and try to characterize when $\mathrm{Nr}(\bar{E}) = N_{K/F}(K^\times)$ holds true. Depending on the halting condition and the output of the recursion, the discussion will be separated into the four cases according to the following table.
Proof. In this case, we have \( \phi \) by the discussion at the bottom of [9, p. 181]. Thus \( \tilde{O} \).

Lemma 4.3.1. \( \Phi \)

Step (III): the case by case study. Keep the notation and assumptions of the previous step.

Lemma 4.3.1. If \( n(O) \leq 2i(B) \), then \( \text{Nr}(\mathcal{E}) \neq \text{N}_{K/F}(K^\times) \).

Proof. In this case, we have \( n(\tilde{O}) \in \{1, 2\} \) and \( i(\tilde{B}) \geq 1 \). If \( n(\tilde{O}) = 2 \), then \( \tilde{O} = O_F + \mathfrak{f}(\mathcal{M}(\tilde{O})) \) by (4.10), which implies that \( \text{Emb}(\tilde{B}, \tilde{O}) = \text{Emb}(\tilde{B}, \mathcal{M}(\tilde{O})) \) by the discussion at the bottom of [9, p. 181]. Thus \( \phi \in \text{Emb}(\tilde{B}, \mathcal{M}(\tilde{O})) \) and \( \tilde{E} = \mathcal{E}(\phi, \tilde{B}, \mathcal{M}(\tilde{O})) \). Replacing \( \tilde{O} \) by \( \mathcal{M}(\tilde{O}) \) if necessary, we may assume that \( n(\tilde{O}) = 1 \) for the remaining proof of this lemma. If \( A \) is split, then \( \tilde{O} \) is an Eichler order of level \( p \), so \( \text{Nr}(\mathcal{N}(\tilde{O})) = F^\times \). It follows from (4.11) and (4.11) that

\[
\text{Nr}(\mathcal{E}) = \text{Nr}(\tilde{\mathcal{E}}) = F^\times \neq \text{N}_{K/F}(K^\times).
\]

If \( A \) is ramified, then \( \tilde{O} \) is the unique maximal order in \( A \). Since \( i(\tilde{B}) \geq 1 \), we have \( \text{Emb}(\tilde{B}, \tilde{O}) = \emptyset \) by [24, Theorem II.3.1], which in turn implies that \( \text{Emb}(B, O) \neq \emptyset \) by the recursion. This contradicts the assumption that \( \text{Emb}(B, O) \neq \emptyset \). Therefore, \( A \) cannot be ramified when \( n(O) \leq 2i(B) \). The lemma is proved.

In the remaining cases, we always have \( n(\tilde{O}) \geq 3 \) and \( i(\tilde{B}) = 1 \). Thanks to (4.11), we may simply assume that \( i(B) = 1 \) and \( n(O) \geq 3 \) at the very beginning. In particular, \( \mathcal{M}(B) = O_K \).

Lemma 4.3.2. If \( i(B) = 1 \) and \( n(O) \geq 5 \), then \( \text{Nr}(\mathcal{E}) = \text{N}_{K/F}(K^\times) \).

Proof. From (4.11), we have

\[
\mathcal{E} = \mathcal{E}(\phi, B, O) \subseteq \mathcal{E}(\phi, O_K, \mathcal{M}^2(O)).
\]

Now \( n(\mathcal{M}^2(O)) \geq 3 \), so it follows from Lemma 4.2.1 that

\[
\text{Nr}(\mathcal{E}(\phi, O_K, \mathcal{M}^2(O))) = \text{N}_{K/F}(K^\times).
\]

We conclude that \( \text{Nr}(\mathcal{E}) = \text{N}_{K/F}(K^\times) \) in this case.

Now we treat the cases that \( i(B) = 1 \) and \( n(O) \in \{3, 4\} \). By the assumption on \( K \), there exists an embedding \( \phi_0 : O_K \to \mathcal{O} \). From the Skolem-Noether theorem, we may write

\[
\phi = z\phi_0 z^{-1} \quad \text{for some} \quad z \in A^\times.
\]

According to Lemma 4.2.2, there is a canonical decomposition

\[
\text{Emb}(O_K, \mathcal{M}^2(O)) = \text{Emb}(O_K, \mathcal{O}) \bigcup \text{Emb}(B, O).
\]
If we define
\[(4.14) \quad C(\varphi, O_K, \mathcal{O}) := \{ g \in A^s \mid \varphi(K) \cap gOg^{-1} = \varphi(O_K) \}, \]
then \(E(\varphi, O_K, M^2(\mathcal{O}))\) decomposes into
\[(4.15) \quad E(\varphi, O_K, M^2(\mathcal{O})) = E(\varphi, B, \mathcal{O}) \cup C(\varphi, O_K, \mathcal{O}). \]
Plugging (4.12) into (4.14), we get
\[(4.16) \quad C(\varphi, O_K, \mathcal{O}) = z \cdot E(\varphi_0, O_K, \mathcal{O}). \]
Since \(n(M^2(\mathcal{O})) \in \{1, 2\}\), we have
\[(4.17) \quad F^x = \text{Nr}(N(M^2(\mathcal{O}))) \subseteq \text{Nr}(E(\varphi, O_K, M^2(\mathcal{O}))) \subseteq F^x. \]
Hence the inclusions are in fact equalities. On the other hand, from Lemma 4.2.1,
\[(4.18) \quad \text{Nr}(E(\varphi_0, O_K, \mathcal{O})) = N_{K/F}(K^x), \]
which has index 2 in \(F^x\). Therefore,
\[(4.19) \quad \text{Nr}(E(\varphi, B, \mathcal{O})) \neq N_{K/F}(K^x) \quad \text{if} \quad \text{Nr}(z) \in N_{K/F}(K^x). \]

Write \(\mathcal{O} = \langle 1, \ x_{\alpha\beta}, \ \pi^r x_1, \ \pi^s x_3 \rangle \) as in (4.8). For simplicity, we identify \(K\) with \(F(x_{\alpha\beta})\) and take \(\varphi_0\) to be the identification map.

**Lemma 4.3.3.** If \(i(B) = 1\) and \(n(\mathcal{O}) = 4\), then \(\text{Nr}(\mathcal{E}) \neq N_{K/F}(K^x)\).

**Proof.** In this case, \(r = 2\) and \(s = 1\) by (5.3). Take \(z = 1 + x_3\). We claim that
\[(4.20) \quad \text{Nr}(z) \in N_{K/F}(K^x) \quad \text{and} \quad zKz^{-1} \cap \mathcal{O} = B. \]
From (3.6), we have \(\text{Tr}(x_3) = 0\) and \(\text{Nr}(x_3) = -\varepsilon \pi\). Therefore,
\[\text{Nr}(z) = (1 + x_3)(1 - x_3) = 1 - \varepsilon \pi \equiv 1 \pmod{p}. \]
This shows that \(\text{Nr}(z) \in O_{F'}^x \subseteq N_{K/F}(K^x)\) by (3.17). Recall that \(O_K = \langle 1, x_{\alpha\beta} \rangle\).
From Lemma 4.2.1 to show that \(zKz^{-1} \cap \mathcal{O} = B\), it is enough to show that \(zx_{\alpha\beta}z^{-1} \notin \mathcal{O}\). Since \(\text{Nr}(z) \in O_{F'}^x\), this is equivalent to show that \(zx_{\alpha\beta}z \notin \mathcal{O}\). Now we compute
\[(4.21) \quad zx_{\alpha\beta}z = (1 + x_3)(ax_1 + x_2 + bx_3)(1 - x_3)
\[= -(1 + \alpha \varepsilon)\pi + (\alpha + 2 + \alpha \varepsilon)\pi x_1
\[\quad + (1 + 2\alpha \varepsilon + \varepsilon \pi)x_2 + (\beta(1 - \varepsilon \pi) - (\alpha + \pi))x_3. \]
If we write \(zx_{\alpha\beta}z = a + bx_{\alpha\beta} + c\pi^2 x_1 + dx_3\), then
\[c\pi^2 = (2 + \alpha \varepsilon)\pi - (2\alpha + \pi)\varepsilon \alpha. \]
Since \(\alpha \in \mathfrak{p}\), we find that \(c\pi^2 \equiv 2\pi \pmod{\pi^2}\), and hence \(c \notin O_F\) because \(F\) is nondyadic. This finishes the verification of our claim. Now the lemma follows from combining (4.20) with (4.19). \(\square\)

**Lemma 4.3.4.** Suppose that \(i(B) = 1\) and \(n(\mathcal{O}) = 3\). Then \(\text{Nr}(\mathcal{E}) \neq N_{K/F}(K^x)\) if one of the following conditions holds:
- \(A\) is split and \(|t| > 5\),
- \(A\) is ramified and \(|t| > 3\).

\(^3\)Note that \(\varphi\) is not an optimal embedding of \(O_K\) into \(\mathcal{O}\), so this set cannot be denoted as \(E(\varphi, O_K, \mathcal{O})\).
Proof. From (3.3), we have \( r = s = 1 \) in this case. Thus \( \mathcal{O} = \{ 1, x_{\alpha\beta}, \pi x_1, \pi x_3 \} \), where \( x_{\alpha\beta} = \alpha x_1 + x_2 + \beta x_3 \). Since \( \alpha \in \mathfrak{p} \), without lose of generality we may assume that \( \alpha = 0 \), so \( x_{\alpha\beta} = x_2 + \beta x_3 \).

First, suppose that \( A \simeq \text{Mat}_2(F) \) and \( |\mathfrak{t}| > 5 \). In this case \( \varepsilon = 0 \), and \( 1 + \beta \in O_F^\times \), so \( x_{\alpha\beta} = \begin{bmatrix} 0 & 1 + \beta \\ \pi & 0 \end{bmatrix} \) by (3.7). Take \( z = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \) for some \( t \in O_F^2 \) with \( 1 - t^2 \notin \mathfrak{p} \).

Such a \( t \) exists because the number of \( a \in \mathfrak{k}^2 \) such that \( a \neq \pm 1 \) is \((|\mathfrak{t}| - 5)/2 > 0 \). We compute

\[
(4.22) \quad z x_{\alpha\beta} z^{-1} = \begin{bmatrix} 0 & t^{-1}(1 + \beta) \\ t \pi & 0 \end{bmatrix} = t x_{\alpha\beta} + \frac{(1 - t^2)(1 + \beta)}{t \pi} \pi x_3 \notin \mathcal{O}.
\]

The lemma in this case follows from combining (4.22) with (4.19).

Next suppose that \( A \) is ramified and \( |\mathfrak{t}| > 3 \). There exists \( a \in \mathfrak{k}^\times \) such that \( 1 - 4a \notin \mathfrak{k}^\times \cdot \mathfrak{k}^2 \) and \( 1 - 2a \neq 0 \). Indeed, the number of choices for such an \( a \) is at least \((|\mathfrak{t}| - 3)/2 > 0 \). Pick \( \varepsilon \in O_F^\times \) to be any element such that \( \varepsilon \) modulo \( \mathfrak{p} \) is equal to \( a \). Then we have \( 1 - 4\varepsilon \in O_F^\times \cdot O_F^2 \), and \( 1 - 2\varepsilon \in O_F^\times \). Take this particular \( \varepsilon \) in (3.5) for the \( F \)-basis \( \{ 1, x_1, x_2, x_3 \} \) of \( A \). Lastly, put \( z = 1 - \varepsilon^{-1}x_1 \). We claim that

\[
(4.23) \quad \text{Nr}(z) = 1 \quad \text{and} \quad z K z^{-1} \cap \mathcal{O} = B.
\]

The first equality follows from a direct calculation. To prove \( z K z^{-1} \cap \mathcal{O} = B \), it is enough to show that \( z x_{\alpha\beta} z^{-1} \notin \mathcal{O} \). We compute

\[
z x_{\alpha\beta} z^{-1} = (1 - \varepsilon^{-1}x_1)(x_2 + \beta x_3)(1 - \varepsilon^{-1} + \varepsilon^{-1}x_1)
= (1 + 2\beta - \varepsilon^{-1} - \varepsilon^{-1}\beta)x_2 + (\beta - 2\varepsilon^{-1} - 3\varepsilon^{-1}\beta + \varepsilon^{-2} + \varepsilon^{-2}\beta)x_3
= (1 + 2\beta - \varepsilon^{-1}(1 + \beta))x_{\alpha\beta} + \pi^{-1}\varepsilon^{-2}(1 - 2\varepsilon)(1 + \beta + \varepsilon\beta^2)(\pi x_3).
\]

By the above choice of \( \varepsilon \) and (3.5), the coefficient \( \pi^{-1}\varepsilon^{-2}(1 - 2\varepsilon)(1 + \beta + \varepsilon\beta^2) \notin O_F \). Our claim is verified. Now the lemma in this case follows from combining (4.23) with (4.19).

Keep the assumption that \( i(B) = 1 \) and \( n(\mathcal{O}) = 3 \). Let \( m(\mathcal{B}, \mathcal{O}, O^\times) \) be the number of \( O^\times \)-conjugacy classes of optimal embeddings of \( B \) into \( \mathcal{O} \) as in (2.4).

Using the assumption that \( F \) is nondyadic, we apply (9) (3.12), (3.13) and (3.15) to obtain

\[
(4.24) \quad m(\mathcal{B}, \mathcal{O}, O^\times) = \begin{cases} 
\frac{((|\mathfrak{t}|^2-|\mathfrak{t}|)\cdot 1 - 2|\mathfrak{t}|)}{|\mathfrak{t}|} & \text{if } A \text{ is split,} \\
\frac{((|\mathfrak{t}|^2+|\mathfrak{t}|)\cdot 1 - 2|\mathfrak{t}|)}{|\mathfrak{t}|} & \text{if } A \text{ is ramified.}
\end{cases}
\]

Assume further that one of the following conditions holds:

- \( A \) is split and \( |\mathfrak{t}| = 5 \),
- \( A \) is ramified and \( |\mathfrak{t}| = 3 \).

Then up to conjugation by \( O^\times \), there are exactly two optimal embeddings of \( B \) into \( \mathcal{O} \), say \( \varphi_1 \) and \( \varphi_2 \). Write \( \varphi_i = z_i \varphi_0 z_i^{-1} \) for \( i = 1, 2 \). We will show that \( \text{Nr}(z_i) \notin N_{K/F}(K^\times) \) for both \( i \). Since \( N_{K/F}(K^\times) \) is a subgroup of index 2 in \( F^\times \), the reduced norm of \( w := z_1 z_2^{-1} \) lies in \( N_{K/F}(K^\times) \), and \( \varphi_2 = w^{-1} \varphi_1 w \). Now

\[
(4.25) \quad E(\varphi_1, B, \mathcal{O}) = \varphi_1(K^\times)O^\times \bigcup \varphi_1(K^\times)wO^\times.
\]

It follows from (3.16) that

\[
(4.26) \quad \text{Nr}(E) = \text{Nr}(E(\varphi_1, B, \mathcal{O})) = N_{K/F}(K^\times).
\]
Lemma 4.3.5. Suppose that \( i(B) = 1 \), \( n(\mathcal{O}) = 3 \), \( A \) is split, and \( |\mathfrak{p}| = 5 \). Then \( \text{Nr}(\mathcal{E}) = N_{K/F}(K^\times) \).

Proof. Similarly as in the proof of Lemma 4.3.4, we take \( x_{\alpha\beta} = \begin{bmatrix} 0 & 1 + \beta \\ 1 & 0 \end{bmatrix} \). Put 
\[
z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad z_2 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.
\]
From (4.22), both \( z_1 \varphi_0 z_1^{-1} \) are optimal embeddings of \( B \) in \( \mathcal{O} \). Thus to finish the proof, it is enough to show that \( z_1 \varphi_0 z_1^{-1} \) and \( z_2 \varphi_0 z_2^{-1} \) are not \( \mathcal{O}^\times \)-conjugate. Suppose otherwise so that there exists \( u \in \mathcal{O} \) satisfying
\[
z_1 \varphi_0 z_1^{-1} \neq u^{-1} z_2 \varphi_0 z_2^{-1} u.
\]
Write \( u = a + bx_{\alpha\beta} + c\pi x_1 + d\pi x_3 \) with \( a, b, c, d \in O_F \). By (4.21), necessarily \( a \in O_F^\times \). But (4.27) holds if and only if \( z_1^{-1} u z_2^{-1} = \gamma + \delta x_{\alpha\beta} \) for some \( \gamma, \delta \in F \). We compute 
\[
z_1^{-1} u z_2 = \begin{bmatrix} a + c\pi & -2b(1 + \beta) - 2d\pi \\ b\pi/2 & -a \end{bmatrix} = \begin{bmatrix} \gamma & \delta(1 + \beta) \\ \delta\pi & \gamma \end{bmatrix}.
\]
Already, this implies that \( c = (-2a)/\pi \notin O_F \), contradiction to the assumption that \( u \in \mathcal{O} \). Therefore, \( z_1 \varphi_0 z_1^{-1} \) and \( z_2 \varphi_0 z_2^{-1} \) indeed represent distinct members of \( \text{Emb}(B, \mathcal{O})/\mathcal{O}^\times \). Since \( N_{K/F}(K^\times) \cap O_F^\times = O_F^2 \) and \( \mathfrak{p} = \mathfrak{F}_5 \), we find \( \pm 2 \notin N_{K/F}(K^\times) \). The lemma is proved.

\[\square\]

Lemma 4.3.6. Suppose that \( i(B) = 1 \), \( n(\mathcal{O}) = 3 \), \( A \) is ramified, and \( |\mathfrak{p}| = 3 \). Then \( \text{Nr}(\mathcal{E}) = N_{K/F}(K^\times) \).

Proof. Since \( |\mathfrak{p}| = 3 \), the assumption that \( 1 - 4\varepsilon \in O_F^\times \setminus O_F^2 \) implies that \( \varepsilon \equiv -1 \pmod{p} \). Thus if we put \( z_1 = x_1^{-1} \) and \( z_2 = x_1 \), then \( \text{Nr}(z_i) = \varepsilon_{z_1} \notin N_{K/F}(K^\times) \). We claim that \( z_i K z_i^{-1} \cap \mathcal{O} = B \) for both \( i = 1, 2 \). If not, then \( z K z^{-1} \cap \mathcal{O} = O_K \) for some \( z \in \{z_1, z_2\} \). Recall that \( \mathcal{N}^0(\mathcal{O}) \) acts transitively by conjugation on the set of embeddings \( \text{Emb}(O_K, \mathcal{O}) \) (cf. (3.17)). Thus there exists \( v \in \mathcal{N}^0(\mathcal{O}) \) such that \( z v z^{-1} \varphi_0 v^{-1} \varphi_0^{-1} \varphi_0 v \). It follows that \( v^{-1} z \in K^\times \), and hence \( \text{Nr}(v^{-1} z) \in N_{K/F}(K^\times) \). Since \( \text{Nr}(\mathcal{N}^0(\mathcal{O})) \subseteq N_{K/F}(K^\times) \), we find that \( \text{Nr}(z) \in N_{K/F}(K^\times) \) as well, contradiction to the choice of \( z_1 \) and \( z_2 \).

Next, we check that \( z_1 \varphi_0 z_1^{-1} \) and \( z_2 \varphi_0 z_2^{-1} \) are not \( \mathcal{O}^\times \)-conjugate. Suppose otherwise so that there exists \( u \in \mathcal{O}^\times \) with \( z_1^{-1} u z_2 \in K \). Write \( u = a + bx_{\alpha\beta} + c\pi x_1 + d\pi x_3 \) with \( a \in O_F^\times \) and \( b, c, d \in O_F \) as before. We compute
\[
z_1^{-1} u z_2 = x_1 u x_1 = -\varepsilon (a + c\pi) + (a + c\pi - c\varepsilon\pi) x_1 + b\varepsilon x_{\alpha\beta} + d\varepsilon\pi x_3.
\]
Thus \( z_1^{-1} u z_2 \in K \) if and only if \( a + c\pi (1 - \varepsilon) = 0 \) and \( d = 0 \). Since \( a \in O_F^\times \) and \( 1 - \varepsilon \in O_F^\times \), we get \( c \notin O_F \) again. This contradiction shows \( z_1 \varphi_0 z_1^{-1} \) and \( z_2 \varphi_0 z_2^{-1} \) indeed represent distinct members of \( \text{Emb}(B, \mathcal{O})/\mathcal{O}^\times \). The lemma is proved.

\[\square\]

End of the proof of Theorem 2.6 Comparing Table 4.1 with Lemmas 4.3.1–4.3.6 it is clear that we have finished the case-by-case study for \( i(B) \geq 1 \). The case \( B = O_K \) has already been treated in Lemma 4.2.1. The proof of Theorem 2.6 is now complete.

\[\square\]

As a by-product of our proof, we obtain the following criterion for nonexistence of local optimal embeddings. Let \( F, \mathcal{O} \) and \( K \) be as in the start of 4.2, except that we only keep the assumption that \( \text{Emb}(O_K, \mathcal{O}) \neq \emptyset \) and drop the assumption that

\(\text{Nr}(N(O)) = N_{K/F}(K^\times)\) (see Lemma 3.6). Write \(E_{\text{ur}}\) for the unique unramified quadratic field extension of \(F\).

**Corollary 4.3.7.** Let \(L/F\) be a semisimple quadratic extension, and \(R\) be an order in \(L\). Assume that \(\text{Hom}_F(L, A) \neq \emptyset\), that is, \(L \neq F \times F\) if \(A\) is ramified. Then \(\text{Emb}(R, O) = \emptyset\) if and only if one of the following holds:

1. \(n(O) < 2i(R)\) and \(A\) is ramified,
2. \(n(O) = 2i(R)\), \(A\) is ramified, and \(L/F\) is ramified,
3. \(n(O) = 2i(R) + 1\), \(L = E_{\text{ur}}\), and \(A = \text{Mat}_2(F)\),
4. \(n(O) = 2i(R) + 1\), \(L = K\), \(A = \text{Mat}_2(F)\) and \(|\mathfrak{t}| = 3\),
5. \(n(O) = 2i(R) + 2\), and either \(L = F \times F\) or \(L = E_{\text{ur}}\),
6. \(n(O) \geq 2i(R) + 3\) and \(L \neq K\).

See [9, Theorem 3.10] for the case \(n(O) = 2\), which holds even if \(F\) is dyadic.

**Proof.** Applying Lemma 4.2.2 recursively to \((R, O)\), we eventually obtain a new pair \((\tilde{R}, \tilde{O})\) for which \(\text{Emb}(\tilde{R}, \tilde{O}) = \text{Emb}(R, O)\). From Table 4.1 the discussion is again separated into four cases.

First, suppose that \(n(O) \leq 2i(R)\) so that \(n(\tilde{O}) \in \{1, 2\}\) and \(i(\tilde{R}) \geq 1\). We further divide it into two subcases according to whether \(n(\tilde{O})\) is equal to 1 or 2.

Suppose that \(n(\tilde{O}) = 1\). If \(A\) is split, then \(\tilde{O}\) is an Eichler order of level \(p\), so \(\text{Emb}(\tilde{R}, \tilde{O}) \neq \emptyset\) by [21, Theorem II.3.2]. If \(A\) is ramified, then \(\tilde{O}\) is the unique maximal order, so \(\text{Emb}(\tilde{R}, \tilde{O}) = \emptyset\) by [21, Theorem II.3.1]. Next, suppose that \(n(\tilde{O}) = 2\). From [9] (3.17), \(\text{Emb}(\tilde{R}, \tilde{O}) \neq \emptyset\) if and only if one of the following conditions holds:

- \(A\) is split,
- \(A\) is ramified, \(L = E_{\text{ur}}\), and \(i(\tilde{R}) = 1\).

Note that \((i(\tilde{R}), n(\tilde{O})) = (1, 2)\) if and only if \(n(\tilde{O}) = 2i(\tilde{R})\). This shows that when \(n(O) \leq 2i(R)\), we have \(\text{Emb}(R, O) = \emptyset\) if and only if either condition (1) or (2) holds.

Now suppose that \(n(O) = 2i(R) + 1\) so that \((i(\tilde{R}), n(\tilde{O})) = (1, 3)\). We further divide it into two subcases according to \(L = K\) or not. First, suppose that \(L \neq K\). Then \(\text{Emb}(\tilde{R}, \tilde{O}) = \text{Emb}(O_L, \mathcal{M}^2(\tilde{O}))\) by Lemma 4.2.2. Since \(n(\mathcal{M}^2(\tilde{O})) = 1\), we find that \(\text{Emb}(O_L, \mathcal{M}^2(\tilde{O})) = \emptyset\) if and only if \(L = E_{\text{ur}}\) and \(A = \text{Mat}_2(F)\). This gives part (3) of the corollary. Next, suppose that \(L = K\). From 4.24, we immediately see that \(\text{Emb}(\tilde{R}, \tilde{O}) = \emptyset\) if and only if \(A = \text{Mat}_2(F)\) and \(|\mathfrak{t}| = 3\). This gives part (4) of the corollary.

Next, suppose that \(n(O) = 2i(R) + 2\) so that \((i(\tilde{R}), n(\tilde{O})) = (1, 4)\). If \(L = K\), we have seen in the proof of Lemma 4.3.3 that \(\text{Emb}(\tilde{R}, \tilde{O}) \neq \emptyset\). Suppose that \(L \neq K\) so that \(\text{Emb}(\tilde{R}, \tilde{O}) = \text{Emb}(O_L, \mathcal{M}^2(\tilde{O}))\) again. Since \(n(\mathcal{M}^2(\tilde{O})) = 2\), it follows from [9] (3.14) that \(\text{Emb}(O_L, \mathcal{M}^2(\tilde{O})) \neq \emptyset\) if and only if \(L/F\) is ramified. This gives part (5) of the corollary.

Lastly, suppose that \(n(O) \geq 2i(R) + 3\) so that \(i(\tilde{R}) = 1\) and \(n(\tilde{O}) \geq 5\). From Lemma 4.2.2, \(\text{Emb}(\tilde{R}, \tilde{O}) \subseteq \text{Emb}(O_L, \mathcal{M}^2(\tilde{O}))\). Thus if \(L \neq K\), then \(\text{Emb}(\tilde{R}, \tilde{O}) = \emptyset\) since \(O_L\) does not embed into \(\mathcal{M}^2(\tilde{O})\) by Lemma 3.5. If \(L = K\), then according to [9] (3.13) and (3.15), we have

\[
m(\tilde{R}, \tilde{O}, \tilde{O}^\times) = \frac{1}{|\mathfrak{t}|} \left( |\mathfrak{t}|^2 \cdot 2|\mathfrak{t}| - 2|\mathfrak{t}| \right) = 2(|\mathfrak{t}|^2 - 1) > 0.
\]
This gives part (6) of the corollary and completes the proof. □

5. Examples

In this section, we construct a family of concrete examples where $B \subset K \subseteq \Sigma_\ell$ and $\text{Emb}(B, \mathcal{O}) \neq \emptyset$ for every $\mathcal{O} \in \mathcal{G}$ (i.e. $B$ is not optimally selective).

Let $p \in \mathbb{N}$ be a prime with $p \equiv 1 \pmod{4}$. Fix an integer $n \geq 3$, and put $r = \lfloor n/2 \rfloor$ and $s = \lfloor (n - 1)/2 \rfloor$ as in \eqref{5.11}. Pick $t \in \mathbb{Z} \cap \mathbb{Z}_p^{\times 2}$, i.e. $t \in \mathbb{Z}$ and is a quadratic residue modulo $p$.

Indeed, $\mathcal{O}_p$ is precisely the order in \eqref{3.8} with $\pi = p$, $\alpha = \beta = 0$. Similarly, we have taken $t = p$ and $\beta = 1$ for $\mathcal{O}_p$. A direct calculation shows that both $\mathcal{O}$ and $\mathcal{O}'$ have index $p^n$ in $\text{Mat}_2(\mathbb{Z})$. Hence

\begin{equation}
\mathcal{O}_\ell = \mathcal{O}'_\ell = \text{Mat}_2(\mathbb{Z}_\ell) \quad \text{for every prime } \ell \neq p.
\end{equation}

Pick $u_p \in \mathbb{Z}_p^\times$ such that $u_p^2 = t$ and put $h_p := \begin{bmatrix} u_p & 0 \\ 0 & 1 \end{bmatrix}$. Then $h_p \mathcal{O}_p h_p^{-1} = \mathcal{O}'_p$. Therefore, $\mathcal{O}$ and $\mathcal{O}'$ belong to the same genus. Let $\mathcal{G}$ be the genus of $\mathcal{O}$ and $\mathcal{O}'$, and $\Sigma_\ell$ be the spinor genus field of $\mathcal{G}$. For simplicity, write $K = \mathbb{Q}(\sqrt{p})$.

Lemma 5.1. $\Sigma_\ell = K = \mathbb{Q}(\sqrt{p})$. In particular, $|\text{Tp}(\mathcal{G})| = 2$.

Proof. From Lemma 3.5, $K_p$ is the unique quadratic extension of $\mathbb{Q}_p$ such that $\mathcal{O}_{K_p}$ embeds into $\mathcal{O}_p$. Since $p \equiv 1 \pmod{4}$, we have $-1 \in \mathbb{Z}_p^{\times 2}$. Thus $K \subseteq \Sigma_\ell$ by Proposition 2.4. On the other hand, $\Sigma_\ell/\mathbb{Q}$ is the compositum of its quadratic subextensions, but $K/\mathbb{Q}$ is the unique quadratic extension unramified outside $p$. We conclude that $\Sigma_\ell = K$.

From \eqref{2.9}, we have $|\text{SG}(\mathcal{G})| = [\Sigma_\ell : \mathbb{Q}] = 2$. Since $A = \text{Mat}_2(\mathbb{Q})$, which clearly satisfies the Eichler condition, $\text{SG}(\mathcal{G})$ is canonically identified with $\text{Tp}(\mathcal{G})$ by Remark 2.2. Therefore, $|\text{Tp}(\mathcal{G})| = 2$. □

Lemma 5.2. The orders $\mathcal{O}$ and $\mathcal{O}'$ are of the same type if and only if $t \in \mathbb{Z}_p^{\times 4}$, that is, $t$ is a quartic residue modulo $p$.

Proof. First, the hereditary closures of both $\mathcal{O}_p$ and $\mathcal{O}'_p$ coincide with $\left[ \begin{array}{cc} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{array} \right]$.

Put $\mathcal{O}' = \left[ \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{array} \right]$. If $g \mathcal{O} g^{-1} = \mathcal{O}'$ for some $g \in \text{GL}_2(\mathbb{Q})$, then necessarily $g \mathcal{O} g^{-1} = \mathcal{O}'$, that is, $g \in \mathcal{N}(\mathcal{O})$. It is well known that

\[ \mathcal{N}(\mathcal{O}) = \mathbb{Q}^\times \mathcal{O}^\times \bigcup_{p \neq \ell} \mathbb{Q}^\times \mathcal{O}^\times \left[ \begin{array}{cc} 0 & 1 \\ p & 0 \end{array} \right]. \]

On the other hand, $\left[ \begin{array}{cc} 0 & 1 \\ p & 0 \end{array} \right] \in \mathcal{N}(\mathcal{O})$. Indeed, clearly $\left[ \begin{array}{cc} 0 & 1 \\ p & 0 \end{array} \right] \in \mathcal{N}(\mathcal{O}_\ell)$ for each prime $\ell \neq p$. Moreover, from \eqref{2.21} we have $\left[ \begin{array}{cc} 0 & 1 \\ p & 0 \end{array} \right] \in \mathcal{N}(\mathcal{O}_p)$ since $x_{1,1} = x_2 = \left[ \begin{array}{cc} 0 & 1 \\ p & 0 \end{array} \right]$ for $\mathcal{O}_p$. Therefore, if there exists $g \in \text{GL}_2(\mathbb{Q})$ such that $\mathcal{O}' = g \mathcal{O} g^{-1}$, then it can be taken inside $\mathcal{O}^\times$. 
Suppose that there exists \( g \in \mathcal{O}^* \) such that \( g\mathcal{O}g^{-1} = \mathcal{O}' \). Then we have \( h_p^{-1} g \in \mathcal{N}(\mathcal{O}_p) \). From Lemma 3.6,

\[
\text{Nr}(h_p^{-1} g) \in \text{Nr}(\mathcal{N}(\mathcal{O}_p)) \cap \mathbb{Z}_p^2 = \mathbb{Z}_p^2.
\]

Since \( \text{Nr}(g) \in \text{Nr}(\mathcal{O}^*) = \{ \pm 1 \} \) and \( p \equiv 1 \pmod{4} \), we get \( u_p = \text{Nr}(h_p) \in \mathbb{Z}_p^2 \), which implies that \( t = u_p^2 \in \mathbb{Z}_p^4 \).

Next, suppose that \( t \in \mathbb{Z}_p^4 \). Then the equation \( x^2 = u_p \) has a solution \( v_p \in \mathbb{Z}_p^4 \). From [16, §6.1], the canonical map \( \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/p^{r+1}\mathbb{Z}) \) is surjective. In particular, there exists \( g \in \text{SL}_2(\mathbb{Z}) \) such that

\[
(5.5) \quad g \equiv \begin{bmatrix} v_p & 0 \\ 0 & v_p^{-1} \end{bmatrix} \pmod{p^{r+1}}.
\]

We claim that \( g\mathcal{O}g^{-1} = \mathcal{O}' \). It is enough to show that \( g\mathcal{O}_p g^{-1} = \mathcal{O}'_p \) for every prime \( \ell \) (including \( \ell = p \)). If \( \ell \neq p \), this is clear from (5.3). At the prime \( p \), observe that \( \mathcal{O}_p \supseteq \mathcal{O}_p := \mathbb{Z}_p + p^{r+1}\text{Mat}_2(\mathbb{Z}_p) \). The choice of \( g \) in (5.5) guarantees that \( h_p^{-1} g \in \mathcal{O}_p^* \subseteq \mathcal{O}_p^* \), which implies that \( g\mathcal{O}_p g^{-1} = \mathcal{O}'_p \). This finishes the verification of our claim and the proof of the lemma.

\[\square\]

**Example 5.3.** Suppose that \( p \equiv 5 \pmod{8} \). Then \(-1 \in \mathbb{Z}_p^2 \neq \mathbb{Z}_p^4 \). Thus if we put \( t = -1 \), then \( \text{Tp}(\mathfrak{f}) \) is represented by \( \mathcal{O} \) and \( \mathcal{O}' \).

**Proposition 5.4.** Suppose that \( t \notin \mathbb{Z}_p^4 \) so that \( \{ \mathcal{O}, \mathcal{O}' \} \) is a complete set of representatives for \( \text{Tp}(\mathfrak{f}) \). Let \( \mathcal{B} \) be an order in \( \mathbb{Q}(\sqrt{p}) \). Suppose that \( n < 2t_p(B) + 3 \), and \( p \neq 5 \) if \( n = 2t_p(B) + 1 \). Then both \( \text{Emb}(\mathcal{B}, \mathcal{O}) \) and \( \text{Emb}(\mathcal{B}, \mathcal{O}') \) are nonempty. In other words, \( \mathcal{B} \) is not optimally selective for the genus \( \mathfrak{f} \).

**Proof.** From (5.3), \( \text{Emb}(B_t, \mathcal{O}_t) \neq \emptyset \) for every prime \( \ell \neq p \). According to Corollary 4.3.7, \( \text{Emb}(B_p, \mathcal{O}_p) \) is nonempty as well. The proposition follows directly from Theorem 2.6. \[\square\]

Lastly, we consider the global number of optimal embeddings up to conjugation as in (2.4). Let \( \mathfrak{f} \) be a number field, and \( \mathfrak{A} \) be a quaternion \( \mathfrak{f} \)-algebra satisfying the Eichler condition. Let \( \mathcal{D}, \mathcal{D}' \subset \mathfrak{A} \) be two orders in the same genus \( \mathfrak{G} \). Suppose that \( e_p(\mathcal{D}) \neq 0 \) for every finite prime \( p \) of \( \mathfrak{f} \). Let \( \mathcal{B} \) be an \( \mathcal{O}_\mathfrak{f} \)-order in a quadratic field extension \( \mathfrak{K}/\mathfrak{f} \) with \( \text{Emb}(\mathcal{B}_p, \mathcal{D}_p) \neq \emptyset \) for every \( p \). Suppose that either \( \mathcal{B} \) is not optimally selective for \( \mathfrak{G} \) or both \( \mathcal{D} \) and \( \mathcal{D}' \) are optimally selected by \( \mathcal{B} \). Then

\[
(5.6) \quad m(\mathcal{B}, \mathcal{D}, \mathcal{D}') = m(\mathcal{B}, \mathcal{D}', \mathcal{D}'^\times).
\]

See [25, Theorem 31.1.7] for the proof in the case of Eichler orders and [26, Proposition 2.15] for the proof in general. Naturally, one asks whether the equality (5.6) still holds true if \( e_p(\mathcal{D}) \) is allowed to be zero at some finite prime \( p \). From [26, Proposition 2.15], inequality is possible if only if \( \mathfrak{K} \subseteq \Sigma_\mathfrak{G} \) and \( \mathcal{B} \) is not optimally selective for \( \mathfrak{G} \). Our family of examples fit this description perfectly, so we ask the following concrete question.

**Question 5.5.** Under the assumption of Proposition 5.4, do we have

\[
(5.7) \quad m(B, \mathcal{O}, \mathcal{O}^\times) = m(B, \mathcal{O}', \mathcal{O}'^\times) ?
\]

"But so far we do not know any such examples."
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