THE INTRINSIC STABLE NORMAL CONE

MARC LEVINE

Abstract. We construct an analog of the intrinsic normal cone of Behrend-Fantechi [3] in the setting of motivic stable homotopy theory. A perfect obstruction theory gives rise to a virtual fundamental class in \( \mathcal{E} \)-cohomology for any cohomology theory \( \mathcal{E} \); this includes the oriented Chow groups of Barge-Morel [2] and Fasel [5].

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Introduction

The various versions of modern enumerative geometry, including Gromov-Witten theory and Donaldson-Thomas theory, are based on two important constructions due to Behrend and Fantechi [3]. The first is the construction of the intrinsic normal cone \( \mathcal{E}_Z \) of a Deligne-Mumford stack \( Z \) over some base-scheme \( B \). The second, based on the first, is the virtual fundamental class \([Z, [\phi]]^{\text{vir}} \in \text{CH}_r(Z)\) associated to a perfect obstruction theory \([\phi] : E_\bullet \to L_{Z/B} \) on \( Z \), with \( r \) the virtual rank of \( E_\bullet \). In case \( r = 0 \) and \( Z \) is proper over a field \( k \), one has the numerical invariant \( \deg_k[Z, [\phi]]^{\text{vir}} \); more generally, one can cut down \([Z, [\phi]]^{\text{vir}} \) to dimension zero by taking so-called descendents, and then taking the degree of the resulting 0-cycle.

Recall that a perfect obstruction theory on \( Z \) is given by a map \([\phi] : E_\bullet \to L_{Z/B} \) in \( D^{\text{perf}}(Z) \) such that \( E_\bullet \) is locally represented on \( Z \) by a two-term complex \( F_1 \to F_0 \) in degrees 0,1 (we use homological notation) and such that the map \([\phi] \) induces an isomorphism on the sheaf \( h_0 \) and a surjection on \( h_1 \).

In case \([\phi] \) admits a global resolution \((F_1 \to F_0) \to L_{Z/B} \), the virtual fundamental class is defined by embedding \( \mathcal{E}_Z \) in the quotient stack \([F^1/F^0] \)
(\(F^i := F^i_\vee\)), pulling back \(\mathcal{C}_Z\) via the quotient map \(F^1 \to [F^1/F^0]\), which gives the subcone \(\mathcal{C}(F_\bullet) \subset F^1\), and then intersecting with the zero-section:

\[
[Z, [\phi]]_{vir} := s^*_\phi([\mathcal{C}(F_\bullet)]).
\]

Here \([\mathcal{C}(F_\bullet)]\) is the fundamental class associated to the closed subscheme \(\mathcal{C}(F_\bullet)\) of \(F^1\). If one wishes to extend this type of construction to more general cohomology theories, there may be a problem in even defining the fundamental class \([\mathcal{C}(F_\bullet)]\). For instance, in algebraic cobordism \(\Omega_*\), fundamental classes of arbitrary schemes do not exist [8, §3].

The main point of this paper is to reinterpret the constructions of the intrinsic normal cone, its fundamental class, and the virtual fundamental class associated to a perfect obstruction theory in the setting of motivic homotopy theory. Rather than taking a DM or Artin stack as our basic object, we work in the \(G\)-equivariant setting, following the current state of affairs in motivic stable homotopy theory. We will assume that \(G\) is tame in the sense of [6]; this includes the case of finite étale group schemes of order prime to all residue characteristics, or reductive group schemes in characteristic zero. For the full theory, we will also need to assume that the base-scheme \(B\) is affine and the \(G\)-scheme \(Z\) carrying the perfect obstruction theory is \(G\)-quasi-projective over \(B\), in other words, there is an equivariant locally closed immersion \(Z \hookrightarrow \mathbb{P}_B(F)\) for some locally free coherent sheaf \(F\) on \(B\), with \(G\)-action.

In spite of these restrictions, we gain a great deal of generality. We construct an “intrinsic stable normal cone” \(\mathcal{C}^{st}_Z\) for each \(G\)-quasi-projective \(B\)-scheme \(Z\), with \(\mathcal{C}^{st}_Z\) defined as an object in the equivariant motivic stable homotopy category \(\text{SH}^G(B)\). \(\mathcal{C}^{st}_Z\) carries a fundamental class \([\mathcal{C}^{st}_Z]\) in cohomotopy \(S^{0,0}_B(\mathcal{C}^{st}_Z)\). Moreover, for a perfect obstruction theory \([\phi] : E_\bullet \to \mathbb{L}_{Z/B}\) on \(Z\), we construct a virtual fundamental class

\[
[Z, [\phi]]_{vir} \in S^{0,0}_B(\pi^*_Z(\Sigma^{E\vee}_* (1_Z))).
\]

Here \(\Sigma^{E\vee}_*\) is the “Thom space operator” on \(\text{SH}^G(Z)\) associated to \(E_\bullet \in D^\text{perf}_G(Z)\) and \(\pi^*_Z : \text{SH}^G(Z) \to \text{SH}^G(B)\) is the exceptional pushforward associated to the structure morphism \(\pi_Z : Z \to B\).

If \(E \in \text{SH}^G(B)\) is a ring spectrum with unit map \(\epsilon_E : S_B \to E\), applying \(\epsilon_E\) to \([\mathcal{C}^{st}_Z]\) or \([Z, [\phi]]_{vir}\) gives us elements

\[
[\mathcal{C}^{st}_Z]_E \in E^{0,0}(\mathcal{C}^{st}_Z);
\]

\[
[Z, [\phi]]_{vir}^E \in E^{0,0}(\pi^*_Z(\Sigma^{E\vee}_* (1_Z))).
\]

For simplicity, we take \(G = \{\text{Id}\}\). If we take \(E = HZ\), the spectrum representing motivic cohomology, then, suitably interpreted, these classes reduce to the classes defined by Behrend-Fantechi. More generally, if \(E\) is orientable, then we can identify the groups \(E^{0,0}(\mathcal{C}^{st}_Z)\) and \(E^{0,0}(\pi^*_Z(\Sigma^{E\vee}_* (1_Z)))\).
as Borel-Moore $\mathcal{E}$-homology, giving classes
\[
[C_E^{st}]_E \in \mathcal{E}_{2dM,dM}(C_{Z \subset M});
\]
\[
[Z, [\phi]]^\text{vir}_E \in \mathcal{E}_{2r,r}(Z).
\]
Here $C_{Z \subset M}$ is the normal cone of $Z$ for a given closed immersion $i : Z \to M$ with $M$ smooth over $B$, $d_M$ is the dimension of $M$ over $B$ (which is the same as the dimension of $C_{Z \subset M}$ over $B$) and $r$ is the virtual rank of $E_\bullet$. Besides motivic cohomology, this includes such oriented theories such as (homotopy invariant) algebraic $K$-theory or algebraic cobordism.

If we work with theories $\mathcal{E}$ that are not oriented, the identification of the group carrying the virtual fundamental class becomes more complicated. However, there are interesting examples of theories, the SL-oriented theories, which admit a Thom isomorphism for bundles with a trivial determinant. One such theory is given by taking cohomology with coefficients the sheaf of Milnor-Witt $K$-groups (see \cite{11}). The part of this theory corresponding to the Chow groups gives the Barge-Morel theory of oriented Chow groups
\[
\widetilde{CH}^n(X) := H^n(X, K_{MW}^n)
\]
a formula reminiscent of Bloch’s formula relating the classical Chow groups with Milnor $K$-theory. There are also twisted versions of the oriented Chow groups
\[
\widetilde{CH}^n(X; L) := H^n(X, K_{MW}^n(L))
\]
for a line bundle $L$ on $X$. These formulas for the oriented Chow groups are only valid for smooth $X$, but one has a straightforward extension to the general case using Borel-Moore homology. The general theory gives us classes
\[
[C_E^{st}]_{K_{MW}^n} \in \widetilde{CH}_{dM}(C_{Z \subset M}; i^*\omega_{M/B});
\]
\[
[Z, [\phi]]^\text{vir}_{E} \in \widetilde{CH}_{r}(Z; \det E_\bullet^\vee).
\]

In this setting the pushforward maps on the oriented Chow groups are restricted to oriented proper morphisms. This still allows one to achieve a refinement of the usual Gromov-Witten-type invariants in case the given perfect deformation theory $E_\bullet$ not only has virtual rank zero, but also has trivial virtual determinant bundle (more generally, the determinant bundle should admit a square root line bundle). In this case, we have
\[
\deg([Z, [\phi]]_{K_{MW}^n}) \in \text{GW}(k)
\]
where $\text{GW}(k)$ is the Grothendieck-Witt group of the base-field $k$. Applying the rank homomorphism $\text{GW}(k) \to \mathbb{Z}$ recovers the classical degree. We hope that this approach will prove useful in studying the enumerative geometry of real varieties.

We conclude with remarking that our approach is essentially formal: our construction uses three ingredients beyond some elementary geometry of normal cones
(1) The existence of Grothendieck’s six operations for the equivariant motivic stable homotopy category $\text{SH}^G(-) : \text{Sch}^{\text{op}} G / B \to \text{Tr}$. Here $\text{Tr}$ is the 2-category of triangulated categories. In particular, for each $G$-vector bundle $V \to X$, we have the automorphism $\Sigma^V : \text{SH}^G(X) \to \text{SH}^G(X)$.

(2) For $X \in \text{Sch}^G / B$, we have the path groupoid $\mathcal{V}^G(X)$ of the $G$-equivariant $K$-theory space of $X$. We need the existence of a natural transformation $\Sigma^\mathcal{V} : \mathcal{V}^G(-) \to \text{Aut}(\text{SH}^G(-))$ extending the map $V \mapsto \Sigma^V$, such that the exceptional pushforward and pullback for a smooth morphism $f : X \to Y$ is given by $f^! = \Sigma^{\mathcal{T}_{X/Y}} \circ f^*$, $f_! = f_# \circ \Sigma^{-\mathcal{T}_{X/Y}}$, where $f_#$ is the left adjoint to $f^*$.

(3) $\mathbb{A}^1$-homotopy invariance: for $p : V \to Z$ an affine space bundle, co-unit of adjunction $pp^! \to \text{Id}_{\text{SH}^G(Z)}$ is an isomorphism.

Presumably many other functor $\text{Sch}^G / B \to \text{Tr}$ have these three properties.

A construction of the fundamental class of the normal cone $C_Z \subset M$ in algebraic cobordism was communicated to us by Parker Lowrey some years ago. Our construction of the fundamental class may be viewed as a generalization of this method, see Example 5.1 for further details. F. Déglise, F. Jin and A. Khan\footnote{private communication} have generalized aspects of the work of Lowrey-Schürg \cite{10}, constructing fundamental classes of quasi-smooth derived schemes in a motivic stable homotopy category of derived schemes; we expect there is a suitable dictionary translating between some of their constructions and some of the ones given here.

1. Background on motivic homotopy theory

We begin by recalling some of the aspects of the six operations on the motivic stable homotopy category. We refer the reader to \cite{1, 4, 7, 12, 14} for details on the non-equivariant case and \cite{6} for the extension to the equivariant setting.

Fix a noetherian affine scheme $U$ with flat, finitely presented linearly reductive group scheme $G_0$ over $U$ (see \cite{6}, Definition 2.14).

We fix a quasi-projective $U$-scheme $B \to U$ (with trivial $G_0$-action) as base-scheme and let $G = G_0 \times_U B$. A $G$-equivariant morphism $q : Y \to X$ of $G$-schemes over $B$ is called $G$-quasi-projective if there is a $G$-vector bundle $V \to X$ and a $G$-equivariant locally closed immersion $i : Y \to \mathbb{P}(V)$ of $X$-schemes. We let $\text{Sch}^G / B$ be full subcategory of $G$-schemes over $B$ with objects the $G$-quasi-projective $B$-schemes and let $\text{Sm}^G / B$ be the full subcategory of smooth $B$-schemes in $\text{Sch}^G / B$.

For $X \in \text{Sch}^G / B$, we have the category $\text{QCoh}^G_X$ of quasi-coherent $O_X$-modules with $G$-action and the full subcategory $\text{Coh}^G_X$ of coherent sheaves. We call $\mathcal{F}$ in $\text{Coh}^G_X$ locally free if $\mathcal{F}$ is locally free and of finite rank as an
\(O_X\)-module. We let \(D^b_G(X)\) denote the bounded derived category of coherent \(G\)-sheaves, \(D_G(X)\) the unbounded derived category of quasi-coherent \(G\)-sheaves and \(D^\text{perf}_G(X)\) the full subcategory of \(D_G(X)\) of complexes isomorphic in \(D_G(X)\) to a bounded complex of locally free sheaves. Such a complex is called a \textit{perfect} complex.

We will use homological notation for complexes: for a homological complex \(C_\bullet\), \(\tau \geq n C_\bullet\) is the complex which is \(C_m\) in degree \(m > n\), 0 in degree \(m < n\) and \(C_n/\partial(C_{n+1})\) in degree \(n\).

We will assume that \(U\) has the \(G_0\)-resolution property, namely, that each \(F \in \text{Coh}^{G_0} U\) admits a surjection \(E \to F\) from a locally free \(E \in \text{Coh}^{G_0} U\) (see [6, Definition 2.7]). This implies that the group scheme \(G\) over \(B\) is \textit{tame} in the sense of [6, Definition 2.26]. Examples of linearly reductive \(G_0\) such that \(U\) has the \(G_0\)-resolution property include

- \(G_0\) is finite locally free of order invertible on \(U\).
- \(G_0\) is of multiplicative type and is isotrivial.
- \(U\) has characteristic zero and \(G_0\) is reductive with isotrivial radical and coradical (e.g., \(G_0\) is semisimple).

See [6, Examples 2.8, 2.16, 2.27].

Hoyois shows [6, Lemma 2.11] that each \(Z \in \text{Sch}^G/B\) has the \(G\)-resolution property. In addition, if \(Z \in \text{Sch}^G/B\) is affine, then a locally free coherent \(G\)-sheaf \(F\) on \(Z\) is projective in \(\text{QCoh}^G X\) [6, Lemma 2.17]. This also implies that a complex \(E_\bullet\) in \(D_G(Z)\) that is locally (on \(Z_{\text{Zar}}\)) a perfect complex is in fact a perfect complex on \(Z\). Similarly, if \(E_\bullet \in D_G(X)\) is locally isomorphic to a complex of locally free \(G\)-sheaves on \(X\) which is 0 in degrees outside a given interval \([a,b]\), then \(E_\bullet\) is isomorphic to a complex of locally free \(G\)-sheaves supported in \([a,b]\).

For \(E \in \text{Coh}^G X\) locally free, we have the associated vector bundle \(p : E^\vee \to X\), with

\[E^\vee := \text{Spec}_{O_X} \text{Sym}^* E.\]

The \(G\)-action on \(E\) gives \(E^\vee\) a \(G\)-action, with \(p : E^\vee \to X\) a \(G\)-equivariant morphism.

We will often drop the “\(G\)” in our notations, speaking of \(B\)-morphisms for \(G\)-equivariant \(B\)-morphisms, vector bundles \(V \to X\) for \(G\)-vector bundles, etc.

Let \(\text{Tr}\) be the 2-category of triangulated categories. Following [6, §6, Theorem 6.18], we have the motivic stable homotopy category

\[\text{SH}^G(-) : \text{Sch}^G/B^{op} \to \text{Tr};\]

for \(f : Y \to X\) in \(\text{Sch}/B\), we have the exact functor \(f^* : \text{SH}^G(X) \to \text{SH}^G(Y)\) with right adjoint \(f_* : \text{SH}^G(Y) \to \text{SH}^G(X)\) and the exceptional pullback \(f^! : \text{SH}^G(X) \to \text{SH}^G(Y)\) with left adjoint \(f_! : \text{SH}^G(X) \to \text{SH}^G(Y)\). If \(f\) is a smooth morphism, \(f^*\) admits the left adjoint \(f_\#\). \(\text{SH}^G(X)\) is a closed symmetric monoidal triangulated category with product denoted \(\wedge_X\).
internal Hom $\text{Hom}_X(-,-); f^*$ is a symmetric monoidal functor and $f_*$ and $f_!$ satisfy projection formulas, that is, for $f : Y \to X, f_*$ and $f_!$ are $\text{SH}^G(X)$-module maps and the same holds for $f_!^\#$ if $f$ is smooth. There is a natural transformation $\eta^f_\# : f_! \to f_*$ which is an isomorphism if $f$ is proper. See also the earlier treatments \cite{1} and \cite{4} for the non-equivariant case.

For the pair of adjoint functors $a_! \dashv a^!$, we let $e_a : a^! a_! \to \text{Id}$ denote the co-unit. For the pair of adjoint functor $a^* \dashv a_*$, we let $u_a : \text{Id} \to a_* a^*$ denote the unit. We will use analogous notation for other adjoint pairs, leaving the context to make the meaning clear.

Besides the equivariant stable motivic homotopy category, Hoyois has defined an equivariant unstable motivic category $\mathcal{H}^G_*(X)$ for $X \in \text{Sch}^G/B$ \cite[§5]{6}, generalizing the constructions of Morel-Voevodsky \cite{12} in the non-equivariant case. There is an infinite $T$-suspension functor $\Sigma^\infty_T : \mathcal{H}^G_*(X) \to \text{SH}^G(X)$; we often simply write $\mathcal{X}$ for $\Sigma^\infty_T \mathcal{X}$ when the context makes the meaning clear. The functors $f^*, f_*$ are $T$-stabilizations of functors $f^* : \mathcal{H}^G_*(X) \to \mathcal{H}^G_*(Y), f_* : \mathcal{H}^G_*(Y) \to \mathcal{H}^G_*(X)$, with $f^*$ left adjoint to $f_*$. If $f : Y \to X$ is smooth, $f_!^\#$ is the $T$-stabilization of $f_! : \mathcal{H}^G_*(Y) \to \mathcal{H}^G_*(X)$, left adjoint to $f^*$. Similarly, if $i : Y \to X$ is a closed immersion, then $i_* : \mathcal{H}^G_*(Y) \to \mathcal{H}^G_*(X)$ admits a left adjoint $i^!$, and the maps $i_! = i^! : \text{SH}^G(Y) \to \text{SH}^G(X), i^! : \text{SH}^G(X) \to \text{SH}^G(Y)$ are the $T$-stabilizations of these unstable versions.

For $p : V \to X$ a vector bundle with zero-section $s : X \to V$, we have the Thom space $\text{Th}_X(V) := p_! s_!(1_V) \in \mathcal{H}^G_*(X)$. The Thom space $\text{Th}_X(V)$ is canonically isomorphic to the cofiber of $V \setminus s(X) \to V$ in $\mathcal{H}^G_*(X)$. $\text{Th}_X(V)$ is invertible in $\text{SH}^G(X)$ with inverse denoted $\text{Tg}_X(-V)$. Let $\Sigma^V : \text{SH}^G(X) \to \text{SH}^G(X)$ denote the functor $\Sigma^V V := \text{Th}_X(V) \wedge_{\mathcal{X}} \alpha$, and define $\Sigma^V_X$ similarly. For $f : Y \to X$ smooth, there are canonical isomorphisms

$$f_! \cong f_!^\# \circ \Sigma^{-T_Y/X}, \quad f^! \cong \Sigma^{T_Y/X} \circ f^*,$$

giving the canonical isomorphism

$$f_! f^! \cong f_!^\# f^*.$$

If $f : V \to X$ is an affine space bundle over $X$, then the $\mathbb{A}^1$-homotopy property shows that the co-unit of the adjunction $f_! \dashv f^*$, $e_f : f_! f^* \to \text{Id}$, is a natural isomorphism.

There are exchange morphisms associated to a cartesian diagram

$$\begin{array}{ccc}
Z & \xrightarrow{q} & Y \\
\downarrow g & & \downarrow f \\
W & \xrightarrow{p} & X
\end{array}$$

as follows:
1. We have $Ex(\Delta_\ast^s) : p^* f_\ast \to g_\ast q^*$ defined as the composition
   \[ p^* f_\ast \xrightarrow{u_p} g_\ast g^* p^* f_\ast = g_\ast q^* f^* f_\ast \xrightarrow{\epsilon_f} g_\ast q^* \]
   $Ex(\Delta_\ast^s)$ is an isomorphism if $p$ is smooth or if $f$ is proper.

2. Suppose that $p$ is smooth. The isomorphism $Ex(\Delta_\ast^s) : q_\# g^* \to f^* p_\#$ is defined as the composition
   \[ q_\# g^* \xrightarrow{u_p} q_\# g^* p_\# = q_\# q^* f^* p_\# \xrightarrow{\epsilon_q} f^* p_\#. \]

3. Suppose $p$ is smooth. We have $Ex(\Delta_\ast^s) : p_\# g_\ast \to f_\ast q_\#$ defined as the composition
   \[ p_\# g_\ast \xrightarrow{u_f} f_\ast f^* p_\# g_\ast \xrightarrow{Ex(\Delta_\ast^s)^{-1}} f_\ast q_\# g_\ast \xrightarrow{\epsilon_g} f_\ast q_\#. \]

4. Suppose $p$ is smooth. We have the isomorphism $Ex(\Delta^s_\ast) : g^* p^! \to q^! f^*$ defined as the composition
   \[ g^* p^! \cong g^* \Sigma^T_{W/X} p^* \cong \Sigma^T_{Z/Y} g^* p^* \cong \Sigma^T_{Z/Y} q^* f^* \cong q^! f^*. \]

5. Suppose $p$ is smooth. We have $Ex(\Delta_\ast^s) : p_\# g_\ast \to f_\ast q_\#$ defined as the composition
   \[ p_\# g_\ast \cong p_\# \Sigma^{-T_{W/X}} g_\ast \cong p_\# g_\ast \Sigma^{-T_{Z/Y}} \xrightarrow{Ex(\Delta_\ast^s)} f_\ast q_\# \Sigma^{-T_{Z/Y}} \cong f_\ast q_\#. \]

For a locally free sheaf $E \in \text{Coh}_{X}^G$, we have the automorphism $\Sigma^{E^\vee}$ of $\text{SH}_G^G(X)$. Ayoub [1, Théoréme 1.5.18] and Riou [13, Proposition 4.1.1] have shown that for $G = \{\text{Id}\}$, the association $E \mapsto \Sigma^{E^\vee}$ extends to a functor
   \[ \Sigma^- : \mathcal{V}(X) \to \text{Aut}(\text{SH}(X)). \]
Here $\mathcal{V}(X)$ is the path groupoid of the $K$-theory space of $X$ and $\text{Aut}(\text{SH}(X))$ is the category with objects the auto-equivalences of $\text{SH}(X)$ and morphisms the natural isomorphisms.

The same arguments extend without problem to the equivariant case: letting $\mathcal{V}^G(X)$ be the path groupoid of the $G$-equivariant $K$-theory space of $X$, there is a functor
   \[ \Sigma^- : \mathcal{V}^G(X) \to \text{Aut}(\text{SH}^G(X)) \]
extending the assignment $E \mapsto \Sigma^{E^\vee}$. Letting $D^\text{perf}_{G,\text{iso}}(X)$ be the subcategory of $D^\text{perf}_G(X)$ with the same objects and morphisms the isomorphisms in $D^\text{perf}_G(X)$, we have the functor $D^\text{perf}_{G,\text{iso}}(X) \to \mathcal{V}^G(X)$. The functor $\Sigma^-$ thus induces the functor
   \[ \Sigma^- : D^\text{perf}_{G,\text{iso}}(X) \to \text{Aut}(\text{SH}^G(X)). \]

We write $\Sigma^{E^\vee}$ for the image of a perfect complex $E_\ast$ under this functor. For each distinguished triangle $E^1_\ast \to E_\ast \to E^2_\ast \to$ there is an isomorphism
   \[ \Sigma^{E^\vee} \cong \Sigma^{E^1^\vee} \circ \Sigma^{E^2^\vee} \]
natural with respect to isomorphisms of distinguished triangles, and for each \( E_\bullet \) an isomorphism \( \Sigma^{E_\bullet \mathbb{1}} \cong (\Sigma^{E_\bullet})^{-1} \). Thus, if \( E_\bullet = (E_n \to \ldots \to E_m) \), supported in \([n, m]\), then \( \Sigma^{E_\bullet} \) is canonically isomorphic to \( \Sigma^{(-1)^m E_n} \circ \ldots \circ \Sigma^{(-1)^n E_m} \).

Suppose we have a closed immersion \( i : Z \to X \) in \( \text{Sch}^G/B \), with open complement \( j : U \to X \). This yields the localization distinguished triangle

\[
j! j^! \to \text{Id}_{\text{SH}^G(X)} \xrightarrow{u} i_* i^*
\]

This gives a canonical isomorphism

\[
\pi_M !(M/(M \setminus Z)) \cong Z/B_{B.M.}
\]

for \( p \) proper, we have the natural transformation (proper pullback)

\[
p^* : \pi_W! \to \pi_Z! \circ p^*
\]

defined as the composition

\[
\pi_W! \xrightarrow{u_{p}} \pi_W! p_* p^* (\eta_{p}^*)^{-1} \xrightarrow{\pi_W! p_* p^* \cong \pi_Z! \circ p^*}
\]

Applying \( p^* \) to \( 1_W \) gives the morphism

\[
p^* : W/B_{B.M.} \to Z/B_{B.M.}
\]
defined by the composition
\[ \pi_! \Sigma_{T\mathcal{Z}/\mathcal{W}} f^* \cong \pi_! f^! e_f \to \pi_! f^! . \]
One checks that \( f \mapsto f_* \) is functorial: For smooth morphisms \( f : Z \to W \), \( g : Y \to Z \), we have
\[ (fg)_* \circ \theta_{Y/Z/W} = f_* \circ [g_* \circ (\Sigma_{T\mathcal{Z}/\mathcal{W}} \circ f^*)] \]
where
\[ \theta_{Y/Z/W} : (\pi_! \Sigma_{T\mathcal{Y}/\mathcal{Z}} g^*) \circ (\Sigma_{T\mathcal{Z}/\mathcal{W}} \circ f^*) \to \pi_! \Sigma_{T\mathcal{Y}/\mathcal{W}} \circ (fg)^* \]
is the isomorphism induced by the exact sequence
\[ 0 \to T_{\mathcal{Y}/\mathcal{Z}} \to T_{\mathcal{Y}/\mathcal{W}} \to g^* T_{\mathcal{Z}/\mathcal{W}} \to 0. \]
Applying \( f_* \) to \( 1_W \) gives the morphism
\[ f_* : \pi_! (\text{Th}_{\mathcal{Z}} (T_{\mathcal{Z}/\mathcal{W}})) \to W/B_{BM}. \]
Suppose a smooth morphism \( f : Z \to W \) admits a section \( s : W \to Z \). We have the canonical isomorphism
(a) \[ f_! s_! \cong (fs)_! \]
which induces the isomorphism on the adjoints
(b) \[ s^! f^! \cong (fs)^! \]
Let \( e_s : s_! s^! \to \text{Id}, e_f : f^! f_! \to \text{Id}, e_{fs} : (fs)_!(fs)^! \to \text{Id} \) be the co-units of adjunction. This gives us the commutative diagram
\[
\begin{array}{ccc}
\text{Id} & \cong & f_! s_! s^! f^! \cong f_! f^! e_f \to \text{Id} \\
\downarrow & (a) \circ (b) & \downarrow \\
\text{Id} & \cong & (fs)_!(fs)^! e_{fs} \to \text{Id},
\end{array}
\]
in other words, \( f_! e_s f^! \) is a right inverse to \( e_f \)
\footnote{I am grateful to F. Déglise and D.C. Cisinski for communicating this argument.}

Define the natural transformation
(1.4) \[ s_* : \pi W! \to \pi Z! \circ \Sigma_{T\mathcal{Z}/\mathcal{W}} \circ f^* \]
as the composition
\[ \pi W! \circ f_* \circ e_{fs} \to \pi W! \circ f ! \circ f^! \cong \pi W! \circ f_! \circ \Sigma_{T\mathcal{Z}/\mathcal{W}} \circ f^* \cong \pi Z! \circ \Sigma_{T\mathcal{Z}/\mathcal{W}} \circ f^* \]

Lemma 1.3. 1. Let \( f : Z \to W \) be a smooth morphism in \( \text{Sch}^G/B \) with a section \( s : W \to Z \). Then the composition
\[ \pi W! \circ s_* \pi Z! \circ \Sigma_{T\mathcal{Z}/\mathcal{W}} \circ f^* \to \pi W! \]
is the identity.

2. If \( f : Z \to W \) is a vector bundle over \( W \), then

\[
f_* : \pi_{Z!} \circ \Sigma^{T_{Z/W}} \circ f^* \to \pi_{W!}
\]

is an isomorphism, with inverse \( s_* \).

Proof. The fact that \( e_f \circ (f_! e_s f^!) = \text{Id} \) implies (1).

For (2), it suffices by (1) to show that \( f_* \) is an isomorphism. Since \( Z \to W \) is a vector bundle, it follows by \( A^1 \) homotopy invariance that \( e_f : f_! f^! = f_# f^* \to \text{Id} \) is an isomorphism; applying \( \pi_{W!} \circ - \) yields (2). \( \square \)

**Lemma 1.4.** Suppose we have a Cartesian diagram in \( \text{Sch}^G/B \)

\[
\begin{array}{ccc}
Z & \xrightarrow{q} & Y \\
\downarrow{g} & & \downarrow{f} \\
W & \xrightarrow{p} & X
\end{array}
\]

with \( p \) proper and \( f \) smooth. Then the diagram

\[
\begin{array}{cccccccc}
\pi_{Y!} \circ \Sigma^{T_{Y/X}} \circ f^* & \xrightarrow{q^* (\Sigma^{T_{Y/X}} \circ f^*)} & \pi_{Z!} \circ g^* \circ \Sigma^{T_{Y/X}} \circ f^* & \cong & \pi_{Z!} \circ \Sigma^{T_{Z/W}} \circ g^* \circ p^*
\\
\downarrow{f_*} & & \downarrow{\pi_{X!}} & & \downarrow{g_* \circ p^*}
\\
\pi_{X!} & \cong & \pi_{X!} & \cong & \pi_{X!}
\end{array}
\]

commutes. In other words, proper pullback commutes with smooth pushforward.

Proof. In what follows we simply write \( \cong \) for isomorphisms that follow from functoriality, such as \( \pi_{X!} f_! \cong \pi_{Y!} \) or that follow from the isomorphisms \( \Sigma^{T_{Y/X}} f^* \cong f^! \) or \( \Sigma^{T_{Z/W}} g^* \cong g^! \).
We fit a number of diagrams together.

(a) \[
\begin{align*}
\pi_Y! \Sigma^{T_Y/X} f^* \xrightarrow{\pi_Y! \eta_Y} & \pi_Y! q_* \Sigma^{T_Y/X} f^* \\
& \downarrow \iota \\
& \pi_Y! q_* \phi^* f^! \\
& \downarrow \iota \\
\pi_X! f! f! \xrightarrow{\pi_X! f! u_q f!} & \pi_X! f! q_* \phi^* f^! \\
& \downarrow \pi_X! Ex(\Delta^e) q_* \phi^* f^! \\
& \pi_X! p_* q_* \phi^* f^! \\
& \downarrow \pi_X! p_* g_q^! p^* \\
\pi_X! \xrightarrow{\pi_X! u} & \pi_X! p_*^! \phi^! p^*
\end{align*}
\]

(b1) \[
\begin{align*}
\pi_Y! q_* \phi^* \Sigma^{T_Y/X} f^* \xrightarrow{\pi_Y! \eta_Y} & \pi_Y! q_* \phi^* \Sigma^{T_Y/X} f^* \\
& \downarrow \iota \\
& \pi_Y! q_* \phi^! f! \\
& \downarrow \iota \\
\pi_X! f! q_* \phi^! f! \xrightarrow{\pi_X! \eta_Y} & \pi_Y! q_* \phi^! f! \\
& \downarrow \pi_X! p_* q_* \phi^! f! \\
\pi_X! Ex(\Delta^e) \xrightarrow{\sim} & \pi_X! p_* q_* \phi^! f! \\
\pi_X! p_* g_q^! f! & \xrightarrow{\sim} \pi_X! p_* g_q^! f!
\end{align*}
\]

(b2) \[
\begin{align*}
\pi_X! p_* g_q^! f! \xrightarrow{\pi_X! \eta_Y} & \pi_X! p_* g_q^! f! \\
& \downarrow \pi_X! p_* g_q^! f! \\
\pi_X! p_* g_q^! f! \xrightarrow{\sim} & \pi_X! p_* g_q^! f! \\
& \downarrow \pi_X! p_* g_q^! f! \\
\pi_X! p_* g_q^! f! \xrightarrow{\sim} & \pi_X! p_* g_q^! f! \\
\pi_X! p_* e_g p^* & \xrightarrow{\sim} \pi_X! p_* e_g p^*
\end{align*}
\]
These fit together as

\[
\pi_Y \circ \Sigma^{T_Y/X} \circ f^* \xrightarrow{q^* \circ (\Sigma^{T_Y/X} \circ f^*)} \pi_Z \circ \Sigma^{T_Z/W} \circ g^* \circ p^*
\]

The four diagrams (a), (b1), (b2) and (c) all commute; this follows from the commutativity of transformations acting on separate parts of a composition of functors, or the naturality of the unit and co-unit of an adjunction, or that fact that the exchange isomorphisms \(Ex(\Delta^1)\) and \(Ex(\Delta_+)\) are derived from the functoriality of composition for \((-)^*\) and \((-)_\ast\), combined with units and co-units of various adjunctions. For instance, the commutativity of the
lower square in (a) is equivalent to the commutativity of the square

We fill this in as follows

The commutativity of (i), (iii) and (vi) is obvious, that of (ii) is the definition of $Ex(\Delta_\#)$ and that of (iv) is the standard identity $(e_q \circ q^* \circ (q^* \circ u_q) = Id$ for the unit and co-unit of an adjunction. The commutativity of (v) reduces to that of
The commutativity of the left side is obvious and, using the definition of $Ex(\Delta^*_\#)$, that of the right side reduces to the commutativity of

$$
\begin{align*}
g\# q^* f^* & \sim g\# g^* p^* \xrightarrow{e_g} p^* \\
e_f & \\
g\# q^* f^* \sim g\# g^* p^* f^* \xrightarrow{e_g} p^* f^* \\
e_f
\end{align*}
$$

Filling this in as

$$
\begin{align*}
g\# q^* f^* & \sim g\# g^* p^* \xrightarrow{e_g} p^* \\
e_f & \\
g\# q^* f^* & \sim g\# q^* f^* \\
e_f & \\
g\# q^* f^* \sim g\# g^* p^* f^* \xrightarrow{e_g} p^* f^*
\end{align*}
$$

we see that the commutativity follows from the identity $(e_f \circ f^*) \circ (f^* \circ u_f) = \text{Id}$.

The commutativity of the remaining diagrams is much easier to verify and we leave the details to the reader. \qed

Remark 1.5. The usual operations on Borel-Moore homology: proper pushforward, smooth pullback, intersection with a section, all follow by applying proper pullback $p^*$, smooth pushforward $f^*$ or section pushforward $s^*$ to morphisms $Z/B^{B,M} \to E$ (or twists thereof). With this translation, Lemma \[\text{[3]}\] is saying that proper pushforward commutes with smooth pullback in twisted Borel-Moore cohomology.

2. The intrinsic stable normal cone

Take $Z \in \textbf{Sch}^G/B$. Since $Z$ is $G$-quasi-projective over $B$, $Z$ admits a closed immersion $i : Z \to M$, with $M \in \textbf{Sm}^G/B$. As in \[\text{[3]}\], we have the normal cone $\mathcal{C}_{Z \subset M}$:

$$
\mathcal{C}_{Z \subset M} := \text{Spec} \mathcal{O}_X(\oplus_{n \geq 0} \mathcal{I}^n_Z / \mathcal{I}^{n+1}_Z),
$$

where $\mathcal{I}_Z$ is the ideal sheaf of $Z \subset M$. Let $p_i : \mathcal{C}_{Z \subset M} \to Z$ be the projection and $\sigma_i : \mathcal{C}_i \to M$ the composition $i \circ p_i$. As before, we denote the structure morphism for $Y \in \textbf{Sch}^G/B$ by $\pi_Y : Y \to B$.

Lemma 2.1. Suppose we have closed immersions $i : Z \to M$, $i' : Z \to M'$. Then there is a canonical isomorphism

$$
\psi_{i,i'} : \pi_{\mathcal{C}_{Z \subset M'}}(\sigma_i^* \text{Th}_{M'}(T_{M'/B})) \to \pi_{\mathcal{C}_{Z \subset M}}(\sigma_i^* \text{Th}_M(T_{M/B})).
$$

If we have a third closed immersion $i'' : Z \to M''$ then $\psi_{i,i'} \circ \psi_{i',i''} = \psi_{i,i''}$. 

Proof. Using the immersion 
\[(i, i') : Z \to M \times_B M'\]
we may assume there is a smooth morphism \(g : M' \to M\) with \(i = g \circ i'\). In this case \(g\) induces a smooth morphism \(\mathcal{C}(g) : \mathcal{C}_{Z \subset M'} \to \mathcal{C}_{Z \subset M}\), giving the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}_{Z \subset M'} & \xrightarrow{\mathcal{C}(g)} & \mathcal{C}_{Z \subset M} \\
\sigma_{i'} & \downarrow & \sigma_i \\
M' & \xrightarrow{g} & M
\end{array}
\]

The projection \(\mathcal{C}(g)\) makes \(\mathcal{C}_{Z \subset M'}\) into a torsor over \(\mathcal{C}_{Z \subset M}\) for the vector bundle \(p_i^* i^* T_{M'}/M\), giving a canonical identification

\[T_{\mathcal{C}_{Z \subset M'}/\mathcal{C}_{Z \subset M}} \cong \sigma_i^* T_{M'/M}.\]

The exact sequence

\[0 \to i^* T_{M'/M} \to i'^* T_{M'/B} \to i^* T_{M/B} \to 0\]
gives the canonical isomorphism

\[\Sigma^{-\sigma_i^* T_{M'/M}} \circ \Sigma^{\sigma_{i'}^* T_{M'/B}} \xrightarrow{\theta_g} \Sigma^{\mathcal{C}(g)^*} \sigma_i^* T_{M/B}\]

Since \(\mathcal{C}(g) : \mathcal{C}_{Z \subset M'} \to \mathcal{C}_{Z \subset M}\) is an affine space bundle, the co-unit

\[\epsilon_{\mathcal{C}(g)} : \mathcal{C}(g)^\# \circ \mathcal{C}(g)^* \to \text{Id}_{SH^G(\mathcal{C}_{Z \subset M})}\]

is an isomorphism. This gives the canonical isomorphisms

\[
\mathcal{C}(g)^! \circ \Sigma^{\sigma_i^* T_{M'/B}} \circ \mathcal{C}(g)^* \cong \mathcal{C}(g)^! \circ \Sigma^{-\sigma_{i'}^* T_{M'/M}} \circ \Sigma^{\sigma_i^* T_{M'/B}} \circ \mathcal{C}(g)^* \\
\cong \mathcal{C}(g)^! \circ \Sigma^{\mathcal{C}(g)^*} \sigma_i^* T_{M/B} \circ \mathcal{C}(g)^* \\
\cong \mathcal{C}(g)^! \circ \mathcal{C}(g)^* \circ \Sigma^{\sigma_i^* T_{M/B}} \\
\cong \Sigma^{\sigma_i^* T_{M/B}}.
\]

Applying this to \(1_{\mathcal{C}_{Z \subset M}}\) gives the canonical isomorphisms

\[\pi_{\mathcal{C}_{Z \subset M}}!(\sigma_i^* \text{Th}_{M'}(T_{M'/B})) \cong \pi_{\mathcal{C}_{Z \subset M}}!(\mathcal{C}(g))!(\sigma_{i'}^* \text{Th}_{M'}(T_{M'/B})) \cong \pi_{\mathcal{C}_{Z \subset M}}!(\mathcal{C}(\theta_g))\]

Note that this composition is just the map \(\mathcal{C}(g)_*\) applied to \(\sigma_i^* \text{Th}_{M}(T_{M/B})\) and composed with \(\pi_{\mathcal{C}_{Z \subset M}}!(\mathcal{C}(\theta_g))\), where \(\mathcal{C}(\theta_g)\) is the isomorphism

\[\sigma_i^* \text{Th}_{M'}(T_{M'/B}) \xrightarrow{\mathcal{C}(\theta_g)} \Sigma^{\mathcal{T}_{\mathcal{C}_{Z \subset M'}/\mathcal{C}_{Z \subset M}}(\mathcal{C}(g)^* \sigma_i^* \text{Th}_{M}(T_{M/B}))}\]

induced by \(\theta_g\). Setting \(\psi_g := \mathcal{C}(g)_* (\sigma_i^* \text{Th}_{M}(T_{M/B})) \circ \pi_{\mathcal{C}_{Z \subset M}}!(\mathcal{C}(\theta_g))\), we have the isomorphism

\[\psi_g : \pi_{\mathcal{C}_{Z \subset M}}!(\sigma_i^* \text{Th}_{M'}(T_{M'/B})) \to \pi_{\mathcal{C}_{Z \subset M}}!(\sigma_i^* \text{Th}_{M}(T_{M/B})).\]
If we have another smooth morphism $g' : M'' \to M'$ and a closed immersion $i'' : Z \to M''$ with $g'' \circ i'' = i'$, then we have the exact sequence

$$0 \to i''^*T_{M''/M'} \to i''^*T_{M''/M} \to i'^*T_{M'/M} \to 0$$

which together with the functoriality of smooth pushforward gives

$$[\mathcal{C}(g)_*(\sigma_i^*\text{Th}_M(T_{M/B})) \circ \pi_{Z_{CM!}}!(\mathcal{C}(\theta_g))]
\circ [\mathcal{C}(g')_*(\sigma_i^*\text{Th}_M(T_{M'/B})) \circ \pi_{Z_{CM''!}}!(\mathcal{C}(\theta_{gg'}))]
= \mathcal{C}(gg')_*(\sigma_i^*\text{Th}_M(T_{M/B})) \circ \pi_{Z_{CM''!}}!(\mathcal{C}(\theta_{gg'}))$$

or in other words,

$$\psi_g \circ \psi_{g'} = \psi_{gg'},$$

and thus

$$\psi_{i,i'} \circ \psi_{i',i''} = \psi_{i,i''}.$$

\[\square\]

**Definition 2.2.** Let $Z$ be in $\text{Sch}^G/B$, admitting a closed immersion $i : Z \to M$ with $M \in \text{Sm}^G/B$. The intrinsic stable normal cone $\mathcal{C}_{Z}^{st} \in \text{SH}^G(B)$ is defined by

$$\mathcal{C}_{Z}^{st} := \pi_{\mathcal{C}Z \subset M}!(\sigma_i^*\text{Th}_M(T_{M/B})) \in \text{SH}^G(B).$$

Lemma 2.1 implies that $\mathcal{C}_{Z}^{st}$ is independent of the choice of closed immersion $i : Z \to M$, up to canonical isomorphism. More precisely, having fixed a closed immersion $i_0 : Z \to M_0$ with $M_0 \in \text{Sm}^G/B$, we set

$$\mathcal{C}_{Z}^{st} := \pi_{\mathcal{C}Z \subset M_0}!(\sigma_i^*\text{Th}_{M_0}(T_{M_0/B})).$$

For each closed immersion $i : Z \to M$ with $M \in \text{Sm}^G/B$, we then have the canonical isomorphism

$$\alpha_i := \psi_{i,i_0} : \mathcal{C}_{Z}^{st} \to \pi_{\mathcal{C}Z \subset M}!(\sigma_i^*\text{Th}_M(T_{M/B}))$$

such that, if $i' : Z \to M'$ is another closed immersion, then the diagram

$$\begin{array}{ccc}
\mathcal{C}_{Z}^{st} & \xrightarrow{\alpha_i} & \pi_{\mathcal{C}Z \subset M}!(\sigma_i^*\text{Th}_M(T_{M/B})) \\
\downarrow{\alpha_{i'}} & & \downarrow{\psi_{i,i'}} \\
\pi_{\mathcal{C}Z \subset M_0}!(\sigma_i^*\text{Th}_{M_0}(T_{M_0/B}))
\end{array}$$

commutes.

**Example 2.3.** Suppose $\pi_{Z} : Z \to B$ is smooth over $B$. Then we may take the identity for $i : Z \to M$, giving $\mathcal{C}_{Z \subset M} = Z$ and $\mathcal{C}_{Z}^{st} = \pi_{Z}!(\text{Th}_Z(T_{Z/B})) = \pi_{Z\#}(1_Z) = Z$ in $\text{SH}^G(B)$. 
Still assuming $Z$ smooth over $B$, we may take a closed immersion $i : Z \to M$ with $M$ smooth over $B$. Then $\mathcal{C}_{Z \subset M}$ is the normal bundle $p_i : N_i \to Z$ and the isomorphism $\psi_{Id_z,i}$ is the composition of the isomorphisms

\[
\pi_{N_i!}(\sigma_i^*\text{Th}_M(T_{M/B})) \cong \pi_{Z!}p_i!(\Sigma^\sigma_i\text{Th}_{T_{M/B}}(1_{N_i})) \\
\cong \pi_{Z!}p_i!(\Sigma^\sigma_iN_i \circ \Sigma^\sigma_iT_{Z/B}(1_{N_i})) \\
\cong \pi_{Z!}p_i!(\Sigma^\sigma_iT_{Z/B}(1_{N_i})) \\
\cong \pi_{Z!}p_i!(\text{Th}_Z(T_{Z/B})) \\
\cong \pi_{Z!}(\text{Th}_Z(T_{Z/B})) \\
\cong \pi_{Z!}(1_{Z}) = Z
\]

The second isomorphism arises from the exact sequence

\[
0 \to T_{Z/B} \to i^*T_{M/B} \to N_i \to 0,
\]

the third from the isomorphism $p_i^*N_i \cong T_{N_i/Z}$ and the fifth from homotopy invariance.

**Lemma 2.4.** Suppose we have a closed immersion $i : Z \to M$ with $M \in \text{Sm}^G/B$ and an affine space bundle $q : V \to M$, giving the cartesian diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{i'} & V \\
\downarrow{q_Z} & & \downarrow{q} \\
Z & \xrightarrow{i} & M
\end{array}
\]

Let $\mathcal{C}(q) : \mathcal{C}_{Z' \subset V} \to \mathcal{C}_{Z \subset M}$ be the morphism induced by $(q_z, q)$ and let

\[
\mathcal{C}_{Z'}^{st}, \mathcal{C}_{Z'}^{st} \xrightarrow{\mathcal{C}(q)} \mathcal{C}_{Z}^{st}
\]

be defined as the composition

\[
\begin{array}{l}
\mathcal{C}_{Z'}^{st} \xrightarrow{\alpha'} \pi_{Z'!}q_{Z'!}(\sigma_{i'}^*\text{Th}_{V}(T_{V/B})) \\
\cong \pi_{Z'!}q_{Z'!}(\Sigma_{Z'}^\sigma q_{Z'!}V(q)^*\sigma_i^*\text{Th}_M(T_{M/B})) \\
\cong \mathcal{C}(q)^*(\sigma_i^*\text{Th}_M(T_{M/B})) \\
\xrightarrow{\alpha^{-1}} \pi_{Z!}p_i!(\sigma_i^*\text{Th}_M(T_{M/B})) \\
\xrightarrow{\mathcal{C}(q)_*} \pi_{Z!}(\sigma_i^*\text{Th}_M(T_{M/B})) \\
\xrightarrow{\mathcal{C}(q)^*(-1)} \mathcal{C}_{Z}^{st}
\end{array}
\]

where the isomorphism $\alpha$ is induced by the exact sequence

\[
0 \to q^*V \to T_{V/B} \to q^*T_{M/B} \to 0.
\]
Then $\mathcal{E}^\text{st}(q)$ is an isomorphism. Moreover, if we have an extension of our diagram to a Cartesian diagram in $\mathbf{Sch}^G/B$

\[
\begin{array}{ccc}
Z'' & \rightarrow & W \\
\downarrow g & & \downarrow g \\
Z' & \rightarrow & V \\
\downarrow q & & \downarrow q \\
\downarrow & & \downarrow \\
Z & \rightarrow & M
\end{array}
\]

with $W \rightarrow V$ an affine space bundle, then

$$\mathcal{E}^\text{st}(q \circ g) = \mathcal{E}^\text{st}(q) \circ \mathcal{E}^\text{st}(g).$$

Proof. We use the closed immersion $i' : Z' \rightarrow V$ to compute $\mathcal{E}^\text{st}_{Z'}$. The morphism

$$\mathcal{E}(q) : \mathcal{E}_{Z' \subset V} \rightarrow \mathcal{E}_{Z \subset M},$$

identifies $\mathcal{E}_{Z' \subset V}$ with $\mathcal{E}_{Z \subset M} \times_M V$ and the homotopy property for affine space bundles thus implies that the co-unit $\mathcal{E}(q) \# \mathcal{E}(q)^* \rightarrow \text{Id}$ of the adjunction is an isomorphism. This gives the series of isomorphisms defining the map $\mathcal{E}^\text{st}(q)$:

\[
\begin{align*}
\mathcal{E}^\text{st}_{Z'} & \xrightarrow{\alpha'} \pi \mathcal{E}_{Z' \subset V}(\sigma_i^* \text{Th}(T_{V/B})) \\
& \xrightarrow{\alpha} \pi \mathcal{E}_{Z' \subset V}(\Sigma \sigma_i^* q^* V \mathcal{E}(q)^* \sigma_i^* \text{Th}(T_{M/B})) \\
& \xrightarrow{(i)} \pi \mathcal{E}_{Z \subset M!} \circ \mathcal{E}(q);(\Sigma \sigma_i^* q^* V \mathcal{E}(q)^* \sigma_i^* \text{Th}(T_{M/B})) \\
& \xrightarrow{(ii)} \pi \mathcal{E}_{Z \subset M!} \circ \mathcal{E}(q) \# (\mathcal{E}(q)^* \sigma_i^* \text{Th}(T_{M/B})) \\
& \xrightarrow{(iii)} \pi \mathcal{E}_{Z \subset M!}(\sigma_i^* \text{Th}(T_{M/B})) \\
& \xrightarrow{\alpha_i^{-1}} \mathcal{E}^\text{st}_{Z}.
\end{align*}
\]

Noting that map $\mathcal{E}(q)_* (\sigma_i^* \text{Th}_M(T_{M/B}))$ is the composition of maps (i), (ii) and (iii), the functoriality $\mathcal{E}^\text{st}(q \circ g) = \mathcal{E}^\text{st}(q) \circ \mathcal{E}^\text{st}(g)$ follows the functoriality of smooth pushforward, the naturality of the isomorphisms $\alpha_-$ and the fact that $V \mapsto \Sigma V$ extends to a functor $\Sigma^- : \text{SH}^G(\cdot) \rightarrow \text{Aut}(\text{SH}^G(\cdot))$. □

Remark 2.5 (Jouanolou covers). Hoyois [6, Proposition 2.20] has shown that the Jouanolou trick extends to the equivariant case: for each $M \in \mathbf{Sch}^G/B$, there is an affine space bundle $\tilde{M} \rightarrow M$ such that $\pi_{\tilde{M}} : \tilde{M} \rightarrow B$ is an affine morphism. We call such a map $\tilde{M} \rightarrow M$ a Jouanolou cover of $M$. For affine $B$, Lemma 2.4 will thus enable us to reduce various constructions to the case of affine $Z \in \mathbf{Sch}^G/B$. 

3. The fundamental class

The next step in the construction is to define a fundamental class in cohomotopy $[\mathcal{C}_Z^m] \in S_B^{0,0}(\mathcal{C}_Z^m)$. We do this by the method of specialization to the normal cone, suitably reinterpreted.

Choose as before a closed immersion $i : Z \to M$ with $M \in \mathrm{Sm}^G/B$. Let

$$\bar{\pi} : \widetilde{M \times A^1} \to M \times A^1$$

be the blow up of $M \times A^1$ along $Z \times 0$, that is

$$\widetilde{M \times A^1} = \text{Proj}_{M \times A^1} \oplus_{n \geq 0} T^n_{Z \times 0}.$$ 

Writing $M \times A^1 = \text{Spec} \mathcal{O}_M[t]$, the element $t \in I_{Z \times 0}$, considered as an element of $\oplus_{n \geq 0} T^n_{Z \times 0}$ of degree one, gives a $G$-invariant section of $\mathcal{O}(1)$, which we denote by $T$. We define

$$\text{Def}(i) := \widetilde{M \times A^1 \setminus (T = 0)} \in \text{Sch}^G/B,$$

so

$$\text{Def}(i) = \text{Spec}_{M \times A^1}(\oplus_{n \geq 0} T^n_{Z \times 0}[T^{-1}])_0,$$

where the subscript 0 denotes the subsheaf of homogeneous sections of degree 0. The projection $p : \text{Def}(i) \to M \times A^1$ is flat, $p^{-1}(M \times (A^1 \setminus \{0\}))$ is isomorphic via $p$ to $M \times (A^1 \setminus \{0\})$ and $p^{-1}(M \times 0) = \mathcal{C}_{Z\infty}$. Thus $\mathcal{C}_{Z\infty}$ is an effective principal Cartier divisor on $\text{Def}(i)$, with ideal $(t)\mathcal{O}_{\text{Def}(i)}$. Let $i : \mathcal{C}_{Z\infty} \to \text{Def}(i)$ be the inclusion, and let $j : M \times (A^1 \setminus \{0\}) \to \text{Def}(i)$ be the open complement.

The localization triangle (1.1), twisted by $\Sigma^{p^*p_1^*T_M/B}$ and pushed forward by $\pi_{\text{Def}(i)}^*$, gives us the distinguished triangle in $\text{SH}^G(B)$

$$\pi_{M \times (A^1 \setminus \{0\})!}(\Sigma^{p^*p_1^*T_M/B}(1_{M \times (A^1 \setminus \{0\})})) \to \pi_{\text{Def}(i)!}(\Sigma^{p^*p_1^*T_M/B}(1_{\text{Def}(i)}))$$

$$\to \pi_{\mathcal{C}_{Z\infty}!}(\sigma^*_i \text{Th}_M(T_M/B)).$$

The isomorphism

$$T_{M \times (A^1 \setminus \{0\})} \cong p_1^*T_M/B \oplus p_2^*T_{A^1 \setminus \{0\}/B}$$

and the canonical isomorphism $T_{A^1 \setminus \{0\}/B} \cong \mathcal{O}_{A^1 \setminus \{0\}}$ gives the isomorphisms

$$\pi_{M \times (A^1 \setminus \{0\})!}(\Sigma^{p^*p_1^*T_M/B}(1_{M \times (A^1 \setminus \{0\})})) \cong \pi_M(1_M) \wedge_B \pi_{A^1 \setminus \{0\}!}(\Sigma^{-1}_T 1_{A^1 \setminus \{0\}})$$

$$\cong \Sigma^{-1}_T M \times (A^1 \setminus \{0\})_+.$$

We have the canonical isomorphism

$$\alpha_i : \mathcal{C}_Z^m \to \pi_{\mathcal{C}_{Z\infty}!}(\sigma^*_i \text{Th}_M(T_M/B)),$$

so our distinguished triangle gives the map in $\text{SH}^G(B)$

$$\partial_{Z,i} : \mathcal{C}_Z^m \to \Sigma^{-1}_T M \times (A^1 \setminus \{0\})_+.$$

Let

$$\tilde{p}_2^M : M \times (A^1 \setminus \{0\})_+ \to \mathcal{G}_m.$$
be the projection $p_2 : M \times (\mathbb{A}^1 \setminus \{0\})_+ \to (\mathbb{A}^1 \setminus \{0\})_+$ followed by the quotient map $(\mathbb{A}^1 \setminus \{0\})_+ \to (\mathbb{A}^1 \setminus \{0\}, \{1\}) = \mathbb{G}_m$.

**Lemma 3.1.** The map

$$\Sigma_{\mathbb{G}_m}^{-1} \mathbb{G}_m \circ \partial_{Z,i} : \mathfrak{c}_Z^i \to \Sigma_{\mathbb{G}_m}^{-1} \mathbb{G}_m \cong S_B$$

is independent of the choice of closed immersion $i : Z \to M$.

**Proof.** We reduce as in the proof of Lemma 2.1 to the case in which we have a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{i'} & M' \\
\downarrow i & & \downarrow g \\
& & M
\end{array}
\]

with $g$ smooth. The map $g$ induces a smooth morphism $Def(g) : Def(i') \to Def(i)$, making the diagram

\[
\begin{array}{ccc}
Def(i') & \xrightarrow{p_{M'}} & M' \times \mathbb{A}^1 \\
\downarrow Def(g) & & \downarrow g \times Id \\
Def(i) & \xrightarrow{p_M} & M \times \mathbb{A}^1
\end{array}
\]

The restriction of $Def(g)$ to $\mathfrak{c}_{Z \subset M'}$ is the map $\mathfrak{c}(g) : \mathfrak{c}_{Z \subset M'} \to \mathfrak{c}_{Z \subset M}$ induced by $g$.

This gives us the map of distinguished triangles (we suppress the isomorphisms on the suspension operations induced by various exact sequences)

$$\begin{array}{ccc}
\pi_{M' \times (\mathbb{A}^1 \setminus \{0\})}((\Sigma p_M^* T_{M'/B}(M' \times (\mathbb{A}^1 \setminus \{0\}))) & \xrightarrow{(g \times Id)_* (\Sigma p_M^* T_{M/B}(1))) & \pi_{M \times (\mathbb{A}^1 \setminus \{0\})}((\Sigma p_M^* T_{M/B}(M \times (\mathbb{A}^1 \setminus \{0\}))) \\
\pi_{Def(i')}((\Sigma p_M^* p_{M'}^* T_{M'/B}(1_{Def(i)})) & \xrightarrow{Def(g)_* (\Sigma p_M^* p_{M'}^* T_{M/B}(1))) & \pi_{Def(i)}((\Sigma p_M^* p_{M'}^* T_{M/B}(1_{Def(i)})) \\
\pi_{\mathfrak{c}_{Z \subset M'}}((\sigma_{\mathfrak{i}'}^* \text{Th}_{M'}(T_{M'/B}))) & \xrightarrow{\mathfrak{c}(g)_* (\sigma_{\mathfrak{i}'}^* \text{Th}_M(T_{M/B}))} & \pi_{\mathfrak{c}_{Z \subset M'}}((\sigma_{\mathfrak{i}'}^* \text{Th}_M(T_{M/B})))
\end{array}$$
which in turn gives the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}^{st}_{\mathcal{Z}} & \xrightarrow{\alpha} & \mathcal{C}^{st}_{\mathcal{Z}} \\
\pi_{\mathcal{Z}_{\mathcal{M}'}!}(\sigma^+_{\mathcal{M}'}\text{Th}_{\mathcal{M}'}(T_{\mathcal{M}'}/\mathcal{B})) & \xrightarrow{v_0} & \pi_{\mathcal{Z}_{\mathcal{M}'}!}(\sigma^+_i\text{Th}_{\mathcal{M}'}(T_{\mathcal{M}/\mathcal{B}})) \\
\phi & \downarrow & \downarrow \partial \\
\Sigma^{-1}_{G_m} \mathcal{M}' \times (\mathbb{A}^1 \setminus \{0\})_+ & \xrightarrow{\Sigma^{-1}_{G_m} g \times \text{Id}} & \Sigma^{-1}_{G_m} \mathcal{M} \times (\mathbb{A}^1 \setminus \{0\})_+ \\
\beta_2^{M'} & \rightarrow & \beta_2^M \\
\downarrow & & \downarrow \\
\mathbb{S}_{\mathcal{B}}, & & \mathbb{S}_{\mathcal{B}}.
\end{array}
\]

completing the proof.

\[\square\]

**Definition 3.2.** Let \( Z \) be in \( \text{Sch}^G/B \). The fundamental class \( [\mathcal{C}^{st}_{\mathcal{Z}}] \in \mathbb{S}^{0,0}_{\mathcal{B}}(\mathcal{C}^{st}_{\mathcal{Z}}) \) is the composition

\[
\mathcal{C}^{st}_{\mathcal{Z}} \xrightarrow{\partial_{Z,i}} \Sigma^{-1}_{G_m} M \times (\mathbb{A}^1 \setminus \{0\})_+ \xrightarrow{\beta_2^M} \Sigma^{-1}_{G_m} \mathcal{M} \cong \mathbb{S}_{\mathcal{B}}.
\]

If \( \mathcal{E} \) is a commutative monoid in \( \text{SH}^G(B) \) with unit \( \epsilon_{\mathcal{E}} : \mathbb{S}_{\mathcal{B}} \rightarrow \mathcal{E} \), we define the fundamental class \( [\text{Th}(\mathcal{C}_Z)]_{\mathcal{E}} \in \mathcal{C}^{0,0}(\text{Th}(\mathcal{C}_Z)) \) by composing \( [\text{Th}(\mathcal{C}_Z)] \) with \( \epsilon_{\mathcal{E}} \).

4. **Perfect obstruction theories and the virtual fundamental class**

We now assume that \( B \) is affine. Using Remark 2.5 for each \( Z \in \text{Sch}^G/B \) and each closed immersion \( Z \rightarrow M \) with \( M \in \text{Sm}^G/B \), there is an affine space bundle \( \tilde{M} \rightarrow M \) with \( \tilde{M} \) affine, and thus \( \tilde{Z} := Z \times_M \tilde{M} \) is affine as well.

Let \( \mathbb{L}_{Z/B} \) be the relative dualizing complex on \( Z \) and let \([\phi] : E_{\bullet} \rightarrow \mathbb{L}_{Z/B} \) be a perfect obstruction theory on \( Z \). Recall that we use homological notation for complexes.

If we choose a closed immersion \( i : Z \rightarrow M \) with \( M \in \text{Sm}^G/B \), we may use the explicit model \( (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^*\Omega_{M/B}) \) for \( \tau_{\geq 1}\mathbb{L}_{Z/B} \). Since \( E_{\bullet} \) is by definition supported in \([0, 1]\) and \( Z \) satisfies the \( G \)-resolution property, we have a global resolution, that is, we have a two-term complex of locally free sheaves in \( \text{Coh}^2_Z \), \( F_{\bullet} := (F_1 \rightarrow F_0) \), and a map of complexes

\[
\phi : (F_1 \xrightarrow{d\phi} F_0) \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^*\Omega_{M/B})
\]

which induces an isomorphism on \( h_0 \) and a surjection on \( h_1 \), representing \([\phi] : E_{\bullet} \rightarrow \mathbb{L}_{Z/B} \). We call a representative \((F_{\bullet}, \phi)\) of \([\phi]\) a normalized representative if the maps \( \phi_0, \phi_1 \) are surjective. If in addition \( F_0 = i^*\Omega_{M/B} \) and \( \phi_0 \) is the identity, we call \( \phi \) reduced.
If we have a map of complexes
\[ \phi : (F_1 \xrightarrow{d_F} F_0) \to (\mathcal{I}_Z / \mathcal{I}_Z^2 \xrightarrow{d} i^* \Omega_{M/B}) \]
as above, inducing an isomorphism on \( h_0 \) and a surjection on \( h_1 \), then by the \( G \)-resolution property for \( Z \), there is a locally free sheaf \( \mathcal{F} \) on \( Z \) and a surjection \( p : \mathcal{F} \to \mathcal{I}_Z / \mathcal{I}_Z^2 \). We may then replace \((F_*, \phi)\) by \( F_1 \oplus F \xrightarrow{d_F \oplus \text{Id}_F} F_0 \oplus F \) and map the copy of \( F \) in degree 0 to \( i^* \Omega_{M/B} \) by \( d \circ p \), giving the map \( \phi' : (F_1 \oplus F \xrightarrow{d_F \oplus \text{Id}_F} F_0 \oplus F) \to (\mathcal{I}_Z / \mathcal{I}_Z^2 \xrightarrow{d} i^* \Omega_{M/B}) \).

The map \( \phi' \) is surjective by construction and the assumption that \( h_0(\phi) \) is an isomorphism implies that \( \phi'_0 \) is surjective as well. Thus, each \([\phi]\) admits a normalized representative.

For a normalized representative \( \phi : F_* \to (\mathcal{I}_Z / \mathcal{I}_Z^2 \xrightarrow{d} i^* \Omega_{M/B}) \), we let \( K_i \subset F_i \) be the kernel of \( \phi_i \). We let \( F_i \to Z \) be the dual vector bundle \( F_i^\vee := \text{Spec}_{\mathcal{O}_Z} \text{Sym}^* F_i \) and similarly define \( K_i = \text{Spec}_{\mathcal{O}_Z} \text{Sym}^* K_i \).

**Lemma 4.1.** Let \( \phi : F_* \to (\mathcal{I}_Z / \mathcal{I}_Z^2 \xrightarrow{d} i^* \Omega_{M/B}) \) be a normalized representative of a perfect obstruction theory \([\phi]\). Let \( K(h_1(F_*)) \) be the kernel of the surjection \( h_1(\phi) : h_1(F_*) \to h_1(\mathbb{L}_{Z/B}) \). Then \( K_0 \) is locally free and in the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K(h_1(F_*)) & \rightarrow & K_1 & \rightarrow & K_0 & \rightarrow & 0 \\
0 & \rightarrow & h_1(F_*) & \xrightarrow{\phi_1} & F_1 & \xrightarrow{d_F} & F_0 & \xrightarrow{\phi_0} & h_0(F_*) & \rightarrow & 0 \\
0 & \rightarrow & h_1(\mathbb{L}_{Z/B}) & \xrightarrow{d} & \mathcal{I}_Z / \mathcal{I}_Z^2 & \xrightarrow{i^* \Omega_{M/B}} & h_0(\mathbb{L}_{Z/B}) & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

all the rows and columns are exact.

*Proof.* Our assumption that \( \phi \) is a normalized representative is just that \( \phi_0, \phi_1 \) and \( h_1(\phi) \) are surjective, and \( h_0(\phi) \) is an isomorphism. The rest follows by the snake lemma. \( \square \)

**Lemma 4.2.** Suppose \( Z \) is affine, choose a closed immersion \( i : Z \to M \) with \( M \in \text{Sm}^G \) and let \([\phi]\) be a perfect obstruction theory on \( Z \). Then \([\phi]\) admits a reduced normalized representative.
Proof. We have already seen that $\phi$ admits a normalized representative

$$
\phi : (F_1 \to F_0) \to (I_Z/I_Z^2 \to i^*\Omega_{M/B})
$$

We use the notation of Lemma 4.1. Since $Z$ is affine, each locally free $F$ on $Z$ is a projective object in $\text{QCoh}_G^Z$. Thus, we may choose a splitting $s_K : K_0 \to K_1$ to the surjection $K_1 \to K_0$. This gives us the commutative diagram

Replacing $F_1$ with $F'_1 := F_1/i_1(K_0)$ and $F_0$ with $i^*\Omega_{M/B} \cong F_0/K_0$, we have the reduced normalized representative

$$
(F'_1 \xrightarrow{d_{F'}} i^*\Omega_{M/B}) \xrightarrow{(\phi'_1, \text{id})} (I_Z/I_Z^2 \to i^*\Omega_{M/B})
$$

for $[\phi]$. □

Lemma 4.3. Suppose $Z$ is affine and choose a closed immersion $i : Z \to M$ with $M \in \text{Sm}^G/B$. If $\phi : F_\bullet \to (I_Z/I_Z^2 \to i^*\Omega_{M/B})$ and $\phi' : F'_\bullet \to (I_Z/I_Z^2 \to i^*\Omega_{M/B})$ are two reduced normalized representatives of a given perfect obstruction theory $[\phi] : E_\bullet \to \mathbb{L}_{Z/B}$ on $Z$, then the induced isomorphism $F_\bullet \cong E_\bullet \cong F'_\bullet$ in $D_{\text{perf}}^B(Z)$ arises from an isomorphism of complexes $\rho_\bullet : F_\bullet \to F'_\bullet$ making the diagram

$$
\begin{array}{ccc}
F_\bullet & \xrightarrow{\rho_\bullet} & F'_\bullet \\
\downarrow & & \downarrow \\
(I_Z/I_Z^2 \to i^*\Omega_{M/B}) & & (I_Z/I_Z^2 \to i^*\Omega_{M/B})
\end{array}
$$

commute. Moreover $\rho_\bullet$ is unique up to chain homotopy.

Proof. We may assume that $[\phi] : E_\bullet \to (I_Z/I_Z^2 \to i^*\Omega_{M/B})$ is given by a map of complexes

$$
\phi_E : E_\bullet \to (I_Z/I_Z^2 \to i^*\Omega_{M/B})
$$

and that the isomorphisms $\alpha : F_\bullet \to E_\bullet$, $\alpha' : F'_\bullet \to E_\bullet$ in $D_{\text{perf}}^B(Z)$ are quasi-isomorphisms of complexes.
Defining \( F''_1 \) as the pullback in the diagram

\[
\begin{array}{ccc}
F''_1 & \xrightarrow{d''} & F_0 \oplus F'_0 \\
\downarrow & & \downarrow \\
E_1 & \rightarrow & E_0,
\end{array}
\]

then \( F''_1 \) is locally free, \((F''_1 \to F_0 \oplus F'_0) \to E_\bullet \) is a quasi-isomorphism and we have monomorphisms

\[
F_\bullet \to (F''_1 \xrightarrow{d''} F_0 \oplus F'_0) \leftarrow F'_\bullet.
\]

Let \( K \cong i^*\Omega_{M/B} \) be the kernel of the sum map \( i^*\Omega_{M/B} \oplus i^*\Omega_{M/B} \to i^*\Omega_{M/B} \). Since \( F_0 = F'_0 = i^*\Omega_{M/B} \), it follows that \( d''(F''_1) \) contains the kernel \( K \), and we may therefore lift \( K \) to \( F''_1 \) so that \( K \to F''_1 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \) is the zero map.

Taking the quotient of \((F''_1 \to F_0 \oplus F'_0)\) by the acyclic complex \( K_\bullet := K \xrightarrow{\text{id}} K \), the maps of \( F_\bullet \) and \( F'_\bullet \) to the quotient complex are both isomorphisms.

The fact that the resulting isomorphism \( \rho_\bullet : F_\bullet \to F'_\bullet \) is unique up to chain homotopy follows from the fact that the image of \( \rho_\bullet \) in \( D^\text{perf}_{\text{et}}(Z) \) is uniquely determined by the given data, and that \( F_\bullet \) is a finite complex of projective objects in \( \text{QCoh}^G_Z \).

We return to case of arbitrary \( Z \in \text{Sch}^G/B \) with a closed immersion \( i_Z : Z \to M, M \in \text{Sm}^G/B \). Suppose we have smooth morphism \( p_M : \tilde{M} \to M \).

Form the Cartesian square

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{i_Z} & \tilde{M} \\
p_Z & & \downarrow p_M \\
Z & \xrightarrow{i_Z} & M
\end{array}
\]

We have the commutative diagram

\[
\begin{array}{cccc}
& & 0 & 0 \\
& & \downarrow & \downarrow \\
p^*_Z(\mathcal{I}_Z/\mathcal{I}_Z^2) & \xrightarrow{i^*_Z} & p^*_Z i^*_Z \Omega_{M/B} & \xrightarrow{i^*_Z} & p^*_Z \Omega_{Z/B} & \rightarrow & 0 \\
& & \downarrow & \downarrow & \downarrow \\
\mathcal{I}_Z/\mathcal{I}_Z^2 & \xrightarrow{i^*_Z} & \tilde{\Omega}_{\tilde{M}/B} & \xrightarrow{i^*_Z} & \tilde{\Omega}_{\tilde{Z}/B} & \rightarrow & 0 \\
& & \downarrow & \downarrow & \downarrow \\
& & i^*_Z \tilde{\Omega}_{\tilde{M}/M} & \xrightarrow{\sim} & \Omega_{\tilde{Z}/Z} & \rightarrow & 0 \\
& & \downarrow & \downarrow & \downarrow \\
& & 0 & 0
\end{array}
\]
with exact rows and columns. If we have a map of complexes \( \phi : (F_1 \xrightarrow{dp} F_0) \to (I_Z/I_Z^2 \xrightarrow{d} i_Z^* \Omega_{M/B}) \) representing a perfect obstruction theory \([\phi] : E \to \tau_{\geq 1}L_{Z/B} \), we define an induced obstruction theory \( p_M^*[\phi] : p_M^*[E] \to \tau_{\geq 1}L_{Z/B} \) to be a perfect obstruction theory represented by

\[
(p^*_Z \phi_1, \phi^*_Z \phi_0 + \rho) : (p^*_Z F_1 \xrightarrow{(dp,0)} p^*_Z F_0 \oplus F_0') \to (I_Z/I_Z^2 \xrightarrow{d} i_Z^* \Omega_{M/B})
\]

where \( \rho : F_0' \to i_Z^* \Omega_{M/B} \) is a map inducing an isomorphism \( F_0' \cong i_Z^* \Omega_{M/B} \) upon composing with the surjection \( i_Z^* \Omega_{M/B} \to i_Z^* \Omega_{M/M} \). Such an obstruction theory exists if and only if \( i_Z^* \Omega_{M/B} \to i_Z^* \Omega_{M/M} \) admits a splitting, and if so, the induced obstruction theories are uniquely determined by a choice of splitting. If for instance, \( Z \) is affine, then \( i_Z^* \Omega_{M/M} \) is a projective object in \( \text{QCoh}^G_Z \) and thus an induced obstruction theory exists. We write \( p_M^*[\phi] \) for any obstruction theory induced by \([\phi]\).

**Lemma 4.4.** Suppose we have a closed immersion \( i_Z : Z \to M \) with \( M \in \text{Sm}^G/B \). Let \( M \to M \) be a Jouanolou cover of \( M \) and form the Cartesian square

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{i_Z} & \tilde{M} \\
p_Z \downarrow & & \downarrow p_M \\
Z & \xrightarrow{i_Z} & M
\end{array}
\]

Let \( (E_\bullet, [\phi]) \) be a perfect obstruction theory on \( Z \). Then an induced obstruction theory \( p_M^*[\phi] \) on \( \tilde{Z} \) exists and for each induced obstruction theory \( p_M^*[\phi] \), there exists a reduced normalized representative for \( p_M^*[\phi] \).

**Proof.** Since \( \tilde{Z} \) is affine, this follows from the discussion in the preceding paragraph. \( \square \)

We return again to the case of a general \( Z \in \text{Sch}^G/B \) with closed immersion \( i_Z : Z \to M, M \in \text{Sm}^G/B \). If \( \phi : F_\bullet \to (I_Z/I_Z^2 \xrightarrow{d} i_Z^* \Omega_{M/B}) \) is a normalized perfect obstruction theory on \( Z \subset M \), the surjection \( \phi_1 : F_1 \to I_Z/I_Z^2 \) induces a closed immersion

\[
i_\phi : C_{Z\subset M} \to F^1 := F_1^v.
\]

Let \( p_{F^1} : F^1 \to Z \) be the projection and let \( s_0 : Z \to F^1 \) be the 0-section.

Noting that \( T_{F^1/Z} \cong p_{F^1}^* F^1 \), we have the sequence of natural transformations

\[
\pi_{Z!} \circ s_0 \circ \pi_{F^1!} \circ \sigma_{F^1!} \circ p_{F^1}^* \circ i_\phi^* \circ \pi_{\epsilon_{Z\subset M}!} \circ i_\phi^* \circ \pi_{\epsilon_{Z\subset M}!} \circ p_{F^1}^* \circ \sigma_{F^1!} \cong \pi_{\epsilon_{Z\subset M}!} \circ p_{F^1}^* \circ \sigma_{F^1!}.
\]

Applying this to \( \Sigma^{-F^1} \circ \Sigma^* T_{M/B}(1_Z) \) and composing with

\[
\alpha_{i!}^{-1} : \pi_{\epsilon_{Z\subset M}!}(\sigma_{i!}^* \text{Th}_M(T_{M/B})) \to C_{Z!}^i,
\]

where
gives the morphism
\[ s_{i_E} : \pi Z! (\Sigma^{-F_1} \circ \Sigma^T_{M/B} (1_{\tilde{Z}})) \to C^q_{\tilde{Z}}. \]

Suppose we have a perfect obstruction theory $[\phi] : E \to \mathbb{L}_{Z/B}$ on some $Z \in \text{Sch}^G/B$. Choose a closed immersion $i : Z \to M$ with $M \in \text{Sm}^G/B$ and take a Jouanolou cover

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{i\tilde{Z}} & \tilde{M} \\
p_{\tilde{Z}} & & p_M \\
Z & \xrightarrow{i_Z} & M
\end{array}
\]

By Lemma 1.14 we have an induced perfect obstruction theory $(\tilde{E}_\bullet, p_M^* [\phi])$ on $\tilde{Z}$ and a reduced normalized representative $\tilde{F}_\bullet$ for $p_M^*[\phi]$. The splitting $i_{\tilde{Z}}^* \Omega_{\tilde{M}/B} \cong p_{\tilde{Z}}^* i_Z^* \Omega_{M/B} \oplus i_{\tilde{Z}}^* \Omega_{M/M}$ and the quasi-isomorphism $\tilde{F}_\bullet \to \tilde{E}_\bullet$ induces a quasi-isomorphism
\[ (\tilde{F}_1 \to p_{\tilde{Z}}^* i_{\tilde{Z}}^* \Omega_{M/B}) \to p_{\tilde{Z}}^* E_\bullet, \]
which in turn induces an isomorphism
\[ \Sigma^{-F_1} \circ \Sigma^p_{i_{\tilde{Z}}^* T_{M/B}} \circ p_{\tilde{Z}}^* \cong p_{\tilde{Z}}^* \circ \Sigma^{E_{\tilde{Z}}}. \]
This, together with homotopy invariance, induces the isomorphism
\[ \partial_{i_Z, p_{\tilde{Z}}, \tilde{F}_\bullet} : \pi Z! \circ \Sigma^{E_{\tilde{Z}}} \to \pi \tilde{Z}! \circ \Sigma^{-F_1} \circ \Sigma^i_{\tilde{Z}} T_{M/B} \circ p_{\tilde{Z}} \]
defined as the composition of isomorphisms
\[
\begin{align*}
\pi Z! \circ \Sigma^{E_{\tilde{Z}}} & \cong \pi Z! \circ p_{Z\#} \circ p_{\tilde{Z}}^* \circ \Sigma^{E_{\tilde{Z}}} \\
& \cong \pi Z! \circ p_{Z\#} \circ (\Sigma^{-F_1} \circ \Sigma^p_{i_{\tilde{Z}}^* T_{M/B}} \circ p_{\tilde{Z}}) \\
& \cong \pi Z! \circ p_{Z\#} \circ (\Sigma^{-F_1} \circ \Sigma^p_{i_{\tilde{Z}}^* T_{M/B}} \circ p_{\tilde{Z}}) \\
& \cong \pi Z! \circ p_{Z\#} \circ (\Sigma^{-F_1} \circ \Sigma^i_{\tilde{Z}} T_{M/B} \circ p_{\tilde{Z}}) \\
& \cong \pi Z! \circ \Sigma^{-F_1} \circ \Sigma^i_{\tilde{Z}} T_{M/B} \circ p_{\tilde{Z}}.
\end{align*}
\]
Applying this to $1_{\tilde{Z}}$ gives the isomorphism
\[ (\tilde{F}_1 \to p_{\tilde{Z}}^* i_{\tilde{Z}}^* \Omega_{M/B} (1_{\tilde{Z}})) \to p_{\tilde{Z}}^* (\Sigma^{-F_1} \circ \Sigma^i_{\tilde{Z}} T_{M/B} (1_{\tilde{Z}})). \]
Lemma 4.5. Let $p_M : \tilde{M} \to M$ be a Jouanolou cover and let $q_M : \tilde{M} \to \tilde{M}$ be a vector bundle. Form the Cartesian diagram

(4.4) \[
\begin{array}{c}
\tilde{Z} \\
\downarrow_{\tilde{q}_Z} \quad \downarrow_{p_M} \\
\tilde{M} \\
\end{array}
\]

Let $[\phi] : E_\bullet \to \mathbb{L}_{Z/B}$ be a perfect obstruction theory, form the induced obstruction theories $p_{Z}^* [\phi]$ and $\tilde{p}_Z^* [\phi]$, and let $\tilde{\phi} : \tilde{F}_\bullet \to \tilde{p}_Z^* [\phi], \phi : \tilde{F}_\bullet \to \tilde{p}_Z^* [\phi]$ be reduced normalized representatives of $p_{Z}^* [\phi]$ and $\tilde{p}_Z^* [\phi]$, respectively. We have the isomorphisms of Lemma 2.4,

\[ \mathcal{C}_{st}(p_M) : \mathcal{C}_{st,Z}^1 \to \mathcal{C}_{st,\tilde{M}}^1, \quad \mathcal{C}_{st}(\tilde{p}_M) : \mathcal{C}_{st,Z}^1 \to \mathcal{C}_{st,\tilde{M}}^1. \]

Then

\[ \mathcal{C}_{st}(p_M) \circ s_\tilde{\phi} \circ \vartheta_{\tilde{z},p_Z,\tilde{F}_\bullet} = \mathcal{C}_{st}(\tilde{p}_M) \circ s_{\phi} \circ \vartheta_{\tilde{z},\tilde{p}_Z,\tilde{F}_\bullet}. \]

Proof. We may assume that $\tilde{F}_1 = q_Z^* \tilde{F}_1$, giving the map of vector bundles over $q_Z, q_F : \tilde{F}^1 \to \tilde{F}^1$, which identifies $\tilde{F}^1$ with the vector bundle $\tilde{Z} \times_{\tilde{Z}} \tilde{M}^1 \to \tilde{F}^1$, which we denote by $\tilde{q}_F : V \to \tilde{F}^1$.

The co-unit

\[ q_{Z}^* q_Z \to \text{Id}_{\tilde{Z}} \]

induces the isomorphism

\[ \alpha : \tilde{p}_Z^* \circ p_{Z}^* = p_{Z}^* \circ q_Z^* q_Z \circ p_{Z}^* \to p_{Z}^* \circ p_{Z}^* \]

and the isomorphism

\[ \beta : \pi_{\tilde{Z}_1} \circ \Sigma^{-\tilde{F}^1} \circ \Sigma^i_{\tilde{z}} T_{\tilde{M}/B} \circ \tilde{p}_Z^* = \pi_{\tilde{Z}_1} \circ \Sigma^{-\tilde{F}^1} \circ \Sigma^i_{\tilde{z}} T_{\tilde{M}/B} \circ q_Z^* q_Z \tilde{p}_Z^* \]

\[ \to \pi_{\tilde{Z}_1} \circ \Sigma^{-\tilde{F}^1} \circ \Sigma^i_{\tilde{z}} T_{\tilde{M}/B} \circ \tilde{p}_Z^* \]

This gives the commutative diagram

(A) \[
\begin{array}{c}
\pi_{\tilde{Z}_1} \circ \Sigma^E_{\tilde{Z}_1} \\
\downarrow_{\pi_{\tilde{Z}_1}} \pi_{\tilde{Z}_1} \circ \Sigma^E_{\tilde{Z}_1} \\
\end{array}
\]

\[ \sim \]

\[ \alpha \]

\[ \sim \]

\[ \beta \]

\[ \beta \circ \vartheta_{\tilde{z},\tilde{p}_Z,\tilde{F}_\bullet} = \vartheta_{\tilde{z},p_Z,\tilde{F}_\bullet} \]

showing that $\beta \circ \vartheta_{\tilde{z},\tilde{p}_Z,\tilde{F}_\bullet} = \vartheta_{\tilde{z},\tilde{p}_Z,\tilde{F}_\bullet}$. 
Next, let \( \hat{s}_0 : Z \rightarrow \hat{F}^1 \), \( \hat{s}_0 : Z \rightarrow \hat{F}^1 \) be the 0-sections, and let \( p_{\hat{F}^1} : \hat{F}^1 \rightarrow \hat{Z} \), \( p_{\hat{F}^1} : \hat{F}^1 \rightarrow \hat{Z} \) be the projections. The map

\[
q_{F*} : \pi_{\hat{F}^1} \circ \Sigma q_{F*} V \circ q_{F*} \rightarrow \pi_{\hat{F}^1}
\]

induces the map

\[
\pi_{\hat{F}^1} \circ \Sigma q_{F*} V \circ q_{F*} \circ \Sigma p_{\hat{F}^1} i_{Z} T_{M/B} \circ p_{\hat{F}^1} \rightarrow \pi_{\hat{F}^1} \circ \Sigma p_{\hat{F}^1} i_{Z} T_{M/B} \circ p_{\hat{F}^1}
\]

The exact sequence of vector bundles on \( \hat{F}^1 \)

\[
0 \rightarrow q_{F*} V \rightarrow p_{\hat{F}^1} i_{Z} T_{M/B} \rightarrow q_{F*} p_{\hat{F}^1} i_{Z} T_{M/B} \rightarrow 0
\]

transforms this map to a morphism

\[
\gamma : \pi_{\hat{F}^1} \circ \Sigma p_{\hat{F}^1} i_{Z} T_{M/B} \circ (\hat{p}_{Z} \circ p_{\hat{F}^1})^* \rightarrow \pi_{\hat{F}^1} \circ \Sigma p_{\hat{F}^1} i_{Z} T_{M/B} \circ (\hat{p}_{Z} \circ p_{\hat{F}^1})^*
\]

giving us the diagram

\[
\begin{array}{ccc}
\pi_{\hat{F}^1} \circ \Sigma q_{F*} V & \xrightarrow{\beta} & \pi_{\hat{F}^1} \circ \Sigma q_{F*} V \\
\hat{s}_0 & \downarrow & \hat{s}_0 \\
\pi_{\hat{F}^1} \circ \Sigma p_{\hat{F}^1} i_{Z} T_{M/B} \circ (\hat{p}_{Z} \circ p_{\hat{F}^1})^* & \xrightarrow{\gamma} & \pi_{\hat{F}^1} \circ \Sigma p_{\hat{F}^1} i_{Z} T_{M/B} \circ (\hat{p}_{Z} \circ p_{\hat{F}^1})^* 
\end{array}
\]

By Lemma 1.3, the maps \( \hat{s}_{0*}, \hat{s}_{0*} \) are inverses to the maps \( p_{\hat{F}^1*}, p_{\hat{F}^1*} \), respectively. The commutativity of this diagram thus follows from the functoriality of smooth pushforward:

\[
q_{Z*} \circ [p_{\hat{F}^1*} \circ (\Sigma q_{F*} V)] = p_{\hat{F}^1*} \circ [q_{F*} \circ (\Sigma q_{F*} V)].
\]

The cartesian diagram (1.4) gives rise to the cartesian diagram

\[
\begin{array}{ccc}
\mathcal{C}_{Z \subset \hat{M}} & \xrightarrow{i_{\hat{Z}}} & \hat{F}^1 \\
\mathcal{C}(q_Z) & \xrightarrow{q_{\hat{Z}}} & \hat{F}^1
\end{array}
\]

The morphism

\[
\mathcal{C}(q_Z)_* : \pi_{\mathcal{C}(q_Z)_* \Sigma q_{F*} V} \circ (\mathcal{C}(q_Z)_* V) \rightarrow \pi_{\mathcal{C}(q_Z)_*}
\]

induces the morphism

\[
\pi_{\mathcal{C}(q_Z)_*} \circ (\mathcal{C}(q_Z)_* V) \circ \Sigma p_{\hat{F}^1} i_{Z} T_{M/B} \rightarrow \pi_{\mathcal{C}(q_Z)_*} \circ \Sigma p_{\hat{F}^1} i_{Z} T_{M/B}.
\]

Using the exact sequence (1.5) again, this gives the morphism

\[
\delta : \pi_{\mathcal{C}(q_Z)_*} \circ \Sigma i_{Z} T_{M/B} \circ \hat{p}_{Z} \rightarrow \pi_{\mathcal{C}(q_Z)_*} \circ \Sigma i_{Z} T_{M/B} \circ \hat{p}_{Z}.
\]
This gives us the diagram

\[(C) \quad \pi_{F^1} \circ \Sigma^{p_{F^1}^* i_2^* T/M/B} \circ (\tilde{p}_Z \circ p_{F^1})^* \xrightarrow{\gamma} \pi_{F^1} \circ \Sigma^{p_{F^1}^* i_2^* T/M/B} \circ (\tilde{p}_Z \circ p_{F^1})^* \]

\[\delta \quad \pi_{\Sigma^{p_{F^1}^* i_2^* T/M/B} \circ \tilde{p}_Z} \circ \delta \quad \pi_{\Sigma^{p_{F^1}^* i_2^* T/M/B} \circ \tilde{p}_Z} \]

It follows from the compatibility of proper pull-back with smooth push-forward in cartesian squares, Lemma 1.4, that this diagram commutes.

Finally, it follows directly from the definitions of the various morphisms involved that the diagram

\[(D) \quad \mathcal{E}^{st} \xrightarrow{\alpha_i} \mathcal{E}^{st} \]

commutes. Putting together the diagrams (A) (1Z), (B) (1Z), (C) (1Z) and (D) gives the identity

\[s_{i_\phi} \circ \vartheta_{i_2, p_Z, \tilde{F}} = \mathcal{E}^{st}(q_M) \circ s_{i_\phi} \circ \vartheta_{i_2, \tilde{p}_Z, \tilde{F}} \]

composing on the left with \(\mathcal{E}^{st}(p_M)\) and using the functoriality

\[\mathcal{E}^{st}(p_M) \circ \mathcal{E}^{st}(q_M) = \mathcal{E}^{st}(\tilde{p}_M)\]

completes the proof. \(\square\)

**Definition 4.6.** Suppose \(B\) is affine, take \(Z \in \text{Sch}^G/B\) and let \([\phi] : E_\bullet \rightarrow \mathbb{L}_{Z/B}\) be a perfect obstruction theory on \(Z\). Choose a closed immersion \(i_Z : Z \rightarrow M\) with \(M \in \text{Sm}^G/B\). Choose a Jouanolou cover \(p_M : \tilde{M} \rightarrow M\) and let \(p_Z : \tilde{Z} \rightarrow Z\) be the pull-back \(\tilde{M} \times_M Z\). Choose a reduced normalized obstruction theory \(\tilde{\phi} : \tilde{F}_\bullet \rightarrow (\mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2 \rightarrow i^* \Omega_{\tilde{M}/B})\) representing \(p_M^*[\phi]\). Define the virtual fundamental class

\[[Z, [\phi]]^{vir} \in S_B^{0,0}(\pi_Z!(\Sigma E^*_\bullet(1Z)))\]

by

\[[Z, [\phi]]^{vir} := \vartheta_{i_Z, p_Z, \tilde{F}_\bullet} s_{i_\phi} \mathcal{E}^{st}(p_M)^* [\mathcal{E}^{st}_Z].\]

If we have a monoid \(E\) in \(\text{SH}^G(B)\) with unit \(\epsilon_E\), we define

\[[Z, [\phi]]^{vir}_E := \epsilon_E([Z, [\phi]]^{vir}) \in \mathcal{E}^{0,0}(\pi_Z!(\Sigma E^*_\bullet(1Z))).\]

We need to check that the virtual fundamental class so defined is independent of the choices involved.

**Lemma 4.7.** Suppose \(B\) is affine. Given a perfect obstruction theory \([\phi] : E_\bullet \rightarrow \mathbb{L}_{Z/B}\) on \(Z \in \text{Sch}^G/B\), the virtual fundamental class \([\mathcal{E}_Z, [\phi]]^{vir} \in S_B^{0,0}(\pi_Z!(\Sigma E^*_\bullet(1Z)))\) is independent of the choice of closed immersion \(i_Z :
Z \to M$, the choice of Jouanolou cover $\tilde{M} \to M$, the choice of induced obstruction theory $p^*_{\tilde{Z}}[\phi]$ and the choice of reduced normalized obstruction theory representing $p^*_{\tilde{Z}}[\phi]$.

Proof. Suppose we have fixed a closed immersion $i_Z : Z \to M$ and a Jouanolou cover $\tilde{M} \to M$, fix an induced obstruction theory $p^*_{\tilde{Z}}[\phi]$, and suppose we have two reduced normalized obstruction theories $(\tilde{F}, \phi)$, $(\tilde{F}', \phi')$ representing $p^*_{\tilde{Z}}[\phi]$. By lemma 4.5 there is an isomorphism $\rho : \tilde{F} \to \tilde{F}'$ of perfect obstruction theories. In particular, the map $\rho^1 : F^1 \to F^1$ satisfies $\rho^1 \circ i_{\phi'} = i_{\phi}$. This gives us the commutative diagram

$$
\begin{array}{ccc}
\pi_{\tilde{Z}} \circ \Sigma F^1(1_{\tilde{Z}}) & \xrightarrow{\theta_{iZ,pZ,\tilde{F}}^1} & \pi_{\tilde{Z}} \circ \Sigma F^1 \circ \Sigma_{\tilde{M}/B}^* T_{\tilde{M}/B}(1_{\tilde{Z}}) \\
\downarrow & & \downarrow \gamma(\rho^1) \\
\pi_{\tilde{Z}} \circ \Sigma F^1(1_{\tilde{Z}}) & \xrightarrow{s_{i\phi'}} & \Sigma F^1 \circ \Sigma_{\tilde{M}/B}^* T_{\tilde{M}/B}(1_{\tilde{Z}}) \\
\end{array}
$$

where $\gamma(\rho^1)$ is the isomorphism induced by $\rho^1$. This yields the independence of $[C_Z, [\phi]]$ on the choice of reduced normalized obstruction theory representing $p^*_{\tilde{Z}}[\phi]$. The choice of the induced obstruction theory $p^*_{\tilde{Z}}[\phi]$ involves only a choice of a splitting to the surjection $\pi : i^*_Z \Omega_{\tilde{M}/B} \to i^*_Z \Omega_{\tilde{M}/M}$. If we have two splittings $s$, $s'$, giving induced obstruction theories $p^*_{\tilde{Z}}[\phi]$ and $p^*_{\tilde{Z}}[\phi]'$, we have reduced normalized obstruction theories $\tilde{F}$ and $\tilde{F}'$ representing $p^*_{\tilde{Z}}[\phi]$ and $p^*_{\tilde{Z}}[\phi]'$, respectively. There is then a splitting of the pull-back of $\pi$ over $\tilde{Z} \times \mathbb{A}^1$, restricting to $s$ at 0 and $s'$ at 1, which will give an $\mathbb{A}^1$ homotopy between $s_{i\phi} \circ \theta_{iZ,pZ,\tilde{F}}$ and $s_{i\phi'} \circ \theta_{iZ,pZ,\tilde{F}'}$.

To show the independence on the choice of Jouanolou cover $p_M : \tilde{M} \to M$ over a fixed closed immersion $i_Z : Z \to M$ we may assume that we are comparing one cover $p_M : \tilde{M} \to M$ with a second cover $\tilde{p}_M : \tilde{M} \to M$ which factors as

$$
\tilde{M} \xrightarrow{q_M} \tilde{M} \xrightarrow{p_M} M
$$

with $q_M : \tilde{M} \to \tilde{M}$ a vector bundle over $\tilde{M}$. The independence here follows from Lemma 4.5.

Finally, suppose we have a smooth morphism $q : N \to M$ and a closed immersion $i'_Z : Z \to N$ with $q \circ i'_Z = i_Z$. We may suppose that $M$ is affine; if we take a Jouanolou cover $\tilde{N} \to N$, then as $Z$ is affine, the cover admits a section over $Z$, so we may assume that $N$ is also affine. With what we have already proven, we may take a reduced normalized representative $\phi : F_{\tilde{Z}} \to (\mathcal{I}_{i'_Z}(Z)/\mathcal{I}_{i'_Z}(Z))^2 \xrightarrow{i'^* \Omega_{N/B}} i'^* \Omega_{N/B}$ for $[\phi]$. This gives the reduced normalized representative for $q'^* [\phi]$

$$
F' := (F_1 \oplus i'^* \Omega_{N/M} \xrightarrow{q'^* \phi + \rho \cdot \text{id}} i'^* \Omega_{N/B}) (\mathcal{I}_{i'_Z}(Z)/\mathcal{I}_{i'_Z}(Z) \xrightarrow{d} i'^* \Omega_{N/B});
$$

here

$$
\rho : i'^* \Omega_{N/M} \to \mathcal{I}_{i'_Z}(Z)/\mathcal{I}_{i'_Z}(Z)
$$

By Lemma 2.1 and Lemma 3.1, the diagram and also as a morphism commutes.

In this case, we have the Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{C}_{Z\subset N} & \xrightarrow{i_{\phi'}} & F^1 \oplus i'^{\ast}T_{N/M} \\
\mathcal{C}(q) & \searrow & \downarrow p_1 \\
\mathcal{C}_{Z\subset M} & \xrightarrow{i_\phi} & F^1
\end{array}
\]

identifying \(\mathcal{C}_{Z\subset N}\) with the bundle

\[\sigma_i^{\ast}T_{N/M} = \mathcal{C}_{Z\subset M} \times Z i'^{\ast}T_{N/M}\]

over \(\mathcal{C}_{Z\subset M}\) and \(i_{\phi'}\) with \(i_\phi \times \text{Id.}\)

We have the isomorphism

\[\psi_q : \pi\epsilon_{Z\subset N}(\sigma_i^{\ast}\text{Th}_N(T_{N/B})) \to \pi\epsilon_{Z\subset M}(\sigma_i^{\ast}\text{Th}_M(T_{M/B})).\]

Represent the fundamental class \([\mathcal{C}_Z]\) as a morphism

\[\zeta_{Z\subset M} : \pi\epsilon_{Z\subset M}\sigma_i^{\ast}\text{Th}_M(T_{M/B}) \to S_B\]

and also as a morphism

\[\zeta_{Z\subset N} : \pi\epsilon_{Z\subset N}(\sigma_i^{\ast}\text{Th}_N(T_{N/B})) \to S_B.\]

By Lemma 2.1 and Lemma 3.1, the diagram

\[
\begin{array}{ccc}
\mathcal{C}_{Z\subset N} & \xrightarrow{\alpha_q} & \pi\epsilon_{Z\subset M}\sigma_i^{\ast}\text{Th}_M(T_{M/B}) \\
\mathcal{C}(q) & \searrow & \downarrow \zeta_{Z\subset M} \\
\pi\epsilon_{Z\subset N}(\sigma_i^{\ast}\text{Th}_N(T_{N/B})) & \xrightarrow{\zeta_{Z\subset N}} & S_B
\end{array}
\]

commutes.

We unpack the definition of the isomorphism \(\psi_q\). By definition, \(\psi_q := \mathcal{C}(q)_*(\sigma_i^{\ast}\text{Th}_M(T_{M/B})) \circ \theta\) where \(\theta\) is the isomorphism

\[\pi\epsilon_{Z\subset N}(\sigma_i^{\ast}\text{Th}_N(T_{N/B})) \cong \pi\epsilon_{Z\subset M}(\Sigma\epsilon_{Z\subset N/\epsilon_{Z\subset M}}(\mathcal{C}(q)^{\ast}\sigma_i^{\ast}\text{Th}_M(T_{M/B})))\]

induced by the exact sequence

(4.6) \[0 \to T_{N/M} \to T_{N/B} \to q^\ast T_{M/B} \to 0\]

and the isomorphism \(\sigma_i^{\ast}T_{N/M} \cong T_{\epsilon_{Z\subset N/\epsilon_{Z\subset M}}}.\) Putting in the definition of the map \(\mathcal{C}(q)_*(\sigma_i^{\ast}\text{Th}_M(T_{M/B}))\), this gives the description of \(\psi_q\) as the composition

\[
\begin{array}{l}
\pi\epsilon_{Z\subset N}(\sigma_i^{\ast}\text{Th}_N(T_{N/B})) \xrightarrow{\theta} \pi\epsilon_{Z\subset M}(\Sigma\epsilon_{Z\subset N/\epsilon_{Z\subset M}}(\mathcal{C}(q)^{\ast}\sigma_i^{\ast}\text{Th}_M(T_{M/B}))) \\
\cong \pi\epsilon_{Z\subset M} \circ (\mathcal{C}(q)_*(\mathcal{C}(q)^{\ast})\sigma_i^{\ast}\text{Th}_M(T_{M/B})) \\
\xrightarrow{\epsilon_{\mathcal{C}(q)}} \pi\epsilon_{Z\subset M}(\sigma_i^{\ast}\text{Th}_M(T_{M/B}))
\end{array}
\]
Via the identification
\[ \pi e_{Z^N}(\sigma^* \text{Th}_N(T_{N/B})) \xrightarrow{\theta} \pi e_{Z^M}(\sigma^* \text{Th}_M(T_{M/B})). \]
the identity \( \zeta_{Z^N} = \zeta_{Z^M} \circ \psi \) shows that we may then identify \( \zeta_{Z^N} \) as the composition
\[ \pi e_{Z^M} \circ (C(q) \# C(q)^*) (\sigma^* \text{Th}_M(T_{M/B})) \]
and the isomorphism
\[ \pi e_{Z^M} \circ (C(q) \# C(q)^*) (\sigma^* \text{Th}_M(T_{M/B})) \xrightarrow{\zeta_{Z^M}} \pi e_{Z^M}(\sigma^* \text{Th}_M(T_{M/B})). \]

We have an isomorphism, defined similarly to the isomorphism \( \theta \),
\[ \pi F^{i!} \circ (p_{F^{i+1}} \circ i^* \text{Th}_N(T_{N/B})) \cong \pi F^{i!} \circ (p_{F^{i+1}} \circ i^* \text{Th}_M(T_{M/B})), \]
and an isomorphism
\[ \pi Z^! (\Sigma^{-F^1} \circ \Sigma^i T_{N/B} (1_Z)) \xrightarrow{\theta'} \pi Z^! (\Sigma^{-F^1} \circ \Sigma^i T_{M/B} (1_Z)). \]
induced by the exact sequence \( [4, 6] \) and the identity \( F^1 \cong F^1 \oplus i^* T_{N/M} \). Via these identifications, we have the diagram
\[
\begin{array}{ccc}
\pi Z^! (\Sigma^{-F^1} \circ \Sigma^i T_{N/B} (1_Z)) & \xrightarrow{\theta'} & \pi Z^! (\Sigma^{-F^1} \circ \Sigma^i T_{M/B} (1_Z)) \\
\downarrow s_{\theta^*} & & \downarrow s_{\theta^*} \\
\pi F^{i!} \circ (p_{F^{i+1}} \circ i^* \text{Th}_M(T_{M/B})) & \xrightarrow{e_{F^1}} & \pi F^{i!} \circ i^* \text{Th}_M(T_{M/B}) \\
\downarrow i_{\theta'} & & \downarrow i_{\theta'} \\
\pi e_{Z^M} \circ (C(q) \# C(q)^*) (\sigma^* \text{Th}_M(T_{M/B})) & \xrightarrow{e_{\theta}} & \pi e_{Z^M} \circ (\sigma^* \text{Th}_M(T_{M/B})).
\end{array}
\]
As for the proof of the commutativity of diagram \( [13] \) in Lemma 4.5 the top square commutes by the functoriality of smooth pushforward. The bottom square commutes by the commutativity of smooth pushforward with proper pullback, Lemma [14].

The fact that sending \( E_* \) to \( \Sigma E_* \) arises from a functor
\[ \Sigma(-) : D_{G_{\text{iso}}}^{\text{perf}}(Y) \to \text{Aut}(\text{SH}^G(Y)) \]
gives us the commutative diagram of isomorphisms
\[
\begin{array}{ccc}
\pi Z^! (\Sigma^{-F^1} \circ \Sigma^i T_{N/B} (1_Z)) & \xrightarrow{\sim} & \pi Z^! (\Sigma^{-F^1} \circ \Sigma^i T_{M/B} (1_Z)) \\
\downarrow \theta_{iZ, \text{id}_M, E_*} & & \downarrow \theta_{iZ, \text{id}_N, E_*} \\
\pi Z^! \circ (\Sigma E_* (1_Z)) & \xrightarrow{\theta_{iZ, \text{id}_M, E_*}} & \pi Z^! \circ (\Sigma E_* (1_Z)).
\end{array}
\]
Putting these two diagrams together gives the identity
\[ \zeta_{Z^M} \circ s_{\phi^i} \circ \theta_{iZ, \text{id}_M, E_*} = \zeta_{Z^N} \circ s_{\phi^i} \circ \theta_{iZ, \text{id}_N, E_*} \]
which completes the proof. \( \square \)
5. Comparisons and examples

We relate our constructions to the classical construction of [3] in case of motivic cohomology/Chow groups, or more generally the case of an oriented theory, for example, $K$-theory or algebraic cobordism. For simplicity, we assume that $G = \{\text{Id}\}$.

5.1. Oriented theories. Let $\mathcal{E}$ be an oriented commutative ring spectrum in $\text{SH}(B)$. This gives us Thom isomorphisms: for $V \to X$ a rank $r$ vector bundle on $X \in \text{Sm}/B$, with $X$ of pure dimension $d_X$ over $B$ and for $\alpha \in \text{SH}^G(X)$, we have natural isomorphisms

$$
\mathcal{E}^{a,b}(\pi_{X!}(\Sigma V\alpha)) \cong \mathcal{E}^{a-2r,b+r}(\pi_{X!}(\alpha)) \cong \mathcal{E}^{a-2r+2d_X,b-r+d_X}(\pi_{X!}(\alpha)).
$$

This extends to arbitrary perfect complexes $E_* \in D^{\text{perf}}(X)$, with $r$ the virtual rank of $E_*$. One can define the Borel-Moore $\mathcal{E}$-homology for an $B$-scheme $Z$ with a closed immersion $i : Z \to M$, $M$ smooth over $B$ of dimension $d_M$ by

$$
\mathcal{E}_{a,b}^{B,M}(Z) := \mathcal{E}^{2d_M-a,d_M-b}(M/M \setminus Z).
$$

This may be interpreted as the $\mathcal{E}$-cohomology of the Borel-Moore motive $Z/B_{B,M} := \pi_{Z!}(1_Z)$, noting that

$$
\pi_{Z!}(1_Z) \cong \pi_{M!}(i_* (1_Z))
\cong \pi_{M\#}(\Sigma^{-T_M/B}(M/M \setminus Z))
$$

which gives the isomorphism

$$
\mathcal{E}^{-a,-b}(Z/B_{B,M}) \cong \mathcal{E}^{2d_M-a,d_M-b}(M/M \setminus Z) = \mathcal{E}_{a,b}^{B,M}(Z).
$$

Thus $\mathcal{E}_{a,b}^{B,M}(Z)$ is well-defined, independent of the choice of closed immersion.

In particular, we have, for $i : Z \to M$ a closed immersion in smooth dimension $d_M$ $B$-scheme

$$
\mathcal{E}^{a,b}(\mathcal{C}^M_Z) \cong \mathcal{E}^{a-2d_M,b,d_M}(i_* \mathcal{C}_{Z\subset M}(1_{\mathcal{C}_{Z\subset M}})) = \mathcal{E}_{2d_M-a,d_M-b}^{B,M}(\mathcal{C}_{Z\subset M}),
$$

and for $E_* \in D^{\text{perf}}(Z)$, we have

$$
\mathcal{E}^{a,b}(\pi_{Z!}(\Sigma^E Z(1_Z))) \cong \mathcal{E}^{a-2r,b-r}(\pi_{Z!}(1_Z)) = \mathcal{E}_{2r-a,r-b}^{B,M}(Z),
$$

where $r$ is the rank of $E_*$.

Noting that $\mathcal{C}_{Z\subset M}$ has pure dimension $d_C = d_M$ over $B$, the fundamental class $[\mathcal{C}^M_Z]_C$ is thus an element of $\mathcal{E}^{0,0}(\mathcal{C}^M_Z) \cong \mathcal{E}_{2d_C,d_C}^{B,M}(\mathcal{C}_{Z\subset M})$ and the virtual fundamental class $[Z,\langle \phi \rangle]_{Z\subset B}^{\text{vir}}$ associated to a perfect obstruction theory $(E_*,\langle \phi \rangle)$ of virtual rank $r$ lives in $\mathcal{E}^{0,0}(\pi_{Z!}(\Sigma^E Z(1_Z))) = \mathcal{E}_{2r,r}^{B,M}(Z)$. 

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5.2. **Fundamental classes.** Let \( \mathcal{E} \) be an oriented theory. Suppose we have an integral \( B \)-scheme \( D \) and a principal effective Cartier divisor \( \mathcal{C} \) on \( D \) such that \( D \setminus \mathcal{C} \) is smooth over \( B \). Suppose we have a codimension \( c \) closed immersion \( i : D \to V \) with \( V \) smooth over \( B \) and that \( \mathcal{C} \) has pure dimension \( d_\mathcal{C} \) over \( B \). Let \( t \in \Gamma(D, \mathcal{O}_D) \) be a generator for \( \mathcal{I}_\mathcal{C} \). The map \( t : D \setminus \mathcal{C} \to \mathbb{G}_m \) determines an element

\[
[t] \in \mathcal{E}^{1,1}(D \setminus \mathcal{C}) \cong \mathcal{E}^{2c+1,c+1}_D(D \setminus \mathcal{C}) = \mathcal{E}^{B,M}_{2d_\mathcal{C}+1,d_\mathcal{C}}(D \setminus \mathcal{C})
\]

We have the localization sequence

\[
\cdots \to \mathcal{E}^{B,M}_{2d_\mathcal{C}+1,d_\mathcal{C}}(D \setminus \mathcal{C}) \xrightarrow{i_*} \mathcal{E}^{B,M}_{2d_\mathcal{C},d_\mathcal{C}}(\mathcal{C}) \xrightarrow{\partial} \mathcal{E}^{B,M}_{2d_\mathcal{C},d_\mathcal{C}}(D) \to \cdots
\]

and we have the fundamental class \([\mathcal{C}] \in \mathcal{E}^{B,M}_{2d_\mathcal{C},d_\mathcal{C}}(\mathcal{C})\) defined by \([\mathcal{C}] := \partial [t]\). This class is independent of the choice of defining equation \( t \).

**Example 5.1.** Take \( B = \text{Spec} \ k \), and \( Z \in \text{Sch}/B \). For \( \mathcal{E} = \text{HZ} \), the ring spectrum representing motivic cohomology, \( \text{HZ}^{B,M}_{2d,d}(Z) \) is the classical Chow group \( \text{CH}_d(Z) \). For \( \mathcal{E} = \text{KGL} \), the ring spectrum representing Quillen \( K \)-theory, \( \text{KGL}^{B,M}_{2d,d}(Z) = G_0(Z) \), the Grothendieck group of coherent sheaves on \( Z \). For \( \mathcal{E} = \text{MGL} \), the ring spectrum representing Voevodsky’s algebraic cobordism, and \( k \) a field of characteristic zero, \( \text{MGL}^{B,M}_{2d,d} = \Omega_d(Z) \), the algebraic cobordism of \( \mathbb{P} \).

For \( \mathcal{E} = \text{HZ} \), the fundamental class \([\mathcal{C}]_\text{HZ} \in \mathcal{E}^{B,M}_{2d,d}(\mathcal{C}) \) is the cycle class associated to the scheme \( \mathcal{C} \). For \( \mathcal{E} = \text{KGL} \), \([\mathcal{C}]_\text{KGL} \) is the class in \( G_0(\mathcal{C}) \) of the structure sheaf \( \mathcal{O}_\mathcal{C} \). For \( \mathcal{E} = \text{MGL} \), \([\mathcal{C}]_\text{MGL} \) is the class associated to the pseudo-divisor \( \text{div}[t] \) on \( D \), applied to any resolution of singularities \( f : \tilde{D} \to D \)

\[
[\mathcal{C}]_\text{MGL} = f_*(\text{div}_D(f^*[t])).
\]

For any oriented theory \( \mathcal{E} \), the fundamental class \([\mathcal{C}]_{Z,M} \mathcal{E} \) as defined in [30] agrees with the definition given here. The construction of the fundamental class of \( \mathcal{E} \subset M \) by this method was described to the author of this paper by Parker Lowrey.

5.3. **Pushforward and intersection with the 0-section.** Let \( p : Y \to X \) be a projective map in \( \text{Sch}/B \). For an oriented theory, the pushforward map

\[
p_* : \mathcal{E}^{a,b}(\pi_Y!(1_Y)) \to \mathcal{E}^{a,b}(\pi_X!(1_X))
\]

induced by \( p^* : \pi_X!(1_X) \to \pi_Y!(1_Y) \) translates to

\[
p_* : \mathcal{E}^{B,M}_{a,b}(Y) \to \mathcal{E}^{B,M}_{a,b}(X).
\]

Suppose we have a rank \( r \) vector bundle \( f : V \to Z \) with a section \( s : Z \to V \). The maps

\[
f^* : \mathcal{E}^{a,b}(\pi_W!(1_W)) \to \mathcal{E}^{a,b}(\pi_Z!\Sigma_{T^V/Z}(1_Z))
\]

\[
s^* : \mathcal{E}^{a,b}(\pi_Z!\Sigma_{T^V/Z}(1_Z)) \to \mathcal{E}^{a,b}(\pi_W!(1_W))
\]

induced by \( f_* \), \( s_* \), respectively, translated to

\[
f^* : \mathcal{E}^{B,M}_{a,b}(W) \to \mathcal{E}^{B,M}_{a+2r,b+r}(V)
\]
Let \( F \) be defined by Behrend-Fantechi, is the quotient stack \( [C_N \to \mathcal{M}] \), also define the normal sheaf \( s \) with \( \phi \in i^*\Omega_{\mathcal{M}/B} \).

**Remark 5.2.** Suppose we have a closed immersion \( i : Z \hookrightarrow M \) with \( M \) smooth of dimension \( d_M \) over \( S \). The intrinsic normal cone \( \mathcal{E}_Z \), as defined by Behrend-Fantechi, is the quotient stack \( [\mathcal{E}_{ZCM}/i^*\mathcal{M}/B] \). They also define the normal sheaf \( \mathcal{N}_{ZCM} := \text{Spec} \mathcal{O}_Z \text{Sym}^*\mathcal{I}_Z/\mathcal{I}_Z^2; \) the surjection \( \text{Sym}^*\mathcal{I}_Z/\mathcal{I}_Z^2 \to \oplus_i \mathcal{I}_Z/\mathcal{I}_Z^{n+1} \) defines the closed immersion \( \mathcal{E}_{ZCM} \hookrightarrow \mathcal{N}_{ZCM} \). This induces the closed immersion of quotient stacks \( \mathcal{E}_Z \hookrightarrow \mathcal{N}_Z := [\mathcal{N}_{ZCM}/i^*\mathcal{M}/B]. \)

Suppose we have a perfect obstruction theory \( \phi \) of virtual rank \( r \) on \( i : Z \hookrightarrow M \), with global resolution \( (F_1 \to F_0) \otimes (\mathcal{I}_Z/\mathcal{I}_Z^2 \to i^*\mathcal{M}/B) \). We may assume that \( (\mathcal{F}_*, \phi) \) is normalized. The assumption that \( \phi \) is a perfect obstruction theory implies that \( \phi \) induces closed immersions

\[
\mathcal{E}_Z \hookrightarrow \mathcal{N}_Z \hookrightarrow [F^1/F^0].
\]

Let \( \mathcal{E}(\mathcal{F}_*) \subset \mathcal{N}(\mathcal{F}_*) \subset F^1 \) be the pullback of this sequence of closed immersions by the quotient map \( F^1 \to [F^1/F^0] \).

One can describe \( \mathcal{N}(\mathcal{F}_*) \) explicitly as follows: We have the commutative diagram

\[
\begin{array}{ccc}
F_1 & \xrightarrow{d_F} & F_0 \\
\phi_1 \downarrow & & \downarrow \phi_0 \\
\mathcal{I}_Z/\mathcal{I}_Z^2 & \xrightarrow{d} & i^*\mathcal{M}/B
\end{array}
\]

Let \( F := \mathcal{I}_Z/\mathcal{I}_Z^2 \times i^*\mathcal{M}/B F_0 \). The map \( (\phi_1, d_F) \) gives a surjection \( \phi_{\mathcal{N}} : F_1 \to F; \mathcal{N}(\mathcal{F}_*) \) is the closed subscheme \( \text{Spec} \mathcal{O}_Z \text{Sym}^*F \) of \( F^1 = \text{Spec} \mathcal{O}_Z \text{Sym}^*F_1 \).

The virtual fundamental class \( [Z, \phi]^{\text{vir}}_{\text{BF}} \) as defined by Behrend-Fantechi is the element of \( \text{CH}_r(Z) \) given by

\[
[Z, \phi]^{\text{vir}}_{\text{BF}} := s_0([\mathcal{E}(\mathcal{F}_*)]).
\]

Suppose that we have a Jouanolou cover \( p_M : \tilde{M} \to M \), with pull-back \( p_Z : \tilde{Z} \to Z \). The perfect obstruction theory \( p_\tilde{Z}^*(\mathcal{F}_*) \to (\mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2 \to i_{\tilde{Z}}^*\mathcal{M}/B) \) is defined, since \( \tilde{Z} \) is affine.

Writing \( (\tilde{T}_1 \to \tilde{T}_0) := p_\tilde{Z}^*(\mathcal{F}_*) \), we have \( \tilde{T}_1 = p_\tilde{Z}^*F_1 \) and \( \tilde{T}_0 = p_\tilde{Z}^*F_0 \oplus \Omega_{\tilde{Z}/Z} \).

Thus, we have the isomorphism of quotients of \( \tilde{T}_1 \)

\[
\tilde{F} := \mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2 \times i_{\tilde{Z}}^*\mathcal{M}/B \tilde{T}_0 \cong p^*_{\tilde{Z}}F
\]

which shows that \( \mathcal{N}(\mathcal{F}_*) \subset \tilde{F}^1 \) is equal to \( p^*_{\tilde{Z}}\mathcal{N}(\mathcal{F}_*) \). Thus

\[
\mathcal{E}(\tilde{F}_*) = p^*_{\tilde{Z}}\mathcal{E}(\mathcal{F}_*) \subset p^*_{\tilde{Z}}F^1 = \tilde{F}^1,
\]
which implies that
\[ p_Z^*([Z, [\phi]]_{BF}) = [\tilde{Z}, p_{\tilde{Z}}^*[\phi]]_{BF} \]
in \( CH_{r+d}(\tilde{Z}) \), where \( d \) is the rank of \( \Omega_{\tilde{Z}/Z} \).

Since \( p_Z : \tilde{Z} \to Z \) induces an isomorphism
\[ p_Z^* : CH_r(Z) \to CH_{r+d}(\tilde{Z}), \]
we may assume that \( Z \) is affine for the purpose of comparing our construction of virtual fundamental classes with that of Behrend-Fantechi.

Assuming then that \( Z \) is affine, with a closed immersion \( i : Z \to M \) into a smooth affine \( B \)-scheme \( M \), we may take a reduced normalized representative \((F_1 \to F_0) \to (I_Z/I_Z^2 \to i^*\Omega_{M/B})\) of a given rank \( r \) perfect obstruction theory \([\phi]\). In this case, we have \( E(F_*) = E_{Z\subset M} \subset F^1 \), and \( d_E = r_0 \). We have already identified our construction of the fundamental class \([E_{Z\subset M}]_{HZ}\) with the cycle class \( E_{Z\subset M} \in CH_{r_0}(E_{Z\subset M}) \); we have also identified the pushforward map \( i_\ast : CH_{r_0}(E_{Z\subset M}) \to CH_{r_0}(F^1) \) and the intersection with the 0-section \( s_0^* : CH_{r_0}(F^1) \to CH_{r_0-1}(Z) \), as defined here, with the classical ones. This gives the identity of virtual fundamental classes
\[ [Z, [\phi]]_{BF} = [Z, [\phi]]_{vir} \]
in \( CH_r(Z) \).

Of course, the Behrend-Fantechi theory is defined for perfect obstruction theories on Deligne-Mumford stacks, whereas the theory presented here is limited to quasi-projective \( B \)-schemes for affine \( B \).

5.4. **GW invariants and descendents.** If \([\phi]\) is a virtual rank zero perfect obstruction theory on some \( Z \in Sch/B \), and \( E \) is an oriented ring spectrum in \( SH(B) \), the virtual fundamental class lives in \( E_{a,b}^{BM}(Z) = E_{0,0}^0(\pi_Z(1_Z)) \).
If \( \pi_Z : Z \to B \) is proper, we have the pushforward map in \( E \)-cohomology
\[ \pi_{Z\ast} : E_{a,b}^{BM}(Z) \to E_{a,b}^{BM}(B) = E_{-a,-b}(B), \]
induced by the map \( \pi_{Z\ast} : 1_B \to \pi_Z(1_Z) \), giving the GW-invariant
\[ deg_E([Z, [\phi]]_{vir}) := \pi_{Z\ast}([Z, [\phi]]_{vir}) \in E_{0,0}^0(B). \]
This is the classical “degree of the virtual fundamental class” in case \( E = HZ \). For more general theories, we may have non-zero invariants for perfect obstruction theories of non-zero ranks which give rise to non-zero degrees:
\[ deg_E([Z, [\phi]]_{vir}) := \pi_{Z\ast}([Z, [\phi]]_{vir}) \in E_{2r,r}^{BM}(B) = E_{-2r,-r}(B) \]
for \([\phi]\) of virtual rank \( r \).

If we have a morphism \( f : Z \to W \), one can twist \([Z, [\phi]]_{vir}\) by classes coming from \( W \); if \( Z \) is proper over \( B \), pushing forward gives the descendent classes in \( E_{s*,r}^{BM}(B) \).
5.5. SL-oriented theories. We now consider theories $\mathcal{E}$ which are not oriented in the sense of the previous section, but are rather SL-oriented. This means that, given a perfect complex of virtual rank $r$, $(E_\bullet, \mathbf{E})$, on some $Z \in \text{Sch}_k$, together with an isomorphism $\alpha : \det E_\bullet \to M_{\otimes 2}$ for some line bundle $M$ on $Z$, there is an isomorphism

$$\lambda_{\alpha, M} : \pi_2 Z \mathcal{E} \wedge \Sigma^{E_\bullet}(1_Z) \to \Sigma_T^{\text{rank}(E_\bullet)} \mathcal{E}$$

This implies that, if $\beta : V \to V$ is an automorphism of a vector bundle $V \to Z$, the induced map

$$\text{Id}_\mathcal{E} \wedge \text{Th}(\beta) : \pi_2 Z \mathcal{E} \wedge \Sigma^V \to \pi_2 Z \mathcal{E} \wedge \Sigma^V$$

induces multiplication by the automorphism $\langle \det \beta \rangle \in \pi_{0,0}(\mathbb{S}_Z)$.

Suppose we have a rank $r$ perfect obstruction theory $((\phi), E_\bullet)$ on some $Z \in \text{Sch}/B$, projective over $B$. The virtual fundamental class $[Z, [\phi]]_{\text{vir}}$ lives in $\mathcal{E}^{0,0}(\pi_Z(\Sigma^{E_\bullet}(1_Z)))$. Given an isomorphism

$$\alpha : \det E_\bullet \to M_{\otimes 2}$$

for some line bundle $M$ on $Z$, we have the isomorphism

$$\lambda_{\alpha} : \mathcal{E}^{0,0}(\pi_Z(\Sigma^{E_\bullet}(1_Z))) \to \mathcal{E}^{B, M}_{2r, r}(Z)$$

so we may push $\lambda_{\alpha}([Z, [\phi]]_{\text{vir}})$ forward to give

$$\deg_{\mathcal{E}}([Z, [\phi]]_{\text{vir}}, \alpha) := \pi_{2r} Z_*(\lambda_{\alpha}([Z, [\phi]]_{\text{vir}})) \in \mathcal{E}^{2r, -r}_r(B) = \mathcal{E}^{B, M}_{2r, r}(B).$$

Example 5.3. We take $B = \text{Spec} k$, $k$ a perfect field, $G = \{\text{Id}\}$ and $\mathcal{E} = H_0(\mathbb{S}_k)$. $H_0(\mathbb{S}_k)$ represents the theory of Milnor-Witt $K$-theory: for $X \in \text{Sm}/k$, there is a canonical isomorphism

$$H_0(\mathbb{S}_k)^{a+b, a+b}(X) \cong H^a_{\text{Nis}}(X, K_b^{\text{MW}})$$

and more generally, for $E_\bullet \in D^{\text{perf}}(X)$,

$$H_0(\mathbb{S}_k)^{a+b, a+b}(\pi_X^{\#} \Sigma^{E_\bullet}(1_X)) \cong H^a_{\text{Nis}}(X, K_b^{\text{MW}}(\det E_\bullet)).$$

This generalizes to

$$H_0(\mathbb{S}_k)^{B, M}_{2m, n}(\pi_Z \Sigma^{E_\bullet}(1_Z)) \cong \text{CH}_n(Z, \det E_\bullet).$$

for $Z \in \text{Sch}_k$, $E_\bullet \in D^{\text{perf}}(Z)$.

For $i : Z \hookrightarrow M$, $M$ smooth of dimension $d_M$ over $k$, and $([\phi], E_\bullet)$ a rank $r$ perfect obstruction theory, this gives us the fundamental class and virtual fundamental class

$$[\mathcal{E}]_{\text{vir}} \in \text{CH}_{d_M}(\mathcal{E} \subset M, \sigma_i^* \omega_{M/k})$$

$$[Z, [\phi]]_{\text{vir}} \in \text{CH}_r(Z, \det E_\bullet).$$

Thus, if we have a rank 0 perfect obstruction theory $((\phi), E_\bullet)$ on some $Z \in \text{Sch}_k$, projective over $k$, and an isomorphism $\alpha : \det E_\bullet \to M_{\otimes 2}$ for some line bundle $M$ on $Z$, we have

$$\deg_{H_0(\mathbb{S}_k)}([Z, [\phi]]_{\text{vir}}, \alpha) := \pi_{2r} Z_*(\lambda_{\alpha}([Z, [\phi]]_{\text{vir}})) \in K^{\text{MW}}_0(k) = \text{GW}(k).$$
Here $\text{GW}(k)$ is the Grothendieck-Witt group of non-degenerate quadratic forms over $k$.

More generally, if $\mathcal{E}$ has virtual rank $r$, we have the canonical isomorphism

$$H_0(S_k)^{0,0}((\pi_1^\vee)^r \otimes 1_Z) \cong H_0(S_k)^{-2r,-r}((\pi_1)^{\det \mathcal{E}^\vee} \otimes \mathcal{O}_Z 1_Z).$$

Thus, if we have a morphism $f : Z \to W$, a line bundle $L$ on $W$, a line bundle $M$ on $Z$ and an isomorphism

$$\alpha : \det \mathcal{E}^\vee \otimes f^* L \to M^\otimes 2$$

then a class

$$\beta \in H_0(S_k)^{2r+s,r+s}((\pi_1)^{\det \mathcal{E}^\vee} \otimes f^* L \otimes \mathcal{O}_Z 1_Z)$$

gives

$$[Z, [\phi]^\text{vir}] \cup f^* \beta \in H_0(S_k)^{s,s}((\pi_1)^{\det \mathcal{E}^\vee} \otimes f^* L \otimes \mathcal{O}_Z 1_Z),$$

and we can define the descendant class

$$\deg H_0(S_k)^{s,s}([Z, [\phi]^\text{vir}] \cup f^* \beta, \alpha) = \pi_{Z,s}([Z, [\phi]^\text{vir}] \cup f^* \beta, \alpha) \in H_0(S_k)^{s,s}(K_{MW}(k)).$$

There is a universal SL-oriented theory, $\text{MSL}$, with $\text{MSL}_n$ the Thom space of the universal bundle $\tilde{E}_n \to B\text{SL}_n$. Just as for $\text{MGL}_n$, $\text{MSL}_{n}^{-2r,-r}(k)$ is non-zero for all $r \geq 0$, so we have a non-trivial target for the degree map for perfect obstruction theories of all non-negative ranks, but having a trivialized determinant bundle (up to a square).

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