SHANNON SAMPLING AND WEAK WEYL’S LAW ON COMPACT RIEMANNIAN MANIFOLDS

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Abstract. The well known Weyl’s asymptotic formula gives an approximation to the number \( N_\omega \) of eigenvalues (counted with multiplicities) on an interval \([0, \omega]\) of the Laplace-Beltrami operator on a compact Riemannian manifold \( M \). In this paper we approach this question from the point of view of Shannon-type sampling on compact Riemannian manifolds. Namely, we give a direct proof that \( N_\omega \) is comparable to cardinality of certain sampling sets for the subspace of \( \omega \)-bandlimited functions on \( M \).

1. Introduction

1.1. Objectives. Spectral geometry concerned with questions which relate spectral properties of operators acting in function spaces on a Riemannian manifold and the geometry of the underlying manifold. One of the most famous results of such kind is the Weyl’s asymptotic formula for the number of eigenvalues of an elliptic (pseudo-)differential operator on a compact Riemannian manifold. The goal of this paper is to demonstrate that in the case of a general compact Riemannian manifold the so-called weak Weyl’s formula closely relates to cardinality of certain sampling sets for bandlimited functions. This fact was first noticed in [4].

1.2. Weyl’s asymptotic formula on compact Riemannian manifolds. Let \( M \) be a compact connected Riemannian manifold without boundary and \( \Delta \) is the Laplace-Beltrami operator. It is given in a local coordinate system by the formula

\[
\Delta f = \sum_{m,k} \frac{1}{\sqrt{\det(g_{ij})}} \partial_m \left( \sqrt{\det(g_{ij})} g^{mk} \partial_k f \right)
\]

where \( g_{ij} \) are components of the metric tensor, \( \det(g_{ij}) \) is the determinant of the matrix \( (g_{ij}) \), \( g^{mk} \) components of the matrix inverse to \( (g_{ij}) \). The operator is second-order differential self-adjoint and non-negative in the space \( L^2(M) \) constructed with respect to Riemannian measure. Domains of the powers \( \Delta^{s/2}, s \in \mathbb{R} \), coincide with the Sobolev spaces \( H^s(M), s \in \mathbb{R} \). Since \( \Delta \) is a second-order differential self-adjoint and non-negative definite operator on a compact connected Riemannian manifold it has a discrete spectrum \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2, ... \) which goes to infinity without any accumulation points and there exists a complete family \( \{u_j\} \) of orthonormal eigenfunctions which form a basis in \( L^2(M) \) [2].

We will need the following definitions.

Definition 1. The space of \( \omega \)-bandlimited functions \( E_\omega(\Delta) \) is defined as the span of all eigenfunctions of \( \Delta \) whose eigenvalues are not greater than \( \omega \). The dimension of the subspace \( E_\omega(\Delta) \) will be denoted as \( N_\omega \).
One can easily verify that \( f \) belongs to \( E_\omega(\Delta) \) if and only if the following Bernstein type inequality holds

\[
\|\Delta^k f\|_{L^2(M)} \leq \omega^k \|f\|_{L^2(M)}
\]

for all natural \( k \).

**Definition 2.** We say that \( M_\rho = \{x_j\}, \ x_j \in M, \ \rho > 0, \) is a metric \( \rho \)-lattice if

1. Balls \( B(x_j, \rho/2) \) are disjoint
   \[ B(x_j, \rho/2) \cap B(x_i, \rho/2) = \emptyset, \quad j \neq i, \]
   but balls \( B(x_\nu, \rho) \) form a cover of \( M \).
2. There exists a constant \( N_M \) such that multiplicity of all such covers \( \{B(x_j, \rho)\} \)
   is bounded by \( N_M \).

One can show [3], [5] existence of metric lattices for sufficiently small \( \rho > 0 \). We reprove this fact in Lemma 2.1 below. Note that \( N_\omega \) is the same as the number of eigenvalues (counting with their multiplicities) which are not greater \( \omega \). According to the Weyl’s asymptotic formula [2] one has for large \( \omega \)

\[
N_\omega \sim A \text{Vol}(M)\omega^{d/2},
\]

where \( d = \text{dim} \ M \) and \( A \) is a constant which is independent on \( M \). To reveal meaning of the right-hand side of this formula let’s rewrite it in the following form

\[
N_\omega \sim A \text{Vol}(M)\omega^{d/2} = A \frac{\text{Vol}(M)}{(\omega^{-1/2})^d}.
\]

Since in the case of a Riemannian manifold \( M \) of dimension \( n \) all the balls of the same radius \( \rho \) have essentially the same volume \( \sim \rho^d \) the last fraction can be interpreted as a number of balls \( B(x_\nu, \omega^{-1/2}) \) whose centers \( \{x_\nu\} \) form a lattice \( M_{\omega^{-1/2}} \).

The main goal of our paper is to present a direct proof of the following Theorem 1.1 (which we call the Weak Weyl’s Law) without using the Weyl’s asymptotic formula (1.2).

**Theorem 1.1.** (Weak Weyl’s Law) In the case of a Riemannian manifold the number \( N_\omega \) of eigenvalues of \( \Delta \) in \( [0, \omega] \) counting with their multiplicities is equivalent to a number of points in a metric lattice \( M_{\omega^{-1/2}} \). Namely, there are constants \( a = a(M) > 0 \) and

\[
0 < \gamma = \gamma(M) < 1,
\]

such that for all sufficiently large \( \omega \) the following double inequality holds

\[
a \sup |M_{\omega^{-1/2}}| \leq N_\omega \leq \inf |M_{\gamma\omega^{-1/2}}|,
\]

where \( \sup \) is taken over all \( \omega^{-1/2} \)-lattices and \( \inf \) is taken over all \( \gamma\omega^{-1/2} \)-lattices and \( |M_s| \) denotes cardinality of a lattice.
2. Covering Lemma

We consider a compact Riemannian manifold $M$, $\dim M = d$, with metric tensor $g$. It is known that the Laplace-Beltrami operator $\Delta$ which is defined in (1.1) is a self-adjoint positive definite operator in the corresponding space $L_2(M)$ constructed from $g$. Domains of the powers $\Delta^{s/2}$, $s \in \mathbb{R}$, coincide with the Sobolev spaces $H^s(M)$, $s \in \mathbb{R}$. To choose norms on spaces $H^s(M)$, we consider a finite cover of $M$ by balls $B(y_\nu, \sigma)$ where $y_\nu \in M$ is the center of the ball and $\sigma$ is its radius. For a partition of unity $\varphi_\nu$ subordinate to the family $\{B(y_\nu, \sigma)\}$ we introduce Sobolev space $H^s(M)$ as the completion of $C^\infty_0(M)$ with respect to the norm

$$
(2.1) \quad \|f\|_{H^s(M)} = \left( \sum_\nu \|\varphi_\nu f\|_{H^s(B(y_\nu, \sigma))}^2 \right)^{1/2}.
$$

The regularity Theorem for the Laplace-Beltrami operator $\Delta$ states that the norm (1.1) is equivalent to the graph norm $\|f\| + \|\Delta^{s/2} f\|$.

The volume of a ball $B(x, \rho)$ will be denoted by $|B(x, \rho)|$. Let us note that in the case of a compact Riemannian manifold of dimension $d$ there exist constants $a_1 = a_1(M), a_2 = a_2(M)$ such that for a ball $B(x, \rho)$ of sufficiently small radius $\rho$ and any center $x \in M$ one has

$$
(2.2) \quad a_1 \rho^d \leq |B(x, \rho)| \leq a_2 \rho^d,
$$

where

$$
|B(x, \rho)| = \int_{B(x, \rho)} dx, \quad d = \dim M.
$$

The inequality (2.2) implies the next inequality with the same $a_1$ and $a_2$:

$$
(2.3) \quad \frac{a_1}{a_2} |B(x_2, \rho)| \leq |B(x_1, \rho)| \leq \frac{a_2}{a_1} |B(x_2, \rho)|, \quad \rho < r,
$$

where $x_1, x_2$ are any two points in $M$ and $r$ is the injectivity radius of the manifold. Since $M$ is compact there exists a constant $c = c(M)$ such that for any $0 < \sigma < \lambda < r/2$ the following inequality holds true

$$
(2.4) \quad |B(x, \lambda)| \leq (\lambda/\sigma)^d c |B(x, \sigma)|.
$$

In what follows we will use the notation

$$
N_M = \frac{12^d c a_2}{a_1}.
$$

The following Covering Lemma plays an important role for the paper.

**Lemma 2.1.** If $M$ satisfy the above assumptions then for any $0 < \rho < r/6$ there exists a finite set of points $\{x_i\}$ such that

1) balls $B(x_i, \rho/4)$ are disjoint,

2) balls $B(x_i, \rho/2)$ form a cover of $M$,

3) multiplicity of the cover by balls $B(x_i, \rho)$ is not greater $N_M$.

**Proof.** Let us choose a family of disjoint balls $B(x_i, \rho/4)$ such that there is no ball $B(x, \rho/4), x \in M$, which has empty intersections with all balls from our family. Then the family $B(x_i, \rho/2)$ is a cover of $M$. Every ball from the family $\{B(x_i, \rho)\}$, that has non-empty intersection with a particular ball $\{B(x_j, \rho)\}$ is contained in
the ball \( \{B(x_j, 3\rho)\} \). Since any two balls from the family \( B(x_i, \rho/4) \) are disjoint, it gives the following estimate for the index of multiplicity \( N \) of the cover \( B(x_i, \rho) \):

\[
(2.5) \quad N \leq \frac{\sup_{y\in M} |B(y, 3\rho)|}{\inf_{x\in M} |B(x, \rho/4)|}.
\]

From here, according to (2.4) we obtain

\[
N \leq \frac{\sup_{y\in M} |B(y, 3\rho)|}{\inf_{x\in M} |B(x, \rho/4)|} \leq 12^d \frac{\sup_{y\in M} |B(y, \rho/4)|}{\inf_{x\in M} |B(x, \rho/4)|} \leq \frac{12^d c a_2}{a_1} = N_M.
\]

\[\square\]

3. Sampling sets for bandlimited functions and the upper estimate on the number of eigenvalues.

3.1. Poincare-type inequality on manifolds. One can prove the following Poincare type inequality (see [2], [3]). We sketch it’s proof for completeness.

**Theorem 3.1.** There exists a constant \( C = C(M, k) \) such that if \( \rho > 0 \) is sufficiently small then for all \( \rho \) lattices \( M_\rho = \{x_j\} \) and all \( f \in H^k(M) \), \( k > d/2 \), \( d = \dim M \),

\[
(3.1) \quad \|f\|_{L^2(M)} \leq C(M, k) \left\{ \rho^{d/2} \left( \sum_{x_j \in M_\rho} |f(x_j)|^2 \right)^{1/2} + \rho^k \|\Delta f\|_{L^2(M)} \right\}.
\]

**Proof.** Let \( M_\rho = \{x_i\} \) be a \( \rho \)-admissible set and \( \{\varphi_\nu\} \) the partition of unity from (1.1). For any \( f \in C^\infty(M) \), every fixed \( B(x_i, \rho) \) and every \( x \in B(x_i, \rho/2) \)

\[
(\varphi_\nu)(x) = (\varphi_\nu)(x_i) + \sum_{1 \leq |\alpha| \leq n-1} \frac{1}{\alpha!} \partial^\alpha (\varphi_\nu)(x_i)(x - x_i)^\alpha +
\]

\[
(3.2) \quad \sum_{|\alpha| = n} \frac{1}{(n - 1)!} \int_0^\tau t^{n-1} \partial^\alpha (\varphi_\nu)(x_i + t t_\theta) d\tau dt,
\]

where \( x = (x_1, ..., x_d), x_i = (x_1^i, ..., x_d^i), \alpha = (\alpha_1, ..., \alpha_d), x - x_i = (x_1 - x_1^i)^{\alpha_1} ... (x_d - x_d^i)^{\alpha_d}, \tau = ||x - x_i||, \theta = (x - x_i)/\tau \). By using the Sobolev embedding Theorem one can prove the following inequality

\[
(3.3) \quad |\partial^\alpha (\varphi_\nu)(x_i)| \leq C_{M, m} \sum_{|\rho| \leq m} \rho^{\mu + |\alpha| - d/2} \|\partial^\mu \alpha (\varphi_\nu, f)\|_{L^2(B(x_i, \rho))},
\]

where \( \mu = (\mu_1, \mu_2, ..., \mu_d), m > d/2 \). It allows the following estimation of the second term in (3.2).

\[
\int_{B(x_i, \rho/2)} \left| \sum_{1 \leq |\alpha| \leq n-1} \frac{1}{\alpha!} \partial^\alpha (\varphi_\nu)(x_i)(x - x_i)^\alpha \right|^2 dx \leq
\]

\[
C_{M, n} \sum_{|\gamma| \leq n+m-1} \rho^{2|\gamma|} \|\partial^\gamma (\varphi_\nu, f)\|_{L^2(B(x_i, \rho))}^2.
\]
Next, to estimate the third term in (3.2) we use the Schwartz inequality and the assumption $n > d/2$

\[
\left| \int_0^\tau t^{n-1} \partial^\alpha (\varphi_\nu f)(x_i + t\vartheta) \vartheta^\alpha dt \right|^2 \leq 
\left( \int_0^\tau t^{n-d/2-1/2} \left| t^{d/2-1/2} \partial^\alpha (\varphi_\nu f)(x_i + t\vartheta) \right| dt \right)^2 \leq 
C_{M,n} \tau^{2n-d} \int_0^\tau t^{d-1} \left| \partial^\alpha (\varphi_\nu f)(x_i + t\vartheta) \right|^2 dt.
\]

We integrate both sides of this inequality over the ball $B(x_i, \rho/2)$ using the spherical coordinate system $(\tau, \vartheta)$.

\[
\int_0^{\rho/2} \tau^{d-1} \left| \int_{|\vartheta|=1} t^{n-1} \partial^\alpha (\varphi_\nu f)(x_i + t\vartheta) \vartheta^\alpha dt \right|^2 d\vartheta d\tau \leq 
C_{M,n} \int_0^{\rho/2} t^{d-1} \left( \int_{|\vartheta|=1} \tau^{2n-d} \left| \partial^\alpha (\varphi_\nu f)(x_i + t\vartheta) \right|^2 \tau^{d-1} d\tau d\vartheta \right) dt \leq 
C_{M,n} \rho^{2n} \| \partial^\alpha (\varphi_\nu f) \|^2_{L^2(B(x_i, \rho))}, \quad \tau = \|x - x_i\| \leq \rho/2, \ |\alpha| = n.
\]

Next, for $n > d/2$ and $k = n + m - 1$,

\[
\| \varphi_\nu f \|^2_{L^2(B(x_i, \rho/2))} \leq C_1(M,k) \left( \rho^d \| f(x_i) \|^2 + \sum_{j=1}^{k} \sum_{1 \leq |\alpha| \leq j} \rho^{2|\alpha|} \| \partial^\alpha (\varphi_\nu f) \|^2_{L^2(B(x_i, \rho))} \right),
\]

where $k > d - 1$ since $n > d/2$ and $m > d/2$. Since balls $B(x_i, \rho/2)$ cover the manifold and the cover by $B(x_i, \rho)$ has a finite multiplicity $\leq N_M$ the summation over all balls gives

\[
\| f \|^2_{L^2(M)} \leq C_2(M,k) \left\{ \rho^d \left( \sum_{i=1}^{\infty} |f(x_i)|^2 \right) + \sum_{j=1}^{k} \rho^{2j} \| f \|^2_{H^j(M)} \right\}, \quad k > d - 1.
\]

Using this inequality and the regularity theorem for Laplace-Beltrami operator we obtain

\[
\| f \|_{L^2(M)} \leq 
C_3(M,k) \left\{ \rho^{d/2} \left( \sum_{i=1}^{\infty} |f(x_i)|^2 \right)^{1/2} + \sum_{j=1}^{k} \rho^{2j} \left( \| f \| + \| \Delta^{j/2} f \| \right) \right\}, \quad k > d - 1.
\]

For the self-adjoint $\Delta$ for any $a > 0, \rho > 0, 0 \leq j \leq k$ we have the following interpolation inequality

\[
\rho^j \| \Delta^{j/2} f \| \leq a^{2k-j} \rho^{2k} \| \Delta^k f \| + c_k a^{-j} \| f \|.
\]

Because in the last inequality we are free to choose any $a > 0$ we are coming to our main claim. \qed
3.2. Sampling sets for bandlimited functions and the upper estimate on the number of eigenvalues. Now, if a bandlimited function $f$ belongs to $E_{\omega}(\Delta)$ the Bernstein inequality implies

$$\rho^k\|\Delta^{k/2}f\|_{L_2(M)} \leq \left(\rho\omega^{1/2}\right)^k \|f\|_{L_2(M)}.$$  

If $C(M, k)$ is the same as in (3.1) and we pick a such $\rho$ for which

$$\rho = \gamma\omega^{-1/2}, \quad \gamma = \frac{(C(M, k))^{1/k}}{2} < 1,$$

we can move the second term on the right side in (3.1) to the left to obtain following Plancherel-Polya-type inequality which shows that in the spaces of $b$ and limited functions $\omega(\Delta)$ the regular $L_2(M)$ norm is controlled by a discrete one (in fact, they are equivalent).

**Theorem 3.2.** There exists a $0 < \gamma = \gamma(M) < 1$ and there exists a constant $C_1 = C_1(M)$ such that for any $\omega > 0$, every metric $\rho$-lattice $M_\rho = \{x_j\}$ with $\rho = \gamma\omega^{-1/2}$ the following inequality holds true

$$\|f\|_{L_2(M)} \leq C_1\rho^{d/2} \left(\sum_{x_j \in M_\rho} |f(x_j)|^2\right)^{1/2}$$

for all $f \in E_{\omega}(\Delta), \quad d = \dim M$.

**Corollary 3.1.** There exists a $0 < \gamma = \gamma(M) < 1$ such that for every $\omega > 0$ and every metric $\rho$-lattice $M_\rho = \{x_j\}$ with $\rho = \gamma\omega^{-1/2}$ the set $M_\rho = \{x_j\}$ is a sampling set for the space $E_{\omega}(\Delta)$.

In other words, every function $f \in E_{\omega}(\Delta)$ is uniquely determined by its values $\{f(x_j)\}$ and can be reconstructed from this set of values in a stable way.

Since dimension $\mathcal{N}_\omega$ of the space $E_{\omega}(\Delta)$ cannot be bigger than cardinality of a sampling set for this space we obtain the following statement.

**Corollary 3.2.** There exists a $0 < \gamma = \gamma(M) < 1$ such that for any $\omega > 0$

$$\mathcal{N}_\omega \leq \inf |M_{\gamma\omega^{-1/2}}|,$$

where $|M_{\gamma\omega^{-1/2}}|$ is the number of points in a lattice $M_{\gamma\omega^{-1/2}}$ and $\inf$ is taken over all such lattices.

4. The lower estimate

4.1. Kernels on compact Riemannian manifolds. Let $\sqrt{\Delta}$ be the positive square root of a second order differential elliptic self-adjoint nonnegative operator $\Delta$ in $L_2(M)$. For any measurable bounded function $F(\lambda), \quad \lambda \in (-\infty, \infty)$ and any $t > 0$ one defines a bounded operator $F(t\sqrt{\Delta})$ by the formula

$$F(t\sqrt{\Delta})f(x) = \int_M K^F_t(x, y)f(y)dy = \left\langle K^F_t(x, \cdot), f(\cdot)\right\rangle,$$

where $f \in L_2(M)$ and

$$K^F_t(x, y) = \sum_{l=0}^{\infty} F(t\sqrt{\lambda_l})u_l(x)\overline{u_l(y)} = K^F_t(y, x).$$
The function $K_F^t$ is known as the kernel of the operator $F(t\sqrt{\Delta})$.

We will need the following lemma.

**Lemma 4.1.** If $0 \leq F_1 \leq F_2$ and both of them are bounded and have sufficiently fast decay at infinity then $K_{t}^{F_1}(x, x) \leq K_{t}^{F_2}(x, x)$ for any $x \in M$ and $t > 0$.

**Proof.** Assume that $0 \leq F_1 \leq F_2$ and that both of them are bounded and have bounded supports. Clearly, $F_2 = F_1 + H$, where $H$ is not negative. By (4.2) we have

$$K_{t}^{F_2}(x, x) = K_{t}^{F_1}(x, x) + K_{t}^{H}(x, x)$$

where each term is non-negative. The lemma is proven.

We are going to make use of the heat kernel

$$p_t(x, y) = \sum_{l=0}^{\infty} e^{-\lambda_l t} u_l(x) u_l(y),$$

which is associated with the heat semigroup $e^{-t \Delta}$ generated by the self-adjoint operator $\Delta$:

$$e^{-t \Delta} f(x) = \int_M p_t(x, y) f(y) dy.$$ 

Note, that in notations (4.1), (4.2)

$$p_t(x, y) = K_{t}^F(x, y), \quad F(\lambda) = e^{-\lambda^2}, \quad e^{-t \Delta} = F(t \sqrt{\Delta}).$$

It is well known that in the case of a compact Riemannian manifold this kernel obeys the following short-time Gaussian estimates:

(4.3) \[ C_1 t^{-d/2} e^{-c_1 \text{dist}(x,y)^2} \leq p_t(x, y) \leq C_2 t^{-d/2} e^{-c_2 \text{dist}(x,y)^2} \]

where $0 < t < 1$, $d = \text{dim} M$ and every constant depends on $M$.

4.2. The lower estimate. We now sketch the proof of the opposite estimate by comparing $|M_{t^{-1/2}}|$ to the number of eigenvalues (counted with multiplicities) in the interval $[0, \omega]$.

Inequalities (4.3) in conjunction with (4.3) it gives for $0 < t < 1$

(4.4) \[ a_1 C_1 |B(x, t^{-1/2})| \leq p_t(x, x) = \sum_{l=0}^{\infty} e^{-\lambda_l^2 t} |u_l(x)|^2 \leq a_2 C_2 |B(x, t^{-1/2})|. \]

**Lemma 4.2.** There exist constants $A_1 = A_1(M) > 0$, $A_2 = A_2(M) > 0$ such that for all sufficiently large $s > 0$

(4.5) \[ \frac{A_1}{|B(x, s^{-1})|} \leq \sum_{l, \lambda_l \leq s} |u_l(x)|^2 \leq \frac{A_2}{|B(x, s^{-1})|}. \]

**Proof.** First, we note that using the right-hand side of (4.4), Lemma 4.1 and the inequality

$$\chi_{[0, s]}(\lambda) \leq e e^{-s^{-2} \lambda^2}$$

we obtain

$$\sum_{l, \lambda_l \leq s} |u_l(x)|^2 \leq e \sum_{l, \lambda_l \leq s} e^{-s^{-2} \lambda_l^2} |u_l(x)|^2 \leq$$
To prove the left-hand side of (4.5) consider the inequality
\[ e^{-t\lambda^2} = e^{-t\lambda^2} \chi[0, s] + \sum_{j \geq 0} \chi[2^j, 2^{j+1}, s](\lambda)e^{-t\lambda^2} \leq \chi[0, s] + \sum_{j \geq 0} \chi[0, 2^j, s](\lambda)e^{-t2^j\lambda^2}, \]
which implies
\[ p_t(x, x) \leq K_1^{X_0}(x, x) + \sum_{j > 0} e^{-t2^j\lambda^2} K_1^j(x, x), \]
where
\[ K_1^{X_0}(x, x) = \sum_{l, \lambda_i \leq s} |u_l(x)|^2, \]
\[ K_1^j(x, y) \text{ being the kernel of the operator } \chi[0, s] \left( \sqrt{A} \right) \text{ and } \]
\[ K_1^j(x, x) = \sum_{l, \lambda_i \leq 2^{j+1}} |u_l(x)|^2, \]
\[ K_1^j(x, y) \text{ being the kernel of the operator } \chi[0, 2^{j+1}](\sqrt{A}). \]
In conjunction with (4.3) it gives
\[ c_3|B(x, t^{-1/2})| \leq p_t(x, x) \leq K_1^{X_0}(x, x) + \sum_{j > 0} e^{-t2^j\lambda^2} K_1^j(x, x) = \]
\[ \sum_{l, \lambda_i \leq s} |u_l(x)|^2 + \sum_{j > 0} e^{-2^j\lambda s^2} \sum_{l, \lambda_i \leq 2^{j+1}} |u_l(x)|^2. \]
Note, that according to (2.4) if \( \rho > 1 \) and \( \rho s^{-1} \) is sufficiently small then
\[ |B(x, \rho s^{-1})| \leq c \rho^d|B(x, s^{-1})|, \quad d = \text{dim } M. \]
Next, given \( s \geq 1 \) and \( m \in \mathbb{N} \) we pick \( t \) such that
\[ s\sqrt{t} = 2^m. \]
The inequality (4.9) and the condition (4.10) imply
\[ \frac{(e^{2md})^{-1}}{|B(x, s^{-1})|} \leq \frac{1}{|B(x, 2^m s^{-1})|} \leq c_1|B(x, t^{-1/2})|, \quad m \in \mathbb{N}, \]
and
\[ \frac{1}{|B(x, 2^{-m-1}s^{-1})|} \leq \frac{c_2^{(m+1)d}}{|B(x, s^{-1})|}, \quad d = \text{dim } M. \]
Thus according to (4.11), (4.12) and (4.2) we obtain that for a certain constant \( c_2 = c_2(M) \)
\[ \frac{(e^{2md})^{-1}}{|B(x, s^{-1})|} \leq c_1|B(x, t^{-1/2})| \leq \]
\[ c_2 \left( \sum_{l, \lambda_i \leq s} |u_l(x)|^2 + \sum_{j > 0} e^{-2^j\lambda s^2} \sum_{l, \lambda_i \leq 2^{j+1}s} |u_l(x)|^2 \right) \leq \]
\[ \frac{A_2}{|B(x, s^{-1})|}, \quad A_2 = A_2(M). \]
(4.13) \[ c_2 \left( \sum_{l, \lambda_l \leq s} |u_l(x)|^2 + \sum_{j > 0} \frac{e^{-2j/2m}}{|B(x, 2^{-j-1}s^-1)|} \right). \]

Using (4.12), (4.10), and (4.12) we obtain that for a certain constant \( a = a(M) \)

\[ \frac{(e^{2^m})^{-1}}{|B(x, s^-1)|} \leq a \left( \sum_{l, \lambda_l \leq s} |u_l(x)|^2 + \sum_{j > 0} \frac{e^{-2j/2m}2^{j+1}}{|B(x, s^-1)|} \right) \]

\[ c_2 \left( \sum_{l, \lambda_l \leq s} |u_l(x)|^2 + \frac{2^d}{|B(x, s^-1)|} \sum_{j > 0} e^{-2j/2m}2^{jd} \right), \]

Since

\[ \frac{2^d}{|B(x, s^-1)|} \sum_{j > 0} e^{-2j/2m}2^{jd} \leq \frac{2^{d-2m}}{|B(x, s^-1)|} \sum_{j > 0} e^{-2j/2m}2^{(j+1)d} \leq \frac{2^{d-2m}}{|B(x, s^-1)|} \sum_{j > 0} e^{-2j/2m}2^{jd}, \]

one has that there are positive constants \( c_3, c_4 \) such that for all sufficiently large \( s \) and \( m \in \mathbb{N} \)

\[ \frac{2^{-md}}{|B(x, s^-1)|} \left( c_3 - c_4 \sum_{j > m} e^{-2j/2^{jd}} \right) \leq \sum_{l, \lambda_l \leq s} |u_l(x)|^2, \]

where expression in parentheses is positive for sufficiently large \( m \in \mathbb{N} \). It proves the left-hand side of (4.5).

We apply this lemma when \( t = \omega \) to obtain the following inequality for sufficiently large \( \omega \):

(4.14) \[ \frac{1}{|B(x, \omega^{-1/2})|} \leq c_5 p_\omega(x, x) \leq c_6 \sum_{l, \lambda_l \leq \omega} |u_l(x)|^2. \]

One has

\[ |M_{\omega^{-1/2}}| = \sum_{x_j \in M_{\omega^{-1/2}}} \frac{|B(x_j, \omega^{-1/2})|}{|B(x_j, \omega^{-1/2})|} = \sum_{x_j \in M_{\omega^{-1/2}}} \frac{1}{|B(x_j, \omega^{-1/2})|} \int_{B(x_j, \omega^{-1/2})} dx, \]

and thanks to (2.3) we also have

\[ \frac{1}{|B(x_j, \omega^{-1/2})|} \int_{B(x_j, \omega^{-1/2})} dx \leq \frac{a_2}{a_1} \int_{B(x_j, \omega^{-1/2})} \frac{dx}{|B(x, \omega^{-1/2})|}. \]

Now, the inequality (4.14) shows that for every sufficiently large \( \omega > 0 \) and every \( \omega^{-1/2}\)-lattice \( M_{\omega^{-1/2}} \) the following inequalities hold true

\[ |M_{\omega^{-1/2}}| \leq \frac{a_2}{a_1} \sum_{x_j \in M_{\omega^{-1/2}}} \int_{B(x_j, \omega^{-1/2})} \frac{dx}{|B(x, \omega^{-1/2})|} \leq \frac{a_2}{a_1} \int_M \frac{dx}{|B(x, \omega^{-1/2})|} \leq \frac{a_2}{a_1} \int_M \left( \sum_{l, \lambda_l \leq \omega} |u_l(x)|^2 \right) dx. \]
Since
\[
\int_{M} \left( \sum_{l, \lambda \leq \omega} |u_l(x)|^2 \right) dx = \sum_{l, \lambda \leq \omega} \int_{M} |u_l(x)|^2 dx = N_\omega.
\]
we receive the inequality
\[
|M_{\omega-1/2}| \leq c_7 N_\omega,
\]
for a certain \( c_7 = c_7(M) > 0 \). Thus there exists an \( a = a(M) > 0 \) such that
\[
a \sup |M_{\omega-1/2}| \leq N_\omega,
\]
where \( |M_{\omega-1/2}| \) is the number of points in a lattice \( M_{\omega-1/2} \) and \( \sup \) is taken over all such lattices. The inequalities (3.5) and (4.16) prove Theorem 1.1.

References

[1] D. Geller and I. Pesenson, Band-limited localized Parseval frames and Besov spaces on compact homogeneous manifolds, J. Geom. Anal. 21 (2011), no. 2, 334-371.
[2] L. Hörmander Hypoelliptic second order differential equations, Acta Math. 119 (1967), 147-171.
[3] I. Pesenson, A sampling theorem on homogeneous manifolds, Trans. Amer. Math. Soc. 352 (2000), no. 9, 4257-4269.
[4] I. Pesenson, An approach to spectral problems on Riemannian manifolds, Pacific J. Math. 215/1 (2004), 183-199.
[5] I. Pesenson, Poincare-type inequalities and reconstruction of Paley-Wiener functions on manifolds, J. of Geometric Analysis, 4, 1, (2004), 101-121.
[6] I. Z. Pesenson, Paley-Wiener approximations and multiscale approximations in Sobolev and Besov spaces on manifolds, J. Geom. Anal. 4/1 (2009), 101-121.

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