On a Class of Markov Order Estimators
Based on PPM and Other Universal Codes

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Abstract

We investigate a class of estimators of the Markov order for stationary ergodic processes which form a slight modification of the constructions by Merhav, Gutman, and Ziv in 1989 as well as by Ryabko, Astola, and Malyutov in 2006 and 2016. All the considered estimators compare the estimate of the entropy rate given by a universal code with the empirical conditional entropy of a string and return the order for which the two quantities are approximately equal. However, our modification, which we call universal Markov orders, satisfies a few attractive properties, not shown by the mentioned authors for their original constructions. Firstly, the universal Markov orders are almost surely consistent, without any restrictions. Secondly, they are upper bounded asymptotically by the logarithm of the string length divided by the entropy rate. Thirdly, if we choose the Prediction by Partial Matching (PPM) as the universal code then the number of distinct substrings of the length equal to the universal Markov order constitutes an upper bound for the block mutual information. Thus universal Markov orders can be also used indirectly for quantification of long memory for an ergodic process.

Keywords:
Markov order; empirical entropy; universal coding; Prediction by Partial Matching; mutual information

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1 Introduction

Throughout this paper, we denote sequences \( x^j_k = (x_i)_{j \leq i \leq k} \) over a finite alphabet \( X = \{a_1, a_2, \ldots, a_D\} \), where \( D \geq 2 \) and \( x^j_{j-1} = \lambda \) equals the empty string. For a stationary probability measure \( P \) on infinite sequences over alphabet \( X \) and random variables \( X_k^n(x_\infty^1) := x^k_1 \), we use the abridged notation \( P(x^n_1) := P(X^n_1 = x^n_1) \) and \( P(x^n_j|x^j_{j-1} = x^j_{j-1}) := P(X_j^n = x^n_j|X^j_{j-1} = x^j_{j-1}) \). For the stationary measure \( P \) as above, we also define the Markov order

\[
M^P := \inf \left\{ k \geq 0 : P(x^n_{k+1}|x^n_1) = \prod_{i=k+1}^n P(x_i|x_{i-1}^{i-1}) \text{ for all strings } x^n_1 \right\}.
\]

(1)

where the infimum of the empty set equals infinity, \( \inf \emptyset := \infty \). If the Markov order \( M^P = M \) is finite, measure \( P \) is called an \( M \)-th order Markov measure.

Several estimators of the Markov order for stationary ergodic measures were exhibited in the literature \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10]\). The goal of this paper is to investigate a class of simple consistent estimators of the Markov order based on universal codes—such as the prefix-free Kolmogorov complexity \([11, 12, 13]\), the Lempel-Ziv code \([14]\), Prediction by Partial Matching (PPM) \([15, 16]\), or one of many grammar-based codes \([17, 18, 19]\). In fact, our estimators are extremely close to the idea initially proposed by Merhav, Gutman, and Ziv \([1]\) and by Ryabko, Astola, and Malyutov \([5, 9]\) but, tampering with fine definition details, we are able to prove easily their almost sure consistency and a few other neat properties which we could not find in \([1, 5, 9]\).

As we have mentioned, the Markov order estimators constructed here are parameterized by universal codes. Simply speaking, our estimators compare the estimate of the entropy rate given by a universal code with the empirical conditional entropy of a string and return the order of the empirical conditional entropy for which the two quantities are roughly equal. Thus, the difference between our estimators and the constructions of \([1, 5, 9]\) is quite fine. Let \( h_k(x^n_1) \) be the empirical conditional entropy of order \( k \), to be formally defined in Section 2, and let \( LZ(x^n_1) \) be the length of the Lempel-Ziv code \([14]\). In Equation (14) of \([1]\) adjusted to our notation the Markov order estimator is defined as

\[
M^\lambda(x^n_1) := \min \left\{ k \geq 0 : h_k(x^n_1) \leq \frac{1}{n} LZ(x^n_1) + \lambda \right\}
\]

(2)

with a parameter \( \lambda > 0 \) fixed. Moreover, in \([1]\), we can find some bounds for the error probability \( P(M^\lambda(X^n_1) \neq k) \) under the hypothesis that \( M^P = k < \infty \). In contrast, in \([2, 9]\), the authors investigated testing the null hypothesis that the probability measure has a Markov order \( M^P \leq M < \infty \) versus the alternative \( M^P > M \). For that goal, they proposed the critical region defined by inequality

\[
(n - M)h_M(x^n_1) \leq LZ(x^n_1) + \log(1/\alpha),
\]

(3)

where \( \alpha \in (0, 1) \). It turned out that the type I error probability is upper bounded by \( \alpha \), whereas the type II error probability tends to 0 for \( n \) going to infinity.

In contrast, in this paper we are interested in the almost surely consistent estimation rather than in hypothesis testing. A respective example of our Markov
order estimators is
\[
M(x^n_1) := \min \left\{ k \geq 0 : (n - k) h_k(x^n_1) \leq LZ(x^n_1) + \log \frac{\pi^2}{6} + 2 \log(n + 1) \right\}.
\]

(4)

In contrast to the original ideas of \[1, 3, 9\], this cosmetic change allows to demonstrate almost sure consistency of the Markov order estimator (4) without much deliberation. In particular, we can substitute the length \(LZ(x^n_1)\) with the length of any universal code to obtain a whole class of strongly consistent estimators. Whereas the idea of using an arbitrary universal code to define critical region (3) was already postulated and proved true in \[3, 9\], we push it somewhat further to obtain a more elegant theory. Since the essential ideas of Markov order estimation based on universal codes have been proposed by \[1, 3, 9\], our merit lies mostly in the aesthetics of construction and proving a few more new properties besides the consistency. We deem that we give a final touch to propositions that may have circulated in the folklore.

We call the particular estimators introduced here universal Markov orders of a string. In the following, we will show that universal Markov orders enjoy several nice properties. To be concrete, in the subsequent three sections, using elementary methods, we will demonstrate the following results:

• In Section 2, we will first define the necessary concepts to introduce a formal definition of universal Markov orders, generalizing the definition of estimator (4). Secondly, we will prove that universal Markov orders are almost surely consistent—for any universal code and for any stationary ergodic measure. Since we will demonstrate the general consistency of universal Markov orders using the Barron lemma \[20, Theorem 3.1\], probably the length of the universal code in (4) cannot be substituted with the consistent estimators of entropy rate introduced in \[21, 22\].

• Section 3 is devoted to demonstrating that universal Markov orders are upper bounded asymptotically by \(h^{-1} \log n\) almost surely, \(h\) being the entropy rate. This result will be contrasted with a naive upper bound given by the maximal repetition length, which is asymptotically greater than \(h^{-1} \log n\) almost surely and whose behavior is better captured by the Rényi entropy rates \[23, 24, 25\].

• Finally, let the vocabulary size of a given order for a string—also called the subword complexity \[26\]—be the number of distinct substrings of the length equal to the order. According to Section 4 if we choose the PPM as the universal code then the vocabulary size of the universal Markov order constitutes an upper bound for the block mutual information. This result strengthens some similar propositions of \[27\] which find their applications in statistical language modeling \[19, 28, 29\]. The analogical results of \[27\] involve the vocabulary size of a larger order, equal to the Krichevsky-Trofimov Markov order estimator, proved to be inconsistent for the uniform measure by Csiszar and Shields \[2\]. In contrast, the universal Markov order applied in Section 4 is a consistent estimator as shown in Section 2 and is usually negligible compared to the vocabulary size because of the bound developed in Section 3. This remains in contrast with the unknown worst-case behavior of the Krichevsky-Trofimov estimator.
In this way, universal Markov orders can be successfully applied not only to consistent estimation of the Markov order for arbitrary stationary ergodic measures but also to other problems connected with quantification of long memory, compare [7]. We hope that these estimators may inspire further constructions. Especially interesting seems the extension to countably infinite alphabets, for which there are known consistent Markov order estimators within the class of Markov measures [4] but there are more general problems with the existence of universal codes [30, 31, 32].

2 Universal Markov orders

In this section, we will define universal Markov orders and we will establish their consistency. First, to fix the notation, let us recall the concepts of the empirical vocabulary, the empirical entropies, the true entropies, and universal coding. Let $X^* = \bigcup_{n \geq 0} X^n$ denote the set of strings of an arbitrary length including the empty string $\lambda \in X^0$. We define the frequency of a substring $w_{k+1}^k \in X^k$ in a string $x^n \in X^n$ where $0 \leq k \leq n$ as

$$N(w_{k+1}^k | x^n) := \sum_{i=1}^{n-k+1} 1 \{ x_i^{i+k-1} = w_{k+1}^k \}. \quad (5)$$

We use this notation also for the empty string $w_{0+1}^0 = \lambda$, where $N(\lambda | x^n) = n + 1$, according to the above definition. Subsequently, we define the empirical vocabulary of a string $x^n$ of order $k$ as

$$V_k(x^n) := \{ w_{k+1}^k \in X^k : N(w_{k+1}^k | x^n) > 0 \}. \quad (6)$$

We use this notation also for $k = 0$, where $V_0(x^n) = \{ \lambda \}$, according to the above definition. Let $\# A$ denote the cardinality of set $A$. The vocabulary size $\# V_k(x^n)$ is also called the subword complexity [26]. Using the empirical vocabulary, the empirical (conditional) entropy of a string $x^n$ of order $k \geq 0$ can be equivalently defined as

$$h_k(x^n) := \sum_{w_{k+1}^k \in V_{k+1}(x^n)} \frac{N(w_{k+1}^k | x^n)}{n-k} \log \frac{N(w_{k+1}^k | x^{n-1})}{N(w_{k+1}^k | x^n)}$$

$$= \frac{1}{n-k} \sum_{i=k+1}^{n} \log \frac{N(x_{i-k}^{i-1} | x^{n-1})}{N(x_{i-k}^{i-1} | x^n)}, \quad (7)$$

where $\log x$ stands for the binary logarithm of $x$.

We notice monotonicity of the empirical entropy.

**Theorem 1** We have $0 \leq h_k(x^n) - h_{k+1}(x^n) \leq \log D$.

**Proof:** Quantity $h_k(x^n) - h_{k+1}(x^n)$ is the conditional mutual information between two random variables, where each assumes $D$ distinct values. □

**Theorem 2** We have $0 \leq h_k(x^n) - \frac{n-k}{n} h_k(x_2^n) \leq \log \min \{2, D\}$.  

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Proof: Quantity $h_k(x^n_1) - \frac{n-1-k}{m-k} h_k(x^n_1) - \frac{1}{m-k} h_k(x^n_{k+1})$ is the conditional mutual information between two random variables, one of which assumes two distinct values and another assumes $D$ distinct values—compare with [27, Theorem A6], which only showed the left inequality. To obtain the claim we notice that $h_k(x^n_{k+1}) = 0$. □

Thus, by Theorems 1 and 2, quantity $(n-k)h_k(x^n_1)$ decreases with $k$ since
\[
(n-k)h_k(x^n_1) \geq (n-k-1)h_k(x^n_2) \geq (n-k-1)h_k(x^n_{k+1}).
\]

(8)

Extending the observation made in Theorem 2, we can also demonstrate superadditivity of the empirical entropy, which will be used in Section 4.

Theorem 3 For $0 \leq k < n, m - n < m$, we have inequality
\[
0 \leq h_k(x^n_1) - \frac{n-k}{m-k} h_k(x^n_1) - \frac{k}{m-k} h_k(x^n_{n-k}) - \frac{m-n-k}{m-k} h_k(x^n_{n+1}) \leq C
\]
where $C = \log \min \{3, D\}$.

Proof: The sandwich-bounded quantity is the conditional mutual information between two random variables, one of which assumes three distinct values and another assumes $D$ distinct values, see [27, Theorem A6]. □

The empirical entropies will be now contrasted with the true conditional entropies. For a stationary probability measure $P$ on infinite sequences over alphabet $X$, let us introduce the conditional entropies $h^P_k$ and the entropy rate $h^P$ defined as
\[
h^P_k := \mathbb{E} \left[ -\log P(X_i|X_{i-k}) \right],
\]
\[
h^P := \inf_{k \in \mathbb{N}} h^P_k = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ -\log P(X^n_i) \right].
\]

In the following assume that $P$ is additionally ergodic. Then by the Birkhoff ergodic theorem [33], we have
\[
\lim_{n \to \infty} h_k(X^n_i) = \lim_{n \to \infty} \frac{1}{n} \left[ -\log \prod_{i=k+1}^n P(X_i|X_{i-k}^{i-1}) \right] = h^P_k \text{ almost surely},
\]
whereas the Shannon-McMillan-Breiman theorem [34] yields
\[
\lim_{n \to \infty} \frac{1}{n} \left[ -\log P(X^n_i) \right] = h^P \text{ almost surely.}
\]

Subsequently, let us approach the problem of universal coding from a more abstract perspective inspired by the concept of algorithmic probability in algorithmic information theory [12]. A semi-distribution is a real function $\Pi$ of strings $w \in X^*$ which is of an arbitrary length (including the empty string) such that $\Pi(w) \geq 0$ and the Kraft inequality $\sum_{w \in X^*} \Pi(w) \leq 1$ is satisfied. Let us write the respective pointwise entropy as $\mathcal{H}(x^n_1) := -\log \Pi(x^n_1)$. The
semidistribution $\Pi$ is called universal if for any stationary ergodic probability measure $P$ on infinite sequences over a finite alphabet $X$ we have
\[
\lim_{n \to \infty} \frac{H(X^n)}{n} = h_P \text{ almost surely.} \tag{14}
\]
Examples of universal semidistributions are well known. In particular, the pointwise entropy $H(x^n)$ can be chosen as the prefix-free Kolmogorov complexity of string $x^n$ \cite{11, 12, 13}, which is uncomputable. In this case, $\Pi(x^n)$ equals approximately the algorithmic probability of $x^n$—by the coding theorem \cite{11, 12}. We stress that the prefix-free Kolmogorov complexity does not require this correction since its Kraft sum equals the halting probability $\Omega$, strictly less than one.

Now we have everything to define the main concept of this paper, i.e., the universal Markov orders—slightly modified with respect to the constructions in \cite{1, 5, 9}.

**Definition 1** Let $\Pi$ be a universal semidistribution and let $H(x^n) := -\log \Pi(x^n)$ be the respective pointwise entropy. The respective universal Markov order of a string $x^n$ is defined as
\[
M(x^n) := \min \{k \geq 0 : (n-k)h_k(x^n) \leq H(x^n)\}. \tag{15}
\]

Subsequently, we observe that the universal Markov order of a string is a consistent estimator of the Markov order of a stationary ergodic probability measure. This proposition complements and strengthens the results of \cite{1, 5, 9} discussed in Section 5.

**Theorem 4** For a stationary ergodic probability measure $P$ on infinite sequences over a finite alphabet and a universal Markov order $M$, we have
\[
\lim_{n \to \infty} M(X^n) = M \text{ almost surely.} \tag{16}
\]

**Proof:** Suppose first that $M^n = M$ is finite. By the non-negativity of the Kullback-Leibler divergence between the empirical distribution and the $k$-th order Markov approximation of a stationary measure $P$, we have
\[
(n - k)h_k(x^n) \leq -\log \prod_{i=k+1}^{n} P(x_i|x_{i-k-1}^{i-1}). \tag{17}
\]

Putting $k = M$, we obtain
\[
(n - M)h_M(x^n) \leq -\log P(x_{M+1}^n|x_1^M) \leq -\log P(x^n). \tag{18}
\]

On the other hand, by the Barron lemma \cite[Theorem 3.1]{20} for any semidistribution $\Pi$ and the respective pointwise entropy, we have
\[
\lim_{n \to \infty} [H(X^n) + \log P(X^n)] = \infty \text{ almost surely.} \tag{19}
\]
Hence almost surely, for sufficiently large \( n \), we obtain
\[
(n - M) h_M(X^n_1) \leq H(X^n_1).
\] (20)

In other words, for these \( n \), we have \( M(X^n_1) \leq M \).

Subsequently, assume an arbitrary \( M' \). By the definition of the Markov order we have \( h^P_k > h^P \) for \( k < M' \). Recall that we have (12) and (13). Hence, almost surely, for each \( k < M' \) and all sufficiently large \( n \), we obtain \( (n - k)h_k(X^n_1) > H(X^n_1) \). Thus \( M(X^n_1) > k \) since \( (n - k)h_k(X^n_1) \) is a decreasing function of \( k \). Combining this result with the observation made in the previous paragraph yields the claim. □

Paying another tribute to the algorithmic information theory, the Barron-lemma-like property (19) for \( H(x^n_1) \) being the prefix-free Kolmogorov complexity is well known to characterize equivalently, via the Schnorr theorem (12), the set of Martin-Löf random sequences. Of course, the probability of this set equals one. Using the effective Birkhoff ergodic theorem (35, 36), it can be shown in fact that convergence (16) holds on all Martin-Löf random sequences for any pointwise entropy function \( H(x^n_1) \) greater than the prefix-free Kolmogorov complexity if the universality condition (14) also holds on all Martin-Löf random sequences. That is, consistency of universal Markov orders for computable universal codes can be easily restated as a so-called effective law of probability—if these codes do not do something crazy on certain Martin-Löf random sequences of a total null measure. Such singular behavior for computable estimators is theoretically possible (Tomasz Steifer, private communication) but the widely used universal codes from (14, 15, 16, 17, 18, 19) decently satisfy universality condition (14) on all Martin-Löf random sequences since their universality rests on the Birkhoff ergodic theorem.

### 3 Three upper bounds

In this section, we will provide three simple upper bounds for universal Markov orders. We will begin with the simplest one. Namely, the better is the code, the larger is the respective universal Markov order. We mention this obvious behavior since it should be contrasted with results like Theorem (11) in Section 4 which conversely give the looser bound for the code-based mutual information when the code compresses better.

**Theorem 5** Consider universal semi-distributions \( \Pi_1 \) and \( \Pi_2 \) such that \( \Pi_1(x^n_1) \geq \Pi_2(x^n_1) \) or equivalently \( H_1(x^n_1) \leq H_2(x^n_1) \). Then \( M_1(x^n_1) \geq M_2(x^n_1) \).

**Proof:** Let \( k = M_1(x^n_1) \). Then \((n - k)h_k(x^n_1) \leq H_1(x^n_1) \leq H_2(x^n_1) \). Hence we obtain \( M_2(x^n_1) \leq k \). □

The second bound for universal Markov orders is the pessimistic upper bound in terms of the maximal repetition length. Let
\[
L(x^n_1) := \max \{ k \geq 0 : \#V_k(x^n_1) < n - k + 1 \}
\] (21)
be the maximal repetition length (20). Observe that \((n - k)h_k(x^n_1) = \log 2 = 1 \) and \( h_k(x^n_1) \leq \log D \) for \( k \leq L(x^n_1) \), whereas \( h_k(x^n_1) = 0 \) for \( k > L(x^n_1) \). Hence we have the following statement.
Theorem 6  For a universal Markov order $M$, we have inequality
\[ M(x^n) \leq L(x^n) + 1. \] (22)

Proof: We have $h_k(x^n) = 0$ for $k > L(x^n)$, whereas $H(x^n) > 0$. □

We will improve the above naive bound in a probabilistic setting. Let us recall two simple bounds for the maximal repetition length. First, since all substrings of $x^n$ of length $L(x^n) + 1$ must be distinct we observe inequality
\[ n - L(x^n) \leq D(L(x^n) + 1), \] whence we obtain the lower bound
\[ L(x^n) \geq \log_D [n - \log_D n] - 1. \] (23)

Moreover for any stationary ergodic measure $P$ on infinite sequences, we have
\[ \liminf_{n \to \infty} \frac{L(X^n)}{\log n} \geq \frac{1}{h_P} \text{ almost surely.} \] (24)

This inequality can be strict and the better bound for the left hand side is given by the inverse (conditional) Rényi entropy rate [23, 24, 25]. In contrast, we will see that universal Markov orders satisfy the converse inequality.

Prior to that, we will state two auxiliary statements. First, analogously to Theorem 2, we can prove
\[ 0 \leq h_k(x^n) - \frac{n-1-k}{n-k} h_k(x^{n-1}) \leq \log \min \{2, D\}, \] (25)
whence we derive
\[ h_l(x^{n+l}) \geq \frac{n-l}{n} h_l(x^n). \] (26)

This inequality can be chained with the subsequent proposition, which says that the infinite series of some empirical entropies is upper bounded.

Theorem 7  We have inequality
\[ \sum_{i=0}^{\infty} h_i(x^{n+i}) \leq \log n. \] (27)

Proof: Notice that
\[ \sum_{i=0}^{k} h_i(x^{n+i}) = \sum_{w_1^{k+1} \in V_{k+1}(x^{n+k})} \frac{N(w_1^{k+1}|x^{n+k})}{n} \log \frac{n}{N(w_1^{k+1}|x^{n+k})} \leq \log n. \] (28)

Taking $k \to \infty$ yields the claim. □

Now we can state the improved third bound for universal Markov orders, which is converse to inequality (24) for the maximal repetition length.

Theorem 8  For a stationary ergodic probability measure $P$ on infinite sequences over a finite alphabet and a universal Markov order $M$, we have
\[ \limsup_{n \to \infty} \frac{M(X^n)}{\log n} \leq \frac{1}{h_P} \text{ almost surely.} \] (29)

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Proof: Observe that \( (n-l) h_l(x^n_1) > H(x^n_1) \) for \( l < M(x^n_1) \). Hence by formulas \(26\) and \(27\), we obtain
\[
\log n \geq \sum_{l=0}^{\infty} h_l(x^n_1) \geq \sum_{l=0}^{M(x^n_1)-1} \frac{(n-l) h_l(x^n_1)}{n} > \frac{M(x^n_1) H(x^n_1)}{n} \tag{30}
\]
In consequence we obtain an upper bound for the universal Markov order,
\[
\frac{M(x^n_1)}{\log n} < \frac{n}{H(x^n_1)} \tag{31}
\]
To complete the proof, we invoke the universality, i.e., property \([14]\). □

As a corollary of Theorems\([3]\) and \([8]\) we obtain this proposition which asserts some convergence of empirical entropies.

**Theorem 9** Consider a stationary ergodic probability measure \( P \) on infinite sequences over a finite alphabet and a universal Markov order \( M \). Then for finite \( M^P \), we have
\[
\lim_{n \to \infty} h_{M(X^n_1)}(X^n_1) = \lim_{n \to \infty} h_{M(X^n_1)} = h^P \text{ almost surely,} \tag{32}
\]
whereas for \( M^P = \infty \) and \( h^P > 0 \), we have
\[
\lim_{n \to \infty} h_{M(X^n_1)-1}(X^n_1) = \lim_{n \to \infty} h_{M(X^n_1)-1} = h^P \text{ almost surely.} \tag{33}
\]

**Proof:** The right equalities in \((32)\) and \((33)\) follow by Theorem \([4]\). It remains to show the left equalities.

Suppose first that \( M^P = M \) is finite. Then for sufficiently large \( n \), we have \( M(X^n_1) = M \) almost surely and by the Birkhoff ergodic theorem, we obtain almost surely
\[
h_{M}(X^n_1) \to h^P_M = h^P. \tag{34}
\]
Thus we derive the left equality in \((32)\).

Subsequently assume that \( M^P = \infty \) and \( h^P > 0 \). Observe that by \((29)\) we have \( \lim_{n \to \infty} M(X^n_1)/n = 0 \) almost surely. Then for any \( k \) and sufficiently large \( n \), we have \( M(X^n_1) - 1 \geq k \) and consequently almost surely
\[
h_{M(X^n_1)-1}(X^n_1) \leq \frac{n-k}{n-M(X^n_1)} h_k(X^n_1) \to h^P_k, \tag{35}
\]
whereas by the universality, i.e., property \([14]\), we have almost surely
\[
h_{M(X^n_1)-1}(X^n_1) > \frac{H(X^n_1)}{n-M(X^n_1)+1} \to h^P. \tag{36}
\]
Hence we infer the left equality in \((33)\) since \( h^P = \inf_{k \in \mathbb{N}} h^P_k \). □
4 PPM code and mutual information

The utility of consistent Markov order estimators exceeds the problem of estimation of the Markov order since they can be fruitfully related to quantification of long memory in stochastic processes, see [1]. Strengthening some results of [27], in this section we will investigate a bound for the power-law growth of block mutual information, which characterizes some non-hidden Markov processes motivated by natural language phenomena [19, 28, 29, 27]. This bound for mutual information applies the concept of a universal Markov order for the universal semi-distribution of the PPM code [15, 16] and the respective vocabulary size. Similar results were obtained in [27] applying a larger empirical order, equal to the Krichevsky-Trofimov order, proved to be an inconsistent estimator of the Markov order for the uniform measure by [2]. Our results have a neat interpretation from the viewpoint of statistical language modeling, which we will comment on at the end of this section.

Following [15, 16, 27], the PPM measure of order $k \geq 0$ is defined as

$$\text{PPM}_k(x^n) := \prod_{i=1}^{n} \text{PPM}_k(x_i|x_i^{-1}), \quad (37)$$

where

$$\text{PPM}_k(x_i|x_i^{-1}) := \begin{cases} \frac{1}{D} & \text{if } k > i - 2, \\ \frac{N(x_{i-k}^i|x_i^{-1}) + 1}{N(x_{i-k}^i|x_i^{-1}) + D} & \text{else.} \end{cases} \quad (38)$$

As we can see, $\text{PPM}_k(x^n)$ is an estimator of the probability of block $x^n$ based on the Markov model of order $k$. We note that these Markov estimators are adaptive, i.e., the transition probabilities $\text{PPM}_k(x_i|x_i^{-1})$ are re-estimated given each new symbol $x_i$. Moreover, we notice that term $\text{PPM}_k(x_i|x_i^{-1})$ is a conditional probability distribution and thus $\text{PPM}_k(x^n)$ is a probability measure,

$$\sum_{x_i \in X} \text{PPM}_k(x_i|x_i^{-1}) = \sum_{x_i^n \in X^n} \text{PPM}_k(x_i^n) = 1. \quad (39)$$

If $k > n - 2$ then $\text{PPM}_k(x^n) = D^{-n}$. Else, by (38), we obtain

$$\text{PPM}_k(x^n) = D^{-k} \prod_{w_k \in V_k(x_i^n^{-1})} \frac{(D - 1)! \prod_{w_{k+1} \in X} N(w_{k+1}^1|x_i^n)!}{(N(w_{k}^1|x_i^n) + D - 1)!}. \quad (40)$$

Hence using the Stirling approximation, the PPM probability can be related to the empirical entropy and the empirical vocabulary of the respective string. In particular, by Theorem A4 in [27], we have

$$\alpha \leq - \log \text{PPM}_k(x^n) - k \log D - (n - k) h_k(x^n) \leq \log \lfloor e^2 n \rfloor, \quad (41)$$

where $\alpha := - \log (D^{-1})!$.

Subsequently, let us consider the PPM semi-distribution

$$\Pi(x^n) := \frac{6^2}{\pi^4} \frac{1}{(n + 1)^2} \sum_{k=0}^{\infty} \frac{\text{PPM}_k(x^n)}{(k + 1)^2}. \quad (42)$$
The series can be computed effectively since $\text{PPM}_k(x^n) = D^{-n}$ for $k > n - 2$. Since the PPM semi-distribution (42) is universal as a consequence of bound (41), see [15], we can consider the respective universal Markov order $M(x^n)$ given by (15) and compare it with the Krichevsky-Trofimov order defined as

$$K(x^n) := \min \left\{ k \geq 0 : \text{PPM}_k(x^n) = \max_{j \geq 0} \text{PPM}_j(x^n) \right\}. \quad (43)$$

Since $\text{PPM}_k(x^n) = D^{-n}$ for $k > n - 2$, the Krichevsky-Trofimov order was shown by [2] to be an inconsistent estimator of the Markov order for the uniform measure. Precisely, we have $\lim_{n \to \infty} K(X^n) = \infty$ almost surely for $P(x^n) = D^{-n}$, which has the Markov order $M = 0$.

This inconsistency result is quite intuitive also because the Krichevsky-Trofimov order is greater than the universal Markov order, which is consistent.

**Theorem 10** Let $\Pi$ be the PPM semi-distribution (42). We have

$$M(x^n) \leq K(x^n). \quad (44)$$

**Proof:** Let $k = K(x^n)$. By (11), we have

$$(n - k)h_k(x^n) < -\log \text{PPM}_k(x^n)$$

$$= -\log \max_{j \geq 0} \text{PPM}_j(x^n) \leq H(x^n) - 2 \log(n + 1) - \log \frac{\pi^2}{6}. \quad (45)$$

Hence $M(x^n) \leq k$. □

Let $I(x^n; x_{n+1}^m) := H(x^n) + H(x_{n+1}^m) - H(x^m)$ be the pointwise mutual information for a semi-distribution $\Pi$. For the PPM semi-distribution (42), let $V_M(x^m) := V_M(x^m_j(x_1^n))$ and $V_K(x^m) := V_K(x^m_j(x_1^n))$ be the respective vocabularies of the universal Markov order and the Krichevsky-Trofimov order. As shown in Theorem A7 in [27], stemming from bound (41), we have inequality

$$I(x^n; x_{n+1}^m) \leq K(x^n) \log D + 2 \left[ D \# V_K(x^m) + 2 \log \frac{\pi^2}{6} + 4 \right] \log[e^2 m], \quad (46)$$

which upper bounds the pointwise mutual information with the size of the Krichevsky-Trofimov order vocabulary. Applying the universal Markov order, this inequality can be strengthened as follows.

**Theorem 11** Let $\Pi$ be the PPM semi-distribution (42). Then for $0 \leq M(x^n) < n, m - n < m$, we have inequality

$$I(x^n; x_{n+1}^m) \leq 2 \left[ D \# V_M(x^m) + \frac{m \log D}{H(x^m)} + 2 \log \frac{\pi^2}{6} + 4 \right] \log[e^2 m]. \quad (47)$$

**Proof:** Let $k = M(x^n)$ and $C = \frac{\pi^2}{6}$. By (42), we obtain

$$H(x^n) \leq \log C + 2 \log(n + 1) + 2 \log(k + 1) - \log \text{PPM}_k(x^n). \quad (48)$$
In consequence, by inequalities (41) and (9) we obtain
\[
\begin{align*}
I(x_1^n; x_{n+1}^m) &= H(x_1^n) + H(x_{n}^{m}) - H(x_1^n) \\
&\leq 2 \log C + 2 \log(n+1) + 2 \log(m-n+1) + 4 \log(k+1) \\
&\quad - \log \text{PPM}_k(x_1^n) - \log \text{PPM}_k(x_{n+1}^m) - (m-k)h_k(x_1^n) \\
&\leq 2 \log C + 2 \log(n+1) + 2 \log(m-n+1) + 4 \log(k+1) + 2k \log D \\
&\quad + (n-k)h_k(x_1^n) + D \# V_k(x_1^n) \log [2^2n] \\
&\quad + (m-n-k)h_k(x_{n+1}^m) + D \# V_k(x_{n+1}^m) \log [2^2(m-n)] \\
&\quad - (m-k)h_k(x_1^n) \\
&\leq 2 \log C + 4 \log(m+1) + 4 \log(k+1) + 2k \log D \\
&\quad + 2D \# V_k(x_1^n) \log [2^2m] \\
&\leq 2 \left[ D \# V_k(x_1^n) + \frac{m \log D}{H(X_1^n)} + \log C + 4 \right] \log [2e^2m]. \\
\end{align*}
\]
(49)

since \(k \leq L(x_1^n) + 1 \leq m\) and \(k < \frac{m \log m}{H(x_1^n)}\) by inequalities (22) and (31). \(\square\)

A similar bound holds in expectation with no restriction on \(n\).

**Theorem 12** Let \(\Pi\) be the PPM semi-distribution (12) and let probability measure \(P\) be arbitrary. Then for any \(n \geq 1\), we have inequality
\[
\mathbb{E}I(X_1^n; X_{n+1}^m) \leq 2 \mathbb{E} \left[ D \# V_M(X_1^n) + \frac{4n \log D}{H(X_1^n)} + 4 \log \frac{\pi^2}{6} + 6 \right] \log [2e^2n].
\]
(50)

**Proof:** Let \(C = \frac{2^4}{\pi^2}\). Since \(\text{PPM}_k(x_1^n) = D^{-n}\) for \(k = n\) then for \(n \geq 1\) we obtain a uniform bound
\[
\log D \leq H(x_1^n) \leq \log C + 4 \log(n+1) + n \log D.
\]
(51)

Hence by Theorem 11 we have
\[
\mathbb{E}I(X_1^n; X_{n+1}^m) \leq 2 \mathbb{E} \left[ D \# V_M(X_1^n) + \frac{2n \log D}{H(X_1^n)} + \log C + 4 \right] \log [2e^2n] \\
+ 2 \left[ \log C + 2 \log(n+1) + n \log D \right] P(M(X_1^n) \geq n).
\]
(52)

But by the Markov inequality and inequality (31),
\[
nP(M(X_1^n) \geq n) \leq \mathbb{E} M(X_1^n) \leq \mathbb{E} \left( \frac{2n}{H(X_1^n)} \right) \log 2n.
\]
(53)

Hence we obtain the claim. \(\square\)

Theorems 11 and 12 complement and somewhat strengthen a series of theorems proved in [27], which allow to effectively upper bound the power-law growth of mutual information for two strings drawn from a stationary ergodic process in terms of the vocabulary size of a computable order. To capture the power-law growth of a real function \(s(n)\) of natural numbers, let us denote the Hilberg exponent
\[
hilb s(n) := \limsup_{n \to \infty} \frac{\log^+ s(n)}{\log n},
\]
(54)
where \( \log^+ x := \log(x + 1) \) for \( x \geq 0 \) and \( \log^+ x := 0 \) for \( x < 0 \) \cite{27}. We have, e.g., \( \text{hilb}_{n \to \infty} n^\beta = \beta \) for \( \beta \geq 0 \).

Subsequently, let \( H^P(x^n) := -\log P(x^n) \) be the true pointwise entropy, let \( H^K(x^n) \) be the prefix-free Kolmogorov complexity of \( x^n \), and let \( H(x^n) := -\log \Pi(x^n) \) for the PPM semi-distribution \cite{12}. Then for any stationary measure \( P \), we have

\[
\begin{align*}
\lim_{n \to \infty} \mathbb{E} [H^P(x^n) - nh^P] &= \lim_{n \to \infty} \mathbb{E} [H^K(x^n) - nh^P] = \lim_{n \to \infty} \mathbb{E} [I(x^n; X_{n+1})],
\end{align*}
\]

as shown in Theorem A1 in \cite{27}.

In the following, we can strengthen Theorem A8 of \cite{27}, which asserts that

\[
\lim_{n \to \infty} \mathbb{E} [I(x^n; X_{n+1})] \leq \lim_{n \to \infty} \mathbb{E} \left[ \frac{1}{2} K(x^n) + \# V_M(x^n) \right].
\]

Our strengthened proposition applies the universal Markov order vocabulary.

**Theorem 13** Consider a stationary probability measure \( P \) on infinite sequences over a finite alphabet. Let \( \Pi \) be the PPM semi-distribution \cite{12}. Denote the Rényi block entropy \( R^\alpha_n := -\log \mathbb{E} P(X_n^1) \). If the Rényi entropy rate is strictly positive, i.e., \( \inf_{n \geq 1} R^\alpha_n / n > 0 \) then

\[
\lim_{n \to \infty} \mathbb{E} [I(x^n; X_{n+1})] \leq \lim_{n \to \infty} \mathbb{E} \# V_M(x^n). \tag{56}
\]

**Proof:** The inequality follows by Theorem \[12\] and the lower bound in \[51\] since using the Markov inequality we can further bound

\[
\begin{align*}
\mathbb{E} \left( \frac{1}{H(X_1^n)} \right) &= \int_0^\infty P \left( \frac{1}{H(x^n)} \geq p \right) dp = \int_{\log D}^\infty P \left( H(x^n) \leq u \right) \frac{du}{u^2} \\
&\leq \int_{\log D}^\infty \min_{\alpha} \left[ P \left( \frac{2^{-H(x^n)}}{P(x^n)2^\alpha} \geq 1 \right) + P \left( \frac{P(x^n)^2 \alpha}{2^\alpha} \geq 1 \right) \right] \frac{du}{u^2} \\
&\leq \int_{\log D}^\infty \min_{\alpha} \left[ 2^{-\alpha} + \mathbb{E} P(X_1^n)2^{u+\alpha} \right] \wedge \frac{2du}{u^2} \\
&\leq 2 \int_{\log D}^R \mathbb{E} P(X_1^n)2^{u+1/2} \wedge \frac{du}{u^2} \\
&\leq 2 \int_0^{R_n^P} \frac{du}{(R_n^P)^2} + 2 \int_{R_n^P}^\infty \frac{du}{u^2} \leq \frac{4}{R_n^P} \tag{58}
\end{align*}
\]

if \( f(R_n^P) \geq f(\log D) \) for the convex function \( f(u) := 2^{u/2}/u^2 \). \( \square \)

Compared to the results of \cite{27} applying the vocabulary size of the Krichevsky-Trofimov order, Theorems \[11\] through \[13\] allow to take the vocabulary size of a smaller order, equal to the universal Markov order. This smaller order enjoys a neat interpretation of a consistent Markov order estimator and is usually negligible compared to the vocabulary size because of the upper bound \[55\]—in contrast to the unknown worst-case behavior of the Krichevsky-Trofimov order. Moreover, according to Theorem \[13\] and inequalities \[55\], if
the true mutual information for a stochastic source grows like a power law, which holds for certain non-finite-state processes \[19, 28, 29, 27\], then the vocabulary size of the universal Markov order must also grow like a power law. Analogous results were obtained not only for the PPM semi-distribution in \[27\] but, earlier, also for minimal grammar-based codes in \[19\], which unfortunately seem to be computationally intractable \[18\].

Let us also add that there is a hypothesis by Hilberg \[37\] that the mutual information for natural language grows like a power-law indeed, which can be further supported not only by large scale computational experiments \[38, 39, 40, 41\] but also by a reasoning linking semantics and non-ergodicity. This semantic interpretation of non-ergodicity provides a lower bound for the mutual information and, combined with propositions such as Theorem \[13\] it yields an intriguing claim that the number of independent timeless facts described by a random text must be roughly smaller than the number of distinct words in the text, see \[19, 27\] for the technical details. Since the number of distinct words grows roughly like a power of the text size for natural language, which is called Herdan’s or Heaps’ law \[42, 43\], one may hope that the number of independent timeless facts described by texts in natural language is also large. In particular, natural language cannot be not a finite-state process.

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