Quantum Lie algebra solitons

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Abstract. We construct a special type of quantum soliton solutions for quantized affine Toda models. The elements of the principal Heisenberg subalgebra in the affinised quantum Lie algebra are found. Their eigenoperators inside the quantized universal enveloping algebra for an affine Lie algebra are constructed to generate quantum soliton solutions.

1. Conformal and affine abelian Toda systems: classical region

1.1. Lie-algebraic setting
Let $M$ be a manifold, $\mathbb{R}^2$ or $\mathbb{C}^1$, with the standard coordinates $z_\pm$. For $\mathbb{C}^1$ we suppose that $z_- = z_+^*$. Let $G$ be a complex simple Lie group [25] of rank $r$ with the Lie algebra $\mathfrak{g}$ endowed with the principal grading. In the decomposition $G = \bigoplus_{m \in \mathbb{Z}} G_m$, the subspace $G_0$ is abelian. Denote by $G_0$ and $G_{\pm 1}$ the subspaces corresponding to $G_0$ and $\bigoplus_{m > 1} G_m$, respectively. Denote by $h_i$ and $x_{\pm i}$ Cartan and Chevalley generators of $\mathfrak{g}$. In the principal grading, the elements of $G_0$ and $G_{\pm 1}$, respectively, satisfy the defining relations

$$[h_i, h_j] = 0, \quad [h_i, x_{\pm j}] = \pm k_{ij} x_{\pm j}, \quad [x_{+i}, x_{-j}] = \delta_{ij} h_i, \quad 1 \leq i, j \leq r,$$

where $k$ is Cartan matrix of $\mathfrak{g}$.

1.2. Conformal Toda systems
Conformal abelian Toda fields $\phi = \sum_{i=1}^r h_i \phi_i$ satisfy the equations [26, 18]

$$\partial_+ \partial_- \phi + \frac{4\eta^2}{\beta} \sum_{i=1}^r m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta \alpha_i \phi} = 0,$$

with some coupling constant $\beta$ and a length scale factor $\eta$. The formal general solution [18, 17] to (2), found in holomorphically factorisable form, is given by

$$e^{-\beta \lambda_j \phi} = \langle \Lambda_j | \gamma_+^{-1} \mu_+^{-1} \mu_- \gamma_- | \Lambda_j \rangle,$$

where $\gamma_\pm (z_\pm)$ and $\mu_\pm (z_\pm)$ are holomorphic and antiholomorphic mappings $M \to G_0$, $M \to G_\pm$, respectively; $|\Lambda_i\rangle$ is the highest vector of the $i$-th fundamental representation of $G$. Mappings $\mu_\pm (z_\pm)$ satisfy the initial value problem

$$\partial_\pm \mu_\pm = \mu_\pm \kappa_\pm, \quad \kappa_\pm (z_\pm) = \sum_{i=1}^r \phi_i^0 x_{\pm i},$$

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where \( \kappa_{\pm} \) are mappings \( M \to \mathcal{G}_{\pm 1} \) that can be also represented as

\[
\kappa_{\pm}(z_{\pm}) = \gamma_{\pm}^{-1}E_{\pm 1} \gamma_{\pm}, \quad E_{\pm 1} = \sum_{i=1}^{r} m_i x_{\pm i},
\]

with some nonzero constants \( m_i \). Note that to obtain parametric solutions from (3), in particular instantons [17], it is pretty enough to take the screening functions \( \phi_{\pm i}^0(z_{\pm}) = c_{\pm i} z_{\pm}^{n_i} \) parametrized by constants \( c_{\pm i} \).

1.4. Solitonic specialisation on classical level

Affine Toda fields satisfy the equations

\[
\partial_+ \partial_- \phi + \frac{4 \eta^2}{\beta} \left( \sum_{i=1}^{r} m_i \frac{\alpha_i}{\alpha_i^+} e^{\beta \alpha_i \phi} - \frac{\psi}{\psi^2} e^{-\beta \psi \phi} \right) = 0.
\]

Here \( \partial_\pm \) stand for the partial derivatives with respect to \( z^\pm \), \( \eta \) conventionally denotes a real inverse length scale, and \( \beta \) is an imaginary coupling constant. The coefficients are arranged in such a way that \( \phi = 0 \) is a constant solution. The formal general solution to (6) was found in [21]

\[
e^{-\beta \lambda_i \phi} = e^{-\beta \lambda_i \phi^0} \frac{\langle \mu_j^1 \mu_j^1 - |A_j| \rangle}{\langle \mu_j^1 \mu_j^1 - |A_j| \rangle^{m_j}}, \quad 1 \leq i \leq r,
\]

with some quite clear different meaning of the ingredients here. Namely, \( \hat{\mathcal{G}} \) is an affine Kac–Moody Lie algebra [14] of rank \( r+1 \), \( k \) (in the relations of type (1)) is an affine (degenerated with the single zero eigenvalue) matrix. For more details on affine Lie algebras see [14]. Moreover, it is convenient to enlarge Cartan subalgebra of \( \hat{\mathcal{G}} \) by the derivative element \( d \), such that \( [d, h_i] = 0, [d, x_{\pm i}] = \pm x_{\pm i} \), and then completed Cartan subalgebra has dimension \( r + 2 \). Positive integers \( m_i \) in (7) are defined as the lowest for which \( \sum_i k_i m_i = 0 \). In terms of these integers the dual Coxeter numbers of \( \hat{\mathcal{G}} \) are written as \( h = \sum_{i=0}^{r} m_i \), while the centre of \( \hat{\mathcal{G}} \) is \( c = \sum_{i=0}^{r} m_i h_i \). Finally, \( |A_i| \) is the highest vector of the \( i \)-th fundamental representation of affine algebra \( \hat{\mathcal{G}} \). Note that, contrarily to those for finite dimensional systems, the general solution (7) has a rather complicated structure. In particular, it can be represented as infinite series, though absolutely convergent ones.

1.3. Affine Toda systems

Affine Toda fields satisfy the equations

\[
\partial_+ \partial_- \phi + \frac{4 \eta^2}{\beta} \left( \sum_{i=1}^{r} m_i \frac{\alpha_i}{\alpha_i^+} e^{\beta \alpha_i \phi} - \frac{\psi}{\psi^2} e^{-\beta \psi \phi} \right) = 0.
\]
with some constants \( \chi_i \) that belong to the root space of \( \tilde{\mathcal{G}} \). Then the generating elements for \( \hat{F}_n^i, \hat{F}_m^i := \hat{F}^i(t) = \sum_{n \in \mathbb{Z}} t^{-n} \hat{F}_n^i \) with a complex parameter \( t \), is an eigenvector of \( \hat{E}_m^i \),

\[
[\hat{E}_m^i, \hat{F}^i(t)] = \chi_i^{(\pm)} t^m \hat{F}^i(t).
\]

The simplest solitonic specialisation found in [22] consists in the following. Let \( \tilde{\mathcal{G}} \) be an infinite-dimensional group with the Lie algebra \( \tilde{\mathcal{G}} \). One chooses mappings \( \gamma_\pm \) in (5) to be unit elements of the subgroup \( \tilde{\mathcal{G}}_0 \) of \( \tilde{\mathcal{G}} \), and writes solutions to (4) as

\[
\mu_\pm = \mu_\pm^0 e^{\eta^\pm z_\pm \hat{E}_\pm^1},
\]

with some constants \( \eta^\pm \), and constant mapping \( \mu_0 \equiv (\mu_+^0)^{-1} \mu_-^0 : \mathbb{R} \to \mathcal{G} \) independent of the coordinates \( z_\pm \). The elements \( \hat{E}_\pm^1 \) can be removed from (7) if one chooses \( \mu_0 \) in the form

\[
\mu_0^0 = \prod_{i=1}^N Q_i e^{\hat{F}_i^1},
\]

where \( Q_i \in \mathbb{R} \) are some constants, and uses relations (8), namely that \([\hat{E}_{\pm 1}^i, \hat{F}_i^j(t)] = \chi_i^{(\pm)} t^m \hat{F}_i^j(t)\). Then (7) delivers an \( N \)-soliton solutions [5] to the affine Toda model characterized by the parameters \( Q_i \) and \( \chi_\pm^i, i = 1, \ldots, N \).

2. Quantum Lie algebras

2.1. The quantised universal enveloping algebra \( U_q(sl_2) \)

In the spirit of [4, 13], the quantised enveloping algebra \( U_q(sl_2) \) is an associative algebra generated by \( X^+, X^-, H \) with \( q \)-deformed commutation relations

\[
X^+ X^- - X^- X^+ = (q^H - q^{-H}) (q - q^{-1})^{-1}, \quad H X^\pm - X^\pm H = \pm 2X^\pm.
\]

It possesses a Hopf algebra structure with the deformed adjoint action

\[
(\text{ad}_{X^\pm})_q a = X^\pm a q^{H/2} - q^{\mp 1} q^{H/2} a X^\pm, \quad (\text{ad}_H)_q a = Ha - aH,
\]

for all \( a \in U_q(sl_2) \).

2.2. The quantum algebra \( (sl_2)_q \)

In [3, 8] a new deformed basis for the generators in the quantised universal enveloping algebra \( U_q(sl_2) \) was introduced

\[
X^\pm_h = \sqrt{2(q + q^{-1})^{-1}} q^{-H/2} X^\pm, \quad H_h = 2(q + q^{-1})^{-1} (q X^+ X^- - q^{-1} X^- X^+) , \quad (9)
\]

which form a three dimensional subspace \( (sl_2)_q \) generated by \( \{X^+_h, X^-_h, H_h\} \) in \( U_q(sl_2) \) closed under the quantum Lie bracket

\[
[a, b]_h := (\text{ad}_a)_q b, \quad a, b \in (sl_2)_q.
\]

The generators (9) possess the following commutation relations:

\[
[H_h, X^+_h]_h = \mp 2q^{-1} X^+_h, \quad [X^+_h, H_h]_h = \mp 2q^{-1} X^+_h, \quad [X^+_h, X^-_h]_h = H_h, \quad [X^-_h, X^+_h]_h = -H_h, \quad [H_h, H_h]_h = 2(q - q^{-1}) H_h, \quad [X^+_h, X^-_h]_h = 0.
\]

The algebra \( (sl_2)_q \) is not a Lie algebra in the standard sense. The generators (9) do not satisfy Jacobi identity and deformed Lie bracket (10) is not skew-symmetric. Although the \( q \)-analogue of Jacobi identity for \( (sl_2)_q \) is missing, nevertheless (10) is \( q \)-skew-symmetric in accordance with [8]. By skew-symmetry we mean a symmetry under \( q \)-conjugation (which we will denote with a tilde), the automorphism of \( (sl_2)_q \) defined by \( q \mapsto 1/q \). Then for an element in \( (sl_2)_q \)

\[
[a, b]_h = [\tilde{b}, \tilde{a}]_h, \quad a, b, c \in \mathbb{C},
\]

the \( q \)-deformed Lie bracket satisfies

\[
[a, b]_h = [\tilde{b}, \tilde{a}]_h.
\]
2.3. The affinisation of \((sl_2)_q\)
In this subsection we introduce the affinisation of the algebra \((sl_2)_q\). Denote \(G = (sl_2)_q\). Let \(L = \mathbb{C}[t, t^{-1}]\) be the algebra of Laurent polynomials in \(t\) and \(L(G) = L \otimes \mathbb{C} G\). Introduce the complex vector space \((sl_2)_q\); \(\tilde{L}(G) = L(G) \oplus \mathbb{C} \otimes \mathbb{C} d\). This is a loop algebra \(L(G)\) completed with the derivation \(d\) (acting as \(t \frac{d}{dt}\) in \(L\) and trivially on \(c\)) extended by one dimensional center \(c\) corresponding to \(\mathbb{C}\)-valued \(q\)-deformed 2-cocycle \(\Psi_q(a, b) = (xy)_h \Phi(P, Q), \Phi(P, Q) = Res_t \frac{dP}{dQ}\) on \(L(G)\). Here \((x|y)_h\) is a non-degenerate bilinear form on \((sl_2)_q\) and \(P, Q\) are polynomials in \(t\).
We define the \(q\)-deformed Lie bracket in this algebra as
\[
[t^m \otimes x \otimes \omega c \otimes \nu d, \ t^n \otimes y \otimes \omega_1 c \otimes \nu_1 d]_h = (t^{m+n} \otimes [x, y]_h + \nu mt^n \otimes y - \nu_1 mt^m \otimes x) \otimes m \delta_{m+n,0} (x|y)_h c,
\]
where \(x, y \in G, \nu, \omega, \nu_1, \omega_1 \in \mathbb{C}\). Now we introduce generators that form the affinisation of the quantum algebra \((sl_2)_q\)
\[
H_1 = 1 \otimes H_h, \ H_0 = 1 \otimes (c - H_h), \ e_1 = 1 \otimes X_h^+, \ e_0 = t \otimes X_h^-, \ f_1 = 1 \otimes X_h^-, \ f_0 = t^{-1} \otimes X_h^+.
\]
Then we derive the adjoint action:
\[
[H_0, e_0]_h = 2e_0q^{-1}, \ [H_0, f_0]_h = -2f_0q^{-1}, \ [H_0, e_1]_h = -qf_1, \ [H_0, f_1]_h = 2f_1q^{-1},
\]
\[
[H_1, e_1]_h = 2e_1q, \ [H_1, f_1]_h = -2f_1q^{-1}, \ [H_1, e_0]_h = 2e_0q^{-1}, \ [H_1, f_0]_h = 2f_0q,
\]
\[
[H_0, H_1]_h = -qH_1, \ [H_0, H_0]_h = 2(q - q^{-1})H_1,
\]
\[
[H_1, H_1]_h = qH_1, \ [H_1, H_0]_h = -qH_1,
\]
\[
[e_0, f_0]_h = H_0, \ [e_1, f_1]_h = H_1, \ [f_0, e_0]_h = -H_0, \ [f_1, e_1]_h = -H_1,
\]
\[
[e_0, H_0]_h = -2q, \ [e_0, H_1]_h = 2q, \ [e_1, H_0]_h = e_1q, \ [e_1, H_1]_h = -2e_1q,
\]
\[
[f_0, H_0]_h = 2q^{-1}f_0, \ [f_0, H_1]_h = -2f_0^{-1}f_0, \ [f_1, H_1]_h = 2qf_1, \ [f_1, H_0]_h = -2qf_1.
\]

2.4. The Heisenberg subalgebra and eigenvectors of the \(q\)-deformed adjoint action
Recall that the basic point in the construction of solitonic solutions [22] to the affine Toda equations (6) is the existence of eigenvectors with respect to elements of the Heisenberg subalgebra of underlying affine algebra. Here we define the elements of the principal Heiseberg subalgebra
\[
\hat{E}_h^+ = 1 \otimes X_h^+ + t \otimes X_h^-, \quad \hat{E}_h^- = 1 \otimes X_h^- + t^{-1} \otimes X_h^+,
\]
so that the following generating series for \(z \in \mathbb{C}\),
\[
F_q = \sum_{k=\pm}^{+\infty} \zeta^k A_k, \quad A_{2m} = (q + q^{-1})^m t^m \otimes (-H_h), \quad A_{2m+1} = (q + q^{-1})^m t^m \otimes q^{-1} X_h^+ - t^{m+1} \otimes q X_h^- + (X_h^- X_h^+)_h, \ m \in \mathbb{Z},
\]
is an eigenvector of \(\hat{E}_h^\pm\) with respect to the bracket (11) with eigenvalues \(z^{\pm1}\). Using \(F_q\) one can find a quantum analogue of solitonic solutions corresponding to the affinisation of the quantum algebra \((sl_2)_q\).

3. Quantum group solutions
3.1. Formal quantum group solutions
There are a few ways to quantize the conformal affine Toda models [6, 9, 10, 11, 12, 15, 16, 20]. In [20] the light cone quantization was performed, appropriate quantum equations and Lax
pairs found, and formal quantum solutions \([20, 6, 15, 16]\) constructed. Previously, in \([15, 16]\) it was shown that the formal quantum solution in terms of Heisenberg field operators and in the Yang–Feldman perturbative formalism for the quantized conformal affine Toda equations can be written in the form \((7), (3)\) but with ingredients replaced by their quantum analogues. Namely the group-like elements \(q^\gamma\), \(q^\mu\) of the quantized universal enveloping algebras \(U(G)\) in the Gauss decomposition \([2]\), and vectors \(|\Lambda\rangle_q\) in highest weight fundamental representations for corresponding quantum group \([1]\).

The quantised affine Toda Heisenberg field operators \((see [20] for the conformal Toda model)\) associated to an affine Lie algebra \(G\) satisfy the equations

\[
\partial_q \partial_{\gamma} \phi^{(q)} + \frac{4\gamma^2}{\beta} \left( \sum_{i=1}^{r} m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta \alpha_i \phi^{(q)}} - \frac{\psi^2}{q^2} : e^{-\beta \psi \phi^{(q)}} : \right) = 0.
\]

We can get quantum group deformed solutions to the affine Toda equations corresponding to \(U_q(G)\). Starting from the general solution \((7)\) to the affine Toda equations for \(G\) we replace the vectors of fundamental highest weight \(G\)-representations of by the fundamental highest weight representation vectors of \(U_q(G)\). One shows \([20]\) that the quantum Toda system possess the Heisenberg field operator solution of the form (dots denoting normal ordering) for \(1 \leq i \leq r\),

\[
e^{-\beta \lambda_j^{(q)} \phi^{(q)}} := e^{-\beta \lambda_j \phi^{(q)}} q \langle \Lambda_j | q \gamma_+^{-1} q \mu_+^{-1} q \mu_- q \gamma_- | \Lambda_j \rangle_q [q \langle \Lambda_0 | q \gamma_+^{-1} q \mu_+^{-1} q \mu_- q \gamma_- | \Lambda_0 \rangle_q]^{-m_j},
\]

where \(\phi^{(q)}\) are free field operators, and \(\partial_{\pm} q \mu_{\pm} = q \mu_{\pm} q \gamma_{\pm}, q \gamma_{\pm}, q \mu_{\pm}, \) and \(q \kappa_{\pm}\) are mappings \(M \rightarrow U(G)\).

3.2. Quantum group soliton solutions

Quantum solutions \([9, 10, 11]\) generated by quantum soliton operators \([24]\) and corresponding to classical soliton solutions \([22]\) can be obtained from the quantum formal solution \((13)\). For that purpose we put \(q \gamma_{\pm} = Id, \) i.e., the unit mapping, and let \(q \kappa_{\pm}(z_{\pm}) = q \gamma_{\pm}^{-1} q \bar{\gamma}_{\pm} q \gamma_{\pm},\) where \(q \bar{\gamma}_{\pm}\) are supposed to be generators of a Heisenberg subalgebra in \(U_q(G)\). Then \(q \mu_{\pm}\) are of the form

\[
q \mu_{\pm} = q \mu_{\pm}^0 e^{\eta_{\pm} \gamma_{\pm}} q \bar{\gamma}_{\pm}.
\]

By letting \(q \mu_{\pm}^0 = (q \mu_{\pm}^+)^{-1} q \mu_{\pm}^0 = e^{Q_F}\) for \(Q \in \mathbb{R}\), we therefore obtain from \((13)\)

\[
e^{-\beta \lambda_j \phi^{(q)}} := e^{-\beta \lambda_j \phi^{(q)}} q \langle \Lambda_j | e^{-q \bar{\gamma}_{\pm}^{-1} e^{Q_F} e^{\eta_{\pm} \gamma_{\pm}} | \Lambda_j \rangle_q [q \langle \Lambda_0 | e^{-q \bar{\gamma}_{\pm}^{-1} e^{Q_F} e^{\eta_{\pm} \gamma_{\pm}} | \Lambda_0 \rangle_q]^{-m_j}.
\]

Here \(|\Lambda_i\rangle_q\) denotes the highest vector in the \(i\)-th fundamental representation of \(U_q(G)\). Though the main fundamental problem remains unsolved, i.e., a suitable analogue for the principal Heisenberg subalgebra of \(U(G)\) is unknown, we are still able to deduce certain soliton solutions from \((14)\) by means of the algebraic considerations in subsection 2. To illustrate that we construct an example (for the case of \((sl_2)_q^+\)) in the next subsection.

3.3. Example: soliton solutions from quantum Lie algebra \((sl_2)_q^+\)

The case \(G = sl_2\) of the affine Toda system corresponds to the sine–Gordon equation \([17]\). We use the quantum Lie algebra constructions given in section 2 to generate quantum group soliton solutions based on \((sl_2)_q^+\). Starting from \((13)\), we put \(t \mu_{\pm} = t \mu_{\pm}^0 e^{\eta_{\pm} \gamma_{\pm}} q \bar{\gamma}_{\pm}, t \mu_{\pm}^0 = (t \mu_{\pm}^0)^{-1} t \mu_{\pm}^0 = e^{Q_F}\) for the group-like mappings associated with \((sl_2)_q^+\), and include the fundamental highest
where we have denoted $\Lambda_0$. Then we obtain for $j = 0, 1$

$$
: e^{-\beta \lambda_j \phi^{(q)}} := e^{-\beta \lambda_j \phi_0^{(q)}} : \frac{t}{q} (\Lambda_j) e^{E^+_h z + e Q^e H e^{-E^-_h z} | \Lambda_j \rangle | \Lambda_0 \rangle} = e^{-\beta \lambda_j \phi_0^{(q)}} : \frac{t}{q} (\Lambda_j) e^{E^+_h z + e Q^e H e^{-E^-_h z} | \Lambda_0 \rangle} \]^{-m_j},
$$
where we have denoted $F = \exp \left( Q e^{-2z+\zeta-2\zeta^{-1}} \right)$. Thus, for $j = 0, 1$ we have

$$
: e^{-\beta \lambda_0 \phi^{(q)}} := e^{-\beta \lambda_0 \phi_0^{(q)}} : \exp \left( -Q e^{-2z+\zeta-2\zeta^{-1}} \Xi(\zeta, t, q) \right),
$$

$$
\Xi(\zeta, t, q) = \sum_{m=-\infty}^{\infty} \zeta^{2m} t^m (q + q^{-1})^m.
$$

Finally, we would like to make several remarks. The construction of a quantum Lie algebra $(\mathcal{G})_h$ finds its extensions to all semisimple Lie algebras $\mathcal{G}$ in [8]. For cases (other than $sl_2$) of quantum Lie algebras associated to $\mathcal{G}$, one can also derive formulae for generators of their Heisenberg subalgebras. In these notes we only discussed the case of the principal grading [17] of $\mathcal{G}$. Similar constructions of Toda systems associated to the homogeneous and non-abelian [23] gradings of corresponding Lie algebras can be found. Conformal affine Toda systems have natural generalizations [7] which involve in the construction generators of higher grading subspaces of Lie algebras. Multi-solitonic solutions for the generalized Toda systems were found in [7]. Generalizations of the constructions mentioned in this paragraph for quantum Lie algebras will be given in a forthcoming publication.

### 4. Appendix: representations for $(sl_2)_h$

#### 4.1. A two-dimensional representation

In [3, 8] a two-dimensional $q$-representation of $(sl_2)_h$ was given in the form

$$
\pi(X^+_h) = \sqrt{(q + q^{-1})}/2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi(H_h) = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, \quad \pi(X^-_h) = \pi(X^+_h)^t,
$$

satisfying $\pi([a, b]_h) = \pi(a)\pi(b) - (\pi(b)\pi(a))$, $\pi(\tilde{a}) = \tilde{\pi}(a)$, with $\tilde{h} = -h$, $h \in sl_2$, $\tilde{q} = q^{-1}$. In [8] it is mentioned that there exist similar construction of $(sl_2)_h$-q-representations of any dimension.

#### 4.2. Fundamental highest weight representations of $(sl_2)_q$

The highest weight vector $v_0$ of the fundamental representation of the quantum Lie algebra $(sl_2)_q$ satisfies the conditions

$$
H_h v_0 = 0, \quad X^+_h v_0 = 0, \quad X^-_h v_0 = v_1.
$$

Thanks to definition (9) of generators $(sl_2)_h$ we have

$$
[H, X^+_h]_h = [H, X^+_h] = 2X^+_h, \quad [H, X^-_h]_h = [H, X^-_h] = -2X^-_h, \quad [H, H_h]_h = [H, H_h] = 0.
$$

The element $H$ lies in Cartan subalgebra of $(sl_2)_q$. Therefore

$$
H_h (H v_0) = (H v_0), \quad H v_0 = \lambda v_0.
$$
Using the definition (9) we find that
\[ H_h = 2(q + q^{-1})^{-1} (qX^+X^- - q^{-1}X^-X^+) = q^H (X^+_hX^-_h - X^-_hX^+_h), \]
and the action of \( H_h \) on \( v_0 \) in such a form gives
\[ q^H (X^+_hX^-_h - X^-_hX^+_h)v_0 = v_0. \]
Finally \( X^+_h v_1 = q^{-\xi}v_0 \), where \( \xi \) can be written in the following fashion
\[ 2(q + q^{-1})^{-1} (qX^+X^- - q^{-1}X^-X^+)v_0 = 2(q + q^{-1})^{-1}[H]v_0 = v_0, \]
\[ 2q(q^\xi - q^{-\xi})(q + q^{-1})^{-1}(q - q^{-1})^{-1} = 1. \]
This procedure can be recurrently continued for all \( v_n \) of the basis of the representation.

4.3. Fundamental highest weight representation for the affinised \((sl_2)_q\)
Above we introduced a quantum affine algebra \((sl_2)^f_q\) as an affinisation of \((sl_2)_q\). The highest weight vector of the \(i\)-th \((i = 1, 2)\) fundamental representation of \((sl_2)^f_q\) possesses the properties similar to the properties of the highest weight vector of fundamental representation of \((sl_2)_q\).

The actions of \( h_{1,2}, e_{1,2} \) and \( f_{1,2} \) generators on highest weight vectors \( v^{(1)}_0 \) and \( v^{(0)}_0 \) are given by
\[
\begin{align*}
 h_1 v^{(1)}_0 &= v^{(1)}_0, \\
 h_1 v^{(0)}_0 &= 0, \\
 h_0 v^{(1)}_0 &= 0, \\
 h_0 v^{(0)}_0 &= v^{(0)}_0, \\
 e_{11} v^{(1)}_0 &= e_{11} v^{(0)}_0 = 0, \\
 e_{11} v^{(1)}_0 &= e_{11} v^{(0)}_0, \\
 f_0 v^{(1)}_0 &= f_0 v^{(0)}_0 = 0, \\
 f_0 v^{(1)}_0 &= f_0 v^{(0)}_0 = v^{(1)}_0, \\
 f_1 v^{(1)}_0 &= f_1 v^{(0)}_0 = v^{(1)}_0, \\
 f_1 v^{(1)}_0 &= f_1 v^{(0)}_0 = v^{(1)}_0,
\end{align*}
\]
where superscripts correspond to representation and subscripts label the vectors of the basis. In the same way as for \((sl_2)_q\), we have
\[
\begin{align*}
 e_{11} v^{(0)}_1 &= q^{-H} v^{(0)}_1, \\
 e_{11} v^{(1)}_1 &= q^{-\lambda^{(1)}} v^{(1)}_0, \\
 h v^{(1)}_1 &= \lambda^{(1)} v^{(1)}_0, \\
 e_{11} v^{(0)}_1 &= 0, \quad h = 1 \otimes H, \\
 e_{01} v^{(0)}_1 &= q^{-H} v^{(0)}_1, \\
 e_{01} v^{(1)}_0 &= q^{-\lambda^{(0)}} v^{(0)}_0, \\
 h v^{(0)}_1 &= \lambda^{(0)} v^{(0)}_0, \\
 e_{01} v^{(1)}_0 &= 0, \\
 H_1 v^{(1)}_1 &= (1 - 2qq^{-\lambda^{(1)}-2})v^{(1)}_1, \\
 H_1 v^{(0)}_1 &= 2q^{-1}q^{2+\lambda^{(0)}-2} v^{(1)}_1.
\end{align*}
\]

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