Quantum Spin Dynamics (QSD) II

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Preprint HUTMP-96/B-352

Abstract

We continue here the analysis of the previous paper of the Wheeler-DeWitt constraint operator for four-dimensional, Lorentzian, non-perturbative, canonical vacuum quantum gravity in the continuum. In this paper we derive the complete kernel, as well as a physical inner product on it, for a non-symmetric version of the Wheeler-DeWitt operator. We then define a symmetric version of the Wheeler-DeWitt operator. For the Euclidean Wheeler-DeWitt operator as well as for the generator of the Wick transform from the Euclidean to the Lorentzian regime we prove existence of self-adjoint extensions and based on these we present a method of proof of self-adjoint extensions for the Lorentzian operator. Finally we comment on the status of the Wick rotation transform in the light of the present results.

1 Complete physical Hilbert space and observables

In this section we will compute the complete kernel of both the Diffeomorphism and the non-symmetric Euclidean and Lorentzian Hamiltonian constraint (for the symmetric Hamiltonian operator, see the next section). The kernel turns out to be spanned by distributions which do not only involve cylindrical functions which live on at most two-valent graphs or on graphs containing vertices with arbitrary valence but such that at each vertex the tangents of incident edges are co-planar. These solutions involve vertices of arbitrary valence and intersection characteristics, do take the curvature term \( F_{ab} \) of the classical constraint fully into account and are not necessarily annihilated by the volume operator. Also they are sensitive to whether they belong to the kernel of the Euclidean or Lorentzian Hamiltonian constraint. This space of distributional solutions inherits a natural inner product coming from \( \mathcal{H} \) via the group averaging method and it turns out that it coincides with the one given in [1]. Furthermore, we will define the notion of an observable and give explicit, non-trivial examples of those.

The key observation is the following:

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Consider the action of $\hat{H}^E(N)$ on a spin-network state $T_{\gamma,j,e}$ defined on a graph $\gamma$. Then $\hat{H}^E(N)T_{\gamma,j,e}$ can be written as a finite linear combination of spin-network states defined on graphs $\gamma_I$ where $\gamma \subset \gamma_I$ and $a_I := \gamma_I - \gamma$ is precisely one of the arcs $a_{ij}(\Delta)$. Moreover, $a_I$ carries spin $j_I = 1/2$ because the arcs $a_{ij}(\Delta)$ do not appear in $\gamma$ but they appear in $\hat{H}^E(N)$ through the fundamental representation of $SU(2)$. The arcs $a_I$ are special edges of $\gamma_I$ in the following sense.

**Definition 1.1** 1) A vertex $v$ of a graph $\gamma$ is called extraordinary provided that
   i) it is tri-valent,
   ii) it is the intersection of precisely two analytic curves $c, c' \subset \gamma$, that is, $v = c \cap c'$, such that $v$ is an endpoint of $c$ but not of $c'$.

2) An edge $e$ of a graph $\gamma$ is called extraordinary provided that
   i) its endpoints $v_1, v_2$ are both extraordinary vertices of $\gamma$,
   ii) there is an at least trivalent vertex $v$ of $\gamma$ which is such that at least three edges incident at it have linearly independent tangents at $v$ and there are two edges $s_1, s_2 \subset \gamma$ respectively which connect $v$ and $v_1, v_2$ respectively and which have linearly independent tangents at $v$. We will call $v$ the typical vertex associated with $e$.

In other words, if $e_1, e_2$ is the connected part of the intersection of the analytic extensions of $s_1, s_2$ with $\gamma$ that contains $s_1, s_2$ then $e_1 \cup e_2 \cup e$ looks like the graph pictured as $\forall$. It is easy to check that $a_I$ is an extraordinary edge for $\gamma_I$ and so a rough description of the action of $\hat{H}^E(N)$ is by saying that it admits a decomposition into spin-network states defined on graphs which differ by one extraordinary edge with spin $1/2$ compared to the original graph.

Next let us look at $\hat{K}$. Since $\hat{K} \propto [V, \hat{H}^E(1)]$ it follows that $\hat{K}$ has the same property. Finally, since $s_i(\Delta)$ are not extraordinary edges of a given graph $\gamma$ it follows that the action of $\hat{T}(N)$ can be described by saying that it admits a decomposition into spin-network states defined on graphs which differ by two, necessarily disjoint, extraordinary edges with spin $1/2$ compared to the original graph. This is because $\hat{T}(N)$ contains two factors of $\hat{K}$.

**Definition 1.2** i) A spin-net is a pair $w = (\gamma, \tilde{j})$ consisting of a graph $\gamma \in \Gamma$ and a colouring of the edges of $\gamma$ with spins $j > 0$ such that the set of vertex contractors compatible with the data $\gamma, \tilde{j}$ is not empty. We will denote the set of all spin-nets by $W$.

ii) The subset $W_0 \subset W$ is defined to be the set of all $(\gamma, \tilde{j}) \in W$ where $\gamma$ is a piecewise analytic graph all of whose extraordinary edges carry a spin $j > 1/2$. We also set $\overline{W_0} := W - W_0$.

iii) Given $w = (\gamma, \tilde{j}) \in W$ there exists a unique spin-net $w_0(w) = ((\gamma_0(\gamma), \tilde{j}_0(\tilde{j}))$, called the source of $w$ and which is defined by the subsequent algorithm:
First, let $\tilde{\gamma}$ be a copy of $\gamma$ which we dye in white.
If $w \notin W_0$ remove all the extraordinary edges $e$ of $\gamma$ which carry spin $1/2$ in $\gamma$ to obtain a graph $\gamma'$. Now, if $s_1, s_2$ are the segments of $\gamma$ which connect the extraordinary edge $e$ with its typical vertex then dye $s_1, s_2$ black in $\tilde{\gamma}$ (no matter which dye they had before) to produce $\tilde{\gamma}'$. Iterate the procedure with $\gamma', \tilde{\gamma}'$ instead of $\gamma, \tilde{\gamma}$. The procedure must come to an end after a finite number of steps because $\gamma$ had only a finite number of edges. The final $\gamma'$ is the searched for $\gamma_0(\gamma)$ which by construction
is unique. Its colouring $\vec{j}_0$ is obtained as follows: Each edge $e$ of $\gamma_0(\gamma)$ has a finite segment $s$ which is dyed in white in the final $\vec{\gamma}$ and which belongs to an edge $e'$ of $\gamma$. We define $\vec{j}_0(\vec{j})$ by requiring that the colour of $e$ is the same as that of $e'$. It is clear that the pair $(\gamma_0, \vec{j}_0)$ defines an element of $W_0$: it is an element of $W$ because the space of vertex contractors associated with a trivalent vertex as that given by the endpoints of an extraordinary edge is one-dimensional and that it lies in $W_0$ follows from the construction.

In order to characterize the complete set of solutions we need one more definition.

**Definition 1.3** a) Let $w_0 = (\gamma_0, \vec{j}_0) \in W_0$. We define inductively finite sets of spin-nets $w = (\gamma, \vec{j}) \in W_0$ with source $w_0$ as follows:

1) Let $W_0(0)(w_0) := \{w_0\}$.
2) Given $W_0(n)(w_0)$ take any $(\gamma, \vec{j}) \in W_0(0)(w_0)$ and construct elements $(\gamma', \vec{j}')$ of $W_0(n+1)(w_0)$ as follows: add precisely one more extraordinary edge $e$ to $\gamma$ in all possible, topologically inequivalent, ways. Furthermore, if $v$ is the typical vertex for $e$ and if $e_i = s_i, i = 1, 2, s_i, s'_i \neq \emptyset$ carries spin $j_i$ where $s_1, s_2$ connect $v$ to the endpoints of $e$ then we define up to four colourings for $\gamma \cup e$ as follows:

i) The extraordinary edge $e$ is coloured with spin $1/2$.

ii) $s'_i$ is coloured with spin $j_i$ as before.

iii) $s_i$ is coloured with spin $j'_i := j_i \pm 1/2$.

iv) The edges of $\gamma - \{e_1, e_2\}$ carry the same spin as in $\gamma$.

v) from the colourings of $\gamma \cup e$ so obtained we keep only those which admit a non-empty set of vertex contractors.

vi) Define $\gamma' := \gamma \cup e, (\gamma - s_1) \cup e, (\gamma - s_2) \cup e, (\gamma - s_1 - s_2) \cup e$ if $(j'_1, j'_2)$ is $(\neq 0, \neq 0), (0, \neq 0), (\neq 0, 0), (0, 0)$ respectively.

The set of data $(\gamma', \vec{j}')$ (at most four) for each $(\gamma, \vec{j})$ and for each $e$ extraordinary for $\gamma$ so obtained comprises the set $W_0(n+1)(w_0)$.

The finite set $W_0(n)(w_0)$ will be called the set of derived spin-nets of level $n$ with source $w_0$.

b) We will denote the associated set of equivalence classes of spin-nets under diffeomorphisms by $[W_0(n)(w_0)]$ which itself, of course, depends only on the equivalence class $[w_0]$ of $w_0$.

Notice that no graph involved in the derived spin-nets can get get disconnected because there must have been $n \geq 3$ edges involved at the typical vertex under question. Therefore the combination $j'_1 = j'_2 = 0$ can actually only occur for $n \geq 4$ because of condition a), v). It follows that we produce only vertices with minimal valence two but then at the next level this is not a typical vertex any longer.

It is therefore clear that for each $w \in W_0$ there is precisely one $n > 0$ such that $w \in W_0(n)(w_0(w))$. In other words, $W_0$ can be derived from $W_0$.

Finally, we recall the definition of diffeomorphism invariant state $\Phi$.

**Definition 1.4** i) Let $T_{\gamma, \vec{j}, \vec{c}}$ be a spin-network state. Its group average is defined by the following well-defined distribution on $\Phi$

$$T_{\gamma, \vec{j}, \vec{c}} := \sum_{\gamma' \in [\gamma]} T_{\gamma', \vec{j}, \vec{c}}$$

(1.1)
where \([\gamma]\) denotes the orbit of \(\gamma\) under smooth diffeomorphisms of \(\Sigma\) which preserve analyticity of \(\gamma\).

ii) The group average \([f]\) of any cylindrical function \(f\) is defined by first decomposing it into spin-network states and then averaging each of the spin-network states separately.

As shown in [9], the distributions of the form \(\Psi := [f]\) provide the general solution to the diffeomorphism constraint. Moreover one can show that

\[
T_{[\gamma],\vec{j},\vec{c}}(T_{\gamma',\vec{j}',\vec{c}'}) = \chi_{[\gamma]}(\gamma')\delta_{\vec{j},\vec{j}'}\delta_{\vec{c},\vec{c}'}
\]

(1.2)

where \(\chi\) denotes the characteristic function. We are now ready to define the complete set of simultaneous solutions to the Diffeomorphism constraint and to the non-symmetric Lorentzian Hamiltonian constraint as well as a physical inner product thereon.

**Theorem 1.1** Each distributional solution to all constraints of Lorentzian quantum gravity is a finite linear combination of states \(\Psi\) of the following two types:

**Type I)**

\(\Psi = [f]\) where \(f\) is an arbitrary linear combination of spin-network states based on spin-nets \(w_0 \in W_0\).

**Type II)**

\(\Psi = [f]\) where \(f\) is a certain linear combination of spin-network states which are constructed from spin-nets in \(W_0\). We will characterize this linear combination precisely in the course of the proof.

**Proof:**

Clearly both types of vectors solve the diffeomorphism constraint.

The basic observation is that if we have a spin-network state \(T_{\gamma,\vec{j},\vec{c}}\) then \(\hat{H}^E(N)T_{\gamma,\vec{j},\vec{c}}\) is a linear combination of spin-network states \(T_{\gamma',\vec{j}',\vec{c}'}, \gamma'\) has precisely one edge \(e\) more than \(\gamma\), moreover, \(e\) is extraordinary edge coloured with spin 1/2. Likewise, \(\hat{T}(N)T_{\gamma,\vec{j},\vec{c}}\) is a linear combination of such spin-network states where \(\gamma'\) has precisely two disjoint edges \(e, f\) more than \(\gamma\), where at least one of them, say \(e\), is an extraordinary edge for \(\gamma'\) coloured with spin 1/2 and where at least one of them, say \(f\), is an extraordinary edge for \(\gamma' - e\) coloured with spin 1/2. It follows that necessarily \(\hat{H}(N)T_{\gamma,\vec{j},\vec{c}}\) is a linear combination of spin-network states which are compatible with spin-nets \(w \in W_0\).

By definition of a solution of the Hamiltonian constraint we have to check that \(\Psi(\hat{H}(N)f) = 0\) for all lapses \(N\) and all cylindrical \(f\) which is clearly equivalent to showing that \(\Psi(\hat{H}(N)T_{\gamma,\vec{j},\vec{c}}) = 0\) for all \(N\) and all \(T_{\gamma,\vec{j},\vec{c}}\).

Now let first \(\Psi = [f]\) be of type I. The condition is trivially met because even if \(f\) contains a spin-network state \(T_{\gamma,\vec{j}',\vec{c}'}\) based on a graph \(\gamma'\) which is diffeomorphic to a graph \(\gamma\) where \(T_{\gamma,\vec{j},\vec{c}}\) is one of the spin-network states into which \(\hat{H}(N)T_{\gamma,\vec{j},\vec{c}}\) can be decomposed, the spin vectors \(\vec{j}',\vec{j}\) are necessarily different in at least one extraordinary edge which carries spin 1/2 in \(\gamma'\) but not in \(\gamma\) and so the inner product (1.2) vanishes. The solutions of type I are in a sense trivial because every operator which extends the graph of a function cylindrical with respect to it by edges of particular topology and spin value will have the same type of solutions.
Consider now solutions of type II. Let $f = \sum_T a_T^{(n)}([w_0])T$ where the sum extends over 1) all spin-nets $w \in W^{(n)}(w_0)$ for some $w_0 = (\gamma_0, \tilde{j}_0) \in W_0$ and some $n > 0$ and 2) over all spin-network states $T$ compatible with precisely one of these $w$ (we will call this set $S^{(n)}(w_0)$). Now, by the explicit expression of $\hat{H}(N)$ (\ref{5.3}), it follows that $\hat{H}^E(N)$ maps precisely all $T \in S^{(n-1)}(w_0)$ into linear combinations of spin-network states which are diffeomorphic with some of the elements $T' \in W^{(n)}(w_0)$ and no other spin-network states do have this property. Likewise, $\hat{T}(N)$ maps precisely all $T \in S^{(n-2)}(w_0)$ into linear combinations of spin-network states which are diffeomorphic with elements $T' \in W^{(n)}(w_0)$ and no other spin-network states do have this property. It follows that we have matrices $m_{[T],[T']}^{(n)}([w_0], [v])$ such that

$$\hat{H}^E(N)T = \sum_{v \in V(\gamma_0), T' \in S^{(n-1)}(w_0)} N_v m_{[T],[T']}^{(n)}([w_0], [v])T' \text{ for } T \in S^{(n-1)}(w_0)$$

$$\hat{T}(N)T = \sum_{v \in V(\gamma_0), T' \in S^{(n-2)}(w_0)} N_v m_{[T],[T']}^{(n)}([w_0], [v])T' \text{ for } T \in S^{(n-2)}(w_0).$$

Here we mean by $T'$ one of the representants of the diffeomorphism class of vectors into which $T$ is mapped. Notice that the matrices $m$ are diffeomorphism invariant which follows from the fact that they can only depend on the $\tilde{j}, \tilde{c}$ involved. It follows that $[f]$ is a solution if and only if

$$\sum_{T' \in S^{(n)}(w_0)} a_T^{(n)}([w_0]) m_{[T],[T']}^{(n)}([w_0], [v]) = 0 \quad \forall \, T \in S^{(n-1)}(w_0) \cup S^{(n-2)}(w_0), \, v \in V(\gamma_0).$$

(1.3)

This is the condition that we looked for.

Since the members of all the $S^{(n)}(w_0)$ for all $w_0$ obviously comprise all the spin-network states compatible with any $w \in W_0$ it follows that we have found the general solution. □

**Corollary 1.1** Every solution of the Lorentzian Hamiltonian constraint solves the Euclidean Hamiltonian constraint as well.

This follows obviously from the proof given above because the two parts $\hat{H}^E, \hat{T}$ of $\hat{H}$ need to vanish separately. It follows that Lorentzian solutions are rather special elements of the bigger set of Euclidean solutions.

A few remarks are in order:

0) Notice that the Diffeomorphism constraint moves the graph of a spin-network state but leaves the spin data $\tilde{j}, \tilde{c}$ invariant. On the other hand, the Hamiltonian constraint is only a condition on the spin-data. It is here where the dynamics is encoded. It is interesting that the two constraints effectively act on different, nicely split, labels of a spin-network state. The solutions of type II) are neatly labelled by the $[W^{(n)}(w_0)]$, that is by the diffeomorphism classes $[W_0]$ and by the number $n$, which can roughly be interpreted as the number of times that $\hat{H}^E(N)$ acts on an element $w_0$ of $W_0$.  

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1) Notice that if we wished to solve the Hamiltonian constraint before the diffeomorphism constraint then we could actually do so: Theorem 1.1 would still hold, we just need to drop the group averaging. Remarkably, the solutions $\psi$ are then not even distributional, they are elements of $\Phi$.

2) Let us then assume that we solve the Hamiltonian constraint before the diffeomorphism constraint. How do our solutions then compare with those found in the literature [19, 20]? The authors of those papers try to compute the kernel of $\hat{H}^E(N)$, that is, the space of solutions to the ordinary eigenvector equation $\hat{H}^E(N)\psi = 0$, albeit only for the Euclidean constraint. That is, the point $\lambda = 0$ of the spectrum is analyzed by treating it as a part of the point spectrum (that is, there exists an eigenvector, which, in particular, is square-integrable, with eigenvalue 0).

Now, although we do not have a complete proof, the fact that $\hat{H}^E(N)$ enlarges the graph of a cylindrical function that it acts on seems to exclude the possibility of a large enough kernel of $\hat{H}^E(N)$ when 0 is considered as a part of the point spectrum. In a sense it is very similar to trying to find the eigenvectors of the creation operator $\hat{a}^\dagger$ of the harmonic oscillator Hamiltonian. The only solution (0) is trivial. The only zero eigenvectors which we find in our approach seem to be related to the solutions found in [19, 20]: they are spanned by functions cylindrical with respect to any graph of arbitrarily high valence but such that the tangents of all edges incident at any of its vertices are co-planar. We conjecture that this is the complete kernel corresponding to the eigenvalue zero. It is clearly too small because these vectors are already annihilated by the volume operator, i.e. they do not take the curvature $F_{ab}$ (except for its anti-symmetry in $a,b$) into account and so are not specific for $\hat{H}^E(N), \hat{H}(N)$. On the other hand, they are the first known non-distributional rigorous solutions also for the Lorentzian Hamiltonian constraint in the continuum (the Lorentzian constraint defined on the lattice considered in [22] blows up on those states because this operator is only defined on states with finite volume). This is because both of $\hat{V}, \hat{H}^E$ and therefore also $\hat{K}$ annihilate such vectors.

Therefore one is naturally led to the viewpoint that 0 should not be considered as a part of the point spectrum: The point 0 of the spectrum is singled out in the sense that even though there maybe zero eigenvectors, they are clearly not in the range of $\hat{H}(N)$ (which is not the case for eigenvalues different from zero). So, neglecting the fact that 0 is an eigenvalue we may treat 0 as part of the residual spectrum, that is, the range of $\hat{H}(N)$ is not dense in $\mathcal{H}$ (notice that by the usual definition the point and residual spectra are automatically disjoint).

The kernel of $\hat{H}(N)$ should then be considered as a subspace of the Hilbert space dual of $\mathcal{H}$, that is we look for $\psi \in \mathcal{H'} = \mathcal{H}$ such that $\psi(\hat{H}(N)f) = 0$ for all $\psi \in \mathcal{H}$ and so we automatically capture the zero eigenvectors as solutions $\psi$. The last condition is equivalent to $\hat{H}(N)^\dagger \psi = 0$, in other words, treating 0 as part of the residual spectrum of $\hat{H}(N)$ is equivalent to treating it as part of the point spectrum of $\hat{H}(N)^\dagger$ in order to get the same kernel (recall that in general, at least for bounded operators $\hat{O}$, the residual spectrum of $\hat{O}$ and the point spectrum of $\hat{O}^\dagger$ coincide). Notice that it was precisely the fact that
the kernel of $\hat{H}(N)$ is not dense in $\mathcal{H}$ which was exploited in Theorem 1.1: since $\hat{H}(N)$ extends the graph of a spin-network state to one with vertices and edges of a special kind and colours its edges in a particular way, its range is not dense. Speaking again in terms of an analogy with the harmonic oscillator, the adjoint of the creation operator, the annihilation operator, has a rich point spectrum, the corresponding eigenvectors are even overcomplete.

**Definition 1.5** i) Consider the vector space of solutions $\mathcal{V} \subset \Phi'$ and complete it with respect to the inner product defined (and extended by sesquilinearity) by

$$<f, f'>_{\text{phys}} := [f](f').$$

where $f, f'$ are any to cylindrical functions. The resulting Hilbert space is called the physical Hilbert space $H_{\text{phys}}$.

ii) An observable $[\hat{O}]$ is defined to be a self-adjoint operator on $H_{\text{phys}}$, densely defined on $\mathcal{V}$. Alternatively, it is a self-adjoint operator densely defined on $\Phi$ such that its extension to $\Phi'$ leaves $\mathcal{V}$ invariant.

A trivial example of an observable is the projector to the type I solutions. That is, viewed as an operator on $\mathcal{H}$ we define for any function $f$ cylindrical with respect to a spin-net $w = (\gamma, \vec{j})$ that $\hat{O}f = 0$ if $w \notin W_0$ and $\hat{O}f = f$ otherwise. $\hat{O}$ is therefore densely defined and it is easy to see that it is self-adjoint. It preserves solutions because if $\psi(\hat{H}(N)f) = 0$ for all $f$ then clearly $\psi(\hat{H}(N)\hat{O}f) = 0$ and since $\hat{O}\hat{H}(N)f = 0$, trivially $\psi(\hat{H}(N)(\hat{O}f) = 0$. An integral kernel representation of $\hat{O}$ is given by $\hat{O}(A, B) = \sum_T T(A)\bar{T}(B)$ where the sum is over all spin-network states compatible with respect to webs in $W_0$. Viewed as an operator defined on $\mathcal{V}$ we merely need to rearrange the last sum and collect sums over diffeomorphic graphs into the group average.

### 2 Method to compute $a_{[T]}^{(n)}([w_0])$

The precise computation of the coefficients $a_{[T]}^{(n)}(w_0)$ is straightforward but rather tedious. We will lay here the computational foundations of an efficient computer code to obtain them. The details of the method are identical to those displayed in [24, 12] and will not be repeated here.

We consider the matrix elements of the volume operator on extended spin-network functions as known through (24). By an extended spin-network function we mean a function of the form $T_{\gamma, \vec{j}, \vec{c}}$ as before, the difference being that each $c_v$ of $\vec{c} = (c_v)_{v \in V(\gamma)}$ now maybe a projector on a non-trivial irreducible representation of $SU(2)$, that is, the state is not gauge invariant.

Let $T_{\gamma, \vec{j}, \vec{c}}$ be an extended spin-network function. The operators $\hat{H}^E(N), \hat{T}(N)$ are gauge invariant but in applying the volume operator involved in them we need extended spin-networks. Consider first $\hat{H}^E_v$ which contains terms of the form $\text{tr}([h_\alpha - h^{-1}_\alpha]h^{-1}_s\hat{V}h_s)$ where $s$ is a segment of an edge $e$ of $\gamma$ starting at $v$ and $\alpha$ is a loop based at $v$ of the form $s' \circ \alpha \circ (s'')^{-1}$ where also $s', s''$ are segments of edges $e', e''$ of $\gamma$ incident at $v$. In order to compute the action of $\hat{V}$ on $(h_s)_{AB}T$ we need to write
depends on the arc \( \alpha \) respectively while the spin of \( s \) is still \( j_e \). So we know how to compute the actions of \( \hat{V}, \hat{H}^E(N) \) and therefore of \( \hat{K} \). Finally, in order to compute the action of \( \hat{T}(N) \) we have to first apply the Clebsh-Gordan decomposition to \( h_s T \) and then evaluate \( \hat{K} \) and so forth. Detailed examples of such a computation will be subject to future publications \([21]\).

3 The symmetric operator

This section is devoted to a detailed analysis of a symmetric version of the Wheeler-DeWitt operator. The definition of such an operator turns out to be a very hard task and the discussion will reveal how tightly the issues of anomaly-freeness, symmetry and the choice of a regularization are interrelated.

3.1 Motivation

We argued that the kernel of the non-symmetric operator \( \hat{H}(N) \), when viewing 0 as an element of the point spectrum, and which consists of cylindrical functions on graphs which are such that the tangents of edges incident at a vertex are co-planar for each vertex, is too small. One might think that this kernel is incomplete since we stuck with square integrable eigenvectors and that it can be enlarged by allowing for more general, distributional solutions \( \psi \in \Phi' \) to \( \hat{H}(N) \psi = 0 \forall N \). In this case, as outlined in sections 2 and 6 of \([1]\) we would like to solve the Hamiltonian constraint before the diffeomorphism constraint. We will now see that even so the triangulation prescription made for the non-symmetric operator seems to allow only for trivial distributional solutions to the Hamiltonian constraint.

The problem already occurs at the level of only the Euclidean Hamiltonian constraint so let us focus our attention only on \( \hat{H}^E(N) \). Let us try to solve the constraint for graphs with valence higher than two. Consider a function \( f \) cylindrical with respect to a graph \( \gamma \) and let \( v \) be a non-trivial (in the sense specified above) vertex of \( \gamma \) with valence three to begin with. Writing out \( \hat{H}^E_v \) in explicit form we have

\[
-3i\ell^2 \hat{H}^E_v f = \text{tr}(\{h_{a12}(\Delta) - h_{a12}^{-1}(\Delta)\} h_{s3}(\Delta) \cdot [h_{s3}(\Delta), \hat{V}]) f + \text{cyclic.} \tag{3.1}
\]

Specifically, let \( f := T_{j_1,j_2,j_3} \) be a spin-network state where the edge \( e_i \) incident at \( v \) carries spin \( j_i > 0 \) (\( s_i(\Delta) \) is a segment of \( e_i \)). It is obvious how the expansion of the right hand side of (3.1) in terms of spin-network states looks like : it is a sum of up to twelve terms : the first four are defined on the graph \( \gamma \cup a_{12}(\Delta) \) where \( a_{12}(\Delta) \) carries spin \( 1/2 \), \( s_1(\Delta) \) and \( s_2(\Delta) \) carry spin \( j_1 \pm 1/2 \) and \( j_2 \pm 1/2 \) respectively while the rest of \( e_1, e_2 \) given by \( s_1(\Delta)^{-1} \circ e_1, s_2(\Delta)^{-1} \circ e_2 \) carries still spin \( j_1, j_2 \) and \( e_3 \) is unchanged and carries spin \( j_3 \). Analogous descriptions hold for
the other two combinations 23, 31. So we see that the original graph got extended.

An ansatz for \( \psi \) consisting of an infinite sum of spin-networks defined on \( \gamma \), that is, \( \psi = \sum_{j_1,j_2,j_3} a(j_1,j_2,j_3) T_{j_1,j_2,j_3} \) does not work for \( \psi \) to be in the kernel which can be seen as follows: First of all, each of the three terms in (3.1) produces a topologically distinct graph so in order for \( \psi \) to vanish each of the three infinite sums corresponding to these three distinct graphs has to vanish separately because spin-network states defined on different graphs are orthogonal. Next, notice that the spins of the “top part” of the edges \( e_1, e_2, e_3 \) are unchanged, therefore actually each \( \hat{H}^E T_{j_1,j_2,j_3} \) has to vanish separately because spin-network states on the same graph but with different spins are orthogonal. That means that the values of the coefficients \( a(j_1,j_2,j_3) \) are completely irrelevant. Finally, each of the twelve terms in the expansion of (3.1) has to vanish separately for the same reason. But one can explicitly check that the coefficients of that expansion are not all simultaneously vanishing. So \( \psi \) is not in the kernel unless \( \psi = 0 \).

The first impulse is that the situation might be improved by choosing \( \psi \) to be diffeomorphism invariant, that is, we take \( \psi := \sum_{j_1,j_2,j_3} a(j_1,j_2,j_3) T_{j_1,j_2,j_3} \) where the bracket indicates that we sum over all spin-network states defined on the set of graphs defined by the orbit of \( \gamma \) under diffeomorphisms \( \Delta \) but with the same spins, as in definition 14. However, one readily sees that this does not help either, again, because of the fact that spin-network states defined on distinct graphs are orthogonal and because if \( \gamma, \phi(\gamma) \) are distinct (\( \phi \) some diffeomorphism of \( \Sigma \)) then \( \gamma \cup \Delta(\gamma), \phi(\gamma) \cup \Delta(\phi(\gamma)) \) are still distinct irrespective of the choice of the assignment \( \Delta(\gamma) \). So diffeomorphism invariance does not help.

The second impulse is that then we should make a more general ansatz for \( \psi \) including infinite sums of spin-network states defined on different graphs not necessarily connected by a diffeomorphism. The simplest guess is to start with two graphs each of them of the form \( \gamma_{ij} = \gamma \cup \alpha_{ij}(\Delta) \) for two distinct choices of \( (ij) \), say (12), (23) and to hope that the terms coming from appending \( a_{23}(\gamma_{12}) \) to \( \gamma_{12} \) and vice versa cancel each other. But that also fails to be true because in appending \( a_{23}(\gamma_{12}) \) the point \( a_{23}(\gamma_{12}) \cap e_2 \) lies topologically closer to \( v \) than \( a_{12}(\gamma) \cap e_2 \) while in appending \( a_{12}(\gamma_{23}) \) the point \( a_{12}(\gamma_{23}) \cap e_2 \) lies topologically closer to \( v \) than \( a_{23}(\gamma) \cap e_2 \). So the resulting graphs are topologically different and the corresponding functions cannot cancel each other.

Obviously the situation does not improve by considering adding even more graphs or by increasing the valence of \( v \). Finally, also considering the full Hamiltonian \( \hat{H}(N) \) rather than only \( \hat{H}^E(N) \) does not help because \( \hat{T}(N) \) contains two factors of \( \hat{K} \) and therefore introduces even more extraordinary three-valent vertices so that there are no cancellations with terms coming from \( \hat{H}^E(N) \) possible.

So it seems that \( \hat{H}(N) \) does not have a bigger space of solutions than the one outlined above and we are naturally led again to consider 0 not as an element of the point spectrum but as a point of the residual spectrum of \( \hat{H}(N) \) (equivalently, as a point of the point spectrum of \( \hat{H}(N)^\dagger \)).

A different factor ordering of the expression of the constraint does not help to expand the kernel of \( \hat{H}(N) \) because the reason of failure to find generalized zero eigenvectors of \( \hat{H}(N) \) does not have to do with the factor ordering, it has to do with the choice of loop-assignment so that it seems that the only way out is to modify the triangulation, the only freedom that we have not exploited yet.
It turns out that the requirement of having a symmetric operator, which is attractive because it removes the quantization ambiguity of whether to choose $\hat{H}(N)$ or $\hat{H}(N)\dagger$ as the constraint, forces us to modify the loop assignment and at the same time enables us to enlarge the (distributional) kernel. We will see that the obstacle to find a symmetric operator is the same as the one that we encountered above: it is the fact that the repeated action of the Hamiltonian constraint enlarges the graph of a cylindrical function without limit.

### 3.2 The symmetric Euclidean Operator

We will prove only those properties of the symmetric operators which are not shared by the non-symmetric ones. The reader can convince himself that the proofs of cylindrical consistency, diffeomorphism covariance and anomaly-freeness as given in the previous paper are entirely unaffected by the modifications introduced in the subsequent subsections.

#### 3.2.1 Symmetry

We still did not show that, with the symmetric version of definition (3.10), $\hat{H}_E^E\gamma$ qualifies as a projection from $\mathcal{H}$ to $\text{Cyl}_\gamma(\mathcal{A}/G)$ of a symmetric operator $\hat{H}_E$ on $\mathcal{H}$. To see the source of the obstruction, observe that if $\hat{H}$ is any self-consistent operator on $\mathcal{H}$ and if $\gamma, g_{\gamma'}$ are two cylindrical functions then we have

\[
< g_{\gamma'}, \hat{H} f_{\gamma} > = < g_{\gamma'}, \hat{H}_{\gamma} f_{\gamma} > = < (\hat{H}_{\gamma})\dagger g_{\gamma'}, f_{\gamma} > = < \hat{H}_{\gamma}\dagger g_{\gamma'}, f_{\gamma} > .
\]

(3.2)

It is important to realize that both adjoint operations involved in (3.2) are with respect to $\mathcal{H}$ and not with respect to the completion $\mathcal{H}_\gamma$ of $\text{Cyl}_\gamma^3(\mathcal{A}/G)$ with respect to the projected measures $\mu_{0,\gamma}$, see section 2 of [1].

Replacing $\hat{H}$ by $\hat{H}_E$ and using its (anticipated) symmetry as well as the one of its projections on $\mathcal{H}$ we find that a necessary and sufficient criterion for $(\hat{H}_E\gamma)^\dagger = \hat{H}_E\gamma$ is

\[
< g_{\gamma'}, \hat{H}_E^E f_{\gamma} >= < \hat{H}_E^{E}\gamma g_{\gamma'}, f_{\gamma} > .
\]

(3.3)

We now will demonstrate that the definition of the triangulation assignment given in section 3.1.3 of [1] fails to satisfy this criterion: In order to see this it is sufficient to check it on a spin-network basis. So, let $\gamma, g_{\gamma'}$ be two spin-network states. Then we see that $\hat{H}_E\gamma f_{\gamma}$ can be written as a finite sum of spin-network states each of which depends on a common graph $\tilde{\gamma}$ which contains $\gamma$ and all the arcs $a_{ij}(\Delta)$ of the tetrahedra with basepoint in one of the vertices. Notice that by choosing the values of the spin quantum numbers involved in $f_{\gamma}$ large enough we can arrange that the dependence of all these spin-networks on all the edges of $\gamma$ and precisely one of the arcs $a_{ij}(\Delta)$ is non-trivial because of the $\hat{p}_{\Delta}$ involved in (3.10). Orthogonality of the spin-network states therefore implies that the left hand side of (3.3) is non-vanishing if and only if $\gamma \subset \gamma' \subset \tilde{\gamma}$. On the other hand, if indeed $\gamma' = \gamma \cup \Delta(\gamma)$ where $\Delta(\gamma)$ is one of the tetrahedra assigned to $\gamma$ such that $< g_{\gamma'}, \hat{H}_E^E(N) f_{\gamma} > \neq 0$ then $\hat{H}_E^E(N) g_{\gamma'}$ can be written as a linear combination of spin-network states each

10
of which is bigger than $\gamma'$ and therefore $\langle \hat{H}_\gamma^E(N)g_{\gamma'}, f_\gamma \rangle \geq 0$ which contradicts symmetry. The reason why with the loop assignment made so far the operator $\hat{H}_\gamma^E(N)$ is not symmetric comes from the requirements 1b) and 1ii) in section 3.1.3 of [1] made for the segments $s_I$ of edges $e_I$ and the arcs $a_{IJ}$ assigned to pairs of edges $e_I, e_J$ of $\gamma$ incident at a vertex $v$: this requirement basically said that $s_I$ only intersects one vertex of $\gamma$, namely $v$, and that $a_{IJ}$ intersects $\gamma$ only in its endpoints. Therefore the assignment made for a graph on which $\hat{H}_\gamma^E(N)f_\gamma$ depends can never coincide with that for $\gamma$ itself.

One could fix the situation as follows: what needs to be done is to compute the matrix elements $H^E(N)_{ff'} := \langle f, \hat{H}_\gamma^E(N)f' \rangle$ for any two cylindrical functions $f, f'$ and then define the matrix elements of a new symmetric operator $\hat{H}_\gamma^E(N)_{symm}$ by $\langle f, \hat{H}_\gamma^E(N)_{symm}f' \rangle := \frac{1}{2}[H^E(N)_{ff'} + \hat{H}_\gamma^E(N)_{ff}]$.

While this is what one should do given the assignment defined in section 3.1.3 of [1] it is a rather indirect procedure because we do not know these matrix elements in explicit form. We prefer to suggest a modification of the assignment and then show that (3.3) follows. At the moment we are able to do that only at the prize of introducing a new structure.

**Definition 3.1**

i) A vertex $v$ of a graph $\gamma$ is said to be exceptional provided that:

a) it has at least valence three

b) all edges of $\gamma$ incident at $v$ have co-linear tangents at $v$ and precisely two of them, call them $s_1, s_2$, are such that $s_1 \circ s_2$ is an analytic edge

c) if $v'$ is any other vertex of $\gamma$ satisfying a) and b) then there exists at most one edge of $\gamma$ such that $v, v'$ are its endpoints.

ii) An analytical edge $e$ of a graph $\gamma$ is said to be exceptional provided that:

a) the two vertices $v, v'$ of $\gamma$ corresponding to the endpoints of $e$ are exceptional

b) there is a vertex $v_0$ of $\gamma$ and outgoing edges $e_1, e_2$ incident at it such that $v, v'$ is the endpoint of $e_1, e_2$ distinct from $v_0$

c) if the orientation of $e$ is such that it starts at $v$ and ends at $v'$ then the tangents of $e, e_1$ at $v$ are parallel and of $e, e_2$ at $v'$ are anti-parallel.

Note that the notion of exceptionality of vertices and edges is diffeomorphism invariant and that an exceptional edge can be an analytical edge. The next definition maps us out of the purely analytical category.

**Definition 3.2**

A smooth exceptional edge $e$ is an edge with all the properties of an analytical exceptional edge but with the following additional feature:

If $v, v'$ are the endpoints of $e$ and $s_1, s_2$ are the edges incident at $v$ mentioned in definition (3.1) i),b) such that $s := s_1 \circ s_2$ is an analytical edge and likewise if $s'_1, s'_2$ are incident at $v'$ such that $s' := s'_1 \circ s'_2$ is an analytical edge then $e$ joins $s, s'$ in a $C^\infty$ fashion.

Notice that a smooth exceptional edge cannot be analytical: since all its derivatives coincide with those of $s, s'$ at $v, v'$ it would follow from analyticity that $s, s', e$ have coinciding maximal analytical extension in contradiction to the fact that $v = e \cap s, v' = e \cap s'$ are only a two points.

The idea of how to achieve symmetry is now clear: the Hamiltonian constraint defined so far adds new edges to a given graph. What one would like to do is to say...
that if \( \gamma' \) is a graph which comes from a smaller graph in the sense that \( \gamma' - \gamma \) is a collection of edges which were added to \( \gamma \) by acting repeatedly with the Hamiltonian constraint then the action of the Hamiltonian constraint on \( \gamma' \) should coincide with that on \( \gamma \). If no such \( \gamma \) exists then one can choose a loop assignment for \( \gamma' \) according to the rules described in section 3.1.3 of [1]. The trouble with this strategy is that

1) it is far from clear that one can construct a consistent loop assignment such that for given \( \gamma' \) there is at most one \( \gamma \subset \gamma' \) such that \( \gamma' \) comes from \( \gamma \) in the sense just explained (so that one would not know how to act with the constraint operator) and

2) since \( \gamma' \) can just be a given graph and does not necessarily arise from acting with \( \hat{H}_E(N) \) it is intuitively wrong to have the “little edge” \( a_{IJ}(\Delta) \) coincide with an edge already existing in \( \gamma' \) because if one would now make the triangulation finer one would need to do that by simultaneously changing the graph itself.

The following modification of the loop-assignment in section 3.1.3 of [1] adapted to the case where the constraint should be a symmetric operator circumvents these problems:

We keep all points 0),2),4),5). However, we introduce the following changes.

6) Anomaly-Freeness:

As we have seen in the main text, a solution to the anomaly-freeness condition can be given by the following quite simple requirement: each tetrahedron \( \Delta, v(\Delta) \in V(\gamma) \) is subject to the condition that the loop \( \alpha_{ij}(\Delta) := s_i(\Delta) \circ a_{ij}(\Delta) \circ s_j(\Delta) \) is a kink with vertex at \( v \). That is, the arc \( a_{ij} \) joins \( s_i, s_j \) in at least a \( C^1 \) fashion. We choose the tangent direction of \( a_{ij} \) such that it is parallel to the one of \( s_i \) at \( s_i \cap a_{ij} \) and antiparallel to the one of \( s_j \) at \( s_i \cap a_{ij} \).

1') Segments and arcs:

Moreover, to satisfy the symmetry requirement we modify point 1) of section 3.1.3 of [1] as follows:

Let \( \Gamma \) again be the set of piecewise analytic graphs. Given \( \gamma_0 \in \Gamma \) let now the edge \( a_{ij}(\Delta) \) be a smooth exceptional edge of the graph \( \gamma \cap \alpha_{ij}(\Delta) \) (thus, requirement 6) is met). We keep all the requirements of section 3.1.3. of [1] for the \( s_i(\Delta(\gamma_0)), a_{ij}(\Delta(\gamma_0)) \).

The image of the \( n-th \) power of \( \hat{H}_E(N) \) on functions cylindrical with respect to piecewise analytical graphs are now functions on graphs \( \gamma_n \) which are piecewise analytic after removing precisely \( n \) smooth exceptional edges. The loop assignment for such graphs \( \gamma_n \) is then defined inductively as follows:

i) if \( e_I \) is a piecewise analytic edge of \( \gamma_n \) necessarily incident at a non-exceptional vertex \( v \) then let \( s_I(\gamma_n) \) incident at \( v \) be chosen such that in case of Situation A: the endpoint of \( e_I \) distinct from \( v \) is not an endpoint of a smooth exceptional edge of \( \gamma_n \); then apply the rules of section 3.1.3 of [1] to choose \( s_I(\gamma_n) \).

Situation B: the endpoint of \( e_I \) distinct from \( v \) is an endpoint of a smooth exceptional edge; then choose \( s_I(\gamma_n) := e_I \).

ii) if \( e_I \) is either a piecewise analytic edge of \( \gamma_n \) incident at an exceptional vertex \( v \) or a smooth exceptional edge, necessarily incident at an exceptional vertex \( v \) then let \( s_I(\gamma_n) \) incident at \( v \) be chosen according to the rules of section 3.1.3 of [1].

iii) if \( e_I, e_J \) are both piecewise analytic edges of \( \gamma_n \) necessarily incident at a
non-exceptional vertex $v$ then there is either a smooth exceptional edge $a_{IJ}$ connecting the endpoints of $e_I, e_J$ distinct from $v$ or there is not. In the former case we choose $a_{IJ} := a_{IJ}$, in the latter we apply the rules of section 3.1.3 of [1] to choose $a_{IJ}$ with the addition that $a_{IJ}$ is a smooth exceptional edge. 

iv) if at least one of the two edges of a pair $e_I, e_J$ incident at $v$ is an exceptional edge then $v$ is necessarily an exceptional vertex and we apply the rules of section 3.1.3 of [1] to choose a smooth exceptional edge $a_{IJ}$.

It will be shown that the exceptional vertices of $\gamma_n$ do not contribute to the action of the constraint. It follows that the repeated action of the Euclidean Hamiltonian constraint produces functions cylindrical with respect to only a finite number of graphs, each of which has the same unique analytic “skeleton” obtained by removing its smooth exceptional edges. The uniqueness property follows from the fact that the exceptional edges are not analytic, they are “marked” and that was the virtue of the construction.

Notice that if we have two graphs $\gamma_n, \gamma'_n$ which come from the n-th power of $\hat{H}_E(N)$ so that they have both $n$ smooth exceptional edges connecting the same pairs of piecewise analytic edges of their common skeleton then $\gamma_n, \gamma'_n$ will in general not coincide but they will be diffeomorphic. This will be shown in the next point 3′).

3′) Diffeomorphism invariant prescription of the position of the arcs $a_{ij}(\Delta)$:

Point 3) of section 3.1.3 of [1] does not quite cover the present situation yet because we introduce exceptional edges which in contrast to section 3.1.3. of [1] always are incident at the same exceptional vertex provided they have an endpoint on a piecewise analytic edge of the skeleton of the graph under investigation. So given a pair of piecewise analytic segments $s_1, s_2$ incident at a non-exceptional vertex $v$ of $\gamma$ the requirements of section 3.1.3 of [1] make already sure that the smooth exceptional arc $a$ connecting $s_1, s_2$ at their endpoints distinct from $v$ does not intersect any other piecewise analytic segment $s$ incident at $v$. Now, if there are already smooth exceptional arcs $a_1, a_2$ between $s_1, s$ and $s_2, s$ respectively, then in the frame adapted to $s_1, s_2$ as indicated in section 3.1.3 of [1] we can further specify the diffeomorphism in such a way that $a_1, a_2$ do not intersect the part of the $x/y$ plane bounded by $s_1, s_2$, except in their endpoints. That this is always possible follows from the fact that we already found a diffeomorphism adapted to $s_1, s_2$ such that $s$ lies either above or below the $x/y$ plane or that it lies outside the part of the $x/y$ plane bounded by $s_1, s_2$. Since we can apply a smooth diffeomorphism to $a_1, a_2$ which preserves the rest of the graph, the assertion follows.

Since the notion of smooth exceptionality is invariant under analyticity preserving diffeomorphisms and since we have shown that the assignment subject to the above modification of our triangulation adapted to a graph is diffeomorphism covariant, none of the properties proved before in [1] are ruined.

Definition 3.3 Consider the range of finite powers of the Euclidean Hamiltonian constraint on functions cylindrical with respect to graphs in $\Gamma$. These functions depend on extended graphs $\gamma$ with an analytic skeleton $\gamma_0 = \gamma - S(\gamma) \in \Gamma$ where $S(\gamma)$ is the set of smooth exceptional edges of $\gamma$. We call $\Gamma_e$ the set of extended
graphs so obtained and $\Gamma_e(\gamma_0)$ is the subset of $\Gamma_e$ consisting of graphs with skeleton $\gamma_0 \in \Gamma_0$.

As we have seen, an immediate consequence of this prescription is that an (extended) graph $\gamma$ does not grow under the repeated action of the Hamiltonian constraint beyond one with a certain finite number of smooth exceptional edges. This is in contrast to the prescription made in section 3.1.3 of [1] and it seems that this property is forced on us by the requirement of symmetry. The dynamical consequence of this is a very different structure of the kernel of the constraint (see next sections).

The reader may feel uneasy with this prescription because once we have left the analytic category of graphs we are losing many of the properties of the holonomy algebra [3, 4] and one worries that the quantum configuration space $A/\Gamma$ is altered. This is because, if we multiply cylindrical functions defined on finite piecewise analytic graphs, the resulting function is a function defined on the union of the two graphs and the analyticity of the graphs prevents this union from being an infinite piecewise analytic graph so that the cylindrical functions form an algebra. Now if we define the extended graphs to be those which have a finite piecewise analytic skeleton after removing a finite number of smooth exceptional edges then it is easy to see that cylindrical functions on extended graphs do not form an algebra. However, we do not want to do that: we view functions cylindrical with respect to extended analytical graphs as states in the Hilbert space $\mathcal{H}$ and as such we cannot multiply them. We still use only functions which are defined on $\Gamma_0$ to define the spectrum $A/\Gamma$. The only source of non-linearity is the inner product. Now, when computing the inner product between functions cylindrical with respect to extended graphs we make use of the fact that in order that the inner product be non-vanishing, their skeletons must coincide and if so, then the smooth exceptional edges are finite in number and mutually non-intersecting and therefore weakly independent [5] so that the inner product can easily be computed. This is different from inner products between functions cylindrical with respect to general smooth graphs and requires more sophisticated techniques as for instance in [17].

We confess, however, that a technique that prevents us from introducing the notion of a smooth exceptional edge and thus leaving the analytical category would be strongly preferred. Unfortunately, at the moment we do not have such a technique at our disposal.

The assertion that with this assignment the family of projections $(\hat{H}_E^E(N))$ qualifies as a symmetric operator now follows from the following lemma.

Lemma 3.1 Let $\gamma$ be a piecewise analytic graph, let $\gamma' \in \Gamma_e(\gamma)$ and let $f$ be any cylindrical function thereon. Then $H^E_{\gamma'} f = \hat{H}^E_{\gamma} f$.

Proof: By construction we just need to check that the edges of $V(\gamma') - V(\gamma)$ do not contribute. Consider a function $f$ cylindrical with respect to $\gamma'$ and let $v \in V(\gamma') - V(\gamma)$. Consider the term $\hat{h}^{\Delta} f$ for any $\Delta$ such that $v(\Delta) = v$. Writing out the anti-commutator involved in this term we get two terms. The first is proportional to

\begin{equation*}
\epsilon^{ijk} \text{tr}(h_{\alpha ij}(\Delta)h_{sk}(\Delta)[h_{sk}^{-1}(\Delta), \hat{V}]f = -\epsilon^{ijk} \text{tr}([h_{\alpha ij}(\Delta)h_{sk}(\Delta)\hat{V}h_{sk}^{-1}(\Delta)]f
\end{equation*}
where we have used the $SU(2)$ Mandelstam identity $\text{tr}(h_\alpha) = \text{tr}(h_\alpha^{-1})$ to simplify the commutator. The volume operator acts on the cylindrical function $h_{s_k(\Delta)}^{-1}f$ which depends on the graph $\gamma'$. Accordingly we can write out $\hat{V} = \sum_{v' \in V(\gamma')} \hat{V}_{v'}$ where $\hat{V}_{v'}$ acts only on those edges of $\gamma'$ which are incident at $v'$, using the notation of $[4]$, (2.8). Take any $v' \neq v$, then the corresponding contribution in the above expression vanishes because then $h_{s_k(\Delta)}^{-1}$ commutes with $\hat{V}_{v'}$ and using the $SU(2)$ Mandelstam identity again we see that the result is zero. Now if $v' = v$ then the contribution vanishes anyway because $v$ is by construction a vertex such that all edges incident at it have mutually colinear tangents.

Let us now turn to the second term. It is proportional to

$$
\epsilon^{ijk} \text{tr}(h_{s_k(\Delta)}[h_{s_k(\Delta)}^{-1}, \hat{V}]) h_{\alpha_{ij}(\Delta)}(f) = -\epsilon^{ijk} \text{tr}(h_{s_k(\Delta)} \hat{V} h_{s_k(\Delta)}^{-1} h_{\alpha_{ij}(\Delta)}) f
$$

where again use was made of the Mandelstam identity. The volume operator now acts on the cylindrical function $h_{s_k(\Delta)}^{-1}h_{\alpha_{ij}(\Delta)}f$ which depends on the graph $\gamma' \cup \Delta$ and accordingly the volume operator is now a sum of terms $\hat{V}_{v'}$ where $v'$ runs through the vertices of $\gamma'$ and the vertices $v_i(\Delta), i = 1, 2, 3$ of $\Delta$ distinct from $v(\Delta) = v$. The only difference to the previous situation is related to the additional vertices $v_i(\Delta)$. But these have the same property as $v$, namely all incident edges have colinear tangents. Therefore this contribution vanishes as well.

We conclude that all the vertices of $V(\gamma') - V(\gamma)$ are ignored by the Hamiltonian constraint and the assertion follows now from the cylindrical consistency of the volume operator.

□

We notice that if we replace $\hat{h}^E_{\Delta}$ by $\hat{H}^E_{\Delta}$ then we find by the same argument (all we used is that the volume operator vanishes at vertices which are such that all edges have incident tangents) that $\hat{H}^E_{\gamma} f$ and $\hat{H}^E_{\gamma} f$ are diffeomorphic for each $\gamma' \subset \gamma \cup \bigcup_{v \in V(\gamma)} \bigcup_{v(\Delta) = v} \Delta$.

Using exactly the same arguments as in Lemma 3.1 we derive the following.

**Corollary 3.1** With the same notation as in $[4]$, (2.8) we have

$$
\hat{h}^E_{\Delta} f = -\frac{1}{3if_p^2} \epsilon^{ijk} \text{tr}(\{h_{\alpha_{ij}(\Delta)}, h_{s_k(\Delta)}[h_{s_k(\Delta)}^{-1}, \hat{V}_{v(\Delta)}]\}) f.
$$

For $\hat{H}^E_{\Delta}$ a similar formula holds (just drop the anticommutator and multiply by 2).

**Theorem 3.1** The system of symmetric projections $\hat{H}^E_{\gamma}(N)$ defined on $D_\gamma$ in $[4]$, (3.10) defines a symmetric operator $\hat{H}^E(N)$ on $D$.

Proof:

First of all, since the symmetric version of $[4]$, (3.10) involves two projectors $\hat{h}^E_{\Delta}$, one before and one after acting with $\hat{h}^E_{\Delta}$, it follows that either $f_{\gamma}$ depends non-trivially on all three $s_i(\Delta)$ before and after acting with $\hat{h}^E_{\Delta}$ or $\hat{h}^E_{\Delta} f_{\gamma} = 0$. Thus the right hand side of (3.3) is non-vanishing if and only if $\gamma \subset \gamma' \in \Gamma_e(\gamma)$. By Lemma 3.1 we may replace $\hat{H}^E_{\gamma}$ by $\hat{H}^E_{\gamma}$ on the left hand side of (3.3). The assertion follows now from the symmetry of the operators $\hat{H}^E(N)_{\gamma}$.

□
Before closing this section we would like to point out the following observation:
The requirement that the loops assigned to an (extended) graph are kinks seems to
be forced on us by anomaly-freeness (compare Theorem 3.1). But as we saw in
the proof of the lemma, the kink property was also important to prove symmetry.
So it seems that symmetry and anomaly-freeness are tightly knit with each other.
We see explicitly that the choice of a triangulation adapted to a graph is not only a
kinematical element of the quantum theory. It is also a very dynamical ingredient.

3.2.2 Self-adjointness

In the sequel an exceptional edge is a smooth exceptional edge and it is understood
that in all cylindrical constructions $\Gamma$ is replaced by $\Gamma_e$.
We have shown that $\hat{H}_E^\gamma$ is a symmetric operator on $\mathcal{H}$ with dense domain $D_\gamma :=$
Cyl$^3(\overline{A/G})$, the thrice differentiable functions cylindrical with respect to the graph $\gamma$. For $\gamma \subset \gamma'$ let $p_{\gamma'}^\gamma :$ Cyl$\gamma(\overline{A/G}) \rightarrow$ Cyl$\gamma'(\overline{A/G})$ be the pull-back of functions
from smaller to bigger graphs. The object $\hat{H}_E^\gamma$ is defined by the family of projec-
tions $(\hat{H}_E^\gamma,D_\gamma)$ and in order to qualify as an operator defined on $D =$ Cyl$^3(\overline{A/G})$
it needs to be cylindrically consistent up to a diffeomorphism, that is, $\left(\hat{H}_E^\gamma\right)|_\gamma,\hat{H}_E^\gamma$
are diffeomorphic and $p_{\gamma'}^\gamma D_\gamma \subset D_{\gamma'}$. In the main text we have shown that this is
indeed the case.
If we could show that each $\hat{H}_E^\gamma[N]$ is an essentially self-adjoint operator on $\mathcal{H}_\gamma$ with
core $D_\gamma$ then we could conclude immediately from theorems proved in [8] that the
self-adjoint extensions are cylindrically consistent. There are two reasons why those
theorems are inapplicable in our case:

1) The range of $\hat{H}_E^\gamma$ on $D_\gamma$ does not lie in $\mathcal{H}_\gamma$ since $\hat{H}_E^\gamma$ enlarges the graph by the
arcs.

2) While one could try to circumvent that problem by considering $\hat{H}_E^\gamma$ as an operator
on $D_{\tilde{\gamma}}$ where $\tilde{\gamma}$ is a graph on which all cylindrical functions in the range of powers
of $\hat{H}_E^\gamma(N)$ on $D_{\tilde{\gamma}}$ depend, according to lemma 3.1, (and it turns out that it then
is symmetric on $D_{\tilde{\gamma}}$) we simply do not know whether that operator is essentially
self-adjoint on $D_{\tilde{\gamma}}$.

The way out is to work directly on the full Hilbert space $\mathcal{H}$ which is the completion
with respect to the obvious inner product of the space $\bigcup_\gamma \mathcal{H}_\gamma$ (again we did not display
identifications due to cylindrical equivalence).

To see that each $\hat{H}_E^\gamma$ has self-adjoint extensions we use a theorem due to von Neu-
mann ([18], p. 143).

**Definition 3.4** An antilinear map $\hat{k} : \mathcal{H} \rightarrow \mathcal{H}$ is called a conjugation if it is
norm-preserving and $\hat{k}^2 = \text{id}_\mathcal{H}$.

**Theorem 3.2 (von Neumann’s theorem)** Let $\hat{H}$ be a symmetric operator on a
Hilbert space with dense domain $D$ and suppose that there exists a conjugation $\hat{k}$
satisfying the following two properties:

1) $\hat{k} D \subset D$ preserves the domain and

2) $\hat{k}\hat{H} = \hat{H}\hat{k}$ on $D$, that is, $\hat{H}$ commutes with the conjugation.

Then $\hat{H}$ has self-adjoint extensions.
The proof follows from the fact that the assumptions imply that $\hat{H}$ has equal deficiency indices.

To apply this theorem to our case we begin by noticing that $D = \text{Cyl}^3(\mathcal{A}/\mathcal{G}) = \bigcup_{\gamma} \text{Cyl}^2(\mathcal{A}/\mathcal{G})$ is a dense domain for $\hat{H}^E$ and that it is spanned by spin-network states. But these states can be expanded, with real coefficients, into traces of the holonomy around closed loops (that is, Wilson loop functionals) and it is a peculiarity of $SU(2)$ that the latter are real valued. The explicit form of (3.10) implies then that the result of applying $\hat{H}^E$ will be a sum of spin-network states with purely imaginary coefficients, that is, the operator $\hat{H}^E$ is imaginary-valued, its matrix elements are purely imaginary and anti-symmetric in a basis of real valued functions like the spin-network basis. Therefore, it is not enough to choose $\hat{k}$ to be just complex conjugation.

Given an extended graph $\gamma$, consider its skeleton $\gamma_0$. Recalling the definition of a smooth exceptional edge, by inspection of (3.10) each of the six terms involved in $\hat{h}^E_{\Delta}$ depends precisely one one smooth exceptional edge $a_{ij}(\Delta)$. Therefore, given a spin-network state $f$ cylindrical with respect to $\gamma$, if we expand the state $\hat{H}^E_{\Delta}f$ as a linear combination of spin-network states, then each of those states depends on a graph $\gamma'$ such that the spin associated with precisely one of the smooth exceptional edges assigned to $\gamma_0$ has changed in $\gamma'$ by $\pm\hbar/2$ as compared to $\gamma$ (to see this, consider $f$ as a state on $\gamma'$). We are going to exploit precisely this fact to construct an appropriate conjugation.

**Theorem 3.3** The operator $\hat{H}^E(N)$, densely defined on $\text{Cyl}^3(\mathcal{A}/\mathcal{G})$, possesses self-adjoint extensions.

Proof:

Denote the exceptional edges of a graph $\gamma$ by $E_0(\gamma)$. Let $e \in E_0(\gamma)$ and $Y^i_e := X^i(h_e)$, where $X^i$ is the right invariant vector field on $SU(2)$, and construct their Laplacians $\Delta_e := \text{tr}(Y^i_eY^i_e)$ which are negative definite operators on $SU(2)$ with usual eigenvalues $-j(j+1)$ on eigenfunctions with spin $j$, in particular on spin-network states. We construct a positive definite spin operator $\hat{J}_e := \sqrt{\frac{1}{4} - \Delta_e} - \frac{1}{2}$ with eigenvalues $j$. Finally we set

$$\hat{P}_{\gamma} := \prod_{e \in E_0(\gamma)} e^{2\pi i J_e} \text{ and } \hat{k}_\gamma := \hat{P}_{\gamma} \hat{c}$$

(3.4)

where $\hat{c}$ is the operator of complex conjugation.

Obviously each $\hat{P}_{\gamma}$ has a domain $\text{Cyl}^2(\mathcal{A}/\mathcal{G})$, dense in $\mathcal{H}_{\gamma}$ and is a bounded (by 1) and symmetric operator thereon (on a spin-network state it corresponds to multiplying the state by $\pm 1$). $\hat{P}_{\gamma}$ is even an essentially self-adjoint operator on $\mathcal{H}_{\gamma}$ with core $\text{Cyl}^2(\mathcal{A}/\mathcal{G})$ : To see this we check the basic criterion of essential self-adjointness. We need to show that $\hat{P}_{\gamma} \pm \text{id}_{\mathcal{H}_{\gamma}}$ has dense range and it will be enough to show that each spin-network $f$ state on $\gamma$ is in the range of that operator when evaluated on its domain. But $[\hat{P}_{\gamma} \pm \text{id}_{\mathcal{H}_{\gamma}}]T = \pm T$ so $T$ is reproduced up to a never vanishing multiplicative factor. That proves that $(\hat{P}_{\gamma}, \text{Cyl}^2(\mathcal{A}/\mathcal{G}))$ is essentially self-adjoint.

Let us check that the family $(\hat{P}_{\gamma}, \text{Cyl}^2(\mathcal{A}/\mathcal{G}))_{\gamma \in \mathcal{G}}$ defines an essentially self-adjoint operator $\hat{P}$ on $\mathcal{H}$ with dense domain $\text{Cyl}^2(\mathcal{A}/\mathcal{G})$. The condition $p_{\gamma'\gamma}\text{Cyl}^2(\mathcal{A}/\mathcal{G}) \subset \text{Cyl}^2(\mathcal{A}/\mathcal{G})$ is certainly satisfied for each $\gamma \subset \gamma'$. Since each $f \in \text{Cyl}^2(\mathcal{A}/\mathcal{G})$ can be expressed in terms of spin-network states $T$ it is sufficient to check cylindrical
consistency on those functions. But if \( \gamma \) is lacking an exceptional edge \( e \) as compared to \( \gamma' \) then \( \hat{J}_e T = 0 \) proving cylindrical consistency. This shows that the closure of \( \hat{P} \) is even a self-adjoint operator on \( H \) since it was shown in ([8]) that each consistent and essentially self-adjoint family is such that the family of self-adjoint extensions is cylindrically consistent.

Finally it is easy to see that \( \hat{P} \) is a linear, norm-preserving operator on \( H \) upon checking in a spin-network base. Moreover, \( \hat{P}^2 = \text{id}_H \) which follows from \( \hat{P}T = \pm T \) for any spin-network state, that is, \( \hat{P} \) acts like a parity operator on exceptional edges.

Finally, it follows that \( \hat{k} := \hat{P}\hat{c} \) is a conjugation on \( H \) which follows from the fact that the phase shift of a spin-network state \( T \) induced by \( \hat{P} \) is real and from the fact that \( \hat{P} \) is linear so that \( \hat{k} \) is anti-linear.

We are now ready to see that \( \hat{H}_{E[N]} \) commutes with \( \hat{k} \). Consider a term \( \hat{H}_{ij,k,\Delta} := -\frac{1}{36p} \text{tr}(\{h_{a_{ij}(\Delta)}, h_{s_k(\Delta)}[h_{s_k^{-1}(\Delta)}(\hat{V}_v)] \}) \) where \( e := a_{ij}(\Delta) \) in \( \alpha_{ij}(\Delta) \) is a smooth exceptional edge. Let \( T \) be a spin-network state with spin \( j_e \) associated with \( e \). Then \( \hat{H}_{ij,k,\Delta} T = f_+ + f_- \) where \( f_\pm \) are sums of spin-network states such that they depend on \( e \) through \( j_e^\pm = j_e \pm 1/2 \) while the spins of all other exceptional edges are unchanged (this is because the other edges \( s_i(\Delta) \) involved in \( \hat{H}_{ij,k,\Delta} \) are non-exceptional edges). It follows easily that \( \hat{P}f_\pm = e^{2\pi j_e \epsilon/2} f_\pm \). Thus

\[
\hat{P}\hat{H}_{ij,k,\Delta} T = -\hat{H}_{ij,k,\Delta} \hat{P}T
\]

and so \([\hat{k}, \hat{H}_{E[N]}] = 0 \) for any \( f \in D \) due to the factor of \( i \) involved in \( \hat{H}_{ij,k,\Delta} \).

This proves only existence, not uniqueness, of self-adjoint extensions for \( \hat{H}_{E[N]} \). We do not know how many extension there are and how to select one in case there are several. We conjecture, that \( \hat{H}_{E[N]} \) is even essentially self-adjoint in which case that extension would be unique and concisely described by the theorems in [8]. A proof for that conjecture is missing, however, at the present stage.

### 3.3 The symmetric Lorentzian operator

Again we will only discuss the points of departure between the symmetric and non-symmetric operators. It is understood that the triangulation as modified in the previous section is applied to the present section as well. Also, as discussed in the main text, without changing formula ([3](4.1)), \( \hat{K} \) is now automatically a symmetric operator and it has self-adjoint extensions.

#### 3.3.1 Symmetry and cylindrical consistency

It turns out that if we choose the ordering

\[
\hat{t}_{\Delta} := -\frac{64}{3(2p)^3} e^{ijk} \text{tr}(h_{s_1(\Delta)} h_{s_1^{-1}(\Delta)}(\hat{V}_v) h_{s_1(\Delta)} h_{s_1^{-1}(\Delta)}(\hat{V}_v)) \}
\]

and use that \( \hat{V}, \hat{K} \) are symmetric operators, as well as the \( SU(2) \) reality conditions, and use this operator in ([4](5.3)) then the operator is already symmetric with domain
Cyl$^3(\mathcal{A}/\mathcal{G})$ on $\mathcal{H}$. Therefore we replace (5.3) by
\[
\hat{t}_\gamma[N] := \sum_{v \in V(\gamma)} \frac{N_v}{E(v)} \hat{i}_v, \quad \hat{i}_v := \sum_{v(\Delta) = v} \hat{i}_\Delta.
\]

In complete analogy with the discussion for $\hat{H}^E$ we now define
\[
\hat{T}_\gamma[N] := \sum_{v \in V(\gamma)} N_v \sum_{v(\Delta) = v} \frac{\hat{p}_\Delta}{\sqrt{\hat{E}(v)}} \frac{\hat{i}_\Delta}{\sqrt{\hat{E}(v)}}
\]
and arrive at a self-consistent family of symmetric operators. To show that this family qualifies as the set of graph projections of a symmetric operator on $\mathcal{H}$ we need an analogue of Lemma 3.1.

Lemma 3.2 With the same notation as in Lemma 3.1 it holds that $\hat{T}_\gamma f = \hat{T}_\gamma f$.

Proof:
The proof follows immediately from the fact that the volume operator vanishes at the vertices of $V(\gamma') - V(\gamma)$ and the explicit expression (3.5) along the same line of argument as in Lemma 3.1.

3.3.2 Self-Adjointness

While we could try to invoke von Neumann’s theorem again to prove that self-adjoint extensions of $\hat{T}$ exist, this is insufficient since self-adjointness does not respect the linear structure of the operator algebra. Rather, given some self-adjoint extension $D(\hat{H}^E)$ of $\hat{H}^E$, what we need is an extension of $\hat{T}$ to the same domain $D(\hat{H}^E)$.

An obvious approach to prove existence of such an extension is suggested by the following theorem [18].

Theorem 3.4 (Kato-Rellich) Suppose that $\hat{H}^E$ is self-adjoint on $\mathcal{H}$ with domain $D(\hat{H}^E)$ and that $\hat{T}$ is symmetric with domain $D(\hat{T})$ such that $D(\hat{H}^E) \subset D(\hat{T})$. Furthermore, suppose that there are real numbers $a, b$ such that for all $\psi \in D(\hat{H}^E)$ it holds that $||\hat{T}\psi|| \leq a||\hat{H}^E\psi|| + b||\psi||$ and that the infimum of all possible $a$ (as $b$ varies) satisfies $a < 1$. Then $\hat{H} := \hat{T} - \hat{H}^E$ is self-adjoint on $D(\hat{H}^E)$.

To apply this theorem we therefore need to perform three steps:
- a) Choose a self-adjoint extension of $\hat{H}^E$,
- b) Check whether there is a domain of $\hat{T}$ which contains the determined domain of $\hat{H}^E$ and
- c) check whether the bound condition mentioned in the theorem (which in the mathematics literature is called “$\hat{T}$ is $\hat{H}^E$-bounded with relative bound < 1”) can be satisfied for some choice of $b$.

Clearly, such an analysis is far from trivial and is beyond the scope of the paper. We will get back to this question in a later paper and just comment on why we can hope to find a relative bound < 1: A dense domain of $\hat{H}^E$ are the finite
linear combinations of spin-network states on which also $\hat{T}$ is symmetric so that it
is plausible that the first condition in the theorem is satisfied. If $N$ is the total
number of edges of a graph $\gamma$ define $j := j_1 + \ldots + j_N$ for a spin network state $\psi$
with spins $j_1, \ldots, j_N$. It follows from elementary angular momentum algebra that
$||\hat{V}\psi|| \leq j^{3/2}||\psi||$ (here we used the boundedness of the matrix elements of an
element of $SU(2)$). Moreover, since $h_e$ changes the spin associated with the edge $e$
by $\pm \hbar/2$, it follows that $||h_e h_e^{-1}, V\psi|| \propto j^{1/2}||\psi||$. We thus expect a behaviour like
$||\hat{H}^E\psi|| \propto j^{1/2}||\psi||$. Next, recall that $\hat{K} \propto [\hat{V}, \hat{H}^E]$ so that we find $||\hat{K}\psi|| \propto j||\psi||$
which means that by a similar argument also $||\hat{T}\psi|| \propto j^{1/2}||\psi||$. So the large spin
behaviour of both $\hat{T}, \hat{H}^E$ is comparable and it is conceivable that a relative bound
$a < 1$ exists given the fact that in $\hat{T}$ a lot more symmetrizations among the edges
are taking place.

3.3.3 Solutions

The detailed analysis of the kernel of $\hat{H}, \hat{H}^E$ will be left for future publications [21].
Here we content ourselves with a qualitative description.

1) The most important property of the symmetric operator is that it does not
extend a given analytic graph $\gamma$ beyond graphs contained in $\Gamma_e(\gamma)$ as described in
Lemma 3.1. If we work on diffeomorphism invariant states then there is even a
maximal, finite graph $\tilde{\gamma}$ on which (diffeomorphic images of) all $\gamma' \in \Gamma_e(\gamma)$ depend.
This implies that we can study the eigenvalue problem on the finite graph $\tilde{\gamma}$, that is,
instead of dealing with $\hat{H}$ we just have to consider its projection $\hat{H}_{\tilde{\gamma}}$ which turns the
spectral analysis into a problem on a Hilbert space with a finite number of degrees
of freedom. In particular, since we know that all the spin-network states on that
graph form a complete set of orthonormal states, this Hilbert space is separable.
In particular, this property is precisely the reason why now an infinite series of spin-
networks on the graph $\tilde{\gamma}$ has a chance to be annihilated by $\hat{H}(N)$ upon choosing the
coefficients of that expansion appropriately. Such a series is a well-defined element
of $\Phi'$ and we see that again the action of the Hamiltonian and Diffeomorphism
constraints on spin-networks is nicely split : the Hamiltonian constraint acts on $\vec{\gamma}$,$\vec{c}$
and leaves $\gamma$ invariant while the Diffeomorphism constraint acts on $\tilde{\gamma}$ only. This
separation between labels on which the two constraints effectively act on is the
deeper reason for the fact that the constraint algebra of the symmetric Hamiltonian
constraint is effectively Abelian.

2) If we can at least prove existence of self-adjoint extensions then we can ex-
ponentiate the Hamiltonian and compute rigorously defined solutions by the group-
averaging method [15, 16]. By the same method we are able to find a scalar product
on the space of solutions. This is possible because the second important property of
the Hamiltonian constraint is that the operators corresponding to different vertices
commute (in the diffeomorphism invariant context) and so far we are only able to
deal with the group averaging method provided we know the group that is generated,
and a special case of this is when we have a finite number of Abelian constraints.
This goes as follows :

On the graph $\tilde{\gamma}$ the Hamiltonian constraint reduces to $\hat{H}_{\tilde{\gamma}}[N] = \sum_{e \in V(\gamma)} N_e \hat{H}_e$.
Suppose we have found a self-adjoint extension for each of the $\hat{H}_e$ then, by Stone’s
theorem, we can exponentiate $\hat{H}_\gamma$ and obtain a unitary operator

$$\hat{U}[\vec{N}] := \prod_{v \in V(\gamma)} e^{iN_v \hat{H}_v}$$

(3.8)

where $\vec{N} = (N_v)_{v \in V(\gamma)}$. Actually we obtain a unitary representation of an $n-$dimensional Abelian group with parameter $\vec{N}$ and group structure $\hat{U}[\vec{M}] \hat{U}[\vec{M}] = \hat{U}[\vec{M} + \vec{N}]$. Then the group average proposal says that we take a physical state to be

$$[f] := \int_S d\mu_H(\{N_v\}_{v \in V(\gamma)}) \prod_{v \in V(\gamma)} e^{iN_v \hat{H}_v} f$$

(3.9)

where $f$ is any function cylindrical with respect to $\gamma$ and $d\mu_H$ is the Haar-measure on the group manifold $S$ coordinatized by the $N_v$. To see that $[f]$ is a solution of the constraint we just verify that

$$\hat{U}[M][f] := \int_S d\mu_H(\{N_v\}) \hat{U}[M_v + N_v] f = [f]$$

(3.10)

since the Haar measure is translation invariant.

The inner product induced by the Hamiltonian constraint is given by

$$< [f], [g] >_{phys} := < f, [g] >$$

(3.11)

where the inner product on the right hand side is the one on $\mathcal{H}$. This inner product has the feature that whenever we have an observable on $\mathcal{H}$ which commutes with $\hat{H}$ strongly, that is, $< f, [\hat{O}, \hat{H}] g > = 0$ for all $f, g \in \text{Cyl}^\infty(\mathcal{A}/G)$ then it projects to an operator $[\hat{O}]$ on $\mathcal{H}_{phys}$ with preserved adjointness relations. Namely from $\hat{U}(\vec{N})^{-1} \hat{O} \hat{U}(\vec{N}) = \hat{O}$ and $< f, g > = < f, [g] >$ we find upon choosing $[\hat{O}][f] := [\hat{O} f]$ that $[\hat{O}]^* = [\hat{O}]$.

All these concepts are explained in more detail in [15, 16, 9].

### 3.4 Wick rotation transform

As explained in [3] (see also [4]) one has also another option to define the Wheeler-DeWitt constraint operator provided that the generator of the Wick rotation transform is self-adjoint. But that we checked to be the case and we can proceed and repeat the main argument.

One can show that there is a classical generator, called the complexifier in [3], of the canonical transformation $(A = \Gamma + K, E) \rightarrow (A = \Gamma - iK, iE)$ and it is just given by $C = (\pi/2)K$. Then one can show that up to a term proportional to the Gauss constraint it holds that

$$H = p(H^E + \{H^E, -iC\} + \frac{1}{2}\{\{H^E, -iC\}, -iC\} + \frac{1}{3!}\{\{\{H^E, -iC\}, -iC\}, -iC\} + ..)$$

(3.12)

where $p$ is a phase depending on how we take the square root of $i^3$, because effectively the real connection $A$ gets replaced by the complex Ashtekar connection. This expression motivates to just define

$$\hat{H} = p(H^E + [H^E, -\hat{C}/\hbar] + \frac{1}{2}[[H^E, -\hat{C}/\hbar], -\hat{C}/\hbar] + \frac{1}{3!}[[[H^E, -\hat{C}/\hbar], -\hat{C}/\hbar], -\hat{C}/\hbar] + ..)$$

(3.13)
or, upon defining the Wick rotation operator

\[ \hat{W} := e^{-\hat{C}/\hbar} \]  

we have

\[ \hat{H} = p\hat{W}^{-1}\hat{H}^{E}\hat{W}. \]  

There are three obvious problems:

1) Although \( \hat{C}, \hat{H}^{E} \) were shown to possess self-adjoint extensions, it is unclear whether they possess self-adjoint extensions to a common domain (in which case we would have a chance that (3.15) makes sense as far as domain questions are concerned).

2) The operator \( \hat{C} \) is far from being positive definite, therefore \( \hat{C} \) is not the generator of a contraction semigroup given formally by \( \hat{W}^t := \exp(-t\hat{C}/\hbar), t > 0 \) and it is unclear whether \( \hat{W} \) can be defined at all on a dense domain of \( \mathcal{H} \). One possible approach would be to restrict the Hilbert space to the “positive frequency subspace” where \( \hat{C} \) is positive definite (indeed, \( 1/\ell_3^3 \hat{C} \) has the dimension of a frequency), however, that could mean that we alter the reality conditions.

3) Whenever \( \hat{W} \) can be defined, it is going to be a symmetric operator. But then \( \hat{H} \) will not even be symmetric and again group averaging methods cannot be immediately applied.

A way to resolve these problems is suggested by recalling a theorem due to Nelson [18].

**Definition 3.5** Let \( \hat{C} \) be an operator on \( \mathcal{H} \). Then \( C^\infty(\hat{C}) := \cap_{n=0}^{\infty} D(\hat{C}^n) \) is called the set of smooth vectors for \( \hat{C} \) and a vector \( \psi \in C^\infty(\hat{C}) \) is called analytic if there exists \( t > 0 \) such that

\[ \sum_{n=0}^{\infty} \frac{||\hat{C}^n\psi||}{n!} \left( \frac{t}{\hbar} \right)^n < \infty. \]

**Theorem 3.5 (Nelson’s analytic vector theorem)** A closed symmetric operator \( \hat{C} \) is self-adjoint if and only if its domain \( D(\hat{C}) \) contains a dense set of analytic vectors.

We have shown already that \( \hat{C} \) (actually its closure) has a self-adjoint extension. Therefore it follows from Nelson’s theorem that there exists a dense set of analytic vectors for \( \hat{C} \) on which \( \hat{W}^t \) actually does converge in norm for some \( t > 0 \). The question then remains if we can choose \( t = 1 \).

On the other hand, even if we can choose \( t = 1 \), we are actually interested in solving the quantum constraint \( \hat{H}\psi = 0 \) and we would like to do that by setting \( \psi = \hat{W}^{-1}\psi^E \) where \( \hat{H}^{E}\psi^E = 0 \) is typically a distributional solution of the Euclidean Hamiltonian constraint. So how can we even hope to solve the Lorentzian constraint by this method? The answer is the following: \( \psi^E \) is an infinite sum of \( L_2 \) vectors (which does not converge in \( \mathcal{H} \) but in \( \Phi' \)). Since the set of analytic vectors is dense in \( \mathcal{H} \), each of these \( L_2 \) vectors can be written as a (infinite) sum of analytic vectors for \( \hat{C} \) which converges in \( \mathcal{H} \). In summary we can write \( \psi^E \) in terms of analytic vectors for \( \hat{C} \) and we can apply \( \hat{W} \) to each of them separately. Since the result of this is
a series of \( L_2(\mathcal{A}/\mathcal{G}, d\mu_0) \) vectors we can hope that it makes sense as a distribution again, provided we can choose \( t = 1 \) in Nelson’s theorem.

If it turns out that we cannot choose \( t = 1 \) to define \( \hat{W} \) or even if it does, that \( \hat{W} \psi^E \not\in \Phi' \) then we may be forced to adopt still another strategy which consists in going to a holomorphic representation \[3\]. The point of this is the following: one maps a cylindrical function \( f \) by \( \hat{W} \) and then analytically continues it. This analytic continuation is done for each term \( \hat{C}^n \psi \) (which is a cylindrical function again and so has a well-defined analytic continuation) separately. While the sum of these terms may not make any sense as an element of \( \mathcal{H} \) before analytically continuing it, after analytic continuation it may make sense as a distribution on a dense subspace of a Hilbert space of functions of complex-valued connections upon choosing a measure thereon which has the necessary stronger fall-off property. In order to satisfy the correct reality conditions this measure needs to be chosen in such a way that \( \hat{U} := \hat{a}\hat{W} \) (where \( \hat{a} \) means analytic continuation) is unitary (see \[3\] for a more detailed discussion).

This could resolve issues 1) and 2) but not 3). One might think that the part of the algebraic quantization programme that concerns the group averaging method is inapplicable because of that. However, while we cannot define the unitary evolution of \( \hat{H} \) immediately by exponentiating it since it is not self-adjoint, we can define the unitary evolution of \( \hat{H}^E \) and then just define \( \exp(it\hat{H}) := \hat{W}^{-1} \exp(it\hat{H}^E)\hat{W} \). The operator \( \exp(it\hat{H}^E) \) can then be used to define the physical inner product by the group averaging method.

The task to answer these questions will be left to future investigations. As it should be clear, to settle these mathematical issues it is again of utmost importance to gain maximum control over the spectrum of the volume operator \[24\].

### 4 Conclusions

Let us now summarize what can be said qualitatively about the action of the Wheeler-DeWitt constraint operator as defined in these two papers.

1) **Action of \( \hat{H}^E \):**

Spin-network states on a fixed graph are labelled by the spin quantum numbers \( j_I \) associated with the edges of the graph and a contraction matrix which turns the associated tensor product of irreducible representations into a gauge invariant function. Consider first the operator \( \hat{H}^E \). Qualitatively, the part \( h_{s_k}^{-1}[h_{s_k}^{-1}, \hat{V}] \) does not change the quantum numbers \( j_I \) at all, it changes the contraction matrix. The part \( h_{s_k}^{-1} \) on the other hand changes the spins \( j_I \) by \( \pm 1/2 \). For instance, on a trivalent graph where for given \( j_1, j_2, j_3 \) the contraction matrix is given uniquely, one can show that the action of the Euclidean Hamiltonian constraint looks like this

\[
\hat{H}_v^E T_{j_1,j_2,j_3} = i \sum_{\mu, \nu = \pm 1/2} c_{3}(\mu, \nu; j_1, j_2, j_3) T_{j_1+\mu,j_2+\nu,j_3} + \text{cyclic} \quad (4.1)
\]

for certain real-valued functions \( c_I \) of \( j_1, j_2, j_3 \) and \( T_{j_1,j_2,j_3} \) is a spin-network function corresponding to the spins \( j_I \) associated with the edges meeting at \( v \) and it is understood that the graph with respect to which \( T' \) is cylindrical
contains one of the arcs $a_{IJ}$.

One sees that the action of $\hat{H}^E$ can be visualized as the annihilation ($\mu = \nu = -1/2$), creation ($\mu = \nu = +1/2$) and rerouting ($\mu \nu = -1/4$) of spin associated with the graph in units of $\Delta j = \pm 1/2$. This picture is insensitive of whether we are dealing with the symmetric or non-symmetric version of the constraint. In other words, the picture we have is quite similar to the one we have in Quantum Electrodynamics (QED): the Hamiltonian of QED is an infinite sum of uncoupled harmonic oscillators, two for each mode (momentum $\vec{k}$). A cylindrical function for QED is a state with a finite number $n_I$ of photons of momentum $k_I$ and polarization $p_I$. On such a cylindrical function the QED Hamiltonian reduces to a finite number of harmonic oscillator Hamiltonians each of which is a polynomial of annihilation and creation operators which act by annihilating and creating the number of photons for the given mode and polarization in units of $\Delta n = \pm 1$. The two objects that correspond to each other in the two theories are first a) the continuous labels $\gamma = \vec{e}$ (the edges) and $\vec{k}, \vec{p}$ and secondly b) the discrete quantum numbers or occupation numbers $j$ and $n$.

The analogy fails in the respect that we cannot associate elementary particles (we do not have gravitons, the analogon of photons) with the elementary excitations of the gravitational field. What is excited are lines of force and the continuous information that they carry is position rather than momentum. Thus this Fock representation is based on position rather than momentum.

2) Action of $\hat{T}$

Let us now consider the operator $\hat{T}$. Since $\hat{K} \propto [\hat{V}, \hat{H}^E]$ it follows from the fact that $\hat{V}$ does not alter representations that also $\hat{K}$ acts by annihilation, creation and rerouting of spin by $\Delta j = \pm 1/2$. Also, it is clear that $h_s[h_s^{-1}, \hat{K}]$ does not modify the qualitative behaviour of $\hat{K}$. It follows then that $\hat{T}$ changes the spin of one edge by $\Delta j = -1, -1/2, 0, 1/2, 1$ because there are two factors of $\hat{K}$ involved and the various terms can act on different edges or the same again. Therefore, the behaviour of $\hat{H}$ and $\hat{H}^E$ are roughly the same, just the numerical coefficients are different, in principle we can describe the Wheeler-DeWitt operator as a low order polynomial of degree two in the creation and annihilation operators associated with the spin of the edges. The computation of the precise coefficients of this polynomial is a tedious but straightforward task. In particular, even for the symmetric operator, it seems that the spectrum can be computed either exactly or with a high degree of precision and that the self-adjoint extensions can be obtained by direct methods.

3) ADM energy is diagonal

The analogy with the Fock representation of QED is further enhanced by noticing that the ADM-Hamiltonian is diagonal on certain linear combinations of spin-network states on one and the same graph, just like the QED Hamiltonian which is diagonal on the photon Fock states. So the ADM-Hamiltonian is essentially an occupation number operator.

To see this recall that $E_{ADM} = \lim_{r \to \infty} \int_{S_r} dS \left( q_{ab,b} - q_{bb,a} \right)$ where $S_r$ is a one-parameter family of two-dimensional surfaces with the topology of $S^2$ and $r$ is the radius of the sphere as measured by a fixed asymptotic flat
background metric. Now it follows immediately from $e_a \propto \{A_a, V\}$ that
$q_{ab}$, when integrated over a two-dimensional surface has the chance to have
a well-defined quantization and that turns out to be correct [23]. Again, the
eigenvalues of $E_{ADM}$ are certain algebraic function of the spins $\vec{j}$. This fact
motivates to call the spin-network representation $|\gamma, \vec{j}, \vec{c}>$. defined abstractly
by $<\gamma, \vec{j}, \vec{c}> = T_{\gamma, \vec{j}, \vec{c}}(A)$, $[A]$ the gauge equivalence class of $A$, where as
usual $<[A]', [A]> = \delta_{[A, [A']]}$, the “non-linear Fock-representation” for
the string-like excitations of the gravitational field.
All these facts motivate to call the dynamical theory obtained “Quantum Spin
Dynamics (QSD)” as opposed to “Quantum Geometrodynamics” or QED.

4) **Final Comments in order**:
   - Both, the non-symmetric [1], (5.5) and the symmetric (3.13) version are quan-
tizations of the Wheeler-DeWitt constraint for Lorentzian, four-dimensional
quantum gravity in the continuum which are well-defined on the whole Hilbert
space $\mathcal{H}$. In that respect they differ considerably from the operator defined in
[22] which a) is given on a lattice rather than in the continuum, b) is a discretization of the rescaled form of the Wheeler-DeWitt constraint with density
weight two which is possible only on a lattice without capturing the singularities
that one will ultimately encounter in any suitable continuum limit and c) is singular on a huge subspace of the lattice Hilbert space in any ordering and
therefore is not even densely defined.
   - Our Euclidean Hamiltonian constraint operator [1], (3.10) also is
entirely different from those proposed in [10, 11] (it is our understanding that those
operators are meant for Euclidean, rather than Lorentzian gravity). The only
thing they share is that the square of the operators in [10, 11], which is singular,
and [1], (3.10) possess classical limits which are proportional to each other.
It is therefore to be expected that the solutions that have been found already
in the literature for the formal square of those operators in [10, 11] (see, for
instance, [19, 20]) are far from being annihilated by our operator. What is
appealing about the operators constructed here is that they present quanti-
zations of [1], (2.1), the original Wheeler-Dewitt constraint, rather than the
square root of a rescaled version thereof.
   - Interestingly, although the classical theory only makes sense for non-degenerate
metrics, the quantum theory does not blow up on states which represent de-
genenerate metrics since the volume operator only occurs in a positive power.
While this has been shown to be possible also in the Ashtekar framework [3]
(that is, after rescaling by $\sqrt{\det(q)}$) we see this effect already in the original
framework without rescaling.
   - There is a lot of freedom involved in the regularization step reflecting the
fact that the quantum theory of a given classical field theory is not unique.
An important but unresolved question is how to select the correct (physically
relevant) regularization procedure. A possible avenue to resolve this question is
to apply the framework to exactly soluble models and to compare the re-
results.

Another interesting question is how much freedom there actually remains in
the regularization step once we imposed our requirements as stated in section

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3.1.2 of [1].

- The final expression of the Wheeler-DeWitt constraint is surprisingly simple: on each cylindrical function it is a low order polynomial in the volume operator and holonomy operators and therefore one can find exact solutions to the Quantum Einstein Equations, perhaps even easier than it is possible to find classical solutions. Remarkably, the spectrum of the Hamiltonian constraint operator at a given vertex is largely determined by the spectrum of the volume operator so that it becomes important to gain control over it [24].

- Our simple trick, which essentially consists in replacing \( e_{i}^{a} \) by \( \{ A_{a}^{i}, V \} \) and so renders the seemingly ill-defined, non-polynomial, non-analytic (in \( E_{a}^{i} \)) operator \( \hat{e}_{a}^{i} \) into a perfectly well-defined quantity can also be applied to making sense out of operators which so far were completely out of reach as they involve \( q_{ab} \) and thus cannot be written as square roots of polynomials in \( E_{a}^{i} \).

This class of operators includes, but does not exhaust, operators that measure the length of a curve [12], the quantum generators of the asymptotic Poincaré group [13] and Hamiltonian operators describing the matter sector, as for instance Yang-Mills theory [14].

- Concluding, we have shown, that there exists a mathematically rigorous and consistent way to non-perturbatively quantize the Lorentzian Wheeler-DeWitt constraint for full four-dimensional vacuum gravity in the continuum. The stage is set to solve the theory, that is, to find explicitly the physical states, observables and to compute their spectra. As outlined above, modulo computing the precise coefficients of the expansion of a solution in terms of diffeomorphism invariant spin-network states (we also have given a method of computation), at least for the non-symmetric operator we already computed the physical Hilbert space. We are now in the position to settle non-perturbatively and rigorously questions that arise, for instance, in black hole physics.

Acknowledgments

This research project was supported in part by DOE-Grant DE-FG02-94ER25228 to Harvard University.

References

[1] T. Thiemann, “Quantum Spin Dynamics (QSD)”, Harvard Preprint HUTMP-96/B-351, The previous paper in this volume or in the gr-qc archive.
[2] A. Ashtekar, Phys. Rev. Lett. **57** 2244 (1986), Phys. Rev. **D36**, 1587 (1987).
[3] T. Thiemann, Class. Quantum Grav. **13** (1996) 1383-1403
[4] A. Ashtekar, “A generalized Wick transform for gravity”, preprint CGPG-95/12-1, gr-qc/9511083
[5] A. Ashtekar and C.J. Isham, Class. & Quan. Grav. **9**, 1433 (1992).
[6] A. Ashtekar and J. Lewandowski, “Representation theory of analytic holonomy \( C^{*}\) algebras”, in Knots and quantum gravity, J. Baez (ed), (Oxford University Press, Oxford 1994)
[7] J. Baez, Lett. Math. Phys. 31, 213 (1994); “Diffeomorphism invariant generalized measures on the space of connections modulo gauge transformations”, hep-th/9305045, in the Proceedings of the conference on quantum topology, D. Yetter (ed) (World Scientific, Singapore, 1994).

[8] A. Ashtekar, J. Lewandowski, Journ. Geo. Physics 17 (1995) 191, J. Math. Phys. 36, 2170 (1995)

[9] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann, “Quantization for diffeomorphism invariant theories of connections with local degrees of freedom”, Journ. Math. Phys. 36 (1995) 519-551

[10] C. Rovelli, L. Smolin, Phys. Rev. Lett. 72 (1994) 446

[11] A. Ashtekar, J. Lewandowski, “Regularization of the Hamiltonian constraint”, in preparation

[12] T. Thiemann, “The length operator in canonical quantum gravity”, Harvard Preprint HUTMP-96/B-394

[13] T. Thiemann, “A Hamiltonian operator for canonical quantum gravity”, Harvard University Preprint

[14] T. Thiemann, “A regularization of canonical Yang-Mills quantum theory”, Harvard University Preprint

[15] A. Higuchi Class. Quant. Grav. 8, 1983 (1991), Class. Quant. Grav. 8, 2023 (1991)

[16] D. Marolf, “The spectral analysis inner product for quantum gravity,” preprint gr-qc/9409030, to appear in the Proceedings of the VIIth Marcel-Grossman Conference, R. Ruffini and M. Keiser (eds) (World Scientific, Singapore, 1995), Class. Quant. Grav. (1995)

“Almost Ideal Clocks in Quantum Cosmology: A Brief Derivation of Time,” preprint gr-qc/9412016.

[17] J. Baez, S. Sawin, “Functional Integration on Spaces of Connections”, q-alg/9507023

[18] M. Reed, B. Simon, “Functional Analysis”, Mod. Meth. Math. Phys. Vol II, Avcademic Press, New York, 1970

[19] B. Brügmann, J. Pullin, Nucl. Phys. B 363 221

[20] B. Brügmann, J. Pullin, R. Gambini, Phys. Rev. Lett. 68 (1992) 431, Nucl. Phys. B 385 (1992) 1199

[21] T. Thiemann, “Spectral Analysis of the Wheeler-DeWitt constraint operator”, Harvard University preprint

[22] R. Loll, “A real alternative to to quantum gravity in loop space”, Preprint DFF 244/02/96, gr-qc/9602041

[23] T. Thiemann, “The ADM Hamiltonian operator for canonical quantum gravity”, Harvard University Preprint

[24] T. Thiemann, “Complete formula for the matrix elements of the volume operator in canonical quantum gravity”, Harvard Preprint HUTMP-96/B-393