A Linearly Relaxed Approximate Linear Program for Markov Decision Processes

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Abstract

Approximate linear programming (ALP) and its variants have been widely applied to Markov Decision Processes (MDPs) with a large number of states. A serious limitation of ALP is that it has an intractable number of constraints, as a result of which constraint approximations are of interest. In this paper, we define a linearly relaxed approximation linear program (LRALP) that has a tractable number of constraints, obtained as positive linear combinations of the original constraints of the ALP. The main contribution is a novel performance bound for LRALP.

Keywords: Markov Decision Processes (MDPs), Approximate Linear Programming (ALP).

I. INTRODUCTION

Markov decision processes (MDPs) have proved to be an indispensable model for sequential decision making under uncertainty with applications in networking, traffic control, robotics, operations research, business, finance, artificial intelligence, health-care and more (see, e.g., [Whi93; Rus96a; FS02; HY07; SB10; BR11; Put94; LL12; AA+15; BD17]). In this paper we adopt the framework of discrete-time, discounted MDPs when a controller steers the stochastically evolving state of a system while receiving rewards that depends on the states visited and actions chosen. The goal is to choose the actions so as to maximize the return, defined as the total discounted expected reward. A controller that uses past state information is called a policy. An optimal policy is one that maximizes the value no matter where the process is started from [Put94]. In this paper we consider planning problems where the goal is to calculate actions of policies

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that give rise to high values and give new error bounds on the quality of solutions obtained by solving linear programs of tractable size. To explain the contributions in more details, we start by describing the computational challenges involved in planning.

The main objective of planning is to compute actions of an optimal policy while interacting with an MDP model. In finite state-action MDPs, assuming access to individual transition probabilities and rewards along transitions, various algorithms are available to perform this computation in time and space that scales polynomially with the number of states and actions. However, in most practical applications, the MDP is compactly represented and if it is not infinite, the number of states scale exponentially with the size of the representation of the MDP. If planners are allowed to perform some fixed amount of calculations for each state encountered, it is possible to use sampling to make the per-state calculation-cost independent of the size of the state space [Rus96b; Sze01; KMN02]. Nevertheless, the resulting methods are still quite limited. In fact, various hardness results show that computing actions of (near-) optimal policies is intractable in various senses and in various compactly represented MDPs [BT00]. Given these negative results, it is customary to adopt the modest goal of efficiently computing actions of a policy that is nearly as good as a policy chosen by a suitable (computationally unbounded, and well-informed) oracle from a given restricted policy class. Here, within some restrictions (see below), the policy class can be chosen by the user. The more flexibility the user is given in this choice, the stronger a planning method is. The problem of planning with limited resources is also one of the key problem in artificial intelligence (AI). The book of [KM12] gives a relatively fresh, algorithm-centered summary of existing methods suitable for planning in MDPs. AI research tend to focus on empirical results through the development of various benchmarks and little if any effort is devoted to the theoretical understanding of the quality-effort tradeoff exhibited by the that the various algorithms that are developed in this field.

A popular approach along these lines, which goes back to Schweitzer and Seidmann [SS85], relies on considering linear approximations to the optimal value function: The idea is that, similarly to linear regression, a fixed sequence of basis functions are combined linearly. The user’s task is to use a priori knowledge of the MDP to choose the basis functions so that a good approximation to the optimal value function will exist in the linear space spanned by the basis functions. The idea then is to design some algorithm to find the coefficients of the basis functions that gives a good approximation, while keeping computation cost in check. Finding a good approximation is sufficient, since at the expense of an extra $O(1/\varepsilon^2)$ randomized computation,
a uniform $O(\varepsilon)$-approximation to the optimal value function can be used to calculate an action of an $O(\varepsilon)$-optimal policy at any given state (e.g., follow the ideas in [Sze01; KMN02]; see also Theorem 3.7 of Kallenberg [Kal17]). Since the number of coefficients can be much smaller than the number of states, the algorithms that search for the coefficients have the potential to run efficiently regardless of the number of states.

Following Schweitzer and Seidmann [SS85], most of the literature considers algorithms that are obtained from restricting exact planning methods to search in the span of the fixed basis functions when performing computations. In this paper we consider the so-called approximate linear programming (ALP) approach, which was heavily studied during the last two decades, e.g., [SP01; Gue+03; FV03; FV04; KH04; PZ09; DFM09; Tay+10; PP11; BFM12; AYBM14]. The basic idea here is to combine a linear program whose solution is the optimal value function (and thus the number of optimization variables in it scales with the number of states) with a linear constraint that restricts the optimization variables to lie in the subspace spanned by the basis functions. As already noted by Schweitzer and Seidmann [SS85], the new LP can still be kept feasible by just adding one special basis function, while by substituting the “value function candidates” with their linear expansions, the number of optimization variables becomes the number of basis functions. As shown by de Farias and Van Roy [FV03], the solution to the resulting LP is within a constant factor of the best approximation to the optimal value function within the span of the chosen bases. However, since the number of constraints in the LP is still proportional to the number of states, it is not obvious whether a solution to the resulting LP can be found in time independent of the number of states (other computations can be done in time independent of the number of states, e.g., using sampling, at the price of a controlled increase of the error, e.g., Theorem 6 of [PZ09]).

Most of the literature is thus devoted to designing methods to select a tractable subset of the constraints while keeping the approximation guarantees, as well as keeping computations tractable. Since a linear objective is optimized by a point on the boundary of the feasible region, knowing the optimizer would be sufficient to eliminate all but as many constraints as the number of optimization variables. The question is how to find a superset of these, or an approximating set, without incurring much computational overhead. Schuurmans and Patrascu [SP01] and Guestrin et al. [Gue+03] propose constraint generation in a setting where the MDP has additional structure (i.e., factorized transition structure). This additional structure is then exploited in designing constraint generation methods which are able to efficiently generate violated constraints. A more
general approach due to de Farias and Van Roy [FV04] is to choose a random subset of the constraints by choosing states to be included at random from a distribution that reflects the “importance” of states. While constraint generation can be powerful, it is not known how solution quality degrades with the budget on the constraints generated (Guestrin et al. note that the number of constraints generated can be at most exponential in a fundamental quantity, the induced width of a so-called cost-network, which may be large and is in general hard to control). For constraint sampling, de Farias and Van Roy [FV04] prove a bound on the suboptimality, but this bound applies only in the unrealistic scenario when the constraints are sampled from an idealized distribution, which is related to the stationary distribution of an optimal policy. While it is possible to extend this result to any sampling distribution, the bound then scales with the mismatch between the sampling and the idealized distributions, which, in general, will be uncontrolled. Another weakness of the bound is related to that when constraints are dropped, the linear program may become unbounded. To prevent this, de Farias and Van Roy [FV04] propose imposing an extra constraint on the optimization variables. The bound they obtain, however, scales with the worst approximation error over this constraint set. While in a specific example it is shown that this error can be controlled, no general results are derived in this direction. Later works, such as that of Desai, Farias, and Moallemi [DFM09] and Bhat, Farias, and Moallemi [BFM12], repeat the analysis of de Farias and Van Roy [FV04] in combinations with other ideas. However, no existing work that we know of addresses the above weaknesses of the result of de Farias and Van Roy [FV04].

Another interesting approach is to consider the dual linear program, where the optimization variables are measures over the state-action space and the feasible set is the set of discounted state-occupation (DSO) measures of all possible policies. By adding an extra linear constraint on the optimization variables, we arrive at an “approximate dual LP”. Feasibility of the resulting LP can be ensured by adding basis functions that represent DSO measures of some select policies. Abbasi-Yadkori, Bartlett, and Malek [AYBM14] considered this approach together and proposed to use a randomized gradient method to minimize a penalized form of the linear objective to approximately enforce the constraints. The algorithm computes the parameters of a measure over the state-action space, from which a policy can be derived by normalization. The main result of Abbasi-Yadkori, Bartlett, and Malek is a bound on the performance loss of this policy relative
to the performance of the best DSO measure in the feasible set of the approximate dual LP,\(^1\)
while iteration cost to obtain an \(\varepsilon\)-competitive solution is \(O(1/\varepsilon^4)\) provided that a number of conditions hold. On the above complexity bound, the constants hidden are instance dependent, but do not depend on the number of states or actions. The conditions under which the result is proven are as follows: (i) the algorithm needs to be able to sample from distributions not too dissimilar to the idealized distributions \(q_1, q_2\), where \(q_1\) is a distribution over state-action pairs, \(q_2\) is a distribution over states and, e.g., \(q_1\) is defined by
\[
q_1(s,a) = \frac{\|\phi(s,a)\|}{\sum_{s',a'} \|\phi(s',a')\|}
\]
with \(\phi(s,a) = (\phi_1(s,a), \ldots, \phi_k(s,a))^\top\) and \(\phi_1, \ldots, \phi_k\) being the chosen basis functions specifying the linear constraints and \((s,a)\) is a state-action pair; (ii) the Markov chains underlying all policies in the MDP are uniformly fast mixing; (iii) for any state \(s' \in S\) and index \(1 \leq i \leq k\), the expression \(\sum_{s,a} \phi_i(s,a)p_a(s,s')\) can be evaluated in \(O(1)\) time, where \(p_a(s,s')\) is the probability of transitioning from state \(s\) to \(s'\) provided action \(a\) is chosen. While the second assumption limits the scope of MDPs that the result can be applied to, the other two assumptions limit the choice of the basis functions. Among other things, it is unclear how feasibility can be ensured while satisfying (i). Nevertheless, Abbasi-Yadkori, Bartlett, and Malek demonstrate promising empirical results on a queuing problem.

Our main contribution is a new suboptimality bound for the case when the constraint system is replaced with a smaller, linearly projected constraint system. We also propose a specific way of adding the extra constraint to keep the resulting LP bounded. Rather than relying on combinatorial arguments (such as those at the heart of de Farias and Van Roy [FV04]), our argument uses previously unexploited geometric structure of the linear programs underlying MDPs. As a result our bound avoids distribution-mismatch terms and we also remove the scaling with worst approximation error. A specific outcome of our general result is the realization that it is beneficial to select states so that the “feature vectors” of all states when scaled with a fixed constant factor are included in the conic hull of the “feature vectors” underlying the selected states. This suggests to choose the basis functions so that this property can be satisfied by selecting only a few states. As we will argue, this property holds for several popular choices of basis functions. A preliminary version of this paper without the theoretical analysis and without the geometric arguments was published in a short conference communication [LB15].

\(^1\)The result shown is more general, allowing to use measures outside of the feasible set. However, for such measures the performance bound degrades very rapidly and hence the greater generality does not seem to add much to the result.
II. BACKGROUND

The purpose of this section is to introduce the necessary background before we can present the problem studied and the main results.

We shall consider finite state-action space, discounted total expected reward MDPs. We note in passing that the assumption that number of states is finite is mainly made for convenience and at the expense of a more technical presentation could be lifted. We will comment later on the assumption concerning the number of actions. Let the set of states, or state space be \( S = \{1, 2, \ldots, S\} \) and let the set of actions be \( A = \{1, 2, \ldots, A\} \). For simplicity, we assume that all actions are admissible in all states. Given a choice of an action \( a \in A \) in a state \( s \in S \), the controller incurs a reward (or gain) of \( g_a(s) \in [0, 1] \) and the state moves to a next state \( s' \in S \) with probability \( p_a(s, s') \). A policy \( u \) is a mapping from states to actions. When a policy is followed, the state sequence evolves as a Markov chain with transition probabilities given by \( P_u \) matrix whose \( (s, s') \)th entry is \( P_u(s, s') \). Along the way the rewards generated from \( g_u \) defined by \( g_u(s) = g_u(s) \). The value of following a policy from a starting state \( s \) is denoted by \( J_u(s) \) and is defined as the expected total reward discounted reward. Thus,

\[
J_u(s) = \sum_{t=0}^{\infty} \alpha^t (P_u^t g_u)(s),
\]

where \( \alpha \in (0, 1) \) is the so-called discount factor. We call \( J_u \) the value function of policy \( u \). The value function of a policy satisfies the fixed-point equation \( J_u = T_u J_u \) where the affine-linear operator \( T_u \) is defined by \( T_u J = g_u + \alpha P_u J \). An optimal policy, is one that maximizes the value simultaneously for all initial states. The optimal value function \( J^* \) is defined by \( J^*(s) = \max_u J_u(s) \) and is known to be the solution of the fixed-point equation \( J^* = T J^* \) where the operator \( T \) is defined by \( (T J)(s) = \max_u (T_u J)(s) \), \( s \in S \), i.e., the maximization is component-wise. Optimal policies exist and in fact any policy \( u \) such that the equation \( T_u J^* = T J^* \) holds is optimal (e.g., Corollary 3.3 of [Kal17]). A policy \( u \) is said to be greedy with respect to (w.r.t.) \( J \) if \( T_u J = T J^* \). Thus, any policy that is greedy w.r.t. \( J^* \) is optimal.

III. THE LINEARLY RELAXED ALP

In this section we introduce the computational model used and the “Linearly Relaxed Approximate Linear Program” a relaxation of the ALP.

2For the scope of this paper, it suffices to restrict our attention to such policies as opposed to considering history dependent policies. See Chapter 3, and specifically Corollary 3.3 of [Kal17].
As discussed in the introduction, we are interested in methods that compute a good approximation to the optimal value function. As noted earlier, at the expense of a modest additional cost, knowing an $O(\varepsilon)$ approximation to $J^*$ at a few states suffices to compute actions of an $O(\varepsilon)$-optimal policy. We will take a more general view, and we will consider calculating good approximations to $J^*$ with respect to a weighted 1-norm, where the weights $c$ form a probability distribution over $S$. Recall that the weighted 1-norm $\|J\|_{1,c}$ of a vector $J \in \mathbb{R}^S$ is defined as $\|J\|_{1,c} = \sum_s c(s)|J(s)|$. Note that here and in what follows we identify elements of $\mathbb{R}^S$ (functions, mapping $S = \{1, \ldots, S\}$ to the reals) with elements of $\mathbb{R}^S$ in the obvious way. This allows us to write e.g. $c^\top J$, which denotes $\sum_s c(s)J(s)$.

To introduce the optimization problem we study, first recall that the optimal value function $J^*$ is the solution of the fixed point equation $TJ^* = J^*$. It follows from the definition of $T$ that $J^* = \max_a T_a J^* \geq T_a J^*$ for any $u$, where $\geq$ is the componentwise partial ordering of vectors ($\leq$ is the reverse relation). With some abuse of notation, we also introduce $T_a$ to denote $T_u$ where $u(s) = a$ for any $s \in S$. It follows that $J^* \geq T_a J^*$ for any $a \in A$ and also that $T = \max_a T_a$, where again the maximization is componentwise. We call a vector $J$ that satisfies $J \geq T_a J$ for any $a \in A$ a superharmonic. Note that this is a set of linear inequalities. By our note on $T$ and $(T_a)_a$, these inequalities can also be written compactly as $J \geq TJ$. It is not hard to show then that $J^*$ is the smallest superharmonic function (i.e., for any $J$ superharmonic, $J \geq J^*$). It also follows that for any $c \in \mathbb{R}_+^S = (0, \infty)^S$, the unique solution to the linear program $\min\{c^\top J : J \geq TJ\} = \min\{c^\top J : J \geq T_a J, a \in A\}$ is $J^*$.

Now, let $\phi_1, \ldots, \phi_k : S \to \mathbb{R}$ be $k$ basis functions. The Approximate Linear Program (ALP) of Schweitzer and Seidmann [SS85] is obtained by adding the linear constraints $J = \sum_{i=1}^k r_i \phi_i$ to the above linear program. Eliminating $J$ gives
\[
\min\\left\{ \sum_i r_i c^\top \phi_i : \sum_i r_i \phi_i \geq g_a + \alpha \sum_i r_i P_a \phi_i, a \in A, r = (r_i) \in \mathbb{R}^k \right\}.
\]

As noted by Schweitzer and Seidmann [SS85], the linear program is feasible as long as 1, defined as the vector with all components being identically equal to one, is in the span of $\{\phi_1, \ldots, \phi_k\}$. For the purpose of computations, it is assumed that the values $c^\top \phi_i$, $i = 1, \ldots, k$ and the values $(P_a \phi_i)(s)$ and $g_a(s)$ can be accessed in constant time. This assumption can be relaxed to assuming that one can access $g_a(s)$ and $\phi_i(s)$ for any $(s, a)$ in constant time, as well as to that one can efficiently sample from $c$, from $P_a(s, \cdot)$ for any $(s, a)$ pair, but the details of this are the beyond the scope of the present work. As shown by de Farias and Van Roy [FV03],
if \( r_{\text{ALP}} \) denotes the solution to the above ALP then for \( J_{\text{ALP}} = \sum_i r_{\text{ALP}}(i)\phi_i = \Phi r_{\text{ALP}} \) it holds that \( \|J_{\text{ALP}} - J^*\|_{1,c} \leq \frac{2\varepsilon}{1-\alpha} \), where \( \varepsilon = \inf_r \|J^* - \Phi r\|_{\infty} \) is the error of approximating the optimal value with the span of the basis functions \( \phi_1, \ldots, \phi_k \) and \( \|J\|_{\infty} = \max_s |J(s)| \) is the maximum norm and \( \Phi \in \mathbb{R}^{S \times k} \) is the matrix formed by \( (\phi_1, \ldots, \phi_k) \). That the error of approximating \( J^* \) with \( J_{\text{ALP}} \) is \( O(\varepsilon) \) is significant: The user can focus on finding a good basis, leaving the search for the “right” coefficients to a linear program solver.

While solving the ALP can be significantly cheaper than solving the LP underlying the MDP and thus it can be advantageous for moderate-scale MDPs, the number of constraints in the ALP is \( SA \), hence the ALP is still intractable for huge-scale MDPs. To reduce the number of constraints, we consider a relaxation of ALP where the constraints are replaced with positive linear combinations of them. Recalling that the constraints took the form \( J \geq g_a + \alpha P_a J \) (with \( J = \Phi r \)), choosing \( m \) to be target number of constraints, for \( 1 \leq i \leq m \), the \( i \)th new constraint is given by \( \sum_a w_{i,a}^T J \geq \sum_a w_{i,a}^T (g_a + \alpha P_a J) \), where the choice of \( m \) and that of the vectors \( w_{i,a} \in \mathbb{R}^S_+ \) is left to the user. Note that this results in a linear program with \( k \) variables and \( m \) constraints, which can be written as

\[
\begin{align*}
\min_{r \in \mathbb{R}^k} & \quad c^T \Phi r \\
\text{s.t.} & \quad \sum_a W_a^T \Phi r \geq \sum_a (W_a^T (g_a + \alpha P_a)) \Phi r ,
\end{align*}
\]

where \( W_a = (w_{1,a}, \ldots, w_{m,a}) \in \mathbb{R}_+^{S \times m} \). Note that the \((i,j)\)th entry of the \( m \times k \) constraint matrix of the resulting LP is \( \sum_a w_{i,a}^T \phi_j - \alpha \sum_a w_{i,a}^T P_a \phi_j \) and assuming that \((w_{i,a})_a\) has \( p \) nonzero elements, this can be calculated in \( O(p) \) time, making the total cost of obtaining the constraint matrix to be \( O(mkp) \) regardless the value of \( S \) and \( A \).

We will call the LP in (1) the linearly relaxed approximate linear program (LRALP). Any LP obtained using any constraint selection/generation process can be represented by choosing an appropriate binary-valued matrix \( W^T = (W_1^T, \ldots, W_A^T) \in \mathbb{R}_+^{m \times SA} \). In particular, when the constraints are selected in a random process as suggested by de Farias and Van Roy [FV04], the matrix \( W \) would be a random, binary-valued matrix.

Note that the LRALP may be unbounded. Unboundedness could be avoided by adding an
extra constraint of the form \( r \in \mathcal{N} \) to the LRALP, for a properly chosen polyhedron \( \mathcal{N} \subset \mathbb{R}^k \).\(^3\)

However, it seems to us that it is downright misleading to think that guaranteeing a bounded solution will also lead to reasonable solutions. Thus we will stick to the above simple form, forcing a discussion of how \( W \) should be chosen to get meaningful results.\(^4\)

Further insight into the choice of \( W \) can be gained by considering the Lagrangians of the ALP and LRALP. To write both LP’s in a similar form let us introduce \( E = (I_{S \times S}, \ldots, I_{S \times S})^\top \), where \( I_{S \times S} \) is the \( S \times S \) identity matrix. Further, let \( H : \mathbb{R}^S \rightarrow \mathbb{R}^{S \times A} \) be the operator defined by

\[
(HJ)^\top = ((T_1 J)^\top, \ldots, (T_A J)^\top).
\]

Note that \( H \), which we call the *linear Bellman operator*, is a linear operator. Then, the ALP can be written as

\[
\min \{ c^\top \Phi r \mid E\Phi r \geq H\Phi r \}, \tag{ALP}
\]

while LRALP takes the form

\[
\min \{ c^\top \Phi r \mid W^\top E\Phi r \geq W^\top H\Phi r \}. \tag{LRALP}
\]

Hence, their Lagrangians are

\[
\mathcal{L}_{\text{ALP}}(r, \lambda) = c^\top \Phi r + \lambda^\top (H\Phi r - E\Phi r) \quad \mathcal{L}_{\text{LRALP}}(r, q) = c^\top \Phi r + q^\top W^\top (H\Phi r - E\Phi r).
\]

Thus, we can view \( Wq \) as a “linear approximation” to the dual variable \( \lambda \in \mathbb{R}^{S \times A} \). This suggests that perhaps \( W \) should be chosen such that it approximates well the optimal dual variable. If \( \Phi \) spans \( \mathbb{R}^S \), the optimal dual variable \( \lambda^* \) is known to be the discounted occupancy measure underlying the optimal policy (Theorem 3.18, \[Kal17\]), suggesting that the role of \( W \) is very similar to the role of \( \Phi \) excepts that the subspace spanned by the columns of \( W \) should ideally be close to \( \lambda^* \).

\(^3\) In particular, to obtain their theoretical result, de Farias and Van Roy \[FV04\] need the assumption that the set \( \mathcal{N} \) is bounded and that it contains \( r_{\text{ALP}} \). In fact, the error bound derived by de Farias and Van Roy depends on the worst error of approximating \( J^* \) with \( \Phi r \) when \( r \) ranges over \( \mathcal{N} \). Hence, if \( \mathcal{N} \) is unbounded, their bound is vacuous. In the context of a particular application, de Farias and Van Roy \[FV04\] demonstrate that \( \mathcal{N} \) can be chosen properly to control this term. However, no general construction is presented to choose \( \mathcal{N} \).

\(^4\) The only question is whether there is some value in adding constraints beyond choosing \( W \) properly. Our position is that the set \( \mathcal{N} \) would most likely be chosen based on very little and general information; the useful knowledge is in choosing \( W \), not in choosing some general set \( \mathcal{N} \). Since randomization does not guarantee bounded solutions, de Farias and Van Roy \[FV03\] must use \( \mathcal{N} \): In their case, \( \mathcal{N} \) incorporates all the knowledge that makes the LP bounded.
IV. Main Results

The purpose of this section is to present our main results. Let \( r_{\text{LRA}} \) be a solution to the LRALP given by (1) and let \( J_{\text{LRA}} = \Phi r_{\text{LRA}} \). When multiple solutions exist, we can choose any of them. For the result, we assume that the LRALP is not unbounded, and hence a solution exist. In fact, we will assume something much stronger. The discussion of why our assumptions are reasonable and how to ensure that they hold is postponed to after the presentation of our results. Our main results bounds the error \( \| J^* - J_{\text{LRA}} \|_{1,c} \).

The bound is given in terms of the approximation error of \( J^* \) with the basis functions \( \Phi = (\phi_1, \ldots, \phi_k) \), as well as the deviation between two functions, \( J_{\text{ALP}}^*, J_{\text{LRA}}^* : \mathcal{S} \rightarrow \mathbb{R} \), which we define next. In particular,

\[
J_{\text{ALP}}^*(s) = \min \{ r^T \phi(s) \mid \Phi r \geq J^*, r \in \mathbb{R}^k \}, \\
J_{\text{LRA}}^*(s) = \min \{ r^T \phi(s) \mid W^T E \Phi r \geq W^T E J^*, r \in \mathbb{R}^k \},
\]

where \( s \in \mathcal{S} \). Recall that \( E : \mathbb{R}^S \rightarrow \mathbb{R}^{SA} \) is defined so that \( (EJ)^T = (J^T, \ldots, J^T) \), i.e., \( E \) stacks its argument \( A \)-fold. Hence, \( W^T E = \sum_a W_a^T \). Our strong assumption is that \( J_{\text{LRA}}^* \) is finite-valued. Note that \( J_{\text{ALP}}^* \geq J^* \) reflects the error due to using the basis functions \( (\phi_j) \), and the magnitude of the deviation \( J_{\text{LRA}}^* - J_{\text{ALP}}^* \) reflects the error introduced due to the relaxed constraint system.

Following de Farias and Van Roy [FV03; FV04], we will quantify the magnitude of the error \( J_{\text{LRA}}^* - J_{\text{ALP}}^* \) and also that of the error of approximating \( J^* \) with the subspace spanned by \( \Phi \), in terms of a \textit{weighted maximum norm}, \( \| J \|_{\infty, \psi} = \max_{s \in \mathcal{S}} |J(s)|/\psi(s) \), where \( \psi : \mathcal{S} \rightarrow \mathbb{R}^{++} \) is a positive-valued weighting function.\(^5\) As also stressed by de Farias and Van Roy, the appropriate choice of \( \psi \) is crucial for MDPs with huge state-spaces: The problem is that if the range of values of \( |J^*(s)| \) in different parts of the state space differ in orders of magnitude, we do not expect to be able to control the error of approximating \( J^* \) uniformly over \( \mathcal{S} \). By choosing the weighting function to reflect the magnitude of \( J^* \), the weighted maximum norm is controlled as soon as the relative errors are and this latter goal may be much easier to achieve than controlling absolute errors.

\(^5\)As opposed to de Farias and Van Roy [FV03] and others, our definition uses division and not multiplication with the weights. We choose this form for mathematical convenience: With this definition, nice duality results hold between weighted 1-norms and weighted maximum norms.
Just like de Farias and Van Roy [FV03], we will also require that $\psi$ is a stochastic Lyapunov-function for the MDP. In particular, we require that the $\alpha$-discounted stability coefficient

$$
\beta_\psi = \alpha \max_a \|P_a \psi\|_{\infty, \psi}
$$

is strictly less than one. This can be seen to imply that $H : (\mathbb{R}^S, \| \cdot \|_{\infty, \psi}) \rightarrow (\mathbb{R}^{SA}, \| \cdot \|_{\infty, \psi})$ is a contraction, where for $J = (J_1^T, \ldots, J_A^T)^T \in \mathbb{R}^{SA}$ we let $\|J\|_{\infty, \psi} = \max_a \|J_a\|_{\infty, \psi}$. That $H$ is a contraction will play a crucial role in our results. Note that the condition $\beta_\psi < 1$ is closely related to the condition that for any policy $u$, $P_u \psi \leq \psi$, which can be viewed as a stability condition on the MDP and which appeared in a slightly altered form in studying the stability of MDPs with infinite state spaces [e.g., CM99]. Note also that one can always choose $\psi = 1$, which gives $\beta_1 = \alpha < 1$. With this, we are ready to state our main result:

**Theorem IV.1 (Error Bound for LRALP).** Assume that $c \in \mathbb{R}_+^S$ is such that $1^T c = 1$ and that $W \in \mathbb{R}_+^{SA \times m}$ is nonnegative valued. Let $\psi \in \mathbb{R}_+^S$ be in the column span of $\Phi$ and assume that the $\alpha$-discounted stability coefficient of $\psi$ is $\beta_\psi < 1$. Let $\varepsilon = \inf_{r \in \mathbb{R}^k} \|J^* - \Phi r\|_{\infty, \psi}$ be the error of approximation $J^*$ using the basis functions in $\Phi$. Then,

$$
\|J^* - J^*_{\text{LRA}}\|_{1, c} \leq \frac{2c^T \psi}{1 - \beta_\psi} (2.5 \varepsilon + \|J^*_{\text{ALP}} - J^*_{\text{LRA}}\|_{\infty, \psi}).
$$

Note that the result implicitly assumes that $J^*_{\text{LRA}}$ exists, because if $J^*_{\text{LRA}}$ does not exist then $J^*_{\text{LRA}}$ is necessarily unbounded, making the last error term infinite. To ensure that $\psi$ is in the span of $\Phi$, after choosing $\psi$, one can add $\psi$ as one of the basis functions. Alternatively, the bound can also be interpreted to hold for any $\psi$ in the span of $\Phi$ with $\beta_\psi < 1$.

As noted earlier, de Farias and Van Roy [FV03] prove a similar error bound for $J^*_{\text{ALP}}$, the solution of the ALP. In particular, their Theorem 3 states that under identical assumptions as in our result, $\|J^* - J^*_{\text{ALP}}\|_{1, c} \leq \frac{2c^T \psi \varepsilon}{1 - \beta_\psi}$ for $\varepsilon$ defined as above (the result we cited previously is a simplified form of this bound). The larger coefficient of $\varepsilon$ is probably an artifact of our analysis. Note that when $W$ does not reduce the constraints, our bound is only a constant factor larger than this previous result. The extra term $\|J^*_{\text{ALP}} - J^*_{\text{LRA}}\|_{\infty, \psi}$ can be seen as the price paid for relaxing the constraints.

From linear programming theory, it follows that primal boundedness is equivalent to dual feasibility. Since the dual of $\min \{ c^T x : Ax \geq b \}$ is $\max \{ y^T b : y \geq 0, c = A^T y \}$, we get that a necessary and sufficient condition for $J^*_{\text{LRA}}$ to be finite-valued is that for any $s \in S$, $\phi(s)$ lies in the conic span, $\{ U \lambda : \lambda \in \mathbb{R}^{SA}_+ \}$, of (the columns) of $U = \Phi^T E^T W$. When $W$
is such that its constituents \( W_1, \ldots, W_A \) are all identical, the conic span of \( U \) is equal to the conic span of \( \Phi^TW_1 \). It is particularly instructive to consider the case when the common matrix \( W_a = (w_{1,a}, \ldots, w_{m,a}) \) “selects” the \( m \) states, i.e., when \( \{w_{1,a}, \ldots, w_{m,a}\} = \{e_s : s \in S_0\} \) for some \( S_0 \subset S \), \( |S_0| \leq m \), where \( e_s \in \{0,1\}^S \) are the \( s \in S \) vectors in the standard Euclidean basis. In this case, the condition that \( \phi(s) \) lies in the conic span of \( U \) is equivalent to that \( \phi(s) \) lies in the conic span of \( \phi(S_0) \triangleq \{\phi(s') : s' \in S_0\} \). Thus, to ensure boundedness of \( J^*_\text{LRA} \), the chosen states should be selected to “conicly cover” all the vectors in \( \phi(S) \subset \mathbb{R}^k \).\(^6\)

The next theorem shows that magnitudes of the coefficients used in the conic cover control the size of \( \|J^*_\text{ALP} - J^*_\text{LRA}\|_{\infty,\psi} \). For the theorem we let \( \Lambda \in \mathbb{R}_{+}^{S \times S_0} \) be the matrix of conic coefficients:

\[
\text{Theorem IV.2. Assume that } W_1 = \cdots = W_A, \{w_{1,a}, \ldots, w_{m,a}\} = \{e_s : s \in S_0\} \text{ and that } \phi(S) \text{ lies in the conic span of } \phi(S_0) \text{ with conic coefficients given by } \Lambda. \text{ Let } \varepsilon = \inf_r \|J^* - \Phi r\|_{\infty,\psi}. \text{ Then,}
\]

\[
\|J^*_\text{ALP} - J^*_\text{LRA}\|_{\infty,\psi} \leq \|J^*_\text{ALP} - J^*\|_{\infty,\psi} + (1 + \|\Lambda \psi\|_{\psi,\infty}) \varepsilon.
\]

\textit{Proof.} Let \( r^* \) be such that \( \|J^* - \Phi r^*\|_{\infty,\psi} = \varepsilon \) (this exists by continuity) and let \( \delta = J^* - \Phi r^* \). Pick any \( s \in S \) and let \( r_s = \arg\min_{r \in \mathbb{R}^k} \{r^T \phi(s) : W^T E \Phi r \geq W^T E J^*, \ r \in \mathbb{R}^k\} \) so that \( J^*_\text{LRA}(s) = r^T_s \phi(s) \). Note that by assumption, for any \( s' \in S_0 \), \( J^*_\text{LRA}(s') = r^T_s \phi(s') \geq J^*(s') \). Now, notice that by definition, \( J^*_\text{LRA} \leq J^*_\text{ALP} \) (the LP defining \( J^*_\text{LRA} \) is the relaxation of the LP defining \( J^*_\text{ALP} \)). Hence,

\[
0 \leq J^*_\text{ALP}(s) - J^*_\text{LRA}(s) = J^*_\text{ALP}(s) - J^*(s) + J^*(s) - J^*_\text{LRA}(s)
\]

and \( J^*_\text{LRA}(s) = r^T_s \phi(s) = r^T_s \sum_{s' \in S_0} \Lambda(s, s') \phi(s') = \sum_{s' \in S_0} \Lambda(s, s') J^*_\text{LRA}(s') \geq \sum_{s' \in S_0} \Lambda(s, s') J^*(s') \). Combining this with the previous inequality we get

\[
0 \leq \frac{J^*_\text{ALP}(s) - J^*_\text{LRA}(s)}{\psi(s)} \leq \frac{J^*_\text{ALP}(s) - J^*(s)}{\psi(s)} + \frac{J^*(s) - \sum_{s' \in S_0} \Lambda(s, s') J^*(s')}{\psi(s)}.
\]

\text{\(^6\)The same implies that, under the same condition on } W, \text{ boundedness of the LRALP holds if and only if } \sum_s c(s) \phi(s) \text{ is in the conic span of } \phi(S_0). \text{ Note that this is easy to fulfill if the support of } c \text{ has a small cardinality by add all states in the support of } c \text{ to } S_0.\)
Plugging in $J^*(s) = \phi(s)^T r^* + \delta(s)$, using again that $\phi(s) = \sum_{s' \in S_0} \Lambda(s, s') \phi(s')$, and also using the triangle inequality after taking absolute values, we get

$$\frac{|J^*(s) - \sum_{s' \in S_0} \Lambda(s, s') J^*(s')|}{\psi(s)} \leq \frac{|\delta(s)|}{\psi(s)} + \frac{\sum_{s' \in S_0} \Lambda(s, s') |\delta(s')|}{\psi(s)} \leq \frac{|\delta(s)|}{\psi(s)} + \frac{1}{\psi(s)} \sum_{s' \in S_0} \Lambda(s, s') \psi(s') |\delta(s')| \leq \frac{|\delta(s)|}{\psi(s)} + \|\delta\|_{\infty,\psi} \frac{\sum_{s' \in S_0} \Lambda(s, s') \psi(s')}{\psi(s)}.$$  

Combining this with the previous display and noting that $\|\delta\|_{\infty,\psi} = \varepsilon$ finishes the proof. □

Given $\phi : S \to \mathbb{R}^k$, what is the minimum cardinality set $S_0$ that conically covers $\phi(S)$ and how to find such a set? Further, how to keep the magnitude of $\|\Lambda \psi\|_{\infty,\psi}$ small? To control this latter quantity it seems essential to make sure $S_0$ contains states with high $\psi$-values. However, if one is content with a bound that depends on $\|\psi\|_\infty$, one can bound $\|\Lambda \psi\|_{\infty,\psi}$ by $\|\psi\|_\infty \zeta$ where $\zeta = \max_s \sum_{s' \in S_0} \Lambda(s, s')$, hence, the second term in the previous bound will be bounded by $(1 + \|\psi\|_\infty \zeta) \varepsilon$.

Let us now return to the problem of finding conic covers. We will proceed by considering some illustrative examples. As a start, consider the case when the basis functions are binary valued. In this case, it is sufficient and necessary to choose one state for each binary vector that appears in $\phi(S) \subset \{0, 1\}^k$. This gives that $m_0 = |S_0| \leq 2^k$ representative states will be sufficient regardless of the cardinality of $S$. Further, in this case $\zeta = 1$. For moderate to large $k$ (e.g., $k \gg 20$), it will quickly become infeasible to keep $2^k$ constraints. In this case we may need to restrict what features are considered to guarantee the conic cover condition. Letting $A_i = \{s \in S : \phi_i(s) = 1\}$, if for a many pairs $i \neq j$, $A_i$ and $A_j$ do not overlap then $N = |\phi(S)|$ can be much smaller than $2^k$. For example, in the commonly used state aggregation procedures $A_i \cap A_j = \emptyset$ for any $i \neq j$, giving $N = k$. In the more interesting case of hierarchical aggregation (when the sets $\{A_i\}$ form a nested hierarchical partitioning of $S$), we have $m_0 \leq D \cdot k$ where $D$ is the depth of the hierarchy.

Another favourable example is the case of separable bases. In this case, the states are assumed to be factored and the basis functions depend only on a few factors. Let us consider a simple illustration. By abusing notation (redefining $S$), let $S = S_1 \times S_2$, let there be $k = 2$ basis functions and assume that $\phi_i(s) = h_i(s)$ for some $h_i : S_i \to \mathbb{R}$, $i = 1, 2$. Assume further that $0 \in h_i(S_i)$ for both $i$ and specifically let $s_{i0}$ be such that $h_i(s_{i0}) = 0$. In this case it is not hard to verify that
if $S_{i0}$ is such that $h_i(S_i)$ is in the conic span of $h_i(S_{i0})$ then $\phi(S)$ is also in the conic span of $S_0 = S_1 \times \{s_{20}\} \cup \{s_{10}\} \times S_2$. The point is that $|S_0| \leq |S_1| + |S_2|$, which is a tolerable increase of growth. This example is not hard to generalize to more general, ANOVA-like basis expansions. The moral is that as long as their limited order of interaction (which is usually necessary for information theoretic reasons as well), the number of constraints may grow moderately with the number of factors (dimensionality) of the state space.

In some cases, finding a conic cover with a small cardinality is not possible. This can already happen in simple examples such as when $S = \{1, \ldots, S\}$ (as before) and $\phi(s) = (1, s, s^2)$. In this case, the only choice is $S_0 = S$. In examples similar to this one one possibility is to quantize the range of $\phi$, which may loose little on approximation quality, while it creates the opportunity to construct a small cardinality conic cover.

Note that the bound of [FV04] and our main result can be seen as largely complementary. Recall that de Farias and Van Roy consider adding an extra constraint $r \in N$, while they propose to select all $A$ constraints from the ALP corresponding to $m$ states chosen at random from some distribution $\mu$. Then, with high probability, they show that, provided that $r_{ALP} \in N$, the extra price paid for relaxing the constraints of the ALP is $O(\rho \epsilon k/m)$, where

$$\rho = \max_s \frac{\mu^*(s)}{\mu(s)}$$

$$\mu^* = (1 - \alpha) e^T (I - \alpha P^*)^{-1},$$

$u^*$ is an optimal policy, and $\epsilon_{\mathcal{N}} = \sup_{r \in \mathcal{N}} \|J^* - \Phi r\|_{\infty, \psi}$.\footnote{The paper presents the results for $\mu = \mu^*$ giving $\rho = 1$, but the analysis easily extends to the general case.}

The bound is nontrivial when $m \geq \rho \epsilon_{\mathcal{N}} k$. In general, it may be hard to control $\rho$, or even $\epsilon_{\mathcal{N}}$ while ensuring that $r_{ALP} \in \mathcal{N}$.

V. PROOF OF THEOREM IV.1

In this section we present the proof of the main result, Theorem IV.1. The proof uses contraction-arguments. We will introduce a novel contraction operator, $\hat{\Gamma} : \mathbb{R}^S \to \mathbb{R}^S$, that captures the distortion introduced by the extra constraint in ALP and the relaxation in LRALP, respectively. Then we relate the solution of LRALP to the fixed point of $\hat{\Gamma}$.

Note that for the proof it suffices to consider the case when $J_{LRA}^*$ is finite-valued because otherwise the bound is vacuous. Also, recall that it was assumed that $\psi$ lies in the column space of $\Phi$, while $\beta_\psi$, the $\alpha$-discounted stability of $\psi$ w.r.t. the MDP (cf. (2)) is strictly below one.

We will let $r_0 \in \mathbb{R}^k$ be such that $\psi = \Phi r_0$. We also assumed that the matrix $W$ is nonnegative valued, while $c$ specifies a probability distribution over $S$: $\sum_s c(s) = 1$ and $c \in \mathbb{R}^S_+$.\footnote{The paper presents the results for $\mu = \mu^*$ giving $\rho = 1$, but the analysis easily extends to the general case.}
The operator $\hat{\Gamma}$ are defined as follows: For $J \in \mathbb{R}^S$, $s \in S$,

$$(\hat{\Gamma} J)(s) = \min \{ r^\top \phi(s) : W^\top \Phi r \geq W^\top E H J, \ r \in \mathbb{R}^k \}.$$ 

Note that $(\hat{\Gamma} J)(s)$ mimics the definition of ALP with $c = e_s$, except that the constraint $J = \Phi r$ is dropped.

Let us now recall some basic results from the theory of contraction maps. First, let us recall the definition of contractions. Let $\| \cdot \|$ be a norm on $\mathbb{R}^S$ and $\rho > 0$. We say that the map $B : \mathbb{R}^S \to \mathbb{R}^S$ is $(\rho, \| \cdot \|)$-Lipschitz if for any $J, J' \in \mathbb{R}^S$, $\| BJ - BJ' \| \leq \rho \| J - J' \|$. We say that $B$ is a $\| \cdot \|$-contraction with factor $\rho$ if it is $(\rho, \| \cdot \|)$-Lipschitz and $\rho < 1$. It is particularly easy to check whether a map is a contraction map with respect to a weighted maximum norm if it is known to be monotone. Here, $B$ is said to be monotone if for any $J \leq J'$, $J, J' \in \mathbb{R}^S$, $BJ \leq BJ'$ also holds, where $\leq$ is the componentwise partial order between vectors. We start with the following characterization of monotone contractions with respect to weighted maximum norms:

**Lemma V.1.** Let $B : \mathbb{R}^S \to \mathbb{R}^S$, $\psi : S \to \mathbb{R}^+$, $\beta \in (0, 1)$. The following are equivalent:

(i) $B$ is a monotone contraction map with contraction factor $\beta$ with respect to $\| \cdot \|_{\psi, \infty}$.

(ii) For any $J, J' \in \mathbb{R}^S$, $t \geq 0$, $J \leq J' + t\psi$ implies that $BJ \leq BJ' + \beta t\psi$.

The proof, which essentially copies that of Lemma 3.1 of [Kal17], is given for completeness:

**Proof.** Introduce $\cdot$ to denote elementwise products: Thus, $(\psi \cdot J)(s) = \psi(s)J(s)$. We also let $\psi^{-1}(s) = 1/\psi(s)$ and we will use the shorthand $\| \cdot \| = \| \cdot \|_{\infty, \psi}$.

Let us first prove (i) $\Rightarrow$ (ii). Thus, assume that $B$ is a monotone contraction map with factor $\beta$. Take any $J, J', t > 0$, $J \leq J' + t\psi$. We have $BJ = B(J + t\psi) - BJ' + BJ' \leq (\psi^{-1} \cdot (B(J + t\psi) - BJ')) \cdot \psi + BJ' \leq \|B(J + t\psi) - BJ'\|\psi + BJ' \leq \beta t\| J - J' \|\psi + BJ'$.

For the reverse direction, note that monotonicity follows by taking $t = 0$. Now, let $\varepsilon = \| J - J' \|$. Then, $J \leq J' + \varepsilon\psi$ and $J' \leq J + \varepsilon\psi$. By monotonicity and the assumed property of $B$ (using $t = \varepsilon \geq 0$), $-\beta \varepsilon\psi \leq BJ - BJ' \leq \beta \varepsilon\psi$, which implies that $\| BJ - BJ' \| \leq \beta$. $\square$

**Corollary V.2.** If $B$ is monotone and there exists some $\beta \in [0, 1)$ such that for any $J \in \mathbb{R}^S$ and any $t > 0$,

$$B(J + t\psi) \leq BJ + \beta t\psi$$

(3)
then $B$ is a $\| \cdot \|_{\infty, \psi}$ contraction with factor $\beta$.

**Proof.** Let $J, J' \in \mathbb{R}^S$, $t \geq 0$ and assume that $J \leq J' + t\psi$. By monotonicity $BJ \leq B(J' + t\psi)$, while by (3), $B(J' + t\psi) \leq BJ' + \beta t\psi$. Hence, $BJ \leq BJ' + \beta t\psi$. This shows that (ii) of Lemma V.1 holds. Hence, by this lemma, $B$ is a contraction with factor $\beta$ with respect to $\| \cdot \|_{\infty, \psi}$.

Let us now return to the proof of our main result. Recall that the goal is to bound $\| J^* - J_{\text{LRA}} \|_1, c$ through relating this deviations from the fixed point of $\hat{\Gamma}$, which was promised to be a contraction. Let us thus now prove this. For this, it suffices to show that $\hat{\Gamma}$ satisfies the conditions of Corollary V.2. In fact, we will see this holds with $\beta = \beta_\psi$.

**Proposition V.3.** The operator $\hat{\Gamma}$ satisfies the conditions of Corollary V.2 with $\beta = \beta_\psi$, and is thus a $\| \cdot \|_{\infty, \psi}$-contraction with coefficient $\beta_\psi$.

**Proof.** First, note that (as it is well known) $H$ is monotone (all the $P_a$ matrices in the definition of $H$ are nonnegative valued) and that it satisfies an inequality similar to (3): For any $t \geq 0$, $J \in \mathbb{R}^S$,

$$H(J + t\psi) \leq HJ + \beta t\psi E \psi.$$  \hfill (4)

This follows again because our assumption on $\psi$ implies that for any $a \in \mathcal{A}$, $\alpha P_a \psi \leq \beta_\psi \psi$.

Let us now prove that $\hat{\Gamma}$ is monotone. Given $J \in \mathbb{R}^S$, let $\mathcal{F}'(J) = \{ \Phi r : W^T E \Phi r \geq W^T HJ, r \in \mathbb{R}^k \}$. Choose any $s \in S$. Since $J_1 \leq J_2$, $W$ is nonnegative valued and $H$ is monotone, we have $W^T HJ_1 \leq W^T HJ_2$. Hence, $\mathcal{F}'_{J_2} \subset \mathcal{F}_{J_1}$ and thus $(\hat{\Gamma} J_1)(s) \leq (\hat{\Gamma} J_2)(s)$. Since $s$ was arbitrary, monotonicity of $\hat{\Gamma}$ follows.

Let us now turn to proving that (3) holds with $\beta = \beta_\psi$. By definition, for $s \in S$, $t \geq 0$, $J \in \mathbb{R}^S$, $(\hat{\Gamma}(J + t\psi))(s) = \min \{ r^T \phi(s) : W^T E \Phi r \geq W^T H(J + t\psi), r \in \mathbb{R}^k \}$. By (4), $H(J + t\psi) \leq HJ + t\beta_\psi E \psi$ and hence $W^T H(J + t\psi) \leq W^T (HJ + t\beta_\psi E \psi)$. Thus, $(\hat{\Gamma}(J + t\psi))(s) \leq \min \{ r^T \phi(s) : W^T E \Phi r \geq W^T (HJ + t\beta_\psi E \psi), r \in \mathbb{R}^k \}$.

To finish, we need the following elementary observation:

**Claim V.4.** Let $A \in \mathbb{R}^{u \times v}$, $b \in \mathbb{R}^u$, $d \in \mathbb{R}^v$ and $b_0 = Ax_0$ for some $x_0 \in \mathbb{R}^v$. Then

$$\min \{ d^T x : Ax \geq b + b_0, x \in \mathbb{R}^u \}$$

$$= \min \{ d^T y : Ay \geq b, y \in \mathbb{R}^v \} + d^T x_0.$$
Proof of Claim V.4. Set \( y = x - x_0 \). \(\square\)

Now, using Claim V.4 with \( A = W^T E \Phi \), \( b = W^T H J \), \( d = \phi(s) \), \( b_0 = t \beta_w W^T E \psi \) and \( x_0 = t \beta_w r_0 \), thanks to \( \Phi r_0 = \psi \) we have \( Ax_0 = b_0 \). Hence the desired statement follows from the claim. \(\square\)

Let us now return to bounding \( \|J^* - J_{\text{LRA}}\|_{1,c} \). For \( x \in \mathbb{R} \), let \( (x)^- \) be the negative part of \( x \): \( (x)^- = \max(-x,0) \). Then, \( |x| = x + 2(x)^- \). For a vector \( J \in \mathbb{R}^S \), we will write \( (J)^- \) to denote the vector obtained by applying the negative part componentwise. We consider the decomposition

\[
\|J_{\text{LRA}} - J^*\|_{1,c} = c^T (J_{\text{LRA}} - J^*) + 2c^T (J_{\text{LRA}} - J^*)^-.
\]  

(5)

Let \( V_{\text{LRA}} \) be the fixed point \( \hat{\Gamma} \). We know claim the following:

Claim V.5. We have \( J_{\text{LRA}} \geq V_{\text{LRA}} \), \( c^T J_{\text{ALP}} \geq c^T J_{\text{LRA}} \).

Proof. The inequality \( c^T J_{\text{ALP}} \geq c^T J_{\text{LRA}} \) follows immediately from the definitions of \( J_{\text{ALP}} \) and \( J_{\text{LRA}} \).

To prove the first part let \( s \in \mathcal{S}, c = e_s \) and let \( r_s \) be a solution to LRALP in (1). For \( s \in \mathcal{S} \), let \( V_0(s) = \min_{s' \in \mathcal{S}} r_s^T \phi(s) \).

It suffices to show that \( V_1(s) = \hat{\Gamma} V_0(s) \leq V_0 \leq J_{\text{LRA}} \). Indeed, if this holds then \( V_{n+1} = \hat{\Gamma} V_n, n \geq 1 \), satisfies \( V_{n+1} \leq V_n \) and \( V_n \to V_{\text{LRA}} \) as \( n \to \infty \) since \( \hat{\Gamma} \) is a monotone contraction mapping.

Since \( r_s^T \phi(s) \geq r_s^T \phi(s) \) also holds for any \( s, s' \in \mathcal{S} \), we have \( V_0(s) = r_s^T \phi(s) \). Also, since \( J_{\text{LRA}}(s) \geq r_s^T \phi(s) \), it follows that \( J_{\text{LRA}} \geq V_0 \). Now, fix some \( s \in \mathcal{S} \) and define \( r_{\epsilon s, V_0} \) be the solution to the linear program defining \( (\hat{\Gamma} V_0)(s) \). We need to show that \( V_1(s) = (\hat{\Gamma} V_0)(s) = (r_{\epsilon s, V_0})^T \phi(s) \leq V_0(s) \). By the definition of \( r_{\epsilon s, V_0} \) we know that \( (r_{\epsilon s, V_0})^T \phi(s) \leq r^T \phi(s) \) holds for any \( r \in \mathbb{R}^k \) such that \( W^T E \Phi r \geq W^T H V_0 \). Thus, it suffices to show that \( r_s \) satisfies \( W^T E \Phi r_s \geq W^T H V_0 \). By definition, \( r_s \) satisfies \( W^T E \Phi r_s \geq W^T H \Phi r_s \). Hence, by the monotone property of \( H \) and since \( W \) is nonnegative valued, it is sufficient if \( \Phi r_s \geq V_0 \). This however follows from the definition of \( V_0 \). \(\square\)

Thanks to the previous claim, \( (J_{\text{LRA}} - J^*)^- \leq (V_{\text{LRA}} - J^*)^- \) and \( c^T J_{\text{LRA}} \leq c^T J_{\text{ALP}} \). Hence, from (5) we get

\[
\|J_{\text{LRA}} - J^*\|_{1,c} \leq c^T (J_{\text{ALP}} - J^*) + 2c^T (V_{\text{LRA}} - J^*)^-.
\]
By Theorem 3 of [FV03], the first term is bounded by $2c^T\psi\varepsilon$, where recall that $\varepsilon = \inf r \|J^* - \Phi r\|_{\infty, \psi}$. Hence, it remains to bound the second term.

For this, note that for any $J \in \mathbb{R}^S$, $(J)^- \leq |J|$ and also that $\|J\|_{1, c} \leq c^T\psi\|J\|_{\infty, \psi}$. Hence, we switch to bounding $\|J^* - V_{LRA}\|_{\infty, \psi}$. A standard contraction argument gives

$$
\|J^* - V_{LRA}\|_{\infty, \psi} = \|J^* - \hat{\Gamma}J^* - \hat{\Gamma}V_{LRA}\|_{\infty, \psi} \\
\leq \|J^* - \hat{\Gamma}J^*\|_{\infty, \psi} + \|\hat{\Gamma}J^* - V_{LRA}\|_{\infty, \psi} \\
\leq \|J^* - \hat{\Gamma}J^*\|_{\infty, \psi} + \beta\psi\|\hat{\Gamma}J^* - V_{LRA}\|_{\infty, \psi}.
$$

Reordering and using another triangle inequality we get

$$
\|J^* - V_{LRA}\|_{\infty, \psi} \leq \frac{\|J^* - J^*_{ALP}\|_{\infty, \psi} + \|J^*_{ALP} - \hat{\Gamma}J^*\|_{\infty, \psi}}{1 - \beta\psi}.
$$

We bound the term $\|J^* - J^*_{ALP}\|_{\infty, \psi}$ in the following lemma:

**Lemma V.6.** We have $\|J^* - J^*_{ALP}\|_{\infty, \psi} \leq 2\varepsilon$, where recall that $\varepsilon = \inf r \in \mathbb{R}^k \|J^* - \Phi r\|_{\infty, \psi}$.

**Proof.** Let $r^* = \arg\min_{r \in \mathbb{R}^k} \|J^* - \Phi r\|_{\infty, \psi}$. First, notice that $J^*_{ALP} \geq J^*$. Hence, $0 \leq J^*_{ALP} - J^*$. Now let $r' = r^* + \varepsilon r_0$. Then, $\Phi r' = \Phi r^* + \varepsilon \psi \geq J^*$, where the equality follows by the definition of $r_0$ and the inequality follows by the definition of $\varepsilon$. Hence, $r'$ is in the feasible set of the LP defining $J^*_{ALP}$ and thus $J^*_{ALP} \leq \Phi r'$. Thus, $0 \leq J^*_{ALP} - J^* \leq \Phi r^* - J^* + \varepsilon \psi$. Dividing componentwise by $\psi$, taking absolute value and then taking maximum of both sides gives the result. \qed

The proof of the main result is finished by noting that $\hat{\Gamma}J^* = J^*_{LRA}$ and the chaining the inequalities we derived.

**VI. Numerical Illustration**

In this section, we show via an example in the domain of controlled queues the consequences of Theorem IV.2, which bounded the error when the constraints are chosen based on selecting a set of representative states (further preliminary experimental results have been reported in [LB15]).

**Model:** We ran the experiments in the context of a queuing model similar to the one in Section 5.2 of [FV03]. We consider a (simple) small scale model so that we can compare with the optimal policy. At the same time, we will use a small number of basis functions and constraints, to “stress-test” the algorithm. The queuing system has a single queue with random
arrivals and departures. The state of the system is the queue length with the state space given by \( S = \{0, \ldots, S - 1\} \), where \( S - 1 \) is the buffer size of the queue. The action set \( \mathcal{A} = \{1, \ldots, A\} \) is related to the service rates. We let \( s_t \) denote the state at time \( t \). The state at time \( t + 1 \) when action \( a_t \in \mathcal{A} \) is chosen is given by \( s_{t+1} = s_t + 1 \) with probability \( p \), \( s_{t+1} = s_t - 1 \) with probability \( q(a_t) \) and \( s_{t+1} = s_t \), with probability \( (1 - p - q(a_t)) \). For states \( s_t = 0 \) and \( s_t = S - 1 \), the system dynamics is given by \( s_{t+1} = s_t + 1 \) with probability \( p \) when \( s_t = 0 \) and \( s_{t+1} = s_t - 1 \) with probability \( q(a_t) \) when \( s_t = S - 1 \). The service rates satisfy \( 0 < q(1) \leq \ldots \leq q(A) < 1 \) with \( q(A) > p \) so as to ensure ‘stabilizability’ of the queue. The reward associated with action \( a \in \mathcal{A} \) and state \( s \in S \) is given by \( g_a(s) = -(s/N + q(a)^3) \) (the idea here is to penalize higher queue lengths and higher service rates).

**Parameter Settings:** We ran our experiments for \( S = 1000 \), \( A = 4 \) with \( q(1) = 0.2 \), \( q(2) = 0.4 \), \( q(3) = 0.6 \), \( q(4) = 0.8 \), \( p = 0.4 \) and \( \alpha = 1 - \frac{1}{q} \). The moderate size of \( S = 1000 \) enabled us to compute the exact value of \( J^* \) (the most expensive part of the computation). We made use of polynomial features in \( \Phi \) (i.e., \( 1, s, \ldots, s^{k-1} \)) since they are known to work reasonably well for this domain [FV03]. Note that the conic span conditions will only be met with some lag, unless all the constraints are selected. Hence, these features allow us to test the limits of the theory. We chose \( k = 4 \), a low number, to counteract that the MDP is small scale.

**Experimental Methodology:** We compare two different sampling strategies \( (i) \) based on the cone conditions, and \( (ii) \) based on constraint sampling. The two strategies are compared via lookahead policies, wherein, the action at state \( s \) is obtained by computing the approximate value functions of the next states and selecting the action that leads to the larger estimated value. The details are as follows: Case \( (i) \): Except for the corner states i.e., \( s = 0 \) and \( s = 999 \), each state \( 0 < s < S - 1 \) has two next states namely \( s' = s - 1 \) and \( s' = s + 1 \). We formulate two separate LRALPs (or just one LRALP for \( s = 0 \) and \( s = S - 1 \)) for next states. When formulating the LRALP for state \( s' \), we let \( c = e_{s'} \) and choose the constraint corresponding to state \( s' \) to ensure the cone condition to be met for LRALP. We choose 5 more constraints corresponding to states \( 1, 200, 400, 600, 800, 999 \) (uniformly spaced across the state space) and compute \( \hat{J}_{e_{s'}} \). The number of constraints is kept very small as a way of emulating that in large-scale problems we cannot expect a dense covering of the state-space when selecting the constraints. The lookahead policy is formulated as \( u_{LRA}(s) = \arg\min_{a \in \mathcal{A}} g_a(s) + \sum_{s' \in S} p_a(s, s') \hat{J}_{e_{s'}}(s') \). Case \( (ii) \): In a manner similar to Case \( (i) \), we formulate two separate LRALPs for next states. However, as opposed to the previous case, when formulating the LRALP for state \( s' \), we sample \( m = 6 \).
states (defining the constraints) from a distribution dependent on \( s' \). We experimented with two sampling distributions that lead to two the lookahead policies that we denote by \( u_{CS-ideal} \) and \( u_{CS} \), respectively. The sampling distribution that defines \( u_{CS-ideal} \) is the sampling distribution that minimizes the upper bound proved by de Farias and Van Roy [FV04]. In particular, the sampling distribution used at state \( s' \) is \( c_{s'} = e_s^\top(1-\alpha)(I-\alpha P_u)^{-1} \), with \( e_s \) denoting the standard basis vector which is 1 in the \( s^{th} \) coordinate and 0 in all the other coordinates. This sampling distribution is used as a baseline; it is unrealistic to assume that one would be able to sample from this distribution without access to the optimal policy \( u^* \), which is the quantity of ultimate interest. As a more realistic approach, we also consider sampling from \( c_{s'}(s) = \kappa(1-\alpha)(\alpha)^{|s'-s|} \), where \( \kappa > 0 \) is a normalization factor that ensures that \( c_{s'} \) is a distribution. Again, we sample \( m = 6 \) states. This leads to the policy \( u_{CS} \).

![Image](image_url)

**Fig. 1.** Results for a single-queue with polynomial features. On both figures the \( x \) axis represents the state space: the length of the queue. The left-hand-side figure shows the value functions of the various policies computed, alongside with the optimal value function (higher values are better), while the right-hand side subfigure shows the underlying policies. “CS” and “CS-ideal” stand for constraint sampling, while LRA stands for choosing the constraints based on geometric principles proposed in the paper. For further details, see the text.

The results are shown in Fig. 1. The right-hand-side figure shows the policies computed, while the left-hand-side figure shows their value functions. Since constraint sampling (CS) produces randomized results, we repeated the simulations 10 times. The results in all cases were quite
close, hence we show the plot for a typical run. The plots show that the CS case with the ideal sampler is slightly worse, which can be attributed to the fact that in the case of ideal sampler, the sampling distribution is concentrated near the start state $s'$ in comparison to the behaviour of the distribution $c_{s'}(s) = (1 - \alpha)(\alpha)^{|s' - s|}$ which distributes the mass more evenly. As can be seen from the figure, choosing the constraints to (approximately) satisfy the constraint of the theoretical results reliably produces better results: In fact, the value functions $J^*$ and $J_{LRA}$ are mostly on the top of each other. We expect that in larger domains, differences between constraints chosen based on the principles discovered in this paper and choosing constraints in more heuristic ways will lead to similar, or even larger differences. However, the study of this is left for future work.

VII. Conclusion

In this paper, we introduced and analyzed the linearly relaxed approximate linear program (LRALP) whose constraints were obtained as positive linear combination of the original constraints of the ALP. The main novel contribution is a theoretical result which gives a geometrically interpretable bound on the performance loss due to relaxing the constraint sets. Possibilities for future work include extending the results to other forms of approximate linear programming in MDPs (e.g., [DFM09]), exploring the idea of approximating dual variables and designing algorithms that use the newly derived results to actively compute what constraints to select.

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