Brauer-friendly modules and slash functors

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Abstract

This paper introduces the notion of Brauer-friendly modules, a generalisation of endo-
p-permutation modules. A module over a block algebra $OGe$ is said to be Brauer-friendly if it is a direct sum of indecomposable modules with compatible fusion-stable endopermu-
tation sources. We obtain, for these modules, a functorial version of Dade’s slash construc-
tion, also known as deflation-restriction. We prove that our slash functors, defined over
Brauer-friendly categories, share most of the very useful properties that are satisfied by
the Brauer functor over the category of $p$-permutation $OGe$-modules. In particular, we
give a parametrisation of indecomposable Brauer-friendly modules, which opens the way
to a complete classification whenever the fusion-stable sources are classified. Those tools
have been used to prove the existence of a stable equivalence between non-principal blocks
in the context of a minimal counter-example to the odd $Z^*_p$-theorem.

Introduction

The $p$-permutation modules, or trivial-source modules, are an essential tool of the local repre-
sentation theory of finite groups. What makes them so useful is the existence of a localising tool,
the Brauer functor, which satisfies nice properties such as transitivity, and provides a simple
classification of indecomposable $p$-permutation modules, as proven by Broué in [4, Theorem
3.2 (3)]. In particular, the properties of the Brauer functor have enabled Rickard, in [13], to
prove that a derived equivalence defined by a so-called splendid complex of bimodules induces
a collection of local derived equivalences.

A possible generalisation of $p$-permutation modules is the notion of endo-$p$-permutation modules, as described by Urfer in [18]. Let $O$ be a discrete valuation ring with positive residual
characteristic $p$. An indecomposable $OG$-module $M$ is an endo-$p$-permutation module if its
source $V$ with respect to a given vertex $P$ is an endopermutation $OP$-module that is fusion-
stable with respect to the fusion system $F_G(P)$ defined by the group $G$ on the $p$-subgroup $P$.
This characterisation suggests that endo-$p$-permutation modules are not the most natural notion
when one wants to take blocks into account. Indeed, a non-principal block $e$ of the group $G$ usually defines a fusion system on a $p$-subgroup $P$ that is finer than the fusion system $F_G(P)$ defined by the group $G$. This is why we introduce a more general notion. We say that
an $OGe$-module $M$ is Brauer-friendly if it is a direct sum of indecomposable modules with endopermutation sources that are fusion-stable with respect to the block $e$, as defined in [8],
and moreover compatible with one another.
Although they do not seem to have been explicitly described before, these modules are ubiquitous in modular representation theory. Indeed, let $M$ be an indecomposable bimodule that induces a Morita or stable equivalence between two block algebras. If $M$ admits a diagonal vertex, it follows from [11, §7] that a source of $M$ must be a fusion-stable endopermutation module, i.e., $M$ must be a Brauer-friendly module. Puig usually studies Brauer-friendly modules by turning them into endopermutation modules over source algebras. However, it seems reasonable to study Brauer-friendly modules directly at the level of block algebras, a more common framework in modular representation theory. Such is the aim of the present paper.

In Sections 1 and 2, we review the theory of subpairs that has been defined in [1]. Then we refine the definition of the Brauer functor, as well as Green’s theory of vertices and sources (following [15]) to take subpairs into consideration. In Section 3, we define the notion of a Brauer-friendly module. We prove that a module $M$ over a block algebra $\mathcal{O}Ge$ with defect group $D$ is Brauer-friendly if, and only if, the corresponding module over a source algebra $A$ of $\mathcal{O}Ge$ is an endopermutation $\mathcal{O}D$-module.

In Sections 4 and 5, we prove that Dade’s localising tool, known as the slash construction or deflation-restriction, can be applied to Brauer-friendly modules. We also show that this construction can be turned into a functor, provided that one focuses on what we call a Brauer-friendly category, i.e., a category of compatible Brauer-friendly modules. This functoriality may have important consequences in local representation theory. For instance, by allowing one to localise a complex of compatible Brauer-friendly modules, it would make it easier to deal with the local properties of a derived equivalence between block algebras, even though this equivalence is not defined by a splendid complex.

In Section 6, we use the properties of slash functors to give a practical parametrisation of indecomposable Brauer-friendly modules. As mentioned above, a Morita or stable equivalence between block algebras is often defined by such a module, so our parametrisation can be an important tool when it comes to finding a bimodule that induces such an equivalence.

Let us give a more precise example of application. The search for a modular proof of the odd $Z^*_p$-theorem is an important open question of the representation theory of finite groups. In the context of a (putative) minimal counter-example to that theorem, thanks to the above tools, we have been able to consider a family of local Morita equivalences and glue them together to obtain a stable equivalence between non-principal block algebras, as appears in [3, Chapter 4].

1 Subpairs and the Brauer functor

We first give a few notations that we use throughout this article. We let $\mathcal{O}$ be a complete discrete valuation ring with algebraically closed residue field $k$ of characteristic $p$. This includes the case $\mathcal{O} = k$, so that every result that is proven over the ring $\mathcal{O}$ remains true over the field $k$.

For any finite group $G$, we denote by $\Delta G = \{(g, g) ; g \in G\}$ the diagonal subgroup of the direct product $G \times G$. We denote by $\mathcal{O}G\text{Mod}$ the category of $\mathcal{O}G$-modules and morphisms of $\mathcal{O}G$-modules, and by $\mathcal{O}G\text{Perm}$ the full subcategory of $p$-permutation $\mathcal{O}G$-modules, i.e., direct summands of $\mathcal{O}G$-modules that are $\mathcal{O}$-free and admit a $G$-stable basis. For an element $g \in G$ and an object $X$, the notation $gX$ stands for the object $gXg^{-1}$ whenever this makes sense. For any element $x$ of the group algebra $\mathcal{O}G$, we denote by $\overline{x}$ its image by the natural projection map $\mathcal{O}G \rightarrow kG$. For any two groups $G$ and $H$, an ($\mathcal{O}G, \mathcal{O}H$)-bimodule $M$ will be considered
as an $O(G \times H)$-module by setting

$$(g, h) \cdot m = g \cdot m \cdot h^{-1}$$

for any $g \in G$, $h \in H$ and $m \in M$.

We recall the definition of subpairs from [1]. Let $G$ be a finite group, and $e$ be a block of the algebra $OG$. A subpair of the group $G$ is a pair $(P,e_P)$, where $P$ is a $p$-subgroup of $G$ and $e_P$ is a block of the group algebra $O_C^G(P)$. The subpair $(P,e_P)$ is an $e$-subpair if $\bar{e}_P \mbox{br}_P(e) \neq 0$, where $\mbox{br}_P : (OG)^P \to kC_G^G(P)$ denotes the Brauer morphism. The idempotent $e_P$ is a block of the algebra $O_H$ whenever $H$ is a local subgroup of $G$ with respect to the subpair $(P,e_P)$, i.e., a subgroup such that $C_O^G(P) \leq H \leq N_G(P,e_P)$.

The group $G$ acts by conjugation on the poset of $e$-subpairs of $G$. This action can be described by the Brauer category $\mbox{Br}(G,e)$, defined as follows. An object is an $e$-subpair $(P,e_P)$. An arrow $\phi : (P,e_P) \to (Q,e_Q)$ is a group morphism $\phi : P \to Q$ such that there exists an element $g \in G$ that satisfies $g(P,e_P) \leq (Q,e_Q)$ and $\phi(x) = gx$ for any $x \in P$. The composition of arrows is the usual composition of group morphisms. The Brauer category $\mbox{Br}(G,e_0)$ of the principal block $e_0$ actually describes the action of the group $G$ on the poset of its $p$-subgroups, since an $e_0$-subpair $(P,e_P)$ is completely determined by the $p$-subgroup $P$. This is called the Frobenius category of the group $G$, and denoted by $\mbox{Fr}(G)$.

Let $(D,e_D)$ be a maximal $e$-subpair, i.e., an $e$-subpair such that the $p$-subgroup $D$ of the block $e$ is a defect group of the block $e$. Let $\mbox{Fr}_{(G,e)}(D,e_D)$ be the full subcategory of $\mbox{Br}(G,e)$ of which the objects are the $e$-subpairs contained in $(D,e_D)$. This subcategory is called the fusion system of the block $e$ with respect to the subpair $(D,e_D)$; it is equivalent to the Brauer category $\mbox{Br}(G,e)$. A source idempotent of the block $e$ with respect to the maximal subpair $(D,e_D)$ is a primitive idempotent $i$ of the algebra $(OG)e_D$ such that $\bar{e}_D \mbox{br}_D(i) \neq 0$. The $D$-interior algebra $A = eOGi$ is called a source algebra of the block $e$. The $(A,OG)$-bimodule $iOG$ induces a Morita equivalence $A \sim OG$. Moreover, the fusion system $\mbox{Fr}_{(G,e)}(D,e_D)$ can be read in the $O(D \times D)$-module $A$. More details on this approach may be found in [2].

We now recall and slightly generalise the classical definition of the Brauer functor. Let $G$ be a finite group and $P$ be a $p$-subgroup of $G$. We write $N_G(P) = N_G(P)/P$. For any $OG$-module $M$, we denote by $\mbox{Br}_P(M)$ the Brauer quotient of $M$, which some authors denote by $M(P)$, i.e., the $kN_G(P)$-module

$$\mbox{Br}_P(M) = M_P / \biggl( \sum_{Q < P} \mbox{Tr}_Q^P(M_Q) + \mathfrak{m}M_P \biggr),$$

where $M_P$ is the submodule of $P$-fixed points in $M$, $\mbox{Tr}_Q^P : M_Q \to M_P$ is the relative trace map and $\mathfrak{m}$ is the maximal ideal of the local ring $O$. We denote by $\mbox{br}_P^M : M_P \to \mbox{Br}_P(M)$ the projection map. Any morphism of $OP$-modules $u : L \to M$ induces a $k$-linear map $\mbox{Br}_P(u) : \mbox{Br}_P(L) \to \mbox{Br}_P(M)$. If $u$ is a morphism of $OG$-modules, then $\mbox{Br}_P(u)$ is a morphism of $kN_G(P)$-modules. This defines a functor

$$\mbox{Br}_P : O_G \mbox{Mod} \to kN_G(P) \mbox{Mod}.$$  

Notice that we write the Brauer functor $\mbox{Br}_P$ with a capital B, and the Brauer map $\mbox{br}_P$ with a lowercase b. This tool can be adapted to take subpairs into consideration. Let $e$ be a block of the algebra $OG$, $(P,e_P)$ be an $e$-subpair of the group $G$, and $M$ be an $OG$-module. We define the Brauer quotient of $M$ with respect to the subpair $(P,e_P)$ by setting

$$\mbox{Br}_{(P,e_P)}(M) = \mbox{Br}_P(e_PM).$$
This generalised Brauer quotient has a natural structure of $k\tilde{N}_G(P,e_P)\bar{e}_P$-module, where we write $\tilde{N}_G(P,e_P) = N_G(P,e_P)/P$. Notice that we slightly abuse notations by identifying the block $\bar{e}_P \in k\tilde{N}_G(P,e_P)$ to a block of the quotient algebra $k\tilde{N}_G(P,e_P)$.

Let $L$ and $M$ be two $OGe$-modules, and let $u : L \to M$ be a morphism of $OP$-modules. Then the map $u$ induces a morphism of $OP$-modules $e_Pue_P : e_PL \to e_PM$. We set $Br_{(P,e_P)}(u) = Br_P(e_Pue_P)$, a $k$-linear map from $Br_{(P,e_P)}(L)$ to $Br_{(P,e_P)}(M)$. If $u : L \to M$ is a morphism of $OGe$-modules, then $Br_{(P,e_P)}(u)$ is a morphism of $k\tilde{N}_G(P,e_P)\bar{e}_P$-modules. This defines a functor $Br_{(P,e_P)} : OGe\text{Mod} \to k\tilde{N}_G(P,e_P)\bar{e}_P\text{Mod}$.

2 Vertex subpairs and the Green correspondence

In this section, we review the notion of a vertex subpair of an indecomposable module, which has been defined in [15]. Let $G$ be a finite group, and $e$ be a block of the algebra $OG$. The following lemma extends Green’s theory of vertices and sources, as well as the Green correspondence, to take $e$-subpairs into consideration.

**Lemma 1.** Let $M$ be an indecomposable $OGe$-module and $(P,e_P)$ be an $e$-subpair of the group $G$. The following conditions are equivalent.

(i) The $p$-subgroup $P$ is contained in a vertex of $M$, and $M$ is isomorphic to a direct summand of the $OGe$-module $eOG_{e_P} \otimes_{OP} V$ for some indecomposable $OP$-module $V$.

(ii) The $OG$-module $M$ is relatively $P$-projective, and the $OP$-module $e_PM$ admits an indecomposable direct summand $V$ with vertex $P$.

(iii) The $ON_G(P,e_P)$-module $e_PM$ admits an indecomposable direct summand $L$ with vertex $P$ such that $M$ is isomorphic to a direct summand of the induced module $Ind_{N_G(P,e_P)}^G L$.

(iv) The $p$-group $P$ is a vertex of $M$, and the Green correspondent of $M$ with respect to this vertex is an $ON_G(P)$-module $M'$ that belongs to the block $e'_P = Tr_{N_G(P)}^{N_G(P,e_P)} e_P$ of the algebra $ON_G(P)$.

If these conditions are satisfied, then the $ON_G(P,e_P)$-module $L$ of (iii) and the $ON_G(P)$-module $M'$ of (iv) satisfy the relations $M' \simeq Ind_{N_G(P,e_P)}^{N_G(P)} L$ and $L \simeq e_PM'$.

**Proof.** With the notations of Condition (iv), we know from [6] Theorem 1.6] that the restriction/induction functors $e_P Res_{N_G(P,e_P)}^{N_G(P)}$ and $Ind_{N_G(P,e_P)}^{N_G(P)}$ define a Morita equivalence between the block algebras $ON_G(P,e_P)e_P$ and $ON_G(P)e_P$. The equivalence (iii)$\iff$(iv), as well as the last statement of the Lemma, immediately follow from this Morita equivalence and the classical definition (and uniqueness) of the Green correspondent of $M$.

We now suppose that (iii) is satisfied. Then the restriction $Res_{N_G(P,e_P)}^{N_G(P)} L$ admits a direct summand $V$ with vertex $P$, so the $OP$-module $e_PM$ also admits $V$ as a direct summand. Moreover, the induced module $Ind_{N_G(P,e_P)}^{N_G(P)} L$ is relatively $P$-projective, so the $OG$-module $M$ is relatively $P$-projective. Since $V$ is also a direct summand of $M$ and admits $P$ as a vertex, it
follows that the $p$-group $P$ is a vertex of $M$. Moreover, the $\mathcal{O}P$-module $V$ is a source of $L$. Thus $L$ is isomorphic to a direct summand of the $\mathcal{O}N_G(P, e_P)_{e_P}$-module $\mathcal{O}N_G(P, e_P)_{e_P} \otimes_{\mathcal{O}P} V$, and $M$ to a direct summand of the $\mathcal{O}Ge$-module

$$e \operatorname{Ind}^{G}_{N_G(P, e_P)} \left( \mathcal{O}N_G(P, e_P)_{e_P} \otimes_{\mathcal{O}P} V \right) \simeq e\mathcal{O}Ge \otimes_{\mathcal{O}P} V.$$  

We have proven that the condition in (iii) implies (i) and (ii).

Next, we suppose that (i) is satisfied. The $\mathcal{O}G$-module $M$ is relatively $P$-projective and the $p$-group $P$ is contained in a vertex of $M$, so $P$ itself is a vertex of $M$. Moreover, $M$ is isomorphic to a direct summand of the $\mathcal{O}Ge$-module

$$\mathcal{O}Ge \otimes_{\mathcal{O}P} V \simeq \operatorname{Ind}^{G}_{N_G(P)} \left( \mathcal{O}N_G(P)_{e_P} \otimes_{\mathcal{O}P} V \right),$$  

so there is an indecomposable direct summand $M'$ of the $\mathcal{O}N_G(P)_{e_P}$-module $\mathcal{O}N_G(P)_{e_P} \otimes_{\mathcal{O}P} V$ such that $M$ is isomorphic to a direct summand of $\operatorname{Ind}^{G}_{N_G(P)} M'$. The $\mathcal{O}N_G(P)$-module $M'$ is therefore a Green correspondent of $M$ with respect to the vertex $P$, so that (i) implies (iv).

Finally, we suppose that (ii) is satisfied. Then the restriction $\operatorname{Res}^{G}_{P} M$ admits an indecomposable direct summand with vertex $P$, and the $p$-group $P$ contains a vertex of $M$, so $P$ itself is a vertex of $M$. Let $M'$ be a Green correspondent of $M$ with respect to this vertex, and let $f$ be the block of the algebra $\mathcal{O}N_G(P)$ such that $fM' = M'$. Then the uniqueness of the Green correspondent implies that no indecomposable direct summand of the $\mathcal{O}P$-module $(1 - f)M$ admits $P$ as a vertex. Since $e_{P} M = f_{e_{P}} M + (1 - f)_{e_{P}} M$, it follows that $f_{e_{P}} \neq 0$, so that $f = \operatorname{Tr}^{N_G(P)}_{N_G(P, e_P)} e_{P}$. Thus (ii) implies (iv), and the four conditions are equivalent.

We derive from the above lemma a few definitions and easy consequences that extend some of Green’s classical definitions.

Let $M$ be an indecomposable $\mathcal{O}Ge$-module. It follows from Nagao’s theorem (see, e.g., [2], Theorem 6.3.1]) that there exists an $e$-subpair $(P, e_{P})$ of the group $G$ that satisfies the statement in Lemma [1](iv), hence the statements in (i), (ii) and (iii). Such an $e$-subpair is called a vertex subpair of the indecomposable module $M$ (this is consistent with [15], Definition 2.6]). A source of $M$ with respect to the vertex subpair $(P, e_{P})$ is an $\mathcal{O}P$-module $V$ that satisfies any one of the equivalent conditions (i) and (ii). A source triple of $M$ is a triple $(P, e_{P}, V)$, where $V$ is a source of $M$ with respect to the vertex subpair $(P, e_{P})$. The source triples of $M$ form an orbit under the action of the group $G$ by conjugation.

A Green correspondent of $M$ with respect to the vertex subpair $(P, e_{P})$ is an $\mathcal{O}N_G(P, e_P)$-module $L$ that satisfies the condition in (iii). The mapping $M \mapsto L$ induces a one-to-one correspondence between the isomorphism classes of indecomposable $\mathcal{O}Ge$-modules with vertex subpair $(P, e_{P})$, and the isomorphism classes of indecomposable $\mathcal{O}N_G(P, e_P)_{e_P}$-modules with vertex $P$.

The following properties of vertex subpairs and sources are closer to the approach of [15].

**Lemma 2.** Let $M$ be an indecomposable $\mathcal{O}Ge$-module and $(P, e_{P}, V)$ be a source triple of $M$.

(i) There exists a primitive idempotent $i$ of the algebra $(\mathcal{O}G)^P$ such that $e_{P} \operatorname{br}(i) \neq 0$ and that $M$ is isomorphic to a direct summand of the $\mathcal{O}Ge$-module $\mathcal{O}G i \otimes_{\mathcal{O}P} V$.

(ii) There exists a defect group $D$ of the block $e$ such that $P \leq D$, there exists a primitive idempotent $j$ of the algebra $(\mathcal{O}G)^D$ such that $\operatorname{br}(j) \neq 0$ and $e_{P} \operatorname{br}(j) \neq 0$, and that $M$ is isomorphic to a direct summand of the $\mathcal{O}Ge$-module $\mathcal{O}G j \otimes_{\mathcal{O}D} \operatorname{Ind}_{P}^{D} V$.  


Proof. By assumption, $M$ is a direct summand of the $OGe$-module $eOGeP \otimes_{OP} V$. Consider a decomposition $ee_P = i_1 + \ldots + i_n$ of the idempotent $ee_P$ into mutually orthogonal primitive idempotents in the algebra $(OG)^P$. This brings
\[ eOGeP \otimes_{OP} V = (OG_{i_1} \otimes V) \oplus \cdots \oplus (OG_{i_n} \otimes V). \]
Since the $OGe$-module $M$ is indecomposable, by the Krull-Schmidt theorem, it must be isomorphic to a direct summand of the $OGe$-module $OG_{i_l} \otimes_{OP} V$ for some $l \in \{1, \ldots, n\}$. If $e_P br_P(i_l) = 0$, then the idempotent $i_l$ lies in $T^T_{Q_0}((OG)^Q)$ for some proper subgroup $Q$ of $P$, so $M$ is relatively $Q$-projective and cannot admit $P$ as a vertex. This contradiction proves (i).

Let $\alpha$ be the point of the algebra $(OGe)^P$ that contains the idempotent $i = i_l$. Since $br_P(i_l) \neq 0$, the pointed group $P_\alpha$ is local. Let $D_\beta$ be a defect pointed group of the block algebra $OGe$ such that $P_\alpha \leq D_\beta$ (see [15, §18]). Then the $p$-group $D$ is a defect group of the block $e$ and there exists an idempotent $j \in D$ such that $br_D(j) \neq 0$ and $ij = ji = i$. Then $e_P br_P(j) = 0$, and the $OGe$-module $OG \otimes_{OP} V$ is a direct summand of
\[ OG_{j} \otimes_{OP} V \cong OG_{j} \otimes_{OD} \text{Ind}^P_{P} V. \]

The following result deals with the behaviour of vertex subpairs and sources with respect to restriction. It is similar to [13, Corollary 2.7], with an additional statement about sources.

**Theorem 3.** Let $M$ be an $OGe$-module, and $H$ be a subgroup of $G$. Let $(Q, e_Q, V)$ be a source triple of an indecomposable direct summand $X$ of the restriction $Res^G_H M$. Assume that the subgroup $H$ contains the centraliser $C_G(Q)$. Then $(Q, e_Q)$ is an $\alpha$-subpair of the group $G$, and there exists a source triple $(R, e_R, W)$ of an indecomposable direct summand of the $OGe$-module $M$ such that $(Q, e_Q) \leq (R, e_R)$ and that $V$ is a direct summand of the $OQ$-module $Res^Q_H W$.

Proof. We may assume that the $OGe$-module $M$ is indecomposable. Let $(P, e_P, U)$ be a source triple of $M$. By Lemma [2] there is a primitive idempotent $i \in (OGe)^P$ such that $e_P br_P(i) \neq 0$ and $X$ is (isomorphic to) an indecomposable direct summand of the $OH$-module $OGi \otimes_{OP} U$. Since $C_G(Q) \leq H$, the block $e_Q$ of $OC_H(Q)$ is also a block of the algebra $OQC_G(Q)$. The $OQC_G(Q)$-module $e_QX$ admits an indecomposable direct summand $X'$ with vertex $Q$ and source $V$. Up to replacing $X$ by $X'$, we may now assume that $H = QC_G(Q)$, and that $X$ is an indecomposable direct summand of the $OH$-module $L = e_QOGi \otimes_{OP} U$.

We consider the $OH$-module $L_0 = OG \otimes_{OP} U$. The map $f : L_0 \to L_0$ defined by $g \otimes u = e_Qgi \otimes u$ is an idempotent of $\text{End}_{OH}(L_0)$ such that $f(L_0) = L$, so $L$ is a direct summand of $L_0$. Let $L_0 = X_0 \oplus \cdots \oplus X_m$ be a Krull-Schmidt decomposition of the $OH$-module $L_0$ such that $X_0 = X$, $L = X_0 \oplus \cdots \oplus X_m$ and $\ker f = X_{m+1} \oplus \cdots \oplus X_n$ for some $m \in \{1, \ldots, n\}$.

We then consider the decomposition $L_0 = \oplus_{g \in R} L_g$, where $R$ is a set of representatives for the double class set $H \setminus G/P$, and $L_g = OHg P \otimes_{OP} U$ for any $g \in R$. This can be refined into a Krull-Schmidt decomposition $L_0 = Y_0 \oplus \cdots \oplus Y_n$ of the $OH$-module $L_0$. From the proof of the Krull-Schmidt theorem in [2], we know that there exists an $l \in \{1, \ldots, n\}$ such that the projection map from $Y_l$ onto $X_0$ along $X_1 \oplus \cdots \oplus X_n$ is an isomorphism. We may assume $l = 0$. Then $f(Y_0)$ is a complement of the direct summand $X_1 \oplus \cdots \oplus X_m$ in the $OH$-module $L$, so the $OH$-modules $f(Y_0)$ and $X_0$ are isomorphic. We may assume $X_0 = f(Y_0)$.

By construction, there exists a $g \in R$ such that the $OH$-module $Y_0$ is a direct summand of $L_g$. The isomorphism
\[ L_g \cong \text{Ind}^H_{H \cap P} Res^{qP}_{H \cap P} gU \]
shows that the $\mathcal{O}H$-module $L_g$, hence also $Y_0$, is relatively $H \cap g P$-projective. Since the vertex $Q$ of the $\mathcal{O}H$-module $Y_0 \cong X_0$ is normal in $H$, it follows that $Q \leq gP$. Assume for a moment that $(Q,e_Q) \not\leq g(P,e_P)$. For any $h \in H$, we have
\[
f(hgP \otimes_{\mathcal{O}P} U) = e_Q(hgP)i \otimes_{\mathcal{O}P} U = e_Q(hgi)(hgP) \otimes_{\mathcal{O}P} U.
\]
The $e$-subpair $(Q,e_Q)$ is normalised by the group $H$, so we have $Q \leq h^gP$ but $(Q,e_Q) \not\leq h^g(P,e_P)$. By definition, $i$ is a primitive idempotent of $(\mathcal{O}Ge)^P$ such that $\bar{e}_P \mathcal{B}r_P(i) \neq 0$. Thus $hgi$ is a primitive idempotent in $(\mathcal{O}Ge)^Q$, and it follows from [16, Lemma 4.1] that $\bar{e}_Q \mathcal{B}r_Q(hgi) = 0$. Then we obtain $f(hgP \otimes_{\mathcal{O}P} U) \subseteq J_Q L_0$, where $J_Q \subseteq (\mathcal{O}Ge)^Q$ is the kernel of the Brauer map $\mathcal{B}r_Q$. This implies
\[
X_0 = f(Y_0) \subseteq f(L_g) \subseteq J_Q L_0.
\]
We know that $X_0$ is a direct summand of the $\mathcal{O}H$-module $L_0$. Consider the endomorphism algebra $A = \text{End}_{\mathcal{O}}(L_0)$, and let $a \in A^H$ be an idempotent such that $X_0 = aL_0$. Then there is an isomorphism $\text{End}_{\mathcal{O}}(X_0) \cong aAa$, which induces an isomorphism $\mathcal{B}r_{\Delta Q}(\text{End}_{\mathcal{O}}(X_0)) \cong \mathcal{B}r_{\Delta Q}(aAa)$. The inclusion $X_0 \subseteq JQL_0$ yields $aAa \subseteq JQL$, hence $\mathcal{B}r_{\Delta Q}(aAa) = 0$. We obtain $\mathcal{B}r_{\Delta Q}(\text{End}_{\mathcal{O}}(X_0)) = 0$, a contradiction since $Q$ is a vertex of $X_0$.

This contradiction proves that $(Q,e_Q) \leq g(P,e_P)$. Then the $\mathcal{O}H$-module $X = X_0$ is isomorphic to a direct summand of the induced module $L_g \cong \text{Ind}_{H \cap g P}^H \mathcal{B}r_{\Delta P}^g \text{Res}_{H \cap g P}^g gU$. As a consequence, the source $V$ of $X$ is isomorphic to a direct summand of the $\mathcal{O}Q$-module $\mathcal{B}r_{\Delta P}^g \Delta_P gU$ for some $h \in H$. Then we set $(R,e_R,W) = h^g(P,e_P,U)$, and the proof is complete.

The application of Theorem 3 that we have in mind is as follows. Let $M$ be an indecomposable $\mathcal{O}Ge$-module, and $(P,e_P)$ be an $e$-subpair of the group $G$. Let $H$ be a subgroup of $G$ such that $PC_G(P) \leq H \leq NC_G(P,e_P)$. Let $(Q,e_Q,V)$ be a source triple of an indecomposable direct summand of the $\mathcal{O}H$-module $e_PM$. If the vertex $Q$ contains the $p$-group $P$, then we have $C_G(Q) \leq C_G(P) \leq H$. Thus the theorem applies: there exists a source triple $(R,e_R,W)$ of $M$ such that $(Q,e_Q) \leq (R,e_R)$ and that $V$ is a direct summand of the $\mathcal{O}Q$-module $\text{Res}_Q^R W$. Notice that this statement is unlikely to be true without the assumption $P \leq Q$.

3 Brauer-friendly modules

Let $P$ be a $p$-group. An endopermutation $\mathcal{O}P$-module, as defined in [3], is an $\mathcal{O}P$-module $V$ such that the endomorphism algebra $\text{End}_\mathcal{O}(V)$ is $\mathcal{O}$-free and admits a $\Delta P$-stable basis. Two endopermutation $\mathcal{O}P$-modules $V$ and $W$ are said to be compatible if the direct sum $V \oplus W$ is an endopermutation $\mathcal{O}P$-module.

In this section, we fix a finite group $G$ and a block $e$ of the algebra $\mathcal{O}G$. We aim at studying certain $\mathcal{O}Ge$-modules with endopermutation sources. The following definition is derived from [8]; notice however that we work over the local ring $\mathcal{O}$ rather than its residue field $k$, and that we replace the language of fusion systems with that of Brauer categories.

**Definition 4.** Let $(P,e_P)$ be an $e$-subpair of the group $G$, and let $V$ be an endopermutation $\mathcal{O}P$-module. We say that $V$ is fusion-stable in the group $G$ with respect to the subpair $(P,e_P)$ if the endopermutation $\mathcal{O}Q$-modules $\mathcal{B}r_{\Delta P}^g V$ and $\mathcal{B}r_{\Delta Q}^g V$ are compatible for any $e$-subpair $(Q,e_Q)$ and any two arrows $\phi_1, \phi_2 : (Q,e_Q) \rightarrow (P,e_P)$ in the Brauer category $\mathcal{B}r(G,e)$. 
In practice, let \((P,e_P)\) is an \(e\)-subpair with respect to \(V\) an endopermutation \(OP\)-module. If we wish to prove that \(V\) is fusion-stable with respect to the subpair \((P,e_P)\), it is enough to check that the endopermutation \(OQ\)-modules \(Res^P_Q V\) and \(Res^P_Q qV\) are compatible, for any \(e\)-subpair \((Q,e_Q)\) contained in \((P,e_P)\) and any element \(g \in G\) such that \((Q,e_Q) \leq (P,e_P)\).

To make it shorter, we say that a triple \((P,e_P,V)\) is a fusion-stable endopermutation source triple in the group \(G\) if \(V\) is an endopermutation \(OP\)-module that is indecomposable, capped (i.e., with vertex \(P\)), and fusion-stable with respect to the subpair \((P,e_P)\) of the group \(G\).

**Definition 5.** We say that two fusion-stable endopermutation source triples \((P_1,e_1,V_1)\) and \((P_2,e_2,V_2)\) are compatible if the endopermutation \(OQ\)-modules \(Res_{\phi_1}^P V_1\) and \(Res_{\phi_2}^P V_2\) are compatible for any \(e\)-subpair \((Q,e_Q)\) and any two arrows \(\phi_1 : (Q,e_Q) \to (P_1,e_1), \phi_2 : (Q,e_Q) \to (P_2,e_2)\) in the Brauer category \(Br\) of the group \(G\).

**Lemma 6.** Let \((P_1,e_1,V_1)\) and \((P_2,e_2,V_2)\) be two source triples in the group \(G\) with respect to the block \(e\).

(i) If \((P_1,e_1,V_1)\) is a fusion-stable endopermutation source triple, then any \(G\)-conjugate of \((P_1,e_1,V_1)\) is a fusion-stable endopermutation source triple. If moreover \((P_2,e_2,V_2)\) is a fusion-stable endopermutation source triple that is compatible with \((P_1,e_1,V_1)\), then it is also compatible with any \(G\)-conjugate of \((P_1,e_1,V_1)\).

(ii) For \(i = 1,2\), let \((Q_i,f_i)\) be a subpair of \((P_i,e_i)\), and let \(W_i\) be a capped indecomposable direct summand of the restriction \(Res_{\phi_i}^P V_i\). If \((P_1,e_1,V_1)\) and \((P_2,e_2,V_2)\) are compatible fusion-stable endopermutation source triples, then \((Q_1,f_1,W_1)\) and \((Q_2,f_2,W_2)\) are compatible fusion-stable endopermutation source triples.

(iii) Let \((R,e_R)\) be an \(e\)-subpair and \(\psi_1 : (P_1,e_1) \to (R,e_R), \psi_2 : (P_2,e_2) \to (R,e_R)\) be arrows in the Brauer category \(Br\). Then \((P_1,e_1,V_1)\) and \((P_2,e_2,V_2)\) are compatible fusion-stable endopermutation source triples if, and only if, the direct sum \(\text{Ind}_{\psi_1}V_1 \oplus \text{Ind}_{\psi_2} V_2\) is an endopermutation \(OR\)-module that is fusion-stable in the group \(G\) with respect to the subpair \((R,e_R)\).

**Proof.** The statements in (i) and (ii) are straightforward consequences of the definition, as well as the “if” part of (iii). The “only if” part of (iii) is more subtle.

With the notations of (iii), we suppose that \((P_1,e_1,V_1)\) and \((P_2,e_2,V_2)\) are compatible fusion-stable endopermutation source triples. Up to replacing these triples by conjugates, we may suppose that the \(e\)-subpairs \((P_1,e_1)\) and \((P_2,e_2)\) are contained in \((R,e_R)\), and that the arrows \(\psi_1\) and \(\psi_2\) are the inclusion maps. We follow the lines of the proof of \[Lemma 6.8\]. The Mackey formula gives an isomorphism of \(O\Delta R\)-modules

\[
\text{Hom}_O(\text{Ind}_{P_1}^R V_1, \text{Ind}_{P_2}^R V_2) \cong \text{Ind}_{\Delta R}^{\Delta P_2} \text{Ind}_{\Delta P_1}^{\Delta R} \text{Hom}_O(V_1, V_2) \cong \bigoplus_{x \in P_2/P_1} \text{Ind}_{\Delta Q_x}^{\Delta P_1} \text{Hom}_O(\text{Res}_{\phi_1}^P V_1, \text{Res}_{\phi_2}^P V_2),
\]

where \(Q_x = P_2 \cap x P_1\), the morphism \(\phi_1^x : Q_x \to P_1\) sends \(y \in Q_x\) to \(x^{-1}y \in P_1\), and \(\phi_2^x : Q_x \to P_2\) is the inclusion map. Let \(e_{Q_x}\) be the block of \(OC_G(Q_x)\) such that \((Q_x,e_{Q_x}) \leq (R,e_R)\). Then the inclusions \((P_2,e_2) \leq (R,e_R)\) and \(Q_x \leq P_2\) imply \((Q_x,e_{Q_x}) \leq (P_2,e_2)\). Since \(x\) lies in \(R\), we also have \(x^{-1}(Q_x,e_{Q_x}) \leq (R,e_R)\) so the inclusions \((P_1,e_1) \leq (R,e_R)\) and \(x^{-1}Q_x \leq P_2\) imply

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is fusion-stable in the Frobenius category $\text{Fr}(G, e)$. Since the triples $(P_1, e_1, V_1)$ and $(P_2, e_2, V_2)$ are compatible, we deduce that $\text{Hom}_O(\text{Ind}_P^R V_1, \text{Ind}_P^R V_2)$ is a permutation $O\Delta R$-module.

Similarly, the endomorphism algebras $\text{End}_O(\text{Ind}_P^R V_1)$ and $\text{End}_O(\text{Ind}_P^R V_2)$ are permutation $O\Delta R$-modules. Thus $\text{Ind}_P^R V_1$ and $\text{Ind}_P^R V_2$ are compatible endopermutation $O\Delta R$-modules.

More generally, let $(Q, e_Q)$ be an $e$-subpair that is contained in $(R, e_R)$ and let $g \in G$ be an element such that $(Q, e_Q) \leq g(R, e_R)$. A similar use of the Mackey formula proves that $\text{Hom}_O(\text{Res}_Q^R \text{Ind}_P^R V_1, \text{Res}_Q^R g \text{Ind}_P^R V_2)$ is a permutation $O\Delta Q$-module. It follows that the restrictions

$$\text{Res}_Q^R \text{Ind}_P^R V_1; \text{Res}_Q^R \text{Ind}_P^R V_2; \text{Res}_Q^R g \text{Ind}_P^R V_1; \text{Res}_Q^R g \text{Ind}_P^R V_2$$

are compatible endopermutation $OQ$-modules. So the direct sum $\text{Ind}_P^R V_1 \oplus \text{Ind}_P^R V_2$ is an endopermutation $OR$-module that is fusion-stable with respect to the subpair $(R, e_R)$. $\square$

We know from Lemma 6 (i) that the notion of compatible fusion-stable endopermutation source triples is invariant by conjugation in the group $G$. The source triples of an indecomposable $OGe$-module are defined up to conjugation, so the following definition is unambiguous.

**Definition 7.** We say that an $OGe$-module $M$ is Brauer-friendly if it is a direct sum of indecomposable $OGe$-modules with compatible fusion-stable endopermutation source triples. We say that two Brauer-friendly $OGe$-modules $L$ and $M$ are compatible if the direct sum $L \oplus M$ is a Brauer-friendly $OGe$-module.

Let us give a few examples. It is clear that the triples $(P, e_P, O)$, where $(P, e_P)$ runs into the set of $e$-subpairs of the group $G$ and $O$ is the trivial $OP$-module, are compatible fusion-stable endopermutation source triples. Thus the $p$-permutation $OGe$-modules are Brauer-friendly and pairwise compatible.

Next, let $M$ be an indecomposable endo-$p$-permutation $OGe$-module, and $(P, e_P, V)$ be a source triple of $M$. By [18] Theorem 1.5], the source $V$ is an endopermutation $OP$-module that is fusion-stable in the Frobenius category $\text{Fr}(G)$ with respect to the $p$-subgroup $P$. *A fortiori*, $V$ is fusion-stable in the Brauer-category $\text{Br}(G, e)$ with respect to the $e$-subpair $(P, e_P)$, so $M$ is a Brauer-friendly $OGe$-module. If $M$ and $N$ are compatible endo-$p$-permutation $OGe$-modules (i.e., the direct sum $M \oplus N$ is an endo-$p$-permutation module), then they are compatible as Brauer-friendly modules.

Conversely, let $M$ be a Brauer-friendly $OGe$-module. If $e = e_0$ is the principal block of the group $G$, then an $e$-subpair $(P, e_P)$ is uniquely determined by the $p$-subgroup $P$ of $G$. Thus it follows from [18] Theorem 1.5] that $M$ is an endo-$p$-permutation $OGe$-module. But the fusion system of an arbitrary block is usually “finer” than the fusion system of the principal block, so $M$ may not be an endo-$p$-permutation module in general.

The following lemma, of which a proof is straightforward from Lemma 6 and the proof of Theorem 8 gives a more general example of Brauer-friendly module.

**Lemma 8.** Let $(P, e_P)$ be an $e$-subpair, and let $V$ be an endopermutation $OP$-module that is fusion-stable with respect to the subpair $(P, e_P)$. Let $i \in (OGe)^p$ be an idempotent such that $e_P \text{br}_P(i) \neq 0$. Assume that, for any subgroup $Q$ of $P$, the idempotent $\text{br}_Q(i)$ lies in a single block of the algebra $kG(Q)$. Then the $OGe$-module $L = OGi \otimes OP V$ is Brauer-friendly.
Notice that the compatibility of endopermutation modules is preserved by the reduction from the local ring $O$ to the residue field $k$. For any Brauer-friendly $OGe$-module $M$, it follows that the reduction $k \otimes_O M$ is a Brauer-friendly $kGe$-module. The notion is also partially compatible with the restriction to a local subgroup, as appears in the following lemma.

**Lemma 9.** Let $M$ be a Brauer-friendly $OGe$-module. Let $(P, e_P)$ be an $e$-subpair of the group $G$, and $H$ be a subgroup of $G$ such that $PC_G(P) \leq H \leq NG_G(P, e_P)$.

(i) The $OHe_P$-module $e_PM$ admits the decomposition $e_PM = L \oplus L'$, where $L$ is a Brauer-friendly $OHe_P$-module and $L'$ is a direct sum of indecomposable $OHe_P$-modules with vertices that do not contain the $p$-subgroup $P$.

(ii) The restriction $Res^H_P L$ is an endopermutation $O_P$-module.

**Proof.** The $OH$-module $e_PM$ certainly admits the decomposition $e_PM = L \oplus L'$, where $L$ is a direct sum of indecomposable $OH$-modules with vertices that contain $P$ and $L'$ is a direct sum of indecomposable $OH$-modules with vertices that do not contain $P$. Let $X$, $X'$ be indecomposable direct summands of the $OHe_P$-module $L$, and let $(Q, f, W)$, $(Q', f', W')$ be respective source triples of $X$, $X'$. Since the $OGe$-module $M$ is Brauer-friendly, it follows from Theorem 3 that $(Q, f, W)$ and $(Q', f', W')$ are compatible fusion-stable source triples. Thus the $OHe_P$-module $L$ is Brauer-friendly.

Moreover, we have $(P, e_P) \leq h(Q, f)$ and $(P, e_P) \leq h'(Q', f')$ for any two elements $h, h' \in H$. So the restrictions $Res^h_P hW$ and $Res^{h'}_P h'W'$ are compatible endopermutation $O_P$-modules. Then it follows from the Mackey formula that the restrictions $Res^h_P Ind^H_Q W$ and $Res^{h'}_P Ind^{H'}_{Q'} W'$ are compatible endopermutation $O_P$-modules. Since the restriction $Res^h_P L$ is a direct sum of direct summands of modules of this kind, it is an endopermutation $O_P$-module. 

We conclude this section with a lemma that connects our notion of Brauer-friendly modules with Linckelmann’s notion of modules with fusion-stable endopermutation sources over a source algebra, which appears in [7, §3]. It should be mentioned here that it was Linckelmann who initially gave us the idea to study such modules.

**Lemma 10.** Let $G$ be a finite group and $e$ be a block of the group $G$. Let $(D, e_D)$ be a maximal $e$-subpair and $i$ be a primitive idempotent of the algebra $(OGe)^D$ such that $e_D br_D((i) \neq 0$, i.e., a source idempotent. Let $A = iOGi$ be the corresponding source algebra. An $OGe$-module $M$ is Brauer-friendly if, and only if, the $A$-module $iM$ is an endopermutation $OD$-module.

**Proof.** Let $M$ be an $OGe$-module, and let $M = M_1 \oplus \cdots \oplus M_n$ be a Krull-Schmidt decomposition. For each integer $l \in \{1, \ldots, n\}$, let $(P_l, e_l, V_l)$ be a source triple of the indecomposable $OGe$-module $M_l$. By Lemma 9 (ii), there exists a defect group $D_l$ and a source idempotent $j_l$ of the block $e$ with respect to the defect group $D_l$ such that $P_l \leq D_l$ and $e_l br_{P_l}(j_l) \neq 0$, and such that $M_l$ is isomorphic to a direct summand of the $OGe$-module $OGe_{j_l} \otimes_O D_l Ind^D_{P_l} V_l$. Up to replacing the defect group $D_l$ and the idempotent $j_l$ by conjugates, we may assume that $D_l = D$ and $j_l = i$. We set $W = Ind^D_{P_l} V_l \oplus \cdots \oplus Ind^D_{P_n} V_n$. Then $M$ is isomorphic to a direct summand of the $OG$-module $L = OGi \otimes_O D W$, and $iM$ is isomorphic to a direct summand of the $A$-module $iL = A \otimes_{OD} W$.

Suppose that the $OGe$-module $M$ is Brauer-friendly. Then, by Lemma 9 (iii), $W$ is an endopermutation $OD$-module that is fusion-stable with respect to the subpair $(D, e_D)$. By [7, Proposition 3.2 (i)], this implies that the $A$-module $iM$ is an endopermutation $OD$-module.
Conversely, suppose that $iM$ is an endopermutation $OD$-module. Let $F$ be the fusion system of the block $e$ with respect to the maximal subpair $(D, e_D)$. We know from [16, §47] or [7, §2] that the fusion system $F$ can be read in the $(OD, OD)$-bimodule structure of the source $A = iOG_i$. Since $iM$ is an $A$-module, it follows that the endopermutation $OD$-module $iM$ is fusion-stable with respect to the subpair $(D, e_D)$. For any $l \in \{1, \ldots, n\}$, the source $V_l$ is isomorphic to a direct summand of the restriction $Res^D_{P^\ell}iM$. Thus, by Lemma 6(ii), the triples $(P_l, e_l, V_l), 1 \leq l \leq n$, are compatible fusion-stable endopermutation source triples. So $M$ is a Brauer-friendly $OG_e$-module.



4 Slash functors

In this section, we extend Dade’s slash construction, which has been defined in [5, Theorem 4.15], to take subpairs and Brauer-friendly modules into account. The slash construction is often called deflation-restriction, e.g. in [7] or [17]. In our context, this would become deflation-truncation-restriction. Let us briefly recall Dade’s original construction.

Let $R$ be a $p$-group and $P$ be a subgroup of $R$. Let $V$ be an endopermutation $OR$-module. The Brauer quotient $Br_{\Delta P}(End_O(V))$ has a natural structure of $N_R(P)/P$-algebra over $k$. Moreover, there exists an endopermutation $kN_R(P)/P$-module $V[P]$ and an isomorphism of $N_R(P)/P$-algebras $Br_{\Delta P}(End_O(V)) \cong End_k(V[P])$. The $kN_R(P)/P$-module $V[P]$, which is unique up to (non-unique) isomorphism, is called a $P$-slashed module relative to the $OR$-module $V$. In particular, if $P = R$ and $V$ is a capped indecomposable endopermutation $OD$-module, then the group $N_R(P)/P$ is trivial, and a slashed module $V[P]$ is just the $k$-vector space $k$.

If $V$ is a permutation $OR$-module, then $V$ is a fortiori an endopermutation $OR$-module, and the Brauer quotient $Br_P(V)$ is a $P$-slashed module relative to the $OR$-module $V$. However, in general, the slash construction appears to be functorial in $End_O(V)$, but not in $V$. In a first step, we prove that this construction can actually be turned into a functor in $V$, provided that it is restricted to a suitable category of compatible endopermutation modules.

We consider a finite $p$-group $P$. If $L$ and $M$ are objects of the category $OP\text{Pern}_{M}$ of permutation $OP$-modules, then one can derive from [14, Lemma 3.3] a natural isomorphism

$$Br_{\Delta P}(\text{Hom}_O(L,M)) \cong \text{Hom}_k(\text{Br}_P(L), \text{Br}_P(M)).$$

More generally, let $OP\text{M}$ be a full subcategory of the category $OP\text{Mod}$ of $OP$-modules. We say that a functor $Sl : OP\text{M} \to k\text{Mod}$ is a $P$-slash functor if, for any two objects $L, M$ in the category $OP\text{M}$, the map $\text{Hom}_{OP\text{M}}(L, M) \to \text{Hom}_k(Sl(L), Sl(M)), u \mapsto Sl(u)$, factors through an isomorphism

$$Br_{\Delta P}(\text{Hom}_O(L,M)) \cong \text{Hom}_k(Sl(L), Sl(M)).$$

**Lemma 11.** Let $P$ be a finite $p$-group and $OP\text{M}$ be a full subcategory of the category $OP\text{Mod}$. Assume that any two capped indecomposable direct summands of objects in $OP\text{M}$ are compatible endopermutation $OP$-modules.

(i) There exists a $P$-slash functor $Sl_{P} : OP\text{M} \to k\text{Mod}$.

(ii) If $Sl : OP\text{M} \to k\text{Mod}$ is another $P$-slash functor, then there exists an isomorphism of functors $Sl_{P} \to Sl$, and this isomorphism is unique up to scalar multiplication.
Proof. If no indecomposable direct summand of an object of the category \( \mathcal{O}_P \mathcal{M} \) is capped, then any \( P \)-slash functor \( SL : \mathcal{O}_P \mathcal{M} \to k \text{Mod} \) is zero and the proof of Lemma \([11]\) is straightforward. Thus we may assume that there exists in \( \mathcal{O}_P \mathcal{M} \) an object \( X \) that admits a capped indecomposable direct summand \( V \).

Let \( M \) be an object of \( \mathcal{O}_P \mathcal{M} \). We consider \( \text{Hom}_\mathcal{O}(V, M) \) as an \( \mathcal{O}\Delta P \)-module and we set \( \text{Sl}_P(M) = \text{Br}_\Delta P(\text{Hom}_\mathcal{O}(V, M)) \). Let \( L, M \) be two object in \( \mathcal{O}_P \mathcal{M} \) and \( u : L \to M \) be a morphism of \( \mathcal{O}_P \)-modules. Then the map \( \text{Hom}_\mathcal{O}(V, u) : \text{Hom}_\mathcal{O}(V, L) \to \text{Hom}_\mathcal{O}(V, M) \) is a morphism of \( \mathcal{O}\Delta P \)-modules, and we set \( \text{Sl}_P(u) = \text{Br}_\Delta P(\text{Hom}_\mathcal{O}(V, u)) : \text{Sl}_P(L) \to \text{Sl}_P(M) \). This clearly defines a functor \( \text{Sl}_P : \mathcal{O}_P \mathcal{M} \to k \text{Mod} \). By construction, for any two objects \( L, M \) in \( \mathcal{O}_P \mathcal{M} \), the functor \( \text{Sl}_P \) induces a \( k \)-linear map

\[
\Phi_{L,M} : \text{Br}_\Delta P(\text{Hom}_\mathcal{O}(L, M)) \to \text{Hom}_k(\text{Br}_\Delta P(\text{Hom}_\mathcal{O}(V, L)), \text{Br}_\Delta P(\text{Hom}_\mathcal{O}(V, M))).
\]

It follows from the assumptions of the lemma that the \( \mathcal{O}_P \)-modules \( L \) and \( M \) decompose as \( L = L' \oplus L'' \) and \( M = M' \oplus M'' \), where \( L', M' \) are direct sums of copies of \( V \) and \( L'', M'' \) are direct sums of non-capped indecomposable \( \mathcal{O}_P \)-modules. This implies \( \text{Br}_\Delta P(\text{Hom}_\mathcal{O}(V, L'')) = \text{Br}_\Delta P(\text{Hom}_\mathcal{O}(V, M'')) = 0 \). Hence we may forget about \( L'', M'' \) and assume that \( L = L', M = M' \). Then \( L \) and \( M \) are finite direct sums of copies of \( V \), so it is enough to prove that \( \Phi_{L,M} \) is an isomorphism in the case \( L = M = V \), which is trivial since \( \text{Br}_\Delta P(\text{Hom}_\mathcal{O}(V, V)) \simeq k \).

It follows that \( \text{Sl}_P \) is a \( P \)-slash functor.

We now consider another \( P \)-slash functor \( SL : \mathcal{O}_P \mathcal{M} \to k \text{Mod} \). The \( \mathcal{O}_P \)-module \( X \) is an object of the category \( \mathcal{O}_P \mathcal{M} \), and there is an idempotent \( i \in \text{End}_{\mathcal{O}_P}(X) \) such that \( V = iX \). We set \( j = SL(i) \in \text{End}_k(SL(X)) \), and \( W = jSL(X) \). The map \( SL^X : \text{End}_L(X) \to \text{End}_k(SL(X)) \) induces an isomorphism \( \text{Br}_{\Delta P}(\text{End}_L(X)) \to \text{End}_k(SL(X)) \) that sends the idempotent \( \text{br}_P(i) \) to \( j \), hence an isomorphism \( \text{Br}_{\Delta P}(\text{End}_L(V)) \to \text{End}_k(W) \). It follows that \( W \) is a 1-dimensional \( k \)-vector space. The choice of an isomorphism \( \zeta : k \to W \) brings a natural isomorphism

\[
\phi_M : \text{Hom}_k(W, SL(M)) \xrightarrow{\sim} SL(M), \quad M \in \text{Ob}(\mathcal{O}_P \mathcal{M}).
\]

By assumption, the \( P \)-slash functor \( SL \) induces an isomorphism \( \text{Br}_{\Delta P}(\text{Hom}_\mathcal{O}(V, M)) \to \text{Hom}_k(W, SL(M)) \), i.e., a natural isomorphism

\[
\psi_M : \text{Sl}_P(M) \xrightarrow{\sim} \text{Hom}_k(W, SL(M)), \quad M \in \text{Ob}(\mathcal{O}_P \mathcal{M}).
\]

So we obtain a natural isomorphism \( \xi = \phi \circ \psi : \text{Sl}_P \to SL \). Moreover the correspondence \( \xi \leftrightarrow \zeta \) is one-to-one, so \( \xi \) is unique up to scalar multiplication. \( \square \)

The second step is to extend the notion of a slash functor to take subpairs into account. In such a situation, what we call a slash functor is actually a little more than a functor.

**Definition 12.** Let \( G \) be a finite group, \( e \) be a block of the group \( G \), and \( \mathcal{O}_G \mathcal{M} \) be a subcategory of the category \( \mathcal{O}_G \text{Mod} \) of \( \mathcal{O}_G \)-modules. Let \( (P, e_P) \) be an \( e \)-subpair of the group \( G \), and \( H \) be a subgroup of \( G \) such that \( PC_G(P) \leq H \leq N_G(P, e_P) \). Write \( H = H/P \). A \( (P, e_P) \)-slash functor \( SL : \mathcal{O}_G \mathcal{M} \to k\hat{H}\hat{e}_P \text{Mod} \) is defined by the following data:

- for each object \( M \) of the category \( \mathcal{O}_G \mathcal{M} \), a \( k\hat{H}\hat{e}_P \)-module \( SL(M) \);
- for each pair \( L, M \) of objects of the category \( \mathcal{O}_G \mathcal{M} \), a map

\[
SL^{L,M} : \text{Hom}_{\mathcal{O}_P}(L, M) \to \text{Hom}_k(SL(L), SL(M));
\]
such that

- $SL^{M,M}(1_{\text{End}_G(M)}) = 1_{\text{End}_k(Sl(M))}$ for any object $M$ of the category $\mathcal{O}_G,M$;
- $SL^{L,N}(v \circ u) = SL^{M,N}(v) \circ SL^{L,M}(u)$ for any three objects $L, M, N$ of the category $\mathcal{O}_G,M$ and any two morphisms of $\mathcal{O}P$-modules $u : L \to M$, $v : M \to N$;
- for any two objects $L, M$ of the category $\mathcal{O}_G,M$, the map $SL^{L,M}$ factors through an isomorphism of $k(C_G(P) \times C_G(P))\Delta H$-modules

$$\text{Br}_{\Delta P}(\text{Hom}_k(e_p L, e_p M)) \longrightarrow \text{Hom}_k(\text{Sl}(L), \text{Sl}(M)).$$

The first example of a $(P, e_p)$-slash functor is the Brauer functor $\text{Br}(P, e_p) : \mathcal{O}_G,\text{Perm} \to k\mathcal{C}_G(P,e_p)\mathcal{M}od$. To obtain a more general example, we need to consider a subcategory $\mathcal{O}_G,M$ of the category $\mathcal{O}_G,\mathcal{M}od$ such that any two objects of $\mathcal{O}_G,M$ are compatible Brauer-friendly $\mathcal{O}G$-modules. This is what we call a Brauer-friendly category of $\mathcal{O}G$-modules.

**Remark 13.** The category of all Brauer-friendly $\mathcal{O}G$-modules does not fit our purposes, because its objects need not be compatible with one another (unless the block $e$ has defect zero). On the contrary, there are usually various Brauer-friendly categories of $\mathcal{O}G$-modules.

For instance, let $(D, e_D)$ be a maximal $e$-subpair, and $F = F_{(G,e)}(D, e_D)$ be the corresponding fusion system. Denote by $\mathcal{D}(D, F)$ the Dade group of that fusion system, as defined in \cite{8}. Any element of that Dade group determines a capped indecomposable endopermutation $\mathcal{O}D$-module $V$ that is fusion-stable in the group $G$ with respect to the subpair $(D, e_D)$, as can be deduced from \cite{17} Theorem 14.2. Then we let $\mathcal{O}_G,M(D, e_D, V)$ be the full subcategory of $\mathcal{O}_G,\mathcal{M}od$ of which the objects are the direct sums of indecomposable $\mathcal{O}G$-modules with source triples compatible with $(D, e_D, V)$. In this way, we obtain a collection of Brauer-friendly categories, indexed by the Dade group $\mathcal{D}(D, F)$. In particular, the trivial element of $\mathcal{D}(D, F)$ corresponds to the category $\mathcal{O}_G,\text{Perm}$ of $p$-permutation $\mathcal{O}G$-modules.

We can easily prove the existence and uniqueness of slash functors over such categories in the case $H = PC_G(P)$. We write $C_G(P) = PC_G(P)/P$.

**Lemma 14.** Let $G$ be a finite group, $e$ be a block of the group $G$, and $\mathcal{O}_G,M$ be a Brauer-friendly category of $\mathcal{O}G$-modules. Let $(P, e_p)$ be an $e$-subpair of the group $G$.

(i) There exists a $(P, e_p)$-slash functor $Sl((P, e_p)) : \mathcal{O}_G,M \to k\mathcal{C}_G(P,e_p)\mathcal{M}od$.

(ii) If $Sl : \mathcal{O}_G,M \to k\mathcal{C}_G(P)e_p,\mathcal{M}od$ is another $(P, e_p)$-slash functor, then there exists an isomorphism of slash functors $Sl((P, e_p)) \rightarrow Sl$, which is unique up to scalar multiplication.

**Proof.** Let $\mathcal{O}_P,M$ be the full subcategory of $\mathcal{O}P,\mathcal{M}od$ over the essential image of the truncation-restriction functor $e_p \text{Res}^G_P$. By Lemma \footnote{11} the category $\mathcal{O}_P,M$ satisfies the assumptions of Lemma \footnote{9} Thus there exists a $P$-slash functor $Sl_P : \mathcal{O}_P,M \rightarrow \mathcal{O}P,\mathcal{M}od$.

For any object $M$ in $\mathcal{O}_G,M$, we have a morphism of algebras $Sl_P^{\times} : \text{End}_{\mathcal{O}P}(e_p M) \rightarrow \text{End}_{k}(Sl_P(e_p M))$, and a natural group morphism $\iota : C_G(P) \rightarrow \text{End}_{\mathcal{O}P}(e_p M)$. The composition $Sl_P^{\times} : C_G(P) \rightarrow \text{End}_{k}(Sl_P(e_p M))$ makes the endomorphism ring $\text{End}_{k}(Sl_P(e_p M))$ a $C_G(P)$-interior algebra. As a consequence, it makes the slashed module $Sl_P(e_p M)$ a $k\mathcal{C}_G(P)e_p$-module, which we denote by $Sl((P, e_p))(M)$.

For any two objects $L, M$ in $\mathcal{O}_G,M$ and any morphism of $\mathcal{O}P$-module $u : L \rightarrow M$, we denote by $e_p u e_p : e_p L \rightarrow e_p M$ the morphism of $\mathcal{O}P$-modules induced by $u$. We consider
Sl\(_{(p,e_P)}\) as a \(k\)-linear map \(\text{Sl}_p(L) \rightarrow \text{Sl}_p(M)\), and we denote it by \(\text{Sl}_u(u)\). If \(u\) is a morphism of \(\mathcal{O}G\_e\)-modules, then it follows from the functoriality of \(\text{Sl}_p\) and from the definition of the \(\mathcal{C}_G(P)\)-interior structures on \(\text{End}_k(\text{Sl}_p(L))\) and \(\text{End}_k(\text{Sl}_p(M))\) that \(\text{Sl}_u(u)\) is a morphism of \(k\mathcal{C}_G(P)\)-modules. This defines a \((p,e_P)\)-slash functor \(\text{Sl}_u : \mathcal{O}G\_e \rightarrow k\mathcal{C}_G(P)\_e\_P\text{-Mod}\).

If \(\text{Sl} : \mathcal{O}G\_e \rightarrow k\mathcal{C}_G(P)\_e\_P\text{-Mod}\) is another \((p,e_P)\)-slash functor, then \(\text{Sl}\) induces a \(\mathcal{P}\)-slash functor \(\text{Sl}' : \mathcal{O}P\_e \rightarrow \mathcal{P}\text{-Mod}\), such that \(\text{Sl}' \circ e_P \circ \text{Res}_P^G = \text{Res}_P^{\mathcal{C}_G(P)} \circ \text{Sl}\). By Lemma \([11]\) there exists an isomorphism of \(\mathcal{O}G\_e\)-modules, the isomorphism of \(k\)-vector spaces \(\xi'^{P,e} : \text{Sl}_p(\mathcal{M}) \rightarrow \text{Sl}'(\mathcal{M})\) may be seen as an isomorphism of \(k\mathcal{C}_G(P)\)-modules \(\text{Sl}(\mathcal{M}) \rightarrow \text{Sl}(\mathcal{M})\), which we denote by \(\xi_M\). By construction, for any two objects \(L,M\) in \(\mathcal{O}G\_e\) and any morphism of \(\mathcal{P}\)-module \(u : L \rightarrow M\), there is a commutative diagram

\[
\begin{array}{ccc}
\text{Sl}_p(L) & \xrightarrow{\xi_L} & \text{Sl}(L) \\
\text{Sl}_p(u) & & \text{Sl}(u) \\
\text{Sl}_p(M) & \xrightarrow{\xi_M} & \text{Sl}(M)
\end{array}
\]

Thus \(\xi : \text{Sl}_u \rightarrow \text{Sl}\) is what we would like to call an isomorphism of \((p,e_P)\)-slash functors. The uniqueness of \(\xi\) up to scalar multiplication follows from the similar uniqueness of \(\xi'\). □

The third and last step is to consider a \((p,e_P)\)-slash functor with a codomain such as the category \(k\mathcal{H}\_e\_P\text{-Mod}\), where \(\mathcal{H}\) is any subgroup of \(G\) with \(\mathcal{P}C_G(P) \leq \mathcal{H} \leq N_G(P,e_P)\). The existence of such a slash functor follows from a deep result proven by Puig in \([9]\). Unfortunately, we lose the uniqueness of a slash functor up to isomorphism. We need to explain how a \((p,e_P)\)-slash functor \(\text{Sl} : \mathcal{O}G\_e \rightarrow k\mathcal{H}\_e\_P\text{-Mod}\) may be twisted by a linear character \(\chi : \mathcal{H}/\mathcal{C}_G(P) \rightarrow k^\times\).

If \(M\) is an object of the category \(\mathcal{O}G\_e\), then \(\text{Sl}(M)\) is a \(k\mathcal{H}\_e\_P\)-module. We set \(\chi \_\text{Sl}(M) = \text{Sl}(M)\) as a \(k\)-vector space, and we endow \(\chi \_\text{Sl}(M)\) with the action \(\cdot\_\chi\) of the group \(\mathcal{H}\) defined by \(h \cdot \chi m = h \cdot \chi(h)m\) for any \(h \in \mathcal{H}\) and \(m \in \text{Sl}(M)\), where the single dot stands for the preexisting action of the group \(\mathcal{H}\) on the \(k\)-vector space \(\text{Sl}(M)\). If \(L,M\) are two objects of \(\mathcal{O}G\_e\) and \(u : L \rightarrow M\) is a morphism of \(\mathcal{P}\)-modules, then we set \(\chi \_\text{Sl}(L,M)(u) = \text{Sl}(L,M)(u)\), considered as a \(k\)-linear map \(\chi \_\text{Sl}(L) \rightarrow \chi \_\text{Sl}(M)\). This defines another \((p,e_P)\)-slash functor \(\chi \_\text{Sl} : \mathcal{O}G\_e \rightarrow k\mathcal{H}\_e\_P\text{-Mod}\). Notice that the slash functors \(\text{Sl}\) and \(\chi \_\text{Sl}\) might be isomorphic.

**Theorem 15.** Let \(G\) be a finite group, \(e\) be a block of the group \(G\), and \(\mathcal{O}G\_e\) be a Brauer-friendly category of \(\mathcal{O}G\)-modules. Let \((p,e_P)\) be an e-superpair of the group \(G\), and \(\mathcal{H}\) be a subgroup of \(G\) such that \(\mathcal{P}C_G(P) \leq \mathcal{H} \leq N_G(P,e_P)\).

(i) There exists a \((p,e_P)\)-slash functor \(\text{Sl}_u : \mathcal{O}G\_e \rightarrow k\mathcal{H}\_e\_P\text{-Mod}\).

(ii) If \(\text{Sl} : \mathcal{O}G\_e \rightarrow k\mathcal{H}\_e\_P\text{-Mod}\) is another \((p,e_P)\)-slash functor, then there exists a linear character \(\chi : \mathcal{H}/\mathcal{C}_G(P) \rightarrow k^\times\) and an isomorphism of slash functors \(\chi \_\text{Sl} : \text{Sl}_u \rightarrow \text{Sl}\).

**Proof.** We know from Lemma \([14]\)(i) that there exists a \((p,e_P)\)-slash functor \(\text{Sl}_u : \mathcal{O}G\_e \rightarrow k\mathcal{H}\_e\_P\text{-Mod}\). We may assume that this slash functor is nonzero, i.e., that there exists an object \(X\) of \(\mathcal{O}G\_e\) such that \(\text{Sl}_u(X) \neq 0\).
Let $L$ and $M$ be two objects of the category $\mathcal{O}_{G_p}\mathcal{M}$. The $\mathcal{O}$-module $\text{Hom}_\mathcal{O}(e_pL, e_pM)$ has a natural structure of $\mathcal{O}(H \times H)$-module, so the Brauer quotient $\text{Br}_\Delta \mathcal{P}(\text{Hom}_\mathcal{O}(e_pL, e_pM))$ admits a natural structure of $k(C_G(P) \times C_G(P))\Delta N_G(P, e_p)$-module. If $L = M$, the Brauer quotient $\text{Br}_\Delta \mathcal{P}(\text{End}_\mathcal{O}(e_pM)) \simeq \text{End}_k(\text{Sl}_{(P, e_p)}(M))$ is more precisely a $C_G(P)$-interior $N_G(P, e_p)$-algebra, as defined in [12].

The main result of [9] implies that $\text{End}_k(\text{Sl}_{(P, e_p)}(M))$ may be extended (non-uniquely) to an $H$-interior algebra. This defines an extension of the slashed module $\text{Sl}_{(P, e_p)}(M)$ to a $kH$-module, which is unique up to twisting by a linear character $\chi : H/\bar{C}_G(P) \to k^\times$. Let us choose, once and for all, such an extension for the slashed module $\text{Sl}_{(P, e_p)}(X)$.

Then, for any object $M$ of $\mathcal{O}_{G_p}\mathcal{M}$, the slashed module $\text{Sl}_{(P, e_p)}(M)$ may admit several extensions to a $k\bar{H}$-module. Each one of them defines a structure of $k(H \times H)$-module on $\text{Hom}_k(\text{Sl}_{(P, e_p)}(X), \text{Sl}_{(P, e_p)}(M))$. By definition, the slash functor $\text{Sl}_{(P, e_p)}$ induces an isomorphism of $k(C_G(P) \times C_G(P))$-modules

$$\text{Br}_\Delta \mathcal{P}(\text{Hom}_\mathcal{O}(e_pX, e_pM)) \xrightarrow{\sim} \text{Hom}_k(\text{Sl}_{(P, e_p)}(X), \text{Sl}_{(P, e_p)}(M)).$$

Since the slashed module $\text{Sl}_{(P, e_p)}(X)$ is nonzero, there is only one extension of $\text{Sl}_{(P, e_p)}(M)$ to a $k\bar{H}$-module such that the above map is a morphism of $k(C_G(P) \times C_G(P))\Delta \bar{H}$-module. We denote this $k\bar{H}$-module by $\text{Sl}_{(P, e_p)}(M)$, in italics. This defines a $(P, e_p)$-slash functor

$$\text{Sl}_{(P, e_p)} : \mathcal{O}_{G_p}\mathcal{M} \to k\bar{H}\mathcal{E}_p\text{Mod}.$$

Let $SL : \mathcal{O}_{G_p}\mathcal{M} \to k\bar{H}\mathcal{E}_p\text{Mod}$ be another $(P, e_p)$-slash functor. Then, by Lemma [14] (ii), the restriction $\text{Res}_{\bar{C}_G(P)}^H k\bar{H}\mathcal{E}_p\text{Mod}$ is isomorphic to the slash functor $\text{Sl}_{(P, e_p)}$. By transport of structure, we may suppose that $\text{Res}_{\bar{C}_G(P)}^H \text{Sl}_{(P, e_p)} = \text{Sl}_{(P, e_p)}$, i.e., the only difference between $\text{Sl}_{(P, e_p)}$ and $\text{Sl}$ is the choice of an extension of the slashed module $\text{Sl}_{(P, e_p)}(X)$ to a $k\bar{H}$-module. Up to twisting $\text{Sl}_{(P, e_p)}$ by a linear character, we can obtain $\text{Sl}_{(P, e_p)}(X) = \text{Sl}(X)$, which implies $\text{Sl}_{(P, e_p)} = \text{Sl}$. This completes the proof of the theorem.

In general, there are several $(P, e_p)$-slash functors on a given Brauer-friendly category, none of which can be singled out. However, in the case of the category $\mathcal{O}_{G_p}\mathcal{P}$ of $p$-permutation modules, there is a canonical choice: the Brauer functor $\text{Br}_{(P, e_p)}$.

5 The essential image of a slash functor

In this section, we prove that the essential image of a slash functor is, as expected, a Brauer-friendly category. This enables us to compose slash functors, and to prove that such a composition results in another slash functor. These results are prepared by two technical lemmas, which will also be used in the next section.

**Lemma 16.** Let $H$ be a finite group and $P$ be a normal $p$-subgroup of $H$. Let $R$ be a $p$-subgroup of $H$ that contains $P$, and $V$ be an $OR$-module. Suppose that the restriction $\text{Res}_R^H \text{Ind}_R^H V$ is an endopermutation $OP$-module. The $C_H(P)$-interior $H$-algebra $\text{Br}_\Delta \mathcal{P}(\text{Ind}_R^H \text{End}_R(V))$ admits a (non-unique) extension to an $H$-interior algebra. Once this extension has been chosen, there is a natural isomorphism of $H$-interior algebras

$$\Phi : \text{Ind}_R^H \text{Br}_\Delta \mathcal{P}(\text{End}_R(V)) \xrightarrow{\sim} \text{Br}_\Delta \mathcal{P}(\text{Ind}_R^H \text{End}_R(V)).$$
where we write $\text{Ind}_H^R$ for the induction of interior algebras (see [16, §16]).

Proof. Let us write $S = \text{End}_O(V)$, an $R$-interior matrix algebra over $O$ that can also be seen as an $O(R \times R)$-module. We may assume that $\text{Br}_P(S) \neq 0$, since the conclusion of the lemma is trivial otherwise. On the one hand, the $H$-interior algebra $A = \text{Ind}_R^H S$ is defined, as an $O(H \times H)$-module, by

$$A = O H \otimes_O S \otimes_O O H.$$  

The map $\phi : S \rightarrow A, s \mapsto 1_H \otimes s \otimes 1_H$ is an embedding of algebras. It induces an isomorphism of $R$-algebras $S \rightarrow \alpha A \alpha$, where $\alpha = \phi(1)$ is an idempotent of the algebra $A^R$. By [16], the $C_H(P)$-interior $H$-algebra $A_P = \text{Br}_P(A)$ admits an extension to a $H$-interior algebra, which makes $A_P$ a $k(H \times H)$-module. Consider the idempotent $\alpha_P = \text{br}_P(\alpha) \in (A_P)^R$, and the $R$-interior subalgebra $\alpha_P A_P \alpha_P$. The decomposition $1_{A_P} = \sum_{g \in H/R} g \alpha_P$ of the unity into mutually orthogonal idempotents brings a decomposition of the $k(R \times R)$-module $A_P$:

$$A_P = \bigoplus_{g, h \in H/R} g \alpha_P A_P h \alpha_P = \bigoplus_{g, h \in H/R} g(\alpha_P A_P h \alpha_P)h^{-1}.$$  

On the other hand, the Brauer quotient $S_P = \text{Br}_P(S)$ is a matrix algebra over $k$, with a natural $R$-interior structure that also makes it a $k(R \times R)$-module. The $H$-interior algebra $B = \text{Ind}_R^H S_P$ is defined, as a $k(H \times H)$-module, by

$$B = k H \otimes_{k R} S_P \otimes_{k R} k H.$$  

The embedding $\psi : S_P \rightarrow B, s \mapsto 1_H \otimes s \otimes 1_H$ induces an isomorphism of $R$-algebras $S_P \rightarrow \beta B\beta$, where $\beta = \psi(1) \in B^R$. The decomposition $1_B = \sum_{g \in H/R} g \beta$ brings a decomposition of the $k(R \times R)$-module $B$:

$$B = \bigoplus_{g, h \in H/R} g \beta B h \beta = \bigoplus_{g, h \in H/R} g(\beta B \beta)h^{-1}.$$  

The Brauer functor $\text{Br}_P$ sends the map $\phi : S \rightarrow A$ to a morphism of algebras $\phi_P : S_P \rightarrow A_P$, which restricts to an isomorphism of $R$-algebras $\phi_P' : S_P \rightarrow \alpha_P A_P \alpha_P$. By uniqueness of the $R$-interior structure on $S_P$, $\phi_P'$ is an isomorphism of $R$-interior algebras. Thus $\phi_P : S_P \rightarrow A_P$ is a morphism of $k(R \times R)$-modules. By the universal property of induced modules, there exists a unique morphism of $k(H \times H)$-modules $\Phi : B \rightarrow A_P$ such that $\Phi \circ \psi = \phi_P$.

By construction, we have $\Phi(\beta) = \alpha_P$ and $\Phi$ induces an isomorphism of $R$-interior algebras $\beta B_\beta \rightarrow \alpha_P A_P \alpha_P$. Since $\Phi$ is a morphism of $k(H \times H)$-modules, it follows from the above decompositions of $B$ and $A_P$ that $\Phi$ is an isomorphism of $k(H \times H)$-modules. Then the definition of induced interior algebras implies that $\Phi$ is an isomorphism of $H$-interior algebras.\[\Box\]

Lemma 17. With the assumptions of Lemma [16] let $i$ be an idempotent of the algebra $(O H)^R$. Write $i P = \text{br}_P(i)$. The isomorphism $\Phi$ induces an isomorphism of $H$-interior algebras

$$\Phi_i : k H i P \otimes_{k R} \text{Br}_P(\text{End}_k(V)) \otimes_{k R} i P k H \rightarrow \text{Br}_P(O H i \otimes_{O R} \text{End}_k(V) \otimes_{O R} i O H).$$  

Proof. With the notations of the proof of Lemma [16] we consider the idempotent $u = \text{Tr}_R^H(i \circ i)$ of the algebra $A^R_H$, and the idempotent $u_P = \text{br}_P(u) = \text{Tr}_R^H(i P \alpha \circ i P \circ i P)$ of the algebra $(A_P)^H$.

A direct computation in the induced algebra $A$ yields

$$u A u = O H i \otimes_{O R} S \otimes_{O R} i O H,$$  

so that $u_P A_P u_P \simeq \text{Br}_P(O H i \otimes_{O R} S \otimes_{O R} i O H)$.  

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Thus $\Phi$ sends $\beta$ to $\alpha_P$ and commutes with the relative trace map, we obtain $\Phi(v) = u_P$. So $\Phi$ induces an isomorphism of $H$-interior algebras $\Phi_i : vBv \rightarrow u_PA_Pu_P$. 

We can now prove the main result of this section.

**Lemma 18.** Let $G$ be a finite group, $e$ be a block of the algebra $\mathcal{O}G$, and $\mathcal{O}Ge, M$ be a Brauer-friendly category of $\mathcal{O}Ge$-modules. Let $(P,e_P)$ be an $e$-subpair of the group $G$, $H$ be a subgroup of $G$ such that $PC_G(P) \leq H \leq N_G(P,e_P)$, and

$$SL_{(P,e_P)} : \mathcal{O}Ge, M \rightarrow kH_ie_P \mathbf{Mod}$$

be a $(P,e_P)$-slash functor. Then there exists a Brauer-friendly category $kH_ie_P \mathbf{Mod}$ of $kH_ie_P$-modules that contains the essential image of $SL_{(P,e_P)}$.

**Proof.** We let $S$ be the set of source triples $(Q,e_Q,V)$ of the group $G$ for which there exists a source triple $(R,e_R,W)$ of an indecomposable direct summand of an object of $\mathcal{O}Ge, M$ such that $(P,e_P) \subseteq (Q,e_Q) \subseteq (R,e_R)$ and $V$ is a direct summand of the $\mathcal{O}Q$-module $\text{Res}_Q^R W$. We let $\tilde{S}$ be the set of triples $(\tilde{Q},\tilde{e}_Q,\tilde{V})$ for which there exists an element $(Q,e_Q,V)$ of the set $S$ such that $\tilde{Q} = Q/P$, $\tilde{e}_Q = \text{br}_{P}(e_Q)$, and the $k\tilde{Q}$-module $\tilde{V}$ is a capped indecomposable direct summand of a $P$-slashed module $V[P]$. We denote by $\mathcal{O}Hie_P \mathbf{Mod}$ (resp. $kHie_P \mathbf{Mod}$) the full subcategory of $\mathcal{O}He_P \mathbf{Mod}$ (resp. $kHe_P \mathbf{Mod}$) of which the objects are the direct sums of indecomposable modules with source triples in $S$ (resp. $\tilde{S}$). These are Brauer-friendly categories.

We know from Lemma 15 that there exists a $(P,e_P)$-slash functor $SL_{(P,e_P)} : \mathcal{O}He_P \mathbf{Mod} \rightarrow kHie_P \mathbf{Mod}$. Let $M$ be an object of the category $\mathcal{O}Ge, M$. By Lemma 9, the $\mathcal{O}He_P$-module $e_PM$ admits the decomposition $e_PM = L \oplus L'$, where $L$ is a Brauer-friendly $\mathcal{O}He_P$-module and $L'$ is a direct sum of indecomposable $\mathcal{O}He_P$-modules with vertices that do not contain the $p$-subgroup $P$. It follows from Lemma 15(ii) that the slashed modules $SL_{(P,e_P)}(M)$ and $SL_{(P,e_P)}(L)$ are isomorphic, up to twisting by a linear character of the group $H/C_G(P)$. Notice that this twist preserves the source triples of an indecomposable $kHie_P$-module.

Let $(Q,e_Q,\tilde{V})$ be a source triple of an indecomposable direct summand of $SL_{(P,e_P)}(M)$, hence of $SL_{(P,e_P)}(L)$. To study this triple, we may first suppose that the $\mathcal{O}He_P$-module $L$ is indecomposable. Then we may suppose, as in the proof of Theorem 3, that $L = \mathcal{O}Hi \otimes_{\mathcal{O}R} W$, with $(R,e_R,W)$ a source triple of the group $H$ that lies in the set $S$, and $i$ a primitive idempotent of the algebra $(\mathcal{O}H)^R$ such that $\tilde{e}_R \text{br}_R(i) \neq 0$. With this assumption, we know from Lemma 17 that the slashed module $SL_{(P,e_P)}(L)$ is isomorphic to the $kHie_P$-module $L_P = kHbr_P(i) \otimes_{kQ/P} W[P]$. Then we deduce from Theorem 3 that the subpair $(\tilde{R},\tilde{e}_R)$ is contained in $(Q/P,\text{br}_P(e_Q))$, and that $W$ is isomorphic to a direct summand of the $k\tilde{R}$-module $\text{Res}_{\tilde{R}}^Q V[P]$. Thus $SL_{(P,e_P)}(M)$ lies in $kHie_P \mathbf{Mod}$. 

Lemma 18 enables us to study the transitivity of slash functors.

**Lemma 19.** Let $G$ be a finite group, $e$ be a block of the group $G$, and $\mathcal{O}Ge, M$ be a Brauer-friendly category of $\mathcal{O}Ge$-modules.
(i) Let \((P,e_P) \trianglelefteq (Q,e_Q)\) be e-subpairs of the group \(G\). Let \(\text{Sl}_{(P,e_P)} : \mathcal{O}_G \mathcal{M} \to k \tilde{N}_G(P,e_P)\text{Mod}\) be a \((P,e_P)\)-slash functor, and \(k \tilde{N}_G(P,e_P)\text{Mod}\) be a Brauer-friendly category of \(k \tilde{N}_G(P,e_P)\text{Mod}\) modules that contains the essential image of \(\text{Sl}_{(P,e_P)}\). Let \(\text{Sl}_{(Q/P,e_Q)} : k \tilde{N}_G(P,e_P)\text{Mod} \to k \tilde{N}_G(P,Q,e_Q)\text{Mod}\) be an \((Q/P,e_Q)\)-slash functor. Then the composition
\[
\text{Sl}_{(Q/P,e_Q)} \circ \text{Sl}_{(P,e_P)} : \mathcal{O}_G \mathcal{M} \to k \tilde{N}_G(P,Q,e_Q)\text{Mod}
\]
is an \((Q,e_Q)\)-slash functor.

(ii) Let \((P,e_P)\) be an e-subpair, and \(\text{Sl}_{(P,e_P)} : \mathcal{O}_G \mathcal{M} \to k \tilde{N}_G(P,e_P)\text{Mod}\) be a \((P,e_P)\)-slash functor. For an element \(g\) of the group \(G\), let \(g_* : k \tilde{N}_G(P,e_P)\text{Mod} \to k \tilde{N}_G(P,e_P)\text{Mod}\) stand for the “twist by \(g\)”. Then the composition
\[
g_* \circ \text{Sl}_{(P,e_P)} : \mathcal{O}_G \mathcal{M} \to k \tilde{N}_G(P,e_P)\text{Mod}
\]
is a \(g(P,e_P)\)-slash functor.

Proof. In order to prove (i), we may assume that \(G = N_G(P,Q,e_Q)\) and \(e = e_P = e_Q\). We fix two compatible Brauer-friendly \(\mathcal{O}_G\)-modules \(L\) and \(M\). Thus \(\text{Hom}_\mathcal{O}(L,M)\) is a \(P\)-permutation \(\mathcal{O}\Delta Q\)-module, and we know from [14 §2.2] that the natural map
\[
\text{Br}_{\Delta Q}(\text{Hom}_\mathcal{O}(L,M)) \to \text{Br}_{\Delta Q/P} \circ \text{Br}_{\Delta P}(\text{Hom}_\mathcal{O}(L,M))
\]
is an isomorphism. This proves (i); the proof of (ii) is similar.

\section{A parametrisation of the indecomposable Brauer-friendly modules}

In this section, we prove that an indecomposable Brauer-friendly \(\mathcal{O}_G\)-module \(X\) is characterized, up to isomorphism, by a conjugacy class of quadruples of the form \((P,e_P,V,X)\), where \((P,e_P,V)\) is a fusion-stable endopermutation source triple of the group \(G\), and \(X\) is a projective indecomposable \(k \tilde{N}_G(P,e_P)\text{mod}\)-module. The precise statement is the following.

\textbf{Theorem 20.} Let \(G\) be a finite group and \(e\) be a block of the group \(G\). Let \((P,e_P,V)\) be a fusion-stable endopermutation source triple of the group \(G\) with respect to the block \(e\). Let \(\mathcal{O}_G \mathcal{M}\) be a Brauer-friendly category of \(\mathcal{O}_G\)-modules that is “big enough”, i.e., such that any finite direct sum of indecomposable \(\mathcal{O}_G\)-modules with source triple \((P,e_P,V)\) is an object of \(\mathcal{O}_G \mathcal{M}\). Let
\[
\text{Sl}_{(P,e_P)} : \mathcal{O}_G \mathcal{M} \to k \tilde{N}_G(P,e_P)\text{Mod}
\]
be a \((P,e_P)\)-slash functor. Then the mapping \(X \mapsto \text{Sl}_{(P,e_P)}(X)\) induces a one-to-one correspondence between the isomorphism classes of indecomposable \(\mathcal{O}_G\)-modules with source triple \((P,e_P,V)\) and the isomorphism classes of projective indecomposable \(k \tilde{N}_G(P,e_P)\text{mod}\)-modules.

Proof. We write \(H = N_G(P,e_P)\). There exists a Brauer-friendly category \(\mathcal{O}_H \mathcal{M}\) of \(OH\text{-mod}\) modules that contains any finite direct sum of indecomposable \(OH\text{-mod}\)-modules with source triple \((P,e_P,V)\), and there exists a \((P,e_P)\)-slash functor \(\text{Sl}_{(P,e_P)}'' : \mathcal{O}_H \mathcal{M} \to k \tilde{H}_e\text{Mod}\).
Let $X$ be an indecomposable $OGe$-module with source triple $(P,e_P,V)$. Let $X'$ be its Green correspondent, an object of the category $OGet \mathcal{M}$. Up to twisting the slash functor $Sl'_(P,e_P)$ by a linear character of the group $\bar{H}/\bar{C}_G(P)$, we may assume that the slashed modules $Sl'_(P,e_P)(X)$ and $Sl'_(P,e_P)(X')$ are isomorphic. Once this assumption has been made for $X$, the same is necessarily true for any other indecomposable $OGe$-module with source triple $(P,e_P,V)$.

By definition of a source triple, $X'$ is isomorphic to a direct summand of the $OHe_P$-module $L = OHe_P \otimes_{O_P} V$. A $P$-slashed module relative to the capped indecomposable endopermutation $OP$-module $V$ is the $k$-vector space $k$. Thus, by Lemma 17 there is an isomorphism of $kH\bar{e}_P$-modules

$$Sl'_(P,e_P)(L) \cong kH\bar{e}_P \otimes_{kP} k \cong kH\bar{e}_P.$$  

Since $L$ is an endo-$p$-permutation $OH$-module, the Brauer map $\text{br}^{End_O(L)}_P \Delta$ induces an epimorphism $[End_O(L)]^H \rightarrow [\text{Br}_P((End_O(L)))]^H$. In other words, the slash functor $Sl'_(P,e_P)$ induces an epimorphism

$$\beta : \text{End}_{OH}(L) \rightarrow \text{End}_{kH}(Sl'_(P,e_P)(L))$$

By a classical result about the lifting of idempotents such as [10, Theorem 3.1], the epimorphism $\beta$ induces a one-to-one correspondence between the conjugacy classes of primitive idempotents of the algebra $\text{End}_{OH}(L)$ and the conjugacy classes of primitive idempotents of $\text{End}_{kH}(Sl'_(P,e_P)(L))$. In other words, the mapping $X' \mapsto Sl'_(P,e_P)(X')$ induces a one-to-one correspondence between the isomorphism classes of indecomposable direct summands of the $OH$-module $L$ and the isomorphism classes of indecomposable direct summands of the $kH$-module $Sl'_(P,e_P)(L)$, i.e., between the isomorphism classes of indecomposable $OHe_P$-modules with source triple $(P,e_P,V)$ and the isomorphism classes of indecomposable projective $kH\bar{e}_P$-modules. Then the above remark concerning Green correspondents completes the proof. 

The one-to-one correspondence of Theorem 20 may be seen as an instance of the Puig correspondence defined in [10]. An indecomposable Brauer-friendly $OGe$-module $X$ is characterised by a quadruple $((P,e_P),V,\bar{X})$, as described at the beginning of this section. The correspondence $X \leftrightarrow \bar{X}$ is usually not canonical; it depends on the isomorphism class of the slash functor $Sl'_(P,e_P)$. This is consistent with what Thévenaz explains in [10] before Example 26.5.

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