On gravity localization in scalar braneworlds with a super-exponential warp factor

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Abstract We show that within single brane tachyonic braneworld models, super-exponential warp factors of the form \( e^{-2f} \sim e^{-2c_1 e^{2|\sigma|}} \) are problematic when dealing with both the finiteness of the effective four-dimensional (4d) Planck mass and the localization of 4d gravity, which can be stated by the requirement that \( \int e^{-2f(\sigma)} d\sigma < \infty \), because this condition necessarily implies that \( c_1 \) and \( c_2 \) should be positive. As a consequence of this fact the tachyonic field \( T \) turns out to be complex in contradiction with the real nature of the starting action for the tachyonic braneworld. Conversely if one requires to have a real tachyon field, 4d gravity will not be localized and the effective gravitational coupling will be infinite. We present several typical examples where this problem occurs: we have analysed this situation for thin as well as thick tachyonic braneworlds with 4d Poincaré symmetry, for the case when a bulk cosmological constant is present, and even for a brane with an induced spatially flat 4d cosmological background, and shown that in all cases the tachyon field \( T \) comes out to be inconsistently complex when imposing localization of 4d gravity on the brane.
On the other hand, when dealing with a further reduction of the hierarchy problem on a two-brane system, one should carefully consider the sign of the constants $c_1$ and $c_2$ to avoid inconsistencies in the tachyonic braneworld model. We also present a similar discussion involving a canonical scalar field in the bulk where none of these problems arise and hence, the mass hierarchy and 4d gravity localization problems can be successfully addressed at once, i.e., with the same warp factor. Finally, the stability analysis of this scalar tensor braneworld model with a super-exponential warp factor is performed.

**Keywords**  Analytical GR · Gravity localization · Hierarchy problem · Randall-Sundrum-type models · General relativity and extra dimensions

### 1 Introduction

There has recently been some interest in braneworld models, in which our universe is embedded in a spacetime with extra dimensions, as a way to solving the mass hierarchy and 4d gravity localization problems [1–4]. It has also become a matter of interest to find smooth braneworld solutions (for interesting reviews on these issues see e.g., [5–8]). Typically such solutions are obtained by introducing one or several scalar fields in the bulk and the large variety of scalar fields that can be used to generate these models gives rise to different scenarios [9–37].

Several authors have chosen a tachyonic scalar field in the bulk and address issues like the hierarchy problem, and localization of gravity and matter fields [35–44]. For instance, in [36,37] a further reduction of the hierarchy between the fundamental Planck scale and the compactification scale ($kr_c \approx 5$), when compared to the two-brane Randall-Sundrum model where $kr_c \approx 10$, is achieved by using a super-exponential warp factor in a two-brane tachyonic braneworld, rendering a model with parameters of the same order. Moreover, all kinds of matter fields as well as gravity can be localized within this super-exponential tachyonic braneworld as long as one is concerned with a two-brane system, since then the higher dimensional volume is always finite. Troubles with gravity localization arise only when one works in the one-brane model, i.e., when the second brane has been removed to infinity along the fifth dimension, since then the volume along the extra dimension can be infinite. On the other hand, when attempting to achieve a further reduction of the hierarchy by implementing a super-exponential warp factor in the two-brane picture, the tachyonic field can also result complex for some choice of the constants $c_1$ and $c_2$ (see Sect. 2). Thus care should be taken when looking at this problem within tachyonic braneworld models. We have analyzed this situation in some detail in a simple case and established the resultant nature of the tachyon field $T$ in Table 1. In general, attempts to solve the highly non-linear field equations give rise to imaginary tachyon field configurations within this framework [36,37,39].

Here we will show that when the braneworld model is generated by gravity and a tachyonic scalar field the conditions for localizing 4d gravity on a single brane as well as the finiteness of the 4d gravitational coupling cannot be fulfilled by proposing super-exponential warp factors of the form $e^{-2f} \sim e^{-2c_1 e^{c_2 |\sigma|}}$. The reason is that the
relevant integral that defines both of these conditions, \( \int e^{-2f(\sigma)}d\sigma < \infty \), diverges when one requires a real tachyonic scalar field in the model. In other words, the finite character of the effective 4d Planck mass and the gravity localization condition on the brane imply that both \( c_1 \) and \( c_2 \) should be positive and as a consequence of this the tachyonic field \( T \) turns out to be complex in contradiction with the real nature of the starting action of the braneworld models under consideration. We have analysed this situation for thin as well as thick branes with a tachyonic scalar field \( T \) in the bulk as in [36–40], with a bulk cosmological constant, and even for a 4d spatially flat cosmological metric induced on the brane. We find that in all cases the tachyon field is inconsistently complex in the case when one requires gravity localization on this single brane super-exponential braneworld model.

Thus, it is not possible to solve the hierarchy problem in the two-brane picture and the localization of gravity in the single brane model with the same warp factor, contrary to the Randall-Sundrum case where the same warp factor is used to tackle both problems. Although, in principle, these are independent problems we believe it is desirable to deal with both of them within the same family of nonfactorizable metrics and thus the same warp factor. In this spirit, in Sect. 4 we briefly discuss the case of a braneworld model generated by a canonical scalar field where none of these problems occur. Moreover, in Sect. 5 we analyze the localization of 4d gravity in our braneworld by proving the existence of a normalizable massless zero mode on the brane. In Sect. 6 the stability analysis for this braneworld under scalar perturbations is also performed. As a result we get a stable scalar field braneworld configuration with no localized massive modes.

2 The simplest tachyonic braneworld model

We start our discussion of the problem with the simplest action we will consider and later extend it in several ways. The action is then given as follows

\[
S = \int d^5x \sqrt{-g} \left( \frac{1}{2\kappa_5^2} R - V(T) \sqrt{1 + g^{MN} \partial_M T \partial_N T} \right),
\]

(1)

where the first term describes the 5d gravity and the second corresponds to the matter in the bulk, in this case a real scalar tachyonic field, while \( V(T) \) is its self-interacting potential [45–50]. The coefficient \( \kappa_5 \) is given by \( \kappa_5^2 = 8\pi G_5 \) where \( G_5 \) is the 5d gravitational coupling constant. The indices take the values \( M, N = 0, 1, 2, 3, 5 \).

The 5d Einstein equations are given by

\[
G_{MN} = \kappa_5^2 T_{MN}^{\text{bulk}}.
\]

(2)

The energy-momentum tensor components read

\[
T_{AB}^{\text{bulk}} = -g_{AB} V(T) \sqrt{1 + (\nabla T)^2} + \frac{V(T)}{\sqrt{1 + (\nabla T)^2}} \partial_A T \partial_B T,
\]

(3)
where \((\nabla T)^2\) is shorthand notation for \(g^{MN} \partial_M T \partial_N T\). The covariant equation for the \(T\) field is

\[
\Box T - \frac{g^{AB} g^{MN} \nabla_A \nabla_M T \nabla_B \nabla_N T}{1 + g^{CD} \nabla_C T \nabla_D T} = \frac{1}{V} \frac{\partial V(T)}{\partial T}.
\]

The background metric is given by

\[
ds^2 = e^{-2f(\sigma)} \eta_{\mu\nu} dx^\mu dx^\nu + d\sigma^2,
\]

corresponding to a warped 5d line element with an induced 3-brane with a flat 4d geometry/metric. The function \(f(\sigma)\) is the warp factor and \((- , + , + , + , + )\) the signature. Using Eq. (5) we obtain the Einstein tensor components

\[
G_{00} = 3 e^{-2f} \left( f'' - 2 f'^2 \right),
\]

\[
G_{ii} = -3 e^{-2f} \left( f'' - 2 f'^2 \right) = -G_{00},
\]

\[
G_{\sigma\sigma} = 6 f'^2,
\]

where \(i\) labels the spatial dimensions \(x^i\) and a prime denotes derivative with respect to the extra dimension \(\sigma\). Since the non-diagonal components of the Einstein tensor vanish, consistency of Einstein equations demands that non-diagonal components of the stress energy tensor should vanish identically. This allows two possibilities: (i) the field \(T\) depends merely on time and not on any of the spatial coordinates—which is the case for a scalar field in an homogeneous and isotropic background as in cosmology; (ii) the field \(T\) depends only on the extra dimension. This simply amounts to a consistent time independence of the tachyon field even in the case when the background is time dependent, and shall be considered here. The tachyon field \(T\) depends then only on the extra dimension \(\sigma\) and Eq. (4) reads

\[
T'' + 4 f'T' (1 + T'^2) = (1 + T'^2) \frac{\partial T V(T)}{V(T)}.
\]

The relevant Einstein equations (2) can then be written as

\[
f'' = \kappa_5^2 \frac{V(T)T'}{3\sqrt{1 + T'^2}},
\]

\[
f'^2 = -\kappa_5^2 \frac{V(T)}{6\sqrt{1 + T'^2}}.
\]

On the gravity localization problem. In this case, we have a single brane configuration with 4d Poincaré symmetry given by Eqs. (7), (8) and (9), and the fifth dimension is allowed to be infinitely large.

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It is easy to invert Eqs. (8) and (9) in terms of $T''^2$ and $V(T)$

\[
T''^2 = \frac{-f'''}{2f'^2},
\]

(10)

\[
V(T) = -\frac{6f'^2}{\kappa_5^2} \sqrt{1 - \frac{f''}{2f'^2}}.
\]

(11)

For a real potential energy we obtain

\[
f'' < 2f'^2,
\]

(12)

however a stronger condition arises from the requirement of having a real tachyonic field

\[
f'' < 0.
\]

(13)

For a super-exponential warp factor of the form $e^{-2f} \sim e^{-2c_1e^{2|\sigma|}}$ where the warp function is given by

\[
f \sim c_1e^{c_2|\sigma|},
\]

(14)

both the finiteness of the effective 4d Planck mass and gravity localization conditions, which can be expressed by the following finite integral [44]

\[
\int e^{-2f(\sigma)} d\sigma < \infty,
\]

(15)

require $c_1, c_2 > 0$. Thus $f$ should be a positive definite function such that in the bulk

\[
f'^2 \sim c_2^2 f^2, \quad f'' \sim c_2^2 f > 0.
\]

(16)

We see that $f''$ turns out to be positive, contrary to Eq. (13) which is the condition for having a real tachyonic field. Thus a super-exponential warp factor is not allowed in the tachyonic braneworld model described by Eq. (1) when one wishes to recover 4d gravity on a single brane, our world, unless we have a complex tachyonic scalar field.

**On the hierarchy solution within the two-brane model.** When looking at the hierarchy problem one should keep in mind the compact nature of the region of variability of $\sigma$. In this case the localization of gravity can be achieved for any sign of the constants $c_1$ and $c_2$ in the warp function $f$ since the condition (15) is always fulfilled. However, in order to obtain a real $T$ field, from (13), (14) and (16) it follows that the constant $c_1$ of the warp function must be negative (see Table 1). This is true even for the two-brane model in which the gauge hierarchy problem is solved.

We then see that we cannot simultaneously solve both the localization of gravity on a single brane and the hierarchy problem in the two-brane picture with the same warp factor, contrary to the Randall-Sundrum case.
Table 1 For the braneworld given by Eqs. (1) and (5) with a super-exponential warp factor (for both one- and two-brane models) we show the nature of the resulting tachyonic field $T$ in the bulk (whether real $R$ or complex $C$) depending on the chosen constant $c_1$ appearing in Eq. (14)

| $c_1$ | $c_2$ | $f''$ | $T$ |
|------|------|------|------|
| $>0$  | $>0$  | $>0$  | $C$  |
| $>0$  | $<0$  | $>0$  | $C$  |
| $<0$  | $>0$  | $<0$  | $R$  |
| $<0$  | $<0$  | $<0$  | $R$  |

While localization of gravity on a single brane requires $c_1$ and $c_2$ positive, a real tachyonic field requires $c_1$ negative. Thus we cannot simultaneously solve both problems with the same warp factor, contrary to the Randall-Sundrum case.

Although the hierarchy and the gravity localization problems are in principle independent, we believe it is desirable to deal with both of them within the same family of nonfactorizable metrics and thus the same warp factor. This has been attempted in [36,37] by using a super-exponential warp factor. While the gauge hierarchy problem has been addressed successfully with a further reduction of the hierarchy when compared to the Randall-Sundrum two-brane model result, the gravity localization problem cannot be dealt with in the single brane picture because the integral in Eq. (15) becomes divergent for a real tachyonic field.

This situation could be avoided if instead of working with a tachyonic field in the bulk we use a canonical scalar field minimally coupled to 5d gravity as argued in Sects. 4–6 below.

3 Extending the simplest model

We generalise and modify our starting action in several ways, initially by introducing a 5d cosmological constant then also by adding a thin brane, and finally by including a spatially flat cosmological background induced on the brane. The action with a 5d cosmological constant is given as follows

$$S = \int d^5 x \sqrt{-g} \left( \frac{1}{2\kappa_5^2} (R - \Lambda_5) - V(T)\sqrt{1 + g^{AB}\partial_A T \partial_B T} \right).$$  \hspace{1cm} (17)

Einstein’s equations (2) are now given by

$$f'' = \kappa_5^2 \frac{V(T)T'^2}{3\sqrt{1 + T^2}},$$  \hspace{1cm} (18)

$$f' = -\kappa_5^2 \frac{V(T)}{6\sqrt{1 + T^2}} - \frac{\Lambda_5}{6}.$$  \hspace{1cm} (19)
When $\sigma$ is large, the tachyon scalar field has the following behavior

$$T'^2 = \frac{-f''}{2\left(f'^2 + \frac{\Lambda_5}{6}\right)} \rightarrow \frac{-f''}{2f'^2} \rightarrow 0^-.$$  

(20)

where the $\rightarrow$ symbol denotes large $\sigma$. For an exponential function of the form given by Eq. (14) the $f'^2$ term dominates and according to the last equation of (16) $T$ again becomes complex when requiring localization of 4d gravity on a single brane.

We use now for the background metric the ansatz of a warped 5d line element with an induced 3-brane with spatially flat cosmological background that reads

$$ds^2 = e^{-2f(\sigma)} \left[-dt^2 + a^2(t)\eta_{ij}dx^i dx^j\right] + d\sigma^2,$$  

(21)

while the Einstein equations (2) can be rewritten in a simple way

$$f'' = \kappa_5^2 \frac{V(T)T'^2}{3\sqrt{1 + T'^2}} + e^{2f} \frac{\ddot{a}}{a},$$  

(22)

$$f'^2 = -\kappa_5^2 \frac{V(T)}{6\sqrt{1 + T'^2}} - \frac{\Lambda_5}{6} + \frac{e^{2f}}{2} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right).$$  

(23)

Thus, we end up with the case of a single brane configuration in a 4d spatially flat cosmological background given by Eqs. (7), (22) and (23). Therefore, from these equations we conclude that we must have a de Sitter 4d cosmological background defined by the scale factor

$$a(t) = e^{Ht},$$  

(24)

where $H$ is the Hubble parameter and an overall constant has been absorbed into a coordinate redefinition of the 3d spatial variables [44]. This fact makes clear the role of the action of the tachyonic scalar field as a non-trivial 5d configuration which leads to a set up in which the $dS_4$ geometry is embedded into $AdS_5$ if $\Lambda_5 < 0$ [7]. It is easy to invert these equations in terms of $T'^2$ and $V(T)$, the result for $T'^2$ is

$$T'^2 = \frac{-f'' + H^2 e^{2f}}{2\left(f'^2 + \frac{\Lambda_5}{6} - H^2 e^{2f}\right)} \rightarrow \frac{H^2 e^{2f}}{-2H^2 e^{2f}} = \frac{1}{2},$$  

(25)

where we have taken into account relations (16). Again, $T$ is a complex field, this is easier to see for large $\sigma$ as indicated by the $\rightarrow$ symbol above. Since this asymptotic behaviour also holds for an arbitrary super-exponential warp factor $e^{2f}$, in principle this result remains valid for tachyonic thick braneworld configurations.

Finally, we shall also work in conformal metric coordinates

$$ds^2 = e^{-2f(w)} \left[-dt^2 + a^2(t)\eta_{ij}dx^i dx^j + dw^2\right],$$  

(26)
obtained from (5) with the aid of the transformation

$$d\sigma = e^{-f(w)}\, dw.$$  

(27)

Following similar steps with the aid of (16), as before we get for the tachyonic field

$$T_2' = \frac{-f'' - f'^2 + \frac{\kappa_5^2}{4} \sum_i \tau_i \delta(w - w_i) e^{-2f} \left( f'^2 + \frac{\Lambda_5}{6} e^{-2f} - H^2 \right)}{2 \left( f'^2 + \frac{\Lambda_5}{6} e^{-2f} - H^2 \right) e^{2f}} \rightarrow -\frac{f'^2}{2f'^2 e^{2f}} = -\frac{1}{2e^{2f}} \rightarrow 0^-,$$

(28)

where now primes stand for derivatives with respect to $w$. We have explicitly written the brane term, which is suppressed for large $w$. In the large $w$ (large-$f$) regime one can easily see the complex nature of the resultant field for any choice of the constants $c_1$ and $c_2$.

To address the hierarchy problem, where the variable $\sigma$ takes values in a finite range only, one should carefully construct tables similar to Table 1 for each case under consideration and see what combinations of constants $c_1$ and $c_2$ give a real tachyonic field in the bulk.

### 4 Canonical scalar field coupled to 5d gravity

As we have seen the tachyon field becomes complex for large $\sigma$ in the models considered above when requiring the localization of 4d gravity on the brane. This problem arises because the variable $\sigma$ (or $w$) is of infinite range when one considers the localization of 4d gravity on the single 3-brane of the setup. As a way out of this problem we can consider instead the action for a canonical scalar field $\phi$ minimally coupled to gravity plus a single 3-brane

$$S = \int d^5 x \sqrt{-g} \left[ \frac{1}{2\kappa_5^2} \left( R - \Lambda_5 \right) - \frac{1}{2} g^{MN} \nabla_M \phi \nabla_N \phi - V(\phi) \right] + S_B,$$

(29)

where $V(\phi)$ is the self-interaction potential of the bulk scalar,

$$S_B = -\int d^4 x d\sigma \sqrt{-g_1} V_1(\phi) \delta(\sigma),$$

(30)

with $V_1$ the brane potential of the 3-brane located at the origin and endowed with an induced metric $g_1$, finally, we have also introduced a 5d cosmological constant. The Einstein equations with a cosmological constant in five dimensions are given by

$$G_{MN} = -\Lambda_5 g_{MN} + \kappa_5^2 T_{MN}^{\text{bulk}}.$$

(31)

\[\text{1 Even when this could seem to be a simple coordinate play, the field equations that $T$ must obey are different for a given function $f$.}\]
By using the metric Eq. (5) we obtain the components of the Einstein tensor as in Eq. (7). The energy-momentum tensor components read

\[ T_{MN}^{\text{bulk}} = \nabla_M \phi \nabla_N \phi - g_{MN} \left( -\frac{1}{2} g^{AB} \nabla_A \phi \nabla_B \phi - V(\phi) \right). \]  

(32)

Following the discussion after Eq. (7) the field \( \phi \) becomes a function of the fifth coordinate only and Einstein’s equations (31) in the bulk are therefore written as

\[ 3 f'' - 6 f'^2 = \kappa_5^2 \left( \frac{1}{2} \phi'^2 + V \right) + \Lambda_5, \]  

(33)

\[ 6 f'^2 = \kappa_5^2 \left( \frac{1}{2} \phi'^2 - V \right) - \Lambda_5, \]  

(34)

under the metric ansatz (5). The equation for the field \( \phi \) is

\[ \phi'' - 4 f' \phi' = \frac{\partial V}{\partial \phi}. \]  

(35)

On the other hand the junction conditions on the brane imply that the brane potential obeys \( V_1|_{\sigma=0} = 6c_1 c_2 / \kappa_5^2 \), implying that the brane tension is positive, while the jump of the first derivative of the scalar field at \( \sigma = 0 \), denoted by \( \left[ \phi' \right] \), reads \( \left[ \phi' \right] = \frac{1}{2} \frac{\partial V_1}{\partial \sigma} |_{\sigma=0} \).

After some algebra we get

\[ \phi'^2 = \frac{3}{\kappa_5^2} f'', \]  

(36)

\[ V(\phi) = \frac{3}{2 \kappa_5^2} \left( f'' - 4 f'^2 - \frac{2}{3} \Lambda_5 \right). \]  

(37)

Thus we choose both \( c_1 \) and \( c_2 \) positive to satisfy the requirement of localization of 4d gravity given by Eq. (15) with a real scalar field \( \phi \). In this case we can consistently address both the hierarchy and the gravity localization problems with the same warp factor in the metric.

For the warp function given by Eq. (14) we get

\[ \phi'^2 = \frac{3 c_2^2}{\kappa_5^2} f, \]  

(38)

\[ V(\phi) = \frac{3 c_2^2}{2 \kappa_5^2} \left( f - 4 f^2 - \frac{2}{3} \frac{\Lambda_5}{c_2^2} \right). \]  

(39)

The solution to Eq. (38) is

\[ \phi = \pm 2 \sqrt{\frac{3 c_1}{\kappa_5^2} c_2^2} |\sigma| = \pm 2 \sqrt{\frac{3}{\kappa_5^2}} f^{1/2}, \]  

(40)
from where it follows that

\[ V(\phi) = -\frac{\Lambda_5}{\kappa_5^2} + \frac{c_2^2}{8}\phi^2 - \frac{c_2^2\kappa_5^2}{24}\phi^4. \] (41)

Thus, the warp factor (14), the canonical scalar field (40) and the self-interaction potential (41) define the complete solution for our scalar–tensor braneworld model.

Note that this potential is unbounded from below, which is common and free of pathologies when studying domain walls in AdS$_5$ supergravity [9,51–53]. This fact was non-perturbatively established in [51,52] by determining the conditions under which the self-interaction potential $V$ of a single scalar field guarantees a stable AdS vacuum, regardless of supersymmetry. It turns out that in $D$ dimensions $V$ must be of the form [52]

\[ V(\phi) = 2(D - 2) \left[ (D - 2) \left( \frac{dW}{d\phi} \right)^2 - (D - 1)W^2 \right], \] (42)

where $W(\phi)$ is the so-called superpotential, an arbitrary function of $\phi$ with at least one critical point.

Our 5D self-interaction potential (41) can be recast into the form of (42) with the aid of the superpotential

\[ W(\phi) = A + B\phi^2, \] (43)

and the following choice of the constants

\[ A = \pm \sqrt{\frac{\Lambda_5}{24\kappa_5^2}} \quad \text{and} \quad B = \pm \frac{c_2^2\kappa_5^2}{24} \]

under the restriction $\Lambda_5 = \frac{3c_2^2(1-\kappa_5^2)^2}{2}$. The superpotential (43) clearly possesses a critical point (a maximum or a minimum depending on the sign of $B$) and then ensures the stability of the corresponding vacuum. This result is in agreement with the study of scalar perturbations that will be performed in Sect. 6 below.

In the next section we shall study the stability properties of the above constructed braneworld model which allows for the localization of 4d gravity in contrast with previously studied tachyonic braneworlds with super-exponential warping in the metric [36,37].

5 Tensor perturbations and localization of 4d gravity

We shall analyze the localization of 4d gravity in our braneworld field configuration following the work presented in [13,14], where a generalization of the Bardeen formalism for metric fluctuations of higher-dimensional backgrounds with non-compact extra dimensions was accomplished (see also [19] and [20] for similar approaches);
moreover, the master equations for the coupled system of metric and scalar perturbations were also derived in a gauge-invariant form.

In order to analyze the dynamics of the metric fluctuations we start with the following ansatz

$$ds^2 = e^{-2f} \left[ (\eta_{\mu\nu} + h_{\mu\nu}) \, dx^\mu \, dx^\nu + dw^2 \right], \quad (44)$$

where $h_{\mu\nu}(x^\mu, w)$ are gauge-invariant metric perturbations when considering the transverse and traceless conditions $\partial^\mu h_{\mu\nu} = h_{\mu\mu} = 0 \ [13, 14]$. We further make use of the following separation of variables $h_{\mu\nu} = C_{\mu\nu} e^{3f/2} e^{ipx} \rho(w)$, where $C_{\mu\nu}$ are arbitrary constants, and the relevant dynamical equation adopts the form of a Schrödinger equation along the fifth dimension

$$-\rho'' + V_{QM}(w) \rho = m^2 \rho, \quad (45)$$

where $m$ is the mass that a 4d observer sees \[12\], whereas the effective quantum mechanical potential reads

$$V_{QM}(w) = \frac{s''}{s} = \mathcal{J}^2 - \mathcal{J}', \quad \text{with} \quad s = e^{-3f/2}. \quad (46)$$

In this equation we have introduced the quantity $\mathcal{J} = -\frac{s'}{s}$, called superpotential within the framework of supersymmetric quantum mechanics. Moreover, this superpotential allows us to express the Schrödinger equation for $\rho$ (45) as follows

$$Q^\dagger Q \rho = m^2 \rho, \quad (47)$$

where the operators $Q^\dagger$ and $Q$ are defined according to

$$Q^\dagger = \left( -\frac{d}{dw} + \mathcal{J} \right), \quad Q = \left( \frac{d}{dw} + \mathcal{J} \right). \quad (48)$$

The fact that the Schrödinger equation (45) can be expressed in the form (47) guarantees that the spectrum of metric fluctuations is positive definite and there are no tachyonic modes with $m^2 < 0$, guaranteeing the stability of the system under the tensorial sector of perturbations.

For the massless zero mode the Schrödinger equation (45) yields $\rho = s = e^{-3f/2}$ with the following normalization condition

$$\int e^{-3f(w)} \, dw < \infty, \quad (49)$$

which transforms into (15) when performing the coordinate transformation (27). Since this integral converges for positive $c_1$ and $c_2$, this guarantees the existence of a normalizable massless zero mode which is interpreted as a 4d graviton localized on the brane.
6 Stability of the brane under scalar perturbations

In this section we shall perform the stability analysis of our braneworld configuration under linear perturbations of the scalar sector following again the line of work of [13,14], where the master equations for the coupled system of scalar perturbations were derived.

Thus we shall consider spin-0 linear perturbations of the scalar-gravity coupled system that generates our braneworld and study their dynamics as well as the structure of the corresponding mass spectra. Although in the previous section we used a gauge-invariant treatment, in the scalar stability analysis it is more convenient to use a specific gauge.

Let us start by considering the perturbed metric for the scalar sector of fluctuations written in conformal coordinates in the so-called longitudinal gauge

\[ ds^2 = e^{2f(z)} \left[ (1 + 2 \psi) \eta_{\mu \nu} dx^\mu dx^\nu + (1 + 2 \xi) d\bar{w}^2 \right], \tag{50} \]

together with the fluctuations of the scalar field \( \phi = \phi + \chi \), where \( \psi, \xi \) and \( \chi \) are small perturbations.

In Giovannini [13,14] it was shown that the corresponding system of coupled differential equations can be reduced to a couple of master equations which can be suitable expressed in a Schrödinger-like form. Moreover, it was determined that there is only one independent degree of freedom, i.e. just one scalar physical mode, after taking into account the following relations \( \xi = 2 \psi \) and \( \chi = -\frac{6}{\phi'} (\psi' - 2 f' \psi) \).

Therefore the equation for the rescaled scalar perturbation \( \psi \) can be recast into the Schrödinger form

\[ \Psi'' - g \left(g^{-1}\right)'' \Psi = m^2_\psi \Psi, \tag{51} \]
after a convenient separation of 4d and 5d variables which define the 4d mass \( m_\psi \) and by setting \( \Psi = e^{-3f/2} \phi' \psi \) (a similar equation holds for the rescaled scalar fluctuation \( \xi \)). In this equation it was introduced a very useful function

\[ g = \frac{e^{-3f/2} \phi'}{f'}, \tag{52} \]

that parameterizes the analogue quantum mechanical potential

\[ U_\psi = g \left(g^{-1}\right)'' \tag{53} \]

On the other hand, the corresponding equation for the scalar perturbation \( \chi \) also transforms into a Schrödinger-like equation

\[ X'' - g^{-1} g'' X = m^2_\chi X, \tag{54} \]
after the definition of the new fluctuation \( X = e^{-3f/2} - g \psi \) and the introduction of the 4d mass of this scalar perturbation \( m_X \), where now the analogue quantum mechanical potential reads

\[
U_X = g^{-1} g''.
\]

These potentials can respectively be written in the following form

\[
U_\Psi = J_\psi^2 - J_\psi', \quad U_X = J_X^2 + J'_X.
\]

with the aid of the superpotentials \( J_i = g' g \), with \( i = \Psi, X \). As in the case of metric fluctuations, these quantities allow us to write the Schrödinger equations for \( \Psi \) and \( X \) as

\[
Q^{\dagger} Q \Psi = m_\psi^2 \Psi, \quad Q^{\dagger} Q X = m_X^2 X,
\]

where the operators \( Q^{\dagger} \) and \( Q \) are defined according to (48).

Again, since the Schrödinger equation for both \( \Psi \) and \( X \) can be written in this form, this means that there are no tachyonic modes with \( m_i^2 < 0 \) in the respective spectra of scalar fluctuations and thus the scalar–tensor braneworld configuration that defines our braneworld model is stable under linear scalar perturbations.

6.1 Delocalization of the scalar modes

In order to study the (de)localization properties of the scalar perturbation sector it is enough to understand the behaviour of the analogue quantum mechanical potentials (53) and (55) of the corresponding Schrödinger equations (51) and (54). These potentials are written in terms of the conformal coordinate \( w \) while our solution is parameterized in the language of the coordinate \( \sigma \). However, the dependence of these potentials can be plotted parametrically once we have obtained

\[
w = \left[ \text{Ei} \left( -c_1 e^{c_2 |\sigma|} \right) - \text{Ei} \left( -c_1 \right) \right] \text{sgn}(\sigma)/c_2 = \left[ \text{Ei} ( -f ) - \text{Ei} ( -c_1 ) \right] \text{sgn}(\sigma)/c_2,
\]

where \( \text{Ei}(x) \) stands for the exponential integral special function, with the aid of the coordinate transformation (27).

It turns out that for our superexponential warp factor (14) and the corresponding background scalar field (40), both \( U_i(w) \) are positive potential barriers distributed along the fifth dimension, a fact that implies that no massless nor massive scalar modes are localized on the brane within our model.

7 Discussion and conclusions

The finiteness of the 4d effective gravitational coupling constant and the gravity localization conditions, which are encoded in the requirement that \( \int e^{-2f(\sigma)} d\sigma < \infty \), imply that both \( c_1 \) and \( c_2 \) should be positive for a tachyonic braneworld model with
super-exponential warp factors of the form $e^{-2f} \sim e^{-2c_1e^{2|\sigma|}}$. As a consequence of this the examples considered within this context show that the tachyonic field $T$ turns out to be complex in contradiction with the real character of the starting tachyonic action. The inverse statement is also valid: if the tachyonic scalar field is required to be real then the 4d gravity is not localized on the brane and the effective 4d Planck mass turns out to be infinitely large. We have analysed this situation for thin as well as thick braneworlds generated by gravity and a tachyonic field $T$, with an additional bulk cosmological constant, and even for a spatially flat 4d cosmological metric induced on the brane and shown that in all cases the tachyon field is inconsistently complex if 4d gravity is required to be localized on a single 3-brane.

On the other hand there has recently been some interest in using super-exponential warp factors to address the hierarchy and localization problems in tachyonic braneworld type models. As originally shown by Randall and Sundrum and by Gogberashvili the hierarchy problem of masses in the standard model of particles can be considerably ameliorated by introducing a factor of the form $e^{-2f(\sigma)}$ corresponding to a warped 5d line element with an induced 3-brane in such a way that higher scales are exponentially suppressed. Thus scales of the order of the Planck mass end up with values a few tens the electroweak scale. This is achieved by using warp functions of the type $f \sim k|\sigma|$. A further reduction of the hierarchy was accomplished in [36,37] with the aid of a super-exponential warp factor with $f \sim c_1e^{c_2|\sigma|}$ which leads to a model with physical parameters of the same order. In general care should be taken when choosing the constants $c_1$ and $c_2$ because the wrong choice can result in an imaginary tachyonic field $T$ in some braneworld models of this type. We also find that it is not possible to solve the hierarchy problem and the localization of gravity with the same warp factor within these models. This is contrary to the Randall-Sundrum model case where the same warp factor is used in both problems.

The gauge hierarchy and the 4d gravity localization problems are independent but since both of them occur in the same universe we consider desirable to solve both of them with the same nonfactorizable metric and therefore the same warp factor. With this motivation we proposed considering a bulk canonical scalar field instead of a tachyonic one in order to generate a braneworld model and show that the above mentioned problems can be solved with the same super-exponential warp factor. Finally, this latter scalar braneworld configuration is shown to be stable under scalar linear fluctuations. Moreover, the scalar modes, both massless and massive, were shown not to be localized on the brane.

It would be interesting to further study the dynamics of the massive KK tensorial fluctuations in the single brane picture and see how do they correct the Newton’s law, for instance. However, this task is not trivial at all from the mathematical point of view and it seems that numerical tools should be implemented to afford it. Notwithstanding, numerical studies render qualitatively results and do not allow one to obtain precise predictions about these corrections within the braneworld paradigm (see [54] and references therein for an example on this issue).

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