Interaction quenches in the two-dimensional fermionic Hubbard model

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The generic non-equilibrium evolution of a strongly interacting fermionic system is studied. For strong quenches, a collective collapse-and-revival phenomenon is found extending over the whole Brillouin zone. A qualitatively distinct behavior occurs for weak quenches where only weak wiggling occurs. Surprisingly, no evidence for prethermalization is found in the weak coupling regime. In both regimes, indications for relaxation beyond oscillatory or power law behavior are found and used to estimate relaxation rates without resorting to a probabilistic ansatz. The relaxation appears to be fastest for intermediate values of the quenched interaction.

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I. INTRODUCTION

Recently refined experimental techniques based on ultracold gases in optical lattices\textsuperscript{14,25} and femtosecond spectroscopy\textsuperscript{26} allow for studies of systems out of equilibrium. Such studies require a very good decoupling from the environment to realize long observation times during which the system is out of equilibrium. One way to push the system far out of equilibrium is to switch intrinsic system parameters abruptly. Such a scenario is called a quench. In interaction quenches the system is prepared initially in eigenstates of a non-interacting Hamiltonian. At a specific time the interaction is suddenly turned on and the state of the system is no longer an eigenstate of the (quenched) Hamiltonian.

Typically, the quenched systems are in highly excited states with respect to the quenched Hamiltonian. Thus their dynamics is governed by processes on all energy scales including high energies. Properties may occur which are totally different from the equilibrium ones. The necessity to include all energy scales makes theoretical calculations, numerical or analytical ones, notoriously difficult. So far, the majority of theoretical investigations were focussed on one-dimensional (1D) systems, on infinite-dimensional (\infty D) systems, and on small finite systems because for these cases powerful tools are available. For 1D systems, the tool box is best: Quantum field theoretical descriptions provide analytical approaches, see, e.g., Refs. \textsuperscript{10,11}. The best understood models remain those which correspond to non-interacting fermionic or bosonic systems\textsuperscript{10,12} or models which are effectively close to non-interacting ones\textsuperscript{13}. Time-dependent density-matrix renormalization is a powerful numerical tool which enables to study non-equilibrium phenomena in 1D systems\textsuperscript{14–16}. The other dimensionality allowing for well-controlled studies is \infty D where dynamical mean-field theory becomes exact\textsuperscript{17,18} and Gutzwiller approaches are well justified\textsuperscript{19}. Exact diagonalization is completely flexible concerning dimensionality, but it is restricted to small systems\textsuperscript{20,21}.

So far, the question to which extent strongly conserved quantities restrict or even prevent relaxation was in the center of interest\textsuperscript{6,12,13,22–24}. Thus, integrable systems and systems close to integrability were studied, which drew the interest to 1D systems. Studies of two-dimensional (2D) models out of equilibrium are still rare. Goth and Assaad studied the sudden turning off of the interaction in a half-filled 2D Hubbard model with 20×20 sites by continuous time quantum Monte Carlo\textsuperscript{25}. The large energy put into the system and the simple dynamics induced by the non-interacting Hamiltonian after the quench lead to a well-understood evolution in agreement with the findings of perturbative approaches. Other 2D studies address the influence of a strong electric field on the dynamics of a single charge carrier in a Mott insulator\textsuperscript{26} and on a bound pair of two carriers\textsuperscript{27}.

In the present work, we study the interaction quench in the 2D Hubbard model far from any integrability. In contrast to the work by Goth and Assaad, the interaction is switched on abruptly. Our goal is to assess the time scale on which relaxation takes place in a generic model between one and infinite dimension. The sensitive quantity which we investigate is the momentum distribution and its jump at the Fermi surface \textsuperscript{\textit{K}_F} in particular.

The article is set up as follows. After this introduction we present the model studied and the method used in Sect. \textbf{II}. In the subsequent Sect. \textbf{III} the results for the momentum distribution and its jump at the Fermi surface will be presented. The scenario for prethermalization will be an important issue as well as an estimate of relaxation rates. In Sect. \textbf{IV} the article is concluded.

II. MODEL AND METHOD

The Hamiltonian under study is

\[
\hat{H} = -J \sum_{\langle \vec{r}, \vec{s} \rangle, \sigma} \hat{c}_{\vec{r}, \sigma}^\dagger \hat{c}_{\vec{s}, \sigma} + h.c.) + U(t) \sum_{\vec{r}} \hat{n}_{\vec{r}, \uparrow} \hat{n}_{\vec{r}, \downarrow} \quad (1)
\]

with the hopping parameter \( J \); \( \vec{r} \) and \( \vec{s} \) denote nearest neighbors on the square lattice. The fermionic annihila-
tion (creation) operator at site $\vec{r}$ with spin $\sigma$ is denoted by $c_{\vec{r} \sigma}^{\dagger}$ and $c_{\vec{r} \sigma}$ counts the fermions at site $\vec{r}$. Profiting from translational invariance we directly address the infinite model in the thermodynamic limit. The colons indicate normal ordering with respect to the Fermi sea, which is the ground state of the non-interacting model and the initial state of the quench. The interaction $U(t) = U \Theta(t) \geq 0$ is suddenly turned on at $t = 0$ so that the time evolution is governed by the interacting Hamiltonian.

We use the band width $W = 8J$ as a natural energy scale and $\hbar$ is set to unity so that time is measured in the inverse band width $1/W$. The time evolution of the jump $\Delta n_k(t)$ is used as sensitive probe for the dynamics after the quench. Its initial value is unity. The momentum distribution is calculated by an expansion of the Heisenberg equations of motion (EoM) for an operator $\hat{A}$

$$\partial_t \hat{A}(\vec{r}, t) = i [\hat{H}, \hat{A}(\vec{r}, t)]$$

to the highest order possible. We consider $\hat{A}(t = 0) = c_{k, \sigma}^{\dagger}$. The EoM are iterated by recursive commutation with $H$ yielding more and more operators which we represent in real space. As each commutation implies one additional order in time $t$ the results obtained after $n$ commutations are exact at least up to $t^n$.

We emphasize, however, that we are not computing a plain series in powers of time. In the algebraic part, we derive a set of differential equations which allow for the determination of the series up to $t^n$. This means that the solution of the approximate differential equations has the same expansion in powers of $t$ as the exact solution. But here we do not present results for the series, but for the full solution of the approximate differential equation which turn out to be more stable and reliable up to longer times than the plain series. Thus the order $n$ of the calculation is a control parameter of the approximation which becomes exact for $n \to \infty$, but it does not refer to the maximum order of a truncated series.

Due to the exponentially rising number of terms to be tracked for increasing $n$ one has to stop at values of $n$ of the order 10. The results are well-controlled for about $t \lesssim n/W$. For the 2D model, up to $n = 9$ commutations are performed and the data is shown up to times for which the results are reliable. This can be inferred from the comparison of the curves for various numbers $n$, which display convergence upon increasing $n$, see also Refs. 31,32.

### III. RESULTS

The approach sketched above is applied to the fermionic creation operator. In this way, the time dependence of expectation values such as $\langle c_{\vec{r} \sigma} c_{\vec{r} \sigma}^{\dagger} \rangle(t)$ becomes accessible. Fourier transformation of these expressions yields the momentum distribution.

### A. Momentum Distribution

We show a complete view on the momentum distribution in the Brillouin zone in Fig. 1 as function of time. Since many points in momentum space have to be evaluated we have to restrict ourselves to $n = 6$ commutations. Thus the data in Fig. 1 is not of the highest accuracy for the longer time, but it renders an excellent overview. All other figures present data for $n = 9$ commutations.

For the relatively large values $U = 2W$, the momentum distribution in Fig. 1 shows oscillations over the whole Brillouin zone. At the instants at which the jump vanishes, see panel $t = 1.7/W$, the distribution is featureless and almost constant indicating a state which is essentially local in real space. But afterwards, the jump re-occurs and the momentum distribution resembles the initial one qualitatively. Thus the total behavior follows a collapse-and-revival scenario.

The results in Fig. 1 suggest that it would be very fascinating to observe such a behavior in fermionic systems experimentally. Note that collapse-and-revival was observed experimentally after the interaction quench in a bosonic system. But there is an essential qualitative difference between the fermionic and the bosonic collapse-and-revival. As seen in Fig. 1 the fermionic one is characterized by the disappearance and re-appearance of the Fermi surface, i.e., a one-dimensional singularity in the two-dimensional Brillouin zone. In contrast, the bosonic collapse-and-revival is related to the disappearance and re-appearance of a zero-dimensional singularity, namely of a $\delta$-function in the bosonic momentum distribution at the center (Γ point) of the Brillouin zone.

![FIG. 1: (Color online) Evolution of the momentum distribution of the half-filled Hubbard model quenched to $U = 2W$. Only one quadrant of the Brillouin zone is shown due to point group symmetry. The data results from $n = 6$ commutations.](image)
i.e., at \( k = (\pm \pi, 0) \) and \( (0, \pm \pi) \), and those at the middle of the edges, i.e., at \( k = (\pm \pi/2, \pm \pi/2) \), appears up to the time-scales investigated. This is illustrated in Fig. 2 for the jump \( \Delta n(t) \) at the given momenta on the Fermi surface, see legend. Note that the difference between \( U \) remains small even for \( U \). In the remainder, we will only show results for \( k = (\pi, 0) \) for simplicity if not stated otherwise.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{(Color online) Jump \( \Delta n(t) \) calculated at two positions on the Fermi surface for various interaction strengths \( U \). The curves for \( U = 1.0W \) change their style for larger times; then we do not consider them fully reliable anymore.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{(Color online) Comparison of the time dependence of the jump \( \Delta n(t) \) for various \( U \) in half-filled Hubbard models: Solid lines show the 2D, the dashed lines of the corresponding gray scale/color show the 1D data.}
\end{figure}

\textbf{B. Comparison to the behavior in 1D}

In Fig. 3 we compare the quench dynamics in 1D and in 2D at half-filling for various values of \( U \). Our findings provide evidence that in 2D the same dynamical transition exists between quenches to weak and to strong interactions that was observed previously in \( \infty \text{D} \)\textsuperscript{21}, by Gutzwiller approach\textsuperscript{22}, and in 1D\textsuperscript{31}. For quenches to stronger interactions \( (U \gtrsim 0.7W) \) one observes dominant oscillations which decay slowly. At half-filling, these oscillations display zeros in the jump \( \Delta n \) as in the previous cases\textsuperscript{21,22,31}. Away from half-filling, the minima still exist, but they are no longer at \( \Delta n = 0 \) (not shown).

For quenches to weak and moderate interactions we observe a decay of \( \Delta n \) with only hardly visible oscillations, cf. the curves for \( U < 0.7W \). These oscillations can be attributed to the finite band width \( W \), i.e., the frequency of oscillations is the band width \( W \). This explanation is supported by the fact that the oscillations are stronger in 1D than in 2D because the Van Hove singularities in 1D (inverse square roots, \( \propto \Delta \omega^{-1/2} \))\textsuperscript{18,31} are much more pronounced than in 2D (jumps, \( \propto \Delta \omega^0 \)). This argument is in line with the observations that no oscillations are observed in the infinite dimensional calculations based on the Bethe lattice with infinite branching ratio displaying even less pronounced singularities (square roots, \( \propto \Delta \omega^{1/2} \))\textsuperscript{21}.

Another remarkable contrast to the 1D curves consists in the much faster decay of the jump in 2D. This feature is striking in the curves in Fig. 3 which all start with the same curvature \( -U^2/2 \) determined by \( U \) alone. We interpret this important qualitative difference by the fact that the decay of the jump in 1D is governed by slowly decreasing power laws\textsuperscript{15,18,32}. The 2D characteristics appears different: The 2D system allows for sufficiently effective scattering mechanisms so that one may expect to observe first signs of true relaxation governed by exponential decay \( \Delta n(t) \propto \exp(\alpha t) \) with relaxation rate \( \alpha > 0 \) for longer times. We will come back to this point below. For intermediate values of \( U \approx 0.7W \) we find a particularly fast decaying jump indicating efficient relaxation indeed, cf. Fig. 3.

\textbf{C. Strong Quenches and their decay}

We proceed to a quantitative analysis by fits. For strong quenches at half-filling we take the oscillations into account and we allow for relaxation to occur. For the oscillations we simply include a cosine term, see Eq. \textsuperscript{33}. The relaxation is trickier for the following reason. In the long time limit it is described by the factor \( \exp(-\alpha t) \) with decay rate \( \alpha > 0 \). But around \( t = 0 \) this behavior does not and cannot appear because the time dependence induced by Hamiltonians, whose local terms are bounded, is analytically smooth. Thus the fit function to describe relaxation must be smooth at \( t = 0 \) and then it must crossover to \( \exp(-\alpha t) \). The simplest function we could think of with the desired property is \( \exp(-\sqrt{a(t)^2 + b^2} + b) \). Obviously, the decay rate at large \( t \) is given by \( a \) while the behavior at small \( t \) is smooth. The crossover from \( t^2 \) behavior to \( |t| \) behavior occurs at
The resulting oscillation periods $T$ for the fit parameters $\omega, a, b$ are fitted and
the third one is determined by the analytic curvature $-U^2/2$ at $t = 0$. Exemplary fits are shown in Fig. 4.

The resulting oscillation periods $T = 2\pi/\omega$ and values for $a$ and $b$ are depicted in Fig. 4 to the right of the vertical dotted lines which mark the region between weak and strong quenches. This region cannot be resolved by our present approach; the weak quenches are considered below in Sect. III E.

Note that the quantitative description of our data with the fitting function (3) works very nicely. We are aware, however, that the parameters $a$ and $b$ resulting from these fits give only estimates for the relaxation rate and the crossover time, respectively. In case another dynamical time scale would govern the behavior at intermediate times it may be that the numbers for $a$ and $b$ are affected by this intermediate time scale and not by the relaxation at long times.

D. Existence of Prethermalization

The natural next issue are weak quenches and their decay. Indeed, we will address it in the following subsection. But before doing so it is worth to recall the leading perturbative result in order $U^2$ derived by Moeckel and Kehrein$^{33,34}$ which reads

$$\Delta n_{\text{stro}}(t) = \cos(\omega t)^2 \exp(-\sqrt{(at)^2 + b^2 + b}). \quad (3)$$

Two of the three fit parameters $\omega, a, b$ are fitted and the third one is determined by the analytic curvature $-U^2/2$ at $t = 0$. Exemplary fits are shown in Fig. 4. The resulting oscillation periods $T = 2\pi/\omega$ and values for $a$ and $b$ are depicted in Fig. 4 to the right of the vertical dotted lines which mark the region between weak and strong quenches. This region cannot be resolved by our present approach; the weak quenches are considered below in Sect. III E.

For the jump at the Fermi surface with

$$f_{kp}(t) = \frac{4}{N^2} \sum_{pp'q} \frac{\sin^2(\Delta ct/2)}{\Delta^2} \left( n_p n_{p'q} - n_{p'q} n_p \right), \quad (5)$$

where $N$ is the number of sites, $k_F$ a wave vector on the Fermi surface, $n_p = 1$ if $p$ is within the Fermi surface and zero otherwise, $n_p = 1 - n_p$, and $\Delta \varepsilon = \varepsilon_{p'} + \varepsilon_p - \varepsilon_{q}$.

In infinite dimensions, the multidimensional integral (5) is shown to yield a constant for $t \to \infty$ so that for weak enough interaction an almost constant plateau appears before relaxation sets in on much longer time scales$^{21,33,34}$. It is expected that this is the generic behavior in finite dimensions as well. But it is also known from bosonization$^{21}$ that the 1D case is special because of particularly strong scattering yielding a logarithmic divergence of $f_{kp}(t)$. So the question arises what happens in two dimensions?

To clarify this issue we evaluated (5) in 2D for the half-filled Hubbard model. This calculation is done in real space up to long times though the limit of infinite time cannot be addressed directly. This is left to future work. Fig. 5 displays the results for $f_{kp}(t)$ at momenta $(\pi, 0)$ and $(\pi/2, \pi/2)$. Unexpectedly, the data indicates a logarithmic divergence for $t \to \infty$ as is revealed by the fits. This suggests that no prethermalization plateaus arise because the perturbative correction diverges for $t \to \infty$ even for arbitrarily small quenched interaction $U$. Of course, this fact influences the quench dynamics decisively.

So far, we cannot prove what the reason is for the non-existence of prethermalization in the 2D Hubbard model at half-filling. But we attribute this non-existence to the perfectly flat stretches of the Fermi surface linking the four points $(\pm \pi, 0)$ and $(0, \pm \pi)$. In the vicinity...
of these flat regions only the perpendicular momentum transfer matters while the parallel one can be integrated over yielding just a certain prefactor. As a result the relevant, perpendicular scattering processes behave as if they were acting in 1D. If this hypothesis turns out to be true, any system doped away from half-filling should show prethermalization because the Fermi surface will be curved. But the time scales, on which the effect of doping becomes visible, are presumably very long for low doping. Thus the interaction values $U$ at which plateaus become discernible will be fairly low. Further work is called for to elucidate this issue.

E. Weak quenches and their decay

We have argued that due the divergence of $f_{kp}(t)$ for $t \to \infty$ the jump in Eq. (4) does not display a prethermalization plateau. A second corollary is that the strict perturbative result becomes unphysical, namely negative, see, e.g., dotted curve in Fig. 7. This happens even for a perturbative result becomes unphysical, namely negative, $26,33$ to reconcile this behavior with the physical fact $\Delta n$ increases only weakly for $a$ sufficiently small $U$ if $t$ is chosen sufficiently large. We have to reconcile this behavior with the physical fact $\Delta n_{kp} \geq 0$ because otherwise there is no way to estimate the relaxation. Based on the analogy to the 1D case $5,18,32$ we propose the hypothesis that the logarithmic divergence is the signature of a power law behavior if there were no relaxation. In other words, only the deviation from the power law behavior can be taken as sign of relaxation.

In order to use this hypothesis we pass from the logarithmically diverging Eq. (4) to the power law behavior

$$\Delta n_{kp, exp}(t) = \exp(-U^2 f_{kp}(t)) + O(U^4),$$

where we omit the corrections $O(U^4)$. This result only uses the leading perturbative result, but extrapolates it as a power law. Indeed, a comparison to a diagrammatic analysis based on dynamic cluster theory shows that weak quenches follow the prediction $\exp(-U^2 f_{kp}(t))$. For illustration, the dashed-dotted curve in Fig. 7 shows the result from Eq. (4) for the case $U = 0.5W$. Note that the solid curves display the full result which can be taken to be exact up to the times shown.

The perturbative result $\Delta n_{kp, exp}(t)$ in Eq. (4) serves as our reference in the following ansatz

$$\Delta n_{weak}(t) = \Delta n_{kp, exp}(t) \exp(-[(at)^4 + b^4]^{-1/4} + b). \quad (7)$$

The last factor is again chosen such that it is compatible with all known properties of $\Delta n(t)$. It starts smoothly, all quadratic dependence in $U$ is contained in the first factor $\Delta n_{kp, exp}(t)$ in all orders in $t$, and the quadratic behavior in $t$ is also fully described by $\Delta n_{kp, exp}(t)$ because the $U^2$ term is sufficient to describe the short-time behavior. This can be concluded from the EoM used here and it was previously concluded based on other techniques. Thus, the minimal relaxation factor $\exp(-[(at)^4 + b^4]^{-1/4} + b)$ is chosen such that it does not alter the exactly known $t^2$ term. Together with the fact that only even powers in time and in $U$ can occur leads to the use of the unusual exponent of $1/4$.

Based on Eq. (7) we fit the EoM data and determine $a$ and $b$ in this way. The results are shown in Fig. 5 for smaller values of $U$, i.e., on the left side of the dashed vertical lines. We are aware that the decay rate $a$ and the crossover time $b/a$ ensuing from this analysis are only estimates in view of the hypothesis necessary to analyze the data. The decay rate $a$ increases only weakly for increasing $U$; our data is consistent with $a \propto U^4$ as it is built-in into the fit function. The $U^2$ dependence is taken into account by the factor $\Delta n_{kp, exp}(t)$.

Our analysis is not unbiased, but relies on certain assumptions. We emphasize that this is also the case in many other approaches on relaxation which rely on a
probabilistic description which has relaxation built-in by
construction, see for instance\textsuperscript{30,31}.

Except for a fairly narrow window between \( U \approx 0.65W \)
and \( 0.7W \) the EoM data allows us to decide whether
a strong quench with oscillatory behavior (Eq. (3)) oc-
curs or whether a weak quench displaying only some
shoulders or wiggles occurs (Eq. (4)). The existence
of these two qualitative different regimes, separated by
a dynamic transition is obvious. This 2D result is in
line with previous observations in \( \infty D \text{21} \), in Gutzwiller
approximation\textsuperscript{22} and in 1D\textsuperscript{23}.

The relaxation as captured by \( a \), see Fig. 5 is by far
largest in the vicinity of the dynamic transition, i.e.,
around \( U = 0.7W \). In this region, the rate is of the order
of the band width. But away from this region, i.e., for
small or for large interaction the relaxation is very weak.
For low \( U \) this is expected as explained above since the
leading order \( U^2 \) does not lead to relaxation so that only
the next-leading \( U^4 \) processes induce relaxation.

For strong values of \( U \) it is remarkable that the relax-
ation becomes small again. We attribute this fascinating
behavior to the dominance of the local Rabi oscillations\textsuperscript{24}
with \( \omega \approx U \), see dashed line in Fig. 5. These oscillations
do not relax at all for \( W = 0 \) so that the conclusion
\( a \propto W \propto U^0 \) suggests itself. The fits shown in Fig. 5 are
consistent with this argument, but they are not particu-
larly stable.

IV. CONCLUSIONS AND OUTLOOK

Concluding, we studied interaction quenches in the 2D
Hubbard model as a generic finite dimensional model
between one and infinite dimension. The momentum
distribution is computed and for strong interactions
collapse-and-revival oscillations of the singular jump in
the momentum distribution at the Fermi surface is found.
Though qualitatively similar to what has been measured
in bosonic systems, the key difference is that the bosonic
collapse-and-revival occurs at a single point, the center,
in the Brillouin zone. For weak interactions, only very
weak oscillations occur so that two qualitatively distinct
regimes are found, separated by a dynamic transition.

Considering the Fermi jump in the momentum dis-
tribution as particularly sensitive probe we have found
that it decreases much faster in two dimensions than
in one dimension. This provides evidence for relaxation
without the bias of a probabilistic ansatz in terms of a
density matrix. In particular for intermediate interac-
tions \( U \approx 0.7W \) a significant relaxation rate of the or-
er of the band width was found. Based on plausible,
though not rigorous assumptions, on the functional form
of the relaxation we estimated the decay rates in the two-
dimensional Hubbard model at half-filling. The results
are summarized in Fig. 5.

By a nonequilibrium extension of dynamic cluster
theory Tsuji and co-workers have also obtained results
for quenches in the two-dimensional Hubbard model\textsuperscript{15}.

Since they are using iterated perturbation theory they
focus on weak quenches (\( U \approx 0.25W \)). One of their
central issues is the differing relaxation rate at different
points on the Fermi surface. A quantitative comparison
to our approach shows that their data is close to what
we obtain within the exponentiated perturbative result
in Eq. (6). But the differences for times up to \( t \approx 6/W \)
between their results and the reliable results of the equa-
tion of motion are larger than the differences between
the results for \((\pi, 0)\) and \((\pi/2, \pi/2)\). Thus we consider it
difficult to draw definite conclusions on this issue at the
present stage.

The approach as it is presented here makes contribu-
tions to nonequilibrium dynamics up to intermediate
times. Moreover, it can contribute to the theoretically
fascinating, but intricate, issue what happens at long
times by gauging other techniques. In particular, no assu-
ption has been made that the evolution of the system
can be described by a statistical mixture even though one
starts from a pure state.

Furthermore, we stress that the approach as it stands
has the potential to provide experimentally relevant
data. Often, experimental data is also restricted to short
and intermediate times due to various disturbing effects
whose detrimental influence grows in relative importance
with time.

The quenches studied in the present article started
from a non-interacting Fermi sea as an initial state which
can be treated according to Wick’s theorem. But it must
be emphasized that this property is not essential for the
approach used. The indispensable prerequisite is to know
the correlations of the initial state in order that the equa-
tion of motion technique can be put to use. Thus, many
different initial states can indeed be treated. Also mix-
tures, for instance the thermal density operator at a cer-
tain temperature \( T > 0 \), can be used to analyse the final
result of the equations of motion.

Beyond short and intermediate times, the employed
technique can be iterated over many short time intervals
to reach long times. The key idea is to assume that a
probabilistic description holds after each short time in-
terval so that one can re-initialize the EoM approach after
each time step. By comparison to the direct results by
EoM, one can investigate to which extent the assumption
that the system is describable as a mixture holds. If satis-
fying agreement is found one can then use the approach
of iterated time steps to reach much longer times.

Even the properties of stationary states can be tackled,
that is, the steady-state that describes the system after
infinite long time. This steady-state can be addressed
by equations of motion if they are combined with the
concept of stationary phases, see for instance Ref. 6. In
this way, the way is paved for the further methodolog-
ical developments which help us to better understand
nonequilibrium physics.
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