Maximal Acceleration Corrections to the Lamb Shift of Muonic Hydrogen

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Abstract
The maximal acceleration corrections to the Lamb shift of muonic hydrogen are calculated by using the relativistic Dirac wave functions. The correction for the 2S − 2P transition is ∼ 0.38 meV and is higher than the accuracy of present QED calculations and of the expected accuracy of experiments in preparation.

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1 Introduction

This paper presents the calculation of maximal acceleration (MA) corrections to the Lamb shift of muonic atoms $p^+\mu^-$, according to the model of Caianiello and collaborators [1], [2]. The view frequently held [3], [4] that the proper acceleration of a particle has an upper limit finds in this model a geometrical interpretation expressed by the line element

$$d\tilde{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{|\ddot{x}|^2}{A_m^2}\right)ds^2 \equiv \sigma^2(x)ds^2,$$

(1.1)

experienced by the accelerating particle along its worldline. In (1.1) $A_m \equiv 2mc^3/\hbar$ is the proper MA of the particle of mass $m$, $\ddot{x} = d^2x^\mu/ds^2$ its acceleration and $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ is the metric due to a background gravitational field. In the absence of gravity, $g_{\mu\nu}$ is replaced by the Minkowski metric tensor $\eta_{\mu\nu}$. Results similar to (1.1) have also been obtained in the context of Weyl space [5].

Eq. (1.1) has several implications for relativistic kinematics [6], the energy spectrum of a uniformly accelerated particle [7], the periodic structure as a function of momentum in neutrino oscillations [7], the Schwarzschild horizon [8], the expansion of the very early universe [9], the classical electrodynamics of a particle [10] and the mass of the Higgs boson [11]. It also makes the metric observer-dependent, as conjectured by Gibbons and Hawking [12], and leads in a natural way to hadron confinement [13].

The extreme large value that $A_m$ takes for all known particles makes a direct test of Eq. (1.1) very difficult. Nonetheless a realistic test has also been suggested [14].

Using the same model, we have recently calculated [15] in a non–relativistic approximation, the MA corrections to the Lamb shift of hydrogenic atoms and found them compatible with experimental results. In particular, the agreement between MA corrections and experiment is very good for the $2S-2P$ Lamb shift in hydrogen ($\sim 7$ kHz) and comparable with the agreement of experiments with standard QED with and without two–loop corrections. The agreement also is good for the $(1/4)L_{1S} - (5/4)L_{2S} + L_{4S}$ Lamb shift in $H$ and comparable, in some instances, with that between experiment and QED ($\sim 30$ kHz). The agreement remains good, in this instance, for $D$ too. For the $L_{1S}$ case in $D$, the MA theory is worse ($\sim -270$ kHz) than the
standard one in reproducing the experimental data when the two-loop corrections are included, but better than QED alone when these are excluded. Finally, the MA corrections improve the agreement between theory and experiment by $\sim 50\%$ for the $2S - 2P$ shift in $He^+$. In this work we extend the calculation of the MA corrections to muonic hydrogen atoms for essentially two reasons. First, the levels of muonic hydrogen are very sensitive to QED, recoil and proton-size effects and may lead to a more precise determination of the proton radius. An accurate measurement of the proton radius would affect all QED tests based on the hydrogen atom and corresponding comparisons with the MA corrections. Second, MA effects are larger in muonic hydrogen because the muon in the ground state is much closer to the proton, hence its acceleration is higher. Unlike Ref. [15], the present calculations are fully relativistic. Section 2 contains the Dirac Hamiltonian, its eigenfunctions and the MA perturbations. The Lamb shifts are calculated in Section 3 and the conclusions are given in Section 4.

2 The Dirac Hamiltonian

The MA corrections due to the metric (1.1) appear directly in the Dirac equation for the muon that must now be written in covariant form and referred to a local Minkowski frame by means of the vierbein field $e_\mu^a(x)$. As in Ref. [15] and [16] one finds $e_\mu^a = \sigma(x)\delta_\mu^a$, where Latin indices refer to the locally inertial frame and Greek indices to a generic non-inertial frame. The covariant matrices $\gamma^\mu(x)$ satisfy the anticommutation relations $\{\gamma^\mu(x), \gamma^\nu(x)\} = 2\delta^{\mu\nu}(x)$, while the covariant derivative $D_\mu \equiv \partial_\mu + \omega_\mu$ contains the total connection $\omega_\mu = \frac{1}{2}\sigma^{ab}\omega_{\mu ab}$, where $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$, $\omega_\mu^a_b = (\Gamma_\mu^a e_\nu^b - \partial_\mu e_\nu^a)e_\nu^b$ and $\Gamma_\mu^a$ represent the usual Christoffel symbols. For conformally flat metrics $\omega_\mu$ takes the form $\omega_\mu = (1/\sigma)\sigma^{ab}\eta_{\mu a}\sigma^b$. By using the transformations $\gamma^\mu(x) = e^a(x)\gamma^a$ so that $\gamma^\mu(x) = \sigma^{-1}(x)\gamma^\mu$, where $\gamma^\mu$ are the usual constant Dirac matrices, the Dirac equation can be written in the form

$$i\hbar\gamma^\mu \left( \partial_\mu + i\frac{e}{\hbar c}A_\mu \right) + i\frac{3\hbar}{2}\gamma^\mu (\ln \sigma)_{,\mu} - mc\sigma(x) = 0.$$ \hspace{1cm} (2.1)

\[\hat{H} = -i\hbar c\vec{\alpha} \cdot \vec{\nabla} + e\gamma^0\gamma^\mu A_\mu(x) - i\frac{3\hbar c}{2}\gamma^0\gamma^\mu (\ln \sigma)_{,\mu} + mc^2\sigma(x)\gamma^0, \] \hspace{1cm} (2.2)
which is in general non–Hermitian [16]. If $\sigma$ varies slowly in time, or is time-independent, as in the present case, the term $(\ln \sigma)_0$ can be neglected and Hermiticity is recovered.

The Lamb shift corrections are calculated by means of relativistic wave functions [17]. For the electric field $E(r) = kZe/r^2 (k = 1/4\pi\varepsilon_0)$, the conformal factor becomes $\sigma(r) = (1 - (\frac{m}{r})^4)^{1/2}$, where $r_0 \equiv (kZe^2/mA_m)^{1/2} \sim 1.59 \cdot 10^{-15}$m and $r > r_0$. The calculation of $\ddot{\sigma}^\mu$ is performed classically. Neglecting contributions of the order $O(A_m^{-4})$ one gets $\sigma(r) \sim 1 - (1/2)(r_0/r)^4$.

This expansion requires that in the following only those values of $r$ be chosen that are above a cut–off $\Lambda$, such that for $r > \Lambda > r_0$ the validity of the expansion is preserved. The actual value of $\Lambda$ is determined by the maximum probability distance of the muon from the proton. Thus $\Lambda \sim a_0$, where $a_0 \equiv \hbar/m_\mu c\alpha$ is the Bohr radius of the muon. The length $r_0$ has no fundamental significance in QED and depends in general on the details of the acceleration mechanism. It is only the distance at which the muon would attain, classically, the acceleration $A_m$ irrespective of the probability of getting there.

By using the expansion for $\sigma(r)$ in (2.2) one finds that all MA effects are contained in the perturbative terms

$$H_{r_0} = -\frac{mc^2}{2} \left( \frac{r_0}{r} \right)^4 \beta + i \frac{3hc}{4} r_0^4 \alpha \cdot \vec{\nabla} \frac{1}{r^2} \equiv \mathcal{H} + \mathcal{H}' .$$ (2.3)

The corrections to the energy levels $2S$ and $2P$ are calculated by using the eigenfunctions of the Dirac Hamiltonian

$$|\psi^{(0)}> = \begin{pmatrix} g_{n,k}(r) \chi_k^\mu \\ if_{n,k}(r) \chi_k^{-\mu} \end{pmatrix} ,$$ (2.4)

where $\chi^\mu$ are the spin functions and $g_{n,k}(r)$ and $f_{n,k}(r)$ are the radial wave functions

$$g_{n,k}(r) = B_k e^{-\rho/2} \rho^{\gamma-1} \left[ \left( k - \frac{Z\alpha}{\lambda\zeta} \right) F_1(-n_r; 2\gamma + 1; \rho) + n_r F_1(-n_r + 1; 2\gamma + 1; \rho) \right] ,$$ (2.5)

$$f_{n,k}(r) = C_k e^{-\rho/2} \rho^{\gamma-1} \left[ \left( k - \frac{Z\alpha}{\lambda\zeta} \right) F_1(-n_r; 2\gamma + 1; \rho) - n_r F_1(-n_r + 1; 2\gamma + 1; \rho) \right] .$$ (2.6)
\[ -n_r F_1(-n_r + 1; 2\gamma + 1; \rho) \] .

\[ F_1(a, b, x) \] are the confluent hypergeometric functions. The constants in Eqs. (2.5) and (2.6) are defined by

\[ B_k \equiv A(n_r, k) \frac{(\lambda_c^{-1} + W)^{1/2}}{k - Z\alpha/\lambda \lambda_c}, \quad C_k \equiv -A(n_r, k) \frac{(\lambda_c^{-1} - W)^{1/2}}{k - Z\alpha/\lambda \lambda_c}, \quad (2.7) \]

\[ A(n_r, k) \equiv \frac{2^{1/2} \lambda_c^{3/2}}{\Gamma(2\gamma + 1)} \left[ \lambda_c \left( \frac{Z\alpha}{\lambda \lambda_c} - k \right) \frac{\Gamma(2\gamma + n_r + 1)}{(\alpha/\lambda \lambda_c)(n_r)!} \right]^{1/2}. \quad (2.8) \]

The quantum numbers \( n_r, k \) and \( \gamma \) are given by

\[ n_r = n - |k|, \quad \gamma = \sqrt{k^2 - (Z\alpha)^2}, \quad \alpha \sim 1/137, \quad (2.9) \]

where \( k \) is related the angular quantum number \( l \) (for instance, \( k = -1 \) for the states \( S \) and \( k = 1 \) for the states \( P \)). \( W \) is defined in terms of the energy \( E_{nlj} \)

\[ W = \frac{E_{nlj}}{\hbar c} = \frac{m c^2}{\hbar c} \left[ 1 + \frac{(Z\alpha)^2}{n - (j + 1/2) + (k^2 - (Z\alpha)^2)/2} \right]^{-1/2}, \quad (2.10) \]

where \( n = 1, 2, 3, \ldots, j = 1/2, 3/2, \ldots \leq n, \quad 0 \leq l \leq n - 1. \) Finally,

\[ \lambda_c = \frac{\hbar}{mc}, \quad \lambda \equiv (\lambda_c^{-2} - W^2)^{1/2}, \quad \rho \equiv 2\lambda r. \quad (2.11) \]

The perturbation due to \( \mathcal{H}' \) vanishes, while for \( \mathcal{H} \) one finds

\[ \Delta E = -\frac{mc^2 r_0^4}{2} \int_\Lambda \frac{1}{r^2} [g_{n_r k}(r)^2 - f_{n_r k}(r)^2] dr. \quad (2.12) \]

### 3 \( p^+ \mu^- \) Lamb Shifts of the States 2\( S_{1/2}, 2P_{1/2} \)

The contribution to the Lamb shift \( 2S - 2P \) is calculated by using Eq. (2.12).

For \( 2S_{1/2} \) states, one has \( (Z=1) \ n = 2, \ n_r = 1, \ k = -1 \), and from Eqs. (2.5) and (2.6) one gets

\[ g_{1,-1}(r) = B_{-1} e^{-\rho/2} \rho^{\gamma-1} \left[ \left( 1 + \frac{\alpha}{\lambda \lambda_c} \right) \frac{\rho}{2\gamma + 1} - \frac{\alpha}{\lambda \lambda_c} \right]. \quad (3.1) \]
\[ f_{1,-1}(r) = C_{-1}e^{-\rho/2}\rho^{\gamma-1} \left[ \left( 1 + \frac{\alpha}{\lambda\lambda_c}\right) \frac{\rho}{2\gamma+1} - \frac{\alpha}{\lambda\lambda_c} - 2 \right], \quad (3.2) \]

where the identities \[ 1 \]

\[ 1 F_1(-1, b; x) = 1 - \frac{x}{b}, \quad 1 F_1(0, b; x) = 1 \quad (3.3) \]

have been used. Inserting Eq. (3.1) and (3.2) into Eq. (2.12) one obtains the correction to \( 2S_{1/2} \)

\[ \Delta E(2S_{1/2}) = a_{-1}I(0) + b_{-1}I(1) + c_{-1}I(2). \quad (3.4) \]

The coefficients \( a_{-1}, b_{-1}, c_{-1} \) and the integral function \( I(q) \) are defined below. Similarly, for the state \( 2P_{1/2} \) one has \( n = 2, n_r = 1, k = 1, \) and

\[ g_{1,1}(r) = B_{1}e^{-\rho/2}\rho^{\gamma-1} \left[ \left( -1 + \frac{\alpha}{\lambda\lambda_c}\right) \frac{\rho}{2\gamma+1} - \frac{\alpha}{\lambda\lambda_c} + 2 \right], \quad (3.5) \]

\[ f_{1,1}(r) = C_{1}e^{-\rho/2}\rho^{\gamma-1} \left[ \left( -1 + \frac{\alpha}{\lambda\lambda_c}\right) \frac{\rho}{2\gamma+1} - \frac{\alpha}{\lambda\lambda_c} \right]. \quad (3.6) \]

Therefore, the correction to the level \( 2P_{1/2} \) is

\[ \Delta E(2P_{1/2}) = a_{1}I(0) + b_{1}I(1) + c_{1}I(2). \quad (3.7) \]

The integral function \( I(q) \) is defined as

\[ I(q) = -mc^2r_0^4\lambda \int_{2\lambda\lambda}^{\infty} d\rho e^{-\rho}\rho^{2\gamma-2-q}, \quad (3.8) \]

while the constant coefficients are

\[ a_{-1} = \frac{B_{-1}^2 - C_{-1}^2}{(2\gamma+1)^2} \left( 1 + \frac{\alpha}{\lambda\lambda_c} \right)^2, \quad (3.9) \]

\[ b_{-1} = -\frac{2}{2\gamma+1} \left( 1 + \frac{\alpha}{\lambda\lambda_c} \right) \left[ (B_{-1}^2 - C_{-1}^2)\frac{\alpha}{\lambda\lambda_c} - 2C_{-1}^2 \right], \quad (3.10) \]

\[ c_{-1} = B_{-1}^2 \left( \frac{\alpha}{\lambda\lambda_c} \right)^2 - C_{-1}^2 \left( \frac{\alpha}{\lambda\lambda_c} + 2 \right)^2. \quad (3.11) \]

\[ a_{1} = \frac{B_{1}^2 - C_{1}^2}{(2\gamma+1)^2} \left( -1 + \frac{\alpha}{\lambda\lambda_c} \right)^2, \quad (3.12) \]
\[ b_1 = -\frac{2}{2\gamma + 1} \left( -1 + \frac{\alpha}{\lambda \lambda_c} \right) \left[ (B_1^2 - C_1^2) \frac{\alpha}{\lambda \lambda_c} - 2B_1^2 \right], \quad (3.13) \]
\[ c_1 = B_1^2 \left( \frac{\alpha}{\lambda \lambda_c} - 2 \right)^2 - C_1^2 \left( \frac{\alpha}{\lambda \lambda_c} \right)^2. \quad (3.14) \]
The integral \( I(q) \) depends strongly on the cut–off \( \Lambda \). For \( \Lambda \sim a_0 \), a numerical evaluation of corrections (3.4) and (3.7) yields
\[ \Delta E(2S_{1/2}) \sim -2.06 \cdot 10^5 \text{MHz}, \quad (3.15) \]
\[ \Delta E(2P_{1/2}) \sim -2.99 \cdot 10^5 \text{MHz}, \quad (3.16) \]
so that the Lamb shift correction for the muonic hydrogen atom is
\[ \Delta E_L = \Delta E(2S_{1/2}) - \Delta E(2P_{1/2}) \sim 9.3 \cdot 10^4 \text{MHz} \sim 0.39 \text{meV}. \quad (3.17) \]

It is interesting to note that repeating the same calculation for \((p^+e^-)\) hydrogen atoms, one finds
\[ \Delta E_L(p^+e^-) \sim 11.37 \text{kHz}, \quad (3.18) \]
in excellent agreement with the result +10.45 kHz calculated in the non–relativistic approximation \[15\].

4 Summary

The results of interest in the present calculation are Eqs. (3.4) and (3.7). When \( \Lambda \sim a_0 \sim 2.6 \cdot 10^{-13} \text{cm} \), the muon Bohr radius, the \( 2S - 2P \) Lamb shift is given by Eq. (3.17). The validity of the calculation is supported by the value (3.18) obtained for the \( H \)-atom, which agrees well with the result \( \Delta E_L(p^+e^-) \sim +10.45 \text{kHz} \) previously calculated using a non–relativistic approximation. The result (3.17) is of opposite sign and much smaller than the Lamb shift from all sources \( E_L = 202\,070(108) \text{meV} \) recently calculated by Pachuki \[19\], but much higher than the estimated 0.01 meV precision level (three–loop vacuum polarization) of his calculation. In fact it ranks higher than all corrections reported in \[19\] with the exception of vacuum polarization to leading order (205.006 meV), two–loop vacuum polarization (1.508 meV) and muon self–energy and vacuum polarization (−0.668 meV). Measurements at the expected level of accuracy \[19\] may provide direct evidence for the MA corrections calculated in the present work.
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