CONVEX FUNCTIONAL AND THE STRATIFICATION OF THE SINGULAR SET OF THEIR STATIONARY POINTS

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ABSTRACT. We prove partial regularity of stationary solutions and minimizers $u$ from a set $\Omega \subset \mathbb{R}^n$ to a Riemannian manifold $N$, for the functional $\int_{\Omega} F(x, u, |\nabla u|^2)dx$. The integrand $F$ is convex and satisfies some ellipticity and boundedness assumptions. We also develop a new monotonicity formula and an $\epsilon$-regularity theorem for such stationary solutions with no restriction on their images. We then use the idea of quantitative stratification to show that the $k$-th strata of the singular set of such solutions are $k$-rectifiable.

1. INTRODUCTION

In this paper we develop the regularity theory for minimizing and stationary points of the energy functional

$$E(u) = \int_{\Omega} F(|\nabla u|^2)dx$$

or more generally

$$E(u) = \int_{\Omega} F(x, i \circ u, |\nabla u|^2)dx$$

where $u$ is in Sobolev space of maps $H^1(\Omega, N)$, $\Omega$ is an open domain with smooth boundary in $\mathbb{R}^n$, and $N = N^m$ is a compact, smooth manifold with

$$\partial N = \emptyset, \ \text{inj}(N) > \rho > 0, \ |\text{sec}_N| < k, \ \text{Diam}_N < D,$$

isometrically embedded in some Euclidean space, $i : N \hookrightarrow \mathbb{R}^q$. Abusing notation in many places in this paper we write $u$ instead of $i \circ u$. For the purpose of regularity, we assume the $C^2$ function $F$

$$F(x, z, p) \text{ with } x \in \mathbb{R}^n, z \in \mathbb{R}^q, \text{ and } p \in \mathbb{R},$$

satisfies some ellipticity and integrability assumptions, i.e.

Assumption A.  

i. For some $B > 1$, $F$ satisfies the ellipticity condition

$$B^{-1} \leq F_{pp}(x, z, p)p + \frac{ng}{2}F_p(x, z, p) \leq B.$$

ii. $|F_{x^i}(x, z, p)|, |F_{z^j}(x, z, p)| < \vartheta p$, for some positive constant $\vartheta$.

iii. $F_{pp}(x, z, p) \geq 0$.

iv. $F$ satisfies the following integrability conditions

$$\int_{1}^{\infty} \sup_{x, z} F_{pp}(x, z, p) \ln p \, dp = \mathcal{C} < \infty.$$

$$\int_{0}^{1} p \sup_{x, z} e(x, z, p^{-2}) \, dp = \mathcal{D} < \infty.$$
Hereafter we always assume $F$ satisfies Assumption A. See also Section 2 for more explanation on condition i. The Euler-Lagrange equation for this energy functional is

$$- \int F_{z_k}(x,u,\vert \nabla u \vert^2)z_k + \int F_p(x,u,\vert \nabla u \vert^2)(\langle \nabla u, \nabla z \rangle) - A(u)(\nabla u, \nabla u)z = 0,$$

(5)

where $F_{z_k}$ denotes the partial derivative with respect to the $k$-th component of $z = (z_1, \ldots, z_q)$ and $F_p$ denotes the derivative with respect to the last component of $F(x, z, p)$. Considering the variation generated by a compactly supported vector field $\zeta$ on $\Omega$, the stationary equation related to this energy functional is

$$- \int F_{x_l}(x,u,\vert \nabla u \vert^2)\zeta_l + \int F_p(x,u,\vert \nabla u \vert^2)(\langle \nabla u, \nabla \zeta \rangle \nabla^i \zeta^l - F(x,u,\vert \nabla u \vert^2) \text{div}(\zeta) = 0,$$

(6)

where $F_{x_l}$ denotes the partial derivative with respect to the $l$-th component of $x = (x_1, \ldots, x_n)$. We call the weak solutions of (5) and (6), stationary $F$-harmonic maps and the minimizers of the functional (1) and (2), minimizing $F$-harmonic maps.

The existence and regularity of minimizing and stationary $F$-harmonic maps have been considered extensively. For example in [Uhl77], Uhlenbeck has shown that under ellipticity assumption on $F$, the weak solutions to equation (5), when $u$ is a map to $\mathbb{R}$, is $C^{1,\alpha}$ regular for some $0 < \alpha < 1$.

In [GM79], the authors have shown that under smallness assumption on the image, and ellipticity and growth condition on $F$, weak solutions to (5) are Hölder continuous outside a set of finite codimension 2 Hausdorff measure. See also the book [Gia83] and the references therein for a complete survey on this subject.

Later Schoen and Uhlenbeck in [SU82] have developed the classical theory of harmonic maps when

$$F(\vert \nabla u \vert^2) = \vert \nabla u \vert^2$$

(7)

and they have shown that the $k$-dimensional stratum of the singular set

$$S^k(u) = \{x \in \Omega \mid \text{no tangent map at } x \text{ is } k\text{-symmetric}\}$$

of the classical stationary harmonic maps, i.e. weak solutions of (5) and (6) for the functional (7), satisfy

$$\dim (S^k(u)) \leq k.$$  

(8)

They also showed that the singular set of the classical minimizing harmonic maps satisfy

$$S^{n-3}(u) = S(u)$$

(9)

where $S(u)$ denotes the singular set of the map $u$,

$$S(u) = \{x \in \Omega \mid \exists r > 0 \text{ such that } u\vert_{B_r(x)} \text{ is Hölder continuous}$$.  

(10)

This was then extended by Lin in [Lin99] where he used the idea of defect measures to prove inequality (8) for the stationary harmonic maps. He also showed that

$$\mathcal{H}^{n-2}(S(u)) = 0.$$  

(11)

In the two latter examples the authors prove a monotonicity formula for $\theta(x, r) = r^{2-n} \int_{B_r(x)} \vert \nabla u \vert^2$,

$$\frac{d}{dr} \theta(x, r) = 2r^{2-n} \int_{\partial B_r(x)} \vert \frac{\partial u}{\partial r} \vert^2,$$
which shows the scale invariant quantity \( \theta(x,r) \) is monotone, and is constant if and only if \( u \) is homogenous. This is the main step of the proof of inequality (8). The proof of (9) and (11) are again based on monotonicity formula and a so called \( \epsilon \)-regularity theorem [Bet93].

Recently in [NV17], Naber and Valtorta have used the idea of quantitative stratification, which first appeared in the work of Almgren [Alm] and was later developed in [CN13a], [CN13b] by Cheeger and Naber, to show that when \( u \) is stationary harmonic

\[
S^k(u) \text{ is } k\text{-rectifiable.} \tag{12}
\]

They have further shown

\[
\mathcal{H}^{n-3}(S^{n-3}(u) \cap B_1(0)) \text{ is finite} \tag{13}
\]

when \( u \) is a minimizing harmonic map.

The goal of this paper is to generalize the results above for minimizing F-harmonic maps and stationary F-harmonic maps. A crucial ingredient in the proof of these results was a suitable monotonicity formula. The analogous results could not be extended to stationary solutions and minimizers of the more general functional (2) due to the absence of monotonicity formula. Furthermore, there is no \( \epsilon \)-regularity type theorem in this context and for a general target manifold \( N \).

As a crucial first step for proving a regularity result we obtain the following monotonicity formula for stationary F-harmonic maps,

\[
\frac{d}{dr} \left( e^{\frac{\vartheta}{c_e} r^{2-n}} \int_{B_r(x_0)} F(x,u,|\nabla u|^2) dx + h(r) \right) \geq \int_{\partial B_r(x)} F_p(x,u,|\nabla u|^2) |\partial u/\partial r|^2 \tag{14}
\]

where \( c_e = nqB^2/2 \) and \( \vartheta \) is a constant depending on \( F \). Here \( h \) is a positive monotone function with \( \lim_{r \to 0} h(r) = 0 \) which will be defined explicitly in terms of \( F \) in Theorem 3.2. For the proof of (14) we prove a Jensen-type inequality for functions with positive first derivatives.

The \( \epsilon \)-regularity theorem for classical stationary harmonic maps in [Bet93] says that if \( \theta(x,r) \) is small enough for some positive number \( r \), then \( u \) is smooth on the ball \( B_{r/2}(x) \). By use of similar techniques as in [Bet93], we prove the following \( \epsilon \)-regularity result. Define

\[
\Theta(x_0,r) = e^{\frac{\vartheta}{c_e} r^{2-n}} \int_{B_r(x_0)} F(x,u,|\nabla u|^2) dx + h(r).
\]

We have

**Theorem 1.1.** There exist \( \epsilon_0, \alpha \geq 0 \) depending only on \( n, N \) and \( F \), such that if \( u \in H^1(B_r(x_0), N) \) is a stationary F-harmonic map with

\[
\Theta(x_0,r) \leq \epsilon_0,
\]

then \( u \) is in \( C^{1,\alpha}(B_r(x_0)) \) with \( |u|_{C^{0,\alpha}} \leq C(n,N,F) \).

As a corollary we show that there exist \( \epsilon_0, \alpha, r_0 > 0 \) depending only on \( n, N \) and \( F \), such that if \( u \) is any stationary F-harmonic map with

\[
r^{2-n} \int_{B_r(x_0)} F(x,u,|\nabla u|^2) \leq \epsilon_0,
\]

for some \( 0 < r < r_0 \), then \( u \) is \( C^{0,\alpha}(B_r(x_0)) \).
Note that in Theorem 1.1 we only assume $N$ satisfy (3) and we do not consider any smallness assumption on the image of the map $u$. By a simple covering argument and Theorem 1.1 we get

$$\mathcal{H}^{n-2}(S(u)) = 0.$$ 

Moreover, the monotonicity formula (3.2) and Theorem 1.1 enable us to generalize (12) for stationary $F$-harmonic maps. More precisely, for a map $u : B_3(0) \subset \Omega \rightarrow N$ with

$$u \in H^1(B_3(0), N), \text{ and } \tilde{\Theta}_u(0, 3) \leq \Lambda.$$ 

(15)

We prove the following result.

**Theorem 1.2.** Let $u$ be a stationary $F$-harmonic map. For every $\epsilon > 0$ there exists, $C_{\epsilon}(n, N, \Lambda, F)$ such that for all $0 < r \leq 1$ we have

$$\text{vol} \left( B_r \left( S^{k}_{\epsilon,r}(u) \right) \cap B_1(0) \right) \leq C_{\epsilon} r^{-k}. $$

(16)

Similarly for $S^k_\epsilon$ we have

$$\text{vol} \left( B_r \left( S^k_\epsilon(u) \right) \cap B_1(0) \right) \leq C_{\epsilon} r^{-k}. $$

(17)

In particular, $\mathcal{H}^k \left( S^k_\epsilon(u) \right) < C_\epsilon$. We also have

$$S^k_\epsilon \text{ is k-rectifiable.} $$

(18)

As a corollary

$$S^k \text{ is k-rectifiable.} $$

(19)

Here $S^k_{\epsilon,r}(u)$ and $S^k_\epsilon(u)$ denote the k-th quantitative strata which classify points on the domain based on $L^2$-closeness of the map $u$ to a k-symmetric map in the balls of certain size around them. See Subsection 4.1 for the exact definitions. For the proof of the above theorem we follow a similar argument as in [NV16], which uses a simpler covering argument compared with [NV17]. Having Theorem 1.2 in hand and by proving a quantitative version of Theorem 1.1, one can conclude (13) for minimizing $F$-harmonic maps and prove the following theorem.

**Theorem 1.3.** Let $u$ be as in (15) and be a minimizing $F$-harmonic map. Then $S(u)$ is $(n-3)$-rectifiable and there exists $C(n, N, \Lambda, F)$ such that

$$\text{vol} \left( B_r \left( S(u) \right) \cap B_1(0) \right) < C r^3.$$ 

Consequently, $\mathcal{H}^{n-3}(S(u) \cap B_1(0)) \leq C.$

We should mention here the results of this paper can be extended to maps from a Riemannian manifold $M$ into a Riemannian manifold $N$, for $N$ as above and where $M$ satisfies $\text{inj}_M > \rho > 0$ and $|\text{sec}_M| < K_M$.

The organization of this paper is as follows. In Section 2 we consider the functional (1) and in Section 3 we generalize the results proven in Section 2, for the functional (2). More precisely in Subsection 2.1, we prove a monotonicity formula, Theorem 2.3, which we generalize in Subsection 3.1 for the general functional (2). Subsection 2.2 is where we prove Theorem 1.1 for (1). We adjust this proof for (2) in Subsection 3.2. Subsection 2.3 is devoted to the proof of a compactness result for solutions of (5) and (6), for the functional (1) (see Proposition 2.11) and some properties of tangent maps in this context (see Lemma 2.14). We generalize these results in Subsection 3.3 for the functional (2). Finally we prove Theorem 1.2 and Theorem 1.3 in Section 4. The proof of Theorem 1.2 requires three additional ingredients.

1. The $L^2$-approximation theorem, Theorem A.12, which relates the $\beta$-Jones’ number and the average of
pinches of the monotone quantity $\bar{\Theta}(x, r)$ on a ball. 2. Rectifiable-Reifenberg theorems, Theorem A.10 and Theorem A.11, which are generalizations of original Rectifiable-Reifenberg result [Rei60]. 3. Two covering lemmas, Lemma A.13 and Lemma A.15. Since the proof of these ingredients are similar to the analogous results for harmonic maps [NV17] and approximate harmonic maps [NV16], we discuss them in Appendix A.

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2. SPECIAL CASE $F(|\nabla u|^2)$

In this section we consider the energy functional

$$E(u) = \int_\Omega G(\nabla u) = \int_\Omega F(|\nabla u|^2)$$

on $H^1(\Omega, N)$. The Euler-Lagrange equation with respect to this energy functional is

$$\int F'(|\nabla u|^2)[\langle \nabla_i u, \nabla_i \zeta \rangle - A(u)(\nabla u, \nabla \zeta)] = 0,$$

or equivalently

$$\text{div}(F'(|\nabla u|^2)\nabla u) - F'(|\nabla u|^2)A(u)(\nabla u, \nabla u) = 0 \quad \text{in the weak sense.}$$

One can find the stationary points for $E$ considering the variation on $\Omega$

$$\frac{d}{dt}|_{t=0} E(u \circ \phi_t) = \frac{d}{dt}|_{t=0} E(\phi_t, g(u)) = 0$$

where $\phi_t$ is the flow generated by a compactly supported vector field $X$ and $g$ is the Euclidean metric on $\Omega$. This will reduce to

$$\int \left[ F(|\nabla u|^2)g_{\alpha \beta} - 2F'(|\nabla u|^2)(u^* h)_{\alpha \beta} \right] (L_X g)^{-1} \gamma_{\alpha \beta}. $$

So the stress energy tensor for this equation is

$$S_{\alpha \beta} = F(|\nabla u|^2)g_{\alpha \beta} - 2F'(|\nabla u|^2)(u^* h)_{\alpha \beta}.$$ 

$S$ is divergence free

$$\nabla^\alpha S_{\alpha \beta} = 0 \quad \text{in the distributional sense.}$$

Therefore the stationary equation for the energy functional (1) is

$$\nabla^\alpha \left( F(|\nabla u|^2)g_{\alpha \beta} - 2F'(|\nabla u|^2)(u^* h)_{\alpha \beta} \right) = 0 \quad \text{in the distributional sense}$$

or

$$\int F(|\nabla u|^2) \text{div}(X) - 2F'(|\nabla u|^2)(u^* h)_{\alpha \beta} \nabla^\alpha X^\beta = 0,$$

for any compactly supported smooth vector field $X$ on $\Omega$. Note that the weak solutions of (21) and (23) are the stationary $F$-harmonic maps for the functional (1).
Without loss of generality we can assume 0 ∈ Ω and B_r(0) ⊂ Ω for some r > 0. Define the vector field X as follows: let ψ_ε(|x|) be a compactly supported smooth function on B_r(0) with ψ_ε(|x|) ≡ 1 for x ∈ B_r(1−ε)(0). Then we define X(x) = ψ_ε(|x|)x. By replacing this vector field in (24) and sending ε to 0 we have
\[
\frac{d}{dr} \left( r^{2-n} \int_{B_r(0)} F(|\nabla u|^2) \right) + 2r^{1-n} \int_{B_r(0)} e(|\nabla u|^2) = 2r^{2-n} \int_{\partial B_r(0)} F'(|\nabla u|^2) \frac{\partial u}{\partial r}^2, \tag{25}
\]
where e(x) = F'(x)x - F(x). We refer to e as the error term. Note that to obtain the above equation we have not used any assumption on F.

Properties of G and F. As we mentioned in introduction, we assume some ellipticity and boundedness assumptions on F. Indeed if we assume the integrand G satisfies the following strong ellipticity and boundedness condition,
\[
4B^{-1} |\zeta|^2 \leq G_{\rho_i \rho_j}(p) \zeta_i \zeta_j \leq 4B |\zeta|^2 \quad \text{for all } \zeta \in M^{n \times q},
\tag{26}
\]
where M^{n \times q} denotes the space of all real valued matrices, we have
\[
G_{\rho_i}(p) = 2F'(|p|^2)p_i^\alpha, \\
G_{\rho_i \rho_j}(p) = 4F''(|p|^2) p_i^\alpha p_j^\beta + 2F'(|p|^2) \delta_{ij} \delta_{\alpha \beta}.
\]
By considering ζ the unit vector in M^{n \times q} we have
\[
B^{-1} \leq F''(x) x + \frac{nq}{2} F'(x) \leq B. \tag{27}
\]
which is equivalent to the condition i in Assumption A for the functional (1). Note that
\[
F''(x) x + \frac{nq}{2} F'(x) = x^{1-\frac{m}{m'}} (x^{\frac{m}{m'}} F'(x))'
\]
and therefore
\[
B^{-1} x^{\frac{m}{m'}} \leq (x^{\frac{m}{m'}} F'(x))' \leq B x^{\frac{m}{m'}} - 1.
\]
By integrating the above inequality
\[
\frac{2B^{-1}}{nq} \leq F'(x) \leq \frac{2B}{nq} \tag{28}
\]
and so
\[
\frac{B^{-1} - B}{nq} \leq F''(x) x \leq \frac{B - B^{-1}}{nq}. \tag{29}
\]
Concerning the error term e, we have and e satisfies the following properties on [0,∞)
\begin{enumerate}
  \item e(x) is bounded for x < C.
  \item \( \lim_{x \to 0} e'(x) = \lim_{x \to \infty} e'(x) = 0. \)
  \item \( \frac{B^{-1} - B}{nq} \leq e'(x) < \frac{2B}{nq}. \)
  \item e'(x) ≥ 0 if and only if F''(x) ≥ 0.
\end{enumerate}
2.1. **Monotonicity formula for the special case.** In this subsection we obtain a monotonicity formula which is the key for our regularity theorem. We first recall Assumption A for the functional (1) and throughout this section we always assume $F$ satisfies the followings.

**Assumption B.**

i. $F$ satisfies the ellipticity condition (27).

ii. The second derivative of $F$ is non-negative

$$F''(x) \geq 0 \text{ on } [0, \infty).$$

iii. $F$ satisfies the following integrability condition

$$
\int_{1}^{\infty} F''(t) \ln(t) dt = C < \infty.
$$

Before we state our main monotonicity theorem we prove the following lemma which is crucial ingredients in the proof of this theorem.

**Lemma 2.1.** Let $g(x)$ be a positive function with $g'(x) \geq 0$ for all $x \geq 0$. Then

a. $g$ satisfies the following Jensen-type inequality

$$
\int_{B_r(x_0)} g(f(y)) dy \leq J_g \left( \int_{B_r(x_0)} f(y) dy \right)
$$

for any non-negative $f \in L^1(\Omega)$ and $B_r(x_0) \subset \Omega \subset \mathbb{R}^n$, where the function $J_g$ is

$$J_g(x) = g(x) + x \int_{x}^{\infty} \frac{g'(t)}{t} dt.$$

b. If $\int_{1}^{\infty} \frac{g'(t)}{t} \ln(t) < \infty$, then

$$
\int_{0}^{1} x^{-1} \int_{x^{-2}}^{\infty} \frac{g'(t)}{t} dt dx < \infty.
$$

**Proof.** To prove part a, for every $y \in B_r(x_0)$ we have

$$g(f(y)) - g(\bar{f}) = \int_{f}^{f(y)} g'(t) dt$$

(30)

where $\bar{f} = \int_{B_r(x_0)} f(x) dx = \frac{1}{\text{vol}(B_r(x_0))} \int f(x) dx$. Define the set

$$U = \left\{ y \in B_r(x_0) \mid f(y) \geq \int f \right\}$$

$$U_t = \{ y \in B_r(x_0) \mid f(y) \geq t \}$$

$$U^c = B_r(x_0) \setminus U.$$
By averaging (30) over the ball $B_r(x_0)$,
\[
\int_{B_r(x_0)} g(f) - g(\bar{f}) = \int_{B_r(x_0)} \int_{J_f(y)} g'(t) dt \, dy
\]
\[
= \frac{1}{\text{vol}(B_r(x_0))} \int_u \int_{J_f(y)} g'(t) dt \, dy - \frac{1}{\text{vol}(B_r(x_0))} \int_{J_f(y)} g'(t) dt \, dy
\]
\[
\leq \frac{1}{\text{vol}(B_r(x_0))} \int_{J_f(y)} g'(t) dt \, dy
\]
\[
= \frac{1}{\text{vol}(B_r(x_0))} \int_{J_f(y)} g'(t) dt \, dy
\]
\[
\leq \frac{1}{\text{vol}(B_r(x_0))} \int_{J_f(y)} g'(t) \int_{U_i} \frac{f(y)}{t} dy \, dt
\]
\[
\leq \frac{1}{\text{vol}(B_r(x_0))} \int_{J_f(y)} g'(t) \int_{U_i} \frac{f(y)}{t} dy \, dt = \int_{J_f(y)} g'(t) \int_{U_i} \frac{f(y)}{t} dy \, dt
\]
Therefore
\[
\int g(f) \leq J(\bar{f}).
\]
For part b we have
\[
\int_0^1 x^{-1} \int_{x^{-2}}^\infty \frac{g'(t)}{t} dt \, dx = \int_1^\infty \frac{g'(t)}{t} dt \int_1^x \frac{dx}{x} \, dt
\]
\[
= \frac{1}{2} \int_1^\infty \frac{g'(t)}{t} \ln(t) < \infty.
\]

**Remark 2.2.** By Assumption B and Lemma 2.1 we have
\[
\int_{B_r(x_0)} e(f(y)) \, dy \leq J_e \left( \int_{B_r(x_0)} f(y) \, dy \right).
\]
Further, since $\int_0^1 xe(x^{-2}) \, dx = -2x^2F(x^{-2})|_0^1 < \infty$ we have
\[
\int_0^1 xJ_\epsilon(x^{-2}) \, dx < \infty.
\]

Now we are able to state and prove our monotonicity formula.

**Theorem 2.3.** Let $u$ be a stationary $F$-harmonic map in $H^1(\Omega, N)$ for the functional (1). Then there exists $A = A(n, N, B, C)$ such that
\[
\frac{d}{dr}(r^2 \mathcal{E}(r) + h(r)) \geq \int_{\partial B_r(x)} F'(|\nabla u|^2) \frac{\partial u}{\partial r}^2
\]
where $\mathcal{E}(r) = \int_{B_r(x)} F(|\nabla u|^2) \, dx$ and $h(r) = 2 \int_0^r tJ_e(2c_\epsilon A^2 r^{-2}) \, dt$ with $c_\epsilon = nqB/2$. 
**Proof.** Recall that we have
\[
\frac{d}{dr} \left( r^2 \int_{B_r(x)} F(|\nabla u|^2) \right) + 2r J \left( e \int_{B_r(x)} F(|\nabla u|^2) \right) \geq \frac{d}{dr} \left( r^2 \int_{B_r(x)} F(|\nabla u|^2) \right) + 2r \int_{B_r(x)} e(|\nabla u|^2)
\]
\[
= 2r^2 - \int_{\partial B_r(x)} F'(|\nabla u|^2) \left| \frac{\partial u}{\partial r} \right|^2.
\]
(32)

where \( c_e = nqB/2 \). First we claim that \( r^2 \mathcal{E}(r) \) is bounded. To prove our claim we will use the following argument. Let
\[
\mathcal{E}(1) \leq A^2 \quad \text{and} \quad r_0 = r_0(A) \quad \text{be the smallest} \quad r \quad \text{s.t.} \quad r^2 \mathcal{E}(r) \leq 2A^2 \quad \text{on} \quad [r_0, 1].
\]

We show that there exists \( A \) such that \( r_0 = 0 \). By (32) and for \( \bar{r} \in [r_0, 1] \) we have
\[
\int_{\bar{r}}^1 \left[ \frac{d}{dr} \left( r^2 \mathcal{E}(r) \right) + 2r J_e(2c_eA^2r^{-2}) \right] \geq 0.
\]
(33)

Put \( \alpha = \sqrt{2c_eA^2} \). We have
\[
\int_{\bar{r}}^1 r J_e(\alpha^2 r^{-2}) dr = \alpha^2 \int_{\bar{r}/\alpha}^{1/\alpha} s J_e(s^{-2}) ds.
\]

Choose \( \varepsilon \ll 1 \). Since
\[
\int_0^1 s J_e(s^{-2}) ds < \infty,
\]
for \( A \) large enough
\[
\int_{\bar{r}/\alpha}^{1/\alpha} s J_e(s^{-2}) ds < \varepsilon/2
\]
and therefore
\[
\int_{\bar{r}}^1 2r J_e(2c_eA^2 r^{-2}) < \varepsilon \alpha^2 \ll A^2.
\]

Finally by (33) we have
\[
\bar{r}^2 \mathcal{E}(\bar{r}) \leq \mathcal{E}(1) + \int_{\bar{r}}^1 2r J_e(2c_eA^2 r^{-2}) < 2A^2.
\]

and since \( r^2 \mathcal{E}(r) \) is continuous on \((0, 1]\), for \( r_0 - \delta \leq r \leq r_0 \), for some small \( \delta > 0 \), we have \( r^2 \mathcal{E}(r) \leq 2A^2 \) which contradicts the fact that \( r_0 \) is the smallest such \( r \). Assume \( A \) is chosen such that \( r_0(A) = 0 \), then for such \( A \), we have
\[
\frac{d}{dr} \left( r^2 \mathcal{E}(r) + h(r) \right) = \frac{d}{dr} \left( r^2 \mathcal{E}(r) \right) + 2r J_e(2c_eA^2 r^{-2}) \geq \int_{\partial B_r(x)} F'(|\nabla u|^2) \left| \frac{\partial u}{\partial r} \right|^2.
\]

Note that \( \lim_{r \to 0} h(r) = 0 \) since \( \int_0^1 r J_e(2c_eA^2 r^{-2}) < \infty \). \( \square \)
Remark 2.4. An example of a functional which satisfies Assumption B is

\[ F_1(x) = x \left( 2 - \frac{1}{(x + 1)^\beta} \right) \]  

for \( \beta < 1 \).

2.2. \( \varepsilon \)-regularity theorem for the special case. In this subsection we prove the \( \varepsilon \)-regularity theorem, Theorem 1.1 for the functional (1). We restate this theorem for this case. Set

\[ \theta(x_0, r) = r^{2-n} \int_{B_r(x_0)} F(|\nabla u|^2)dx, \]

\[ \Theta(x_0, r) = \theta(x_0, r) + h(r). \]  

Theorem 2.5. There exist \( \varepsilon_0, \alpha \geq 0 \) depending only on \( n, N \) and \( F \) such that if \( u \in H^1(B_r(x_0), N) \) is a stationary \( F \)-harmonic map for the functional (1) with

\[ \Theta(x_0, r) \leq \varepsilon_0 \]

then \( u \) is in \( C^{0,\alpha}(B_\frac{r}{2}(x_0)) \) with \( |u|_{C^{0,\alpha}} \leq C(n, N, F) \).

Here \( |u|_{C^{0,\alpha}(B_{r/2}(x_0))} = \sup_{x,y \in B_{r/2}(x_0)} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \). Before we prove Theorem 2.5, we recall some background material which we need for the proof.

2.2.1. Background. Let \( \Omega \subset \mathbb{R}^n \) be an open domain with smooth boundary. Let \( \phi \) be a positive \( L^2 \) function on \( \Omega \). We will briefly review Hodge theory on the space \((\Omega, g, \phi dx)\) where \( g \) denotes the Euclidean metric, and the Hardy and BMO spaces with respect to the measure \( \phi dx \).

Hodge theory on \((\Omega, g, \phi dx)\). Let \( X \) be a smooth vector field on \( \Omega \). Then

\[ \int_{\Omega} \text{div}(\phi X)dx = \int_{\Omega} \text{div}(X)\phi dx + \int_{\Omega} \langle \nabla \ln \phi, X \rangle \phi dx \]

We define

\[ \text{div}_\phi(X) = \text{div}(X) + \langle \nabla \ln \phi, X \rangle = \frac{1}{\phi} \text{div}(\phi X). \]

In a similar way, we define the adjoint operator \( \tilde{\delta} = \delta_\phi \) of differential operator \( d \) with respect to the measure \( \phi dx \) by

\[ \int_{\Omega} \langle d\alpha, \beta \rangle \phi dx = \int_{\Omega} \langle \alpha, \delta(\phi \beta) \rangle dx \]

\[ = \int_{\Omega} \langle \alpha, \delta \beta \rangle \phi dx + \int_{\Omega} \langle \alpha, i\nabla \ln \phi \beta \rangle \phi dx \]

\[ = \int_{\Omega} \langle \alpha, \tilde{\delta} \beta \rangle \phi dx \]

where \( \tilde{\delta} \beta = \delta \beta + i\nabla \ln \phi \beta \).
Hardy and BMO spaces. There is a vast literature on analysis on spaces of homogeneous type, including Euclidean spaces with doubling measure. These spaces arise in harmonic analysis in the study of Hardy-Littlewood maximal functions and Hardy spaces, and duality of Hardy and BMO spaces (see [CW71, CW77]). Many properties of the classical BMO space have been shown to hold for doubling metric spaces. These include the Calderón-Zygmund decomposition, the John-Nirenberg inequality (see [Buc99]).

The Hardy and BMO spaces on $\mathbb{R}^n$ with respect to the doubling measure $\phi \, dx$ are defined in a similar way to their original definition, but instead of the Lebesgue measure on $\mathbb{R}^n$, the measure $\phi \, dx$ is used. We use the notation $H^1_\phi$ and $\text{BMO}_\phi$ to distinguish them with their original counterpart. In this context we also have the following theorems where $\phi \, dx$ is a doubling measure.

**Theorem 2.6.** Suppose $u \in L^\infty$ and $v \in H^1_\phi(\mathbb{R}^n) \cap L^1_\phi(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} uv \phi \, dx \leq C [u]_{\text{BMO}_\phi(\mathbb{R}^n)} |v|_{H^1_\phi(\mathbb{R}^n)}.$$  

**Theorem 2.7.** Let $f$ be in $H^1_\phi(\mathbb{R}^n)$ and $\omega$ be a 1-form in $L^2_\phi(\mathbb{R}^n)$. Let $\delta \omega = 0$ in the distributional sense. Then the function $v = df \cdot \omega$ is in the Hardy space $H^1_\phi(\mathbb{R}^n)$. Moreover, there exists a constant $C_0$ depending only on $n$ such that

$$|v|_{H^1_\phi(\mathbb{R}^n)} \leq C_0 |\omega|_{L^2_\phi(\mathbb{R}^n)} |df|_{L^2_\phi(\mathbb{R}^n)}.$$  

By $L^p_\phi$ and $H^1_\phi$ we mean $L^p$ and $H^1$ spaces with respect to measure $\phi \, dx$. The proof of the above two theorems follows from the proof of their original counterparts with the Lebesgue measure.

2.2.2. *Proof of the Theorem 2.5.* In this part we prove Theorem 2.5. The proof is very similar to the proof of original $\epsilon$-regularity theorem for stationary harmonic maps as in [Bet93] (see also [Mos05]). Without loss of generality we assume $B_1 (0) \subset \Omega$. We denote the inner product on $B_1 (0)$ and the space of 1-forms on $\mathbb{R}^n$ by $\langle \ , \ \rangle$ and the inner product on $N$ by $(\ , \ )$.

By a similar argument to the one used in [H90, Bet93] we can show there exists an orthonormal tangent frame field $\{ e_1 \circ u, \ldots, e_m \circ u \}$ along the map $u$ which minimizes

$$\frac{1}{2} \sum_{i=1}^m \int_{B_r(x_0)} |\nabla e_i|^2 \phi \, dx$$

where $\phi = F'(|\nabla u|^2)$ and $B_r(x_0) \subset B_1 (0)$. Such a minimizer satisfies the following Euler-Lagrange equation in the weak sense

$$\text{div}((\nabla e_i, e_j) \phi) = 0 \quad \text{in} \quad B_r(x_0),$$  

with the Neumann boundary condition

$$\langle e_i, \frac{\partial e_j}{\partial \nu} \rangle = 0.$$  

Furthermore, the minimizer satisfies

$$\sum_{i=1}^k \int_{B_r(x_0)} |\nabla e_i|^2 \phi \, dx \leq C \int_{B_r(x_0)} |\nabla u|^2 \phi \, dx.$$  

(39)
Define
\[ \gamma_{ij} = \begin{cases} 
\langle \nabla e_i, e_j \rangle & x \in B_r(x_0) \\
0 & x \notin B_r(x_0).
\end{cases} \] (40)

Then for the 1-form \( \gamma_{ij} \) we have
\[ \text{div}_\phi \gamma_{ij} = 0. \] (41)

Since \( u \) satisfies (21) then for the 1-form \( \omega_i = \langle du, e_i \rangle \) we have
\[ \text{div} (\phi \omega_i) = \phi \langle \nabla u, e_i \rangle \cdot \gamma_{ij} = \phi \omega_j \cdot \gamma_{ij} \]
and therefore
\[ \tilde{\omega}_i = \omega_j \cdot \gamma_{ij}, \]
\[ d\omega_i = \omega_j \wedge \gamma_{ij}. \] (42)

The following lemma is the main step in the proof of Theorem 2.5.

**Lemma 2.8.** There exists a constant \( C \) depending on \( n, N \) and \( F \) such that the following holds. Suppose \( u \in H^1(B_r(x_0), N) \) satisfies equation (21) with
\[ r^{2-n} \int_{B_r(x_0)} F'(|\nabla u|^2) |\nabla u|^2 \leq \epsilon. \]

Then
\[ (kr)^{1-n} \int_{B_{kr}(x_0)} |\nabla u| F'(|\nabla u|^2) dx \leq C k^{1-n} [u]_{\text{BMO}} (B_{kr}(x_0)) (\epsilon + C \sqrt{\epsilon}) \]
\[ + C kr^{1-n} \int_{B_r(x_0)} |\nabla u| F'(|\nabla u|^2) \ dx \]
for any \( \kappa \in (0, 1) \).

**Proof.** Consider a compactly supported cut-off function \( \eta \in C_0^\infty(B_r(x_0)) \) satisfying \( \eta \equiv 1 \) in \( B_{r/2}(x_0) \) and \( 0 \leq \eta \leq 1 \) in \( B_r(x_0) \), such that \( |\nabla \eta| \leq \frac{1}{r} \). We apply the Hodge decomposition theorem to
\[ \tilde{\omega}_i = \langle d(\eta(u - \bar{u})), e_i \rangle, \]
where \( \bar{u} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u \), with respect to the measure \( \phi dx \). Therefore, there exist \( \alpha_i \) and \( \beta_i \) such that
\[ \tilde{\omega}_i = \alpha_i + \beta_i \]
where
\[ d\alpha_i = \tilde{\delta} \beta_i = 0 \]
and
\[ \sum_{i=1}^k \int_{\mathbb{R}^n} |\alpha_i|^2 + |\beta_i|^2 \phi dx \leq C \int_{B_r(x_0)} |\nabla u|^2 \phi dx \]
and on $B_{r/2}(x_0)$

$$|\nabla u| \leq C \sum_{i=1}^{k} |\alpha_i| + |\beta_i|.$$  

Further, on $B_{r/2}(x_0)$ we have

$$\tilde{\delta} \alpha_i = \tilde{\delta} \omega_i = \omega_j \cdot \gamma_{ij},$$

$$d \beta_i = d \omega_i = \omega_j \wedge \gamma_{ij}.$$  

Note that $\alpha_i = d \tilde{\alpha}_i$ and $\beta_i = \tilde{\delta} \beta_i$ and therefore

$$\tilde{\Lambda} \tilde{\alpha}_i = \omega_j \cdot \gamma_{ij},$$

$$\tilde{\Lambda} \beta_i = \omega_j \wedge \gamma_{ij}.$$  

We also have

$$\int_{\mathbb{R}^n} \alpha_i \cdot \beta_i \, \phi \, dx = 0.$$  

**Estimate for $\beta_i$.** We have

$$\int_{\mathbb{R}^n} |\beta_i|^2 \, \phi \, dx = \int \beta_i \cdot \tilde{\omega}_i \, \phi \, dx = \int \beta_i \cdot (d(\eta(u - \bar{u})) \cdot e_i) \, \phi \, dx$$

$$= \int (\eta(u - \bar{u}), \tilde{\delta}(\beta_i \otimes e_i)) \cdot \phi \, dx = \int (\eta(u - \bar{u}), \beta_i \cdot de_i) \, \phi \, dx$$

$$= \int \beta_i \cdot \gamma_{ij} \langle \eta(u - \bar{u}), e_j \rangle \, \phi \, dx \leq C [u]_{\text{BMO}_{\mathbb{R}^n}(B(0))} \| \beta_i \cdot \gamma_{ij} \|_{L^1_{\mathbb{R}^n}}$$

$$\leq C [u]_{\text{BMO}_{\mathbb{R}^n}(B(0))} \| \beta_i \|_{L^2_{\mathbb{R}^n}} \| \nabla e_i \|_{L^2_{\mathbb{R}^n}}.$$  

The two last inequalities on the right hand side of (43) will follow from Theorem 2.7, Theorem 2.6, and the fact that $[\eta(u - \bar{u})]_{\text{BMO}_{\mathbb{R}^n}} \leq C_1 [u]_{\text{BMO}_{\mathbb{R}^n}(B(0))}$, where $C_1$ depends only on $m$ and $n$.  

We extend $e_i$ to $\mathbb{R}^n$ such that

$$|\nabla e_i| \leq C \int_{B_{r/2}(x_0)} |\nabla u|^2 \, \phi \, dx.$$  

Consider now a new cut-off function $\zeta \in C^0_0(B_{2\frac{r}{2}}(x_0))$, $0 \leq \zeta \leq 1$ with $\zeta \equiv 1$ on $B_{\frac{r}{2}}(x_0)$ such that $|\nabla \zeta| \leq \frac{8}{r}$. Then we have

$$\int_{\mathbb{R}^n} \zeta |\beta_i| \, \phi \, dx \leq \left( \int_{\mathbb{R}^n} \zeta^2 \, \phi \, dx \right)^{1/2} \left( \int_{\mathbb{R}^n} |\beta_i|^2 \, \phi \, dx \right)^{1/2}$$

$$\leq C \left( \frac{r}{2} \right)^{n/2} \left( \int_{\mathbb{R}^n} |\beta_i|^2 \, \phi \, dx \right)^{1/2}$$

$$\leq C \left( \frac{r}{2} \right)^{n/2} [u]_{\text{BMO}_{\mathbb{R}^n}(B_{r/2}(x_0))} \left( \int_{B_{r/2}(x_0)} |\nabla u|^2 \, \phi \, dx \right)^{1/2},$$  

where $C$ depends on the upper bound for $F'$ and $n$.  

**CONVEX FUNCTIONAL 13**
Estimate for $a_i$. Recall that $a_i = d\tilde{a}_i$. Decompose

$$\tilde{a}_i = \tilde{a}_i^1 + \tilde{a}_i^2,$$

where

$$\begin{cases}
\tilde{\Delta} \tilde{a}_i^1 = 0 & \text{in } B_{\tau/2}(x_0) \\
\tilde{a}_i^1 = \tilde{a}_i & \text{on } \partial B_{\tau/2}(x_0),
\end{cases}$$

(46)

and

$$\begin{cases}
\tilde{\Delta} \tilde{a}_i^2 = \tilde{\Delta} \tilde{a}_i = \omega_j \cdot \gamma_{ij} & \text{in } B_{\tau/2}(x_0) \\
\tilde{a}_i^2 = 0 & \text{on } \partial B_{\tau/2}(x_0).
\end{cases}$$

(47)

We estimate first $d\tilde{a}_i^2$ as follows

$$\int_{B_{\tau/2}(x_0)} |\nabla \tilde{a}_i^2| \phi dx \leq \int_{B_{\tau/2}(x_0)} \nabla \tilde{a}_i^2 \cdot \nabla \tilde{a}_i^2 \phi dx \leq \int_{B_{\tau/2}(x_0)} \tilde{a}_i^2 \text{div}_\phi \left( \frac{\nabla \tilde{a}_i^2}{|\nabla \tilde{a}_i^2|} \right) \phi dx.$$

(48)

Let $\psi_i$ be the solution to

$$\begin{cases}
\tilde{\Delta} \psi_i = \text{div}_\phi \left( \frac{\nabla \tilde{a}_i^2}{|\nabla \tilde{a}_i^2|} \right) & \text{in } B_{\tau/2}(x_0) \\
\psi_i = 0 & \text{on } \partial B_{\tau/2}(x_0).
\end{cases}$$

(49)

By the Hodge decomposition theorem for $\frac{\nabla \tilde{a}_i^2}{|\nabla \tilde{a}_i^2|}$ and the fact that $\frac{|\nabla \tilde{a}_i^2|}{|\nabla \tilde{a}_i^2|} = 1$, we have

$$|D\psi_i|_{L^{\infty}_\phi} \leq \left| \frac{\nabla \tilde{a}_i^2}{|\nabla \tilde{a}_i^2|} \right|_{L^q_\phi} \leq C r^{m/q}.$$

(50)

The Sobolev embedding theorem implies

$$|\psi_i|_{L^{\infty}(B_{\tau/2}(x_0))} \leq C r.$$

(51)
So by (48) we have
\[
\int_{B_{r/2}(x_0)} |\nabla \bar{\alpha}_i^2| \phi \, dx \leq \int_{B_{r/2}(x_0)} \bar{\alpha}_i^2 \Delta \psi_i \phi \, dx
\]

\[
= \int_{B_{r/2}(x_0)} \bar{\alpha}_i^2 \psi_i \phi \, dx = \int_{B_{r/2}(x_0)} \omega_j \cdot \gamma_{ij} \psi_i \phi \, dx
\]

\[
= \int_{B_{r/2}(x_0)} \langle u, e_j \rangle \cdot \gamma_{ij} \phi \, dx = \int_{B_{r/2}(x_0)} \langle u - \bar{u}, \tilde{\delta}(\gamma_{ij} \otimes \psi_i) \rangle \phi \, dx
\]

\[
= \int_{B_{r/2}(x_0)} \langle u - \bar{u}, d(\psi_i e_i) \cdot \gamma_{ij} \rangle \phi \, dx
\]

(52)

\[
\leq C[u]_{\text{BMO}_{\phi}(B_r(x_0))} \left( \int_{B_r(x_0)} |\nabla u|^2 \phi \, dx \right)^{1/2} \left( \int_{B_r(x_0)} |d(\psi_i e_i)|^2 \phi \, dx \right)^{1/2}
\]

\[
\leq C[u]_{\text{BMO}_{\phi}(B_r(x_0))} \left( \int_{B_r(x_0)} |\nabla u|^2 \phi \, dx \right)^{1/2} \left( \rho^2 \int_{B_r(x_0)} |\nabla u|^2 \phi \, dx + C r^n \right)^{1/2}.
\]

Now we estimate $d\bar{\alpha}_i^1$. We have $\bar{\alpha}_i^1 = 0$. Therefore by the mean value formula in this setting we have
\[
\int_{B_{r/2}(x_0)} |d\bar{\alpha}_i^1| \phi \, dx \leq C \kappa \int_{B_r(x_0)} |d\bar{\alpha}_i^1| \phi \, dx.
\]

(53)

By (45), (53), (75) and for $\kappa \in (0, \frac{1}{2})$ we have
\[
\int_{B_{r/2}(x_0)} |\nabla u| \phi \, dx \leq \sum_{i=1}^{m} \int_{B_{r/2}(x_0)} (|d\bar{\alpha}_i^1| + |d\bar{\alpha}_i^2| + |\beta_i|) \phi \, dx
\]

\[
\leq \sum_{i=1}^{m} \int_{B_{r/2}(x_0)} (C \kappa |d\bar{\alpha}_i^1| + |d\bar{\alpha}_i^2| + |\beta_i|) \phi \, dx
\]

\[
\leq \sum_{i=1}^{m} \int_{B_{r/2}(x_0)} (C \kappa |\nabla u| + |d\bar{\alpha}_i^2| + |\beta_i|) \phi \, dx
\]

\[
\leq C_1[u]_{\text{BMO}_{\phi}(B_r(x_0))} \left( \int_{B_r(x_0)} |\nabla u|^2 \phi \, dx \right)^{1/2} \left( \rho^2 \int_{B_r(x_0)} |\nabla u|^2 \phi \, dx + C_2 r^n \right)^{1/2}
\]

\[+ C_3 \kappa \int_{B_r(x_0)} |\nabla u| \phi \, dx.
\]

Therefore if $r^{2-n} \int_{B_r(x_0)} |\nabla u|^2 \phi \, dx \leq \epsilon$, then we have
\[
(kr)^{-n} \int_{B_{r/2}(x_0)} |\nabla u| \phi \, dx \leq C_1 \kappa^{-n}[u]_{\text{BMO}_{\phi}(B_r(x_0))} \left( r^{2-n} \int_{B_r(x_0)} |\nabla u|^2 \phi \, dx \right)^{1/2} \left( r^{2-n} \int_{B_r(x_0)} |\nabla u|^2 \phi \, dx + C_2 \right)^{1/2}
\]

\[+ C_3 r^{1-n} \kappa \int_{B_r(x_0)} |\nabla u| \phi \, dx
\]

\[
\leq C_1 \kappa^{-n}[u]_{\text{BMO}_{\phi}(B_r(x_0))}(\epsilon + C_2 \sqrt{\epsilon}) + C_3 kr^{1-n} \int_{B_r(x_0)} |\nabla u| \phi \, dx.
\]
Define

$$M(u, x_0, r) = \sup_{B_r(x_0) \subset B_r(x_0)} \left( s^{1-m} \int_{B_r(x_0)} |\nabla u| \, \phi \, dx \right).$$

**Lemma 2.9.** There exist $\epsilon_1$ and $\kappa \in (0, 1)$, depending on $n$, $N$ and $F$, such that the following holds. Suppose $u \in H^1(B_r(x_0), N)$ is a stationary $F$-harmonic map with

$$\Theta(x_0, r) \leq \epsilon_0.$$

Then

$$M(u, x_0, kr) \leq \frac{1}{2} M(u, x_0, r).$$

**Proof.** By the monotonicity formula and for every $s \leq \frac{r}{2}$ and $x_1 \in B_r(x_0)$ we have

$$\theta(x_1, s) \leq \Theta(x_1, s) \leq \Theta(x_1, \frac{r}{2}) \leq c(n) \Theta(x_0, r) \leq c(n) \epsilon_0.$$

First by (28) we have

$$s^{2-n} \int_{B_r(x_1)} |\nabla u|^2 \, \phi \, dx < Kc(n) \epsilon_0 \quad (54)$$

where $K$ depends on $F$ and $n$. Define $\epsilon_1 = Kc(n) \epsilon_0$. By the Poincaré inequality

$$[u]_{BMO(B_r(x_1))} \leq C \rho M(u, x_0, r).$$

By Lemma 2.8,

$$(\kappa s)^{1-n} \int_{B_r(x_1)} |\nabla u|^2 + |\nabla u|^2 \, dx \leq \left[ C (\kappa s)^{1-n} (\epsilon_1 + C \sqrt{\epsilon_1}) + C \kappa \right] M(u, x_0, r).$$

We can choose $\epsilon_1$ and $\kappa$ such that $C (\kappa s)^{1-n} (\epsilon_1 + C \sqrt{\epsilon_1}) + C \kappa \leq 1/2$, completing the proof of Lemma 2.9. \qed

Now we are ready to complete the proof of Theorem 2.5.

**Proof of Theorem 2.5.** By Lemma 2.9 we have

$$M(u, x_0, kr) \leq \frac{1}{2} M(u, x_0, r).$$

Applying this lemma repeatedly, and since $M(u, x_0, r)$ is bounded, we have

$$M(u, x_1, s) \leq C s^\alpha,$$

for $x_1 \in B_{r/2}(x_0)$ and all $s \in (0, r/2)$, where $\alpha$ and $C$ do not depend on $x_1$ and $s$. In particular

$$\sup_{x_1 \in B_{r/2}(x_0)} \sup_{0 < s \leq r/2} \left( s^{1-m-\alpha} \int_{B_s(x_1)} |\nabla u| \phi \, dx \right) < \infty.$$

Since $\phi$ is bounded and by the Morrey decay lemma (see for example Lemma 2.1 in [Mos05]), $u \in C^{0, \alpha}(B_{r/2}(x_0))$ and $|u|_{C^{0, \alpha}} \leq C(n, N, F)$. \qed

Here we have another version of Theorem 2.5.
Lemma 2.10. There exist $\epsilon_0, r_0, \alpha \geq 0$ depending only on $n, N$ and $F$ such that if $u \in H^1(B_r(x_0), N)$ is any stationary $F$-harmonic map with

$$\theta(x_0, r) \leq \epsilon_0$$

for $r \leq r_0$, then $u$ is in $C^{0,\alpha}(B_{r_0}(x_0))$ with $|u|_{C^{0,\alpha}} \leq C(n, N, F)$.

2.3. Compactness for the special case. In order to define the quantitative strata we first need to build the notion of tangent maps. Recall that for a map $u \in H^1(\Omega, N)$ we define the regular points and the singular points of $u$ as follows:

$$\mathcal{R}eg(u) = \{ x \in \Omega \mid u \text{ is } C^{0,\alpha} \text{ in a neighborhood of } x \} ,$$

$$\mathcal{S}(u) = \mathcal{S}ing(u) = \Omega \setminus \mathcal{R}eg(u),$$

where $\alpha$ is the minimum of Hölder constants $\alpha$ in Theorem 2.5 and Lemma 2.10. By a simple covering argument and Theorem 2.5, one can easily show that

$$\mathcal{H}^{n-2}(\mathcal{S}(u)) = 0.$$

In this subsection we first study the convergence of sequence $s$ of maps which satisfy (21) and (23) under a uniform bound on their energy functional, and then we define the notion of tangent maps. See for example [Sch84], [Lin99] for the similar results for harmonic maps and stationary harmonic maps. More precisely, we have a sequence of maps $u_i$ which satisfies

$$\int_{B_3(0)} \int_{B_3(0)} F_i(|\nabla u_i|^2) < \Lambda$$

where $F_i$ satisfies Assumption B. We also assume $u_i$ satisfies

$$\text{div} \left( F_i'(|\nabla u_i|^2) \nabla u_i \right) - F_i'(|\nabla u_i|^2) A(u_i)(\nabla u_i, \nabla u_i) = 0,$$

$$\nabla^\alpha (F_i(|\nabla u_i|^2) g_{\alpha \beta} - 2 F_i'(|\nabla u_i|^2) u_i^r h_{\alpha \beta}) = 0,$$

in the weak sense.

Proposition 2.11. Let $u_i$ and $F_i$ be as above. Then there exists a subsequence of $u_i$ (which we still denote by $u_i$) such that

a. $u_i$ converges weakly in $H^1(B_3(0))$ to some $u$ and $u_i$ converges strongly to $u$ in $L^2(B_3(0))$.

b. Define

$$\Sigma = \bigcap_{0 < r < r_0} \left\{ x \in B_1(0) \mid \liminf_i \Theta_{u_i}(x, r) \geq \epsilon_0 \right\},$$

where $r_0$ is as in Lemma 2.10 and $\epsilon_0$ is the minimum of $\epsilon_0$ in Theorem 2.5 and Lemma 2.10. Then $\Sigma$ is a closed set and has finite $(n-2)$-packing content. The maps $u_i$ converge strongly in $H^1_{\text{loc}}(B_1(0) \setminus \Sigma, N) \cap C^{0,\alpha}_{\text{loc}}(B_1(0) \setminus \Sigma, N)$ to $u$.

c. $u$ satisfies equation (57) weakly with $F_\infty$.

d. The Radon measures $\mu_i = F_i(|\nabla u_i|^2) dx$ on $B_1(0)$ converge weakly as Radon measures to $\mu$,

$$\mu_i \rightharpoonup \mu.$$
Remark 2.12. By Fatou’s Lemma

\[ \mu = F_\infty(|\nabla u|^2)dx + \nu \]

where \( \nu \) is a non-negative measure on \( B_1(0) \) which is supported on \( \Sigma \),

\[ \Sigma = \text{spt} \nu \cup \mathcal{D}(u). \]

Define

\[ \theta_{\mu_i}(x_0, r) = r^{2-n}\mu_i(B_r(x_0)), \]
\[ \Theta_{\mu_i}(x_0, r) = \theta_{\mu_i}(x_0, r) + h_i(r), \]

where \( h_i \) is as in Theorem 2.3. Then

\[ \theta_{\mu_i}(x_0, r) \to \theta_{\mu}(x_0, r) = \theta_{\mu_{\infty}}(x_0, r) + \theta_{\nu}(x_0, r) \]
\[ \Theta_{\mu_i}(x_0, r) \to \Theta_{\mu}(x_0, r) \]

where \( \mu_{\infty} = F_\infty(|\nabla u|^2)dx \). Therefore we have

\[ \Sigma = \bigcap_{0 < \epsilon_0} \left\{ x \in B_1(0) \mid \liminf_i \Theta_{\mu_i}(x, r) \geq \epsilon_0 \right\} \]
\[ = \left\{ x \in B_1(0) \mid \Theta_{\mu}(x) \geq \epsilon_0 \right\} \]
\[ = \left\{ x \in B_1(0) \mid \theta_{\mu}(x) \geq \epsilon_0 \right\}. \]

Here \( \Theta_{\mu}(x, r) \) is monotone increasing with respect to \( r \) and

\[ \Theta_{\mu}(x) = \lim_{r \to 0} \Theta_{\mu_i}(x, r) = \lim_{r \to 0} \theta_{\mu_i}(x, r). \]

Note that we do not know if \( \theta_{\mu_i}(x, r) \) is monotone increasing with respect to \( r \) but its limit exists as \( r \) goes to zero.

The proof of the above proposition is similar to Proposition 2.7 in [NV16]. The key point in the proof of the above proposition is the following lemma and we leave the rest of the proof to the reader.

Lemma 2.13. Let \( u_i \) and \( F_i \) be as above. Let \( \Theta_{\mu_i}^F = \Theta_{\mu_i}^F \leq \epsilon_0 \) where \( \epsilon_0 \) is the same as in Theorem 2.5. If \( u_i \to u \) in \( H^1(B_1(0), N) \) then \( u_i \) converges strongly to \( u \) in \( H^1(B_1(0), N) \) and \( u \) satisfies

\[ \text{div} \left( F_i'(\|\nabla u_i\|^2)\nabla u_i \right) - F_i'(\|\nabla u_i\|^2)A(u)(\nabla u_i, \nabla u) = 0 \]

on \( B_1(0) \), in the distributional sense.

Proof. First by Theorem 2.5, we have that \( |u_i|_{C^0(B_1(0))} \leq C \), with a uniform bound independent of \( i \). Since \( N \) is a compact manifold, we also have that \( |u_i|_{L^\infty(B_1(0))} \) is uniformly bounded. Thus \( u_i \) converges to \( u \) in \( C^{0,\alpha/2}(B_1(0)) \). For the strong \( L^2 \) convergence we show that

\[ \int_{B_1(0)} |\nabla(u_i - u)|^2 F_i'(\|\nabla u_i\|^2)\zeta \to 0 \]

for any \( \zeta \in C^\infty_c(B_1(0)) \). We have

\[ \int_{B_1(0)} |\nabla(u_i - u)|^2 F_i'(\|\nabla u_i\|^2)\zeta \]
\[ = \int (\nabla(u_i - u), \nabla u_i)F_i'(\|\nabla u_i\|^2)\zeta + \int (\nabla(u_i - u), \nabla u)F_i'(\|\nabla u_i\|^2)\zeta. \]
The second integral converges to 0 because of the uniform $C^1$-norm bound on $F_i$ and since $u_i$ weakly converges to $u$ in $H^1(B_3(0))$. Further, we have
\[
\int \langle \nabla (u_i - u), \nabla u_i \rangle F_i'(|\nabla u_i|^2) \zeta \leq \int \langle u_i - u, \text{div}(F_i'(|\nabla u_i|^2) \nabla u_i) \rangle \zeta + C \int \langle u_i - u, \nabla u_i \cdot \nabla \zeta \rangle F_i'(|\nabla u_i|). \tag{59}
\]
This term also converges to zero by the fact $u_i - u$ converges to zero in $L^\infty$ and since
\[
\left| \int \text{div}(F_i'(|\nabla u_i|^2) \nabla u_i) \right| = \left| F_i'(|\nabla u_i|^2) A(u_i)(\nabla u_i, \nabla u_i) \right| \leq C |A|_{L^\infty} \int |\nabla u_i|^2. \tag{60}
\]
To see that $u$ satisfies (21), note that for every $\zeta$ in $C^\infty_c(B_1(0))$
\[
\int F_i'(|\nabla u_i|^2) (\nabla u_i, \nabla \zeta) = \lim_{i \to \infty} \int F_i'(|\nabla u_i|^2) (\nabla u_i, \nabla \zeta) = \lim_{i \to \infty} \int F_i'(|\nabla u_i|^2) A(u_i)(\nabla u_i, \nabla u_i) \zeta = \int F_i'(|\nabla u_i|^2) A(u)(\nabla u, \nabla u) \zeta. \tag{61}
\]
□

2.3.1. Tangent map. Let $u \in H^1(B_3(0), N)$ be a stationary $F$-harmonic map. Define the map $u_{x,\lambda}(y) = u(x + \lambda y)$ for $x \in B_1(0)$ and $\lambda \leq 1$. Then the map $u_{x,\lambda}$ satisfies the following equations:
\[
\int [\text{div}(F_i'(|\nabla u_{x,\lambda}|^2) \nabla u_{x,\lambda}) - F_i'(|\nabla u_{x,\lambda}|^2) A(u_{x,\lambda})(\nabla u_{x,\lambda}, \nabla u_{x,\lambda})] \zeta = 0
\]
\[
\int F_i'(|\nabla u_{x,\lambda}|^2) \text{div}(\zeta) - 2F_i'(|\nabla u_{x,\lambda}|^2) (u^*_{x,\lambda} h)_{\alpha\beta} \nabla^\alpha \zeta^\beta = 0 \tag{62}
\]
where $F_i(x) = \lambda^2 F(i \frac{x}{\lambda})$ and so $F_i'(x) = F'(i \frac{x}{\lambda})$. Note that the corresponding $G_\lambda$ for $F_\lambda$ satisfies Assumption B. Then $\theta^1$ and $\Theta^1$ for the function $F_\lambda$ will be as follows
\[
\theta^1_{x,\lambda}(x_0, r) = r^{2-n} \int_{B_r(x_0)} F_\lambda(|\nabla u|^2) dx,
\]
\[
\Theta^1_{x,\lambda}(x_0, r) = \theta^1_{x,\lambda}(x_0, r) + h_\lambda(r).
\]
One can easily check that $h_\lambda(r) = h(\lambda r)$, and thus
\[
\theta^1_{x,\lambda}(0, r) = \theta(u(x, \lambda r)),
\]
\[
\Theta^1_{x,\lambda}(0, r) = \Theta(u(x, \lambda r)).
\]
By the monotonicity formula for $\theta^1$ we have $\int_{B_r(0)} F_\lambda(|\nabla u_{x,\lambda}|^2)$ is uniformly bounded.

Therefore there exist a subsequence of $u_{x,\lambda}$ (denoted again by $u_{x,\lambda}$) which converges weakly in $H^1(B_1(0))$ to a map $u_\star$ as $\lambda$ goes to zero. We have
\[
\lim_{\lambda \to 0} F_\lambda(x) = \lim_{\lambda \to \infty} F'(t)x = F'(\infty)x,
\]
\[
\lim_{\lambda \to 0} h_\lambda(r) = 0.
\]
For a measure $\mu$ we define $\mu_{x,\lambda}(A) = \lambda^{n-2} \mu(x + \lambda A)$. For $\mu = F(|\nabla u|^2)dx$, then we have
\[
\mu_{x,\lambda} = F_\lambda(|\nabla u_{x,\lambda}|^2)dx \to \mu_\star = F'(\infty)|\nabla u_{\star}|^2 dx + v_\star
\]
\[ \theta_{\mu_\ast}(0, r) \to \theta_{\mu_\ast}(0, r) = \theta_{\mu_{0\ast}}(0, r) + \theta_{1, \ast}(0, r) \]

where \( \mu_{0\ast} = F'(\infty)|\nabla u|^2 \, dx \). Note that \( \Theta_{\mu_\ast}(0, r) = \Theta_{\mu_\ast}(0, r) \) and therefore \( \theta_{\mu_\ast}(0, r) \) is monotone increasing with respect to \( r \). We further have

\[ \theta_{\mu_\ast}(0, r) = \lim_{r \to 0} \theta_{\mu_\ast}(x, r) = \theta_{\mu_\ast}(x). \]  (63)

We call \( u_\ast \) a tangent map for \( u \) at \( x \) and we have the following result.

**Lemma 2.14.** The tangent map \( u_\ast \) satisfies the following properties:

a. \( u_\ast \) is homogeneous, i.e. \( (u_\ast)_0, 1 = u_\ast \).

b. \( u_\ast \) is a weakly harmonic map.

c. The measures \( \mu_\ast \) and \( \nu_\ast \) are homogeneous measures.

**Proof.** For Part a, the 0-homogeneity of the tangent map is because by monotonicity formula (31) we have

\[ \int_{B_r(0) \setminus B_1(0)} 2r^{2-n} F'(|\nabla u_{x, \ast}|^2) \left[ \frac{\partial u_{x, \ast}}{\partial r} \right]^2 \leq \Theta^1_{u_{x, \ast}}(0, 1) - \Theta^1_{u_{x, \ast}}(0, t) \]  (64)

and therefore

\[ \int_{B_r(0) \setminus B_1(0)} 2r^{2-n} F'(|\nabla u_{x, \ast}|^2) \left[ \frac{\partial u_{x, \ast}}{\partial r} \right]^2 \leq \theta_{\mu_\ast}(0, 1) - \theta_{\mu_\ast}(0, t) = 0. \]  (65)

This shows that \( \frac{\partial u_{x, \ast}}{\partial r} = 0 \) for almost every \( r \). Part b is obtained by Proposition 2.11. Part c follows by a similar argument to the one in [Lin99], Lemma 1.7 (ii).

3. **GENERAL CASE** \( F(x, u, |\nabla u|^2) \)

In this section we consider the general case

\[ E(u) = \int_{\Omega} F(x, u, |\nabla u|^2) \]

for \( u \in H^1(\Omega, N) \). Recall that the Euler-Lagrange equation with respect to this energy functional is

\[ -\int F_{x_i}(x, u, |\nabla u|^2) \zeta^i + \int F_p(x, u, |\nabla u|^2) \left[ \langle \nabla_i u, \nabla_j \zeta \rangle - A(u)(\nabla u, \nabla u) \zeta \right] = 0, \]

and the stationary equation related to this energy functional is

\[ -\int F_{x_i}(x, u, |\nabla u|^2) \zeta^i + 2 \int F_p(x, u, |\nabla u|^2)(\nabla_i u, \nabla_j \zeta) \nabla^i \zeta^j - F(x, u, |\nabla u|^2) \text{div}(\zeta) = 0. \]

Again by considering \( \zeta \) as in Section 2, for \( q \in \Omega \) and \( r > 0 \) such that \( B_r(q) \subset \Omega \) we have

\[ \frac{d}{dr} \left( r^{2-n} \int_{B_r(x_0)} F(x, u, |\nabla u|^2) \right) + 2r^{1-n} \int_{B_r(x_0)} e(x, u, |\nabla u|^2) - r^{1-n} \int_{B_r(x_0)} F_p(x, u, |\nabla u|^2)(x_l - q_l) \]

\[ = 2r^{2-n} \int_{\partial B_r(x_0)} F_p(x, u, |\nabla u|^2) \frac{\partial u}{\partial r}^2, \]  (66)

where

\[ e(x, u, |\nabla u|^2) = F_p(x, u, |\nabla u|^2) |\nabla u|^2 - F(x, u, |\nabla u|^2). \]  (67)
By a similar argument to that in the special case $F = F(|\nabla u|^2)$, the ellipticity condition in Assumption A imposes the following conditions on $F$:

$$\frac{2^B - 1}{nq} \leq F_p(x, z, p) \leq \frac{2B}{nq},$$

$$\frac{B - 1}{nq} \leq F_{pp}(x, z, p) \leq \frac{B - 1}{nq}.$$  \hfill (68)

### 3.1. Monotonicity formula for the general case.

We use a similar argument to that in Lemma 2.1 and Theorem 2.3 to prove our monotonicity formula for the general case.

**Lemma 3.1.** The error term $e$ satisfies the following Jensen-type inequality

$$\int_{B_r(x_0)} e(x, u, |\nabla u|^2) dx \leq J \left( \int_{B_r(x_0)} |\nabla u|^2 dy \right)$$

for any map $u \in H^1(\Omega, \mathbb{N})$ and $B_r(x_0) \subset \Omega$, where the function $J$ is

$$J(y) = \tilde{E}(y) + y \int_y^\infty \frac{E(t)}{t} dt$$

with $\tilde{E}(y) = \sup_{x, z} e(x, z, p)$ and $E(y) = \sup_{x, z} e_p(x, z, y)$. Furthermore

$$\int_0^1 y J(y^{-2}) dy < \infty.$$

**Proof.** For every $x \in B_r(x_0)$ we have

$$e(x, u(x), f(x)) - e(x, u(x), \tilde{f}) = \int_{\tilde{f}}^{f(x)} e_t(x, u, t) dt$$

where $f = |\nabla u|^2$ and $\tilde{f} = \frac{1}{\text{vol}(B_r(x_0))} \int_{B_r(x_0)} |\nabla u|^2 dx$. Define the set

$$U = \left\{ x \in B_r(x_0) \mid f(x) \geq \int f \right\},$$

$$U_t = \left\{ x \in B_r(x_0) \mid f(x) \geq t \right\}.$$
By averaging (69) over ball $B_r(x_0)$
\[
\int e(x, u(x), f(x))dx - \int e(x, u(x), \bar{f}) = \int \int_{f} e_t(x, u, t)dt dx
\]
\[
\leq \frac{1}{\text{vol}(B_r(x_0))} \int_{U_r} \int_{f} e_t(x, u, t)dt dx
\]
\[
= \frac{1}{\text{vol}(B_r(x_0))} \int_{f} \int_{U_r} e_t(x, u, t)dy dt
\]
\[
\leq \frac{1}{\text{vol}(B_r(x_0))} \int_{f} \int_{U_r} e_t(x, u, t)\frac{f(x)}{t} dx dt
\]
\[
\leq \frac{1}{\text{vol}(B_r(x_0))} \int_{B_r(x_0)} f(x) \int_{f} \frac{e_t(x, u, t)}{t} dx dt
\]
\[
\leq \tilde{f} \int_{f} \frac{E(t)}{t} dt,
\]
where $E(t) = \sup_{x, u} e_t(x, u, t)$. Define $\tilde{E}(t) = \sup_{x, u} e(x, u, t)$. Therefore
\[
\int e(x, u(x), f(x))dx \leq \int e(x, u(x), \bar{f}) + \tilde{f} \int_{f} \frac{E(t)}{t} dt
\]
\[
\leq \tilde{E}(\tilde{f}) + \tilde{f} \int_{f} \frac{E(t)}{t} dt.
\]
Finally we have
\[
\int e(x, u, |\nabla u|^2) \leq J(\int |\nabla u|^2),
\]
where $J(y) = \tilde{E}(y) + y \int_{y}^{\infty} \frac{E(t)}{t} dt$. The second part of this theorem follows from by Lemma 2.1.

**Theorem 3.2.** Let $u$ be a stationary $F$-harmonic map in $H^1(\Omega, N)$. Then there exists $A = A(n, N, B, C, D)$ such that
\[
\frac{d}{dr} \left( \varepsilon^2 r^2 \mathcal{E}(r) + h(r) \right) \geq \int_{\partial B_r(x_0)} F_\rho(x, u, |\nabla u|^2) \frac{\partial u}{\partial r}^2,
\]
where $\mathcal{E}(r) = \int_{B_r(x_0)} F(x, u, |\nabla u|^2) dx$ and $h(r) = 2 \int_0^r t J(2c_e A^2 r^{-2}) dt$ with $c_e = nqB/2$.

**Proof.** Recall that by (66) we have
\[
\frac{d}{dr} \left( r^{2-n} \int_{B_r(x_0)} F(x, u, |\nabla u|^2) \right) + 2r^{1-n} \int_{B_r(x_0)} e(x, u, |\nabla u|^2) - r^{1-n} \int_{B_r(x_0)} F_{\xi}(x, u, |\nabla u|^2) \xi_l
\]
\[
= 2r^{2-n} \int_{\partial B_r(x_0)} F_\rho(x, u, |\nabla u|^2) \frac{\partial u}{\partial r}^2
\]
where $e(x, u, |\nabla u|^2) = F_\rho(x, u, |\nabla u|^2) |\nabla u|^2 - F(x, u, |\nabla u|^2)$. By Assumption B we have
\[
\left| -r^{1-n} \int_{B_r(x_0)} F_{\xi}(x, u, |\nabla u|^2) \xi_l \right| \leq \frac{\theta}{c_e} r^{2-n} \int_{B_r(x_0)} F(x, u, |\nabla u|^2).
\]
Therefore
\[ \frac{d}{dr} \left( e^{\frac{\kappa}{r^2}} r^2 \int_{B_i(x_0)} F(x, u, |\nabla u|^2) \right) + 2r J \left( c_e \int_{B_i(x_0)} F(x, u, |\nabla u|^2) \right) \geq 0. \] (72)

The rest of the proof follows by the exact same argument as in the proof of Theorem 2.3. \( \square \)

3.2. \( \epsilon \)-regularity for the general case. In this subsection we prove Theorem 1.1. Roughly speaking, we show that for a map \( u \in H^1(\Omega, N) \) which satisfies (5) and (6) where the energy (2) is small, \( u \) is Hölder continuous. The argument here will be a slight generalization of argument in Subsubsection 2.2.2. First we generalize Lemma 2.8.

**Lemma 3.3.** Suppose \( u \in H^1(B_r(x_0), N) \) satisfies equation (5) with
\[ \int_{B_i(x_0)} F_p(x, u, |\nabla u|^2) \leq \epsilon. \]

Then there exists a constant \( C = C(n, N, F, \Lambda) \) such that the following holds:
\[ (\kappa r)^{1-n} \int_{B_{\kappa r}(x_0)} |\nabla u|^p F_p(x, u, |\nabla u|^2) dx \leq C \kappa^{1-n} \left( u_{BMO_r(x_0)}(\epsilon + C \sqrt{\epsilon}) \right) \]
\[ + C \kappa r^{1-n} \int_{B_r(x_0)} |\nabla u|^p F_p(x, u, |\nabla u|^2) dx + C \kappa^{1-n} r \]

for any \( \kappa \in (0, 1) \).

**Proof.** We follow a similar argument to that in Subsubsection 2.2.2 and we only mention the changes we need to consider in this general case. First there exists an orthonormal tangent frame \( (e_1, \ldots, e_m) \) along \( u \) which satisfies (39) and the forms \( \gamma_{ij} \) satisfy (41). For the 1-form \( \omega_i \) we get
\[ \delta \omega_i = \omega_j \cdot \gamma_{ij} + \frac{1}{\phi} (F_{\zeta}(x, u, |\nabla u|^2), e_i), \]
\[ d\omega_i = \omega_j \wedge \gamma_{ij}, \]

where \( \phi = F_p(x, u, |\nabla u|^2) \). Following the argument in the proof of Lemma 2.8 we then have
\[ \delta \alpha_i = \delta \omega_i = \omega_j \cdot \gamma_{ij} + \frac{1}{\phi} (F_{\zeta}(x, u, |\nabla u|^2), e_i), \]
\[ d\beta_i = d\omega_i = \omega_j \wedge \gamma_{ij}. \]

The estimate for \( \beta_i \) will remain similar to that in Subsubsection 2.2.2 but the estimate for \( \alpha_i \) changes. For estimate on \( \alpha_i \), define \( \tilde{\alpha}_i^1 \) similar to that in Subsubsection 2.2.2 and let \( \tilde{\alpha}_i^2 \) satisfy the following.
\[
\begin{cases}
\Delta \tilde{\alpha}_i^2 = \Delta \tilde{\alpha}_i = \omega_j \cdot \gamma_{ij} + \frac{1}{\phi} (F_{\zeta}(x, u, |\nabla u|^2), e_i), & \text{in } B_{r/2}(x_0) \\
\tilde{\alpha}_i^2 = 0 & \text{on } \partial B_{r/2}(x_0).
\end{cases}
\] (74)

While the estimate for \( \tilde{\alpha}_i^1 \) will remain the same as in the proof Lemma 2.8, for \( \tilde{\alpha}_i^2 \) we have
\[ \int_{B_{r/2}(x_0)} |\nabla \tilde{\alpha}_i^2|^2 \phi dx \leq C[u]_{BMO_r(x_0)} \left( \int_{B_r(x_0)} |\nabla u|^2 \phi dx \right)^{1/2} \left( \int r^2 \int_{B_r(x_0)} |\nabla u|^2 \phi dx + C r^2 \right)^{1/2} \]
\[ + \int_{B_r(x_0)} \langle F_{\zeta}(x, u, |\nabla u|^2), e_i \rangle \psi_i dx. \] (75)
By (51) and Assumption A we have
\[
(kr)^{1-n} \int_{B_r(x_0)} |\nabla u| \phi dx \leq C_1 r^{1-n} |u|_{\text{BMO}_0(B_r(x_0))}(\epsilon + C_2 \sqrt{\epsilon}) + C_3 kr^{1-n} \int_{B_r(x_0)} |\nabla u| \phi dx \\
+ \kappa^{1-n} \frac{\theta}{C_e} r^{2-n} \int_{B_r(x_0)} |\nabla u|^2 \phi dx.
\]

\[\square\]

Now we prove Theorem 1.1.

**Proof of Theorem 1.1.** Define
\[
M(u, x_0, r) = \sup_{B_s(x_0) \subset B_r(x_0)} \left( s^{1-m} \int_{B_s(x_0)} |\nabla u| \phi dx \right)
\]

By a similar argument to that in Lemma 2.9 we can choose \( \epsilon_0 \) and \( \kappa \) small enough such that
\[
M(u, x_0, kr) \leq \frac{1}{2} M(u, x_0, r) + \frac{1}{2}.
\]

Applying the Lemma 3.3 repeatedly we have
\[
M(u, x_1, s) \leq C s^a
\]

for \( x_1 \in B_{r/2}(x_0) \) and all \( s \in (0, r/2] \) where \( a \) and \( C \) do not depend on \( x_1 \) and \( s \). In particular
\[
\sup_{x_1 \in B_{r/2}(x_0) \cap [0, r/2]} \sup_{0 < s \leq r/2} \left( s^{1-m-a} \int_{B_s(x_1)} |\nabla u| \phi dx \right) < \infty.
\]

Since \( \phi \) is bounded and by the Morrey decay lemma, \( u \in C^{0,\alpha}(B_{r/2}(x_0)) \).

\[\square\]

3.3. **Compactness for the general case.** In this subsection we discuss the proof of Proposition 2.11 and Lemma 2.14. We do not state these results again here. We only consider sequences \( u_i \) which satisfy
\[
\int_{B_1(0)} F_i^j(x, u_i, |\nabla u_i|^2) < \Lambda, \tag{76}
\]

where \( F_i^j \) satisfies Assumption A, and
\[
-F_i^j(x, u_i, |\nabla u_i|^2) + \text{div} \left( F_i^j(x, u_i, |\nabla u_i|^2) \nabla u_i \right) - F^p_i(x, u_i, |\nabla u_i|^2)A(u_i)(\nabla u_i, \nabla u_i) = 0 \tag{77}
\]

\[
F^i_{x_i}(x, u_i, |\nabla u_i|^2) + \nabla^a \left( F^i(x, u_i, |\nabla u_i|^2) g_{a\beta} - 2 F^i_p(x, u_i, |\nabla u_i|^2) u^*_{i} h_{a\beta} \right) = 0 \tag{78}
\]
in the weak sense.

The only main change in the proof of Proposition 2.11 for the general case compared to the special case happens in the proof of Lemma 2.13. Here we present the changes which we should consider in the proof of Lemma 2.13 for the general case.

For the strong \( L^2 \) convergence we show that
\[
\int_{B_{1}(0)} |\nabla(u_i - u)|^2 F_i^j(x, u_i, |\nabla u_i|^2) \zeta \to 0
\]

for any \( \zeta \in C_c^\infty(B_1(0)) \). The argument is the same as in the proof of Lemma 2.13 except in (60) we have
\[
\left| \int \text{div} (F_i^j(x, u_i, |\nabla u_i|^2) \nabla u_i) \right| = F^j_i(x, u_i, |\nabla u_i|^2) |A(u_i)(\nabla u_i, \nabla u_i)| + F^j_{x_i}(x, u_i, |\nabla u_i|^2) \leq C |A|_{L^\infty} + \theta \int |\nabla u_i|^2.
\]
Considering the properties of tangent maps in Lemma 2.14, first notice that the maps $u_{x,t}$ in the general case satisfy
\[
-F^3_{\partial}(x, u_{x,t}, |\nabla u_{x,t}|^2) + \text{div} \left(F^3_p(x, u_{x,t}, |\nabla u_{x,t}|^2) \nabla u_{x,t} \right) - F^3_{\partial}(x, u_{x,t}, |\nabla u_{x,t}|^2)A(u_{x,t})(\nabla u_{x,t}, \nabla u_{x,t}) = 0
\]
\[
F^3_{\partial}(x, u_{x,t}, |\nabla u_{x,t}|^2) + \nabla^p \left(F^3(x, u_{x,t}, |\nabla u_{x,t}|^2) \right)g_{\alpha\beta} - 2F^3_{\partial}(x, u_{x,t}, |\nabla u_{x,t}|^2)u_{x,t}^*h_{\alpha\beta} = 0
\]
in the weak sense, where
\[
F^3(x, z, p) = \lambda^2 F(\lambda x, z, \frac{p}{\lambda^2}).
\]
We also have $\theta^3_{\alpha}(x_0, r) = \frac{C}{r} r^{2-n} \int_{B_r(x_0)} F^3(x, u, |\nabla u|^2)$. The limit function is given by
\[
\lim_{t \to 0} F_{\lambda}(x, z, p) = \lim_{t \to 0} F_{p}(0, z, t)p = F_{p}(0, z, \infty)p.
\]
Finally with the argument similar to the one in Lemma 2.14, we can conclude $u_\ast, \mu_\ast$ and $v_\ast$ are homogeneous and that $u_\ast$ weakly satisfies
\[
-F^3_{\partial}(0, u_\ast, \infty) \nabla^k u_\ast |\nabla u_\ast|^2 + \text{div} \left(F^3_p(0, u_\ast, \infty) \nabla u_\ast \right) - F^3_p(0, u_\ast, \infty)A(u_\ast)(\nabla u_\ast, \nabla u_\ast) = 0.
\]

4. Stratification of the Singular Set

In this section we prove Theorem 1.2 and Theorem 1.3. The proof of Theorem 1.2 is very similar to the proof of Theorem 3.1 in [NV16]. We explain the necessary background material for the proof of Theorem 1.2 in Appendix A and we use the notations from Appendix A. We first recall the definitions of quantitative strata and their properties (see [CN13b] and [NV17]).

4.1. Quantitative singular set. Here we give the definition of $k$-th singular strata and its quantitative version.

**Definition 4.1.** Given a map $h \in H^1(\mathbb{R}^n, N)$, we say that
a. $h$ is homogeneous with respect to the point $p$ if $h(p + \lambda v) = h(p + v)$ for all $\lambda > 0$ and $v \in \mathbb{R}^n$.
b. $h$ is $k$-symmetric if it is homogeneous with respect to the origin and it has an invariant $k$-dimensional subspace, i.e., if there exists a linear subspace $V \subset \mathbb{R}^n$ of dimension $k$ such that $h(x + v) = h(x)$ for all $x \in \mathbb{R}^n$ and $v \in V$.

A map $h$ is 0-symmetric if and only if it is homogeneous with respect to the origin.

**Definition 4.2.** Given a map $u \in H^1(\Omega, N)$, we say that $B_r(x) \subset \Omega$ is $(k, \epsilon)$-symmetric for $u$ if there exists a $k$-symmetric function $h$ such that
\[
\int_{B_r(x)} |u_{x,y}(y) - h(y)|^2 \, dy \leq \epsilon.
\]

Now we define a stratification for the singular set $S(u)$ of a stationary map $u$ in $H^1(\Omega, N)$.

**Definition 4.3.** The $k$-th strata for $u$ which we denote by $S^k(u)$ is
\[
S^k(u) = \{x \in \Omega \mid \text{no tangent map at } x \text{ is } k\text{-symmetric} \}.
\]
Definition 4.4. Given a map $u \in H^1(\Omega, N)$, and $r, \epsilon > 0$ and $k \in \{0, \ldots, n\}$ we define the $k$-th $(\epsilon, r)$-stratification $S^k_{\epsilon, r}(u)$ by

$$S^k_{\epsilon, r}(u) = \{ x \in \Omega \mid \text{for no } r \leq s < 1, \ B_s(x) \text{ is } (k + 1, \epsilon)\text{-symmetric w.r.t. } u \}. \quad (81)$$

Note that $S^k_{\epsilon, r}(u)$ has the following property

$$k' \leq k, \ \epsilon' \geq \epsilon, \ r' \leq r \Rightarrow S^{k'}_{\epsilon', r'}(u) \subset S^k_{\epsilon, r}(u). \quad (82)$$

Using this fact we define the $k$th $\epsilon$-stratification by

$$S^k_{\epsilon}(u) = \bigcap_{r > 0} S^k_{\epsilon, r}(u) \quad (83)$$

$$= \{ x \in \Omega \mid \text{for no } 0 < r < 1, \ B_r(x) \text{ is } (k + 1, \epsilon)\text{-symmetric w.r.t. } u \}$$

Note that

$$S^k(u) = \bigcup_{\epsilon > 0} S^k_{\epsilon}(u). \quad (84)$$

See Lemma 4.3 in [NV16] for the proof of the above equalities.

4.2. Proof of Theorem 1.2. First we prove the Minkowski estimates (16) and (17).

Proof of Minkowski estimates. We will give the proof of Theorem 1.2 only for the sets $S \subset S^k_{\epsilon, r}$. Since $\delta$ is a constant depending on $(n, N, \Lambda, F, \epsilon)$ this does not effect the conclusion for $S \subset S^k_{\epsilon, r}$ except for the size of the constant $C'$. Therefore we will show

$$\text{vol}\left( B_r \left( S^k_{\epsilon, r}(u) \right) \cap B_1(0) \right) \leq C'_r r^{n-k}. \quad (85)$$

where $\delta = \min \{ \delta, \bar{\delta} \}$. We put $S = S^k_{\epsilon, r}(u) \cap B_1(0)$ and by the monotonicity formula for $\bar{\Theta}$ we have

$$\forall x \in B_1(0) \text{ and } \forall r \in [0, 1], \ \bar{\Theta}(x, r) \leq \Lambda' = c(n)\Lambda + c(n, C). \quad (86)$$

Let $E = \sup_{x \in \bar{S}} \bar{\Theta}(x, 1) \leq \Lambda'$. We refine the covering in Lemma A.15 through an inductive process to get the following covering on $S$

$$S \subset \bigcup_{x \in \hat{C}} B_{r_x}(x) \text{ with } \sum_{x \in \hat{C}} r_x^k \leq c(n)C_F(n), \quad (87)$$

$$\forall x \in \hat{C}, \ r_x \leq r \text{ or } \forall y \in S \cap B_{2r_x}(x), \ \bar{\Theta}(y, r_x) \leq E - i\delta. \quad (88)$$

First step of induction. This will follow by Lemma A.15.

Inductive step. Assume now that we have a covering which satisfies (87) and (88) for $i = j$. We leave the balls with property $r_x \leq r$ as they are and we use a rescaled version of the Lemma A.15 to cover again the balls centered at $\hat{C}_j$ which satisfy the drop condition $\bar{\Theta}(y, r_x) \leq E - j\delta$ for all $y \in B_{2r_x}(x) \cap S$. By Lemma A.15 we have

$$S \cap B_{r_x}(x) \subset \bigcup_{y \in \hat{C}_j} B_{r_y}(y) \text{ and } \sum_{x \in \hat{C}_j} r_y^k \leq C_F(n)r_x^k,$$

where for $y \in \hat{C}_j$ either $r_y = r$ or

$$\forall z \in B_{2r_y}(y) \cap S, \ \bar{\Theta}(z, r_y/10) \leq E - (j + 1)\delta.$$
For the latter case we again cover again the ball $B_{r_{y}}(y)$ by the minimal set of balls of radius $\rho(n)r_{y}$, $\left\{ B_{\rho(n)r_{y}}(z_{i}) \right\}_{i=1}^{c(n)}$. We then get

$$\mathcal{C}_{j+1} \subseteq \bigcup_{x \in \mathcal{C}'} \bigcup_{y \in \mathcal{C}_{j}'} \bigcup_{i=1}^{c(n)} z_{i}.$$

and so we have

$$\sum_{x \in \mathcal{C}'} r_{x}^{k} \leq \sum_{x \in \mathcal{C}'} \sum_{y \in \mathcal{C}_{j}'} r_{y}^{k} \leq (c(n)C_{F}(n))^{j+1}.$$

**Conclusion.** We continue the induction at most $\lceil E/\delta \rceil + 1$ steps. Then we will have $C'_{e} = (c(n)C_{F}(n))^{\lceil E/\delta \rceil + 1}$. The proof of (17) then follow by (16).

We now prove $S_{e}^{k}$ and $S^{k}$ are rectifiable.

**Proof of Rectifiability.** To prove that $S_{e}^{k}$ is rectifiable we use Theorem A.10 and Lemma A.12. Fix $S \subseteq S_{e}^{k}$. For each $\delta > 0$ there exists a subset $E_{\delta} \subseteq S$ with $\mathcal{H}^{k}(E_{\delta}) \leq \delta \mathcal{H}^{k}(S)$ such that $F_{\delta} = S \setminus E_{\delta}$ is $k$-rectifiable. To see this first note that by monotonicity formula, for every $\delta$ there exist $\bar{r}$ and measurable subset $E \subseteq S$ with the following property:

$$\mathcal{H}^{k}(E) \leq \delta \mathcal{H}^{k}(S) \tag{89}$$

$$\forall x \in F_{\delta} = S \setminus E_{\delta}, \ \Theta(x, 10\bar{r}) - \Theta(x, 0) \leq \delta. \tag{90}$$

See [NV16] for the proof of this statement. We cover $F_{\delta}$ by balls $B_{r}(x)$ and then on $F_{\delta} \cap B_{r}(x)$ we apply Lemma A.12. This is possible because $F_{\delta} \subseteq S_{e}^{k}$ and in view of (90). For simplicity we renormalize the ball $B_{r}(x)$ to the unit ball $B_{1}(0)$. For all $x \in F_{\delta}$ and $s \leq 1$ and $\mu = \mathcal{H}^{k} |_{F_{\delta}}$ we have

$$\left( \beta_{2\mu}^{k}(x, s) \right)^{2} \leq C_{L} s^{-k} \int_{B_{1}(x)} W_{s}(y) d\mu(y).$$

We integrate the above and by the fact that $\mathcal{H}^{k}(S \cap B_{r}(x)) \leq C_{e} r^{k}$ and for $p \in B_{1}(0)$, $s \leq r \leq 1$ we have

$$\int_{B_{1}(p)} \left( \beta_{2\mu}^{k}(x, s) \right)^{2} d\mu(x) \leq C_{L} s^{-k} \int_{B_{1}(p)} \int_{B_{1}(x)} W_{s}(y) d\mu(y) d\mu(x) \leq C_{L} C_{e} \int_{B_{1}(p)} W_{s}(x) d\mu(x).$$

Integrating again we get

$$\int_{B_{1}(p)} \int_{0}^{\gamma_{2\mu}^{k}(x, s)} \left( \beta_{2\mu}^{k}(x, s) \right)^{2} d\mu(x) \frac{ds}{s} \leq C_{L} C_{e} \int_{B_{1}(p)} \left[ \Theta(x, 8r) - \Theta(x, 0) \right] d\mu(x) \leq c(n)C_{L} C_{e}^{2} \delta(2r)^{k}.$$

Choosing $\delta \leq \frac{\delta_{e}}{C_{L} C_{e}^{2} c(n)}$ prove the $k$-rectifiability of $F_{\delta}$. Sending $\delta$ to zero we get the rectifiability of $S_{e}^{k}$. Since $S^{k} = \bigcup_{i} S_{1/i}^{k}(u)$ we conclude that $S^{k}(u)$ is rectifiable.
4.3. **Minimizing maps.** In this subsection we study the singularities of minimizing F-harmonic maps and prove Theorem 1.3. first we recall the definition of minimizing F-harmonic maps.

**Definition 4.5.** We say a map \( u \in H^1(\Omega, N) \) is minimizing F-harmonic map, if for any ball \( B_r(p) \subset \Omega \) and for any \( w \in H^1(B_r(p), N) \) with \( w \equiv u \) in a neighborhood of \( \partial B_r(p) \),

\[
\int_{B_r(p)} F(x,u,|\nabla u|^2) \leq \int_{B_r(p)} F(x,w,|\nabla w|^2).
\]

Note that a minimizing F-harmonic map is a stationary F-harmonic map. In what follows, we develop the quantitative version of the \( \varepsilon \)-regularity theorem, Theorem 1.1, which combined with Theorem 1.2 leads to the proof of Theorem 1.3.

4.3.1. **Quantitative \( \varepsilon \)-regularity.** First we define the regularity scale \( r_u(x) \) of a map \( u : \Omega \to N \) at a point \( x \), which measures how far \( x \) is from the singular set of \( u \). Define \( r_{0,u}(x) \) to be the maximum of \( r > 0 \) such that \( u \) is \( C^{0,\alpha} \) on \( B_r(x) \).

**Definition 4.6.** Define the regularity scale \( r_u(x) \) by

\[
 r_u(x) = \max \left\{ 0 < r \leq r_{0,u}(x) \mid r^\alpha \sup_{p,q \in B_r(x)} \frac{|u(p) - u(q)|}{|p-q|^\alpha} \right\},
\]

where \( \alpha \) is as in Theorem 1.1.

Note that \( r_u(x) \) is scale invariant. Now we are able to state the quantitative \( \varepsilon \)-regularity theorem minimizing F-harmonic maps. Let

\[
 H_\Lambda = \{ u \in H^1(B_3(0), N) \mid \int_{B_3(0)} F(x,u,|\nabla u|^2) < \Lambda \}.
\]

**Theorem 4.7.** Let \( u : H_\Lambda \) be a minimizing F-harmonic map. Then there exists \( \varepsilon(n,N,\Lambda,F) \) such that if \( B_r(p) \) is not \((n-2,\varepsilon)\)-symmetric, then

\[
 r_u(p) \geq r/2.
\]

The proof is similar to the proof of Theorem 2.4 in [CN13b]. For the sake of completeness we mention the steps of the proof as each step is an interesting result on its own.

**Proof.**

**Step 1.** Let \( u_i \in H_\Lambda \) be minimizing F-harmonic maps. Then \( u_i \) converges strongly in \( H^1(B_3(0), N) \) to a map \( u \) which is again minimizing F-harmonic map.

Arguing as in the case of classical minimizing harmonic maps (see for example Lemma 1 Section 2.9 in [Sim96]) we can show that \( u \) is minimizing. The strong \( L^2 \) convergence will follow by similar argument as the one in Proposition 4.6 in [SU82].

**Step 2.** For all \( \bar{\varepsilon} \), there exists \( \delta(n,N,\Lambda,\bar{\varepsilon}) \) such that if \( B_r(p) \subset B_3(0) \) is \((n-2,\delta)\)-symmetric for the map \( u \in H_\Lambda \), then \( B_r(p) \) is \((n,\bar{\varepsilon})\)-symmetric. Consequently we have

\[
 S^{n-3}_{\bar{\varepsilon},r}(u) \subset S^{n-1}_{\delta,r}(u).
\]

The proof is similar to the proof of Lemma 2.5 in [CN13b].
Step 3. Let \( u \in H_\Lambda \) be a minimizing \( F \)-harmonic map. Then there exists \( \epsilon_0(n,N,\Lambda,F) \), such that if there exists \( c \in N \) with
\[
\int_{B_r(p)} |u - c|^2 < \epsilon_0,
\]
then \( u \) is in \( C^{0,\alpha}(B_{r/2}(p)) \).

This step follows by a simple contradictory argument.

Now we are able to prove Theorem 4.7. We argue by contradiction. Suppose there exists a sequence of minimizing \( F \)-harmonic maps \( u_i \in H_\Lambda \) such that \( B_r(p) \) is \((n-2,1/2)\)-symmetric but \( r_{u_i}(p) < r/2 \). By Step 1 the sequence \( u_i \) converges strongly to a minimizing \( F \)-harmonic map \( u \) in \( H^1 \) and the ball \( B_r(p) \) is \((n-2,0)\)-symmetric for \( u \), and therefore by Step 2, \( B_r(p) \) is \((n,\epsilon_0)\)-symmetric. Finally by Step 3, \( u \in C^{0,\alpha}(B_{r/2}(p)) \) which is a contradiction. \( \square \)

Now we are able to prove Theorem 1.3.

Proof of Theorem 1.3. By Theorem 4.7 we know
\[ S(u) \cap B_1(0) \subset S_{\epsilon}^{n-3}(u). \]
Then by Theorem 1.2,
\[ \text{vol} \left( B_r(S(u)) \cap B_1(0) \right) \leq \text{vol} \left( B_r(S_{\epsilon}^{n-3}(u)) \right) < C_\epsilon r^3, \]
which shows that the \((n-3)\)-Minkowski dimension, and therefore the \((n-3)\)-Hausdorff dimension of \( S(u) \cap B_1(0) \) is finite. \( \square \)

Appendix A. Background for the Proof of the Theorem 1.2

In this part we explain the background material for the proof of Theorem 1.2 from [NV16] and for many details we refer the reader to [NV16]. Throughout this Appendix we assume that
\[ u \] is a stationary \( F \)-harmonic map and satisfies (15). (92)

Before we state the results in this section, we draw your attention to the following remark.

Remark A.1. In Proposition 2.7(4) in [NV16], the authors prove a so called unique continuation property. They use this property in the proof of technical results in Subsection A.1 to show
\[ \tilde{\Theta}_u(x,r) - \tilde{\Theta}_u(x,r/2) = 0 \] if and only if \( \frac{\partial u}{\partial t} = 0 \) for a.e. \( t \in (0,r] \). (93)

One can avoid the unique continuation property and use
\[ \tilde{\Theta}_u(x,r) = \int_0^\infty \tilde{\Theta}_u(x,s) \psi\left(\frac{s}{r}\right) \frac{ds}{r} \] (94)
instead of \( \tilde{\Theta}_u(x,r) \) in the proof of Theorem 1.2. Here \( \psi \) is a test function in \( C_c^\infty([0,1]) \) which is equal to 1 on \([\epsilon, 1 - \epsilon] \) for small enough \( \epsilon \). Note that
\[ \frac{d}{dr} \tilde{\Theta}_u(x,r) = \int_0^\infty \frac{d}{ds} \tilde{\Theta}_u(x,s) \psi\left(\frac{s}{r}\right) \frac{sd}{r} \] (95)
which shows that $\tilde{\Theta}_u(x,r)$ is also monotone in $r$. Moreover we have

$$\tilde{\Theta}_u(x,r) - \tilde{\Theta}_u(x,r/2) = 0 \text{ if and only if } \frac{du}{dt} = 0 \text{ for a.e. } t \in (0,r). \quad (96)$$

In the proof of Theorem 1.2 and without lose of generality we assume $\tilde{\Theta}$ satisfy (93).

A.1. **Quantitative symmetry.** Here we recall the adapted version of the quantitative rigidity theorem and the cone splitting theorem (see [CN13a]) in our context.

**Proposition A.2.** *(Quantitative rigidity)* Let $u$ satisfies (92). Then for every $\epsilon > 0$, there exist $\delta_0(n,N,F,\Lambda,\epsilon)$ such that if for some $x$ in $B_1(0)$

$$\tilde{\Theta}_u(x,r) - \tilde{\Theta}_u(x,\frac{r}{2}) < \delta_0,$$

then $B_r(x)$ is $(0,\epsilon)$-symmetric.

See the proof of Proposition 4.1 in [NV16]. This proposition says that if $\tilde{\Theta}(x, \cdot)$ is sufficiently pinched on two consecutive scales (i.e. $\tilde{\Theta}_u(x,r) - \tilde{\Theta}_u(x,r/2)$ is small enough), then $B_r(x)$ will be $(0,\epsilon)$-symmetric. We call the point $x$ a pinched point for $\tilde{\Theta}_u$. The following definitions express the quantitative version of linear independence.

**Definition A.3.** We say that $\{x_i\}_{i=0}^k$, $x_i \in B_1(0)$, $\rho$-effectively span the $k$-dimensional affine subspace

$$V = \text{Span}\{x_1 - x_0, \ldots, x_k - x_0\}$$

if for all $i = 1, \ldots, k$

$$x_i \notin B_{2\rho}(\text{Span}\{x_1 - x_0, \ldots, x_{i-1} - x_0\}).$$

**Definition A.4.** Given $K \subset B_1(0)$, we say $K$, $\rho$-effectively spans a $k$-dimensional subspace if there exist $\{x_0, \ldots, x_k\} \subset K$ that $\rho$-effectively spans a $k$-dimensional subspace.

The following theorem is a generalization of Proposition A.2 where we have $k+1$ distinct pinching points.

**Proposition A.5.** *(Cone splitting)* Let $u$ satisfies (92). Then for every $\epsilon, \rho > 0$, there exist $\delta_1(n,N,F,\Lambda,\epsilon,\rho)$ such that if for some $\{x_i\}_{i=0}^k \subset B_r(x)$ with $x$ in $B_1(0)$ and $0 < r \leq 1$ we have

a. $\{x_i\}_{i=0}^k$ $\rho$-effectively span a $k$-dimensional subspace $V$,

b. $\tilde{\Theta}(x_i,r) - \tilde{\Theta}(x_i,r/2) < \delta_1$ for all $i$,

then $B_r(x)$ is $(k,\epsilon)$-symmetric.

The proof is similar to that Proposition A.2. See also Proposition 4.6 and its proof in [NV16]

**Proposition A.6.** *(Quantitative dimension reduction)* Let $u$ satisfies (92). For $\rho, \epsilon > 0$ there exists $\delta_2(n,N,F,\Lambda,\epsilon,\rho)$ such that the following holds. Let

$$H = \left\{ y \in B_2(0) \mid \tilde{\Theta}(y,1) - \tilde{\Theta}(y,\rho) < \delta_2 \right\}.$$

If $H$ is $\rho$-effectively spanned by a $k$-dimensional subspace $V$, then

$$S_{\epsilon,\delta_2}^k(u) \subset B_{2\rho}(V).$$

See also Proposition 4.7 in [NV16]. The following theorem says that $\tilde{\Theta}$ is almost constant on the pinched points.
Lemma A.7. Let $u$ satisfies (92). Assume $\bar{\Theta}(y, 1) < E$ for all $y$ in $B_1(0)$. Then for $\rho, \eta > 0$ there exists $\delta_3(n, N, F, \Lambda, \eta, \rho)$ such that the following holds. If

$$H = \{ y \in B_1(0) \mid \bar{\Theta}(y, \rho) < E - \delta_3 \}.$$  

is $\rho$-effectively spanned by a $k$-dimensional subspace $V$, then

$$\forall x \in V \cap B_2(0), \bar{\Theta}(x, \rho) < E - \eta.$$  

Moreover if $k \geq n - 1$, then $E \geq \eta$.

See Lemma 4.10 and its proof in [NV16]. We also need the following technical lemma for the proof of Lemma A.13.

Lemma A.8. Let $u$ satisfies (92). For $\rho, \epsilon > 0$ there exists $\delta_4(n, N, F, \Lambda, \epsilon, \rho)$ such that if

$$\bar{\Theta}(0, 1) - \bar{\Theta}(0, 1/2) < \delta_4$$

and there exists $y$ in $B_3(0)$ such that

a. $\bar{\Theta}(y, 1) - \bar{\Theta}(y, 1/2) < \delta_4$,

b. for some $r \in [\rho, 2]$, $B_r(y)$ is not $(k + 1, \epsilon)$-symmetric.

Then $B_r(0)$ is not $(k + 1, \epsilon/2)$-symmetric.

A.2. Generalized Reifenberg theorem. In this part we recall two versions of Reifenberg’s theorem from [NV17]. First we define the Jones’ $\beta_2$ number.

Definition A.9. Let $\mu$ be a non-negative Radon measure on $B_3(0)$. For any $r > 0$ and $k \in N$, the $k$-dimensional Jones’ $\beta_2$ number, $\beta_{2, \mu}^k$, is defined to measure how close the support of $\mu$ is to a $k$-dimensional affine subspace. More precisely

$$\beta_{2, \mu}^k(x, r) = \left( r^{-2-k} \min_{L^k \subset \mathbb{R}^n} \int_{B_r(x)} d^2(y, L^k) \, d\mu(y) \right)^{1/2}. \quad (97)$$

Here $L^k$ denotes the set of $k$-dimensional affine subspaces of $\mathbb{R}^n$.

Now we are ready to state the generalized Reifenberg’s theorems. See Theorem 3.3 in [NV17] for the proof.

Theorem A.10. There exist dimensional constants $\delta_R(n)$ and $C_R(n)$ such that for every $\mathcal{H}^k$-measurable subset $S \subset B_1(0)$ which satisfies

$$\int_{B_r(p)} \int_0^r \left( \beta_{2, \mu}^k(x, s) \right)^2 \frac{ds}{s} \, d\mu(x) \leq \delta_R(n) r^k \quad (98)$$

for each $p \in B_1(0)$ and $r \leq 1$ and $\mu = \mathcal{H}^k|_S$ we have

$$\mathcal{H}^k(S) < C_R(n) r^k \text{ and } S \text{ is } k\text{-rectifiable}. \quad (99)$$

We also need the following discrete version of Reifenberg’s theorem. Here we assume the set $S$ to be a discrete subset of $B_1(0)$ such that the balls $\{B_{r_s}(x)\}_{x \in S}$ are pairwise disjoint balls where $B_{r_s}(x) \subset B_2(0)$. Define

$$\mu = \omega_k \sum_{x \in S} r_s^k \delta_x.$$
Theorem A.11. There exist dimensional constant $\delta_R(n)$ and $C_R(n)$ such that if $\mu$ satisfies
\[
\int_{B_r(0)} \int_0^r \left( \frac{\beta_{2,\mu}^k(x,s)}{s} \right)^2 \frac{ds}{s} d\mu(x) \leq \delta_R(n) r^k
\]
for all $B_r(p) \subset B_2(0)$, then we have
\[
\sum_{x \in S} r_x^k < C_R(n).
\]
See Theorem 3.4 in [NV17] for the proof.

A.3. $L^2$-approximation theorem. In this subsection we state the $L^2$-approximation theorem which together with a covering argument, are the main ingredients of the proof of Theorem 1.2. This theorem controls the Jone's $\beta$ number from above by the averages of pinches on the ball $B_r(x)$.

Theorem A.12. Let $u$ satisfies (92). Let $B_r(x)$ be a ball with $x \in B_1(0)$ and $r \in (0, 1]$. For every $\epsilon > 0$ there exists a constant $C_L(n, N, \Lambda, E, F, \epsilon)$ such that if $B_{\delta r}(x)$ is $(0, \delta_3)$-symmetric but not $(k + 1, \epsilon)$-symmetric then
\[
\left( \beta_{2,\mu}^k(x,r) \right)^2 \leq C_L r^{-k} \int_{B_{r}(x)} W_r(y) d\mu(y),
\]
where $\mu$ is a non-negative finite measure on $B_r(x)$ and
\[
W_r(x) = 2 \int_{A_{x,\delta r}(x)} s^{2-a} F_p(x, u, |\nabla u|^2) \left| \frac{\partial u}{\partial s} \right|^2 \text{dvol} = \tilde{\Theta}_{\delta r}(x) - \tilde{\Theta}_r(x).
\]

Proof. The proof is similar to that of Theorem 6.1 in [NV17]. \qed

Note that $\delta_3$ in the above theorem is the same as $\delta_3$ in Lemma A.7.

A.4. Covering lemmas. In this subsection we discuss the two covering lemmas as in [NV16]. In the first covering lemma we refine our covering by keeping to refine the cover of the so called good balls. In the second covering lemma we refine our cover by keeping to refine the cover of so called bad balls.

Lemma A.13 (Covering lemma I). Suppose $u$ satisfies in (92). Fix $\epsilon > 0$ and $\rho \leq \rho(n) < 100^{-1}$ and $r_0 \in (0, 1]$. There exist $\delta(n, N, F, \Lambda, \epsilon, \rho)$ and a dimensional constant $C_V(n)$ such that the following is true. Let
\[
S \subset S^k_{\epsilon, \delta r_0} \text{ and } E = \sup_{x \in B_2(0) \cap S} \tilde{\Theta}(x, 1).
\]
Assume $E \leq \Lambda$. There exists a covering of $S \cap B_1(0)$ such that
\[
S \cap B_1(0) \subset \bigcup_{x \in \mathcal{C}} B_{r_x}(x) \text{ with } r_x \geq r_0 \text{ and } \sum_{x \in \mathcal{C}} r_x^k \leq C_V(n).
\]
Moreover for each $x \in \mathcal{C}$ one of the following is satisfied
a. $r_x = r_0$
b. The set $H_x = \{ y \in S \cap B_{2r_x}(x) \mid \tilde{\Theta}(y, \rho r_x/10) > E - \delta \}$ is contained in $B_{\rho r_x/5}(V_x)$ where $V_x$ is an affine subspace of dimension at most $k - 1$.

Remark A.14. Without loss of generality we will consider $\rho = 2^{-a}$ and $r_0 = \rho^j$ for $a, j \in \mathbb{N}$.
Proof of Covering Lemma I. The proof will follow by an inductive covering argument as follows. We will start with $\hat{\delta}$ as in Proposition A.6 and then we will determine $\hat{\delta}$ in the induction process. We assume our inductive covering at step $j$ satisfies the followings.

$$S \subset \bigcup_{x \in C_j} B_{\delta^j}(x) = \bigcup_{x \in C^j_b} B_{\delta^j}(x) \cup \bigcup_{x \in C^j_g} B_{\delta^j}(x)$$

The balls with centers in $C^j_b$ are called bad balls and the ones with centers in $C^j_g$ are called good balls.

i. If $x \in C^j_b$ then $r^j_x \geq \rho^j$ and $H_x = \{ y \in S \cap B_{2r^j_x}(x) \mid \hat{\Theta}(y, \rho^j / 10) > E - \hat{\delta} \}$ is contained in $B_{p(\rho^j / 10)}(V_x)$, where $V_x$ is a $k - 1$-dimensional affine subspace.

ii. If $x \in C^j_g$ then $r^j_x = \rho^j$ and $H_x = \{ y \in S \cap B_{2r^j_x}(x) \mid \hat{\Theta}(y, \rho^j / 10) > E - \hat{\delta} \}$ is contained in $B_{p(\rho^j / 10)}(V_x)$, where $V_x$ is a $k$-dimensional affine subspace.

iii. For all $x \neq y$, $B_{r^j_x}(x) \cap B_{r^j_y}(y) = \emptyset$.

iv. For all $x \in C^j$ we have $\hat{\Theta}(x, r^j) > E - \eta$.

v. For all $x \in C^j$ and for all $s \in [r^j_x, 1]$, $B_s(x)$ is not $(k + 1, \epsilon/2)$-symmetric.

First step of the Induction. Consider the ball $B_1(0)$. Let

$$H = \{ y \in B_2(0) \cap S \mid \hat{\Theta}(y, \rho / 10) > E - \hat{\delta} \}.$$  

(105)

If there exists no $k$ dimensional subspace $V$ such that $H$ is contained in $B_{\rho / 5}(V)$ then we call $B_1(0)$ a bad ball and we stop the induction process. Otherwise $B_1(0)$ is a good ball and by Proposition A.6, for $\delta \leq \delta^2$

$$S_{\epsilon, \delta^2}(u) \cap B_1(0) \subset B_{\rho / 5}(V).$$

Now we cover $B_{\rho / 5}(V)$ by balls $\{ B_{\rho}(x) \}_{x \in C}$ such that

i. $x \in V \cap B_1(0)$

ii. if $x \neq y$, $B_{\rho / 5}(x) \cap B_{\rho / 5}(y) = \emptyset$.

Then by Lemma A.7 and for every $\eta$ there exists $\delta_3(\eta)$ such that if $\hat{\delta} \leq \min\{\delta_2, \delta_3(\eta)\}$ and for every $x \in C$ we have

$$\hat{\Theta}(x, \rho / 10) > E - \eta.$$  

Next by Lemma A.8 and for $\rho = \hat{\delta} r_0$ and every $\epsilon$ there exists $\delta_4(\rho, \epsilon)$ such that if $\eta < \min\{\delta_4, \delta_2\}$ we get $B_s(x)$ for $s > \rho$ is not $(k + 1, \epsilon/2)$-symmetric. Therefore we have properties i-v for the first step.

Inductive step. In this step we assume we have properties i-v for step $j$ and we prove it for step $j + 1$. This is very similar to the first step and we refer the reader to [NV16].

Conclusion. We stop inductive covering when $j = \bar{j}$ (recall $r_0 = \rho^j$).

Volume estimate. Now we prove the volume estimate

$$\sum_{x \in C} r^k_x \leq C_V(n).$$

(106)

Define

$$\mu = \omega_k \sum_{x \in C} r^k_x \delta_x.$$
Therefore it is enough to prove
\[ \mu(B_r(x)) \leq C_V(n)r^k \]  
(107)
for \( x \in B_1(0) \) and \( r \leq 1 \). The proof of (107) will be based on an inductive process as follows. For all \( t \in (0, 1] \), set
\[ C_t = \{ x \in C \mid r_x \leq t \}, \]
\[ \mu_t = \mu \mid_{C_t} \leq \mu. \]
First we have \( \mu_1 = \mu \). For \( t_l = 2^l r_0 \leq 1/8 \) we show by induction on \( t \geq 0 \) that
\[ \mu_t(B_{t_l}(x)) \leq C_R(n)t_l^k, \]
(108)
for the constant \( C_R(n) \) as in Theorem A.11. We then cover \( B_1(0) \) with \( c(n) \) balls of radius 1/8. We put then \( C_V(n) = c(n)C_R(n) \).

The first step of induction is clear since
\[ C_{r_0} = \{ x \in C \mid r_x = r_0 \} \]
and they are at least \( 2r_0/5 \) away from each other and therefore
\[ \mu_{r_0}(B_{r_0}(x)) \leq c(n)r_0^k. \]
Now assume (108) is true for \( t \leq t_l \), we want to show (108) for \( t_{l+1} = 2t_l \). First we show the following weaker estimate for \( \mu_{t_{l+1}} \) and then we improve our estimate by use of \( L^2 \)-approximation Theorem and Reifenberg Theorem. We claim
\[ \mu_{t_{l+1}}(B_{t_{l+1}}(x)) \leq c(n)C_R(n)t_{l+1}^k. \]
(109)
To prove above we set
\[ \mu_{t_{l+1}} = \mu_t \mid_{C_{t_{l+1}}} + \tilde{\mu}_{t_{l+1}}, \]
where \( \tilde{\mu}_{t_{l+1}} = \mu \mid_{\{ x \in C_{t_{l+1}} \mid r_x > t_l \}} \). Take a cover \( B_{2t_l}(x) \) by balls \( \{ B_{t_l}(y_i) \} \) such that \( \{ B_{t_l/2}(y_i) \} \) are disjoint. There are \( c(n) \) of such balls. Then
\[ \mu_t(B_{t_{l+1}}(x)) \leq \sum_i \mu_t(B_{t_l}(y_i)) \leq c(n)C_R(n)t_l^k. \]
For \( \tilde{\mu}_{t_{l+1}} \) we have
\[ \tilde{\mu}_{t_{l+1}}(B_{t_{l+1}}(x)) \leq c(n)t_{l+1}^k \]
since our covering balls \( \{ B_{t_l}(x) \}_{x \in C} \) are \( 2r_1/5 \)-away from each other. Therefore we have (109).

Now we use our inductive assumption in (108) and (109) and Theorem A.11, Theorem A.12 to finish the proof of the volume estimate (107). Set
\[ \bar{\mu} = \mu_{t_{l+1}} \mid_{B_{t_{l+1}}(x)}. \]
We use Theorem A.12 to show
\[ \left( \beta_{2d}^k(y, s) \right)^2 \leq C_L s^{-k} \int_{B(y)} \hat{W}(q) d\bar{\mu}(q) \]
(110)
for \( y \in \text{supp}(\bar{\mu}) \) and \( s \in (0, 1] \) where
\[ \hat{W}_s(q) = \begin{cases} 0 & s \leq r_q \\ W_s(q) & s > r_q. \end{cases} \]
We show (110) first for the case where \( s \in [r_y, 1/8] \). For every \( y \in \text{supp} (\tilde{\mu}) \) and \( s \in [r_y, 1/8] \) we know that \( B_{8s}(y) \) is not \((k+1, \epsilon/2)\)-symmetric. Moreover, \( \tilde{\Theta}(y, r_y) > E - \eta \), and therefore
\[ \tilde{\Theta}(y, 8s) - \tilde{\Theta}(y, 4s) < \eta. \]
By choosing \( \eta \leq \delta_0 \) where in Proposition A.2 we use \( \epsilon = \delta_3 \), we conclude that \( B(y, 8s) \) is \((0, \delta_3)\)-symmetric. Finally, for every \( q \in B_s(y) \) and since \( s > r_y \) we have \( r_q < s \), and (110) follows for the case where \( s \in [r_y, 1/8] \). Inequality (110) is straightforward for \( s \leq r_y \), since we have \( B_s(y) \cap \text{supp} (\tilde{\mu}) = \{ y \} \) and so \( \beta_{2, \tilde{\mu}}(y, s) = 0 \).

In order to finish our induction we use (110) to show that for \( y \in B_{t_{t+1}}(x) \) and \( r < t_{t+1} \),
\[ \int_{B_{s}(y)} \int_{0}^{r} \left( \beta_{2, \tilde{\mu}}^{k}(z, s) \right)^{2} \frac{ds}{s} d\tilde{\mu}(z) \leq C_{L} C(n) C_{R}^{2} \eta^{k}. \] (111)
Then by Theorem A.11 and for \( \eta < \frac{\delta_{r}}{C_{L} C(n) C_{W}} \) we have
\[ \mu_{t_{t+1}}(B_{t_{t+1}}(x)) \leq C_{R}(n) t_{t+1}. \]
For the proof of (111), by (110) and for all \( s \leq r \),
\[ \int_{B_{s}(y)} \left( \beta_{2, \tilde{\mu}}^{k}(z, s) \right)^{2} d\tilde{\mu}(z) \leq C_{L} s^{-k} \int_{B_{s}(y)} \int_{B_{s}(z)} \hat{W}_{s}(q) d\mu(q) d\tilde{\mu}(z). \] (112)
By (109) for \( s \leq t_{t+1} \),
\[ \tilde{\mu}(B_{s}(z)) \leq \mu_{s}(B_{s}(z)) \leq c(n) C_{R} s^{k}. \]
Thus we have
\[ \int_{B_{s}(y)} \left( \beta_{2, \tilde{\mu}}^{k}(z, s) \right)^{2} d\tilde{\mu}(z) \leq c(n) C_{R} C_{L} \int_{B_{s}(y)} \hat{W}_{s}(z) d\tilde{\mu}(z) \] (113)
and
\[ \int_{0}^{r} \int_{B_{s}(y)} \left( \beta_{2, \tilde{\mu}}^{k}(z, s) \right)^{2} d\tilde{\mu}(z) \frac{ds}{s} \leq c(n) C_{R} C_{L} \int_{0}^{r} \int_{B_{s}(y)} \hat{W}_{s}(z) d\tilde{\mu}(z) \frac{ds}{s} \]
\[ = c(n) C_{R} C_{L} \int_{B_{s}(y)} \int_{r}^{1/8} \hat{W}_{s}(z) \frac{ds}{s} d\tilde{\mu}(z) \]
\[ \leq c(n) C_{R} C_{L} \int_{B_{s}(y)} \int_{r}^{1/8} \hat{W}_{s}(z) \frac{ds}{s} d\tilde{\mu}(z) \leq c(n) C_{R} C_{L} \left[ \tilde{\Theta}(y, 1) - \tilde{\Theta}(y, r_{y}) \right] \leq c(n) C_{R} C_{L} c_{\eta} (2r)^{k}. \]
(114)
Therefore if we choose \( \eta \leq \frac{\delta_{r}}{c(n) C_{R} C_{L}} \), then by Theorem A.11, we get (108).

In our second covering lemma we refine the covering of the bad balls from Covering Lemma I.

**Lemma A.15** (Covering lemma II). Let \( u \) satisfies (92). Fix \( \varepsilon > 0 \) and \( r_{0} \in (0, 1] \). There exist \( \tilde{\delta} = \tilde{\delta}(n, N, \Lambda, F, \varepsilon) \) and a dimensional constant \( C_{F}(n) \) such that the following is true. Let
\[ S \subset S_{\varepsilon, \tilde{\delta}_{0}} \text{ and } E = \sup_{x \in B_{2}(0) \cap S} \tilde{\Theta}(x, 1). \]
Assume \( E \leq \Lambda \). There exists a covering of \( S \cap B_{1}(0) \) such that
\[ S \cap B_{1}(0) \subset \bigcup_{x \in \mathcal{E}} B_{r_{x}}(x) \text{ with } r_{x} \geq r_{0} \text{ and } \sum_{x \in \mathcal{E}} r_{x}^{k} \leq C_{F}(n). \]
Moreover for each \( x \in \mathcal{C} \) one of the following is satisfied:

\( a. \ r_x = r_0, \)

\( b. \) or we have the following drop

\( \forall y \in B_{2r_x}(x) \cap \mathcal{S}, \ \hat{\Theta}(y, r_x/10) \leq E - \tilde{\delta}. \)  \( (115) \)

**Proof of Covering Lemma II.** We will refine the covering of Lemma A.15 through an inductive process. At the step \( j \) of our induction we have

i. For all \( j \)

\[ \mathcal{S} \subset \bigcup_{x \in \mathcal{C}(j, r_0)} B_{r_0}(x) \cup \bigcup_{x \in \mathcal{C}(j, f)} B_r(x) \cup \bigcup_{x \in \mathcal{C}(j, b)} B_{r_0}(x). \]  \( (116) \)

ii. For all \( x \in \mathcal{C}(j, r_0), r_x = r_0. \) On these balls condition \( a \) of Lemma A.15 is satisfied and we stop the inductive process.

iii. For all \( x \in \mathcal{C}(j, f), \) and all \( z \in B_{2r_x}(x) \) we have \( \bar{\Theta}(z, r_x/10) \leq E - \tilde{\delta}. \) On these balls condition \( b \) of Lemma A.15 is satisfies and we stop the inductive process.

iv. For all \( x \in \mathcal{C}(j, b), \) \( r_0 < r_x \leq \rho^j \) and neither condition \( a \) nor condition \( b \) satisfies and we continue our inductive process.

v. For some constant \( C_F(n) \) we have

\[ \sum_{x \in \mathcal{C}(j, r_0) \cup \mathcal{C}(j, f)} r_x^k \leq C_F(n) \sum_{l=1}^{j} 2^{-l} \]

\[ \sum_{x \in \mathcal{C}(j, b)} r_x^k \leq 2^{-j}. \]  \( (117) \)

**First step of induction.** Consider \( \tilde{\delta} \leq \hat{\delta}, \) where the exact value of \( \tilde{\delta} \) will be determined during the proof. Recall that from Lemma A.13 that we have the following covering of \( \mathcal{S} \subset \mathcal{S}_{\varepsilon, \delta r_0} \)

\[ \mathcal{S} \subset \bigcup_{x \in \mathcal{C}} B_r(x) = \bigcup_{x \in \mathcal{C}_r_0} B_r(x) \cup \bigcup_{x \in \mathcal{C}_{r_0}^*} B_r(x), \]  \( (118) \)

where

\[ \mathcal{C}_{r_0} = \{ x \in \mathcal{C} \mid r_x = r_0 \} \text{ and } \mathcal{C}_{r_0}^* = \{ x \in \mathcal{C} \mid r_x > r_0 \} \]  \( (119) \)

and

\[ \sum_{x \in \mathcal{C}_{r_0} \cup \mathcal{C}_{r_0}^*} r_x^k < C_V(n), \]

and for every \( x \in \mathcal{C}_{r_0}^* \) we have

\[ H_x = \{ y \in B_{2r_x}(x) \cap \mathcal{S} \mid \hat{\Theta}(y, \rho r_x/10) > E - \hat{\delta} \} \subset B_{\rho/5}(V_x) \]

for a subspace \( V_x \) of dimension at most \( k - 1. \) We include the balls \( \{ B_{r_x}(x) \}_{x \in \mathcal{C}_{r_0}} \) in our final covering. In fact we let

\[ \mathcal{C}^{(1, r_0)} = \mathcal{C}_{r_0}. \]

For the balls \( \{ B_{r_x}(x) \}_{x \in \mathcal{C}_{r_0}^*} \) we use a finer cover as follows.
If $H_x = \emptyset$ then every point in $B_{r_x}(x)$ satisfies the drop condition (115) in Lemma A.15. We cover $B_{r_x}(x)$ by balls of radius $\{B_{pr_x}(y)\}_{y \in f}$. The number of these balls is bounded by a constant $c(n)\rho^{-n}$.

If $H_x \neq \emptyset$ then

$$H_x \subset B_{\rho/5}(V_x)$$

where $V_x$ is a subspace with dimension at most $k-1$. Then $B_{r_x}(x) \setminus B_{\rho/5}(V_x)$ can be covered by balls of radius $\{B_{pr_x}(y)\}_{y \in f}$ as above. On balls $\{B_{pr_x}(y)\}_{y \in f}$, the energy drop condition (115) satisfied. We cover $B_{\rho/5}(V_x)$ by balls $\{B_{pr_x}(y)\}$ and there are at most $c(n)\rho^{1-k}$ such balls. These balls either satisfy the stopping condition $\rho r_x = r_0$, in which case we include them in $\{B_{pr_x}(y)\}_{y \in \bar{f}}$, or they satisfy $\rho r_x > r_0$ where they need more refinement and we include them in $\{B_{pr_x}(y)\}_{y \in \bar{f}}$. More precisely,

$$\mathcal{C}^{(1, f)} \subset \bigcup_{x \in \bar{f}} \mathcal{C}^{(1, f)}_x$$

and

$$\sum_{z \in \mathcal{C}^{(1, f)}} r_z^k = \sum_{x \in \bar{f}} \sum_{y \in \bar{f}} (\rho r_x)^k \leq C_V(n)c(n)\rho^{k-n},$$

$$\mathcal{C}^{(1, b)} \subset \bigcup_{x \in \bar{f}, \rho r_x = r_0} \mathcal{C}^{(1, b)}_x$$

and

$$\sum_{z \in \mathcal{C}^{(1, b)}} r_z^k = \sum_{x \in \bar{f}} \sum_{y \in \bar{f}} (\rho r_x)^k \leq C_V(n)c(n)\rho,$$

$$\mathcal{C}^{(1, r_0)} = \mathcal{C}_{r_0} \cup \bigcup_{x \in \bar{f}, \rho r_x = r_0} \mathcal{C}^{(1, r_0)}_x$$

and

$$\sum_{z \in \mathcal{C}^{(1, r_0)}} r_z^k \leq \sum_{x \in \bar{f}} \sum_{y \in \bar{f}} r_x^k + \sum_{x \in \bar{f}} \sum_{y \in \bar{f}} r_0^k \leq C_V(n) + C_V(n)c(n)\rho.$$

So we choose $\rho = \rho(n) \leq \min \{100^{-1}, \frac{1}{2}C_V^{-1}(n)c^{-1}(n)\}$ and then we have

$$\sum_{z \in \mathcal{C}^{(1, b)}} r_z^k \leq 1/2,$$

$$\sum_{z \in \mathcal{C}^{(1, r_0)} \cup \mathcal{C}^{(1, f)}} r_z^k \leq C_F(n).$$

Also for each $y$ in $\mathcal{C}^{(1, b)}$, we have $r_y < \rho$ and therefore the first step of our inductive covering satisfies the conditions i-v above.

**Inductive step.** In this step we assume that we have the properties i-v for step $j$ and we prove then for step $j + 1$. Basically we only refine the covering for the balls in $\mathcal{C}^{(j, b)}$. This is very similar to the first step and we omit the details. See [NV16].

**Conclusion.** By property iv above $r_0 < r_x \leq \rho^j$ for $x \in \mathcal{C}^{(j, b)}$. But there will be a step $\bar{j}$ such that $\rho^j \leq r_0$ and therefore $\mathcal{C}^{(j, b)} = \emptyset$.

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