Differential relations for almost Belyi maps

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Abstract

Several kinds of differential relations for polynomial components of almost Belyi maps are presented. Saito’s theory of free divisors give particularly interesting (yet conjectural) logarithmic action of vector fields. The differential relations implied by Kitaev’s construction of algebraic Painlevé VI solutions through pull-back transformations are used to compute almost Belyi maps for the pull-backs giving all genus 0 and 1 Painlevé VI solutions in the Lisovyy-Tyhyy classification.

1 Introduction

Importance of Belyi maps was highlighted in the *l’Esquisse d’une programme* by Grothendieck [9]. Since then, Belyi maps attract increasing attention in algebraic geometry, number theory, mathematical physics. One elementary application of Belyi maps is pull-back transformations of hypergeometric differential equations to Fuchsian equations with a small number of singularities, and corresponding transformations of special functions [11], [28], [29].

Recall that a *Belyi map* is an algebraic covering \( \varphi: C \to \mathbb{P}^1 \) that branches only above \( \{0, 1, \infty\} \subset \mathbb{P}^1 \). In particular, a genus 0 Belyi map (with \( C \cong \mathbb{P}^1 \)) is defined by a rational function \( \varphi(x) \in \mathbb{C}(x) \) such that all branching points \( \{ x : \varphi'(x) = 0 \} \) lie in the fibers \( \varphi(x) \in \{0, 1, \infty\} \).

Almost Belyi maps were assertively introduced by Kitaev [18], [19] in the context of algebraic solutions of the Painlevé VI equation.

**Definition 1.1.** An *almost Belyi map* (or an *AB-map*, for shorthand) is an algebraic covering \( \varphi: C \to \mathbb{P}^1 \) that has exactly one simple branching point outside the fibers \( \{0, 1, \infty\} \subset \mathbb{P}^1 \). (Recall that simple branching points have the branching order 2.)

Kitaev constructed algebraic Painlevé VI functions using the Jimbo-Miwa correspondence [14] to isomonodromic \( 2 \times 2 \) Fuchsian systems with 4 singularities. The corresponding Fuchsian systems are generated by pull-backs of the Gauss-Euler hypergeometric equation with respect to AB-maps. In the context of Picard-Fuchs equations, the same pull-back method with AB-maps was employed by Doran [6], Movasatti, Reiter [23].

Recently [15], algebraic Painlevé VI solutions and AB-maps found application in Saito’s singularity theory [25], [26] and Dubrovin’s theory of Frobenius manifolds [7].

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This direction motivates computation of new examples of AB-maps. In particular, a list of AB-maps giving pull-backs to all cases of algebraic Painlevé VI solutions in the Lisovyy-Tykhyy classification [21] (up to Schlessinger gauge transformations) is desirable.

The problem of computing AB-maps and the mentioned applications give an interesting set of differential relations for AB-maps and their polynomial components. In particular,

- Usefulness of differentiation in computing Belyi maps was noticed by several authors [27, §2.5]. Further, the very fact of implied pull-back transformations of Fuchsian equations gives additional differential and algebraic restrictions. The same techniques apply to computation of AB-maps, as we demonstrate in §2.3.

- Kitaev’s basic construction entails differentiation with respect to the “isomonodromic” parameter (rather than with respect to the independent variable), leading to differential relations between the coefficients of an AB-map. The straightforward case of Kitaev’s RS-transformations is summarized in Theorem 2.8.

- Saito’s construction of free divisors gives action of vector fields that relates differentiation both with respect to the independent variable and the “isomonodromic” parameter. Remarkably, we observe existence of vector fields that are logarithmic along each hypersurface defined by the polynomial components of an AB-map, leading us to Conjecture 3.5.

Analysis of these differential relations (in §2.3, §2.4, §3, respectively) is the main contribution of this article. Additionally, Section 4 presents computational results of AB-maps for all genus 0 and 1 cases of the Lisovyy-Tykhyy classification [21] of algebraic Painlevé VI solutions.

2 Preliminaries

Here we introduce application of Belyi maps and AB-maps to pull-back transformations between Fuchsian equations; basic methods for computing these maps and differential relations they employ.

2.1 Nomenclature for AB-maps

This paper studies AB-maps of genus 0. We are thus looking at rational functions \( \varphi(x) \in \mathbb{C}(x) \) such that all branching points \( \{ x : \varphi'(x) = 0 \} \) except one lie in the fibers \( \varphi(x) \in \{ 0, 1, \infty \} \). The extra branching point has the branching order 2 (thus \( \varphi'' \neq 0 \) at that branching point if it is not \( \infty \)).

Important distinctions between Belyi maps and AB-maps are:

\( i \) Belyi maps form discrete (0-dimensional) Hurwitz spaces. AB-maps form 1-dimensional Hurwitz spaces; that is, there are 1-dimensional families of them parametrized by algebraic curves.
By Hurwitz theorem, a Belyi map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of genus 0, degree $d$ has exactly $d + 2$ distinct points in the 3 fibers $\varphi(x) \in \{0, 1, \infty\}$. An AB-map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of genus 0, degree $d$ has exactly $d + 3$ points in the 3 fibers.

**Example 2.1.** An example of a AB-map of degree 6 is

$$\varphi_1(x) = \frac{(wx^3 + 15x^2 + 20x + 8)^2}{64(x + 1)^5}. \quad (1)$$

The parameter $w$ appears only once. We can compute:

$$\varphi_1(x) - 1 = \frac{x^3(w^2x^3 + 2(15w - 32)x^2 + 5(8w - 19)x + 16w - 40)}{64(x + 1)^5},$$

$$\varphi'_1(x) = \frac{x^2(wx^3 + 15x^2 + 20x + 8)(wx + 6w - 15)}{64(x + 1)^5}.$$

The root $x = q_1 = -6 + 15/w$ of $(wx + 6w - 15)$ is the only branching point outside the fibers $\varphi(x) \in \{0, 1, \infty\}$.

**Notation 2.2.** Let $\varphi \in \mathbb{C}(x)$ be a rational function of degree $d$. The **branching pattern** in a fiber $\varphi = C$ is given by a partition of $d$. We choose the multiplicative notation $1^{n_1}2^{n_2}\ldots$ for a branching pattern, meaning $n_1$ non-branching points, $n_2$ branching points of order 2, etc. For example, we write the branching pattern of the fiber $\varphi_1 = 1$ of the AB-map in (1) as $1^33$ rather than $1 + 1 + 1 + 3$. The partition fact is expressed by $\sum k n_k = d$.

The collection $[P_1/P_2/P_3]$ of the branching patterns $P_1, P_2, P_3$ in the fibers $\varphi = 0, \varphi = 1, \varphi = \infty$ is called at the **passport** of $\varphi$. For example, the passport of $\varphi_1$ in (1) is $[2^3/3^1 5/1]$, keeping in mind the point $x = \infty$ in the fiber $\varphi = \infty$. The order of branching patterns in the passport is not significant to us, as permutation of the 3 fibers is realized by the fractional-linear expressions $\varphi/(\varphi - 1), 1 - \varphi, 1/\varphi, 1/(1 - \varphi), (\varphi - 1)/\varphi$.

### 2.2 Pull-backs of Fuchsian equations

One application of Belyi maps is pull-back transformations of the hypergeometric equation

$$\frac{d^2 y(z)}{dz^2} + \left(\frac{c}{z} + \frac{a + b - c + 1}{z - 1}\right) \frac{dy(z)}{dz} + \frac{ab}{z(z - 1)} y(z) = 0. \quad (2)$$

to Fuchsian equations with a few singularities (e.g., Heun, other hypergeometric equations). The pull-back transformations have the form

$$z \mapsto \varphi(x), \quad y(z) \mapsto Y(x) = \theta(x) y(\varphi(x)), \quad (3)$$

where $\varphi(x)$ is a rational function, and $\theta(x)$ is a Liouvillian (e.g., power) function. The rational function $\varphi(x)$ is typically a special Belyi map. Applicable Belyi maps are characterized using the following definition [11, Definition 1.2].
Definition 2.3. Given positive integers $k, \ell, m, n$, a Belyi map $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ is called $(k, \ell, m)$-minus-$n$ regular if, with exactly $n$ exceptions in total, all points above $z = 1$ have branching order $k$, all points above $z = 0$ have branching order $\ell$, and all points above $z = \infty$ have branching order $m$.

The singularities and the local exponents of the pulled-back Fuchsian equation are straightforwardly determined from the pull-back (3) and Riemann's $P$-symbol

$$
P \left\{ \begin{array}{ccc}
0 & 1 & \infty \\
0 & 0 & a \\
1 - c & c - a - b & b \\
\end{array} \right\}
$$

of hypergeometric equation (2). For the pulled-back Fuchsian equation to have only $n$ singularities, we usually need the local exponent differences $c - a - b, 1 - c, b - a$ to be inverse integers $\pm 1/k, \pm 1/\ell, \pm 1/m$, and the covering $z = \varphi(x)$ to be a $(k, \ell, m)$-minus-$n$ regular Belyi map. The canonical Fuchsian equations with $n \leq 4$ are hypergeometric and Heun equations.

We extend Definition 2.3 to AB-maps.

Definition 2.4. Given positive integers $k, \ell, m, n$, an AB-map $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ is called $(k, \ell, m)$-minus-$n$ regular if, with exactly $n$ exceptions in total, all points above $z = 1$ have branching order $k$, all points above $z = 0$ have branching order $\ell$, and all points above $z = \infty$ have branching order $m$.

Example 2.5. The AB-map $\varphi_1(x)$ in Example 2.1 is $(3, 2, 5)$-minus-4 regular. The 4 exceptional points are $x = \infty$ and the 3 simple roots of $\varphi_1(x) - 1$.

Remark 2.6. Recently, van Hoeij and Kunwar classified $(2, 3, \infty)$-minus-5 regular AB-maps in [10]. Here $\infty$ means that all points in the third fiber are counted as exceptional (towards 5). These maps have degree $\leq 12$. A portion of the AB-maps $N_1, \ldots, N_{68}$ in [10, Table 1] are applicable as $(2, 3, m)$-minus-4 maps to the Fuchsian equations considered here; see the fifth column in Table 4.1.

Pull-back transformations with respect to $(k, \ell, m)$-minus-$n$ regular AB-maps can transform hypergeometric equation (2) with the local exponent differences $1/k, 1/\ell, 1/m$ to Fuchsian equations with an apparent singularity and $n$ other singularities. The apparent singular point will have the local exponents $0, 2$, rather than $0, 1$ for regular points. Since AB-maps are parametrized by algebraic curves, a generic pull-back transformation will give isomonodromic families of Fuchsian equations with these singularities.

An important case is Fuchsian ordinary differential equations with an apparent singularity and $n = 4$ other singular points. Isomonodromic families of these equations are parametrized by solutions of the Painlevé VI equation

$$
\frac{d^2q}{dt^2} = \frac{1}{2} \left( \frac{1}{q} + \frac{1}{q - 1} + \frac{1}{q - t} \right) \left( \frac{dq}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{q - t} \right) \frac{dq}{dt}
+ \frac{q(q - 1)(q - t)}{t^2(t - 1)^2} \left( \alpha + \beta \frac{t}{q^2} + \gamma \frac{t - 1}{(q - 1)^2} + \delta \frac{t(t - 1)}{(q - t)^2} \right).
$$

(4)
By the Jimbo-Miwa correspondence [14], a solution $q(t)$ parametrizes isomonodromic $2 \times 2$ Fuchsian systems $dY/dx = A(x,t)Y$ with the singularities $x = 0, x = 1, x = t, x = \infty$ and the local monodromy differences $\theta_0, \theta_1, \theta_t, \theta_\infty$ such that
\begin{equation}
\alpha = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = 1 - \frac{\theta_t^2}{2}.
\end{equation}

An equivalent isomonodromic family of ODEs has 1 apparent and 4 other singularities. To write down the parametric Fuchsian ODE explicitly, one can use a specification of the Painlevé VI equation in terms of the Hamiltonian system
\begin{equation}
\frac{dq}{dt} = \frac{\partial H_0}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_0}{\partial q}.
\end{equation}

with
\begin{align*}
H_0 &= \frac{q(q-1)(q-t)}{t(t-1)} \left( p^2 - \left( \frac{\theta_0}{q} + \frac{\theta_1}{q-1} + \frac{\theta_t}{q-t} \right) p + \frac{\Theta}{q(q-1)} \right), \\
\Theta &= \frac{(\theta_0 + \theta_1 + \theta_t - \theta_\infty)(\theta_0 + \theta_1 + \theta_t + \theta_\infty - 2)}{4}.
\end{align*}

The Painlevé VI equation is obtained by eliminating $p$. The corresponding Fuchsian ODE is
\begin{equation}
\frac{d^2Y(x)}{dx^2} + \left( \frac{1 - \theta_0}{x} + \frac{1 - \theta_1}{x-1} + \frac{1 - \theta_t}{x-t} - \frac{1}{x-q} \right) \frac{dY(x)}{dx} + W_1 Y(x) = 0
\end{equation}
with
\begin{align*}
W_1 &= \frac{\Theta}{x(x-1)} + \frac{q(q-1)p}{x(x-1)(x-q)} - \frac{t(t-1)H_0}{x(x-1)(x-t)},
\end{align*}
see [13] pg. 169–173] with $n = 1$.

**Notation 2.7.** Let $P_{VI}(\theta_0, \theta_1, \theta_t, \theta_\infty)$ denote the Painlevé VI equation [6] with the parameters [5]. Similarly, let $E(1-c,c-a-b,b-a)$ denote the hypergeometric equation [2] by the local exponent differences.

### 2.3 Computational methods

As considered in [11] §5.2, a $(k, \ell, m)$-minus-$n$ regular Belyi map has the forms
\begin{align}
\varphi(x) &= r_1 \frac{P^t F}{Q^m G} \\
&= 1 + r_2 \frac{R^k H}{Q^m G},
\end{align}
where $P, Q, R$ are monic polynomials without multiple roots; $F, G, H$ are monic polynomials with $n$ or $n - 1$ distinct roots in total; and $r_1, r_2$ are constants. We refer to the polynomials $P, Q, R, F, G, H$ as *polynomial components* of $\varphi$.

The total number of distinct roots of the 6 polynomial components (including $x = \infty$ if one of the 3 terms in the polynomial identity is of lower degree) equals
deg $\varphi + 2$, by §2.1(ii). The two expressions (9)–(10) are equivalent to the polynomial identity

$$r_1 P^\ell F = Q^m G + r_2 R^k H.$$  \hfill (11)

A $(k, \ell, m)$-minus-$n$ regular AB-map has the same shape, but the total number of roots in the terms (including $x = \infty$) equals $\deg \varphi + 3$ rather than $\deg \varphi + 2$. The polynomial components and the constants $r_1, r_2$ may then depend on a continuous parameter.

Polynomial identity (11) gives a system of necessary polynomial equations for the undetermined coefficients of $P, Q, R$ and perhaps of $F, G, H$. If the degree of the target Belyi map significantly exceeds 10, the algebraic system is too complicated, with too many degenerate (parasytic) solutions to be solved by Gröbner basis techniques efficiently. Simpler algebraic systems are obtained by considering the logarithmic derivatives

$$\frac{\varphi'(x)}{\varphi(x)} = k \frac{R'}{R} + \frac{H'}{H} - m \frac{Q'}{Q} - \frac{G'}{G}.$$  \hfill (12)

$$\frac{\varphi'(x) - 1}{\varphi(x) - 1} = k \frac{R'}{R} + \frac{H'}{H} - m \frac{Q'}{Q} - \frac{G'}{G}.$$  \hfill (13)

The roots of $\varphi'(x)/\varphi(x)$ are the branching points outside the fibers $\varphi(x) \in \{0, \infty\}$, with the multiplicity reduced by 1. This consideration gives the alternative expressions

$$\frac{\varphi'(x)}{\varphi(x)} = h_1 \frac{R^{k-1} H}{P Q S}, \quad \frac{\varphi'(x) - 1}{\varphi(x) - 1} = h_2 \frac{P^{\ell-1} F}{Q R S}.$$  \hfill (14)

If $\varphi(x)$ is supposed to be a Belyi map, $S$ here equals the product of irreducible monic factors of $F G H$, each to the power 1. If $\varphi(x)$ is an AB-map, $S$ equals this product divided by $x - q$, where $q$ is the (undetermined) extra branching point. If $x = \infty$ is in the $\varphi = \infty$ fiber, then $h_1, h_2$ are equal to the branching order at $x = \infty$; otherwise they are (undetermined) constants. The obtained algebraic system for the coefficients is typically over-determined, with fewer degenerate solutions. According to [5, 27], these differential relations for Belyi maps were noticed by Fricke, Atkin, Swinnerton-Dyer.

Additional algebraic equations are obtained by considering implied pull-back transformations of second order Fuchsian equations. In particular [11, Lemma 5.1], the pull-back

$$z \mapsto \varphi(x), \quad y(z) \mapsto Y(x) = (Q^m G)^a y(\varphi(x))$$  \hfill (15)

transforms the hypergeometric equation (2) with

$$a = \frac{1}{2} \left( 1 - \frac{1}{k} - \frac{1}{\ell} - \frac{1}{m} \right), \quad b = \frac{1}{2} \left( 1 - \frac{1}{k} - \frac{1}{\ell} + \frac{1}{m} \right), \quad c = 1 - \frac{1}{\ell}$$

to the Fuchsian equation

$$\frac{d^2 Y(x)}{dx^2} + \left( \frac{S'}{S} - \frac{F'}{F} - \frac{G'}{G} - \frac{H'}{H} \right) \frac{dY(x)}{dx} + W_2 Y(x) = 0$$  \hfill (16)
with
\[ W_2 = a \left[ b \left( h_1 h_2 P^{\ell - 2} R^{k - 2} F H - \frac{m^2 Q'^2}{Q^2} - \frac{G'^2}{G^2} \right) + \frac{m Q''}{Q} + \frac{G''}{G} + \right. \\
\left. + \left( \frac{1}{k} + \frac{1}{\ell} \right) \frac{m Q' G'}{Q G} + \left( \frac{m Q'}{Q} + \frac{G'}{G} \right) \left( \frac{S'}{S} - \frac{F'}{F} - \frac{G'}{G} - \frac{H'}{k H} \right) \right]. \]

In the context of pull-back transformations to isomonodromic Fuchsian system with one apparent singularity and 4 other singularities, this equation can be compared with (8).

### 2.4 Relation to algebraic Painlevé VI solutions

Kitaev [18], [19] initiated study of AB-maps with the purpose of constructing algebraic Painlevé VI solutions. The relevant AB maps are \((k, \ell, m)\)-minus-4 regular, as they induce pull-back transformations of to isomonodromic \(2 \times 2\) Fuchsian systems with 4 singularities (or the corresponding ODEs) by the Jimbo-Miwa correspondence [14] of these systems to Painlevé VI solutions. Kitaev’s basic construction gives the following result.

**Theorem 2.8.** Let \(\varphi(X)\) denote a \((k, \ell, m)\)-minus-4 regular AB-map. Suppose that its irregular branching points are \(X = 0, X = 1, X = \infty, X = t\). Let \(X = q\) denote the extra branching point of order 2. Then \(q(t)\) is an algebraic Painlevé VI solution with the parameters \(\theta_j = a_j/K_j\) for \(j \in \{0, 1, t\}\), and \(\theta_\infty = 1 - a_\infty/K_\infty\). Here \(K_j, K_\infty \in \{k, \ell, m\}\) depending on the fiber of each of the 4 singularities, and \(a_j, a_\infty\) are the branching orders at them.

**Proof.** The Jimbo-Miwa correspondence [14] and explicit consideration of a pull-back from \(E(1/\ell, 1/k, 1/m)\). This is the particular case \(\varepsilon = 1\) of [18, Theorem 2.1].

This theorem gives differential relations between coefficients of AB-maps. The relation between \(t\) and \(q\) is algebraic because the Hurwitz space is one-dimensional.

**Example 2.9.** Consider the polynomials
\[
P = x^4 + 4wx^2 - 6wx + w^2, \\
R = 2x^6 + 12wx^4 - 18wx^3 + 15w^2x^2 - 36w^2x - w^2(2w - 27), \\
G_1 = x - 1, \\
G_2 = 4x^3 + wx^2 + 18wx + w(4w - 27). \tag{17}
\]

Reminiscent to (11), we have a polynomial identity \(4P^3 = R^2 + r_0 G_1^2 G_2\) with \(r_0 = 27w^3\). It defines a \((2, 3, 7)\)-minus-4 regular AB-map
\[
\varphi_2(x) = \frac{4P^3}{r_0 G_1^2 G_2} = 1 + \frac{R^2}{r_0 G_1^2 G_2} \tag{18}
\]
of degree 12, with the branching pattern \([2^6/3^4/7 2 1^3]\). The extra branching point is \(x = q_2 = (9 - 2w)/7\). To obtain an algebraic Painlevé VI solution of \(PVI(1/7, 1/7, 2/7, 6/7)\) by Theorem 2.8 we first reparametrize
\[
w \mapsto -\frac{(s^2 + 3)^3}{(s - 1)^2(s + 1)^2} \tag{19}
\]
so that $G_2$ has rational roots:

$$x_1 = \frac{(s^2 + 3)(s^2 + 15)}{4(s - 1)(s + 1)}, \quad x_2 = \frac{(s^2 + 3)(2s^2 + 3s + 3)}{(1 - s)(s + 1)^2}, \quad x_3 = \frac{(s^2 + 3)(2s^2 - 3s + 3)}{(s - 1)^2(s + 1)}.$$  

We move these points to the locations $X_1 = \infty$, $X_2 = 0$, $X_3 = 1$ by the Möbius $x$-transformation

$$x \mapsto \frac{(s^2 + 3)}{s^2 - 1} \frac{4s^3(s^2 + 15)X - (s - 3)^3(2s^2 + 3s + 3)}{16s^3X + (s + 1)(s - 3)^3}. \quad (20)$$

The root of $G_1$ is transformed to $X = t_2$ with

$$t_2 = \frac{(s - 3)^3(s^2 + s + 2)^2}{2s^3(s^2 + 7)^2}, \quad (21)$$

and the transformed location of the extra branching point is

$$q_2 = \frac{(s + 1)(3 - s)(s^2 + s + 2)}{2s(s^2 + 7)}. \quad (22)$$

This parametrizes an algebraic solution $q_2(t_2)$ of $P_{V1}(1/7, 1/7, 2/7, 6/7)$. The fractional-linear transformation $t_2(q_2 - t_2)/q_2 - t_2$ permutes the singularities $0 \leftrightarrow 1$, $t \leftrightarrow \infty$, and gives the Kleinian solution of Boalch [2]. Kitaev derived this solution by the pull-back construction [18], also followed in [32], §3.4.3, also followed in [32], §5.

**Example 2.10.** Consider the polynomials

$$P = x^3 + (w - 6)x^2 + 24x - 48,$$

$$R = x^5 + 2(w - 6)x^4 + (w^2 - 12w + 72)x^3 + 36(w - 8)x^2 - 72(w - 9)x - 864,$$

$$F = x + w - 6,$$

$$G = wx^3 + (w^2 - 6w - 3)x^2 + 8(3w + 1)x - 16(4w + 3). \quad (23)$$

We have a polynomial identity $P^3 F = R^2 + 1728G$. It defines a $(2, 3, 7)$-minus-4 regular AB-map $\varphi_3(x)$ of degree 10, with the branching pattern $[2^5/3^31/71^3]$. The extra branching point is $x = q_3 = -4(w^2 - 6w - 6)/(7w)$. The curve $G(x, w) = 0$ defines a genus 0 curve; a parametrization of it gives a substitution after which the polynomial $G(x)$ has a rational root:

$$w \mapsto \frac{(s + 2)(s^2 + 2s + 9)}{(s - 1)^2}. \quad (24)$$

Complete factorization of $G$ is achieved on the genus 1 curve $y^2 = s(s^2 + s + 7)$. Here are the roots of $G$:

$$x_1 = \frac{(1 - s)(s + 3)}{s + 2}, \quad x_2 = \frac{4(2s^2 + 2s + 5 + 3y)}{(s - 1)(3 - y)}, \quad x_3 = \frac{4(2s^2 + 2s + 5 - 3y)}{(s - 1)(3 + y)}.$$  

The three roots are mapped to $X_1 = \infty$, $X_2 = 0$, $X_3 = 1$ by the Möbius $x$-transformation

$$x \mapsto \frac{4(1 - s)}{8y(s + 2)(s^2 + s + 7)} \frac{(2y(s + 3)(s^2 + s + 7)(2X - 1) - 3s^4 - 34s^3 - 114s^2 - 252s - 245)}{(2X - 1) - s^6 - 2s^5 + 9s^4 + 64s^3 + 221s^2 + 210s + 147}. \quad (25)$$
The root of $F$ is transformed to $X = t_3$ with
\[
t_3 = \frac{1}{2} + \frac{s^6 - 84s^5 - 378s^4 - 1512s^3 - 5208s^2 - 7236s - 8127s - 874}{432 (s + 1)^2 (s^2 + s + 7) y},
\] (25)
and the transformed location of the extra branching point is
\[
q_3 = \frac{1}{2} - \frac{s (s^4 + 2s^3 + 12s^2 + 20s + 73)}{12 (s + 1) (s + 2) y}.
\] (26)

This parametrizes an algebraic solution $q_3(t_3)$ of $PV_1(1/7, 1/7, 1/3, 6/7)$, of genus 1. An equivalent solution $t_3(q_3 - 1)/(q_3 - t_3)$ of $PV_1(1/7, 1/7, 1/7, 2/3)$ was first found by Kitaev [19] §3 by the pull-back method.

More generally, Kitaev’s method [18] allows further Schlesinger gauge transformations to obtain multiple algebraic Painlevé VI solutions from the same pull-back transformation. These transformations are matrix analogues of (3) with $\varphi(x) = x$. They shift local exponent differences (including $\theta_0, \theta_1, \theta_2, \theta_\infty$) by integers; the total shift sum must be even. The whole construction is called RS-transformations, where R stands for a Rational pull-back, and S stands for a Schlesinger transformation.

**Example 2.11.** Examples 2.9, 2.10 implicitly employ pull-backs of the hypergeometric equation $E(1/2, 1/3, 1/7)$ to isomonodromic Fuchsian equations with 4 singularities at the roots of $V_1, V_2$ (or $U, V$, respectively) and an apparent singularity at $x = q_2$ (or $x = q_3$). This lead to algebraic solutions of $PV_1(1/7, 1/7, 2/7, 6/7)$ and $PV_1(1/7, 1/7, 1/3, 6/7)$ by Theorem 2.8. The same pull-back transformations can be applied to the hypergeometric equations $E(1/2, 1/3, 2/7)$ and $E(1/2, 1/3, 3/7)$, as suggested by Kitaev [18], [19]. The pull-backs of $E(1/2, 1/3, 2/7)$ have the same 4 + 1 singularities, plus a new apparent singularity at $x = \infty$. Schlesinger transformations neutralizing this singularity give algebraic solutions of $PV_1(2/7, 2/7, 4/7, 2/7)$, $PV_1(2/7, 2/7, 1/3, 2/7)$, as demonstrated in [32]. Similarly, the pull-backs of $E(1/2, 1/3, 3/7)$ have the same 4 + 1 singularities, plus a new singularity at $x = \infty$ with the monodromy difference 3. Neutralizing Schlesinger transformations lead to algebraic solutions of $PV_1(3/7, 3/7, 6/7, 4/7)$ and $PV_1(3/7, 3/7, 1/3, 4/7)$, as shown in [32].

It is worth recalling here the Okamoto (also called Bäcklund) transformations [24] that convert $q(t)$ to rational functions of $q(t), dq/dt$ and $t$. The basic transformation acts on the parameters of the Painlevé VI equation as follows:
\[
(\theta_0, \theta_1, \theta_2, \theta_\infty) \mapsto (\Theta - \theta_0, \Theta - \theta_1, \Theta - \theta_2, \Theta - \theta_\infty),
\] (27)
with $\Theta = (\theta_0 + \theta_1 + \theta_2 + \theta_\infty)/2$. Special cases are transformations that shift $(\theta_0, \theta_1, \theta_2, \theta_\infty)$ by integer vectors, with the total shift even. They can be realized by Schlesinger gauge transformations of the Fuchsian equations.

Note that $PV_1(\pm \theta_0, \pm \theta_1, \pm \theta_2, 1 \pm \theta_\infty)$ is the same Painlevé VI equation, hence [27] defines 16 “neighbouring” Painlevé VI equations by Okamoto transformations. A set of fractional linear transformations permutes the 4 singular points. All together [24], these transformations form an affine Weyl group of type $E_6$. Up to the integer shifts and permutation of the singular points, a generic Okamoto orbit contains three distinct Painlevé VI solutions.
Example 2.12. The equations
\[ PV_1(1/7, 7/1, 2/7, 6/7), \quad PV_1(2/7, 4/7, 2/7) \]
\[ PV_1(3/7, 7/6, 4/7) \]
in Example 2.11 and their algebraic solutions are related by the Okamoto transformations. But the equations
\[ PV_1(1/7, 1/7, 1/3, 6/7), \quad PV_1(2/7, 1/3, 2/7), \quad PV_1(3/7, 7/1, 1/3, 4/7) \]
are not related by the Okamoto transformations. For example, the Okamoto orbit of \( PV_1(1/7, 1/7, 1/3, 6/7) \) consists of Schlessinger and fractional-linear transformations of itself and \( PV_1(17/42, 17/42, 17/42, 23/42) \).

3 Differentiation relations from free divisors

Theorem 2.8 gives differential relations between coefficients of AB-maps. Here we observe differential relations with differentiations both with respect to the variable \( x \) and a parameter \( w \).

3.1 Free divisors, logarithmic vector fields

As presented in [15], interesting examples of flat structures, free divisors in the sense of Saito [24] can be constructed from algebraic Painlevé VI solutions. In Dubrovin’s context [7] of Frobenius manifolds, the potentials which are solutions of the Witten-Dijkgraaf-Verlinde-Verlinde equations play a similar key role.

As discussed in §2.4, the use of AB-maps is one of the methods to construct algebraic Painlevé VI solutions. For these reasons, it is meaningful to study a relationship between AB-maps and free divisors. As an observation by comparing AB-maps with free divisors, we recognized that after a suitable homogenization of variables of \((k, \ell, m)\)-minus-4 regular AB-maps, polynomials which define free divisors appear as polynomial components of AB-maps. We explain this observation by taking the following example.

Example 3.1. We homogenize the AB-map \( \varphi_1 \) of Example 2.1 by \( w = v/u^3 \), \( x = uX/v^2 \) with the variables \( u, v, X \) of weights 1, 3, 5, respectively. The weighted-homogeneous polynomials are
\[ P = X, \quad Q = uX + v^2, \]
\[ R = X^3 + 15u^2vX^2 + 20uv^3X + 8v^5, \]
\[ F = X^3 + 2u^2(15v - 32u^3)X^2 + 5uv^2(8v - 19u^3)X + 8(2v - 5u^3)v^4. \]

Correspondingly, they satisfy \( P^3F + 64Q^5 = R^2 \). Let us consider the vector fields
\[ V_1 = u \frac{\partial}{\partial u} + 3v \frac{\partial}{\partial v} + 5X \frac{\partial}{\partial X}, \quad (28) \]
\[ V_2 = -2(v - 3u^3) \frac{\partial}{\partial u} + (X + 3u^2v) \frac{\partial}{\partial v}, \quad (29) \]
\[ V_3 = 3(X + 27u^2v - 64u^3) \frac{\partial}{\partial u} + 8u(7v - 12u^3)v \frac{\partial}{\partial v} - 40v^3 \frac{\partial}{\partial X}. \quad (30) \]
They are logarithmic along the hypersurface $F = 0$, meaning that their action on the polynomial $F$ coincides with some polynomial multiplication:

$$V_1 F = 15F, \quad V_2 F = 30u^2 F, \quad V_3 F = 60(3v - 16u^3)F.$$  \hfill (31)

Consider the matrix

$$M = \begin{pmatrix} 
    u & 3v & 5X \\
    -2(v - 3u^3) & X + 3u^2v & 0 \\
    3(X + 27u^2v - 64u^3) & 8u(7v - 12u^3)v & -40u^3 
\end{pmatrix}$$  \hfill (32)

where the rows represent the vector fields, so that

$$\begin{pmatrix} 
    V_1 \\
    V_2 \\
    V_3 
\end{pmatrix} = M \begin{pmatrix} 
    \partial/\partial u \\
    \partial/\partial v \\
    \partial/\partial X 
\end{pmatrix}.$$  

Then $\det M = -15F$. Existence of 3 logarithmic vector fields along $F = 0$, and the identification of $F$ with $\det M$ up to a constant multiple means that the hypersurface $F = 0$ is a free divisor. More conceptually [22], a characteristic property is that the logarithmic vector fields form a free module over $\mathbb{C}[u, v, X]$.

The Euler vector field $V_1$ acts on the other polynomial components $P, Q, R$ as multiplication by the weighted-homogeneous degrees 5, 6, 9 (respectively). Remarkably, the vector field $V_2$ is logarithmic along the hypersurfaces $P = 0, Q = 0, R = 0$ as well:

$$\begin{align*}
    V_2 P &= 0, \\
    V_2 Q &= 6u^2 Q, \\
    V_2 R &= 15u^2 R. 
\end{align*}$$  \hfill (33)

This special role of $V_2$ is unexpected.

The isomonodromic Fuchsian system can be elegantly expressed in terms of the vector fields

$$\tilde{V}_2 = V_2 - 2u^2 V_1, \quad \tilde{V}_3 = V_3 + 32u^2 V_2 - 12uv V_1.$$  \hfill (34)

The action on the AB-map

$$\tilde{\varphi}_1 = -\frac{P^3 F}{64 Q^5}$$  \hfill (35)

is

$$V_1 \tilde{\varphi}_1 = 0, \quad \tilde{V}_2 \tilde{\varphi}_1 = 0, \quad \tilde{V}_3 \tilde{\varphi}_1 = -\frac{15 R}{PQ} \tilde{\varphi}_1,$$  \hfill (36)

and the pulled-back hypergeometric function

$$f = Q^{1/12} F^{\lambda/15} {}_2F_1 \left( \begin{array}{c} -1/60, 11/60 \\ 2/3 \end{array} \bigg| \tilde{\varphi}_1 \right)$$  \hfill (37)

satisfies the differential system

$$\begin{align*}
    V_1 f &= (\lambda + \frac{1}{2}) f, \\
    \tilde{V}_2 f &= -\frac{1}{2} u^2 f, \\
    \tilde{V}_3^2 f &= -((9v + 20u^3)X + 30u^2 v^2) f. 
\end{align*}$$  \hfill (38)

The last equation has order 2, just as the hypergeometric equation.
As free divisors and AB-maps are defined for many algebraic Painlevé VI solutions, we checked that attractive differential systems like (38) for pulled-back hypergeometric solutions exists in every computed (and homogenized) case. The computed cases are presented in §4. Existence of “universally” logarithmic vector fields as in (39) was observed as well.

**Observation 3.2.** For every computed AB-map \( \varphi(X:u:v) \) in weighted-homogeneous variables \( u, v, X \) of the respective weights \( N_X, N_u, N_v \), there is a vector field

\[
\tilde{A}(X,u,v) \frac{\partial}{\partial X} + \tilde{B}(X,u,v) \frac{\partial}{\partial u} + \tilde{C}(X,u,v) \frac{\partial}{\partial v}
\]  

(39)

linearly independent from the Euler vector field

\[
N_x x \frac{\partial}{\partial x} + N_u u \frac{\partial}{\partial u} + N_v v \frac{\partial}{\partial v}
\]  

(40)

that acts by polynomial multiplication on all polynomial components of \( \varphi \).

This observation is remarkable. As a weaker implication, it says that there are low degree syzygies between \( \partial U/\partial X, \partial U/\partial u, \partial U/\partial v \) and \( U \) for any polynomial component \( U \in \{P,Q,R,F,G,H\} \) as in (9–10). We found the exceptional vector fields by computing the lowest degree syzygy between the derivatives of \( R \), and checking the observation on other polynomial components. The chosen syzygy is always much smaller than alternatives.

### 3.2 Dehomogenization

Observation 3.2 can be modified to apply to non-homogeneous AB-maps \( \varphi(x,w) \). The modified claim is that there exists a single vector field that acts by polynomial multiplication on all polynomial components of \( \varphi \). If de-homogenization of \( \varphi(X:u:v) \) is simply \( u = 1 \), one can find the “universally” logarithmic vector field as a linear combination of (39) and (40) with eliminated \( \partial/\partial u \), and then specialize to \( u = 1 \).

**Example 3.3.** Recall Example 2.1 and consider the vector field

\[
\mathcal{L}_1 = -2x(x+1) \frac{\partial}{\partial x} + (wx + 6w - 15) \frac{\partial}{\partial w}.
\]  

(41)

This vector field acts by polynomial multiplication on all polynomial components of \( \varphi_1(x) \), including on \( x \) and \( x + 1 \). To derive this vector field from Example 3.1 we substitute

\[
\frac{\partial}{\partial u} = x \frac{\partial}{v^2} - \frac{3v}{u^4} \frac{\partial}{\partial w} = \frac{1}{u} \left( x \frac{\partial}{\partial x} - 3w \frac{\partial}{\partial w} \right),
\]

\[
\frac{\partial}{\partial v} = -\frac{2ux}{v^3} \frac{\partial}{\partial x} + \frac{1}{u^4} \frac{\partial}{\partial w} = \frac{1}{v} \left( -2x \frac{\partial}{\partial x} + u \frac{\partial}{\partial w} \right)
\]

into \( V_2 \), and recognize \( \mathcal{L}_1 \) after multiplication by \( u/v \).
Example 3.4. For Example 2.9, the vector field
\[ L_2 = (x - 1)(3x + w) \frac{\partial}{\partial x} + w(7x + 2w - 9) \frac{\partial}{\partial w} \] (42)
acts on \( P, R, G_1, G_2 \) and even on \( r_0 \) by polynomial multiplication. Considering \( \varphi_2(x, s) \) after the substitution (19), the vector field
\[ \tilde{L}_2 = -14(s^2 + 3)x(x - 1) \frac{\partial}{\partial x} + (2s(s^2 + 7)x + (s + 1)(s - 3)(s^2 + s + 2)) \frac{\partial}{\partial s} \]
is logarithmic for every polynomial component (with cleared denominators \( \in \mathbb{Q}[s] \)). As \( G_2 \) factors \((x - x_1)(x - x_2)(x - x_3)\) over \( \mathbb{Q}(s) \), the vector field is logarithmic even along the hypersurfaces \( x - x_k = 0 \) for \( k \in \{1, 2, 3\} \).

For Example 2.10, the vector field
\[ L_3 = (x^2 - 2(w + 3)x + 24) \frac{\partial}{\partial x} + (7wx + 4w^2 - 24w - 24) \frac{\partial}{\partial w} \] (43)
acts on \( P, R, F, G \) by polynomial multiplication.

If a vector field is logarithmic along two hypersurfaces \( F = 0, G = 0 \), it is logarithmic along \( FG = 0 \) as well. In the observed examples, the exceptional vector fields annihilate the AB-maps \( \varphi_j \). Consequently, those vector fields can be normalized to
\[ A(x, w) \frac{\partial}{\partial x} + B(x, w) \frac{\partial}{\partial w}, \] (44)
with \( A(x, w) = \partial \varphi_j / \partial w \) and \( B(x, w) = -\partial \varphi_j / \partial x \). This explains why the coefficient to \( \partial / \partial w \) or \( \partial / \partial s \) is linear in \( x \) in the above examples, and the roots are the extra branching points \( q_j \) for \( j \in \{1, 2, 3\} \). The extra branching point is the only root of \( \partial \varphi_j / \partial x \) that is not a root of \( \partial \varphi_j / \partial w \).

Observation 3.2 becomes simpler in a dehomogenized form. With more specificity, we formulate the following conjecture.

Conjecture 3.5. For any AB-map \( \varphi(x, w) \) with a field of definition \( K = \mathbb{Q}(w) \), there exists a vector field (44) that acts on every \( K[x] \)-irreducible factor of the numerators and the denominators of \( \varphi \) and \( \varphi - 1 \) by polynomial multiplication. The vector field annihilates \( \varphi(x, w) \).

As exemplified above, the conjecture implies that \( B(x, w) \) is linear in \( x \), and its root gives the extra branching point of \( \varphi \) outside the critical fibers \( \{0, 1, \infty\} \). By the asymptotics at \( x = \infty \), the degree of \( A(x, w) \) in \( x \) is at most 2.

We checked the conjecture for all known AB-maps, including the \((2, 3, \infty)-\text{minus-5 maps from} [15] \) that we mentioned in Remark 2.6. Explicit prior knowledge of these vector fields should be very useful in speeding up computation of a desired AB-map, by utilizing new algebraic equations for undetermined coefficients.
4 Algebraic Painlevé VI solutions

Algebraic solutions of the Painlevé VI equation were recently classified by Lisovyy and Tykhyy [21]. Apart from infinite families of rational or Picard’s $P_{VI}(0,0,0,1)$ solutions presented in [21] Propositions 49, 51 and their Okamoto orbits, there is a finite list (up to Okamoto transformations) of 3 parametric and 45 non-parametric solutions. The non-parametric solutions were already derived by Dubrovin, Mazzocco [8], Kitaev [18], [19] and Boalch [2], [3], [4] in 2000–2007.

4.1 AB-maps for algebraic Painlevé VI solutions

Kitaev conjectured [17] that all algebraic solutions of the Painlevé VI equation can be obtained from pull-back transformations by $(k,\ell,m)$-minus-4 regular AB-maps, up to Okamoto and Schlessinger transformations. By checking the Lisovyy-Tykhyy classification we see that this conjecture is true for the 3 + 45 solutions in [21]:

(i) The 3 Okamoto orbits #II–#IV of parametric solutions have corresponding pull-back transformations, as first established in [11].

(ii) The Lisovyy-Tykhyy solutions #8, #33 are obtained by the pull-back maps $\varphi_2(x)$, $\varphi_3(x)$ of Examples 2.9, 2.10. The similar solutions #32, #34 solve $P_{VI}(2/7,2/7,1/3,2/7)$ and $P_{VI}(3/7,3/7,1/3,4/7)$. They are obtained from $\varphi_3(x)$ by additional Schlessinger transformations described in Example 2.11.

(iii) The other solutions in [21] correspond (up to Okamoto transformations) to isomonodromic Fuchsian equations with finite monodromy. Existence of pull-backs is implied by celebrated Klein’s theorem [20]: any second order Fuchsian equations with finite monodromy is a pull-back of a hypergeometric equation with finite monodromy.

In (iii), there are 33 Okamoto orbits corresponding to Fuchsian systems with the icosahedral monodromy group; and 7 octahedral (#4, #5, #9, #10, #20, #21, #30), 1 tetrahedral (#3) cases. As Schlessinger transformations do not change monodromy of Fuchsian equations, the exponent differences $\theta_0, \theta_1, \theta_t, \theta_\infty$ can be shifted by integers. This gives infinitely many Kleinian pull-backs by AB-maps of unbounded degree in these Okamoto orbits. Okamoto transformations are necessary, as (for example, #16, #17, #31 in [21]) the Dubrovin-Mazzocco solutions of $P_{VI}(0,0,0,4/5)$, $P_{VI}(0,0,0,2/5)$, $P_{VI}(0,0,0,2/3)$ correspond to Fuchsian systems with logarithmic singularities and cannot be obtained directly by a pull-back transformation.

With the construction of Theorem 2.8 in mind, we computed AB-maps for all Lisovyy-Tykhyy cases algebraic Painlevé VI solutions of genus 0 or 1. The results are presented in Table 4.1 with the genus 0 and 1 cases separated by a horizontal line. The first column gives the enumeration in [21].

The second column of Table 4.1 gives the branches permutation monodromy of the Painlevé VI solutions, using the fact that in a parametrization $(q(s),t(s))$ of those algebraic solutions, $t(s)$ is a Belyi map (by the Painlevé property). The second column gives the passport of that Belyi map (without the [ ] delimiters), but the notation is compacted when branching patterns in 2 or all 3 fibers is the same. The
Table 1: AB-maps for algebraic Painlevé VI solutions of genus 0 (in the upper part) or genus 1 (in the lower part).
repetition is indicated by the number of //’s. For example, $3^2/2^1$ for the solution #3 means the passport $[3^2/3^2/2^1]$, and $3^22^2/2$ for the solution #15 means the passport $[3^22^2/3^2/2^2]$, etc. The algebraic degree of the Painlevé VI solution can be quickly determined from the passport.

The third column gives the exponent differences of representative Painlevé VI equations $P_{VI}(\theta_0, \theta_1, \theta_t, \theta_{\infty})$. Two distinct Painlevé VI equations are given for the parametric solution IV, because they are generally not related by Shlessinger and fractional-linear transformations, and AB-maps (of degree 3 and 6) exist for both of them. The case #30 is represented by two Painlevé VI equations solutions for the same reason, while #3, #5 take two lines each because two AB-maps for them are already known.

The fourth column gives the passport of a $(2, 3, m)$-minus-4 regular AB-map giving an algebraic solution of $P_{VI}(\theta_0, \theta_1, \theta_t, \theta_{\infty})$ by Theorem 2.8. The three fibers are ordered to match the order of the $\theta_j$’s in the third column conveniently. The fifth column either gives the degree $d$ of the AB-map if it was not computed previously, or gives references to [10, Table 1] (by the $N_j$-label) and other publications [1], [18], [19], [32]. Given $\theta_0 > 0, \theta_1 > 0, \theta_t > 0, \theta_{\infty} < 1$, the degree of the pull-back map from $E(1/2, 1/3, 1/m)$ equals

$$d = \frac{\theta_0 + \theta_1 + \theta_t - \theta_{\infty}}{2 + \frac{1}{3} + \frac{1}{m} - 1}.$$ (45)

This follows from the Hurwitz theorem, or (assuming the AB-map is defined over $\mathbb{R}$) by geometric consideration of spherical or hyperbolic areas in analytic continuation of pulled-back hypergeometric functions by the Schwarz reflection principle [32, Lemma 6.2, etc.].

The last two columns characterize an important Belyi map derived from each AB-map $\varphi(x, w)$. All presented AB-maps are parametrized (as Hurwitz spaces of dimension 1) by algebraic curves of genus 0, with $w$ as a minimal projective parameter of those curves. The fourth fiber $\psi(w) = \varphi(q, w)$ of the extra branching point $x = q$ is a function of $w$ that is intrinsic to $\varphi(x, w)$. It gives the braid group action on $\varphi(x, w)$ as the fourth fiber is moved continuously around the other three fibers. The function $\psi(w)$ is a Belyi map [10, Remark 5.3], and is a good measure of complexity of the AB-map. The passport and degree $d^*$ of $\psi(w)$ are given in the last two columns of Table 4.1. For the $w$-values in the three critical fibers of $\psi(w) \in \{0, 1, \infty\}$, the AB-map specializes to Belyi maps of degree $\leq d$.

**Remark 4.1.** The cases #32, #34 are skipped in Table 4.1, because Schlessinger transformations are necessary to obtain those Painlevé VI solutions. As we discussed in Example 2.11 the AB-map of #33 has to be applied for a pull-back from $E(1/2, 1/3, 2/7)$ or $E(1/2, 1/3, 3/7)$. Kitaev [18] stresses that pairs of icosahedral cases with the same monodromy (such as #6, #7; see the second column in Table 4.1) can be similarly obtained by pull-backs with respect to a common AB-map applied to $E(1/2, 1/3, 1/5)$ and $E(1/2, 1/3, 2/5)$, with a Schlessinger transformation necessary after one or other pull-back.

Examples of AB-maps for the solutions #40 – #45 in [21] of genus 2, 3 or 7 remain to be computed. But even these cases can be considered as handled if we allow Kitaev’s quadratic transformations [16] of Painlevé VI solutions and corresponding
From their leading coefficients we can consequently eliminate all coefficients of \(a\) except \(a\) and \(H\) ansatz (12)–(14) with \(P\) with solution #15, we are looking for a polynomial identity

Example 4.2. To find an AB-map with the passport \([3^6/5^321/2^81^2]\) for the algebraic solution #15, we are looking for a polynomial identity

\[
P^3 + r_0 Q^5 G = R^2 H
\]

with \(P = x^6 + a_1 x^5 + \ldots + a_6, Q = x^3 + b_1 x^2 + b_2 x + b_3, R = x^8 + c_1 x^7 + \ldots + c_8, G = x\) and \(H = x^2 + d_1 x + d_2\). After clearing denominators in the logarithmic derivative ansatz (12)–(14) with \(h_1 = h_2 = 2, S = GH/(x - q)\), we get the equations

\[
0 = (2q + 7b_1 - a_1 - 2c_1) x^8 + (12b_2 - 4a_2 - 2c_2 + 2q c_1 + 4a_1 b_1) x^7 + \ldots
\]

\[
0 = (2q + 7b_1 - 4a_1 + d_1) x^{12} + (12b_2 - 4a_2 - 2c_2 + 4aq_1 + 5b_1 c_1 + \ldots) x^{11} + \ldots
\]

From their leading coefficients we can consequently eliminate all coefficients of \(P, R\) except \(a_2\). Next we compute the pull-back (15)–(16), with \(k = 3, \ell = 2, m = 5\), thus \(a = -1/60, b = 11/60\). The coefficient \(W_2\) in (15) equals

\[
\frac{27 x^9 + (11a_1 + 82b_1 - 104d_1 - 289q)x^8 + (11a_2 + 282b_2 - 224d_2 - \frac{11}{4} b_1^2 + \ldots)x^7 + \ldots}{900 (q - x) H G^2 Q^2}.
\]

To compute the corresponding equation (8), we start with this solution \(q_{15}(t_{15})\) of \(P_{VI}(1/5, 1/2, 1/2, 3/5)\):

\[
q_{15} = -\frac{2s(s - 1)(s - 5)^2(s^2 - 3)(s^2 + 4s + 5)}{(s + 1)^2(s + 5)(s^2 - 4s + 5)(s^4 + 6s^2 - 75)},
\]

\[
t_{15} = -\frac{(s - 1)^3(s - 5)^3(s^2 + 4s + 5)^2}{(s + 1)^3(s + 5)^3(s^2 - 4s + 5)^2}.
\]

It differs from the solution of \(P_{VI}(1/2, 1/5, 1/2, 2/5)\) in (21) by the fractional-linear transformation \((q_{15}, t_{15}) \mapsto (1 - q_{15}, 1 - t_{15})\). We express the entities in (13)–(16) in the parametrized form:

\[
p_{15} = -\frac{s(s + 1)^2(s + 5)(s^2 - 4s + 5)(s^4 + 6s^2 - 75)}{10(s - 1)(s - 5)^2(s^4 - 25)(s^2 + 4s + 5)}, \quad \Theta = -\frac{3}{100}, \quad \text{etc.}
\]
The symmetry between \( x = 1 \) and \( x = t_{15} \) is realized by \( s \mapsto -s \). To identify \((x - 1)(x - t_{15})\) with the irreducible polynomial \( H \), we scale \( x \mapsto x/K \) with

\[
K = s(s + 1)^3(s + 5)^3(s^2 - 4s + 5)^2.
\]

The coefficient \( W_1 \) in \((4)\) is thereby divided by \( K^2 \) (along with the substitution of \( x \) and becomes a function of the invariant \( u = s^2 \):

\[
W_1 = \frac{3x^2 + 6u(41u^6 - 900u^5 + \ldots + 46875)}{u^2 + 6u - 75} x + \frac{4u^2(u - 1)(u - 3)(u - 25)^2(u^2 - 6u + 25)(5u^5 + \ldots - 9375)}{u^2 + 6u - 75},
\]

with explicitly

\[
H = x^2 - 4u(5u^4 - 80u^3 + 678u^2 - 2000u + 3125)x - u(u - 1)^3(u - 25)^3(u^2 - 6u + 25)^2,
\]

\[
q = -\frac{2u(u - 1)(u - 3)(u - 25)^2(u^2 - 6u + 25)}{u^2 + 6u - 75}.
\]

This parametrizes \( d_1, d_2, q \). The remaining coefficients \( a_2, b_1, b_2, b_3 \) are obtained from the identification \( W_1 = W_2 \). After clearing denominators, we get a polynomial expression of degree 8 in \( x \). The leading coefficients gives immediately

\[
b_1 = -\frac{8u(u^6 - 15u^5 - 14u^4 + 3326u^3 - 29575u^2 + 100625u - 187500)}{u^2 + 6u - 75}.
\]

The coefficient to \( x^7 \) is linear in \( a_2, b_2 \), and the next two coefficients are linear in \( b_3 \). After elimination of \( b_2, b_3 \), we get a quadratic polynomial in \( a_2 \) that factorizes. We check both candidates for \( a_2 \) on another equation, and the correct value is

\[
a_2 = -\frac{64u(u - 25)^3(11u^6 - 165u^5 + 968u^4 - 3082u^3 + 6875u^2 - 20625u + 3125)}{u^2 + 6u - 75},
\]

\[
b_2 = \frac{512u^2(u - 3)(u - 25)^6(u^2 - 6u + 25)^2}{u^2 + 6u - 75}
\]

and the other coefficients. The factor \( r_0 \) can be determined by dividing the left-hand side of \((16)\) by \( H \) with respect to \( x \), and looking at the remainder. We find \( r_0 = 27u(u^2 + 6u - 75)^5 \).

Simplification of the obtained AB-map to a presentable size is a tedious, less automated task that may take much more time than the above computation. The basic ideas are to simplify the Belyi map \( \varphi(q(u), w) \) stated in the last two columns in Table \((4,3)\) (of degree 60); simplification of elliptic surfaces such as \( y^2 = GQ \); and considering factorization of the discriminants, resultants of \( P, Q, R, H \) with respect to \( x \). For example, the transformation \( u = 5v, x = 100x + 500v(v - 5)^2(5v^2 - 6v + 5) \) is useful for a start, introducing high powers of \((v - 1)\) in the coefficients while keeping the powers of \( v, v - 5, 5v^2 - 6v + 5 \).
Example 4.3. To find an AB-map with the passport \([3^4 1^2 5^2 13^2 7^2]\) for the Painlevé VI solution \#22, we are looking for a polynomial identity

\[ P^4 F + r_0 Q^5 G = R^2 \]  

(51)

with \( P = x^4 + a_1 x^3 + \ldots + a_4, Q = x^2 + b_1 x + b_2, R = x^7 + c_1 x^6 + \ldots + c_7, \) \( F = x^2 + d_1 x + d_2 \) and \( G = x + e_1 \). We do not hurry with setting \( e_1 = 0 \) by choosing the point \( x = 0 \). In the logarithmic derivative ansatz we have \( h_1 = h_2 = 3, S = FG/(x - q) \). It allows to eliminate straightforwardly all coefficients of \( P, R \) except \( a_3 \). We calculate the coefficient \( W_2 \) in (13).

To compute the Fuchsian equation (12), we use the Painlevé VI solution of \( P_{VI}(1/3, 1/3, 1/5, 2/5) \) from [21], with \( z = \sqrt{3(s + 1)(8s^2 - 9s + 3)} \):

\[
q_{22} = \frac{1}{2} + \frac{140s^6 + 1029s^5 - 1023s^4 + 360s^3 - 288s^2 + 27s + 27}{18z(s + 1)(7s^3 - 3s^2 - s + 1)},
\]

(52)

\[
t_{22} = \frac{1}{2} + \frac{40s^6 + 540s^5 - 765s^4 + 540s^3 - 270s^2 + 27}{6z(s + 1)^2(8s^2 - 9s + 3)}.
\]

(53)

We wish to utilize the symmetry \( z \rightarrow -z, (q_{22}, t_{22}) \rightarrow (1 - q_{22}, 1 - t_{22}) \) while identifying \( FG \) with \( x(x - 1)(x - t_{22}) \). For this purpose we find an elliptic surface that is described over \( \mathbb{Q}(t(1 - t)) \) and has the same \( j \)-invariant as the Legendre family \( y^2 = x(x - 1)(x - t) \). The following elliptic surface has these properties:

\[
y^2 = (x - t) (x - 1 + t) (x - 2t(1 - t)).
\]

(54)

Therefore we identify

\[
F = (x - t_{22}) (x - 1 + t_{22}) = x^2 - x + t_{22} (1 - t_{22}),
\]

(55)

\[
G = x - 2 t_{22} (1 - t_{22})
\]

initially. Here \( t_{22} (1 - t_{22}) \) is not dependent on \( z \):

\[
t_{22} (1 - t_{22}) = -\frac{16s^5(s - 3)^5(5s - 3)^2}{27(s + 1)^4(5s + 1)(8s^2 - 9s + 3)^2}.
\]

(56)

Additionally, we transform

\[
x \mapsto \frac{1}{2} + \frac{40s^6 + 540s^5 - 765s^4 + 540s^3 - 270s^2 + 27}{54(s + 1)^4(5s + 1)(8s^2 - 9s + 3)^3} x
\]

(57)

to get the simpler

\[
F = x^2 - 27(s + 1)^4(5s + 1)(8s^2 - 9s + 3)^3,
\]

(58)

\[
G = x + 40s^6 + 540s^5 - 765s^4 + 540s^3 - 270s^2 + 27.
\]

This parametrizes \( d_1, d_2, e_1 \). An isomorphism from the Legendre curve to (54) is given by \( x \mapsto (x - t)/(1 - 2t) \). The composition of this isomorphism (with \( t = t_{22} \)) and (57) is the transformation

\[
x \mapsto Kx + \frac{1}{2}, \quad \text{with} \quad K = -\frac{z}{18(s + 1)^2(5s + 1)(8s^2 - 9s + 3)^2}.
\]

(59)
After this whole transformation, the coefficient $W_1$ in (8) equals

$$W_1 = \frac{77x^2 + \frac{8(30625s^9 + \ldots - 2673s^2 + 2673)}{3(7s^3 - 3s^2 - s + 1)}x + \frac{(s+1)(8s^2 - 9s + 3)(6664000s^{12} + \ldots + 136323)}{3(7s^3 - 3s^2 - s + 1)}}{900(q-x)FG}$$

with

$$q = -\frac{(s+1)(8s^2 - 9s + 3)(140s^6 + 1029s^5 - 1023s^4 + 360s^3 - 288s^2 + 27s + 27)}{3(7s^3 - 3s^2 - s + 1)}.$$

The identification $W_1 = W_2$ leads to a polynomial of degree 6 in $x$ after clearing the denominators. Its 3 leading coefficients give straightforwardly

$$b_1 = \frac{2(8s^2 - 9s + 3)^2(16s^4 - 8s^3 + 8s^2 + 15s + 3)}{7s^3 - 3s^2 - s + 1},$$

$$b_2 = -\frac{(s+1)^2(8s^2 - 9s + 3)^3(625s^6 + 1386s^5 - 567s^4 + 540s^3 - 27s^2 - 162s - 27)}{7s^3 - 3s^2 - s + 1},$$

$$a_3 = -2(8s^2 - 9s + 3)^3(192500s^{10} + 300697s^9 + 68513s^8 + 41532s^7 + 297588s^6 - 867785s^5 + 57510s^4 + 43740s^3 - 19440s^2 - 10935s - 1215).$$

The logarithmic derivative ansatz already gave expressions of the other coefficients in terms of $a_3, b_1, b_2, d_1, d_2, e_1, q$. With all coefficients parametrized, we find $r_0 = 13824(5s + 1)(7s^3 - 3s^2 - s + 1)^5$.

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