DEFORMATIONS OF VECTOR BUNDLES ON COISOTROPIC SUBVARIEITIES VIA THE ATIYAH CLASS

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Abstract. Using the Atiyah class we give a criterion for a vector bundle on a coisotropic subvariety, \( Y \), of an algebraic Poisson variety \( X \) to admit a first and second order noncommutative deformation. We also show noncommutative deformations of a vector bundle are governed by a curved dg Lie algebra which reduces to the classical relative Hochschild complex when the Poisson structure on \( X \) is trivial.

1. Introduction

Let \( X \) be a smooth algebraic variety and let \( \mathcal{O}_X \) denote the sheaf of regular functions on \( X \). Recall a deformation quantization of order 2 of \( X \) is a flat sheaf \( \mathcal{A}_2 \) of algebras over \( k[\epsilon]/\epsilon^3 \) such that on affine subsets \( U_i \) of a Zariski open covering of \( X \) we have \( \mathcal{A}_2|_{U_i} \simeq (\mathcal{O}_X \oplus \epsilon \mathcal{O}_X \oplus \epsilon^2 \mathcal{O}_X)|_{U_i} \) as sheaves of \( k[\epsilon]/\epsilon^3 \)-modules. The product is locally given by

\[
a \ast_i b = ab + \epsilon \alpha_1^{Xi}(a, b) + \epsilon^2 \alpha_2^{Xi}(a, b)
\]

where \( a, b \) are local sections of \( \mathcal{O}_{U_i} \) and \( \alpha_1^{Xi}(a, b) = \frac{1}{2} P(da, db) \) for a globally defined bivector \( P \in H^0(X, \wedge^2 T_X) \) and \( \alpha_2^{Xi} \) is a bidifferential operator. On double intersection \( U_i \cap U_j \) the restrictions are identified by sending a regular function \( f \) to \( f + \epsilon \beta_1^{Xij}(f) + \epsilon^2 \beta_2^{Xij}(f) \) where \( \beta_1^{Xij}, \beta_2^{Xij} \) are differential operators from \( \mathcal{O}_{U_i \cap U_j} \) to \( \mathcal{O}_{U_i \cap U_j} \). In a far fancier language than we will need here \( U \mapsto \mathcal{A}_2(U) \) is a presheaf of algebroids [18].

In this paper we consider the higher rank version of [4]: let \( Y \subset X \) be a smooth closed coisotropic subvariety of a smooth Poisson variety \( X \) and \( E \) a vector bundle on \( Y \). Viewing \( E \) as coherent \( \mathcal{O}_X \)-module a natural question to ask is when does \( E \) admit a flat second order deformation to an \( \mathcal{A}_2 \)-module. This means, we want a coherent sheaf \( \mathcal{E}_2 \) which splits locally on an affine open cover \( \{U_i\} \), with a module action given by

\[
a \ast_i e = ae + \epsilon \alpha_1^i(a, e) + \epsilon^2 \alpha_2^i(a, e)
\]

and transition functions on \( U_i \cap U_j \) given by

\[
e \mapsto e + \epsilon \beta_1^{ij}(e) + \epsilon^2 \beta_2^{ij}(e)
\]

where \( a, e \) are local sections of \( \mathcal{O}_X \), and \( E \), respectively, and \( \alpha_1^i, \alpha_2^i \) and \( \beta_1^{ij}, \beta_2^{ij} \) are (bi)differential operators.

For simplicity we set \( F(E) := F \otimes_{\mathcal{O}_Y} \text{End}_{\mathcal{O}_Y}(E) \) where \( F \) is a coherent sheaf on \( X \). Using spectral sequences there are three obstructions to the existence of \( \mathcal{E}_1 \) in \( H^0(Y, \wedge^2 N(E)) \), \( H^1(Y, N(E)) \) and \( H^2(Y, \mathcal{O}_Y(E)) \) where \( N \) is the normal bundle of \( Y \) in \( X \). The first obstruction measures whether \( \mathcal{E}_1 \) exists locally in the Zariski/étale topology. If an infinitesimal deformation exists locally the class...
in $H^1(Y, N(E))$ is well-defined and its vanishing is equivalent to the existence of transition functions $\beta_{ij}^k$ which agree with the module structure. The class in $H^2(Y, \mathcal{O}_Y(E))$ is well-defined when the previous class vanishes and this class vanishes precisely when the transition functions satisfy the cocycle condition on each triple intersection $U_i \cap U_j \cap U_k$. The class in $H^2(Y, \mathcal{O}_Y(E))$ for $Y = X$ has been studied in [4]. When $Y \subset X$ the author believes it is connected to Rozanksy-Witten invariants but we leave this for future study cf. [3].

In [3] it was shown the first obstruction class is the image of $P$ in $H^0(Y, \wedge^2 N(E))$ and its vanishing is equivalent to $Y$ be coisotropic cf. Lemma 2.2. Recall, that $Y$ is coisotropic if $P(I_Y, I_Y) \subset I_Y$ where $I_Y$ is the sheaf of regular functions vanishing on $Y$. With this in mind, we now assume that $Y$ is coisotropic. By coisotropness of $Y$ the bivector $P$ defines a morphism $p : N^\vee \to T_Y$ along with its adjoint $p^* : \Omega^1_Y \to N$. Throughout the paper we fix a line bundle $L$ which admits a first/second order deformation. In the case when $\beta_1^X = 0$ and $\alpha_2^X$ is symmetric we can take $L = (\det N)^{1/2}$ [3].

Denote by

\[ at_N(E \otimes \mathcal{O}_Y L^\vee) := p^*(at(E \otimes \mathcal{O}_Y L^\vee)) \]

the Yoneda product of $p^*$ and $at(E \otimes L^\vee) \in H^1(Y, \Omega^1_Y(E))$. Where $at(M)$ is the Atiyah class of a vector bundle $M$ [2].

**Theorem 1.1.** Let $X$ be a smooth algebraic variety with a bivector $P$ and $Y$ a smooth coisotropic subvariety with a vector bundle $E$. If $E$ admits a first order deformation $\mathcal{E}_1$ then

\[ at_N(E \otimes \mathcal{O}_Y L^\vee) = 0 \]

in $H^1(Y, N(E))$. If, in addition, $H^2(Y, \mathcal{O}_Y(E)) = 0$ the above equality in $H^1(Y, N(E))$ is also sufficient for the existence of a first order deformation. In particular, a first order deformation exists when $X$ and $Y$ are affine. Moreover, in the affine case there is a globally split deformation, i.e. $\mathcal{E}_1 \simeq E \oplus \epsilon E$ as sheaves of $k[\epsilon]/\epsilon^2$-modules.

A first order deformation up to isomorphism is given by a collection of operators $\gamma^i_F : N^\vee \to \mathcal{D}^\vee_{\mathcal{O}}(E)$ on a Zariski open cover $\{U_i\}$ which satisfy a gluing condition on $U_i \cap U_j$ cf. Proposition 2.1. $\mathcal{D}^\vee_{\mathcal{O}}(E)$ are first order differential operators with scalar principal symbol. Using this we give an explicit connection on $E \otimes L^\vee$ which represents the class in Theorem 1.1.

With regards to second order deformations we tacitly assume $\beta_1^X \equiv 0$. For $A_2$ a second order deformation $\{a, b\}_P := 2\alpha_1^X(a, b)$ is a Lie bracket. By coisotropness of $Y$ the conormal bundle $N^\vee = I_Y/I_Y^2$ becomes a Lie algebra where $I_Y$ is the ideal of functions that vanish on $Y$. By Proposition 2.1 a first order deformation gives a global operator $\gamma$ which defines a morphism between Lie algebras which will not respect the bracket in general (we are using the assumption that $\beta_1^X \equiv 0$). The curvature, $c(\gamma)$, measures the failure of $\gamma$ to be a morphism of Lie algebras. We define the normal complex of $E$ as

\[ (1.1) \quad \mathcal{N}^\bullet_E : \left\{ 0 \to \mathcal{O}_Y(E) \xrightarrow{d_{NE}} N(E) \xrightarrow{d_{NE}} \wedge^2 N(E) \xrightarrow{d_{NE}} \ldots \right\} \]
where the odd derivation is given by
\[
d_{N_E} \omega(x_0, \ldots, x_{n+1}) = \sum_{j=0}^{n+1} (-1)^j \left[ \gamma(x_j, \cdot), \omega(x_0, \ldots, \hat{x_j}, \ldots, x_{n+1}) \right] + \sum_{i<j} (-1)^{i+j} \omega(\{x_i, x_j\}p, x_0, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_{n+1})
\]
and the \(x_i\)'s are local sections of \(N^\vee\). A straightforward but tedious calculation shows \(c(\gamma) \in H^0(Y, \wedge^2 N(E))\). Another standard computation using the Jacobi identity for \(\{,\}_p\) gives \(d_{N_E}^2 \omega = [c(\gamma), \omega]\) and \(d_{N_E} c(\gamma) = 0\) (second Bianchi identity) which make \(N_E\) into a curved dg Lie algebra \[20\]. When \(E \subset N_E\) is weakly obstructed meaning \([c(\gamma), \omega] = 0\) for all \(\omega\). In the case when \(E\) is rank 1 the complex is automatically \("weakly obstructed."

For ease of the notation we make a definition similar to Deligne’s \(\lambda\)-connections.

**Definition 1.2.** A \((\lambda, \mu)\)-connection on a vector bundle \(E\) is a \(k\)-linear operator \(\nabla : N^\vee \to \mathcal{D}_\mathcal{O}(E)\) whose principal symbols are
\[
\nabla(ax, e) - a \nabla(x, e) = \lambda p(x)(a)e; \quad \nabla(x, ae) - a \nabla(x, e) = \mu p(x)(a)e
\]
We denote the set of \((\lambda, \mu)\)-connections by \(\mathcal{D}^1_{(\lambda, \mu)}(E)\). A \((0, 1)\)-connection is an \(N^\vee\)-connection from \([4]\).

**Theorem 1.3.** Assume \(E\) admits a second order deformation \(E_2\) and \(\beta^X_1 \equiv 0\) then \(c(\nabla) = 0\)
where \(\nabla\) is the \((0, 1)\)-connection on \(E \otimes L^\vee\) given by the first order deformation. If \(H^1(Y, N(E)) = H^2(Y, \mathcal{O}_Y(E)) = 0\) this equality also implies the existence of a second order deformation. For affine \(X\) and \(Y\) the deformation of \(E\) may be chosen globally split: \(E_2 \simeq E \oplus E \oplus E^2\) as sheaves of \(k[\epsilon]/\epsilon^3\)-modules.

The outline of the paper is as follows: In section 2 we give the proofs of Theorems 1.1 and 1.2. Section 3 we show a flat bundle can be deformed to second order after twisting by a line bundle which admits a second order deformation. The last section shows deforming a module is not governed by dg Lie algebra but a curved dg Lie algebra defined over ring of formal power series. We have also included an appendix with the module equations to order 2 and a version of the HKR theorem which we use throughout the paper.

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## 2. First and second order deformations

### 2.1. First Order.
Suppose there is a first order deformation \(E_1\) of a vector bundle \(E\). This means there is an affine open cover \(\{U_i\}\) of \(X\) such that \(E_1 \simeq (E \oplus \epsilon E)|_{U_i}\) for all \(i\) as sheaves of \(k[\epsilon]/\epsilon^2\)-modules. In particular by \[1.3\] the operator \(\alpha_1^i\) vanishes on \(I^2_E \otimes \hat{E}\). Denote by \(\gamma_E^1\) the restriction of \(\alpha_1^i\) to \(N^\vee \otimes E \to E\) which we view as a bidifferential operator on \(Y \cap U_i\). Applying \[1.5\] twice implies \(\gamma_E^1\) is a \((1/2, 1)\)-connection. On double intersections \[1.7\] reduces to

\[
(2.1) \quad \gamma_E^1(x, e) - \gamma_E^1(x, e) + \beta^X_{1i}(x)e = 0
\]
The following proposition proven in \[3\] determines a first order deformation up to isomorphism.
Proposition 2.1. The collection of $(1/2, 1)$-connections $\{\gamma^i_E\}$ defines $\mathcal{E}_1$ uniquely up to isomorphism if $H^1(Y, \mathcal{O}_Y(E)) = 0$. Conversely, if $Y$ satisfies $H^2(Y, \mathcal{O}_Y(E)) = 0$, for any such collection of $(1/2, 1)$-connections satisfying (2.1), there exists a first order deformation $\mathcal{E}_1$ inducing it.

A local first order deformation exists if and only if $P$ projects to zero in $H^0(Y, \wedge^2 N(E))$ which is a priori weaker than being coisotropic. However, an easy application of the HKR theorem shows a first order deformation locally exists if and only if $Y$ is coisotropic.

Lemma 2.2. The projection of $P$ is contained in $H^0(Y, \wedge^2 N) \subset H^0(Y, \wedge^2 N(E))$. Locally there exists a first order deformation if and only if the projection of $P$ vanishes, that is $Y$ is coisotropic.

Proof. The following proof is an application Lemma [A.1] which will be used repeatedly, so we will give all the details. A first order deformation locally exists if and only if there is an $\alpha_1^Y$ which satisfies (A.3). Applying Lemma [A.1] we must show that $G(a, b, e) := \alpha_1^Y(a, b)e$ is symmetric when restricted to $I \otimes I \otimes E \rightarrow E$ and is a cocycle i.e.

$$aG(b, c, e) - G(ab, e, e) + G(a, bc, e) - G(a, b, ce) = 0$$

This holds since $\alpha_1^Y$ is a cocycle in $C^*(\mathcal{O}_X, \mathcal{O}_X)$ [19]. The projection of $P$ in $H^0(Y, \wedge^2 N(E))$ is the anti-symmetrization of $G(x, y, e)$ for $x, y \in \mathcal{I}_Y$. This is a scalar endomorphism since $G(a, b, e)$ is a scalar endomorphism. The cocycle $\alpha_1^Y$ comes from the Poisson structure hence is antisymmetric. Since a cocycle is symmetric if and only if the anti-symmetrization vanishes we must have $2\alpha_1^Y(x, y)e = 0$ for all $x, y \in I$ and $e \in E$. This implies $\alpha_1^X(x, y) \in I$ for all $x, y \in I$ which is the coisotropic condition.

We first need a couple of lemmas which will be useful in the proof of Theorem [1.1] and later in the text.

Lemma 2.3. Let $E, F$ be two vector bundles on $Y$ with connections $\gamma_E \in D(\lambda, \mu)(E)$ and $\gamma_F \in D(\lambda, -\mu)(F)$. Then

$$(2.2) \quad \gamma_{E \otimes F}(x, e \otimes f) := \gamma_E(x, e) \otimes f + e \otimes \gamma_F(x, f)$$

defines an element of $D(\lambda+\lambda', -\mu')(E \otimes \mathcal{O}_Y F)$. The curvature of $\gamma_{E \otimes F}$ is given by

$$c(\gamma_{E \otimes F})(x, y)(e \otimes f) = c(\gamma_E)(x, y)(e) \otimes f + e \otimes c(\gamma_F)(x, y)(f)$$

Proof. The proof is by direct calculation: let $a \in \mathcal{O}_Y$, $x \in N^\vee$ then

$$\gamma_{E \otimes F}(ax, e \otimes f) - a\gamma_{E \otimes F}(x, e \otimes f) = \gamma_E(ax, e) \otimes f + e \otimes \gamma_F(ax, f) - a\gamma_E(x, e) \otimes f - ae \otimes \gamma_F(x, f)$$

$$= \lambda p(x)(a)e \otimes f + \lambda' p(x)(ae) \otimes f$$

$$= (\lambda + \lambda') p(x)(a)e \otimes f$$

The other symbol gives

$$\gamma_{E \otimes F}(x, ae \otimes f) - a\gamma_{E \otimes F}(x, e \otimes f) = \gamma(x, ae) \otimes f + ae \otimes \gamma_F(x, f) - a\gamma_E(x, e) \otimes f - ae \otimes \gamma_F(x, f)$$

$$= \mu p(x)(a)e \otimes f$$

The curvature formula is standard from differential geometry. \qed

Lemma 2.4. Let $L$ be as above then there exists a collection $\{\gamma^i_{L^\vee}\}$ of $(−1/2, 1)$-connections on $L^\vee$ such that on double intersections we have $\gamma^i_{L^\vee}(x, l^\vee) - \gamma^j_{L^\vee}(x, l^\vee) = \beta^i_{Xij}(x) l^\vee = 0$.
Proof. Fix a section \( l \) of \( L \) and let \( l^\vee \) be the dual section of \( L^\vee \) under the non-degenerate \( \mathcal{O}_Y \)-bilinear pairing \( \langle \cdot, \cdot \rangle: L \otimes_{\mathcal{O}_Y} L^\vee \to \mathcal{O}_Y \). Define an operator on \( L^\vee \) via the Leibniz rule

\[
\partial_x(l, l^\vee)) = \gamma_L(x, l) \otimes l^\vee + l \otimes \gamma_{L^\vee}(x, l^\vee)
\]

By Lemma 2.3 \( \gamma_{L^\vee} \) is a \((-1/2, 1)\)-connection. The formula on double intersections holds since the left hand side is a global connection. \( \square \)

Proof of theorem 1.1. Suppose \( \gamma_E^i \) exists and on \( U_i \) we define \( \nabla^i : N^\vee \to \mathcal{D}^1_{\mathcal{O}_Y}(E \otimes L^\vee) \) by

\[
\nabla^i(x, e \otimes l^\vee) = \gamma^i_E(x, e) \otimes l^\vee + e \otimes \gamma^i_{L^\vee}(x, l^\vee)
\]

which is \((0, 1)\)-connection by the previous two lemmas. It is also easy to check on double intersections that \( \nabla^i - \nabla^j = 0 \). The cocycle \( \nabla^i - \nabla^j \) represents the class at \( N(E \otimes L^\vee) \in H^1(Y, N(E)) \). Since the connections \( \nabla^i \) are chosen up to a section of \( H^0(U_i, N(E)) \) we see that

\[
\text{at}_N(E \otimes_{\mathcal{O}_Y} L^\vee) = 0
\]

Conversely, if the equality holds, we can find \((0, 1)\)-connections \( \nabla^i \) on \( U_i \) which glue to a global connection. We now define \( \gamma_E^i \) to be

\[
\gamma_E^i(x, e) = \frac{\nabla^i(x, e \otimes l^\vee) - e \otimes \gamma_{L^\vee}^i(x, l^\vee)}{l^\vee}
\]

where \( l^\vee \) is any local section of \( L^\vee \). We now apply the previous proposition. \( \square \)

Remark. In the case when \( H^2(Y, \mathcal{O}_Y(E)) = 0 \) there is a bijection

\[
\text{\{first order deformations of } E, \mathcal{E}_1 \text{\} } / \sim \leftrightarrow \text{\{(0, 1)-connections on } E \otimes_{\mathcal{O}_Y} L^\vee \text{\}}
\]

To any \((0, 1)\)-connection on \( E \otimes_{\mathcal{O}_Y} L^\vee \) there is a collection \((1/2, 1)\)-connection on \( E \) by Lemma 2.3 which satisfy (2.1). Applying Lemma 2.1 shows there is a bijection. There is a \((0, 1)\)-connection when \( \text{at}_N(E \otimes_{\mathcal{O}_Y} L^\vee) = 0 \) in \( H^1(Y, N(E)) \).

Proposition 2.5. Given a first order deformation \( \mathcal{E}_1 \), its group of automorphisms restricting to the identity modulo \( \epsilon \) is isomorphic to \( H^0(Y, \mathcal{O}_Y(E)) \). If \( H^1(Y, \mathcal{O}_Y(E)) = H^2(Y, \mathcal{O}_Y(E)) = 0 \) and the condition imposed on \( \text{at}(E \otimes_{\mathcal{O}_Y} L^\vee) \) holds, then the set of isomorphism classes of first order deformations is a torsor over \( H^0(Y, N(E)) \).

Proof. Let \( \phi : \mathcal{E}_1 \to \mathcal{E}_1 \) be an automorphism which restricts to the identity modulo \( \epsilon \) then \( 1 - \phi \) takes values in \( \epsilon \mathcal{E}_1 \) and hence descends to \( \mathcal{E}_1 / \epsilon \mathcal{E}_1 \simeq E \to E \simeq \epsilon E \). The map \( 1 - \phi \) gives a section \( \phi_1 \in H^0(Y, \mathcal{O}_Y(E)) \) since it is \( \mathcal{O}_Y \)-linear. Therefore \( \phi = 1 + \epsilon \phi_1 \).

By Proposition 2.1 the vanishing of the cohomology groups implies the isomorphism class is uniquely determined by the choice of \( \gamma_i \). The difference of two \((1/2, 1)\)-connections will be an \( \mathcal{O}_Y \)-bilinear map \( N^\vee \times E \to E \). This means the difference will be a section of \( N(E) \) over \( U_i \). Moreover, equation (2.1) shows that such a section will glue on \( U_i \cap U_j \). \( \square \)
2.2. Second Order. In this subsection assume that $\beta_1^X \equiv 0$. When $\mathcal{A}_2$ exists locally there are bidifferential operators $\alpha_2^{X_i}$ which satisfy (A.2) along with gluing conditions on double intersections (A.4). By skew-symmetry of $\alpha_1^X$ and (A.2) the anti-symmetrization $\mathcal{A}_2^{ij}(a,b) := \alpha_2^{X_i}(a,b) - \alpha_2^{X_i}(b,a)$ satisfies
\[a\mathcal{A}_2^{ij}(b,c) - \mathcal{A}_2^{ij}(a,c) + \mathcal{A}_2^{ij}(a,b) - \mathcal{A}_2^{ij}(a,b)c = 0\]
The HKR isomorphism shows $\mathcal{A}_2^{ij}$ is given by a bivector in $\mathcal{H}^0(U_i, \wedge^2 T_X)$. Since, $\beta_1^X \equiv 0$ the collection $\{\mathcal{A}_2^{ij}\}$ glues to a global bivector $\mathcal{A}_2 \in \mathcal{H}^0(X, \wedge^2 T_X)$.

Proof of theorem (A.3) Recall, we now assume $\beta_1^X \equiv 0$. Let $X$ be affine and suppose we are given a first order deformation $\mathcal{E}_1 \simeq E \oplus \alpha E$ from the previous theorem. To extend to $\mathcal{E}_2$ we need to find $\alpha_2$ which solves (A.3). By Lemma A.1 the existence of $\alpha_2$ is equivalent to the vanishing of the antisymmetrization of (A.6) when restricted to $I_Y$, i.e. we must solve
\[\mathcal{A}_2(x,y)e = \gamma(x,x) - \gamma(y,y) - \gamma(x,y) - \gamma(x,y)\]
By assumption, $L$ has a second order deformation which implies $c(\gamma_L)(x,y)(l) = \mathcal{A}_2(x,y)l$ by the previous paragraph. The left hand side of (2.3) is a flat connection therefore $c(\gamma_L\gamma)(x,y)(l^y) = -\mathcal{A}_2(x,y)l^y$. Using Lemma 2.3 we see $c(\nabla) = 0$.

In general, the same reasoning implies the existence of operators $\alpha_2^{ij}(a,e)$ on affine subsets $U_i$. By Lemma A.1 the existence of $\beta_2^{ij}$ satisfying (A.8) is equivalent to
\[\beta_2^{ij}(x,e) - \beta_2^{ij}(x,e) + \beta_2^{X^{ij}}(x,e) + \alpha_2^{ij}(x,\beta_1^{ij}(e)) - \beta_1^{ij}(\alpha_1^{ij}(x,e)) = 0\]
A straightforward calculation shows the RHS of (A.8) is a Hochschild cocycle and hence $\mathcal{O}_X$-bilinear when restricted to $I \otimes E$ by lemma A.2. Moreover, it vanishes for $x \in I_Y$ and therefore descends to $N^\vee$ defining a section of $c_{ij} \in \mathcal{H}^0(U_i \cap U_j, N(E))$. If this class vanishes in $\mathcal{H}^1(Y, N(E))$ then there are sections $c_i \in \mathcal{H}^0(U_i, N(E))$ such that $c_{ij} = c_i - c_j$ on $U_i \cap U_j$. For fixed $i$, the conormal sequence
\[0 \to N^\vee \to \Omega^1_X|_Y \to \Omega^1_Y \to 0\]
 splits since $U_i \cap Y$ is affine and the three sheaves in the sequence are locally free. Denote the surjection by $\pi_i : \Omega^1_X|_Y \to N^\vee$. The expression, $c_i(x)e$ can then be lifted to an operator $\psi_i(a,e) = c_i(\pi_i(da))e$ which is a Hochschild cocycle. We now replace $\alpha_i(a,e)$ with $\alpha_i(a,e) - \psi_i(a,e)$ to ensure (A.8) holds. Since $H^2(Y, \mathcal{O}_Y(E)) = 0$ we can adjust $\beta_2^{ij}$ by adding $\mathcal{O}_Y$-linear operators $E \to E$ on $U_i \cap U_j$ so the cocycle condition for gluing functions hold on $U_i \cap U_j \cap U_k$. This completes the proof. 

Remark 2.6. If $X$ is affine and $\alpha_2^X$ is symmetric the two previous theorems imply any vector bundle supported on a coisotropic subvariety along with a flat connection along the null foliation has a second order quantization. When the subvariety, $Y$, is Lagrangian any vector bundle on $Y$ with a flat (0,1)-connection has a second order quantization.

Proposition 2.7. Assume $H^1(Y, \mathcal{O}_Y(E)) = 0$. (a) Let $E$ be a vector bundle which admits a second order deformation $\mathcal{E}_2$ and let $\phi : \mathcal{E}_1 \to \mathcal{E}_1$ be an automorphism restricting to the identity modulo $\epsilon$. If $\phi_1 \in H^0(Y, \mathcal{O}_Y(E))$ is the regular section corresponding to $\phi$ via Proposition 2.5 then $\phi$ extends to a second order automorphism $\phi_2 : \mathcal{E}_2 \to \mathcal{E}_2$ if and only if $d_{N^\vee} \phi_1 = 0$. In this case the set of all extensions $\phi_2$ is a torsor over $H^0(Y, \mathcal{O}_Y(E))$.

(b) Assume that $E$ has two second order deformations $\mathcal{E}_2$ and $\mathcal{E}_2'$ such that for their first order truncations we have
\[\mathcal{E}_1 = \mathcal{E}_1 + \zeta; \quad \zeta \in H^0(Y, N(E))\]
in the sense of the torsor structure of Proposition 2.5. Then \( d_{N_E} \zeta + [\zeta, \zeta] = 0 \) in \( H^0(Y, \wedge^2 N(E)) \). If \( H^2(Y, \mathcal{O}_Y(E)) = H^1(Y, N(E)) = 0 \) and \( \zeta \) satisfying \( d_{N_E} \zeta + [\zeta, \zeta] = 0 \) is fixed, then for a given \( \mathcal{E}_2 \) and \( \mathcal{E}_1' \) the set of isomorphism classes of \( \mathcal{E}_2' \) is a torsor over \( H^0(Y, N(E)) \).

**Proof.** Locally the automorphism is of the form \( e \mapsto e + \epsilon \phi_1(e) + \epsilon^2 \eta(e) \). If we write out the equation \( \phi(a \ast e) = a \ast \phi(e) \) then we see that

\[
\eta(a e) - a \eta(e) = a_1(a, \phi_1(e)) - \phi_1(a_1(a, e))
\]

By Lemma 3.1 such an \( \eta \) exists if and only if the RHS vanishes for \( a \in I_Y \). This is precisely the condition \( d_{N_E} \phi_1 = 0 \).

We now show that two open sets \( U_i \) and \( U_j \) are related by the transition \( e \mapsto e + \epsilon \beta^{ij}_1(e) + \epsilon^2 \beta^{ij}_2(e) \). This leads to

\[
\eta_i(e) - \eta_j(e) - \beta^{ij}_1(\phi_1(e)) + \phi_1(\beta^{ij}_1(e)) = 0
\]

which may not hold with the original \( \eta_i, \eta_j \) but these may be adjusted by an \( \mathcal{O}_Y \)-linear endomorphism of \( E \) on \( U_i \) and \( U_j \). The defining equations for \( \eta_i \) equation (A.7), and that \( \phi_1 \) is an \( \mathcal{O}_Y \)-linear endomorphism imply the LHS is \( \mathcal{O}_Y \)-linear and defines a cocycle in \( H^1(Y, \mathcal{O}_Y(E)) \). Since \( H^1(Y, \mathcal{O}_Y(E)) = 0 \) the \( \eta_i \)'s can adjusted to ensure the local automorphisms agree on double intersections. The only remaining ambiguity for \( \eta_i \) is the addition of a globally defined \( \mathcal{O}_Y \)-linear endomorphism of \( E \).

To prove (b) we recall that if \( H^1(Y, N(E)) = 0 \) then a first order deformation is determined by a collection of \((1/2, 1)\)-connections. Given two second order deformations \( \mathcal{E}_2 \) and \( \mathcal{E}_2' \) whose first order deformations satisfy

\[
\gamma^{ij}_E(x, e) - \gamma^{ij}_E(x, e) = \zeta(x) e
\]

for some \( \zeta \in H^0(Y, N(E)) \) implies \( c(\gamma) = c(\gamma' + \zeta) \). A quick calculation shows \( c(\gamma' + \zeta) = c(\gamma') + d_{N_E} \zeta + [\zeta, \zeta] \). Since \( \gamma \) and \( \gamma' \) extend to second order theorem 1.3 shows their curvatures are equal \( c(\gamma) = c(\gamma') \).

Conversely, if \( d_{N_E} \zeta + [\zeta, \zeta] = 0 \) then the above calculation shows that \( \gamma^{ij}_E(x, e) + \zeta(x) e \) satisfies the curvature equation of Theorem 1.3. Hence there exists local operators \( \alpha^{ij}_2(x, e) \) satisfying (A.6). The assumptions \( H^2(Y, \mathcal{O}_Y(E)) = H^1(Y, N(E)) = 0 \) imply that all obstructions to existence of \( \beta^{ij}_2 \) which satisfy (A.8) vanish. The second order deformation corresponding to \( \zeta \) can be found. \( \square \)

### 3. Deforming flat vector bundles

Throughout this section we will assume that \( X, Y \) are affine varieties. The statements can be generalized to the non-affine case with suitable cohomology vanishing which we leave to the motivated reader. Furthermore, we also assume that \( P \) is non-degenerate i.e. symplectic. In this case \( p : N' \rightarrow T_Y \) is an embedding of vector bundles. The image, \( T_F \), is the null-foliation of \( Y \).

#### 3.1. Flat vector bundles.

The main result of this section is the following theorem:

**Theorem 3.1.** There is a bijection

\[
\mathcal{M}_F(Y) \xrightarrow{\text{quant}} Q_2(Y)
\]

where \( \mathcal{M}_F(Y) \) is the set of vector bundles on \( Y \) which admit a \((0, 1)\)-connection flat along the null foliation and \( Q_2(Y) \) is the set of vector bundles on \( Y \) which admit a second order deformation.
Proof. For $M$ a vector bundle with a connection $\nabla_M$ that is flat along $T_F$ let $\text{quant}(M) := M \otimes_{\mathcal{O}_Y} L$. Define a $(1/2, 1)$-connection on $M \otimes_{\mathcal{O}_Y} L$ via (2.2). Therefore by Proposition 2.3 $M \otimes_{\mathcal{O}_Y} L$ admits a first order deformation. By Lemma 2.3 $c(\gamma_M) = \omega_2$ then Theorem 1.3 shows $M \otimes L$ admits a second order deformation.

If $M$ is a bundle which admits a second order deformation then

$$\nabla_{M \otimes L^\vee}(x, m \otimes l^\vee) = \gamma_M(x, m) \otimes l^\vee + m \otimes \gamma_{L^\vee}(x, l^\vee)$$

is a flat $(0, 1)$-connection on $\text{dequant}(M) := M \otimes_{\mathcal{O}_Y} L^\vee$ again by Lemma 2.3. □

Remark 3.2. Let $X$ be a smooth variety then to any $\mathcal{D}$-module one can associate a coisotropic subvariety $Y \subset T_X$ i.e. the singular support. Let $W \subset X$ be a smooth subvariety with a local system which we view as a coherent $\mathcal{D}$-module on $X$ via the direct image. The singular support is then $N_{X/W}^* \subset T_X^*$ which is a Lagrangian subvariety with a local system induced by the local system on $W$. Denote by $\pi : T_X^* \rightarrow X$ the projection map. Using the sequence

$$0 \rightarrow \pi^* T_W^* \rightarrow N_{T_X^*/W}^* \rightarrow \pi^* N_{X/W} \rightarrow 0$$

we see that $\wedge^* N_{T_X^*/W}^*$ has a second order deformation. By the above theorem, the local system on $N_{X/W}$ can be deformed to second order over the deformation quantization of $\mathcal{O}_{T^*X}$ given by the standard symplectic form on $T_X$ after twisting by $\wedge^* N_{T_X^*/W}^*$.

In unpublished work, Dmitry Kaledin has proven the same theorem for smooth $\mathcal{D}$-modules with smooth support but for infinite order deformations using completely different methods. [16].

A direct corollary of the above proof shows that $\mathcal{Q}_2(Y)$ is a symmetric monoidal category

Corollary 3.3. If $Y \subset X$ are affine then the category of second order deformations, $\mathcal{Q}_2(Y)$ is a symmetric monoidal category.

Proof. Define a tensor product via

$$\otimes : \mathcal{Q}_1(Y) \times \mathcal{Q}_1(Y) \rightarrow \mathcal{Q}_1(Y)
\begin{align*}
(E_1, E_2) & \rightarrow E_1 \otimes_{\mathcal{O}_Y} E_2 \otimes_{\mathcal{O}_Y} L^\vee
\end{align*}$$

The identity element is given by $L$. It is then clear $E_1 \otimes E_2$ admits a first/second order deformation with the above hypotheses. Moreover, it is easy to check that $\otimes$ is symmetric and associative using Lemma 2.3. □

In the case when $Y$ is lagrangian i.e. $\dim Y = 1/2 \dim X$, there is an isomorphism $p : N^\vee \simeq T_Y$. To any vector bundle with a flat connection there corresponds module over $\mathcal{A}_2$ which splits as sheaf of $k[\epsilon]/\epsilon^3$-modules. When $\alpha_2^X$ is symmetric we can take $L = (\det N)^{1/2} = K_Y^{1/2}$ if it exists. The quantization map is given by twisting by $K_Y^{1/2}$.

3.2. Atiyah algebra. Define the null foliation Atiyah algebra $At_F(E) \subset \mathcal{D}(E)$ to be those operators operators whose symbol belongs to $T_F \subset \text{End}_{\mathcal{O}_Y}(E) \otimes_{\mathcal{O}_Y} T_F$. By coisotropness $At_F(E)$ is a Lie algebra since $T_F$ is involutive. We can also define $At_F(E)$ by the following null foliation Atiyah sequence

$$0 \rightarrow \text{End}_{\mathcal{O}_Y}(E) \rightarrow At_F(E) \rightarrow T_F \rightarrow 0$$
Theorem 3.4. If \( P \) is non-degenerate along \( Y \), existence of a first order deformation is equivalent to the existence of a \( k \)-linear splitting of the null foliation Atiyah sequence which is a \((1/2,1)\)-connection. Furthermore, if \( \alpha_2^X \) is symmetric then a second order deformation exists if and only if the splitting agrees with the bracket.

Proof. The first part is a restatement of Theorem 1.1. By definition a splitting, \( \gamma \), commutes with the bracket when \( c(\gamma) = 0 \). Since \( \alpha_2^X \) is symmetric this happens if and only if there is a second order deformation. \( \square \)

4. Curved DGLA on Hochschild complex

4.1. \( L_\infty \)-algebras. In this section we define strongly homotopy Lie algebras, commonly known as \( L_\infty \)-algebras. We give the definitions and results in the curved \( L_\infty \) case, for lack of a convenient reference.

Let \( A \) be a graded vector space over a commutative ring \( k \) (not necessarily a field) which contains the rational numbers as a subring. The homogenous elements of degree \( n \) are denoted by \( A^n \).

The suspension of graded vector space is the graded vector space, \( A[1] \), such that \( A[1]^n := A^{n+1} \). Consider the cofree coassociative cocommutative counital coaugmented coalgebra generated by \( A[1] \)

\[
S(A[1]) := \oplus_{n \geq 0} S^n(A[1]) \quad \text{where} \quad S^n(A[1]) := (\otimes^n A[1])^{\Sigma_n} \simeq (\wedge^n A)[n] \quad \text{i.e. the set of tensors which are invariant under the natural action of the symmetric group on \( n \) elements.}
\]

Recall, a counital coalgebra is coaugmented if there exists a coalgebra morphism \( \eta : k \to C(V) \). The notion of a dg-coalgebra morphism will be defined shortly. The coalgebra structure is given by

\[
\Delta(a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^n \sum_{\sigma \in \Sigma_{i,n-i}} e(\sigma)(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(i)}) \otimes (a_{\sigma(i+1)} \wedge \cdots \wedge a_{\sigma(n)})
\]

where \( \Sigma_{i,n-i} \) is the set of \((i, n-i)\)-shuffles of \( \Sigma_n \) i.e. \( \sigma \in \Sigma_n \) such that \( \sigma(1) < \cdots < \sigma(i) \) and \( \sigma(i+1) < \cdots < \sigma(n) \). The sign is determined by the Koszul rule.

A curved \( L_\infty \)-algebra structure on \( A \) is a codifferential \( Q \) of degree 1 on \( S(A[1]) \) i.e. a linear map

\[
Q : S(A[1]) \to S(A[1])[1]
\]

such that \( \Delta Q = (Q \otimes \text{id})\Delta + (\text{id} \otimes Q)\Delta \) and \( Q^2 = 0 \). In other words \( S(A[1]) \) has the structure of a dg-coalgebra. Any coderivation is completely determined by the values on the cogenerators given by the composition

\[
\ell_n : S^n(A[1]) \to S(A[1]) \xrightarrow{Q} S(A[1])[1] \to A[2]
\]

for all \( n \geq 0 \). The \( \{\ell_k\}_{k \geq 0} \) are known as higher brackets. The condition \( Q^2 = 0 \) implies an infinite set of quadratic equations that \( \{\ell_n\}_{n \geq 0} \) must satisfy known as higher Jacobi relations. If \( Q \) agrees with the coaugmentation i.e. \( Q\eta = 0 \) we simply say \( A \) is an \( L_\infty \)-algebra. In the curved case the quadratic relation implies \( \ell_1^2 = \ell_2\ell_0 \) which is nonzero in general hence cohomology is not defined. Furthermore, for an \( L_\infty \)-algebra \( A \) we set \( H^*(A) := H^*(A, \ell_1) \).

If \( \ell_n = 0 \) for \( n \neq 2 \) then \( A \) is a graded Lie algebra. A dg Lie algebra is an \( L_\infty \) algebra with \( \ell_n = 0 \) for \( n \neq 1, 2 \). A curved dg (CDG) Lie algebra is an \( L_\infty \)-algebra with \( \ell_n = 0 \) for \( n \geq 3 \). The quadratic relation implies \( \ell_1\ell_0 = 0, \ell_1\ell_1 = \ell_2\ell_0, \ell_2 \) satisfies the Jacobi identity and \( \ell_1 \) is a derivation with respect to \( \ell_2 \).
Let $(C_i, Q_i)$ be two dg-coalgebras a dg-coalgebra morphism is a morphism of vector spaces $F : C_1 \rightarrow C_2$ which is equivariant with respect to the codifferentials i.e. $FQ_1 = Q_2F$. In then case when $C_i = S(A_i[1])$ for a graded vector space $A_i$ then such a morphism is determined by a sequence of maps $F_n : \wedge^n A_1 \rightarrow A_2[1 - n]$ for $n \geq 0$ which satisfy an infinite set of equations coming from the compatibility with the codifferentials. The explicit formulae for a DGLA are in \cite{Kadeishvili}. In this case $F_1$ is a morphism of Lie algebras only up to a homotopy. In particular the category of dg Lie algebras with dg Lie algebra morphisms is not a full subcategory of the $L_\infty$ category.

An $L_\infty$-morphism $F : (S(A_1[1]), Q_1) \rightarrow (S(A_2[1]), Q_2)$ is a quasi-isomorphism if its first component $F_1 : A_1 \rightarrow A_2$ is an isomorphism on cohomology. Here we are assuming $\ell_0 = 0$ so cohomology is defined. An important theorem due to Kadeishvili says an $L_\infty$-algebra is quasi-isomorphic to its cohomology.

**Theorem 4.1.** \cite{Kadeishvili} There exists a quasi-isomorphism of $L_\infty$-algebras $H^*(A) \rightarrow A$ which lifts the identity of $H^*(A)$.

A dg Lie algebra $A$ formal if the induced brackets $\ell_n$ on $H^*(A)$ are 0 for $n \geq 3$.

The zeroes of $Q$ are solutions of the Maurer-Cartan equation and put $\text{Zero}(Q) := MC(A)$. In terms of the higher brackets $b \in A^1$ is an element of $MC(A)$ if and only if

\[(4.1) \quad \sum_{k=0}^{\infty} \frac{1}{k!} \ell_k(b, \ldots, b) = 0 \]

If $A$ is a dg Lie algebra then (4.1) is the usual Maurer-Cartan equation i.e. $db + \frac{1}{2}[b, b] = 0$. A dg-coalgebra morphism $F : C_1 \rightarrow C_2$ induces a mapping on solutions of the Maurer-Cartan equation, $F_* : MC(C_1) \rightarrow MC(C_2)$ since it commutes with the codifferentials.

**4.2. Homotopy theory of $L_\infty$-algebras.** The primary difficulty in dealing with curved $L_\infty$-algebras is that quasi-isomorphism no longer has any meaning i.e. cohomology is no longer defined. A replacement is homotopy equivalence which is more general than quasi-isomorphism. We follow the terminology and exposition of \cite{Pech} where the case of $A_\infty$-algebras was worked out in detail but little needs to be changed for curved $L_\infty$-algebras. The proofs though are contained in \cite{Pech}.

In the category of topological spaces the notion of homotopy is very well-known. In particular, a homotopy between two morphisms $f, f' : X \rightarrow Y$ is another morphism $H : [0, 1] \times X \rightarrow Y$ which lives in the category of topological spaces. Motivated by this there is similar notion of a homotopy in the category of $L_\infty$-algebras. But first we must make sense of the what it means to take the product an $L_\infty$-algebra, $A$, with the unit interval. This is called a model of $[0, 1] \times A$ in \cite{Pech}.

**Definition 4.2.** Define an $L_\infty$-algebra $A[1] \otimes k[t, dt]$ where $A[t]$ is the polynomial ring with coefficients in $A$. An element of $A[1] \otimes k[t, dt]$ is written as a sum $a(t) + b(t)dt$ where $a(t), b(t) \in A[t]$. Also define $\deg dt = 1$ and set $\ell_0(1) = \ell_0(1) + 0dt$. The higher brackets are given by

\[\ell_1(a(t) + b(t)dt) = \ell_1(a(t)) - \ell_1(b(t))dt - \frac{db}{dt}dt\]

\[\ell_k(a_1(t) + b_1(t)dt, \ldots, a_k(t) + b_k(t)dt) = \ell_k(a_1(t), \ldots, a_k(t)) + \sum_{j=1}^{k}(-1)^{|a_1|+\cdots+|a_j-1|+j} \ell_k(a_1(t), \ldots, b_j(t), \ldots, a_k(t))dt\]
We define the $L_\infty$ evaluation homomorphism $\text{Eval}_{t=t_0} : A[1] \otimes k[t, dt] \to A[1]$ as

$$\text{Eval}_{t=t_0}(a(t) + b(t)dt) = a(t_0)$$

for $t_0 \in \mathbb{R}$.

**Definition 4.3.** Two $L_\infty$ morphisms $f, g : A \to A'$ are homtopic, denoted by $f \sim g$, if there exists an $L_\infty$ homomorphism $H : A \to A' \otimes k[t, dt]$ such that $\text{Eval}_{t=0} \circ H = f$ and $\text{Eval}_{t=1} \circ H = g$.

**Definition 4.4.** An $L_\infty$ morphism $f : A \to A'$ between $L_\infty$-algebras is a homotopy equivalence if there exists an $L_\infty$ morphism $g : A' \to A$ such that $fg \sim id$ and $gf \sim id$. Furthermore, we say $A$ and $A'$ are homotopy equivalent if there exists a homotopy equivalence as above.

If $t_0 = 0$ then a quasi-isomorphism is the same as a homotopy equivalence. In the category of $L_\infty$-algebras with $L_\infty$ homomorphisms a quasi-isomorphism has a homotopy inverse which is also a quasi-isomorphism. This is not true in the category of dg Lie algebras with dg Lie algebra homomorphisms as there exist quasi-isomorphisms without inverses. This is one of the reasons to enlarge the dg Lie algebra category to the homotopy $L_\infty$-category.

We now introduce a covariant functor $MC(A)$ from the category of Artin $k$-local algebras to the category of sets. Let $\mathcal{R}$ be such an algebra, which we consider as a graded algebra concentrated in degree 0, and $m_\mathcal{R}$ the maximal ideal. Since $\mathcal{R}$ is concentrated in degree 0 we have $(A \otimes m_\mathcal{R})^i = A^i \otimes m_\mathcal{R}$. Define the functor $MC(A)(\mathcal{R}) := MC(A \otimes m_\mathcal{R})$. If $\psi : \mathcal{R} \to \mathcal{R}'$ is a morphism of algebras and $b \in MC(A \otimes \mathcal{R})$ then $(1 \otimes \psi)(b) \in MC(A \otimes m_{\mathcal{R}'})$. Hence there is a morphism $MC(A)(\mathcal{R}) \to MC(A)(\mathcal{R}')$ which makes $MC(A)$ into a covariant functor. However, the set $MC(A)(\mathcal{R})$ is too large to be homotopy invariant so instead we look at equivalence classes in $MC(A)(\mathcal{R})$.

**Definition 4.5.** Let $b, b' \in MC(A)(\mathcal{R})$ then $b$ and $b'$ are gauge equivalent denoted by $b \sim b'$ if there exists an element $\tilde{b} \in MC(A \otimes k[t, dt])(\mathcal{R})$ such that $\text{Eval}_{t=0} \circ \tilde{b} = b_0$, $\text{Eval}_{t=1} \circ \tilde{b} = b_1$.

The proof that gauge equivalence is an equivalence relation is found in [11]. Using this define the deformation set as

$$\text{Def}(A)(\mathcal{R}) := MC(A)(\mathcal{R}) / \sim$$

For $\psi : \mathcal{R} \to \mathcal{R}'$ there is a morphism $\psi_* : MC(A)(\mathcal{R}) \to MC(A)(\mathcal{R}')$. Thus there is a deformation functor $\text{Def}(A)$ from algebras as above to the category of sets. The following theorem provides justification for taking gauge equivalence classes of solutions to the Maurer-Cartan equation.

**Theorem 4.6.** [10] Theorem 2.2.2 If $A$ is homotopy equivalent to $A'$ then the deformation functor $\text{Def}(A)$ is equivalent to $\text{Def}(A')$.

One can extend the above discussion to projective limits of Artin local algebras. In this paper we will consider the usual projective limit: $\epsilon k[[\epsilon]] = \lim \epsilon k[[\epsilon]]/\epsilon^r k[[\epsilon]]$ cf. [19].

By definition $\tilde{b} = a(t) + b(t)dt \in MC(A \otimes k[t, dt])(\mathcal{R})$ if and only if

$$\frac{da}{dt} + \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \ell_k(b(t), a(t), \ldots, a(t)) = 0$$

$$\sum_{k=0}^{\infty} \ell_k(a(t), \ldots, a(t)) = 0$$
For the first equation we used skew-symmetry of the $\ell_k$. This implies $a(t) \in \mathcal{MC}(A)(\mathcal{R})$ for all $t$. If $A$ is a dg Lie algebra gauge equivalence reduces to $da/dt = \ell_1(b) + \ell_2(b, a)$ cf. [19].

As noted above the difficulty in dealing with curved $L_\infty$-algebras is cohomology is not defined. To overcome this we can twist by a Maurer-Cartan element to an $L_\infty$-algebra with $\ell_0 = 0$. Suppose $b \in \mathcal{MC}(A)$ and define

$$\ell^b_k(a_1, \ldots, a_k) = \sum_{j=0}^{\infty} \frac{1}{j!} \ell_{k+j}(b, \ldots, b, a_1, \ldots, a_k)$$

In particular,

$$\ell^b_0(1) = \sum_{j=0}^{\infty} \frac{1}{j!} \ell_j(b, \ldots, b) = 0$$

It is then a straightforward calculation to see $(A, \ell^b_k)$ is an $L_\infty$-algebra which we call the $b$-twist of $A$.

**Definition 4.7.** The $b$-twisted cohomology of an $L_\infty$-algebra $A$ is

$$H^*_b(A) := H^*(A, \ell^b_1)$$

**Proposition 4.8.** [11, Proposition 4.3.16] If $b_0 \sim b_1$ then there is a homotopy equivalence $f : (A, \ell^b_0) \to (A, \ell^b_1)$. This implies $b$-twisted cohomology depends only on the gauge equivalence class of $b$.

### 4.3. Commutative Deformations

Let $A$ be a ring and $E$ be an $A$-module the Hochschild complex, $\mathfrak{g}_E^k := \text{Hom}_k(A^\otimes n \otimes E, E)$, is a dg Lie algebra. The differential is given by

$$d_{\text{Hoch}}(a_1, \ldots, a_{k+1}, e) := a_1\alpha(a_2, \ldots, a_{k+1}, e) + \sum_{i=1}^{k} (-1)^i \alpha(a_1, \ldots, a_i a_{i+1}, \ldots, a_{k+1}, e)$$

and the Lie bracket is

$$[\alpha_1, \alpha_2]_G := \alpha_1 \circ \alpha_2 - (-1)^{kl} \alpha_2 \circ \alpha_1$$

where $\alpha, \alpha_1 \in \mathfrak{g}^k$ and $\alpha_2 \in \mathfrak{g}^l$ and

$$\alpha_1 \circ \alpha_2(a_1, \ldots, a_{k+l}, e) = \alpha_1(a_1, \ldots, a_k, \alpha_2(a_{k+1}, \ldots, a_{k+l}, e))$$

We will just write $\mathfrak{g}$ instead of $\mathfrak{g}_E$ when there is no confusion. It is the well known in this case that [23, Lemma 9.1.9]

$$H^*(\mathfrak{g}) = \text{Ext}_A^*(E, E)$$

In this section we do not assume $Y$ is coisotropic. Let $X$ be an affine scheme and $Y$ a subvariety with a vector bundle $E$. A commutative deformation of $E$ as a coherent $\mathcal{O}_X$-module is a flat deformation to an $\mathcal{O}_X$-module. The module structure is given by

$$a \ast e = a e + \epsilon \alpha_1(a, e) + \epsilon^2 \alpha_2(a, e) + \cdots$$
If we define $\alpha^E := \epsilon_1 + \epsilon^2 \alpha_2 + \cdots \in g[[\epsilon]]$ then associativity of $[12]$ is equivalent to the Maurer-Cartan equation in $g[[\epsilon]]$

\begin{equation}
  d_{\text{Hoch}} \alpha^E + \frac{1}{2} [\alpha^E, \alpha^E]_G = a \alpha^E(b,e) - \alpha^E(ab,e) + \alpha^E(a,be) + \alpha^E(a,a^E(b,e))
\end{equation}

Two solutions $\alpha^E, (\alpha^E)'$ are gauge equivalent denoted by $\alpha^E \sim (\alpha^E)'$ if there exists a $\phi \in g^0[[\epsilon]]$ such that

\begin{equation}
  \phi(a \star e) = a \star' \phi(e)
\end{equation}

which restricts to the identity modulo $\epsilon$. Such a $\phi$ is of the form $\phi = id + \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots$ and (4.4) is equivalent to

\begin{equation}
  \phi_n(ae) + \alpha_n(a,e) + \sum_{j+k=n-1} \phi_j(\alpha_k(a,e)) = a\phi_n(e) + \alpha_n'(a,e) + \sum_{j+k=n-1} \alpha_j'(a,\phi_k(e))
\end{equation}

for all $n \geq 1$. It is then straightforward to check that gauge equivalence as defined above is equivalent to gauge equivalence defined in the previous subsection cf. [19, Section 3.2]. The main result of this section is the following formality theorem in this setting cf. [11, Conjecture 2.36]:

**Theorem 4.9.** In the above setting the dg Lie algebra $(g, d_{\text{Hoch}}, [\cdot])$ is quasi-isomorphic to the abelian Lie algebra $(\wedge^* N(E), 0)$ i.e. $g_E$ is formal.

First we need a lemma to compute the cohomology of $g_E$.

**Lemma 4.10.** Let $X$ be a smooth affine variety and $Y$ a subvariety with a vector bundle $E$. Then there is an isomorphism

\[ H^*(g) \simeq \wedge^* N(E) \]

**Proof.** By [23, Lemma 9.1.9]

\[ H^p(g) = \mathcal{E}xt_{\mathcal{O}_X}^p(E, E) \]

The rest of the lemma is a special case of a more general calculation found in [8]. They compute $\mathcal{E}xt_{\mathcal{O}_X}(E_1, E_2)$ where $E_i$ are vector bundles supported on possibly distinct subvarieties. In our case the calculation simplifies dramatically so we include it for completeness.

To calculate $\mathcal{E}xt_{\mathcal{O}_X}^p(E, E)$ for $E$ a vector bundle we use the change of ring spectral sequence:

\[ \ell_* \mathcal{E}xt_{\mathcal{O}_Y}^p(E, \mathcal{E}xt_{\mathcal{O}_X}^q(\mathcal{O}_Y, E)) \Rightarrow \mathcal{E}xt_{\mathcal{O}_X}^{p+q}(E, E) \]

Since $X$ is affine and $E$ is locally free over $Y$ the spectral sequence becomes

\[ \mathcal{E}xt_{\mathcal{O}_X}^p(E, E) = \mathcal{H}om_{\mathcal{O}_Y}(E, \mathcal{E}xt_{\mathcal{O}_X}^p(\mathcal{O}_Y, E)) = E^* \otimes_{\mathcal{O}_Y} \mathcal{E}xt_{\mathcal{O}_X}^p(\mathcal{O}_Y, E) \]

A lemma is needed in order to compute $\mathcal{E}xt_{\mathcal{O}_X}^p(\mathcal{O}_Y, E)$

**Lemma 4.11.** $H_k L^* t_* \mathcal{O}_Y = \wedge^k N^*_Y / X$

**Proof.** First notice

\begin{equation}
  \ell_* L^* t_* \mathcal{O}_Y = \ell_* (L^* t_* \mathcal{O}_Y L) \mathcal{O}_Y = \ell_* \mathcal{O}_Y \mathcal{O}_Y \ell_* \mathcal{O}_Y
\end{equation}
where the second equality is the projection formula. Since \( \iota \) is a closed embedding the underived pullback of the direct image fixes the sheaf. This gives
\[
H^k L^* \iota_* \mathcal{O}_Y = t^* H^k L^* \iota_* \mathcal{O}_Y = L^* H^k (\iota_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \iota_* \mathcal{O}_Y) = \wedge^k N^*_Y / X
\]
The first equality is the above remark about \( \iota \) being a closed embedding, the second uses \( \iota^* \) is an exact functor, third is (4.6) from above. The last equality uses the well known fact that \( \iota^* \text{Tor}_k^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \wedge^k N^*_Y / X \).

Returning to the original calculation
\[
\mathcal{E}xt^k_{\mathcal{O}_X}(\iota_* \mathcal{O}_Y, \iota_* E) = H^k R\text{Hom}_{\mathcal{O}_X}(\iota_* \mathcal{O}_Y, \iota_* E) = H^k \iota_* R\text{Hom}_{\mathcal{O}_Y}(L^* \iota_* \mathcal{O}_Y, E) = \iota_* H^k R\text{Hom}_{\mathcal{O}_Y}(L^* \iota_* \mathcal{O}_Y, E)
\]
The second equality is the adjoint relation \( L^* \dashv \iota_* \) \([13, 2.5.10]\) and the last equality is exactness of \( \iota_* \). The Grothendieck spectral sequence in this case yields
\[
H^p R\text{Hom}_{\mathcal{O}_Y}(H^q L^* \iota_* \mathcal{O}_Y, E) \Rightarrow H^{p+q} R\text{Hom}_{\mathcal{O}_Y}(L^* \iota_* \mathcal{O}_Y, E)
\]
By the above
\[
\iota_* H^p R\text{Hom}_{\mathcal{O}_Y}(H^q L^* \iota_* \mathcal{O}_Y, E) = \iota_* H^p R\text{Hom}_{\mathcal{O}_Y}(\wedge^q N^*_Y / X, E) = \iota_* H^p (Y, \wedge^q N^*_Y / X \otimes_{\mathcal{O}_Y} E)
\]
This implies since \( Y \) has no higher cohomology that
\[
\mathcal{E}xt^k_{\mathcal{O}_X}(\iota_* \mathcal{O}_Y, \iota_* E) = \iota_* H^0 (Y, \wedge^k N^*_Y / X \otimes_{\mathcal{O}_Y} E) = \iota_* (\wedge^k N^*_Y / X \otimes_{\mathcal{O}_Y} E)
\]

**Proof of Theorem 4.9.** First construct a contraction from \( g \xrightarrow{\pi} \wedge^* N(E) \) which exists since the ground ring contains the rational numbers as a subring. By comparing symmetry properties the map
\[
g \otimes g \xrightarrow{[\cdot, \cdot]} g \xrightarrow{\pi} \wedge^* N(E)
\]
is identically 0. Now use the arguments from \([14, \text{Section 2}]\) to see that \( \wedge^* N(E) \) has no higher brackets. This is the definition that \( g \) is formal. \( \square \)

Applying Theorem 4.6 and using that a homotopy equivalence is the same as a quasi-isomorphism we get the following corollary

**Corollary 4.12.** *There is a bijection between commutative deformations up to equivalence and section of \( N(E) \).*
4.4. Noncommutative Deformations. Once Poisson structures are introduced the problem is far more elaborate. Let $X$ be an affine Poisson variety with Poisson bivector $P \in \Gamma(X, \wedge^2 T_X)$. A deformation quantization of $\mathcal{O}_X$ is an associative product of the form

$$a \ast b = ab + \epsilon \alpha_1^X(a, b) + \epsilon^2 \alpha_2^X(a, b) + \cdots$$

where $a, b \in \mathcal{O}_X$ and $\alpha_1^X$ are bi-differential operators. Associativity of $\ast$ is equivalent to

$$(4.7) \quad a \alpha^X(b, c) - \alpha^X(ab, c) + \alpha^X(a, bc) - \alpha^X(a, b)c - \alpha^X(\alpha^X(a, b), c) + \alpha^X(a, \alpha^X(b, c)) = 0$$

where $\alpha^X := \epsilon^2 \alpha_2^X + \cdots$ and $a, b, c \in \mathcal{O}_X$. Equation (4.7) is the Maurer-Cartan equation in the Hochschild complex of $\mathcal{O}_X$ with the usual Hochschild differential and Gerstenhaber bracket $[19]$. Given a subvariety $Y \subset X$ and a vector bundle $E$ on $Y$ which we view as a coherent $\mathcal{O}_X$-module define a quantization of $E$ as a flat coherent $\mathcal{A}$-module, $\mathcal{E}$. Immediately from the above $Y$ must coisotropic. The module action is still given by (4.2) but associativity of the action is

$$aa^E(b, e) - a^E(ab, e) + a^E(a, be) - a^E(a, b)e - a^E(\alpha^X(a, b), e) + a^E(a, \alpha^E(b, e)) = 0$$

We define gauge equivalence as in (4.3) which is still equivalent to (4.5) for all $n \geq 1$. Unlike the commutative case, noncommutative deformations are not governed by a dg Lie algebra but a curved dg Lie algebra. Define

1. $\ell_0(1) := -\alpha_X \otimes \text{id}_E$
2. $\ell_1(\alpha)(a_1, \ldots, a_{k+1}, e) := d_{Hoch} \alpha(a_1, \ldots, a_{k+1}, e) + \sum_{j=1}^{k} (-1)^j \alpha(a_1, \ldots, \alpha_X(a_j, a_{j+1}), a_{j+2}, \ldots, a_{k+1}, e)$
3. $\ell_2(\alpha_1, \alpha_2) := [\alpha_1, \alpha_2]_G$

The fact these make $\mathfrak{g}$ into a curved dg Lie algebra is a straightforward computation. In particular, $\ell_1(\ell_0(1)) = 0$ is precisely the Maurer-Cartan equation in $C^* (\mathcal{O}_X, \mathcal{O}_X)[[\epsilon]]$. Associativity of (4.2) is then given by solutions of the Maurer-Cartan equation

$$(4.8) \quad \ell_0(1) + \ell_1(\alpha) + \frac{1}{2} \ell_2(\alpha, \alpha) = 0$$

That gauge equivalence is equivalent to that defined in section 4.2 follows since $\ell_1(\beta) = d_{Hoch}(\beta)$ for $\beta \in \mathfrak{g}_0^0$. Hence the deformation space controls noncommutative module deformations up to gauge equivalence.

Given a solution, $\alpha$, of (4.8) define a deformed derivation by $\ell_1^\alpha(\beta) := \ell_1(\beta) + \ell_2(\alpha, \beta)$. It is easy to check that $\ell_1^\alpha \ell_1^\alpha = 0$ is equivalent to (4.8) so $\ell_1^\alpha$ defines a differential on $\mathfrak{g}[[\epsilon]]$. There is a deformed dg Lie algebra structure on $\mathfrak{g}[[\epsilon]]$ given by $(\ell_1^\alpha, \ell_2)$. Definition 4.13. Let $E$ be a vector bundle on $Y$ which has a deformation quantization, $\alpha^E$. The Poisson-Hochschild cohomology of $E$ is defined as

$$(4.9) \quad HP^* (Y, E, \alpha^E) := H^* (\mathfrak{g}[[\epsilon]], \ell_1^\alpha)$$

When $P$ is nondegenerate and $Y$ is Lagrangian with a line bundle $L$ the “classical” limit of $HP^* (Y, L, \alpha^L)$ recovers the de Rham cohomology of $Y$. In a subsequent paper we will discuss the construction of a category consisting of pairs $(Y, E)$ where $Y$ is a coisotropic subvariety and $E$ is a vector bundle supported on $Y$ which has a deformation quantization. The endomorphisms will be the Poisson-Hochschild cohomology of $E$ cf. [9] [17] [22].
APPENDIX A.

A.1. Local equations for deformations. Here we collect the standard formulas describing a second order deformation $\mathcal{A}_2$. The first two formulas must hold on each open subset $U_i$ of an affine covering. To unload notation we write $\alpha_2^X$ instead of $\alpha_2^{Xi}$.

\begin{equation}
\alpha_1^X(b, c) - \alpha_1^X(ab, c) + \alpha_1^X(a, bc) - \alpha_1^X(a, b)c = 0
\end{equation}

\begin{equation}
\alpha_2^X(b, c) - \alpha_2^X(ab, c) + \alpha_2^X(a, bc) - \alpha_2^X(a, b)c = \alpha_1^X(\alpha_1^X(a, b), c) - \alpha_1^X(a, \alpha_1^X(b, c))
\end{equation}

The next two formulas must hold on each double intersection $U_i \cap U_j$; we write $\beta_1^X$ and $\beta_2^X$ instead of $\beta_1^{Xij}$ and $\beta_2^{Xij}$, respectively.

\begin{equation}
\beta_1^X(ab) - a\beta_1^X(b) - \beta_1^X(a)b = 0
\end{equation}

\begin{equation}
\beta_2^X(ab) - a\beta_2^X(b) - \beta_2^X(a)b = \alpha_2^{Xj}(a, b) - \alpha_2^{Xi}(a, b) + \beta_1^X(a)\beta_1^X(b) - \beta_1^X(\alpha_1^X(a, b)) + \alpha_1^X(\beta_1^X(a), b) + \alpha_1^X(a, \beta_1^X(b))
\end{equation}

We also give similar equations for the module action:

\begin{equation}
a\alpha_1(b, e) - \alpha_1(ab, e) + \alpha_1(a, be) = \alpha_1^X(a, b)e
\end{equation}

\begin{equation}
a\alpha_2(b, e) - \alpha_2(ab, e) + \alpha_2(a, be) = \alpha_2^X(a, b)e + \alpha_1(\alpha_1^X(a, b), e) - \alpha_1(a, \alpha_1(1, b, e))
\end{equation}

\begin{equation}
\beta_1(ae) - a\beta_1(e) = \alpha_1^X(a, e) - \alpha_1^X(a, e) + \beta_1^X(a)e
\end{equation}

\begin{equation}
\beta_2(ae) - a\beta_2(e) = \alpha_2^X(a, e) - \alpha_2^X(a, e) + \beta_2^X(a)e + \alpha_1^X(\beta_1(a, e)) - \beta_1(\alpha_1^X(a, e)) + \beta_1^X(a)\beta_1(e) + \alpha_1^X(\beta_1^X(a), e)
\end{equation}

In addition, there should be equalities on the triple intersections $U_i \cap U_j \cap U_k$ saying that the transition functions satisfy the cocycle condition. Only the module version of these equations is relevant to this paper:

\[ \beta_1^{kj} + \beta_1^{ji} - \beta_1^{ki} = 0 \quad \beta_2^{kj} + \beta_2^{ji} - \beta_2^{ki} = \beta_2^{kj} \circ \beta_1^{ji} \]

However, we will avoid dealing with these equations directly by assuming that $H^2(Y, O_Y(E)) = 0$. In our applications the difference $LHS - RHS$ is always $O_Y$-linear and satisfies the cocycle condition on the fourfold intersections. Since the 2-cocycle can always be resolved due to the assumption, we can adjust $\beta_3^{ij}$ to ensure that the last two equations hold as well.

**Lemma A.1.** Let $A$ be the ring of regular functions on a smooth affine variety $X$ and $E$ a projective module of finite rank over the quotient ring $B$ corresponding to a smooth affine subvariety $Y$. Let $R : A \otimes_k E \to E$ be a $k$-linear map. Then $\beta(ae) - a\beta(e) = R(a, e)$ for some $\beta \in Hom_k(E, E)$ if and only if $R$ vanishes in $I \otimes_k E$ and also satisfies

\[ aR(b, e) - R(ab, e) + R(a, be) = 0 \]

Similarly, if $G : A \otimes_k A \otimes_k E \to E$ is a $k$-linear map then $ap(b, e) - \rho(ab, e) + \rho(a, be) = G(a, b, e)$ for some $\rho : A \otimes_k E \to E$ if and only if the restriction of $G$ to $I \otimes_k I \otimes_k E \to E$ is symmetric in the first two arguments and

\[ aG(b, c, e) - G(ab, c, e) + G(a, bc, e) - G(a, b, ce) = 0 \]
Moreover, if \( R \), resp. \( G \) is an algebraic differential operator in each of its arguments then one can choose \( \beta \), resp. \( \rho \), with the same property.

**Lemma A.2.** Let \( \beta \in \mathfrak{g}_E \) (see section 4.3 for the notation) be a cocycle for \( i = 2, 3 \) then the antisymmetrization of \( \beta \) when restricted to \( I_Y \) is \( \mathcal{O}_X \) polylinear.

**Proof.** The conclusion for \( i = 2 \) is clear. For \( i = 3 \) we have
\[
\alpha \beta(b, c, e) - \beta(ab, c, e) + \beta(a, bc, e) - \beta(a, b, ce) = 0
\]
for all \( a, b, c \in A \). Hence for \( a \in A, x, y \in I \)
\[
\alpha \beta(x, y, e) - \alpha \beta(y, x, e) - \beta(ax, y, e) + \beta(y, ax, e) = -\beta(a, xy, e) - \beta(y, x, e) + \beta(a, ye, x)
\]
and
\[
\alpha \beta(x, y, e) - \alpha \beta(y, x, e) - \beta(xy, a, e) + \beta(ya, x, e) = -\beta(a, xy, e) + \beta(a, ye, x) = 0
\]
\[\square\]

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