A VOLUME COMPARISON THEOREM FOR ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. We define a notion of renormalized volume of an asymptotically hyperbolic manifold. Moreover, we prove a sharp volume comparison theorem for metrics with scalar curvature at least $-6$. Finally, we show that the inequality is strict unless the metric is isometric to one of the Anti-deSitter-Schwarzschild metrics.

1. INTRODUCTION

Let $(\bar{M}, \bar{g})$ denote the standard three-dimensional hyperbolic space, so that

$$\bar{g} = \frac{1}{1 + s^2} ds \otimes ds + s^2 g_{S^2}.$$ 

Let us consider a Riemannian metric $g$ on $M = \bar{M} \setminus K$, where $K$ is a bounded domain with smooth connected boundary. We assume that $g$ is asymptotically hyperbolic in the sense that $|g - \bar{g}|_{\bar{g}} = O(s^{-2-4\delta})$ for some $\delta \in (0, \frac{1}{4})$ and $|\bar{D}(g - \bar{g})|_{\bar{g}} = o(1)$. We define the renormalized volume of $(M, g)$ by

$$V(M, g) := \lim_{i \to \infty} (\text{vol}(\Omega_i \cap M, g) - \text{vol}(\Omega_i, \bar{g})), $$

where $\Omega_i$ is an arbitrary exhaustion of $\bar{M}$ by compact sets. The condition $|g - \bar{g}|_{\bar{g}} = O(s^{-2-4\delta})$ guarantees that the quantity $V(M, g)$ does not depend on the choice of the exhaustion $\Omega_i$. Clearly, $V(\bar{M}, \bar{g}) = 0$.

As an example, let us consider the Anti-deSitter-Schwarzschild manifold with mass $m > 0$. To that end, let $s_0 = s_0(m)$ denote the unique positive solution of the equation $1 + s_0^2 - m s_0^{-1} = 0$. We then consider the manifold $\bar{M}_m = \bar{M} \setminus \{s \leq s_0(m)\}$ equipped with the Riemannian metric

$$\bar{g}_m = \frac{1}{1 + s^2 - m s^{-1}} ds \otimes ds + s^2 g_{S^2}.$$ 

The boundary $S^2 \times \{s_0(m)\}$ is an outermost minimal surface, which is referred to as the horizon. Moreover, it is easy to see that $|\bar{g}_m - \bar{g}|_{\bar{g}} = O(s^{-3})$, so $\bar{g}$ satisfies the asymptotic assumptions above. The renormalized volume $V(\bar{M}_m, \bar{g}_m)$ is a well-defined quantity.
of \((\bar{M}_m, \bar{g}_m)\) is given by

\[
V(\bar{M}_m, \bar{g}_m) = \lim_{r \to \infty} \left( \int_{s_0(m)}^{r} \frac{4\pi s^2}{\sqrt{1 + s^2 - ms}} ds - \int_{0}^{r} \frac{4\pi s^2}{\sqrt{1 + s^2}} ds \right).
\]

We now state the main result of this paper.

**Theorem 1.** Let us consider a Riemannian metric \(g\) on \(M = \bar{M} \setminus K\), where \(K\) is a compact set with smooth connected boundary. We assume that \(g\) has the following properties:

- The manifold \((M, g)\) is asymptotically hyperbolic in the sense that
  \[|g - \bar{g}|_{\bar{g}} = O(s^{-2-\delta})\]  and \[|D(g - \bar{g})|_{\bar{g}} = o(1)\].
- The scalar curvature of \(g\) is at least \(-6\).
- The boundary \(\partial M\) is an outermost minimal surface with respect to \(g\), and we have
  \[\text{area}(\partial M, g) \geq \text{area}(\partial \bar{M}_m, \bar{g}_m)\]  for some \(m > 0\).

Then \(V(M, g) \geq V(\bar{M}_m, \bar{g}_m)\). Moreover, if equality holds, then \(g\) is isometric to \(\bar{g}_m\).

We note that our asymptotic assumptions are quite weak: in particular, \(g\) and \(\bar{g}_m\) may have different mass at infinity. An immediate consequence of Theorem 1 is that the function \(m \mapsto V(\bar{M}_m, \bar{g}_m)\) is strictly monotone increasing. This fact is not obvious, as \(s_0(m)\) is an increasing function of \(m\).

Theorem 1 is motivated in part by Bray’s volume comparison theorem [1] for three-manifolds with scalar curvature at least 6, as well as by a rigidity result due to Llarull [9]. A survey of this and other rigidity results involving scalar curvature can be found in [4].

The proof of Theorem 1 uses two main ingredients. The first is the weak inverse mean curvature flow, which was introduced in the ground-breaking work of Huisken and Ilmanen [8] on the Riemannian Penrose inequality (see also [2], where an alternative proof is given). The inverse mean curvature flow has also been considered as a possible tool for proving a version of the Penrose inequality for asymptotically hyperbolic manifolds; see [10], [11]. More recently, the inverse mean curvature flow was used in [6] to prove a sharp Minkowski-type inequality for surfaces in the Anti-deSitter-Schwarzschild manifold.

The second ingredient in our argument is an isoperimetric principle which asserts that a coordinate sphere in the standard Anti-deSitter-Schwarzschild manifold has smallest area among all surfaces that are homologous to the horizon and enclose the same amount of volume. This inequality was established in [7] following earlier work by Bray [1]. In fact, it is known that the coordinate spheres are the only embedded hypersurfaces with constant mean curvature in the Anti-deSitter-Schwarzschild manifold (see [5]).

Our approach also shares common features with a result of Bray and Miao [3], which gives a sharp bound for the capacity of a surface in a three-manifold of nonnegative scalar curvature.
2. Proof of Theorem

Let $(M, g)$ be a Riemannian manifold which satisfies the assumptions of Theorem 1 and let $(M_m, g_m)$ be an Anti-deSitter-Schwarzschild manifold satisfying $\text{area}(\partial M, g) \geq \text{area}(\partial M_m, g_m)$. For abbreviation, let $A = \text{area}(\partial M, g)$ and $\bar{A} = \text{area}(\partial M_m, g_m)$.

Let $\Sigma_t$ denote the weak solution of the inverse mean curvature flow in $(M, g)$ with the initial surface $\Sigma_0 = \partial M$. For each $t$, we denote by $\Omega_t \subset \bar{M}$ the region bounded by $\Sigma_t$.

Proposition 2. Let $\delta \in (0, \frac{1}{4})$ be as above. Then we have $\{ s \leq e^{\frac{(1-\delta)t}{2}} \} \subset \Omega_t$ for $t$ sufficiently large.

Proof. The coordinate sphere $S^2 \times \{s\}$ has mean curvature $2 + o(1)$ for $s$ large. Hence, we can find a real number $t_0$ such that the surfaces $S_t = \{ s = e^{\frac{(1-2\delta)t}{2}} \}$ move with a speed less than $\frac{1}{H}$ for $t \geq t_0$. By the Weak Existence Theorem 3.1 in [8], the regions $\Omega_t$ will eventually contain every given compact set. Hence, we can find a real number $\tau$ such that $\{ s \leq e^{\frac{(1-2\delta)t}{2}} \} \subset \Omega_\tau$. By the maximum principle (cf. Theorem 2.2 in [8]), we have $\{ s \leq e^{\frac{(1-2\delta)(t-\tau+t_0)}{2}} \} \subset \Omega_t$ for $t \geq \tau$. From this, the assertion follows.

Since the boundary $\partial M$ is an outermost minimal surface, we have $\text{area}(\Sigma_t, g) = e^t A$. Moreover, it is well known that the quantity

$$m_H(\Sigma_t) = \text{area}(\Sigma_t, g)^{\frac{1}{2}} \left( 16\pi - \int_{\Sigma_t} (H_g^2 - 4) \, d\mu_g \right)$$

is monotone increasing in $t$.

Proposition 3. For each $\tau \geq 0$, we have

$$\text{vol}(\Omega_\tau \cap M, g) \geq \int_0^\tau e^{\frac{4A}{\bar{A}}} A^{\frac{1}{2}} (4e^t A - 16\pi - e^{-\frac{\bar{A}}{2}} A^{-\frac{1}{2}} m_H(\Sigma_t))^{-\frac{1}{2}} \, dt.$$ 

Proof. By the co-area formula, we have

$$\int_{\Omega_\tau \cap M} \psi H_g \, d\vol_g = \int_0^\tau \left( \int_{\Sigma_t} \psi \, d\mu_g \right) \, dt$$

for every nonnegative measurable function $\psi$. Hence, if we put

$$\psi = \begin{cases} \frac{1}{\mu_g} & \text{if } H_g > 0 \\ \infty & \text{if } H_g = 0, \end{cases}$$

then we obtain

$$\text{vol}(\Omega_\tau \cap M, g) \geq \int_0^\tau \left( \int_{\Sigma_t} \psi \, d\mu_g \right) \, dt.$$
Moreover, it follows from Hölder’s inequality that

$$
\int_{\Sigma_t} \psi \, d\mu_g \geq \text{area}(\Sigma_t, g)^{\frac{3}{2}} \left( \int_{\Sigma_t} H_g^2 \, d\mu_g \right)^{-\frac{1}{2}}
$$

$$
= \text{area}(\Sigma_t, g)^{\frac{3}{2}} \left( 4 \text{area}(\Sigma_t, g) + 16\pi - \text{area}(\Sigma_t, g)^{-\frac{1}{2}} m_H(\Sigma_t) \right)^{-\frac{1}{2}}
$$

$$
= e^{\frac{3t}{2}} A_{\Sigma_t}^3 \left( 4 e^t A + 16\pi - e^{-\frac{t}{2}} A^{-\frac{1}{2}} m_H(\Sigma_t) \right)^{-\frac{1}{2}}.
$$

Putting these facts together, the assertion follows.

**Corollary 4.** We have

$$
2 \text{vol}(\Omega \cap M, g) \geq \int_0^T e^t \left( (1 - e^{-\frac{4t}{3}}) A + 4\pi (e^{-t} - e^{-\frac{4t}{3}}) \right)^{-\frac{1}{2}} dt.
$$

**Proof.** Using the monotonicity of $m_H(\Sigma_t)$, we obtain

$$
m_H(\Sigma_t) \geq m_H(\Sigma_0) = 4 A_{\Sigma_0}^\frac{3}{2} (A + 4\pi).
$$

This implies

$$
2 \text{vol}(\Omega \cap M, g) \geq \int_0^T e^{\frac{3t}{2}} A_{\Sigma_t}^3 \left( (e^t - e^{-\frac{t}{2}}) A + 4\pi (1 - e^{-\frac{t}{2}}) \right)^{-\frac{1}{2}} dt.
$$

From this, the assertion follows.

**Proposition 5.** Let $\Omega$ be a domain in $\bar{M}$ such that $\{ s \leq s_0(m) \} \subset \Omega$, and let $\Sigma$ denote the boundary of $\Omega$. Then

$$
2 \text{vol}(\Omega \cap \bar{M}, \bar{g}_m) \leq \int_0^\bar{\tau} e^t \bar{A}_{\Sigma_t}^3 \left( (1 - e^{-\frac{4t}{3\bar{\tau}}} \bar{A} + 4\pi (e^{-t} - e^{-\frac{4t}{3\bar{\tau}}}) \right)^{-\frac{1}{2}} dt,
$$

where $\bar{\tau}$ is defined by $\text{area}(\Sigma, \bar{g}_m) = e^{\bar{\tau}} \bar{A}$.

**Proof.** If $\Sigma$ is a coordinate sphere in $(\bar{M}, \bar{g}_m)$, then we have

$$
2 \text{vol}(\Omega \cap \bar{M}, \bar{g}_m) = \int_0^\tau e^t \bar{A}_{\Sigma_t}^3 \left( (1 - e^{-\frac{4t}{3\tau}} \bar{A} + 4\pi (e^{-t} - e^{-\frac{4t}{3\tau}}) \right)^{-\frac{1}{2}} dt,
$$

where $\tau$ is defined by $\text{area}(\Sigma, \bar{g}_m) = e^\tau \bar{A}$. On the other hand, it is known (cf. [7], Theorem 4.2) that the coordinate spheres in $(\bar{M}, \bar{g}_m)$ enclose the largest volume for any given surface area. Putting these facts together, the assertion follows.

Let us consider a sequence of times $\tau_i \to \infty$. Moreover, we define a sequence of times $\bar{\tau}_i \to \infty$ by $\text{area}(\Sigma_{\tau_i}, \bar{g}_m) = e^{\bar{\tau}_i} \bar{A}$. By Proposition [2] we have $s \geq e^{\frac{t}{2} - \frac{1}{6}\bar{\tau}_i}$ on $\Sigma_t$ if $t$ is large enough. This implies

$$
|g - \bar{g}_m|_{\bar{g}_m} \leq O(s^{-2 - 4\delta}) \leq O(e^{-\frac{(1-\delta)(1+2\delta)t}{2}}).
$$
at each point on $\Sigma_t$. From this, we deduce that

$$e^{\tau_t}A = \text{area}(\Sigma_{\tau_t}, g)$$

$$= \text{area}(\Sigma_{\tau_t}, \bar{g}_m) (1 + O(e^{-(1-\delta)(1+2\delta)\tau_t}))$$

$$= e^{\bar{\tau}_t} \bar{A} (1 + O(e^{-(1-\delta)(1+2\delta)\tau_t})).$$

Thus, we conclude that

$$\tau_t = \bar{\tau}_t - \alpha + O(e^{-(1-\delta)(1+2\delta)\tau_t}),$$

where $\alpha = \log(A/\bar{A}) \geq 0$. Note that $(1-\delta)(1+2\delta) > 1$ since $\delta \in (0, \frac{1}{4})$.

By Corollary [4] we have

$$2 \text{vol}(\Omega_{\tau_t} \cap M, g) \geq \int_0^{\tau_t} e^t A^\frac{3}{2} ((1 - e^{-\frac{3t}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3t}{2}}))^{-\frac{1}{2}} dt$$

$$= \int_\alpha^{\tau_t + \alpha} e^{t-\alpha} A^\frac{3}{2} ((1 - e^{-\frac{3(t-\alpha)}{2}}) \bar{A} + 4\pi (e^{-t+\alpha} - e^{-\frac{3(t-\alpha)}{2}}))^{-\frac{1}{2}} dt$$

$$= \int_\alpha^{\tau_t + \alpha} e^{t-\alpha} \bar{A}^\frac{3}{2} ((1 - e^{-\frac{3(t-\alpha)}{2}}) \bar{A} + 4\pi (e^{-t+\alpha} - e^{-\frac{3(t-\alpha)}{2}}))^{-\frac{1}{2}} dt.$$

On the other hand, we have

$$2 \text{vol}(\Omega_{\tau_t} \cap \bar{M}_m, \bar{g}_m) \leq \int_0^{\bar{\tau}_t} e^t \bar{A}^\frac{3}{2} ((1 - e^{-\frac{3t}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3t}{2}}))^{-\frac{1}{2}} dt$$

by Proposition [5]. Putting these facts together, we obtain

$$2 (V(M, g) - V(\bar{M}_m, \bar{g}_m))$$

$$= \limsup_{i \to \infty} (\int_\alpha^{\tau_t + \alpha} e^{t-\alpha} A^\frac{3}{2} ((1 - e^{-\frac{3(t-\alpha)}{2}}) \bar{A} + 4\pi (e^{-t+\alpha} - e^{-\frac{3(t-\alpha)}{2}}))^{-\frac{1}{2}} dt$$

$$- \int_0^{\tau_t} e^t A^\frac{3}{2} ((1 - e^{-\frac{3t}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3t}{2}}))^{-\frac{1}{2}} dt)$$

$$= \limsup_{i \to \infty} \left( \int_\alpha^{\tau_t} e^t \bar{A}^\frac{3}{2} ((1 - e^{-\frac{3t}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3t}{2}}))^{-\frac{1}{2}} dt$$

$$- \int_0^{\tau_t} e^t \bar{A}^\frac{3}{2} ((1 - e^{-\frac{3t}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3t}{2}}))^{-\frac{1}{2}} dt \right)$$

$$= A^\frac{3}{2} I(\alpha),$$

where

$$I(\alpha) = \int_\alpha^\infty e^t \left[ ((1 - e^{-\frac{3t}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3t}{2}}))^{-\frac{1}{2}}$$

$$- ((1 - e^{-\frac{3t}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3t}{2}}))^{-\frac{1}{2}} \right] dt$$

$$- \int_0^\infty e^t ((1 - e^{-\frac{3t}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3t}{2}}))^{-\frac{1}{2}} dt.$$
It is shown in the appendix that the function $I(\alpha)$ is positive for all $\alpha > 0$. Thus, we conclude that $V(M, g) \geq V(\bar{M}_m, \bar{g}_m)$.

Finally, we analyze the case of equality. Suppose that $V(M, g) = V(\bar{M}_m, \bar{g}_m)$. Then $I(\alpha) \leq 0$, which implies that $\alpha = 0$. Moreover, the difference

$$2 \operatorname{vol}(\Omega_t \cap M, g) - \int_0^{\tau_i} e^t A_\frac{t}{2} ((1 - e^{-\frac{3t}{2}}) A + 4\pi (e^{-t} - e^{-\frac{3t}{2}}))^{-\frac{3}{2}} dt$$

must converge to 0 as $i \to \infty$. Using Proposition 3, we conclude that $m_H(\Sigma_t) = m_H(\Sigma_0)$ for all $t$. This implies that $g$ is the isometric to one of the standard Anti-deSitter-Schwarzschild metrics. Since $\alpha = 0$, the manifolds $(M, g)$ and $(\bar{M}_m, \bar{g}_m)$ have the same boundary area. Therefore, they are isometric.

**Appendix A. Positivity of the function $I(\alpha)$**

In this section, we show that $I(\alpha) > 0$ for all $\alpha > 0$. We begin with a lemma:

**Lemma 6.** Let $\varepsilon$ and $\mu$ be positive real numbers. If the ratio $\frac{\varepsilon}{\mu}$ is sufficiently small, then we have

$$3\mu \int_0^\infty e^{-\frac{t}{2}} (\varepsilon + (1 - e^{-\frac{3t}{2}})\mu)^{-\frac{3}{2}} dt \geq 4 e^{-\frac{t}{2}} + \mu^{-\frac{3}{2}}.$$

**Proof.** It is elementary to check that

$$e^t \geq 1 + \frac{2}{3} (1 - e^{-\frac{3t}{2}}),$$

hence

$$e^{-\frac{t}{2}} \geq e^{-\frac{3t}{2}} + \frac{2}{3} e^{-\frac{3t}{2}} (1 - e^{-\frac{3t}{2}})$$

for all $t \geq 0$. This implies

$$\int_0^1 e^{-\frac{t}{2}} (\varepsilon + 1 - e^{-\frac{3t}{2}})^{-\frac{3}{2}} dt$$

$$\geq \int_0^1 e^{-\frac{3t}{2}} (\varepsilon + 1 - e^{-\frac{3t}{2}})^{-\frac{3}{2}} dt + \frac{2}{3} \int_0^1 e^{-\frac{3t}{2}} (1 - e^{-\frac{3t}{2}}) (\varepsilon + 1 - e^{-\frac{3t}{2}})^{-\frac{3}{2}} dt$$

$$= \int_0^1 e^{-\frac{3t}{2}} (\varepsilon + 1 - e^{-\frac{3t}{2}})^{-\frac{3}{2}} dt + \frac{2}{3} \int_0^1 e^{-\frac{3t}{2}} (1 - e^{-\frac{3t}{2}})^{-\frac{1}{2}} dt - o(1)$$

$$= \frac{4}{3} e^{-\frac{1}{2}} - \frac{4}{3} (\varepsilon + 1 - e^{-\frac{3}{2}})^{-\frac{1}{2}} + \frac{8}{9} (1 - e^{-\frac{3}{2}})^{\frac{1}{2}} - o(1)$$
for $\varepsilon > 0$ sufficiently small. Hence, we obtain
\[
\int_0^\infty e^{-\frac{t}{2}} (\varepsilon + 1 - e^{-\frac{3\varepsilon}{2}})^{-\frac{3}{2}} dt \\
\geq \frac{4}{3} \varepsilon^{-\frac{1}{2}} - \frac{4}{3} (\varepsilon + 1 - e^{-\frac{3\varepsilon}{2}})^{-\frac{1}{2}} + \frac{8}{9} (1 - e^{-\frac{3\varepsilon}{2}})^{\frac{1}{2}} + (\varepsilon + 1)^{-\frac{3}{2}} \int_1^\infty e^{-\frac{t}{2}} dt - o(1) \\
\geq \frac{4}{3} \varepsilon^{-\frac{1}{2}} - \frac{4}{3} (1 - e^{-\frac{3\varepsilon}{2}})^{-\frac{1}{2}} + \frac{8}{9} (1 - e^{-\frac{3\varepsilon}{2}})^{\frac{1}{2}} + 2 e^{-\frac{1}{2}} - o(1) \\
\geq \frac{4}{3} \varepsilon^{-\frac{1}{2}} + \frac{1}{3}
\]
if $\varepsilon > 0$ is small enough. This proves the assertion for $\mu = 1$. The general case follows by scaling.

We now consider the function
\[
I_\varepsilon(\alpha) = \int_\alpha^\infty e^t \left[ (\varepsilon + (1 - e^{-\frac{3\varepsilon}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3\varepsilon}{2}}))^{-\frac{1}{2}} \\
- (\varepsilon + (1 - e^{-\frac{3\varepsilon}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3\varepsilon}{2}}))^{-\frac{1}{2}} \right] dt \\
- \int_0^\alpha e^t (\varepsilon + (1 - e^{-\frac{3\varepsilon}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3\varepsilon}{2}}))^{-\frac{1}{2}} dt.
\]

Then
\[
\frac{d}{d\alpha} I_\varepsilon(\alpha) = \int_\alpha^\infty e^t \frac{d}{d\alpha} \left[ (\varepsilon + (1 - e^{-\frac{3\varepsilon}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3\varepsilon}{2}}))^{-\frac{1}{2}} \right] dt - e^\alpha \varepsilon^{-\frac{1}{2}} \\
= \frac{1}{4} (3 e^{\frac{3\alpha}{2}} \bar{A} + 4\pi e^\frac{\alpha}{2}) \\
\cdot \int_\alpha^\infty \frac{1}{2} e^\frac{1}{2} (\varepsilon + (1 - e^{-\frac{3\varepsilon}{2}}) \bar{A} + 4\pi (e^{-t} - e^{-\frac{3\varepsilon}{2}}))^{-\frac{3}{2}} dt - e^\alpha \varepsilon^{-\frac{1}{2}} \\
= \frac{e^\alpha}{4} (3 \bar{A} + 4\pi e^{-\alpha}) \\
\cdot \int_0^\infty e^\frac{1}{2} (\varepsilon + (1 - e^{-\frac{3\varepsilon}{2}}) \bar{A} + 4\pi e^{-\alpha} (e^{-t} - e^{-\frac{3\varepsilon}{2}}))^{-\frac{3}{2}} dt - e^\alpha \varepsilon^{-\frac{1}{2}} \\
\geq \frac{e^\alpha}{4} (3 \bar{A} + 4\pi e^{-\alpha}) \\
\cdot \int_0^\infty e^{-\frac{1}{2}} (\varepsilon + (1 - e^{-\frac{3\varepsilon}{2}}) (\bar{A} + \frac{4\pi}{3} e^{-\alpha}))^{-\frac{3}{2}} dt - e^\alpha \varepsilon^{-\frac{1}{2}},
\]
where in the last step we have used the inequality $e^{-t} - e^{-\frac{3\varepsilon}{2}} \leq \frac{1}{3} (1 - e^{-\frac{3\varepsilon}{2}})$. Hence, if the ratio $\frac{\varepsilon}{\bar{A} + \frac{4\pi}{3} e^{-\alpha}}$ is sufficiently small, then
\[
\frac{d}{d\alpha} I_\varepsilon(\alpha) \geq \frac{e^\alpha}{4} (\bar{A} + \frac{4\pi}{3} e^{-\alpha})^{-\frac{1}{2}}
\]
by Lemma [6]. Since $I(\alpha) = \lim_{\varepsilon \to 0} I_\varepsilon(\alpha)$ for each $\alpha \geq 0$, we conclude that the function $I(\alpha)$ is strictly monotone increasing. In particular, $I(\alpha) > 0$ for all $\alpha > 0$. 

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