Theory of and Experiments on Minimally Invasive Stability Preservation in Changing Two-Sided Matching Markets

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Abstract

Following up on purely theoretical work of Bredereck et al. [AAAI 2020], we contribute further theoretical insights into adapting stable two-sided matchings to change. Moreover, we perform extensive empirical studies hinting at numerous practically useful properties. Our theoretical extensions include the study of new problems (that is, incremental variants of ALMOST STABLE MARRIAGE and HOSPITAL RESIDENTS), focusing on their (parameterized) computational complexity and the equivalence of various change types (thus simplifying algorithmic and complexity-theoretic studies for various natural change scenarios). Our experimental findings reveal, for instance, that allowing the new matching to be blocked by a few pairs significantly decreases the necessary differences between the old and the new stable matching.

1 Introduction

In our dynamic world, change is omnipresent in society and business. Typically, there is no permanent stability. We address this issue in the context of stable matchings in two-sided matching markets and their adaptivity to change. Consider as an example the dynamic nature of centrally assigning students to public schools. Here, students are matched to schools, trying to accommodate the students’ preferences over the schools as well as possible. However, due to students reallocating or deciding to visit a private school, according to Feigenbaum et al. [2020], in New York typically around 10% of the students drop out after a first round of assignments, triggering some readjustments in the school-student matchings in a further round.

Matching students to schools can be modeled as an instance of the HOSPITAL RESIDENTS problem, where we are given a set of residents and hospitals each with preferences over the agents from the other set. One wants to find a “stable” assignment of each resident to at most one hospital such that a given capacity for each hospital is respected. To model the task of adjusting a matching to change, Bredereck et al. [2020] introduced the problem, given a stable matching with respect to some initial preference profile, to find a new matching which is stable with respect to an updated preference profile (where some agents performed swaps in their preferences) and which is as similar as possible to the given matching. They referred to this as the “incremental” scenario and studied the computational complexity of this question for STABLE MARRIAGE and STABLE ROOMMATES (both being one-to-one matching problems).

1Motivated by this, Boehmer and Niedermeier [2021] recently challenged the computational social choice community to adapt classical models to also account for dynamic aspects.
In this work, we address multiple so-far unstudied aspects of our introductory school choice example. First, we theoretically and experimentally relate different types of changes to each other, including swapping two agents in some preference list (as studied by Bredereck et al. [2020]) and deleting an agent (as in our introductory example). Second, we initiate the study of the incremental variant of many-to-one stable matchings (HOSPITAL RESIDENTS). Third, as perfect stability might not always be essential, for instance, in large markets, we introduce the incremental variant of ALMOST STABLE MARRIAGE (where the new matching is allowed to be blocked by few agent pairs) and study its computational complexity and practical impact. Fourth, we experimentally analyze how many adjustments are typically needed when a certain amount of change occurs; moreover, we give some recommendations to market makers for adapting stable matchings to change.

1.1 Related Work

We are closest to the purely theoretical work of Bredereck et al. [2020], using their formulation of incremental stable matching problems (we refer to their related work section for an extensive discussion of related and motivating literature before 2020). Among others, they proved that INCREMENTAL STABLE MARRIAGE is polynomial-time solvable but is NP-hard (and W[1]-hard parameterized by the allowed change between the two matchings) if the preferences may contain ties. We complement and enhance some of Bredereck et al.’s findings, focusing on two-sided markets and contributing extensive experiments.

Besides the work of Bredereck et al. [2020], there are several other works dealing with adapting a (stable) matching to a changing agent set or changing preferences [Bhattacharya et al., 2015, Feigenbaum et al., 2020, Gajulapalli et al., 2020, Ghosal et al., 2020, Kanade et al., 2016, Nimbhorkar and Rameshwar, 2019]. Closest to our work, Gajulapalli et al. [2020] designed polynomial-time algorithms for two variants of an incremental version of HOSPITAL RESIDENTS where the given matching is always resident-optimal (unlike in our setting where the given matching can be an arbitrary stable matching) and in the updated instance either new residents are added or the quotas of some hospitals are modified.

Sharing a common motivation with our work, there is a rich body of studies concerning dynamic matching markets mostly driven by economists [Akbarpour et al., 2020, Baccara et al., 2020, Damiano and Lari, 2005, Liu, 2021]. In the context of matching under preferences, a frequently studied exemplary (online) problem is that agents arrive over time and want to be matched as soon as possible in an—also in the long run—stable way (reassignments are not allowed) [Doval, 2021, Liu, 2021].

Lastly, instead of trying to adapt an already implemented matching to change, it is also possible to try to construct the initial stable matching to be as robust as possible, i.e., to pick a stable matching that remains stable even if the instance is slightly changed or that can be easily adapted to a stable matching after some changes have been performed [Boehmer et al., 2021a, Chen et al., 2021, Genc et al., 2017a,b, 2019, Mai and Vazirani, 2018].

1.2 Our Contributions

On the theoretical side, while Bredereck et al. [2020] focused on swapping adjacent agents in preference lists, we consider three further natural types of changes: the deletion and addition of agents and the complete replacement of an agent’s preference list. These different change types model different kinds of real-world scenarios; however, as one of our main theoretical results, we prove in Section 3 that all four change types are equivalent from a theoretical perspective, thus allowing us to transfer both algorithmic and computational hardness results from one type to another.

Motivated by the polynomial-time algorithm of Bredereck et al. [2020] for INCREMENTAL STABLE MARRIAGE (ISM), in Section 4, we study the related problem INCREMENTAL ALMOST STABLE MARRIAGE (IASM) (where the new matching may admit few blocking pairs). We show that INCREMENTAL
Almost Stable Marriage is NP-hard and establish parameterized tractability and intractability results. Moreover, motivated by the observation that, in practice, also many-to-one matching markets may change, we consider Incremental Hospital Residents (IHR) in Section 5. We show that the problem is polynomial-time solvable. However, if preferences may contain ties, then it becomes NP-hard and \(W[1]\)-hard when parameterized by the number of hospitals; still, we can identify several (fixed-parameter) tractable cases. See Table 1 for an overview of our results.

On the experimental side (Section 6), we perform an extensive study, among others taking into account the four different change types discussed above. For instance, we investigate the relation between the number of changes and the symmetric difference between the old and new stable matching. We observe that often already very few random changes require a major restructuring of the matching. One way to circumvent this problem is to allow that the new matching might be blocked by a few agent pairs. Moreover, reflecting its popularity, we compute the input matching using the Gale-Shapley algorithm [Gale and Shapley, 1962] and observe that, in this case, computing the output matching also with \(FPT\) wrt. \(t_U + t_W\) (Pr. 5)

|                | without ties          | with ties               |
|----------------|-----------------------|-------------------------|
| ISM            | \(P\)                 | \(W[1]\)-h. wrt. \(k\) even if \(|P_1 \oplus P_2| = 1\)\)          |
|                | \(FPT\) wrt. \(t_U + t_W\) (Pr. 5) | \(W[1]\)-h. wrt. \(k\) even if \(|P_1 \oplus P_2| = 1\)\)          |
| IAM            | \(W[1]\)-h. wrt. \(k + b + |P_1 \oplus P_2|\) (Th. 2) | \(W[1]\)-h. wrt. \(k\) for \(b = 0\) and \(|P_1 \oplus P_2| = 1\)\) \(XP\) wtw. \(k\) (Pr. 5) |
|                | \(XP\) wtw. \(k\) or \(b\) or \(|P_1 \oplus P_2|\) (Pr. 4) | \(XP\) wtw. \(k\) (Pr. 5) |
| IHR            | \(P\) (Pr. 2)         | \(W[1]\)-hard wtw. \(m\) even if \(|P_1 \oplus P_2| = 1\) (Th. 5) |
|                | FPT wtw. \(n\) (Pr. 3) | \(XP\) wtw. \(m\) (Pr. 4) |

Table 1: Overview of our results. For definitions of our parameters, see Section 2. All \(W[1]\)-hardness results imply NP-hardness. Results marked with \(\dagger\) were proven by [Bredereck et al., 2020].

2 Preliminaries

An instance of the Stable Marriage with Ties (SM-T) problem consists of two sets \(U\) and \(W\) of agents and a preference profile \(P\) containing a preference relation for each agent. Following conventions, we refer to the agents from \(U\) as men and to the agents from \(W\) as women. We denote the set of all agents by \(A := U \cup W\). Each man \(m \in U\) accepts a subset \(Ac(m) \subseteq W\) of women, and each woman \(w\) accepts a subset \(Ac(w) \subseteq U\) of men. The preference relation \(\succeq_a\) of agent \(a \in A\) is a weak order of the agents \(Ac(a)\) that agent \(a\) accepts. For two agents \(a', a'' \in Ac(a)\), agent \(a\) weakly prefers \(a'\) to \(a''\) if \(a' \succeq_a a''\). If \(a\) both weakly prefers \(a'\) to \(a''\) and \(a''\) to \(a'\), then \(a\) is indifferent between \(a'\) and \(a''\) and we write \(a' \sim a\). If \(a\) weakly prefers \(a'\) to \(a''\) but does not weakly prefer \(a''\) to \(a'\), then \(a\) strictly prefers \(a'\) to \(a''\) and we write \(a' \succ a\). If the preference relation of an agent \(a\) is a strict order, that is, there are no two agents such that \(a\) is indifferent between the two, then we say that \(a\) has strict preferences and denote \(a\)'s preference relation as \(\succ a\). In this case, we use the terms “strictly prefer” and “prefer” interchangeably.

Stable Marriage (SM) is the special case of SM-T where all agents have strict preferences. For two preference relations \(\succeq\) and \(\succeq'\), the swap distance between \(\succeq\) and \(\succeq'\) is the number of agent pairs that are ordered differently by the two relations, i.e., \(|\{\{a, b\} : a \succ b \land b \succeq' a\}| + |\{\{a, b\} : a \sim b \land a \sim b\}|\); if both relations are defined on different sets, then we define the swap distance to be infinity. For two strict preference relations \(\succ\) and \(\succ'\), the swap distance yields the minimum number of swaps of adjacent agents needed to transform \(\succ\) into \(\succ'\). For two preference profiles \(P_1\) and \(P_2\) on the same set of agents, \(|P_1 \oplus P_2|\) denotes the summed swap distance between the two preference relations of each agent.
A matching $M$ is a set of pairs $\{m, w\}$ with $m \in Ac(w)$ and $w \in Ac(m)$ where each agent appears in at most one pair. For two matchings $M$ and $M'$, the symmetric difference is $M \triangle M' = (M' \setminus M) \cup (M \setminus M')$. In a matching $M$, an agent $a$ is matched if $a$ appears in one pair, i.e., $\{a, a'\} \in M$ for some $a' \in A \setminus \{a\}$; otherwise, $a$ is unmatched. A matching is perfect if each agent is matched. For a matching $M$ and a matched agent $a \in A$, we denote by $M(a)$ the partner of $a$ in $M$, i.e., $M(a) = a'$ if $\{a, a'\} \in M$. For an unmatched agent $a \in A$, we set $M(a) := \emptyset$. All agents $a \in A$ strictly prefer any agent from $Ac(a)$ to being unmatched (thus, we have $a' \succ_a \emptyset$ for $a' \in Ac(a)$).

A pair $\{u, w\}$ with $u \in U$ and $w \in W$ blocks a matching $M$ if $m$ and $w$ accept each other and strictly prefer each other to their partners in $M$, i.e., $m \in Ac(w)$, $w \in Ac(m)$, $m \succ_w M(w)$, and $w \succ_m M(m)$. A matching $M$ is stable if it is not blocked by any pair. SM and SM-T ask whether there is a stable matching of the agents $A$ with respect to preference profile $P$. For a matching $M$, we denote as $bp(M, P)$ the set of pairs that block $M$ in preference profile $P$.

We also consider a generalization of SM called almost stable marriage (ASM), where as an additional part of the input we are given an integer $b$ and the question is whether there is a matching admitting at most $b$ blocking pairs. Furthermore, we study the hospital residents (HR) problem, a generalization of SM where we are given a set $R$ of residents and a set $H$ of hospitals and agents from both sets have preferences over a set of acceptable agents from the other set and each hospital $h \in H$ has an upper quota $u(h)$. A matching then consists of resident-hospital pairs $\{r, h\}$ with $r \in Ac(h)$ and $h \in Ac(r)$, where each resident can appear in at most one pair, while each hospital $h$ can appear in at most $u(h)$ pairs. In this context, we slightly adapt the definition of a blocking pair and say that a resident-hospital pair $\{r, h\}$ blocks a matching $M$ if both $r$ and $h$ accept each other, $r$ prefers $h$ to $M(r)$, and $h$ is matched to less than $u(h)$ residents in $M$ or prefers $r$ to one of the residents matched to it.

Our work focuses on “incrementalized versions” of the discussed two-sided stable matching problems. For SM(SM-T), this reads as follows:

**Incremental Stable Marriage [with Ties] (ISM/[ISM-T])**

**Input:** A set $A = U \cup W$ of agents, two preference profiles $P_1$ and $P_2$ containing the strict [weak] preferences of all agents, a stable matching $M_1$ in $P_1$, and an integer $k$.

**Question:** Is there a matching $M_2$ that is stable in $P_2$ such that at most $k$ edges appear in only one of $M_1$ and $M_2$, i.e., $|M_1 \triangle M_2| \leq k$?

IHR and IHR-T are defined analogously. IASM [IASM-T] is defined as ISM [ISM-T] with the difference that we are given an additional integer $b$ as part of the input and the question is whether there is a matching $M_2$ that admits at most $b$ blocking pairs in $P_2$ such that $|M_1 \triangle M_2| \leq k$.

### 3 Equivalence of Different Types of Changes

Bredereck et al. [2020] focused on the case where the preference profile $P_2$ arises from $P_1$ by performing some swaps of adjacent agents in the preferences of some agents (we refer to this as Swap). However, there are many more types of changes: Allowing for more radical changes, denoted by Replace, we count the number of agents whose preferences changed (here in contrast to Swap, we also allow that the set of acceptable partners may change). Next, recall that in our introductory example from school choice children leave the matching market, which corresponds to agents getting deleted. We denote this type of change by Delete—formally, we model the deletion of an agent by setting its set of acceptable partners in $P_2$ to $\emptyset$. Moreover, children leaving one market might enter a new one, which corresponds to agents getting added (Add). Formally, we model the addition of an agent $a$ by already including it in $P_1$, but with $Ac(a) = \emptyset$ in $P_1$. The goal of this section is to show that these four natural possibilities of how $P_2$ may arise from $P_1$ actually result in equivalent computational problems. More formally, we say that a type of change $\mathcal{X} \in \{\text{Delete, Add, Swap, Replace}\}$ linearly reduces to a type of change $\mathcal{Y} \in \{\text{Delete, Add, Swap, Replace}\}$ if any instance $I = (A, P_1, P_2, M_1, k)$ of ISM(-T)
where \( P_1 \) and \( P_2 \) differ by \( x \) changes of type \( \mathcal{X} \) can be transformed in linear time to an equivalent instance \( \mathcal{I}' = (A', P'_1, P'_2, M'_1, k') \) of ISM(-T) with \( P'_1 \) and \( P'_2 \) differing by \( \mathcal{O}(x) \) changes of type \( \mathcal{Y} \). We call two change types \( \mathcal{X} \) and \( \mathcal{Y} \) linearly equivalent if both \( \mathcal{X} \) linearly reduces to \( \mathcal{Y} \) and \( \mathcal{Y} \) linearly reduces to \( \mathcal{X} \).

**Theorem 1.** Swap, Replace, Delete, and Add are linearly equivalent for ISM and ISM-T.

We show the equivalence of all considered different types of changes using a circular reasoning. First, we observe that Swap is a special case of Replace since every swap can be performed by a Replace operation.

**Observation 1.** Swap can be linearly reduced to Replace.

Next, we show how Delete can be linearly reduced to Swap.

**Lemma 1.** Delete can be linearly reduced to Swap.

**Proof.** Let \( \mathcal{I} = (A, P_1, P_2, M_1, k) \) be an instance of ISM(-T) for Delete. Let \( A_{\text{delete}} \) be the set of agents with empty preferences in \( P_2 \) and non-empty preferences in \( P_1 \) (i.e., the set of “deleted” agents). We create an instance \( \mathcal{I}' = (A', P'_1, P'_2, M'_1, k') \) for Swap as follows. To create \( A' \), for each agent \( a \in A \), we add an agent \( a' \) and set \( a' \)'s preferences in both \( P'_1 \) and \( P'_2 \) to \( a \)'s preferences in \( P_1 \). For each \( a \in A_{\text{delete}} \), we further add two agents \( a'' \) and \( a''' \) to \( A' \). In \( P'_1 \), agent \( a'' \) prefers agent \( a''' \) to \( a' \), while \( a''' \) only considers \( a'' \) acceptable. Moreover, we modify the preferences of agent \( a' \) such that it prefers \( a'' \) to all other agents in \( P'_1 \) and \( P'_2 \). In \( P'_2 \), agent \( a'' \) performs a swap in its preferences and now prefers \( a' \) to \( a''' \). We set \( M'_1 := \{\{a', b\}' : \{a, b\} \in M_1 \} \cup \{\{a'', a'''\}' : a \in A_{\text{delete}} \} \) and \( k' := k + 2|A_{\text{delete}}| \).

The correctness easily follows from the observation that every stable matching for \( P'_1 \) contains edge \( \{a'', a'''\}' \) for every \( a \in A_{\text{delete}} \), while every stable matching for \( P'_2 \) contains edge \( \{a', a'''\}' \) for every \( a \in A_{\text{delete}} \), and \( a''' \) is unmatched. \( \square \)

We continue by observing that Add can be linearly reduced to Delete.

**Lemma 2.** Add can be linearly reduced to Delete.

**Proof.** Let \( \mathcal{I} = (A, P_1, P_2, M_1, k) \) be an instance of ISM(-T) for Add. Let \( A_{\text{add}} \) be the set of agents with empty preferences in \( P_1 \) and non-empty preferences in \( P_2 \) (i.e., the set of “added” agents). We create an instance \( \mathcal{I}' = (A', P'_1, P'_2, M'_1, k') \) for Delete as follows. To create \( A' \), for each agent \( a \in A \), we add an agent \( a' \) and set \( a' \)'s preferences in both \( P'_1 \) and \( P'_2 \) to \( a \)'s preferences in \( P_2 \). For each \( a \in A_{\text{add}} \), we add an agent \( a'' \) to \( A' \). Agent \( a'' \) only finds \( a' \) acceptable in \( P'_1 \), while the preferences of \( a'' \) in \( P'_2 \) are empty (\( a'' \) gets deleted). In \( P'_1 \) and \( P'_2 \), we modify the preferences of \( a' \) by adding \( a'' \) at the first position. We set \( M'_1 := \{\{a, b\}' : \{a, b\} \in M_1 \} \cup \{\{a', a''\}' : a \in A_{\text{add}} \} \) and \( k' := k + |A_{\text{add}}| \). The correctness of the reduction follows from the observation that every stable matching in \( \mathcal{P}'_1 \) contains edge \( \{a', a''\}' \) for every \( a' \in A_{\text{add}} \), while \( a'' \) is unmatched in every stable matching in \( \mathcal{P}'_2 \) (and cannot form a blocking pair in any matching). \( \square \)

Finally, we show that Replace can be reduced to Add.

**Lemma 3.** Replace can be linearly reduced to Add.

**Proof.** Let \( \mathcal{I} = (A = U \cup W, P_1, P_2, M_1, k) \) be an instance of ISM(-T) for Replace. From this, we construct an instance \( \mathcal{I}' = (A' = U' \cup W', P'_1, P'_2, M'_1, k') \) of ISM(-T) for Add as follows. Let \( A_{\text{repl}} \) be the set of agents with different preferences in \( P_1 \) and \( P_2 \), and let \( A_{\text{repl}} := A_{\text{repl}} \cup \{M_1(a)' : a \in A_{\text{repl}} \land M_1(a) \neq \emptyset \} \) be the set of these agents and their partners in \( M_1 \). To construct \( \mathcal{I}' \), we start by adding all agents from \( A \) to \( A' \) and set the preferences of all agents in \( \mathcal{P}'_1 \) and \( \mathcal{P}'_2 \) to be their preferences.
in \( P_1 \) (the preferences of some of these agents will be modified slightly in the following). Moreover, for each \( a \in A_{\text{repl}} \), we add to \( A' \) a “binding” agent \( b_a \) and a “clone” \( c_a \). Agent \( c_a \) has empty preferences in \( P'_1 \) and has \( a \)’s preferences from \( P_2 \) in \( P'_2 \). We modify the preferences of all so far added agents such that \( c_a \) appears directly before \( a \) (or is tied with \( a \) if we have an instance with ties). Agent \( b_a \) has empty preferences in \( P'_1 \), only finds \( a \) acceptable in \( P'_2 \), and we modify the preferences of \( a \) in both \( P'_1 \) and \( P'_2 \) such that \( a \) prefers \( b_a \) to all other agents.

The idea behind the construction is as follows. We add \( b_a \) in \( P'_2 \) which forces \( M'_2 \) to contain \( \{a, b_a\} \) and further add agent \( c_a \), who “replaces” \( a \) in \( P'_2 \) and has \( a \)’s changed preferences. However, this construction does not directly work: Let \( m \in A'_{\text{repl}} \cap U \) and \( w = M_1(m) \). Unfortunately, adding the edge \( \{m, w\} \) to \( M_2 \) corresponds to adding the edge \( \{c_m, c_w\} \) to \( M'_2 \), which leads to an increase of \( |M'_1 \triangle M'_2| \) but not of \( |M_1 \triangle M_2| \). In order to cope with this, we replace the edge \( \{c_m, c_w\} \) by an edge gadget consisting of multiple agents: For each \( m \in A'_{\text{repl}} \cap U \) matched by \( M_1 \) to a woman \( w \), we introduce agents as depicted in Figure 1 and modify the preferences of \( c_m \) and \( c_w \) by replacing \( w \) and \( m \) by \( a_{\text{lm}} \) and \( a_{\text{rm}} \), respectively. The newly introduced agents from this gadget have empty preferences in \( P'_1 \) and preferences as depicted in Figure 1 in \( P'_2 \) except for agents \( a_{\text{lm}} \) and \( a_{\text{rm}} \) who have their depicted preferences in both \( P'_1 \) and \( P'_2 \). We set \( M'_1 := M_1 \cup \{ \{a_{\text{lm}}, a_{\text{rm}}\} : \{m, w\} \in M_1 \land m \in A'_{\text{repl}} \cap U \} \) and \( k := k + |A'_{\text{repl}}| + 3k^* \) with \( k^* := |\{ \{m, w\} : \{m, w\} \in M_1 \land m \in A'_{\text{repl}} \}| \).

Next, we show the correctness of the forward direction of our reduction. Given a stable matching \( M_2 \) in \( P_2 \), we construct a stable matching \( M'_2 \) in \( P'_2 \) with \( |M'_1 \triangle M'_2| = |M_1 \triangle M_2| + |A'_{\text{repl}}| + 3k^* \) as follows. We start with \( M'_2 := M'_1 \). We first implement the adjustments corresponding to edges from \( M_1 \triangle M_2 \): Let \( \{m, w\} \in M_2 \setminus M_1 \). We delete the edges containing \( m \) and \( w \) from \( M'_2 \) (if there are any). Moreover, if \( m, w \notin A^*_{\text{repl}} \), then we add \( \{m, w\} \) to \( M'_2 \). If \( m \in A^*_{\text{repl}} \) and \( w \notin A^*_{\text{repl}} \), then we add \( \{c_m, w\} \). If \( w \in A^*_{\text{repl}} \) and \( m \notin A^*_{\text{repl}} \), then we add \( \{m, c_w\} \). If \( m, w \in A^*_{\text{repl}} \), then we add \( \{c_m, c_w\} \). After these adjustments, it holds that \( |M'_1 \triangle M'_2| = |M_1 \triangle M_2| \).

We now turn to extending matching \( M'_2 \) to include edges from the edge gadgets. For every edge \( \{m, w\} \in M_1 \cap M_2 \) with \( m, w \in A^*_{\text{repl}} \), we delete \( m, w \) from \( M'_2 \) and add edges \( \{c_m, a_{\text{lm}}\}, \{a_{\text{ml}}, a_{\text{m}}\}, \{a_{\text{lt}}, a_{\text{lt}}\}, \{a_{\text{lb}}, a_{\text{lb}}\}, \{a_{\text{rm}}, c_w\} \). This contributes seven edges to \( M'_1 \triangle M'_2 \) (note that the pair \( \{m, w\} \) has already been deleted from \( M'_2 \) and thus already contributed to \( |M'_1 \triangle M'_2| \)). For every edge \( \{m, w\} \in M_1 \setminus M_2 \) with \( m, w \in A^*_{\text{repl}} \), we first delete edge \( \{a_{\text{lm}}^*, a_{\text{lm}}^*\} \) from \( M'_2 \). Subsequently, we make a case distinction based on whether \( m \) strictly prefers \( w \) to \( M_2(m) \). If yes, then the stability of \( M_2 \) implies that \( w \) does not strictly prefer \( m \) to \( M_2(w) \). Thus, we can add the edges \( \{a_{\text{ml}}^*, a_{\text{ml}}^*\}, \{a_{\text{lt}}^*, a_{\text{lt}}^*\}, \{a_{\text{lb}}^*, a_{\text{lb}}^*\}, \{a_{\text{rm}}, a_{\text{lm}}^*\}, \{a_{\text{rm}}, a_{\text{rm}}^*\}, \{a_{\text{rm}}, a_{\text{rm}}^*\}, \{a_{\text{rm}}, a_{\text{lm}}^*\}, \{a_{\text{rm}}, a_{\text{lm}}^*\}, \) and \( \{a_{\text{rm}}, a_{\text{lm}}^*\} \), and the resulting matching is not

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\(^2\)We remark that this gadget is a concatenation of two parallel-edges gadgets used by Cechlárová and Fleiner [2005].

Figure 1: The edge gadget for edge \( e = \{c_m, c_w\} \), where \( m \in U \), \( w \in W \), and \( m \) ranks \( w \) at the \( i \)-th rank, and \( w \) ranks \( m \) at the \( j \)-th rank. Squared agents have empty preferences in \( P'_1 \). The edge contained in \( M'_1 \) is bold. The numbers on the edges indicate the preferences of the agents: The number \( x \) closer to an agent \( a \) means that \( a \) ranks the other endpoint \( a' \) of the edge at rank \( x \), i.e., there are \( x - 1 \) agents which \( a \) prefers to \( a' \).

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Reference:
Cechlárová, A., & Fleiner, T. (2005).
contributing seven edges, we have

\[
M'_1 \triangle M'_2 = |M'_1 \triangle M'_2| + 7k^* \]

Also blocks.

Lastly, for every \( a \in A'_{\text{repl}} \), we add the edge \( \{a, b_a\} \) to \( M'_2 \), which contributes \( |A'_{\text{repl}}| \) edges to \( |M'_1 \triangle M'_2| \) leading to an overall symmetric difference of \( |M'_1 \triangle M'_2| = |M'_1 \triangle M'_2| + |A'_{\text{repl}}| + 7k^* \). It is easy to verify that \( M'_2 \) is stable in \( \mathcal{P}'_2 \).

Vice versa, given a stable matching \( M'_2 \in \mathcal{P}'_2 \), we construct a stable matching \( M_2 \in \mathcal{P}'_2 \) with \( |M_1 \triangle M_2| = |M'_1 \triangle M'_2| - |A'_{\text{repl}}| - 7k^* \) as follows. We add edge \( \{m, w\} \) to \( M_2 \) if one of the following conditions hold:

- \( m, w \notin A'_{\text{repl}} \) and \( \{m, w\} \in M'_2 \);
- \( m \in A^*_\text{repl} \) and \( \{m, w\} \in M'_2 \);
- \( m \notin A^*_\text{repl} \) and \( \{m, w\} \notin M'_2 \).

First, we show that for each edge \( \{m, w\} \in M_1 \) with \( m, w \in A^*_\text{repl} \), we have \( \{m, a^m_m\} \in M'_2 \) if and only if \( \{c_m, a^{lm}_m\} \). If \( M'_2 \) contains \( \{c_m, a^{lm}_m\} \), then the stability of \( M'_2 \) implies that \( M'_2 \) also contains edges \( \{a^*_m, a^{lm}_m\}, \{a^{lb}_m, a^{lt}_m\}, \{a^{lb}_m, a^{rt}_m\}, \{a^{lt}_m, a^{rt}_m\}, \{a^{lb}_m, a^{rb}_m\}, \{a^{lb}_m, a^{rb}_m\}, \{a^{rt}_m, a^{lt}_m\}, \{a^{rt}_m, a^{rb}_m\}, \{a^{rt}_m, a^{rb}_m\}, \{a^{rt}_m, a^{lm}_m\}, \{a^{rt}_m, a^{lm}_m\}, \{a^{rt}_m, a^{lm}_m\} \). (An analogous argument also works if \( \{c_m, a^{lm}_m\} \in M'_2 \).) Next, we show the stability of \( M_2 \). Assume for a contradiction that \( \mathcal{P}'_2 \) contains a blocking pair \( \{m, w\} \) for \( M_2 \). If neither \( m \) nor \( w \) is contained in \( A^*_\text{repl} \), then \( \{m, w\} \) also blocks \( M'_2 \), a contradiction. In the following, we assume that \( m \in A^*_\text{repl} \) (the case \( w \in A^*_\text{repl} \) is symmetric). If \( \{m, w\} \notin M'_1 \), then \( \{c_m, a^{lm}_m\} \) (if \( w \notin A^*_\text{repl} \)) or \( \{c_m, c_w\} \) (if \( w \in A^*_\text{repl} \)) blocks \( M'_2 \) in \( \mathcal{P}'_2 \), a contradiction. Thus, we have \( \{m, w\} \in M_1 \). Because \( \{m, w\} \) blocks \( M_2 \), it follows that \( \{m, w\} \notin M_2 \) and by our initial observation that \( \{c_m, a^{lm}_m\} \notin M'_1 \) and \( \{a^{lm}_m, a^{lm}_m\} \notin M'_1 \). As \( \{m, w\} \) blocks \( M_2 \), man \( m \) prefers \( w \) to \( M_2(m) \) in \( \mathcal{P}'_2 \) and woman \( w \) prefers \( m \) to \( M_2(w) \) in \( \mathcal{P}'_2 \). Thus, \( c_m \) prefers \( a^{lm}_m \) to \( M'_2(m) \) in \( \mathcal{P}'_2 \) and \( w \) prefers \( a^{lm}_m \) to \( M'_2(w) \) in \( \mathcal{P}'_2 \). For \( \{c_m, a^{lm}_m\} \) and \( \{a^{lm}_m, a^{lm}_m\} \) not to block \( M'_2 \), woman \( a^{lm}_m \) needs to be matched better than \( c_m \) and man \( a^{lm}_m \) needs to be matched better than \( c_w \) in \( M'_2 \). Thus, \( M'_2 \) contains the edges \( \{a^{lb}_m, a^{lm}_m\} \) and \( \{a^{lm}_m, a^{lb}_m\} \), and consequently also \( \{a^{lm}_m, a^{lm}_m\} \) and \( \{a^{lm}_m, a^{lm}_m\} \). However, it follows that \( \{a^{lm}_m, a^{lm}_m\} \) blocks \( M'_2 \), a contradiction to the stability of \( M'_2 \). Thus, \( M_2 \) is stable.

It remains to show that \( |M_1 \triangle M_2| \leq |M'_1 \triangle M'_2| - |A'_{\text{repl}}| - 7k^* \). Note that apart from replacing a by \( c_a \) for \( a \in A^*_\text{repl} \), matchings \( M_2 \) and \( M'_2 \) differ by the \( |A^*_\text{repl}| \) edges \( \{a, b_a\} \) for \( a \in A^*_\text{repl} \) and the edges contained in the edge gadget for replacing edges \( \{m, w\} \) from \( M_1 \) with \( m, w \in A^*_\text{repl} \).

We now describe all edges that are part of \( M'_1 \triangle M'_2 \): For each edge \( \{m, w\} \in M_1 \) with \( m, w \in A^*_\text{repl} \), we identify seven edges containing agents of this edge gadget in the symmetric difference \( M'_1 \triangle M'_2 \): If \( \{m, w\} \in M_2 \), then (as observed above) \( M'_2 \) contains seven edges from this edge gadget (including \( \{a^{lm}_m, a^{lm}_m\} \)). All these edges apart from \( a^{lm}_m, a^{lm}_m \) are part of \( M'_1 \triangle M'_2 \). Additionally, edge \( \{m, w\} \) is contained in \( M'_1 \triangle M'_2 \). If \( \{m, w\} \notin M_2 \), then \( M'_2 \) contains the edges \( \{a^{lb}_m, a^{lb}_m\}, \{a^{lb}_m, a^{lb}_m\}, \{a^{lt}_m, a^{rt}_m\}, \{a^{lt}_m, a^{rt}_m\}, \{a^{rt}_m, a^{lt}_m\}, \{a^{rt}_m, a^{lt}_m\}, \{a^{rm}_m, a^{rm}_m\}, \{a^{rm}_m, a^{rm}_m\}, \{a^{rt}_m, a^{rm}_m\}, \{a^{rt}_m, a^{rm}_m\}, \{a^{rt}_m, a^{rm}_m\}, \{a^{rm}_m, a^{rm}_m\} \). In all three cases, we have that this edge gadget contributes seven edges to \( M'_1 \triangle M'_2 \). Finally, for each edge \( e = \{m, w\} \in M_1 \triangle M_2 \), we get an edge in \( M'_1 \triangle M'_2 \) (different from the edges that we have already identified to be part of \( M'_1 \triangle M'_2 \): If \( e \in M_1 \cap M_2 \), then also \( e \in M'_1 \setminus M'_2 \). If \( e \in M_2 \setminus M_1 \), then, depending on whether \( m \) or \( w \) are contained in \( A^*_\text{repl} \), edge \( e \) (if \( m, w \notin A^*_\text{repl} \)), edge \( \{m, c_w\} \) (if \( m \notin A^*_\text{repl} \) and \( w \in A^*_\text{repl} \)), edge \( \{c_m, w\} \) (if
m, w ∈ A∗ Repl) is contained in M′ 2 \ M′ 1. Note that in the last case, edge \{c_m, c_w\} exists as \{m, w\} ∉ M_1. Summing up, we get that \(|M_1 ∆ M_2| \leq |M_1′ ∆ M_2′| - |A∗ Repl| - 7k∗ ≤ k.

Now, Theorem 1 directly follows from Observation 1 and Lemmas 1 to 3.

Theorem 1 allows us to transfer algorithmic and hardness results for one type of change to another type. For example, the polynomial-time algorithm of Bredereck et al. [2020] for ISM for Swap implies that ISM can also be solved in polynomial time for Add, Delete, and Replace. Using similar constructions as in our proofs, it is also possible to prove that the different types of changes are equivalent for IHR (although, here, to model \(x\) changes of type \(X\) more than \(O(x)\) changes of type \(Y\) may be needed; e.g., in the above reduction from Replace to Add, modeling the replacement of a hospital \(h\) would need \(u(h)\) binding residents \(b_h\) and STABLE ROOMMATES (which is a generalization of SM where agents are not partitioned into men and women). However, Theorem 1 does not directly transfer to IASM; for instance, in the reduction from Lemma 6, \(M′ 2\) might “ignore” the added edge gadgets by allowing few of the edges to block \(M′ 2\).

4 Almost Stable Marriage

Sometimes, it may be acceptable that “few” agent pairs block an implemented matching (for instance, in very large markets where agents might not even be aware that they are part of a blocking pair). In Section 6, we experimentally show that allowing that \(M_2\) may be blocked by few agent pairs significantly decreases the number of necessary adjustments. We now show that, in contrast to ISM [Bredereck et al. 2020], IASM is computationally intractable:

**Theorem 2.** IASM is NP-hard and W[1]-hard when parameterized by \(k + b + |P_1 ⊕ P_2|\).

**Proof.** To show Theorem 2, we devise a polynomial-time many-one reduction from LOCAL SEARCH ASM. In LOCAL SEARCH ASM, we are given an SM instance \((U, W, P)\), a stable matching \(N\) in \(P\), and integers \(q, t,\) and \(z,\) and the question is whether there is a matching \(N^∗\) of size at least \(|N| + t\) admitting at most \(z\) blocking pairs such that \(|N ∆ N^∗| ≤ q\). Gupta et al. [2020, Theorem 3] proved that LOCAL SEARCH ASM is W[1]-hard with respect to the combined parameter \(q + t + z\). Notably, their hardness result even holds if the number of men and women is the same (we denote this number as \(n\)) and \(|N| + t = n\), i.e., \(N^∗\) needs to be a perfect matching and exactly \(t\) men and \(t\) women are unmatched in \(N\). We reduce from this regularized version in the following.

Given an instance \(I′ = (U′ = \{m_1′, \ldots, m_n′\}, W′ = \{w_1′, \ldots, w_n′\}, P, N, q, t, z)\) of LOCAL SEARCH ASM, we assume without loss of generality that \(m_1′, \ldots, m_n′\) and \(w_1′, \ldots, w_n′\) are the agents that are not matched by \(N\).

From \(I′\), we now construct an instance \(I = (U, W, P_1, P_2, b, k)\) of IASM as follows. We set \(b := z\) and \(k := q + t\). We start constructing the set of agents. First of all, for each \(i \in [n]\), we add a man \(m_i\) modeling man \(m_i′\) from the given instance \(I′\) and a woman \(w_i\) modeling woman \(w_i′\). We refer to these agents as original agents. Further, we add \(t\) catch men \(m_1^∗, \ldots, m_t^∗\) (one for each unmatched original woman). Additionally, we insert a penalizing component consisting of \(j + 2\) layers of \(b + 1\) men and women each. For each layer \(j ∈ [k + 1]\), we denote the agents of the penalizing component in layer \(j\) as \(\tilde{m}_1^j, \ldots, \tilde{m}_{b+1}^j\) and \(\tilde{w}_1^j, \ldots, \tilde{w}_{b+1}^j\).

The intuition behind the construction is that in \(M_1\) all original agents are matched as they are matched in \(N\) and each unmatched original woman is matched to her designated catch man. In \(P_2\), we modify the preferences of the original women unmatched by \(N\) such that they prefer the \(b + 1\) men from the first layer of the penalizing component to their catch man. We construct the penalizing component in a way such that as soon as one agent from the component is matched outside of the component the resulting change is larger than \(k\). This enforces that each original woman needs to be matched to...
an original man in $M_2$, as each original women only prefers original man to agents from the penalizing component in $P_2$.

The preferences of the agents in $P_1$ are as follows. For every $i \in [n]$, man $m_i$ has the preferences of $m_i$ where each woman $w_j$ is replaced by $w_j$. For every $i \in [t]$, woman $w_i$ has the preferences of $w_i$ where every man $m_j$ is replaced by $m_j$ and additionally $m_i \succ m_1 \succ \cdots \succ m_{b+1}$ is appended at the end of the preferences of $w_i$. For every $i \in [t+1, n]$, woman $w_i$ has the preferences of $w_i$ where every man $m_j$ is replaced by $m_j$ and additionally $\tilde{m}_1 \succ \cdots \succ \tilde{m}_{b+1}$ is appended at the end of the preferences of $w_i$. The preferences of all other agents are as follows:

\[
\begin{align*}
  m_i^* & : w_i, & i \in [t]; \\
  \tilde{m}_i & : w_1 \succ \cdots \succ w_n \succ \tilde{w}_1, & i \in [b+1]; \\
  \tilde{w}_i^{k+1} & : \tilde{m}_i^{k+1}, & i \in [b+1]; \\
  \tilde{w}_i^j & : \tilde{w}_i^{j-1} \succ \cdots \succ \tilde{w}_i^{j+1} \succ \tilde{w}_i^j, & i \in [b+1], j \in [2, k+1]; \\
  \tilde{w}_i^j & : \tilde{m}_i \succ \tilde{m}_i^{j+1} \succ \cdots \succ \tilde{m}_i^{b+1}, & i \in [b+1], j \in [k];
\end{align*}
\]

The preferences of all agents are the same in $P_2$ as in $P_1$ except for the women $w_1, \ldots, w_t$ who all swap down the catch man in their preferences to the last place, i.e., the preferences of $w_i$ in $P_2$ arise from the preferences of $w_i'$ by replacing every man $m_j$ by $m_j$ and appending $\tilde{m}_1 \succ \cdots \succ \tilde{m}_{b+1} \succ m_i^*$ at the end. Thus, it holds that $|P_1 \cup P_2| = t \cdot (b+1)$. Lastly, we set the matching $M_1$ to:

\[
M_1 = \{\{m_i, w_j\} \mid \{m_i', w_j'\} \in N\} \cup \{\{m_i^*, w_i\} \mid i \in [t]\} \cup \{\{\tilde{m}_i, \tilde{w}_i\} \mid i \in [b+1], j \in [k]\}.
\]

The matching $M_1$ is stable in $P_1$, as $N$ is stable in $I'$, each original woman is matched to her catch man which they prefer to all men from the penalizing component, all catch men are matched to their top choice, and the agents in the penalizing component are matched in a stable way.

We now prove that the construction described above is indeed a correct parameterized reduction from Local Search ASM parameterized by $q+t+z$ to IASM parameterized by $k+b+|P_1 \cup P_2|$. The reduction clearly runs in polynomial time and $k+b+|P_1 \cup P_2| = z+q+t+z \cdot (z+1)$, that is, the new parameter combination only depends on the old one. It remains to prove the correctness of the reduction:

$(\Rightarrow)$: Given a matching $N^*$ of size $|N|+t$ with at most $z$ blocking pairs in $I'$ and with $|N \Delta N^*| \leq q$, we set matching $M_2$ to be the following:

\[
M_2 := \{\{m_i, w_j\} \mid \{m_i', w_j'\} \in N^*\} \cup \{\{\tilde{m}_i, \tilde{w}_i\} \mid i \in [b+1], j \in [k]\}.
\]

Each original woman is matched to an original man and thereby to a man they prefer to their catch man and all men from the penalizing component. As the agents in the penalizing component are matched in a stable way, this implies that all blocking pairs need to involve two original agents and thus that the number of blocking pairs for $M_2$ and $N^*$ is the same and thus at most $b$. It remains to examine the symmetric difference between $M_1$ and $M_2$. That is $M_1 \Delta M_2 = N \Delta N^* \cup \{\{m_i^*, w_i\} \mid i \in [t]\}$. Thus, it holds that $|M_1 \Delta M_2| \leq q+t$.

$(\Leftarrow)$: Assume that we are given a solution $M_2$ to the constructed IASM instance. We claim that $M_2$ cannot contain a pair involving an original woman and a man from the penalizing component. To prove this, observe that for all $j \in [k]$ it cannot be the case that there is a woman $\tilde{w}_i^j$ in layer $j$ who is not matched to $\tilde{m}_i^j$, and each man $\tilde{m}_i^{j+1}$ is matched to $\tilde{w}_i^{j+1}$ for all $i \in [b+1]$. For the sake of contradiction, assume that this situation occurs. As $\tilde{w}_i^j$ is not matched to $\tilde{m}_i^j$, and also not matched to one of $\tilde{m}_1^{j+1}, \ldots, \tilde{m}_{b+1}^{j+1}$, she needs to be unmatched. Thus, all $b+1$ men $\tilde{m}_1^{j+1}, \ldots, \tilde{m}_{b+1}^{j+1}$ form a blocking pair with $\tilde{w}_i^j$, contradicting that $M_2$ admits only at most $b$ blocking pairs. It follows that if a man in some layer $j$ is matched differently in $M_1$ and $M_2$, then also a man in layer $j+1$ is matched
differently in $M_1$ and $M_2$. Let us assume now that there exists a pair in $M_2$ consisting of an original woman and a man $m_i^*$ for some $i \in [b+1]$. Using our above observation this, however, implies that at least one man from each of the $k$ layers of the penalizing component is matched differently in $M_1$ and $M_2$, contradicting that $|M_1 \triangle M_2| \leq k$. Hence, no original woman can be matched to a man from the penalizing component in $M_2$. From this it also follows that no original woman $w_i$ can be unmatched or matched to a catch man in $M_2$, as otherwise $w_i$ forms blocking pairs with all $b+1$ men from the first layer of the penalizing component. Hence, all original women need to be matched to an original man in $M_2$. This implies that the matching $N^*$ defined as the matching $M_2$ restricted to original agents is a perfect matching of these agents which admits at most $z = b$ blocking pairs with $|N \triangle N^*| \leq q$, as $|M_1 \triangle M_2| = |N \triangle N^*| + |\{\{m_i^*, w_i\} | i \in [t]\}|.$

On the positive side, we provide XP-algorithms for all three single parameters:

**Proposition 1.** IASM is in XP when parameterized by any of $k$ or $b$ or $|P_1 \oplus P_2|$.

**Proof.** We give a separate proof for each parameter.

**Parameter $k$.** For the allowed size $k$ of the symmetric difference between $M_1$ and $M_2$, we guess the set $F$ up to $k$ edges in $M_1 \triangle M_2$ and check whether $M_1 \triangle F$ is a stable matching.

**Parameter $b$.** For the number $b$ of blocking pairs $M_2$ is allowed to admit, we start by guessing the blocking pairs. For each guessed blocking pair $\{m, w\}$, we modify the preferences of $m$ in $P_2$ by deleting $w$ from his preferences and from $Ac(m)$. Similarly to the polynomial-time algorithm for ISM, we now compute a maximum-weight stable matching $M$ for $P_2$, where we set the weight of an edge to be 2 if it is contained in $M_1$, and 0 otherwise. If the weight weight$(M)$ of $M$ is at least $|M_1| + |M| - k$, then we have $|M \triangle M_1| = |M| + |M_1| - 2|M \cap M_1| = |M| + |M_1| - \text{weight}(M) \leq |M| + |M_1| - (|M_1| + |M| - k) = k$, so $M$ is a solution to the ISM instance. Otherwise, there is no matching $M_2$ with exactly the guessed set of blocking pairs and $|M_1 \cap M_2| \leq k$ (note that due to the the Rural Hospitals Theorem all stable matchings after the deletion of the blocking pairs have the same size).

**Parameter $|P_1 \oplus P_2|$.** Note that each swap in some agent’s preferences can only create a single blocking pair for the initial matching $M_1$. Thus, if $|P_1 \oplus P_2| \leq b$, then we can simply set $M_2 := M_1$. Otherwise, we use the XP-algorithm for the number of $b$ of blocking pairs.

We finally remark that while the XP-algorithm for the parameter $k$ also works for IASM-T, IASM-T is NP-hard even for $b = 0$ and $|P_1 \oplus P_2| = 1$ (as Bredereck et al. [2020] proved that ISM-T is NP-hard for $|P_1 \oplus P_2| = 1$).

## 5 Incremental Hospital Residents

We start our study of the incremental variant of Hospital Residents by observing that one can reduce IHR to the polynomial-time solvable Weighted Stable Marriage problem [Feder, 1992]; this yields:

**Proposition 2.** IHR is solvable in $O(n^{2.5} \cdot m^{1.5})$ time, where $n$ is the number of residents and $m$ is the number of hospitals.

**Proof.** Let $I = (A = R \cup H, P_1, P_2, M_1, k)$ be the given instance of IHR. We reduce the problem to an instance of Weighted Stable Marriage where given an SM instance $(U \cup W, P)$, a weight
function weight \( : U \times W \to \mathbb{Q} \) on the edges, and an integer \( z \), the question is whether there is a stable matching \( M \) in \( \mathcal{P} \) of weight at least \( z \), i.e., \( \sum_{e \in M} \text{weight}(e) \geq z \).

We construct an instance of **Weighted Stable Marriage** as follows. The set \( U \) of men consists of all residents and the set \( W \) of women consists of \( u(h) \) copies \( h^1, \ldots, h^{u(h)} \) of each hospital \( h \in H \) with upper quota \( u(h) \). The preferences of all \( r \in U \) in \( \mathcal{P} \) are the same as in \( \mathcal{P}_2 \), where we replace a hospital \( h \) by \( h^1 \succ \cdots \succ h^{u(h)} \). The preferences of a \( h^i \in W \) are the same as the preferences of \( h \) in \( \mathcal{P}_2 \).

For the weight function weight, for each edge \( \{r, h^i\} \in M \), we assign the edges \( \{r, h^j\} \) for \( i \in [u(h)] \) weight two, and all other edges weight zero. Finally, we set \( z := n_1 + n_2 - k \), where \( n_1 \) is the size of a stable matching in \( \mathcal{P}_1 \) and \( n_2 \) is the size of a stable matching in \( \mathcal{P}_2 \) (note that \( n_1 \) and \( n_2 \) are well-defined due to the Rural Hospitals Theorem, which says that the number of residents matched to a hospital is the same in all stable matchings in an HR instance).

As we may assume that \( u(h) \leq n \) for every hospital \( h \in H \), the constructed instance has \( \mathcal{O}(n) \) men and \( \mathcal{O}(nm) \) women. Consequently, there are \( \mathcal{O}(n^2m) \) many acceptable man-woman pairs. Using the algorithm for **Weighted Stable Marriage** of Feder [1992] (which solves the problem in \( \mathcal{O}(\sqrt{N}p) \) time if the weight of every edge is bounded by a constant, where \( N \) is the number of agents and \( p \) is the number of acceptable pairs), our algorithm runs in \( \mathcal{O}(\sqrt{nm} \cdot n^2m) = \mathcal{O}(n^{2.5} \cdot m^{1.5}) \) time.

It remains to show the correctness of our reduction. Assume that there is a stable matching \( M_2 \) in \( \mathcal{P}_2 \) with \( |M_1 \triangle M_2| \leq k \). For a hospital \( h \in H \) with \( M_2(h) = \{r_1, \ldots, r_k\} \) (i.e., exactly the agents \( r_1, \ldots, r_k \) are matched to \( h \) in \( M_2 \) with \( r_1 \succ_h r_2 \succ_h \cdots \succ_h r_k \)), we define \( M(h) := \{h^i, r_i\} : i \in \{1, \ldots, k\} \). We claim that \( M := \bigcup_{h \in H} M(h) \) is a stable matching for \( \mathcal{P} \) of weight at least \( z \). First note that \( \text{weight}(M) = 2|M_1 \cap M_2| = |M_1| + |M_2| - |M_1 \triangle M_2| \geq z \). It remains to show that \( M \) is stable.

Assume for a contradiction that there is a blocking pair \( \{r, h^i\} \). Because \( h^i \) prefers \( M(h^i) \) to \( r \), it follows that \( M_2(r) \neq h \). Thus, \( \{r, h^i\} \) blocks \( M_2 \), a contradiction to the stability of \( M_2 \).

To show the other direction, we consider a stable matching \( M \) for \( \mathcal{P}_2 \) with \( \text{weight}(M) \geq z \). We define \( M_2 := \{\{r, h\} : \{r, h^i\} \in M\} \). We have \( |M_1 \triangle M_2| = |M_1| + |M_2| - 2|M_1 \cap M_2| = |M_1| + |M_2| - \text{weight}(M) \leq |M_1| + |M_2| - (n_1 + n_2 - k) = k \), so it remains to show that \( M_2 \) is stable. Assume for a contradiction that there exists a blocking pair \( \{r, h\} \) for \( M_2 \). Then in \( M \) there exists some \( i \) so that \( h^i \) is unmatched or \( h^i \) prefers \( r \) to \( M(h^i) \). This implies that \( \{r, h^i\} \) blocks \( M \), a contradiction to the stability of \( M \).

In the rest of this section, we focus on IHR-T. As IHR-T generalizes ISM-T, the results of Bredereck et al. [2020] imply that IHR-T is NP-hard and W[1]-hard parameterized by \( k \) even for \( |\mathcal{P}_1 \cup \mathcal{P}_2| = 1 \). Thus, we focus on the parameters number \( n \) of residents and number \( m \) of hospitals.

For the number \( n \) of residents, we can bound the number of “relevant” hospitals by \( \mathcal{O}(n^2) \). Subsequently guessing for each resident the hospital it is matched to yields:

**Proposition 3.** IHR-T is solvable in \( \mathcal{O}(n^{2n} \cdot nm) \) time.

**Proof.** We will reduce the given instance of IHR-T to an instance whose size is upper-bounded in the number \( n \) of residents. Observe that in a stable matching \( M \) for a resident \( r \in R \) there can be at most \( n - 1 \) hospitals which \( r \) strictly prefers to \( M(r) \). This implies that \( r \) needs to be matched to one of its \( n \) most preferred hospitals (which due to ties can be more than \( n \) hospitals). However, as it is irrelevant for the stability of the matching to which hospital from a single tie a resident is matched, in the modified instance we keep all hospitals a resident is matched to in \( M_1 \) and all hospitals that appear in one of the preference lists of residents on one of the first \( n \) positions (breaking ties arbitrarily). Afterwards, we can apply a brute-force algorithm by guessing the partner of each resident to solve the problem.

**Proposition 3** means fixed-parameter tractability with respect to \( n \). In contrast to this, the number of hospitals is (presumably) not sufficient to gain fixed-parameter tractability, even if the two preference profiles differ only in one swap.
Theorem 3. Parameterized by the number $m$ of hospitals, IHR-T is W[1]-hard even if $|\mathcal{P}_1 \oplus \mathcal{P}_2| = 1$.

Proof. We reduce from the COM HR-T problem: Given an instance of HR-T, decide whether there is a stable matching which matches all residents. [Boehmer and Heeger 2021, Proposition 8] showed that COM HR-T is W[1]-hard when parameterized by the number $m$ of hospitals. Given an instance $\mathcal{I} = (R = \{r_1, \ldots, r_n\} \cup H = \{h_1, \ldots, h_m\}, \mathcal{P})$ of COM HR-T, let $N$ be an arbitrary stable matching in $\mathcal{I}$ (we assume that $N$ does not match all residents, as we otherwise know that $\mathcal{I}$ is a yes-instance).

To construct an instance of IHR-T, we first add $R$ to the set of residents and $H$ to the set of hospital. Subsequently, we add a penalizing component consisting of two hospitals $h_1^*$ and $h_2^*$, both with upper quota one, and two hospitals $\tilde{h}_1$ and $\tilde{h}_2$ both with upper quota $n + 1$. We additionally add a resident $r^*$ and two sets of $n + 1$ residents $\tilde{r}_1, \ldots, \tilde{r}_{n+1}$ and $\tilde{r}_1', \ldots, \tilde{r}_n' + 1$.

Turning to the agents’ preferences in $\mathcal{P}_1$, all agents from $R \cup H$ have their preferences from $\mathcal{P}$, except that, for each resident, $h^*$ is added at the end of her preferences. The preferences of the agents from the penalizing component are:

- $h_1^*: r_1 \succ \cdots \succ r_n \succ r^*$;
- $\tilde{h}_1: r^* \succ \tilde{r}_1' \succ \cdots \succ \tilde{r}_n' \succ \tilde{r}_i \succ \cdots \succ \tilde{r}_{n+1}$;
- $h_2^*: r^*$;
- $\tilde{h}_2: \tilde{r}_1 \succ \cdots \succ \tilde{r}_{n+1} \succ \tilde{r}_1' \succ \cdots \succ \tilde{r}_n' + 1$;
- $\tilde{r}_i: \tilde{h}_1 \succ \tilde{r}_i', \tilde{r}_i': \tilde{h}_2 \succ \tilde{h}_1$, $i \in [n + 1]$.

Profile $\mathcal{P}_2$ equals $\mathcal{P}_1$ except that we swap $h_2^*$ and $\tilde{h}_1$ in the preferences of $r^*$. Let $i^*$ be the smallest index of a resident unmatched in $N$. We set $k := 2(n + 1)$ and

$$M_1 := N \cup \{\{r_{i^*}, h_1^*\}, \{r^*, h_2^*\}\} \cup \{\{\tilde{r}_i, \tilde{h}_1\}, \{\tilde{r}_i', \tilde{h}_2\} | i \in [n + 1]\}.$$ 

It is easy to see that the matching $M_1$ is stable. There is clearly no blocking pair involving an original agent, all residents $\tilde{r}_i$ and $\tilde{r}_i'$ are matched to their top choice, and $r^*$ does not form a blocking pair with $h_1^*$ as $h_1^*$ is matched better.

Note that the construction of the instance of IHR-T can be done in polynomial time since the matching $N$ can be computed in linear time by the Gale-Shapley algorithm.

It remains to prove that the given instance $\mathcal{I}$ of COM HRT is a yes-instance if and only if the constructed instance $\mathcal{I}'$ of IHR-T is a yes-instance.

$(\Rightarrow)$: Assume that $N^*$ is a perfect stable matching in $\mathcal{I}$. Let $M_2$ be the following matching:

$$M_2 = N^* \cup \{\{r^*, h_1^*\}\} \cup \{\{\tilde{r}_i, \tilde{h}_1\}, \{\tilde{r}_i', \tilde{h}_2\} | i \in [n + 1]\}.$$ 

Matching $M_2$ is stable in $\mathcal{P}_2$, as $N^*$ is stable in $\mathcal{P}$ and all residents from the penalizing component are matched to their top choice. Furthermore, it holds that $|M_1 \triangle M_2| = |N^* \triangle N| + |\{\{r_{i^*}, h_1^*\}, \{r^*, h_2^*\}, \{r^*, h_1^*\}\}| \leq 2(n + 1)$.

$(\Leftarrow)$: Let $M_2$ be a solution to the constructed IHR-T instance $\mathcal{I}'$. We claim that it needs to hold that $\{r^*, h_1^*\} \in M_2$. For the sake of contradiction, assume that this is not the case. Then, as $r^*$ is the top choice of $\tilde{h}_1$, which is the second-most preferred hospital of $r^*$ (after $h_1^*$), it needs to hold that $\{r^*, h_1^*\} \in M_2$. However, as both $\tilde{h}_1$ and $h_2^*$ have upper quota $n + 1$, this implies that one resident among the residents $\tilde{r}_1, \ldots, \tilde{r}_{n+1}$ and $\tilde{r}_1', \ldots, \tilde{r}_n'$ needs to be unmatched in $M_2$. This needs to be resident $\tilde{r}_{n+1}$, as all other of these residents appear in one of the first $n + 1$ positions in the preferences of either $\tilde{h}_1$ or $h_2^*$. For $\tilde{r}_{n+1}$ to be unmatched and not to form a blocking pair with $\tilde{h}_1$ or $h_2^*$, it needs to hold that for all $i \in [n]$, $\{\tilde{r}_i, \tilde{r}_i' \tilde{h}_1\} \in M_2$ and thereby also that for all $i \in [n + 1]$, $\{\tilde{r}_i, \tilde{h}_2\} \in M_2$. This in turn implies that $|M_1 \triangle M_2| \geq |\{\{\tilde{r}_i, \tilde{h}_1\}, \{\tilde{r}_i, \tilde{h}_2\} | i \in [n + 1]\}| \cup \{\{\tilde{r}_i, \tilde{h}_1\}, \{\tilde{r}_i, \tilde{h}_2\} | i \in [n]\}| \cup \{\{r^*, h_2^*\}, \{r^*, h_1^*\}\} + 2(n + 1)$, a contradiction. Thus, it needs to hold that $\{r^*, h_1^*\} \in M_2$. This implies that all original residents need to be matched to original hospitals, as an unmatched original resident would form a blocking pair together with $h_1^*$. Consequently, matching $M_2$ restricted to original agents induces a perfect stable matching in the given COM HRT instance $\mathcal{I}$.

\[\square\]
We leave open whether the (above shown) $W[1]$-hardness of $IHR-T$ upholds for the parameter $m + k + |\mathcal{P}_1 \cup \mathcal{P}_2|$.

On the positive side, devising an Integer Linear Program whose number of variables is upper-bounded in a function of $m$ and some guessing as preprocessing, $IHR-T$ admits an XP-algorithm for the number $m$ of hospitals:

**Proposition 4.** $IHR-T$ is in XP when parameterized by the number $m$ of hospitals.

**Proof.** We construct an algorithm very similar to the XP algorithm for the number of hospitals of [Boehmer and Heeger 2021, Proposition 10] for the Hospital Residents Problem with Ties and Lower and Upper Quotas. In the following, we describe the main ideas of the algorithm for our problem.

We start by guessing the subset $H_{\text{open}} \subseteq H$ of hospitals to which we assign at least one resident in the matching to be found. Further, we guess for each hospital $h \in H_{\text{open}}$ a worst resident $r_h$ matched to it (i.e., $h$ weakly prefers all residents matched to it to $r_h$). For each resident $r$ and hospital $h \in H_{\text{open}}$, let $z_h^r$ be 1 if $h$ strictly prefers $r$ to $r_h$, 0 if $h$ is indifferent among $r$ and $r_h$, and $-1$ if $h$ strictly prefers $r_h$ to $r$. Now, for the purpose of solving our problem, each resident is fully characterized by the hospital $h$ it is matched to in $M_1$, its preferences over hospitals (there are $O(m \cdot m! \cdot 2^m)$ possibilities) and by $(z_h^r)_{h \in H_{\text{open}}}$ (there are $3^m$ possibilities). Thus, we can bound the number of different “types” of residents in $O(m^2 \cdot m! \cdot 2^m \cdot 3^m)$. Using this, one can formulate the problem as an Integer Linear Program (ILP) whose number of variables is upper-bounded in a function of $m$, and subsequently employ Lenstra’s algorithm [Kannan, 1987, Lenstra, 1983]. For this, let $\succcurlyeq_1, \ldots, \succcurlyeq_q$ be a list of all weak incomplete orders over $H$. For each $z \in \{−1, 0, 1\}^{H_{\text{open}}}$, $i \in [q]$, and $h \in H$, let $n_{i,h}^r$ be the number of residents $r \in R$ with preference order $\succcurlyeq_i$ and $(z_h^r)_{h \in H_{\text{open}}} = z$ who are matched to $h$ in $M_1$. For each $z \in \{−1, 0, 1\}^{H_{\text{open}}}$, $i \in [q]$, and $h, h' \in H$, we introduce a variable $x_{i,h,h'}^z$ which denotes the number of residents $r \in R$ with preference order $\succcurlyeq_i$ and $(z_h^r)_{h \in H_{\text{open}}} = z$ who are matched to $h$ in $M_1$ and to $h'$ in the matching to be found. To minimize the size of the symmetric difference between $M_1$ and the matching to be found, note that $|M_1 \triangle M| = |M_1| + |M| - 2 \cdot |M_1 \cap M|$; thus, we minimize

$$\sum_{h \in H, h' \in H_{\text{open}}, i \in [q], z \in \{1, 0, −1\}^{H_{\text{open}}}} x_{i,h,h'}^z - 2 \sum_{h \in H_{\text{open}}, i \in [q], z \in \{1, 0, −1\}^{H_{\text{open}}}} x_{i,h,h'}^z.$$

Lastly, we add linear constraints ensuring that the current guess is respected and that the resulting matching is feasible and stable as done by [Boehmer and Heeger 2021, Proposition 10].

In an SM-T instance, we say that two agents are of the same agent type if they have the same preference relation and all other agents are indifferent between them. One can interpret a hospital in an instance of $HR-T$ as $u(h)$ agents of the same type and thus an $HR-T$ instance as an instance of SM-T where agents from one side are of only $m$ different agent types. This interpretation raises the question what happens when we parameterize ISM-T by the total number of agent types on both sides (and not only by the number of agent types on one of the sides as done in Theorem 5 and Proposition 4). We show that, in fact, this is enough to establish fixed-parameter tractability:

**Proposition 5.** ISM-T is solvable in $O(2^{(t_U + 1) \cdot (t_W + 1)} \cdot n^{2.5})$ time, where $t_U$ respectively $t_W$ is the number of agent types of men respectively women in $\mathcal{P}_2$.

**Proof.** Let $T_U$ respectively $T_W$ be the set of agent types of men respectively women in $\mathcal{P}_2$ with $t_W := |T_W|$ and $t_U := |T_U|$, $n_m$ the number of men, $n_w$ the number of women, and, for $\alpha \in T_U \cup T_W$, let $\succcurlyeq_\alpha$ be the preference relation of agents of type $\alpha$ and $A_\alpha \subseteq A$ be the agents of type $\alpha$. Slightly abusing notation, for types $\alpha, \beta, \gamma \in T_U \cup T_W$, we write $\beta \succcurlyeq_\alpha \gamma$ if agents of type $\alpha$ (weakly) prefer agents of type $\beta$ to agents of type $\gamma$. To simplify the algorithm, we modify the given instance by adding (to $T_U$...
and \( U \) a dummy men type consisting of \( n_w \) men who are indifferent among all women in \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) and (to \( T_W \) and \( W \)) a dummy women type consisting of \( n_w \) women who are indifferent among all men in \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \). We insert the dummy men type at the end of the preferences of all women and the dummy women type at the end of the preferences of all men in \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \).

The algorithm iterates over all undirected bipartite graphs \( G \) on \( T_U \cup T_W \) (their number is \( 2^{(t_U + 1) \cdot (t_W + 1)} \)). We say that a matching \( M \) is compatible with \( G \) if agents of type \( \alpha \in T_U \) and \( \beta \in T_W \) are only matched to each other by \( M \) if there is an edge between \( \alpha \) and \( \beta \) in \( G \). We reject \( G \) if a matching \( M \) compatible with \( G \) can be unstable. To be precise, we reject \( G \) if there are two types \( \alpha \in T_U \) and \( \beta \in T_W \) such that there is an edge between \( \alpha \) and some \( \beta' \in T_W \) and an edge between \( \beta \) and some \( \alpha' \in T_U \) such that \( \beta \succ_\alpha \beta' \) and \( \alpha \succ_\beta \alpha' \). If \( G \) is not rejected, then we construct a graph \( G' \) on \( A \) from it by connecting agents of type \( \alpha \in T_U \) and type \( \beta \in T_W \) if and only if there is an edge between \( \alpha \) and \( \beta \) in \( G \). Moreover, we give all edges in \( G' \) that appear in \( M_1 \) weight 1, all edges that do not appear in \( M_1 \) and do not contain a non-dummy agent weight \(-1\), and all remaining edges (i.e., those involving at least one dummy agent) weight 0. We compute a perfect maximum weight matching \( M \) in \( G' \) in \( O(n^{2.5}) \) time \[\text{Duan and Su, 2012}\] and return yes if \( M \) has weight at least \(|M_1| - k\), and otherwise continue with the next graph \( G \).

Assume that the algorithm returns yes because we found a matching \( M \). Let \( M_2 \) be the matching \( M \) restricted to all agents which are not of one of the two dummy types. We first prove that \( M_2 \) is always stable. As dummy types appear only at the end of the preferences of each agent, it is sufficient to argue that \( M \) is stable. Assume for the sake of contradiction that the returned matching \( M \) is blocked by \( \{m, w\} \), where \( m \in A_\alpha \) and \( w \in A_\beta \) for some \( \alpha \in T_U \) and \( \beta \in T_W \). Let \( \beta' \in T_W \) be the type of woman \( M(m) \) and \( \alpha' \in T_U \) the type of man \( M(w) \). Then, \( G \) contains edges \( \{\alpha, \beta'\} \) and \( \{\alpha', \beta\} \) and as \( \{m, w\} \) blocks \( M \) it holds \( \beta \succ_\alpha \beta' \) and \( \alpha \succ_\beta \alpha' \). Thus, \( G \) was rejected, a contradiction. Further, as \( M \) is of weight at least \(|M_1| - k\), it holds that \(|M_1 \bigtriangleup M_2| = |M_1| - |M_1 \cap M_2| + |M_2 \setminus M_1| \leq |M_1| - (|M_1| - k) = k \), where we use that \( M \) has weight at least \(|M_1| - k\) for the inequality.

It remains to argue why the algorithm always finds a solution if one exists. Let \( M_2 \) be a stable matching in \( \mathcal{P}_2 \) with \(|M_1 \bigtriangleup M_2| \leq k \). Let \( M \) be the matching \( M_2 \) where we match all agents that are unmatched in \( M_2 \) to agents from the dummy types. Further, let \( G' \) be the graph on \( T_U \cup T_W \) where \( \alpha \in T_U \) and \( \beta \in T_W \) are connected if and only if an agent of type \( \alpha \) is matched to an agent of type \( \beta \) in \( M \). If the graph \( G \) would have been rejected because of types \( \alpha' \in T_U \) and \( \beta' \in T_W \), then agents of these types form a blocking pair for \( M \), a contradiction. Moreover, matching \( M \) is clearly a perfect matching of weight at least \(|M_1 \cap M_2| - |M_2 \setminus M_1| = |M_1| - |M_1 \bigtriangleup M_2| \geq |M_1| - k \) in the constructed graph \( G' \).

Notably, the above algorithm with minor modifications also works for the incremental variant of \text{Stable Roommates with ties}.

### 6 Experiments

In this section, we consider different practical aspects of incremental stable matching problems. To keep the setup of our experiments simple, we focus on ISM, our most basic model, and its variant IASM.

After having analyzed the theoretical relationship between different types of changes in Section 3 in this section, we compare the impact of the following three different types of changes (we always assume that all agents have complete and strict preferences):

**Reorder** A Reorder operation consists of permuting the preference list of an agent uniformly at random.

**Delete** A Delete operation consists of deleting an agent from the instance.
A swap consists of swapping two adjacent agents in the preference relation of an agent. As sampling preference profiles that are at a certain swap distance from a given one turns out to be practically infeasible with more than 40 agents [Boehmer et al., 2021a], we always perform the same number of swaps in the preferences of each agent: If we are to perform $i$ Swap operations, then for each agent separately we replace its preferences by uniformly at random sampled preferences that are at swap distance $i$ from its original preferences (using the procedure described by Boehmer et al. [2021a]).

In Section 6.1, we present formulations of ISM and IASM as Integer Linear Programs (ILPs) which we use to solve these problems. Our subsequent experiments are divided into three parts: In Section 6.2 we analyze the relationship between the difference between $\mathcal{P}_1$ and $\mathcal{P}_2$ and the size $|M_1 \triangle M_2|$ of the symmetric difference between $M_1$ and $M_2$ for different methods to compute a stable matching $M_2$ in $\mathcal{P}_2$. In Section 6.3 we consider the relationship between the number of agent pairs that block $M_1$ in $\mathcal{P}_2$ and the minimum symmetric difference between $M_1$ and a stable matching $M_2$ in $\mathcal{P}_2$. In Section 6.4 we study the trade-off between allowing $M_2$ to be blocked by some pairs and $|M_1 \triangle M_2|$.

### 6.1 (ILP) Formulation of Incremental (Almost) Stable Marriage

In our experiments, to compute the new matching $M_2$, we solve an (integer) linear programming formulation of Incremental (Almost) Stable Marriage using Gurobi Optimization, LLC [2021]. We now start by presenting an ILP formulation for IASM and afterwards point out how it can be adapted to an LP formulation for ISM. We write $m \succ_{\mathcal{P}_2}^w m'$ to denote that $w$ weakly prefers $m$ to $m'$ in profile $\mathcal{P}_2$.

For our ILP, we introduce two binary variables $x_{m,w}$ and $y_{m,w}$ for each pair $(m, w) \in U \times W$. Setting $x_{m,w}$ to one corresponds to matching $m$ to $w$ in $M_2$ and setting $y_{m,w}$ to one allows $m$ and $w$ to form a blocking pair for the matching $M_2$ in $\mathcal{P}_2$. Given an instance $\mathcal{I} = (A = U \cup W, \mathcal{P}_1, \mathcal{P}_2, M_1, k, b)$ of IASM, we solve the following ILP:

$$
\min \sum_{(m,w) \in U \times W} x_{m,w} - 2 \cdot \sum_{(m,w) \in M_1} x_{m,w} \quad \text{such that} \quad (1)
$$

$$
\sum_{w' \in W: \ w' \succ_{\mathcal{P}_2}^w w} x_{m,w'} + \sum_{m' \in U: \ m' \succ_{\mathcal{P}_2}^w m} x_{m',w} \geq 1 - y_{m,w}, \forall (m, w) \in U \times W \quad (2)
$$

$$
\sum_{w \in W} x_{m,w} \leq 1, \forall m \in U \quad \sum_{m \in U} x_{m,w} \leq 1, \forall w \in W \quad (3)
$$

$$
\sum_{(m,w) \in U \times W} y_{m,w} \leq b \quad (4)
$$

From a solution to the ILP, we construct a matching $M_2$ by including a pair $(m, w)$ if and only if $x_{m,w} = 1$. By Constraint (1), in $M_2$, each agent is matched to at most one partner. Moreover, by Constraint (2), a pair $(m, w)$ can only block $M_2$ if $y_{m,w} = 1$, as in case $y_{m,w} = 0$, Constraint (2) enforces that either $m$ or $w$ are matched to an agent they weakly prefer to $w$ or $m$, respectively. In case $y_{m,w} = 1$, Constraint (2) is trivially fulfilled for this man-woman pair. Constraint (3) imposes that at most $b$ man-woman pairs may block $M_2$. It is easy to see that each matching $M$ that admits at most $b$ blocking pairs corresponds to a feasible solution of the ILP. It remains to argue why $M_2$ minimizes the symmetric difference with $M_1$. This is ensured by Line (1) which, as $|M_1|$ is fixed, is equivalent to minimizing $|M_1 \triangle M_2| = |M_1| + |M_2| - 2|M_1 \cap M_2|$.

To adapt the ILP from above to ISM, we need to set $b$ to zero. Moreover, we replace Line (1) by $\max \sum_{(m,w) \in M_1} x_{m,w}$ (if we want to find the solution minimizing the symmetric difference with $M_1$) and by $\min \sum_{(m,w) \in M_1} x_{m,w}$ (if we want to find the solution maximizing the symmetric difference with $M_1$).
with \(M_1\), as the size of all stable matchings in \(P_2\) is the same. As proven by Vande Vate [1989, Theorem 1], the LP relaxation of the resulting ILP has integral extreme points, implying that we can solve the formulation as an LP.

### 6.2 Relationship between \(|P_1 \oplus P_2|\) and \(|M_1 \triangle M_2|\)

In this section, we analyze the relationship between the number of changes that are applied to \(P_1\) to obtain \(P_2\) and the size of the symmetric difference between the given matching \(M_1\) that is stable in \(P_1\) and a stable matching \(M_2\) in \(P_2\) for different methods to compute \(M_2\). We start in Section 6.2.1 by considering instances with 100 agents having uniformly at random sampled preferences. Afterwards, we vary our setting in three different ways: We analyze in Section 6.2.2 the influence of the structure of the preferences on our results, and in Section 6.2.3 the influence of the number of agents. Lastly, in Section 6.2.4 with Reorder (inverse) and Add we consider two further types of changes in addition to Reorder, Delete, and Swap.

#### 6.2.1 Basic Setup: 100 Agents with Random Preferences

We start in this subsection with our basic setup with 100 agents having random preferences, addressing the following fundamental question:

**Research Question.** What is the relationship between the number of changes that are applied to \(P_1\) and the difference between \(M_1\) and \(M_2\)? Can only few changes already require that the matching needs to be fundamentally restructured?

**Experimental Setup.** For each of the three considered types of changes, for \(r \in \{0, 0.01, 0.02, \ldots, 0.3\}\) we sampled 200 STABLE MARRIAGE instances with 50 men and 50 women with random preferences (collected in the preference profile \(P_1\)). For each of these instances, we set \(M_1\) to be the men-optimal matching. Afterwards, we applied a uniformly at random sampled \(r\)-fraction of all possible changes of the considered type to profile \(P_1\) to obtain profile \(P_2\). Subsequently, we computed a stable matching \(M_2\) in \(P_2\) with minimum/maximum normalized symmetric difference \(\frac{|M_1 \triangle M_2|}{|M_1| + |M_2|}\) to \(M_1\). We denote the solution with minimum symmetric difference as “Best” and the solution with maximum symmetric difference as “Worst”. Moreover, we computed the men-optimal matching in \(P_2\) using the Gale-Shapley algorithm and denote this as “Gale-Shapley”. The results of this experiment are depicted in Figure 2a.

**Evaluation.** We start by focusing on the optimal solution (“Best”; solid line in Figure 2a). What stands out from Figure 2a is that already very few or even one change in \(P_1\) requires a fundamental restructuring of the given matching \(M_1\). To be precise, for Reorder, one reordering (which corresponds to a 0.01 fraction of changes) results in an average normalized symmetric difference between \(M_1\) and \(M_2\) of 0.1. For Swap, a 0.01-fraction of all swaps, which corresponds to making twelve random swaps per preference order (the total number of swaps is \(n(n-1)/2\)), results in an average normalized symmetric difference of 0.28, whereas a single swap per preference order already results in an average normalized symmetric difference of 0.05. For Delete, the effect was strongest, as deleting a single agent leads to an average normalized symmetric difference of 0.38. Ashlagi et al. [2017], Cai and Thomas [2021], Knuth et al. [1990] and Pittel [1989] offer some theoretical intuition of this phenomenon for Delete:

---

*We used the implementation of the Gale-Shapley algorithm of Wilde et al. [2020].

*Note that the denominator is independent of the selected matchings \(M_1\) and \(M_2\), as all stable matching in \(P_1\) and \(P_2\) have the same size by the Rural Hospitals theorem.
Assuming that agents have random preferences (as in our experiments), with high probability in a men-optimal matching the average rank that a man has for the woman matched to him is $\log(n)$ [Knuth et al., 1990, Pittel, 1989], whereas in an instance with $n$ men and $n - 1$ women the average rank a man has for the woman matched to him in any stable matching is $\frac{n}{3}\log(n)$ [Ashlagi et al., 2017, Cai and Thomas, 2021]. Thus, if we delete a single woman from the instance (which happens with 50% probability when we delete a single agent), then already only to realize these average ranks, the given matching needs to be fundamentally restructured. Notably, if we delete two agents from the instance, which results only with a 25% probability in a higher number of men than women, then the minimum normalized symmetric difference between $M_1$ and $M_2$ is only $0.28$.

While ISM is solvable in polynomial time, in a matching market in practice, decision makers might simply rerun the initially employed matching algorithm (the popular Gale-Shapely algorithm in our case) to compute the new matching $M_2$. In Figure 2a, in the dotted line, we indicate the normalized symmetric difference between $M_1$, which is the men-optimal matching in $P_1$, and the men-optimal matching in $P_2$. Overall, for all three types of changes and independent of the applied fraction of changes, the normalized symmetric difference between the two men-optimal matchings is quite close to the minimum achievable normalized symmetric difference, being, on average, always only at most 0.05 higher (i.e., 5 edges larger) than for the optimal solution.

Since the Gale-Shapley solution has such a good quality, one might conjecture that all stable matchings in $P_2$ are roughly similarly different from $M_1$. To check this hypothesis, in Figure 2a in the dashed line, we display the average normalized symmetric difference of $M_1$ and the stable matching in $P_2$ that is furthest away from $M_1$. For Delete, the above hypothesis actually gets confirmed: after few changes, the worst, the men-optimal, and the best stable matching in $P_2$ have a similar distance to $M_1$, indicating that after randomly deleting some agents it does not really matter which stable matching in $P_2$ is chosen. In contrast to this, for the other two types of changes, there is a significant difference between the best and worst solution.

On a theoretical level, a possible explanation for this is a result of [Ashlagi et al. 2017], who proved that in SM instances with an unequal number of men and women and random preferences, stable matchings are “essentially unique.”
6.2.2 Influence of the Structure of Preferences

Now, we analyze whether our observations from Figure 2a are still applicable beyond the case where agents have random preferences.

Research Question. How does the structure of the preferences influence our study of the relationship between \(|P_1 \oplus P_2|\) and \(|M_1 \triangle M_2|\)?

Experimental Setup I. In our first experiment in this subsection, we consider the situation which is as different as possible from our setup with random preferences, that is, we assume that all agents from one side have the same preference relation. For this, we reran the experiments from Figure 2a but instead of drawing the preferences of agents uniformly at random from the set of all preferences, we only drew one preference relation over men, respectively, women and set the preferences of all women, respectively, men to this order. The results of this experiments can be found in Figure 2b.

Evaluation I. Comparing Figure 2a for random preferences and Figure 2b for identical preferences, there are three major differences.

First, for identical preferences few changes have an even stronger effect than for random preferences. That is, a \(0.01\) fraction of changes here makes it necessary to replace on average more than half of all edges in \(M_1\). To get a feeling for why this is the case (for Delete), observe that in a STABLE MARRIAGE instance where all agents from one side have identical preferences, there exists only one stable matching, namely, the one where the man appearing in the \(i\)th position of the women’s preference list is matched to the woman appearing in the \(i\)th position of the men’s preference list. If we now delete, without loss of generality, a man from the instance who appears on position \(j\) of the women’s preferences, then there is still only a single stable matching. In this matching, for \(i \in [1, j - 1]\), the man on the \(i\)th position of the women’s preference list is matched to the woman on the \(i\)th position of the men’s preference list, whereas for \(i \in [j, n - 1]\), the woman on the \(i\)th position is matched to the man appearing on the \(i + 1\)st position (and the last woman on the men’s preference list remains unmatched). Thus, both matchings share \(j - 1\) edges while there are \(n - (j - 1)\) edges unique to \(M_1\) and \(n - j\) edges unique to \(M_2\). Observing that \(j\) is, on average, \(n/2\), it follows that half of the matching, on average, needs to be replaced; this also fits Figure 2b. For Reorder and Swap the situation is less clean, yet a similar intuition applies.

Second, for identical preferences, the size of the symmetric difference to \(M_1\) of all matchings that are stable in \(P_2\) is quite similar (as the best and worst solution are nearly indistinguishable), which can be intuitively explained by the fact that, initially, there is only a single stable matching and even after some changes have been applied, a large part of the matching is still fixed.

Third, while for random preferences performing some number of Delete operations requires more adjustments than performing the same number of Reorder operations, for identical preferences, both produce nearly identical results.

Experimental Setup II. In addition to exploring the extremes, that is, random preferences and instances where all men or all women have the same preferences, we are also interested in what happens if the agent’s preferences have some “structure”. For this, we generated agent’s preferences using the Mallows model [Mallows, 1957], which is parameterized by a central preference order \(\succ^*\) and a so-called dispersion parameter \(\phi \in [0, 1]\). In the Mallows model, the probability of drawing a preference order \(\succ\) is proportional to \(\phi^{\text{swap}(\succ^*, \succ)}\), where \(\text{swap}(\succ^*, \succ)\) is the swap distance between \(\succ\) and \(\succ^*\).

As pointed out by [Boehmer et al., 2021b], one drawback of the Mallows model is that there exists no natural interpretation of \(\phi\) and that a uniform distribution of \(\phi\) does not lead to a uniform coverage of the spectrum of preferences. That is why, as proposed by [Boehmer et al., 2021b], we use a normalized
Normalized dispersion parameter $\phi \in [0, 1]$ which is internally converted into a value of $\phi$ such that the expected swap distance between the central order and a preference order sampled from the Mallows model with parameter $\phi$ is $\frac{\phi}{2}$ times the number of possible swaps. Notably, $\phi = 1$ results in all preference orders being sampled with the same probability, while $\phi = 0$ results in all agents from one side having the same preferences and $\phi = 0.5$ results in a model that is, in some sense, exactly between the two.

For our experiment, we proceed analogously to our basic setup (Section 6.2.1): For our three different types of changes, for $\phi \in \{0, 0.05, \ldots, 0.95, 1\}$, we sampled 200 Stable Marriage instances with 50 men and 50 women whose preferences $P_1$ are drawn from the Mallows model with the same central order and normalized dispersion parameter $\phi$. Subsequently, we compute $M_1$ as the men-optimal matching and apply a 0.1 fraction of changes sampled uniformly at random to $P_1$ to obtain $P_2$ (we also performed this experiment with a 0.05/0.15/0.2/0.25 fraction of changes and observed similar results). We visualize the results of this experiment in Figure 3.

**Evaluation II.** For Reorder and Swap, the more unstructured the preferences of agents are the lower is the minimum symmetric difference of $M_1$ and $M_2$ (which might be intuitively surprising). For the Gale-Shapley solution and even more the worst solution, the more unstructured the preferences of agents are, the larger becomes the gap between these two solutions and the best solution (this effect becomes particularly strong if the normalized dispersion parameter goes beyond 0.5). This can be explained by the fact that if the preferences of agents are similar to each other, then there might exist only few stable matchings being quite similar to each other (and in the extreme case with each agent having the same preferences only a unique stable matching), while for random preferences there is more flexibility when choosing a stable matching.

In contrast to the other two types of changes, for Delete the minimum achievable symmetric difference between $M_1$ and $M_2$ first sharply decreases until around $\phi = 0.3$ and afterwards steadily increases again. Moreover, as already observed before, there is only a small difference between the worst and best solution.
6.2.3 Influence of the Number of Agents

Now, we analyze whether our observations from Figure 2a are still applicable for varying number of agents.

Research Question. How do the number of agents influence our study of the relationship between $|P_1 \oplus P_2|$ and $|M_1 \Delta M_2|$?

Experimental Setup. For our three different types of changes, for $n \in \{10, 20, \ldots, 140, 150\}$, we sampled 200 instances of ISM with $n$ men and $n$ women having random preferences, where a 0.1 fraction of changes is performed (as described in Section 6.2.1). Again as in Section 6.2.1 we computed the stable matching in $P_2$ with maximum/minimum symmetric difference with $M_1$ and a stable matching in $P_2$ using the Gale-Shapely algorithm. Our results can be found in Figure 4. We also repeated the same experiment where a 0.05/0.15/0.2/0.25 fraction of changes is applied producing very similar results.

Evaluation. The general trends we observed in Figure 2a, e.g., concerning the relationships between the different types of changes or between the three different types of solutions examined, can still be found for different numbers of agents. However, while for Reorder the minimum normalized symmetric difference between $M_1$ and a stable matching $M_2$ in $P_2$ and the normalized symmetric difference between $M_1$ and the Gale-Shapely matching in $P_2$ stays more or less constant for increasing number of agents, for Delete and even more for Swap, the average normalized symmetric difference between $M_1$ and $M_2$ for all three ways of computing $M_2$ slowly increases. For our sampled ISM instances, we also measured the fraction of pairs that block $M_1$ in $P_2$. For each of the three types of changes, this value is, on average, the same for all considered numbers of agents (which is in contrast to our previous observations that the fraction of necessary adjustments increases for Delete and Swap when the number of agents increases).

6.2.4 Two Further Types of Changes

Finally, we briefly discuss two further types of change, i.e., Reorder (inverse) where we reverse the preferences of one agent, and Add where we add an agent with random preferences. We repeated the
Figure 5: For different types of changes and ways to compute $M_2$, average normalized symmetric difference between $M_1$ and $M_2$ for a varying fraction of change between $P_1$ and $P_2$ when agents have random preferences.

We performed the experiment from Figure 2a for these two types of changes and display the results in Figure 5. The reason why we do not explicitly consider these two types of changes in our other experiments is that Reorder (inverse) produces results similar to Reorder (typically and intuitively requiring few more adjustments) and Add produces results similar to Delete. This effect is also visible in Figure 5.

6.3 Correlation Between the Number of Blocking Pairs of $M_1$ in $P_2$ and the Symmetric Difference Between $M_1$ and $M_2$

Motivated by the hypothesis that in practice a once implemented matching may not be changed even if the instance slightly changes, we ask the following question:

Research Question I. By how many pairs is matching $M_1$ blocked in $P_2$ if more and more changes are performed?

Experimental Setup I. For each of the three considered types of changes, for $r \in \{0, 0.01, 0.02, \ldots, 0.3\}$, we sampled 1000 Stable Marriage instances with 50 men and 50 women with random preferences collected in $P_1$. For each of these instances, we computed as $M_1$ the men-optimal matching and applied an $r$-fraction of all possible changes of the considered type to $P_1$ to obtain preference profile $P_2$. Subsequently, we computed the set bp$(M_1, P_2)$ of pairs that block matching $M_1$ in instance $P_2$. In Figure 6, we depict the average and 90th quantile of the fraction of man-woman pairs that block $M_1$ in $P_2$ depending on the fraction of applied changes.

Evaluation I. Comparing the results from Figure 6 to the results from Figure 2a, several things stand out. First, examining the range $[0, 0.05]$ of changes, there are only “few” blocking pairs (around a 0.02 fraction of all pairs), which (as observed in Section 6.2.1) are nevertheless in most cases enough to make it necessary to fundamentally restructure the given matching $M_1$. Second, the ordering of Delete and Swap is reversed here compared to the needed adjustments. Third, while the fraction of pairs that block $M_1$ in $P_2$ constantly grows—nevertheless always staying on a surprisingly low level—the minimum symmetric difference between $M_1$ and a stable matching in $P_2$ grows significantly slower after a certain fraction of changes have been applied (see Figure 2a).
The Pearson correlation coefficient is a measure for the linear correlation between two quantities $y$ and $y$. Typically, a correlation between 0 and 0.3 is considered as a weak correlation, while a correlation between 0.3 and 0.5 is considered as a moderate correlation, while a correlation between 0.5 and 0.7 is considered as a strong correlation, and a correlation between 0.7 and 1 as a very strong correlation [Schober et al., 2018]. For Reorder, the correlation coefficient is 0.8, for Delete the correlation coefficient is 0.55, and for Swap the correlation coefficient is 0.81. Thus, for all three types of changes there is a noticeable correlation, which is particularly strong for Swap and Reorder and a bit weaker for Delete. To get a feeling for the correlation, in Figure 7 for Reorder, we represent each instance by a point whose $x$-coordinate is the fraction of pairs that block $M_1$ in $P_2$, whose $y$-coordinate is the minimum normalized symmetric difference between $M_1$ and a stable matching $M_2$ in $P_2$, and whose color reflects the applied fraction of changes.

Examining Figure 7, it seems that the points in one color (that is, instances where a similar number of changes changed) are correlated: for all three types there is a noticeable correlation, which is particularly strong for Reorder, bit weaker for Delete, and weakest for Swap.
of changes have been applied) exhibit a weaker correlation than the collection of all points. In fact, for Reorder, while for instances with a change between $[0, 0.05]$ the correlation coefficient is 0.62, it is only 0.44 for instances from the interval $(0.05, 0.1]$, 0.39 for $(0.1, 0.15]$, and 0.42 for $(0.15, 0.2]$. It is quite remarkable that for all four groups the correlation coefficient is (significantly) below the overall correlation coefficient and, so far, we have no explanation for this. For the two other types of changes a similar but less strong effect is present.

### 6.4 Almost Stable Marriage

As featured in Section 4, we now analyze the trade-off between the number of pairs that are allowed to block $M_2$ and the minimum symmetric difference between $M_1$ and $M_2$.

#### Experimental Setup I.

For our three different types of changes, for $r \in \{0, 0.01, 0.02, \ldots, 0.3\}$, and for $\beta \in \{0, 0.005, 0.05\}$, as in Section 6.2.1, we prepared 200 instances consisting of 50 men and 50 women with random preferences collected in $P_1$ and $P_2$. Then, for a given fraction of possible changes and an allowed blocking pairs for $M_2$, we computed the minimum symmetric difference between $M_1$ and a matching $M_2$ in $P_2$ for which at most a $\beta$-fraction of all 50 · 50 man-woman pairs is blocking. Figure 8 shows the results of this experiment.

#### Evaluation I.

We observe that independent of the type and fraction of change, allowing for few blocking pairs for $M_2$ enables a significantly larger overlap of $M_2$ with $M_1$. That is, allowing for a 0.005 fraction of pairs to be blocking increases the average normalized symmetric difference by around 0.2. We also examined the effect of doubling the fraction of blocking pairs and allowing for a 0.01 fraction, which gives an additional decrease by 0.1. If we allow for a 0.05 fraction of pairs to be blocking, then, for Swap and Reorder, until a 0.2 fraction of changes, $M_2$ can be chosen to be almost identical to $M_1$.

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6The y-axis in Figure 8 is labeled differently than in the previous figures: Here, we divide $|M_1 \Delta M_2|$ by the size of a stable matching in $P_1$ plus the size of a stable matching in $P_2$. As all stable matchings have the same size, this is the same as $|M_1 \Delta M_2| / (|M_1| + |M_2|)$. If $M_2$ is a stable matching, however, different almost stable matchings may have different sizes.
Experimental Setup II. While in the previous experiment we have focused on allowing $M_2$ to be blocked by a “fixed” number of pairs (independent of $|bp(M_1, P_2)|$), we now want to explore the spectrum between allowing that $M_2$ is not blocked by any pairs and that $M_2$ is blocked by all pairs that block $M_1$ in $P_2$, which allows to set $M_2 := M_1$. For this, we conducted the following experiment: For each of our three types of changes, we computed 200 instances with 50 men and 50 women having random preferences with a $0.1$ fraction of all changes applied uniformly at random between $P_1$ and $P_2$. Subsequently, for each $i \in \{0, 0.02, 0.04, \ldots, 0.98, 1\}$, we computed a matching $M_2$ with minimum symmetric difference with $M_1$ that is blocked by at most $i \cdot |bp(M_1, P_2)|$ pairs in $P_2$. Note that the results can be found in Figure 9. We also repeated this experiment with different fractions of changes between $P_1$ and $P_2$ producing very similar results.

Evaluation II. Examining Figure 9 confirms our observation from Figure 8 that the first $x$ blocking pairs that $M_2$ is allowed to admit have a higher impact than the second $x$ blocking pairs, as the symmetric difference between $M_1$ and $M_2$ decreases particularly quickly for smaller fractions (up to around 0.3). The reason why for Delete allowing for $|bp(M_1, P_2)|$ many blocking pairs does not lead to a symmetric difference of zero is because the deleted agents are not part of any pair in $M_2$ and thus their edges from $M_1$ are always part of the symmetric difference.

7 Conclusion

This paper extends the study of adapting stable matchings in two-sided matching markets to change in various directions: We systematically analyzed how different types of changes relate to each other in theory and in practice, initiated the study of incremental versions of two further two-sided stable matching problems and investigated experimentally practical aspects of adapting stable matchings.

For future work, on the experimental side, it would be interesting to perform experiments with methods that are used in practice to deal with settings discussed in the paper on real-world data. Moreover, extending our experiments to STABLE ROOMMATES is a natural next step. Finally, it might also be interesting to investigate how the complexity of the considered problems changes if the first matching is not given to us, i.e., we are given two preference profiles and the task is to find a stable matching for each profile which are close to each other. Notably, [Chen et al., 2018, Theorem 4.6] already proved that finding a matching that is stable in two different given preference profiles is NP-hard.

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