GROUPS ACTING ON MANIFOLDS: AROUND THE ZIMMER PROGRAM

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To Robert Zimmer on the occasion of his 60th birthday.

ABSTRACT. This paper is a survey on the Zimmer program. In its broadest form, this program seeks an understanding of actions of large groups on compact manifolds. The goals of this survey are (1) to put in context the original questions and conjectures of Zimmer and Gromov that motivated the program, (2) to indicate the current state of the art on as many of these conjectures and questions as possible and (3) to indicate a wide variety of open problems and directions of research.

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1. Prologue

Traditionally, the study of dynamical systems is concerned with actions of $\mathbb{R}$ or $\mathbb{Z}$ on manifolds. I.e. with flows and diffeomorphisms. It is natural to consider instead dynamical systems defined by actions of larger discrete or continuous groups. For non-discrete groups, the Hilbert-Smith conjecture is relevant, since the conjecture states that non-discrete, locally compact totally disconnected groups do not act continuously by homeomorphisms on manifolds. For smooth actions known results suffice to rule out actions of totally disconnected groups, the Hilbert-Smith conjecture has been proven for Hölder diffeomorphisms \cite{152,200}. For infinite discrete groups, the whole universe is open. One might consider the “generalized Zimmer program” to be the study of homomorphisms $\rho : \Gamma \to \text{Diff}(M)$ where $\Gamma$ is a finitely generated group and $M$ is a compact manifold. In this survey much emphasis will be on a program proposed by Zimmer. Here one considers a very special class of both Lie groups and discrete groups, namely semi-simple Lie groups of higher real rank and their lattices.

Zimmer’s program is motivated by several of Zimmer’s own theorems and observations, many of which we will discuss below. But in broadest strokes, the motivation is simpler. Given a group whose linear and unitary representations are very rigid or constrained, might it also be true that the group’s representations into $\text{Diff}^\infty(M)$ are also very rigid or constrained at least when $M$ is a compact manifold? For the higher rank lattices considered by Zimmer, Margulis’ superrigidity theorems classified finite dimensional representations and the groups enjoy property $(T)$ of Kazhdan, which makes unitary representations quite rigid.

This motivation also stems from an analogy between semi-simple Lie groups and diffeomorphism groups. When $M$ is a compact manifold, not only is $\text{Diff}^\infty(M)$ an infinite dimensional Lie group, but its connected component is simple. Simplicity of the connected component of $\text{Diff}^\infty(M)$ was proven by Thurston using results of Epstein and Herman.
Herman had used Epstein’s work to see that the connected component of Diff∞(T^n) is simple and Thurston’s proof of the general case uses this. See also further work on the topic by Banyaga and Mather [7, 164, 163, 165, 166], as well as Banyaga’s book [8].

A major motivation for Zimmer’s program came from his cocycle superrigidity theorem, discussed in detail below in Section 5. One can think of a homomorphism \( \rho : \Gamma \to \text{Diff}(M) \) as defining a virtual homomorphism from \( \Gamma \) to \( GL(\dim(M), \mathbb{R}) \). The notion of virtual homomorphisms, now more commonly referred to in our context as cocycles over group actions, was introduced by Mackey [151].

In all of Zimmer’s early papers, it was always assumed that we had a homomorphism \( \rho : \Gamma \to \text{Diff}_\infty(M, \omega) \) where \( \omega \) is a volume form on \( M \). A major motivation for this is that the cocycle superrigidity theorem referred to in the last paragraph only applies to cocycles over measure preserving actions. That this hypothesis is necessary was confirmed by examples of Stuck, who showed that no rigidity could be hoped for unless one had an invariant volume form or assumed that \( M \) was very low dimensional [220]. Recent work of Nevo and Zimmer does explore the non-volume preserving case and will be discussed below, along with Stuck’s examples and some others due to Weinberger in Section 10.

Let \( G \) be a semisimple real Lie group, all of whose simple factors are of real rank at least two. Let \( \Gamma \) in \( G \) be a lattice. The simplest question asked by Zimmer was: can one classify smooth volume preserving actions of \( \Gamma \) on compact manifolds? At the time of this writing, the answer to this question is still unclear. There certainly are a wider collection of actions than Zimmer may have initially suspected and it is also clear that the moduli space of such actions is not discrete, see [12, 77, 125] or section 9 below. But there are still few enough examples known that a classification remains plausible. And if one assumes some additional conditions on the actions, then plausible conjectures and striking results abound.

We now discuss four paradigmatic conjectures. To make this introduction accessible, these conjectures are all special cases of more general conjectures stated later in the text. In particular, we state all the conjectures for lattices in \( SL(n, \mathbb{R}) \) when \( n > 2 \). The reader not familiar with higher rank lattices can consider the examples of finite index subgroups of \( SL(n, \mathbb{Z}) \), with \( n > 2 \), rather than for general lattices in \( SL(n, \mathbb{R}) \). We warn the reader that the algebraic structure of \( SL(n, \mathbb{Z}) \) and its finite index subgroups makes many results easier for these groups than for other lattices in \( SL(n, \mathbb{R}) \). The conjectures we state concern, respectively, (1) classification of low dimensional actions,
(2) classification of geometric actions (3) classification of uniformly hyperbolic actions and (4) the topology of manifolds admitting volume preserving actions.

A motivating conjecture for much recent research is the following:

**Conjecture 1.1** (Zimmer’s conjecture). For any \( n > 2 \), any homomorphism \( \rho : SL(n, \mathbb{Z}) \rightarrow \text{Diff}(M) \) has finite image if \( \dim(M) < n - 1 \). The same for any lattice \( \Gamma \) in \( SL(n, \mathbb{R}) \).

The dimension bound in the conjecture is clearly sharp, as \( SL(n, \mathbb{R}) \) and all of its subgroups act on \( \mathbb{P}^{n-1} \). The conjecture is a special case of a conjecture of Zimmer which concerns actions of higher rank lattices on low dimensional manifolds which we state as Conjecture 4.12 below. Recently much attention has focused on these conjectures concerning low dimensional actions.

In our second, geometric, setting (a special case of) a major motivating conjecture is the following:

**Conjecture 1.2** (Affine actions). Let \( \Gamma < SL(n, \mathbb{R}) \) be a lattice. Then there is a classification of actions of \( \Gamma \) on compact manifolds which preserve both a volume form and an affine connection. All such actions are algebraically defined in a sense to be made precise below. (See Definition 2.4 below.)

The easiest example of an algebraically defined action is the action of \( SL(n, \mathbb{Z}) \) (or any of its subgroups) on \( \mathbb{T}^n \). We formulate our definition of algebraically defined action to include all trivial actions, all isometric actions, and all (skew) products of other actions with these.

The conjecture is a special case of a conjecture stated below as Conjecture 6.15. In addition to considering more general acting groups, we will consider more general invariant geometric structures, not just affine connections. A remark worth making is that, for the geometric structures we consider, the automorphism group is always a finite dimensional Lie group. The question of when the full automorphism group of a geometric structure is large is well studied from other points of view, see particularly [132]. However, this question is generally most approachable when the large subgroup is connected and much less is known about discrete subgroups. In particular, geometric approaches to this problem tend to use information about the connected component of the automorphism group and give much less information about the group of components particularly if the connected component is trivial.

One is also interested in the possibility of classifying actions under strong dynamical hypotheses. The following conjecture is motivated
by work of Feres-Labourie and Goetze-Spatzier \cite{60, 104, 105} and is similar to conjectures stated in \cite{107, 117}:

**Conjecture 1.3.** Let $\Gamma < SL(n, \mathbb{R})$ be a lattice. Then there is a classification of actions of $\Gamma$ on a compact manifold $M$ which preserve both a volume form and where one element $\gamma \in \Gamma$ acts as an Anosov diffeomorphism. All such actions are algebraically defined in a sense to be made precise below. (Again see Definition 2.4.)

In the setting of this particular conjecture, a proof of an older conjecture of Franks concerning Anosov diffeomorphisms would imply that any manifold $M$ as in the conjecture was homeomorphic to an infranil-manifold on which $\gamma$ is conjugate by a homeomorphism to a standard affine Anosov map. Even assuming Franks’ conjecture, Conjecture 1.3 is open. One can make a more general conjecture by only assuming that $\gamma$ has some uniformly partially hyperbolic behavior. Various versions of this are discussed in §7, see particularly Conjecture 7.9.

We end this introduction by stating a topological conjecture about all manifold admitting smooth volume preserving actions of a simple higher rank algebraic group. Very special cases of this conjecture are known and the conjecture is plausible in light of existing examples. More precise variants and a version for lattice actions will be stated below in §8.

**Conjecture 1.4.** Let $G$ be a simple Lie group of real rank at least two. Assume $G$ has a faithful action preserving volume on a compact manifold $M$. Then $\pi_1(M)$ has a finite index subgroup $\Lambda$ such that $\Lambda$ surjects onto an arithmetic lattice in a Lie group $H$ where $H$ locally contains $G$.

The conjecture says, more or less, that admitting a $G$ action forces the fundamental group of $M$ to be large. Passage to a finite index subgroup and a quotient is necessary, see subsection 9 and §83 for more discussion. The conjecture might be considered the analogue for group actions of Margulis’ arithmeticity theorem.

This survey is organized on the following lines. We begin in §2 by describing in some detail the kinds of groups we consider and examples of their actions on compact manifolds. In §3 we digress with a prehistory of motivating results from rigidity theory. Then in §4 we discuss conjectures and theorems concerning actions on “low dimensional” manifolds. In this section, there are a number of related results and conjectures concerning groups not covered by Zimmer’s original conjectures. Here low dimensional is in two senses (1) compared to the group as in Conjecture 1.1 and (2) absolutely small, as in dimension.
being between 1 and 4. This discussion is simplified by the fact that many theorems conclude with the non-existence of actions or at least with all actions factoring through finite quotients. In section 5 we further describe Zimmer’s motivations for his conjectures by discussing the cocycle superrigidity theorem and some of its consequences for smooth group actions. In §6 and §7 we discuss, respectively, geometric and dynamical conditions under which a simple classification might be possible. In Section 8 we discuss another approach to classifying $G$ and $\Gamma$ actions using topology and representations of fundamental groups in order to produce algebraically defined quotients of actions. Then in §9 we describe the known “exotic examples” of the acting groups we consider. This constructions reveals some necessary complexity of a high dimensional classification. We then describe known results and examples of actions not preserving volume in §10 and some rather surprising group actions on manifolds in §11. Finally we end the survey with a collection of questions about the algebraic and geometric structure of finitely generated subgroups of $\text{Diff}(M)$.

Some remarks on biases and omissions. Like any survey of this kind, this work is informed by its authors biases and experiences. There is an additional bias that this paper emphasizes developments that are close to Zimmer’s own work and conjectures. In particular, the study of rigidity of group actions often focuses on the low dimensional setting where all group actions are conjectured to be finite or trivial. While Zimmer did substantial work in this setting, he also proved many results and made many conjectures in more general settings where any potential classification of group actions is necessarily, due to the existence of examples, more complicated.

Another omission is that almost nothing will be said here about local rigidity of groups actions, since the author has recently written another survey on that topic [76]. While that survey could already use updating, that update will appear elsewhere.

For other surveys of Zimmer’s program and rigidity of large group actions the reader is referred to [67, 141, 255]. The forthcoming book by Witte Morris and Zimmer, [255], is a particularly useful introduction to ideas and techniques in Zimmer’s own work. Also of interest are (1) a brief survey of rigidity theory by Spatzier with a more geometric focus [217], (2) an older survey also by Spatzier, with a somewhat broader scope, [216], (3) a recent problem list by Margulis on rigidity theory with a focus on measure rigidity [161] and (4) a more recent survey by Lindenstrauss focused on recent developments in measure rigidity [144]. Both of the last two mentioned surveys are particularly oriented
towards connections between rigidity and number theory which are not
tioned at all in this survey.

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2. A BRIEF DIGRESSION: SOME EXAMPLES OF GROUPS AND
   ACTIONS

In this section we briefly describe some of the groups that will play
important roles in the results discussed here. The reader already famil-
lar with semi-simple Lie groups and their lattices may want to skip to
the second subsection where we give descriptions of group actions. The
following convention is in force throughout this paper. For definitions
of relevant terms the reader is referred to the following sub sections.

Convention: In this article we will have occasion to refer to three
overlapping classes of lattices in Lie groups which are slightly different.
Let $G$ be a semisimple Lie group and $\Gamma < G$ a lattice. We call $\Gamma$
a higher rank lattice if all simple factors of $G$ have real rank at least 2.
We call $\Gamma$ a lattice with $(T)$ if all simple factors of $G$ have property $(T)$.
Lastly we call $\Gamma$ an irreducible higher rank lattice if $G$ has real rank at
least 2 and $\Gamma$ is irreducible.

2.1. Semi-simple groups and their lattices. By a simple Lie group,
we mean a connected Lie group all of whose normal subgroups are
discrete, though we make the additional convention that $\mathbb{R}$ and $S^1$
are not simple. By a semi-simple Lie group we mean the quotient of a
product of simple Lie groups by some subgroup of the product of their
centers. Note that with our conventions, the center of a simple Lie
group is discrete and is in fact the maximal normal subgroup. There is
an elaborate structure theory of semi-simple Lie groups and the groups
are completely classified, see [112] or [131] for details. Here we merely
derive some examples, all of which are matrix groups. All connected
semisimple Lie groups are discrete central extensions of matrix groups,
so the reader will lose very little by always thinking of matrix groups.

(1) The groups $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$ and $SL(n, \mathbb{H})$ of $n$ by $n$ matrices
of determinant one over the real numbers, the complex numbers
or the quaternions.
(2) The group $SP(2n, \mathbb{R})$ of $2n \times 2n$ matrices of determinant one which preserve a real symplectic form on $\mathbb{R}^{2n}$.

(3) The groups $SO(p, q), SU(p, q)$ and $SP(p, q)$ of matrices which preserve inner products of signature $(p, q)$ where the inner product is real linear on $\mathbb{R}^{p+q}$, hermitian on $\mathbb{C}^{p+q}$ or quaternionic hermitian on $\mathbb{H}^{p+q}$ respectively.

Let $G$ be a semi-simple Lie group which is a subgroup of $GL(n, \mathbb{R})$. We say that $G$ has real rank $k$ if $G$ has a $k$ dimensional abelian subgroup which is conjugate to a subgroup of the real diagonal matrices and no $k + 1$ dimensional abelian subgroups with the same property. The groups in (1) have rank $n - 1$, the groups in (2) have rank $n$ and the groups in (3) have rank $\min(p, q)$.

Since this article focuses primarily on finitely generated groups, we are more interested in discrete subgroups of Lie groups than in the Lie groups themselves. A discrete subgroup $\Gamma$ in a Lie group $G$ is called a lattice if $G/\Gamma$ has finite Haar measure. The lattice is called cocompact or uniform if $G/\Gamma$ is compact and non-uniform or not cocompact otherwise. If $G = G_1 \times \cdots \times G_n$ is a product then we say a lattice $\Gamma < G$ is irreducible if its projection to each $G_i$ is dense. It is more typical in the literature to insist that projections to all factors are dense, but this definition is more practical for our purposes. More generally we make the same definition for an almost direct product, by which we mean a direct product $G$ modulo some subgroup of the center $Z(G)$. Lattices in semi-simple Lie groups can always be constructed by arithmetic methods, see [18] and also [235] for more discussion. In fact, one of the most important results in the theory of semi-simple Lie groups is that if $G$ is a semi-simple Lie group without compact factors, then all irreducible lattices in $G$ are arithmetic unless $G$ is locally isomorphic to $SO(1, n)$ or $SU(1, n)$. For $G$ of real rank at least two, this is Margulis’ arithmeticity theorem, which he deduced from his super-rigidity theorems [156, 195, 159]. For non-uniform lattices, Margulis had an earlier proof which does not use the superrigidity theorems, see [155, 157]. This earlier proof depends on the study of dynamics of unipotent elements on the space $G/\Gamma$, and particularly on what is now known as the “non-divergence of unipotent flows”. Special cases of the super-rigidity theorems were then proven for $Sp(1, n)$ and $F_{4}^{\pm}$ by Corlette and Gromov-Schoen, which sufficed to imply the statement on arithmeticity given above [39, 110]. As we will be almost exclusively concerned with arithmetic lattices, we do not give examples of non-arithmetic lattices here, but refer the reader to [159] and [235] for
A formal definition of arithmeticity, at least when $G$ is algebraic is:

**Definition 2.1.** Let $G$ be a semisimple algebraic Lie group and $\Gamma \leq G$ a lattice. Then $\Gamma$ is arithmetic if there exists a semi-simple algebraic Lie group $H$ such that

1. there is a homomorphism $\pi : H^0 \to G$ with compact kernel,
2. there is a rational structure on $H$ such that the projection of the integer points of $H$ to $G$ are commensurable to $\Gamma$, i.e. $\pi(H(\mathbb{Z})) \cap \Gamma$ is of finite index in both $H(\mathbb{Z})$ and $\Gamma$.

We now give some examples of arithmetic lattices. The simplest is to take the integer points in a simple (or semi-simple) group $G$ which is a matrix group, e.g. $SL(n, \mathbb{Z})$ or $Sp(n, \mathbb{Z})$. This exact construction always yields lattices, but also always yields non-uniform lattices. In fact the lattices one can construct in this way have very special properties because they will contain many unipotent matrices. If a lattice is cocompact, it will necessarily contain no unipotent matrices. The standard trick for understanding the structure of lattices in $G$ which become integral points after passing to a compact extension is called change of base. For much more discussion see [159, 235, 242]. We give one example to illustrate the process. Let $G = SO(m,n)$ which we view as the set of matrices in $SL(n+m, \mathbb{R})$ which preserve the inner product

$$\langle v, w \rangle = \left( -\sqrt{2} \sum_{i=1}^{m} v_i w_i \right) + \left( \sum_{i=m+1}^{n+m} v_i w_i \right)$$

where $v_i$ and $w_i$ are the $i$th components of $v$ and $w$. This form, and therefore $G$, are defined over the field $\mathbb{Q}(\sqrt{2})$ which has a Galois conjugation $\sigma$ defined by $\sigma(\sqrt{2}) = -\sqrt{2}$. If we looks at the points $\Gamma = G(\mathbb{Z}[\sqrt{2}])$, we can define an embedding of $\Gamma$ in $SO(m,n) \times SO(m+n)$ by taking $\gamma$ to $(\gamma, \sigma(\gamma))$. It is straightforward to check that this embedding is discrete. In fact, this embeds $\Gamma$ in $H = SO(m,n) \times SO(m+n)$ as integral points for the rational structure on $H$ where the rational points are exactly the points $(M, \sigma(M))$ where $M \in G(\mathbb{Q}(\sqrt{2}))$. This makes $\Gamma$ a lattice in $H$ and it is easy to see that $\Gamma$ projects to a lattice in $G$, since $G$ is cocompact in $H$. What is somewhat harder to verify is that $\Gamma$ is cocompact in $H$, for which we refer the reader to the list of references above.

Similar constructions are possible with $SU(m,n)$ or $SP(m,n)$ in place of $SO(m,n)$ and also with more simple factors and fields with
more Galois automorphisms. There are also a number of other constructions of arithmetic lattices using division algebras. See [187, 235] for a comprehensive treatment.

We end this section by defining a key property of many semisimple groups and their lattices. This is property \((T)\) of Kazhdan, and was introduced by Kazhdan in [128] in order to prove that non-uniform lattices in higher rank semi-simple Lie groups are finitely generated and have finite abelianization. It has played a fundamental role in many subsequent developments. We do not give Kazhdan’s original definition, but one which was shown to be equivalent by work of Delorme and Guichardet [43, 111].

**Definition 2.2.** A locally compact group \(\Gamma\) has property \((T)\) of Kazhdan if \(H^1(\Gamma, \pi) = 0\) for every continuous unitary representation \(\pi\) of \(\Gamma\) on a Hilbert space. This is equivalent to saying that any continuous isometric action of \(\Gamma\) on a Hilbert space has a fixed point.

**Remarks 2.3.**

1. Kazhdan’s definition is that the trivial representation is isolated in the Fell topology on the unitary dual of \(\Gamma\).
2. If a continuous group \(G\) has property \((T)\) so does any lattice in \(G\). This result was proved in [128].
3. Any semi-simple Lie group has property \((T)\) if and only if it has no simple factors locally isomorphic to \(SO(1,n)\) or \(SU(1,n)\). For a discussion of this fact and attributions, see [42]. For groups with all simple factors of real rank at least three, this is proven in [128].
4. No noncompact amenable group, and in particular no noncompact abelian group, has property \((T)\). An easy averaging argument shows that all compact groups have property \((T)\).

Groups with property \((T)\) play an important role in many areas of mathematics and computer science.

2.2. Some actions of groups and lattices. Here we define and give examples of a general class of actions. A major impetus in Zimmer’s work is determining optimal conditions for actions to lie in this class. The class we describe is slightly more general than the class Zimmer termed “standard actions” in e.g. [246]. Let \(H\) be a Lie group and \(L < H\) a closed subgroup. Then a diffeomorphism \(f\) of \(H/L\) is called affine if there is a diffeomorphism \(\tilde{f}\) of \(H\) such that \(f([h]) = \tilde{f}(h)\) where \(\tilde{f} = A \circ \tau_h\) with \(A\) an automorphism of \(H\) with \(A(L) = L\) and \(\tau_h\) is left translation by some \(h\) in \(H\). Two obvious classes of affine diffeomorphisms are left translations on any homogeneous space and linear
automorphisms of tori, or more generally automorphisms of nilmanifolds. A group action is called affine if every element of the group acts by an affine diffeomorphism. It is easy to check that the full group of affine diffeomorphisms $\text{Aff}(H/L)$ is a finite dimensional Lie group and an affine action of a group $D$ is a homomorphism $\pi : D \to \text{Aff}(H/L)$. The structure of $\text{Aff}(H/L)$ is surprisingly complicated in general, it is a quotient of a subgroup of the group $\text{Aut}(H) \times H$ where $\text{Aut}(H)$ is a the group of automorphisms of $H$. For a more detailed discussion of this relationship, see [80, Section 6]. While it is not always the case that any affine action of a group $D$ on $H/L$ can be described by a homomorphism $\pi : D \to \text{Aut}(H) \times H$, this is true for two important special cases:

(1) $D$ is a connected semi-simple Lie group and $L$ is a cocompact lattice in $H$,

(2) $D$ is a lattice in a semi-simple Lie group $G$ where $G$ has no compact factors and no simple factors locally isomorphic to $SO(1,n)$ or $SU(1,n)$, and $L$ is a cocompact lattice in $H$.

These facts are [80, Theorem 6.4 and 6.5] where affine actions as in (1) and (2) above are classified.

The most obvious examples of affine actions of large groups are of the following forms, which are frequently referred to as standard actions:

(1) Actions of groups by automorphisms of nilmanifolds. I.e. let $N$ be a simply connected nilpotent group, $\Lambda < N$ a lattice (which is necessarily cocompact) and assume a finitely generated group $\Gamma$ acts by automorphisms of $N$ preserving $\Lambda$. The most obvious examples of this are when $N = \mathbb{R}^n$, $\Lambda = \mathbb{Z}^n$ and $\Gamma < SL(n,\mathbb{Z})$, in which case we have a linear action of $\Gamma$ on $\mathbb{T}^n$.

(2) Actions by left translations. I.e. let $H$ be a Lie group and $\Lambda < H$ a cocompact lattice and $\Gamma < H$ some subgroup. Then $\Gamma$ acts on $H/\Lambda$ by left translations. Note that in this case $\Gamma$ need not be discrete.

(3) Actions by isometries. Here $K$ is a compact group which acts by isometries on some compact manifold $M$ and $\Gamma < K$ is a subgroup. Note that here $\Gamma$ is either discrete or a discrete extension of a compact group.

We now briefly define a few more general classes of actions, which we need to formulate most of the conjectures in this paper. We first fix some notations. Let $A$ and $D$ be topological groups, and $B < A$ a closed subgroup. Let $\rho : D \times A/B \to A/B$ be a continuous affine action.

**Definition 2.4.** (1) Let $A, B, D$ and $\rho$ be as above. Let $C$ be a compact group of affine diffeomorphisms of $A/B$ that commute
with the $D$ action. We call the action of $D$ on $C\setminus A/B$ a generalized affine action.

(2) Let $A$, $B$, $D$ and $\rho$ be as in 1 above. Let $M$ be a compact Riemannian manifold and $i : D \times A/B \to \text{Isom}(M)$ a $C^1$ cocycle. We call the resulting skew product $D$ action on $A/B \times M$ a quasi-affine action. If $C$ and $D$ are as in 2, and we have a smooth cocycle $\alpha : D \times C \setminus A/B \to \text{Isom}(M)$, then we call the resulting skew product action of $D$ on $C \setminus A/B \times M$ a generalized quasi-affine action.

Many of the conjectures stated in this paper will end with the conclusion that all actions satisfying certain hypotheses are generalized quasi-affine actions. It is not entirely clear that generalized quasi-affine actions of higher rank groups and lattices are much more general than generalized affine actions. The following discussion is somewhat technical and might be skipped on first reading.

One can always take a product of a generalized affine action with a trivial action on any manifold to obtain a generalized quasi-affine action. One can also do variants on the following. Let $H$ be a semisimple Lie group and $\Lambda < H$ a cocompact lattice. Let $\pi : \Lambda \to K$ be any homomorphism of $\Lambda$ into a compact Lie group. Let $\rho$ be a generalized affine action of $G$ on $C \setminus H/\Lambda$ and let $M$ be a compact manifold on which $K$ acts. Then there is a generalized quasi-affine action of $G$ on $(C \setminus H \times M)/\Lambda$.

**Question 2.5.** Is every generalized quasi-affine action of a higher rank simple Lie group of the type just described?

The question amounts to asking for an understanding of compact group valued cocycles over quasi-affine actions of higher rank simple Lie groups. We leave it to the interested reader to formulate the analogous question for lattice actions. For some work in this direction, see the paper of Witte Morris and Zimmer [171].

### 2.3. Induced actions.

We end this section by describing briefly the standard construction of an *induced or suspended action*. This notion can be seen as a generalization of the construction of a flow under a function or as an analogue of the more algebraic notion of inducing a representation. Given a group $H$, a (usually closed) subgroup $L$, and an action $\rho$ of $L$ on a space $X$, we can form the space $(H \times X)/L$ where $L$ acts on $H \times X$ by $h \cdot (l, x) = (hl^{-1}, \rho(h)x)$. This space now has a natural $H$ action by left multiplication on the first coordinate. Many properties of the $L$ action on $X$ can be studied more easily in
terms of properties of the $H$ action on $(H \times X)/L$. This construction is particularly useful when $L$ is a lattice in $H$.

This notion suggests the following principle:

**Principle 2.6.** Let $\Gamma$ be a cocompact lattice in a Lie group $G$. To classify $\Gamma$ actions on compact manifolds it suffices to classify $G$ actions on compact manifolds.

The principle is a bit subtle to implement in practice, since we clearly need a sufficiently detailed classification of $G$ actions to be able to tell which one’s arise as induction of $\Gamma$ actions. While it is a bit more technical to state and probably more difficult to use there is an analogous principle for non-cocompact lattices. Here one needs to classify $G$ actions on manifolds which are not compact but where the $G$ action preserves a finite volume. In fact, one needs only to study such actions on manifolds that are fiber bundles over $G/\Gamma$ with compact fibers.

The lemma begs the question as to whether or not one should simply always study $G$ actions. While in many settings this is useful, it is not always. In particular, many known results about $\Gamma$ actions require hypotheses on the $\Gamma$ action where there is no useful way of rephrasing the property as a property of the induced action. Or, perhaps more awkwardly, require assumptions on the induced action which cannot be rephrased in terms of hypotheses on the original $\Gamma$ action. We will illustrate the difficulties in employing Principle 2.6 at several points in this paper.

A case where the implications of the principle are particularly clear is a negative result concerning actions of $SO(1,n)$. We will make clear by the proof what we mean by:

**Theorem 2.7.** Let $G = SO(1,n)$. Then one cannot classify actions of $G$ on compact manifolds.

**Proof.** For every $n$ there is at least one lattice $\Gamma < G$ which admits homomorphisms onto non-abelian free groups, see e.g. [147]. And therefore also onto $\mathbb{Z}$. So we can take any action of $F_n$ or $\mathbb{Z}$ and induce to a $G$ action. It is relatively easy to show that if the induced actions are isomorphic, then so are the actions they are induced from, see e.g. [177]. A classification of $\mathbb{Z}$ actions would amount to a classification of diffeomorphisms and a classification of $F_n$ actions would involve classifying all $n$-tuples of diffeomorphisms. As there is no reasonable classification of diffeomorphisms there is also no reasonable classification of $n$-tuples of diffeomorphisms. 

We remark that essentially the same theorem holds for actions of $SU(1,n)$ where homomorphisms to $\mathbb{Z}$ exist for certain lattices [129].
Much less is known about free quotients of lattices in $SU(1, n)$, see [145] for one example. For a surprising local rigidity result for some lattices in $SU(1, n)$, see [71, Theorem 1.3].

3. Pre-history

3.1. Local and global rigidity of homomorphisms into finite dimensional groups. The earliest work on rigidity theory is a series of works by Calabi–Vesentini, Selberg, Calabi and Weil, which resulted in the following:

**Theorem 3.1.** Let $G$ be a semi-simple Lie group and assume that $G$ is not locally isomorphic to $SL(2, \mathbb{R})$. Let $\Gamma < G$ be an irreducible cocompact lattice, then the defining embedding of $\Gamma$ in $G$ is locally rigid, i.e. any embedding $\rho$ close to the defining embedding is conjugate to the defining embedding by a small element of $G$.

**Remarks 3.2.**

1. If $G = SL(2, \mathbb{R})$ the theorem is false and there is a large, well studied space of deformation of $\Gamma$ in $G$, known as the Teichmüller space.
2. There is an analogue of this theorem for lattices that are not cocompact. This result was proven later and has a more complicated history which we omit here. In this case it is also necessary to exclude $G$ locally isomorphic to $SL(2, \mathbb{C})$.

This theorem was originally proven in special cases by Calabi, Calabi–Vesentini and Selberg. In particular, Selberg gives a proof for cocompact lattices in $SL(n, \mathbb{R})$ for $n \geq 3$ in [206], Calabi–Vesentini give a proof when the associated symmetric space $X = G/K$ is Kähler in [28] and Calabi gives a proof for $G = SO(1, n)$ where $n \geq 3$ in [27]. Shortly afterwards, Weil gave a complete proof of Theorem 3.1 in [226, 227].

In all of the original proofs, the first step was to show that any perturbation of $\Gamma$ was discrete and therefore a cocompact lattice. This is shown in special cases in [27, 28, 206] and proven in a somewhat broader context than Theorem 3.1 in [227].

The different proofs of cases of Theorem 3.1 are also interesting in that there are two fundamentally different sets of techniques employed and this dichotomy continues to play a role in the history of rigidity. Selberg’s proof essentially combines algebraic facts with a study of the dynamics of iterates of matrices. He makes systematic use of the existence of singular directions, or Weyl chamber walls, in maximal diagonalizable subgroups of $SL(n, \mathbb{R})$. Exploiting these singular directions is essential to much later work on rigidity, both of lattices in higher rank groups and of actions of abelian groups. It seems possible
to generalize Selberg’s proof to the case of $G$ an $\mathbb{R}$-split semi-simple Lie group with rank at least 2. Selberg’s proof, which depended on asymptotics at infinity of iterates of matrices, inspired Mostow’s explicit use of boundaries in his proof of strong rigidity \[174\]. Mostow’s work in turn provided inspiration for the use of boundaries in later work of Margulis, Zimmer and others on rigidity properties of higher rank groups.

The proofs of Calabi, Calabi–Vesentini and Weil involve studying variations of geometric structures on the associated locally symmetric space. The techniques are analytic and use a variational argument to show that all variations of the geometric structure are trivial. This work is a precursor to much work in geometric analysis studying variations of geometric structures and also informs later work on proving rigidity/vanishing of harmonic forms and maps. The dichotomy between approaches based on algebra/dynamics and approaches that are in the spirit of geometric analysis continues through much of the history of rigidity and the history of rigidity of group actions in particular.

Shortly after completing this work, Weil discovered a new criterion for local rigidity \[228\]. In the context of Theorem 3.1 this allows one to avoid the step of showing that a perturbation of $\Gamma$ remains discrete. In addition, this result opened the way for understanding local rigidity of more general representations of discrete groups than the defining representation.

**Theorem 3.3.** Let $\Gamma$ be a finitely generated group, $G$ a Lie group and $\pi : \Gamma \to G$ a homomorphism. Then $\pi$ is locally rigid if $H^1(\Gamma, \mathfrak{g}) = 0$. Here $\mathfrak{g}$ is the Lie algebra of $G$ and $\Gamma$ acts on $\mathfrak{g}$ by $\text{Ad}_G \circ \pi$.

Weil’s proof of this result uses only the implicit function theorem and elementary properties of the Lie group exponential map. The same theorem is true if $G$ is an algebraic group over any local field of characteristic zero. In \[228\], Weil remarks that if $\Gamma \lhd G$ is a cocompact lattice and $G$ satisfies the hypothesis of Theorem \[3.1\] then the vanishing of $H^1(\Gamma, \mathfrak{g})$ can be deduced from the computations in \[226\]. The vanishing of $H^1(\Gamma, \mathfrak{g})$ is proven explicitly by Matsushima and Murakami in \[167\].

Motivated by Weil’s work and other work of Matsushima, conditions for vanishing of $H^1(\Gamma, \mathfrak{g})$ were then studied by many authors. See particularly \[167\] and \[196\]. The results in these papers imply local rigidity of many linear representations of lattices.

3.2. **Strong and super rigidity.** In a major and surprising development, it turns out that in many instances, local rigidity is just the tip
of the iceberg and that much stronger rigidity phenomena exist. We discuss now major developments from the 60’s and 70’s.

The first remarkable result in this direction is Mostow’s rigidity theorem, see [172, 173] and references there. Given $G$ as in Theorem 3.1 and two irreducible cocompact lattices $\Gamma_1$ and $\Gamma_2$ in $G$, Mostow proves that any isomorphism from $\Gamma_1$ to $\Gamma_2$ extends to an isomorphism of $G$ with itself. Combined with the principal theorem of [227] which shows that a perturbation of a lattice is again a lattice, this gives a remarkable and different proof of Theorem 3.1, and Mostow was motivated by the desire for a “more geometric understanding” of Theorem 3.1 [173]. Mostow’s theorem is in fact a good deal stronger, and controls not only homomorphisms $\Gamma \to G$ near the defining homomorphism, but any homomorphism into any other simple Lie group $G'$ where the image is lattice. As mentioned above, Mostow’s approach was partially inspired by Selberg’s proof of certain cases of Theorem 3.1 [174]. A key step in Mostow’s proof is the construction of a continuous map between the geometric boundaries of the symmetric spaces associated to $G$ and $G'$. Boundary maps continue to play a key role in many developments in rigidity theory. A new proof of Mostow rigidity, at least for $G_i$ of real rank one, was provided by Besson, Courtois and Gallot. Their approach is quite different and has had many other applications concerning rigidity in geometry and dynamics, see e.g. [13, 14, 38].

The next remarkable result in this direction is Margulis’ superrigidity theorem. Margulis proved this theorem as a tool to prove arithmeticity of irreducible uniform lattices in groups of real rank at least 2. For irreducible lattices in semi-simple Lie groups of real rank at least 2, the superrigidity theorems classifies all finite dimensional linear representations. Margulis’ theorem holds for irreducible lattices in semi-simple Lie groups of real rank at least two. Given a lattice $\Gamma < G$ where $G$ is simply connected, one precise statement of some of Margulis results is to say that any linear representation $\sigma$ of $\Gamma$ almost extends to a linear representation of $G$. By this we mean that there is a linear representation $\tilde{\sigma}$ of $G$ and a bounded image representation $\bar{\sigma}$ of $\Gamma$ such that $\sigma(\gamma) = \tilde{\sigma}(\gamma)\bar{\sigma}(\gamma)$ for all $\gamma$ in $G$. Margulis’ theorems also give an essentially complete description of the representations $\tilde{\sigma}$, up to some issues concerning finite image representations. The proof here is partially inspired by Mostow’s work: a key step is the construction of a measurable “boundary map”. However the methods for producing the boundary map in this case are very dynamical. Margulis’ original proof used Oseledec Multiplicative Ergodic Theorem. Later proofs were given by both Furstenberg and Margulis using the theory of group boundaries as developed by Furstenberg from his study of random walks on groups.
Furstenberg’s probabilistic version of boundary theory has had a profound influence on many subsequent developments in rigidity theory. For more discussion of Margulis’ superrigidity theorem, see [156, 158, 159, 242].

Margulis theorem, by classifying all linear representations, and not just one’s with constrained images, leads one to believe that one might be able to classify all homomorphisms to other interesting classes of topological groups. Zimmer’s program is just one aspect of this theory, in other directions, many authors have studied homomorphisms to isometry groups of non-positively curved (and more general) metric spaces. See e.g. [23, 97, 169].

3.3. Harmonic map approaches to rigidity. In the 90’s, first Corlette and then Gromov-Schoen showed that one could prove major cases of Margulis’ superrigidity theorems also for lattices in $Sp(1,n)$ and $F_{4-20}$ [39, 110]. Corlette considered the case of representations over Archimedean fields and Gromov and Schoen proved results over other local fields. These proofs used harmonic maps and proceed in three steps. First showing a harmonic map exists, second showing it is smooth, and third using certain special Bochner-type formulas to show that the harmonic mapping must be a local isometry. Combined with earlier work of Matsushima, Murakami and Raghunathan, this leads to a complete classification of linear representations for lattices in these groups [167, 196]. It is worth noting that the use of harmonic map techniques in rigidity theory had been pioneered by Siu, who used them to prove generalizations of Mostow rigidity for certain classes of Kähler manifolds [214]. There is also much later work on applying harmonic map techniques to reprove cases of Margulis’ superrigidity theorem by Jost-Yau and Mok-Siu-Yeung among others, see e.g. [119, 168]. We remark in passing that the general problem of existence of harmonic maps for non-compact, finite volume locally symmetric spaces has not been solved in general. Results of Saper and Jost-Zuo allow one to prove superrigidity for fundamental groups of many such manifolds, but only while assuming arithmeticity [120, 204]. The use of harmonic maps in rigidity was inspired by the use of variational techniques and harmonic forms and functions in work on local rigidity and vanishing of cohomology groups. The original suggestion to use harmonic maps in this setting appears to go back to Calabi [213].

3.4. A remark on cocycle super-rigidity. An important impetus for the study of rigidity of groups acting on manifolds was Zimmer’s proof of his cocycle superrigidity theorem. We discuss this important
result below in section 5 where we also indicate some of its applications to group actions. This theorem is a generalization of Margulis’ super-rigidity theorem to the class of virtual homomorphisms corresponding to cocycles over measure preserving group actions.

3.5. Margulis’ normal subgroup theorem. We end this section by mentioning another result of Margulis that has had tremendous importance in results concerning group actions on manifolds. This is the normal subgroups theorem, which says that any normal subgroup in a higher rank lattice is either finite or of finite index see e.g. [159, 242] for more on the proof. The proof precedes by a remarkable strategy. Let N be a normal subgroup of Γ, we show Γ/N is finite by showing that it is amenable and has property (T). This strategy has been applied in other contexts and is a major tool in the construction of simple groups with good geometric properties see [6, 24, 25, 36]. The proof that Γ/N is amenable already involves one step that might rightly be called a theorem about rigidity of group actions. Margulis shows that if G is a semisimple group of higher rank, P is a minimal parabolic, and Γ is a lattice then any measurable Γ space X which is a measurable quotient of the Γ action on G/P is necessarily of the form G/Q where Q is a parabolic subgroup containing P. The proof of this result, sometimes called the projective factors theorem, plays a fundamental role in work of Nevo and Zimmer on non-volume preserving actions. See §9 below for more discussion. It is also worth noting that Dani has proven a topological analogue of Margulis’ result on quotients [41]. I.e. he has proven that continuous quotients of the Γ action on G/P are all Γ actions on G/Q.

The usual use of Margulis’ normal subgroup theorem in studying rigidity of group actions is usually quite straightforward. If one wants to prove that a group satisfying the normal subgroups theorem acts finitely, it suffices to find one infinite order element that acts trivially.

4. Low dimensional actions: conjectures and results

We begin this section by discussing results in particular, very low dimensions, namely dimensions 1, 2 and 3.

Before we begin this discussion, we recall a result of Thurston that is often used to show that low dimensional actions are trivial [222].

**Theorem 4.1** (Thurston Stability). Assume Γ is a finitely generated group with finite abelianization. Let Γ act on a manifold M by $C^1$ diffeomorphisms, fixing a point p and with trivial derivative at p. Then the Γ action is trivial in a neighborhood of p. In particular, if M is connected, the Γ action is trivial.
The main point of Theorem 4.1 is that to show an action is trivial, it often suffices to find a fixed point. This is because, for the groups we consider, there are essentially no non-trivial low dimensional linear representations and therefore the derivative at a fixed point is trivial. More precisely, the groups usually only have finite image low dimensional linear representations, which allows one to see that the action is trivial on a subgroup of finite index.

4.1. Dimension one. The most dramatic results obtained in the Zimmer program concern a question first brought into focus by Dave Witte Morris in [231]: can higher rank lattices act on the circle? In fact, the paper [231] is more directly concerned with actions on the line \( \mathbb{R} \). A detailed survey of results in this direction is contained in the paper by Witte Morris in this volume, so we do not repeat that discussion here. We merely state a conjecture and a question.

**Conjecture 4.2.** Let \( \Gamma \) be a higher rank lattice. Then any continuous \( \Gamma \) action on \( S^1 \) is finite.

This conjecture is well known and first appeared in print in [100]. By results in [231] and [143], the case of non-cocompact lattices is almost known. The paper [231] does the case of higher \( \mathbb{Q} \) rank. The latter work of Lifchitz and Witte Morris reduces the general case to the case of quasi-split lattices in \( SL(3, \mathbb{R}) \) and \( SL(3, \mathbb{C}) \). It follows from work of Ghys or Burger-Monod that one can assume the action fixes a point, see e.g. [21, 22, 98, 100, 101], so the question is equivalent to asking if the groups act on the line. An interesting approach might be to study the induced action of \( \Gamma \) on the space of left orders on \( \Gamma \), for ideas about this approach, we refer the reader to work of Navas and Witte Morris [170, 176].

Perhaps the following should also be a conjecture, but here we only ask it is a question.

**Question 4.3.** Let \( \Gamma \) be a discrete group with property \((T)\). Does \( \Gamma \) admit an infinite action by \( C^1 \) diffeomorphisms on \( S^1 \)? Does \( \Gamma \) admit an infinite action by homeomorphisms on \( S^1 \)?

By a result of Navas, the answer to the above question is no if \( C^1 \) diffeomorphisms are replaced by \( C^k \) diffeomorphisms for any \( k > \frac{3}{2} \) [177]. A result noticed by the author and Margulis and contained in a paper of Bader, Furman, Gelander, and Monod allows one to adapt this proof for values of \( k \) slightly less than \( \frac{3}{2} \) [5, 81]. By Thurston’s Theorem 4.1 in the \( C^1 \) case it suffices to find a fixed point for the action.
We add a remark here pointed out to the author by Navas, that perhaps justifies only calling Question 4.3 a question. In [178], Navas extends his results from [177] to groups with relative property (T). This means that no such group acts on the circle by $C^k$ diffeomorphisms where $k > \frac{3}{2}$. However, if we let $\Gamma$ be the semi-direct product of a finite index free subgroup of $SL(2, \mathbb{Z})$ and $\mathbb{Z}^2$, this group is left orderable, and so acts on $S^1$, see e.g. [170]. This is the prototypical example of a group with relative property (T). This leaves open the possibility that groups with property (T) would behave in a similar manner and admit continuous actions on the circle and the line.

Other possible candidate for a group with property (T) acting on the circle by homeomorphisms are the more general variants of Thompson’s group $F$ constructed by Stein in [219]. The proofs that $F$ does not have (T) do not apply to these groups [57].

A closely related question is the following, suggested to the author by Andrés Navas along with examples in the last paragraph.

**Question 4.4.** Is there a group $\Gamma$ which is bi-orderable and does not admit a proper isometric action on a Hilbert space? I.e. does not have the Haagerup property?

For much more information on the fascinating topic of groups of diffeomorphisms of the circle see both the survey by Ghys [101], the more recent book by Navas [175] and the article by Witte Morris in this volume [234].

**4.2. Dimension 2.** Already in dimension 2 much less is known. There are some results in the volume preserving setting. In particular, we have

**Theorem 4.5.** Let $\Gamma$ be a non-uniform irreducible higher rank lattice. Then any volume preserving $\Gamma$ action on a closed orientable surface other than $S^2$ is finite. If $\Gamma$ has Q-rank at least one, then the same holds for actions on $S^2$.

This theorem was proven by Polterovich for all surfaces but the sphere and shortly afterwards proven by Franks and Handel for all surfaces [86, 85, 189]. It is worth noting that the proofs use entirely different ideas. Polterovich’s proof belongs very clearly to symplectic geometry and also implies some results for actions in higher dimensions. Franks and Handel use a theory of normal forms for $C^1$ surface diffeomorphisms that they develop in analogy with the Thurston theory of normal forms for surface homeomorphisms with finite fixed sets. The proof of Franks and Handel can be adapted to a setting where one assume much less regularity of the invariant measure [87].
There are some major reductions in the proofs that are similar, which we now describe. The first of these should be useful for studying actions of cocompact lattices as well. Namely, one can assume that the homomorphism \( \rho : \Gamma \to \text{Diff}(S) \) defining the action takes values in the connected component. This follows from the fact that any \( \rho : \Gamma \to \text{MCG}(S) \) is finite, which can now be deduced from a variety of results \[16, 21, 52, 121\]. This result holds not only for the groups considered in Theorem 4.5, but also for cocompact lattices and even for lattices in \( \text{SP}(1, n) \) by results of Sai-Kee Yeung \[236\]. It may be possible to show something similar for all groups with property (\( T \)) using recent results of Andersen showing that the mapping class group does not have property (\( T \)) \[2\]. It is unrealistic to expect a simple analogue of this result for homomorphisms to \( \text{Diff}(M)/\text{Diff}(M)^0 \) for general manifolds, see section 4.4 below for a discussion of dimension 3.

Also, in the setting of Theorem 4.5, Margulis normal subgroup theorem implies that it suffices to show that a single infinite order element of \( \Gamma \) acts trivially. The proofs of Franks-Handel and Polterovich then use the existence of distortion elements in \( \Gamma \), a fact established by Lubotzky, Mozes and Raghunathan in \[138\]. The main result in both cases shows that \( \text{Diff}(S, \omega) \) does not contain exponentially distorted elements and the proofs of this fact are completely different. An interesting result of Calegari and Freedman shows that this is not true of \( \text{Diff}(S^2) \) and that \( \text{Diff}(S^2) \) contains subgroups with elements of arbitrarily large distortion \[29\]. These examples are discussed in more detail in section 9.

Polterovich’s methods also allow him to see that there are no exponentially distorted elements in \( \text{Diff}(M, \omega)^0 \) for certain symplectic manifolds \( (M, \omega) \). Again, this yields some partial results towards Zimmer’s conjecture. On the other hand, Franks and Handel are able to work with a Borel measure \( \mu \) with some properties and show that any map \( \rho : \Gamma \to \text{Diff}(S, \mu) \) is finite.

We remark here that a variant on this is due to Zimmer, when there is an invariant measure supported on a finite set.

**Theorem 4.6.** Let \( \Gamma \) be a group with property (\( T \)) acting on a compact surface \( S \) by \( C^1 \) diffeomorphisms. Assume \( \Gamma \) has a periodic orbit on \( S \), then the \( \Gamma \) action is finite.

The proof is quite simple. First, pass to a finite index subgroup which fixes a point \( x \). Look at the derivative representation \( d\rho_x \) at the fixed point. Since \( \Gamma \) has property (\( T \)), and \( \text{SL}(2, \mathbb{R}) \) has the Haagerup property, it is easy to prove that the image of \( d\rho_x \) is bounded. Bounded
subgroups of $SL(2, \mathbb{R})$ are all virtually abelian and this implies that the image of $d\rho_x$ is finite. Passing to a subgroup of finite index, one has a fixed point where the derivative action is trivial of a group which has no cohomology in the trivial representation. One now applies Thurston Theorem 4.1.

The difficulty in combining Theorem 4.6 with ideas from the work of Franks and Handel is that while Franks and Handel can show that individual surface diffeomorphisms have large sets of periodic orbits, their techniques do not easily yield periodic orbits for the entire large group action.

We remark here that there is another approach to showing that lattices have no volume preserving actions on surfaces. This approach is similar to the proof that lattices have no $C^1$ actions on $S^1$ via bounded cohomology [21, 22, 98]. That $\text{Diff}(S, \omega)^0$ admits many interesting quasi-morphisms, follows from work of Entov-Polterovich, Gambaudo-Ghys and Py [48, 95, 192, 193, 192]. By results of Burger and Monod, for any higher rank lattice and any homomorphism $\rho : \Gamma \to \text{Diff}(S, \omega)^0$, the image is in the kernel of all of these quasi-morphisms. (To make this statement meaningful and the kernel well-defined, one needs to take the homogeneous versions of the quasi-morphisms.) What remains to be done is to extract useful dynamical information from this fact, see the article by Py in this volume for more discussion [191].

In our context, the work of Entov-Polterovich mentioned above really only constructs a single quasi-morphism on $\text{Diff}(S^2, \omega)^0$. However, this particular quasimorphism is very nice in that it is Lipschitz in metric known as Hofer’s metric on $\text{Diff}(S^2, \omega)^0$. This construction does apply more generally in higher dimensions and indicates connections between quasimorphisms and the geometry of $\text{Diff}(M, \omega)^0$, for $\omega$ a symplectic form, that are beyond the scope of this survey. For an introduction to this fascinating topic, we refer the reader to [188].

Motivated by the above discussion, we recall the following conjecture of Zimmer.

Conjecture 4.7. Let $\Gamma$ be a group with property (T), then any volume preserving smooth $\Gamma$ action on a surface is finite.

Here the words “volume preserving” or at least “measure preserving” are quite necessary. As $SL(3, \mathbb{R})$ acts on a $S^2$, so does any lattice in $SL(3, \mathbb{R})$ or any irreducible lattice in $G$ where $G$ has $SL(3, \mathbb{R})$ as a factor. The following question seem reasonable, I believe I first learned it from Leonid Polterovich.

Question 4.8. Let $\Gamma$ be a higher rank lattice (or even just a group with property (T)) and assume $\Gamma$ acts by diffeomorphisms on a surface $S$. Is
it true that either (1) the action is finite or (2) the surface is $S^2$ and the action is smoothly conjugate to an action defined by some embedding $i: \Gamma \to SL(3, \mathbb{R})$ and the projective action of $SL(3, \mathbb{R})$ on $S^2$?

This question seems quite far beyond existing technology.

4.3. Dimension 3. We now discuss briefly some work of Farb and Shalen that constrains actions by homeomorphisms in dimension 3 [54]. This work makes strong use of the geometry of 3 manifolds, but uses very little about higher rank lattices and is quite soft. A special case of their results is the following:

**Theorem 4.9.** Let $M$ be an irreducible 3 manifold and $\Gamma$ be a higher rank lattice. Assume $\Gamma$ acts on $M$ by homeomorphisms so that the action on homology is non-trivial. Then $M$ is homeomorphic to $T^3$, $\Gamma \subset SL(3, \mathbb{Z})$ with finite index and the $\Gamma$ action on $H^1(M)$ is the standard $\Gamma$ action on $\mathbb{Z}^3$.

Farb and Shalen actually prove a variant of Theorem 4.9 for an arbitrary 3 manifold admitting a homologically infinite action of a higher rank lattice. This can be considered as a significant step towards understanding when, for $\Gamma$ a higher rank lattice and $M^3$ a closed three manifold, $\rho : \Gamma \to Diff(M^3)$ must have image in $Diff(M^3)^0$. Unlike the results discussed above for dimension two, the answer is not simply “always”. This three dimensional result uses a great deal of the known structure of 3 manifolds, though it does not use the full geometrization conjecture proven by Perelman, but only the Haken case due to Thurston. The same sort of result in higher dimensions seems quite out of reach. A sample question is the following. Here we let $\Gamma$ be a higher rank action and $M$ a compact manifold.

**Question 4.10.** Under what conditions on the topology of $M$ do we know that a homomorphism $\rho : \Gamma \to Diff(M)$ has image in $Diff(M)^0$?

4.4. Analytic actions in low dimensions. We first mention a direction pursued by Ghys that is in a similar spirit to the Zimmer program, and that has interesting consequences for that program. Recall that the Zassenhaus lemma shows that any discrete linear group generated by small enough elements is nilpotent. The main point of Ghys’s article [99] is to attempt to generalize this result for subgroups of $Diff^\omega(M)$. While the result is not actually true in that context, Ghys does prove some intriguing variants which yield some corollaries for analytic actions of large groups. For instance, he proves that $SL(n, \mathbb{Z})$ for $n > 3$ admits no analytic action on the two sphere. We remark that the attempt to prove the Zassenhaus lemma for diffeomorphism groups...
suggests that one can attempt to generalize other facts about linear groups to the category of diffeomorphism groups. We discuss several questions in this direction, mainly due to Ghys, in Section 13.

For the rest of this subsection we discuss a different approach of Farb and Shalen for showing that real analytic actions of large groups are finite. This method is pursued in [53, 55, 56].

We begin by giving a cartoon of the main idea. Given any action of a group $\Gamma$, an element $\gamma \in \Gamma$ and the centralizer $Z(\gamma)$, it is immediate that $Z(\gamma)$ acts on the set of $\gamma$ fixed points. If the action is analytic, then the fixed sets are analytic and so have good structure and are “reasonably close” to being submanifolds. If one further assumes that all normal subgroups of $\Gamma$ have finite index, then this essentially allows one to bootstrap results about $Z(\gamma)$ not acting on manifolds of dimension at most $n-1$ to facts about $\Gamma$ not having actions on manifolds of dimension $n$ provided one can show that the fixed set for $\gamma$ is not empty. This is not true, as analytic sets are not actually manifolds, but the idea can be implemented using the actual structure of analytic sets in a way that yields many results.

For example, we have:

**Theorem 4.11.** Let $M$ be a real analytic four manifold with zero Euler characteristic, then any real analytic, volume preserving action of any finite index subgroup in $SL(n, \mathbb{Z})$ for $n \geq 7$ is trivial.

This particular theorem also requires a result of Rebelo [198] which concerns fixed sets for actions of nilpotent groups on $\mathbb{T}^2$ and uses the ideas of [99].

The techniques of Farb and Shalen can also be used to prove that certain cocompact higher rank lattices have no real analytic actions on surfaces of genus at least one. For this result one needs only that the lattice contains an element $\gamma$ whose centralizer already contains a higher rank lattice. In the paper [53] a more technical condition is required, but this can be removed using the results of Ghys and Burger-Monod on actions of lattices on the circle. (This simplification was first pointed out to the author by Farb.) The point is that $\gamma$ has fixed points for topological reasons and the set of these fixed points contains either (1) a $Z(\gamma)$ invariant circle or (2) a $Z(\gamma)$ invariant point. Case (1) reduces to case (2) after passing to a finite index subgroup via the results on circle actions. Case (2) is dealt with by the proof of Theorem 4.6.

4.5. Zimmer’s full conjecture, partial results. We now state the full form of Zimmer’s conjecture. In fact we generalize it slightly to
include all lattices with property \( (T) \). Throughout this subsection \( G \) will be a semisimple Lie group with property \( (T) \) and \( \Gamma \) will be a lattice in \( \Gamma \). We define two numerical invariants of these groups. First, for any group \( F \), let \( d(F) \) be the lowest dimension in which \( F \) admits an infinite image linear representation. We note that the superrigidity theorems imply that \( d(\Gamma) = d(G) \) when \( \Gamma \) is a lattice in \( G \) and either \( G \) is a semisimple group property \( (T) \) or \( \Gamma \) is irreducible higher rank. The second number, \( n(G) \) is the lowest dimension of a homogeneous space \( K/C \) for a compact group \( K \) on which a lattice \( \Gamma \) in \( G \) can act via a homomorphism \( \rho : \Gamma \to K \). In Zimmer’s work, \( n(G) \) is defined differently, in a way that makes clear that there is a bound on \( n(G) \) that does not depend on the choice of \( \Gamma \). Namely, using the superrigidity theorems, it can be shown that \( n \) satisfies

\[
\frac{n(n+1)}{2} \geq \min \{ \dim_C G' \text{ where } G' \text{ is a simple factor of } G \}.
\]

The more fashionable variant of Zimmer’s conjecture is the following, first made explicitly by Farb and Shalen for higher rank lattices in [53].

**Conjecture 4.12.** Let \( \Gamma \) be a lattice as above. Let \( b = \min \{ n, d \} \) and let \( M \) be a manifold of dimension less than \( b - 1 \), then any \( \Gamma \) action on \( M \) is trivial.

It is particularly bold to state this conjecture including lattices in \( Sp(1, n) \) and \( F_4^{-20} \). This is usually avoided because such lattices have abundant non-volume preserving actions on certain types of highly regular fractals. I digress briefly to explain why I believe the conjecture is plausible in that case. Namely such a lattice \( \Gamma \), being a hyperbolic group, has many infinite proper quotients which are also hyperbolic groups [108]. If \( \Gamma' \) is a quotient of \( \Gamma \) by an infinite index, infinite normal subgroup \( N \) and \( \Gamma' \) is hyperbolic, then the boundary \( \partial \Gamma' \) is a \( \Gamma \) space with interesting dynamics and a good (Ahlfors regular) quasi-invariant measure class. However, \( \partial \Gamma' \) is only a manifold when it is a sphere. It seems highly unlikely that this is ever the case for these groups and even more unlikely that this is ever the case with a smooth boundary action. The only known way to build a hyperbolic group which acts smoothly on its boundary is to have the group be the fundamental group of a compact negatively curved manifold \( M \) with smoothly varying horospheres. If smooth is taken to mean \( C^\infty \), this then implies the manifold is locally symmetric [13] and then superrigidity results make it impossible for this to occur with \( \pi_1(M) \) a quotient of a lattice by an infinite index infinite normal subgroup. If smooth only means \( C^1 \), then even in this context, no result known rules out \( M \) having fundamental
group a quotient of a lattice in $Sp(1,n)$ or $F_{4}^{-20}$. All results one can prove using harmonic map techniques in this context only rule out $M$ with non-positive complexified sectional curvature. We make the following conjecture, which is stronger than what is needed for Conjecture 4.12.

**Conjecture 4.13.** Let $\Gamma$ be a cocompact lattice in $Sp(1,n)$ or $F_{4}^{-20}$ and let $\Gamma'$ be a quotient of $\Gamma$ which is Gromov hyperbolic. If $\partial\Gamma'$ is a sphere, then the kernel of the quotient map is finite and $\partial\Gamma' = \partial\Gamma$.

For background on hyperbolic groups and their boundaries from a point of view relevant to this conjecture, see [130]. As a cautionary note, we point the reader to subsection 11.3 where we recall a construction of Farrell and Lafont that shows that any Gromov hyperbolic group has an action by homeomorphisms on a sphere.

The version of Zimmer’s conjecture that Zimmer made in [246] and [247] was only for volume preserving actions. Here we break it down somewhat explicitly to clarify the role of $d$ and $c$.

**Conjecture 4.14 (Zimmer).** Let $\Gamma$ be a lattice as above and assume $\Gamma$ acts smoothly on a compact manifold $M$ preserving a volume form. Then if $\dim(M) < d$, the $\Gamma$ action is isometric. If, in addition, $\dim(M) < n$ or if $\Gamma$ is non-uniform, then the $\Gamma$ action is finite.

Some first remarks are in order. The cocycle super-rigidity theorems (discussed below) imply that, when the conditions of the conjecture hold, there is always a measurable invariant Riemannian metric. Also, the finiteness under the conditions in the second half of the conjecture follow from cases of Margulis’ superrigidity theorem as soon as one knows that the action preserves a smooth Riemannian metric. So from one point of view, the conjecture is really about the regularity of the invariant metric.

We should also mention that the conjecture is proven under several additional hypotheses by Zimmer around the time he made the conjecture. The first example is the following.

**Theorem 4.15.** Conjecture 4.14 holds provided the action also preserves a rigid geometric structure, e.g. a torsion free affine connection or a pseudo-Riemannian metric.

This is proven in [243] for structures of finite type in the sense of Elie Cartan, see also [247]. The fact that roughly the same proof applies for rigid structures in the sense of Gromov was remarked in [84]. The point is simply that the isometry group of a rigid structure acts properly on some higher order frame bundle and that the existence of...
the measurable metric implies that \( \Gamma \) has bounded orbits on all frame bundles as soon as \( \Gamma \) has property \((T)\). This immediately implies that \( \Gamma \) must be contained in a compact subgroup of the isometry group of the structure.

Another easy version of the conjecture follows from the proof of Theorem 4.6. That is:

**Theorem 4.16.** Let \( G \) be a semisimple Lie group with property \((T)\) and \( \Gamma < G \) a lattice. Let \( \Gamma \) act on a compact manifold \( M \) by \( C^1 \) diffeomorphisms where \( \dim(M) < d(G) \). If \( \Gamma \) has a periodic orbit on \( S \), then the \( \Gamma \) action is finite.

A more difficult theorem of Zimmer shows that Conjecture 4.14 holds when the action is distal in a sense defined in [244].

### 4.6. Some approaches to the conjectures.

#### 4.6.1. Discrete spectrum of actions. A measure preserving action of a group \( D \) on a finite measure space \( (X, \mu) \) is said to have discrete spectrum if \( L^2(X, \mu) \) splits as sum of finite dimensional \( D \) invariant subspaces. This is a strong condition that is (quite formally) the opposite of weak mixing, for a detailed discussion see [91]. It is a theorem of Mackey (generalizing earlier results of Halmos and Von Neumann for \( D \) abelian) that an ergodic discrete spectrum \( D \) action is measurably isomorphic to one described by a dense embedding of \( D \) into a compact group \( K \) and considering a \( K \) action on a homogeneous \( K \)-space. The following remarkable result of Zimmer from [250] is perhaps the strongest evidence for Conjecture 4.14. This result is little known and has only recently been applied by other authors, see [82, 90].

**Theorem 4.17.** Let \( \Gamma \) be a group with property \((T)\) acting by smooth, volume preserving diffeomorphisms on a compact manifold \( M \). Assume in addition that \( \Gamma \) preserves a measurable invariant metric. Then the \( \Gamma \) action has discrete spectrum.

This immediately implies that no counterexample to Conjecture 4.14 can be weak mixing or even admit a weak mixing measurable factor.

The proof of the theorem involves constructing finite dimensional subspaces of \( L^2(M, \omega) \) that are \( \Gamma \) invariant. If enough of these subspaces could be shown to be spanned by smooth functions, one would have a proof of Conjecture 4.14. Here by “enough” we simply mean that it suffices to have a collection of finite dimensional \( D \) invariant subspaces that separate points in \( M \). These functions would then specify a \( D \) equivariant smooth embedding of \( M \) into \( \mathbb{R}^N \) for some large value of \( N \).
To construct finite dimensional invariant subspaces of $L^2(M)$, Zimmer uses an approach similar to the proof of the Peter-Weyl theorem. Namely, he constructs $\Gamma$ invariant kernels on $L^2(M \times M)$ which are used to define self-adjoint, compact operators on $L^2(M)$. The eigenspaces of these operators are then finite dimensional, $\Gamma$ invariant subspaces of $L^2(M)$. The kernels should be thought of as functions of the distance to the diagonal in $M \times M$. The main difficulty here is that for these to be invariant by “distance” we need to mean something defined in terms of the measurable metric instead of a smooth one. The construction of the kernels in this setting is quite technical and we refer readers to the original paper.

It would be interesting to try to combine the information garnered from this theorem with other approaches to Zimmer’s conjectures.

4.6.2. Effective invariant metrics. We discuss here briefly another approach to Zimmer’s conjecture, due to the author, which seems promising.

We begin by briefly recalling the construction of the space of “$L^2$ metrics” on a manifold $M$. Given a volume form $\omega$ on $M$, we can consider the space of all (smooth) Riemannian metrics on $M$ whose associated volume form is $\omega$. This is the space of smooth sections of a bundle $P \to M$. The fiber of $P$ is $X = SL(n, \mathbb{R})/SO(n)$. The bundle $P$ is an associated bundle to the $SL(n, \mathbb{R})$ sub-bundle of the frame bundle of $M$ defined by $\omega$. The space $X$ carries a natural $SL(n, \mathbb{R})$-invariant Riemannian metric of non-positive curvature; we denote its associated distance function by $d_X$. This induces a natural notion of distance on the space of metrics, given by $d(g_1, g_2)^2 = \int_M d_X(g_1(m), g_2(m))^2 d\omega$. The completion of the sections with respect to the metric $d$ will be denoted $L^2(M, \omega, X)$; it is commonly referred to as the space of $L^2$ metrics on $M$ and its elements will be called $L^2$ metrics on $M$. That this space is $\text{CAT}(0)$ follows easily from the fact that $X$ is $\text{CAT}(0)$. For more discussion of $X$ and its structure as a Hilbert manifold, see e.g. [78]. It is easy to check that a volume preserving $\Gamma$ action on $M$ defines an isometric $\Gamma$ action on $L^2(M, \omega, X)$. Given a generating set $S$ for $\Gamma$ and a metric $g$ in $L^2(M, \omega, X)$, we write $\text{disp}(g) = \max_{\gamma \in S} d(\gamma g, g)$.

Given a group $\Gamma$ acting smoothly on $M$ preserving $\omega$, this gives an isometric $\Gamma$ action on $L^2(M, \omega, X)$ which preserves the subset of smooth metrics. Let $S$ be a generating set for $\Gamma$. We define an operator $P : L^2(M, \omega, X) \to L^2(M, \omega, X)$ by taking a metric $g$ to the barycenter of the measure $\sum_S \delta_{\gamma g}$. The first observation is a consequence of the (standard, finite dimensional) implicit function theorem.

Lemma 4.18. If $g$ is a smooth metric, then $Pg$ is also smooth.
Moreover, we have the following two results.

**Theorem 4.19.** Let $M$ be a surface and $\Gamma$ a group with property $(T)$ and finite generating set $S$. Then there exists $0 < C < 1$ such that the operator $P$ satisfies:

1. $\text{disp}(Pg) < C \text{disp}(g)$
2. for any $g$, the $\lim_n (P^n g)$ exists and is $\Gamma$ invariant.

A proof of this theorem can be given by using the standard construction of a negative definite kernel on $H^2$ to produce a negative definite kernel on $L^2(S, \mu, \mathbb{H}^2)$. The theorem is then proved by transferring the first property from the resulting $\Gamma$ action on a Hilbert space. The second property is an obvious consequence of the first and completeness of the space $L^2(M, \omega, X)$.

**Theorem 4.20.** Let $G$ be a semisimple Lie group all of whose simple factors have property $(T)$ and $\Gamma < G$ a lattice. Let $M$ be a compact manifold such that $\dim(M) < d(G)$. Then the operator $P$ on $L^2(M, \omega, X)$ satisfies the conclusions of Theorem 4.19.

This theorem is proven from results in [79] using convexity of the distance function on $L^2(M, \omega, X)$. For cocompact lattices a proof can be given using results in [135] instead.

The problem now reduces to estimating the behavior of the derivatives of $P^n g$ for some initial smooth $g$. This expression clearly involves derivatives of random products of elements of $\Gamma$, i.e derivatives of elements of $\Gamma$ weighted by measures that are convolution powers of equidistributed measure on $S$. The main cause for optimism is that the fact that $\text{disp}(P^n g)$ is small immediately implies that the first derivative of any $\gamma \in S$ must be small when measured at that point in $L^2(M, \omega, X)$. One can then try to use estimates on compositions of diffeomorphisms and convexity of derivatives to control derivatives of $P^n g$. The key difficulty is that the initial estimate on the first derivative of $\gamma$ applied to $P^n g$ is only small in an $L^2$ sense.

**4.7. Some related questions and conjectures on countable subgroups of $\text{Diff}(M)$.** In this subsection, we discuss related conjectures and results on countable subgroups of $\text{Diff}(M)$. All of these are motivated by the belief that countable subgroups of $\text{Diff}(M)$ are quite special, though considerably less special than say linear groups. We defer positive constructions of non-linear subgroups of $\text{Diff}(M)$ to Section 11 and a discussion of possible algebraic and geometric properties shared by all finitely generated subgroups of $\text{Diff}(M)$ to Section 13. Here we concentrate on groups which do not act on manifolds, either by theorems or conjecturally.
4.7.1. Groups with property (T) and generic groups. We begin by focusing on actions of groups with property (T). For a finitely generated group $\Gamma$, we recall that $d(\Gamma)$ be the smallest dimension in which $\Gamma$ admits an infinite image linear representation. We then make the following conjecture:

**Conjecture 4.21.** Let $\Gamma$ be a group with property (T) acting smoothly on a compact manifold $M$, preserving volume. Then if $\dim(M) < d(\Gamma)$, the action preserves a smooth Riemannian metric.

For many groups $\Gamma$ with property (T), one can produce a measurable invariant metric, see [82]. In fact, in [82], Silberman and the author prove that there are many groups with property (T) with no volume preserving actions on compact manifolds. Key steps include finding the invariant measurable metric, applying Zimmer’s theorem on discrete spectrum from §4.6 and producing groups with no finite quotients and so no linear representations at all.

A result of Furman announced in [89] provides some further evidence for the conjecture. This result is analogous to Proposition 5.1 and shows that any action of a group with property (T) either leaves invariant a measurable metric or has positive random entropy. We refer the reader to [89] for more discussion. While the proof of this result is not contained in [89], it is possible to reconstruct it from results there and others in [90].

Conjecture 4.21 and the work in [82] are motivated in part by the following conjecture of Gromov:

**Conjecture 4.22.** There exists a model for random groups in which a “generic” random group admits no smooth actions on compact manifolds.

It seems quite likely that the conjecture could be true for random groups in the density model with density more than $\frac{1}{3}$. These groups have property (T), see Ollivier’s book [186] for discussion on random groups. For the conjecture to be literally true would require that a random hyperbolic group have no finite quotients and it is a well-known question to determine if there are any hyperbolic groups which are not residually finite, let alone generic ones. If one is satisfied by saying the generic random group has only finite smooth actions on compact manifolds, one can avoid this well-known open question.

4.7.2. Universal lattices. Recently, Shalom has proven that the groups $SL(n,\mathbb{Z}[X])$ have property (T) when $n > 3$. (As well as some more general results.) Shalom refers to these groups as universal lattices. An interesting and approachable question is:
Question 4.23. Let \( \Gamma \) be a finite index subgroup in \( SL(n, \mathbb{Z}[X]) \) and let \( \Gamma \) acting by diffeomorphisms on a compact manifold \( M \) preserving volume. If \( \dim(M) < n \) is there a measurable \( \Gamma \) invariant metric?

If one gives a positive answer to this question, one is then clearly interested in whether or not the metric can be chosen to be smooth. It is possible that this is easier for these “larger” groups than for the lattices originally considered by Zimmer. One can ask a number of variants of this question, including trying to prove a full cocycle superrigidity theorem for these groups, see below. The question just asked is particularly appealing as it can be viewed as a fixed point problem for the \( \Gamma \) action on the space of metrics (see \( \S 4.6 \) for a definition).

In fact, one knows one has the invariant metric for the action of any conjugate of \( SL(n, \mathbb{Z}) \) and only needs to show that there is a consistent choice of invariant metrics over all conjugates. One might try to mimic the approach from [210], though a difficulty clearly arises at the point where Shalom applies a scaling limit construction. Scaling limits of \( L^2(M, \omega, X) \) can be described using non-standard analysis, but are quite complicated objects and not usually isomorphic to the original space.

4.7.3. Irreducible actions of products. Another interesting variant on Zimmer’s conjecture is introduced in [90]. In that paper they study obstructions to irreducible actions of product groups. An measure preserving action of a product \( \Gamma_1 \times \Gamma_2 \) on a measure space \( (X, \mu) \) is said to be irreducible if both \( \Gamma_1 \) and \( \Gamma_2 \) act ergodically on \( X \). Furman and Monod produce many obstructions to irreducible actions, e.g. one can prove from their results that:

**Theorem 4.24.** Let \( \Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3 \). Assume that \( \Gamma_1 \) has property \((T)\) and no unbounded linear representations. Then there are no irreducible, volume preserving \( \Gamma \) actions on compact manifolds. The same is true for \( \Gamma = \Gamma_1 \times \Gamma_2 \) if \( \Gamma_1 \) is as above and \( \Gamma_2 \) is solvable.

This motivates the following conjecture.

**Conjecture 4.25.** Let \( \Gamma = \Gamma_1 \times \Gamma_2 \). Assume that \( \Gamma_1 \) has property \((T)\) and no unbounded linear representations and that \( \Gamma_2 \) is amenable. Then there are no irreducible, volume preserving \( \Gamma \) actions on compact manifolds.

One might be tempted to prove this by showing that no compact group contains both a dense finitely generated amenable group and a dense Kazhdan group. This is however not true. The question was
raised by Lubotzky in [146] but has recently been resolved in the negative by Kassabov [124].

In the context of irreducible actions, asking that groups have property (T) is perhaps too strong. In fact, there are already relatively few known irreducible actions of $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$! If one element of $\mathbb{Z}^2$ acts as an Anosov diffeomorphism, then it is conjectured by Katok and Spatzier that all actions are algebraic [127]. Even in more general settings where one assumes non-uniform hyperbolicity, there are now hints that a classification might be possible, though clearly more complicated than in the Anosov case, see [122] and references there. In a sense that paper indicates that the “exotic examples” that might arise in this context may be no worse than those that arise for actions of higher rank lattices.

4.7.4. Torsion groups and Homeo($M$). A completely different and well studied aspect of the theory of transformation groups of compact manifolds is the study of finite subgroups of Homeo($M$). It has recently been noted by many authors that this study has applications to the study of “large subgroups” of Homeo($M$). Namely, one can produce many finitely generated and even finitely presented groups that have no non-trivial homomorphisms to Homeo($M$) for any compact $M$. This is discussed in e.g. [19, 82, 229]. As far as I know, this observation was first made by Ghys as a remark in the introduction to [99]. In all cases, the main trick is to construct infinite, finitely generated groups which contains infinitely many conjugacy classes of finite subgroups, usually just copies of $(\mathbb{Z}/p\mathbb{Z})^k$ for all $k > 1$. A new method of constructing such groups was recently introduced by Chatterji and Kassabov [37].

As far as the author knows, even if one fixes $M$ in advance, this is the only existing method for producing groups $\Gamma$ with no non-trivial (or even no infinite image) homomorphisms to Homeo($M$) unless $M = S^1$. For $M = S^1$ we refer back to subsection 4.1 and to [234].

5. Cocycle superrigidity and immediate consequences

5.1. Zimmer’s cocycle super-rigidity theorem and generalizations. A main impetus for studying rigidity of group actions on manifolds came from Zimmer’s theorem on superrigidity for cocycles. This theorem and its proof were strongly motivated by Margulis’ work. In fact, Margulis’ theorem reduces to the special case of Zimmer’s theorem for a certain cocycle $\alpha : G \times G/\Gamma \to \Gamma$. In order to avoid technicalities, we describe only a special case of this result, essentially avoiding boundedness and integrability assumptions on cocycles that are automatic fulfilled in any context arising from a continuous action on a compact
manifold. Let $M$ be a compact manifold, $H$ a matrix group and $P$ an $H$ bundle over $M$. For readers not familiar with bundle theory, the results are interesting even in the case where $P = M \times H$. Now let a group $\Gamma$ act on $P$ continuously by bundle automorphisms, i.e. such that there is a $\Gamma$ action on $M$ for which the projection from $P$ to $M$ is equivariant. Further assume that the action on $M$ is measure preserving and ergodic. The cocycle superrigidity theorem says that if $\Gamma$ is a lattice in a simply connected, semi-simple Lie group $G$ all of whose simple factors are noncompact and have property (T) then there is a measurable map $s : M \rightarrow H$, a representation $\pi : G \rightarrow H$, a compact subgroup $K < H$ which commutes with $\pi(G)$ and a measurable map $\Gamma \times M \rightarrow K$ such that

$$\gamma \cdot s(m) = k(m, \gamma)\pi(\gamma)s(\gamma \cdot m).$$

It is easy to check from this equation that the map $K$ satisfies the equation that makes it into a cocycle over the action of $\Gamma$. One should view $s$ as providing coordinates on $P$ in which the $\Gamma$ action is almost a product. For more discussion of this theorem, particularly in the case where all simple factors of $G$ have higher rank, the reader should see any of \cite{64, 65, 80, 94, 242}. (The version stated here is only proven in \cite{80}, previous proofs all yielded somewhat more complicated statements that require passing to finite ergodic extensions of the action.) For the case of $G$ with simple factors of the form $Sp(1, n)$ and $F_{4}^{\times}$, the results follows from work of the author and Hitchman \cite{79}, building on earlier results of Korevaar-Schoen and Corlette-Zimmer \cite{40, 135, 136, 137}.

As a sample application, let $M = \mathbb{T}^{n}$ and let $P$ be the frame bundle of $M$, i.e. the space of frames in the tangent bundle of $M$. Since $\mathbb{T}^{n}$ is parallelizable, we have $P = \mathbb{T}^{n} \times GL(n, \mathbb{R})$. The cocycle super-rigidity theorem then says that “up to compact noise” the derivative of any measure preserving $\Gamma$ action on $\mathbb{T}^{n}$ looks measurably like a constant linear map. In fact, the cocycle superrigidity theorems apply more generally to continuous actions on any principal bundle $P$ over $M$ with fiber $H$, an algebraic group, and in this context produces a measurable section $s : M \rightarrow P$ satisfying equation (1). So in fact, cocycle superrigidity implies that for any action preserving a finite measure on any manifold the derivative cocycle looks measurably like a constant cocycle, up to compact noise. That cocycle superrigidity provides information about actions of groups on manifolds through the derivative cocycle as was first observed by Furstenberg in \cite{94}. Zimmer originally proved cocycle superrigidity in order to study orbit equivalence of group actions. For recent surveys of subsequent developments concerning orbit
equivalence rigidity and other forms of superrigidity for cocycles, see \cite{88,190,209}.

5.2. First applications to group actions and the problem of regularity. The following result, first observed by Furstenberg, is an immediate consequence of cocycle superrigidity.

**Proposition 5.1.** Let $G$ be a semisimple Lie group with no compact factors and with property (T) of Kazhdan and let $\Gamma < G$ be a lattice. Assume $G$ or $\Gamma$ acts by volume preserving diffeomorphisms on a compact manifold $M$. Then there is a linear representation $\pi : G \to G L(\dim(M), \mathbb{R})$ such that the Lyapunov exponents of $g \in G$ or $\Gamma$ are exactly the absolute values of the eigenvalues of $\pi(g)$.

This Proposition is most striking to those familiar with classical dynamics, where the problem of estimating, let alone computing, Lyapunov exponents, is quite difficult.

Another immediate consequence of cocycle super-rigidity is the following:

**Proposition 5.2.** Let $G$ or $\Gamma$ be as above, acting by smooth diffeomorphisms on a compact manifold $M$ preserving volume. If every element of $G$ or $\Gamma$ acts with zero entropy, then there is a measurable invariant metric on $M$. In particular, if $G$ admits no non-trivial representations of $\dim(M)$, then there is a measurable invariant metric on $M$.

As mentioned above in subsection 4, this reduces Conjecture 4.14 to the following technical conjecture.

**Conjecture 5.3.** Let $G$ or $\Gamma$ acting on $M$ be as above. If $G$ or $\Gamma$ preserves a measurable Riemannian metric, then they preserve a smooth invariant Riemannian metric.

We recall that first evidence towards this conjecture is Theorem 4.15. We remark that in the proof of that theorem, it is not the case that the measurable metric from Proposition 5.2 is shown to be smooth, but instead it is shown that the image of $\Gamma$ in Diff$(M)$ lies in a compact subgroup. In other work, Zimmer explicitly improves regularity of the invariant metric, see particularly \cite{244}.

In general the problem that arises immediately from cocycle superrigidity in any context is understanding the regularity of the straightening section $\sigma$. This question has been studied from many points of view, but still relatively little is known in general. For certain examples of actions of higher rank lattices on compact manifolds, discussed below in section 9, the $\sigma$ that straightens the derivative cocycle cannot
be made smooth on all of $M$. It is possible that for volume preserving actions on manifolds and the derivative cocycle, $\sigma$ can always be chosen to be smooth on a dense open set of full measure.

We will return to theme of regularity of the straightening section in Section 7. First we turn to more geometric contexts in which the output of cocycle superrigidity is also often used but usually more indirectly.

6. Geometric actions and rigid geometric structures

In this section, we discuss the role of rigid geometric structures in the study of actions of large groups. The notion of rigid geometric structure was introduced by Gromov, partially in reaction to Zimmer’s work on large group actions.

The first subsection of this section recalls the definition of rigid geometric structure, gives some examples and explains the relation of Gromov’s rigid geometric structures to other notions introduced by Cartan. Subsection 6.2 recalls Gromov’s initial results relating actions of simple groups preserving rigid geometric structures on $M$ to representations of the fundamental group of $M$, and extensions of these results to lattice actions due to Zimmer and the author. The third section concerns a different topic, namely the rigidity of connection preserving actions of a lattice $\Gamma$, particularly on manifolds of dimension not much larger than $d(G)$ as defined in section 4. The fourth subsection recalls some obstructions to actions preserving geometric structures, particularly a result known as Zimmer’s geometric Borel density theorem and some recent related results of Bader, Frances and Melnick. We discuss actions preserving a complex structure in subsection 6.5 and end with a subsection on questions concerning geometric actions.

6.1. Rigid structures and structures of finite type. In this subsection we recall the formal definition of a rigid geometric structure. Since the definition is somewhat technical, some readers may prefer to skip it, and read on keeping in mind examples rather than the general notion. The basic examples of rigid geometric structures are Riemannian metrics, pseudo-Riemannian metrics, conformal structures defined by either type of metric and affine or projective connections. Basic examples of geometric structures which are not rigid are a volume form or symplectic structure. An intermediate type of structure which exhibits some rigidity but which is not literally rigid in the sense discussed here is a complex structure.

If $N$ is a manifold, we denote the $k$-th order frame bundle of $N$ by $F^k(N)$, and by $J^{s,k}(N)$ the bundle of $k$-jets at 0 of maps from $\mathbb{R}^s$ to $N$. If $N$ and $N'$ are two manifolds, and $f : N \to N'$ is a map between
them, then the $k$-jet $j^k(f)$ induces a map $J^{s,k}N \to J^{s,k}N'$ for all $s$. We let $D^k(N)$ be the bundle whose fiber $D^k_p$ at a point $p$ consists of the set of $k$-jets at $p$ of germs of diffeomorphisms of $N$ fixing $p$. We abbreviate $D^k_0(\mathbb{R}^n)$ by $D^k_0$ or simply $D^k$; this is a real algebraic group. For concreteness, one can represent each element uniquely, in terms of standard coordinates $(\xi_1, \ldots, \xi_n)$ on $\mathbb{R}^n$, in the form

$$(P_1(\xi_1, \ldots, \xi_n), \ldots, P_n(\xi_1, \ldots, \xi_n))$$

where $P_1, P_2, \ldots, P_n$ are polynomials of degree $\leq k$. We denote the vector space of such polynomial maps of degree $\leq k$ by $P_{n,k}$.

The group $D^k_n$ has a natural action on $F^k(N)$, where $n$ is the dimension of $N$. Suppose we are given an algebraic action of $D^k_n$ on a smooth algebraic variety $Z$. Then following Gromov (109), we make the following definition:

**Definition 6.1.**

1. An $A$-structure on $N$ (of order $k$, of type $Z$) is a smooth map $\phi : F^k(N) \to Z$ equivariant for the $D^k_n$ actions.

2. With notation as above, the $r$-th prolongation of $\phi$, denoted $\phi^r$, is the map $\phi^r : F^{k+r}(N) \to J^{n,r}(Z)$ defined by $\phi^r = j^r(\phi) \circ i^{r+k}_k$ where $i^{r+k}_k : F^{k+r}(N) \to J^n(F^k(N))$ is the natural inclusion and $j^k(h) : J^{n,r}(F^k(N)) \to J^{n,r}(Z)$ is as before; this is an $A$-structure of type $J^{n,r}(Z)$ and order $k + r$.

Equivalently, an $A$-structure of type $Z$ and order $k$ is a smooth section of the associated bundle $F^k(N) \times_{D^k} Z$ over $N$. Note that an $A$-structure on $N$ defines by restriction an $A$-structure $\phi|_U$ on any open set $U \subset N$.

**Remark 6.2.** $A$-structures were introduced in [109]; a good introduction to the subject, with many examples, can be found in [9]. A comprehensive and accessible discussion the results of [109] concerning actions of simple Lie groups can be found in [66].

Note that if $N$ and $N'$ are $n$-manifolds, and $h : N \to N'$ is a diffeomorphism, then $h$ induces a bundle map $j^k(h) : F^k(N) \to F^k(N')$.

**Definition 6.3.**

1. If $\phi : F^k(N) \to Z$, $\phi' : F^k(N') \to Z$ are $A$-structures, a diffeomorphism $h : N \to N'$ is an isometry from $\phi$ to $\phi'$ if $\phi' \circ j^k(h) = \phi$.

2. A local isometry of $\phi$ is a diffeomorphism $h : U_1 \to U_2$, for open sets $U_1, U_2 \subset N$, which is an isometry from $\phi|_{U_1}$ to $\phi|_{U_2}$.

For $p \in M$ denote by $Is^l_p(\phi)$ the pseudogroup of local isometries of $\phi$ fixing $p$, and, for $l \geq k$, we denote by $Is^l_p(\phi)$ the set of elements $j^l_p(h) \in D^l_p$ such that $j^l_p(\phi \circ j^k_p(h)) = \phi^l - k$, where both sides are considered as
maps $F^{k+l}(N) \to J^{l-k}(Z)$. Note that $Is_p^l(\phi)$ is a group, and there is a natural homomorphism $r_p^{l+m} : Is_p^l(\phi) \to Is_p^m(\phi)$ for $m < l$; in general, it is neither injective nor surjective.

**Definition 6.4.** The structure $\phi$ is called $k$-rigid if for every point $p$, the map $r_p^{k+1:k}$ is injective.

A first remark worth making is that an affine connection is a rigid geometric structure and that any generalized quasi-affine action on a manifold of the form $K\setminus H/\Lambda \times \mathcal{M}$ with $\Lambda$ discrete preserves an affine connection and therefore also a torsion free affine connection. In order to provide some examples, we recall the following lemma of Gromov.

**Lemma 6.5.** Let $V$ be an algebraic variety and $G$ a group acting algebraically on $V$. For every $k$, there is a tautological $G$ invariant geometric structure of order $k$ on $V$, given by $\omega : P^k(V) \to P^k(V)/G$. This structure is rigid if and only if the action of $G$ on $P^k(V)$ is free and proper.

The conclusion in the first sentence is obvious. The second sentence is proven in section 0.4, pages 69-70, of [109].

**Examples**

1. The action of $G = SL_n(\mathbb{R})$ on $\mathbb{R}^n$ is algebraic. So is the action of $G$ on the manifold $N_1$ obtained by blowing up the origin. The reader can easily verify that the action of $G$ on $P^2(N_1)$ is free and proper.

2. We can compactify $N_1$ by $N_2$ by viewing the complement of the blow up as a subset of the projective space $P^n$. Another description of the same action, which may make the rigid structure more visible to the naked eye, is as follows. $SL_{n+1}(\mathbb{R})$ acts on $P^n$. Let $G$ be $SL_n(\mathbb{R}) < SL_{n+1}(\mathbb{R})$ as block diagonal matrices with blocks of size $n$ and 1 and $1 \times 1$ block equal to 1. Then $G$ acts on $P^n$ fixing a point $p$. We can obtain $N_2$ by blowing up the fixed point $p$. The $G$ actions on both $P^n$ and $N_2$ are algebraic and again the reader can verify that the action is free and proper on $P^2(N_2)$.

3. In the construction from 2 above, there is an action of a group $H$ where $H = SL_n(\mathbb{R}) \times \mathbb{R}^n$ and $G = SL_n(\mathbb{R}) < SL_{n+1}(\mathbb{R}) < SL_{n+1}(\mathbb{R})$. The $H$ action fixes the point $p$ and so also acts on $N_2$ algebraically. However, over the exceptional divisor, the action is never free on any frame bundle, since the subgroup $\mathbb{R}^n$ acts trivially to all orders at the exceptional divisor.
The behavior in example 3 above illustrates the fact that existence of invariant rigid structures is more complicated for algebraic groups which are not semisimple, see the discussion in section 0.4.C. of [109].

**Definition 6.6.** A rigid geometric structure is called a finite type geometric structure if $V$ is a homogeneous $D_k^k$ space.

This is by no means the original definition which is due to Cartan and predates Gromov’s notion of a rigid geometric structure by several decades. It is equivalent to Cartan’s definition of a finite type structure by work of Candel and Quiroga [30, 31, 194]. Candel and Quiroga also give a development of rigid geometric structures that more closely parallels the older notion of a structure of finite type as presented in e.g. [132].

All standard examples of rigid geometric structures are structures of finite type. However, example (2) above is not a structure of finite type. The notion of structure of finite type was first given in terms of prolongations of Lie algebras and so yields a criterion that is, in principle, computable. The work of Candel and Quiroga extends this computable nature to general rigid geometric structures.

### 6.2. Rigid structures and representations of fundamental groups.

In this subsection, we restrict our attention to actions of simple Lie groups $G$ and lattices $\Gamma < G$. Many of the results of this section extend to semisimple $G$, though the formulations become more complicated.

A major impetus for Gromov’s introduction of rigid geometric structures is the following theorem from [109].

**Theorem 6.7.** Let $G$ be a simple noncompact Lie group and $M$ a compact real analytic manifold. Assume $G$ acts on $M$ preserving an analytic rigid geometric structure $\omega$ and a volume form $\nu$. Further assume the action is ergodic. Then there is a linear representation $\rho : \pi_1(M) \rightarrow GL(n, \mathbb{R})$ such that the Zariski closure of $\rho(\pi_1(M))$ contains a group locally isomorphic to $G$.

We remark that, by the Tits’ alternative, the theorem immediately implies that actions of the type described cannot occur on manifolds with amenable fundamental group or even on manifolds whose fundamental groups do not contain a free group on two generators [223]. Expository accounts of the proof can be found in [66, 252, 255]. The representation $\rho$ is actually defined on Killing fields of the lift $\tilde{\omega}$ of $\omega$ to $\tilde{M}$.

This is a worthwhile moment to indicate the weakness of Lemma 2.6. If we start with a cocompact lattice $\Gamma < G$ and an action of $\Gamma$
on a compact manifold $M$ satisfying the assumptions of Theorem 6.7, we can induce the action to a $G$ action on the compact manifold $N = (G \times M)/\Gamma$. However, in this setting, there is an obvious representation of $\pi_1(N)$ satisfying the conclusion of Theorem 6.7. This is because there is a surjection $\pi_1(N) \to \Gamma$. In fact for most cocompact lattices $\Gamma < G$ there are homomorphisms $\sigma : \Gamma \to K$ where $K$ is compact and simply connected so that the we can have $\Gamma$ act on $K$ by left translation and obtain examples where $\pi_1(N) = \Gamma$. For $G$ of higher rank, the following theorem of Zimmer and the author shows that this is the only obstruction to a variant of Theorem 6.7 for lattices.

**Theorem 6.8.** Let $\Gamma < G$ be a lattice, where $G$ is a simple group and $\mathbb{R} - \text{rank}(G) \geq 2$. Suppose $\Gamma$ acts analytically and ergodically on a compact manifold $M$ preserving a unimodular rigid geometric structure. Then either

1. the action is isometric and $M = K/C$ where $K$ is a compact Lie group and the action is by right translation via $\rho : \Gamma \to K$, a dense image homomorphism, or

2. there exists an infinite image linear representation $\sigma : \pi_1(M) \to GL_n\mathbb{R}$, such that the algebraic automorphism group of the Zariski closure of $\sigma(\pi_1(M))$ contains a group locally isomorphic to $G$.

The proof of this theorem makes fundamental use of a notion that does not occur in it’s statement: the entropy of the action. A key observation is the following formula for the entropy of the restriction of the induced action to $\Gamma$:

$$h_{(G \times M)/\Gamma}(\gamma) = h_{G/\Gamma}(\gamma) + h_M(\gamma)$$

We then use the fact that the last term on the right hand side is zero if and only if the action preserves a measurable Riemannian metric as already explained in Proposition 5.1. In this setting, it then follows from Theorem 4.15 that the action is isometric and the other conclusions in (1) are simple consequences of ergodicity of the action.

When $h_M(\gamma) > 0$ for some $\Gamma$, we use relations between the entropy of the action and the Gromov representation discovered by Zimmer [254]. We discuss some of these ideas below in Section 8.

6.3. **Affine actions close to the critical dimension.** In this section we briefly describe a direction pursued by Feres, Goetze, Zeghib and Zimmer that classify connection preserving actions of lattices in dimensions close to, but not less than, the critical dimension $d$. A representative result is the following:
Theorem 6.9. Let \( n > 2 \), \( G = SL(n, \mathbb{R}) \) and \( \Gamma < G \) a lattice. Let \( M \) be a compact manifold with \( \dim(M) \leq n + 1 = d(G) + 1 \). Assume that \( \Gamma \) acts on \( M \) preserving a volume form and an affine connection. Then either

1. \( \dim(M) < n \) and the action is isometric
2. \( \dim(M) = n \) and either the action is isometric or, upon passing to finite covers and finite index subgroups, smoothly conjugate to the standard \( SL(n, \mathbb{Z}) \) action on \( \mathbb{T}^n \)
3. \( \dim(M) = n + 1 \) and either the action is isometric or, upon passing to finite covers and finite index subgroups, smoothly conjugate to the action of \( SL(n, \mathbb{Z}) \) on \( \mathbb{T}^{n+1} = \mathbb{T}^n \times \mathbb{T} \) where the action on the second factor is trivial.

Variants of this theorem under more restrictive hypotheses were obtained by Feres, Goetze and Zimmer [62, 103, 245]. The theorem as stated is due to Zeghib [240]. The cocycle superrigidity theorem is used in all of the proofs, mainly to force vanishing of curvature and torsion tensors of the associated connection.

One would expect similar results for a more general class of acting groups and also for a wider range of dimensions. Namely if we let \( d_2(G) \) be the dimension of the second non-trivial representation of \( G \), we would expect a similar result to (3) for any affine, volume preserving action on a manifold \( M \) with \( d(G) < d_2(G) \). With the further assumption that the connection is Riemannian (and still only for lattices in \( SL(n, \mathbb{R}) \)) this is proven by Zeghib in [239].

Further related results are contained in other papers of Feres [59, 63]. No results of this kind are known for \( \dim(M) \geq d_2(G) \). A major difficulty arises as soon as one has \( \dim(M) \geq d_2(G) \), namely that more complicated examples can arise, including affine actions on nilmanifolds and left translation actions on spaces of the form \( G/\Lambda \). All the results mentioned in this subsection depend on the particular fact that flat Riemannian manifolds have finite covers which are tori.

6.4. Zimmer’s Borel density theorem and generalizations. An obvious first question concerning \( G \) actions is the structure of the stabilizer subgroups. In this direction we have:

Theorem 6.10. Let \( G \) be a simple Lie group acting essentially faithfully on a compact manifold \( M \) preserving a volume form, then the stabilizer of almost every point is discrete.

As an application of this result, one can prove the following:
**Theorem 6.11.** Let $G$ be a simple Lie group acting essentially faithfully on a compact manifold, preserving a volume form and a homogeneous geometric structure with structure group $H$. Then there is an inclusion $\mathfrak{g} \to \mathfrak{h}$.

In particular, the theorem provides an obstruction to a higher rank simple Lie group having a volume preserving action on a compact Lorentz manifold. For more discussion of these theorems, see [255]. This observation lead to a major series of works studying automorphism groups of Lorentz manifolds, see e.g. [1, 138, 237, 238]. We remark here that Theorem 6.11 does not require that the homogeneous structure be rigid.

More recently some closely related phenomena have been discovered in a joint work of Bader, Frances and Melnick [224]. Their work uses yet another notion of a geometric structure, that of a Cartan geometry. We will not define this notion rigorously here, it suffices to note that any rigid homogeneous geometric structure defines a Cartan geometry, as discussed in [224, Introduction]. The converse is not completely clear in general, but most classical examples of rigid geometric structures can also be realized as Cartan geometries. A Cartan geometry is essentially a way of saying that a manifold is infinitesimally modeled on some homogeneous space $G/P$. To recapture the notion of a Riemannian connection, $G = O(n) \ltimes \mathbb{R}^n$ and $P = O(n)$, to recapture the notion of an affine connection $G = Gl(n, \mathbb{R}) \ltimes \mathbb{R}^n$ and $P = Gl(n, \mathbb{R})$. Cartan connections come with naturally defined curvatures which vanish if and only if the manifold is locally modeled on $G/P$. We refer the reader to the book [211] for a general discussion of Cartan geometries and their use in differential geometry.

Given a connected linear group $L$, we define its real rank, $rk(L)$ as before to be the dimension of a maximal $\mathbb{R}$-diagonalizable subgroup of $L$, and let $n(L)$ denote the maximal nilpotence degree of a connected nilpotent subgroup. One of the main results of [224] is:

**Theorem 6.12.** If a group $L$ acts by automorphisms of a compact Cartan geometry modeled on $G/P$, then

1. $rk(AdL^0) \leq rk(AdP)$
2. $n(AdL^0) \leq n(AdP)$

The main point is that $L$ is not assumed either simple or connected. This theorem is deduced from an embedding theorem similar in flavor to Theorem 6.10.

In addition, when $G$ is a simple group and $P$ is a maximal parabolic subgroup, Bader, Frances and Melnick prove rigidity results classifying
all possible actions when the rank bound in Theorem 6.12 is achieved. This classification essentially says that all examples are algebraic, see [224] for detailed discussion.

An earlier paper by Feres and Lampe also explored applications of Cartan geometries to rigidity and dynamical conditions for flatness of Cartan geometries [61].

6.5. Actions preserving a complex structure. We mention here one recent result by Cantat and a few active related directions of research. In [247], Zimmer asked whether one had restrictions on low dimensional holomorphic actions of lattices. The answer was obtained in [35] and is:

Theorem 6.13. Let $G$ be a connected simple Lie group of real rank at least 2 and suppose $\Gamma < G$ is a lattice. If $\Gamma$ admits an action by automorphisms on a compact Kähler manifold $M$, then the rank of $G$ is at least the complex dimension of $M$.

The proof of the theorem depends primarily on results concerning $\text{Aut}(M)$, particularly recent results on holomorphic actions of abelian groups by Dinh and Sibony [46]. It is worth noting that while the theorem does depend on Margulis’ superrigidity theorem, it does not depend on Zimmer’s cocycle superrigidity theorem.

More recently Cantat, Deserti and others have begun a program of studying large subgroups of automorphism groups of complex manifolds, see e.g. [34, 45]. In particular, Cantat has proven an analogue of conjecture 4.7 in the context of birational actions on complex surfaces.

Theorem 6.14. Let $S$ be a compact Kähler surface and $G$ an infinite, countable group of birational transformations of $S$. If $G$ has property $(T)$, then there is a birational map $j: S \to \mathbb{P}^2(\mathbb{C})$ which conjugates $G$ to a subgroup of $\text{Aut}(\mathbb{P}^2(\mathbb{C}))$.

6.6. Conjectures and questions. A major open problem, generalizing Conjecture 1.2 is the following:

Conjecture 6.15 (Gromov-Zimmer). Let $D$ be a semisimple Lie group with all factors having property $(T)$ or a lattice in such a Lie group. Then any $D$ action on a compact manifold preserving a rigid geometric structure and a volume form is generalized quasi-affine.

A word is required on the attribution. Both Gromov and Zimmer made various less precise conjectures concerning the classification of actions as in the conjecture [109, 246]. The exact statement of the correct conjecture was muddy for several years while it was not known
if the Katok-Lewis and Benveniste type examples admitted invariant rigid geometric structures. See Section 9 for more discussion of these examples. In the context of the results of [11], where Benveniste and the author prove that those actions do not preserve rigid geometric structures, this version of the classification seems quite plausible. It is perhaps more plausible if one assumes the action is ergodic or that the geometric structure is homogeneous.

A possibly easier question that is relevant is the following:

**Question 6.16.** Let $M$ be a compact manifold equipped with a homogeneous rigid geometric structure $\omega$. Assume $\text{Aut}(M, \omega)$ is ergodic or has a dense orbit. Is $M$ locally homogeneous?

Gromov’s theorem on the open-dense implies that a dense open set in $M$ is locally homogeneous even if $\omega$ is not homogeneous. The question is whether this homogeneous structure extends to all of $M$ if $\omega$ is homogeneous. If $\omega$ is not homogeneous, the examples following Lemma 6.5 show that $M$ need not be homogeneous.

It seems possible to approach the case of Conjecture 6.15 where $\omega$ is homogeneous and $D$ acts ergodically by answering Question 6.16 positively. At this point, one is left with the problem of classifying actions of higher rank groups and lattices on locally homogeneous manifolds. Induction easily reduces one to considering $G$ actions. The techniques Gromov uses to produce the locally homogeneous structure gives slightly more precise information in this setting: one is left trying to classify homogeneous manifolds modeled on $H/L$ where $G$ acts via an inclusion in $H$ centralizing $L$. In fact this reduces one to problems about locally homogeneous manifolds studied by Zimmer and collaborators, see particularly [139, 140, 142, 253]. For a more general survey of locally homogeneous spaces, see also [133] and [134].

The following question concerns a possible connection between preserving a rigid geometric structure and having uniformly partially hyperbolic dynamics.

**Question 6.17.** Let $D$ be any group acting on a compact manifold $M$ preserving a volume form and a rigid geometric structure. Is it true that if some element $d$ of $D$ has positive Lyapunov exponents, then $d$ is uniformly partially hyperbolic?

This question is of particular interest for us when $D$ is a semisimple Lie group or a lattice, but would be interesting to resolve in general. The main reason to believe that the answer might be yes is that the action of $D$ on the space of frames of $M$ is proper. Having a proper
action on tangent bundle minus the zero section implies that an action is Anosov, see Mañé’s article for a proof [153].

7. Topological super-rigidity: regularity results from hyperbolic dynamics and applications

A major area of research in the Zimmer program has been the application of hyperbolic dynamics. This area might be described by the maxim: in the presence of hyperbolic dynamics, the straightening section from cocycle superrigidity is often more regular. Some major successes in this direction are the work of Goetze-Spatzier and Feres-Labourie. In the context of hyperbolic dynamical approaches, there are two main settings. In the first one considers actions of higher rank lattices on a particular class of compact manifolds and a second in which one makes no assumption on the topology of the manifold acted upon.

In this section we discuss a few such results, after first recalling some facts about the stability of hyperbolic dynamical systems in subsection 7.1. We then discuss best known rigidity results for actions on tori in subsection 7.2 and after this discuss work of Goetze-Spatzier and Feres-Labourie in the more general context in subsection 7.3.

The term “topological superrigidity” for this area of research was coined by Zimmer, whose early unpublished notes on the topic dramatically influenced research in the area [241].

This aspect of the Zimmer program has also given rise to a study of rigidity properties for other uniformly hyperbolic actions of large groups, most particularly higher rank abelian groups. See e.g [123, 127, 203].

The following subsection recalls basic notions from hyperbolic dynamics that are needed in this section.

7.1. Stability in hyperbolic dynamics. A diffeomorphism $f$ of a manifold $X$ is said to be Anosov if there exists a continuous $f$ invariant splitting of the tangent bundle $TX = E^u_f \oplus E^s_f$ and constants $a > 1$ and $C, C' > 0$ such that for every $x \in X$,

1. $\|Df^n(v^u)\| \geq Ca^n\|v^u\|$ for all $v^u \in E^u_f(x)$ and,
2. $\|Df^n(v^s)\| \leq C'a^{-n}\|v^s\|$ for all $v^s \in E^s_f(x)$.

We note that the constants $C$ and $C'$ depend on the choice of metric, and that a metric can always be chosen so that $C = C' = 1$. There is an analogous notion for a flow $f_t$, where $TX = TO \oplus E^u_{ft} \oplus E^s_{ft}$, where $TO$ is the tangent space to the flow direction and vectors in $E^u_{ft}$ (resp. $E^s_{ft}$) are uniformly expanded (resp. uniformly contracted) by the flow. This notion was introduced by Anosov and named after Anosov by
Smale, who popularized the notion in the United States \[3, 215\]. One of the earliest results in the subject is Anosov’s proof that Anosov diffeomorphisms are *structurally stable*, i.e. that any $C^1$ perturbation of an Anosov diffeomorphism is conjugate back to the original diffeomorphism by a homeomorphism. There is an analogous result for flows, though this requires that one introduce a notion of time change that we will not consider here. Since Anosov also showed that $C^2$ volume preserving Anosov flows and diffeomorphisms are ergodic, structural stability implies that the existence of an open set of “chaotic” dynamical systems.

The notion of an Anosov diffeomorphism has had many interesting generalizations, for example: Axiom A diffeomorphisms, non-uniformly hyperbolic diffeomorphisms, and diffeomorphisms admitting a dominated splitting. A notion that has been particularly useful in the study of rigidity of group actions is the notion of a partially hyperbolic diffeomorphism as introduced by Hirsch, Pugh and Shub. Under strong enough hypotheses, these diffeomorphisms have a weaker stability property similar to structural stability. More or less, the diffeomorphisms are hyperbolic relative to some foliation, and any nearby action is hyperbolic to some nearby foliation. To describe more precisely the class of diffeomorphisms we consider and the stability property they enjoy, we require some definitions.

The use of the word *foliation* varies with context. Here a *foliation by $C^k$ leaves* will be a continuous foliation whose leaves are $C^k$ injectively immersed submanifolds that vary continuously in the $C^k$ topology in the transverse direction. To specify transverse regularity we will say that a foliation is transversely $C^r$. A foliation by $C^k$ leaves which is transversely $C^k$ is called simply a $C^k$ foliation. (Note our language does not agree with that in the reference \[115\].)

Given an automorphism $f$ of a vector bundle $E \to X$ and constants $a > b \geq 1$, we say $f$ is \((a, b)\)-*partially hyperbolic* or simply *partially hyperbolic* if there is a metric on $E$, a constant and $C \geq 1$ and a continuous $f$ invariant, non-trivial splitting $E = E^u_f \oplus E^c_f \oplus E^s_f$ such that for every $x$ in $X$:

\begin{enumerate}
  \item $\|f^n(v^u)\| \geq Ca^n\|v^u\|$ for all $v^u \in E^u_f(x)$,
  \item $\|f^n(v^s)\| \leq C^{-1}a^{-n}\|v^s\|$ for all $v^s \in E^s_f(x)$ and
  \item $C^{-1}b^{-n}\|v^0\| < \|f^n(v^0)\| \leqCb^0\|v^0\|$ for all $v^0 \in E^c_f(x)$ and all integers $n$.
\end{enumerate}

A $C^1$ diffeomorphism $f$ of a manifold $X$ is \((a, b)\)-*partially hyperbolic* if the derivative action $Df$ is \((a, b)\)-partially hyperbolic on $TX$. We remark that for any partially hyperbolic diffeomorphism, there always
exists an adapted metric for which $C = 1$. Note that $E^c_f$ is called the central distribution of $f$, $E^u_f$ is called the unstable distribution of $f$ and $E^s_f$ the stable distribution of $f$.

Integrability of various distributions for partially hyperbolic dynamical systems is the subject of much research. The stable and unstable distributions are always tangent to invariant foliations which we call the stable and unstable foliations and denote by $W^s_f$ and $W^u_f$. If the central distribution is tangent to an $f$ invariant foliation, we call that foliation a central foliation and denote it by $W^c_f$. If there is a unique foliation tangent to the central distribution we call the central distribution uniquely integrable. For smooth distributions unique integrability is a consequence of integrability, but the central distribution is usually not smooth. If the central distribution of an $(a,b)$-partially hyperbolic diffeomorphism $f$ is tangent to an invariant foliation $W^c_f$, then we say $f$ is $r$-normally hyperbolic to $W^c_f$ for any $r$ such that $a > b^r$. This is a special case of the definition of $r$-normally hyperbolic given in [115].

Before stating a version of one of the main results of [115], we need one more definition. Given a group $G$, a manifold $X$, two foliations $\mathcal{F}$ and $\mathcal{F}'$ of $X$, and two actions $\rho$ and $\rho'$ of $G$ on $X$, such that $\rho$ preserves $\mathcal{F}$ and $\rho'$ preserves $\mathcal{F}'$, following [115] we call $\rho$ and $\rho'$ leaf conjugate if there is a homeomorphism $h$ of $X$ such that:

1. $h(\mathcal{F}) = \mathcal{F}'$ and
2. for every leaf $\mathcal{L}$ of $\mathcal{F}$ and every $g \in G$, we have $h(\rho(g)\mathcal{L}) = \rho'(g)h(\mathcal{L})$.

The map $h$ is then referred to as a leaf conjugacy between $(X, \mathcal{F}, \rho)$ and $(X, \mathcal{F}', \rho')$. This essentially means that the actions are conjugate modulo the central foliations.

We state a special case of some the results of Hirsch-Pugh-Shub on perturbations of partially hyperbolic actions of $\mathbb{Z}$, see [115]. There are also analogous definitions and results for flows. As these are less important in the study of rigidity, we do not discuss them here.

**Theorem 7.1.** Let $f$ be an $(a,b)$-partially hyperbolic $C^k$ diffeomorphism of a compact manifold $M$ which is $k$-normally hyperbolic to a $C^k$ central foliation $W^c_f$. Then for any $\delta > 0$, if $f'$ is a $C^k$ diffeomorphism of $M$ which is sufficiently $C^1$ close to $f$ we have the following:

1. $f'$ is $(a',b')$-partially hyperbolic, where $|a - a'| < \delta$ and $|b - b'| < \delta$, and the splitting $TM = E^u_{f'} \oplus E^c_{f'} \oplus E^s_{f'}$, for $f'$ is $C^0$ close to the splitting for $f$;
(2) there exist \( f' \) invariant foliations by \( C^k \) leaves \( \mathcal{W}_{f'}^s \) tangent to \( E_{f'}^s \), which is close in the natural topology on foliations by \( C^k \) leaves to \( \mathcal{W}_f^s \),

(3) there exists a (non-unique) homeomorphism \( h \) of \( M \) with \( h(\mathcal{W}^c_f) = \mathcal{W}^c_{f'} \), and \( h \) is \( C^k \) along leaves of \( \mathcal{W}^c_f \), furthermore \( h \) can be chosen to be \( C^0 \) small and \( C^k \) small along leaves of \( \mathcal{W}^c_f \).

(4) the homeomorphism \( h \) is a leaf conjugacy between the actions \((M, \mathcal{W}^c_f, f)\) and \((M, \mathcal{W}^c_{f'}, f')\).

Conclusion (1) is easy and probably older than [115]. One motivation for Theorem 7.1 is to study stability of dynamical properties of partially hyperbolic diffeomorphisms. See the survey, [26], by Burns, Pugh, Shub and Wilkinson for more discussion of that and related issues.

7.2. Uniformly hyperbolic actions on tori. Many works have been written considering local rigidity of actions with some affine, quasi-affine and generalized quasi-affine actions with hyperbolic behavior. For a discussion of this, we refer to [76]. Here we only discuss results which prove some sort of global rigidity of groups acting on manifolds. The first such results were contained in papers of Katok-Lewis and Katok-Lewis-Zimmer [125, 126]. As these are now special cases of later more general results, we do not discuss them in detail here.

In this section we discuss results which only provide continuous conjugacies to standard actions. This is primarily because these results are less technical and easier to state. In this context, one can improve regularity of the conjugacy given certain technical dynamical hypotheses on certain dynamical foliations.

**Definition 7.2.** An action of a group \( \Gamma \) on a manifold \( M \) is weakly hyperbolic if there exist elements \( \gamma_1, \ldots, \gamma_k \) each of which is partially hyperbolic such that the sum of the stable sub-bundles of the \( \gamma_i \) spans the tangent bundle to \( M \) at every point, i.e. \( \sum_i E_{\gamma_i} = TM \).

To discuss the relevant results we need a related topological notion that captures hyperbolicity at the level of fundamental group. This was introduced in [83] by Whyte and the author. If a group \( \Gamma \) acts on a manifold with torsion free nilpotent fundamental group and the action lifts to the universal cover, then the action of \( \Gamma \) on \( \pi_1(M) \) gives rise to an action of \( \Gamma \) on the Malcev completion \( N \) of \( \pi_1(M) \) which is a nilpotent Lie group. This yields a representation of \( \Gamma \) on the Lie algebra \( n \).

**Definition 7.3.** We say an action of \( \Gamma \) on a manifold \( M \) with nilpotent fundamental group is \( \pi_1 \)-hyperbolic if for the resulting \( \Gamma \) representation
on \( n \), we have finitely many elements \( \gamma_1, \ldots, \gamma_k \) such that the sum of their eigenspaces with eigenvalue of modulus less than one is all of \( n \).

One can make this definition more general by considering \( M \) where \( \pi_1(M) \) has a \( \Gamma \) equivariant nilpotent quotient, see [83]. We now discuss results that follow by combining work of Margulis-Qian with later work of Schmidt and Fisher-Hitchman [78, 162, 205].

**Theorem 7.4.** Let \( M = N/\Lambda \) be a compact nilmanifold and let \( \Gamma \) be a lattice in a semisimple Lie group with property (T). Assume \( \Gamma \) acts on \( M \) such that the action lifts to the universal cover and is \( \pi_1(M) \) hyperbolic, then the action is continuously semi-conjugate to an affine action.

This theorem was proven by Margulis and Qian, who noted that if the \( \Gamma \) action contained an Anosov element, then the conjugacy could be taken to be a homeomorphism. In [83], the author and Whyte point out that this theorem extends easily to the case of any compact manifold with torsion free nilpotent fundamental group. In [83], we also discuss extensions to manifolds with fundamental group with a quotient which is nilpotent.

Margulis and Qian asked whether the assumption of \( \pi_1 \) hyperbolicity could be replaced by the assumption that the action on \( M \) was weakly hyperbolic. In the case of actions of Kazhdan groups on tori, Schmidt proved that weak hyperbolicity implies \( \pi_1 \) hyperbolicity, yielding:

**Theorem 7.5.** Let \( M = \mathbb{T}^n \) be a compact torus and let \( \Gamma \) be a lattice in a semisimple Lie group with property (T). Any weakly hyperbolic \( \Gamma \) action \( M \) and that lifts to the universal cover is continuously semi-conjugate to an affine action.

**Remarks:**

1. The contribution of Fisher-Hitchman in both theorems is just in extending cocycle superrigidity to a wider class of groups, as discussed above.

2. The assumption that the action lifts to the universal cover of \( M \) is often vacuous because of results concerning cohomology of higher rank lattices. In particular, it is vacuous for cocompact lattices in simple Lie groups of real rank at least 3.

It remains an interesting, open question to take this result and prove that the semiconjugacy is always a conjugacy and is also always a smooth diffeomorphism.

7.3. **Rigidity results for uniformly hyperbolic actions.** We begin by discussing some work of Goetze and Spatzier. To avoid technicalities
we only discuss some of their results. We begin with the following definition.

**Definition 7.6.** Let $\rho : \mathbb{Z}^k \times M \to M$ be an action and $\gamma_1, \ldots, \gamma_l$ be a collection of elements which generate for $\mathbb{Z}^k$. We call $\rho$ a Cartan action if

1. each $\rho(\gamma_i)$ is an Anosov diffeomorphism,
2. each $\rho(\gamma_i)$ has one dimensional strongest stable foliation,
3. the strongest stable foliations of the $\rho(\gamma_i)$ are pairwise transverse and span the tangent space to the manifold.

It is worth noting that Cartan actions are very special in three ways. First we assume that a large number of elements in the acting group are Anosov diffeomorphisms, second we assume that each of these has one dimensional strongest stable foliation and lastly we assume these one dimensional directions span the tangent space. All aspects of these assumptions are used in the following theorem. Reproving it even assuming two dimensional strongest stable foliations would require new ideas.

**Theorem 7.7.** Let $G$ be a semisimple Lie group with all simple factors of real rank at least two and $\Gamma$ in $G$ a lattice. Then any volume preserving Cartan action of $\Gamma$ is smoothly conjugate to an affine action on an infranilmanifold.

This is slightly different than the statement in Goetze-Spatzier, where they pass to a finite cover and a finite index subgroup. It is not too hard to prove this statement from theirs. The proof spans the two papers [104] and [105]. The first paper [104] proves, in a somewhat more general context, that the $\pi$-simple section arising in the cocycle superrigidity theorem is in fact Hölder continuous. The second paper makes use of the resulting Hölder Riemannian metric in conjunction with ideas arising in other work of Katok and Spatzier to produce a smooth homogeneous structure on the manifold.

The work of Feres-Labourie differs from other work on rigidity of actions with hyperbolic properties in that does not make any assumptions concerning existence of invariant measures. Here we state only some consequences of their results, without giving the exact form of cocycle superrigidity that is their main result.

**Theorem 7.8.** Let $\Gamma$ be a lattice in $SL(n, \mathbb{R})$ for $n \geq 3$ and assume $\Gamma$ acts smoothly on a compact manifold $M$ of dimension $n$. Further assume that for the induced action $N = (G \times M)/\Gamma$ we have

1. every $\mathbb{R}$-semisimple 1-parameter subgroup of $G$ acts transitively on $N$ and
(2) some element $g$ in $G$ is uniformly partially hyperbolic with $E_s \oplus E_w$ containing the tangent space to $M$ at any point, then $M$ is a torus and the action on $M$ is a standard affine action.

These hypotheses are somewhat technical and essentially ensure that one can apply the topological version of cocycle superrigidity proven in [60]. The proof also uses a deep result of Benoist and Labourie classifying Anosov diffeomorphisms with smooth stable and unstable foliations [10].

The nature of the hypotheses of Theorem 7.8 make an earlier remark clear. Ideally one would only have hypotheses on the $\Gamma$ action, but here we require hypotheses on the induced action instead. It is not clear how to reformulate these hypotheses on the induced action as hypotheses on the original $\Gamma$ action.

Another consequence of the work of Feres andLabourie is a criterion for promoting invariance of rigid geometric structures. More precisely they give a criterion for a $G$ action to preserve a rigid geometric structure on a dense open subset of a manifold $M$ provided a certain type of subgroup preserves a rigid geometric structure on $M$.

7.4. Conjectures and questions on uniformly hyperbolic actions. We begin with a very general variant of Conjecture 1.3.

**Conjecture 7.9.** Let $G$ be a semisimple Lie group all of whose simple factors have property $(T)$, let $\Gamma < G$ be a lattice. Assume $G$ or $\Gamma$ acts smoothly on a compact manifold $M$ preserving volume such that some element $g$ in the acting group is non-trivially uniformly partially hyperbolic. Then the action is generalized quasi-affine.

There are several weaker variants on this conjecture, where e.g. one assumes the action is volume weakly hyperbolic. Even the following much weaker variant seems difficult:

**Conjecture 7.10.** Let $M = \mathbb{T}^n$ and $\Gamma$ as in Conjecture 7.9. Assume $\Gamma$ acts on $\mathbb{T}^n$ weakly hyperbolicly and preserving a smooth measure. Then the action is affine.

This conjecture amounts to conjecturing that the semiconjugacy in Theorem 7.5 is a diffeomorphism. In the special case where some element of $\Gamma$ is Anosov, the semiconjugacy is at least a homeomorphism. If this is true and enough dynamical foliations are one and two dimensional and $\Gamma$ has higher rank, one can then deduce smoothness of the conjugacy from work of Rodriguez-Hertz on rigidity of actions of abelian groups [203]. Work in progress by the author, Kalinin and Spatzier seems likely to provide a similar result when $\Gamma$ contains many
commuting Anosov diffeomorphisms without any assumptions on dimensions of foliations.

Finally, we recall an intriguing question from [83] which arises in this context.

**Question 7.11.** Let $M$ be a compact manifold with $\pi_1(M) = \mathbb{Z}^n$ and assume $\Gamma < SL(n, \mathbb{Z})$ has finite index. Let $\Gamma$ act on $M$ fixing a point so that the resulting $\Gamma$ action on $\pi_1(M)$ is given by the standard representation of $SL(n, \mathbb{Z})$ on $\mathbb{Z}^n$. Is it true that $\text{dim}(M) \geq n$?

The question is open even if the action on $M$ is assumed to be smooth. The results of [83] imply that there is a continuous map from $M$ to $\mathbb{T}^n$ that is equivariant for the standard $\Gamma$ action on $M$. Since the image of $M$ is closed and invariant, it is easy to check that it is all of $\mathbb{T}^n$. So the question amounts to one about the existence of equivariant “space filling curves”, where curve is taken in the generalized sense of continuous map from a lower dimensional manifold. In another contexts there are equivariant space filling curves, but they seem quite special. They arise as surface group equivariant maps from the circle to $S^2$ and come from three manifolds which fiber over the circle, see [33].

8. **Representations of fundamental groups and arithmetic quotients**

In this section we discuss some results and questions related to topological approaches to classifying actions particularly some related to Conjecture [14]. In the second subsection, we also discuss related results and questions concerning maximal generalized affine quotients of actions.

8.1. **Linear images of fundamental groups.** This section is fundamentally concerned with the question:

**Question 8.1.** Let $G$ be a semisimple Lie group all of whose factors are not compact and have property $(T)$. Assume $G$ acts by homeomorphisms on a manifold $M$ preserving a measure. Can we classify linear representations of $\pi_1(M)$? Similarly for actions of $\Gamma < G$ a lattice on a manifold $M$.

We remark that in this context, it is possible that $\pi_1(M)$ has no infinite image linear representations, see discussion in [83] and Section 9 below. In all known examples where this occurs there is an “obvious” infinite image linear representation on some finite index subgroup.
Also as first observed by Zimmer [248] for actions of Lie groups, under mild conditions on the action, representations of the fundamental group become severely restricted. Further work in this direction was done by Zimmer in conjunction with Spatzier and later Lubotzky [149, 150, 218]. Analogous results for lattices are surprisingly difficult in this context and constitute the authors dissertation [72, 73].

We recall a definition from [248]:

**Definition 8.2.** Let $D$ be a Lie group and assume that $D$ acts on a compact manifold $M$ preserving a finite measure $\mu$ and ergodically. We call the action **engaging** if the action of $\tilde{D}$ on any finite cover of $M$ is ergodic.

There is a slightly more technical definition of engaging for non-ergodic actions which says there is no loss of ergodicity on passing to finite covers. I.e. that the ergodic decomposition of $\mu$ and its lifts to finite covers are canonically identified by the covering map. There are also two variants of this notion **totally engaging** and **topologically engaging**, see e.g. [255] for more discussion.

It is worth noting that for ergodic $D$ actions on $M$, the action on any finite cover has at most finitely many ergodic components, in fact at most the degree of the cover many ergodic components. We remark here that any generalized affine action of a Lie group is engaging if it is ergodic. The actions constructed below in Section 9 are not in general engaging.

**Theorem 8.3.** Let $G$ be a simple Lie group of real rank at least 2. Assume $G$ acts by homeomorphisms on a compact manifold $M$, preserving a finite measure $\mu$ and engaging. Assume $\sigma : \pi_1(M) \to GL(n, \mathbb{R})$ is an infinite image linear representation. Then $\sigma(\pi_1(M))$ contains an arithmetic group $H_Z$ where $H_R$ contains a group locally isomorphic to $G$.

For an expository account of the proof of this theorem and a more detailed discussion of engaging conditions for Lie groups, we refer the reader to [255].

The extension of this theorem to lattice actions is non-trivial and is in fact the author’s dissertation. Even the definition of engaging requires modification, since it is not at all clear that a discrete group action lifts to the universal cover.

**Definition 8.4.** Let $D$ be a discrete group and assume that $D$ acts on a compact manifold $M$ preserving a finite measure $\mu$ and ergodically. We call the action engaging if for every

1. finite index subgroup $D'$ in $D$,
(2) finite cover $M'$ of $M$, and
(3) lift of the $D'$ action to $M'$,
the action of $D'$ of $M'$ is ergodic.

The definition does immediately imply that every finite index sub-
group $D'$ of $D$ acts ergodically on $M$. In [73], a definition is given
which does not require the $D$ action to be ergodic, but even in that
context the ergodic decomposition for $D'$ is assumed to be the same as
that for $D$. Definition 8.4 is rigged to guarantee the following lemma:

Lemma 8.5. Let $G$ be a Lie group and $\Gamma < G$ a lattice. Assume $\Gamma$
acts on a manifold $M$ preserving a finite measure and engaging, then
the induced $G$ action on $(G \times M)/\Gamma$ is engaging.

We remark that with our definitions here, the lemma only makes
sense for $\Gamma$ cocompact, but this is not an essential difficulty. We can
now state a first result for lattice actions. We let $\Lambda = \pi_1((G \times M)/\Gamma)$.

Theorem 8.6. Let $G$ be a simple Lie group of real rank at least 2 and
$\Gamma < G$ a lattice. Assume $\Gamma$ acts by homeomorphisms on a compact
manifold $M$, preserving a finite measure $\mu$ and engaging. Assume $\sigma$:
$\Lambda \to GL(n, \mathbb{R})$ is a linear representation whose restriction to $\pi_1(M)$ has
infinite image. Then $\sigma(\pi_1(M))$ contains an arithmetic group $H\mathbb{Z}$ where
$Aut(H\mathbb{Z})$ contains a group locally isomorphic to $G$.

This theorem is proven by inducing actions, applying Theorem 8.3
and analyzing the resulting output carefully. One would like to assume
$\sigma$ a priori only defined on $\pi_1(M)$, but there seems no obvious way to
extend such a representation to the linear representation of $\Lambda$ required
by Theorem 8.3 without a priori information on $\Lambda$. This is yet another
example of the difficulties in using induction to study lattice actions.
As a consequence of Theorem 8.6 we discover that at least $\sigma(\Lambda)$ splits
as a semidirect product of $\Gamma$ and $\sigma(\pi_1(M))$. But this is not clear a
priori and not clear a posteriori for $\Lambda$.

8.2. Arithmetic quotients. In the context of Theorems 8.3 and The-
orem 8.6 one can obtain much greater dynamical information con-
cerning the relation of $H\mathbb{Z}$ and the dynamics of the action on $M$. In
particular, there is a compact subgroup $C < H\mathbb{R}$ and a measurable
equivariant map $\phi : M \to C\backslash H\mathbb{R}/H\mathbb{Z}$, which we refer to as a measurable
arithmetic quotient. The papers [72] and [149] prove that there is
always a canonical maximal quotient of this kind for any action of $G$
or $\Gamma$ on any compact manifold, essentially by using Ratner’s theorem
to prove that every pair of arithmetic quotients are dominated by a
The results we mention from [73] and [150] then show that there are “lower bounds” on the size of this arithmetic quotient, provided that the action is engaging, in terms of the linear representations of the fundamental group of the manifold. In particular, one obtains arithmetic quotients where $H_{\mathbb{R}}$ and $H_{\mathbb{Z}}$ are essentially determined by $\sigma(\pi_1(M))$. In fact $\sigma(\pi_1(M))$ contains $H_{\mathbb{Z}}$ and is contained in $H_{\mathbb{Q}}$.

The book [255] provides a good description of how to produce arithmetic quotients for $G$ actions and the article [74] provides an exposition of the relevant constructions for $\Gamma$ actions.

Under even stronger hypotheses, the papers [84] and [254] imply that the arithmetic quotient related to the “Gromov representation” discussed above in subsection 6.2 has the same entropy as the original action. This means that, in a sense, the arithmetic quotient captures most of the dynamics of the original action.

### 8.3. Open questions.

As promised in the introduction, we have the following analogue of Conjecture 1.4 for lattice actions.

**Conjecture 8.7.** Let $\Gamma$ be a lattice in a semisimple group $G$ with all simple factors of real rank at least 2. Assume $\Gamma$ acts faithfully, preserving volume on a compact manifold $M$. Further assume the action is not isometric. Then $\pi_1(M)$ has a finite index subgroup $\Lambda$ such that $\Lambda$ surjects onto an arithmetic lattice in a Lie group $H$ where $\text{Aut}(H)$ locally contains $G$.

The following questions about arithmetic quotients are natural.

**Question 8.8.** Let $G$ be a simple Lie group of real rank at least 2 and $\Gamma < G$ a lattice. Assume that $G$ or $\Gamma$ act faithfully on a compact manifold $M$, preserving a smooth volume.

1. Is there a non-trivial measurable arithmetic quotient?
2. Can we take the quotient map $\phi$ smooth on an open dense set?

Due to a construction in [83], one cannot expect that every $M$ admitting a volume preserving $G$ or $\Gamma$ action has an arithmetic quotient that is even globally continuous or has $\pi_1(M)$ admitting an infinite image linear representation. However, the difficulties created in those examples all vanish on passage to a finite cover.

**Question 8.9.** Let $G$ be a simple Lie group of real rank at least 2 and $\Gamma < G$ a lattice. Assume that $G$ or $\Gamma$ act smoothly on a compact manifold $M$, preserving a smooth volume.
(1) Is there a finite cover of $M'$ of $M$ such that $\pi_1(M')$ admits an infinite image linear representation?

(2) Can we find a finite cover $M'$ of $M$, a lift of the action to $M'$ (on a subgroup of finite index) and an arithmetic quotient where the quotient map $\phi$ is continuous and smooth on an open dense set?

The examples discussed in the next subsection imply that $\phi$ is at best Hölder continuous globally. It is not clear whether the questions and conjectures we have just discussed are any less reasonable for $SP(1,n)$, $F_4^{-20}$ and their lattices.

9. Exotic volume preserving actions

In this subsection, I discuss what is known about what are typically called “exotic actions”. These are the only known smooth volume preserving actions of higher rank lattices and Lie groups which are not generalized affine algebraic. These actions make it clear that a clean classification of volume preserving actions is out of reach. In particular, these actions have continuous moduli and provide counter-examples to any naive conjectures of the form “the moduli space of actions of some lattice $\Gamma$ on any compact manifold $M$ are countable.” In particular, I explain examples of actions of either $\Gamma$ or $G$ which have large continuous moduli of deformations as well as manifolds where these moduli have multiple connected components.

Essentially all of the examples given here derive from the simple construction of “blowing up” a point or a closed orbit, which was introduced to this subject in [125]. The further developments after that result are all further elaborations on one basic construction. The idea is to use the “blow up” construction to introduce distinguished closed invariant sets which can be varied in some manner to produce deformations of the action. The “blow up” construction is a classical tool from algebraic geometry which takes a manifold $N$ and a point $p$ and constructs from it a new manifold $N'$ by replacing $p$ by the space of directions at $p$. Let $\mathbb{R}P^l$ be the $l$ dimensional projective space. To blow up a point, we take the product of $N \times \mathbb{R}P^{\dim(N)}$ and then find a submanifold where the projection to $N$ is a diffeomorphism off of $p$ and the fiber of the projection over $p$ is $\mathbb{R}P^{\dim(N)}$. For detailed discussion of this construction we refer the reader to any reasonable book on algebraic geometry.

The easiest example to consider is to take the action of $SL(n,\mathbb{Z})$, or any subgroup $\Gamma < SL(n,\mathbb{Z})$ on the torus $\mathbb{T}^n$ and blow up the fixed point, in this case the equivalence class of the origin in $\mathbb{R}^n$. Call the
resulting manifold $M$. Provided $\Gamma$ is large enough, e.g. Zariski dense in $SL(n, \mathbb{R})$, this action of $\Gamma$ does not preserve the measure defined by any volume form on $M$. A clever construction introduced in [125] shows that one can alter the standard blowing up procedure in order to produce a one parameter family of $SL(n, \mathbb{Z})$ actions on $M$, only one of which preserves a volume form. This immediately shows that this action on $M$ admits perturbations, since it cannot be conjugate to the nearby, non-volume preserving actions. Essentially, one constructs different differentiable structures on $M$ which are diffeomorphic but not equivariantly diffeomorphic.

After noticing this construction, one can proceed to build more complicated examples by passing to a subgroup of finite index, and then blowing up several fixed points. One can also glue together the “blown up” fixed points to obtain an action on a manifold with more complicated topology. In particular, one can achieve a fundamental group which is an essentially arbitrary free product with amalgamation or HNN extension of the fundamental group of the original manifold over the fundamental group of (blown-up) orbits. In the context described in more detail below, of blowing up along closed orbits instead of points, it is not hard to do the blowing up and gluing in way that guarantees that there are no linear representations of the fundamental group of the “exotic example”. To prove non-existence of linear representations, one chooses examples where all groups involved are higher rank lattices and have very constrained linear representation theory. See [83, 125] for discussion of the topological complications one can introduce.

While these actions do not preserve a rigid geometric structure, they do preserve a slightly more general object, an almost rigid structure introduced by Benveniste and the author in [11] and described below.

In [12] it is observed that a similar construction can be used for the action of a simple group $G$ by left translations on a homogeneous space $H/\Lambda$ where $H$ is a Lie group containing $G$ and $\Lambda < H$ is a cocompact lattice. Here we use a slightly more involved construction from algebraic geometry, and “blow up” the directions normal to a closed submanifold. I.e. we replace some closed submanifold $N$ in $H/\Lambda$ by the projective normal bundle to $N$. In all cases we consider here, this normal bundle is trivial and so is just $N \times \mathbb{R}P^l$ where $l = \dim(H) - \dim(N)$.

Benveniste used his construction to produce more interesting perturbations of actions of higher rank simple Lie group $G$ or a lattice $\Gamma$ in $G$. In particular, he produced volume preserving actions which admit volume preserving perturbations. He does this by choosing $G < H$ such that not only are there closed $G$ orbits but so that the centralizer
Z = Z_H(G) of G in H has no-trivial connected component. If we take a closed G orbit N, then any translate zN for z in Z is also closed and so we have a continuum of closed G orbits. Benveniste shows that if we choose two closed orbits N and zN to blow up and glue, and then vary z in a small open set, the resulting actions can only be conjugate for a countable set of choices of z.

This construction is further elaborated in [77]. Benveniste’s construction is not optimal in several senses, nor is his proof of rigidity. In [77], I give a modification of the construction that produces non-conjugate actions for every choice of z in a small enough neighborhood. By blowing up and gluing more pairs of closed orbits, this allows me to produce actions where the space of deformations contains a submanifold of arbitrarily high, finite dimension. Further, Benveniste’s proof that the deformation are non-trivial is quite involved and applies only to higher rank groups. In [77], I give a different proof of non-triviality of the deformations, using consequences of Ratner’s theorem due to Shah and Witte Morris [197, 207, 232]. This shows that the construction produces non-trivial perturbations for any semisimple G and any lattice Γ in G.

As mentioned above, in [11] we show that none of these actions preserve any rigid geometric structure in the sense of Gromov but that they do preserve a slightly more complicated object which we call an almost rigid structure. Both rigid and almost rigid structures are easiest to define in dimension 1. In this context a rigid structure is a non-vanishing vector field and an almost rigid structure is a vector field vanishing at isolated points and to finite degree.

We continue to use the notation of section 6.1 in order to give the precise definition of almost rigid geometric structure.

**Definition 9.1.** An A-structure ψ is called (j, k)-almost rigid (or just almost rigid) if for every point p, r_p^{k,k-1} is injective on the subgroup r^{k+j,k}(Is^{k+j}) ⊂ Is^k.

Thus k-rigid structures are the (0, k)-almost rigid structures.

**Basic Example:** Let V be an n-dimensional manifold. Let X_1, . . . , X_n be a collection of vector fields on M. This defines an A-structure ψ of type R^n on M. If X_1, . . . , X_n span the tangent space of V at every point, then the structure is rigid in the sense of Gromov. Suppose instead that there exists a point p in V and X_1 ∧ . . . ∧ X_n vanishes to order ≤ j at p in V. Then ψ is a (j, 1)-almost rigid structure. Indeed, let p ∈ M, and let (x_1, . . . , x_n) be coordinates around p. Suppose that in terms of these coordinates, X_l = a_m^l ∂ / ∂x_m. Suppose that f ∈ Is_{p+1}. We
must show that \( r_{j+1,1}^p(f) \) is trivial. Let \((f^1, \ldots, f^n)\) be the coordinate functions of \(f\). Then \(f \in Is_{j+1}^p\) implies that

\[
(2) \quad a_k^l - a_k^m \frac{\partial f^l}{\partial x^m}
\]

vanishes to order \(j + 1\) at \(p\) for all \(k\) and \(l\). Let \((b^k_l)\) be the matrix so that \(b^m_k a^l_m = \det(a^*_s) \delta^l_k\). Multiplying expression \((2)\) by \((b^k_l)\), we see that \(\det(a^*_s)(\delta^l_k - \frac{\partial f^l}{\partial x^k})\) vanishes to order \(j + 1\). But since by assumption \(\det(a^*_s)\) vanishes to order \(\leq j\), this implies that \(\frac{\partial f^l}{\partial x^k}(p) = \delta^l_k\), so \(r_{j+1,1}^p(f)\) is the identity, as required.

If confused by the notation, the interested reader may find it enlightening to work out the basic example in the trivial case \(n = 1\). Similar arguments can be given to show that frames that degenerate to subframes are also almost rigid, provided the order of vanishing of the form defining the frame is always finite.

**Question 9.2.** Does any smooth (or analytic) action of a higher rank lattice \(\Gamma\) admit a smooth (analytic) almost rigid structure in the sense of [11]? More generally does such an action admit a smooth (analytic) rigid geometric structure on an open dense set of full measure?

This question is, in a sense, related to the discussion above about regularity of the straightening section in cocycle superrigidity. In essence, cocycle superrigidity provides one with a measurable invariant connection and what one wants to know is whether one can improve the measurable connection to a smooth geometric structure with some degeneracy on a small set. The examples described in this section show, among other things, that one cannot expect the straightening section to be smooth in general, though one might hope it is smooth in the complement of a closed submanifold of positive codimension or at least on an open dense set of full measure.

We remark that there are other possible notions of almost rigid structures. See the article in this volume by Dumitrescu for a detailed discussion of a different useful notion in the context of complex analytic manifolds [47]. Dumitrescu’s notion is strictly weaker than the one presented here.

10. Non-volume preserving actions

This section describes what is known for non-volume preserving actions. The first two subsections describe examples which show that a classification in this setting is not in any sense possible. The last subsection describes some recent work of Nevo and Zimmer that proves surprisingly strong rigidity results in special settings.
10.1. **Stuck’s examples.** The following observations are from Stuck’s paper [220]. Let \( G \) be any semisimple group. Let \( P \) be a minimal parabolic subgroup. Then there is a homomorphism \( P \to \mathbb{R} \). As in the proof of Theorem 2.7, one can take any \( \mathbb{R} \) action, view it is a \( P \) action and induce to a \( G \) action. If we take an \( \mathbb{R} \) action on a manifold \( M \), then the induced action takes place on \((G \times M)/P\). We remark that the \( G \) action here is not volume preserving, simply because the \( G \) action on \( G/P \) is proximal. The same is true of the restriction to any \( \Gamma \) action when \( \Gamma \) is a lattice in \( G \). This implies that classifying \( G \) actions on all compact manifolds implicitly involves classifying all vector fields on compact manifolds. It is relatively easy to convince oneself that there is no reasonable sense in which the moduli space of vector fields can be understood up to smooth conjugacy.

10.2. **Weinberger’s examples.** This is a variant on the Katok-Lewis examples obtained by blowing up a point and is similar to a blowing up construction common in foliation theory. The idea is that one takes an action of a subgroup \( \Gamma \) of \( SL(n, \mathbb{Z}) \) on \( M = \mathbb{T}^n \), removes a fixed or periodic point \( p \), retracts onto a manifold with boundary \( \bar{M} \) and then glues in a copy of the \( \mathbb{R}^n \) compactified at infinity by the projective space of rays. It is relatively easy to check that the resulting space admits a continuous \( \Gamma \) action and even that there are many invariant measures for the action, but no invariant volume. One can also modify this construction by doing the same construction at multiple fixed or periodic points simultaneously and by doing more complicated gluings on the resulting \( \bar{M} \).

This construction is discussed in [52] and a variant for abelian group actions is discussed in [122]. In the abelian case it is possible to smooth the action, but this does not seem to be the case for actions of higher rank lattices.

As far as I can tell, there is no obstruction to repeating this construction for closed orbits as in the case of the algebro-geometric blow-up, but this does not seem to be written formally anywhere in the literature.

Also, a recent construction of Hurder shows that one can iterate this construction infinitely many times, taking retracts in smaller and smaller neighborhoods of periodic points of higher and higher orders. The resulting object is a kind of fractal admitting an \( SL(n, \mathbb{Z}) \) action [116].

10.3. **Work of Nevo-Zimmer.** In this subsection, we describe some work of Nevo and Zimmer from the sequence of papers [180, 181, 184].
For more detailed discussion see the survey by Nevo and Zimmer as well as the following two articles for related results.

Given a group $G$ acting on a space $X$ and a measure $\mu$ on $G$, we call a measure $\nu$ on $X$ stationary if $\mu \ast \nu = \nu$. We will only consider the case where the group generated by the support $\mu$ is $G$, such measures are often called admissible. This is a natural generalization of the notion of an invariant measure. If $G$ is an amenable group, any action of $G$ on a compact metric space admits an invariant measure. If $G$ is not amenable, invariant measures need not exist, but stationary measures always do. We begin with the following cautionary example:

**Example 10.1.** Let $G = SL(n, \mathbb{R})$ acting on $\mathbb{R}^{n+1}$ by the standard linear action on the first $n$ coordinates and the trivial action on the last. Then the corresponding $G$ action on $\mathbb{P}(\mathbb{R}^{n+1})$ has the property that any stationary measure is supported either on the subspace $\mathbb{P}(\mathbb{R}^n)$ given by the first $n$ coordinates or on the subspace $\mathbb{P}(\mathbb{R})$ given by zeroing the first $n$ coordinates.

The proof of this assertion is an easy exercise. The set $\mathbb{P}(\mathbb{R})$ is a collection of fixed points, so clearly admits invariant measures. The orbit of any point in $\mathbb{P}(\mathbb{R}^{n+1}) \setminus (\mathbb{P}(\mathbb{R}^n) \cup \mathbb{P}(\mathbb{R}))$ is $SL(n, \mathbb{R})/(SL(n, \mathbb{R}) \times \mathbb{R}^n)$. It is straightforward to check that no stationary measures can be supported on unions of sets of this kind. This fact should not be a surprise as it generalizes the fact that invariant measures for amenable group actions are often supported on minimal sets.

To state the results of Nevo and Zimmer, we need a slightly stronger notion of admissibility. We say a measure $\mu$ on a locally compact group $G$ is strongly admissible if the support of $\mu$ generates $G$ and $\mu^k$ is absolutely continuous with respect to Haar measure on $G$ for some positive $k$. In the papers of Nevo and Zimmer, this stronger notion is called admissible.

**Theorem 10.2.** Let $X$ be a compact $G$ space where $G$ is a semisimple Lie group with all factors of real rank at least 2. Then for any admissible measure $\mu$ on $G$ and any $\mu$-stationary measure $\nu$ on $X$, we have either

1. $\nu$ is $G$ invariant or
2. there is a non-trivial measurable quotient of the $G$ space $(X, \nu)$ which is of the form $(G/Q, \text{Lebesgue})$ where $Q \subset G$ is a parabolic subgroup.

The quotient space $G/Q$ is called a projective quotient in the work of Nevo and Zimmer. Theorem 10.2 is most interesting for us for minimal actions, where the measure $\nu$ is necessarily supported on all of $X$ and
the quotient therefore reflects the $\Gamma$ action on $X$ and not some smaller set. Example [10.1] indicates the reason to be concerned, since there the action on the larger projective space is not detected by any stationary measure.

We remark here that Feres and Ronshaunen have introduced some interesting ideas for studying group actions on sets not contained in the support of any stationary measure [68]. Similar ideas are developed in a somewhat different context by Deroin, Kleptsyn and Navas [44].

In [179], Nevo and Zimmer show that for smooth non-measure preserving actions on compact manifolds which also preserve a rigid geometric structure, one can sometimes prove the existence of a projective factor where the factor map is smooth on an open dense set.

We also want to mention the main result of Stuck’s paper [220], which we have already cited repeatedly for the fact that non-volume preserving actions of higher rank groups on compact manifolds cannot be classified.

**Theorem 10.3.** Let $G$ be a semisimple Lie group with finite center. Assume $G$ acts minimally by homeomorphisms on a compact manifold $M$. Then either the action is locally free or the action is induced from a minimal action by homeomorphisms of a proper parabolic subgroup of $G$ on a manifold $N$.

Stuck’s theorem is proven by studying the Gauss map from $M$ to the Grassman variety of subspaces of $\mathfrak{g}$ defined by taking a point to the Lie algebra of its stabilizer. This technique also plays an important role in the work of Nevo and Zimmer.

**11. Groups which do act on manifolds**

Much of the work discussed so far begs an obvious question: “are there many interesting subgroups of $\text{Diff}(M)$ for a general $M$?” So far we have only seen “large” subgroups of $\text{Diff}(M)$ that arise in a geometric fashion, from the presence of a connected Lie group in either $\text{Diff}(M)$ or $\text{Diff}(\bar{M})$. In this section, we describe two classes of examples which make it clear that other phenomena exist. The following problem, however, seems open:

**Problem 11.1.** For a compact manifold $M$ with a volume form $\omega$ construct a subgroup $\Gamma < \text{Diff}(M, \omega)$ such that $\Gamma$ has no linear representations.

An example that is not often considered in this context is that $\text{Aut}(F_n)$ acts on the space $\text{Hom}(F_n, K)$ for $K$ any compact Lie group.
Since \( \text{Hom}(F_n, K) \cong K^n \) this defines an action of \( \text{Aut}(F_n) \) on a manifold. This action clearly preserves the Haar measure on \( K^n \), see [106]. This action is not very well studied, we only know of [75, 96]. Similar constructions are possible, and better known, with mapping class groups, although in that case one obtains a representation variety which is not usually a manifold. Since \( \text{Aut}(F_n) \) has no faithful linear representations, this yields an example of truly "non-linear" action of a large group. This action is still very special and one expects many other examples of non-linear actions. We now describe two constructions which yield many examples if we drop the assumption that the action preserves volume.

11.1. **Thompson’s groups.** Richard Thompson introduced a remarkable family of groups, now referred to as Thompson’s groups. These come in various flavors and have been studied from several points of view, see e.g. [32, 114] For our purposes, the most important of these groups are the one’s typically denoted \( T \). One description of this group is the collection of piecewise linear diffeomorphisms of a the circle where the break points and slopes are all dyadic rationals. (One can replace the implicit 2 here with other primes and obtain similar, but different, groups.) We record here two important facts about this group \( T \) of piecewise linear homeomorphisms of \( S^1 \).

**Theorem 11.2** (Thompson, see [32]). The group \( T \) is simple.

**Theorem 11.3** (Ghys-Sergiescu [102]). The defining piecewise linear action of the group \( T \) on \( S^1 \) is conjugate by a homeomorphism to a smooth action.

These two facts together provide us with a rather remarkable class of examples of groups which act on manifolds. As finitely generated linear groups are always residually finite, the group \( T \) has no linear representations whatsoever. A simpler variant of Problem 11.4 is:

**Problem 11.4.** Does \( T \) admit a volume preserving action on a compact manifold?

It is easy to see that compactness is essential in this question. We can construct a smooth action of \( T \) on \( S^1 \times \mathbb{R} \) simply by taking the Ghys-Sergiescu action on \( S^1 \) and acting on \( \mathbb{R} \) by the inverse of the derivative cocycle. Replacing the derivative cocycle with the Jacobian cocycle, this procedure quite generally converts non-volume preserving actions on compact manifolds to volume preserving one’s on non-compact manifolds, but we know of no real application of this in the present context. Another variant of Problem 11.4 is
**Problem 11.5.** Given a compact manifold $M$ and a volume form $\omega$ does $\text{Diff}(M,\omega)$ contain a finitely generated, infinite discrete simple group?

This question is reasonable for any degree of regularity on the diffeomorphisms.

In Ghys survey on groups acting on the circle, he points out that Thompson’s group can be realized as piecewise $SL(2,\mathbb{Z})$ homeomorphisms of the circle [101]. In [69], Whyte and the author point out that the group of piecewise $SL(n,\mathbb{Z})$ maps on either the torus or the real projective space is quite large. The following are natural questions, see [69] for more discussion.

**Question 11.6.** Are there interesting finitely generated or finitely presented subgroups of piecewise $SL(n,\mathbb{Z})$ maps on $T^n$ or $\mathbb{P}^{n-1}$? Can any such group which is not a subgroup of $SL(n,\mathbb{Z})$ be made to act smoothly?

11.2. **Highly distorted subgroups of $\text{Diff}(S^2)$.** In [29], Calegari and Freedman construct a very interesting class of subgroups of $\text{Diff}^\infty(S^2)$. Very roughly, they prove:

**Theorem 11.7.** There is a finitely generated subgroup $G$ of $\text{Diff}^\infty(S^2)$ which contains a rotation $r$ as an arbitrarily distorted element.

Here by *arbitrarily distorted*, we mean that we can choose the group $G$ so that the function $f(n) = \|r^n\|_G$ grows more slowly than any function we choose. It is well-known that for linear groups, the function $f(n)$ is at worst a logarithm, so this theorem immediately implies that we can find $G$ with no faithful linear representations. This also answered a question raised by Franks and Handel in [87].

More recently, Avila has constructed similar examples in $\text{Diff}^\infty(S^1)$ [4]. This answers a question raised in [29] where such subgroups were constructed in $\text{Diff}^1(S^1)$.

We are naturally led to the following questions. We say a diffeomorphism has *full support* if the complement of the fixed set is dense.

**Question 11.8.** For which compact manifolds $M$ does $\text{Diff}^\infty(M)$ contain arbitrarily distorted elements of full support? The same question for $\text{Diff}^\omega(M)$? The same question for $\text{Diff}^\infty(M,\nu)$ where $\nu$ is a volume form on $M$? For the second two questions, we can drop the hypothesis of full support.

The second and third questions here seem quite difficult and the answer could conceivably be “none”. The only examples where anything
is known in the volume preserving setting are compact surfaces of genus at least one, where no element is more than quadratically distorted by a result of Polterovich \cite{189}. However this result depends heavily on the fact that in dimension two, preserving a volume form is the same as preserving a symplectic structure.

11.3. **Topological construction of actions on high dimensional spheres.** In this subsection, we recall a construction due to Farrell and Lafont which allows one to construct actions of a large class of groups on closed disks, and so by doubling, to construct actions on spheres \cite{58}. The construction yields actions on very high dimensional disks and spheres and is only known to produce actions by homeomorphisms.

The class of groups involved is the set of groups which admit an $EZ$-structure. This notion is a modification of an earlier notion due to Bestvina of a $Z$-structure on a group \cite{15}. We do not recall the precise definition here, but refer the reader to the introduction to \cite{58} and remark here that both torsion free Gromov hyperbolic groups and CAT(0)-groups admit $EZ$-structures.

The result that concerns us is:

**Theorem 11.9.** Given a group $\Gamma$ with an $EZ$-structure, there is an action of $\Gamma$ by homeomorphisms on a closed disk.

In fact, Farrell and Lafont give a fair amount of information concerning their actions which are quite different from the actions we are usually concerned with here. In particular, the action is properly discontinuous off a closed subset $\Delta$ of the boundary of the disk. So from a dynamical viewpoint $\Delta$ carries the “interesting part” of the action, e.g. is the support of any $\Gamma$ stationary measure. Farrell and Lafont point out an analogy between their construction and the action of a Kleinian group on the boundary of hyperbolic space. An interesting general question is:

**Question 11.10.** When can the Farrell-Lafont action on a disk or sphere be chosen smooth?

12. **Rigidity for other classes of acting groups**

In this section, we collect some results and questions concerning actions of other classes of groups. In almost all cases, little is known.

12.1. **Lattices in semi-direct products.** While it is not reasonable to expect classification results for arbitrary actions of all lattices in all Lie groups, there are natural broader classes to consider. To pick a reasonable class, a first guess is to try to exclude Lie groups whose
lattices have homomorphisms onto $\mathbb{Z}$ or larger free groups. There are many such groups. For example, the groups $\text{Sl}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ which are lattices in $\text{Sl}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ have property $(T)$ as soon as $n > 2$. As it turns out, a reasonable setting is to consider perfect Lie groups with no compact factors or factors locally isomorphic to $SO(1, n)$ or $SU(1, n)$. Any such Lie group will have property $(T)$ and therefore so will its lattices. Many examples of such groups are described in [225]. Some first rigidity results for these groups are contained in Zimmer’s paper [249]. The relevant full generalization of the cocycle superrigidity theorem is contained in [233]. In this context it seems that there are probably many rigidity theorems concerning actions of these groups already implicit in the literature, following from the results in [233] and existing arguments.

12.2. Universal lattices. As mentioned above in subsection 4.7, the groups $SL(n, \mathbb{Z}[X])$ for $n > 2$ have property $(T)$ by a result of Shalom [210]. His proof also works with larger collections of variables and some other arithmetic groups. The following is an interesting problem.

**Problem 12.1.** Prove cocycle superrigidity for universal lattices.

For clarity, we indicate that we expect that all linear representations and cocycles (up to “compact noise”) will be described in terms of representations of the ambient groups, e.g. $SL(n, \mathbb{R})$, and will be determined by specifying a numerical value of $X$.

Some partial results towards superrigidity are known by Farb [51] and Shenfield [212]. The problem of completely classifying linear representations in this context does seem to be open.

12.3. $SO(1, n)$ and $SU(1, n)$ and their lattices. It is conjectured that all lattices in $SO(1, n)$ and $SU(1, n)$ admit finite index subgroups with surjections to $\mathbb{Z}$, see [17]. The conjecture is usually attributed to Thurston and sometimes to Borel for the case of $SU(1, n)$. If this is true, it immediately implies that actions of those lattices can never be classified.

There are still some interesting results concerning actions of these lattices. In [208], Shalom places restrictions on the possible actions of $SO(1, n)$ and $SU(1, n)$ in terms of the fundamental group of the manifold acted upon. This work is similar in spirit to work of Lubotzky and Zimmer described in Section 8, but requires more restrictive hypotheses.

In [77], the author exhibits large moduli spaces of ergodic affine algebraic actions are constructed for certain lattices in $SO(1, n)$. These moduli spaces are, however, all finite dimensional. In [70], I construct
an infinite dimensional moduli of deformations of an isometric action of $SO(1,n)$. Both of these constructions rely on a notion of bending introduce by Johnson and Millson in the finite dimensional setting \cite{118}.

I do not formulate any precise questions or conjectures in this direction as I am not sure what phenomenon to expect.

12.4. **Lattices in other locally compact groups.** Much recent work has focused on developing a theory of lattices in locally compact group other than Lie groups. This theory is fully developed for algebraic groups over other local fields. Though we did not mention it here, some of Zimmer’s own conjectures were made in the context of $S$-arithmetic groups, i.e. lattices in products of real and p-adic Lie groups. The following conjecture is natural in this context and does not seem to be stated in the literature.

**Conjecture 12.2.** Let $G$ be semisimple algebraic group defined over a field $k$ of positive characteristic and $\Gamma < G$ a lattice. Further assume that all simple factors of $G$ have $k$-rank at least 2. Then any $\Gamma$ action on a compact manifold factors through a finite quotient.

The existence of a measurable invariant metric in this context should be something one can deduce from the cocycle superrigidity theorems, though it is not clear that the correct form of these theorems is known or in the literature.

There is also a growing interest in lattices in locally compact groups that are not algebraic. We remark here that Kac-Moody lattices typically admit no non-trivial homomorphisms even to Homeo($M$), see \cite{82} for a discussion.

The only other interesting class of lattices known to the author is the lattices in the isometry group of a product of two trees constructed by Burger and Mozes \cite{25,24}. These groups are infinite simple finitely presented groups.

**Problem 12.3.** Do the Burger-Mozes lattices admit any non-trivial homomorphisms to Diff($M$) when $M$ is a compact manifold.

That these groups do act on high dimensional spheres by the construction discussed in subsection \cite{113} was pointed out to the author by Lafont.

12.5. **Automorphism groups of free and surface groups.** We briefly mention a last set of natural questions. There is a longstanding analogy between higher rank lattices and two other classes of groups.
These are mapping class groups of surfaces and the outer automorphism group of the free group. See [20] for a detailed discussion of this analogy. In the context of this article, this raises the following question. Here we denote the mapping class group of a surface by $\text{MCG}(\Sigma)$ and the outer automorphism group of $F_n$ by $\text{Out}(F_n)$.

**Question 12.4.** Assume $\Sigma$ has genus at least two or that $n > 2$. Does $\text{MCG}(\Sigma)$ or $\text{Out}(F_n)$ admit a faithful action on a compact manifold? A faithful action by smooth, volume preserving diffeomorphisms?

By not assuming that $M$ is connected, we are implicitly asking the same question about all finite index subgroups of $\text{MCG}(\Sigma)$ and $\text{Out}(F_n)$. We recall from section 11 that $\text{Aut}(F_n)$ does admit a volume preserving action on a compact manifold. This makes the question above particularly intriguing.

13. **Properties of subgroups of $\text{Diff}(M)$**

As remarked in subsection 4.4, in the paper [99], Ghys attempts to reprove the classical Zassenhaus lemma for linear groups for groups of analytic diffeomorphisms. While the full strength of the Zassenhaus lemma does not hold in this setting, many interesting results do follow. This immediately leaves one wondering to what extent other properties of linear groups might hold for diffeomorphism groups or at least what analogues of many theorems might be true. This direction of research was initiated by Ghys and most of the questions below are due to him.

It is worth noting that some properties of finitely generated linear groups, like residual finiteness, do not appear to have reasonable analogues in the setting of diffeomorphism groups. For residual finiteness, the obvious example is the Thompson group discussed above, which is simple.

13.1. **Jordan’s theorem.** For linear groups, there is a classical (and not too difficult) result known as Jordan’s theorem. This says that for any finite subgroup of $\text{GL}(n, \mathbb{C})$, there is a subgroup of index at most $c(n)$ that is abelian. One cannot expect better than this, as $S^1$ has finite subgroups of arbitrarily large order and is a subgroup of $\text{GL}(n, \mathbb{C})$. For proofs of Jordan’s theorem as well as the theorems on linear groups mentioned in the next subsection, we refer the reader to e.g. [202].

**Question 13.1.** Given a compact manifold $M$ and a finite subgroup $F$ of $\text{Diff}(M)$, is there a constant $c(M)$ such that $F$ has an abelian subgroup of index $c(M)$.
As above, one cannot expect more than this, simply because one can construct actions of $S^1$ on $M$ and therefore finite abelian subgroups of $\text{Diff}(M)$ of arbitrarily large order.

For this question, it may be more natural to ask about finite groups of homeomorphisms and not assume any differentiability of the maps. Using the results in e.g. [154], one can show that at most finitely many simple finite groups act on a given compact manifold. To be clear, one can show this using the classification of finite simple groups. It would be most interesting to resolve Question 13.1 without reference to the classification.

A recent preprint of Mundet i Rieri provides some evidence for a positive answer to the question [201].

13.2. **Burnside problem.** A group is called periodic if all of its elements have finite order. We say a periodic group $G$ has bounded exponent if every element has order at most $m$. In 1905 Burnside proved that finitely generated linear groups of bounded exponent are finite and in 1911 Schur proved that finitely generated periodic linear groups are finite. For a general finitely generated group, this is not true, counterexamples to the Burnside conjecture were constructed in a sequence of works by many authors, including Novikov, Golod, Shafarevich and Ol’shanskii. We refer the reader to the website: [http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Burnside_problem.html](http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Burnside_problem.html) for a detailed discussion of the history. This page also discusses the restricted Burnside conjecture resolved by Zelmanov.

In our context, the following questions seem natural:

**Question 13.2.** Are there infinite, finitely generated periodic groups of diffeomorphisms of a compact manifold $M$? Are there infinite, finitely generated bounded exponent groups of diffeomorphisms of a compact manifold $M$?

Some first results in this direction are contained in the paper of Rebelo and Silva [199].

13.3. **Tits’ alternative.** In this section, we ask a sequence of questions related to a famous theorem of Tits’ and more recent variants on it [223].

**Theorem 13.3.** Let $\Gamma$ be a finitely generated linear group. Then either $\Gamma$ contains a free subgroup on two generators or $\Gamma$ is virtually solvable.

The following conjecture of Ghys is a reasonable alternative to this for groups of diffeomorphisms.
Conjecture 13.4. Let $M$ be a compact manifold and $\Gamma$ a finitely generated group of smooth diffeomorphisms of $M$. Then either $\Gamma$ contains a free group on two generators or $\Gamma$ preserves some measure on $M$.

The best evidence for this conjecture to date is a theorem of Margulis which proves the conjecture for $M = S^1$ [160]. Ghys has also asked if the more exact analogue of Tits’ theorem might be true for analytic diffeomorphisms.

A recent related line of research for linear groups concerns uniform exponential growth. A finitely generated group $\Gamma$ has uniform exponential growth if the number of elements in a ball of radius $r$ in a Cayley graph for $\Gamma$ grows at least as $\lambda^r$ for some $\lambda > 1$ that does not depend on the choice of generators. For linear groups, exponential growth implies uniform growth by a theorem of Eskin, Mozes and Oh [50]. There are examples of groups having exponential but not uniform exponential growth due to Wilson [230]. This raises the following question.

Question 13.5. Let $M$ be a compact manifold and $\Gamma$ a finitely generated subgroup of $\text{Diff}(M)$. If $\Gamma$ has exponential growth does $\Gamma$ have uniform exponential growth?

The question seems most likely to have a positive answer if one further assumes that $\Gamma$ is non-amenable.

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