EXTENDABILITY OF FUNCTIONS WITH PARTIALLY VANISHING TRACE

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Abstract. Let $\Omega \subset \mathbb{R}^d$ be an open set and $D$ be a closed part of its boundary. Under very mild assumptions on $\Omega$, we construct a bounded Sobolev extension operator for the Sobolev spaces $W^{1,p}_D(\Omega)$, $1 \leq p \leq \infty$, containing all $W^{1,p}(\Omega)$-functions that vanish in some sense on $D$. In comparison to other constructions of Brewster et al. [2] and Haller-Dintelmann et al. [14] the main focus of this work is to generalize the geometric conditions at the boundary dividing $D$ and $\partial \Omega \setminus D$ that ensure the existence of an extension operator.

1. Introduction

Sobolev spaces that contain functions that only vanish on a portion of the boundary of some given domain $\Omega \subset \mathbb{R}^d$ play an eminent role in the study of the mixed problem for second-order elliptic operators, see, e.g., [1, 2, 5–10, 14, 19, 20]. These spaces enter the game as the spaces where the solution to the mixed problem is sought for. Thus, if a homogeneous Dirichlet boundary condition is imposed on a closed set $D \subset \partial \Omega$, then the solutions are sought in a space $W^{1,p}_D(\Omega)$, which encodes the property that functions vanish in some sense on $D$. See Section 2.1 for the precise definition.

In this article, we are concerned with the construction of a bounded and linear extension operator $E : W^{1,p}_D(\Omega) \to W^{1,p}_D(\mathbb{R}^d)$, i.e., an operator that satisfies $E f|_{\Omega} = f$. The constructions of extension operators for the Sobolev spaces $W^{1,p}(\Omega)$ on Lipschitz domains by Stein [18, pp. 180–192] and Calderón [3] or on $(\varepsilon, \delta)$-domains by Jones [16] and Rogers [17] already yield the existence of a bounded and linear extension operator $E : W^{1,p}_D(\Omega) \to W^{1,p}_D(\mathbb{R}^d)$ under some conditions on $D$. Indeed, the authors above construct linear and bounded extension operators $E'$ that extend also functions in $W^{1,p}_D(\Omega)$ as this is a subspace of $W^{1,p}(\Omega)$. Moreover, the constructed operators $E'$ do not depend on $p$, and thus one verifies under a mild measure theoretic condition on $D$, that $E'$ maps compactly supported and smooth functions whose support stays away from $D$ into functions whose trace on $D$ vanishes. This eventually yields that $E' : W^{1,p}_D(\Omega) \to W^{1,p}_D(\mathbb{R}^d)$.

However, aiming only at extending functions in $W^{1,p}_D(\Omega)$ (and not all functions in the larger space $W^{1,p}(\Omega)$) allows for some freedom in the choice of the underlying geometry of $\Omega$. Indeed, if we choose a point $x$ that is an interior point of $D$, it should be easy to extend a function which vanishes on $D$ by setting the extension to be zero outside $\Omega$ and near $x$. As this extension by zero does not require any boundary regularity, this suggests that the conditions necessary to allow extension should become weaker as we approach $D$.

The construction of an extension operator for the spaces $W^{1,p}_D(\Omega)$ has been studied, e.g., by Haller-Dintelmann et al. [14] and by Brewster et al. [2]. In both works, an extension was constructed by locally extending functions in a neighborhood $U$ of $\partial \Omega \setminus D$ and by extending by zero outside of $U$. This approach requires regularity properties of the portion of $\partial \Omega$ inside $U$ and...
as $U$ is a neighbourhood of the closure of $\partial \Omega \setminus D$ a small portion of $D$ has to share this regularity property as well. Thus, despite all its simplicity, this approach does not seem to yield optimal geometric assumptions at the boundary portion that divides $D$ and $\partial \Omega \setminus D$ as the condition imposed should become less restrictive as we approach $D$.

This work adapts Jones’ seminal paper [16] to the spaces $W^{1,p}_D(\Omega)$ and provides a geometric condition that pursues the following philosophy: If a point is close to $\partial \Omega \setminus D$ and far from $D$, then the usual $(\varepsilon, \delta)$-condition must be fulfilled. If a point is close to $D$ and far from $\partial \Omega \setminus D$, no condition is imposed. In between, it should still be possible to connect two nearby points by an $(\varepsilon, \delta)$-path. This path, however, is only supposed to keep some distance to $\partial \Omega \setminus D$ and it is allowed to cross the Dirichlet boundary. The fact that the $(\varepsilon, \delta)$-path has to keep some distance only to $\partial \Omega \setminus D$ and not to a relatively open neighborhood of $\partial \Omega \setminus D$ in $\partial \Omega$ (as it is imposed by Brewster et. al.) allows for outward cusps in $D$ that are close to $\partial \Omega \setminus D$. Moreover, the fact that the $(\varepsilon, \delta)$-path connecting two nearby points can cross the Dirichlet boundary allows for inward cusps in $D$ that are close to $\partial \Omega \setminus D$.

There is one final remark concerning the geometric condition that is considered here. Namely, an $(\varepsilon, \delta)$-path is not allowed to be too far from $\Omega$ relative to $\partial \Omega \setminus D$. This distance is measured with respect to a quasihyperbolic metric, see Section 2 and especially Assumption 2.2 for the precise formulation of the geometric setup. The purpose of this distance condition on the $(\varepsilon, \delta)$-path is to exclude the existence of certain cusps that lie directly on the boundary dividing $D$ and $\partial \Omega \setminus D$, see Figure 3. This is necessary as the domain depicted in Figure 3 is not a $W^{1,p}_D(\Omega)$-extension domain for any $1 < p < \infty$. For a further discussion on how sharp the geometric condition imposed in this paper is, see Section 4.

We shortly outline the structure of the paper. In Section 2 we introduce the geometric setting and the functional framework and then state the main result in Section 3. In Section 4 we compare our results to the existing results and give examples of domains that meet or do not meet our geometric condition. In Section 5 properties of quasihyperbolic distances are connected with properties of Whitney decompositions. Section 6 deals with the reflection of Whitney cubes under our geometric setting and in Section 7 the extension operator is constructed and its mapping properties are proved. Notice that Sections 6 and 7 are inspired by Jones [16].

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2. Notation and general assumptions

Throughout this article, the dimension $d \geq 2$ of the underlying Euclidean space $\mathbb{R}^d$ is fixed. The closure, interior, and complement of a set $A \subset \mathbb{R}^d$ are denoted by $\overline{A}$, $A^\circ$, and $A^c$, respectively. The Euclidean norm of a complex vector as well as the Lebesgue measure of a measurable set in $\mathbb{R}^d$ are denoted by $| \cdot |$. If not otherwise stated, integration is with respect to the Lebesgue measure on $\mathbb{R}^d$ and for a measurable set $A \subset \mathbb{R}^d$ with $|A| > 0$ and an integrable function $f$ on $A$ the mean value is denoted by $(\int_A f)_A := \int_A f = |A|^{-1} \int_A f$. The distance of two sets $A, B \subset \mathbb{R}^d$ is denoted by $d(A, B)$ and in the case $A = \{x\}$ the distance is abbreviated by $d(x, B)$. The diameter of an arbitrary subset of $\mathbb{R}^d$ is denoted by $\text{diam}(\cdot)$. Finally, we follow the standard convention that the infimum over the empty set is $+\infty$.

2.1. Sobolev spaces. Let $\Omega \subset \mathbb{R}^d$ be an open set and let $D \subset \partial \Omega$ be closed. Define the space of smooth, compactly supported functions which vanish in a neighborhood of $D$ by

$$C^\infty_D(\Omega) := \{ \varphi |_{\Omega} : \varphi \in C^\infty(\mathbb{R}^d) \text{ and } d(\text{supp}(\varphi), D) > 0 \}.$$
The space of bounded $C^1$-functions with bounded derivative that vanish in a neighborhood of $D$ is defined as

$$BC_1^D(Ω) := \{ φ | Ω : φ ∈ BC^1(ℝ^d) \text{ and } d(\text{supp}(φ), D) > 0 \}.$$  

For $p ∈ [1, ∞)$ define the Sobolev space $W_1^1p(D, Ω)$ of functions vanishing on $D$ as the closure of $C_0^∞(Ω)$ in $W_1^1p(Ω)$ with the norm $\| u \|_{W_1^1p(D, Ω)}^p := \| u \|_{L_1p(Ω)}^p + \| ∇ u \|_{L_1p(Ω)}^p$. Notice that $W_1^1p(D, Ω)$ is a separable Banach space which is reflexive whenever $1 < p < ∞$.

The case $p = ∞$ is considered in form of the space of Lipschitz continuous functions that vanish on $D$, which is given by

$$\text{Lip}_D(Ω) := \{ u : \overline{Ω} → ℝ : u \text{ Lipschitz and } u = 0 \text{ on } D \}$$

with norm $\| u \|_{\text{Lip}_D(Ω)} := \max(\| u \|_{L_∞(Ω)}, | u |_{\text{Lip}(Ω)})$. Here, $| u |_{\text{Lip}(Ω)}$ is defined as

$$| u |_{\text{Lip}(Ω)} := \sup_{x, y ∈ Ω, x ≠ y} \frac{| u(x) - u(y) |}{| x - y |}.$$

We abbreviate $\text{Lip}_D(Ω)$ by $\text{Lip}(Ω)$.

Finally, by $W_1^1∞(Ω)$ we denote the usual Sobolev space of weakly differentiable functions on $Ω$ such that $u$ and $∇ u$ lies in $L_1∞(Ω)$. Notice that $\text{Lip}(Ω)$ always continuously embeds into $W_1^1∞(Ω)$ but that the contrary does only hold true under particular geometric assumptions on $Ω$. The continuous inclusion of $\text{Lip}(Ω)$ into $W_1^1∞(Ω)$ can be understood as follows. Let for $f ∈ \text{Lip}(Ω)$ denote $f_∞ ∈ \text{Lip}(ℝ^d)$ its Whitney extension [18, Thm. VI.3] satisfying $f_∞|Ω = f$ and $∥ f_∞ ∥_{\text{Lip}(ℝ^d)} ≤ C ∥ f ∥_{\text{Lip}(Ω)}$ for some constant $C > 0$ depending only on $d$. Since $\text{Lip}(ℝ^d) = W_1^1∞(ℝ^d)$ with equal norms, this implies the inequality

$$∥ f ∥_{W_1^1∞(Ω)} ≤ ∥ f_∞ ∥_{W_1^1∞(ℝ^d)} ≤ C ∥ f ∥_{\text{Lip}(Ω)} \quad (f ∈ \text{Lip}(Ω)),$$

where $C$ denotes the constant from the estimate of the Whitney extension. The following approximation lemma for functions in $\text{Lip}_D(Ω)$ is a modified version of an argument of Stein [18, p. 188] and will prove to be very useful.

**Lemma 2.1.** Let $f ∈ \text{Lip}_D(Ω)$. Then there exists a sequence $(φ_n)_{n ∈ N} ⊂ C^∞(ℝ^d) ∩ W_1^1∞(Ω)$ that satisfies for all $n ∈ N$ that $d(\text{supp}(φ_n), D) > 0$ and

$$∥ φ_n - f ∥_{L_∞(Ω)} → 0 \quad as \quad n → ∞.$$

Furthermore, there exists a constant $C > 0$ depending only on $d$ such that

$$∥ φ_n ∥_{W_1^1∞(Ω)} ≤ C ∥ f ∥_{\text{Lip}_D(Ω)}.$$

**Proof.** Let $f ∈ \text{Lip}_D(Ω)$. By Whitney’s extension theorem [18, Thm. VI.3] there exists a Lipschitz continuous extension $f_∞$ of $f$ to $ℝ^d$ which vanishes on $D$ and satisfies $∥ f_∞ ∥_{W_1^1∞(ℝ^d)} ≤ C ∥ f ∥_{\text{Lip}(Ω)}$ for some constant $C > 0$ that depends only on $d$. For $n ∈ N$ define the set

$$D_n := \{ x ∈ ℝ^d : d(x, D) < \frac{1}{n} \}.$$

By convolution construct a smooth cut-off function $ψ_n$ with $0 ≤ ψ_n ≤ 1$, $ψ_n ≡ 1$ on $D_2n$, $ψ_n ≡ 0$ on $D_1n$, and $∥ ∇ ψ_n ∥_{L_∞} ≤ C_n$, where $C > 0$ depends only on $d$. Furthermore, let $ρ$ be a standard mollifier with supp$(ρ) ⊂ B(0, 1)$ and for $ε > 0$ define $ρ_ε(x) := ε^{-d}ρ(ε^{-1}x)$. Then, $φ_n := ρ_1n ∗ [f_∞(1 - ψ_n)]$ is smooth and vanishes in a neighborhood of $D$. Moreover, for $x ∈ ℝ^d$ and $y ∈ D$ with $| x - y | = d(x, D)$ we have

$$| f_∞(x) - f_∞(y) | = ψ_n(x)| f_∞(x) − f_∞(y) | ≤ ∥ ∇ f_∞ ∥_{L_∞(ℝ^d)} ψ_n(x) d(x, D).$$

(2.1)
Thus, because $\psi_n$ vanishes outside $D_n$
\[ \|f_W - f_W(1 - \psi_n)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{\|\nabla f_W\|_{L^\infty(\mathbb{R}^d)}}{n} \to 0 \quad \text{as} \quad n \to \infty. \]

Consequently, by Young’s inequality for convolutions
\[ \|f - \varphi_n\|_{L^\infty(\Omega)} \leq \|f_W - f_W \ast \rho_{\frac{1}{n}}\|_{L^\infty(\mathbb{R}^d)} + \|f_W - f_W(1 - \psi_n)\|_{L^\infty(\mathbb{R}^d)} \to 0 \quad \text{as} \quad n \to \infty. \]
The same calculation also proves the bound on the $L^\infty$-norm of $\varphi_n$. Furthermore, by Young’s inequality and the same trick as in \([2.1]\)
\[ \|\nabla \varphi_n\|_{L^\infty(\Omega)} \leq \|\nabla f_W(1 - \psi_n)\|_{L^\infty(\mathbb{R}^d)} + \|f_W \nabla \psi_n\|_{L^\infty(\mathbb{R}^d)} \leq C\|\nabla f_W\|_{L^\infty(\mathbb{R}^d)}. \quad \Box \]

2.2. The geometry. Let $\Xi \subset \mathbb{R}^d$ be open. For two points $x, y \in \Xi$ their quasihyperbolic distance, first introduced by Gehring and Palka \([12]\), is given by
\[ k_{\Xi}(x, y) := \inf_\gamma \int_\gamma \frac{1}{d(z, \partial \Xi)} |dz|, \]
where the infimum of the path integrals is taken over all rectifiable curves $\gamma$ in $\Xi$ joining $x$ and $y$. Notice that its value might be $+\infty$ if there is no path connecting $x$ and $y$. The function $k_{\Xi}$ is called the quasihyperbolic metric. If $\Xi' \subset \Xi$ define
\[ k_{\Xi}(x, \Xi') := \inf\{k_{\Xi}(x, y) : y \in \Xi'\} \quad (x \in \Xi). \]

To construct a Sobolev extension operator related to the spaces defined in Section 2.1 we will make the following geometric assumption.

**Assumption 2.2.** Let $\Omega \subset \mathbb{R}^d$ be open, $D \subset \partial \Omega$ be closed, and define $\Gamma := \partial \Omega \setminus D$. We assume that there exist $\varepsilon, \delta, K > 0$ such that for all points $x, y \in \Omega$ with $|x - y| < \delta$ there exists a rectifiable curve $\gamma$ that joins $x$ and $y$ and takes values in $\Xi := \mathbb{R}^d \setminus \Gamma$ and satisfies
\begin{align*}
\text{(2.2)} & \quad \text{length}(\gamma) \leq \varepsilon^{-1}|x - y| \\
\text{(2.3)} & \quad d(z, \Gamma) \geq \varepsilon \frac{|x - z||y - z|}{|x - y|} \quad (z \in \gamma) \\
\text{(2.4)} & \quad k_{\Xi}(z, \Omega) \leq K \quad (z \in \gamma).
\end{align*}
Notice that $\text{length}(\gamma)$ is measured with respect the standard metric of $\mathbb{R}^d$. For the radius of $\Omega$, assume that there exists $r_0 > 0$ such that
\[ \text{radius}_\Gamma(\Omega) := \inf_{x \in \Omega} \sup_{\Omega_m} \{\varepsilon > 0 : \exists x \in \Omega_m \text{ with } \Omega_m \subset B(x, r)\} \geq r_0, \]
where the first infimum is taken over all connected components $\Omega_m$ of $\Omega$ with $\partial \Omega_m \cap \Gamma \neq \emptyset$.

Henceforth, $\Xi$ always denotes the set $\mathbb{R}^d \setminus \overline{\Gamma}$, where $\Gamma$ is as in Assumption \([2.2]\) given by $\Gamma = \partial \Omega \setminus D$. Let $(\Xi_m)_{m \in \mathcal{I}}$ denote the connected components of $\Xi$, where $\mathcal{I}$ is an at most countable index set. For each $m \in \mathcal{I}$ it is clear that
\[ d(z, \partial \Xi_m) = d(z, \partial \Xi) \quad (z \in \Xi_m), \]
so that the following properties hold true
- $k_{\Xi}(x, y) = k_{\Xi_m}(x, y)$ for all $x, y \in \Xi_m$,
- $k_{\Xi}(x, y) = \infty$ for all $x \in \Xi_m$ and $y \in \Xi_n$ with $m \neq n$,
- $k_{\Xi}(x, y) = 0$ for all $x, y \in \mathbb{R}^d$.

**Remark 2.3.**
1. In Euclidean space, the shortest path connecting two points $x, y$ has length $|x - y|$, so that $\varepsilon \in (0, 1]$. 
To formulate our main result, we say that a linear operator $E : L^1_{\text{loc}}(\Omega) \to L^1_{\text{loc}}(\mathbb{R}^d)$ is an extension operator if it satisfies $[Ef]_\Omega = f$ for all $f \in L^1_{\text{loc}}(\Omega)$.

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^d$ be an open set with $\overline{\Omega} \neq \mathbb{R}^d$ and $D \subset \partial \Omega$ be closed such that $\Omega$ and $D$ are subject to Assumption 2.2. Then there exists an extension operator $E$ such that for all $1 \leq p < \infty$ one has

$$E \in \mathcal{L}(L^p(\Omega), L^p(\mathbb{R}^d)) \cap \mathcal{L}(W^{1,p}_D(\Omega), W^{1,p}_D(\mathbb{R}^d))$$

and

$$E \in \mathcal{L}(L^\infty(\Omega), L^\infty(\mathbb{R}^d)) \cap \mathcal{L}(\text{Lip}_D(\Omega), \text{Lip}_D(\mathbb{R}^d)).$$

Furthermore, for $f \in BC^1_1(\Omega)$ we have $d(\text{supp}(Ef), D) > 0$ and the operator norms of $E$ only depend on $d$, $p$, $K$, $\varepsilon$, $\delta$, and $r_0$ and they are uniform with respect to $r_0$ whenever $r_0 \geq 1$.

As a corollary, we obtain the existence of an extension operator on even more general but very inexplicit geometries.

**Corollary 3.2.** Let $\Omega \subset \mathbb{R}^d$ be an open set with $\overline{\Omega} \neq \mathbb{R}^d$, $D \subset \partial \Omega$ be closed, and define $\Gamma := \partial \Omega \setminus D$. Further, assume that there exists an open set $\Omega_\Gamma \supset \Omega$ such that $\Gamma$ is contained in $\partial \Omega_\Gamma$ and is relatively open with respect to $\partial \Omega_\Gamma$. Finally, assume that $\Omega_\Gamma$ and $D' := \partial \Omega_\Gamma \setminus \Gamma$ satisfy Assumption 2.2. Then there exists an extension operator $E$ such that for all $1 \leq p < \infty$ one has

$$E \in \mathcal{L}(L^p(\Omega), L^p(\mathbb{R}^d)) \cap \mathcal{L}(W^{1,p}_D(\Omega), W^{1,p}_D(\mathbb{R}^d))$$

and

$$E \in \mathcal{L}(L^\infty(\Omega), L^\infty(\mathbb{R}^d)) \cap \mathcal{L}(\text{Lip}_D(\Omega), \text{Lip}_D(\mathbb{R}^d)).$$

The operator norms of $E$ only depend on $d$, $p$, $K$, $\varepsilon$, $\delta$, and $r_0$ and they are uniform with respect to $r_0$ whenever $r_0 \geq 1$. Here, the quantities $K$, $\varepsilon$, $\delta$, and $r_0$ are measured with respect to $\Omega_\Gamma$.

**Proof.** Throughout this proof, let $\varepsilon, \delta > 0$ be such that $\Omega_\Gamma$ and $D'$ satisfy Assumption 2.2.

Let $E_0 : L^1_{\text{loc}}(\Omega) \to L^1_{\text{loc}}(\Omega_\Gamma)$ be the operator, that extends functions by zero. It is clear that for $E_0$ is bounded from $L^p(\Omega)$ to $L^p(\Omega_\Gamma)$ for all $1 \leq p \leq \infty$. Let $E_\Gamma$ denote the extension operator subject to $\Omega_\Gamma$ which exists due to Theorem 3.1 and define the operator $E := E_0 E_\Gamma$. We show in the following, that $E$ is the desired extension operator. Notice that the boundedness properties from $L^p(\Omega)$ to $L^p(\mathbb{R}^d)$ are clear by construction.

Let $f \in \text{Lip}_D(\Omega)$. We claim that $E_0 f$ is Lipschitz continuous on $\Omega_\Gamma$. To this end, let without loss of generality $x \in \Omega$ and $y \in \Omega_\Gamma \setminus \Omega$ with $|x - y| < \delta$ and let $\gamma$ be the path connecting $x$ to $y$ subject to Assumption 2.2. By virtue of (2.3), the path $\gamma$ has no intersection point with $\Gamma$.
so that the intermediate value theorem implies that there exists \( z \in \gamma \cap D \). Now, by Lipschitz continuity of \( f \), the fact that \( f \) vanishes on \( D \), and by (2.2) one estimates

\[
|E_0 f(x) - E_0 f(y)| = |f(x) - f(y)| \leq |f|_{\text{Lip}(\Omega)} |x - y| \leq |f|_{\text{Lip}(\Omega)} \text{length}(\gamma) \leq \frac{|f|_{\text{Lip}(\Omega)}}{\varepsilon} |x - y|.
\]

Notice further, that \( E_0 f \) vanishes on \( D' \) so that \( E_0 f \in \text{Lip}_D(\Omega_T) \). Thus, by Theorem 3.1, \( Ef \in \text{Lip}_D(\mathbb{R}^d) \) and

\[
\|Ef\|_{\text{Lip}(\mathbb{R}^d)} \leq C\|f\|_{\text{Lip}(\Omega)}.
\]

To show that \( E \) maps into \( \text{Lip}_D(\mathbb{R}^d) \), notice that

\[
D = [D \cap D'] \cup [D \cap \Omega_T],
\]

Thus, by the mapping properties of \( E_T \), \( Ef \) vanishes on \( D \cap D' \). On \( D \cap \Omega_T \), one uses that \( E_T \) is an extension operator, so that

\[
Ef(x) = E_0 f(x) = 0 \quad (x \in D \cap \Omega_T).
\]

It follows that \( E : \text{Lip}_D(\Omega) \to \text{Lip}_D(\mathbb{R}^d) \) is a bounded extension operator.

Let \( 1 \leq p < \infty \). To show that \( E_0 \) maps \( W^{1,p}_D(\Omega) \) into \( W^{1,p}_D(\Omega_T) \), let \( f \in W^{1,p}_D(\Omega) \). Then there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^d) \) such that \( d(\text{supp}(f_n), D) > 0 \) and such that \( f_n|_\Omega \to f \) in \( W^{1,p}(\Omega) \) as \( n \to \infty \). In the following, we write \( E_0 f_n \) to denote the application of \( E_0 \) to \( f_n|_\Omega \).

Since \( f_n \in \text{Lip}_D(\Omega) \), we find \( E_0 f_n \in \text{Lip}_D(\Omega_T) \) by the previous discussion. In particular, \( E_0 f_n \) is weakly differentiable with \( \nabla E_0 f_n = E_0 \nabla f_n \). It follows that

\[
\|E_0 f_n\|_{W^{1,p}(\Omega_T)} = \|f_n\|_{W^{1,p}(\Omega)}
\]

and that there exists \( g \in W^{1,p}(\Omega_T) \) such that \( E_0 f_n \to g \) in \( W^{1,p}(\Omega_T) \) as \( n \to \infty \). We claim that each \( E_0 f_n \) lies in \( W^{1,p}_D(\Omega) \) and thus that \( g \in W^{1,p}_D(\Omega_T) \). Since \( E_0 f_n \in \text{Lip}_D(\Omega_T) \) there exists a corresponding sequence \( \{\varphi_{n,k}\}_{k \in \mathbb{N}} \) subject to Lemma 2.4. Since \( E_0 f_n \) has compact support one can multiply each \( \varphi_{n,k} \) by a smooth cutoff function being one on a huge ball containing the support of \( E_0 f_n \) without changing the properties of the approximating sequence stated in Lemma 2.4. Thus, without loss of generality, we may assume that there exists \( R > 0 \) such that for each for each \( k \) the support of \( \varphi_{n,k} \) is contained in \( B(0,R) \). Consequently, due to the properties stated in Lemma 2.4 it follows that \( \varphi_{n,k} \in C_0^\infty(\Omega_T) \) for each \( k \in \mathbb{N} \). Let \( 0 < q < \infty \). Due to the continuous inclusions \( W^{1,\infty}(\Omega_T \cap B(0,R)) \subset W^{1,q}(\Omega_T \cap B(0,R)) \), we thus find \( \varphi_{n,k} \to E_0 f_n \) in \( L^q(\Omega_T) \) as \( k \to \infty \) and that \( \|\varphi_{n,k}\|_{W^{1,q}_D(\Omega_T)} \) is bounded with respect to \( k \). By reflexivity of \( W^{1,\infty}_D(\Omega_T) \), there exists a subsequence of \( \{\varphi_{n,k}\}_{k \in \mathbb{N}} \) that weakly converges to some \( F_n \in W^{1,\infty}_D(\Omega_T) \). By the \( L^q \)-convergence of \( \varphi_{n,k} \) to \( E_0 f_n \) it follows that \( E_0 f_n = F_n \) and by the compact support of \( E_0 f_n \) that \( E_0 f \in W^{1,p}_D(\Omega_T) \).

\[
\Box
\]

Remark 3.3. Notice that Assumption 2.2 is an explicit assumption that uses only information on points in \( \Omega \). To the contrary of that, the geometry described in Corollary 3.2 has an inexplicit nature, as it is a priori not clear how to construct such a set \( \Omega_T \). However, in some particular examples it proves to be a condition that can be fulfilled in the “blink of an eye”, see Example 4.9.

4. Comparison with other results and examples

This section is devoted to compare our results with existing results. The most general geometric setup to construct a Sobolev extension operator for the spaces \( W^{1,p}_D(\Omega) \) was considered in the work of Brewster, Mitrea, Mitrea, and Mitrea [2, Thm. 1.3, Def. 3.4] and reads as follows.

Assumption 4.1. Let \( \Omega \subset \mathbb{R}^d \) be an open, nonempty, and proper subset of \( \mathbb{R}^d \), \( D \subset \partial \Omega \) be closed, and let \( \Gamma := \partial \Omega \setminus D \). Let \( \epsilon, \delta > 0 \) be fixed. Assume there exist \( r_0 > 0 \) and an at most countable family \( \{O_j\}_{j \in J} \) of open subsets of \( \mathbb{R}^d \) satisfying

\[
\text{and (ii) } O_j \cap O_k = \emptyset \quad (j, k \in J) .
\]
Proof. Let \( \varepsilon, \delta, r > 0 \) be as in Assumption \( 4.1 \). Define \( \kappa := r/8 \), \( \delta' := \min\{\varepsilon, \varepsilon r/8\} \), and \( U_\kappa := \{x \in \mathbb{R}^d : d(x, \Gamma) \leq \kappa\} \). Recall that \( \varepsilon \in (0, 1) \) by Remark \( 2.3 \) so that in particular \( \delta' < r/8 \).

Let \( x, y \in U_\kappa \cap \Omega \) with \( |x - y| < \delta' \) and let \( x_0 \in \overline{\Gamma} \) be such that \( d(x, \overline{\Gamma}) = |x - x_0| \). In this case, we have \( x, y \in B(x_0, r) \subset O_j \) for some \( j \in J \). Let \( \Omega_j \) be the corresponding \((\varepsilon, \delta)\)-domain, so that \( x, y \subset B(x_0, r) \cap \Omega = B(x_0, r) \cap \Omega_j \) (cf. Assumption \( 4.1 \)). Notice that this implies that \( x_0 \in \partial \Omega_j \). Let \( \gamma \) denote the \((\varepsilon, \delta)\)-path subject to \( 2.2 \) and \( 2.3 \) that connects \( x \) and \( y \). For \( z \in \gamma \), we have by \( 2.2 \) and the choice of \( \delta' \)

\[
|x_0 - z| \leq |x_0 - x| + |x - z| < \frac{r}{8} + \text{length}(\gamma) \leq \frac{r}{4}.
\]

Thus, \( \gamma \) takes its values in \( B(x_0, r/4) \cap \Omega \). In particular, it holds

\[
d(z, \partial \Omega_j) = d\left(z, \partial \Omega_j \cap B\left(x_0, \frac{r}{2}\right) \right) = d\left(z, \partial \Omega \cap B\left(x_0, \frac{r}{2}\right) \right)
\]

and thus we further have by \( 2.3 \)

\[
d(z, \overline{\Gamma}) \geq d\left(z, \partial \Omega \cap B\left(x_0, \frac{r}{2}\right) \right) = d(z, \partial \Omega_j) \geq \varepsilon \frac{|x - y||y - z|}{|x - y|}.
\]

Since \( \gamma \) takes its values inside \( \Omega \), it satisfies also \( 2.4 \) with \( K = 0 \) and thus all conditions subject to Assumption \( 2.2 \).

If \( x, y \in U_{\kappa/2} \cap \Omega \) with \( |x - y| < \kappa/4 \), let \( \gamma : [0, 1] \to \mathbb{R}^d \) denote the path that connects \( x \) and \( y \) by a straight line. In this case, we have length(\( \gamma \)) = \( |x - y| \) (this is \( 2.2 \)) and we find for \( z \in \gamma \)

\[
d(z, \overline{\Gamma}) \geq d(x, \overline{\Gamma}) - \text{length}(\gamma) \geq \frac{\kappa}{4}.
\]

Now, if without loss of generality \( |x - z| \leq |x - y|/2 \), then

\[
\varepsilon \frac{|x - z||y - z|}{|x - y|} \leq \frac{\varepsilon}{2} |y - z| < \frac{\varepsilon \kappa}{8},
\]

which yields \( 2.3 \).

Finally, to control the quasihyperbolic distance of a point \( z = \gamma(t) \) (for some \( t \in (0, 1) \)) to \( \Omega \) with respect to \( \Xi := \mathbb{R}^d \setminus \overline{\Gamma} \) define the path \( \overline{\gamma} : [0, t] \to \mathbb{R}^d \) by \( \overline{\gamma}(s) = \gamma(s) \). Then

\[
k_{\Xi}(z, \Omega) \leq \int_0^t \frac{|\overline{\gamma}'(s)|}{d(\overline{\gamma}(s), \overline{\Gamma})} \, ds \leq \frac{4|x - y|}{\kappa} \leq 1.
\]

Thus, for all \( x, y \in \Omega \) with \( |x - y| < \min\{\delta, \delta_L, \kappa/2\} \) we find a path \( \gamma \) that satisfies the conditions required in \( 2.2 \), \( 2.3 \), and \( 2.4 \). \( \square \)

A common geometric setup, which is used in many works, see, e.g., [1, 5–8, 10, 19, 20] dealing with mixed Dirichlet/Neumann boundary conditions requires Lipschitz charts around points on the closure of \( \Gamma \) and is presented in the following assumption.
Figure 1. A generic picture of a domain described in Example 4.5.

Assumption 4.3. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $D \subset \partial \Omega$ be closed and assume that $\Gamma$ has a Lipschitz boundary in the sense, that around each point $x \in \Gamma$ there exists a neighborhood $U_x$ of $x$ and a bi-Lipschitz homeomorphism $\Phi_x : U_x \to (-1,1)^d$ such that $\Phi_x(x) = 0$, $\Phi_x(U_x \cap \Omega) = (-1,1)^{d-1} \times (0,1)$, and $\Phi_x(U_x \cap \Omega) = (-1,1)^{d-1} \times \{0\}$.

We then have the following proposition.

Proposition 4.4. Assumption 4.3 implies Assumption 4.1.

Proof. By [4, Lem. 2.2.20], for all $x_0 \in \Gamma$ the sets $\Omega_{x_0} := U_{x_0} \cap \Omega$ are $(\varepsilon, \delta)$-domains in the sense of Jones. Here, $\varepsilon$ and $\delta$ only depend on $d$ and the Lipschitz constant. The compactness of $\Gamma$ implies that there exist finitely many $x_1, \ldots, x_m \in \Gamma$, such that $\Gamma \subset \bigcup_{j=1}^m U_{x_j}$. Define $O_j := U_{x_j}$ and $\Omega_j := \Omega_{x_j}$ for $j = 1, \ldots, m$. Due to the finiteness of the family $\{\Omega_j\}_{j=1}^m$, the constants $\varepsilon$ and $\delta$ can be chosen to be uniform in $j$. Finally, if $r > 0$ is the Lebesgue number of this covering, then for all $x_0 \in \Gamma$ there exists $1 \leq j \leq m$ such that $B(x_0, r) \subset O_j$. Thus, all properties required in Assumption 4.1 are fulfilled. \qed

We close this section by giving an example of a two-dimensional domain that satisfies the conditions of Assumption 2.2 but not of Assumption 4.1. We further show that, within this configuration, the geometry described in Assumption 2.2 is the most general to hope for the existence of a bounded $W_{d,p}^1$-extension operator.

Example 4.5. Let $\theta \in (0, \pi)$ and let $S_\theta \subset \mathbb{R}^2$ denote the open sector symmetric about the positive $x$-axis with opening angle $2\theta$. Let $\Omega \subset \mathbb{R}^2$ be any domain satisfying

$$\Omega \cap S_\theta = \{(x, y) \in S_\theta : y < 0\}$$

and define

$$D := \partial \Omega \cap [\mathbb{R}^2 \setminus S_\theta] \quad \text{and} \quad \Gamma := \partial \Omega \setminus D = (0, \infty) \times \{0\}.$$  

Essentially, this means that inside the sector $S_\theta$ the domain $\Omega$ looks like the lower half-space and the half-space boundary that lies inside $S_\theta$ is $\Gamma$. In the complement of the sector $S_\theta$, $\Omega$ could be any open set and the boundary of $\Omega$ in the complement of $S_\theta$ is defined to be $D$. See Figure 1 for an example of such a configuration.
To verify that such a domain fulfills the geometric setup described in Assumption 2.2, notice that
\[ \Delta_\theta := S^c_\theta \cup \{(x, y) \in \mathbb{R}^2 : y < 0\} \]
is an \((\varepsilon, \delta)\)-domain for some values \(\varepsilon, \delta > 0\). Notice that the boundary of \(\Delta_\theta\) is the union of the positive \(x\)-axis and the boundary part of \(\partial S_\theta\) that lies in the upper half-plane and notice that \(\Omega \subset \Delta_\theta\). Since \(\Delta_\theta\) is an \((\varepsilon, \delta)\)-domain, we find for all \(x, y \in \Delta_\theta\) (so especially also for all \(x, y \in \Omega\)) with \(|x - y| < \delta\) a path \(\gamma\) that takes its values in \(\Delta_\theta\) such that
\[
\text{length}(\gamma) \leq \varepsilon^{-1}|x - y| \quad \text{and such that for all } z \in \gamma \quad d(z, \partial \Delta_\theta) \geq \varepsilon \frac{|x - z||y - z|}{|x - y|}.
\]
(4.1)

Since \(\Gamma \subset \partial \Delta_\theta\) it especially holds \(d(z, \Gamma) \geq d(z, \partial \Delta_\theta)\), so that (4.1) is valid with \(\partial \Delta_\theta\) replaced by \(\Gamma\).

To conclude the example, we show that there exists \(K > 0\) such that for all \(z \in \Delta_\theta\) it holds with \(\Xi := \mathbb{R}^2 \setminus \Gamma\)
\[
k_{\Xi}(\Omega, z) \leq K.
\]
(4.2)

Since the paths obtained above take their values only in \(\Delta_\theta\) this will establish the remaining condition (2.4). Notice that since \(S_\theta \cap \{(x, y) \in \mathbb{R}^2 : y < 0\} \subset \Omega\) it suffices to show that there exists \(K > 0\) such that for all \(z \in \Delta_\theta\) it holds
\[
k_{\Xi}(S_\theta \cap \{(x, y) \in \mathbb{R}^2 : y < 0\}, z) \leq K.
\]

In the following, we only describe one particular situation, since all other arising cases are similar. Assume that \(\theta < \pi/2\) and let for example \(z = (v, w) \in \Delta_\theta\) with \(v \geq 0\) and \(w > 0\). Choose \((x, y) \in \partial S_\theta\) such that \(y := -w\) and let \(\gamma := \gamma_1 + \gamma_2 + \gamma_3\) with
\[
\begin{align*}
\gamma_1 : [0, 1] \rightarrow \mathbb{R}^2, \quad t \mapsto (x, y) + t(y - x, 0), \\
\gamma_2 : [0, 1] \rightarrow \mathbb{R}^2, \quad t \mapsto (y, y) + t(0, w - y), \\
\gamma_3 : [0, 1] \rightarrow \mathbb{R}^2, \quad t \mapsto (y, w) + t(v - y, 0).
\end{align*}
\]

See Figure 2 for this configuration. The path \(\gamma\) then connects \((x, y)\) to \((v, w)\) and

![Figure 2](image_url)
\[
\kappa_{\Xi}(x,y), v, w) \leq \int_0^1 \frac{|y-x|}{|y|} \, dt + \int_0^1 \frac{|w-y|}{|y|} \, dt + \int_0^1 \frac{|v-y|}{w} \, dt = 2 + \frac{|y-x| + v + w}{w}.
\]

Notice that \( x = w/\tan(\theta) \) and that \( w \geq v \tan(\theta) \), so that
\[
\kappa_{\Xi}(x,y), (v, w) \leq 2 \left( 2 + \frac{1}{\tan(\theta)} \right).
\]

Finally, notice that in the remaining cases \( v < 0 \) and \( w \geq 0 \), \( v < 0 \) and \( w < 0 \), or \( v \geq 0 \) and \( w < 0 \) the quasihyperbolic distance to \( \Omega \) would only be smaller. This proves the validity of (4.2) and thus that \( \Omega \) fulfills Assumption 2.2.

**Remark 4.6.** Notice that the geometric setup considered by Brewster et al. (see Assumption 4.1) imposes boundary regularity in an open neighborhood of \( \Gamma \), while in the situation described in Example 4.5 the portion \( D \) of \( \partial \Omega \) can be arbitrarily irregular as long as it remains outside of \( S_\theta \).

We conclude this section by giving examples of domains where the boundary portion \( D \) fails to remain outside of a sector \( S_\theta \) and show that the \( W^{1,p}_D \)-extension property fails for these types of domains. These examples essentially show, that interior cusps that lie directly on the interface separating \( D \) and \( \Gamma \) destroy the \( W^{1,p}_D \)-extension property. The same happens with 'interior cusps at infinity', i.e., if \( D \) and \( \Gamma \) approach themselves at infinity at a certain rate. To present the examples, let \( Q(x,r) \) denote a cube in \( \mathbb{R}^2 \) with center \( x \) and sidelength \( 2r \).

**Example 4.7** (Interior boundary cusp in zero). Let \( \alpha \in (1, \infty), R > 0 \), and let \( \Omega \subset \mathbb{R}^2 \) be such that

\[
\Omega \cap Q(0,R) = [\mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : x \geq 0 \text{ and } -x^\alpha \leq y \leq 0\}] \cap Q(0,R).
\]

Define \( D \) and \( \Gamma \) such that

\[
D \cap Q(0,R) = \{(x,y) \in \mathbb{R}^2 : x \geq 0 \text{ and } -x^\alpha = y \} \cap Q(0,R)
\]

and

\[
\Gamma \cap Q(0,R) = \{(x,0) \in \mathbb{R}^2 : x > 0\} \cap Q(0,R) = (0,R) \times \{0\}.
\]

See Figure 3 for such a configuration. To prove that the \( W^{1,p}_D \)-extension property fails, let \( 1 < p < \infty \) and \( 0 < r < R/2 \). Let \( f_r \) be a smooth function, that is supported in

\[
Q_r := \{(x,y) \in \mathbb{R}^2 : r/2 \leq x \leq 2r \text{ and } 0 \leq y \leq r\},
\]
satisfies \( 0 \leq f_r \leq 1 \), and is identically 1 on

\[
R_r := \{(x,y) \in \mathbb{R}^2 : 3r/4 \leq x \leq 3r/2 \text{ and } 0 \leq y \leq r/2\}.
\]

Moreover, let \( f_r \) be such that \( \|\nabla f_r\|_{L^\infty} \leq Cr^{-1} \) for some constant \( C > 0 \). In this case

\[
\|f_r\|_{W^{1,p}(\Omega)}^p \leq C(r^2 + r^{2-p}).
\]

Next, employ the fundamental theorem of calculus and a density argument to conclude that for all \( F \in W^{1,p}(\mathbb{R}^d) \) it holds

\[
\int_{3r/4}^{3r/2} F(x,0) \, dx - \int_{3r/4}^{3r/2} F(x,-x^\alpha) \, dx = \int_{3r/4}^{3r/2} \int_{-x^\alpha}^0 \partial_y F(x,y) \, dy \, dx.
\]

If there exists a bounded extension operator \( E : W^{1,p}_D(\Omega) \to W^{1,p}_D(\mathbb{R}^2) \), put \( F := Ef_r \) and conclude that the second integral on the left-hand side vanishes since \( Ef \in W^{1,p}_D(\mathbb{R}^2) \). Using further that by construction the trace of \( Ef_r \) onto the set \((3r/4,3r/2) \times \{0\}\) is identically 1 one concludes

\[
\frac{3r}{4} \leq \int_{3r/4}^{3r/2} \int_{-x^\alpha}^0 |\partial_y Ef_r(x,y)| \, dy \, dx \leq C r^{(\alpha+1)/p'} \|Ef_r\|_{W^{1,p}(\mathbb{R}^2)}.
\]
Dividing by \( r \) and using that \( E \) is bounded delivers together with (4.3) the relation (4.4) 
\[
1 \leq C r^{(\alpha + 1)/p' - 1/2} (r^{2/p} + r^{2/p-1}),
\]
which results for \( r \to 0 \) in the condition
\[
\frac{\alpha + 1}{p'} + \frac{2}{p} - 2 \leq 0 \quad \Leftrightarrow \quad \alpha \leq 1.
\]
This is a contradiction since \( \alpha \) is assumed in \((1, \infty)\). Thus, there cannot be a bounded extension operator \( E : W^{1,p}_D(\Omega) \to W^{1,p}_D(\mathbb{R}^2) \).

**Example 4.8** (Interior boundary cusp at infinity). Let \( \alpha \in (0, \infty) \) and \( R > 0 \) and let \( \Omega \subset \mathbb{R}^2 \) be such that
\[
\Omega \cap \{(x, y) \in \mathbb{R}^2 : x > R\} = \{(x, y) \in \mathbb{R}^2 : x > R \text{ and } y < -x^{-\alpha} \text{ or } y > 0\}.
\]
Define \( D \) and \( \Gamma \) such that
\[
D \cap \{(x, y) \in \mathbb{R}^2 : x > R\} = \{(x, y) \in \mathbb{R}^2 : x > R \text{ and } -x^{-\alpha} = y\}
\]
and
\[
\Gamma \cap \{(x, y) \in \mathbb{R}^2 : x > R\} = \{(x, 0) \in \mathbb{R}^2 : x > R\} = (R, \infty) \times \{0\}.
\]
See Figure 4 for such a configuration. The proof that in this situation for no \( p \in (1, \infty) \) there exists a bounded extension operator \( E \) from \( W^{1,p}_D(\Omega) \) to \( W^{1,p}_D(\mathbb{R}^2) \) is similar to Example 4.7. The only modification is that \( Q_r \) and \( R_r \) are replaced for \( 0 < \beta < 1 \) and \( r > 2R \) by
\[
Q_r^\beta := \{(x, y) \in \mathbb{R}^2 : r/2 \leq x \leq 2r \text{ and } 0 \leq y \leq r^\beta\}
\]
and
\[
R_r^\beta := \{(x, y) \in \mathbb{R}^2 : 3r/4 \leq x \leq 3r/2 \text{ and } 0 \leq y \leq r^\beta/2\}.
\]
Take a smooth function \( f_r^\beta \) that satisfies \( 0 \leq f_r^\beta \leq 1 \), that is identically 1 on \( R_r^\beta \), supported in \( Q_r^\beta \), and satisfies \( \| \nabla f_r^\beta \|_{L^\infty} \leq C r^{-\beta} \) for some constant \( C > 0 \). As in Example 4.7 one derives a relation similar to (4.4), which reads in the present situation
\[
1 \leq C r^{(1 - \alpha)/p' - 1/2} (r^{1 + \beta}/p + r(1 + \beta)/p - \beta).
\]
For \( r \to \infty \) this results in the condition
\[
\frac{1 - \alpha}{p'} - 1 + \frac{1 + \beta}{p} \geq 0 \quad \Leftrightarrow \quad \alpha \leq \frac{\beta p'}{p}.
\]
As \( \beta \in (0, 1) \) was arbitrary, for \( \beta \to 0 \) this yields \( \alpha \leq 0 \). Thus, there cannot exist a bounded extension operator \( E : W^{1,p}_D(\Omega) \to W^{1,p}_D(\mathbb{R}^2) \).

**Example 4.9** (Exterior boundary cusps at zero or at infinity). Let \( \Omega \) be a domain that has an exterior boundary cusp either at zero or at infinity, as it is informally depicted in Figure 5. In this case, \( \Omega \) is an \( W^{1,p}_D \)-extension domain as a simple reflection argument shows. However, it seems to be not so clear of how to verify, if possible, the validity of Assumption 2.2. Nevertheless, it is simple to verify the validity of the geometric setting stated in Corollary 3.2. Indeed, simply take as \( \Omega := \mathbb{R}^2 \) and notice that the parameter \( K \) in Assumption 2.2 can be set to zero, as \( \mathbb{R}^2 \) is already an \((\varepsilon, \delta)\)-domain.

We can even go further and extend the geometric setting from Example 4.5 to the following one (see Figure 5).

Let \( \theta \in (0, \pi) \) and let \( S_\theta \subset \mathbb{R}^2 \) denote the open sector symmetric about the positive \( x \)-axis with opening angle \( 2\theta \). Define \( T_\theta := S_\theta \cap \{(x, y) \in \mathbb{R}^2 : y > 0\} \). Let \( \Omega \subset \mathbb{R}^2 \) be a domain satisfying \( T_\theta \subset \Omega^c \) and define
\[
\Gamma := (0, \infty) \times \{0\} \quad \text{and} \quad D := \partial \Omega \setminus \Gamma.
\]
Assume further, that $\Omega$ is such that $D$ is closed (this avoids that $D$ touches $\Gamma$ from below). To apply Corollary 3.2 take $\Omega_F := (T^c_\delta)^\circ$. As this is an $(\varepsilon, \delta)$-domain, it satisfies Assumption 2.2 with $K = 0$ and thus, for all $1 \leq p < \infty$ there exists a bounded extension operator $E : W^{1,p}_D(\Omega) \rightarrow W^{1,p}_D(\mathbb{R}^d)$.

5. Whitney decompositions and the quasihyperbolic distance

In this section, we introduce the Whitney decomposition of an open subset of $\mathbb{R}^d$ and show how condition (2.4) relates to properties of Whitney cubes. A cube $Q \subset \mathbb{R}^d$ is always closed and is said to be dyadic if there exists $k \in \mathbb{Z}$, such that $Q$ coincides with a cube of the mesh determined by the lattice $2^{-k}\mathbb{Z}^d$. Two cubes are said to touch, if a face of one cube lies in a face of the other cube. The sidelength of a cube is denoted by $l(Q)$. For a number $\alpha > 0$ the dilation of $Q$ about its center by the factor $\alpha$ is denoted by $\alpha Q$. 
Let $F \subset \mathbb{R}^d$ be a non-empty closed set. Then, by [18, Thm. VI.1] there exists a collection of cubes $\{Q_j\}_{j \in \mathbb{N}}$ with pairwise disjoint interiors such that

(i) $\bigcup_{j \in \mathbb{N}} Q_j = \mathbb{R}^d \setminus F$,
(ii) $\text{diam}(Q_j) \leq d(Q_j, F) \leq 4 \text{diam}(Q_j)$ for all $j \in \mathbb{N}$.

By an inspection of the proof of [18, Thm. VI.1] and by [18, Prop. VI.1] one further gets

(iii) the cubes $\{Q_j\}_{j \in \mathbb{N}}$ are dyadic,
(iv) $\frac{1}{4} \text{diam}(Q_j) \leq \text{diam}(Q_k) \leq 4 \text{diam}(Q_j)$ if $Q_j \cap Q_k \neq \emptyset$.

The collection $\{Q_j\}_{j \in \mathbb{N}}$ are called Whitney cubes will be referred to as $\mathcal{W}(F)$. We say, that a collection of cubes $Q_1, \ldots, Q_m \in \mathcal{W}(F)$ is a *touching chain* if $Q_j$ and $Q_{j+1}$ are touching cubes and it is an *intersecting chain* if $Q_j \cap Q_{j+1} \neq \emptyset$ for all $j = 1, \ldots, m - 1$. The *length* of a chain is the number $m$.

The following lemma translates (2.4) to the existence of intersecting chains of uniformly bounded length. Notice that if $\Xi = \mathbb{R}^d \setminus \Gamma$, Gehring and Osgood [11, Lem. 1] proved that for any two points $x, y \in \Xi_m$ there exists a quasi-hyperbolic geodesic $\gamma_{x,y}$ with endpoints $x$ and $y$ satisfying

$$k_{\Xi}(x, y) = \int_{\gamma_{x,y}} \frac{1}{d(z, \partial \Xi)} |dz|.$$ 

Trivially, if $\Xi = \mathbb{R}^d$, then any path connecting $x$ and $y$ is a quasihyperbolic geodesic.

**Lemma 5.1.** Fix $k > 0$. There exists a constant $N = N(d, k) \in \mathbb{N}$ such that for all $x, y \in \Xi$ with $k_{\Xi}(x, y) \leq k$ there exists an intersecting chain $Q_1, \ldots, Q_m \in \mathcal{W}(\Gamma)$ with $x \in Q_1$ and $y \in Q_m$ and $m \leq N$.

Conversely, if for $x, y \in \Xi$ there exists an intersecting chain connecting $x$ and $y$ of length less than $N \in \mathbb{N}$, then there exists a constant $k = k(N) > 0$ such that $k_{\Xi}(x, y) \leq k$.

**Proof.** Notice that $k_{\Xi}(x, y) < \infty$ implies that $x$ and $y$ lie in the same connected component. Assume first that

$$|x - y| \leq \frac{1}{10\sqrt{d}} \min\{d(x, \Gamma), d(y, \Gamma)\}.$$ 

Let $Q_x, Q_y \in \mathcal{W}(\Gamma)$ with $x \in Q_x$ and $y \in Q_y$, and let $\tilde{Q}_x$ denote the region occupied by $Q_x$ and all its intersecting Whitney cubes and similarly let $\tilde{Q}_y$ denote its counterpart for $Q_y$. Then by (iv)

$$d(x, \tilde{Q}_x) \geq \frac{1}{4\sqrt{d}} \text{diam}(Q_x) \quad \text{and} \quad d(y, \tilde{Q}_y) \geq \frac{1}{4\sqrt{d}} \text{diam}(Q_y).$$

Moreover, by (iii) and the triangle inequality it holds

$$\text{diam}(Q_x) \geq \frac{1}{4} d(Q_x, \Gamma) \geq \frac{1}{4} [d(x, \Gamma) - \text{diam}(Q_x)],$$

so that by (5.1) it follows

$$d(x, \tilde{Q}_x) \geq \frac{1}{4\sqrt{d}} \text{diam}(Q_x) \geq \frac{1}{2} |x - y|.$$ 

By symmetry, the same is valid for $y$ instead of $x$. Consequently, $\tilde{Q}_x$ and $\tilde{Q}_y$ have a common point and thus, $x$ and $y$ can be connected by an intersecting chain of length at most 4.

Now, let

$$|x - y| > \frac{1}{10d} \min\{d(x, \Gamma), d(y, \Gamma)\}.$$ 

Assume without loss of generality that $d(x, \Gamma) \leq d(y, \Gamma)$. Then, Herron and Koskela [15, Prop. 2.2] ensures the existence of points $y_0 := x, y_1, \ldots, y_k \in \mathbb{R}^d \setminus \Gamma$ such that the quasihyperbolic geodesic
\( \gamma_{x,y} \) (provided by [11, Lem. 1]) is contained in the closure of \( \bigcup_{i=0}^{k} B_{i} \), where \( B_{i} := B(y_{i}, r_{i}) \) with \( r_{i} := d(y_{i}, \Gamma)/(10d) \), and such that

\[
(5.2) \quad k \leq 20d k_{\Xi}(x, y).
\]

Next, we estimate the number of Whitney cubes that cover each of these balls. Denote the number of Whitney cubes that cover each of these balls. Denote the Whitney decomposition delivers so that by definition of \( i \)

\[
\text{diam}(Q) \geq \frac{1}{4} d(Q, \Gamma) \geq \frac{1}{4} [d(y_{i}, \Gamma) - r_{i} - \text{diam}(Q)],
\]

so that by definition of \( r_{i} \)

\[
\text{diam}(Q) \geq \frac{(10d - 1) d(y_{i}, \Gamma)}{50d}.
\]

Moreover,

\[
\text{diam}(Q) \leq d(Q, \Gamma) \leq d(B_{i} \cap Q, \Gamma) \leq d(y_{i}, \Gamma) + \frac{d(y_{i}, \Gamma)}{10d} = \left[ 1 + \frac{1}{10d} \right] d(y_{i}, \Gamma).
\]

Consequently,

\[
W_{i} \left[ \frac{(10d - 1) d(y_{i}, \Gamma)}{50d} \right]^{d} \leq \sum_{Q \in W_{i}(\Gamma) \cap B_{i} \neq \emptyset} |Q| \leq \left| B \left( y_{i}, \left[ 1 + \frac{1}{5d} \right] d(y_{i}, \Gamma) \right) \right|,
\]

what proves that \( W_{i} \) is controlled by a constant depending only on \( d \). We conclude by (5.2) and by the bound on each \( W_{i} \) that there exists an intersecting chain connecting \( x \) and \( y \) of length bounded by a constant depending only on \( d \) and \( k \).

For the other direction, let \( Q_{1}, \ldots, Q_{m} \) be an intersecting chain with \( m \leq N \). Thus, by definition \( Q_{j} \cap Q_{j+1} \neq \emptyset \). Let \( \gamma \) be a path connecting \( x \) and \( y \) which is constructed by linearly connecting a point in \( Q_{j-1} \cap Q_{j} \) with a point in \( Q_{j} \cap Q_{j+1} \). Thus, employing Property [ii] of the Whitney decomposition delivers

\[
k_{\Xi}(x, y) \leq \sum_{j=1}^{m} \int_{\gamma \cap Q_{j}} \frac{1}{d(Q_{j}, \Gamma)} |dz| \leq \sum_{j=1}^{m} \frac{\text{diam}(Q_{j})}{\text{diam}(Q_{j})} \leq m. \quad \square
\]

6. Cubes and Chains

In this section, we describe how to 'reflect' cubes at \( \Gamma \) if \( \Omega \) is subject to Assumption 2.2 and establish some natural properties of these 'reflections'. This is an adaption of an argument of Jones presented in [16]. Throughout, assume in Sections 6 and 7 that \( \Omega \) is an open set subject to Assumption 2.2 which satisfies \( \overline{\Omega} \neq \emptyset \). We will from now on assume that radius \( \Gamma \) \( \geq 1 \), i.e., that \( r_{0} = 1 \). For general \( r_{0} > 0 \), Theorem 3.1 then follows by scaling. Moreover, without loss of generality assume that \( \delta \leq 1 \).

Lemma 6.1. We have \( |\Gamma| = 0 \).

Proof. Fix \( x_{0} \in \Gamma \) and \( y \in \Omega \) with \( |x_{0} - y| < \frac{\delta}{2} \). Let \( Q \) be any cube in \( \mathbb{R}^{d} \) centered in \( x_{0} \) with \( l(Q) \leq \frac{1}{2}|x_{0} - y| \). We will show that \( [\mathbb{R}^{d} \setminus \Gamma] \cap Q \) has Lebesgue measure comparable to that of \( Q \). Let \( x \in \Omega \) with \( |x - x_{0}| \leq \frac{1}{2} l(Q) \). Then, we have

\[
|x - y| \geq \frac{15}{8} l(Q) \quad \text{and} \quad |x - y| \leq \frac{17}{16} |x_{0} - y|.
\]

Let \( \gamma \) be a path connecting \( x \) and \( y \) subject to Assumption 2.2. By virtue of \( (6.1) \), the intermediate value theorem implies that there exists \( z \in \gamma \) with \( |x - z| = \frac{1}{8} l(Q) \). This point lies in \( \frac{1}{2} Q \) by
construction. Moreover, together with \(|y - z| \geq |x - y| - |x - z|\) implies
\[
d(z, \overline{\Gamma}) \geq \frac{\varepsilon l(Q)}{8} |x - y| - |x - z| \geq \frac{7\varepsilon}{60} l(Q).
\]
Thus, \(\limsup_{n(Q) \to 0} \frac{|\mathbb{R}^d \cap \Gamma|}{l(Q)} > 0\), where the \(\limsup\) is taken over all cubes centered at \(x_0\). Since \(\chi_{\mathbb{R}^d \setminus \Gamma}(x_0) = 0\) and \(\chi_{\mathbb{R}^d \setminus \Gamma} \in L^1_{\text{loc}}(\mathbb{R}^d)\) Lebesgue’s differentiation theorem implies \(|\Gamma| = 0\).

To proceed, we define two families of cubes. The family of interior cubes is given by
\[
\mathcal{W}_i := \{Q \in \mathcal{W}(\overline{\Gamma}) : Q \cap \Omega \neq \emptyset\}.
\]
These interior cubes will be the reflections of exterior cubes \(\mathcal{W}_e\). To define \(\mathcal{W}_e\) choose numbers \(A > 0\) and \(B > 2\) whose values are to be fixed during this section and define
\[
\mathcal{W}_e := \{Q \in \mathcal{W}(\overline{\Omega}) : \text{diam}(Q) \leq A\delta \text{ and } \text{d}(Q, \overline{\Gamma}) < B \text{d}(Q, \partial \Omega \setminus \Gamma)\}.
\]

**Remark 6.2.** First of all, notice that \(\mathcal{W}_e\) is empty if and only if the condition on the distance and the diameter are not fulfilled. Notice that due to the relative openness of \(\Gamma\) and Property [ii] of the Whitney decomposition this condition is not fulfilled if and only if \(D = \partial \Omega\). Second, if \(D \neq \partial \Omega\) for a cube \(Q \in \mathcal{W}_e\) we have
\[
d(Q, \overline{\Omega}) = \min\{d(Q, \overline{\Gamma}), d(Q, \partial \Omega \setminus \Gamma)\} \geq B^{-1} d(Q, \overline{\Gamma})
\]
what implies that for all \(Q \in \mathcal{W}_e\) it holds
\[
d(Q, \overline{\Omega}) \leq d(Q, \overline{\Gamma}) \leq B d(Q, \overline{\Gamma}).
\]
Thus the diameter of \(Q\) is comparable to its distance to \(\Gamma\).

For the rest of this section, we assume that \(\Gamma \neq \emptyset\). Before we present how to ‘reflect’ cubes, we prove a technical lemma that, given an exterior cube \(Q \in \mathcal{W}_e\), allows us to find a connected component of \(\Omega\) whose boundary intersects \(\Gamma\) and which is not too far away from \(Q\).

**Lemma 6.3.** Let \(Q \in \mathcal{W}_e\). Then there exists a connected component \(\Omega_m\) of \(\Omega\) with \(\Gamma \cap \partial \Omega_m \neq \emptyset\) and \(x \in \Omega_m\) with
\[
d(x, Q) \leq 5B \text{diam}(Q).
\]

**Proof.** By Property [ii] of the Whitney decomposition and Remark 6.2 there exists \(x' \in \overline{\Gamma}\) such that \(d(x', Q) \leq 4B \text{diam}(Q)\). Since \(x' \in \overline{\Gamma}\) there is \(x'' \in \Gamma\) with \(d(x'', Q) \leq \frac{9}{2} B \text{diam}(Q)\).

Denote the at most countable family of connected components of \(\Omega\) whose boundary has a non-empty intersection with \(\Gamma\) by \(\{\Omega_m\}_m\) and the connected components whose boundary has an empty intersection with \(\Gamma\) by \(\{\Upsilon_m\}_m\).

If there is \(\Omega_m\) with \(x'' \in \partial \Omega_m\), then the proof is finished. If not, we establish the existence of a sequence \((x_n)_{n \in \mathbb{N}}\) and indices \(m_n\) with \(x_n \in \Omega_{m_n}\) and \(x_n \to x''\) as \(n \to \infty\), whose existence still concludes the proof. To this end, assume to the contrary that there exists \(\eta > 0\) such that for all \(x \in \bigcup_m \Omega_m\)
\[
|x - x''| \geq \eta.
\]
However, because \(x'' \in \partial \Omega\) there exists a sequence \((x_n)_{n \in \mathbb{N}}\) in \(\Omega\) with \(x_n \to x''\) as \(n \to \infty\). For \(n\) large there must therefore exist indices \(m_n\) with \(x_n \in \Upsilon_{m_n}\). If \(x'' \in \partial \Upsilon_{m_n}\) for some \(n\) we arrive at a contradiction, because \(\Gamma \cap \partial \Upsilon_{m_n} = \emptyset\). Hence, \(x'' \in \Upsilon_{m_n}\). Now, by connecting \(x''\) and \(x_n\) by a straight line, the intermediate value theorem implies the existence of a point \(x'_n \in \partial \Upsilon_{m_n}\) with
\[
|x'' - x'_n| \leq |x'' - x_n|.
\]
Passing to the limit \(n \to \infty\) yields \(x'' \in D\) by the closedness of \(D\) and thus a contradiction. \(\square\)
The following lemma assigns to every cube in $\mathcal{W}_c$ a ‘reflected’ cube in $\mathcal{W}_i$. For the rest of Sections 6 and 7 we will reserve the letter $N$ to denote the constant $N$ appearing in Lemma 5.1 applied with $k = 2K$, where $K$ is the number from Assumption 2.2. Notice that $N$ solely depends on $d$ and $K$. For the rest of the paper, make the following agreement.

**Agreement 6.4.** If $X$ and $Y$ are two quantities and if there exists a constant $C$ depending only on $d$, $p$, $K$, $\varepsilon$, and $\delta$ such that $X \leq CY$ holds, then we will write $X \lesssim Y$ or $Y \gtrsim X$. If both $\frac{Y}{C} \leq X \leq CY$ holds, then we will write $X \simeq Y$.

**Remark 6.5.** The dependence on the parameter $p$ and $\delta$ only occurs in Section 4.

**Lemma 6.6.** There exists a constant $C = C(N, \varepsilon) > 0$ such that if $AB \leq C$, then for every $Q \in \mathcal{W}_c$ there exists a cube $R \in \mathcal{W}_i$ satisfying

$$\text{(6.3)} \quad \text{diam}(Q) \leq \text{diam}(R) \lesssim (1 + B + (AB)^{-1}) \text{diam}(Q)$$

and

$$\text{(6.4)} \quad \text{d}(R, Q) \lesssim (1 + B + (AB)^{-1}) \text{diam}(Q).$$

**Proof.** Fix $Q \in \mathcal{W}_c$ and recall that $\text{diam}(Q) \leq A\delta$ by definition of $\mathcal{W}_c$. By Lemma 6.3 there exists a connected component $\Omega_m$ of $\Omega$ with $\Gamma \cap \partial \Omega_m \neq \emptyset$ and $x \in \Omega_m$ with $d(x, Q) \leq 5B \text{diam}(Q)$. Since $(AB)^{-1} \text{diam}(Q) \leq \delta B^{-1} < 1$ the assumption $\text{radius}_1(\Omega) \geq 1$ implies the existence of $y \in \Omega_m$ satisfying

$$\text{(6.5)} \quad |x - y| = (AB)^{-1} \text{diam}(Q) \quad \text{and} \quad |x - y| < \delta.$$

Let $\gamma$ be a path subject to Assumption 2.2 connecting $x$ and $y$ and let $z \in \gamma$ with $|x - z| = \frac{1}{2} |x - y|$. Estimate by virtue of (2.3)

$$\text{(6.6)} \quad d(z, \Gamma) \geq \frac{\varepsilon}{2} |y - z| \geq \frac{\varepsilon}{2} (|x - y| - |x - z|) = \frac{\varepsilon}{2} (AB)^{-1} \text{diam}(Q).$$

By Assumption 2.2 we have $k\varepsilon(z, \Omega) \leq K$, hence there exists $z' \in \Omega$ with $k\varepsilon(z, z') \leq 2K$. Thus, by Lemma 5.1 there exists an intersecting chain $Q_1, \ldots, Q_m \in \mathcal{W}(\Gamma)$ with $Q_m \cap \Omega \neq \emptyset$, $z \in Q_1$, and $m \leq N$. Choose the reflected cube as $R := Q_m$. Using Properties (iii) and (iv) of the Whitney decomposition one gets

$$4 \text{diam}(R) \geq d(R, \Gamma) \geq d(z, \Gamma) - \sum_{j=1}^{m} \text{diam}(Q_j) \geq d(z, \Gamma) - \sum_{j=1}^{m} 4^{m-j} \text{diam}(R).$$

Thus, by (6.6) and $m \leq N$

$$\frac{11 + 4^N}{3} \text{diam}(R) \geq \frac{\varepsilon}{2} (AB)^{-1} \text{diam}(Q).$$

Consequently, there exists $C = C(N, \varepsilon) > 0$ such that $AB \leq C$ implies $\text{diam}(Q) \leq \text{diam}(R)$.

In order to control $\text{diam}(R)$ by $\text{diam}(Q)$, employ Properties (ii) and (iv) of the Whitney decomposition and the triangle inequality to deduce

$$4^{1-m} \text{diam}(R) \leq \text{diam}(Q_1) \leq d(z, \Gamma) \leq d(z, Q) + \text{diam}(Q) + d(Q, \Gamma).$$

The right-hand side is estimated by the triangle inequality, followed by (6.2) and Property (ii) of the Whitney decomposition, the choice $|x - z| = \frac{1}{2} |x - y|$ combined with (6.5), and $d(x, Q) \leq 5B \text{diam}(Q)$, yielding

$$d(z, Q) + \text{diam}(Q) + d(Q, \Gamma) \leq |z - x| + d(x, Q) + \text{diam}(Q) + B d(Q, \Gamma) \leq ((2AB)^{-1} + 1 + 9B) \text{diam}(Q).$$
Taking into account that \(d(z, R) \leq \text{diam}(R)(4^m - 1)/3\), the distance from \(R\) to \(Q\) is estimated similarly, yielding

\[
d(R, Q) \leq \text{diam}(R) + \text{diam}(Q) + |x - z| + d(z, R) + d(x, Q) \\
\leq (1 + (2AB)^{-1} + 5B) \text{diam}(Q) + \frac{4^m + 2}{3} \text{diam}(R).
\]

Together with the previous estimate, this concludes the proof. \(\square\)

For the rest of this article, we fix the notation that if \(Q \in \mathcal{W}_e\) and \(R \in \mathcal{W}_i\) is the cube constructed in Lemma 6.6, then \(R\) is denoted by \(R = Q^*\) and \(Q^*\) is called the reflected cube of \(Q\). The next lemma gives a bound on the distance of reflected cubes of two intersecting cubes. Its proof is a direct consequence of Lemma 6.6 and Property (iv) of the Whitney decomposition and is thus omitted.

**Lemma 6.7.** If \(Q_1, Q_2 \in \mathcal{W}_e\) with \(Q_1 \cap Q_2 \neq \emptyset\), then

\[
d(Q_1^*, Q_2^*) \lesssim (1 + B + (AB)^{-1}) \text{diam}(Q_1).\]

In the proof of the boundedness of the extension operator, one needs to connect Whitney cubes by appropriate touching chains. The following lemma presents a basic principle of how to build a chain out of a path \(\gamma\) and how the quantities \(\text{length}(\gamma)\) and \(d(\gamma, \Gamma)\) translate into the length of the chain and the distance of the cubes of the chain to \(\Gamma\).

**Lemma 6.8.** Let \(R_1, R_2 \in \mathcal{W}((\Gamma))\) with \(R_1 \neq R_2\) and let \(x \in R_1, y \in R_2,\) and \(\gamma\) be a rectifiable path in \(\mathbb{R}^d \setminus \Gamma\) connecting \(x\) and \(y\). Assume that there exist constants \(C_1, C_2 > 0\) such that \(\text{length}(\gamma) \leq C_1 \text{diam}(R_1)\) and \(d(z, \Gamma) \geq C_2 \text{diam}(R_1)\) for all \(z \in \gamma\), then there exists a touching chain of cubes \(R_1 = S_1, \ldots, S_m = R_2\) in \(\mathcal{W}((\Gamma))\), where \(m\) is bounded by a number depending only on \(d, C_1,\) and \(C_2\). Moreover,

\[
\frac{C_2}{5} \text{diam}(R_1) \leq \text{diam}(S_i) \leq (5 + C_1) \text{diam}(R_1) \quad (i = 1, \ldots, m).
\]

**Proof.** Let \(S\) be the finite set of cubes in \(\mathcal{W}((\Gamma))\) intersecting \(\gamma\). For \(S \in S\) one finds by Property (ii) of the Whitney decomposition and by assumption that \(\text{diam}(S) \geq C_2/5 \text{diam}(R_1)\). Fix \(z \in S \cap \gamma\), then

\[
d(z, \Gamma) \leq d(x, \Gamma) + |x - z| \leq 5 \text{diam}(R_1) + \text{length}(\gamma) \leq (5 + C_1) \text{diam}(R_1),
\]

so that \(\text{diam}(S) \leq (5 + C_1) \text{diam}(R_1)\) by Property (ii) of the Whitney decomposition. This, together with \(\text{length}(\gamma) \leq C_1 \text{diam}(R_1)\) implies that \(S \subseteq B(x, (5 + 2C_1) \text{diam}(R_1))\). Because all elements of \(S\) are mutually disjoint one finds

\[
\sharp(S) \leq \frac{|B(x, (5 + 2C_1) \text{diam}(R_1))|}{\left(\frac{5 \text{diam}(R_1)}{C_2}\right)^d} = \omega_d \left(\frac{5 \sqrt{d}(5 + 2C_1)}{C_2}\right)^d,
\]

where \(\sharp(S)\) denotes the cardinality of \(S\) and \(\omega_d := |B(0, 1)|\). By Property (iii) of the Whitney decomposition one finds that \(S\) are dyadic and thus one finds a subset of \(S\) which is a touching chain starting at \(R_1\) and ending at \(R_2\). \(\square\)

**Lemma 6.9.** There exist constants \(C_1, C_2 > 0\) depending only on \(\varepsilon, d,\) and \(K\) such that if \(A \leq C_1\) and \(B \geq C_2\) and if \(Q_j, Q_k \in \mathcal{W}_i\) with \(Q_j \cap Q_k \neq \emptyset\), then there exists a touching chain \(F_{j,k} = \{Q_j^* = S_1, \ldots, S_m = Q_k^*\}\) of cubes in \(\mathcal{W}(\Gamma)\) connecting \(Q_j^*\) and \(Q_k^*\), where \(m\) can be bounded uniformly by a constant depending only on \(\varepsilon, d, K, A,\) and \(B\). Moreover, there exist \(K_1, K_2 > 0\) depending only on \(\varepsilon, d, K, A,\) and \(B\) such that

\[
K_1 \text{diam}(Q_j) \leq \text{diam}(S_i) \leq K_2 \text{diam}(Q_j) \quad (i = 1, \ldots, m).
\]
Proof. If $Q_j^* = Q_k^*$, there is nothing to show. Thus, assume $Q_j^* \neq Q_k^*$. We show in the following that the assumptions of Lemma 6.8 are satisfied.

Fix $x \in Q_j^* \cap \Omega$ and $y \in Q_k^* \cap \Omega$. Since $\text{diam}(Q_j) \leq A\delta$ one obtains by Lemmas 6.6 and 6.7

$$|x - y| \leq d(Q_j^*, Q_k^*) + \text{diam}(Q_j^*) + \text{diam}(Q_k^*) \lesssim (1 + B + (AB)^{-1}) \text{diam}(Q_j).$$

Estimating $\text{diam}(Q_j)$ by $A\delta$ (this is possible since $Q_j \in \mathcal{W}_\varepsilon$) shows that choosing first $B$ large enough and then $A$ small enough ensures $|x - y| < \delta$. Let $\gamma$ be a path connecting $x$ and $y$ according to Assumption 2.2. By (2.2), (6.7), and (6.3) one finds

$$\text{length}(\gamma) \lesssim (1 + B + (AB)^{-1}) \text{diam}(Q_j^*).$$

To estimate the distance between each $z \in \gamma$ and $\Gamma$, notice that if $|x - z| \leq \frac{1}{2} \text{diam}(Q_j^*)$, then $d(z, \Gamma) \geq \frac{1}{2} \text{diam}(Q_j^*)$. Analogously, but by employing additionally (6.3) twice and Property (iv) of the Whitney decomposition, if $|y - z| \leq \frac{1}{2} \text{diam}(Q_k^*)$, then

$$d(z, \Gamma) \geq \frac{1}{2} \text{diam}(Q_k^*) \geq \frac{1}{2} \text{diam}(Q_j) \gtrsim (1 + B + (AB)^{-1})^{-1} \text{diam}(Q_j^*).$$

In the remaining case, one estimates by (2.3), the calculation performed in (6.8), and (6.7) that

$$d(z, \Gamma) \gtrsim \frac{\text{diam}(Q_j^*)^2}{(1 + B + (AB)^{-1})|x - y|^2} \gtrsim \frac{\text{diam}(Q_j^*)}{(1 + B + (AB)^{-1})^2}. \quad \square$$

The following lemma provides the existence of chains that ‘escape $\Omega$’ for reflections of cubes $Q \in \mathcal{W}(\overline{\Omega})$ that are close to a relatively open portion of $D$. These chains will be important to obtain a Poincaré inequality with a quantitative control of the constants.

Lemma 6.10. Assume that $\partial \Omega \setminus \Gamma \neq \emptyset$. There exist constants $C_1, C_2 > 0$ depending only on $\varepsilon$, $d$, and $K$ such that if $A \leq C_1$ and $B \geq C_2$ and if $Q \in \mathcal{W}(\overline{\Omega}) \setminus \mathcal{W}_\varepsilon$ satisfies $\text{diam}(Q) \leq A\delta$ and has a non-empty intersection with a cube in $\mathcal{W}_\varepsilon$ of $Q$, there exists a touching chain $F_{P,j} = \{Q_j = S_1, \ldots, S_m\}$ of cubes in $\mathcal{W}(\overline{\Gamma})$, where $m$ is bounded by a constant depending only on $\varepsilon$, $d$, $K$, $A$, and $B$ and $S_m$ satisfies

$$|S_m \cap Q_j| \gtrsim \text{diam}(Q_j)^d.$$

Furthermore, all $S_i \in F_{P,j}$ satisfy

$$K_1 \text{diam}(Q_j) \leq \text{diam}(S_i) \leq K_2 \text{diam}(Q_j) \quad (i = 1, \ldots, m).$$

The constants $K_1, K_2 > 0$ depend only on $\varepsilon$, $d$, $K$, $A$, and $B$.

Proof. Let $Q_j \in \mathcal{W}_\varepsilon$ be an intersecting cube of $Q$. Then, by virtue of Property (iv) and $Q \notin \mathcal{W}_\varepsilon$, one estimates

$$Bd(Q_j, \partial \Omega \setminus \Gamma) \leq 6Bd(Q, \partial \Omega \setminus \Gamma) \leq 6d(Q, \Gamma) \leq 36 \text{diam}(Q_j, \Gamma).$$

Let $B \geq 720$, then (6.9) implies that $d(Q_j, \partial \Omega \setminus \Gamma) = d(Q_j, \overline{\Omega})$ and by virtue of (6.9) and Property (iv) of the Whitney decomposition one finds that $d(Q_j, \Gamma) \geq \frac{B}{36} \text{diam}(Q_j)$. Let $x_0 \in \partial \Omega \setminus \overline{\Gamma}$ be such that $d(x_0, Q_j) = d(Q_j, \overline{\Omega})$. The properties collected above then imply

$$d(x_0, \Gamma) \geq d(Q_j, \Gamma) - d(x_0, Q_j) - \text{diam}(Q_j) \geq (36^{-1}B - 5) \text{diam}(Q_j)$$

and if $y$ is any point from $B(x_0, 5 \text{diam}(Q_j))$, then the previous estimate combined with (6.3) delivers

$$d(y, \Gamma) \geq d(x_0, \Gamma) - 5 \text{diam}(Q_j) \geq (36^{-1}B - 10) \text{diam}(Q_j).$$

Notice that the midpoint $z$ of $Q_j$ is contained in $B(x_0, 5 \text{diam}(Q_j))$. Thus, each point on the line segment $\gamma_1$ connecting a point $y \in B(x_0, 5 \text{diam}(Q_j)) \cap \Omega$ to $z$ has at least a distance which is larger than $(36^{-1}B - 10) \text{diam}(Q_j)$ to $\overline{\Gamma}$. Notice that this distance is comparable to $\text{diam}(Q_j)$ by virtue of (6.3).
For \( x \in Q_j^* \cap \Omega \) Lemma \( \text{6.6} \) together with \( \{y\} \cup Q_j \subset B(x_0, 5 \operatorname{diam}(Q_j)) \) implies
\[
|x - y| \leq d(y, Q_j) + d(Q_j, Q_j^*) + \operatorname{diam}(Q_j) + \operatorname{diam}(Q_j^*) \lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q_j).
\]

Recall that by definition of \( W_\varepsilon \) it holds \( \operatorname{diam}(Q_j) \leq A\delta \). Choosing \( B \) large enough and \( A \) small enough delivers then that \( |x - y| < \delta \). Let \( \gamma_2 \) be the path connecting \( x \) and \( y \) subject to Assumption \( 2.2 \). Since \( d(Q, \Gamma) \geq C \operatorname{diam}(Q_j^*) \) for some \( Q \in W(\Gamma) \) with \( y \in Q \) and \( C > 0 \) depending only on \( \varepsilon, K, d, A, \) and \( B \), one concludes as in the proof of Lemma \( \text{6.9} \) that the path \( \gamma_2 \), and hence, by the consideration above, also the path \( \gamma = \gamma_1 + \gamma_2 \) satisfies the assumptions of Lemma \( \text{6.8} \) with constants depending only on \( \varepsilon, K, d, A, \) and \( B \). If \( S_m \) denotes the cube provided by Lemma \( \text{6.8} \) which contains \( z \), then one distinguishes the following cases.

Since \( Q_j \cap S_m \neq \emptyset \) and since Whitney cubes are dyadic, it either holds \( S_m \subset Q_j \) or \( Q_j \subset S_m \). If \( Q_j \subset S_m \) the proof is finished. If \( S_m \subset Q_j \), then
\[
4 \operatorname{diam}(S_m) \geq d(S_m, \overline{\Gamma}) \geq d(Q_j, \overline{\Gamma}) \geq 36^{-1} B \operatorname{diam}(Q_j)
\]
so that \( |S_m \cap Q_j| \gtrsim \operatorname{diam}(Q_j)^4 \). \( \square \)

The next lemma shows that for a fixed cube \( R \in W_i \), there are only finitely many cubes in \( W_\varepsilon \), whose reflected cube is \( R \).

**Lemma 6.11.** There is a constant \( C \in \mathbb{N} \) such that for each \( R \in W_i \) there are at most \( C \) cubes \( Q \in W_\varepsilon \) such that \( Q^* = R \), where \( C \) solely depends on \( d, K, A, B, \) and \( \varepsilon \).

**Proof.** Let \( \alpha \) denote the constant from \( \text{(6.4)} \), i.e., it holds \( d(R, Q) \leq \alpha \operatorname{diam}(Q) \) if \( R \) is the reflected cube of \( Q \). Thus, since \( \operatorname{diam}(Q) \leq \operatorname{diam}(R) \), a cube \( Q \in W_\varepsilon \) satisfying \( d(R, Q) > \alpha \operatorname{diam}(R) \) cannot have \( R \) as a reflected cube. So, if \( x_R \) denotes the center of \( R \), every cube \( Q \) with \( Q^* = R \) must be contained in \( B(x_R, (\alpha + \frac{1}{2}) \operatorname{diam}(R)) \). Because for those cubes \( \operatorname{diam}(Q) \) is controlled from below by \( \operatorname{diam}(R) \) according to \( \text{(6.3)} \) and because cubes from \( W_\varepsilon \) have disjoint interiors, the lemma follows by a counting argument. \( \square \)

7. **The extension operator**

This section is devoted to the construction of the extension operator, whose existence is stated in Theorem \( \text{3.1} \). The numbers \( A \) and \( B \), which were introduced in Section \( \text{6} \), will be considered as fixed numbers depending only on \( N \) and \( \varepsilon \) such that all the statements in Section \( \text{6} \) are valid. Therefore, due to the correspondence between the numbers \( K \) and \( N \), all constants from Section \( \text{6} \) depend only on \( \varepsilon, d, \) and \( K \).

Since \( W_\varepsilon \) is countable there exists an enumeration of \( W_\varepsilon \) which is denoted by \( \{Q_j\}_{j \in \mathbb{N}} \) and which is fixed for the rest of this article. Take a partition of unity \( \{\varphi_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^d) \) satisfying \( \operatorname{supp}(\varphi_j) \subset \frac{17}{16} Q_j, 0 \leq \varphi_j \leq 1, \sum_{Q_j \in W_\varepsilon} \varphi_j = 1 \) on \( \bigcup_{Q_j \in W_\varepsilon} Q_j \) and
\[
\|\nabla \varphi_j\|_{L^\infty} \leq C \operatorname{diam}(Q_j)^{-1} \quad (j \in \mathbb{N})
\]
with an absolute constant \( C > 0 \).

Let \( f \in BC^1_D(\Omega) \) and \( F \in BC^1_D(\mathbb{R}^d) \) be such that \( F|_\Omega = f \). If \( G \subset \mathbb{R}^d \) is a set with \( G \cap \Gamma = \emptyset \), then one can extend \( f \) from \( \Omega \cap G \) to a \( C^1 \)-function on \( G \) by extending by zero. The relevant example for such a set \( G \) is a cube from the collection \( W_i \). More precisely, one defines the extended function by
\[
E_G f := F \chi_\Omega.
\]

Note that \( \|E_G f\|_{L^p(G)} = \|f\|_{L^p(\Omega \cap G)} \) and \( \|\nabla E_G f\|_{L^p(G)} = \|\nabla f\|_{L^p(\Omega \cap G)} \) holds and that the only non-zero values of \( E_G f \) are in \( \Omega \cap G \). Thus, this definition is independent of the choice of \( F \).
Recall, that \((u)_S\) denotes the average of a function \(u\) on a measurable set \(S\). Define an extension operator \(E\) by

\[
Ef(x) := \begin{cases} 
  f(x), & x \in \Omega, \\
  0, & x \in D, \\
  \sum_{Q_j \in W} (EQ_j f)Q_j \varphi_j(x), & x \in \Omega^c.
\end{cases}
\]

Notice that due to the properties of \((\varphi_j)_{j \in \mathbb{N}}\) and \((Q_j)_{j \in \mathbb{N}}\) the sum in the previous definition contains for each fixed \(x \in \Omega^c\) only finitely many terms. Moreover, according to Lemma 6.1, \(Ef\) is defined almost everywhere on \(\mathbb{R}^d\). If \(W_\varepsilon\) is empty, i.e., if \(D = \partial \Omega\) according to Remark 6.2 then the sum is empty and its value is zero. The goal of this section is to show that \(E\) satisfies the properties stated in Theorem 8.1.

To begin with, we record the following proposition which will be important for the rest of this section and which is established by combining [13, Lem. 7.12, Lem. 7.16].

**Proposition 7.1.** Let \(\Upsilon \subset \mathbb{R}^d\) be open, bounded, and convex and \(S \subset \Upsilon\) be measurable with \(|S| > 0\). Then for all \(1 \leq p \leq \infty\) the inequality

\[
\|u - (u)_S\|_{L^p(\Upsilon)} \leq \frac{\omega_d^{-\frac{1}{n}} \text{diam}(\Upsilon)^{\frac{d}{n}} \|\nabla u\|_{L^p(\Upsilon)}}{|S|} (u \in W^{1,p}(\Upsilon))
\]

holds true, where \(\omega_d = |B(0,1)|\).

**Lemma 7.2.** Let \(f \in BC_p(\Omega)\). If \(S_1, \ldots, S_m\) is a touching chain provided either by Lemma 6.9 or Lemma 6.10 then

\[
\|(ES_1 f)_{S_1} - (ES_m f)_{S_m}\|_{L^p(S_1)} \lesssim \text{diam}(S_1) \|\nabla f\|_{L^p(S_1 \cap (2S_1) \cap \Omega)}.
\]

**Proof.** Note that \((ES_1 f)_{S_r}\) is just a number for \(r = 1, \ldots, m\) and that \(m\) is bounded by a constant that depends only on \(\varepsilon, K,\) and \(d\). Moreover, throughout the proof, we will use that \(|S_1| \approx |S_r|\) (which follows from Lemmas 6.9 and 6.10). Then

\[
\|(ES_1 f)_{S_1} - (ES_m f)_{S_m}\|_{L^p(S_1)} \\
\leq \sum_{r=1}^{m-1} \|(ES_1 f)_{S_r} - (ES_{r+1} f)_{S_{r+1}}\|_{L^p(S_r)} \\
\leq \sum_{r=1}^{m-1} \|(ES_1 f)_{S_r} - (ES_{r+1} f)_{S_{r+1}}\|_{L^p(S_r)} + \|(ES_{r+1} f)_{S_{r+1}} - (ES_{r+1} f)_{S_{r+1} \cup S_{r+1}}\|_{L^p(S_{r+1})} \\
\leq \sum_{r=1}^{m-1} \|(ES_1 f)_{S_r} - ES_1 f\|_{L^p(S_r)} + \|(ES_{r+1} f)_{S_{r+1}} - (ES_{r+1} f)_{S_{r+1} \cup S_{r+1}}\|_{L^p(S_{r+1})} \\
\leq \sum_{r=1}^{m-1} \|(ES_1 f)_{S_r} - ES_1 f\|_{L^p(S_r)} + \|(ES_{r+1} f)_{S_{r+1}} - ES_{r+1} f\|_{L^p(S_{r+1})} + \|(ES_{r+1} f)_{S_{r+1} \cup S_{r+1}} - ES_{r+1} f\|_{L^p(S_{r+1} \cap \Omega)}
\]

By virtue of Proposition 7.1 (applied either with \(S = \Upsilon = S_r\) or \(S = \Upsilon = S_{r+1}\), the first two terms in the sum on the right-hand side are controlled by \(\text{diam}(S_1) \|\nabla f\|_{L^p(S_1 \cap \Omega)}\) and \(\text{diam}(S_1) \|\nabla f\|_{L^p(S_{r+1} \cap \Omega)}\). For the third term note that since \(S_r\) and \(S_{r+1}\) are touching cubes so that \(S_r \cup S_{r+1}\) is convex if \(\text{diam}(S_r) = \text{diam}(S_{r+1})\). Hence, Proposition 7.1 also applies in this case. Let without loss of generality \(\text{diam}(S_{r+1}) < \text{diam}(S_r)\). Then \(\text{diam}(S_{r+1}) \leq \frac{1}{2} \text{diam}(S_r)\)
since Whitney cubes are dyadic and this implies that $S_r \cup S_{r+1} \subset 2S_r$. Moreover,
\[
d(2S_r, \Gamma) \geq d(S_r, \Gamma) - \frac{1}{2} \text{diam}(S_r) \geq \frac{1}{2} \text{diam}(S_r),
\]
by Property (iii) of the Whitney decomposition. Note that $E_{2S_r} f$ is an extension to $2S_r$ of 
$E_{S_r \cup S_{r+1}} f$ so that in particular $(E_{S_r \cup S_{r+1}} f)_{S_r \cup S_{r+1}} = (E_{2S_r} f)_{S_r \cup S_{r+1}}$. Invoking Proposition 7.1
again, yields
\[
\|E_{S_r \cup S_{r+1}} f - (E_{S_r \cup S_{r+1}} f)_{S_r \cup S_{r+1}}\|_{L^p(S_r \cup S_{r+1})} \leq \|E_{2S_r} f - (E_{2S_r} f)_{S_r \cup S_{r+1}}\|_{L^p(2S_r)} \lesssim \text{diam}(S_r) \||\nabla f||_{L^p((2S_r) \cap \Omega)}.
\]
\[\square\]

Fix for a moment $Q_j \in \mathcal{W}_e$ and let $Q_k \in \mathcal{W}_e$ be an intersecting cube. Recall that $F_{j,k}$
denotes the chain obtained by Lemma 6.9. Since intersecting cubes have comparable diameter
by Property (iv) of the Whitney decomposition and are also mutually disjoint, it should be no
surprise that the number of intersecting cubes of a fixed cube depends only on the dimension.
More precisely, there are at most $12^d$ intersecting cubes of $Q_j$, see [18, p. 169]. Consequently, if we define $H_{j,k} := \bigcup_{S_r \in F_{j,k}} 2S_r$, then for fixed $j$ we get
\begin{equation}
\| \sum_{Q_k \in \mathcal{W}_e, Q_j \cap Q_k \neq \emptyset} \chi_{H_{j,k}} \|_{L^\infty} \leq 12^d.
\end{equation}

The same holds true if replacing $Q_j \in \mathcal{W}_e$ by a cube in $\mathcal{W}(\Gamma)$ adjacent to $\mathcal{W}_e$ and by replacing $F_{j,k}$
with the chain $F_{P,j}$ from Lemma 6.10. Define
\[
F(Q_j) := \bigcup_{Q_k \in \mathcal{W}_e, Q_j \cap Q_k \neq \emptyset} H_{j,k} \quad \text{and} \quad F_P(Q) := \bigcup_{Q_j \in \mathcal{W}_e} \bigcup_{Q_j \cap Q_k \neq \emptyset} 2S_r,
\]
where $Q \in \mathcal{W}(\Gamma) \setminus \mathcal{W}_e$ is such that $Q$ touches a cube in $\mathcal{W}_e$ and satisfies \text{diam}(Q) \leq A\delta$. Next, we count how many of the ‘extended’ chains $F(Q_j)$ and $F_P(Q)$, respectively, can intersect a
fixed point $x \in \mathbb{R}^d$. We give an exemplary proof for $F(Q_j)$, the proof for $F_P(Q)$ is similar. By Lemma 6.9 we know that the chain $F_{j,k}$ has length less than a constant $M$, which depends only
on $d$, $K$, and $\varepsilon$. If $x \in F(Q_j)$, then there exists $S_r \in F_{j,k}$ with $x \in 2S_r$. If $R \in \mathcal{W}(\Gamma)$ is any cube
such that $2S_r \cap 2R \neq \emptyset$, there exists $z \in 2S_r \cap 2R$. By Property (iii) of the Whitney decomposition
elementary geometric consideration one then infers for $z \in S_r$ that
\[
4 \text{diam}(R) \geq d(R, \Gamma) \geq d(x, \Gamma) - \sup \{|x - y| : y \in R\} \geq d(z, \Gamma) - |x - z| - \frac{3}{2} \text{diam}(R).
\]
Notice that $z$ can be chosen such that $|x - z| \leq \text{diam}(S_r)/2$ so that in total
\[
4 \text{diam}(R) \geq d(S_r, \Gamma) - \frac{1}{2} \text{diam}(S_r) - \frac{3}{2} \text{diam}(R) \geq \frac{1}{2} \text{diam}(S_r) - \frac{3}{2} \text{diam}(R).
\]
By symmetry (interchanging $S_r$ and $R$) this implies that
\begin{equation}
11^{-1} \text{diam}(S_r) \leq \text{diam}(R) \leq 11 \text{diam}(S_r).
\end{equation}
So, if $R \in F_{\alpha, \beta}$ for some $\alpha, \beta \in \mathbb{N}$, then $R$ can be connected to $Q^*_\alpha$ by a touching chain of length
at most $M$. Thus, by (7.3) and Property (iv) of the Whitney decomposition any cube $R'$ in the
chain $F_{\alpha, \beta}$ satisfies
\[
11^{-1} 4^{-2M} \leq \frac{\text{diam}(Q^*_\alpha)}{\text{diam}(R')} \leq 11 \cdot 4^{2M} \quad \text{and} \quad \text{d}(Q^*_\alpha, R') \leq 11(2M + 1)4^{2M} \text{diam}(Q^*_\alpha).
\]
This combined with Lemma 6.11 implies that there exists a constant $C > 0$ that depends only on $d$, $K$, and $\varepsilon$ such that
\begin{equation}
\sum_{Q_j \in W_e} \chi_{F(Q_j)}(x) \leq C.
\end{equation}

**Lemma 7.3.** Let $f \in BC^1_2(\Omega)$. If $Q_j \in W_e$, then
\begin{equation}
\|\nabla Ef\|_{L^p(Q_j)} \lesssim \|f\|_{L^p(H) \cap \Omega} + \|\nabla f\|_{L^p(F(Q_j) \cap \Omega)}.
\end{equation}

**Proof.** By definition, it holds $Ef = \sum_{Q_j \in W_e} (E_{Q_j} f)_{Q_j} \varphi_k$ on $Q_j$ and $\sum_{Q_j \in W_e} \varphi_k = 1$ on $Q_j$. Consequently,
\begin{equation}
\nabla \sum_{Q_j \in W_e, Q_j \cap Q_k \neq \emptyset} (E_{Q_j} f)_{Q_j} \varphi_k \leq \sum_{Q_j \in W_e, Q_j \cap Q_k \neq \emptyset} \left[ (E_{Q_j} f)_{Q_j} - (E_{Q_j} f)_{Q_j} \right] \nabla \varphi_k \|_{L^p(Q_j)} =: I.
\end{equation}
Proceed by first employing (7.1) combined with Property (iv) of the Whitney decomposition and then Lemma 7.2 and (7.2) yielding
\begin{equation}
I \lesssim \sum_{Q_j \in W_e, Q_j \cap Q_k \neq \emptyset} \text{diam}(Q_k)^{-1} \|E_{Q_j} f\|_{Q_j} - \|E_{Q_j} f\|_{Q_j} \lesssim \|\nabla f\|_{L^p(F(Q_j) \cap \Omega)}.
\end{equation}

**Lemma 7.4.** Let $f \in BC^1_2(\Omega)$. If $Q \in W(\Omega) \setminus W_e$ intersects a cube in $W_e$, then
\begin{equation}
\|\nabla Ef\|_{L^p(Q)} \lesssim \|f\|_{L^p(\bigcup_{Q_j \in W_e, Q_j \cap Q \neq \emptyset} Q_j) \cap \Omega} + \|\nabla f\|_{L^p(F(Q) \cap \Omega)},
\end{equation}
where the second term on the right-hand side just appears if $\text{diam}(Q) \lesssim A\delta$.

**Proof.** If $\text{diam}(Q) > A\delta$, then by (7.1) and from Property (iv) of the Whitney decomposition one obtains
\begin{equation}
\nabla \sum_{Q_j \in W_e, Q_j \cap Q \neq \emptyset} (E_{Q_j} f)_{Q_j} \varphi_j \|_{L^p(Q)} \lesssim \|f\|_{L^p(\bigcup_{Q_j \in W_e, Q_j \cap Q \neq \emptyset} Q_j) \cap \Omega}.
\end{equation}

If $\text{diam}(Q) \lesssim A\delta$, then $Q$ satisfies the assumptions of Lemma 6.10. Let $Q_j \in W_e$ be an intersecting cube of $Q$, let $Q_j^* = S_{1j}, \ldots, S_{mj}$ be the corresponding chain. Note that $|S_{1j}| \approx |S_{2j}|$ and recall that $(E_{S_{1j}} f)_{S_{1j}}$ are constant numbers for all $r = 1, \ldots, m_j$, which permits to change the domain of integration in the $L^p$-norm. Then
\begin{equation}
\nabla \sum_{Q_j \in W_e, Q_j \cap Q \neq \emptyset} (E_{Q_j} f)_{Q_j} \varphi_j \|_{L^p(Q)} \lesssim \sum_{Q_j \in W_e, Q_j \cap Q \neq \emptyset} \text{diam}(Q_j^*)^{-1} \|E_{S_{1j}} f\|_{S_{1j}} \|L^p(S_{1j})
\begin{equation}
\lesssim \sum_{Q_j \in W_e, Q_j \cap Q \neq \emptyset} \text{diam}(Q_j^*)^{-1} \left[ \|E_{S_{1j}} f\|_{S_{1j}} - \|E_{S_{mj}} f\|_{S_{mj}} \right] \|L^p(S_{1j})
\begin{equation}
+ \|E_{S_{mj}} f\|_{S_{mj}} \|L^p(S_{mj})\bigg].
\end{equation}

By virtue of Lemma 7.2 and the version of (7.2) for the chains $F_{P,j}$ the first term in the sum is controlled by $\|\nabla f\|_{L^p((\cup_{j=1}^{mj}) (2S_{mj}) \cap \Omega)}$. For the second term in the sum, note that $E_{S_{mj}} f \equiv 0$ on $S_{mj} \cap Q_j$ and $|S_{mj} \cap Q_j| \approx \text{diam}(Q_j)^d$ (by Lemma 6.10) so that first by Hölder’s inequality and then by Proposition 7.1 one estimates
\begin{equation}
\|E_{S_{mj}} f\|_{S_{mj}} \|L^p(S_{mj})\bigg] \lesssim \|E_{S_{mj}} f\|_{L^p(S_{mj})} = \|E_{S_{mj}} f - (E_{S_{mj}} f)_{S_{mj}} \cap Q_j\|_{L^p(S_{mj})}
\begin{equation}
\lesssim \text{diam}(Q_j)^d \|\nabla f\|_{L^p(S_{mj} \cap \Omega)}.
\end{equation}
By virtue of (7.2), for the chain $F_{P,j}$ this concludes the proof.

**Proposition 7.5.** For all $1 \leq p \leq \infty$ there exists a constant $C > 0$ depending only on $d$, $\varepsilon$, $\delta$, $p$, and $K$ such that for all $f \in BC^1_D(\Omega)$ one has

$$\|Ef\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \quad \text{and} \quad \|\nabla Ef\|_{L^p(\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)}.$$  

**Proof.** The estimate for the gradient in the case $p < \infty$ is deduced by the following calculation based on Lemmas 7.3 and 7.4:

$$\|\nabla Ef\|_{L^p(\Omega)} = \sum_{Q_j \in W_n} \|\nabla Ef\|_{L^p(Q_j)} + \sum_{Q \in W(\Omega) \setminus W_n} \|\nabla Ef\|_{L^p(Q)}$$

$$\leq \sum_{Q_j \in W_n} \|\nabla f\|_{L^p(Q_j \cap \Omega)} + \|\nabla f\|_{L^p(F_{P,j} \cap \Omega)}$$

$$+ \sum_{Q \in W(\Omega) \setminus W_n} \|\nabla f\|_{L^p(Q \cap \Omega)}$$

$$+ \sum_{Q \in W(\Omega) \setminus W_n} \|\nabla f\|_{L^p(Q \cap \Omega)}.$$

The estimate then follows from Lemma 6.11 and from (7.4), which holds also true for the chains $F_{P,j}$ as mentioned in the discussion previous to (7.4).

In the case $p = \infty$, the estimate for the gradient follows by Lemmas 7.3 and 7.4 employed in the calculation:

$$\|\nabla Ef\|_{L^\infty(\Omega)} = \sup_{Q_j \in W_n} \|\nabla Ef\|_{L^\infty(Q_j)} + \sup_{Q \in W(\Omega) \setminus W_n} \|\nabla Ef\|_{L^\infty(Q)}$$

$$\leq \sup_{Q_j \in W_n} \|f\|_{L^\infty(Q_j \cap \Omega)} + \|\nabla f\|_{L^\infty(F_{P,j} \cap \Omega)}$$

$$+ \sup_{Q \in W(\Omega) \setminus W_n} \|f\|_{L^\infty(Q \cap \Omega)}.$$

The $L^p$-estimate follows as well by decomposing $\Omega$ by means of the Whitney decomposition $W(\Omega)$, followed by an application of Hölder’s inequality to the mean values building the extension operator $E$ and an application of Lemma 6.11.

**Proposition 7.6.** If $f \in BC^1_D(\Omega)$, then $Ef$ has a locally Lipschitz continuous representative $g$ which satisfies $d(\text{supp}(g), D) > 0$.

**Proof.** Let $F \in BC^1_D(\mathbb{R}^d)$ with $F|_{\Omega} = f$. Define the function $g := F\chi_\Omega + (Ef)\chi_{\Omega^c}$. By definition it is clear that $Ef$ and $g$ coincide in $\Omega \cup \Omega^c \cup D$ and thus by Lemma 6.1, $g$ is a representative of $Ef$. Since $F$ is smooth with bounded derivative on all of $\mathbb{R}^d$, $g$ is Lipschitz continuous on $\Omega$. Applying Proposition 7.5 for $p = \infty$ one infers that $g$ is locally Lipschitz continuous on $\Omega^c$.

Recall that $d(\text{supp}(F), D) > 0$. If now $Q \in W_n$, then Lemma 6.6 and Property (ii) of the Whitney decomposition yield

$$d(Q^*, D) + \text{diam}(Q^*) \lesssim d(Q, D) + d(Q, Q^*) + \text{diam}(Q) \lesssim d(Q, D) + d(Q, \Omega) \lesssim d(Q, D).$$
Hence, by construction of the extension operator we find \( d(\text{supp}(g), D) > 0 \). Thus, \( g \) is Lipschitz continuous on small balls centered in \( D \). Consequently, it remains to show that \( g \) is Lipschitz continuous on small balls centered in \( \Gamma \).

To this end, fix \( x_0 \in \Gamma \) and let \( r > 0 \). If \( x, y \in \overline{\Omega} \cap B(x_0, r) \) it is clear that the Lipschitz estimate holds by Lipschitz continuity of \( F \). Consequently, one has to consider the case \( x \in \overline{\Omega} \cap B(x_0, r) \) and \( y \in B(x_0, r) \). Note that if \( r < \frac{1}{2} d(x_0, D) \) one finds for \( z \in B(x_0, r) \) that

\[
d(z, D) \geq d(x_0, D) - |x_0 - z| > |x_0 - z| \geq d(z, \Gamma),
\]

so that each point in \( B(x_0, r) \) realizes its distance to \( \partial \Omega \) in \( \Gamma \). If \( x \in \overline{\Omega} \cap B(x_0, r) \) and \( y \in B(x, d(x, \Gamma)) \cap B(x_0, r) \), the Lipschitz estimate is also clear since \( x \) and \( y \) can be connected by a straight line in \( \overline{\Omega} \) and because the derivative of \( g \) is bounded in \( \overline{\Omega} \). Concluding, it remains to investigate the case \( x, y \in B(x_0, r) \) with \( x \in \overline{\Omega} \) and \( y \notin B(x, d(x, \Gamma)) \).

If \( y \notin \Omega \), take \( z \in B(x, 2d(x, \Gamma)) \cap \Omega \), and estimate

\[
|g(x) - g(y)| \leq |g(x) - g(z)| + |g(z) - g(y)|
\]

and

\[
|y - z| \leq |x - y| + |x - z|, \quad |x - z| \leq 2d(x, \Gamma) \leq 2|x - y|.
\]

Since \( |x - z| < 3r \), everything is reduced to the situation where \( x \in \overline{\Omega} \cap B(x_0, r) \) and \( y \in \Omega \cap B(x_0, 3r) \). To this end, let \( Q \in \mathcal{W}(\overline{\Omega}) \) with \( x \in Q \). If \( r \leq \delta \) it holds

\[
diam(Q) \leq d(Q, \overline{\Gamma}) \leq |x - x_0| < r \leq \delta.
\]

This, together with

\[
d(Q, \Gamma) \leq d(x, \Gamma) \leq d(x, D) \leq d(x, \partial \Omega \setminus \Gamma) \leq diam(Q) + d(Q, \partial \Omega \setminus \Gamma) \leq 2d(Q, \partial \Omega \setminus \Gamma)
\]

implies that \( Q \in \mathcal{W}_e \). Let \( x_Q \) denote the midpoint of the cube \( Q \). Since \( Q \subset \overline{\Omega} \) and since \( diam(Q) \leq |x - y| \) (by Property (i) of the Whitney decomposition), one finds

\[
|g(x) - g(x_Q)| \lesssim \|f\|_{W^{1, \infty}(\Omega)} |x - y|.
\]

Finally, notice that if \( r \) is small enough, then it holds \( Q^* \subset \Omega \) by Lemma 6.6. Moreover, notice that \( |EF|(x_Q) = (F)_{Q^*} \) since \( x_Q \) is the midpoint of \( Q \). Thus, by the Lipschitz continuity of \( F \) and another application of Lemma 6.6 we find

\[
|g(x_Q) - g(y)| \leq \frac{1}{|Q^*|} \int_{Q^*} F(x') \, dx' - F(y) \leq \|\nabla F\|_{L^\infty(\mathbb{R}^d)} \left( d(y, Q^*) + \text{diam}(Q^*) \right)
\]

\[
\lesssim \|\nabla F\|_{L^\infty(\mathbb{R}^d)} (|x - y| + \text{diam}(Q)) \leq 2\|\nabla F\|_{L^\infty(\mathbb{R}^d)} |x - y|.
\]

We are now in the position to present to proof of Theorem 3.1.

Proof of Theorem 3.1. Let \( 1 \leq p < \infty \). If \( f \in BC_D^{1,p}(\Omega) \cap W^{1,p}(\Omega) \), Proposition 7.6 implies that \( Ef \) is Lipschitz continuous and that \( d(\text{supp}(Ef), D) > 0 \). The Lipschitz continuity of \( Ef \) in turn implies the weak differentiability of \( Ef \) and then Proposition 7.5 yields that \( Ef \in W^{1,p}(\mathbb{R}^d) \). The positive distance of the support of \( Ef \) to \( D \) implies \( Ef \in \text{Lip}_D(\mathbb{R}^d) \) and by convolution we conclude that \( Ef \in W^{1,p}_D(\mathbb{R}^d) \). By virtue of Proposition 7.5 and density, \( E \) extends to a unique bounded operator from \( W^{1,p}_D(\Omega) \) into \( W^{1,p}_D(\mathbb{R}^d) \). It remains to consider the case \( f \in \text{Lip}_D(\Omega) \).

To this end, let \( \{\varphi_n\}_{n \in \mathbb{N}} \) denote the sequence of functions constructed in Lemma 2.1. Notice that \( E \) is bounded from \( L^\infty(\Omega) \) into \( L^\infty(\mathbb{R}^d) \) according to Proposition 7.5. Thus, for all \( x, y \in \mathbb{R}^d \) we have

\[
|Ef(x) - Ef(y)| = \lim_{n \to \infty} |E\varphi_n(x) - E\varphi_n(y)|.
\]
By Proposition 7.6, $E \psi_n$ is locally Lipschitz and hence
\[ \lim_{n \to \infty} |E \varphi_n(x) - E \varphi_n(y)| \leq \liminf_{n \to \infty} \| \nabla E \varphi_n \|_{L^\infty(\mathbb{R}^d)} |x - y|. \]
Again, by Proposition 7.5 and Lemma 2.1, we obtain
\[ \liminf_{n \to \infty} \| \nabla E \varphi_n \|_{L^\infty(\mathbb{R}^d)} \lesssim \liminf_{n \to \infty} \| \varphi_n \|_{W^{1, \infty}(\Omega)} \lesssim \| f \|_{\text{Lip}(\mathbb{R}^d)}. \]

Finally, Proposition 7.6 implies the validity of the stated property on the support. This concludes the proof of the theorem. □

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