Relativistic quantum dynamics of spin-0 system of the DKP oscillator in a Gödel-type space-time

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Abstract

In this article, we study the DKP equation for the oscillator in a Gödel-type space-time background. We derive the final form of this equation in a flat class of Gödel-type space-time and solve it analytically, and evaluate the eigenvalues and corresponding eigenfunctions, in detail.

Keywords: Gödel-type space-time, DKP equation, energy spectrum, wave functions, Oscillator

1. Introduction

In relativistic quantum mechanics, spin-0 and spin-\(1/2\) particles have been investigated via many analytical and numerical techniques such as the Nikiforov-Uvarov method (NU), supersymmetric quantum mechanics (SQM), alternative iteration method (AIM) etc. The Duffin-Kemmer-Petiau (DKP) equation [1–4] (see also [5, 6]) provides a good theoretical approach for describing spinless and spin-one particles. This equation is an extension of the Dirac equation [7, 8] in which \(\gamma^\mu\) matrices are replaced by \(\beta^\mu\) matrices.

The Dirac oscillator was studied in [9–12], where in the free Dirac equation the momentum operator \(\vec{p} \rightarrow \vec{p} - im\omega\gamma^\mu\vec{\rho}\), with \(\vec{\rho}\) being the position vector, \(m\) the mass of the particle, \(\gamma^\mu, (\gamma^0)^2 = 1\) as the Dirac matrix, and \(\omega\) being the frequency of oscillator. Similarly, the DKP oscillator system is constructed by replacing \(\vec{p} \rightarrow \vec{p} - im\omega\beta^\mu\vec{\rho}\), where the additional term is linear in \(\vec{\rho}\), \(\eta = 2(\beta^0)^2 = I\) with \(\eta^2 = I\) where, \(I\) is the \(5 \times 5\) unit matrix. A new oscillator model with a different form of non-minimal substitution within the framework of DKP equation, were presented in [13]. Bound state energy eigenvalues for the relativistic DKP oscillator by using an exact quantization rule investigated in [14]. Recently, there has been a growing interest on the so called DKP oscillator [15–31], with or without potentials.

For a charge-free particle of spin-0 system, the well-known first order relativistic DKP equation is [1–4]

\[ (i\beta^\mu \partial_\mu - m)\Psi = 0 \quad (\hbar = 1 = c), \]  

where \(m\) is the mass of the particle and \(\beta^j (j = 0, 1, 2, 3)\) are the DKP matrices satisfying the DKP algebra [32].

In this work, the DKP oscillator in a topologically trivial flat class of Gödel-type space-time is studied. This paper is comprised as follows: in section 2, spin-0 system of the DKP oscillator in a flat class of Gödel-type space-time is studied and the eigenvalues are obtained, and conclusions are within section 3.

2. DKP oscillator in a Gödel-type space-time

Consider the following stationary space-time [33] (see [32, 34–36])

\[ ds^2 = -(dt + \alpha_0 x_i dx^i)^2 + h_{ij} dx^i dx^j \quad \text{(2)} \]

\(i, j = 1, 2, 3\),

where \(\alpha_0 > 0\) is a real number and \(h_{ij}\) is the spatial part of the Minkowski metric. The properties of the above space-time have been studied in detail in [33]. The different classes of Gödel-type geometries have been studied in [36], and was shown that the above space-time belongs to a linear or flat class of Gödel-type space-time. The determinant of the above metric tensor is \(\det(g_{\mu\nu}) = -1\).

We have defined the tetrad basis \(e_\mu^a, e_\nu^a\) for the space-time in [32] which must satisfy the following relation:

\[ g_{\mu\nu} = e_\mu^a(x)e_\nu^b(x)\eta_{ab}, \]  

where \(\eta_{ab}\) is the 5 \(\times\) 5 unit matrix and \(\eta_{00} = -1, \eta_{ij} = 1 (i, j = 1, 2, 3)\).
where \( \eta_{k,b} = \text{diag}(-1, 1, 1, 1) \) is the Minkowski metric tensor.

For the DKP equation in curved space, partial derivative \( \partial_{\mu} \) is replaced by covariant derivatives \( \nabla_{\mu} = \partial_{\mu} + \Gamma_{\mu} \), where \( \Gamma_{\mu} \) is called the spinorial affine connection. Therefore, the DKP equation in curved space in the background curvature is given by

\[
[i \beta^\mu(x) (\partial_\mu + \Gamma_\mu(x)) - m - \xi R] \Psi = 0, \tag{4}
\]

where \( \xi \) is the non-minimal coupling constant, \( R \) is the scalar curvature. For the space-time (2), the Ricci scalar curvature is given \( R = 2\Omega^2 \).

The DKP oscillator of the DKP field \( \Psi \) of mass \( m \) in curved space-time is described by

\[
[i \beta^\mu(x) (\partial_\mu + \Gamma_\mu(x)) + m \omega X_\mu \eta^0) - m - \xi R] \Psi = 0, \tag{5}
\]

where

\[
X_\mu = (0, x, 0, 0), \quad \eta^0 = 2(\beta^0)^2 - I. \tag{6}
\]

In [32], we have constructed beta matrices \( \beta^\mu(x) \) and the spin-connections \( \Gamma^\mu(x) \). Since, the metric (2) is independent of \( t, y, z \). Suppose, the general wave-function to be

\[
\Psi(t, x, y, z) = e^{i(E t + l y + k z)} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}, \tag{7}
\]

where \( E \) is the total energy, and \( l, k \) are constants. Substituting (7) into the equation (5), we arrive at the following differential equations:

\[
E \psi_2 - i \omega \psi_3 + l(\alpha_0 x \psi_2 + \psi_3) + k \psi_4 - i(m - \omega) x \psi_3 = (m + 2\Omega^2) \psi_1, \tag{8}
\]

\[
(E + \alpha_0 l \psi_1 = (m + 2\Omega^2) \psi_2, \tag{9}
\]

\[
i \psi'_3 + \omega \alpha \psi_3 = (m + 2\Omega^2) \psi_3, \tag{10}
\]

\[
-k \psi_1 = (m + 2\Omega^2) \psi_4, \tag{11}
\]

\[
-k \psi_1 = (m + 2\Omega^2) \psi_4, \tag{12}
\]

in which a prime means a derivative w. r. t. x. These equations can be decoupled after a little algebra, which results in

\[
\psi_2 = \frac{1}{m + 2\Omega^2} (E + \alpha_0 l \psi_1, \tag{13}
\]

\[
\psi_3 = \frac{i}{m + 2\Omega^2} (\psi'_3 + m \omega \psi_3), \tag{14}
\]

\[
\psi_4 = -\frac{l}{m + 2\Omega^2} \psi_3, \tag{15}
\]

\[
\psi_4 = -\frac{k}{m + 2\Omega^2} \psi_1. \tag{16}
\]

Substituting equations (13)–(16) into the equation (8), we get

\[
\psi' + ax \psi'_j + b x \psi_1 + cx^2 \psi_1 = \lambda \psi_1, \tag{17}
\]

where

\[
a = \frac{3 \alpha_0^2}{2} = 6\Omega^2, \quad b = 2 \alpha_0 E l = 4\Omega E l, \quad c = \alpha_0^2 l^2 + m \omega(a - m \omega) = 4\Omega^2 l^2 + m \omega(6\Omega^2 - m \omega), \quad \lambda = (m + 2\Omega^2)^2 + l^2 + k^2 - E^2 - m \omega. \tag{18}
\]

Following a similar technique to the one used in [32], one will arrive at the following eigenvalue equation:

\[
\frac{\beta^1}{\sqrt{\beta^2 - c}} = (2n + 1), \tag{19}
\]

with the energy eigenvalues associated with \( nth \) radial modes are

\[
E_{n,l} = \pm \frac{\eta_0}{\sqrt{\eta_0^2 + \frac{4l^2}{9\Omega^2}}} \left[ 3\Omega^2 \{ 1 + \eta_0(2n + 1) \} + (m + 2\Omega^2)^2 + l^2 + k^2 - m \omega \right], \tag{20}
\]

where

\[
\eta_0 = \sqrt{1 - \frac{4l^2}{9\Omega^2}} = \frac{m \omega(6\Omega^2 - m \omega)}{9\Omega^4}, \tag{21}
\]

with \( n = 0, 1, 2, 3, \ldots \) This result shows that the discrete set of DKP energies are symmetrical about \( E = 0 \) and this is irrespective of the sign of \( l \). This fact is associated to the fact that the DKP oscillator embedded in a class of flat Gödel-type space-time background does not distinguish particles from antiparticles. At this stage, due to invariance under rotation along the \( z \)-direction, without loss of generality we can fix \( k = 0 \). Note that for \( \omega \to 0 \), the energy eigenvalues reduces to the result obtained in [32]. Equation (20) is the relativistic energy eigenvalues of the DKP oscillator of spin-0 system in a Gödel-type space-time. The corresponding wave-functions are similar to those obtained in [32].

In [30], authors studied the DKP oscillator in a cosmic string space-time. The energy eigenvalues for spin-0 system is given by (see equation (57) in [30])

\[
E_{n,l} = \pm \sqrt{M^2 + k^2 + 2 \omega \left( 2n + \frac{|l|}{\alpha} \right)}, \tag{22}
\]

where \( M \) is the mass, \( \omega \) is the oscillator frequency, \( \alpha \) is the string parameter, \( n = 0, 1, 2, 3, \ldots \) and \( l = \pm 1, \pm 2, \ldots \). For \( \alpha \to 1 \), cosmic string space-time reduces to the Minkowski metric in cylindrical coordinates \( (t, r, \phi, z) \). Therefore, the energy eigenvalues (22) becomes

\[
E_{n,l} = \pm \sqrt{M^2 + k^2 + 2 \omega (2n + |l|)}. \tag{23}
\]

Equation (23) is the energy eigenvalues of the DKP oscillator for spin-0 system in Minkowski flat space metric in cylindrical system.
For $\Omega \to 0$, the space-time (2) reduces to the Minkowski metric in Cartesian coordinates $(x, y, z)$. In that case, the parameters $a = 0 = b$ and therefore from condition (19), we have the following eigenvalues

$$E_{n,l} = \pm \sqrt{m^2 + k^2 + l^2 + 2m\omega n},$$

($n = 0, 1, 2, 3,...$). \hspace{1cm} (24)

Note that equation (23) is the energy eigenvalues of the DKP oscillator for spin-0 system in Minkowski metric in Cylindrical coordinates whereas, the equation (24) is the energy eigenvalues of the same system in Cartesian system.

Therefore from equations (13)–(16), we obtain

$$\psi_{2n} = \frac{1}{m + 2\xi_0^2(E_n + 2\Omega l)\psi_{1n}},$$

$$\psi_{3n} = \frac{i}{m + 2\xi_0^2}(\psi_{1n}^l + m \omega \psi_{1n}),$$

$$\psi_{4n} = -\frac{l}{m + 2\xi_0^2}\psi_{1n},$$

$$\psi_{5n} = -\frac{k}{m + 2\xi_0^2}\psi_{1n}. \hspace{1cm} (25)$$

Let us study the eigenvalues and corresponding wavefunctions one by one. We set $n = 0, 1, 2$ and others are in the same way.

(i) $n = 0$

\begin{align*}
E_0 &= \pm \frac{\eta_0}{\sqrt{\eta_0^2 + 4l^2}}[3\Omega^2(1 + \eta_0) \\
&\quad + (m + 2\xi_0^2)^2 + l^2 + k^2 - m \omega l^2], \\
\psi_{10} &= \left(\frac{3\Omega^2}{\pi}\right)^{\frac{1}{2}}e^{-\frac{\eta_0 l^2}{2}}, \hspace{1cm} (29)
\end{align*}

(ii) $n = 1$

\begin{align*}
E_1 &= \pm \frac{\eta_0}{\sqrt{\eta_0^2 + 4l^2}}[3\Omega^2(1 + 3\eta_0) \\
&\quad + (m + 2\xi_0^2)^2 + l^2 + k^2 - m \omega l^2], \\
\psi_{11} &= \left(\frac{3\Omega^2}{4\pi}\right)^{\frac{1}{2}}xe^{-\frac{\eta_0 l^2}{2}}, \hspace{1cm} (30)
\end{align*}

(iii) $n = 2$

\begin{align*}
E_2 &= \pm \frac{\eta_0}{\sqrt{\eta_0^2 + 4l^2}}[3\Omega^2(1 + 5\eta_0) \\
&\quad + (m + 2\xi_0^2)^2 + l^2 + k^2 - m \omega l^2], \\
\psi_{12} &= \left(\frac{3\Omega^2}{4\pi}\right)^{\frac{1}{2}}(2x^2 - 1)e^{-\frac{\eta_0 l^2}{2}}, \hspace{1cm} (31)
\end{align*}

3. Conclusions

In [30], the quantum dynamics of scalar bosons embedded in the background of a cosmic string were addressed. Authors investigated scalar bosons described by the Duffin-Kemmer-Petiau oscillator in this background and obtained the energy eigenvalues. In [32], we have studied the DKP equation for spin-0 system in a Gödel-type space-time in a linear or flat cases ($\mu^2 = 0$), and evaluated the energy eigenvalues and eigenfunctions.

In this work, the DKP equation for the oscillator in spin-0 system in a Gödel-type space-time is studied. The study space-time belongs to a class of flat Gödel-type metrics, and reduces to four-dimensional Minkowski metric for zero vorticity parameter ($\Omega \to 0$). We could derived the final form of the DKP oscillator in this flat Gödel-type space-time with background curvature. We have solved it analytically, and evaluated the energy eigenvalues equations (20)–(21). We have seen that both particle and antiparticle energy levels are members of the spectrum and are symmetrical about $E = 0$. This fact implies that there is no channel for spontaneous boson-antiboson creation. We have shown that for $\omega \to 0$, zero oscillator frequency, the energy eigenvalues reduce to the result obtained in [32]. A special case that corresponds to $\Omega \to 0$ has also been addressed, and the energy eigenvalues (24) is obtained, which is found to be different from the result obtained in [30] for $\alpha \to 1$.

In the context of quantum chromodynamics, Cosmology, gravity and particle and nuclear physics [28, 37, 38], the DKP equation has been examined. The results of this paper could be used in condensed matter physics and to Bose–Einstein condensates [39, 40], and for neutral atoms as well. It is well known that condensates can be exploited for building a coherent source of neutral atoms [41], which in turn can be used to study entanglement and quantum information processing [42]. So, we have presented some results which, in addition to those obtained in [30], present many interesting effects.

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