Moment based estimation of supOU processes and a related stochastic volatility model

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Abstract: After a quick review of superpositions of OU (supOU) processes, integrated supOU processes and the supOU SV model we estimate these processes by using the generalized method of moments. We show that the GMM approach yields consistent estimators and that it works very well in practice. Moreover, we discuss the influence of long memory effects.

Key Words: generalized method of moments; Ornstein-Uhlenbeck type process; Lévy basis; long memory; stochastic volatility; superpositions;

1 Introduction

Lévy driven Ornstein-Uhlenbeck processes, short (OU) processes, are a widely studied class of stochastic processes. When used to describe the volatility in a financial model the resulting stochastic volatility model covers many of the stylized facts as heavy tails, volatility clustering, jumps, etc. (see [Cont & Tankov, 2004]). A Lévy driven Ornstein-Uhlenbeck process $Y_t = (Y_t)_{t \in \mathbb{R}}$ is the solution of the stochastic differential equation

$$dY_t = -aY_t dt + dL_t,$$

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where \( a \in \mathbb{R} \) and \( L = (L_t)_{t \in \mathbb{R}} \) is a Lévy process. Under the assumptions \( a > 0 \) and \( \mathbb{E}(\log(|L| \vee 1)) < \infty \) there exists a unique stationary solution of \( (1) \) which is given by

\[
Y_t = \int_{-\infty}^t e^{-a(t-s)} dL_s.
\]

These Lévy driven Ornstein-Uhlenbeck processes are popular mean-reverting jump processes. Since the mean-reverting parameter \( a \) is constant, these processes have always the same exponential decay at all times. Likewise the autocorrelation function is simply \( e^{-ah} \). Typically, however, the autocovariance function of the squared returns of financial prices decays much faster in the beginning than at higher lags. An obvious generalization would be a random mean-reverting parameter, i.e. we substitute the constant \( a \) by a random variable \( A \) which is different for every jump of the Lévy process. This feature allows to model more flexible autocovariance functions. The works of [Barndorff-Nielsen, 2001] and [Barndorff-Nielsen & Stelzer, 2011] focus on that generalization and ended up with a superposition of (OU) processes, called supOU process. Furthermore, it turned out that supOU processes may have the nice feature of exhibiting long range dependence, i.e. they may have a slowly polynomially decaying autocovariance function. In [Barndorff-Nielsen & Stelzer, 2013] the authors went a step further and studied a stochastic volatility model in which the volatility process is modeled by a positive supOU process and call it a supOU SV model. Moreover, they showed that long range dependence in the volatility process yields long range dependence in the squared log-returns of a supOU SV model. This may be a desirable stylized fact of the log-returns which can only be modeled by few models. This makes the supOU processes and the supOU SV model very interesting in the modeling of financial data.

The modeling of financial data demands statistical estimation procedures for supOU processes and for the supOU SV model. Unfortunately, the classical and efficient Maximum-Likelihood approach seems not applicable, since the density of supOU processes is not known. Because of that we propose in this paper the generalized method of moments which leads to a consistent estimation of supOU processes, integrated supOU processes and of the supOU SV model. In a semiparametric framework we consider in detail examples in which the random mean-reverting parameter \( A \) is Gamma distributed and we calculate the moment functions in closed form. Afterwards we show how to estimate the parameters and we discuss the estimation approach in a simulation study. We use a 2-step iterated GMM estimator i.e. we weight all moments equally in the first step and in the second step we weight the different moments according to the estimation result of the step before. In the illustrations we find out that the GMM estimator works very well and yields very good and well-balanced estimators. In the presence of long range dependence we make the experience that the estimation is slightly positively biased. This is not a surprising outcome because such effects occur often when long range dependence is present.

This paper is organized as follows. In the second section we give a short review of supOU processes, integrated supOU processes and the supOU SV model. Moreover we give the second order structure of these processes and consider a special case in which
we discuss the occurrence of long range dependence. In Section 3 we introduce the generalised method of moments, give the moment functions and show that the GMM approach yields consistent estimators. In the next section we illustrate how good the GMM approach works. In the last section we give a short conclusion.

2 Review of supOU processes

In this section we give a short review and some intuition on supOU processes, integrated supOU processes and of the supOU stochastic volatility model. For a comprehensive study we refer to [Barndorff-Nielsen, 2001], [Barndorff-Nielsen & Stelzer, 2011] and [Barndorff-Nielsen & Stelzer, 2013].

2.1 supOU and integrated supOU processes

To introduce a random mean-reverting parameter $A$ for the jumps of an OU process we generalize the driving Lévy process to a so called Lévy basis which is also known as infinitely divisible independent scattered random measure (abbreviated i.d.i.s.r.m.). In the following $R_-$ denotes the set of negative real numbers and $B_b(R_- \times R)$ denotes the bounded Borel sets of $R_- \times R$.

**Definition 2.1** A family $\Lambda = \{\Lambda(B) : B \in B_b(R_- \times R)\}$ of real-valued random variables is called a real-valued Lévy basis on $(R_- \times R)$ if:

- the distribution of $\Lambda(B)$ is infinitely divisible for all $B \in B_b(R_- \times R)$,
- for any $n \in \mathbb{N}$ and pairwise disjoint sets $B_1, ..., B_n \in B_b(R_- \times R)$ the random variables $\Lambda(B_1), ..., \Lambda(B_n)$ are independent,
- for any sequence of pairwise disjoint sets $B_i \in B_b(R_- \times R)$ with $i \in \mathbb{N}$ satisfying $\bigcup_{n \in \mathbb{N}} B_n \in B_b(R_- \times R)$ the series $\sum_{n=1}^{\infty} \Lambda(B_n)$ converges almost surely and $\Lambda(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n=1}^{\infty} \Lambda(B_n)$.

As in [Barndorff-Nielsen & Stelzer, 2011] we consider only Lévy bases whose characteristic functions have the following form:

$$E(\exp(iu\Lambda(B))) = \exp(\phi(u)\Pi(B))$$

for all $u \in \mathbb{R}$ and all $B \in B_b(\mathbb{R}_- \times \mathbb{R})$, where $\Pi = \pi \times \lambda$ is the product of a probability measure $\pi$ on $\mathbb{R}_-$ and the Lebesgue measure on $\mathbb{R}$ and

$$\phi(u) = iu\gamma - \frac{1}{2}\sum a^2 + \int_{\mathbb{R}} \left( e^{iux} - 1 - iux1_{\{|x| \leq 1\}} \right) \nu(dx)$$
is the cumulant transform of an infinitely divisible distribution on $\mathbb{R}$ with Lévy-Khintchine triplet $(\gamma, \Sigma, \nu)$. We call the quadruple $(\gamma, \Sigma, \nu, \pi)$ the generating quadruple, since it determines completely the distribution of the Lévy basis. It follows that the Lévy process $L$ defined by

$$L_t = \Lambda(\mathbb{R}_- \times (0, t]) \text{ and } L_{-t} = \Lambda(\mathbb{R}_- \times (-t, 0))$$

has characteristic triplet $(\gamma, \Sigma, \nu)$ and is called the underlying Lévy process. Using such a Lévy basis we finally end up with a superposition of Ornstein-Uhlenbeck processes which is called supOU process.

**Theorem 2.2 (supOU processes)** [Fasen & Klüppelberg, 2007, Proposition 2.1]

Let $\Lambda$ be a real-valued Lévy basis on $(\mathbb{R}_- \times \mathbb{R})$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ which satisfies

$$\int_{x>1} \log(|x|) \nu(dx) < \infty \text{ and } \int_{\mathbb{R}_-} -\frac{1}{A} \pi(dA) < \infty .$$

Then the process $(X_t)_{t \in \mathbb{R}}$ given by

$$X_t = \int_{\mathbb{R}_-} \int_{-\infty}^{t} e^{A(t-s)} \Lambda(dA, ds)$$

is well defined for all $t \in \mathbb{R}$ and stationary. We call the process $X$ a supOU process.

To apply supOU processes in a practical or a financial framework, we need to estimate them. The Maximum Likelihood or a similar approach is not feasible since the density of a supOU process is not known. Hence, we propose here a moment based estimation for which the second order structure of supOU processes is needed.

**Theorem 2.3** [Barndorff-Nielsen & Stelzer, 2011, Th. 3.9]

Let $X$ be a stationary real-valued supOU process driven by a Lévy basis $\Lambda$ satisfying the conditions of Theorem 2.2. If

$$\int_{x>1} x^2 \nu(dx) < \infty \quad (2)$$

then $X$ has finite second moments and it holds

$$\mathbb{E}(X_0) = -\mu \int_{\mathbb{R}_-} \frac{1}{A} \pi(dA) , \quad \text{var}(X_0) = -\sigma^2 \int_{\mathbb{R}_-} \frac{1}{2A} \pi(dA) ,$$

$$\text{cov}(X_h, X_0) = -\sigma^2 \int_{\mathbb{R}_-} \frac{e^{Ah}}{2A} \pi(dA) ,$$

where $\mu := \mathbb{E}[L_1] = \gamma + \int_{|x|>1} x \nu(dx)$, $\sigma^2 := \text{var}(L_1) = \Sigma + \int_{\mathbb{R}} x^2 \nu(dx)$ and $L$ is the underlying Lévy process.
SupOU processes may cover the very interesting stylized fact of a slowly decaying autocorrelation function. More precisely, a stochastic process is said to have long memory effects (or long range dependence) if the autororrelation function $\rho(h)$ satisfies

$$\rho(h) \sim l(h)h^{-H} \quad \text{for } h \to \infty,$$

where $H \in (0,1)$ and the function $l$ is slowly varying, i.e. $\lim_{t \to \infty} \frac{l(xt)}{l(t)} = 1 \ \forall x > 0$. Long memory effects are discussed in detail in [Cont, 2010] and [Taqqu, 2003]. In Section 2.2 we focus on these long memory effects and present a special case in which supOU processes have such a slowly decaying autocorrelation function.

In some empirical studies, see e.g. [Cont, 2001] or [Guillaume et al., 1997], it is suggested that the prices of financial assets may have long memory effects. Positive Ornstein-Uhlenbeck type processes are convenient to model squared volatility processes of stochastic volatility models, see e.g. [Barndorff-Nielsen & Shephard, 2002], but they do not yield long memory effects. Therefore [Barndorff-Nielsen & Stelzer, 2013] replaced the positive Ornstein-Uhlenbeck type process by a positive supOU process and called it a supOU SV model. In their work it was shown that long memory effects in the volatility process yield long memory effects in the squared log-returns which makes this stochastic volatility model very interesting.

As before it seems appropriate to use moment estimators for estimating a supOU SV model. Later we will see that the integrated supOU process, which we introduce now, can be used to determine the moments of such a supOU SV model. We only consider positive integrated supOU processes as they are mainly of interest in connection with stochastic volatility models where they are naturally positive. However, the results remain true in general as an inspection of the proofs in [Barndorff-Nielsen & Stelzer, 2011] shows.

**Definition 2.4 (integrated supOU process)**

Let $X$ be a supOU process with generating quadruple $(\gamma, 0, \nu, \pi)$ such that such that $\gamma_0 := \gamma - \int_{|x| \leq 1} x\nu(dx) > 0$, $\int_{|x| \leq 1} |x|\nu(dx) < \infty$ and $\nu(\overline{\mathbb{R}}_-) = 0$ hold. Assume that $(V_n)_{n \in \mathbb{N}}$ is given by

$$V_n := \int_{(n-1)\Delta}^{n\Delta} X_s ds$$

where $\Delta$ is a fixed positive number. Then we call the process $V$ an integrated supOU process.

Again we want to estimate such processes via moment estimators and therefore need the second order structure of them.
Theorem 2.5 [Barndorff-Nielsen & Stelzer, 2013, Th. 3.4]

Let $V$ be an integrated supOU process such that (2) holds for the underlying supOU process. Then the process $(V_n)_{n \in \mathbb{N}}$ is stationary and square-integrable with

$$E(V_1) = -\Delta \mu \int_{\mathbb{R}_-} \frac{1}{A} \pi(dA),$$

$$\text{var}(V_1) = -\sigma^2 \int_{\mathbb{R}_-} \frac{1}{A^2} \left( \frac{e^{A\Delta}}{A} - 1 - \Delta \right) \pi(dA),$$

$$\text{cov}(V_{h+1}, V_1) = -\sigma^2 \int_{\mathbb{R}_-} \frac{1}{2A^3} (f_{h+1} - 2f_h + f_{h-1}) \pi(dA),$$

where $f_h := e^{A\Delta_h}$, $\mu := E[L_1] = \gamma_0 + \int_{\mathbb{R}} x \nu(dx)$, $\sigma^2 := \text{var}(L_1) = \int_{\mathbb{R}} x^2 \nu(dx)$ and $L$ is the underlying Lévy process.

2.2 Long memory effects

In this section we present the first and second order structure of a supOU process $X$ and an integrated supOU process $V$ under the assumption that the stochastic mean-reverting parameter $A$ is Gamma distributed. Furthermore, we investigate in which cases supOU processes have long memory effects.

Let us consider a semiparametric framework in which we assume that $\pi$ is the distribution of $BR$ where $B \in \mathbb{R}_-$ and $R \sim \Gamma(\alpha_\pi, 1)$. Furthermore, we emphasize that setting the second parameter of the Gamma distribution equal to one does not restrict the model since this is equivalent to varying $B$, cp. Barndorff-Nielsen & Stelzer, 2013. From Example 3.1 in Barndorff-Nielsen & Stelzer, 2011 we get that the supOU process $X$ has finite second moments and applying Theorem 2.5 yields

$$E(X_0) = -\frac{\mu}{B(\alpha_\pi - 1)}, \quad \text{var}(X_0) = -\frac{\sigma^2}{2B(\alpha_\pi - 1)},$$

$$\text{cov}(X_0, X_h) = -\frac{\sigma^2(1 - Bh)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)},$$

where $\mu$ and $\sigma^2$ denote the expectation and the variance of the underlying Lévy process, respectively. Moreover, the moment structure depends only on the parameter vector $\beta := (\mu, \sigma^2, \alpha_\pi, B)$ and the autocorrelation function

$$\rho(h) = (1 - Bh)^{1-\alpha_\pi}$$

exhibits long memory effects for $\alpha_\pi \in (1, 2)$ as one can see immediately.

In the case of the corresponding integrated supOU process $V$ we get from Barndorff-Nielsen & Stelzer, 2013 Theorem 3.4 that
\[
\begin{align*}
E(V_0) &= -\frac{\Delta \mu}{B(\alpha_\pi - 1)}, \\
\text{var}(V_0) &= -\frac{\sigma^2(1 - B\Delta)^3 - \alpha_\pi - 1 - \Delta B(\alpha_\pi - 3)}{B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)}, \\
\text{cov}(V_0, V_h) &= -\frac{\sigma^2(f_{h+1} - 2f_h + f_{h-1})}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)},
\end{align*}
\]

where \( f_h := (1 - B\Delta h)^{3 - \alpha_\pi} \) and \( \mu, \sigma^2 \) denote the expectation and the variance of the underlying Lévy process, respectively. As in the case of a supOU process, the moments depend only on the parameter vector \( \beta := (\mu, \sigma^2, \alpha_\pi, B) \) and for \( \alpha_\pi \in (1, 2) \) the process exhibits long memory effects, see [Barndorff-Nielsen & Stelzer, 2011, Example 3.1].

### 2.3 SupOU SV model

Stochastic volatility models in which the volatility process is a positive Ornstein-Uhlenbeck type process capture most of the stylized facts as heavy tails, volatility clustering, jumps, etc. If we model the volatility process by a positive supOU process we may add the feature of long memory effects as described in Section 2.2.

**Definition 2.6** Let \( W \) be a standard Brownian motion independent of the Lévy basis and \( \Sigma \) be a supOU process with generating quadruple \( (\gamma, 0, \nu, \pi) \) such that \( \gamma - \int_{|x|\leq 1} x\nu(dx) > 0, \int_{|x|\leq 1} |x|\nu(dx) < \infty \) and \( \nu(\mathbb{R}_-) = 0 \) hold. Then we define \((X_t)_{t \geq 0}\) by

\[
dX_t = \sqrt{\Sigma_t}dW_t, \quad X_0 = 0,
\]

and say that the process \( X \) follows a supOU type SV model. In the following we abbreviate the supOU type SV model by \( \text{SVsupOU}(\gamma, 0, \nu, \pi) \).

There is no drift in the supOU SV model included. The reason is compared to [Barndorff-Nielsen & Stelzer, 2011] that in the presence of a drift one has no longer an explicit formula for the second order structure available. To obtain meaningful estimates one thus should apply our estimation procedure to demeaned observations.

In financial markets one typically observes the log-returns on a discrete-time basis. This suggests that we focus on the log-returns \((Y_n)_{n \in \mathbb{N}}\) which are given by

\[
Y_n := X_{n\Delta} - X_{(n-1)\Delta} = \int_{(n-1)\Delta}^{n\Delta} \sqrt{\Sigma_t}dW_t, \quad \text{for some fixed } \Delta > 0.
\]

Using the Itô Isometry as in [Pigorsch & Stelzer, 2009], it turns out that the second order structure of the supOU SV model can be determined by using the second order structure of the integrated supOU process. This was the main reason why we considered integrated supOU processes before.
Theorem 2.7 [Barndorff-Nielsen & Stelzer, 2013, Th. 3.4]
Let $X, \Sigma$ be an $SV_{OV}^{supOU}(\gamma, 0, \nu, \pi)$ model satisfying (2). Then $(Y_n)_{n \in \mathbb{N}}$ as well as $(Y^2_n)_{n \in \mathbb{N}}$ are stationary and square integrable with

$$E(Y_1) = 0, \quad \text{var}(Y_1) = E(V_1), \quad \text{cov}(Y_{h+1}, Y_1) = 0 \quad \forall h > 0,$$

(3)

$$E(Y^2_1) = E(V_1), \quad \text{var}(Y^2_1) = 3\text{var}(V_1) + 2E(V_1)^2,$$

(4)

$$\text{cov}(Y^2_{h+1}, Y^2_1) = \text{cov}(V_{h+1}, V_1) \quad \forall h > 0.$$  

(5)

Due to equation (5) long memory effects carry over from the integrated supOU process to the squared log-returns. Hence the squared log-returns exhibit long memory effects if $\pi$ is the distribution of $BR$ where $B \in \mathbb{R}_-$, $R \sim \Gamma(\alpha_\pi, 1)$ and $\alpha_\pi \in (1, 2)$.

3 Moment based estimation under a Gamma distributed mean reverting parameter

In this section we study a moment based estimation approach of supOU processes, integrated supOU processes and of the supOU SV model. For a comprehensive introduction to the generalized method of moments we refer to [Hansen, 1982], [Mátýás, 1999] or [Hall, 2005].

Assumption 3.1 Let $\pi$ be the distribution of $BR$ where $B \in \mathbb{R}_-$ and $R \sim \Gamma(\alpha_\pi, 1)$.

We recall that Assumption[3.1] implies that we are in a semiparametric setting and that we estimate the parameter vector $\beta = (\mu, \sigma^2, \alpha_\pi, B)$.

Let $X(\omega) = (X_t(\omega))_{t \in \mathbb{R}}$ be a realization of the underlying process and $(X_1, X_2, ..., X_N)$ be a vector of $N \in \mathbb{N}$ equidistant observations. We introduce the vector

$$Y_t := (X_t, ..., X_{t+m}) \quad \text{for } t \in \{1, ..., N - m\}$$

since the estimation procedure will include autocovariances up to a lag $m \geq 2$. In the first step, we have to find a measurable function $f : \mathbb{R}^{m+1} \times W \rightarrow \mathbb{R}^{m+2}$, called moment function, such that

$$E[f(Y_t, \beta)] = 0 \quad \iff \quad \beta = \beta_0,$$

where $W \subset \mathbb{R}^4$ denotes the compact parameter space which includes the true parameter vector $\beta_0$. In the second step we estimate $\beta_0$ by minimizing the objective

$$\beta \rightarrow g_{N,m}(\omega, \beta)'Ig_{N,m}(\omega, \beta)$$

(6)
where \( g_{N,m}(\omega, \beta) = \frac{1}{N-m} \sum_{i=1}^{N-m} f(Y_t, \beta) \) and \( I \) is a positive definite matrix to weight the different moments of \( g_{N,m} \). It is well-known that there exists an optimal choice of the weighting matrix \( I \), but determining that matrix in the forefront of the estimation is in practice mostly impossible. Because of that we use a 2-step iterated GMM estimator which is easy to implement and improves the estimates. For more details on that topic we refer to [Hall, 2005, Section 3.5 and 3.6].

**Theorem 3.2** Let \( X \) be either a supOU process, an integrated supOU process or a supOU SV model. Moreover let \( t \in \mathbb{R} \) be fixed and \( E[f(Y_t, \beta)] \) be a function only depending on the parameter vector \( \beta \). If the true parameter vector \( \beta_0 \) is identifiable, i.e. \( E[f(Y_t, \beta)] = 0 \) if and only if \( \beta = \beta_0 \), then we have a consistent moment estimator.

**Proof:** From [Fuchs & Stelzer, 2011] we know that supOU processes, integrated supOU processes and the supOU SV model are ergodic. Hence we have ergodicity of the mean and the result follows by [Mátéyás, 1999, Theorem 1.1]. \( \square \)

**Proposition 3.3** (Moment function for supOU processes)

Let \( X \) be a supOU process as introduced in Section 2.1, \( m \geq 2 \) be a fixed integer and

\[
 f_X(Y_t, \beta) = \begin{pmatrix} f_E(Y_t, \beta) \\ f_{\text{var}}(Y_t, \beta) \\ f_1(Y_t, \beta) \\ \vdots \\ f_m(Y_t, \beta) \end{pmatrix}
\]

where

\[
 f_E(Y_t, \beta) = X_t + \frac{\mu}{B(\alpha_\pi - 1)} \\
 f_{\text{var}}(Y_t, \beta) = \left( X_t + \frac{\mu}{B(\alpha_\pi - 1)} \right)^2 + \frac{\sigma^2}{2B(\alpha_\pi - 1)} \\
 f_h(Y_t, \beta) = \left( X_t + \frac{\mu}{B(\alpha_\pi - 1)} \right) \left( X_{t+h} + \frac{\mu}{B(\alpha_\pi - 1)} \right) + \frac{\sigma^2(1-Bh)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)}.
\]

Then the parameter vector \( \beta_0 \) is identifiable.

**Proof:** To prove the identifiability of the parameter vector \( \beta_0 \) it is enough to consider four equations, the equation with the expectation, with the variance and with the autocovariances with lag \( h_1, h_2 \) where \( h_1 \neq h_2 \) and \( h_1, h_2 > 0 \). From the autocovariances it follows that \( \frac{\log \rho(h_1)}{\log \rho(h_2)} = \frac{\log(1-Bh_1)}{\log(1-Bh_2)} \). Defining \( c := \frac{\log \rho(h_1)}{\log \rho(h_2)} \) gives us \( (1-Bh_2)^c + Bh_1 - 1 = 0 \). The left hand side of the last equation is a function in \( B \) which has a positive second derivative. Hence, it is a strictly convex function which has at most two zeros. Because one zero is at zero, the parameter \( B \) is the unique strictly negative zero. With that uniquely determined \( B \) we are able to determine \( \alpha_\pi \) uniquely by
\[ \alpha_\pi = 1 - \frac{\log \rho(h_1)}{\log(1 - Bh_1)}. \]

The expectation and the variance equations yield unique \( \mu \) and \( \sigma^2 \) which completes the identifiability of \( \beta_0 \). Now Theorem 3.2 yields the result. \( \square \)

In the case of an integrated supOU process we have not been able to show the identifiability based on a finite number of moments. Instead we can show an asymptotically identifiability.

**Definition 3.4** A parameter vector \( \beta \) of a stochastic process \( Y \) is said to be asymptotically identifiable if the mapping \( \beta : W \to \mathbb{R}^N \) with \( \beta \to \mathbb{E}(f_k(Y_t, \beta)) \) is injective where \( f_k \) is the \( k \)-th component of the moment function \( f : \mathbb{R}^N \times W \to \mathbb{R}^N \).

**Proposition 3.5** (Moment function for integrated supOU processes)
Let \( X \) be a supOU process as introduced in Section 2.1 and \( V = (V_n)_{n \in \mathbb{N}} \) the corresponding integrated supOU process and

\[
 f_V(Y_t, \beta) = \begin{pmatrix}
 f_{\mathbb{E}}(Y_t, \beta) \\
 f_{\text{var}}(Y_t, \beta) \\
 f_1(Y_t, \beta) \\
 \vdots \\
 f_m(Y_t, \beta)
\end{pmatrix}
\]

where

\[
 f_{\mathbb{E}}(Y_t, \beta) = V_t + \frac{\Delta \mu}{B(\alpha_\pi - 1)} \\
 f_{\text{var}}(Y_t, \beta) = \left(V_t + \frac{\Delta \mu}{B(\alpha_\pi - 1)}\right)^2 + \sigma^2 \left(1 - B\Delta\right)^{3 - \alpha_\pi} - 1 - \Delta B(\alpha_\pi - 3) \\
 f_h(Y_t, \beta) = \left(V_t + \frac{\Delta \mu}{B(\alpha_\pi - 1)}\right) \left(V_{t+h} + \frac{\Delta \mu}{B(\alpha_\pi - 1)}\right) + \frac{\sigma^2 (f_{h+1} - 2f_h + f_{h-1})}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)}.
\]

Then the parameter vector \( \beta_0 \) is asymptotically identifiable.

Proof: To get a consistent estimator for the integrated supOU process \( (V_n)_{n \in \mathbb{N}} \) we have to consider the infinitely dimensional vector \( Y_t = (V_t, V_{t+1}, V_{t+2}, \ldots)' \) of observations. Let \( \alpha_\pi \neq 2, 3 \). From [Barndorff-Nielsen & Stelzer, 2013, Example 3.1 and Prop. 3.5 (i)] we get
\[
\lim_{h \to \infty} \frac{\text{cov}(V_{t+h}, V_1)}{\text{cov}(V_{t+2h}, V_1)} = \lim_{h \to \infty} \frac{-\Delta^{3-\alpha} \int_0^\infty x^2 \nu(dx) (-Bh)^{1-\alpha} - \Delta^{3-\alpha} \int_0^\infty x^2 \nu(dx) (-2Bh)^{1-\alpha}}{2B(\alpha-1)} = \left( \frac{1}{2} \right)^{1-\alpha}.
\]

This yields a unique \(\alpha_\pi\). Using again Barndorff-Nielsen & Stelzer, 2013 Example 3.1 and Prop. 3.5 (i) we also obtain

\[
\rho(h) \sim \tilde{\rho}_{B,\alpha}(h) := \frac{\Delta^{3-\alpha}(\alpha - 2)(\alpha - 3)h^{1-\alpha}(-B)^{3-\alpha}}{2((1 - B\Delta)^{3-\alpha} - 1 - B(\alpha - 3))}.
\]

The derivative of \(\tilde{\rho}_{B,\alpha}(h)\) with respect to \(B\) is a monotone function in \(B\). Hence there can only be one \(B < 0\) such that \(E[f_V(Y_t, \beta)] = 0\). The uniqueness of \(\mu\) and \(\sigma^2\) follow from \(f_{\text{E}}(Y_t, \beta) = 0\) and \(f_{\text{var}}(Y_t, \beta) = 0\), respectively. Again Theorem 3.2 yields the claim. The remaining cases \(\alpha_\pi = 2\) and \(\alpha_\pi = 3\) can be treated similar. \(\Box\)

Due to (4) and (5) we are able to deduce the moment function for the supOU SV model easily.

**Corollary 3.6**  
(Moment function for supOU stochastic volatility model)

Let \(X, \Sigma\) be a supOU SV model as introduced in Section 2.3, \((V_n)_{n \in \mathbb{N}}\) be the integrated supOU process of \(\Sigma\) and

\[
f_{SV}(Y_t, \beta) = \begin{pmatrix}
f_{\text{var}}(Y_t, \beta) \\
f_{\text{var}}(Y_t^2, \beta) \\
f_1(Y_t^2, \beta) \\
\vdots \\
f_m(Y_t^2, \beta)
\end{pmatrix}
\]

where

\[
f_{\text{var}}(Y_t^2, \beta) = \text{var}(Y_t) - E[V_t], \quad f_{\text{var}}(Y_t^2, \beta) = \text{var}(Y_t^2) - 3\text{var}(V_t) - 2(E[V_t])^2, \\
f_h(Y_t, \beta) = \text{cov}(Y_t^{2+h}, Y_t^2) - \text{cov}(V_{t+h}, V_t).
\]

Then the parameter vector \(\beta_0\) is asymptotically identifiable.

**Proof:** The identifiability of the true parameter vector \(\beta_0\) can be directly deduced from the integrated supOU process \((V_n)_{n \in \mathbb{N}}\), see Section 2.3. Hence Theorem 3.2 yields the claim. \(\Box\)

From Proposition 3.5 and 3.6 one conjectures that for reasonable big finite \(m\) the model is identifiable. Unfortunately, it seems very hard to prove this. Thus for large values of \(m\) in practice the procedure should give consistent estimators. In the simulated
examples in Section 4 we see that $m$, which corresponds to the highest used order of the autocovariance function, has not to be chosen oversized to get good estimation results.

4 Illustrative examples

In this section we illustrate that the moment estimators are working very well. For that, we assume the semiparametric framework of Section 2.2 and that the underlying Lévy process $L$ of the supOU process $X$ is a compound Poisson process. Applying the Lévy-Itô decomposition to the supOU process $X$ (see [Barndorff-Nielsen & Stelzer, 2013, Theorem 2.2]) yields

$$X_t = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{-\infty}^{t} e^{A(t-s)} x \mu(dx, dA, ds)$$

where $\mu$ is a Poisson random measure. From that representation it follows that $X$ can be written as

$$X_t = \sum_{i \geq 1, \tau_i \leq t} e^{A_i(t-\tau_i)} U_i + \sum_{i=1}^{\infty} e^{A_{-i}(t+\tau_{-i})} U_{-i},$$

where

$$\tau_i := \sum_{j=1}^{i} T_j \text{ and } \tau_{-i} := \sum_{j=1}^{i} T_{-j} \quad \forall \ i \in \mathbb{N},$$

and $(T_i)_{i \in \mathbb{Z}\{0\}}$, $(U_i)_{i \in \mathbb{Z}\{0\}}$ and $(A_i)_{i \in \mathbb{Z}\{0\}}$ are independent sequences of iid distributed random variables with $T_i \sim \exp(\nu(\mathbb{R}))$, $U_i \sim \frac{1}{\nu(\mathbb{R})} \nu$ and $A_i \sim \pi$. Now we are able to simulate the introduced stochastic processes easily and to illustrate moment estimators. We simulate the processes 1000 times with 2000 observations in each run which corresponds to an observation period of 8 years for financial data. To avoid burn-in effects we simulate 500 time points more and cut them at the beginning. Afterwards we compute the empirical moments and solve the optimization problem (6) separately for each of the 1000 paths. To weight the moments appropriately we use the 2-step iterated GMM Estimation as described in [Hall, 2005, Section 3.6]. This means that the weighting matrix $I$ equals the identity matrix in the first step and in the second step the weighting matrix $I$ is an approximation of $S^{-1}$ where

$$S = \lim_{n \to \infty} \text{var} \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} f(Y_t, \beta_1) \right)$$

and $\beta_1$ is the estimation result of the first step. To simplify the estimation of $S$ we choose $n = (N - m)$, transform the right hand side of (7) as in [Hall, 2005, Formula 12]
and focus only on the diagonal, i.e.

\[ \hat{S} := \text{diag} \left( \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} (f(Y_t, \beta_1) - \hat{\mu})(f(Y_s, \beta_1) - \hat{\mu})' \right) \]

where \( \hat{\mu} = \frac{1}{n} \sum_{t=1}^{n} f(Y_t, \beta_1) \). In the illustrations below we see that this simplified 2-step iterated GMM Estimation works very well. Throughout the illustrations we concentrate on two different cases. In the first case we assume a parameter vector \( \beta_0 = (9, 36, 4, -2) \) to cover the case of short memory effects and in the second one we assume a parameter vector \( \beta_0 = (9, 36, 3/2, -5) \) to cover the case of long memory effects. Finally, the results are illustrated in histograms where the thick line indicates the true parameter value.

### 4.1 Estimation results

#### 4.1.1 SupOU processes

In Figure 2 we see the estimation results for a supOU process. The upper set of plots shows the short memory case and the lower set of plots shows the long memory case where the thick line indicates the true parameter values in all pictures. Obviously the estimation procedure works very well and the estimates are evenly distributed around the true parameter values. In the long memory case we have a little skewness in the parameter estimation of the Gamma distribution but this effect vanishes when rising the number of observations, see Figure 5 - Figure 8.

#### 4.1.2 Integrated supOU processes

In Figure 3 we see the estimation results of an integrated supOU process. Again, the upper set of plots shows the short memory case and the lower one shows the long memory case where the thick line indicates the true parameter values. The estimation procedure works again very well and the estimates are evenly distributed around the true parameter values, too. In the long memory case the skewness of the parameter estimation of the Gamma distribution is more distinct compared to the long memory case of the supOU process before.

#### 4.1.3 The supOU SV model

In Figure 4 we see the estimation results of a supOU SV model with short (upper set of plots) and long memory effects (lower set of plots). Although we have an additional noise in the model - the Brownian motion - we get excellent estimation results.
estimation quality seems due to a nice minimization problem in the case of a supOU SV model. It turned out that we get basically the same estimation quality if we weighted all moments equally, i.e. we can use a 1-step GMM Estimator where $I$ equals the identity matrix. This fact makes the 2-step iterated GMM Estimator unnecessary. Also in the case of long memory effects, when estimation is often challenging, our estimators behaves quite good.

4.2 Convergence illustration

We present the convergence of the estimates of a supOU process when the number of observations is rising. The estimation procedure is done for 500, 1000, 5000, 10,000 observations where 1000 paths are considered in each case. The results are plotted in Figure 5 to Figure 8 where the first two figures show the short memory case and the last two figures the long memory case. The convergence to the true parameters can be clearly seen in both cases. In particular, this shows that the skewness, which we have mentioned before, is vanishing for a rising number of observations. Moreover, the plots suggest nice distribution limits of the estimators (also in the long memory case). In the short memory case we actually are led to expect that the estimators are asymptotically normally distributed. Furthermore, at least for the parameters $\mu$ and $B$, the convergence seems to be slower in the long memory case.

4.3 Empirical data illustration

In an illustrative application to empirical data we estimate the S&P 500 via a supOU SV model under Assumption 3.1. We use the daily time series from 03/29/2010 to 03/20/2013 which corresponds to 750 observations. After the estimation of

$$(\hat{\mu}, \hat{\sigma}, \hat{\alpha}_\pi, \hat{B}) = (0.144, 0.025, 9.23, -2.36)$$

we plotted the empirical and the estimated autocorrelation function of the squared log returns in Figure 1. As we can see, the empirical autocorrelation function is well approximated by the estimated autocorrelation function. At the beginning we have a small sinusoidal effect in the empirical autocorrelation function which the model cannot capture. The power decay at rate $h^{-8.23}$ of the model autocorrelation function clearly fits well with the rather slow decay of the empirical autocorrelation, but recall that the estimated model has no long memory effects as $\hat{\alpha}_\pi > 2$. 
5 Conclusion

This paper developed a new estimation method for supOU processes and supOU SV models which are of particular interest because of the possibility of long memory effects. In a simulation study the estimation behaved very well and the results indicate that one has not only consistent (as shown in the paper) but also nice distribution limits, probably asymptotic normality in the case of short memory effects.

How the estimators are distributed (e.g. asymptotic normality) is future work beyond the scope of the present paper. First one needs to show central limit theorems for supOU processes. The standard way via strong mixing appears very hard since supOU processes are non-Markovian.

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![autocorrelation function of the squared log returns](image)

Figure 1: Illustration of the empirical and estimated autocorrelation function of the squared log returns. The dots display the autocorrelation function of the estimated supOU SV model and the bars display the empirical one.
Figure 2: Estimation of a supOU process with short (upper set of plots) and with long memory effects (lower set of plots).
Figure 3: Estimation of an integrated supOU process with short (upper set of plots) and with long memory effects (lower set of plots).
Figure 4: Estimation of the supOU SV model with short (upper set of plots) and with long memory effects (lower set of plots).
Figure 5: Convergence illustration with parameter vector \((\mu, \sigma, \alpha, \pi, B) = (9, 36, 4, -2)\). Estimation of the parameter \(\mu\) (upper set of plots) and \(\sigma^2\) (lower set of plots) of a supOU process with 500, 1000, 5000 and 10,000 observations.
Figure 6: Convergence illustration with parameter vector $(\mu, \sigma, \alpha_\pi, B) = (9, 36, 3/2, -5)$. Estimation of the parameter $\alpha_\pi$ (upper set of plots) and $B$ (lower set of plots) of a supOU process with 500, 1000, 5000 and 10,000 observations.
Figure 7: Convergence illustration with parameter vector $(\mu, \sigma, \alpha_\pi, B) = (9, 36, 3/2, -5)$. Estimation of the parameter $\mu$ (upper set of plots) and $\sigma^2$ (lower set of plots) of a supOU process with 500, 1000, 5000 and 10,000 observations.
Figure 8: Convergence illustration with parameter vector \((\mu, \sigma, \alpha_\pi, B) = (9, 36, 3/2, -5)\). Estimation of the parameter \(\alpha_\pi\) (upper set of plots) and \(B\) (lower set of plots) of a supOU process with 500, 1000, 5000 and 10,000 observations.