Convergence of perturbation series for renormalization constants in Kraichnan model with "frozen" velocity field

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Abstract.
Instanton was found for Kraichnan model with 'frozen' velocity field. Large order asymptotic of quantum-field perturbation expansion for renormalization constant $Z_\nu$ was investigated. It was shown that this expansion is convergent one. The radius of convergence was calculated.

1. Introduction

Large-order asymptotic analysis of quantum-field perturbation expansions is an actual problem of modern statistical physics. Direct perturbation calculations are cumbersome and difficult. Knowledge of large-order asymptotic and choice of the corresponding resummation procedure allow to get more or less good estimation on the base of a few first terms of perturbation series. Application of the resummation procedure without knowledge about large-order asymptotic behaviour can lead to inaccurate results.

Article [1] can be considered as a first attempt to solve this problem. The asymptotes were estimated here merely by number of graphs in the perturbation expansion order. The correct large-order asymptotic investigation proposed by Lipatov [2] is based on the saddle-point calculation of path integrals. It was used for all main quantum-field theory models and static models of critical behaviour, see [3]. Large-order analysis also was consistently constructed for the dynamic models with equilibrium static limit [4, 5] using standard Martin-Siggia-Rose (MSR) [6] variables. For all models mentioned above the instanton was found and divergent character of perturbation series was proved. Then there is common opinion that the instanton existence always leads to divergent series.
This paper accounts the Kraichnan model with 'frozen' velocity field. It describes the turbulent diffusion in stationary random field [7, 8, 9] and well known problem of random walks in random media [10] – [15]. The renormalization group (RG) approach was used for investigation of the scaling behavior in this model [8, 9]. The objects of interest are renormalization constants. In this paper large order asymptotic of perturbation expansion of viscosity renormalization constant in this model is investigated on the base of instanton found and it is shown that the coefficients of perturbation series grow essentially slower than \( N! \). The series of perturbation theory turn out to be convergent. The radius of convergence is calculated.

Then our result contradicts the common opinion about connection between instanton existence and perturbation expansion divergence. The more or the less general explanations of this phenomenon is given in Sec. 4. Let us note that the examples of instanton analysis of convergent series are known. One of these is the convergent perturbation series introduced in [16]. The instanton analysis of the convergence of this expansion was fulfilled in [17]. Another example is standard Kraichnan model [18, 19].

The large order asymptotic analysis is more difficult in dynamic models then in static ones. Usually there is no instanton in natural class of MSR variables here. For example the absence of instanton in Kraichnan model of turbulent diffusion was proven in [20]. Fortunately, the Lagrange variables can be used in this model [21, 22]. In these variables instanton was found and the large-order asymptotic behaviour was investigated [22]. The perturbation series in this model appeared to have a finite radius of convergence. Note that a specific feature of standard Kraichnan model is a proportionality of the velocity field correlator to \( \delta(t) \)-function. Then a lot of graphs of perturbation theory are absent here [23] that could explain the convergence of series discussed.

Because of the difficulties of instanton analysis the large-order asymptotic form was estimated in some papers merely by the number of graphs at large order of perturbation expansion in spirit of [1]. This approach produces accurate results for models with scalar and vector fields without derivations in interaction. But we will show that such estimation [7] may cause a mistake. Kraichnan model with frozen velocity field discussed in this article is a good example of this fact. In contrast with standard Kraichnan model, here the velocity field correlator doesn’t depend on time, so the number of perturbation diagrams demonstrates a factorial behaviour \( N! \) while the perturbation series is convergent. Simplified example of model considered with the constant velocity was considered in [24], where the number of perturbation diagrams demonstrates a factorial behaviour too, the instanton was found and the perturbation series convergence was proved by exact solution of the model.

This paper is organized as follows. The Kraichnan model with a "frozen" velocity field is described briefly in Sec. 2. Also the response function to be studied using MSR-formalism is introduced. The composite operator in Lagrange variables which is used for calculation of renormalization constant \( Z_\nu \) is introduced in Sec. 3. Instanton approach for this composite operator is presented in Sec. 4. A particular solution
of stationary equations used in the following analyses is investigated in Sec. 5. The
renormalization of the Green function with the composite operator is described in Sec.
6. Large-order asymptotic for expansion of $\ln Z_\nu$ is calculated in Sec. 7 using the replica
trick. Cumbersome stationarity equations for some numerical parameters are discussed
in Appendix.

2. Kraichnan model with a ‘frozen’ velocity field

Kraichnan model describes the turbulent advection of passive scalar admixture in
$d$-dimensional fluid. It is based on a stochastic equation

$$\partial_t + g \nabla_i V_j(x, t) - \nu \Delta \varphi(x, t) = \xi(x, t).$$ (1)

Here $x \in \mathbb{R}^d$ and $t$ are space and time variables, $\varphi(x, t)$ is a passive scalar field, $V(x, t)$
is a random vector velocity field, $\xi(x, t)$ is a random force, $\nu$ is a viscosity, $g$ is a coupling
constant. Laplacian and gradient refer to the space variable $x$; here and henceforth all
derivatives in squared brackets act on the field $\varphi$ as well. For shortness we introduce
$\partial_t \equiv \partial/\partial t$. The convolution with respect to repeating subscripts is implied here and
henceforth.

The random values $\xi$ and $V$ are supposed to be distributed by Gauss law. It is
known that the results obtained in RG analysis are independent from
$D_\xi$ [8, 9] then one can state the correlator $D_\xi$ has an arbitrary form. In contrast with the standard
Kraichnan model in the model discussed the velocity field correlator doesn’t depend on
time

$$\langle V_i(x, t)V_j(x', t') \rangle = D_{ij}(x - x')$$

(compare with $\langle V_i(x, t)V_j(x', t') \rangle = D_{ij}(x - x')\delta(t - t')$ in the standard Kraichnan
model), in other words one can consider $V$-field as a time independent. The velocity
field correlator in the momentum representation has the power-like form [8, 9]

$$D_{ij}^F(q) \equiv \int d^d z D_{ij}(z) \exp(iqz) = \lambda_T \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) \frac{1}{q^{2\alpha}} + \lambda_L \frac{q_i q_j}{q^2} \frac{1}{q^{2\alpha}},$$ (2)

that was used for RG-analyses of the model. Then in the coordinate representation

$$D_{ij}(z) = a_1 \frac{\delta_{ij}}{z^{2\beta}} + a_2 \frac{z_i z_j}{z^{2\beta+2}}, \quad \beta = d/2 - \alpha = 1 - \varepsilon/2,$$ (3)

where the parameters $a_1$, $a_2$ are known:

$$a_1 = \frac{\Gamma(\beta)}{2^{2\alpha+1}\pi^{d/2}\Gamma(\alpha + 1)}(\lambda_T (2\alpha - 1) + \lambda_L),$$ (4)

$$a_2 = (\lambda_T - \lambda_L) \frac{\Gamma(\beta + 1)}{2^{2\alpha}\pi^{d/2}\Gamma(\alpha + 1)}.$$ (5)

$\lambda_T$ and $\lambda_L$ are transverse and longitudinal coupling constants. The parameter $\lambda_L$
corresponds to compressibility of fluid.

The model considered is important for the description of diffusion in random fluids
[8, 9]. Moreover it relates to the developed turbulence problem [25]. In fact the model
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(1, 2) has two coupling constants $\lambda_T$, $\lambda_L$. Nevertheless for large order asymptote of $\varepsilon$ expansion investigation these can be reduced to the only one coupling constant $g$. Indeed in a fixed point all coupling constants are proportional to a small parameter of regular expansion $\varepsilon = 2 + 2\alpha - d$, [8, 9] $\lambda_T \sim \lambda_L \sim g^2$. The similar situation was observed in dynamic models C-H (A, B, C,... are a common nomination for particular dynamic models introduced in [26]) with equilibrium $\varphi^4$ static limit [27, 5].

The infrared behaviour of the model was investigated by means of RG method. The renormalization yields the substitution $\nu \to \nu Z_{\nu}$, $g \to g Z_g$, the renormalization constant $Z_{\nu}$ is calculated by means of perturbation theory, then it has a form of series in coupling constant. The properties of this series are the main subject of this paper.

Usually MSR-formalism [6, 27] is used to transform stochastic models to quantum-field ones. The base equation (1) is represented in a form of path integration in auxiliary field $\varphi'$; path integrations in $\varphi$ and $V$ fields are introduced to study averaged characteristics of fluid. Then the expression for the response function in an arbitrary velocity field has a form

$$G_{V}^{(1, 2)} = \frac{\int \mathcal{D}\varphi \mathcal{D}\varphi' \varphi(x_1, t_1)\varphi'(x_2, t_2) \exp(S_{msr}^{msr})}{\int \mathcal{D}\varphi \mathcal{D}\varphi' \exp(S_{msr}^{msr})},$$

with the renormalized action [8, 9]

$$S_{msr}^{msr} = \frac{\varphi'(x, t)D\varphi'(x', t')}{2} + \varphi'(x, t)[\partial_t + gZ_g \nabla_i V_i(x) - \nu Z_\nu \Delta] \varphi(x, t).$$

usual standard agreements for dynamic models [27] and all integrations needed are implied henceforth. The normalization factor in Exp. (6) corresponds to a free model (at $g = 0$).

After the integration in $V$ field one obtains MSR representation for the renormalized response function

$$< \varphi(x_1, t_1)\varphi'(x_2, t_2) = \frac{\int \mathcal{D}V \mathcal{D}V G_{V}^{(1, 2)} e^{-V_i(x)D^{-1}_{ij}(x-x')V_j(x')/2}}{\int \mathcal{D}Ve^{-V_i(x)D^{-1}_{ij}(x-x')V_j(x')/2}}.$$

3. Lagrange variables

As it was stated in papers [22, 20] there is no instanton for Kraichnan model in the framework of MSR formalism. Similar arguments are correct for the model with the frozen velocity field. But Lagrange variables [21, 22] can be used for instanton analysis of the model considered as in standard Kraichnan model.

Seems these variables have an origin in the quantum-fields methods application in random walks and macromolecules problems (see [28]) in analogy with Hamiltonian form of the standard Feynman-Kac path integral. Lagrange variables can be introduced successfully in the dynamic models with linear in main field $\varphi$ stochastic equation (1) only. Then the dynamic equation for the response function (6) can be considered as Schrodinger equation [21] or as Fokker-Plank equation [22], and response function in an
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arbitrary velocity field can be represented in the form

\[ G_v^{(1,2)} = \frac{\Theta(t_2 - t_1)}{(4\pi \nu)^{d/2}} \int \mathcal{D}c \mathcal{D}c' \exp(S_{LGR}^{(t_2 - t_1)}) \exp(S_{LGR}^{(t_1 - t_2)}), \]  

(9)

\[ S_{LGR} = \int_{t_1}^{t_2} d\tau \left(-\nu Z_{\nu} c'^2(\tau) + i c'(\tau) \partial_\tau c(\tau) + g Z_{\nu} c(\tau) V(c(\tau))\right). \]

The boundary conditions

\[ c(t_1) = x_1, \quad c(t_2) = x_2 \]

are implied for the path integration in c field in the numerator. The integration in c' fields is supposed to have free boundary conditions. The normalization path integral corresponds to a free model with g = 0 and may be calculated at zero boundary conditions for c, c' fields. Vector fields c(\tau), c'(\tau) play a role of coordinates and momenta of fluid particles and depend on time only.

Let us note that this representation produces one Green function of the model only, namely the response Green function in an arbitrary velocity field. Then the MSR action (7) can be obtained from (9) by no change of variables and the statement about an instanton absence in MSR variables is not correct for Lagrange ones.

Lagrange variables can be used to investigate renormalization constant \( Z_{\nu} \) in theory (7). Indeed let’s differentiate Exp. (8) with respect to \( \nu \) in order to extract \( Z_{\nu} \) constant. Then one has two point Green function with the \( \varphi' \nu Z_{\nu} \Delta \varphi \) composite operator insertion. Using the response function (8) this Green function

\[ Z_{\nu} < \varphi(x_1, t_1) \varphi'(x_2, t_2) \int dx_0 dt_0 \varphi'(x_0, t_0) \Delta \varphi(x_0, t_0) > \equiv G \]

can be rewritten as

\[ G = Z_{\nu} \int dx_0 dt_0 \]

\[ \frac{\int \mathcal{D}V G_v^{(1,2)}(x_1, t_1, x_0, t_0) \Delta G_v^{(1,2)}(x_0, t_0, x_2, t_2) e^{-V_i(x)D_{ij}^{-1}(x-x')V_j(x')/2}}{\int \mathcal{D}V e^{-V_i(x)D_{ij}^{-1}(x-x')V_j(x')/2}}. \]

The renormalization constant \( Z_{\nu} \) has poles in \( \varepsilon \) at \( \varepsilon \to 0 \) that have to cancel UV divergences of the model. As we consider a renormalized response function (8) Exp. (10) must be finite at \( \varepsilon \to 0 \). Then

\[ \text{res}_{\varepsilon \to 0} \ln Z_{\nu} = -\text{res}_{\varepsilon \to 0} \int dx_0 dt_0 G \]

(11)

and the Green function G contains all information needed about the poles of renormalization constant \( Z_{\nu} \).

The diagrams for G include the internal loop part and the external full propagator part without divergences. The last does not contribute into Exp. (11) then one will discuss now the amputated diagrams for G.
The Green function $G$ can be easily written in Lagrange variables. The following Gaussian path integration in field $V$ and Fourier transforms in $x_2 - x_1$ and $t_2 - t_1$ variables yield the action to be studied

$$S = -i\mathbf{q}(x_2 - x_1) - \nu Z_\nu (c_1'^2 + c_2'^2) + i c_1' \partial c_1 + i c_2' \partial c_2 + Z_u S_u,$$

where $\mathbf{q}$ is a momentum. Frequency $\omega = 0$, that is sufficient for the renormalization constant investigation, because this constant is frequency independent. A nonlinear part of the action is collected in the term

$$S_u = -\frac{u}{2} \left( c_{11}'(\tau_1) D_{ij}(c_1(\tau_1) - c_1(\tau'_1)) c_{1j}'(\tau'_1) + c'_{2i}(\tau_2) D_{ij}(c_2(\tau_2) - c_2(\tau'_2)) c'_{2j}(\tau'_2) + 2 c'_{1i} D_{ij}(c_1 - c_2) c'_{2j} \right), \quad u \equiv g^2.$$

Here and henceforth one implies that the fields $c_i, c'_i$ ($l = 1, 2$) with argument omitted depend on $\tau_l$. All necessary integrations in $\tau_l$ and the ranges of integration

$$t_1 \leq \tau_1, \tau'_1 \leq t_0 \leq \tau_2, \tau'_2 \leq t_2$$

are assumed. Finally one gets for Fourier transformed $G$ function the following expression in Lagrange variables:

$$G = \int d(x_2 - x_1) \int d(t_2 - t_1) \frac{1}{16\pi \nu \sqrt{(t_2 - t_0)(t_2 - t_1)d}}$$

where fore-exponential factor

$$W = -\frac{d\tau_1 d\tau_2 (ic'_1 + (t_1 - \tau_2)\mathcal{F}_1) \int d\tau_1 d\tau_2 (ic'_2 - (t_2 - \tau_2)\mathcal{F}_1)}{t_1 - t_0} +$$

$$+ \frac{d\tau_1 d\tau_2 (t_1 - \tau_1)(t_2 - \tau_2)\mathcal{F}_2}{(t_1 - t_0)(t_2 - t_0)},$$

$$\mathcal{F}_s \equiv u Z_u c'_1 \frac{\partial^s}{\partial c_1^s} D_{ij}(c_1 - c_2) c_{2j}$$

is produced by Laplace operator in (10). Let us note that the non-trivial boundary conditions for $c_1, c_2$ fields

$$c_1(t_1) = x_1, \quad c_2(t_2) = x_2, \quad c_1(t_0) = c_2(t_0) = x_0$$

mean the path integration with fixed boundary conditions. The integration in $c'_1$ fields is supposed to have free boundary conditions. The normalization factor $\mathcal{N}$ corresponds to a free model with $u = 0$ and may be calculated at zero boundary conditions for $c_1$:

$$\mathcal{N} = \int Dc_1 Dc_2 Dc'_1 Dc'_2 \exp (S|_{g=0}).$$
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For shortness we introduce the parameters

\[ T \equiv t_2 - t_1, \quad T_1 \equiv t_0 - t_1, \quad T_2 \equiv t_2 - t_0, \]
\[ x \equiv x_2 - x_1, \quad x^{(1)} \equiv x_0 - x_1, \quad x^{(2)} \equiv x_2 - x_0, \]

the values \( x, x_0, x_1, x_2, x^{(1)}, x^{(2)} \) are the modules of corresponding vectors.

4. Instanton analysis

In order to extract the \( N \)-th term of perturbation series the Cauchy formula is traditionally used [2]:

\[ G^{[N]} = \frac{1}{2\pi i} \oint \frac{G(u)}{u^{N+1}} du, \quad G(u) = \sum_{N=0}^{\infty} G^{[N]} u^N, \]

the integration is produced along a closed contour containing zero in a complex plane.

Let us extract the large \( N \) parameter from the action \( S \) (12) by the following scaling

\[ \{ \mathbf{c}, \mathbf{c}' \} \rightarrow \{ N^{1/2} \mathbf{c}, N^{1/2} \mathbf{c}' \}, \quad \mathbf{x} \rightarrow \sqrt{N} \mathbf{x}, \quad u \rightarrow N^\beta u, \]

the scaling of \( \mathbf{x} \) variable is necessary due to the connection between \( \mathbf{c} \) and \( \mathbf{x} \) based on (15). The same scaling in normalization factor \( \mathcal{N} \) cancels the determinant corresponding to this change of variables. The momentum \( \mathbf{q} \) is scaled \( \mathbf{q} \rightarrow N^{1/2} \mathbf{q} \) also. It is possible because the renormalization constant investigated is momentum independent.

The integrals in \( \mathbf{c}', \mathbf{c} \) and \( u \) at large \( N \) can be calculated by the saddle-point approach. The main contribution at \( N \rightarrow \infty \) is given by the integration near the instanton that is a special realization of variables \( \mathbf{c}_{st}, \mathbf{c}'_{st}, u_{st} \). The action has an extremum at the instanton.

Let us mark that the scaling procedure is an essential in the determination of the connection between instanton existence and divergence or convergence of perturbation theory. Usually the quantum-field theory action has a form \( S = S_0 + S_{int} \), where \( S_0 \) is a free part of the action, \( S_0 = \Phi K \Phi / 2 \) where \( \Phi \) denotes a field or a set of fields of the theory considered, \( K \) is a linear operator. \( S_{int} \) is an interaction of the form \( S_{int} = \lambda \Phi^k \) \((k > 2) \) in a local theory. Presence of derivations in the interaction does not affect analysis presented below. \( \lambda \) is a coupling constant and the expansion parameter. To extract large \( N \) parameter from the action the scaling \( \Phi \rightarrow \sqrt{N} \Phi, \quad \lambda \rightarrow \lambda / N^{(k-2)/2} \) is necessary. Then the main exponential contribution in expression similar to (19) due to instanton is proportional to

\[ G^{[N]} \sim N^{k-2} e^{-NS_{st}}, \]

where \( S_{st} \) is an action in the stationary point. This contribution demonstrates the divergence of the perturbation expansion in \( \lambda \) due to \( k > 2 \). Then the divergence of the series is connected with the scaling of the coupling constant. In [3] another instanton analysis scheme without Cauchy formula using was proposed. But the results of analysis presented above will be the same in both schemes.
Note that the model considered (12) in Lagrangian variables differs from the general case. The coupling constant is scaled here by positive power of \( N \) due to the nonlocal character of the interaction, then instanton analysis can lead to convergent series as in standard Kraichnan model [22] or simplified example of Kraichnan model with constant velocity field [24], where the instanton was found and the perturbation series convergence was proved by exact solution of the model.

Let us describe instanton calculation in the model (12) in more details. It’s quite reasonable to simplify problem by taking into account the symmetry of the model that is initially violated by \( \mathbf{x} \) vector only. Then we suppose that the fields \( c_l, c'_l \) are parallel to \( \mathbf{x} \) and the only modules \( c_l, c'_l \) must be found. Let us mark that the stationarity equations are non-linear differential ones. The existence and the uniqueness of the solution are not proved in general case. We propose to use the solution with the same symmetry as Green function investigated. Note that the spherical symmetry of the instanton in Lipatov work [2] based on the same ideas. The possibility of other solutions existence with other contributions to large order asymptotes is an open question. The same situation is observed in every case of instanton analysis. The supposition about the symmetry of solution used here was proven in [24] for simplified Kraichnan model with known exact solution. For standard Kraichnan model considered in [22] this supposition yields the result coinciding with the exact known anomalous dimensions of a set of composite operators.

All stationarity equations are supposed then to be projected on the direction of \( \mathbf{x} \) vector. It also simplifies the tensor structure of \( D \) correlator that has more compact form now:

\[
D(\mathbf{x}) = \frac{D_0}{|\mathbf{x}|^{2\beta}}, \quad D_0 = a_1 + a_2. \tag{21}
\]

Except the regular in \( \varepsilon \) terms the action \( S \) (12) contains poles in \( \varepsilon \) due to \( Z_\nu, Z_\eta \) constants. It was shown in [29] that while a renormalization constant is investigated the corresponding singularities must be extracted before any instanton calculations and later they contribute only in a fore-exponential factor of the saddle-point method. This extraction of singularities is necessitated by existence of two large parameters, namely \( 1/\varepsilon \) connected with regularization and the saddle-point method parameter \( N \). Due to the renormalization approach the value \( N\varepsilon \) must be considered as a small one in the framework of instanton analyses [29]. Then the exponential term in (13) must be presented in a form

\[
\exp(S) = \exp(S_{\text{reg}} + S_{\text{sing}}) = \exp(S_{\text{reg}}) \sum_{p=0}^{\infty} \frac{1}{p!}(S_{\text{sing}})^p, \tag{22}
\]

\[
S_{\text{reg}} = S \bigg|_{Z_\nu = 1, Z_\eta = 1}, \quad S_{\text{sing}} = -\nu(Z_\nu - 1)(c_1^2 + c'_2) + (Z_u - 1)S_u
\]

and only the term \( S_{\text{reg}} \) in the exponent must be variated. In standard Kraichnan model this approach was proven in [22] by the comparison of the radius of convergence calculated with the exact known results. Mention should be made that the l.h.s and the
r.h.s of the identity (22) are essentially different under the integral $\oint du / u^{N+1}$ and they yield different results of the saddle-point method due to the competing of parameters $N$ and $1/\varepsilon$.

Let’s remind the renormalization constants $Z_\nu$, $Z_u$ in minimal subtraction (MS) scheme have a form $1 + \text{poles in } \varepsilon$ terms, so that $(Z - 1)$ contain the pure singularities at $\varepsilon \to 0$ only. Exp. (22) shows that the divergences in $\varepsilon$ make a sense in the framework of perturbation theory and diagrammatic expansion only. Then the path integration must be interpreted as a sum of perturbation terms, the renormalization is supposed to be a cancellation of divergences.

Then let us set $Z_\nu = 1, Z_g = 1$ in the expression for the action (12) in order to write the regular instanton equations. The variations of action $S$ involve the integral operators of the form

$$[D_{lk}c'_k](\zeta) \equiv \int d\tau_k D(c_l(\zeta) - c_k)c'_k, \quad l, k = 1, 2.$$  

For example the variation in $c_1$ yields

$$\frac{\delta S}{\delta c_1(\zeta)} = 0 \quad \Rightarrow \quad uc'_1(\zeta)\partial_\zeta ([D_{11}c'_1](\zeta) + [D_{12}c'_2](\zeta)) = -i\partial_\zeta c'_1\partial_\zeta c_1, \quad (23)$$

as well the variation in $c'_1$

$$\frac{\delta S}{\delta c'_1(\zeta)} = 0 \quad \Rightarrow \quad -2\nu c'_1(\zeta) + i\partial_\zeta c_1 - u([D_{11}c'_1](\zeta) + [D_{12}c'_2](\zeta)) = 0. \quad (24)$$

The contribution of integral operators in eq. (23) can be excluded with the help of eq. (24). For this purpose we should differentiate (23) in $\zeta$

$$\partial_\zeta \frac{\delta S}{\delta c'_1(\zeta)} = 0 \quad \Rightarrow \quad -2\nu \partial_\zeta c'_1 + i\partial_\zeta^2 c_1 - u([D_{11}c'_1](\zeta) + [D_{12}c'_2]) = 0,$$

express $[D_{11}c'_1] + [D_{12}c'_2]$ and substitute it in the eq. (23). It yields the 2nd order differential instanton equation

$$i\partial_\zeta^2 c_1 - 2\nu \partial_\zeta c'_1 + i\frac{\partial_\zeta c'_1}{c'_1(\zeta)} = 0 \quad (25)$$

that can be solved with respect to $\partial_\zeta c(\zeta)$. The same calculation can be done for $c_2$ field. The solution of equation (25) has a form

$$\partial_\zeta c_l(\zeta) = \frac{F}{c'_l(\zeta)} - i\nu c'_l(\zeta), \quad l = 1, 2$$

and contains the arbitrary parameter $F$. Each numeric value of $F$ constant corresponds to a particular solution with its own boundary conditions.

The following calculations at arbitrary $F$ can not be performed analytically, the simplest case at $F = 0$ allows to reduce the instanton to quadrature only. Nevertheless the corresponding particular solution can be used to determine the asymptotic behaviour of the renormalization constant investigated.
5. Particular solution

Let us consider the particular solution with $F = 0$ and determine its boundary conditions. The solution
\[
c_l'(\zeta) = \frac{i\partial_\zeta c_l}{\nu}, \quad l = 1, 2
\] (26)
can be substituted into the variation equations with respect to $c_l'$. Using the identity
\[
d\tau \frac{\partial}{\partial c_l} = \frac{dc_l}{\partial c_l}
\]
the result can be written in a form
\[
-\partial_\zeta c_l = \frac{u}{\nu} \int_{x_1}^{x_2} D(c_l(\zeta) - z)dz, \quad l = 1, 2.
\]
The last differential equation can be easily integrated and this leads to the solution for the $c_l(\zeta)$ fields in quadrature:
\[
\int_{x_1}^{c_1(\zeta)} \frac{dc}{\int_{x_1}^{x_2} D(c - z)dz} = -\frac{u(\zeta - t_1)}{\nu}, \quad \int_{x_0}^{c_2(\zeta)} \frac{dc}{\int_{x_1}^{x_2} D(c - z)dz} = -\frac{u(\zeta - t_0)}{\nu}.
\] (27)

Note that an analytic regularization is assumed in (27) that eliminates the singularity in $z = c$ point.

After the substitution of explicit form $D(c)$ (21) Exp. (27) produces the boundary condition interested
\[
\frac{1}{T_1} \int_0^{x^{(1)}/x} f(v)dv = \frac{uD_0}{x^{2-\varepsilon}(1-\varepsilon)\nu} = \frac{1}{T_2} \int_0^{x^{(2)}/x} f(v)dv,
\] (28)
where the function $f(v)$ introduced is
\[
f(v) = \frac{v^{1-\varepsilon}(1-v)^{1-\varepsilon}}{v^{1-\varepsilon} + (1-v)^{1-\varepsilon}}.
\]

One sees that the case $F = 0$ is simple enough to give a quadrature representation (27) for $c_1$, $c_2$ fields with the boundary conditions (15).

Initially the problem consists in calculation of instanton for an arbitrary boundary condition $x(T)$. Exp. (28) solves the problem in the specific case with a special value of $x^{(1)}$, $x^{(2)}$, and $x$. Then one has instanton for functional integral in $c$, $c'$ fields. But the object investigated (19, 12) includes integrations in $x$, $T$, $x_0$, $t_0$ and $u$ also. Our main idea here is to explore the independence of the renormalization constant investigated on the momentum $q$. Let us include the integrals in variables $x$, $t$, $x_0$, $t_0$ into the saddle-point method. Let’s choose the momentum $q$ so that the solution of a stationarity equation for $x$ variable be exactly equal to the result obtained with the help of boundary condition (28) corresponding to the case $F = 0$. This choice gives us a chance to explore the particular solution constructed analytically and to solve the problem without numerical calculations. Moreover constant $F$ is not a free parameter in this approach, then one has no problem of zero modes connected with $F$ arbitrariness.
The stationarity equations for $x$, $x_0$, $t_0$ variables are too cumbersome to be written down here. Nevertheless, as shown in Appendix of the article, these can be simplified significantly using particular solution (26), its properties, and integration by parts. Then the equations solution calculated corresponds to the case $x^{(1)} = x^{(2)}$, $T_1 = T_2$ and the action has the following form in stationary point (26)

$$S_{st} = -iqx - \frac{uD_0}{\nu^2\varepsilon(1 - \varepsilon)}x^\varepsilon. \quad (29)$$

Now the stationary equations for $x$ in $F = 0$ case:

$$\frac{\delta S}{\delta x} = 0 \Rightarrow \frac{iq\nu^2}{u} = \int_0^x D(z)dz. \quad (30)$$

Besides we have the boundary conditions (28) imposed by our choice of particular solution $F = 0$. By solving equation (30) one obtains the proper value for $q$ in case $F = 0$:

$$q = q_0 = \frac{iD_0u}{(1 - \varepsilon)x^{1-\varepsilon}\nu^2}, \quad x^{2-\varepsilon} = x_{st}^{2-\varepsilon} = \frac{uD_0T/2}{(1 - \varepsilon)\nu^{1/2}} \int_0^1 f(v)dv \quad (31)$$

Combining (29) and (31) finally, one obtains the action $S$ (12) at the stationarity solution and $Z_\nu = 1$, $Z_u = 1$:

$$S_{st} = -\frac{uD_0x_{st}^\varepsilon}{\nu^2\varepsilon}. \quad (32)$$

6. Simple poles in $\varepsilon$

Due to (11) simple poles in $\varepsilon$ of $G$ contain all the necessary information about the poles of renormalization constant $Z_\nu$ and the corresponding critical indices.

Since $D_0(\varepsilon) = a_1 + a_2$ (21) and (4,5) $D_0(\varepsilon)$ function can be presented in a form

$$D_0(\varepsilon) \equiv A + \varepsilon B(\varepsilon), \quad B(\varepsilon) = B_0 + B_1\varepsilon + O(\varepsilon^2). \quad (33)$$

It now follows that the action (32) has a form

$$S_{st} = -\frac{uA}{\nu^2\varepsilon} - \frac{uA(x_{st}^\varepsilon - 1)}{\nu^2\varepsilon} - \frac{uB(\varepsilon)x_{st}^\varepsilon}{\nu^2}. \quad (34)$$

The first term here is singular in $\varepsilon \to 0$, then as well as $S_{sing}$ term discussed in Section 4 it must be presented in fore-exponent form (22). The second term seems to be finite at small $\varepsilon$. Nevertheless its logarithmic behaviour in $x_{st}$ results in singular in $\varepsilon$ contribution to the large order asymptote. We will discuss this at the end of this Section. As a result the only regular term of action $S_{st}$ (34) is the third one. Then Exp. (22) must be corrected by changing $S_{reg} \to \bar{S}_{reg}$, $S_{sing} \to \bar{S}_{sing}$,

$$\bar{S}_{sing} = S_{sing} - \frac{uA}{\nu^2\varepsilon} - \frac{uA(x_{st}^\varepsilon - 1)}{\nu^2\varepsilon}. \quad (35)$$
Combining Exp. (34) with the result for \( x_{st} \) (31) we get

\[
\bar{S}_{reg} = (uT^{\varepsilon/2})^{2/(2-\varepsilon)} P(\varepsilon), \quad P(\varepsilon) \equiv -\frac{B(\varepsilon)}{\nu^2} \left( \frac{D_0}{(1-\varepsilon)\nu^2} \int_0^{1/2} f(v)dv \right)^{\varepsilon/(2-\varepsilon)}.
\]

Thus the Green function studied is of the form

\[
G[^N] = N^{-N(1-\varepsilon)} \int \frac{du}{u^{N+1}} \int dT \mathcal{Z}(T, u, \varepsilon) \exp( NS_{reg}(\varepsilon, T)) \times \\
\times \sum_{p=0}^{\infty} \frac{1}{p!} (\bar{S}_{sing})^p (1 + O(N^{-1})).
\] (35)

The \( \mathcal{Z} \) factor stays for a Gaussian fluctuations contribution and fore-exponential factor \( W(14) \), the corresponding integration is normalized by \( N \) factor. \( O(N^{-1}) \) term shows the accuracy of calculation at \( N \to \infty \).

The amputated Green function with composite operator considered must be dimensionless. The factor \( T^{-1} \) restoring this zero dimension has to be produced by \( \mathcal{Z} \) and we will extract \( T^{-1} \) from \( \mathcal{Z} \) in order to stress this fact.

The integration in \( T \) then diverges as a logarithm and produces singularities. It can not be treated by the saddle-point approach. The analogous situation exists in a well-developed instanton analyses for static \( \phi^4 \) model where the role of divergent integration parameter \( T \) plays the scale parameter in the coordinate space [2, 29].

Let us change variables

\[
\bar{u} \equiv uT^{\varepsilon/2},
\]

the new variable \( \bar{u} \) is dimensionless. As a result the expression

\[
G[^N] = N^{-N(1-\varepsilon)} \int \frac{dT}{T^{1-N\varepsilon/2}} \int \frac{d\bar{u}}{\bar{u}} \mathcal{Z}(\bar{u}, \varepsilon) \exp \left( N \left[ P(\varepsilon)\bar{u}^{2\varepsilon/(2-\varepsilon)} - \ln \bar{u} \right] \right) \times \\
\times \sum_{p=0}^{\infty} \frac{1}{p!} (\bar{S}_{sing})^p (1 + O(N^{-1}))
\] (36)

can be integrated over \( T \) in UV region (small \( T \)). This yields a simple pole \( 2/(N\varepsilon) \). The convergence of integral at large \( T \) is provided by IR regularization assumed. Note the factor \( \mathcal{Z} \) does not contribute to the stationary equations as it does not depend on \( N \). In the same way this factor did not affect the simple pole in \( N\varepsilon \) at least at principal order in \( 1/N \).

The integration in \( \bar{u} \) is investigated by the saddle-point approach. The stationary equation with respect to \( \bar{u} \) is

\[
\frac{\partial}{\partial \bar{u}} \left[ P(\varepsilon)\bar{u}^{2\varepsilon/(2-\varepsilon)} - \ln \bar{u} \right] = 0, \quad \text{then} \quad \bar{u}_{st}^{2\varepsilon/(2-\varepsilon)} = \frac{1-\varepsilon/2}{P(\varepsilon)}.
\]

So we obtain the leading order in \( N \)

\[
G[^N] = C(\varepsilon)N^{-N(1-\varepsilon)+\rho} \frac{2}{N\varepsilon} e^{(\varepsilon/2-1)} \left( \frac{P(\varepsilon)e}{1-\varepsilon/2} \right)^{(N+1)(1-\varepsilon/2)} \sum_{p=0}^{\infty} \frac{1}{p!} (\bar{S}_{sing})^p.
\]
The factor $C(\varepsilon) N^0$ appears due to $Z(\bar{u}_{\text{st}}, \varepsilon)$ contribution and the fluctuation integration in $\bar{u}$; $\rho$ is a constant.

Let us discuss the residue in $\varepsilon$ calculation. Simple poles in $\varepsilon$ can appear if higher poles of $S_{\text{sing}}$ in the fore-exponent are multiplied by regular in $N \varepsilon$ contribution of the exponent term $\exp(S_{\text{reg}}(\varepsilon))$. Therefore in the MS scheme chosen all $p \neq 0$ terms of the sum contribute to the result. Fortunately a finite renormalization could help us to map out all this terms. Indeed let’s scale the expansion parameter

$$uD_0(\varepsilon) \to \kappa(\varepsilon)uD_0(\varepsilon).$$

Such a renormalization is equivalent to a scaling of the velocity field correlator and doesn’t affect the scaling dimensions. Let’s choose $\kappa(\varepsilon)$ so that the regular part of action loses its dependence on $N \varepsilon$, namely

$$\left( \frac{P(\varepsilon)e}{1 - \varepsilon/2} \right)^{(1 - \varepsilon/2)} \to \left( \frac{P(\varepsilon)e}{1 - \varepsilon/2} \right)^{(1 - \varepsilon/2)} \bigg|_{\varepsilon=0} \equiv K_0.$$

The corresponding renormalization can be easily written as a perturbation series

$$\kappa(\varepsilon) = 1 + \left( \frac{B_0}{2A} \ln \left[ - \frac{B_0}{6A\nu} \right] - \frac{B_1}{A} \right) \varepsilon^2 + O(\varepsilon^3),$$

the parameters $A, B_0, B_1$ are introduced in (33). After the scaling all poles in $\varepsilon$ contained in the sum with respect to $p$ don’t contribute to the simple pole. As a result the residue discussed demonstrates the asymptotic behaviour

$$\text{res}_{\varepsilon \to 0} G^{[N]} = \text{Const} N^\text{Const} K_0^N / N!, \quad N \to \infty$$

that corresponds to a finite radius of convergence for the perturbation series of $G$ function.

Now let’s show that the second term in (34) doesn’t change our answer. Indeed its behaviour in $x$ is logarithmic. Then the presence of additional factor $\ln x \sim \ln(T)$ in (36) is equivalent to the additional operation $d/d(N\varepsilon)$ of $T^{N\varepsilon/2}$ factor in (36). This operation can’t produce simple pole in $\varepsilon$ as our expression doesn’t contain logarithmic in $\varepsilon$ contributions. It was shown in details for the similar problem in [29].

The following step is to calculate $\ln G(u)$, extract the residue in $\varepsilon = 0$ of the $N$-th order of perturbation theory and explore formula (11). It’s useful to transform the logarithm of the composite operator $G$ with the help of replica trick.

7. The replica trick

The simplest way to present a logarithm of arbitrary integral expression in an integral form is to use the formula

$$\ln \int dT f(T) = \lim_{r \to 0} \frac{\partial}{\partial r} \prod_{\alpha=0}^{r-1} \int dT_\alpha f(T_\alpha).$$

(37)
As a result the variable $T$ becomes an $r$-dimensional vector in a replica space [27].

Performing such a procedure with respect to $G$ one gets the expression similar to (35) where the integrals must be rewritten as follows

$$\oint \frac{du}{u^{N+1}} \int \prod_{a=0}^{r-1} dT_a \exp(N \bar{S}_{\text{reg}}) Z({\{T_a\}_{a=0}^{r-1}}),$$

$$\bar{S}_{\text{reg}} = \sum_{\alpha=0}^{r-1} (uT^2/2)^{2/(2-\varepsilon)} P(\varepsilon),$$

the factor $Z$ depends on the replica variables $\{T_\alpha\}$. Nevertheless one knows that the expression investigated with the help of replica trick is also dimensionless due to $Z$ contribution. Besides the expression must be proportional to $r$ at small $r$. Then the operation $\lim_{r \to 0} \partial / \partial r$ in (37) yields a non-trivial finite result.

It is easy to see that the saddle-point approach can not be applied to all integrations in (38). At least one of integration considered has a non-saddle-point structure. We observed the same situation in the previous section in case of integral in $T$.

Let us exclude the integration in $T_0$ from the saddle-point approach consideration. The set of other variables $\{T_\alpha\}_{\alpha=1}^{r-1}$ has a zero solution at the saddle point. So the stationarity solution for $u_{st}$ depends on $T_0$ only. As this non-saddle-point mode in the replica space is chosen in an arbitrary manner, in fact we have constructed $r$ identically instanton solutions in the replica space. This results in the factor $r$ which allows us to produce the operation $\lim_{r \to 0} \partial / \partial r$ correctly.

The integration in $T_0$ must be treated in the same way as the integration in $T$ in the previous section and yields the same result

$$\text{res} \ln Z_\nu = \text{Const} N^{\text{Const} K_0^N / N!}, \quad N \to \infty.$$ (39)

The calculation of $\text{Const}$ in this formula could not be produced without an explicit calculation of the fluctuation integral. But this is difficult and cumbersome problem that is not to be solved here. Thus the asymptotic form (39) demonstrates that the series investigated has a finite radius of convergence.

In fact this section can be resumed as follows. We have shown the saddle-point method yields the appropriate result namely the circle of convergence for function $\ln G$ is determined by the same singularity as function $G$. The properties of this singularity were calculated by saddle-point method. Other singularities for $\ln G$ could exist in principle but these have non-saddle-point structure.

8. Conclusions

We have constructed the family of instantons in Kraichnan model with "frozen" velocity field and found one of them in explicit form. Considering the asymptotic behaviour of the renormalization constant at large order of perturbation expansion we have demonstrated that the corresponding perturbation series has finite convergence.
radius. Furthermore, we have disproved the common statement that the behaviour of the series may be defined by quantity of diagrams at large order of perturbation.

Our results can be used for an improvement of resummation procedures constructed for the Kraichnan model.

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9. Appendix

Let us consider the stationarity equations for $x$, $x_0$, $t_0$ variables. The linear translation of fields

$$c_1(\tau_1) = x_1 + x^{(1)}(\frac{\tau_1 - t_1}{T_1}) + \bar{c}_1(\tau_1), \quad c_2(\tau_2) = x_0 + x^{(2)}(\frac{\tau_2 - t_0}{T_2}) + \bar{c}_2(\tau_2)$$

shows explicitly the dependence of action $S$ on $x_1$, $x_2$. The boundary conditions for new fields $\bar{c}_l$ are assumed to be zero. The differentiation of action $S$ in $x_0$, $t_0$ produces cumbersome terms due to the interaction part $S_u$ of the action. These terms are of the form

$$\frac{u}{2} \int d\tau_1 d\tau'_1 c'_1(\tau'_1) D'(c_1(\tau'_1) - c_1(\tau_1)) c'_1(\tau_1) \cdot \frac{\tau_1 - \tau'_1}{T_1},$$

where $D'$ is a derivative of correlator $D$ on its argument (21).

Due to $D$ correlator is an even function, it's enough to calculate in Exp. (41) only the term corresponding to the $\tau_1$ contribution. Substitution of particular solution $c'_1$ (26) yields

$$-\frac{u}{\nu^2 T_1} \int d\tau_1 \frac{\partial c_1(\tau'_1)}{\partial \tau'_1} \int d\tau_1 \frac{\partial D(c_1(\tau'_1) - c_1(\tau_1))}{\partial \tau_1}.$$

The inner integral can be calculated by parts. Then the integral term is proportional to $\int d\tau'_1 [D_{11} c'_1(\tau'_1)]$ and surface terms can be calculated trivially. The result turns out to be calculated with the help of the following identities based on the stationarity equations

$$\int d\zeta \frac{\delta S}{\delta c'_l(\zeta)} = 0 \Rightarrow \int d\zeta u[D_u c'_l(\zeta)] + u[D_{12}(c'_1 + c'_2 - c'_l)](\zeta) =$$

$$= \int d\zeta (-2\nu c'_l(\zeta) + i\partial_\zeta c_l) = -i\nu x^{(l)}, \quad l = 1, 2.$$

The last equality is written using the eqns. (26).

In this way the stationarity equation discussed has the form simplified by $F = 0$ condition:

$$\frac{\delta S}{\delta t_0} = 0 \Rightarrow \left[ x^{(2)}(T_2) - x^{(1)}(T_1) \right] \int_{-x^{(1)}}^{x^{(2)}} D(z) dz = 0,$$

$$\frac{\delta S}{\delta x} = 0 \Rightarrow \frac{i\nu^2}{u} = \int_0^x D(z) dz.$$
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Besides we have the boundary conditions (28) imposed by our choice of particular solution \( F = 0 \). Let’s note that the non-trivial instanton equation with respect to \( x_0 \) variable turn out to be an identity at \( F = 0 \). Indeed the substitution of (26) into the action \( S (12) \) yields \( S (x) \) as the function in the unique \( x \equiv x_2 - x_1 \) space variable. But the boundary conditions (28) give an opportunity to determine stationary value of \( x_0 \).

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