EISENSTEIN COCYCLES FOR GL(n) AND VALUES OF L-FUNCTIONS IN IMAGINARY QUADRATIC EXTENSIONS

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Abstract. In this paper we generalize R. Sczech’s Eisenstein cocycle for GL(n) over totally real extensions of Q to extensions of an imaginary quadratic field. Using this, we parametrize values of certain Hecke L-functions considered by P. Colmez, giving a cohomological interpretation of his algebraicity result on special values of the L-functions.

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1. Introduction

1.1. Motivation. Let F be an imaginary quadratic field, and K a degree n extension of F. Consider the ideal character defined on principal integral ideals (a) in K,

$$\Phi_k((a)) = \varphi(a)(\psi^k \circ N_{K/F}(a)),$$

where \(\varphi\) is a Dirichlet character of K and \(\psi\) a Hecke character of F with infinity type \(\overline{z}\), with \(k \geq 1\). Then P. Colmez \([\text{Col89, Théorème 5}]\) showed that for an integer \(t\), the associated Hecke L-function

$$L(t, \Phi_k) = \Gamma(t)^n \sum_a \Phi_k(a)N_{F/Q}(a)^{-t} \in \Omega_{\infty}^n R^n(t-k)[Q],$$

in the cases where (i) \(n = 2\) with \(1 \leq t \leq k\), and (ii) \(n \geq 3\) with \(t = k\), and \(\text{Gal}(K/K)\) acts on the embeddings of F into \(\mathbb{C}\) by either the permutation group \(S_n\) or \(A_n\). Here \(\Omega_{\infty}\) is the real period of an elliptic curve over \(\mathbb{Q}\) with complex multiplication by F. This algebraicity result can be viewed as a special case of more general conjectures about the arithmetic nature of special values of L-functions, for example by P. Deligne, A. Beilinson, S. Bloch, and K. Kato (see for example \([\text{Sch90}]\) and the references therein).
Based on these conjectures, one expects moreover that these special values are related to the cohomology of some algebraic variety. In this paper, we explicitly construct a group cocycle $\Psi_s$ representing a class in the group cohomology of $GL_n(F)$ with coefficients in a certain space of functions; then paired with an explicit cycle, we show that the cocycle parametrizes certain values of $L(s, \Phi_k)$. Indeed, in the cases where these values coincide with (i) and (ii), this can be viewed as a cohomological interpretation of Colmez’s algebraicity result.

We refer to this cocycle as the Eisenstein cocycle, for the reason that our construction follows closely that of R. Sczech for $L$-functions of totally real fields [Scz92, Scz93], which gives a different proof of the Klingen-Siegel theorem on the rationality of certain partial zeta values. Sczech names this cocycle after Eisenstein, as it generalizes the classical Dedekind sum in a way that involves a technique known as Eisenstein summation (see, for example, [Wei76, p.9]). Indeed, the special values obtained are shown to be related to periods of Eisenstein series.

Note, however, that in more recent work of others building on Sczech, [CD14, DS16], this cocycle is renamed the Sczech cocycle. In [CD14], the authors refine Sczech’s cocycle to an $l$-smoothed cocycle, and as a result prove certain integrality properties of values of a smoothed partial zeta function, and interpolate these values to a $p$-adic zeta function, previously considered by Deligne and Ribet [DR80], Cassou-Nogus [CN79], and Barsky [Bar78].

1.2. Main result. The setting that we work in is the precise analogue of the totally real extensions of degree $n$ considered by Sczech in the sense that the rank of the unit group of $K$ is again $n - 1$, and these units of infinite order play a key role in the parametrization of $L$-values. Our main result can be summarized as follows, and stated as Corollary 4.8 below:

**Theorem 1.1.** Let $K$ be a degree $n$ extension of an imaginary quadratic field $F$. Consider the Hecke character $\chi((a)) = \varphi(a)^{N_{K/F}(a)^k}N_{K/F}(a)^{-l}$ with conductor $\mathfrak{f}$, where $k \geq 0$ and $l > 0$ are integers. Then for a fixed $u \in K$, $M \in GL_n(K)$ whose columns are conjugate over $F$, there exists a cocycle $\Psi_s$ and cycle $\mathcal{E}$ such that

$$L(s, \chi) = \sum_{\mathfrak{b}} \frac{\chi(b)}{N_{K/F}(b)^s} \sum_{(r) \in I(\mathfrak{f})} \varphi(r)\Psi_s(\mathcal{E}[\mathfrak{b}, M])(P^{l-1}, u, M)$$

for $\text{Re}(s) > 1 + \frac{k}{2}$. The outer sum is taken over integral ideals $\mathfrak{b}$ prime to $\mathfrak{f}$, and the inner sum is taken over principal ideals prime to $\mathfrak{f}$ with representatives in $\mathfrak{b}^{-1}$. Moreover, for $n = 2$ the identity holds for all $s$.

Initially, $L(s, \chi)$ converges absolutely for $\text{Re}(s) > 1 + \frac{k-l}{2}$, and by Hecke one knows that it has analytic continuation to the entire complex plane. We expect that the region of convergence in the theorem above can be extended further, and in particular to include the point $s = 0$, which is important for applications.

Sczech’s work over totally real fields does not generalize immediately to our setting due to the following problems: One, the convergence of the cocycle is no longer obtained using the so-called $Q$-summation trick of Sczech, but instead we introduce a convergence factor inspired by Colmez’s work. This greatly simplifies our exposition as a major part of [Scz93] is devoted to this $Q$-summation. Two, for extensions of an imaginary quadratic field one can no longer choose a system of
totally positive units, a key condition in the parametrization in [Scz93, Lemma 6]. We prove the analogue of this in Lemma 4.6 below, which represents the technical heart of this paper. As discussed in [Scz93, p.597], a similar parametrization is obtained in [Col89, Chapitre III] but is ‘rather complicated’ as it does not use the cocycle property. In fact, our proof holds for any number field; for this reason it will be interesting to ask how Sczech’s construction might apply to general number fields.

Finally, we note that the case where $K$ is a quadratic extension of $F$ was first considered in the unpublished thesis of R. Obaisi [Oba00], whose work we summarize in Section 2 below. Most importantly, in this case $\Psi_s$ parametrizes all values of $L(s, \chi)$, whereas for general $n$ we are restricted to the region of convergence above.

This is in contrast to the result of Sczech for totally real fields, where Eisenstein cocycles parametrize the partial zeta functions only at integer arguments, whereas our construction holds for complex arguments. This added flexibility is due to the introduction of the convergence factor mentioned above.

This paper is organized as follows: Section 2 describes the case $n = 2$, which will set the stage for the general case. Section 3 constructs the Eisenstein cocycle and determines a region of absolute convergence for the cocycle. In Section 4, we introduce the $L$-functions of interest, and construct the cycle on which the cocycle is to be evaluated. Section 5 finally proves the key Lemma 4.6 required for the parametrization.

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2. The case $n = 2$

In this section we describe the case $n = 2$, in preparation for the general case. We recall that this case was studied in the unpublished thesis of Obaisi [Oba00], and note that her result is stronger than our case because the Eisenstein cocycle is shown to have analytic continuation to all of $\mathbb{C}$, due to prior work of Colmez. We caution the reader that the cocycle described here is an inhomogeneous cocycle, whereas in the general case it is more convenient to work with a homogeneous cocycle, as does [Scz93].

2.1. The rational cocycle. Let $\sigma_1, \sigma_2$ be two nonzero column vectors in $\mathbb{C}^2$ and $\sigma = (\sigma_1, \sigma_2)$, and $x$ a row vector in $\mathbb{C}^2$. Now consider the function,

\begin{equation}
(2.1) \quad f(\sigma_1, \sigma_2)(x) = \frac{\det(\sigma_1, \sigma_2)}{\langle x, \sigma_1 \rangle \langle x, \sigma_2 \rangle}
\end{equation}

defined outside the hyperplanes $\langle x, \sigma_1 \rangle = 0$ and $\langle x, \sigma_2 \rangle = 0$. For any $A$ in $GL_2(\mathbb{C})$, we see that $f(A\sigma)(x) = \det(A)f(\sigma)(xA).$

Let $H \subset \mathbb{C}[x_1, x_2]$ be the space of homogeneous polynomials of degree $g$. Then for any $P$ in $H$, we extend $f$ to $H \times \mathbb{C}^2 \setminus \{0\}$ as follows:

\begin{equation}
(2.2) \quad f(\sigma)(P, x) = P(-\partial_{x_1}, -\partial_{x_2})f(\sigma)(x),
\end{equation}

where $\partial_{x_i}$ denotes the partial derivative with respect to $x_i$. It satisfies the property

\begin{equation}
(2.3) \quad Af(\sigma)(P, x) = \det(A)f(\sigma)(A^TP, xA)
\end{equation}

where the action of $A \in GL_n(\mathbb{C})$ on $P$ is defined to be $AP(x) = P(xA).$
Now define a left 1-cocycle $\psi$ with values in the space of complex-valued functions on $H \times \mathbb{C}^2 \setminus \{0\}$ by
\begin{equation}
(2.4) \quad \psi(A)(P, x) = f(e_1, Ae_1)(P, x)
\end{equation}
for $x$ nonzero. Note that this definition holds only if the denominators $(x, \sigma_1), (x, \sigma_2)$ are nonzero, otherwise one introduces an alternative definition so that one obtains a well-defined expression. See [Scz92, p.371] for the case of $GL_2(\mathbb{Q})$ (see [Oba00, p.16]).

2.2. The Eisenstein cocycle. Fix a quadratic extension $K$ of $F$. We want to sum the 1-cocyle $\psi$ to produce the desired 1-cocycle $\Psi_s$. Let $\Lambda_1, \Lambda_2$ be lattices in $F$ with the same multiplier ring $\mathcal{O} \neq \mathbb{Z}$. Let $R_2$ be the set of matrices $M$ in $GL_2(K)$ whose columns are conjugate over $F$. Let $S'$ be the space of complex-valued function $f$ on $R_2 \times F/(\Lambda_1 \times \Lambda_2) \times H$, and define the action of $GL_2(F)$ on $S'$ by
\begin{equation}
(2.5) \quad A f(P, u, M) = \det(A) f(A^T P, u A, A^{-1} M).
\end{equation}
Define the Eisenstein cocycle
\begin{equation}
(2.6) \quad \Psi_s(A)(P, u, M) = \sum_{x \in \Lambda + u} \psi(A)(P, x) \Omega^k_s(x, M)
\end{equation}
where $u \in F^2$, $\Lambda = \Lambda_1 \times \Lambda_2$, and
\begin{equation}
(2.7) \quad \Omega^k_s(x, M) = \prod_{i=1}^2 \frac{x M_i^k}{|x M_i|^{2s}}
\end{equation}
which, by a result of Colmez [Col89], converges absolutely for $Re(s) \gg k$, and in fact has analytic continuation as a function of $s$. It follows that $\Psi_s$ is a 1-cocycle for all $s$, thus represents a cohomology class in $H^1(GL_2(F), S')$.

2.3. Parametrizing the $L$-function. Using this, one can relate the Eisenstein cocyle to the values of certain Hecke $L$-functions associated to quadratic extensions $K$ of $F$.

Let $\mathfrak{f}$ be an integral ideal of $K$. Consider a Hecke character $\chi$ on fractional ideals prime to $\mathfrak{f}$ satisfying
\begin{equation}
\chi((a)) = \varphi(a) \lambda(a)
\end{equation}
where $\varphi$ is a residue class character on $(\mathcal{O}_K/\mathfrak{f})^\times$, and $\lambda = \frac{N_{K/F}(a)^k}{\mathcal{O}_K^k} N_{K/F}(a)^{-l}$, with $k, l$ positive integers such that $\lambda(\epsilon) \equiv 1$ for all $\epsilon \in \mathcal{O}_K^\times$ such that $\epsilon \equiv 1 \mod \mathfrak{f}$.

Define the Hecke $L$-function
\begin{equation}
L(s, \chi) = \sum_a \chi(a) N_{K/Q}(a)^{-s}
\end{equation}
where the sum is taken over integral ideals $a$ prime to $\mathfrak{f}$. It converges absolutely for $Re(s) > 1 + \frac{d-1}{2}$. Also define the partial Hecke $L$-function by
\begin{equation}
L(b, r, s) = \sum_a \lambda(a) N_{K/Q}(a)^{-s}
\end{equation}
where the sum is now taken over representatives $a$ in $\mathfrak{f} b^{-1} + r$ where $b$ is an integral ideal in $K$ prime to $\mathfrak{f}$, and $r$ belonging to $b^{-1}$.

Then we may rewrite
\begin{equation}
(2.9) \quad L(s, \chi) = \sum_b \chi(b) N_{K/Q}(b)^{-s} \sum_{(r) \in I(\mathfrak{f}), r \in b^{-1}} \varphi(r) L(b, r, s)
\end{equation}
where \( b \) runs over integral ideals in \( K \) prime to \( f \) and \( I(f) \) denotes the integral ideal prime to \( f \).

Next, let \( V \) be a subgroup of finite index in \( U_f \), the group of units in \( K \) congruent to 1 mod \( f \). In particular, we choose \( V \) to be cyclic with generator \( \epsilon \) with relative norm 1. Then choosing \( A = M \text{diag}(\epsilon, \epsilon')M^{-1} \), \( \epsilon' \) being the conjugate of \( \epsilon \) in \( F \), and \( P(x) = ((xM_1^{-T})(xM_2^{-T}))^{k-1} \), the Eisenstein cocycle can be expressed as

\[
\Psi_s(A)(P, u, M) = -[U_f : V][(l - 1)!^2 \det(M)\mathcal{L}(b, r, s)]
\]

for all \( s \in \mathbb{C} \).

Thus, putting the two equations together we conclude that values of \( \mathcal{L}(b, r, s) \), and hence \( L(s, \chi) \), are parametrized by the cocycle \( \Psi_s \) for all values of \( s \).

### 2.4. Algebraicity

As a corollary, certain special values of the \( L \)-function

\[
L(n, \chi), \quad 1 \leq n \leq k,
\]

for a given \( \chi \) with positive integers \( k, l \), are shown to be algebraic up to a transcendental factor, as in \((1.1)\). In particular, the special values are expressed as finite linear combinations of the elliptic function

\[
E^l_k(u, \Lambda, s) = \sum_{w \in \Lambda + u}^{'} \frac{\tilde{w}^k}{w^l |w|^{2s}}
\]

that are known to be algebraic by Damerell [Dam70] (see also [Wei76, Ch.XIII]).

In [Scz93], the rationality of special values of the partial zeta function is obtained by expressing the Eisenstein cocycle in terms of a generalization of Dedekind sums, which are shown to be rational up to a transcendental factor, thus recovering the classical rational result of Klingen and Siegel. The Eisenstein cocycle \( \Psi_s \) constructed in our case is related instead to the elliptic Dedekind sums, also defined by Sczech, expressed in terms of the cotangent functions appearing in classical Dedekind sums.

### 3. The Eisenstein cocycle \( \Psi_s \)

Having sketched the case \( n = 2 \), we are now ready for the general setting. Throughout we will let \( F \) be a fixed imaginary quadratic extension of \( \mathbb{Q} \).

#### 3.1. The rational cocycle

We first construct an \((n - 1)\)-cocycle of \( GL_n(\mathbb{C}) \) acting on a certain space of functions defined below, following the construction of Sczech for \( GL_n(\mathbb{Q}) \), which will allow us to construct the Eisenstein cocycle on \( GL_n(F) \).

**Definition 3.1.** Let \( \sigma_1, \ldots, \sigma_n \) be the columns of a \( n \times n \) matrix \( \sigma \) in \( GL_n(\mathbb{C}) \). Similar to \( n = 2 \) case, we start with the rational function

\[
f(\sigma)(x) = \frac{\det(\sigma_1, \ldots, \sigma_n)}{\langle x, \sigma_1 \rangle \ldots \langle x, \sigma_n \rangle}
\]

of a row vector \( x \) in \( \mathbb{C}^n \), and extend \( f \) to \( H \times \mathbb{C}^n \setminus \{0\} \)

\[
f(\sigma)(P, x) = P(-\partial_x_1, \ldots, -\partial_x_n)f(\sigma)(x)
\]

where \( H \) is the space of homogeneous polynomials \( P \) of degree \( g \geq 1 \) in \( n \) variables over \( \mathbb{C} \), and where the \( \partial_x_j \) are the partial derivatives with respect to the variable \( x_j \). This function is well-defined outside the hyperplanes \( \langle x, \sigma_j \rangle = 0 \), for all \( j = 1, \ldots, n \). When \( \langle x, \sigma_j \rangle = 0 \) we set \( f(\sigma)(x) = 0 \). Notice that \( f(\sigma) \) does not change if \( \sigma_j \) is
replaced by $\lambda \sigma_j$ for any $\lambda \in \mathbb{C}^\times$, so we may view $\sigma_j$ as points in projective space $\mathbb{P}^{n-1}(\mathbb{C})$.

We define the action of $A \in GL_n(\mathbb{C})$ on a polynomial $P(x) \in H$ by

$$AP(x) = P(xA), \quad x = (x_1, \ldots, x_n).$$

The following lemma gives some properties of $f$:

**Lemma 3.2.** The function $f(\sigma)(P, x)$ satisfies the following properties:

1. Let $\sigma_0, \ldots, \sigma_n$ be nonzero column vectors in $\mathbb{C}^n$. Then

$$\sum_{i=0}^{n} (-1)^i f(\sigma_0, \ldots, \hat{\sigma}_i, \ldots, \sigma_n) = 0.$$

2. Given $A \in GL_n(\mathbb{C})$, we have

$$Af(\sigma)(P, x) = f(A\sigma)(P, x),$$

where the action is given by $Af(\sigma)(P, x) = \det(A)f(\sigma)(A^T P, xA)$.

**Proof.** These properties follow the same argument as in [Scz93, pp.586-587], observing that the argument is independent of the coefficient field. For (2), we use the fact that $f(\sigma)(x) = \det(\sigma)\sum_{r} P_r(\sigma) \prod_{j=1}^{n} \frac{r_j!}{(x, \sigma_k)^{r_j+1+r_j}}$

which follows directly from the definition of $f$. \qed

**Remark 3.3.** As a corollary to property (2), our function can be expressed as

$$f(\sigma)(P, x) = \det(\sigma) \sum_{r} P_r(\sigma) \prod_{j=1}^{n} y_{ij}^{r_j},$$

where $r$ runs over all partitions of $\deg(P) = r_1 + \cdots + r_n$ into nonnegative parts $r_j \geq 0$, and $P_r(\sigma)$ is a homogeneous polynomial in $\sigma_{ij}$ defined by the series expansion

$$P(\sigma^T) = \sum_{r} P_r(\sigma) \prod_{j=1}^{n} y_{ij}^{r_j}.$$ 

with $\sigma^T$ denoting the transpose of the matrix $\sigma$. In the case of a divided power $P(x) = x_1^{(g_1)} \cdots x_n^{(g_n)}$ with $x^{(k)} = x^k/k!$, we have

$$P_r(\sigma) = \sum_{r_j, i, j=1}^{n} \sigma_{ij}^{r_j},$$

where the sum runs over simultaneous decompositions $r_j = r_{i1} + \cdots + r_{nj}$, $r_{ij} \geq 0$, for $j = 1, \ldots, n$ satisfying $r_{i1} + \cdots + r_{in} = g_i$.

**Definition 3.4.** (The rational cocycle.) Next define an $n$-tuple $(A_1, \ldots, A_n)$ with $A_i \in GL_n(\mathbb{C})$, and write $A_{i,j}$ for the $j$-th column of $A_i$. Given any $k = 1, \ldots, n$ and nonzero vector $x \in \mathbb{C}^n$, there is a smallest index $j_k$ such that $\langle x, A_{k,j_k} \rangle \neq 0$.

Consider the space of complex valued functions

$$S_0 = \{ f : H \times \mathbb{C}^n \to \mathbb{C} \}.$$ 

An element $A \in GL_n(\mathbb{C})$ acts on $S_0$ as before by

$$Af(\sigma)(P, x) = \det(A)f(\sigma)(A^T P, xA),$$
and \( AP(X) = P(XA) \) for \( X = (X_1, \ldots, X_n) \). Now, define the rational cocycle by a map on \( n \) copies

\[
\psi : GL_n(\mathbb{C}) \times \cdots \times GL_n(\mathbb{C}) \to S_0
\]

sending

\[
(A_1, \ldots, A_n) \mapsto f(A_{1j_1}, \ldots, A_{nj_n})(P, x)
\]

which we will denote for short by \( \psi(A_1, \ldots, A_n)(P, x) = \psi(\mathfrak{A})(P, x) \), where \( \mathfrak{A} \) stands for the \( n \)-tuple \( (A_1, \ldots, A_n) \). If \( x = 0 \), we set \( \psi(A_1, \ldots, A_n)(P, x) = 0 \).

By Lemma 3.2, \( \psi \) is a homogenous \((n-1)\)-cocycle on \( GL_n(\mathbb{C}) \), i.e.,

\[
(3.10) \quad \psi(AA') = A \psi(A),
\]

and

\[
(3.11) \quad \sum_{i=0}^{n} (-1)^i \psi(A_0, \ldots, \hat{A}_i, \ldots, A_n) = 0.
\]

This \( \psi \) is the analogue of the rational cocycle constructed by R. Sczech for \( GL_n(\mathbb{Q}) \). We will use this cocycle to construct the Eisenstein cocycle for an imaginary quadratic field.

**Theorem 3.5.** The map \( \psi \) represents a cohomology class in \( H^{n-1}(GL_n(\mathbb{C}), S_0) \).

**Proof.** This follows immediately from the properties of \( \psi \) described in Lemma 3.2. \( \square \)

**Remark 3.6.** (Alternate expression for \( \psi \).) We may rewrite \( \psi \) as follows: let \( d = (d_1, \ldots, d_n) \) be an \( n \)-tuple of integers with \( 1 \leq d_j \leq n \), and consider the space \( A(d) \subset \mathbb{C}^n \) generated by all columns \( A_{ij} \) with \( j < d_i \), and let \( A(d) \perp \) be the orthogonal complement. Then define

\[
(3.12) \quad X(d) = A(d)\perp \cup_{i=1}^{n} A_{id_i}^\perp,
\]

so that if \( X(d) \) is nonempty, then it is a subspace of \( \mathbb{C}^n \) with a finite number of codimension one subspaces removed. Let

\[
(3.13) \quad D = D(A_1, \ldots, A_n) = \{d : X(d) \neq \emptyset\}.
\]

Then by construction of \( X(d) \), we can associate with \( (A_1, \ldots, A_n) \) a finite decomposition of \( \mathbb{C}^n \) into a disjoint union

\[
\mathbb{C}^n \{0\} = \bigcup_{d \in D} X(d).
\]

In terms of this decomposition, the definition of \( \psi \) can be restated as

\[
(3.14) \quad \psi(A_1, \ldots, A_n)(P, x) = f(A_{1d_1}, \ldots, A_{nd_n})(P, x)
\]

for \( x \) in \( X(d) \). The cardinality of \( D \), i.e., the number of nonempty sets \( X(d) \) depends in general on the matrices \( A_i \), but we simply observe that the cardinality of \( D \) is bounded above by the generic case, that is, when

\[
\dim A(d) = \sum_{i} (d_i - 1)
\]

for all nonempty \( X(d) \). In this case every \( d \in D \) satisfies \( \sum d_i > 2n \), hence \( D \) is finite.
3.2. The Eisenstein cocycle. We now use $\psi$ to construct the cocycle $\Psi_s$ that will parametrize the values of Hecke $L$-functions.

**Definition 3.7.** (The Eisenstein cocycle.) Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ be a product of $n$ lattices in $\mathbb{C}$ with the same ring of multipliers $O_F$ in $F$. Consider the space $S$ of functions

$$S = \{ f : H \times F^n / \Lambda \times R_n \rightarrow \mathbb{C} \}$$

where $R_n$ is the set of matrices $M$ in $GL_n(\mathbb{C})$ such that there exists a degree $n$ extension $K/F$ such that the columns $M_1, \ldots, M_n$ are conjugate over $F$. That is,

$$M = \begin{pmatrix} \rho_1(m_1) & \cdots & \rho_n(m_1) \\ \vdots & \ddots & \vdots \\ \rho_1(m_n) & \cdots & \rho_n(m_n) \end{pmatrix}$$

where $m_1, \ldots, m_n \in K$ and the $\rho_i$ are distinct embeddings of $K$ into $\mathbb{C}$, fixing $F$. In particular, all the coefficients of $M$ are nonzero. The action of $A$ in $GL_n(F)$ on any $f$ in $S$ is given by

$$Af(P, u, M) = \det(A) f(A^TP, uA, A^{-1}M).$$

Now define the *Eisenstein cocycle* to be a map $\Psi_s$ from $n$ copies of $GL_n(F)$ to the space of complex valued functions on $S$,

$$\Psi_s : GL_n(F) \times \cdots \times GL_n(F) \rightarrow S$$

by

$$\Psi_s(\mathfrak{A})(P, u, M) = \sum_{x \in \Lambda + u} \psi(\mathfrak{A})(P, x) \Omega_s^k(x, M),$$

where $u \in F^n$ and

$$\Omega_s^k(x, M) = \prod_{i=1}^n \frac{xM_i^i}{|xM_i|^2s}.$$

The factor (3.19) will be referred to as the *convergence factor* and $k \in \mathbb{Z}$. This map is well-defined for $s$ large enough, and we shall determine a range of absolute convergence after the following theorem (cf. Theorem 3.12).

**Theorem 3.8.** The Eisenstein cocycle $\Psi_s$ represents a nontrivial cohomology class in $H^{n-1}(GL_n(F), S)$, which we denote by $[\Psi_s]$.

**Proof.** Let $A_0, \ldots, A_n$ be elements in $GL_n(F)$. Since $\psi$ is a $(n-1)$-cocycle, we know that

$$\sum_{i=0}^n (-1)^i \psi(A_0, \ldots, \hat{A}_i, \ldots, A_n)(P, x) = 0$$

for a fixed $P$ and $x$. Multiplying by the convergence factor $\Omega_s^k(x, M)$ and summing over $x$ in $\Lambda + u$ we have

$$\sum_{x \in \Lambda + u} \sum_{i=0}^n (-1)^i \psi(A_0, \ldots, \hat{A}_i, \ldots, A_n)(P, x) \Omega_s^k(x, M) = 0,$$
where the sum converges for $\text{Re}(s)$ large enough. Then interchanging the sums and using the definition of $\Psi_s$ we have

\begin{equation}
(3.22) \sum_{i=0}^{n} (-1)^i \Psi_s(A_0, \ldots, \hat{A}_i, \ldots, A_n)(P, x) = 0,
\end{equation}

therefore $\Psi_s$ is a cohomology class in $H^{n-1}(GL_n(F), S)$.

To see that $\Psi_s$ is nontrivial, we use the property that $\Psi_s$ parametrizes values of the $L$-function and the cap product as follows. Fix $f, b, P, u$ and $M$ as in Theorem 4.7. Then pairing $\Psi_s(\cdot)(P, u, M) \in H^{n-1}(GL_n(F), \mathbb{C})$ with the cycle $\mathcal{E} = [b, M] \in H_{n-1}(GL_n(F), \mathbb{Z})$ (cf. (4.20)) via the nondegenerate bilinear pairing

\begin{equation}
(\cdot, \cdot) : H^{n-1}(GL_n(F), \mathbb{C}) \times H_{n-1}(GL_n(F), \mathbb{Z}) \rightarrow H_0(GL_n(F), \mathbb{C}) = \mathbb{C}
\end{equation}

we have, after Theorem 4.7, that

\begin{equation}
(3.23) \langle [\Psi_s(\cdot)(P, u, M)], [\mathcal{E}] \rangle = \Psi_s(\mathcal{E})(P, u, M)
\end{equation}

parametrizes special values of the $L$-function $L(b, r, s)$. In particular, since $L(b, r, s)$ is non-trivial, then $\langle [\Psi_s(\cdot)(P, u, M)], [\mathcal{E}] \rangle \neq 0$ and so the cocycle $\Psi_s$ is non-trivial as well. 

### 3.3. Convergence

We conclude this section by determining a domain of absolute convergence for the cocycle $\Psi_s$, with the aid of the convergence factor. First, we make the following observation:

**Lemma 3.9.** Let $x$ be a row vector in $\mathbb{C}^n$, $M \in R_n$ and $\sigma \in GL_n(O_F)$. Then

\begin{equation}
(3.24) \Omega_\sigma^k(x, \sigma M) = \Omega_\sigma^k(x, \sigma M).
\end{equation}

**Proof.** Observe that since the entries of $\sigma$ are in $O_F$, then $\sigma M$ is also in $R_n$. Now, from the associativity of the product of matrices we have $x(\sigma M) = (x\sigma)M$. Therefore by comparing the $i$-th entry on each side we get $\langle x\sigma, M_i \rangle = \langle x, (\sigma M)_i \rangle$. The lemma follows. 

**Remark 3.10.** Using the partition $X(d)$ in (3.12), we may rearrange the sum (3.18) as in [Scz93, p.598] to obtain the expression

\begin{equation}
(3.25) \Psi_s(\mathcal{A}) = \sum_{d \in D} \det(\sigma) \sum_{r} P_r(\sigma) G_r(\sigma),
\end{equation}

where

\begin{equation}
(3.26) G_r(\sigma) = \sum_{x \in X(d)} \Omega_\sigma^k(x, M) \prod_{j=1}^{n} \frac{r_j!}{\langle x, \sigma_j \rangle^{1+r_j}}.
\end{equation}

The following result of Colmez, from Collaraire 1 of [Col89, p.196], plays a key role in the proof of absolute convergence. We will consider $\mathbb{C}^n$ endowed with the sup norm, i.e.,

\[ ||w|| = \max_{i=1,\ldots,n} \{ |w_i| \} \]

for any $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$. 


Lemma 3.11 (Colmez). Let $\Lambda$ be a lattice in $\mathbb{C}^n$. Assume that for all $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that if $w \in \Lambda$ and $\prod_{i=1}^{n} |w_i| \leq C(\epsilon) ||w||^{-\epsilon}$ then $w = 0$. Then the series
\begin{equation}
\sum_{w \in \Lambda \atop w \neq 0} \prod_{i=1}^{n} |w_i|^{-2\gamma} ||w||^{-\mu}
\end{equation}
converges if $\mu > 0$ and $\gamma > 1$.

Theorem 3.12. Fix $\mathfrak{A}, P, \mathbf{u}, \mathbf{M}$. Then $\Psi_s$ converges absolutely for $\text{Re}(s) > 1 + \frac{k}{2}$.

Proof. From the decomposition $\mathbb{C}^n = \bigcup_{d \in D} X(d)$, we can partition the lattice $\Lambda + \mathbf{u}$ as $\bigcup_{d \in D} \Lambda_d$, where $\Lambda_d = (\Lambda + \mathbf{u}) \cap X(d)$. Thus
\begin{equation}
\Psi_s(\mathfrak{A})(P, \mathbf{u}, \mathbf{M}) = \sum_{d \in D} \sum_{x \in \Lambda_d} \psi(\mathfrak{A})(P, x) \Omega^k_s(x, M).
\end{equation}
Since $D$ is finite, it suffices to show that for a fixed $d \in D$ the sum
\begin{equation}
\sum_{x \in \Lambda_d} \psi(\mathfrak{A})(P, x) \Omega^k_s(x, M)
\end{equation}
is convergent for $\text{Re}(s) > 1 + \frac{k}{2}$.

Observe that for $x \in \Lambda_d$, $\psi(\mathfrak{A})(P, x)$ can be expressed as $f(A_{d_1}, \ldots, A_{d_n})(P, x)$. Let us denote the matrix $(A_{d_1}, \ldots, A_{d_n})$ by $\sigma = (\sigma_1, \ldots, \sigma_n)$. From (3.5) we have that
\begin{equation}
\sum_{x \in \Lambda_d} \psi(\mathfrak{A})(P, x) \Omega^k_s(x, M) = \sum_{\sigma} \det(\sigma) P_r(\sigma) G_r(\sigma),
\end{equation}
where
\begin{equation}
G_r(\sigma) = \sum_{x \in \Lambda_d} \Omega^k_s(x, M) \prod_{j=1}^{n} \frac{r_j!}{(x, \sigma_j)^{1+r_j}}.
\end{equation}
We may assume that $\det(\sigma) \neq 0$, otherwise the above sum is zero. Also, by a change of variables $w = xM$, and letting $M' = M^{-1}\sigma$ and $\Lambda'_d = \Lambda_d M$, we obtain the simplified expression
\begin{equation}
G_r(\sigma) = \sum_{w \in \Lambda'_d} \prod_{j=1}^{n} \frac{r_j!}{|wM'_j|^{1+r_j} |w_j|^{2\sigma_j}}.
\end{equation}

Then it suffices to bound $G_r(\sigma)$. We have trivially by the triangle inequality
\begin{equation}
|G_r(\sigma)| \leq \sum_{w \in \Lambda'_d} \prod_{j=1}^{n} \frac{r_j!}{|wM'_j|^{1+r_j} |w_j|^{2\text{Re}(\sigma)-k}}.
\end{equation}
By Lemma 3.13 below, the constant
\begin{equation}
C = \max_{w \in \Lambda'_d} \left\{ |wM'_j| \prod_{j=1}^{n} \frac{1}{|wM'_j|^{1+r_j}} \right\} = \max_{w \in \Lambda'_d} \left\{ ||wM'|| \prod_{j=1}^{n} \frac{1}{|wM'_j|^{1+r_j}} \right\}
\end{equation}
is finite. Then
\[
|G_r(\sigma)| \leq C \sum_{w \in \Lambda} \prod_{j=1}^{n} \frac{r_j!}{|w_j|^{2\text{Re}(s)-k} ||w||}
\]
\[
\leq ||M'||C \sum_{w \in \Lambda} \prod_{j=1}^{n} \frac{r_j!}{|w_j|^{2\gamma} ||w||}
\]
(3.35)
where \( \gamma = \text{Re}(s) - \frac{k}{2} \), and \( ||M'|| \) is a constant such that \( ||w|| \leq ||M'|| ||w'|| \) for all \( w \).

Then by Lemma 3.11 we see that (3.35) converges absolutely for \( \gamma > 1 \), which shows that \( G_r(\sigma) \) is bounded. \( \square \)

To conclude the proof, we obtain the constant \( C \) as required above.

**Lemma 3.13.** Let \( \Lambda \) be a lattice in \( \mathbb{C}^n \) and real numbers \( t_i \geq 0, \ i = 1, \ldots, n \). Let \( \Lambda' := \{ w \in \Lambda : \prod_{i=1}^{n} w_i \neq 0 \} \). Then the supremum
\[
\sup_{w \in \Lambda} \left( \prod_{i=1}^{n} |w_i|^{-t_i} \right)
\]
is finite.

**Proof.** Since \( \Lambda \) is a lattice there exists a constant \( C > 0 \) such that \( C < |w_i| \) for all \( w \in \Lambda' \). Then
\[
C^{t_1 + \cdots + t_n} < \prod_{i=1}^{n} |w_i|^{t_i}
\]
(3.37)
for all \( w \in \Lambda' \) and the result follows. \( \square \)

4. Values of \( L \)-functions

In this section, we introduce the Hecke \( L \)-functions whose special values we will parametrize using the Eisenstein cocycle. In particular, we will construct a cycle \( \mathcal{E}[b, M] \) on which the Eisenstein cocycle will be evaluated. The value of the cocycle on \( \mathcal{E}[b, M] \) will then be related to the \( L \)-function. As the proof of this relation is rather technical, for the sake of exposition we delay it to the Section 5. Indeed, assuming this relation it is straightforward to parametrize the \( L \)-function, which we do in this section.

4.1. An expression for the Hecke \( L \)-function. For the rest of this paper, we fix a degree \( n \) extension \( K \) of an imaginary quadratic extension \( F \).

**Definition 4.1.** (The Hecke \( L \)-function \( L(s, \chi) \)) Let \( \mathfrak{f} \) be an integral ideal of \( K \) and \( I(\mathfrak{f}) \) be the group of fractional ideals relatively prime to \( \mathfrak{f} \). Let \( \chi : I(\mathfrak{f}) \to \mathbb{C}^\times \) be an ideal character such that
\[
\chi((a)) = \varphi(a)\lambda(a),
\]

1This constant \( ||M'|| \) exists since the matrix \( M' \) is invertible, so as an operator its inverse is bounded.

2The hypothesis of Lemma 3.11 is satisfied for \( \Lambda_d \) for the following reason. For a fix \( 1 \leq i \leq n \), let \( t_i = 1 + \epsilon \) and \( t_j = 1 \) for \( 1 \leq j \leq n, j \neq i \). Then, by Lemma 3.13, there exists a constant \( C_1(\epsilon) > 0 \) such that \( C_1(\epsilon) < \prod_{j=1}^{n} |w_j|^{t_j} \) for every \( w \in \Lambda_d \). Let \( C(\epsilon) = \min_{1 \leq i \leq n} C_i(\epsilon) \), then \( C(\epsilon) < ||w||^{t_i} \prod_{j=1}^{n} |w_j| \), for every \( w \in \Lambda_d \).
for any principal ideal \((a)\) in \(I(\mathfrak{f})\), where \(\varphi : (\mathcal{O}_K/\mathfrak{f})^\times \to \mathbb{C}^\times\) is a residue class character and \(\lambda : K^\times \to F^\times\) is a character defined by
\[
\lambda(a) = \frac{N_{K/F}(a)^k}{N_{K/F}(a)^l}.
\]
Here \(k \geq 0\) and \(l > 0\) are both integers such that
\[
\lambda(\epsilon) = 1
\]
for all units \(\epsilon\) in the group
\[
U_f = \{ \epsilon \in \mathcal{O}_K^\times : \epsilon \equiv 1 \pmod{\mathfrak{f}} \}.
\]
Then, the Hecke \(L\)-function associated to \(\chi\) is
\[
L(s, \chi) = \sum_a \chi(a) N_{K/Q}(a)^{-s}
\]
where the sum is taken over integral ideals \(a\) in \(K\) prime to \(\mathfrak{f}\). It converges absolutely for \(\text{Re}(s) > 1 + \frac{k-1}{2}\), and it is furnished by the usual analytic continuation and functional equation.

**Definition 4.2.** (The partial \(L\)-function \(L(b, r, s)\)) Let \(\{b\}\) be a complete system of representatives of integral ideals prime to \(\mathfrak{f}\). For a fixed ideal \(b\) in this system and any element \(r\) in \(b^{-1}\), we define the partial Hecke \(L\)-function as
\[
L(b, r, s) = \sum_{a \in (a) \sim b} \lambda(a) N_{K/Q}(a)^{-s}
\]
where \(a \sim b\) denotes the equivalence class of \(a\) in \(I(\mathfrak{f})\) modulo \(a \in b^{-1} + r\).

The second equality follows from the observation that for two principal ideals \((a), (b)\) where \(a, b \in fb^{-1} + r\), we have \((a) = (b)\) if and only if \(a = bw\) for some \(w\) in \(U_f\).

The following proposition expresses \(L(s, \chi)\) in terms of \(L(b, r, s)\).

**Proposition 4.3.** The Hecke \(L\)-function \(L(s, \chi)\) with \(\chi\) as in Definition 4.1 can be written in terms of the partial \(L\)-function as
\[
L(s, \chi) = \sum_b \frac{\chi(b)}{N_{K/Q}(b)^s} \sum_{(r) \in I(\mathfrak{f})} \varphi(r) L(b, r, s).
\]
The outer sum is taken over integral ideals \(b\) of \(K\) prime to \(\mathfrak{f}\).

**Proof.** Since \(b \in I(\mathfrak{f})\), we can write
\[
L(s, \chi) = \sum_b \frac{\chi(b)}{N_{K/Q}(b)^s} \sum_{a \sim b} \frac{\chi(ab^{-1})}{N_{K/Q}(ab^{-1})^s},
\]
where $a \sim b$ if $ab^{-1} = (\alpha)$ for some $\alpha$ in $b^{-1}$. We can write the inner sum as

\[
\sum_{a \sim b} \chi(ab^{-1}) = \sum_{(\alpha) \in \mathcal{I}(f)} \varphi(\alpha) \lambda(\alpha) \sum_{a \in b^{-1} + r} \lambda(a) N_{K/Q}(a)^{-s} 
\]

Finally, notice that for $r_1, r_2 \in b^{-1}$, $(r_1) = (r_2)$ if and only if $r_1 \equiv r_2 \pmod{fb^{-1}}$. Thus the identity follows. $\square$

It is clear from the proposition above that to parametrize the value $s$ of $L(s, \chi)$ it is enough to parametrize the values of $L(b, r, s)$ for every $b$ in $\{b\}$. This is what we accomplish in the following section.

4.2. The cycle $\mathcal{E}[b, M]$ and the parametrization of $L(b, r, s)$. Let us first introduce the notation we will use for the rest of the paper.

We fix an ideal $b$ in the system of representatives $\{b\}$ of integral ideals prime to $f$ and also fix an element $r$ of $b^{-1}$. By [O’M63, Theorem 81.3], for the ideal $fb^{-1}$ there exists a basis $m_1, \ldots, m_n$ of the extension $K/F$, as well as ideals $\Lambda_1, \ldots, \Lambda_n$ of $\mathcal{O}_F$\(^3\), such that

\[
fb^{-1} = \Lambda_1 m_1 + \cdots + \Lambda_n m_n.
\]

For this given basis, there exists elements $u_1, \ldots, u_n \in F$ such that $r = u_1 m_1 + \cdots + u_n m_n$. Thus

\[
fb^{-1} + r = (\Lambda_1 + u_1) m_1 + \cdots + (\Lambda_n + u_n) m_n.
\]

Let $M_i$ denote the column vector $(\rho_1(m_1), \ldots, \rho_i(m_n))^T$, $1 \leq i \leq n$, where $\rho_1, \ldots, \rho_n$ denote the embeddings of $K$ into $\mathbb{C}$, fixing the imaginary quadratic field $F$. Let

\[
M = (M_1, \ldots, M_n), \quad u = (u_1, \ldots, u_n), \quad \text{and} \quad \Lambda = \Lambda_1 \times \cdots \times \Lambda_n.
\]

Associated to the basis $m_1, \ldots, m_n$ there is also a representation $\varrho : \mathcal{O}_K^\times \to GL_n(F)$ defined as

\[
\varrho(\eta) = M\delta(\eta)M^{-1},
\]

where $\delta(\eta)$ is the diagonal matrix

\[
\delta(\eta) = \begin{pmatrix} 
\rho_1(\eta) & & \\
& \ddots & \\
& & \rho_n(\eta)
\end{pmatrix}.
\]

\(^3\)Moreover, by [O’M63, 81.5] we can further assume that $\Lambda_i = \mathcal{O}_F$, for $2 \leq i \leq n$. However, this stronger assumption is not needed for our purposes.
The transpose of the matrix (4.12) corresponds to the matrix representation of the linear transformation from $K$ to $K$ given by multiplication-by-$\eta$ in $K$ with respect to the basis $m_1, \ldots, m_n$ of the extension $K/F$. Thus, it indeed lies in $GL_n(F)^\dagger$. 

Additionally, there are also two norm forms associated to $m_1, \ldots, m_n$, namely 

\begin{equation}
Q(x) = \prod_{i=1}^{n} x M_i \quad \text{and} \quad P(x) = \prod_{i=1}^{n} x M_i^{-T}, \quad x \in \mathbb{C}^n.
\end{equation}

Here $M_i$ and $M_i^{-T}$ denote the $i$th column of the matrices $M$ and $M^{-T}$, respectively, defined in (4.11). Notice that, for $x = (x_1, \ldots, x_n) \in F^n$, we have 

\begin{equation}
Q(x) = N_{K/F}(\xi), \quad \xi = \sum_{i=1}^{n} x_i m_i.
\end{equation}

Furthermore, a simple calculation gives the following relationships 

\begin{equation}
(\varphi(\alpha)Q)(x) = N_{F/Q}(\alpha)Q(x) \quad \text{and} \quad (\varphi(\alpha)^T P)(x) = N_{F/Q}(\alpha)P(x),
\end{equation}

for any $\alpha$ in $\mathcal{O}_K^\times$ and any $x$ in $\mathbb{C}^n$.

Finally, we let $V_f$ be the free part of the group $U_f^{(1)} = \{ \epsilon \in U_f : N_{K/F}(\epsilon) = 1 \}$, i.e., the subgroup generated by the elements of infinite order in $U_f^{(1)}$. Each one of the groups 

\begin{equation*}
V_f \subset U_f^{(1)} \subset U_f \subset \mathcal{O}_K^\times,
\end{equation*}

is of rank $n - 1$ since they have finite index in $\mathcal{O}_K^\times$ and rank$\mathcal{O}_K^\times = n - 1$, by the Dirichlet unit theorem\footnote{To be more specific, from the fact that $\eta f^{-1} = f^{-1}$ for any unit $\eta \in \mathcal{O}_K^\times$, it follows after (4.9) that $\varphi(\eta)$ has entries in}

Thus we will choose a fix set of generators $\epsilon_1, \ldots, \epsilon_{n-1}$ of $V_f$, i.e., 

\begin{equation}
V_f = \langle \epsilon_1, \ldots, \epsilon_{n-1} \rangle \quad \text{and} \quad N_{K/F}(\epsilon_i) = 1, \quad i = 1, \ldots, n - 1.
\end{equation}

With these observations, the sum (4.4) can now be expressed as 

\begin{equation}
\mathcal{L}(b, r, s) = |U_f : V_f|^{-1} \sum_{x \in \mathcal{B}^{-1} + r/V_f} \lambda(a) N_{K/F}((a))^{-s}.
\end{equation}

Let $\mathfrak{V} = \varphi(V_f)$. From (4.17), this is a subgroup of $\text{SL}_n(F)$ generated by the matrices 

\begin{equation}
A_i = \varphi(\epsilon_i). \quad i = 1, \ldots, n - 1.
\end{equation}

**Remark 4.4.** Note that in most cases $V_f$ coincides with the group $U_f^{(1)} = \{ \epsilon \in U_f : N_{K/F}(\epsilon) = 1 \}$. This is true, for example, when $N_{K/F}(\hat{f})$ is large enough. Indeed, this last condition forces $U_f^{(1)}$ to have no roots of unity, thus being free. To see this, just take $f$ such that $N_{K/F}(\hat{f})$ is larger than any of the elements in the finite set 

\begin{equation*}
\{ |N_{K/F}(\zeta - 1) : \zeta \in (\mathcal{O}_K^\times)/(\mathfrak{a}) \}.
\end{equation*}

\[5\text{Since } K \text{ is a degree } n \text{ extension of the imaginary quadratic extension } F, \text{ then } n = r_2 \text{ where } r_2 \text{ is the number of embeddings of } K \text{ into } \mathbb{C} \text{ fixing } F.\]
where \((\mathcal{O}_K)\)_{tor} is the group of roots of unity in \(K\), which we know is finite.

4.2.1. The cycle \(\mathcal{E}[\mathfrak{b}, M]\). In this section we will construct the cycle \(\mathcal{E} = \mathcal{E}[\mathfrak{b}, M]\), which depends on \(\mathfrak{b}\) and \(M\), with which the Eisenstein cocycle will be paired, giving the parametrization of \(L\)-values.

**Definition 4.5.** (The cycle) Define the chain in \(C_{n-1}((\mathfrak{m}, \mathbb{Z}))\),

\[
\mathcal{E} = \mathcal{E}[\mathfrak{b}, M] = \rho \sum_{\pi} \text{sign}(\pi) [A_{\pi(1)} | \cdots | A_{\pi(n-1)}]
\]

where \([A_1 | \cdots | A_n] = (1, A_1, A_1 A_2, \ldots, A_1 A_2 \cdots A_n)\) and \(\rho = (-1)^{n-1} \text{sign}(R_{K/F})\). Here \(R_{K/F}\) is the regulator \(\det(2 \log |\rho_j(\epsilon_j)|), 1 \leq i, j \leq n - 1, \text{ and } \pi \text{ runs over all permutations of the set } \{1, \ldots, n-1\}\).

From Lemma 5 of [Scz93, p.592], one sees that this chain defines a cycle in \(H_{n-1}((\mathfrak{m}, \mathbb{Z}))\), denoted by \([\mathcal{E}]\), whose homology class is independent of the set of generators of \(V_f\).

In general, if \(V\) is a subgroup of finite index in \(V_f\) with generators \(\epsilon'_1, \ldots, \epsilon'_{n-1}\) we define

\[
\mathcal{E}[\mathfrak{b}, M, V] = \rho \sum_{\pi} \text{sign}(\pi) [A'_{\pi(1)} | \cdots | A'_{\pi(n-1)}]
\]

where \(A'_i = \varrho(\epsilon'_i), i = 1, \ldots, n-1\). This also defines a homology class in \(H_{n-1}(\varrho(V), \mathbb{Z})\) independent of the set of generators of \(V\).

We now derive the following expression, which will allow us to express values of the \(L\)-function using the Eisenstein cocycle. This identity is the key to parametrizing values of the \(L\)-functions using the Eisenstein cocycle.

**Lemma 4.6.** With notation as above we have

\[
\sum_{A \in \mathfrak{m}} \psi(A \mathcal{E}[\mathfrak{b}, M])(P^{l-1}, x) = \det(M)(l - 1)! \left(\frac{l}{Q(x)}\right)^n,
\]

for any nonzero \(x \in \Lambda + \mathfrak{u}\) and any positive integer \(l\). \(^6\)

**Proof:** See Section 5. \(\square\)

We remark that the proof of the identity follows the ideas of Lemma 6 of [Scz93, p.594], and is related to the proof of the Dirichlet unit theorem. Indeed, the main ingredient of the proof is a certain geometric construction that realizes a cancellation in the sum using the cocycle property of \(\psi\). But in our case one no longer has a system of totally positive units, which is crucial to the Sczech’s proof. Instead, we modify Sczech’s construction so as to avoid this assumption.

4.2.2. Parametrization of \(\mathcal{L}(r, s)\). We are now ready to parametrize the values of the partial \(L\)-function, which will lead us to the main result.

**Theorem 4.7.** Let \(\mathfrak{f}, \mathfrak{b}\) be relatively prime integral ideals of \(K\) and \(r \in \mathfrak{b}^{-1}\). Let \(M, \mathfrak{u}\) and \(\Lambda\) as in (4.11) and \(P\) as in (4.14). Then

\[
\Psi_s(\mathcal{E}[\mathfrak{b}, M])(P^{l-1}, \mathfrak{u}, M) = \det(M)(l - 1)! \left(\frac{l}{Q(x)}\right)^n |U_f : V_f| \mathcal{L}(\mathfrak{b}, r, s)
\]

for \(\text{Re}(s) > 1 + \frac{k}{2}\) and any positive integer \(l\). \(^7\)

---

\(^6\)In Lemma 5.1 we will show that the left-hand side of this identity is independent of any subgroup \(V\) of finite index in \(V_f\) and any choice of generators for \(V\).

\(^7\)This identity also holds if we replace \(V_f\) by any of its subgroups of finite index \(V\). In this case, \(\mathcal{E}[\mathfrak{b}, M]\) has to be replaced by \(\mathcal{E}[\mathfrak{b}, M, V]\), and [\(U_f : V_f]\] by [\([U_f : V]\).
Proof. Let $\mathcal{E} = \mathcal{E}[b, M]$. We start by expressing the left-hand side of the identity as

$$
\Psi_s(\mathcal{E})(P^{l-1}, u, M) = \sum_{x \in \Lambda + u} \psi(\mathcal{E})(P^{l-1}, x) \Omega^k_s(x, M)
= \sum_{x \in (\Lambda + u)/\mathfrak{U}} \sum_{A \in \mathfrak{U}} \psi(\mathcal{E})(P^{l-1}, xA) \Omega^k_s(xA, M)
= \sum_{x \in (\Lambda + u)/\mathfrak{U}} \Omega^k_s(x, M) \sum_{A \in \mathfrak{U}} \psi(\mathcal{E})(P^{l-1}, xA),
$$

(4.24)

where the last equality follows from

$$
\Omega^k_s(xA, M) = \Omega^k_s(x, M), \quad A \in \mathfrak{U} = \varrho(V_f).
$$

This last identity is in turn, a consequence of Lemma 4.28 and Equation (4.25); together with the fact that $N_{K/F}(\epsilon) = 1$ for every $\epsilon$ in $V_f$, and also $\Omega^k_s(x, M) = \frac{Q(x)^k}{|Q(x)|^{2s}}$. Next, by the definition of the action of $GL_n$ on $S_0$ and the fact that $A^T \cdot P^{l-1} = P^{l-1}$ for $A \in \mathfrak{U}$, which can be verified from (4.19) and (4.16), it follows that

$$
\psi(\mathcal{E})(P^{l-1}, xA) = A \psi(\mathcal{E})(P^{l-1}, x).
$$

Furthermore, since the cocycle is homogeneous, we have

$$
A \psi(\mathcal{E})(P^{l-1}, x) = \psi(A \mathcal{E})(P^{l-1}, x).
$$

Altogether this gives

$$
\Psi_s(\mathcal{E})(P^{l-1}, u, M) = \sum_{x \in \Lambda + u/\mathfrak{U}} \Omega^k_s(x, M) \sum_{A \in \mathfrak{U}} \psi(A \mathcal{E})(P^{l-1}, x).
$$

(4.28)

Applying Lemma 4.6, this expression equals

$$
\det(M)((l - 1)!)^n \sum_{x \in \Lambda + u/\mathfrak{U}} \frac{\Omega^k_s(x, M)}{Q(x)^l}.
$$

(4.29)

Finally, we show how the partial $L$-function is expressed in terms of the last expression obtained above. First, notice that (4.15) implies

$$
|Q(x)|^2 = |N_{K/F}(\xi)|^2 = |N_{F/Q}(N_{K/F}(\xi))| = |N_{K/Q}(\xi)| = N_{K/Q}(\xi).
$$

(4.30)

for $\xi = \sum x_i m_i$, with $x = (x_1, \ldots, x_n) \in \Lambda + u$. Second, from (4.10) we have that as $x$ runs through $\Lambda + u/\mathfrak{U}$, $\xi$ runs through $\mathfrak{b}^{-1} + r/V_f$. These two observations, together with (4.18) implies

$$
[U_f : V_f] L(b, r, s) = \sum_{\xi \in (\mathfrak{b}^{-1} + r)/V_f} \frac{N_{K/F}(\xi)^k}{Q(x)^l} \frac{Q(x)^{-l}|Q(x)|^{-2s}}{Q(x)^l} \frac{\Omega^k_s(x, M)}{Q(x)^l},
$$

(4.31)

and the identity follows. \hfill \square

Thus, we obtain the following parametrization for the Hecke $L$-function:
Corollary 4.8. Let \( L(s, \chi) \) be the Hecke \( L \)-function as defined at the beginning of this section. Then, we have

\[
(4.32) \quad L(s, \chi) = \sum_{b} \frac{\chi(b)}{N_{K/Q}(b)} \sum_{r \in \mathbb{A}^{-1}} \varphi(r) \Psi_s(\mathcal{E}[b, M])(P^{l-1}, u, M)
\]

for \( \text{Re}(s) > 1 + \frac{k}{2} \).

Proof. This follows immediately from Proposition 4.3 and Theorem 4.7 above. \( \square \)

Remark 4.9. We point out that the cocycle constructed by Sczech in [Scz93] parametrizes the values of Hecke \( L \)-function for totally real extensions of \( \mathbb{Q} \) at integer values, while the Eisenstein cocycle constructed in this paper parametrizes all complex values of the relevant \( L \)-function in an entire half-plane. This ability to parametrize complex values in an entire half-plane is due to use of the convergence factor following the work of Colmez.

5. Proof of Lemma 4.6

We now provide the proof of the key lemma. The method of proof is similar to that of [Scz93], i.e., to construct a simplex (5.5) and compare each side of (4.22) with this simplex. After taking a limiting process we will obtain the equality of both sides of (4.22). The novelty of our proof is that we exploit the fact that the infinite sum on the left hand-side of (4.22) is independent of the subgroup \( V \) and its set of generators (cf. Lemma 5.1) in order to make the comparison between the infinite sum and the simplex (5.5) much easier.

Proof of Lemma 4.6. We first find an expression equivalent to (4.22). We will work with the units \( \epsilon_i \) rather than the matrices \( A_i \). Using the bar notation we have the relation

\[
[A_{\pi(1)} \cdots A_{\pi(n-1)}] = M[\delta(\epsilon_{\pi(1)})] \cdots \delta(\epsilon_{\pi(n-1)})]M^{-1}.
\]

Then letting \( \mathfrak{A}_{\eta,\pi} = \delta(\eta)[\delta(\epsilon_{\pi(1)})] \cdots \delta(\epsilon_{\pi(n-1)})] \) for \( \eta \in V_1 \), and using the action of \( M \) on \( \psi \) (cf. (3.7)), we can rewrite the equation (4.22) as

\[
(5.1) \quad \rho \sum_{\eta \in V_1} \sum_{\pi} \text{sign}(\pi) \psi(\mathfrak{A}_{\eta,\pi} M^{-1})(M^T P^{l-1}, \xi) = \frac{(l-1)!)^n}{N_{K/F}(\xi)^n}
\]

where \( \xi = \sum x_i m_i \) as in (4.15) and \( \xi = (\xi_1, \ldots, \xi_n) \) with \( \xi_i = \rho_i(\xi) \), for \( i = 1, \ldots, n \), so that \( \xi = xM \).

To give the expression for the right hand side, we introduce the following construction: consider the subgroup of diagonal matrices in \( GL_n(\mathbb{C}) \) with determinant 1, and consider the logarithm map \( \ell \) of this subgroup into \( \mathbb{R}^n \) given by

\[
(5.2) \quad \ell : \text{diag}(x_1, \ldots, x_n) \mapsto (2 \log |x_1|, \ldots, 2 \log |x_n|).
\]

The image of this subgroup under this map is a hypersurface \( H \subset \mathbb{R}^n \) defined by \( \sum \log |x_i| = 0 \). In particular, the element \( \mathfrak{A}_{\eta,\pi} \) defines an oriented simplex \( S_{\eta,\pi} \) in \( H \), where for any \( k = 1, \ldots, n \), the \( k \)-th vertex is the image of the \( k \)-th component of \( \mathfrak{A}_{\eta,\pi} \) under the map (5.2). If \( v_1, \ldots, v_n \) are the vertices of any oriented \( (n-1) \)-simplex in \( H \), define its orientation to be the sign of

\[
(5.3) \quad \det(e, v_2 - v_1, \ldots, v_n - v_{n-1}), \quad e = (1, \ldots, 1).
\]
By direct computation we find that this determinant is equal to \((-1)^{n-1}nR_{K/F}\), and thus by this definition we see that the orientation of \(S_{n,π}\) is the same as that of \(ρ(π)\). Now let

\[H_j = \{y ∈ H : y_i < 0 < y_j \text{ for all } i \neq j\}\]

where \(j = 1, \ldots, n\). By the proof of the Dirichlet’s unit theorem\(^8\) [Lan94] we can find \(n\) units \(θ_j\) in \(V^9\) such that \(l(δ(θ_j)) ∈ H_j\) for every \(j\). We claim that the simplex with vertices \(l(δ(θ_j))\) is positively oriented for all \(j\). Indeed, by continuity, it follows that the sign of the determinant (5.3) is independent of the particular choice of the \(θ_j\) so long as the condition that \(l(δ(θ_j)) ∈ H_j\) is maintained. To calculate the sign, for each \(j\) we pass to a limit \(v_j = \lim l(δ(θ_j)) ∈ H_j\) such that

\[v_{nj} < 0, \quad v_{jj} > 0, \quad v_{ji} = 0, \quad i \neq j, n.\]

Then by direct computation one sees that for such \(v_j\) the determinant (5.3) is indeed positive.

Next, we evaluate the following limit:

\[
\lim_{a → ∞} ψ(ℳM^{-1})(MTP^{l-1}, ξ)
\]

where

\[
ℳ = ℳ(a) = (δ(θ_{i1}^a), \ldots, δ(θ_{in}^a))
\]

and \(a\) is a positive integer. We begin with the observation that the rows of \(M^{-1}\) are conjugated over \(F\). It follows that \(δ(θ_{i}^a)M^{-1}\) modulo projective equivalence converges as \(a → ∞\) to a matrix whose nonzero entries are exactly the entries in the \(j\)-th row (cf. Definition 3.1). According to the definition of \(P\),

\[
M^TP^{l-1}(ξ_1, \ldots, ξ_n) = (ξ_1 \ldots ξ_n)^{l-1},
\]

and therefore by (3.2) and (3.9), the limit (5.4) is equal to

\[
(-1)^{n(l-1)}(∂ξ_1 \ldots ∂ξ_n)^{l-1}f(1_n)(ξ) = \frac{(l - 1)!}{{N_{K/F}(ξ)^l}}.
\]

This is the expression for the right hand side. It remains to show that the left hand side of (5.1) is equal to the limit (5.4). To this end we consider the simplex \(S = S(a)\) in \(H\) corresponding to \(ℳ(a)\) under the map \(l\) for a fixed positive integer \(a\).

Since rank \(O_K^∞ = n - 1\), then there exist integers \(b_1, \ldots, b_n\), not all zero, such that

\[
θ_{i1}^{b_1} \ldots θ_{in}^{b_n} = 1.
\]

Assume, without loss of generality, that \(b_1 ≠ 0\). Let \(b\) be a positive integer large enough so that \(|b_1b_2 + \cdots + b_n| > 0\). Then, replacing \(θ_i\) by \(θ_i^b\), for every \(2 ≤ i ≤ n\), the equation (5.7) becomes \(θ_{i1}^{b_1} \ldots θ_{in}^{b_n} = 1\). Now let \(ε_i = θ_{i1+1}θ_{i1}^{-1}, 1 ≤ i ≤ n - 1\). Then, the group generated by the \(ε_i\)-s, denoted by \(V\), is a finite index subgroup of \(V_f\). This is true since it contains a non-negative power of each \(θ_i\). Indeed, it contains a positive power of \(θ_1\), namely \(β = |b_1b_2 + \cdots + b_n| > 0\); therefore \(θ_i^β \in V\), for \(1 ≤ i ≤ n\).

\(^8\)The existence of these units, constitutes a crucial step in the classical proof of the Dirichlet’s unit theorem.

\(^9\)In principle, these units belong to \(O_K^∞\), but since \(V_f\) has finite index in \(O_K^∞\), we can replace \(θ_i\) by a positive power of it.
Since from Lemma 5.1 the left-hand side of (5.1) is the same for any subgroup of finite index of \( V_1 \) and any choice the generators, we will work with \( V \) and generators \( \epsilon_i, 1 \leq i \leq n - 1 \), such that
\[
\epsilon_1 = \epsilon_1, \epsilon_2 = \epsilon_1 \epsilon_2, \ldots, \epsilon_{n-1} = \epsilon_1 \cdots \epsilon_{n-1}.
\]
Then the simplex
\[
(5.9) \quad \left( \delta(\theta_{1}^{\beta N}), \ldots, \delta(\theta_{n}^{\beta N}) \right) = \delta(\theta_{1}^{\beta N}) \left[ \delta(\epsilon_1^{\beta N}) \mid \ldots \mid \delta(\epsilon_1^{\beta N} \cdots \epsilon_{n-1}^{\beta N}) \right]
\]
for any positive integer \( N \) is, by Lemma 5.3, homologous to a sum of simplicies \( S_{n, \pi} \). Therefore, letting \( N \to \infty \), the equation (5.4) coincides with the right-hand side of (5.1). This follows since the simplex \( S(\beta N) \) exhausts the hypersurface \( H \) as \( N \to \infty \) and so every oriented simplex \( S_{n, \pi} \) will eventually be contained in \( S(\beta N) \), which in turn is composed by simplicies \( S_{n, \pi} \). Therefore every term in the sum (5.4) will eventually be caught in \( \psi(\mathcal{S}(\beta N)M^{-1})(MTP^{l-1}, \xi) \) for \( N \) large enough.

Lemma 5.1. The infinite sum in equation (4.22)
\[
(5.10) \quad \sum_{A \in \mathfrak{A}} \psi(A \mathcal{E}[b, M])(P^{l-1}, x)
\]
is independent of the set of generators of the group \( V_1 \). Moreover, the sum is the same if we replace \( V_1 \) by any subgroup \( V \) of finite index in \( V_1 \), i.e.,
\[
(5.11) \quad \sum_{A \in \mathfrak{A}} \psi(A \mathcal{E}[b, M])(P^{l-1}, x) = \sum_{A \in \mathfrak{A}} \psi(A \mathcal{E}[b, M, V])(P^{l-1}, x),
\]
where \( \mathfrak{A} = \mathfrak{c}(V) \).

Proof. First, suppose \( \epsilon_1', \ldots, \epsilon_{n-1}' \) is another set of generators for \( V_1 \) and let
\[
(5.12) \quad \mathcal{E}' = \rho \sum_{\pi} \text{sign}(\pi)|A'_\pi(1)| \ldots |A'_\pi(n-1)|,
\]
where \( A'_\pi \) is defined as in (4.19) with \( \epsilon_i \) replaced by \( \epsilon_i' \). By Lemma 5 of [Scz93, p.592], we know that \( [\mathcal{E}] \) and \( [\mathcal{E}'] \) generate the same homology class in \( H_{n-1}(\mathfrak{A}, \mathbb{Z}) \).

In other words, \( \mathcal{E} - \mathcal{E}' \) belongs to the image of the boundary map
\[
(5.13) \quad \partial \otimes_{\mathfrak{A}} \mathbb{Z} : \mathbb{Z}[\mathfrak{A}^{n}] \otimes_{\mathfrak{A}} \mathbb{Z} \to \mathbb{Z}[\mathfrak{A}^{n-1}] \otimes_{\mathfrak{A}} \mathbb{Z},
\]
which is obtained from the boundary map \( \partial : \mathbb{Z}[\mathfrak{A}^{n}] \to \mathbb{Z}[\mathfrak{A}^{n-1}] \) after tensoring by \( \mathbb{Z} \). Therefore there exists an element \( \mathcal{X} \) in \( I_{\mathfrak{A}} \mathbb{Z}[\mathfrak{A}^{n-1}] \), where \( I_{\mathfrak{A}} \) is the augmentation ideal of \( \mathbb{Z}[\mathfrak{A}] \), such that \( \mathcal{E} - \mathcal{E}' \equiv \mathcal{X} \) (mod \( \text{Im}(\partial) \)); this follows from the fact that
\[
(5.14) \quad \mathbb{Z}[\mathfrak{A}^{n-1}] \otimes_{\mathfrak{A}} \mathbb{Z} \simeq \mathbb{Z}[\mathfrak{A}^{n-1}]/I_{\mathfrak{A}} \mathbb{Z}[\mathfrak{A}^{n-1}].
\]

Without loss of generality we may assume that \( \mathcal{X} = (1_n - \rho(\beta))\mathcal{Y} \), where \( 1_n \) is the \( n \times n \) identity matrix, \( \beta \) is a unit in \( V \) and \( \mathcal{Y} \in \mathbb{Z}[\mathfrak{A}^{n-1}] \). Since \( \psi \) is a cocyle, it vanishes on \( \text{Im}(\partial) \). Thus after multiplying on the left by an arbitrary \( A \in \mathfrak{A} \), we obtain from the homogeneity of \( \psi \), cf. (3.10), that
\[
(5.15) \quad \psi(A\mathcal{E}) - \psi(A\mathcal{E}') = \psi(A\mathcal{Y}) = (1_n - \rho(\beta)) \psi(A\mathcal{Y}).
\]
Now summing over all \( A \in \mathfrak{A} \), the right hand side vanishes \(^{10}\) and we obtain
\[
(5.16) \quad \sum_{A \in \mathfrak{A}} \psi(A\mathcal{E})(P^{l-1}, x) - \sum_{A \in \mathfrak{A}} \psi(A\mathcal{E}')(P^{l-1}, x) = 0.
\]

\(^{10}\)Indeed, as \( \eta \) runs through \( V_1 \), then both \( A = \rho(\eta) \) and \( A\rho(\beta) = \rho(\eta\beta) \) run through every element of \( \mathfrak{A} \). Thus, every element in the right hand side will eventually cancel out.
As for the second assertion, let $V$ be a subgroup of finite index of $V_1$. By the first assertion, we may assume that a set of generators for $V$ is given by $e_1^{b_1}, \ldots, e_{n-1}^{b_{n-1}}$.

For this specific subgroup and generators, let $\tilde{E} = E[\bar{b}, M, V]$ be as in (4.21). Also denote by $\mathfrak{D}$ the subgroup $g(V)$, i.e., the subgroup in $S\text{L}_n(F)$ generated by the $g(e_i^{b_i})$’s. Then

$$\tilde{E} = \sum_{0 \leq k_i \leq b_i \leq 1, i = 1, \ldots, n-1} g(e_1^{k_1} \ldots e_{n-1}^{k_{n-1}})E$$

is homologous to an element $\tilde{X}$ in $I_{\mathfrak{D}}Z[\mathfrak{M}^{n-1}]$ (see equation (5.22) in Remark 5.2 below). Thus, after multiplying (5.22) on the left by an element $A \in \mathfrak{D}$, we have

$$\psi(A\tilde{E} \tilde{E}) = \sum_{0 \leq k_i \leq b_i \leq 1, i = 1, \ldots, n-1} g(e_1^{k_1} \ldots e_{n-1}^{k_{n-1}}) \psi(AE) \psi(\tilde{X}) = \psi(A\tilde{X}).$$

Summing now over all $A \in \mathfrak{D}$, then noticing that $g(e_1^{k_1} \ldots e_{n-1}^{k_{n-1}})A$ will be running over all elements of $\mathfrak{D}$ and that the right-hand side vanishes since $\tilde{X}$ lies in the augmentation ideal $I_{\mathfrak{D}}Z[\mathfrak{M}^{n-1}]$ 11, we have

$$\sum_{A \in \mathfrak{D}} \psi(A\tilde{E} \tilde{E})(P^{l-1}, x) = \sum_{A \in \mathfrak{D}} \psi(AE)(P^{l-1}, x) = 0.$$

as desired. \hfill \square

**Remark 5.2.** The homology classes $[\tilde{E}]$ and

$$\sum_{0 \leq k_i \leq b_i \leq 1, i = 1, \ldots, n-1} g(e_1^{k_1} \ldots e_{n-1}^{k_{n-1}})E$$

are well defined in $H_{n-1}(\mathfrak{D}, M_{\mathfrak{D}})$, where $M_{\mathfrak{D}} = Z[\mathfrak{M}]/I_{\mathfrak{D}}Z[\mathfrak{M}]$ 12 and $I_{\mathfrak{D}}$ is the augmentation ideal of $Z[\mathfrak{M}]$. Moreover, they generate the same element in this homology group. This implies that the difference (5.17) belongs to the image of the boundary map

$$\partial \otimes_{\mathfrak{D}} M_{\mathfrak{D}} : Z[\mathfrak{M}] \otimes_{\mathfrak{D}} M_{\mathfrak{D}} \rightarrow Z[\mathfrak{M}^{n-1}] \otimes_{\mathfrak{D}} M_{\mathfrak{D}}.$$

Here $\partial \otimes_{\mathfrak{D}} M_{\mathfrak{D}}$ denotes the boundary map $\partial : Z[\mathfrak{M}] \rightarrow Z[\mathfrak{M}^{n-1}]$ after tensoring by $M_{\mathfrak{D}}$. Notice that $Z[\mathfrak{M}^{n-1}] \otimes_{\mathfrak{D}} M_{\mathfrak{D}} \cong Z[\mathfrak{M}^{n-1}]/I_{\mathfrak{D}}Z[\mathfrak{M}^{n-1}]$. Therefore, there exists an element $\tilde{X} \in I_{\mathfrak{D}}Z[\mathfrak{M}^{n-1}]$ such that

$$\tilde{E} - \sum_{0 \leq k_i \leq b_i \leq 1, i = 1, \ldots, n-1} g(e_1^{k_1} \ldots e_{n-1}^{k_{n-1}})E \equiv \tilde{X} \pmod{\text{Im}(\partial)}.$$

**Lemma 5.3.** For any permutation $\tau \in S_{n-1}$, the simplex

$$\epsilon_{m, \tau} = \left[ \delta(e_1^{m_{\tau(1)}}) \cdots \delta(e_{n-1}^{m_{\tau(n-1)}}) \right] \quad (m \geq 1)$$

is homologous to a sum of simplices of the form

$$\alpha_{\eta, \pi} = \delta(\eta)\left[ \delta(e_1^{\pi(1)}) \cdots \delta(e_{n-1}^{\pi(n-1)}) \right] \quad (\eta \in V_1, \pi \in S_{n-1}).$$

---

11Without loss of generality, we may assume $\tilde{X} = (1 - g(\bar{\beta})\bar{\gamma})$, for $\beta \in V$ and $\gamma \in Z[\mathfrak{M}^{n-1}]$. Once again, as $\eta$ runs through $V$, then both $A = g(\bar{\eta})$ and $Ae(\bar{\beta}) = g(\bar{\eta} \bar{\beta})$ run through every element of $\mathfrak{D}$. Thus, every element in the right-hand side will eventually cancel out.

12$M_{\mathfrak{D}}$ is isomorphic to $Z[\mathfrak{M}/\mathfrak{D}]$ as $Z[\mathfrak{M}]$-modules.
Proof. We will use the same notation of Lemma 4.6. Let \( l \) be the map defined in (5.2) and let \( E_{m, \tau} \) be an oriented simplex in \( H \), whose \( k \)-th vertex, \( k = 1, \ldots, n \), is the image of the \( k \)-th coordinate of \( \xi_{m, \tau} \) under \( l \). Given \( \tau \in S_{n-1} \), define

\[
V_\tau = \{ l(\delta(e_{\tau(1)}^1 \cdots e_{\tau(n-1)}^{n-1})) : m \geq i_1 \geq \cdots \geq i_{n-1} \geq 0 \}.
\] (5.25)

This is the set of lattice points contained in the simplex (5.23). Now let \( \Xi_{m, \tau} \) be the sum of all simplices \( \mathfrak{A}_{\eta, \pi} \) such that \( S_{\eta, \pi} \) is contained inside \( E_{m, \tau} \) (or, equivalently, whose entries all lie in \( V_\tau \)), and denote by \( T_{m, \tau} \) the sum of all the simplices \( S_{\eta, \pi} \) that are contained in \( E_{m, \tau} \). We will show that \( \Xi_{m, \tau} \) is homologous to \( E_{m, \tau} \) by showing that \( \text{vol}(T_{m, \tau}) = \text{vol}(E_{m, \tau}) \); here vol is understood to be the usual Lebesgue measure on \( H \simeq \mathbb{R}^n \). Clearly, since \( T_{m, \tau} \subset E_{m, \tau} \), then

\[
\text{vol}(T_{m, \tau}) \leq \text{vol}(E_{m, \tau}).
\] (5.26)

To show equality, it is enough to show that

\[
\sum_{\tau \in S_{n-1}} \text{vol}(T_{m, \tau}) = \sum_{\tau \in S_{n-1}} \text{vol}(E_{m, \tau}).
\] (5.27)

But this last equality is clear since the right-hand side represents the volume of the \((n-1)\)-dimensional cube generated by \( l(\delta(e_i^m)) \), \( i = 1, \ldots, n-1 \), which is

\[
\text{vol}(T_{m, \tau}) = m^{n-1} \text{vol}(E_{m, \tau}).
\] (5.28)

and the expression on the left-hand side is the volume of each \( S_{\eta, \pi} \), which is

\[
\text{vol}(T_{m, \tau}) = (n-1)! \text{vol}(E_{m, \tau}).
\] (5.29)

times the number of such simplices, which is \( m^{n-1}(n-1)! \).

\[\square\]

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