Lax-Phillips Scattering Theory 
of a 
Relativistic Quantum Field Theoretical Lee-Friedrichs Model 
and 
Lee-Oehme-Yang-Wu Phenomenology

Y. Strauss and L.P. Horwitz*
School of Physics
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University
Ramat Aviv 69978, Israel

Abstract: The one-channel Wigner-Weisskopf survival amplitude may be dominated by exponential type decay in pole approximation at times not too short or too long, but, in the two channel case, for example, the pole residues are not orthogonal, and the pole approximation evolution does not correspond to a semigroup (experiments on the decay of the neutral $K$-meson system support the semigroup evolution postulated by Lee, Oehme and Yang, and Yang and Wu to very high accuracy). The scattering theory of Lax and Phillips, originally developed for classical wave equations, has been recently extended to the description of the evolution of resonant states in the framework of quantum theory. The resulting evolution law of the unstable system is that of a semigroup, and the resonant state is a well-defined function in the Lax-Phillips Hilbert space. In this paper we apply this theory to a relativistically covariant quantum field theoretical form of the (soluble) Lee model. We show that this theory provides a rigorous underlying basis for the Lee-Oehme-Yang-Wu construction.

* Also at Department of Physics, Bar Ilan University, Ramat Gan 52900, Israel. e-mail:larry@ccsg.tau.ac.il
1. Introduction.

The theory of Lax and Phillips\(^1\) (1967), originally developed for the description of resonances in electromagnetic or acoustic scattering phenomena, has been used as a framework for the construction of a description of irreversible resonant phenomena in the quantum theory\(^2−5\) (which we will refer to as the quantum Lax-Phillips theory). This leads to a time evolution of resonant states which is of semigroup type, i.e., essentially exponential decay. Semigroup evolution is necessarily a property of irreversible processes\(^6\). It appears experimentally that elementary particle decay, to a high degree of accuracy, follows a semigroup law, and hence such processes seem to be irreversible.

The theory of Weisskopf and Wigner\(^7\), which is based on the definition of the survival amplitude of the initial state \(\phi\) (associated with the unstable system) as the scalar product of that state with the unitarily evolved state,

\[
(\phi, e^{-iHt}\phi)
\]

\((1.1)\)

cannot have exact exponential behavior\(^8\). One can easily generalize this construction to the problem of more than one resonance\(^9,10\). If \(P\) is the projection operator into the subspace of initial states (\(N\)-dimensional for \(N\) resonances), the reduced evolution operator is given by

\[
P e^{-iHt} P.
\]

\((1.1')\)

Since the Laplace transform of this operator has a cut and not just poles, this operator cannot be an element of an exact semigroup.\(^8\)

Experiments on the decay of the neutral \(K\)-meson system\(^11\) show clearly that the phenomenological description of Lee, Oehme and Yang\(^12\), and Wu and Yang\(^13\), by means of a \(2 \times 2\) effective Hamiltonian which corresponds to an exact semigroup evolution of the unstable system, provides a very accurate description of the data. Is can be proved that the Wigner-Weisskopf theory cannot provide a semigroup evolution law\(^8\) and, thus, the effective \(2 \times 2\) Hamiltonian cannot emerge in the framework of this theory. Furthermore, it has been shown, using estimates based on the quantum mechanical Lee-Friedrichs model\(^14\), that the experimental results appear to rule out the application of the Wigner-Weisskopf theory to the decay of the neutral \(K\)-meson system. While the exponential decay law can be exhibited explicitly in terms of a Gel’fand triple\(^15\) (rigged Hilbert space), the representation of the resonant state in this framework is in a space which does not coincide with the quantum mechanical Hilbert space, and does not have the properties of a Hilbert space, such as scalar products and the possibility of calculating physical properties associated with expectation values.

The seminal work of Lax and Phillips\(^1\) has provided us with the basic ideas necessary for the construction of a fundamental theoretical description, in the framework of the quantum theory\(^2−5\), of a resonant system which has exact semigroup evolution, and represents the resonance as a state in a Hilbert space. In the following, we describe briefly the structure of this theory, a rather straightforward generalization of standard quantum scattering theory, and give some physical interpretation for the states of the Lax-Phillips Hilbert space.

The Lax-Phillips theory is defined in a Hilbert space \(\mathcal{H}\) of states which contains two distinguished subspaces, \(D_\pm\), called “outgoing” and “incoming”. There is a unitary
evolution law which we denote by \( U(\tau) \), for which these subspaces are invariant in the following sense:

\[
U(\tau)D_+ \subset D_+ \quad \tau \geq 0 \\
U(\tau)D_- \subset D_- \quad \tau \leq 0
\] (1.2)

The translates of \( D_\pm \) under \( U(\tau) \) are dense, i.e.,

\[
\bigcup_\tau U(\tau)D_\pm = \overline{H}
\] (1.3)

and the asymptotic property

\[
\bigcap_\tau U(\tau)D_\pm = \emptyset
\] (1.4)

is assumed. It follows from these properties that

\[
Z(\tau) = P_+ U(\tau) P_-, \tag{1.5}
\]

where \( P_\pm \) are projections into the subspaces orthogonal to \( D_\pm \), is a strongly contractive semigroup\(^1\), i.e.,

\[
Z(\tau_1)Z(\tau_2) = Z(\tau_1 + \tau_2)
\] (1.6)

for \( \tau_1, \tau_2 \) positive, and \( \|Z(\tau)\| \to 0 \) for \( \tau \to 0 \). It follows from (1.2) that \( Z(\tau) \) takes the subspace \( \mathcal{K} \), the orthogonal complement of \( D_\pm \) in \( \overline{H} \) (associated with the resonances in the Lax-Phillips theory), into itself\(^1\), i.e.,

\[
Z(\tau) = P_\mathcal{K} U(\tau) P_\mathcal{K}. \tag{1.7}
\]

The relation (1.7) is of the same structure as (1.1\'); there is, as we shall see in the following, an essential difference in the way that the subspaces associated with resonances are defined. The argument that (1.1\') cannot form a semigroup is not valid\(^3\) for (1.7); the generator of \( U(\tau) \) restricted to \( \mathcal{K} \) is not self-adjoint.

A Hilbert space with the properties that there are distinguished subspaces satisfying, with a given law of evolution \( U(\tau) \), the properties (1.2), (1.3), (1.4) has a foliation\(^16\) into a one-parameter (which we shall denote as \( s \)) family of isomorphic Hilbert spaces, which are called auxiliary Hilbert spaces, \( \{H_s\} \) for which

\[
\overline{H} = \int_\oplus H_s. \tag{1.8}
\]

Representing these spaces in terms of square-integrable functions, we define the norm in the direct integral space as

\[
\|f\|^2 = \int_{-\infty}^{\infty} ds \|f_s\|^2_H,
\] (1.9)

where \( f \in \overline{H} \) represents a vector in the \( L^2 \) function space \( \overline{H} = L^2(-\infty, \infty, H) \); \( f_s \) is an element of \( H_s \), the \( L^2 \) function space (the auxiliary space) corresponding to \( H_s \) for any \( s \).
[we shall not add in what follows a subscript to the norm or scalar product symbols for scalar products of elements of the auxiliary Hilbert space associated to a point \( s \) on the foliation axis since these spaces are all isomorphic].

There are representations for which the action of the full evolution group \( U(\tau) \) on \( L^2(-\infty, \infty; H) \) is translation by \( \tau \) units. Given \( D_\pm \) there is such a representation, called the \textit{incoming representation} \(^1\), for which the set of all functions in \( D_- \) have support in \((-\infty, 0)\) and constitute the subspace \( L^2(-\infty, 0; H) \) of \( L^2(-\infty, \infty; H) \); there is another representation, called the \textit{outgoing representation}, for which functions in \( D_+ \) have support in \((0, \infty)\) and constitute the subspace \( L^2(0, \infty; H) \) of \( L^2(-\infty, \infty; H) \). The fact that \( Z(\tau) \) in Eq. (1.7) is a semigroup is a consequence of the definition of the subspaces \( D_\pm \) in terms of support properties on intervals along the foliation axis in the \textit{outgoing} and \textit{incoming} translation representations respectively. The non-self-adjoint character of the generator of the semigroup \( Z(\tau) \) is a consequence of this structure. \(^3\)

Lax and Phillips \(^1\) show that there are unitary operators \( W_\pm \), called wave operators, which map elements in \( H \) to these representations. They define an \( S \)-matrix,

\[
S = W_+ W_-^{-1}
\]

which connects the incoming to the outgoing representations; it is unitary, commutes with translations, and maps \( L^2(-\infty, 0; H) \) into itself. Since \( S \) commutes with translations, it is diagonal in Fourier (spectral) representation. As pointed out by Lax and Phillips \(^1\), according to a special case of a theorem of Fourès and Segal \(^17\), an operator with these properties can be represented as a multiplicative operator-valued function \( S(\sigma) \) which maps \( H \) into itself, and satisfies the following conditions:

\begin{enumerate}
  \item \( S(\sigma) \) is the boundary value of an operator-valued function \( S(z) \) analytic for \( \text{Im} z > 0 \).
  \item \( \| S(z) \| \leq 1 \) for all \( z \) with \( \text{Im} z > 0 \).
  \item \( S(\sigma) \) is unitary for almost all real \( \sigma \).
\end{enumerate}

An operator with these properties is known as an inner function \(^18\); such operators arise in the study of shift invariant subspaces, the essential mathematical content of the Lax-Phillips theory. The singularities of this \( S \)-matrix, in what is called \textit{spectral representation} (defined in terms of the Fourier transform on the foliation variable \( s \)), correspond to the spectrum of the generator of the semigroup characterizing the evolution of the unstable system.

In the framework of quantum theory, one may identify the Hilbert space \( H \) with a space of physical states, and the variable \( \tau \) with the laboratory time (the semigroup evolution is observed in the laboratory according to this time). The representation of this space in terms of the foliated \( L^2 \) space \( \mathcal{H} \) provides a natural probabilistic interpretation for the auxiliary spaces associated with each value of the foliation variable \( s \), i.e., the quantity \( \| f_s \|^2 \) corresponds to the probability density for the system to be found in the neighborhood of \( s \). For example, consider an operator \( A \) defined on \( \mathcal{H} \) which acts pointwise, i.e., contains no shift along the foliation. Such an operator can be represented as a direct integral

\[
A = \int \oplus A_s.
\]

\(^4\)
It produces a map of the auxiliary space $H$ into $H$ for each value of $s$, and thus, if it is self-adjoint, $A_s$ may act as an observable in a quantum theory associated to the point $s^4$; the expectation value of $A_s$ in a state in this Hilbert space defined by the vector $\psi_s$, the component of $\psi \in \overline{H}$ in the auxiliary space at $s$, is

\[ \langle A_s \rangle_s = \frac{(\psi_s, A_s \psi_s)}{\|\psi_s\|^2} \]  \hspace{1cm} (1.12)

Taking into account the \textit{a priori} probability density $\|\psi_s\|^2$ that the system is found at this point on the foliation axis, we see that the expectation value of $A$ in $\overline{H}$ is

\[ \langle A \rangle = \int ds \langle A_s \rangle_s \|\psi_s\|^2 = \int ds (\psi_s, A_s \psi_s), \]  \hspace{1cm} (1.13)

the direct integral representation of $(\psi, A\psi)$.

As we have remarked above, in the translation representations for $U(\tau)$ the foliation variable $s$ is shifted (this shift, for sufficiently large $|\tau|$, induces the transition of the state into the subspaces $D_\pm$). It follows that $s$ may be identified as an intrinsic time associated with the evolution of the state; since it is a variable of the measure space of the Hilbert space $\overline{H}$, this quantity itself has the meaning of a quantum variable.

We are presented here with the notion of a virtual history. To understand this idea, suppose that at a given time $\tau_0$, the function which represents the state has some distribution $\|\psi_{\tau_0}\|^2$. This distribution provides an \textit{a priori} probability that the system would be found at time $s$ (greater or less than $\tau_0$), if the experiment were to be performed at time $s$ corresponding to $\tau = s$ on the laboratory clock. The state of the system therefore contains information on the structure of the \textit{history} of the system as it is inferred at $\tau_0$.

We shall assume the existence of a unitary evolution on the Hilbert space $\overline{H}$, and that for

\[ U(\tau) = e^{-iK\tau}, \]  \hspace{1cm} (1.14)

the generator $K$ can be decomposed as

\[ K = K_0 + V \]  \hspace{1cm} (1.15)

in terms of an unperturbed operator $K_0$ with spectrum $(-\infty, \infty)$ and a perturbation $V$, under which this spectrum is stable. We shall, furthermore, assume that wave operators exist, defined on some dense set, as

\[ \Omega_\pm = \lim_{\tau \to \pm \infty} e^{iK\tau} e^{-iK_0\tau}. \]  \hspace{1cm} (1.16)

In the soluble model that we shall treat as an example in this paper, the existence of the wave operators is assured.

With the help of the wave operators, we can define translation representations for $U(\tau)$. The translation representation for $K_0$ is defined by the property

\[ 0\langle s, \alpha|e^{-iK_0\tau} f \rangle = 0\langle s - \tau, \alpha|f \rangle, \]  \hspace{1cm} (1.17)
where $\alpha$ corresponds to a label for the basis of the auxiliary space. Noting that

$$K\Omega_{\pm} = \Omega_{\pm}K_0$$  \hfill (1.18)

we see that

$$\langle s, \alpha | e^{-iK\tau}f \rangle = \langle s - \tau, \alpha | f \rangle,$$  \hfill (1.19)

where

$$\langle s, \alpha | f \rangle = 0 \langle s, \alpha | \Omega^\dagger_{\pm} f \rangle$$  \hfill (1.20)

It will be convenient to work in terms of the Fourier transform of the in and out translation representations; we shall call these the in and out spectral representations, \textit{i.e.},

$$\langle \sigma, \alpha | f \rangle = \int_{-\infty}^{\infty} e^{-i\sigma s} \langle s, \alpha | f \rangle.$$  \hfill (1.21)

In these representations, (1.20) is

$$\langle \sigma, \alpha | f \rangle = 0 \langle \sigma, \alpha | \Omega^\dagger_{\pm} f \rangle$$  \hfill (1.22)

and (1.19) becomes

$$\langle \sigma, \alpha | e^{-iK\tau} f \rangle = e^{-i\sigma\tau} \langle \sigma, \alpha | f \rangle.$$  \hfill (1.23)

Eq. (1.17) becomes, under Fourier transform

$$0 \langle \sigma, \alpha | e^{-iK_0\tau} f \rangle = e^{-i\sigma\tau} 0 \langle \sigma, \alpha | f \rangle.$$  \hfill (1.24)

For \( f \) in the domain of \( K_0 \), (1.23) implies that

$$0 \langle \sigma, \alpha | K_0 f \rangle = \sigma_0 \langle \sigma, \alpha | f \rangle.$$  \hfill (1.25)

With the solution of (1.25), and the wave operators, the \textit{in} and \textit{out} spectral representations of a vector \( f \) can be constructed from (1.24).

We are now in a position to construct the subspaces \( D_{\pm} \), which are not given \textit{a priori} (as they are in the classical theory\textsuperscript{1}) in the Lax-Phillips quantum theory. We shall define \( D_+ \) as the set of functions with support in \((0, \infty)\) in the \textit{outgoing} translation representation. Similarly, we shall define \( D_- \) as the set of functions with support in \((-\infty, 0)\) in the \textit{incoming} translation representation. The corresponding elements of \( \mathcal{H} \) constitute the subspaces \( D_{\pm} \). By construction, \( D_{\pm} \) have the required invariance properties under the action of \( U(\tau) \).

The \textit{outgoing spectral representation} of a vector \( g \in \mathcal{H} \) is

$$\langle \sigma, \alpha | g \rangle = 0 \langle \sigma, \alpha | \Omega_{\pm}^{-1} g \rangle = \int d\sigma' \sum_{\alpha'} 0 \langle \sigma, \alpha | S | \sigma', \alpha' \rangle 0 \langle \sigma', \alpha' | \Omega_{\pm}^{-1} g \rangle$$

$$= \int d\sigma' \sum_{\alpha'} 0 \langle \sigma, \alpha | S | \sigma', \alpha' \rangle 0 \langle \sigma', \alpha' | g \rangle.$$  \hfill (1.26)
where we call
\[ S = \Omega_+^{-1}\Omega_. \] (1.27)
the quantum Lax-Phillips $S$-operator. We see that the kernel $0\langle \sigma\alpha|S|\sigma'\alpha'\rangle_0$ maps the incoming to outgoing spectral representations. Since $S$ commutes with $K_0$, it follows that
\[ 0\langle \sigma\alpha|S|\sigma'\alpha'\rangle_0 = \delta(\sigma - \sigma')S^{\alpha\alpha'}(\sigma)\] (1.28)
It follows from (1.16) and (1.22), in the standard way\(^{19}\), that
\[ 0\langle \sigma\alpha|S|\sigma'\alpha'\rangle_0 = \lim_{\epsilon \to 0} \delta(\sigma - \sigma')\{\delta^{\alpha\alpha'} - 2\pi i_0\langle \sigma\alpha|T(\sigma + i\epsilon)|\sigma'\alpha'\rangle_0\}, \] (1.29)
where
\[ T(z) = V + VG(z)V = V + VG_0(z)T(z). \] (1.30)
We remark that, by this construction, $S^{\alpha\alpha'}(\sigma)$ is analytic in the upper half plane in $\sigma$. The Lax-Phillips $S$-matrix\(^{1}\) is given by the inverse Fourier transform,
\[ S = \{0\langle s\alpha|S|s'\alpha'\rangle_0\}; \] (1.31)
this operator clearly commutes with translations.

From (1.29) it follows that the inner function property (a) of $S(\sigma)$ above is true. Property (c), unitarity for real $\sigma$, is equivalent to asymptotic completeness, a property which is stronger than the existence of wave operators. For the relativistic Lee model, which we shall treat in this paper, this condition is satisfied. In the model that we shall study here, we shall see that there is a wide class of potentials $V$ for which the operator $S(\sigma)$ satisfies the property (b).

In the next section, we review briefly the structure of the relativistic Lee model\(^{19}\), and construct explicitly the Lax-Phillips spectral representations and $S$-matrix. Introducing auxiliary space variables, we then characterize the properties of the finite rank Lee model potential which assure that the $S$-matrix is an inner function, i.e., that property (b) listed above is satisfied.

2. The multi-channel relativistic Lee-Friedrichs model

The multi-channel relativistic Lee-Friedrichs model is defined in terms of bosonic quantum fields on spacetime. These fields, which emerge from the second quantization of the Stueckelberg covariant quantum theory\(^{20}\), evolve with an invariant evolution parameter$^5 \tau$ (which we identify here with the evolution parameter of the Lax-Phillips theory discussed above); at equal $\tau$, they satisfy the commutation relations (with $\psi^\dagger_i$ as the canonical conjugate field to $\psi_i$; the fields $\psi_i$, which satisfy first order evolution equations as for nonrelativistic Schrodinger fields, are just annihilation operators)
\[ [\psi_{i\tau}(x), \psi^\dagger_{j\tau}(x')] = \delta^4(x - x')\delta_{ij}. \] (2.1)
Transforming to momentum space, in which we have
\[
\psi_{i\tau}(p) = \frac{1}{(2\pi)^2} \int d^4x e^{-ipx^\mu}\psi_{i\tau}(x),
\]  

(2.2)

relation (2.1) becomes

\[
[\psi_{i\tau}(p), \psi_{j\tau}^\dagger(p')] = \delta^4(p-p')\delta_{ij}.
\]  

(2.3)

The manifestly covariant spacetime structure of these fields is admissible when \(E, p\) are not \textit{a priori} constrained by a sharp mass-shell relation. In the mass-shell limit (for which the variation in \(m^2\) defined by \(E^2 - p^2\) is small), multiplying both sides of (2.3) by \(\Delta E = \Delta m^2/2E\), one obtains the usual commutation relations for on shell fields,

\[
[\tilde{\psi}_{i\tau}(p), \tilde{\psi}_{j\tau}^\dagger(p)] = 2E\delta^3(p-p')\delta_{ij},
\]  

(2.4)

where \(\tilde{\psi}_{i\tau}(p) = \sqrt{\Delta m^2}\psi_{i\tau}(p)\). In this limit, \(t\) and \(\tau\) coincide. The generator of evolution

\[
K = K_0 + V
\]  

(2.5)

for which the Heisenberg picture fields are

\[
\psi_{i\tau}(p) = e^{iK_\tau}\psi_{i0}(p)e^{-iK_\tau}
\]  

(2.6)

is given, in this model, as (we write \(p^2 = p_\mu p^\mu, k^2 = k_\mu k^\mu\) in the following)*

\[
K_0 = \sum_{i=1,2} \left\{ \int d^4p \frac{p^2}{2MV_i}b_i^\dagger(p)b_i(p) + \int d^4p \frac{p^2}{2MN_i}a_{N_i}^\dagger(p)a_{N_i}(p) \right\}
\]  

\[
+ \sum_{i=1,2} \int d^4p \frac{p^2}{2M_\theta}a_{\theta_i}^\dagger(p)a_{\theta_i}(p)
\]  

(2.7)

and

\[
V = \sum_{i,j=1,2} \int d^4p \int d^4k \left( f_{ij}(k)b_{i\dagger}(p)a_{N_j}(p-k)a_{\theta_j}(k) + f_{ij}^*(k)b_i(p)a_{N_j}^\dagger(p-k)a_{\theta_j}^\dagger(k) \right)
\]  

(2.8)

This model describes the process \(V_i \rightarrow N_j + \theta_j\). We assume that the fields associated with different particles commute. The fields \(b_i(p), a_{N_i}(p)\) and \(a_{\theta_i}\) are annihilation operators for the \(V_i, N_i\) and \(\theta_i\) particles, respectively. We take \(M_{V_i}, M_{N_i}\) and \(M_{\theta_i}\) to be the mass parameters for the fields\(^\text{19,22}\). We restrict our development to the two channel case in the following. The generalization to any number of channels is straightforward.

The following operators are conserved

* We remark that Antoniou, \textit{et al} \(^\text{21}\), have constructed a relativistic Lee model of a somewhat different type; their field equation is second order in derivative with respect to the variable conjugate to the mass.
\[ Q_1 = \sum_{i=1,2} \int d^4p \left( b_i^\dagger(p)b_i(p) + a_{N_i}^\dagger(p) a_{N_i}(p) \right) \]
\[ Q_2 = \int d^4p \left( a_{N_1}^\dagger(p) a_{N_1}(p) - a_{\theta_1}^\dagger(p) a_{\theta_1}(p) \right) \]
\[ Q_3 = \int d^4p \left( a_{N_2}^\dagger(p) a_{N_2}(p) - a_{\theta_2}^\dagger(p) a_{\theta_2}(p) \right) \]

(2.9)

This fact enables us to decompose the Fock space to sectors. We study the sector with \( Q_1 = 1, Q_2 = 0, Q_3 = 0 \). This is identified as a sector containing either one \( V_i \) particle or one \( N_j \) together with one \( \theta_j \) particle. It follows from the commutativity of the fields that the states \( |V_1\rangle, |V_2\rangle \), as well as \( |N_1\theta_1\rangle, |N_2\theta_2\rangle \), which exist in this sector, are orthogonal. In this sector the generator of evolution \( K \) can be rewritten in the form

\[ K = \int d^4p K^p = \int d^4p (K_0^p + V^p) \]

where

\[ K_0^p = \sum_{i=1,2} \left\{ \frac{p^2}{2M_{V_i}} b_i^\dagger(p)b_i(p) + \int d^4k \left( \frac{(p-k)^2}{2M_{V_i}} + \frac{k^2}{2M_{\theta_i}} \right) a_{N_i}^\dagger(p-k) a_{\theta_i}^\dagger(k) a_{\theta_i}(k) a_{N_i}(p-k) \right\} \]

and

\[ V^p = \sum_{i,j=1,2} \int d^4k \left( f_{ij}(k)b_i^\dagger(p)a_{N_j}(p-k)a_{\theta_j}(k) + f_{ij}^*(k)b_i(p)a_{N_j}^\dagger(p-k)a_{\theta_j}^\dagger(k) \right) \]

In this form it is clear that both \( K \) and \( K_0 \) have a direct integral structure. This implies a similar structure for the wave operator \( \Omega_{\pm} \) and the possibility of defining restricted wave operators \( \Omega_{\pm}^p \) for each value of \( p \). We see from the expression for \( K_0^p \) that \( |V_i(p)\rangle = b_i^\dagger(p)|0\rangle \) can be regarded as a set of discrete eigenstates of \( K_0^p \) (for each \( p \)) which span a subspace which is, therefore, annihilated by the restricted wave operators \( \Omega_{\pm}^p \). This implies immediately that \( \Omega_{\pm}|V(p)\rangle = 0 \) for every \( p \) (an explicit demonstration of this fact is given in appendix A).

In order to construct the Lax-Phillips incoming and outgoing spectral representations for the model presented here it is necessary, according to the discussion following equation (1.25), to obtain appropriate expressions for the wave operators \( \Omega_{\pm}^1 \) and the spectral representation for the generator \( K_0 \) of free evolution, i.e, the solution of equation (1.25).

We begin our discussion with a derivation of the appropriate expressions, for the model considered here, of the wave operators \( \Omega_{\pm} \). We first calculate the following matrix elements of \( \Omega_+ \)

\[ \langle V_m(q)|\Omega_+|N_n(p),\theta_n(k)\rangle \quad \langle N_m(p'),\theta_m(k')|\Omega_+|N_n(p),\theta_n(k)\rangle \]
Equation (1.16) can be rewritten, following the standard procedure, in the integral form

\[ \Omega_+ = 1 + i \lim_{\epsilon \to 0} \int_0^{+\infty} U^\dagger(\tau) V U_0(\tau) e^{-\epsilon \tau} d\tau \]  

(2.10)

where \( U(\tau) = e^{-iK\tau} \), \( U_0(\tau) = e^{-iK_0\tau} \). Using (2.7), we have

\[ \Omega_+ |N_n(p_1), \theta_n(p_2)\rangle = |N_n(p_1), \theta_n(p_2)\rangle + i \lim_{\epsilon \to 0} \int_0^{+\infty} d\tau e^{-\epsilon \tau} U^\dagger(\tau) V U_0(\tau) a^\dagger_{N_n}(p) a^\dagger_{\theta_n}(k) |0\rangle \]

\[ = |N_n(p_1), \theta_n(p_2)\rangle + i \lim_{\epsilon \to 0} \int_0^{+\infty} d\tau e^{-i(\omega_{N_n}(p_1) + \omega_{\theta_n}(p_2) - i\epsilon)\tau} U^\dagger(\tau) V a^\dagger_{N_n}(p) a^\dagger_{\theta_n}(k) |0\rangle \]  

(2.11)

where \( \omega_{N_n}(p) = p^2/2M_{N_n} \), \( \omega_{\theta_n}(p) = p^2/2M_{\theta_n} \). Using (2.8) we find

\[ V a^\dagger_{N_n}(p_1) a^\dagger_{\theta_n}(p_2) |0\rangle = \sum_{k=1,2} f_{kn}(p_2) b^\dagger_k(p_1 + p_2) |0\rangle \]  

(2.12)

Inserting (2.12) into (2.11) and changing the integration variable from \( \tau \) to \( -\tau \) it follows that

\[ \Omega_+ |N_n(p_1), \theta_n(p_2)\rangle = \]

\[ = |N_n(p_1), \theta_n(p_2)\rangle - i \lim_{\epsilon \to 0} \sum_{k=1,2} \int_0^{-\infty} d\tau e^{i(\omega_{N_n}(p_1) + \omega_{\theta_n}(p_2) - i\epsilon)\tau} U(\tau) f_{kn}(p_2) b^\dagger_k(p_1 + p_2) |0\rangle \]  

(2.13)

In order to continue with the evaluation of the integral in (2.13) we find the time evolution of some arbitrary state \( \chi \) under the action of \( U(\tau) \)

\[ \psi(\tau) = U(\tau) \chi = e^{-iK\tau} \chi \]  

(2.14)

In the sector of the Fock space that we are considering, the state \( \psi(\tau) \) at any time \( \tau \) can be expanded as

\[ \psi(\tau) = \sum_{i=1,2} \int d^4q \ A_i(q, \tau) b^\dagger_i(q) |0\rangle + \sum_{i=1,2} \int d^4p \ \int d^4k \ B_i(p, k, \tau) a^\dagger_{N_i}(p) a^\dagger_{\theta_i}(k) |0\rangle \]  

(2.15)

In particular, we see that the initial conditions for the evolution in (2.13), where the state \( \chi \) is taken to be \( \psi_0 = \sum_k f_{kn}(p_2) b^\dagger_k(p_1 + p_2) |0\rangle \), are

\[ A_i(q, 0) = f_{in}(p_2) \delta^4(q - p_1 - p_2) \quad B_i(p, k, 0) = 0 \]  

(2.16)

The equations of evolution for the coefficients \( A(q, \tau) \) and \( B(p, k, \tau) \) are then obtained from (2.14) and (2.15), i.e.,
\[
\frac{i}{\tau} \frac{\partial B_i(p-k, k, \tau)}{\partial \tau} = B_i(p-k, k, \tau) \left( \frac{(p-k)^2}{2M_N} + \frac{k^2}{2M_\theta} \right) + \sum_{j=1,2} f_{ji}^*(k) A_j(p, \tau) \\
\frac{i}{\tau} \frac{\partial A_i(p, \tau)}{\partial \tau} = \frac{p^2}{2M_{V_i}} A_i(p, \tau) + \sum_{j=1,2} \int d^4k f_{ij}(k) B_j(p-k, k, \tau)
\]  

(2.17)

These equations can be solved algebraically\textsuperscript{10,19} by performing Laplace transforms and defining

\[
\tilde{B}_i(p, k, z) = \int_{-\infty}^{0} d\tau e^{iz\tau} B_i(p, k, \tau) \quad \text{Im} z < 0 \\
\tilde{A}_i(p, z) = \int_{-\infty}^{0} d\tau e^{iz\tau} A_i(p, \tau) \quad \text{Im} z < 0
\]  

(2.18)

Equations (2.17) are transformed into

\[
\tilde{B}_i(p-k, k, z) \left( z - \frac{(p-k)^2}{2M_N} - \frac{k^2}{2M_\theta} \right) = iB_i(p-k, 0) + \sum_{j=1,2} f_{ji}^*(k) \tilde{A}_j(p, z) \\
\tilde{A}_i(p, z)(z - \frac{p^2}{2M_{V_i}}) = iA_i(p, 0) + \sum_{j=1,2} \int d^4k f_{ij}(k) \tilde{B}_j(p-k, k, z)
\]  

(2.19)

Using the initial conditions (2.16) we obtain the following expressions for the Laplace transformed coefficients

\[
\tilde{A}_k(p, z) = i \sum_{i=1}^{n} W_{ki}^{-1}(z, p) A_i(p, 0) \\
\tilde{B}_j(p-k, k, z) = i \left( \frac{z - (p-k)^2}{2M_{N_j}} - \frac{k^2}{2M_{\theta_j}} \right)^{-1} \left[ \sum_{k, i=1,2}^{n} f_{kj}^*(k) W_{ki}^{-1}(z, p) A_i(p, 0) \right]
\]  

(2.20)

where

\[
W_{ik}(z, p) = \delta_{ik} (z - \frac{p^2}{2M_{V_i}}) - \sum_{j=1,2} \int d^4k \frac{f_{ij}(k) f_{kj}^*(k)}{z - \frac{(p-k)^2}{2M_{N_j}} - \frac{k^2}{2M_{\theta_j}}}
\]  

(2.21)

The Laplace transform of \(\psi(\tau)\) is then

\[
\psi(z) = i \sum_{i,k=1,2} \int d^4q W_{ik}^{-1}(z, q) A_k(q, 0) b_i^\dagger(q) |0\rangle
\]
\[ +i \sum_{i,j,k=1,2} \int d^4p \int d^4k \frac{f_{ki}(k)W_{kj}^{-1}(z,p+k)A_j(p+k,0)}{(z - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta})} a_N^\dagger(p)a_{\theta_i}(k)|0\rangle \quad (2.22) \]

From (2.13), (2.22) and the initial conditions Eq.(2.16) we get

\[ \Omega_+|N_n(p_1), \theta_n(p_2)\rangle = |N_n(p_1), \theta_n(p_2)\rangle + i \sum_{i,k=1,2} W_{ik}^{-1}(\omega_n - i\epsilon, p_1 + p_2) f_{kn}(p_2)b_{i}^\dagger(p_1 + p_2)|0\rangle \]

\[ + i \sum_{i,j,k=1,2} \int d^4k \frac{f_{ki}(k)W_{kj}^{-1}(\omega_n - i\epsilon, p_1 + p_2) f_{jn}(p_2)}{\omega_n - i\epsilon - (p_1 + p_2 - k)^2} a_N^\dagger(p_1 + p_2 - k)a_{\theta_i}(k)|0\rangle \quad (2.23) \]

where we denote \( \omega_n \equiv \omega_{N_n}(p_1) + \omega_{\theta_n}(p_2) \). We can now evaluate the desired matrix elements of the wave operator \( \Omega_+ \). From (2.23) one finds

\[ \langle V_m(p)|\Omega_+|N_n(p_1), \theta_n(p_2)\rangle = \sum_{k=1,2} W_{mk}^{-1}(\omega_n - i\epsilon, p_1 + p_2) f_{kn}(p_2)\delta^4(p - p_1 - p_2) \quad (2.24) \]

and

\[ \langle N_m(\tilde{p}_1), \theta_m(\tilde{p}_2)|\Omega_+|N_n(p_1), \theta_n(p_2)\rangle = \delta_{mn}\delta^4(\tilde{p}_1 - p_1)\delta^4(\tilde{p}_2 - p_2) \]

\[ + \sum_{k,j=1,2} \frac{f_{km}(\tilde{p}_2)W_{kj}^{-1}(\omega_n - i\epsilon, p_1 + p_2) f_{jn}(p_2)}{\omega_n - i\epsilon - \frac{\tilde{p}_1^2}{2M_N} - \frac{\tilde{p}_2^2}{2M_\theta}} \delta^4(p_1 + p_2 - \tilde{p}_1 - \tilde{p}_2) \quad (2.25) \]

In a similar fashion one can find the corresponding matrix elements of the wave operator \( \Omega_- \):

\[ \langle V_m(p)|\Omega_-|N_n(p_1), \theta_n(p_2)\rangle = \sum_{k=1,2} W_{mk}^{-1}(\omega_n + i\epsilon, p_1 + p_2) f_{kn}(p_2)\delta^4(p - p_1 - p_2) \quad (2.26) \]

and

\[ \langle N_m(\tilde{p}_1), \theta_m(\tilde{p}_2)|\Omega_-|N_n(p_1), \theta_n(p_2)\rangle = \delta_{mn}\delta^4(\tilde{p}_1 - p_1)\delta^4(\tilde{p}_2 - p_2) \]

\[ + \sum_{k,j=1,2} \frac{f_{km}^*(\tilde{p}_2)W_{kj}^{-1}(\omega_n + i\epsilon, p_1 + p_2) f_{jn}(p_2)}{\omega_n + i\epsilon - \frac{\tilde{p}_1^2}{2M_N} - \frac{\tilde{p}_2^2}{2M_\theta}} \delta^4(p_1 + p_2 - \tilde{p}_1 - \tilde{p}_2) \quad (2.27) \]

According to eq. (1.22), (1.25) the complete transformation to the incoming or outgoing representations requires us to solve for the (improper) eigenvectors with spectrum \( \{\sigma\} \) on \(( -\infty, +\infty)\) of \( K_0 \). The complete set of these states is decomposed into two subsets corresponding to the quantum numbers for states containing \( N \) and \( \theta \) particles and states
containing a $V$ particle. These quantum numbers are denoted $\sigma, \alpha$ (for the $N+\theta$ states) and $\sigma, \beta$ ($V$ states) respectively. For the projections into these two subspaces, we have

$$|\sigma, \alpha\rangle_0 = \sum_{n=1,2} \int d^4p \int d^4k |N_l(p), \theta_l(k)\rangle \langle N_l(p), \theta_l(k)|\sigma, \alpha\rangle_0$$

$$|\sigma, \beta\rangle_0 = \sum_{m=1,2} \int d^4p |V_m(p)\rangle \langle V_m(p)|\sigma, \beta\rangle_0$$

(2.28)

It is convenient to define

$$O_{n,p,k}^{\sigma,\alpha} \equiv \langle N_n(p), \Theta_n(k)|\sigma, \alpha\rangle_0$$

$$O_{m,p}^{\sigma,\beta} \equiv \langle V_m(p)|\sigma, \beta\rangle_0$$

(2.29)

With the help of these definitions we can rewrite (2.28) as

$$|\sigma, \alpha\rangle_0 = \sum_{n=1,2} \int d^4p \int d^4k O_{n,p,k}^{\sigma,\alpha} |N_l(p), \theta_l(k)\rangle$$

$$|\sigma, \beta\rangle_0 = \sum_{m=1,2} \int d^4p O_{m,p}^{\sigma,\beta} |V_m(p)\rangle$$

(2.30)

It follows from equations (1.25) and (2.30) that

$$K_0|\sigma, \alpha\rangle_0 = \sum_{n=1,2} \int d^4p \int d^4k (\omega_{N_n}(p) + \omega_{\theta_n}(k))O_{n,p,k}^{\sigma,\alpha} |N_n(p), \Theta_n(k)\rangle = \sigma|\sigma, \alpha\rangle_0$$

$$K_0|\sigma, \beta\rangle_0 = \sum_{m=1,2} \int d^4p \omega_{V_m}(p)O_{m,p}^{\sigma,\beta} |V_m(p)\rangle = \sigma|\sigma, \beta\rangle_0$$

(2.31)

From the orthogonality of the final state channels, it follows that we must have

$$O_{n,p,k}^{\sigma,\alpha} = \delta(\sigma - \omega_{N_n}(p) - \omega_{\theta_n}(k))\tilde{O}_{n,p,k}^{\sigma,\alpha}$$

$$O_{m,p}^{\sigma,\beta} = \delta(\sigma - \omega_{V_m}(p))\tilde{O}_{m,p}^{\sigma,\beta}$$

(2.32)

to satisfy the kinematic conditions imposed by eq. (2.31). A more detailed analysis of the structure of the matrix elements (2.32) requires further knowledge regarding the nature of the variables $\alpha, \beta$. We will postpone the discussion of this point to later and remark here only that orthogonality and completeness requires that

$$\sum_{\alpha} \int d\sigma \left( O_{n,p,k}^{\sigma,\alpha} \right)^* O_{n',p',k'}^{\sigma',\alpha'} = \delta^4(p - p')\delta^4(k - k')\delta_{nn'}$$

$$\sum_{n=1,2} \int d^4p \int d^4k \left( O_{n,p,k}^{\sigma,\alpha} \right)^* O_{n,p,k}^{\sigma',\alpha'} = \delta(\sigma - \sigma')\delta_{\alpha,\alpha'}$$

(2.33)
\[
\sum_{\beta} \int d\sigma \ (O_{m,p}^{\sigma,\beta})^* O_{m',p'}^{\sigma,\beta} = \delta^4(p - p')\delta_{mm'}
\]
\[
\sum_{m=1,2} \int d^4p \ (O_{m,p}^{\sigma,\beta})^* O_{m',p'}^{\sigma',\beta'} = \delta(\sigma - \sigma')\delta_{\beta,\beta'}
\]

(2.34)

To complete the transformation to the outgoing spectral representation we have to calculate, according to eq. (1.22), the following quantities

\[
\langle V_m(p)|\Omega_+|\sigma,\beta\rangle_0 \quad \langle N_n(p),\theta_n(k)|\Omega_+|\sigma,\beta\rangle_0
\]

and

\[
\langle V_m(p)|\Omega_+|\sigma,\alpha\rangle_0 \quad \langle N_n(p),\Theta_n(k)|\Omega_+|\sigma,\alpha\rangle_0
\]

From the second of equations (2.31), the discussion following eq. (2.9) and the results of Appendix A, it is clear that the first two transformation matrix elements are identically zero (since $\Omega_+|V(p)\rangle = 0$). We obtain expressions for the second pair with the help of (2.24), (2.25) and (2.32). For the first matrix element in the second pair above we have

\[
\langle V_m(p)|\Omega_+|\sigma,\alpha\rangle_0 = \sum_{n=1,2} \int d^4p_1 \int d^4p_2 \langle V_m(p)|\Omega_+|N_n(p_1),\theta_n(p_2)\rangle \langle N_n(p_1),\theta_n(p_2)|\sigma,\alpha\rangle_0 = \\
= \sum_{n=1,2} \int d^4p_1 \int d^4p_2 \sum_{k=1,2} W_{mk}^{-1}(\omega_n - i\epsilon, p_1 + p_2) f_{kn}(p_2)\delta^4(p - p_1 - p_2)O_{n,p_1,p_2}^{\sigma,\alpha} = \\
= \sum_{k=1,2} W_{mk}^{-1}(\sigma - i\epsilon, p) \sum_{n=1,2} \int d^4p_2 f_{kn}(p_2)O_{n,p-p_2,p_2}^{\sigma,\alpha} = \sum_{k=1,2} W_{mk}^{-1}(\sigma - i\epsilon, p) F_k^\alpha(\sigma, p)
\]

(2.35)

where we have used (2.30) and the definition

\[
F_k^\alpha(\sigma, p) \equiv \sum_{n=1,2} \int d^4p' f_{kn}(p')O_{n,p-p',p'}^{\sigma,\alpha}
\]

(2.36)

For the second matrix element we get in similar way

\[
\langle N_m(p_1),\theta_m(p_2)|\Omega_+|\sigma,\alpha\rangle_0 = \\
= \sum_{n=1,2} \int d^4p'_1 \int d^4p'_2 \langle N_m(p_1),\theta_m(p_2)|\Omega_+|N_n(p'_1),\theta_n(p'_2)\rangle \langle N_n(p'_1),\theta_n(p'_2)|\sigma,\alpha\rangle_0 = \\
= O_{m,p_1,p_2}^{\sigma,\alpha} + \sum_{k,j=1,2} f_{kn}(p_2) W_{kj}^{-1}(\sigma - i\epsilon, p_1 + p_2) F_j^\alpha(\sigma, p_1 + p_2)
\]

(2.37)
Following the same steps we obtain for the matrix elements of the wave operator Ω-

\[ \langle V_m(p) | \Omega_- | \sigma, \alpha \rangle_0 = \sum_{k=1,2} W_{mk}^{-1}(\sigma + i\epsilon, p) F_k^\alpha(\sigma, p) \]  (2.38)

and

\[ \langle N_m(p_1), \theta_m(p_2) | \Omega_- | \sigma, \alpha \rangle_0 = O^\sigma,\alpha_{m,p_1,p_2} + \left( \sigma + i\epsilon - \frac{p_1^2}{2MN_m} - \frac{p_2^2}{2Mq_m} \right)^{-1} \]

\[ \times \sum_{k,j=1,2} f_{km}^* (p_2) W_{kj}^{-1}(\sigma + i\epsilon, p_1 + p_2) F_j^\alpha(\sigma, p_1 + p_2) \]  (2.39)

This completes the calculation of the Lax-Phillips wave operators providing the transformation to the incoming and outgoing (spectral) representations. Given these transformations it is possible in principle to construct the subspaces \( D_{\pm} \) according to the method described in the introduction. We can now calculate the Lax-Phillips S-matrix mapping the incoming representation into the outgoing representation. If this S-matrix satisfies the conditions (a), (b), (c) given in the introduction then there exist incoming and outgoing subspaces \( D_{\pm} \) orthogonal to each other and the Lax-Phillips structure is complete.

From eq. (1.26) we see that the Lax-Phillips S-matrix is given by \( 0 \langle \sigma', \alpha' | S | \sigma, \alpha \rangle_0 \). We can calculate explicitly the Lax-Phillips S-matrix for the model presented here with the help of the following useful expression (valid in the sector of the Fock space in which we are working)

\[ 0 \langle \sigma', \alpha' | S | \sigma, \alpha \rangle_0 = \sum_{m=1,2} \int d^4 p_1 \int d^4 p_2 \langle \sigma', \alpha' | \Omega_+ \rangle_N(p_1, \theta_m(p_2)) \langle N_m(p_1), \theta_m(p_2) | \Omega_- | \sigma, \alpha \rangle_0 \]  (2.40)

Using the expressions obtained for the wave operators Eqs. (2.35), (2.37), (2.38), (2.39) and the definition (2.36) we get

\[ 0 \langle \sigma', \alpha' | S | \sigma, \alpha \rangle_0 = \sum_{m=1,2} \int d^4 p_1 \sum_{k,k'=1,2} W_{mk}^{-1}(\sigma' + i\epsilon, p) F_k^{\alpha'}(\sigma', p) \]

\[ + \delta(\sigma' - \sigma) \delta_{\alpha \alpha'} + \frac{1}{\sigma - \sigma'} + i\epsilon \sum_{k,j} F_{kj}^{\alpha'}(\sigma', p_1) W_{kj}^{-1}(\sigma + i\epsilon, p_1) F_j^\alpha(\sigma, p_1) \]

\[ + \frac{1}{\sigma' - \sigma'} + i\epsilon \sum_{k,j} F_k^\alpha(\sigma, p_1) W_{kj}^{-1}(\sigma' + i\epsilon, p_1) F_j^{\alpha'^*}(\sigma', p_1) \]

\[ + \sum_{m=1,2} \int d^4 p_1 \int d^4 p_2 \left[ \sum_{k,j,k',j'=1,2} f_{km}(p_2) W_{kj}^{-1}(\sigma' + i\epsilon, p_1 + p_2) F_{kj}^{\alpha'^*}(\sigma', p_1 + p_2) \right] \frac{\sigma + i\epsilon - \frac{p_1^2}{2MN_m} - \frac{p_2^2}{2Mq_m}}{\sigma' + i\epsilon - \frac{p_1^2}{2MN_m} - \frac{p_2^2}{2Mq_m}} \]
\[ f^*_{km}(p_2)W^{-1}_{k'j'}(\sigma + i\epsilon, p_1 + p_2)F^\alpha_j(\sigma, p_1 + p_2) \times \frac{f^*_{km}(p_2)W^{-1}_{k'j'}(\sigma + i\epsilon, p_1 + p_2)F^\alpha_j(\sigma, p_1 + p_2)}{\sigma + i\epsilon - \frac{p_1^2}{2M_{NM}} - \frac{p_2^2}{2M_{Nm}}} \] 

(2.41)

The last term in Eq. (2.41) can be put into a simpler form by the following manipulation

\[
\sum_{m=1,2} \int d^4p_1 \int d^4p_2 \left[ \sum_{k,j,k',j'=1,2} \frac{f_{km}(p_2)W^{-1}_{k'j'}(\sigma' + i\epsilon, p_1 + p_2)F^\alpha_{j'}(\sigma', p_1 + p_2)}{\sigma' + i\epsilon - \frac{p_1^2}{2M_{NM}} - \frac{p_2^2}{2M_{Nm}}} \times f^*_{km}(p_2)W^{-1}_{k'j'}(\sigma + i\epsilon, p_1 + p_2)F^\alpha_j(\sigma, p_1 + p_2) \right] = 
\]

\[
= \int d^4p_1 \sum_{k,j,k',j'=1,2} \frac{1}{\sigma - \sigma'} [\delta_{kk'}(\sigma' - \sigma) - W_{kk'}(\sigma' + i\epsilon, p_1) + W_{kk'}(\sigma + i\epsilon, p_1)] 
\times W^{-1}_{k'j'}(\sigma' + i\epsilon, p_1)F^\alpha_{j'}(\sigma', p_1)W^{-1}_{k'j'}(\sigma + i\epsilon, p_1)F^\alpha_j(\sigma, p_1) = 
\]

\[
= - \int d^4p_1 \sum_{k,j,j'=1,2} W^{-1}_{k'j'}(\sigma' + i\epsilon, p_1)F^\alpha_{j'}(\sigma', p_1)W^{-1}_{k'j'}(\sigma + i\epsilon, p_1)F^\alpha_j(\sigma, p_1) 
+ P \frac{1}{\sigma - \sigma'} \int d^4p_1 \sum_{j,j'=1,2} F^\alpha_j(\sigma, p_1)W^{-1}_{j'j}(\sigma' + i\epsilon, p_1)F^\alpha_{j'}(\sigma', p_1) 
- P \frac{1}{\sigma - \sigma'} \int d^4p_1 \sum_{j,j'=1,2} F^\alpha_{j'}(\sigma', p_1)W^{-1}_{jj'}(\sigma + i\epsilon, p_1)F^\alpha_j(\sigma, p_1) \tag{2.42}
\]

where \( P \) stands for the principal part and we have performed a partial fraction decomposition at the second step in (2.41) and used the definition Eq. (2.21) of \( W_{ik}(z, p) \). Combining (2.42) and (2.41) we find for the Lax-Phillips S-matrix *

\[
0(\sigma', \alpha' | S | \sigma, \alpha)_0 = 
\]

\[
\delta(\sigma - \sigma') \left[ \delta_{\alpha\alpha'} - 2\pi i \int d^4p \sum_{k,j} F^\alpha_k(\sigma, p)W^{-1}_{k'j'}(\sigma' + i\epsilon, p)F^\alpha_{j'}(\sigma', p_1) \right] = 
\]

\[
\delta(\sigma - \sigma') \left[ \delta_{\alpha\alpha'} - 2\pi i \int d^4p \sum_{k,j} F^\alpha_{j'}(\sigma, p)W^{-1}_{jj'}(\sigma + i\epsilon, p)F^\alpha_j(\sigma, p) \right] \tag{2.43}
\]

We observe that in eq. (2.43) the quantity \( F^\alpha_{j'}(\sigma, p) \) can be considered, for each fixed value of \( \tilde{n}_0 \), as a vector-valued function on the independent variable \( \sigma \), taking its values in

* In (2.42) we use a partial fraction decomposition of the denominators of the form
\[
(\sigma + i\epsilon - A)^{-1} \times (\sigma' + i\epsilon_2 - A)^{-1} = (\sigma - \sigma' + i(\epsilon_2 - \epsilon_1))^{-1} \times ((\sigma' + i\epsilon_1 - A)^{-1} - (\sigma + i\epsilon_2 - A)^{-1}) = (P(\sigma - \sigma')^{-1} \pm i\pi \delta(\sigma - \sigma')) \times ((\sigma' + i\epsilon_1 - A)^{-1} - (\sigma + i\epsilon_2 - A)^{-1}) = P(\sigma - \sigma')^{-1} \times ((\sigma' + i\epsilon_1 - A)^{-1} - (\sigma + i\epsilon_2 - A)^{-1})
\]
an auxiliary Hilbert space defined by the variables $\alpha$. We write it as (see equations (2.29) and (2.36))

$$F^{\alpha*}(\sigma, p) \equiv (|n_j\rangle_{\sigma, p})^{\alpha}$$  \hspace{1cm} (2.44)

where (for a fixed value of $p$) $(|n_j\rangle_{\sigma, p})^{\alpha}$ is the $\alpha$ component of the vector valued function $|n_j\rangle_{\sigma, p}$. With this notation we have (we suppress the auxiliary Hilbert space variables $\alpha$)

$$S(\sigma) = 1 - 2\pi i \int d^4p \sum_{k,j} |n_j\rangle_{\sigma, p} W^{-1}_{jk}(\sigma + i\epsilon, p)_{\sigma, p} \langle n_k|$$  \hspace{1cm} (2.45)

Further simplification of the expression given here for the $S$-matrix can be achieved by identifying the auxiliary Hilbert space variables $\alpha$. This results in an observation of the direct integral structure of the $S$-matrix on the center of momentum $P$ and the definition of the reduced $S$-matrix $S_P(\sigma)$ for each value of $P$. Another important result is the fact that the requirement that the Lax-Phillips $S$-matrix is an inner function implies that an analysis of its action involves a consideration of only a two dimensional subspace of the auxiliary Hilbert space. These simplifications in the structure of the Lax-Phillips $S$-matrix is the subject of next section.

3. The auxiliary Hilbert space and characterization of the Lax-Phillips $S$-matrix

The auxiliary Hilbert space of the Lax-Phillips representation of the relativistic Lee-Friedrichs model acquires a complete characterization when an exact specification of the variables $\alpha$ in the transformation matrix $O_{n,p,k}^{\sigma,\alpha}$ of equation (2.29) is given. To achieve this goal we proceed in two steps. The first one is to define a new set of independent variables $\{n, p, k\} \rightarrow \{n, P, p_{rel}\}$ by the following linear combination of $p$ and $k$

$$a. \quad P = p + k \quad b. \quad p_{rel} = \frac{M_{\theta_n} p - M_n k}{M_{\theta_n} + M_n}$$  \hspace{1cm} (3.1)

These momentum space variables correspond to the following configuration space variables

$$a. \quad X_{c.m} = \frac{M_n x_1 + M_{\theta_n} x_2}{M_n + M_{\theta_n}} \quad b. \quad x_{rel} = x_1 - x_2$$

From eq. (2.32) we know that

$$O_{n,p,k}^{\sigma,\alpha} = \delta(\sigma - \omega_{N_n}(p) - \omega_{\Theta_n}(k)) \tilde{O}_{n,p,k}^{\sigma,\alpha}$$

This implies that

$$\sigma = \frac{P^2}{2M_n} + \frac{k^2}{2M_{\theta_n}} = \frac{P^2}{2M_n} + \frac{p_{rel}^2}{2\mu_n}$$  \hspace{1cm} (3.2)

where $M_n = M_{N_n} + M_{\theta_n}$ and $\mu = M_{N_n} M_{\theta_n} / (M_{N_n} + M_{\theta_n})$. We take $\sigma$ and $P$ to be independent variables. In this case $p_{rel}^2$ is a dependent variable with a value given by
\[ p_{\text{rel}} = 2\mu_n(\sigma - \frac{P^2}{2M_n}) \]

To complete the set of independent quantum numbers we have to find a complete set of commuting operators that commute with \( p_{\text{rel}}^2 \) and \( P \). Since \( p_{\text{rel}}^2 \) is a Casimir of the Poincaré group on the relative coordinates, we may take for the set of commuting operators the full set of quantum numbers corresponding to the latter three operators. We then have \( \{\sigma, \alpha\} \equiv \{\sigma, n, P, \gamma\} \). It follows from eq. (2.32) and (3.1a) that

\[ O_{n,p,k}^{\sigma,\alpha} \equiv O_{n,p,k}^{\sigma, P, \gamma, i} = \delta(\sigma - p^2/2M_n - k^2/2M_\theta_n)\delta_{n_1}\delta^4(P - p - k)\hat{O}_{n,p,k}^{\sigma, P, \gamma, i} \left| p_{\text{rel}}^2 = 2\mu_n(\sigma - p^2/2M_n) \right| \]

Inserting this into the definition of \( F_k^\sigma(\sigma, p) (\equiv F_k^{\sigma, P, \gamma, i}(\sigma, p)) \), eq. (2.36) we get

\[ F_k^{P, \gamma, i}(\sigma, p) \equiv \delta^4(P - p) \sum_{n=1,2} \int d^4p' f_{kn}(p')\delta_{n_1}\delta(\sigma - (p - p')^2/2M_n) - \frac{p'^2}{2M_n}\hat{O}_{n,p,k}^{\sigma, P, \gamma, i} \left| p_{\text{rel}}^2 = 2\mu_n(\sigma - p^2/2M_n) \right| \]

\[ = \delta^4(P - p) \sum_{n=1,2} \int d^4p_{\text{rel}} f_{kn}(M\theta_n P/M_n - p_{\text{rel}})\delta_{n_1}\delta(\sigma - \frac{P^2}{2M_n} - \frac{p_{\text{rel}}^2}{2\mu_n})\hat{O}_{n,p,k}^{\sigma, P, \gamma, i} \]

We define the following \( P \)-dependent vector valued function

\[ (|n_k\rangle_{\sigma, p})^{\gamma, i} \equiv \sum_{n=1,2} \int d^4p_{\text{rel}} f_{kn}(M\theta_n P/M_n - p_{\text{rel}})\delta_{n_1}\delta(\sigma - \frac{P^2}{2M_n} - \frac{p_{\text{rel}}^2}{2\mu_n})\hat{O}_{n,p,k}^{\sigma, P, \gamma, i} \]

so that \( F_k^{P, \gamma, i}(p, \sigma)^* = \delta^4(P - p)(|n_k\rangle_{\sigma, p})^{\gamma, i} \). When this form of \( F_k^\sigma(p, \sigma) \) is used in eq. (2.43) we get

\[ 0\langle \sigma', \alpha'|S|\sigma, \alpha\rangle_0 = 0\langle \sigma', P', \gamma', i'|S|\sigma, P, \gamma, i\rangle_0 = \delta(\sigma' - \sigma)\delta(P' - P)S_{P'}^{\gamma', i', \gamma, i}(\sigma) \]

where we define the reduced \( S \)-matrix, for a specified value of the center of momentum 4-vector \( P \), to be

\[ S_{P'}^{\gamma', i', \gamma, i}(\sigma) = \left[ 1 - 2\pi i \sum_{k,j=1,2} |n_j\rangle_{\sigma, p}W_{jk}^{-1}(\sigma + i\epsilon, P)_{\sigma, p}|n_k\rangle \right]^{\gamma', i', \gamma, i} \]

The form of \( S_P(\sigma) \) allows for a further simplification. For each value of \( \sigma \) the two vectors \( |n_k\rangle_{\sigma, p} \), \( k = 1,2 \) span a two dimensional subspace of the auxiliary Hilbert space. These vectors are, in general, not orthogonal. We find the orthogonal projection onto the
two dimensional subspace using these non-orthogonal vectors by finding linear combinations, denoted \( \sigma, P \langle F_i \rangle \) such that

\[
\sigma, P \langle F_i | n_j \rangle_{\sigma, P} = \delta_{ij} \tag{3.8}
\]

Denoting the projection operator on the subspace spanned by \( |n_k\rangle_{\sigma, P} \), \( k = 1, 2 \) by \( P_2(\sigma, P) \) we have

\[
P_2(\sigma, P) = \sum_{i=1,2} |n_i\rangle_{\sigma, P} \langle F_i |_{\sigma, P} \tag{3.9}
\]

With this projection we construct the unit operator \( 1_{\sigma, P} \) on the auxiliary Hilbert space and write

\[
1_{\sigma, P} = (1_{\sigma, P} - P_2(\sigma, P)) + P_2(\sigma, P)
\]

Multiplying \( S_P(\sigma) \) of eq. (3.7) by this unit operator from the right we obtain (here, and in the sequel, we suppress reference to the auxiliary Hilbert space variables \( \gamma', i, \gamma, i \))

\[
S_P(\sigma) = 1 - P_2(\sigma, P)
\]

\[
+ \sum_i \left[ |n_i\rangle_{\sigma, P} \langle F_i | - 2\pi i \sum_{k, j} |n_j\rangle_{\sigma, P} W_{jk}^{-1}(\sigma + i\epsilon, P)_{\sigma, P} \langle n_k | n_i \rangle_{\sigma, P} \langle F_i | \right] = \tag{3.10}
\]

\[
= 1 - P_2(\sigma, P) + \sum_{i, j} |n_j\rangle_{\sigma, P} \left[ \delta_{ji} - 2\pi i \sum_k W_{jk}^{-1}(\sigma + i\epsilon, P)_{\sigma, P} \langle n_k | n_i \rangle_{\sigma, P} \right]_{\sigma, P} \langle F_i | \]

We now write the Kronecker delta \( \delta_{ij} \) in the form \( \delta_{ji} = \sum_k W_{jk}^{-1}(\sigma + i\epsilon, P) W_{ki}(\sigma + i\epsilon, P) \) and get

\[
S_P(\sigma) = 1 - P_2(\sigma, P) + \sum_{i, j, k} |n_j\rangle_{\sigma, P} W_{jk}^{-1}(\sigma + i\epsilon, P) \left[ W_{ki}(\sigma + i\epsilon, P) - 2\pi i \langle n_k | n_i \rangle_{\sigma, P} \right]_{\sigma, P} \langle F_i | \] \tag{3.11}

In order to proceed at this point it is necessary to evaluate explicitly the expression \( \sigma, P \langle n_k | n_i \rangle_{\sigma, P} \). Using the definition Eq. (3.5) we obtain

\[
\sigma, P \langle n_k | n_i \rangle_{\sigma, P} = \sum_j \int d^4 p_{rel} \int d^4 p_{rel}' f_{kj}^* (M_{\theta, j} P/M_j - p_{rel}) f_{ij} (M_{\theta, j} P/M_j - p_{rel}')
\]

\[
\times \delta(\sigma - \frac{P^2}{2M_j} - \frac{p_{rel}^2}{2\mu_j}) \delta(\frac{p_{rel}^2}{2\mu_j} - \frac{p_{rel}'^2}{2\mu_j}) \sum_\gamma (\hat{O}_{j, p_{rel}}^\gamma \cdot \gamma) \hat{O}_{j, p_{rel}}^\gamma \]

\[
= \sum_j \int d^4 p_{rel} \delta(\sigma - \frac{P^2}{2M_j} - \frac{p_{rel}^2}{2\mu_j}) f_{kj}^* (M_{\theta, j} P/M_j - p_{rel}) f_{ij} (M_{\theta, j} P/M_j - p_{rel})
\]

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\[ S_P(\sigma) = 1 - P_2(\sigma, P) + \sum_{i,j,k} |n_j\rangle_{\sigma,P} W^{-1}_{jk}(\sigma + i\epsilon, P) W_{ki}(\sigma - i\epsilon, P)_{\sigma,P} \langle F_i | (3.14) \]

The operator valued function \( P_2(\sigma, P) \) defined in eq. (3.9) is a projection operator for each value of \( \sigma \)

\[ P_2(\sigma, P)P_2(\sigma, P) = P_2(\sigma, P) \]

It is, therefore, a bounded positive operator on the real \( \sigma \) axis. In order to characterize \( P_2(\sigma, P) \) we need several definitions and results from operator theory on positive operator valued functions. We give these in the appendix, where we prove that \( P_2(\sigma, P) \) is an outer function \(^{18}\) and that it is actually independent of \( \sigma \), that is

\[ P_2(\sigma, P) = P_{2,P} \quad (3.15) \]

where \( P_{2,P} \) is a projection operator on some fixed two dimensional subspace of the auxiliary space. This proof rests on the properties of \( S_P(\sigma) \) as an inner function. \(^{18}\) We shall assume that the functions \( f_{ij}(k) \) are such that the operator valued function defined by Eq. (3.14) has the appropriate analytic properties in the upper half plane.

In eq. (3.8) and (3.9) the vectors \( |n_i\rangle_{\sigma,P} \) and \( \langle F_i |_{\sigma,P} \) may depend on \( \sigma \), but this dependence is such that the projection operator \( P_2(\sigma, P) \) projects on a fixed two dimensional subspace of the auxiliary space for each and every value of \( \sigma \). Eq. (3.14) can be written then in the form

\[ S_P(\sigma) = 1 - P_{2,P} + \sum_{i,j,k} |n_j\rangle_{\sigma,P} W^{-1}_{jk}(\sigma + i\epsilon, P) W_{ki}(\sigma - i\epsilon, P)_{\sigma,P} \langle F_i | \quad (3.16) \]

and we see that the \( S \)-matrix \( S_P(\sigma) \) acts in a non trivial way only on a two dimensional subspace of the auxiliary space.

We now complete the characterization of the Lax-Phillips \( S \)-matrix \( S_P(\sigma) \). Eq. (3.15) implies that the projection valued function \( P_2(\sigma, P) \) projects the Hilbert space \( L^2(-\infty, +\infty; H) \) on the subspace \( L^2(-\infty, +\infty; H_2) \) of vector valued functions taking their
values in some fixed two dimensional subspace $H_2$ of the auxiliary Hilbert space. We use
again the notation $P_{2,p}$ to denote the projection $P_2(\sigma, P)$ as an operator valued function
projecting on $L^2(-\infty, +\infty; H_2)$, that is

$$P_{2,p} : L^2(-\infty, +\infty; H) \rightarrow L^2(-\infty, +\infty; H_2) \quad (3.17)$$

We denote by $P_{1-2,p}$ the operator projecting on the subspace of functions with a range in
$H \ominus H_2$. We have

$$P_{1-2,p} : L^2(-\infty, +\infty; H) \rightarrow L^2(-\infty, +\infty; H \ominus H_2) \quad (3.18)$$

It is obvious from eq. (3.17),(3.18) that

$$L^2(-\infty, +\infty; H) = P_{2,p}L^2(-\infty, +\infty; H) \oplus P_{1-2,p}L^2(-\infty, +\infty; H) \quad (3.19)$$

In particular, if $U(\tau)$ is the operator of right translation by $\tau$ units then any left translation
invariant subspace $I_H^{-H_2} \subset H_2^1(\Pi)$ can be written as

$$I_H^{-H_2} = P_{2,p}I_H^- \oplus P_{1-2,p}I_H^- \quad (3.20)$$

The translation $U(\tau)$ commutes with the projections $P_{2,p}$, $P_{1-2,p}$ and, since $I_H^{-H_2}$ is a left
translation invariant subspace, we have $U(\tau)I_H^{-H_2} \subset I_H^{-H_2}$. Denoting $I_H^{-H_2} = P_{2,p}I_H^-$ we find

$$U(\tau)I_H^{-H_2} = U(\tau)P_{2,p}I_H^- = P_{2,p}U(\tau)I_H^- \subset P_{2,p}I_H^- = I_H^{-H_2} \quad (3.21)$$

We see that if $I_H^{-H_2}$ is a left translation invariant subspace then $I_H^{-H_2} = P_{2,p}I_H^-$ is a two
dimensional invariant subspace under left translations.

In the Lax-Phillips theory the Lax-Phillips $S$-matrix is an inner function that generates
a left translation invariant subspace from the Hardy class $H_2^1(\Pi)$ (this corresponds to the
stability property of $D_-$). In this case we can write

$$I_H^- = S^{LP}H_2^1(\Pi), \quad (3.22)$$

where $S^{LP}$ is the Lax-Phillips $S$-matrix. From eq. (3.16) we see that in the case of the
two channel relativistic Lee-model we have ($S_P(\sigma)$ is the realization of $S^{LP}$ in terms of an
operator valued function)

$$[S^{LP}, P_{2,p}] = 0 \quad (3.23)$$

From eq. (3.22), (3.23) and the definition of $I_{H_2}^-$ we find that

$$I_{H_2}^- = P_{2,p}I_H^- = P_{2,p}S^{LP}H_2^1(\Pi) = S^{LP}P_{2,p}H_2^1(\Pi) = S^{LP}H_{H_2}^2(\Pi)$$

where $H_{H_2}^2(\Pi) \equiv P_{2,p}H_{H_2}^2(\Pi)$. We can write this result in the form

$$I_{H_2}^- = P_{2,p}S^{LP}P_{2,p}H_{H_2}^2(\Pi) \quad (3.24)$$
From this we see that $P_2, p S^{LP} P_2, p$, when it acts on the Hardy space $H^2_{\mathcal{H}}(\Pi)$, generates a two dimensional left translation invariant subspace. From eq. (3.16) we get (if $A$ is an operator on a Hardy class $H^2_{\mathcal{H}}(\Pi)$ or $H^2_{\mathcal{H}}(\Pi)$ then $T(A)$ is its realization in terms of an operator valued function)

$$T(P_2, p S^{LP} P_2, p) = \delta(\sigma - \sigma') \sum_{i,j,k} |n_j)_{\sigma, p} W^{-1}_{jk}(\sigma + i \epsilon, P) W_{ki}(\sigma - i \epsilon, P)_{\sigma, p} \langle F_i| (3.25)$$

According to eq. (3.24) this immediately implies that the right hand side of eq. (3.25) is an inner function acting on the Hardy space $H^2_{\mathcal{H}}(\Pi)$ consisting of vector valued functions taking their values in some fixed two dimensional subspace of the auxiliary Hilbert space. This observation allows for a complete characterization of the Lax-Phillips $S$-matrix, eq. (3.16). Such an inner function can be represented as a product of a rational inner function containing the poles and zeros of $S_p(\sigma)$ and a factor which is an inner function with non-vanishing determinant. If the latter factor is bounded exponentially, it corresponds to a trivial inner factor and does not change the spectrum of the semigroup. In the following, we consider the case of a purely rational $S$-matrix.

4. The resonant states for a rational $S$-matrix

In this section we shall identify the resonant states of the relativistic two channel Lee-model in the Lax-Phillips outgoing translation representation for the case of a rational $S$-matrix of the form

$$S(\sigma) = 1 + \left( \frac{\text{Res} S(z_1)}{\sigma - z_1} + \frac{\text{Res} S(z_2)}{\sigma - z_2} \right) \quad \text{Im} z_1, \text{Im} z_2 < 0. \quad (4.1)$$

We also have

$$S^\dagger(\sigma) = 1 + \left( \frac{\text{Res} S^\dagger(\overline{z}_1)}{\sigma - \overline{z}_1} + \frac{\text{Res} S^\dagger(\overline{z}_2)}{\sigma - \overline{z}_2} \right) \quad \text{Im} \overline{z}_1, \text{Im} \overline{z}_2 > 0 \quad (4.2)$$

A rational $S$-matrix of this form implies the property, as assumed in the remarks following Eq. (3.15), that $S(\sigma)$ is an inner factor. There are simple conditions, which we shall discuss elsewhere, for which the converse is true, i.e., that an inner function is rational.

In order to identify the resonant states we obtain, in the outgoing translation representation, an expression for the generator of the Lax-Phillips semigroup. We then find the eigenfunctions of this generator. Lax and Phillips then assert that these are the resonant states associated with the poles of the Lax-Phillips $S$-matrix.

The Lax-Phillips semigroup is defined as $Z(\tau) = P_+ U(\tau) P_-$, $\tau > 0$. The generator of the semigroup is given by

$$B = i \lim_{\tau \to 0^+} \frac{Z(\tau) - Z(0)}{\tau} \quad (4.3)$$

In the outgoing translation representation we have
\[
\text{out}(s, \beta | B | s', \beta')_{\text{out}} = i \lim_{\tau \to 0^+} \frac{1}{\tau} \sum_{\gamma, \gamma'} \int d\eta \int d\eta' \!
\]

\[
\left[ \text{out}(s, \beta | P_+ | \eta, \gamma)_{\text{out}} \text{out}(\eta, \gamma | U(\tau) | \eta', \gamma')_{\text{out}} \text{out}(\eta', \gamma' | P_- | s', \beta')_{\text{out}} - \text{out}(s, \beta | P_+ | \eta, \gamma)_{\text{out}} \text{out}(\eta, \gamma | U(0) | \eta', \gamma')_{\text{out}} \text{out}(\eta', \gamma' | P_- | s', \beta')_{\text{out}} \right] \quad (4.4)
\]

In this representation the subspace \( D_+ \) is given by \( L^2(-\infty, +\infty; H) \), i.e. it defined in a simple way by its support properties. Therefore, the operator \( P_+ \), the projection into the subspace \( K \oplus D_- \) is given simply by

\[
\text{out}(s, \beta | P_+ | \eta, \gamma)_{\text{out}} = \Theta(-s) \delta(s - \eta) \delta_{\gamma, \gamma'}
\]

Furthermore, in the outgoing translation representation the evolution is just translation

\[
\text{out}(\eta, \gamma | U(\tau) | \eta', \gamma')_{\text{out}} = \delta(\eta - \tau - \eta') \delta_{\gamma, \gamma'}
\]

Then (4.4) becomes

\[
\text{out}(s, \beta | B | s', \beta')_{\text{out}} =
\]

\[
= i \lim_{\tau \to 0^+} \frac{1}{\tau} [\Theta(-s)_{\text{out}} \text{out}(s - \tau, \beta | P_- | s', \beta')_{\text{out}} - \Theta(-s)_{\text{out}} \text{out}(s, \beta | P_- | s', \beta')_{\text{out}}] \quad (4.5)
\]

We use the fact that the subspace \( D_- \) is given in the incoming translation representation in terms of its support properties. This allows us to write

\[
P_- = \sum_{\gamma} \int d\eta \eta, \gamma_{\text{in}} \Theta(\eta) \langle \eta, \gamma | s'| \sum_{\gamma} \int d\eta \Omega_- | \eta, \gamma f \Theta(\eta) \langle \eta, \gamma | f \Omega^\dagger
\]

In the outgoing translation representation we have

\[
\text{out}(s, \beta | P_- | s', \beta')_{\text{out}} = \sum_{\gamma} \int d\eta \text{out}(s, \beta | \Omega_- | \eta, \gamma f \Theta(\eta) \langle \eta, \gamma | f \Omega^\dagger | s', \beta')_{\text{out}}
\]

\[
= \sum_{\gamma} \int d\eta f(s, \beta | \Omega^\dagger_+ \Omega_- | \eta, \gamma | f \Theta(\eta) \langle \eta, \gamma | f \Omega^\dagger_- | \Omega^\dagger_+ s', \beta')_f
\]

\[
= \sum_{\gamma} \int d\eta f(s, \beta | S | \eta, \gamma | f \Theta(\eta) \langle \eta, \gamma | f S^\dagger | s', \beta')_f
\]

In this expression we would like to represent the scattering operator \( S \) and its adjoint \( S^\dagger \) in the spectral representation. Performing the appropriate Fourier transforms we get
\[
\text{out}(s, \beta)|P-|s', \beta'\rangle_{\text{out}} = \int d\sigma \int d\sigma' \sum_\alpha \int d\eta e^{i\sigma s} S^{\beta, \alpha}(\sigma) e^{-i\eta \sigma} \Theta(\eta) e^{i\eta \sigma'} S^{\dagger}(\sigma')^{\alpha, \beta} e^{-i\sigma' s'}
\]

\[
= \frac{-i}{4\pi^2} \int d\sigma \int d\sigma' \sum_\alpha e^{i\sigma s} \frac{S^{\beta, \alpha}(\sigma) S^{\dagger}(\sigma')^{\alpha, \beta'}}{\sigma - (\sigma' + i\epsilon)} e^{-i\sigma' s'}
\]

(4.7)

The operator valued function \(S(\sigma)\) is analytic in the upper half of the complex \(\sigma\) plane. Its adjoint \(S^{\dagger}(\sigma)\) is analytic in the lower half plane. We assume that \(S(\sigma)\) has the form (4.1) and has two poles in the upper half plane, located at \(z_1\) and \(z_2\). The poles of \(S^{\dagger}(\sigma)\) are thus at \(\overline{z}_1\) and \(\overline{z}_2\). The form of \(S(\sigma)\) and of \(S^{\dagger}(\sigma)\) allows the integrals in (5.7) to be performed by contour integration, according to the various possible signs of \(s\) and \(s'\). The result is (through the rest of this section we suppress in our notation the auxiliary Hilbert space variables)

\[
\text{out}(s)|P-|s'\rangle_{\text{out}} = \Theta(s)\delta(s - s') + \frac{1}{2\pi} \Theta(-s) \times \left[ e^{iz_1 s} \text{Res} S(z_1) \int d\sigma' \frac{S^{\dagger}(\sigma')}{z_1 - (\sigma' + i\epsilon)} e^{-i\sigma' s'} + e^{iz_2 s} \text{Res} S(z_2) \int d\sigma' \frac{S^{\dagger}(\sigma')}{z_2 - (\sigma' + i\epsilon)} e^{-i\sigma' s'} \right] = \\
\Theta(s)\delta(s - s') - i\Theta(-s)\Theta(s') \left[ e^{iz_1 s} \text{Res} S(z_1) S^{\dagger}(z_1) e^{-i\overline{z}_1 s'} \right.

\]

\[
\left. + e^{iz_2 s} \text{Res} S(z_2) S^{\dagger}(z_2) e^{-i\overline{z}_2 s'} \right] + i\Theta(-s)\Theta(-s') \left[ e^{iz_1 s} \text{Res} S(z_1) \left\{ \frac{\text{Res} S^{\dagger}(\overline{z}_1)}{z_1 - \overline{z}_1} e^{-i\overline{z}_1 s'} + \frac{\text{Res} S^{\dagger}(\overline{z}_2)}{z_1 - \overline{z}_2} e^{-i\overline{z}_2 s'} \right\} \right.

\]

\[
\left. + e^{iz_2 s} \text{Res} S(z_2) \left\{ \frac{\text{Res} S^{\dagger}(\overline{z}_1)}{z_2 - \overline{z}_1} e^{-i\overline{z}_1 s'} + \frac{\text{Res} S^{\dagger}(\overline{z}_2)}{z_2 - \overline{z}_2} e^{-i\overline{z}_2 s'} \right\} \right] \quad (4.8)
\]

The Lax-Phillips \(S\)-matrix is analytic in the upper half-plane. Its analytic continuation into the lower half-plane is given by \(S(\sigma) \equiv (S^{\dagger}(\overline{\sigma}))^{-1}\), \(\text{Im} \sigma < 0\). Similarly, \(S^{\dagger}(\sigma)\) is analytic in the lower half-plane and its analytic continuation to the upper half-plane is given by \(S^{\dagger}(\sigma) \equiv (S(\sigma))^{-1}\), \(\text{Im} \sigma > 0\). At any point in the complex plane we have

\[
S(\sigma) S^{\dagger}(\sigma) = S^{\dagger}(\sigma) S(\sigma) = 1 \quad (4.9)
\]

This relation is obtained by analytic continuation and does not imply unitarity of the \(S\)-matrix off the real axis. From (4.2) and (4.9) we have

\[
S(\sigma) \left[ 1 + \left( \frac{\text{Res} S^{\dagger}(\overline{z}_1)}{\sigma - \overline{z}_1} + \frac{\text{Res} S^{\dagger}(\overline{z}_2)}{\sigma - \overline{z}_2} \right) \right] = 1
\]

In the limit as \(\sigma\) goes to \(\overline{z}_1\) or to \(\overline{z}_2\) we then get
\[ S(\sigma) \text{Res} S^\dagger(\bar{z}_1) \simeq A_1(\sigma - \bar{z}_1) \quad \sigma \to \bar{z}_1 \]
\[ S(\sigma) \text{Res} S^\dagger(\bar{z}_2) \simeq A_2(\sigma - \bar{z}_2) \quad \sigma \to \bar{z}_2 \]
\[ \text{Res} S^\dagger(\bar{z}_1) S(\sigma) \simeq \hat{A}_1(\sigma - \bar{z}_1) \quad \sigma \to \bar{z}_1 \]
\[ \text{Res} S^\dagger(\bar{z}_2) S(\sigma) \simeq \hat{A}_2(\sigma - \bar{z}_2) \quad \sigma \to \bar{z}_2 \]

(4.10)

for some fixed (i.e., independent of \( \sigma \)) operators \( A_1, A_2, \hat{A}_1, \hat{A}_2 \). From (4.1) and (4.9) we have

\[
S^\dagger(\sigma) \left[ 1 - \left( \frac{\text{Res} S(z_1)}{\sigma - z_1} + \frac{\text{Res} S(z_2)}{\sigma - z_2} \right) \right] = 1
\]

and in the limit as \( \sigma \) approaches \( z_1 \) or \( z_2 \) we get

\[
S^\dagger(\sigma) \text{Res} S(z_1) \simeq -B_1(\sigma - z_1) \quad \sigma \to z_1
\]
\[
S^\dagger(\sigma) \text{Res} S(z_2) \simeq -B_2(\sigma - z_2) \quad \sigma \to z_2
\]
(4.11)

\[
\text{Res} S(z_1) S^\dagger(\sigma) \simeq -\hat{B}_1(\sigma - z_1) \quad \sigma \to z_1
\]
\[
\text{Res} S(z_2) S^\dagger(\sigma) \simeq -\hat{B}_2(\sigma - z_2) \quad \sigma \to z_2
\]

With the help of equations (4.11) we find that the second term of eq. (4.8) vanishes and the representation of \( P_- \) in the outgoing translation representation to be

\[
\left. \langle out, s, \beta | P_- | s', \beta' \rangle \right|_{out} = \Theta(s)\delta(s - s') - \frac{1}{2\pi} \Theta(-s)
\]

\[
x \left[ e^{iz_1 s} \text{Res} S(z_1) \int d\sigma' \frac{S^\dagger(\sigma')}{\bar{z}_1 - (\sigma' + i\epsilon)} e^{-i\sigma' s'} + e^{iz_2 s} \text{Res} S(z_2) \int d\sigma' \frac{S^\dagger(\sigma')}{\bar{z}_2 - (\sigma' + i\epsilon)} e^{-i\sigma' s'} \right]
\]

\[
= \Theta(s)\delta(s - s')
\]
\[
+ i\Theta(-s)\Theta(-s') \left[ e^{iz_1 s} \text{Res} S(z_1) \left( \frac{\text{Res} S^\dagger(\bar{z}_1)}{\bar{z}_1 - \bar{z}_1} e^{-i\bar{z}_1 s'} + \frac{\text{Res} S^\dagger(\bar{z}_2)}{\bar{z}_1 - \bar{z}_2} e^{-i\bar{z}_2 s'} \right) + e^{iz_2 s} \text{Res} S(z_2) \left( \frac{\text{Res} S^\dagger(\bar{z}_1)}{\bar{z}_2 - \bar{z}_1} e^{-i\bar{z}_1 s'} + \frac{\text{Res} S^\dagger(\bar{z}_2)}{\bar{z}_2 - \bar{z}_2} e^{-i\bar{z}_2 s'} \right) \right]
\]
(4.12)

Inserting (4.12) in (4.5) we get for the generator of the semigroup

\[
\left. \langle out, s, \beta | B | s', \beta' \rangle \right|_{out} =
\]
\[
= i \lim_{\tau \to 0^+} \frac{1}{\tau} \left[ \Theta(-s) \left\langle out, s - \tau, \beta | P_- | s', \beta' \right\rangle_{out} - \Theta(-s) \left\langle out, s, \beta | P_- | s', \beta' \right\rangle_{out} \right] =
\]
\[
= i \lim_{\tau \to 0^+} \frac{1}{\tau} \left\{ \Theta(-s) \left( \Theta(s - \tau)\delta(s - \tau - s') \right) + i\Theta(-s + \tau)\Theta(-s') \left[ e^{iz_1 (s - \tau)} \text{Res} S(z_1) K_1(s') + e^{iz_2 (s - \tau)} \text{Res} S(z_2) K_2(s') \right] \right\}
\]

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\[-\Theta(-s) \left( \Theta(s) \delta(s - s') + i \Theta(-s) \Theta(-s') \left[ e^{iz_1 s} \text{Res} \left( z_1 \right) K_1(s') + e^{iz_2 s} \text{Res} \left( z_2 \right) K_2(s') \right] \right)\]

\[= i \Theta(-s) \Theta(-s') \]

\[\times \lim_{\tau \to 0^+} \frac{1}{\tau} \left[ \left( e^{iz_1 (s - \tau)} - e^{iz_2 s} \right) \text{Res} \left( z_1 \right) K_1(s') + \left( e^{iz_2 (s - \tau)} - e^{iz_2 s} \right) \text{Res} \left( z_2 \right) K_2(s') \right]\]

\[= \Theta(-s) \Theta(-s') \left[ z_1 e^{iz_1 s} \text{Res} \left( z_1 \right) K_1(s') + z_2 e^{iz_2 s} \text{Res} \left( z_2 \right) K_2(s') \right] \quad (4.13)\]

where we have denoted

\[K_1(s') = \frac{\text{Res} \left( z_1 \right)}{z_1 - z_1} e^{-i z_1 s'} + \frac{\text{Res} \left( z_2 \right)}{z_1 - z_2} e^{-i z_2 s'} \]

\[K_2(s') = \frac{\text{Res} \left( z_1 \right)}{z_2 - z_1} e^{-i z_1 s'} + \frac{\text{Res} \left( z_2 \right)}{z_2 - z_2} e^{-i z_2 s'} \quad (4.14)\]

We show that, in the outgoing translation representation, the eigenvectors of the generator \( B \) of the Lax-Phillips semigroup are

\[\psi_1(s) = \Theta(-s) \text{Res} \left( z_1 \right) e^{iz_1 s}, \quad \psi_2(s) = \Theta(-s) \text{Res} \left( z_2 \right) e^{iz_2 s} \quad (4.15)\]

in the sense that a vector in \( \mathcal{H} \) given by \( \psi_\beta(s) = \Theta(-s) e^{iz_1 s} (\text{Res} (z_1))^{\beta'} \beta' \phi' \) or by \( \psi_\beta(s) = \Theta(-s) e^{iz_2 s} (\text{Res} (z_2))^{\beta'} \beta' \phi' \) where \( \phi \in H \), is an eigenvector of the generator of the semigroup. This is achieved by demonstrating that these vectors satisfy the eigenvalue equation

\[\int ds' \text{out} \langle s, \beta | B | s', \beta' \rangle_{\text{out}} \psi_\beta'(s') = z_1,2 \psi_\beta'(s') \quad (4.16)\]

We verify eq. (4.16) for \( \psi_1(s) \). Inserting (4.13) into (4.16) and performing the integration we find for the second term, containing the factor \( z_1 \),

\[\text{Res} \left( z_2 \right) \int_{-\infty}^{0} \left( \frac{\text{Res} \left( z_1 \right)}{z_2 - z_1} e^{-i z_1 s'} + \frac{\text{Res} \left( z_2 \right)}{z_2 - z_2} e^{-i z_2 s'} \right) e^{iz_1 s'} \text{Res} \left( z_1 \right) =\]

\[\frac{i \text{Res} \left( z_2 \right)}{z_2 - z_1} \left( \frac{\text{Res} \left( z_1 \right)}{z_2 - z_1} + \frac{\text{Res} \left( z_1 \right)}{z_1 - z_1} + \frac{\text{Res} \left( z_2 \right)}{z_2 - z_2} + \frac{\text{Res} \left( z_2 \right)}{z_2 - z_1} \right) \text{Res} \left( z_1 \right) =\]

\[\text{Res} \left( z_2 \right) (S^\dagger (z_2) - S^\dagger (z_1)) \text{Res} \left( z_1 \right) = 0 \quad (4.17)\]

and for the first term containing the factor \( z_1 \)

\[\text{Res} \left( z_1 \right) \int_{-\infty}^{0} \left( \frac{\text{Res} \left( z_1 \right)}{z_1 - z_1} e^{-i z_1 s'} + \frac{\text{Res} \left( z_2 \right)}{z_1 - z_2} e^{-i z_2 s'} \right) e^{iz_1 s'} \text{Res} \left( z_1 \right) =\]

\[-i \text{Res} \left( z_1 \right) \left( \frac{\text{Res} \left( z_1 \right)}{(z_1 - z_1)^2} + \frac{\text{Res} \left( z_2 \right)}{(z_1 - z_2)^2} \right) \text{Res} \left( z_1 \right) =\]

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\[ i \text{Res} \left. S(z_1) \frac{dS^\dagger(\sigma)}{d\sigma} \right|_{\sigma = z_1} \text{Res} \left. S(z_1) \right) \tag{4.18} \]

In order to simplify this last expression we need two identities, the first of which is obtained by exploiting the unitarity of \( S(\sigma) \) for real \( \sigma \). Taking the derivative \( d/d\sigma (S^\dagger(\sigma)S(\sigma)) \) we can write

\[ \frac{dS^\dagger(\sigma)}{d\sigma} = -S^\dagger(\sigma) \frac{dS(\sigma)}{d\sigma} S^\dagger(\sigma) \tag{4.19} \]

The second identity is obtained with the help of equation (5.1)

\[ \text{Res} \ S(z_1) = S(\sigma)S^\dagger(\sigma) \text{Res} \ S(z_1) = \left( 1 - \frac{\text{Res} \ S(z_1)}{\sigma - z_1} - \frac{\text{Res} \ S(z_2)}{\sigma - z_2} \right) S^\dagger(\sigma) \text{Res} \ S(z_1) \]

From this identity and eq. (4.10) we get, for small values of \( |\sigma - z_1| \)

\[ \text{Res} \ S(z_1)S^\dagger(\sigma) \text{Res} \ S(z_1) \simeq -\text{Res} \ S(z_1)(\sigma - z_1) \tag{4.20} \]

When eq. (4.19) and (4.20) are used in (4.18) we get

\[ i \text{Res} \left. S(z_1) \frac{dS^\dagger(\sigma)}{d\sigma} \right|_{\sigma = z_1} \text{Res} \left. S(z_1) \right) = (-i) \lim_{\sigma \to z_1} \text{Res} \ S(z_1)S^\dagger(\sigma) \frac{dS(\sigma)}{d\sigma} S^\dagger(\sigma) \text{Res} \ S(z_1) = \]

\[ = -i \lim_{\sigma \to z_1} \text{Res} \ S(z_1)S^\dagger(\sigma) \left( \frac{\text{Res} \ S(z_1)}{(\sigma - z_1)^2} + \frac{\text{Res} \ S(z_2)}{(\sigma - z_2)^2} \right) S^\dagger(\sigma) \text{Res} \ S(z_1) = \]

\[ -i \lim_{\sigma \to z_1} \frac{1}{(\sigma - z_1)^2} \text{Res} \ S(z_1)S^\dagger(\sigma) \text{Res} \ S(z_1)S^\dagger(\sigma) \text{Res} \ S(z_1) = -i \text{Res} \ S(z_1) \tag{4.21} \]

Making use of the results (4.21), (4.17) it is easy to verify Eq. (4.16) for \( \psi_1(s) \). A similar calculation shows that \( \psi_2(s) \) also satisfies Eq. (4.16).

A rational Lax-Phillips S-matrix is a rational, operator valued inner function. Such an operator can be written as\(^{25}\)

\[ S(\sigma) = \frac{\sigma - z_1 P_1}{\sigma - z_1 P_1} \frac{\sigma - z_2 P_2}{\sigma - z_2 P_2} \tag{4.22} \]

where \( P_1 = |n_1\rangle \langle n_1| \) and \( P_2 = |n_2\rangle \langle n_2| \) are projectors on one dimensional subspaces of the auxiliary Hilbert space (we take \( |n_1\rangle \) and \( |n_2\rangle \) to be normalized and that, in general, \( P_1 P_2 \neq 0 \)). This S-matrix can be rewritten in a form corresponding to Eq. (4.1) as

\[ S(\sigma) = 1 + \frac{1}{\sigma - z_1} \left[ (z_1 - \bar{z}_1) P_1 \frac{z_1 - \bar{z}_2 P_2}{z_1 - z_2 P_2} \right] + \frac{1}{\sigma - z_2} \left[ \frac{z_2 - \bar{z}_1 P_1}{z_2 - z_1 P_1} (z_2 - \bar{z}_2 P_2) \right] \tag{4.23} \]

From Eq. (4.23) we we identify the two residues
\[ \text{Res } S(z_1) = (z_1 - \overline{z}_1)P_1(1 + \frac{z_2 - \overline{z}_2}{z_1 - z_2}P_2) \]
\[ \text{Res } S(z_2) = (1 + \frac{z_1 - \overline{z}_1}{z_2 - z_1}P_1)(z_2 - \overline{z}_2)P_2 \] (4.24)

Inserting in (4.24) the expressions for \( P_1 \) and \( P_2 \) in terms of \( |n_1\rangle \) and \( |n_2\rangle \) we find
\[ \text{Res } S(z_1) = (z_1 - \overline{z}_1)|n_1\rangle \langle n_1| + \frac{z_2 - \overline{z}_2}{z_1 - z_2}|n_1\rangle \langle n_2| \langle n_2| \] (4.25)
\[ \text{Res } S(z_2) = (|n_2\rangle + \frac{z_1 - \overline{z}_1}{z_2 - z_1}|n_1\rangle \langle n_2|)(z_2 - \overline{z}_2)\langle n_2| \]

The eigenvectors of the generator \( B \) of the semigroup, which we denote by \( |\chi_1\rangle \) and \( |\chi_2\rangle \), can now be immediately identified, in light of the remarks following Eq. (4.15), from Eq. (4.25).
\[ |\chi_1\rangle = \Theta(-s)(z_1 - \overline{z}_1)|n_1\rangle e^{iz_1 s} \]
\[ |\chi_2\rangle = \Theta(-s)(|n_2\rangle + \frac{z_1 - \overline{z}_1}{z_2 - z_1}|n_1\rangle \langle n_2|)e^{iz_2 s} \] (4.26)

Once the residues of the \( S \)-matrix and the eigenvectors \( |\chi_1\rangle, |\chi_2\rangle \) are given explicitly in Eq. (4.25) and (4.26) we can insert these expressions into Eq. (4.13) to achieve an explicit expression for the generator \( B \) of the semigroup. We find that
\[ B = z_1|\chi_1\rangle \langle \tilde{\chi}_1| + z_2|\chi_2\rangle \langle \tilde{\chi}_2| \] (4.27)
where \( \langle \chi_i|\chi_j\rangle = \delta_{ij} \) and
\[ \langle \tilde{\chi}_1| = a_1|\chi_1\rangle + b_1|\chi_2\rangle \]
\[ \langle \tilde{\chi}_2| = a_2|\chi_1\rangle + b_2|\chi_2\rangle \] (4.28)

with the coefficients \( a_1, b_1, a_2, b_2 \) given by
\[ a_1 = |\langle n_1|n_2\rangle|^2\frac{(z_1 - \overline{z}_2)(\overline{z}_2 - z_2)}{z_1 - z_2} + 1 \]
\[ b_1 = \frac{\overline{z}_2 - z_2}{z_1 - z_2}\langle n_1|n_2\rangle \]
\[ a_2 = \frac{z_2 - \overline{z}_2}{\overline{z}_2 - z_1}\langle n_2|n_1\rangle \]
\[ b_2 = \overline{z}_2 - z_2 \] (4.29)

Eq. (4.27) has the diagonalized form of the Lee-Oehme-Yang-Wu phenomenological Hamiltonian in the subspace of the two resonance channel containing, in this case, the \( K^0 \) and \( \overline{K}^0 \) (or \( K_S, K_L \)) states. One sees from Eqs. (3.12) and (3.13) that the jump function containing the essential parameters of the \( S \)-matrix in this subspace contain the matrix elements \( \{ f_{ij} \} \) of the perturbation. These transition matrix elements coincide in form with the quantities calculated in quantum field theoretical models for the vertex for
neutral $K$ meson decay. The theory that we have given here explains how the neutral $K$ meson corresponds to a state in the quantum mechanical Hilbert space (even though it is relatively short-lived) with an exact semigroup decay law, as seen to high accuracy in experiment\textsuperscript{11}.

5. Discussion and Conclusions

We have shown that the quantum mechanical formulation of Lax-Phillips theory for the description of resonances and decay\textsuperscript{5} can be generalized to a system with a finite discrete set of resonances. If this set of resonances spans the unstable system subspace, the most general form of the $S$-matrix is that of a rational inner function\textsuperscript{25}, treated in detail in Section 4 for the two-dimensional case.

The eigenstates corresponding to the poles of the $S$ matrix are well-defined vectors in the full Hilbert space $\mathcal{H}$, and the left and right eigenvectors are orthogonal with respect to the scalar product of $\mathcal{H}$. They span a two-dimensional subspace of $\mathcal{H}$; the $S$-matrix acts non-trivially on a two dimensional subspace of the auxiliary space $\mathcal{H}$ for each value of the foliation parameter $\sigma$ (independently of $\sigma$). This corresponds to an ideal form of “resonance dominance.”

The relation between the eigenvectors of the generator of the semigroup in the space $\mathcal{H}$ and the vectors spanning the two-dimensional subspace of $\mathcal{H}$ is very simple (see Eq. (4.15)). We are therefore able to construct a model completely within the two dimensional subspace, containing an effective non-Hermitian generator of the semigroup, and a set of vectors in a two dimensional space with scalar products taking the same value as the corresponding vectors in the full space. This two dimensional (in general, $N$-dimensional) space and the generator of the semigroup acting on it coincides with the Lee-Oehme-Yang-Wu model. Moreover, as we have seen in the simple Lee model which we have studied here, the matrix elements of the model Hamiltonian are related to the perturbation formally in the same way as in the framework of the Wigner-Weisskopf model.

Appendix A.

We show that $\Omega_\pm|V_i(\tilde{p})\rangle = 0$ applying the methods used in section 2. The procedure is explicitly performed for $\Omega_+|V_i(\tilde{p})\rangle = 0$. The result for $\Omega_-|V_i(\tilde{p})\rangle = 0$ is obtained in a similar way.

We start with the integral representation of the wave operator (see eq. (2.10))

$$\Omega_+ = 1 + i \lim_{\epsilon \to 0} \int_0^{+\infty} U^\dagger(\tau) V U_0(\tau) e^{-\epsilon \tau} d\tau$$

(A.1)

applying this operator to $|V_i(\tilde{p})\rangle$ we get

$$\Omega_+|V_i(\tilde{p})\rangle = |V(\tilde{p})\rangle + i \lim_{\epsilon \to 0} \int_0^{+\infty} d\tau U^\dagger(\tau) V U_0(\tau) e^{-\epsilon \tau} b^\dagger_i(\tilde{p}) |0\rangle$$

$$= |V_i(\tilde{p})\rangle - i \lim_{\epsilon \to 0} \int_0^{-\infty} d\tau U(\tau) V e^{i(\omega V_i(\tilde{p}) - \epsilon \tau)} b^\dagger_i(\tilde{p}) |0\rangle$$

(A.2)
Where \( \omega_{V_i}(p) = p^2/2M_{V_i} \). As in section 2, we want to evaluate the time evolution in the integral and perform a Laplace transform. The result of the action of the potential operator, given in eq. (2.8) to \( |V_i(p)\rangle \), is

\[
V |V_i(\tilde{\rho})\rangle = V b_i^\dagger(\tilde{\rho}) |0\rangle = \sum_{j=1,2} \int d^4k f_{ij}^*(k) a_{N_j}^\dagger(\rho - k) a_{B_j}^\dagger(k) |0\rangle \tag{A.3}
\]

A general form of a state in the sector of the Fock space with \( Q_1 = 1, Q_2 = 0 \) is given in eq. (2.20). From eq. (A.3) we find, at time \( \tau = 0 \),

\[
A_j(q, 0) = 0 \quad B_j(p, k, 0) = f_{ij}^*(k) \delta^4(\rho - p - k) \tag{A.4}
\]

Defining the Laplace transformed coefficients \( A_j(q, z) \) and \( B_j(p, k, z) \) as in eq. (2.18), we use eq. (2.19) and the fact that in eq. (A.4) \( A_j(q, 0) = 0 \) to obtain

\[
\tilde{A}(p, z)(z - \frac{p^2}{2M_{V_i}}) = \sum_{j=1,2} \int d^4k f_{ij}^\dagger(k) B_j(p - k, k, z)
\]

\[
\tilde{B}(p - k, k, z)(z - \frac{(p - k)^2}{2M_{N_i}} - \frac{k^2}{2M_{\theta_i}}) = iB_i(p - k, k, 0) + \sum_{j=1,2} f_{j\dagger}(k) A_j(p, z) \tag{A.5}
\]

Solving for \( \tilde{A}(p, z) \) we get

\[
\tilde{A}(p, z) = i \sum_{i=1,2} W_{li}^{-1}(z, p) \sum_{j=1,2} \int d^4k \frac{f_{ij}B_j(p - k, k, 0)}{z - \frac{(p - k)^2}{2M_{N_i}} - \frac{k^2}{2M_{\theta_i}}}
\]

\[
\tilde{B}(p - k, k, z) = \left( z - \frac{(p - k)^2}{2M_{N_i}} - \frac{k^2}{2M_{\theta_i}} \right)^{-1} \left[ iB_i(p - k, k, 0) + \sum_{j=1,2} f_{j\dagger}(k) A_j(p, z) \right] \tag{A.6}
\]

Inserting the initial condition for \( B_i(p - k, k, 0) \) from eq. (A.4) in eq. (A.6) we have

\[
\tilde{A}(p, z) = i \left[ W_{li}^{-1}(p, z) \left( z - \frac{p^2}{2M_{V_i}} \right) - \delta_{li} \right] \delta^4(p - \rho)
\]

\[
\tilde{B}(p - k, k, z) = \left( z - \frac{(p - k)^2}{2M_{N_i}} - \frac{k^2}{2M_{\theta_i}} \right)^{-1} \sum_{j=1,2} i f_{j\dagger}(k) W_{ji}^{-1}(p, z) \left( z - \frac{p^2}{2M_{V_i}} \right) \delta^4(p - \rho) \tag{A.7}
\]

Performing the Laplace transform of eq. (2.15) implied by eq. (A.2), we use the coefficients from eq. (A.7) and evaluate the resulting expression at the point \( z = \omega_{V_i}(\rho) + i\epsilon = \tilde{\rho}^2/2M_{V_i} - i\epsilon \). This procedure gives the simple answer

\[
\lim_{\epsilon \to 0} \int_0^{\infty} d\tau U(\tau)V e^{i(\omega_{V_i}(\rho) - i\epsilon)} b_i^\dagger(\tilde{\rho}) |0\rangle = -ib_i^\dagger(\tilde{\rho}) |0\rangle = -i|V_i(\rho)\rangle \tag{A.8}
\]
and this implies the desired result.

**Appendix B.**

We prove here that the value taken by the projection valued function \( P_{2,P}(\sigma) \) is actually a projection operator, for all values of \( \sigma \), on a fixed two dimensional subspace of the auxiliary Hilbert space of the Lax-Phillips representation of the relativistic Lee-Friedrichs model, this projection operator is denoted \( P_{2,P} \), i.e., we prove that

\[
P_{2,P}(\sigma) = P_{2,P}
\]

We start with the observation made at the beginning of section 4 (see Eq. (4.1) and the discussion following it) that the operator valued function \( P_{2,P}(\sigma) \), defined in eq. (3.14), is a projection operator for each value of \( \sigma \)

\[
P_{n,P}(\sigma)P_{n,P}(\sigma) = P_{n,P}(\sigma)
\]

It is, therefore, a bounded positive operator on the real \( \sigma \) axis.

In order to proceed we need several definitions and results from the theory of operator valued functions. We denote the upper half plane of the complex \( \sigma \) plane by \( \Pi \). If \( b \) is some separable Hilbert space, we denote by \( B(b) \) the set of bounded linear operators on \( b \). We define the following sets of \( B(b) \) valued functions

**Definition A:**

(i) A holomorphic \( B(b) \) valued function \( f(\sigma) \) on \( \Pi \) is of bounded type on \( \Pi \) if \( \log^+ |f(\sigma)|_{B(b)} \) has a harmonic majorant on \( \Pi \). The class of all such functions is denoted \( N_{B(b)}(\Pi) \).

(ii) If \( \phi \) is any strongly convex function, then by \( H_{\phi,B(b)}(\Pi) \) we mean the class of all holomorphic \( B(b) \) valued functions \( f(\sigma) \) on \( \Pi \) such that \( \phi(\log^+ |f(z)|_{B(b)}) \) has a harmonic majorant on \( \Pi \).

(iii) We define \( N_{B(b)}^+(\Pi) = \bigcup H_{\phi,B(b)}(\Pi) \), where the union is over all strongly convex functions \( \phi \).

(iv) By \( H_{B(b)}^\infty(\Pi) \) we mean the set of all bounded holomorphic \( B(b) \) valued functions on \( \Pi \).

Here \( \log^+ t = \max(\log t, 0) \) for \( t > 0 \) and \( \log 0 = -\infty \). The sets \( N_{B(b)} \) and \( N_{B(b)}^+ \) are called Nevanlinna classes and \( H_{\phi,B(b)}(\Pi) \) is a Hardy-Orlicz class.

We will need the following theorems and definitions:

**Theorem A:** The following

\[
H_{B(b)}^\infty(\Pi) \subseteq H_{\phi,B(b)}(\Pi) \subseteq N_{B(b)}^+(\Pi) \subseteq N_{B(b)}(\Pi)
\]

is a valid sequence.

**Definition B:** Let \( u, v \) be nonzero scalar valued functions in \( N^+(R) \) \( (N^+(R) \) is the boundary function for a scalar Nevanlinna class function). A \( B(b) \)-valued function \( F \) on \( R \) is of class \( M(u_i, v_i) \) if \( uF, vF^* \in N_{B(b)}^+(R) \).

**Definition C:** If \( A \in H_{B(b)}^\infty(\Pi) \) then:
(i) A is an inner function if the operator
\[ T(A): f \rightarrow Af, \quad f \in H^2_b(\Pi) \]
is a partial isometry on \( H^2_b(\Pi); \)
(ii) A is an outer function if
\[ \bigcup \{ Af: f \in H^2_b(\Pi) \} = H^2_M(\Pi) \]
for some subspace \( M \) of \( b. \)

The main theorem which we will apply here is the following:

**Theorem B:** Let \( v \) be any nonzero scalar function in \( N^+(R) \). If \( F \) is any nonnegative \( B(b) \)-valued function of class \( \mathcal{M}(v, v) \) on \( R \) then
\[ F = G^*G \]
on \( R \), where \( G \) is an outer function of class \( \mathcal{M}(1, v) \) on \( R \). The factorization of \( F \) is essentially unique.

As we have remarked above, we have assumed that the functions \( f_{ij}(k) \) of the Lee model are such that \( S_P(\sigma) \) is an inner function. Since \( P_{n,P}(\sigma) \) is a bounded operator then, from definition A(iv), the relation (3.14) and Theorem A, we see that
\[ P_{n,P}(\sigma) \in N^+_{B(H)}(\Pi). \]

where \( H \) is the auxiliary Hilbert space of the Lax-Phillips representation of the relativistic Lee-Friedrichs model, defined by the variables \( \gamma \) in eq. (3.3) (or eq. (3.5),(3.6)). Furthermore, the projection operator \( P_{2,P}(\sigma) \) satisfies \( (P_{2,P}(\sigma))^* = P_{2,P}(\sigma) \) and, from definition B we immediately have
\[ P_{2,P}(\sigma) \in \mathcal{M}(1, 1) \]
We can apply theorem B with the result that there is a unique decomposition of \( P_{2,P}(\sigma) \)
\[ P_{2,P}(\sigma) = G^*G = (P_{2,P}(\sigma))^*P_{2,P}(\sigma) = P_{2,P}(\sigma)P_{2,P}(\sigma) \]
and that \( G = P_{2,P}(\sigma) \) is an outer function. We denote by \( P \) the operator on \( H^2_H(\Pi) \) for which the realization is the operator valued function \( P_{2,P}(\sigma) \). From definition C(ii) we therefore have
\[ \bigcup \{ Pf: f \in H^2_H(\Pi) \} = H^2_M(\Pi) \quad (B.2) \]
where \( M \) is a subspace of the auxiliary Hilbert space \( H \).

Now \( P_{2,P}(\sigma) \) is a projection operator for each value of \( \sigma \). We have that the range of \( P_{2,P}(\sigma) \) is a two dimensional subspace of the auxiliary Hilbert space \( H \) for each \( \sigma \). We denote \( M(\sigma) = \text{Im} P_{2,P}(\sigma) \). Define
\[ \bar{M} = \sum_{\sigma} M(\sigma) \]

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For each vector valued function $f \in H^2_H(\Pi)$ we have
\[ P_{2,P}(\sigma)f(\sigma) \in \hat{M}. \] (B.3)

Furthermore, there is no subspace of $\hat{M}$ that has the property (B.3). Since $(Pf)(\sigma) = P_{2,P}(\sigma)f(\sigma) \in \hat{M}$ for $f \in H^2_H(\Pi)$ we must have
\[ \bigcup Pf : f \in H^2_H(\Pi) \bigg\} = H^2_{\hat{M}}(\Pi) \] (B.4)

We conclude that $\hat{M}$ must be a two dimensional subspace of the auxiliary Hilbert space $H$. If it has a higher dimension we consider two different values of $\sigma$, say $\sigma_1 \neq \sigma_0$ and $\sigma_0$ such that $P_{2,P}(\sigma_1) \neq P_{2,P}(\sigma_0)$. We then take a vector $v_0 \in P_{2,P}(\sigma_0)H$, $v_0 \in (P_{2,P}(\sigma_1)H)^\perp$, a scalar valued Hardy class function $g(\sigma) \in H^2(\Pi)$ and construct the vector valued function $j(\sigma) = g(\sigma)v_0$. Clearly, $j \in H^2_{\hat{M}}(\Pi)$ (where we denote by $j$ the vector valued function taking the value $j(\sigma)$ at the point $\sigma$) but $j \notin \bigcup Pf : f \in H^2_H(\Pi)$, since for any $f \in H^2_H(\Pi)$ we have $j(\sigma_1) = g(\sigma_1)v_0 \perp (Pf)(\sigma_1) = P_{2,P}(\sigma_1)f(\sigma_1)$. Therefore we have
\[ \{ Pf : f \in H^2_H(\Pi) \bigg\} \subset H^2_{\hat{M}}(\Pi) \] (B.4)
and we have a contradiction with eq. (B.3).

Since Dim $\hat{M} = 2$ we must have $P_{2,P}(\sigma) = P_{2,P}(\sigma')$ for arbitrary $\sigma$ and $\sigma'$ and we may write
\[ P_{2,P}(\sigma) = P_{2,P} \]

where $P_{2,P}$ is a projection operator on some fixed (independent of $\sigma$) two dimensional subspace of $H$, which is the desired result.

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