ON AN INITIAL AND FINAL VALUE PROBLEM FOR FRACTIONAL NONCLASSICAL DIFFUSION EQUATIONS OF KIRCHHOFF TYPE

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Dedicated to Tomás Caraballo on his 60th birthday.

Abstract. We study for nonlinear Kirchhoff’s model of pseudo parabolic type by considering its two different problems.
• For initial value problem, we obtain the results on existence and regularity of solutions. Moreover, we also prove that the solutions $u$ corresponding with $\beta < 1$ of the problem convergence to $u$ for $\beta = 1$.
• For final value problem, we show that the ill-posed property in the sense of Hadamard is occurring. Using the Fourier truncation method to regularize the problem. We establish some stability estimates in the $H^1$ and $L^p$ norms under some a-priori conditions on the sought solution.

1. Introduction. We are interested in considering the following nonlinear pseudo parabolic with Kirchhoff’s model type (NPPK for short)

$$u_t - a\Delta u_t + M \left( \|\nabla u\|_{L^2(\Omega)} \right) (-\Delta)\beta u = F(x,t) \quad \text{in} \; \Omega \times (0,T),$$

(1)

and the Dirichlet boundary condition

$$u(x,t) = 0, \quad \text{on} \; \partial\Omega \times (0,T),$$

(2)

by adding some initial conditions

(Initial conditions): $u(x,t) = f(x), \quad \text{in} \; \Omega \times \{0\},$

(3)

or

(Final conditions): $u(x,t) = g(x), \quad \text{in} \; \Omega \times \{T\}.$

(4)

Here, $T, \beta$ are given positive numbers; $\Omega \subset \mathbb{R}^N, N \geq 1$ be a bounded and connected domain with a smooth boundary $\partial\Omega$. The functions $f, g$ are given and the source term $F$ is defined later. Function $M \in C^1(\mathbb{R})$ satisfies (H1)-(H3) below. Equation (1) is a mathematical model that has many applications in the study of population dynamics of biological species. For example, one might describe the population density of a naturally occurring population of organisms (e.g., viruses, bacteria). The diffusion coefficient $M$ expresses the dependence on the global population density.

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in the environment instead of expressing the density at a local location, that is, the information considered global state of a required coefficient. From here, we denote the problem (1)-(2)-(3) by \((P_f)\) and the problem (1)-(2)-(4) by \((P_g)\).

In recent studies, the problem of non-local diffusion has become a topic of concern. A typical case of this type is the “Kirchhoff” model with non-local diffusion coefficients such as equation 1, which has many important applications in phenomena such as nonlinear elasticity, Image recovery, electrophoresis fluid, biology, and ecology,... (see [1,3,4,9,13,26,29,31] and references therein).

The initial value problems (IVPs) with the type of nonlocal diffusion are quite classic and has been studied in recent decades. More specific,

• In [8], T. Caraballo et al. studied the following problem

\[
\begin{aligned}
&u_t - M(\mathcal{L}(u))\Delta u = F(u) + G(t), \quad \text{in} \quad \Omega \times (\delta, +\infty), \\
u = 0, \quad &\text{on} \quad \partial\Omega \times (\delta, +\infty), \\
u = u_\delta, \quad &\text{in} \quad \Omega \times \{\delta\}.
\end{aligned}
\]

Here \(0 < M_0 \leq M(s) \leq M_1, \forall s \in (-\infty, \infty);\) the coefficient \(M\) satisfies the locally Lipschitz conditions (continuous and depends on the global population density). In this work, the results of asymptotic behavior of the solutions to equation (5) are considered. Next, in [11], T. Caraballo et al. replaced the Laplace \(\Delta\) operator with a more complex operator \(p\)-laplacian \(\Delta_p, \ p \geq 2\) in equation (5). For this equation, the authors studied on the existence of attractors in \(L^2(\Omega)\) and \(L^p(\Omega)\). An extended form of the problem (5) has also been interested in researching by T. Caraballo on the following equation [7,9,10]

\[
u_t - (1 - \epsilon)M(\mathcal{L}(u))\Delta u = F(u) + \epsilon G(t), \quad \text{in} \quad \Omega \times (\delta, +\infty).
\]

It is known as the perturbed nonautonomous nonlocal reaction-diffusion equation. Under appropriate assumptions, as the parameter \(\epsilon\) tends to zero, the authors pointed out that the convergence of pullback attractors to global compact attractor.

• M. Gobbino [22] concerned the following equation

\[
\begin{aligned}
&u_t + M\left(\left\|\left(-L\right)^{\frac{1}{2}} u\right\|_H^2(t)\right)(-L)u = 0, \quad t \in (0, +\infty), \\
u(0) = u_0 \in H.
\end{aligned}
\]

where the operator \((-L)\) is a linear, non-negative and self-adjoint on the Hilbert space \(H\). Based on suitable conditions on \(u_0\) and the hypotheses on \(M\), the author has shown that the problem (7) has at least or does not exist a global solution.

• There are many interesting results on the (IVP) for parabolic equations with nonlocal term (Kirchhoff type) such as [6,12,15–19,19–21,24,25,27,28,30,35] and references therein.

For considering the final value problems (FVPs), in [33], Tuan et al. investigated the problem:

\[
u_t = M(\left\|\nabla u\right\|_{L^2(\Omega)}\Delta u + F(x, t; u), \quad \text{in} \quad \Omega \times (0, T),
\]

accompanied with the conditions

\[
\begin{aligned}
u(x, t) = 0, \quad &\text{on} \quad \partial\Omega \times (0, T), \\
u(x, t) = g(x), \quad &\text{in} \quad \Omega \times \{T\}.
\end{aligned}
\]

They have proposed the regularization methods as Fourier truncation and Quasi-reversibility to establish the regularized problems. Moreover the authors given the
stability estimates in \( H^1 \) norm. See also the results of Tuan et al. \[2,32,34\] on the nonlocal problems.

Up to now, as far as we know, there has not been work involving pseudo parabolic with the nonlocal diffusion term \( \mathcal{M} \) in form of \( \mathcal{M}(\|\nabla u\|_{L^2(\Omega)}(t)) \) (as Problem (1)). The solution of problem (1) can be represented by the nonlinear integral equation. However, this equation actually causes a complexity because the diffusion coefficient \( \mathcal{M} \) contains the gradient term of \( u \). Our new results and main contributions in this paper are described as follows.

- For (IVP), the Problem (\( \mathcal{P}_f \)) becomes much more difficult caused by the diffusion coefficient \( \mathcal{M} \) and operator \((-\Delta)^\beta\), we set up the Sobolev embeddings to obtain the existence and regularity of solutions. Moreover, we prove that the solutions \( u_\beta (\beta < 1) \) of (\( \mathcal{P}_f \)) convergence to \( u (\beta = 1) \).
- For (FVP), an interesting thing happened, for \( \beta < 1 \), the problem (\( \mathcal{P}_g \)) is well-posedness, we prove the existence of the solution in an interpolation space. For \( \beta > 1 \), the problem is ill-posed in the sense of Hadamard, so the regularization methods are required to regularize Problem (\( \mathcal{P}_g \)). Using the truncation regularization method to regularize Problem (\( \mathcal{P}_g \)) with the help of the Sobolev embeddings, we establish the new error estimates in \( H^1 \) and \( L^p \)-norms.

The paper is designed as follows

* Section 2. Preliminaries: abstract framework.
* Section 3. Consider the (IVP): we proposed the existence, uniqueness and regularity results of the solution to Problem (\( \mathcal{P}_f \)).
* Section 4. Consider the (FVP): we propose the formulation solution of the Problem (\( \mathcal{P}_g \)). For \( \beta < 1 \), we prove the existence and uniqueness of a mild solution to Problem (\( \mathcal{P}_g \)) in an interpolation space. For \( \beta > 1 \), we prove the ill-posedness of the Problem (\( \mathcal{P}_g \)) and we apply the truncation regularization method to regularize the problem. The errors analysis in \( H^1 \) and \( L^p \) norms have been investigated.

2. Preliminaries. Let the operator \( \mathcal{A} = -\Delta \) on \( \mathbb{V} := H^1(\Omega) \cap H^2(\Omega) \), and assume that \( \mathcal{A} \) has the eigenvalues \( \lambda_n \) such that \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \) which approach \( \infty \) as \( n \) goes to \( \infty \), and \( \lambda_n \geq C_n 2^N \) for all \( n \geq 1 \). Denoted by \( e_n \in \mathbb{V} \) is the corresponding eigenfunctions. For all \( k \geq 0 \), we define by \( \mathcal{A}^k \) (fractional powers of \( \mathcal{A} \)) as the following operator

\[
\begin{aligned}
\mathcal{A}^k e_n(x) &= \lambda_n^k e_n(x), \quad x \in \Omega, \\
e_n(x) &= 0, \quad x \in \partial\Omega, \quad k > 0, \; n \in \mathbb{N}^*.
\end{aligned}
\]  

The notation \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L^2(\Omega) \) and \( (u(t), e_n) \) denotes by \( u_n(t) \). The notation \( \| \cdot \|_B \) indicates for the norm in the Banach space \( B \). We denote the Banach space \( L^p(0, T; B) \), \( p \geq 1 \) of real-valued measurable functions \( v: (0, T) \to B \) corresponding to the norm

\[
\|v\|_{L^p(0, T; B)} = \left( \int_0^T \|v(t)\|^p_B \, dt \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty,
\]  

\[
\|v\|_{L^\infty(0, T; B)} = \sup_{0 < t < T} \|v(t)\|_B, \quad \text{for } p = \infty.
\]
Denoted by $C^m([0, T]; B)$, $0 \leq m \leq \infty$ the space of continuous functions (including time derivatives of order less than or equal to $m$) and its norm is denoted by

$$\|v\|_{C^m([0, T]; B)} = \sum_{i=0}^{m} \sup_{t \in [0, T]} \|v^{(i)}(t)\|_B < \infty.$$  \hspace{1cm} (13)

For $s \geq 0$, we define the following space (as the type of Hilbert scale)

$$D(A^s) = \left\{ L^2(\Omega) \ni v = \sum_{n=1}^{\infty} e_n(x) (v, e_n) \text{ s.t. } \sum_{n=1}^{\infty} (v, e_n)^2 \lambda_n^{2s} < \infty \right\},$$

which is endowed with the norm $\|v\|_{D(A^s)} = \left( \sum_{n=1}^{\infty} (v, e_n)^2 \lambda_n^{2s} \right)^{\frac{1}{2}}$. Obviously, we have $D(A^0) \equiv L^2(\Omega)$ if $s = 0$ and $D(A^s) \equiv W^{1,2}_{0}(\Omega)$. We denote by $D(A^{-s})$ is a Hilbert space with respect to the norm

$$\|v\|_{D(A^{-s})} = \left( \sum_{n=1}^{\infty} (v, e_n)_{s} \lambda_n^{-s} \right)^{\frac{1}{2}},$$  \hspace{1cm} (14)

for $v \in D(A^{-s})$ where $-s \langle \cdot, \cdot \rangle_s$ is the dual product between $D(A^{-s})$ and $D(A^s)$. We note that

$$-s (v_1, v_2)_s = (v_1, v_2), \text{ for } v_1 \in L^2(\Omega), v_2 \in D(A^{-s}).$$

We introduce the Gevrey space containing a class of functions of index $\zeta_1, \zeta_2 > 0$ (see [5]), is defined by

$$G_{\zeta_1, \zeta_2}(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} (v, e_n)^2 \lambda_n^{2\zeta_1} e^{2\zeta_2 \lambda_n} < \infty \right\},$$  \hspace{1cm} (15)

and associated with the following norm

$$\|v\|_{G_{\zeta_1, \zeta_2}(\Omega)} = \left( \sum_{n=1}^{\infty} (v, e_n)^2 \lambda_n^{2\zeta_1} e^{2\zeta_2 \lambda_n} \right)^{\frac{1}{2}}.$$

We need to establish the following hypotheses:

*(Hyp1)*: The functions $M$ are measurable and non-negative such that the mapping

$$\nu \mapsto M(\nu),$$

is continuous for $\nu \in \mathbb{R}$;

*(Hyp2)*: There are positive constants $M_0, M_1$ such that

$$M_0 \leq M(\nu) \leq M_1, \text{ for all } \nu \in \mathbb{R};$$  \hspace{1cm} (H2)

*(Hyp3)*: Assume that $M(0) = 0$ and there exist positive constant $K_M$ such that

$$|M(\nu_1) - M(\nu_2)| \leq K_M |\nu_1 - \nu_2|.$$  \hspace{1cm} (H3)

In the next lemmas, we present some useful embeddings between the spaces mentioned above.
Lemma 2.1. (See [23]) For $N \geq 1$, and $-\frac{N}{4} < \beta \leq 0 \leq \alpha < \frac{N}{4}$, we have the following embeddings hold:

\[
\begin{align*}
L^p(\Omega) &\hookrightarrow D(A^\beta), \quad \text{if} \quad -\frac{N}{4} < \beta \leq 0, \quad p \geq \frac{2N}{N - 4\beta}, \\
D(A^\alpha) &\hookrightarrow L^p(\Omega), \quad \text{if} \quad 0 \leq \alpha < \frac{N}{4}, \quad p \leq \frac{2N}{N - 4\alpha}.
\end{align*}
\]

(16)

We are now in a position to consider the main results of the paper. The results are achieved on Problem $(P_f)$ (known as well-posed problems) and Problem $(P_g)$ (widely known as ill-posed problems).

3. The initial value problem. Consider the following problem

\[
\begin{align*}
\begin{cases}
\frac{du}{dt} - a\Delta u + M(\|\nabla u\|_{L^2(\Omega)}) (-\Delta)^\beta u = F(x,t) & \text{in } \Omega \times (0,T), \\
u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\
u(x,0) = f(x), & \text{in } \Omega.
\end{cases}
\end{align*}
\]

(\(P_f\))

We express the solution \(u\) in the form of Fourier series as \(u(x,t) = \sum_{n=1}^{\infty} u_n(t) e_n(x)\), with \(u_n(t) = (u(\cdot,t), e_n)\). Then from which we arrive at the following ordinary differential equations

\[
u_n(t) + a\lambda_n u_n(t) + \lambda_n^\beta M(\|\nabla u\|_{L^2(\Omega)}) u_n(t) = F_n(t), \quad u_n(0) = f_n = (f, e_n),
\]

(17)

where \(F_n(t) = \langle F(\cdot,t), e_n \rangle\). The equation is equivalent to the following equation

\[
u_n(t) + \lambda_n^\beta (1 + a\lambda_n)^{-1} M(\|\nabla u\|_{L^2(\Omega)}) u_n(t) = F_n(t)(1 + a\lambda_n)^{-1}.
\]

(18)

Solving the latter equation, we get that

\[
u_n(t) = \exp\left(-\frac{\lambda_n^\beta}{1 + a\lambda_n} \int_0^t \mathcal{M}(\|\nabla u(\cdot,s)\|_{L^2(\Omega)}) ds\right) u_n(0)
\]

\[+
\frac{1}{1 + a\lambda_n} \int_0^t \exp\left(-\frac{\lambda_n^\beta}{1 + a\lambda_n} \int_s^t \mathcal{M}(\|\nabla u(\cdot,\eta)\|_{L^2(\Omega)}) d\eta\right) F_n(s) ds.
\]

(19)

This yields immediately that

\[
u(x,t)
\]

\[= \sum_{n=1}^{\infty} \exp\left(-\frac{\lambda_n^\beta}{1 + a\lambda_n} \int_0^t \mathcal{M}(\|\nabla u(\cdot,s)\|_{L^2(\Omega)}) ds\right) u_n(0) e_n(x)
\]

\[+
\sum_{n=1}^{\infty} \left\{\frac{1}{1 + a\lambda_n} \int_0^t \exp\left(-\frac{\lambda_n^\beta}{1 + a\lambda_n} \int_s^t \mathcal{M}(\|\nabla u(\cdot,\eta)\|_{L^2(\Omega)}) d\eta\right) F_n(s) ds\right\} e_n(x).
\]

(20)

Theorem 3.1. Let the initial data \(f \in D(A^{\nu + \beta - 1}) \cap D(A^\alpha)\) and \(F\) belongs to \(L^2(0,T; D(A^{\nu - 1})) \cap L^2(0,T; D(A^{\nu + \beta - 2}))\), here we assume that \(\nu \geq 1/2\). Then Problem \((P_f)\) has a mild (unique) solution in \(L^\infty_t(0,T; D(A^\nu))\). In addition, for \(\epsilon > 0\), we have that \(u \in C^\epsilon([0,T]; D(A^\nu))\).

Proof. For any \(\theta > 0\), denote by \(L^\infty_\theta(0,T; D(A^\nu))\) the function space \(L^\infty(0,T; D(A^\nu))\) associated with the norm

\[\|w\|_{\theta,\nu} := \max_{0 \leq t \leq T} \|\exp(-\theta t)w(\cdot,t)\|_{D(A^\nu)}, \quad \forall w \in L^\infty_\theta(0,T; D(A^\nu)).\]
Let $Q$ be defined by $Q(t, u) = S(t, u) + \mathcal{P}(t, u)F$ where we denote by

$$ S(t, u) = \sum_{n=1}^{\infty} \exp \left( - \frac{\lambda_n^2}{1 + a\lambda_n} \int_0^t \mathcal{M}(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \right) f_n e_n(x) $$

(21)

$$ \mathcal{P}(t, u)F = \sum_{n=1}^{\infty} \left[ \frac{1}{1 + a\lambda_n} \int_0^t \exp \left( - \frac{\lambda_n^2}{1 + a\lambda_n} \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds \right] e_n(x) \right]. $$

(22)

Now, we divide the proof into some steps.

**Step 1.** Estimate $\|S(t, u) - S(t, v)\|_{p, \nu}$.

From (21), using the basic inequality $|e^{-a} - e^{-b}| \leq |a - b|$, for any $a, b \in \mathbb{R}$, $a > 0$, $b > 0$ and (H3), we get

$$ \exp(-2pt)|S(t, u) - S(t, v)|^2_{D(A^p)} $$

$$ = \sum_{n=1}^{\infty} \lambda_n^{2\nu} \left[ \exp \left( - \frac{\lambda_n^2}{1 + a\lambda_n} \int_0^t \mathcal{M}(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \right) \right] - \exp \left( - \frac{\lambda_n^2}{1 + a\lambda_n} \int_0^t \mathcal{M}(\|\nabla v(\cdot, s)\|_{L^2(\Omega)}) ds \right] \right]^2 f_n^2 $$

$$ \leq \sum_{n=1}^{\infty} \frac{\lambda_n^{2\nu+2\beta}}{(1 + a\lambda_n)^2} \left( \int_0^t e^{-2p(t-s)} e^{-2ps} \mathcal{M}(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds - \int_0^t \mathcal{M}(\|\nabla v(\cdot, s)\|_{L^2(\Omega)}) ds \right) ^2 f_n^2 $$

$$ \leq K^2 \sum_{n=1}^{\infty} \frac{\lambda_n^{2\nu+2\beta}}{(1 + a\lambda_n)^2} \left( \int_0^t e^{-2p(t-s)} ds \right) \sup_{0 \leq s \leq T} \left| \int_0^t \exp(-ps)\|\nabla u - v\|_{L^2(\Omega)} ds \right|^2 f_n^2. $$

(23)

Since $\nu \geq \frac{1}{2}$, we know that the Sobolev embedding $D(A^\nu) \hookrightarrow D(A^{\frac{1}{2}})$ holds. We compute

$$ \sup_{0 \leq t \leq T} \left| \int_0^t \exp(-ps)\|\nabla u - v\|_{L^2(\Omega)} ds \right|^2 $$

$$ \leq \sup_{0 \leq t \leq T} \int_0^t \exp(-2ps)\|u - v\|_{D(A^{\frac{1}{2}})}^2 ds $$

$$ \leq \|u - v\|_{p, \nu}^2. $$

(24)

Combining (23) and (24), for $t \in [0, T]$ we find that

$$ \exp(-pt)|S(t, u) - S(t, v)|_{D(A^p)} \leq \frac{K^2}{2p} \|f\|_{D(A^\nu+\beta-1)} \|u - v\|_{p, \nu}. $$

(25)

Here we note that $\int_0^t \exp(-2p(t-s)) ds \leq \frac{1}{2p}$. So, we get immediately that

$$ \|S(t, u) - S(t, v)\|_{p, \nu} \leq \frac{K^2}{2p} \|f\|_{D(A^\nu+\beta-1)} \|u - v\|_{p, \nu}. $$

(26)

**Step 2.** Estimate $\|\mathcal{P}(t, u) - \mathcal{P}(t, v)\|_{p, \nu}$.

From (22), we compute

$$ \exp(-pt)\|\mathcal{P}(t, u) - \mathcal{P}(t, v)\|_{D(A^p)} $$

$$ \leq \exp(-pt) \int_0^t \sqrt{\sum_{n=1}^{\infty} \frac{\lambda_n^{2\nu}}{(1 + a\lambda_n)^2} |J_n(t, s, u, v)|^2 F_n^2(s)} ds. $$

(27)
where
\[ J_n(t, s, u, v) = \exp \left( -\frac{\lambda_n^{\beta} \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) \, d\eta}{1 + a\lambda_n} \right) \]
\[ - \exp \left( -\frac{\lambda_n^{\beta} \int_s^t \mathcal{M}(\|\nabla v(\cdot, \eta)\|_{L^2(\Omega)}) \, d\eta}{1 + a\lambda_n} \right). \]

Using (H3), we continue to get that
\[ J_n(t, s, u, v) \leq \frac{\lambda_n^{\beta}}{1 + a\lambda_n} \left| \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) \, d\eta - \int_s^t \mathcal{M}(\|\nabla v(\cdot, \eta)\|_{L^2(\Omega)}) \, d\eta \right| \]
\[ \leq K\mathcal{M} \frac{\lambda_n^{\beta}}{1 + a\lambda_n} \int_s^t \|\nabla(u - v)(\cdot, \eta)\|_{L^2(\Omega)} \, d\eta \]
\[ \leq K\mathcal{M} \frac{\lambda_n^{\beta}}{1 + a\lambda_n} \int_s^t \|u - v\|_{D(A^{\frac{\beta}{2}})}(\eta) \, d\eta \]
\[ \leq K\mathcal{M} \frac{\lambda_n^{\beta}}{1 + a\lambda_n} \int_0^t \|u - v\|_{D(A^{\frac{\beta}{2}})}(\eta) \, d\eta. \] (28)

For \( t \in [0, T] \), we conclude that
\[ \exp(-pt) \|\mathcal{P}(t, u) - \mathcal{P}(t, v)\|_{D(A^{\frac{\beta}{2}})} \]
\[ \leq \left( \int_0^T \left| \sum_{n=1}^{\infty} \frac{\lambda_n^{2\nu+2\beta-4}}{(\int_0^T \mathcal{F}_n(s) \, ds)^2} \right| \int_0^t \exp(-p(t - s)) \exp(ps) \|u - v\|_{D(A^{\frac{\beta}{2}})}(s) \, ds \right) \]
\[ \leq \left( \int_0^T e^{-p(t-s)} \, ds \right) \left( \int_0^T \|F(s, \cdot)\|_{D(A^{\frac{\beta}{2}})}(s) \, ds \right) \]
\[ \times \left( \sup_{0 \leq s \leq T} \int_0^t \exp(-ps) \|u(s) - v(s)\|_{D(A^{\frac{\beta}{2}})} \, ds \right) \]
\[ \leq \frac{\|F\|_{L^2(0,T;D(A^{\frac{\beta}{2}}))}}{2p} \|u - v\|_{p,\nu}. \] (29)

This implies immediately that
\[ \|\mathcal{P}(t, u) - \mathcal{P}(t, v)\|_{p,\nu} \leq \frac{\|F\|_{L^2(0,T;D(A^{\frac{\beta}{2}}))}}{2p} \|u - v\|_{p,\nu}. \] (30)

Combining (26) and (30), we obtain
\[ \|Q(t)u - Q(t)v\|_{p,\nu} \leq \frac{K\mathcal{M}}{2p} \|f\|_{D(A^{\frac{\beta}{2}})} \|u - v\|_{p,\nu} + \frac{\|F\|_{L^2(0,T;D(A^{\frac{\beta}{2}}))}}{2p} \|u - v\|_{p,\nu}. \] (31)

Let \( v_0 = 0 \) then
\[ Q(t)v_0 = f + \sum_{n=1}^{\infty} \left[ \int_0^t F_n(s) \, ds \right] e_n(x). \]
By a simple computing and using Hölder inequality, we have the following bound
\[
\left\| \sum_{n=1}^{\infty} \frac{1}{1 + a\lambda_n} \int_{0}^{t} F_n(s) \, ds \right\|_{D(A^{\nu})}^2 = \sum_{n=1}^{\infty} \frac{\lambda_n^{2\nu}}{(1 + a\lambda_n)^2} \left( \int_{0}^{t} F_n(s) \, ds \right)^2 \leq \frac{T^2}{a^2} \| F \|^2_{L^2(0, T; D(A^{\nu-1}))},
\]
\[\text{(32)}\]
Since \( f \in D(A^{\nu+\beta-1}) \cap D(A^\nu) \) and \( F \in L^2(0, T; D(A^{\nu-1})) \cap L^2(0, T; D(A^{\nu+\beta-2})) \), we deduce that \( Q(t)u \in L^\infty_p(0, T; D(A^\nu)) \) for any \( u \in L^\infty_p(0, T; D(A^\nu)) \). Letting \( p \) such that
\[ p \geq K_M \| f \|_{D(A^{\nu+\beta-1})} + \| F \|_{L^2(0, T; D(A^{\nu+\beta-2}))}.
\]
Then, we get immediately that \( Q(t) \) has a fixed point \( u \in L^\infty_p(0, T; D(A^\nu)) \), i.e. the function \( u \) be a unique solution to the equation \( Q(t)w = w \). The inequality (31) implies that the following bound
\[
\| u(p, \nu) \|_{D(A^\nu)} \leq \| Q(t)u - f \|_{D(A^\nu)} + e^{-pt} \| f \|_{D(A^\nu)}
\]
\[
\leq \left( \frac{K_M \| f \|_{D(A^{\nu+\beta-1})} + \| F \|_{L^2(0, T; D(A^{\nu+\beta-2}))}}{2p} \right) \| u \|_{p, \nu}
\]
\[
+ \| f \|_{D(A^\nu)} + \frac{T}{a} \| F \|_{L^2(0, T; D(A^{\nu-1}))}.
\]
This leads to
\[
\| u(\cdot, t) \|_{D(A^\nu)} \leq \frac{e^{pt} \left( \| f \|_{D(A^\nu)} + \frac{T}{a} \| F \|_{L^2(0, T; D(A^{\nu-1}))} \right)}{1 - \frac{K_M \| f \|_{D(A^{\nu+\beta-1})} + \| F \|_{L^2(0, T; D(A^{\nu+\beta-2}))}}{2p}}.
\]
\[\text{(33)}\]
By choose
\[ p = K_M \| f \|_{D(A^{\nu+\beta-1})} + \| F \|_{L^2(0, T; D(A^{\nu+\beta-2}))},
\]
we get immediately that for \( t \in [0, T] \)
\[
\| u(\cdot, t) \|_{D(A^\nu)} \leq 2 \exp \left( TK_M \| f \|_{D(A^{\nu+\beta-1})} + t \| F \|_{L^2(0, T; D(A^{\nu+\beta-2}))} \right)
\]
\[
\times \left( \| f \|_{D(A^\nu)} + \frac{T}{a} \| F \|_{L^2(0, T; D(A^{\nu-1}))} \right)
\]
\[
\leq \mathcal{C}(f, F, T) \left( \| f \|_{D(A^\nu)} + \frac{T}{a} \| F \|_{L^2(0, T; D(A^{\nu-1}))} \right).
\]
\[\text{(35)}\]
Here, we denote that
\[ \mathcal{C}(f, F, T) = \exp \left( TK_M \| f \|_{D(A^{\nu+\beta-1})} + T \| F \|_{L^2(0, T; D(A^{\nu+\beta-2}))} \right).
\]
Some above observations, we imply
\[
\| u \|_{L^\infty_p(0, T; D(A^\nu))} \leq \mathcal{C}(f, F, T) \left( \| f \|_{D(A^\nu)} + \frac{T}{a} \| F \|_{L^2(0, T; D(A^{\nu-1}))} \right).
\]
\[\text{(36)}\]
**Step 3.** \( C^\nu([0, T]; D(A^m)) \) regularity.
For \( 0 < \epsilon < \min(\gamma, 1 - \gamma) \), let us introduce the following space to be used
\[
C^\nu([0, T]; D(A^m)) = \left\{ v \in C([0, T]; D(A^m)) : \sup_{0 \leq t \leq s \leq T} \frac{\| v(\cdot, t) - v(\cdot, s) \|_{D(A^m)}}{|t - s|^\epsilon} < \infty \right\}.
\]
\[\text{(37)}\]
First, we know that \( u(t) = S(t, u) + P(t, u) \). So, we find that
\[
(38) \quad u(x, t + \rho) - u(x, t) = (\mathcal{S}(t + \rho, u) - \mathcal{S}(t, u)) + (\mathcal{P}(t + \rho, u) - \mathcal{P}(t, u)).
\]

Now, we estimate the term \((I)\). By a direct calculation, thanks to the inequality \(|e^{-a} - e^{-b}| \leq C_\varepsilon |a - b|^\varepsilon\), for any \( \varepsilon > 0 \), we infer that
\[
\frac{n}{n} \exp \left( - \frac{\lambda_n^\alpha}{1 + a\lambda_n} \int_t^{t + \rho} \mathcal{M}(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \right)
\leq C^2 \sum_{n=1}^{\infty} \frac{n^2 \lambda_n^\alpha}{1 + a\lambda_n} \left( \int_t^{t + \rho} \mathcal{M}(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \right)^{2\varepsilon} \frac{f_n^2}{\rho}.
\]

Since \( \nu > \frac{1}{2} \) and for any \( s \in [0, T] \), we compute
\[
\|\nabla u(\cdot, s)\|_{L^2(\Omega)} \leq \|u\|_{L^\infty(0, T; D(\mathcal{A}^\nu))} \leq \mathcal{C}(f, F, T) \left( \|f\|_{D(\mathcal{A}^\nu)} + \frac{T}{a} \|F\|_{L^2(0, T; D(\mathcal{A}^\nu - 1))} \right),
\]
and using \((H3)\), which gives that
\[
\int_t^{t + \rho} \mathcal{M}(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \leq K_M \int_t^{t + \rho} \|\nabla u(\cdot, s)\|_{L^2(\Omega)} ds
\leq K_M \mathcal{C}(f, F, T) \left( \|f\|_{D(\mathcal{A}^\nu)} + \frac{T}{a} \|F\|_{L^2(0, T; D(\mathcal{A}^\nu - 1))} \right)^{\rho}.
\]

By combining \((39)\) and \((41)\), we derive that
\[
\|\mathcal{I}\|_{D^{(\mathcal{A}^\nu)}} \leq \frac{C_cK_M \mathcal{C}(f, F, T)}{a} \left( \|f\|_{D(\mathcal{A}^\nu)} + \frac{T}{a} \|F\|_{L^2(0, T; D(\mathcal{A}^\nu - 1))} \right)^{\rho}. \tag{42}
\]

The term \((II)\) can be rewritten as follows
\[
(II) = \sum_{n=1}^{\infty} \left[ \frac{1}{1 + a\lambda_n} \int_0^{t} \exp \left( - \frac{\lambda_n^\alpha}{1 + a\lambda_n} \int_s^{t + \rho} \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds \right] e_n(x)
- \sum_{n=1}^{\infty} \left[ \frac{1}{1 + a\lambda_n} \int_0^{t} \exp \left( - \frac{\lambda_n^\alpha}{1 + a\lambda_n} \int_s^{t} \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds \right] e_n(x)
+ \sum_{n=1}^{\infty} \left[ \frac{1}{1 + a\lambda_n} \int_t^{t + \rho} \exp \left( - \frac{\lambda_n^\alpha}{1 + a\lambda_n} \int_s^{t + \rho} \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds \right] e_n(x)
= (III) + (IV).
\]

\[\text{IVP & FVP FOR FRACTIONAL NONCLASSICAL DIFFUSION EQUATIONS}  \]
Using Hölder inequality, the term \( (III) \) can be bounded by

\[
\| (III) \|_{D(A^m)}^2 \leq \sum_{n=1}^{\infty} \frac{\lambda_n^{2m-2}}{a^2} \left[ \int_0^T \left( \exp \left( - \frac{\lambda_n^\beta \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta}{1 + a\lambda_n} \right) \right) F_n(s) \right] ds \]

\[
- \exp \left( - \frac{\lambda_n^\beta \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta}{1 + a\lambda_n} \right) \right)^2 |F_n(s)|^2 ds. \tag{44}
\]

We have in view of \(|e^{-a} - e^{-b}| \leq C_\epsilon |a - b|^{\epsilon} \), with \( \epsilon > 0 \) and (H3) that

\[
\left| \exp \left( - \frac{\lambda_n^\beta \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta}{1 + a\lambda_n} \right) - \exp \left( - \frac{\lambda_n^{\beta \epsilon} \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta}{1 + a\lambda_n} \right) \right|
\]

\[
\leq C_\epsilon \lambda_n^{(\beta - 1)\epsilon} \left( \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right)^{\epsilon}
\]

\[
\leq C_\epsilon \left( \frac{K_M \mathcal{C}(f, F, T) (\|f\|_{D(A^\nu)} + \frac{T}{a} \|F\|_{L^2(0, T; D(A^\nu - 1)})} \right)^{\epsilon} \lambda_n^{(\beta - 1)\epsilon} \rho^\epsilon, \tag{45}
\]

where in the last inequality, we have to used (41). This latter estimate together with (44) leads to

\[
\| (III) \|_{D(A^m)}^2 \leq \frac{T^2 D^2}{a^2} \rho^{2\epsilon} \int_0^T \left( \sum_{n=1}^{\infty} \frac{\lambda_n^{2m - 2 + 2\beta \epsilon - 2\epsilon} F_n^2(s)}{a^2} \right) ds
\]

\[
= \frac{T^2 D^2}{a^2} \| F \|_{L^2(0, T; D(A^m_{\beta - \epsilon - 1 - \epsilon})}^2 \rho^{2\epsilon}, \tag{46}
\]

which allows us to get the following estimate

\[
\| (III) \|_{D(A^m)} \leq \frac{T D}{a} \rho^{\epsilon} \| F \|_{L^2(0, T; D(A^m_{\beta - \epsilon - 1 - \epsilon})}. \tag{47}
\]

Using the inequality \( e^{-z} \leq z^{\epsilon - 1} \) for \( 0 < \epsilon < 1 \), (40) and (H3), we continue for the estimate of \( (IV) \) as

\[
\| (IV) \|_{D(A^m)} \]

\[
\leq \sum_{n=1}^{\infty} \frac{\lambda_n^{2m - 2}}{a^2} \left[ \int_0^T \lambda_n^\beta \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right] \frac{\epsilon - 1}{\epsilon} F_n(s) ds \right]^2
\]

\[
\leq \sum_{n=1}^{\infty} \frac{\lambda_n^{2m + 2\beta \epsilon - 2\epsilon - 2\beta}}{a^2} \left[ \int_0^T \left( \int_s^t \|\nabla u(\cdot, \eta)\|_{L^2(\Omega)} d\eta \right)^{\epsilon - 1} F_n(s) ds \right]^2
\]

\[
\leq \sum_{n=1}^{\infty} \frac{\lambda_n^{2m + 2\beta \epsilon - 2\epsilon - 2\beta}}{a^2} K_n^{2m - 2} \mathcal{C}(f, F, T)^{2\epsilon - 2} \left( \|f\|_{D(A^\nu)} + \frac{T}{a} \|F\|_{L^2(0, T; D(A^\nu - 1))} \right)^{2\epsilon - 2}.
\]
Hence we get

$$\|\langle IV\rangle\|_{\mathcal{D}(\mathcal{A}^m)}$$

$$\leq C_4 \frac{K^{\kappa-1}}{\alpha^{\kappa-1}} |\mathcal{V}(f, F, T)|^{\kappa-1} \left( \|f\|_{\mathcal{D}(\mathcal{A}^\nu)} + \frac{T}{\alpha} \|F\|_{L^2(0, T; \mathcal{D}(\mathcal{A}^{\nu-1}))} \right)^{\kappa-1} \|F\|_{L^2(0, T; \mathcal{D}(\mathcal{A}^{m-\beta-\epsilon-\hat{\nu} - \epsilon}))}^{\kappa-1} \|F\|_{L^2(0, T; \mathcal{D}(\mathcal{A}^{m-\beta}))}. \quad (48)$$

Some above observations, we get that $u \in C^\nu([0, T]; \mathcal{D}(\mathcal{A}^m))$. \hfill \Box

### 3.1. Convergence of the solution when $\beta \to 1$. Now, we derive the convergence of the solutions when $\beta \to 1$. Let us assume $u_\beta$ be the solution of $\mathcal{P}_f$ for $0 < \beta < 1$ and $u^*$ is the solution of $\mathcal{P}_f$ in the case $\beta = 1$. Then we find that

$$u_\beta(x, t) = \sum_{n=1}^{\infty} \exp \left( - \frac{\lambda^\beta_n}{1 + \alpha \lambda_n} \int_0^t \mathcal{M}(\|\nabla u_\beta(x, s)\|_{L^2(\Omega)}) ds \right) f_n e_n(x)$$

$$+ \sum_{n=1}^{\infty} \left[ \frac{1}{1 + \alpha \lambda_n} \int_0^t \exp \left( - \frac{\lambda^\beta_n}{1 + \alpha \lambda_n} \int_0^s \mathcal{M}(\|\nabla u_\beta(x, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds \right] e_n(x)$$

$$= I_1 + I_2.$$ \quad (49)

and

$$u^*(x, t) = \sum_{n=1}^{\infty} \exp \left( - \frac{\lambda_n}{1 + \alpha \lambda_n} \int_0^t \mathcal{M}(\|\nabla u^*(x, s)\|_{L^2(\Omega)}) ds \right) f_n e_n(x)$$

$$+ \sum_{n=1}^{\infty} \left[ \frac{1}{1 + \alpha \lambda_n} \int_0^t \exp \left( - \frac{\lambda_n}{1 + \alpha \lambda_n} \int_0^s \mathcal{M}(\|\nabla u^*(x, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds \right] e_n(x)$$

$$= T_1 + T_2.$$ \quad (50)

**Theorem 3.2.** For $\beta \in (0, 1]$, assume that $u_\beta$ and $u^*$ are the solutions of $\mathcal{P}_f$ for $0 < \beta < 1$ and $\beta = 1$, respectively. Then, we get that $u_\beta \to u^*$ as $\beta \to 1$.

**Proof.** Step 1. Estimate the term $I_1 - T_1$.

In this step, we treat the term $I_1 - T_1$. Indeed, by a simple calculation, we get that

$$I_1 - T_1 = \sum_{n=1}^{\infty} \exp \left( - \frac{\lambda_n}{1 + \alpha \lambda_n} \int_0^t \mathcal{M}(\|\nabla u^*(x, s)\|_{L^2(\Omega)}) ds \right) f_n e_n(x)$$

$$- \sum_{n=1}^{\infty} \exp \left( - \frac{\lambda_n}{1 + \alpha \lambda_n} \int_0^t \mathcal{M}(\|\nabla u(x, s)\|_{L^2(\Omega)}) ds \right) f_n e_n(x)$$

$$+ \sum_{n=1}^{\infty} \left[ \exp \left( - \frac{\lambda_n}{1 + \alpha \lambda_n} \right) - \exp \left( - \frac{\lambda_n}{1 + \alpha \lambda_n} \right) \right] \int_0^t \mathcal{M}(\|\nabla u(x, s)\|_{L^2(\Omega)}) ds f_n e_n(x).$$ \quad (51)
This follows from the inequality \((c + d)^2 \leq 2c^2 + 2d^2\) that
\[
\|I_1 - T_t\|^2_{\mathcal{H}^1(\Omega)} \\
\leq 2 \sum_{n=1}^{\infty} \lambda_n^2 \left[ \exp \left( \frac{-\lambda_n}{1 + a\lambda_n} \int_{0}^{t} \mathcal{M}(\|\nabla u_\beta(\cdot, s)\|_{L^2(\Omega)}) ds \right) \\
- \exp \left( \frac{-\lambda_n}{1 + a\lambda_n} \int_{0}^{t} \mathcal{M}(\|\nabla u^*(\cdot, s)\|_{L^2(\Omega)}) ds \right) \right] f_n^2 \\
+ 2 \sum_{n=1}^{\infty} \lambda_n^2 \left[ \exp \left( \frac{-\lambda_n}{1 + a\lambda_n} \right) - \exp \left( \frac{-\lambda_n}{1 + a\lambda_n} \right) \right]^2 \left( \int_{0}^{t} \mathcal{M}(\|\nabla u^*(\cdot, s)\|_{L^2(\Omega)}) ds \right)^2 f_n^2 \\
= I_3 + I_4.
\] (52)

Using (H3), it is easy to get that the following estimate for the term \(I_3\)
\[
I_3 \leq \frac{2}{a^2} \sum_{n=1}^{\infty} \lambda_n^{2\beta} \left( \int_{0}^{t} \mathcal{M}(\|\nabla u^*(\cdot, s)\|_{L^2(\Omega)}) ds \right)^2 f_n^2 \\
\leq \frac{2K_M^2}{a^2} \|f\|_{L^2(\Omega)^3} \left( \int_{0}^{t} \|\nabla u^*(\cdot, s)\|_{L^2(\Omega)} ds \right)^2 f_n^2,
\] (53)
where we have relied on the inequality \(|e^{-a} - e^{-b}| \leq |a - b|\) for any positive \(a, b\).

For \(I_4\), using (H3), we find
\[
\left( \int_{0}^{t} \mathcal{M}(\|\nabla u^*(\cdot, s)\|_{L^2(\Omega)}) ds \right)^2 \leq K_M^2 T^2 \int_{0}^{t} \|\nabla u^*(\cdot, s)\|_{L^2(\Omega)}^2 ds \\
\leq K_M^2 T^2 \|\nabla u^*\|_{L^2(\Omega)}^2.
\] (54)

Using the inequality \(|e^{-a} - e^{-b}| \leq |z - y|\), we have
\[
I_4 \leq 2K_M^2 T^2 \|\nabla u^*\|_{L^2(\Omega)}^2 \sum_{\lambda_n > 1} \lambda_n^{2\beta} (1 - \lambda_n^{1-\beta})^2 f_n^2 \\
+ 2K_M^2 T^2 \|\nabla u^*\|_{L^2(\Omega)}^2 \sum_{\lambda_n < 1} \left( 1 - \lambda_n^{\beta-1} \right)^2 \lambda_n^{2\beta} f_n^2 = I_5 + I_6.
\] (55)

Now, if \(\lambda_n > 1\) then for \(\mu > 0\), there is a positive constant \(C_\mu > 0\) such that
\[
1 - \lambda_n^{1-\beta} = 1 - e^{-(1-\beta)\log(\lambda_n)} \leq C_\mu (1 - \beta)^\mu \log(\lambda_n) \leq C_\mu (1 - \beta)^\mu \lambda_n^\mu.
\] (56)

So, we get that the following inequality
\[
\sum_{\lambda_n > 1} \lambda_n^{2\beta}(1 - \lambda_n^{1-\beta})^2 f_n^2 \leq C_\mu a^{-2}(1 - \beta)^{2\mu} \sum_{\lambda_n > 1} \lambda_n^{2\mu+2\beta} f_n^2.
\] (57)

Hence, using (H3), we find that
\[
I_5 \leq \frac{2K_M^2 T^2 \|\nabla u^*\|_{L^2(\Omega)}^2}{a^2} \sum_{n=1}^{\infty} \lambda_n^{2\mu+2\beta} f_n^2 \\
= \frac{2K_M^2 T^2 \|\nabla u^*\|_{L^2(\Omega)}^2}{a^2} C_\mu a^{-2}(1 - \beta)^{2\mu} \|f\|^2_{D(A^{\mu+\beta})}.
\] (58)
It is easy to prove that

\[
I_6 \leq 2K_M^2 T^2 \| \nabla u^* \|_{L^2(0,T;L^2(\Omega))}^2 \sum_{n=1}^{\lambda_n<1} \frac{(1-\lambda_n^{\beta-1})^2}{a^2} \lambda_n^{2\beta} f_n^2
\]

\[
= 2K_M^2 T^2 \| \nabla u^* \|_{L^2(0,T;L^2(\Omega))}^2 \frac{(1-\lambda_1^{\beta-1})^2}{a^2} \| f \|_{D(A^\beta)}^2.
\]

If the set \( \{ n \in \mathbb{N}, \lambda_n < 1 \} \) is empty set then \( I_6 = 0 \). Therefore, we obtain

\[
I_4 \leq I_5.
\]

We only consider \( I_5 \). If the set \( \{ n \in \mathbb{N}, \lambda_n \geq 1 \} \) is non-empty set then we must treat the term \( I_5 \) and \( I_6 \). From some above observations, we deduce that

\[
I_4 \leq \frac{2K_M^2 T^2 \| \nabla u^* \|_{L^2(0,T;L^2(\Omega))}^2}{a^2} C_\mu^2 a^{-2} (1-\beta) \| f \|_{D(A^{\alpha+\beta})}^2
\]

\[
+ 2K_M^2 T^2 \| \nabla u^* \|_{L^2(0,T;L^2(\Omega))}^2 \max \left( \left( \frac{(1-\lambda_1^{\beta-1})^2}{a^2} \right), 0 \right) \| f \|_{D(A^\beta)}^2.
\]

**Step 2.** Estimate the term \( I_2 - T_2 \).

It is easily be seen that

\[
I_2 - T_2 = \sum_{n=1}^{\infty} \left[ \frac{1}{1 + a\lambda_n} \int_0^t \exp \left( -\frac{\lambda_n^\beta}{1 + a\lambda_n} \right) \int_s^t \mathcal{M}(\| \nabla u^*(\cdot,\eta) \|_{L^2(\Omega)}) d\eta F_n(s) ds \right] e_n(x)
\]

\[
- \sum_{n=1}^{\infty} \left[ \frac{1}{1 + a\lambda_n} \int_0^t \exp \left( -\frac{\lambda_n^\beta}{1 + a\lambda_n} \right) \int_s^t \mathcal{M}(\| \nabla u^*(\cdot,\eta) \|_{L^2(\Omega)}) d\eta F_n(s) ds \right] e_n(x)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \frac{1}{1 + a\lambda_n} \int_0^t \mathcal{H}(\lambda_n, a, \beta) \int_s^t \mathcal{M}(\| \nabla u^*(\cdot,\eta) \|_{L^2(\Omega)}) d\eta F_n(s) ds \right] e_n(x),
\]

where we denote by

\[
\mathcal{H}(\lambda_n, a, \beta) = \exp \left( -\frac{\lambda_n^\beta}{1 + a\lambda_n} \right) - \exp \left( -\frac{\lambda_n}{1 + a\lambda_n} \right).
\]

By applying the inequality \((c + d)^2 \leq 2c^2 + 2d^2\), we find that

\[
\| I_2 - T_2 \|_{H^1(\Omega)}^2 \leq 2 \sum_{n=1}^{\infty} \lambda_n^2 \left[ \frac{1}{1 + a\lambda_n} \int_0^t \exp \left( -\frac{\lambda_n^\beta}{1 + a\lambda_n} \right) \right]
\]

\[
\times \left( \int_s^t \left( \mathcal{M}(\| \nabla u^*(\cdot,\eta) \|_{L^2(\Omega)}) - \mathcal{M}(\| \nabla u^*(\cdot,\eta) \|_{L^2(\Omega)}) \right) d\eta F_n(s) ds \right)^2
\]

\[
+ 2 \sum_{n=1}^{\infty} \lambda_n^2 \left[ \frac{1}{1 + a\lambda_n} \int_0^t \mathcal{H}(\lambda_n, a, \beta) \int_s^t \mathcal{M}(\| \nabla u^*(\cdot,\eta) \|_{L^2(\Omega)}) d\eta F_n(s) ds \right] = T_3 + T_4.
\]
By using Hölder inequality and (H3), we get immediately that
\[
T_3 \leq 2T^2 \sum_{n=1}^{\infty} \frac{\lambda_n^2}{(1 + a\lambda_n)^2} \left( \int_0^t \exp \left( -\frac{2\lambda_n^\beta}{1 + a\lambda_n} \right) |F_n(\tau)|^2 \, d\tau \right) 
\times \int_0^t (\mathcal{M}(\|\nabla u^\beta(\cdot, \eta)\|_{L^2(\Omega)}) - \mathcal{M}(\|\nabla u^*(\cdot, \eta)\|_{L^2(\Omega)}))^2 \, d\eta 
\lesssim K^2 M^2 \|F\|^2_{L^2(0,T;L^2(\Omega))} \int_0^t \|\nabla u^\beta(\cdot, \tau; s) - \nabla u^*(\cdot, \tau; s)\|^2_{L^2(\Omega)} \, ds. \tag{64}
\]

We next bound the term $T_4$ as follows
\[
T_4 \leq 2T^2 \sum_{n=1}^{\infty} \frac{\lambda_n^2}{(1 + a\lambda_n)^2} \left( \int_0^t \left| \mathcal{H}(\lambda_n, a, \beta) \right|^2 F_n(\tau)^2 \, d\tau \right) \left( \int_0^t \mathcal{M}^2 \|\nabla u^*(\cdot, \eta)\|_{L^2(\Omega)} \, d\eta \right). \tag{65}
\]

By the estimate (54), we find that
\[
T_6 \leq K^2 M^2 T^2 \|\nabla u^*\|^2_{L^2(0,T;L^2(\Omega))}. \tag{66}
\]

For $T_5$, we observe that
\[
T_5 = \sum_{n=1}^{\infty} \frac{\lambda_n^2}{(1 + a\lambda_n)^2} \left( \int_0^t \left| \mathcal{H}(\lambda_n, a, \beta) \right|^2 F_n(\tau)^2 \, d\tau \right) 
\leq \sum_{n=1}^{\lambda_n > 1} \frac{\lambda_n^2}{(1 + a\lambda_n)^2} \int_0^t \left| \mathcal{H}(\lambda_n, a, \beta) \right|^2 F_n(\tau)^2 \, d\tau 
+ \sum_{n=1}^{\lambda_n \leq 1} \frac{\lambda_n^2}{(1 + a\lambda_n)^2} \int_0^t \left| \mathcal{H}(\lambda_n, a, \beta) \right|^2 F_n(\tau)^2 \, d\tau 
\leq \sum_{n=1}^{\lambda_n > 1} \frac{\lambda_n^2(1 - \lambda_n^{-\beta})^2}{a^4 \lambda_n^2} \int_0^T F_n(\tau)^2 \, d\tau + \sum_{n=1}^{\lambda_n \leq 1} \frac{\lambda_n^2(1 - \lambda_n^{-\beta})^2}{a^4 \lambda_n^2} \int_0^T F_n(\tau)^2 \, d\tau 
\leq C^2 \mu (1 - \beta)^2 \sum_{n=1}^{\lambda_n > 1} \lambda_n^{2\beta - 2} \lambda_n^2 \int_0^T F_n(\tau)^2 \, d\tau + \frac{1}{a^2} \max \left( \frac{(1 - \lambda_n^{-\beta - 1})^2}{\lambda_n^{2\beta}}, 0 \right) \int_0^T F_n(\tau)^2 \, d\tau \tag{67}
\]

Hence, we get
\[
T_5 \leq C^2 \mu (1 - \beta)^2 \|F\|^2_{L^2(0,T;D(A^\beta + \mu - 1))} + \frac{1}{a^2} \max \left( \frac{(1 - \lambda_n^{-\beta - 1})^2}{\lambda_n^{2\beta}}, 0 \right) \|F\|^2_{L^2(0,T;D(A^\beta + \mu - 1))}. \tag{68}
\]

Combining the results of Step 1-2, from the Grönwall’s inequality we get $\|u^\beta(\cdot, T) - u^*(\cdot, T)\|_{H^1(\Omega)} = 0$ as $\beta \to 1$, we finish the proof. \hfill \Box

4. Final value problem. Consider the following final value problem
\[
\begin{aligned}
&\{ u_t - a\Delta u_t + \mathcal{M}(\|\nabla u\|_{L^2(\Omega)}) (\Delta)^\beta u = F(x,t) \quad \text{in} \ \Omega \times (0,T), \\
&u(x,t) = 0, \quad \text{on} \ \partial\Omega \times (0,T), \\
&u(x,T) = g(x),
\end{aligned}
\tag{F\beta}
\]
Let us assume Problem (P₉) has a solution u. Then from (19), we have
\[
\begin{align*}
  u_n(t) &= \exp \left( -\frac{\lambda_n^\beta}{1+a\lambda_n} \int_0^t \mathcal{M}(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \right) u_n(0) \\
  &\quad + \frac{1}{1+a\lambda_n} \int_0^t \exp \left( -\frac{\lambda_n^\beta}{1+a\lambda_n} \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds.
\end{align*}
\]
Let \( t = T \) into the above display which allow us to obtain that
\[
\begin{align*}
  u_n(0) &= \exp \left( \frac{\lambda_n^\beta}{1+a\lambda_n} \int_0^T \mathcal{M}(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \right) u_n(T) \\
  &\quad - \frac{1}{1+a\lambda_n} \int_0^T \exp \left( -\frac{\lambda_n^\beta}{1+a\lambda_n} \int_s^T \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds.
\end{align*}
\]
From two preceding equality, we find that
\[
\begin{align*}
  u_n(t) &= \exp \left( \frac{\lambda_n^\beta}{1+a\lambda_n} \int_0^T \mathcal{M}(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \right) u_n(T) \\
  &\quad - \frac{1}{1+a\lambda_n} \int_0^T \exp \left( -\frac{\lambda_n^\beta}{1+a\lambda_n} \int_s^T \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds \\
  &\quad + \frac{1}{1+a\lambda_n} \int_0^t \exp \left( -\frac{\lambda_n^\beta}{1+a\lambda_n} \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds.
\end{align*}
\]
After some simple calculation, we derive that
\[
\begin{align*}
  u(x, t) &= \sum_{n=1}^{\infty} \left[ \exp \left( \frac{\lambda_n^\beta}{1+a\lambda_n} \int_0^T \mathcal{M}(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \right) g_n \right] e_n(x) \\
  &\quad - \sum_{n=1}^{\infty} \left[ \frac{1}{1+a\lambda_n} \int_0^t \exp \left( -\frac{\lambda_n^\beta}{1+a\lambda_n} \int_s^t \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds \right] e_n(x).
\end{align*}
\]

4.1. **The case \( \beta < 1 \).** Now, we are in the position to introduce the existence result of the solution to Problem (P₉). For \( \vartheta > 0 \), we consider the following set
\[
\mathcal{Z}_{\vartheta,s} := \left\{ w : [0, T] \to L^2(\Omega), \left\| \exp(-\vartheta(T-t))w(\cdot, t) \right\|_{D(A^*)} < \infty, 0 \leq t \leq T \right\},
\]
associated with the following norm \( \|w\|_{\mathcal{Z}_{\vartheta,s}} := \max_{0 \leq t \leq T} \left\| \exp(-\vartheta(T-t))w(\cdot, t) \right\|_{D(A^*)} \). This set plays an important role in the proof of the next theorem.

**Theorem 4.1.** Let \( 0 < \beta < 1 \). Then, the Problem (P₉) has a unique solution \( u \) in \( \mathcal{Z}_{\vartheta,s}([0; T]; D(A^*)) \) with some \( \vartheta > 0 \).
Proof. We define the operator \( \overline{Q} \) by \( \overline{Q}(t)u = \overline{S}(t,u) + \overline{P}(t,u) \) where we denote by
\[
\overline{S}(t,u)g = \sum_{n=1}^{\infty} \exp \left( \frac{\lambda_n^\beta}{1 + a\lambda_n} \int_t^T M(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \right) g_n e_n(x),
\]
(74)
\[
\overline{P}(t,u)F = \sum_{n=1}^{\infty} \left[ \frac{1}{1 + a\lambda_n} \int_t^T \exp \left( - \frac{\lambda_n^\beta}{1 + a\lambda_n} \int_t^T M(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) d\eta \right) F_n(s) ds \right] e_n(x).
\]
(75)
and \( g_n = (g, e_n) \). Now, we divide into two steps.

**Step 1.** Estimate \( \|\overline{S}(\cdot, u)g - \overline{S}(\cdot, v)g\|_{D(A^\alpha)} \).

First, since the fact that \( \frac{\lambda_n^\beta}{1 + a\lambda_n} \leq a^{-1}\lambda_n^\beta_1 \) and using (H2), we note that
\[
\exp \left( \frac{\lambda_n^\beta}{1 + a\lambda_n} \int_t^T M(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}) ds \right) \leq \exp \left( a^{-1}\lambda_n^\beta_1 M_1(T - t) \right),
\]
(76)
and
\[
\exp \left( \frac{\lambda_n^\beta}{1 + a\lambda_n} \int_t^T M(\|\nabla v(\cdot, s)\|_{L^2(\Omega)}) ds \right) \leq \exp \left( a^{-1}\lambda_n^\beta_1 M_1(T - t) \right) = B.
\]
(77)
Let \( \vartheta \) be any positive number which is defined later. Using the inequality \( |e^z - e^y| \leq \max(e^z, e^y)|z - y| \) for any \( z, y \in \mathbb{R}^+ \) and from two preceding estimates, we get
\[
\exp(-2\vartheta(T - t))\|\overline{S}(t,u)g - \overline{S}(t,v)g\|_{D(A^\alpha)}^2
\]
\[
= \sum_{n=1}^{\infty} \lambda_n^{2\alpha} \left[ \exp \left( \frac{\lambda_n^\beta}{1 + a\lambda_n} \int_t^T M(\|\nabla u(\cdot, \tau)\|_{L^2(\Omega)}) d\tau \right) 
\]
\[
- \exp \left( \frac{\lambda_n^\beta}{1 + a\lambda_n} \int_t^T M(\|\nabla v(\cdot, \tau)\|_{L^2(\Omega)}) d\tau \right) \right] |g_n|^2
\]
\[
\leq \sum_{n=1}^{\infty} \frac{B\lambda_n^{2\alpha+2\beta}}{(1 + a\lambda_n)^2} \left( \int_t^T \exp(-\vartheta \tau) \exp(-\vartheta(T - \tau)) \left( M(\|\nabla u(\cdot, \tau)\|_{L^2(\Omega)}) - M(\|\nabla v(\cdot, \tau)\|_{L^2(\Omega)}) \right) d\tau \right)^2 |g_n|^2
\]
\[
\leq \frac{B}{2} \sum_{n=1}^{\infty} \frac{\lambda_n^{2\alpha+2\beta}}{(1 + a\lambda_n)^2} \left( \int_t^T \exp(-2\vartheta \tau) d\tau \right)^2
\]
\[
\sup_{0 \leq t \leq T} \left| \int_t^T \exp(-\vartheta(T - \tau)) \|\nabla(u - v)(\cdot, \tau)\|_{L^2(\Omega)} d\tau \right|^2 |g_n|^2.
\]
(78)
Since \( s \geq \frac{1}{2} \), we know that the Sobolev embedding \( D(A^\alpha) \hookrightarrow D(A^{\frac{1}{2}}) \) holds. One has
\[
\sup_{0 \leq t \leq T} \int_t^T \exp(-\vartheta(T - \tau)) \|\nabla(u - v)(\cdot, \tau)\|_{L^2(\Omega)} d\tau
\]
\[
\leq \sup_{0 \leq t \leq T} \int_t^T \exp(-\vartheta(T - \tau)) \|u(\cdot, \tau) - v(\cdot, \tau)\|_{D(A^{\frac{1}{2}})}^2 d\tau
\]
\[
\leq T \|u - v\|_{\alpha, s}^2.
\]
(79)
Step 2. Estimate $\|\mathcal{P}(\cdot, u) F - \mathcal{P}(\cdot, v) F\|_{\sigma,s}$.

From (75), we observe that
\[
\exp(-\vartheta(T - t)) \left\| \mathcal{P}(t, u) F - \mathcal{P}(t, v) F \right\|_{D(A^s)} \leq \exp(-\vartheta(T - t)) \int_T^T \left( \sum_{n=1}^{\infty} \lambda_n^2 \frac{L(H_n(t, \tau, u, v)^2 F_n^2(\tau) \, d\tau)\right) dt.
\]

where we denote by
\[
H_n(t, \tau, u, v)
\]
\[
= \exp\left(-\frac{\lambda_n^2}{1 + a\lambda_n} \int_\tau^T \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) \, d\eta\right) - \exp\left(-\frac{\lambda_n^2}{1 + a\lambda_n} \int_\tau^T \mathcal{M}(\|\nabla v(\cdot, \eta)\|_{L^2(\Omega)}) \, d\eta\right)
\]
for $0 \leq t \leq \tau$. Using (H3), let us continue to treat the term $H_n(t, \tau, u, v)$ by the following estimation
\[
H_n(t, \tau, u, v)
\]
\[
\leq \frac{\lambda_n^2}{1 + a\lambda_n} \left| \int_t^\tau \mathcal{M}(\|\nabla u(\cdot, \eta)\|_{L^2(\Omega)}) \, d\eta - \int_t^\tau \mathcal{M}(\|\nabla v(\cdot, \eta)\|_{L^2(\Omega)}) \, d\eta\right|
\]
\[
\leq K_M \frac{\lambda_n^2}{1 + a\lambda_n} \int_t^\tau \|\nabla(u - v)(\cdot, \eta)\|_{L^2(\Omega)} \, d\eta
\]
\[
\leq \frac{K_M \lambda_n^2}{1 + a\lambda_n} \int_{\mathcal{M}} \|u - v\|_{D(A^\frac{1}{2})} d\eta \leq \frac{K_M \lambda_n^2}{1 + a\lambda_n} \int_t^\tau \|u - v\| d\eta
\]

This together with (82) leads to
\[
\exp(-\vartheta(T - t)) \left\| \mathcal{P}(t, u) F - \mathcal{P}(t, v) F \right\|_{D(A^s)} \leq (\int_0^T \sum_{n=1}^{\infty} \lambda_n^{2+2\beta-4} \mathcal{F}_n^2(\tau) \, d\tau) \int_T^T \exp(-\vartheta(\tau - t)) \exp(-\vartheta(T - \tau)) \|u - v\| \, d\tau
\]
we deduce that

\[ g \in D(A^{s+\beta-2}) \]

Since the term in the right hand side of (85) is independent of \( t \), so we immediately get that

\[ \| \mathcal{P}(\cdot, u) F - \mathcal{P}(\cdot, v) F \|_{\vartheta, s} \leq \frac{\| F \|_{L^2(0,T;D(A^{s+\beta-2}))}}{\vartheta} \| u - v \|_{\vartheta, s}. \]  (86)

Combining (81) and (86), we obtain that

\[ \| \overline{Q}(\cdot, u) - \overline{Q}(\cdot, v) \|_{\vartheta, s} \leq \| \mathcal{S}(\cdot, u) g - \mathcal{S}(\cdot, v) g \|_{\vartheta, s} + \| \mathcal{P}(\cdot, u) F - \mathcal{P}(\cdot, v) F \|_{\vartheta, s} \]

\[ \leq \frac{BT}{\vartheta} \left( \| g \|_{D(A^{s+\beta-1})} \| u - v \|_{\vartheta, s} + \| F \|_{L^2(0,T;D(A^{s+\beta-2}))} \| u - v \|_{\vartheta, s} \right) \]

\[ = \left( \frac{BT}{\vartheta} \| g \|_{D(A^{s+\beta-1})} + \| F \|_{L^2(0,T;D(A^{s+\beta-2}))} \right) \| u - v \|_{\vartheta, s}. \]  (87)

Let \( w_0 = 0 \) then from (74) and (75), we get

\[ \overline{Q}(t, w_0) = g + \sum_{n=1}^{\infty} \left[ \frac{1}{1 + a\lambda_n} \int_t^T F_n(\tau) d\tau \right] e_n(x). \]

By a simple computing and using Hölder inequality, we find

\[ \left\| \sum_{n=1}^{\infty} \left[ \frac{1}{1 + a\lambda_n} \int_t^T F_n(\tau) d\tau \right] e_n(x) \right\|_{D(A^s)}^2 \]

\[ = \sum_{n=1}^{\infty} \frac{\lambda_n^{2s}}{(1 + a\lambda_n)^2} \left[ \int_t^T F_n(\tau) d\tau \right]^2 \]

\[ \leq \frac{T^2}{a^2} \| F \|_{L^2(0,T;D(A^{s-1}))}^2. \]  (88)

Since \( g \in D(A^{s+\beta-1}) \cap D(A^s) \) and \( F \in L^2(0,T;D(A^{s-1})) \cap L^2(0,T;D(A^{s+\beta-2})) \), we deduce that \( \overline{Q}(t, u) \in L^p(0,T;D(A^s)) \) for any \( u \in L^p(0,T;D(A^s)) \). Let us choose \( \vartheta \) enough large, we get that

\[ \frac{BT}{\vartheta} \| g \|_{D(A^{s+\beta-1})} + \frac{\| F \|_{L^2(0,T;D(A^{s+\beta-2}))}}{\vartheta} < 1. \]

This implies that \( \overline{Q} \) have a fixed point in \( Z_{\vartheta, s} \).

4.2. The case \( \beta > 1, F = 0 \). In this subsection, we will consider two parts.

4.2.1. Existence of the mild solution of Problem (\( \mathcal{P}_g \)) for the case \( \beta > 1 \).

**Theorem 4.2.** Let \( \beta > 1 \) and let \( g \) such that the following condition

\[ \sum_{n=1}^{\infty} \lambda_n^{2\beta-1} e^{2TM_1\lambda_n^{2\beta-1}} (g, e_n)^2 = D_\beta < \infty. \]  (89)

Then the Problem (\( \mathcal{P}_g \)) has a unique mild solution \( u \in C([0,T]; H^1(\Omega)) \) which satisfies that

\[ u(x,t) = \sum_{n=1}^{\infty} \exp \left( \frac{\lambda_n^{\beta}}{1 + a\lambda_n} \int_t^T \mathcal{M}(\| \nabla u(\cdot, \tau) \|_{L^2(\Omega)}) d\tau \right) (g, e_n) e_n(x). \]  (90)
Moreover, we show that this solution is not stable in the $L^2$-norm.

**Proof.** From (72) and for $F = 0$, we recall the mild solution $u$ of Problem $(\mathbb{P}_\delta)$ which is given as follows

$$u(x, t) = \sum_{n=1}^{\infty} \exp \left( \frac{\lambda_n^\beta}{1 + a \lambda_n} \int_t^T \mathcal{M}(\|\nabla u(\cdot, \tau)\|_{L^2(\Omega)}) d\tau \right) (g, e_n) e_n(x). \quad (91)$$

Now, we show the nonlinear equation (91) admits a unique mild solution in $C([0, T]; H^1(\Omega))$. In fact, we consider the function

$$\mathcal{Y}(v)(x, t) = \sum_{n=1}^{\infty} \exp \left( \frac{\lambda_n^\beta}{1 + a \lambda_n} \int_t^T \mathcal{M}(\|\nabla v(\cdot, \tau)\|_{L^2(\Omega)}) d\tau \right) (g, e_n) e_n(x). \quad (92)$$

By using the principle of induction, for $v_1, v_2 \in C([0, T]; H^1(\Omega))$, for $m \geq 1$ we need to prove that

$$\left\| \mathcal{Y}^m(v_1)(\cdot, t) - \mathcal{Y}^m(v_2)(\cdot, t) \right\|_{H^1(\Omega)} \leq \frac{\big(C \delta K^2 M(T - t)\big)^m}{m!} \left\| v_1 - v_2 \right\|_{C([0, T]; H^1(\Omega))}. \quad (93)$$

For $m = 1$, using (H3), the basic inequality $|e^a - e^b| \leq |a - b| \max(e^a, e^b)$, for $a, b \in \mathbb{R}$ and noting (89), we have

$$\left\| \mathcal{Y}(v_1) - \mathcal{Y}(v_2) \right\|_{H^1(\Omega)}^2 = \sum_{n=1}^{\infty} \lambda_n \left[ \exp \left( \frac{\lambda_n^\beta}{1 + a \lambda_n} \int_t^T \mathcal{M}(\|\nabla v_1(\cdot, \tau)\|_{L^2(\Omega)}) d\tau \right) \right]^2 (g, e_n)^2$$

$$\leq \sum_{n=1}^{\infty} \lambda_n^{2\beta - 1} \left[ \int_t^T \mathcal{M} \left( \|\nabla v_1(\cdot, \tau)\|_{L^2(\Omega)} \right) d\tau \right]^2 e^{2(T - t) M_1 \lambda_n^{\beta - 1}} (g, e_n)^2$$

$$\leq D_\beta K^2 M \int_t^T \|\nabla(v_1 - v_2)(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau$$

$$\leq CD_\beta K^2 M \int_t^T \|v_1(\cdot, \tau) - v_2(\cdot, \tau)\|_{H^1(\Omega)}^2 d\tau, \quad (94)$$

where, we used the inequality $\|\nabla v\|_{L^2(\Omega)} \leq C\|v\|_{H^1(\Omega)}$. Now, suppose that (93) holds for $m = p$. We prove that (93) holds for $m = p + 1$. Indeed, we have

$$\left\| \mathcal{Y}^{p+1}(v_1) - \mathcal{Y}^{p+1}(v_2) \right\|_{H^1(\Omega)}^2 = \sum_{n=1}^{\infty} \lambda_n \left[ \exp \left( \frac{\lambda_n^\beta}{1 + a \lambda_n} \int_t^T \mathcal{M}(\|\nabla \mathcal{Y}^p(v_1)(\cdot, \tau)\|_{L^2(\Omega)}) d\tau \right) \right]^2 (g, e_n)^2$$

$$- \exp \left( \frac{\lambda_n^\beta}{1 + a \lambda_n} \int_t^T \mathcal{M}(\|\nabla \mathcal{Y}^p(v_2)(\cdot, \tau)\|_{L^2(\Omega)}) d\tau \right) \right]^2 (g, e_n)^2$$

$$\leq C D_\beta K^2 M \int_t^T \|v_1(\cdot, \tau) - v_2(\cdot, \tau)\|_{H^1(\Omega)}^2 d\tau.$$
This implies that unique solution in and example. For any NGUYEN HUY TUAN to see that the function Thus, we have immediately that point of Y.

Hence, according to the principle of induction, (93) holds for all v₁, v₂ ∈ C([0, T]; H¹(Ω)). Since

\[ \lim_{m \to +\infty} \frac{(CDβK₄M)²T}{m!} = 0, \]

there is a number m₀ ∈ N⁺ such that

\[ \frac{(CDβK₄M)²T}{m!} < 1, \quad \text{for any } m \geq m₀, \]

and Yᵐ₀ is a contraction map and Yᵐ₀(u) = u. We also have Yᵐ₀(Y(u)) = Y(Yᵐ₀(u)) = Y(u). We infer that Y(u) is a fixed point of Yᵐ₀. Because the fixed point of Yᵐ₀ is unique, we have that Y(u) = u or the equation Y(v) = v has a unique solution in C([0, T]; H¹(Ω)).

Next, we show a mild solution of the problem (P₉) is unstable through a simple example. For any k ∈ N, let us set gₖ as

\[ gₖ(x) = \frac{e_k(x)}{λₖ}. \]

Thus, we have immediately that

\[ \sum_{n=1}^{∞} λₙ^{2β-1} e^{2TM₁λₙβ-1} (gₙ, e_n)^2 = λₖ^{2β-3} e^{2Tβ₁ₖ} < ∞. \]

It verifies that (89) holds. Using Theorem (4.2), we know that (95) has a unique solution uₖ ∈ C([0, T]; H¹(Ω)). And from the fact that gₖ(x) = e_k(x)/λₖ, it is obvious to see that the function uₖ satisfies the following integral equation

\[ uₖ(x, t) = \sum_{n=1}^{∞} \exp \left( \frac{λₙ^{2β}}{1 + αλₙ} \int_t^T \mathcal{M}(\|∇u(\cdot, τ)\|_{L²(Ω)})dτ \right) \langle gₙ, e_n \rangle e_n(x) \]

\[ \exp \left( \frac{λₖ^{2β}}{1 + αλₖ} \int_t^T \mathcal{M}(\|∇u(\cdot, τ)\|_{L²(Ω)})dτ \right) eₖ(x). \]  

Moreover, since the bound \( \mathcal{M}(z) ≥ M₀ ∀z ∈ \mathbb{R} \), we obtain that

\[ \|uₖ(\cdot, t)\|^₂_{L²(Ω)} = \exp \left( \frac{2λₖ^{β-1} M₀}{1 + αλₖ} (T - t) \right) \geq \exp \left( \frac{2λₖ^{β-1} M₀}{1 + αλₖ} (T - t) \right). \]

This implies that

\[ \|uₖ\|_{C([0, T]; L²(Ω))} ≥ \sup_{0 ≤ t ≤ T} \exp \left( \frac{2λₖ^{β-1} M₀}{1 + αλₖ} (T - t) \right) = \exp \left( \frac{2λₖ^{β-1} M₀T}{1 + αλₖ} \right). \]
As $k \to +\infty$, we easily to see that
\[
\lim_{k \to +\infty} \|g^k\|_{L^2(\Omega)} = \lim_{k \to +\infty} \frac{1}{\lambda_k} = 0
\]
\[
\lim_{k \to +\infty} \|u^k\|_{C([0,T];L^2(\Omega))} = \lim_{k \to +\infty} \frac{\exp\left(2\lambda_k^{\beta-1} \frac{M_n T}{a}\right)}{\lambda_k^2} = +\infty.
\]
From that, the problem $(F_g)$ is ill-posed in $L^2$-norm via the sense of Hadamard. \(\square\)

4.2.2. Fourier truncation method. We establish the regularized solution as follows
\[
W^{N(\delta)}(x,t) = \sum_{n=1}^{N(\delta)} \exp\left(-\frac{\lambda_n^\beta}{1+a\lambda_n}\right) \int_t^T M\left(\|\nabla W^{N(\delta)}(\cdot,\tau)\|_{L^2(\Omega)}\right) d\tau \left\langle g^\delta, e_n \right\rangle e_n(x).
\]
Here, $N := N(\delta) > 0$ be the parameter regularization which is chosen later.

(Hyp4): For $g \in L^2(\Omega)$ and $g^\delta \in L^2(\Omega)$ represent the exact and measured data, respectively, $\delta > 0$ be a noise level satisfying
\[
\|g^\delta - g\|_{L^2(\Omega)} \leq \delta.
\]
We are now ready to present the following theorem.

**Theorem 4.3.** Let any $g^\delta \in L^2(\Omega)$ be as (H4). Then, the equation $(96)$ has a unique solution $W^{N(\delta),\delta} \in C([0,T];H^1(\Omega))$. Let us choose $N := N(\delta)$ such that
\[
\lim_{\delta \to 0} N(\delta) = +\infty, \quad \lim_{\delta \to 0} \lambda_{N(\delta)} \exp\left(\frac{2TM_1\lambda_{N(\delta)}^{\beta-1}}{a}\right) \delta^2 = 0.
\]

Let $g$ be as Theorem 4.2. Thus we have
\[
\|W^{N(\delta),\delta}(\cdot,\cdot) - u(\cdot,\cdot)\|_{H^1(\Omega)}^2 \
\leq \left[3\lambda_{N(\delta)} \exp\left(\frac{2TM_1\lambda_{N(\delta)}^{\beta-1}}{a}\right)\delta^2 + 3\lambda_{N(\delta)}^{2-2\beta} D_\beta\right] \exp\left(3CD_\beta K^2 M(T-t)\right).
\]

**Remark 4.1.** Since $\lambda_N \sim N^{\frac{\beta}{\beta}}$, choose $N \in \mathbb{N}^*$ such that
\[
\frac{1-b}{2TB_1} \ln\left(\frac{1}{\delta}\right) \leq \lambda_N \leq \frac{1-b}{TB_1} \ln\left(\frac{1}{\delta}\right), \quad 0 < b < 1.
\]
Then we have the error $\|W^{N(\delta),\delta}(\cdot,\cdot) - u(\cdot,\cdot)\|_{H^1(\Omega)}$ is of following logarithmic order
\[
\max\left(\left[\ln\left(\frac{1}{\delta}\right)\right]^{-2}, \delta^{2b} \ln\left(\frac{1}{\delta}\right)\right).
\]

**Proof.** Part 1. The existence, uniqueness results to the equation $(96)$. Let $v \in C([0,T];H^1(\Omega))$, we consider
\[
\mathcal{J}(v)(x,t) = \sum_{n=1}^{N(\delta)} \exp\left(-\frac{\lambda_n^\beta}{1+a\lambda_n}\right) \int_t^T M\left(\|\nabla v(\cdot,\tau)\|_{L^2(\Omega)}\right) d\tau \left\langle g^\delta, e_n \right\rangle e_n(x).
\]
By induction we prove that if \( v_1, v_2 \in C([0, T]; H^1(\Omega)) \) then we get

\[
\|J^m(v_1)(\cdot, t) - J^m(v_2)(\cdot, t)\|_{H^1(\Omega)}
\leq \left( \frac{C\lambda^{\beta-1}_N \exp \left( \frac{2TM_1\lambda^\beta_N}{a} \right) K_M^2\|g\|^2_{L^2(\Omega)} (T-t)^m}{m!} \right)^{1/2} \|w_1 - w_2\|_{C([0, T]; H^2(\Omega))}, \quad \forall m \geq 1.
\]

(98)

We begin to prove with \( m = 1 \), from the inequality \(|e^a - e^b| \leq |a - b| \max(e^a, e^b)\) for any \( a, b \in \mathbb{R} \) and (H3), we have

\[
\|J(v_1) - J(v_2)\|_{H^1(\Omega)}^2
= \sum_{n=1}^{N(\delta)} \lambda_n \left[ \exp \left( \frac{\lambda_n^\beta}{1 + a\lambda_n} \int_t^T (m \nabla v_1(\cdot, s))_L^2(\Omega) \right) ds \right]
- \exp \left( \frac{\lambda_n^\beta}{1 + a\lambda_n} \int_t^T (m \nabla v_2(\cdot, s))_L^2(\Omega) \right) \rangle \langle g^\delta, e_n \rangle^2
\leq \sum_{n=1}^{N(\delta)} \lambda_n^{2\beta-1} \left[ \int_t^T (m \nabla w_1(\cdot, s))_L^2(\Omega) \right] ds
- \exp \left( \frac{2(T-t)M_1\lambda^\beta_N}{a} \right) \langle g^\delta, e_n \rangle^2
\leq \lambda^{2\beta-1}_{N(\delta)} \exp \left( \frac{2TM_1\lambda^\beta_N}{a} \right) K_M^2\|g\|^2_{L^2(\Omega)} \int_t^T (m \nabla (w_1 - w_2)(\cdot, s))_L^2(\Omega) ds
\leq C\lambda^{2\beta-1}_{N(\delta)} \exp \left( \frac{2TM_1\lambda^\beta_N}{a} \right) K_M^2\|g\|^2_{L^2(\Omega)} \int_t^T \|v_1(\cdot, s) - v_2(\cdot, s)\|_{H^1(\Omega)}^2 ds.
\]

Assuming that (98) holds with \( m = p \). We will show that (98) holds with \( m = p+1 \). Indeed, we have

\[
\|J^{p+1}(v_1)(\cdot, t) - J^{p+1}(v_2)(\cdot, t)\|_{H^1(\Omega)}^2
= \sum_{n=1}^{N(\delta)} \lambda_n \left[ \exp \left( \frac{\lambda_n^\beta}{1 + a\lambda_n} \int_t^T (m \nabla J^p(v_1)(\cdot, s))_L^2(\Omega) \right) ds \right]
- \exp \left( \frac{\lambda_n^\beta}{1 + a\lambda_n} \int_t^T (m \nabla J^p(v_2)(\cdot, s))_L^2(\Omega) \right) \rangle \langle g^\delta, e_n \rangle^2
\leq \lambda^{2\beta-1}_{N(\delta)} \exp \left( \frac{2TM_1\lambda^\beta_N}{a} \right) K_M^2\|g\|^2_{L^2(\Omega)} \int_t^T \|J^\delta(v_1)(\cdot, s) - J^\delta(v_2)(\cdot, s)\|_{H^1(\Omega)}^2 ds
\leq C\lambda^{2\beta-1}_{N(\delta)} \exp \left( \frac{2TM_1\lambda^\beta_N}{a} \right) K_M^2\|g\|^2_{L^2(\Omega)}
\int_t^T (C\lambda^{2\beta-1}_{N(\delta)} \exp \left( \frac{2TM_1\lambda^\beta_N}{a} \right) K_M^2\|g\|^2_{L^2(\Omega)} (T-s)^p) \|v_1 - v_2\|_{C([0, T]; H^2(\Omega))} ds
\leq \left( \frac{C\lambda^{2\beta-1}_{N(\delta)} \exp \left( \frac{2TM_1\lambda^\beta_N}{a} \right) K_M^2\|g\|^2_{L^2(\Omega)} (T-t)^{p+1}}{(p+1)!} \right) \|v_1 - v_2\|_{C([0, T]; H^2(\Omega))}^2.
\]
From the induction principle, we infer that (98) fulfilled for all \( v_1, v_2 \in C([0, T]; H^1(\Omega)) \). Since the fact that

\[
\lim_{m \to +\infty} \left( C \lambda_{N(\delta)}^{2\beta - 1} \exp \left( \frac{2TM_m \lambda_{N(\delta)}^{a + 1}}{a} \right) K_M^2 \| g^\delta \|_{L^2(\Omega)}^2 T \right)^m = 0,
\]

there exists the number \( m_0 \in \mathbb{N}^* \) such that

\[
\left( C \lambda_{N(\delta)}^{2\beta - 1} \exp \left( \frac{2TM_m \lambda_{N(\delta)}^{a + 1}}{a} \right) K_M^2 \| g^\delta \|_{L^2(\Omega)}^2 T \right)^{m_0} < 1,
\]

which allows us to obtain that \( \mathcal{J}^{m_0} = w \) has a unique solution \( W^{N(\delta), \delta} \in C([0, T]; H^1(\Omega)) \). By an argument similar to that used for the proof of Theorem 4.2, we imply that (96) has a unique solution \( W^{N(\delta), \delta} \in C([0, T]; H^1(\Omega)) \).

**Part 2.** Estimate \( \| W^{N(\delta), \delta}(\cdot, t) - u(\cdot, t) \|_{H^1(\Omega)} \). We observe that

\[
W^{N(\delta), \delta}(x, t) - u(x, t)
\]

\[
= \sum_{n=1}^{N(\delta)} \exp \left( \frac{\lambda_n^\beta}{1 + a \lambda_n} \int_t^T M \left( \| \nabla W^{N(\delta), \delta}(\cdot, \tau) \|_{L^2(\Omega)} \right) d\tau \right) \langle g^\delta - g, e_n \rangle e_n(x)
\]

\[
+ \sum_{n=1}^{N(\delta)} \left[ \exp \left( \frac{\lambda_n^\beta}{1 + a \lambda_n} \int_t^T M \left( \| \nabla W^{N(\delta), \delta}(\cdot, \tau) \|_{L^2(\Omega)} \right) d\tau \right) - \exp \left( \frac{\lambda_n^\beta}{1 + a \lambda_n} \int_t^T M \left( \| \nabla u(\cdot, \tau) \|_{L^2(\Omega)} \right) d\tau \right) \right] \langle g, e_n \rangle e_n(x)
\]

\[
- \sum_{n=N(\delta)+1}^{\infty} \exp \left( \frac{\lambda_n^\beta}{1 + a \lambda_n} \int_t^T M \left( \| \nabla u(\cdot, \tau) \|_{L^2(\Omega)} \right) d\tau \right) \langle g, e_n \rangle e_n(x).
\]

The latter equality implies that

\[
\| W^{N(\delta), \delta}(\cdot, t) - u(\cdot, t) \|_{H^1(\Omega)}^2
\]

\[
\leq 3 \sum_{n=1}^{N(\delta)} \lambda_n \exp \left( \frac{2\lambda_n^\beta}{1 + a \lambda_n} \int_t^T M \left( \| \nabla W^{N(\delta), \delta}(\cdot, \tau) \|_{L^2(\Omega)} \right) d\tau \right) \langle g^\delta - g, e_n \rangle^2
\]

\[
+ 3 \sum_{n=1}^{N(\delta)} \lambda_n \left[ \exp \left( \frac{2\lambda_n^\beta}{1 + a \lambda_n} \int_t^T M \left( \| \nabla W^{N(\delta), \delta}(\cdot, \tau) \|_{L^2(\Omega)} \right) d\tau \right) - \exp \left( \frac{2\lambda_n^\beta}{1 + a \lambda_n} \int_t^T M \left( \| \nabla u(\cdot, \tau) \|_{L^2(\Omega)} \right) d\tau \right) \right] \langle g, e_n \rangle^2
\]

\[
+ 3 \sum_{n=N(\delta)+1}^{\infty} \lambda_n \exp \left( \frac{2\lambda_n^\beta}{1 + a \lambda_n} \int_t^T M \left( \| \nabla u(\cdot, \tau) \|_{L^2(\Omega)} \right) d\tau \right) \langle g, e_n \rangle^2
\]

\[
= \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3.
\]
Now, we look at the term $L_1$. Due to (H2), we estimate $L_1$ as follows
\[
L_1 \leq 3\lambda_N(\delta) \exp \left( \frac{2TM_1\beta^{-1}}{a} \right) \sum_{n=1}^{N(\delta)} (g^n - g, e_n)^2
\]
\[
\leq 3\lambda_N(\delta) \exp \left( \frac{2TM_1\beta^{-1}}{a} \right) \|g - g\|_{L^2(\Omega)}^2
\]
\[
\leq 3\lambda_N(\delta) \exp \left( \frac{2TM_1\beta^{-1}}{a} \right) \delta^2. \tag{101}
\]
For the quantity $L_2$, using the inequality $|e^a - e^b| \leq |a - b| \max(e^a, e^b)$ and (H3), we obtain the following estimate
\[
L_2 \leq 3 \sum_{n=1}^{N(\delta)} \lambda_n^{2\beta - 1} \left[ \int_t^T \mathcal{M} \left( \|\nabla W^{N(\delta),\delta}(\cdot, s)\|_{L^2(\Omega)} \right) ds - \int_t^T \mathcal{M} \left( \|\nabla u(\cdot, s)\|_{L^2(\Omega)} \right) ds \right]^2 \exp \left( \frac{2(T-t)M_1\beta^{-1}}{a} \right) (g, e_n)^2
\]
\[
\leq 3 \sum_{n=1}^{N(\delta)} \lambda_n^{2\beta - 1} (g, e_n)^2 \exp \left( \frac{2(T-t)M_1\beta^{-1}}{a} \right) K_M^2 \int_t^T \|\nabla(W^{N(\delta),\delta}(\cdot, s) - u(\cdot, s))\|_{L^2(\Omega)}^2 ds
\]
\[
\leq 3D_\beta K_M^2 \int_t^T \|\nabla(W^{N(\delta),\delta}(\cdot, s) - u(\cdot, s))\|_{L^2(\Omega)}^2 ds. \tag{102}
\]
Since the fact that $\beta > 1$, the term $L_3$ is bounded by
\[
L_3 = 3 \sum_{n=N(\delta)+1}^{\infty} \lambda_n^{2\beta - 1} \sum_{j=1}^{\infty} \lambda_n^{2\beta - 1} \exp \left( \frac{2TM_1\beta^{-1}}{a(1 + a\lambda_n)} \right) \int_t^T \mathcal{M} \left( \|\nabla u(\cdot, s)\|_{L^2(\Omega)} \right) ds (g, e_n)^2
\]
\[
\leq 3\lambda_N^{2-2\beta} \sum_{n=N(\delta)+1}^{\infty} \lambda_n^{2\beta - 1} \exp \left( \frac{2TM_1\beta^{-1}}{a} \right) (g, e_n)^2 \leq 3\lambda_N^{2-2\beta} D_\beta. \tag{103}
\]
Combining (100), (101), (102), (103), we get that
\[
\|W^{N(\delta),\delta}(\cdot, t) - u(\cdot, t)\|_{H^1(\Omega)}^2
\]
\[
\leq 3\lambda_N(\delta) \exp \left( \frac{2TM_1\beta^{-1}}{a} \right) \delta^2
\]
\[
+ 3\lambda_N^{2-2\beta} D_\beta + 3D_\beta K_M^2 \int_t^T \|\nabla(W^{N(\delta),\delta}(\cdot, s) - u(\cdot, s))\|_{L^2(\Omega)}^2 ds
\]
\[
\leq 3\lambda_N(\delta) \exp \left( \frac{2TM_1\beta^{-1}}{a} \right) \delta^2
\]
\[
+ 3\lambda_N^{2-2\beta} D_\beta + 3CD_\beta K_M^2 \int_t^T \|W^{N(\delta),\delta}(\cdot, s) - u(\cdot, s)\|_{H^1(\Omega)}^2 ds.
\]
From Gronwall’s inequality to obtain
\[
\|u^{N,\delta}(\cdot, t) - u(\cdot, t)\|_{H^1(\Omega)}^2
\]
\[
\leq \left[ 3\lambda_N(\delta) \exp \left( \frac{2TM_1\beta^{-1}}{a} \right) \delta^2 + 3\lambda_N^{2-2\beta} D_\beta \right] \exp \left( 3CD_\beta K_M^2(T-t) \right).
\]
The proof of the theorem is done. \qed
4.3. The case $\beta > 1$, $F = F(x, t)$.

(Hypothesis 5): For $g \in L^p(\Omega)$ and $g^\delta \in L^p(\Omega)$ represent the exact data and the measured data, respectively, here $\delta > 0$ is a noise level and satisfying
\[
\|g^\delta - g\|_{L^p(\Omega)} \leq \delta.
\]

We let $F^\delta \in L^\infty(0, T; \mathcal{G}_{\sigma_1, \sigma_2}(\Omega))$ to be the perturbed functions of
\[
F \in L^\infty(0, T; \mathcal{G}_{\sigma_1, \sigma_2}(\Omega))
\]

where the constant $\delta$ is a noise level and satisfying
\[
\|F^\delta - F\|_{L^\infty(0, T; \mathcal{G}_{\sigma_1, \sigma_2}(\Omega))} \leq \delta.
\]

We propose the regularized solution $W^{N(\delta), \delta}(x, t)$ given as
\[
W^{N(\delta), \delta}(x, t) = \sum_{n=1}^{N(\delta)} \exp \left( -\frac{\lambda_n^\delta}{1 + a_n} \int_t^T M \left( \|\nabla W^{N(\delta), \delta}(\cdot, \tau)\|_{L^2(\Omega)} \right) d\tau \right) \langle g^\delta, e_n \rangle e_n(x)
\]

The next theorem will be the main result on the error estimate in $L^{\frac{2N}{N-2\alpha}}(\Omega)$.

**Theorem 4.4** ($L^{\frac{2N}{N-2\alpha}}$-Estimate). Let $N, \alpha, \beta \geq 1$, for $\alpha' \in (-\frac{N}{4}, 0]$ and assume that the final data $g^\delta, g \in L^p(\Omega)$, for $p \geq \frac{2N}{N-4\alpha}$ satisfying
\[
\|g^\delta - g\|_{L^p(\Omega)} \leq \delta, \quad \text{for } \delta > 0,
\]

and $F, F^\delta \in L^\infty(0, T; \mathcal{G}_{\sigma_1, \sigma_2}(\Omega))$ for $\sigma_1 \geq \alpha + \beta - 2 > 0$ and $\sigma_2 \geq \frac{TM_1}{a\lambda_1}$ satisfying
\[
\|F^\delta - F\|_{L^\infty(0, T; \mathcal{G}_{\sigma_1, \sigma_2}(\Omega))} \leq \delta, \quad \text{for } \delta > 0,
\]

For $1 < \alpha \leq \min\left\{ \frac{N}{4}, \frac{\beta}{2} \right\}$, let $N(\delta) > 0$ such that
\[
\lim_{\delta \to 0^+} N(\delta) = \infty, \quad \text{and} \quad \lim_{\delta \to 0^+} \lambda_n^{2\alpha - 2\alpha'} \exp \left( \frac{a\lambda_n^{2\alpha - 1}}{2TM_1} \right) \delta^2 = \lim_{\delta \to 0^+} \lambda_n^{4\alpha - 2\beta} G_{\alpha, \beta} = 0,
\]

with
\[
G_{\alpha, \beta} := \sum_{n=1}^{\infty} \lambda_n^{2\beta - 2\alpha} \exp \left( \frac{2TM_1\lambda_n^{2\alpha - 1}}{a} \right) \langle g, e_n \rangle^2 < \infty.
\]

Then, we have for all $t \in [0, T]$
\[
\|W^{N(\delta), \delta}(\cdot, t) - u(\cdot, t)\|_{L^{\frac{2N}{N-2\alpha}}(\Omega)} \leq CB_\delta \exp \left( \frac{3CG_{\alpha, \beta}K_2^2(T - t)}{2} \right),
\]

where the constant $B_\delta > 0$ depends on $\delta$.

**Proof.** Noting that the proof of the existence and uniqueness of the equation (104) are completely the same as in Theorem 4.3. For the regularized solution $W^{N(\delta), \delta}$ (defined in (96)) and the exact solution $u$ (defined in (90)) to Problem ($\mathbb{P}_g$), we observe that
\[
\|W^{N(\delta), \delta}(\cdot, t) - u(\cdot, t)\|_{D(A^\alpha)}^2 \leq \sum_{i=1}^{5} 5P_i.
\]

(108)
Now, we look at the term $\mathcal{P}_1$. Due to $M(z) \leq M_1, \forall z \in \mathbb{R}$, we estimate $\mathcal{P}_1$ as follows

$$
\mathcal{P}_1 \leq \lambda_{2N(\delta)}^{2\alpha - 2\alpha'} \exp \left( \frac{2TM_1\lambda_{N(\delta)}^{\beta - 1}}{a} \right) \sum_{n=1}^{N(\delta)} \lambda_n^{2\alpha'} \langle g^\beta - g, e_n \rangle^2
$$

$$
\leq \lambda_{2N(\delta)}^{2\alpha - 2\alpha'} \exp \left( \frac{2TM_1\lambda_{N(\delta)}^{\beta - 1}}{a} \right) \|g^\beta - g\|^2_{\mathcal{D}(A^{\alpha'})}.
$$

For $-\frac{N}{4} < \alpha' \leq 0$ and $p \geq \frac{2N}{N-4\alpha'}$, we have the Sobolev embedding $L^p(\Omega) \hookrightarrow D(A^{\alpha'})$. Thus, we get

$$
\mathcal{P}_1 \leq C\lambda_{N(\delta)}^{2\alpha - 2\alpha'} \exp \left( \frac{2TM_1\lambda_{N(\delta)}^{\beta - 1}}{a} \right) \|g^\beta - g\|^2_{L^p(\Omega)} \leq C\lambda_{N(\delta)}^{2\alpha - 2\alpha'} \exp \left( \frac{2TM_1\lambda_{N(\delta)}^{\beta - 1}}{a} \right) e^2.
$$

(109)

For the quantity $\mathcal{P}_2$, using the inequality $|e^a - e^b| \leq |a - b| \max(e^a, e^b)$ and (H3), we obtain the following estimate

$$
\mathcal{P}_2 \leq \sum_{n=1}^{N(\delta)} \lambda_{n}^{2\beta - 2\alpha} \left[ \int_t^T \left\{ \mathcal{M} \left( \|\nabla W^{N(\delta), \delta}(\cdot, s)\|_{L^2(\Omega)} \right) - \mathcal{M} \left( \|\nabla u(\cdot, s)\|_{L^2(\Omega)} \right) \right\} ds \right]^2
$$

$$
\times \exp \left( \frac{2(T-t)M_1\lambda_n^{\beta - 1}}{a} \right) \langle g, e_n \rangle^2
$$

$$
\leq \sum_{n=1}^{N(\delta)} \lambda_{n}^{2\beta - 2\alpha} \langle g, e_n \rangle^2 \exp \left( \frac{2(T-t)M_1\lambda_n^{\beta - 1}}{a} \right) K_M^2 \int_t^T \|\nabla(W^{N(\delta), \delta}(\cdot, s) - u(\cdot, s))\|^2_{L^2(\Omega)} ds
$$

$$
\leq G_{\alpha, \beta} K_M^2 \int_t^T \|\nabla(W^{N(\delta), \delta}(\cdot, s) - u(\cdot, s))\|^2_{L^2(\Omega)} ds.
$$

(110)
Since the fact that \( \beta > 1 \), the term \( P_3 \) is bounded by

\[
P_3 = \sum_{n=1}^{\infty} \lambda_n^{4\alpha - 2\beta} \lambda_n^{2\beta - 2\alpha} \exp\left(\frac{2\lambda_n^2}{1 + a\lambda_n} \int_t^T \mathcal{M}\left(\|\nabla u(\cdot, s)\|_{L^2(\Omega)}\right) ds\right) \langle g, e_n \rangle^2
\]

\[
\leq \lambda_n^{4\alpha - 2\beta} \sum_{j=1}^{\infty} \lambda_j^{2\beta - 2\alpha} \exp\left(\frac{2TM_1\lambda_n^\beta - 1}{a}\right) \langle g, e_n \rangle^2 \leq \lambda_n^{4\alpha - 2\beta} G_{\alpha, \beta}.
\]

For the term \( P_4 \), using the Hölder inequality, we deduce that

\[
P_4 \leq P_{41} + P_{42}.
\]

where

\[
P_{41} := 2 \sum_{n=1}^{N(\delta)} \frac{\lambda_n^{2\alpha}}{(1 + a\lambda_n)^2} \int_t^T \exp\left(- \frac{\lambda_n^3}{1 + a\lambda_n} \int_s^t \mathcal{M}(\|\nabla W^{N(\delta), \delta}(\cdot, \tau)\|_{L^2(\Omega)}) d\tau\right)
\]

\[
\times \left( F_n(s) - F_n(s) \right)^2 ds
\]

\[
P_{42} := 2 \sum_{n=1}^{N(\delta)} \frac{\lambda_n^{2\alpha}}{(1 + a\lambda_n)^2} \int_t^T \left| \exp\left(- \frac{\lambda_n^3}{1 + a\lambda_n} \int_s^t \mathcal{M}(\|\nabla u(\cdot, \tau)\|_{L^2(\Omega)}) d\tau\right) - \exp\left(- \frac{\lambda_n^3}{1 + a\lambda_n} \int_s^t \mathcal{M}(\|\nabla u(\cdot, \tau)\|_{L^2(\Omega)}) d\tau\right) \right|^2 |F_n(s)|^2 ds.
\]

We continue to estimate the term \( P_{41} \) as follow

\[
P_{41} \leq \frac{2}{a^2} \int_t^T \|F^\delta(s) - F(s)\|^2_{H_{1, \sigma_1, \sigma_2}(\Omega)} ds \leq \frac{2T}{a^2} \|F^\delta - F\|^2_{L^\infty(0, T; H_{1, \sigma_1, \sigma_2}(\Omega))} \leq \frac{2T\delta^2}{a^2}
\]

Using the basic inequality \( 1 - e^{-z} \leq z \) and (H3), we estimate the term \( P_{42} \) as

\[
P_{42} \leq 2 \sum_{n=1}^{N(\delta)} \frac{\lambda_n^{2\alpha}}{(1 + a\lambda_n)^2} \int_t^T \left( 2(s-t)M_1\lambda_n^{\beta - 1} \right)
\]

\[
\times \left| 1 - \exp\left(\frac{\lambda_n^3}{1 + a\lambda_n} \int_s^t \left( \mathcal{M}(\|\nabla u(\cdot, \tau)\|_{L^2(\Omega)}) - M_1 \right) d\tau\right) \right|^2 |F_n(s)|^2 ds
\]

\[
\leq 2 \sum_{n=1}^{N(\delta)} \frac{\lambda_n^{2\alpha + 2\beta}}{(1 + a\lambda_n)^2} \int_t^T \exp\left(\frac{2(s-t)M_1\lambda_n^{\beta}}{a\lambda_1}\right) \int_s^t \left( K_M \|\nabla u(\cdot, \tau)\|_{L^2(\Omega)} - M_1 \right) |F_n(s)|^2 ds
\]

\[
\leq \frac{4(K_M T \|u\|^2_{L^2(0, T; H^1(\Omega))} + M_1^2 T)}{a} \sum_{n=1}^{N(\delta)} \lambda_n^{2\alpha + 2\beta - 4} \exp\left(\frac{2TM_1\lambda_n^{\beta}}{a\lambda_1}\right) |F_n(s)|^2 ds
\]

\[
\leq \frac{4T(K_M T \|u\|^2_{L^2(0, T; H^1(\Omega))} + M_1^2 T)}{a} \||F||^2_{L^\infty(0, T; G_{\sigma_1, \sigma_2}(\Omega))}.
\]

Similar to the previous one, we get

\[
P_5 \leq \sum_{n=N(\delta)+1}^{\infty} \lambda_n^{\beta} \left[ \frac{1}{1 + a\lambda_n} \int_t^T \exp\left(- \frac{\lambda_n^3}{1 + a\lambda_n} \int_s^t \mathcal{M}(\|\nabla u(\cdot, \tau)\|_{L^2(\Omega)}) d\tau\right) F_n(s) ds \right]^2
\]

\[
\leq \lambda_n^{\beta} \frac{T}{a} \||F||^2_{L^\infty(0, T; G_{\sigma_1, \sigma_2}(\Omega))}.
\]
Combining (108)- (115), and put
\[ 0 < B_\delta := 5C_{\lambda N(\delta)}^{2T-2a^2} \exp \left( \frac{2TM_1\lambda^\beta N(\delta)}{a} \right) \delta^2 + 5\lambda^{4n-2\alpha}G_{\alpha,\beta} + \frac{10T\delta^2}{a^2} \]
\[ + 20T(K_MT \|u\|_{L^2(0,T;H^1(\Omega))}^2 + M^2T) \|F\|_{L^\infty(0,T;G_{\sigma_1,\sigma_2}(\Omega))}^2 \]
\[ + \frac{5\lambda^{2}\beta}{a} \|F\|_{L^\infty(0,T;G_{\sigma_1,\sigma_2}(\Omega))}^2, \]
we deduce that
\[ \left\| W^{N(\delta),\delta}(\cdot, t) - u(\cdot, t) \right\|_{D(A^\alpha)}^2 \]
\[ \leq B_\delta + 5G_{\alpha,\beta}K_M^2 \int_t^T \left\| W^{N(\delta),\delta}(\cdot, s) - u(\cdot, s) \right\|_{H^1(\Omega)}^2 ds \]
\[ \leq B_\delta + CG_{\alpha,\beta}K_M^2 \int_t^T \left\| W^{N(\delta),\delta}(\cdot, s) - u(\cdot, s) \right\|_{D(A^\alpha)}^2 ds, \]
where for \( \alpha > 1 \), we have the Sobolev embedding \( D(A^\alpha) \hookrightarrow H^1(\Omega) \). By using the Gronwall’s inequality to obtain that
\[ \left\| W^{N(\delta),\delta}(\cdot, t) - u(\cdot, t) \right\|_{D(A^\alpha)} \leq B_\frac{1}{2} \exp \left( \frac{CG_{\alpha,\beta}K_M^2(T-t)}{2} \right). \]
For \( \alpha \in [1, \frac{N}{4}) \), we have that \( D(A^\alpha) \hookrightarrow L^\frac{2N}{N+2}(\Omega) \), which implies that
\[ \left\| W^{N(\delta),\delta}(\cdot, t) - u(\cdot, t) \right\|_{L^\frac{2N}{N+2}(\Omega)} \leq C \left\| W^{N(\delta),\delta}(\cdot, t) - u(\cdot, t) \right\|_{D(A^\alpha)} \]
\[ \leq C B_\frac{1}{2} \exp \left( \frac{CG_{\alpha,\beta}K_M^2(T-t)}{2} \right), \] (117)
for all \( t \in [0, T) \), then (107) followed. The proof is done.

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\section*{REFERENCES}

[1] R. M. P. Almeida, S. N. Antontsev and J. C. M. Duque, On a nonlocal degenerate parabolic problem, *Nonlinear Anal. RWA*, 27 (2016), 146–157.
[2] V. V. Au, M. Kirane and N. H. Tuan, On a terminal value problem for a system of parabolic equations with nonlinear-nonlocal diffusion terms, *Discrete Contin. Dyn. Syst. Ser. B*, 174 (2020), 27 pages.
[3] G. Autuori and P. Pucci, Kirchhoff systems with dynamic boundary conditions, *Nonlinear Anal.*, 73 (2010), 1952-1965.
[4] G. Autuori, P. Pucci and M. C. Salvatori, Global nonexistence for nonlinear Kirchhoff systems, *Arch. Ration. Mech. Anal.*, 196 (2010), 489–516.
[5] C. Cao, M. A. Rammaha and E. S. Titi, The Navier-Stokes equations on the rotating 2-D sphere: Gevrey regularity and asymptotic degrees of freedom, *Z. Angew. Math. Phys.*, 50 (1999), 341–360.
[6] T. Caraballo, H. Crauel, J. A. Langa and J. C. Robinson, The effect of noise on the Chafee-Infante equation: A nonlinear case study, *Proc. Amer. Math. Soc.*, 135 (2007), 373–382.
[7] T. Caraballo, M. Herrera-Cobos and P. Marín-Rubio, Global attractor for a nonlocal p-Laplacian equation without uniqueness of solution, *Discrete Contin. Dyn. Syst. Ser. B*, 22 (2017), 1801–1816.
[8] T. Caraballo, M. Herrera-Cobos and P. Marín-Rubio, Long-time behavior of a non-autonomous parabolic equation with nonlocal diffusion and sublinear terms, *Nonlinear Anal.*, 121 (2015), 3–18.
[9] T. Caraballo, M. Herrera-Cobos and P. Marín-Rubio, Robustness of nonautonomous attractors for a family of nonlocal reaction-diffusion equations without uniqueness, *Nonlinear Dynam.*, 84 (2016), 35–50.
[10] T. Caraballo, M. Herrera-Cobos and P. Marín-Rubio, Global attractor for a nonlocal $p$-Laplacian equation without uniqueness of solution, *Discrete Contin. Dyn. Syst. Ser. B.*, 22 (2017), 1801–1816.
[11] T. Caraballo, M. Herrera-Cobos and P. Marín-Rubio, Asymptotic behaviour of nonlocal $p$-Laplacian reaction-diffusion problems, *J. Math. Anal. Appl.*, 459 (2018), 997–1015.
[12] T. Caraballo, J. A. Langa and J. Valero, Extremal bounded complete trajectories for nonautonomous reaction-diffusion equations with discontinuous forcing term, *Rev. Mat. Complut.*, 33 (2020), 583–617.
[13] A. S. Carasso, J. G. Sanderson and J. M. Hyman, Digital removal of random media image degradations by solving the diffusion equation backwards in time, *SIAM J. Numer. Anal.*, 15 (1978), 344–367.
[14] A. N. Carvalho, J. A. Langa and J. C. Robinson and A. Suárez, Characterization of non-autonomous attractors of a perturbed infinite-dimensional gradient system, *J. Differential Equations*, 236 (2007), 570–603.
[15] N.-H. Chang and M. Chipot, Nonlinear nonlocal evolution problems, *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, 97 (2003), 423–445.
[16] M. Chipot and B. Lovat, Some remarks on nonlocal elliptic and parabolic problems, *Proceedings of the Second World Congress of Nonlinear Analysts, Part 7* (Athens, 1996), *Nonlinear Analysis: TMA*, 30 (1997), 4619–4627.
[17] I. Chueshov, Long-time dynamics of Kirchhoff wave models with strong nonlinear damping, *J. Differential Equations*, 252 (2012), 1229–1262.
[18] L. Dawidowski, The quasilinear parabolic Kirchhoff equation, *Open Math.*, 15 (2017), 382–392.
[19] Y. Fu and M. Xiang, Existence of solutions for parabolic equations of Kirchhoff type involving variable exponent, *Appl. Anal.*, 95 (2016), 524–544.
[20] M. Ghisi and M. Gobbino, Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: Time-decay estimates, *J. Differential Equations*, 245 (2008), 2979–3007.
[21] M. Ghisi and M. Gobbino, Hyperbolic-parabolic singular perturbation for nondegenerate Kirchhoff equations with critical weak dissipation, *Math. Ann.*, 354 (2012), 1079–1102.
[22] M. Gobbino, Quasilinear degenerate parabolic equations of Kirchhoff type, *Math. Meth. Appl. Sci.*, 22 (1999), 375–388.
[23] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, 840. Springer-Verlag, Berlin-New York, 1981.
[24] S. Kundu, K. A. Pani and M. Khebchareon, On Kirchhoff's model of parabolic type, *Numer. Funct. Anal. Optim.*, 37 (2016), 719–732.
[25] Z. Liu and S. Guo, On ground states for the Kirchhoff-type problem with a general critical nonlinearity, *J. Math. Anal. Appl.*, 426 (2015), 267–287.
[26] L. A. Medeiros, J. Limaco and S. B. Menezes, Vibrations of elastic strings: Mathematical aspects, *I., J. Comput. Anal. Appl.*, 4 (2002), 91–127.
[27] X. Mingqi, V. D. Rădulescu and B. Zhang, Nonlocal Kirchhoff diffusion problems: Local existence and blow-up of solutions, *Nonlinearity*, 31 (2018), 3228–3250.
[28] X. Peng, Y. Shang and X. Zheng, Pullback attractors of nonautonomous nonclassical diffusion equations with nonlocal diffusion, *Z. Angew. Math. Phys.*, 69 (2018), Paper No. 110, 14 pp.
[29] C. A. Raposo, M. Sepúlveda, O. V. Villagrán, D. C. Pereira and M. L. Santos, Solution and asymptotic behaviour for a nonlocal coupled system of reaction-diffusion, *Acta Appl. Math.*, 102 (2008), 37–56.
[30] J. Simsen and J. Ferreira, A global attractor for a nonlocal parabolic problem, *Nonlinear Stud.*, 21 (2014), 405–416.
[31] T. H. Skaggs and Z. J. Kabala, Recovering the history of a groundwater contaminant plume: Method of quasi-reversibility, *Water Resources Research.*, 31 (1995), 2669–2673.
[32] N. H. Tuan, V. A. Khoa and V. A. Vo, Analysis of a quasi-reversibility method for a terminal value quasi-linear parabolic problem with measurements, *SIAM J. Math. Anal.*, 51 (2019), 60–85.
[33] N. H. Tuan, D. H. Q. Nam and T. M. N. Vo, On a backward problem for the Kirchhoff’s model of parabolic type, *Comput. Math. Appl.*, **77** (2019), 15–33.

[34] N. H. Tuan, V. A. Vo, V. A. Khoa and D. Lesnic, Identification of the population density of a species model with nonlocal diffusion and nonlinear reaction, *Inverse Problems*, **33** (2017), 055019, 40 pp.

[35] S. Zheng and M. Chipot, Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms, *Asymptot. Anal.*, **45** (2005), 301–312.

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