Operational approach to the Uhlmann holonomy

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I. INTRODUCTION

If a quantum system depends on a slowly varying external parameter, Berry [1] showed that there is a geometric phase factor associated to the path that an eigenvector of the corresponding Hamiltonian traverses during the evolution. These geometric phase factors were later generalized by Wilczek and Zee [2] to holonomies, i.e., unitary state changes associated with the motion of a degenerate subspace of the parameter-dependent Hamiltonian. In view of the Berry phase and Wilczek-Zee holonomy, one may ask if a phase or a holonomy can be associated with families of mixed states. This was answered in the affirmative by Uhlmann [3] by introducing “amplitudes” of density operators, and a condition for parallelity of amplitudes along a family of density operators (for other approaches to geometric phases and holonomies of mixed states and their relation to the Uhlmann approach, see Refs. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]).

As mentioned above, the Berry phases and the non-Abelian holonomies can be given a clear physical and operational interpretation in terms of the evolution caused by adiabatically evolving quantum systems. One may also consider the evolution as caused by a sequence of projective measurements of observables with nondegenerate or degenerate eigenvalues, giving rise to a Berry phase or a non-Abelian holonomy, respectively. The physical interpretation of the Uhlmann amplitudes and their parallel transport is less clear. One interpretation of the Uhlmann amplitude [12, 13, 14, 15, 16], which corresponds to the state vector of a purification on a combined system and ancilla. Here we suggest another interpretation, where the amplitude corresponds to an “off-diagonal block” of a density operator with respect to two orthogonal subspaces. In this framework, we address the question of how to obtain an explicitly operational approach to the Uhlmann holonomy, which, to the knowledge of the authors, has not been previously considered [17, 18, 19].

The structure of the paper is as follows. In Sec. III we give a brief introduction to the Uhlmann holonomy. In Sec. IV we introduce our interpretation of the Uhlmann amplitude and show that it is possible to determine the amplitude using an interferometric approach. Given the interpretation of the amplitude we consider an operational implementation of the parallelity condition in Sec. V and in Sec. VI we use the parallelity condition to establish the parallel transport. In Sec. VII we present a technique to generate the states needed in the parallel transport procedure. We generalize the approach to sequences of not faithful density operators (operators not of full rank) and introduce a preparation procedure for the density operators needed in the generalized case in Sec. VII. The paper ends with the conclusions in Sec. VII.

II. UHL Mann HOLonomy

Consider a sequence of density operators $\sigma_1, \sigma_2, \ldots, \sigma_K$ on a Hilbert space $\mathcal{H}_I$. A sequence of amplitudes of these states are operators $W_1, W_2, \ldots, W_K$ on $\mathcal{H}_I$, such that $\sigma_k = W_k W_k^\dagger$. In Uhlmann’s terminology [3], a density operator $\sigma$ is faithful if its range $R(\sigma)$ coincides with the whole Hilbert space, i.e., if $R(\sigma) = \mathcal{H}_I$ (Note that what we refer to as a faithful operator is often referred to as an operator of “full rank”). For the present we shall assume that all density operators are faithful, and return to the question of unfaithful operators in Sec. VII. Using the polar decomposition [20], the amplitudes can be written $W_k = \sqrt{\sigma_k} V_k$, where $V_k$ is unitary. The gauge-freedom in the Uhlmann approach is the freedom to choose the unitary operators $V_k$. For faithful density operators $\sigma_k$ adjacent amplitudes are parallel if and only if $W_k^\dagger W_{k+1} > 0$. Given an initial amplitude $W_1$ and the corresponding unitary operator $V_1$, the parallelity condition uniquely determines the sequence of amplitudes $W_1, W_2, \ldots, W_K$, and unitaries $V_1, V_2, \ldots, V_K$. The Uhlmann holonomy of the sequence of density operators is defined as $U_{\text{Uhl}} = V_K V_1^\dagger [21]$. The Uhlmann approach fits naturally within the framework of differential geometry. The amplitudes are the
elements of the total space of the fiber bundle, with the
set of faithful density operators as the base manifold, and
the set of unitary operators $U(N)$ as the fibers. More-
over, $WW^\dagger = \sigma$ gives the projection from the total space
down to the base manifold. Finally, given a sequence
in the base manifold of density operators, the parallelli-
ty condition $W_k^\dagger W_k > 0$ induces a unique sequence in the
total space, leading to an element of the fiber as the re-
sulting holonomy.

III. INTERPRETATION OF THE UHLmann AMPLITUDE

As mentioned above, our first task is to find a phys-
ically meaningful interpretation of the Uhlmann ampli-
tude. We regard the density operators in the given se-
cquence as operators on a Hilbert space $\mathcal{H}_I$ of finite di-
ension $N$. In addition, we append a single qubit with Hilbert space $\mathcal{H}_S = \text{Sp}(|0\rangle, |1\rangle)$, with $|0\rangle$ and $|1\rangle$ or-
thonormal, and where Sp denotes the linear span. The
total Hilbert space we denote $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_S$. Note that $\mathcal{H}$ can be regarded as the state space of a single particle
in the two paths of a Mach-Zehnder interferometer, where $\mathcal{H}_I$ corresponds to the internal degrees of freedom (e.g., spin or polarization) of the particle and $|0\rangle$ and $|1\rangle$ correspond to the two paths.

We let $Q(\sigma^{(0)}, \sigma^{(1)})$ denote the set of density operators $\rho$ on $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_S$ such that

$$
\langle 0|\rho|0 \rangle = \frac{1}{2}\sigma^{(0)}, \quad \langle 1|\rho|1 \rangle = \frac{1}{2}\sigma^{(1)},
$$

i.e., $Q(\sigma^{(0)}, \sigma^{(1)})$ consists of those states that have the
prescribed “marginal states” $\sigma_0$ and $\sigma_1$, each found with probability one half. We span $Q(\sigma^{(0)}, \sigma^{(1)})$ by varying
the “off-diagonal” operator $\langle 0|\rho|1 \rangle$. What freedom do we
have in the choice of the operator $\langle 0|\rho|1 \rangle$? This question turns out to have the following answer.

**Proposition 1.** $\rho \in Q(\sigma^{(0)}, \sigma^{(1)})$ if and only if there exists an operator $\tilde{V}$ on $\mathcal{H}_I$ such that

$$
\rho = \frac{1}{2}\sigma^{(0)} \otimes |0\rangle\langle 0| + \frac{1}{2}\sigma^{(1)} \otimes |1\rangle\langle 1|
+ \frac{1}{2}\sqrt{\sigma^{(0)}\sigma^{(1)}} |0\rangle\langle 1| + \frac{1}{2}\sqrt{\sigma^{(1)}\sigma^{(0)}} |1\rangle\langle 0|,
$$

and

$$
\tilde{V}\tilde{V}^\dagger \leq \hat{I}_I.
$$

To prove this we use the following (see Lemma 13 in Ref. [22]): Let $A$, $B$, and $C$ be operators on $\mathcal{H}_I$, and let $A \geq 0$, $B \geq 0$, and

$$
F = A \otimes |0\rangle\langle 0| + C \otimes |0\rangle\langle 1| + C^\dagger \otimes |1\rangle\langle 0| + B \otimes |1\rangle\langle 1|.
$$

Then $F$ is positive semidefinite if and only if

$$
P_{R(A)}CP_{R(B)} = C, \quad A \geq CB^\ominus C^\dagger,
$$

where $P_{R(A)}$ and $P_{R(B)}$ denote the projectors onto the ranges $R(A)$ and $R(B)$ of $A$ and $B$, respectively. In
Eq. (5) the symbol $B^\ominus$ denotes the Moore-Penrose (MP)
pseudo inverse [20] of $B$. The reason why the MP inverse
is used is to allow us to handle those cases when $A$ and $B$
have ranges that are proper subspaces of $\mathcal{H}_I$. Note that
when $B$ is invertible, the MP inverse coincides with the
ordinary inverse.

To prove Proposition 1 we first note that if $\rho$ can be
written as in Eq. (2), then $\text{Tr}(\rho) = 1$ and $\rho$ satisfies Eq. (4). If we compare Eqs. (2) and (4), we can identify $A$, $B$, and $C$, and see that they satisfy the conditions
in Eq. (5). From this follows that $\rho$ is positive semidefinite.

We can thus conclude that $\rho$ is a density operator and
an element of $Q(\sigma^{(0)}, \sigma^{(1)})$.

Now, we wish to show the converse, i.e., if $\rho \in Q(\sigma^{(0)}, \sigma^{(1)})$ then it can be written as in Eq. (2). By definition it follows that we can identify $A = \sigma^{(0)}/2$ and $B = \sigma^{(1)}/2$ in Eq. (3). Since $\rho$ is positive semidefinite
it follows that $C$ has to satisfy the conditions in Eq. (4)
and thus

$$
\frac{1}{2}\sigma^{(0)} \geq 2C\sigma^{(1)} \sigma^{\dagger},
$$

Define $\tilde{V} = 2\sqrt{\sigma^{(0)}\sigma^{(1)}}C\sqrt{\sigma^{(1)}}^{-1}$. From Eq. (6) it follows that $\tilde{V}$ satisfies $\tilde{V}\tilde{V}^\dagger \leq 1$. Moreover,

$$
\frac{1}{2}\sqrt{\sigma^{(0)}\sigma^{(1)}} \sqrt{\sigma^{(1)}} = P_{R(\sigma^{(0)})}CP_{R(\sigma^{(1)})} = C,
$$

where the last equality follows from Eq. (5). Thus we have shown that $\rho \in Q(\sigma^{(0)}, \sigma^{(1)})$ if and only if $\rho$ can be
written as in Eq. (2). This proves Proposition 1.

Now, consider the set of density operators $Q(\sigma, 1_I/N)$, i.e., when one of the marginal states is the maximally mixed state. According to Eq. (2) it follows that $\langle 0|\rho|1 \rangle = \sqrt{\text{Tr}(\rho)}/(2\sqrt{N})$. Note that the condition in Eq. (3) allows us to choose $\tilde{V}$ as an arbitrary unitary operator, and we thus obtain

$$
\rho \equiv D(\sigma, W) = \frac{1}{2\sqrt{N}} W \otimes |0\rangle\langle 1|
+ \frac{1}{2\sqrt{N}} W^\dagger \otimes |1\rangle\langle 0|
$$

where $W$ is an arbitrary Uhlmann amplitude of the
density operator $\sigma$, i.e., $\sigma = WW^\dagger$. We thus have a physical realization of the Uhlmann amplitude as corre-
sponding to the off-diagonal operator $\langle 0|\rho|1 \rangle$. Note that $Q(\sigma, 1_I/N)$ contains more states than those corresponding
to amplitudes of $\sigma$. As will be seen later, these other states have an important role when we consider sequences of
density operators that are not faithful.
Let us note some of the differences between the above interpretation of the Uhlmann amplitude and the purification interpretation in terms of purifications \[12, 13, 14, 15, 16\]. In the latter, the amplitude corresponds to a pure state on a combination of the system and an ancilla, such that the density operator \(\sigma\) is retained when the ancilla is traced over. Similarly as for the purification interpretation, we consider here an extension to a larger Hilbert space, but in a different manner. In the purification approach we extend the Hilbert space as \(\mathcal{H}_I \otimes \mathcal{H}_s\), while in the present approach we extend the space as \(\mathcal{H}_I \otimes \mathcal{H}_s\). Since \(\mathcal{H}_s\) is two-dimensional it follows that \(\mathcal{H}_I \otimes \mathcal{H}_s \simeq \mathcal{H}_I \oplus \mathcal{H}_I\), i.e., the space we use to represent the state and its amplitude is isomorphic to an orthogonal sum of two copies of \(\mathcal{H}_I\) (one for each path of the interferometer). With respect to these two subspaces the amplitude is essentially carried by the off-diagonal operator, rather than the whole density operator \(D(\sigma, W)\). In some sense the amplitude describes the nature of the superposition between these two subspaces, or equivalently, the coherence of the particle between the two paths of the interferometer. One can also note from Eq. (8) that the total state \(\rho\) is isomorphic to an orthogonal sum of two copies of \(\mathcal{H}_I\) and \(\mathcal{H}_s\) can be experimentally determined by first applying onto the state \(\hat{V}\). Next, a measurement to determine the probability \(p\) to find the state \(|0\rangle\langle0|\). This probability turns out to be

\[
U_{tot} = 1_I \otimes |0\rangle\langle0| + U \otimes |1\rangle\langle1|,
\]

where \(U\) is a variable unitary operation on \(\mathcal{H}_I\). Next, a Hadamard gate is applied onto \(\mathcal{H}_s\), followed by a measurement to determine the probability \(p\) to find the state \(|0\rangle\langle0|\). This probability turns out to be

\[
p = \frac{1}{2} + \frac{1}{2\sqrt{N}} \text{ReTr}(\sqrt{\sigma}VU^\dagger).
\]

IV. REALIZATION OF THE UHL Mann HOLONOMY

A. Parallelity

Here, we address the question of implementing the parallelity condition between two amplitudes. Consider two faithful density operators \(\sigma_a\) and \(\sigma_b\). As mentioned above the corresponding amplitudes are parallel if and only if \(W_b^\dagger W_a > 0\). Let \(|\chi_k\rangle\}_k\) be an arbitrary orthonormal basis of \(\mathcal{H}_I\). We denote \(|\chi_k\rangle = |\chi_k\rangle|x\rangle\) and \(P_x = \hat{1}_I \otimes |x\rangle\langle x|\) for \(x = 0, 1\). Since we use the Hilbert space \(\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_s\) to represent a density operator and its amplitude, we consider two copies of \(\mathcal{H}\) in order to compare the amplitudes of two different density operators. On \(\mathcal{H} \otimes \mathcal{H}\) we define the following unitary and Hermitian operator:

\[
Z = \sum_{kl} |\chi_k,0\rangle\langle\chi_l,1| + \sum_{kl} |\chi_l,1\rangle\langle\chi_k,0|
\]

\[
+ \sum_{kl} |\chi_l,0\rangle\langle\chi_k,0| + \sum_{kl} |\chi_k,0\rangle\langle\chi_l,1|
\]

\[
+ P_0 \otimes P_0 + P_1 \otimes P_1.
\]

For \(\rho_a = D(\sigma_a, W_a)\) and \(\rho_b = D(\sigma_b, W_b)\),

\[
E = \text{Tr}(Z \rho_b \otimes \rho_a)
\]

\[
= \frac{1}{2} + \frac{1}{2N} \text{ReTr}(W_b^\dagger W_a).
\]

This means that the maximal value of the real and non-negative quantity \(E\) is reached when \(W_b\) is parallel to \(W_a\).

Now we use the fact that \(Z\) is a unitary operator in order to test the degree of parallelity between two amplitudes (see Fig. 1). Consider an “extra” qubit \(e\) whose Hilbert space \(\mathcal{H}_e\) is spanned by the orthonormal basis \(|0_e\rangle, |1_e\rangle\) (not to be confused with \(\mathcal{H}_s\) and the corresponding qubit in the construction of \(D(\sigma, W)\)). We first prepare the state \(|0_e\rangle \otimes |0_e\rangle \otimes \rho_e \otimes \rho_a\) on the total Hilbert space \(\mathcal{H}_e \otimes \mathcal{H} \otimes \mathcal{H} = \mathcal{H}_e \otimes \mathcal{H}_I \otimes \mathcal{H}_s\), followed by an application of the unitary operation

\[
U_Z = |0_e\rangle \otimes Z + |1_e\rangle \otimes \hat{1} \otimes \hat{1},
\]

i.e., an application of the unitary operation \(Z\), conditioned on the qubit \(e\). Finally, we apply the Hadamard gate on qubit \(e\), and measure the probability to find \(e\) in state \(|0_e\rangle \otimes |0_e\rangle\). This procedure results in the detection probability

\[
p = \frac{1}{2} + \frac{1}{2} E = \frac{3}{4} + \frac{1}{4N} \text{ReTr}(W_b^\dagger W_a).
\]

Thus, the probability \(p\) is maximal when \(W_b\) is parallel to \(W_a\) in the Uhlmann sense. In other words, given the state \(\rho_a = D(\sigma_a, W_a)\) we prepare various states \(\rho_b = D(\sigma_b, W_b)\) until we find the amplitude \(W_b\) that maximizes the probability \(p\) [28]. We have thus obtained an operational method to find parallel amplitudes. One may note the similarity between this procedure and the method introduced in [24] to estimate the trace of products of density operators.

The above approach is based on the fact that \(Z\) is a unitary operator and consequently corresponds to a
After the final step $K$ we have prepared the state $\rho_K$ containing the amplitude $W_K = \sqrt{\sigma_K U_{\text{Uhl}}} V_1$, where $U_{\text{Uhl}}$ is the Uhlmann holonomy and $V_1$ is the unitary part of the chosen initial amplitude $W_1$. The state $\rho_K$ can be modified by applying the unitary operator

$$U_{\text{mod}} = \mathbb{1}_I \otimes |0\rangle\langle 0| + V_1 \otimes |1\rangle\langle 1|,$$  \hspace{1cm} (15)

which results in the new state

$$\rho_K = U_{\text{mod}} \rho_K U_{\text{mod}}^\dagger = D(\sigma_K, \sqrt{\sigma_K} U_{\text{Uhl}}),$$  \hspace{1cm} (16)

and hence $\langle 0|\rho_K|1\rangle = \sqrt{\sigma_K} U_{\text{Uhl}}/(2\sqrt{N})$. Given this state we obtain the Uhlmann holonomy $U_{\text{Uhl}}$ as the unitary operator that yields the maximal detection probability, as described by Eq. (10).

Although the iterative procedure described above is realizable in principle, it is no doubt the case that it would be challenging in practice, since at each step of the procedure we must implement an optimization to find parallel amplitudes. However, the functions we optimize over have rather favorable properties. In the case of faithful density operators one can show that the function (taken over all amplitudes $W_k$) defined by Eq. (14) is such that there is no local maximum except for the global maximum. In the case of not faithful density operators the global maximum is not unique, but any of them gives the desired result, as shown in Sec. [11]. Moreover, it still remains the case that every local maximum is a global maximum. Hence, in both the faithful and unfaithful case we can apply local optimization methods (see, e.g., [33]). The fact that local optimization techniques are applicable is favorable for practical implementations, and improve the chances to find efficient procedures. However, a more detailed analysis would be required to determine what efficiency that ultimately can be obtained. This question is, however, not considered here.

\section*{V. STATE PREPARATION}

Since the parallel transport procedure involves repeated preparations of states $D(\sigma, W)$, with arbitrary amplitudes $W$ of $\sigma$, we here consider preparation techniques for such states (see Fig. 2). First, we show how to prepare the state $\rho = D(\sigma, \sqrt{\sigma})$. Consider the following orthogonal but not normalized vectors:

$$|\psi_k\rangle = \sqrt{\lambda_k/2}|k\rangle|0\rangle + |k\rangle |1\rangle/\sqrt{2N},$$  \hspace{1cm} (17)

where $\lambda_k$ and $|k\rangle$ are eigenvalues and corresponding orthonormal eigenvectors of $\sigma$. One can check that $\sum_k |\psi_k\rangle \langle \psi_k| = \rho$. The probability distribution $(1/N, \ldots, 1/N)$ is majorized by the vector $(\lambda_1, \ldots, \lambda_N)$. Thus, there exists a unitary matrix $U$ such that $U_{jk}^2 \lambda_k = 1/N$ for all $j = 1, \ldots, N$ (see also Refs. [13, 36]). Define the vectors

$$|\eta_j\rangle = \sqrt{N} \sum_k U_{jk} |\psi_k\rangle.$$  \hspace{1cm} (18)
One can check that these vectors are normalized. Since $\mathbf{U}$ is unitary it follows that $N^{-1}\sum_j |\eta_j\rangle\langle \eta_j| = \rho$. Thus, $\rho$ is the result if we prepare $|\eta_j\rangle$ with probability $1/N$. One can check that $\langle \eta_j|P_0\rangle |\eta_j\rangle = 1/2$. Thus there exist normalized vectors $|\eta_0\rangle, |\eta_1\rangle \in H_I$, such that

$$|\eta_j\rangle = \frac{1}{\sqrt{2}} |\eta_0\rangle |0\rangle + \frac{1}{\sqrt{2}} |\eta_1\rangle |1\rangle. \quad (19)$$

For any normalized $|\eta\rangle \in H_I$ there exist unitary operators $U_{j}^{(0)}$ and $U_{j}^{(1)}$ such that $U_{j}^{(0)} |\eta\rangle = |\eta_0\rangle$ and $U_{j}^{(1)} |\eta\rangle = |\eta_1\rangle$. The state $\rho$ is prepared if we apply a Hadamard gate to the state $|\eta\rangle |0\rangle$, followed by the application of the unitary operator $U_{j}^{(0)} \otimes |0\rangle \langle 0| + U_{j}^{(1)} \otimes |1\rangle \langle 1|$ with probability $p_j = 1/N$. In terms of an interferometric approach we thus apply a pair of unitary operations $U_{j}^{(0)}, U_{j}^{(1)}$, one in each path of the interferometer, with the choice of pair based on the output of a random generator shared between the two paths. This procedure leads to the output density operator $\rho = \mathcal{D}(\sigma, \sqrt{\sigma})$. To obtain a state that corresponds to an arbitrary amplitude, i.e., $\mathcal{D}(\sigma, \sqrt{\sigma}V)$ with $V$ unitary, we simply apply the unitary operation $1_I \otimes |0\rangle \langle 0| + V \otimes |1\rangle \langle 1|$ onto $\rho$.

VI. ADMISSIBLE SEQUENCES

So far we have assumed that the density operators are faithful. Here we consider the generalization to admissible sequences (defined below) of not faithful density operators $\mathbf{3}$. When the assumption of faithfulness is removed we have to review all the steps in the procedure. First, we note that Eqs. (4) - (8) are true irrespective of whether the involved density operators are faithful or not. By using the polar decompositions

$$\sqrt{\sigma_{k+1}} \sqrt{\sigma_k} = \sqrt{\sigma_{k+1} \sigma_k \sqrt{\sigma_{k+1} U_{k+1,k}}}, \quad (20)$$

the Uhlmann holonomy can be reformulated as $U_{\text{Uhl}} = U_{k,k-1} \ldots U_{2,1} \mathbf{21}$. If the density operators are not faithful then Eq. (20) does not determine $U_{k+1,k}$ uniquely. However, if we require $U_{k+1,k}$ to be a partial isometry $\mathbf{3}$ with initial space $\mathcal{R}(\sqrt{\sigma_{k+1} \sqrt{\sigma_k}})$ and final space $\mathcal{R}(\sqrt{\sigma_{k+1} \sqrt{\sigma_k}})$, then Eq. (20) uniquely determines $U_{k+1,k}$ to be the partial isometry

$$U_{k+1,k} = \sqrt{\sqrt{\sigma_{k+1} \sigma_k} \sqrt{\sigma_{k+1} \sqrt{\sigma_{k+1} U_{k+1,k}}}} \otimes \sqrt{\sigma_{k+1} \sqrt{\sigma_k}}. \quad (21)$$

If the sequence of density operators is such that the final space of $U_{k+1,k}$ matches the initial space of $U_{k+2,k+1}$, then we may define the Uhlmann holonomy as the partial isometry $\mathbf{3}$ with final space $\mathcal{R}(\sqrt{\sigma_{k+1} \sqrt{\sigma_k}})$ and initial space $\mathcal{R}(\sqrt{\sigma_{k+1} \sqrt{\sigma_k}})$.

Another way to express the condition for an admissible sequence is

$$\mathcal{R}(\sqrt{\sigma_{k+1} \sqrt{\sigma_k}}) = \mathcal{R}(\sqrt{\sigma_{k+2} \sqrt{\sigma_{k+1} \sqrt{\sigma_k}}} \otimes \sqrt{\sigma_{k+1} \sqrt{\sigma_k}}) \equiv \mathcal{R}(\sqrt{\sigma_{k+1} \sqrt{\sigma_k}}), \quad (22)$$

for $k = 1, \ldots, K - 2$.

Now we introduce some terminology and notation. We say that an operator $\tilde{W}$ on $\mathcal{H}$ is a subamplitude of $\sigma$ if $\tilde{W} \tilde{W}^\dagger \leq \sigma$. It can be shown that $\tilde{W}$ is a subamplitude if and only if it can be written $\tilde{W} = \sqrt{\sigma V}$, where $\tilde{V} \tilde{V}^\dagger \leq 1_I$. One may note that the physical interpretation we have constructed encompasses these subamplitudes. Given a density operator $\sigma$ and one of its subamplitudes $\tilde{W}$, we let $\mathcal{D}(\sigma, \tilde{W})$ denote the density operator in Eq. (8) with the amplitude $W$ replaced by the subamplitude $\tilde{W}$. One can see that when $W$ is varied over all subamplitudes, then $\mathcal{D}(\sigma, \tilde{W})$ spans all of $\mathcal{B}(\sigma, 1_I/N)$.

The following modified procedure results in the Uhlmann holonomy for an arbitrary admissible sequence of density operators. Let $\sigma_1, \ldots, \sigma_K$ be an admissible ordered sequence of density operators. Assume $\tilde{\rho}_1 = \mathcal{D}(\sigma_1, \sqrt{\sigma_1} V_1)$ is given, where we assume that $\sqrt{\sigma_1} V_1$ is an amplitude (not a subamplitude). For $k = 1, \ldots, K - 1$:

- Prepare $\rho_k = \mathcal{D}(\sigma_k, \sqrt{\sigma_k} \tilde{V}_k)$.
- Vary the preparation of $\rho = \mathcal{D}(\sigma_{k+1}, \sqrt{\sigma_{k+1}} \tilde{V})$ with $\tilde{V} \tilde{V}^\dagger \leq 1_I$ until the maximum of $\text{Tr}(Z \rho \otimes \rho_k)$ is reached.
- Let $\tilde{V}_{k+1} = P_{\mathcal{R}(\sqrt{\sigma_{k+1} \sqrt{\sigma_k}})} \tilde{V}$. 

FIG. 2: (Color online) Preparation method to obtain the states $\rho = \mathcal{D}(\sigma, W)$ that represent density operators $\sigma$ and their amplitudes $W$, as defined in Eq. (8). The output state $\rho$ describes both the path state and the internal state of the particle. All states $\mathcal{D}(\sigma, W)$ can be prepared by letting a particle in a pure internal reference state $|\eta\rangle$ and path state $|0\rangle$ fall onto a 50-50 beam-splitter, followed by unitary operations acting separately in the two paths on the internal state of the particle. The application of the unitary operations have to be coordinated by a shared output of a random generator, implementing unitary operators $U_{j}^{(0)}$ and $U_{j}^{(1)}$ in respective path, with probability $p_j = 1/N$. By application of a final unitary $V$ in path 1 we can obtain any desired amplitude $W = \sqrt{\sigma V}$. 

One may note that the physical interpretation we have constructed encompasses these subamplitudes. Given a density operator $\sigma$ and one of its subamplitudes $\tilde{W}$, we let $\mathcal{D}(\sigma, \tilde{W})$ denote the density operator in Eq. (8) with the amplitude $W$ replaced by the subamplitude $\tilde{W}$. One can see that when $W$ is varied over all subamplitudes, then $\mathcal{D}(\sigma, \tilde{W})$ spans all of $\mathcal{B}(\sigma, 1_I/N)$.
After the final step
\[ U_{\text{Uhl}} = \tilde{V}_K \tilde{V}_1^\dagger. \] (23)

Note that we may reformulate the second step as a variation of \( \rho \) over all \( Q(\sigma_{k+1}, I_1/N) \), and thus we vary over all possible subamplitudes of \( \sigma_{k+1} \). Note also that, by the very nature of the problem, the sequence of density operators \( \sigma_1, \ldots, \sigma_K \) is known to us. Thus, the projectors \( P_{\mathcal{R}(\sqrt{\sigma_{k+1}})} \), that we are supposed to apply in each step of the preparation procedure, are also known to us. After the last step we “extract” the Uhlmann holonomy as described below.

To outline of the proof of the modified procedure we first note the following fact.

**Lemma 1.** Let \( A \) be an arbitrary operator on \( \mathcal{H}_I \). If \( \tilde{V} \) is such that it maximizes \( \text{ReTr}(AV^\dagger) \) among all operators on \( \mathcal{H}_I \) that satisfies \( \tilde{V} \tilde{V}^\dagger \leq \hat{1}_I \), then
\[ \tilde{V} = \sqrt{AA^\dagger} + Q, \] (24)
where \( Q \) satisfies \( P_{\mathcal{R}(A)}^\perp Q P_{\mathcal{R}(A)}^\perp = Q \), and where \( P_{\mathcal{R}(A)}^\perp \) denotes the projector onto the orthogonal complement of the range \( \mathcal{R}(A) \) of \( A \).

The following lemma is convenient for the proof of the modified procedure.

**Lemma 2.** Let \( \sigma_1, \sigma_2, \ldots, \sigma_K \) be an admissible sequence of density operators, and suppose that the operator \( \tilde{V}_k \) satisfies
\[ \tilde{V}_k \tilde{V}_k^\dagger = P_{\mathcal{R}(\sqrt{\sigma_k})}. \] (25)
It follows that if \( \tilde{V} \) maximizes \( \text{ReTr}(\sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \tilde{V} \tilde{V}^\dagger) \) among all \( \tilde{V} \tilde{V}^\dagger \leq \hat{1}_I \), then
\[ \tilde{V}_{k+1} \equiv P_{\mathcal{R}(\sqrt{\sigma_{k+1}} \sqrt{\sigma_k})} \tilde{V} = U_{k+1,k} \tilde{V}_k \] (26)
is uniquely determined and satisfies
\[ \tilde{V}_{k+1} \tilde{V}_{k+1}^\dagger = P_{\mathcal{R}(\sqrt{\sigma_{k+1}} \sqrt{\sigma_k})}. \] (27)

**Proof.** We have to prove that if there exists an operator \( \tilde{V} \) that maximizes \( \text{ReTr}(\sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \tilde{V} \tilde{V}^\dagger) \) and is such that \( \tilde{V} \tilde{V}^\dagger \leq \hat{1}_I \), then this operator satisfies Eq. 26. According to Lemma 1 (with \( A = \sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \tilde{V}_k \)) it follows that we can write
\[ \tilde{V} = \sqrt{\sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \tilde{V}_k \tilde{V}_k^\dagger} \sqrt{\sigma_k} \sqrt{\sigma_k} + Q, \] (28)
where
\[ P_{\mathcal{R}(\sqrt{\sigma_k})}^\perp Q P_{\mathcal{R}(\sqrt{\sigma_k})}^\perp = Q. \] (29)
If we combine Eq. 25 with the assumption that the sequence is admissible, and thus \( P_{\mathcal{R}(\sqrt{\sigma_k})}^\perp = P_{\mathcal{R}(\sqrt{\sigma_k})}^\perp = P_{\mathcal{R}(\sqrt{\sigma_k})}^\perp = P_{\mathcal{R}(\sqrt{\sigma_k})}^\perp \), it can be shown that
\[ \sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \tilde{V}_k \tilde{V}_k^\dagger = \sqrt{\sigma_k} \sqrt{\sigma_k} \sqrt{\sigma_k} + 1. \] (30)
If Eq. 30 is inserted into Eq. 28 we obtain \( \tilde{V} = U_{k+1,k} \tilde{V}_k + Q \). By the properties of the operator \( Q \) in Eq. 29, together with \( \mathcal{R}(\sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \tilde{V}_k) = \mathcal{R}(\sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \tilde{V}_k) \), it follows that \( \tilde{V} \) satisfies Eq. 26.

Now we have to prove that \( \tilde{V}_{k+1} \) satisfies Eq. 27. We again make use of the assumption that the sequence is admissible, and we find that
\[ \tilde{V}_{k+1} \tilde{V}_{k+1}^\dagger = U_{k+1,k} \tilde{V}_k \tilde{V}_k^\dagger U_{k+1,k} = U_{k+1,k} \tilde{V}_k \tilde{V}_k^\dagger U_{k+1,k} = P_{\mathcal{R}(\sqrt{\sigma_k} \sqrt{\sigma_k})}, \] (31)
where in the last equality we have used that the initial space of \( U_{k+1,k} \) is \( \mathcal{R}(\sqrt{\sigma_k} \sqrt{\sigma_k}) \). Note that Eq. 31 implies that \( \tilde{V}_{k+1} \tilde{V}_{k+1}^\dagger \leq \hat{1}_I \). We have now proved that if there exists a maximizing operator \( \tilde{V} \), then this operator satisfies Eqs. 26 and 27. We finally have to prove that there actually does exist such an operator. If we let \( \tilde{V} = U_{k+1,k} \tilde{V}_k \), then the assumption of admissible sequences can be used to show that
\[ \text{ReTr}(\sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \tilde{V} \tilde{V}^\dagger) = \text{ReTr}(\sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \tilde{V} \tilde{V}^\dagger) = \text{Tr}(\sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \sqrt{\sigma_{k+1}}), \] (32)
which is the maximal value of \( \text{ReTr}(\sqrt{\sigma_{k+1}} \sqrt{\sigma_k} \tilde{V} \tilde{V}^\dagger) \) under the assumption that \( \tilde{V} \tilde{V}^\dagger \leq \hat{1}_I \). This proves the lemma.

Lemma 2 can be used to prove the modified procedure in an iterative manner. We begin with the first step of the procedure. Thus, we are given the state \( \rho_1 = D(\sigma_1, \sqrt{\sigma_1} V_1) \), where \( \sqrt{\sigma_1} V_1 \) is an amplitude of \( \sigma_1 \), and hence \( V_1 \tilde{V}_1^\dagger = P_{\mathcal{R}(\sigma_1)} \). If we now let \( \rho = D(\sigma_2, W) \), and vary \( W = \sqrt{\sigma_2} \tilde{V} \) over all subamplitudes of \( \sigma_2 \), we find the maximum of \( \text{Tr}(\rho \rho \otimes \rho_1) \) to be obtained when we reach the maximum of
\[ \text{ReTr}(W^\dagger W_1) = \text{ReTr}(\sqrt{\sigma_1} \sqrt{\sigma_1} V_1 \tilde{V}_1^\dagger). \] (33)
According to Lemma 2 every maximizing \( \tilde{V} \) is such that \( P_{\mathcal{R}(\sqrt{\sigma_k} \sqrt{\sigma_k})} \tilde{V} = U_{2,1} \tilde{V}_1 \). Now we let \( \tilde{V}_2 = P_{\mathcal{R}(\sqrt{\sigma_k} \sqrt{\sigma_k})} \tilde{V} \) and prepare the state \( \tilde{V}_2 = D(\sigma_2, \sqrt{\tau_2} \tilde{V}_2) \). We can repeat the above procedure in an iterative manner to find that
\[ \tilde{V}_K = U_{K,K-1} \ldots U_{2,1} \tilde{V}_1 = U_{\text{Uhl}} \tilde{V}_1. \] (34)
Note that \( \tilde{V}_1 \) is a partial isometry and thus may be completed to a unitary operator \( \tilde{V}_1 \). If we apply \( U_{\text{mod}} \) in Eq. 13, but with \( V_1 \) replaced by \( \tilde{V}_1 \), the resulting state is \( D(\sigma_K, \sqrt{\sigma_K} U_{\text{Uhl}}) \). (Note that \( \tilde{V}_1 \) is not unique, but this does not matter since \( \tilde{V}_1 \tilde{V}_1^\dagger = \tilde{V}_1 \tilde{V}_1^\dagger \) for all such extensions.) Now we shall extract the Uhlmann holonomy from the state \( D(\sigma_K, \sqrt{\sigma_K} U_{\text{Uhl}}) \). Note that \( U_{\text{Uhl}} \) is a partial isometry and that \( \sqrt{\sigma_K} U_{\text{Uhl}} \) in general is a
subamplitude of $\sigma_K$. In order to make sure that we indeed extract the Uhlmann holonomy when we apply the procedure described in Sec. III we first apply the projection $P_{\mathcal{R}(\sqrt{\rho_K}, \sqrt{\rho_{K-1}})} \otimes |0\rangle \langle 0| + 1_f \otimes |1\rangle \langle 1|$ onto the state $\mathcal{D}(\sigma_K, \sqrt{\sigma_K}U_{\text{Uhl}})$. By this filtering (post selection) we obtain a new normalized state $\tilde{\rho}$, for which

$$
\langle 0 | \tilde{\rho} | 1 \rangle = N P_{\mathcal{R}(\sqrt{\rho_K}, \sqrt{\rho_{K-1}})} \sqrt{\sigma_K} U_{\text{Uhl}}
$$

$$
= N P_{\mathcal{R}(\sqrt{\rho_K}, \sqrt{\rho_{K-1}})} \sqrt{\sigma_K} P_{\mathcal{R}(\sqrt{\rho_K}, \sqrt{\rho_{K-1}})} U_{\text{Uhl}},
$$

where the constant $N$ is a real nonnegative number, and where the last equality follows since $\mathcal{R}(\sqrt{\rho_K}, \sqrt{\rho_{K-1}})$ is the final space of $U_{\text{Uhl}}$. If we apply the extraction procedure described in Sec. III we find that the unitary operator $U$ that gives the maximal detection probability is not uniquely determined. However, by using Lemma 1 one can show that every maximizing unitary operator $U$ can be written $U = U_{\text{Uhl}} + Q$, where $P_{\mathcal{R}(\sqrt{\rho_K}, \sqrt{\rho_{K-1}})} Q P_{\mathcal{R}(\sqrt{\rho_K}, \sqrt{\rho_{K-1}})} = Q$. Hence, $P_{\mathcal{R}(\sqrt{\rho_K}, \sqrt{\rho_{K-1}})} U = U_{\text{Uhl}}$ is uniquely defined by this procedure. We have thus found a modified procedure to obtain the Uhlmann holonomy for admissible sequences of density operators.

**Preparation procedures for unfaithful density operators.** As a final note concerning the generalization to unfaithful density operators we show that the preparation procedure described in Eqs. (13) and (14) to obtain the states $\mathcal{D}(\sigma, W)$ with $W$ an amplitude of $\sigma$, can be modified to obtain states $\mathcal{D}(\sigma, \tilde{W})$, with $\tilde{W}$ an arbitrary subamplitude of $\sigma$. All subamplitudes $\tilde{W} = \sqrt{\sigma V}$ can be reached via $\tilde{V}$ such that $\tilde{V}V^\dagger \leq 1_f$. The set of operators $\tilde{V}$ on $\mathcal{H}_f$ such that $\tilde{V}V^\dagger \leq 1_f$, forms a convex set whose extreme points are the unitary operators on $\mathcal{H}_f$, which follows from Lemma 21 in Ref. [22]. Thus, for every choice of $\tilde{V}$ there exist probabilities $\mu_n$ and unitaries $V_n$, such that $\tilde{V} = \sum_n \mu_n V_n$. Hence, instead of applying the unitary operator $1_f \otimes |0\rangle \langle 0| + V \otimes |1\rangle \langle 1|$ at the end of the preparation procedure, we can instead apply $1_f \otimes |0\rangle \langle 0| + V_n \otimes |1\rangle \langle 1|$ with probability $\mu_n$. This modified procedure results in the desired state $\rho = \mathcal{D}(\sigma, \tilde{W})$.

**VII. CONCLUSIONS**

In conclusion, we present an interpretation of the Uhlmann amplitude that gives it a clear physical meaning and makes it a measurable object. In contrast to previous approaches where the amplitude resides in the total pure state of a twofold copy of the original system (a purification), we suggest an alternative where the amplitude is represented by the coherences of a mixed state on a composite system. This gives a more compact representation and also allows for a direct interferometric determination of the Uhlmann parallelity condition. Based on this, we reformulate the parallelity condition entirely in operational terms, which enables an implementation of parallel transport of amplitudes along a sequence of density operators through an iterative procedure. At the end of this transport process the Uhlmann holonomy can be identified as a unitary mapping that gives the maximal detection probability in an interference experiment.

In this paper, we consider the Uhlmann holonomy concomitant to sequences of density operators, i.e., discrete families of density operators. However, the Uhlmann holonomy can also be associated to a smooth path of density operators $\mathcal{R}$, e.g., the time evolution of a quantum system. The parallel transport procedure discussed here is by its very nature iterative, but we can form successively refined discrete approximations of the desired path and obtain the Uhlmann holonomy within any non-zero error bound. The question is whether it is possible to find an operational parallel transport procedure formulated explicitly for smoothly parameterized families of density operators. We hope that the framework we suggest in this paper may serve as a starting point for such an attempt. Given a family of density operators $\sigma(s)$, one could consider a differential equation for the evolution of $\mathcal{D}(\sigma(s), W(s))$ (defined in Eq. (3)) such that $W(s)$ becomes the parallel transported amplitudes of $\rho(s)$. However, it is far from clear whether such a differential equation could be given a reasonable physical and operational interpretation.

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[38] If the operator $V_1$ on $\mathcal{H}$ is a partial isometry with initial space $\mathcal{I}$ and final space $\mathcal{F}$, it can be "completed" into a unitary operator by adding to it any partial isometry with initial space $\mathcal{I}^\perp$ and final space $\mathcal{F}^\perp$.