A sufficient condition for the continuity of permanental processes with applications to local times of Markov processes and loop soups

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Abstract

We provide a sufficient condition for the continuity of real valued permanental processes. When applied to the subclass of permanental processes which consists of squares of Gaussian processes, we obtain the sufficient condition for continuity which is also known to be necessary. Using an isomorphism theorem of Eisenbaum and Kaspi which relates Markov local times and permanental processes we obtain a general sufficient condition for the joint continuity of the local times. We show that for certain Markov processes the associated permanental process is equal in distribution to the loop soup local time.

1 Introduction

Let $T$ be an index set and \( \{G(x), x \in T\} \) be a mean zero Gaussian process with covariance \( u(x, y), x, y \in T \). It is remarkable that for certain Gaussian processes, called associated processes, the process \( G^2 = \{G^2(x), x \in T\} \) is closely related to the local times of a strongly symmetric Borel right process with zero potential density \( u(x, y) \). This connection was first noted in the Dynkin Isomorphism Theorem [5, 6] and has been studied by several probabilists including the authors and N. Eisenbaum and H. Kaspi. Our book [24] presents several results about local times that are obtained using this relationship.

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The process \( G^2 \) can be defined by the Laplace transform of its finite joint distributions

\[
E \left( \exp \left( \frac{1}{2} \sum_{i=1}^{n} \alpha_i G^2(x_i) \right) \right) = \frac{1}{|I + \alpha U|^{1/2}} \tag{1.1}
\]

for all \( x_1, \ldots, x_n \) in \( T \), where \( I \) is the \( n \times n \) identity matrix, \( \alpha \) is the diagonal matrix with \( (\alpha_{ii} = \alpha_i) \), \( \alpha_i \in \mathbb{R}^+ \) and \( U = \{u(x_i, x_j)\} \) is an \( n \times n \) matrix, that is symmetric and positive definite.

In 1997, D. Vere-Jones, \[30\], introduced the permanental process \( \theta := \{\theta_x, x \in T\} \), which is a real valued positive stochastic process with finite joint distributions that satisfy

\[
E \left( \exp \left( -\frac{1}{2} \sum_{i=1}^{n} \alpha_i \theta_{x_i} \right) \right) = \frac{1}{|I + \alpha \Gamma|^\beta}. \tag{1.2}
\]

where \( \Gamma = \{\Gamma(x_i, x_j)\}_{i,j=1}^{n} \) is an \( n \times n \) matrix and \( \beta > 0 \). (It would be better to refer to \( \theta \) as a \( \beta \)-permanental process.) In most of this paper, in analogy with (1.1), we consider these processes only for \( \beta = 1/2 \) and refer to them as permanental processes. The generalization here is that \( \Gamma \) need not be symmetric or positive definite. In Section \[8\] we consider the general class of \( \beta \)-permanental processes.

Even in (1.1) the matrix \( U \) is not unique. The determinant

\[
|I + \alpha U| = |I + \alpha M U M|
\]

for any signature matrix \( M \). (A signature matrix is a diagonal matrix with entries \( \pm 1 \).)

The non-uniqueness is even more evident in (1.2). If \( D \) is any diagonal matrix with non-zero entries we have

\[
|I + \alpha \Gamma| = |I + \alpha D^{-1} \Gamma D| = |I + \alpha D^{-1} \Gamma^T D|. \tag{1.4}
\]

For a very large class of irreducible matrices \( \Gamma \), it is known that these are the only sources of non-uniqueness; see [18]. On the other hand, in certain extreme cases, for example, if \( \Gamma_1 \) and \( \Gamma_2 \) are \( n \times n \) matrices with the same diagonal elements and all zeros below the diagonal, then \( |I + \alpha \Gamma_1| = |I + \alpha \Gamma_2| \).

For this reason we refer to a matrix \( \Gamma \) for which (1.2) holds as a kernel of \( \theta \), (rather than as the kernel of \( \theta \)).

When \( \Gamma \) is not symmetric and positive definite, it is not at all clear what kernels \( \Gamma \) allow an expression of the form (1.2). (In [30] necessary and sufficient
conditions on $\Gamma$ for (1.2) to hold are given but they are very difficult to verify. There are very few concrete examples of permanental processes in [30].

It follows from the results in [30] that a sufficient condition for (1.2) to hold is that all the real non-zero eigenvalues of $\Gamma$ are positive and that $r \Gamma(I + r \Gamma)^{-1}$ has only non-negative entries for all $r > 0$. In [8], Eisenbaum and Kaspi note that this is the case when $\Gamma(x, y)$, $x, y \in T$, is the potential density of a transient Markov process on $T$. This enables them to find a Dynkin type isomorphism for the local times of Markov processes that are not necessarily symmetric, in which the role of $G^2$ is taken by the permanental process $\theta$.

Both Eisenbaum and Kaspi have asked us if we could find necessary and sufficient conditions for the continuity and boundedness of permanental processes. In this paper we give a sufficient condition for the continuity of permanental processes. When applied to the subclass of permanental processes which consists of squares of Gaussian processes, it is, effectively, the sufficient condition for continuity which is also known to be necessary. We use our sufficient condition for the continuity of permanental processes and an isomorphism theorem for permanental processes given by Eisenbaum and Kaspi in [8, Theorem 3.2], to extend a sufficient condition they obtain in [7, Theorem 1.1] for the continuity of local times of Markov processes, to a larger class of Markov processes.

In Section 3 we review several properties of permanental processes. In particular, a key property of permanental processes is that $\Gamma(x, x) \geq 0$ and

\[ 0 \leq \Gamma(x, y)\Gamma(y, x) \leq \Gamma(x, x)\Gamma(y, y), \quad \forall x, y \in T. \tag{1.5} \]

This allows us to define

\[ d(x, y) = 4\sqrt{2/3} \left( \Gamma(x, x) + \Gamma(y, y) - 2 (\Gamma(x, y)\Gamma(y, x))^{1/2} \right)^{1/2}. \tag{1.6} \]

Let $D = \sup_{s, t \in T} d(s, t)$. $D$ is called the $d$ diameter of $T$. We say that $T$ is separable for $d$, if there exists a countable subset $T' \subseteq T$, such that for any $s \in T$ and $u > 0$, there is a $t \in T'$ with $d(s, t) \leq u$.

Let $(T, \rho)$ be a separable metric or pseudometric space. Let $B_{\rho}(t, u)$ denote the closed ball in $(T, \rho)$ with radius $u$ and center $t$. For any probability measure $\mu$ on $(T, \rho)$ we define

\[ J_{T, \rho, \mu}(a) = \sup_{t \in T} \int_{0}^{a} \left( \log \frac{1}{\mu(B_{\rho}(t, u))} \right)^{1/2} du. \tag{1.7} \]

We occasionally omit some of the subscripts $T, \rho$ or $\mu$, if they are clear from the context.
In general, \( d(x, y) \) is not a metric or pseudometric on \( T \). Nevertheless, we can still define the sets \( B_d(s, u) = \{ t \in T \mid d(s, t) \leq u \} \). We can then define \( J_{T, d, \mu}(a) \) as in (1.7), for any probability measure \( \mu \) on \( B(T, d) \), the \( \sigma \)-algebra generated by the sets \( B_d(s, u) \).

**Theorem 1.1** Let \( \theta = \{ \theta_x : x \in T \} \) be a permanental process with kernel \( \Gamma \) satisfying \( \sup_{x \in T} \Gamma(x, x) < \infty \). Let \( D \) denote the \( d \) diameter of \( T \) and assume that \( T \) is separable for \( d \), and that there exists a probability measure \( \mu \) on \( B(T, d) \) such that
\[
J_d(D) < \infty.
\] (1.8)

Then there exists a version \( \theta' = \{ \theta'_x, x \in T \} \) of \( \theta \) which is bounded almost surely.

If
\[
\lim_{\delta \to 0} J_d(\delta) = 0,
\] (1.9)

there exists a version \( \theta' = \{ \theta'_x, x \in T \} \) of \( \theta \) such that
\[
\lim_{\delta \to 0} \sup_{\delta, t \leq \delta} \left| \frac{\theta'_s(\omega) - \theta'_t(\omega)}{d(s, t) / 2} \right| = 0, \quad \text{a.s.}
\] (1.10)

If (1.9) holds and
\[
\lim_{\delta \to 0} \frac{J_d(\delta)}{\delta} = \infty,
\] (1.11)

then
\[
\lim_{\delta \to 0} \sup_{s, t \in T, d(s, t) \leq \delta} \frac{|\theta'_s - \theta'_t|}{J_d(d(s, t) / 2)} \leq 30 \left( \sup_{x \in T} \theta'_x \right)^{1/2} \quad \text{a.s.}
\] (1.12)

Let
\[
d_\theta(x, y) = \left( E (\theta_x - \theta_y)^2 \right)^{1/2}.
\] (1.13)

It follows from Lemma 5.2 that (1.10) implies that \( \theta' = \{ \theta'_x, x \in T \} \) is almost surely continuous on \( (T, d_\theta) \). However, we prefer to state our basic result as (1.10) and explore its implications in the next two corollaries.

**Corollary 1.1** Let \( T \) be a separable topological space and let \( \theta = \{ \theta_x : x \in T \} \) be a permanental process with kernel \( \Gamma \), with \( \sup_{x \in T} \Gamma(x, x) < \infty \). Assume that \( d(x, y) \) is continuous on \( T \times T \) and that there exists a probability measure \( \mu \) on \( T \) such that (1.9) holds. Then there exists a version \( \theta' = \{ \theta'_x : x \in T \} \) of \( \theta \) that is continuous almost surely.
We show in Lemma 3.2 that when $\theta$ is continuous on $T$ almost surely, then $d(x, y)$ is continuous on $T \times T$. Therefore, the condition that $d(x, y)$ is continuous on $T \times T$ in Corollary 1.1 is perfectly reasonable.

We say that a metric or pseudometric $d_1$ dominates $d$ on $T$ if

$$d(x, y) \leq d_1(x, y), \quad \forall x, y \in T. \quad (1.14)$$

In the Section 5 we give several natural metrics that dominate $d$.

**Corollary 1.2** Let $\theta = \{\theta_x : x \in T\}$ be a permanental process with kernel $\Gamma$ satisfying $\sup_x \Gamma(x, x) < \infty$. Let $d$ be given by (1.6) and let $d_1(x, y)$ be a metric or pseudo-metric on $T$ that dominates $d(x, y)$ and is such that $(T, d_1)$ is separable and has finite diameter $D$. If there exists a probability measure $\mu$ on $(T, d_1)$ such that

$$J_{d_1}(D) < \infty, \quad (1.15)$$

then there exists a version $\theta' = \{\theta'_x, x \in T\}$ of $\theta$ which is bounded almost surely.

If

$$\lim_{\delta \to 0} J_{d_1}(\delta) = 0, \quad (1.16)$$

there exists a version $\theta'$ of $\theta$ which is uniformly continuous on $(T, d_1)$, almost surely.

If (1.16) holds and

$$\lim_{\delta \to 0} \frac{J_{d_1}(\delta)}{\delta} = \infty, \quad (1.17)$$

then

$$\lim_{\delta \to 0} \sup_{s, t \in T \atop d_1(s, t) \leq \delta} \frac{|\theta'_s - \theta'_t|}{J_{d_1}(d_1(s, t)/2)} \leq 30 \left(\sup_{x \in T} \theta'_x\right)^{1/2} a.s. \quad (1.18)$$

Other useful inequalities for permanental processes are given in Section 3 (see, in particular, Lemma 3.4).

In Section 4 we give a version of (1.18) for $|\theta'_s - \theta'_t|$ for fixed $t_0 \in T$, which provides a local modulus of continuity for permanental processes.

Let $X = (\Omega, X_t, P^x)$ be a transient Borel right process with state space $S$ and 0-potential density $u(x, y)$. We assume that $S$ is a locally compact topological space, and that $u(x, y)$ is continuous. This guarantees that $X$ has local times. (See e.g. [24, Theorem 3.6.3].) It is shown in [8, Theorem 3.1] that there exists a permanental process $\theta = \{\theta_y ; y \in S\}$, with kernel $u(x, y)$, which they refer to as the permanental process associated with $X$. 

In [8, Theorem 3.2] an isomorphism theorem is given that relates the local times of \( X \) and \( \theta \). In the next theorem we use this isomorphism together with Theorem 1.1 in this paper, to obtain a sufficient condition for the joint continuity of the local times of \( X \). When applied to strongly symmetric Markov processes, we obtain the sufficient condition for joint continuity, that is known to be necessary; see [24, Theorem 9.4.11]. Applied to Lévy processes, which need not be symmetric, we also obtain the sufficient condition for the joint continuity of local times, that is known to be necessary; see [2].

As usual, we use \( \zeta \) to denote the death time of \( X \).

**Theorem 1.2** Let \( S \) be a locally compact topological space with a countable base. Let \( X = (\Omega, X_t, P^x) \) be a recurrent Borel right process with state space \( S \) and continuous, strictly positive 1-potential densities \( u^1(x,y) \). Define \( d(x,y) \) as in (1.6) for the kernel \( u^1(x,y) \). Suppose that for every compact set \( K \subseteq S \), we can find a probability measure \( \mu_K \) on \( K \), such that

\[
\lim_{\delta \to 0} J_{K,d,\mu_K}(\delta) = 0, \tag{1.19}
\]

then \( X \) has a jointly continuous local time \( \{L^y_t; (y,t) \in S \times R_+\} \).

Let \( X \) be a transient Borel right process with state space \( S \) and continuous, strictly positive 0-potential densities \( u(x,y) \). If (1.19) holds for every compact set \( K \subseteq S \), with \( d(x,y) \) defined as in (1.6) for the kernel \( u(x,y) \), \( X \) has a local time \( \{L^y_t; (y,t) \in S \times R_+\} \) which is jointly continuous on \( S \times [0, \zeta) \).

Note that Theorem 1.2 gives continuity on \( S \times R_+ \) for recurrent processes. For transient processes it only gives continuity on \( S \times [0, \zeta) \). As pointed out in [7], if \( X \) is transient, by an argument due to Le Jan we can always find a recurrent process \( Y \) such that \( X \) is \( Y \) killed the first time it hits the cemetery state \( \Delta \). Of course, this changes the potentials, see [4, (78.5)]), and hence the condition (1.19). We leave it to the interested reader to work out the details.

It is interesting to place Theorem 1.2 in the history of results on the joint continuity of local times of Markov processes. A good discussion is given in [7]. We make a few comments here. In [2] Barlow gives necessary and sufficient condition for the joint continuity of local times of Lévy processes. Local times are difficult to work with. He works hard to obtain many of their properties. In [22] we use the Dynkin Isomorphism Theorem (DIT) to obtain necessary and sufficient condition for the joint continuity of local times of strongly symmetric Borel right processes, which, obviously, includes symmetric Lévy processes. Using the DIT enables us to infer properties of local times from those of Gaussian processes. These processes are well understood and easier to work
with than local times. Although the results in [22] only give the results in [2] for symmetric Lévy processes, they apply to a much larger class of symmetric Markov processes.

In [7], Eisenbaum and Kaspi extend Barlow’s approach to obtain sufficient conditions for the joint continuity of local times of a large class of recurrent Borel right processes and also give a modulus of continuity for the local times. In Theorem 1.2, using a proof similar to the one in [22], we use Eisenbaum and Kaspi’s isomorphism theorem for permanental processes [8, Theorem 3.2], to extend their results in [7]. (In [7], they require the existence of a Borel right dual process. This is not needed in Theorem 1.2. In Section 7 we show how to obtain [8, Theorem 3.2] from Theorem 1.2.)

We also obtain results about the boundedness of local times, and when they are continuous, about their uniform modulus of continuity.

**Theorem 1.3** Let $S$ be a locally compact topological space with a countable base. Let $X = (Ω, X_t, P^x)$ be a Borel right process with continuous, strictly positive $1$-potential density $u_1(x, y)$. Let $\{L^y_t, (t, y) ∈ S × R_+\}$ be the local time of $X$ and define $d(x, y)$ as in (1.6) for the kernel $u_1(x, y)$.

Let $C$ be a countable subset of $S$. If $J_{C, d, \mu_C}(D) < ∞$ for some probability measure $\mu_C$ on $C$ and some $D > 0$, then

$$\sup_{y ∈ C} L^y_t < ∞ \quad (1.20)$$

almost surely, for each $t < ζ$.

Let $K ⊂ S$ be compact. Let $d_1$ be a continuous metric or pseudo-metric that dominates $d$ on $K × K$. If for some probability measure $\mu_K$ on $K$

$$\lim_{δ → 0} J_{K, d_1, \mu_K}(δ) = 0, \quad \text{and} \quad \lim_{δ → 0} \frac{J_{K, d_1, \mu_K}(δ)}{δ} = ∞ \quad (1.21)$$

then

$$\lim_{δ → 0} \sup_{x, y ∈ K \atop d_1(x, y) ≤ δ} \frac{|L^x_t - L^y_t|}{J_{K, d_1, \mu_K}(d_1(x, y)/2)} \leq 30 \sup_{y ∈ K} (L^y_t)^{1/2} \quad \text{for almost all } t ∈ [0, ζ] \text{ a.s.} \quad (1.22)$$

A local modulus of continuity for local times is given in Theorem 6.2.

In [15] Theorem 9] Le Jan shows that squares of certain associated Gaussian processes, labeled $G^2$ in the first paragraph of this section, are equal in
distribution to the occupation fields of Poissonian ensembles of Markov loops, also called ‘loop soup local times’. We explain this in detail in Section 8. As we know, $G^2$ is a permanental process with a symmetric kernel. In Theorem 8.1 we extend Le Jan’s result to a large class of associated permanental processes. Thus we can identify permanental processes as processes that have already received attention in other contexts, (see [21]), and use Theorem 1.1, Corollary 1.1 and Corollary 1.2 to obtain sufficient conditions for the almost sure continuity and moduli of continuity for loop soup local times.

2 Some basic continuity theorems

For $p \geq 1$, let $\psi_p(x) = \exp(x^p) - 1$ and $L^{\psi_p}(\Omega, \mathcal{F}, P)$ denote the set of random variables $\xi : \Omega \rightarrow \mathbb{R}$ such that $E\psi_p(|\xi|/c) < \infty$ for some $c > 0$. $L^{\psi_p}(\Omega, \mathcal{F}, P)$ is a Banach space with norm given by

$$\|\xi\|_{\psi_p} = \inf \{ c > 0 : E\psi_p(|\xi|/c) \leq 1 \}.$$  

We shall only be concerned with the cases $p = 1$ and 2.

We obtain Theorem 1.1 with the help of the following basic continuity theorems. They are, essentially, best possible sufficient conditions for continuity and boundedness of Gaussian process. However, it is well known that they hold for any stochastic process satisfying certain conditions with respect to the Banach space $L^{\psi_2}$.

**Theorem 2.1** Let $X = \{X(t) : t \in T\}$ be a stochastic process such that $X(t, \omega) : T \times \Omega \mapsto [-\infty, \infty]$ is $\mathcal{A} \times \mathcal{F}$ measurable for some $\sigma$-algebra $\mathcal{A}$ on $T$. Suppose $X(t) \in L^{\psi_2}(\Omega, \mathcal{F}, P)$ and let

$$\tilde{d}(t, s) := \|X(t) - X(s)\|_{\psi_2}.$$  

(Note that the balls $B_{\tilde{d}}(s, u)$ are $\mathcal{A}$ measurable).

Suppose that $(T, \tilde{d})$ has finite diameter $D$, and that there exists a probability measure $\mu$ on $(T, \mathcal{A})$ such that

$$J_{\tilde{d}}(D) < \infty.$$  

Then there exists a version $X' = \{X'(t), t \in T\}$ of $X$ such that

$$E \sup_{t \in T} X'(t) \leq C J_{\tilde{d}}(D),$$  

 (**2.4**)
for some \( C < \infty \). Furthermore for all \( 0 < \delta \leq D \),
\[
\sup_{s,t \in T, d(s,t) \leq \delta} |X'(s,\omega) - X'(t,\omega)| \leq 2Z(\omega) J_{\hat{d}}(\delta),
\] (2.5)
almost surely, where
\[
Z(\omega) := \inf \left\{ \alpha > 0 : \int_T \psi_2(\alpha^{-1}|X(t,\omega)|) \mu(dt) \leq 1 \right\}
\] (2.6)
and \( \|Z\|_{\psi_2} \leq K \), where \( K \) is a constant.

In particular, if
\[
\lim_{\delta \to 0} J_{\hat{d}}(\delta) = 0,
\] (2.7)
\( X' \) is uniformly continuous on \((T, \hat{d})\) almost surely.

Remark 2.1 Theorem 2.1 is well known. It contains ideas that originated in an important early paper by Garcia, Rodemich and Rumsey Jr., \[11\] and were developed further by Preston, \[26, 27\] and Fernique, \[9\]. We present a generalization of it in \[25, Theorem 3.1\]. Unfortunately, the statement of \[25, Theorem 3.1\] makes it appear that (2.7), in this paper, is required for (2.5), in this paper, to hold. This is not the case as one can see from going through the proof of \[25, Theorem 3.1\]. However, an easier way to see that (2.5), in this paper, holds is to note that it follows immediately from \[24, Theorem 6.3.3\].

Again, unfortunately, the hypothesis of \[24, Theorem 6.3.3\] requires that \( X \) is a Gaussian process. A reading of the proof shows that it actually only requires that \( X(t) \in L^{\psi_2}(\Omega, P) \) and \( \|X(t) - X(s)\|_{\psi_2} \leq d(s,t) \) for all \( s,t \in T \).

It is easy to see that the results of Theorem 2.1 hold if (2.2) is replaced by
\[
\|X(t) - X(s)\|_{\psi_2} \leq \hat{d}(t,s) \quad s,t \in T,
\] (2.8)
for a general symmetric function \( \hat{d}(t,s) \), if we assume that \( B_{\hat{d}}(s,u) =: \{ t \in T | \hat{d}(s,t) \leq u \} \), \( s,u \in T \times R_+ \), are \( \mathcal{A} \) measurable. To see this, note that \( X(t) \in L^{\psi_2}(\Omega, \mathcal{F}, P) \) implies that \( \rho(s,t) := \|X(t) - X(s)\|_{\psi_2} \) exists. Consequently, Theorem 2.1 gives (2.3)–(2.7) with \( \hat{d} \) replaced by \( \rho \). Using (2.8) it is easy to see that this implies that they hold as stated.

The inequality in (2.5) is not quite enough to give a best possible uniform modulus of continuity for \( X' \). Instead we use the following lemma due to B. Heinkel, \[13, Proposition 1\].
**Lemma 2.1** Let \((T, \hat{d})\) be a metric or pseudo-metric space with finite diameter \(D\) and \(\mu\) be a probability measure on \(T\) with the property that \(\mu(B_{\hat{d}}(t, u)) > 0\) for all \(u, t \in T\), \(u \neq t\). Assume that (2.7) holds. Let \(\{f(t), t \in T\}\) be continuous on \((T, \hat{d})\) and set

\[
\tilde{f}(s, t) = \frac{f(s) - f(t)}{\hat{d}(s, t)} I_{\{u, v; \hat{d}(u, v) \neq 0\}}(s, t). \tag{2.9}
\]

Then if

\[
c_{\mu, T}(\tilde{f}) = \int_{T \times T} \psi_2(\tilde{f}(s, t)) \, d\mu(s) \, d\mu(t) < \infty, \tag{2.10}
\]

we have that for all \(x, y \in T\)

\[
|f(x) - f(y)| \leq 20 \sup_{t \in T} \int_0^{\hat{d}(x, y)/2} \left( \log \left( \frac{c_{\mu, T}(\tilde{f}) + 1}{\mu^2(B_{\hat{d}}(t, u))} \right) \right)^{1/2} \, du. \tag{2.11}
\]

**Corollary 2.1** Under the hypotheses of Lemma 2.1 assume further that

\[
\lim_{\delta \to 0} \frac{J_{\hat{d}}(\delta)}{\delta} = \infty. \tag{2.12}
\]

Then

\[
\lim_{\delta \to 0} \sup_{x, y \in T, \hat{d}(x, y) \leq \delta} \frac{|f(x) - f(y)|}{J_{\hat{d}}(\hat{d}(x, y)/2)} \leq 30. \tag{2.13}
\]

**Proof** This follows immediately from (2.11) and (2.12) once we note that the second line of (2.11)

\[
\leq 10 \hat{d}(x, y) \left( \log \left( \frac{c_{\mu, T}(\tilde{f}) + 1}{\mu^2(B_{\hat{d}}(t, u))} \right) \right)^{1/2} + 30 J_{\hat{d}}(\hat{d}(x, y)/2). \tag{2.14}
\]

The above proof as well as the proof of Theorem 2.2 follows ideas in \[1\] pages 30 and 31.

**Theorem 2.2** Under the hypotheses of Theorem 2.1 assume that (2.7) and (2.12) hold. Then there exists a version \(X' = \{X'(t), t \in T\}\) of \(X\) such that

\[
\lim_{\delta \to 0} \sup_{s, t \in T, \hat{d}(s, t) \leq \delta} \frac{|X'(s) - X'(t)|}{J_{\hat{d}}(\hat{d}(s, t)/2)} \leq 30 \quad \text{a.s.} \tag{2.15}
\]
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Proof By Theorem 2.1 we can assume that \( X = \{ X(t), t \in T \} \) is continuous on \(( T, \hat{d} )\) almost surely. Define \( \tilde{X} \) as in (2.9). Note that by Fubini’s Theorem

\[
E \left( \int_{T \times T} \psi_2(\tilde{X}(s,t)) \, d\mu(s) \, d\mu(t) \right) \leq E \left( \int_{T \times T} \psi_2 \left( \frac{X(t) - X(s)}{\|X(t,s)\|_\psi_2} \right) \, d\mu(s) \, d\mu(t) \right) = 1.
\]

Consequently

\[
\int_{T \times T} \psi_2(\tilde{X}(s,t)) \, d\mu(s) \, d\mu(t) < \infty \quad \text{a.s.} \quad (2.17)
\]

Let \( \Omega' \) be the set of measure 1 in the probability space for which this is finite and for which \( X(t, \omega) \) is continuous. For each \( \omega \in \Omega' \)

\[
c_{\mu,T}(\tilde{X}) = \int_{T \times T} \psi_2(\tilde{X}(s,t,\omega)) \, d\mu(s) \, d\mu(t) < \infty. \quad (2.18)
\]

Therefore, we can apply Lemma 2.1 and Corollary 2.1 to \( X(t, \omega) \) to get (2.15) for \( X'(t, \omega) \). We do this for all \( \omega \in \Omega' \) to get (2.15) as stated.

We get a similar result for the local modulus of continuity but it is more delicate. We take this up in Section 4.

3 Proof of Theorem 1.1

We begin with some observations about permanental processes. It is noted in [30], and immediately obvious from (1.2), that the univariate marginals of a permanental process are squares of normal random variables. A key observation used in the proof of Theorem 1.1 which also follows from (1.2), is that the bivariate marginals of a permanental process are squares of bivariate normal random variables. We proceed to explain this.

For \( n = 2, (1.2) \) takes the form

\[
E \left( \exp \left( -\frac{1}{2} (\alpha_1 \theta_x + \alpha_2 \theta_y) \right) \right) = \frac{1}{|I + \alpha \Gamma|^{1/2}} = \left( 1 + \alpha_1 \Gamma(x, x) + \alpha_2 \Gamma(y, y) \right. \left. + \alpha_1 \alpha_2 \left( \Gamma(x, x) \Gamma(y, y) - \Gamma(x, y) \Gamma(y, x) \right) \right)^{-1/2}.
\]
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Taking $\alpha_1 = \alpha_2$ sufficiently large, this implies that

$$\Gamma(x, x)\Gamma(y, y) - \Gamma(x, y)\Gamma(y, x) \geq 0. \quad (3.2)$$

If we set $\alpha_2 = 0$ in (3.1) we see that for any $x \in T$

$$\Gamma(x, x) \geq 0. \quad (3.3)$$

In addition, by [30, p. 135, last line], for any pair $x, y \in T$

$$\Gamma(x, y)\Gamma(y, x) \geq 0. \quad (3.4)$$

It follows from (3.2)–(3.4) that for any pair $x, y \in T$, the matrix

$$\begin{bmatrix}
\Gamma(x, x) & (\Gamma(x, y)\Gamma(y, x))^{1/2} \\
(\Gamma(x, y)\Gamma(y, x))^{1/2} & \Gamma(y, y)
\end{bmatrix}$$

is positive definite, so that we can construct a mean zero Gaussian vector \{G(x), G(y)\} with covariance matrix

$$E(G(x)G(y)) = (\Gamma(x, y)\Gamma(y, x))^{1/2}. \quad (3.5)$$

Note that

$$\left( E(G(x) - G(y))^2 \right)^{1/2} = \frac{\sqrt{3/2}}{4} d(x, y), \quad (3.6)$$

defined in (1.6).

**Lemma 3.1** Suppose that $\theta := \{\theta_x, x \in T\}$ is a permanental process for $\Gamma$ as given in (1.2). Then for any pair $x, y$,

$$\{\theta_x, \theta_y\} \xrightarrow{\mathcal{L}} \{G^2(x), G^2(y)\} \quad (3.7)$$

where $\{G(x), G(y)\}$ is a mean zero Gaussian random variable with covariance matrix given by (3.5).

**Proof** By (3.1) the Laplace transform of $\{\theta_x, \theta_y\}$ is the same as the Laplace transform of $\{G^2(x), G^2(y)\}$.

**Lemma 3.2** When $\theta$ is continuous on $T$ almost surely, $d(x, y)$ is continuous on $T \times T$.  

Proof By Lemma 3.1

\[ E(θ_x) = Γ(x, x) \quad \text{and} \quad \text{cov}\{θ_x, θ_y\} = 2Γ(x, y)Γ(y, x). \quad (3.8) \]

In addition, since the univariate marginals of θ are the squares of Gaussian random variables, θ_x and θ_y are locally uniformly bounded in any \( L^p \) space. □

The next lemma is the critical ingredient in the proof of Theorem 1.1.

Lemma 3.3 Let \( \theta := \{θ_x, x ∈ T\} \) be a permanental process with kernel \( Γ \). Then for all \( x, y ∈ T \) and \( 0 < λ < ∞ \)

\[ ρ_λ(x, y) := \left\| \frac{θ_x ∧ λ}{λ^{1/2}} − \frac{θ_y ∧ λ}{λ^{1/2}} \right\|_ψ ≤ d(x, y). \quad (3.9) \]

Proof By Lemma 3.1 it suffices to show this with \( \{θ_x, θ_y\} \) replaced by \( \{G^2(x), G^2(y)\} \). We have

\[ \left| \frac{G^2(x) ∧ λ}{λ^{1/2}} − \frac{G^2(y) ∧ λ}{λ^{1/2}} \right| = \frac{1}{λ^{1/2}}|G(x) ∧ λ^{1/2} − |G(y) ∧ λ^{1/2}|\left|G(x) ∧ λ^{1/2} + |G(y) ∧ λ^{1/2}|\right| \leq 2|G(x) ∧ λ^{1/2} − |G(y) ∧ λ^{1/2}| \leq 2|G(x) − |G(y)| \right| \leq 2|G(x) − G(y)|. \quad (3.10) \]

Consequently, it follows from (3.6) that

\[ \left\| \frac{G^2(x) ∧ λ − G^2(y) ∧ λ}{2λ^{1/2} d(x, y)} \right\|_ψ ≤ \left\| \frac{G(x) − G(y)}{d(x, y)} \right\|_ψ = \frac{\sqrt{3/2}}{4} ||θ||_ψ = 1/2. \quad (3.11) \]

The proof of Theorem 1.1 depends on the estimates in the following lemma.

Lemma 3.4 Under the hypotheses of Theorem 1.1 assume that (1.8) holds. Then there exists a version \( \theta' := \{θ'_x, x ∈ T\} \) of \( \theta \) such that for any \( x_0 ∈ T \)

\[ \left\| \sup_{x ∈ T} θ'_x \right\|_ψ ≤ 4\left\| θ'_{x_0} \right\|_ψ + C \left( \sup_{x ∈ T} Γ(x, x) \right) J^2_δ(D), \quad (3.12) \]
where $C$ is a constant.

Furthermore, there exists a version $\theta' = \{\theta'_x, x \in T\}$ of $\theta$ such that for any $x_0 \in T$

$$
\sup_{s,t \in T, d(s,t) \leq \delta} |\theta'_s(\omega) - \theta'_t(\omega)| \leq 4 \left( \sup_{x \in T} \theta'_x \right)^{1/2} Z(\omega) J_d(\delta) \quad a.s.,
$$

(3.13)

where $\|Z\|_{\psi_2} \leq K$, for some constant $K$.

**Proof of Lemma 3.4** By hypothesis $T$ is separable for $d$. Therefore, it follows from Lemma 5.4 that $(T, d)$ is a separable metric or pseudometric space, and that $\mathcal{B}(T, d) = \mathcal{B}(T, d_\theta)$, the Borel $\sigma$-algebra for $(T, d_\theta)$. By [3, Theorem 2] we may assume that $\theta = \{\theta_x, x \in T\}$ is measurable and separable with respect to $(T, d_\theta)$. (But for this assumption we must allow $\theta_x$ to take the value $\infty$). More explicitly, measurability means that $\theta_x(\omega) : T \times \Omega \rightarrow [0, \infty]$ is $\mathcal{B}(T, d_\theta) \times \mathcal{F}$ measurable and separability means that we can find a countable subset $T'$ of $T$ and a $P$-null set $N \in \mathcal{F}$ such that for all $\omega \not\in N$ and $x \in T - T'$, $\theta_x(\omega) = \lim_{y \in T'} \theta_y(\omega)$.

Set

$$
Y_\lambda(x) = \frac{\theta_x \wedge \lambda}{\lambda^{1/2}}, \quad x \in T.
$$

(3.14)

By Lemma 3.3

$$
\rho_\lambda(x, y) = \|Y_\lambda(x) - Y_\lambda(y)\|_{\psi_2} \leq d(x, y).
$$

(3.15)

Note also that

$$
Y_\lambda^2(x) = \frac{(\theta_x \wedge \lambda)^2}{\lambda} = \frac{(\theta_x \wedge \lambda)(\theta_x \wedge \lambda)}{\lambda} \leq \frac{\theta_x \lambda}{\lambda} = \theta_x.
$$

(3.16)

We now consider

$$
Z_\lambda(\omega) := \inf\{\alpha > 0 : \int_T \left( \exp \left( \frac{Y_\lambda^2(t)}{\alpha^2} \right) - 1 \right) \mu(dt) \leq 1\}
$$

(3.17)

and

$$
Z(\omega) := \inf\{\alpha > 0 : \int_T \left( \exp \left( \frac{\theta_t}{\alpha^2} \right) - 1 \right) \mu(dt) \leq 1\}.
$$

(3.18)

It follows from (3.16) that

$$
Z_\lambda \leq Z, \quad \forall \lambda > 0.
$$

(3.19)
Following the argument in [24] page 258 we see that for all $u \geq 0$ and $p \geq 1$,
\[ P(Z > u) \leq 2^{-p} E \left( \int_T \exp \left( \frac{p \theta t}{u^2} \right) \mu(dt) \right). \] (3.20)

Let $\Gamma^* := \sup_x \Gamma(x, x)$. By Lemma 3.1, for any $p < u^2/(2\Gamma^*)$,
\[ E \left( \exp \left( \frac{p \theta t}{u^2} \right) \right) = E \left( \exp \left( \frac{p G^2(t)}{u^2} \right) \right) \leq (1 - 2p\Gamma^*/u^2)^{-1/2}. \] (3.21)

Using this in (3.20) we see that
\[ P(Z > u) \leq 2^{-p} (1 - 2p\Gamma^*/u^2)^{-1/2}. \] (3.22)

For $u > \sqrt{\Gamma^*(2 + 1/\log 2)}^{1/2}$, this last term is minimized by $p = u^2/(2\Gamma^*) - 1/(2 \log 2)$. Thus we see that for $u > \sqrt{\Gamma^*(2 + 1/\log 2)}^{1/2}$
\[ P(Z > u) \leq (e \log 2)^{1/2} \left( u/\sqrt{\Gamma^*} \right) 2^{-u^2/2\Gamma^*}. \] (3.23)

Let $\beta = 2 + 1/\log 2$. It follows from (3.23) that for $p \geq 1$
\[ E(Z^{2p}) \leq \int_0^\infty P(Z > u) \, d(u^{2p}) \] (3.24)
\[ \leq (\beta \Gamma^*)^p + (e \log 2)^{1/2} \int_{\beta\Gamma^*}^\infty \frac{u}{\sqrt{\Gamma^*}} 2^{-u^2/2\Gamma^*} \, d(u^{2p}) \]
\[ = (\beta \Gamma^*)^p + (e \log 2)^{1/2}(\Gamma^*)^p \int_{\beta}^\infty v^{-v^2/2} \, d(v^{2p}) \]
\[ \leq (\beta \Gamma^*)^p + 2 \left( e \sqrt{2\pi \log 2} \right)^{1/2} \Gamma^p E(\eta^{2p}) \]
\[ \leq C p(\Gamma^*)^p E(\eta^{2p}) \leq E \left( (2C\Gamma^* \eta^2)^p \right), \]

where $\eta$ is $N(0, 1)$ and $C$ is a constant.

For later use we note that (3.24) implies that
\[ \|Z^2\|_{\psi_1} \leq C' \Gamma^*, \] (3.25)

for some absolute constant $C'$.

It follows from (3.15) and (1.8) that we can apply Theorem 2.1 with $X = Y_{\lambda}$, $A = B(D, d\theta)$ and $d = \rho_{\lambda}$. Let $T'$ be a separability set for $\theta$. By (2.5), (3.15) and (3.19), we see that for any fixed $\lambda$ and any $x_0 \in T'$
\[ \sup_{x \in T'} |Y_{\lambda}(x, \omega) - Y_{\lambda}(x_0, \omega)| \leq 2Z(\omega)J_{\lambda}(D) \leq 2Z(\omega)J_{d}(D), \quad a.s. \] (3.26)
By (3.14), for any fixed $\lambda$
\[
\sup_{x \in T'} \left| \theta_x(\omega) \wedge \lambda \right| \leq \theta_{x_0}(\omega) \wedge \lambda + 2\lambda^{1/2} Z(\omega)J_d(D) \quad \text{a.s.} \tag{3.27}
\]

Let $A_\lambda = \{ \omega : \sup_{x \in T'} |\theta_x(\omega)| \geq \lambda \}$. For all $\omega \in A_\lambda$
\[
\lambda \leq \theta_{x_0}(\omega) + 2\lambda^{1/2} Z(\omega)J_d(D). \tag{3.28}
\]

Barlow points out in [1, page 31], that when $\lambda^2 \leq A + \lambda B$, then $\lambda^2 \leq 2A + B^2$.

Therefore, for all $\omega \in A_\lambda$,
\[
\lambda \leq 2\theta_{x_0}(\omega) + 4(Z(\omega)J_d(D))^2. \tag{3.29}
\]

This shows that
\[
\text{Pr} \left( \sup_{x \in T'} \theta_x \geq \lambda \right) \leq \text{Pr} \left( 4\theta_{x_0} \geq \lambda \right) + \text{Pr} \left( 8(Z(\omega)J_d(D))^2 \geq \lambda \right). \tag{3.30}
\]

Using the formula $E|X|^p = \int_0^\infty P(|X| > u) \, d(\lambda^p)$, we obtain
\[
E \left( \sup_{x \in T'} \sum_{x_0}^\lambda \theta_x \right) \leq E \left( (4\theta_{x_0})^p \right) + E \left( 8J_d^2(D)Z^2 \right)^p. \tag{3.31}
\]

Therefore
\[
\| \sup_{x \in T'} \theta_x \|_\psi \leq 4\| \theta_{x_0} \|_\psi + 8J_d^2(D)\| Z^2 \|_\psi. \tag{3.32}
\]

Using (3.25) and separability we get (3.12).

We return to (2.5) and now use it together with (3.15) and (3.19) to see that after restricting to $T'$, for any $\lambda > 0$
\[
\sup_{x,y \in T'} d(x,y) \leq \lambda - \theta_y(\omega) \wedge \lambda \leq 2\lambda^{1/2} Z(\omega) J_d(\delta), \quad \text{a.s.} \tag{3.33}
\]

Using (3.15) again we see that
\[
\sup_{x,y \in T'} \left| \theta_x(\omega) \wedge \lambda - \theta_y(\omega) \wedge \lambda \right| \leq 2\lambda^{1/2} Z(\omega) J_d(\delta), \quad \text{all rational } \lambda, \quad \text{a.s.} \tag{3.34}
\]

Let $\Omega'$ be the set on which (3.34) holds. For a given $\omega \in \Omega'$, choose $\lambda$ to be some rational number satisfying $\sup_{x \in T'} \theta'_x(\omega) \leq \lambda \leq 2\sup_{x \in T'} \theta'_x(\omega)$. Doing this for all $\omega \in \Omega'$ we get
\[
\sup_{x,y \in T'} \left| \theta_x(\omega) - \theta_y(\omega) \right| \leq 4 \left( \sup_{x \in T'} \theta_x(\omega) \right)^{1/2} Z(\omega) J_d(\delta) \quad \text{a.s.} \tag{3.35}
\]
To obtain (3.13) it suffices to show that for any $x, y \in T$ we can find sequences $x_n, y_n \in T'$ such that $d(x_n, y_n) \to d(x, y)$ and for all $\omega \in N$, $\lim_{n \to \infty} \theta_{x_n}(\omega) = \theta_{x}(\omega)$ and $\lim_{n \to \infty} \theta_{y_n}(\omega) = \theta_{y}(\omega)$. To this end we first choose sequences $x_n, y_n \in T'$ such that $d_\theta(x_n, x) \to 0$ and $d_\theta(y_n, y) \to 0$. Using separability we see that for all $\omega \notin N$, $\lim_{n \to \infty} \theta_{x_n}(\omega) = \theta_{x}(\omega)$ and $\lim_{n \to \infty} \theta_{y_n}(\omega) = \theta_{y}(\omega)$ and by Lemma 5.3 and (5.6) and (5.2),

$$|d(x_n, y_n) - d(x, y)| \leq C \left( d_\theta^{1/4}(y_n, y) + d_\theta^{1/4}(x_n, x) \right) \to 0,$$

as $d_\theta(x_n, x) \to 0$ and $d_\theta(y_n, y) \to 0$. 

**Proof of Theorem 1.1** Since both random variables on the right-hand side of (3.13) are finite almost surely we see that (1.9) implies (1.10). To obtain (1.12) we use (3.15) and the hypotheses of this theorem we get (2.15) with $X'$ replaced by $Y_\lambda$ and $T$ replaced by $T'$. We complete the proof as we did in the previous paragraph.

**Proof of Corollary 1.2** We proceed exactly as in the proof of Theorem 1.1 except that now, we replace $(T, d)$ by $(T, d_1)$, and by (1.14), in place of (3.15) we have

$$\rho_\lambda(x, y) = \|Y_\lambda(x) - Y_\lambda(y)\|_{\psi_2} \leq d_1(x, y).$$

4 Local moduli of continuity

In this section we give a basic theorem for moduli of continuity of processes in $L^{\psi_2}$ in the spirit of Section 2 and apply it to permanental processes, as we do for the uniform modulus of continuity in Section 3.

**Lemma 4.1** Let $(T, \hat{d})$ be a separable metric or pseudometric space with finite diameter $D$. Suppose that there exists a probability measure $\mu$ on $(T, \hat{d})$ such that $J_{T, \hat{d}, \mu}(D) < \infty$.

For any $t_0 \in T$ and $\delta > 0$, let $T_\delta := \{s : \hat{d}(s, t_0) < \delta/2\}$. Suppose $0 < \delta \leq \delta_0 < D$ which implies that $T_\delta \subseteq T_D$. Consider the probability measures
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\[ \mu_\delta(\cdot) := \mu(\cdot \cap T_\delta)/\mu(T_\delta), \quad 0 < \delta \leq \delta_0 \text{ and assume that } c_{\mu_\delta, T_\delta}(\bar{f}) < \infty, \text{ for each } 0 < \delta \leq \delta_0; \text{ (see (2.10) for the definition of } c_{\mu_\delta, T_\delta}). \]

Then

\[
\sup_{\bar{d}(s,t_0) < \delta/2} |f(s) - f(t_0)| \leq 20 \sup_{t \in T_\delta} \int_0^{\delta/4} \left( \log \left( \frac{c_{\mu_\delta, T_\delta}(\bar{f}) + 1}{\mu_\delta^2(B_{\bar{d}}(t,u))} \right) \right)^{1/2} du. \quad (4.1)
\]

**Proof** The condition that \( J_{T, d, \mu}(D) < \infty \) implies that \( \mu(B_{\bar{d}}(t,u)) > 0 \) for all \( u, t \in T, u \neq t \). Since \( T_\delta \) is open for every \( t \in T_\delta \) there exists a ball say \( B'_{\bar{d}}(t,u) \subset T_\delta \). Consequently

\[
\mu_\delta(B'_{\bar{d}}(t,u)) = \frac{\mu(B'_{\bar{d}}(t,u))}{\mu(T_\delta)} > 0 \quad (4.2)
\]

for all \( u, t \in T, u \neq t \). Therefore, (4.1) follows from Lemma 2.1.

**Corollary 4.1** Let

\[
H_{T_\delta, d, \mu_\delta, \delta}(\bar{f})
\]

\[
= \delta \left( \log(c_{\mu_\delta, T_\delta}(\bar{f}) + 1) \right)^{1/2} + \sup_{t \in T_\delta} \int_0^{\delta/4} \left( \log \left( \frac{1}{\mu_\delta(B_{\bar{d}}(t,u))} \right) \right)^{1/2} du
\]

Under the hypotheses of Lemma 4.1

\[
\lim_{\delta \to 0} \sup_{\bar{d}(s,t_0) \leq \delta/2} \sup_{f \in H_{T_\delta, d, \mu_\delta, \delta}} |f(s) - f(t_0)| \leq 30 \quad \text{a.s.} \quad (4.4)
\]

**Theorem 4.1** Under the hypotheses of Theorem 2.1 assume that (2.7) holds. Define \( \mu_\delta \) and \( T_\delta \) as in Lemma 4.1. Then

\[
\lim_{\delta \to 0} \sup_{\bar{d}(s,t_0) \leq \delta/2} \left| X'(s) - X'(t_0) \right| \leq 30 \quad \text{a.s.} \quad (4.5)
\]

**Example 4.1** Theorem 4.1 seems very abstract. We show here how it gives the familiar iterated logarithm behavior for fairly regular processes on nice spaces.

To begin let us consider the first term on the right-hand side of (4.3), with \( \bar{f} \) replaced by \( \bar{X} \). It is simply bounded by a constant times \( \delta \) unless
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\[ \limsup_{\delta \to 0} c_{\mu_{\delta},T_{\delta}}(\bar{X}) = \infty \] on a set of positive measure. Let us assume this is the case. As in (2.16), \( E c_{\mu_{\delta},T_{\delta}}(\bar{X}) = 1 \). Therefore, for \( \epsilon > 0 \),

\[
P \left( \log c_{\mu_{\delta},T_{\delta}}(\bar{X}) \geq (1 + \epsilon)u \right) \leq P \left( c_{\mu_{\delta},T_{\delta}}(\bar{X}) \geq e^{(1+\epsilon)u} \right) \leq e^{-(1+\epsilon)u}. \tag{4.6}
\]

It follows from the Borel-Cantelli Lemma that for all \( \beta < 1 \)

\[
\limsup_{k \to \infty} \log c_{\mu_{\beta k},T_{\beta k}}(\bar{X}) \leq \log \log \frac{1}{\beta^k}. \tag{4.7}
\]

We would like to extend this to get

\[
\limsup_{\delta \to 0} \frac{\delta \left( \log c_{\mu_{\delta},T_{\delta}}(\bar{X}) \right)^{1/2}}{\delta \left( \log \log 1/\delta \right)^{1/2}} \leq C. \tag{4.8}
\]

Note that for \( \beta^{k+1} < \delta \leq \beta^k \)

\[
c_{\mu_{\delta},T_{\delta}}(\bar{X}) = \frac{1}{\mu^2(T_{\delta})} \int_{T_{\delta} \times T_{\delta}} \psi_2(\bar{X}) \, d\mu(s) \, d\mu(t) \leq \frac{1}{\mu^2(T_{\delta})} \int_{T_{\beta^k} \times T_{\beta^k}} \psi_2(\bar{X}) \, d\mu(s) \, d\mu(t) \leq \frac{\mu^2(T_{\beta^k})}{\mu^2(T_{\beta^{k+1}})} c_{\mu_{\beta k},T_{\beta k}}(\bar{X}). \tag{4.9}
\]

Consequently, if

\[
\limsup_{k \to \infty} \frac{\mu(T_{\beta^k})}{\mu(T_{\beta^{k+1}})} \leq C, \tag{4.10}
\]

we can use (4.7) to get (4.8).

We now make many regularity assumptions. Take \( T \) to be the unit interval in \( R^1 \). Assume that

\[
d(s,t_0) = \phi(|s-t_0|), \quad \text{for} \quad 0 < |s-t_0| \leq \delta_0, \tag{4.11}
\]

for some \( \delta_0 > 0 \), and some continuous increasing function \( \phi \). Now take \( \mu \) to be Lebesgue measure. In this case

\[
\mu(T_{\delta}) = \phi^{-1}(\delta/2), \tag{4.12}
\]
so that, for example, (4.10) holds if \( \phi \) is regularly varying. In addition, it follows from [24, (7.94)], that the second term on the right-hand side of (4.18), with \( \hat{f} \) replaced by \( \hat{X} \) is bounded by a constant times

\[
\delta + \int_0^1 \frac{\phi(\phi^{-1}(\delta/2)u)}{u(\log 2/u)^{1/2}} \, du. \tag{4.13}
\]

Note that under (4.11) we can replace \( \hat{d}(s,t_0) \leq \delta/2 \) in (4.15) by \( |s - t_0| \leq \phi^{-1}(\delta/2) \). Then, replacing \( \phi^{-1}(\delta/2) \) by \( \delta' \) and making a change of variables, as in [24, (7.96)], and using (4.8) we get

\[
\lim_{\delta' \to 0} \sup_{|s-t_0| \leq \delta'} \frac{|X'(s) - X'(t_0)|}{\tilde{H}(\delta')} \leq C \quad \text{a.s.} \tag{4.14}
\]

where

\[
\tilde{H}(\delta) = \phi(\delta)(\log \log 1/\delta)^{1/2} + \int_0^1 \frac{\phi(2\delta u)}{u(\log 2/u)^{1/2}} \, du. \tag{4.15}
\]

By [24, (7.128)], if \( \phi \) is regularly varying,

\[
\lim_{\delta \to 0} \frac{\tilde{H}(\delta)}{\phi(\delta)(\log 1/\delta)^{1/2}} = 1. \tag{4.16}
\]

In the same vein, under (4.10) and the assumption that \( \phi \) is regularly varying, it follows from [22] and the material in [24] pages 298 and 299] that

\[
\lim_{\delta \to 0} \sup_{|s-t| \leq \delta} \frac{|X'(s) - X'(t)|}{\phi(\delta)(\log 1/\delta)^{1/2}} \leq C \quad \text{a.s.} \tag{4.17}
\]

We have the following results for the local moduli of continuity of permanental processes.

**Theorem 4.2** Under the hypotheses of Theorem 1.1 assume that (1.9) and (4.10) hold. Then if \( \theta_{t_0} \neq 0 \) almost surely there exists a version \( \theta' = \{\theta'_x, x \in T\} \) such that

\[
\lim_{\delta \to 0} \sup_{\hat{d}(s,t_0) \leq \delta/2} \frac{|\theta'_s - \theta'_{t_0}|}{\overline{H}_{T_\delta,d,\mu_4}(\delta/4)} \leq C \theta_{t_0}^{1/2} \quad \text{a.s.} \tag{4.18}
\]

where

\[
\overline{H}_{T_\delta,d,\mu_4}(\delta/4) := \delta(\log 1/\delta)^{1/2} + J_{T_\delta,d,\mu_4}(\delta/4). \tag{4.19}
\]
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(See Lemma 4.1 for the definitions of other terms.)

If \( \theta_{t_0} \equiv 0 \) there exists a version \( \theta' = \{ \theta'_x, x \in T \} \) such that

\[
\lim_{\delta \to 0} \sup_{d(s,t_0) \leq \delta/2} \frac{\theta'_s}{(H_{T_{\delta},d,\mu}(\delta/4))^2} \leq C \quad \text{a.s.} \tag{4.20}
\]

**Proof** By (3.15) and the hypotheses of this theorem we get (4.5) with \( X' \) replaced by \( Y' \) and \( H_{T_{\delta},d,\mu}(\delta) \) replaced by \( H_{T_{\delta},d,\mu}(\delta/4) \). As we do at the end of the proof of Lemma 3.4, for a given \( \omega \in \Omega \), we choose \( \lambda \) to be some rational number satisfying

\[
\sup_{x \in B_{d}(t_0,\delta')} \theta'_x(\omega) \leq \lambda \leq 2 \sup_{x \in B_{d}(t_0,\delta')} \theta'_x(\omega)
\]

for some \( \delta' > 0 \). Doing this for all \( \omega \in \Omega \) we get

\[
\lim_{\delta \to 0} \sup_{d(s,t_0) \leq \delta/2} \frac{|\theta'_s - \theta'_t_0|}{H_{T_{\delta},d,\mu}(\delta/4)} \leq C \sup_{x \in B_{d}(t_0,\delta')} (\theta'_x)^{1/2} \quad \text{a.s..} \tag{4.21}
\]

Since this holds for all \( \delta' > 0 \) we get (4.18). To get (4.20) we simply note that when \( \theta_{t_0} \equiv 0 \)

\[
\frac{\theta_s \wedge \lambda}{\lambda^{1/2}} - \frac{\theta_{t_0} \wedge \lambda}{\lambda^{1/2}} = \frac{\theta_s \wedge \lambda}{\lambda^{1/2}} \leq \theta_s^{1/2} \wedge \lambda^{1/2}. \tag{4.22}
\]

In this case, instead of (4.21) we get

\[
\lim_{\delta \to 0} \sup_{d(s,t_0) \leq \delta/2} \frac{(\theta'_s)^{1/2}}{H_{T_{\delta},d,\mu}(\delta/4)} \leq C \quad \text{a.s.} \tag{4.23}
\]

**Remark 4.1** Note that if \( \theta \) is the square of Gaussian process, \( H_{T_{\delta},d,\mu}(\cdot) \) is equivalent to the correct local modulus of continuity of the Gaussian process.

5 Dominating metrics for permanental processes

We exhibit several interesting metrics and other functions that dominate \( d \) or are even equivalent to \( d \). (\( d_1 \) is equivalent to \( d \), \( d_1 \approx d_2 \) if there exist constants \( 0 < c_1 \leq c_2 < \infty \) such that \( c_1 d \leq d_1 \leq c_2 d \).) Note that for \( C \neq 0 \)

\[
J_{T,Cd,\mu}(a) = C J_{T,d,\mu}(a/C). \tag{5.1}
\]

Therefore, multiplying a metric or related function by a constant alters our results in an acceptable way.
We consider several scenarios. To simplify the exposition we work with
\[ \overline{d}(x, y) := d(x, y) / 4 \sqrt{2/3} = \left( \Gamma(x, x) + \Gamma(y, y) - 2 \left( \Gamma(x, y) \Gamma(y, x) \right)^{1/2} \right)^{1/2}. \] (5.2)

1. The kernel \( \Gamma \) is symmetric and positive definite. This is the classical case in which the permanental process is the square of a Gaussian process and the kernel is the covariance matrix of a Gaussian process, say \( G = \{ G(x), x \in T \} \), which when squared is equal to the permanental process. Let
\[ d_G(x, y) = \left( E (G(x) - G(y))^2 \right)^{1/2}. \] (5.3)

Since
\[ (\Gamma(x, y) \Gamma(y, x))^{1/2} = |\Gamma(x, y)| \geq \Gamma(x, y), \] (5.4)
we get \( \overline{d}(x, y) \leq d_G(x, y) \). Obviously, \( d_G \) is a metric. It is well known that \( J_{T, d_G, \mu}(D) \) is finite if (and only if) \( G \) is bounded, and that \( \lim_{\delta \to 0} J_{T, d_G, \mu}(\delta) = 0 \) if (and only if) \( G \) is a continuous on \((T, d_G)\). In Section 7.2 we explain how to use Theorem 1.1 to show that \( \lim_{\delta \to 0} J_{T, d_G, \mu}(\delta) = 0 \) is a sufficient condition for the almost sure continuity of \( G \) on \((T, d_G)\).

2. Possibly \( \overline{d} \) itself is a metric even when \( \Gamma(t, s) \neq \Gamma(s, t) \) for some \( s, t \in T \). To verify this it suffices to show that all the \( 3 \times 3 \) determinants of the matrices \( \{ (\Gamma(x_i, x_j) \Gamma(x_j, x_i))^{1/2}, 1 \leq i, j \leq 3 \} \), are greater than or equal to zero, for all distinct triples \( (x_1, x_2, x_3 \in T) \), since this implies that \( (G_{x_1}, G_{x_2}, G_{x_3}) \) is a Gaussian vector in \( \mathbb{R}^3 \) and therefore, \( d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3) \).

3. Conditions under which \( \overline{d} \) is equivalent to natural metrics for \( \theta \).

**Lemma 5.1** Let \( d_\theta \) be as given in (1.13) and define
\[ \hat{d}_\theta(x, y) = \left( E \left( (\theta_x - E\theta_x) - (\theta_y - E\theta_y) \right)^2 \right)^{1/2}. \] (5.5)

Then
\[ \frac{\sqrt{2}}{\sqrt{2} + 1} d_\theta(x, y) \leq \hat{d}_\theta(x, y) \leq 2 d_\theta(x, y) \] (5.6)
and
\[ K (\Gamma(x, x) + \Gamma(y, y))^{1/2} \overline{d}(x, y) \leq \hat{d}_\theta(x, y) \leq 2 (\Gamma(x, x) + \Gamma(y, y))^{1/2} \overline{d}(x, y), \] (5.7)
where \( K = \sqrt{2} / (\sqrt{2} + 1) \).
Remark 5.1 By (5.7)
\[ c_1 \bar{d}(x,y) \leq \hat{d}_\theta(x,y) \leq c_2 \bar{d}(x,y). \]  
(5.8)
where \( c_1 = 2 \inf_{x \in T} \Gamma^{1/2}(x,x) \) and \( c_2 = 2 \sqrt{2} \sup_{x \in T} \Gamma^{1/2}(x,x) \). In particular, if
\[ 0 < \inf_{x \in T} \Gamma(x,x) \leq \sup_{x \in T} \Gamma(x,x) < \infty, \]
then \( d \) is equivalent to \( \hat{d}_\theta \) and \( d_\theta \).

Proof By Lemma 3.2
\[ \bar{d}_\theta^2(x,y) = 2 \left( \Gamma^2(x,x) + \Gamma^2(y,y) - 2 \Gamma(x,y) \Gamma(y,x) \right). \]  
(5.9)
Let
\[ \tilde{d}^2(x,y) := (E\theta_x - E\theta_y)^2 = (\Gamma^2(x,x) + \Gamma^2(y,y) - 2 \Gamma(x,y) \Gamma(y,x)) \]  
(5.10)
By (3.2)
\[ \tilde{d}(x,y) \leq \frac{1}{\sqrt{2}} \hat{d}_\theta(x,y). \]  
(5.11)
By the Cauchy-Schwarz Inequality
\[ \bar{d}(x,y) \leq d_\theta(x,y). \]  
(5.12)
Using this and the triangle inequality we see that
\[ \hat{d}_\theta(x,y) \leq d_\theta(x,y) + \bar{d}(x,y) \leq 2d_\theta(x,y) \]  
(5.13)
and
\[ \hat{d}_\theta(x,y) \geq d_\theta(x,y) - \tilde{d}(x,y), \]  
(5.14)
which, along with (5.11), implies that
\[ (1 + \frac{1}{\sqrt{2}}) \hat{d}_\theta(x,y) \geq d_\theta(x,y). \]  
(5.15)
Thus we get (5.6).

By (5.9) and (5.15)
\[ \bar{d}_\theta^2(x,y) \leq 2 \left( (\Gamma(x,x) + \Gamma(y,y))^2 - 4 \Gamma(x,y) \Gamma(y,x) \right) \]  
(5.16)
\[ = 2 \left( \Gamma(x,x) + \Gamma(y,y) - 2 \sqrt{\Gamma(x,y) \Gamma(y,x)} \right) \times \left( \Gamma(x,x) + \Gamma(y,y) + 2 \sqrt{\Gamma(x,y) \Gamma(y,x)} \right). \]
This gives the upper bound in (5.7).

For the lower bound we note that

\[
d^2_\theta (x,y) = E \left( (G(x) - G(y))^2 (G(x) + G(y))^2 \right) \\
= E (G(x) - G(y))^2 E (G(x) + G(y))^2 + 2 \left( E \{G^2(x) - G^2(y)\}\right)^2 \\
\geq E (G(x) - G(y))^2 E (G(x) + G(y))^2 \\
\geq \left( \Gamma(x,x) + \Gamma(y,y) - 2\sqrt{\Gamma(x,y}\Gamma(y,x)} \right) \\
\times \left( \Gamma(x,x) + \Gamma(y,y) + 2\sqrt{\Gamma(x,y}\Gamma(y,x)} \right)
\]

Consequently

\[
d_\theta (x,y) \geq (\Gamma(x,x) + \Gamma(y,y))^{1/2} \hat{d}(x,y).
\]

(5.18)

Using (5.6) we get the lower bound in (5.7).

\[\square\]

**Lemma 5.2**

\[
d(x,y) \leq d^{1/2}_\theta (x,y).
\]

(5.19)

**Proof**

By (5.17)

\[
d^2_\theta (x,y) \geq (\Gamma(x,x) + \Gamma(y,y))^2 - 4\Gamma(x,y)\Gamma(y,x)).
\]

(5.20)

Consequently

\[
d_\theta (x,y) \geq \left( (\Gamma(x,x) + \Gamma(y,y)) - 2(\Gamma(x,y)\Gamma(y,x))^{1/2} \right).
\]

(5.21)

Taking the square root again we get (5.19).

\[\square\]

**Lemma 5.3**

\[
|\hat{d}(x,y) - \hat{d}(x,z)| \leq C \left( 1 + \sup_{u,v\in T} \Gamma(u,u) \right) \left( \hat{d}^{3/4}_\theta (y,z) + \hat{d}^{1/2}_\theta (y,z) \right).
\]

(5.22)

**Proof**

\[
|\hat{d}(x,y) - \hat{d}(x,z)|
\]

(5.23)

\[
\leq |\hat{d}^2(x,y) - \hat{d}^2(x,z)|^{1/2}
\leq |\Gamma(y,y) - \Gamma(z,z)|^{1/2} + 2|\Gamma(x,z)\Gamma(z,x)\Gamma(y,z)\Gamma(y,x)|^{1/2} - (\Gamma(x,y)\Gamma(y,x))^{1/2}|^{1/2}
\leq |\Gamma(y,y) - \Gamma(z,z)|^{1/2} + 2\Gamma(x,z)\Gamma(z,x) - \Gamma(x,y)\Gamma(y,x)|^{1/4}.
\]
By (5.9)
\[ |\Gamma(x, z)\Gamma(z, x) - \Gamma(x, y)\Gamma(y, x)| \leq C \left( |\hat{d}_\theta^2(x, z) - \hat{d}_\theta^2(x, y)| + |\Gamma^2(y, y) - \Gamma^2(z, z)| \right), \tag{5.24} \]
and by (5.11)
\[ |\Gamma(y, y) - \Gamma(z, z)| \leq \hat{d}_\theta(y, z). \tag{5.25} \]

In addition
\[ |\hat{d}_\theta^2(x, z) - \hat{d}_\theta^2(x, y)| \leq 2 \sup_{u,v \in T} \hat{d}_\theta(u, v)|\hat{d}_\theta(x, z) - \hat{d}_\theta(x, y)| \tag{5.26} \]
\[ \leq 8 \sup_{u \in T} \Gamma(u, u)\hat{d}_\theta(y, z). \tag{5.27} \]

Putting these together we get (5.22).

**Lemma 5.4** Assume that \( \sup_{u \in T} \Gamma(u, u) < \infty \). Then the sets \( b_{\hat{d}}(x, u) = \{ y \in T \mid \hat{d}(x, y) < u \}, x \in T, u \in R_+ \) form the base for the \( \hat{d}_\theta \) (and equivalently the \( d_\theta \)) metric topology.

**Proof** Let \( f_x(y) = \hat{d}(x, y) \). By (5.22) we have that \( f_x \) is continuous with respect to \( \hat{d}_\theta \), and hence \( b_{\hat{d}}(x, u) = f_x^{-1}([0, u]) \) is open with respect to \( \hat{d}_\theta \). We now show that for any \( x \in T, u \in R_+ \), and any \( y \in b_{\hat{d}}(x, u) \), we can find \( v > 0 \) such that \( b_{\hat{d}}(y, v) \subseteq b_{\hat{d}}(x, u) \). To see this, first choose \( w > 0 \) such that \( b_{\hat{d}}(y, w) \subseteq b_{\hat{d}}(x, u) \). It then follows from (5.8) that \( b_{\hat{d}}(y, c_2^{-1}w) \subseteq b_{\hat{d}}(y, w) \). By (5.6) the same argument applies with \( \hat{d}_\theta \) replaced by \( d_\theta \).

Let \( \Sigma(x, y) = \Gamma(x, y)\Gamma(y, x) \). It follows from Lemma 3.2 that \( \{ \Sigma(x, y), x, y \in T \} \) is positive definite. Therefore it is the covariance of a mean zero Gaussian process which we denote by \( \{ S(x), x \in T \} \). Clearly
\[ \hat{d}_\theta(x, y) = \left( E'(S_x - S_y)^2 \right)^{1/2}. \tag{5.27} \]

4. **Conditions under which \( \hat{d} \) is equivalent to a function that may be a metric for a Gaussian process** We suppose that
\[ |\Gamma(x, y)| \vee |\Gamma(y, x)| \leq \Gamma(y, y) \wedge \Gamma(x, x). \tag{5.28} \]

Let
\[ d_2(x, y) = \{ \Gamma(x, x) + \Gamma(y, y) - (|\Gamma(x, y)| + |\Gamma(y, x)|) \}^{1/2}. \tag{5.29} \]
Lemma 5.5  When \((5.28)\) holds

\[
\frac{1}{\sqrt{2}} \bar{d}(x, y) \leq d_2(x, y) \leq \bar{d}(x, y).
\]

(5.30)

In general when \(\Gamma(x, y)\) is the potential density of a Borel right process \(X\), in place of \((5.28)\) we only have

\[
0 \leq \Gamma(x, y) \leq \Gamma(y, y) \quad \text{and} \quad 0 \leq \Gamma(y, x) \leq \Gamma(x, x).
\]

(5.31)

(See, e.g. [24, Lemma 3.3.6] where this is proved for symmetric potential densities and note that the proof also works when the densities are not symmetric.)

Set \(\bar{\Gamma}(x, y) = \Gamma(y, x)\). This is the potential density of \(\bar{X}\), the dual process of \(X\). Therefore, if \(\bar{X}\) is also a Borel right process, using \((5.31)\), we actually get \((5.28)\). In [7] it is shown that for certain Borel right processes \(X\) with potential density \(\Gamma(x, y)\), \(d_2(x, y)\) is a metric. (See Section 7 for details.)

Proof of Lemma 5.5  We have

\[
\bar{d}^2(x, y) = d_2^2(x, y) + ||\Gamma(x, y)||^{1/2} - ||\Gamma(y, y)||^{1/2} ||\Gamma(y, x)||^{1/2} \leq d_2^2(x, y) + ||\Gamma(x, y)| - |\Gamma(y, x)||.
\]

(5.32)

By \((5.28)\) if \(|\Gamma(x, y)| - |\Gamma(y, x)|| \geq 0\) then

\[
||\Gamma(x, y)| - |\Gamma(y, x)|| \leq \Gamma(y, y) - |\Gamma(y, x)| \leq (|\Gamma(y, x)| + |\Gamma(x, x)|)
\]

(5.33)

Interchanging \(x\) and \(y\) we also get that when and if \(|\Gamma(y, x)| - |\Gamma(x, y)|| \geq 0\)

\[
|\Gamma(y, x)| - |\Gamma(x, y)| \leq d_2^2(x, y).
\]

(5.34)

Therefore

\[
\bar{d}^2(x, y) \leq 2d_2^2(x, y).
\]

(5.35)

Using this and the first line of \((5.32)\) we get \((5.30)\).

Remark 5.2  Suppose that \(\Gamma(x, y)\) is the potential density of a Borel right process but that we don’t require that the process has a strong dual. By \((5.31)\)

\[
d_2(x, y) := \{\Gamma(x, x) + \Gamma(y, y) - (\Gamma(x, y) + \Gamma(y, x))\}^{1/2}
\]

(5.36)
is well defined and we have

\[
\overline{d}^2(x, y) = d_2^2(x, y) + \Gamma(x, y) + \Gamma(y, x) - 2 (\Gamma(x, y)\Gamma(y, x))^{1/2} \quad (5.37)
\]

\[
= d_2^2(x, y) + \left(\sqrt{\Gamma(x, y)} - \sqrt{\Gamma(y, x)}\right)^2.
\]

Define

\[
d_3(x, y) = |\sqrt{\Gamma(x, y)} - \sqrt{\Gamma(y, x)}|.
\quad (5.38)
\]

thus we have

\[
d(x, y) \geq d_2(x, y) \geq d_3(x, y) \quad \text{and} \quad d(x, y) \leq d_2(x, y) + d_3(x, y).
\quad (5.39)
\]

This gives the following lemma:

**Lemma 5.6** For fixed \(\mu\) and \(T\) consider \(J_d(a), J_{d_2}(a)\) and \(J_{d_3}(a)\) for any \(0 < a \leq D\). Then

\[
J_d(a) < \infty \quad \text{if and only if both} \quad J_{d_2}(a) < \infty \quad \text{and} \quad J_{d_3}(a) < \infty.
\quad (5.40)
\]

**Proof** Only if is a trivial consequence of the first two inequalities in (5.39). For the other direction, let

\[
A_t(u) := \{u : d_2(t, u) \leq d_3(t, u)\} \quad \text{and} \quad B_t(u) := \{u : d_3(t, u) < d_2(t, u)\}.
\quad (5.41)
\]

Therefore,

\[
\overline{d}(t, u) \leq 2d_3(t, u) \quad \text{on} \quad A_t(u) \quad \text{and} \quad \overline{d}(t, u) \leq 2d_2(t, u) \quad \text{on} \quad B_t(u).
\quad (5.42)
\]

This implies that

\[
J_{T,\overline{d},\mu}(a) \leq \int_{A_t(u)} I_{[0, a]}(u) \left(\log \frac{1}{\mu(B_{2d_3}(t, u))}\right)^{1/2} du \quad (5.43)
\]

\[
+ \int_{B_t(u)} I_{[0, a]}(u) \left(\log \frac{1}{\mu(B_{2d_2}(t, u))}\right)^{1/2} du.
\]

\[
\leq \int_0^a \left(\log \frac{1}{\mu(B_{2d_3}(t, u))}\right)^{1/2} du + \int_0^a \left(\log \frac{1}{\mu(B_{2d_2}(t, u))}\right)^{1/2} du.
\]

Using (5.1) completes the proof. \(\square\)

The point here is that (5.28) makes \(d_3\) irrelevant, but for general Borel right processes it must be taken into account. Using \(d\) in Theorem 1.1 takes it into account.
Remark 5.3 Let \( \{\Gamma(x,y), x, y \in T\} \) be the kernel of a permanental process. In \cite{8} an important role is played by the family of related permanental processes \( \Gamma_\delta := \{\Gamma(x,y) + \delta, x, y \in T\}, \delta \geq 0 \). Let \( d_\delta \) be the function defined in (5.2) with \( \Gamma \) replaced by \( \Gamma_\delta \). We note that when \( \Gamma(x,y) \geq 0 \) for all \( x, y \in T \)
\( d_\delta(x,y) \) decreases as \( \delta \) increases. (5.44)

To see this we use the first line of (5.32) to get
\[
d_\delta^2(x,y) = d_2^2(x,y) + \left( (\Gamma(x,y) + \delta)^{1/2} - (\Gamma(y,x) + \delta)^{1/2} \right)^2,
\]
(5.45)
since, obviously, \( d_2 \) doesn’t change when \( \Gamma \) replaced by \( \Gamma_\delta \). It is easy to see that the last term in (5.45) is decreasing as \( \delta \) increases.

6 Local times of Borel right processes

Our primary motivation for obtaining sample path properties of permanental processes was to use them, along with the following isomorphism theorem, to obtain sample path properties of the local times of Borel right processes, paralleling our use of Dynkin’s isomorphism theorem in \cite{22}, to obtain sample path properties of the local times of strongly symmetric Borel right processes.

Let \( X = (\Omega, X_t, P^x) \) be a Borel right process with 0-potential density \( u(x,y) \). Let \( h_x(z) = u(z,x) \) and assume that \( h_x(z) > 0 \) for all \( x, z \in S \). Recall that the expectation operator \( E^{z/h_x} \) for the \( h_x \)-transform of \( X \) is given by
\[
E^{z/h_x} (F_{\{t < \zeta\}}) = \frac{1}{h_x(z)} E^z (F h_x(X_t)) \quad \text{for all } F \in bF_0,
\]
(6.1)
where \( F_0 \) is the \( \sigma \)-algebra generated by \( \{X_r, 0 \leq r \leq t\} \). (See e.g. \cite{24} (3.211).)

Recall that on page 3 we wrote that Eisenbaum and Kaspi pointed out that the 0-potential of a transient Markov process was a kernel for a permanental process. Using this they establish the following isomorphism theorem.

**Theorem 6.1 (Eisenbaum and Kaspi, \cite{8})** Let \( X = (\Omega, X_t, P^x) \) be a Borel right process with 0-potential density \( u(x,y) \), and let \( L = \{L^y_t; (y, t) \in S \times R_+\} \) denote the local times for \( X \), normalized so that
\[
E^v(L^y_\infty) = u(v,y).
\]
(6.2)
Let \( x \) denote a fixed element of \( S \), and assume that \( u(x,x) > 0 \). Set
\[
h_x(z) = u(z,x).
\]
(6.3)
Let \( \theta = \{ \theta_y ; y \in S \} \) denote the permanental process with kernel \( u(x,y) \). Then, for any countable subset \( D \subseteq S \),

\[
\left\{ L^y_\infty + \frac{1}{2} \theta_y ; y \in D, P^{x/h_x} \times P_\theta \right\} \overset{\text{law}}{=} \left\{ \frac{1}{2} \theta_y ; y \in D, \frac{\theta_x}{u(x,x)} P_\theta \right\}. \tag{6.4}
\]

Equivalently, for all \( x_1, \ldots, x_n \) in \( S \) and bounded measurable functions \( F \) on \( \mathbb{R}^n \), for all \( n \),

\[
E^{x/h_x} E_\theta \left( F \left( L^x_\infty + \frac{\theta_{x_i}}{2} \right) \right) = E_\theta \left( \frac{\theta_x}{u(x,x)} F \left( \frac{\theta_{x_i}}{2} \right) \right). \tag{6.5}
\]

(Here we use the notation \( F(f(x_i)) := F(f(x_1), \ldots, f(x_n)) \).)

Theorem 6.1 is only a partial analogue of Dynkin’s isomorphism theorem for strongly symmetric Borel right processes, [24, Theorem 8.1.3], which holds with measures \( P^{x/h} \), for a much wider class of functions \( h \) than those in (6.3). In addition, note that Theorem 6.1 can only give a version of \( \{ \hat{L}^y_t ; (y,t) \in S \times \mathbb{R}^+ \} \) which is jointly continuous with respect to the measures \( P^{x/h_x} \). In order to use this to obtain joint continuity with respect to the measures \( P^x \) we use (6.1) with \( z = x \). Therefore, since we require that \( h_x(z) > 0 \) for all \( z \in S \), when \( P^{x/h_x}(A, t < \zeta) = 0 \) for some \( A \in \mathcal{F}^0 \), we also have \( P^x(A, t < \zeta) = 0 \).

When we say that a stochastic process \( \hat{L} = \{ \hat{L}^y_t ; (y,t) \in S \times \mathbb{R}^+ \} \) is a version of the local time of a Markov process \( X \) we mean more than the traditional statement that one stochastic process is a version of the other. Besides this we also require that the version is itself a local time for \( X \), i.e. that for each \( y \in S \), \( \hat{L}^y \) is a local time for \( X \) at \( y \). To be more specific, suppose that \( L = \{ L^y_t ; (y,t) \in S \times \mathbb{R}^+ \} \) is a local time for \( X \). When we say that we can find a version of the local time which is jointly continuous on \( S \times \mathbb{T} \), where \( T \subset \mathbb{R}^+ \), we mean that we can find a stochastic process \( \hat{L} = \{ \hat{L}^y_t ; (y,t) \in S \times \mathbb{R}^+ \} \) which is continuous on \( S \times \mathbb{T} \) for all \( x \in S \) and which satisfies, for each \( x, y \in S \)

\[
\hat{L}^y_t = L^y_t \quad \forall t \in \mathbb{R}^+, \quad P^x \text{ a.s.} \tag{6.6}
\]

Following convention, we often say that a Markov process has a continuous local time, when we mean that we can find a continuous version for the local time.

**Proof of Theorem 1.2** The proof follows the general lines of the proof for symmetric Markov processes in [22, Section 6]. However, there are significant differences, so we give a self contained proof.
Since $S$ is a locally compact topological space with a countable base, we can find a metric $\rho$ which induces the topology of $S$. We first consider the case where $X$ is a transient Borel right process with state space $S$ and continuous, strictly positive 0-potential densities $u(x,y)$. We take $\theta$ to be the permanental process with kernel $u(x,y)$.

Fix a compact set $K \subseteq T$ and some $x \in K$. By (1.19), Corollary 1.1 and Lemma 3.4 we can find a version of $\theta$ which is continuous on $K$ almost surely and such that for each $p$

$$E \sup_{x \in K} \theta^p_x < \infty. \quad (6.7)$$

We work with this version.

It follows from [22] (4.30) and (4.31) that for any $z, y \in S$

$$E^z/h_x (L^y_\infty) = \frac{u(z,y)h_x(y)}{h_x(z)}. \quad (6.8)$$

We shall use the fact that $X_t$ is a right continuous simple Markov process under the measures $P^z/h_x$, [24] Lemma 3.9.1.

To begin, we first show that $L$ is jointly continuous on $K \times \mathbb{R}_+^{+}$, almost surely with respect to $P^x/h_x$. By [24] Lemma 3.9.1 we can assume that the local times $L^y_t$ are $\mathcal{F}_t^0$ measurable. Consider the martingale

$$A^y_t = E^{x/h_x}(L^y_\infty | \mathcal{F}_t^0). \quad (6.9)$$

Let $\tau_t$ denote the shift operator on $\Omega$. Then

$$L^y_\infty = L^y_t + L^y_\infty \circ \tau_t = L^y_t + 1_{(t<r)} L^y_\infty \circ \tau_t. \quad (6.10)$$

Therefore

$$A^y_t = L^y_t + E^{x/h_x} (1_{(t<r)} L^y_\infty \circ \tau_t | \mathcal{F}_t^0) = L^y_t + 1_{(t<r)} E^{x/h_x} (L^y_\infty \circ \tau_t | \mathcal{F}_t^0) = L^y_t + 1_{(t<r)} E^{x/h_x} (L^y_\infty), \quad (6.11)$$

where we use the simple Markov property described above. It follows from (6.8), using the convention that $1/h(\Delta) = 0$, that

$$A^y_t = L^y_t + \frac{u(X_t,y)h_x(y)}{h_x(X_t)}. \quad (6.12)$$

Since $X_t$ is right continuous for $P^x/h_x$, $A^y_t$ is also right continuous. Let $D$ be a countable dense subset of $K$ and $F$ a finite subset of $D$. Since

$$\sup_{\rho(x,y) \leq \delta} A^y_t - A^z_t = \sup_{\rho(x,y) \leq \delta} |A^y_t - A^z_t| \quad (6.13)$$
is a right continuous, non-negative submartingale, we have, for any \( \epsilon > 0 \)
\[
P^{x/h} \left( \sup_{t \geq 0} \sup_{\rho(y,z) \leq \delta} A^y_t - A^z_t \geq \epsilon \right)
\leq \frac{1}{\epsilon} E^{x/h} \left( \sup_{\rho(y,z) \leq \delta} L^y_t - L^z_t \right)
\leq \frac{1}{\epsilon} E^{x/h} \left( \sup_{\rho(y,z) \leq \delta} L^y_\infty - L^z_\infty \right).
\] (6.14)

It follows from (6.5) that
\[
E^{x/h} \left( \sup_{\rho(y,z) \leq \delta} L^y_\infty - L^z_\infty \right) \leq E_\theta \left( \sup_{\rho(y,z) \leq \delta} \left| \frac{\theta_y}{2} - \frac{\theta_z}{2} \right| \right)
+ \frac{1}{u(x,x)} \left( E_\theta \left( \sup_{\rho(y,z) \leq \delta} \left| \frac{\theta_y}{2} - \frac{\theta_z}{2} \right|^2 \right) E_\theta(\theta_x^2) \right)^{1/2}.
\] (6.15)

It follows from the uniform continuity of \( \theta \) on \( K \) and (6.7) that for any \( \bar{\epsilon} > 0 \), we can choose a \( \delta > 0 \) such that the right hand side (6.15) is less that \( \bar{\epsilon} \). Combining (6.12)–(6.15) we get
\[
P^{x/h} \left( \sup_{t \geq 0} \sup_{\rho(y,z) \leq \delta} L^y_t - L^z_t \geq 2\epsilon \right)
\leq \bar{\epsilon} + P^{x/h} \left( \sup_{t \geq 0} \frac{1}{h(X_t)} \sup_{\rho(y,z) \leq \delta} \left( u(X_t, y) h_x(y) - u(X_t, z) h_x(z) \right) \geq \epsilon \right)
\leq \bar{\epsilon} + P^{x/h} \left( \sup_{t \geq 0} \frac{1}{h(X_t)} \geq \frac{\epsilon}{\gamma(\delta)} \right),
\] (6.16)

where
\[
\gamma(\delta) = \sup_{x \in S} \sup_{\rho(y,z) \leq \delta} \sup_{y,z \in D} |u(x, y) h_x(y) - u(x, z) h_x(z)|
= \sup_{x \in K} \sup_{\rho(y,z) \leq \delta} \sup_{y,z \in D} |u(x, y) h_x(y) - u(x, z) h_x(z)|.
\] (6.17)

The last equality follows from [24, (3.69)], since the proof does not require that \( u(x, y) \) is symmetric.

It follows easily from (6.1) and the fact that \( X_t \) is a simple Markov process under the measures \( P^{x/h} \), that \( 1/h_x(X_t) \) is a supermartingale with respect to
\( P_{x/h_x} \). Since \( 1/h_x(X_t) \) is also right continuous and non-negative, we have

\[
P_{x/h_x} \left( \sup_{t \geq 0} \frac{1}{h_x(X_t)} \geq \frac{\epsilon}{\gamma(\delta)} \right) \leq \frac{\gamma(\delta)}{\epsilon} E^{x/h_x} \left( \frac{1}{h_x(X_0)} \right) = \frac{\gamma(\delta)}{\epsilon} E^{x/h_x} \left( x \right).
\]

(6.18)

Since both \( h \) and \( u \) are bounded and uniformly continuous on \( K \), it follows from (6.17) that by choosing \( \delta > 0 \) sufficiently small we can make the right-hand-side of (6.18) less than \( \bar{\epsilon} \). By this observation and (6.16), and taking the limit over a sequence of finite sets increasing to \( D \) we see that for any \( \epsilon \) and \( \bar{\epsilon} > 0 \) we can find a \( \delta > 0 \) such that

\[
P_{x/h_x} \left( \sup_{t \geq 0} \sup_{\rho(y,z) \leq \delta} (y,z \in D) \right) \leq \gamma(\delta) \epsilon h_x(x) = \gamma(\delta) \epsilon.
\]

(6.19)

It follows by the Borel–Cantelli Lemma that we can find a sequence \( \{\delta_i\}_{i=1}^{\infty} \), \( \delta_i > 0 \), such that \( lim_{i \to \infty} \delta_i = 0 \) and

\[
\sup_{t \geq 0} \sup_{\rho(y,z) \leq \delta_i} L^y_t - L^z_t \leq \frac{1}{2^i}
\]

(6.20)

for all \( i \geq I(\omega) \), almost surely with respect to \( P_{x/h_x} \).

Fix \( T < \infty \). We will now show that \( L^y_t \) is uniformly continuous on \([0, T] \times D\), almost surely with respect to \( P_{x/h_x} \). That is, for each \( \omega \in \Omega' \subseteq \Omega \), with \( P_{x/h_x}(\Omega') = 1 \), we can find an \( I(\omega) \), such that for \( i \geq I(\omega) \)

\[
\sup_{|s-t| \leq \delta_i} \sup_{\rho(y,z) \leq \delta_i} \rho(y,z) \in D} \left| L^y_s - L^z_t \right| \leq \frac{1}{2^i},
\]

(6.21)

where \( \{\delta_i\}_{i=1}^{\infty} \) is a sequence of real numbers such that \( \delta_i > 0 \) and \( \lim_{i \to \infty} \delta_i = 0 \).

To prove (6.20), fix \( \omega \) and assume that \( i \geq I(\omega) \), so that (6.19) holds. Let \( Y = \{y_1, \ldots, y_n\} \) be a finite subset of \( D \) such that

\[
K \subseteq \bigcup_{j=1}^{n} B_{\rho}(y_j, \delta_{i+2}).
\]

By definition each \( L^y_j(\omega), j = 1, \ldots, n \), is uniformly continuous on \([0, T]\). Therefore we can find a finite increasing sequence \( t_1 = 0, t_2, \ldots, t_{k-1} < T, t_k \geq T \) such that \( t_m - t_{m-1} = \delta^u_{i+2} \) for all \( m = 1, \ldots, k \) where \( \delta^u_{i+2} \) is chosen so that

\[
\left| L^y_{t_{m+1}}(\omega) - L^y_{t_{m-1}}(\omega) \right| \leq \frac{1}{2^{i+2}} \quad \forall j = 1, \ldots, n \quad \forall m = 1, \ldots, k-1.
\]

(6.21)
Let $s_1, s_2 \in [0, T]$ and assume that $s_1 \leq s_2$ and that $s_2 - s_1 \leq \delta_{i+2}$. There exists an $1 \leq m \leq k - 1$, such that
\[ t_{m-1} \leq s_1 \leq s_2 \leq t_{m+1}. \]

If $y, z \in D$ satisfy $\rho(y, z) \leq \delta_{i+2}$ we can find a $y_j \in Y$ such that $y \in B_\rho(y_j, \delta_{i+2})$. If, in addition, $L^\rho_{s_2}(\omega) \geq L^\rho_{s_1}(\omega)$ we have
\[ \begin{align*}
0 & \leq L^\rho_{s_2}(\omega) - L^\rho_{s_1}(\omega) \\
& \leq L^\rho_{t_{m+1}}(\omega) - L^\rho_{t_{m-1}}(\omega) \\
& \leq |L^\rho_{t_{m+1}}(\omega) - L^\rho_{t_{m+1}}(\omega)| + |L^\rho_{t_{m-1}}(\omega) - L^\rho_{t_{m-1}}(\omega)| \\
& \quad + |L^\rho_{t_{m-1}}(\omega) - L^\rho_{t_{m-1}}(\omega)| + |L^\rho_{y_j}(\omega) - L^\rho_{z}(\omega)|,
\end{align*} \]
where the second inequality uses the fact that local time is non-decreasing in $t$. The second term to the right of the last inequality in (6.22) is less than or equal to $2^{-i-2}$ by (6.21). The other three terms are also less than or equal to $2^{-i-2}$ by (6.19) since $\rho(y, y_j) \leq \delta_{i+2}$ and $\rho(z, y_j) \leq \delta_{i+2}$. Taking $\delta'_i = \delta'_{i+2} \land \delta_{i+2}$ we get (6.20) on the larger set $[0, T'] \times D$ for some $T' \geq T$. Obviously this implies (6.20) as stated in the case when $L^\rho_{s_2}(\omega) \geq L^\rho_{s_1}(\omega)$. A similar argument gives (6.20) when $L^\rho_{s_2}(\omega) \leq L^\rho_{s_1}(\omega)$. Thus (6.20) is established.

In what follows we say that a function is locally uniformly continuous on a measurable set $A$ in a locally compact metric space if it is uniformly continuous on $A \cap K$ for all compact subsets $K \subseteq S$. Let $K_n$ be a sequence of compact subsets of $S$ such that $S = \bigcup_{n=1}^\infty K_n$, and let $D'$ be a countable dense subset of $S$. Let
\[ \hat{\Omega} = \{ \omega \mid L^\rho_t(\omega) \text{ is locally uniformly continuous on } [0, \zeta) \times D' \} \]
Let $Q$ denote the rational numbers. Then
\[ \hat{\Omega}_c = \bigcup_{1 \leq n \leq \infty} \{ \omega \mid L^\rho_t(\omega) \text{ is not uniformly continuous on } [0, s] \times (K_n \cap D'); s < \zeta \} \]
(6.23)
Since $h_x > 0$ it follows from (6.20) and (6.1) that $P^x(\hat{\Omega}_c) = 0$ for all $x \in S$, or equivalently, that
\[ P^x(\hat{\Omega}) = 1 \quad \forall x \in S. \]
(6.24)
We now construct a stochastic process $\hat{L} = \{ \hat{L}^\rho_t, (t, y) \in R_+ \times S \}$ which is continuous on $[0, \zeta) \times S$ and which is a version of $L$. For $\omega \in \hat{\Omega}$, let $\{ \hat{L}^\rho_t(\omega), (t, y) \in [0, \zeta) \times S \}$
[0, \zeta) \times S$ be the continuous extension of $\{L^y_t(\omega), (t, y) \in [0, \zeta) \times D'\}$ to $[0, \zeta) \times S$. Set

$$\hat{L}^y_t(\omega) = \tilde{L}^y_t(\omega) \quad \text{if } t < \zeta(\omega) \quad (6.25)$$

$$\hat{L}^y_t(\omega) = \liminf_{s \uparrow \zeta(\omega)} \tilde{L}^y_s(\omega) \quad \text{if } t \geq \zeta(\omega) \quad (6.26)$$

and for $\omega \in \hat{\Omega}$ set

$$\hat{L}^y_t(\omega) \equiv 0 \quad \forall t, y \in R_+ \times S.$$ 

The stochastic process $\{\hat{L}^y_t, (t, y) \in R_+ \times S\}$ is well defined and, clearly, is jointly continuous on $[0, \zeta) \times S$.

We now show that $\hat{L}$ is a local time by showing that for each $x, y \in S$

$$\hat{L}^y_t = L^y_t, \quad \forall t \in R_+, \quad P^x \text{ almost surely.} \quad (6.27)$$

Recall that for each $z \in D'$, $\{L^y_t, t \in R_+\}$ is increasing, $P^x$ almost surely. Hence, the same is true for $\{\hat{L}^y_t, t < \zeta\}$ and so the limit inferior in (6.26) is actually a limit, $P^x$ almost surely. Thus $\{\hat{L}^y_t, t \in R_+\}$ is continuous and constant for $t \geq \zeta$, $P^x$ almost surely. Similarly, $L^y_t$, the local time for $X$ at $y$, is, by definition, continuous in $t$ and constant for $t \geq \zeta$, $P^x$ almost surely. Now let us note that we could just as well have obtained (6.20) with $D'$ replaced by $D' \cup \{y\}$ and hence obtained (6.24) with $D'$ replaced by $D' \cup \{y\}$ in the definition of $\hat{\Omega}$. Therefore if we take a sequence $\{y_i\}_{i=1}^\infty$ with $y_i \in D' \cup \{y\}$ such that $\lim_{i \to \infty} y_i = y$ we have that

$$\lim_{i \to \infty} L^y_{t_i} = L^y_t \quad \text{locally uniformly on } [0, \zeta), \quad P^x \text{ a.s.} \quad (6.28)$$

By the definition of $\hat{L}$ we also have

$$\lim_{i \to \infty} \hat{L}^y_{t_i} = \hat{L}^y_t \quad \text{locally uniformly on } [0, \zeta), \quad P^x \text{ a.s.} \quad (6.29)$$

This shows that

$$\hat{L}^y_t = L^y_t \quad \forall t < \zeta, \quad P^x \text{ a.s.} \quad (6.30)$$

Since $\hat{L}^y_t$ and $L^y_t$ are continuous in $t$ and constant for $t \geq \zeta$ we get (6.27). This completes the proof of Theorem 1.2 when $X$ is a transient Borel right process.

Now let $X$ be a recurrent Borel right process with state space $S$ and continuous, strictly positive 1-potential densities $u^1(x, y)$. Let $Y$ be the Borel right process obtained by killing $X$ at an independent exponential time $\lambda$ with mean
one. The 0-potential densities for $Y$ are the 1-potential densities for $X$. Thus we have a transient Borel right process $Y$ with continuous, strictly positive 0-potential densities $u^1(x, y)$. It is easy to see that $L_t^y$ is a local time for $Y$. Therefore, by what we have just shown for transient processes, $L_t^y$ is continuous on $S \times [0, \lambda)$, $P^x \times \nu$ almost surely, where $\nu$ is the probability measure of $\lambda$. It now follows by Fubini’s Theorem that $L_t^y$ is continuous $[0, q_i) \times S$ for all $q_i \in Q$, $P^x$ almost surely, where $Q$ is a countable dense subset of $R_+$. This gives the proof when $X$ is recurrent.

**Proof of Theorem 1.3** We use (6.5) for the process obtained by killing $X$ at an independent exponential time $\rho$ with mean 1 to see that for any countable set $C \subseteq T$

$$E^{x/\rho} \left( \sup_{y \in C} L_t^y \right) \leq E^{x/\rho} E_\theta \left( \sup_{y \in C} \left( L_t^y + \frac{\theta y}{2} \right) \right)$$

$$= E_\theta \left( \frac{\theta_x}{u(x, x)} \sup_{y \in C} \left( \frac{\theta y}{2} \right) \right)$$

$$\leq \frac{1}{u(x, x)} \left\{ E_\theta \left( \theta_x^2 \right) E_\theta \left( \sup_{y \in C} \left( \frac{\theta y}{2} \right)^2 \right) \right\}^{1/2}.$$  

By Lemma 3.4

$$E^{x/\rho} \left( \sup_{y \in C} L_t^y \right) < \infty,$$

so, consequently,

$$\sup_{y \in C} L_t^y < \infty, \quad P^{x/\rho} \text{ almost surely.}$$  

By Fubini’s Theorem this implies that

$$\sup_{y \in C} L_t^y < \infty, \quad P^{x/\rho} \text{ almost surely}$$

for almost all $t \in [0, \zeta)$. As in the preceding proof, we can deduce that this also holds $P^x$ almost surely. Since $L_t^y$ is non-decreasing in $t$ we get (1.20).

For the proof of (1.22) we require the next lemma. Fix an element $0 \in S$. We use the notation $\|x||_{\psi_2, 0/h_0}$ to denote the Orlicz space norm with respect to $E^{0/h_0}$. 


Lemma 6.1 Under the hypotheses of Theorem 6.1 let
\[ \kappa(x, y) = \left( u(x, x) + u(y, y) - 2(u(x, y)u(y, x))^2 \right)^{1/2}. \] (6.35)

Then
\[ \| L_x^x \wedge \lambda - L_y^y \wedge \lambda \|_{\psi_2, 0/h_0} \leq C_0 \lambda^{1/2} \kappa(x, y), \] (6.36)
where \( C_0 = 2(\sqrt{2} + \sqrt{2}/\sqrt{3}) \).

Proof It follows from (6.5), Lemma 3.1 and (3.10) that
\[ E_0/h_0 E_0 \left( \psi_2 \left( \frac{2(L_x^x \wedge \lambda - L_y^y \wedge \lambda) + (\theta_x \wedge \lambda - \theta_y \wedge \lambda)}{2\alpha \lambda^{1/2} \kappa(x, y)} \right) \right) \] (6.37)
\[ = E_0 \left( \frac{\theta_0}{u(0, 0)} \psi_2 \left( \frac{\theta_x \wedge \lambda - \theta_y \wedge \lambda}{2\alpha \lambda^{1/2} \kappa(x, y)} \right) \right) \]
\[ \leq \frac{1}{u(0, 0)} \left\{ E_0 \left( \psi_0^2 \left( \frac{\theta_x \wedge \lambda - \theta_y \wedge \lambda}{2\alpha \lambda^{1/2} \kappa(x, y)} \right) \right) \right\}^{1/2} \]
\[ \leq \sqrt{3} \left\{ E \left( \psi_0^2 (\eta/a) \right) \right\}^{1/2}. \]

Note that
\[ E \left( \psi_0^2 (\eta/a) \right) = E \left( \exp \left( 2\eta^2/a^2 \right) \right) - 2E \left( \exp \left( \eta^2/a^2 \right) \right) + 1 \] (6.38)
\[ = \frac{1}{\sqrt{1 - 4/a^2}} - 2 \cdot \frac{1}{\sqrt{1 - 2/a^2}} + 1. \]

In particular, if we set \( a^2 = 8 \), \( E_0 \left( \psi_0^2 (\eta/a) \right) \leq .2. \) Consequently
\[ \| 2(L_x^x \wedge \lambda - L_y^y \wedge \lambda) + (\theta_x \wedge \lambda - \theta_y \wedge \lambda) \|_{\psi_2, 0/h_x} \leq 4\sqrt{2} \lambda^{1/2} \kappa(x, y). \] (6.39)

Using the triangle inequality and Lemma 3.3 we get (6.36).

Proof of Theorem 1.3 continued We use (6.36) for the process obtained by killing \( X \) at an independent exponential time \( \rho \) of mean 1 to see that
\[ \| L_{x, \rho}^x \wedge \lambda - L_{y, \rho}^y \wedge \lambda \|_{\psi_2, 0/h_0} \leq C_0 \lambda^{1/2} d(x, y). \] (6.40)

In the same way we obtained (1.18), this gives
\[ \lim_{\delta \to 0} \sup_{x,y \in K, d_1(x,y) \leq \delta} \frac{|L_{x, \rho}^x - L_{y, \rho}^y|}{J_{d_1}(d_1(x,y)/2)} \leq C_1 (\sup_{x \in K} L_{x, \rho}^x)^{1/2}, \quad P^{0/h_0} \text{ a.s.} \] (6.41)
By Fubini’s Theorem this implies that
\[
\lim_{\delta \to 0} \sup_{x, y \in K, d_1(x, y) \leq \delta} \frac{|L^x_t - L^y_t|}{J_{d_1}(d_1(x, y)/2)} \leq C_1 \left( \sup_{x \in K} L^x_t \right)^{1/2}, \tag{6.42}
\]
for almost all \( t \in [0, \zeta) \), \( P^0/h_0 \) almost surely. As in the last proof, we can deduce that this also holds \( P^0 \) almost surely.

**Theorem 6.2** Let \( X = (\Omega, \mathcal{X}_t, P^x) \) be a Borel right process that satisfies all the hypotheses in Theorem 1.3 used to obtain (1.22). Let \( T_\delta \) and \( \mu_\delta \) be as in Lemma 4.1 and assume that (4.10) holds. Then
\[
\lim_{\delta \to 0} \sup_{d(x, x_0) \leq \delta/2} \frac{|L^x_t - L^{x_0}_t|}{H_{T_\delta, d_1, \mu_\delta}(\delta/4)} \leq 30 \left( L^{x_0}_t \right)^{1/2} \quad \text{a.s.} \tag{6.43}
\]
where \( H_{T_\delta, d_1, \mu_\delta}(\delta/4) \) is given in (4.19).

**Proof** This follows from (4.18) in the same way (1.22) follows from (1.18).

\[\square\]

### 7 Further discussion

#### 7.1 **Theorem 1.2** gives the continuity results in [7, Theorem 1.1]

Let \( X \) be a recurrent Borel right process with state space \( S \) and strictly positive \( \alpha \)-potential densities with respect to some reference measure. Let 0 be a distinguished point in \( S \) and let \( u_{T_0}(x, y) \) denote the potential densities of the Borel right process \( Y \), which is \( X \) killed the first time it hits 0. In [7], the authors show that when \( X \) has a dual Borel right process, \( u_{T_0}(x, y) + u_{T_0}(y, x) \) is positive definite, so that
\[
\kappa(x, y) = \left( u_{T_0}(x, x) + u_{T_0}(y, y) - u_{T_0}(x, y) - u_{T_0}(y, x) \right)^{1/2} \tag{7.1}
\]
is a metric on \( S \). In [7, Theorem 1.1] they show that if for every compact set \( K \subseteq S \), one can find a probability measure \( \mu_K \) on \( K \), such that
\[
\lim_{\delta \to 0} J_{K, \kappa, \mu_K}(\delta) = 0, \tag{7.2}
\]
then the local times of \( X \) are jointly continuous.
We show how this result follow from Theorem 1.2. Let \( \{L^0_t; (y, t) \in S \times R_+\} \) denote the local times of \( X \). Let \( \tau(t) = \inf\{s \geq 0 | L^0_s > t\} \) be the inverse local time at 0, and let \( \lambda \) be an independent exponential random variable with mean 1. Let \( u_{\tau(\lambda)}(x, y) \) denote the potential densities for the Borel right process \( Z \), which is \( X \) killed at \( \tau(\lambda) \). It follows from \([24, (3.193)]\) that

\[
u(x, y) := u_{\tau(\lambda)}(x, y) + 1.
\]

Let \( d(x, y) \) be the function defined in (1.6) for the kernel \( u_{\tau(\lambda)}(x, y) \).

Therefore, it follows from Theorem 1.2, that \( X \) has continuous local times on \( S \times [0, \tau(\lambda)) \). Using Fubini’s Theorem, as in the last paragraph of the proof of Theorem 1.1, and the fact that \( \lim_{t \to \infty} \tau(t) = \infty \), we see that \( X \) has jointly continuous local times on \( S \times [0, \infty) \).

### 7.2 On the continuity of squares of Gaussian processes

Let \( \{G(t), t \in \mathcal{T}\} \) be a Gaussian process with covariance \( u(x, y) \). Let

\[
d_G(x, y) := \left( E(G(x) - G(y))^2 \right)^{1/2}
\]

\[
= (u(x, x) + u(y, y) - 2u(x, y))^{1/2}.
\]

The well known necessary and sufficient condition for the almost sure continuity of \( G \) on the metric space \((\mathcal{T}, d_G)\) is that there exists a probability measure \( \mu \) on \( \mathcal{T} \) such that

\[
\lim_{\delta \to 0} J_{\mathcal{T}, d_G, \mu}(\delta) = 0.
\]

We now explain how to use Theorem 1.1 to show that (7.7) is a sufficient condition for the almost sure continuity of \( G \) on \((\mathcal{T}, d_G)\).

Since continuity is a local property and the covariance \( u(x, y) \) is continuous on \((\mathcal{T}, d_G)\), we can assume that \( |u(x, y)| \) is bounded on \( \mathcal{T} \). We can therefore choose an \( 0 < M < \infty \) such that

\[
v(x, y) := u(x, y) + M > 0, \quad \forall x, y \in \mathcal{T}.
\]
Set $\tilde{G}(t) = G(t) + M\eta$, where $\eta$ is an independent standard normal. Note that $u(x, y) + M$ is the covariance of the Gaussian process $\{\tilde{G}(t), t \in T\}$. Therefore $\{(\tilde{G}(t))^2, t \in T\}$ is the permanental process with kernel $v(x, y)$. Since $v(x, y)$ is positive and symmetric, the function $d(x, y)$ determined by $v(x, y)$ in (1.6) is

$$d(x, y) = 4\sqrt{2/3} ((u(x, x) + M) + (u(y, y) + M) - 2(u(x, y) + M))^{1/2}$$

(7.9)

Thus if (7.7) holds, Theorem 1.1 shows that $\{(\tilde{G}(t))^2, t \in T\}$ is almost surely continuous on the metric space $(T, d_G)$. We now explain why this implies the almost sure continuity of $\{\tilde{G}(t), t \in T\}$, which, obviously, is equivalent to the almost sure continuity of $\{G(t), t \in T\}$.

Clearly if a Gaussian process $G$ is continuous so is $G^2$. The converse is not true without additional conditions. For example consider the Gaussian process $G(t) = \eta f(t), t \in [0, 1]$, where $\eta$ is a normal random variable with mean 0 and variance 1, and where $f^2(t)$ is continuous but $f(t)$ is not. In this case $G^2(t)$ has continuous paths on $[0, 1]$, in the usual topology, but $G(t)$ does not. Note that in this example the covariance of $G(t)$ is not continuous. In the following lemma we add this condition.

**Lemma 7.1** Let $(T, d)$ be a compact metric space and let $\{G(t), t \in T\}$ be a Gaussian process with covariance continuous in some neighborhood of $(t_0, t_0)$. Then if $G^2$ is continuous at $t_0$ so is $G$. If the covariance is continuous on all of $T$, and $\{G^2(t), t \in T\}$ is continuous then $\{G(t), t \in T\}$ is continuous.

**Proof** Because the covariance of $G$ is continuous, the discontinuities of $G$ have certain regularity properties. When $G$ has a discontinuity at $t_0$ with positive probability, i.e.

$$0 < \alpha(t_0, \omega) := \lim_{\delta \to 0} \sup_{t \in B_d(t_0, \delta)} |G(t, \omega) - G(t_0, \omega)| \leq \infty,$$

(7.10)

for $\omega \in \Omega' \subseteq \Omega$, with $P(\Omega') > 0$. Then, $P(\Omega') = 1$, and $\alpha(t, \omega) = \bar{\alpha}(t)$, is a deterministic function of $t$, on $\Omega'$. Furthermore,

$$(\bar{\alpha}(t_0))^2 = \lim_{\delta \to 0} \sup_{t \in B_d(t_0, \delta)} (G(t, \omega) - G(t_0, \omega))^2.$$  

(7.11)

(See, e.g. [24, Theorem 5.3.7].) This shows that $G^2(t)$ is not continuous at $t_0$.

It follows from the preceding paragraph that if the covariance is continuous on all of $T$ and $\{G^2(t), t \in T\}$ is continuous, then $G(t)$ is continuous at each
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t ∈ T almost surely. By [24, Corollary 5.3.8, (1)] this implies that \{G(t), t ∈ T\} is continuous almost surely.

8 Loop soup local times of certain Markov processes

Certain Markov processes \(Y_t\) have bridge measures, \(P_t^{x,x}\), which serve as regular \(P^x\) conditional measures for \(Y_s, 0 ≤ s ≤ t\), given that \(Y_t = x\). Rather than strive for the greatest possible generality, we consider the processes studied in [10]. Our work is based on [15] and [16], which deals with symmetric Markov processes.

Let \(S\) be a locally compact set with a countable base. Let \(Y = (\Omega, Y_t, P^x)\) be a recurrent Borel right process with state space \(S\), cadlag paths, and transition densities \(p_t(x, y)\), with respect to some \(\sigma\)-finite measure \(m\) on \(S\). We assume that the 1-potential densities \(u^1(x, y) = \int_0^\infty e^{-t}p_t(x, y)\) are continuous. We do not require that the process is symmetric.

Assume that \(Y\) has a dual Borel right process. It then follows that \(Y\) has jointly measurable transition densities \(p_t(x, y)\) with respect to \(m\). Assume, furthermore, that \(0 < p_t(x, x) < \infty\), for all \(0 < t < \infty\) and \(x \in S\). It then follows from [10] that for all \(0 < t < \infty\), and \(x \in S\), there exists a finite measure \(P_t^{x,x}\) on \(\mathcal{F}_{t^-}\), of total mass \(p_t(x, x)\), such that

\[
P_t^{x,x}(F) = P^x(F | p_{t-s}(Y_s, x)), \tag{8.1}
\]

for all \(F \in \mathcal{F}_s\) with \(s < t\), and that \(P_t^{x,x}\) is supported on \(\{Y_0 = Y_{t^-} = x\}\).

For \(\Delta \notin S\), let \(\Omega_\Delta\) denote the set of cadlag paths \(\omega\) in \(S \cup \Delta\) with \(\omega_t = \Delta\) for all \(t ≥ \zeta\). We set \(X_t(\omega) = \omega_t\) and as usual \(\zeta = \inf\{t > 0, X_t = \Delta\}\). We define a \(\sigma\)-finite measure \(\mu\) on \((\Omega_\Delta, \mathcal{F})\) by the following ‘disintegration’ formula, (see [3, p. 78-III]),

\[
\mu(A) = \int_0^\infty e^{-t} \int P_t^{x,x}(A \cap \{\zeta = t\}) \, dm(x) \, dt, \quad A \in \mathcal{F}. \tag{8.2}
\]

Note that \(\mu\) is supported on

\[
\mathcal{L} = \{X : X_{\zeta^-} = X_0\}. \tag{8.3}
\]

Consequently we refer to \(\mu\) as the loop measure associated with the Markov process \(Y\).

The next lemma describes the finite dimensional distributions of \(\mu\). As usual, if \(F\) is a function, we often write \(\mu(F)\) for \(\int F \, d\mu\). (We already used this notation in (8.1)).
Lemma 8.1 For any $0 < t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty$ and bounded Borel measurable functions $f_1, \ldots, f_k$ on $S \cup \Delta$ with $f_j(\Delta) = 0$, $j = 1, \ldots, k$,

\[
\mu \left( \prod_{j=1}^{k} f_j(X_{t_j}) \right) = \int_{t_k}^{\infty} e^{-t} \int \cdots \cdot \int f_1(y_1)p_{t_2-t_1}(y_1, y_2)f_2(y_2) \cdots \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k)f_k(y_k)p_{t_1+t-t_k}(y_k, y_1) \; dm(y_1) \cdots dm(y_k) \; dt.
\]

Proof Since $f_j(\Delta) = 0$, $j = 1, \ldots, k$,

\[
\prod_{j=1}^{k} f_j(X_{t_j})1_{\{\zeta = t\}} = 1_{\{t_k < t\}} \prod_{j=1}^{k} f_j(X_{t_j}).
\]

Therefore, by (8.4)

\[
\int_{t_k}^{\infty} e^{-t} \int \cdots \cdot \int f_1(y_1)p_{t_2-t_1}(y_1, y_2)f_2(y_2) \cdots \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k)f_k(y_k)p_{t_1+t-t_k}(y_k, x) \; dm(y_1) \cdots dm(y_k),
\]

so that

\[
\int_{t_k}^{\infty} e^{-t} \int \cdots \cdot \int f_1(y_1)p_{t_2-t_1}(y_1, y_2)f_2(y_2) \cdots \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k)f_k(y_k)p_{t_1+t-t_k}(y_k, y_1) \; dm(y_1) \cdots dm(y_k).
\]

Using the definition of $\mu$ in (8.2) we get (8.6).

We have the following consequence of Lemma 8.1.
Lemma 8.2 Let $f_1, \ldots, f_k$ be bounded Borel measurable functions on $S \cup \Delta$ with $f_j(\Delta) = 0$, $j = 1, \ldots, k$, and let $u^1$ be the 1-potential density of $Y$. Then

\[
\mu \left( \prod_{j=1}^{k} \left( \int_0^\infty f_j(X_t) \, dt \right) \right) = \frac{1}{k} \sum_{\pi \in P_k} \int f_{\pi(1)}(y_1)u^1(y_1,y_2)f_{\pi(2)}(y_2) \cdots u^1(y_{k-1},y_k)f_{\pi(k)}(y_k)u^1(y_k,y_1) \prod_{j=1}^{k} dm(y_j),
\]

where $P_k$ denotes the set of permutations of $[1,k]$.

Proof It follows from (8.8) that

\[
\mu \left( \int_{\{0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty\}} \prod_{j=1}^{k} f_j(X_t) \, dt_j \right) = \int_{\{0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty\}} e^{-t} \sum_{\pi \in P_k} \int f_{\pi(1)}(y_1) \cdots f_{\pi(k)}(y_k) \prod_{j=1}^{k} dm(y_j) \, dt dt_1 \cdots dt_k.
\]

We make the change of variables $(t, t_2, \ldots, t_k) \rightarrow (r_1 = t_1 + t - t_k, r_2 = t_2 - t_1, \ldots, r_k = t_k - t_{k-1})$, and integrate on $t_1$ to obtain

\[
\mu \left( \int_{\{0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty\}} \prod_{j=1}^{k} f_j(X_t) \, dt_j \right) = \int \frac{e^{-(r_1 + \cdots + r_k)}}{r_1 + \cdots + r_k} \left( \int f_{\pi(1)}(y_1)pr_2(y_1,y_2)f_2(y_2) \cdots \right.
\]

\[
\left. \cdots pr_k(y_{k-1},y_k)f_k(y_k)pr_1(y_k,y_1) \prod_{j=1}^{k} dm(y_j) \left( \int \frac{r_1}{1} dt_1 \right) \prod_{j=1}^{k} dr_j \right).
\]

\[
= \int \frac{r_1 e^{-(r_1 + \cdots + r_k)}}{r_1 + \cdots + r_k} \int f_{\pi(1)}(y_1)pr_2(y_1,y_2)f_2(y_2) \cdots \right.
\]

\[
\left. \cdots pr_k(y_{k-1},y_k)f_k(y_k)pr_1(y_k,y_1) \prod_{j=1}^{k} dm(y_j) dr_j. \right)
\]
Hence
\[
\mu \left( \prod_{j=1}^{k} \left( \int_{0}^{\infty} f_j(X_t) \, dt \right) \right) \tag{8.11}
\]
\[
= \sum_{\pi \in P_k} \mu \left( \int_{\{0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty\}} \prod_{j=1}^{k} f_{\pi(j)}(X_{t_j}) \, dt \right)
\]
\[
= \int \frac{r_1 e^{-(r_1 + \cdots + r_k)}}{r_1 + \cdots + r_k} \sum_{\pi \in P_k} \int f_{\pi(1)}(y_1)p_{\pi(2)}(y_1, y_2)f_{\pi(2)}(y_2) \cdots
\]
\[
\cdots p_{\pi k}(y_{k-1}, y_k)f_{\pi(k)}(y_k)p_{\pi_1}(y_k, y_1) \prod_{j=1}^{k} dm(y_j) \, dr_j.
\]
Since this last integral is unchanged if \( r_1 \) and \( r_j, j \neq 1 \) are interchanged we see that (8.11)
\[
= \frac{1}{k} \int e^{-(r_1 + \cdots + r_k)} \sum_{\pi \in P_k} \int f_{\pi(1)}(y_1)p_{\pi(2)}(y_1, y_2)f_{\pi(2)}(y_2) \cdots
\]
\[
\cdots p_{\pi k}(y_{k-1}, y_k)f_{\pi(k)}(y_k)p_{\pi_1}(y_k, y_1) \prod_{j=1}^{k} dm(y_j) \, dr_j,
\]
which gives (8.8). \( \square \)

Let \( f_{x, \delta} \) be an approximate \( \delta \)-function at \( x \). As usual we define the total local time of \( \{X_t, t \in \mathbb{R}_+\} \) at \( x \) by
\[
L^x_\infty = \lim_{\delta \to 0} \int_{0}^{\infty} f_{x, \delta}(X_t) \, dt. \tag{8.13}
\]
(Sometimes we emphasize that this depends on the paths \( \omega \) by writing \( L^x_\infty(\omega) \).) Following the argument of [24, Theorem 3.6.3] we can show that this limit exists almost surely and in all \( L^p \).

**Lemma 8.3**
\[
\mu \left( \prod_{j=1}^{k} L^x_\infty \right) \tag{8.14}
\]
\[
= \sum_{\pi \in P_{k-1}} u^1(x_k, x_{\pi(1)}) \cdots u^1(x_{\pi(k-2)}, x_{\pi(k-1)}) u^1(x_{\pi(k-1)}, x_k),
\]
Proof

By (8.8)

\[
\mu \left( \prod_{j=1}^{k} L_{x_j}^\infty \right) = \lim_{\delta \to 0} \mu \left( \prod_{j=1}^{k} \left( \int_0^\infty f_{x_j,\delta}(X_t) \, dt \right) \right)
\]

\[
= \frac{1}{k} \sum_{\pi \in \mathcal{P}_k} \lim_{\delta \to 0} \int f_{x_{\pi(1)},\delta}(y_1) u^1(y_1, y_2) f_{x_{\pi(2)},\delta}(y_2) \cdots
\]

\[
\cdots u^1(y_{k-1}, y_k) f_{x_{\pi(k)},\delta}(y_k) u^1(y_k, y_1) \prod_{j=1}^{k} dm(y_j)
\]

\[
= \frac{1}{k} \sum_{\pi \in \mathcal{P}_k} u^1(x_{\pi(1)}, x_{\pi(2)}) \cdots u^1(x_{\pi(k-1)}, x_{\pi(k)}) u^1(x_{\pi(k)}, x_{\pi(1)})
\]

\[
= \sum_{\pi \in \mathcal{P}_{k-1}} u^1(x_{k}, x_{\pi(1)}) \cdots u^1(x_{\pi(k-2)}, x_{\pi(k-1)}) u^1(x_{\pi(k-1)}, x_{k}).
\]

For the last equation we use the fact that we are permuting \( k \) points on a circle.

Let \( \mathcal{L}_\alpha \) be a Poisson point process on \( \Omega_\Delta \) with intensity measure \( \alpha \mu \). Note that \( \mathcal{L}_\alpha \) is a random variable; each realization of \( \mathcal{L}_\alpha \) is countable subset of \( \Omega_\Delta \). To be more specific, let

\[
N(A) := \# \{ \mathcal{L}_\alpha \cap A \}, \quad A \subseteq \Omega_\Delta.
\]

Then for any disjoint measurable subsets \( A_1, \ldots, A_n \) of \( \Omega_\Delta \), the random variables \( N(A_1), \ldots, N(A_n) \), are independent, and \( N(A) \) is a Poisson random variable with parameter \( \alpha \mu(A) \), i.e.

\[
P(N(A) = k) = \frac{(\alpha \mu(A))^k}{k!} e^{-\alpha \mu(A)}.
\]

The Poisson point process \( \mathcal{L}_\alpha \) is called the ‘loop soup’ of the Markov process \( Y \). We define the ‘loop soup local time’, \( \widehat{L}_x^x \), of \( Y \), by

\[
\widehat{L}_x^x = \sum_{\omega \in \mathcal{L}_\alpha} L_{x_\omega}^\infty(\omega).
\]

The terms ‘loop soup’ and ‘loop soup local time’ are used in [16, Theorem 9]. In [15] they are referred to, less colorfully albeit more descriptively, as Poissonian ensembles of Markov loops, and occupation fields of Poissonian ensembles of Markov loops.

The next theorem is given for associated Gaussian squares in [16, Theorem 9].
Theorem 8.1 Let $Y$ be a recurrent Borel right process with state space $S$, as described in the beginning of this section, and let $u^1(x,y)$, $x,y \in S$ denote its 1-potential density. Let $\{\hat{L}_\alpha^x, x \in S\}$ be the loop soup local time of $Y$. Then $\{2\hat{L}_\alpha^x, x \in S\}$, is an $\alpha$-permanental process with kernel $u^1(x,y)$.

Proof This theorem can be proved using the methods developed in [15]. We give a simple proof using the method of moments.

Let $\theta_x = \{\theta_x, x \in S\}$ be an $\alpha$-permanental process with kernel $u^1(x,y)$. It follows from [30, Proposition 4.2] that for any $x^1, \ldots, x^n \in S$

$$E\left( \prod_{j=1}^{n} \frac{\theta_{x_j}}{2} \right) = \sum_{\pi \in \mathcal{P}_n} \alpha^{c(\pi)} \prod_{j=1}^{n} u^1(x_j, x_{\pi(j)}), \quad (8.19)$$

where $c(\pi)$ is the number of cycles in the permutation $\pi$ of $[1,n]$. In addition by, [30, p. 128], the moment generating function of $\sum_{j=1}^{n} z_j \theta_{x_j}$ has a non-zero radius of convergence. Consequently, an $\alpha$-permanental process is determined by its moments. Therefore to prove this theorem it suffices to show that

$$E\left( \prod_{j=1}^{n} \hat{L}_{\alpha}^{x_j} \right) = \sum_{\pi \in \mathcal{P}_n} \alpha^{c(\pi)} \prod_{j=1}^{n} u^1(x_j, x_{\pi(j)}). \quad (8.20)$$

By the master formula for Poisson processes, [17, (3.6)],

$$E\left( e^{\sum_{j=1}^{n} z_j \hat{L}_{\alpha}^{x_j}} \right) = \exp \left( \int_{\Omega_\Delta} \left( e^{\sum_{j=1}^{n} z_j \hat{L}_{\alpha}^{x_j}(\omega)} - 1 \right) d\mu(\omega) \right). \quad (8.21)$$

Differentiating each side of (8.21) with respect to $z_1, \ldots, z_n$ and then setting $z_1, \ldots, z_n$ equal to zero, we get

$$E\left( \prod_{j=1}^{n} \hat{L}_{\alpha}^{x_j} \right) = \sum_{l=1}^{n} \sum_{\cup_{i=1}^{l} B_i = [1,n]} \alpha^l \prod_{i=1}^{l} \mu\left( \prod_{j \in B_i} L_{\alpha}^{x_j} \right), \quad (8.22)$$

where the second sum is over all partitions $B_1, \ldots, B_l$ of $[1,n]$. Using (8.14) it is easily seen that this is the same as the right-hand side of (8.20). \hfill \square

Remark 8.1 The results given in Section 1 provide sufficient conditions for the almost sure continuity and moduli of continuity for the loop soup local times corresponding to $\alpha = 1/2$. These continuity results also hold for the loop soup local times corresponding to all $\alpha < 1/2$, which by Theorem 8.1 are
α-permanent processes with kernels \( u^1(x, y) \). To see this let \( \mathcal{L}_\alpha \) and \( \mathcal{L}_{1/2-\alpha} \) be independent loop soups with local times \( \{ \hat{L}_\alpha^x, x \in S \} \) and \( \{ \hat{L}_{1/2-\alpha}^x, x \in S \} \). Since \( \mathcal{L}_\alpha \) is the Poisson process in \( \Omega_\Delta \) with intensity measure \( \alpha \mu \) it follows that
\[
\{ \hat{L}_\alpha^x, x \in S \} + \{ \hat{L}_{1/2-\alpha}^x, x \in S \} \overset{d}{=} \{ \hat{L}_{1/2}^x, x \in S \},
\]
(8.23)
(see (8.21)). Since the processes on the left-hand side of (8.23) are independent, continuity of \( \{ \hat{L}_{1/2}^x, x \in S \} \) implies the continuity of both \( \{ \hat{L}_\alpha^x, x \in S \} \) and \( \{ \hat{L}_{1/2-\alpha}^x, x \in S \} \).

Let \( \theta = \{ \theta_x, x \in S \} \) be an \( \alpha \)-permanent process with kernel \( u^1(x, y) \), \( x, y \in S \), as considered in Theorem 8.1. By [8, Theorem 3.1], \( \theta \) is infinitely divisible. Furthermore, in [8, Corollary 3.4], Eisenbaum and Kaspi show that the Lévy measure of \( \{ \theta_x/2, x \in S \} \) is given by the law of \( \{ L^\infty_x, x \in S \} \) under the \( \sigma \)-finite measure
\[
\frac{\alpha u^1(y, y)}{L^\infty_y} P_{y/h_y} \tag{8.24}
\]
for any \( y \in S \). (Recall the definition of \( E^{x/h_x} \) given in (6.1) and that we are assuming that \( u^1(y, y) > 0 \).) However it follows from Theorem 8.1 that the loop measure \( \alpha \mu \) is the Lévy measure of \( \{ \theta_x/2, x \in S \} \). Therefore
\[
\mu = \frac{u^1(y, y)}{L^\infty_y} P_{y/h_y}, \tag{8.25}
\]
as measures on \( \{ L^\infty_x, x \in S \} \), for any \( y \in S \). This fact is also an immediate consequence of Lemma 8.3 as we now show.

**Lemma 8.4** For any \( x_1, \ldots, x_k \in S \),
\[
\mu \left( \prod_{j=1}^k L^x_j \right) = u^1(x_k, x_k) \, E^{x_k/h_{x_k}} \left( \prod_{j=1}^{k-1} L^x_j \right). \tag{8.26}
\]

**Proof** It follows from [22] (4.28)-(4.33) that
\[
E^{x_k/h_{x_k}} \left( \prod_{j=1}^{k-1} L^x_j \right) \tag{8.27}
= \sum_{\pi \in \mathcal{P}_{k-1}} \frac{1}{h_{x_k}(x_k)} u^1(x_k, x_{\pi(1)}) \cdots u^1(x_{\pi(k-2)}, x_{\pi(k-1)}) h_{x_k}(x_{\pi(k-1)})
= \sum_{\pi \in \mathcal{P}_{k-1}} \frac{1}{u^1(x_k, x_k)} u^1(x_k, x_{\pi(1)}) \cdots u^1(x_{\pi(k-2)}, x_{\pi(k-1)}) u^1(x_{\pi(k-1)}, x_k).
Using \((8.14)\) this gives \((8.26)\). \hfill \Box

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