L-Fuzzy Filters of a Poset

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Abstract—Many generalizations of ideals and filters of a lattice to an arbitrary poset have been studied by different scholars. The authors of this paper introduced several generalizations of L-fuzzy ideal of a lattice to an arbitrary poset in [1]. In this paper, we introduce several L-fuzzy filters of a poset which generalize the L-fuzzy filter of a lattice and give several characterizations of them.

Index Terms—Poset, Filter, L-fuzzy closed filter, L-fuzzy Frink filter, L-fuzzy V-Filter, L-fuzzy semi-filter, L-fuzzy filter, l-L-fuzzy filter.

I. INTRODUCTION

We have found several generalizations of ideals and filters of a lattice to arbitrary poset (partially ordered set) in a literature. Birkhoff in [2, p. 59] introduced a closed or normal ideals who gives accredit to the work of Stone in [3]. Next, in 1954 the second type of ideal and filter of a poset called Frink ideal and Frink filter have been introduced by O. Frink [4]. Following this P. V. Venkatanarasimhan developed the theory of semi ideals and semi filter in [5] and ideals and filters for a poset in [6], in 1970. These ideals (respectively, filters) are called ideals (respectively, filter) in the sense of Venkatanarasimhan or V-ideals (V-filters) for short. Later Haláš [7], in 1994, introduced a new ideal and filter of a poset which seems to be a suitable generalization of the usual concept of ideal and filter in a lattice. We will simply call it ideal (respectively, filter) in the sense of Haláš.

Moreover, the concept of fuzzy ideals and filters of a lattice has been studied by different authors in series of papers [8], [9], [10], [11] and [12]. The aim of this paper is to notify several generalizations of L-fuzzy filters of a lattice to an arbitrary poset whose truth values are in a complete lattice satisfying the infinite meet distributive law and give several characterizations of them. We also prove that the set of all L-fuzzy filters of a poset forms a complete lattice with respect to point-wise ordering “⊆”. Throughout this work, L means a non-trivial complete lattice satisfying the infinite meet distributive law: \( x \land \sup S = \sup \{x \land s : s \in S\} \) for all \( x \in L \) and for any subset \( S \) of \( L \).

II. PRELIMINARIES

We briefly recall certain necessary concepts, terminologies and notations from [2], [13] and [14]. A binary relation “≤” on a non-empty set \( Q \) is called a partial order if it is reflexive, anti-symmetric and transitive. A pair \((Q, ≤)\) is called a partially ordered set or simply a poset if \( Q \) is a non-empty set and “≤” is a partial order on \( Q \). When confusion is unlikely, we use simply the symbol \( Q \) to denote a Poset \((Q, ≤)\). Let \( Q \) be a poset and \( S ⊆ Q \). An element \( x \) in \( Q \) is called a lower bound (respectively, an upper bound) of \( S \) if \( x ≤ a \) (respectively, \( x ≥ a \)) for all \( a ∈ S \). We denote the set of all lower bounds and upper bounds of \( S \) by \( S^l \) and \( S^u \), respectively. That is \( S^l = \{x ∈ Q : x ≤ a \land a ∈ S\} \) and \( S^u = \{x ∈ Q : x ≥ a \land a ∈ S\} \). \( S^u \) shall mean \( \{S^u\}^l \) and \( S^l \) shall mean \( \{S^l\}^u \). Let \( a, b ∈ Q \). Then \( \{a\}^u \) is simply denoted by \( a^u \) and \( \{a,b\}^u \) is denoted by \( (a,b)^u \). Similar notations are used for the set of lower bounds. We note that \( S ⊆ S^u \) and \( S ⊆ S^l \) and if \( S ⊆ T \) in \( Q \) then \( S^u ⊆ T^u \) and \( S^l ⊆ T^l \). Moreover, \( S^u = S^u \), \( S^l = S^l \), \( (a^u)^l = a \) and \( (a^l)^u = a^l \). An element \( x_0 \) in \( Q \) is called the least upper bound of \( S \) or supremum of \( S \), denoted by sup\( S \) (respectively, the greatest lower bound of \( S \) or infimum of \( S \), denoted by inf\( S \) if \( x_0 ∈ S^u \) and \( x_0 ≤ x \forall x ∈ S^u \) (respectively, if \( x_0 ∈ S^l \) and \( x ≤ x_0 \forall x ∈ S^l \)). An element \( x_0 \) in \( Q \) is called the largest (respectively, the smallest) element if it exists in \( Q \) is denoted by 1 (respectively, by 0). A poset \((Q, ≤)\) is called bounded if it has 0 and 1. Note that if \( S = \emptyset \) we have \( S^u = (\emptyset)^l = \emptyset^u \) which is equal to the empty set or the singleton set \( \{1\} \) if \( Q \) has the largest element 1.

Now we recall definitions of filters of a poset that are introduced by different scholars.

**Definition 2.1 (Dual of [2]):** A subset \( F \) of a poset \((Q, ≤)\) is said to be a closed or a normal filter in \( Q \) if \( F^u ⊆ F \).

**Definition 2.2 ([4]):** A subset \( F \) of a poset \((Q, ≤)\) is said to be a Frink filter in \( Q \) if \( F^u ⊆ F \) whenever \( S \) is a finite subset of \( F \).

**Definition 2.3 ([5]):** A non-empty subset \( F \) of a poset \((Q, ≤)\) is called a semi-filter or an order filter of \( Q \) if \( a ≤ b \) and \( a ∈ F \) implies \( b ∈ F \).

**Definition 2.4 ([6]):** A subset \( F \) of a poset \((Q, ≤)\) is said to be a V-filter or a filter in the sense of Venkatanarasimhan if \( F \) is a semi-filter and for any nonempty finite subset \( S \) of \( F \), if inf\( S \) exists, then inf\( S \) ∈ \( F \).

**Definition 2.5 ([7]):** A subset \( F \) of a poset \((Q, ≤)\) is called a filter in \( Q \) in the sense of Haláš if \( (a,b)^u \) contained in \( F \) whenever \( a, b ∈ F \).

Note that every filter of a poset \( Q \) defined above contains \( Q^u \).

**Remark 2.6:** The following remarks are due to R. Haláš and J. Rachůnek [15].

1. If \((Q, ≤)\) is a lattice then a non-empty subset \( F \) of \( Q \) is a filter as a poset if and only if it is a filter as a lattice \((Q, ≤)\).
2. If a poset does not have the largest element then the empty subset \( \emptyset \) is a filter in \((Q, ≤)\) (since \( \emptyset^u = (\emptyset)^l = \emptyset^u = \emptyset \)).
Definition 2.7: Let $A$ be any subset of a poset $Q$. Then the smallest filter containing $A$ is called a filter generated by $A$ and is denoted by $\langle A \rangle$. The filter generated by a singleton set \{a\}, is called a principal filter and is denoted by $[a]$. Note that for any subset $S$ of $Q$ if $\inf S$ exists then $S^lu = [\inf S]$.

The followings are some characterizations of filters generated by a subset $S$ of a poset $Q$. We write $T \subset S$ to mean $T$ is a finite subset of $S$.

1) The closed or normal filter generated by $S$, denoted by $[S]c$, is $[S]c = \bigcup\{T^lu : T \subset S\}$ where the union is taken over all subsets $T$ of $S$.

2) The Frink filter generated by $S$, denoted by $[S]f$, is $[S]f = \bigcup\{T^lu : T \subset S\}$, where the union is taken over all finite subsets $T$ of $S$.

3) Define $B_1 = \bigcup\{(a,b)^lu : a,b \in S\}$ and $B_n = \bigcup\{(a,b)^lu : a,b \in S_{n-1}\}$ for each positive integer $n \geq 2$, inductively.

Then the filter generated by $S$ in the sense of Halaš, denoted by $[S]h$, is $[S]h = \bigcup\{B_n : n \in \mathbb{N}\}$ where $\mathbb{N}$ denotes the set of positive integers.

4) If $a \in Q$ then $[a] = \{x \in Q : x \leq a\} = d^l$ is the principal ideal generated by $a$.

Definition 2.8 (177): A filter $F$ of a poset $Q$ is called an $l$-filter if (x,y)^l \cap F \neq \emptyset$ for all x,y \in F.

Note that an easy induction shows that F is an l-filter if B^l \cap F \neq \emptyset for every non-empty subset B of Q.

Theorem 2.9 (177): Let $\mathcal{F}(Q)$ be the set of filters of a poset $Q$ and $A$ and $B$ be $l$-filters of $Q$. Then the supremum $A \lor B$ of $A$ and $B$ in $\mathcal{F}(Q)$ is $A \lor B = \bigcup\{(a,b)^lu : a \in A, b \in B\}$.

Definition 2.10 (167): An $l$-fuzzy subset $\eta$ of a poset $Q$ is a function from $Q$ into $L$. Note that if $L$ is a unit interval of real numbers $[0,1]$, then the $l$-fuzzy subset $\eta$ is the fuzzy subset of Q which is introduced by L. Zadeh [17]. The set of all $l$-fuzzy subsets of $Q$ is denoted by $L^Q$.

Definition 2.11 (111): Let $\eta \in L^Q$. Then for each $a \in L$ the set $\eta_a = \{x : \eta(x) \geq a\}$ is called the level subset or level cut of $\eta$ at $a$.

Lemma 2.12 (197): Let $\eta \in L^Q$. Then $\eta(x) = \sup\{a \in L : x \in \eta_a\}$ for all $x \in Q$.

Definition 2.13 (167): Let $\nu, \sigma \in L^Q$. Define a binary relation $\subseteq^L$ on $L^Q$ by $\nu \subseteq^L \sigma$ if and only if $\nu(x) \leq \sigma(x)$ for all $x \in Q$.

It is simple to verify that the binary relation $\subseteq^L$ on $L^Q$ is a partial order and it is called the pointwise ordering.

Definition 2.14 (187): Let $\theta$ and $\eta$ be in $L^Q$. Then the union of fuzzy subsets $\theta$ and $\eta$ of $X$, denoted by $\theta \cup^L \eta$, is a fuzzy subset of $Q$ defined by $(\theta \cup^L \eta)(x) = \theta(x) \lor \eta(x)$ for all $x \in Q$ and the intersection of fuzzy subsets $\theta$ and $\eta$, denoted by $\theta \cap^L \eta$, is a fuzzy subset of $X$ defined by $(\theta \cap^L \eta)(x) = \theta(x) \land \eta(x)$ for all $x \in Q$.

More generally, the union and intersection of any family $\{\eta_i\}_{i \in \Delta}$ of $l$-fuzzy subsets of $Q$, denoted by $\bigcup_{i \in \Delta} \eta_i$ and $\bigcap_{i \in \Delta} \eta_i$ respectively, are defined by:

$(\bigcup_{i \in \Delta} \eta_i)(x) = \sup_{i \in \Delta} \eta_i(x)$ and $(\bigcap_{i \in \Delta} \eta_i) = \inf_{i \in \Delta} \eta_i(x)$ for all $x \in Q$, respectively.

Definition 2.15 (1107): A $l$-fuzzy subset $\eta$ of a lattice $Q$ with 1 is said to be an $L$-fuzzy filter of $Q$, if $\eta(1) = 1$ and $\eta(a \land b) = \eta(a) \land \eta(b)$ for all $a, b \in Q$.

Definition 2.16: Let $\eta$ be an $L$-fuzzy subset of a poset $Q$. Then the smallest $L$-fuzzy filter of $Q$ containing $\eta$ is called a $L$-fuzzy filter generated by $\eta$ and is denoted by $[\eta]$.

III. $l$-FUZZY FILTERS OF A POSET

In this section, we notify the concept of $L$-fuzzy filters of a poset and give several characterizations of them. Throughout this paper, $Q$ stands for a poset $(Q, \leq)$ with 1 unless otherwise stated. We begin with the following

Definition 3.1: An $L$-fuzzy subset $\eta$ of $Q$ is called an $L$-fuzzy closed filter if it fulfills the following conditions:

1) $\eta(1) = 1$
2) For any subset $S$ of $Q$, $\eta(x) \geq \inf \{\eta(a) : a \in S\}$ for all $x \in S^lu$.

Lemma 3.2: A subset $F$ of $Q$ is a closed filter of $Q$ if and only if it has a characteristic map $\chi_F$ which is an $L$-fuzzy closed filter of $Q$.

Proof: Suppose $F$ is a closed filter of $Q$. Then we have $\chi_F(1) = 1$. Let $S$ be any subset of $Q$ and $x \in S^lu$. Then if $S \subseteq F$, we have $S^lu \subseteq F^lu \subseteq F$ and $\chi_F(a) = 1$ for all $a \in S$. Therefore $\chi_F(x) = 1 = \inf \{\chi_F(a) : a \in S\}$. Again if $S \not\subseteq F$, then there is $c \in S$ such that $c \not\in F$ and hence $\chi_F(c) = 0$ and hence $\chi_F(x) \geq 0 = \inf \{\chi_F(a) : a \in S\}$. Thus in either cases, $\chi_F(x) \geq \inf \{\chi_F(a) : a \in S\}$. This implies in either cases, $\chi_F(x) \geq \inf \{\chi_F(a) : a \in S\} = 1$. Hence $F^lu \subseteq F$ and hence $F$ is a closed filter. This proves the result.

The following result characterizes the $L$-fuzzy closed filter of $Q$ in terms of its level subsets.

Lemma 3.3: Let $\eta$ be in $L^Q$. Then $\eta$ is an $L$-fuzzy closed filter of $Q$ if and only if $\eta_{\alpha}$ is a closed filter of $Q$ for all $\alpha \in \Lambda$.

Proof: Let $\eta$ be an $L$-fuzzy closed filter of $Q$ and $\alpha \in \Lambda$. Then $\eta(1) \geq \alpha$ and hence $1 \in \eta_{\alpha}$, i.e., $\{1\} = Q^lu \subseteq \eta_{\alpha}$. Again let $x \in (\eta_{\alpha})^lu$. Then $\eta(x) \geq \inf \{\eta(a) : a \in \eta_{\alpha}\} \geq \alpha$ and hence $x \in \eta_{\alpha}$. Therefore $(\eta_{\alpha})^lu \subseteq \eta_{\alpha}$ and hence $\eta_{\alpha}$ is a closed filter. Conversely, let $\eta_{\alpha}$ be a closed filter of $Q$ for all $\alpha \in \Lambda$. In particular $\eta_1$ is a closed filter. Since $1 \in (\eta_1)^lu \subseteq \eta_1$, we have $\eta(1) = 1$.

Again let $S$ be any subset of $Q$. Put $\alpha = \inf \{\eta(a) : a \in S\}$. Then $\eta(a) \geq \alpha \forall a \in S$ and hence $S \subseteq \mu_\alpha$. This implies $S^lu \subseteq \mu_{\alpha} \subseteq \mu_\alpha$. Now $x \in S^lu \Rightarrow x \in \eta_{\alpha} \Rightarrow \eta(x) \geq \alpha = \inf \{\eta(a) : a \in S\}$. Therefore $\eta$ is an $L$-fuzzy closed filter of $Q$. This proves the result.

Lemma 3.4: Let $\eta$ be fuzzy closed filter of a poset $Q$. Then $\eta$ is iso-tone, in the sense that $\eta(x) \leq \eta(y)$ whenever $x \leq y$.

Proof: Let $x, y \in Q$ such that $x \leq y$. Put $\eta(x) = \alpha$. Since $\eta$ is a fuzzy closed filter, $\eta_{\alpha}$ is a closed filter of $Q$ and hence $(\eta_{\alpha})^lu \subseteq \eta_{\alpha}$. Now $\eta(x) = \alpha \Rightarrow x \in \eta_{\alpha} \Rightarrow x \in (\eta_{\alpha})^lu \subseteq (\eta_{\alpha})^lu \subseteq \eta_{\alpha}$. Thus $x \leq y \Rightarrow y \in (\eta_{\alpha})^lu \Rightarrow y \in \eta_{\alpha}$. Therefore $\eta(x) = \alpha \leq \eta(y)$. This proves the result.

Theorem 3.5: Let $(Q, \leq)$ be a lattice. Then an $L$-fuzzy subset $\eta$ of $Q$ is an $L$-fuzzy closed filter in the poset $Q$ if and only if it is an $L$-fuzzy filter in the lattice $Q$.

Proof: Let $\eta$ be an $L$-fuzzy filter in the poset $Q$ and $a, b \in Q$. Then $\eta(1) = 1$ and since $S = \{a, b\} \subseteq Q$ and $a \land b \in S^lu$. 
we have $\eta(a \land b) \geq \inf\{\eta(x) : x \in S\} = \eta(a) \land \eta(b)$. Again since $\eta$ is iso-tone, we have $\eta(a \land b) \leq \eta(a)$ and $\eta(a \land b) \leq \eta(b)$ and hence we have $\eta(a) \land \eta(b) \leq \eta(a,b)$. Therefore $\eta(a,b) = \eta(a) \land \eta(b)$ and hence $\eta$ is an L-fuzzy filter in the lattice $Q$. Conversely suppose $\mu$ be an L-fuzzy filter in the lattice $Q$. Then $\eta(1) = 1$ and $\eta(a \land b) = \eta(a) \land \eta(b)$ $\forall a, b \in Q$. Let $S \subseteq Q$ and $x \in (S)^I$. Then $x$ is an upper bound of $(S)^I$. Since $\inf \in \langle (\alpha)^I \rangle$, we have $x \geq \inf S$ and hence we have $\eta(x) \geq \eta(\inf S) = \inf\{\eta(a) : a \in S\}$. Therefore $\eta$ is an L-fuzzy closed filter in the poset $Q$. This proves the result.

**Lemma 3.6:** The intersection of any family of L-fuzzy closed filters is an L-fuzzy closed filter.

**Theorem 3.7:** Let $[S]_C$ be a closed filter generated by a subset $S$ of $Q$ and $\chi_S$ be its characteristic functions. Then the $[\chi_S] = [\chi_{[S]_C}]$.

**Proof:** Since $[S]_C$ is a closed filter of $Q$ containing $S$, by Lemma 3.2, we have $\chi_{[S]_C}$ is a fuzzy closed filter. Again since $S \subseteq [S]_C$, clearly we have $\chi_S \subseteq \chi_{[S]_C}$. Now, we show that it is the smallest L-fuzzy closed filter containing $\chi_S$. Let $\eta$ be an L-fuzzy closed filter such that $\chi_S \subseteq \eta$. Then $\eta(a) = 1$ for all $a \in S$. Now we claim $\chi_{[S]_C} \subseteq \eta$. Let $x \in Q$. If $x \notin [S]_C$, then $\chi_{[S]_C}(x) = 0 \leq \eta(x)$. If $x \in [S]_C$, then $x \in T^I$ for some subset $T$ of $S$ and hence $\inf\{\eta(a) : a \in T\} = 1 = \chi_{[S]_C}(x)$. Hence in either cases, $\chi_{[S]_C}(x) \leq \eta(x)$ for all $x \in Q$ and hence $\chi_{[S]_C} \subseteq \eta$. This proves the theorem.

In the following theorem we characterize a fuzzy closed filter generated by a fuzzy subset of $Q$ in terms of its level closed filters.

**Theorem 3.8:** Let $\eta \in L^Q$. Then the L-fuzzy subset $\hat{\eta}$ of $Q$ defined by $\hat{\eta}(x) = \sup\{\alpha \in L : x \in \eta(a)\}$ for all $x \in Q$ is a fuzzy closed filter generated by $\eta$, where $\{\mu(a)\}$ is a closed filter generated by $\eta_a$.

**Proof:** Now we show $\hat{\eta}$ is the smallest fuzzy closed filter containing $\eta$. Let $x \in Q$ and put $\hat{\eta}(x) = \beta$. Then $x \in \eta_\beta \subseteq \eta_{\hat{\eta}(x)} = \beta$ $\in \{\alpha \in L : x \in \eta(a)\}$. Thus $\hat{\eta}(x) = \beta \leq \sup\{\alpha \in L : x \in \eta(a)\} = \hat{\eta}(x)$ and hence $\hat{\eta} \subseteq \beta$. Again since $\{1\} = Q^I \subseteq \{\eta_\alpha\}$ for all $\alpha \in L$, clearly we have $\hat{\eta}(1) = 1$. Let $S$ be any subset of $Q$ and $x \in S^I$. Now inf $\hat{\eta}(a) : a \in S\} = \inf\{\sup\{\eta_\alpha : a \in \eta(a)\} : a \in S\} = \inf\{\sup\{\eta_\alpha : a \in S\} : a \in \eta(a)\}$. Thus $\lambda = \inf\{\alpha_\alpha : a \in S\}$ and hence $\eta_\lambda \subseteq \eta_a$. Therefore $\{\eta_\lambda\}$ is a closed filter generated by $\eta_a$.

**Theorem 3.9:** Let $\eta \in L^Q$. Then the fuzzy subset $\hat{\eta}$ defined by

$$\hat{\eta}(x) = \begin{cases} 1 & \text{if } x = 1 \\ \sup\{\inf\{\eta(a) : x \in S^I \} \subseteq S \} & \text{if } x \neq 1 \end{cases}$$

is a fuzzy closed filter of $Q$ generated by $\eta$.

**Proof:** It is easy to show that $\hat{\eta} = \hat{\eta}$ where $\hat{\eta}$ is an L-fuzzy subset given in the above theorem. Let $x \in Q$. If $x = 1$, then $\hat{\eta}(x) = 1 = \hat{\eta}(x)$. Let $x \neq 0$. Put $A_1 = \{\inf\{\eta(a) : x \in S^I \} \subseteq S \}$ and $B_1 = \{\alpha \in L : x \in \eta(a)\}$. Now we show $\sup A_1 = \sup B_1$. Let $\alpha \in A_1$. Then $\alpha = \inf\{\eta(a) : x \in S^I \}$ for some subset $S$ of $Q$ such that $x \in S^I$. This implies that $\alpha \leq \eta(a)$ for all $a \in S$ and hence $S \subseteq \eta(a) \subseteq \{\eta_\alpha\}$. Thus $S^I \subseteq \{\eta_\alpha\}$ and hence $x \in \{\eta_\alpha\}$. Therefore $\alpha \leq \eta$. Thus $A_1 \subseteq B_1$ and hence $\sup A_1 \leq \sup B_1$. Again let $\alpha \in B_1$. Then $x \in \{\eta_\alpha\}$. Since $\{\mu(a)\} = \bigcup\{S^I : x \in \{\eta_\alpha\}\}$, we have $x \in S^I$ for some subset $S$ of $\eta_\alpha$. This implies $\alpha \geq \eta(a)$ for all $a \in S$ and hence $\inf\{\eta(a) : x \in S^I \} \geq \alpha$. Thus $\beta = \inf\{\eta(a) : a \in S\} \subseteq \eta(a)$. Therefore $\alpha \geq \eta$. For each $\alpha \in B_1$ we get $\beta \in A_1$ such that $\beta \leq \eta$ and hence $\sup A_1 \geq \sup B_1$. Therefore $\sup A_1 = \sup B_1$. Hence $\hat{\eta} = \hat{\eta}$.

The above result yields the following.

**Theorem 3.10:** Let $\mathcal{F} \subseteq \mathcal{F}(Q)$ be the set of all L-fuzzy closed filters of $Q$. Then $(\mathcal{F} \subseteq \mathcal{F}(Q))$ forms a complete lattice with respect to the point wise ordering ” $\subseteq$”, in which the supremum $\sup\{\eta_i\}_{\eta_i = \eta}$ and the infimum $\inf\{\eta_i\}$ of any family $\{\eta_i : i \in I\}$ in $(\mathcal{F} \subseteq \mathcal{F}(Q))$ are given by:

$$\sup\{\eta_i\} = \bigcup_{i \in I} \{\eta_i\}$$

$$\inf\{\eta_i\} = \bigcap_{i \in I} \{\eta_i\}$$

**Corollary 11:** For any L-fuzzy closed filters $\eta$ and $\nu$ of $Q$, the supremum $\eta \lor \nu$ and the infimum $\eta \land \nu$ of $\eta$ and $\nu$ in $(\mathcal{F} \subseteq \mathcal{F}(Q))$ respectively are:

$$\eta \lor \nu = \overline{\eta \lor \nu}$$

$$\eta \land \nu = \overline{\eta \land \nu}$$

Now we introduce the fuzzy version of a filter (dual ideal) of a poset introduced by O. Frink [4].

**Definition 3.12:** An L-fuzzy subset $\eta$ of $Q$ is an L-fuzzy Frink filter if it satisfies the following conditions:

1) $\eta(1) = 1$ and
2) for any finite subset $F$ of $Q$, $\eta(x) \geq \inf\{\eta(a) : a \in F\}$ $\forall x \in F^I$

**Lemma 13:** Let $\eta \in L^Q$. Then $\eta$ is an L-fuzzy Frink filter of $Q$ if and only if $\eta$ is a Frink filter of $Q$ for all $a \in L$.

**Lemma 14:** Let $\eta$ be fuzzy Frink filter of a poset $Q$. Then $\eta$ is iso-tone, in the sense that $\eta(x) \leq \eta(y)$ whenever $x \leq y$.

**Corollary 15:** A subset $S$ of $Q$ is a Frink filter of $Q$ if and only if its characteristic map $\chi_S$ is an L-fuzzy Frink filter of $Q$.

**Theorem 16:** Let $(Q, \leq)$ be a lattice and $\eta \in L^Q$. Then $\eta$ is an L-fuzzy Frink filter in the poset $Q$ if and only if it is an L-fuzzy filter in the lattice $Q$.

**Lemma 17:** The intersection of any family of L-fuzzy Frink-filters is an L-fuzzy Frink filter.

**Theorem 18:** Let $[S]_C$ be a Frink-filter generated by subset $S$ of $Q$ and $\chi_S$ be its characteristic functions. Then $[\chi_S] = [\chi_{[S]_C}]$.

The following theorems, we give characterizations of L-Fuzzy Frink filters generated by fuzzy subset of $Q$.

**Theorem 19:** Let $\eta \in L^Q$. Define a fuzzy subset $\hat{\eta}$ of $Q$ by $\hat{\eta}(x) = \sup\{\alpha \in L : x \in \eta(a)\}$ for all $x \in Q$ where $\{\eta_a\}$ a
Frink filter generated by $\eta_\alpha$, where $[\eta_\alpha]_F$ is a Frink filter generated by $\eta_\alpha$. Then $\hat{\eta}$ is an $L$-fuzzy Frink filter of $Q$ generated by $\eta$.

In the following, we give an algebraic characterization of $L$-fuzzy Frink filters generated by fuzzy subset of $Q$.

**Theorem 3.20**: Let $\eta$ be a fuzzy subset of $Q$. Then the fuzzy subset $\overrightarrow{\eta}$ defined by

$$\overrightarrow{\eta}(x) = \begin{cases} 1 & \text{if } x = 1 \\ \sup\{\inf_{a\in F} \eta(a) : F \subseteq Q, x \in F^{lu}\} & \text{if } x \neq 1 \end{cases}$$

is a Frink fuzzy filter of $Q$ generated by $\eta$.

**Theorem 3.21**: Let $\mathcal{F} \subseteq \mathcal{F}(Q)$ be the set of all $L$-fuzzy Frink filters of $Q$. Then $(\mathcal{F}, \mathcal{F}(Q), \subseteq)$ forms a complete lattice with respect to pointwise ordering “$\subseteq$”, in which the supremum and the infimum of any family $\{\eta_i : i \in \Lambda\}$ in $\mathcal{F}(Q)$ respectively are: $\sup_{i \in \Lambda} \eta_i = \bigcup_{i \in \Lambda} \{\eta_i\}$ and $\inf_{i \in \Lambda} \eta_i = \bigcap_{i \in \Lambda} \eta_i$.

**Corollary 3.22**: For any $L$-fuzzy Frink ideals $\eta$ and $\nu$ of $Q$ in the supremum $\eta \vee \nu$ and the infimum $\eta \wedge \nu$ of $\eta$ and $\nu$ in $\mathcal{F}(Q)$ respectively are: $\eta \vee \nu = \eta \wedge \nu = \nu \wedge \eta = \nu \wedge \eta$.

Now we introduce the fuzzy version of semi-filters and V-filters of a poset introduced by P.V. Venkataramasimhan [5] and [6].

**Definition 3.23**: $\eta$ in $L^Q$ is said to be an $L$-fuzzy semi-filter or $L$-fuzzy order filter if $\eta(x) \leq (y)$ whenever $x \leq y$ in $Q$.

**Definition 3.24**: $\eta$ in $L^Q$ is said to be an $L$-fuzzy $V$-filter if it satisfies the following conditions:
1) for any $x, y \in Q$ $\eta(x) \leq (y)$ whenever $x \leq y$ and
2) for any non-empty finite subset $B$ of $Q$, if $\inf B$ exists then $\eta(\inf B) \geq \inf \{\eta(b) : b \in B\}$.

**Theorem 3.25**: Every $L$-fuzzy Frink filter is an $L$-fuzzy $V$-filter.

**Proof**: Let $\eta$ be an $L$-fuzzy Frink filter and let $x, y \in Q$ such that $x \leq y$. Put $\eta(\alpha) = \alpha$. Since $\eta$ is an $L$-fuzzy Frink filter, $\eta_\alpha$ is a Frink filter of $Q$. Now $\eta(\alpha) = \alpha \Rightarrow x \in \eta_\alpha \Rightarrow \{x\} \subseteq \eta_\alpha$. Now $x \leq y \Rightarrow y \in x^L = x^L \subseteq \eta_\alpha \Rightarrow \eta(\alpha) = \alpha \leq (y)$. Again let $B$ be any nonempty subset of $Q$ such that $\inf B$ exists in $Q$. Then $\inf B \in B^L$ and hence $\eta(\inf B) \geq \inf \{\eta(a) : a \in B\}$. Therefore $\eta$ is an $L$-fuzzy $V$-filter.

Now we introduce the fuzzy version filters of a poset introduced by Halaš [7] which seems to be a suitable generalization of the usual concept of $L$-fuzzy filter of a lattice.

**Definition 3.26**: $\eta$ in $L^Q$ is called an $L$-fuzzy filter in the sense of Halaš if it fulfills the following:
1) $\eta(1) = 1$ and
2) for any $a, b \in Q$, $\eta(x) \geq \eta(a) \land \eta(b)$ for all $x \in (a, b)^L$.

In the rest of this paper, an $L$-fuzzy filter of a poset will mean an $L$-fuzzy filter in the sense of Halaš.

**Lemma 3.27**: $\eta$ in $L^Q$ is an $L$-fuzzy filter of $Q$ if and only if $\eta_\alpha$ is a filter of $Q$ in the sense of Halaš for all $\alpha \in L$.

**Corollary 3.28**: A subset $S$ of $Q$ is a filter of $Q$ in the sense of Halaš if and only if its characteristic map $\chi_S$ is an $L$-fuzzy filter of $Q$.

**Lemma 3.29**: If $\eta$ is an $L$-fuzzy filter of $Q$, then the following assertions hold:
1) for any $x, y \in Q$, $\eta(x) \leq (y)$ whenever $x \leq y$.
2) for any $x, y \in Q$, $\eta(x \land y) \geq \mu(x) \land \eta(y)$ whenever $x \land y$ exists.

**Theorem 3.30**: Let $(Q, \subseteq)$ be a lattice. Then an $L$-fuzzy subset $\eta$ of $Q$ is an $L$-fuzzy filter in the poset $Q$ if and only if an $L$-fuzzy filter is in the lattice $Q$.

**Theorem 3.31**: Let $[\delta_{S}]$ be a filter generated by subset $S$ of $Q$ in the sense of Halaš and $\chi_S$ be its characteristic functions. Then $[\delta_{S}] = \chi_{[\delta_{S}]}$.

**Lemma 3.32**: The intersection of any family of $L$-fuzzy filters is an $L$-fuzzy filter.

Now we give characterization of an $L$-fuzzy filter generated by a fuzzy subset of a poset $Q$.

**Definition 3.33**: Let $\eta$ be a fuzzy subset of $Q$ and $\mathcal{N}$ be a set of positive integers. Define fuzzy subsets of $Q$ inductively as follows:

$$B_n^1(x) = \sup\{\eta(a) \land \eta(b) : x \in (a, b)^L\}$$

and

$$B_n^\eta(x) = \sup\{B_{n-1}^\eta(a) \land B_{n-1}^\eta(b) : x \in (a, b)^L\}$$

for each $n \geq 2$ and $a, b \in Q$.

**Theorem 3.34**: The set $\{B_n^\eta : n \in \mathcal{N}\}$ forms a chain and the fuzzy subset $\hat{\eta}$ defined by $\hat{\eta}(x) = \sup\{B_n^\eta(x) : x \in \mathcal{N}\}$ is a fuzzy filter generated by $\eta$.

**Proof**: Let $x \in Q$ and $n \in \mathcal{N}$. Then

$$B_{n+1}^\eta(x) = \sup\{B_n^\eta(a) \land B_n^\eta(b) : x \in (a, b)^L\}$$

$$\geq B_n^\eta(x) \land B_n^\eta(x) \text{ (since } x \in (a, b)^L)$$

$$= \eta(x) \land \eta(x) \Rightarrow x \in Q.$$

Therefore $B_n^\eta \subseteq B_{n+1}^\eta$ for each $n \in \mathcal{N}$ and hence $\{B_n^\eta : n \in \mathcal{N}\}$ is a chain. Now we show $\hat{\eta}$ is the smallest fuzzy filter containing $\eta$.

Since

$$\hat{\eta}(x) = \sup\{B_n^\eta(x) : n \in \mathcal{N}\}$$

$$= B_1^\eta(x) \geq \sup\{\eta(a) \land \eta(b) : x \in (a, b)^L\}$$

$$\geq \eta(x) \land \eta(x) \text{ (since } x \in (a, b)^L)$$

$$= \eta(x) \land \eta(x) \Rightarrow x \in Q.$$

Therefore $\eta \subseteq \hat{\eta}$. Let $a, b \in L$ and $x \in (a, b)^L$.

Now

$$\hat{\eta}(x) = \sup\{B_n^\eta(x) : n \in \mathcal{N}\}$$

$$\geq B_n^\eta(x) \text{ for all } n \in \mathcal{N}$$

$$= \sup\{B_{n-1}^\eta(y) \land B_{n-1}^\eta(z) : x \in (y, z)^L\}$$

for all $n \geq 2$.

$$\geq B_n^\eta(x) \land B_{n-1}^\eta(b) \forall n \geq 2$$

$$= B_{n+1}^\eta(a) \land B_{n+1}^\eta(b) \forall n \in \mathcal{N}$$

Thus

$$\hat{\eta}(a) \land \hat{\eta}(b).$$

Therefore $\hat{\eta}$ is a fuzzy filter. Again let $\theta$ be any $L$-fuzzy filter of $Q$ such that $\theta \subseteq \theta$. Now let $a, b \in Q$ and $x \in (a, b)^L$. Then $\theta(x) \geq \sup\{\eta(a) \land \eta(b) : x \in (a, b)^L\} = B_1^\eta(x)$. Therefore $\theta(x) \geq B_1^\eta(x)$ for all $x \in (a, b)^L$. Again for any $x \in (a, b)^L$ we have $\theta(x) \geq \sup\{\eta(a) \land \eta(b) : x \in (a, b)^L\} = B_1^\eta(x)$. This implies

$$\theta(x) \geq \sup\{B_1^\eta(a) \land B_1^\eta(b) : x \in (a, b)^L\} = B_2^\eta(x).$$
induction we have $\theta(x) \geq B_n^j(x)$ $\forall n \in N$ and $\forall x \in (a,b)^{lu}$. Thus for any $x \in Q$, we have

$\hat{\eta}(x) = \sup\{B_n^j(x) : n \in N\} = \sup\{B_n^j(a) \land B_n^j(b) : n \in N, x \in (a,b)^{lu}\} \leq \sup\{\theta(a) \land \theta(b) : x \in (a,b)^{lu}\}$

(since, $a,b \in (a,b)^{lu}$.)

$\leq \theta(x)$

Therefore $\theta \geq \hat{\eta}$. This proves the theorem.

The above result yields the following.

**Theorem 3.35:** Let $\mathcal{F}(Q)$ be the set of all $L$-fuzzy filter of $Q$. Then $\mathcal{F}(Q)$ forms a complete lattice with respect to the point wise ordering $\subseteq$, in which the supremum and the infimum of any family $\{\eta_i : i \in A\}$ in $\mathcal{F}(Q)$ respectively are: $\left(\sup_{i \in A}\eta_i\right)(x) = \sup\{B_i(x) : n \in N\}$ and $\left(\inf_{i \in A}\eta_i\right)(x) = \left(\bigcap_{i \in A}\eta_i\right)(x)$ for any $x \in Q$.

**Corollary 3.36:** For any $L$-fuzzy filter $\eta$ and $\nu$ of $Q$, the supremum $\eta \lor \nu$ and the infimum $\eta \land \nu$ of $\eta$ and $\nu$ in $\mathcal{F}(Q)$ respectively are: $\left(\eta \lor \nu\right)(x) = \sup\{B_i \lor B_j(x) : n \in N\}$ and $\left(\eta \land \nu\right)(x) = \left(\bigcap_{i \in A}\eta_i\right)(x)$ for any $x \in Q$.

**Theorem 3.37:** The following implications hold, where all of them are not equivalent:

1) $L$-fuzzy closed filter $\implies$ $L$-fuzzy Frink filter $\implies$ $L$-fuzzy $V$-filter $\implies$ $L$-fuzzy semi-filter.

2) $L$-fuzzy closed filter $\implies$ $L$-fuzzy Frink filter $\implies$ $L$-fuzzy filter $\implies$ $L$-fuzzy semi-filter.

The following examples show that the converse of the above implications do not hold in general.

**Example 3.38:** Consider the Poset $([0,1], \leq)$ with the usual ordering. Define a fuzzy subset $\eta : [0,1] \to [0,1]$ by

\[
\eta(x) = \begin{cases} 1 & \text{if } x \in [\frac{1}{2},1] \\ 0 & \text{if } x \in [0,\frac{1}{2}] \end{cases}
\]

Then $\eta$ is an $L$-fuzzy Frink filter but not an $L$-fuzzy closed filter.

**Example 3.39:** Consider the poset $(Q, \leq)$ depicted in the figure below. Define a fuzzy subset $\nu : Q \to [0,1]$ by $\nu(1) = \nu(a) = 1, \nu(b) = \nu(c) = \nu(d) = \nu(0) = 0.2, \nu(b') = 0.6, \nu(c') = 0.5$ and $\nu(d') = 0.7$. Then $\nu$ is an $L$-fuzzy filter but not an $L$-fuzzy Frink-filter.

**Example 3.40:** Consider the poset $(Q, \leq)$ depicted in the figure below. Define a fuzzy subset $\theta : Q \to [0,1]$ by $\theta(U) = 1, \theta(L) = \theta(M) = 0.8$ and $\theta(N) = 0.6$. Then $\theta$ is an $L$-fuzzy $V$-filter but not an $L$-fuzzy Frink-filter.

**Example 3.41:** Consider the poset $(Q, \leq)$ depicted in the figure below. Define a fuzzy subset $\sigma : Q \to [0,1]$ by $\sigma(1) = 1, \sigma(a) = 0.8, \sigma(b) = 0.9$ and $\sigma(0) = 0.2$.

Then $\sigma$ is an $L$-fuzzy semi-filter but not an $L$-fuzzy filter.

**Theorem 3.42:** Let $x \in Q$ and $\alpha \in L$. Define an $L$-fuzzy subset $\alpha^x$ of $Q$ by

\[
\alpha^x(y) = \begin{cases} 1 & \text{if } y \in [x] \\ \alpha & \text{if } y \notin [x] \end{cases}
\]

for all $y \in Q$. Then $\alpha^x$ is an $L$-fuzzy filter.

**Proof:** By the definition of $\alpha^x$, we clearly have $\alpha_x(1) = 1$. Let $a,b \in Q$ and $y \in (a,b)^{lu}$. Now if $a,b \in [x]$, then we have $(a,b)^{lu} \subseteq [x]$ and $\alpha^x(a) = \alpha^x(b) = 1$. Thus $\alpha^x(y) = 1 = 1 \land 1 = \alpha^x(a) \land \alpha^x(b)$. Again if $a \notin [x]$ or $b \notin [x]$, we have $\alpha^x(a) \land \alpha^x(b) = \alpha$ and hence $\alpha^x(y) \geq \alpha = \alpha^x(a) \land \alpha^x(b)$. Therefore in either cases we have $\alpha^x(y) \geq \alpha^x(a) \land \alpha^x(b)$ for all $y \in (a,b)^{lu}$ and hence $\alpha^x$ is an $L$-fuzzy filter.

**Definition 3.43:** The $L$-fuzzy filter $\alpha^x$ defined above is called the $\alpha$-level principal fuzzy filter corresponding to $x$.

**Definition 3.44:** An $L$-fuzzy filter $\mu$ of a poset $Q$ is called an $l$-$L$-fuzzy filter if for any $a,b \in Q$, there exists $x \in (a,b)^l$ such that $\mu(x) = \mu(a) \land \mu(b)$.

**Lemma 3.45:** An $L$-fuzzy filter $\mu$ of $Q$ is an $l$-$L$-fuzzy filter of $Q$ if and only if $\mu_a$ is an $l$-filter of $Q$ for all $\alpha \in L$.

**Proof:** Suppose $\mu$ is an $l$-$L$-fuzzy filter and $\alpha \in L$. Since $\mu$ is an $L$-fuzzy filter, $\mu_a$ is a filter of $Q$. Let $a,b \in \mu_a$. Then $\mu(a) \geq \alpha$ and $\mu(b) \geq \alpha$ and hence $\mu(a) \land \mu(b) \geq \alpha$. Also since $\mu$ is an $l$-$L$-fuzzy filter there exists $x \in (a,b)^l$ such that $\mu(x) = \mu(a) \land \mu(b)$ and hence $\mu(x) \geq \alpha$. Therefore $x \in \mu_a \land (a,b)^l$.
and hence $\mu_a \cap (a,b)^l \neq \emptyset$. Therefore $\mu_a$ is an $l$-filter of a poset $Q$. Conversely suppose $\mu_a$ is an $l$-filter of a poset $Q$ for all $\alpha \in L$. Then $\mu$ is an $L$-fuzzy filter. Let $a, b \in Q$ and put $\alpha = \mu(a) \wedge \mu(b)$. Then $\mu_a \cap (a,b)^l \neq \emptyset$. Let $x \in \mu_a \cap (a,b)^l$. Then $x \in \mu_a$ and $x \in (a,b)^l$. This implies $x(a) = \mu(a \wedge b)$ and $x \leq a$, $x \leq b$. Since $\mu$ is iso-tone we have $x \leq \mu(a)$ and $\mu(x) \leq \mu(b)$ and hence $x \leq \mu(a \wedge b)$. Therefore there exists $x \in (a,b)^l$ such that $x(a) = \mu(a \wedge b)$ and hence $\mu$ is an $L$-fuzzy filter.

Corollary 3.46: Let $(Q, \leq)$ be a poset with 0 and let $x \in Q$ and $\alpha \in L$. Then the $\alpha$-level principal fuzzy filter corresponding to $x$ is an $l$-fuzzy filter.

Remark 3.47: Every $L$-fuzzy filter is not an $l$-fuzzy filter. For example consider the poset $(Q, \leq)$ depicted in the figure below and define a fuzzy subset $\mu : Q \rightarrow [0,1]$ by $\mu(1) = 1$, $\mu(e) = \mu(d) = 0.9$, $\mu(a) = \mu(b) = \mu(0) = 0.7$. Then $\mu$ is an $L$-fuzzy filter but not an $l$-fuzzy filter.

![Fig. 4. A Poset.](image)

Theorem 3.48: Every $L$-fuzzy filter is an $l$-fuzzy Frink filter.

Proof: Suppose $\eta$ is an $l$-fuzzy filter. Let $F$ be a finite subset of $Q$. Then there exists $y \in F^l$ such that $\eta(y) = \inf \{ \eta(a) : a \in F \}$.

Again $x \in F^l$ implies $s \leq x \forall s \in F^l$.

$\Rightarrow y \leq x$ (since $y \in F^l$)

$\Rightarrow \eta(x) \geq \eta(y) = \inf \{ \eta(a) : a \in F \}$

$\Rightarrow \eta(x) \geq \inf \{ \eta(a) : a \in F \}$

Therefore $\eta$ is an $L$-fuzzy Frink filter.

Theorem 3.49: Let $\eta$ and $\theta$ be $l$-fuzzy filters of $Q$. Then the supremum $\eta \lor \theta$ of $\eta$ and $\theta$ in $\mathcal{F}(Q)$ is given by:

$(\eta \lor \theta)(x) = \sup \{ \eta(a) \lor \theta(b) : x \in (a,b)^l \}$ for all $x \in Q$.

Proof: Let $\sigma$ be an $L$-fuzzy subset of $Q$ defined by $\sigma(x) = \sup \{ \eta(a) \lor \theta(b) : x \in (a,b)^l \}$ $\forall x \in Q$. Now we claim $\sigma$ is the smallest $L$-fuzzy filter of $Q$ containing $\eta \lor \theta$. Let $x \in Q$.

Now $\sigma(x) = \sup \{ \eta(a) \lor \theta(b) : x \in (a,b)^l \}$

$\geq \eta(x) \lor \theta(1)$, (since $x \in (x,1)^l$)

$= \eta(x) \lor 1 = \eta(x)$

and hence $\sigma \supset \eta$. Similarly we can show $\sigma \supset \theta$ and hence $\sigma \supset \eta \lor \theta$.

Let $a,b \in Q$ and $x \in (a,b)^l$. Now

$\sigma(a) \lor \sigma(b) = \sup \{ \eta(c) \lor \theta(d) : a \in (c,d)^l \} \lor \sup \{ \eta(e) \lor \theta(f) : b \in (e,f)^l \}$

$= \sup \{ \eta(c) \lor \theta(d) \lor \eta(e) \lor \theta(f) : a \in (c,d)^l, b \in (e,f)^l \}$

$\leq \sup \{ \eta(c) \lor \eta(e) \lor \theta(d) \lor \theta(f) : a \in (c,d)^l, b \in (e,f)^l \}$

Again since $\eta$ and $\theta$ are $l$-fuzzy filters, for each $c,e$ and $d,f$ there are $r \in (c,e)^l$ and $s \in (d,f)^l$ such that $\eta(r) = \eta(c) \lor \eta(e)$ and $\theta(s) = \theta(d) \lor \theta(f)$. Now

$r \in (c,e)^l$ and $s \in (d,f)^l$ $\Rightarrow \{ c,d,e,f \}^l \subseteq \{ s,r \}^l$

$= a,b \in \{ s,r \}^l$

$= (a,b)^l \subseteq \{ s,r \}^l$

$= x \in \{ s,r \}^l$

Thus $\sigma(a) \lor \sigma(b) \leq \sup \{ \eta(c) \lor \eta(e) \lor \theta(d) \lor \theta(f) : a \in (c,d,e,f)^l \} \leq \sup \{ \eta(r) \lor \theta(s) : x \in (r,s)^l \} \leq \sigma(x)$ for all $x \in (a,b)^l$ and hence $\sigma$ is an $L$-fuzzy filter.

Let $\phi$ be any $L$-fuzzy filter of $Q$ such that $\eta \lor \theta \subseteq \phi$. Now for any $x \in Q$, we have

$\sigma(x) = \sup \{ \eta(a) \lor \theta(b) : x \in (a,b)^l \}$

$\leq \sup \{ \phi(a) \lor \phi(b) : x \in (a,b)^l \}$

$\leq \phi(x)$

and hence $\sigma \subseteq \phi$. Therefore $\sigma = (\eta \lor \theta) = \eta \lor \theta$, that is $\sigma$ is the supremum of $\eta$ and $\theta$ in $\mathcal{F}(Q)$.

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