Computing Subset Feedback Vertex Set via Leafage∗

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Abstract

Chordal graphs are characterized as the intersection graphs of subtrees in a tree and such a representation is known as the tree model. Restricting the characterization results in well-known subclasses of chordal graphs such as interval graphs or split graphs. A typical example that behaves computationally different in subclasses of chordal graph is the Subset Feedback Vertex Set (SFVS) problem: given a vertex-weighted graph $G = (V, E)$ and a set $S \subseteq V$, the Subset Feedback Vertex Set (SFVS) problem asks for a vertex set of minimum weight that intersects all cycles containing a vertex of $S$. SFVS is known to be polynomial-time solvable on interval graphs, whereas SFVS remains NP-complete on split graphs and, consequently, on chordal graphs. Towards a better understanding of the complexity of SFVS on subclasses of chordal graphs, we exploit structural properties of a tree model in order to cope with the hardness of SFVS. Here we consider variants of the leafage that measures the minimum number of leaves in a tree model. We show that SFVS can be solved in polynomial time for every chordal graph with bounded leafage. In particular, given a chordal graph on $n$ vertices with leafage $\ell$, we provide an algorithm for SFVS with running time $n^{O(\ell)}$. We complement our result by showing that SFVS is W[1]-hard parameterized by $\ell$. Pushing further our positive result, it is natural to consider a slight generalization of leafage, the vertex leafage, which measures the smallest number among the maximum number of leaves of all subtrees in a tree model. However, we show that it is unlikely to obtain a similar result, as we prove that SFVS remains NP-complete on undirected path graphs, i.e., graphs having vertex leafage at most two. Moreover, we strengthen previously-known polynomial-time algorithm for SFVS on rooted path graphs that form a proper subclass of undirected path graphs and graphs of mim-width one.

1 Introduction

Several fundamental optimization problems are known to be intractable on chordal graphs, however they admit polynomial time algorithms when restricted to a proper subclass of chordal graphs such as interval graphs. Typical examples of this type of problems are domination or induced path problems [2, 5, 12, 23, 24, 30]. Towards a better understanding of why many intractable problems on chordal graphs admit polynomial time algorithms on interval graphs, we consider the algorithmic usage of the structural parameter named leafage. Leafage, introduced by Lin et al. [28], is a graph parameter that captures how close is a chordal graph of being an interval graph. As it concerns chordal graphs, leafage essentially measures the smallest number of leaves in a clique tree, an intersection representation of the given graph [19]. Here we are

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concerned with the Subset Feedback Vertex Set problem, SFVS for short: given a vertex-weighted graph and a set $S$ of its vertices, compute a vertex set of minimum weighted size that intersects all cycles containing a vertex of $S$. Although Subset Feedback Vertex Set does not fall to the themes of domination or induced path problems, it is known to be $\text{NP}$-complete on chordal graphs [16], whereas it becomes polynomial-time solvable on interval graphs [32]. Thus our research study concerns to what extent the structure of the underlying tree representation influences the computational complexity of Subset Feedback Vertex Set.

An interesting remark concerning Subset Feedback Vertex Set, is the fact that its unweighted and weighted variants behave computationally different on hereditary graph classes. For example, Subset Feedback Vertex Set is $\text{NP}$-complete on $H$-free graphs for some fixed graphs $H$, while its unweighted variant admits polynomial time algorithm on the same class of graphs [7, 33]. Thus the unweighted and weighted variants of Subset Feedback Vertex Set do not align. This comes in contrast even to the original problem of Feedback Vertex Set which is obtained whenever $S = V(G)$. Subset Feedback Vertex Set remains $\text{NP}$-complete on bipartite graphs [37] and planar graphs [18], as a generalization of Feedback Vertex Set. Notable differences between the two latter problems regarding their complexity status is the class of split graphs and $4P_1$-free graphs for which Subset Feedback Vertex Set is $\text{NP}$-complete [16, 33], as opposed to the Feedback Vertex Set problem [11, 36, 7]. Inspired by the $\text{NP}$-completeness on chordal graphs, Subset Feedback Vertex Set restricted on (subclasses of) chordal graphs has attracted several researchers to obtain faster, still exponential-time, algorithms [21, 34].

On the positive side, Subset Feedback Vertex Set can be solved in polynomial time on restricted graph classes [7, 6, 32, 33]. Related to the structural parameter mim-width, Bergougnoux et al. [1] recently proposed an $n^{O(w^3)}$-time algorithm that solves Subset Feedback Vertex Set given a decomposition of the input graph of mim-width $w$. As leaf power graphs admit a decomposition of mim-width one [25], from the later algorithm Subset Feedback Vertex Set can be solved in polynomial time on leaf power graphs if an intersection model is given as input. However, to the best of our knowledge, it is not known whether the intersection model of a leaf power graph can be constructed in polynomial time. Moreover, even for graphs of mim-width one that do admit an efficient construction of the corresponding decomposition, the exponent of the running time given in [1] is relatively high.

Habib and Stacho [22] showed that the leafage of a connected chordal graph can be computed in polynomial time. Their described algorithm also constructs a corresponding clique tree with the minimum number of leaves. Regarding other problems that behave well with the leafage, we mention the Minimum Dominating Set problem for which Fomin et al. [17] showed that the problem is $\text{FPT}$ parameterized by the leafage of the given graph. Here we show that Subset Feedback Vertex Set is polynomial-time solvable for every chordal graph with bounded leafage. In particular, given a chordal graph with a tree model having $\ell$ leaves, our algorithm runs in $O(n^{2\ell+1})$ time. Thus, by combining the algorithm of Habib and Stacho [22], we deduce that Subset Feedback Vertex Set is in $\text{XP}$, parameterized by the leafage.

One advantage of leafage over mim-width is that we can compute the leafage of a chordal graph in polynomial time, whereas we do not know how to compute in polynomial time the mim-width of a chordal graph. However we note that a graph of bounded leafage implies a graph of bounded mim-width and, further, a decomposition of bounded mim-width can be computed in polynomial time [17]. This can be seen through the notion of $H$-graphs which are exactly the intersection graphs of connected subgraphs of some subdivision of a fixed graph $H$. The intersection model of subtrees of a tree $T$ having $\ell$ leaves is a $T'$-graph where $T'$ is obtained from $T$ by contracting nodes of degree two. Thus the size of $T'$ is at most $2\ell$, since $T$ has $\ell$ leaves. Moreover, given an $H$-graph and its intersection model, a (linear) decomposition of mim-width at most $2|E(H)|$ can be computed in polynomial time [17]. Therefore, given a graph of leafage
there is a polynomial-time algorithm that computes a decomposition of mim-width $O(\ell)$. Combined with the algorithm via mim-width [1], one can solve SUBSET FEEDBACK VERTEX SET in time $n^{O(\ell^2)}$ on graphs having leafage $\ell$. Notably, our $n^{O(\ell)}$-time algorithm is a non-trivial improvement on the running time obtained from the mim-width approach.

We complement our algorithmic result by showing that SUBSET FEEDBACK VERTEX SET is W[1]-hard parameterized by the leafage of a chordal graph. Thus we can hardly avoid the dependence of the exponent in the stated running time. Our reduction is inspired by the W[1]-hardness of FEEDBACK VERTEX SET parameterized by the mim-width given in [26]. However we note that our result holds on graphs with arbitrary vertex weights and we are not unaware if the unweighted variant of SUBSET FEEDBACK VERTEX SET admits the same complexity behavior.

Our algorithm works on an expanded tree model that is obtained from the given tree model and maintains all intersecting information without increasing the number of leaves. Then in a bottom-up dynamic programming fashion, we visit every node of the expanded tree model in order to compute partial solutions. At each intermediate step, we store all necessary information of subsets of vertices that are of size $O(\ell)$. As a byproduct of our dynamic programming scheme and the expanded tree model, we show how our approach can be extended in order to handle rooted path graphs. Rooted path graphs are the intersection graphs of rooted paths in a rooted tree. They form a subclass of leaf powers and have unbounded leafage (through their underlying tree model). Although rooted path graphs admit a decomposition of mim-width one [25] and such a decomposition can be constructed in polynomial time [14, 20], the running time obtained through the bounded mim-width approach is rather unpractical, as it requires to store a table of size $O(n^{13})$ even in this particular case [1]. By analyzing further subsets of vertices at each intermediate step, we manage to derive an algorithm for SUBSET FEEDBACK VERTEX SET on rooted path graphs that runs in $O(n^2m)$ time. Observe that the stated running time is comparable to the $O(nm)$-time algorithm on interval graphs [32] and interval graphs form a proper subclass of rooted path graphs.

Moreover, inspired by the algorithm on bounded leafage graphs we consider its natural generalization concerning the vertex leafage of a graph. Chaplick and Stacho [10] introduced the vertex leafage of a graph $G$ as the smallest number $k$ such that there exists a tree model for $G$ in which every subtree corresponding to a vertex of $G$ has at most $k$ leaves. As leafage measures the closeness to interval graphs (graphs with leafage at most two), vertex leafage measures the closeness to undirected path graphs which are the intersection graphs of paths in a tree (graphs with vertex leafage at most two). We prove that the unweighted variant of SUBSET FEEDBACK VERTEX SET is NP-complete on undirected path graphs and, thus, the problem is para-NP-complete parameterized by the vertex leafage. An interesting remark of our NP-completeness proof is that our reduction comes from the MAX CUT problem as opposed to known reductions for SUBSET FEEDBACK VERTEX SET which are usually based on, more natural, covering problems [16, 33]. Thus we obtain a complexity dichotomy of the problem restricted on the two comparable classes of rooted and undirected path graphs. Our findings are summarized in Figure 1.

2 Preliminaries

All graphs considered here are finite undirected graphs without loops and multiple edges. We refer to the textbook by Bondy and Murty [4] for any undefined graph terminology and to the recent book of [13] for the introduction to Parameterized Complexity. For a positive integer $p$, we use $[p]$ to denote the set of integers $\{1, \ldots, p\}$. For a graph $G = (V_G, E_G)$, we use $V_G$ and $E_G$ to denote the set of vertices and edges, respectively. We use $n$ to denote the number of vertices of a graph and use $m$ for the number of edges. Given $x \in V_G$, we denote by $N_G(x)$ the neighborhood of $x$. The degree of $x$ is the number of edges incident to $x$. Given $X \subseteq V_G$, we
| unbounded vertex leafage | chordal graphs NP-hard [16] |
|--------------------------|-----------------------------|
| vertex leafage at most \(v_{\ell} \geq 2\) | \(\subseteq\) undirected path graphs NP-hard, Theorem 19 |
| vertex leafage at most 1 | \(\subseteq\) \(O(n^{2\ell+1})\), Theorem 10 |
| \(\equiv\) interval graphs \(O(nm)\) [33] | \(\subseteq\) \(O(n^2m)\), Theorem 16 |
| leafage at most 2 | \(\subseteq\) leafage at most \(\ell \geq 3\) | \(\subseteq\) unbounded leafage |
| \(\equiv\) rooted path graphs \(O(n^2m)\), Theorem 16 |

Figure 1: Computational complexity of the SFVS problem parameterized by leafage and vertex leafage.

denote by \(G - X\) the graph obtained from \(G\) by the removal of the vertices of \(X\). If \(X = \{u\}\), we also write \(G - u\). The subgraph induced by \(X\) is denoted by \(G[X]\), and has \(X\) as its vertex set and \(\{uw \mid u, v \in X\text{ and } uv \in E_G\}\) as its edge set. A clique is a set \(K \subseteq V_G\) such that \(G[K]\) is a complete graph.

Given a collection \(C\) of sets, the graph \(G = (C, \{\{X, Y\} : X, Y \in C\text{ and } X \cap Y \neq \emptyset\})\) is called the intersection graph of \(C\). Structural properties and recognition algorithms are known for intersection graphs of (directed) paths in (rooted) trees [9, 29, 31]. Depending on the collection \(C\), we say that a graph is

- **chordal** if \(C\) is a collection of subtrees of a tree,
- **undirected path** if \(C\) is a collection of paths of a tree,
- **rooted path** if \(C\) is a collection of rooted paths of a rooted tree, and
- **interval** if \(C\) is a collection of subpaths of a path.

For any undirected tree \(T\), we use \(L(T)\) to denote the set of its leaves, i.e., the set of nodes of \(T\) having degree at most one. If \(T\) contains only one node then we let \(L(T) = \emptyset\). Let \(T\) be a rooted tree. We assume that the edges of \(T\) are directed towards the root. If there is a (directed) path from node \(v\) to node \(w\) in \(T\), we say that \(v\) is a descendant of \(w\) and that \(w\) is an ancestor of \(v\). The leaves of a rooted tree \(T\) are exactly the nodes of \(T\) having out-degree one and in-degree zero. Observe that for an undirected tree \(T\) with at least one edge we have \(|L(T)| \geq 2\), whereas in a rooted tree \(T\) with at least one edge \(||L(T)\| \geq 1\) holds.

A binary relation, denoted by \(\leq\), on a set \(V\) is called partial order if it is transitive and anti-symmetric. For a partial order \(\leq\) on a set \(V\), we say that two elements \(x\) and \(y\) of \(V\) are comparable if \(x \leq y\) or \(y \leq x\); otherwise, \(x\) and \(y\) are called incomparable. If \(x \leq y\) and \(x \neq y\) then we simply write \(x < y\). Given \(X, Y \subseteq V\), we write \(X \leq Y\) if for any \(x \in X\) and \(y \in Y\), we have \(x \leq y\); if \(X\) and \(Y\) are disjoint then \(X \leq Y\) is denoted by \(X < Y\). Given a rooted tree \(T\), we define a partial order on the nodes of \(T\) as follows: \(x \leq_T y\) if \(x\) is a descendant of \(y\). It is not difficult to see that if \(x \leq_T y\) and \(x \leq_T z\) then \(y\) and \(z\) are comparable, as \(T\) is a rooted tree.
Leafage and vertex leafage  A tree model of a graph \( G = (V_G, E_G) \) is a pair \( (T, \{T_v\}_{v \in V_G}) \) where \( T \) is a tree, called a host tree\(^1\), each \( T_v \) is a subtree of \( T \), and \( uv \in E_G \) if and only if \( V(T_u) \cap V(T_v) \neq \emptyset \). We say that a tree model \( (T, \{T_v\}_{v \in V_G}) \) realizes a graph \( H \) if its corresponding graph \( G \) is isomorphic to \( H \). It is known that a graph is chordal if and only if it admits a tree model \([8, 19]\). The tree model of a chordal graph is not necessarily unique. The leafage of a chordal graph \( G \), denoted by \( \ell(G) \), is the minimum number of leaves of the host tree among all tree models that realize \( G \), that is, \( \ell(G) \) is the smallest integer \( \ell \) such that there exists a tree model \( (T, \{T_v\}_{v \in V_G}) \) of \( G \) with \( \ell = |L(T)| \) \([28]\). Moreover, every chordal graph \( G \) admits a tree model for which its host tree \( T \) has the minimum \( |L(T)| \) and \( |V(T)| \leq n \) \([10, 22]\); such a tree model can be constructed in \( O(n^3) \) time \([22]\). Thus the leafage \( \ell(G) \) of a chordal graph \( G \) is computable in polynomial time.

A generalization of leafage is the vertex leafage introduced by Chaplick and Stacho \([10]\). The vertex leafage of a chordal graph \( G \), denoted by \( v\ell(G) \), is the smallest integer \( k \) such that there exists a tree model \( (T, \{T_v\}_{v \in V_G}) \) of \( G \) where \( |L(T_v)| \leq k \) for all \( v \in V_G \). Clearly, we have \( v\ell(G) \leq \ell(G) \).

Although leafage was originally introduced for connected chordal graphs, as opposed to the vertex leafage, hereafter we relax the connectedness restriction on leafage to avoid confusion between the two notions and we assume that the considered tree model realizes any chordal graph. Moreover, we will impose that the host tree \( T \) is a rooted tree without affecting structural and algorithmic consequences. Under these terms, observe that \( \ell(G) = 0 \) iff \( G \) is a disjoint union of cliques, \( \ell(G) \leq 2 \) iff \( G \) is an interval graph, \( v\ell(G) \leq 1 \) iff \( G \) is a rooted path graph, and \( v\ell(G) \leq 2 \) iff \( G \) is an undirected path graph.

S-forests and S-triangles  By an induced cycle of \( G \) we mean a chordless cycle. A triangle is a cycle on 3 vertices. Hereafter, we consider subclasses of chordal graphs, that is graphs that do not contain induced cycles on more than 3 vertices.

Given a graph \( G \) and \( S \subseteq V(G) \), we say that a cycle of \( G \) is an \( S \)-cycle if it contains a vertex in \( S \). Moreover, we say that an induced subgraph \( F \) of \( G \) is an \( S \)-forest if \( F \) does not contain an \( S \)-cycle. Thus an induced subgraph \( F \) of a chordal graph is an \( S \)-forest if and only if \( F \) does not contain any \( S \)-triangle. Typically, the Subset Feedback Vertex Set problem asks for a vertex set of minimum (weight) size such that its removal results in an \( S \)-forest. The set of vertices that do not belong to an \( S \)-forest is referred to as subset feedback vertex set. In our dynamic programming algorithms, we focus on the equivalent formulation of computing a maximum weighted \( S \)-forest.

For a collection \( C \) of sets, we write \( \max_{weight} \{C \in C\} \) to denote \( \arg \max_{weight \in C} \{weight(C)\} \), where \( weight(C) \) is the sum of weights of the vertices in \( C \). The collection of \( S \)-forests of a graph \( G \), is denoted by \( F_S \). Let \( X, Y \subseteq V_G \) such that \( X \cap Y = \emptyset \) and \( G[Y] \in F_S \). Then, \( A^0_X = \max \{U \subseteq X : G[U \cup Y] \in F_S\} \).

- Our desired optimal solution is \( A^0_{V_G} = \max \{U \subseteq V_G : G[U] \in F_S\} \). We will subsequently show that in order to compute \( A^0_{V_G} \) it is sufficient to compute \( A^0_X \) for a polynomial number of sets \( X \) and \( Y \).

Let \( G = (V_G, E_G) \) be a chordal graph and let \( X, Y \subseteq V_G \) such that \( X \cap Y = \emptyset \) and \( G[Y] \in F_S \). A partition \( \mathcal{P} \) of \( X \) is called nice if for any \( S \)-triangle \( S_t \) of \( G[X \cup Y] \), there is a partition class \( P_t \in \mathcal{P} \) such that \( V(S_t) \cap X \subseteq P_t \). In other words, any \( S \)-triangle of \( G[X \cup Y] \) is involved with at most one partition class of a nice partition \( \mathcal{P} \) of \( X \). With respect to the optimal defined solutions \( A^0_X \), we observe the following:

\(^1\)The host tree is also known as a clique tree, usually when we are concerned with the maximal cliques of a chordal graph \([19]\).
Observation 1. Let $G = (V_G, E_G)$ be a chordal graph and let $X, Y \subseteq V_G$ such that $X \cap Y = \emptyset$ and $G[Y] \in \mathcal{F}_S$. Then, the following hold:

1. $A_X^Y = A_{X'}^Y$ for any $Y \supseteq Y' \supseteq Y \cap N(X')$ where $X' = X \setminus \{u \in X \mid S \cap N(u) \subseteq Y \setminus S\}$.
2. $A_X^Y = \bigcup_{X' \in \mathcal{P}} A_{X'}^Y$, for any nice partition $\mathcal{P}$ of $X$.

Proof. For the first statement, observe that $G[Y'] \in \mathcal{F}_S$, as an induced subgraph of an $S$-forest. Also, notice that any $S$-triangle in $G[X \cup Y]$ remains an $S$-triangle in $G[X \cup Y']$. Consider an $S$-triangle $\{a, b, x\}$ in $G[X \cup Y]$ with $x \in X$ and $a \in Y$. We show that $a \in Y'$ and $b \in X \cup Y'$. If $x \in X'$, then $a \in Y'$ and $b \in X \cup Y'$ by the fact that $Y \cap N(X') \subseteq Y'$. Suppose that $x \in X \setminus S$ such that $Y \cap N(x) \subseteq Y \setminus S$. This means that the only vertex of $S$ in the $S$-triangle is $b$. In particular, we have $b \in X \cap S$ and, since $a \in N(b)$, we conclude that $a \in Y'$. Thus, any $S$-triangle in $G[X \cup Y]$ remains an $S$-triangle in $G[X \cup Y']$, which shows the claim.

For the second statement, assume that there is an $S$-triangle $S_i$ in $G[X \cup Y]$. Then it must contain a vertex $v$ of some partition class $P_i$, as $G[Y]$ is an $S$-forest. By the definition of a nice partition $\mathcal{P}$, we have $V(S_i) \cap X \subseteq P_i$. Therefore, we deduce $A_X^Y \cap P_i = A_{X'}^Y$, which concludes the proof. 

By Observation 1, we search for nice partitions of the vertex set $X$ in order to consider smaller instances of $A_X^Y$. More precisely, Observation 1 (ii) suggests how to consider the natural sets $X'$ of a nice partition of $X$, whereas Observation 1 (i) indicates which vertices of $Y$ are relative to each set $X'$.

3 Expanded tree model and related vertex subsets

Given a tree model of a chordal graph, we are interested in defining a partial order on the vertices of the graph that takes advantage of the underlying tree structure. For this reason, it is more convenient to consider the tree model as a natural rooted tree and each of its subtrees to correspond to at most one maximal vertex. Here we show how a tree model can be altered in order to capture the appropriate properties in a formal way. We assume that $G$ is a chordal graph that admits a tree model $(T, \{T_v\}_{v \in V_G})$ such that $|L(T)| = \ell(G)$. We will concentrate on the case in which $|L(T)| \geq 2$ and $T$ contains a non-leaf node. The rest of the cases (i.e., $|V(T)| \leq 2$) are handled by the algorithm on interval graphs [32] in a separate way. For this purpose we say that a chordal graph $G$ is non-trivial if $|V(T)| > 2$.

Definition. A tree model $(T, \{T_v\}_{v \in V_G})$ of $G$ is called expanded tree model if

- the host tree $T$ is rooted (and, consequently, all of its subtrees are rooted),
- for every $v \in V_G$, $L(T_v) \neq \emptyset$ holds, and
- every node of $T$ is either the root or a leaf of at most one subtree $T_v$ that corresponds to a vertex $v$ of $G$.

We show that any non-trivial chordal graph admits an expanded tree model that is close to its tree model. In fact, we provide an algorithm that, given a tree model of a non-trivial chordal graph $G$, constructs an expanded tree model that realizes $G$.

Lemma 2. For any tree model $(T, \{T_v\}_{v \in V_G})$ of $G$ with $|L(T)| = \ell \geq 2$ and $|L(T_v)| \leq v\ell \leq \ell$ for all $v \in V_G$, there is an expanded tree model $(T', \{T'_v\}_{v \in V_G})$ of $G$ such that:

- $|L(T')| = \ell$, 

Moreover, given \((T, \{T_v\}_{v \in V_G})\), the expanded tree model can be constructed in time \(O(n^2)\).

**Proof.** We root \(T\) at a non-leaf node of \(T\), resulting in a rooted tree \(T'\) with \(|L(T')| = \ell\). Moreover, we root every \(T_v\) at the node of \(T_v\) which is closer to \(r(T')\), resulting in a rooted subtree \(T'_v\). Notice that \(|L(T'_v)| - 1 \leq |L(T'_v)| \leq |L(T_v)|\), as \(r(T'_v)\) may be a leaf of \(T_v\). In what follows, we assume that \(T\) and all of its subtrees \(\{T_v\}_{v \in V_G}\) are rooted trees.

![Figure 2: We replace node \(x\) of \(T\) by the directed path \(x_{-k_1}, \ldots, x_0, \ldots, x_{k_r}\) such that the nodes \(N^-(x)\) now point to \(x_{-k_1}\) and node \(N^+(x)\) is now pointed by \(x_{k_r}\).](image)

Consider a node \(x\) of \(T\). Assume that \(x\) is the root of \(k_r\) subtrees \(T_{v_{1}}, \ldots, T_{v_{k_r}}\) and a leaf of \(k_l\) subtrees \(T_{v_{-1}}, \ldots, T_{v_{-k_l}}\) of \(\{T_v\}_{v \in V_G}\) where \(k_r + k_l \geq 2\). In this context, for every \(v \in V_G\) corresponding to \(T_v = \{x\}\), we consider \(x\) as being both the root and a leaf of \(T_v\). We replace the node \(x\) in \(T\) by the gadget shown in Figure 2. We also modify every subtree \(T_v\) of \(\{T_v\}_{v \in V_G}\) as follows:

- If \(T_v = T_{v_i}\) for some \(i \in \{1, \ldots, k_r\}\) and \(T_v \neq T_{v_j}\) for all \(j \in \{-1, \ldots, -k_l\}\), then we replace \(x\) in \(T_v\) by the part of the gadget involving the vertices \(x_{-k_1}, \ldots, x_0, \ldots, x_{i}\).
- If \(T_v \neq T_{v_i}\) for all \(i \in \{1, \ldots, k_r\}\) and \(T_v = T_{v_j}\) for some \(j \in \{-1, \ldots, -k_l\}\), then we replace \(x\) in \(T_v\) by the part of the gadget involving the vertices \(x_j, \ldots, x_0, \ldots, x_{k_r}\).
- If \(T_v = T_{v_i}\) for some \(i \in \{1, \ldots, k_r\}\) and \(T_v = T_{v_j}\) for some \(j \in \{-1, \ldots, -k_l\}\), then we replace \(x\) in \(T_v\) by the part of the gadget involving the vertices \(x_{j}, \ldots, x_0, \ldots, x_{i}\).
- If \(T_v \neq T_{v_i}\) for all \(i \in \{1, \ldots, k_r\}\) and \(T_v \neq T_{v_j}\) for all \(j \in \{-1, \ldots, -k_l\}\), then \(T'_v = T_v\).

To see that this new model indeed realizes \(G\), observe that for every \(T_u, T_w \in \{T_v\}_{v \in V_G}\):

- if \(x \in V(T_u) \cap V(T_w)\), then \(x_0 \in V(T'_u) \cap V(T'_w)\), and
- if \(x \notin V(T_u) \cap V(T_w)\), then \(x_{-k_1}, \ldots, x_{k_r} \notin V(T'_u) \cap V(T'_w)\).

Thus the intersection graph of \((T', \{T'_v\}_{v \in V_G})\) is isomorphic to \(G\). Notice that any node \(x_i\), \(i \in \{-k_1, \ldots, k_r\}\) is either the root or a leaf of at most one subtree of \(T'\) and, in particular, \(|L(T'_v)| = 1\) for any \(T_v = \{x\}\). Iteratively applying the above modifications to \(T\) and \(\{T_v\}_{v \in V_G}\), results in an expanded tree model \(T'\) of \(G\) that satisfies the claimed properties.

To bound \(|V(T')|\), observe that the first step adds at most \(n\) new nodes in \(T\), so that \(|V(T')| \leq |V(T)| + n\). Further notice that every subtree \(T_v\) has at most \(v\ell\) leaves. In the worst case, all subtrees of \(\{T_v\}_{v \in V_G}\) are rooted in the same node and all their leaves are contained in a set of \(|\max_{v \in V_G} L(T_v)| = v\ell\) nodes, so our preprocessing algorithm will add \((1 + v\ell)(n - 1)\) nodes to \(T\). Moreover, as we need to update \(n + 1\) trees by adding at most a total of \((1 + v\ell)(n - 1)\) new nodes and \(|V(T)| \leq n\), the total running time is \(O(n^2)\). \(\square\)
Hereafter we assume that \((T, \{T_v\}_{v \in V_G})\) is an expanded tree model of a non-trivial chordal graph \(G\). For any vertex \(u\) of \(G\), we denote the root of its corresponding rooted tree \(T_u\) in \(T\) by \(r(u)\).

We define the following partial order on the vertices of \(G\): for all \(u, v \in V_G\), \(u \leq v \iff r(u) \leq_T r(v)\). In other words, two vertices of \(G\) are comparable (with respect to \(\leq\)) if and only if there is a directed path between their corresponding roots in \(T\). For all \(u \in V_G\), we define \(U_u = \{u' \in V_G : u' \leq u\}\).

**Observation 3.** Let \(u, v, w, z \in V_G\). Then, the following hold:

1. If \(uv \in E_G\), then \(u\) and \(v\) are comparable.
2. If \(u \leq v, w \leq z\), and \(u\) and \(w\) are comparable, then \(v\) and \(z\) are comparable.
3. If \(u < v < w\) and \(uw \in E_G\), then \(vw \in E_G\).

**Proof.** Assume that \(x \in V(T_u) \cap V(T_v)\), which exists as \(uv \in E_G\). Then there are paths \(x \to r(u)\) and \(x \to r(v)\). Since \(T\) is a rooted tree, any node besides its root has a unique parent. This implies that the shortest of the aforementioned paths is a subpath of the longest. Assume, without loss of generality, that \(x \to r(u)\) is the shortest path. Then \(x \to r(v) = x \to r(u) \to r(v)\), so that \(u\) and \(v\) are indeed comparable.

For the second statement, observe that all ancestors of a node of \(T\) are pairwise comparable with respect to \(\leq_T\). Assume that \(u \leq w\). Then \(r(u) \leq r(w) \leq r(z)\) because \(w \leq z\), so \(u \leq z\) in addition to \(u \leq v\). Now assume that \(w \leq u\). Then \(r(w) \leq r(u) \leq r(v)\) because \(u \leq v\), so \(w \leq v\) in addition to \(w \leq v\). In both cases we conclude that \(v\) and \(z\) are comparable.

For the third statement, observe that \(u < v < w\) implies that \(r(u) < r(v) < r(w)\) which in turn implies that \(r(u) \to r(v) \to r(w)\). We show that \(r(u) \in V(T_w)\). Since \(u\) and \(w\) are adjacent, there exists a node \(x \in V(T_u) \cap V(T_w)\). As shown in the proof of the first statement, there exists a path \(x \to r(u) \to r(w)\). Then all the nodes of this path are in \(V(T_w)\) because its endpoints are in \(V(T_w)\) and \(T_w\) is connected. Thus we have \(r(u) \in V(T_w)\). With the same argumentation, we conclude that all nodes of the path \(r(u) \to r(v) \to r(w)\) are in \(V(T_w)\), so that \(r(v) \in V(T_w)\) holds. Therefore, \(v\) and \(w\) are adjacent.

**Lemma 4.** For every \(u \in V_G\), we have \(N(V_u) \subseteq N(u)\).

**Proof.** Let \(w \in N(V_u)\). Then there is a vertex \(v \in V_u\) such that \(vw \in E_G\). It suffices to show that \(w \in N(u)\). Assume that \(u \neq v\), as otherwise the claim trivially holds. Then \(v < u\), because \(v \in V_u\). Moreover, Observation 3 (1) implies that \(w < v\) or \(v < w\). Since \(w \notin V_u\), we conclude \(u < v < w\). Therefore Observation 3 (3) shows that \(uw \in E_G\).

For all \(u \in V_G\), we denote the set of all maximal proper predecessors of \(u\) by \(\prec u\). Notice that such vertices correspond to the maximal descendants of \(r(u)\). For all \(U \subseteq V_G\), we define \(V_U = \{V_u : u \in U\}\). We extend the previous case of a single vertex, on subsets of vertices with respect to an edge. For all \(u, v \in V_G\) such that \(uv \in E_G\), we denote by \(\prec uv\) the set of all maximal vertices of \(V_G\) that are proper predecessors of both \(u\) and \(v\) but are not adjacent to both, so \(\prec uv = \max_G((V_u \cap V_v) \setminus (N[u] \cap N[v]))\). Recall that for any edge \(uv \in E_G\), either \(u < v\) or \(v < u\) by Observation 3 (1). If \(u < v\) holds, then \(\prec uv = \max_G(V_u \setminus (N[u] \cap N(v)))\).

The following two lemmas are crucial for our algorithms, as they provide natural partitions into smaller instances.

**Lemma 5.** For every \(u \in V_G\), the collection \(V_{\prec u}\) is a partition of \(V_u \setminus \{u\}\) into pairwise disconnected sets. For every \(u, v \in V_G\) such that \(u < v\) and \(uv \in E_G\), \(V_{\prec uv}\) is a partition of \(V_u \setminus (N[u] \cap N(v))\) into pairwise disconnected sets.
Proof. We prove the first statement. The proof of the second statement is completely analogous. Firstly notice that, by definition, the vertices of \( \prec u \) are pairwise incomparable. Consider two vertices \( u'_1 \) and \( u'_2 \) such that \( u'_1 \leq u_1 \) and \( u'_2 \leq u_2 \) where \( u_1 \) and \( u_2 \) are two vertices of \( \prec u \). Clearly, \( u'_1 \in V_{u_1} \) and \( u'_2 \in V_{u_2} \). By Observation 3 (1–2), it follows that \( u'_1 \) and \( u'_2 \) are distinct and non-adjacent. \( \square \)

Lemma 6. For every \( u \in V_G \), the collection \( V_{\prec u} \) is a nice partition of \( V_u \setminus \{u\} \). For every \( u, v \in V_G \) such that \( u \prec v \) and \( uv \in E_G \), the collection \( V_{\prec uv} \) is a nice partition of \( V_u \setminus (N[u] \cap N(v)) \).

Proof. We prove the first statement. The proof of the second statement is completely analogous. Let \( X = V_u \setminus \{u\} \) and \( Y \subseteq V_G \) such that \( X \cap Y = \emptyset \). Suppose that \( S_t \) is an \( S \)-triangle of \( G[X \cup Y] \) for which the intersection of \( V(S_t) \) and a class of \( V_{\prec uv} \) is non-empty for at least two such classes. Assume that \( P_1 \) and \( P_2 \) are two of those classes and let \( u_1 \in V(S_t) \cap P_1 \) and \( u_2 \in V(S_t) \cap P_2 \). Then \( u_1 \) and \( u_2 \) must be adjacent, which is in contradiction to Lemma 5. \( \square \)

Having defined the necessary predecessors (maximal descendants) of \( u \), we next analyze specific solutions described in \( A_{V_u}^Y \) with respect to the vertices of \( \prec u \). Both statements follow by carefully applying Lemma 4 and Lemma 6.

Lemma 7. Let \( Y \subseteq V_G \setminus V_u \). (i) If \( u \notin A_{V_u}^Y \), then \( A_{V_u}^Y = \bigcup_{u' \in \prec u} A_{V_{u'}}^{Y \cap N(u')} \).

(ii) Moreover, \( A_{V_u}^0 = \max_{\text{weight}} \left\{ \bigcup_{u' \in \prec u} A_{V_{u'}}^0, \{u\} \cup \bigcup_{u' \in \prec u} A_{V_{u'}}^{(u) \cap N(u')} \right\} \).

Proof. We first show claim (i). Since \( u \notin A_{V_u}^Y \), we have \( A_{V_u}^Y = A_{V_u \setminus \{u\}}^Y \). According to Lemma 6, the collection \( V_{\prec u} \) is a nice partition of \( V_u \setminus \{u\} \). Thus, by Observation 1 and Lemma 4, we have

\[
A_{V_u \setminus \{u\}}^Y = \bigcup_{X \in V_{\prec u}} A_X^Y = \bigcup_{u' \in \prec u} A_{V_{u'}}^{Y \cap N(u')}.
\]

For the second claim, we distinguish two cases depending on whether \( u \) is in \( A_{V_u}^0 \) or not. If \( u \notin A_{V_u}^0 \) then claim (i) shows the desired formula.

Assume that \( u \in A_{V_u}^0 \). Then \( A_{V_u}^0 = \{u\} \cup A_{V_{u} \setminus \{u\}}^{(u)} \). Recall that the collection \( V_{\prec u} \) is a nice partition of \( V_u \setminus \{u\} \). Moreover, if \( u \) has no neighbor in \( V_{u'} \) then \( u \notin N(u') \) by Lemma 4. Thus, we get the desired formula:

\[
A_{V_u \setminus \{u\}}^{(u)} = \bigcup_{X \in V_{\prec u}} A_X^{(u)} = \bigcup_{u' \in \prec u} A_{V_{u'}}^{(u) \cap N(u')}.
\]

\( \square \)

4 SFVS on graphs with bounded leafage

In this section we concern ourselves with chordal graphs that have an intersection model tree with at most \( \ell \) leaves. Our goal is to show that SFVS can be solved in polynomial time on chordal graphs with bounded leafage. In particular, we show that SFVS is in XP parameterized by \( \ell \). In the case of \( \ell \leq 2 \), the input graph is an interval graph, so SFVS can be solved in \( O(nm) \) time [32]. We subsequently assume that we are given a chordal graph \( G \) that admits an expanded tree model \( (T, \{T_v\}_{v \in V_G}) \) with \( \ell = L(T) \geq 2 \), due to Lemma 2.

Given a subset of vertices of \( G \), we collect the leaves of their corresponding subtrees: for every \( U \subseteq V_G \), we define \( L(U) = \bigcup_{u \in U} L(T_u) \). Notice that for any non-empty \( U \subseteq V_G \), we have
\(L(U) \neq \emptyset\), since \((T, \{T_v\}_{v \in V_G})\) is an expanded tree model. Moreover, we associate the nodes of \(T\) with the vertices of \(G\) for which the nodes appear as leaves in their corresponding subtrees: for every \(V \subseteq V_T\), we define \(L^{-1}(V)\) to be the set \(\{u \in V_G : L(T_u) \cap V \neq \emptyset\}\). For \(V \subseteq V_T\), we denote by \(\min_T V\) the subset of minimal nodes of \(V\) with respect to \(\leq_T\). Observe that \(\min_T V\) is a set of pairwise incomparable nodes, so \(|\min_T V| \leq |\min_T V_T| \leq \ell\).

**Lemma 8.** Let \(U \subseteq V_G\) and \(V \subseteq L(U)\). Then \(L^{-1}(V) \subseteq U\).

**Proof.** The fact that \(V \subseteq L(U)\) yields \(L^{-1}(V) \subseteq L^{-1}(L(U))\). We will show that \(L^{-1}(L(U)) \subseteq U\).

Let \(u\) be a vertex of \(G\) such that \(u \notin U\). Then \(L(T_u) \cap L(U) = \emptyset\), because \((T, \{T_v\}_{v \in V_G})\) is an expanded tree model. Thus \(u \notin L^{-1}(L(U))\).

Instead of manipulating with the actual vertices of \(U\), our algorithm deals with the representatives of \(U\) which contain the vertices of \(L^{-1}(\min_T L(U))\). In particular, we are interested in the set of vertices \(F_{\leq 2}(U) = F_1(U) \cup F_2(U)\), where \(F_1(U) = L^{-1}(\min_T \{L(U)\})\) and \(F_2(U) = L^{-1}(\min_T \{L(U \setminus F_1(U))\})\), for any vertex \(u' \in U\). We show that the representatives hold all the necessary information needed from their actual vertices.

**Lemma 9.** Let \(u \in V_G\) and \(W \subseteq V_G \setminus V_u\) such that \(W \neq \emptyset\), \(G[\{u\} \cup W]\) is a clique, and \(G[W] \in \mathcal{F}_S\), and let \(u \in A^W_{V_u}\).

- If \(\{\{u\} \cup W\} \cap S \neq \emptyset\) then \(W = \{u\}\) and no vertex of \(V_u \cap N(u) \cap N(w)\) belongs to \(A^w_{V_u}\).

- If \(\{\{u\} \cup W\} \cap S = \emptyset\) then \(A^W_{V_{u'}} = A^{F_{\leq 2}(\{\{u\} \cup W\}) \cap N(u')}_{V_{u'}}\), for any vertex \(u' \in U\).

**Proof.** Assume that some vertex of \(\{u\} \cup W\) is in \(S\) and \(|W| \geq 2\). Then there are \(w_1, w_2 \in W\) such that \(\{u, w_1, w_2\} \cap S \neq \emptyset\). Since \(\{u \cup W\}\) induces a clique, we have that \(\{u, w_1, w_2\}\) induces a \(S\)-triangle, contradicting that \(u\) belongs to \(A^W_{V_u}\). Thus \(W = \{u\}\) because \(W \neq \emptyset\). Then \(A^w_{V_u} = \{u\} \cup A^w_{V_{u'} \setminus \{u\}}\) by definition. Observe that for any \(u' \in V_u \cap N(u) \cap N(w)\) the vertex set \(\{u', u, w\}\) induces an \(S\)-triangle, since \(u\) and \(w\) are adjacent. Thus, no vertex of \(V_u \cap N(u) \cap N(w)\) is in \(A^w_{V_u}\).

Assume that no vertex of \(\{u\} \cup W\) is in \(S\). Consider a vertex \(u' \in U\). Observe that for any two vertices \(a \in V_{u'}\) and \(b \in V_G \setminus V_{u'}\) to be adjacent, since \(r(a) \leq r(u') < r(b)\) already holds, \(l < r(a)\) must hold for some \(l \in L(T_u)\). Let \(W' = \{\{u\} \cup W\} \cap N(u')\) and \(F = F_{\leq 2}(W')\). We will show that \(F\) is a representation of \(W'\) on \(V_{u'}\).

- Assume there are two vertices \(u'' \in V_{u'}\) and \(u'' \in V_{u'}\) and a vertex \(w' \in W'\) such that \(\{u'' \in V_{u'}\}\) and \(\{u'' \in V_{u'}\}\) are adjacent, so without loss of generality we may assume that \(r(u'') < r(u'')\). Let \(l\) be a node of \(L(T_{w'})\) such that \(l < r(u'')\). There is a vertex \(w'' \in F\) such that \(l' \leq l\) for some \(l' \in L(T_{w''})\). This implies that the set \(\{u'', u''\}\) also induces an \(S\)-triangle.

- Assume there is a vertex \(w'' \in V_{u'}\) and two vertices \(w', w' \in W'\) such that \(\{w', w'\}\) induces an \(S\)-triangle. Let \(l_1\) and \(l_2\) be nodes of \(L(T_{w'1})\) and \(L(T_{w'2})\) respectively such that \(l_1, l_2 < r(u'')\). Then, there are two distinct vertices \(w_1', w_2' \in F\) such that \(l_1 \leq l_1\) and \(l_2 \leq l_2\) for some \(l_1' \in L(T_{w'1})\) and some \(l_2' \in L(T_{w'2})\). This implies that the set \(\{w', w', w''\}\) also induces an \(S\)-triangle.

We conclude that \(A^W_{V_{u'}} = A^F_{V_{u'}}\).

We next show that Lemma 7 (ii) and Lemma 9 are enough to develop a dynamic programming scheme. As the size of the representatives is bounded with respect to \(\ell\) by Lemma 8, we are able to store a bounded number of partial suboptima. In particular we show that we only need to compute \(A^X\) such that \(|X| = O(n)\) and \(|Y| \leq 2\ell + 1\).
Theorem 10. There is an algorithm that, given a connected chordal graph $G$ with leafage $\ell \geq 2$ and an expanded tree model of $G$, solves Subset Feedback Vertex Set in $O(n^{2\ell+1})$ time.

Proof. Let $\{T, \{T_v\}_{v \in V_G}\}$ be an expanded tree model of $G$ and let $r$ be the root of $T$. Our task is to solve SFVS by computing $A^W_{V_r}$. To do so, we construct a dynamic programming algorithm that visits the nodes on $T$ in a bottom-up fashion, starting from the leaves and moving towards the root $r$. At each node $u$ of $T$, we store the values corresponding to $A^W_{V_u}$ and $A^F_{V_u}$ for every $W \subseteq V_G \setminus V_u$ such that $W \not= \emptyset$, $G[W] \in \mathcal{F}_S$, and $G\{u\} \cup W$ is a clique. In order to compute $A^W_{V_u}$, we apply Lemma 7 (ii) by collecting all corresponding values on the necessary descendants of $u$. For computing $A^F_{V_u}$, we apply Lemma 9 by looking at the values stored on the necessary descendants of $u$. In particular, we deduce the following formulas, where $W_S = \{\{u\} \cup W) \cap S$:

- If $W_S \not= \emptyset$ and $|W| \geq 2$ then $A^W_{V_u} = \bigcup_{u' \in \cup u} A^{W \cap N(u')}_{V_u'}$. Lemma 9 implies that $u \not\in A^W_{V_u}$. Thus by Lemma 7 (i) we get the claimed formula.

- If $W_S \not= \emptyset$ and $W = \{w\}$ then $A^W_{V_u} = \max_{\text{weight}} \left\{ \bigcup_{u' \in \cup u} A^{(u \cap N(u'))}_{V_u'}, \{u\} \cup \bigcup_{u' \in \cup u} A^{(u \cup w \cap N(u'))}_{V_u'} \right\}$. According to Lemma 6, the collection $\mathcal{V} \cup u \cup u$ is a nice partition of $V_u \setminus (N[u] \setminus N(u))$. Thus, by Observation 1 we get the desired formula.

- If $W_S = \emptyset$ then $A^W_{V_u} = \max_{\text{weight}} \left\{ \bigcup_{u' \in \cup u} A^{W \cap N(u')}_{V_u'}, \{u\} \cup \bigcup_{u' \in \cup u} A^{F_{\leq 2}((\{u\} \cup W) \cap N(u'))}_{V_u'} \right\}$. According to Lemma 6, the collection $\mathcal{V} \cup u$ is a nice partition of $V_u \setminus \{u\}$. Thus, Observation 1 and Lemma 4 imply the corresponding formula.

Notice that we compute $A^W_{V_u}$ for $n^{O(\ell)}$ sets $W_1, \ldots, W_l \subseteq V_G \setminus V_u$ such that $W$ is represented by a set $W_i$ (i.e., there exists $W_i$ such that $A^W_{V_u} = A^{W_i}_{V_u}$). At the root $r$ of $T$, we only compute $A^\emptyset_{V_r}$ by applying Lemma 7 (ii).

Regarding the correctness of the algorithm, we show that the recursive formulas given in Lemma 7 (ii) and Lemma 9 require only sets that are also computed via these formulas. The formula given in Lemma 7 (ii) requires sets $A^W_{V_u}$, where $u' < u$ and either $W' = \emptyset$ or $W' = \{w\}$ such that $u' < u'$ and $u'w' \in E_G$. In the second case, it is not difficult to see that $u'$ and $W'$ satisfy the hypothesis of Lemma 9 as they induce a graph in $\mathcal{F}_S$. Notice that an induced subgraph of a clique is also a clique and an induced subgraph of a graph in $\mathcal{F}_S$ is also a graph in $\mathcal{F}_S$. The formulas given in Lemma 9 require sets $A^{W'}_{V_{u'}}$ of the following three cases:

- Sets such that $u' \in \cup u$ and $W' = W \cap N(u') \subseteq W$. These sets are clearly computed via the formulas of Lemma 7 (ii) or 9 according to whether $W$ is empty or not.

- Sets such that $u' \in \cup uw$ and $W' = \{u, w\} \cap N(u)$. Since $u'$ is only adjacent to at most one of $u$ and $w$, we have either $W' = \emptyset$ or $W' = \{w\}$ such that $u' < u'$ and $u'w' \in E_G$.

- Sets such that $u' \in \cup u$ and $W' = F_{\leq 2}((\{u\} \cup W) \cap N(u')) \subseteq (\{u\} \cup W) \cap N(u) \subseteq \{u\} \cup W$ where $(\{u\} \cup W) \cap S = \emptyset$. Since $G\{u\} \cup W \in \mathcal{F}_S$ and is a clique, we obtain that $G[W'] \in \mathcal{F}_S$ and $G\{u\} \cup W'$ is a clique.

We conclude that in all cases the sets required by a formula of Lemma 9 are computed via a formula given in Lemma 7 (ii) or Lemma 9.

We now analyze the running time of our algorithm. We begin by determining for every pair of nodes $x, y$ of $T$ whether $x < y$, $y < x$ or they are incomparable. Since for any one pair this can be done in $O(n)$ time, we complete this task in $O(n^3)$ time. Notice that, since the input
Proof. For every connected component corresponding tree model $T$ with leafage at most $\ell$, consider a set $A$ of pairwise incomparable nodes. This means that any set of pairwise incomparable nodes is of size at most $\ell$. This fact implies that $|F_{\leq 2}(U)| \leq 2\ell$ for any $U \subseteq V_G$. Due to the recursion, we only need to compute $F_{\leq 2}(U)$ for sets $U$ such that $|U| \leq 2\ell + 1$. Computing any such set requires at most $(2\ell + 1)^2$ comparisons and consequently constant time, so the total preprocessing time is $O(n^{2\ell+1})$. Now consider a set $A_X$. The parts of any partition of $X$ that we use in our formulas are rooted in pairwise incomparable nodes. This means that any set $A_X$ is computed in $O(\ell)$ time. Thus we conclude that total running time of our algorithm is $O(n^{2\ell+1})$.

If we let the leafage of a chordal graph to be the maximum over all of its connected components then we reach to the following result.

**Corollary 11.** **Subset Feedback Vertex Set** can be solved in time $n^{O(\ell)}$ for chordal graphs with leafage at most $\ell$.

**Proof.** For every connected component $C$ of a chordal graph $G$, we compute its leafage and the corresponding tree model $T(C)$ by using the $O(n^3)$-time algorithm of Habib and Stacho [22]. If the leafage of $C$ is less than two, then $C$ is an interval graph and we can compute $A^V(C)$ in $n^{O(1)}$ time by running the algorithm for SFVS on interval graphs given in [32]. Otherwise, we compute the expanded tree model $T'(C)$ from $T(C)$ by Lemma 2 in $O(n^2)$ time. Applying Theorem 10 on $T'(C)$ shows that $A^V(C)$ can be computed in $n^{O(\ell)}$. Since the connected components of $G$ form a nice partition of $V(G)$, Observation 1 implies that $A^V(G)$ is the union of all $A^V(C)$ for every connected component $C$ of $G$. Therefore all steps can carried out in $n^{O(\ell)}$ time.

We next prove that we can hardly avoid the dependence of the exponent in the stated running time, since we show that **Subset Feedback Vertex Set** is $W[1]$-hard parameterized by the leafage of a chordal graph. Our reduction is inspired by the $W[1]$-hardness of **Feedback Vertex Set** parameterized by the mim-width given by Jaffke et al. in [26].

**Theorem 12.** **Subset Feedback Vertex Set** on chordal graphs is $W[1]$-hard when parameterized by its leafage.

**Proof.** We provide a reduction from the **Multicolored Clique** problem. Given a graph $G = (V, E)$ and a partition $\{V_i\}_{i \in [k]}$ of $V$ into $k$ parts, the **Multicolored Clique** (MCC) problem asks whether $G$ has a clique that contains exactly one vertex of $V_i$ for every $i \in [k]$. It is known that MCC is $W[1]$-hard when parameterized by $k$ [15, 35].

Let $(G = (V, E), \{V_i\}_{i \in [k]})$ be an instance of MCC. We assume that $k \geq 10$ and without loss of generality that there exists $p \in \mathbb{N}$ such that $V_i = \{v_{ij}\}_{j \in [p]}$ for every $i \in [k]$. We consider the $\frac{k}{2}(k + 3)$-star $T$ with root $r$ and leaves $x_i^+, x_i^-$ for every $i \in [k]$ and $y_{ij}$ for every $i, j \in [k]$ such that $i < j$. We modify the star $T$ as follows: for every $i \in [k]$, we transform the edge $\langle r, x_i^+ \rangle$ through edge subdivisions into $\langle r = x_i^0, x_i^1, \ldots, x_i^{p} = x_i^+ \rangle$ and in a similar way we replace the edge $\langle r, x_i^- \rangle$ by $\langle r = x_i^0, x_i^{-1}, \ldots, x_i^{−p} = x_i^- \rangle$. Given a set $X$ of vertices of $T$, we write $T(X)$ to denote the minimal subtree of $T$ containing all vertices of $X$. A certain subtree of $T$ is depicted in Figure 4. We define the following subtrees of $T$:

![Figure 3: The subtree $T(\{x_i^+, x_i^-, y_{ij}, x_j^+, x_j^-\})$ of $T$ for some $i, j \in [k]$ such that $i < j$.]
• For every $i, j \in [k]$ such that $i < j$ and for every $a, b \in [p]$ such that $v_i^a v_j^b \in E$, we define $e_{ij}^{ab} = T(\{x_i^a, x_i^{a-p}, y_{ij}, x_j^b, x_j^{b-p}\})$. We denote by $R$ the set of all these subtrees. For all $i, j \in [k]$ such that $i < j$, we denote by $R_{ij}$ the set of all subtrees in $R$ with subscript $ij$.
  
  For all $i \in [k]$:
  
  - We denote the set $\{e_{ij}^{ab} \in R\} \cup \{e_{ji}^{ba} \in R\}$ by $R_i$.
  
  - For all $a_i \in [p]$, we denote the set $\{e_{ij}^{aib} \in R\} \cup \{e_{ji}^{bai} \in R\}$ by $R_{a_i}$.

• For every $i \in [k]$ and $a \in [p]$, we define $s_i^{a,1} = s_i^{a,2} = T(\{x_i^a\})$ and $s_i^{-a,1} = s_i^{-a,2} = T(\{x_i^{-a}\})$. We denote by $S_i$ the set of all these subtrees. For all $i \in [k]$:
  
  - We denote by $S_i$ the set of all subtrees in $S_i$ with subscript $i$.
  
  - For all $a_i \in [p]$, we denote the set $\{s_i^{a,1} \in S_i : |a_i - p| \leq a \leq a_i\}$ by $S_{a_i}$.

• For every $i, j \in [k]$ such that $i < j$, we define $s_{ij} = T(\{y_{ij}\})$. We denote by $S_E$ the set of all these subtrees.

We further denote by $S$ the set $S_V \cup S_E$ and by $T$ the collection $R \cup S$. We construct a graph $G'$ that is the intersection graph of $(T, T)$. Notice that $G'$ is a chordal graph of leafage at most $\frac{1}{2}(k + 3)$. We identify the vertices of $G'$ with their corresponding subtrees in $T$. By the construction of $(T, T)$, regarding adjacencies between vertices of $G'$ we observe the following:

• $R$ induces a clique, because all its elements contain the node $r$.

• For every $i \in [k]$ and $a \in [p]$, the vertices $s_i^{a,1}$ and $s_i^{-a,1}$ are adjacent to $s_i^{a,2}$ and $s_i^{-a,2}$ respectively.

• For every $i \in [k]$ and $a_i \in [p]$, we have $N(e) \cap S_i = S_{a_i}$ for all $e \in R_{a_i}$.

• For every $i, j \in [k]$ such that $i < j$, we have $N(s_{ij}) = R_{ij}$.

We set the weight of all vertices of $R$, $S_V$ and $S_E$ to be $\frac{3}{2}$, 1 and $\frac{2}{3}m$ respectively. We will show that $(G, \{V_i\}_{i \in [k]})$ is a Yes-instance of MCC if and only if there exists a solution to SFVS on $(G', S)$ of weight $\frac{3}{2}(m - \frac{k}{2}(k - 9))$.

For the reverse direction, let $\{v_i^{a_1}, \ldots, v_k^{a_k}\}$ be a solution to MCC on $(G, \{V_i\}_{i \in [k]})$. We define $R_C = \{e_{ij}^{a_1} : i, j \in [k], i < j\}$. Observe that $R_C$ contains exactly one element of $R_{ij}$ for each $i, j \in [k]$ such that $i < j$. We further define the set $U = (R \setminus R_C) \cup (\cup\{S_{a_i}\}_{i \in [k]})$. Now observe that in $G - U$ each of the remaining vertices of $S$ has exactly one neighbour. Thus $U$ a solution to SFVS on $(G', S)$ of weight $\frac{3}{2}(m - \frac{k}{2}(k - 1)) + 2pk = \frac{3}{2}(m - \frac{k}{2}(k - 9))$.

For the reverse direction, let $U$ be a solution to SFVS on $(G', S)$ of weight $\frac{3}{2}(m - \frac{k}{2}(k - 9))$. Notice that no element of $S_E$ can be in $U$. Consequently, for every $i, j \in [k]$ such that $i < j$, $|R_{ij} \setminus U| \leq 1$, since any two elements of $R_{ij}$ along with $s_{ij}$ form an $S$-triangle of $G'$. Any one of the remaining $S$-triangles of $G'$ is formed by

• either an element of $R_{a_i}$ and both $s_i^{a,1}$ and $s_i^{a,2}$

• or an element of $R_{a_i}$, an element of $R_{a_i}$ and either $s_i^{a,1}$ or $s_i^{a,2}$

for some $i \in [k]$ and for some $a, a_i, a_i' \in [p]$ such that $\max\{a_i, a_i'\} - p \leq a \leq \min\{a_i, a_i'\}$. Let $i \in [k]$.

Claim 12.1. If $|R_i \setminus U| \geq 1$, then $|S_i \cap U| \geq p$.
Proof: Assume that $R_i \backslash U \supseteq \{e\}$ and $e \in R_i^{a_i}$ for some $a_i \in [p]$. Then for every integer $a \neq 0$ such that $a_i - p < a \leq a_i$ at least one of $s_i^{a_i,1}, s_i^{a_i,2}$ must be in $U$, so $|S_i \cap U| \geq p$.

Claim 12.2. If $|R_i \backslash U| \geq 2$, then $|S_i \cap U| \geq 2p$.

Proof: Assume that $R_i \backslash U \supseteq \{e, e'\}$ and $e \in R_i^{a_i}$ and $e' \in R_i^{a'_i}$ for some $a_i, a'_i \in [p]$ such that $a_i \leq a'_i$. Then

- for every integer $a \neq 0$ such that $a'_i - p \leq a \leq a_i$ both $s_i^{a_i,1}$ and $s_i^{a_i,2}$ must be in $U$ and
- for every integer $a \neq 0$ such that $a_i - p \leq a < a'_i - p$ or $a_i < a \leq a'_i$ at least one of $s_i^{a_i,1}, s_i^{a_i,2}$ must be in $U$, so $|S_i \cap U| \geq 2(a_i + (p - a'_i)) + 1((a'_i - a_i) + (a'_i - a_i)) = 2p$.

Claim 12.3. If $|R_i \backslash U| \geq 3$, then $|S_i \cap U| = 2p$ only if $R_i \backslash U \subseteq R_i^{a_i}$ for some $a_i \in [p]$.

Proof: Assume that $R_i \backslash U \supseteq \{e, e', e''\}$ and $e \in R_i^{a_i}, e' \in R_i^{a'_i}$ and $e'' \in R_i^{a''_i}$ for some $a_i, a'_i, a''_i \in [p]$ such that $a_i \leq a'_i \leq a''_i$. Then

- for every integer $a \neq 0$ such that $a'_i - p \leq a \leq a_i$ both $s_i^{a_i,1}$ and $s_i^{a_i,2}$ must be in $U$ and
- for every integer $a \neq 0$ such that $a_i - p \leq a < a'_i - p$ or $a_i < a < a''_i$ at least one of $s_i^{a_i,1}, s_i^{a_i,2}$ must be in $U$, so $|S_i \cap U| \geq 2p + 1((a'_i - a_i) + (a''_i - a'_i)) = 2p + (a''_i - a_i)$. We conclude that for $|R_i \backslash U|$ to be $2p$, all elements of $R_i \backslash U$ must be elements of the same $R_i^{a_i}$ for some $a_i \in [p]$.

Assume that $|\{i \in [k] : |R_i \backslash U| = 1\}| = k'$ and $|\{i \in [k] : |R_i \backslash U| \geq 2\}| = k''$. Then $|R \backslash U| \leq k' + k''(k'' - 2)$ and $|S \cap U| \geq p(k' + 2k'')$, so the weight of $U$ must be at least

$$p \left( m - k' - \frac{k''}{2}(k'' - 1) \right) + p(k' + 2k'') = p \left( m + k' - \frac{k''}{2}(k'' - 9) \right) = B(k', k'').$$

Clearly, $k', k'' \in \{0, \ldots, k\}$. Regarding the values of $B$, we obsserve the following:

- $B(0, k'') \leq B(k', k'')$ for all $k', k'' \in \{0, \ldots, k\}$,
- $B(0, 9) \leq B(0, k'')$ for all $k'' \in \{0, \ldots, 9\}$ and
- $B(0, k'') < B(0, k'' - 1)$ for all $k'' \in \{10, \ldots, k\}$.

We conclude that a weight of $\frac{p}{2}(m - \frac{k}{2}(k - 9)) = B(0, k)$ is within bounds only for $k' = 0$ and $k'' = k$ and can be achieved only when $|R \backslash U| = \frac{k}{2}(k - 1)$ and $|S \cap U| = 2p$ for all $i \in [k]$. For every $i \in [k]$, let $a_i \in [p]$ be such that $R_i \backslash U \subseteq R_i^{a_i}$. Then the set $\{v_1^{a_1}, \ldots, v_k^{a_k}\}$ is a solution to MCC on $(G, \{V_i\}_{i \in [k]})$.

5 SFVS on rooted path graphs

Here we show how to extend our previous approach for SFVS on rooted path graphs. Rooted path graphs are exactly the intersection graphs of rooted paths on a rooted tree. Notice that rooted path graphs have unbounded leafage. Our main goal is to derive a recursive formulation for $A_X^Y$, similar to Lemma 9. In particular, we show that it is sufficient to consider sets $Y$ containing at most one vertex.

For any vertex $u$ of $G$, we denote the leaf of its corresponding rooted path in $T$ by $l(u)$. We need to define further special vertices and subsets. Let $u, v \in V_G$ such that $u \prec v$. The (unique) maximal predecessor $u'$ of $v$ such that $l(u') \prec r(u) \leq r(u')$ is denoted by $u \triangleleft v$. Moreover, for every $V_1, V_2, V_3 \subseteq V_G$, we define the following sets:
Lemma 14. Let \( V_{(v_1|v_2|v_3)} \) = \( \{ u \in V_G : r(v_1) < l(u) < r(v_2) < r(u) \leq r(v_3) \} \) for some \( v_1 \in V_i, i \in \{ 1, 2, 3 \} \)
- \( V_{(v_1|v_3)} \) = \( \{ u \in V_G : l(u) < r(v_2) < r(u) \leq r(v_3) \} \) for some \( v_1 \in V_i, i \in \{ 2, 3 \} \)
- \( V_{(v_1|v_3)} \) = \( \{ u \in V_G : r(v_1) < l(u) < r(u) \leq r(v_3) \} \) for some \( v_1 \in V_i, i \in \{ 1, 3 \} \)

The vertical bars indicate the placements of \( l(u) \) and \( r(u) \) with respect to \( V_1, V_2, V_3 \).

**Lemma 13.** Let \( u, w \in V_G \setminus S \) such that \( u < w \) and \( uw \in E_G \). Then, the collection

\[
\{ V_{\langle uw \rangle \setminus u} \} \cup \{ V_{u'} \cup (V_{\langle u' \rangle \setminus u} \setminus S) \}_{u' \in \langle uw \rangle}
\]

is a nice partition of \( X = V_u \setminus (\{ u \} \cup (N(u) \cap N(w) \cap S)) \) with respect to any \( Y \subseteq V_G \setminus X \) such that \( Y \cap S = \emptyset \).

**Proof.** We first show that this collection is indeed a partition of \( X \). Recall that the vertices of \( \langle uw \rangle \) induce an independent set by Lemma 5. Consider a vertex \( u'' \in X \). Then exactly one of the following statements holds, implying the claimed partition:

\[
\begin{align*}
\exists u' \in \langle uw \rangle : r(u'') \leq r(u') \\
\exists u' \in \langle uw \rangle : l(u'') < r(u') < r(u'') \\
\exists u' \in \langle uw \rangle : r(u') < l(u'')
\end{align*}
\]

Now let \( Y \subseteq V_G \setminus X \) such that \( Y \cap S = \emptyset \) and consider an \( S \)-triangle \( S_t \) of \( G[X \cup Y] \). Then there is some vertex \( u' \in \langle uw \rangle \) such that \( V(S_t) \cap V_{u'} \cap S \neq \emptyset \). Since \( S_t \) is a triangle, every vertex in \( V(S_t) \setminus V_{u'} \) must be adjacent to every vertex in \( V(S_t) \cap V_{u'} \). By Lemma 4, a vertex \( u'' \in V_G \setminus V_{u'} \) that is adjacent to a vertex of \( V_{u'} \) must be adjacent to \( u' \), so \( l(u'') < r(u') < r(u'') \) must hold. We conclude that \( V(S_t) \subseteq V_{u'} \cup (V_{\langle u' \rangle \setminus u} \setminus S) \).

For every appropriate \( u, v \), we will denote the set \( V_u \cup (V_{\langle u \rangle \setminus u} \setminus S) \) by \( V_{u,v} \). Observe that the set \( V_{u,u} \) is simply \( V_u \). First we consider the set \( A_{V_u}^{(w)} \) for which \( u < w \) and \( uw \in E_G \).

- If \( u \notin A_{V_u}^{(w)} \) then \( A_{V_u}^{(w)} = \bigcup_{u' \in \langle uw \rangle} A_{V_{u'}}^{(w)} \cap N(u') \) by Lemma 7 (i).

Also, recall that \( A_{V_u}^{(w)} \) is described by the formula given in Lemma 7 (ii). We derive the following result that handles the cases in which \( u \in A_{V_u}^{(w)} \).

**Lemma 14.** Let \( u, w \in V_G \) such that \( u < w \) and \( uw \in E_G \), and let \( u \in A_{V_u}^{(w)} \).

- If \( u \in S \) or \( w \in S \) then \( A_{V_u}^{(w)} = \{ u \} \cup \bigcup_{u' \in \langle uw \rangle} A_{V_{u'}}^{(w)} \cap N(u') \).

- If \( u, w \notin S \) then \( A_{V_u}^{(w)} = \{ u \} \cup (V_{\langle uw \rangle \setminus u} \setminus S) \cup \bigcup_{u' \in \langle uw \rangle} A_{V_{u'}}^{(w)} \cap N(u') \).

**Proof.** Observe that \( A_{V_u}^{(w)} = \{ u \} \cup A_{V_u}^{(u,w)} \) by definition. Since \( \langle uw \rangle = \max_G(V_u \setminus (N[u] \cap N(w))) \), any vertex \( u' \in \langle uw \rangle \) is adjacent to at most one of \( u \) and \( w \). Regarding triangles of \( G[V_u \cup \{ w \}] \), we observe the following property:

\((P1)\) By the hypothesis, the vertices \( u \) and \( w \) are adjacent. Thus, for any \( u' \in V_u \cap N(u) \cap N(w) \), the vertex set \( \{ u', u, w \} \) induces a triangle.
If $u \in S$ or $w \in S$, then no vertex of $V_u \cap N(u) \cap N(w)$ is in $A^{(w)}_{V_u}$ because of (P1). According to Lemma 6, the collection $V_{<uw}$ is a nice partition of $V_u \setminus (N(u) \cap N(w))$. Thus, by Observation 1 we get the desired formula: $A^{(u,w)}_{V_u} = \{u\} \cup \bigcup_{u' \in <uw} A^{(u,w)\cap N(u')}_{V_{u'}}$.

If $u, w \not\in S$, then no vertex of $V_u \cap N(u) \cap N(w) \cap S$ is in $A^{(w)}_{V_u}$ because of (P1). Thus, by definition, we have $A^{(u,w)}_{V_u} = A^{(u,w)}_{V_u \setminus \{u\} \cap \{N(u) \cap N(w) \cap S\}}$. Let $X = (V_u \setminus \{u\}) \setminus (N(u) \cap N(w) \cap S)$. By Lemma 13, the collection $\{V_{<uw} \cup \{u\} \setminus S\} \cup \{V_{u',u''<uw} \cap \{v\}\}$ is a nice partition of $X$ with respect to $\{u, w\}$. Thus, Observation 1 and Lemma 4 imply

$$A^{(u,w)}_X = A^{(u,w)}_{V_{<uw} \cup \{u\} \setminus S} \cup \bigcup_{u' \in <uw} A^{(u,w) \cap N(u')}_{V_{u'}},$$

$$= (V_{<uw} \cup \{u\} \setminus S) \cup \bigcup_{u' \in <uw} A^{(u,w) \cap N(u')}_{V_{u'}},$$

We next deal with the sets $A^{(w)}_{V_{u,v}}$, for which $u < v$, $|Y| \leq 1$ and no vertex of $\{v\} \cup Y$ belongs to $S$. Observe that $V_{u,v}$ is not necessarily described by a set $V_u$ for some $v \in V_G$. Thus we need appropriate formulas that handle such sets. For doing so, notice that

- $V_{u,v} \setminus \{v\} = V_{u,v} \cap \{\{u\}\} \setminus S$ and $u \leq u < v$.

This means that if $v \not\in A^{(w)}_{V_{u,v}}$, we have $A^{(w)}_{V_{u,v}} = A^{(w)}_{V_{u,v} \setminus \{v\}}$.

With the next result we consider the corresponding case in which $v \in A^{(w)}_{V_{u,v}}$. Notice that given a partition $P$ of a set $X$ and a set $X' \subseteq X$, the collection $P' = \{P \cap X'\}_{P \in P}$ is a partition of $X'$. Furthermore, observe that if $P$ is a nice partition of $X$ with respect to a set $Y \subseteq V_G \setminus X$ such that $Y \cap S = \emptyset$, then $P'$ is a nice partition of $X'$ with respect to $Y$.

**Lemma 15.** Let $u \in V_G$ and $v, w \in V_G \setminus S$ such that $u < v < w$ and $\{u, v, w\}$ induce a clique and let $v \in A^{(w)}_V$. Then, $A^{(w)}_{V_{u,v}} = \{v\} \cup (V_{<uw} \cup \{u\}) \setminus S \cup \bigcup_{u' \in <uw} A^{(v,w) \cap N(u')}_{V_{u',u''<uw}}$.

**Proof.** By definition, we have $A^{(w)}_{V_{u,v}} = \{v\} \cup A^{(v,w)}_{V_{u,v} \setminus \{v\}} = \{v\} \cup A^{(v,w)}_{V_{u,v} \setminus \{v\}}$. Regarding triangles of $G[V_{u,v} \cup \{w\}]$, we observe the following property:

(P2) By the hypothesis, the vertices $v$ and $w$ are adjacent. Thus, for any $u' \in V_{u,v} \cap N(v) \cap N(w)$, the vertex set $\{u', v, w\}$ induces a triangle.

Since $v, w \not\in S$, no vertex of $V_{u,v} \cap N(v) \cap N(w) \cap S$ is in $A^{(w)}_{V_{u,v}}$ because of (P2). Thus $A^{(v,w)}_{V_{u,v} \cap \{N(v) \cap N(w) \cap S\}}$. Let $X = (V_u \setminus \{v\}) \setminus (N(v) \cap N(w) \cap S)$ and $X' = V_{u,v} \setminus (N(v) \cap N(w) \cap S)$. Now, notice that $X' \subseteq X$. Applying Lemma 13 on $X$ and $Y = \{v, w\}$ shows that the collection $\{V_{<uw} \cup \{u\} \setminus S\} \cup \{V_{u',u''<uw} \cup \{v\}\}$ is a nice partition of $X'$ with respect to $Y$. Hence, Observation 1 and Lemma 5 imply

$$A^{(v,w)}_{X'} = A^{(v,w)}_{V_{<uw} \cup \{u\}} \setminus S \cup \bigcup_{u' \in V_{u,v} \setminus \{u\}} A^{(v,w) \cap N(u')}_{V_{u',u''<uw}},$$

$$= (V_{<uw} \cup \{u\} \setminus S) \cup \bigcup_{u' \in V_{u,v} \setminus \{u\}} A^{(v,w) \cap N(u')}_{V_{u',u''<uw}}.$$

Now we are in position to state our claimed result, which is obtained in a similar fashion with the algorithm given in Theorem 10.

**Theorem 16.** *Subset Feedback Vertex Set* can be solved on rooted path graphs in $O(n^2m)$ time.
Proof. We first describe the algorithm. Given a rooted path graph $G$, we construct its tree model $\{T, \{T_v\}_{v \in V_G}\}$ in $O(n + m)$ time [14, 20]. If $G$ is an interval graph then SFVS can be solved by the algorithm described in [32] that runs in $O(nm)$ time. We assume henceforth that $G$ is not an interval graph, so that $L(T) \geq 2$. We apply Lemma 2 and obtain an expanded tree model $\{T', \{T'_v\}_{v \in V_G}\}$ in $O(n^2)$ time. As any host tree $T$ of $G$ has at most $n$ nodes [10, 22], the expanded host tree $T'$ has $O(n)$ nodes by the third property of Lemma 2. Moreover, observe that all subtrees $T'_v$ are rooted paths by the second property of Lemma 2. Then we solve SFVS by computing $A^0_{V'_r}$ for the root $r$ of $T'$.

For this purpose, we construct a dynamic programming algorithm for computing $A^0_{V'_v}$. The algorithm works on $T'$ in a bottom-up fashion, starting from the leaves and moving towards the root $r$. As $T'$ is the host tree of an expanded tree model, there is a mapping between the vertices of $G$ and their corresponding root nodes in $T'$. We start with defining the tables of data that the algorithm stores for each node $u \neq r$ of $T'$. The constructed tables correspond to the sets $A^0_{V'_v}, A^0_{V'_u}, A^0_{V'_u,v}, A^0_{V'_v,u}$. In particular, we get the following formulas for the described sets.

- Let $u, w \in V_G$ such that $u < w$ and $uw \in E_G$. Lemma 7 (i) and Lemma 14 imply the following:

$$A^0_{V'_v} = \max_{\text{weight}} \left\{ \bigcup_{u' \in u} A^0_{V'_u,v} \cap N(u'), \{u\} \cup \bigcup_{u' \in uw} A^0_{V'_u,v} \cap N(u') \right\}.$$  

- If $u, w \notin S$ then

$$A^0_{V'_u} = \max_{\text{weight}} \left\{ \bigcup_{u' \in u} A^0_{V'_u'} \cap N(u'), \{u\} \cup (V_{\{u \cap uw\}} \setminus S) \cup \bigcup_{u' \in uw} A^0_{V'_u,v} \cap N(u') \right\}.$$  

- Let $u \in V_G$ and $v \in V_G \setminus S$ such that $u < v$ and $uv \in E_G$. Lemma 7 (i) and the description of $V_{u,v} \setminus \{v\}$ imply the following:

$$A^0_{V_{u,v}} = \max_{\text{weight}} \left\{ A^0_{V_{u,u,v}} \cap N(v), \{v\} \cup A^0_{V_{u,u,v}} \right\}.$$  

- Let $u \in V_G$ and $v, w \in V_G \setminus S$ such that $u < v < w$ and $\{u, v, w\}$ induce a clique. Lemma 7 (i) and Lemma 15 imply the following:

$$A^0_{V_{u,v}} = \max_{\text{weight}} \left\{ A^0_{V_{u,u,v}} \cap N(v), \{v\} \cup (V_{\{u \cap uw\}} \setminus S) \cup \bigcup_{u' \in uw} A^0_{V_{u',u,v}} \cap N(u') \right\}.$$  

Let $v_1, \ldots, v_k$ be the neighbors of $u$ on the path from $u$ towards the root $r$ of $T$ such that $v_i < v_{i+1}, 1 \leq i < k$. We compute $A^0_{V'_v}$ and $A^0_{V'_v}$, for each $1 \leq i \leq k$, according to Lemma 7 (ii) and Lemma 14, respectively, by collecting the data stored on descendants of $u$. For every $1 \leq i < j \leq k$, with $v_i, v_j \notin S$ and $v_iv_j \in E_G$, we compute $A^0_{V_{u,v}}$ according to Lemma 15. Observe that $A^0_{V_{u,v}}$ and $A^0_{V_{u,v}}$ are computed by table entries that correspond to values of $A^0_{V_{v,v'}}$ with $u' \leq u, v' < v_i$, and $v' < w' \leq v_j$. When reaching the root $r$ of $T'$, it is enough to compute $A^0_{V'_v}$ by Lemma 7 (ii).

To evaluate the running time of the algorithm, we assume that the input graph is a connected rooted path graph having at least one cycle, so that $n \leq m$. For this, observe that we can simply run our algorithm on each connected component and any tree has a trivial solution as it does not contain any $S$-cycle. Now let us first determine the number of table entries required by our dynamic programming algorithm. Consider the entries corresponding to $A^0_{V'_v}$. The sets $X$ are either $V_u$ or $V_{u,v}$ for some $u, v \in V_G$ such that $u < v$ and $uv \in E_G$ of which there are in total $n$ and $m$, respectively. The sets $Y$ are either $\emptyset$ or $\{w\}$ for some $w \in V_G$ of which there are in total $m$.
n + 1. We conclude that our table entries are $O(n(n + m))$. Calculating a single entry requires to collect values of $O(n)$ entries. Those entries are determined via the vertex sets $\langle v, \langle vw and $V_v \cap \langle uw$ and the vertices $u \langle v$, which are precalculated. Observe that these objects are also $O(n(n + m))$ in total. To calculate $\langle v$ and $u \langle v$ we need only to traverse the host tree once for every $v \in V_G$. As there are $O(n)$ nodes in $T'$, such a computation takes $O(n^2)$ time in total. Similarly, to calculate $\langle vw$ and $V_v \cap \langle uw$ we need only to traverse the host tree once for every $v, w \in V_G$ such that $v < w$ and $vw \in E_G$. Thus the total preprocessing time can be accomplished in $O(n^2 + nm)$ time. Therefore, the total running time of our algorithm is $O(n^2 m)$.

6 Vertex leakage to cope with SFVS

Due to Theorem 10 and Corollary 11, it is interesting to ask whether our results can be further extended on larger classes of chordal graphs. Here we consider graphs of bounded vertex leafage. Here we consider graphs of bounded vertex leafage. Due to Theorem 10 and Corollary 11, it is interesting to ask whether our results can be further extended on larger classes of chordal graphs. Here we consider graphs of bounded vertex leafage.

Towards the claimed reduction, for any graph $G$ on $n$ vertices and $m$ edges, we will associate a graph $H_G$ on $12n^2 + 4n + 2m$ vertices. First we describe the vertex set of $H_G$. For every vertex $v \in V(G)$ we have the following sets of vertices:

- $X(v) = \{x_v^1, \ldots, x_v^{2n}\}$ and $\overline{X}(v) = \{\overline{x}_v^1, \ldots, \overline{x}_v^{2n}\}$,
- $Y(v) = \{y_v^1, \ldots, y_v^{2n+1}\}$ and $\overline{Y}(v) = \{\overline{y}_v^1, \ldots, \overline{y}_v^{2n+1}\}$,
- $Z(v) = \{z_v^1, \overline{z}_v^1, \ldots, z_v^{2n+1}, \overline{z}_v^{2n+1}\}$, and
- $E(v) = \{(v, x) \mid \{v, x\} \in E(G)\}$.

Observe that for every edge $\{u, v\} \in E(G)$ there are two vertices in $H_G$ that correspond to the ordered pairs $(u, v)$ and $(v, u)$. We denote by $E(v)$ the set of vertices $(x, v) \in H_G$ such that $\{x, v\} \in E(G)$. The edge set of $H_G$ contains precisely the following:

- all edges required for the set $\bigcup_{v \in V(G)} (Y(v) \cup \overline{Y}(v) \cup E(v))$ to form a clique and
- for every vertex $v \in V(G)$,
  - all elements of the sets $E(X(v), Y(v))$, $E(\overline{X}(v), \overline{Y}(v))$, $E(X(v), E(v))$, and $E(\overline{X}(v), \overline{E}(v))$;
  - $\{x_v^i, x_v^{n+i}\}$, $\{\overline{x}_v^i, \overline{x}_v^{n+i}\}$ for each $i \in [n]$;
  - $\{y_v^j, z_v^j\}$, $\{\overline{y}_v^j, \overline{z}_v^j\}$, $\{\overline{y}_v^j, \overline{z}_v^j\}$ for each $j \in [2n+1]$.

This completes the construction of $H_G$. Observe that the vertices of each pair $(x_v^i, x_v^{n+i})$ and $(\overline{x}_v^i, \overline{x}_v^{n+i})$ are true twins, whereas $(z_v^i, \overline{z}_v^i)$ are false twins. An example of $H_G$ is given in Figure 4.
Figure 4: Illustrating the undirected path graph $H_G$. On top left we show a graph $G$ on three vertices and on the bottom part we illustrate the corresponding graph $H_G$. A tree model $T(H_G)$ for $H_G$, is given on the top right part. The vertices of $H_G$ that lie on the grey area form a clique.

**Lemma 17.** For any graph $G$, $H_G$ is an undirected path graph.

**Proof.** In order to show that $H_G$ is an undirected path graph, we construct a tree model $T(H_G)$ for $H_G$ such that the vertices of $H_G$ correspond to particular undirected paths of $T(H_G)$. To distinguish the vertex sets between $G$ and $T(H_G)$, we refer to the vertices of $T(H_G)$ as nodes. In order to construct $T(H_G)$, starting from a particular node $r$, we create the following paths:

- for each $v \in V(G)$, $P_X(v) = (r, x_1^{(v)}, \ldots, x_n^{(v)})$ and $P_{\overline{X}}(v) = (r, \overline{x}_1^{(v)}, \ldots, \overline{x}_n^{(v)})$;
- for each $v \in V(G)$ and $j \in [2n + 1]$, $P_Z(v, j) = (r, z_1^{(v,j)}, z_2^{(v,j)})$. 

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Let \( \text{Lemma 18.} \) contain any \( u, v \) such that \( u \in H \). Now it is not difficult to see that the intersection graph of the above constructed undirected \( H \) is isomorphic to \( H \). More precisely, all paths containing node \( r \) correspond to the vertices of the clique \( \bigcup_{v \in V(G)} Y(v) \cup \overline{Y}(v) \cup E(v) \). Moreover all subpaths of \( P_X(v) \) and \( P_X(v) \) correspond to the vertices of \( X(v) \) and \( \overline{X}(v) \), respectively, while the subpaths of \( P_Z(v, j) \) correspond to the vertices of \( Z(v) \). Therefore, \( H \) is an undirected path graph.

Let us now show that there is a subset feedback vertex set in \( H \) that is related to a cut-set in \( G \). We let \( X = \bigcup X(v) \), \( \overline{X} = \bigcup \overline{X}(v) \), and \( Z = \bigcup Z(v) \).

**Lemma 18.** Let \( G \) be a graph with \( A \subseteq V(G) \) and let \( H \) be the undirected path graph of \( G \). For the set of vertices \( S = X \cup \overline{X} \cup Z \) of \( H \), there is a subset feedback vertex set \( U \) of \( (H, S) \) such that \( |U| = 4n^2 + n + 2m - k \), where \( k \) is the size of the cut-set of \( A \) in \( G \).

**Proof.** We describe the claimed set \( U \). For simplicity, we let \( \overline{A} = V(G) \setminus A \) and \( \overline{E} = \{(u, v) \mid u \in A, v \in \overline{A}, \{u, v\} \in E(G)\} \). Clearly, \( k = |\overline{E}| \).

- For every vertex \( v \in A \), let \( U(v) = X(v) \cup \overline{Y}(v) \).
- For every vertex \( v \in \overline{A} \), let \( \overline{U}(v) = \overline{X}(v) \cup Y(v) \).
- For every vertex \( v \in V(G) \), let \( E_U(v) = E(v) \setminus \overline{E} \).

Now \( U \) contains all the above sets of vertices, that is, \( U = \bigcup_{v \in V(G)} U(v) \cup \overline{U}(v) \cup E_U(v) \). To show that \( U \) is indeed a subset feedback vertex set, we claim that the graph \( H_G - U \) does not contain any \( S \)-cycle. Assume for contradiction that there is an \( S \)-cycle \( C \) in \( H_G - U \). Then \( C \) contains at least one vertex from \( S \). We consider the following cases:

- Let \( z \in C \cap Z \). Any vertex \( z \in Z \) has exactly two neighbors \( y, y' \) in \( H \). Thus \( C \) must pass through both \( y \) and \( y' \). By construction, we know that \( y \in Y(v) \) and \( y' \in \overline{Y}(v) \) for a vertex \( v \in V(G) \). Hence we reach a contradiction to the fact that \( y, y' \) are both vertices of \( H_G - U \), since either \( Y(v) \subseteq U \) and \( \overline{Y}(v) \subseteq V(H_G) \setminus U \) or vice versa.

- Let \( x \in C \cap X \). Observe that there exists \( v \in \overline{A} \) such that \( x \in X(v), \) since \( X(v') \subseteq U \) for all \( v' \in A \). Also notice that all the vertices of \( X(v) \) have exactly one neighbor in \( X(v) \). Thus there is a vertex \( w \in (H_G - X(v)) \cap C \) that is a neighbor of \( x \) in \( C \). By construction, all vertices of \( X(v) \) are adjacent to every vertex of \( Y(v) \) and to every vertex of the form \( (v, a) \) for which \( \{v, a\} \in E(G) \). Since \( v \in \overline{A} \), we know that \( Y(v) \subseteq U \) and \( (v, a) \in \overline{E} \) so that \( (v, a) \in U \). Hence, in both cases, we reach a contradiction to the fact that \( w \in C \).
• Let \( x \in C \cap \overline{X} \). Arguments that are completely symmetrical to the ones employed in the previous case yield a contradiction to the fact that \( w \in C \).

Therefore, there is no cycle that passes through a vertex of \( S \), so that \( U \) is indeed a subset feedback vertex set. Regarding the size of \( U \), observe that \( |U(v)| = |\overline{U}(v)| = 4n + 1 \) and \( |\overline{E}| = k \). Thus the described set \( U \) fulfils the claimed properties. \( \square \)

Now we are ready to show our main result of this section. One direction follows from the previous lemma. The reverse direction is achieved through a series of claims. The key part is to force at least \( 2m - k \) particular \( S \)-cycles of \( H_G \) to appear within the corresponding \( k \) cut-edges in \( G \).

**Theorem 19.** Unweighted Subset Feedback Vertex Set is NP-complete on undirected path graphs.

**Proof.** We provide a polynomial reduction from the NP-complete Max Cut problem. Given a graph \( G \) on \( n \) vertices and \( m \) edges for the Max Cut problem, we construct the graph \( H_G \). Observe that the size of \( H_G \) is polynomial and the construction of \( H_G \) can be done in polynomial time. By Lemma 17, \( H_G \) is an undirected path graph. According to the terminology explained earlier for the vertices of \( H_G \), we let \( S = X \cup \overline{X} \cup Z \). We claim that \( G \) admits a cut-set of size at least \( k \) if and only if \( (H_G, S) \) admits a subset feedback vertex set of size at most \( 4n^2 + n + 2m - k \).

Lemma 18 shows the forward direction. The reverse direction is achieved through a series of claims. In what follows, we are given a subset feedback vertex set \( U \) of \( (H_G, S) \) with \( |U| \leq 4n^2 + n + 2m - k \). Based on the structure of \( H_G \), it is not difficult to see that any \( S \)-triangle has one of the following forms for some \( v \in V(G) \).

• \((x_i^v, x_{i+j}^v, w)\) for some \( i \in [n] \) and \( w \in Y(v) \cup E(v) \);

• \((x, w, w')\) for some \( x \in X(v) \) and \( w, w' \in Y(v) \cup E(v) \);

• \((\overline{x}_i^v, \overline{x}_{i+j}^v, \overline{w})\) for some \( i \in [n] \) and \( \overline{w} \in \overline{Y}(v) \cup \overline{E}(v) \);

• \((\overline{x}, \overline{w}, \overline{w}')\) for some \( \overline{x} \in \overline{X}(v) \) and \( \overline{w}, \overline{w}' \in \overline{Y}(v) \cup \overline{E}(v) \);

• \((y_i^v, \overline{y}_{i+j}^v, \tilde{z})\) for some \( j \in [2n + 1] \) and \( \tilde{z} \in \{z_i^v, \overline{z}_i^v\} \)

We next define the following sets of vertices.

• \( \overline{E} = \bigcup_{v \in V(G)} (E(v) \cup \overline{E}(v)) \),

• \( \overline{E}(A, B) = (\bigcup_{a \in A} E(a)) \cap \bigcup_{b \in B} \overline{E}(b) \) for every \( A, B \subseteq V(G) \),

• \( A_U = \{ v \in V(G) \mid Y(v) \subseteq V(H_G) \setminus U \} \) for every \( U \subseteq V(H_G) \), and

• \( \overline{A}_U = \{ v \in V(G) \mid \overline{Y}(v) \subseteq V(H_G) \setminus U \} \) for every \( U \subseteq V(H_G) \).

Our task is to show that there exists a subset feedback vertex set \( U \) such that \( |U| \leq 4n^2 + n + 2m - k \) and \( U(A_U) = U \), that is, \( U \) satisfies the following properties:

(1) \( Z \subseteq V(H_G) \setminus U \).

(2) For all \( v \in V(G) \), either \( X(v) \subseteq U \) or \( X(v) \subseteq V(H_G) \setminus U \) and either \( \overline{X}(v) \subseteq U \) or \( \overline{X}(v) \subseteq V(H_G) \setminus U \).
With completely symmetric arguments, there is a tier-1 sfvs from the fact that the subgraph induced by the vertices of one of \(w, w\) because both vertices have at most one neighbor in \(H_G - U\). Therefore, the number of vertices of such terms, our goal is to show that there exists a tier-3 sfvs and \(|\{z^j_v, z^j_u\} \cap U | \geq 1\) and \(|\| \{y^j_v, z^j_u\} \cap U \| \leq 1\), so that \(|U'| \leq |U|\). Also notice that \(U'\) is a tier-0 sfvs because both vertices \(z^j_v\) and \(z^j_u\) have at most one neighbor in \(H_G - U\). Iteratively applying the same argument for each \(v \in V(G)\) and for each \(j \in [2n + 1]\) such that \(\{z^j_v, z^j_u\} \cap U \neq \emptyset\), we obtain a tier-0 sfvs \(U'\) such that \(Z \subseteq V(H_G) \setminus U'\). \(\) 

**Claim 19.1.** For every tier-0 sfvs \(U\), there is a tier-1 sfvs \(U'\) such that \(|U'| \leq |U|\).

**Proof:** If \(Z \cap U = \emptyset\), then \(U\) is already a tier-1 sfvs. Assume that \(\{z^j_v, z^j_u\} \cap U \neq \emptyset\) for some \(v \in V(G)\) and some \(j \in [2n+1]\). We construct the set \(U' = (U \setminus \{z^j_v, z^j_u\}) \cup \{y^j_v\}\). Notice that \(|\{z^j_v, z^j_u\} \cap U | \geq 1\) and \(|\{y^j_v\} \cap U | \leq 1\), so that \(|U'| \leq |U|\). Also notice that \(U'\) is a tier-0 sfvs because both vertices \(z^j_v\) and \(z^j_u\) have at most one neighbor in \(H_G - U\). Iteratively applying the same argument for each \(v \in V(G)\) and for each \(j \in [2n+1]\) such that \(\{z^j_v, z^j_u\} \cap U \neq \emptyset\), we obtain a tier-0 sfvs \(U'\) such that \(Z \subseteq V(H_G) \setminus U'\). \(\)

**Claim 19.2.** For every tier-1 sfvs \(U\), \(|\{Y(v) \cup \overline{Y}(v)\} \cap U | \geq 2n + 1\) holds for every \(v \in V(G)\).

**Proof:** Consider the sets \(Y(v)\) and \(\overline{Y}(v)\) of a vertex \(v \in V(G)\). By the fact that \(U\) is a Z-solution, we have \(Z(v) \subseteq V(H_G) \setminus U\). Since \(Z(v) \subseteq S\), the cycles \(\{y^j_v, z^j_u\}, j \in [2n+1]\) are \(2n+1\) vertex-disjoint \(S\)-cycles. Therefore, the number of vertices of \(Y(v) \cup \overline{Y}(v)\) in \(U\) must be at least \(2n + 1\). \(\)

**Claim 19.3.** For every tier-1 sfvs \(U\), there is a tier-2 sfvs \(U'\) such that \(|U'| \leq |U|\).

**Proof:** We consider the set \(X(v)\) for a vertex \(v \in V(G)\). Assume that there are vertices \(x \in X(v) \cap U\) and \(x' \in X(v) \cap (V(H_G) \setminus U)\). Observe that \(N(x') \setminus \{x\} \cap (V(H_G) \setminus U)\). Consider any two vertices \(w, w' \in Y(v) \cup E(v)\). Since \(x' \in S\) and \(Y(v) \cup E(v)\) induces a clique, we have that at most one of \(w, w'\) is in \(V(H_G) \setminus U\). We construct the set \(U' = (U \setminus X(v)) \cup (Y(v) \cup E(v))\). Notice that \(|X(v) \cap U | \geq 1\) and \(|\{Y(v) \cup E(v)\} \cap U | \leq 1\), so that \(|U'| \leq |U|\). Thus \(Y(v) \cup E(v) \subseteq U'\) and \(X(v) \setminus U' = \emptyset\). Moreover the constructed set \(U'\) is indeed a tier-1 sfvs. This follows from the fact that the subgraph induced by the vertices of \(X(v)\) is acyclic and \(N(X(v)) \subseteq U'\). With completely symmetric arguments, there is a tier-1 sfvs \(U''\) such that either \(X(v) \subseteq U''\) or \(X(v) \cap U'' = \emptyset\) holds, and \(|U''| \leq |U'|\). Iteratively applying these arguments for each \(v \in V(G)\), we obtain a tier-2 sfvs \(U''\) such that \(|U'| \leq |U|\). \(\)

**Claim 19.4.** For every tier-2 sfvs \(U\), there is a tier-3 sfvs \(U'\) such that \(|U'| \leq |U|\).

**Proof:** Consider a vertex \(v \in V(G)\). By the fact that \(U\) is a tier-2 sfvs, exactly one of the following holds:

1. \(X(v) \cup \overline{X}(v) \subseteq V(H_G) \setminus U\)
2. \(X(v) \subseteq U\) and \(\overline{X}(v) \subseteq V(H_G) \setminus U\)
3. \(X(v) \subseteq V(H_G) \setminus U\) and \(\overline{X}(v) \subseteq U\)
4. \(X(v) \cup \overline{X}(v) \subseteq U\)
Assume that (1) holds. Then \( Y(v) \cup \overline{Y}(v) \subseteq U \) must hold, so the set \( U' = (U \setminus Y(v)) \cup X(v) \) is also a tier-2 sfvs and additionally \( X(v) \cup \overline{Y}(v) \subseteq U' \) and \( \overline{X}(v) \cup Y \subseteq V(H_G) \setminus U' \) hold. Since \( |Y(v)| = |X(v)| + 1 \), it follows that \(|U'| < |U|\).

Now assume that (2) holds. Then \( \overline{Y}(v) \subseteq U \) must hold, so the set \( U' = U \setminus Y(v) \) is also a tier-2 sfvs and additionally \( X(v) \cup \overline{Y}(v) \subseteq U' \) and \( \overline{X}(v) \cup Y \subseteq V(H_G) \setminus U' \) hold. For the case where (3) holds, completely symmetrical arguments yield that \( U' = U \setminus \overline{Y}(v) \) is also a tier-2 sfvs and additionally \( \overline{X}(v) \cup Y \subseteq U' \) and \( X(v) \cup \overline{Y}(v) \subseteq V(H_G) \setminus U' \) hold.

Lastly assume that (4) holds. By Claim 19.2 we have \( |Y(v) \cup \overline{Y}(v)| \leq 2n + 1 \). Without loss of generality, assume that \( |\overline{Y}(v) \setminus U| \leq n \). Then the set \( U' = (U \setminus \overline{X}(v)) \cup (\overline{Y}(v) \cup \overline{E}(v)) \) is also a tier-2 sfvs and additionally \( X(v) \cup \overline{Y}(v) \subseteq U' \) and \( \overline{X}(v) \cup Y \subseteq V(H_G) \setminus U' \) hold. Since \( |\overline{X}(v)| = 2n > |(\overline{Y}(v) \cup \overline{E}(v)) \setminus U| \), it follows that \(|U'| < |U|\).

Iteratively applying the arguments stated in the appropriate case above for each \( v \in V(G) \), we obtain a tier-3 sfvs \(|U'|\) such that \(|U'| \leq |U|\).

**Claim 19.5.** For every tier-3 sfvs \( U \), there is a tier-4 sfvs \( U' \) such that \(|U'| \leq |U|\).

**Proof:** Let \( U \) be a tier-3 sfvs. Clearly \((A_U, \overline{A}_U)\) is a partition of \( V(G) \). Consider a vertex \( e \in \overline{E} \setminus \overline{E}(A_U, \overline{A}_U) \). Then \( e \in E(v) \cap \overline{E}(v') \) for some \( v, v' \in V(G) \) such that \( v \notin A_U \) or \( v' \notin \overline{A}_U \), which implies that \( X(v) \subseteq V(H_G) \setminus U \) or \( \overline{X}(v') \subseteq V(H_G) \setminus U \). Since \((x^n, x^n, e)\) and \((x^n, x^n, e)\) are \( S \)-cycles, it follows that \( e \) must be in \( U \). Now consider a vertex \( e \in \overline{E}(A_U, \overline{A}_U) \). Then \( e \in E(v) \cap \overline{E}(v') \) for some \( v \in A_U \) and some \( v' \in \overline{A}_U \), which implies that \( N(e) \cap S = X(v) \cup \overline{X}(v') \subseteq U \). It follows that the set \( U' = U \setminus \overline{E}(A_U, \overline{A}_U) \) is a tier-4 sfvs.

To conclude our proof, let \( U \) be an tier-4 sfvs such that \(|U| \leq 4n^2 + n + 2m - k\) that exists by Claim 19.5. By definition, for any \( v \in V(G) \) exactly one of the following holds:

- \((X(v) \cup \overline{Y}(v)) \setminus U = \emptyset \) and \((\overline{X}(v) \cup Y(v)) \cap U = \emptyset \)
- \((\overline{X}(v) \cup Y(v)) \setminus U = \emptyset \) and \((X(v) \cup \overline{Y}(v)) \cap U = \emptyset \).

This yields \(|U \cap (X \cup \overline{X} \cup Y \cup \overline{Y} \cup Z)| = 4n^2 + n\), so that \(|U \cap \overline{E}| \leq 2m - k\). Therefore we deduce that \(|\overline{E}(A_U, \overline{A}_U)| \geq k\), which means that \( A_U \) provides a desired cut-set in \( G \).

\[ \square \]

### 7 Concluding Remarks

We provided a systematic and algorithmic study towards the classification of the complexity of **Subset Feedback Vertex Set** on subclasses of chordal graphs. We considered the structural parameters of leafage and vertex leafage as natural tools to exploit insights of the corresponding tree representation. Our proof techniques revealed a fast algorithm for the class of rooted path graphs. Naturally, it is interesting to settle whether the unweighted **Subset Feedback Vertex Set** problem is **FPT** when parameterized by the leafage of a chordal graph. Towards this direction, it is likely that the unweighted and weighted variants of the problem behave computationally different, as occurs in other cases [7, 33]. We also believe that our NP-hardness proof on undirected path graphs carries along the class of directed path graphs which are the intersection graphs of directed paths taken from an oriented tree (i.e., the underlying undirected graph is a tree). Further, in order to have a more complete picture on the behavior of the problem on subclasses of chordal graphs, strongly chordal graphs seems a candidate family of chordal graphs as they are incomparable to (vertex) leafage.

Moreover it would be interesting to consider the close related problem **Subset Odd Cycle Transversal** in which the task is to hit all odd \( S \)-cycles. Preliminary results indicate that the two problems align on particular hereditary classes of graphs [6, 7]. As a byproduct, it is notable that all of our results for **Subset Feedback Vertex Set** are still valid for **Subset
**Odd Cycle Transversal**, as any induced cycle is an odd induced cycle (triangle) in chordal graphs. More generally, an interesting direction for further research along the leafage is to consider induced path problems that admit complexity dichotomies on interval graphs and split graphs, respectively [2, 23, 24, 30].

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