THE CONIFOLD POINT

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Abstract. Consider a Laurent polynomial with real positive coefficients such that the origin is strictly inside its Newton polytope. Then it is strongly convex as a function of real positive argument. So it has a distinguished Morse critical point — the unique critical point with real positive coordinates.

As a consequence we obtain a positive answer to a question of Ostrover and Tyomkin: the quantum cohomology algebra of a toric Fano manifold contains a field as a direct summand. Moreover, it gives a good evidence that the same statement holds for any Fano manifold.

Let \( W \) be a Laurent polynomial of variables \( x_1, \ldots, x_d \) with complex coefficients \( a_n \in \mathbb{C} \):

\[
W = \sum_{n \in \mathbb{Z}^d} a_n x^n = \sum_{(n_1, \ldots, n_d) \in \mathbb{Z}^d} a_{n_1, \ldots, n_d} x_1^{n_1} x_d^{n_d}.
\]

Recall that the Newton polytope \( \Delta_W \) of a Laurent polynomial \( W \) is defined as the convex hull in \( \mathbb{R}^d \) of \( n \in \mathbb{Z}^d \) such that \( a_n \neq 0 \). Assume further that \( \Delta_W \) is \( d \)-dimensional and the origin 0 lies strictly inside \( \Delta_W \). In [6] Duistermaat and van der Kallen proved that there are infinitely many natural \( k \) such that \( k \)-th moments \( M_k(W) \) (defined as the constant terms of a Laurent polynomial \( W^k \)) does not vanish. Moreover, they proved that the generating function \( \hat{G}_W(t) = \sum_{k \geq 0} t^k M_k(W) \) has finite radius of convergence \( 0 < R < \infty \), and function \( \hat{G}_W(t) \) has logarithmic monodromy around some point \( t_0 \) with \( |t_0| = R \). They have to use Hironaka’s resolution of singularities in order to prove these theorems. In this note we restrict to the case where coefficients are real and non-negative \( a_n \in \mathbb{R}_+ \). This condition greatly simplifies the picture, existence of infinitely many non-vanishing moments is almost obvious, and additionally there is a very distinguished critical point \( W \) that is non-degenerate.

Let \( u_1, \ldots, u_d \) be the coordinates on \( \mathbb{C}^d \), the analytic map \( x_i = \exp(u_i) \) is an \( \mathbb{Z}^d \)-covering \( \exp : \mathbb{C}^d \to (\mathbb{C}^*)^d \). For a pullback

\[
\exp^* W = \sum_{n \in \mathbb{Z}^d} a_n e^{(u,n)} = \sum_{(n_1, \ldots, n_d) \in \mathbb{Z}^d} a_{n_1, \ldots, n_d} e^{(u_1 n_1 + \cdots + u_d n_d)}
\]

the partial derivation \( \frac{\partial}{\partial u_i} \) coincides with the pullback of the logarithmic derivation \( x_i \frac{\partial}{\partial u_i} \). Note that \( \exp \) establishes an isomorphism between the domain \( \mathbb{R}^d \subset \mathbb{C}^d \) (where all \( u_i \) are real) and the domain \( \mathbb{R}_+^d \subset (\mathbb{C}^*)^d \) (where all \( x_i \) are real and positive), in what follows we consider these identified domains as a topological manifold \( T_+ \).

Lemma. Under the assumptions above:

1. The Hessian matrix \( H_{ij} = \frac{\partial^2 W}{\partial u_i \partial u_j} \) is positive-semidefinite on \( T_+ \), so \( W : T_+ \to \mathbb{R} \) is a convex function.
2. Moreover \( H_{ij} \) is positive-definite on \( T_+ \), so \( W : T_+ \to \mathbb{R} \) is strictly convex.
3. Moreover, the function \( W : T_+ \to \mathbb{R} \) is strongly convex and attains a global minimum at some point \( P \in T_+ \).
4. Point \( P \) is the unique critical point of \( W \) in domain \( T_+ \), i.e. \( dW_{u=u_0} = 0 \iff u_0 = P \) for any \( u_0 \in T_+ \).
5. Point \( P \) is Morse i.e. the Hessian matrix of \( W \) at \( P \) is non-degenerate.

In what follows we’ll refer to \( P \) as the conifold point.

Proof. Clearly, partial derivatives of \( W \) are given by

\[
\frac{\partial W}{\partial u_i} = \sum_n n_i a_n x^n
\]

and

\[
H_{ij} = \frac{\partial^2 W}{\partial u_i \partial u_j} = \sum_n n_i n_j a_n x^n.
\]
Proof. By Proposition 3.3 of [16] the algebra $X, \omega$ contains a field as a direct summand. Nevertheless, for toric Fano manifolds $X$ be a toric Fano manifold with a toric symplectic form $\omega$. Then the small quantum cohomology algebra $QH(X, \omega)$ coincides with the Jacobi ring of a particular combinatorially constructed Laurent polynomial $W_X, \omega$. Moreover, as explained in Subsection 3.3 of loc.cit. to prove the Theorem for any $\omega$ it would suffice to consider the monotone case (i.e. $[\omega] = c_1(X)$). In the monotone case the respective Laurent polynomial equals $W_X = \sum v^n$, here $v$ runs over all primitive generators of all rays of the fan of $X$, and $x^n$ is the respective monomial. Clearly, $W_X$ satisfies the conditions of the Lemma, so $W_X$ has a non-degenerate critical point $P$; point $P$ contributes a field as a direct summand of the Jacobi ring $J_W$.

Theorem (about toric Fano manifolds). Let $X$ be a toric Fano manifold with a toric symplectic form $\omega$. Then the small quantum cohomology algebra $QH(X, \omega)$ contains a field as a direct summand.

Proof. By Proposition 3.3 of [16] the algebra $QH(X, \omega)$ coincides with the Jacobi ring

$$J_W = C[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]/(\frac{dW}{dx_1}, \ldots, \frac{dW}{dx_d})$$

of a particular combinatorially constructed Laurent polynomial $W_X, \omega$. Moreover, as explained in Subsection 3.3 of loc.cit. to prove the Theorem for any $\omega$ it would suffice to consider the monotone case (i.e. $[\omega] = c_1(X)$). In the monotone case the respective Laurent polynomial equals $W_X = \sum v^n$, here $v$ runs over all primitive generators of all rays of the fan of $X$, and $x^n$ is the respective monomial. Clearly, $W_X$ satisfies the conditions of the Lemma, so $W_X$ has a non-degenerate critical point $P$; point $P$ contributes a field as a direct summand of the Jacobi ring $J_W$.

The Theorem above gives a positive answer to a question of Ostrover–Tyomkin [16]. Their question was in turn raised as a modification of a question of Entov–Polterovich [8], for which Proposition B of [16] gives a negative answer: there are monotone symplectic toric Fano 4-folds $(\gamma)$ such that algebra $QH(X, \omega)$ is not semi-simple. Nevertheless, for toric Fano manifolds $X$ the algebra $QH(X, \omega)$ is semi-simple for a generic choice of $\omega$: see Corollary 5.12 in [13], Proposition 7.6 in [9] and Theorem A in [16]. For non-toric Fano manifolds even generic semi-simplicity usually fails due to strong homological obstructions: Theorem 1.8.1 of [3] and Theorem 1.3 of [11] imply that if $QH(X, \omega)$ is semi-simple then $h^{p,q}(X) = 0$ unless $p = q$. In contrast, next Remark aims to explain that the analogue of the Theorem should also hold for non-toric symplectic Fano manifolds.

Remark regarding non-toric Fano manifolds. The argument combines pictures of SYZ (Strominger-Yau-Zaslow [14]) and HMS (homological mirror symmetry [14]) with ideas of [12, 4, 9], and very sketchy it goes as follows (see [2, 13] for at least some details). Start from an arbitrary (non necessarily toric) Fano manifold $Y$. Degenerate $Y$ to a (singular) toric Fano variety $X_0$. The moment map $\mu : X_0 \to B^0$ gives a special Lagrangian tori fibration over the interior $\mu : X_0^0 \to B^0$. The symplectic transport (given by the distribution of the orthogonal of the fibers of degeneration with respect to symplectic form) establishes a symplectomorphism between $X_0^0$ and an open subset in $Y$ thus producing many special Lagrangian tori $L_b \subset Y$. The potential $W$ is then constructed as Fukaya-Oh-Ohta-Ono’s obstruction $m_0(L_b, \nabla)$ for a (monotone) fiber $L_b$ equipped with a flat $U(1)$-connection $\nabla$, that is a generating function for Maslov index two pseudoholomorphic discs bounded on $L_b$. The case of a smooth toric Fano manifold $Y$ is explicitly computed in [4], and the case of small degenerations in [15]. The positivity of coefficients of $W$ is the geometrically evident fact, that sometimes could be proved, e.g. if there are no pseudoholomorphic discs of negative Maslov index, or if those discs are away from $L_b$. Thus by Lemma there is the conifold point $P$. Section 6 of [2] and [9] explains how to identify the respective field summand of the Jacobi ring with a subalgebra of $QH(Y)$. In particular, Theorem 6.1 of [2] ensures that in smooth toric case, the set of all critical values of $W$ coincides with the set of eigenvalues of the quantum multiplication operator $*_{c_1}(Y) : QH(Y)$.

The Theorem and the Remark above help to partially resolve the following conjecture of [16] (Section 3.1). For a Fano manifold $Y$ consider the set $U_Y$ of all eigenvalues $u_i$ of the quantum multiplication operator $*_{c_1}(Y)$, denote $T = \max |u_i|$. The Conjecture $\mathcal{O}$ says: $T$ lies in $U_Y$, for any $u_i \in U_Y$ if $|u_i| = T$ then $u_i/T$ is a root of unity, and the multiplicity in $U_Y$ of the eigenvalue $T$ equals one.

We formulated it together with Hiroshi Iritani and Vasily Goryachev. It appeared as a part of our investigation of the Gamma-conjectures (the relation between the (asymptotic) Apery class, the Gamma class, and the integral structure in quantum cohomology), which turned out to be a close relative of Conjecture 4.2.2(3) of Dubrovin [5].
Number $T_Y$ is a real positive algebraic integer which can be considered as a symplectic invariant of a monotone Fano manifold $Y$. The Lemma easily implies that if $Y$ is a toric Fano manifold, then $T_Y$ is bounded from above by $\dim Y + b_2(Y)$. Non-toric Fano manifolds usually have $T_Y$ (high) above this bound. On the other hand, we were not able to prove any lower bound for $T_Y$, even in the toric case. A plausible conjecture for the lower bound is $T_Y \geq \dim Y + 1$, with equality only for the projective space $Y \simeq \mathbb{P}^d$.

Relation to other work. The conifold point is explicitly constructed in the proof (due to Iritani) of Proposition 12.3 of [15].

Apparently it was observed by van Enckevort and van Straten as the Hypothesis 1 (H2) of [7]. In some particular cases, such as for the mirror dual family of quintic threefolds the terminology for the conifold point is well-established and its origin was not questioned.

The numerical conjecture $\mathcal{O}$ above constrains the geometry of the set $U_Y \subset \mathbb{C}$ of the critical values, but does not say much about the set of the critical points. In case $W$ is mirror dual to a Fano manifold $Y$ we expect that $P$ is the unique singular point in the fiber $W^{-1}(W(P))$. In contrast, such uniqueness sometimes fails for orbifolds, e.g. for the global quotient $\mathbb{P}^2/(\mathbb{Z}/3\mathbb{Z})$ the mirror dual is $W = \frac{x^2}{x^2} + \frac{z^2}{z^2} + \frac{1}{x^2}z^2$, and it has exactly three conifold points over each of its three critical values.

In the discussion above we relate the $A$-model (quantum cohomology and other symplectic invariants) of the Fano manifold $Y$ to the $B$-model (Jacobi ring which can be thought as an algebro-geometric invariant) of the mirror-dual potential $W$. Another direction of HMS relates the $B$-model of $Y$ (e.g. the bounded derived category of coherent sheaves $\mathcal{D}^{b}_{coh} Y$) to the $A$-model of $W$ (the wrapped Fukaya category, or Fukaya–Seidel category of vanishing Lagrangian cycles). Flip the direction of HMS to obtain the conjecture that explains the terminology: the structure sheaf $\mathcal{O}_Y$ (as an exceptional object in $\mathcal{D}^{b}_{coh} Y$) corresponds under HMS to the real positive locus $T_\mathcal{L} \subset (\mathbb{C}^*)^d$ (considered as a Lefschetz thimble). In this form we heard it from Denis Auroux. Now folklore Conjecture 1 in [1] is its relative, it can be deduced by combining SYZ, HMS, and the fact that the equality $\text{RHom}(\mathcal{L}, \mathcal{O}_{\mathcal{L}}) = \mathcal{C}(0)$ for any structure sheaf $\mathcal{O}_Y$ of a point $y \in Y$ implies that $\mathcal{L}$ is a line bundle.

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References

[1] Mohammed Abouzaid: Morse Homology, Tropical Geometry, and Homological Mirror Symmetry for Toric Varieties, arXiv:math/0610004, Selecta Mathematica, 15(2), 189-270.
[2] Denis Auroux: Mirror symmetry and $T$-duality in the complement of an anticanonical divisor, arXiv:0706.3207, J. Gökova Geom. Topol. 1 (2007) 51–91.
[3] Arend Bayer, Yuri Manin: (Semi)simple exercises in quantum cohomology, arXiv:math/0103164, The Fano Conference, Univ. Torino, Turin, 2004, 143–173.
[4] Cheol-Hyun Cho, Yong-Geun Oh: Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds, arXiv:0908225, Asian J. Math. Vol. 10, No. 4 (2006), 773–814.
[5] Boris Dubrovin: Geometry and analytic theory of Frobenius manifolds, arXiv:math/980734, Proceedings of the International Congress of Mathematicians (Vol. 2, pp. 315–326).
[6] J.J. Duistermaat, Wilberd van der Kallen: Constant terms in powers of a Laurent polynomial, Indagationes Mathematicae, Volume 9, Issue 2, 15 June 1998, Pages 221–231.
[7] Christian van Enckevort, Duco van Straten: Monodromy calculations of fourth order equations of Calabi–Yau type, arXiv:math/0412539 in “Mirror Symmetry V”, the BIRS Proc. on Calabi–Yau Varieties and Mirror Symmetry, AMS/IP.
[8] Michael Entov, Leonid Polterovich: Symplectic quasi-states and semi-simplicity of quantum homology, arXiv:0705.3735.
[9] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono: Lagrangian Floer theory on compact toric manifolds I, arXiv:0802.1703.
[10] Maxim Kontsevich: Homological algebra of mirror symmetry, arXiv:alg-geom/9411018. Proceedings of the International Congress of Mathematicians, Vol. 1, 2, pp. 120–139. Lectures at ENS Paris, Spring 1998.
[11] Vasily Golyshev: Toric degenerations of Gelfand–Cetlin systems and potential functions, arXiv:0810.3470, Advances in Mathematics 224.2 (2010): 648–706.
[16] Yaron Ostrover, Ilya Tyomkin: *On the quantum homology algebra of toric Fano manifolds*, arXiv:0804.0270, Selecta Math. (N.S.) 15 (2009), no. 1, 121–149.

[17] Andrew Strominger, Shing-Tung Yau, Eric Zaslow: *Mirror Symmetry is T-Duality*, arXiv:hep-th/9606040, Nucl.Phys.B479:243–259,1996.

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