Zero energy bound states in many-particle systems

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Received 17 April 2012, in final form 20 August 2012
Published 11 September 2012
Online at stacks.iop.org/JPhysA/45/395302

Abstract

It is proved that the eigenvalues in the $N$-particle system are absorbed at the zero-energy threshold, if none of the subsystems has a bound state with $E \leq 0$ and none of the particle pairs has a zero-energy resonance. The pair potentials are allowed to take both signs.

PACS numbers: 03.65.Ge, 03.65.Db, 21.45.–v, 67.85.–d, 02.30.Tb

1. Introduction

In [1], it was proved that the three-body system, which is at the three-body coupling constant threshold, has a square integrable state at zero energy if none of the two-body subsystems is bound or has a zero-energy resonance. The condition on the absence of two-body zero-energy resonances is essential, that is, the three-body ground state at zero energy can be at most a resonance and not an $L^2$ state if at least one pair of particles has a zero-energy resonance [1]. One of the restrictions on pair potentials in [1] was their being non-positive. The aim of this paper is to generalize the result of [1] to the case of many particles and get rid of the restriction on the sign of pair potentials. The main result is expressed in theorems 1 and 2, which state that the eigenvalues in the $N$-particle system are absorbed at the zero-energy threshold, if none of the subsystems has a bound state with $E \leq 0$ and none of the particle pairs has a zero-energy resonance.

In physics there are numerous examples when systems exhibit a large spatial extension, when they are bound in the ground state but have a small binding energy. One could mention neutron halos in nuclear physics, Rydberg and Efimov states, etc [2–4]. In particular, the case of Borromean nuclei [2, 3] is interesting, where two neutrons produce a dilute density distribution around a tightly bound nuclear core, thereby forming the so-called halo; see also [1]. Under the halo one understands [2] that more than 50% of the wavefunction is located in the classically forbidden region. The present results help us understand the conditions, which would allow large size objects (like multi-neutron halos). A large spatial extension...
in a bound multi-particle system with the zero dissociation threshold can only be achieved when the energy of the system is close to the threshold and the energies of some subsystems are resonant (in the sense that some of the subsystems are ‘almost bound’). Otherwise, the wavefunction of the system would form a bound state at the threshold and would not totally spread in space; see [1].

Throughout the paper we use the following operator notation. $A \geq 0$ means that $(f, Af) \geq 0$ for all $f \in D(A)$, and $A \not\geq 0$ means that there exists $f_0 \in D(A)$ such that $(f_0, Af_0) < 0$.

We consider the $N$-particle Schrödinger operator

$$H(\lambda) = H_0 + \lambda \sum_{1 \leq i < j \leq N} V_{ij}(r_i - r_j), \quad (1)$$

where $\lambda > 0$ is a coupling constant, $H_0$ is a kinetic energy operator with the centre of mass removed, $r_i \in \mathbb{R}^3$ are particle position vectors; the pair potentials are real and $V_{ij} \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. The operator $H(\lambda)$ is self-adjoint on $D(H_0) \subset L^2(\mathbb{R}^{3N-3})$, and the set of relative coordinates in $\mathbb{R}^{3N-3}$ we shall denote as $\xi$. Throughout the paper, we shall assume that

$$\sigma_{\text{ess}}(H(\lambda)) = [0, \infty). \quad (2)$$

which, of course, restricts possible values of $\lambda$. Here we shall extensively use the term critical coupling. In the literature, one finds several related definitions: critically bound [5], critical coupling [6], coupling constant threshold [7, 8], the virtual level at the threshold [9], etc. To avoid possible confusion, we list three of the most popular definitions and indicate the relations between them.

**Definition 1.** $H(\lambda)$ is at critical coupling if $H(\lambda) \geq 0$ and $H(\lambda) + \epsilon \sum_{i < j} V_{ij} \not\geq 0$ for any $\epsilon > 0$.

In the terminology of [7, 8], definition 1 implies that $H(\lambda)$ is at the coupling constant threshold. The next definition due to scaling arguments is fully equivalent to definition 1.

**Definition 2.** $H(\lambda)$ is at critical coupling if $H(\lambda) \geq 0$ and $H(\lambda) - (1 - \epsilon)H_0 \not\geq 0$ for any $0 < \epsilon < 1$.

So under the term critical coupling we shall mean any of definitions 1 and 2. The next definition can be found, for example, in [9].

**Definition 3.** $H(\lambda)$ is said to have a virtual level at zero energy if $H(\lambda) \geq 0$ and $H(\lambda) - \epsilon V_R \not\geq 0$ for any $\epsilon > 0$, where $V_R := 1/(1 + |\xi|^2)$.

In the case of $N = 2$, it is easy to show that all three definitions are equivalent to the definition of a two-particle zero-energy resonance, cf [10, 11]. Note that, in general, for $N \geq 3$ definitions 1–3 are not equivalent. The difference lies in the fact that the perturbation in definition 3 does not move the lower bound of the essential spectrum, since $V_R$ is a relatively $H_0$-compact perturbation, contrary to the perturbations in definitions 1 and 2, where the lower bound of the essential spectrum can be moved, if some of the subsystems are at critical coupling.

**Proposition 1.** A system of $N$ particles is at critical coupling if it has a virtual level at zero energy.
Proof. Suppose that the system is not at critical coupling. Then there must exist \( \epsilon_0 > 0 \), such that \( H - \epsilon_0 H_0 \geq 0 \). By the Courant identity \([12, 14]\) there exists \( \kappa > 0 \), such that \( H_0 - \kappa V_R \geq 0 \). Hence,

\[
H - \epsilon_0 \kappa V_R = H - \epsilon_0 H_0 + \epsilon_0 (H_0 - \kappa V_R) \geq 0,
\]

which means that the system does not have a virtual level at zero energy. \( \square \)

As already mentioned, the converse of proposition 1 is in general not true. Note, however, that if a system has a zero-energy bound state, then it automatically has a virtual level at zero energy.

2. Main result

For the formulation of theorem 1, we need to impose the following requirement.

There exists a sequence of coupling constants \( \lambda_n \in \mathbb{R}_+ \) such that \( \lim_{n \to \infty} \lambda_n = \lambda_{cr} \in \mathbb{R}_+ \), and \( H(\lambda_n)\psi_n = E_n\psi_n \), where \( \psi_n \in D(H_0) \), \( \| \psi_n \| = 1 \), \( E_n < 0 \), \( \lim_{n \to \infty} E_n = 0 \).

Further in this section we shall prove the following.

**Theorem 1.** Suppose \( H(\lambda) \) defined in \((1)\) for \( N \geq 3 \) satisfies R1, \( H(\lambda_{cr}) \) and \( H(\lambda_{cr}) \) have no subsystems, which have a bound state with \( E \leq 0 \), and no particle pairs at critical coupling. Then there exists normalized \( \psi_0 \in D(H_0) \) such that \( H(\lambda_{cr})\psi_0 = 0 \).

The next statement can be considered as a corollary to theorem 1.

**Theorem 2.** Suppose that \( N \geq 3 \) and \( H(\lambda_{cr}) \) is at critical coupling. Suppose also that \( H(\lambda_{cr}) \) has no subsystems, which have a bound state with \( E \leq 0 \), and no particle pairs at critical coupling. Then there exists normalized \( \psi_0 \in D(H_0) \) such that \( H(\lambda_{cr})\psi_0 = 0 \).

**Proof.** Let us assume that none of the subsystems is at critical coupling. On one hand, from the HVZ theorem \([13, 14]\) it follows that there exists \( \epsilon_0 > 0 \) such that for \( \lambda_n = \lambda_{cr}(1 + \epsilon_0/n) \) and \( n = 1, 2, \ldots \) we have \( \inf \sigma_{eq}(H(\lambda_n)) = 0 \). We also choose \( \epsilon_0 \) small enough to guarantee that \( H(\lambda_n) \) has no subsystems that are either bound or at critical coupling. On the other hand, \( H(\lambda_n) \notin 0 \). Therefore, there are \( \psi_n \in D(H_0) \) such that \( H(\lambda_n)\psi_n = E_n\psi_n \), where \( E_n < 0 \), \( \| \psi_n \| = 1 \) and \( E_n \to 0 \). Now the statement follows from theorem 1. It remains to get rid of the assumption that there are no subsystems at critical coupling. If there would be such case, then it is always possible to pass to the corresponding subsystem (call it \( \mathcal{S} \)).

Following \([1]\) let us introduce the operator \( B_{\tau_1 \tau_2}(z) \), where \( 1 \leq \tau_1 < \tau_2 \leq N \). We construct \( B_{12}(z) \); for other particle pairs the construction is analogous.

We use Jacobi coordinates \([15]\) \( z = (x, y_1, y_2, \ldots, y_{N-2}) \), where \( x, y_i \in \mathbb{R}^3 \). We set \( x = \alpha^{-1}(r_2 - r_1) \) and \( y_1 = (\sqrt{2M_1} \alpha)(r_3 - m_1/(m_1 + m_2)r_1 - m_2/(m_1 + m_2)r_2) \), where \( \alpha := h/\sqrt{2M_1} \), \( M_{12} := (m_1 + m_2)m_3/(m_1 + m_2 + m_3) \) and \( \mu_1 := m_1m_3/(m_1 + m_2) \) is the reduced mass. For \( N = 4 \), this choice of coordinates is illustrated in figure 1 (left). The coordinate \( y_i \in \mathbb{R}^3 \) is proportional to the vector pointing from the centre of mass of the particles \([1, 2, \ldots, i + 1]\) to the particle \( i + 2 \), and the scale is set to make the kinetic energy operator take the form

\[
H_0 = -\Delta_x - \sum_i \Delta_{y_i}.
\]
Let $\mathcal{F}_{12}$ denote the partial Fourier transform in $L^2(\mathbb{R}^{3N-3})$ acting as follows:

$$\hat{f}(x, p_y) = \mathcal{F}_{12} f = \frac{1}{(2\pi)^{3N-6}/2} \int d^{3N-6}\tilde{y} e^{-ip_y \cdot \tilde{y}} f(x, y),$$

(5)

where $y = (y_1, \ldots, y_{N-2})$, $p_y = (p_{y_1}, p_{y_2}, \ldots, p_{y_{N-2}}) \in \mathbb{R}^{3N-6}$. Then $B_{12}(z)$ is defined through

$$B_{12}(z) = 1 + z + \mathcal{F}_{12}^{-1} t(p_y) \mathcal{F}_{12},$$

(6)

where

$$t(p_y) = (\sqrt{|p_y|} - 1) 1_{|p_y| \leq 1},$$

(7)

$|p_y| = (\sum p_y^2)^{1/2}$ and $1_{\Omega}$ denotes the characteristic function of the set $\Omega$. Let us transform the coordinates through $y_1 = \sum \xi \delta x_1$, where $T_{\delta x_1}$ is any orthogonal $(N-2) \times (N-2)$ matrix. It is easy to check that the construction of $B_{12}(z)$ is invariant with respect to these coordinate transformations. That is,

$$B_{12}(z) = 1 + z + \mathcal{F}_{12}^{-1} t(\tilde{p}_y) \mathcal{F}_{12},$$

(8)

where $\mathcal{F}_{12}$ is defined through

$$\hat{f}(x, \tilde{p}_y) = \mathcal{F}_{12} f = \frac{1}{(2\pi)^{3N-6}/2} \int d^{3N-6}\tilde{y} e^{-i\tilde{p}_y \cdot \tilde{y}} f(x, \tilde{y}).$$

(9)

Similarly, one defines $B_{r_1 r_2}(z)$ for all particle pairs. $B_{r_1 r_2}(z)$ and $B_{r_1 r_2}^{-1}(z)$ are analytic on Re $z > 0$.

**Proof of theorem 1.** By contradiction, let us assume that the zero-energy bound state does not exist. Then by theorem 1 in [1], $\psi_n$ totally spreads and $\psi_n \to 0$ (for the definition of spreading see [1]). Let $r = 1, 2, \ldots, N(N-1)/2$ for $N \geq 4$ label all particle pairs and $r_1 < r_2$ label the particle numbers entering the pair $r$. We shall denote $v_r := V_{r_1 r_2}$. It is helpful to split $v_r$ into the positive and negative parts $v_r = (v_r)_+ - (v_r)_-$, where $(v_r)_+ := \max[0, v_r]$ and $(v_r)_- := \max[0, -v_r]$. On one hand, the Schrödinger equation for $\psi_n$ reads

$$(H_0 + \lambda_n U_+ + k_+^2) \psi_n = \lambda_n \sum_r (v_r)_+ \sqrt{(v_r)_-(v_r)_+} \psi_n,$$

(10)

where we set

$$U_+ := \sum_r (v_r)_+.$$

(11)
Acting on the last equation with an inverse operator gives
\[
\psi_n = \lambda_n \sum \frac{1}{H_0 + \lambda_n U_+ + k_n^2} \sqrt{(v_\tau)_-} \left[ H_0 + \lambda_n (v_\tau)_+ + k_n^2 \right]^{-1} \psi_n.
\] (12)

On the other hand, we can rearrange the terms in the Schrödinger equation as follows:
\[
[H_0 + \lambda_n (v_\tau)_+ + k_n^2] \psi_n = \lambda_n (v_\tau)_- \psi_n - \lambda_n \sum_{\delta \neq \tau} v_\delta \psi_n,
\] (13)

where index \(\delta\) runs through all particle pairs. This gives us
\[
\psi_n = \lambda_n \left[ H_0 + \lambda_n (v_\tau)_+ + k_n^2 \right]^{-1} (v_\tau)_- \psi_n = \lambda_n (v_\tau)_- \psi_n - \lambda_n \sum_{\delta \neq \tau} \sqrt{(v_\tau)_-} \left[ H_0 + \lambda_n (v_\tau)_+ + k_n^2 \right]^{-1} v_\delta \psi_n.
\] (14)

Using (14), we obtain the following expression for the last term in brackets in (12):
\[
\sqrt{(v_\tau)_-} \psi (\lambda_n) = -\lambda_n \sum_{\delta \neq \tau} \left[ 1 - \lambda_n \sqrt{(v_\tau)_-} [H_0 + \lambda_n (v_\tau)_+ + k_n^2]^{-1} \sqrt{(v_\tau)_-} \right]^{-1}
\times \sqrt{(v_\tau)_-} \left[ H_0 + \lambda_n (v_\tau)_+ + k_n^2 \right]^{-1} v_\delta \psi_n.
\] (15)

The fact that the inverse of the operator in curly brackets makes sense would be shown in lemma 1 below. Substituting (15) into (12) yields the equation
\[
\psi_n = -\lambda_n \sum_{\delta \neq \tau} (H_0 + \lambda_n U_+ + k_n^2)^{-1} \sqrt{(v_\tau)_-} \left[ 1 - \lambda_n \sqrt{(v_\tau)_-} [H_0 + \lambda_n (v_\tau)_+ + k_n^2]^{-1} \sqrt{(v_\tau)_-} \right]^{-1}
\times \sqrt{(v_\tau)_-} \left[ H_0 + \lambda_n (v_\tau)_+ + k_n^2 \right]^{-1} v_\delta \psi_n.
\] (16)

All operators under the sum except \(v_\delta\) are positivity preserving, see [13, 14, 16]. The inverse of the operator in curly brackets being positivity preserving can be seen from its expansion in the von Neumann series; see lemma 12 in [1] and lemma 1 of this paper. Thus, we can transform (16) into the following inequality:
\[
|\psi_n| \leq \frac{\lambda_n}{\tau} \sum_{\delta \neq \tau} (H_0 + \lambda_n U_+ + k_n^2)^{-1} \sqrt{(v_\tau)_-} \left[ 1 - \lambda_n \sqrt{(v_\tau)_-} [H_0 + \lambda_n (v_\tau)_+ + k_n^2]^{-1} \sqrt{(v_\tau)_-} \right]^{-1}
\times \sqrt{(v_\tau)_-} \left[ H_0 + \lambda_n (v_\tau)_+ + k_n^2 \right]^{-1} |v_\delta| |\psi_n|.
\] (17)

Note that \((H_0 + \lambda_n U_+ + k_n^2)^{-1}\) and \([H_0 + \lambda_n (v_\tau)_+ + k_n^2]^{-1}\) are the integral operators, see [16], and positivity preserving operators, see, for example, [13] (example 3 from section IX.7 in volume 2 and theorem XIII.44 in volume 4). By resolvent identities
\[
(H_0 + k_n^2)^{-1} - (H_0 + \lambda_n U_+ + k_n^2)^{-1} = \lambda_n (H_0 + \lambda_n U_+ + k_n^2)^{-1} U_+ (H_0 + k_n^2)^{-1},
\] (18)

\[
(H_0 + k_n^2)^{-1} - (H_0 + \lambda_n (v_\tau)_+ + k_n^2)^{-1} = \lambda_n (H_0 + \lambda_n (v_\tau)_+ + k_n^2)^{-1} (v_\tau)_+ (H_0 + k_n^2)^{-1},
\] (19)

the differences on the lhs of (18) and (19) are positivity preserving operators. Therefore, we can rewrite (17) as
\[
|\psi_n| \leq \frac{\lambda_n}{\tau} \sum_{\delta \neq \tau} (H_0 + k_n^2)^{-1} \sqrt{(v_\tau)_-} \left[ 1 - \lambda_n \sqrt{(v_\tau)_-} [H_0 + \lambda_n (v_\tau)_+ + k_n^2]^{-1} \sqrt{(v_\tau)_-} \right]^{-1}
\times [H_0 + k_n^2]^{-1} |v_\delta| |\psi_n|.
\] (20)
We use the notation $B_\tau(z) \equiv B_{\tau}(z)$, where $B_{\tau}(z)$ was defined above. Inserting into (20) the identity $1 = B_\tau(k_0)B_\tau^{-1}(k_0)$ and using that $[B_\tau(k_0), H_0] = 0$ and $[B_\tau(k_0), (v_\tau)_n] = 0$ we obtain

$$|\psi_n| \leq \lambda_n^{2} \sum_{\tau} \sum_{\delta \neq \tau} A_\tau(k_0)R_\tau(k_0)D_{\tau, \delta}(k_0)\sqrt{|v_\delta|}||\psi_n||,$$

(21)

where we defined the operators

$$A_\tau(k_0) := (H_0 + k_0^2)^{-1}\sqrt{(v_\tau)} B_\tau(k_0),$$

(22)

$$R_\tau(k_0) := \{1 - \lambda_n \sqrt{(v_\tau)} - [H_0 + \lambda_n(v_\tau)_+ + k_0^2]^{-1}\sqrt{(v_\tau)}\}^{-1},$$

(23)

$$D_{\tau, \delta}(k_0) := \sqrt{(v_\tau)} [H_0 + k_0^2]^{-1} B_\tau^{-1}(k_0)\sqrt{|v_\delta|} \quad (\tau \neq \delta).$$

(24)

Note that $H(\lambda_{n})$ satisfies the conditions of theorem 3 in the appendix. It is easy to see that $(\psi_n, H(\lambda_{n})\psi_n) \to 0$, where $\psi_n$ totally spreads. Hence, by theorem 3, we have $\|\sqrt{|v_\delta|}\psi_n\| = (\psi_n, |v_\delta|\psi_n) \to 0$. Applying lemmas 1 and 2 to the rhs of (21) tells us that it goes to zero in norm, which is a contradiction, since $|\psi_n| = 1$ by R1.

**Lemma 1.** The operators $A_\tau(k_0)$ and $R_\tau(k_0)$ given by (22) and (23) are uniformly norm bounded.

**Proof.** Without loosing generality, we can consider the pair $\tau = (1, 2)$, where $\tau_1 = 1$ and $\tau_2 = 2$. The proof that $\|A_{12}(k_0)\|$ is uniformly bounded follows the same pattern as the proof of lemma 6 in [1] and we omit it here. The proof for $R_{\tau}(k_0)$ uses the Birman–Schwinger principle in the form suggested in [17]. Note that for self-adjoint operators $A, B \geq 0$, where $A^{-1}$ and $A^{-1/2}B^{1/2}$ are bounded, one has

$$\|A^{-1/2}B A^{-1/2}\| = \|B^{1/2}A^{-1}B^{1/2}\|,$$

(25)

which follows from $\|C^*C\| = \|CC^*\|$ for any bounded $C$, see for example [18]. Due to conditions of theorem 1, there exists $0 < \alpha_0' < 1$ (independent of $n$) such that $(1 - \alpha')H_0 + \lambda_n(v_\tau)_+ \geq 0$, or, equivalently

$$H_0 + \lambda_n(v_\tau)_+ \geq \alpha' H_0.$$  

(26)

By standard estimates, there must exist $\gamma_0 > 0$ such that $H_0 - \gamma_0 \lambda_n(v_\tau)_- \geq 0$ for all $n$. Together with (26), this means that there exists $\omega > 0$ independent of $n$ such that

$$H_0 + \lambda_n(v_\tau)_- - \lambda_n \omega(v_\tau)_- \geq 0.$$  

(27)

To use identity (25), let us set

$$A := H_0 + \lambda_n(v_\tau)_+ + k_0^2,$$

(28)

$$B := \lambda_n(v_\tau)_-.$$  

(29)

Because $A - (1 + \omega)B \geq 0$ for any $\phi \in D(H_0)$ we have

$$(\phi, [A - (1 + \omega)B]\phi) = (\tilde{\phi}, \{1 - (1 + \omega)A^{-1/2}B A^{-1/2}\}\phi) \geq 0,$$

(30)

where $\tilde{\phi} := A^{1/2}\phi$. For $\phi \in D(H_0)$, the functions $\tilde{\phi}$ span a dense set in $L^2(\mathbb{R}^{3N-3})$ since $D(H_0)$ is dense and $A^{-1/2}$ is bounded. Hence, $(1 + \omega)A^{-1/2}B A^{-1/2} \leq 1$ and $\|A^{-1/2}B A^{-1/2}\| \leq 1/(1 + \omega)$. By identity (25), we obtain

$$\|\lambda_n(v_\tau)_- [H_0 + \lambda_n(v_\tau)_+ + k_0^2]^{-1} \sqrt{(v_\delta)}\| = \|B^{1/2}A^{-1/2}\| \leq (1 + \omega)^{-1},$$

(31)

which means that $R_{\tau}(k_0)$ in (24) is correctly defined and uniformly norm-bounded.  

$\square$
Lemma 2. The operators $D_{12,\delta}(k_n)$ given by (24) are uniformly norm bounded.

Proof. Again it suffices to consider the pair $\tau = (1, 2)$, where $\tau_1 = 1$ and $\tau_2 = 2$. We split $D_{12,\delta}(k_n)$ as follows:

$$D_{12,\delta}(k_n) = D_{12,\delta}^{(1)}(k_n) + D_{12,\delta}^{(2)}(k_n),$$

(32)

$$D_{12,\delta}^{(1)}(k_n) := \sqrt{(v_{12})^{-1}[H_0 + k_n^2]^{-1}} \left( \frac{1}{k_n + 1} \right) \sqrt{|v_{12}|},$$

(33)

$$D_{12,\delta}^{(2)}(k_n) := (k_n + 1)^{-\frac{1}{2}} \sqrt{(v_{12})^{-1}[H_0 + k_n^2]^{-1}} \sqrt{|v_{12}|},$$

(34)

For the operator in (34), we obtain (see equations (43) and (44) in [1])

$$\|D_{12,\delta}^{(2)}(k_n)\| \lesssim \|\sqrt{(v_{12})^{-1}[H_0 + k_n^2]^{-1}}\|^{1/2} \|\sqrt{|v_{12}|[H_0 + k_n^2]^{-1}}\|^{1/2},$$

(35)

where both norms in the product are uniformly bounded. (This can be easily shown after making an appropriate Fourier transform.) It remains to prove that $D_{12,\delta}^{(1)}(k_n)$ is uniformly norm-bounded. Let us first consider two cases: (a) $\delta_1 = 2, \delta_2 = 3$ and (b) $\delta_1 = 3, \delta_2 = 4$. The proof for the case (a) almost repeats the one in lemma 9 in [1]. Indeed, we need to show that $\|K_n\|$ is uniformly bounded, where

$$K_n = \mathcal{F}_{12}D_{12,\delta}^{(1)}(k_n)\mathcal{F}_{12}^{-1}.$$  

(36)

For convenience, we denote $p_{yi} := (p_{y1}, p_{y2}, \ldots, p_{yN-2}) \in \mathbb{R}^{3N-9}$. The integral operator $K_n$ acts on $\phi(x, p_{yi}, p_{yi}) \in L^2(\mathbb{R}^{3N-3})$ as follows:

$$K_n \phi(x, p_{yi}, p_{yi}) = \int d^3x d^3p_{yi} K_n(x, x', p_{yi}, p_{yi}', p_{yi}) \phi(x', p_{yi}', p_{yi}).$$

(37)

where the integral kernel has the form [1]

$$K_n(x, x', p_{yi}, p_{yi}', p_{yi}) = \frac{1}{2\pi \delta_n \pi \delta_n p_{yi}\delta_n} \left[ \frac{1}{k_n + 1 + i(p_{yi})} - \frac{1}{k_n + 1} \right] (V_{12} - (\alpha x))^{1/2}$$

$$\times \frac{e^{i\sqrt{p_{yi}^2 + k_n^2}|x - x'|}}{|x - x'|} \exp \left\{ \frac{\beta}{\gamma} (p_{yi} - p_{yi}') \right\} |V_{23}|^{1/2} ((p_{yi} - p_{yi}')/\gamma),$$

(38)

$$\beta := -m_2 h/(\sqrt{m_1 + m_2} \sqrt{M_{12}}) \quad \text{and} \quad \gamma := h/\sqrt{M_{12}}.$$  

Using the estimate

$$\|K_n\|^2 \leq \sup_{p_{yi}} \int d^3x d^3x' d^3p_{yi} d^3p_{yi}' |K_n(x, x', p_{yi}, p_{yi}', p_{yi})|^2,$$

(39)

we obtain

$$\|K_n\|^2 \leq C_0 \sup_{|p_{yi}| \leq 1/\|p_{yi}'\|} \int_{|p_{yi}| \leq 1} d^3p_{yi} \left[ \frac{1}{k_n + \sqrt{|p_{yi}|}} - \frac{1}{k_n + 1} \right]^2 \frac{1}{\sqrt{p_{yi}^2 + k_n^2}},$$

(40)

where the constant

$$C_0 := \frac{1}{2^{3N-3 \pi \delta_n \delta_n \gamma}} \left( \int d^3x (V_{12}) - (\alpha x) \right) \left( \int d^3s |V_{23}|^{1/2} (s/\gamma) \right)^2$$

(41)

is finite. Continuing (40)

$$\|K_n\|^2 \leq C_0 \int_{|p_{yi}| \leq 1} d^3p_{yi} \frac{1}{|p_{yi}|^2} = 4\pi C_0.$$  

(42)

In the case (b), we make the orthogonal transformation of Jacobi coordinates, where $x$ remains the same, $y_1 := \alpha^{-1}(r_4 - r_3)$ and $\alpha' := h/\sqrt{2M_{34}}$. We denote other transformed coordinates as $\tilde{y}_i := (\tilde{y}_2, \ldots, \tilde{y}_{N-2}) \in \mathbb{R}^{3N-9}$ and (4) reads $H_0 = -\Delta_\tau - \sum_i \Delta_{\tilde{y}_i}$. The coordinates $(x, y_1, \ldots, y_{N-2})$ and $(x, \tilde{y}_1, \ldots, \tilde{y}_{N-2})$ are connected through an orthogonal
transformation. Thus, there must exist an orthogonal \((N - 2) \times (N - 2)\) matrix \(T_{ik}\) with
real entries such that \(\tilde{y}_i = \sum_{k} T_{ik} y_k\). This choice of coordinates for \(N = 4\) is illustrated
in figure 1 (right). We need to prove that \(\|L_n\|\) is uniformly bounded, where
\[
L_n = \tilde{F}_{12} \sqrt{(v_{12})_{-}[H_0 + k_n^2]^{-1}} \{B_{12}^{-1}(k_n) - 1\} \sqrt{|v_{34}|} \tilde{F}_{12}^{-1},
\]
and \(\tilde{F}_{12}\) is defined as in (9). The operator \(L_n\) acts on \(\phi(x, \tilde{p}_{\tilde{y}_1}, \tilde{p}_{\tilde{y}_2}) \in L^2(\mathbb{R}^{3N-3})\) as
\[
L_n \phi(x, \tilde{p}_{\tilde{y}_1}, \tilde{p}_{\tilde{y}_2}) = \int d^3x' d^3p'_{\tilde{y}_1} L_n(x, x', p'_{\tilde{y}_1}, p'_{\tilde{y}_2}; \tilde{p}_{\tilde{y}_1}, \tilde{p}_{\tilde{y}_2}) \phi(x', \tilde{p}_{\tilde{y}_1}, \tilde{p}_{\tilde{y}_2}),
\]
where the integral kernel is
\[
L_n(x, x', \tilde{p}_{\tilde{y}_1}, \tilde{p}'_{\tilde{y}_1}; \tilde{p}_{\tilde{y}_2}) = \frac{1}{2\pi \alpha'} (\alpha')^3 \left\{ \frac{1}{k_n + 1} - \frac{1}{k_n + 1} \right\} |(V_{12})_{-}(\alpha x)|^{1/2}
\times e^{-\sqrt{|p'_r + k_n^2|}|x-x'|} |V_{34}|^{1/2} \left( \frac{|\tilde{p}_{\tilde{y}_1} - \tilde{p}'_{\tilde{y}_1}|}{\alpha'}. \right)
\]
Now the proof that \(\|L_n\|\) is uniformly bounded is identical to the one in the case (a) and so we
omit it. The general case of \(D^3_{12,3}\) follows from (a) and (b) by making an orthogonal coordinate
transformation, which corresponds to the appropriate permutation of the particle numbers. □

**Acknowledgment**

The author would like to thank Professor Walter Greiner for the warm hospitality at FIAS.

**Appendix. The no-clustering theorem**

Below we prove the statement, which we call the no-clustering theorem. In the following
\(\chi_L : \mathbb{R}^3 \to \mathbb{R}\) denotes the function such that \(\chi_L(r) = 1\) if \(0 \leq |r| \leq L\) and zero otherwise.
We shall make use of the following lemma concerning minimizing sequences [19].

**Lemma 3** (Zhislin). Suppose that \(H \geq 0\) is given by (1), where \(\lambda = 1\) and \(V_{ij} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)\). Suppose additionally that there is a normalized minimizing sequence
\(f_n \in D(H_0)\) such that \((f_n, H f_n) \to 0\). If \(f_n\) does not totally spread then there exists a
normalized \(\phi_0 \in D(H_0)\) such that \(H \phi_0 = 0\).

**Proof.** Since \(f_n\) does not spread totally, there must exist a subsequence such that
\(\|\chi_{|r| \leq R} f_n\| > a\) for some \(R > 0\) and \(a > 0\). We can assume that \(f_n \to \phi \in L^2(\mathbb{R}^{3N-3})\);
otherwise, we could pass to the weakly converging subsequence, which exists by the Banach–
Alaoglu theorem. Thus, for any \(g \in D(H_0)\) we have
\[
(H g, \phi) = \lim_{k \to \infty} \langle H f_n, f_n \rangle = \lim_{k \to \infty} \langle H^{1/2} g, H^{1/2} f_n \rangle = 0, \quad (A.1)
\]
because \(\|H^{1/2} f_n\| = (f_n, H f_n) \to 0\) by condition of the lemma. From (A.1), it follows that
\(\phi \in D(H_0)\) and \(H \phi = 0\). That \(\|\phi\| \neq 0\) follows from lemma 3 in [1]. Setting \(\phi_0 = \phi/\|\phi\|\) we
prove the lemma. □

**Theorem 3.** Suppose that \(H\) is given by (1), where \(\lambda = 1\) and \(V_{ij} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)\), and
none of the subsystems has an eigenstate with the energy less or equal to zero. Let \(\psi_n \in D(H_0)\)
be a totally spreading sequence such that \((\psi_n, H \psi_n) \to 0\). Then \((\psi_n, F(r_i - r_j) \psi_n) \to 0\) for
all particle pairs \((i, j)\) and any given \(F \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)\).
Proof. Note that \( \| H_0 \| \psi_n \| \) is uniformly bounded, cf lemma 1 in [1] and \( \psi_n \xrightarrow{w} 0 \) because \( \psi_n \) totally spreads. By lemma 4, it is enough to prove the statement for \( F(r) = \chi_L(r) \) and all \( L > 0 \). For \( N = 2 \), the statement becomes trivial. For \( N \geq 3 \), we prove the theorem by induction assuming that it holds for \( N - 1 \) particles. Without losing generality, it is enough to show that \( (\psi_n, \chi_L(r_1 - r_2) \psi_n) \rightarrow 0 \) for all \( L > 0 \).

A proof by contradiction. Let us assume that

\[
\limsup_{n \to \infty} (\psi_n, \chi_L(r_1 - r_2) \psi_n) = a',
\]

for some \( L > 0 \) and \( a' > 0 \). From now on, we assume that \( \psi_n \in C^\infty_0(\mathbb{R}^{3N-3}) \); otherwise, we can pass to an appropriate sequence using that \( C^\infty_0(\mathbb{R}^{3N-3}) \) is dense in \( D(H_0) \) equipped with the norm \( \| f \|_{H_0} := \| H_0^{1/2} f \| + \| f \| \), see [20].

Let \( J_s \in C^1(\mathbb{R}^{3N-3}) \) denote the Ruelle–Simon partition of unity, see definition 3.4 and proposition 3.5 in [14]. For \( s = 1, 2, \ldots, N \), one has \( J_s \geq 0 \), \( \sum_s J_s^2 = 1 \) and \( J_s(\lambda x) = J_s(x) \) for \( \lambda \geq 1 \) and \( |x| = 1 \). Besides there exists \( C > 0 \) such that for \( i \neq s \)

\[
\text{supp } J_i \cap \{ |x| > 1 \} \subset \{ |x| | r_i - r_s | \geq C |x| \}.
\]

By the IMS formula (theorem 3.2 in [14])

\[
H = \sum_{s=1}^N J_s H J_s + K,
\]

where

\[
K := \sum_s \sum_{l \neq s} V_{ls} |J_s| + \sum_s |\nabla J_s|^2,
\]

\[
H_s := H - \sum_{l \neq s} V_{ls}.
\]

The operator \( K \) is relatively \( H_0 \) compact, see lemma 7.11 in [21]. \( H_s \) is the same operator as \( H \); except that the pair interactions that involve the particle \( s \) are switched off. By (A.2) we obtain

\[
\limsup_{n \to \infty} \sum_s \langle \phi_n^{(s)}, \chi_L(r_1 - r_2) \psi_n^{(s)} \rangle = a',
\]

where we define \( \psi_n^{(s)} := J_s \psi_n \in C^\infty_0(\mathbb{R}^{3N-3}) \). The operators \( J_1 \chi_L(r_1 - r_2) \) and \( J_2 \chi_L(r_1 - r_2) \) are relatively \( H_0 \) compact; hence, \( \| \chi_L(r_1 - r_2) \psi_n^{(s)} \| \to 0 \) and \( \| \chi_L(r_1 - r_2) \psi_n^{(s)} \| \to 0 \) by lemma 2 in [1]. Thus, there must exist \( s_0 \geq 3 \) such that

\[
\limsup_{n \to \infty} \left( \phi_n^{(s_0)}, \chi_L(r_1 - r_2) \psi_n^{(s_0)} \right) = 2a,
\]

where \( 0 < a \leq 1/2 \) is a constant. Let \( \xi \in \mathbb{R}^{3N-6} \) denote the internal Jacobi coordinates for the particles \( \{ 1, 2, \ldots, s_0 - 1, s_0 + 1, \ldots, N \} \) and \( y \in \mathbb{R}^3 \) the coordinate, which points from the particle \( s_0 \) to the centre of mass of other particles. We choose the scales so that \( H_0 = -\Delta_\xi - \Delta_y \). It is convenient to introduce

\[
H_0^{(s_0)} := -\Delta_\xi + V(\xi),
\]

\[
V(\xi) := \sum_{i < k} V_{ik} - \sum_{I \neq s_0} V_{ls}.
\]

Clearly,

\[
H_0 \geq H_0^{(s_0)} \geq 0.
\]
The operator $H^{(n)}$ is the Hamiltonian of the particles $\{1, 2, \ldots, n_0 - 1, n_0 + 1, \ldots, N\}$ and can be considered on the domain $D(-\Delta_{\zeta}) \subset L^2(\mathbb{R}^{3N-6})$ as well. We have $\psi_n \to 0$ because $\psi_n$ totally spreads. Because $K$ in (A.5) is relatively $H_0$ compact, we have $K\psi_n \to 0$, see lemma 2 in [1]. Using $(\psi_n, H\psi_n) \to 0$ and $H_0 \geq 0$, we infer from (A.4) that $(\psi_n^{(s)}, H\psi_n^{(s)}) \to 0$ for all $s$. Hence, by (A.10)
\[
\left(\psi_n^{(s)}, H^{(s)}\psi_n^{(s)}\right) \to 0. \tag{A.11}
\]
Looking at (A.8) and (A.11), we conclude that there exists a subsequence $\psi_{n_0}^{(s)}$ such that
\[
\left(\psi_{n_0}^{(s)}, \chi_L(r_1 - r_2)\psi_{n_0}^{(s)}\right) \geq a, \tag{A.12}
\]
\[
\left(\psi_{n_0}^{(s)}, H^{(s)}\psi_{n_0}^{(s)}\right) \to 0. \tag{A.13}
\]
From (A.12), it follows that $\sqrt{a} \leq \|\psi_{n_0}^{(s)}\| \leq 1$. Thus, defining $g_k := \psi_{n_0}^{(s)} / \|\psi_{n_0}^{(s)}\|$ we obtain
\[
g_k, \chi_L(r_1 - r_2)g_k \geq a, \tag{A.14}
\]
\[
e_k := (g_k, H^{(s)}g_k) \to 0, \tag{A.15}
\]
where $g_k \in C_0^\infty(\mathbb{R}^{3N-3})$ and $\|g_k\| = 1$. For $f(\xi, y), h(\xi, y) \in L^2(\mathbb{R}^{3N-3})$, let us introduce the notation
\[
(f, h)_\xi := \int_{\mathbb{R}^3} d^{3N-6}_\xi f^*(\xi, y)h(\xi, y),
\]
where $(f, h)_\xi$ depends on $y \in \mathbb{R}^3$.

Now we define the following subsets of $\mathbb{R}^3$:
\[
\mathcal{M}_k := \{y|(g_k, g_k)_\xi > 0\} \cap \{y|(g_k, H^{(s)}g_k)_\xi < \sqrt{\varepsilon_k}(g_k, g_k)_\xi\}, \tag{A.17}
\]
\[
\mathcal{N}_k := \{y|(g_k, \chi_L(r_1 - r_2)g_k)_\xi \geq (a/2)(g_k, g_k)_\xi\}. \tag{A.18}
\]
By standard results $\mathcal{M}_k, \mathcal{N}_k$ are the Borel sets due to $\psi_n \in C_0^\infty(\mathbb{R}^{3N-3})$. Below we prove that there exists $k_0$ such that $\mathcal{N}_k \cap \mathcal{M}_k \neq \emptyset$ for $k \geq k_0$. For any Borel set $X \subset \mathbb{R}^3$, we define
\[
\mu_k(X) := \int_X d^3y \, (g_k, g_k)_\xi. \tag{A.19}
\]
Because $g_k$ is normalized, we have $\mu_k(\mathbb{R}^3) = 1$. On one hand, using (A.15) and (A.17)
\[
\mu_k(\mathbb{R}^3/\mathcal{M}_k) = \int_{\mathbb{R}^3/\mathcal{M}_k} d^3y \, (g_k, g_k)_\xi \leq \frac{1}{\sqrt{\varepsilon_k}} \int_{\mathbb{R}^3/\mathcal{M}_k} d^3y \, (g_k, H^{(s)}g_k)_\xi \leq \sqrt{\varepsilon_k}.
\]
Hence,
\[
\mu_k(\mathcal{M}_k) \geq 1 - \sqrt{\varepsilon_k}. \tag{A.20}
\]
On the other hand, using that according to (A.14) $\int d^3y \, (g_k, \chi_L(r_1 - r_2)g_k)_\xi \geq a$, we obtain
\[
\mu_k(\mathcal{N}_k) \geq \int_{\mathcal{N}_k} d^3y \, (g_k, \chi_L(r_1 - r_2)g_k)_\xi \geq a - \int_{\mathbb{R}^3/\mathcal{N}_k} d^3y \, (g_k, \chi_L(r_1 - r_2)g_k)_\xi
\]
\[
\geq a - \frac{a}{2}\mu_k(\mathbb{R}^3/\mathcal{N}_k) \geq a - \frac{a}{2}, \tag{A.21}
\]
where we applied (A.18) and $\mu_k(\mathbb{R}^m/\mathcal{N}_k) \leq 1$. Now it is clear that there exists $k_0$ such that $\mathcal{N}_k \cap \mathcal{M}_k \neq \emptyset$ for $k \geq k_0$. Otherwise, according to (A.21) and (A.22) we would have
\[
1 = \mu_k(\mathbb{R}^m) \geq \mu_k(\mathcal{M}_k) + \mu_k(\mathcal{N}_k) \geq 1 + \frac{a}{2} \geq \sqrt{\varepsilon_k}.
\]
which is a contradiction since $\varepsilon_k \to 0$. Now we construct the minimizing sequence for $H^{(s)}$ (considered now on $D(-\Delta_{\zeta})$) taking any $y_k \in \mathcal{N}_k \cap \mathcal{M}_k$ for $k \geq k_0$ and setting
\[
\phi_k(\xi) := g_k(y_k, \xi) \left(\int d^{3N-6}_\xi |g_k(y_k, \xi)|^2\right)^{-1/2}.
\]
\[
(\phi_k^{(s)}, H^{(s)}\phi_k^{(s)}) \to 0. \tag{A.24}
\]
Due to (A.17) and (A.18), the sequence \( \phi_k(\zeta) \in C_0^\infty(\mathbb{R}^{3N-6}) \) has the following properties: 
\[ \|\phi_k\| = 1, \ (\phi_k, H^{(n)}\phi_k) \to 0 \quad \text{and} \quad (\phi_k, \chi_L(r_1 - r_2)\phi_k) \geq a/2. \]  
(A.25)

By lemma 3, \( \phi_k \) must totally spread because \( H^{(n)} \) is not allowed to have zero-energy bound states. Since \( H^{(n)} \) is the Hamiltonian of \( N - 1 \) particles by the induction assumption it follows that:
\[ (\phi_k, \chi_L(r_1 - r_2)\phi_k) \to 0, \]  
(A.26)
which contradicts (A.25).

The following corollary describes the behaviour of wavefunctions corresponding to an infinite sequence of levels like in the case of Efimov effect [4, 11].

**Corollary 1.** Let \( H \) be given by (1), where \( \lambda = 1 \) and \( V_{ij} \in L^2 + L^\infty_{\infty} \), and none of the subsystems has an eigenstate with an energy less or equal to zero. Suppose \( \psi_n \in D(H_0) \) for \( n = 1, 2, \ldots \) is an infinite orthonormal sequence of eigenfunctions such that \( H\psi_n = e_n\psi_n \), where \( e_n < 0 \). Then \( (\psi_n, F(r_i - r_j)\psi_n) \to 0 \) for all particle pairs \((i, j)\) and any given \( F \in L^2(\mathbb{R}^3) + L^\infty_{\infty}(\mathbb{R}^3) \).

**Proof.** Clearly, \( \psi_n \xrightarrow{\infty} 0 \) because \( \psi_n \) form an infinite orthonormal set. By lemmas 1 and 3 in [1] \( \psi_n \) totally spreads. Due to the location of the essential spectrum \( e_n \to 0 \). Hence, theorem 3 applies.

**Lemma 4.** Let \( f_n(x) \in D(H_0) \subset L^2(\mathbb{R}^{3N-3}) \) and \( \|f_n\| + \|H_0f_n\| \leq 1 \). Suppose that \( \|\chi_{x[i|r_i - r_j|<q]}f_n\| \xrightarrow{\infty} 0 \) for some fixed \( i \neq j \) and any \( q > 0 \). Then \( (f_n, F(r_i - r_j)f_n) \to 0 \) for any given \( F \in L^2(\mathbb{R}^3) + L^\infty_{\infty}(\mathbb{R}^3) \).

**Proof.** Obviously, it suffices to consider \( F \in L^2(\mathbb{R}^3) \):

\[
\|F\|^{1/2}f_n \leq (\chi_{x[i|r_i - r_j|<q]}f_n, |F|f_n) + (\chi_{x[i|r_i - r_j|>q]}f_n, |F|f_n) \\
= (\chi_{x[i|r_i - r_j|<q]}f_n, |F|f_n) + (f_n, \chi_{x[i|r_i - r_j|>q]}|F|(H_0 + 1)^{-1}(H_0 + 1)f_n) \\
\leq \|\chi_{x[i|r_i - r_j|<q]}f_n\| \|F\|f_n\| + \|\chi_{x[i|r_i - r_j|>q]}|F|(H_0 + 1)^{-1}\|f_n\|. 
\]  
(A.29)

The first term in (A.29) goes to zero because \( |F(r_i - r_j)| \) is relatively \( H_0 \) bounded. The second term is an operator norm, which can be made as small as pleased by setting \( q \) large enough, see lemma 5 in [1].

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