\section*{Abstract}
The presence of autonomous vehicles on the roads presents an opportunity to compensate the unstable behaviour of conventional vehicles. Vehicles should (i) be able to recover their equilibrium speed, and (ii) react so as not to propagate but absorb perturbations. In this paper, we address the stability analysis of platoon systems consisting of heterogeneous vehicles updating their dynamics according to realistic behavioural car-following models. First, definitions of all types of stability that are of interest in the platoon system -asymptotic, input-output, weak and strict string stability- are carefully presented based on recent studies. Then, frequency domain linear stability analyses are conducted after linearisation of the modelled system of vehicles. They lead to conditions for input-output stability, strict and weak string stability over the behavioural parameters of the system, for finite and infinite platoons of homogeneous and heterogeneous vehicles. This provides a solid basis that was missing for car-following model-based control design in mixed traffic environments. After visualisation of the theoretical results in simulation, we formulate an optimisation strategy with LMI constraints to tune the behavioural parameters of the (partially) autonomous vehicles in mixed and heterogeneous traffic in order to maximise the stability of the traffic flow while considering the comfort of autonomous driving. We show that very few autonomous vehicles are enough to prevent the propagation of realistic disturbances.

\section*{I. Introduction}

With the advent of Automated Driving Systems (ADS)-equipped vehicles, interest is growing in how to control autonomous vehicles to increase traffic flow stability and safety in mixed traffic contexts, i.e. when (partially) autonomous vehicles and conventional vehicles coexist on the road. A recent report has underlined the unsafe nature of autonomous vehicles, which are five times as likely to crash as conventional vehicles, even though they are almost never to blame when a crash does occur \cite{1}. In this context, it seems fair to assume that ADS-equipped vehicles should behave similarly to surrounding drivers in order not to surprise them and, when automation is only partial, similarly to the in-vehicle driver in order to increase driving comfort and facilitate driving task switching between automation and conventional mode. As a result, understanding of vehicle behaviour is critical. The longitudinal microscopic behaviour of vehicles is usually approximated using car-following models, where vehicles only react to the behaviour of their leaders. Car-following models are continuous time models that can be either delay-free \cite{2}, \cite{3} or time-delayed \cite{4}, \cite{5}. These models have been shown to accurately represent traffic flow features such as speed and headway distributions, stop and go waves using real world datasets \cite{6}, \cite{7}, \cite{8} and are therefore a solid base for further developments on stability analyses and control applications.

Stability analysis of conventional and autonomous vehicle dynamics has caught the attention of many researchers in the traffic flow and control theory communities, as it characterises the ability of a system to recover an equilibrium -asymptotic stability- and to prevent the propagation of perturbations -string stability. Pioneering work was made in the context of the Automated Highway System (AHS) effort of the California Path Program \cite{9}, \cite{10}. Among the contributions were the formulation of lateral and longitudinal control laws for stable, safe and comfortable platooning manoeuvres, see \cite{11}, \cite{12} for example, the development of a control system architecture for platooning, see \cite{13} for instance, and the San Diego demonstration in 1997 \cite{14}. Since then, several control schemes have been proposed over the years to address stability and string stability \cite{15}, \cite{16} in autonomous platoon systems. Recently, algorithms were presented to achieve stable platooning considering communication delays \cite{17}, and to achieve string stable behaviour via $H_\infty$ control \cite{18} and energy-based arguments \cite{19}. One issue regards the multiple existing definitions of string stability. Since early work \cite{20}, it has been agreed that in the case of scalar perturbation outputs, string stability refers to a decrease of the absolute value of the output as we move downstream the platoon of vehicles. In \cite{21}, \cite{22}, \cite{23}, the string stability definition is extended to the consideration of multiple outputs and a sufficient condition to achieve it is for the $H_\infty$ norm of the transfer function between two successive vehicle outputs to be less than 1. The lack of a unified definition was identified in a recent work \cite{24}, where a definition of $L_\infty$ string stability was proposed for interconnected systems, and several of its implications were discussed. Regarding conventional vehicle dynamics, one key work is the one of \cite{25}, where the behaviour of a uniform flow of vehicles with car-following dynamics was investigated. The solution of the speed perturbation equation is decomposed into Fourier modes with discrete wave numbers associated to each vehicle, and the behaviour of the solution is assessed as the system size grows to infinity: if the solution grows with time and space, the system is said to be string unstable. One of the early works to use this approach...
appeared in [26], and this was recently utilised to study weakly non-linear congestion patterns of bilateral multi-
anticipative traffic [27] as well as multi-anticipative traffic
introducing time-delays [28]. However, the string stability of
car-following dynamics remains not well understood in the
case of finite systems -platoons- of homogeneous vehicles
as well as in the case of heterogeneous platoons of vehicles,
i.e. groups of vehicles exhibiting different car-following
behaviour.

In mixed traffic environments, one way of progressing
towards a more efficient and safer traffic flow is to ensure
input-output stability and string stability for platoons of
vehicles. In particular, (partially) autonomous vehicles could
be used to compensate the unstable driving behaviour of
conventional vehicles. This requires, however, insights into
the stability properties of heterogeneous platoons. In this
paper, we present a general framework for investigating the
stability of heterogeneous platoons of vehicles, and propose
an optimisation policy to tune the behavioural parameters of
each (partially) autonomous vehicle present in the platoon
to maximise traffic flow stability in the platoon, as well as
to minimise the driving discomfort potentially caused by
autonomous driving. Section II lists the notation used in
the paper. Section III recalls the general form of car-following
models, and describes the linearisation of the heterogeneous
platoon system. Section IV presents the definition of input-
output stability, weak string stability and string stability for
heterogeneous finite and infinite platoons of vehicles. We
introduce the definition of weak stability, particularly
relevant to mixed traffic environments. In Section V the
definitions are applied to the system of vehicles providing
a list of necessary and sufficient conditions when possible,
sufficient conditions otherwise, for input-output, strict and
weak string stability, for both homogeneous and heteroge-
neous platoons. In Section VI we visualise these results
in simulation, and discuss linear vs non-linear stability. In
Section VII we provide an optimisation strategy to tune the
behavioural parameters of the (partially) autonomous
vehicles in the platoon so as to increase weak string stability
while considering the comfort of driving. We formulate the
constraints as linear Matrix Inequalities (LMI), and solve
the optimisation using its convex structure. We show that using
our approach autonomous vehicles consistently contribute to
increasing traffic flow stability of heterogeneous platoons.

We conclude with a summary of our findings.

II. NOTATIONS

The notation used throughout this paper is as follows.

\( \mathbb{N} \) Set of natural numbers.
\( \mathbb{N}^* \) Set of strictly positive natural numbers.
\( \mathbb{R} \) Set of real numbers.
\( \mathbb{R}_+ \) Set of positive real numbers.
\( \mathbb{R}_+^* \) Set of strictly positive real numbers.
\( \mathcal{M}_{p,q} \) Set of matrices defining a linear map from \( \mathbb{R}^p \) to \( \mathbb{R}^q \) with coefficients in \( \mathbb{R} \), with \( p \) and \( q \in \mathbb{N}^* \).
\( ||.||_2 \) \( L_2 \) norm of a vector in \( \mathbb{R}^n \), with \( n \in \mathbb{N}^* \).

\( \mathbb{L}_p \) Space of functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( t \to |f(t)|^p \) is integrable over \( \mathbb{R} \), here \( p = 2, \infty \).
\( ||.||_{\mathbb{L}_p} \) \( \mathbb{L}_p \) norm of a \( \mathbb{L}_p \) function, here \( p = 2, \infty \).
\( \mathbb{Y}_i(s) \) Laplace transform of \( y_i(t) \).
\( \mathbb{U}_i(s) \) Laplace transform of \( u_i(t) \).
\( ||.||_{\mathcal{H}_\infty} \) \( \mathcal{H}_\infty \) norm of a defined Laplace transform.
\( \mathbb{R} e(z) \) Real part of complex number \( z \).
\( \mathbb{K} \) Class of continuous functions \( h \), defined such as \( h(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \), \( h(0) = 0 \) and \( h(\cdot) \) is strictly increasing.

II. GENERAL FORM OF CAR-FOLLOWING MODELS AND
CORRESPONDING PLATOON EQUATION

Consider a platoon of \( m \in \mathbb{N} \setminus \{0,1\} \) vehicles with indices \( n \in \{1,...,m\} \). The first vehicle of the platoon \( n = 1 \) follows a virtual reference vehicle \( (n = 0) \) which keeps a
constant speed \( \dot{x}_0(t) = v_{eq} \). The other vehicles \( (1 \leq n \leq m) \)
behave according to a car-following model. Disturbances can
occur as external acceleration inputs \( u_n \) for any vehicle in
the platoon.

A. Car-following models

Car-following models describe the dynamics of a vehicle
in response to the trajectory of its leading vehicle, taking into
account the technical features of the car and the behaviour
of the driver through vehicle behavioural parameters.

1) Selection of the state vector: before presenting the
models we introduce \( x_n \in \mathbb{R}^2 \) the state vector of vehicle
\( n \) defined as:

\[
x_n = \begin{pmatrix} \Delta x_n \\ v_n \end{pmatrix},
\]

where \( \Delta x_n = x_{n-1} - x_n \) is the space headway or distance
between vehicle \( n \) and its leading vehicle \( n - 1 \), and \( v_n \)
the speed of vehicle \( n \).

2) Delay-free car-following models: delay-free continu-
ous time models are the most well-known type of car-
following models in the literature. For a vehicle \( n \), the vehicle
dynamics are as follows:

\[
\Delta \dot{x}_n = v_{n-1} - v_n,
\]
\[
\dot{v}_n = f_n(v_n, \Delta x_n, v_{n-1} - v_n, \theta_n) + u_n,
\]

where \( \dot{v}_n \) is the acceleration of the vehicle \( n \in \{1,...,m\} \),
\( \Delta x_n = x_{n-1} - x_n \) the distance to the vehicle in front
(headway), \( \dot{x}_n = \dot{x}_{n-1} - \dot{x}_n \) the relative velocity, \( \theta_n \in \mathbb{R}^l \)
the vector of the behavioural parameters, with \( l \in \mathbb{N}^* \)
the number of parameters, and \( f_n \) the acceleration function
of the car following model which captures the non linear
dynamics of the vehicle. Input \( u_n \) captures the effect of
external disturbances.

Note that, in this paper, we restrict our analysis to time-
delayed car-following models, but the presented approach
could be readily extended to the consideration of time-
delayed car-following models.
B. Linearisation of the vehicle state

For any vehicle $n \in \{1,...,m\}$, an acceleration input perturbation on vehicle $i \in \{1,...,n\}$ generates responses $\dot{x}_n, \Delta x_n, \Delta \dot{x}_n$ that can be written as perturbations about the equilibrium values $\dot{x}_{n,eq}, \Delta x_{n,eq}, \Delta \dot{x}_{n,eq}$:

\[
\dot{x}_n(t) = x_{n,eq}(t) + y_n(t),
\]
\[
\dot{x}_n(t) = \dot{x}_{n,eq} + \dot{y}_n(t),
\]
\[
\Delta x_n(t) = \Delta x_{n,eq}(t) + \Delta y_n(t),
\]
where perturbations $y_n(t) = x_n(t) - x_{n,eq}(t)$ and $\Delta y_n(t) = y_{n-1}(t) - y_n(t)$. At equilibrium, we also have $\Delta \dot{x}_{n,eq} = \Delta \dot{y}_n = 0$ and $\dot{x}_{n,eq} = v_{eq}$. Focusing on continuous time car-following models, when $(\dot{y}_n, \Delta y_n, \Delta \dot{y}_n)$ is small enough, the following approximation is valid:

\[
\begin{align*}
\dot{x}_n &= f_n(\dot{x}_n, \Delta x_n, \Delta \dot{x}_n) + u_n \\
&= f_n(\dot{x}_{n,eq} + \dot{y}_n, \Delta x_{n,eq} + \Delta y_n, 0 + \Delta \dot{y}_n) + u_n \\
&\approx f_n(\dot{x}_{n,eq}, \Delta x_{n,eq}, 0) + \dot{y}_n f_{n,1} + \Delta y_n f_{n,2} + \Delta \dot{y}_n f_{n,3} + u_n,
\end{align*}
\]

where

\[
\begin{align*}
f_{n,1} &= \frac{\partial f_n(\dot{x}_{n,eq}, \Delta x_{n,eq}, 0)}{\partial \dot{x}_n}, \\
f_{n,2} &= \frac{\partial f_n(\dot{x}_{n,eq}, \Delta x_{n,eq}, 0)}{\partial \Delta x_n}, \\
f_{n,3} &= \frac{\partial f_n(\dot{x}_{n,eq}, \Delta x_{n,eq}, 0)}{\partial \Delta \dot{x}_n}.
\end{align*}
\]

Observe that $f_n(\dot{x}_{n,eq}, \Delta x_{n,eq}, 0) = 0$ and $\dot{x}_n = \dot{y}_n$, equation (7) can be rewritten as

\[
\dot{y}_n \approx \dot{y}_n f_{n,1} + \Delta y_n f_{n,2} + \Delta \dot{y}_n f_{n,3} + u_n.
\]

Note that, as mentioned in [25], equation (11) has a physical meaning in traffic. When the speed deviation $\dot{y}_n$ is increased, the vehicle tends to decelerate to return to the equilibrium speed; when the relative distance $\Delta y_n$ or relative speed $\Delta \dot{y}_n$ is increased, it tends to accelerate to return to the equilibrium speed. These considerations lead to the following conditions: \forall n \in \{1,...,m\},

\[
f_{n,1} < 0, \ f_{n,2} > 0, \ f_{n,3} > 0.
\]

C. State-space dynamics

By analogy with (13), we have $y_n \in \mathbb{R}^2$ the perturbed state vector defined as:

\[
y_n = \begin{pmatrix} \Delta y_n \dot{y}_n \end{pmatrix}.
\]

The relation (7) can then be rewritten equivalently as:

\[
\dot{y}_n = a_n y_{n-1} + b_n y_n + b_n u_n,
\]

where $u_n \in \mathbb{R}$ is the external acceleration input, $b_n \in \mathbb{R}^2$ is the following vector,

\[
b_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

and the $a_{n,0}$ and $a_{n,1} \in \mathcal{M}_{2,2}(\mathbb{R})$ are the matrices:

\[
a_{n,0} = \begin{pmatrix} 0 & 1 \\ 0 & f_n,3 \end{pmatrix},
\]
\[
a_{n,1} = \begin{pmatrix} 0 & -1 \\ f_n,2 & f_n,1 - f_n,3 \end{pmatrix}.
\]

Note that the form of vector $b_n$ enforces a zero speed input, as the actuator is assumed to be the throttle pedal which only acts upon acceleration.

D. Linearisation of the platoon equation

Let $u \in \mathbb{R}^m$ be the vector of inputs:

\[
u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix},
\]

and $y \in \mathbb{R}^{2m}$ the platoon state vector:

\[
y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.
\]

For the leader of the platoon ($n = 1$), there is only a fictitious vehicle ahead ($n = 0$). The equation of motion of vehicle 0 is:

\[
y_0 = \dot{a} y_0,
\]

with $\dot{a} \in \mathcal{M}_{2,2}$ the matrix:

\[
\dot{a} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

The linearised dynamics of the platoon are:

\[
\dot{y} = ay + bu,
\]
\[
h = cy,
\]

where $h \in \mathbb{R}^{2m}$ is the vector of outputs, $c \in \mathcal{M}_{2m,2m}$ is the observation matrix, and the matrix $b \in \mathcal{M}_{2m,m}$ is:

\[
b = \begin{pmatrix} b_v & 0 & \cdots & 0 \\ 0 & b_v & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix},
\]

where $b_v$ is given by equation (13) and matrix $a \in \mathcal{M}_{2m,2m}$ has a $2 \times 2$ block form:

\[
a = \begin{pmatrix} a_{1,0} & a_{1,1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & a_{m,0} & a_{m,1} \end{pmatrix}.
\]

The equilibrium state of the linear system is defined as $y_{eq}$, which is also the zero vector $\in \mathbb{R}^{2m}$ (since $u$ is zero in equilibrium), and $h_{eq} = cy_{eq}$ is the equilibrium output. $h_n \in \mathbb{R}^2$ is the output of vehicle $n$, and $h_{n,eq} \in \mathbb{R}^2$ is the equilibrium output of vehicle $n$. Note that the form of the observation matrix $c$ depends on the characteristics of the observer. For a centralised observer $c$ would be the identity matrix $\in \mathcal{M}_{2m,2m}$. However, a decentralised observer such as a vehicle only sees a local part of the full state vector.
IV. STABILITY OF THE SYSTEM: DEFINITIONS AND REMARKS

In this section, the definitions of stability, exponential stability, input-output stability and strict string stability are recalled for linear systems, and we propose a definition of weak string stability. In brief, exponential stability means that when there is no disturbance the state vector converges exponentially quickly to the equilibrium state; string stability means that there is no disturbance amplification in the upstream direction. We assume \( y_{eq} = 0 \) for the sake of simplicity.

**Definition 1 (Stability and exponential stability), see [29].** Consider the linear system defined in equations (22), (23). The platoon system of \( m \in \mathbb{N} \setminus \{0, 1\} \) vehicles is said to be stable if for a given \( t_0 > 0 \), \( \forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0 \) such that when \( ||y(t_0)||_2 < \delta \), then \( \forall t \geq t_0, ||y||_{L_p} < \epsilon \); and the system is said to be exponentially stable if for every \( \delta > 0 \) there exists \( \alpha, \beta \in \mathbb{R}_+^\ast \) such that if \( ||y(t_0)||_2 < \delta \), then \( ||y||_{L_p} \leq \alpha ||y(t_0)||_2 e^{-\beta(t-t_0)} \).

Note that in the case of the linear system of equation (22), (23), the system is exponentially stable iff \( \forall \lambda \in \{ \text{Spectrum}(a) \}, \Re(\lambda) < 0 \), see [30] for instance.

**Definition 2 (Input-output stability), see [30], [24].** Consider the linear system defined in equations (22), (23). The platoon system of \( m \in \mathbb{N} \setminus \{0, 1\} \) vehicles is said to be input-output stable if there exists class \( K \) functions \( \alpha : [0, a) \rightarrow [0, \infty) \) and \( \beta : [0, b) \rightarrow [0, \infty) \), \( a, b \in \mathbb{R} \), such that, for any initial state \( h_{0i} \in \mathbb{R}^{2m} \) and any input such that \( ||u||_{L_p} < u \forall t \geq t_0, u \in \mathbb{R} \), then \( \forall h \in \{1, ..., m\} \),

\[
||h_{n}||_{L_p} \leq \alpha(u) + \beta(||h(0)||_2)
\]  

**Remark 1.** For linear systems when input-output stability holds element-wise then it holds for all inputs, i.e. when the system is input-output stable for inputs \( u \in \mathbb{R}^{m} \) such that \( u_i, i \in \{1, ..., m\} \), is the only non-zero component of \( u \) then it also holds for all inputs (by superposition).

**Remark 2.** If the input-output property holds for any platoon size \( \forall m \in \mathbb{N} \setminus \{0, 1\} \), i.e. the class \( K \) functions \( \alpha \) and \( \beta \) do not depend on \( m \), then the platoon system is said to be scalable [24], [31].

We now give the definition of \( L_p \) strict string stability, adapted from [24], with \( p = 2 \) or \( p = \infty \), see Section [12]. Note that, as mentioned in [32] for instance, the major part of the string stability analyses in the literature deal with the \( L_2 \) norm which is easier to work with, as it can be rewritten immediately as a condition on the \( H_\infty \) norm of the corresponding transfer function. However, the \( L_\infty \) norm has a stronger physical meaning as it deals with the peak of the deviations, and therefore \( L_\infty \) strict string stability can be directly related to a condition for collision avoidance.

**Definition 3 (\( L_p \) strict string stability).** We apply the definition of strict string stability in [24] to the linear system defined in equations (22), (23). The platoon system of \( m \in \mathbb{N} \setminus \{0, 1\} \) vehicles is said to be \( L_p \) strictly string stable if it is input-output stable and if, in addition, for inputs \( u \in \mathbb{R}^{m} \) such that \( u_i, \forall i \in \{1, ..., m\} \), is the only non-zero component and \( h(t_0) = h_{eq} \), then \( \forall n \in \{1, ..., m\} \),

\[
||h_{n}||_{L_p} \leq ||h_{n-1}||_{L_p}.
\]  

We now formalise the definition of weak string stability which was first intuitively mentioned in [23].

**Definition 4 (\( L_p \) weak string stability).** Consider the linear system defined in equations (22), (23). Let the platoon system of \( m \in \mathbb{N} \setminus \{0, 1\} \) vehicles be input-output stable and consider inputs \( u \in \mathbb{R}^{m} \) such that \( u_i, \forall i \in \{1, ..., m\} \), is the only non-zero component and \( h(t_0) = h_{eq} \). Then, for given \( l \in \{1, ..., m\} \) and \( n \in \{1, ..., m\} \), the platoon system is said to be \((l, n)\) weakly string stable if

\[
||h_{n}||_{L_p} \leq ||h_{l}||_{L_p}.
\]  

**Remark 3.** For linear systems when string stability holds element-wise then it holds for all inputs, i.e. when the system is string stable for inputs \( u \in \mathbb{R}^{m} \) such that \( u_i, \forall i \in \{1, ..., m\} \), is the only non-zero component of \( u \) then it also holds for all inputs (by superposition).

**Remark 4.** The weak string stability definition is introduced to handle mixed traffic situations where platoons are composed of conventional vehicles and autonomous vehicles. The idea is to achieve the weak string stability condition by only acting upon the controllable autonomous vehicles. Note that for a given \( i \in \{1, ..., m - 2\} \), we may have \((i, i + 2)\) weak string stability, i.e. \( ||h_{i+2}||_{L_2} \leq ||h_{i}||_{L_2} \), but strict string stability does not hold.

V. STABILITY RESULTS

In this section we investigate the stability properties of the linearised platoon dynamics. We consider an input \( u \) such that only element \( u_i \) is non-zero, i.e. \( u_n(t) = 0 \) for \( n \neq i \), and note that \( y_j(t) = 0 \) for \( j < i \) since there is no input to these vehicles and the initial conditions are zero.

A. Laplace transforms of the vehicle platoon dynamics

Taking Laplace transforms and rearranging, equation (14) yields:

\[
Y_i(s) = (sI - a_{i,1})^{-1}b_iU_i(s),
\]  

i.e.

\[
Y_i(s) = G_i(s)U_i(s),
\]  

where \( I \in \mathbb{M}_{2,2} \) is the identity matrix, and where transfer function \( G_i(s) \) is

\[
G_i(s) = \frac{1}{D_i(s)} \left( -1 \right),
\]  

where

\[
D_i(s) = s^2 + s(f_{i,3} - f_{i,1}) + f_{i,2}.
\]  

For vehicles \( n > i \) we also have

\[
Y_n(s) = \Gamma_n(s)Y_{n-1}(s) = \left( \prod_{j=i+1}^{n} \Gamma_j(s) \right) Y_i(s),
\]  

(33)
where \( \Gamma_n(s) = (sI - a_{n,1})^{-1}a_{n,0} \). (34)

Hence,

\[
Y_n(s) = \prod_{j=i+1}^{n} \Gamma_j(s) G_i(s) U_i(s). \tag{35}
\]

**B. Input-output stability**

In the general case where \( G_i \) and \( \Gamma_n \) are not known, we know that, see page 26 of [33] for instance, the \( \mathcal{L}_2 \)-induced norm of the linear map \( u_i \rightarrow y_i \), is the \( \mathcal{H}_\infty \) norm of \( G_i \). By definition, the \( \mathcal{H}_\infty \) norm of \( G_i \) is

\[
||G_i||_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}}(G_i(j\omega)), \tag{36}
\]

where \( \sigma_{\text{max}} \) is the maximum singular value of the matrix. The inequality

\[
||y_i||_{\mathcal{L}_2} \leq ||G_i(s)||_{\mathcal{H}_\infty} ||u_i||_{\mathcal{L}_2} \tag{37}
\]

holds, and \( ||y_i||_{\mathcal{L}_2} \) can be made arbitrarily close to \( ||G_i(s)||_{\mathcal{H}_\infty} ||u_i||_{\mathcal{L}_2} \) by appropriate selection of input \( ||u_i||_{\mathcal{L}_2} \). In addition, for \( n > i \), the inequality

\[
||y_n||_{\mathcal{L}_2} \leq ||\Gamma_n||_{\mathcal{H}_\infty} ||y_{n-1}||_{\mathcal{L}_2} \tag{38}
\]

follows from equation (33), and we have,

\[
||y_n||_{\mathcal{L}_2} \leq \prod_{j=i+1}^{n} ||\Gamma_j||_{\mathcal{H}_\infty} ||G_i||_{\mathcal{H}_\infty} ||u_i||_{\mathcal{L}_2}. \tag{39}
\]

Therefore, the existence of \( \max_{i \in \mathbb{N}^+} ||G_i(s)||_{\mathcal{H}_\infty} \) and \( \max_{i \in \mathbb{N}^+} ||\Gamma_i(s)||_{\mathcal{H}_\infty} \) is a sufficient condition for input-output stability of the platoon system. In addition, with regard to Remark 2, the existence of \( \max_{i \in \mathbb{N}^+} ||G_i(s)||_{\mathcal{H}_\infty} \) and \( \sup_{i \in \mathbb{N}^+} \prod_{j=2}^{n} ||\Gamma_j||_{\mathcal{H}_\infty} \) is a sufficient condition for the scalability of the platoon system. Finally, in the case of an homogeneous platoon with Single Input Single Output (SISO) transfer functions, we recall that the conditions \( ||G_i||_{\mathcal{H}_\infty} \) bounded and \( ||\Gamma_i||_{\mathcal{H}_\infty} \leq 1 \) become necessary and sufficient conditions for input-output stability, see the work of [24].

In the case of our system, following equations (22-25), the system matrix \( a \) is block diagonal and so the eigenvalues of \( a \) are the eigenvalues of the \( m \) block matrices on the diagonal. The eigenvalues of the matrices \( (a_{n,1})_{1 \leq i \leq m} \) are \( (f_{n,1} - f_{n,3} \pm \sqrt{\Delta_n})/2 \), where \( \Delta_n = (f_{n,1} - f_{n,3})^2 - 4f_{n,2} \). If we consider the rational driving constraints of equation (12), the eigenvalues have negative real parts, i.e. \( (a_{n,1})_{1 \leq i \leq m} \) are Hurwitz matrices, and therefore the system is always exponentially stable. If not, the sufficient and necessary conditions for exponential stability are:

\[
f_{n,1} - f_{n,3} + Re\left(\sqrt{(f_{n,1} - f_{n,3})^2 - 4f_{n,2}}\right) < 0. \tag{40}
\]

**C. String stability of the vehicle platoon**

We start the discussion with \( \mathcal{L}_2 \) string stability before addressing \( \mathcal{L}_\infty \) string stability.

*1) \( \mathcal{L}_2 \) strict string stability*  

Applying Definition 3, if we have input-output stability, a sufficient condition for \( \mathcal{L}_2 \) strict string stability follows from equation (38), as mentioned in [21]:

\[
||\Gamma_n||_{\mathcal{H}_\infty} \leq 1, \ \forall n \in \{2, ..., m\}. \tag{41}
\]

In the case of our system, following equations (22, 25), we obtain by developing equation (42):

\[
\Gamma_n(s) = \frac{1}{D_n(s)} \begin{pmatrix} 0 & s - f_{n,1} \\ 0 & f_{n,2} + s f_{n,3} \end{pmatrix}. \tag{42}
\]

The first column of \( \Gamma_n(s) \) is made of zeros so the structure of the transfer function can be summarised in Figure 1 for a perturbation \( u_1 \) on the first vehicle of the platoon. Note that, \( Y_n \) being the Laplace transform of \( y_n \), we write \( Y_{n,1} \) the Laplace transform of \( \Delta y_n \) and \( Y_{n,2} \) the Laplace transform of \( y_n \). We also write \( \Gamma_{n,(i,j)} \) the element of \( \Gamma_n \) on the \( i \)th row and \( j \)th column, with \( i, j \in \{1, 2\}, n \geq 2 \), and \( G_{1,i} \) the \( i \)th element of \( G_1 \), \( i \in \{1, 2\} \).

![Fig 1: SISO and MIMO transfer functions of the studied cascaded system.](image)

\[ a) \text{Single Input Single Output (SISO) systems: we are interested in capturing the headway and speed perturbations. For that, we first look at the SISO headway and speed to speed transfer functions, defined as } Y_{n,k} = \Gamma_{n,k}Y_{n-1,k}, \text{ with } k \in \{1, 2\}, n \geq 2. \text{ As visible on Figure 1 we immediately have the speed to speed transfer function:} \]

\[
\Gamma_{n,2} = \frac{s f_{n,3} + f_{n,2}}{s^2 + s (f_{n,3} - f_{n,1}) + f_{n,2}}. \tag{43}
\]

However, we need to express \( Y_{n-1,2} \) as a function of \( Y_{n-1,2} \) to have an expression of \( \Gamma_{n,1} \). We have

\[
Y_{n-1,2} = \frac{s \Gamma_{n-1,2} Y_{n-1,1}}{1 - \Gamma_{n-1,2}}, \tag{44}
\]

which reduces to

\[
Y_{n-1,2} = \frac{s f_{n,1} + f_{n-1,2} Y_{n-1,1}}{s - f_{n-1,2}}. \tag{45}
\]

We therefore have the headway to headway transfer function:

\[
\Gamma_{n,1} = \frac{(s - f_{n,3}) (s f_{n,1} + f_{n-1,2})}{(s - f_{n-1,2}) (s^2 + s (f_{n,3} - f_{n,1}) + f_{n,2})}, \tag{46}
\]

which is the same condition as equation (43) when vehicles \( n \) and \( n - 1 \) have identical behavioural parameters. We now focus on the speed to speed transfer function \( \Gamma_{n,1} \), as it has second order dynamics, and therefore we can readily get an analytical condition for \( \mathcal{L}_2 \) strict string stability. The \( \mathcal{H}_\infty \) norm of \( \Gamma_{n,2} \) is the maximum gain \( ||\Gamma_{n,2}(j\omega)|| \) across the frequencies. We have

\[
||\Gamma_{n,2}(j\omega)|| = \frac{\omega^2 f_{n,3}^2 + f_{n,2}^2}{(f_{n,2} - \omega^2)^2 + \omega^2 (f_{n,3} - f_{n,1})^2}. \tag{47}
\]
Condition $|\Gamma_{n,2}(j\omega)| \leq 1$ leads to equation
\[ \omega^4 + \omega^2(f_{n,1}^2 - 2f_{n,3}f_{n,1} - 2f_{n,2}) \geq 0. \] (48)

That is, the $L_2$ strict string stability condition is a simple condition on the partial derivatives of the system. $\forall \omega \in \mathbb{R}^+$, $\forall n \in \mathbb{N} > i$,
\[ |\Gamma_{n,2}(j\omega)| \leq 1 \iff f_{n,1}^2 - 2f_{n,1}f_{n,3} - 2f_{n,2} \geq 0. \] (49)

We can observe that these conditions are the same conditions as the well-known string stability conditions that are usually derived from an infinite homogeneous traffic [25].

b) *Multiple Inputs Multiple Outputs (MIMO) systems:* another way to capture the propagation of the headway and speed perturbations is to look at the full MIMO system represented by transfer function $\Gamma_n$, see Figure 1. We calculate the singular values of $\Gamma_n(j\omega)$ to obtain its $H_\infty$ norm, following the definition of equation (36). After some manipulation, see Appendix B, sufficient conditions for strict string stability are:
\[ f_{n,1} = 0, \] (50)
\[ -2f_{n,2} - 1 \geq 0. \] (51)

Comparing equations (50), (51) with equation (49), it can be seen that the sufficient conditions for strict string stability are more conservative in the MIMO case. Besides, they are not compatible with the physical rational conditions presented in equation (12).

2) $L_2$ weak string stability: applying Definition 4, for a given $l \in \{i, ..., m\}$ and $n \in \{l, ..., m\}$, a sufficient condition for $(l, n)$ weak string stability is:
\[ \left\| \prod_{i=l+1}^{n} \Gamma_i \right\|_{H_\infty} \leq 1. \] (52)

As an example, consider a platoon system consisting of 3 vehicles numbered 1, 2 and 3. A perturbation is introduced to the first vehicle. We work with the following partial derivative values: $f_{21} = -0.075$, $f_{22} = 0.091$, $f_{23} = 0.55$, and $f_{31} = -0.26$, $f_{32} = 0.10$, $f_{33} = 0.64$, which are obtained taking realistic parameter values of the Intelligent Driver Model (IDM), a well-known physical model for reproducing realistic traffic [2]. We compute the $L_2$ speed gains and we find $\|\Gamma_{2,2}\|_{H_\infty} = 1.06$, $\|\Gamma_{3,2}\|_{H_\infty} = 1$ and $\|\Gamma_{2,2}\|_{H_\infty} = 1$, as we can observe in Figure 2. The platoon is therefore (1,3) weakly string stable but not proved to be string stable. Parameter tuning of controllable vehicles (here vehicle 3) could compensate instabilities generated by conventional vehicles (here vehicle 2). However, analytical conditions for achieving weak string stability are not easy to obtain as solving inequality (52) requires solving an 8 degree polynomial equation.

**Remark 5.** Note that, $\forall k \in \{1, 2\}$, $\left\| \prod_{i=k+1}^{n} \Gamma_i \right\|_{H_\infty} \leq 1$ is equivalent to $\left\| \prod_{i=k+1}^{n} \Gamma_i \right\|_{H_\infty} = 1$, as $\forall i \in \{1, ..., m\}$, we have $|\Gamma_i(0)| = 1$.

3) $L_\infty$ strict string stability: as discussed in Section IV, $L_\infty$ string stability is more practical than $L_2$ string stability as it deals with the peak values of the perturbations. One of the early works to introduce the $L_\infty$-induced norm of a linear map, that is the $L_1$ norm of its impulse response, are the ones of [34], [35]. This means that the condition to guarantee $L_\infty$ strict string stability is to have the $L_1$ norm of the impulse response less than 1. It is known from [36] that the $H_\infty$ norm is upper bounded by the $L_\infty$-induced norm, and that for non-negative impulse responses, those norms are identical. Therefore, if we look at the speed to speed transfer function of equation (43), sufficient conditions for having a monotonic step response are non-imaginary poles and positive zeros, which leads to:
\[ (f_{n,3} - f_{n,1})^2 - 4f_{n,2} \geq 0, \] (53)
\[ -f_{n,2} < 0. \] (54)

Note that the conditions (54) is always verified due to the physical relation (12). It is interesting to investigate which equation is the most conservative between (49) and (53). In fact, by substracting equation (49) from equation (53), we can verify that the $L_2$ strict stability condition is stronger than the condition for the equality of the norms iff $f_{2,3}^2 \geq 2f_{n,2}$, which we expect to be almost always verified for realistic parameter values. In summary, we have:
\[ (L_\infty \text{ stability} = L_2 \text{ stability}) \iff f_{2,3}^2 \geq 2f_{n,2}, \] (55)
and if condition (55) is not verified we only have $L_\infty$ stability $\Rightarrow L_2$ stability. To our knowledge, whereas equation (49) is well-known to the traffic flow theory community, equations (46), (53) and (55) are novel conditions for the investigated car-following dynamics of equations (2) and (3).
D. Case of a closed platoon

Closed platoons, where vehicle 1 in the platoon follows vehicle \( m \), have often been studied to interpret congestion, as it seems intuitive that the asymptotic instability of the closed platoon is linked to the string stability of the open platoon, see [25] for instance. Here we provide a frequency-based analysis of that relation.

By analogy with equation (25), the system matrix \( a_c \in \mathcal{M}_{2(m+1),2(m+1)} \) for a closed platoon can be written in 2x2 block form:

\[
a_c = \begin{pmatrix}
a_{1,1} & 0 & \cdots & 0 & a_{1,0} \\
a_{2,0} & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & a_{m,0} & a_{m,1}
\end{pmatrix}.
\]

(56)

For acceleration input \( u_i \) at vehicle \( i \), where \( i \in \{2, \ldots, m\} \), we no longer have \( y_n(t) = 0 \) for \( n < i \) since now the vehicles \( n < i \) are affected by the disturbance on vehicle \( i \). The dynamics of vehicle \( i \) are now written as

\[
Y_i(s) = \Gamma_i(s)Y_{i-1}(s) + (sI - a_{i-1,1})^{-1}U_i(s).
\]

(57)

In the case where the acceleration input \( u_1 \) in on vehicle 1, we have \( Y_1(s) = \Gamma_1(s)Y_m(s) + (sI - a_{1,1})^{-1}U_1(s) \), where \( \Gamma_1(s) \) is the transfer function for vehicle \( m \) to vehicle 1. We then have

\[
Y_1(s) = \prod_{j=1}^{m} \Gamma_j(s)Y_j(s) + (sI - a_{1,1})^{-1}U_1(s),
\]

(58)

and finally, with \( u_i, \forall i \in \{1, \ldots, m\} \), being the only non-zero component, we have

\[
Y_i(s) = \left( I - \prod_{j=1}^{m} \Gamma_j(s) \right)^{-1}G_i(s)U_i(s).
\]

(59)

1) SISO homogeneous case: we investigate the particular case where \( u_j \in \mathbb{R}, \forall j \geq 1, y_n \in \mathbb{R}, \forall n \in \{1, \ldots, m\} \), \( \forall k \in \{1, 2\}, \Gamma_{n,k} = \Gamma_1 \). Following Remark 1 and equation (59), as \((a_{n,1})\) is a Hurwitz matrix, exponential stability is achieved when the poles of \((1 - \Gamma_1^n)^{-1}\) have negative real parts. We factorise \((1 - \Gamma_1^n)\) as

\[
1 - \Gamma_1^n = (1 - \Gamma_1) \prod_{k=1}^{m-1} \left( \Gamma_1 - e^{2ik\pi/m} \right),
\]

(60)

and developing from equation (43), the denominator \( D_c \) of \((1 - \Gamma_1^n)^{-1}\) is equal to

\[
D_c = \prod_{k=0}^{m-1} \left( s^2 - s \left( f_1 + f_3 \left( e^{2ik\pi/m} - 1 \right) \right) - f_2 \left( e^{2ik\pi/m} - 1 \right) \right).
\]

(61)

Note that this expression closely resembles the condition for string stability in the infinite homogeneous case derived using the Fourier perturbation technique [25]: the infinite homogeneous platoon system is said to be stable iff \( \forall k \in [0, 2\pi], s^2 - s \left( f_1 + f_3 \left( e^{ik} - 1 \right) \right) - f_2 \left( e^{ik} - 1 \right) \) has negative real parts. This condition can be shown to be equivalent to the \( L_2 \) strict stability condition of equation (49), see [27].

2) General case: in the general case, exponential stability is achieved when the transfer function in equation (59) has poles with negative real parts. From equation (42), we have

\[
\prod_{j=1}^{m} \Gamma_j(s) = \frac{1}{\prod_{j=1}^{m} \gamma_j(s)} \begin{pmatrix} 0 & P_1(s) \\ 0 & P_2(s) \end{pmatrix},
\]

(62)

where \( \gamma_i(s) = s^2 + s(f_{j,3} - f_{j,1}) + f_{j,2} \), and \( P_1(s) \) and \( P_2(s) \) are polynomials of degree \( m \), with

\[
P_2(s) = \prod_{i=1}^{m} (f_{i,2} + s f_{i,3}).
\]

(63)

Given that \( \forall i \in \{1, \ldots, m\} \), the solutions of \( \gamma_i(s) = 0 \) have negative real parts, the system is exponentially stable iff the solutions of the following equation

\[
\prod_{i=1}^{m} \gamma_i(s) - P_2(s) = 0,
\]

(64)

have negative real parts.

Remark 6. Consider a finite closed platoon. There are situations for which the platoon is asymptotically stable and the strict string stability condition of equation (49) is not verified. For example, in the case of a platoon of 3 vehicles, choosing \( f_{n,1} = -0.075 \), \( f_{n,2} = 0.091 \), \( f_{n,3} = 0.55 \) as in Section V-C.2, with \( n \in \{1, \ldots, 3\} \), we have \(|\Gamma_3|_{\infty} > 1\) while the eigenvalues of matrix \( a_c \) have negative real parts.

VI. SIMULATION

In this section we illustrate the previous analytical results regarding string stability using simulations. We focus on \( L_2 \) string stability as we showed its equivalence with \( L_{\infty} \) string stability, see equation (55).

A. Model selection and parameter distributions

The Intelligent Driver Model (IDM) [37] defines function \( f_n \) of equation (41) as:

\[
f_n(\dot{x}_n, \Delta x_n, \Delta \dot{x}_n) = a \left[ 1 - \left( \frac{\dot{x}_n}{V_{\text{max},n}} \right)^4 - \left( \frac{s^*(\dot{x}_n, \Delta \dot{x}_n)}{\Delta x_n - l_n} \right)^2 \right],
\]

(65)

where

\[
s^*(\dot{x}_n, \Delta \dot{x}_n) = s_{0,n} + \max \left( 0, \dot{x}_n T_n - \frac{\dot{x}_n \Delta \dot{x}_n}{2\sqrt{a_n b_n}} \right),
\]

(66)

where the following behavioural parameters are specific to vehicle \( n \): \( V_{\text{max},n} \) is the desired free-flow speed, \( T_n \) is the safe time headway, \( a_n \) is the maximum tolerated acceleration, \( b_n \) is the comfortable deceleration, and \( s_{0,n} \) is the minimum stopping distance.

In order to reproduce realistic heterogeneous traffic in simulation, we have developed a complete methodology to perform robust offline parameter identification starting from noisy trajectory data [38], which involves sensitivity analysis, point estimation and interval estimation. The parameter estimates we use here are the outputs of this methodology.
for the 3 most left lanes of the well-known US101 NGSIM dataset [39], during morning peak time (7:50am to 8:05am). The estimates were found to fit log-normal distributions for parameters $a$ and $b$ and normal distributions for parameters $T$ and $s_0$. The mean and standard deviations are $m_a = 0.77 \text{ m s}^{-2}$, $\sigma_a = 0.42 \text{ m s}^{-2}$, $m_b = 1.1 \text{ m s}^{-2}$, $\sigma_b = 0.43 \text{ m s}^{-2}$, $m_T = 1.5 \text{ s}$, $\sigma_T = 0.57 \text{ s}$, $m_{s_0} = 2 \text{ m}$, and $\sigma_{s_0} = 0.5 \text{ m}$. To reproduce heterogeneous traffic, the parameters are sampled from these distributions truncated at the physical bounds chosen as in the literature: $a \in [0.3, 3]$, $b \in [0.3, 3]$, $T \in [0.3, 3]$, $s_0 \in [0.5, 3.5]$ [40]. With $V_{\text{max}} = 33 \text{ m s}^{-1}$ roughly corresponding to the speed limit of the section, see [40], and taking for instance an equilibrium speed of $V_{eq} = V_{\text{max}}/2$, this gives, following relations (8), (9), (10), $f_{n,1}^2 - 2f_{n,1}f_{n,3} - 2f_{n,2} = -0.063$, meaning that parameters are distributed so that the $L_2$ string stability condition is not verified for a number of vehicles. In the remainder of this paper we write $S_n := f_{n,1}^2 - 2f_{n,1}f_{n,3} - 2f_{n,2}$.

B. Homogeneous traffic: $L_2$ strict string stability

We focus on homogeneous traffic first, i.e. when the behavioural parameters are the same for all vehicles.

1) Relevance of the strict string stability condition: since the autonomous vehicles obey the IDM car-following model, it is of interest to investigate the parameter space for which the model exhibits strict and weak string stable behaviour. In this subsection we focus on the evolution of the two most sensitive parameters, parameters $a$ and $T$, see [40], to gain insights on the possibilities to reach string stable behaviour with realistic parameters values. The rest of the parameters are chosen to have realistic values, see Section VI-A, i.e. $b = 1.1 \text{ m s}^{-2}$, $s_0 = 2 \text{ m}$ and $V_{\text{max}} = 33 \text{ m s}^{-1}$.

Figure 3 plots the contour lines of the string stability coefficient $S_n$. It can be seen that the limit between string stability and string instability depends on the traffic equilibrium speed, i.e. the traffic regime. It appears that, for low equilibrium speeds of $V_{\text{eq}}$, see equation (49), and taking for instance an equilibrium speed of $V_{eq} = V_{\text{max}}/3$, this gives, following relations (8), (9), (10), $f_{n,1}^2 - 2f_{n,1}f_{n,3} - 2f_{n,2} = -0.063$, meaning that parameters are distributed so that the $L_2$ string stability condition is not verified for a number of vehicles. In the remainder of this paper we write $S_n := f_{n,1}^2 - 2f_{n,1}f_{n,3} - 2f_{n,2}$.

number 30 and higher it starts to increase again. Figure 5 displays the evolution of the time and headway time histories as we move downstream in the platoon. The string stability property means that the perturbation fades away. It is also noticeable that the two curves are almost identical, which is not surprising as the sufficient condition for a decrease of the $L_2$ norm is the same for headway and speed perturbations, see equation (49).

A last remark should be made in the light of Figures 4 and 5. It is observed that the perturbation does not completely vanish, i.e. the bounded disturbance is not attenuated to a perfect zero $L_2$ norm as we move downstream the platoon. This is related to the fact that $||\Gamma_{n,2}||_{\infty}$ asymptotically converges towards $\Gamma_{n,2}(0) = 1$, see equation (49) and Figure 2 and that (38) is not a strict inequality, which means that the strict string stability condition does not require long-wave perturbations to be attenuated at a specific rate.
Therefore, a stronger condition than $\mathcal{L}_2$ strict string stability would be to force a sharper decrease of the Bode plot for low frequencies, see Figure 2.

3) Nonlinear vs linear string stability: empirical observations: let us now briefly discuss the dependence of string stability on the size of the acceleration input. For the situation where $a = 1.55 \text{ m/s}^{-2}$, and $T = 0.8 \text{ s}$, $b = 1.7 \text{ m/s}^{-2}$, we have $s_{np} = 0.0038 > 0$. Figure 6a presents the evolution of the time-position diagram for an acceleration input of $-7 \text{ m/s}^{-2}$ between $t_1$ and $t_2$ and Figure 6b presents the evolution of the $L_2$ norm of the speed perturbation for varying acceleration inputs.

It can be seen that, for acceleration inputs of $-5 \text{ m/s}^{-2}$ and $-7 \text{ m/s}^{-2}$, the values of the $L_2$ norm of the speed perturbation are growing as we move downstream the platoon. This contradicts the string stability condition (49). We see that, for an acceleration input of $-7 \text{ m/s}^{-2}$, the perturbation is being amplified until the vehicles completely stop, and the $L_2$ norm seems to be unbounded. For other acceleration inputs, Figure 6b shows that the slope of the $L_2$ norm curve seems to be decreasing as we move downstream the platoon. Finally, when the intensity of the acceleration input is kept within realistic values, i.e. $A < -5 \text{ m/s}^{-2}$, the $L_2$ norms appear to remain bounded as the number of vehicles in the platoon is increased.

Such observations are related to nonlinear effects and to the non-validity of the linearisation hypothesis. If the low speed area spreads through the time-space, the linearised dynamics which satisfy the string stability condition is not valid anymore, and another linearisation about a lower equilibrium speed would indicate string instability, as Figure 3 suggests.
Fig. 6: Evolution of the (a) time vs position diagram for a string stable platoon following an acceleration input of \( A = -7 \) m s\(^{-2}\); (b) \( L_2 \) norm of the speed perturbation for a string stable platoon and varying accelerations inputs.

Fig. 7: Evolution of the speed perturbation for successive vehicles in the case of: (a) \((0,30)\) weak string stability; (b) \((0,30)\) weak string instability.

C. Heterogeneous traffic: \( L_2 \) weak strict string stability

In this section we present an example that highlights the relevance of verifying the weak string stability condition, see equation (52). We consider a platoon composed of 30 vehicles having different behaviour, and we introduce an acceleration input of \(-1\) m s\(^{-2}\) on vehicle 1 between \( t_1 \) and \( t_2 \). The variability in the platoon is introduced by sampling parameters \( a \) and \( T \) from truncated distributions as described in Section VI-A. For instance, we can get a \((0,30)\) weakly string stable platoon, i.e. \( \prod_{i=1}^{30} ||\Gamma_{i,2}||_{H_\infty} \leq 1 \), when \( a \) and \( T \) are sampled from the distributions presented in Section VI-A but defining \( a \in [0.5,3] \) and \( T \in [1.1,3] \); and we can get a \((0,30)\) weakly string unstable platoon, i.e. \( \prod_{i=1}^{30} ||\Gamma_{i,2}||_{H_\infty} = 1.94 > 1 \), by defining \( a \in [0.3,1] \) and \( T \in [0.3,2] \). Note that the \((0,30)\) weakly string stable and weakly string unstable platoons are obtained for particular samples of the truncated distributions.

Besides, the car-following formulation we consider does not deal with the zero speed constraint, i.e. the fact that vehicles cannot have negative speeds.
VII. PARAMETER OPTIMISATION

In this section, the autonomous vehicles update their longitudinal dynamics according to the IDM car-following model. We formulate the following optimisation problem: the vehicle behavioural parameters of each autonomous vehicle are picked to maximise the strict/weak string stability of the platoon, while minimising the distance between their parameter values and the vehicle behavioural parameters when there is no control, e.g. in the case of a partially autonomous vehicle with autonomous and non-autonomous driving modes, when the autonomous mode is deactivated.

Remark 7. Note that we make the choice to rely on in-vehicle sensors to estimate the behavioural parameters of a limited number of leaders and followers, and to use those estimated parameters to design the optimisation policy so as to increase string stability. Another way of increasing string stability is to utilise the car-following model structure itself to integrate V2V communication, see [27], [28] for instance, however by doing that the safe structure of car-following dynamics is lost, i.e. collisions may occur. Our approach of optimising the car-following parameters is key to preserving the safe structure of the car-following dynamics. This enables the design of a new kind of safe ACC system that takes into consideration the driving behaviour of the surrounding vehicles in a heterogeneous platoon as well as the driving comfort of the driver inside the autonomous vehicle.

A. Generic formulation of the optimisation problem

Let \( T \) denote the joint distribution of the car-following parameters. The vector of parameters \( \theta_i \in \mathbb{R}^l, i \in \{1, \ldots, n\} \), defining the dynamics of each vehicle, is sampled from this distribution. We write the covariance matrix of \( T \) as \( \Sigma_T \in \mathcal{M}_{k,k}(\mathbb{R}) \). When there is no correlation between parameters, as in Section VI-A, \( \Sigma_T \) is diagonal, and the elements of the diagonal are the inverses of the standard deviations of each parameter.

For each autonomous vehicle indexed \( n \), we seek to optimise the \( k \in \{1, \ldots, l\} \) parameters \( \theta_n \), with \( \Theta \subset \mathbb{R}^k \) denoting the admissible set of parameter values, e.g. the physical bounds of the parameters defined in section VI-A. Therefore we have \( \theta_n \in \Theta \). In the case of a partially-autonomous vehicle, \( \hat{\theta}_n \) denotes the estimated behavioural parameters for vehicle \( n \) when the autonomous mode is deactivated; in the case of a fully-autonomous vehicle, \( \hat{\theta}_n \) designates average comfortable driving parameters.

1) Relaxation of weak string stability: in Section V-C.2 we discussed how autonomous vehicles can be used to achieve weak string stability. However, there exist situations for which this is not possible: for example, consider a platoon of 3 vehicles with parameters \( a_1 = 0.58 \text{ m/s}^2 \), \( a_2 = 0.35 \text{ m/s}^2 \), \( a_3 = 0.39 \text{ m/s}^2 \), and \( T_1 = 1.76 \text{ s}, T_2 = 1.26 \text{ s}, T_3 = 1.43 \text{ s} \). The rest of the parameters are chosen as \( V_{\text{max}} = 33 \text{ m/s} \), \( b = 1.1 \text{ m/s}^2 \), and \( s_0 = 2 \text{ m} \) as in Section VI-A and \( V_{eq} = V_{\text{max}}/3 \). For this situation, we have \( \prod_{i=1}^3 ||\Gamma_{i,2}||_{\mathcal{H}_\infty} = 1.12 > 1 \), and after numerical simulations we find no values of \( a_4 \in [0.3,3] \) and \( T_4 \in [0.3,3] \) leading to \( \prod_{i=1}^4 ||\Gamma_{i,2}||_{\mathcal{H}_\infty} = 1 \). This means that the weak string stability constraint of equation (52) can not be used as a hard constraint to an optimisation policy. Consequently, in the next sections we relax constraint \( \prod_{i=1}^4 ||\Gamma_{i,2}||_{\mathcal{H}_\infty} = 1 \) to \( \prod_{i=1}^4 ||\Gamma_{i,2}||_{\mathcal{H}_\infty} < \gamma \), with \( \gamma \in \mathbb{R}, \gamma \geq 1 \).

2) Optimisation problem: let \( i \) and \( j \) be the farthest upstream and downstream vehicles for which parameter estimates \( \hat{\theta}_i \) and \( \hat{\theta}_j \) are known. We have \( 1 \leq i \leq n \leq j \leq m \). If there is no knowledge of the behaviour of upstream and downstream vehicles, then \( i = j = n \). Constraints are placed on the \( L_2 \) gain between the speed perturbation of vehicle \( i-1 \) and the speed perturbation of vehicle \( j \), i.e. reflecting our aim of achieving \( (i-1,j) \) weak string instability, see equation (52). The decision variables are the behavioural parameters \( \theta_n \) of the partially-autonomous vehicle \( n \).

We propose the following optimisation problem to capture these design requirements:

\[
\min_{\theta_n, \gamma} \frac{1}{k} \left( \theta_n - \hat{\theta}_n \right)^T \left( \theta_n - \hat{\theta}_n \right) + \frac{1}{\gamma} \left( \theta_n - \hat{\theta}_n \right)^T \left( \Gamma_{i,2} \right) \left( \theta_n - \hat{\theta}_n \right),
\]

s.t. \( \forall (i, j) \in \mathcal{N}_n, \quad ||\Gamma_{i,2} \Gamma_{j,2}||_{\mathcal{H}_\infty} \leq \gamma \)

where the objective is to minimise the distance between the optimised parameters \( \theta_n \) and the vector of parameters of the vehicle \( \hat{\theta}_n \) when the automatic mode is deactivated, as well as to minimise \( \gamma \). Constant \( \alpha \in \mathbb{R}_+^* \) is a design parameter. \( \mathcal{N}_n \) designates the set of pairs of neighbouring upstream and downstream vehicles for which parameter estimates \( \hat{\theta}_j \) are known, with \( i \leq n \leq j \). The \( (i-1,j) \) weak string stability condition is relaxed as discussed in Section VII-

A.1

Remark 8. Note that the minimisation of the \( \mathcal{H}_\infty \) norm of the input-output transfer function \( ||\Gamma_{i,2} \Gamma_{j+1,2} \cdots \Gamma_{j,2}||_{\mathcal{H}_\infty} \) could be formulated as another constraint.

Remark 9. Regarding the values of \( i \) and \( j \), in practice, autonomous vehicles are equipped with sensors which can enable parameter estimation for only a few leading/following vehicles. Looking at equation (52), we need to track the relative position and speeds between vehicle \( n \) and vehicle \( n-1 \) to estimate \( \theta_n \), i.e. considering that only the positions and speeds of 2 upstream and downstream vehicles can be tracked with in-vehicle sensors, we rarely have \( i < n-1 \) and \( j > n+2 \) unless behavioural parameter data is transmitted via communication channels.

3) Limitations of weak string stability: when the knowledge of behavioural parameters is limited to only a few leaders and followers, there exist situations for which the \( (i-1,j) \) weak string stability constraint is verified but the \( (i-1-i_1,j+j_1) \) weak string stability constraint is not, for given \( i_1, j_1 \). For example, taking \( i = j = n \), we have \( ||\Gamma_{i,n}||_{\mathcal{H}_\infty} = 1 \) and \( ||\Gamma_{i,n}^2 \Gamma_{i,n-1}||_{\mathcal{H}_\infty} > 1 \) for the following parameter values: \( a_n = 0.9 \text{ m/s}^2 \), \( b_n = 0.9 \text{ m/s}^2 \), \( T_n = 2.5 \text{ s}, a_{n-1} = 0.5 \text{ m/s}^2 \), \( b_{n-1} = 1.7 \text{ m/s}^2 \), \( T_{n-1} = 0.8 \text{ s} \), with \( V_{\text{max}} = 35 \text{ m/s} \), \( s_0 = 2 \text{ m} \), and \( V_{eq} = V_{\text{max}}/3 \). This means that the verification of the \( (i-1,j) \) weak string stability is not sufficient to ensure a \((0,m)\) weakly string
stable platoon. However, there exist two ways to address this issue in order to provide a more stable platoon dynamics. The first one consists in considering the minimisation of the input-output $L_2$ gain as well, that is the minimisation of $\|G_{n,n}\|_{\mathcal{H}_2}$. The second one consists in adding one (or various) fictitious unstable leading or following vehicle(s), i.e worst case vehicle(s), which will eventually lead to more extreme parameter values compensating the fictitious instabilities. For instance, let us consider the case where the parameters of only vehicle $n$ and vehicle $n - 1$ are known. Then, introducing a worst case vehicle with parameters $\theta_{wc}$, we now perform the minimisation of $\|\Gamma_{w_2,2} \Gamma_{n,n-1} \|_{\mathcal{H}_\infty}$.

**B. LMI formulation of the optimisation problem**

We can rewrite constraints (68) as Linear Matrix Inequalities [41]. Starting from equation (14) and combining the linearised dynamics of cars $i, ..., j$, with $i, j \in \mathcal{N}_s$ and $i \leq j$, we write the following linearised platoon dynamics:

$$
\begin{align*}
\dot{y}_{i,j} &= A_{i,j}y_{i,j} + b_{i,j}u_{i,j}, \quad (69) \\
b_{i,j} &= c_{i,j}y_{i,j}, \quad (70)
\end{align*}
$$

where $y_{i,j}^T = [y_i^T, y_{i+1}^T, ... , y_j^T]$, $u_{i,j}^T = [u_i^T, u_{i+1}^T, ... , u_j^T]$, $b_{i,j} \in \mathcal{M}_{2(j-i+1),2(j-i+1)}$ and $c_{i,j} \in \mathcal{M}_{2(j-i+1),2(j-i+1)}$ are the input weighting and observation matrices, and

$$
A_{i,j} = \begin{pmatrix}
a_{1,1} & 0 & \cdots & 0 \\
a_{i+1,0} & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & a_{j,0} \\
0 & \cdots & 0 & a_{j,1}
\end{pmatrix}. \quad (71)
$$

As the speed $\dot{y}_j$ is observed, we have

$$
c_{i,j} = \begin{pmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}. \quad (72)
$$

The stability constraint is on the $L_2$ gain between speed perturbation $\dot{y}_{i-1}$ and speed perturbation $\dot{y}_j$. We consider the input as $y_{i-1}$. Following equation (14), since the first column of matrix $a_{i,0}$ consists of zeros, only $\dot{y}_{i-1}$ acts as input to the $y_j$ dynamics, i.e. the $\Delta y_{i-1}$ term has no effect since it is multiplied by the zeros in the first column of $a_{i,0}$, which makes it a SISO system. We can therefore write

$$
b_{i,j,1} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}. \quad (73)
$$

Therefore, using the LMI characterisation of the $L_2$ gain, see [41], [42], we can reformulate the optimisation problem (67), (68) as:

$$
\min_{\delta_n, x_{i,j}, \gamma} \alpha \gamma + \frac{1}{K} \left( \delta_n - \theta_n \right) \Sigma_T^{-1} \left( \delta_n - \theta_n \right) \quad (74)
$$

$$
\begin{align*}
\theta_n &\in \Theta, \\
\forall (i,j) &\in \mathcal{N}_n, \\
\left[ A_{i,j}^T X_{i,j} + X_{i,j} A_{i,j} - \gamma I_{i,j} \right] &< 0, \\
X_{i,j} &> 0,
\end{align*} \quad (75)
$$

where $X_{i,j} \in \mathcal{M}_{2(j-i+1),2(j-i+1)}$, and $I_{i,j}$ is the identity matrix of dimension $2(j-i+1) \times 2(j-i+1)$. The optimisation problem is convex in $X_{i,j}$, $\gamma$, but not in the parameters of the car-following model $\theta_n$, and not jointly convex in $A_{i,j}$ and $X_{i,j}$. Even after linearising or convexifying the car-following model, assuming $\Sigma_T^{-1}$ to be positive semidefinite, we would still be facing a biconvex optimisation problem. In this paper, we explore the car-following model parameter space using simulated annealing and solve the convex part of the optimisation using cvx [43] to obtain $X_{i,j}$ and $\gamma$ at each iteration. Note that other heuristics such as the Alternate Convex Search (ACS) [44] may be of use.

**Remark 10.** The constraint concerning the minimisation of the $L_2$ gain between the acceleration input $u_i$ and the speed perturbation $\dot{y}_j$, mentioned in Remark 8, can also be formulated as LMI.

**C. Simulation analyses and main results**

1) Scenario and stochastic variables: although we performed numerous simulation experiments, in this section, we only show the results obtained for a representative example. We consider a platoon composed of 30 vehicles, i.e. $m = 30$, and vehicle $n = 0$ evolving at an equilibrium speed $V_{eq} = V_{max}/3$, as roughly observed in the NGSIM data set, see section VI-B. We introduce an acceleration perturbation to vehicle 1, which is forced to be a non-autonomous vehicle. This perturbation takes the form of a PRBS input sequence of amplitude $[-1, +1]$, which remains constant over time intervals ranging from 2 s to 5 s and has a duration of 1 min. The simulation length is set to 4 min as, given the considered perturbation, this is the time needed to cover all of the effects of perturbation propagation on the 30 vehicle trajectories.

The stochastic variables are the sampled parameters of the 30 vehicles, the acceleration PRBS inputs, the position of the autonomous vehicles in the platoon, and the number of autonomous vehicles in the platoon. Then, we perform $25 \times 4$ simulations: we repeat the simulation 25 times to consider the effects of stochastic variables; and for each of the 25 simulations we fix the seed of the introduced randomness, and consider 4 different configurations of the optimisation strategy (75), e.g. different proportions of autonomous vehicles.
2) Evolution of the $L_2$ norm of the speed perturbation and distribution of optimised parameters: we focus on tuning parameters $a$, $b$, and $T$ for the autonomous vehicles according to optimisation (75), i.e. the tolerated acceleration, comfortable deceleration and safe time headway parameters. We choose $\alpha = 10^3$, and work with $i = n - 1$ and $j = n + 2$, see Section VII-A.2.

First, Figure 8 displays the influence of the optimisation strategy (75) on the evolution of the $L_2$ norm of the speed perturbation in the platoon, following the introduced PRBS acceleration inputs, for different proportion of autonomous vehicles. We plot the average values and errors bars of $\pm 1$ standard deviation over the 25 simulations. The error bars show the impact of the stochastic variables over the outcome of the minimisation. The positive effects are clearly visible: an increasing percentage of autonomous vehicles consistently leads to lower average values and standard deviations of the $L_2$ norm of the speed perturbation. Here the (0,30) weak string stability condition, i.e. decrease of the $L_2$ norm of the speed perturbation, is verified when 30% vehicles are autonomous, which makes sense as the $(n-2, n+2)$ weak string stability condition of (75) involves a total of 4 vehicles, which means that an average of $1/4 = 25\%$ of autonomous vehicles should be enough to guarantee (0,30) weak string stability provided the autonomous vehicles are well dispersed. Note that the PRBS input considered is actually a linear combination of step inputs of $\pm 2$ m s$^{-2}$, which are strong deceleration/acceleration inputs in realistic traffic. Note also that it was observed in section VI-B.3 and Figure 6 that for such inputs the verification of the strict string stability for homogeneous traffic still leads to a decrease of the $L_2$ norm of the speed perturbation.

Second, it is worth exploring whether the proposed optimisation strategy systematically leads to positive outcomes. Figure 9 displays the average, minimum and maximum values of the deviation from the value of the $L_2$ norm of the speed perturbation without any autonomous vehicles, for 3, 6 and 9 autonomous vehicles, i.e. proportions of 10%, 20% and 30%. We observe that the autonomous vehicles with parameters derived from the optimisation strategy (75) contribute to systematically decrease the value of the $L_2$ norm of the speed perturbation, i.e. negative relative $L_2$ norms. In that sense, the proposed optimisation algorithm (75) consistently increases the traffic flow stability of the heterogeneous platoon.

Finally, we look at the distributions of the optimised parameters $\tilde{a}$, $\tilde{T}$ and $\tilde{b}$, displayed in Figures 10, 11, 12. As might be expected, the distributions are shifted but are still realistic, i.e. lead to reasonable driving behaviour by the autonomous vehicles. We observe that the optimisation strategy (75) pushes parameters $a$ and $T$ towards higher values and parameter $b$ towards lower values. This is in accordance
with physical considerations: a longer safe time headway $T$ gives more time for vehicles to damp perturbations; a higher tolerated acceleration helps recover the equilibrium speed faster; a smaller comfortable deceleration results in less sharp braking and helps to smooth perturbations.

We can also observe that parameters $b$ and $T$ are sometimes pushed towards the limits of the selected physical bounds, i.e. in this case $T = 3\text{s}$ and $b = 0.3\text{m.s}^{-2}$. When this is the case, one idea to provide more flexibility to guarantee weak string stability, see Section VII-A.1, is to increase the upper bound $T_{\text{up}}$ of the safe time headway parameter $T$, which is not a critical parameter as it does not depend on the capabilities of the vehicle or does not affect driving comfort as much as other parameters.

3) Enforcing more stable dynamics with very few autonomous vehicles: with very few autonomous vehicles in the platoon, and when the parameters of only a few leading and following vehicles are known, we might be interested in investigating how to enforce an even more stable platoon dynamics. To do so we can introduce fictitious unstable vehicles, see section VII-A.3.

We consider $i = n - 1$ and $j = n - 2$, and 3 autonomous vehicles in the platoon, i.e. $\text{NbAut} = 3$. A fictitious worst case unstable vehicle $n - 2$ is introduced with parameters $a_{n-2} = 0.3\text{ m.s}^{-2}$, $T_{n-2} = 0.3\text{ s}$, $b_{n-2} = 3\text{ m.s}^{-2}$.

Figure 13 shows the results of the optimisation without and with the fictitious introduced vehicle, for 2 different admissible upper values of parameter $T_{\text{up}}$, $T_{\text{up}} = 3\text{ s}$ and $T_{\text{up}} = 5\text{ s}$. It is visible that adding one fictitious unstable vehicle consistently increases weak string stability in the platoon. However, the limitations in the stabilisation effect of autonomous vehicles come from the admissible set of parameter values $\Theta$. By increasing the upper bound of the admissible safe time headway $T_{\text{sn}}$, we are able to bypass this limitation and reach $(0, 30)$ weak string stability with only 3 autonomous vehicles in the platoon. This strategy may lead to less realistic and less comfortable driving behaviour, as parameters tend to move towards their upper/lower admissible bounds. However the physical bounds can be selected appropriately, as for instance parameter $T$ is less critical than parameters $a$ and $b$ in terms of vehicle capabilities and driving comfort, although increasing $T$ may encourage vehicles to change lanes and enter the empty slots created in the platoon.

Finally, the overall conclusion is that the number of autonomous vehicles needed to prevent perturbations from growing can be reduced depending on the following parameters of the optimisation strategy: the number of vehicles for which the behavioural parameters are known, i.e. parameters $i$ and $j$, the set of admissible parameters $\Theta$, and the parameters of introduced fictitious unstable vehicles. Given
a string of vehicles, a small number of autonomous vehicles is enough to damp the effect of realistic perturbations that would otherwise grow.

VIII. CONCLUSION

This paper applies \( L_2 \) linear control theory to linearised systems of vehicles moving accordingly to realistic car-following models. The contributions are the following: (i) a general framework for investigating the stability of heterogeneous platoons in the frequency domain is introduced (most previous studies assume homogeneous traffic); (ii) the definition of weak stability is introduced and its relevance in a traffic environment with a mix of autonomous and non-autonomous vehicles is highlighted; (iii) conditions for input-output stability and string stability are given for heterogeneous traffic, and for single and multiple outputs; (iv) the equivalence between string stability and asymptotic stability is shown not to hold for closed loop systems; (v) simulations underline the critical feature of nonlinearities; (vi) an optimisation strategy to tune the behavioural parameters is proposed which gives

\[
\sigma_{\text{max}}(\Gamma_n(j\omega)) = \sqrt{\lambda_2(\omega)}. 
\]  

(80)

We recall that, by definition, see equation (36), we have \( ||\Gamma_n||_{H_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}}(\Gamma_n(j\omega)) \). The sufficient condition for strict string stability is written \( ||\Gamma_n||_{H_\infty} \leq 1 \), see equation (41), which is equivalent to \( \lambda_2(\omega) \leq 1 \).

Developing \( \lambda_2(\omega) \leq 1 \), and writing \( \Omega = \omega^2 \), we obtain a polynomial of order 2 in \( \Omega \):

\[
\Omega^2 + \Omega(f_{n,3}^2 - 2f_{n,1}f_{n,3} - 2f_{n,2} - 1) - f_{n,1}^2 \geq 0. 
\]  

(81)

As this inequality must be verified \( \forall \Omega \in \mathbb{R}_+ \), we must have \( f_{n,1} = 0 \), and the following sufficient conditions follow:

\[
f_{n,1} = 0, 
\]  

(82)

\[
-2f_{n,2} - 1 \geq 0. 
\]  

(83)

ACKNOWLEDGMENT

This work was supported by SFI grants 11/PI/1177, 13/RC/2077 and 10/IN.1/I2980.

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