THE GENERALIZED CARTAN DECOMPOSITION FOR CLASSICAL RANDOM MATRIX ENSEMBLES

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ABSTRACT. We present a completed classification of the classical random matrix ensembles: Hermite (Gaussian), Laguerre (Wishart), Jacobi (MANOVA) and Circular by introducing the concept of the generalized Cartan decomposition and the double coset space.

Previous authors [40, 8] associate a symmetric space $G/K$ with a random matrix density on the double coset structure $K \backslash G/K$. However this is incomplete.

Complete coverage requires the double coset structure $A = K_1 \backslash G/K_2$, where $G/K_1$ and $G/K_2$ are two symmetric spaces. Furthermore, we show how the matrix factorization obtained by the generalized Cartan decomposition $G = K_1 A K_2$ plays a crucial role in sampling algorithms and the derivation of the joint probability density of $A$.

1. Introduction

In pioneering work by Zirnbauer [40] and Dueñez [8] connections are made between random matrix ensembles and the symmetric spaces [19, 21] studied in differential geometry, representation theory, and harmonic analysis. Unfortunately, symmetric spaces alone do not account for a rich enough set of Jacobi ensemble densities (See Figure 1). This work fills this significant gap.

Noting that more general tools are needed, we not only take a closer look at the KAK decomposition from the theory of symmetric spaces, but we propose the incorporation of the more recent generalized Cartan decomposition (or the $K_1 A K_2$ decomposition) [15, 24, 14, 30] especially for complete coverage of the Jacobi ensembles.

We show that double coset spaces are rich enough to cover these important matrix ensembles. Furthermore, the key object is not the joint density, per se, but the matrix factorization. To be clear, in the compact cases, one should take a natural matrix factorization such as the eigenvalue or CS decomposition [36, 6] of a Haar measure unitary. To this end, we introduce the ODO [17] and a new QDQ decomposition for the circular $\beta = 1, 4$ cases in Sections 4.1 and 4.3. Similarly in the noncompact case, one should take a natural matrix factorization such as the eigenvalue, SVD or Hyperbolic CS decomposition [23] of matrices with sufficiently many independent Gaussian elements. In many cases, the forces of history have shoehorned these ensembles into the eigenvalue format, and this has obscured rather than revealed structure.

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Figure 1. While symmetric spaces served as a classification tool for random matrices for a long time, a slide from the first author’s seminar at MIT [11] in 2014 asks where are the missing Jacobi ensembles and clarification on the Laguerre ensembles. This paper will complete the theory concentrating on the first six rows as examples, but the theory applies to the entire table.

This work also endeavors to make the Lie theory more widely accessible, by rewriting key ideas and reworking the classical proofs in [20] in a way that the authors hope may be more insightful for a much larger audience. Cartan’s theory [3, 4, 5] as developed by Helgason [21] is a crowning mathematical achievement, and it is our hope to open up this theory for the benefit of all. Indeed, in [35, p. 428], Sigurdur Helgason writes about the difficulty of understanding Cartan’s writings:

_Then there was Élie Cartan’s work. But his work was, for one thing, relatively little understood, in spite of its great importance. He was one of the great mathematicians of the period, but his papers were quite a challenge. Hermann Weyl, in reviewing a book by Cartan from 1937 writes: “Cartan is undoubtedly the greatest living master in differential geometry... I must admit that I found the book like most of Cartan’s papers, hard reading.”_

In the same vein, while we are admirers of Helgason’s extensive work, we authors must admit that we in turn found [21] hard reading as well, and this paper attempts to introduce the theory accessible to a larger community, by couching the ideas in terms of what we call ping pong operators, etc.

Summarizing our work, we provide the following:

- A complete classification of the classical ensembles, including the full coverage of the Jacobi ensemble. (Table 1)
• Recognition and classification of matrix factorizations hidden in the (generalized) Cartan decomposition (including the hyperbolic CS decomposition in Section 8).
• Translating some of the key concepts in the Lie theory and the theory of symmetric spaces into easier to follow linear algebra. (Section 3)
• Identification of new matrix factorizations (the QDQ decomposition) relevant to the sampling of the circular ensembles. (Section 4)

The full correspondence with double coset spaces, matrix factorizations, and the classical random matrix ensembles may be found in Table 1.

| Cartan Type | $G/K$ | KAK(K1AK2) decomposition | Decomposition $p = kak^{-1}$ (See $\Phi$ in Remark 2.5) | Random $\beta$-Matrix Ensemble |
|-------------|-------|--------------------------|----------------------------------------------------|-------------------------------|
| Compact     |       |                          |                                                    |                               |
| AI          | $U(n)/O(n)$ | ODO Decomposition | Unitary Symmetric Eigendecomposition | Circular $\beta = 1$ (COE) |
| A           | $U(n)$   | -                        | -                                                  | $\beta = 2$ (CUE)             |
| AII         | $U(2n)/Sp(n)$ | QDQ Decomposition | Unitary Self-Dual Eigendecomposition | $\beta = 4$ (CSE) |
| BDI-I       | $O(n)/\text{O}(p) \times \text{O}(q)$ | Real CSD | CSD of Real Grassmannian (Bi-Stiefel Decomp [22]) | $\beta = 1$ |
| AIII-III    | $U(n)/\text{U}(p) \times \text{U}(q)$ | Complex CSD | CSD of Complex Grassmannian | $\beta = 2$ |
| CII-II      | $Sp(n)/\text{Sp}(p) \times \text{Sp}(q)$ | Quaternion CSD | CSD of Quaternionic Grassmannian | $\beta = 4$ |
|             |         |                          |                                                    |                               |
| Non-Compact |       |                          |                                                    |                               |
| AI          | $\text{GL}(n,\mathbb{R})/\text{O}(n)$ | Real Square SVD | Symmetric (Pos. Def) Eigendecomposition | $\beta = 1$ (GOE) |
| A           | $\text{GL}(n,\mathbb{C})/\text{U}(n)$ | Complex Square SVD | Hermitian (Pos. Def) Eigendecomposition | $\beta = 2$ (GUE) |
| AII         | $\text{GL}(n,\mathbb{H})/\text{Sp}(n)$ | Quaternion Square SVD | Self-Dual (Pos. Def) Eigendecomposition | $\beta = 4$ (GSE) |
| BDI         | $O(p,q)/\text{O}(p) \times \text{O}(q)$ | Real HCSD | HCSD of Real Hyperbolic Grassmannian ( = Real SVD) | $\beta = 1$ |
| AIII        | $U(p,q)/\text{U}(p) \times \text{U}(q)$ | Complex HCSD | HCSD of Complex Hyperbolic Grassmannian ( = Complex SVD) | $\beta = 2$ |
| CII         | $\text{Sp}(p,q)/\text{Sp}(p) \times \text{Sp}(q)$ | Quaternion HCSD | HCSD of Quaternion Hyperbolic Grassmannian ( = Quaternion SVD) | $\beta = 4$ |

Table 1. Correspondence between double coset spaces (which may be a symmetric space), matrix factorizations, and the classical random matrix ensembles.
2. Background

Our interest in Lie Theory emanates from the fact that the famous $K$’s, $P$’s, and the refinement of the $P$’s to the $A$’s generate so many of the important factorizations of applied mathematics. With symmetric spaces we only have Jacobians as products of sines of the “restricted roots”, but random matrix theory requires cosines in the case of Jacobi, which is where the generalized Cartan decomposition comes in.

We start out by introducing the theory related to the generalized Cartan decomposition. For readers without preliminary knowledge in Lie theory, we recommend skipping to Section 3 which follows a more modern linear algebra approach.

2.1. Generalized Cartan Decomposition. Let $(G, K_\tau), (G, K_\sigma)$ be two symmetric pairs with real reductive Lie group $G$ such that $\text{Lie}(K_\tau) = \{g \in \mathfrak{g} | \tau(g) = g\}, \text{Lie}(K_\sigma) = \{g \in \mathfrak{g} | \sigma(g) = g\}$. If $G$ is noncompact, let $\sigma$ be its Cartan involution. Assume $\tau \sigma = \sigma \tau$. Define $p_\tau := \{g \in \mathfrak{g} | \tau(g) = -g\}$ and let $p_\sigma$ be defined similarly. Let $a$ be a maximal abelian subalgebra of $p_\tau \cap p_\sigma$ and define $A := \exp(a)$. We start with the generalized Cartan decomposition.

**Theorem 2.1** (Generalized Cartan Decomposition, $K_1AK_2$ Decomposition). Any reductive Lie group $G$ with $\tau$ and $\sigma$ as above can be decomposed as,

$$G = K_\tau AK_\sigma,$$

That is, for any $g \in G$, we have $k_1 \in K_\tau, k_2 \in K_\sigma$ and $a \in A$ such that $g = k_1ak_2$.

From the space of linear functionals $a^*$, we collect eigenvalues of an adjoint representation (the commutator) of $a$ on $\mathfrak{g}$ and call these eigenvalues the roots of the $K_1AK_2$ decomposition. By fixing the Weyl chamber, we obtain a set of positive roots $\Sigma^+$. Details of the theory of the generalized Cartan decomposition and its root system can be found in Flensted-Jensen [15, 14], Matsuki [30, 31, 32] and Kobayashi [29]. The generalized Cartan decomposition is also studied in the context of spherical harmonics and intertwining functions [24]. Double coset spaces also appear in [25]. Now let $m_\alpha^\pm$ be the dimensions of the root space of the root $\alpha$ refined by eigenvalues $\pm 1$ of $\sigma \tau$. Let $dk_\sigma, dk_\tau$ be Haar measures of $K_\sigma, K_\tau$, respectively. Assume for a moment $G$ is noncompact.

**Theorem 2.2.** Let $dg$ be the Haar measure on $G$ and let $H \in \mathfrak{a}$ be such that $a = \exp(H)$. Then we have the Jacobian and the integration corresponding to the change of variables associated with the decomposition $(2.1)$, $g = k_\sigma ak_\tau$,

$$\int_G f(g)dg = \int_{K_\tau} \int_A \int_{K_\sigma} f(k_\sigma ak_\tau) \, d\mu(a) \, dk_\sigma \, dk_\tau,$$

where

$$d\mu(a) \propto \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha^+}(\cosh \alpha(H))^{m_\alpha^-}dH.$$

The proof of (2.2) is attached in the Appendix. The result when $G$ is compact is identical upon replacing $\sinh \alpha(H)$ (and $\cosh \alpha(H)$) with $\sin \alpha(H)$ (and $\cos \alpha(H)$). If $\tau = \sigma$, we recover the famous KAK decomposition. Results regarding the KAK decomposition and restricted roots of symmetric spaces can be found in standard Lie group textbooks [23, 19, 28, 18, 2]. In this case, the Jacobian (2.2) reduces down to $\prod (\sinh \alpha(H))^{m_\alpha}$ as it can be found in several references [27, 20, 26]. Since
the invariant measure of $G/K$ inherits the Haar measure of $G$, the same Jacobian is obtained for the decomposition of a symmetric space.

**Theorem 2.3.** For a noncompact symmetric space $G/K$ we have the map $\Phi$,

$$\Phi : (K, A) \to G/K, \quad \Phi(k, a) = kaK.$$  

Let $H \in a$ with $a = \exp(H)$. For the $G$-invariant measure $dx$ of $G/K$, Haar measure $dk$ of $K$ and Euclidean measure $dH$ on $H$, relationship $dx = dk d\mu(a)$ holds with

$$d\mu(a) \propto \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha} dH. \quad (2.3)$$

Similar result is obtained for the compact case.

**Theorem 2.4.** For a compact symmetric space $G/K$ we have the map $\Phi$,

$$\Phi : (K, A) \to G/K, \quad \Phi(k, a) = kaK.$$  

Let $H \in a$ with $a = \exp(H)$. For the $G$-invariant measure $dx$ of $G/K$, the Haar measure $dk$ of $K$ and the Euclidean measure $dH$ on $H$, the relationship $dx = dk d\mu(a)$ holds with

$$d\mu(a) \propto \prod_{\alpha \in \Sigma^+} (\sin \alpha(H))^{m_\alpha} dH. \quad (2.4)$$

**Remark 2.5.** Some authors \cite{27, 20, 21, 15} use $\prod \sinh \alpha(H)$ as the Jacobian, whereas some authors \cite{20, 8, 37} use $\prod \sin(\alpha(H)/2)$. This gap is due to the difference in the realization of a symmetric space $G/K$ as a subset $P \subset G$. The former uses the right coset representative, i.e., $G/K \to P$ as $gK = p$ where $g = pk$ is its (generalized) polar decomposition. Then the $G$-action of $G$ on $G/K$ is given as $(g_1, g_2K) \mapsto g_1 g_2 K$. Latter authors use the map $G/K \to P$ such that $gK \mapsto g(\sigma g)^{-1}$ where $\sigma$ is the group level involution. The $G$-action is $(g_1, g_2) \mapsto g_1 g_2 (\sigma g_1)^{-1}$, $g_1 \in G, g_2 \in P$. In terms of Thm 2.3, the latter gives the map $\Phi$ such that $(k, a) \mapsto ka^2 k^{-1}$ since

$$g(\sigma g)^{-1} = p k (\sigma p k)^{-1} = pk(p^{-1} k)^{-1} = p k^{-1} p = p^2 = kak^{-1} k^{-1} = ka^2 k^{-1}$$

and this explains the extra factor $\frac{1}{2}$ applied to $H$. More importantly, under these identifications the map $\Phi$ can be defined in a more intuitive way, $\Phi : (K, A) \to P$, such that

$$\Phi(k, a) \mapsto kak^{-1} \quad \text{or} \quad \Phi(k, a) \mapsto ka^2 k^{-1} \quad (2.5)$$

(depending on the author’s notational choice) and this $\Phi$ (with either choice) is listed in the column $p = kak^{-1}$ in Table 1.

Equipped with these tools, we work with 12 double coset spaces which are connected to four famous numerical matrix factorizations: Unitary Eigendecomposition, CS Decomposition, Hermitian Eigendecomposition and SVD (or Hyperbolic CSD). These four factorizations imply the four most famous classical random joint densities: Circular, Hermite(Gaussian), Laguerre(Wishart) and Jacobi(MANOVA) Ensembles. Starting from Section 4 the detailed classification and connections will be made. For the double coset space $K_\tau \backslash G/K_\tau$, we name the double coset decomposition after the two types of involution from Cartan’s Classification, e.g., $O(n) \backslash U(n)/O(n)$ is denoted compact AI-I.
3. Cartan’s idea with Linear Algebra

The Jacobian of the KAK Decomposition (and the generalized Cartan decomposition), equivalently the determinant of the differential of the map \( \Phi : K \times A \rightarrow P \), (See Thm 2.3 [4], 2.4 and Remark 2.5) is computed with Lie theoretic tools in the Appendix. The proof derives (2.2) by extending the pioneering work of Cartan and Helgason [5]. However the proof can be inaccessible to some audiences. Meanwhile, individual cases of the KAK decomposition, recognized as matrix factorizations, show up in many areas of mathematics, and some were discovered in various formats by specialists in numerical linear algebra. Motivated by Random Matrix Theory (and sometimes perturbation theory in numerical analysis), Jacobians of these factorizations were often computed case-by-case using the matrix differentials and wedging of independent elements [16, 9, 13, 33].

In this section, we provide a generalization of such individual Jacobian computations and compare it to the general technique Helgason proposed. With appropriate translation of terminologies and maps in Lie theory into linear algebra, we observe both methods are indeed the same process but have been illustrated in different languages for a long time. We start out by introducing some important concepts in Lie theory accessible to an audience with a good background in linear algebra and perhaps some basic geometry. Then, in Table 3, we present a line-by-line correspondence between Helgason’s derivation and the proof by matrix differentials.

3.1. The ping pong operator, ping pong vectors and ping pong subspaces.

We will start with a concrete \( 2 \times 2 \) linear operator so as to establish the notions of the ping pong operator, ping pong vectors, ping pong subspaces and the relationship to eigenvectors. Then we will define a “bigger” linear operator \( \text{ad}_H \) that acts on \( 2 \times 2 \) spaces exactly in the manner we are about to describe.

We introduce the \( 2 \times 2 \) matrix 

\[
M := \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

which we will call a 2 by 2 ping pong operator and we will call \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) the ping pong vectors of \( M \), in that \( M \) bounces these two vectors into \( \alpha \) times the other,

\[
M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad M \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Furthermore \( M \) has eigenvectors \( \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \), with eigenvalues \( \alpha, -\alpha \). We will call the eigenvalue a root of \( M \).

Also worth pointing out are the matrix exponential and matrix sinh of \( M \),

\[
e^M = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \quad \text{and} \quad \sinh M = \frac{1}{2} (e^M + e^{-M}) = \sinh \alpha \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

and thus we see that \( \sinh M \) is another ping pong operator with scaling \( \sinh \alpha \).

Figure 2 plots the action of a ping pong matrix and its exponential, with notations that we will use in the next sections, i.e, the ping pong operator is denoted \( \text{ad}_H \), \( p_j \) and \( k_j \) are the ping pong vectors, and \( x_j \) and \( \theta x_j \) are the eigenvectors. The right side of Figure 2 shows the action of \( e^M \) and portrays \( \sinh(M) \) as a projection of \( e^M \) on the \( p_j \) direction.
We now go beyond $2 \times 2$ matrices, and suggest the more general $2N \times 2N$ ping pong matrix $M_N$, with $N$ roots, $\alpha_1, \ldots, \alpha_N$, $N$ pairs of ping pong vectors $(k_1, p_1), \ldots, (k_N, p_N)$ along with eigenvectors $(x_1, y_1), \ldots, (x_N, y_N)$.

\begin{equation}
M_N = \begin{pmatrix}
0 & \alpha_1 \\ 
\alpha_1 & 0 \\
\vdots & \ddots \\
0 & \alpha_N \\
\alpha_N & 0
\end{pmatrix}
\end{equation}

\begin{align}
k_j, p_j, x_j, y_j &= \begin{bmatrix}
\vdots \\
1 \\
0 \\
1 \\
\vdots
\end{bmatrix}, \\
x_j &= \begin{bmatrix}
\vdots \\
1 \\
1 \\
\vdots
\end{bmatrix}, \\
y_j &= \begin{bmatrix}
\vdots \\
1 \\
-1 \\
\vdots
\end{bmatrix},
\end{align}

where the $2j - 1$ and $2j$ positions are $0$ or $\pm 1$ and all other entries of these vectors are $0$. The matrices $\exp(M_N)$ and $\sinh M_N$ are block versions of the $2 \times 2$ case.

We may define the subspaces, $\mathfrak{k}$ and $\mathfrak{p}$ (using the “mathfrak” Fraktur letters “k” and “p”) to be the span of the $k_j$ and $p_j$ respectively. Notice that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal complements as subspaces. A key “ping pong” relationship between these subspaces is that $M_N k \in \mathfrak{p}$ if $k \in \mathfrak{k}$,

$M_N p \in \mathfrak{k}$ if $p \in \mathfrak{p}$.

Thus, if we consider $M_N|_{\mathfrak{k}}$, the restriction of $M_N$ to $\mathfrak{k}$ we have an operator from $\mathfrak{k}$ to $\mathfrak{p}$. Evidently, $M_N|_{\mathfrak{k}}$ as a matrix may be obtained by taking the even rows and odd columns of $M_N$. Similarly $\sinh(M_N)|_{\mathfrak{k}}$ is a diagonal matrix with $\sinh(\alpha_j)$ on the diagonal. We then get the important result that

$$\det(\sinh(M_N)|_{\mathfrak{k}}) = \prod_{j=1}^{N} \sinh \alpha_j,$$

the product of the hyperbolic sines of the roots.

Given a linear operator $\mathcal{L}$ on a vector space with nonzero eigenvalues $\pm \lambda$, the following lemma constructs a pair of ping pong vectors from $\mathcal{L}$.

**Lemma 3.1.** For a linear operator $\mathcal{L}$ defined on any vector space, assume $\pm \lambda$ are both nonzero eigenvalues of $\mathcal{L}$. Let $x$ and $y$ be the corresponding eigenvectors, i.e., $\mathcal{L}x = \lambda x$ and $\mathcal{L}y = -\lambda y$. Define two vectors $k := x + y, p := x - y$. Then, $k, p$ are ping pong vectors. Furthermore we have for the operator $\exp(\mathcal{L})$,

$$e^{\mathcal{L}}k = \cosh \lambda k + \sinh \lambda p, \quad e^{\mathcal{L}}p = \sinh \lambda k + \cosh \lambda p.$$

The proof is a straightforward extension of the discussion in previous paragraphs.

**Remark 3.2.** For the reader who wants to know the upcoming significance of this fact for Jacobians of matrix factorizations, it turns out (or maybe as the reader already observed in Section 2) that the Jacobian will be the product of $\sinh \alpha$’s. Just as the matrix $\sinh \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ takes one of the ping pong vectors to $\sinh \alpha$
times the other, the key piece of the differential map will consist of multiple ping pong relationships, each one sending one ping pong vector to another.

3.2. The Kronecker product, linear operator \( \text{ad}_X \) and its exponential.

Lie theory picks out operators \( \mathcal{L} \) that exactly have the properties in Section 3.1. Our vector spaces are now matrix spaces, and our operators are linear operators on a matrix space. We introduce the Lie bracket, denoted by \([X,Y]\), defined as \([X,Y] = XY - YX\) (the commutator). The Kronecker product notation is very helpful in this context. We define the Kronecker product notation \( I \) as a linear operator on a matrix space.

\[
(A \otimes B)X = BXAT.
\]

With this, we can express the Lie bracket with Kronecker products.

\[
(I \otimes X - X^T \otimes I)Y = XY - YX.
\]

Consider the Lie bracket as a linear operator (determined by \( X \)) applied to \( Y \), and call this operator \( \text{ad}_X \). (abbreviation for “adjoint”)

\[
\text{ad}_X(Y) = [X,Y].
\]

This will be the important ping pong operator \( L \). The operator exponential of \( \text{ad}_X \) (equivalently, the matrix exponential of \( I \otimes X - X^T \otimes I \)) is given in the following.

Lemma 3.3. For the linear operator \( \text{ad}_X \), the following holds for \( e^{\text{ad}_X} := \sum_{j=0}^{\infty} \frac{(\text{ad}_X)^j}{j!} \) and \( \sinh \text{ad}_x = \frac{e^{\text{ad}_x} - e^{-\text{ad}_x}}{2} \):

\[
e^{\text{ad}_X} Y = e^X Y e^{-X} \quad \text{and} \quad (\sinh \text{ad}_x)Y = (e^X Y e^{-X} - e^{-X} Y e^X)/2.
\]

Proof. The proof is straightforward by the identity \(3.4\). \( e^X Y e^{-X} = ((e^{-X})^T \otimes e^X)Y \) and \( e^{\text{ad}_X} Y = \exp(I \otimes X - X^T \otimes I) Y \). It is left to prove \( (e^{-X})^T \otimes e^X = \exp(I \otimes X - X^T \otimes I) \). Since \( I \otimes X \) commutes with \( X^T \otimes I \), we have

\[
\exp(I \otimes X - X^T \otimes I) = e^{\otimes X} e^{-X^T \otimes I} = (I \otimes e^X)((e^{-X})^T \otimes I) = (e^{-X})^T \otimes e^X,
\]

proving the result. The sinh result follows trivially.

3.3. Antisymmetric and symmetric matrices: an important first example of symmetric space as ping pong spaces. In our first example, our vector space is \( n \times n \) real matrices. Consider

\[
\mathfrak{t} = \text{Antisymmetric matrices}
\]

\[
\mathfrak{p} = \text{Symmetric matrices}.
\]

The ping pong operator that will bounce \( \mathfrak{t} \) and \( \mathfrak{p} \) around will be \( \text{ad}_H = I \otimes H - H^T \otimes I \), where \( H \) is the diagonal matrix

\[
H = \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix}.
\]

\(^1\)Many authors would write \( \text{vec}(BXAT) = (A \otimes B)\text{vec}(X) \), but we omit the “vec” as we believe it is always clear from context. In a computer language such as Julia, one would write \( \text{kron}(A,B) * \text{vec}(X) = \text{vec}(B*X*A') \).
Notice that the operator $\text{ad}_H$ sends an antisymmetric matrix to a symmetric matrix, and a symmetric matrix to an antisymmetric matrix.

What does this have to do with Jacobians of matrix factorizations such as the symmetric positive definite eigenvalue factorization? Consider a perturbation of $Q$ when forming $S = QAQ^T$. An infinitesimal antisymmetric perturbation $Q^T dQ$ is mapped into a $dS$, a infinitesimal symmetric perturbation. This is the very linear map from the tangent space of $Q$ to that of $S$ that we wish to understand, so perhaps it is not surprising we would want to restrict our ping pong operator from $\mathfrak{k}$ to $\mathfrak{p}$. We invite the reader to check that the corresponding eigenmatrices and ping pong matrices of $\text{ad}_H$ which may be found in the first column of Table 2.

3.4. General $\mathfrak{k}$ and $\mathfrak{p}$ arise from an involution $\theta$. We proceed to construct more important general operators $\mathcal{L}$ that have the property in the assumption of Lemma 3.1. This is where the theory of Lie groups and symmetric spaces need to be brought in. Upon doing so, we will obtain two linear spaces of matrices $\mathfrak{k}$, $\mathfrak{p}$ and also a space $\mathfrak{a}$.

For the reader not familiar with Lie groups, one need only imagine a continuous set of matrices which are a subgroup of real, complex, or quaternion matrices. The tangent space $\mathfrak{g}$ is just a vector space of matrix differentials at the identity. One key example is the compact Lie group $O(n)$ (the group of square orthogonal matrices) and its tangent space at the identity $\mathfrak{g}_{O(n)}$: the set of anti-symmetric matrices. Another key example is all $n$-by-$n$ invertible matrices $GL(n, \mathbb{R})$ (a noncompact Lie group), and its tangent space $\mathfrak{g}_{GL(n, \mathbb{R})}$, consisting of all $n$-by-$n$ matrices.

Cartan noticed that important matrix factorizations start with two ingredients: the tangent space $\mathfrak{g}$ (at the identity) of a Lie group $G$ and an involution $\theta$ on $\mathfrak{g}$. (i.e., $\theta^2 = 1$ and $\theta [X, Y] = [\theta X, \theta Y]$) An example of $\theta$ is $\theta(X) = -X^T$ on $\mathfrak{g}$, for $G = GL(n, \mathbb{R})$. Among matrices in $\mathfrak{g}$, we select two kinds of matrices. The ones fixed by the involution $\theta$ and the ones negated by $\theta$. Denote each set by $\mathfrak{k}$ and $\mathfrak{p}$.

$$\mathfrak{k} := \{ g \in \mathfrak{g} \mid \theta(g) = g \}, \quad \mathfrak{p} := \{ g \in \mathfrak{g} \mid \theta(g) = -g \}. \quad (3.8)$$

(For $GL(n, \mathbb{R})$, these are the anti-symmetric and symmetric matrices respectively.)

The next important player is $\mathfrak{a} \subset \mathfrak{p}$. Readers familiar with the singular value decomposition know the special role of diagonal matrices in the svd as they list the very important “singular values.” Diagonal matrices have the nice property that linear combinations are still diagonal, they commute (the Lie bracket of any two are zero), and they are symmetric (the $\mathfrak{p}$ of our first example). The generalization of this is to take a $\mathfrak{p}$, and find a maximal subalgebra where every matrix commutes. This is the maximal subspace $\mathfrak{a} \subset \mathfrak{p}$ such that for all $a_1, a_2 \in \mathfrak{a}$, $[a_1, a_2] = 0$.

If $H \in \mathfrak{a}$, then $S = QAQ^T$ is a symmetric positive definite eigendecomposition, with $\Lambda = e^H$. In the rest of the section we will be focusing on factorizations of the form $QAQ^{-1}$ where $\Lambda$ is a matrix exponential of $H \in \mathfrak{a}$. (These will be more general than eigendecompositions, as $Q$ may not be orthogonal, and $\Lambda$ may not be diagonal.) In particular, we will compute the Jacobian of perturbations with respect to $Q$, holding $H$ constant, and thus necessarily the Jacobian will be defined in terms of $H$.

From here we assume that the Lie group $G$ is noncompact. The compact case will be discussed after completing the noncompact case. Pick $H \in \mathfrak{a}$, and recall that $\text{ad}_H$ is a linear operator on $\mathfrak{g}$. The operator $\text{ad}_H$ will play the role of $\mathcal{L}$, the ping pong operator. We decompose $\mathfrak{g}$ into the eigenspaces of $\text{ad}_H$. For any eigeneigenvectors...
(α_j, x_j) of ad_H, i.e., \(\text{ad}_H(x_j) = [H, x_j] = \alpha_j x_j\), we observe (for \(\alpha_j \neq 0\))
\[
\text{ad}_H(\theta x_j) = [H, \theta x_j] = -[\theta H, \theta x_j] = -\theta([H, x_j]) = -\alpha_j \theta x_j,
\]
which implies the eigenvalues ±α_j always exist in pairs, with corresponding eigenmatrices \(x_j\) and \(\theta x_j\). This satisfies the assumption of Lemma 3.1, from which we can now construct our ping pong matrices,
\[
(3.9) \quad k_j := x_j + \theta x_j \quad p_j := x_j - \theta x_j,
\]
with the ping pong relationship by the operator \(\text{ad}_H\),
\[
(3.10) \quad \text{ad}_H k_j = \alpha_j p_j \quad \text{ad}_H p_j = \alpha_j k_j.
\]
Also the relationship by the operator \(e^{\text{ad}_H}\) follows,
\[
(3.11) \quad e^{\text{ad}_H} k_j = \cosh \alpha_j k_j + \sinh \alpha_j p_j,
\]
\[
(3.12) \quad e^{\text{ad}_H} p_j = \sinh \alpha_j k_j + \cosh \alpha_j p_j.
\]
The ping pong matrices \(k_j, p_j\), eigenmatrices \(x_j, \theta x_j\) and the relationships (3.10), (3.11) are illustrated in Figure 2.

![Figure 2](image-url)

**Figure 2.** The eigenmatrices \(x_j, \theta x_j\) and ping pong matrices \(k_j, p_j\) in the tangent space \(\mathfrak{g}\). The operators are illustrated in blue lines.

The operator \(\text{ad}_H\) and ping pong relationship (left, (3.10)), the operator \(e^{\text{ad}_H}\) on \(k_j\) to \(p_j\) (right, (3.11)).

As we mentioned in Remark 3.2 and Section 3.3, the role of ping pong matrices \(k_j, p_j\) are crucial. The map \(e^{\text{ad}_H}\) (particularly, \(\sinh \text{ad}_H\)) is the main ingredient constructing the differential map \(d\Phi\) of the factorization \(\Phi : (Q, \Lambda) \mapsto QAQ^{-1}\). The operator \(e^{\text{ad}_H}\) is applied to \(k_j\) and then projected to the span of \(p_j\) as in Figure 2 (right), leaving only the \(\sinh \alpha_j\) factor.

We now compute the full basis of \(\mathfrak{t}\) and \(\mathfrak{p}\). The collection \(\cup_j \{x_j, \theta x_j\}\) is a full basis for the union of eigenspaces with nonzero eigenvalues. Since \(\text{span}(\{x_j, \theta x_j\}) = \text{span}(\{k_j, p_j\})\) for any \(j\), \(\cup_j \{k_j, p_j\}\) is another full basis for the eigenspaces with nonzero eigenvalues. Interestingly, we observe \(\theta k_j = k_j\) and \(\theta p_j = -p_j\), which identifies \(\cup_j \{k_j\}\) and \(\cup_j \{p_j\}\) as subsets of the basis of \(\mathfrak{t}\) and \(\mathfrak{p}\) respectively. The remaining case is the zero eigenspace. When \(\alpha_j = 0\), there are two possibilities.
Firstly, if $x_j$ and $\theta x_j$ are independent of each other, we can still obtain $k_j$ and $p_j$ as before and add them to $\cup_j \{k_j\}$ and $\cup_j \{p_j\}$. Secondly, if $x_j$ and $\theta x_j$ are collinear, $\theta x_j$ is either $x_j$ or $-x_j$. If $\theta x_j = x_j$ we collect such $x_j$ and name the set $K$. Similarly, if $\theta x_j = -x_j$ then we put them in $P_z$. Since we analyzed both nonzero and zero eigenspaces, we have obtained a full basis of $g$, which is $(\cup_j \{k_j\}) \cup K_z \cup P_z$. Refining once more, span($\cup_j \{k_j\}$) $\cup K_z$ = $\mathfrak{t}$ and span($\cup_j \{p_j\}$) $\cup P_z$ = $\mathfrak{p}$.

3.5. The operators $ad_H, e^{ad_H}$, and the subspaces $\mathfrak{t}, \mathfrak{p}$. In Section 3.4 we obtained the basis of $\mathfrak{t}$ and $\mathfrak{p}$, in terms of ping pong matrices, by linearly combining eigenmatrices of the operator $ad_H$. We now illustrate the relationship of the basis of $\mathfrak{t}$ and $\mathfrak{p}$ under $e^{ad_H}$, just like we illustrated the operator $M_N$ in Section 3.1. In the $k_1, \ldots, k_N$ and $p_1, \ldots, p_N$ basis we have the following.

\[
e^{ad_H} \begin{bmatrix} k_1 \\ p_1 \\ \vdots \\ k_N \\ p_N \end{bmatrix} = \begin{bmatrix} \cosh \alpha_1 & \sinh \alpha_1 \\ \sinh \alpha_1 & \cosh \alpha_1 \\ & \ddots \\ \cosh \alpha_N & \sinh \alpha_N \\ \sinh \alpha_N & \cosh \alpha_N \end{bmatrix} \begin{bmatrix} k_1 \\ p_1 \\ \vdots \\ k_N \\ p_N \end{bmatrix}
\]

We are now ready to carefully investigate the map $d\Phi$, using (3.13).

Remark 3.4. Results in Lie theory imply that the eigenmatrices $x_j$ and $\theta x_j$ of $ad_H$ are independent of the choice of $H \in \mathfrak{a}$. In other words, the complete basis of $g$ and $\mathfrak{t}$, $\mathfrak{p}$ obtained above does not care about the specific choice of $H$. Furthermore, the eigenvalues $\pm \alpha_j$ are functions of $H \in \mathfrak{a}$ and these eigenvalue assigning functions $\alpha_j : H \mapsto \alpha_j \in \mathbb{R}$ are more properly called the restricted roots. It can be inferred from the separation of the basis that $\mathfrak{t}$ and $\mathfrak{p}$ together form the whole tangent space $g$.

\[
g = \mathfrak{t} \oplus \mathfrak{p}.
\]

3.6. Symmetric Spaces. The reader may have noticed that our discussions have focused on the Lie algebras rather than the Lie groups themselves. It is point of fact, that Lie groups are mostly useful to define the factorizations of our interest, but Lie algebras are where the Jacobian “lives” and hence this is the most important place to concentrate. For the interested reader, the subgroup $K$ of $G$ is picked such that its tangent space is exactly $\mathfrak{t}$ (one easy way to imagine such a subgroup is to define $K := \exp(\mathfrak{t})$, and we now obtain the Symmetric Space $G/K$.

It can be proven that for the noncompact Lie group, there exists a unique involution $\theta$ such that the subgroup $K$ is the maximal compact subgroup of $G$. We call $\theta$ the Cartan Involution and (3.14) is called the Cartan Decomposition. Furthermore the subset $P := \exp(\mathfrak{p})$ plays an important role as its elements serve as representatives of the cosets in $G/K$. Regarding the identification of $G/K$ as elements in $P$, refer to the remark 2.5 where we point out as an example, taking $G/K = GL(n, \mathbb{R})/O(n)$ that an element of $G/K$ has the form of a coset $gK$, then $gg^T$ may be a representative of the coset in $\mathfrak{p}$. While some authors use $(gg^T)^{1/2}$, the key point being each choice is well-defined independent of choice of representative.

3.7. When $G$ is a compact Lie group. Upon considering the compact cases, it is helpful to make use of a certain duality between compact and noncompact symmetric spaces. We again start with a Lie group $G_C$(to emphasize that now the group is compact) and an involution $\theta$. In the tangent space $\mathfrak{g}_C$, we define $\mathfrak{t}_C$ and...
\[ p_C \text{ exactly the same way we did in the noncompact case. Then, we think about a new tangent space,} \]

\[ g := k_C \oplus ip_C, \]

where \( i \) is the imaginary unit. Another result in Lie theory implies that the new vector space \( g \) is the tangent space of some noncompact Lie group \( G \). (See Figure 3.) Furthermore, the vector spaces in the previous section, \( k \) and \( p \), correspond to \( k_C \) and \( ip_C \). Matrixwise, the ping pong matrices \( k_j \in k, p_j \in p \) of \( g \) are brought back to a new set of ping pong matrices \( k_j \in k_C, ip_j \in p_C \) in \( g_C \). Let’s denote them by \( \tilde{k}_j := k_j \) and \( \tilde{p}_j := ip_j \). The role of the subspace \( a \) is now played by \( ia \). replacing

| \( G \) | \( \frac{GL(n, \mathbb{R})}{O(n)} \) | \( \frac{U(n)}{O(n)} \) | \( \frac{O(p, q)}{O(p) \times O(q)} \) | \( \frac{O(n)}{O(p) \times O(q)} \) |
|---|---|---|---|---|
| \( x_l \) |  |  |  |  |
| \( j \quad k \)
| \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ -1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ k' \end{pmatrix} \)
| \( \begin{pmatrix} j' \quad k' \\ -1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j' \quad k' \\ -1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j ' \quad k ' \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j ' \quad k ' \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j ' \quad k ' \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j ' \quad k ' \\ 1 & 0 \end{pmatrix} \)

| \( \theta x_l \) |  |  |  |  |
|---|---|---|---|---|
| \( j \quad k \)
| \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ -1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ k' \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ -1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ k' \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 1 & 0 \end{pmatrix} \)

| \( k_l \) |  |  |  |  |
|---|---|---|---|---|
| \( j \quad k \)
| \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 0 & 1 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ -1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ -1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ -1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 1 & 0 \end{pmatrix} \)

| \( p_l \) |  |  |  |  |
|---|---|---|---|---|
| \( j \quad k \)
| \( \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 0 & i \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ i & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j \quad k \\ -1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j ' \quad k ' \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j ' \quad k ' \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j ' \quad k ' \\ 1 & 0 \end{pmatrix} \)
| \( \begin{pmatrix} j ' \quad k ' \\ 1 & 0 \end{pmatrix} \)

Table 2. Examples of eigenmatrices \( x_l, \theta x_l \) and ping pong matrices \( k_l, p_l \). By \( x_l + \theta x_l \) and \( p_l = x_l - \theta x_l \) as defined in (3.9). \( k_l, p_l \) are normalized to have \( \pm 1 \) entries. A block structure on row/columns \( j, k \) and \( j' := p + j \) and \( k' := p + k \) are filled up with 0 and \( \pm 1 \).
ad_H by ad_iH. We deduce a set of similar relationships for \( \tilde{k}_j, \tilde{p}_j \) under ad_iH.

\[
\begin{align*}
\text{ad}_H(\tilde{k}_j) &= \alpha_j \tilde{p}_j, \\
\text{ad}_H(\tilde{p}_j) &= -\alpha_j \tilde{k}_j.
\end{align*}
\]

In matrix form,

\[
\begin{align*}
\text{ad}_H \begin{pmatrix} \tilde{k}_j \\ \tilde{p}_j \end{pmatrix} &= \begin{pmatrix} 0 & \alpha_j \\ -\alpha_j & 0 \end{pmatrix} \begin{pmatrix} \tilde{k}_j \\ \tilde{p}_j \end{pmatrix},
\end{align*}
\]

which leads to the compact version of (3.11) and (3.12),

\[
\begin{align*}
\exp(\text{ad}_H) \begin{pmatrix} \tilde{k}_j \\ \tilde{p}_j \end{pmatrix} &= \begin{pmatrix} \cos \alpha_j & \sin \alpha_j \\ -\sin \alpha_j & \cos \alpha_j \end{pmatrix} \begin{pmatrix} \tilde{k}_j \\ \tilde{p}_j \end{pmatrix}.
\end{align*}
\]

At the group level, the symmetric spaces \( G/K \) and \( G_C/K \) are called the duals of each other, and they appear in the same row of standard symmetric space charts. An example of eigenmatrices \( x_j, \theta x_j \) and ping pong matrices for some symmetric spaces and their duals are presented in Table 2.

3.8. **Jacobian of the map \( \Phi \)**. We provide a generalized algorithm for finding a Jacobian of the decomposition \( \Phi(Q, \Lambda) = QAQ^{-1} \) (as we defined in (2.3)) where \( \Lambda \in A := \exp(a), Q \in K \). The \( \mathfrak{f} \) and \( \mathfrak{p} \) from the previous section are the tangent spaces of \( K \) and \( P \), respectively. As mentioned, we follow Helgason’s derivation (20, Ch.I, Thm 5.8) and start by directly translating his proof into simple linear algebra terms. In Table 3 we have Helgason’s derivation (Left) compared in the same row with linear algebra (Right). Table 3 is using the noncompact symmetric space \( G/K \) but the compact case are identical with replacing \( \sin \alpha_j \) by \( \sinh \alpha_j \).

From the last line of Table 3 we can finish the story with two different directions, depending on the choice of the volume measure. First, if we use a **G-invariant measure** (the “canonical measure”) of \( p \), the measure is invariant under the map \( d\tilde{r} \) or \( d\tilde{r} (\text{by definition of the invariant measure}) \). Thus we can disregard \( \tilde{r}(Q\Lambda \frac{1}{2}) \) (or \( \tilde{r}(ka) \)) so that the Jacobian of \( d\Phi \) (or \( d\Phi \)) only depends on the differential map \( k_j \mapsto (\sinh \alpha_j) p_{\alpha} \). Since \( \cup \{k_j\} \) and \( \cup \{p_j\} \) are both orthonormal bases, we obtain the Jacobian

\[
\prod_{\alpha \in \Sigma^+} \sinh \alpha(H).
\]

The second choice of measure is the **Euclidean measure**, which is a wedge product of independent entrywise differentials. In this case the procedure is identical up to the factor \( \sin \alpha_j \), but the map \( d\tilde{r}(Q\Lambda \frac{1}{2}) \) (equivalently \( d\tilde{r}(ka) \)) cannot be ignored. One needs to carefully compute the differential map \( d\tilde{r}(Q\Lambda \frac{1}{2}) p_j = QA_j p_j \Lambda \frac{1}{2} Q^{-1} \) under the Euclidean measure. We can further use the fact that conjugation by the matrix \( Q \) always preserves the Euclidean measure, since the subgroup \( K \) is always a set of matrices with an Orthogonal/Unitary type of property. Thus, one needs to compute the map \( p_j \mapsto \Lambda \frac{1}{2} p_j \Lambda \frac{1}{2} \) and multiply its Jacobian by \( \prod_{\alpha \in \Sigma^+} \sinh \alpha(H) \).

**Remark 3.5.** For the compact Lie group \( G \), we have \( \sin \alpha_j \) replaced by \( \sinh \alpha_j \) everywhere. Moreover, the last Jacobian computation step \( p_j \mapsto \Lambda \frac{1}{2} p_j \Lambda \frac{1}{2} \) can be omitted for the compact cases, since \( \Lambda \frac{1}{2} \) is an orthogonal/Unitary matrix for the compact cases. The map \( d\tilde{r}(\Lambda \frac{1}{2}) \) preserves the Euclidean measure as \( d\tilde{r}(Q) \).
Table 3. Translating the Lie Theoretic proof to linear algebra (non-compact)

3.9. Extension to the Generalized Cartan Decomposition. In the previous paragraphs, we studied the Jacobian of the usual Cartan decomposition. We now proceed to consider the generalized Cartan Decomposition \([\Gamma, o]\), its Jacobian \([\Gamma, o]\) and the extension of Table 3. The derivations are analogous, analyzing subspaces of \(g\) but one should now proceed with four tangent subspaces, \(\mathfrak{t}_\tau \cap \mathfrak{t}_\sigma\), \(\mathfrak{t}_\tau \cap \mathfrak{p}_\tau\), \(\mathfrak{p}_\tau \cap \mathfrak{t}_\tau\), \(\mathfrak{p}_\tau \cap \mathfrak{p}_\tau\). Earlier work on these Jacobian related derivations may be found in [14]. The maximal subspace \(a\) is now defined inside \(\mathfrak{p}_\tau \cap \mathfrak{p}_\sigma\). We start with the same strategy: the tangent space \(g\) is decomposed into the eigenspaces of the linear operator \(ad_H\) with \(H \in a\). The eigenvalues \(\pm \alpha_j\) still come in pairs but we have two eigenmatrices \(x_j, \tau\sigma x_j\) for eigenvalue \(\alpha_j\), and two eigenmatrices \(\tau x_j, \sigma x_j\) for eigenvalue \(-\alpha_j\). We define four vectors \(v_1, v_2, w_1, w_2\) with the same roles as \(k_j\) and \(p_j\) played before,

\[
\begin{align*}
v_1 &:= x_j + \tau x_j + \sigma x_j + \tau \sigma x_j \in \mathfrak{t}_\tau \cap \mathfrak{t}_\sigma, \\
v_2 &:= x_j - \tau x_j - \sigma x_j + \tau \sigma x_j \in \mathfrak{p}_\tau \cap \mathfrak{p}_\sigma, \\
w_1 &:= x_j - \tau x_j + \sigma x_j - \tau \sigma x_j \in \mathfrak{p}_\tau \cap \mathfrak{t}_\tau, \\
w_2 &:= x_j + \tau x_j - \sigma x_j - \tau \sigma x_j \in \mathfrak{t}_\tau \cap \mathfrak{p}_\sigma,
\end{align*}
\]

and these have similar ping pong relationships by \(ad_H\) like \(k_j\) and \(p_j\),

\[
\begin{align*}
ad_H(v_1) &= \alpha_j v_2 \\
ad_H(v_2) &= \alpha_j v_1 \\
ad_H(w_1) &= \alpha_j w_2 \\
ad_H(w_2) &= \alpha_j w_1.
\end{align*}
\]
We can similarly extend (3.13) and other relationships, and proceed as in Table 3 to obtain a result (2.2). For further details please refer to the Appendix.

4. Compact AI-I, A, AII-II

Remark 4.1. Some of the results in Section 4, 5 may be found in the nice thesis of Dueñez [7]. We include the results here for completeness, and also cover the KAK factorizations with discussions about sampling methodologies.

The joint probability distribution of the \( \beta \)-Circular ensemble is defined as,

\[
E_{n, \beta}^{\beta}(\theta) = C_{n, \beta} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta.
\]

Circular ensembles \( \beta = 1, 2, 4 \) (COE, CUE, CSE) arise as eigenvalues of particular unitary matrices and they can be precisely represented as probability measures endowed on double coset spaces of compact AI-I, A, AII-II, respectively. The \( K_1AK_2 \) decomposition is the usual KAK decomposition and the restricted root system(and dimensions) of AI, A, AII are, \( 1 \leq j < k \leq n \)

\[
\alpha(H) \frac{\alpha_m(\pm(h_j - h_k))}{\beta}
\]

and since they are compact symmetric spaces, we use (2.4) with these root systems.

4.1. AI-I(=AI), \( \beta = 1 \) COE. The involution on the tangent space of the group \( G = U(n) \) is \( X \mapsto -X^T \) and \( G/K = U(n)/O(n) \). The involution has no free parameters, reducing the AI-I \( K_1AK_2 \) decomposition to the KAK decomposition of \( U(n)/O(n) \). The Abelian torus \( A \) is,

\[
A = \{ \text{Diagonal matrices with entries } e^{ih_j}, \text{ where } h_j \in \mathbb{R} \}.
\]

From Thm 2.1, we obtain \( U = O_1DO_2 \), a factorization of a unitary matrix into the product of two orthogonal \( O_1, O_2 \in O(n) \) and a unit complex diagonal matrix \( D \in A \). We call this the ODO decomposition, and it first appears in [17]. The corresponding Jacobian factor(up to constant) using (2.4) and (4.2) with \( \beta = 1 \) is, (with change of variable \( \theta_j = 2h_j \))

\[
\left( \prod_{j<k} \sin(h_j - h_k) \right) dh_1 \cdots dh_n \propto \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}| d\theta_1 \cdots d\theta_n,
\]

which exactly is the joint density of the Circular Orthogonal Ensemble(COE). In other words, doubled angles in \( D \) of the ODO decomposition of Haar distributed unitary matrices is the COE distribution. Moreover since we can identify \( G/K \) as a set of unitary symmetric matrices(following remark 2.5 and (2.5)), we obtain a factorization \( S = OAO^T \), an eigendecomposition of a unitary symmetric matrix \( S \) with real eigenvectors. Speaking of the algorithm, we can utilize both matrix factorizations to obtain the COE, as follows:

- Two times the angles in \( D \) of the ODO decomposition of \( U \in \text{Haar}(U(n)) \).
- The angles of (unit) eigenvalues of the unitary symmetric matrix obtained from \( UU^T \), with Haar-distributed unitary matrix \( U \)

Remark 4.2. The second algorithm above would be obvious since the days of Dyson [10] while we are not aware of the first algorithm appearing in the literature.
4.2. $A$, $\beta = 2$ CUE. For the symmetric space of compact type $A$, $G/K = U(n) \times U(n)/U(n)$ and the corresponding tangent space involution is $X \mapsto -X^T$, an identity map. The restricted root system returns to the usual root system $A_n$ of classical Lie algebra. The maximal torus of $U(n)$ is a Cartan subalgebra of $U(n)$. The Jacobian of Weyl’s integration formula agrees with endowing the Circular Unitary Ensemble(CUE), which is just the eigenvalues of unitary matrices with normalized Haar measure. The derivation of the CUE is found in many standard random matrix textbooks.\cite{[1, 16, 33]}

4.3. $AII-II(=AII)$, $\beta = 4$ CSE. As in AI-I, the involution $X \mapsto -J_n^T X J_n^T$, $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ on the tangent space of $U(2n)$ has no free parameters. The double coset space is $Sp(n) \backslash U(2n)/Sp(n)$ where $Sp(n) = Sp(n, \mathbb{C}) \cap U(2n)$. The abelian torus $A$ is

$$A = \{ \text{diag}(\tilde{D}, \tilde{D}^{-1}) \mid \tilde{D} = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}), \ h_j \in \mathbb{R} \}.$$ 

Again from the KAK decomposition we obtain $U = Q_1 D Q_2$, a factorization of a unitary matrix into the product of two unitary symplectic matrices $Q_1, Q_2 \in Sp(n)$ and a unit complex diagonal matrix $D \in A$. We call this the QDQ decomposition. From \cite{[2, 4]} and \cite{[12]} with $\beta = 4$ the probability density (up to a constant) on $A$ is,

$$\left( \prod_{j<k} \sin^4(h_j - h_k) \right) dh_1 \cdots dh_n \propto \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^4 d\theta_1 \cdots d\theta_n,$$

with the change of variables $\theta_j = 2h_j$. This is the Circular Symplectic Ensemble(CSE). Similarly, an eigendecomposition of unitary skew-hamiltonian matrices obtained by $UJ_n X J_n^T U$ is equivalent to Thm \cite{[2, 4]} and the map \cite{[2, 5]}. Two numerical algorithms to obtain CSE are,

- Two times the first $n$ angles in $D$ of the QDQ factorization of $U \in \text{Haar}(U(2n))$
- The first $n$ angles of unit eigenvalues of a unitary skew-hamiltonian matrix, which can be obtained by $UJ_n X J_n^TU$ with $U \in \text{Haar}(U(2n))$.

5. COMPACT BDI-I, AIII-III, CII-II

The joint probability distribution of the $\beta$-Jacobi ensemble is defined as,

$$J_{a,b}^{\beta,s}(x) = C_{a,b,\beta,s} \prod_{j<k} |x_j - x_k|^{a+b-s} \prod_{j=1}^s x_j^{(s+1)-1}(1-x_j)^{2(b+1)-1}.$$ 

Jacobi ensembles with $\beta = 1, 2, 4$ arise as a probability density of the CS values of Haar measured orthogonal/unitary/symplectic matrices \cite{[12]}. This is reconfirmed in this section, since the $K_1AK_2$ decomposition of the compact type BDI-I, AIII-III, CII-II are exactly the CS decomposition(CSD) \cite{[36, 6]} of orthogonal/unitary/symplectic matrices. We assume $r \geq p \geq q \geq s$, and $n = p+q = r+s$ throughout the section. With the KAK decomposition, only the limited cases ($p = r, q = s$) of the Jacobi ensembles can be obtained. The root system of the double coset space and their dimensions are,

$$\begin{array}{ccc}
\alpha(H) & \pm(\theta_j \pm \theta_k) & \pm\theta_j \\
ma, 1 & \beta & \beta(p-s) & \beta-1 \\
ma, -1 & 0 & \beta(q-s) & 0 \\
\end{array}$$
with $1 \leq j < k \leq s$. For all three $\beta$ we have an identical maximal abelian subgroup $A$,

$$A = \{n\text{-by-}n \text{ Matrices with block structure } \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \}$$

with $C, S \in \mathbb{R}^{s \times s}$ being diagonal matrices with cosine and sine values of $\theta_1, \ldots, \theta_s$.

5.1. **BDI-I, $\beta = 1$ Jacobi.** Let $I_{pq} := \text{diag}(I_p, -I_q)$. With the involution $X \mapsto I_{pq}X I_{pq}$ on the tangent space of $O(n)$ we obtain the symmetric space $\text{BDI}, G/K = O(n)/O(p) \times O(q)$. With two symmetric pairs $(O(n), O(p) \times O(q))$ and $(O(n), O(r) \times O(s))$, we obtain the $K_1AK_2$ decomposition which is the CSD of an orthogonal matrix,

$$\left( \begin{array}{c} \text{n-by-n} \\ \text{Orthogonal} \end{array} \right) = \begin{pmatrix} O_p & O_q \\ C & -S \\ -S & C \end{pmatrix} \begin{pmatrix} O_r & O_s \\ -S & C \end{pmatrix} \begin{pmatrix} -S & C \\ C & -S \end{pmatrix}.$$

From (2.4) and using the $\beta = 1$ roots (5.2), we obtain the measure on $A$,

$$d\mu(a) = \prod_{j<k} (\sin(\theta_j - \theta_k) \sin(\theta_j + \theta_k)) \prod_j (\sin \theta_j)^{(p-s)} (\cos \theta_j)^{(q-s)} d\theta_1 \ldots d\theta_s,$$

and by using trigonometric identities with the change of variables $x_j = \cos^2 \theta_j = \frac{1 + \cos(2\theta_j)}{2}$, $a = q - s$ and $b = p - s$ we obtain

$$d\mu(a) \propto \prod_j x_j^{(n+1)-1} (1 - x_j)^{\frac{1}{2}(b+1)-1} \prod_{j<k} |x_j - x_k| dx_1 \ldots dx_s,$$

which exactly is a joint density of the $\beta$-Jacobi ensemble $J_{a,b}^{1,s}$. This result agrees with Thm 1.5 of [12], where squared cosine values of the CSD of a Haar distributed orthogonal matrix gives the eigenvalue distribution of the $\beta$-Jacobi ensemble with $\beta = 1$. Moreover, recall the fact that the QR decomposition of an $n$-by-$n$ independent Gaussian matrix is used to obtain a Haar distributed orthogonal matrix. Since the Generalized Singular Value Decomposition (GSVD) is equivalent to the combination of a QR decomposition and a CSD, one can take the GSVD of a matrix with independent Gaussian entries to obtain the $\beta$-Jacobi ensemble. Speaking of the numerical algorithm, the following are the ways to obtain $J_{a,b}^{1,s}$.

- Squared cosine CS values of a Haar distributed $m$-by-$m$ orthogonal matrix ($m = 2s + a + b$) with partitions $(s + a, s + b)$ and $(s, s + a + b)$
- Squared cosine values, where tangent values are the generalized singular values of Gaussian distributed $(s + a)$-by-$s$ and $(s + b)$-by-$s$ matrices.

5.2. **AIII-III, $\beta = 2$ Jacobi.** Two symmetric pairs of compact AIII are $(U(n), U(p) \times U(q))$ and $(U(n), U(r) \times U(s))$. The $K_1AK_2$ decomposition is exactly the CSD of unitary matrices. Using (2.4) on $\beta = 2$ root system (5.2) and a similar change of variable as above, we obtain the measure $d\mu(a)$,

$$\prod_{j<k} (\sin(\theta_j - \theta_k) \sin(\theta_j + \theta_k))^2 \prod_j ((\sin \theta_j)^{2(p-s)} (\cos \theta_j)^{2(q-s)} \sin(2\theta_j)) d\theta_1 \ldots d\theta_s,$$

$$\propto \prod_j x_j^{q-s} (1 - x_j)^{p-s} \prod_{j<k} |x_j - x_k|^2 dx_1 \ldots dx_s,$$

which has normalized Jacobian equivalent to $J_{a,b}^{2,s}(a = q - s, b = p - s)$ Numerically,
• The squared cosine CS values of a Haar distributed \(m\)-by-\(m\) unitary matrix 
\((m = 2s + a + b)\) with partitions \((s + a, s + b)\) and \((s, s + a + b)\)
• The squared cosine values, where the tangent values are the GSVD values 
of (complex)Gaussian distributed \((s + a)\)-by-\(s\) and \((s + b)\)-by-\(s\) matrices.

5.3. CII-II, \(\beta = 4\) Jacobi. The \(\beta\)-Jacobi ensemble with \(\beta = 4\) is similarly obtained from two symmetric spaces \(\text{Sp}(n)/\text{Sp}(p) \times \text{Sp}(q)\) and \(\text{Sp}(n)/\text{Sp}(r) \times \text{Sp}(s)\).
However we identify \(\text{Sp}(n)\) as the quaternionic unitary group, \(U(n, \mathbb{H}) := \{g \in \text{GL}(n, \mathbb{H}) | g^Dg = I_n\}\). The generalized Cartan decomposition gives the CSD of quaternionic unitary matrices and using (2.4) with \(\beta = 4\) roots (5.2), we obtain the measure on \(A\),
\[
\prod_{j<k} (\sin(\theta_j - \theta_k) \sin(\theta_j + \theta_k))^4 \prod_j ((\sin \theta_j)^{4(p-s)}(\cos \theta_j)^{4(q-s)} \sin^3(2\theta_j)) d\theta_1 \ldots d\theta_s
\]
\[
\propto \prod_j x_j^{2(q-s)+1}(1-x_j)^{2(p-s)+1} \prod_{j<k} |x_j - x_k|^4 dx_1 \ldots dx_s,
\]
which is a \(\beta\)-Jacobi ensemble \(J^{4,s}_{a,b}\), when normalized. Again the algorithm is,
• The squared cosine CS values of Haar distributed \(m\)-by-\(m\) quaternion unitary matrix 
\((m = 2s + a + b)\) with partition \((s + a, s + b)\) and \((s, s + a + b)\) is the Jacobi ensemble \(J^{4,s}_{a,b}\)

Remark 5.1. One can use the GSVD on quaternionic unitary matrices to obtain the \(\beta = 4\) Jacobi ensemble density.

6. Noncompact Lie groups and probability measure

Hermite and Laguerre eigenvalue distributions arise as a result of (2.3), if applied to a decomposition of a noncompact Lie group or symmetric space. As opposed to compact Lie groups and compact symmetric spaces where the Haar measure and \(G\)-invariant measures are normalized by a constant to a probability measure, corresponding measures on noncompact manifolds cannot be normalized to one by a constant. A normalizing factor \(S\) should be introduced to complete the construction of a probability measure. Therefore, random matrices on noncompact manifolds face an innate problem:

• The choice of the probability measure on noncompact \(G\) or \(G/K\) is not unique.

As we push the measure forward to the subgroup \(A\), the resulting measure should be a symmetric function of independent generators of \(A\). Hence a choice of probability measure \(I(g)\) of a random matrix is Haar or \(G\)-invariant measure on \(G\) or \(G/K\), multiplied by a symmetric function \(S\) on \(A\),
\[
I(g) = S(a)\mu(g),
\]
where \(g = k_1ak_2\) or \(g = kak^{-1}\) and \(\mu(g)\) is an invariant measure. Using (2.3), the measure on \(A\) is induced,
\[
I(g) = dk \cdot S(a) \left( \prod_{\alpha \in \Sigma^+} \sinh(\alpha(H)) \right) dH_1 \cdots dH_{\text{dim}(A)},
\]
which means even though the measure \(I\) changes, the measure on \(A\) still differs only by a normalization function.
The traditional choice of $S$ has been made such that $I(g)$ can be constructed from independent Gaussian distributions endowed on matrix entries. Two classical ensembles, Hermite and Laguerre are equipped with independent Gaussian distributions on the entries of a noncompact symmetric space $G/K$. For the Hermite(Gaussian) ensembles, the symmetric spaces $G/K$ are sets of Symmetric/Hermitian/Self-Dual positive definite matrices, and $I(g)$ are independent Gaussian measures imposed on upper triangular entries,

$$I(g) = \exp(-\operatorname{Tr}(g^*g)/2) \prod_{j \leq k} dg_{jk} = \prod_{j \leq k} (e^{-g_{jk}^2/(2+2\delta_{jk})} dg_{jk}).$$

The variance 2 on the diagonal makes $S(g)$ a symmetric function of entries of $a \in A$. For the Laguerre(Wishart) ensemble, coset representatives of $G/K$ are,

$$\{ \left( \sqrt{I+XX^*} \atop X^* \right) \left( X \atop \sqrt{I+X^*X} \right) \mid X \in \mathbb{R}^{p \times q} \}$$

and the measure $I(g)$ are independent Gaussian measures imposed on the upper right $p$ by $q$ block,

$$I(g) = \prod_{\substack{j \leq p \\
                    k \leq q}} e^{-X_{jk}^2/2} dX_{jk}, \quad \text{where } g = \left( \sqrt{I+XX^*} \atop X^* \right) \frac{X}{\sqrt{I+X^*X}}.$$

(6.1)

7. **Noncompact AI, A, AII**

The joint probability distribution of the $\beta$-Hermite ensemble is defined as,

$$H_n^\beta(\lambda) = C_{n,\beta} \prod_{j < k} |\lambda_j - \lambda_k|^\beta e^{-\sum \lambda_j^2/2}.$$

Hermite ensembles with $\beta = 1, 2, 4$(GOE, GUE, GSE) arise as Gaussian measures endowed on symmetric/Hermitian/Self-Dual matrices. Decompositions of the symmetric spaces of noncompact type AI, A, AII are exactly the eigendecompositions, used for computing the eigenvalue density of Gaussian ensembles. The restricted root systems and their dimensions are, $(1 \leq j < k \leq n)$

$$\alpha(H) \begin{pmatrix} \pm(h_j - h_k) \\ \beta \end{pmatrix}$$

with an identical double coset structure $A$ for all three cases,

$$A = \{ \text{Diagonal matrices with entries } e^{h_i}, \text{ where } h_j \in \mathbb{R} \}.$$
invariant measure on $S_n$ is $\mu(g) = (\det g)^{-\frac{(n+1)}{2}} \prod_{j \leq k} dg_{jk}$. Using (2.3) on $\mu$ with (7.2) $\beta = 1$,

$$I(g) = \exp(-\text{Tr}(g^T g)/2)(\det g)^{\frac{(n+1)}{2}} \mu(g)$$

$$= \exp(-\text{Tr}(g^T g)/2)(\det g)^{\frac{(n+1)}{2}} \prod_{j < k} \sinh \frac{h_j - h_k}{2} = e^{-\sum \lambda_j^2 / 2} \prod_{j < k} |\lambda_j - \lambda_k|,$$

where $\lambda_j$ are the eigenvalues of $g$. This is the eigenvalue density of the GOE.

**Remark 7.1.** An identical result can also be carried out by starting from the measure $I$ on $G$, not $G/K$. The equivalent $I(g)$, for $g \in G$ is $(\det g) \exp(-||g^T g||^2) \prod (dg_{jk})$. Since the Haar measure of $G$ is $\mu(g) = (\det g)^{-n} \prod dg_{jk}$ and the KAK decomposition is the square SVD, a change of variables ends up with the measure on the singular values $A$ resulting in the GOE eigenvalue density. However this is less ideal for sampling, so we continue the rest of the section just using the measure $I$ on $G/K$.

**Remark 7.2.** Another point we wish to discuss here is the positivity of $\lambda_j$ in the above derivation with symmetric spaces whereas the Gaussian ensembles have no such restriction. One way to handle this disparity is to notice that the Euclidean measure is invariant under translations, so that a shift by a multiple of the identity gives the exact same Jacobian formula. By shifting any open set of symmetric matrices to an open set in $G/K$ (or using a sequence with decreasing tails) the Jacobian of the eigenvalues of the Gaussian ensembles can be derived. An alternative approach suggested by [40] is to place a Gaussian measure directly on the Lie algebra which has no positivity constraint.

**7.2. A, $\beta = 2$ GUE.** The noncompact symmetric space $A$ is $G/K = \text{GL}(n, \mathbb{C})/\text{U}(n)$, represented by $\mathcal{H}_n$, the set of Hermitian positive definite matrices. Again, the initial measure for $g \in \mathcal{H}_n$ is

$$I(g) = \exp(-\text{Tr}(g^T g)/2) \prod_{j \leq k} dg_{jk} = \exp(-\text{Tr}(g^T g)/2)(\det g)^{n} \mu(g),$$

where $\mu(g)$ is the $G$-invariant measure and $dg_{jk} = d\text{Re}(g)d\text{Im}(g)$. Using (2.3) and roots from (7.2) $\beta = 2$,

$$I(g) = \exp(-\text{Tr}(g^T g)/2)(\det g)^{n} \prod_{j < k} \sinh^2 \frac{h_j - h_k}{2} = e^{-\sum \lambda_j^2 / 2} \prod_{j < k} |\lambda_j - \lambda_k|^2,$$

which is an eigenvalue density of the GUE.

**7.3. AII, $\beta = 4$ GSE.** The noncompact symmetric space $A\text{II}$ is given as $G/K = \text{GL}(n, \mathbb{H})/\text{U}(n, \mathbb{H})$. We use $\text{U}(n, \mathbb{H})$ instead of $\text{Sp}(n)$ to clarify the quaternionic representation. $G/K$ can be represented by a set of Quaternionic Hermitian positive definite matrices, $\mathcal{QH}_n$. The initial probability measure is

$$I(g) = \exp(-\text{Tr}(g^T g)/2) \prod_{j \leq k} dg_{jk} = \exp(-\text{Tr}(g^T g)/2)(\det g)^{2n - 1} \mu(g),$$

for $g \in \mathcal{QH}_n$ and the $G$-invariant measure $\mu$. For a quaternion entry $g_{jk} \in \mathbb{H}$, $dg_{jk}$ is the Euclidean measure on $\mathbb{R}^4$, i.e.,

$$dg_{jk} = d\text{Re}(g)d\text{Im}_1(g)d\text{Im}_2(g)d\text{Im}_3(g).$$
we obtain the eigenvalue density of the GSE.

8. Noncompact BDI, AII, CII

The joint probability distribution of the $\beta$-Laguerre(Wishart) ensemble is,

$$L_{a,q}^\beta(\lambda) = C_{a,q,\beta} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{j=1}^q \lambda_j^{\beta(a+1)-1} e^{-\sum \lambda_j/2}.$$

Wishart ensembles with $\beta = 1, 2, 4$ and parameters $p, q$ arise from independent Gaussian measures endowed on the matrix $A \in \mathbb{F}^{p \times q}$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Equivalently, the eigenvalues of the matrix $A^*A \in \mathbb{F}^{q \times q}$ is frequently used for sampling, where $^*$ is the (conjugate) transpose of the Real/Complex/Quaternion matrix. The KAK decomposition of the noncompact symmetric spaces BDI, AIII, CII is the Hyperbolic CS decomposition (HCSD) [23], and the map $\Phi$ of (2.5) on the symmetric space $G/K$ is equivalent to the SVD of a $p$-by-$q$ matrix with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. In the SVD form, these decompositions sample square roots of the eigenvalues of the Wishart ensemble. The structure of $A$ for all $\beta$ is,

$$A = \begin{pmatrix} C & I_{n-2q} \\ S & C \end{pmatrix} \in \mathbb{R}^{n \times n}[C^2 - S^2 = I_q],$$

for $p+q = n, p \geq q$. $C, S$ are diagonal matrices with $\cosh, \sinh$ values of $h_1, \ldots, h_q$, where the restricted root systems of the noncompact BDI, AIII, CII ($\beta = 1, 2, 4$) are,

$$\begin{array}{c|c|c|c}
\alpha(H) & \pm(h_j \pm h_k) & \pm h_j & \pm h_j \\
m_\alpha & \beta & \beta(p-q) & \beta - 1 \\
\end{array}$$

8.1. BDI, $\beta = 1$ Laguerre. The noncompact symmetric space BDI is $G/K = O(p,q)/O(p) \times O(q)$. The KAK decomposition of the indefinite orthogonal(pseudo-orthogonal, J-Orthogonal) group $O(p,q)$ is the HCSD of J-Orthogonal matrices in numerical linear algebra. The map $\Phi$ of (2.5) on the symmetric space is the HCSD of the symmetric J-Orthogonal matrix,

$$\left( \begin{array}{cc}
(I + gg^T)^{\frac{1}{2}} & g \\
I + g^Tg & \frac{1}{2} \\
\end{array} \right) = \begin{pmatrix} O_p \tilde{C} O_T & O_p S O_T^T \\
O_q S T O_p^T & O_q C O_q^T \\
\end{pmatrix},$$

which is isomorphic to the SVD of the upper right block $g \in \mathbb{R}^{p \times q}$, $g = O_p S O_T^T$. Here the matrices $\tilde{C}$ and $S$ are defined as $p \times p$ and $p \times q$ matrices, to match the block sizes. The probability measure endowed on $G/K$ is a product of independent Gaussian measures on the entries of $g$, $I(g) = \exp(-\operatorname{Tr}(g^2)) \prod dg_{jk}$. In terms of the $G$-invariant measure $\mu$, $I(g) = \exp(-\operatorname{Tr}(g^2)) \prod \cosh h_j \mu(g)$. Using (2.3) with (8.2), $\beta = 1$ and applying hyperbolic trigonometric identities,

$$I(g) \propto e^{-\sum \lambda_j/2} \prod_j \lambda_j^{\frac{n-q-1}{2}} \prod_{j<k} (\lambda_j - \lambda_k) d\lambda_1 \cdots d\lambda_q,$$
where \( \lambda_j = \sinh^2 h_j \), squared singular values of \( g \). With normalization, this is exactly the Laguerre distribution \( L^1_{p-q,q} \).

8.2. AIII, \( \beta = 2 \) Laguerre. The noncompact symmetric space AIII is \( G/K = U(p,q)/U(p) \times U(q) \). Similarly, the KAK decomposition is the HCSD of a J-Unitary matrix and Thm 23 is equivalent to the complex SVD on the upper left block. The initial measure is \( I(g) = \exp(-\text{Tr}(g^H g)/2) \prod dg_{jk} = \exp(-\text{Tr}(g^H g)/2)\mu(g) \) for \( g \in \mathbb{C}^{p \times q} \) and the \( G \)-invariant measure \( \mu \) on \( G/K \). Again using (2.3) with (8.2) \( \beta = 2 \) and the change of variables \( \lambda_j = \sinh^2 h_j \),

\[
I(g) \propto e^{-\sum \lambda_j/2} \prod_j \lambda_j^{p-q} \prod_{j<k} |\lambda_j - \lambda_k|^2 d\lambda_1 \ldots d\lambda_q,
\]

which gives us the eigenvalue density of the complex Laguerre ensemble \( L^2_{p-q,q} \), as squared singular values of the complex Gaussian random matrix \( g \).

8.3. CII, \( \beta = 4 \) Laguerre. The noncompact symmetric space CII is \( G/K = Sp(p,q)/Sp(p) \times Sp(q) \). Note that in this section we identify \( Sp(p,q) \) as a subgroup of \( \mathbb{H}^{n \times n} \), and \( Sp(p), Sp(q) \) as quaternionic unitary groups, \( U(p, \mathbb{H}), U(q, \mathbb{H}) \). Analogously, we observe the KAK decomposition is the HCSD of the quaternionic J-Unitary matrix and Thm 23 as the quaternionic SVD [19]. The initial measure on \( g \in \mathbb{H}^{p \times q} \) is endowed as

\[
I(g) = \exp(-\text{Tr}(g^D g)/2) \prod dg_{jk} = \exp(-\text{Tr}(g^D g)/2) \prod \cosh^{-2} h_j \mu(g).
\]

Using (2.3) with the roots (8.2) \( \beta = 4 \),

\[
I(g) \propto e^{-\sum \lambda_s/2} \prod_s \lambda_s^{2(p-q)+1} \prod_{s<t} |\lambda_s - \lambda_t|^4 d\lambda_1 \ldots d\lambda_q,
\]

which is the joint eigenvalue density of the quaternion Laguerre ensemble \( L^4_{p-q,q} \).

9. Appendix : Proof of Thm 2.2

This appendix provides an introduction [20, 31, 22] on the theory of the generalized Cartan decomposition and the detailed derivation of (2.2) by expanding [15]. The proof follows the notations and flow used in [20].

Let \( G \) be a noncompact semisimple Lie group and \( g \) be its Lie algebra. Let \( (\tau, \sigma) \) be two involutions (including the Cartan involution) of \( g \) which commute and \( (G, K_\tau), (G, K_\sigma) \) be two symmetric spaces obtained by the two involutions, respectively. Let

\[
g = g^{\tau} \oplus g^{-\tau} = g^{\sigma} \oplus g^{-\sigma}
\]

be the decomposition of \( g \) into the +1 and -1 eigenspaces of \( \tau, \sigma \), respectively. Let \( a \) be a maximal abelian subspace of \( g^{-\tau} \cap g^{-\sigma} \). Then we define root spaces similar as before,

\[
g_\alpha = \{ X \in g \mid [Y, X] = \alpha(Y)X \text{ for all } Y \in a \},
\]

where \( \alpha : a \to i\mathbb{R} \). Let \( \Sigma \) be the set of all nonzero roots with nontrivial root spaces,

\[
\Sigma = \{ \alpha \in ia^* - \{0\} \mid g_\alpha \neq \{0\} \}.
\]

In [15], it has been shown that we have a root space decomposition

\[
g = \bigoplus_{\alpha \in \Sigma \cup \{0\}} g_\lambda.
\]
Using eigenspaces of the involution $\sigma \tau$, we get

$$g = \bigoplus_{\alpha \in \Sigma, \lambda = \pm 1} \mathfrak{g}_{\alpha, \lambda},$$

where $\mathfrak{g}_{\alpha, \lambda} = \{X \in \mathfrak{g}_\alpha \mid \sigma \tau X = \lambda X\}$. Additionally, let $m_{\alpha, \lambda}$ be the dimension of $\mathfrak{g}_{\alpha, \lambda}$, which adds up to the root space dimension $m_\alpha$.

The symmetric space $G/K_\tau$ admits a polar coordinate map,

$$\Phi : (K_\tau/M, A) \to G/K_\sigma, \quad (kM, a) \mapsto kaK_\sigma$$

which makes it diffeomorphic to $(K_\tau/M) \times A$, where $A = \exp(a)$ and $M$ is a centralizer of $A$ inside $K_\tau$. Now we present the proof of Thm. 2.2 by mimicking [20], p187.

**Proof of Thm 2.2** First let’s refine $\mathfrak{g}_{\alpha, \lambda}$. It is clear that

$$(\mathfrak{g}_\sigma \cap \mathfrak{g}_\tau) \oplus (\mathfrak{g}_{-\sigma} \cap \mathfrak{g}_{-\tau}) = \bigoplus_{\alpha \in \Sigma \setminus \{0\}} \mathfrak{g}_{\alpha, 1}.$$

The map $X \mapsto X + \sigma X$ sends $\mathfrak{g}_{\alpha, 1}$ to $\mathfrak{g}_{\alpha, +} := \{X \in \mathfrak{g}_\sigma \cap \mathfrak{g}_\tau \mid \text{ad}(H)^2 X = \alpha(H)^2 X, \forall H \in \mathfrak{a}\}$. If we fix a positive Weyl chamber then for $\alpha \in \Sigma^+$ this map is a bijection from $\mathfrak{g}_{\alpha, 1}$ to $\mathfrak{g}_{\alpha, +}$. Applying the same map to $\mathfrak{g}_{\alpha, -}$ we obtain $\mathfrak{g}_{\alpha, -}$. Similarly, $\mathfrak{g}_{\alpha, +}$ and $\mathfrak{g}_{\alpha, -}$ can be obtained from $\mathfrak{g}_{\alpha, -}$ and $\mathfrak{g}_{\alpha, 1}$, respectively, by the map $X \mapsto X - \sigma X$. Note that $\mathfrak{g}_{\alpha, +} \subset \mathfrak{g}_\sigma \cap \mathfrak{g}_\tau$, $\mathfrak{g}_{\alpha, -} \subset \mathfrak{g}_\sigma \cap \mathfrak{g}_{-\tau}$, $\mathfrak{g}_{\alpha, +} \subset \mathfrak{g}_{-\sigma} \cap \mathfrak{g}_\tau$, and $\mathfrak{g}_{\alpha, -} \subset \mathfrak{g}_{-\sigma} \cap \mathfrak{g}_{-\tau}$. Also $\text{ad}(H)$ is a bijective map between $\mathfrak{g}_{\alpha, +}$ and $\mathfrak{g}_{\alpha, -}$, and between $\mathfrak{g}_{\alpha, +}$ and $\mathfrak{g}_{\alpha, -}$.

We now proceed as in [20]. Let $e_1^\alpha, \ldots, e_{\dim \mathfrak{g}_{\alpha, +}}^\alpha$ and $f_1^\alpha, \ldots, f_{\dim \mathfrak{g}_{\alpha, -}}^\alpha$ be an orthonormal basis of $\mathfrak{g}_{\alpha, +}$ and $\mathfrak{g}_{\alpha, -}$, respectively. Let $H_1, \ldots, H_{\dim \mathfrak{a}}$ be an orthonormal basis of $\mathfrak{a}$. Let $t(g_0)$ be the translation map on the homogeneous space $G/K$ by $t(g_0) : gK \mapsto g_0gK$, and let $\pi : G \to G/K_\sigma$ be the natural quotient map.

Choose a vector $e \in \{e_1^\alpha\} \cup \{f_1^\alpha\}$, and consider the curve $s \mapsto k \exp(se)M$ in $K_\tau/M$ which has the tangent vector $dt(k)e$ at $s = 0$. For $a = \exp(H), H \in \mathfrak{a}^+$,

$$\Phi(k \exp(se)M, a) = k \exp(se)a \cdot o = ka \exp(s\text{Ad}(a^{-1})e) \cdot o,$$

where $o$ is the identity element of a coset space $G/K_\tau$. Also for the tangent map $d\Phi$,

$$d\Phi_{(kM, a)}(dt(k)e, 0) = dt(ka)d\pi(\text{Ad}(a^{-1})e).$$

Now since $\text{ad}(H)^2 e = \alpha(H)^2 e$, if $e \in \{e_1^\alpha\}$ we have

$$dt(ka)d\pi(\text{Ad}(a^{-1})e_1^\alpha) = dt(ka)d\pi\left(\frac{1}{2} \text{Ad}(a^{-1})e_1^\alpha - \sigma \text{Ad}(a^{-1})e_2^\alpha\right)$$

$$= dt(ka)d\pi\left(\frac{1}{2} \exp(-\text{ad}H)e_1^\alpha - \exp(\text{ad}H)e_2^\alpha\right)$$

$$= dt(ka)d\pi\left(-\alpha(H)^{-1}[H, e_1^\alpha] \sinh(\alpha(H))\right).$$
and if \( e \in \{ f_j^\alpha \} \) we have
\[
dt(ka)d\pi(\text{Ad}(a^{-1})f_j^\alpha) = \frac{1}{2}(\text{Ad}(a^{-1})f_j^\alpha - \sigma\text{Ad}(a^{-1})f_j^\alpha)
\]
\[
= \frac{1}{2}(\exp(-\text{ad}H)f_j^\alpha + \exp(\text{ad}H)f_j^\alpha)
\]
\[
= \frac{1}{2}(\exp(\text{ad}H)(\text{Ad}(a^{-1})f_j^\alpha) - \sigma\exp(\text{ad}H)f_j^\alpha).
\]

Also for the curve \( s \mapsto a \exp sH_j \) in \( A \) which has the tangent vector \( dt(ka)H_j \) at \( s = 0 \),
\[
\Phi(kM,a \exp sH_j) = ka \exp(sH_j) \cdot a.
\]
Similarly for \( d\Phi \),
\[
d\Phi_{(kM,a)}(0,dt(ka)H_j) = dt(ka)d\pi(H_j).
\]
\( \{-\alpha(H)^{-1}[H,e_j^\alpha]\} \) and \( \{H_j\} \) form an orthonormal basis of \( g^\sigma \cap g^\tau \) and \( \{f_j^\alpha\} \) is an orthonormal basis of \( g^\sigma \cap g^\tau \). So together they form an orthonormal basis of \( g^\sigma \) and since \( d\pi, dt(ka) \) are isometries we conclude
\[
J(a) = | \det d\Phi_{(kM,a)} | = \prod_{\alpha \in \Sigma^\pm} (\sin i\alpha(H))^{m_+^\alpha} (\cos i\alpha(H))^{m_-^\alpha}.
\]

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