Rational maps are $d$-adic Bernoulli

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Abstract

Freire, Lopes and Mañé proved that for any rational map $f$ there exists a natural invariant measure $\mu_f$ [5]. Mañé showed there exists an $n > 0$ such that $(f^n, \mu_f)$ is measurably conjugate to the one-sided $d^n$-shift, with Bernoulli measure $(\frac{1}{d^n}, \ldots, \frac{1}{d^n})$ [15]. In this paper we show that $(f, \mu_f)$ is conjugate to the one-sided Bernoulli $d$-shift. This verifies a conjecture of Freire, Lopes and Mañé [5] and Lyubich [11].

1. Introduction

Let $f(z) = P(z)/Q(z)$ be a rational map of the Riemann sphere, $\bar{C}$, of degree $d \geq 2$. In [5], Freire, Lopes and Mañé proved the existence of a natural invariant measure $\mu_f$ for the map $f$. Namely, the measure is the asymptotic distribution of preimages of any point $z$, except at most two exceptional points. Furthermore, it is the unique measure of maximal entropy. The uniqueness was shown by both Lyubich [11] and Mañé [14]. These properties and others are explicitly laid out in Section 2. Freire, Lopes and Mañé [5] and Lyubich [11] all conjectured that the system $(f, \mu_f)$ is conjugate to the one-sided Bernoulli $d$-shift. In this paper we give an affirmative answer to this conjecture.

Let $X = \{0, \ldots, d-1\}^\mathbb{N}$, $\mathcal{B}$ be the Borel $\sigma$-algebra and $\sigma$ be the shift. In other words, $\sigma(x)_n = x_{n+1}$. Let $\mu$ be product measure, where the weight on each of the $d$ symbols is uniform, namely $\frac{1}{d}$. This system is called the one-sided Bernoulli $d$-shift. An endomorphism $f$ on a measure space $(Y, \mathcal{C}, \nu)$ is $d$-adic Bernoulli if $f$ is measurably conjugate to the one-sided Bernoulli $d$-shift. This is equivalent to the existence of a partition $P$ of $Y$ into $d$ sets of equal measure such that $\{f^{-n}P\}_{n \geq 0}$ are mutually independent and $P$ generates $\mathcal{C}$, i.e., $\bigvee_{n \geq 0} f^{-n}P = \mathcal{C}$. An example of a rational map that is $d$-adic Bernoulli is $f(z) = z^d$. The invariant measure $\mu_f$ is normalized arclength on the unit circle, $\{z \mid |z| = 1\}$. More generally, an endomorphism is one-sided Bernoulli if it is conjugate to some one-sided Bernoulli shift, where the weights on the symbols may vary.
Any measure-preserving endomorphism (noninvertible map) has a natural measure theoretic two-sided (invertible) extension. The two-sided extension of the one-sided Bernoulli $d$ shift is the Bernoulli $d$-shift, the shift automorphism on $\mathbb{X} = \{0, \ldots, d-1\}^\mathbb{Z}$ with uniform product measure on $\mathbb{X}$. An automorphism (invertible map) is Bernoulli if it is conjugate to a Bernoulli shift. If an endomorphism is one-sided Bernoulli then its two-sided extension is Bernoulli. The converse is not true. For example, there exist Markov endomorphisms which are not one-sided Bernoulli but whose two-sided extension are Bernoulli [1], [4].

Historically, determining whether an endomorphism is one-sided Bernoulli is a much harder question than whether an automorphism is Bernoulli. This is because there is a reasonable condition for determining whether an automorphism is Bernoulli called very weak Bernoulli (VWB) [16]. Ornstein and Weiss proved that an automorphism is Bernoulli if and only if it is very weak Bernoulli [16], [18]. Many natural systems have been shown to be either Bernoulli or not Bernoulli by checking this condition. Examples of Bernoulli systems are toral automorphisms and geodesic flows on spaces of constant negative curvature [10], [19]. In the other direction, Kalikow showed that the $[T, T^{-1}]$ system is not Bernoulli [9].

Recently, Hoffman and Rudolph developed a condition analogous to VWB called tree very weak Bernoulli (tree vwB) [7]. In this paper they showed that tree vwB is equivalent to one-sided Bernoulli. We define and explain this condition in Section 3. With this condition there is no need to construct a conjugacy or to find a $d$ set independent generating partition.

In [15] Mañé proved that there exists an $n > 0$ such that the system $(f^n, \mu_f)$ is conjugate to the one-sided Bernoulli $d^n$-shift. (Throughout this paper, $f^n$ denotes iteration.) Since $(f^n, \mu_f)$ is one-sided Bernoulli, its two-sided extension is isomorphic to a Bernoulli automorphism. By Ornstein’s theory, the two-sided extension of $(f, \mu_f)$ is isomorphic to a Bernoulli shift [17]. It is important to note that, in contrast to the case of automorphisms, the fact that $f^n$ is one-sided Bernoulli does not imply that $f$ itself is one-sided Bernoulli [8].

One possible way to prove that $f$ is $d$-adic Bernoulli is as follows. Since $f$ has finitely many critical values (the image under $f$ of points with $f' = 0$), it is possible to connect them with an arc $\gamma$ such that $\mu_f(\gamma) = 0$. Now $f^{-1}(\gamma^c)$ is an open set with $d$ connected components. Let $P$ be the partition whose sets are these connected components. It can be shown that $\{f^{-i}P\}_{i \geq 0}$ is independent and for every $p \in P$, $\mu_f(p) = \frac{1}{d}$. However it is not clear that $P$ generates. This is the general approach that Mañé used in [15].

There are two main tools we use in the proof that $(f, \mu_f)$ is $d$-adic Bernoulli. The first is the tree vwB condition described in Section 3. The second is the construction of the unique measure of maximal entropy [5]. We show that for
any two points \( z, z' \) in the support of this measure, the set \( \{ f^{-n}(z) \} \) can be matched in a one-to-one manner with the points in \( \{ f^{-n}(z') \} \) in such a way that the matching preserves the underlying tree structure of \( \{ f^{-n}(z) \} \) and so that most paired points are close together. We then use this matching to show that the endomorphism is tree \( \nu \)-B.

### 2. Rational endomorphisms

In this section we present some basic facts about rational maps. For more information about this see [3]. Then we describe the unique invariant measure of maximal entropy and some of its properties.

Let \( f(z) \) be a rational endomorphism of \( \bar{C} \). Then \( f(z) = P(z)/Q(z) \) where \( P \) and \( Q \) are relatively prime polynomials. Define the degree of \( f \) to be the maximum degree of \( P \) and \( Q \). Throughout we assume the degree of \( f \) is at least 2. We use \( \text{dist}(x, y) \) to represent the distance between \( x \) and \( y \) on the Riemann sphere. Thus \( \text{dist}(x, y) \) is bounded by 1.

The Julia set, denoted by \( J(f) \) is the set of all points \( z \in \bar{C} \) such that for every neighborhood \( U \) of \( z \), \( \{ f^n|_U \} \) is not a normal family. That is, no subsequence of this family of functions converges uniformly on compact subsets. \( J(f) \) is a nonempty compact invariant set. Moreover, it is the closure of the repelling periodic orbits. The simplest example of a Julia set is for the map \( f(z) = z^2 \) when \( J(f) \) is the unit circle. For the map \( f(z) = 1 - 2/z^2 \) the Julia set is \( \bar{C} \). However for other rational functions the Julia set can be very complex. For the map \( f(z) = z^2 + 1 \), \( J(f) \) is a totally disconnected set. In this case it is also fractal and conformally self-similar. There are also examples of rational functions for which the Julia is connected but not locally connected, as well as functions for which the Julia set is neither connected nor totally disconnected.

Fatou and Julia introduced a set \( \text{Exc}(f) \subset \bar{C} \). This set is the maximal finite set which is invariant under \( f \) and \( f^{-1} \). They proved it contains at most two points. The measure \( \mu_f \) is the weak star limit of measures uniformly supported on \( \{ f^{-n}(a) \} \) for any \( a \notin \text{Exc}(f) \). Brolin first introduced this measure for the case when \( f \) is a polynomial [2]. Lyubich [12] and Freire, Lopes and Mañé [5] generalized this to the case of rational functions. This limit does not depend on the choice of \( a \). More precisely, define the \( n \)-preimages of \( a \) to be the set \( \{ f^{-n}(a) \} \). (The cardinality of \( \{ f^{-n}(a) \} \) is \( d^n \) as elements are listed possibly multiple times according to their multiplicities. We will use this convention throughout the paper.) Order the \( n \)-preimages of \( a \) as \( z_i^n(a), i = 1, \ldots, d^n \). Define

\[
\mu_n(a) = \frac{1}{d^n} \sum \delta_{z_i^n(a)}
\]
where $\delta_{z^{n_i}(a)}$ is Dirac measure supported on $z^{n_i}(a)$. The space of invariant probability measures on $\bar{C}$ is endowed with the weak star topology.

**Theorem 2.1 ([5]).** There exists an $f$ invariant probability measure $\mu_f$ satisfying the following properties.

1. $\lim \mu_n(a) = \mu_f$ for all $a \notin \text{Exc}(f)$. Moreover this convergence is uniform as $a$ varies over a compact subset of $\text{Exc}(f)^c$.

2. Support $\mu_f = J(f)$.

3. $f$ is exact.

4. For all Borel sets $A$ such that $f|_A$ is injective, $\frac{1}{d} \mu_f(fA) = \mu_f(A)$.

5. $h_{\mu_f}(f) = \log d$.

Properties 3–5 are conjugacy invariants. They are necessary conditions in order for $(f, \mu_f)$ to be measure theoretically isomorphic to the one-sided Bernoulli $d$-shift, which also satisfies these properties. The endomorphism $f$ is exact if $\cap_{n \geq 0} f^{-n}(A)$ is the trivial $\sigma$-algebra, where $A$ is the Borel $\sigma$-algebra on $\bar{C}$. This property implies the endomorphism is mixing but is much stronger. The last property listed implies that $\mu_f$ is a measure of maximal entropy. This follows from Gromov’s result that the topological entropy of $f$ is $\log d$ [6]. Lyubich [11], [13] and Mañé [14] showed that $\mu_f$ is the unique measure of maximal entropy.

3. Tree very weak Bernoulli

In this section we outline the ergodic theory necessary for this paper. Let $T$ be a measure-preserving endomorphism on a probability space $(Y, \mathcal{C}, \nu)$. It causes no loss of generality to assume that $Y$ is a compact metric space with metric $D$. Also, $T$ is $d$-adic if almost every atom of $T^{-1}\mathcal{C}$ consists of $d$ points in $\mathcal{C}$, and the conditional measure of these points given the atom is $1/d$ almost everywhere. For completeness we include the proof of the following well known fact.

**Lemma 3.1.** If $f$ is a rational map of degree $d$ then $(f, \mu_f)$ is a $d$-adic system.

**Proof.** In order to show $(f, \mu_f)$ is $d$-adic, it suffices to show that there exists a $d$ set partition $P$ satisfying the following properties. Let $\mathcal{A}$ be the Borel $\sigma$-algebra of $\bar{C}$. For all $p \in P$, $\mu_f(p) = \frac{1}{d}$ and $P$ and $f^{-1}\mathcal{A}$ are independent.
Consider the $d$ set partition $P$ constructed in the introduction. For any $A \in \mathcal{A}$ and $p \in P$, it suffices to show
\[ \mu_f(f^{-1}A \cap p) = \mu_f(p)\mu_f(f^{-1}A) = \frac{1}{d}\mu_f(A). \]

This follows from the fact that $f$ is injective on $f^{-1}A \cap p$ and that $f(f^{-1}A \cap p) = A$. Hence, $\mu_f(f^{-1}A \cap p) = d(\mu_f(f^{-1}A \cap p))$ and the desired equality follows.

For any $d$-adic system, define an $N$-tree of a point $z$ to be a $d$-ary tree with $N + 1$ levels, 0 through $N$, whose $d^n$ vertices at the $n$-th level are identified with the $d^n$ elements of $T^{-n}(z)$. Furthermore, the vertices are labeled in such a way that if $x$ and $y$ have the same parent vertex, then $f(x) = f(y)$. We define an automorphism of the $N$-tree of $z$ to be a map $A$ from the vertices of the $N$-tree of $z$ to themselves that preserves the tree structure. In particular, if $x = f(y)$, then $A(x) = f(Ay)$. The labeling of the vertices is not unique because applying an automorphism of the $N$-tree of $z$ results in a different labeled tree which satisfies the definition. This ambiguity will not make a difference. Define a tree of a point $z$ to be an infinite $d$-ary tree, whose vertices are identified with the preimages of $z$ in the same way as before.

Given two points $z$ and $w$ we now define a metric between the $N$ trees of $z$ and $w$. For each $n$ choose an ordering of $T^{-n}(z)$ and $T^{-n}(w)$. Call them $z^N_i$ and $w^N_i$, $i = 1, \ldots, d^n$. Set
\[ t^*_N(z, w) = \min_A \frac{1}{N} \sum_{n=1}^{N} \frac{1}{d^n} \sum_{i=1}^{d^n} D(z^N_i, A(w^N_i)) \]
where $A$ ranges over all automorphisms of the $N$-tree of $w$. Each automorphism $A$ generates a pairing of the vertices of the $N$-trees of $z$ and $w$. The quantity that we are minimizing is a weighted average of the distance between paired vertices. Thus this metric is a minimum distance between two sets of preimages, subject to the constraint that the pairing of the two sets respects the tree structures of the two sets.

Another way to view this quantity is to notice that for any $z, w$, and $A$
\[ \frac{1}{N} \sum_{n=1}^{N} \frac{1}{d^n} \sum_{i=1}^{d^n} D(z^N_i, A(w^N_i)) = \frac{1}{d^N} \sum_{i=1}^{d^N} \frac{1}{N} \sum_{j=0}^{N-1} D(T^j(z^N_i), T^j(A(w^N_i)))). \]

Each automorphism $A$ generates a pairing of $T^{-N}(z)$ and $T^{-N}(w)$. The first summand is the average distance between images of $z_i^N$ and $A(w_i^N)$ under $T^0, T^1, \ldots, T^{N-1}$. The outer sum averages this quantity over $T^{-N}(z)$. Viewed
in this manner the definition of tree vwB is close to the definition of very weak Bernoulli for two-sided extensions of \( d \)-adic endomorphisms. See [7] for a discussion of this definition difference.

**Definition 3.1.** A \( d \)-adic endomorphism \( T \) acting on \((Y, \mathcal{C}, \nu)\) is tree very weak Bernoulli (tree vwB) if for all \( \varepsilon > 0 \), there exist \( N \) and a set \( G \) such that

1. \( \nu(G) > 1 - \varepsilon \).
2. For all \( z, w \in G \), \( t^D_N(z, w) < \varepsilon \).

The main result (Theorem 5.5) of [7] is the following.

**Theorem 3.1.** Let \( T \) be a \( d \)-adic endomorphism acting on \((Y, \mathcal{C}, \nu)\) which is tree vwB. Then \( T \) acting on \((Y, \mathcal{C}, \nu)\) is \( d \)-adic Bernoulli.

### 4. Rational maps are Bernoulli

In this section we verify the tree vwB condition for all rational maps with the dist metric. The main tool is Lemma 4.3.

**Definition 4.1.** Two points \( z, w \in \bar{C} \) are \((N, \varepsilon)\) tree related if there exists an invertible map

\[ \phi: \text{vertices of the tree of } z \to \text{vertices of the tree of } w \]

and for each \( n \) there exists an ordering of \( f^{-n}(z) \), \( \{z^n_i\}_{i=1}^{d^n} \), such that

1. \( \phi(f(z^n_i)) = f(\phi(z^n_i)) \) for all \( n \) and \( i \).
2. \( \text{dist}(z^n_i, \phi(z^n_i)) < \varepsilon \), for all \( n \geq N \) and \( i, 1 \leq i \leq (1 - \varepsilon)d^n \).

The main idea of this section is to show that any two points in the Julia set are \((N, \varepsilon)\) tree related. This will imply the tree vwB condition. The next lemma shows the relationship between \((N', \varepsilon)\) tree related and the \( t^\text{dist}_N \) metric.

**Lemma 4.1.** If \( z \) and \( w \) are \((N', \varepsilon)\) tree related and \( N > N'/\varepsilon \) then

\[ t^\text{dist}_N(z, w) < 3\varepsilon. \]

**Proof.** The map \( \phi \) generates an automorphism of the \( N \) tree of \( w \) by

\[ A(w^n_i) = \phi(z^n_i). \]

Thus if \( n > N' \) and \( i \leq (1 - \varepsilon)d^n \) then

\[ \text{dist}(z^n_i, A(w^n_i)) < \varepsilon. \]
Thus
\[
\frac{1}{N} \sum_{n=1}^{N} \frac{1}{d^n} \sum_{i=1}^{d^n} \text{dist}(z_i^n, A(w_i^n))
\]
\[
\leq \frac{1}{N} \left( \sum_{n=1}^{N} \frac{1}{d^n} \sum_{i=1}^{d^n} \text{dist}(z_i^n, A(w_i^n)) + \sum_{n=N'+1}^{N} \frac{(1-\varepsilon)d^n}{d^n} \sum_{i=1}^{d^n} \text{dist}(z_i^n, A(w_i^n)) \right)
\]
\[
\leq \frac{1}{N} \left( \sum_{n=1}^{N} \frac{1}{d^n} \sum_{i=1}^{d^n} 1 + \sum_{n=N'+1}^{N} \frac{1}{d^n} \sum_{i=1}^{d^n} \frac{1}{(1-\varepsilon)d^n+1} \sum_{i=1}^{d^n} 1 \right)
\]
\[
\leq \frac{1}{N} (N' + \varepsilon N + \varepsilon N)
\]
\[
\leq 3\varepsilon.
\]

For the remainder of the paper we say the map \(g : U \to V\) is a \(k\)-to-one map if \(g\) is a branched covering of finite degree such that, for each \(v \in V\), \(k\) is equal to the sum over \(g^{-1}(v) \cap U\) of the multiplicity of the solution of \(g(u) = v\).

**Definition 4.2.** A topological disk \(U\) is \((N, \varepsilon)\) tree adapted if there exist topological disks, \(S^0 = U, \{S^n_i\}, 1 \leq i \leq l_n,\) and integers \(k^n_i\), for every \(n \geq 1\) such that
1. \(\sum_{i=1}^{l_n} k^n_i = d^n\) for all \(n \leq N\).
2. \(\sum_{i=1}^{l_n} k^n_i \geq (1-\varepsilon)d^n\) for all \(n \geq 1\).
3. For all \(n \leq N\) for all \(j\), \(f^n|_{S^n_j}\) is a \(k^n_j\)-to-one map onto \(S^0\). Moreover, for all \(n \leq N\) for all \(j\), there exists \(i\) such that \(f|_{S^n_i}\) is a finite to one map of \(S^n_{i-1}\).
4. For all \(n > N\) for all \(j\) there exists \(i\) such that \(f\) maps \(S^n_j\) homeomorphically onto \(S^n_{i-1}\).
5. \(\lim_{n \to \infty} \sup_i \text{diam } S^n_i = 0\).

The following lemma ties together the concepts of tree adapted and tree related.

**Lemma 4.2.** Given an \((N, \varepsilon)\) tree adapted set \(U\), there exists an \(M\) such that for any two points \(z, w \in U\), the points \(z\) and \(w\) are \((M, \varepsilon)\) tree related.
Proof. Fix $z, w \in U$, an $(N, \varepsilon)$ tree adapted set. There exists a collection of sets, $\{S^n_i\}$, such that $\lim_{n \to \infty} \sup_i \text{diam} S^n_i = 0$. Pick $M$ such that $\text{diam} S^n_i < \varepsilon$ for all $n \geq M$ and $i \leq l_n$. Define $\phi(z) = w$, and define $\phi : \{f^{-1}(z)\} \to \{f^{-1}(w)\}$ in a one-to-one manner so that if $z^1_i \in S^i_k$, then $\phi(z^1_i) \in S^i_k$. Define $\phi$ on the rest of the tree by induction. Assume $\phi$ has been constructed on the first $n$ levels such that if $z^1_i \in S^i_k$, then $\phi(f(z^1_i)) = f(w^1_i)$ for all $i, k$ and for $j \leq n$. Also assume that if $z^j_i \in S^j_k$, then $\phi(z^j_i) \in S^j_k$ for all $k$ and $j \leq n$.

For $n + 1$, pick $z^{n+1}_i$ which is contained in some $S^{n+1}_k$. Then $f(z^{n+1}_i) \in S^n_j$ for some $j$, and $\phi(f(z^{n+1}_i)) \in S^n_j$. Notice that

$$\#\{f^{-1}(f(z^{n+1}_i)) \cap S^{n+1}_k\} = \#\{f^{-1}(\phi(f(z^{n+1}_i))) \cap S^{n+1}_k\}$$

since $f|S^{n+1}_k$ is a branched covering of finite degree. Define $\phi$ to be any one-to-one map between these sets. On the preimages that lie inside some $S^{n+1}_k$, $\phi$ fulfills the requirements. On preimages not lying in one of these sets, define $\phi$ to be any map that preserves the tree structure, namely, any map that sends $f^{-1}(f(z^{n+1}_i))$ to $f^{-1}(\phi(f(z^{n+1}_i)))$. Thus $\phi(f(z^{n+1}_i)) = f(\phi(z^{n+1}_i))$. Furthermore, for all $n \geq M$, for all $i$ such that $z^n_i \in \bigcup_{j=1}^n S^n_j$,

$$\text{dist}(z^n_i, \phi(z^n_i)) < \varepsilon,$$

since $z^n_i, \phi(z^n_i) \in S^n_j$ for some $j$ and $\text{diam} S^n_j < \varepsilon$. The cardinality of the set $\{i \mid z^n_i \in \bigcup_{j=1}^n S^n_j\}$ is at least $(1 - \varepsilon)d^n$ since $\sum_{i=1}^n k^n_i \geq (1 - \varepsilon)d^n$. Thus the desired orderings exist.

The fundamental lemma of this section is the following.

**Lemma 4.3.** Given $\varepsilon > 0$, $z \notin \text{Exc}(f)$, and an arc $\gamma$ containing $z$ such that $\gamma \setminus \{z\}$ does not contain any critical values of $f^n$ for all $n \geq 1$, there exists an $(N, \varepsilon)$ tree adapted set $U$ containing $\gamma$ for some $N \geq 1$.

We leave the proof of this lemma until the end of the section. The fundamental lemma gives us the following lemma.

**Lemma 4.4.** Given $\varepsilon > 0$ there exists $N$ such that all pairs of points $x, y \in J(f)$ are $(N, \varepsilon)$ tree related.

**Proof.** First we show that for any two points $x, y \in J(f)$, there exists an $M$ such that $x, y$ are $(M, \varepsilon)$ tree related. Given $\varepsilon > 0$ pick arcs $\gamma_x$ and $\gamma_y$ satisfying the hypothesis of the fundamental lemma, such that $x \in \gamma_x$, $y \in \gamma_y$, and such that $\gamma_x \cap \gamma_y \neq \emptyset$. By the fundamental lemma, there exist topological disks $U_x$ and $U_y$ containing $\gamma_x$ and $\gamma_y$ that are $(N, \frac{\varepsilon}{2})$ tree adapted. Pick $M$ such that any two points in $U_x$ (and $U_y$) are $(M, \frac{\varepsilon}{2})$ tree related. This implies that $x, y$ are $(M, \varepsilon)$ tree related since $U_x \cap U_y \neq \emptyset$. In particular, suppose $z \in U_x \cap U_y$, $\phi_x$ is the map from the tree of $z$ to that of $x$ and $\phi_y$ is the map
from the tree of $z$ to the tree of $y$. Then $\phi = \phi_x \phi_y^{-1}$ is a map from the tree of $y$ to the tree of $x$ preserving the tree structure. Furthermore, if $x_i^n \in S_j^n$, for some $j \leq l_n$, and $n \geq M$, then $\text{dist}(x_i^n, \phi(x_i^n)) < \varepsilon$. This implies that the cardinality of the set $\{i \mid \text{dist}(x_i^n, \phi(x_i^n)) < \varepsilon\}$ is at least $(1 - \varepsilon)\delta^n$ since $\sum_{i=1}^{l_n} k_i^n \geq (1 - \varepsilon)\delta^n$.

Now define $\tilde{M} : J(f) \times J(f) \to \mathbb{N}$ by the following property. Let $\tilde{M}(z, w)$ be the minimum $N \geq 0$ such that there are neighborhoods $U_z$ and $U_w$ such that every point in $U_z$ is $(N, \varepsilon)$ tree related to every point in $U_w$. The previous argument shows that this is well defined. Furthermore, this function is upper semicontinuous. Since $J(f)$ is compact $\tilde{M}$ is bounded. Let $N$ be an upper bound. $N$ satisfies the required property.

**Theorem 4.1.** If $f$ is a rational map of degree $d \geq 2$ then the system $f$ acting on $(\bar{C}, A, \mu_f)$ is $d$-adic Bernoulli.

**Proof.** Given $\varepsilon > 0$. Set $G = J(f)$. By Lemma 4.4 pick $N'$ such that for all $z, w \in J(f)$ are $(N', \varepsilon/3)$ tree related. Choose $N > 3N'/\varepsilon$. By Lemma 4.1

$$t_N^{\text{dist}}(z, w) < \varepsilon.$$ 

Thus $f$ acting on $(\bar{C}, A, \mu_f)$ is tree vwB. By Theorem 3.1, $f$ acting on $(\bar{C}, A, \mu_f)$ is $d$-adic Bernoulli. \qed

In order to prove the fundamental lemma we need a preliminary lemma from [5] that we state but do not prove. For any $z \in \bar{C}$ define

$$m_n(z) = \text{the maximum multiplicity of any } n\text{-preimage of } z.$$ 

**Lemma 4.5.** For all $z \notin \text{Exc}(f)$ there exists $N > 0$ and $d_0$, $1 \leq d_0 < d$ such that $m_n(z) \leq (d_0)^n$ for all $n \geq N$.

**Proof of Lemma 4.3.** Pick $N_0$ such that $m_{N_0}(z)N_0d^{-N_0} < \varepsilon/2$. Such an $N_0$ exists by the previous lemma. Furthermore, pick $N_0$ such that

$$4m_{N_0}(z)d^{-N_0} \frac{d^3}{d - 1} < \frac{\varepsilon}{4}$$

and

$$2m_{N_0}(z)N_0d^{-N_0}(1 + \sum_{j=1}^{\infty} j \frac{d^j}{d^j}) < \frac{\varepsilon}{4}.$$ 

Let $m_{N_0}(z) = m$. Since the only possible critical value of $f^{N_0}$, contained in $\gamma$, is $z$, it follows that the connected components $\{\gamma_i^n\}_{n=1}^{N_0}$ of $f^{-n}(\gamma)$ are either arcs or unions of arcs with a unique point of intersection. Therefore, each $\gamma_i^n$ is simply connected. We can then take a topological disk $U_0 \supset \gamma$ so
thin that for each \( n \leq N_0 \) there exist disjoint topological disks \( U^n_i \) containing \( \gamma^n_i \), such that for all \( n > N_0 \), for all \( i \), there exists \( j \) such that \( f(U^n_i) = U^{n-1}_j \). Furthermore, \( f|U^n_i \) is a \( k^n_i \)-to-one map (counting multiplicity) onto \( U_0 \).

Set \( \varepsilon_{N_0} = 2mN_0d^{-N_0} \) and \( \varepsilon_{n+1} = \varepsilon_{n} + 4md^{-(n+1)}d^2 + 2md^{-(n+1)}(n + 1) \) for \( n \geq N_0 \). Observe that

\[
\varepsilon_{n} \leq 4md^2 \sum_{j=N_0}^{\infty} \frac{1}{d^j} + 2m \sum_{j=N_0}^{\infty} \frac{j}{d^j} \leq 4md^{-N_0} \frac{d^3}{d-1} + 2mN_0d^{-N_0}(1 + \sum_{j=1}^{\infty} \frac{j}{d^j}) \leq \frac{\varepsilon}{2}.
\]

We claim that for all \( n \geq N_0 \), \( f^{-n}(U_0) \) contains a union of topological disks, \( W^n_i, i = 1, \ldots, l_n \) such that \( f^n(W^n_i) = U_0 \) is a \( k^n_i \)-to-one map, where \( k^n_i, l_n \) are integers satisfying

1. \( 1 \leq k^n_i \leq m \) and
2. \( \sum_{i=1}^{l_n} k^n_i \geq (1 - \frac{\varepsilon}{2})d^n \).

Furthermore \( \lambda(W^n_i) < \frac{1}{n} \), where \( \lambda \) denotes Lebesgue measure. For \( N_0 \), Let \( W^n_i = U^n_i \). Since \( \{U^n_i \} \) are disjoint, there are at most \( N_0 \) of these disks such that \( \lambda(U^n_i) \geq \frac{1}{N_0} \). Throw these disks away, so that if \( 1 \leq i \leq l_{N_0} \), then \( \lambda(U^n_i) < \frac{1}{N_0} \). Now

\[
\sum_{i=1}^{l_n} k^n_i \geq d^{N_0} - N_0m = d^{N_0}(1 - mN_0d^{-N_0}) \geq d^{N_0}(1 - \frac{\varepsilon}{2}).
\]

The proof of the claim is completed by induction. Suppose there exists the collection of topological disks \( W^n_i \) and integers \( k^n_i, i = 1, \ldots, l_n \). Let \( H \) be the set of integers \( t \) between 1 and \( l_n \) such that \( W^n_i \) contains no critical values of \( f \). For every \( t \in H \) there is a disk that maps homeomorphically onto \( W^n_i \). Define this disk as \( W^n_{i+1} \) and only keep those \( i \) such that \( \lambda(W^n_{i+1}) < \frac{1}{n+1} \). These will be the values \( 1 \leq i \leq l_{n+1} \). Then

\[
\sum_{i=1}^{l_{n+1}} k^{n+1}_i \geq d(\sum_{i \in H} k^n_i) - (n + 1)m \geq d(\sum_{i=1}^{l_n} k^n_i - \sum_{i \notin H} k^n_i) - (n + 1)m.
\]
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\[ \geq d \sum_{i=1}^{l_n} k_i^{n+1} - dm(l_n - \#H) - m(n + 1) \]

But $l_n - \#H$ is bounded by the number of critical values of $f$, which is bounded by $2d$. Hence

\[ \sum_{i=1}^{l_{n+1}} k_i^{n+1} \geq d^{n+1}(1 - \frac{\varepsilon_n}{2}) - 2d^2m - m(n + 1) \]

By the way, the $W_i^n$ are constructed, for $n > N_0$, for all $i$, there exists a $j$ such that $f$ restricted to $W_i^n$ is a homeomorphism onto $W_j^{n-1}$. Furthermore, if we omit some $W_i^n$ because the diameter of the set is too large or because $W_i^n$ contains a critical value of $f$, then no subsequent preimage of the set is included in the collection. We have now shown the first four properties. In order to show the last property, we need to shave down the set $U_0$ to a set $U$ and the sets $W_i^n$ to sets $S_i^n$ and apply Koebe’s distortion theorem. We do this exactly as in [5]. The proof is included for completeness.

We show that for any topological disk $U$ whose closure is contained in $U_0$,

\[ \lim_{n \to \infty} (\sup_{i} \text{diam}(f^{-n}(U) \cap W_i^n)) = 0. \]

If this property is true, the lemma is proved taking $U$ containing $\gamma$ and with closure contained in $U_0$. Then we define

\[ S_i^n = W_i^n \cap f^{-n}(U). \]

By the way the sets $W_i^n$ are constructed, $f^{-n-N_0}|_{W_i^n}$ is a conformal representation onto some $W_j^{N_0}$. Let $\phi_i^n : W_j^{N_0} \to W_i^n$ be its inverse. Set $D_r = \{ z \mid |z| \leq r \}$. Let $\alpha_j : D_1 \to W_j^{N_0}$ be a conformal representation. Define $\psi_i^n : D_1 \to W_i^n$ as $\psi_i^n = \phi_i^n \alpha_j$. We shall prove

\[ \lim_{n \to \infty} (\sup_{i} \text{diam}(\psi_i^n(D_r))) = 0 \]

for all $r$, $0 < r < 1$. This implies the result because $\psi_i^n(D_r) \supset f^{-(n-N_0)}(U) \cap W_i^n$ for all $n \geq N$, if $r$ is near enough to 1. To prove the result, recall Koebe’s distortion theorem for univalent functions. This theorem says that for all $0 < r < 1$ there exists $K(r)$ such that every univalent function $\phi : D_1 \to \mathbb{C}$ satisfies $|\phi'(a)/\phi'(b)| \leq K(r)$ for all $a$ and $b$ in $D_r$. In particular,

\[ \lambda(\phi(D_r)) \geq K(r)^{-1}|\phi'(a)|\lambda(D_r) \]
for all \( a \in D_r \). In our case

\[
\frac{1}{n} \geq \lambda(W^n_i) \geq \lambda(\psi^n_i(D_r)) \geq K(r)^{-1} \lambda(D_r) |(\psi^n_i)'(z)|
\]

for all \( 0 < r < 1 \), \( z \in D_r \). Then

\[
\lim_{n \to \infty} \sup_{i, z \in D_r} |(\psi^n_i)'(z)| = 0
\]

which implies the result.

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