Competition between $\pi$–coupling and FFLO modulation in SF/SF atomic thickness bilayers

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Abstract

We present the detailed theoretical study of a heterostructure comprising of two coupled ferromagnetic superconducting layers. Our model may be also applicable to the layered superconductors with alternating interlayer coupling in a parallel magnetic field. It is demonstrated that such systems exhibit a competition between the nonuniform Larkin-Ovchinnikov-Fulde-Ferrel (FFLO) state and the $\pi$ superconducting state where the sign of the superconducting order parameter is opposite in adjacent layers. We determine the complete temperature-field phase diagram. In the case of low interlayer coupling we obtain a new $\pi$ phase inserted within the FFLO phase and located close to the usual tricritical point, whereas for strong interlayer coupling the bilayer in the $\pi$ state reveals a very high paramagnetic limit and the phenomenon of field-induced superconductivity.
I. INTRODUCTION

The question of coexistence of singlet superconductivity and magnetism has been addressed for many years. It was found that the superconducting order parameter is destroyed by a magnetic field both via the orbital effect [1] and the paramagnetic effect [2]. In the usual case of an isotropic three-dimensional (3D) superconductor under an external magnetic field, the orbital effect prevails and leads to the well-known temperature-field phase diagram of conventional type I or II superconductors [3]. In contrast, superconductivity is essentially suppressed by the paramagnetic effect in the presence of a ferromagnetic exchange interaction. This is also true for quasi-two-dimensional (2D) superconductors under in-plane magnetic field and for heavy fermions materials wherein the orbital effect is partially quenched. In the whole paper, the magnetism is characterized by an internal exchange field $h$ (given in energy units) which may arise either from an externally applied magnetic field or from ferromagnetic ordering. Note that ferromagnetism must be weak in order to avoid complete suppression of superconductivity. This is realized in rare-earth metals or actinides in which the indirect exchange interaction leads to Curie temperatures of a few degrees.

Superconductors with internal homogeneous exchange field $h$ exhibit a very special behaviour. According to Chandrasekhar [4] and Clogston [5] at zero temperature uniform superconductivity should be destroyed when the polarization energy of the free electron gas exceeds the energy gain due to Cooper pairing in the BCS ground state. This criterion gives the exchange field $h_p(T = 0) = \Delta_0/\sqrt{2}$ where the superconductor should undergo a first-order transition to the normal state, $\Delta_0 = 1.76T_c$ being the zero temperature superconducting gap. Larkin and Ovchinnikov [6] and Fulde and Ferrell [7] (FFLO) predicted the existence of a nonuniform superconducting state with higher critical exchange field $h_{3D}^{FFLO}(T = 0) = 0.755\Delta_0 > h_p(T = 0)$ and second-order transition to the normal state. This prediction was made for 3D superconductors. In quasi-2D superconductors the critical exchange field of the FFLO state is even higher, namely $h_{2D}^{FFLO}(T = 0) = \Delta_0$, [8] while in quasi-one-dimensional systems there is no paramagnetic limit at all [9]. The appearance of the modulated FFLO state is related to the pairing of electrons with opposite spins which do not have the opposite momenta anymore due to the Zeeman splitting. From now on we focus on the 2D case for which a generic temperature-exchange-field phase diagram has been established [8]. At low field and temperature, the ground state is characterized by a
uniform superconducting order parameter. A tricritical point, located at $h^* = 1.07T_{c,0}$ and $T^* = 0.56T_{c,0}$, is the meeting point of three transition lines separating the normal metal, the uniform and the nonuniform superconductors. At $T < 0.56T_{c,0}$, the (low-field) uniform superconductor is separated from the (high-field) normal metal by a narrow FFLO nonuniform superconducting phase. In contrast, at $T > 0.56T_{c,0}$, the system undergoes merely a second-order phase transition from the uniform superconductor to the normal metal when increasing the exchange field. The nonuniform FFLO state is settled in a small region of the phase diagram and is very sensitive to impurities,[10, 11] making it difficult to observe experimentally. Nevertheless, several evidences of the FFLO state have been obtained recently in organic superconductors [12, 13] and in heavy fermions compounds, see Martin et al.[14] and references therein.

In the context of organic and high-$T_c$ superconductors, layered systems made of conducting atomic planes have been extensively studied.[15] In order to investigate the interplay of superconductivity and magnetism in such anisotropic systems,[16] Andreev et al. considered a periodic array of alternating ferromagnetic and superconducting 2D planes.[17] Solving the corresponding Gor'kov equations, these authors established the existence of a $\pi$ state wherein each F layer separates superconducting planes with opposite order parameter. This is relevant for the ruthenocuprate compound RuSr$_2$GdCu$_2$O$_8$ which comprises CuO$_2$ superconducting planes and RuO$_2$ magnetic planes.[18, 19] A related system is an isolated F/S/F trilayer which exhibits the so-called superconducting spin-valve effect. Namely, its critical temperature is higher in the antiparallel (AP) orientation of the layers magnetizations than in the parallel (P) orientation both for thick layers[20, 21] and atomic size layers.[22, 23] Surprisingly, in the atomic thickness limit, the superconducting gap at zero temperature is higher for P orientation of the magnetizations.[22–25] Hence one expects a transition from AP to P orientation by cooling the system below a finite crossing temperature. The recent progress in molecular beam epitaxy[26] enables to fabricate such F/S/F trilayer with atomic thicknesses.

In this paper, we consider a periodic array of SF bilayers. Each bilayer is made of two atomic planes coupled by single electron tunneling. Both exchange fields and BCS superconducting pairing are present in each SF plane. The possibility of $\chi = \pi$ phase difference between the planes inside each bilayer is also taken into account. In the whole paper, we assume that the coupling $t'$ between successive bilayers is considerably weaker than the
intra-bilayer coupling $t$. Our study is performed within the framework of the BCS theory of $s$-wave superconductivity. Solving exactly the Gor'kov equations in the limit $t'/t \to 0$, we first derive the critical temperature and the superconducting gap both for parallel (P) and antiparallel (AP) orientation of the magnetizations. We show that the critical temperature is higher for the AP orientation than for the P orientation whereas it is the opposite for the zero temperature gap. We also investigate the interlayer Josephson current in the small coupling limit: the current increases as a function of the exchange field for AP orientation whereas it is field-independent for the P orientation. Furthermore, we find that for low exchange fields and high temperatures, the ground state corresponds to identical superconducting order parameters on adjacent layers. For high enough fields and/or low enough temperatures, the $\pi$ phase ground state is favored and competes with the FFLO state. For the P orientation, the full temperature-exchange field phase diagram is constructed in the two limits of extremely low and high coupling between the planes. As expected, for perturbative coupling between two SF planes, the phase diagram is very close to the quasi-2D superconductor’s phase diagram. Nevertheless an important change arises. Indeed a new $\pi-$phase is inserted inside the usual FFLO phase close to the tricritical point. For higher tunneling coupling $t \geq T_{c0}$, this $\pi-$phase is pushed to low temperatures $T \leq T_{c0}^2/t$ and high fields $h \approx t$. In this unusual superconducting phase, the Zeeman splitting is compensated by the bonding/antibonding energy splitting due to single-electron tunneling between the planes.[27] As a result, field-induced superconductivity and enhanced paramagnetic limit are realized in this simple model. These new phenomena are encountered due to the introduction of an additional discrete degree of freedom, here the layer index $j$. The layer index acts as a pseudo-spin and thus enlarges the usual spin-space for singlet pairing. This idea was introduced by Kulic and Hofmann[28] in the context of two-bands superconductivity for which the pseudo-spin was the band index. Nevertheless, these authors did not investigate the presently studied $\pi$ state.

The outline of the paper is the following. In Sec.II, we present the model, derive the corresponding Gor’kov equations and give their exact solutions. In Sec.III, we investigate the critical temperature, the gap and the interlayer Josephson current in the small exchange field regime for which there are only uniform superconducting phases. In the last two sections the temperature-exchange field phase diagram of the bilayer is studied thoroughly. In Sec.IV, we first construct a Ginzburg-Landau functional to determine the transitions between the
different phases in the low interlayer coupling limit. Sec.V is devoted to the opposite limit of strong interlayer coupling. In conclusion, we discuss the conditions for the observation of field-induced superconductivity.

II. ATOMIC THICKNESS SF/SF BILAYER

We consider a superconducting ferromagnetic bilayer (see Fig.1) constituted of two superconducting atomic layers, labeled as \( j = 1 \) and \( j = 2 \). In the whole article, we assume \( t \ll E_F \) where \( t \) is the interlayer coupling energy and \( E_F \) the Fermi energy. As a consequence, Cooper pairs are localized within each plane.[15] Each layer \( j \) supports a superconducting singlet BCS coupling with the energy gap \( \Delta_j \) and an internal exchange field \( h_j \). The Hamiltonian of the system can be written as

\[
H = \sum_{j=1,2} \left[ H^0_j + H^{BCS}_j + \frac{1}{|\lambda|} \int d^2 r \Delta_j^2(r) \right] + H_t, \tag{1}
\]

where \( \lambda \) is the attractive BCS interaction constant and \( r \) is the two-dimensional coordinate within each layer. For the layer \( j \) the kinetic and Zeeman parts of the Hamiltonian are written together as

\[
H^0_j = \sum_{\mathbf{p}} \xi_{j\sigma\sigma'}(\mathbf{p}) \psi_{j\sigma}^+(\mathbf{p}) \psi_{j\sigma'}(\mathbf{p}), \tag{2}
\]

in which summation over repeated spin indexes \( \sigma \) and \( \sigma' \) is implied. Creation (resp. annihilation) operator of an electron with spin \( \sigma \) and two-dimensional momentum \( \mathbf{p} \) in the layer \( j \) is denoted \( \psi_{j\sigma}(\mathbf{p}) \) (resp. \( \psi^+_{j\sigma}(\mathbf{p}) \)). The exchange fields \( h_j \) are assumed to be either equal \( (h_1 = h_2 = h) \) or opposite \( (h_1 = -h_2 = h) \). As a consequence the matrix \( \xi_{j\sigma\sigma'} \) is spin-diagonal, and the Zeeman effect manifests itself in breaking the spin degeneracy of the electronic energy levels according to

\[
\xi_{j\sigma\sigma'}(\mathbf{p}) = \delta_{\sigma\sigma'} [\xi(\mathbf{p}) - \sigma h_j], \tag{3}
\]

where \( \xi(\mathbf{p}) = \mathbf{p}^2/2m - E_F \). The s-wave singlet superconductivity is represented by the standard mean-field Hamiltonian

\[
H^{BCS}_j = \sum_{\mathbf{p}} \left[ \Delta_j^* (\mathbf{q}) \psi^+_{j\downarrow}(\mathbf{p}) \psi_{j\uparrow}(-\mathbf{p}) + h.c. \right], \tag{4}
\]
and the layers are coupled together by the hopping Hamiltonian

$$H_t = t \sum_{\mathbf{p}, \sigma} \left[ \psi_{1\sigma}^+(\mathbf{p}) \psi_{2\sigma}^-(\mathbf{p}) + h.c. \right].$$  

(5)

In order to investigate the occurrence of modulated superconducting phases (FFLO), we choose the following spatial dependence for the superconducting order parameter

$$\Delta_1(\mathbf{r}) = \Delta e^{i\mathbf{q}\cdot\mathbf{r} + i\chi/2}, \quad \Delta_2(\mathbf{r}) = \Delta e^{i\mathbf{q}\cdot\mathbf{r} - i\chi/2},$$  

(6)

where $\mathbf{q}$ is the FFLO modulation wave vector and $\chi$ the superconducting phase difference between the layers.

The above model can be solved exactly using the Green functions

$$F_{jk}^+(\mathbf{p}, \mathbf{p}') = \langle \psi_{jk}^+(\mathbf{p}) \psi_{k1}^+(\mathbf{p}') \rangle = \delta(\mathbf{p} + \mathbf{p}') F_{jk}^+(\mathbf{p}),$$

$$G_{jk}(\mathbf{p}, \mathbf{p}') = -\langle \psi_{j1}^+(\mathbf{p}) \psi_{k1}^+(\mathbf{p}') \rangle = \delta(\mathbf{p} - \mathbf{p}' + \mathbf{q}) G_{jk}(\mathbf{p}),$$  

(7)

where $j$ and $k$ are the layer’s indexes. The brackets mean statistical averaging over grand-canonical distribution.[29]

We obtain the following Gor’kov equations in the Fourier representation:

$$\begin{pmatrix}
i\omega - \xi_{11}(\mathbf{p} + \mathbf{q}) & -t & \Delta_1 & 0 \\
-t & i\omega - \xi_{21}(\mathbf{p} + \mathbf{q}) & 0 & \Delta_2 \\
\Delta_1^* & 0 & i\omega + \xi_{11}(\mathbf{p}) & t \\
0 & \Delta_2^* & t & i\omega + \xi_{21}(\mathbf{p})
\end{pmatrix}
\begin{pmatrix}
G_{11}(\mathbf{p} + \mathbf{q}) \\
G_{21}(\mathbf{p} + \mathbf{q}) \\
F_{11}(\mathbf{p}) \\
F_{21}(\mathbf{p})
\end{pmatrix} = \begin{pmatrix}1 \\
0 \\
0 \\
0\end{pmatrix},$$  

(8)

where $\omega = (2n + 1)\pi T$ are the fermionic Matsubara frequencies.

In quasi-2D superconductors [8, 30] the maximal FFLO modulation amplitude is of the order of $(\xi_0)^{-1}$, $\xi_0$ being the typical superconducting coherence length. This means that with a good approximation we can consider $\xi_{j1}(\mathbf{p} + \mathbf{q}) = \xi(\mathbf{p}) - h_j + \mathbf{v}_F \cdot \mathbf{q}$, $\mathbf{v}_F$ being the Fermi velocity vector in the plane.

Solving the Gor’kov equations (8) yields the anomalous Gor’kov Green function for the $j = 1$ SF layer

$$F_{11}^+ = \frac{\alpha_2 \Delta_1^* + t^2 \Delta_2^*}{\alpha_1 \alpha_2 + (2 \Delta^2 \cos \chi - \beta)t^2 + t^4},$$

(9)

where

$$\alpha_j = \Delta^2 - \omega_{j+} + \bar{\omega}_{j-} \quad \text{and} \quad \beta = \omega_{1+} \omega_{2+} + \bar{\omega}_{1-} \bar{\omega}_{2-},$$  

(10)
with $\omega_{j\pm} = i\omega \pm \xi(p) - h_j$ and $\tilde{\omega}_{j\pm} = i\omega \pm \xi(p + q) - h_j$. Similar equation holds for $F_{22}^\dagger$. Note that in the case where a lattice made of such SF/SF bilayers is considered, the generalized anomalous Green function is obtained by replacing $t^2$ by $|t + t'e^{-ip \cdot a}|^2$ in Eq.(9), $p_z$ being the projection of the momentum $p$ along the $z$ axis and $a$ the period of the lattice. As a consequence, a finite inter-bilayer coupling $t'$ introduces an anisotropy in the dispersion relation which leads to a broadening of the electronic excitation levels.

In the absence of tunneling $t = 0$ we retrieve from Eq.(9) the anomalous Green function of a quasi-2D superconductor with the exchange field $h_1$

$$F_{11}^\dagger = \frac{\Delta_1^*}{\alpha_1} = \frac{\Delta_1^*}{\Delta^2 - (i\omega + \xi_{1\dagger})(i\omega - \xi_{1\uparrow})}.$$  
(11)

Although the dependence on momentum has been removed for simplicity, notice that $\xi_{j\uparrow} = \xi_{j\uparrow}(p + q)$ and $\xi_{j\downarrow} = \xi_{j\downarrow}(p)$. The set of basic equations (8) must be completed by the self-consistency equation

$$\Delta_j^* = |\lambda| N(0)T \sum_\omega \int_{-\infty}^{+\infty} d\xi F_{jj}^\dagger.$$
(12)

Close to the critical temperature $T_c$ of the second-order phase transition, the order parameters $\Delta_j$ are small and Eq.(12) can be written as

$$\ln \frac{T_c}{T_{c0}} = 2T_c \sum_{\omega > 0} \int_{-\infty}^{+\infty} d\xi \left( \frac{\text{Re} F_{jj}^\dagger}{\Delta_j^*} - \frac{\pi}{\omega} \right),$$
(13)

where $T_{c0}$ is the critical temperature for the 2D superconducting single layer in the absence of exchange field, namely for $h = t = 0$. At zero temperature, it is convenient to write Eq.(12) as

$$\ln \frac{\Delta}{\Delta_0} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} d\xi \left( \frac{F_{jj}^\dagger}{\Delta_j^*} - \frac{1}{\omega^2 + \xi^2 + \Delta^2} \right),$$
(14)

where $\Delta_0 = \Delta(T = 0, h = 0, t = 0)$ is the superconducting order parameter at $T = 0$ in the absence of exchange field and interlayer coupling.

III. UNIFORM SUPERCONDUCTING STATES

In this section, we investigate phases with uniform superconductivity within each layer. We obtain the critical temperature of the second-order superconducting ($S$) to normal metal ($N$) phase transition and the order parameter $\Delta(T, h, t)$ as a function of the temperature, the exchange field $h$ and the interlayer coupling $t$. We consider both parallel (P) and antiparallel
(AP) orientations of the magnetizations, the superconducting phase difference being either \( \chi = 0 \) or \( \chi = \pi \). We also calculate the Josephson interlayer current when the bilayer is connected to external superconducting leads. Most of these results are obtained in the perturbative limit of small coupling between the layers \( t \ll T_{c0} \). The field is also assumed to be sufficiently small to prevent the occurrence of a spatial modulation of the superconductivity within the planes. Study of nonuniform phases and strong coupling \( t \gg T_{c0} \) are respectively postponed to Sec. IV and Sec. V.

A. Critical temperature

We consider the second-order phase transition between the normal metal and the uniform BCS superconductor. Thus the order parameters \( \Delta_1 \) and \( \Delta_2 \) are small and the anomalous Gor’kov Green function (9) can be linearized in the following form

\[
F_{11}^+ = \frac{-\omega_2 + \bar{\omega}_2 - \Delta_1^* + t^2 \Delta_2^*}{(\bar{\omega}_1 - \omega_2 - t^2)(\omega_1 + \bar{\omega}_2 + t^2)},
\]

where \( \xi_{j\sigma} = \xi - \sigma h_j \). Similar equation may be found for \( F_{22}^+ \). We first consider the parallel \((P)\) orientation of the magnetizations, namely \( h = h_1 = h_2 \). The first possibility is \( \chi = 0 \) wherein the layers have the same superconducting order parameters \( \Delta_1 = \Delta_2 = \Delta \). In this situation the anomalous Green function obtained from Eq.(15) is denoted \( (F_{11}^+)^{P,0} \). The identity

\[
\int_{-\infty}^{+\infty} d\xi (F_{11})^{P,0} = \frac{\pi \Delta^*}{\omega - i\hbar},
\]

and Eq.(13) yield the following implicit equation for the critical temperature \( T_{c}^{P,0} \)

\[
\ln \frac{T_c^{P,0}}{T_{c0}} = \Psi \left( \frac{1}{2} \right) - \text{Re} \left( \frac{1}{2} + i \frac{h}{2\pi T_c^{P,0}} \right),
\]

where \( \Psi(x) \) denotes the Euler digamma function. Therefore the interlayer coupling disappears from the self-consistency equation and Eq.(17) is identical to that for the 2D monolayer in a uniform exchange field: the bilayer is equivalent to a single layer in the neighborhood of superconducting to normal state transition.[2] The critical temperature of the bilayer decreases when the exchange field \( h \) increases. The equation (17) describes the second-order phase transition between the normal metal and the uniform superconductor which is realised only for fields smaller than the tricritical one \( h^* = 1.07T_{c0} \). For larger fields, superconductivity becomes nonuniform.
A second possibility is the \textit{P orientation with }$\chi = \pi$ \textit{phase difference between the layers.} Now the anomalous Gor’kov Green function is denoted $(F_{11}^+)_{P,\pi}$. Then

\[ \int_{-\infty}^{+\infty} (F_{11}^+)_{P,\pi} \, d\xi = \frac{\pi}{2} \left[ \frac{\Delta^*}{\omega - i(h + t)} + \frac{\Delta^*}{\omega - i(h - t)} \right], \tag{18} \]

and the self-consistency relation Eq.(13) yield a critical temperature $T_{c,P,\pi}^0$ given by

\[ \ln \frac{T_{c,P,\pi}^0}{T_{c,0}} = \Phi \left( \frac{1}{2} \right) - \frac{1}{2} \sum_{a=\pm 1} \text{Re} \Phi \left( \frac{1}{2} + i \frac{h + at}{2\pi T_{c,P,\pi}^0} \right). \tag{19} \]

From this expression one may notice that superconductivity in the $\pi$ state is destroyed by a combination of two effective exchange fields $h \pm t$. In the small interlayer coupling limit $t \ll T_c$, Eq.(19) becomes

\[ \ln \frac{T_{c,P,\pi}^0}{T_{c,0}} = \Phi \left( \frac{1}{2} \right) - \frac{1}{2} \sum_{a=\pm 1} \text{Re} \Psi \left( \frac{1}{2} + i \frac{h + at}{2\pi T_{c,P,\pi}^0} \right), \tag{20} \]

where the function $K_3(x)$ is defined and represented in Appendix A. In the regime of low magnetic fields, namely for $h/2\pi T_c^0 < h^*/2\pi T^* = 0.3$, the factor $K_3(h/2\pi T_c^0)$ is positive and thus the critical temperature is smaller in the $\pi$ superconducting state than in the 0 state. However the situation may be inverted if $h/2\pi T_c^0 > 0.3$. Moreover, along the critical line, the value $h^*/2\pi T^* = 0.3$ corresponds to the tricritical point, $h^* \approx 1.07 T_{c,0}$ and $T^* \approx 0.56 T_{c,0}$, where FFLO nonuniform states appear. As a consequence one expects competition between the $\pi$ superconducting phase and FFLO phases in the neighborhood of the tricritical point. This competition will be detailed in Sec. IV.

Let us focus on the case of \textit{AP orientation }$h = h_1 = -h_2$. Following the same procedure as previously, the equations for the critical temperatures $T_{c,AP,\chi}^0$ are obtained. In the limit $t \ll T_{c,0}$, it reads

\[ \frac{T_{c,AP,0}^0 - T_{c,P,0}^0}{T_{c,P,0}^0} = 2\pi T_{c,AP,0}^0 \sum_{\omega > 0} \frac{h^2}{(h^2 + \omega^2)^2 \omega}, \tag{21} \]

for $\chi = 0$, and

\[ \frac{T_{c,AP,\pi}^0 - T_{c,P,0}^0}{T_{c,P,0}^0} = -2\pi T_{c,AP,\pi}^0 \sum_{\omega > 0} \frac{\omega}{(h^2 + \omega^2)^2} \tag{22} \]

for $\chi = \pi$. From Eqs.(21,22) the critical temperature is clearly higher in the 0 phase than in the $\pi$ phase. Therefore the 0 phase is the more stable in this region of the $(T,h)$ phase diagram, \textit{i.e.} in the vicinity of the critical temperature and for low fields $h < h^* = 1.07 T_{c,0}$.  

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In conclusion, the bilayer is always in the 0 superconducting state for temperatures close to the critical temperature, whatever the relative orientation of magnetizations is. A spin-valve effect is also present: the critical temperature is higher for the AP orientation than for the P orientation of the magnetizations.

B. Zero temperature superconducting gap

For $t = 0$ and low fields $h < \Delta_0/\sqrt{2}$, it is well-known that the zero temperature gap $\Delta(T = 0, h, t = 0) = \Delta_0$ is field-independent.[3] For small interlayer coupling $t \ll T_c$, the anomalous Green function (9) may be expanded to the second order in $t$ as

$$F_{11}^+= \frac{1}{\alpha_1} + t^2 \frac{\alpha_1 e^{i\chi} - 2\Delta^2 \cos \chi + \beta}{\alpha_1^2 \alpha_2},$$

where the full nonlinear dependence on $\Delta$ is kept in $\alpha_1, \alpha_1$ and $\beta$. Then self-consistency relation (14) becomes

$$\ln \frac{\Delta}{\Delta_0} = t^2 \int \int \frac{d\omega}{2\pi} d\xi \frac{(\alpha_1 - 2\Delta^2) \cos \chi + \beta}{\alpha_1^2 \alpha_2},$$

where $\Delta = \Delta(T = 0, h, t)$ and $\chi = 0$ or $\chi = \pi$. Using the preceding equation in the P orientation we obtain $\Delta^{P,0} = \Delta^{P,\pi} = \Delta_0$, either for 0 or $\pi$ phase difference. As a result, the superconducting gap $\Delta(T = 0, h, t)$ is not affected by a small interlayer coupling, at least at the order of $t^2$. The superconducting condensation energy gain has also been calculated and the zero state found to be more stable than $\pi$ state.

For the AP orientation, the superconducting gap $\Delta^{AP,0} = \Delta^{AP,0}(T = 0, h, t)$ is given by

$$\ln \frac{\Delta^{AP,0}}{\Delta_0} = \frac{t^2}{2} \left[ \frac{1}{\Delta_0^2 - h^2} \left( \frac{\Delta_0^2 - 2h^2}{h(\Delta_0^2 - h^2)^{3/2}} \arcsin \frac{h}{\Delta_0} \right) \right],$$

for zero phase difference. The unphysical divergence at $h \rightarrow \Delta_0$ is removed by terms of higher order in $t$. Expression (24) is the main result of this paragraph and reduces to

$$\ln \frac{\Delta^{AP,0}}{\Delta_0} = -\frac{4t^2 h^2}{3\Delta_0^4},$$

in the small field regime $h \ll \Delta_0$. Therefore in the $\chi = 0$ state and for AP orientation, the order parameter is suppressed by the exchange field in the small coupling limit. This is surprising because AP orientation was expected to weaken the effective exchange field and thus
enhance superconducting properties. Nevertheless such a decrease of the superconducting order parameter has already been found in a ballistic atomic-scaled F/S/F trilayer.[22, 23]

For the AP orientation and π phase difference, the gap \( \Delta^{AP,\pi}(T = 0, h, t) = \Delta_0 \) is field and coupling independent. Moreover the energy of the \( \chi = \pi \) state does not depend on the relative orientation of the magnetizations.

To summarize, the lowest energy corresponds to the \((P, 0)\) phase. The \((P, \pi)\) and \((AP, \pi)\) phases are degenerate with a somewhat higher energy than the \((P, 0)\) phase. Although we have not performed the energy calculation in the case where the magnetizations are antiparallel and the phase difference is 0, we believe that the highest energy corresponds to the \((AP, 0)\) phase since its order parameter is the smallest one.

C. Superconducting gap versus temperature: inversion of the proximity effect

We now extend our investigation of the superconducting gap to finite temperatures. In order to determine the gap \( \Delta(T, h, t) \) as a function of temperature \( T \), exchange field \( h \) and coupling \( t \), we analyse numerically the self-consistency relation (14) using the exact anomalous Gor’kov Green function (9). The result is shown schematically in Fig.2. For the \( P \) orientation and \( \chi = 0 \), the superconducting gap \( \Delta^{P,0}(T, h, t) \) is the same as the gap \( \Delta(T, h, t = 0) \) of a single layer whereas for \( \chi = \pi \) the superconducting gap \( \Delta^{P,\pi}(T, h, t) \) is lowered by finite interlayer coupling. For the AP orientation and \( \chi = 0 \), the gap is smaller than \( \Delta(T, h, t = 0) \) for \( T < T_i \) and larger for \( T > T_i \) where the inversion temperature \( T_i = T_i(h) \) depends only on the exchange field in the small interlayer coupling limit (see Fig.2 inset). This phenomenon has been called inversion of the proximity effect.[22, 23]

Moreover, the gap \( \Delta^{P,0} \) is larger than \( \Delta^{AP,\pi} \) for all temperatures.

According to these results, one may suggest several experiments. First we consider a bilayer with magnetizations pinned in the AP mutual orientation. By lowering the temperature, a 0-π transition is expected at some temperature \( T_\pi \). In the small interlayer coupling limit, this temperature \( T_\pi(h) \) is a function of the exchange field only (see Fig.2 inset). In contrast, the 0 state is more favorable energetically for all temperatures in the case of magnetizations pinned in the P orientation. As another illustration we consider samples where the relative orientation of magnetizations is free. Then the orientation is chosen by the system to minimize its energy. Cooling such a bilayer will result in a switching from the AP
orientation to the P orientation at the inversion temperature $T_i(h)$. The same prediction was made recently in a ballistic F/S/F trilayer.[22, 23]

D. Josephson current at T=0

Here we consider that the SF/SF bilayer is connected to superconducting electrodes. In this set-up, one may impose an arbitrary superconducting phase difference $\chi$ between the SF layers, and thus a non dissipative Josephson current flows through the bilayer in the direction perpendicular to the planes. This interlayer Josephson current is evaluated here in the tunneling limit $t \ll \Delta_0$ and at zero temperature. Within the Green functions formalism, the general formula for the interlayer Josephson current is

$$j = \frac{2ie t N_{2D}}{\hbar} \iiint d\omega d\xi (G_{12} - G_{21}),$$

where $N_{2D} = m/(2\pi\hbar^2)$ is the two-dimensional density of state per spin direction and unit surface. Solving exactly the Gor’kov equations (8) leads to

$$G_{21} = \frac{t(\omega_1 + \omega_2 - \Delta_1^* \Delta_2)}{\alpha_1 \alpha_2 + (2\Delta^2 \cos \chi - \beta)t^2 + t^4}.$$ (27)

The function $G_{12}$ is obtained from Eq.(27) by permuting the layer indexes $1 \leftrightarrow 2$. The corresponding anharmonic current-phase relationship is given by

$$j = \frac{2ie t^2 N_{2D}}{\hbar} \iiint d\omega d\xi \frac{2\Delta^2 \sin \chi}{\alpha_1 \alpha_2 + (2\Delta^2 \cos \chi - \beta)t^2 + t^4}$$

In the tunneling limit $t \ll \Delta_0$, the interlayer Josephson current becomes sinusoidal as a function of the phase difference,

$$j = j_0 \iiint d\omega d\xi \frac{2\Delta^2 \sin \chi}{(\Delta^2 + \xi^2 + (\omega + i\Delta_1))(\Delta^2 + \xi^2 + (\omega + i\Delta_2))},$$ (28)

where $j_0 = 2e N_{2D} t^2 / \hbar$. The second harmonic $\sin 2\chi$ has also been evaluated and is smaller than the first one by a factor $(t/\Delta_0)^2$. The preceding equation (28) yields the current-phase relation both for parallel (P) and antiparallel (AP) orientation of magnetizations. In the parallel case the critical current does not depend on the field as already reported in other systems since $j_P = j_0 \sin \chi$.[31, 32] For the antiparallel orientation and to the lowest order in $t$, the current-phase relation reads

$$j_{AP} = j_0 f_1 \left( \frac{\Delta}{h} \right) \sin \chi,$$ (29)
where \( f_1(x) = \arcsin \frac{x}{(x \sqrt{1 - x^2})} \). Therefore the critical current increases with the exchange field \( h \) and even diverges for \( h = \Delta = \Delta^{AP}(T = 0) \). Of course this divergence is unphysical and should disappear if all orders in \( t \) were taken into account. In Fig.3, the critical current is shown as a function of the exchange field both for P and AP orientations. Recently, the issue of the Josephson coupling between two clean SF layers through an insulating layer was considered using Eilenberger equations [31] or Bogoliubov-de Gennes formalism.[32] Similar results as ours were obtained: the critical current increases with \( h \) only if three conditions are met: low temperature, very weak coupling between the SF layers and AP orientation. Otherwise the presence of an exchange interaction suppresses the Josephson current. Using Usadel equations, Krivoruchko demonstrated that this statement holds in the diffusive regime for which the divergence for \( h = \Delta \) is replaced by a regular peak.[33]

IV. PHASE DIAGRAM OF THE WEAKLY COUPLED SF/SF BILAYER

From now on, we consider the SF/SF bilayer only for the parallel (P) orientation. Hence the results obtained in the next sections may be also applied to a superconducting bilayer in an external in-plane magnetic field. The present section is devoted to the weak coupling regime \( t \ll T_c \). In contrast to the low field restriction of Sec. III, regions of the phase diagram with \( h/(2\pi T) > 0.3 \) are also investigated here. Then competition between the FFLO and \( \pi \) phases is expected to take place. Particular attention is paid to the vicinity of the tricritical point given by \( h \approx 1.07T_{c_0} \) and \( T \approx 0.56T_{c_0} \).[2] In order to examine this narrow region of the \((T, h)\) plane, we construct a Ginzburg-Landau (GL) functional from the Gor’kov equations used in the previous sections. In the past Buzdin and Kachkachi [34] derived a generalized Ginzburg-Landau (GL) functional for a single SF layer that describes the FFLO superconducting state near the tricritical point. Here we extend this functional to a SF/SF bilayer for which it is possible to have not only FFLO modulation within the planes but also \( \chi = \pi \) superconducting phase difference between the planes. For \( \chi = 0 \), the physics of the bilayer is independent of the coupling and thus the Buzdin-Kachkachi GL functional is retrieved. In contrast for \( \chi = \pi \), we obtain a free energy functional which depends on the interlayer coupling \( t \) and leads to the presence of a superconducting \( \pi \) phase.
A. Ginzburg-Landau free energies

The free energy of the SF/SF bilayer in a uniform superconducting state with $\chi = 0$ ($U - 0$ state) is given by (see details in Appendix B)

$$F_{U-0}(\tilde{\Delta}, \tilde{\tau}) = \tilde{\tau}\left|\tilde{\Delta}\right|^2 - \epsilon\left|\tilde{\Delta}\right|^4 + b\left|\tilde{\Delta}\right|^6,$$

with

$$\tilde{\tau}(h, T) = \ln \frac{T}{T_c} - K_1(\tilde{h}), \epsilon = -\frac{K_3(\tilde{h})}{4}, b = -\frac{K_5(\tilde{h})}{8},$$

where $\tilde{h} = h/2\pi T$ and $\tilde{\Delta} = \Delta/2\pi T$ are respectively the reduced exchange field and the reduced order parameter. Here we retrieve the well-known case of a single SF layer. For reduced exchange fields lower than the tricritical one $\tilde{h} < \tilde{h}^*$, the transition between the superconducting and the normal states is a second-order one since $\epsilon < 0$. The critical line is given by the equation $\tilde{\tau}(h, T) = 0$. For higher fields, the transition becomes a first-order one because $\epsilon > 0$ and $b > 0$. As for any first-order transition, two conditions must be fulfilled. On one hand, the free energy (30) is minimized, $(\partial F_{U-0}/\partial \tilde{\Delta})_{\tilde{\Delta}=\tilde{\Delta}_1} = 0$, and on the other hand the free energies of the superconducting and normal phases are equal, $F_{U-0}(\tilde{\Delta}_1) = 0$. Hence in the $(T, h)$ plane the equation for this first-order line is $\tilde{\tau}_1(h, T) = \epsilon^2/(4b)$, and the jump of the superconducting gap at the transition is given by $\left|\tilde{\Delta}_1\right|^2 = \epsilon/(2b)$. It is well-known that this scenario is not realized because it is replaced by a transition between normal metal and nonuniform superconductivity.[8] Nevertheless, this first-order transition provides a useful energy scale $\tilde{\Delta}_1$ and a reference line in the $(T, h)$ plane that will be used to construct a universal phase diagram, namely a $t$-independent phase diagram valid for any weakly coupled SF/SF bilayers, see Sec. IV.C.

The SF/SF bilayer may support opposite order parameters on the layers, superconductivity being still uniform within each SF plane. In this so-called $U - \pi$ state, the free energy of the bilayer depends on the reduced interlayer coupling $\tilde{t} = t/2\pi T$ according to

$$F_{U-\pi}(\tilde{\Delta}, \tilde{\tau}, \tilde{t}) = \left(\tilde{\tau} - 4e\tilde{t}^2 + 8b\tilde{t}^4\right)\left|\tilde{\Delta}\right|^2 - (\epsilon - 12b\tilde{t}^2)\left|\tilde{\Delta}\right|^4 + b\left|\tilde{\Delta}\right|^6.$$
For low reduced fields $\tilde{h} < \tilde{h}^*$, $F_{U-0}(\Delta, \tau) < F_{U-\pi}(\Delta, \tau, \tilde{t})$. Hence the uniform superconducting phase with $\chi = 0$ is more stable than the $\pi$ phase, as already found in the Sec. III. Interestingly for higher reduced fields $\tilde{h} > \tilde{h}^*$, this $\pi$ phase is in competition with FFLO nonuniform superconducting phases having either $\chi = 0$ ($FFLO-0$) or $\chi = \pi$ ($FFLO-\pi$).

According to Buzdin and Kachkachi the order parameter $\Delta(x) = \Delta \cos qx$ leads to the lowest energy.[34] For the $FFLO-0$ phase, the corresponding free energy reads

$$F_{LO-0}(\tilde{\Delta}, Q, \tau) = \left(\frac{\tau}{2} - 2\epsilon Q^2 + 6bQ^4\right) |\tilde{\Delta}|^2 - \left(\frac{3}{8} \epsilon + \frac{5b}{16} Q^2\right) |\tilde{\Delta}|^4 + \frac{5}{16} b |\tilde{\Delta}|^6,$$

$$\text{whereas for the } FFLO-\pi \text{ phase, the free energy depends on the interlayer coupling } t \text{ in the following manner}$$

$$F_{LO-\pi}(\tilde{\Delta}, Q, \tau, \tilde{t}) = F_{LO-0}(\tilde{\Delta}, Q, \tau) + \left(-2\epsilon \tilde{t}^2 + 4b\tilde{t}^4 + 24b\tilde{t}^2 Q^2\right) |\tilde{\Delta}|^2 + \frac{9}{2} b \tilde{t}^2 |\tilde{\Delta}|^4.$$  

The notation $Q = v_Fq/(4\sqrt{2}\pi T)$ is introduced in Appendix B.

### B. Competition between FFLO-0 and U-\pi phases

Now we proceed to analyse the above free energies in order to determine the critical line between the normal and the superconducting states. We will also describe the nature of the various superconducting states and what kinds of transitions are encountered. We focus on the vicinity of the tricritical point. Then $\epsilon = \tau(\tilde{h} - \tilde{h}^*)$ is a linear function of the exchange field with $\tau > 0$, whereas $b > 0$ is nearly field and temperature independent.

For high reduced fields $\tilde{h} > \tilde{h}^*$, it appears that the $U-0$ and the $FFLO-\pi$ never lead to the highest critical temperature. Then we emphasize the competition between the two remaining phases, namely $U-\pi$ and $FFLO-0$. The minimization of the GL functional (34) leads to a modulation wave vector given by $Q^2 = \epsilon/(6b)$ in the limit $\tilde{\Delta} \rightarrow 0$, i.e. near
the critical line. The \( U - \pi \) phase is more stable than the FFLO\(-0\) phase under the energy condition \( F_{U-\pi}(\tilde{\Delta}, \tau, \tilde{t}) < F_{LO-0}(\tilde{\Delta}, \sqrt{\epsilon/(6b)}, \tau) \), or equivalently for

\[
2(3 - \sqrt{3})bt^2 < \epsilon < 2(3 + \sqrt{3})bt^2.
\]  

(38)

Therefore the uniform \( \pi \) phase is "inserted" within the usual FFLO superconducting state. The upper and lower values of \( \tilde{h} \) between which this new \( \pi \) phase is stable depend on the particular value of the coupling \( t \). It is convenient to define a dimensionless generalized coordinate \( \eta = \epsilon/(2b\tilde{t}^2) \) that quantifies the "distance" from the tricritical point along the S/N transition line. Indeed \( \eta = 0 \) at the tricritical point and the \( \pi \) phase is settled in the region \( (3 - \sqrt{3}) < \eta < (3 + \sqrt{3}) \). As shown on Fig. 4, going along the critical line from low to high fields, one expects the sequence of superconducting states: uniform in the planes with \( \chi = 0 \) for \( \eta < 0 \), FFLO modulation along the planes with \( \chi = 0 \) for \( 0 < \eta < (3 - \sqrt{3}) \), then uniform \( \pi \) state for \( (3 - \sqrt{3}) < \eta < (3 + \sqrt{3}) \) and finally FFLO modulation along the planes with \( \chi = 0 \) for \( \eta > (3 + \sqrt{3}) \). In all cases, the sign of the \( \tilde{\Delta}^4 \) coefficient in the GL free energy is always positive and thus transitions between these superconducting states and the normal metal are second-order ones.

C. Universal \((\tau, \eta)\) phase diagram

We now construct the phase diagram around the tricritical point. Because each value of the coupling leads to different transition lines, we introduce the following mapping of the thermodynamic variables

\[
\delta = \frac{\tilde{\Delta}}{\Delta_1}, \tau = \frac{\tau}{\tau_1},
\]

(39)
in order to obtain a universal phase diagram valid in the small coupling regime. This mapping makes use of the energy scale \( \tilde{\Delta}_1 \) and of the function \( \tau_1 \) related to the first-order transition between the normal state and the uniform superconducting state, see Sec. IV.A. Then the free energies for the uniform superconducting phases Eqs.(30,32) become

\[
\frac{F_{U-0}(\delta, \tau)}{F_0} = \tau |\delta|^2 - 2|\delta|^4 + |\delta|^6,
\]

(40)

and
\[
\frac{F_{U-\pi}(\delta, \tau, \eta)}{F_0} = \left( \tau - 8 \left( \frac{1}{\eta} - \frac{1}{\eta^2} \right) \right) |\delta|^2
- 2 \left( 1 - \frac{6}{\eta} \right) |\delta|^4 + |\delta|^6 ,
\]
\[
(41)
\]
\[
\frac{F_{U-\pi}(\delta, \tau)}{F_0} = \left( \tau - 8 \left( \frac{1}{\eta} - \frac{1}{\eta^2} \right) \right) |\delta|^2
- 2 \left( 1 - \frac{6}{\eta} \right) |\delta|^4 + |\delta|^6 ,
\]
\[
(42)
\]
where \( F_0 = \epsilon^3/(8b^2) \). First it is straightforward to minimize \( F_{U-\pi}(\delta, \tau) \) and \( F_{U-\pi}(\delta, \tau, \eta) \) with respect to \( \delta \). Then replacing the reduced gap \( \delta \) by its equilibrium value, one obtains the equilibrium energies \( F_{U-\pi}(\tau) \) and \( F_{U-\pi}(\tau, \eta) \) of each superconducting phase. These energies are functions of both field and temperature via the dimensionless thermodynamical variables \( \tau \) and \( \eta \).

Using the same scaling Eq.(39) the free energies of the \( FFLO - 0 \) phase,

\[
\frac{F_{LO-0}(\delta, Q, \tau)}{F_0} = \left( \frac{\tau}{2} - \frac{8b}{\epsilon} Q^2 - \frac{24b^2}{\epsilon^2} Q^4 \right) |\delta|^2
- \left( \frac{3}{4} + \frac{5b}{8\epsilon} Q^2 \right) |\delta|^4 + \frac{5}{16} |\delta|^6 ,
\]
\[
(43)
\]
and of the \( FFLO - \pi \) phase,

\[
\frac{F_{LO-\pi}(\delta, Q, \tau, \eta)}{F_0} = \frac{F_{LO-0}(\delta, Q, \tau)}{F_0}
+ \left( -\frac{4}{x} + \frac{4}{\eta^2} + \frac{48b}{\eta \epsilon} Q^2 \right) |\delta|^2
+ \frac{9}{2\eta} |\delta|^4 ,
\]
\[
(44)
\]
are also obtained for the order parameter \( \Delta(x) = \Delta \cos qx \). The equilibrium energies \( F_{LO-0}(\tau) \) and \( F_{LO-\pi}(\tau, \eta) \) of the modulated phases are obtained after minimization of Eqs.(43,45) with respect to \( \delta \) and \( Q \). Then Eq.(43) enables to study the second-order phase transition between the normal metallic state and the nonuniform FFLO state. Under the assumption of second-order phase transition, it is sufficient to consider the GL free energy up to the \( \delta^2 \) order. Then the free energy (43) is minimal for \( Q^2 = \epsilon/(6b) \). For this particular modulation, the critical FFLO/N line is given by \( \tau = 4/3 \).

We now consider the transition lines between the various superconducting states obtained in the previous paragraph, in particular the \( FFLO - 0/U - \pi \), the \( FFLO - 0/U - 0 \) and the \( U - \pi/U - 0 \) transitions. Let us focus on the transition between the uniform phases \( U - 0 \)
and \( U - \pi \). Solving \( F_{U-0}(\tau) = F_{U-\pi}(\tau, \eta) \), we obtain the critical line \( \tau(\eta) \) which corresponds to a first order \( U - 0/U - \pi \) phase transition. However, this transition is not realized (see Fig.4) because the transition to the nonuniform superconducting state occurs before.

The \( FFLO - 0/U - 0 \) transition line is obtained in a similar way. The equation \( F_{U-0}(\tau) = F_{LO-0}(\tau) \) has the solution \( \tau \approx 0.913 \). At this value of \( \tau \), the system undergoes a first order phase transition from the uniform state to the modulated FFLO state. Adding higher harmonics to the order parameter \( \Delta(x) = \Delta \cos qx + \Delta' \cos 3qx + ... \), gives a more accurate evaluation, namely \( \tau \approx 0.859 \).

Finally, the \( FFLO - 0/U - \pi \) transition line is obtained from \( F_{LO-0}(\tau) = F_{U-\pi}(\tau, \eta) \) and shown in Fig.4 in the \((\tau, \eta)\) plane. This transition is a first-order one.

Using the mapping \((T, h, t) \rightarrow (\tau, \eta)\) of the thermodynamical variables, we have obtained a universal phase diagram Fig.4 of all weakly coupled SF/SF bilayers in the vicinity of the tricritical point. An important feature of this phase diagram is the presence of a superconducting \( \pi \)-phase. As an example, the phase diagram has been redrawn in the \((T, h)\) plane on Fig.5 for a particular value of the coupling \( t \).

V. PHASE DIAGRAM OF THE STRONGLY COUPLED SF/SF BILAYER

Now we consider the ballistic SF/SF bilayer in the regime of strong interlayer coupling limit \( t \gg T_{c0} \) and low temperature. A very unusual \( \pi \)-superconducting state is found between a lower \( h_{\text{low}}^{(II)} = t - \Delta_0^2/4t \) and an upper \( h_{\text{up}}^{(II)} = t + \Delta_0^2/4t \) critical exchange field, and below a maximal temperature of the order of \( T_{c0}^2/t \). Therefore field-induced superconductivity is obtained above \( h_{\text{low}}^{(II)} \) within the BCS theory of superconductivity. The underlying physical mechanism is the compensation of the Zeeman splitting by the energy splitting between bonding and antibonding electronic states of the bilayer, see Fig.6.[27] Thus the new zero temperature paramagnetic limit \( h_{\text{up}}^{(II)} = t + \Delta_0^2/4t \) may be tuned far above the usual one \([8] h = \Delta_0 \) merely by increasing the interlayer coupling. This compensation also occurs for small coupling, but the \( \pi \)-superconducting state is then less energetically favorable than the usual 0-superconducting phase as demonstrated in Sec.IV. Therefore the \((T, h)\) phase diagrams are topologically distinct in the opposite limits of small (Sec.IV) and strong (Sec.V) coupling. We first analyse the second-order superconducting/normal phase transition in Sec.V.A. Then the first-order transition between uniform superconductivity and the normal
A. Second-order phase transition

Here we study the second-order phase transition between the $\pi$ superconducting state and the normal metal state, as a function of the field. We start from the the linearized anomalous Green function (9) for arbitrary coupling $t$ and $\pi$ superconducting phase difference,

$$F_{11}^+ = \frac{(i\omega + h + \xi)(\xi - i\omega - h) - t^2}{[t^2 - (i\omega + h - \xi)^2][t^2 - (i\omega + h + \xi)^2]}.$$  \hspace{1cm} (46)

From this equation and the self-consistency relation (12), the critical exchange field $h$ is shown to satisfy

$$\left|h_{c} + t + \sqrt{(h_{c} + t)^2 - X^2}\right| \cdot \left|h_{c} - t + \sqrt{(h_{c} - t)^2 - X^2}\right| = 4h_0^2,$$ \hspace{1cm} (47)

where $X = |q| v_F$, and $h_0 = \Delta_0/2$ is the critical exchange field for the second-order superconducting phase transition in a two-dimensional monolayer. One must then find the value of $X$ which maximizes the critical field $h_c$. If the $\pi$ phase is assumed to be uniform inside each plane, namely if $q = 0$, Eq.(47) merely reduces to $|h_{c}^{2} - t^2| = h_0^2$. The lower and upper critical fields are respectively given by $h_{c} = t \pm h_0^2/2t$, in the limit $t \gg h_0$. Thus at zero temperature and strong enough coupling, the superconductivity destruction follows a very special scenario. At low fields, superconductivity is first suppressed as usual at the paramagnetic limit $h_{2D}^{FFLO} = \Delta_0$ leading to the normal metal phase. Then further increase of the field leads to a normal to superconducting phase transition at the lower critical field. This superconducting $\pi$ phase is finally suppressed at the upper critical field. This is a new paramagnetic limit which may be tuned far above the usual one merely by choosing the coupling $t$ greater than $\Delta_0$. Thorough analysis of Eq.(47) shows that the upper critical field is even increased by an in-plane modulation in analogy with the case of the two-dimensional FFLO phase.[8] The upper critical field is maximal for the choice $X = |q| v_F/2 = |h_{c} - t|$, and then Eq.(47) reduces to

$$|h_{c} - t| \cdot \left|h_{c} + t + 2\sqrt{h_{c}t}\right| = 4h_0^2,$$ \hspace{1cm} (48)

that gives the upper and lower fields $h_{up,low}^{(I)} = t \pm h_0^2/t$ in the $t \gg \Delta_0$ limit. Note that the period of the modulated order parameter $|q|^{-1} = \xi_0(t/\Delta_0)$ is larger than the corresponding...
period in the two-dimensional FFLO phase which coincides with the ballistic coherence length $\xi_0 = v_F/\Delta_0$. [8]

Furthermore one may derive the full temperature-field phase diagram using Eqs.(13,46) and the result is shown in Fig.7. When the temperature is increased, the lower critical field increases whereas the upper one decreases. Along the upper (resp. lower) critical line the FFLO modulation is lost at some temperature $T_{up}^*$ (resp. $T_{low}^*$). For higher temperatures a uniform $\pi$ phase ($U - \pi$) is recovered and the temperature dependence of the critical field is given by

$$\ln \frac{T}{T_c} = \frac{1}{2} \sum_{a=\pm 1} \text{Re} \left[ \Psi \left( \frac{1}{2} \right) - \Psi \left( \frac{1}{2} + i \frac{h_c(T) + at}{2\pi T} \right) \right],$$

(49)

where $\Psi(x)$ is the Digamma function and $\Psi(1/2) = -C - 2\ln 2 \simeq -1.963$, $C$ being the Euler constant. Finally the lower and upper critical lines merge at field $h_c = t$ and temperature $T_M = \pi e^{-C}T_{c0}^2/(4t)$ in the limit $t \gg T$. Therefore the field-induced $\pi$ superconductivity is confined to temperatures lower than $T_M$. The structure of these $U - \pi$ and the FFLO $- \pi$ phases is reminiscent of the corresponding $U - 0$ and the FFLO $- 0$ phases although the former are shifted to higher fields and lower temperatures than the later.

Above results were obtained for relatively strong coupling. For lower coupling $t \simeq \Delta_0$, the $U - \pi$ and the FFLO $- \pi$ phases merge continuously into the usual $\chi = 0$ phases as shown in Fig.8, and finally disappear for $t$ slightly smaller than $\Delta_0$. From an experimental point of view, one might choose a system with intermediate coupling $t$ small enough to settle the $\pi$ phase island in an available range of temperatures but also large enough to separate the $\pi$ phase island from the usual superconducting phases with $\chi = 0$. In the general SF multilayer case the inter-bilayer coupling constant $t'$ needs to be sufficiently high to prevent from superconductivity destruction by 2D fluctuations but also sufficiently low to preserve the effect of field-induced superconductivity.[27, 35]

B. First-order phase transition

In the following we investigate the first-order $U - \pi/N$ transition to determine whether it is more or less favorable than the above studied second-order transition. The zero-temperature superconducting order parameter $\Delta = \Delta(t, h, T = 0)$ is calculated from the self-consistency
equation (14) for P orientation of magnetizations and $\chi = \pi$ phase difference. At zero temperature, the difference between the energy $E_S$ of the superconducting state and the energy $E_N$ of the normal metal state is given by [29]

$$E_S - E_N = \int_0^\delta \frac{\partial}{\partial \delta} \left[ \int \left( \frac{F^+(h,t,\omega,\xi,\delta)}{\delta^*} \right) \frac{d\omega}{2\pi} d\xi \right] d\delta$$  \hspace{1cm} (50)

In the limit $t \to 0$, we retrieve the well-known case of the single SF layer.[2, 3] Then the self-consistency relation (14) admits two branches of solutions. The lower branch $\Delta = \sqrt{\Delta_0 (2h - \Delta_0)}$, labelled (2) in the inset of Fig.9, corresponds to a positive energy cost $E_S - E_N$. Thus this superconducting solution is never realized. The actual superconducting gap is given by the upper horizontal branch $\Delta = \Delta_0$, (1) in the inset of Fig.9, which corresponds to the energy difference

$$E_S - E_N = -\frac{\pi}{2} \left( \Delta_0^2 - 2h^2 \right).$$  \hspace{1cm} (51)

Hence the superconducting phase is settled for low fields $h \leq \Delta_0 / \sqrt{2}$ with a field-independent order parameter $\Delta = \Delta_0$. For higher fields $h > \Delta_0 / \sqrt{2}$, the system is in the normal phase $\Delta = 0$. Finally the zero temperature gap exhibits a jump at $h = \Delta_0 / \sqrt{2}$ which reveals the first-order transition from the uniform superconducting phase to the normal phase.

In the opposite limit of strong interlayer coupling, we have obtained in Sec.V.A. that field-induced superconductivity with $\chi = \pi$ phase difference occurs for fields close to $t$ and at low temperatures. From the self-consistency equation (14) one obtains several possible solutions for the zero-temperature superconducting gap $\Delta = \Delta(t,h,T = 0)$ as a function of the exchange field $h$, see Fig.9. For relatively low fields $h_{2D}^{FFLO} < h < h_-$ and for high fields $h > h_+$, the bilayer is in the normal phase $\Delta = 0$. The limiting fields $h_\pm$ are solutions of

$$\left( \frac{h_\pm - t}{\Delta_0} \right)^4 - 2 \left( \frac{t^2 - h_\pm^2}{\Delta_0^2} \right) + 1 = 0.$$  \hspace{1cm} (52)

For intermediate fields ranging between $h_-$ and $h_+$ there are three superconducting branches. Two of them, (2') and (2'') are never realized owing to their energy cost $E_S - E_N > 0$. The third branch (1') requires more detailed analysis. Namely, it is given by the equation

$$\frac{\Delta^4}{\Delta_0^4} - 2 \frac{(h + t)\Delta}{\Delta_0^2} + 1 = 0,$$  \hspace{1cm} (53)
and the corresponding energy cost is

\[
E_S - E_N = -\frac{\pi \Delta^2}{2} + \frac{\pi (h + t)^2}{2} \left[ 1 - \sqrt{1 - \left( \frac{\Delta}{h + t} \right)^2} \right] + \frac{\pi (h - t)^2}{2}.
\]

Analysis of these equations reveals that \( E_S - E_N \) is negative for \( h_{\text{low}}^{(I)} < h < h_{\text{up}}^{(I)} \) where \( h_{\text{low}}^{(I)} = \sqrt{t^2 - \Delta_0^2/(2\sqrt{2})} \) and \( h_{\text{up}}^{(I)} = \sqrt{t^2 + \Delta_0^2/(2\sqrt{2})} \). Hence the SF bilayer undergoes first-order transition at \( h = h_{\text{low}}^{(I)} \) and \( h = h_{\text{up}}^{(I)} \). This scenario is quite similar than the one for \( t = 0 \), but with a smaller order parameter jump at the transition. Moreover there are two first-order transitions, respectively at \( h_{\text{low}}^{(I)} \) and \( h_{\text{up}}^{(I)} \) instead of one at \( h = \Delta_0/\sqrt{2} \).

In order to generalize the above gap calculations to finite temperatures and determine the first-order S/N transition line, we have solved numerically together the self-consistency equation (13) and the condition \( E_S - E_N = 0 \). The result is given in the inset of Fig.7.

Collecting results from Sec.V.A and B. we obtain the full \((T, h)\) phase diagram for the field-induced \( \pi \) superconductivity. Note that this \( \pi \) superconductivity reproduces the structure of the phase diagram in quasi-2D superconductors [8] although it is shifted to higher fields and lower temperatures.

VI. CONCLUSION

In this paper we have studied a periodic array of SF/SF bilayers in the limit of small coupling between the different bilayers. The corresponding Gor’kov equations have been solved exactly, taking into account both in-plane FFLO modulation and arbitrary superconducting phase difference between SF layers. The superconducting state with zero phase difference is always settled in the low field regime, \( h < T_{c0} \) for parallel (P) orientation of the magnetizations. For antiparallel (AP) orientation, the \( \pi \) state predominates at low temperatures over the 0 state which is settled in the neighborhood of the critical line. Consequently if the system is pinned in the antiparallel orientation, we predict a transition from the usual \( \chi = 0 \) superconducting state to the \( \pi \) state by cooling.

While the critical temperature is higher for the AP orientation, the zero temperature order parameter is larger for the P orientation. This results in a crossing temperature \( T_i(h) \) below
which the P orientation is more suitable for superconductivity. This temperature has been calculated as a function of the exchange field. In an experiment where the magnetizations might be easily reversed, one therefore expects a transition from the AP to the P orientation by cooling the system below this crossing temperature.

In the low interlayer coupling limit, a Ginzburg-Landau functional has been derived from the exact expression of the anomalous Gor’kov Green function. As a main result, we have obtained a $\pi$ superconducting state located in the vicinity of the tricritical point $(h^*, T^*)$. Details of the bilayer phase diagram are obtained in this framework, including the first-order transition lines between superconducting phases. Since increasing the interlayer coupling enlarges the $\pi$ phase region, experimental observation of such details of the phase diagram requires the use of SF layers with large enough interlayer coupling, namely $t \approx 0.1T_{c0}$.

Finally the case of even stronger interlayer coupling, namely $t \gg T_{c0}$ has been also investigated. It appears that at low temperatures the $\pi$ superconducting state is settled for exchange fields of the order of $t$, which are well above the Chandrasekhar-Clogston paramagnetic limit. Thus this new paramagnetic limit may be tuned by varying the interlayer coupling. In the present article we have reported the detailed structure of the phase diagram in this regime of high magnetic field. The first-order $U - \pi/N$ transition line is also derived.

We expect that our results may be applicable to compounds like Bi$_2$Sr$_2$CaCu$_2$O$_8$ under a magnetic field. Indeed such perovskite superconductors comprise tightly coupled superconducting CuO planes separated by BiO layers. However observing the field-induced superconductivity in a reasonable range of magnetic field requires relatively low critical temperatures which are realized in the heavily doped or underdoped regimes. Finally the latter effect is solely related to the compensation of the energy shift in the two layers systems by the Zeeman splitting. So it should be quite general and might appear also in two band superconductors or in weakly coupled superconducting grains. Note that the inhomogeneous superconductivity has been obtained in the absence of magnetic field in two-bands superconductors.[28] However since the $\pi$ state was not considered in this latter work no field-induced superconductivity had been noticed.

Bulaevskii [8] studied thoroughly Josephson coupling in periodic layered structures with one SF plane as unit cell. Here we have demonstrated that systems with several SF planes as unit cell exhibit qualitatively new phenomena like field-induced superconductivity. The simplest case, two planes per unit cell, has been studied here. It may be regarded as a
basic approach to understand the properties of more complex ferromagnetic superconducting compounds or artificial heterojunctions.

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VII. APPENDIX

A. Definition and properties of the functions $K_{\mu}(\tilde{h})$

We define the function $K_1(\tilde{h})$ by

$$K_1(\tilde{h}) = \sum_{n=0}^{\infty} \text{Re} \left[ \frac{1}{n + \frac{1}{2} + i\tilde{h}} - \frac{1}{n + \frac{1}{2}} \right],$$

and for any integer $\mu \geq 2$ the function $K_{\mu}(\tilde{h})$ is given by

$$K_{\mu}(\tilde{h}) = \sum_{n=0}^{\infty} \text{Re} \left( \frac{1}{n + \frac{1}{2} + i\tilde{h}} \right)^\mu.$$  

The variations of $K_1$, $K_3$, $K_5$, with $\tilde{h}$ are represented in Fig.10. One can notice that in the vicinity of the tricritical point, i.e. $\tilde{h} \approx \tilde{h}^* \approx 0.3$, the functions $K_1(\tilde{h})$ and $K_5(\tilde{h})$ are negative and of the order of unity. $K_3(\tilde{h})$ cancels exactly at $\tilde{h} = \tilde{h}^*$ and becomes negative in the domain $\tilde{h} \geq \tilde{h}^*$, which is studied Sec.IV.

B. Ginzburg-Landau functional

This part of the Appendix refers to Sec. IV of the paper. In the Ginzburg-Landau theory, the free energy is expanded in terms of the gap $\Delta$, i.e. the order parameter, assuming the temperature close to $T_c$. Originally it was introduced as a phenomenological theory for superconductivity before the BCS theory. Here we derive the Ginzburg-Landau free energy from the full microscopic knowledge of our model in order to analyze the vicinity of the tricritical point. To do this, we consider the simplest case where the FFLO gap modulation is exponential, namely $\Delta(x) = \Delta e^{iqx}$, $q$ being the in-plane modulation wave vector. It is known that this modulation structure is not realized to the benefit of the cosine modulation
discussed in the article’s body. However, In Sec.II, the Gor’kov Green functions of the SF/SF bilayer were derived for a modulated order parameter $\Delta(x) = \Delta e^{i\eta x}$ and $\chi = 0$ or $\pi$. This modulation structure is then convenient to calculate the coefficients of the generalized GL functional because the exact expression of the anomalous Green function (see Eq.(9)) is valid for this gap modulation structure, whereas it is unknown with the cosine structure. We first expand the exact anomalous Green function (9) and the self-consistency relation in powers of the gap $\Delta$ and the FFLO wave vector $q$. Then this self-consistency relation is interpreted as the stationarity condition for the Ginzburg-Landau free energy, which allows (by identification) to determine the coefficient of every term of the GL functional.

In the $\chi = 0$ case, the expansion of the anomalous Green function reads

$$
\frac{F^{+}_{11}(t)}{\Delta^*} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} |\Delta|^{2n-2}}{(\xi - a(t, q))^n (\xi + a(-t, 0))^n} + (t \leftrightarrow -t),
$$

(57)

where $a(t, q) = i\omega - h + t - \mathbf{v}_F \cdot \mathbf{q}$. We first consider the case of uniform superconductivity, i.e. $\mathbf{q} = 0$. After integration over $\xi$, we obtain:

$$
\int_{-\infty}^{\infty} \frac{d\xi \ F^{+}_{11}(t)}{2\pi \Delta^*} = sgn(\omega) \sum_{n=0}^{\infty} (-1)^n b_n \frac{|\Delta|^{2n}}{2 (\omega + i\eta)^{2n+1}},
$$

(58)

with $b_n = \frac{(2n)!}{n! 2^{2n}} = \frac{\Gamma(n+1/2)}{\Gamma(1/2)n!}$. Note that the interlayer coupling $t$ has disappeared in Eq.(58). We are now able to write down the self-consistency equation (13) as an expansion in powers of $\Delta$

$$
\ln \frac{T}{T_{c0}} - \sum_{n=0}^{\infty} (-1)^n b_n K_{2n+1}(h) \left| \tilde{\Delta} \right|^{2n} = 0,
$$

(59)

where the functions $K_{\mu}$ are those defined in Appendix A, and $\tilde{\Delta} = \Delta/2\pi T$. This self-consistency relation may be interpreted as the stationnary condition

$$
\frac{\partial F_{U,0}}{\partial \Delta} = 0
$$

(60)

for the Ginzburg-Landau free energy with uniform order parameter within each superconducting plane and $\chi = 0$ phase difference between the planes. Close to the tricritical point, $\tilde{\Delta}$ is small and it is enough to retain only the first term in this infinite expansion as

$$
\ln \frac{T}{T_{c0}} - b_0 K_1(h) + b_1 K_3(h) \left| \tilde{\Delta} \right|^2 - b_2 K_5(h) \left| \tilde{\Delta} \right|^4 = 0,
$$

(61)
where \( \tilde{h} = h/(2\pi T) \). By identification with Eq.(60) we obtain the GL free energy for the \( U-0 \) phase as a function of the variational parameter \( \tilde{\Delta} \) and the thermodynamical variable \( \tilde{h} = h/(2\pi T) \):

\[
F_{U-0}(\tilde{\Delta}) = \left[ \ln \frac{T}{T_{c0}} - b_0 K_1(\tilde{h}) \right] |\tilde{\Delta}|^2 + b_1 K_3(\tilde{h}) \frac{|\tilde{\Delta}|^4}{2} - b_2 K_5(\tilde{h}) \frac{|\tilde{\Delta}|^6}{3}
\]

\[
= \left[ \ln \frac{T}{T_{c0}} - K_1(\tilde{h}) \right] |\tilde{\Delta}|^2 + K_3(\tilde{h}) \frac{|\tilde{\Delta}|^4}{4} - K_5(\tilde{h}) \frac{|\tilde{\Delta}|^6}{8}
\]

which corresponds to Eq.(30). Note that in this usual 0 state the same coefficients have been already reported in Ref.[34]

The same procedure may be followed when the phase difference is \( \pi \). The anomalous Green function is then

\[
F_{11}^+ = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{|\Delta|^{2n-2}}{(\xi - a(t, q))^n (\xi + a(t, 0))^n} + (t \leftrightarrow -t),
\]

and leads to the self-consistency relation which contains explicitly the coupling \( t \), via the normalized coupling \( \tilde{t} = t/\pi T \):

\[
\ln \frac{T}{T_{c0}} - \sum_{n=0}^{\infty} (-1)^n b_n \left[ \frac{K_{2n+1}(\tilde{h} + \tilde{t}) + K_{2n+1}(\tilde{h} - \tilde{t})}{2} \right] |\tilde{\Delta}|^{2n} = 0.
\]

From the latter expression we deduce that the coefficients of the GL free energy for the \( \chi = \pi \) state can be directly obtained using the coefficient of \( F_{U-0}(\tilde{\Delta}) \) in which we replace \( K_{2n+1}(\tilde{h}) \) by \( (K_{2n+1}(\tilde{h} + \tilde{t}) + K_{2n+1}(\tilde{h} - \tilde{t}))/2 \). Finally the free energy of the \( U-\pi \) state is

\[
F_{U-\pi}(\tilde{\Delta}) = \left[ \ln \frac{T}{T_{c0}} - \frac{K_1(\tilde{h} + \tilde{t}) + K_1(\tilde{h} - \tilde{t})}{2} \right] |\tilde{\Delta}|^2 + \frac{K_3(\tilde{h} + \tilde{t}) + K_3(\tilde{h} - \tilde{t})}{8} |\tilde{\Delta}|^4 - \frac{K_5(\tilde{h} + \tilde{t}) + K_5(\tilde{h} - \tilde{t})}{16} |\tilde{\Delta}|^6
\]

which yields Eq.(32) in the small interlayer coupling limit \( \tilde{t} \ll \tilde{h} \).

We have developed in a similar way the GL free energy in the case where the order parameter is modulated within each superconducting plane. Using the expressions (57) and (63) for the anomalous Green function of the bilayer with \( \Delta(x) = \Delta e^{iqx} \) FFLO modulation respectively in the \( \chi = 0 \) and \( \chi = \pi \) cases, one obtains the expansion of the self-consistency
equation in powers of $\Delta$ and of the FFLO wave vector $q$. Finally, after averaging over all possible orientations of the FFLO modulation vector, the self-consistency equation reads:

\[ \ln \frac{T}{T_{c0}} - \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{n+p} c_{n,p} K_{2(n+p)+1}(\tilde{h}) \left| \tilde{\Delta} \right|^{2n} \left( \frac{v_F q}{4\pi T} \right)^{2p} = 0 \]  

(66)

for $\chi = 0$. The coefficients $c_{n,p}$ are symmetric with respect to the expansion indexes $n$ and $p$

\[ c_{n,p} = \frac{\Gamma(n + p + 1/2)(n + p)!}{\Gamma(1/2)(n!)^2(p!)^2}. \]  

(67)

and related to the coefficients $b_n$ by $b_n = c_{n,0}$.

From Eq.(66) the GL free energy can be constructed using the method described in the previous paragraph for uniform phases. We retrieve all the coefficients already obtained by Buzdin and Kachkachi,[34] including the coefficients of the gradient terms of the generalized functional. To derive the free energy of the $FFLO - 0$ phase, we have therefore used the BK functional with the cosine modulation which is effectively realized in each superconducting layer. As a result, it reads

\[ F_{LO-0}(\tilde{\Delta}, Q, \tau) = \left( \frac{\tau}{2} - 2\epsilon Q^2 + 6bQ^4 \right) \left| \tilde{\Delta} \right|^2 \]

\[ - \left( \frac{3}{8} \epsilon + \frac{5}{16} bQ^2 \right) \left| \tilde{\Delta} \right|^4 + \frac{5}{16} b \left| \tilde{\Delta} \right|^6 \]

(68)

where $Q = v_F q/(4\sqrt{2}\pi T)$. In the $\chi = \pi$ state the free energy has been derived from the BK functional in which the replacement

\[ K_{2(n+p)+1}(\tilde{h}) \rightarrow \frac{K_{2(n+p)+1}(\tilde{h} + \tilde{t}) + K_{2(n+p)+1}(\tilde{h} - \tilde{t})}{2} \]

(69)

has been done in order to obtain the modified coefficients. Finally the free energy of the $FFLO - \pi$ phase can be written as

\[ F_{LO-\pi}(\tilde{\Delta}, Q, \tau) = \left( \frac{\tau}{2} - 4\epsilon \tilde{t}^2 + 8b\tilde{t}^4 \right) \left| \tilde{\Delta} \right|^2 \]

\[ - \left( \frac{3}{8} (\epsilon - 12b\tilde{t}^2) + \frac{5}{16} b\tilde{t}^2 \right) \left| \tilde{\Delta} \right|^4 + \frac{5}{16} b \left| \tilde{\Delta} \right|^6 \]

(70)

In the article body more convenient forms of Eqs.(68,70) involving the reduced quantities $\tau$, $\delta$ and $\eta$ are used in order to derive the universal phase diagram.

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VIII. FIGURE CAPTIONS

FIG.1: SF/SF bilayer. The interlayer coupling constant is denoted $t$. The exchange fields $h_j$ can be either equal (parallel orientation) or opposite (antiparallel orientation). The superconducting phase difference between $\Delta_1$ and $\Delta_2$ can be either 0 ($\Delta_1 = \Delta_2$) or $\pi$ ($\Delta_1 = -\Delta_2$).

FIG.2: Schematic representation of the superconducting gap as a function of temperature for P orientation and $\chi = 0$ (thicker solid line), AP orientation and $\chi = 0$ (intermediate thickness line), and AP orientation and $\chi = \pi$ (thinner line). All curves are given for the same value of the exchange field that is smaller $T_c$ (in energy units). The inversion of the proximity effect occurs at the temperature $T_i(h)$ and the transition from 0 state to $\pi$ state in the AP orientation at the temperature $T_\pi(h)$. Temperatures $T_i(h)$ and $T_\pi(h)$ as a function of the field are shown in the inset.

FIG.3: Enhancement of the critical current $j_c$ with the field in the AP orientation (dashed line). In the P orientation $j_c$ does not depend on the exchange field $h$ (solid line).

FIG.4: Universal phase diagram for weakly coupled bilayers, in $(\tau, \eta)$ coordinates. The critical line (solid line) corresponds either to a $U - \pi/N$ or to a $FFLO - 0/N$ transition depending on $\eta$. The transition between the nonuniform $FFLO - 0$ superconducting phase and uniform $U - 0$ (resp. $U - \pi$) phase is represented with dash-dotted (resp. dashed) line.
FIG.5: Phase diagram in \((T, h)\) coordinates, for \(t/(2\pi T_{c0}) = 0.07\). Only the neighborhood of the tricritical point is represented. The lines have the same meaning than in Fig.4. Note that the \(\pi\) phase is settled in a very narrow region of the phase diagram.

FIG.6: Excitation spectrum. Usual singlet pairing (thin line circles) between opposite-spin electrons occupying the same orbital is affected by Zeeman effect. In contrast, \(\pi\) coupling (thick line) between two electrons occupying a bonding and an antibonding orbitals may lead to the cancellation of the Zeeman splitting.

FIG.7: Phase diagram for \(t = 3\Delta_0 \approx 5.3T_{c0}\). Thick (resp. thin) solid lines represents second-order transition between \(U - \chi\) (resp. \(FFLO - \chi\)) and normal metal phase \(\mathcal{N}\) for \(\chi = 0\) and \(\pi\). We expect the \(U - \chi/FFLO - \chi\) transition lines (not calculated) to be in the vicinity of the (virtual) first order \(U - \chi/\mathcal{N}\) lines (dash-dotted).

FIG.8: Phase diagram for \(t = \Delta_0 \approx 1.76T_{c0}\). All lines have the same meaning than in Fig.7.

FIG.9: Order parameter at \(T = 0\) in the \((P, \pi)\) state for \(t = 3\Delta_0\) as a function of the exchange field (thick solid line).

FIG.10: Functions \(K_1(\tilde{h})\), \(K_3(\tilde{h})\) and \(K_5(\tilde{h})\) defined in Appendix B.
\[ \frac{h}{2\pi T_{c0}} \]

\[ h = t \]

\[ \frac{h}{2\pi T_{c0}} \]

\[ \frac{T}{T_{c0}} \]
Line $\Delta = t-h$

Line $\Delta = h-t$

$\Delta / \Delta_0$

$h / \Delta_0$

$\Delta / \Delta_0$

$h^{(1)}_{\text{low}}$

$h^{(1)}_{\text{up}}$

$h_+$

$h_-$

2.9

$2'$

$2''$

$1'$

$\Delta = h$

$\Delta = t$
