HARNACK ESTIMATES FOR CONJUGATE HEAT KERNEL ON EVOLVING MANIFOLDS

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Abstract. In this article we derive Harnack estimates for conjugate heat kernel in an abstract geometric flow. Our calculation involves a correction term $D$. When $D$ is nonnegative, we are able to obtain a Harnack inequality. Our abstract formulation provides a unified framework for some known results, in particular including corresponding results of Ni [8], Perelman [11] and Tran [12] as special cases. Moreover it leads to new results in the setting of Ricci-Harmonic flow and mean curvature flow in Lorentzian manifolds with nonnegative sectional curvature.

Contents

1. Introduction 1
1.1. Main Results 2
2. Preliminaries 3
2.1. Evolution Equations 3
2.2. Asymptotic Behavior and Reduced Geometry 4
2.3. Entropy Formulas 6
3. Estimates on the Heat Kernel 8
3.1. A Gradient Estimate 8
3.2. $L_\infty$ Bound 10
3.3. Proofs of Main Results 12
References 15

1. INTRODUCTION

Assume that $M$ is an $n$-dimensional closed manifold endowed with a one-parameter family of Riemannian metrics $g(t)$, $t \in [0, T]$, evolving by

$$\frac{\partial g(t, x)}{\partial t} = -2\alpha(t, x).$$

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Here $\alpha(t, x)$ is a one-parameter family of smooth symmetric 2-tensors on $M$. In particular, when $\alpha = \text{Rc}$, Eq. (1.1) is R. Hamilton’s Ricci flow. Let

$$S(t, x) \triangleq g^{ij}\alpha_{ij}$$

be the trace of $\alpha$ with respect to the time-dependent metric $g(t)$.

In [7], R. Müller studied reduced volumes for the abstract flow (1.1) and defined the following quantity for tensor $\alpha$ and vector $V$,

$$D_{\alpha}(V) \triangleq \frac{\partial S}{\partial t} - \Delta S - 2|\alpha|^2 + 2 (\text{Rc} - \alpha) (V, V) + \langle 4 \text{Div}(\alpha) - 2\nabla S, V \rangle,$$

(1.2)

where $\text{Div}$ is the divergence operator defined by $\text{Div}(\alpha)_k = g^{ij}\nabla_i\alpha_{jk}$ (in local coordinates). Under the assumption that $D_{\alpha} \geq 0$, Müller obtained monotonicity of the reduced volumes [7]. Most recently, in [6], the authors proved monotonicity for the entropy and the lowest eigenvalue. In [5], a Harnack inequality for positive solutions of the conjugate heat equation and heat equation with potential has been proved.

The main purpose of this article is to derive Harnack inequalities for a conjugate heat kernel in the abstract setting with $D_{\alpha} \geq 0$.

1.1. Main Results. We consider $(M, g(t))$, $0 \leq t \leq T$, to be a solution of (1.1) and $\tau \triangleq T - t$,

$$\Box^* \triangleq -\frac{\partial}{\partial t} - \Delta + S = \frac{\partial}{\partial \tau} - \Delta + S.$$ 

A function $u = (4\pi\tau)^{-n/2}e^{-f}$ is a solution to the conjugate heat equation if,

(1.3) \hspace{1cm} \Box^* u = 0.

We also denote

$$H(x, t; y, T) = (4\pi(T - t))^{-n/2}e^{-h} = (4\pi\tau)^{-n/2}e^{-h}$$

to be a heat kernel. That is, based at a fixed $(x, t)$, $H$ is the fundamental solution of heat equation $\Box H = 0$, and similarly for fixed $(y, T)$ and conjugate heat equation $\Box^* H = 0$. Our first result is computational.

Theorem 1.1. Let

$$v = (\tau (2\Delta f - |\nabla f|^2 + S) + f - n) u,$$
then we have

\[ \Box^* v = -2\tau u \left| \alpha + \nabla \nabla f - \frac{g}{2\tau} \right|^2 - \tau u D_\alpha(\nabla f). \]  

(1.4)

Secondly, we obtain the following Harnack estimate.

**Theorem 1.2.** If \( D_\alpha \geq 0 \), then the following inequality holds,

\[ \tau (2\Delta h - |\nabla h|^2 + S) + h - n \leq 0. \]  

(1.5)

**Remark 1.3.** For the Ricci flow, where \( \alpha = \text{Rc} \), one has \( D = 0 \); \( (1.5) \) has been proved by G. Perelman [10]. On a static Riemannian manifold where \( \alpha = 0 \) one has \( D = \text{Rc} \), and \( (1.5) \) has been proved by L. Ni in [8] for static manifolds with nonnegative Ricci curvature. Another special case of \( (1.5) \) was recently proved by the third author for the extended Ricci flow in [12]. \( (1.5) \) is new for Müller’s Ricci-Harmonic flow and mean curvature flow in a Lorentzian manifold with nonnegative sectional curvature. The detailed calculations of \( D \) can be found in [7].

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2. Preliminaries

2.1. Evolution Equations. In this section, we collect several evolution equations and prove Theorem 1.1.

For the Laplace-Beltrami operator \( \Delta \) with respect to \( g(t) \) we have,

\[ \left( \frac{\partial}{\partial t} \Delta \right) f = 2\langle \alpha, \nabla \nabla f \rangle + \langle 2 \text{Div}(\alpha) - \nabla S, \nabla f \rangle, \]  

(2.6)

where \( f \) is any smooth function on \( M \). This formula can be found in standard textbooks, for instance [3].

Now we assume \( u \) is a solution to the conjugate heat equation. The operator \( -\Box^* \) acting on the term \( u \log u \) produces,

\[ -\Box^* u \log u = u|\nabla \log u|^2 + uS. \]  

(2.7)

The same operator acts once more and we have,

\[ -\Box^* (u|\nabla \log u|^2 + uS) \]

\[ = 2u\alpha(\nabla \log u, \nabla \log u) + 4\langle \nabla S, \nabla u \rangle + u \frac{\partial S}{\partial t} + 2u|\nabla \nabla \log u|^2 \]
\( 2 u |\nabla \nabla \log u - \alpha|^2 + 4 u \langle \alpha, \nabla \nabla \log u \rangle + 2 u \alpha (\nabla \log u, \nabla \log u) \)

\[ \begin{aligned} &- 2 u |\alpha|^2 + 4 \langle \nabla S, \nabla u \rangle + u \frac{\partial S}{\partial t} + 2 u \text{Rc}(\nabla \log u, \nabla \log u) + u \Delta S. \end{aligned} \]

Notice that,

\[ -\Box^* (\Delta u) = 2 \langle \alpha, \nabla \nabla \log u \rangle + \langle 2 \text{Div}(\alpha) - \nabla S, \nabla u \rangle + 2 \langle \nabla S, \nabla u \rangle + u \Delta S. \]

Thus, by (1.2), we have, for \( V = -\nabla \log u \),

\[ \begin{aligned} &-\Box^* \left( u |\nabla \nabla \log u|^2 + u S - 2 \Delta u \right) = 2 u |\nabla \nabla \log u - \alpha|^2 + u D_\alpha(V). \end{aligned} \]

Moreover,

\[ \begin{aligned} &-\Box^* \left( \tau \left( u |\nabla \nabla \log u|^2 + u S - 2 \Delta u \right) - u \log u - \frac{nu}{2} \log \tau \right) \\
&= 2 \tau u |\alpha - \nabla \nabla \log u - \frac{g}{2 \tau}|^2 + \tau u D_\alpha(-\nabla \log u). \end{aligned} \]

In the calculation above, if we add a normalization term \( c_n u \) to the left hand side, we get the same on the right hand side since \( \Box^* u = 0 \). Thus, we have the following result.

**Lemma 2.1.**

(2.12)

\[ \Box^* \left[ \tau \left( u |\nabla \nabla \log u|^2 + u S - 2 \Delta u \right) - u \log u - \frac{nu}{2} \log \left( 4 \pi \tau \right) - nu \right] \\
= -2 \tau u \left| \alpha - \nabla \nabla \log u - \frac{g}{2 \tau} \right|^2 + \tau u D_\alpha(-\nabla \log u). \]

Theorem 1.1 follows by realizing that \( f \equiv -\log u - \frac{n}{2} \log (4 \pi \tau) \).

**2.2. Asymptotic Behavior and Reduced Geometry.** Let’s recall the asymptotic behavior of the heat kernel as \( t \to T \).

**Theorem 2.2.** \([1]\) Theorem 24.21] For \( \tau = T - t \),

\[ H(x, t; y, T) \sim e^{-\frac{d^2_{(x,y)}}{4\tau} - \Sigma_{j=0}^{\infty} \tau^i u_j(x, y, \tau)}. \]

More precisely, there exist \( t_0 > 0 \) and a sequence \( u_j \in C^\infty(M \times M \times [0, t_0]) \) such that,

\[ H(x, t; y, T) - e^{-\frac{d^2_{(x,y)}}{4\tau} - \Sigma_{j=0}^{k} \tau^i u_j(x, y, T - l)} = w_k(x, y, \tau), \]
with
\[ u_0(x, x, 0) = 1, \]
\[ w_k(x, y, \tau) = O(\tau^{k+1-\frac{n}{2}}), \]
as \( \tau \to 0 \) uniformly for all \( x, y \in M \).

Then following [7], we can define reduced length and distance.

**Definition 2.3.** Given \( \tau(t) = T - t \), we define the \( L \)-length of a curve \( \gamma: [\tau_0, \tau_1] \mapsto N, [\tau_0, \tau_1] \subset [0, T] \) by,

\[ \mathcal{L}(\gamma) := \int_{\tau_0}^{\tau_1} \sqrt{\tau(S(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2)}d\tau. \]

For a fixed point \( y \in N \) and \( \tau_0 = 0 \), the backward reduced distance is defined as,

\[ \ell(x, \tau_1) := \inf_{\gamma \in \Gamma} \left\{ \frac{1}{2\tau_1} \mathcal{L}(\gamma) \right\}, \]

where \( \Gamma = \{ \gamma: [0, \tau_1] \mapsto M, \gamma(0) = y, \gamma(\tau_1) = x \} \).

The backward reduced volume is defined as

\[ V(\tau) := \int_M (4\pi \tau)^{-n/2} e^{-\ell(y, \tau)}d\mu_\tau(y). \]

The next result, mainly from [7], relates the reduced distance defined in (2.14) with the distance at time \( T \).

**Lemma 2.4.** Let \( L(x, \tau) = 4\tau \ell(x, \tau) \) then we have the followings:

a. Assume that there exists \( k_1, k_2 \geq 0 \) such that \( -k_1 g(t) \leq \alpha(t) \leq k_2 g(t) \) for \( t \in [0, T] \), then \( L \) is smooth almost everywhere and a local Lipschitz function on \( N \times [0, T] \). Furthermore,

\[ e^{-2k_1 \tau}d_T^2(x, y) - \frac{4k_1 n}{3} \tau^2 \leq L(x, \tau) \leq e^{2k_2 \tau}d_T^2(x, y) + \frac{4k_2 n}{3} \tau^2. \]

b. If \( D_\alpha \geq 0 \), then \( \Box^{*}\left(\frac{\ell(x, \tau)}{(4\pi \tau)^{n/2}}\right) \leq 0. \)

c. For the same point \( y \) in the definition of reduced distance and \( H(x, t; y, T) = (4\pi \tau)^{-n/2} e^{-h} \), then \( h(x, t; y, T) \leq \ell(x, T - t) \).

**Proof.** Parts a. and b. follow from [7] Lemmas 4.1, 5.15 respectively.

For part c. we provide a brief argument here (for more details, see [2] Lemma 16.49).

We first observe that part a. implies \( \lim_{\tau \to 0} L(x, \tau) = d_T^2(y, x) \) and,

\[ \lim_{\tau \to 0} \frac{e^{-Lw(x, \tau)}}{4\pi \tau} = \delta_y(x), \]
since Riemannian manifolds locally look like Euclidean. It then follows from part \( b \) and maximum principle that,
\[
H(x, t; y, T) \geq e^{-\frac{L(x, \tau)}{4\tau}} = \frac{e^{-\frac{L(x, T-t)}{4\tau}}}{(4\pi(T-t))^{n/2}}.
\]
Hence,
\[
h(x, t; y, T) \leq \frac{L(x, \tau)}{4\tau} = \ell(x, \tau) = \ell(x, T-l).
\]

2.3. Entropy Formulas. In this subsection, we define several functionals and collect their properties.

**Definition 2.5.** Along flow (1.1), for \( h \) satisfying
\[
\int_M (4\pi \tau)^{-n/2} e^{-h} d\mu = 1,
\]
we define
\[
W_\alpha(g, \tau, h) = \int_M \left( \tau(|\nabla h|^2 + S) + (h - n) \right) (4\pi \tau)^{-n/2} e^{-h} d\mu.
\]
Associated functionals are defined as follows:
\[
\mu_\alpha(g, \tau) = \inf_h W_\alpha(g, h, \tau),
\]
\[
v_\alpha(g) = \inf_{\tau > 0} \mu_\alpha(g, \tau).
\]

**Remark 2.6.** Since \( \alpha \) is a \((2, 0)\)-tensor, \( S \) scales like the inverse of the metric. Thus, these functionals satisfy diffeomorphism invariance and the following scaling rules:
\[
W_\alpha(g, \tau, h) = W_\alpha(cg, c\tau, h),
\]
\[
\mu_\alpha(g, \tau) = \mu_\alpha(cg, c\tau),
\]
\[
v_\alpha(g, u) = v_\alpha(cg).
\]

Next, we collect some useful results.

**Lemma 2.7.** On a closed Riemannian manifold \((M, g(t)), t \in [0, T]\), evolved by (1.1), with \( D_\alpha \geq 0 \), let \( \tau = T-t \), the following holds:

\( a \). \( W_\alpha(g, \tau, h) \) is non-decreasing in time \( t \) (non-increasing in \( \tau \)).

\( b \). There exists a smooth minimizer \( h_\tau \) for \( W_\alpha(g, \tau, .) \) which satisfies
\[
\tau(2\Delta h_\tau - |\nabla h_\tau|^2 + S) + h_\tau - n = \mu_\alpha(g, \tau).
\]
In particular, \( \mu_\alpha(g, \tau) \) is finite.

\( c \). \( \mu_\alpha(g, \tau) \) is non-decreasing in time \( t \).

\( d \). \( \lim_{\tau \to 0^+} \mu_\alpha(g, \tau) = 0 \).
Proof. Part a. follows from [6, Theorem 5.2].

Part b. is deducted from the regularity theory for elliptic equations based on Sobolev spaces. The details can be found in [1, Proposition 17.24]. Replacing R by S, the argument works exactly the same.

Part c. is an immediate consequence of the monotonicity formula (part a.) and the existence of a minimizer realizing the $\mu_\alpha$ functional (part b.).

The proof of part d. is mostly identical to that of [11, Prop 3.2] (also [1, Prop 17.19, 17.20]), but it is subtle so we give a brief argument here.

Assume that the flow exists for $\tau \in [0, \overline{\tau}]$. The idea is to construct cut-off functions reflecting the local geometry which looks like Euclidean. Then it is shown that the limit of $\mathcal{W}_\alpha$ functional on these functions is 0 if a certain parameter approaches 0. Thus, by the monotonicity of $\mu_\alpha$ and L. Gross’s logarithmic-Sobolev inequality on an Euclidean space [4], the result then follows.

The construction of cut-off functions follows [11, Prop 3.2]. Let $\tau_0 = \overline{\tau} - \epsilon$ for small $\epsilon$. Using normal coordinates at a point p on $(M, g(\tau_0))$, we define a cut-off function

$$f_1 = \begin{cases} \frac{|x|^2}{4\epsilon} & \text{if } d(x, p) = |x| < \rho, \\ \frac{\rho^2}{4\epsilon} & \text{elsewhere}, \end{cases}$$

where $\rho$ is a positive number smaller than the injectivity radius (which exists since $M$ is closed). Then by the choice of our coordinate,

$$\text{d} \mu(\tau_0) = 1 + O(|x|^2), |x| << 1,$$

and let

$$e^{-C} \div \int_M (4\pi\epsilon)^{-n/2}e^{f_1},$$

then $C \to 0$ as $\epsilon \to 0$.

Let $f = f_1 + C$ then,

$$u \div (4\pi\epsilon)^{-n/2}e^{-f};$$

$$1 = \int_M (4\pi\epsilon)^{-n/2}e^{-f}d\mu(\tau_0);$$

$$|\nabla f|^2 = |\nabla f_1|^2 = |\nabla \frac{|x|^2}{4\epsilon}|^2 = \frac{|x|^2}{4\epsilon}, \text{ for } |x| < \rho.$$
The solution clearly depends on the choice of $\epsilon$. Now using (2.16), we calculate,

\[
\mathbb{W}(g(\tau_0), \bar{\tau} - \tau_0, f(\tau_0)) = \int_{|x|<\rho} \left( \epsilon (S + \frac{|x|^2}{4\epsilon^2}) + \frac{|x|^2}{4\epsilon^2} + C - n \right) ud\mu \\
+ \int_{d(x,p)\geq \rho} (\epsilon S + \frac{\rho^2}{4\epsilon} + C - n) ud\mu \\
= \int_{|x|<\rho} (\frac{|x|^2}{2\epsilon} - n) ud\mu + \epsilon \int_{M} Sud\mu \\
+ C \int_{M} ud\mu + \int_{d(x,p)\geq \rho} (\frac{\rho^2}{4\epsilon} - n) ud\mu \\
= I + II + III + IV.
\]

By a change of variable, as $\epsilon \to 0$, we have,

\[
II + III \to 0;
\]

\[
IV = \int_{d(x,p)\geq \rho} (\frac{\rho^2}{4\epsilon} - n - C) e^{-\frac{|x|^2}{4\epsilon}} \to 0;
\]

\[
I = e^{-C} \int_{|y|\leq \frac{\rho}{2\sqrt{\pi}}} (\frac{|y|^2}{2} - n)(2\pi)^{-n/2} e^{-|y|^2/4} (1 + O(\epsilon |y|^2)) dy \\
\to \int_{\mathbb{R}^n} (\frac{|y|^2}{2} - n)(2\pi)^{-n/2} e^{-|y|^2/4} dy = 0.
\]

Thus, by part a. and b, $\mu_\alpha(g(t), \bar{\tau} - t) \leq 0$ for any $t \leq \bar{\tau}$. The proof that the limit is actually 0 when $\tau \to 0^+$ follows from a rather standard blow-up argument whose details can be found in either [11, Prop 3.2] or [1, Prop 17.20].

3. Estimates on the Heat Kernel

In this section, we obtain several estimates on the heat kernel using maximum principle and the monotone framework. Particularly, we derive a gradient estimate and an upper bound for positive solutions of the conjugate heat equation. Then we prove our main result.

3.1. A Gradient Estimate. We first establish a space-only gradient estimate. Recall that,

\[
\Box^* = \frac{\partial}{\partial \tau} - \Delta + S.
\]
Lemma 3.1. Assume there exist $k_1, k_2, k_3, k_4 > 0$ such that the follow-
ings hold on $N \times [0, T]$,
\[
\begin{align*}
\text{Rc}(g(t)) &\geq -k_1 g(t), \\
\alpha &\geq -k_2 g(t), \\
|\nabla S|^2 &\leq k_3, \\
|S| &\leq k_4.
\end{align*}
\]
Let $q$ be any positive solution to the conjugate heat equation on $M \times [0, T]$, i.e., $\Box^* q = 0$, and $\tau = T - t$. If $q < Q$ for some constant $Q$ then there exist $C_1, C_2$ depending on $k_1, k_2, k_3, k_4$ and $n$, such that for
$0 < \tau \leq \min\{1, T\}$, we have
\[
(3.19) \hspace{1cm} \tau \frac{\nabla q^2}{q^2} \leq (1 + C_1 \tau)(\ln \frac{Q}{q} + C_2 \tau).
\]

Proof. We compute that
\[
\begin{align*}
(-\frac{\partial}{\partial t} - \Delta)\frac{\nabla q^2}{q} &= \text{S} \frac{\nabla q^2}{q} + \frac{1}{q} (-\frac{\partial}{\partial t} - \Delta)\nabla q^2 + 2\nabla q^2 \nabla^1 \frac{1}{q} \\
&\quad - 2\nabla |\nabla q|^2 \nabla^1 \frac{1}{q}, \\
\frac{1}{q} (-\frac{\partial}{\partial t} - \Delta)\nabla q^2 &= \frac{1}{q} \left[ -2(\alpha + \text{Rc})(\nabla q, \nabla q) - 2\nabla q \nabla(Sq) - 2|\nabla^2 q|^2 \right], \\
2\nabla q^2 \nabla^1 \frac{1}{q} \nabla \ln q &= -\frac{2 |\nabla q|^4}{q^2}, \\
-2\nabla |\nabla q|^2 \nabla^1 \frac{1}{q} &= \frac{4 \text{S}^2 q(\nabla q, \nabla q)}{q^2}.
\end{align*}
\]
Thus
\[
\begin{align*}
(-\frac{\partial}{\partial t} - \Delta)\frac{\nabla q^2}{q} &= -\frac{2}{q} |\nabla^2 q - \frac{dq \otimes dq}{q}|^2 + \text{S} \frac{|\nabla q|^2}{q} \\
&\quad + \frac{-2(\alpha + \text{Rc})(\nabla q, \nabla q) - 2\nabla q \nabla(Sq) - 2|\nabla^2 q|^2}{q} \\
&\quad \leq [2(k_1 + k_2) + nk_2] \frac{|\nabla q|^2}{q} + 2|\nabla q||\nabla S| \\
&\quad \leq [2k_1 + (2 + n)k_2 + 1] \frac{|\nabla q|^2}{q} + k_3 q.
\end{align*}
\]
Furthermore, we have
\[
(-\frac{\partial}{\partial t} - \Delta)(q \ln \frac{Q}{q}) = -Sq \ln \frac{Q}{q} + Sq + \frac{|\nabla q|^2}{q}
\]
\[ \frac{\nabla q}{q} \geq \frac{|\nabla q|^2}{q} - nk_2 q - k_4 q \ln \frac{Q}{q}. \]

Let \( \Phi = a(\tau) \frac{|\nabla q|^2}{q} - b(\tau) q \ln \frac{Q}{q} - c(\tau) q \), then
\[
(- \frac{\partial}{\partial t} - \Delta) \Phi \leq a' \left( a(\tau) + a(\tau) (2k_1 + (2 + n)k_2 + 1) - b(\tau) \right) + q \ln \frac{Q}{q} \left( k_4 b(\tau) - b'(\tau) \right) + q \left( k_3 a(\tau) - c'(\tau) + nk_2 b(\tau) + c(\tau) k_4 \right).
\]

We can now choose \( a, b \) and \( c \) appropriately such that \((- \partial_t - \Delta) \Phi \leq 0\).

For example, take
\[
a = \frac{\tau}{1 + (2k_1 + (2 + n)k_2 + 1)\tau},
\]
\[
b = e^{k_4 \tau},
\]
\[
c = (e^{k_5 k_4 \tau} nk_2 + k_3) \tau,
\]
for \( k_5 = 1 + \frac{k_3}{nk_2} \). Then by maximum principle, noticing that \( \Phi \leq 0 \) at \( \tau = 0 \), we arrive at
\[
a \frac{|\nabla q|^2}{q} \leq b(\tau) q \ln \frac{Q}{q} + cq.
\]

The result then follows from simple algebra.

\( \square \)

3.2. \( L_\infty \) Bound. Second, we shall derive an upper bound for positive conjugate heat solutions. Our main statement says that any normalized solution can not blow up too fast.

**Lemma 3.2.** Let \( q \) be any normalized positive solution to the conjugate heat equation on \( M \times [0, T] \), i.e., \( \Box^* q = 0 \) with \( \int q d\mu_{g(0)} = 1 \). Let \( \tau = T - t \), then there exists a constant \( C \) depending on the geometry of \( g(t)_{t \in [0, T]} \), such that
\[
q(y, \tau) \leq \frac{C}{\tau^{n/2}}.
\]

**Proof.** The proof is modeled after [9, Lemma 2.2] (also see [2, Lemma 16.47]). As the solution and the flow is well defined in \( M \times [0, T] \), there exists \( y_0, \tau_0 \) such that
\[
\sup_{M \times [0, \min\{1, T\}]} \tau^{n/2} q(y, \tau) = \tau_0^{n/2} q(y_0, \tau_0).
\]
In particular,

\[ \sup_{M \times [\tau_0/2, \tau_0]} q(y, \tau) \leq \frac{\tau_0^{n/2}}{\tau^{n/2}} q(y_0, \tau_0) \leq 2^{n/2} q(y_0, \tau_0) := Q. \]

Applying Lemma 3.1 to \( q(y, \tau) \) on \( M \times [\tau_0/2, \tau_0] \) we obtain,

\[ \frac{\tau_0}{2} \left| \nabla q \right|^2 (y, \tau_0) \leq (1 + C_1 \frac{\tau_0}{2}) (\log \left( \frac{Q}{q(y, \tau_0)} \right) + C_2 \frac{\tau_0}{2}). \]

Let \( G(y, \tau_0) := \log \left( \frac{Q}{q(y, \tau_0)} \right) + C_2 \frac{\tau_0}{2} \), then the inequality above can be rewritten as

\[ |\nabla \sqrt{G}|^2 = \frac{1}{2 \sqrt{G}} \frac{\nabla G}{\sqrt{G}} = \frac{1}{4 \sqrt{G}} \frac{\left| \nabla q \right|^2}{q^2} \leq 1 + C_1 \frac{\tau_0}{2}. \]

Therefore, with \( B_{r}(y, r) \) denoting the ball of radius \( r \) measured by \( g(\tau) \) around the point \( y \), we have

\[ \sup_{B_{\tau_0}(y_0, \sqrt{\frac{\tau_0}{1 + C_1 \frac{\tau_0}{2}}})} \sqrt{G}(y, \tau_0) \leq \sqrt{G}(y_0, \tau_0) + \frac{1}{\sqrt{2}} = \sqrt{\frac{n}{2} \log 2 + C_2 \frac{\tau_0}{2} + \frac{1}{\sqrt{2}}}. \]

Writing the above inequality in terms of \( q(y, \tau_0) \) yields,

\[ q(y, \tau_0) \geq q(y_0, \tau_0) \exp \left\{ -\frac{1}{2} \frac{n}{2} \log 2 - \frac{2}{\sqrt{2}} \sqrt{n} \frac{\log 2 + C_2}{2} \right\} := C_3 q(y_0, \tau_0). \]

Now we observe that there exists a constant \( C_4 \) depending on the geometry of \( (M, g(\tau_0)) \), such that

\[ \text{Vol}_{g(\tau_0)} \left( B_{\tau_0}(y_0, \sqrt{\frac{\tau_0}{1 + C_1 \frac{\tau_0}{2}}}) \right) \geq C_4 \tau_0^{n/2}. \]

Therefore we have,

\[ q(y_0, \tau_0) \leq \frac{1}{C_3 C_4 \tau_0^{n/2}} \int_{B_{\tau_0}(y_0, \sqrt{\frac{\tau_0}{1 + C_1 \frac{\tau_0}{2}}})} q(y, \tau_0) d\mu_{\tau_0}(y) := C_5 \int_{M} q(y, \tau_0) d\mu_{\tau_0}(y). \]

By our choice of \( y_0, \tau_0 \) and the fact that \( \int_{M} q(y, \tau) d\mu_{\tau}(y) \) remains constant along the flow, the statement follows.

**Remark 3.3.** It is interesting to note that in [13], a Harnack inequality is used to obtain an off-diagonal bound, while here the argument goes the opposite direction.
3.3. Proofs of Main Results. Finally, we are ready to finish our proof of the main theorem.

Lemma 3.4. Let \( H(x,t; y, T) = (4\pi T)^{-n/2}e^{-h} \) be a heat kernel and \( \Phi \) be any positive solution to the heat equation. Then we have
\[
\int_M hH\Phi d\mu \leq \frac{n}{2} \Phi(y, T), \quad \text{i.e.,} \quad \int_M (h - \frac{n}{2}) H\Phi d\mu \leq 0.
\]

Proof. By Lemma 2.4 we have
\[
\limsup_{\tau \to 0} \int_M hH\Phi d\mu \leq \limsup_{\tau \to 0} \int_M \ell(x, \tau) H\Phi d\mu(x) \leq \limsup_{\tau \to 0} \int_M \frac{d_T^2(x, y)}{4\tau} H\Phi d\mu(x).
\]
Using Theorem 2.2, it follows that,
\[
\lim_{\tau \to 0} \int_M \frac{d_T^2(x, y)}{4\tau} H\Phi d\mu(x) = \lim_{\tau \to 0} \int_M \frac{d_T^2(x, y)}{4\tau} e^{-\frac{d_T^2(x, y)}{4\tau}} (4\pi \tau)^{n/2} \Phi d\mu(x).
\]
Either by differentiating twice under the integral sign or using these following identities on Euclidean spaces
\[
\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}},
\]
we obtain that
\[
\int_{\mathbb{R}^n} |x|^2 e^{-a|x|^2} dx = n\left(\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-ax^2} dx\right)^{n-1} = \frac{n}{2a} \left(\frac{\pi}{a}\right)^{n/2}.
\]
Therefore,
\[
\lim_{\tau \to 0} \frac{d_T^2(x, y)}{4\tau} e^{-\frac{d_T^2(x, y)}{4\tau}} (4\pi \tau)^{n/2} = \frac{n}{2} \delta_y(x),
\]
hence
\[
\lim_{\tau \to 0} \int_M \frac{d_T^2(x, y)}{4\tau} e^{-\frac{d_T^2(x, y)}{4\tau}} (4\pi \tau)^{n/2} \Phi d\mu_N(x) = \frac{n}{2} \Phi(y, T).
\]
Thus the result follows. \(\square\)

The following result implies that the equality actually holds.

Proposition 3.5. Let \( H(x,t; y, T) = (4\pi T)^{-n/2}e^{-h} \) be a heat kernel and \( \Phi \) be any positive solution to the heat equation. Then for
\[
v = \left[(T - t)(2\Delta h - \nabla^2 h) + h - n\right] H,
\]
\[
\rho_{\Phi}(t) = \int_M v\Phi d\mu,
\]
we have
\[ \lim_{t \to T} \rho_\Phi(t) = 0. \]

**Proof.** Integrating by parts yields that
\[
\rho_\Phi(t) = \int_M \left[ \tau(2\Delta h - |\nabla h|^2 + S) + h - n \right] H\Phi d\mu
\]
\[
= -\int_M 2\tau \nabla h \nabla (H\Phi) d\mu - \int_M \tau |\nabla h|^2 H\Phi d\mu + \int_M (\tau S + h - n) H\Phi d\mu
\]
\[
= \int_M \tau |\nabla h|^2 H\Phi d\mu - 2\tau \int_M \nabla \Phi \nabla h H d\mu + \int_M (\tau S + h - n) H\Phi d\mu
\]
\[
= \int_M \tau |\nabla h|^2 H\Phi d\mu - 2\tau \int_M H \nabla \Phi d\mu + \int_M (\tau S + h - n) H\Phi d\mu
\]
\[
= \int_M \tau |\nabla h|^2 H\Phi d\mu + \int_M h H\Phi d\mu - 2\tau \int_M H \nabla \Phi d\mu + \int_M (\tau S - n) H\Phi d\mu.
\]

Notice that, except the first two terms, the rest approaches \(-n\Phi(y, T)\) as \(\tau \to 0\). For the first term, using Lemmas 3.2 and 3.1 for any space-\((\tau)\) time point on \(M \times \left[\frac{T}{2}, \tau\right]\) we arrive at
\[
\tau \int_M |\nabla h|^2 H\Phi d\mu \leq (2 + C_1 \tau) \int_M (\ln \left(\frac{C_3}{H^{\frac{n}{2}}\tau} + C_2 \tau\right)) H\Phi d\mu
\]
\[
\leq (2 + C_1 \tau) \int_M (\ln C_3 + h + \frac{n}{2} \ln(4\pi) + C_2 \tau) H\Phi d\mu,
\]
with \(C_1, C_2\) defined as in Lemma 3.1 while \(C_3\) is a constant depending on the geometry of \(g(t)\), \(\frac{T}{2} \leq T - t \leq \tau\). As \(\tau \to 0\), \(\ln C_3 + \frac{n}{2} \ln(4\pi)\) is bounded from above by another constant \(C_4\) also depending on the geometry of \(g(t)\), \(t \in [0, T]\). Consequently, by Lemma 3.4, which claims the finiteness of \(\int_M h H\Phi d\mu\),
\[
\lim_{\tau \to 0} \left( \int_M \tau |\nabla h|^2 d\mu + \int_M h H\Phi d\mu \right) \leq 3 \int_M h H\Phi d\mu + 2 \ln C_4 \Phi(x, T)
\]
\[
\leq \left(\frac{3n}{2} + 2 \ln C_4\right) \Phi(x, T).
\]
Thus we have
\[
\lim_{t \to T} \rho_\Phi(t) \leq C_5 \Phi(x, T).
\]

Since \(\Phi\) is a positive solution satisfying \(\partial_t \Phi = \Delta \Phi\), applying Theorem 1.1 yields that,
\[
(3.22) \quad \partial_t \rho_\Phi(t) = \partial_t \int v \Phi d\mu = \int (\Box \Phi v - \Phi \Box v) d\mu \geq 0.
\]
The above implies that there exists $\beta$, such that
\[ \lim_{t \to T} \rho_{\Phi}(t) = \beta. \]

Hence
\[ \lim_{\tau \to 0} (\rho_{\Phi}(T - \tau) - \rho_{\Phi}(T - \frac{\tau}{2})) = 0. \]

By the above equation (3.22), Theorem 1.1, and the mean-value theorem, there exists a sequence $\tau_i \to 0$, such that
\[ \lim_{\tau_i \to 0} \tau_i^2 \int_M \left( |\alpha + \text{Hess} - \frac{g}{2\tau_i}|^2 + \frac{1}{2} D_{\alpha}(\nabla h) \right) H \Phi d\mu = 0. \]

Now standard inequalities yield that,
\[ \left[ \int_M \tau_i (S + \Delta h - \frac{n}{2\tau_i}) H \Phi d\mu \right]^2 \leq \left[ \int_M \tau_i^2 (S + \Delta h - \frac{n}{2\tau_i})^2 H \Phi d\mu \right] \left[ \int_M H \Phi d\mu \right] \]
\[ \leq n \left[ \int_M \tau_i^2 |S + \text{Hess} - \frac{g}{2\tau_i}|^2 H \Phi d\mu \right] \left[ \int_M H \Phi d\mu \right]. \]

Since
\[ \lim_{\tau_i \to 0} \int_M H \Phi d\mu = \Phi(y, T) < \infty, \]
and $\frac{1}{2} D_{\alpha}(\nabla h) \geq 0$, we derive that
\[ \lim_{\tau_i \to 0} \int_M \tau_i (S + \Delta h - \frac{n}{2\tau_i}) H \Phi d\mu = 0. \]

Therefore, by Lemma 3.3,
\[ \lim_{t \to T} \rho_{\Phi}(t) = \lim_{\tau_i \to 0} \int_M \left[ \tau_i (2\Delta h - |\nabla h|^2 + S) + h - \frac{n}{2} \right] H \Phi d\mu \]
\[ = \lim_{\tau_i \to 0} \int_M \left[ \tau_i (\Delta h - |\nabla h|^2) + h - \frac{n}{2} \right] H \Phi d\mu \]
\[ = \lim_{\tau_i \to 0} \left[ \int_M -\tau_i H \Delta \Phi d\mu + \int_M (h - \frac{n}{2}) H \Phi d\mu \right] \]
\[ = \lim_{\tau_i \to 0} \int_M (h - \frac{n}{2}) H \Phi d\mu \leq 0. \]

So $\beta \leq 0$. To show that equality holds, we proceed by contradiction. Without loss of generality, we may assume $\Phi(y, T) = 1$. Let $H \Phi = (4\pi\tau)^{-n/2} e^{-\tilde{h}}$ (that is, $\tilde{h} = h - \ln \Phi$), then integrating by parts yields,
\[ \rho_{\Phi}(t) = \mathbb{W}_{\alpha}(g, \tau, \tilde{h}) + \int_M \left( \tau \left( \frac{|\nabla \Phi|^2}{\Phi} \right) - \Phi \ln \Phi \right) H d\mu. \]
By the choice of $\Phi$ the last term converges to 0 as $\tau \to 0$. So if $\lim_{t \to T} \rho_\beta(t) = \beta < 0$ then $\lim_{\tau \to 0} \mu_\alpha(g, \tau) < 0$ and, thus, contradicts Lemma 2.7. Therefore $\beta = 0$. □

Now Theorem 1.2 follows immediately.

**Proof.** (Theorem 1.2) Recall from inequality (3.22)

$$
\partial_t \int_M v\Phi d\mu = \int_M (\Box \Phi v - \Phi \Box^* v) d\mu \geq 0.
$$

By Proposition 3.5

$$
\lim_{t \to T} \int_M v\Phi d\mu = 0.
$$

Since $\Phi$ is arbitrary, $v \leq 0$. □

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