Moving quantum agents in a finite environment

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Abstract

We investigate an all-quantum-mechanical spin network, in which a subset of spins, the \( K \) “moving agents”, are subject to local and pair unitary transformations controlled by their position with respect to a fixed ring of \( M \) “environmental”-spins. We demonstrate that a “flow of coherence” results between the various subsystems. Despite entanglement between the agents and between agent and environment, local (non-linear) invariants may persist, which then show up as fascinating patterns in each agent’s Bloch-sphere. Such patterns disappear, though, if the agents are controlled by different rules. Geometric aspects thus help to understand the interplay between entanglement and decoherence.

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INTRODUCTION

Originally, coherence refers to a strictly defined phase-relationship between (linear) wave-components. Such phase-relationships also underly superposition states in quantum mechanics (with respect to a given set of basis states). The superposition principle can be extended to composite systems (N subsystems), for which we usually refer to product-states as the pertinent basis set. Corresponding coherent superpositions then include “local coherent states” in which case the product character remains untouched, and so-called “entangled” states for which the strict product form is lost: The reduced subsystem-states appear “non-pure” for the latter case, entanglement acts as a source of local entropy. As opposed to local coherence, entanglement has no classical analogue.

Decoherence may generally be seen as a suppression of superposition. In the strong sense this suppression results from (approximate) disjointness due to super-selection rules (leading to a classical domain proper). The effect of superposition may (partially) be quenched for collective observables (such as the total spin), when, due to phase-randomness, the individual spins, though each well-defined, virtually cancel each other (in this case coherence may be recovered, cf. spin echo). Roughly speaking, non-selective local properties of such incoherent ensembles are “ill-defined”, when used as an input to appropriate interferometers, no interference fringes are observed. “Ill-defined” properties also occur in individual quantum objects as a result of incompatibility (cf. Heisenberg uncertainty relations). Entanglements in composite systems implies that local properties become “incompatible” for the state considered, the superposition character of reduced subsystem-states is suppressed: Local coherence may be said “to have moved where nobody looks” (while coherence as such has not been destroyed). In this paper we investigate local state changes resulting from an iterative map applied to a composite system whose state evolves in the (in terms of N) exponentially large Hilbert-space. This map may be visualized as arising from a circular motion of spins (“agents”) on a closed chain of fixed environment spins.

DEFINITION OF NETWORK SPACE

The quantum network to be considered here is composed of N pseudo-spins with state \( |j(\mu)⟩, j = 0, 1; \mu = 1, 2, \cdots, N \), and the transition-operators \( \hat{P}_{ij}^{(\mu)} = |i(\mu)⟩\langle j(\mu)| \). The \( 2^N \) product-states are

\[
|jk\cdots I⟩ = |j⟩^{(1)} \otimes |k⟩^{(2)} \otimes \cdots \otimes |I⟩^{(N)}
\]

on which the “c-cluster-operators”

\[
\hat{Q}_{jk\cdots I} := \hat{\lambda}_j^{(1)} \otimes \hat{\lambda}_k^{(2)} \otimes \cdots \otimes \hat{\lambda}_I^{(N)}
\]

act. Here and in the following the upper indices (in parenthesis) refer to the subsystem-numbers, lower indices to the component/type of property. The \( \hat{\lambda}_i^{(\mu)} \) are the following SU(2)-operators

\[
\hat{\lambda}_1^{(\mu)} = \hat{P}_{01}^{(\mu)} + \hat{P}_{10}^{(\mu)} \\
\hat{\lambda}_2^{(\mu)} = i(\hat{P}_{01}^{(\mu)} - \hat{P}_{10}^{(\mu)})
\]
\[ \hat{\lambda}_3^{(\mu)} = \hat{P}_{11}^{(\mu)} - \hat{P}_{00}^{(\mu)} \]
\[ \hat{\lambda}_0^{(\mu)} = \hat{1}^{(\mu)} = \hat{P}_{11}^{(\mu)} + \hat{P}_{00}^{(\mu)} \]  
(3)

and \( c \) is the number of indices \( \neq 0 \) within the given set of the lower indices \( \{j k \cdots l\} \). \( \hat{Q}_{jk\cdots l} \) thus operates on \( c \) subsystems out of \( N \). Full description of the network-state \( |\psi\rangle \) requires the specification of all the \( 2^N - 1 \) expectation values
\[ -1 \leq \langle \psi|\hat{Q}_{jk\cdots l}|\psi\rangle =: Q_{jk\cdots l} \leq 1. \]  
(4)

Here, \( Q_{00\cdots 0} = 1 \), the \( c = 1 \) expectation values are the so-called Bloch-vectors for the individual spins, the \( c > 1 \) terms constitute \( c \)-point correlation functions. Dispersion-free expectation values are \( \pm 1 \). Largest quantum mechanical uncertainty pertains to operators with expectation-value zero. Therefore, as a convenient measure for the “weight” of well-defined properties of a given cluster of type \( c \) we take the respective “cluster-sum” \( Y \)

\( c = 1: \) \( Y_1^{(1)} = \sum_{i=1}^{3} Q_{i00\cdots 0} Q_{i00\cdots 0} \), \( Y_1^{(2)} = \sum_{i=1}^{3} Q_{i0i0\cdots 0} Q_{i0i0\cdots 0} \) etc.

\( c = 2: \) \( Y_2^{(12)} = \sum_{i,j=1}^{3} Q_{ij0\cdots 0} Q_{ij0\cdots 0} \) etc.

The \( Y_1 \)-terms are identical with the square of the respective Bloch-vector-length. These cluster-sums obey the inequalities
\[ Y_c \leq Z_c = \begin{cases} 2^{(c-1)} & c = \text{odd} \\ 2^{(c-1)} + 2 & c = \text{even} \end{cases} \]  
(5)

For a product-state each cluster-sum equals 1, irrespective of its size. There are \( 2^N - 1 \) different clusters, implying
\[ 2^N - 1 = \sum_{c=1}^{N} \sum_{\mu,\nu,\cdots} Y_c^{(\mu,\nu,\cdots)}. \]  
(6)

One easily shows that this sum-rule must hold for any pure state! As a consequence only few clusters can exploit their full allowance of \( Z_c \) (which increases exponentially with \( c \! \)); in most cases \( Y_c < Z_c \), which we call (partial) “\( c \)-decoherence”. A given 2-cluster has surplus correlation (“entanglement”), if for its partition \( Y_2^{(12)} > Y_1^{(1)} Y_2^{(1)} \) holds. The various clusters have thus to “compete for weight”. Dynamically we may say that coherence “flows” from one part to another (without getting lost, though), under the influence of unitary transformations.

**ARCHITECTURE**

We first split our network into 2 subgroups, \( N = K + M \), where \( M \) pseudo-spins constitute the “environment”, \( \mu = 1, 2, \cdots, M \), and the remaining spins are the “agents”, \( Sk = S1, S2, \cdots, SK \) (Fig 1). The behavior of the network is specified in terms of a discrete set of unitary transformations [3,4].
\[ \hat{U}^{(Sk)} = \hat{1}^{(Sk)} \cos (\alpha_k/2) - \hat{\lambda}_1^{(Sk)} i \sin (\alpha_k/2) \]

\[ \hat{U}_0^{(Sk,\mu)} = \hat{P}_{00}^{(Sk)} \otimes \hat{\lambda}_1^{(\mu)} + \hat{P}_{11}^{(Sk)} \otimes \hat{1}^{(\mu)} \]

\[ \hat{U}_\pi^{(Sk,\mu)} = \hat{P}_{00}^{(Sk)} \otimes i \hat{\lambda}_2^{(\mu)} + \hat{P}_{11}^{(Sk)} \otimes \hat{1}^{(\mu)}. \] (7)

\( \hat{U}^{(Sk)} \) is a local rotation around the x axis by angle \( \alpha_k \), \( \hat{U}_\theta^{(Sk,\mu)} \) is the quantum-controlled-NOT-operation (QCNOT) with \( (\hat{U}_\theta)^2 = \hat{1} \) for \( \theta = 0 \), \( (\hat{U}_\theta)^4 = \hat{1} \) for \( \theta = \pi \). The sequence of transformations may be interpreted to result from cyclic and discretized motions of the \( K \) agents along a circular chain of environment pseudo-spins: Controlled by position a local transformation \( \hat{U}^{(Sk)} \) is followed by a pair-interaction \( \hat{U}_\theta^{(Sk,\mu)} \), \( \theta = 0, \pi \), and vice versa. For any given agent \( Sk \), the subscript \( \theta \) (“type of the agent”) is assumed to be fixed. The various agents may move in a constant or in a changing sequential order. In the former case a strict iterative map results from \( 2M \) unitary transformations for each agent (equivalent to one full cycle \( p = 1, 2, \ldots \)). For \( K = 1 \) this architecture may be viewed as a quantum-Turing-machine [9–13], with the agent-spin being the Turing-head and the environment acting as the Turing-tape. Most results will be presented for this simplest scenario. In this case the resulting iterative map can be specified as

\[ |\psi(m_1)\rangle = \hat{U}^{(S1)}(m_1) \cdots \hat{U}^{(S1)}(1) (\hat{U}^{(S1)}(2M) \cdots \hat{U}^{(S1)}(1))^{p_1}|\psi(0)\rangle \]

\[ \equiv \hat{T}^{(S1)}(m_1)|\psi(0)\rangle \] (8)

where \( n_1 = 1, 2, \ldots, 2M; m_1 = n_1 + 2M(p_1 - 1); p_1 \) being the number of completed cycles, \( m_1 \) the step number, and

\[ \hat{U}^{(S1)}(2\mu - 1) := \hat{U}^{(S1)}_{\alpha}\]

\[ \hat{U}^{(S1)}(2\mu) := \hat{U}^{(S1)}_{\theta}. \] (9)

**PURE-STATE TRAJECTORIES FOR \( K = 1, \theta = 0 \)**

We restrict ourselves to the reduced state-space dynamics (our “system of interest”)

\[ \lambda_i^{(S1)}(m_1) = Q_{i00}^{(S1)}(m_1) = \langle \psi(m_1)|\tilde{Q}_{i00}|\psi(m_1)\rangle \] (10)

Due to entanglement we will, in general, see the apparent decoherence,

\[ Y_1^{(S1)} = \sum_{i=1}^{3} |\lambda_i^{(S1)}|^2 < 1. \] (11)

However, for specific initial states \( |\psi(0)\rangle \) the state of the Turing-head \( S1 \) will remain pure: As \( |\pm\rangle^{(\mu)} = \frac{1}{\sqrt{2}} (|0\rangle^{(\mu)} \pm |1\rangle^{(\mu)}) \) are the eigenstates of \( \hat{\lambda}_1^{(\mu)} \) with \( \hat{\lambda}_1^{(\mu)} |\pm\rangle^{(\mu)} = \pm |\pm\rangle^{(\mu)} \), the action of the QCNOT reduces to

\[ \hat{U}_0^{(S1,\mu)} |\varphi^{(S1)}\rangle \otimes |+\rangle^{(\mu)} = |\varphi^{(S1)}\rangle \otimes |+\rangle^{(\mu)} \]

\[ \hat{U}_0^{(S1,\mu)} |\varphi^{(S1)}\rangle \otimes |-\rangle^{(\mu)} = \hat{\lambda}_3^{(S1)} |\varphi^{(S1)}\rangle \otimes |-\rangle^{(\mu)}. \] (12)
As a consequence, for the initial product-state
\[ |\psi(0)\rangle = |\varphi(0)\rangle^{(S_1)} \otimes |P^j(0)\rangle \]
with
\[ |\varphi(0)\rangle^{(S_1)} = \cos(\varphi_0/2)|0\rangle^{(S_1)} - i \sin(\varphi_0/2)|1\rangle^{(S_1)} \]
\[ |P^j(0)\rangle \in \{|\pm\rangle^{(1)} \otimes |\pm\rangle^{(2)} \otimes \cdots \otimes |\pm\rangle^{(M)}\} , \]
the state \(|\psi(m)\rangle\) remains a product-state at all steps m ("primitives"):
\[ |\psi(m|P^j)\rangle = |\varphi(m|P^j)\rangle^{(S_1)} \otimes |P^j(0)\rangle . \]

Here \(|\varphi(m|P^j)\rangle^{(S_1)}\) is the Turing-head state at step m conditioned by \(|P^j(0)\rangle\) and for a given head-state there are \(2^M\) primitives. The Turing-head Bloch-vectors
\[ \lambda_k^{(S_1)}(m|P^j) = \langle \psi(m|P^j)|\hat{\lambda}_k^{(S_1)} \otimes \hat{1}^{(1)} \otimes \hat{1}^{(2)} \cdots \otimes \hat{1}^{(M)}|\psi(m|P^j)\rangle \]
are all confined to the \(k = 2, 3\)-plane and obey the relation \(|\bar{\lambda}_k^{(S_1)}(m|P^j)| = 1\). In Fig 2 we show the 4 primitives \(\lambda_k^{(S_1)}(m|P^j), j = 1, 2, 3, 4\), for \(M = 2\) and \(\alpha = \pi/\sqrt{3}\). The primitives are either periodic (Floquet-states) and then independent of \(\alpha\), or aperiodic (here: \(P^{++}\)) and then controlled by \(\alpha\) (the latter orbits are also periodic if \(\alpha\) is a rational multiple of \(\pi\)). A similar analysis holds for \(\theta = \pi\).

**SUPERPOSITION OF PRIMITIVES FOR \(K = 1, \theta = 0\)**

Now let the initial state be
\[ |\psi(0)\rangle = \sum_{j=1}^{2^M} a_j |\varphi(0)\rangle^{(S_1)} \otimes |P^j(0)\rangle . \]

Then we find at step m
\[ |\psi(m)\rangle = \sum_{j=1}^{2^M} a_j |\varphi(m|P^j)\rangle^{(S_1)} \otimes |P^j(0)\rangle \]
and, observing the orthogonality of the \(|P^j(0)\rangle\),
\[ \lambda_k^{(S_1)}(m) = \langle \psi(m)|\hat{\lambda}_k^{(S_1)} \otimes \hat{1}^{(1)} \otimes \hat{1}^{(2)} \cdots \otimes \hat{1}^{(M)}|\psi(m)\rangle \]
\[ = \sum_{j=1}^{2^M} |a_j|^2 \lambda_k^{(S_1)}(m|P^j) . \]

This trajectory of agent \(S_1\) represents a non-orthogonal pure-state decomposition with weights \(|a_j|^2\) independent of \(m\), and the decomposition can be seen as an intuitive example for quantum parallelism: The individual Turing-head performs exponentially many primitive trajectories "in parallel". Different initial states with the same \(|a_j|^2\) show the same reduced dynamics for \(S_1\).
Special superpositions are

\[ |a_j|^2 = \begin{cases} 
\text{const.} & \text{for } j \in \{\text{periodic orbits}\} \\
0 & \text{otherwise} 
\end{cases} \]

and the complementary type

\[ |a_j|^2 = \begin{cases} 
\text{const.} & \text{for } j \in \{\text{aperiodic orbits}\} \\
0 & \text{otherwise} 
\end{cases} \]

The example for \( M = 3 \) is shown in Fig 3. In this case each superposition consists of 4 primitives. The trend towards reduced Bloch-vector lengths is easily recognized; however, there are no privileged basis states (in which case the points would be on a single straight line).

Finally, starting from the ground state, the typical initial state for quantum computation \[14,15\]

\[ |\psi(0)\rangle = |0\rangle^{(S1)} \otimes |00\cdots0\rangle \]

all \( 2^M \) pure state trajectories contribute with equal weight

\[ |a_j|^2 = \frac{1}{2^M}. \]

Calculations for \( M = 3,10 \) are shown in Fig 4. The resulting pattern (“quasi-1-dimensional point manifolds”) shows the existence of a local invariant (for \( M = 1,2, \text{ cf. } [7] \)), which is a consequence of the underlying primitives.

**COMPUTATIONAL REDUCIBILITY**

Knowing the \( 2^M \) primitives we can calculate the reduced Turing-head dynamics from eq. (19) for any initial tape-state. This procedure, to be sure, becomes prohibitive for large \( M \), as the number of those primitives grows exponentially. It turns out that a complementary problem is much easier to solve: To calculate all possible iterative maps (for any \( M \) and any control-angles \( \alpha \)) for a selected initial tape-state (here: the ground state). This can be done by the recursion relation

\[ \begin{align*}
\lambda_1^{(S1)}(m) &= 0 \\
\lambda_2^{(S1)}(m) &= Y_{m,M} \\
\lambda_3^{(S1)}(m) &= Z_{m,M}
\end{align*} \]

where \( Y_{m,M} \) and \( Z_{m,M} \) are specified in Table II.
SEVERAL AGENTS, $K \geq 2$

We start by noting the commutator relations:

$$[\hat{U}_\theta^{(Sk,\mu)}, \hat{U}_{\pi}^{(Sk',\mu')}] = 0$$

$$[\hat{U}_0^{(Sk,\mu)}, \hat{U}_{\pi}^{(Sk',\mu')}] = \begin{cases} -2\hat{P}_{00}^{(Sk)} \otimes \hat{\lambda}_3^{(\mu)} \delta_{\mu\mu'} & \text{for } Sk = Sk' \\ -2\hat{P}_{00}^{(Sk)} \otimes \hat{B}_{00}^{(Sk')} \otimes \hat{\lambda}_3^{(\mu)} \delta_{\mu\mu'} & \text{for } Sk \neq Sk' \end{cases}$$

(25)

This means that for any agent of the same type all unitary transformations between different agents $Sk \neq Sk'$ commute, i.e. the (time-)ordering is irrelevant! Thus for $K = 2$, e.g., one finds (cf. eq. (8))

$$|\psi(m_1, m_2)⟩ = \cdots \hat{U}^{(S2)}(4) \hat{U}^{(S2)}(3) \hat{U}^{(S1)}(4) \hat{U}^{(S1)}(3) \times \hat{U}^{(S2)}(2) \hat{U}^{(S2)}(1) \hat{U}^{(S1)}(2) \hat{U}^{(S1)}(1) |\psi_0⟩$$

$$= \hat{T}^{(S2)}(m_2) \hat{T}^{(S1)}(m_1) |\psi(0)⟩$$

(26)

where $\hat{U}^{(S2)}(2\mu-1), \hat{U}^{(S2)}(2\mu)$ are equivalent to $\hat{U}^{(S1)}(2\mu-1), \hat{U}^{(S1)}(2\mu)$ (eq. (4)) respectively, so that the local agent properties are independent of each other:

$$\lambda_j^{(S1)}(m_1) = \langle \psi(0) \hat{T}^{(S1)}(m_1) |\hat{T}^{(S2)}(m_2) |\hat{\lambda}_j^{(S1)} |\hat{T}^{(S2)}(m_2) \hat{T}^{(S1)}(m_1) \psi(0)⟩$$

$$= \langle \psi(0) \hat{T}^{(S1)}(m_1) |\hat{\lambda}_j^{(S1)} |\hat{T}^{(S1)}(m_1) \psi(0)⟩ \quad .$$

(27)

Corresponding relations hold for $S2$. As a consequence the patterns of each agent are not influenced by the presence of the other, even though both agents become entangled, in addition to the entanglement between each agent and the environment (tape)!

Things change dramatically if the two agents are of different type. Then the order of the unitary transformations with respect to each agent matters, and the actions for each agent can no longer be grouped together. In Fig. 5 we show the result for $K = 2, M = 2$. Now the patterns for each agent show an (apparent) randomness, there are no longer obvious local invariants visible.

SUMMARY AND CONCLUSIONS

Open systems are well-known examples exhibiting loss of coherence with respect to certain privileged basis states (the “measurement basis”). Here we have considered an all-quantum-mechanical spin-network subject to a discretized unitary transformation (iterative map). We deliberately choose to “look” only at a subset of spins, the $K$ “agents”, which interact with a closed chain of $M$ “environment”-spins, one after the other. For this pure-state dynamics the “flow of coherence” is explicitly demonstrated as well as the entanglement-induced decoherence. For agents of the same type fascinating patterns result in their respective Bloch-spheres (a quantum version of Poincaré-cuts). These patterns can be understood
as being based on a set of pure-state trajectories ("primitives"). For specific initial states even explicit recursion relations (in the agent Bloch-vector-space) exist. This remarkable computational reducibility works for any network-size $N = K + M$. However, if the agents are of different type (non-commuting), they disturb each other via their common environment, and the resulting Bloch-sphere patterns show apparent randomness without obvious signs for invariants. This indicates that geometrical aspects can be a useful supplement to conventional algebraic results for the discussion of decoherence and entanglement.

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**FIGURE CAPTIONS**

Fig. 1 The network architecture: The moving agents $S_1, S_2$ move along the circular environment (spins 1, 2, 3, 4) in discrete steps, thus iterating between local transformations (arrows) and pair interactions (when in touch with an environment spin).

Fig. 2 The primitives $P_0^+$ (aperiodic) and $P_0^-$ (periodic) for $K = 1, \theta = 0, M = 1$, and $P_0^{++}$ (aperiodic), $P_0^{--}, P_0^{-+}, P_0^{+-}$ (periodic) for $K = 1, l = 1, M = 2; \alpha = \pi/\sqrt{3}$, $\varphi(0) = \pi/6$.

Fig. 3 Equal-weight superpositions ($a_i = 1/2$) of 4 periodic (4 aperiodic) orbits for $|\psi_0\rangle = |0\rangle^{(S_1)} \otimes |000\rangle$, $K = 1, \theta = 0, M = 3$, and step numbers $m \leq 3000$. The equal-weight superposition $(1/\sqrt{2})$ of these two, in turn, generates the pattern for $|\psi_0\rangle = |0\rangle^{(S_1)} \otimes |000\rangle$ (see Fig. 4 for $M = 3$); $\alpha = \pi/\sqrt{3}$.

Fig. 4 Turing-head-patterns for $|\psi_0\rangle = |0\rangle^{(S_1)} \otimes |00\cdots0\rangle$, $K = 1, l = 1, M = 3, 10$, and step numbers $m \leq 3000$; $\alpha = \pi/\sqrt{3}$.

Fig. 5 Local Bloch-vector patterns, $\lambda_y = \lambda_y^{(S_1)}, \lambda_z = \lambda_z^{(S_1)}$. First row: $K = 1, M = 1; \theta = 0$ for $S_1$ (left), $\theta = \pi$ (right); second row: $K = 2, \theta = 0$ for $S_1$, $\theta = \pi$ for $S_2$; $M = 1$ (left), $M = 2$ (right); last row: $K = 1, M = 2; \theta = 0$ for $S_1$ (left), $\theta = \pi$ (right); $|\psi_0\rangle = |00\cdots0\rangle; \alpha = \pi/\sqrt{3}$ in each case, $m_i \leq 4500, i = S_1, S_2$. 
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TABLE I. Recursion relations for the reduced state evolution of $S_1$ in the case of $K = 1$, $	heta = 0$, $|\psi_0\rangle = |0\rangle^{(S_1)} \otimes |00 \cdots 0\rangle$. Let $Y_m = Y_{m,M}$, $Z_m = Z_{m,M}$, $Z_{m,0} := -1$, and $m' := m - 4p + 2$, where $p$ is the cycle number for step $m$; $m = n + 2M(p - 1)$, $n = 1, 2, \cdots, 2M$. $Y_0 = 0$, $Y_1 = \sin \alpha$, $Z_0 = -1$, $Z_1 = -\cos \alpha$.

| $Y_m = -Y_1Z_{m-1} - Z_1Y_{m-1}$ | $n = \text{odd}$ |
|-----------------------------------|-----------------|
| $Y_{m,M} = Y_{m-1,M} + Y_1Z_{m',M-2}$ | $n = \text{even} \neq 2M$ |
| $Y_{m,M} = Y_{m-1,M} - Y_1(-Z_1)^{M-1}$ | $n = 2M$, $p = \text{odd}$ |
| $Y_{m,M} = Y_{m-1,M}$ | $n = 2M$, $p = \text{even}$ |
| $Z_m = -Z_1Z_{m-1} + Y_1Y_{m-1}$ | $n = \text{odd}$ |
| $Z_m = -Z_1Z_{m-2} + Y_1Y_{m-2}$ | $n = \text{even}$ |
FIGURES

\[ S_1 \quad S_2 \quad \alpha \quad \alpha \quad \alpha \quad \alpha \quad \theta = 0 \quad \theta = \pi \]

\[ \begin{array}{ccc}
1 & 2 & \cdots \quad \text{Umgebung} \\
\text{S1} & \text{S2} & \text{Agenten}
\end{array} \]
