Standard modules, Jones-Wenzl projectors, and the valenced Temperley-Lieb algebra

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This article concerns a generalization of the Temperley-Lieb algebra, motivated by applications to conformal field theory. We call this algebra the valenced Temperley-Lieb algebra. We prove salient facts concerning this algebra and its representation theory, which are both of independent interest and used in our subsequent work [FP18a+, FP18b+, FP18c+], where we uniquely and explicitly characterize the monodromy invariant correlation functions of certain conformal field theories.

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1. INTRODUCTION

The Temperley-Lieb algebra is ubiquitous in the mathematics and physics literature. Named after its discoverers H. Temperley and E. Lieb, it initially found its role as an algebra related to transfer matrices in integrable statistical mechanics models [TL71, Mar91, Bax07]. Later, V. Jones independently discovered this algebra as a tool for constructing invariants of knots and links [Jon83, Jon89]. This new application established the Temperley-Lieb algebra as a key ingredient in the theory of quantum groups [Jim86, Kau87, CP94, Kas95, GRAS96, KRT97] and topological quantum computation [Tur94, CKL08].

One of the most important aspects of the Temperley-Lieb algebra, especially in applications to physics, is its representation theory, now already well-understood. The pioneering works include the book [Mar91] of P. Martin and the articles [Wen87, GW93] of F. Goodman and H. Wenzl, of combinatorial nature, the more algebraic work of B. Westbury [Wes95], as well as the rather general framework of cellular algebras developed by J. Graham and G. Lehrer in [GL96, GL98]. As a very concrete approach, the recent survey [RSA14] by D. Ridout and Y. Saint-Aubin is perhaps the most comprehensive and accessible treatment of this topic.

The purpose of the present article is to consider a natural generalization of the Temperley-Lieb algebra, which we call the “valenced Temperley Lieb algebra,” and to study its representation theory following the concrete approach in [RSA14]. This algebra is motivated by applications to conformal field theory. It will be crucial in our subsequent work [FP18a+, FP18b+, FP18c+], where we uniquely and explicitly characterize the monodromy invariant correlation functions of certain conformal field theories.

To outline the content of this article, we organize the introduction as follows. First, in section 1A we collect important results about the Temperley-Lieb algebra and its representation theory. Next, in section 1B, we introduce a special subalgebra of the Temperley-Lieb algebra, which we call the “Jones-Wenzl algebra.” In section 1C, we use it to introduce the “valenced Temperley Lieb algebra,” and we list all of the main results of this article regarding this algebra and its representation theory, presenting them in parallel to the already well-known results about the Temperley-Lieb algebra stated in section 1A. Then, in section 1D we briefly discuss our motivation from conformal
field theory. This article is the first in a series of four articles, and in section 1 D we also briefly describe the contents of the other three forthcoming articles [FP18a⁺, FP18b⁺, FP18c⁺]. Finally, we outline the organization of this article and give some literary remarks in sections 1 E and 1 F respectively.

A. Background: Temperley-Lieb algebra

In this section, we review definitions and basic properties of the Temperley-Lieb algebra and its standard modules. First, for each integer \( n \in \mathbb{Z}_{\geq 0} \), we define an \( n \)-link diagram to be any planar geometric object comprising two vertical lines, \( n \) distinct marked points ("nodes") on each line, and \( n \) simple, nonintersecting, planar curves ("links") between the lines, joining the nodes pairwise. The links are determined up to homotopy. Examples of link diagrams are

\[
\begin{align*}
&\quad \\
&\quad 
\end{align*}
\]

(1.1)

We sort the links of a link diagram into two types, crossing links and turn-back links, as follows:

\[
\begin{align*}
&\text{crossing link} \\
&\text{turn-back link}
\end{align*}
\]

(1.2)

We consider the complex vector space \( \mathbb{T}_n \) of all tangles, that is, formal linear combinations of \( n \)-link diagrams. We can concatenate two link diagrams in this vector space in a natural manner, as exemplified below:

\[
\begin{align*}
&\quad \\
&\quad = \\
&\quad =
\end{align*}
\]

(1.3)

(1.4)

The concatenation forms a number \( k \in \mathbb{Z}_{\geq 0} \) of internal loops. We remove the loops and multiply the resulting tangle by \( \nu^k \), where \( \nu \) is a complex number, called the loop fugacity. Thus, for instance, (1.4) becomes

\[
\begin{align*}
&\quad \\
&\quad = \\
&\quad = 
\end{align*}
\]

(1.5)

This concatenation recipe endows the vector space \( \mathbb{T}_n \) with an associative multiplication, making it an associative, unital algebra, called the Temperley-Lieb algebra and denoted as \( \mathbb{T}_n(\nu) \). Its unit is

\[
\begin{array}{c|cccc}
1 & 2 & \ddots & \\
1 & i - 1 & \ddots & \\
i & i & \ddots & \\
i + 1 & i + 1 & \ddots & \\
i + 2 & \ddots & \ddots & \\
n & \ddots & \ddots & \\
\end{array}
\]

(1.6)
To study the representation theory of $\text{TL}_n(\nu)$, we introduce link patterns. For this purpose, we take any $n$-link diagram having $s$ crossing links, where $s$ is necessarily any number in the set

$$E_n := \{ n \text{ mod } 2, (n \text{ mod } 2) + 2, \ldots, n \},$$

(1.7)
divide it vertically in half, discard the right half, and rotate the left half by $\pi/2$ radians:

$$\text{link diagram} \quad \mapsto \quad \text{link pattern}.$$  

(1.8)

We call the remaining left half an $(n, s)$-link pattern, and we call each of the broken links in it a defect, above. We also call a formal linear combination of $(n, s)$-link patterns with complex coefficients a $(n, s)$-link state.

As (1.3–1.5), we can concatenate an $n$-link diagram to an $(n, s)$-link pattern (rotated back $-\pi/2$ radians) from the right to form a new link pattern. Again, we remove any $k$ loops formed by the concatenation and multiply the result by $\nu^k$. Also, turn-back links may appear. We regard diagrams containing turn-back links as equaling zero:

$$\begin{align*}
\text{link pattern} & \quad \mapsto \quad \nu \times \text{link pattern}, \\
\text{link pattern} & \quad \mapsto \quad 0 \times \text{link pattern} = 0.
\end{align*}$$

(1.9)

This concatenation defines an action of $\text{TL}_n(\nu)$ on the complex vector space of $(n, s)$-link states. We call this $\text{TL}_n(\nu)$-module a standard module and denote it by $L_n^{(s)}$. We also define the link state module

$$L_n := \bigoplus_{s \in E_n} L_n^{(s)}.$$  

(1.11)

A certain bilinear form, and in particular its radical, is key to understanding the representation theory of $\text{TL}_n(\nu)$. In section 3 A, we define this bilinear form on $L_n$ by pairwise concatenation of link patterns as exemplified below:

$$\begin{align*}
\left( \begin{array}{c}
\text{link pattern} \\
\text{link pattern}
\end{array} \right) & \quad = \quad \nu \times \text{link pattern}, \\
\nu \times \text{link pattern} & \quad = \quad 0 \times \text{link pattern} = 0.
\end{align*}$$

(1.12)

As before, we replace each internal loop by a factor of $\nu$ and each turn-back link by a factor of zero, and now, we also replace each through-path by a factor of one, thus arriving with a complex number:

$$\begin{align*}
\text{link pattern} & \quad = \quad \nu \times 0 \times 0 = 0, \\
\text{link pattern} & \quad = \quad \nu \times 1 \times 1 = \nu.
\end{align*}$$

(1.13)

(1.14)

We also define the radical of $L_n$ with respect to the bilinear form to be the vector space

$$\text{rad} L_n := \{ \alpha \in L_n \mid (\alpha \mid \beta) = 0 \text{ for all } \beta \in L_n \}.$$  

(1.15)
The radical $\text{rad} L_n$ is a $\text{TL}_n(\nu)$-submodule of $L_n$ and it equals a direct sum of the radicals of the standard modules $L_n^{(s)}$, which themselves are $\text{TL}_n(\nu)$-submodules of $L_n^{(s)}$,

$$\text{rad} L_n = \bigoplus_{s \in E_n} \text{rad} L_n^{(s)}, \quad \text{where} \quad \text{rad} L_n^{(s)} := \{ \alpha \in L_n^{(s)} \mid (\alpha \mid \beta) = 0 \text{ for all } \beta \in L_n^{(s)} \}. \quad (1.16)$$

For each standard module, we denote the corresponding quotient module as

$$Q_n^{(s)} := L_n^{(s)}/\text{rad} L_n^{(s)}. \quad (1.17)$$

To state key properties of the Temperley-Lieb algebra that are relevant to this article, we introduce some notation.

1. We parameterize the loop fugacity parameter $\nu \in \mathbb{C}$ by a nonzero complex number $q \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ as follows:

$$\nu = -q - q^{-1}. \quad (1.18)$$

2. For each $q \in \mathbb{C}^\times$, we define

$$p(q) := \begin{cases} \infty, & q \text{ is not a root of unity}, \\ p, & q = e^{\pi ip/p} \text{ for coprime } p, p' \in \mathbb{Z}_{>0}, \end{cases} \quad \bar{p}(q) := \begin{cases} \infty, & q \in \{\pm1\}, \\ p(q), & q \notin \{\pm1\}. \end{cases} \quad (1.19)$$

3. For each $k \in \mathbb{Z}_{\geq 0}$, we define $\Delta_k$ to be the following integer:

$$\Delta_k = \Delta_k(q) := \begin{cases} -1, & k = 0 \text{ and } p(q) = \infty, \\ kp(q) - 1, & \text{otherwise}. \end{cases} \quad (1.20)$$

4. For each $s \in \mathbb{Z}_{\geq 0}$, we define $k_s \in \mathbb{Z}_{\geq 0}$ and $R_s \in \{0, 1, \ldots, p(q) - 1\}$ to be the unique integers such that

$$s = \Delta_{k_s} + R_s. \quad (1.21)$$

5. We define the set

$$\text{Non}_n^{(s)} = \{ q \in \mathbb{C}^\times \mid R_s = 0 \text{ or } n + s \leq 3k_s p(q) + 2p(q) - 5 \}, \quad (1.22)$$

whose complement within $\mathbb{C}$ has Lebesgue measure zero. We also define

$$\text{Non}_n := \bigcap_{s \in E_n} \text{Non}_n^{(s)} = \{ q \in \mathbb{C}^\times \mid n < \bar{p}(q), \text{ or } q = \pm i \text{ if } n \text{ is odd} \}. \quad (1.23)$$

In section 5C, lemma 5.27 translates the condition $n < \bar{p}(q)$ to a condition involving $\nu$. Finally, we define

$$\text{Tot}_n^{(s)} := \begin{cases} \emptyset, & s \neq 0, \\ \{\pm1\}, & s = 0. \end{cases} \quad (1.24)$$

The goal of this article is to obtain analogues of the following well-known properties of the Temperley-Lieb algebra for a more general diagram algebra discussed shortly (which includes the former as a special case):

**TL$_n$1. [RSA14, above theorem 2.4]:** We have $\dim L_n^{(s)} = D_n^{(s)}$, where $D_n^{(s)}$ is the unique solution to the recursion

$$D_n^{(s)} = \sum_{r \in E_{n-1} \cap \{\pm1\}} D_r^{(r)} + \begin{cases} D_{n-1}^{(1)}, & s = 0, \\ D_{n-1}^{(n-1)} + D_{n-1}^{(s+1)}, & s \in \{1, 2, \ldots, n-1\}, \\ D_{n-1}^{(n-1)}, & s = n, \end{cases} \quad (1.25)$$

**TL$_n$2. [RSA14, above theorem 2.4]:** With $C_n := \frac{1}{n+1} \binom{2n}{n} = D_{2n}^{(0)}$ denoting the $n$:th Catalan number, we have

$$\dim \text{TL}_n(\nu) = C_n = \sum_{s \in E_n} \left( \dim L_n^{(s)} \right)^2. \quad (1.26)$$
\textbf{TL}_3. \textit{[RSA14, theorem 2.4]}: Let $W_n(\nu)$ be the associative, unital algebra with generators $\{U_i\}_{i=1}^{n-1}$ and relations

\begin{align}
U_i U_{i+1} U_i &= U_i, & 1 &\leq i &\leq n-1, \\
U_i^2 &= \nu U_i, \\
U_i U_j &= U_j U_i, & |i-j| &> 1,
\end{align}

for all $i, j \in \{1, 2, \ldots, n-1\}$. Then there is a unique isomorphism of algebras $f_n: W_n(\nu) \rightarrow \mathbb{T}L_n(\nu)$ such that

\begin{align}
f_n(U_i) &= \\
&= \begin{pmatrix}
1 \\
2 \\
\vdots \\
i-1 \\
i \\
i+1 \\
i+2 \\
\vdots \\
n
\end{pmatrix},
\end{align}

for all $i \in \{1, 2, \ldots, n-1\}$. In an abuse of notation, we let $U_i$ denote the diagram on the right side of (1.28).

\textbf{TL}_4. \textit{[RSA14, proposition 3.3]}: Suppose $\text{rad} L_n^{(s)} \neq L_n^{(s)}$. Then the following are true:

(a): the quotient module $Q_n^{(s)}$ is simple, or equivalently, $\text{rad} L_n^{(s)}$ is the unique maximal proper submodule of $L_n^{(s)}$, and

(b): the standard module $L_n^{(s)}$ is indecomposable.

\textbf{TL}_5. \textit{[RSA14, corollary 3.7]}: Suppose $\text{rad} L_n^{(s)} \neq L_n^{(s)}$ and $\text{rad} L_n^{(r)} \neq L_n^{(r)}$. Then we have

\begin{align}
L_n^{(s)} \cong L_n^{(r)} &\iff s = r \\
Q_n^{(s)} \cong Q_n^{(r)} &\iff s = r.
\end{align}

\textbf{TL}_6. (see, e.g., corollary 3.8): The link state representation of $\mathbb{T}L_n(\nu)$ on $L_n$ induced by the action

\begin{align}
(T, \alpha) \quad \mapsto \quad T\alpha
\end{align}

for all tangles $T \in \mathbb{T}L_n(\nu)$ and all link states $\alpha \in L_n$ is faithful if and only if $\text{rad} L_n = \{0\}$.

\textbf{TL}_7. \textit{[RSA14, proposition 5.1]}: We have $\dim \text{rad} L_n^{(s)} = D_n^{(s)}$, where $D_n^{(s)}$ is the unique solution to the recursion

\begin{align}
D_n^{(s)} &= \begin{cases}
0, & R_s = 0, \\
D_{n-1}^{(s)} + D_{n-1}^{(s+1)}, & R_s = p(q) - 1, \\
D_{n-1}^{(s)} + D_{n-1}^{(s+1)}, & R_s \in \{1, 2, \ldots, p(q) - 2\},
\end{cases}
\end{align}

In particular, this implies that we have

\begin{align}
\text{rad} L_n^{(s)} = \{0\} &\iff q \in \text{Non}_n^{(s)},
\end{align}

and this in turn implies that $\text{rad} L_n$ is trivial if and only if $q \in \text{Non}_n$. Also, the above implies that

\begin{align}
\text{rad} L_n^{(s)} = L_n^{(s)} &\iff q \in \text{Tot}_n^{(s)}.
\end{align}

\textbf{TL}_8. \textit{[RSA14, theorem 8.1]}: We have the following:

(a): If $\nu = 0$ and $n \in 2\mathbb{Z}_{>0}$, then the collection $\{Q_n^{(s)} \mid s \in E_n, s \neq 0\}$ is the complete set of all simple $\mathbb{T}L_n^{(\nu)}$-
modules.

(b): If $\nu \neq 0$ or $n \notin 2\mathbb{Z}_{>0}$, then the collection $\{Q_n^{(s)} \mid s \in E_n\}$ is the complete set of all simple $\mathbb{T}L_n^{(\nu)}$-
modules.

\textbf{TL}_9. \textit{[RSA14, corollary to theorem 8.1]}: The Temperley-Lieb algebra $\mathbb{T}L_n(\nu)$ is semisimple if and only if $q \in \text{Non}_n$.

Other key results about the Temperley-Lieb algebra, such as a complete classification of its principal indecomposable modules in the nonsemisimple cases, also appear in [RSA14].
B. Restriction: Jones-Wenzl algebra

The purpose of this article is to obtain results analogous to items TL_{n}^{1–9} for a diagram algebra that we view as a natural generalization of the Temperley-Lieb algebra. As we discuss briefly in section 1D, we use this algebra extensively in our forthcoming articles [FP18a+, FP18b+, FP18c+]. We define this diagram algebra as a subalgebra of TL_{n}(\nu), using the “Jones-Wenzl projector of size s” [Wen87, Jon83, KL94], the nonzero tangle P_{(s)} in TL_{s}(\nu) uniquely defined by properties P1 and P2 stated later in section 2C (assuming s < \bar{p}(\nu)). For example, we have

\[ P_{(0)} = \text{the empty tangle}, \quad P_{(1)} = 1_{TL_{1}(\nu)}, \quad P_{(2)} = 1_{TL_{2}(\nu)} - \nu^{-1}U_{1}. \] (1.34)

We use the following diagrammatic representation for the Jones-Wenzl projectors. Within a tangle, we call a collection of s parallel links a “cable of size s,” and we illustrate it as one link with label “s” next to it. For example,

\[ s \{ \quad \text{---} \quad = \quad \text{s} . \] (1.35)

We represent the tangle P_{(s)} as an empty box, called a “projector box,” with a cable of size s passing into either side:

\[ s \{ \quad \text{---} \quad = \quad \text{s} . \] (1.36)

For example, using (1.34), we write the size-two projector box as follows:

\[ \begin{array}{c}
\text{---} \\
\text{---} \\
\text{---} \\
\end{array} = \begin{array}{c}
\text{---} \\
\text{---} \\
\text{---} \\
\end{array} - \nu^{-1} \times \text{---} . \] (1.37)

If s ≤ n, then we may embed a projector box of size s into a tangle in TL_{n}(\nu). For example, the diagram

\[ \begin{array}{c}
i - 1 \\
\text{s} \\
n - s - i + 1 \\
\end{array} \] (1.38)

represents the tangle in TL_{n}(\nu) obtained by replacing the box of size s with the tangle P_{(s)} within the larger diagram.

With \( \varsigma = (s_1, s_2, \ldots, s_d) \) denoting a multiindex of nonnegative entries such that \( s_1 + s_2 + \cdots + s_d = n \) and \( \max \varsigma < \bar{p}(\nu) \), the “Jones-Wenzl algebra” JW_{\varsigma}(\nu) is the collection of all tangles in TL_{n}(\nu) of the form

\[ \begin{array}{c}
s_1 \\
s_2 \\
\vdots \\
s_d \\
\end{array} \quad T \quad \begin{array}{c}
s_1 \\
s_2 \\
\vdots \\
s_d \\
\end{array} \] (1.39)

with \( T \in TL_{n}(\nu) \) (we will see in section 2 that some of these tangles are in fact zero, namely the ones having a link with both endpoints at the same projector box). We call each tangle of the form (1.39) a “\( \varsigma \)-Jones-Wenzl tangle”. Examples of (1,2)-Jones-Wenzl tangles are

\[ \begin{array}{c}
\text{---} \\
\text{---} \\
\end{array}, \quad \begin{array}{c}
\text{---} \\
\end{array}, \quad \text{and} \quad \begin{array}{c}
\text{---} \\
\text{---} \\
\end{array} \begin{array}{c}
\text{---} \\
\end{array} \begin{array}{c}
\text{---} \\
\end{array} = 0. \] (1.40)
The Jones-Wenzl algebra $JW_\varsigma(\nu)$ is an associative subalgebra of $TL_n(\nu)$, with unit given in (2.104) in section 2 F. To investigate the representation theory of the subalgebra $JW_\varsigma(\nu)$, we define $W_\varsigma^{(s)}$ to be the subspace of the module $L_n^{(s)}$ comprising all “$(\varsigma, s)$-Jones-Wenzl link states,” that is, $(n, s)$-link states of the form

$$\alpha$$

with $\alpha \in L_n^{(s)}$. Examples of $((3, 2, 2), 3)$-Jones-Wenzl link states are

$$\text{and}$$

Finally, we define the direct sum $JW_\varsigma(\nu)$-module

$$W_\varsigma := \bigoplus_{s \in E_\varsigma} W_\varsigma^{(s)},$$

where $E_\varsigma$ denotes the set of all integers $s \in \mathbb{Z}_{\geq 0}$ such that the module $W_\varsigma^{(s)}$ is nontrivial. (See section 2, in particular equation (2.25) and lemmas 2.1–2.3, for a complete determination of the set $E_\varsigma$.)

C. Main results: valenced Temperley-Lieb algebra

Instead of the Jones-Wenzl algebra, we usually work with another diagram algebra that is isomorphic to it and that we call the “valenced Temperley Lieb algebra.” To define it, we consider the terminus of a cable of size $s$ within a tangle. This terminus comprises $s$ adjacent nodes, each hosting exactly one endpoint of a link within the cable. From now on, we allow the possibility that multiple links terminate at a common node. We illustrate this as follows:

$$\begin{align*}
\text{\vdots} & \quad \Rightarrow \quad \text{\vdots} \\
\text{\vdots} & \quad \Rightarrow \quad \text{\vdots} \\
\text{\vdots} & \quad \Rightarrow \quad \vdots \\
\end{align*}$$

We call the number $s$ of links anchored to a node the “valence” of that node. We let $\varsigma = (s_1, s_2, \ldots, s_d)$ be a multiindex of nonnegative entries and $s_1 + s_2 + \cdots + s_d = n$. We call every diagram of the form

$$\begin{align*}
\begin{array}{c}
\text{\vdots} \\
\text{\vdots} \\
\text{\vdots} \\
\vdots \\
\text{\vdots} \\
\end{array} \quad \Rightarrow \quad \begin{array}{c}
\text{\vdots} \\
\text{\vdots} \\
\text{\vdots} \\
\vdots \\
\text{\vdots} \\
\end{array} \\
\begin{array}{c}
\text{\vdots} \\
\text{\vdots} \\
\text{\vdots} \\
\vdots \\
\text{\vdots} \\
\end{array} \quad \Rightarrow \quad \begin{array}{c}
\text{\vdots} \\
\text{\vdots} \\
\text{\vdots} \\
\vdots \\
\text{\vdots} \\
\end{array} \\
\begin{array}{c}
\text{\vdots} \\
\text{\vdots} \\
\text{\vdots} \\
\vdots \\
\text{\vdots} \\
\end{array} \\
\begin{array}{c}
\text{\vdots} \\
\text{\vdots} \\
\text{\vdots} \\
\vdots \\
\text{\vdots} \\
\end{array} \\
\end{align*}$$

with $T \in TL_n(\nu)$, a “$\varsigma$-valenced tangle.” Examples of $(1, 2)$-valenced tangles are

$$\text{and}$$

(1.41)

(1.42)

(1.43)

(1.44)

(1.45)

(1.46)
We restrict our attention to \( \varsigma \)-valenced tangles lacking “loop links,” i.e., links with both endpoints at the same node; thus, we regard such \( \varsigma \)-valenced tangles as zero. For example,

\[
\begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array} = 0.
\]

(1.47)

We let \( TL_{\varsigma}(\nu) \) denote the space of all such \( \varsigma \)-valenced tangles. When \( \max \varsigma < \bar{p}(q) \), this space is an associative unital algebra isomorphic to the Jones-Wenzl algebra \( JW_{\varsigma}(\nu) \). The isomorphism from \( JW_{\varsigma}(\nu) \) to \( TL_{\varsigma}(\nu) \) sends each Jones-Wenzl tangle \( 1.39 \) to the corresponding \( \varsigma \)-valenced tangle \( 1.45 \). In particular, the algebra \( TL_{\varsigma}(\nu) \) inherits from \( JW_{\varsigma}(\nu) \) its multiplication by concatenation. We call \( TL_{\varsigma}(\nu) \) the “valenced Temperley-Lieb algebra.”

The isomorphism from \( JW_{\varsigma}(\nu) \) to \( TL_{\varsigma}(\nu) \) is only a cosmetic change. As such, the reader may wonder why do we introduce two notations for what are morally identical algebras. Here are some partial answers to this question:

- We may think of \( JW_{\varsigma}(\nu) \) and \( TL_{\varsigma}(\nu) \) as the collections of all intertwiners of two isomorphic but otherwise different \( U_{q}(sl_{2}) \)-modules. (Here, \( U_{q}(sl_{2}) \) is a certain Hopf algebra \[Jim85, KRT97, FP18a^{+}\].) Because the modules are different, we distinguish their algebras of intertwiners via the modest notation difference from \( 1.39 \) to \( 1.45 \).
- Our work in this article is motivated by a problem in conformal field theory, as we discuss in section 1D. In our application, elements of certain modules of the two algebras \( JW_{\varsigma}(\nu) \) and \( TL_{\varsigma}(\nu) \) are viewed as different correlation functions, ones of which are certain limits of the other ones. We investigate such functions in detail in \[FP18b\].
- Our plan in this article is not to define \( TL_{\varsigma}(\nu) \) from \( JW_{\varsigma}(\nu) \) but to define it formally by suitably generalizing the definition of \( TL_{n}(\nu) \) and then prove that the former diagram algebra is in fact isomorphic to \( JW_{\varsigma}(\nu) \). (See lemma 2.11 in section 2.) We feel that this approach is natural.

The main purpose of this article is to understand the representation theory of the valenced Temperley-Lieb algebra \( TL_{\varsigma}(\nu) \). For this purpose, we construct \( TL_{\varsigma}(\nu) \)-modules \( L_{\varsigma}^{(s)} \), which we call “valenced standard modules,” from the \( JW_{\varsigma}(\nu) \)-modules \( W_{\varsigma}^{(s)} \) via an isomorphism similar to the one described below \( 1.47 \). Using notation \( 1.44 \), we identify each Jones-Wenzl link state \( 1.41 \) associated with \( \alpha \in L_{n}^{(s)} \) with the “\((\varsigma, s)\)-valenced link state”

\[
\begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}
\]

(1.48)

Examples of \((3, 2, 2), 3\)-valenced link states are

\[
\begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}.
\]

(1.49)

Again, we restrict our attention to \((\varsigma, s)\)-valenced link states lacking loop links, and we let \( L_{\varsigma}^{(s)} \) denote the space of all such objects. We also define the “valenced link state module” to be the direct sum

\[
L_{\varsigma} := \bigoplus_{s \in E_{\varsigma}} L_{\varsigma}^{(s)}.
\]

(1.50)

When \( \max \varsigma < \bar{p}(q) \), the space \( L_{\varsigma}^{(s)} \) is a \( TL_{\varsigma}(\nu) \)-module isomorphic to the \( JW_{\varsigma}(\nu) \)-module \( W_{\varsigma}^{(s)} \). The isomorphism from \( L_{\varsigma}^{(s)} \) to \( W_{\varsigma}^{(s)} \) sends each \((\varsigma, s)\)-valenced link state \( 1.48 \) to the corresponding \((\varsigma, s)\)-Jones-Wenzl link state \( 1.41 \). In particular, the module \( L_{\varsigma}^{(s)} \) inherits its \( TL_{\varsigma}(\nu) \)-action from the corresponding \( JW_{\varsigma}(\nu) \)-action on \( W_{\varsigma}^{(s)} \) by concatenation.

By obvious extension, the space \( L_{\varsigma} \) is also a \( TL_{\varsigma}(\nu) \)-module isomorphic to the \( JW_{\varsigma}(\nu) \)-module \( W_{\varsigma} \) whenever \( \max \varsigma < \bar{p}(q) \). The \( TL_{\varsigma}(\nu) \)-module \( L_{\varsigma} \) also inherits a natural bilinear form from the \( JW_{\varsigma}(\nu) \)-module \( W_{\varsigma} \). It is given by

\[
(\alpha \mid \beta) = (\text{preimage of } \alpha \text{ in } W_{\varsigma} \mid \text{preimage of } \beta \text{ in } W_{\varsigma})
\]

(1.51)
for all valenced link states $\alpha, \beta \in L_\varsigma$; the detailed definition is given in section 3 A. For example, we have
\[
\left(\begin{array}{c|c}
\includegraphics[height=1cm]{fig1.png} & \includegraphics[height=1cm]{fig2.png} \\
\end{array}\right) = \includegraphics[height=1cm]{fig3.png} = \includegraphics[height=1cm]{fig4.png}.
\]

We also define the radical of this bilinear form to be the vector space
\[
\text{rad } L_\varsigma := \{ \alpha \in L_\varsigma \mid (\alpha \mid \beta) = 0 \text{ for all } \beta \in L_\varsigma \}.
\]

The goal of this article is to find results for the valenced Temperley-Lieb algebra that generalize the analogous results about the Temperley-Lieb algebra stated as items $\text{TL}_n^1$–$\text{TL}_n^8$ above. The following is a list of our findings:

\begin{enumerate}
\item $\text{Lemma 2.8:}$ We have $\dim L_\varsigma^{(s)} = D_\varsigma^{(s)}$, where $\varsigma = (s_1, s_2, \ldots, s_d)$, $\hat{\varsigma} := (s_1, s_2, \ldots, s_{d-1})$, and $t := s_d$, and $D_\varsigma^{(s)}$ are defined for each $s \in E_\varsigma$ by the recursion
\[
D_\varsigma^{(s)} = \sum_{r \in E_\hat{\varsigma} \cap E_{(s,t)}} D_\varsigma^{(r)}, \quad \text{and} \quad D^{(t)} = 1.
\]

\item $\text{Special case of corollary 2.7:}$ We have $\dim \text{TL}_\varsigma^{(s)} = \sum_{s \in E_\varsigma} \left( \dim L_\varsigma^{(s)} \right)^2$.

\item $\text{Proposition 2.9:}$ Suppose $n < \bar{p}(q)$. Then the unit
\[
1_{\text{TL}_\varsigma^{(s)}} = \includegraphics[height=4cm]{fig5.png}
\]

together with all $\varsigma$-valenced tangles of the form
\[
\includegraphics[height=4cm]{fig6.png}
\]

with $i \in \{1, 2, \ldots, d-1\}$ forms a minimal generating set for the valenced Temperley-Lieb algebra $\text{TL}_\varsigma^{(s)}$.\)
The statements. Proving that this is true is a possible direction for future research. (See also the discussion in section 6.)

... and this in turn implies that \( \text{rad } \mathcal{L}_\xi \) is trivial if and only if \( q \in \text{Non}_\xi := \bigcap_{\alpha \in \mathcal{E}} \text{Non}_\xi^{(s)} \). Also, with the set \( \text{Tot}_{\xi}^{(s)} \) (with Lebesgue measure zero and not dense in \( \mathbb{C} \)) defined via (5.132) in section 5 D, the above implies that

\[
\text{rad } \mathcal{L}_\xi^{(s)} = \mathcal{L}_\xi^{(s)} \iff q \in \text{Tot}_{\xi}^{(s)}. 
\]

D. Motivation: correlation functions of conformal field theory

For us, the main reason to introduce the valenced Temperley-Lieb algebra is its value in applications to conformal field theory (CFT). In this section, we briefly explain the particular application treated in forthcoming work [FP18c+]. Mathematically, many aspects of CFT are still poorly understood and are presently the subject of active research. However, the problem that we are interested in is well-posed and our solution to it is rigorous. We invite the reader to consult the physics literature for background on CFT, for example [DMS97, Hen99, Rib14].

... and where \( \Delta_\xi \) is defined in (1.20) and \( \xi \) and \( t \) are defined in item TLc1. With the set \( \text{Non}_\xi^{(s)} \) (whose complement in \( \mathbb{C} \) has Lebesgue measure zero) defined via (5.114, 5.116) in section 5 C, this implies that

\[
\text{rad } \mathcal{L}_\xi = \{0\} \iff \text{rad } \mathcal{L}_n = \{0\} \iff q \in \text{Non}_{\xi}^{(s)},
\]

We anticipate that, like items TLc8 and TLc9 of section 1 A, items TLc8 and TLc9 actually hold as if-and-only-if statements. Proving that this is true is a possible direction for future research. (See also the discussion in section 6.)

\[
\nu = -2 \cos \left( \frac{4\pi}{\kappa} \right) \iff c = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa},
\]
and we assume that \( \kappa \in (0, 8) \) is irrational (i.e., \( q = \exp(4\pi i/\kappa) \) is not a root of unity). Under this assumption, the radical of any standard module is then trivial by (1.32) from item TL\_7 above.

First, let us consider a special case of the problem we are interested in. We denote by \( \psi_1 \) the spinless primary conformal field whose (holomorphic and antiholomorphic) conformal weight equals the quantity

\[
b_1 := \frac{6 - \kappa}{2\kappa},
\]

that is, the Kac weight \( h_{1,2} \) or \( h_{2,1} \) indexed by the second entry in the first row or column of the Kac table. We consider the following \( n \)-point CFT correlation function, with \( n \in 2\mathbb{Z}_{>0} \):

\[
F_n(z, \bar{z}) = \langle \psi_1(z_1, \bar{z})\psi_1(z_2, \bar{z}) \cdots \psi_1(z_n, \bar{z}_n) \rangle,
\]

where we treat \( z_i \) and \( \bar{z}_i \) as independent variables, rather than complex conjugates. The domain of this function is the set of all points \( (z, \bar{z}) \in \mathbb{C}^n \) with \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \) and \( \bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n) \in \mathbb{C}^n \), where no two coordinates of \( z \) (resp. \( \bar{z} \)) are equal. Correlation functions of type (1.67) should satisfy the following key properties:

1. The following two decoupled systems of partial differential equations of BPZ type [BPZ84]:

\[
\begin{align*}
\frac{\kappa}{4} \Delta F + \sum_{j \neq i} \left( \frac{1}{z_j - z_i} \frac{\partial}{\partial z_j} - \frac{b_1}{(z_j - z_i)^2} \right) F(z, \bar{z}) &= 0, \quad \text{for all } i \in \{1, 2, \ldots, n\}, \tag{1.68}
\end{align*}
\]

\[
\begin{align*}
\frac{\kappa}{4} \Delta F + \sum_{j \neq i} \left( \frac{1}{\bar{z}_j - \bar{z}_i} \frac{\partial}{\partial \bar{z}_j} - \frac{b_1}{(\bar{z}_j - \bar{z}_i)^2} \right) F(z, \bar{z}) &= 0, \quad \text{for all } i \in \{1, 2, \ldots, n\}. \tag{1.69}
\end{align*}
\]

2. Invariance under all monodromy transformations, defined below, and coordinate permutations.

3. Covariance under all conformal transformations \( \varphi : \mathbb{C} \to \mathbb{C} \), that is,

\[
F(\varphi(z), \varphi(\bar{z})) = \left( \prod_{i=1}^{n} \partial \varphi(z_i)^{-b_1} \partial \varphi(\bar{z}_i)^{-b_1} \right) F(z, \bar{z}).
\]

4. There exist numbers \( C, p \in \mathbb{R}_{>0} \) such that the magnitude of correlation function (1.67) is globally bounded by

\[
C \prod_{i<j}^{n} \max \{ |(z_i - z_j)^{p}(\bar{z}_i - \bar{z}_j)^{p}|, |(z_i - z_j)^{-p}(\bar{z}_i - \bar{z}_j)^{-p}| \}.
\]

In item 2 above, we use the convention that the monodromy transformation that winds \( z_j \) counterclockwise around \( z_j \) also simultaneously winds \( \bar{z}_i \) clockwise around \( \bar{z}_j \), as illustrated in the figure in (1.76).

The following question motivates our work in this article, and we answer it in our forthcoming article [FP18c+]:

**Question 1.1.** *What is the dimension of the space of all functions with properties 1-4? Can we find an explicit basis?*

Using representation theory and recent results from the series of articles [FK15a, FK15b, FK15c, FK15d, KP18, KP16], in [FP18c+] we prove that this space is one-dimensional and we obtain an explicit formula for a function that spans it. To find this function, we use \((n,0)\)-link states. First, for each \((n,0)\)-link pattern, we define

\[
\mathcal{H}_n(\alpha)(z) := \left( \prod_{i<j}^{n} (z_i - z_j)^{2/\kappa} \right) \int_{\Gamma_{\alpha}} \left( \prod_{i=1}^{n} (w_i - z_j)^{-4/\kappa} \right) \left( \prod_{1 \leq i < j \leq n/2} (w_i - w_j)^{8/\kappa} \right) dw,
\]

where the integration surface \( \Gamma_{\alpha} \) is the product of simple contours which, after identifying the \( i \)-th node of \( \alpha \) from the left with the coordinate \( z_i \) for each \( i \in \{1, 2, \ldots, n\} \), follow the paths of the links in \( \alpha \): for example,

\[
\alpha = \quad \implies \quad \Gamma_{\alpha} = \quad z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6.
\]

We define the function \( \mathcal{H}_n(\alpha) \) by linearly extending (1.72) from all \((n,0)\)-link patterns \( \alpha \) to all link states \( \alpha \in L(0)_{\alpha} \). All such functions satisfy the system (1.68) and the properties stated in items 3 and 4 with \( \bar{z} \) dropped [Dub06, KP18].

Before continuing, we clarify some technical details that arise with the definition of \( \mathcal{H}_n(\alpha) \):
So far, our choice of the integration surface $\Gamma_\alpha$ makes sense only when $\Re(z_1) < \Re(z_2) < \cdots < \Re(z_n)$. Nevertheless, we can use functions of the form (1.72) to construct a correlation function of type (1.67) that is single-valued when analytically continued from this starting region into its full domain along any path.

- We have not explicitly specified a branch choice for the factors in the integrand of (1.72). However, such a choice affects the function $H_n[\alpha]$ by a single multiplicative factor, which will end up being irrelevant in our application.

- If $\kappa \in (0, 4)$, then the improper integrals in the formula (1.72) for $H_n[\alpha]$ diverge. However, we can renormalize these divergent quantities by replacing their integration contours with Pochhammer contours, without affecting our results. (See appendix A for a definition of this contour, and see [FK15c, section II] for a full explanation.)

Next, we let $\alpha^\vee$ denote the dual of $\alpha$ with respect to bilinear form (1.12) defined on $L_n^{(0)}$ and some specified basis $B$ for this module:

$$(\alpha^\vee \mid \beta) = \delta_{\alpha, \beta} \quad \text{for all } \alpha, \beta \in B. \quad (1.74)$$

The assumption that $\kappa$ is irrational implies, by item TL1,7, that the radical of $L_n^{(0)}$ is trivial. Hence, $\alpha^\vee$ is well-defined.

**Claim 1.2.** The following sum spans the one-dimensional space of functions that satisfy the above properties 1–4:

$$\sum_{\alpha \in B} H_n[\alpha](z) H_n[\alpha^\vee](\bar{z}). \quad (1.75)$$

In [FP18c], we prove claim 1.2 using the key fact [FFK89, MR89, GS90, FW91] that the response of the function $H_n[\alpha](z) H_n[\alpha^\vee](\bar{z})$ to simultaneously permuting its $i$:th and $(i + 1)$:st holomorphic and antiholomorphic coordinates via the half-twists

$$12 \ldots i-1 i i+1 i+2 \ldots n$$

matches the response of the corresponding tensor product link state $\alpha \otimes \alpha^\vee$ to the following actions of $\mathbf{TL}_n(\nu)$: an action on $\alpha$, corresponding to the holomorphic coordinates of $z$, by the “braid generator” tangle

$$\begin{array}{ccc}
1 & 2 & \ldots \\
\vdots & \vdots & \\
i-1 & i & i+1 & i+2 & \ldots & n
\end{array}
\begin{array}{cc}
q^{1/2} & \\
\times & \\
\end{array}
\begin{array}{ccc}
1 & 2 & \ldots \\
\vdots & \vdots & \\
i-1 & i & i+1 & i+2 & \ldots & n
\end{array}
$$

and a simultaneous action on $\alpha^\vee$, corresponding to the antiholomorphic coordinates of $\bar{z}$, by the “inverse braid”

$$\begin{array}{ccc}
1 & 2 & \ldots \\
\vdots & \vdots & \\
i-1 & i & i+1 & i+2 & \ldots & n
\end{array}
\begin{array}{cc}
q^{-1/2} & \\
\times & \\
\end{array}
\begin{array}{ccc}
1 & 2 & \ldots \\
\vdots & \vdots & \\
i-1 & i & i+1 & i+2 & \ldots & n
\end{array}
$$

$$\begin{array}{ccc}
1 & 2 & \ldots \\
\vdots & \vdots & \\
i-1 & i & i+1 & i+2 & \ldots & n
\end{array}
\begin{array}{cc}
q^{1/2} & \\
\times & \\
\end{array}
\begin{array}{ccc}
1 & 2 & \ldots \\
\vdots & \vdots & \\
i-1 & i & i+1 & i+2 & \ldots & n
\end{array}
$$

$$\begin{array}{ccc}
1 & 2 & \ldots \\
\vdots & \vdots & \\
i-1 & i & i+1 & i+2 & \ldots & n
\end{array}
\begin{array}{cc}
q^{-1/2} & \\
\times & \\
\end{array}
\begin{array}{ccc}
1 & 2 & \ldots \\
\vdots & \vdots & \\
i-1 & i & i+1 & i+2 & \ldots & n
\end{array}
$$
Now, a crucial observation is that, by item TLₙ,3, tangles (1.77) or (1.78) generate all of TLₙ(ν). Thus, invariance of (1.75) under monodromy transformations and coordinate permutations is morally equivalent to invariance of its link-state representation

\[ \sum_{\alpha \in B} \alpha \otimes \alpha^\vee \]  

(1.79)

under the corresponding TLₙ(ν)-action on it.

After realizing all monodromy transformations and coordinate permutations of the function \( \mathcal{H}_n[\alpha](z) \mathcal{H}_n[\alpha](\bar{z}) \) as TLₙ(ν)-actions on the corresponding tensor product link state \( \alpha \otimes \alpha^\vee \), we prove in [FP18c⁺] that there is a unique one-dimensional subspace of link states in \( L^{(0)}_{\nu} \otimes L^{(0)}_{\nu} \) that is invariant under the TLₙ(ν)-action on it, spanned by (1.75). This claim provides a key ingredient for concluding that the space of functions satisfying properties 1–4 above is also one-dimensional. Its proof is an almost routine application of Schur’s lemma, and the main detail that allows us to use this lemma is the fact that the standard module \( L^{(0)}_{\nu} \) is simple, as stated above in item 3.

Next, using ideas from the above discussion, we explain how to construct monodromy invariant and coordinate-permutation invariant multi-point CFT correlation functions comprising a more general class of Kac operators. We denote by \( \psi_s \) the spinless primary conformal field whose (holomorphic and antiholomorphic) conformal weight equals

\[ b_s = \frac{s(2s + 4 - \kappa)}{2\kappa}, \]

(1.80)
namely the Kac weight \( h_{1,s+1} \) or \( h_{s+1,1} \) indexed by the \((s + 1)\)th entry in the first row or column of the Kac table. Then, we consider the following \( d \)-point CFT correlation function with respect to the multiindex \( \zeta = (s_1, s_2, \ldots, s_d) \):

\[ \mathcal{C}_i(z, \bar{z}) := \langle \psi_{s_1}(z_1, \bar{z}_1) \psi_{s_2}(z_2, \bar{z}_2) \cdots \psi_{s_d}(z_d, \bar{z}_d) \rangle. \]

(1.81)

This function should satisfy properties similar to 1–4 above, except that in item 1, we replace PDE system (1.68, 1.69) by a more complicated collection of BPZ partial differential equations [BPZ84, BSA88], and in item 3, we replace the conformal weight \( b_1 \) with the more general conformal weight \( b_s \).

In [FP18a⁺, FP18b⁺, FP18c⁺], we study question 1.1 in this more general setting. Again, we find that the space of functions with the desired properties is one-dimensional. A key ingredient to the proof of this is the fact that the standard module \( L^{(0)}_{\nu} \) is simple, as stated above in item 3.

For each nonzero \((\zeta, 0)\)-valenced link pattern \( \alpha \) (defined in section 2 A), where \( \Gamma_\alpha \) is a certain \( \alpha \)-dependent integration contour, and choosing an arbitrary basis \( B \) for \( L^{(0)}_{\nu} \), the following function spans this one-dimensional solution space:

\[ \sum_{\alpha \in B} \mathcal{H}_\alpha[\alpha](z) \mathcal{H}_\alpha[\alpha^\vee](\bar{z}). \]

(1.83)

To prove this claim, we realize the monodromy of the function (1.83) as an appropriate TLₙ(ν)-action on the tensor product valenced link state \( \alpha \otimes \alpha^\vee \), via the generating set from item TLₙ,3 above: e.g., the “braid” tangles

\[ \begin{array}{ccc}
\vdots & \vdots & \vdots \\
\vdots & s_{i-1} & \vdots \\
\vdots & s_i & \vdots \\
\vdots & s_{i+1} & \vdots \\
\vdots & s_{i+2} & \vdots \\
\end{array} \]

(1.84)
with $i \in \{1, 2, \ldots, d - 1\}$ and $c_k \in \mathbb{C}$, give rise to the monodromy of the holomorphic coordinates $z$. We explain this in more detail in forthcoming work [FP18a+].

In this section, we have explained how the determination of a monodromy invariant and coordinate-permutation invariant correlation function (1.81) motivates the work and key results that we present in this article. In the second article [FP18a+] in the series, we discuss the so-called “quantum Schur-Weyl duality” [Jim86, Mar92, MMA92] between the valenced Temperley-Lieb algebra $\mathbb{T}_L(\nu)$ and the Hopf algebra $U_q(\mathfrak{sl}_2)$. This duality endows certain tensor product representations of the latter algebra with a bimodule structure, where the two algebras $U_q(\mathfrak{sl}_2)$ and $\mathbb{T}_L(\nu)$ have commuting actions. From the above discussion, we know that the action of $\mathbb{T}_L(\nu)$ is closely related to the monodromy of correlation function (1.67). In the next article [FP18b+], we discuss solution spaces of the PDE systems of BPZ type. Finally, in the last article [FP18c+], we use the results of the articles [FP18a+] and [FP18b+] together with the results of the present article to prove claim of type 1.2 for any multiindex $\zeta \in \{(0)\} \cup \mathbb{Z}^\#$.

E. Organization

In section 2, we define the diagram algebras which are the principal concern of this article, introduce notations, and present some key results. In detail, in section 2 A we define valenced link diagrams, tangles, link patterns, and link spaces, and the diagram spaces $\mathbb{T}_L(\nu)$ and $\mathbb{L}_L(\nu)$. In the next section 2 B, we collect simple but useful observations of combinatorial nature. Section 2 C contains the definition of the Jones-Wenzl projectors. In section 2 D (and appendix F) we briefly discuss a categorical framework for the diagram spaces $\mathbb{T}_L(\nu)$, realizing their elements as morphisms of a certain tensor category, with composition by diagram concatenation. Section 2 E concerns the valenced Temperley-Lieb algebra $\mathbb{T}_L(\nu)$, which is central to this article. A key result of section 2 E is proposition 2.9, which gives a minimal generating set for the algebra $\mathbb{T}_L(\nu)$. We prove this result in appendix D. Finally, in section 2 F we discuss the Jones-Wenzl algebra $\mathbb{JW}_L(\nu)$, a subalgebra of $\mathbb{T}_L(\nu)$ which is isomorphic to $\mathbb{T}_L(\nu)$. In appendix C we discuss the features of the isomorphism between the two algebras $\mathbb{T}_L(\nu)$ and $\mathbb{JW}_L(\nu)$ in detail.

In section 3, we focus on the representation theory of the valenced Temperley-Lieb algebra $\mathbb{T}_L(\nu)$. Much of this theory depends on a certain symmetric, invariant bilinear form of the link state $\mathbb{T}_L(\nu)$-module $\mathbb{L}_L(\nu)$. Understanding this bilinear form, and in particular its radical, is a major undertaking of this article. We define the bilinear form in section 3 A via network evaluations from Temperley-Lieb recoupling theory [KL94]. In section 3 B, we present some fundamental properties of the standard $\mathbb{T}_L(\nu)$-modules $\mathbb{L}_L(\nu)$. In particular, in proposition 3.3 we show that generically, these modules are indecomposable and that their quotients by their radicals are simple. Moreover, in proposition 3.6 we show that generically, no two standard modules, and no two of their quotients, are isomorphic. Finally, in corollary 3.8 in section 3 C we prove that the link state representation of $\mathbb{T}_L(\nu)$ on $\mathbb{L}_L(\nu)$ is faithful if and only if the radical of the latter is trivial.

In section 4, we study the Gram matrix $\mathcal{G}_L(\nu)$ of the bilinear form restricted to the standard module $\mathbb{L}_L(\nu)$, with respect to the basis $\mathbb{LP}_L(\nu)$ of $(\zeta, s)$-valenced link patterns (defined in section 2 A). Our motivation to consider $\mathcal{G}_L(\nu)$ is the fact that fundamental properties of the radical of $\mathbb{L}_L(\nu)$, such as its dimension, and fundamental properties of the Gram matrix, such as its nullity, are often interdependent; understanding the latter gives useful information about the former. In section 4 A, we define “trivalent link states” (which correspond to conformal blocks in CFT, via the “spin-chain Coulomb gas map” [FP18b+]), and in section 4 B we disseminate their key properties. One of these properties, stated in proposition 4.6, is that the trivalent link states form an orthogonal basis if, for example, $q \in \mathbb{C}^\times$ in (1.18) is not a root of unity. Then, in proposition 4.8 in section 4 C we make use of the orthogonality property to give an explicit formula for the determinant of $\mathcal{G}_L(\nu)$. Finally, in section 4 D we present recursions and useful formulas for the determinant of this Gram matrix.

In section 5, we completely determine a basis for and the dimension of the radical of the standard module $\mathbb{L}_L(\nu)$ with respect to the bilinear form from section 3 A. From the explicit formula for the determinant of the Gram matrix $\mathcal{G}_L(\nu)$, found in the previous section 4, it follows that this radical is trivial if, for example, $q \in \mathbb{C}^\times$ in (1.18) is not a root of unity. Therefore, we assume that $q$ is a root of unity throughout section 5. In section 5 A, we derive a basis for and the dimension of each of the standard modules $\mathbb{L}_L(\nu)$ (proposition 5.7 and corollary 5.9), and in section 5 B for each of the standard modules $\mathbb{L}_L(\nu)$ for any multiindex $\zeta \in \{(0)\} \cup \mathbb{Z}^\#$ (proposition 5.18 and corollary 5.20). Using these results, we determine in section 5 C all values of $q \in \mathbb{C}^\times$ for which the radical of $\mathbb{L}_L(\nu)$ is trivial (corollary 5.23). Finally, in section 5 D we study cases where the radical of $\mathbb{L}_L(\nu)$ equals the entire space, a phenomenon outlawed as a precondition to most results presented in section 3.

In the final section 6, we present a condition sufficient to guarantee that the valenced Temperley-Lieb algebra $\mathbb{T}_L(\nu)$ is semisimple and that the collection of all standard modules $\mathbb{L}_L(\nu)$ is, in fact, the complete set of all simple $\mathbb{T}_L(\nu)$-
modules. We state these results in section 6 A as theorem 6.2 and proposition 6.1 respectively. Using proposition 6.4, which is an important result borrowed from [RSA14], we present in section 6 B a necessary and sufficient condition for the semisimplicity of $\text{TL}_n(\nu)$. This last result, stated as theorem 6.5 in this article, is implied, e.g., by results in [RSA14, GL98] and ostensibly well-known. However, to our knowledge, it is not correctly stated anywhere in the literature. (See section 1 F for more on this.)

In the appendices, we give background and proofs for technical results needed in this article, which are not so useful to include in the bulk. Some of these results already appear in the literature. Nevertheless, we present them here either for clarity, to stay self-contained, or because we believe that we have a new and interesting proof. First, in appendix A we derive explicit formulas for all coefficients of Jones-Wenzl projectors by computing entries in the inverse of the meander matrix [DGG97]. Our technique for this is to manipulate integration contours of Coulomb-gas-type integral formulas in the spirit of [FK97, FKK98, FK15c, FK15d]. Next, in appendix B we present known results from Temperley-Lieb recoupling theory [KL94, CFS95], for use throughout this article. In appendix C we discuss the relation of the two algebras $\text{TL}_n(\nu)$ and $\text{JW}_n(\nu)$, and in appendix D we prove proposition 2.12 concerning the generators of these algebras. Although it is long, we intend that appendix to be accessible because this result, while not central to this article, supports some of our main results in the sequel [FP18a+]. In appendix E, we address a technical detail concerning the definition of the trivalent link states at roots of unity. Finally, in appendix F, we briefly discuss the categorical framework for the diagram algebras.

F. Relation to previous work

In this section, we survey results that are similar to those in this article and that already appear in the literature.

In the work [GL96, GL98] of J. Graham and G. Lehrer, the authors develop and use a general theory of “cellular algebras,” a powerful category-theoretic approach for obtaining strong results about the representation theory of diagram algebras that include the Temperley-Lieb algebra $\text{TL}_n(\nu)$ as a special case. Among other results, the authors prove that all simple $\text{TL}_n(\nu)$-modules are quotients of the standard modules by the radical of the link state bilinear form and that all such quotients are simple. They also obtain explicit dimensions of these modules. In recent work [ILZ17], K. Iohara, G. Lehrer, and R. Zhang consider a semisimple quotient of the Temperley-Lieb algebra $\text{TL}_n(\nu)$ when $q$ in (1.19) is a root of unity and relate it to a fusion category of certain quantum $\mathfrak{sl}_2$-tilting modules.

In the article [RSA14] of D. Ridout and Y. Saint-Aubin, similar results are obtained with more concrete but related techniques, previously used also in the article [Wes95] of B. Westbury (see also the earlier paper [JM79] of G. James and G. Murphy). In particular, in [RSA14], the authors identify all simple and, at roots of unity, principal indecomposable modules of the Temperley-Lieb algebra.

In our work, we follow the concrete approach of [RSA14]. Combining ideas from [Wes95, GL98, RSA14] with explicit diagram calculations of Temperley-Lieb recoupling theory of Kauffman [KL94, CFS95], our approach provides elementary tools to understand the representation theory of the valenced Temperley-Lieb algebra.

We also mention another abstract but very powerful approach for analyzing representation theory, known as “categorification.” (See e.g. [BFK99, FKS06, CK12, FSS12, SS14] and references therein.) The rough idea is to associate to an algebra a category with certain properties and to use this abstract framework to study the representation theory of this algebra. From this approach, very general results follow in an elegant way that avoids explicit calculations.

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2. Diagram Algebras

In this section, we give some fundamental definitions and prove some fundamental results concerning the diagram algebras that we use throughout this article and the sequel [FP18a]. In section 2 A, we define valenced link diagrams, valenced tangles, valenced link patterns, and valenced link states. In section 2 B, we collect key results about them, of a combinatorial nature, for use throughout this article. Then, in section 2 C, we define the Jones-Wenzl projectors, and in section 2 D, we use these projectors to define bilinear maps via concatenation of diagrams. In section 2 E, we define and discuss the valenced Temperley-Lieb algebra $\mathcal{TL}_\varsigma(\nu)$, the main object of study of this article. Finally, in section 2 F, we define and discuss the Jones-Wenzl algebra $\mathcal{JW}_\varsigma(\nu)$ and prove that the two algebras $\mathcal{JW}_\varsigma(\nu)$ and $\mathcal{TL}_\varsigma(\nu)$ are isomorphic. We give minimal generating sets for these two algebras in propositions 2.9 and 2.12 respectively.

A. Valenced tangles and link states

In this section, we formally define valenced link diagrams, valenced tangles, valenced link patterns, and valenced link states. For this purpose, we denote

$$Z_{\geq 0}^\# := \mathbb{Z}_{>0} \cup \mathbb{Z}_{>0}^2 \cup \mathbb{Z}_{>0}^3 \cup \cdots,$$

and we let $\varsigma$ and $\wp$ denote two multiindices with $d_\varsigma$ and $d_{\wp}$ nonnegative integer entries respectively,

$$\varsigma = (s_1, s_2, \ldots, s_{d_\varsigma}) \in \{(0)\} \cup Z_{\geq 0}^\#,$$

and such that $n_\varsigma + n_{\wp} = 0 \pmod{2}$, where $n_\varsigma$ and $n_{\wp}$ denote the respective sums

$$n_\varsigma := s_1 + s_2 + \cdots + s_{d_\varsigma}, \quad n_{\wp} := p_1 + p_2 + \cdots + p_{d_{\wp}}.$$

Then we define a $(\varsigma, \wp)$-valenced link diagram to be any collection of the following planar geometric objects:

1. two vertical lines,
2. $d_\varsigma$ distinct marked points, called nodes, on the left line and $d_{\wp}$ nodes on the right line, and
3. $\frac{1}{2}(n_\varsigma + n_{\wp})$ planar curves, called links, that may intersect themselves or each other only at their endpoints and arranged such that $s_i$ (resp. $p_j$) endpoints reside at the $i$:th left (resp. $j$:th right) node. Each link is determined only up to a homotopy that preserves its endpoints.

We call the $i$:th entry $s_i$ of $\varsigma$ (resp. $j$:th entry $p_j$ of $\wp$) the valence of the $i$:th left (resp. $j$:th right) node. As in (1.44), we illustrate a node with valence $s$ as a small box that sits over the node itself, with a cable of size $s$ exiting it:

$$s \quad \Rightarrow \quad \begin{array}{c} \cdots \\ \cdots \\ \cdots \end{array} \quad s \quad \Rightarrow \quad \begin{array}{c} \cdots \end{array} \quad s \quad \Rightarrow \quad \begin{array}{c} \cdots \end{array}. \quad (2.4)$$

As we do with ordinary link diagrams, we sort the links of each valenced link diagram into two types, called “crossing links” and “turn-back links”, as follows:

$$\text{crossing link} \quad \Rightarrow \quad \text{turn-back link}. \quad (2.5)$$

Also, we define a loop link to be a link with both endpoints at the same node:

$$\text{loop link}. \quad (2.6)$$
Now, for any two multiindices \( \varsigma, \varpi \in \{(0)\} \cup \mathbb{Z}_{>0} \), we define

\[
\text{pre-LD}_\varsigma^\varpi := \text{the collection of all } (\varsigma, \varpi)\text{-valenced link diagrams}, \tag{2.7}
\]
\[
\text{pre-TL}_\varsigma^\varpi := \text{the complex vector space of all formal linear combinations of } (\varsigma, \varpi)\text{-valenced link diagrams}, \tag{2.8}
\]

and we call an element of \( \text{pre-TL}_\varsigma^\varpi \) a \((\varsigma, \varpi)\)-valenced tangle. Hence, \( \text{pre-LD}_\varsigma^\varpi \) is a basis for \( \text{pre-TL}_\varsigma^\varpi \). Rather than working with the vector space \( \text{pre-TL}_\varsigma^\varpi \), we exclude diagrams containing loop links. Thus, we define

\[
\LD_\varsigma^\varpi := \text{the collection of all } (\varsigma, \varpi)\text{-valenced link diagrams that do not have a loop link}, \tag{2.9}
\]
\[
\TL_\varsigma^\varpi := \text{span } \LD_\varsigma^\varpi. \tag{2.10}
\]

It is also useful to distinguish diagrams with a given number of crossing links:

\[
\LD_\varsigma^{\varpi:(s)} := \{ \text{all valenced link diagrams in } \LD_\varsigma^\varpi \text{ with exactly } s \text{ crossing links} \}, \tag{2.11}
\]
\[
\TL_\varsigma^{\varpi:(s)} := \text{span } \LD_\varsigma^{\varpi:(s)}. \tag{2.12}
\]

With these definitions, we have the \( s \)-grading

\[
\TL_\varsigma^\varpi = \bigoplus_{s \in \mathbb{Z}_\varsigma^\varpi} \TL_\varsigma^{\varpi:(s)}. \tag{2.13}
\]

Next, we define a \((\varsigma, s)\)-valenced link pattern to be any planar geometric object formed by

1. dividing a \((\varsigma, \varpi)\)-valenced link diagram with exactly \( s \) crossing links in half,
2. discarding the right half, and
3. rotating the left half by \( \pi/2 \) radians.

The division breaks the \( s \) crossing link in half, resulting in a valenced link pattern with \( s \) defects and a number of links. For example, we have

\[
\begin{array}{c|c}
\text{\((\varsigma, \varpi)\)-valenced link diagram} & \rightarrow & \text{\((\varsigma, s)\)-valenced link pattern.} \\
\end{array} \tag{2.14}
\]

Now, for any multiindex \( \varsigma \in \{(0)\} \cup \mathbb{Z}_{\geq 0} \) and \( s \in \mathbb{Z}_{\geq 0} \) such that there exists a \((\varsigma, s)\)-valenced link pattern, we set

\[
\text{pre-LP}_\varsigma^{(s)} := \text{the collection of all } (\varsigma, s)\text{-valenced link patterns}, \tag{2.15}
\]
\[
\text{pre-L}_\varsigma^{(s)} := \text{the complex vector space of all formal linear combinations of } (\varsigma, s)\text{-valenced link patterns}, \tag{2.16}
\]

and we call an element of \( \text{pre-L}_\varsigma^{(s)} \) a \((\varsigma, s)\)-valenced link state. As before, rather than working with the vector space \( \text{pre-L}_\varsigma^{(s)} \), we exclude link states containing loop links, such as

\[
\text{loop link} \quad \begin{array}{c}
\text{\((\varsigma, s)\)-valenced link state.} \\
\end{array} \tag{2.17}
\]

Thus, we define

\[
\LP_\varsigma^{(s)} := \text{the collection of all } (\varsigma, s)\text{-valenced link patterns that do not have a loop link}, \tag{2.18}
\]
\[
L_\varsigma^{(s)} := \text{span } \LP_\varsigma^{(s)}. \tag{2.19}
\]

We also denote

\[
\LP_\varsigma := \bigcup_{s \in \mathbb{E}_\varsigma} \LP_\varsigma^{(s)} \quad \text{and} \quad L_\varsigma := \bigoplus_{s \in \mathbb{E}_\varsigma} L_\varsigma^{(s)}. \tag{2.20}
\]
In both cases, we omit the arrow over the multiindex \( \vec{n} \) whenever it appears as a superscript or subscript. We also notice that if \( \vec{\zeta} = \vec{n} \) and \( \vec{\varpi} = \vec{m} \) for some \( n, m \in \mathbb{Z}_{\geq 0} \), then the links join the nodes pairwise, so we have

\[
\text{LD}_n^m = \text{pre}-\text{LD}_n^m, \quad \text{TL}_n^m = \text{pre}-\text{TL}_n^m, \quad \text{LP}_n^s = \text{pre}-\text{LP}_n^s, \quad \text{and} \quad \text{L}_n^s = \text{pre}-\text{L}_n^s. \tag{2.22}
\]

We call elements of \( \text{LD}_n^m \) and \( \text{TL}_n^m \) \((n, m)\)-link diagrams and \((n, m)\)-tangles, respectively. From section 1A, we recall that elements of \( \text{LP}_n^s \) and \( \text{L}_n^s \) are called \((n, s)\)-link patterns” and “\((n, s)\)-link states,” and that \((n, n)\)-link diagrams and \((n, n)\)-tangles are called “\(n\)-link diagrams” and “\(n\)-tangles,” respectively.

### B. Basic combinatorial properties

With valenced link diagrams and link patterns defined, we next study their combinatorial properties (excluding all link diagrams and patterns containing loop links (2.6)). First, for all multiindices \( \varsigma \in \{0\} \cup \mathbb{Z}_{\geq 0}^\# \), we define

\[
\mathcal{E}_\varsigma := \{ s \in \mathbb{Z}_{\geq 0} \mid \text{there exists a} \ (\varsigma, s)-\text{valenced link pattern} \ \alpha \ \text{in} \ \text{LP}_\varsigma^s \}. \tag{2.23}
\]

It is also useful to extend the definition of \( \mathcal{E}_\varsigma \) to include all multiindices \( \varsigma \) with some zero entries: denoting

\[
\mathbb{Z}_{\geq 0}^\# := \mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{\geq 0}^2 \cup \mathbb{Z}_{\geq 0}^3 \cup \cdots,
\]

we recursively define \( \mathcal{E}_\varsigma \) for any multiindex \( \varsigma \in \mathbb{Z}_{\geq 0}^\# \) to be the set \( \mathcal{E}_{\vec{\vartheta}} \), where \( \vec{\vartheta} \) is any multiindex obtained by dropping a zero entry from \( \varsigma \). We also observe that in the special case of \( \varsigma = \vec{n} \), definition (2.23) for \( \mathcal{E}_n \) agrees with that (1.7) given in section 1A.

By breaking links into pairs of defects, it becomes evident that there are integers \( s_{\min}(\varsigma), s_{\max}(\varsigma) \in \mathbb{Z}_{\geq 0} \) such that

\[
\mathcal{E}_\varsigma = \{ s_{\min}(\varsigma), s_{\min}(\varsigma) + 2, \ldots, s_{\max}(\varsigma) \}. \tag{2.25}
\]

Hence, to determine this set, it suffices to find its extreme values. We establish this in lemmas 2.1 and 2.3 below.

**Lemma 2.1.** Suppose \( r, s, t \in \mathbb{Z}_{\geq 0} \). Then we have

\[
\mathcal{E}_{(s)} = \{ s \} \quad \text{and} \quad \mathcal{E}_{(r,t)} = \{ |r - t|, |r - t| + 2, \ldots, r + t \}. \tag{2.26}
\]

Furthermore, we have the symmetry relations

\[
s \in \mathcal{E}_{(r,t)} \iff r \in \mathcal{E}_{(s,t)} \iff t \in \mathcal{E}_{(s,r)}. \tag{2.27}
\]

**Proof.** The proof of (2.26) is a simple combinatorial exercise. Then (2.27) immediately follows from (2.26). \( \square \)

In the next lemma, we give a recursion formula for the set \( \mathcal{E}_\varsigma \). To state the recursion, we use the following notation, frequently appearing throughout this article:

\[
\varsigma = (s_1, s_2, \ldots, s_{d_\zeta}) \quad \Rightarrow \quad \hat{\varsigma} := (s_1, s_2, \ldots, s_{d_\zeta} - 1) \quad \text{and} \quad t = s_{d_\zeta}. \tag{2.28}
\]

**Lemma 2.2.** We have the recursion

\[
\mathcal{E}_\varsigma = \bigcup_{r \in \mathcal{E}_\varsigma} \mathcal{E}_{(r,t)}. \tag{2.29}
\]
Proof. For each \( s \in E \), we may write any valenced link pattern \( \alpha \in \text{LP}_\varsigma \) in the form
\[
\text{(2.30)}
\]
for a unique valenced link pattern \( \hat{\alpha} \in \text{LP}_\hat{\varsigma} \) that depends on \( \alpha \). With \( r \) defects leaving \( \hat{\alpha} \), we must have \( r \in E \). Furthermore, after removing \( \hat{\alpha} \) from this valenced link pattern, we obtain the simpler \((r, t, s)\)-valenced link pattern
\[
\text{(2.31)}
\]
whose existence implies that \( s \in E_{(r, t)} \). Altogether, we thus have
\[
\text{(2.32)}
\]
On the other hand, let \( s \in E_{(r, t)} \) for some \( r \in E \). Then, inserting into (2.30) any valenced link pattern \( \hat{\alpha} \in \text{LP}_{\hat{\varsigma}} \) determines a unique valenced link pattern \( \alpha \in \text{LP}_\varsigma \). This shows that \( s \in E \), so
\[
\text{(2.33)}
\]
This finishes the proof.

Now we are ready to determine \( s_{\text{max}}(\varsigma) \) and present a recursion formula that determines \( s_{\text{min}}(\varsigma) \). To state the latter formula and for later use throughout this article, it is useful to split \( \varsigma \) into two parts thus:
\[
\hat{\varsigma}_i := (s_1, s_2, \ldots, s_i), \quad \tilde{\varsigma}_i := (s_i, s_{i+1}, \ldots, s_d \varsigma).
\]

Lemma 2.3. Let \( \varsigma = (s_1, s_2, \ldots, s_d \varsigma) \). Then

1. we have
\[
s_{\text{max}}(\varsigma) = n_{\varsigma} := s_1 + s_2 + \cdots + s_d \varsigma, \quad \text{(2.35)}
\]
2. and for each \( i \in \{1, 2, \ldots, d_{\varsigma} - 1\} \), we have the recursion
\[
s_{\text{min}}(\hat{\varsigma}_1) = s_1, \quad s_{\text{min}}(\hat{\varsigma}_{i+1}) = \begin{cases} 
\min \{s_{\text{min}}(\hat{\varsigma}_i) - s_{i+1}, s_i + 1 \} & s_i + 1 \leq s_{\text{min}}(\hat{\varsigma}_i), \\
(s_{\text{min}}(\hat{\varsigma}_i) - s_{i+1}) \mod 2, & s_{\text{min}}(\hat{\varsigma}_i) < s_i + 1 < s_{\text{max}}(\hat{\varsigma}_i), \\
(s_{\text{min}}(\hat{\varsigma}_i) - s_{i+1}) & s_{\text{max}}(\hat{\varsigma}_i) \leq s_i + 1.
\end{cases}
\quad \text{(2.36)}
\]
In particular, with \( s_{\text{min}}(\varsigma) = s_{\text{min}}(\tilde{\varsigma}_{d_{\varsigma}}) \), this recursion formula with \( i = d_{\varsigma} - 1 \) determines \( s_{\text{min}}(\varsigma) \).

Proof. Item 1 immediately follows from considering the \((\varsigma, n_{\varsigma})\)-valenced link state with only defects and no links. For item 2, we observe that by lemma 2.2, we have
\[
s_{\text{min}}(\hat{\varsigma}_{i+1}) = \min_{r \in E_{\hat{\varsigma}_i}} \min_{E_{(r, s_{i+1})}} = \min_{r \in E_{\hat{\varsigma}_i}} |r - s_{i+1}|.
\]
Because \( E_{\hat{\varsigma}_i} \) has the form (2.25), this result simplifies to the right side of (2.36).
We note that using the form (2.25) for $E_\varsigma$, it is straightforward to show that lemma 2.3 implies that
\[ E_\varsigma \subset E_{n_\varsigma}. \] (2.38)

We conclude our investigation of $s_{\min}(\varsigma)$ with the following lemma, which we use in section 5D and appendix D.

**Lemma 2.4.** Suppose $\alpha \in \text{LP}^{(s)}_\varsigma$ with $s = s_{\min}(\varsigma)$. Then

1. all $s_{\min}(\varsigma)$ defects of $\alpha$ attach to a common node of $\alpha$, and
2. if all defects of $\alpha$ attach to its $i$:th node, then $s_{\min}(\varsigma) < s_i$ if $d_\varsigma > 1$, and $s_{\min}(\varsigma) = s_1$ if $d_\varsigma = 1$.

In particular, items 1 and 2 together imply that
\[ s_{\min}(\varsigma) \begin{cases} < \max \varsigma, & d_\varsigma > 1, \\ = \max \varsigma, & d_\varsigma = 1. \end{cases} \] (2.39)

**Proof.** To prove item 1, we assume the contrary. Then, replacing two adjacent defects attached to different boxes by a link creates a valenced link pattern with fewer than $s_{\min}(\varsigma)$ links, contradicting the definition of $s_{\min}(\varsigma)$:
\[ \longrightarrow \]
(2.40)
This proves item 1 by contradiction.

Item 2 is straightforward to prove for $d_\varsigma \in \{1, 2\}$. For $d_\varsigma \geq 3$, we prove item 2 by contradiction. Thus, we assume the contrary: $s_{\min}(\varsigma) = s_i$. With no defects attached to the other boxes, we may replace (here, $i = 1$)
\[ \longrightarrow \]
(2.41)
creating a valenced link pattern with $s_{\min}(\varsigma)$ defects not all attached to a common box. This contradicts item 1. \(\square\)

Next, in lemmas 2.5 and 2.6, we obtain two isomorphisms relating the vector space $TL^{(\varsigma)}$ with spaces of link states. Using them, in corollary 2.7 we determine the dimension of $TL^{(\varsigma)}$. Finally, in lemma 2.8, we find a recursion formula for counting $(\varsigma, s)$-valenced link patterns.

We let $\tilde{\alpha}$ denote the valenced link state obtained by reflecting $\alpha$ about a vertical axis. For example,
\[ \alpha \quad \Rightarrow \quad \tilde{\alpha} \]
(2.42)
Thus, if $\alpha \in L_\varsigma$, then $\tilde{\alpha} \in L_\varsigma$, where the multiindex $\varsigma$ is given by
\[ \varsigma = (s_1, s_2, \ldots, s_{d_\varsigma-1}, s_{d_\varsigma}) \quad \Rightarrow \quad \tilde{\varsigma} := (s_{d_\varsigma}, s_{d_\varsigma-1}, \ldots, s_2, s_1). \] (2.43)
We note that
\[ \dim L^{(s)}_\varsigma = \dim L^{(s)}_{\tilde{\varsigma}} \quad \text{and} \quad E_{\varsigma} = E_{\tilde{\varsigma}}. \] (2.44)

Above, we construct valenced link patterns from valenced link diagrams. Conversely, we may construct any $(\varsigma, \pi)$-valenced link diagram with $s$ crossing links from a $(\varsigma, s)$-valenced link pattern $\alpha$ and a $(\pi, s)$-valenced link pattern $\beta$ as follows:

1. we flip $\beta \mapsto \tilde{\beta}$,
2. we position $\alpha$ to the left of $\tilde{\beta}$ in the plane,
3. we rotate $\alpha$ and $\tilde{\beta}$ by $-\pi/2$ and $\pi/2$ radians respectively, and
4. we join the $s$ defects of $\alpha$ and $\tilde{\beta}$ together pairwise top-to-bottom.
We denote the \((\varsigma, \wp)\)-valenced link diagram thus obtained by \( | \alpha \beta | \): \[
\begin{align*}
| \alpha \beta | & = \\
\begin{array}{c}
s_1 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
p_{4m} \\
p_{4m-1} \\
p_2 \\
p_1
\end{array}
\end{align*}
\] (2.45)

For example, we have
\[
\begin{align*}
\alpha & \in \text{LP}(3)_{(1,1,1,1)} \\
\beta & \in \text{LP}(3)_{(2,1,1,1)} \\
\rightarrow & \quad \begin{array}{c}
s_1 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
p_{4m} \\
p_{4m-1} \\
p_2 \\
p_1
\end{array}
\end{align*}
\] (2.46)

We note that a \((\varsigma, \wp)\)-valenced link diagram with \(s\) crossing links and no loop links exists if and only if the set \(E_{\wp \varsigma} := E_{\varsigma} \cap E_{\wp}\) is not empty. We can extend this recipe to establish an isomorphism of vector spaces from \(L_{\varsigma}^{(\varsigma)} \otimes L_{\wp}^{(\wp)}\) to \(\text{TL}_{\varsigma}^{\wp}\).

**Lemma 2.5.** The linear map \( | \cdot \cdot | : \bigoplus_{s \in E_{\wp}} L_{\varsigma}^{(\varsigma)} \otimes L_{\wp}^{(\wp)} \rightarrow \text{TL}_{\varsigma}^{\wp}\) defined by linear extension of \(\alpha \otimes \beta \rightarrow | \alpha \beta |\), for all valenced link patterns \(\alpha \in \text{LP}^{(\varsigma)}\) and \(\beta \in \text{LP}^{(\wp)}\), is an isomorphism of vector spaces.

**Proof.** This map sends the collection \(\{ \alpha \otimes \beta | s \in E_{\wp}, \alpha \in \text{LP}^{(\varsigma)}, \beta \in \text{LP}^{(\wp)} \}\), which is a basis for its domain, to \(\text{LD}_{\varsigma}^{\wp} = \{ | \alpha \beta | | s \in E_{\wp}, \alpha \in \text{LP}^{(\varsigma)}, \beta \in \text{LP}^{(\wp)} \}\), which is a basis for its codomain. Hence, the claim follows. \(\square\)

With notation (2.2), we let \(\oplus\) denote the operation that concatenates two multiindices \(\varsigma, \wp \in \{\bar{0}\} \cup \mathbb{Z}^\#_{>0}\),
\[
\begin{cases}
\varsigma \oplus \bar{0} := \varsigma, \\
\bar{0} \oplus \wp := \wp, \\
\varsigma \oplus \wp := (s_1, s_2, \ldots, s_d, p_1, p_2, \ldots, p_d).
\end{cases}
\] (2.50)

Now, we obtain another useful isomorphism, from \(\text{TL}_{\varsigma}^{\wp}\) to \(L_{\varsigma \oplus \wp}^{(0)}\), by sending each valenced tangle \( | \alpha \beta | \in \text{TL}_{\varsigma}^{\wp}\) to a valenced link state in \(L_{\varsigma \oplus \wp}^{(0)}\) formed by joining all of the \(s \in E_{\wp}\) defects of \(\alpha \otimes \beta \in L_{\varsigma}^{(\varsigma)} \otimes L_{\wp}^{(\wp)}\) together:
\[
\begin{array}{c}
| \alpha \beta | \rightarrow \\
\begin{array}{c}
s_1 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
p_{4m} \\
p_{4m-1} \\
p_2 \\
p_1
\end{array}
\end{array}
\] (2.51)

**Lemma 2.6.** The following map is an isomorphism of vector spaces from \(\text{TL}_{\varsigma}^{\wp}\) to \(L_{\varsigma \oplus \wp}^{(0)}\):
\[
| \alpha \beta | \rightarrow \\
\begin{array}{c}
\begin{array}{c}
\alpha \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
p_{4m} \\
p_{4m-1} \\
p_2 \\
p_1
\end{array}
\end{array}
\] (2.52)
Proof. The map (2.52) from valenced link diagrams in \( \text{LD}_\xi^\varpi \), a basis for \( \text{TL}_\xi^\varpi \), to valenced link patterns in \( \text{LP}_{\xi \oplus \varpi}^{(0)} \), a basis for \( \text{L}_{\xi \oplus \varpi}^{(0)} \), is a bijection, so the claim follows.

Among other uses in section 3, we may use valenced link patterns to find the dimension of the vector space \( \text{TL}_\xi^\varpi \).

Corollary 2.7. We have

\[
\dim \text{L}_{\xi \oplus \varpi}^{(0)} = \dim \text{TL}_\xi^\varpi = \sum_{s \in \mathbb{E}_\xi^\varpi} \left( (\dim L_\xi^{(s)}) (\dim L_\varpi^{(s)}) \right). \tag{2.53}
\]

Proof. The first equality follows from lemma 2.6 with (2.44), and the second equality follows from lemma 2.5.

Thus, to determine the dimension of \( \text{TL}_\xi^\varpi \), it is sufficient to determine the dimensions of the vector spaces \( L_\xi^{(s)} \). Using the notation introduced in (2.28), we define the numbers \( D_\xi^{(s)} \) to be the unique solution to the recursion

\[
D_\xi^{(s)} = \sum_{r \in \mathbb{E}_\xi \cap \mathbb{E}_{t \oplus \varpi}} D_\xi^{(s-r)}, \quad \text{and} \quad D_\xi^{(0)} = 1. \tag{2.54}
\]

In lemma 2.8, we show that \( D_\xi^{(s)} \) equals the dimension of the vector space \( L_\xi^{(s)} \).

In the special case that \( \xi = \tilde{n} \) for some integer \( n \in \mathbb{Z}_{\geq 0} \), we denote these numbers by \( D_\tilde{n}^{(n)} := D_\tilde{n}^{(n)} \). Then for all integers \( s \in \mathbb{E}_\tilde{n} \) (so \( n \) and \( s \) have the same parity), recursion (2.54) reduces to

\[
D_\tilde{n}^{(n)} = \sum_{r \in \mathbb{E}_{n-1} \cap \mathbb{E}_{t \oplus \varpi}} D_\tilde{n}^{(n-r)} = \begin{cases} D_\tilde{n}^{(n-1)}, & s = 0, \\ D_\tilde{n}^{(n-1)} + D_\tilde{n}^{(n+1)}, & s \in \{1, 2, \ldots, n-1\}, \\ D_\tilde{n}^{(n-1)}, & s = n, \end{cases} \quad \text{and} \quad D_\tilde{n}^{(1)} = 1. \tag{2.55}
\]

(We note that \( D_\tilde{n}^{(n)} = 1 \) for all \( n \in \mathbb{Z}_{\geq 0} \).) The following well-known formula gives the unique solution to (2.55):

\[
D_\tilde{n}^{(n)} = \frac{2(s+1)}{n+s+2} \binom{n}{\frac{n+s}{2}}. \tag{2.56}
\]

We also observe that the \( n \):th Catalan number \( C_n \) arises in a special instance of these numbers:

\[
C_n := \frac{1}{n+1} \binom{2n}{n} = D_\tilde{n}^{(0)}. \tag{2.57}
\]

Lemma 2.8. We have

\[
\dim \text{L}_\xi^{(s)} = \# \text{LP}_\xi^{(s)} = D_\xi^{(s)}. \tag{2.58}
\]

Proof. With \( \dim \text{L}_\xi^{(s)} = \# \text{LP}_\xi^{(s)} \) by definition, we only need to prove the second equality. We use the notation introduced in (2.28). As in the proof of lemma 2.2, we may write any valenced link pattern \( \alpha \in \text{LP}_\xi^{(s)} \) in the form (2.30), for a unique valenced link pattern \( \tilde{\alpha} \in \text{LP}_\xi \) that depends on \( \alpha \), with

\[
\tilde{\alpha} \in \mathbb{E}_\xi \cap \mathbb{E}_{t \oplus \varpi}. \tag{2.59}
\]

On the other hand, inserting into (2.30) any valenced link pattern \( \tilde{\alpha} \in \text{LP}_\xi^{(r)} \) with \( r \) as in (2.59) determines a unique valenced link pattern \( \alpha \in \text{LP}_\xi^{(s)} \). This establishes a bijection

\[
\text{LP}_\xi^{(s)} \leftrightarrow \bigcup_{r \in \mathbb{E}_\xi \cap \mathbb{E}_{t \oplus \varpi}} \text{LP}_\xi^{(r)}. \tag{2.60}
\]

Therefore, the cardinalities \( \# \text{LP}_\xi^{(s)} \) satisfy the recursion problem of (2.54), including its initial condition: \( \# \text{LP}_\xi^{(t)} = 1 \) for all \( t \in \mathbb{Z}_{\geq 0} \). Because the solution to this recursion problem is unique, we must have \( \# \text{LP}_\xi^{(s)} = D_\xi^{(s)} \).
Corollary 2.7 and lemma 2.8 together imply the following identity:

\[ \sum_{s \in \mathbb{E}_{\mathbb{E}}} D^{(s)}_{\varsigma} D^{(s)}_{\varpi} = D^{(0)}_{\varsigma \oplus \varpi}. \]  

(2.61)

If \( \varsigma = \vec{n} \) and \( \varpi = \vec{m} \) for some \( n, m \in \mathbb{Z}_{\geq 0} \), then this reduces to the identity

\[ \sum_{s \in \mathbb{E}_{\mathbb{E}}} D^{(s)}_{n} D^{(s)}_{m} = D^{(0)}_{n+m} = C_{(n+m)/2}, \]

(2.62)

which is more explicit thanks to (2.56).

C. Jones-Wenzl projectors

Recalling definition (1.19) of \( \bar{p}(q) \), we define the Jones-Wenzl projector of size \( s \in \{0, 1, \ldots, \bar{p}(q) - 1\} \) to be the nonzero tangle \( P_{(s)} \in TL_{n}(\nu) \) uniquely determined by the following two properties [Wen87, Jon83, KL94]:

P1. \( P_{(s)}^{2} = P_{(s)} \),

P2. \( U_{i} P_{(s)} = P_{(s)} U_{i} = 0 \) for all \( i \in \{1, 2, \ldots, s-1\} \).

We recall from section 1 that we represent the tangle \( P_{(s)} \) as the empty projector box (1.36), with a cable of size \( s \), i.e., \( s \) parallel links (1.35), entering its left side and a cable of size \( s \) exiting its right side:

\[ \begin{array}{c}
\text{i - 1} \\
\text{s} \\
n - s - i + 1
\end{array} \]

(2.63)

Furthermore, we can insert this projector box within any larger link diagram, as in (1.38):

\[ s \begin{array}{c}
\text{1} \\
\text{s}
\end{array} \]

(2.64)

Abusing notation, we also let the symbol \( P_{(s)} \in TL_{n}(\nu) \) denote this tangle in \( TL_{n}(\nu) \) with \( i = 1 \). Then, the various projectors \( P_{(1)} = 1_{TL_{\mathbb{E}}(\nu)} \), \( P_{(2)} \), \ldots, \( P_{(n)} \) in \( TL_{n}(\nu) \) satisfy the following recursive relation [Wen87, Jon83, KL94]:

\[ P_{(s)}^{(1)} = 1_{TL_{n}(\nu)}, \quad P_{(s+1)} = P_{(s)} + \left( \frac{[s][s+1]}{[s+1]} \right) P_{(s)} U_{s} P_{(s)}, \quad \text{for all } s \in \{1, 2, \ldots, n-1\}, \]

(2.65)

where \([k]\) denotes the \( k:\text{th quantum integer} \), defined as

\[ [k] = [k]_{q} := \frac{q^{k} - q^{-k}}{q - q^{-1}}. \]

(2.66)

We note that our parameterization (1.18) of the loop fugacity parameter \( \nu \in \mathbb{C} \) reads

\[ \nu = -q - q^{-1} = -[2], \]

(2.67)

and in terms of diagrams, recursion relation (2.65) reads

\[ s \begin{array}{c}
\text{1} \\
\text{s}
\end{array} = s \begin{array}{c}
\text{1} \\
\text{s}
\end{array} + \frac{[s]}{[s+1]} \times 1 \begin{array}{c}
\text{1} \\
\text{s}
\end{array}. \]

(2.68)
For example, the first $P_1$, second $P_2$, and third $P_3$ Jones-Wenzl projectors are

\[
\begin{align*}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\end{align*}
\quad =
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad ,
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\end{align*}
\quad =
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad +
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad +
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad ,
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\end{align*}
\quad =
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad +
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad +
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad ,
\end{align*}
\]

Next, we state a few more facts concerning the Jones-Wenzl projectors, for use throughout this article. First, we let $T^\dagger$ denote the tangle obtained by reflecting $T$ about a vertical axis. For example,

\[
T = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \quad \implies \quad T^\dagger = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}.
\]

Recursion rule (2.65) inductively gives the reflection symmetry

\[
P^\dagger_{(s)} = P_{(s)}.
\]

With the graphical representation, properties P1 and P2 respectively translate to the diagram identities

\[
\begin{align*}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\end{align*} = \begin{align*}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad = \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad = \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad = \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} = 0.
\end{align*}
\]

In fact, property P1 can be strengthened to say that $P_{(s)}P_{(t)} = P_{(t)}P_{(s)} = P_{(s)}$ whenever $t \leq s$ [KL94]:

\[
\begin{align*}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\end{align*} = \begin{align*}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad = \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad = \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad = \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}.
\end{align*}
\]

Finally, as a tangle in $\mathbf{TL}_s(\nu)$, the Jones-Wenzl projector $P_{(s)}$ equals a linear combination of the link diagrams in $\mathbf{LD}_s$. Property P1 implies that the coefficient of the identity $1_{\mathbf{TL}_s(\nu)}$ in this linear combination equals one. Hence, we have

\[
P_{(s)} = 1_{\mathbf{TL}_s(\nu)} + \sum_{T \in \mathbf{LD}_s, T \neq 1_{\mathbf{TL}_s(\nu)}} \left( \text{coefficient} \right) T,
\]

for some coefficients $\text{coefficient} \in \mathbb{C}$ (whose values depend on $q \in \mathbb{C}^\times$). In fact, in [Mor15] S. Morrison derives an explicit formula for these coefficients. In appendix A, we give a new, alternative derivation of his formula.

D. Composition of valenced tangles and valenced link states

Now we use the Jones-Wenzl projectors to generalize concatenation rules (1.3–1.5) for $n$-tangles and $(n,s)$-link states (defined in section 1A) to concatenation rules for $(\varsigma, \bar{\omega})$-valenced tangles and $(\varsigma, s)$-valenced link states. For this
purpose, we realize the vector spaces $\text{TL}_\varsigma^\varsigma$ and $L^{(s)}$ respectively as certain quotients of the spaces of all $(\varsigma, \varpi)$-valenced tangles and all $(\varsigma, s)$-valenced link states, by setting diagrams containing loop links to zero. To formalize this, we define the following equivalence relations on the latter spaces $\text{pre-}\text{TL}_\varsigma^\varpi$ and $\text{pre-}L^{(s)}$:

\[ T \sim S \iff T = S + K, \]

where $K \in \text{pre-}\text{TL}_\varsigma^\varpi$ is a linear combination of link diagrams, each having a loop link (2.75)

\[ \alpha \sim \beta \iff \alpha = \beta + \gamma, \]

where $\gamma \in \text{pre-}L^{(s)}$ is a linear combination of link patterns, each having a loop link (2.76)

and we set

\[ \text{TL}_\varsigma^\varpi := \text{pre-}\text{TL}_\varsigma^\varpi / \sim \quad \text{and} \quad \Gamma^{(s)} := \text{pre-}L^{(s)} / \sim. \] (2.77)

Then, each equivalence class in $\overline{\text{TL}}_\varsigma^\varpi$ (resp. $\overline{\Gamma}^{(s)}$) contains a unique valenced tangle in $\text{TL}_\varsigma^\varpi$ (resp. link state in $L^{(s)}$), and the map that sends each equivalence class to this tangle (resp. link state) is an isomorphism of vector spaces:

\[ \overline{\text{TL}}_\varsigma^\varpi \cong \text{TL}_\varsigma^\varpi \quad \text{and} \quad \overline{\Gamma}^{(s)} \cong L^{(s)}. \] (2.78)

Thus, from now on, we identify $(\varsigma, \varpi)$-valenced tangles in $\text{TL}_\varsigma^\varpi$ with their equivalence classes in $\overline{\text{TL}}_\varsigma^\varpi$. For example,

\[ \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tangle1}}
\end{array}
\end{array} \quad \leftrightarrow \quad \left[ \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tangle2}}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tangle3}}
\end{array}
\end{array} \right] \sim. \] (2.79)

Similarly, from now on, we identify $(\varsigma, s)$-valenced link states in $L^{(s)}$ with equivalence classes in $\overline{\Gamma}^{(s)}$. For example,

\[ \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{link1}}
\end{array}
\end{array} \quad \leftrightarrow \quad \left[ \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{link2}}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{link3}}
\end{array}
\end{array} \right] \sim. \] (2.80)

In particular, we identify all of the $(\varsigma, \varpi)$-valenced tangles and $(\varsigma, s)$-valenced link states which contain loop links with the zero tangle and the zero link state, respectively; for example,

\[ \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{link4}}
\end{array}
\end{array} \sim \leftrightarrow 0 \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{link5}}
\end{array}
\end{array} \sim \leftrightarrow 0. \] (2.81)

Next, we define a bilinear map $\mu_{\nu}: \text{TL}_\varsigma^\varpi \times \text{TL}_\varsigma^\varpi \rightarrow \text{TL}_\varsigma^\varpi$ for any complex number $\nu \in \mathbb{C}$, parameterized by $q \in \mathbb{C}^\times$ as in (1.18), and for any multiindex $\varepsilon \in \{0\} \cup \mathbb{Z}_{>0}$ satisfying the condition

\[ \max \varepsilon < \overline{p}(q). \] (2.82)

We define this map by linear extension of the following recipe: writing $\varepsilon = (e_1, e_2, \ldots, e_d)$,

\begin{enumerate}
\item $\mu_1$. we concatenate the $(\varsigma, \varepsilon)$-valenced link diagram $T$ to the $(\varepsilon, \varpi)$-valenced link diagram $U$ from the left,
\item $\mu_2$. we replace the node of size $e_i$ with a Jones-Wenzl projector box of size $e_i$ for each $i \in \{1, 2, \ldots, d_\varepsilon\}$, and
\item $\mu_3$. we decompose each projector box and, in each resulting term, we replace each loop with a factor of $\nu$ and each diagram containing a loop link with zero, to arrive with the $(\varsigma, \varpi)$-valenced tangle
\[ T U := \mu_{\nu}(T, U). \] (2.83)
\end{enumerate}

We write the vector space $\text{TL}_\varsigma^\varpi$ as $\text{TL}_\varsigma^\varpi(\nu)$ to emphasize the chosen value of $\nu$. Pictorially, steps $\mu_1$ and $\mu_2$ are
Followed by step \( \mu_3 \), the above concatenation evaluates to the following:

\[
\begin{align*}
\text{Diagram 1} & \quad := \quad \text{Diagram 2} \\
(2.71) & = \quad \frac{2}{3} \times \text{Diagram 3} + \frac{2}{3} \times \text{Diagram 4} + \frac{2}{3} \times \text{Diagram 5} \\
& = \quad \frac{2}{3} \cdot (1 - [3]) \times \text{Diagram 6} - \frac{4}{3} \times \text{Diagram 7} \\
(2.85) & = (2.70) + 1 \times \text{Diagram 8} = \left(\frac{1}{2} - \frac{2}{3}\right) \times \text{Diagram 9} = -\frac{3}{2} \times \text{Diagram 10},
\end{align*}
\]

where we used the identity \([4] = [2][3] - 1\) (see, e.g., item 1 of lemma B.6 in appendix B).

Next, we define another bilinear map \( \lambda^{(s)}: \mathcal{T}_{\mathcal{L}}(\nu) \times \mathcal{L}^{(s)}(\nu) \rightarrow \mathcal{L}^{(s)}(\nu) \) by linear extension of the following recipe:

\(\lambda_1\). We concatenate the \((\varsigma, \wp)-valenced\) link diagram \(T\) to the \((\wp, s)-valenced\) link pattern \(\alpha\) (rotated by \(-\pi/2\) radians) from the left,

\(\lambda_2\). We replace the node of size \(p_j\) with a Jones-Wenzl projector box of size \(p_j\) for each \(j \in \{1, 2, \ldots, d_\wp\}\), and

\(\lambda_3\). We decompose each projector box, multiply each resulting term by

\[
\begin{align*}
0, & \quad \text{there is a turn-back link in the concatenated diagram}, \\
\nu^{k}, & \quad \text{there is no turn-back link in the concatenated diagram},
\end{align*}
\]

and replace each diagram containing a loop link with zero, to arrive with the \((\varsigma, s)-valenced\) link state

\[
T\alpha := \lambda^{(s)}_\nu(T, \alpha).
\]

Pictorially, steps \(\lambda_1\) and \(\lambda_2\) are

\[
\begin{align*}
\text{Diagram 11} & \quad := \quad \text{Diagram 12} \\
(2.90) & = \quad \text{Diagram 13}.
\end{align*}
\]

Followed by step \(\lambda_3\), the above concatenation evaluates to the following:

\[
\begin{align*}
\text{Diagram 14} & \quad := \quad \text{Diagram 15} \\
(2.70) & = \quad \frac{1}{2} \times \text{Diagram 16} = \left(\frac{1}{2} - \frac{2}{3}\right) \times \text{Diagram 17} = -\frac{3}{2} \times \text{Diagram 18},
\end{align*}
\]

where we used the identity \([3] = [2]^2 - 1\) (see, e.g., item 1 of lemma B.6 in appendix B).

Finally, we remark that valenced tangles arise as morphisms of a certain category \(\mathcal{TL}(\nu)\) [GL98, Tur94, Kas95]. We discuss this category further in our forthcoming article [FP18a]. In appendix F of the present article, we consider a subcategory of \(\mathcal{TL}(\nu)\), which we call the “Temperley-Lieb category,” and we give a minimal set of “generators” and “relations” for morphisms of this category. The object class and morphism class of the larger category \(\mathcal{TL}(\nu)\) are respectively

\[
\text{Ob} \mathcal{TL}(\nu) = \{\varsigma \in \{0\} \cup \mathbb{Z}_{\geq 0} \mid \max \varsigma < \bar{p}(q)\}
\]

(2.92)
\[
\text{Hom } \mathcal{T}L(\nu) = \{ \mathcal{T}L_{\zeta}(\nu) \mid \zeta, \varpi \in \text{Ob } \mathcal{T}L(\nu) \text{ with } n_\zeta + n_\varpi = 0 \text{ (mod 2)} \}.
\] (2.93)

The source and target associated with the tangle \( T \in \mathcal{T}L_{\zeta}(\nu) \) are the objects \( \varpi \) and \( \zeta \) respectively, and the identity morphism associated with the object \( \zeta \) is the unit of the valenced Temperley-Lieb algebra \( \mathcal{T}L_{\zeta}(\nu) \); see (2.97) below. The composition of two morphisms \( T, U \in \text{Hom } \mathcal{T}L(\nu) \) is given by \( \mu_\nu(T, U) \).

In fact, \( \mathcal{T}L(\nu) \) is a tensor category, with identity object \( \vec{0} \). Concatenation of two multiindices \( \zeta, \varpi \in \text{Ob } \mathcal{T}L(\nu) \) is given by (2.50), and the tensor product for all objects \( \zeta, \varpi \in \text{Ob } \mathcal{T}L(\nu) \) and morphisms \( T, U \in \text{Hom } \mathcal{T}L(\nu) \) is given by

\[
\zeta \otimes \varpi := \zeta \oplus \varpi \quad \text{and} \quad T \otimes U := \begin{array}{c} T \\ \hline \end{array} \begin{array}{c} U \\ \hline \end{array}
\] (2.94)

It is also sometimes useful to identify the tensor product \( \alpha \otimes \beta \) of two valenced link states \( \alpha \in \mathcal{L}(s) \) and \( \beta \in \mathcal{L}(t) \) with the link state \( \gamma \in \mathcal{L}(s+t) \) obtained by concatenating \( \alpha \) to the left of \( \beta \):

\[
\alpha \otimes \beta := \begin{array}{c} \alpha \\ \hline \end{array} \begin{array}{c} \beta \\ \hline \end{array}
\] (2.95)

E. Valenced Temperley-Lieb algebra

In the special case that \( \varpi = \zeta \), the vector space \( \mathcal{T}L_{\zeta}(\nu) =: \mathcal{T}L_{\zeta}(\nu) \) is the valenced Temperley-Lieb algebra, already appearing in section 1. A generic valenced tangle in \( \mathcal{T}L_{\zeta}(\nu) \) is of the form

\[
\begin{array}{c} s_1 \\ \vdots \\ s_{d_{\zeta}} \\ s_{d_{\zeta}} \\ \vdots \\ s_2 \\ s_1 \\ T \\ \vdots \\ s_{d_{\zeta}} \\ s_{d_{\zeta}} \\ \vdots \\ s_1 \\ \end{array}
\] (2.96)

for some ordinary tangle \( T \in \mathcal{T}L_{n_\zeta}(\nu) \). We call elements of \( \mathcal{T}L_{\zeta}(\nu) \) \( \zeta \)-valenced tangles. If \( T \) is an \( n_\zeta \)-link diagram such that (2.96) does not vanish (i.e., does not contain loop links), then we call (2.96) a \( \zeta \)-valenced link diagram.

The valenced Temperley-Lieb algebra \( \mathcal{T}L_{\zeta}(\nu) \) has an associative multiplication given by the bilinear map \( \mu_\nu \) (defined in recipe \( \mu_1-\mu_3 \)), and the \( \zeta \)-valenced tangle

\[
\begin{array}{c} s_1 \\ \vdots \\ s_{d_{\zeta}} \\ \vdots \\ s_2 \\ s_1 \\ 1_{\mathcal{T}L_{\zeta}(\nu)} = \\\\ \vdots \\ s_{d_{\zeta}} \\ s_{d_{\zeta}} \\ \vdots \\ s_1 \\ \end{array}
\] (2.97)

is the unit of \( \mathcal{T}L_{\zeta}(\nu) \). From (2.82), we see that the multiplication map of \( \mathcal{T}L_{\zeta}(\nu) \) is defined only if we have

\[
\max \zeta < \bar{p}(q).
\] (2.98)
As a complex vector space, $\text{TL}_\varsigma(\nu)$ has the basis

$$LD_\varsigma := \text{LD}_\varsigma^\varsigma.$$  \hfill (2.99)

By setting $\varsigma = \varpi$ in corollary 2.7, we obtain the following expressions for the dimension of this algebra:

$$D_{\varsigma \in \varsigma}^{(0)} \text{dim TL}_\varsigma(\nu) \overset{(2.53)}{=} (2.58) \sum_{s \in E_\varsigma} \left( \text{dim L}_\varsigma^{(s)} \right)^2.$$  \hfill (2.100)

The bilinear map $\lambda_\varsigma^{(s)} : \text{TL}_\varsigma(\nu) \times L_\varsigma^{(s)} \rightarrow L_\varsigma^{(s)}$ (defined in recipe $\lambda 1$–$\lambda 3$) endows the vector space $L_\varsigma^{(s)}$ with a $\text{TL}_\varsigma(\nu)$-module structure. In this context, we call $L_\varsigma^{(s)}$ a \textit{valenced standard module}, or simply a “standard module.” As we will see in section 3, these $\text{TL}_\varsigma(\nu)$-modules play an essential role in its representation theory. Also, we call the direct sum $L_\varsigma$ from (2.20), now a $\text{TL}_\varsigma(\nu)$-module too, the \textit{valenced link state module} of $\text{TL}_\varsigma(\nu)$, or simply the “link state module.”

Next we present two minimal generating sets for the valenced Temperley-Lieb algebra $\text{TL}_\varsigma(\nu)$, assuming that $n_\varsigma < \overline{\rho}(q)$. In the second generating set (2.102), we use the definition of a closed three-vertex, given by (4.36) in section 4A. This result is crucial in our work [FP18a+] for obtaining a generalization of the quantum Schur-Weyl duality, a result essential to determining unique monodromy invariant CFT correlation functions in [FP18c+].

\textbf{Proposition 2.9.} Suppose $n_\varsigma < \overline{\rho}(q)$. Then the unit (1.58) together with all $\varsigma$-valenced link diagrams of the form

\begin{center}
\begin{tikzpicture}
    \draw (0,0) -- (0,2);
    \draw (0,0) -- (2,0);
    \node at (1,1) {$1$};
    \node at (0.5,0.5) {$s_{i-1}$};
    \node at (1.5,0.5) {$s_i$};
    \node at (2,0) {$s_{i+1}$};
    \node at (2,2) {$s_{i+2}$};
    \node at (2,0) {$s_{i+1}$};
    \node at (2,2) {$s_{i+2}$};
    \node at (2,0) {$s_{i+1}$};
    \node at (2,2) {$s_{i+2}$};
\end{tikzpicture}
\end{center}

with $i \in \{1, 2, \ldots, d_\varsigma - 1\}$, forms a minimal generating set for the valenced Temperley-Lieb algebra $\text{TL}_\varsigma(\nu)$. Alternatively, the collection of all $\varsigma$-valenced tangles of the form

\begin{center}
\begin{tikzpicture}
    \draw (0,0) -- (0,2);
    \draw (0,0) -- (2,0);
    \node at (1,1) {$s$};
    \node at (0.5,0.5) {$s_{i-1}$};
    \node at (1.5,0.5) {$s_i$};
    \node at (2,0) {$s_{i+1}$};
    \node at (2,2) {$s_{i+2}$};
    \node at (2,0) {$s_{i+1}$};
    \node at (2,2) {$s_{i+2}$};
    \node at (2,0) {$s_{i+1}$};
    \node at (2,2) {$s_{i+2}$};
\end{tikzpicture}
\end{center}

with $s \in E_{(s, s_{i+1})}$ and $i \in \{1, 2, \ldots, d_\varsigma - 1\}$, form a minimal generating set for $\text{TL}_\varsigma(\nu)$.

\textbf{Proof.} We prove this proposition beneath proposition 2.12, stated below. The proof primarily relies on the veracity of proposition 2.12, which we prove in appendix D.

We expect the above result to still hold if we relax the condition $n_\varsigma < \overline{\rho}(q)$ to $\max \varsigma < \overline{\rho}(q)$ (see also conjecture 2.13):

\textbf{Conjecture 2.10.} \textit{Proposition 2.9 holds whenever $\max \varsigma < \overline{\rho}(q)$}.
F. Jones-Wenzl algebra

Rather than working directly with the valenced Temperley-Lieb algebra $\text{TL}_\varsigma(\nu)$, at times, it is useful to work instead with a subalgebra of the ordinary Temperley-Lieb algebra $\text{TL}_n(\nu)$, isomorphic to $\text{TL}_\varsigma(\nu)$ whenever $\max \varsigma < \bar{p}(\nu)$. This subalgebra, previously discussed in section 1B, is called the Jones-Wenzl algebra. It is defined as

$$ JW_\varsigma(\nu) := P_\varsigma \text{TL}_n(\nu) P_\varsigma = \{ P_\varsigma T P_\varsigma | T \in \text{TL}_n(\nu) \}, $$

where

$$ P_\varsigma := P_{s_1} \cdots P_{s_{d-1}} P_{s_d}, $$

is the Jones-Wenzl composite projector, defined whenever $\max \varsigma < \bar{p}(\nu)$ (i.e., whenever all projector boxes in it exist). The algebra $JW_\varsigma(\nu)$ is the collection of all tangles in $\text{TL}_n(\nu)$ of the form

$$ T $$

with $T \in \text{TL}_n(\nu)$. We call an element of $JW_\varsigma(\nu)$ a $\varsigma$-Jones-Wenzl tangle. If $T$ is an $n_\varsigma$-link diagram such that (2.105) does not vanish, then we call (2.105) a $\varsigma$-Jones-Wenzl link diagram.

The Jones Wenzl algebra is a unital, associative algebra. Indeed, it inherits associativity from the Temperley-Lieb algebra $\text{TL}_n(\nu)$, and property P1 of the Jones-Wenzl projectors implies that $P_\varsigma$ is its unit: for all tangles $T \in \text{TL}_n(\nu)$, we have

$$ P_\varsigma T = TP_\varsigma = T. $$

It is straightforward to see that the following vector spaces are $JW_\varsigma(\nu)$-modules, and we respectively call them a (Jones-Wenzl) standard module and the (Jones-Wenzl) link state module:

$$ W^{(s)}_\varsigma := P_\varsigma L^{(s)}_n = \{ P_\varsigma \alpha | \alpha \in L^{(s)}_n \}, $$

$$ W_\varsigma := P_\varsigma L_n = \{ P_\varsigma \alpha | \alpha \in L_n \} = \bigoplus_{s \in E_\varsigma} W^{(s)}_\varsigma. $$
A generic element of $W^{(s)}_\zeta$, called a $(\zeta, s)$-Jones-Wenzl link state, has the form

\[ (2.109) \]

for some ordinary link state $\alpha \in L^{(s)}_{n_\zeta}$. If $\alpha$ is a $(n_\zeta, s)$-link pattern such that (2.109) does not vanish, then we call (2.109) a $(\zeta, s)$-Jones-Wenzl link pattern.

As discussed in section 1, the Jones-Wenzl algebra is isomorphic to the valenced Temperley-Lieb algebra. Furthermore, the link state modules of these two algebras are isomorphic too. We give the explicit isomorphisms in the next lemma, and in appendix C, we prove a more detailed version of this lemma.

**Lemma 2.11.** Suppose $\max \zeta < \bar{p}(q)$. Then

1. the linear map sending $TL^{(s)}_\zeta(\nu) \rightarrow JW^{(s)}_\zeta(\nu)$ via

\[ (2.110) \]

where $T \in TL^{(s)}_{n_\zeta}(\nu)$, is an isomorphism of unital, associative algebras, and

2. the linear map sending $L^{(s)}_\zeta(\nu) \rightarrow W^{(s)}_\zeta(\nu)$ via

\[ (2.111) \]

where $\alpha \in L^{(s)}_{n_\zeta}$, is an isomorphism of modules (from a $TL^{(s)}_\zeta(\nu)$-module to a $JW^{(s)}_\zeta(\nu)$ module).

**Proof.** We only prove item 1; the proof of item 2 is similar. In light of the fact that the diagrams of $TL^{(s)}_\zeta(\nu)$ do not contain loop links and property (P2) of Jones-Wenzl projectors, it is evident that the map of (2.110) is an isomorphism of vector spaces. The claim in item 1 then follows from the definition of the multiplication in $TL^{(s)}_\zeta(\nu)$.

In the next proposition, parallel to proposition 2.9, we present two minimal generating sets for the Jones-Wenzl algebra $JW^{(s)}_\zeta(\nu)$, assuming that $n_\zeta < \bar{p}(q)$. Again, in the second generating set (2.114), we use the definition of a closed three-vertex, given by (4.36) in section 4A. We prove this proposition in appendix D.
Proposition 2.12. Suppose $n_\varsigma < \bar{p}(q)$. Then the unit (2.104) together with all $\varsigma$-Jones-Wenzl link diagrams of the form

\[ P_\varsigma U_{s_1+s_2+\ldots+s_i} P_\varsigma = \] (2.112)

with $i \in \{1, 2, \ldots, d_\varsigma - 1\}$, forms a minimal generating set for the Jones-Wenzl algebra $JW_\varsigma(\nu)$. Explicitly, this generating set is

\[ G_\varsigma := P_\varsigma \{ 1_{n_\varsigma}(\nu), U_{s_1}, U_{s_1+s_2}, \ldots, U_{s_1+s_2+\ldots+s_{d_\varsigma-1}} \} P_\varsigma. \] (2.113)

Alternatively, the collection of all $\varsigma$-Jones-Wenzl tangles of the form

\[ \] (2.114)

with $s \in E_{(s_i, s_{i+1})}$ and $i \in \{1, 2, \ldots, d_\varsigma - 1\}$, form a minimal generating set for $JW_\varsigma(\nu)$.

**Proof.** The entirety of appendix D gives the proof of this proposition.

It is natural to conjecture that proposition 2.12 holds if we drop the condition $n_\varsigma < \bar{p}(q)$, or equivalently, relax it to the minimal condition $\max_\varsigma < \bar{p}(q)$ necessary for $JW_\varsigma(\nu)$ to be defined:

Conjecture 2.13. Proposition 2.12 holds whenever $\max_\varsigma < \bar{p}(q)$.

Unfortunately, our proof of proposition 2.12 in general relies heavily on the condition that $n_\varsigma < \bar{p}(q)$, and it seems to provide little insight to proving this conjecture, other than to suggest that such a proof could be rather different. However, conjecture 2.13 is known to hold at least in the special case $\varsigma = (1, 1, \ldots, 1, k)$ with $\max_\varsigma = k < \bar{p}(q)$. A proof for it is given in [MDRR15, appendix C]. Also, in appendix D, corollary D.6, we prove proposition 2.12 in the case of $d_\varsigma = 2$ with the weaker assumption $\max(s_1, s_2) < \bar{p}(q)$ instead of $s_1 + s_2 < \bar{p}(q)$.

With proposition 2.12, it is now straightforward to prove proposition 2.9.

**Proof of proposition 2.9.** The claim immediately follows by combining proposition 2.12 with lemma 2.11. \qed
3. STANDARD MODULES

Now we begin to investigate the representation theory of the valenced Temperley-Lieb algebra $\mathrm{TL}_\varsigma(\nu)$. The standard modules $L^{(\varsigma)}_s$, spanned by valenced link patterns, and their radicals play key roles in this theory. We follow the approach of [JM79, Wes95, GL98, RSA14]: we study the representation theory of $\mathrm{TL}_\varsigma(\nu)$ via a natural bilinear form $(\cdot | \cdot) : L^{(\varsigma)}_s \times L^{(\varsigma)}_s \rightarrow \mathbb{C}$.

A key result in this section is proposition 3.3, a simple extension of [RSA14, proposition 3.3]. It says that if the bilinear form $(\cdot | \cdot)$ is not identically zero on it, then the standard module $L^{(\varsigma)}_s$ is indecomposable and the radical

$$\text{rad} L^{(\varsigma)}_s := \{ \alpha \in L^{(\varsigma)}_s | (\alpha | \beta) = 0 \text{ for all } \beta \in L^{(\varsigma)}_s \}$$

is the maximal proper submodule of $L^{(\varsigma)}_s$. In particular, $L^{(\varsigma)}_s$ is simple if and only if its radical is trivial, and otherwise, its quotient by its radical is simple. This is the first step to determining all of the simple modules of $\mathrm{TL}_\varsigma(\nu)$.

Another key result is corollary 3.8 in section 3C, which says that the link state representation of the valenced Temperley-Lieb algebra $\mathrm{TL}_\varsigma(\nu)$ on the link state module $L^{(\varsigma)}_s$ is faithful if and only if the radical of $L^{(\varsigma)}_s$ is trivial. We use this result in our forthcoming article [FP18a⁺].

Most of the results stated in this section depend on properties of the radical (3.1). Hence, to understand the scope of these results, we completely determine these radicals in section 5 (proposition 5.7 and theorem 5.18).

A. Networks and the link state bilinear form

We first define the link state bilinear form for the special case when $\varsigma = \vec{n}$ for some integer $n \in \mathbb{Z}_{\geq 0}$. For this purpose, we introduce the notion of a network: a collection of nonintersecting, non-self-intersecting planar loops and paths within a rectangle. A path in a network can be a through-path, which is a curve that respectively enters and exits the network at the bottom and top sides of the rectangle, or a turn-back path, which enters and exits the network at the same side of the rectangle, either top or bottom:

We define the evaluation of a network $T$ to be the following complex number: we assign all loops, through-paths, and turn-back paths in $T$ the following weights in $\mathbb{C}$:

- loop weight (fugacity): $\circ$ and $\bigcirc$ and $\bigcirc$ etc. $= \nu$, (3.3)
- through-path weight: $\left| \right.$ and $\left| \right.$ and $\left| \right.$ etc. $= 1$, (3.4)
- turn-back path weight: $\bigcup$ and $\bigcup$ and $\bigcup$ etc. $= 0$. (3.5)

Then with “$\# \text{ loops}$” equaling the number of loops in the network $T$, the evaluation of $T$ is defined to be

$$(T) := \prod \{ \text{the weights of all objects in the network } T \}$$

(3.6)
Now, using the notion of a network and its evaluation, we define a bilinear form on the link state module $L_n$. For two link patterns $\alpha, \beta \in \text{LP}_n$, we horizontally reflect $\alpha$ so it is upside down, we concatenate it to $\beta$ from below, and delete the overlapping horizontal lines of $\alpha$ and $\beta$. The resulting diagram is a network $\alpha \mid \beta$. For instance, we have

$$\alpha = \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array}, \quad \beta = \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array} \implies \alpha \mid \beta = \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}, \quad (3.8)$$

$$\alpha = \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array}, \quad \beta = \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array} \implies \alpha \mid \beta = \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}, \quad (3.9)$$

$$\alpha = \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array}, \quad \beta = \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array} \implies \alpha \mid \beta = \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}. \quad (3.10)$$

The map $(\cdot \mid \cdot): L_n \times L_n \rightarrow \mathbb{C}$ given by linear extension of the following rule for each pair of link patterns $\alpha, \beta \in \text{LP}_\varsigma$ is a bilinear form on $L_n$, called the link state bilinear form:

$$(\alpha, \beta) \mapsto (\alpha \mid \beta). \quad (3.11)$$

If $\alpha, \beta \in L_0$, then the product in (3.6) is empty, so we take $(\alpha \mid \beta) = 1$. As examples, the bilinear forms $(\alpha \mid \beta)$ of the link patterns $\alpha$ and $\beta$ in (3.8, 3.9, 3.10) respectively evaluate to the following:

$$(\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc) = \nu^2, \quad \left(\begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array}\right) = \nu, \quad \left(\begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array}\right) = 0. \quad (3.12)$$

Then the bilinear form $(\alpha \mid \beta)$ for any pair of link states $\alpha, \beta \in L_n$ is given via bilinear extension of rule (3.11).

In order to generalize the above definition to give a bilinear form on the valenced link state module $L_\varsigma$, we first define the Jones-Wenzl composite embedder to be the following tangle (assuming $\max \varsigma < \bar{p}(q)$):

$$I_\varsigma := \begin{array}{c}
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array} \\
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array} \\
\vdots \\
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array}
\end{array} \quad (3.13)$$

Then, the following map is a linear injection, by item 2 of lemma C.1 in appendix C:

$$I_\varsigma(\cdot): L_\varsigma \rightarrow L_{n_\varsigma}, \quad \alpha \mapsto I_\varsigma \alpha. \quad (3.14)$$
Also, we denote the reflection $I_\top$ of $I_\top$ about a vertical axis by

$$
\hat{P}_\top := \quad (3.15)
$$

We immediately have

$$
\hat{P}_\top I_\top = 1_{TL_\top(\nu)}, \quad \text{and} \quad I_\top \hat{P}_\top = P_\top. \quad (3.16)
$$

Now, assuming that $\max_\top < \overline{p}(q)$ and using the linear injection $I_\top(\cdot) : L_\top \rightarrow L_n$, we define the link state bilinear form $(\cdot | \cdot) : L_\top \times L_\top \rightarrow \mathbb{C}$ on the valenced link state module $L_\top$ to be

$$(\alpha | \beta) := (I_\top \alpha | I_\top \beta), \quad (3.17)$$

for all valenced link states $\alpha, \beta \in L_\top$, where the right side is the bilinear form on $L_n$. For example, we have

$$
\left( \begin{array}{c|c}
\alpha & \beta \\
\end{array} \right) = \quad (3.18)
$$

As a consequence of rule (3.5), the standard modules $L_\top^{(s)}$ and $L_\top^{(r)}$ are orthogonal if $s \neq r$, so we have

$$
\text{rad } L_\top^{(s)} := \{ \alpha \in L_\top^{(s)} | (\alpha | \beta) = 0 \text{ for all } \beta \in L_\top^{(s)} \}, \quad (3.19)
$$

$$
\text{rad } L_\top := \{ \alpha \in L_\top | (\alpha | \beta) = 0 \text{ for all } \beta \in L_\top \} = \bigoplus_{s \in E_\top} \text{rad } L_\top^{(s)}. \quad (3.20)
$$

In the following lemma, similar to [RSA14, lemma 3.1], we give some basic properties of the bilinear form $(\cdot | \cdot)$. We say that the bilinear form $(\cdot | \cdot)$ is symmetric because of property (3.21) and invariant because of property (3.22).

**Lemma 3.1.** Suppose $\max_\top < \overline{p}(q)$. Then for all valenced link patterns $\alpha, \beta \in L_\top$ and valenced tangles $T \in TL_\top(\nu)$, we have

$$
(\alpha | \beta) = (\beta | \alpha), \quad (3.21)
$$

$$
(\alpha | T\beta) = (T^\dagger \alpha | \beta). \quad (3.22)
$$

**Proof.** In light of definition (3.17), we may assume that $\zeta = \mathbf{n}$ for some integer $n \in \mathbb{Z}_{>0}$. In this case, identity (3.21) is immediate from definition (3.11), and identity (3.22) also follows easily from the definitions: for example, for

$$
\alpha = \quad \text{and} \quad T = \quad (3.23)
$$

the following network represents either quantity $(\alpha | T\beta)$ or $(T^\dagger \beta | \alpha)$:

$$
\left[ \begin{array}{c}
\beta \\
\end{array} \right], \quad \left[ \begin{array}{c}
T \\
\end{array} \right], \quad \text{and} \quad \left[ \begin{array}{c}
\alpha \\
\end{array} \right]. \quad (3.24)
$$

\qed
In spite of its simplicity, identity (3.25) in the next lemma is a powerful tool for determining representation-theoretic properties of the standard modules. This lemma and its proof are natural generalizations of [RSA14, lemma 3.2]:

Lemma 3.2. Suppose \( \max(\varsigma, \varpi) < \tilde{p}(q) \). Then for all valenced link states \( \alpha \in L_\varsigma^{(s)} \) and \( \beta, \gamma \in L_\varpi^{(s)} \), we have

\[
| \alpha \beta | \gamma = (| \beta | \gamma)\alpha. \tag{3.25}
\]

Proof. First, we prove identity (3.25) for the case \( \varsigma = \vec{n} \) and \( \varpi = \vec{m} \), with \( \alpha, \beta, \) and \( \gamma \) ordinary link states. By linearity, we may assume that they are link patterns. We consider two scenarios:

1. All defects of \( \beta \) join with all defects of \( \gamma \). In this case, we readily simplify \( | \alpha \beta | \gamma \) to \( \alpha \) multiplied by the number of loops in the diagram for \( (\beta | \gamma) \), which equals \( (\beta | \gamma)\alpha \). Therefore, identity (3.25) holds for this case.
2. Some defects of \( \beta \) do not join with defects of \( \gamma \). In this case, a link of \( \beta \) must join two defects of \( \gamma \) together in the diagram for \( (\beta | \gamma) \). By rule (3.5), identity (3.25) holds also for this case.

By linearity, this proves (3.25) for all link states \( \alpha \in L_\varsigma^{(s)} \) and \( \beta, \gamma \in L_\varpi^{(s)} \).

Second, we prove identity (3.25) for the general case. Now, we have \( I_\varsigma \alpha \in L_\varsigma^{(s)} \) and \( I_\varpi \beta, I_\varpi \gamma \in L_\varpi^{(s)} \), so the already proved identity (3.25) for them gives

\[
| I_\varsigma \alpha \ I_\varpi \beta | I_\varpi \gamma = (I_\varpi \beta | I_\varpi \gamma)I_\varsigma \alpha. \tag{3.26}
\]

On the other hand, by drawing a picture, it is straightforward to see that

\[
I_\varsigma \ | \alpha \beta | I_\varpi = | I_\varsigma \alpha \ I_\varpi \beta |. \tag{3.27}
\]

Using these identities and item 5 of lemma C.1 from appendix C, we obtain

\[
| \alpha \beta | \gamma \overset{(3.16)}{=} \hat{P}_c I_\varsigma | \alpha \beta | \gamma \overset{(C.16)}{=} \hat{P}_c (I_\varsigma | \alpha \beta | \hat{P}_\varpi (I_\varpi \gamma)) \overset{(3.27)}{=} \hat{P}_c | I_\varsigma \alpha \ I_\varpi \beta | \gamma \overset{(3.26)}{=} (I_\varpi \beta | I_\varpi \gamma) \hat{P}_c I_\varsigma \alpha \overset{(3.16)}{=} (\beta | \gamma)\alpha, \tag{3.28}
\]

which proves (3.25) for all valenced link states \( \alpha \in L_\varsigma^{(s)} \) and \( \beta, \gamma \in L_\varpi^{(s)} \).

\( \square \)

B. Standard modules and their radicals

Now we use the link state bilinear form (3.17) to study the structure of the standard modules \( L_\varsigma^{(s)} \), their radicals \( \text{rad} L_\varsigma^{(s)} \), and their quotients \( Q_\varsigma^{(s)} \) by these radicals (defined in (1.55)). We say that the radical \( \text{rad} L_\varsigma^{(s)} \) is totally degenerate if \( \text{rad} L_\varsigma^{(s)} = L_\varsigma^{(s)} \). Throughout this section, we assume that the radicals are not totally degenerate. This is usually true, but not always. In section 5D, we determine when this assumption is violated.

The following proposition is a key result in this section. Its proof is a straightforward adaptation of the proof of [RSA14, proposition 3.3].

Proposition 3.3. Suppose \( \max\varsigma < \tilde{p}(q) \). If \( \text{rad} L_\varsigma^{(s)} \neq L_\varsigma^{(s)} \), then the following are true:

1. the quotient module \( Q_\varsigma^{(s)} \) is simple, or equivalently, \( \text{rad} L_\varsigma^{(s)} \) is the unique maximal proper submodule of \( L_\varsigma^{(s)} \), and
2. the standard module \( L_\varsigma^{(s)} \) is indecomposable.

Proof. For a valenced link state \( \alpha \in L_\varsigma^{(s)} \), we let \( [\alpha] \in Q_\varsigma^{(s)} \) denote its equivalence class in the quotient module (1.55). We prove items 1 and 2 as follows:

1. Let \( \gamma \in L_\varsigma^{(s)} \setminus \text{rad} L_\varsigma^{(s)} \). Then, we may choose a valenced link state \( \beta \in L_\varsigma^{(s)} \) such that \( (\beta \ | \ \gamma) = 1 \). Now, identity (3.25) of lemma 3.2 shows that we can construct any \( [\alpha] \in Q_\varsigma^{(s)} \) through multiplying \( [\gamma] \) by a tangle in \( \text{TL}_\varsigma(\nu) \):

\[
| \alpha \beta | [\gamma] = | [\alpha \beta \ | \gamma] = (\beta \ | \gamma)[\alpha] = [\alpha]. \tag{3.29}
\]

This shows that \( Q_\varsigma^{(s)} \) is cyclic with generator \( [\gamma] \). Because \( \gamma \in L_\varsigma^{(s)} \setminus \text{rad} L_\varsigma^{(s)} \) can be chosen arbitrarily, we see that any nonzero element of \( Q_\varsigma^{(s)} \) generates \( Q_\varsigma^{(s)} \), so \( Q_\varsigma^{(s)} \) is simple. This proves item 1.
2. Suppose \( L^{(\nu)}_\varsigma \) can be decomposed as a direct sum of two \( \mathbb{T}_\varsigma(\nu) \)-submodules \( U \) and \( V \), i.e., \( L^{(\nu)}_\varsigma = U \oplus V \). To prove that \( L^{(\nu)}_\varsigma \) is indecomposable, we need to show that one of the submodules, \( U \) or \( V \), is trivial. Now, the same argument that we used to prove item 1 shows that \( L^{(\nu)}_\varsigma \) is cyclic, generated by any nonzero valenced link state \( \gamma \notin \text{rad} L^{(\nu)}_\varsigma \). We choose one such \( \gamma \) and write it as \( \gamma = \gamma_U + \gamma_V \) with \( \gamma_U \in U \) and \( \gamma_V \in V \). Then we consider two cases:

(a): Both \( \gamma_U \) and \( \gamma_V \) belong to the radical \( \text{rad} L^{(\nu)}_\varsigma \). In this case, the \( \mathbb{T}_\varsigma(\nu) \)-submodules of \( L^{(\nu)}_\varsigma \) generated by \( \gamma_U \) and \( \gamma_V \) are also \( \mathbb{T}_\varsigma(\nu) \)-submodules of \( \text{rad} L^{(\nu)}_\varsigma \subset L^{(\nu)}_\varsigma \), and so is their direct sum. We get a contradiction:

\[
L^{(\nu)}_\varsigma = \mathbb{T}_\varsigma(\nu) \gamma \subset \mathbb{T}_\varsigma(\nu) \gamma_U \oplus \mathbb{T}_\varsigma(\nu) \gamma_V \subset \text{rad} L^{(\nu)}_\varsigma \subset L^{(\nu)}_\varsigma \quad \implies \quad \text{rad} L^{(\nu)}_\varsigma = L^{(\nu)}_\varsigma. \tag{3.30}
\]

(b): Either \( \gamma_U \) or \( \gamma_V \) does not belong to the radical \( \text{rad} L^{(\nu)}_\varsigma \). Without loss of generality, we assume that \( \gamma_U \notin \text{rad} L^{(\nu)}_\varsigma \). Then, \( \gamma_U \) generates the whole module \( L^{(\nu)}_\varsigma \). Thus, we have \( U = L^{(\nu)}_\varsigma \) and \( V = \{0\} \).

This concludes the proof.

\( \square \)

**Remark 3.4.** We make a remark on the prospect of obtaining an analogue of proposition 3.3 when the radical \( \text{rad} L^{(\nu)}_\varsigma \) is totally degenerate. In [RSA14], this is successfully done in the case that \( \varsigma = \bar{n} \) for some \( n \in \mathbb{Z}_{>0} \). To establish this, the authors first show that \( \text{rad} L^{(\nu)}_\varsigma \) is totally degenerate if and only if \( \nu = 0 \) (i.e., \( \bar{p}(q) = 2 \) by (1.19)) and \( s = 0 \). Then, with \( \nu = 0 = s \), they use the renormalized bilinear form

\[
\langle \cdot, \cdot \rangle' := \lim_{\nu \to 0} \nu^{-1}\langle \cdot, \cdot \rangle \tag{3.31}
\]

to prove an analogue [RSA14, proposition 3.5] of proposition 3.3. Later, they show that the radical of \( L^{(0)}_\varsigma \) with respect to the new bilinear form is trivial [RSA14, proposition 4.9], concluding that \( L^{(0)}_\varsigma \) is a simple module when \( \nu = 0 \).

Unfortunately, the method of [RSA14] seems to not extend to general multiindices \( \varsigma \in \{0\} \cup \mathbb{Z}_{>0} \). However, it may not be necessary to pursue such an extension after all. Indeed, the authors go on to prove that the radical of \( L^{(\nu)}_\varsigma \) (with respect to the usual bilinear form (3.11)) is either the trivial module or a simple module [RSA14, theorem 7.2]. To prove this claim in the special case that \( \nu = 0 = s \), the authors simply use the facts that \( \text{rad} L^{(0)}_\varsigma = L^{(0)}_\varsigma \) and the latter module is simple. Generalizing the result of [RSA14] to all multiindices \( \varsigma \in \{0\} \cup \mathbb{Z}_{>0} \), if one proved that the radical of \( L^{(\nu)}_\varsigma \) is either the trivial module or a simple module, for all \( s \in \mathbb{E}_\varsigma \) and whenever \( \max \varsigma < \bar{p}(q) \), then in all cases when it is totally degenerate, one could infer that \( L^{(\nu)}_\varsigma \) is a simple module. Knowing this, there would be no reason to obtain a generalization of proposition 3.3.

One key purpose of this article is to initiate the construction of a complete set of all nonisomorphic simple \( \mathbb{T}_\varsigma(\nu) \)-modules. The following proposition 3.5 and corollary 3.6 help us in this endeavor. Their proofs are straightforward adaptations of the proofs of [RSA14, proposition 3.6] and [RSA14, corollary 3.7], respectively.

**Proposition 3.5.** Suppose \( \max \varsigma < \bar{p}(q) \). If \( \text{rad} L^{(\nu)}_\varsigma \neq L^{(\nu)}_\varsigma \) and \( M \) and \( N \) are submodules of \( L^{(\nu)}_\varsigma \) and \( L^{(\nu)}_{\bar{\varsigma}} \) respectively with \( s < r \), then the only homomorphism sending \( L^{(\nu)}_\varsigma/M \to L^{(\nu)}_{\bar{\varsigma}}/N \) is the zero homomorphism.

**Proof.** We let \( [\alpha] \in L^{(\nu)}_\varsigma/M \) (resp. \( [\alpha] \in L^{(\nu)}_{\bar{\varsigma}}/N \)) denote the equivalence class of \( \alpha \in L^{(\nu)}_\varsigma \) (resp. \( \alpha \in L^{(\nu)}_{\bar{\varsigma}} \)). Then for any \( \beta, \gamma \in L^{(\nu)}_\varsigma \) chosen such that \( \langle \beta, \gamma \rangle = 1 \), and for any homomorphism \( \theta : L^{(\nu)}_\varsigma/M \to L^{(\nu)}_{\bar{\varsigma}}/N \), we have

\[
[\alpha] \quad \beta \quad \theta([\gamma]) = \theta([\alpha] \quad \beta \quad [\gamma]) \quad (3.29) \quad \theta([\alpha]). \tag{3.32}
\]

We let \( \delta \in L^{(\nu)}_{\bar{\varsigma}} \) be such that \( \theta([\gamma]) = [\delta] \). Then, because \( \alpha, \beta \in L^{(\nu)}_\varsigma \) and \( \delta \in L^{(\nu)}_{\bar{\varsigma}} \) with \( s < r \), the tangle \( [\alpha] \quad \beta \quad \delta \) necessarily joins two defects of \( \delta \) together, so \( [\alpha] \quad \beta \quad \delta = 0 \) by rule (3.5). It follows that

\[
\theta([\alpha]) \quad (3.32) \quad [\alpha] \quad \beta \quad \theta([\gamma]) = [\alpha] \quad \beta \quad [\delta] = [\alpha] \quad \beta \quad [\delta] = 0. \tag{3.33}
\]

This implies that \( \theta \) is the zero homomorphism.

\( \square \)

**Corollary 3.6.** Suppose \( \max \varsigma < \bar{p}(q) \). If \( \text{rad} L^{(\nu)}_\varsigma \neq L^{(\nu)}_\varsigma \) and \( \text{rad} L^{(\nu)}_{\bar{\varsigma}} \neq L^{(\nu)}_{\bar{\varsigma}} \), then we have

\[
L^{(\nu)}_\varsigma \cong L^{(\nu)}_{\bar{\varsigma}} \iff s = r \quad \text{and} \quad Q^{(\nu)}_\varsigma \cong Q^{(\nu)}_{\bar{\varsigma}} \iff s = r. \tag{3.34}
\]

**Proof.** To obtain the first equivalence, we set \( M = N = \{0\} \) in proposition 3.5, and to obtain the second, we set \( M = \text{rad} L^{(\nu)}_\varsigma \) and \( N = \text{rad} L^{(\nu)}_{\bar{\varsigma}} \) in proposition 3.5.

\( \square \)
The assumption that the radical is not totally degenerate is essential in the above results. For example, as pointed out in [RSA14], it is straightforward to verify that
\[
\text{rad } L_2^{(0)} = L_2^{(0)} \iff \bar{p}(q) = 2 \quad \text{(i.e., } \nu = 0) \iff L_2^{(0)} \cong L_2^{(2)}.
\] (3.35)

C. Faithfulness of the link state representations

We say that the radical \( \text{rad } L_\zeta^{(s)} \) (resp. \( \text{rad } L_\zeta \)) is nondegenerate if it is trivial, i.e., \( \text{rad } L_\zeta^{(s)} = \{0\} \) (resp. \( \text{rad } L_\zeta = \{0\} \)). In this section, we prove that the link state representation of \( \text{TL}_\zeta(\nu) \) on \( L_\zeta \) is faithful if and only if \( \text{rad } L_\zeta \) is nondegenerate. We use this result in [FP18a+]. We obtain it as a corollary of proposition 3.7 below, in the special case that \( \varpi = \zeta \).

In the proof of proposition 3.7, we use the Gram matrix \( \mathcal{G}_\zeta^{(s)} \) of the bilinear form \( (3.17) \) with respect to the basis \( L_\zeta^{(s)} \) of valenced link patterns. This matrix is given by
\[
[\mathcal{G}_\zeta^{(s)}]_{\alpha,\beta} := (\alpha \mid \beta), \quad \text{for all } \alpha, \beta \in L_\zeta^{(s)}.
\] (3.36)

In the proof, we only use the elementary fact that the matrix \( \mathcal{G}_\zeta^{(s)} \) is invertible if and only if \( \text{rad } L_\zeta^{(s)} \) is nondegenerate. We study this matrix in greater detail later in section 4.

**Proposition 3.7.** Suppose \( \max \varpi < \bar{p}(q) \). Then \( \text{rad } L_\zeta^{(s)} \neq \{0\} \) for some \( s \in E_\varpi^- \) if and only if there exists a nonzero valenced tangle \( T \in \text{TL}_\zeta(\nu) \) such that \( T \gamma = 0 \) for all valenced link states \( \gamma \in L_{\varpi}^{(s)} \) with \( t \in E_\varpi^- \).

**Proof.** First, we prove that if there exists a nonzero valenced tangle \( T \in \text{TL}_\zeta(\nu) \) such that \( T \gamma = 0 \) for all valenced link states \( \gamma \in L_{\varpi}^{(s)} \) with \( t \in E_\varpi^- \), then the radical of \( L_\zeta^{(s)} \) is nondegenerate for some \( s \in E_- \). For this, we write
\[
T = \sum_{r \in E_\varpi^-} T_r, \quad \text{where } T_r \in \text{TL}_\zeta^{(r)(\nu)} \text{ as in } (2.12),
\] (3.37)
and we expand each valenced tangle \( T_r \) in the link pattern basis \( L_\varpi^{(s)} \) of \( \text{TL}_\zeta^{(r)(\nu)} \) given in (2.11), writing
\[
T_r = \sum_{\alpha \in L_\varpi^{(r)}, \beta \in L_\varpi^{(s)}} c_{\alpha,\beta}^{(r)} \mid \alpha \beta \mid
\] (3.38)
for some constants \( c_{\alpha,\beta}^{(r)} \in \mathbb{C} \). Now, we note that if \( \gamma \in L_\varpi^{(s)} \) and \( r < t \), then each term in \( T_r \gamma \) contains a turn-back link. By this observation, identity \( (3.5) \), and linearity, we therefore have
\[
\gamma \in L_\varpi^{(s)} \quad \implies \quad T_r \gamma = 0 \text{ for all } r < t.
\] (3.39)

Also, because \( T \neq 0 \), we may choose \( s \in E_\varpi^- \) to be the largest number such that \( c_{\alpha,\beta}^{(s)} \neq 0 \) in (2.46) for some pair of valenced link patterns \( \alpha \in L_\varpi^{(s)} \) and \( \beta \in L_\varpi^{(s)} \). Then, for all valenced link states \( \gamma \in L_{\varpi}^{(s)} \), using lemma 3.2, we get
\[
0 = T \gamma = \sum_{r \leq s} T_r \gamma = T_\gamma = \sum_{\beta \in L_\varpi^{(s)}} \sum_{\alpha \in L_\varpi^{(s)}} c_{\alpha,\beta}^{(s)} \mid \alpha \beta \gamma
\] (3.40)
\[
= \sum_{\alpha \in L_\varpi^{(s)}} \left( \Sigma_\alpha \mid \gamma \right) \alpha, \quad \text{where } \Sigma_\alpha := \sum_{\beta \in L_\varpi^{(s)}} c_{\alpha,\beta}^{(s)} \beta \in L_\varpi^{(s)}.
\] (3.41)

With the set \( L_\varpi^{(s)} \) linearly independent, this implies that for every \( \alpha \in L_\varpi^{(s)} \), we have \( \left( \Sigma_\alpha \mid \gamma \right) = 0 \), for all \( \gamma \in L_\varpi^{(s)} \). That is, \( \Sigma_\alpha \) lies in the radical of \( L_\varpi^{(s)} \), so this radical is nondegenerate.

Second, we prove that if the radical of \( L_{\varpi}^{(s)} \) is nondegenerate for some \( s \in E_\varpi^- \), then there exists a nonzero valenced tangle \( T \in \text{TL}_\zeta(\nu) \) such that \( T \gamma = 0 \) for all \( \gamma \in L_{\varpi}^{(s)} \) with \( t \in E_\varpi^- \). For this, we let \( s \) be the smallest number such that \( \text{rad } L_{\varpi}^{(s)} \neq \{0\} \), choose arbitrary nonzero valenced link states \( \delta \in \text{rad } L_{\varpi}^{(s)} \) and \( \epsilon \in L_{\varpi}^{(s)} \), and form the valenced tangle
\[
T := \mid \epsilon \delta \mid + \sum_{r < s} \sum_{\alpha \in L_\varpi^{(r)}} \sum_{\beta \in L_\varpi^{(s)}} c_{\alpha,\beta}^{(r)} \mid \alpha \beta \mid,
\] (3.42)
where the coefficients $c_{\alpha,\beta}^{(r)} \in \mathbb{C}$ are to be determined later. We immediately note that $T \neq 0$ because $\delta, \epsilon \neq 0$.

To finish, by linearity it suffices to show that there exists a $t$-independent choice of coefficients for $T$ such that $T_{\gamma} = 0$ for all valenced link patterns $\gamma \in \text{LP}^{(t)}_{\infty}$ with $t \in \mathbb{E}_{\infty}$. First, if $t \geq s$, then we immediately have $T_{\gamma} = 0$. Indeed, if $t > s$, then this follows from (3.39), and if $t = s$, then with $\delta \in \text{rad} \mathbb{L}^{(s)}$, we have

$$T_{\gamma} = \epsilon \delta \gamma + \sum_{r < s} \sum_{\alpha \in \text{LP}^{(r)}_{\infty}} \sum_{\beta \in \text{LP}^{(r)}_{\infty}} c_{\alpha,\beta}^{(r)} \alpha \beta \gamma = 0.$$  

(3.43)

Hence, we suppose $t < s$ from now on. We will prove recursively that there exists a $t$-independent choice of coefficients for $T$ such that $T_{\gamma} = 0$ for all valenced link patterns $\gamma \in \text{LP}^{(t)}_{\infty}$. Thus, we assume that for all $r \in \mathbb{E}_{\infty}$ such that $t < r < s$, and for all valenced link patterns $\alpha \in \text{LP}^{(r)}_{\infty}$ and $\beta \in \text{LP}^{(r)}_{\infty}$, there exist constants $b_{\alpha,\beta}^{(r)}$ such that the following holds:

$$\begin{cases} 
\gamma' \in \text{LP}^{(r)}_{\infty}, & \text{for any } r \in \mathbb{E}_{\infty} \text{ such that } t < r < s, \\
\gamma = b_{\alpha,\beta}^{(r)}, & \text{for all } r \in \mathbb{E}_{\infty} \text{ such that } t < r < s, \text{ and } \alpha \in \text{LP}^{(r)}_{\infty}, \beta \in \text{LP}^{(r)}_{\infty}, \implies T_{\gamma'} = 0. 
\end{cases}$$

(3.44)

Setting in the valenced tangle $T$ the constants $c_{\alpha,\beta}^{(r)} = b_{\alpha,\beta}^{(r)}$ for all of the above $r$, $\alpha$, and $\beta$, we have

$$T_{\gamma} = \epsilon \delta \gamma + \sum_{r = t+2}^{s-2} \sum_{\alpha \in \text{LP}^{(r)}_{\infty}} \sum_{\beta \in \text{LP}^{(r)}_{\infty}} b_{\alpha,\beta}^{(r)} \alpha \beta \gamma + \sum_{r < t} \sum_{\alpha \in \text{LP}^{(r)}_{\infty}} \sum_{\beta \in \text{LP}^{(r)}_{\infty}} c_{\alpha,\beta}^{(r)} \alpha \beta \gamma.$$  

(3.45)

Expanding this in the valenced link pattern basis $\text{LP}^{(t)}_{\infty}$, we may write the first and the second term in (3.45) in the following form, for some coefficients $c_{\alpha,\gamma} \in \mathbb{C}$ indexed by valenced link patterns $\alpha \in \text{LP}^{(t)}_{\infty}$ and $\gamma \in \text{LP}^{(t)}_{\infty}$:

$$\epsilon \delta \gamma + \sum_{r = t+2}^{s-2} \sum_{\alpha \in \text{LP}^{(r)}_{\infty}} b_{\alpha,\beta}^{(r)} \alpha \beta \gamma = \sum_{\alpha \in \text{LP}^{(t)}_{\infty}} c_{\alpha,\gamma} \gamma.$$  

(3.46)

After inserting this into (3.45), applying (3.25) to the third term in (3.45), and observing that the last term in (3.45) equals zero by (3.39), we obtain

$$T_{\gamma} = \sum_{\alpha \in \text{LP}^{(t)}_{\infty}} c_{\alpha,\gamma} \gamma + \sum_{\alpha \in \text{LP}^{(t)}_{\infty}} \sum_{\beta \in \text{LP}^{(t)}_{\infty}} c_{\alpha,\beta}^{(t)} \beta \gamma \alpha.$$  

(3.47)

Now, the requirement that (3.47) equals zero for all valenced link patterns $\gamma \in \text{LP}^{(t)}_{\infty}$ determines a linear system of equations for each valenced link pattern $\alpha \in \text{LP}^{(t)}_{\infty}$. Each system is of the form

$$G^{(t)}_{\infty} v_{\alpha} = -v'^{t}_{\alpha},$$  

(3.48)

where $v_{\alpha}$ and $v'^{t}_{\alpha}$ are respectively the vectors with components $c_{\alpha,\beta}^{(t)}$ and $c_{\alpha,\gamma}^{(t)}$ indexed by the valenced link patterns $\beta \in \text{LP}^{(t)}_{\infty}$. Now, because $t < s$ and $s$ is the smallest integer such that the radical of $\mathbb{L}^{(s)}_{\infty}$ is nondegenerate, we have

$$\text{rad} \mathbb{L}^{(t)}_{\infty} = \{0\} \implies \text{det} G^{(t)}_{\infty} \neq 0.$$  

(3.49)

Therefore, each linear system of this form has a unique solution. With $b_{\alpha,\beta}^{(t)}$ denoting the component indexed by $\beta$ of the solution $v_{\alpha}$ to the particular system indexed by $\alpha$, (3.44) now expands to say that

$$\begin{cases} 
\gamma' \in \text{LP}^{(r)}_{\infty}, \text{ for any } r \in \mathbb{E}_{\infty} \text{ such that } t \leq r < s, \\
\gamma = b_{\alpha,\beta}^{(t)}, \text{ for all } r \in \mathbb{E}_{\infty} \text{ such that } t \leq r < s, \text{ and } \alpha \in \text{LP}^{(r)}_{\infty}, \beta \in \text{LP}^{(r)}_{\infty}, \implies T_{\gamma'} = 0. 
\end{cases}$$

(3.50)

This shows that we may recursively determine coefficients $c_{\alpha,\beta}^{(t)}$ for $T$ in (3.42), indexed by $r \in \mathbb{E}_{\infty}$ and valenced link patterns $\alpha \in \text{LP}^{(r)}_{\infty}$ and $\beta \in \text{LP}^{(r)}_{\infty}$, in such a way that $T_{\gamma} = 0$ for all valenced link patterns $\gamma \in \text{LP}^{(t)}_{\infty}$ with $t \in \mathbb{E}_{\infty}$.  \[\square\]
The following corollary is the main result of this section, and it immediately follows from proposition 3.7.

**Corollary 3.8.** Suppose \( \max \varsigma < \bar{p}(q) \). Then, the link state representation of \( \mathbb{T} L_\varsigma(\nu) \) on \( L_\varsigma \) is faithful if and only if \( \text{rad} L_\varsigma = \{0\} \).

**Proof.** This follows from proposition 3.7 specialized to the case \( \omega = \varsigma \) and decomposition (3.20).

We note that corollary 3.8 does not hold if we replace \( L_\varsigma \) with \( L_\varsigma^{(s)} \) in it. Indeed, to paraphrase (3.39), if \( T \in \mathbb{T} L_\varsigma^{(s)}(\nu) \) for some \( r < s \), then we have \( T\alpha = 0 \) for all \( \alpha \in L_\varsigma^{(s)} \).

## 4. TRIVALENT LINK STATES AND GRAM MATRIX

Proposition 3.3 shows that the quotient \( Q_\varsigma^{(s)} \), if not trivial, is a simple \( \mathbb{T} L_\varsigma(\nu) \)-module. It is natural to ask what is the dimension of the simple module \( Q_\varsigma^{(s)} \) and when do we have the equality \( Q_\varsigma^{(s)} = L_\varsigma^{(s)} \), or equivalently, \( \text{rad} L_\varsigma^{(s)} = \{0\} \). The answer hinges on a complete understanding of the radical of \( L_\varsigma^{(s)} \).

In section 5, we completely determine the radical of \( L_\varsigma^{(s)} \) for all multiindices \( \varsigma \in \{0\} \cup \mathbb{Z}_{>0}^s \) and integers \( s \in E_\varsigma \). That is, we determine bases for and the dimensions of all of these radicals. In the present section, we introduce tools that we use in section 5 to complete these tasks. A key tool is the Gram matrix \( G_\varsigma^{(s)} \) of the bilinear form (3.17):

\[
[w_{\alpha \beta}]_{\alpha, \beta} := (\alpha \mid \beta), \quad \text{for all } \alpha, \beta \in L_\varsigma^{(s)}. \tag{4.1}
\]

The Gram matrix encodes valuable information about the radical of \( L_\varsigma^{(s)} \). Specifically, the dimension of \( \text{rad} L_\varsigma^{(s)} \) equals the nullity of \( G_\varsigma^{(s)} \), so \( \text{rad} L_\varsigma^{(s)} \) is trivial if and only if \( \det G_\varsigma^{(s)} \neq 0 \). Hence, we are interested in zeros of the determinant of \( G_\varsigma^{(s)} \). In proposition 4.8, we give an explicit formula for \( \det G_\varsigma^{(s)} \). We use this formula to prove the main result of this section, proposition 4.9, which says that \( \det G_\varsigma^{(s)} \neq 0 \) (thus, \( \text{rad} L_\varsigma^{(s)} \) is trivial) for all \( s \in E_\varsigma \) if \( n_\varsigma < \bar{p}(q) \).

However, this key result is not enough to determine all parameters \( q \in \mathbb{C}^x \) such that the radical of \( L_\varsigma^{(s)} \) trivial, let alone to find a basis for it. Because of its complexity, the formula appearing in proposition 4.8 for \( \det G_\varsigma^{(s)} \) is difficult to use when \( \bar{p}(q) \leq n_\varsigma \). For example, if \( \bar{p}(q) \neq 2 \), then proposition 4.9 in fact holds as an if-and-only-if statement (corollary 5.24), but one direction of it does not follow easily by using only the formula for the determinant.

Thus, in section 5 we turn to other methods for determining the radical of \( L_\varsigma^{(s)} \): we use special types of valenced link states that we call “trivalent link states.” They form an orthogonal basis for the standard module \( L_\varsigma^{(s)} \) if \( n_\varsigma < \bar{p}(q) \), as we prove in proposition 4.6. We define and study these link states in sections 4 A and 4 B. First, we give a simple definition for the trivalent link states, assuming \( n_\varsigma < \bar{p}(q) \). Then, for cases where this inequality does not hold, we give an alternative, more complicated definition, which reduces to the simpler one if \( n_\varsigma < \bar{p}(q) \).

In section 4 C, we use the basis of trivalent link states to diagonalize the Gram matrix \( G_\varsigma^{(s)} \), compute its determinant (proposition 4.8), and prove that this determinant does not vanish if \( n_\varsigma < \bar{p}(q) \) (proposition 4.9). In section 4 D, we derive various recursion formulas for the determinant of \( G_\varsigma^{(s)} \), to be used in section 5 and in subsequent work [FP18b+].

### A. Definition of the trivalent link states

In this section, we introduce trivalent link states. In our forthcoming article [FP18a+], we identify them with conformal blocks of CFT via relations that we call “the link-state spin-chain correspondence” [FP18b+] and the “spin-chain – Coulomb gas correspondence” [KP18].

To begin, we define the trivalent link states under the assumption that \( n_\varsigma < \bar{p}(q) \). In section 4 C, we use them to diagonalize the Gram matrix \( G_\varsigma^{(s)} \) and compute its determinant. We present the definition first and explain later why the assumption that \( n_\varsigma < \bar{p}(q) \) is needed. Using the open three-vertex notation

\[
\begin{align*}
\text{for } s \in E_{(r,t)},
\quad i &= \frac{r + s - t}{2}, \\
\quad j &= \frac{s + t - r}{2}, \\
\quad k &= \frac{t + r - s}{2},
\end{align*}
\tag{4.2}
\]
we write each $\varsigma$-valenced link pattern $\alpha$ in the following generic form of a trivalent graph with open vertices,

$$\alpha = \begin{array}{c}
\bullet & \bullet & \bullet & \bullet \\
& r_2 = 3 & r_3 = 2 & r_4 = 2 \\
& r_5 = 3 & & & \\
& s_1 = 2 & s_2 = 3 & s_3 = 3 & s_4 = 4 & s_5 = 3 \\
\end{array} \begin{array}{c}
\bullet & \bullet & \bullet & \bullet \\
& r_1 = 2 & & \\
& & r_2 = 3 & r_3 = 2 & r_4 = 2 \\
& & & r_5 = 3 \\
& & s_1 = r_1 & s_2 & s_3 & s_4 & s_5 = 3 \\
\end{array}
$$

which we call the walk representation of $\alpha$, and we let

$$\varrho_\alpha = (r_1, r_2, \ldots, r_\varsigma)$$

(4.4)
denote the multiindex of cable sizes in (4.3). Item 1 of lemma 4.1 below says that the multiindex $\varrho_\alpha$ is a walk over $\varsigma = (s_1, s_2, \ldots, s_\varsigma)$, that is, a multiindex $\varrho = (r_1, r_2, \ldots, r_\varsigma)$ whose entries, each called a height, satisfy the following two conditions relative to $\varsigma$:

$$r_0 = 0, \quad r_{i+1} \in E_{(r_i, s_{i+1})} = \{ |r_i - s_{i+1}|, |r_i - s_{i+1}| + 2, \ldots, r_i + s_{i+1} \} \quad \text{for all } i \in \{1, 2, \ldots, d_\varsigma - 1\}. \quad (4.5)$$
As a notation convention, we do not explicitly show the zeroth height \( r_0 = 0 \) of the walk \( \varrho = (r_1, r_2, \ldots, r_d) \) so the length of \( \varrho \) matches that of \( \varsigma \). However, it is convenient to implicitly include this entry for later use. We observe that
\[
r_0 = 0 \quad \text{and} \quad r_1 \in E(r_0, s_1). \tag{4.6}
\]

We may visualize a walk by joining the points \((j, r_j)\) and \((j+1, r_{j+1})\) with a line segment for each \( j \in \{0, 1, \ldots, d_\varsigma - 1\} \). For example, figure 1 shows a walk over the multiindex \( \varsigma = (2, 3, 3, 4, 3) \), representing a link pattern in \( \text{LP}(2, 3, 3, 4, 3) \).

We casually refer to the \( j \):th vertex as the \( j \):th "step" of the walk. For each walk \( \varrho \) over \( \varsigma \), there are two related walks over \( \vec{n}_\varsigma \) that are useful to consider:
\[
\varrho^{\uparrow} := \text{the unique highest walk over} \ \vec{n}_\varsigma \ \text{that touches the walk} \ \varrho \ \text{over} \ \varsigma \ \text{at all steps of the latter},
\tag{4.7}
\]
\[
\varrho^{\downarrow} := \text{the unique lowest walk over} \ \vec{n}_\varsigma \ \text{that touches the walk} \ \varrho \ \text{over} \ \varsigma \ \text{at all steps of the latter}. \tag{4.8}
\]

For example, figure 2 shows an illustration of the walks \( \varrho^{\uparrow} \) and \( \varrho^{\downarrow} \) relative to the walk \( \varrho \) of figure 1. For each walk \( \varrho \) over \( \varsigma \), we also define the following quantities:
\[
h_{\min,j}(\varrho) := \begin{cases} \frac{r_j + r_{j+1} - s_{j+1}}{2}, & j \in \{0, 1, \ldots, d_\varsigma - 1\}, \\ r_{d_\varsigma}, & j = d_\varsigma, \end{cases} \tag{4.9}
\]
\[
h_{\max,j}(\varrho) := \begin{cases} \frac{r_j + r_{j+1} + s_{j+1}}{2}, & j \in \{0, 1, \ldots, d_\varsigma - 1\}, \\ r_{d_\varsigma}, & j = d_\varsigma, \end{cases} \tag{4.10}
\]

and for each \( j \in \{0, 1, \ldots, d_\varsigma - 1\} \), we call \( h_{\max,j}(\varrho) \) the apex of the \((j + 1)\):st step of \( \varrho \), and we call \( h_{\max,d_\varsigma}(\varrho) = r_{d_\varsigma} \) the defect of the walk \( \varrho \). We note that
\[
0 \leq h_{\min,j}(\varrho) \leq h_{\max,j}(\varrho). \tag{4.11}
\]

for all \( j \in \{0, 1, \ldots, d_\varsigma \} \). Finally, for each \( j \in \{1, 2, \ldots, d_\varsigma \} \), using notation from (2.34), we define
\[
\mu_j(\varsigma) := \max\{s_{\min}(\varsigma_j), s_{\min}(\varsigma_j)\}, \tag{4.12}
\]
\[
M_j(\varsigma) := \min\{s_{\max}(\varsigma_j), s_{\max}(\varsigma_j)\}. \tag{4.13}
\]

**Lemma 4.1.** The following are true:

1. For each valenced link pattern \( \alpha \in \text{LP}_\varsigma \), the multiindex \( \varrho_\alpha \), defined in (4.4), is a walk over \( \varsigma \).

2. The map \( \alpha \mapsto \varrho_\alpha \) is a bijection from \( \text{LP}_\varsigma \) to the set of all walks over \( \varsigma \).

3. For each valenced link pattern \( \alpha \in \text{LP}_\varsigma \), the defect of \( \alpha \) equals the defect of \( \varrho_\alpha \).
4. The number $D^{(s)}_\zeta$ and the set $E_\zeta$, defined respectively in (2.54, 2.23), satisfy

\begin{align}
D^{(s)}_\zeta &= \#\{\text{walks over } \zeta \text{ with defect } s\}, \tag{4.14} \\
E_\zeta &= \{s \in \mathbb{Z}_{\geq 0} \mid \text{there exists a walk over } \zeta \text{ with defect } s\}. \tag{4.15}
\end{align}

5. If there exists a walk $\varrho$ over $\zeta$ with defect zero (i.e., $0 \in E_\zeta$ by (4.15)), then the set $E_{\zeta_j} \cap E_{\zeta_j}$ is nonempty and equals

\begin{equation}
E_{\zeta_j} \cap E_{\zeta_j} = \{\mu_j(\varsigma), \mu_j(\varsigma) + 2, \ldots, M_j(\varsigma)\}, \tag{4.16}
\end{equation}

for each $j \in \{1, 2, \ldots, d_\zeta\}$. Furthermore, the $j$:th height of any walk $\varrho$ over $\zeta$ with defect zero is an element of this set, and conversely, every element of this set equals the $j$:th height of some walk $\varrho$ over $\zeta$ with defect zero.

6. For each $j \in \{1, 2, \ldots, d_\zeta\}$, at the $j$:th step, the minimum height over all walks $\varrho$ over $\zeta$ that have defect zero equals

\begin{align}
\min_\varrho r_j = \mu_j(\varsigma) := \max\{s_{\min}(\varsigma_j), s_{\min}(\varsigma_j)\}. \tag{4.17}
\end{align}

7. For each walk $\varrho$ over $\zeta$ and for each $j \in \{1, 2, \ldots, d_\zeta - 1\}$,

(a): the maximal height of $\varrho^j$ between the heights $r_j$ and $r_{j+1}$ of $\varrho$ equals $h_{\max,j}(\varrho)$,

(b): the minimal height of $\varrho^j$ between the heights $r_j$ and $r_{j+1}$ of $\varrho$ equals $h_{\min,j}(\varrho)$.

8. For each $j \in \{0, 1, \ldots, d_\zeta - 1\}$, at the $(j+1)$:st step, the minimum apex over all walks $\varrho$ over $\zeta$ with defect zero is

\begin{equation}
\min_\varrho h_{\max,j}(\varrho) = \max\left\{\frac{\mu_j(\varsigma) + \mu_{j+1}(\varsigma) + s_{j+1}}{2}, s_{j+1}\right\}. \tag{4.18}
\end{equation}

Proof. We prove items 1–8 as follows:

1. From the definition (4.2) of the open three-vertex, it is clear that the entries of $\varrho_\alpha = (r_1, r_2, \ldots, r_{d_\zeta})$ satisfy the defining conditions (4.5) of a walk over $\zeta$.

2. By item 1, for each valenced link pattern $\alpha \in \text{LP}_\zeta$, the multiindex $\varrho_\alpha$ (4.4) is a walk over $\zeta$. On the other hand, we may substitute any walk $\varrho = (r_1, r_2, \ldots, r_{d_\zeta})$ over $\zeta$ into (4.3) to obtain a valenced link pattern $\alpha \in \text{LP}_\zeta$.

3. Each link in $\alpha$ contributes no net gain to the defect of $\varrho_\alpha$, but each defect of $\alpha$ contributes a gain of one to $\varrho_\alpha$.

4. Lemma 2.8 implies that $\# \text{LP}^{(s)}_\zeta = D^{(s)}_\zeta$, and items 2 and 3 imply that $\# \text{LP}^{(s)}_\zeta$ equals the number of walks over $\zeta$ with defect $s$. Now, (4.14) follows from these two facts, and (4.15) follows from this with definition (2.23) of $E_\zeta$.

5. If a walk $\varrho$ over $\zeta$ has defect zero, then we split it into two pieces: $(0 = r_0, r_1, r_2, \ldots, r_j)$, which is a walk over $\hat{\varsigma}_j$, and $(0 = r_{d_\zeta}, r_{d_\zeta-1}, r_{d_\zeta-2}, \ldots, r_j)$, which is a walk over $\tilde{\varsigma}_j$. Because $r_j$ is the defect of either walk, item 4 implies

\begin{equation}
r_j \in E_{\hat{\varsigma}_j} \cap E_{\tilde{\varsigma}_j}. \tag{4.19}
\end{equation}

This shows that the set $E_{\hat{\varsigma}_j} \cap E_{\tilde{\varsigma}_j}$ is nonempty and, by (2.25), it has the form of (4.16). Furthermore, for each element $s$ of this set, (4.15) of item 4 combined with (2.44) implies that there exists a walk over $\hat{\varsigma}_j$ with defect $s$ and another walk over $\tilde{\varsigma}_j$ also with defect $s$. After reflecting the latter about a vertical axis and joining it to the former from the right, we obtain a walk $\varrho$ over $\zeta = \hat{\varsigma}_j \oplus \tilde{\varsigma}_j$ with defect zero and with its $j$:th height $r_j$ equaling $s$.

6. After combining (4.16, 4.19) and taking the minimum over all walks $\varrho$ over $\zeta$ with defect zero, we infer that

\begin{align}
\mu_j(\varsigma) := \max\{s_{\min}(\varsigma_j), s_{\min}(\varsigma_j)\} \leq \min_{\varrho} r_j. \tag{4.20}
\end{align}

Because the left side is an element of $E_{\hat{\varsigma}_j} \cap E_{\tilde{\varsigma}_j}$, it follows from item 5 that there exists a walk $\varrho$ over $\zeta$ with defect zero and with its $j$:th height $r_j$ equaling this left side. Hence, (4.20) is really an equality, which gives (4.17).

7. The proof of item 7 only uses simple geometry. We leave the details to the reader.
8. To begin, we observe that by definition (4.5), for any walk \( q = (r_1, r_2, \ldots, r_{d_\varsigma}) \) over \( \varsigma \), we have \( r_{j+1} \in E_{(r_j, s_{j+1})} \), which by lemma 2.1 translates to

\[
s_{j+1} \in E_{(r_j, r_{j+1})} \quad \overset{(2.26)}{=} \quad \{ |r_j - r_{j+1}|, |r_j - r_{j+1}| + 2, \ldots, r_j + r_{j+1} \}.
\]  

Therefore, we have

\[
\min_{q} h_{\text{max}, j}(q) \overset{(4.10)}{=} \min_{q} \left( \frac{r_j + r_{j+1} + s_{j+1}}{2} \right) \overset{(4.21)}{\geq} s_{j+1}.
\]  

On the other hand, by item 6, we also have

\[
\min_{q} \left( \frac{r_j + r_{j+1} + s_{j+1}}{2} \right) \geq \frac{(\min_{q} r_j) + (\min_{q} r_{j+1}) + s_{j+1}}{2} \overset{(4.17)}{=} \frac{\mu_j(\varsigma) + \mu_{j+1}(\varsigma) + s_{j+1}}{2},
\]  

so altogether, (4.22–4.23) imply

\[
\min_{q} h_{\text{max}, j}(q) \geq \max \left\{ \frac{\mu_j(\varsigma) + \mu_{j+1}(\varsigma) + s_{j+1}}{2}, s_{j+1} \right\}.
\]  

Hence, to prove (4.18), it remains to prove the reverse inequality of (4.24). For this, we consider two cases:

(a): \( s_{j+1} \leq \mu_j(\varsigma) + \mu_{j+1}(\varsigma) \): Item 5 shows that there exist two walks \( q = (r_1, r_2, \ldots, r_{d_\varsigma}) \) and \( q' = (r'_1, r'_2, \ldots, r'_{d_\varsigma}) \) over \( \varsigma \), both with defect zero and such that we have

\[
\mu_j(\varsigma) - \mu_{j+1}(\varsigma) \overset{(4.17)}{\leq} r'_j - r'_{j+1} \leq s_{j+1} \quad \text{and} \quad \mu_{j+1}(\varsigma) - \mu_j(\varsigma) \overset{(4.17)}{\leq} r_{j+1} - r_j \leq s_{j+1}.
\]  

On the other hand, we observe that concatenating pieces of the walks \( q \) and \( q' \) we obtain a new walk \( (r_1, r_2, \ldots, r_j, r'_{j+1}, r'_2, \ldots, r'_{d_\varsigma}) \) over \( \varsigma \) with defect zero: indeed, lemma 2.1 shows that

\[
\left| \mu_j(\varsigma) - \mu_{j+1}(\varsigma) \right| \overset{(4.26)}{\leq} s_{j+1} \leq \mu_j(\varsigma) + \mu_{j+1}(\varsigma)
\]  

\[
\overset{(2.26)}{\Rightarrow} \quad s_{j+1} \in E_{(\mu_j(\varsigma), \mu_{j+1}(\varsigma))} \overset{(2.26)}{=} \{ |\mu_j(\varsigma) - \mu_{j+1}(\varsigma)|, \ldots, \mu_j(\varsigma) + \mu_{j+1}(\varsigma) \}
\]  

\[
\overset{(2.27)}{\Rightarrow} \quad \mu_{j+1}(\varsigma) \in E_{(\mu_j(\varsigma), s_{j+1})} \overset{(4.25)}{=} r'_{j+1} \in E_{(r_j, s_{j+1})}.
\]  

Therefore, we obtain

\[
\min_{q} h_{\text{max}, j}(q) \overset{(4.10)}{=} \frac{r_j + r'_{j+1} + s_{j+1}}{2} \overset{(4.25)}{=} \frac{\mu_j(\varsigma) + \mu_{j+1}(\varsigma) + s_{j+1}}{2}.
\]  

By our assumption \( s_{j+1} \leq \mu_j(\varsigma) + \mu_{j+1}(\varsigma) \), the right hand side of (4.30) equals the right hand side of (4.18). Thus, combining (4.24) and (4.30) gives asserted equality (4.18) in this case.

(b): \( \mu_j(\varsigma) + \mu_{j+1}(\varsigma) < s_{j+1} \): We observe that the sets

\[
E := E_{s_{j+1}} \cap E_{s_{j+1}} \overset{(4.16)}{=} \{ \mu_j(\varsigma), \mu_j(\varsigma) + 2, \ldots, M_j(\varsigma) \},
\]  

\[
F := s_{j+1} - (E_{s_{j+1}} \cap E_{s_{j+1}}) \overset{(4.16)}{=} \{ s_{j+1} - M_j(\varsigma), s_{j+1} - M_j(\varsigma) + 2, \ldots, s_{j+1} - M_j(\varsigma) \}
\]  

satisfy \( E \cap F \neq \emptyset \), because we have \( 0 \leq \mu_j(\varsigma) \leq s_{j+1} - \mu_j(\varsigma) \) by assumption and \( M_j(\varsigma) \geq s_{j+1} - M_j(\varsigma) \) by definition (4.13) (in fact, we have \( M_j(\varsigma) \geq s_{j+1} \)). Hence, item 5 shows that there exist two walks \( q = (r_1, r_2, \ldots, r_{d_\varsigma}) \) and \( q' = (r'_1, r'_2, \ldots, r'_{d_\varsigma}) \) over \( \varsigma \), both with defect zero and such that we have

\[
r_j = s_{j+1} - r'_{j+1}.
\]
Furthermore, the walk \((r_1, r_2, \ldots, r_j, r_{j+1}', r_{j+2}', \ldots, r_{d_\varsigma}')\) obtained by concatenating pieces of the walks \(\varrho\) and \(\varrho'\) is a walk over \(\varsigma\) with defect zero. Therefore, we have

\[
\min_{\varrho} h_{\max,j}(\varrho) \leq \frac{r_j + r_{j+1}' + s_{j+1}}{2} \overset{(4.10)}{=} s_{j+1}.
\]

By our assumption \(\mu_j(\varsigma) + \mu_{j+1}(\varsigma) < s_{j+1}\), the right hand side of (4.34) equals the right hand side of (4.18). Thus, combining (4.24) and (4.34) gives asserted equality (4.18) in this case.

This concludes the proof.

Replacing \(\varsigma\) with the symbol \(\vartheta\) for now, items 5 and 6 of lemma 4.1 may seem to be useful only in the narrow case that \(0 \in E_\vartheta\). However, by lemma 2.3, we have

\[
\{ s \in E_\vartheta \mid \varsigma = \vartheta \oplus (s) \implies 0 = s_{\min}(\varsigma) \overset{(2.25)}{\in} E_\varsigma, \]

which allows us to adapt items 5 and 6 to useful statements when \(0 \not\in E_\vartheta\). We leave the details to the reader.

Now we construct an orthogonal basis of \(L_\varsigma\), assuming that \(n_\varsigma < \bar{p}(q)\). A key element of our construction is the following closed three-vertex notation [KL94, MV94, CFS95]:

For each \(\varsigma\)-valenced link pattern \(\alpha\), we define the trivalent link state \(\alpha^\varsigma\) to be the following \(\varsigma\)-valenced link state:

\[
\alpha^\varsigma := \alpha_{r_1, r_2, \ldots, r_{d_\varsigma}}^{s_1, s_2, \ldots, s_{d_\varsigma}}
\]

that is, to obtain \(\alpha^\varsigma\) from \(\alpha\), we replace the \(j\):th open vertex in the walk representation (4.3) of the link pattern \(\alpha\) with a closed vertex for each step \(j \in \{1, 2, \ldots, d_\varsigma - 1\}\) of the walk. Replacing the \(j\):th open vertex with a closed vertex inserts three projector boxes on the cables of respective sizes \(r_j, s_{j+1}, \) and \(r_{j+1}\). Because we have

\[
\begin{align*}
0 \leq r_j & \overset{(2.26)}{\leq} r_{j-1} + s_j & \leq & \overset{(2.26)}{\leq} r_{j-2} + s_{j-1} + s_j & \overset{(2.26)}{\leq} \cdots \leq & \overset{(2.26)}{\leq} s_1 + s_2 + \cdots + s_j & \overset{(2.3)}{<} n_\varsigma < \bar{p}(q) \overset{(4.39)}{\leq} \bar{p}(q) - 1,
\end{align*}
\]

for all \(j \in \{1, 2, \ldots, d_\varsigma - 1\}\), these projector boxes exist. Moreover, by lemma B.2 of appendix B, we may freely omit the projector box across the \(r_{d_\varsigma} = s\) defects. Finally, it is evident that we have

\[
\alpha \in L_{\varsigma}^{\varsigma} \implies \alpha^\varsigma \in L_{\varsigma}^{\varsigma}.
\]

If \(\bar{p}(q) \leq n_\varsigma\), then our definition of the trivalent link state \(\alpha^\varsigma\) may be invalid because it uses a projector box with size greater than \(\bar{p}(q) - 1\), which does not exist. However, we can overcome this problem and give a useful definition for \(\alpha^\varsigma\) whenever \(\max \varsigma < \bar{p}(q)\). We use this definition to investigate the radical of \(L_{\varsigma}^{\varsigma}\) in section 5.
To define trivalent link states \( \mathcal{A} \) that make sense under the weaker condition \( \max \varsigma < \bar{p}(q) \), we need some more terminology. First, we recall definitions (1.20, 1.21) of the symbols \( \Delta_k \) and \( R_{\varsigma} \) from section 1 and observe that

\[
\begin{align*}
\varsigma \in \Delta_{k_{\varsigma}}, \Delta_{k_{\varsigma} + 1}, \ldots, \Delta_{k_{\varsigma} + 1} - 1. \quad (4.41)
\end{align*}
\]

Next, for each walk \( q = (r_1, r_2, \ldots, r_\varsigma) \) over \( \varsigma \) with defect \( s \), we let \( J \) denote the special index

\[
J = J_q(q) := \sup \left\{ j \in \mathbb{Z}_{>0} \mid h_{\min,j}(q) \leq \Delta_{k_{\varsigma}} \text{ or } \Delta_{k_{\varsigma} + 1} \leq h_{\max,j}(q) \right\},
\]

with the convention that \( \sup \emptyset = -\infty \). If \( J \geq 0 \), then we divide \( q \) into two pieces, called the head of \( q \) and tail of \( q \):

\[
(\varrho, q) \quad \Rightarrow \quad \begin{cases} 
\text{head}(q) := (r_0, r_1, \ldots, r_{J-1}), \\
\text{tail}(q) := (r_J, r_{J+1}, \ldots, r_\varsigma), 
\end{cases}
\]

where \( J = J_q(q) \). (4.43)

We also define the head and tail of a link pattern \( \alpha \) to be the head and tail of its corresponding walk \( q_{\alpha} \), and we write

\[
J = J_\alpha(q) := J_{q_\alpha}(q), \quad \text{head}(\alpha) := \text{head}(q_\alpha), \quad \text{and} \quad \text{tail}(\alpha) := \text{tail}(q_\alpha). \quad (4.44)
\]

Now we are ready to define the trivalent link states \( \mathcal{A} \) for all \( \varsigma \)-valenced link patterns \( \alpha \) with \( \max \varsigma < \bar{p}(q) \).

**Definition 4.2.** To obtain the trivalent link state \( \mathcal{A} \) from \( \alpha \), we write the walk representation of \( \alpha \) in a different way: for each \( j \in \{1, 2, \ldots, d_{\varsigma} - 1\} \), we replace the \( j \)-th open vertex of \( \alpha \) with \( s_{j+1} \) adjacent vertices as follows:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\infty \quad (4.45)
\end{array}
\end{array}
\end{align*}
\]

As shown, each link from the cable of size \( s_{j+1} \) enters its own vertex. Making the replacement (4.45) at each open vertex in the original walk representation (4.3) of \( \alpha \) gives a new walk representation for \( \alpha \). Now, for each \( \varsigma \)-valenced link state \( \alpha \), the lowest walk \( q^+_{\alpha} \) is the path connecting all of the open vertices in this new walk representation of \( \alpha \). Figure 3 shows an example of these two walk representations of a link pattern \( \alpha \).

Next, starting from the rightmost vertex and proceeding leftwards, we replace each open vertex in the walk \( q^+_{\alpha} \) of the new walk representation of \( \alpha \) with a closed vertex. If \( J = -\infty \), then we make this replacement at all vertices. Otherwise, the last vertex replacement occurs between the \( J \)-th and \((J + 1)\)-st steps of \( q_\alpha \) at the first time that

1. we arrive at a step of the walk \( q^+_{\alpha} \) whose height equals \( \Delta_{k_{\varsigma} + 1} \), or
2. we arrive at the \( J \)-th step of the walk \( q_\alpha \), with height \( r_J \) satisfying \( \Delta_{k_{\varsigma}} < r_J < \Delta_{k_{\varsigma} + 1} \), or
3. we arrive at a step of the walk \( q^+_{\alpha} \) whose height equals \( \Delta_{k_{\varsigma}} \).

After making the last vertex replacement, we arrive with \( \mathcal{A} \). Figures 4, 5, and 6 show examples of trivalent link states \( \mathcal{A} \) derived from each of these stopping conditions.

One can check that if \( J = 0 \), then item 1 always gives the stopping condition. On the other hand, if \( n_{\varsigma} < \bar{p}(q) \), then we have \( J_\alpha(q) = -\infty \) for all \( \varsigma \)-valenced link patterns \( \alpha \), and in this case, the new definition 4.2 of \( \mathcal{A} \) reduces to the old definition (4.37) for it, because definition (4.36) of the closed three-vertex and property (P1') of the Jones-Wenzl
FIG. 3: Two walk representations of a link pattern $\alpha \in \text{LP}_{(2,3,3,4,3)}$ and the associated walks: the black walk is $\varrho_\alpha$ over $\varsigma = (2,3,3,4,3)$, associated to the walk representation of type (4.3), and the blue walk $\varrho^\downarrow$ over $\bar{\varsigma} = \bar{15}$ gives an alternative walk representation of $\alpha$ via replacements (4.45).
FIG. 4: Tail of a trivalent link state $\epsilon$ associated to a $(\varsigma, s)$-valenced link pattern $\alpha$ when stopping condition 1 occurs. The lowest walk $\varrho_{\alpha}^{\downarrow}$ and the associated walk representation is depicted in blue, the highest walk $\varrho_{\alpha}^{\uparrow}$ in red, and the walk $\varrho_{\alpha}$ in black.

projectors show that

$$\begin{align*}
\varrho_{j} & = 
\begin{cases}
\varrho_{j+1} & \text{if } j \neq n_{\varsigma} - 1 \\
0 & \text{if } j = n_{\varsigma} - 1
\end{cases}
\end{align*}$$

(4.46)

Remark 4.3. In contrast to (4.37), in this new definition of $\epsilon$ we do not require $n_{\varsigma} < \bar{p}(q)$. If $\bar{p}(q) \leq n_{\varsigma}$, then the trivalent link state $\alpha$ may contain projector boxes of size at least $\bar{p}(q)$, which are undefined. In this situation, we let $\alpha_{q'}$ denote the trivalent link state $\alpha$ with $q$ perturbed to some value $q' \in \mathbb{C}^\times$ with $\bar{p}(q') = \infty$ (so projector boxes of all sizes exist) while holding $J$ fixed (so $J = J_{\alpha}(q) \neq J_{\alpha}(q')$). Then we define $\epsilon$ to be the limit of $\alpha_{q'}$ as $q' \to q$ along a sequence not containing roots of unity. We show that this limit exists in appendix E, lemma E.1.

B. Properties of the trivalent link states

In this section, we determine important properties of the trivalent link states. Most importantly, we show in item 3 of proposition 4.6 that the set $\{\epsilon | \alpha \in LP^{(s)}_{\varsigma}\}$ is a basis for $L^{(s)}_{\varsigma}$, and if $n_{\varsigma} < \bar{p}(q)$, then this basis is orthogonal.
FIG. 5: Tail of a trivalent link state $\alpha$ associated to a $(\varsigma, s)$-valenced link pattern $\alpha$ when stopping condition 2 occurs. The lowest walk $\varrho_{\alpha}^\downarrow$ and the associated walk representation is depicted in blue, the highest walk $\varrho_{\alpha}^\uparrow$ in red, and the walk $\varrho_{\alpha}$ in black.

FIG. 6: Tail of a trivalent link state $\alpha$ associated to a $(\varsigma, s)$-valenced link pattern $\alpha$ when stopping condition 3 occurs. The lowest walk $\varrho_{\alpha}^\downarrow$ and the associated walk representation is depicted in blue, the highest walk $\varrho_{\alpha}^\uparrow$ in red, and the walk $\varrho_{\alpha}$ in black.

Lemma 4.4. Suppose $\max \varsigma < \overline{p}(q)$. Let $B_\varsigma$ be a basis for $L_\varsigma$, all of whose elements $\alpha$ may be written in the form

\[
\begin{array}{c}
\alpha_2 \\
\vdots \\
\alpha_1 \\
\vdots \\
\alpha_s
\end{array}
\]

(4.47)
for some integers \(i \in \{1, 2, \ldots, d_\varsigma - 1\}\) and \(j \in \{i + 1, i + 2, \ldots, d_\varsigma\}\) common to all elements of \(B_\varsigma\), and (with \(d = d_\varsigma\))

\[
t_\alpha \in E_{(s_{i+1}, s_{i+2}, \ldots, s_j)}; \quad \alpha_1 \in L_{(t_\alpha)}^{(s_{i+1}, s_{i+2}, \ldots, s_j)}; \quad \alpha_2 \in L_{(s_1, s_2, \ldots, s_{i}, s_\alpha, s_{j+1}, s_{j+2}, \ldots, s_d)}.
\] (4.48)

Also, let \(T: L_\varsigma \rightarrow L_\varsigma\) be the linear extension of the map sending each element \(\alpha \in B_\varsigma\), represented as (4.47), to

\[
\begin{aligned}
\alpha_2 \\
\vdots \\
\alpha_1 \\
t_1 \\
\vdots \\
t_k
\end{aligned}
\]

(4.49)

for some integers \(k, t_1, t_2, \ldots, t_k \in \mathbb{Z}_{\geq 0}\) depending on \(\alpha\) and such that \(t_1 + t_2 + \cdots + t_k = t_\alpha\), with \(k\) vanishing only if \(t_\alpha = 0\). Then \(T\) has an upper triangular matrix representation, whose all diagonal entries equal one.

**Proof.** The following relation endows the basis \(B_\varsigma\) with a strict partial ordering:

\[
t_\alpha < t_\beta \implies \alpha < \beta.
\] (50)

Now, for \(\alpha \in B_\varsigma\) represented as (4.47), we decompose the \(k\) projector boxes over their internal link diagrams. Using recursion relation (2.65), we arrive with a sum of the form

\[
T(\alpha) \stackrel{(2.65)}{=} \sum_{\beta \in B_\varsigma} T_{\beta, \alpha} \beta, \quad \text{with } T_{\beta, \alpha} \begin{cases} \in \mathbb{C}, & \alpha > \beta, \\
1, & \alpha = \beta, \\
0, & \text{otherwise.} \end{cases}
\] (51)

The coefficients \(T_{\beta, \alpha}\) form a matrix representation of the linear operator \(T\) with respect to the basis \(B_\varsigma\). If we arrange the elements of \(B_\varsigma\) in such a way that the column for \(\alpha\) is left of the column for \(\beta\) if \(\alpha < \beta\), then \((T_{\beta, \alpha})\) is an upper triangular matrix whose all diagonal entries equal one.

**Corollary 4.5.** Suppose \(\max \varsigma < \bar{p}(q)\). Then the self-map of \(L_\varsigma\) given by linear extension of \(\alpha \mapsto \uparrow \uparrow\) is an automorphism of vector spaces with an upper triangular matrix representation, whose all diagonal entries equal one.

**Proof.** This map is a composition of linear maps of the type in (4.49) of lemma 4.4, each of which has an upper triangular matrix representation, whose all diagonal entries equal one. Hence, the claim follows.

Now we use corollary 4.5 to prove that the set \(\{\alpha | \alpha \in LP_\varsigma, \max \varrho_\alpha < \bar{p}(q)\}\) is orthogonal and linearly independent. We use lemmas B.4 and B.5 of appendix B, the latter containing the evaluation of the Theta network from lemma B.7 of the same appendix:

\[
\Theta(r, s, t) = \frac{(-1)^{r+s+t} \left[ \frac{r+s+t}{2} + 1 \right]! \left[ \frac{r+s-t}{2} \right]! \left[ \frac{s+t-r}{2} \right]! \left[ \frac{t+r-s}{2} \right]!}{[r]![s]![t]!}.
\] (52)

**Proposition 4.6.** Suppose \(\max \varsigma < \bar{p}(q)\). Then the following are true:

1. For any valenced link patterns \(\alpha, \beta \in LP_\varsigma\) with \(J_\alpha(q) = J_\beta(q) = -\infty\), writing \(\varrho_\alpha = (r_1, r_2, \ldots, r_{d_\varsigma})\), we have

\[
\langle \alpha | \not=_\beta \rangle = \delta_{\alpha, \beta} \prod_{j=1}^{d_\varsigma-1} \frac{\Theta(r_j, r_{j+1}, s_{j+1})}{(-1)^{r_j+s_j} [r_j+s_j+1]},
\] (53)

where \(\Theta(r, s, t)\) is given in (4.52).

2. The collection \(\{\alpha | \alpha \in LP_\varsigma, J_\alpha(q) = -\infty\}\) is orthogonal and linearly independent.

3. The collection \(\{\alpha | \alpha \in LP_\varsigma^{(\varsigma)}\}\) is a basis for \(L_\varsigma^{(\varsigma)}\), and if \(n_\varsigma < \bar{p}(q)\), then this basis is orthogonal.
Proof. We prove items 1–3 as follows:

1. For $\alpha, \beta \in \mathbb{L}_\zeta$ with $J_\alpha(q) = J_\beta(q) = -\infty$, the bilinear form $\langle \alpha | \beta \rangle$ equals the evaluation of the following network:

![Network Diagram](4.54)

To evaluate this network, we use lemmas B.4 and B.5 of appendix B to recursively erase the smallest (i.e., the leftmost) loop in it, eventually arriving with (4.53).

2. Orthogonality of the collection $\{\alpha | \alpha \in \mathbb{L}_\zeta, J_\alpha(q) = -\infty\}$ follows from (4.53). Linear independence of this collection follows from corollary 4.5 and the fact that $\mathbb{L}_\zeta$ is a basis for $L_\zeta$.

3. That the set $\{\alpha | \alpha \in \mathbb{L}_\zeta^{(s)}\}$ is a basis for $L_\zeta^{(s)}$ follows from corollary 4.5 and the fact that $\mathbb{L}_\zeta^{(s)}$ is a basis for it. Orthogonality follows from item 1 with the fact that $J_\alpha(q) = -\infty$ for all $\zeta$-valenced link patterns $\alpha$ if $n_\zeta < p(q)$.

This concludes the proof. \qed

Now we study the product (4.53) in item 1 of proposition 4.6 when the assumption that $J_\alpha(q) = -\infty$ may not hold (but we still have $\max \zeta < p(q)$). In the next lemma, we show that certain factors in the product (4.53) are finite and nonzero. We use this result to prove proposition 4.9 below, and to determine the radical of the $L_\zeta^{(s)}$ in section 5.

Lemma 4.7. Suppose $\max \zeta < p(q)$. Let $\varrho = (r_1, r_2, \ldots, r_{d_\zeta})$ be a walk over $\zeta$, and let $J := \max(J, 0)$. Then for all $j \in \{J + 1, J + 2, \ldots, d_\zeta - 1\}$, we have

$$0 < \left| \frac{\Theta(r_j, r_{j+1}, s_{j+1})}{(-1)^{r_{j+1}}[r_{j+1} + 1]} \right| < \infty. \quad (4.55)$$

Proof. We fix $j \in \{J + 1, J + 2, \ldots, d_\zeta - 1\}$ and consider the different factors in

$$\left| \frac{\Theta(r_j, r_{j+1}, s_{j+1})}{(-1)^{r_{j+1}}[r_{j+1} + 1]} \right| \overset{(4.52)}{=} \left| \frac{[r_j + r_{j+1} + s_{j+1} + 1]! [r_j + r_{j+1} - s_{j+1} + 1]! [r_j + r_{j+1} + 1]! [s_{j+1} + r_{j+1} + 1]!}{[r_j]! [r_{j+1} + 1]! [s_{j+1}]!} \right|. \quad (4.56)$$

First, we recall from (4.5) that $r_{i+1} \in E_{(r_i, s_{i+1})}$ for any $i \in \{0, 1, \ldots, d_\zeta - 1\}$. After combining this with (2.26, 2.27) of lemma 2.1, we obtain

$$|r_i - r_{i+1}| \leq s_{i+1}, \quad r_{i+1} \leq r_i + s_{i+1}, \quad \text{and} \quad r_i \leq r_{i+1} + s_{i+1}, \quad (4.57)$$

for any $i \in \{0, 1, \ldots, d_\zeta - 1\}$. Thus, with $\max \zeta < p(q)$, we have the following for any index $i$ in this set:

$$0 \overset{(4.57)}{\leq} \max \left( \frac{s_{i+1} - (r_j - r_{i+1})}{2}, \frac{s_{i+1} - (r_{i+1} - r_i)}{2} \right) \overset{(4.57)}{\leq} s_{i+1} < \max \zeta < p(q). \quad (4.58)$$

By definition (2.66), the quantum integer $[k]$ does not vanish if $k \in \{0, 1, \ldots, p(q) - 1\}$, so (4.58) shows that the quantum factorials $[s_{j+1}]!$, $[\frac{1}{2}(r_{j+1} + s_{j+1} - r_j)]!$, and $[\frac{1}{2}(s_{j+1} + r_j - r_{j+1})]$! in (4.56) are nonzero.
Second, we show that the zeros of the remaining factors in (4.56) cancel, so that their ratio is also finite and nonzero:

$$0 < \left| \frac{[r_j + r_{j+1} + s_{j+1}]!}{[r_j + r_{j+1} - s_{j+1}]!} \right| < \infty.$$  \hfill (4.59)

To see this, we observe that, for any \( j \in \{J + 1, J + 2, \ldots, d_c - 1\} \), we have

$$k_s p(q) = \Delta_{k_s} + 1 \geq h_{\min, j}(q) \geq \frac{r_j + r_{j+1} - s_{j+1}}{2} \geq \min(r_j, r_{j+1}) < \max(r_j, r_{j+1}) + 1$$

$$\geq \frac{r_j + r_{j+1} + s_{j+1}}{2} + 1 \geq h_{\max, j}(q) + 1 \geq \Delta_{k_s+1} \geq (k_s + 1)p(q) - 1.$$  \hfill (4.60)

From definition (2.66), we see that \([k]q = 0\) if and only if \( p(q) \mid k \), and these zeros of \([k]q\) as a function of \( q \in \mathbb{C}^* \) are of first order. Therefore, (4.60) implies (4.59), which implies (4.55) and concludes the proof.

\[\square\]

C. Determinant of the Gram matrix

Now we use proposition 4.6 to find a formula for the determinant of the Gram matrix \( G_s^{(s)} \), defined in (4.1). We use this formula to prove that if \( n_c < \bar{p}(q) \), then \( \det G_s^{(s)} \neq 0 \), i.e., the radical of \( L_s^{(s)} \) trivial, for all \( s \in \mathbb{E}_c \).

**Proposition 4.8.** Suppose \( \max \varsigma < \bar{p}(q) \). Then we have

$$\det G_s^{(s)} = \prod_{\varsigma} \prod_{j=1}^{d_c-1} \Theta(r_j, r_{j+1}, s_{j+1}) (-1)^{r_{j+1}[r_{j+1} + 1]}.$$  \hfill (4.61)

where the first product is over all walks \( q = (r_1, r_2, \ldots, r_{d_c}) \) over \( \varsigma \) with defect \( r_{d_c} = s \) and \( \Theta(r, s, t) \) is given in (4.52).

**Proof.** For all \( \alpha, \beta \in L_s^{(s)} \), the bilinear form \( (\alpha \mid \beta) \) is a continuous function of \( q \in \{q \in \mathbb{C}^* \mid \max \varsigma < \bar{p}(q)\} \). Thus, the determinant of the Gram matrix \( G_s^{(s)} \), defined in terms of this bilinear form in (4.1), is also continuous on this set. Hence, we may assume that \( n_c < \bar{p}(q) \). Then by corollary 4.5 and items 1 and 3 of proposition 4.6, we have

$$\det G_s^{(s)} = \det[(\alpha \mid \beta)]_{\alpha, \beta \in L_s^{(s)}} \geq \prod_{\alpha \in L_s^{(s)}} \prod_{j=1}^{d_c-1} \Theta(r_j, r_{j+1}, s_{j+1}) (-1)^{r_{j+1}[r_{j+1} + 1]}.$$  \hfill (4.62)

To obtain the determinant formula (4.61) from this expression, we use the bijection of item 2 in lemma 4.1, sending \( \alpha \mapsto g_\alpha \) for all \( \alpha \in L_s^{(s)} \), to index the product by all walks over \( \varsigma \) with defect \( s \).

\[\square\]

In some cases, an alternative formula for the determinant \( \det G_s^{(s)} \) is known. For example, in the case with \( \varsigma = \bar{n} \) and \( s = 0 \) (so \( n \) is necessarily even by (1.7)), [DGG97, equation (5.6)] gives an explicit formula for the determinant of the Gram matrix \( G_n^{(0)} \), now called the meander matrix. Next, in the case with \( \varsigma = \bar{n} \), [RSA14, theorem 4.7] gives an alternative formula for the determinant of \( G_n^{(s)} \) (see also [GL98, Corollary 4.7] for a more general case). We state this formula as lemma 4.11 in section 4 D below, and we derive it from our formula (4.61) for use in section 5. The last special case that we know of has \( \varsigma = (1, 1, \ldots, 1, k) \) for some \( k \in \mathbb{Z}_{\geq 0} \). In this case, [MDRR15, proposition D.4] gives an alternative formula for the determinant of \( G_s^{(1, \ldots, 1, k)} \).

Now, using the explicit formula (4.61) for the determinant of \( G_s^{(s)} \) from proposition 4.8, we deduce that this determinant is not zero if \( n_c < \bar{p}(q) \). This is the main result of the present section.

**Proposition 4.9.** Suppose \( n_c < \bar{p}(q) \). Then we have \( \det G_s^{(s)} \neq 0 \) for all \( s \in \mathbb{E}_c \).
Then, in lemma 4.13, we derive a formula for the determinant of another Gram matrix

\[ \text{Proof.} \]

If \( p(q) = 1 \) (so \( q \in \{ \pm 1 \} \) and \( \bar{p}(q) = \infty \) by (1.19)), then definition (2.66) shows that we have \( [k] \neq 0 \) for all \( k \in \mathbb{Z}_{>0} \). Then the claim follows from this fact and the explicit formula (4.61) for the determinant of \( G^{(j)} \).

Throughout the rest of the proof, we therefore assume that \( p(q) \neq 1 \), so \( \bar{p}(q) = p(q) \) by (1.19), and we let \( s \) be an arbitrary integer in \( E_s \). Then we have

\[ \begin{cases} s \in E_s \\ n_s < \bar{p}(q) = p(q) \end{cases} \]

(4.25) \[ s \leq n_s < p(q) \]

(4.12) \[ k_s = 0. \]

(4.63)

We divide the proof into two cases:

1. \( s < p(q) - 1 \): For any walk \( q = (r_1, r_2, \ldots, r_d) \) over \( \varsigma \) with defect \( s \) and for all \( j \in \{0, 1, \ldots, d_\varsigma - 2\} \), we have

\[ \Delta_{k_s} \]

(4.20) \[ -1 < 0 \]

(4.63) \[ \leq h_{\text{min}, j}(q) \]

(4.11) \[ \leq h_{\text{max}, j}(q) \]

(4.5) \[ \leq r_j + s_{j+1} \]

(4.38) \[ n_s \leq p(q) - 1 \]

(4.12) \[ \Delta_{k_{s+1}}. \]

(4.64)

Moreover, for \( j = d_\varsigma - 1 \), we use the fact that \( s < p(q) - 1 \) to obtain

\[ \Delta_{k_s} \]

(4.20) \[ -1 < 0 \]

(4.63) \[ \leq h_{\text{min}, d_\varsigma - 1}(q) \]

(4.11) \[ \leq h_{\text{max}, d_\varsigma - 1}(q) \]

(4.10) \[ \leq \frac{n_s + s}{2} \]

(4.38) \[ p(q) - 1 \]

(4.63) \[ \Delta_{k_{s+1}}. \]

(4.65)

and finally, for \( j = d_\varsigma \), we again use the fact that \( s < p(q) - 1 \) to obtain

\[ \Delta_{k_s} \]

(4.20) \[ -1 < 0 \]

(4.63) \[ \leq s \]

(4.9) \[ = h_{\text{min}, d_\varsigma}(q) \]

(4.49) \[ \leq h_{\text{max}, d_\varsigma}(q) \]

(4.10) \[ s < p(q) - 1 \]

(4.63) \[ \Delta_{k_{s+1}}. \]

(4.66)

Altogether, (4.64–4.66) imply that \( J(q) = -\infty \) for any walk \( q \) over \( \varsigma \) with defect \( s \). Then lemma 4.7 implies that the product in the formula (4.61) for the determinant of \( G^{(s)} \) does not vanish.

2. \( s = p(q) - 1 \): We have

\[ \begin{cases} s \leq n_\varsigma \\ s = p(q) - 1 \implies s = p(q) - 1 = n_\varsigma. \\ n_s < p(q) \end{cases} \]

(4.67)

Now, there exists only one walk over \( \varsigma \) with defect \( s = n_\varsigma \): that with heights \( r_j = s_1 + s_2 + \cdots + s_j \) for all \( j \in \{1, 2, \ldots, d_\varsigma\} \). Moreover, it follows from (4.52) that for any pair of integers \( r, t \in \mathbb{Z}_{>0} \), we have

\[ \Theta(r, t, r + t) = 1. \]

(4.68)

Combining these facts with (4.61), we see that the determinant of \( G^{(s)} \) equals one in this case.

This concludes the proof. \( \square \)

D. Recursion formulas for the Gram determinant

In this section, we gather more results on the determinant of the Gram matrix \( G^{(s)} \). First, in lemma 4.10 we give a general recursion formula for the determinant of \( G^{(s)} \) and in lemma 4.11, we employ it to derive an alternative formula for this determinant when \( \varsigma = n \) for some integer \( n \in \mathbb{Z}_{>0} \). This formula already appears, e.g., in [Wes95, RSA14]. Then, in lemma 4.13, we derive a formula for the determinant of another Gram matrix \( G^{(s)} \), related to the original
\( \mathcal{G}_\xi^{(s)} \) and defined below. We use these results later in section 5. Finally, in lemma 4.14 we obtain yet another recursion formula for the determinant of \( \mathcal{G}_n^{(s)} \) in the case that \( \varsigma = \bar{n} \), which we use in our forthcoming article [FP18b⁺].

In the case that \( \varsigma = \bar{n} \), explicit formulas and recursions similar to those appearing in this section have been obtained previously in [GL98, RSA14].

For the next lemma, we use the notations \( \hat{\varsigma} = (s_1, s_2, \ldots, s_{d-1}) \) and \( \bar{t} = s_d \).

**Lemma 4.10.** Suppose \( \max \varsigma < \bar{p}(q) \). Then we have the recursion

\[
\det \mathcal{G}_\varsigma^{(s)} = \prod_{r' \in \mathcal{E}_\varsigma \cap \mathcal{E}_{(t,s)}} (\det \mathcal{G}_r^{(t)})^{D^{(r)}_{\bar{t}}} \det \mathcal{G}_\varsigma^{(s)}. \tag{4.69}
\]

**Proof.** The following fact is evident from the definition (4.5) of a walk \( \varrho \) over \( \varsigma \):

\[
\varrho = (r_1, r_2, \ldots, r_d) \text{ is a walk over } \varsigma \text{ if and only if } \hat{\varrho} = (r_1, r_2, \ldots, r_{d-1}) \text{ is a walk over } \hat{\varsigma}. \tag{4.70}
\]

Therefore, by item 4 of lemma 4.1, the penultimate height \( r_{d-1} \) of a walk \( \varrho \) over \( \varsigma \) is an element of \( \mathcal{E}_\varsigma \). Moreover, definition (4.5) immediately gives

\[
\varrho = (r_1, r_2, \ldots, r_{d-1}) \text{ is a walk over } \varsigma \text{ with defect } r_{d-1} = s \implies r_{d-1} \in \mathcal{E}_\varsigma. \tag{4.71}
\]

We conclude that if \( \varrho = (r_1, r_2, \ldots, r_{d-1}) \) is a walk over \( \varsigma \) with defect \( s \), then \( r := r_{d-1} \in \mathcal{E}_\varsigma \cap \mathcal{E}_{(t,s)} \). Hence, with \( \varrho_r \) denoting a walk over \( \varsigma \) with penultimate height \( r \) and defect \( s \), we may write the determinant formula (4.61) as

\[
\det \mathcal{G}_\varsigma^{(s)} = \prod_{r' \in \mathcal{E}_\varsigma \cap \mathcal{E}_{(t,s)}} \prod_{\hat{\varrho}_r \in \mathcal{E}_\hat{\varsigma}} \prod_{j=1}^{d_r-1} \left( \frac{(\Theta(r_j, r^{j+1}, s^{j+1}))}{(-1)^{r^{j+1}[r^{j+1}+1]}} \right). \tag{4.72}
\]

The \( j = d_r - 1 \) factor in the product over \( j \in \{1, 2, \ldots, d_r - 1\} \) depends only on the last two heights \( r \) and \( s \) of a walk over \( \varsigma \), which are the same for all walks \( \varrho_r \). Hence, we may factor it out of the product, obtaining

\[
\det \mathcal{G}_\varsigma^{(s)} = \prod_{r' \in \mathcal{E}_\varsigma \cap \mathcal{E}_{(t,s)}} \left( \frac{(\Theta(r, s, t))}{(-1)^{s[t+1]}} \right)^{D^{(r)}_{\bar{t}}} \prod_{\varrho_r} \prod_{j=1}^{d_r-2} \left( \frac{(\Theta(r, r^{j+1}, s^{j+1}))}{(-1)^{r^{j+1}[r^{j+1}+1]}} \right), \tag{4.73}
\]

where the power \( D^{(r)}_{\bar{t}} \) follows from the fact that there are \( \Theta(r, s, t) \) many distinct walks \( \hat{\varrho}_r \) over \( \hat{\varsigma} \) with defect \( r \). Finally, after recalling the formula (4.61) for \( \det \mathcal{G}_\varsigma^{(s)} \), we observe that (4.73) matches the sought recursion formula (4.69). \( \square \)

In the next lemma, we give another formula for the determinant of the Gram matrix \( \mathcal{G}_n^{(s)} \). This formula appears in [Wes95], and a proof for it appears as [RSA14, theorem 4.7]. (See also [GL98, Corollary 4.7].) We use this formula to prove lemma 5.3 in section 5, a crucial ingredient for completely and explicitly characterizing the radical of \( L_{\xi}^{(s)} \) for any multindex \( \varsigma \in \{0\} \cup \mathbb{Z}_{\geq 0}^d \) and any integer \( s \in \mathcal{E}_\varsigma \).

**Lemma 4.11.** [RSA14, Theorem 4.7] We have

\[
\det \mathcal{G}_n^{(s)} = \prod_{j=1}^{n-s} \left( \frac{[s+j+1]}{(-1)^{s+j+1}[s]} \right)^{D^{(s+j)}_{\bar{n}}}. \tag{4.74}
\]

**Proof.** For the same reason that was given in proof of proposition 4.8, we may assume that \( n < \bar{p}(q) \) throughout. For convenience, we also substitute \( q \mapsto -q \) throughout. With \( [s]_{-q} = (-1)^{s-1}[s]_q \) by (2.66), we may write (4.74) as

\[
\det \mathcal{G}_n^{(s)} = \prod_{j=1}^{n-s} \left( \frac{[s+j+1]_{-q}}{[j]_{-q}} \right)^{D^{(s+j)}_{\bar{n}}}. \tag{4.75}
\]

We prove formula (4.75) by induction on \( n \in \mathbb{Z}_{\geq 0} \). First, it trivially holds for \( n = 0 \). Next, assuming that (4.75) holds with \( n = m - 1 \) for some integer \( m \geq 2 \), we prove that it holds with \( n = m \). Now, lemma 4.10 with \( \varsigma = \bar{m} \) gives

\[
\det \mathcal{G}_m^{(s)} = \det \mathcal{G}_{m-1}^{(s-1)} \left( \det \mathcal{G}_{(s-1,1)}^{(s-1)} \right)^{D^{(s-1)}_{\bar{n}-1}} \det \mathcal{G}_{m-1}^{(s+1)} \left( \det \mathcal{G}_{(s+1,1)}^{(s)} \right)^{D^{(s+1)}_{\bar{n}-1}}. \tag{4.76}
\]
Lemma 4.13. Suppose of the link state bilinear form (3.11) with respect to the basis \( L_P \). Using proposition 4.8, we have

\[
\varrho \quad \text{obviously equals the dimension of} \quad \{ \text{box set across the} \quad \L_P \text{valenced link pattern} \quad \text{Proof.} \quad \text{We also let} \quad \alpha \quad \text{LP}_{\L_P} \text{is a subset of} \quad \L_{\L_P} \text{LP}_{\L_P} \text{defects of the associated trivalent link state} \quad \text{LP}_{\L_P} \text{is a basis of} \quad \L_{\L_P} \text{LP}_{\L_P} \text{size} \quad s_{d_{L_P}} + v \text{and attaching v defects to the extended box. For example,}
\]

Next, we prove another recursion formula, lemma 4.13, for later use in section 5B. To state it, we define the set \( \LP_{\L_P}^{(s)} \) to be the collection of valenced link patterns obtained by increasing the size \( s_{d_L} \) of the rightmost valenced node in every \((s, s)\)-valenced link pattern \( \alpha \) to size \( s_{d_L} + v \) and attaching \( v \) defects to the extended box. For example,

\[
\begin{align*}
\text{LP}_{\L_P}^{(s)} = & \quad \text{LP}_{\L_P}^{(s)} \quad \L_{\L_P}^{(s)}.
\end{align*}
\]

We also let \( \L_{\L_P}^{(s)} \) denote the complex vector space with basis \( \LP_{\L_P}^{(s)} \).

Lemma 4.12. Suppose \( \max_{s} \zeta < p(q) \). Then we have

\[
\alpha \in \LP_{\L_P}^{(s)} \quad \implies \quad \zeta \in \L_{\L_P}^{(s)}.
\]

and the set \( \{ \alpha \mid \alpha \in \LP_{\L_P}^{(s)} \} \) is a basis of \( \L_{\L_P}^{(s)} \).

Proof. Lemma B.2 of appendix B implies that, for each valenced link pattern \( \alpha \in \LP_{\L_P}^{(s)} \), discarding the projector box set across the \( s + v \) defects of the associated trivalent link state \( \zeta \) does not alter \( \zeta \). Therefore, the collection \( \{ \zeta \mid \alpha \in \LP_{\L_P}^{(s)} \} \) is a subset of \( \L_{\L_P}^{(s)} \). Furthermore, this collection is linearly independent by lemma 4.5, and its cardinality obviously equals the dimension of \( \L_{\L_P}^{(s)} \). Thus, the set \( \{ \alpha \mid \alpha \in \LP_{\L_P}^{(s)} \} \) is actually a basis of \( \L_{\L_P}^{(s)} \).

Now, we find the determinant of the Gram matrix

\[
[\mathcal{G}^{(s)}]_{\alpha, \beta} := (\alpha \mid \beta), \quad \text{for all} \quad \alpha, \beta \in \LP_{\L_P}^{(s)},
\]

of the link state bilinear form (3.11) with respect to the basis \( \LP_{\L_P}^{(s)} \). (4.82)

Lemma 13. Suppose \( \max_{s} \zeta < p(q) \). Then we have

\[
\det \mathcal{G}^{(s)} = \det \mathcal{G}^{(s)} \prod_{e} \left[ \frac{s_{d_{L_P}} + s_{d_{L_P}} + s + v + 1}{s_{d_{L_P}} + s_{d_{L_P}} + s + 1} \right] \left[ \frac{s_{d_{L_P}} + s_{d_{L_P}} + s - r_{d_{L_P}} - 1}{s_{d_{L_P}} + s_{d_{L_P}} + s - r_{d_{L_P}} - 1} \right] \left[ \frac{s_{d_{L_P}} + s_{d_{L_P}} + s - r_{d_{L_P}} - 1}{s_{d_{L_P}} + s_{d_{L_P}} + s - r_{d_{L_P}} - 1} \right] \left[ \frac{s_{d_{L_P}} + v}{s_{d_{L_P}} + v} \right],
\]

where the product is over all walks \( q = (r_1, r_2, \ldots, r_{d_{L_P}}) \) over \( \zeta \) with defect \( r_{d_{L_P}} = s \).
Proof. For the same reason that was given in proof of proposition 4.8, we may assume that \( n_\varsigma < \bar{p}(\varsigma) \) throughout. Then, lemma 4.12 shows that \( \{ \bullet \mid \alpha \in \mathbb{L}^{(s)}_{\varsigma,0} \} \) is a basis for \( \mathbb{L}^{(s)}_{\varsigma,0} \), so corollary 4.5 with item 1 of proposition 4.6 imply
\[
\det \mathcal{G}_{\varsigma,0}^{(a)} = \det (\bullet \mid \bullet \in \mathbb{L}^{(s)}_{\varsigma,0}) = \prod_{\mathcal{G}_{\varsigma,0}^{(a)}} \prod_{j=1}^{d} \frac{\Theta(r_j, r_{j+1}, s_{j+1})}{(s_j + v + 1)} \frac{\Theta(r_{d-1}, s + v, s_d + v)}{(s + v + 1)},
\]
where the product is over all walks \( \varrho_\alpha = (r_1, r_2, \ldots, r_{d-1}, s + v) \) corresponding to some valenced link pattern \( \alpha \in \mathbb{L}^{(s)}_{\varsigma} \). Now, in light of item 2 in lemma 4.1 and the definition of the set \( \mathbb{L}^{(s)}_{\varsigma,0} \), the map
\[
\varrho_\alpha = (r_1, r_2, \ldots, r_{d-1}, s + v) \mapsto \rho = (r_1, r_2, \ldots, r_{d-1}, r_d)
\]
for all valenced link patterns \( \alpha \in \mathbb{L}^{(s)}_{\varsigma} \) is a bijection onto the set of all walks \( \rho = (r_1, r_2, \ldots, r_{d-1}, r_d) \) over \( \varsigma \) with defect \( r_d = s \). Hence, using proposition 4.8, we may write
\[
\det \mathcal{G}_{\varsigma}^{(a)} = \prod_{\rho} (-1)^{s+1} \frac{\Theta(r_{d-1}, s + v, s_d + v)}{(s + v + 1)}
\]
while inserting (4.52) into (4.86) and simplifying, we arrive with (4.83).

After inserting (4.52) into (4.86) and simplifying, we arrive with (4.83). \( \square \)

Last, we derive a recursion formula for \( \det \mathcal{G}_{\varsigma}^{(a)} \) different from the one obtained from lemma 4.10, for use in [FP18b]. To obtain this recursion, we group the nodes of each link pattern in \( \mathbb{L}^{(s)}_{\varsigma} \) into \( d \) adjacent bins of sizes \( s_1, s_2, \ldots, s_d \), and we partition the link pattern into a unique collection of \( d + 1 \) sub-link patterns thus:
\[
\alpha = \begin{array}{c}
\alpha_0 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_d \\
\hline
s_1 & s_2 & \ldots & s_d
\end{array}
\]
where for all indices \( i \in \{1, 2, \ldots, d\} \), we have
\[
\alpha_i \in \mathbb{L}^{(t_i)}_{s_i} \quad \text{for some } t_i \in \mathbb{E}_{s_i}, \quad \text{and} \quad \alpha_0 \in \mathbb{S}^{(s)}_{\varrho_0} \quad \text{with } \varrho = (t_1, t_2, \ldots, t_d).
\]
and where \( \mathbb{S}^{(s)}_{\varrho_0} \) is the set of link patterns that do not have turn-back links joining two left nodes or two right nodes in a common group in (4.87). (See (C.11) in appendix C.)

Let \( \varsigma = (s_1, s_2, \ldots, s_d) \) be the multiindex consisting of the sizes of the bins. To avoid too many subscripts, we denote \( d = d_\varsigma \). The above type of partitioning of link patterns into link sub-patterns implies the existence of an isomorphism of vector spaces that sends the vector space \( \mathbb{L}^{(s)}_{n} \) onto the vector space
\[
\bigoplus_{t_1 \in \mathbb{E}_{s_1}} \bigoplus_{t_2 \in \mathbb{E}_{s_2}} \cdots \bigoplus_{t_d \in \mathbb{E}_{s_d}} \left( \text{span } \mathbb{S}^{(s)}_{\varrho_0} \right) \otimes \mathbb{L}^{(t_1)}_{s_1} \otimes \mathbb{L}^{(t_2)}_{s_2} \otimes \cdots \otimes \mathbb{L}^{(t_d)}_{s_d} \quad \text{with } \varrho = (t_1, t_2, \ldots, t_d),
\]
by mapping \( \alpha \) to the tensor product \( \alpha_0 \otimes \alpha_1 \otimes \cdots \alpha_d \), where \( \alpha_i \) is defined relative to \( \alpha \) in (4.87). Next, assuming \( \max \varsigma < \bar{p}(\varsigma) \), we consider the linear self-map of \( \mathbb{L}^{(s)}_{n} \) sending the link pattern \( \alpha \in \mathbb{L}^{(s)}_{n} \) to the similar link state
\[
\alpha = \begin{array}{c}
\alpha_0 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_d \\
\hline
s_1 & s_2 & \ldots & s_d
\end{array}
\]
with a projector box between \( \alpha_0 \) and each \( \alpha_i \). Lemma 4.4 implies that this map is an automorphism of vector spaces with an upper triangular matrix representation whose all diagonal entries equal one. Hence, we have

\[
\det \mathcal{G}_n^{(s)} = \det \mathcal{G}_n^{(s')} \quad \text{where} \quad \mathcal{G}_n^{(s)} := \left[ (\alpha \mid \beta) \right]_{\alpha, \beta \in \mathcal{LP}_n^{(s)}}.
\]  

(4.91)

With these observations, we are ready to state the new recursion for \( \det \mathcal{G}_n^{(s)} \).

**Lemma 4.14.** Suppose \( \max \varsigma < \bar{p}(q) \), and denote \( n = n_\varsigma \) and \( d = d_\varsigma \). Then we have the recursion

\[
\det \mathcal{G}_n^{(s)} = \prod_{t_1 \in E_{s_1}} \prod_{t_2 \in E_{s_2}} \ldots \prod_{t_d \in E_{s_d}} \left( \det \mathcal{G}_\theta^{(s)} \right)^{D_{s_1}^{(t_1)} \ldots D_{s_d}^{(t_d)}} \prod_{i=1}^{d} \left( \det \mathcal{G}_{s_i}^{(t_i)} \right)^{p_i},
\]

(4.92)

where

\[
\theta := (t_1, t_2, \ldots, t_d) \quad \text{and} \quad p_i := D_\theta^{(s)} \prod_{j \neq i}^{d} D_{s_i}^{(t_i)},
\]

(4.93)

and \( D_\theta^{(s)}, D_{s_i}^{(t_i)} \) are the numbers determined by recursions (2.54) and (2.55).

**Proof.** To prove the lemma, we show that the determinant of \( \mathcal{G}_n^{(s)} \) equals the right side of (4.92) and use (4.91). To determine \( \det \mathcal{G}_n^{(s)} \), we consider the generic form of the network

\[
\begin{array}{c}
\begin{array}{c}
\alpha_0 \\
\beta_0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{|c|c|c|}
\hline
\alpha_1 & \beta_1 & \alpha_2 \\
\hline
\alpha_0 & \beta_0 & \alpha_1 \\
\hline
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{|c|c|c|}
\hline
\beta_2 & \ldots & \beta_{d_\varsigma} \\
\hline
\alpha_2 & \ldots & \alpha_{d_\varsigma} \\
\hline
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\beta_{d_\varsigma}
\end{array}
\end{array}
\end{array}
\]

(4.94)

After using lemma B.4 of appendix B to factor out the evaluation of the network \( \alpha \mid \beta \) sandwiched between the \( i \)-th left and right projector boxes for each \( i \in \{1, 2, \ldots, d\} \), we find that for all \( \alpha, \beta \in \mathcal{LP}_n^{(s)} \), we have the factorization

\[
(\alpha \mid \beta) = (P_\theta \alpha_0 \mid P_\theta \beta_0)(\alpha_1 \mid \beta_1)(\alpha_2 \mid \beta_2) \ldots (\alpha_d \mid \beta_d).
\]

(4.95)

Because the spaces \( L_{s_i}^{(r_1)} \) and \( L_{s_i}^{(t_1)} \) are orthogonal if \( r_i \neq t_i \) and the linear map sending

\[
\alpha \longrightarrow \alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_d
\]

(4.96)

for all \((n, s)\)-link patterns \( \alpha \) is an isomorphism of vector spaces from \( L_{s_i}^{(r_1)} \) to the space (4.89), we infer from factorization (4.95) that the matrix \( \mathcal{G}_n^{(s)} \) equals the following direct sum of tensor products of Gram matrices:

\[
\mathcal{G}_n^{(s)} = \bigoplus_{t_1 \in E_{s_1}} \bigoplus_{t_2 \in E_{s_2}} \ldots \bigoplus_{t_d \in E_{s_d}} \mathcal{G}_\theta^{(s)} \otimes \mathcal{G}_{s_1}^{(t_1)} \otimes \mathcal{G}_{s_2}^{(t_2)} \otimes \cdots \otimes \mathcal{G}_{s_d}^{(t_d)}.
\]

(4.97)

After taking the determinant of both sides, using the well-known formula for the determinant of the tensor product of matrices, and recalling from (4.91) that \( \det \mathcal{G}_n^{(s)} = \det \mathcal{G}_n^{(s')} \), we finally arrive with (4.92).

\[ \square \]

5. RADICAL OF THE LINK STATE FORM

In this section, we determine the dimension of and a basis for the radical of the standard module \( L_{s_i}^{(s)} \) for all multiindices \( \varsigma \in \{0\} \cup Z_{\mathbb{E}_0}^s \) and integers \( s \in \mathbb{E}_\varsigma \). Corollaries 5.1 and 5.2 below treat the trivial cases with \( n_\varsigma < \bar{p}(q) \).
For the nontrivial cases, we divide the problem into two parts: the specific case with \( \zeta = \bar{n} \) for some integer \( n \in \mathbb{Z}_{\geq 0} \), treated in section 5 A, and the case of a general multiindex \( \zeta \in \{ \bar{0} \} \cup \mathbb{Z}_{>0}^\# \), treated in section 5 B. In section 5 C, we use these results to prove the remarkable fact that \( \text{rad} \, L^{(s)} \) is trivial if and only if \( \text{rad} \, L^{(s)}_{\bar{n}} \) is trivial, and we then use this fact with our other results to determine all pairs \((\zeta, s)\) such that \( \text{rad} \, L^{(s)}_{\bar{n}} \) is trivial. Finally, in section 5 D, we study cases in which and conditions under which \( \text{rad} \, L^{(s)}_{\bar{n}} \) equals the entire module \( L^{(s)} \).

**Corollary 5.1.** Suppose \( n, < \bar{p}(q) \). Then we have \( \text{rad} \, L^{(s)} = \{0\} \) for all \( s \in \mathbb{E} \), and thus, \( \text{rad} \, L^{(s)} = \{0\} \).

**Proof.** The claim immediately follows from proposition 4.9 and decomposition (3.20).

**Corollary 5.2.** Suppose \( \bar{p}(q) = \infty \). Then we have \( \text{rad} \, L^{(s)} = \{0\} \) for all \( \zeta \in \{ \bar{0} \} \cup \mathbb{Z}_{>0}^\# \).

**Proof.** The claim immediately follows from corollary 5.1.

Corollary 5.2 settles the case that \( \bar{p}(q) = \infty \); all radicals of \( L^{(s)} \) are trivial. However, if \( \bar{p}(q) < \infty \), then the radical of these standard modules may not be trivial for certain multiindices \( \zeta \in \mathbb{Z}_{>0}^\# \) and integers \( s \in \mathbb{E} \). In the remainder of this section, we completely determine these radicals for all values of \( \bar{p}(q) \), all multiindices \( \zeta \in \mathbb{Z}_{>0}^\# \), and all integers \( s \in \mathbb{E} \). These forthcoming results include the case \( \bar{p}(q) = \infty \), already settled above, as a special instance.

### A. Radical at roots of unity

First, we find the dimension of and a basis for the radical of \( L^{(s)}_{\bar{n}} \) for all integers \( n \in \mathbb{Z}_{\geq 0} \) and \( s \in \mathbb{E} \). Throughout, we define the integers \( k_s \) and \( R_s \) as in (1.21). The following lemma gives the case of \( R_s = 0 \) (i.e., \( s = \Delta_{k_s} \)):

**Lemma 5.3.** [RSA14, corollary 4.8] Suppose \( \bar{p}(q) \mid (s + 1) \). Then \( \text{rad} \, L^{(s)}_{\bar{n}} = \{0\} \).

**Proof.** First, we assume that \( q \neq \pm 1 \). By definition (2.66), for all nonzero integers \( k \), we have \( [k]_q = 0 \) if and only if \( \bar{p}(q) \mid k \), and the zeros of \( [k]_q \) as a function of \( q \in \mathbb{C}^\times \) are of first order. Also, our assumption that \( \bar{p}(q) \mid (s + 1) \) implies that \( j \) is a multiple of \( \bar{p}(q) \) if and only if \( s + j + 1 \) is a multiple of \( \bar{p}(q) \). Hence, we have

\[
[j]_q = 0 \iff [s + j + 1]_q = 0 \quad \text{and} \quad \lim_{q \to q'} \frac{[s + j + 1]_{q'}}{[j]_{q'}} = \text{const.} \neq 0. \tag{5.1}
\]

From these facts and the formula (4.74) for \( \det \mathcal{G}^{(s)}_{\bar{n}} \) in lemma 4.11, we see that \( \det \mathcal{G}^{(s)}_{\bar{n}} \neq 0 \), so \( \text{rad} \, L^{(s)}_{\bar{n}} = \{0\} \).

Last, we assume that \( q = \pm 1 \), in which case we have \( \bar{p}(q) \mid (r + 1) \) for all integers \( r \in \mathbb{Z}_{\geq 0} \). If \( q = \pm 1 \), then clearly no quantum integer (2.66) except \( [0] \) vanishes. Hence, it is evident from (4.74) that \( \det \mathcal{G}^{(s)}_{\bar{n}} \neq 0 \), so \( \text{rad} \, L^{(s)}_{\bar{n}} = \{0\} \).

Now we use lemma 5.3 to determine the radical of \( L^{(s)}_{\bar{n}} \) when \( \bar{p}(q) < \infty \). First, we recall from section 4 A the definition (4.42, 4.44) of the tail of a link pattern \( \alpha \) and definition 4.2 of the corresponding trivalent link state \( \underline{\alpha} \). In the special case that \( \zeta = \bar{n} \) for some integer \( n \in \mathbb{Z}_{\geq 0} \), stopping condition 1 in definition 4.2 occurs, with \( k_s \) given in (1.21). Second, a moderate tail has \( r_j = \Delta_{k_s} \) (i.e., stopping condition 3 in definition 4.2 occurs), again with \( k_s \) given in (1.21). We let \( R_n^{(s)} \) and \( M_n^{(s)} \) respectively denote the collection of all radical tails and all moderate tails of \((n, s)\)-link patterns. Because stopping condition 2 in definition 4.2 cannot occur if \( \zeta = \bar{n} \) for some integer \( n \in \mathbb{Z}_{\geq 0} \), we have

\[
R_n^{(s)} \cup M_n^{(s)} = T_n^{(s)}. \tag{5.3}
\]
We also note that, for each tail \( \tau = (r_{J+1}, r_{J+2}, \ldots, r_n) \in T^{(s)}_n \), the following properties hold:

\[
\begin{align*}
    r_J &= \begin{cases} 
        \Delta_{k+1}, & \tau \in R^{(s)}_n, \\
        \Delta_{k}, & \tau \in M^{(s)}_n, 
    \end{cases} \\
    &\quad \text{and} \quad \\
    r_{J+1} &= \begin{cases} 
        r_{J} - 1, & \tau \in R^{(s)}_n, \\
        r_{J} + 1, & \tau \in M^{(s)}_n. 
    \end{cases}
\end{align*}
\]

(5.4)

Or next goal, proposition 5.7, is to prove that \( \{ \alpha \mid \alpha \in L^{(s)}_n \} \) is a basis for the radical of \( L^{(s)}_n \). For this, we first observe that link patterns with different tails span orthogonal subspaces of \( L^{(s)}_n \).

**Lemma 5.4.** For all link patterns \( \alpha, \beta \in L^{(s)}_n \), we have

\[
\text{tail}(\alpha) \neq \text{tail}(\beta) \implies (\alpha \mid \beta) = 0.
\]

(5.5)

**Proof.** Assuming that \( \alpha, \beta \in L^{(s)}_n \) satisfy \( \text{tail}(\alpha) \neq \text{tail}(\beta) \), with \( g_\alpha = (r_1, r_2, \ldots, r_n) \) and \( g_\beta = (r'_1, r'_2, \ldots, r'_n) \), we set \( I = \max\{ j \in \mathbb{Z}_{\geq 0} \mid r_j \neq r'_j \} \). Then we have \( \max(J_\alpha(q), J_\beta(q)) \leq I \), so the network \( \alpha \mid \beta \) has the following form:

\[
\begin{array}{c}
\alpha \mid \beta \\
\hline
\alpha \mid \beta
\end{array}
\]

(5.6)

The sizes \( r_I \) and \( r'_I \) of the two leftmost projector boxes are different by definition of \( I \). Hence, there exists a link with both of its endpoints touching the larger of these two boxes, so we have \( (\alpha \mid \beta) = 0 \).

Orthogonality of link patterns with different tails immediately gives a direct sum decomposition of the radical:

**Corollary 5.5.** We have the following direct sum decomposition:

\[
\text{rad} L^{(s)}_n = \bigoplus_{\tau \in T^{(s)}_n} \text{rad span} \{ \alpha \mid \alpha \in L^{(s)}_n, \text{tail}(\alpha) = \tau \}.
\]

(5.7)

**Proof.** Item 3 of proposition 4.6 implies the following direct sum decomposition:

\[
L^{(s)}_n = \bigoplus_{\tau \in T^{(s)}_n} \text{span} \{ \alpha \mid \alpha \in L^{(s)}_n, \text{tail}(\alpha) = \tau \}.
\]

(5.8)

Lemma 5.4 implies that the spans in the direct sum (5.8) are orthogonal. Hence, (5.7) follows from (5.8).

Now, to determine the radical of \( L^{(s)}_n \), we only need to understand the summands of (5.7).

**Lemma 5.6.** Suppose \( \tau \in T^{(s)}_n \). Then we have

\[
\text{rad span} \{ \alpha \mid \alpha \in L^{(s)}_n, \text{tail}(\alpha) = \tau \} = \begin{cases} 
    \text{span} \{ \alpha \mid \alpha \in L^{(s)}_n, \text{tail}(\alpha) = \tau \}, & \tau \in R^{(s)}_n, \\
    \{0\}, & \tau \in M^{(s)}_n.
\end{cases}
\]

(5.9)

**Proof.** Let \( \beta \) and \( \gamma \) be two link states in the span of \( \{ \alpha \mid \alpha \in L^{(s)}_n, \text{tail}(\alpha) = \tau \} \), and denote their common tail by \( \tau = (r_J, r_{J+1}, \ldots, r_n) \), with (5.4). Then, the bilinear form \( (\beta \mid \gamma) \) equals the evaluation of the following network:

\[
\begin{array}{c}
\beta \mid \gamma \\
\hline
\beta \mid \gamma
\end{array}
\]

(5.10)
where $\beta'$ and $\gamma'$ are $(J, r_J)$-link states. We use lemmas B.4 and B.5 of appendix B to evaluate this network:

$$
\begin{aligned}
(\beta \mid \gamma) & \overset{(B.7)}{=} (\beta' \mid \gamma') \times \\
& \overset{(B.10)}{=} \left( \begin{array}{c}
1 \\
\vdots \\
1 \\
\end{array} \right)
\begin{pmatrix}
\Theta(r_j, r_{j+1}, 1) \\
(-1)^{r_{j+1}} [r_{j+1} + 1]
\end{pmatrix} \prod_{j=J}^{n-1} \Theta(r_j, r_{j+1}, 1)
\end{aligned}
$$

(5.11)

Now, lemma 4.7 shows that the product of the factors in (5.11) with $j > J$ is finite and does not vanish:

$$
0 < \left| \prod_{j=J+1}^{n-1} \frac{\Theta(r_j, r_{j+1}, 1)}{(-1)^{r_{j+1}} [r_{j+1} + 1]} \right| < \infty.
$$

(5.12)

Then using (4.52, 5.4), we find that the factor with $j = J$ equals

$$
\Theta(r_J, r_{J+1}, 1) = \begin{cases} 
0, & \tau \in R_n^{(s)}, \\
1, & \tau \in M_n^{(s)}.
\end{cases}
$$

(5.13)

Next, according to lemma 5.3 together with (5.4), we have $\text{rad} L_J^{(r_J)} = \{0\}$. Hence, for each nonzero link state $\beta' \in L_J^{(r_J)}$, there exists a companion link state $\gamma'_\beta \in L_J^{(r_J)}$ such that

$$
(\beta' \mid \gamma'_\beta) \neq 0.
$$

(5.14)

We define $\gamma_\beta$ to be the link state obtained by setting $\gamma' = \gamma'_\beta$ in

$$
\begin{pmatrix}
\gamma \\
\gamma'
\end{pmatrix} = 
\begin{pmatrix}
\gamma \\
\gamma'
\end{pmatrix}
\begin{pmatrix}
r_J \\
r_{J+1} \\
\vdots \\
r_{n-1} \\
r_n = s
\end{pmatrix}
$$

(5.15)

Now, we may combine (5.11–5.14) to arrive with the following result: for every link state $\beta$ in the span of $\{ \omega \mid \alpha \in LP_n^{(s)}, \text{tail}(\alpha) = \tau \}$, we have

$$
(\beta \mid \gamma) \begin{cases} 
= 0 \text{ for all } \gamma \in \text{span} \{ \omega \mid \alpha \in LP_n^{(s)}, \text{tail}(\alpha) = \tau \}, & \tau \in R_n^{(s)}, \\
\neq 0 \text{ if } \gamma = \gamma_\beta, & \tau \in M_n^{(s)}.
\end{cases}
$$

(5.16)

This final result is equivalent to (5.9). \qed

**Proposition 5.7.** The collection

$$
\{ \omega \mid \alpha \in LP_n^{(s)}, \text{tail}(\alpha) = R_n^{(s)} \}
$$

(5.17)

is a basis for $\text{rad} L_n^{(s)}$.

**Proof.** After combining corollary 5.5 with lemma 5.6, we obtain the direct sum decomposition

$$
\text{rad} L_n^{(s)} \overset{(5.7)}{=} \bigoplus_{\tau \in R_n^{(s)}} \text{span} \{ \omega \mid \alpha \in LP_n^{(s)}, \text{tail}(\alpha) = \tau \} = \text{span} \{ \omega \mid \alpha \in LP_n^{(s)}, \text{tail}(\alpha) = R_n^{(s)} \}. 
$$

(5.18)

Item 3 of proposition 4.6 implies that the collection (5.17) is linearly independent. Thus, it is basis for $\text{rad} L_n^{(s)}$. \qed
Using proposition 5.7, we next determine the dimension of \( \text{rad} L_n^{(s)} \). We recall from lemma 2.8 that the dimension of the standard module \( L_n^{(s)} \) is \( D_n^{(s)} \), the unique solution to recursion problem (2.55). Then, we define the numbers \( D_n^{(s)} \) for all integers \( s \in E_n \) to be the unique solution to the recursion

\[
D_n^{(s)} = \begin{cases} 
0, & R_s = 0, \\
D_{n-1}^{(s)} + D_{n-1}^{(s+1)}, & R_s = p(q) - 1, \\
D_{n-1}^{(s)} + D_{n-1}^{(s+1)}, & R_s \in \{1, 2, \ldots, p(q) - 2\},
\end{cases}
\]

and \( D_1^{(1)} = D_{m-1}^{(1)} = 0 \) for all \( m \in \mathbb{Z}_{\geq 0} \). (5.19)

This recursion is equivalent to the recursion problem stated in [RSA14, Proposition 4.5].

The following lemma is similar to (4.14) in item 4 of lemma 4.1 with \( \zeta = \bar{n} \).

**Lemma 5.8.** We have

\[
D_n^{(s)} = \# \left\{ \text{walks } \varrho \text{ over } \bar{n} \text{ with defect } s \text{ and such that, when followed backward, hit height } \Delta_k + 1 \text{ before height } \Delta_{k+1} \right\}. \quad (5.20)
\]

**Proof.** We recall that by (4.14), the quantity \( D_n^{(s)} \) equals the number of walks over \( \bar{n} \) with defect \( s \). By considering the last step of an arbitrary walk in the set appearing on the right side of (5.20), we see that the cardinality of this set satisfies recursion (5.19). If \( \zeta = (1) \) and \( s = 1 \), then there are no such walks, so the initial condition also holds. We conclude that the right side of the asserted equation (5.20) equals the unique solution \( D_n^{(s)} \) to recursion (5.19).

By (1.20, 1.21) we have \( \Delta_k = \Delta_0 = -1 \) if \( s + 1 < p(q) \). Because a walk over \( \zeta \) cannot have negative height, it follows that if \( s + 1 < p(q) \), then \( D_n^{(s)} \) equals the number of walks \( \varrho \) over \( \zeta \) with defect \( s \) and that hit height \( p(q) - 1 \).

**Corollary 5.9.** We have

\[
\dim \text{rad } L_n^{(s)} = \# R_n^{(s)} = D_n^{(s)}. \quad (5.21)
\]

**Proof.** The first equality in (5.21) immediately follows from proposition 5.7, and the second equality from (5.20) of lemma 5.8 and the definition of a radical tail.

\section*{B. Valenced radical at roots of unity}

In this section, we find the dimension of and a basis for the radical of \( L_n^{(s)} \) for all multiindices \( \zeta \in \{0\} \cup \mathbb{Z}_{\geq 0}^{k} \) and integers \( s \in E_\zeta \). For convenience, we assume that \( p(q) < \infty \), although this condition is not necessary for the results in this section to be true. Before we define the integers \( k_\alpha \) and \( R_\alpha \) as in (1.21).

The module \( W_\zeta \) (defined in (2.107)) has a natural bilinear form, given by restricting the bilinear form for its parent module \( L_n \) to this subspace. We define the radical of this bilinear form to be the vector space

\[
\text{rad } W_\zeta := \{ \alpha \in W_\zeta \mid (\alpha \mid \beta) = 0 \text{ for all } \beta \in W_\zeta \}. \quad (5.22)
\]

The radical \( \text{rad } W_\zeta \) is a \( JW_\zeta(\nu) \)-submodule of \( W_\zeta \), it equals a direct sum of the radicals of its submodules,

\[
\text{rad } W_\zeta = \bigoplus_{s \in E_\zeta} \text{rad } W_\zeta^{(s)}, \quad \text{where} \quad \text{rad } W_\zeta^{(s)} := \{ \alpha \in W_\zeta^{(s)} \mid (\alpha \mid \beta) = 0 \text{ for all } \beta \in W_\zeta^{(s)} \}, \quad (5.23)
\]

and \( \text{rad } W_\zeta^{(s)} \) is a \( JW_\zeta(\nu) \)-submodule of \( W_\zeta^{(s)} \).

First, we invoke from lemma C.1 of appendix C the facts that the map \( I_\zeta(\cdot) : L_\zeta \longrightarrow L_n^{(s)} \), defined in (C.1) is a linear injection with image \( W_\zeta \), it preserves the \( s \)-grading of its domain, and by definition (3.17), it preserves the bilinear form defined in section 3. Thus, we have

\[
I_\zeta \text{rad } L_\zeta^{(s)} = \text{rad } W_\zeta^{(s)}. \quad (5.24)
\]

In particular, it suffices to determine the radical of \( W_\zeta^{(s)} \) in order to determine the sought radical of \( L_\zeta^{(s)} \). Now, by definition (2.107), we have

\[
W_\zeta^{(s)} = P_\zeta L_n^{(s)}. \quad (5.25)
\]

In fact, a similar property holds after we replace \( W_\zeta^{(s)} \) and \( L_n^{(s)} \) in this equation by their radicals:
Lemma 5.10. Suppose \( \max \zeta < \bar{p}(q) \). Then we have
\[
\text{rad } W^{(s)}_\zeta = W^{(s)}_\zeta \cap \text{rad } L^{(a)}_{n_\zeta} = P_\zeta \text{rad } L^{(a)}_{n_\zeta}. 
\]  
(5.26)

Proof. To begin, we prove the containment
\[
\text{rad } W^{(s)}_\zeta \subset W^{(s)}_\zeta \cap \text{rad } L^{(a)}_{n_\zeta}. 
\]  
(5.27)

Indeed, using invariance property (3.22) of the bilinear form from (3.22) of lemma 3.1, and the property \( P^1_\zeta = \hat{P}_\zeta \) from (2.73) and the idempotent property \( \hat{P}_\zeta \) from (P1) of the composite projector \( P_\zeta \), we obtain (5.27):
\[
\alpha \in \text{rad } W^{(s)}_\zeta \quad \implies \quad \alpha \in W^{(s)}_\zeta \text{ and } (\alpha \mid \gamma) = 0 \text{ for all } \gamma \in W^{(s)}_\zeta, 
\]  
(5.28)

\[
\implies \quad \alpha = P_\zeta \beta \text{ for some } \beta \in L^{(a)}_{n_\zeta} \text{ and } (P_\zeta \beta \mid P_\zeta \delta) = 0 \text{ for all } \delta \in L^{(a)}_{n_\zeta},
\]  
(5.29)

\[
(3.22) \quad \implies \quad \alpha = P_\zeta \beta \text{ for some } \beta \in L^{(a)}_{n_\zeta} \text{ and } (\alpha \mid \delta) = (P_\zeta \beta \mid \delta) = 0 \text{ for all } \delta \in L^{(a)}_{n_\zeta},
\]  
(5.30)

\[
\implies \quad \alpha \in W^{(s)}_\zeta \cap \text{rad } L^{(a)}_{n_\zeta}. 
\]  
(5.31)

Next, we prove the containment
\[
W^{(s)}_\zeta \cap \text{rad } L^{(a)}_{n_\zeta} \subset P_\zeta \text{rad } L^{(a)}_{n_\zeta}. 
\]  
(5.32)

Indeed, by the idempotent property \( P^2_\zeta = \hat{P}_\zeta \) from (P1) of the composite projector \( P_\zeta \), we obtain (5.32):
\[
\alpha \in W^{(s)}_\zeta \cap \text{rad } L^{(a)}_{n_\zeta} \quad \implies \quad \alpha = P_\zeta \beta \in \text{rad } L^{(a)}_{n_\zeta} \text{ for some } \beta \in L^{(a)}_{n_\zeta}, 
\]  
(5.33)

\[
\implies \quad \alpha = P_\zeta \beta = P^2_\zeta \beta = P_\zeta \alpha \in P_\zeta \text{rad } L^{(a)}_{n_\zeta}. 
\]  
(5.34)

To finish, we prove the containment
\[
P_\zeta \text{rad } L^{(a)}_{n_\zeta} \subset \text{rad } W^{(s)}_\zeta.
\]  
(5.35)

Indeed, using invariance property (3.22) of the bilinear form from (3.22) of lemma 3.1, we have
\[
\alpha \in P_\zeta \text{rad } L^{(a)}_{n_\zeta} \quad \implies \quad \alpha = P_\zeta \beta \text{ for some } \beta \in \text{rad } L^{(a)}_{n_\zeta}, 
\]  
(5.36)

\[
\implies \quad \alpha \in W^{(s)}_\zeta \text{ and } (\alpha \mid \gamma) = (P_\zeta \beta \mid \gamma) \overset{(3.22)}{=} (\beta \mid P^1_\gamma) = 0 \text{ for all } \gamma \in W^{(s)}_\zeta, 
\]  
(5.37)

\[
\implies \quad \alpha \in \text{rad } W^{(s)}_\zeta,
\]  
(5.38)

which proves (5.35). Finally, combining (5.27, 5.32, 5.35) gives the sought equalities (5.26).

\[\square\]

Corollary 5.11. Suppose \( \max \zeta < \bar{p}(q) \). Then we have
\[
\text{rad } L^{(s)}_\zeta = \hat{P}_\zeta \text{rad } L^{(a)}_{n_\zeta}, 
\]  
(5.39)

Proof. By lemma C.1 of appendix C, the map \( L_\zeta(\cdot) : L_\zeta \to L_{n_\zeta} \) defined in (C.1) is a linear injection with image \( W_\zeta \) that respects the \( s \)-grading (2.20) of its domain and the bilinear form (3.17) of \( L_\zeta \). Applying its inverse, i.e., \( \hat{P}_\zeta \), to (5.26) of lemma 5.10 gives
\[
\text{rad } L^{(s)}_\zeta = \hat{P}_\zeta \text{rad } W^{(s)}_\zeta = \hat{P}_\zeta P_\zeta \text{rad } L^{(a)}_{n_\zeta}. 
\]  
(5.40)

By the idempotent property (P1) for Jones-Wenzl projectors, we evidently have \( \hat{P}_\zeta P_\zeta = \hat{P}_\zeta \). After inserting this last simplification into (5.40), we finally arrive with (5.39).

\[\square\]

Corollary 5.12. Suppose \( \max \zeta < \bar{p}(q) \). If \( p(q) \mid (s + 1) \), then \( \text{rad } L^{(s)}_\zeta = \{0\} \).

Proof. This immediately follows from lemma 5.3 with corollary 5.11.

\[\square\]
Now we use these results to determine the radical of \( L^{(s)}_\varsigma \). As in section 5A, we first recall from section 4A the definition (4.43, 4.44) of the tail of a valenced link pattern \( \alpha \), and we denote
\[
T^{(s)}_\varsigma := \{ \text{tail}(\alpha) \mid \alpha \in LP^{(s)}_\varsigma \}. \tag{5.41}
\]
Recalling definition 4.2 of the trivalent link state \( \varsigma \) for any \((\varsigma, s)\)-valenced link pattern \( \alpha \), we say that the tail of \( \alpha \) is

1. **type-one radical tail** if condition 1 in definition 4.2 occurs in the determination of the trivalent link state \( \varsigma \),
2. **type-two radical tail** if condition 2 in definition 4.2 occurs in the determination of the trivalent link state \( \varsigma \),
3. **“moderate tail”** if condition 3 in definition 4.2 occurs in the determination of the trivalent link state \( \varsigma \).

Figures 4, 5, and 6 show examples of these tails. We also say that the tail of \( \alpha \) is a moderate tail if none of these stopping condition occurs, i.e., if \( J_\alpha(q) = -\infty \). Finally, we let
\[
R^{(s)}_{\varsigma,1} := \{ \text{tail}(\alpha) \mid \alpha \in LP^{(s)}_\varsigma, \text{tail}(\alpha) \text{ is a type-one radical tail} \}, \tag{5.42}
\]
\[
R^{(s)}_{\varsigma,2} := \{ \text{tail}(\alpha) \mid \alpha \in LP^{(s)}_\varsigma, \text{tail}(\alpha) \text{ is a type-two radical tail} \}, \tag{5.43}
\]
\[
M^{(s)}_\varsigma := \{ \text{tail}(\alpha) \mid \alpha \in LP^{(s)}_\varsigma, \text{tail}(\alpha) \text{ is a moderate tail} \}, \tag{5.44}
\]
we denote
\[
R^{(s)}_\varsigma := R^{(s)}_{\varsigma,1} \cup R^{(s)}_{\varsigma,2}, \tag{5.45}
\]
and we call an element of this set simply a “radical tail.” It is clear from the definitions that the union of these sets equals the collection of all tails \( T^{(s)}_\varsigma \):
\[
T^{(s)}_\varsigma = R^{(s)}_\varsigma \cup M^{(s)}_\varsigma. \tag{5.46}
\]

Our next goal, theorem 5.18, is to prove that the set \( \{ \varsigma \mid \alpha \in LP^{(s)}_\varsigma, \text{tail}(\alpha) \in R^{(s)}_\varsigma \} \) is a basis for the radical of \( L^{(s)}_\varsigma \). The logic of this work is similar to the proof of proposition 5.7: we decompose the radical of \( L^{(s)}_\varsigma \) into a direct sum of certain subspaces labeled by either type-one radical tails, or type-two radical tails, or moderate tails, and we explicitly determine these subspaces. To do this, we employ proposition 5.7 from section 5A, which already gives the radical of the related standard module \( L^{(s)}_{\nu_\varsigma} \).

**Lemma 5.13.** Suppose \( \max \varsigma < \bar{p}(q) \). Then we have
\[
\{ \varsigma \mid \alpha \in LP^{(s)}_\varsigma, \text{tail}(\alpha) \in R^{(s)}_{\varsigma,1} \} \subset \text{rad } L^{(s)}_\varsigma. \tag{5.47}
\]

**Proof.** Let \( \beta \in LP^{(s)}_\varsigma \) be the link pattern obtained by separating the \( i \)-th node of \( \alpha \in LP^{(s)}_\varsigma \) into \( s_i \) adjacent nodes for each \( i \in \{1, 2, \ldots, d_\varsigma \} \) without changing the connectivities of the links in \( \alpha \). Then we have
\[
\begin{align*}
\alpha &= \tilde{P}_\varsigma \beta \\
\text{tail}(\alpha) &= R^{(s)}_{\varsigma,1} & \Rightarrow & \varsigma = \tilde{P}_\varsigma \beta. \tag{5.48}
\end{align*}
\]
Now, if the valenced link pattern \( \alpha \) has a type-one radical tail, then the link pattern \( \beta \) necessarily has a radical tail. Hence, by proposition 5.7 and corollary 5.11, we have
\[
\beta \in \text{rad } L^{(s)}_{\nu_\varsigma} \tag{5.39} \Rightarrow \varsigma = \tilde{P}_\varsigma \beta \in \text{rad } L^{(s)}_\varsigma, \tag{5.49}
\]
so any trivalent link state \( \varsigma \) derived from a \((\varsigma, s)\)-valenced link state \( \alpha \) with type-one radical tail belongs to \( \text{rad } L^{(s)}_\varsigma \).

**Lemma 5.14.** Suppose \( \max \varsigma < \bar{p}(q) \). Then the following are equivalent:

1. We have \( s = \Delta_k \varsigma \) (i.e., \( R_s = 0 \)).
2. \( J_\alpha(q) = d_\varsigma \) for all valenced link patterns \( \alpha \in LP^{(s)}_\varsigma \).
3. \( J_\alpha(q) = d_\varsigma \) for some valenced link pattern \( \alpha \in LP^{(s)}_\varsigma \).
Furthermore, if any one of items 1–3 hold, then we have
\[
T_\varsigma^{(s)} = M_\varsigma^{(s)}, \quad R^{(s)}_{\varsigma,1} = R^{(s)}_{\varsigma,2} = \emptyset, \quad \text{and} \quad \text{rad} \, L^{(s)}_\varsigma = \{0\}. \quad (5.50)
\]

**Proof.** First, it is evident from definition (4.42, 4.44) of the index \( J_\varsigma(q) \) and definitions (4.9, 4.10) of the heights \( h_{\min,d_\varsigma}(q_\alpha) \) and \( h_{\max,d_\varsigma}(q_\alpha) \) that item 1 implies item 2. Also, it is obvious that item 2 implies item 3.

Next, we prove that item 3 implies item 1. By definition (4.42, 4.44) of the index \( J_\varsigma(q) \) and the fact that \( d_\varsigma \) is the maximal value that this quantity may equal, we have
\[
J_\varsigma(q) = J_{\varsigma_\alpha}(q) = d_\varsigma \quad \text{(4.42)} \quad \text{or} \quad d_\varsigma \quad \text{(5.51)}
\]
By (4.41), it is impossible to have \( \Delta_{\varsigma_{k+1}} \leq s \) or \( s < \Delta_{\varsigma_k} \). Hence, (5.51) implies item 1.

Finally, with \( s = \Delta_{\varsigma_k} \), we have by definition that \( \text{tail}(\alpha) \in M_\varsigma^{(s)} \) for all \( (\varsigma, s) \)-valenced link patterns \( \alpha \). Furthermore, the equality \( s = \Delta_{\varsigma_k} \) implies that \( p(q) \mid (s + 1) \), so \( \text{rad} \, L^{(s)}_\varsigma = \{0\} \) by corollary 5.12.

Now we can decompose the radical of \( L^{(s)}_\varsigma \) into a direct sum of radicals of three subspaces, specified by the three types of tails according to the decomposition (5.45, 5.46):

**Lemma 5.15.** Suppose \( \max \varsigma < \bar{p}(q) \). Then we have
\[
\text{rad} \, L^{(s)}_\varsigma = \text{rad} \, \text{span} \{ \varnothing \mid \alpha \in \mathbb{L}(\varsigma) \}, \quad s \in \mathbb{M}(\varsigma), \quad \text{tail}(\alpha) \in \mathbb{R}^{(s)}_\varsigma.
\]

**Proof.** To begin, we prove the lemma when \( p(q) \mid (s + 1) \), or equivalently, \( \Delta_{\varsigma_k} = s \). In this case, all tails in \( T^{(s)}_\varsigma \) are moderate by lemma 5.14. This fact with item 3 of proposition 4.6 shows that
\[
L^{(s)}_\varsigma = \text{span} \{ \varnothing \mid \alpha \in \mathbb{L}(\varsigma) \} \quad \text{and} \quad \text{span} \{ \varnothing \mid \alpha \in \mathbb{M}(\varsigma), \text{tail}(\alpha) \in \mathbb{R}^{(s)}_\varsigma \}, \quad \text{if} \ \Delta_{\varsigma_k} = s. \quad (5.53)
\]
After taking the radical of both sides and invoking (5.50), we arrive with asserted formula (5.52).

Now we prove the lemma when \( p(q) \nmid (s + 1) \), or equivalently, \( \Delta_{\varsigma_k} \neq s \). In this case, lemma 5.13 readily implies that the following subspaces are orthogonal to one another:
\[
\text{rad} \, \text{span} \{ \varnothing \mid \alpha \in \mathbb{L}(\varsigma), \text{tail}(\alpha) \in \mathbb{R}^{(s)}_\varsigma \} \perp \text{rad} \, \text{span} \{ \varnothing \mid \alpha \in \mathbb{M}(\varsigma), \text{tail}(\alpha) \in \mathbb{R}^{(s)}_\varsigma \}, \quad (5.54)
\]
\[
\text{rad} \, \text{span} \{ \varnothing \mid \alpha \in \mathbb{L}(\varsigma), \text{tail}(\alpha) \in \mathbb{R}^{(s)}_{\varsigma,1} \} \perp \text{rad} \, \text{span} \{ \varnothing \mid \alpha \in \mathbb{L}(\varsigma), \text{tail}(\alpha) \in \mathbb{R}^{(s)}_{\varsigma,2} \}. \quad (5.55)
\]
Thus, it remains to prove that
\[
\text{rad} \, \text{span} \{ \varnothing \mid \alpha \in \mathbb{L}(\varsigma), \text{tail}(\alpha) \in \mathbb{M}(\varsigma) \} \perp \text{rad} \, \text{span} \{ \varnothing \mid \alpha \in \mathbb{L}(\varsigma), \text{tail}(\alpha) \in \mathbb{R}^{(s)}_{\varsigma,2} \}. \quad (5.56)
\]
For this, we let \( \alpha \) and \( \beta \) be arbitrary \( \varsigma \)-valenced link patterns with moderate and type-two radical tail, respectively:
\[
\text{tail}(\alpha) \in \mathbb{M}^{(s)}_\varsigma \quad \text{and} \quad \text{tail}(\beta) \in \mathbb{R}^{(s)}_{\varsigma,2}. \quad (5.57)
\]
and we write \( g_\alpha = (r_1, r_2, \ldots, r_{d_\varsigma}) \) and \( g_\beta = (r'_1, r'_2, \ldots, r'_{d_\varsigma}) \) for their respective walks over the multiindex \( \varsigma \). We set
\[
I := \max\{j \in \mathbb{Z}_{>0} \mid r_j \neq r'_j\} \quad \text{and} \quad K := \max\{J_\alpha(q), J_\beta(q)\}. \quad (5.58)
\]
Because the tail of \( \beta \) is not moderate, we have \( J_\beta(q) \geq 0 \) by definition. Moreover, we have \( J_\alpha(q), J_\beta(q), K \leq d_\varsigma \) by definition. However, if \( K = d_\varsigma \), then \( s = \Delta_{\varsigma_k} \) by lemma 5.14, which contradicts our initial assumption. Thus, we have
\[
0 \leq K \leq d_\varsigma - 1. \quad (5.59)
\]
In the two respective cases that \( J_\alpha < J_\beta = K \) or \( J_\beta < J_\alpha = K \), the network \( \varnothing \mid \varnothing \) has the following form:

```
\[
\text{When } J_\alpha < J_\beta, \quad (5.60)
\]```
Because the tails of $\alpha$ and $\beta$ are different, we also have $K \leq I$. If $K < I$, then we arrive with $(\alpha \parallel \beta) = 0$ by reusing the arguments in the proof of lemma 5.4. Hence, we assume $K = I$ throughout and, to lighten notation, we write

\[ r := r_K, \quad r' := r'_K, \quad t := s_{K+1}, \quad u := r_{K+1} = r'_{K+1}. \]  

Inequality (5.59) guarantees that these quantities exist. Now, there are three cases to consider: either $J_\alpha < J_\beta (= K)$, or $J_\alpha = J_\beta (= K)$, or $J_\beta < J_\alpha (= K)$. We illustrate the part of the walk $\varrho_\alpha$ that goes from height $r$ to height $u$ and the part of the walk $\varrho_\beta$ that goes from height $r'$ to height $u$ respectively as

\begin{align*}
\varrho_\alpha : & \quad \begin{array}{c}
\Delta_{k_s + 1} \\
\Delta_{k_s}
\end{array} \\
\varrho_\beta : & \quad \begin{array}{c}
\Delta_{k_s + 1} \\
\Delta_{k_s}
\end{array}
\end{align*}

where the top sides and bottom sides of the rhombus are respectively parts of the walks $\varrho_\alpha$ or $\varrho_\beta$ and $\varrho_\alpha$ or $\varrho_\beta$. We note that in all of the cases, the following inequality holds:

\[ r < r' : \]  

1. $J_\alpha \leq J_\beta = K$: From illustrations (5.63, 5.64) we see that

\[ \frac{r + u + t}{2} = h_{\text{max},K}(\varrho_\alpha) \leq \Delta_{k_s + 1} - 1 < \Delta_{k_s + 1} \leq h_{\text{max},K}(\varrho_\beta) = \frac{r' + u + t}{2}, \]

which implies that $r < r'$, so (5.66) holds in this case.

2. $K = J_\alpha > J_\beta$: Similarly, from illustration (5.65), we see that

\[ \frac{r + u - t}{2} = h_{\text{min},K}(\varrho_\alpha) \leq \Delta_{k_s} < \Delta_{k_s} + 1 \leq h_{\text{min},K}(\varrho_\beta) = \frac{r' + u - t}{2}, \]

which implies that $r < r'$, so (5.66) holds also in this case.

Now with $r < r'$, the same argument that we used in the proof of lemma 5.4 shows that in the network $\alpha \parallel \beta$, there must exist a turn-back link with both endpoints touching the projector box of size $r'$. Hence, we have $(\alpha \parallel \beta) = 0$, which proves (5.56) and implies the claim (5.52).

Our next task is to determine all three of the radicals appearing in decomposition (5.52). We begin with moderate tails, in which case the radical is trivial.
Lemma 5.16. Suppose $\max_{\zeta} \varsigma < \bar{p}(q)$. Then we have
\[
\text{rad span} \{ \alpha \mid \alpha \in \text{LP}_{\zeta}^{(s)}, \text{tail}(\alpha) \in \text{M}_{\zeta}^{(s)} \} = \{0\}. \tag{5.69}
\]

Proof. To begin, we prove the lemma when $p(q) \mid (s + 1)$, or equivalently, $\Delta_{k_s} = s$. In this case, we recall the work in the first paragraph in the proof of lemma 5.15. After taking the radical of both sides of (5.53) and invoking corollary 5.12, we arrive with (5.69).

Now we prove the lemma when $p(q) \nmid (s + 1)$, or equivalently, $\Delta_{k_s} \neq s$. In this case, lemma 5.14 shows that for any $(\varsigma, s)$-valenced link pattern $\alpha$, we have
\[
J := J_{\alpha}(q) < d_{\varsigma}. \tag{5.70}
\]
Therefore, with inequality (5.70) satisfied, we may denote
\[
\text{tail}(\alpha) = (r_{J}, r_{J+1}, r_{J+2}, \ldots, r_{d_{\varsigma}}) \quad \Rightarrow \quad \text{tail}(\alpha) := (r_{J+1}, r_{J+2}, \ldots, r_{d_{\varsigma}}), \tag{5.71}
\]
and with this notation, we can write
\[
\text{span} \{ \alpha \mid \alpha \in \text{LP}_{\zeta}^{(s)}, \text{tail}(\alpha) \in \text{M}_{\zeta}^{(s)} \} = \bigoplus_{\tau \in \mathbb{Z}^+_{\geq 0}} \text{span} \{ \alpha \mid \alpha \in \text{LP}_{\zeta}^{(s)}, \text{tail}(\alpha) = \tau \}. \tag{5.72}
\]
Reusing the arguments from the proof of lemma 5.4, we see that the subspaces in this direct sum are orthogonal, so
\[
\text{rad span} \{ \alpha \mid \alpha \in \text{LP}_{\zeta}^{(s)}, \text{tail}(\alpha) \in \text{M}_{\zeta}^{(s)} \} = \bigoplus_{\tau \in \mathbb{Z}^+_{\geq 0}} \text{rad span} \{ \alpha \mid \alpha \in \text{LP}_{\zeta}^{(s)}, \text{tail}(\alpha) = \tau \}. \tag{5.73}
\]
Now we show that each summand in the direct sum (5.73) is trivial. Selecting an arbitrary summand, let $\beta$ and $\gamma$ be two link states in the span of $\{ \alpha \mid \alpha \in \text{LP}_{\zeta}^{(s)}, \text{tail}(\alpha) \in \text{M}_{\zeta}^{(s)} \}$, and $\tau = (r_{J+1}, r_{J+2}, \ldots, r_{d_{\varsigma}})$ with
\[
J = J_{\beta}(q) = J_{\gamma}(q). \tag{5.74}
\]
Then, the bilinear form $(\beta \mid \gamma)$ equals the evaluation of the following network:
where $\beta'$ and $\gamma'$ are appropriate valued link states, described in greater detail below. We use lemmas B.4 and B.5 of appendix B to evaluate this network: 

$$
(\beta \mid \gamma) \overset{(B.7)}{=} (\beta' \mid \gamma' ) \times \left( \frac{d_{r} - 1}{d_{r} - 1} \prod_{j = J + 1}^{d_{r} - 1} \Theta(r_{j}, r_{j+1}, s_{j+1}) \frac{(-1)^{r_{j+1}}[r_{j+1} + 1]}{[t]} \right).
$$

Lemma 4.7 shows that the product in (5.77) is finite and nonzero:

$$
0 < \left| \prod_{j = J + 1}^{d_{r} - 1} \Theta(r_{j}, r_{j+1}, s_{j+1}) \frac{(-1)^{r_{j+1}}[r_{j+1} + 1]}{[t]} \right| < \infty.
$$

Next, we focus on the bilinear form $(\beta' \mid \gamma')$ in (5.77). To understand this quantity, we first define:

$$
u := \Delta_{k_{r}}, \quad v := r_{J+1} - u \geq 0, \quad t := s_{J+1} - v \geq 0, \quad \omega := (s_{1}, s_{2}, \ldots, s_{J}, t).
$$

By the definition (4.42, 4.44) of $J$, the valued link states $\beta', \gamma'$ are elements of $L_{\omega,v}^{(u)}$ (defined beneath (4.80)), with $v$ defects anchored to the rightmost valued node of size $s_{J+1}$:

Now we prove that rad $L_{\omega,v}^{(u)} = \{0\}$ by showing that the determinant $\det \mathcal{G}_{\omega,v}^{(u)}$ does not vanish. To do this, we use lemma 4.13 with replacements $d_{r} \mapsto J + 1$, $\varsigma \mapsto \omega$, and $s \mapsto u$ to write

$$
\det \mathcal{G}_{\omega,v}^{(u)} \overset{(4.83)}{=} \det \mathcal{G}_{\omega}^{(u)} \prod_{\epsilon} \frac{[\epsilon + t + u + v + 1][\epsilon + t + u + v][\epsilon + u + 1]}{[\epsilon + t + u + v + 1][\epsilon + t + u + v][\epsilon + u + 1][\epsilon + v + 1]}.
$$

where $u, v, t$ are fixed in (5.79) and the product is over all walks $\epsilon$ over $\omega$ with defect $u$, and where $r$ denotes the penultimate height of $\epsilon$. Next, we prove that $\det \mathcal{G}_{\omega,v}^{(u)} \neq 0$. According to lemma 5.12, we have

$$
p(q) \mid (u + 1) \text{ by (1.20, 5.79)} \quad \implies \quad \text{rad } L_{\omega}^{(u)} = \{0\} \quad \implies \quad \det \mathcal{G}_{\omega,v}^{(u)} \neq 0.
$$

To show that the product over $\epsilon$ in (5.81) does not vanish either, we gather some inequalities. First, we trivially have

$$
0 \leq t \leq t + v \overset{(5.79)}{=} s_{J+1} \leq \max \varsigma \leq \hat{p}(q) - 1 \implies \frac{[t]!}{[t + v]!} \neq 0.
$$

Furthermore, by definition (4.42) of $J = J_{\beta}(q) = J_{\gamma}(q)$, we have $r_{J+1} < \Delta_{k_{r}+1}$. Hence,

$$
k_{s}p(q) = \Delta_{k_{r}+1} \overset{(1.20)}{=} (5.79) u + 1 \overset{(5.79)}{=} u + v + 1.
$$
Therefore, we have

\[
\frac{r_{J+1} + 1}{u + v + 1} \leq \Delta_{k_x + 1} \leq (k_x + 1)p(q) - 1 \implies 0 < \frac{[u + 1]!}{[u + v + 1]!} < \infty.
\] (5.84)

Next, with \( u \in E_{(r,t)} \), we have

\[
\frac{r_{J+1} \leq \Delta_{k_x + 1} \leq u + 1}{2} \leq \frac{r + t + u + 1}{2} \leq \frac{r - t + u + t + v + 1}{2} \leq \frac{\beta}{\gamma}
\]

\[
\leq \frac{\gamma}{p(q)} \leq \frac{\gamma}{(k_x + 1)p(q) - 1} \implies 0 < \frac{[r + t + u + v + 1]!}{[r + t + u + 1]!} < \infty,
\] (5.85)

and finally, with \( t \in E_{(r,u)} \), we have

\[
0 \leq \frac{t - (r - u)}{2} \leq \frac{t - (r - u)}{2} \leq t + v \leq \frac{\bar{p}(q) - 1}{2} \implies 0 < \frac{[t + u - v + 1]!}{[t + u - v]!} < \infty.
\] (5.86)

Combining (5.82–5.87), we conclude from (5.81) that \( \det \rho^{(e)}_{w,v} \neq 0 \). Hence, we have \( \text{rad span } L^{(u)}_{w,v} = \{0\} \).

Now we are ready to finish the proof. Because the radical of \( L^{(u)}_{w,v} \) is trivial, it follows that each nonzero valenced link state \( \beta' \in L^{(u)}_{w,v} \) has a companion link state \( \gamma_\beta \in L^{(u)}_{w,v} \) such that

\[
(\beta' \mid \gamma_\beta) \neq 0.
\] (5.88)

We define \( \gamma_\beta \) to be the link state obtained by setting \( \gamma' = \gamma_\beta \) in

\[
\gamma = \gamma', \quad \tau = t_{J+2} r_{J+3} \cdots r_{d_u} = s
\]

(5.89)

Now, we may combine (5.77, 5.78, 5.88) to conclude that for every nonzero link state \( \beta \) in the span of \( \{ \leftrightarrow \mid \alpha \in \mathcal{L}_\zeta^{(s)}, \text{tail}(\alpha) = \tau \} \), we have

\[
(\beta \mid \gamma_\beta) = (\beta' \mid \gamma_\beta) \prod_{j = J+1}^{d_u - 1} \frac{\Theta(r_j, r_{j+1}, s_{j+1})}{(-1)^{r_j + 1}[r_{j+1} + 1]} \neq 0.
\] (5.90)

Therefore, we have

\[
\text{rad span } \{ \leftrightarrow \mid \alpha \in \mathcal{L}_\zeta^{(s)}, \text{tail}(\alpha) = \mathcal{M}_\zeta^{(e)}, \text{tail}(\alpha) = \tau \} = \{0\}.
\] (5.91)

Because \( \tau \) is arbitrary, (5.73, 5.91) combine to give (5.69). This finishes the proof. \( \square \)

We continue the determination of the radical (5.52), now addressing the case of radical tails.

**Lemma 5.17.** Suppose \( \max \zeta < \bar{p}(q) \). Then for \( i \in \{1, 2\} \), we have

\[
\text{rad span } \{ \leftrightarrow \mid \alpha \in \mathcal{L}_\zeta^{(s)}, \text{tail}(\alpha) = \mathcal{R}_\zeta^{(e)}, \text{tail}(\alpha) = \mathcal{R}_\zeta^{(w,i)} \} = \text{span } \{ \leftrightarrow \mid \alpha \in \mathcal{L}_\zeta^{(s)}, \text{tail}(\alpha) = \mathcal{R}_\zeta^{(w,i)} \}.
\] (5.92)

**Proof.** Lemma 5.13 already gives (5.92) for \( i = 1 \), so it only remains to prove the case \( i = 2 \). For this, we let \( \beta \) or \( \gamma \) be any valenced link states in the span of \( \{ \leftrightarrow \mid \alpha \in \mathcal{L}_\zeta^{(s)}, \text{tail}(\alpha) = \mathcal{R}_\zeta^{(w,i)} \} \). Without loss of generality, we assume that

\[
J_{\chi}(q) \leq J := J_{\beta}(q).
\] (5.93)

By definition and lemma 5.14, we have

\[
0 \leq J < d_\zeta,
\] (5.94)
for the tails of $\beta$ and $\gamma$ are not moderate. The bilinear form $(\beta \mid \gamma)$ equals the evaluation of the following network:

\[
\beta \mid \gamma = (\beta' \mid \gamma')
\]

where $\beta'$ and $\gamma'$ are are appropriate valenced link states. We use lemmas B.4 and B.5 of appendix B to evaluate this network:

\[
(\beta \mid \gamma) \overset{(B.7)}{=} (\beta' \mid \gamma') \times \left( \Theta(r_J, r_{J+1}, s_{J+1}) \Theta(-1)^{r_{J+1}[r_{J+1} + 1]} \right)
\]

First, we consider the factor of the product in (5.97) with $j = J$. By definition (4.42, 4.44) of $J = J_\beta(q)$, we have

\[
\max(r_J, r_{J+1}) < \Delta_{k_{s+1}}.
\]

This gives us

\[
k_s q(p)^{(1.20)} \Delta_{k_s} + 1 \overset{(4.42)}{\leq} h_{\min,J}(q)^{(4.9)} \frac{r_J + r_{J+1} - s_{J+1}}{2} \overset{(4.57)}{\leq} \min(r_J, r_{J+1}) < \max(r_J, r_{J+1}) + 1
\]

\[
< \Delta_{k_{s+1}} \overset{(1.20)}{=} (k_s + 1)q(p) - 1
\]

\[
< h_{\max,J}(q)^{(4.10)} \frac{r_J + r_{J+1} + s_{J+1}}{2} + 1.
\]

Similarly to our reasoning in the proof of lemma 4.7, formula (4.52) of the Theta network with (5.99) gives

\[
\Theta(r_J, r_{J+1}, s_{J+1}) \Theta(-1)^{r_{J+1}[r_{J+1} + 1]} = 0.
\]
On the other hand, lemma 4.7 says that the factors in (5.97) with $j \in \{J + 1, J + 2, \ldots, d_s - 1\}$ are finite, so

$$0 < \left| \prod_{j=J+1}^{d_s-1} \frac{\Theta(r_j, r_{j+1}, s_j+1)}{(-1)^{r_{j+1}}[r_{j+1}+1]} \right| < \infty.$$ (5.101)

After inserting (5.100, 5.101) into (5.97), we arrive with ($\beta \mid \gamma$) = 0. Because $\beta$ and $\gamma$ were arbitrary valenced link states in the span of the collection $\{\alpha \mid \alpha \in L_{\xi}^{(s)}, \text{tail}(\alpha) \in \mathcal{R}_{\chi_{\xi}}^{(s)}\}$, we conclude that (5.92) holds for $i = 2$. \qed

Now we are finally ready to collect our results and finish the complete determination of the radical $\text{rad} L_{\xi}^{(s)}$.

**Theorem 5.18.** Suppose $\max \zeta < \bar{p}(q)$. Then the collection

$$\{\alpha \mid \alpha \in L_{\xi}^{(s)}, \text{tail}(\alpha) \in \mathcal{R}_{\chi_{\xi}}^{(s)}\}$$ (5.102)

is a basis for $\text{rad} L_{\xi}^{(s)}$.

**Proof.** After combining lemma 5.15 with lemmas 5.16 and 5.17, we obtain

$$\text{rad} L_{\xi}^{(s)}$$ (5.103)

Also, item 3 of proposition 4.6 implies that the set (5.102) is linearly independent. Thus, it is basis for $\text{rad} L_{\xi}^{(s)}$. \qed

To end this section, we determine the dimension of the radical $\text{rad} L_{\xi}^{(s)}$. We recall from lemma 2.8 that the dimension of the standard module $L_{\xi}^{(s)}$ is $D_{\xi}^{(s)}$, the unique solution to recursion problem (2.54). Then, analogously to (5.19), with $\Delta_k$ defined in (1.20) and denoting $\zeta = (s_1, s_2, \ldots, s_{d_s})$, $\xi := (s_1, s_2, \ldots, s_{d_s-1})$, and $t := s_{d_s}$, we define the numbers $D_{\xi}^{(s)}$ to be the unique solution to the recursion

$$D_{\xi}^{(s)} = \sum_{r \in \mathcal{E}_{\xi} \cap \mathcal{E}_{\xi_{r_{s_{d_s-1}}}}} \left( \mathbb{I}\{\Delta_{k_s} < \frac{r + s - t}{2}\} \mathbb{I}\{\frac{r + s + t}{2} < \Delta_{k_{s+1}}\}\right) D_{\xi_{r_{s_{d_s}+1}}}^{(s)} + \mathbb{I}\{\Delta_{k_{s+1}} \leq \frac{r + s + t}{2}\} D_{\xi_{r_{s_{d_s}+1}}}^{(s)},$$ (5.105)

The following lemma is similar to (4.14) in item 4 of lemma 4.1. Before stating it, it is useful to make the following observation: for any walk $\varphi$ over $\zeta$ with defect $s$ and each $j \in \{0, 1, \ldots, d_s - 1\}$, we have

$$0 \leq h_{\max,j}(\varphi) - h_{\min,j}(\varphi) \overset{(4.9)}{=} h_{s+1}(\varphi) \leq \max \zeta < p(q) = \Delta_{k_{s+1}} - \Delta_{k_s}.$$ (5.106)

Thus, walks $\varphi^+$ and $\varphi^-$ cannot simultaneously hit heights $\Delta_{k_{s+1}}$ and $\Delta_{k_s}$ respectively on the same step of $\varphi$.

**Lemma 5.19.** We have

$$D_{\xi}^{(s)} = \# \rightbracebrace{\text{walks } \varphi \text{ over } \zeta \text{ with defect } s \text{ and such that, when followed backward, } \varphi^+ \text{ hits height } \Delta_{k_{s+1}} \text{ before } \varphi^- \text{ hits height } \Delta_{k_s}}.$$ (5.107)

**Proof.** The proof is the same as the proof of lemma 5.8, except that $\zeta$ may be any multiindex in $\{\vec{0}\} \cup \mathbb{Z}^{d_s}_{\geq 0}$. \qed

Extending the comment following lemma 5.8, if $s + 1 < p(q)$, then $D_{\xi}^{(s)}$ equals the number of walks $\varphi$ over $\zeta$ with defect $s$ and such that $\varphi^+$ hits height $p(q) - 1$ (or equivalently, with maximum apex at or above $p(q) - 1$).

**Corollary 5.20.** Suppose $\max \zeta < \bar{p}(q)$. Then we have

$$\dim \text{rad} L_{\xi}^{(s)} = \# \mathcal{R}_{\chi_{\xi}}^{(s)} = D_{\xi}^{(s)}.$$ (5.108)

**Proof.** The first equality in (5.108) immediately follows from theorem 5.18, and the second equality from (5.107) of proposition 5.19 and the definition of a radical tail. \qed

We observe that lemma 5.19 and corollary 5.20 reduce to lemma 5.8 and corollary 5.9 respectively if $\zeta = \vec{n}$. 
C. Nondegenerate cases

We recall that the radical of the standard module $L^{(s)}_ς$ is said to be “nondegenerate” if it is trivial, i.e., $\text{rad} L^{(s)}_ς = \{0\}$. In section 3 B, proposition 3.3 implies that the standard module $L^{(s)}_ς$ is simple if and only if its radical is nondegenerate. Thus, for the purpose of classifying all simple $TL_1(ν)$-modules, it is worthwhile to determine all $q \in \mathbb{C}^\times$ for which the radical of a given standard module is nondegenerate. To this end, we begin with the following lemma. To state it, it is first helpful to recall the containment $E_ς \subset E_{n_ς}$ from (2.38).

**Lemma 5.21.** Suppose $\max ζ < \bar{p}(q)$. Then for each $s \in E_ς$, we have

$$\mathcal{D}^{(s)}_ς = 0 \iff \mathcal{D}^{(s)}_{n_ς} = 0.$$  \hspace{1cm} (5.109)

**Proof.** We consider the tallest walk $\varrho_{\max}$ over $ς$ with defect $s$, and the tallest walk $\varrho_{\max}^\dagger$ over $\bar{n}_ς$ with defect $s$: e.g.,

$$\text{Because } \varrho_{\max} \text{ is the tallest walk over } \varsigma \text{ with defect } s, \text{ if there exists a walk over } \varsigma \text{ with defect } s \text{ and with radical tail, then the tail of } \varrho_{\max} \text{ is also radical. A similar fact holds for } \varrho_{\max}^\dagger \text{ and walks over } \bar{n}_ς \text{ with defect } s. \text{ Hence, we have}

$$\text{the tail of } \varrho_{\max} \text{ is radical } \iff \mathcal{D}^{(s)}_ς = 0 \hspace{1cm} (5.111)$$

$$\text{the tail of } \varrho_{\max}^\dagger \text{ is radical } \iff \mathcal{D}^{(s)}_{n_ς} = 0.$$ \hspace{1cm} (5.112)

Now from the definition of a radial tail, we see that the tail of $\varrho_{\max}$ is radical if and only if the tail of $\varrho_{\max}^\dagger$ is radical. Combined with (5.111), this last fact implies that $\mathcal{D}^{(s)}_ς = 0$ if and only if $\mathcal{D}^{(s)}_{n_ς} = 0$. \hfill □

**Corollary 5.22.** Suppose $\max ζ < \bar{p}(q)$. Then for each $s \in E_ς$, we have

$$\text{rad} L^{(s)}_ς = \{0\} \iff \text{rad} L^{(s)}_{n_ς} = \{0\}.$$ \hspace{1cm} (5.113)

**Proof.** This immediately follows from corollaries 5.9 and 5.20 with lemma 5.21. \hfill □

Now we determine all $q \in \mathbb{C}^\times$ such that the radical of $L^{(s)}_ς$ is nondegenerate. In light of corollary 5.22, we only need to consider the case $ζ = \bar{n}$ for any $n \in \mathbb{Z}_{\geq 0}$. Here, the pairs $(n, s) \in \mathbb{Z}_{\geq 0} \times E_n$ live on the square lattice, within a semi-infinite triangle bound between the lines $s = 0$ and $s = n$, as illustrated in figure 7. Certain points of the lattice shown in figure 7 are of special interest:

- points $(n, s) = (n, \Delta k)$ on a dashed line, each at height $\Delta k = \Delta k(q)$ for some $k \in \mathbb{Z}_{\geq 0}$,
- points $(n, s)$ on the pink triangle with corners at $(0, 0)$, $(2\Delta_1 - 2, 0)$, and $(\Delta_1 - 1, \Delta_1 - 1)$, and
- points $(n, s)$ on the pink triangle with corners at $(\Delta_k, \Delta_k)$, $(2\Delta_{k+1} - 2, \Delta_k)$, and $(\Delta_{k+1} - 1, \Delta_{k+1} - 1)$ for $k \in \mathbb{Z}_{\geq 0}$.

Using this, we define

$$\text{Non}^{(s)}_n := \left\{ q \in \mathbb{C}^\times \left| \begin{array}{c} (n, s) \text{ lies on a pink triangle} \\ \text{or on a dashed line in figure 7} \end{array} \right. \right\}$$

$$\left\{ q \in \mathbb{C}^\times \left| \begin{array}{c} p(q) | (s + 1) \text{ or } n + s \leq 3k_s p(q) + 2p(q) - 5 \end{array} \right. \right\}.$$  \hspace{1cm} (5.114)

We note that the complement of this set within $\mathbb{C}$ has Lebesgue measure zero. We also define

$$\text{Non}_n := \bigcap_{s \in E_n} \text{Non}^{(s)}_n.$$ \hspace{1cm} (5.115)
FIG. 7: Illustration of the pairs \((n, s) \in \mathbb{Z}_{\geq 0} \times E_n\). The set of \(q \in \mathbb{C}^n\) such that \((n, s)\) is on either a pink triangle or a horizontal dashed line is denoted by \(\text{Non}_n\).

Finally, for each \(s \in E_{\varsigma}\), we define
\[
\text{Non}_{\varsigma} := \text{Non}_{n_{\varsigma}} \quad \text{and} \quad \text{Non}_{\varsigma} := \bigcap_{s \in E_{\varsigma}} \text{Non}_{\varsigma}.
\]

We consider these sets in more detail in the end of this section (lemmas 5.26 and 5.27).

**Corollary 5.23.** Suppose \(\max \varsigma < \bar{\rho}(q)\). Then \(\text{rad} L_{\varsigma} = \{0\} \iff q \in \text{Non}_{\varsigma}\).

**Proof.** It is evident that \(D_{n_{\varsigma}} = 0\) if and only if \((n_{\varsigma}, s)\) lies in the closure of a pink triangle or on a dashed line in the lattice in figure 7. Hence, the claim follows from corollaries 5.9 and 5.22 and the definition of \(\text{Non}_{\varsigma}\).

We can use corollary 5.23 to strengthen corollaries 5.1 and 5.2 to if-and-only-if statements:

**Corollary 5.24.** Suppose \(\max \varsigma < \bar{\rho}(q)\). Then we have \(\text{rad} L_{\varsigma} = \{0\} \iff q \in \text{Non}_{\varsigma}\).

**Proof.** Corollary 5.23 with corollary 5.22 implies that \(\text{rad} L_{\varsigma} = \{0\}\) for all \(s \in E_{\varsigma}\), or equivalently by (3.20) that \(\text{rad} L_{n_{\varsigma}} = \{0\} = \text{rad} L_{\varsigma}\), if and only if all of the points \((n_{\varsigma}, s)\) with \(s \in E_{\varsigma}\) lie in the closures of the pink triangles or on the dashed lines in figure 7. This happens if and only if \(q \in \text{Non}_{\varsigma}\).

**Corollary 5.25.** Suppose \(\max \varsigma < \bar{\rho}(q)\). Then we have \(\text{rad} L_{\varsigma} = \{0\}\) for all \(\varsigma \in \hat{0} \cup \mathbb{Z}_{\geq 0}^\#\) if and only if \(\bar{\rho}(q) = \infty\).

**Proof.** This immediately follows from corollary 5.24.
The containment $E_\varsigma \subset E_{n_\varsigma}$ implies that $\text{Non}_{n_\varsigma} \subset \text{Non}_\varsigma$. In fact, this containment becomes an equality when intersected with the set $\{ q \in \mathbb{C}^\times \mid \max \varsigma < \bar{p}(q) \}$:

**Lemma 5.26.** We have

\[
\text{Non}_\varsigma \cap \{ q \in \mathbb{C} \mid \max \varsigma < \bar{p}(q) \} = \text{Non}_{n_\varsigma} \cap \{ q \in \mathbb{C} \mid \max \varsigma < \bar{p}(q) \}
\]

\[
= \{ q \in \mathbb{C}^\times \mid n_\varsigma < \bar{p}(q), \text{ or } q = \pm i \text{ and } \varsigma = \bar{n}_\varsigma \text{ if } n_\varsigma \text{ is odd} \}. \tag{5.117}
\]

**Proof.** To prove the lemma, we show that each $q \in \mathbb{C}^\times$ either belongs to both sets on either side of the equality (5.117) or belongs to neither. This approach will indirectly yield the explicit form (5.118) for these two sets.

Throughout this proof, we assume that $q \in \mathbb{C}^\times$ is such that $\max \varsigma < \bar{p}(q)$. Proving (5.117) first, we also initially assume that $q \neq \pm 1$, so $2 \leq p(q) = \bar{p}(q)$ by (1.19). Under these assumptions, we consider two cases:

1. $n_\varsigma \geq \bar{p}(q)$: In this case, we first observe that

\[
d_\varsigma = 1 \implies \varsigma = (s) \text{ for some } s \in \mathbb{Z}_{\geq 0} \implies n_\varsigma = s = \max \varsigma < \bar{p}(q), \tag{5.119}
\]

a contradiction. Hence, we must have $d_\varsigma > 1$ whenever $n_\varsigma \geq \bar{p}(q)$. In light of this observation, we may invoke lemma 2.4 to say that the minimum value $s_{\text{min}}(\varsigma)$ of the set $E_\varsigma$ (2.25) satisfies

\[
s_{\text{min}}(\varsigma) \overset{\text{(2.39)}}{\leq} \max \varsigma < \bar{p}(q) - 1 \overset{\text{(1.19)}}{=} p(q) - 1 \overset{\text{(1.20)}}{=} \Delta_1. \tag{5.120}
\]

Furthermore, lemma 2.4 implies that the maximum value $s_{\text{max}}(\varsigma)$ of the set $E_\varsigma$ satisfies

\[
\Delta_1 \overset{\text{(1.20)}}{<} \bar{p}(q) \overset{\text{(2.35)}}{\leq} n_\varsigma \overset{\text{max}}{=} s_{\text{max}}(\varsigma). \tag{5.121}
\]

Assuming that $q \neq \pm i$, so $\bar{p}(q) \geq 3$, it is straightforward to see that, under (5.120, 5.121), there is a lattice point $(n_\varsigma, s)$ off the pink triangles and dashed lines in figure 7 and with $s \in E_\varsigma \subset E_{n_\varsigma}$. Thus, we have $q \notin \text{Non}_\varsigma \cup \text{Non}_{n_\varsigma}$.

On the other hand, if $q = \pm i$, then $\bar{p}(q) = 2$, and $\max \varsigma < \bar{p}(q) = 2$ implies that $\varsigma = n_\varsigma'$. Also, if $n_\varsigma$ is odd, then by (2.25), $\bar{p}(q) = p(q) = 2$ divides $s + 1$ for each $s \in E_{n_\varsigma}$. By the containment $E_\varsigma \subset E_{n_\varsigma}$, the same holds for each $s \in E_\varsigma$. Hence, $\pm i \in \text{Non}_\varsigma \cap \text{Non}_{n_\varsigma}$ if $n_\varsigma$ is odd. On the other hand, if $n_\varsigma$ is even, then $p(q) = 2$ divides no element of either sets $E_{n_\varsigma}$ or $E_\varsigma$. Reasoning as in the previous paragraph, we then see that $\pm i \notin \text{Non}_\varsigma \cup \text{Non}_{n_\varsigma}$.

2. $n_\varsigma < \bar{p}(q)$: By (2.25), it is evident that for each $s \in E_{n_\varsigma}$, the lattice point $(n_\varsigma, s)$ is on the bottommost pink triangle in figure 7, with one exception: if $s = n_\varsigma = p(q) - 1$, then $(n_\varsigma, s)$ lies on the lowest dashed line, at height $\Delta_1$. By the containment $E_\varsigma \subset E_{n_\varsigma}$, the same holds for every $s \in E_\varsigma$. Therefore, we have $q \notin \text{Non}_\varsigma \cap \text{Non}_{n_\varsigma}$.

Finally, from (1.19, 5.114, 5.115), it is straightforward to see that $\pm 1$ is an element of the sets in (5.117, 5.118). From this and items 1 and 2 above, we conclude that equality (5.117) holds, and we infer (5.118).

It is sometimes useful to understand the domain $\text{Non}_{n_\varsigma}$ determined mainly by the condition $n < \bar{p}(q)$, in terms of the fugacity $\nu$. For this, we observe that $q = \pm i$ if and only if $\nu = 0$, and we invoke the following simple lemma:

**Lemma 5.27.** Suppose $\nu = -q - q^{-1}$ and $\bar{p}(q)$ is given by (1.19). Then we have

\[
n < \bar{p}(q) \iff \nu^2 \neq 4 \cos^2 \left( \frac{\pi p'}{p} \right) \text{ for any } p', p \in \mathbb{Z}_{>0} \text{ coprime and satisfying } 0 < p' < p \leq n. \tag{5.122}
\]

**Proof.** To prove (5.122), we observe that with $p'$ any positive integer coprime with and less than $p$, we have

\[
p := \bar{p}(q) \leq n \iff q = \pm e^{\pi ip'/p}. \tag{5.123}
\]

Relation (5.122) follows from this and our chosen parameterization (1.18).
D. Totally degenerate cases

We recall that the radical of the standard module $L_{\lambda}^{(s)}$ is said to be “totally degenerate” if $\text{rad} L_{\lambda}^{(s)} = L_{\lambda}^{(s)}$. In section 3B, propositions 3.3 and 3.5 and corollary 3.6 all assume that $\text{rad} L_{\lambda}^{(s)}$ is not totally degenerate. Because these results are fundamental to understanding the structure of the standard modules, it is worthwhile to determine all $q \in \mathbb{C}^\times$ for which the radical of a given standard module is totally degenerate. We establish this in proposition 5.29.

Lemma 5.28. Suppose $\max \varsigma < \overline{p}(q)$. If $\overline{p}(q) \leq s + 1$, then there exists a walk $\rho$ over $\varsigma$ with defect $s$ such that, when followed backward, $\rho^\dagger$ hits height $\Delta_{k_s}$ before $\rho^\dagger$ hits height $\Delta_{k_{s+1}}$.

Proof. Because $s$ is finite and $\overline{p}(\pm 1) = \infty$ by (1.19), we must assume that $q \neq \pm 1$ throughout, so $\overline{p}(q) = p(q)$. We prove the lemma by induction on the length $d_\varsigma \in \mathbb{Z}_{>0}$ of the multiindex $\varsigma$. Assuming first that $d_\varsigma = 1$, we have

$$\varsigma = (s_1) \implies s_1 = \max \varsigma < p(q) \quad (5.124)$$

by the assumption in the lemma. Furthermore, there is exactly one walk $\rho = (s_1)$ over $\varsigma = (s_1)$, trivially with defect $s = s_1$. Thus, with $p(q) < s + 1$ by assumption, we have

$$p(q) - 1 \leq s + 1 = s_1 + 1 \quad (5.124) \implies p(q) = s_1 + 1 = s + 1 \quad (1.20) \implies \begin{cases} s = \Delta_{k_s}, \\ k_s = 1. \end{cases} \quad (5.125)$$

Thus, it is trivially true that when followed backward, $\rho^\dagger$ hits height $\Delta_{k_s}$ before $\rho^\dagger$ hits height $\Delta_{k_{s+1}}$.

Next, we prove that if the lemma holds for all multiindices in $\mathbb{Z}^{d-1}_{>0}$ for some $d \in \{2, 3, \ldots\}$, then it holds for all multiindices $\varsigma \in \mathbb{Z}^{d-1}_{>0}$. In light of the lemma made immediately beneath the proof of lemma 4.1, item 5 of that lemma implies that there exists a walk $\rho$ over $\varsigma$ with defect $s$ whose penultimate height is

$$r := r_{d-1} = \min(E_{\varsigma} \cap E_{(s_1, s)}) \quad (4.16) \quad \max(s_{\min}(\varsigma), |s - s_d|). \quad (5.126)$$

Now there are two scenarios to consider:

1. $r \leq \Delta_{k_s}$: With the penultimate height of $\rho$ equaling $r$, it is evident that $\rho^\dagger$ hits height $\Delta_{k_s}$ while, as we observe in (5.106), $\rho^\dagger$ does not hit height $\Delta_{k_{s+1}}$ in the last step of $\rho$.

2. $r > \Delta_{k_s}$: First, for the last step of $\rho$, we note that by the assumptions of this lemma and by lemma 2.4, we have

$$\begin{cases} p(q) \leq s + 1 \\ s_d \leq \max \varsigma < p(q) \end{cases} \implies |s - s_d| = s - s_d \leq s \quad (1.21) \quad k_s = 1. \quad (5.127)$$

and that

$$s_{\min}(\varsigma) \quad (2.39) \leq \max \varsigma \leq \max \varsigma < p(q) - 1 \quad (1.20) \quad \Delta_{k_s} < r, \quad (5.128)$$

which together show that

$$r \quad (5.126) = \max(s_{\min}(\varsigma), |s - s_d|) \quad (5.127) \quad (5.128) = s - s_d. \quad (5.129)$$

This implies that the apex of $\rho$ at its last step is less than $\Delta_{k_{s+1}}$:

$$h_{s_d, d-1}(\rho) \quad (4.10) = \frac{r + s_d + s}{2} \quad (5.129) = s \quad (1.21) \quad \Delta_{k_{s+1}}. \quad (5.130)$$

Second, we consider $\rho$ from its first to its penultimate step, or equivalently, we consider $\hat{\rho}$. Here, we observe that

$$p(q) - 1 \quad (1.20) = \Delta_1 \leq \Delta_{k_s} < r \quad (5.129) \quad (5.130) \quad s \quad (1.21) \quad \Delta_{k_{s+1}}, \quad (5.131)$$

which implies two facts. First, by (1.21), we have $k_r = k_s$. Second, with $p(q) < r + 1$, the induction hypothesis says that we can choose the walk $\overline{\rho} = (\hat{\rho}, s)$ such that $\hat{\rho}$ is a walk over $\varsigma$ with defect $r$ and with the property that, when followed backward, $\hat{\rho}^\dagger$ hits height $\Delta_{k_s} = \Delta_{k_s}$ before $\rho^\dagger$ hits height $\Delta_{k_{s+1}} = \Delta_{k_{s+1}}$. In light of (5.131), this implies that $\rho$, a walk over $\varsigma$ with defect $s$, has the same property when followed backward from its penultimate step. It follows from this fact and (5.130) that $\rho^\dagger$ hits height $\Delta_{k_s}$ before $\rho^\dagger$ hits height $\Delta_{k_{s+1}}$ as we follow $\rho$ backward.
This concludes the proof. □

Next, we state a condition that is both necessary and sufficient for the radical of \( L_\zeta^{(s)} \) to be totally degenerate. For this purpose, we define the following set, with Lebesgue-measure zero in \( \mathbb{C} \):

\[
\text{Tot}_\zeta^{(s)} := \left\{ q \in \mathbb{C}^\times \mid s + 1 < \bar{p}(q) \leq \min_{\epsilon} \max_{0 \leq j < d_\zeta} h_{\max,j}(q) + 1 \right\},
\]

where the maximum is taken over all walks \( \rho \) over \( \zeta \) with defect \( s \). Stated in other words, we have \( q \in \text{Tot}_\zeta^{(s)} \) if and only if \( q \in \mathbb{C}^\times \) and the maximum apex of each walk \( \rho \) over \( \zeta \), with defect \( s < \bar{p}(q) - 1 \), is at or above height \( \bar{p}(q) - 1 \).

**Proposition 5.29.** \( \max \zeta < \bar{p}(q) \). Then \( \text{rad} L_\zeta^{(s)} = L_\zeta^{(s)} \) if and only if \( q \in \text{Tot}_\zeta^{(s)} \).

**Proof.** If \( q \in \{ \pm 1 \} \), then \( q \not\in \text{Tot}_\zeta^{(s)} \) and the radical of \( L_\zeta^{(s)} \) is nondegenerate by (1.19) and corollary 5.1. Thus, we take \( q \not\in \{ \pm 1 \} \) throughout the proof. As such, we have \( \bar{p}(q) = p(q) \) throughout.

First, we assume that \( q \in \text{Tot}_\zeta^{(s)} \). Then by the comment beneath (5.132), the comment beneath corollary 5.19, and item 4 of lemma 4.1, this implies that \( D_\zeta^{(s)} = D_\zeta^{(s)} \), or equivalently by lemma 2.8 and corollary 5.20, that \( \text{rad} L_\zeta^{(s)} = L_\zeta^{(s)} \).

Next, we assume that \( \text{rad} L_\zeta^{(s)} = L_\zeta^{(s)} \), or equivalently by lemma 2.8 and corollary 5.20, that \( D_\zeta^{(s)} = D_\zeta^{(s)} \). Now, item 4 of lemma 4.1, lemma 5.19, and lemma 5.28 combine to say that if \( p(q) \leq s + 1 \), then \( D_\zeta^{(s)} < D_\zeta^{(s)} \), a contradiction.

Hence, we have \( s + 1 < \bar{p}(q) \). From this inequality, the equality \( D_\zeta^{(s)} = D_\zeta^{(s)} \), the comment following corollary 5.19, and the comment following (5.132), we conclude that \( q \in \text{Tot}_\zeta^{(s)} \). □

Now we consider the special case that \( \zeta = \bar{n} \) for some \( n \in \mathbb{Z}_{>0} \):

**Lemma 5.30.** We have

\[
\text{Tot}_n^{(s)} = \begin{cases} 
\emptyset, & s \neq 0, \\
\{ \pm i \}, & s = 0.
\end{cases}
\]

**Proof.** Let first \( s \in \mathbb{E}_n \setminus \{ 0 \} \), and let \( \rho \) be the walk over \( \bar{n} \) such that, when followed backward, descends from height \( s \) until it hits height zero and then jumps back and forth between heights one and zero for the rest of its length. For this walk, the maximum apex \( h_{\max,j}(\rho) \) over \( j \in \{ 0, 1, \ldots, d_\zeta - 1 \} \) equals \( s \). Therefore, by (5.132), we have

\[
\text{Tot}_n^{(s)} = \{ q \in \mathbb{C}^\times \mid s + 1 < \bar{p}(q) \leq s + 1 \} = \emptyset.
\]

On the other hand, if \( s = 0 \) and \( \rho \) is the walk over \( \bar{n} \) that jumps back and forth between heights one and zero, then the maximum apex \( h_{\max,j}(\rho) \) over \( j \in \{ 0, 1, \ldots, d_\zeta - 1 \} \) equals one and we have

\[
\text{Tot}_n^{(0)} = \{ q \in \mathbb{C}^\times \mid 1 < \bar{p}(q) \leq 2 \} = \{ \pm i \}.
\]

This finishes the proof. □

From the above lemma, we recover a result of D. Ridout and Y. Saint-Aubin:

**Corollary 5.31.** [RSA14, proposition 3.5] We have \( \text{rad} L_n^{(s)} = L_n^{(s)} \) if and only if \( s = 0 \) and \( \bar{p}(q) = 2 \) (i.e., \( q \in \{ \pm i \} \)).

**Proof.** This immediately follows from proposition 5.29 and lemma 5.30. □

We finish this section by some further observations concerning totally degenerate radicals. First, thanks to the condition \( s + 1 < \bar{p}(q) \), we can use the following lemma to “zero-out” the defect height in all cases where the radical of \( L_\zeta^{(s)} \) is totally degenerate:

**Lemma 5.32.** \( \max \zeta < \bar{p}(q) \). Then we have

\[
\text{rad} L_\zeta^{(s)} = L_\zeta^{(s)} \implies \text{rad} L_\zeta^{(0)} = L_\zeta^{(0)}.
\]

(5.136)
Proof. If $q \in \{\pm 1\}$, then the radical of $\text{rad} L^{(s)}_\varsigma$ is nondegenerate by (1.19) and corollary 5.1. Thus, we take $q \notin \{\pm 1\}$ throughout the proof. As such, we have $\tilde{p}(q) = \tilde{p}(q)$ throughout.

With $\text{rad} L^{(s)}_\varsigma = L^{(s)}_\varsigma$, we have $s + 1 < \tilde{p}(q)$ by lemma 5.29 and (5.132). In light of this fact, lemma B.3 of appendix B says that the evaluations of the following networks are equal for any two valenced link states $\alpha, \beta \in L^{(s)}_\varsigma$:

\[
\begin{array}{cc}
I_\alpha | I_\beta & \text{and} & \frac{(-1)^s}{|s + 1|} I_\alpha | I_\beta \\
\end{array}
\]

(5.137)

For any pair of valenced link states $\gamma, \delta \in L^{(0)}_{\varsigma \ominus (s)}$, we may write the network $\gamma \mid \delta$ in the form on the right side of (5.137) for some corresponding pair of valenced link states $\alpha, \beta \in L^{(s)}_\varsigma$. Then with $\text{rad} L^{(s)}_\varsigma = L^{(s)}_\varsigma$, the network on the left side of (5.137) vanishes for any pair of valenced link states $\alpha, \beta \in L^{(s)}_\varsigma$, and with $s + 1 < \tilde{p}(q)$, we have $|s + 1| \neq 0$. Thus, the network $\gamma \mid \delta$ on the right side of (5.137) also vanishes for any pair of valenced link states $\gamma, \delta \in L^{(0)}_{\varsigma \ominus (s)}$.

We observe that a straightforward adaptation of the proof of lemma 5.32 gives a near converse to this lemma:

**Corollary 5.33.** Suppose $\max \varsigma < \tilde{p}(q)$. Then we have

\[
\begin{align*}
\text{rad} L^{(0)}_{\varsigma \ominus (s)} &= L^{(0)}_{\varsigma \ominus (s)} \\
\Rightarrow & \quad \text{rad} L^{(s)}_\varsigma = L^{(s)}_\varsigma.
\end{align*}
\]

(5.138)

The following two lemmas assume that $s = 0$. Using lemma 5.32 and the comment beneath the proof of lemma 4.1, it is straightforward to extend them to all cases in which this assumption is not true.

**Lemma 5.34.** Suppose $\max \varsigma < \tilde{p}(q)$. If there exists a walk over $\varsigma$ with defect zero (i.e., $0 \in E_\varsigma$ (4.15)), then

\[
\tilde{p}(q) \leq \max_{0 \leq j < d_\varsigma} \max \left\{ \frac{\mu_j(\varsigma) + \mu_{j+1}(\varsigma) + s_{j+1}}{2}, s_{j+1} \right\} + 1 \quad \Rightarrow \quad \text{rad} L^{(s)}_\varsigma = L^{(s)}_\varsigma.
\]

(5.139)

Proof. For each $j \in \{0, 1, \ldots, d_\varsigma - 1\}$ and each walk $q$ over $\varsigma$ with defect zero, we have

\[
\min_{q'} h_{\max,j}(q') \leq h_{\max,j}(q) \leq \max_{0 \leq k < d_\varsigma} h_{\max,k}(q),
\]

(5.140)

where the minimum is over all walks $q'$ over $\varsigma$ with defect zero. Taking the minimum and maximum of (5.140) over all $j \in \{0, 1, \ldots, d_\varsigma - 1\}$ and all walks $q$ over $\varsigma$ with defect zero respectively and using item 8 of lemma 4.1, we find

\[
\tilde{p}(q) \leq \max_{0 \leq j < d_\varsigma} \max \left\{ \frac{\mu_j(\varsigma) + \mu_{j+1}(\varsigma) + s_{j+1}}{2}, s_{j+1} \right\} + 1 \quad \Rightarrow \quad \text{rad} L^{(s)}_\varsigma = L^{(s)}_\varsigma.
\]

(5.141)

Taking this together with (5.132) and corollary 5.29 with $s = 0$ and the fact that $1 < \tilde{p}(q)$ for all $q \in \mathbb{C}^\varsigma$ by (1.19), we finally arrive with (5.139).

**Corollary 5.35.** Suppose $\max \varsigma < \tilde{p}(q)$. If $\tilde{p}(q) = s_1 + 1$ for some $i \in \{1, 2, \ldots, d_\varsigma\}$ and there exists a walk over $\varsigma$ with defect zero (i.e., $0 \in E_\varsigma$ (4.15)), then we have $\text{rad} L^{(s)}_\varsigma = L^{(s)}_\varsigma$.

**Proof.** This follows from lemma 5.34 with the fact that

\[
\tilde{p}(q) = s_1 + 1 \leq \max_{0 \leq j < d_\varsigma} \max \left\{ \frac{\mu_j(\varsigma) + \mu_{j+1}(\varsigma) + s_{j+1}}{2}, s_{j+1} \right\} + 1.
\]

(5.142)

(Alternatively, one may use lemma B.1 of appendix B with $s = s_1 = \tilde{p}(q) - 1$ and $r = 0$.)
6. SEMISIMPLICITY OF THE VALENCE TEMPERLEY-LIEB ALGEBRA

In this section, we give a semisimplicity criterion for the valenced Temperley-Lieb algebra, using corollary 5.24.

A. Semisimplicity of the valenced Temperley-Lieb algebra

Proposition 6.1. Suppose \( \max \varsigma < \bar{p}(q) \). If \( q \in \text{Non}_\varsigma \), then the collection \( \{L^{(s)}_\varsigma\}_{s \in E_\varsigma} \) is the complete set of all simple \( TL_\varsigma(\nu) \)-modules.

Proof. Proposition 3.3 and corollary 5.24 together imply that \( L^{(s)}_\varsigma \) is simple for each \( s \in E_\varsigma \). Corollary 3.6 says that these modules are not isomorphic for different \( s \in E_\varsigma \). Finally, corollary 2.7 with \( \varpi = \varsigma \) gives

\[
\sum_{s \in E_\varsigma} \left( \dim L^{(s)}_\varsigma \right)^2 (2.53) = \dim TL_\varsigma(\nu). \]

By [Sha05, Chapter 10, Theorem XV], from this “sum-of-squares formula” (6.1), we conclude that the collection \( \{L^{(s)}_\varsigma\}_{s \in E_\varsigma} \) contains all simple \( TL_\varsigma(\nu) \)-modules. \( \square \)

Theorem 6.2. Suppose \( \max \varsigma < \bar{p}(q) \). If \( q \in \text{Non}_\varsigma \), then the valenced Temperley-Lieb algebra \( TL_\varsigma(\nu) \) is semisimple.

Proof. This immediately follows from the sum-of-squares formula (6.1) and [Sha05, Chapter 10, Theorem XV]. \( \square \)

We believe that this theorem can be expanded to an if-and-only-if statement on the semisimplicity of \( TL_\varsigma(\nu) \):

Conjecture 6.3. Suppose \( \max \varsigma < \bar{p}(q) \). The valenced Temperley-Lieb algebra \( TL_\varsigma(\nu) \) is semisimple if and only if \( q \in \text{Non}_\varsigma \).

To prove this conjecture, one could use the same strategy as in the proof of theorem 6.5 below. However, one would first need to obtain a complete set of all simple \( TL_\varsigma(\nu) \)-modules. After doing this, the proof of theorem 6.5 could be modified in obvious ways to prove the conjecture.

B. Semisimplicity of the Temperley-Lieb algebra

Next we show that conjecture 6.3 holds for the case of the ordinary Temperley-Lieb algebra \( TL_n(\nu) \). It was proved in [GL98, RSA14] that the quotients (1.55) constitute the complete set of simple modules for \( TL_n(\nu) \).

Proposition 6.4. [RSA14, theorem 8.1] The following are true:

1. If \( \nu \neq 0 \) or \( n \notin 2\mathbb{Z}_{>0} \), then the collection \( \{Q^{(s)}_n\mid s \in E_n\} \) contains all simple \( TL_n(\nu) \)-modules.
2. If \( \nu = 0 \) and \( n \in 2\mathbb{Z}_{>0} \), then the collection \( \{Q^{(s)}_n\mid s \in E_n, s \neq 0\} \) contains all simple \( TL_n(\nu) \)-modules.

We can use this result to strengthen theorem 6.2 with \( \varsigma = \bar{u} \) to an if-and-only-if statement concerning the semisimplicity of \( TL_n(\nu) \). We note that this well-known result is implicit in [RSA14] and a general version of it appears in [GL98, corollary (4.8)].

Theorem 6.5. The Temperley-Lieb algebra \( TL_n(\nu) \) is semisimple if and only if \( q \in \text{Non}_n \).

Proof. Because the \( TL_n(\nu) \)-module \( Q^{(s)}_n \) is the quotient of \( L^{(s)}_n \) by its radical, we have \( \dim Q^{(s)}_n \leq \dim L^{(s)}_n \), with equality if and only if the radical of \( L^{(s)}_n \) is trivial. Therefore, we generically have the inequality

\[
\sum_{s \in E_n} \left( \dim Q^{(s)}_n \right)^2 (1.55) \leq \sum_{s \in E_n} \left( \dim L^{(s)}_n \right)^2 (1.26) \Rightarrow \dim TL_n(\nu). \]
First, we assume that \( q \in \text{Non}_n \). Then by corollary 5.24 with \( \zeta = \tilde{n} \), we have \( \text{rad} \, L_n^{(\nu)} = \{0\} \), so \( \dim Q_n^{(\nu)} = \dim L_n^{(\nu)} \), for all \( s \in E_n \). In light of this fact, (6.2) becomes an equality:

\[
\sum_{s \in E_n} (\dim Q_n^{(s)})^2 = \dim TL_n(\nu),
\]

(6.3)

Also, definition (1.23) and item 1 of proposition 6.4 imply that the sum on the left side is over all simple modules. Hence, by [Sha05, Chapter 10, Theorem XV], \( TL_n(\nu) \) is semisimple.

Second, we assume that \( TL_n(\nu) \) is semisimple and consider the two cases of proposition 6.4:

1. \( \nu \neq 0 \) or \( n \notin 2\mathbb{Z}_{>0} \): By semisimplicity of \( TL_n(\nu) \), the sum-of-squares formula (6.3) holds, and by (1.26), it implies that \( \dim Q_n^{(\nu)} = \dim L_n^{(\nu)} \), so \( \text{rad} \, L_n^{(\nu)} = \{0\} \) for all \( s \in E_n \). Thus, corollary 5.24 with \( \zeta = \tilde{n} \) implies that \( q \in \text{Non}_n \).

2. \( \nu = 0 \) and \( n \in 2\mathbb{Z}_{>0} \): In this case, we have \( q \notin \text{Non}_n \), so by corollary 5.24, \( \text{rad} \, L_n^{(\nu)} \) is nontrivial. Hence, we have \( \dim Q_n^{(\nu)} < \dim L_n^{(\nu)} \) for some \( s \in E_n \), and (6.2) becomes a strict inequality. We write this inequality as

\[
\sum_{s \in E_n, s \neq 0} (\dim Q_n^{(s)})^2 \leq \sum_{s \in E_n} (\dim Q_n^{(s)})^2 < \sum_{s \in E_n} (\dim L_n^{(s)})^2 \overset{(1.55)}{=} \dim TL_n(\nu).
\]

(6.4)

According to proposition 6.4, the leftmost sum in (6.4) is over all simple modules of \( TL_n(\nu) \). Therefore, [Sha05, Chapter 10, Theorem XV] shows that \( TL_n(\nu) \) cannot be semisimple.

This concludes the proof.

In spite of the fact that theorem 6.5 is not new, the necessary and sufficient condition for semisimplicity of \( TL_n(\nu) \) that we give in it seems to be rarely stated and even misstated in the literature. For example, in [BR99, Theorem B8.4], the condition for semisimplicity is stated, without proof, to be

\[
\frac{1}{\nu^2} \neq 4 \cos^2(\pi/p) \quad \text{for all } p \in \{2, 3, \ldots, n\},
\]

(6.5)

which is similar but not identical to our condition (5.122).

APPENDICES

A. COEFFICIENTS OF THE JONES-WENZL PROJECTOR

In this appendix, we derive formulas for certain entries of the inverse of the Gram matrix \( \mathcal{G}^{(0)}_{2n} \) and from these, we obtain formulas for the coefficients of the Jones-Wenzl projector in expansion (2.74). The former appear to us to be new but the latter have been derived already in the article [Mor15] of S. Morrison. We also mention that in [GL98, corollary 3.7], J. Graham and G. Lehrer obtain a closed formula for the Jones-Wenzl projector in the exceptional cases when \( q \) in (1.18) is a root of unity.

It is customary to call the Gram matrix \( \mathcal{G}^{(0)}_{2n} \) the meander matrix [DGG97]. We index its entries by the basis \( L_n^{(0)} \) consisting of link patterns,

\[
[\mathcal{G}^{(0)}_{2n}]_{\alpha,\beta} = (\alpha \ | \ \beta), \quad \text{for all } \alpha, \beta \in L_n^{(0)},
\]

(A.1)

and, as in (2.74), we expand the Jones-Wenzl projector in the basis \( LD_n \) of link diagrams as

\[
P_{(\alpha)} = \sum_{T \in \text{LD}_n} (\text{coef}_T) T.
\]

(A.2)

In corollary A.6, we determine the coefficients \( \text{coef}_T \) of this expansion and in proposition A.9, we give explicit formulas in important special cases used frequently in this article. Lemma A.2 shows that the coefficients are given by certain entries of the inverse of the meander matrix \( \mathcal{G}^{(0)}_{2n} \), and lemma A.5 determines these matrix entries.
1. Preliminary observations

As a special case of lemma 2.6, the linear extension of the following map is an isomorphism of vector spaces from the Temperley-Lieb algebra $\mathcal{T}_n(\nu)$ to the space $L^{(0)}_{2n}$ of link states without defects:

$$T = \begin{array}{c}
\begin{array}{c}
\vdots \\
T \\
\vdots
\end{array}
\end{array} \quad \mapsto \quad \alpha_T := \begin{array}{c}
\begin{array}{c}
\vdots \\
T \\
\vdots
\end{array}
\end{array} \quad \text{(A.3)}$$

To begin, we prove that the following *rainbow link pattern* behaves very nicely under the isomorphism (A.3):

$$\ominus_n := \alpha_{1_{\mathcal{T}_n(\nu)}} \quad \text{(A.4)}$$

**Lemma A.1.** Suppose $n + 1 < \bar{p}(q)$. Then $\text{rad} \ L^{(0)}_{2n} = \{0\}$, and the bijection (A.3) sends $P_{(\alpha)}$ to $(-1)^n[n+1] \ominus_n$. Equivalently, we have

$$(-1)^n[n+1] \ominus_n = (P_{(\alpha)} \otimes 1_{\mathcal{T}_n(\nu)}) \ominus_n = (1_{\mathcal{T}_n(\nu)} \otimes P_{(\alpha)}) \ominus_n = (P_{(\alpha)} \otimes P_{(\alpha)}) \ominus_n, \quad \text{(A.5)}$$

or in terms of diagrams,

$$(-1)^n[n+1] = \begin{array}{c}
\begin{array}{c}
\vdots \\
\ominus_n \\
\vdots
\end{array}
\end{array} \quad = \begin{array}{c}
\begin{array}{c}
\vdots \\
\ominus_n \\
\vdots
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\vdots \\
\ominus_n \\
\vdots
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\vdots \\
\ominus_n \\
\vdots
\end{array}
\end{array} \quad \text{(A.6)}$$

**Proof.** That $\text{rad} \ L^{(0)}_{2n} = \{0\}$ follows from lemma 5.23. With the radical $\text{rad} \ L^{(0)}_{2n}$ trivial, the bilinear form is nondegenerate, so the dual link pattern $\ominus_n^*$ is well-defined.

Now, by property (P1) of the Jones-Wenzl projectors, it is evident that all three diagrams in (A.6), and hence all three expressions in (A.5), are equivalent. Also, if $\alpha \in \mathcal{LP}_{2n}^{(0)}$ but $\alpha \neq \ominus_n$, then the network $(P_{(\alpha)} \otimes P_{(\alpha)}) \ominus_n \mid \alpha$ must have a link touching two nodes of a projector box and therefore vanishes. For example,

$$\begin{array}{c}
\begin{array}{c}
\vdots \\
\ominus_n \\
\vdots
\end{array}
\end{array} \quad = \begin{array}{c}
\begin{array}{c}
\vdots \\
\ominus_n \\
\vdots
\end{array}
\end{array} \quad = \begin{array}{c}
\begin{array}{c}
\vdots \\
\ominus_n \\
\vdots
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\vdots \\
\ominus_n \\
\vdots
\end{array}
\end{array} \quad = \begin{array}{c}
\begin{array}{c}
\vdots \\
\ominus_n \\
\vdots
\end{array}
\end{array} \quad \text{if } \alpha \not\in \mathcal{LP}_{2n}^{(0)} \setminus \{\ominus_n\}. \quad \text{(A.7)}$$

Therefore, we have $((P_{(\alpha)} \otimes P_{(\alpha)}) \ominus_n \mid \alpha) = 0$ if $\alpha \in \mathcal{LP}_{2n}^{(0)} \setminus \{\ominus_n\}$. Finally, it immediately follows from (B.1) of lemma B.1 that $(P_{(\alpha)} \otimes 1_{\mathcal{T}_n(\nu)}) \ominus_n \mid \ominus_n) = (-1)^n[n+1]$. Hence, the claim that $\ominus_n^*$ is given by (A.5) follows. \hfill $\Box$
Lemma A.2. Suppose \( n + 1 < \bar{p}(q) \). Then the meander matrix \( \mathcal{G}^{(0)}_{2n} \) is invertible, and we have

\[
P_{(n)} = \sum_{T \in \mathcal{LD}_{n}} (-1)^{n}[n + 1]\left[ (\mathcal{G}^{(0)}_{2n})^{-1} \right]_{\bar{\alpha},\alpha} T \tag{A.8}
\]

\[
= \sum_{T \in \mathcal{LD}_{n}} \left( \left[ \left( (\mathcal{G}^{(0)}_{2n})^{-1} \right)_{\bar{\alpha},\alpha} \right]_{\bar{\alpha},\alpha} \right) T. \tag{A.9}
\]

Proof. We recall that \( \mathcal{G}^{(0)}_{2n} \) is invertible if and only if \( \text{rad} \mathcal{G}^{(0)}_{2n} = \{0\} \), and the latter condition holds if \( n + 1 < \bar{p}(q) \) according to lemma A.1. Therefore, \( \mathcal{G}^{(0)}_{2n} \) is invertible if \( n + 1 < \bar{p}(q) \).

Next, we prove (A.8). Using the nondegenerate bilinear form on \( \mathcal{LP}^{(0)}_{2n} \) and definition (A.1) we see that for all link patterns \( \alpha, \beta \in \mathcal{LP}^{(0)}_{2n} \), the following two equations are both true and equivalent:

\[
\alpha = \sum_{\beta \in \mathcal{LP}^{(0)}_{2n}} \left[ \mathcal{G}^{(0)}_{2n} \right]_{\alpha,\beta} \beta' \iff \beta' = \sum_{\alpha \in \mathcal{LP}^{(0)}_{2n}} \left[ \left( \mathcal{G}^{(0)}_{2n} \right)^{-1} \right]_{\beta,\alpha} \alpha. \tag{A.10}
\]

After substituting \( \beta = \bar{\alpha} \), into the second equation of (A.10), multiplying both sides of it by \( (-1)^{n}[n + 1] \) and indexing the link patterns in \( \mathcal{LP}^{(0)}_{2n} \) by the link diagrams in \( \mathcal{LD}_{n} \) via the bijection (A.3), we arrive with

\[
(-1)^{n}[n + 1] 2\bar{\alpha} = \sum_{T \in \mathcal{LD}_{n}} \left( (-1)^{n}[n + 1] \left[ \left( \mathcal{G}^{(0)}_{2n} \right)^{-1} \right]_{\bar{\alpha},\alpha} \alpha \right) T. \tag{A.11}
\]

Now, lemma A.1 says that the bijection (A.3) sends \( P_{(n)} \) to \( (-1)^{n}[n + 1] \bar{\alpha} \). Thus after applying the inverse of this bijection (A.3) to both sides of (A.11), we arrive with the sought result (A.8).

Finally, to obtain (A.9) from (A.8), we examine the term in the sum of (A.8) with link diagram \( T = \text{id}_{\mathcal{L}_{n}(\nu)} \). According to (2.74), the coefficient of this term necessarily equals one. Combining this fact with (A.4), we find

\[
(-1)^{n}[n + 1] \left[ \left( \mathcal{G}^{(0)}_{2n} \right)^{-1} \right]_{\bar{\alpha},\alpha} \alpha = 1. \tag{A.12}
\]

Using this identity, we immediately obtain (A.9) from (A.8). This finishes the proof. \( \square \)

Corollary A.3. Suppose \( n + 1 < \bar{p}(q) \). Then we have

\[
\left[ \left( \mathcal{G}^{(0)}_{2n} \right)^{-1} \right]_{\bar{\alpha},\alpha} = \frac{(-1)^{n}}{[n + 1]} \tag{A.13}
\]

Proof. This follows immediately from (A.12). \( \square \)

2. Formulas for entries of the inverse meander matrix: ideas

In light of (A.9), we may determine the coefficients of the Jones-Wenzl projector (A.2) by computing the entries of the inverse of the meander matrix \( \mathcal{G}^{(0)}_{2n} \) with row index \( \bar{\alpha} \). In our method, we assume that \( q = e^{4\pi i/\kappa} \) for some irrational \( \kappa \in (4, 8) \). Because the entries of \( \mathcal{G}^{(0)}_{2n} \) are analytic functions of \( q \in \mathbb{C} \times \) with \( n + 1 < \bar{p}(q) \), we do not lose generality with this assumption.

For any \( \kappa \in (4, 8) \), any points \( x_{1} < x_{2} < \ldots < x_{2n} \), and any collection \( \{\Gamma_{1}, \Gamma_{2}, \ldots, \Gamma_{n-1}\} \) of contours in \( \mathbb{C} \), we define a Coulomb gas integral function by

\[
\mathcal{J} \left( \Gamma_{1}, \Gamma_{2}, \ldots, \Gamma_{n-1} \mid x_{1}, x_{2}, \ldots, x_{2n} \right) = \int_{\mathcal{R}_{n-1}} \text{d} u_{n-1} \int_{\mathcal{R}_{n-2}} \text{d} u_{n-2} \cdots \int_{\mathcal{R}_{2}} \text{d} u_{2} \int_{\mathcal{R}_{1}} \text{d} u_{1} \left( \prod_{l=1}^{2n-1} \prod_{m=1}^{n-1} (x_{l} - u_{m})^{-4/\kappa} \right) \left( \prod_{m=1}^{n-1} (x_{2n} - u_{m})^{12/\kappa-2} \right) \left( \prod_{r<s} (u_{r} - u_{s})^{8/\kappa} \right), \tag{A.14}
\]

where the branch of the multivalued integrand is specified later.
We use two classes of functions employing Coulomb gas integrals (A.14). Both classes are linearly independent collections indexed by link patterns, and we denote them by \( \{ F_\alpha \mid \alpha \in LP^{(0)}_{2n} \} \) and \( \{ \Pi_\alpha \mid \alpha \in LP^{(0)}_{2n} \} \). To define the function \( F_\alpha \), we enumerate all links of the link pattern \( \alpha \in LP^{(0)}_{2n} \) from one to \( n \) and in such a way that the link anchored to the \( 2n \)-th node of \( \alpha \), is always the \( n \)-th link. Then, the function \( F_\alpha \) is defined by [FK15c, definition 4]

\[
F_\alpha(x_1, x_2, \ldots, x_{2n}) = \text{const.} \times J(\Gamma_m = m:th \text{ link of } \alpha, \text{ for all } m \in \{1, 2, \ldots, n-1\} \mid x_1, x_2, \ldots, x_{2n}),
\]

(A.15)

where the constant is finite and independent of \( \alpha \) (but depends on \( \kappa \)), and where the branch of the multivalued integrand in (A.14) is fixed in such a way that the integrand is real-valued and positive when

\[
x_1 < x_2 < \cdots < x_{n+1} < u_1 < x_{n+2} < u_2 < \cdots < x_{2n-1} < u_{n-1} < x_{2n}.
\]

(A.16)

The functions \( \Pi_\alpha \) in the second aforementioned class are called “connectivity weights” or “multiple SLE pure partition functions.” They are related to the functions \( F_\alpha \) via the meander matrix [FK15d, definition 4]:

\[
F_\alpha = \sum_{\beta \in \text{LP}^{(0)}_{2n}} [\mathcal{G}^{(0)}_{2n}]_{\alpha, \beta} \Pi_\beta \quad \text{for all } \alpha \in \text{LP}^{(0)}_{2n}.
\]

(A.17)

With \( \kappa \) irrational, we have \( \bar{p}(q) = \infty \), so the matrix \( \mathcal{G}^{(0)}_{2n} \) is invertible and we may solve this system for the rainbow link pattern connectivity weight \( \Pi_{\theta_n} \), finding

\[
\Pi_{\theta_n} = \sum_{\alpha \in \text{LP}^{(0)}_{2n}} \left( (\mathcal{G}^{(0)}_{2n})^{-1} \right)_{\theta_n, \alpha} F_\alpha.
\]

(A.18)

These coefficients of the \( F_\alpha \) on the right side also appear in the formula (A.9) for the Jones-Wenzl projector coefficients.

We may use (A.18) and the explicit formula [FSK15, equation (56)] for the rainbow connectivity weight \( \Pi_{\theta_n} \) to solve for all coefficients that appear on the right side of (A.18). We give these coefficients in lemma A.5. Then, upon inserting the result into (A.9), in corollary A.6 we arrive with explicit formulas for all Jones-Wenzl projector coefficients in (A.2).

To begin, we let \( \mathcal{P}(A, B) \) denote the “Pochhammer contour” surrounding two simply connected regions \( A, B \subset \mathbb{C} \):

\[
\mathcal{P}(A, B) = \begin{array}{c}
\includegraphics[width=0.3\textwidth]{pochhammer_contour.png}
\end{array}
\]

(A.19)

We note that \( \mathcal{P}(A, B) \) is a loop on the universal cover of the set \( \mathbb{C} \setminus (A \cup B) \). Now, denoting \( \Gamma_0 := \{x_{2n}\} \), the formula for \( \Pi_{\theta_n} \) according to [FSK15, equation (56)] reads

\[
\Pi_{\theta_n}(x_1, x_2, \ldots, x_{2n}) = \text{const.} \times J(\Gamma_m = \mathcal{P}(x_{2n-m}, \Gamma_{m-1}), \text{ for all } m \in \{1, 2, \ldots, n-1\} \mid x_1, x_2, \ldots, x_{2n}),
\]

(A.20)

where again, the \( \kappa \)-dependent constant is irrelevant to our purpose. In our definition (A.20) of \( \Pi_{\theta_n} \), the branch of the multivalued integrand in (A.14) is fixed in such a way that the integrand is real-valued and positive when all the integration variables \( u_1, u_2, \ldots, u_{n-1} \) are at the points indicated by red dots, as in (A.16):

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{pochhammer_contour_2.png}
\end{array}
\]

(A.21)

Combining (A.15, A.18, A.20), we infer that for some nonzero \( \kappa \)-dependent constant, we have

\[
J(\Gamma_m = \mathcal{P}(x_{2n-m}, \Gamma_{m-1}), \text{ for all } m \in \{1, 2, \ldots, n-1\}) = \text{const.} \times \sum_{\alpha \in \text{LP}^{(0)}_{2n}} \left( (\mathcal{G}^{(0)}_{2n})^{-1} \right)_{\theta_n, \alpha} F_\alpha.
\]

(A.22)

Our goal in this appendix is to find a formula in terms of \( q \) for all entries of the inverse of the meander matrix \( \mathcal{G}^{(0)}_{2n} \) that appear in (A.22). The text between equations (48–51) of [FSK15, section II E] details how to do this, and
we survey the highlights here. The idea is to deform the Pochhammer contours \( \mathcal{P}(x_{2n-m}, \Gamma_{m-1}) \) in (A.22) and write the integrals (A.14) around them in terms of the integrals appearing in (A.22) with some (non-zero) multiplicative factors depending on \( \kappa \) (i.e., \( q \)). These multiplicative factors arise from the multi-valuedness of the integrand of \( J \) in (A.14) when deforming the contours. The overall multiplicative constants "const." that were not specified above are fixed by formula (A.13) of corollary (A.3). Collecting all of the constants, we finally obtain the sought entries of the inverse meander matrix.

3. Formulas for entries of the inverse meander matrix: method

We begin by writing the outermost Pochhammer contour \( \Gamma_{n-1} \), which surrounds the point \( x_{n+1} \) and the other integration contours \( \Gamma_1, \Gamma_2, \ldots, \Gamma_{n-2} \) (represented here by a gray disc), in the following form:

\[
\Gamma \bigoplus \Gamma_{n-1} = (1 - q^{-2}) \\quad (A.23)
\]

where the red dot represents the choice of branch of the multivalued integrand of \( J \) in (A.14). The phase factor in the second term comes from the integration variable \( u_{n-1} \) winding around the branch cut at \( x_{n+1} \) (there is also a branch cut at the gray disc, but \( u_{n-1} \) winds around it both in the clockwise and in the counter-clockwise direction, so the phase factors cancel).

On the other hand, we consider a clockwise-oriented simple loop \( \Gamma \) surrounding the interval \([x_1, x_{2n}]\) and the integration contours \( \Gamma_1, \Gamma_2, \ldots, \Gamma_{n-2} \). Because the residue of the integrand of \( J \) in (A.14) at infinity equals zero, the integration around \( \Gamma \) gives zero:

\[
\Gamma = 0 \quad (A.24)
\]

Pinching \( \Gamma \) to itself at the point \( x_{n+1} \) and dividing it into a left half \( \Gamma_l \) and a right half \( \Gamma_r \) at the pinch point, we can write (A.24) in the form

\[
\Gamma = \Gamma_l + \Gamma_r \quad (A.25)
\]

\[
\Gamma_l = q^{n+1} + \frac{q^{n+2}}{q - q^{-1}} \quad (A.26)
\]

Next, we decompose \( \Gamma_l \) into a linear combination of contours \([x_j, x_{j+1}]^+\), for \( j \in \{1, 2, \ldots, n-1\} \), which are links in the upper half-plane joining the points \( x_j \) to \( x_{j+1} \):

\[
[x_j, x_{j+1}]^+ := x_j \rightarrow x_{j+1} \quad (A.28)
\]

We obtain

\[
\Gamma_l = (q - q^{-1}) \sum_{j=1}^{n} [j] \quad (A.29)
\]
where \([j]\) is the \(j\):th quantum integer (2.66). Including the overall multiplicative factor \((q - q^{-1})\) into the (yet unspecified) constant, we write (A.22) as

\[
\mathcal{J} \left( \Gamma_m = \mathcal{P}(x_{2n-m}, \Gamma_{m-1}) \text{ for all } m \in \{1, 2, \ldots, n - 1\} \right) = \text{const.} \times \sum_{j=1}^{n} [j] \mathcal{J} \left( \Gamma_m = \mathcal{P}(x_{2n-m}, \Gamma_{m-1}) \text{ for all } m \in \{1, 2, \ldots, n - 2\}, \text{ and } \Gamma_{n-1} = [x_j, x_{j+1}]^+ \right),
\]

where the \(\kappa\)-dependent nonzero constant includes not only the factor \((q - q^{-1})\) from (A.29) but also the further phase factor \((-q^{-n-2}(q - q^{-1}))\) from (A.27).

Now, to find the Jones-Wenzl projector coefficients, we repeat this process for each term in (A.31):

- We write the outermost contour \(\Gamma_{n-2}\) in the form (A.23).
- Via (A.23, A.27), we identify it with the right half \(\Gamma_r\) of a simple clockwise-oriented loop \(\Gamma\) surrounding the interval \([x_1, x_{2n}]\) and the contours \(\Gamma_1, \Gamma_2, \ldots, \Gamma_{n-3}\) and \([x_j, x_{j+1}]^+\).
- With the residue of the integrand of \(\mathcal{J}\) in (A.14) at infinity equaling zero, the integration around \(\Gamma\) gives zero (A.24), and we may thus replace the integration around \(\Gamma_r\) by integration around the left half \(\Gamma_l\) of \(\Gamma\).
- We decompose \(\Gamma_l\) into a linear combination of link-type contours, analogously to (A.29). We make use of [FK15c, lemma 10], which implies that the decomposition of \(\Gamma_l\) in the \(j\):th term in (A.31) takes place as if the contour \([x_j, x_{j+1}]^+\) and its endpoints are invisible: thus, only links \([x_k, x_{k+1}]^+\) for \(k \in \{1, 2, \ldots, j - 2, j + 3, \ldots, n + 2\}\) and \([x_{j-1}, x_{j+2}]^+\) appear in this decomposition. We clarify this below in section A 4 with an example.

After repeating the above process for all Pochhammer contours of (A.22), we finally arrive with an equation of the form (A.22). Using formula (A.13) of corollary (A.3) to fix the overall multiplicative constant, we can identify the entries of the inverse of the meander matrix \(\mathcal{M}_2^{(0)}\) as explicit formulas in terms of \(\kappa\) (i.e., \(q\)). We give the result of this procedure in lemma A.5, but first, we present an example for illustration.

4. Formulas for entries of the inverse meander matrix: examples

The contour deformation recipe given above is somewhat vague, and we invite the reader to investigate the details. To clarify some of this vagueness, we give an example of this recipe’s successful implementation. The example is the simplest one that illustrates all features of this recipe: the case with \(n = 3\).

To determine the coefficients in (A.32), we begin by deforming the Pochhammer contour \(\Gamma_2 = \mathcal{P}(x_4, \Gamma_1)\). According to (A.30–A.31) we obtain

\[
\mathcal{J} \left( \Gamma_1 = \mathcal{P}(x_5, x_6), \ \Gamma_2 = \mathcal{P}(x_4, \Gamma_1) \right) = \text{const.} \times \sum_{\alpha \in \mathbb{LP}_5^{(0)}} \left[ (\mathcal{M}_5^{(0)})^{-1} \right]_{\alpha_3, \alpha_2} \mathcal{F}_n.
\]

To determine the coefficients in (A.32), we begin by deforming the Pochhammer contour \(\Gamma_2 = \mathcal{P}(x_4, \Gamma_1)\). According to (A.30–A.31) we obtain

\[
\mathcal{J} \left( \Gamma_1 = \mathcal{P}(x_5, x_6), \ \Gamma_2 = \mathcal{P}(x_4, \Gamma_1) \right) = \text{const.} \times \sum_{j=1}^{3} [j] \mathcal{J} \left( \Gamma_1 = \mathcal{P}(x_5, x_6), \ \Gamma_2 = [x_j, x_{j+1}]^+ \right).
\]

Next, we deform the Pochhammer contour \(\Gamma_1 = \mathcal{P}(x_5, x_6)\) in each term on the right side of (A.33). To arrive with a formula analogous to (A.30–A.31), we first write \(\Gamma_1\) in the form (A.23) and, as in (A.24–A.27), identify it with a constant multiple of the left half \(\Gamma_l\) of a simple clockwise-oriented loop \(\Gamma\) surrounding the interval \([x_1, x_6]\) and the contours \([x_j, x_{j+1}]^+\). Then, we decompose \(\Gamma_l\) into a linear combination of link-type contours. By [FK15c, lemma 10], for the \(j\):th term on the right side of (A.33), the decomposition of \(\Gamma_l\) takes place as if the contour \([x_j, x_{j+1}]^+\) and its endpoints are invisible. The terms \(j = 1, 2, 3\) are the following (including a factor \((q - q^{-1})\) into the overall multiplicative constants):

1. For the \(j = 1\) term, we obtain

\[
\mathcal{J} \left( \Gamma_1 = \mathcal{P}(x_5, x_6), \ \Gamma_2 = [x_1, x_2]^+ \right) = \text{const.} \times \sum_{k=1}^{2} [k] \mathcal{J} \left( \Gamma_1 = [x_{k+2}, x_{k+3}]^+, \ \Gamma_2 = [x_1, x_2]^+ \right).
\]
2. For the \( j = 2 \) term, we obtain

\[
\mathcal{J}\left( \Gamma_1 = \mathcal{P}(x_5, x_6), \Gamma_2 = [x_2, x_3]^+ \right) = \text{const.} \times \left( [1] \mathcal{J}\left( \Gamma_1 = [x_1, x_4]^+, \Gamma_2 = [x_2, x_3]^+ \right) + [2] \mathcal{J}\left( \Gamma_1 = [x_4, x_5]^+, \Gamma_2 = [x_2, x_3]^+ \right) \right),
\]

where the contour \([x_1, x_4]^+\) arcs over the contour \([x_2, x_3]^+\) in the upper half-plane.

3. The \( j = 3 \) term is similar to the \( j = 2 \) term. We obtain

\[
\mathcal{J}\left( \Gamma_1 = \mathcal{P}(x_5, x_6), \Gamma_2 = [x_3, x_4]^+ \right) = \text{const.} \times \left( [1] \mathcal{J}\left( \Gamma_1 = [x_1, x_2]^+, \Gamma_2 = [x_3, x_4]^+ \right) + [2] \mathcal{J}\left( \Gamma_1 = [x_2, x_5]^+, \Gamma_2 = [x_3, x_4]^+ \right) \right),
\]

where again, the contour \([x_2, x_5]^+\) arcs over the contour \([x_3, x_4]^+\) in the upper half-plane.

Inserting (A.34–A.36) into (A.33) and recalling (A.15), we obtain

\[
\mathcal{J}\left( \Gamma_1 = \mathcal{P}(x_5, x_6), \Gamma_2 = \mathcal{P}(x_4, \Gamma_1) \right) = \text{const.} \times \left( [2][3] \mathcal{F} + [2][2] \mathcal{F} + [2] \mathcal{F} + [2] \mathcal{F} + ([1] + [3]) \mathcal{F} \right).
\]

After identifying this final result with the target equation (A.32), we find the entries in the row of the inverse of the meander matrix \( G_6^{(0)} \) indexed by \( \bar{\mu}_3 = \overrightarrow{\bullet\bullet\bullet} \), up to a common multiplicative constant:

\[
\left[ (G_6^{(0)})^{-1} \right]_0 = \text{const.} \times [2][3] \quad \text{(A.39)}
\]
\[
\left[ (G_6^{(0)})^{-1} \right]_0 = \text{const.} \times [2][2] \quad \text{(A.40)}
\]
\[
\left[ (G_6^{(0)})^{-1} \right]_0 = \text{const.} \times [2] \quad \text{(A.41)}
\]
\[
\left[ (G_6^{(0)})^{-1} \right]_0 = \text{const.} \times [2] \quad \text{(A.42)}
\]
\[
\left[ (G_6^{(0)})^{-1} \right]_0 = \text{const.} \times ([1] + [3]) \quad \text{(A.43)}
\]

Finally, we use (A.13) from corollary A.3 with \( n = 3 \) to solve for the overall multiplicative constant:

\[
- \frac{1}{4} \overset{\text{(A.13)}}{=} \left[ (G_6^{(0)})^{-1} \right]_{\bar{\mu}_3, \bar{\mu}_3} = \left[ (G_6^{(0)})^{-1} \right]_0 \overset{\text{(A.39)}}{=} \text{const.} \times [3][2] \quad \implies \quad \text{const.} = - \frac{1}{4!} \quad \text{(A.44)}
\]

Inserting this normalization into (A.39–A.43) finally gives us explicit formulas in terms of \( q \) of all five entries in the row of the inverse of the meander matrix \( G_6^{(0)} \) indexed by \( \bar{\mu}_3 = \overrightarrow{\bullet\bullet\bullet} \):

\[
\left[ (G_6^{(0)})^{-1} \right]_0 = - \frac{1}{4} \quad \text{(A.45)}
\]
\[
\left[ (G_6^{(0)})^{-1} \right]_0 = - \frac{1}{3}[4] \quad \text{(A.46)}
\]
\[
\left[ (G_6^{(0)})^{-1} \right]_0 = - \frac{1}{3}[4] \quad \text{(A.47)}
\]
\[
\left[ (G_6^{(0)})^{-1} \right]_0 = - \frac{1}{3}[4] \quad \text{(A.48)}
\]
\[
\left[ (G_6^{(0)})^{-1} \right]_0 = - \frac{1}{4!}([1] + [3]) \quad \text{(A.49)}
\]
5. Formulas for entries of the inverse meander matrix: general case

Generalizing the above work to arbitrary $n \in \mathbb{Z}_{>0}$, we obtain an explicit formula in terms of $q$ of all entries in the row of the inverse of the meander matrix $G_{2n}^{(0)}$ indexed by $\mathfrak{m}_n$. To write this formula, we use the following recipe:

**Recipe A.4.** For a link pattern $\alpha \in \text{LP}_{2n}^{(0)}$, we let $a_i, b_i \in \{1, 2, \ldots, 2n\}$ respectively denote the label of the left and right endpoint of the $i$:th link of $\alpha$. We enumerate the links of $\alpha$ from 1 to $n$ in such a way that

1. for all $i, j \in \{1, 2, \ldots, n\}$, if $i < j$, then the $i$:th link does not nest the $j$:th link, and
2. $b_i \leq i + n$ for all $i \in \{1, 2, \ldots, n\}$ or equivalently by item 1, $a_i \leq i + n - 1$ for all $i \in \{1, 2, \ldots, n\}$.

We denote $\vartheta = (a_1, a_2, \ldots, a_n)$, and we let

$$S(\alpha) = \{ \vartheta \in \mathbb{Z}_{n}^{\geq 0} \mid \vartheta \text{ arises from an enumeration of the links of } \alpha \text{ satisfying items 1 and 2} \}.$$  \hfill (A.50)

Finally, for each $\vartheta \in S(\alpha)$, we let $\gamma(\vartheta)$ be a multiindex with $n$ entries and whose $i$:th entry is given by

$$\gamma(\vartheta)_i = 2 \times \# \{1 \leq j \leq i - 1 \mid a_j < a_i\}.$$  \hfill (A.51)

For example, the rainbow link pattern $\mathfrak{m}_n$ has only one enumeration (here, circled numbers indicate the label of the link while uncircled numbers still indicate the size of a cable):

$$\Rightarrow \quad S(\mathfrak{m}_n) = \{(n, n - 1, n - 2, \ldots, 2, 1)\}.$$  \hfill (A.52)

Other link patterns may have several different enumerations. For example, the links of the following link pattern $\alpha \in \text{L}_{8}^{(0)}$ may be enumerated according to recipe A.4 in and only in any one of the following three ways:

$$\Rightarrow \quad S(\alpha) = \{(1, 4, 3, 7), (4, 3, 1, 7), (4, 1, 3, 7)\}.$$  \hfill (A.53)

**Lemma A.5.** Suppose $n + 1 < \hat{p}(q)$, and recall the notations and definitions of recipe A.4. Then the entries of the inverse of the meander matrix $G_{2n}^{(0)}$ along the row indexed by $\mathfrak{m}_n$ are given by the formula

$$[(G_{2n}^{(0)})^{-1}]_{\mathfrak{m}_n, \alpha} = \frac{(-1)^n}{(n + 1)!} \sum_{\vartheta \in S(\alpha)} \prod_{i=1}^{n} [a_i - \gamma(\vartheta)_i].$$  \hfill (A.55)

**Proof.** By continuing our work from (A.32), we eventually find the following formula for the coefficients $[(G_{2n}^{(0)})^{-1}]_{\mathfrak{m}_n, \alpha}$ appearing in (A.32),

$$[(G_{2n}^{(0)})^{-1}]_{\mathfrak{m}_n, \alpha} = \text{const.} \sum_{\vartheta \in S(\alpha)} \prod_{i=1}^{n} [a_i - \gamma(\vartheta)_i].$$  \hfill (A.56)

We leave the details for the reader. To find the constant, we set $\alpha = \mathfrak{m}_n$ and use (A.13) from corollary A.3. \hfill \Box

6. Formulas for coefficients of the Jones-Wenzl projector

Using lemma A.5, we recover the following formula [Mor15] for the Jones-Wenzl projector coefficients in (A.2):
Corollary A.6. [Mor15, Proposition 5.1] Suppose \( n < \bar{p}(q) \). Then for all \( T \in LD_n \), we have

\[
\text{coef}_T = \frac{1}{|n|!} \sum_{\theta \in S(n,T)} \prod_{i=1}^{n} [a_i - \gamma(\vartheta)_i].
\] (A.57)

Proof. For \( n + 1 < \bar{p}(q) \), formula (A.57) follows from combining (A.8) of lemma A.2 with (A.55) of lemma A.5. Because the coefficients \( \text{coef}_T \) are analytic functions of \( q \) away from the poles \( \{ q \in \mathbb{C}^\times | n < \bar{p}(q) \} \), this formula extends continuously to cases with \( n + 1 = \bar{p}(q) \) too. (Alternatively, [Mor15] gives the original proof of this corollary.) \( \square \)

To finish, we give explicit formulas for certain important special cases of the Jones-Wenzl projector coefficients. These formulas are needed frequently in the present article. To derive them, we use the following simple identity.

Lemma A.7. Suppose \( b,k \in \mathbb{Z}_{\geq 0} \) and \( i \in \{1,2,\ldots,b-1\} \). Then we have the sum formula

\[
\sum_{a=0}^{k} \frac{[a+i-1]![b-i+a-1]!}{[a]![a+b]!} = \frac{1}{[i]![i+b+k]!}.
\] (A.58)

Proof. We prove the claim for any \( b \in \mathbb{Z}_{\geq 0} \) by induction on \( k \in \mathbb{Z}_{\geq 0} \). It is easy to see that (A.58) holds for \( k = 0 \). Now, we assume that (A.58) holds with \( k = l-1 \) for some \( l \in \mathbb{Z}_{\geq 0} \). Denoting by \( S_l \) the left side of (A.58), we calculate

\[
S_l = S_{l-1} + \frac{[l+i-1]![b-i+l-1]!}{[l]![b+l]!} - \frac{[l+i-1]![b-i+l-1]!}{[l]![b+l]!} + \frac{[l+i-1]![b-i+l-1]!}{[l]![b+l]!}.
\] (A.59)

This proves formula (A.58) with \( k = l \) and concludes the proof. \( \square \)

Lemma A.8. Suppose \( n < \bar{p}(q) \). Let \( \alpha \in \text{LP}_{2n}^{(0)} \) be the link pattern

\[
\alpha = \begin{array}{c}
m \\
\circ \hspace{1cm} \circ \\
i & k & j \\
l
\end{array}
\]

with \( i + j + k + l + m = n \). (A.63)

Then we have

\[
[(\varrho_{2n}^{(0)})^{-1}]_{m,n,\alpha} = (-1)^n \frac{[i+k]! [i+m]! [i+j+k+m]! [j+l+m]!}{[i]! [k]! [i+j+m]! [m]!}.
\] (A.64)

Proof. To explicitly show the dependence on the sizes \( i \) and \( k \) of the cables of \( \alpha \) thus labeled in (A.63), we write \( \alpha_{ik} \) for the link pattern \( \alpha \) in (A.63). We also let

\[
\Upsilon_{ik} = \sum_{\vartheta \in S(\alpha_{ik})} \prod_{\ell=1}^{n} [a_{\ell} - \gamma(\vartheta)_\ell] \quad \Rightarrow \quad [(\varrho_{2n}^{(0)})^{-1}]_{m,n,\alpha_{ik}} \overset{(A.55)}{=} \frac{(-1)^n \Upsilon_{ik}}{[n+1]}.
\] (A.65)

To prove the lemma, we recursively compute the \( \Upsilon_{ik} \). From item 1 of recipe A.4, we see that the links have to be enumerated in order from innermost to outermost. Furthermore, by item 2 of recipe A.4, the link having label 1 has to be either the innermost of the cable of size \( i \) or the innermost of the cable of size \( k \). In either case, for some
a \in \{0, 1, \ldots, k\}, the innermost \(a\) links in the cable of size \(k\) are enumerated from 1 to \(a\) (in order from innermost to outermost), and the innermost link in the cable of size \(i\) is enumerated as \(a + 1\):

\[
\begin{array}{c}
\begin{array}{c}
\text{(A.66)}
\end{array}
\end{array}
\]

After enumerating the links of \(\alpha_{ik}\) in this way, if we drop the links with labels 1 through \(a + 1\) from \(\alpha_{ik}\), and reduce the labels of the remaining links by \(a + 1\), then we obtain an enumeration for links of \(\alpha_{i-1,k-a}\). This gives

\[
\Upsilon_{ik} = \sum_{a=0}^{k} \frac{1}{[i + m][2i + j + k + m]|[2i + j + k + m - 1] \cdots [2i + j + k + m - a + 1]| \Upsilon_{i-1,k-a}}{[i + m][2i + j + k + m]! \sum_{a=0}^{k} \frac{\Upsilon_{i-1,a}}{[2i + j + a + m]!}}.
\]  

(A.67)

where the factor (resp. factors) under the first (resp. second) brace follow (resp. follows) from the innermost link (resp. a links) dropped from the cable of size \(i\) (resp. \(k\)) in \(\alpha_{ik}\). We use recursion (A.67) to obtain the closed formula

\[
\Upsilon_{ik} = \frac{[i + k]!![i + m]!![i + j + k + m]!![j + l + m]!![i + j + m]!!m!}{[i]!![i + j + m]!!m!}.
\]  

(A.68)

We prove this formula by induction on \(i \in \mathbb{Z}_{\geq 0}\). For \(i = 0\), we may compute \(\Upsilon_{0k}\) explicitly from (A.65). With the cable of size \(i\) empty, the links in the cable of size \(k\) in \(\alpha_{0k}\) are necessarily enumerated one through \(a\). Thus,

\[
\Upsilon_{ik} = \frac{1}{[j + m + 1][j + m + 2] \cdots [j + m + k][j + m + 1][j + m + 2] \cdots [j + m + l]}.
\]  

(A.69)

where the factors under the first (resp. second, resp. third) brace follow from the cable of size \((j + m)\) (resp. \(k\), resp. \(l\)) in \(\alpha_{0k}\). This confirms formula (A.68) for the case \(i = 0\). Now, assuming that (A.68) holds for \(\Upsilon_{i-1,k}\), we prove that it holds for \(\Upsilon_{ik}\) too. Indeed, using lemma A.7 with \(b = 2i + j + m\), we have

\[
\Upsilon_{ik} \overset{(A.67)}{=} \frac{[i + a - 1][i + m - 1][i + j + a + m - 1][j + l + m]!![i + j + a + m]!![i + j + m]!!m!}[2i + j + a + m]!!m!.
\]  

(A.71)

This proves formula (A.68) for \(\Upsilon_{ik}\). After inserting this into (A.65), we finally obtain (A.64).

\section*{Proposition A.9} \textbf{[FK97, Proposition 3.10]} Suppose \(n < \mathfrak{p}(q)\). Let \(T \in \text{TL}_n(\nu)\) be the tangle

\[
T = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{or}
\end{array}
\end{array}
\end{array}
\]

with \(2i + j + k + m = n\).

(A.74)

Then we have

\[
\text{coef}_T = \frac{[i + k]!![i + m]!![i + j + k + m]!![i + j + k]!![i + j + m]!![i + j]!![i + m]!![i]!![i + k]!![i + j + m]!![i + j]!![i]!!}{[n][i][k][m]!!}.
\]  

(A.75)
Proof. The sum in formula (A.57) for $\text{coef}_T$ equals the sum $\Upsilon_{ik}$, defined in (A.65) in the proof of lemma A.8, with $l = i$. In that proof, we computed a closed formula (A.68) for this sum, and after inserting this formula into (A.57), we obtain (A.75). (Alternatively, to prove the lemma, we may use (A.9) of lemma A.2 with (A.64) of lemma A.8.)

B. DIAGRAM SIMPLIFICATIONS

The purpose of this appendix is to collect auxiliary results needed in this article, using diagram calculus known as Temperley-Lieb recoupling theory [Pen69, KL94, CFS95]. We include the proofs for convenience of the reader, but all results of this appendix already appear in some forms in the literature. We recall that the evaluation ($T$) of a network $T$ is defined in (3.6) in section 3 as the product of weights (3.3–3.5) of all objects in $T$.

We begin with some diagram simplifications. We use the following extraction rule in section 3 and appendix D.

Lemma B.1. Suppose $s + r < \bar{p}(q)$. Then we have the following extraction rule:

\[
\begin{align*}
\begin{array}{c}
\text{s} \\
\end{array}
\quad = \quad (-1)^t \frac{r + s + 1}{r + 1} \times \\
\begin{array}{c}
\text{r}
\end{array}
\end{align*}
\]

(B.1)

Proof. We prove formula (B.1) by induction on $s \in \mathbb{Z}_{\geq 0}$. It obviously holds for $s = 0$. Now, assuming that (B.1) holds for all $s \leq t - 1$ for some integer $t \geq 2$, using the induction hypothesis first for $s = 1$ and then for $s = t - 1$, we get

\[
\begin{align*}
\begin{array}{c}
\text{t} \\
\end{array}
\quad = \quad \frac{1}{t - 1} \times \\
\begin{array}{c}
\text{r}
\end{array}
\end{align*}
\]

(B.2)

\[
\begin{align*}
\begin{array}{c}
\text{r}
\end{array}
\quad = \quad -\frac{r + t + 1}{r + 1} \times \\
\begin{array}{c}
\text{t - 1}
\end{array}
\end{align*}
\]

(B.3)

This proves (B.1) with $s = t$, finishing the induction step.

We use the next simple observation in section 4 and later in this appendix.

Lemma B.2. We may insert a projector box above and/or below any network as follows:

\[
\begin{align*}
\begin{array}{c}
\text{T}
\end{array}
\quad = \quad T
\end{align*}
\]

(B.4)
In particular, we have
\[
\begin{pmatrix}
  s
\end{pmatrix} = 1.
\] (B.5)

Proof. On the right side of (B.4), each internal link diagram of the upper (resp. lower) projector box with a turn-back link has weight zero by (3.5). Thus, only the unit link diagram (1.6) contributes, and replacing the projector box with only this diagram is the same as removing the box altogether. Identity (B.5) then follows from (3.4).

We remark that because only the unit link diagram (1.6) contributes to the right side of (B.4) and its coefficient equals one in (2.74), we do not need to restrict the size of the inserted projector boxes to less than \( \bar{p}(q) \). Hence, we did not include this condition in the statement of lemma B.2.

We use the following network evaluation rule in section 3 and later in this appendix.

**Lemma B.3.** Suppose \( s < \bar{p}(q) \). Let \( T \) be a network with \( s \) links passing through the top side of the rectangle and \( s \) links passing through the bottom side. Then the evaluations of the following networks are equal:
\[
\begin{pmatrix}
  s
\end{pmatrix} + 1\] and \[
\begin{pmatrix}
  s
\end{pmatrix}.
\] (B.6)

Proof. Assuming without loss of generality that \( T \) is a link diagram, there are two scenarios to consider:

1. **T contains a turn-back link:** then, the left side of (B.6) contains a turn-back link, which vanishes by rule (3.5), and the right side of (B.6) has a link attached to two nodes of the projector box, which also vanishes by property (P2).
2. **T does not contain a turn-back link:** then, all the \( s \) links passing through the top and bottom side of \( T \) are through-links. We recall from (3.4) that through-links have weight one. Thus, lemma B.1 with \( r = 0 \) shows that the evaluations of the link diagrams in (B.6) are equal.

This concludes the proof.

The next extraction rule, even though simple, is very useful in sections 4 and 5, as well as in appendix D.

**Lemma B.4.** Suppose \( s < \bar{p}(q) \). Then, for any network \( T \) contained between two projector boxes within a larger network, we have the following extraction rule:
\[
(T) \times \begin{pmatrix}
  s
\end{pmatrix}.
\] (B.7)
Proof. We consider two cases:

1. *T contains a turn-back link:* then, the left side of (B.7) vanishes by property (P2), and the right side of (B.7) vanishes because \((T) = 0\) by rule (3.5). Hence, equality in (B.7) holds for this case, with both sides equaling zero.

2. *T does not contain a turn-back link:* then, the network \(T\) comprises only through-links and some number \(p\) of loops. After replacing each loop by a factor \(\nu\) on the left side of (B.7), we obtain

\[
T = \nu^p \times \quad \text{(B.8)}
\]

However, by (3.7), the factor \(\nu^p\) on the right side equals the evaluation \((T)\) of \(T\), so the right side of (B.8) equals the right side of (B.7). Hence, equality in (B.7) holds also for this case.

This concludes the proof.

We define the *Theta network* [KL94] to be the tangle

\[
\begin{align*}
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{align*}
\]

We denote the evaluation of the Theta network by \(\Theta(r, s, t)\).

Together with lemma B.4, the following lemma B.5 is a crucial tool in section 4 and, in particular, in section 5.

**Lemma B.5.** *Suppose \(\max(r, s, s', t) < \bar{p}(q)\). Then we have*

\[
\begin{pmatrix} s' \\ r \\ s \\ t \end{pmatrix} = \Theta(r, s, t) (-1)^{s+1} = \delta_{s, s'} \frac{\Theta(r, s, t)}{(-1)^s[s+1]}.
\]

**Proof.** First, we show that if \(s \neq s'\) in (B.10), then both sides vanish. The right side of (B.10) is clearly zero if \(s \neq s'\). Also, if \(s' < s\) (resp. \(s < s'\)), then on the left side of (B.10), a turn-back link touches two nodes of the projector box with size \(s\) in the lower (resp. upper) vertex, so the network vanishes by property (P2).

Thus, we may assume \(s = s'\). Then, after substituting (4.36), simplifying via (P1), and applying lemmas B.2
and B.3, the network on the left side of (B.10) goes to

Taking evaluations of both sides and recalling identity (B.5) from lemma B.2 finishes the proof.

In lemma B.7, we find an explicit formula for the evaluation of the Theta network. For this purpose, we need the following simple identities.

**Lemma B.6.**

1. The following identity holds for all \(i,j,k \in \mathbb{Z}\):
   \[
   [i][j-k] + [j][k-i] + [k][i-j] = 0. \tag{B.12}
   
   \]

2. The following identity holds for all \(i,k \in \mathbb{Z}_{\geq 0}\) and \(j \in \mathbb{Z}\):
   \[
   \sum_{m=0}^{\min(i,k)} (-1)^m \frac{[i+k-m]!}{[j+m+1][i-m][k-m][m]!} = \frac{[j]![i+j+k+1]!}{[i+j+1][j+k+1]!}. \tag{B.13}
   
   \]

**Proof.** The proof of identity (B.12) in item 1 is a straightforward exercise, using definition (2.66) of the \(q\)-integers.

For item 2, we first observe by a straightforward calculation using identity (B.12) of item 1 that both sides of asserted equality (B.13) satisfy the recursion equation

\[
[k]A_{i-1,k}^j - [i]A_{i,k-1}^j = [k-i]A_{i-1,k-1}^{j+1}, \quad A_{0,k}^j = A_{i,0}^j = \frac{1}{[j+1]}, \quad \text{for all } i,k \in \mathbb{Z}_{\geq 0} \text{ and } j \in \mathbb{Z} \tag{B.14}
\]

(with \(A_{i-1,k}^j = 0 = A_{i,1}^j\)). It follows immediately from (B.14) that (B.13) holds for all \(k \in \mathbb{Z}_{\geq 0}\) and \(j \in \mathbb{Z}\), with \(i = 0\). Then, assuming that asserted equality (B.13) holds for all \(k \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}\), and \(i \in \{0,1,\ldots,n-1\}\) for some \(n \in \mathbb{Z}_{>0}\), it remains to conclude that by (B.14) and induction, it also holds when \(i = n\).  

**Lemma B.7.** Suppose \(\max(r,s,t) < \bar{p}(q)\). Then we have

\[
\Theta(r,s,t) = \frac{(-1)^{r+s+t} \frac{r+s+t}{2} [r+s-t]! [s+t-r]! [t+r-s]!}{[r]![s]![t]!}. \tag{B.15}
\]

**Proof.** According to the proof of lemma B.5, we may write the evaluation of the Theta network in the following form:

\[
\Theta(r,s,t) = (-1)^{s+1} \times \begin{pmatrix}
  \frac{1}{(-1)^{s}[s+1]} \\
  \frac{1}{(-1)^{s}[s+1]} \\
\end{pmatrix}.
\tag{B.16}
where \(i, j\) and \(k\) are given in (B.9). Decomposing the projector box of size \(r\) as in (2.74) and using formula (A.75) from proposition A.9 for the coefficients of this decomposition gives the following sum formula for this evaluation:

\[
(-1)^i+j[i + j + 1][i]![k]! \sum_{m=0}^{\min(i,k)} \frac{[i + k - m]!}{[i - m]![k - m]!} \times \left(\begin{array}{c}
k - m \\
j + m \\
t \\
j + m
\end{array}\right) .
\] (B.17)

Now, we evaluate the networks on the right side of (B.17) using lemmas B.1 and B.2:

\[
\left(\begin{array}{c}
k - m \\
j + m \\
t \\
j + m
\end{array}\right) = \frac{(-1)^{k-m}[j+k+1]}{[j+m+1]}.
\] (B.18)

Then (B.17) becomes

\[
(-1)^{i+j+k}[i + j + 1][j + k + 1][i]![k]! \sum_{m=0}^{\min(i,k)} \frac{[i + k - m]!}{[i - m]![k - m]!} (-1)^m [j + m + 1].
\] (B.19)

Using identity (B.13) from lemma B.6 and plugging in the values of \(i, j\) and \(k\) from (B.9) gives the asserted result. \(\square\)

To end this appendix, we prove some further diagram identities that we need in the next appendix D. To state these identities, we define the Tetrahedral network [KL94] and its evaluation as

\[
T = \begin{array}{ccc}
B & D \\
F & C \\
A & E
\end{array} \quad \Rightarrow \quad (T) = \text{Tet} \left[ \begin{array}{ccc}
A & B & C \\
D & E & F
\end{array} \right] .
\] (B.20)

**Lemma B.8.** We have the following network evaluations:

1. For \(s < \bar{p}(q)\), we have

\[
\left(\begin{array}{c}
i \\
s - i
\end{array}\right) = (-1)^s[s + 1]^{-1} .
\] (B.21)

2. For \(s < \bar{p}(q)\), we have

\[
\left(\begin{array}{c}
i \\
s - i \\
\frac{s + k}{k + 1} \\
\frac{s - j}{k + 1}
\end{array}\right) = (-1)^k[s + 1]^2 \left[\frac{s}{\frac{s - j + k}{2}}\right]^{-1} .
\] (B.22)
3. For $\max(A, B + 2, F) < \bar{p}(q)$, we have

$$\text{Tet} \left[ \begin{array}{ccc} A & B & 2 \\ B + 2 & A & F \end{array} \right] = \frac{1}{|A|} \left[ \frac{A - B + F}{2} \right] \Theta(A, F, B + 2).$$

(B.23)

**Proof.** We prove formulas (B.21)–(B.23) in items 1–3 as follows:

1. Decomposing the upper projector box over all internal link diagrams as in (2.74), we see by rule (P2) that only tangle (A.74) in proposition A.9 with $m = k = 0$ contributes a nonvanishing term. Thus, using (A.75), we get

$$\text{(A.75)} = \left[ \begin{array}{c} s \\ i \end{array} \right]^{-1} \times \left\{ \begin{array}{c} s - 2i \\ i \end{array} \right|_{s \geq 2i} \times \left\{ \begin{array}{c} s - i \\ 2i - s \end{array} \right|_{s \leq 2i}. \tag{B.24}$$

Now, simplification rule (B.1) from lemma B.1 with $r = 0$ gives asserted formula (B.21).

2. We decompose the three-vertices via definition (4.36), obtaining

$$\text{(4.36)} = \left( \begin{array}{c} i \\ s - i \\ j \\ s - j \end{array} \right|_{k}, \tag{B.25}$$

where

$$a = \frac{i + j - k}{2}, \quad b = \frac{2s - i - j - k}{2}, \quad c = \frac{i + k - j}{2}, \quad d = \frac{j - i + k}{2}, \quad e = \frac{k + j - i}{2}, \quad f = \frac{i + k - j}{2}. \tag{B.26}$$

Using identity (B.1) from lemma B.1, we find

$$\text{(B.1)} = (-1)^{a+b} \frac{[s + 1]}{[s - a - b + 1]} \times i \left( \begin{array}{c} s \\ s \end{array} \right|_{s - i}. \tag{B.27}$$

After inserting evaluation (B.21) with $i = c$ of the tangle on the right side into (B.27), we obtain (B.22).

3. Finally, to prove item 3, we first write asserted equality (B.23) in terms of networks:

$$\text{(B.23)} = \frac{1}{|A|} \left[ \frac{A - B + F}{2} \right] \times \text{Tet} \left[ \begin{array}{ccc} A & F \\ B + 2 \end{array} \right]. \tag{B.28}$$
By definition (4.36) the top and bottom three-vertices of the Tetrahedral network respectively simplify to

\[ B + 2 \quad = \quad B \]

\[ A \quad = \quad A - 1 \]

(4.36) (B.29) (B.30)

Using (B.29–B.30) and rule (P1), we simplify the Tetrahedral network on the left side of (B.28) to

\[ i = B + F - A \]

\[ j = \frac{B + A - F}{2} \]

\[ k = \frac{A + F - B}{2} \]

(B.32)

Now, when decomposing the bottom left projector box of size \( A \) in (B.31), rule (P2) shows that the only nonzero contribution comes from the term with exactly one turn-back link. Again, we find the coefficient of this term using (A.75) of proposition A.9. Thus, network (B.31) further simplifies to

\[ \frac{1}{|A|} \left[ \frac{A - B + F}{2} \right] \times \]

(B.33)

It remains to note that the network in (B.33) is exactly the Theta network appearing on the right side of (B.28). This proves item 3 and concludes the proof.
C. RELATION OF THE TWO ALGEBRAS $\text{TL}_\varsigma(\nu)$ AND $\text{JW}_\varsigma(\nu)$

In this appendix, we detail the relationship of the valenced Temperley-Lieb algebra $\text{TL}_\varsigma(\nu)$ and the Jones-Wenzl algebra $\text{JW}_\varsigma(\nu)$. Throughout, we assume that $\max(\varsigma, \bar{\nu}) < \bar{p}(q)$. With $\alpha$ and $\beta$ respectively denoting an arbitrary $\varsigma$-valenced link state and $n_\varsigma$-link state, we define the maps

\begin{equation}
I_\varsigma(\cdot) : L_\varsigma \rightarrow L_{n_\varsigma}, \quad \alpha \mapsto I_\varsigma \alpha, \quad (C.1)
\end{equation}

\begin{equation}
\hat{P}_\varsigma(\cdot) : L_{n_\varsigma} \rightarrow L_\varsigma, \quad \beta \mapsto \hat{P}_\varsigma \beta, \quad (C.2)
\end{equation}

\begin{equation}
P_\varsigma(\cdot) : L_{n_\varsigma} \rightarrow L_{n_\varsigma}, \quad \beta \mapsto P_\varsigma \beta, \quad (C.3)
\end{equation}

where $I_\varsigma$, $\hat{P}_\varsigma$, and $P_\varsigma$ are respectively the tangles (3.13), (3.15), and (2.104). Next, with $T$ and $U$ respectively denoting an arbitrary $(\varsigma, \bar{\nu})$-valenced tangle and $(n_\varsigma, n_{\bar{\nu}})$-tangle, we define the maps

\begin{equation}
I_\varsigma(\cdot) \hat{P}_\varsigma : \text{TL}_\varsigma^{\bar{\nu}}(\nu) \rightarrow \text{TL}_{n_\varsigma}^{n_{\bar{\nu}}}(\nu), \quad T \mapsto I_\varsigma T \hat{P}_\varsigma, \quad (C.4)
\end{equation}

\begin{equation}
\hat{P}_\varsigma(\cdot) I_\varsigma : \text{TL}_{n_\varsigma}^{n_{\bar{\nu}}}(\nu) \rightarrow \text{TL}_\varsigma^{\bar{\nu}}(\nu), \quad U \mapsto \hat{P}_\varsigma U I_\varsigma, \quad (C.5)
\end{equation}

\begin{equation}
P_\varsigma(\cdot) P_\varsigma : \text{TL}_{n_\varsigma}^{n_{\bar{\nu}}}(\nu) \rightarrow \text{TL}_{n_\varsigma}^{n_{\bar{\nu}}}(\nu), \quad U \mapsto P_\varsigma U P_\varsigma. \quad (C.6)
\end{equation}

In lemma C.1, we give a commuting diagram that relates these maps together and states elementary properties about them, including their kernels. To explicate these kernels, we give more definitions, and for this, we group the left nodes and right nodes of an $(n_\varsigma, n_{\bar{\nu}})$-link diagram into the respective left and right “blocks” of nodes

left: \quad \{1, 2, \ldots, s_1\}, \quad \{s_1 + 1, s_1 + 2, \ldots, s_1 + s_2\}, \quad \{s_1 + s_2 + 1, s_1 + s_2 + 2, \ldots, s_1 + s_2 + s_3\}, \quad \ldots \quad (C.7)

right: \quad \{1, 2, \ldots, p_1\}, \quad \{p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\}, \quad \{p_1 + p_2 + 1, p_1 + p_2 + 2, \ldots, p_1 + p_2 + p_3\}, \quad \ldots \quad (C.8)

Then we define a special link diagram to be a link diagram in $\text{LD}_{n_\varsigma}^{n_{\bar{\nu}}}$ that lacks a turn-back link joining two left nodes or two right nodes in a common block of (C.7, C.8), and we denote

\begin{equation}
\text{SD}_\varsigma^{\bar{\nu}} = \{\text{special link diagrams in } \text{LD}_{n_\varsigma}^{n_{\bar{\nu}}}\}. \quad (C.9)
\end{equation}

For example, below, the left figure is a special link diagram in $\text{SD}_\varsigma^{\bar{\nu}}$ with $\varsigma = (2, 2, 3, 2)$ and $\bar{\nu} = (2, 3, 2)$, but the right figure is not such a link diagram:

\begin{equation}
\begin{aligned}
s_1 &= 2 \quad \{1\} & \quad p_1 &= 2 \quad \{1\} \\
s_2 &= 2 \quad \{2\} & \quad p_2 &= 3 \quad \{3\} \\
s_3 &= 3 \quad \{3\} & \quad p_3 &= 2 \quad \{2\} \\
s_4 &= 2 \quad \{2\} & & \end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
s_1 &= 2 \quad \{1\} & \quad p_1 &= 2 \quad \{1\} \\
s_2 &= 2 \quad \{2\} & \quad p_2 &= 3 \quad \{3\} \\
s_3 &= 3 \quad \{3\} & \quad p_3 &= 2 \quad \{2\} \\
s_4 &= 2 \quad \{2\} & & \end{aligned}
\end{equation}

We also define a special link pattern to be a link pattern in $\text{LP}_{n_\varsigma}$ that lacks a turn-back linking two left nodes or two right nodes in a common block of (C.7), and we denote

\begin{equation}
\text{SP}_\varsigma^{(s)} := \{\text{special link patterns in } \text{L}_{n_\varsigma}^{(s)}\} \quad \text{and} \quad \text{SP}_\varsigma := \bigcup_{s \in E_{n_\varsigma}} \text{SP}_\varsigma^{(s)}. \quad (C.10)
\end{equation}

Lemma C.1. Suppose $\max(\varsigma, \bar{\nu}) < \bar{p}(q)$. Then the following are true:

1. Commuting diagrams:

\begin{align*}
\text{L}_{n_\varsigma} & \xrightarrow{I_\varsigma(\cdot)} \text{L}_{n_\varsigma} \\
\text{L}_{n_\varsigma} & \xrightarrow{\hat{P}_\varsigma(\cdot)} \text{L}_{n_\varsigma} \\
\text{L}_{n_\varsigma} & \xrightarrow{P_\varsigma(\cdot)} \text{L}_{n_\varsigma} \\
\text{TL}_{n_\varsigma}^{n_{\bar{\nu}}}(\nu) & \xrightarrow{I_\varsigma(\cdot) \hat{P}_\varsigma(\cdot)} \text{TL}_{n_\varsigma}^{n_{\bar{\nu}}}(\nu) \\
\text{TL}_{n_\varsigma}^{n_{\bar{\nu}}}(\nu) & \xrightarrow{\hat{P}_\varsigma(\cdot) I_\varsigma(\cdot)} \text{TL}_{n_\varsigma}^{n_{\bar{\nu}}}(\nu) \\
\text{TL}_{n_\varsigma}^{n_{\bar{\nu}}}(\nu) & \xrightarrow{P_\varsigma(\cdot) P_\varsigma(\cdot)} \text{TL}_{n_\varsigma}^{n_{\bar{\nu}}}(\nu)
\end{align*}
2. Basic properties:

(a): The maps $I_\varsigma(\cdot) : L_\varsigma \rightarrow L_{n_\varsigma}$ (C.1) and $I_\varsigma(\cdot)\hat{P}_\infty : TL_{n_\varsigma}^\infty(\nu) \rightarrow TL_{n_\varsigma}^{n_\varsigma}(\nu)$ (C.4) are linear injections.

(b): The maps $\hat{P}_\varsigma(\cdot) : L_{n_\varsigma} \rightarrow L_\varsigma$ (C.2) and $\hat{P}_\varsigma(\cdot)I_\infty : TL_{n_\varsigma}^{n_\varsigma}(\nu) \rightarrow TL_{n_\varsigma}^\infty(\nu)$ (C.5) are linear surjections.

(c): The maps $P_\varsigma(\cdot) : L_{n_\varsigma} \rightarrow L_{n_\varsigma}$ (C.3) and $P_\varsigma(\cdot)P_\infty : TL_{n_\varsigma}^{n_\varsigma}(\nu) \rightarrow TL_{n_\varsigma}^{n_\varsigma}(\nu)$ (C.6) are linear projections.

3. Images:

(a): We have
\[ \text{im } I_\varsigma(\cdot) = \text{im } P_\varsigma(\cdot) \quad \text{and} \quad I_\varsigma L_{n_\varsigma} = P_\varsigma SP_\varsigma, \] (C.12)
and the latter is a basis for the former.

(b): We have
\[ \text{im } I_\varsigma(\cdot)\hat{P}_\infty = \text{im } P_\varsigma(\cdot)P_\infty \quad \text{and} \quad I_\varsigma LD_{n_\varsigma} = P_\varsigma SP_\varsigma P_\infty, \] (C.13)
and the latter is a basis for the former.

4. Kernels:

(a): We have
\[ \ker \hat{P}_\varsigma(\cdot) = \ker P_\varsigma(\cdot) \] (C.14)
and the set $LP_{n_\varsigma} \setminus SP_\varsigma$ is a basis for this kernel.

(b): We have
\[ \ker \hat{P}_\varsigma(\cdot)I_\infty = \ker P_\varsigma(\cdot)P_\infty \] (C.15)
and the set $LD_{n_\varsigma} \setminus SD_{n_\varsigma}$ is a basis for this kernel.

5. Homomorphism properties:

(a): For all valenced tangles $T \in TL_{n_\varsigma}^\infty(\nu)$ and for all valenced link patterns $\alpha \in L_\infty$, we have
\[ I_\varsigma(T\alpha) = (I_\varsigma T\hat{P}_\infty)(I_\infty\alpha). \] (C.16)

(b): For all valenced tangles $T \in TL_{n_\varsigma}^\infty(\nu)$ and $U \in TL_{n_\varsigma}^\infty(\nu)$ with $\max \epsilon < \bar{p}(q)$, we have
\[ I_\varsigma(TU)\hat{P}_\infty = (I_\varsigma T\hat{P}_\varsigma)(I_\varphi U\hat{P}_\varsigma). \] (C.17)

6. s-grading preservation:

(a): The maps of (C.1–C.3) respect the s-grading of their domains:
\[ I_\varsigma L_{n_\varsigma}^{\langle s \rangle} \subset L_{n_\varsigma}^{\langle s \rangle}, \quad \hat{P}_\varsigma L_{n_\varsigma}^{\langle s \rangle} = L_{\varsigma}^{\langle s \rangle}, \quad P_\varsigma L_{n_\varsigma}^{\langle s \rangle} \subset L_{n_\varsigma}^{\langle s \rangle}. \] (C.18)

(b): The maps of (C.4–C.6) respect the s-grading of their domains:
\[ I_\varsigma TL_{n_\varsigma}^{\langle s \rangle}(\nu)\hat{P}_\infty \subset TL_{n_\varsigma}^{\langle s \rangle}(\nu), \quad \hat{P}_\varsigma TL_{n_\varsigma}^{\langle s \rangle}(\nu)I_\infty = TL_{n_\varsigma}^{\langle s \rangle}(\nu), \quad P_\varsigma TL_{n_\varsigma}^{\langle s \rangle}(\nu)P_\infty \subset TL_{n_\varsigma}^{\langle s \rangle}(\nu). \] (C.19)

Proof. We prove items 1–6 as follows:

1. That the diagrams commute immediately follows from the property $I_\varsigma \hat{P}_\varsigma = P_\varsigma$ from (3.16).

2. All the maps in the assertion are clearly linear. Furthermore,

(a): with $\hat{P}_\varsigma I_\varsigma = 1_{TL_{n_\varsigma}^\infty(\nu)}$ in (3.16), the map $I_\varsigma(\cdot) : L_\varsigma \rightarrow L_{n_\varsigma}$ is invertible,

(c): with $P_\varsigma^2 = P_\varsigma$ due to (2.106), the map $P_\varsigma(\cdot) : L_{n_\varsigma} \rightarrow L_{n_\varsigma}$ is a projection, and
Proposition 2.12. Suppose \( \rho, \beta \in \mathbb{L} \mathbb{P}_\varsigma \) with link pattern \( \beta \in \mathbb{L} \mathbb{P}_{n_\varsigma} \) created by separating the \( i \)-th node of \( \alpha \) into \( s_i \) adjacent nodes for each \( i \in \{1, 2, \ldots, d_\varsigma\} \). Hence, \( \hat{P}_\varsigma : \mathbb{L}_{n_\varsigma} \rightarrow \mathbb{L}_\varsigma \) is a surjection.

The proofs of the asserted properties for \( I_\varsigma (\cdot) \hat{P}_\varsigma, \hat{P}_\varsigma (\cdot) I_\varsigma \), and \( P_\varsigma (\cdot) P_\varsigma \) are nearly identical to the above.

3. Because \( \hat{P}_\varsigma \) is a surjection and the left diagram in item 1 commutes, we immediately have \( \text{im} I_\varsigma (\cdot) = \text{im} P_\varsigma (\cdot) \). Moreover, it is straightforward to verify that \( I_\varsigma \mathbb{L} \mathbb{P}_\varsigma = P_\varsigma \mathbb{S} \mathbb{P}_\varsigma \). Finally, because \( \mathbb{L} \mathbb{P}_\varsigma \) is a basis for \( \mathbb{L}_\varsigma \) and \( P_\varsigma (\cdot) \) is an injection, it follows that \( I_\varsigma \mathbb{L} \mathbb{P}_\varsigma \) is a basis for \( \text{im} I_\varsigma (\cdot) \). This proves item 3. The proof of item 3 is similar.

4. Because \( I_\varsigma \) is an injection and the left diagram in item 1 commutes, we immediately have \( \ker \hat{P}_\varsigma (\cdot) = \ker P_\varsigma (\cdot) \). Also, because the set \( \mathbb{L} \mathbb{P}_{n_\varsigma} = \mathbb{S} \mathbb{P}_\varsigma \cup (\mathbb{L} \mathbb{P}_{n_\varsigma} \setminus \mathbb{S} \mathbb{P}_\varsigma) \) is a basis for \( \mathbb{L}_{n_\varsigma} \), and the set \( P_\varsigma \mathbb{S} \mathbb{P}_\varsigma \) is a basis for \( \text{im} P_\varsigma \) by item 3, it follows that the set \( \mathbb{L} \mathbb{P}_{n_\varsigma} \setminus \mathbb{S} \mathbb{P}_\varsigma \) is a basis for \( \ker P_\varsigma \). This proves item 4. The proof of item 4 is similar.

5. By idempotent property of \( P_\varsigma \) implied by (2.106), and by (3.16), for any valenced link state \( \alpha \in \mathbb{L} \mathbb{P}_\varsigma \) and any valenced tangle \( T \in \mathbb{S} \mathbb{L}_\varsigma (\nu) \), we have

\[
T \alpha (2.106) \Rightarrow T \mathbb{P}_\varsigma \alpha \quad \Rightarrow \quad I_\varsigma (T \alpha) = I_\varsigma (I_\varsigma \mathbb{P}_\varsigma \alpha) \overset{\text{(3.16)}}{=} (I_\varsigma \mathbb{P}_\varsigma) (I_\varsigma \alpha).
\]

This proves item 5, and the proof of item 5 is similar.

6. Item 6 is immediate.

This concludes the proof.

Combining these definitions with item 6 of lemma C.1, we may write the commuting diagrams in item 1 as

\[
\begin{array}{ccc}
L_\varsigma (\cdot) & \mapsto & \mathbb{W}_\varsigma (\cdot) = P_\varsigma L_\varsigma (\cdot) \\
I_\varsigma (\cdot) & \mapsto & \mathbb{W}_\varsigma (\cdot) = P_\varsigma L_\varsigma (\cdot)
\end{array}
\]

Corollary C.2. (c.f. lemma 2.11) Suppose \( \max \varsigma < \tilde{p}(q) \). Then the Jones-Wenzl algebra \( \mathbb{J} \mathbb{W}_\varsigma (\nu) \) is isomorphic to the valenced Temperley-Lieb algebra \( \mathbb{S} \mathbb{L}_\varsigma (\nu) \).

Proof. By lemma C.1, the map \( I_\varsigma (\cdot) \hat{P}_\varsigma : \mathbb{S} \mathbb{L}_\varsigma (\nu) \rightarrow \mathbb{S} \mathbb{L}_{n_\varsigma} (\nu) \) is a linear injection with image \( \mathbb{J} \mathbb{W}_\varsigma (\nu) \), and by property (C.17) in lemma C.1, this map is also a homomorphism of algebras. Thus, the claim follows.

D. PROOF OF PROPOSITION 2.12

The purpose of this appendix is to prove the main result of section 2:

Proposition 2.12. Suppose \( n_\varsigma < \tilde{p}(q) \). Then the unit (2.104) together with all \( \varsigma \)-Jones-Wenzl link diagrams of the form

\[
P_\varsigma U_{s_1 + s_2 + \ldots + s_\varsigma} P_\varsigma = \begin{array}{c}
\vdots \\
\vdots \\
s_{i-1} \\
s_i \\
s_{i+1} \\
\vdots \\
s_{i+2} \\
\vdots
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\vdots \\
st_{\varsigma} \\
1 \\
s_{\varsigma+1} \\
\vdots \\
\vdots
\end{array}
\]
with \( i \in \{1, 2, \ldots, d_\zeta - 1\} \), forms a minimal generating set for the Jones-Wenzl algebra \( \text{JW}_\zeta(\nu) \). Explicitly, this generating set is

\[
G_\zeta := P_\zeta \left\{ 1_{\text{TL}_{n_\zeta}(\nu)}, U_{s_1}, U_{s_1 + s_2}, \ldots, U_{s_1 + s_2 + \ldots + s_{d_\zeta - 1}} \right\} P_\zeta.
\] (2.113)

Alternatively, the collection of all \( \zeta \)-Jones-Wenzl tangles of the form

\[
\begin{array}{c}
\vdots \\
(2.114) \\
\vdots \\
\end{array}
\]

with \( s \in E(s_i, s_{i+1}) \) and \( i \in \{1, 2, \ldots, d_\zeta - 1\} \), form a minimal generating set for \( \text{JW}_\zeta(\nu) \).

We have divided the proof into several parts in order to clarify its structure and emphasize the main ideas. The proof constitutes rather involved diagram calculations presented in this appendix, but no prerequisites are needed. Section D 1 contains some easy observations needed in the proof. In section D 2, we prove proposition 2.12 for the case of two nodes, \( d_\zeta = 2 \). Then, in section D 3 we construct certain basis tangles of \( \text{JW}_\zeta(\nu) \) from the claimed generators \( G_\zeta \), by induction in the number \( d_\zeta \) of nodes. In section D 4, we finish the proof using the work in sections D 2 and D 3.

Throughout, we fix \( \zeta = (s_1, s_2, \ldots, s_{d_\zeta}) \) as in (2.2, 2.3), with \( n_\zeta < \overline{p}(q) \). We consider tangles in the Jones-Wenzl algebra \( \text{JW}_\zeta(\nu) \), with \( \nu \in \mathbb{C} \) parameterized as in (1.18). We recall from section 2 F the notation (2.107) for the space of Jones-Wenzl link states, and we also use the following notation for the sets of Jones-Wenzl link patterns and Jones-Wenzl link diagrams:

\[
\begin{align*}
W_\zeta^{(s)} &:= P_\zeta \mathcal{L}_{n_\zeta}^{(s)}, & W_\zeta := P_\zeta \mathcal{L}_{n_\zeta} = \bigoplus_{s \in E_\zeta} W_\zeta^{(s)}, \\
WP_\zeta^{(s)} &:= P_\zeta \mathcal{L}P_{n_\zeta}^{(s)}, & WP_\zeta := P_\zeta \mathcal{L}P_{n_\zeta} = \bigoplus_{s \in E_\zeta} WP_\zeta^{(s)}, \\
WD_\zeta &:= P_\zeta \mathcal{L}D_{n_\zeta}^{(s)} P_\zeta.
\end{align*}
\] (D.1)

\[
\begin{align*}
\mathcal{W}_\zeta &:= \mathcal{W}_{n_\zeta}^{(s)} = \mathcal{W}_{\zeta}, & W_\zeta := \mathcal{W}_{n_\zeta} = \bigoplus_{s \in E_\zeta} \mathcal{W}_\zeta^{(s)}, \\
\mathcal{W}P_\zeta &:= \mathcal{WP}_{n_\zeta}^{(s)} = \mathcal{WP}_\zeta, & WP_\zeta := \mathcal{WP}_{n_\zeta} = \bigoplus_{s \in E_\zeta} \mathcal{WP}_\zeta^{(s)}, \\
\mathcal{W}D_\zeta &:= \mathcal{WD}_{n_\zeta}^{(s)} P_\zeta.
\end{align*}
\] (D.2)

\[
\begin{align*}
\mathcal{W}D_\zeta &:= \mathcal{WD}_{n_\zeta}^{(s)}.
\end{align*}
\] (D.3)

1. Preliminary results

It is convenient to denote the number of defects in a Jones-Wenzl link state \( \alpha \) by \( u_\alpha \), that is,

\[
\alpha \in W_\zeta^{(s)} \implies u_\alpha := s.
\] (D.4)

To begin, we collect some useful change of basis operations in lemma D.1. For this, we define the index set

\[
R_{\alpha, \beta} := \left\{ 0, 1, \ldots, \min \left( s_{d_\zeta} - \frac{|u_\alpha - u_\beta|}{2}, u_\alpha, u_\beta \right) \right\}, \quad \text{for all } \alpha, \beta \in WP_\zeta.
\] (D.5)

We also frequently use the notations \( \hat{\zeta} = (s_1, s_2, \ldots, s_{d_\zeta}) \) and \( t = s_{d_\zeta} \) from (2.28), and we recall that

\[
n_\zeta = s_1 + s_2 + \cdots + s_{d_\zeta - 1} = s_{\text{max}}(\zeta).
\] (D.6)
Now, the following set is a basis for the Jones-Wenzl algebra \( JW(\nu) \):

\[
\text{WD}_0 := \begin{cases}
\alpha_1, \alpha_2 \in WP(\varsigma), \\
r \in R_{\alpha_1, \alpha_2}, \\
u := \min(u_{\alpha_1}, u_{\alpha_2}) - r, \\
v := \frac{|u_{\alpha_1} - u_{\alpha_2}|}{2}, \\
w := r + \frac{|u_{\alpha_1} - u_{\alpha_2}|}{2}
\end{cases},
\]

(D.7)

Using lemma 4.4 from section 4B, we may construct new bases for \( JW(\nu) \).

**Lemma D.1.** Suppose \( n_\varsigma < p(q) \). Then each of the following sets is a basis for the Jones-Wenzl algebra \( JW(\nu) \):

\[
\text{WD}_1 := \begin{cases}
\alpha_1, \alpha_2 \in WP(\varsigma), \\
r \in R_{\alpha_1, \alpha_2}, \\
u := \min(u_{\alpha_1}, u_{\alpha_2}) - r, \\
v := \frac{|u_{\alpha_1} - u_{\alpha_2}|}{2}, \\
w := r + \frac{|u_{\alpha_1} - u_{\alpha_2}|}{2}
\end{cases},
\]

(D.8)

\[
\text{WD}_2 := \begin{cases}
\alpha_1, \alpha_2 \in WP(\varsigma), \\
j \in E(\alpha_1, t) \cap E(\alpha_2, t), \\
with \ t = s_\varsigma
\end{cases},
\]

(D.9)

\[
\text{WD}_3 := \begin{cases}
\alpha_1, \alpha_2 \in WP(\varsigma), \\
i \in E(\alpha_1, \alpha_2) \cap E(i, t), \\
with \ t = s_\varsigma
\end{cases}.
\]

(D.10)
Proof. First, we label the left and right link states of (D.7) by $a$ and $c$ respectively, and we label the left-bottom and right-bottom projector boxes $b$ and $d$ respectively. Then we obtain $WD_1$ (D.8) from (D.7) by sending

\[
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In the next lemmas D.2 and D.3, we collect further technical results of combinatorial nature. Lemma D.2 is needed in the proofs of lemmas D.3, D.4 and D.10, and lemma D.3 is needed in the proof of lemma D.11.

From now on, we denote \( \zeta := (s_2, s_3, \ldots, s_{d_\ell}) \) and note that \( s_{\max}(\zeta) = s_2 + s_3 + \cdots + s_{d_\ell} \).

**Lemma D.2.** For any Jones-Wenzl link pattern \( \alpha \in \text{WP}_\zeta \), the following holds:

1. All elements of the union \( E_{(\alpha,s_1)} \cup E_\zeta \) have the same parity.
2. Upon writing \( \alpha \) in the form
   \[
   \alpha = \begin{array}{c}
   \vline \\
   \hline \\
   | \\
   \hline \\
   \end{array}
   \begin{array}{c}
   s_1 \\
   \mid \\
   \hat{\alpha} \\
   \hline \\
   \end{array}
   \]
   \[
   \text{(D.19)}
   \]
   we have \( u_\hat{\alpha} \in E_{(\alpha,s_1)} \cap E_\zeta \).
3. The intersection \( E_{(\alpha,s_1)} \cap E_\zeta \) is not empty, and it is given by
   \[
   E_{(\alpha,s_1)} \cap E_\zeta = \left\{ \max(s_{\min}(\zeta), |u_\alpha - s_1|), \max(s_{\min}(\zeta), |u_\alpha - s_1|) + 2, \ldots, \min(s_{\max}(\zeta), u_\alpha + s_1) \right\}. \tag{D.20}
   \]

**Proof.** We prove items 1–3 as follows:

1. The sets \( E_{(\alpha,s_1)} \) and \( E_\zeta \) each comprise a sequence of integers that increases in increments of two. Explicitly,
   \[
   E_{(\alpha,s_1)} \equiv \{ |u_\alpha - s_1|, |u_\alpha - s_1| + 2, \ldots, u_\alpha + s_1 \}, \quad E_\zeta \equiv \{ s_{\min}(\zeta), s_{\min}(\zeta) + 2, \ldots, s_{\max}(\zeta) \}. \tag{D.21}
   \]
   Hence, if any two elements \( a \in E_{(\alpha,s_1)} \) and \( b \in E_\zeta \) have the same parity, which we write as \( a \sim b \), then all elements of the union \( E_{(\alpha,s_1)} \cup E_\zeta \) have the same parity. Thus, to prove item 1, we only have to note that \( u_\alpha \in E_\zeta \) implies
   \[
   u_\alpha \sim n_\zeta = s_1 + s_2 + \cdots + s_{d_\ell} \quad \Rightarrow \quad u_\alpha + s_1 \sim \sum_{i=1}^{d} s_i = s_2 + s_3 + \cdots + s_{d_\ell} = s_{\max}(\zeta). \tag{D.22}
   \]
2. With \( u_\hat{\alpha} \in E_\zeta \) by definition, the claim in item 2 amounts to showing that \( u_\hat{\alpha} \in E_{(\alpha,s_1)} \). Now, the defects of \( \hat{\alpha} \) either are defects of \( \alpha \) or attach to the nodes of the projector box of size \( s_1 \). Hence, we have
   \[
   u_\hat{\alpha} \leq u_\alpha + s_1 \quad \text{with equality achieved for}
   \begin{array}{c}
   \vline \\
   \hline \\
   | \\
   \hline \\
   \end{array}
   \begin{array}{c}
   s_1 \\
   \mid \\
   \hat{\alpha} \\
   \hline \\
   \end{array}
   \]
   \[
   \text{(D.23)}
   \]
   Also, the defects of \( \alpha \) either attach to the nodes of the projector box of size \( s_1 \) or are themselves defects of \( \hat{\alpha} \), so
   \[
   u_\alpha \leq s_1 + u_\hat{\alpha} \quad \text{with equality achieved for}
   \begin{array}{c}
   \vline \\
   \hline \\
   | \\
   \hline \\
   \end{array}
   \begin{array}{c}
   s_1 \\
   \mid \\
   \hat{\alpha} \\
   \hline \\
   \end{array}
   \]
   \[
   \text{(D.24)}
   \]
   Moreover, both links and defects of \( \alpha \) may attach to the projector box of size \( s_1 \), and all links that attach to this projector box must connect to defects of \( \hat{\alpha} \). Hence, we have
   \[
   s_1 \leq u_\hat{\alpha} + u_\alpha \quad \text{with equality achieved for}
   \begin{array}{c}
   \vline \\
   \hline \\
   | \\
   \hline \\
   \end{array}
   \begin{array}{c}
   s_1 \\
   \mid \\
   \hat{\alpha} \\
   \hline \\
   \end{array}
   \]
   \[
   \text{(D.25)}
   \]
   Altogether, (D.23–D.25) imply that \( |u_\alpha - s_1| \leq u_\hat{\alpha} \leq u_\alpha + s_1 \). Moreover, with \( u_\hat{\alpha} \in E_\zeta \), item 1 says that \( u_\hat{\alpha} \) has the same parity as the elements of \( E_{(\alpha,s_1)} \). Hence, we conclude from (D.21) that \( u_\hat{\alpha} \in E_{(\alpha,s_1)} \), proving item 2.
3. Expressions (D.21) for the sets $E_{(\alpha,s_1)}$ and $E_\varsigma$ combined with item 2 imply expression (D.20) for the intersection $E_{(\alpha,s_1)} \cap E_\varsigma$. This proves item 3.

This concludes the proof. \hfill \qed

**Lemma D.3.** For all Jones-Wenzl link patterns $\alpha_1, \alpha_2 \in WP_\varsigma$ such that $u_{\alpha_2} = u_{\alpha_1} + 2$, we have

$$E_{(\alpha_1,s_1)} \cap E_{(\alpha_2,s_1)} \cap E_\varsigma \neq \emptyset.$$  \hspace{1cm} (D.26)

**Proof.** We recall from item 3 of lemma D.2 that for any Jones-Wenzl link pattern $\alpha \in WP_\varsigma$, we have

$$\max(E_{(\alpha,s_1)} \cap E_\varsigma) \overset{(D.20)}{=} \min(s_{\max}(\varsigma), u_\alpha + s_1),$$  \hspace{1cm} (D.27)

$$\min(E_{(\alpha,s_1)} \cap E_\varsigma) \overset{(D.20)}{=} \max(s_{\min}(\varsigma), |u_\alpha - s_1|).$$  \hspace{1cm} (D.28)

By (D.27), we clearly have $\max(E_{(\alpha_1,s_1)} \cap E_\varsigma) \leq s_{\max}(\varsigma)$, which leads us to consider two cases:

1. $\max(E_{(\alpha_1,s_1)} \cap E_\varsigma) = s_{\max}(\varsigma)$: In this case, we trivially have $s_{\max}(\varsigma) \in E_{(\alpha_1,s_1)} \cap E_\varsigma$. Furthermore, (D.27) with $\alpha = \alpha_1$ gives $s_{\max}(\varsigma) \leq u_{\alpha_1} + s_1$, and with $u_{\alpha_2} = u_{\alpha_1} + 2$, this also implies that

$$s_{\max}(\varsigma) < u_{\alpha_1} + s_1 + 2 = u_{\alpha_2} + s_1 \implies s_{\max}(\varsigma) \overset{(D.27)}{=} \max(E_{(\alpha_2,s_1)} \cap E_\varsigma).$$  \hspace{1cm} (D.29)

We conclude that $s_{\max}(\varsigma) \in E_{(\alpha_2,s_1)} \cap E_\varsigma$ too. Thus, we have $s_{\max}(\varsigma) \in E_{(\alpha_1,s_1)} \cap E_{(\alpha_2,s_1)} \cap E_\varsigma$, so this set is not empty.

2. $\max(E_{(\alpha_1,s_1)} \cap E_\varsigma) < s_{\max}(\varsigma)$: In this case, (D.27) with $\alpha = \alpha_1$ gives $u_{\alpha_1} + s_1 < s_{\max}(\varsigma)$. Thus, we have

$$u_{\alpha_1} + s_1 = \min(s_{\max}(\varsigma), u_{\alpha_1} + s_1) \overset{(D.27)}{=} \max(E_{(\alpha_1,s_1)} \cap E_\varsigma) \implies u_{\alpha_1} + s_1 \in E_{(\alpha_1,s_1)} \cap E_\varsigma.$$  \hspace{1cm} (D.30)

To finish, we prove that $u_{\alpha_1} + s_1 \in E_{(\alpha_2,s_1)}$. For this purpose, by lemma D.2 we only need to show that

$$\min E_{(\alpha_2,s_1)} \leq u_{\alpha_1} + s_1 \leq \max E_{(\alpha_2,s_1)}.$$  \hspace{1cm} (D.31)

First, with $u_{\alpha_2} = u_{\alpha_1} + 2$, we have

$$u_{\alpha_1} + s_1 < u_{\alpha_1} + s_1 + 2 = u_{\alpha_2} + s_1 \overset{(D.21)}{=} \max E_{(\alpha_2,s_1)}.$$  \hspace{1cm} (D.32)

Second, we observe that

$$\begin{cases} s_1 - u_{\alpha_2} \leq s_1 \leq u_{\alpha_1} + s_1 \\ u_{\alpha_2} - s_1 \leq u_{\alpha_2} - s_1 + 2(s_1 - 1) = u_{\alpha_1} + s_1 \end{cases} \implies \min E_{(\alpha_2,s_1)} \overset{(D.21)}{=} |u_{\alpha_2} - s_1| \leq u_{\alpha_1} + s_1.$$  \hspace{1cm} (D.33)

we have

$$u_{\alpha_1} + s_1 \in E_{(\alpha_2,s_1)} \overset{(D.30)}{=} u_{\alpha_1} + s_1 \in E_{(\alpha_1,s_1)} \cap E_{(\alpha_2,s_1)} \cap E_\varsigma.$$  \hspace{1cm} (D.34)

Together, items 1 and 2 show that the intersection $E_{(\alpha_1,s_1)} \cap E_{(\alpha_2,s_1)} \cap E_\varsigma$ is never empty. \hfill \qed

We remark that by linearity, lemmas D.2 and D.3 may be applied to all Jones-Wenzl link states $\alpha, \alpha_1, \alpha_2 \in WP_\varsigma$.

As the last preliminary result, we construct new tangles by concatenating basis tangles of $JW_\varsigma(\nu)$ from $WD_0\varsigma$ (D.7) and $WD_3\varsigma$ (D.10). This enables induction in the number $d_\varsigma$ of nodes later on in the proofs of lemmas D.10 and D.11.

To state the next lemma, we denote by $\mathbb{1}(A)$ the indicator function on the statement $A$, equaling one if $A$ is true and zero if $A$ is false. We also use the notation $\varsigma := (s_2, s_3, \ldots, s_{d_\varsigma - 1})$ and $t = s_{d_\varsigma}$.  

Lemma D.4. Suppose $n_\varsigma < \bar{p}(q)$. We let

$$\alpha_1, \alpha_2 \in \mathbb{W}_{\hat{p}} \bar{\varsigma}, \quad u_1 \in E_{(\alpha_1, s_1)} \cap E_\varsigma, \quad u_2 \in E_{(\alpha_2, s_1)} \cap E_\varsigma, \quad \beta_1, \gamma_1 \in \mathbb{W}_\varsigma^{(u_1)}, \quad \beta_2, \gamma_2 \in \mathbb{W}_\varsigma^{(u_2)},$$

and, for each index

$$i \in E_{(u_1, u_2)} \cap E_{(t, t)} = \{|u_1 - u_2|, |u_1 - u_2| + 2, \ldots, \min(u_1 + u_2, 2t)\},$$

we let $T_i \left( \frac{\alpha_1}{\alpha_2}, \frac{\beta_1}{\beta_2}, \frac{\gamma_1}{\gamma_2} \right)$ denote the following product of three tangles:

Then we have

$$T_i \left( \frac{\alpha_1}{\alpha_2}, \frac{\beta_1}{\beta_2}, \frac{\gamma_1}{\gamma_2} \right) = c_i \left( \frac{\alpha_1}{\alpha_2}, \frac{\beta_1}{\beta_2}, \frac{\gamma_1}{\gamma_2} \right)$$

where the coefficient is

$$c_i \left( \frac{\alpha_1}{\alpha_2}, \frac{\beta_1}{\beta_2}, \frac{\gamma_1}{\gamma_2} \right) = \text{Tet} \left[ \begin{array}{ccc} u_1 & u_{\alpha_1} & i \\ u_{\alpha_2} & u_2 & s_1 \end{array} \right] \left( \Theta(i, u_{\alpha_1}, u_{\alpha_2}) \right) \mathbb{1}\{i \in E_{(u_1, u_2)} \cap E_{(u_1, u_2)} \cap E_{(t, t)}\}.$$

Proof. To begin, we verify that the lemma statement makes sense. First, we observe that by lemma D.2 with $\varsigma \mapsto \hat{\varsigma}$, the sets $E_{(\alpha_1, s_1)} \cap E_\varsigma$ and $E_{(\alpha_2, s_1)} \cap E_\varsigma$ are not empty and all elements of their union have the same parity. Therefore, the elements of $E_{(u_1, u_2)}$ are even, so the intersection $E_{(u_1, u_2)} \cap E_{(t, t)}$ is not empty. Finally, conditions (D.35) on $u_1$ and $u_2$ guarantee that the three-vertices of the left and right tangles in the product (D.37) exist, and the condition $i \in E_{(u_1, u_2)} \cap E_{(t, t)}$ guarantees that the three-vertices of the middle tangle in this product exist.

Now we compute the product tangle $T_i$ of (D.37), for all $i$ as in (D.36). On the one hand, lemma B.4 shows that the tangle (D.37) simplifies to

$$T_i \left( \frac{\alpha_1}{\alpha_2}, \frac{\beta_1}{\beta_2}, \frac{\gamma_1}{\gamma_2} \right) = (\beta_1 \mid \gamma_1)(\beta_2 \mid \gamma_2)$$
On the other hand, we may express the tangle in (D.40) as a linear combination of tangles of the basis $\text{WD}_3(\varsigma)$ (D.10) of lemma D.1 with the same left/right link patterns $\alpha_1, \alpha_2 \in \text{WP}_\varsigma$ as in this tangle. We then have

$$T_i\left(\begin{array}{ccc} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{array}\right) = \sum_{j \in E(\alpha_1, \alpha_2) \cap E(t,t)} c_{ij} \left(\begin{array}{ccc} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{array}\right)$$

for some coefficients $c_{ij} \in \mathbb{C}$. We find them equating (D.40) with (D.41) and inserting the tangles of either side into the “dual” tangle (with $s_{\max}(\varsigma) < n_\varsigma < p(q)$, corollary 5.1 implies that the dual link states $\alpha_1^\vee, \alpha_2^\vee \in L_\varsigma$ exist):

$$k \in E(\alpha_1, \alpha_2) \cap E(t,t) = \{|u_{\alpha_1} - u_{\alpha_2}|, |u_{\alpha_1} - u_{\alpha_2}| + 2, \ldots, \min(u_{\alpha_1} + u_{\alpha_2}, 2t)\},$$

thereby closing all links into loops. Thus, we arrive with

$$\begin{align*}
(\beta_1 | \gamma_1)(\beta_2 | \gamma_2) & \quad (\beta_2 | \gamma_2) \\
& = \sum_{j \in E(\alpha_1, \alpha_2) \cap E(t,t)} c_{ij} \left(\begin{array}{ccc} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{array}\right)
\end{align*}$$

for each
Here, we have not labeled the sizes of all cables in these networks. One may infer those sizes from (D.37). Using lemmas B.4 and B.5 of appendix B, we delete the lower loop of either network, finding

\[
\delta_{ik} (\beta_1 | \gamma_1)(\beta_2 | \gamma_2) \frac{\Theta(i, t, t)}{(-1)^{[i+1]} [j + 1]} \]

(D.45)

\[
= \sum_{j \in E(s_1, s_2) \cap E(t, t)} c_{ij} \left( \frac{\alpha_1}{\alpha_2} \frac{\beta_1}{\beta_2} \frac{\gamma_1}{\gamma_2} \right) \delta_{jk} \frac{\Theta(j, t, t)}{(-1)^{[j+1]}} [j + 1].
\]

(D.46)

The left (resp. right) side has a Tetrahedral (B.20) (resp. Theta (B.9)) network. Hence, we have

\[
c_{ik} \left( \frac{\alpha_1}{\alpha_2} \frac{\beta_1}{\beta_2} \frac{\gamma_1}{\gamma_2} \right) = \delta_{ik} \text{Tet} \left( \frac{u_1}{u_2} \frac{u_{\alpha_1}}{u_{\alpha_2}} \frac{i}{s_1} \left( \frac{\beta_1}{\beta_2} \frac{\gamma_1}{\gamma_2} \right) \Theta(i, u_{\alpha_1}, u_{\alpha_2}) \right).
\]

(D.47)

After inserting this result into (D.41) and recalling (D.36, D.43), we arrive with the asserted identities (D.38, D.39), where the coefficients are \( c_i := c_{ii} \).

2. Case of two nodes

Now we prove proposition 2.12 for \( \varsigma = (s_1, s_2) \). Our task is to construct all tangles in \( \text{JW}_{(s_1, s_2)}(\nu) \) from

\[
\mathbf{G}_{(s_1, s_2)} := P_{(s_1, s_2)} \{1_{\text{T}L_{s_1 + s_2}(\nu), U_{s_1}}\} P_{(s_1, s_2)}.
\]

(D.48)

In fact, in this case, we obtain a stronger result: the statement of corollary D.6 is proposition 2.12 with \( \varsigma = (s_1, s_2) \) under the weaker assumption \( \max(s_1, s_2) < \bar{p}(q) \) instead of \( s_1 + s_2 < \bar{p}(q) \).

**Lemma D.5.** Suppose \( \max(s_1, s_2) < \bar{p}(q) \). Then every tangle of the form

\[
\text{(D.49)}
\]

with \( r \in \{0, 1, \ldots, \min(s_1, s_2)\} \), is a polynomial in the elements of \( \mathbf{G}_{(s_1, s_2)} \).

**Proof.** We prove the claim by induction on \( r \in \mathbb{Z}_{\geq 0} \). It is clearly true if \( r \in \{0, 1\} \). Assuming that the claim holds for tangles in (D.49) with cables of size \( r \) joining the two boxes, we form the product

\[
\text{(P2)}
\]

(D.50)

Next, we decompose the upper-middle projector box of (D.50) over all internal link diagrams. By rule (P2), the only nonvanishing terms of (D.50) with the upper-middle box decomposed are

\[
\text{(D.51)}
\]
where the coefficient on the second term follows from (A.75) of proposition A.9. In the first tangle of (D.51), one loop both enters and exits the bottom-middle box from opposite sides. Using identities (P1', B.1), we get

\[
= -\frac{[s_2 + 1]}{[s_2]} \left[ s_2^2 + 1 \right]
\]

(D.52)

In the second tangle of (D.51), all but two terms vanish after we decompose the lower-middle box. Using (A.75) of proposition A.9 to find the coefficients of the tangles in the box decomposition, these terms are

\[
= \left[ s_2 - r + 1 \right] \frac{[s_2]}{[s_2]}
\]

(D.53)

\[
+ \frac{[s_2 - r]}{[s_2]}
\]

(D.55)

After combining (D.52, D.53) and using identity (B.12) of lemma B.6, we find that the product (D.50) equals

\[
= -\frac{[s_1 + s_2 - r + 1]}{[s_1][s_2]}
\]

(D.54)

\[
+ \frac{[s_1 - r][s_2 - r]}{[s_1][s_2]}
\]

(D.55)

By the induction hypothesis, the two tangles in (D.54) are polynomials in the tangle \( P_{(s_1, s_2)} U_{s_1} P_{(s_1, s_2)} \). Therefore, so does the tangle in (D.55). This finishes the induction step and concludes the proof.

Corollary D.6. Suppose \( \max(s_1, s_2) < \overline{p}(q) \). Then the Jones-Wenzl algebra \( JW_{(s_1, s_2)}(\nu) \) is generated by the collection \( \mathcal{G}_{(s_1, s_2)} \) of tangles. Moreover, \( P_{(s_1, s_2)} \) is the unit of \( JW_{(s_1, s_2)}(\nu) \).

Proof. The collection of tangles in lemma D.5 is a basis for \( JW_{(s_1, s_2)}(\nu) \), so it follows from lemma D.5 that \( \mathcal{G}_{(s_1, s_2)} \) generates \( JW_{(s_1, s_2)}(\nu) \). By idempotent property (2.106), \( P_{(s_1, s_2)} \) is the unit of \( JW_{(s_1, s_2)}(\nu) \).
3. Induction step

In this section, we perform the core of the induction step to prove proposition 2.12 for a general multiindex \( \varsigma \). We use induction on the number \( d_\varsigma \) of nodes, the initial case being the content of the previous section D 2. The present section contains a series of lemmas, and in the next section D 4, we finally summarize the proof of proposition 2.12.

**Induction Hypothesis D.7.** We let \( d_\varsigma \geq 3 \) and we assume that proposition 2.12 is true for any multiindex in the set \( \{ \vec{0} \} \cup \mathbb{Z}_{>0} \cup \mathbb{Z}_{>0}^2 \cup \cdots \cup \mathbb{Z}_{>0}^{d_\varsigma-1} \). With

\[
\begin{align*}
G_{\varsigma} &:= P_{\varsigma} \{ 1_{\mathrm{TL}_{n_{\varsigma}}(\nu)}, U_{s_1}, U_{s_1+s_2}, \ldots, U_{s_1+s_2+\cdots+s_{d_\varsigma-1}} \} P_{\varsigma} \quad (D.56) \\
G_{\varsigma}' &:= P_{\varsigma} \{ 1_{\mathrm{TL}_{n_{\varsigma}}(\nu)}, U_{s_1}, U_{s_1+s_2}, \ldots, U_{s_1+s_2+\cdots+s_{d_\varsigma-2}} \} P_{\varsigma} \quad (D.57) \\
G_{\varsigma}'' &:= P_{\varsigma} \{ 1_{\mathrm{TL}_{n_{\varsigma}}(\nu)}, U_{s_2}, U_{s_2+s_3}, \ldots, U_{s_2+s_3+\cdots+s_{d_\varsigma-1}} \} P_{\varsigma}, \quad (D.58)
\end{align*}
\]

this assumption implies that \( \mathbf{JW}_\varsigma(\nu) \subset \mathbf{JW}_\varsigma(\nu) \) and \( \mathbf{JW}_\varsigma(\nu) \subset \mathbf{JW}_\varsigma(\nu) \) are generated by the respective collections \( G_{\varsigma} \) and \( G_{\varsigma}' \) of tangles. Specifically, this is equivalent to assuming that all of the tangles

\[ T \in \mathrm{TL}_{\max(\varsigma)}(\nu) \quad \text{and} \quad U \in \mathrm{TL}_{\max(\varsigma)}(\nu), \]

where \( T \in \mathrm{TL}_{\max(\varsigma)}(\nu) \) and \( U \in \mathrm{TL}_{\max(\varsigma)}(\nu) \), are polynomials in the elements of the collections \( G_{\varsigma} \) and \( G_{\varsigma}' \), respectively.

**Claim D.8.** Suppose \( n_\varsigma < \overline{p}(q) \). If induction hypothesis D.7 holds, then the Jones-Wenzl algebra \( \mathbf{JW}_\varsigma(\nu) \) is generated by the collection \( G_{\varsigma} \) of tangles. Equivalently, each tangle of the form

\[ T \in \mathrm{TL}_{n_{\varsigma}}(\nu), \]

is a polynomial in the elements of the collection \( G_{\varsigma} \).

Assuming that induction hypothesis D.7 holds, in the next lemmas and corollaries D.9–D.18, we apply induction on \( d_\varsigma \) to construct certain simple basis tangles in \( \mathbf{WD}_{1_{\varsigma}} \) (D.8) from the claimed generator set \( G_{\varsigma} \) of \( \mathbf{JW}_\varsigma(\nu) \). After this, we prove claim D.8 in corollary D.19 by showing that every tangle in \( \mathbf{WD}_{1_{\varsigma}} \) is a polynomial in the claimed generator set \( G_{\varsigma} \). Then, it only remains to conclude with the proof of proposition 2.12.

A. Constructing simple basis tangles

We begin by constructing the basis tangles of type \( \mathbf{WD}_{1_{\varsigma}} \) (D.8) with \( u_{\alpha_1} = u_{\alpha_2} \) and \( v = 0 \), and \( r = w = 1 \). First, in lemma D.9 (resp. lemma D.10) we construct such tangles with maximal (resp. minimal) number of crossing links.
Later, in lemma D.13 and corollary D.15, we construct such tangles with any number of crossing links. To prove the latter results, we use certain simple tangles in the basis $WD_3$ (D.10), treated in lemma D.11 and corollary D.12.

**Lemma D.9.** Suppose $n_\varsigma < \bar{p}(q)$. If induction hypothesis D.7 holds, then the following tangle is a polynomial in the elements of $G_\varsigma$:

\[ (D.61) \]

Proof. We generate the sought tangle (D.61) from the following product:

\[ (D.62) \]

By induction hypothesis D.7, the left (resp. middle, resp. right) tangle of this product is a polynomial in the elements of $G_\varsigma$ (resp. $G_\varsigma$, resp. $G_\varsigma$). With $G_\varsigma \cup G_\varsigma = G_\varsigma$, the claim follows.

**Lemma D.10.** Suppose $n_\varsigma < \bar{p}(q)$. If induction hypothesis D.7 holds, then for all Jones-Wenzl link states $\alpha_1$ and $\alpha_2$ in $\bigcup_{s \in E_\varsigma} W_\varsigma^{(s)}$, the following tangle is a polynomial in the elements of $G_\varsigma$:

\[ (D.63) \]

Proof. Lemma D.5 already gives the case with $d_\varsigma = 2$, so we assume that $d_\varsigma \geq 3$. We note that our assumptions give

\[ d_\varsigma \geq 3 \quad \text{and} \quad n_\varsigma < \bar{p}(q) \implies s_1, s_{\min(\varsigma)} < \bar{p}(q) - 2 \quad \text{and} \quad \bar{p}(q) > 3. \]  

\[ (D.64) \]

By linearity, we may also assume that $\alpha_1$ and $\alpha_2$ are link patterns. The idea of the proof is to generate the sought tangle (D.63) by forming the product (D.37) of lemma D.4 with judiciously chosen link states $\beta_1, \beta_2, \gamma_1, \gamma_2$. 
Writing $\alpha_1$ and $\alpha_2$ in the form of (D.19), $\alpha_1'$ and $\alpha_2'$ being the corresponding sub-link patterns, item 2 of lemma D.2 says that $u_{\alpha_1'} \in E_{(\alpha_1,s_1)}$ and $u_{\alpha_2'} \in E_{(\alpha_2,s_1)}$. According to lemma 2.4, with $u_{\alpha_1} = u_{\alpha_2} = s_{\min}(\zeta)$, all defects of $\alpha_1$ and $\alpha_2$ attach to a single projector box. We denote below $t := s_{d_c}$. There are three cases to consider:

1. **All defects of $\alpha_1$ and $\alpha_2$ attach to the first box:** In this case, we have $u_{\alpha_1'} = s_1 - s_{\min}(\zeta) = u_{\alpha_2'}$. Hence, we may form the product (D.37) of lemma D.4 with

$$
\begin{align*}
\beta_1 &= \alpha_1', \\
\gamma_1 &= (\alpha_1')^\vee, \\
\beta_2 &= (\alpha_2')^\vee, \\
\gamma_2 &= \alpha_2', \\
u_1 &= u_{\alpha_1'}, \\
u_2 &= u_{\alpha_2'}
\end{align*}
$$

(we note that with $s_{\max}(\zeta) < n_\zeta < \bar{n}(q)$, corollary 5.1 implies that the dual elements exist). Also, we have

$$
\begin{align*}
\begin{cases}
u_1 \overset{(D.65)}{=} u_{\alpha_1'} = s_1 - s_{\min}(\zeta) = u_{\alpha_2'} \overset{(D.65)}{=} u_2, \\
s_1 - s_{\min}(\zeta) \geq 1 \text{ by item 2 of lemma 2.4},
\end{cases}
\implies u_1 = u_2, \quad u_1 + u_2 = 2u_1 \geq 2
\implies 2 \in E_{(u_1,u_2)} \overset{(2.26)}{=} \{0, 2, \ldots, 2u_1\},
\end{align*}
$$

and $2 \in E_{(t,t)} = \{0, 2, \ldots, 2t\}$ because $t = s_{d_c} \geq 1$. Altogether, we may therefore set $i = 2 \in E_{(u_1,u_2)} \cap E_{(t,t)}$ in addition to (D.65) in the product tangle (D.37) to arrive with

\[
\text{(D.68)}
\]

which by lemma B.4 with $((\alpha_1')^\vee | \alpha_1') = 1 = ((\alpha_2')^\vee | \alpha_2')$ immediately simplifies to

\[
\text{(D.69)}
\]

Now with $u = u_{\alpha_1} = u_{\alpha_2} = s_{\min}(\zeta)$ and $t = s_{d_c}$, we expand this tangle over the basis $WD_{3\zeta}$ (D.10):
To find the coefficients $c_j \in \mathbb{C}$, we proceed as in the proof of lemma D.4; we insert both sides of (D.70) into the "dual" tangle (D.42), thereby closing all links into loops. After simplifying the result, (D.70) becomes

$$
= \sum_{j \in E(u,u) \cap E(t,t)} c_j
$$

where we have not labeled the sizes of all cables in these networks: one may infer those sizes from (D.70). Then, using lemmas B.4 and B.5, we delete the lower loop of either network to find

$$
\delta_{2,k} \frac{\Theta(2, t, t)}{[3]} = \sum_{j \in E(u,u) \cap E(t,t)} c_j \delta_{jk} \frac{\Theta(j, t, t)}{(-1)^j [j + 1]}
$$

The network on the left side evaluates to (B.22) by lemma B.8, and the network on the right side is the Theta network (B.9) with $(r, s, t) = (k, s_{\min}(\hat{\varsigma}), s_{\min}(\hat{\varsigma}))$. Thus, using lemma B.7 we arrive with

$$
c_k = \delta_{2,k} \frac{[s_1 + 1]^2}{[3][s_1] \Theta(2, s_{\min}(\hat{\varsigma}), s_{\min}(\hat{\varsigma}))} = \delta_{2,k} \frac{[s_1 + 1]^2[s_{\min}(\hat{\varsigma})][s_{\min}(\hat{\varsigma}) - 2]!}{[3][s_1][2][s_{\min}(\hat{\varsigma}) + 1][s_{\min}(\hat{\varsigma}) + 2]}
$$

By (D.64), these coefficients are finite, and only $c_2$ is not zero. After inserting (D.73) into (D.70), using (2.70, 4.36) to decompose the three-vertices on the right side of (D.70), and rearranging, (D.70) becomes

$$
= \nu
$$
By induction hypothesis D.7, the first tangle on the right side of (D.74) is a polynomial in the elements of the collection $G_\varsigma$, and the left (resp. middle, resp. right) tangle in the product on the right side is a polynomial in the elements of $G_\varsigma$ (resp. $G_\varsigma$, resp. $G_\varsigma$). With $G_\varsigma \cup G_\varsigma = G_\varsigma$, this proves the lemma for this special case.

2. All defects of $\alpha_1$ and $\alpha_2$ attach to the $j$:th and $k$:th box respectively, with $j, k \in \{2, 3, \ldots, M - 1\}$: As in item 1, we form the product (D.37) of lemma D.4 with substitutions (D.65, D.67). To justify this choice, we note that by our assumption that all defects of $\alpha_1$ and $\alpha_2$ attach to the $j$:th and $k$:th box we have $u_{\alpha_1'} = s_1 + s_{\text{min}}(\varsigma) = u_{\alpha_2'}$, so

$$
\begin{align*}
    u_1 &= u_{\alpha_1'} = s_1 + s_{\text{min}}(\varsigma) = u_{\alpha_2'} = u_2, \\
    s_1 + s_{\text{min}}(\varsigma) &\geq 2,
\end{align*}
$$

$$
\Rightarrow u_1 = u_2, \quad u_1 + u_2 = 2u_1 \geq 2
$$

$$
\Rightarrow 2 \in E(u_1, u_2) = \{0, 2, \ldots, 2u_1\}, \quad (D.75)
$$

and $2 \in E(t, t) = \{0, 2, \ldots, 2t\}$ because $t = s_{d_1} \geq 1$. With substitutions (D.65, D.67) into (D.37), we arrive with $$
\begin{align*}
\text{(D.76)}
\end{align*}
$$

which by lemma B.4 with $((\alpha_1')^\vee | \alpha_1') = 1 = ((\alpha_2')^\vee | \alpha_2')$ immediately simplifies to

$$
\begin{align*}
\text{(D.77)}
\end{align*}
$$
With $u = u_{\alpha_1} = u_{\alpha_2} = s_{\min}(\hat{\varsigma})$ and $t = s_{d_\varsigma}$, we expand this tangle over the basis $\text{WD}_\varsigma$ (D.10):

\begin{align}
(D.77) & = \sum_{j \in E_{(u,u)} \cap E_{(t,t)}} c_j

(D.78)
\end{align}

and to find the coefficients $c_j \in \mathbb{C}$, we proceed as in item 1. In the present situation, inserting both sides of (D.78) into the “dual” tangle (D.42), and simplifying the result gives

\begin{align}
(D.79) & = \sum_{j \in E_{(u,u)} \cap E_{(t,t)}} c_j

(D.80)
\end{align}

Again, using lemmas B.4 and B.5, we delete the lower loop of either network to find

\begin{align}
(D.81) & = \sum_{j \in E_{(u,u)} \cap E_{(t,t)}} c_j \Theta(j, t, t) (-1)^{j[1]}[j + 1] \\
(D.82)
\end{align}

It is easy to see that the networks on both sides are Theta networks defined in (B.9). They respectively evaluate to $\Theta(2, s_{\min}(\hat{\varsigma}) + s_1, s_{\min}(\hat{\varsigma}) + s_1)$ and $\Theta(k, s_{\min}(\hat{\varsigma}), s_{\min}(\hat{\varsigma}))$. Thus, we arrive with

\begin{align}
c_k & = \delta_{2, k} \Theta(2, s_{\min}(\hat{\varsigma}) + s_1, s_{\min}(\hat{\varsigma}) + s_1) \\
& = \delta_{2, k} (-1)^{s_1} \frac{[s_{\min}(\hat{\varsigma}) + s_1 + 1][s_{\min}(\hat{\varsigma}) + s_1 + 2][s_{\min}(\hat{\varsigma})][s_{\min}(\hat{\varsigma}) - 2]!}{[s_{\min}(\hat{\varsigma}) + 1][s_{\min}(\hat{\varsigma}) + 2][s_{\min}(\hat{\varsigma}) + s_1][s_{\min}(\hat{\varsigma}) + s_1 - 2]!}.
\end{align}

Using (D.64) and the fact from (D.76) that

\begin{equation}
s_{\min}(\hat{\varsigma}) + s_1 \leq s_{\max}(\hat{\varsigma}) = n_\varsigma - (s_{d_\varsigma} + s_1) < \bar{p}(q) - 2,
\end{equation}

we see that these coefficients are finite, and only $c_2$ is not zero. Now, proceeding with (D.78, D.81) as we do with (D.70, D.73) just beneath (D.73) of item 1, we finish the proof of the lemma for this special case.
3. The defects of $\alpha_1$ attach to the first box, and the defects of $\alpha_2$ attach to the $i$:th box for some $i \in \{2, 3, \ldots, M - 1\}$ (or vice versa): In this case, we form the product (D.37) with substitutions $i = 2$ and 

$$\beta_1 = \beta_2 = \alpha_1', \quad \gamma_1 = \gamma_2 = (\alpha_1')^\vee \quad \implies \quad u_1 = u_{\alpha_1'} = u_2.$$  

(D.84)

These substitutions into (D.37) give

![Diagram](image)

Repeating the analysis of item 1 above proves the lemma for this special case.

This concludes the proof. \qed

Now we construct certain simple basis tangles in $\text{WD}_3$ (D.10) from the claimed generator set $G_\varsigma$ of $\text{JW}_\varsigma(\nu)$. We make use of them below in the proof of lemma D.13 and corollary D.15.

**Lemma D.11.** Suppose $n_\varsigma < \bar{p}(q)$. If induction hypothesis D.7 holds, then for all Jones-Wenzl link states $\alpha_1$ and $\alpha_2$ in \( \bigcup_{s \in E} W^{(s)}_\varsigma \bigcup_{E} E^{(s)}_\varsigma \) such that $u_{\alpha_2} = u_{\alpha_1} + 2$, the following tangle is a polynomial in the elements of $G_\varsigma$:

![Diagram](image)

(D.86)

**Proof.** It is apparent from (4.36) that equality in (D.86) holds for $u_{\alpha_2} = u_{\alpha_1} + 2$. To generate the left tangle of (D.86), we form the product (D.37) of lemma D.4 as follows: we choose 

$$\gamma_1 = \beta_1', \quad \gamma_2 = \beta_2'$$  

(D.87)

(with $s_{\max}(\varsigma) < n_\varsigma < \bar{p}(q)$, corollary 5.1 implies that the dual elements exist, and we set 

$$u := u_1 = u_2 \in E_{(\alpha_1, s_1)} \cap E_{(\alpha_2, s_1)} \cap E_{\varsigma}.$$  

(D.88)

(Lemma D.3 implies that we can make this choice.) Then, in (D.38) we have (again with $t = s_{d_\varsigma}$) 

$$i = 2 \in E_{(u_1, u_2)} \cap E_{(t, t)} = \{0, 2, \ldots, 2 \min(u, t)\}.$$  

(D.89)
Now, by lemma D.4, the product (D.37) with these substitutions evaluates to

\[
\text{Tet} \left[ \begin{array}{cc} u & u_{\alpha_1} \\ u_{\alpha_2} & u \\ 2 & 1 \end{array} \right] \frac{\mathbb{1}\{2 \in E_{(\alpha_1,\alpha_2)} \cap E_{(u,u)} \cap E_{(t,t)}\}}{\Theta(2, u_{\alpha_1}, u_{\alpha_2})} 
\]

By (D.36, D.43) with \( u_{\alpha_2} = u_{\alpha_1} + 2 \), the indicator \( \mathbb{1} \) in (D.90) equals one. Combining this observation with item 3 of lemma B.8, we find that the coefficient in (D.90) equals

\[
\text{Tet} \left[ \begin{array}{cc} u & u_{\alpha_1} \\ u_{\alpha_2} & u \\ 2 & 1 \end{array} \right] \frac{1}{\Theta(2, u_{\alpha_1}, u_{\alpha_2})} = \frac{1}{[u]} \frac{1}{[u]} \frac{1}{[u]} \frac{1}{[u]} \Theta(u, s_1, u_{\alpha_1} + 2) \Theta(u + s_1 + u_{\alpha_1} + 2, \frac{u + s_1 + u_{\alpha_1} - u}{2}, \frac{s_1 + u_{\alpha_1} - u}{2} + 1) \Theta(u + s_1 + u_{\alpha_1} + 2, \frac{s_1 + u_{\alpha_1} - u}{2} + 1) !. \tag{D.91} \]

Using the facts that \( s_1, u < \bar{p}(q) \) and

\[
u_{\alpha_1} + 2 \leq s_{\text{max}}(\tilde{s}) = n_{\sigma} - s_{d_\tau} < \bar{p}(q) - 1 \implies u_{\alpha_1} < \bar{p}(q) - 3, \tag{D.93} \]

we see that the denominator of (D.92) is finite and not zero, and using the fact from (D.37) that

\[
u + s_1 \leq s_{\text{max}}(\tilde{s}) = n_{\sigma} - s_{d_\tau} < \bar{p}(q) - 1, \tag{D.94} \]

we see that the numerator of (D.92) is finite and not zero. Hence, the product (D.37) with substitutions (D.87)–(D.89) gives the left tangle \( T \) of (D.86) up to a nonzero constant.

Now to see that \( T \) is a polynomial in the elements of \( G_\zeta \), we observe that the left (resp. middle, resp. right) tangle of the product (D.37) giving \( T \) is a polynomial in the elements of \( \hat{G}_\zeta \) (resp. \( \check{G}_\zeta \), resp. \( \hat{G}_\zeta \)) by induction hypothesis D.7. With \( G_\zeta \cup G_\zeta = G_\zeta \), it follows that \( T \) is a polynomial in the elements of \( G_\zeta \).

**Corollary D.12.** Suppose \( n_\zeta < \bar{p}(q) \). If induction hypothesis D.7 holds, then for all Jones-Wenzl link states \( \alpha_1 \) and \( \alpha_2 \) in \( \bigcup_{s \in E_\zeta} W^{(s)}_\zeta \) such that \( u_{\alpha_1} = u_{\alpha_2} + 2 \), the following tangle is a polynomial in the elements of \( G_\zeta \):

\[
= \tag{D.95} \]

**Proof.** We obtain this result by vertically reflecting the tangles of lemma D.11 and exchanging \( \alpha_1 \) and \( \alpha_2 \).

Now we are ready to construct basis tangles of type WD1_\zeta (D.8) with \( u_{\alpha_1} = u_{\alpha_2} \), and \( v = 0 \), and \( r = w = 1 \), with any number of crossing links. The next lemma D.13 and corollary D.15 thus generalize lemmas D.9 and D.10.
Lemma D.13. Suppose \( n_\zeta < \bar{p}(q) \) and \( s_d \leq s_1 \). If induction hypothesis D.7 holds, then for all Jones-Wenzl link states \( \alpha_1 \) and \( \alpha_2 \) in \( \bigcup_{s \in E_\zeta} W_\zeta(s) \) such that

\[
s_{\min}(\zeta) + 2 \leq u := u_{\alpha_1} = u_{\alpha_2} \leq s_{\max}(\zeta) - 2,
\]

the following tangle is a polynomial in the elements of \( G_\zeta \):

\[
\text{(D.96)}
\]

Proof. Lemma D.5 already gives the case with \( d_\zeta = 2 \), so we assume that \( d_\zeta \geq 3 \) throughout. If \( d_\zeta = 3 \), then we must have \( s_2 > 1 \). Indeed, if otherwise, then with \( \zeta = (s_1, s_2) = (s_1, 1) \) and

\[
u := u_{\alpha_1} = u_{\alpha_2} \in E_\zeta \quad \text{where} \quad E_\zeta = E_{(s_1,1)} \overset{(2.26)}{=} \{ s_1 \pm 1 \},
\]

we have \( u = s_1 - 1 = s_{\min}(\zeta) \), or \( u = s_1 + 1 = s_{\max}(\zeta) \), contradicting the condition (D.96) on \( u \). We observe that this condition guarantees that

\[
u \pm 2 \in E_\zeta \quad \implies \quad \text{WP}^{(u \pm 2)}_\zeta \neq \emptyset,
\]

which we frequently use in this proof. By linearity, we also assume that \( \alpha_1 \) and \( \alpha_2 \) are link patterns.

First, we take the tangle (D.86) of lemma D.11, \( \alpha_1 \) in (D.86) being the link pattern consisting of \( u - 2 \) defects, and its vertical reflection with the replacement \( \alpha_2 \mapsto \alpha_1 \). Multiplying them, with the latter on the left, gives

\[
\text{(D.100)}
\]

Next, we decompose the bottom-middle projector box of this result. By property (P2), only two tangles in this decomposition, each of the form (A.74) with \( j = k = 0 \) and \( i \in \{0,1\} \), give nonzero terms. Using (A.75), we get

\[
\text{(D.101)}
\]
Second, we take the tangle (D.86) of lemma D.11, $\alpha_2$ in (D.86) being the link pattern consisting of $u + 2$ defects, and its vertical reflection with replacement $\alpha_1 \mapsto \alpha_2$. Multiplying them now with the latter on the right, we obtain

\[
(D.102)
\]

Again, we decompose the bottom-middle projector box of this product. As before, we have exactly two nonzero terms:

\[
(D.103)
\]

\[
(D.104)
\]

Next, we use identity (B.1) from lemma B.1 to remove the loop passing through the middle projector box of (D.103), and we decompose that box. Once again, there are only two nonvanishing terms. Using (A.75), we find that

\[
(D.103) = -\left[\frac{u + 3}{u + 2}\right]
\]

\[
(D.105)
\]
Then we decompose the middle projector box in the tangle of (D.104), obtaining exactly three nonzero terms:

\[
(D.104) = - \frac{[u + 3][u]}{[u + 2][u + 1]} \\
\]

\[
(D.107) = \left[ \frac{sd_c - 1}{sd_c} \right] \\
\]

\[
(D.108) = \frac{[sd_c - 1][u][2]}{[sd_c][u + 2]} \\
\]

\[
(D.109) = \frac{[sd_c - 1][u][u - 1]}{[u + 2][u + 1]} \\
\]
After replacing (D.103) by (D.105, D.106) and (D.104) by (D.107–D.109), we obtain

\[(D.102) = \left(\frac{s_{d_c} - 1}{s_{d_c}} - \frac{u + 3}{u + 2}\right)\]

\[(D.103) + \left(\frac{s_{d_c} - 1}{s_{d_c}} - \frac{u + 3}{u + 2}\right)\]

\[\left(\frac{u}{u + 2} \frac{u - 1}{u + 1}\right) + \left(\frac{u + 3}{u + 2} \frac{u - 1}{u + 1}\right)\]

Now, we can solve the tangle in (D.111) from the two equations (D.101) and (D.110–D.112). We obtain
\[
\begin{align*}
&= \left( \frac{[s_{d_c} - 1]}{[s_{d_c}]} - \frac{[u + 3]}{[u + 2]} \right) \\
&\quad + \frac{[u][u - 1]}{[u + 2][u + 1]} \\
&\quad - 1 \times
\end{align*}
\]

By induction hypothesis D.7, lemma D.11, and corollary D.12, all tangles on the right side are polynomials in the elements of \( G_\zeta \). Then, using identity (B.12) of lemma B.6, we simplify the coefficient on the left side of (D.113) to

\[
\frac{[u][u - 1]}{[u + 2][u + 1]} + \frac{[u + 3][u]}{[u + 2][u + 1]} - \frac{[s_{d_c} - 1][2][u]}{[s_{d_c}][u + 2]} = \frac{[u][2s_{d_c} + 2]}{[u + 2][s_{d_c}][s_{d_c} + 1]}.
\]

(D.114)

Now, provided that coefficient does not vanish, we may conclude that the tangle on the left side of (D.113) is a polynomial in the elements of \( G_\zeta \) too, thus proving the lemma. To show that this is the case, we observe that

\[
\begin{align*}
\left\{ \begin{array}{l}
s_j \geq 1 \text{ for all } j = 1, 2, \ldots, d_c, \\
s_2 > 1 \text{ if } d_c = 3 \text{ by the first paragraph of this proof,}
\end{array} \right. \\
\implies s_1 + s_{d_c} + 2 \leq n_\zeta < \bar{p}(q).
\end{align*}
\]

(D.115)

Therefore, using the assumption \( s_{d_c} \leq s_1 \) of the lemma, we see that

\[
2s_{d_c} + 2 \leq s_1 + s_{d_c} + 2 \leq \bar{p}(q) \implies [2s_{d_c} + 2] \neq 0 \text{ and } [s_{d_c}], [s_{d_c} + 1] \neq 0.
\]

(D.116)

Also, assumption (D.96) shows that

\[
u \leq s_{\text{max}}(\zeta) - 2 = n_\zeta - s_{d_c} - 2 < \bar{p}(q) - 3 \implies [u] \neq 0 \text{ and } [u + 2] \neq 0.
\]

(D.117)

We conclude that the coefficient (D.114) indeed does not vanish (nor blow up). This proves the lemma. \( \square \)
Remark D.14. The assumption \( s_d \leq s_1 \) in lemma D.13 is not restrictive. Indeed, if \( s_1 \leq s_d \) instead, then we may horizontally flip all tangles and repeat our work with \( \zeta \mapsto \bar{\zeta} \) and \( s_d \mapsto s_1 \) for the flipped tangles.

Corollary D.15. Suppose \( n_\zeta < \bar{p}(q) \). If induction hypothesis D.7 holds, then for all Jones-Wenzl link states \( \alpha_1 \) and \( \alpha_2 \) in \( \bigcup_{s \in E_\zeta} W^{(s)}_\zeta \), the following tangle is a polynomial in the elements of \( G_\zeta \):

\[
\begin{align*}
&s_1 \\
&s_2 \\
&\vdots \\
&s_{d_\zeta - 1} \\
&s_{d_\zeta}
\end{align*}
\]

Proof. This immediately follows from combining lemmas D.9, D.10, and D.13 and remark D.14.

B. Constructing basis tangles without diagonal cables

We continue by constructing the basis tangles of type WD1_\zeta (D.8) with \( u_{\alpha_1} = u_{\alpha_2} \), and \( v = 0 \), and \( r = w \geq 2 \).

Lemma D.16. Suppose \( n_\zeta < \bar{p}(q) \). If induction hypothesis D.7 holds, then for all Jones-Wenzl link states \( \alpha_1 \) and \( \alpha_2 \) in \( \bigcup_{s \in E_\zeta} W^{(s)}_\zeta \), the following tangle is a polynomial in the elements of \( G_\zeta \):

\[
\begin{align*}
&s_1 \\
&s_2 \\
&\vdots \\
&s_{d_\zeta - 1} \\
&s_{d_\zeta}
\end{align*}
\]

Proof. By remark D.14, we may assume \( s_d \leq s_1 \) without loss of generality. We prove the claim by induction on \( r \in \mathbb{Z}_{\geq 0} \). Induction hypothesis D.7 gives the initial case \( r = 0 \) and corollary D.15 the case \( r = 1 \). Assuming that the claim holds if the cables joining the upper and lower boxes in (D.119) have size \( r \), we form the product

\[
\begin{align*}
&s_1 \\
&s_2 \\
&\vdots \\
&s_{d_\zeta - 1} \\
&s_{d_\zeta}
\end{align*}
\]

and proceed exactly as in the proof of lemma D.5, starting at (D.50).

\( \square \)
Finally, in the next lemma D.17 and corollary D.18, we construct the general basis tangles of type WD1$_c$ (D.8).

**Lemma D.17.** Suppose $n_c < \bar{p}(q)$. If induction hypothesis D.7 holds, then for all Jones-Wenzl link states $\alpha_1$ and $\alpha_2$ in $\bigcup_{s \in E_c} W^{(s)}_t$ such that $u_{\alpha_1} \leq u_{\alpha_2}$, the following tangle is a polynomial in the elements of $G_c$:

\[
\text{(D.121)}
\]

where $r \in R_{\alpha_1,\alpha_2}$, $u_{\alpha_2} = u_{\alpha_1} + 2v$, $u + r = u_{\alpha_1}$, and $w = r + v$.

**Proof.** By remark D.14, we may assume $s_{d_c} \leq s_1$ without loss of generality. We prove the claim by induction on $v \in \mathbb{Z}_{\geq 0}$. Lemma D.16 gives the initial case $v = 0$. Next, we assume that the claim holds if the diagonal cable in (D.121) has size $v - 1$, we form the product

\[
\text{(D.122)}
\]

By property (P1'), we absorb the upper-middle projector box into the upper-right one, simplifying (D.122) to

\[
\text{(D.123)}
\]

Next, we decompose the bottom-middle projector box of this product. By property (P2), only one tangle in this decomposition is nonzero: the one of the form (A.74) with $i = 1$, $j = r + v - 2$, $m = s_{d_c} - r - v$, and $k = 0$. 
Using (A.75), we find that

\[(D.123) = \left[ s_{d_\xi} - r - v + 1 \right] \left[ s_{d_\xi} \right] \] \[(D.124) \]

The tangles of (D.122–D.123) are all equal and, by the induction hypothesis applied to (D.122), they equal a product of tangles that are polynomials in the elements of $G_\xi$. Also, with $s_{d_\xi} - r - v + 1 < s_{d_\xi} < \bar{p}(q)$, the coefficient in (D.124) does not vanish (or blow up). Hence, the tangle in (D.124), which matches (D.121), is such a polynomial too. This finishes the induction step and concludes the proof.

**Corollary D.18.** Suppose $n_\xi < \bar{p}(q)$. If induction hypothesis D.7 holds, then for all Jones-Wenzl link states $\alpha_1$ and $\alpha_2$ in $\bigcup_{s \in E_\xi} W_{\xi}^{(s)}$ such that $u_{\alpha_2} \leq u_{\alpha_1}$, the following tangle is a polynomial in the elements of $G_\xi$:

\[(D.125) \]

where $r \in R_{\alpha_1, \alpha_2}$, $u_{\alpha_1} = u_{\alpha_2} + 2v$, $u + r = u_{\alpha_2}$, and $w = r + v$.

**Proof.** We obtain this result by vertically reflecting the tangle (D.121) of lemma D.17 and exchanging $\alpha_1$ and $\alpha_2$. \qed

### 4. Finishing the induction step and the proof of proposition 2.12

Now we are ready to finish the induction step, i.e., claim D.8, and then the proof of proposition 2.12.

**Corollary D.19.** Suppose $n_\xi < \bar{p}(q)$. If induction hypothesis D.7 holds, then claim D.8 holds.

**Proof.** By lemma D.17 and corollary D.18, every tangle in the collection WD$_1$ (D.8) is a polynomial in the elements of $G_\xi$. Because this set is a basis for JW$_\xi(\nu)$ by lemma D.1, this property linearly extends from the basis elements to all tangles in JW$_\xi(\nu)$. This proves claim D.8. \qed

**Proof of proposition 2.12.** First, the claim that $G_\xi$ generates JW$_\xi(\nu)$ immediately follows by induction on $d_\xi \in \mathbb{Z}_{>0}$: corollary D.6 gives the initial case $d_\xi = 2$, and corollary D.19 (i.e., claim D.8) completes the induction step.
Finally, to prove that the collection of tangles of the form

\[
\text{(D.126)}
\]

with \( s \in E(s_i, s_{i+1}) \) and \( i \in \{1, 2, \ldots, d_\varsigma - 1\} \) generates \( JW_\varsigma(\nu) \), we use definition (4.36), properties (P2, P1'), and proposition A.9 to obtain the following upper-triangular system of equations for each \( i \in \{1, 2, \ldots, d_\varsigma - 1\} \):

\[
\min(s_i, s_{i+1}) \sum_{k=1}^{2} \frac{2}{s_i + s_{i+1} - s} \text{coef}_{s,k} = (D.127)
\]

where \( \text{coef}_{s,k} := \frac{[s_i + s_{i+1} - s]! [s_i + s_{i+1} + s - k]!}{[s_i - k]! [s_i + 1 - k]!} \).

Now, with \( s \in E(s_i, s_{i+1}) \) and \( n_\varsigma < \bar{p}(q) \), we see that all of the coefficients \( \text{coef}_{s,k} \) are finite and none of the coefficients of the diagonal terms, i.e., terms with \( k = \frac{1}{2}(s_i + s_{i+1} - s) \), vanish. Hence, the system (D.127) is invertible for each \( i \in \{1, 2, \ldots, d_\varsigma - 1\} \). Because the collection of tangles appearing on the right side of (D.127) generates \( JW_\varsigma(\nu) \), so too does the collection of tangles of the form (D.126), appearing on the left side of (D.127). This finishes the proof.

**E. TRIVALENT LINK STATES AT ROOTS OF UNITY**

We recall from section 4 A that for each link state \( \alpha \in L_{n_\varsigma} \), the trivalent link state \( \alpha \in L_{n_\varsigma} \) is defined by replacing open vertices by closed ones, beginning from the rightmost vertex and proceeding leftwards, until encountering one of the situations in definition 4.2. The procedure terminates at a special index \( J = J_\alpha(q) \) defined in (4.44). The purpose of this appendix is to prove that the trivalent link states are well-defined when \( n_\varsigma \geq \bar{p}(q) \), as we state in remark 4.3.

We fix \( q \in \mathbb{C}^\times \) and \( \alpha \in L_{n_\varsigma} \) throughout. For \( q' \in \mathbb{C}^\times \) with \( \bar{p}(q') = \infty \), we let \( \alpha_{q'} \) denote the trivalent link state \( \alpha \) with \( q \) perturbed to \( q' \) but the special index \( J \) fixed as in (4.44), so that \( J = J_\alpha(q) \neq J_\alpha(q') = -\infty \). We note that because \( \bar{p}(q') = \infty \), projector boxes of all sizes exist for \( \alpha_{q'} \). Our goal is to show that we may define \( \alpha \) as the limit of \( \alpha_{q'} \) as \( q' \to q \) along a sequence not containing roots of unity. For this purpose, we define the following norm on \( L_{n_\varsigma} \):

\[
\alpha = \sum_{\beta \in LP_{n_\varsigma}} c_\beta \beta \in L_{n_\varsigma} \quad \text{for some } c_\beta \in \mathbb{C} \quad \Rightarrow \quad \|\alpha\| := \max_{\beta \in LP_{n_\varsigma}} c_\beta. \quad (E.1)
\]
Now, given a sequence \((\alpha_{q'})\) of valenced link states, we write \(\alpha_{q'} \to \alpha\) as \(q' \to q\), or equivalently \(\lim_{q' \to q} \alpha_{q'} = \alpha\), if we have \(\|\alpha_{q'} - \alpha\| \to 0\) as \(q' \to q\). It is straightforward to see that if
\[
\alpha_{q'} = \sum_{\beta \in LP_n} c_\beta(q') \beta \quad \text{and} \quad \alpha = \sum_{\beta \in LP_n} c_\beta \beta,
\]for some constants \(c_\beta(q'), c_\beta \in \mathbb{C}\), then we have
\[
\lim_{q' \to q} \alpha_{q'} = \alpha \iff \lim_{q' \to q} c_\beta(q') = c_\beta \quad \text{for all } \beta \in LP_n.
\]

**Lemma E.1.** The limit \(\lim_{q' \to q} \alpha_{q'}\) exists for any sequence \((q'_k)_{k \in \mathbb{N}}\) tending to \(q\) such that \(\overline{p}(q'_k) = \infty\) for all \(k \in \mathbb{N}\).

**Proof.** For each \(j \in \{J, J + 1, \ldots, n_\varsigma - 1\}\), starting with \(j = J\), we decompose the projector box of size \(r_j\) in the \(j\)th closed vertex of the link state \(\alpha_{q'_k}\) over its internal link diagrams. Now, as we decompose the box of size \(r_j\), we observe that the \(j\)th closed vertex is one of two types:

\[
r_{j+1} = r_j + 1: \quad \begin{array}{c}
1 \quad r_j \\
\end{array} \quad \begin{array}{c}
r_{j+1} = r_j + 1 \\
1 \\
\end{array},
\]
\[
r_{j+1} = r_j - 1: \quad \begin{array}{c}
1 \quad r_j \\
\end{array} \quad \begin{array}{c}
r_{j+1} = r_j - 1 \\
1 \\
\end{array}.
\]

By property (P2) of the Jones-Wezl projector, for (E.4), only the internal link diagram (1.6) with \(r_j\) through-paths and no turn-back links gives a nonzero contribution, whereas for (E.5), only internal link diagrams with either no turn-back links or one turn-back link joining the two lowest, right nodes give a nonzero contribution. Thus, we have

\[
\begin{array}{c}
1 \\
r_j \\
\end{array} \quad \begin{array}{c}
r_{j+1} \\
1 \\
\end{array} (P1') = (E.4)
\]
\[
\begin{array}{c}
1 \\
r_j \\
\end{array} \quad \begin{array}{c}
r_{j+1} \\
1 \\
\end{array} (P1') = (E.5)
\]
\[
\begin{array}{c}
1 \\
r_j \\
\end{array} \quad \begin{array}{c}
1 \\
r_j \\
\end{array} + \sum_{T \in LD_{r_j}, T \neq 1_{n_{r_j}}} (\text{coef}_T) \times \begin{array}{c}
1 \\
r_j \\
\end{array} (A.75) = (E.7)
\]

Using proposition A.9, we find the coefficients (E.7):

\[
T = T_i = \begin{array}{c}
1 \\
1 \\
i \\
\end{array} \quad \text{with } i \in \{0, 1, \ldots, r_j - 2\}
\]

(E.8)
Now for all $j \in \{J + 1, J + 2, \ldots, n_{\kappa} - 1\}$, we have $[r_{j + 1} + 1]q \neq 0$ because $\bar{p}(q) \not \mid (r_{j + 1} + 1)$. Hence, the limit as $q'_{k} \to q$ of each coefficient in (E.8) exists. From this fact with (E.3), it follows that the limit $\lim_{q'_{k} \to q} \epsilon_{g'_{k}}$ exists. □

F. CATEGORY OF VALENCED TANGLES AND VALENCED LINK STATES

In this appendix, we discuss a subcategory of $TL(\nu)$, which we denote by $TL_{1}(\nu)$. The object class of $TL_{1}(\nu)$ comprises the special multiindices of all valences equal to one:

$$\text{Ob} \ TL_{1}(\nu) = \{\vec{n} \in \mathbb{Z}_{\geq 0}\},$$

where $\vec{0} := (0)_{n}$ times for $n \in \mathbb{Z}_{> 0}$, and its morphisms are

$$\text{Hom} \ TL_{1}(\nu) = \{TL_{m}^{n}(\nu) \mid n, m \in \mathbb{Z}_{\geq 0} \text{ with } n + m = 0 \pmod{2}\}.$$

The composition of two morphisms $T, U \in \text{Hom} \ TL_{1}(\nu)$ is given by concatenation via the map $\mu_{\nu}(T, U)$, defined in recipe $(\mu_{1} - \mu_{3})$ in section 2. The identity morphism associated with the object $\vec{n}$ is simply the unit (1.6) of the corresponding Temperley-Lieb algebra $TL_{n}(\nu) = TL_{n}^{n}(\nu)$.

For later use in the sequel [FP18a+], we determine a minimal collection of generators for the morphism class $\text{Hom} \ TL_{1}(\nu)$. These constitute the left and right generators, defined for each $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ as

$$L_{i} := \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\vdots \\
i - 1 \\
i \\
i + 1 \\
i + 2 \\
n
\end{array}
\end{array} \in TL_{n}^{n-2} \quad \text{and} \quad R_{j} := \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\vdots \\
\vdots \\
j - 1 \\
j \\
j + 1 \\
j + 2 \\
m
\end{array}
\end{array} \in TL_{m}^{m-2}.$$

(F.3)

In the literature, these are also known as the evaluation and coevaluation maps.

Let $T$ be an arbitrary $(n, m)$-link diagram with $s$ crossing links. Then, we can construct $T$ by an insertion of all $(n - s)/2$ left links of $T$ into the unit diagram $1_{TL_{2}(\nu)}$ by repeated application of the left generators $L_{i}$, followed by an insertion of all $(m - s)/2$ right links of $T$ by repeated application of the right generators $R_{j}$:

$$T = L_{i_{(n-s)/2}}L_{i_{(n-s)/2-1}} \cdots L_{i_{2}}L_{i_{1}}1_{TL_{2}(\nu)}R_{j_{1}}R_{j_{2}} \cdots R_{j_{(m-s)/2-1}}R_{j_{(m-s)/2}}.$$

(F.4)

For example,
gives the tangle
\[ g \in TL_8. \] (F.6)

As shown, we include the unit in the middle of the product to emphasize that \( L_i \) is an \((s + 2, s)\)-link diagram and \( R_j \) is an \((s, s + 2)\)-link diagram, in spite of the following obvious relations which render this inclusion frivolous:

\[ L_i 1_{TL_j(\nu)} = L_i \quad \text{and} \quad 1_{TL_j(\nu)} R_j = R_j. \] (F.7)

In \((F.4)\), we order the left generators \( L_i \) such that if the upper endpoint of one left link of \( T \) is above the the upper endpoint of another left link, then the former is inserted before the latter, and similarly for the \( R_j \). This implies that

\[ i_1 < i_2 < \ldots < i_{(n-s)/2}, \quad \text{and} \quad j_1 < j_2 < \ldots < j_{(m-s)/2}. \] (F.8)

We say that any product of left and right generators of the form in \((F.4, F.8)\) is in standard form.

**Lemma F.1.** Each \((n,m)\)-link diagram equals a unique product of left and right generators in standard form \((F.4, F.8)\), and every such product equals a unique \((n,m)\)-link diagram.

**Proof.** It is evident that every product of left and right generators of the form \((F.4, F.8)\) equals a unique \((n,m)\)-link diagram. Also, by the above discussion, every \((n,m)\)-link diagram \( T \) equals a product of left and right generators of the form \((F.4)\), and ordering rule \((F.8)\) uniquely encodes the top-to-bottom ordering and nesting of the left and right links of \( T \). Hence, \( T \) equals a unique product of left and right generators in standard form \((F.4, F.8)\). \( \square \)

**Lemma F.2.** The following is a complete list of all independent relations satisfied by the left and right generators:

\[ R_j L_i = \begin{cases} 1_{TL_j(\nu)}, & i = j \pm 1, \\ i_{TL_j(\nu)}, & i = j, \\ L_i R_j = L_j L_i = L_{i+2} L_j, & j \leq i, \\ L_i R_j = L_j, & i \leq j, \\ L_j = L_{i+2} L_j, & j \leq i, \\ R_j R_{i-2} = R_{i-2} R_j, & j \leq i, \\ R_j = R_{i-2} R_j, & j \leq i, \\ L_{i-2} R_j = R_{i-2} R_j, & j \leq i, \end{cases} \] (F.9)

where \( s \) is the number of crossing links in \( L_i \) and \( R_j \).

**Proof.** Each of these relations is easy to verify with a diagram. To see that these are all of the independent relations, we let

\[ \sum_k c_k A_{k,1} A_{k,2} \cdots A_{k,m_k} = 0 \quad \text{with} \quad c_k \in \mathbb{C} \quad \text{and} \quad A_{k,l} \in \{ L_i, R_j \mid i, j \in \mathbb{Z}_{>0} \} \quad \text{for all} \quad k, l \in \mathbb{Z}_{>0} \] (F.10)

be a relation where all terms in are in standard form. Then by lemma F.1, each term in \((F.10)\) is multiple of a link diagram particular to that term. Because the link diagrams are linearly independent, the coefficients \( c_k \) must vanish, so relation \((F.10)\) is trivial. This proves the assertion. \( \square \)

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