Non-commutative Hardy–Littlewood maximal operator on symmetric spaces of $\tau$-measurable operators

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Abstract

In this paper, we investigate the Hardy–Littlewood maximal operator (in a sense of Bekjan) on non-commutative symmetric spaces. We obtain an upper distributional estimate (by means of the Cesàro operator) of a generalized singular number of the non-commutative Hardy–Littlewood maximal operator. We also show boundedness of the Hardy–Littlewood maximal operator from a general non-commutative symmetric space to another.

Keywords Symmetric spaces of functions and operators · Hardy–Littlewood maximal operator · von Neumann algebra · (Non-commutative) Lorentz and Marcinkiewicz spaces

Mathematics Subject Classification 46E30 · 47B10 · 46L51 · 46L52 · 44A15 · 47L20 · 47C15

1 Introduction

In this paper we investigate the Hardy–Littlewood maximal operator $M$, which is given (in classic literature) by the following formula:
where $B(x, r)$ denotes a (closed) ball in $\mathbb{R}^d (d \geq 1)$ with centre at $x$ and of radius $r > 0$, and $\lambda B(x, r)$ denotes its Lebesgue measure. In other words, $M$ is a non-linear operator which sends $f : \mathbb{R}^d \to \mathbb{C}$ to a function $Mf$, that is the maximal value that $f$ can attain on balls $B(x, r)$. There is an uncountable number of papers devoted to investigation of the Hardy–Littlewood maximal operator defined by the formula (1). For instance, see [2, 6, 18] and references therein.

It is well known that there are positive constants $c_1$ and $c_2$, depending only on $d$, such that

$$c_1 \mu(Mf) \leq C \mu(f) \leq c_2 \mu(Mf)$$

for every locally integrable function $f$ on $\mathbb{R}^d$ (see [2, Theorem III. 3.8, p.122]), where $C$ denotes the Cesàro operator (see (5) below) and $\mu(f)$ is a decreasing rearrangement of the function $|f|$.

In this paper, we mainly deal with the non-commutative version of the double inequality (2). In fact, for our purposes it is sufficient to consider only the first inequality in (2). We complete and refine the main results of Bekjan [1] and Shao [15]. To be precise, in this work we study the boundedness properties of the non-commutative Hardy–Littlewood maximal operator on symmetric spaces of $\tau$-measurable operators. Recently, there was also a special interest to a study of non-commutative maximal inequalities, see, for example, works of Junge and Xu [7, 8]. Afterwards, Mei [14] came up with a slightly different version of the non-commutative Hardy–Littlewood maximal inequality for an operator-valued function. Another version of the (non-commutative) Hardy–Littlewood maximal operator (for $\tau$-measurable operators) was introduced by Bekjan [1]. Moreover, in his paper T. Bekjan obtained weak $(1, 1)$-type and $(p, p)$-type inequalities for the Hardy–Littlewood maximal operator (in new settings). Later Shao investigated the Hardy–Littlewood maximal operator (in a sense of T. Bekjan) on non-commutative Lorentz spaces associated with a finite atomless von Neumann algebra (see [15]). Using the techniques in [6], J. Shao obtained the $(p, q) - (p, q)$-type inequality for the Hardy–Littlewood maximal operator on non-commutative Lorentz spaces. In a sense of T. Bekjan, the Hardy–Littlewood maximal operator of $A$ is defined by (see [1])

$$MA(x) = \sup_{r > 0} \frac{1}{\tau(E_{[x-r,x+r]}(|A|))} \tau(|A| E_{[x-r,x+r]}(|A|)), \quad x \geq 0,$$

where an operator $A$ is affiliated with a general semi-finite von Neumann algebra $\mathcal{M}$. To compare with the classical definition, we recall that the classical Hardy–Littlewood maximal operator of a measurable real-valued function $f$ is defined by

$$Mf(x) = \sup_{r > 0} \frac{1}{m([x-r,x+r])} \int_{[x-r,x+r]} |f(t)| dt, \quad x \geq 0,$$
where $m$ denotes the usual Lebesgue measure, see [18]. Note $|A|$ may be represented as

$$ |A| = \int_{\sigma(|A|)} tdE_t $$

from the spectral theory point of view, also $MA(|A|)$ is represented as $MA(x)$. Thus, for an operator $A$, Bekjan’s consideration is that $MA(|A|)$ is defined as the operator analogue of the Hardy–Littlewood maximal operator in the classical case. Hence, roughly speaking, $MA(|A|)$ stands in relation to $A$ as $Mf(x)$ stands in relation to $f$ in classical analysis, that is

$$ MA(|A|) = \int_{\sigma(|A|)} MA(\lambda) dE_\lambda(|A|), $$

(3)

where $\sigma(|A|)$ is the spectrum of $|A|$. Our purpose is to investigate the non-commutative Hardy–Littlewood maximal operator $M$ in a sense of Bekjan (see [1]).

Followed by introductory Sect. 1, we provide definitions, notations and some auxiliary results needed in the sequel in Preliminaries (Sect. 2).

In Sect. 3 we obtain an upper estimate of a generalized singular number of the non-commutative Hardy–Littlewood maximal operator. In particular, we show that the generalized singular number of the Hardy–Littlewood maximal operator is estimated from above by means of the Cesàro operator, which we address in Theorem 3.1 below. This, in turn, entails specific corollaries concerning the Hardy–Littlewood maximal operator. We also refine Theorem 1 in [1] by showing that the non-commutative Hardy–Littlewood maximal operator is bounded when acting from a non-commutative $L_1$ space to a separable part of (non-commutative) weak-$L_1$ space (see Theorem 3.2 below).

In Sect. 4, we extend some results obtained by Shao (see [15]) from a finite von Neumann algebra $\mathcal{M}$ to a general semi-finite $\mathcal{M}$. Moreover, while Shao considered the boundedness of the non-commutative Hardy–Littlewood maximal operator on Lorentz spaces, we rather show its boundedness from a general non-commutative symmetric space to another one (see Theorem 4.1 below). In particular, we show its boundedness from some non-commutative Lorentz space to another non-commutative Lorentz space (see Proposition 4.1 below) and from some non-commutative Marcinkiewicz space to another non-commutative Marcinkiewicz space (see Proposition 4.2 below). To do so, we use an approach similar to the commutative case. The key results, which we apply in the paper, may be found in [1]. Finally, we provide specific examples showing that the non-commutative Hardy–littlewood maximal operator is bounded from one particular non-commutative Lorentz space to another, and from one particular non-commutative Marcinkiewicz space to another as well.
2 Preliminaries

We denote by \((\mathbb{R}_+,m)\) (where \(\mathbb{R}_+ = (0, \infty)\)) the measure space equipped with the Lebesgue measure \(m\). Let \(L(\mathbb{R}_+,m)\) be the space of all measurable real-valued functions on \(\mathbb{R}_+\) equipped with the Lebesgue measure \(m\). Define \(S(\mathbb{R}_+,m)\) to be the subset of \(L(\mathbb{R}_+,m)\), which consists of all functions \(f\) (in what follows we denote functions by \(f\) or \(g\)) such that \(m(\{t : |f(t)| > s\}) < \infty\) for some \(s > 0\). For \(f \in S(\mathbb{R}_+,m)\) define the decreasing rearrangement \(\mu(f)\) of the function \(|f|\) is defined by

\[
\mu(t,f) = \inf\{s \geq 0 : m(\{|f| > s\}) \leq t\}, \quad t > 0.
\]

The Hardy–Littlewood–Pólya submajorization of the functions \(g\) and \(f\) (written \(g \ll f\)) is defined by

\[
\int_0^t \mu(s,g)ds \leq \int_0^t \mu(s,f)ds \quad \text{for all} \quad t \geq 0.
\]

We denote by \(\mathcal{M}\) a semifinite von Neumann algebra on a separable Hilbert space \(H\) equipped with a faithful normal semifinite trace \(\tau\). A closed and densely defined operator \(A\) affiliated with \(\mathcal{M}\) is called \(\tau\)-measurable if \(\tau(E_A(s,\infty)) < \infty\) for sufficiently large \(s \geq 0\). We denote the set of all \(\tau\)-measurable operators by \(S(\mathcal{M}, \tau)\). Let \(\text{Proj}(\mathcal{M})\) denote the lattice of all projections in \(\mathcal{M}\). For every \(A \in S(\mathcal{M}, \tau)\), we define its singular value function \(\mu(A)\) by setting

\[
\mu(t,A) = \inf\{\|A(1-P)\|_{\mathcal{L}_{\infty}(\mathcal{M})} : P \in \text{Proj}(\mathcal{M}), \quad \tau(P) \leq t\}, \quad t > 0,
\]

where the norm \(\| \cdot \|_{\mathcal{L}_{\infty}(\mathcal{M})}\) is the usual operator (uniform) norm. Equivalently, for positive self-adjoint operators \(A \in S(\mathcal{M}, \tau)\), we have

\[
n_A(s) = \tau(E_A(s,\infty)), \quad s > 0,
\]

\[
\mu(t,A) = \inf\{s : n_A(s) < t\}, \quad t > 0.
\]

An operator \(A\) in \(S(\mathcal{M}, \tau)\) is called \(\tau\)-compact if \(\mu(\infty,A) = 0\). This notion is a direct generalization of the ideal of compact operators on a Hilbert space \(H\). For more details on generalised singular value functions and \(\tau\)-compact operators, we refer the reader to [5, 13]. Let \(\mathcal{L}_{\text{loc}}(\mathcal{M}, \tau)\) be the set of all \(\tau\)-measurable operators \(A\) such that

\[
\tau(|A|E_I(|A|)) < +\infty
\]

for all bounded intervals \(I \subset [0, +\infty)\).

Let \(A,B \in \mathcal{L}_{\text{loc}}(\mathcal{M}, \tau)\), then we say that \(B\) is submajorized by \(A\) (in the sense of Hardy–Littlewood–Pólya), denoted by \(\mu(B) \ll \mu(A)\), if

\[
\int_0^t \mu(s,B)ds \leq \int_0^t \mu(s,A)ds \quad \text{for all} \quad t \geq 0.
\]

**Definition 2.1** [1, Definition 1] For \(A \in \mathcal{L}_{\text{loc}}(\mathcal{M}, \tau)\), we define the maximal operator of \(A\) by
$MA(x) = \sup_{r>0} \frac{1}{\tau(E_{[x-r,x+r]}(|A|))} \tau(|A| E_{[x-r,x+r]}(|A|))$, $x \geq 0,$

(let $\frac{0}{0} = 0$). $M$ is called the non-commutative Hardy–Littlewood maximal operator.

### 2.1 Symmetric (quasi-)Banach function and operator spaces

**Definition 2.2** We say that $(E, \| \cdot \|_E)$ is a symmetric (quasi-)Banach function space on $I$ if the following hold:

1. $E$ is a subset of $S(\mathbb{R}_+, m)$;
2. $(E, \| \cdot \|_E)$ is a (quasi-)Banach space;
3. If $f \in E$ and if $g \in S(\mathbb{R}_+, m)$ are such that $\mu(g) \leq \mu(f)$, then $g \in E$ and $\|g\|_E \leq \|f\|_E$;

For the general theory of symmetric (quasi-)Banach function spaces, we refer the reader to [2, 11, 12].

We say that a symmetric (quasi-)Banach space $E$ has Fatou norm, if the norm closed unit ball $B_E$ of $E$ is closed in $E$ with respect to almost everywhere convergence.

Let $0 < p < \infty$ and $0 < q \leq \infty$. If $q < \infty$, we define the Lorentz space $L_{p,q}(\mathbb{R}_+)$ on the measure space $(\mathbb{R}_+, m)$ as the space of all real-valued measurable functions $f$ on $\mathbb{R}_+$ such that the following quasi-norm is finite

$$\|f\|_{L_{p,q}(\mathbb{R}_+)} := \left( \int_{\mathbb{R}_+} \left( \frac{1}{t^q} \mu(t,f) \right)^\frac{q}{p} \frac{dt}{t} \right)^\frac{1}{q}$$

and, if $q = \infty$,

$$\|f\|_{L_{p,\infty}(\mathbb{R}_+)} := \sup_{t>0} \left( t^\frac{1}{p} \mu(t,f) \right).$$

It is also conventional to set $L_{\infty,\infty}(\mathbb{R}_+) = L_{\infty}(\mathbb{R}_+)$. In particular, for $p = 1$, so-called weak-$L_1$ space $L_{1,\infty}(\mathbb{R}_+)$ is defined by setting

$$L_{1,\infty}(\mathbb{R}_+) = \{ f \in S(\mathbb{R}_+) : \sup_{t>0} t \mu(t,f) < \infty \},$$

and equipped with the quasi-norm

$$\|f\|_{L_{1,\infty}(\mathbb{R}_+)} = \sup_{t>0} t \mu(t,f), \quad f \in L_{1,\infty}(\mathbb{R}_+).$$

Also, note that when $p = q$, we obtain usual $L_p$-spaces. It is well known that all of these spaces are basic examples of symmetric (quasi-)Banach function spaces. For more details we refer the reader to [6].
Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $H$ equipped with a faithful normal semifinite trace $\tau$. A von Neumann algebra $\mathcal{M}$ is also denoted by $L_\infty(\mathcal{M})$, and for all $A \in L_\infty(\mathcal{M})$ we denote $\|A\|_{L_\infty(\mathcal{M})} := \|A\|$. Moreover, $\|A\|_{L_\infty(\mathcal{M})} = \|\mu(A)\|_{L_\infty(\mathbb{R}_+)} = \mu(0,A)$, $A \in L_\infty(\mathcal{M})$.

Let $E$ be a symmetric (quasi-)Banach function space on $\mathbb{R}_+$. Set

$$E(\mathcal{M}, \tau) = \left\{ A \in S(\mathcal{M}, \tau) : \mu(A) \in E \right\}.$$ 

We equip $E(\mathcal{M}, \tau)$ with a natural norm

$$\|A\|_{E(\mathcal{M}, \tau)} = \|\mu(A)\|_E, \quad A \in E(\mathcal{M}, \tau).$$

This becomes a (quasi-)Banach space with the (quasi-)norm $\|\cdot\|_{E(\mathcal{M}, \tau)}$ and is called the (non-commutative) symmetric operator space associated with $(\mathcal{M}, \tau)$ corresponding to $(E, \|\cdot\|_E)$. An extensive discussion of the various properties of such spaces can be found in [10, 13] (see also [20–22]). For simplicity, in what follows we denote $E(\mathcal{M}, \tau)$ by $E(\mathcal{M})$.

For $0 < p < \infty$ and $0 < q < \infty$, define the non-commutative Lorentz space

$$L_{p,q}(\mathcal{M}) := \{ A \in S(\mathcal{M}, \tau) : \mu(A) \in L_{p,q}(\mathbb{R}_+) \}$$

equipped with the quasi-norm

$$\|A\|_{L_{p,q}(\mathcal{M})} := \|\mu(A)\|_{L_{p,q}(\mathbb{R}_+)},$$

and, for $q = \infty$,

$$\|A\|_{L_{p,\infty}(\mathcal{M})} := \sup_{t > 0} \frac{1}{t^p} \mu(t,A).$$

In particular, when $p = 1$, we obtain the non-commutative weak-$L_1$ space, which is defined by

$$L_{1,\infty}(\mathcal{M}) := \{ A \in S(\mathcal{M}, \tau) : \mu(A) \in L_{1,\infty}(\mathbb{R}_+) \}$$

equipped with the quasi-norm

$$\|A\|_{L_{1,\infty}(\mathcal{M})} := \sup_{t > 0} t \mu(t,A)$$
(see [13, 19]). For $1 \leq p = q < \infty$, we set

$$L_p(\mathcal{M}) = \{ A \in S(\mathcal{M}, \tau) : \tau(|A|^p) < \infty \}, \quad \|A\|_{L_p(\mathcal{M})} = (\tau(|A|^p))^{\frac{1}{p}}.$$ 

The Banach spaces $(L_p(\mathcal{M}), \|\cdot\|_{L_p(\mathcal{M})})$ ($1 \leq p < \infty$) are separable symmetric spaces [13].

In what follows, the following notation $A \approx B$ means that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 A \leq B \leq c_2 A$. 

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2.2 \( L_1 \cap L_\infty \) and \( L_1 + L_\infty \) spaces

The space \((L_1 \cap L_\infty)(\mathbb{R}_+) = L_1(\mathbb{R}_+) \cap L_\infty(\mathbb{R}_+)\) consists of all bounded summable functions \(f\) on \(\mathbb{R}_+\) with the norm

\[
\|f\|_{(L_1 \cap L_\infty)(\mathbb{R}_+)} = \max\{\|f\|_{L_1(\mathbb{R}_+)}, \|f\|_{L_\infty(\mathbb{R}_+)}\}, \quad f \in (L_1 \cap L_\infty)(\mathbb{R}_+).
\]

The space

\[
(L_1 + L_\infty)(\mathbb{R}_+) = L_1(\mathbb{R}_+) + L_\infty(\mathbb{R}_+) := \{f \in S(\mathbb{R}_+) : f = f_1 + f_2, f_1 \in L_1(\mathbb{R}_+), f_2 \in L_\infty(\mathbb{R}_+)\}
\]

is equipped with the norm given by

\[
\|f\|_{(L_1 + L_\infty)(\mathbb{R}_+)} = \inf\{\|f_1\|_{L_1(\mathbb{R}_+)} + \|f_2\|_{L_\infty(\mathbb{R}_+)} : f = f_1 + f_2, f_1 \in L_1(\mathbb{R}_+), f_2 \in L_\infty(\mathbb{R}_+)\}.
\]

These are examples of symmetric Banach spaces. For more details we refer to [2, Chapter I], [11, Chapter II]. We recall that every symmetric Banach function space \(E\) on \(\mathbb{R}_+\) (with respect to the Lebesgue measure) satisfies

\[
(L_1 \cap L_\infty)(\mathbb{R}_+) \subset E(\mathbb{R}_+) \subset (L_1 + L_\infty)(\mathbb{R}_+)
\]

with continuous embeddings (see for instance [11, Theorem II. 4.1. p. 91]).

Define the space \(L_1(\mathcal{M}) + L_\infty(\mathcal{M})\) as the class of those operators \(A \in S(\mathcal{M}, \tau)\) for which

\[
\|A\|_{L_1(\mathcal{M}) + L_\infty(\mathcal{M})} := \inf\{\|A_1\|_{L_1(\mathcal{M})} + \|A_2\|_{L_\infty(\mathcal{M})} : A = A_1 + A_2, A_1 \in L_1(\mathcal{M}), A_2 \in L_\infty(\mathcal{M})\} < \infty.
\]

In particular, if \(A \in S(\mathcal{M}, \tau)\), then \(A \in L_1(\mathcal{M}) + L_\infty(\mathcal{M})\) if and only if \(\mu(A) \in (L_1 + L_\infty)(\mathbb{R}_+)\) and

\[
\|A\|_{L_1(\mathcal{M}) + L_\infty(\mathcal{M})} = \|\mu(A)\|_{(L_1 + L_\infty)(\mathbb{R}_+)}., \quad A \in L_1(\mathcal{M}) + L_\infty(\mathcal{M}).
\]

We notice that the space \(L_1(\mathcal{M}) + L_\infty(\mathcal{M})\) is also denoted by \((L_1 + L_\infty)(\mathcal{M})\). Similarly, one may define the intersection of Banach spaces \(L_1(\mathcal{M})\) and \(L_\infty(\mathcal{M})\) as follows:

\[
(L_1 \cap L_\infty)(\mathcal{M}) = \{A \in S(\mathcal{M}, \tau) : \|A\|_{(L_1 \cap L_\infty)(\mathcal{M})} < \infty\},
\]

where the norm on \((L_1 \cap L_\infty)(\mathcal{M})\) is defined by

\[
\|A\|_{(L_1 \cap L_\infty)(\mathcal{M})} = \max\{\|A\|_{L_1(\mathcal{M})}, \|A\|_{L_\infty(\mathcal{M})}\}, \quad A \in (L_1 \cap L_\infty)(\mathcal{M}).
\]

It is easy to see that \((L_1 \cap L_\infty)(\mathcal{M})\) is a Banach space with respect to this norm. Note that if \(A \in S(\mathcal{M}, \tau)\), then \(A \in (L_1 \cap L_\infty)(\mathcal{M})\) if and only if \(\mu(A) \in (L_1 + L_\infty)(\mathbb{R}_+)\). Moreover,
\[ \|A\|_{(L_1 \cap L_\infty)(\mathcal{M})} = \|\mu(A)\|_{(L_1 \cap L_\infty)(\mathbb{R}_+)} \quad A \in (L_1 \cap L_\infty)(\mathcal{M}). \]

For more information on these spaces, we refer to [4, Chapter III] (see also [3]).

Define the Cesàro operator \( C : (L_1 + L_\infty)(\mathbb{R}_+) \to S(\mathbb{R}_+) \) (see [13, Chapter 9, p.278]) by

\[
(Cf)(t) := \frac{1}{t} \int_0^t f(s)ds, \quad f \in L_1(\mathbb{R}_+), \quad t > 0. \tag{5}
\]

### 2.3 Lorentz spaces

A function \( \varphi \) on \([0, \infty)\) satisfying the following conditions

1. \( \varphi(t) = 0 \iff t = 0 \)
2. \( \varphi(t) \) is positive and increasing for \( t > 0 \)
3. \( \frac{\varphi(t)}{t} \) is decreasing for \( t > 0 \)

is called quasiconcave (see [11, Definition II. 1.1, p.49]).

It is easy to see that every nonnegative concave function on \([0, \infty)\) that vanishes only at origin is quasiconcave. The reverse, however, is not always true.

For an increasing concave function \( \varphi : [0, \infty) \to [0, \infty) \) for which \( \lim_{t \to 0^+} \varphi(t) = 0 \) (or simply \( \varphi(0^+) = 0 \)), we define the Lorentz space \( A_\varphi(\mathbb{R}_+) \) by setting

\[
A_\varphi(\mathbb{R}_+) := \left\{ f \in S(\mathbb{R}_+) : \int_{\mathbb{R}_+} \mu(s,f)d\varphi(s) < \infty \right\},
\]

and equip with the norm

\[
\|f\|_{A_\varphi(\mathbb{R}_+)} := \int_{\mathbb{R}_+} \mu(s,f)d\varphi(s).
\]

These spaces are examples of symmetric Banach function spaces. For more details on Lorentz spaces, we refer the reader to [2, Chapter II.5] and [11, Chapter II.5].

As in the commutative case, for an increasing concave function \( \varphi \) on \([0, \infty)\) vanishing at origin only, define the corresponding non-commutative Lorentz space by setting

\[
A_\varphi(\mathcal{M}) := \left\{ A \in S(\mathcal{M}, \tau) : \int_{\mathbb{R}_+} \mu(s,A)d\varphi(s) < \infty \right\},
\]
equipped with the norm

\[ \|A\|_{A^\psi(M)} := \int_{\mathbb{R}_+} \mu(s, A)d\varphi(s). \]

These operator spaces become symmetric operator spaces (see [13]). Let \( \psi \) be a quasiconcave function on \((0, \infty)\). The space

\[ M_\psi(\mathbb{R}_+) = \{ f \in S(\mathbb{R}_+) : \|f\|_{M_\psi} < \infty \}, \]

where the norm is defined by

\[ \|f\|_{M_\psi(\mathbb{R}_+)} = \sup_{t > 0} \frac{\psi(t)}{t} \cdot \int_0^t \mu(s, f)ds \]

is also a symmetric space. The space \((M_\psi, \| \cdot \|_{M_\psi})\) is called the Marcinkiewicz space.

Similarly, define the non-commutative Marcinkiewicz space

\[ M_\psi(M) := \left\{ A \in S(M, \tau) : \sup_{t > 0} \frac{\psi(t)}{t} \cdot \int_0^t \mu(s, A)ds < \infty \right\}, \]

equipped with the norm

\[ \|A\|_{M_\psi(M)} = \sup_{t > 0} \frac{\psi(t)}{t} \cdot \int_0^t \mu(s, A)ds. \]

### 3 Upper estimate of a generalised singular number of the non-commutative Hardy–Littlewood maximal operator

In this section we estimate a generalised singular number of the non-commutative Hardy–Littlewood maximal operator. We also show some applications of this result.

To prove the principal result we will need the following lemma.

**Lemma 3.1** Every operator \( A \) in \( \mathcal{L}_{\text{loc}}(M, \tau) \) is \( \tau \)-compact.

**Proof** Assume, without loss of generality, that \( A \) is positive. Since \( A \) is \( \tau \)-measurable, it follows that \( \tau(E_{(N, \infty)}(A)) < \infty \) for sufficiently large number \( N \). Fix \( \varepsilon > 0 \) and consider \( \tau(E_{(\varepsilon, N)}(A)) \). Since \( \frac{1}{\varepsilon}AE_{(\varepsilon, N)}(A) > E_{(\varepsilon, N)}(A) \), it follows from (7) that

\[ \tau(E_{(\varepsilon, N)}(A)) \leq \frac{1}{\varepsilon} \tau(AE_{(\varepsilon, N)}(A)) < \infty. \]
Thus, \( \tau(E_{\epsilon,\infty}(A)) < \infty \) for any \( \epsilon > 0 \). Hence, \( A \) is \( \tau \)-compact.

The following theorem is the main result of this section, whose commutative counterpart is the classical inequality for the decreasing rearrangement \( \mu \) of the Hardy–Littlewood maximal function \( M(f) \) given by

\[
\mu(t,Mf) \leq c_{\text{obs}} \cdot \frac{1}{t} \int_0^t \mu(s,f)ds.
\]

**Theorem 3.1** For every \( A \in L_{\text{loc}}(\mathcal{M}, \tau) \), we have

\[
\mu(t,MA(|A|)) \leq 32 \cdot (C\mu(A))(t), \quad \forall t > 0.
\]

**Proof** Let \( A \in L_{\text{loc}}(\mathcal{M}, \tau) \). Then by Lemma 3.1, \( A \) is \( \tau \)-compact. Hence, \( A \in (L_1 + L_\infty)(\mathcal{M}) \), or equivalently \( \mu(A) \in (L_1 + L_\infty)(\mathbb{R}_+) \). First, we prove the following inequality

\[
\|A_1\|_{L_1(\mathcal{M})} + t\|A_2\|_{L_\infty(\mathcal{M})} \leq 2t(C\mu(A))(t), \quad t > 0. \tag{6}
\]

Fix \( \xi_0 = \mu(\xi_0,A) > 0 \). Since \( C : (L_1 + L_\infty)(\mathbb{R}_+) \to S(\mathbb{R}_+) \), it follows that \( C\mu(A)(t) \) is finite almost everywhere.

Set \( A_1 \) and \( A_2 \) as follows \( A_1 = AE_{(0,\xi_0]}(|A|) \) and \( A_2 = A - A_1 \). Then

\[
\|A_1\|_{L_1(\mathcal{M})} = \int_0^{\xi_0} \mu(s,A)ds = \xi_0(C\mu(A))(\xi_0) < \infty \tag{7}
\]

and

\[
\|A_2\|_{L_\infty(\mathcal{M})} = \mu(0,A_2) \leq \mu(\xi_0,A) < \infty.
\]

Since \( \mu(s,A) \) is decreasing, we obtain

\[
\xi_0 \cdot \|A_2\|_{L_\infty(\mathcal{M})} \leq \xi_0 \cdot \mu(\xi_0,A) \leq \int_0^{\xi_0} \mu(s,A)ds.
\]

Therefore,

\[
\|A_1\|_{L_1(\mathcal{M})} + \xi_0\|A_2\|_{L_\infty(\mathcal{M})} \leq 2\int_0^{\xi_0} \mu(s,A)ds = 2\xi_0(C\mu(A))(\xi_0).
\]

Since \( \xi_0 \) is arbitrary, the inequality (6) holds.

Following the argument in the proof of Lemma 4 in [1], we obtain
Here, taking supremum over \( r > 0 \), we obtain
\[
MA(x) \leq MA_1(x) + MA_2(x).
\]

It follows from (3) that
\[
MA(|A|) \leq MA_1(|A_1|) + MA_2(|A_2|).
\]

Indeed,
\[
< MA(|A|) f, f > = \int_{\sigma(|A|)} MA(\lambda) \, dE_f f, f >
\]
\[
\leq \int_{\sigma(|A_1|)} MA_1(\lambda) \, dE_f f, f >
\]
\[
+ \int_{\sigma(|A_2|)} MA_2(\lambda) \, dE_f f, f >
\]
\[
= < MA_1(|A_1|) f, f > + < MA_2(|A_2|) f, f >, \quad \forall f \in D(A).
\]

Here \( E_f^{(1)} \) and \( E_f^{(2)} \) are spectral projections of \(|A_1|\) and \(|A_2|\), respectively. Then, by (9) we have (see also Lemma 2.5 in [5] for the second inequality)
\[
\mu(s, MA(|A|)) \leq \mu(s, MA_1(|A_1|) + MA_2(|A_2|))
\]
\[
\leq \mu\left( \frac{s}{2}, MA_1(|A_1|) \right) + \mu\left( \frac{s}{2}, MA_2(|A_2|) \right), \quad s > 0.
\]

Let us estimate both of the latter terms separately. Since \( A_1 \in \mathcal{L}_1(M) \) (see (7)), it follows from [1, Theorem 1] that
\[
\mu\left( \frac{s}{2}, MA_1(|A_1|) \right) \leq \frac{16}{s} ||A_1||_{\mathcal{L}_1(M)},
\]
and since \( A_2 \in \mathcal{L}_\infty(M) \), by [1, Lemma 1 (ii)] we have
\[
\mu\left( \frac{s}{2}, MA_2(|A_2|) \right) \leq \mu(0, MA_2(|A_2|)) = ||MA_2(|A_2|)||_{\mathcal{L}_\infty(M)} \leq ||A_2||_{\mathcal{L}_\infty(M)}.
\]

\[
\frac{1}{\tau(E_{[x-r,x+r]}(|A|))} \tau(|A| E_{[x-r,x+r]}(|A|)) \leq \frac{1}{\tau(E_{[x-r,x+r]}(|A|))} \tau(|A| E_{[x-r,x+r]}(|A_1|))
\]
\[
+ \frac{1}{\tau(E_{[x-r,x+r]}(|A|))} \tau(|A_2| E_{[x-r,x+r]}(|A_2|))
\]
\[
\leq \frac{1}{\tau(E_{[x-r,x+r]}(|A_1|))} \tau(|A_1| E_{[x-r,x+r]}(|A_1|))
\]
\[
+ \frac{1}{\tau(E_{[x-r,x+r]}(|A_2|))} \tau(|A_2| E_{[x-r,x+r]}(|A_2|)).
\]

Hence, taking supremum over \( r > 0 \), we obtain
\[
MA(x) \leq MA_1(x) + MA_2(x).
\]

Indeed,
\[
< MA(|A|) f, f > = \int_{\sigma(|A|)} MA(\lambda) \, dE_f f, f >
\]
\[
\leq \int_{\sigma(|A_1|)} MA_1(\lambda) \, dE_f f, f >
\]
\[
+ \int_{\sigma(|A_2|)} MA_2(\lambda) \, dE_f f, f >
\]
\[
= < MA_1(|A_1|) f, f > + < MA_2(|A_2|) f, f >, \quad \forall f \in D(A).
\]

Here \( E_f^{(1)} \) and \( E_f^{(2)} \) are spectral projections of \(|A_1|\) and \(|A_2|\), respectively. Then, by (9) we have (see also Lemma 2.5 in [5] for the second inequality)
\[
\mu(s, MA(|A|)) \leq \mu(s, MA_1(|A_1|) + MA_2(|A_2|))
\]
\[
\leq \mu\left( \frac{s}{2}, MA_1(|A_1|) \right) + \mu\left( \frac{s}{2}, MA_2(|A_2|) \right), \quad s > 0.
\]

Let us estimate both of the latter terms separately. Since \( A_1 \in \mathcal{L}_1(M) \) (see (7)), it follows from [1, Theorem 1] that
\[
\mu\left( \frac{s}{2}, MA_1(|A_1|) \right) \leq \frac{16}{s} ||A_1||_{\mathcal{L}_1(M)},
\]
and since \( A_2 \in \mathcal{L}_\infty(M) \), by [1, Lemma 1 (ii)] we have
\[
\mu\left( \frac{s}{2}, MA_2(|A_2|) \right) \leq \mu(0, MA_2(|A_2|)) = ||MA_2(|A_2|)||_{\mathcal{L}_\infty(M)} \leq ||A_2||_{\mathcal{L}_\infty(M)}.
\]
Combining (10) and (11) together, we infer
\[ \mu(s, MA(|A|)) \leq \frac{16}{s} \|A_1\|_{L_1(\mathcal{M})} + \|A_2\|_{L_\infty(\mathcal{M})}, \quad s > 0. \]

Letting \( s = t \) and using (6), we obtain
\[ \mu(t, MA(|A|)) \leq \frac{16}{t} \|A_1\|_{L_1(\mathcal{M})} + \|A_2\|_{L_\infty(\mathcal{M})} \leq \frac{16}{t} (\|A_1\|_{L_1(\mathcal{M})} + t\|A_2\|_{L_\infty(\mathcal{M})}) \leq 32 \cdot (C\mu(A))(t). \]

Since \( A \) is arbitrary, we obtain the desired inequality. \qed

We extend the main result of [15] (see [15, Theorem 3.2]) to a general semi-finite von Neumann algebra \( \mathcal{M} \).

**Corollary 3.1** Let \( 0 < q < \infty, \ 1 < p, p_0, p_1 < \infty \) with \( p_0 \neq p_1 \) be such that \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \) for some \( 0 < \theta < 1 \). Then there exists a constant \( c_{pq} > 0 \) such that for all \( A \in L_{p,q}(\mathcal{M}) \) we have
\[ \| MA(|A|) \|_{L_{p,q}(\mathcal{M})} \leq c_{pq} \cdot \|A\|_{L_{p,q}(\mathcal{M})}. \]

**Proof** By Hardy’s inequality in [9] (see also [2, Lemma 3.9, p.124]), we have
\[ \| Cf \|_{L_{p,q}(\mathbb{R}^+)} \leq \tilde{c}_{p,q} \cdot \|f\|_{L_{p,q}(\mathbb{R}^+)}, \quad \forall f \in L_{p,q}(\mathbb{R}^+). \quad (12) \]

By Theorem 3.1, we obtain
\[ \| MA(|A|) \|_{L_{p,q}(\mathcal{M})} = \| \mu(MA(|A|)) \|_{L_{p,q}(\mathbb{R}^+)} \leq 32 \cdot \| C\mu(A) \|_{L_{p,q}(\mathbb{R}^+)} \]
\[ \leq 32 \cdot \tilde{c}_{p,q} \cdot \|\mu(A)\|_{L_{p,q}(\mathbb{R}^+)} = c_{pq} \cdot \|A\|_{L_{p,q}(\mathcal{M})}, \quad (3.7) \]

where \( c_{pq} = 32 \cdot \tilde{c}_{p,q} \). This concludes the proof. \qed

In particular, when \( p = q \), we obtain the result of Theorem 2 in [1].

**Corollary 3.2** For \( 1 < p < \infty \) and \( A \in L_p(\mathcal{M}) \), there is a constant \( c_p > 0 \) such that
\[ \| MA(|A|) \|_{L_p(\mathcal{M})} \leq c_p \cdot \|A\|_{L_p(\mathcal{M})}. \]

**Proof** The proof follows directly from the previous corollary. \qed

Let us denote
\[ (L_{1,\infty}(\mathbb{R}_+))^0 := \{ x \in L_{1,\infty}(\mathbb{R}_+) : \lim_{t \to 0^+} t \mu(t,x) = 0 \} \]

Note that this space coincides with the closure of all bounded functions in \( L_{1,\infty}(\mathbb{R}_+) \).

Also,\[ \mathcal{L}_{1,\infty}(\mathcal{M})^0 =: \{ A \in S(\mathcal{M}, \tau) : \mu(A) \in (L_{1,\infty}(\mathbb{R}_+))^0 \}. \tag{13} \]

The following result refines the result of Theorem 1 in [1].

**Theorem 3.2** The non-commutative Hardy–Littlewood maximal operator
\[ MA(| \cdot |) : \mathcal{L}_1(\mathcal{M}) \to (\mathcal{L}_{1,\infty}(\mathcal{M}))^0 \]

is bounded.

**Proof** It was shown in [16, Remark 3.5.] that \( C : L_1(\mathbb{R}_+) \to (L_{1,\infty}(\mathbb{R}_+))^0 \) is bounded. Hence, by Theorem 3.1 we obtain the desired result. Indeed, let \( A \in \mathcal{L}_1(\mathcal{M}, \tau) \). Then by the definition \( \mu(A) \in L_1(\mathbb{R}_+) \). Since \( C \mu(A) \in (L_{1,\infty}(\mathbb{R}_+))^0 \), it follows from Theorem 3.1 that \( \mu(MA(|A|)) \in (L_{1,\infty}(\mathbb{R}_+))^0 \), which by (13) means that \( MA(|A|) \in (\mathcal{L}_{1,\infty}(\mathcal{M}))^0 \). Since \( A \) is arbitrary, this completes the proof. \( \square \)

4 The Hardy–Littlewood maximal operator on non-commutative symmetric spaces

Let \( E(\mathbb{R}_+) \) be a symmetric space of functions. In [16, Theorem 2.3], [17, Theorem 3.5] the authors provide the minimal (the smallest) symmetric space
\[ F(\mathbb{R}_+) := \{ f \in (L_1 + L_\infty)(\mathbb{R}_+) : \mu(f) \ll C \mu(g) \text{ for some } g \in E(\mathbb{R}_+) \} \tag{14} \]

with Fatou norm such that the Cesàro operator defined in (5) \( C : E(\mathbb{R}_+) \to F(\mathbb{R}_+) \) is bounded.

In particular, if \( \varphi \) is a quasiconcave function satisfying
\[ \varphi(t) \geq c_{abs} \cdot t \log(1 + 1/t), \tag{15} \]

then the Cesàro operator (see [16, Theorem 2.3])
\[ C : A_\varphi(\mathbb{R}_+) \to F(A_\varphi) := \{ f \in (L_1 + L_\infty)(\mathbb{R}_+) : \exists g \in A_\varphi(\mathbb{R}_+) \quad \mu(f) \ll C \mu(g) \} \]
is bounded. Similarly to a symmetric operator space \( \mathcal{E}(\mathcal{M}) \), we define a symmetric operator space
\[ \mathcal{F}(\mathcal{M}) := \{ A \in S(\mathcal{M}, \tau) : \mu(A) \in F(\mathbb{R}_+) \}, \]
equipped with the norm
\[ ||A||_{\mathcal{F}(\mathcal{M})} := ||\mu(A)||_{F(\mathbb{R}_+)}, \]
where $F(\mathbb{R}_+)$ is defined as in (4.1).

The following theorem is the main result of this section, which shows that the non-commutative Hardy–Littlewood maximal operator from a general symmetric operator space to another is bounded, thereby extending the result of Shao [15].

**Theorem 4.1** The non-commutative Hardy–Littlewood maximal operator

$$MA(\cdot) : \mathcal{E}(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$$

is bounded.

**Proof** Let $A \in \mathcal{E}(\mathcal{M})$. Then $\mu(A) \in E(\mathbb{R}_+)$. Since $C : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$ is bounded, it follows from Theorem 3.1 that

$$||MA(|A|)||_{\mathcal{F}(\mathcal{M})} = ||\mu(MA(|A|))||_{F(\mathbb{R}_+)} \leq 32||C\mu(A)||_{F(\mathbb{R}_+)} \leq c_{abs}||\mu(A)||_{E(\mathbb{R}_+)} = c_{abs}||A||_{\mathcal{E}(\mathcal{M})}.$$

Since $A$ is arbitrary the assertion follows. \qed

As an immediate corollary we obtain:

**Proposition 4.1** Let $\varphi$ be an increasing concave function such that $\varphi(+0) = 0$ and satisfying (15) and let $\psi$ be an increasing concave function such that $\psi(+0) = 0$ and

$$\int_t^\infty \frac{\psi(s)}{s^2} \, ds \leq c_{abs} \frac{\varphi(t)}{t}. \tag{16}$$

Then the non-commutative Hardy–Littlewood maximal operator

$$MA(\cdot) : A_\varphi(\mathcal{M}) \to A_\psi(\mathcal{M})$$

is bounded.

**Proof** It is known that if $\varphi$ satisfies (15), the Cesàro operator (see [17, Proposition 4.4])

$$C : A_\varphi(\mathbb{R}_+) \to A_\psi(\mathbb{R}_+)$$

is bounded if and only if (16) holds. Moreover, $A_\varphi(\mathbb{R}_+)$ is minimal (smallest) among such symmetric Banach function spaces. Therefore, similarly to the proof of Theorem 4.1, it follows from Theorem 3.1 that the non-commutative Hardy–Littlewood maximal operator

$$MA(\cdot) : A_\varphi(\mathcal{M}) \to A_\psi(\mathcal{M})$$

is bounded. \qed

Now, we consider the boundedness of the Hardy–Littlewood maximal operator on non-commutative Marcinkiewicz spaces, which are dual spaces to Lorentz spaces.
Similarly, we obtain the following boundedness criterion for Marcinkiewicz spaces.

**Proposition 4.2** Let \( \phi \) be a quasiconcave function such that \( 1/\phi \) is locally integrable at zero. Then the non-commutative Hardy–Littlewood maximal operator

\[
MA(\cdot) : M_{\phi}(\mathcal{M}) \rightarrow M_{\psi}(\mathcal{M})
\]

is bounded. Here

\[
\psi(t) = t \cdot \left( \int_0^t \frac{ds}{\phi(s)} \right)^{-1}, \quad t > 0.
\] (17)

**Proof** In [17, Theorem 4.7] it was proved that the Cesàro operator \( C : M_{\phi}(\mathbb{R}_+) \rightarrow M_{\psi}(\mathbb{R}_+) \) is bounded if and only if \( 1/\phi \) is locally integrable at zero. Moreover, in this case \( M_{\psi}(\mathbb{R}_+) \) is the smallest space among the symmetric spaces such that \( C : M_{\phi}(\mathbb{R}_+) \rightarrow M_{\psi}(\mathbb{R}_+) \). Therefore, by applying Theorem 3.1 we conclude the proof. \( \square \)

Now we provide two examples showing that the non-commutative Hardy–Littlewood maximal operator is bounded from one particular non-commutative Lorentz space to another, and also from one particular non-commutative Marcinkiewicz space to another.

**Example 1** Let \( \varphi(t) = t \log^2(1 + \frac{1}{\sqrt{t}}), t > 0. \) Then it is easy to show that \( \varphi \) satisfies (15). Let \( \psi(t) = t \log(1 + \frac{1}{t}). \) Then

\[
\int_0^\infty \frac{\psi(s)}{s^2} ds \leq c_{\text{abs}} \frac{\varphi(t)}{t}, \quad t > 0.
\]

The latter follows directly from Example 4.5 in [17] with \( \alpha = 1. \)

Hence, by Proposition 4.1 we obtain that

\[
MA(\cdot) : A_{t \log^2 (1+ \frac{1}{\sqrt{t}})}(\mathcal{M}) \rightarrow A_{t \log (1+ \frac{1}{t})}(\mathcal{M})
\]

is bounded.

**Example 2** Let us consider a function \( \phi(t) = \max\{1, t\}, t > 0. \) Then by (17), we obtain \( \psi(t) \approx \frac{t}{\log(1+t)}. \) Since \( M_{\phi}(\mathcal{M}) = (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}) \), it follows from Proposition 4.2 that

\[
MA(\cdot) : (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}) \rightarrow M_{\frac{t}{\log(1+t)}}(\mathcal{M})
\]

is bounded.
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