MEROMORPHIC SOLUTIONS TO A DIFFERENTIAL–DIFFERENCE EQUATION DESCRIBING CERTAIN SELF-SIMILAR POTENTIALS

ALEXANDER TOVBIS

Abstract. In this paper we prove the existence of meromorphic solutions to a nonlinear differential difference equation that describe certain self-similar potentials for the Schrödinger operator.

1. Introduction

Let \( L = -\partial_x^2 + u(x) \) be a Schrödinger operator that we factorize as \( L = A^+A^- + \lambda \), where \( A^\pm = \mp\partial_x + f(x) \) and \( x, \lambda \in \mathbb{C} \). Then the function \( f(x) \) satisfies the Riccati equation \( f^2(x) - f'(x) + \lambda = u(x) \). If \( \tilde{L} = A^-A^+ + \lambda \) denotes a new Schrödinger operator, obtained from \( L \) by permuting the operator factors \( A^\pm \), then the potential \( \tilde{u}(x) \) of \( \tilde{L} \) is given by \( \tilde{u}(x) = f^2(x) + f'(x) + \lambda \). The differential-difference equation (DDE)

\[
[f(x) + f(x+a)]' + f^2(x) - f^2(x+a) = \mu,
\]

where \( a \) and \( \mu \) are some complex constants, was derived in [1, 2] to describe potentials of the Schrödinger operator \( L \) satisfying the self-similarity constraint \( \tilde{L} = ULU^{-1} + \mu \), where \( U \) is the translation operator \( Uf(x) = f(x+a) \). In the case \( \mu = 0 \) equation (1) is satisfied by (1)

\[
f(x) = -\frac{1}{2} \frac{P'(x-x_0) - P'(a)}{P(x-x_0) - P(a)},
\]

where \( P \) denotes the Weierstrass elliptic function and \( x_0 \in \mathbb{C} \) is an arbitrary constant. The function (2) is a meromorphic function with only simple poles.

1991 Mathematics Subject Classification. 34M, 34K, 81Q.
Key words and phrases. differential difference equations, formal solutions, meromorphic continuation, Laplace transform.
The aim of the present paper is to prove existence of meromorphic solutions with simple
poles to (1) in the case \( \mu \neq 0 \). Without any loss of generality, we assume \( a \) to be a positive
real number. Indeed, if \( a = re^{i\phi} \), where \( \phi \in \mathbb{R} \) and \( r > 0 \), then the transformation \( x \to e^{i\phi}x \),
\( f \to e^{-i\phi}f \) reduces (1) to an equation of the same type with the step \( r \in \mathbb{R}^+ \). Our approach,
which utilizes some ideas of [To], consists of essentially three statements: 1) there exist two
different formal power series solutions (in powers of \( x - \frac{1}{2} \)) to (1); 2) for any formal solution
\( \hat{f}(x) \) there exists an actual solution \( f(x) \), analytic (when \( |x| \) is sufficiently large) in a sector \( S \)
on the complex \( x \)-plane of opening greater than \( \pi \) and having the asymptotic expansion \( \hat{f}(x) \)
in \( S \); 3) any such solution can be meromorphically continued onto \( \mathbb{C} \) so that \( f(x) \) may have
only first order poles. These statements are proven in Sections 3 and 2 respectively. The
author wants express his gratitude to V. Spiridonov for interesting discussions held at the
NATO ASI “Special functions 2000”, Tempe, Arizona and for the following correspondence.
The author also want to use this opportunity to thank the organizers of the NATO ASI
“Special functions 2000”.

2. Meromorphic continuation

Let \( R \in \mathbb{C} \) be a simply-connected domain, bounded by piece-wise smooth curves \( \eta_1(\xi) \)
and \( \eta_2(\xi) \), where \( x = \xi + i\eta \) is a complex number and \( \eta_2(\xi) > \eta_1(\xi) \) for all \( \xi \in (-\infty, +\infty) \).
The values \( \eta_2 = +\infty \) and \( \eta_1 = -\infty \) are allowed. We say that the domain \( R \) is \( a \)-wide on the
interval \( I \subset \mathbb{R} \) if \( \eta_2(\xi) - \eta_1(\xi) > a \) for all \( \xi \in I \).

Theorem 2.1. A solution \( f(x) \) to the equation (1) that is analytic in some domain \( R \) that is
\( a \)-wide on \( \mathbb{R} \) admits a meromorphic continuation on the whole complex plane and all possible
singularities of \( f(x) \) are first order poles.

Proof. Let \( R_1 = R \) and let \( R_n \) denote the domain bounded by the curves \( \eta_1(\xi) \) and \( \eta_2(\xi) + (n - 1)a \), where \( n = 2, 3, \ldots \). We first prove that \( f(x) \) can be meromorphically continued
to the domain \( R_{\infty} = \bigcup_1^{\infty} R_n \), i.e., to the right of the domain \( R \), by considering (1) as the
Riccati equation

\[
(3) \quad f'(x) = f^2(x) + h(x),
\]
where \( h(x) = \mu - f(x - a)' - f^2(x - a) \). The latter equation is equivalent to the second order linear differential equation

\[
(4) \quad u''(x) + h(x)u(x) = 0
\]

through the standard transformations

\[
(5) \quad u(x) = \exp \int_{\tilde{x}}^{x} f(t) dt \quad \text{and} \quad f(x) = -\frac{u'(x)}{u(x)},
\]

where \( \tilde{x} \in R \) is a point such that \( \tilde{x} - a \in R \). Note that, according to (5), the function \( u(x) \) is analytic in \( R \).

Consider (4) in the region \( Q_1 \), where \( Q_n = R_{n+1} \setminus R_n, n \in \mathbb{Z}^+ \). According to the assumption of the theorem, the function \( h(x) \) is analytic in \( Q_1 \), so that solutions of the linear equation (4) are analytic in \( Q_1 \). Thus, we can analytically continue \( u(x) \) onto \( R_2 \). If \( u(x) \) does not attain zero value in \( Q_1 \), we get an analytic continuation of \( f(x) \) onto \( R_2 \) by (5). However, if \( u(x_0) = 0 \) for some \( x_0 \in Q_1 \), then \( f(x) \) has a first order pole in \( x_0 \), which is an isolated singularity of \( f(x) \). Thus, we obtained the required meromorphic continuation of \( f(x) \) onto \( R_2 \). This process can be continued to the domains \( R_3, R_4, \) etc. in the same fashion. However, now we have to consider a possibility that \( h(x) \) has a singularity at \( x_0 + a \). Then, \( u(x) \) and, correspondingly, \( f(x) \) may have singularities at \( x_n = x_0 + na \), where \( n \in \mathbb{Z}^+ \). We need to show that these possible singularities of \( f(x) \) are first order poles only.

Let

\[
(6) \quad h(x) = h_0 + h_1(x - x_0) + h_2(x - x_0)^2 + \ldots \quad \text{and} \quad u(x) = (x - x_0) + u_2(x - x_0)^2 + u_3(x - x_0)^3 + \ldots
\]

be the Taylor expansions of \( h(x) \) and \( u(x) \) near \( x = x_0 \) in (4). Comparing the like powers of \( x - x_0 \) in (5), we obtain

\[
(7) \quad u_2 = 0, \quad u_3 = -\frac{h_0}{6}, \quad u_4 = -\frac{h_1}{12}, \ldots,
\]

so that the principle part of \( f(x) \) at \( x = x_0 \) is \( -(x - x_0)^{-1} \).

Combining the expression for \( h(x) \) with (4), we obtain

\[
(8) \quad h(x + a) = \mu - h(x) - 2f^2(x) = \mu - h(x) - 2[(\ln u(x))']^2.
\]
Direct computations show that

\[ (9) \quad [\ln u(x)]' = \frac{1}{x-x_0} - \frac{h_0}{3}(x-x_0) - \frac{h_1}{4}(x-x_0)^2 + O(x-x_0)^3 , \]

and

\[ (10) \quad -2([\ln u(x)]')^2 = -\frac{2}{(x-x_0)^2} + \frac{4h_0}{3} + h_1(x-x_0) + O(x-x_0)^2 , \]

so that

\[ (11) \quad h_1(x+a) = h(x+a) = \frac{-2}{(x-x_0)^2} + (\mu + \frac{h_0}{3}) + O(x-x_0)^2 . \]

Considering now this equation near the point \( x_1 \in Q_2 \), we obtain

\[ (12) \quad h_1(x) = \frac{-2}{(x-x_1)^2} + (\mu + \frac{h_0}{3}) + O(x-x_1)^2 . \]

The proof that all the points \( x_n \) are either regular points or first order poles of \( f(x) \) follows by induction from the following two lemmas.

**Lemma 2.2.** The differential equation

\[ (13) \quad u''(x) + h_n(x)u(x) = 0 , \]

where the coefficient \( h_n(x) \) has the form

\[ (14) \quad h_n(x) = -\frac{N(N-1)}{(x-x_n)^2} + A_0 + A_2(x-x_n)^2 + \cdots + A_{2N-2}(x-x_n)^{2N-2} + O(x-x_n)^{2N-1} , \]

possesses two linearly independent solutions

\[ (15) \quad u(x) = (x-x_n)^N \left[ 1 + u_2(x-x_n)^2 + \cdots + u_{2N}(x-x_n)^{2N} + O(x-x_n)^{2N+1} \right] \]

\[ (16) \quad v(x) = (x-x_n)^{-N+1} \left[ 1 + v_2(x-x_n)^2 + \cdots + u_{2N}(x-x_n)^{2N} + O(x-x_n)^{2N+1} \right] \]

that are analytic in a neighborhood of \( x = x_n \). Here \( A_k, u_k, v_k \in \mathbb{C} \) and \( n, N \in \mathbb{N} \) with \( n+1 \geq N \geq 2 \).
Proof. Frobenius multipliers of (13) at \( x = x_n \) are \( N \) and \( -N + 1 \), so the first terms of \( u(x) \) and \( v(x) \) are \( (x - x_n)^N \) and \( (x - x_n)^{-N+1} \) respectively. Suppose,

\[
(17) \quad u(x) = (x - x_n)^N + \sum_{k=1}^{\infty} u_k(x - x_n)^{N+k}.
\]

Computing \( u_k \), we see that the odd coefficients \( u_1 = u_3 = \cdots = u_{2N-1} = 0 \), so that \( u(x) \) is in the form (15). Similar arguments work for the second solution \( v(x) \).

Arguments of Lemma 2.2 show that solutions to (13) have no branching at \( x = x_n \) since Frobenius multipliers are integer and there are no logarithms. Note that the general solution to (13) has the form

\[
(18) \quad u(x) = (x - x_n)^{-N+1} \left[ u_0 + u_2(x - x_n)^2 + \cdots + u_{2N-2}(x - x_n)^{2N-2} + O(x - x_n)^{2N-1} \right]
\]

where \( u_0 \neq 0 \). The only nontrivial special solution to (13) is proportional to \( u(x) \) given by (14). In any case, the point \( x_n \) is a simple pole of the solution \( f(x) = -[\ln u(x)]' \) to (4). The following Lemma shows that the new coefficient

\[
(19) \quad h_{n+1}(x) = \mu - h_n(x - a) - 2[\ln u(x)]' \]

where \( u(x) \) is a solution to (13), has the form (14) with another \( N \). Thus, the point \( x_{n+1} \) is also a simple pole or a regular point of the solution \( f(x) \). (Note than \( x_{n+1} \) is a regular point of \( h_{n+1} \) if the new \( N + 1 \).

Lemma 2.3. The new coefficient \( h_{n+1}(x) \) has the form (14) with the new \( N \) equal to \( N - 1 \) if \( u(x) \) is given by (18) or equal to \( N + 1 \) if \( u(x) \) is given by (15).

Proof. Consider, for example, the case when \( u(x) \) is given by (15). Using expansion (14) for \( h_n(x) \), we obtain the first odd coefficient \( u_{2N+1} \) in (13) is

\[
(20) \quad u_{2N+1} = -\frac{A_{2N+1}}{4N(2N+1)}.
\]

On the other hand, we see that \( \ln u(x) = N \ln(x - x_n) + u_2(x - x_n)^2 + \ldots \), where the leading odd term of \( \ln u(x) \) is \( u_{2N+1}(x - x_n)^{2N+1} \). Then

\[
[\ln u(x)]' = (x - x_n)^{-1} \left[ N + 2u_2(x - x_n)^2 + \ldots \right]
\]
where the leading odd term in the square brackets is \((2N + 1)u_{2N+1}(x - x_n)^{2N+1}\). Finally, we get the leading odd term of \(-2(\ln u(x))'\) as \(-4N(2N + 1)u_{2N+1}(x - x_n)^{2N-1}\). So, according to (13) and (20), the leading odd term of \(h_{n+1}(x)\) at \(x_{n+1}\) is of the order \((x - x_{n+1})^{2N+1}\). The leading term of \(h_{n+1}(x)\) is \(N(N-1) - 2N^2 = -(N+1)N\). So, \(h_{n+1}(x)\) has the form (14) with the new \(N\) equal to \(N + 1\). The case when \(u(x)\) is given by (18) can be considered in a similar way.

Consider, for example, the singular point \(x_1\). According to (12), we have \(N = 2\) at this point. Solving the corresponding initial value problem for (13), we get the solution \(u(x)\) of either type (15) or (18). In any case, \(f(x)\) has a first order pole. The corresponding function \(h_2(x)\), according to Lemma 2.3, has a second order pole at \(x = x_2\) with the principal part \(-6(\ln u(x))'\) if \(u(x)\) is proportional to (13), and is regular at \(x = x_2\) if \(u(x)\) is given by (18).

We can continue these arguments to show that all points \(x_n\) are either first order poles or regular points. Thus, we proved meromorphic continuation of \(f(x)\) on \(R_\infty\).

To prove the meromorphic continuation to the left of the domain \(R_\infty\) let us note that the transformation \(x = -t - a, g(t) = f(-t)\) reduces the equation (1) to the equation of the same type

\[
(g(t) + g(t + a))' + g^2(t) - g^2(t + a) = -\mu,
\]

which have an analytic solution \(g(t)\) on \(t \in -R_\infty\). We can now use the previous arguments to continue \(g(t)\) to the right on the whole complex plane.

**Corollary 2.4.** Let \(I \subset \mathbb{R}\) be an interval and let \(f(x)\) be a solution to the equation (1) that is analytic in a domain \(R_\infty\) that is \(a\)-wide on \(I\). Then \(f(x)\) admits a meromorphic continuation on the strip \(\Im x \in I\) and all possible singularities of \(f(x)\) are first order poles.

3. Asymptotic solutions

In this section we rewrite (1) as

\[
\square f'(z) = \mu + \Delta f^2(z),
\]

where \(z = x + b, b = \frac{a}{2}\) and the operators \(\square, \Delta\) act on a function \(g(z)\) as

\[
\square g(z) = g(z + b) + g(z - b) \quad \text{and} \quad \Delta g(z) = g(z + b) - g(z - b).
\]
Some important for us properties of $\Box, \Delta$ are given by identities
\begin{equation}
2\Box[fg] = \Box f \cdot \Box g + \Delta f \cdot \Delta g \quad \text{and} \quad 2\Delta[fg] = \Delta f \cdot \Box g + \Box f \cdot \Delta g,
\end{equation}
where $f, g$ are given functions.

In this section we construct two different formal power series solutions $\hat{f}^\pm(z)$ to (1) and prove existence of corresponding actual solutions $f_n^\pm(z)$. These solutions are analytic in the corresponding sectors $S_n^\pm$, $n = 0, 1, 2$, specified below, when $|z|$ is sufficiently large and admit asymptotic expansions
\begin{equation}
f_n^\pm(z) \sim \hat{f}^\pm(z), \quad z \to \infty, \quad z \in S_n^\pm.
\end{equation}
This fact together with Theorem 2.1 prove existence of non-trivial meromorphic solutions to (1).

**Proposition 3.1.** Equation (22) possesses two formal power series solutions
\begin{equation}
\hat{f}^\pm(z) = \pm \lambda \sqrt{z} + \frac{1}{2b} + \sum_{k=2}^{\infty} y_k^\pm z^{-\frac{k}{2}},
\end{equation}
where $\lambda = \sqrt{-\frac{\mu}{2b}}$ and the coefficients $y_k^\pm$ are defined uniquely.

**Proof.** The substitution $f = \lambda z^{\frac{1}{2}} + c + \hat{y}(z)$ reduces the equation (22) to
\begin{equation}
\lambda \Box z^{-\frac{1}{2}} + \Box \hat{y}' = \mu + (\lambda \Delta z^{\frac{3}{2}} + \Delta \hat{y})(\lambda \Box z^{\frac{1}{2}} + 2c + \Box \hat{y}).
\end{equation}
Comparing leading coefficients and taking into account $\Delta z^{\frac{3}{2}} \Box z^{\frac{1}{2}} = 2b$, we get $\lambda = \pm \sqrt{-\frac{\mu}{2b}}$. Taking into account (24), the latter equation can be now rewritten as
\begin{equation}
2\lambda \Delta [z^{\frac{1}{2}} \hat{y}] = \Box \hat{y}' - \Delta \hat{y}^2 - 2c \Delta \hat{y} + \lambda \left( \frac{1}{2} \Box z^{-\frac{1}{2}} - 2c \Delta z^{\frac{1}{2}} \right).
\end{equation}
To expand $\hat{y}$ in powers of $z^{-\frac{1}{2}}$ we need the free term of (28) to be of order $O(z^{-1})$ or less. Thus, we obtain $c = \frac{1}{2b}$, so that the free term is of the order $O(z^{-\frac{3}{2}})$. Equation (28) can be now rewritten as
\begin{equation}
\Delta [z^{\frac{1}{2}} \hat{y}] = \frac{1}{2\lambda} \left( \Box \hat{y}' - \Delta \hat{y}^2 - \frac{1}{b} \Delta \hat{y} \right) + \frac{\Box z^{-\frac{1}{2}}}{4} - \frac{\Delta z^{\frac{1}{2}}}{2b}.
\end{equation}
It is clear that the expression in the left hand side is the dominant term of the latter equation and that the substitution $\hat{y} = \sum_{k=2}^{\infty} y_k z^{-\frac{k}{3}}$, satisfying (29), defines the coefficients $y_k$ uniquely.

Let $\hat{f}(z) = \lambda \sqrt{z} + \frac{1}{2\theta} + \ldots$ denote one of the formal solutions $\hat{f}^\pm(z)$. Given $\lambda \in \mathbb{C}$, we define angles
\begin{equation}
\beta_n = \frac{2}{3} \pi (1 + 2n) - \frac{2}{3} \arg \lambda,
\end{equation}
where $n = 0, 1, 2$ and $\arg \lambda \in [0, 2\pi)$. If $\beta_n + \pi$ is not a multiple of $2\pi$, we define the sector $S_n^+$ on the Riemann surface of $z^{1/3}$ by extending a small sector bisected by $\arg z = \beta_n$ independently in both positive and negative directions until it either hits the negative real direction or the ray $\arg z = \beta_n \pm \pi$ respectively. If $\beta_n$ is not a multiple of $2\pi$, the sector $S_n^-$ on the Riemann surface of $z^{1/3}$ is defined by extending a small sector bisected by $\arg z = \beta_n$ independently in both positive and negative directions until it either hits the positive real direction or the ray $\arg z = \beta_n \pm \pi$ respectively. In the case $\beta_n = 2\pi k, k \in \mathbb{Z}$ or $\beta_n = (2k+1)\pi, k \in \mathbb{Z}$, the sectors $S_n^-$ or $S_n^+$ respectively are considered to be empty.

**Theorem 3.2.** If $\hat{f}(z)$ is a formal solution to the equation (1) and $S_n^+$ is a nonempty sector described above, then there exists and actual solution $f(z)$ to (1) that is analytic in sufficiently remote part of any proper subsector $S$ of $S_n^+$ and
\begin{equation}
f(z) \sim \hat{f}(z), \quad z \to \infty, \quad z \in S.
\end{equation}

**Proof.** Our main idea is to reduce the considered DDE to an integro-differential equation (IDE) and to show that the latter equation can be solved by successive iterations in a proper sectorial neighborhood of infiniti. Substituting $f(z) = \lambda \sqrt{z} + \frac{1}{2\theta} + y(z)$, we obtain equation (29) for $y(z)$. The inverse Laplace transform $L^{-1}$, applied to (29), yields (see, for example, [Ob])
\begin{equation}
-2 \sinh(pb)L^{-1}[z^{\frac{1}{3}}y](p) = \frac{1}{2\lambda} \left( -2p \cosh(pb)Y(p) + 2 \sinh(bp) \left[ Y^{*2}(p) + \frac{1}{b} Y(p) \right] \right) + \frac{\sinh bp}{2\sqrt{\pi}p} (\coth bp - \frac{1}{bp}) ,
\end{equation}
(32)
where $Y(p) = \mathcal{L}^{-1}[y](p)$ and $F(p)^{\ast 2} = \int_0^p F(p - q)F(q)\,dq$. After a simple algebra, the latter equation becomes

$$
\mathcal{L}^{-1}[z^{\frac{1}{2}}y](p) = \frac{\Xi(bp)}{2\lambda b}Y(p) - \frac{1}{2\lambda}Y^{\ast 2}(p) - \frac{\Xi(bp)}{2\sqrt{\pi bp^{3/2}}} ,
$$

(33)

where $\Xi(x) = x \coth x - 1$. Note that $\Xi(x)$ is a meromorphic function with simple poles at the points $i\pi k$, where $k \in \mathbb{Z}\{0\}$, and that $\Xi(x)$ has not more than linear growth in any non-vertical direction $\arg x = \text{const}$.

Separating the linear part $bp$ of $\Xi(bp)$ along the positive real axis, we can rewrite (33) as

$$
\mathcal{L}^{-1}[z^{\frac{1}{2}}y](p) - \frac{p}{2\lambda}Y(p) = \frac{\Xi(bp) - bp}{2\lambda b}Y(p) - \frac{1}{2\lambda}Y^{\ast 2}(p) - \frac{\Xi(bp)}{2\sqrt{\pi bp^{3/2}}} ,
$$

(34)

where the function $\Xi(bp) - bp$ is bounded on $[0, \infty)$. Applying now the Laplace transform $\mathcal{L}$ to (34), we reduce (29) to the IDE

$$
y'(z) + 2\lambda z^{\frac{1}{2}}y(z) = \frac{1}{b}\xi(z) \ast y(z) - y^2(z) - \frac{\lambda}{b\sqrt{\pi}}r(z) ,
$$

(35)

where $\xi(z) = \mathcal{L}[\Xi(bp) - bp](z)$, $r(z) = \mathcal{L}[\Xi(bp)p^{-3/2}](z)$ and the convolution is defined by

$$
f(z) \ast g(z) = \frac{1}{2\pi i} \int_{A-\infty}^{A+i\infty} f(s)g(z - s)ds
$$

(36)

with a sufficiently large $A > 0$. Considering (35) as a perturbed linear equation $y'(z) + 2\lambda z^{\frac{1}{2}}y(z) = 0$, we rewrite the former as

$$
y(z) = e^{-\frac{4}{3}\lambda z^{3/2}} \int_{\gamma(z)} e^{\frac{4}{3}\lambda t^{3/2}}W[y(t)]\,dt ,
$$

(37)

where the nonlinear operator $W[y]$ denotes the right hand side of (35) and the contour of integration $\gamma(z)$ is to be specified below. Equation (37) can be rewritten in the operator form as $y = I\mathcal{W}[y]$.

Let us assume for a while that $\frac{\pi}{2} < \arg \lambda < \frac{7\pi}{2}$ and consider sector $S_0^+$. According to (31), this choice of $\lambda$ allows us to find a proper closed subsector $S \subset S_0^+$ that contains the right half-plane. We want to solve the IDE (33) in a sufficiently remote part of the sector $S$ by successive approximations. In order to formulate the statement more precisely we need to introduce the following notations.
Let $\Sigma$ be the image of the sector $S$ under the transformation $\zeta = z^{3/2}$. Let $z_0$ be a sufficiently remote point on the ray $\arg z = \beta_0$ and let $\Sigma_{\zeta_0}$, where $\zeta_0 = z_0^{3/2}$ denote the parallel shift of $\Sigma$ so that the vertex of $\Sigma$ is shifted to $\zeta_0$. For every $\zeta \in \Sigma_{\zeta_0}$ we define a contour $\Gamma(\zeta)$ as a ray emanating from $\zeta$ and such that: $\Gamma(\zeta) = \{ \tau : \arg(\tau - \zeta) = \frac{3}{2}\beta_0 \}$ if $|\arg \zeta - \frac{3}{2}\beta_0| < \frac{\pi}{2}$; $\Gamma(\zeta) = \{ \tau : \arg(\tau - \zeta) = \arg \zeta + \frac{\pi}{2} \}$ if $\arg \zeta - \frac{3}{2}\beta_0 < -\frac{\pi}{2}$ and; $\Gamma(\zeta) = \{ \tau : \arg(\tau - \zeta) = \arg \zeta - \frac{\pi}{2} \}$ if $\arg \zeta - \frac{3}{2}\beta_0 > \frac{\pi}{2}$. Then the region $S_{z_0}$ and the contour $\gamma(z)$ are the images of $\Sigma_{\zeta_0}$ and $\Gamma(\zeta)$ under the transformation $z = \zeta^{2/3}$.

Let $y_0(z) \equiv 0$, $y_k(z) = I \circ W[y_{k-1}](z)$, $k = 1, 2, \ldots$ and $\delta y_k = y_k - y_{k-1}$. We will show that the solution to (37) is given by

\[
y(z) = \sum_{k=1}^{\infty} \delta y_k ,
\]

where the series converges absolutely and uniformly in $S_{z_0}$ for sufficiently large $|z_0|$. This can be done by introducing the Banach space $B$ of functions $h(z)$, such that $h$ is analytic on $S_{z_0}$ and satisfy $|h(z)| \leq B|z^{-2}|$ there with some constant $B > 0$. (Note that $B$ depends on $h$.) The norm of $h \in B$ is $\sup_{z \in S_{z_0}}|z^{-2}h(z)|$. According to Lemma 14.2 from [Wa], the integral operator $I : B \to B$ is a bounded linear operator, where $\|I\|$ is proportional to $|z_0^{-1/2}|$. We start to study the nonlinear operator $W[y]$ by considering the convolution

\[
\xi(z) * y(z) = \mathcal{L}[\Xi(bp) - bp)Y(p)](z) = \mathcal{L}[\{bp(coth bp - 1) - e^{-2bp}\}Y(p)](z)
\]

\[
+ \mathcal{L}[\{e^{-2bp} - 1\}Y(p)](z) = \mu(z) * y(z) + y(z + 2b) - y(z) ,
\]

where, according to [OB],

\[
\mu(z) = \mathcal{L}[\{bp(coth bp - 1) - e^{-2bp}\} = \frac{1}{2b} \Psi'(1 + \frac{z}{2b}) - \frac{1}{z + 2b} .
\]

Here $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ denotes the logarithmic derivative of the Gamma-function. It is clear that $\mu(z)$ is a meromorphic function with double poles at $z = -2kb$, $k \in \mathbb{Z}^+$. The asymptotic expansion

\[
\Psi'(x) \sim \frac{1}{z} + \frac{1}{2z^2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{z^{2k+1}} , \quad z \to \infty , \quad |\arg z| < \pi ,
\]
where $B_{2k}$ are Bernoulli numbers, follows from the Stirling formula (see [GR]). Combining the latter facts, we obtain the estimate

$$|\mu(z)| \leq \frac{b\Psi_0}{|z + 2b|^2}, \quad z \in S,$$

where the constant $\Psi_0$ depends only on the sector $S$.

**Proposition 3.3.** If $y \in \mathcal{B}$ then $\mu(z) * y(z) \in \mathcal{B}$ and

$$\|\frac{1}{b} \mu * y\| \leq M \|y\|,$$

where $M > 0$ does not depend on $y$ and $z_0$.

**Proof.** Consider first the case $z$ belongs to the right half-plane $\hat{S} = \{z : \Re z \geq mz_0\}$, where the number $m > 0$ is chosen so that $\hat{S} \subset S_{z_0}$. Note that $m$ does not depend on $|z_0|$. Setting $A = mz_0$ in (36), we obtain

$$\mu(z) * y(z) = \frac{1}{2\pi i} \int_{A-i\infty}^{A+i\infty} y(s) \mu(z - s) ds$$

Note that we are integrating over the boundary of $\hat{S}$ and thus $\Re (z - s) \geq 0$. The fact that $\mu(z) * y(z)$ is analytic in $\hat{S}$ follows from the properties of $\mu(z)$ and $y(z)$ immediately. Utilizing (42) and the fact that $y \in \mathcal{B}$, we obtain

$$\|\frac{1}{b} \mu * y\| \leq \Psi_0 \frac{2}{\pi} \int_{A-i\infty}^{A+i\infty} \frac{|ds|}{|s|^2 |z + 2b - s|^2}$$

where $s = A + i\eta$ and $z + 2b = u + iv$ with $\eta, u, v \in \mathbb{R}$.

Let $I$ be the latter integral and $g(\eta)$ denote its integrand, which has simple poles at the points $\eta = iA$ and $\eta = v + i(u - A)$ in the upper half-plane. Computing $I$ via the residues of $g(\eta)$ in the upper half-plane, we obtain

$$I = 2\pi i \sum \text{Res } g(\eta) = \frac{\pi}{v + i(u - 2a)} \left[ \frac{1}{A(v - iu)} + \frac{1}{(u - A)(v + iu)} \right]$$

$$= \frac{\pi}{A|z + 2b|^2} \left( 1 + \frac{A}{u - A} \right)$$
Thus

\[ \frac{1}{b|y|} |\mu(z) * y(z)| \leq \frac{\Psi_0}{2b|z|^2} \]

(47)
since \(A\) can be taken greater than \(2b\).

Consider now the half-plane \(\hat{S}_{\phi}\) that is obtained by rotating \(\hat{S}\) on the angle \(\phi\), where \(\phi\) is chosen in such a way that \(\hat{S}_{\phi} \subset S_{z_0}\). Define now another convolution \(m *_{\phi} y\) by (44), where the contour of integration is the boundary of \(\hat{S}_{\phi}\). Using the same arguments as above, we can obtain the estimate

\[ \frac{1}{b|y|} |\mu(z) *_{\phi} y(z)| \leq M_{\phi} \frac{1}{|z|^2}, \]

(48)
where \(z \in \hat{S}_{\phi}\) and \(M_{\phi}\) continuously depends on \(\phi\). However, \(m(z) *_{\phi} y(z)\) is an analytic continuation of \(m(z) * y(z)\), since the functions coincide on \(\hat{S}_{\phi} \cap \hat{S}\). Taking \(M_{\phi} = \max M_{\phi}\), where \(M_0 = \frac{\Psi_0}{2b}\), we complete the proof of the proposition.

To complete the proof of the theorem we use the standard technique to show the convergence of iterations (38). Using properties of the Laplace transform and of the operator \(I\), one can show that \(y_1 \in B\). Let \(K = \|y_1\|\), and let us prove by induction that \(\|\delta y_n\| \leq 2^{1-n}K\) if \(|z_0|\) is sufficiently large. Indeed, according to the estimate of the convolution,

\[ \|\delta y_n\| \leq \|I\| \left( M + 2 + \frac{4K}{|z|^2} \right) \|\delta y_{n-1}\|, \]

(49)
where the induction assumption

\[ \|y_{n-1}^2 - y_{n-2}^2\| \leq \|y_{n-1} + y_{n_2}\| \frac{|\delta y_{n-1}|}{|z|^2} \leq 4K \frac{|\delta y_{n-1}|}{|z|^2} \]

(50)
was used to estimate the nonlinear term of \(\mathcal{W}\). It remains to choose \(z_0\) so that \(\|I\| \left( M + 2 + \frac{4K}{|z|^2} \right) < \frac{1}{2}\) for \(z \in S_{z_0}\) to complete the proof of the theorem for \(\frac{\pi}{4} < \arg \lambda < \frac{7\pi}{4}\) the sector \(S_0^+\).

Let us now consider the general case \(0 \leq \arg \lambda < 2\pi\). The sector \(S_0^+\) is given by \(-\frac{\pi + 2\arg \lambda}{3} < \arg z < \pi\) if \(\arg \lambda \in [0, \pi]\) and by \(-\pi < \arg z < \frac{\pi - 2\arg \lambda}{3}\) if \(\arg \lambda \in [\pi, 2\pi]\). Note that the opening of the sector \(S_0^+\) is greater than \(\pi\) for any \(\arg \lambda\) and that \(S_0^+\) contains the right half-plane if \(\frac{\pi}{4} < \arg \lambda < \frac{7\pi}{4}\). Let us now choose a proper closed subsector \(S \subset S_0^+\) of opening
greater than \( \pi \), and let \( \arg z = \alpha \) be the bisector of \( S \). Clearly, \( |\alpha| < \frac{\pi}{2} \). Let

\begin{equation}
\mathcal{L}_\alpha[Y](z) = \int_0^{e^{-i\alpha}\infty} e^{-zp}Y(p)dp
\end{equation}

define the Laplace transform along the ray \( \arg p = -\alpha \). The contour for the corresponding inverse Laplace transform as well as for the corresponding convolution \( \ast_\alpha \) is a straight line perpendicular to \( \arg z = \alpha \). Therefore, we can use our previous arguments to show the uniform and absolute convergence of iterations (38) in a properly constructed \( S_{z_0} \subset S \). In the same fashion the theorem can be proven for any nonempty sector \( S_{n}^{\pm}, n \in \mathbb{Z} \).

Recall that the function \( \Xi(bp) \) has poles on the imaginary axis and, therefore, \( \mathcal{L}_\alpha \Xi \) is not defined for \( \alpha = \pm \frac{\pi}{2} \). However, we can define \( \mathcal{L}_\alpha \Xi \) for \( \alpha \) such that \( |\alpha - \pi| < \frac{\pi}{2} \) and repeat the previous arguments for sectors \( S_{n}^{-}, n \in \mathbb{Z} \).

\[ \square \]

**Corollary 3.4.** Let \( f_n^{\pm}, n = 0,1,2 \) denote a solution of (1), analytic in some remote part of \( S_n^\pm \) as described in 3.2. Then, according to 2.1, \( f_n^{\pm} \) is a meromorphic function on \( \mathbb{C} \) that can have only simple poles.

**References**

[GR] I.S Gradstein and I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, N.Y., 1980
[OB] F. Oberhettinger and L. Badii, Tables of Laplace Transforms, Springer-Verlag, N.Y., 1973
[S1] V.P. Spiridonov, (1992) Exactly solvable potentials and quantum algebras, *Phys. Rev. Lett.* 69, 398-401; (1993) Symmetries of the self-similar potentials, Commun. Theor. Phys. (Allahabad) 2, 149-163.
[S2] V.P. Spiridonov, (1995) Coherent states of the \( q \)-Weyl algebra, *Lett. Math. Phys.* 35, 179-185; Universal superpositions of coherent states and self-similar potentials, *Phys. Rev.* A52, 1909-1935; (E) A53, 2903.
[To] A. Tovbis, (1993) Asymptotic solutions of the reduced Kac-van Moerbeke equation, *Phys. Lett.* A180, 215-220.
[Wa] W. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, Dover, N.Y., 1976.