A SPECTRAL STUDY OF THE SECOND-ORDER EXCEPTIONAL $X_1$-JACOBI DIFFERENTIAL EXPRESSION AND A RELATED NON-CLASSICAL JACOBI DIFFERENTIAL EXPRESSION

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Abstract. The exceptional $X_1$-Jacobi differential expression is a second-order ordinary differential expression with rational coefficients; it was discovered by Gómez-Ullate, Kamran and Milson in 2009. In their work, they showed that there is a sequence of polynomial eigenfunctions $\{\hat{P}_n^{(\alpha,\beta)}\}_{n=1}^\infty$ called the exceptional $X_1$-Jacobi polynomials. There is no exceptional $X_1$-Jacobi polynomial of degree zero. These polynomials form a complete orthogonal set in the weighted Hilbert space $L^2((-1,1);\hat{w}_{\alpha,\beta})$, where $\hat{w}_{\alpha,\beta}$ is a positive rational weight function related to the classical Jacobi weight. Among other conditions placed on the parameters $\alpha$ and $\beta$, it is required that $\alpha, \beta > 0$. In this paper, we develop the spectral theory of this expression in $L^2((-1,1);\hat{w}_{\alpha,\beta})$. We also consider the spectral analysis of the ‘extreme’ non-exceptional case, namely when $\alpha = 0$. In this case, the polynomial solutions are the non-classical Jacobi polynomials $\{P_n^{(-2,\beta)}\}_{n=2}^\infty$. We study the corresponding Jacobi differential expression in several Hilbert spaces, including their natural $L^2$ setting and a certain Sobolev space $S$ where the full sequence $\{P_n^{(-2,\beta)}\}_{n=0}^\infty$ is studied and a careful spectral analysis of the Jacobi expression is carried out.

1. Introduction

In 2009, Gómez-Ullate, Kamran, and Milson [11] (see also [12] [13] [14] [15] [16] [17]) characterized all polynomial sequences $\{p_n\}_{n=1}^\infty$, with $\deg p_n = n \geq 1$, which satisfy the following conditions:

(i) there exists a second-order differential expression

$$\ell[y](x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x),$$

and a sequence of complex numbers $\{\lambda_n\}_{n=1}^\infty$ such that $y = p_n(x)$ is a solution of

$$\ell[y](x) = \lambda_n y(x) \quad (n \in \mathbb{N});$$

each coefficient $a_i(x)$, $i = 0, 1, 2$, is a function of the independent variable $x$ and does not depend on the degree of the polynomial eigenfunctions;

(ii) if $C$ is any non-zero constant, $y(x) \equiv C$ is not a solution of $\ell[y](x) = \lambda y(x)$ for any $\lambda \in \mathbb{C}$;

(iii) there exists an open interval $I$ and a positive Lebesgue measurable function $w(x)$ ($x \in I$) such that

$$\int_I p_n(x)p_m w(x) dx = K_n \delta_{n,m},$$

where $K_n > 0$ for each $n \in \mathbb{N}$ and $\delta_{n,m}$ is the standard Kronecker delta symbol; that is to say, $\{p_n\}_{n=1}^\infty$ is orthogonal with respect to $w$ on the interval $I$;

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(iv) all moments \( \{\mu_n\}_{n=0}^\infty \) of \( w \), defined by

\[
\mu_n = \int_I x^n w(x) dx \quad (n = 0, 1, 2, \ldots),
\]

exist and are finite.

Up to a complex linear change of variable, the authors in [11] show that the only solutions to this classification problem are the exceptional \( X_1 \)-Laguerre and \( X_1 \)-Jacobi polynomials. Their results are spectacular and remarkable; indeed, it was believed, due to the ‘Bochner’ classification (see [4], [23] and [28]), that among the class of all orthogonal polynomials, only the Hermite, Laguerre, and Jacobi polynomials, satisfy second-order differential equations and are orthogonal with respect to a positive-definite inner product of the form

\[
(p, q) = \int_\mathbb{R} p(x)q(x)W(x)dx.
\]

We remark that two excellent texts dealing with the subject of orthogonal polynomials are the classical texts of [5] and [29].

Even though the authors in [11] introduce the notion of exceptional polynomials via Sturm-Liouville theory, the path that they followed to their discovery was motivated by their interest in quantum mechanics, specifically with their intent to extend exactly solvable and quasi-exactly solvable potentials beyond the Lie algebraic setting. It is important to note as well that the work in [11] was not originally motivated by orthogonal polynomials although they set out to construct potentials that would be solvable by polynomials which fall outside the realm of the classical theory of orthogonal polynomials. To further note, their work was inspired by the paper of Post and Turbiner [26] who formulated a generalized Bochner problem of classifying the linear differential operators in one variable leaving invariant a given vector space of polynomials.

The \( X_1 \)-Laguerre and \( X_1 \)-Jacobi polynomials, as well as subsequent generalizations, are exceptional in the sense that they start at degree \( \ell \) (\( \ell \geq 1 \)) instead of degree 0, thus avoiding the restrictions of the Bochner classification, but still satisfy second-order differential equations of spectral type. Reformulation within the framework of one-dimensional quantum mechanics and shape invariant potentials is considered by various other authors; for example, see [23] and [27]. Furthermore, the two second-order differential equations that they discover in their \( X_1 \) classification are important examples illustrating the Stone-von Neumann theory [7, Chapter 12] and the Glazman-Krein-Naimark theory (see [1] and [24, Section 18]) of differential operators.

In this paper, we study the exceptional \( X_1 \)-Jacobi expression for all possible parameter choices in various Hilbert spaces. We also consider this expression, the corresponding orthogonal polynomials and the self-adjoint theory for the extreme choice of parameters \( \alpha = 0 \) or \( \beta = 0 \). The corresponding operators and their spectral analysis are not captured by the generalized Bochner classification and we apply a multitude of techniques to accomplish our goals.

The contents of this paper are as follows. In Section 2 we introduce the exceptional \( X_1 \)-Jacobi polynomials and differential expression and briefly review properties of these polynomials. Section 3 deals with standard properties of the exceptional \( X_1 \)-Jacobi differential expression \( \hat{T}_{\alpha,\beta}[] \) in its natural setting \( L^2((-1,1);\hat{w}_{\alpha,\beta}) \), where \( \hat{w}_{\alpha,\beta} \) is the orthogonalizing weight function for the exceptional \( X_1 \)-Jacobi polynomials. This leads to the construction, in Section 4 of a certain self-adjoint operator \( \hat{T}_{\alpha,\beta} \), generated by \( \hat{T}_{\alpha,\beta}[] \), in \( L^2((-1,1);\hat{w}_{\alpha,\beta}) \) (see Theorem 4.1). In Section 5 we begin our analysis of the ‘extreme’ case \( \alpha = 0 \). This choice gets us closer to the realm of classical orthogonal polynomials; indeed the weight function in this case simplifies to \( w_{-2,\beta}(x) = (1-x)^{-2}(1+x)^\beta \), which is the weight function for the non-classical Jacobi polynomials \( \{P_n^{-2,\beta}\} \). Various important
facts about the associated Jacobi differential expression, which we denote by \( m_{-2,\beta}[] \), are discussed in Section 6. These properties are used in Section 7 to construct the self-adjoint operator \( T_{-2,\beta} \), generated by \( m_{-2,\beta}[] \), having the Jacobi polynomials \( \{ P_{n}^{(-2,\beta)} \}_{n=2}^{\infty} \) as eigenfunctions; see Theorem 7.1. We remark that it is not possible for the Jacobi polynomials \( P_{n}^{(-2,\beta)} \) of degrees 0 and 1 to belong to \( L^{2}((-1,1);w_{-2,\beta}) \). Also, in Section 7, we show (Theorem 7.5) that \( T_{-2,\beta} \) is bounded below by the identity operator \( I \) in \( L^{2}((-1,1);w_{-2,\beta}) \). This result will be critical for our analysis in the last two sections of the paper. Section 8 gives a short description of abstract left-definite theory, a subject that is instrumental in the last two sections. Kwon and Littlejohn [21] discovered a Sobolev inner product in which the entire Jacobi sequence \( \{ P_{n}^{(-2,\beta)} \}_{n=0}^{\infty} \) is orthogonal but, for reasons that will be made clearer later, we must require \( \beta \neq 0 \). This inner product and properties of the corresponding Sobolev space \( S \) are discussed in Section 9. Lastly, in Section 10, we construct (Theorem 10.5) a self-adjoint operator \( T \), generated by the differential expression \( m_{-2,\beta}[] \), having the Jacobi polynomials \( \{ P_{n}^{(-2,\beta)} \}_{n=0}^{\infty} \) as eigenfunctions. This construction, essentially, uses all of the results proven in the previous sections.

2. The Exceptional \( X_{1} \)-Jacobi Polynomials

The exceptional \( X_{1} \)-Jacobi differential expression is defined to be

\[
\hat{\ell}_{\alpha,\beta}[y](x) := (x^{2} - 1)y''(x) + 2a \left( \frac{1 - bx}{b - x} \right) (x - c)y'(x) - y(x) \quad (x \in (-1,1)),
\]

where

\[
\alpha, \beta \in (-1, \infty), \quad \alpha \neq \beta, \quad \text{and} \quad \text{sgn}(\alpha) = \text{sgn}(\beta),
\]

and

\[
a := \frac{1}{2}(\beta - \alpha), \quad b := \frac{\beta + \alpha}{\beta - \alpha}, \quad c := b + \frac{1}{a} = \frac{\beta + \alpha + 2}{\beta - \alpha}.
\]

Notice that, from (2.2), that it is not possible for \( \alpha = 0 \) or \( \beta = 0 \). Later, in Section 5 and onwards, we do allow for \( \alpha = 0 \) or \( \beta = 0 \).

Observe that the conditions in (2.2) imply that \( |b| > 1 \). Indeed suppose, to the contrary, that \( |b| \leq 1 \); that is to say,

\[-1 \leq \frac{\beta + \alpha}{\beta - \alpha} \leq 1.
\]

If \( \alpha > \beta \), we see that the above inequality yields \( -\beta + \alpha \geq \beta + \alpha \geq \beta - \alpha \), which in turn implies \( \beta \leq 0 \) and \( \alpha \geq 0 \). Since the case \( \alpha = 0 \) or \( \beta = 0 \) is not possible, we see that \( \text{sgn}(\beta) = -\text{sgn}(\alpha) \), contradicting (2.2). The case \( \alpha < \beta \) can be dealt with similarly.

The exceptional \( X_{1} \)-Jacobi polynomials \( \{ \tilde{P}_{n}^{(\alpha,\beta)} \}_{n=1}^{\infty} \) are eigenfunctions of \( \hat{\ell}_{\alpha,\beta}[] \); specifically

\[
\hat{\ell}_{\alpha,\beta}[\tilde{P}_{n}^{(\alpha,\beta)}](x) = (n - 1)(\alpha + \beta + n)\tilde{P}_{n}^{(\alpha,\beta)}(x) \quad (n \in \mathbb{N}).
\]

Moreover, they show that \( \{ \tilde{P}_{n}^{(\alpha,\beta)} \}_{n=1}^{\infty} \) forms a complete orthogonal set in the Hilbert space

\[
L^{2}((-1,1);w_{\alpha,\beta}) := \left\{ f : (-1,1) \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \| f \|_{\tilde{w}_{\alpha,\beta}} < \infty \right\},
\]
with norm and inner product defined, respectively, by
\begin{equation}
\|f\|_{\hat{w}_{\alpha,\beta}} := \left( \int_{-1}^{1} |f(x)|^2 \hat{w}_{\alpha,\beta}(x) \, dx \right)^{1/2} \quad (f \in L^2((-1,1); \hat{w}_{\alpha,\beta}))
\end{equation}
and
\begin{equation}
(f,g)_{\hat{w}_{\alpha,\beta}} := \int_{-1}^{1} f(x) g(x) \hat{w}_{\alpha,\beta}(x) \, dx \quad (f, g \in L^2((-1,1); \hat{w}_{\alpha,\beta})),
\end{equation}
where
\begin{equation}
\hat{w}_{\alpha,\beta}(x) := \frac{(1-x)^\alpha(1+x)^\beta}{(x-b)^2} \quad (x \in (-1,1)).
\end{equation}

Since $|b| > 1$, the term $(x-b)^{-2}$ in the weight function $\hat{w}_{\alpha,\beta}$ is bounded on $[-1,1]$; consequently the moments of $\hat{w}_{\alpha,\beta}$ all exist and are finite for all $\alpha$ and $\beta$ satisfying the conditions in (2.2).

**Remark 2.1.** The term $x-b$ that appears in the denominator of both (2.1) and (2.2) is a multiple of the degree one Jacobi polynomial $P_1(-\alpha-\beta+1)(x)$; in fact
\[ x-b = \frac{2}{\beta-\alpha} P_1(-\alpha-\beta+1)(x). \]

In [17] and [25], the authors study more general exceptional $X_m$-Jacobi polynomials; these polynomials are orthogonal with respect to the weight function
\[ \hat{w}_{\alpha,\beta,m}(x) = \frac{(x-1)^\alpha(1+x)^\beta}{(P_n^{(-\alpha-\beta+1)}(x))^2}. \]

Notice that, when $m = 1$, this weight reduces, essentially, to (2.2).

These exceptional $X_1$-Jacobi polynomials are explicitly given by
\begin{equation}
\hat{P}_n^{(\alpha,\beta)}(x) = -\frac{1}{2} (x-b) P_{n-1}^{(\alpha,\beta)}(x) + \frac{bP_n^{(\alpha,\beta)}(x) - P_{n-2}^{(\alpha,\beta)}(x)}{\alpha + \beta + 2n - 2} \quad (n \in \mathbb{N}; P_{-1}^{(\alpha,\beta)}(x) = 0),
\end{equation}
where $\left\{ P_n^{(\alpha,\beta)} \right\}_{n=1}^\infty$ are the classical Jacobi polynomials, defined by
\begin{equation}
P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k}.
\end{equation}

For the sake of completeness, we list a few of these exceptional $X_1$-Jacobi polynomials:
\begin{align*}
P_1^{(\alpha,\beta)}(x) &= -\frac{1}{2} x - \frac{\alpha + \beta + 2}{2(\alpha - \beta)} \times \\
P_2^{(\alpha,\beta)}(x) &= -\frac{\alpha + \beta + 2}{4} x^2 - \frac{\alpha(\alpha + 2) + \beta(\beta + 2)}{2(\alpha - \beta)} x - \frac{\alpha + \beta + 2}{4} \times \\
P_3^{(\alpha,\beta)}(x) &= -\frac{(\alpha + \beta + 3)(\alpha + \beta + 4)}{16} x^3 - \frac{(\alpha + \beta + 3)(3\alpha^2 + 6\alpha - 2\alpha\beta + 3\beta^2 + 6\beta)}{16(\alpha - \beta)} x^2 \\
\ &\quad - \frac{(3\alpha^2 + 9\alpha + 2\alpha\beta + 3\beta^2 + 9\beta)}{16} x \\
\ &\quad - \frac{(\alpha^2 + \alpha^2 - 6\alpha - \alpha^2\beta - 6\alpha\beta - \alpha^2\beta - \alpha\beta^2 + \beta^3 + \beta^2)}{16(\alpha - \beta)}.
\end{align*}
The norms of these polynomials are explicitly given by
\[ \left\| \hat{P}_n^{(\alpha,\beta)} \right\|_{\tilde{w}_{\alpha,\beta}}^2 = \left( \frac{2^{\alpha+\beta+1}(\alpha+n)(\beta+n)}{4(\alpha+n+1)(\beta+n-1)(\alpha+\beta+2n-1)} \right) \frac{(\Gamma(\alpha+n)\Gamma(\beta+n))}{\Gamma(n)\Gamma(\alpha+\beta+n)} \quad (n \in \mathbb{N}). \]

In [18], the authors establish the location and asymptotic behavior of the roots of the exceptional $X_1$-Jacobi polynomials. Indeed, they show that there are $n - 1$ simple roots of $\hat{P}_n^{(\alpha,\beta)}(x)$ ($n \in \mathbb{N}_0$) lying in the interval $(-1, 1)$ and there is exactly one negative root. Asymptotically, as $n \to \infty$, the $n - 1$ roots of $\hat{P}_n^{(\alpha,\beta)}(x)$ in $(-1, 1)$ converge to the roots of the classical Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ while the negative root of $\hat{P}_n^{(\alpha,\beta)}(x)$ converges to the root of $P_1^{(-\alpha-1,\beta-1)}(x)$; for further details, see [18] Proposition 5.3, Proposition 5.4 and Corollary 5.1, page 493.

3. Properties of the Exceptional $X_1$-Jacobi Differential Expression

Some properties of this expression were developed by Everitt in [8]; we reproduce some of his results in this section. Both endpoints $x = \pm 1$ are regular singular endpoints, in the sense of Frobenius, of the exceptional $X_1$-Jacobi differential expression $\hat{\ell}_{\alpha,\beta}[\cdot]$. The Frobenius indicial equation at $x = 1$ is $r(r + \alpha) = 0$. Therefore, two linearly independent solutions of $\hat{\ell}_{\alpha,\beta}[y] = 0$ behave asymptotically near $x = 1$ like
\[ z_1(x) = 1 \quad \text{and} \quad z_2(x) = (x - 1)^{-\alpha}. \]
For all feasible values of $\alpha$ and $\beta$, we have
\[ \int_0^1 |z_1(x)|^2 \tilde{w}_{\alpha,\beta}(x) dx < \infty. \]
However,
\[ \int_0^1 |z_2(x)|^2 \tilde{w}_{\alpha,\beta}(x) dx < \infty, \]
only when $-1 < \alpha < 1$. Consequently, at $x = 1$, the expression $\hat{\ell}_{\alpha,\beta}[\cdot]$ is limit-point for $\alpha \geq 1$ and limit-circle when $-1 < \alpha < 1$. The analysis at $x = -1$ is similar, in this case, $\hat{\ell}_{\alpha,\beta}[\cdot]$ is limit-point for $\beta \geq 1$ and limit-circle in the case $-1 < \beta < 1$.

In Lagrangian symmetric form, the $X_1$-Jacobi differential expression (2.1) is given by
\[ (3.1) \quad \hat{\ell}_{\alpha,\beta}[y](x) = \frac{1}{\tilde{w}_{\alpha,\beta}(x)} \left[ - \left( \frac{(1 - x)^{\alpha+1}(1 + x)^{\beta+1}}{(x - b)^2} y'(x) \right)' + \frac{2\alpha(\alpha - c)(bx - 1)(1 - x)^{\alpha}(1 + x)^{\beta}}{(x - b)^3} y(x) \right]. \]

The maximal domain associated with $\hat{\ell}_{\alpha,\beta}[\cdot]$ in the Hilbert space $L^2((-1, 1); \tilde{w}_{\alpha,\beta})$ is
\[ (3.2) \quad \hat{\Delta} := \left\{ f : (0, \infty) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1, 1); f, \hat{\ell}_{\alpha,\beta}[f] \in L^2((-1, 1); \tilde{w}_{\alpha,\beta}) \right\}. \]
The associated maximal operator
\[ \hat{\ell}_{\text{max}} : \mathcal{D}(\hat{\ell}_{\text{max}}) \subset L^2((-1, 1); \tilde{w}_{\alpha,\beta}) \to L^2((-1, 1); \tilde{w}_{\alpha,\beta}), \]
is defined by
\begin{equation}
\hat{T}_{\text{max}}f = \hat{\ell}_{\alpha,\beta}[f] \quad f \in \mathcal{D}(\hat{T}_{\text{max}}) := \hat{\Delta}.
\end{equation}

For \(f, g \in \hat{\Delta}\), Green’s formula can be written as
\begin{equation}
\int_{-1}^{1} \hat{\ell}_{\alpha,\beta}[f](x)\overline{\hat{g}}(x)\hat{w}_{\alpha,\beta}(x)dx = [f, g]_{\hat{w}_{\alpha,\beta}}(x) |_{x=-1}^{x=1} + \int_{-1}^{1} f(x)\hat{\ell}_{\alpha,\beta}[\overline{g}](x)\hat{w}_{\alpha,\beta}(x)dx,
\end{equation}
where \([\cdot, \cdot]\) is the sesquilinear form defined by
\begin{equation}
[f, g]_{\hat{w}_{\alpha,\beta}}(x) := \frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{(x-b)^2} (f(x)\overline{g}'(x) - f'(x)\overline{g}(x)) \quad (-1 < x < 1),
\end{equation}
and
\begin{equation}
[f, g]_{\hat{w}_{\alpha,\beta}}(x) |_{x=-1}^{x=1} := [f, g]_{\hat{w}_{\alpha,\beta}}(1) - [f, g]_{\hat{w}_{\alpha,\beta}}(-1).
\end{equation}
By definition of \(\hat{\Delta}\), and the classical Hölder’s inequality, notice that the limits
\begin{equation}
[f, g]_{\hat{w}_{\alpha,\beta}}(-1) := \lim_{x \to -1^{+}} [f, g]_{\hat{w}_{\alpha,\beta}}(x) \quad \text{and} \quad [f, g]_{\hat{w}_{\alpha,\beta}}(1) := \lim_{x \to 1^{-}} [f, g]_{\hat{w}_{\alpha,\beta}}(x)
\end{equation}
both exist and are finite for each \(f, g \in \hat{\Delta}\).

By standard classical arguments, the maximal domain \(\hat{\Delta}\) is dense in \(L^2((-1,1); \hat{w}_{\alpha,\beta})\); consequently, the adjoint of \(\hat{T}_{\text{max}}\) exists as a densely defined operator in \(L^2((-1,1); \hat{w}_{\alpha,\beta})\). For obvious reasons, the adjoint of \(\hat{T}_{\text{max}}\) is called the minimal operator associated with \(\hat{\ell}_{\alpha,\beta}[-]\) and is denoted by \(\hat{T}_{\text{min}}\). From [11] or [24], this minimal operator \(\hat{T}_{\text{min}} : \mathcal{D}(\hat{T}_{\text{min}}) \subset L^2((-1,1); \hat{w}_{\alpha,\beta}) \to L^2((-1,1); \hat{w}_{\alpha,\beta})\) is defined by
\begin{equation}
\hat{T}_{\text{min}}f = \hat{\ell}_{\alpha,\beta}[f] \quad f \in \mathcal{D}(\hat{T}_{\text{min}}) := \{ f \in \hat{\Delta} \mid [f, g]_{\hat{w}_{\alpha,\beta}} |_{x=-1}^{x=1} = 0 \text{ for all } g \in \hat{\Delta} \}.
\end{equation}

The minimal operator \(\hat{T}_{\text{min}}\) is a closed, symmetric operator in \(L^2((-1,1); \hat{w}_{\alpha,\beta})\); furthermore, because the coefficients of \(\hat{\ell}_{\alpha,\beta}[-]\) are real, \(\hat{T}_{\text{min}}\) necessarily has equal deficiency indices \(m\), where \(m\) is an integer satisfying \(0 \leq m \leq 2\). Therefore, from the general Stone-von Neumann [7] theory of self-adjoint extensions of symmetric operators, \(\hat{T}_{\text{min}}\) has self-adjoint extensions. We seek to find the self-adjoint extension \(\hat{T}\) in \(L^2((-1,1); \hat{w}_{\alpha,\beta})\), generated by \(\hat{\ell}_{\alpha,\beta}[-]\), which has the \(X_1\)-Jacobi polynomials \(\{ \hat{P}_n^{(\alpha,\beta)} \}_n \) as eigenfunctions. From the Frobenius analysis discussed at the beginning of this section, the following Proposition follows immediately.

**Proposition 3.1.** Consider the minimal operator \(\hat{T}_{\text{min}}\) in \(L^2((-1,1); \hat{w}_{\alpha,\beta})\), as defined in (3.6), generated by the exceptional \(X_1\)-Jacobi differential expression \(\hat{\ell}_{\alpha,\beta}[-]\).

(a) For \(\alpha, \beta \geq 1\), the minimal operator \(\hat{T}_{\text{min}}\) has deficiency index \((0,0)\).
(b) For \(\alpha \geq 1\), and \(\beta < 1\), the minimal operator \(\hat{T}_{\text{min}}\) has deficiency index \((1,1)\). The same is true for \(\alpha < 1\) and \(\beta \geq 1\).
(c) For \(\alpha, \beta < 1\), the minimal operator \(\hat{T}_{\text{min}}\) has deficiency index \((2,2)\).
4. A CERTAIN EXCEPTIONAL $X_1$-JACOBI SELF-ADJOINT OPERATOR

Proposition 3.1 puts us in a position to define the self-adjoint operator $\hat{T}_{\alpha,\beta}$ in $L^2((-1,1);\hat{\omega}_{\alpha,\beta})$ having the exceptional $X_1$-Jacobi polynomials $\{\hat{P}_n^{(\alpha,\beta)}\}_{n=1}^\infty$ as eigenfunctions; this operator is found by a direct application of the so-called Glazman-Krein-Naimark theory (see [1] and [24]). The one boundary function, when needed, that we choose to generate the appropriate boundary condition is $g(x) = 1$. When we substitute this function into the sesquilinear form (3.5) associated with $\hat{\ell}_{\alpha,\beta}[\cdot]$, we see that

$$[f,1]_{\hat{\omega}_{\alpha,\beta}}(x) = -\frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{(x-b)^2}f'(x);$$

moreover, notice that the boundary condition $\lim_{x\to1^-}[f,1]_{\hat{\omega}_{\alpha,\beta}}(x) = 0$ simplifies to

$$\lim_{x\to1^-}(1-x)^{\alpha+1}f'(x) = 0.$$

An analogous argument works for $x \to -1^+$. We are now ready to state the following theorem.

**Theorem 4.1.** The self-adjoint operator $\hat{T}_{\alpha,\beta}$ in $L^2((-1,1);\hat{\omega}_{\alpha,\beta})$, generated by the exceptional $X_1$-Jacobi differential expression $\hat{\ell}_{\alpha,\beta}[\cdot]$, having the exceptional $X_1$-Jacobi polynomials $\{\hat{P}_n^{(\alpha,\beta)}\}_{n=1}^\infty$ as eigenfunctions is explicitly given by

$$\hat{T}_{\alpha,\beta}f = \hat{\ell}_{\alpha,\beta}[f]$$

where

$$f \in D(\hat{T}_{\alpha,\beta}),$$

in $L^2((-1,1);\hat{\omega}_{\alpha,\beta})$. Furthermore the spectrum $\sigma(\hat{T}_{\alpha,\beta})$ of $\hat{T}_{\alpha,\beta}$ is pure discrete spectrum consisting of the simple eigenvalues

$$\sigma(\hat{T}_{\alpha,\beta}) = \sigma_p(\hat{T}_{\alpha,\beta}) = \{(n-1)(\alpha + \beta + n) | n \in \mathbb{N}\}.$$

5. THE ‘EXTREME’ CASE $\alpha = 0$ AND $\beta > -1$: NON-CLASSICAL JACOBI POLYNOMIALS

We now study the situation when $\alpha = 0$ and $\beta > -1$ in the exceptional $X_1$-Jacobi case; the reader will recall that this situation was not allowed in our earlier analysis from the conditions given in (2.2). There is the analogous case $\beta = 0$ and $\alpha > -1$ which we will not address in this paper. We remark that there do not appear to be any interesting extreme cases for exceptional $X_m$-Jacobi or $X_m$-Laguerre polynomials when $m > 1$. There is an interesting extreme case for the exceptional $X_1$-Laguerre polynomials. This was reported on, albeit in incomplete details, in [3].

When $\alpha = 0$ and $\beta > -1$, we see from (2.2) that

$$a = \beta/2, \quad b = 1, \quad c = (\beta + 2)/\beta.$$
With these choices, we note that the differential expression \((2.1)\) becomes
\[
\tilde{\ell}_{0,\beta}[y](x) = (x^2 - 1)y''(x) + (\beta x - \beta - 2)y'(x) - \beta y(x) \quad (x \in (-1, 1)).
\]
For reasons that will be made clearer later, we perturb the coefficient of \(y\) (by adding \((1 + \beta)y(x)\)) and we will instead study the Jacobi expression
\[
m_{-2,\beta}[y](x) := (x^2 - 1)y''(x) + (\beta x - \beta - 2)y'(x) + y(x) \quad (x \in (-1, 1)).
\]
Indeed, adding this term will affect only the spectrum but not the eigenfunctions. The weight function \((2.6)\) in this case becomes
\[
w_{-2,\beta}(x) := (1 - x)^{-2}(1 + x)^\beta \quad (x \in (-1, 1)).
\]
This differential expression and weight are precisely the Jacobi differential expression and Jacobi weight for the non-classical Jacobi case \((\alpha, \beta) = (-2, \beta)\).

Even though this is a non-classical Jacobi case, the differential equation
\[
m_{-2,\beta}[y](x) = \lambda_n y
\]
do not have a polynomial solution \(y = P_n^{(-2,\beta)}(x)\) of degree \(n\) for each \(n \in \mathbb{N}_0\). If fact,
\[
P_n^{(-2,\beta)}(x) = \begin{cases} 
1 & \text{if } n = 0 \\
\frac{\beta x - \beta - 2}{(n + \beta)(n + \beta - 1)} & \text{if } n = 1 \\
\frac{(n + \beta)(n + \beta - 1)}{4n(n - 1)}(1 - x)^2P_{n-2}^{(2,\beta)}(x) & \text{if } n \geq 2,
\end{cases}
\]
where \(\{P_n^{(2,\beta)}\}_{n=0}^\infty\) are the classical Jacobi polynomials defined in \((2.8)\). Moreover
\[
m_{-2,\beta}[P_n^{(-2,\beta)}](x) = \lambda_n P_n^{(-2,\beta)}(x),
\]
where
\[
\lambda_n = n^2 + (\beta - 1)n + 1 \quad (n \in \mathbb{N}_0).
\]

**Remark 5.1.** Letting \(\alpha = 0\) in the explicit representation \((2.4)\) of \(\tilde{P}_n^{(\alpha,\beta)}(x)\), we find that
\[
\tilde{P}_n^{(0,\beta)}(x) = -\frac{1}{2}(x - 1)P_{n-1}^{(0,\beta)}(x) + \frac{P_{n-1}^{(0,\beta)}(x) - P_{n-2}^{(0,\beta)}(x)}{\beta + 2n - 2}.
\]
We omit the details but it can be shown that, for \(n \geq 1\), \(\tilde{P}_n^{(0,\beta)}(x)\) is a multiple of the non-classical Jacobi polynomial \(P_n^{(-2,\beta)}(x)\), defined in \((5.4)\).

**Remark 5.2.** In \((5.4)\), the non-classical Jacobi polynomials \(P_n^{(-2,\beta)}\), for \(n \geq 2\), are expressed in terms of the classical Jacobi polynomials \(P_{n-2}^{(-2,\beta)}\); this is a well-known connection (see [29] Chapter 4, (4.22.2))). These Jacobi polynomials \(\{P_n^{(-2,\beta)}\}_{n=2}^\infty\) satisfy the orthogonality relationship
\[
\int_{-1}^{1} P_n^{(-2,\beta)}(x)P_m^{(-2,\beta)}(x)w_{-2,\beta}(x)dx = \frac{\beta}{n\Gamma(n + \beta - 1)\Gamma(n + \beta + 1)}\delta_{n,m} \quad (n, m \geq 2).
\]

**Remark 5.3.** Beginning in Section 9, we will require that the set \(\{P_n^{(-2,\beta)}\}_{n=0}^\infty\) is algebraically complete; that is, \(\deg(P_n^{(-2,\beta)}) = n\) for each \(n \in \mathbb{N}_0\) so \(\{P_n^{(-2,\beta)}\}_{n=0}^\infty\) is a basis for the space \(\mathcal{P}\) of all real-valued polynomials. From \((5.4)\), in order for \(\deg(P_1^{(-2,\beta)}) = 1\), we need \(\beta \neq 0\). Thus, starting in Section 9, we will additionally assume \(\beta \neq 0\).
Let $L^2((-1, 1); w_{-2, \beta})$ be the Hilbert space defined by
\[ L^2((-1, 1); w_{-2, \beta}) = \{ f : (-1, 1) \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \| f \|_{w_{-2, \beta}} < \infty \}, \]
where the norm is
\[ \| f \|_{w_{-2, \beta}} = \left( \int_{-1}^{1} |f(x)|^2 w_{-2, \beta}(x) dx \right)^{1/2} \quad (f \in L^2((-1, 1); w_{-2, \beta})) \]
and inner product is
\[ (f, g)_{w_{-2, \beta}} = \int_{-1}^{1} f(x) \overline{g(x)} w_{-2, \beta}(x) dx \quad (f, g \in L^2((-1, 1); w_{-2, \beta})). \]

**Theorem 5.1.** The polynomials $P^{(-2, \beta)}_j \notin L^2((-1, 1); w_{-2, \beta})$ for $j = 0, 1$. However, $\{ P^{(-2, \beta)}_n \}_{n=2}^{\infty} \subset L^2((-1, 1); w_{-2, \beta})$; moreover, $\text{span} \{ P^{(-2, \beta)}_n \}_{n=2}^{\infty}$ is a complete orthogonal set in $L^2((-1, 1); w_{-2, \beta})$. The last statement is equivalent to saying
\[ \text{span} \{ p \in \mathcal{P} \mid p \text{ is a polynomial of deg } \geq 2 \text{ with } p(1) = p'(1) = 0 \} \]
is dense in $L^2((-1, 1); w_{-2, \beta})$.

**Proof.** The singular term $(1 - x)^{-2}$ in the weight function $w_{-2, \beta}(x)$ prevents $P^{(-2, \beta)}_j$ (when $\beta \neq 0$) from belonging to $L^2((-1, 1); w_{-2, \beta})$ when $j = 0, 1$. The equivalence of the two statements in this theorem is immediate from [5.4]; we will prove the second statement. Let $\varepsilon > 0$ and $f \in L^2((-1, 1); w_{2, \beta})$. Note that
\[ \int_{-1}^{1} |f(x)|^2 (1 - x)^{-2}(1 + x)\beta dx = \int_{-1}^{1} \left| \frac{f(x)}{(1 - x)^2} \right|^2 (1 - x)^{2}(1 + x)^{\beta} dx; \]
by letting $w_{2, \beta}(x) = (1 - x)^2(1 + x)^{\beta}$, we see that
\[ f \in L^2((-1, 1); w_{-2, \beta}) \text{ if and only if } \frac{f}{(1 - x)^2} \in L^2((-1, 1); w_{2, \beta}). \]
Since polynomials are dense in $L^2((-1, 1); w_{2, \beta})$, there exists $p \in \mathcal{P}$ such that
\[ \varepsilon^2 > \int_{-1}^{1} \left| \frac{f(x)}{(1 - x)^2} - p(x) \right|^2 w_{2, \beta}(x) dx. \]
Define $q(x) = p(x)(1 - x)^2$ so $q \in \mathcal{P}$ and $q(1) = q'(1) = 0$. Moreover,
\[ \| f - q \|_{w_{-2, \beta}}^2 = \int_{-1}^{1} \left| \frac{f(x)}{(1 - x)^2} - p(x) \right|^2 w_{2, \beta}(x) dx < \varepsilon^2, \]
proving the desired result. \( \square \)

At this point, we remark that Littlejohn and Kwon [21] showed that the entire sequence of non-classical Jacobi polynomials $\{ P^{(-2, \beta)}_n \}_{n=0}^{\infty}$ are orthogonal with respect to the Sobolev inner product
\[ (5.6) \quad \phi(f, g) := f(1)\overline{g}(1) + \frac{2}{\beta} (f'(1)\overline{g}(1) + f(1)\overline{g}'(1)) + \left(1 + \frac{4}{\beta^2}\right) f'(1)\overline{g}'(1) + \int_{-1}^{1} f''(x)\overline{g}''(x)(1 + x)^{\beta+2} dx. \]
Since
\[ \phi(f, g) = \left( f(1) + \frac{2}{\beta} f'(1) \right) \left( g(1) + \frac{2}{\beta} g'(1) \right) + f'(1)g'(1) + \int_{-1}^{1} p''(x)g''(x)(1 + x)^{\beta+2}dx, \]
it is clear that \( \phi(\cdot, \cdot) \) is an inner product. Also, it is a straightforward exercise to show that \( \{ P_n^{(-2, \beta)} \}_{n=0}^\infty \) is orthogonal with respect to \( \phi(\cdot, \cdot) \). Later in this paper, we do a further study of these Jacobi polynomials under this inner product. In particular, we will identify the appropriate Sobolev space \( S \) in which \( \{ P_n^{(-2, \beta)} \}_{n=0}^\infty \) is a complete orthogonal set. Moreover, we also construct a self-adjoint operator \( T \), generated by \( m_{-2, \beta}[\cdot] \), in \( S \) that has \( \{ P_n^{(-2, \beta)} \}_{n=0}^\infty \) as eigenfunctions. This operator \( T \) is partially constructed from the self-adjoint operator \( T^{\min} \), which we now discuss, in \( L^2((-1, 1); w_{-2, \beta}) \) having the Jacobi polynomials \( \{ P_n^{(-2, \beta)} \}_{n=2}^\infty \) as eigenfunctions.

6. Properties of the Non-classical Jacobi Differential Expression \( m_{-2, \beta}[\cdot] \)

We now focus our attention to the study of \( m_{-2, \beta}[\cdot] \), defined in (5.2), in the Hilbert space \( L^2((-1, 1); w_{-2, \beta}) \) which is the natural ‘right-definite’ setting for an analytic study.

The Lagrangian symmetric form of \( m_{-2, \beta}[\cdot] \) is given by
\[ m_{-2, \beta}[y](x) = \frac{1}{w_{-2, \beta}(x)} \left( -((1 - x)^{-1}(1 + x)^{\beta+1}y'(x))' + w_{-2, \beta}(x)y(x) \right). \]

In this case, the maximal domain of \( m_{-2, \beta}[\cdot] \) in \( L^2((-1, 1); w_{-2, \beta}) \) is given by
\[ \Delta := \{ f : (-1, 1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}((-1, 1); f, m_{-2, \beta}[f] \in L^2((-1, 1); w_{-2, \beta}) \}. \]

For \( f, g \in \Delta \), Green’s formula is
\[ \int_{-1}^{1} m_{-2, \beta}[f](x)g(x)w_{-2, \beta}(x)dx - \int_{-1}^{1} m_{-2, \beta}[g](x)f(x)w_{-2, \beta}(x)dx = [f, g]_{w_{-2, \beta}} |_{x=1}^{x=-1}, \]
where \([\cdot, \cdot] \) is the sesquilinear form defined by
\[ [f, g]_{w_{-2, \beta}}(x) := (1 - x)^{-1}(1 + x)^{\beta+1}(f(x)g'(x) - f'(x)g(x)) \quad (x \in (-1, 1); f, g \in \Delta) \]
and
\[ [f, g]_{w_{-2, \beta}}(\pm 1) = \lim_{x \to \pm 1^\pm}[f, g]_{w_{-2, \beta}}(x) \quad (f, g \in \Delta). \]
Moreover, for \( f, g \in \Delta \) and \(-1 < x, y < 1 \), Dirichlet’s formula reads
\[ \int_{x}^{y} m_{-2, \beta}[f](t)g(t)w_{-2, \beta}(t)dt = (1 - t)^{-1}(1 + t)^{\beta+1}f'(t)g(t) |_{x}^{y} \]
\[ = \int_{x}^{y} f'(t)g'(t)(1 - t)^{-1}(1 + t)^{\beta+1}dt + \int_{x}^{y} f(t)g(t)(1 - t)^{-2}(1 + t)^{\beta}dt. \]

The maximal operator \( T^{\max} \) in \( L^2((-1, 1); w_{-2, \beta}) \), associated with \( m_{-2, \beta}[\cdot] \), is defined as
\[ T^{\max}[f] = m_{-2, \beta}[f] \quad f \in D(T^{\max}) := \Delta \]
and the minimal operator \( T^{\min} \), the adjoint of \( T^{\max} \), is given by
\[ T^{\min}[g] = m_{-2, \beta}[g] \quad f \in D(T^{\min}) := \{ g \in \Delta \mid [g, f]_{w_{-2, \beta}} |_{x=1}^{x=-1} = 0 \text{ for all } f \in \Delta \}. \]
The endpoints \( x = \pm 1 \) are regular singular endpoints of \( m_{-2,\beta}[\cdot] \) in the sense of Frobenius. Elementary calculations show that the Frobenius indicial equations at, respectively, \( x = 1 \) and \( x = -1 \) are \( r(r-2) = 0 \) and \( r(r+\beta) = 0 \). It follows that \( m_{-2,\beta}[\cdot] \) is in the limit-point case at \( x = 1 \) while \( m_{-2,\beta}[\cdot] \) is in the limit-circle case at \( x = -1 \) when \(-1 < \beta < 1\) and in the limit-point case at \( x = -1 \) when \( \beta \geq 1 \). Applying the Glazman-Krein-Naimark theory, all self-adjoint operators \( S \) in \( L^2((-1,1);w_{-2,\beta}) \), generated by \( m_{-2,\beta}[\cdot] \), have the form

\[
S[f] = m_{-2,\beta}[f]
\]

for \( f \in \mathcal{D}(S) \) where

\[
f \in \mathcal{D}(S) := \begin{cases} 
\Delta & \text{if } \beta \geq 1 \\
\{ f \in \Delta \mid \lim_{x \to -1^+} [f,g_S](x) = 0 \} & \text{if } -1 < \beta < 1
\end{cases}
\]

and where \( g_S \in \Delta \setminus \mathcal{D}(T_{\text{min}}) \) (such a \( g_S \) is called a Glazman boundary function).

7. A Certain Self-Adjoint Operator Generated by \( m_{-2,\beta}[\cdot] \)

We are interested in the particular self-adjoint operator which has the Jacobi polynomials \( \{P_n^{(-2,\beta)}\}_{n=2}^{\infty} \) as eigenfunctions and has spectrum \( \{n^2 + (\beta - 1)n + 1 \mid n \geq 2\} \).

Let \( \tilde{g} : [-1,1] \to \mathbb{R} \) be a twice continuously differentiable function such that

\[
(7.1) \quad \tilde{g}(x) = \begin{cases} 
1 & \text{if } x \text{ is near } -1 \\
0 & \text{if } x \text{ is near } +1.
\end{cases}
\]

It is clear that \( \tilde{g} \in \Delta \). We claim that there exists an \( \tilde{f} \in \Delta \) such that

\[
[\tilde{g},\tilde{f}]_{w_{-2,\beta}}(1) - [\tilde{g},\tilde{f}]_{w_{-2,\beta}}(-1) \neq 0.
\]

Of course, this would mean that \( \tilde{g} \notin \mathcal{D}(T_{\text{min}}) \). Let \( \tilde{f}(x) = (1-x)^2(1+x)^{-\beta} \). Remarkably,

\[
m_{-2,\beta}[(1-x)^2(1+x)^{-\beta}] = (-2\beta + 2)(1-x)^2(1+x)^{-\beta}
\]

and, since \(-1 < \beta < 1\), we see that \( \tilde{f} \in \Delta \). Moreover, a calculation shows that

\[
[\tilde{g},\tilde{f}]_{w_{-2,\beta}}(1) - [\tilde{g},\tilde{f}]_{w_{-2,\beta}}(-1) = 2\beta \neq 0.
\]

Hence \( \tilde{g}(x) \) is a Glazman boundary function. Moreover, for \( f \in \Delta \), observe that

\[
(7.2) \quad 0 = - \lim_{x \to -1^+} [f,\tilde{g}](x) \iff 0 = \lim_{x \to -1^+} (1-x)^{\beta+1}(f'(x)\tilde{g}(x) - f(x)\tilde{g}'(x)) \iff 0 = \lim_{x \to -1^+} (1+x)^{\beta+1}f'(x).
\]

Furthermore, a calculation shows that, for \( n \geq 2 \),

\[
\lim_{x \to -1^+} [P_n^{(-2,\beta)},\tilde{g}](x) = - \lim_{x \to -1^+} (1-x)^{\beta+1}(P_n^{(-2,\beta)}(x))' = 0.
\]

Consequently, from (7.2) and Theorem 5.1, the following theorem is immediate from the general Glazman-Krein-Naimark theory [24].
Theorem 7.1. Suppose $\beta > -1$. The operator

$$T_{-2,\beta} : L^2((-1,1); w_{-2,\beta}) \to L^2((-1,1); w_{-2,\beta})$$

defined by

$$T_{-2,\beta}[f] = m_{-2,\beta}[f]$$

for $f \in \mathcal{D}(T_{-2,\beta})$, where

\[
\mathcal{D}(T_{-2,\beta}) := \begin{cases} \Delta & \text{if } \beta \geq 1 \\ \{ f \in \Delta \mid \lim_{x \to -1^+} (1-x)^{\beta+1} f'(x) = 0 \} & \text{if } -1 < \beta < 1 \end{cases}
\]

is self-adjoint. Furthermore, the non-classical Jacobi polynomials $\{ P_n^{(-2,\beta)} \}_{n=2}^{\infty}$ form a complete orthogonal set of eigenfunctions of $T_{-2,\beta}$ in $L^2((-1,1); w_{-2,\beta})$. The spectrum $\sigma(T_{-2,\beta})$ is discrete and consists of the simple eigenvalues

$$\sigma(T_{-2,\beta}) = \sigma_p(T_{-2,\beta}) = \{ n^2 + (\beta - 1)n + 1 \mid n \geq 2 \}.$$ 

Remark 7.1. In a non-rigorous sense, the operator $T_{-2,\beta}$, given above in Theorem 7.1, can be viewed as a ‘limit’ (as $\alpha \to 0$) of the exceptional $X_1$-Jacobi self-adjoint operator $\hat{T}_{\alpha,\beta}$ given in Theorem 4.1; that is to say,

$$\lim_{\alpha \to 0} \hat{T}_{\alpha,\beta} = T_{-2,\beta}.$$ 

Notice that the boundary conditions (4.1) and (7.3) for both operators coincide; however, there is one significant difference. Indeed, the boundary condition given in (4.1) (specifically the one when $0 < \alpha < 1$ and $0 < \beta < 1$) is determined using the Glazman boundary function $g(x) = 1$ on $(-1,1)$ while the boundary condition in (7.3) is determined using the Glazman boundary function $\tilde{g}$ defined in (7.1). This latter function $\tilde{g}$ is only 1 near $x = -1$. In fact, we cannot use $g(x) \equiv 1$ to obtain $T_{-2,\beta}$ since this function does not belong to $L^2((-1,1); w_{-2,\beta})$. 

We now turn our attention to showing that $T_{-2,\beta}$ is a positive operator in $L^2((-1,1); w_{-2,\beta})$; specifically, we prepare to show that

\[
(T_{-2,\beta} f, f)_{w_{-2,\beta}} \geq (f, f)_{w_{-2,\beta}} \quad (f \in \mathcal{D}(T_{-2,\beta})).
\]

It is precisely this reason that we perturbed the Jacobi expression $\hat{T}_{\alpha,\beta}[]$ in (5.1) and shifted our study to $m_{-2,\beta}[]$ in (5.2). Once we establish (7.4), then we can apply the general left-definite theory of Littlejohn and Wellman [22] to construct a self-adjoint operator, generated by $m_{-2,\beta}[]$, in the Sobolev space $S$ having inner product $\phi(\cdot, \cdot)$, defined in (5.0). We establish this positivity (in Theorem 7.5 below) after proving two key technical theorems (Theorems 7.2 and 7.4), which concern the regularity, at the endpoints $x = \pm 1$, of functions from the domain of $T_{-2,\beta}$ and from the maximal domain $\Delta$.

Theorem 7.2. Suppose $\beta > -1$ and $T_{-2,\beta}$ is the self-adjoint operator defined in Theorem 7.1. Let $f, g \in \mathcal{D}(T_{-2,\beta})$. Then

(a) $\lim_{x \to -1} (1+x)^{\beta+1} f'(x) = 0$;
(b) $\lim_{x \to -1} (1+x)^{(\beta+1)/2} f' \in L^2(-1,0)$;
(c) $\lim_{x \to -1} (1-x)^{-1} (1+x)^{\beta+1} f'(x) \tilde{g}(x) = 0$.

Proof. (a): This limit is evident in the case $-1 < \beta < 1$ (see (7.3)) so suppose $\beta \geq 1$. Since $m_{-2,\beta}[]$ is in the limit-point case at $x = -1$, the general Weyl theory (see [19], Chapter 18) states that

\[
\lim_{x \to -1^+} (1-x)^{-1} (1+x)^{\beta+1} (f'(x) \tilde{g}(x) - f(x) \tilde{g}'(x)) = 0 \quad (f, g \in \mathcal{D}(T_{-2,\beta})).
\]
In particular, this limit is zero for all \( f \in \mathcal{D}(T_{-2,\beta}) \) and the special choice \( g \) defined by

\[
g(x) = \begin{cases} 
  1 & \text{if } -1 \leq x \leq -1/2 \\
  16x^3 + 12x^2 & \text{if } -1/2 < x \leq 0 \\
  0 & \text{if } 0 < x \leq 1.
\end{cases}
\]

A calculation shows that substitution of this \( g \) into (7.5) yields the required result.

(b): Assume, without loss of generality, that \( f \) is real-valued. For \(-1 < x \leq 0\),

\[
(7.6) \quad \int_x^0 m_{-2,\beta}[f](t)f(t)(1-t)^{-2}(1+t)^{\beta} \, dt - \int_x^0 |f(t)|^2(1-t)^{-2}(1+t)^{\beta+1}dt + f'(0)f(0)
\]

\[ - (1-x)^{-1}(1+x)^{\beta+1}f'(x)f(x) = \int_x^0 |f'(t)|^2(1-t)^{-1}(1+t)^{\beta+1}dt. \]

As \( x \to -1^+ \), the two integral terms on the left-hand side of (7.6) both converge and are finite. If \((1+x)^{(\beta+1)/2}f' \notin L^2(-1,0)\), then

\[
\int_{-1}^0 |f'(t)|^2(1-t)^{-1}(1+t)^{\beta+1}dt = \infty.
\]

It follows from (7.6) that

\[
\lim_{x \to -1^+} (1-x)^{-1}(1+x)^{\beta+1}f'(x)f(x) = -\infty.
\]

Hence there exists \( x^* \in (-1,0) \) such that \((1-x)^{-1}(1+x)^{\beta+1}f'(x)f(x) \leq -1 \) for \( x \in (-1,x^*) \). Notice, by part (a), that \(((1+x)^{\beta+1}f'(x))^t \neq 0 \) for \( x \in (-1,x^*) \). Without loss of generality, suppose that \((1+x)^{\beta+1}f'(x) > 0 \) and \( f(x) < 0 \) for \( x \in (-1,x^*) \) so that

\[
(7.7) \quad - \left|\left((1+x)^{\beta+1}f'(x)\right)'\right| f(x) \geq \frac{\left|\left((1+x)^{\beta+1}f'(x)\right)'\right|}{(1+x)^{\beta+1}f'(x)} \quad (x \in (-1,x^*)).
\]

Then, for \(-1 < x \leq x^* \),

\[
\infty > - \int_{-1}^1 |m_{-2,\beta}[f](t)f(t) - f^2(t)| (1-t)^{-2}(1+t)^{\beta+1}dt
\]

\[
= - \int_{-1}^1 \left|\left((1-t)^{-1}(1+t)^{\beta+1}f'(t)\right)' f(t)\right| dt \geq \int_{-1}^1 \frac{\left|\left((1+t)^{\beta+1}f'(t)\right)'\right|}{(1+t)^{\beta+1}f'(t)} dt
\]

\[
\geq \int_x^{x^*} \frac{\left|\left((1+t)^{\beta+1}f'(t)\right)'\right|}{(1+t)^{\beta+1}f'(t)} dt \geq \int_x^{x^*} \frac{\left|\left((1+t)^{\beta+1}f'(t)\right)'\right|}{(1+t)^{\beta+1}f'(t)} dt
\]

\[
= \left|\ln\left((1+t)^{\beta+1}f'(t)\right)\right|_x^{x^*} = \left|K - \ln((1+x)^{\beta+1}f'(x))\right|
\]

\[
\to \infty \text{ since } (1+x)^{\beta+1}f'(x) \to 0 \text{ as } x \to -1^+.
\]

This contradiction establishes part (b).

(c): The argument to prove this result mirrors closely the above proof of part (b). Assume both \( f \)
and $g$ are real-valued. From the identity
\[
\int_{0}^{x} m_{-2, \beta}[f](t)g(t)(1-t)^{-2}(1+t)^{\beta}dt - \int_{x}^{0} f(t)g(t)(1-t)^{-2}(1+t)^{\beta}dt + f'(0)g(0)
\]
exists and is finite. Suppose this limit equals $c_0$. Then there exists $c, g$ such that $c > 0$. Then there exists $x^* \in (-1, 0)$ such that
\[
(1-x)^{-1}(1+x)^{\beta+1}f'(x)g(x) \geq \frac{c}{2} \quad (x \in (-1, x^*)]
\]
exists and is finite. Suppose this limit equals $c$; if $c \neq 0$, suppose, without loss of generality, that $c > 0$. Then there exists $x^* \in (-1, 0)$ such that
\[
(1-x)^{-1}(1+x)^{\beta+1}f'(x)g(x) \geq \frac{c}{2} \quad (x \in (-1, x^*)]
\]
where, without loss of generality, $f'(x) > 0$ and $g(x) > 0$ for $x \in (-1, x^*)$. The case when $f'(x) < 0$ and $g(x) < 0$ for $x \in (-1, x^*)$ follows in analogy. Hence
\[
g(x) \geq \frac{c}{2} \cdot \frac{1}{(1-x)^{-1}(1+x)^{\beta+1}f'(x)} \quad (x \in (-1, x^*)]
\]
so that
\[
\left| \left( (1-x)^{-1}(1+x)^{\beta+1}f'(x) \right)^\prime \right| g(x) \geq \frac{c}{2} \cdot \frac{\left| (1-x)^{-1}(1+x)^{\beta+1}f'(x) \right|}{(1-x)^{-1}(1+x)^{\beta+1}f'(x)} \quad (x \in (-1, x^*)].
\]
Integrate to obtain
\[
\infty > \int_{-1}^{1} |m_{-2, \beta}[f](t) - \frac{1}{f(t)}| g(t)(1-t)^{-2}(1+t)^{\beta}dt
\]
\[
= \int_{-1}^{1} \left| \left( (1-t)^{-1}(1+t)^{\beta+1}f'(t) \right)^\prime \right| g(t)dt
\]
\[
\geq \frac{c}{2} \int_{-1}^{x^*} \frac{\left| (1-t)^{-1}(1+t)^{\beta+1}f'(t) \right|}{(1-t)^{-1}(1+t)^{\beta+1}f'(t)} dt
\]
\[
= \frac{c}{2} \left. \ln \left( (1-t)^{-1}(1+t)^{\beta+1}f'(t) \right) \right|_{-1}^{x^*} \rightarrow \infty \quad \text{as} \quad x \rightarrow -1^+.
\]
This contradiction shows that $c = 0$ and establishes part (c). This completes the proof of the theorem. \hfill \Box

Before we can prove Theorem 7.4, which describes properties of functions at $x = 1$ in the maximal domain $\Delta$, we need to recall an ‘$L^2$ inequality’ due to Chisholm, Everitt and Littlejohn (see [6]).

**Theorem 7.3.** Let $I = (a, b)$ where $-\infty \leq a < b \leq \infty$ and let $\omega$ be a positive Lebesgue measurable function on $I$. Let $\varphi, \psi : I \rightarrow \mathbb{C}$ satisfy:

(a) $\varphi, \psi \in L^2_{\operatorname{loc}}(I; \omega)$;
(b) There exists $c \in (a, b)$ such that $\varphi \in L^2((a, c]; \omega)$ and $\psi \in L^2((c, b]; \omega)$. (In this case, we say that $\varphi$ is $L^2$ near $a$ and $\psi$ is $L^2$ near $b$, both with respect to the weight $\omega$);
(c) For all $[\delta, \gamma] \subset I$,
\[
\int_{a}^{b} |\varphi|^2 \omega dx > 0 \quad \text{and} \quad \int_{\gamma}^{b} |\psi|^2 \omega dx > 0.
\]
Define $A, B : L^2(I; \omega) \rightarrow L^2_{\text{loc}}(I; \omega)$ by

$$(Af)(x) := \varphi(x) \int_x^b \psi(x)f(x)\omega(x)dx \quad (x \in (a, b) \text{ and } f \in L^2((a, b); \omega))$$

$$(Bg)(x) := \psi(x) \int_a^x \varphi(x)g(x)\omega(x)dx \quad (x \in (a, b) \text{ and } g \in L^2((a, b); \omega))$$

and $K : (a, b) \rightarrow (0, \infty)$ by

$$K(x) := \left\{ \int_a^x |\varphi|^2 \omega \right\}^{1/2} \left\{ \int_x^b |\psi|^2 \omega \right\}^{1/2} \quad (x \in (a, b))$$

and the number $K \in (0, \infty]$ by

$$K := \sup \{ K(x) : x \in (a, b) \}.$$

Then a necessary and sufficient condition for both $A$ and $B$ to be bounded linear operators into $L^2(I; \omega)$ is for $K$ to be finite. Furthermore, in this case,

$$||Af||_\omega \leq 2K||f||_\omega \text{ and } ||Bf||_\omega \leq 2K||f||_\omega \quad (f \in L^2(I; \omega)).$$

**Theorem 7.4.** Suppose $\beta > -1$ and $T_{-2, \beta}$ is the self-adjoint operator defined in Theorem 7.1. Let $f, g \in \Delta$ (see (6.22)). Then

(a) $f' \in L^2[0, 1]$;
(b) $\lim_{x \to 1-} f(x)$ exists and is finite and $f \in AC_{\text{loc}}(-1, 1)$;
(c) $f(1) = 0$;
(d) $(1-x)^{-1}f' \in L^2[0, 1] \subset L^1[0, 1]$;
(e) $\lim_{x \to 1-} (1-x)^{-1}(1+x)^{\beta+1}f'(x)\overline{g}(x) = 0$.

**Proof.** (a): For $0 \leq x < 1$, note that

$$(7.8) \quad f'(x) = (1-x)(1+x)^{-\beta} \int_0^x \frac{(1-t)^{-1}(1+t)^{\beta/2}}{(1-t)^{-1}(1+t)^{\beta/2}} \left( (1-t)^{-1}(1+t)^{\beta+1}f'(t) \right)' dt$$

$$+ (1-x)(1+x)^{-\beta-1}f'(0).$$

Clearly the term $(1-x)(1+x)^{-\beta-1}f'(0) \in L^2(0, 1)$. By definition of $\Delta$, we see that

$$(1-t)^{-1}(1+t)^{\beta/2} \left( (1-t)^{-1}(1+t)^{\beta+1}f'(t) \right)' \in L^2(0, 1).$$

We apply Theorem 7.3 using $\psi(x) = (1-x)(1+x)^{-\beta-1}$, $\varphi(x) = (1-x)(1+x)^{-\beta/2}$, $\omega(x) = 1$ and $(a, b) = (0, 1)$. We see that $\varphi$ is $L^2$ near 0 and $\psi$ is $L^2$ near 1. Moreover, since

$$\int_0^x |\varphi|^2 dt \cdot \int_x^1 |\psi|^2 dt \quad (0 \leq x \leq 1)$$

is bounded on $[0, 1)$, we conclude from Theorem 7.3 that $f' \in L^2[0, 1)$.

(b): Note that from part (a), $f' \in L^1(0, 1)$. For $0 \leq x < 1$, we can write $f(x) = f(0) + \int_0^x f'(t) dt$.

Then, $\lim_{x \to 1-} f(x)$ exists and is finite. We define $f(1) := \lim_{x \to 1-} f(x)$. In this case we see that $f \in AC_{\text{loc}}[0, 1]$. Since $f \in \Delta$, it follows that $f \in AC'(-1, 1)$.
(c): Suppose \( f(1) \neq 0 \). Without loss of generality, assume \( f(x) \) is real valued and \( f(1) = c > 0 \). This implies that there exists \( x^* \in (0,1) \) such that \( f(x) \geq \frac{c}{2} \) for all \( x \in [x^*, 1] \). Then,
\[
\infty > \int_{-1}^{1} |f(x)|^2 (1-x)^{-2}(1+x)^{\beta} \, dx \geq \int_{x^*}^{1} |f(x)|^2 (1-x)^{-2}(1+x)^{\beta} \, dx
\]
\[
\geq \left( \frac{c}{2} \right)^2 \int_{x^*}^{1} (1-x)^{-2}(1+x)^{\beta} \, dx = \infty ,
\]
a contradiction. Hence \( f(1) = 0 \).

(d): Note from (7.8) that
\[
(1-x)^{-1}f'(x) = (1+x)^{\beta-1} \int_{0}^{x} \frac{(1-t)^{-\frac{1}{2}(1+t)^{\beta/2}}}{(1-t)^{-\frac{1}{2}(1+t)^{\beta/2}}} \left( (1-t)^{-\frac{1}{2}(1+t)^{\beta+1}}f(t) \right) \, dt + (1+x)^{-\beta-1}f'(0).
\]
Applying Theorem 7.3 with the same \( \varphi \) and \( \psi \), part (d) follows similarly to part (a).

(e): Suppose that \( f, g \in \Delta \) are both real-valued. The reader can check the following variant of Dirichlet’s formula: for \( 0 \leq x \leq 1 \), we have
\[
(7.9) \quad \int_{0}^{x} f(t)g'(t)(1-t)^{-1}(1+t)^{\beta+1} \, dt = -f'(0)g(0) + (1-x)^{-1}(1+x)^{\beta+1}f'(x)g(x)
\]
\[
+ \int_{0}^{x} m_{-2,\beta}[f](t)g(t)(1-t)^{-2}(1+t)^{\beta} \, dt - \int_{0}^{x} f(t)g(t)(1-t)^{-2}(1+t)^{\beta} \, dt.
\]
By part (d), \( (1-x)^{-1}f' \in L^2(0,1) \); furthermore since \( g \in AC[0,1] \) and \( (1+x)^{\beta+1} \) is bounded on \( [0,1] \), we can say that
\[
(1-x)^{-1}(1+x)^{\beta+1}f'g \in L^1(0,1).
\]
Similarly, from part (a), we see that
\[
(1-x)^{-1}(1+x)^{\beta+1}f'g' \in L^1(0,1).
\]
Consequently, we see that each of the integral terms in (7.9) converge as \( x \to 1^- \). Hence, from (7.9),
\[
\lim_{x \to 1^-} (1-x)^{-1}(1+x)^{\beta+1}f'(x)g(x)
\]
exists and is finite. Suppose that this limit equals \( c \) but \( c \neq 0 \); without loss of generality, we can assume that \( c > 0 \). Then there exists \( x^* \in [0,1) \) such that, without loss of generality, \( f'(x) > 0 \), \( g(x) > 0 \) and
\[
(1-x)^{-1}(1+x)^{\beta+1}f'(x) \geq c \cdot \frac{1}{g(x)} \quad (x \in [x^*, 1)).
\]
But then
\[
\infty > \int_{0}^{1} (1-t)^{-1}(1+t)^{\beta+1}f'(t) |g'(t)| \, dt
\]
\[
\geq \int_{x^*}^{1} (1-t)^{-1}(1+t)^{\beta+1}f'(t) |g'(t)| \, dt \geq \frac{c}{2} \int_{x^*}^{1} |g(t)| \, dt
\]
\[
\geq \left| \int_{x^*}^{1} \frac{g'(t)}{g(t)} \, dt \right| = \frac{c}{2} |\ln(g(t))|_{x^*} = \infty \quad \text{since} \quad g(1) = 0.
\]
This contradiction completes the proof of part (d) and the theorem. \(\square\)
**Theorem 7.5.** For $f \in \mathcal{D}(T_{-2,\beta})$, the positivity inequality in (7.4) holds; that is to say $T_{-2,\beta}$ is bounded below in $L^2((-1,1);w_{-2,\beta})$ by the identity operator $I$ so

$$(T_{-2,\beta}[f],f)_{w_{-2,\beta}} \geq (f,f)_{w_{-2,\beta}} \quad (f \in \mathcal{D}(T_{-2,\beta})).$$

**Proof.** Let $f \in \mathcal{D}(T_{-2,\beta})$. Let $g = f$ in (6.3) and let $x \to -1^+$ and $y \to 1^-$. From Property (c) in Theorem 7.2 and Property (e) in Theorem 7.4, the result readily follows. □

In fact, if $f,g \in \mathcal{D}(T_{-2,\beta})$ and, in (6.3), we let $x \to -1^+, y \to 1^-$, we obtain

$$\tag{7.10} (T_{-2,\beta}f,g)_{w_{-2,\beta}} = \int_{-1}^{1} \left( (1-t)^{-1}(1+t)^{\beta+2}f'(t)\overline{g'}(t) + (1-t)^{-2}(1+t)^{\beta}f(t)\overline{g(t)} \right) dt.$$

The right-hand side of (7.10) is an inner product; in fact, it is the first left-definite inner product associated with the pair $(T_{-2,\beta},L^2((-1,1);w_{-2,\beta}))$; see Section 3 and [22] for further information.

Since $T_{-2,\beta}$ is positive, we can apply the left-definite theory that Littlejohn and Wellman developed in [22]. Without going into detail, their general results show that $\mathcal{D}(T_{-2,\beta})$ is equal to the second left-definite space $V_2$ associated with $(T_{-2,\beta},L^2((-1,1),w_{-2,\beta}))$. More specifically,

**Theorem 7.6.** The domain of the operator $T_{-2,\beta}$ is given by

$$\mathcal{D}(T_{-2,\beta}) = \left\{ f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); f(j) \in L^2((-1,1);(1-x)^{-2}(1+x)^{\beta+j}) \quad (j = 0, 1, 2) \right\}.$$

In particular, for $f \in \mathcal{D}(T_{-2,\beta})$, we see that $f'' \in L^2((-1,1);(1+x)^{\beta+2})$. Since $(1+x)^{\beta+2}$ is bounded on $[0,1]$, we then have $f'' \in L^2[0,1]$. Hence, $f' \in AC[0,1]$; in fact,

$$\tag{7.11} f' \in AC_{loc}(-1,1).$$

In particular, $f'(1)$ exists and is finite. In fact, it is necessary that

$$\tag{7.12} f'(1) = 0.$$

For suppose $f'(1) \neq 0$; without loss of generality, we can assume that $f'(1) = c > 0$. Hence, there exists $x^* \in (0,1)$ such that $f'(x) \geq \frac{c}{2}$ on $[x^*, 1)$. But then

$$\int_{x^*}^{1} (1-t)^{-1}f'(t)dt \geq \frac{c}{2} \int_{x^*}^{1} (1-t)^{-1}dt = \infty,$$

contradicting part (d) of Theorem 7.4. We note that property (7.12) will be useful to us later in this paper.

8. A Primer on Left-Definite Operator Theory

Now that we have established that $T_{-2,\beta}$ is a self-adjoint operator which is bounded below in $L^2((-1,1);w_{-2,\beta})$ by $I$ (see (7.4)), we can apply the general left-definite theory developed by Littlejohn and Wellman in [22]. This theory will be important as we continue our study of $m_{-2,\beta}[]$ in the Sobolev space generated by the inner product (5.3). We now briefly discuss this theory.

Let $H$ be a Hilbert space with inner product $(\cdot,\cdot)$ and suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below in $H$ by $kI$ for some $k > 0$; that is,

$$(Af,f) \geq k(f,f) \quad (f \in \mathcal{D}(A)).$$

It follows that $A^*$ is self-adjoint and bounded below in $H$ by $k^2I$ for each $r > 0$.

**Theorem 8.1.** Suppose $r > 0$. 

(a) Let
\begin{align}
V_r &= \mathcal{D}(A^{r/2}) \\
(f,g)_r &= (A^{r/2}f, A^{r/2}g) \\
H_r &= (V_r, (\cdot, \cdot)_r).
\end{align}

Then
\begin{align}
\begin{cases}
(i) & H_r \text{ is a Hilbert space;} \\
(ii) & \mathcal{D}(A^r) \text{ is a subspace of } V_r; \\
(iii) & \mathcal{D}(A^r) \text{ is dense in } V_r; \\
(iv) & (f,f)_r \geq k^r (f,f) \quad (f \in V_r); \\
v) & (f,g)_r = (A^r f,g) \quad (f \in \mathcal{D}(A^r), g \in V_r).
\end{cases}
\end{align}

(b) The operator \(A_r : \mathcal{D}(A_r) \subset H_r \to H_r\) given by
\[
\begin{cases}
A_r x &= Ax \\
x \in \mathcal{D}(A_r) &= V_{r+2}
\end{cases}
\]
is self-adjoint in \(H_r\) and has spectrum \(\sigma(A_r) = \sigma(A)\) and is bounded below in \(H_r\) by \(k^r I\).
Furthermore, if \(\{\phi_n\}\) is a complete set of eigenfunctions of \(A\) in \(H\), then \(\{\phi_n\}\) is a complete set of eigenfunctions of \(A_r\) in \(H_r\).

The space \(H_r\) is called the \(r\)th left-definite space associated with the pair \((A,H)\). Notice, from \(\text{S.1}\) that \(\mathcal{D}(A) = V_2\); this new characterization of the domain of \(A\) has proven to be useful in several applications. The operator \(A_r\) is called the \(r\)th left-definite operator associated with \((A,H)\).

The term ‘left-definite’ owes its name to spectral theory of differential operators. Indeed, if \(A\) is self-adjoint, bounded below and generated by a differential expression \(\ell[\cdot]\), property (v) in \(\text{S.2}\) says that the study will be in the space whose inner product is generated by the \(r\)th power \(\ell^r[\cdot]\) of \(\ell[\cdot]\) which, of course, is on the left side of the differential equation \(\ell^r[y] = \lambda y\).

In our situation, it is not difficult to establish that the \(r\)th left-definite space, when \(r \in \mathbb{N}\), associated with \(T_{-2,\beta}, L^2((-1,1); w_{-2,\beta})\) is \(H_r = (V_r, (\cdot, \cdot)_r)\), where
\begin{align}
V_r &= \{ f : (-1,1) \to \mathbb{C} \mid f, f', \ldots, f^{(r-1)} \in AC_{\text{loc}}(-1,1); \\
&\quad f^{(j)} \in L^2((-1,1); (1-x)^{j-2}(1+x)^{\beta+j}), \quad j = 0,1,\ldots,r \},
\end{align}
and
\begin{align}
(f,g)_r &= \sum_{j=0}^{n} c_j^{(-2,\beta)}(n) \int_{-1}^{1} f^{(j)}(x)\overline{g}^{(j)}(x)(1-x)^{j-2}(1+x)^{\beta+j} dx;
\end{align}
here, the numbers \(\{c_j^{(-2,\beta)}\}\) are the so-called Jacobi-Stirling numbers; see \([2]\) and \([9]\). When \(r = 2\), the inner product in \(\text{S.1}\) is specifically given by
\begin{align}
(f,g)_2 &= \int_{-1}^{1} \left( (1+t)^{\beta+2} f''(t)\overline{g}''(t) + (\beta + 2)(1-t)^{-1}(1+t)^{\beta+1} f'(t)\overline{g}'(t) \\
&\quad +(1-t)^{-2}(1+t)^{\beta} f(t)\overline{g}(t) \right) dt.
\end{align}
Notice also when \(r = 2\) in \(\text{S.3}\), we obtain the characterization given in Theorem \(7.6\).
Another left-definite space which will be useful to us later in this paper is
\begin{equation}
V_4 = \{ f : (-1, 1) \to \mathbb{C} \mid f, f', f'', f''' \in AC_{\text{loc}}(-1, 1); \\
f^{(j)} \in L^2((-1, 1); (1 - x)^{j-2}(1 + x)^{\beta+j}, j = 0, 1, 2, 3, 4 \}\.
\end{equation}
This space will turn out to be instrumental in constructing a certain self-adjoint operator \( T_2 \) in the Sobolev space \( S \) which we now introduce.

9. The Sobolev space \((S, \phi(\cdot, \cdot))\)

Recall the Sobolev inner product \( \phi(\cdot, \cdot) \) given in (5.6). The full sequence of non-classical Jacobi polynomials \( \{P_n^{(-2, \beta)}\}_{n=0}^{\infty} \), for \( \beta > -1 \) but \( \beta \neq 0 \) (see Remark 5.3), are orthogonal with respect to this inner product. Let
\[ S := \left\{ f : (-1, 1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1, 1); f'' \in L^2((-1, 1); (1 + x)^{\beta+2}) \right\} \]
and let \( \|\cdot\|_{\phi} \) be the usual norm associated with \( \phi(\cdot, \cdot) \); notice that
\begin{equation}
\|f\|_{\phi} = |f'(1)|^2 + \left| f(1) + \frac{2}{\beta} f'(1) \right|^2 + \int_{-1}^{1} |f''(x)|^2 (1 + x)^{\beta+2} \, dx \quad (f \in S).
\end{equation}
We want to construct a self-adjoint operator \( T \), generated by \( m_{-2, \beta}[\cdot] \), in \( S \) that has the Jacobi polynomials \( \{P_n^{(-2, \beta)}\}_{n=0}^{\infty} \) as eigenfunctions and has spectrum \( \sigma(T) = \{n^2 + (\beta - 1)n + 1 \mid n \in \mathbb{N}_0\} \).

Before we do this, we must discuss certain properties of this Sobolev space \( S \).

**Theorem 9.1.** The space \((S, \phi(\cdot, \cdot))\) is a Hilbert space.

**Proof.** Suppose \( \{f_n\} \subseteq S \) is a Cauchy sequence. Note that
\[
\|f_n - f_m\|_{\phi}^2 = \phi(f_n - f_m, f_n - f_m) \\
= \left| f_n(1) - f_m(1) + \frac{2}{\beta} \left( f'_n(1) - f'_m(1) \right) \right|^2 \\
+ \left| f'_n(1) - f'_m(1) \right|^2 + \int_{-1}^{1} \left| f''_n(x) - f''_m(x) \right|^2 (1 + x)^{\beta+2} \, dx.
\]
From this identity, we see that \( \{f''_n\}_{n=0}^{\infty} \) is Cauchy in \( L^2((-1, 1); (1 + x)^{\beta+2}) \), and that the sequences \( \{f_n(1)\}_{n=0}^{\infty} \) and \( \{f'_n(1)\}_{n=0}^{\infty} \) are both Cauchy in \( \mathbb{C} \). Therefore, from the completeness of the spaces \( L^2((-1, 1); (1 + x)^{\beta+2}) \) and \( \mathbb{C} \), there exists a function \( g \in L^2((-1, 1); (1 + x)^{\beta+2}) \) and scalars \( a, b \in \mathbb{C} \) such that \( \{f''_n\}_{n=0}^{\infty} \) converges to \( g \) in \( L^2((-1, 1); (1 + x)^{\beta+2}) \), \( \{f'_n(1)\}_{n=0}^{\infty} \) converges to \( a \) in \( \mathbb{C} \), and \( \{f_n(1)\}_{n=0}^{\infty} \) converges to \( b \) in \( \mathbb{C} \).

Define the function \( f : (-1, 1) \to \mathbb{C} \) by
\[ f(x) := ax + (b - a) + \int_{-1}^{1} \int_{-1}^{1} g(u) \, du \, dt \]
Then \( f, f' \in AC(-1, 1) \) with \( f(1) = b \) and \( f'(1) = a \). Moreover, \( f'' = g \in L^2((-1, 1); (1 + x)^{\beta+2}) \) so \( f \in S \). Moreover, it is straightforward to see that
\[
\|f_n - f\|_{\phi}^2 \to 0 \quad (n \to \infty),
\]
completing the proof of the theorem. \( \square \)
Theorem 9.2. The set $\mathcal{P}$ of all polynomials is dense in $(S, \phi(\cdot, \cdot))$. Equivalently, the Jacobi polynomials $\left\{ P_n^{(-2, \beta)} \right\}_{n=0}^\infty$ form a complete orthogonal set in $S$.

Proof. Let $f \in S$. Then, $f'' \in L^2((-1, 1); (1 + x)^{\beta+2})$. Since $\mathcal{P}$ is dense in $L^2((-1, 1); (1 + x)^{\beta+2})$, there exists $p \in \mathcal{P}$ such that

$$
\int_{-1}^1 \left| f''(x) - p(x) \right|^2 (1 + \beta)^{\beta+2} \, dx < \varepsilon^2.
$$

With the polynomial $q$ defined by

$$
q(x) := f'(1)x + (f(1) - f'(1)) + \int_x^1 \int_t^1 p(u) \, du \, dt,
$$

we see that $f(1) = q(1)$, $f'(1) = q'(1)$. Moreover, by (9.2) we see that

$$
\|f - q\|_S^2 = \left| f(1) - q(1) + \frac{2}{\beta} (f'(1) - q'(1)) \right|^2
$$

$$
+ \left| f'(1) - q'(1) \right|^2 + \int_{-1}^1 \left| f''(x) - q''(x) \right|^2 (1 + x)^{\beta+2} \, dx
$$

$$
= \int_{-1}^1 \left| f''(x) - q''(x) \right|^2 (1 + x)^{\beta+2} \, dx < \varepsilon^2.
$$

This completes the proof of the theorem. \qed

For reasons that will be made clearer shortly, we now define two subspaces of $S$.

$$
S_1 := \text{span} \left\{ P_0^{(-2, \beta)}, P_1^{(-2, \beta)} \right\} = \{ f \in S \mid f''(x) = 0 \}, \text{ and }
$$

$$
S_2 := \text{span} \left\{ P_n^{(-2, \beta)} \right\}_{n=2}^\infty = \{ f \in S \mid f(1) = f'(1) = 0 \}.
$$

Theorem 9.3. $S = S_1 \oplus S_2$.

Proof. Let $f \in S$. We can write $f(x)$ as

$$
f(x) = g_1(x) + g_2(x),
$$

where

$$
g_1(x) = f'(1)x + f(1) - f'(1) \text{ and } g_2(x) = f(x) - f'(1)x - f(1) + f'(1).
$$

It is clear that $g_i \in S_i$ for $i = 1, 2$ so $S = S_1 + S_2$. To show that $S_1 \perp S_2$, suppose $f_1 \in S_1$ and $f_2 \in S_2$ so $f_2(1) = f_2'(1) = 0$ and $f_2''(x) = 0$. Then

$$
\phi(f_1, f_2) = f_1(1)f_2(1) + \frac{2}{\beta} \left( f_1'(1)f_2(1) + f_1(1)f_2'(1) \right) + \left( 1 + \frac{4}{\beta^2} \right) f_1'(1)f_2'(1)
$$

$$
+ \int_{-1}^1 f_1''(x)f_2''(x)(1 + x)^{\beta+2} \, dx
$$

$$
= 0.
$$

This completes the proof of the theorem. \qed
We remark that, since $S_1$ and $S_2$ are closed subspaces of $S$, both $(S_1, \phi(\cdot, \cdot))$ and $(S_2, \phi(\cdot, \cdot))$ are Hilbert spaces.

In order to construct the self-adjoint operator $T$ in $S$, we will construct two self-adjoint operators $T_1$ and $T_2$, both generated by $m_{-2,\beta}[\cdot]$, in $S_1$ and $S_2$ respectively. The operator $T = T_1 \oplus T_2$, the direct sum of $T_1$ and $T_2$, will be the self-adjoint operator in $S$ that has the properties we desire.

10. The Construction of the Operators $T_1$, $T_2$ and $T$

Define $T_1 : \mathcal{D}(T_1) \subset S_1 \rightarrow S_1$ by

$$T_1 f = m_{-2,\beta}[f]$$

$$f \in \mathcal{D}(T_1) : = S_1.$$

(10.1)

It is straightforward to show that $T_1$ is symmetric with respect to the inner product $\phi(\cdot, \cdot)$ and, since $S_1$ is two-dimensional, it follows that $T_1$ is self-adjoint in $S_1$. Moreover, it is clear that

$$\sigma(T_1) = \{ n^2 + (\beta - 1)n + 1 \mid n = 0, 1 \}.$$

We now focus our attention on the construction of $T_2$. It is remarkable that the left-definite theory associated with $T_{-2,\beta}$ plays a very significant role in this construction.

**Theorem 10.1.** $S_2 = \mathcal{D}(T_{-2,\beta}) = V_2$.

**Proof.** We already know from Theorem [8.4] that $\mathcal{D}(T_{-2,\beta}) = V_2$.

$S_2 \subset V_2$: Let $f \in S_2$. In particular, we see that $f'' \in L^2((-1,1); (1+x)^{\beta+2})$ which, since $(1+x)^{\beta+2}$ is bounded on $[0,1]$ implies that $f'' \in L^2[0,1]$. Since $f'(1) = 0$, we see that

$$f'(x) = - \int_1^x f''(t) dt$$

and

$$(1-x)^{-1/2}(1+x)^{\beta+1/2} f'(x) = -(1-x)^{-1/2}(1+x)^{\beta+1/2} \int_1^x f''(t) dt.$$  

We now use Theorem [7.3] to show that $f' \in L^2((-1,1); (1-x)^{-1}(1+x)^{\beta+1})$. Let $\varphi(x) = 1, \psi(x) = -(1+x)^{-1/2}(1+x)^{\beta+1/2}$; both of these functions satisfy the conditions of Theorem [7.3] since

$$\int_1^x 1 dt \cdot \int_{-1}^x (1+x)^{-1}(1+t)^{\beta+1} dt$$

is bounded on $(-1,1)$, we can conclude from Theorem [7.3] that $f' \in L^2((-1,1); (1-x)^{-1}(1+x)^{\beta+1})$.

A similar application of Theorem [7.3] shows that

$$f \in L^2((-1,1); (1-x)^{-2}(1+x)^{\beta}).$$

This proves that $S_2 \subset V_2$.

$V_2 \subset S_2$: Let $f \in V_2 = \mathcal{D}(T_{-2,\beta})$. From part (b) of Theorem [7.4] and (7.11), we see that $f, f' \in Ac^l_{loc}(-1,1)$. From Theorem [7.6] we find that $f'' \in L^2((-1,1); (1+x)^{\beta+2})$. Finally, part (c) of Theorem [7.4] and (7.12), we see that $f(1) = f'(1) = 0$. This shows $V_2 \subset S_2$ and completes the proof of the theorem. \( \square \)

Now that we have established that $S_2 = V_2$, what can we say about the inner products $\phi(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle_2$, where $\langle \cdot, \cdot \rangle_2$ is the second left-definite inner product defined in (8.5)? Remarkably, the answer is given in the following theorem.
Theorem 10.2. The inner products \( \phi(\cdot, \cdot) \) and \( (\cdot, \cdot)_2 \) are equivalent on the Hilbert spaces \((S_2, \phi(\cdot, \cdot))\) and \((V_2 = S_2, (\cdot, \cdot)_2)\).

Proof. Let \( f \in S_2 = V_2 \). Then
\[
\|f\|_2^2 = (f, f)_2 = \int_{-1}^{1} \left( (1 + t)^{\beta+2} |f''(t)|^2 + (\beta + 2)(1 - t)^{-1}(1 + t)^{\beta+1} |f'(t)|^2 \right) dt \\
\geq \int_{-1}^{1} (1 + t)^{\beta+2} |f(t)|^2 dt = \|f\|_\phi^2,
\]
where the latter identity follows from (10.1) and by definition of \( S_2 \). The Open Mapping Theorem (see [20, Chapter 4.12, Problem 9]) now implies that the two inner products are equivalent. \( \square \)

We now are in position to construct the self-adjoint operator \( T_2 \) in \( S_2 \) which has the Jacobi polynomials \( \{P_n^{(-2, \beta)}\}_{n=2}^{\infty} \) as eigenfunctions. Define \( T_2 : D(T_2) \subset S_2 \to S_2 \) by
\[
T_2 f = m_{-2, \beta}[f] \\
f \in D(T_2) = V_4
\]
where \( V_4 \) is defined in (8.6). Notice that, other than the inner product being different (but equivalent), this operator is essentially the second left-definite operator associated with the pair \( (T_{-2, \beta}, L^2((-1, 1); w_{-2, \beta})) \). From the left-definite theory [22], \( T_2 \) is self-adjoint in the second left-definite space \( H_2 = (V_2, (\cdot, \cdot)_2) \). However, it requires work to show it is self-adjoint in \( S_2 \).

Theorem 10.3. Let \( T_2 \) be the operator defined in (10.2).

(a) \( T_2 \) is densely defined and closed in \((S_2, \phi(\cdot, \cdot))\);
(b) \( T_2 \) is symmetric in \((S_2, \phi(\cdot, \cdot))\); that is,
\[
\phi(T_2 f, g) = \phi(f, T_2 g) \quad (f, g \in D(T_2));
\]
(c) \( T_2 \) is self-adjoint in \((S_2, \phi(\cdot, \cdot))\) and has the Jacobi polynomials \( \{P_n^{(-2, \beta)}\}_{n=2}^{\infty} \) as eigenfunctions. The spectrum of \( T_2 \) is \( \sigma(T_2) = \{n^2 + (\beta - 1)n + 1 \mid n \geq 2\} \).

Proof. Somewhat surprisingly, the difficult part of the proof is in establishing the symmetry, not the self-adjointness, of \( T_2 \). The domain \( D(T_2) \) being dense in \( S_2 \) follows by direct analysis or from part (b) of Theorem 8.1 since \( \{P_n^{(-2, \beta)}\}_{n=2}^{\infty} \subset D(T_2) \). We now show that \( T_2 \) is closed in \( S_2 \). Suppose \( \{f_n\} \subset D(T_2) \) with
\[
f_n \to f \text{ in } (S_2, \phi(\cdot, \cdot)) \text{ and } T_2 f_n \to g \text{ in } (S_2, \phi(\cdot, \cdot)).
\]
We need to show that \( f \in D(T_2) \) and \( T_2 f = g \). Since equivalent inner products have the same convergent sequences, it is clear that \( f_n \to f \) and \( T_2 f_n \to g \) in \( H_2 = (V_2, (\cdot, \cdot)) \). Moreover, since self-adjoint operators are closed and since \( T_2 \) is self-adjoint in \( H_2 \), we see that \( T_2 \) is closed. Hence \( f \in V_4 = D(T_2) \) and \( T_2 f = g \). This proves part (a). Since a closed, symmetric operator having a complete set of eigenfunctions in a Hilbert space is self-adjoint (see [19, Theorem 3, page 173 and Theorem 6, page 184]), we see that the self-adjointness of \( T_2 \) will follow as soon as we establish the symmetry of \( T_2 \). To that end, for \( f, g \in D(T_2) \), a laborious calculation shows that
\[
\phi(T_2 f, g) - \phi(f, T_2 g) = [f, g]_\phi(1) - [f, g]_{\phi(-1)},
\]
where
where $\langle \cdot, \cdot \rangle_\phi$ is the sesquilinear form given by
\[ [f, g]_\phi(x) = (1 - x)(1 + x)^{\beta+3}(f''(x)g''(x) - f''(x)g''(x)) (f, g \in D(T_2)) \]
and
\[ [f, g]_\phi(\pm 1) = \lim_{x \to \pm 1^+} [f, g]_\phi(x). \]
These limits both exist and are finite by definition of $D(T_2)$. We will show that, in fact,
\[ [f, g]_\phi(\pm 1) = 0 \quad (f, g \in D(T_2)). \]
We show the details at $x = +1$.

Claim #1: $\lim_{x \to 1^-} (1 - x)f'''(x) = 0$ for $f \in D(T_2)$.
Without loss of generality, suppose $f$ is real-valued. Let
\[ \hat{g}(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ -704x^7 + 1208x^6 - 707x^5 + 140x^4 & \text{if } 0 < x \leq \frac{1}{2} \\ \frac{1}{2}(x - 1)^2 & \text{if } \frac{1}{2} < x \leq 1. \end{cases} \]
It is clear that $\hat{g} \in V_4$. A calculation shows that
\[ \lim_{x \to 1^-} [f, \hat{g}]_\phi(x) = \lim_{x \to 1^-} (1 - x)(1 + x)^{\beta+3}f''(x). \]
Consequently, we see that
\[ \lim_{x \to 1^-} (1 - x)f'''(x) \quad (f \in D(T_2)). \]
exists and is finite. If this limit, say $c$, is not zero, we can suppose that $c > 0$. Hence there exists $x^* \in (0, 1)$ such that
\[ f'''(x) \geq \frac{c}{2} \cdot \frac{1}{1 - x} \quad (x \in [x^*, 1]) \]
and consequently
\[ (1 - x)^{1/2}(1 + x)^{(\beta+3)/2}f'''(x) \geq \frac{c}{2} \cdot \frac{(1 + x)^{(\beta+3)/2}}{(1 - x)^{1/2}} \quad (x \in [x^*, 1]). \]
Since $f''' \in L^2((-1, 1); (1 - x)(1 + x)^{\beta+3})$, we see that
\[ \infty > \int_{x^*}^1 \left| f'''(x) \right|^2 (1 - x)(1 + x)^{\beta+3}dx \geq \frac{c^2}{4} \int_{x^*}^1 \frac{(1 + x)^{\beta+3}}{1 - x}dx = \infty. \]
This contradiction proves the claim.

Claim #2: $f''' \in L^2(0, 1)$; consequently, $f, f', f'' \in AC[0, 1]$ and, in particular, $f''(1)$ exists and is finite. To see this, note that
\[ f'''(x) = f'''(0) + \int_0^x \frac{f^{(4)}(t)(1 - t)(1 + t)^{(\beta+4)/4}}{(1 - t)(1 + t)^{(\beta+4)/4}}dt \quad (0 \leq x < 1). \]
By definition of $V_4$, $(1 - t)(1 + t)^{(\beta+4)/2}f^{(4)} \in L^2(-1, 1)$. We apply Theorem 7.3 using
\[ \varphi(x) = \frac{1}{(1 - x)(1 + x)^{(\beta+4)/4}} \quad \text{and} \quad \psi(x) = 1 \quad (0 \leq x < 1). \]
A calculation shows that
\[ \int_0^x \varphi^2(t)dt \cdot \int_x^1 \psi^2(t)dt \]
is bounded on $[0, 1]$. Thus, it follows that $f''' \in L^2(0, 1)$. 

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Claim #3: For all \( f, g \in \mathcal{D}(T_2) \),
\[
\lim_{x \to 1^-} [f, g]_\phi(x) = 0.
\]
It suffices to show that
\[
\lim_{x \to 1^-} (1 - x) (1 + x)^{\beta+3} f''(x) \overline{g''}(x) = 0.
\]
We apply Claims 1 and 2 and see that
\[
\lim_{x \to 1^-} (1 - x)(1 + x)^{\beta+3} f''(x) \overline{g''}(x)
= \lim_{x \to 1^-} (1 + x)^{\beta+3} \lim_{x \to 1^-} (1 - x) f''(x) \lim_{x \to 1^-} \overline{g''}(x)
= 0.
\]
A similar analysis shows
\[
\lim_{x \to 1^+} [f, g]_\phi(x) = 0 \quad (f, g \in \mathcal{D}(T_2)).
\]
Referring to (10.3), we see that \( T_2 \) is symmetric in \( S_2 \) and this completes the proof of the theorem. \( \square \)

The following theorem was shown in [10, Theorem 11.1].

**Theorem 10.4.** Suppose \( H \) is a Hilbert space with the orthogonal decomposition
\[
H = H_1 \oplus H_2,
\]
where \( H_1 \) and \( H_2 \) are closed subspaces of \( H \). Suppose \( A_1 : \mathcal{D}(A_1) \subset H_1 \to H_1 \) and \( A_2 : \mathcal{D}(A_2) \subset H_2 \to H_2 \) are self-adjoint operators in \( H_1 \) and \( H_2 \), respectively. For \( f_1 \in \mathcal{D}(A_1) \) and \( f_2 \in \mathcal{D}(A_2) \), write
\[
f = f_1 + f_2,
\]
and let \( A = A_1 \oplus A_2 : \mathcal{D}(A) \subset H \to H \) be the operator defined by
\[
Af = A_1 f_1 + A_2 f_2,
\]
\( f \in \mathcal{D}(A) := \mathcal{D}(A_1) \oplus \mathcal{D}(A_2) \).

Then \( A \) is self-adjoint in \( H \).

We remark that if these operators \( A_1 \) and \( A_2 \) are both generated by, say, a linear differential expression \( m[\cdot] \), then so is \( A = A_1 \oplus A_2 \). Indeed, if \( f = f_1 + f_2 \in \mathcal{D}(A_1) \oplus \mathcal{D}(A_2) \), then
\[
Af = A_1 f_1 + A_2 f_2 = m[f_1] + m[f_2] = m[f_1 + f_2] = m[f].
\]

We are now in position to state the main result of this section.

**Theorem 10.5.** Let \( T_1 \) and \( T_2 \) be the self-adjoint operators defined in, respectively, (10.1) and (10.2). Let \( T : \mathcal{D}(T) \subset S \to S \) be the operator defined by
\[
T = T_1 \oplus T_2
\]
\( \mathcal{D}(T) := \mathcal{D}(T_1) \oplus \mathcal{D}(T_1) \).

Then \( T \) is a self-adjoint operator, generated by the Jacobi differential expression \( m_{-2,\beta}[\cdot] \), in the Sobolev space \( S \). The Jacobi polynomials \( \{P_n^{(-2,\beta)}\}_{n=0}^{\infty} \) form a complete set of eigenfunctions of \( T \). The spectrum of \( T \) is discrete and given specifically by \( \sigma(T) = \{n^2 + (\beta - 1)n + 1 \mid n \in \mathbb{N}_0\} \).
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