SMOOTH SIEGEL DISCS WITHOUT NUMBER THEORY: A REMARK ON A PROOF BY BUFF AND CHÉRITAT

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Abstract. In [BC1], Buff and Chéritat proved that there are quadratic polynomials having Siegel discs with smooth boundaries. Based on a simplification of Avila, we give yet another simplification of their proof. The main tool used is a harmonic function introduced by Yoccoz whose boundary values are the sizes of the Siegel discs. The proof also applies to some other families of polynomials, entire and meromorphic functions.

1. Introduction and Statement of Results

In 1970 Katok asked whether there exist analytic circle diffeomorphisms which are $C^\infty$ but not analytically linearizable. This question was answered affirmatively by Pérez-Marco in [PM], through the construction of Siegel discs with smooth boundaries. Pérez-Marco also announced a proof of the existence of smooth Siegel discs in the quadratic family $P_\lambda(z) = \lambda z(1 - z)$. Later Buff and Chéritat found a different proof of this result [BC1], which was subsequently simplified by Avila [Av], culminating in the joint paper [ABC]. The present note gives yet another simplification of the proof, based on representing the size of the Siegel discs as boundary values of a certain harmonic function in the unit disc, introduced by Yoccoz in [Yo]. Yoccoz used this function to give a short proof that Siegel discs exist for almost all rotation numbers, and later Carleson and Jones used it to prove that the critical point is on the boundary of the Siegel disc for almost all rotation numbers. The method of Carleson and Jones was later applied by Rippon in [Ri] to show that almost all Siegel discs in the exponential family $E_\lambda(z) = \lambda(e^z - 1)$ are unbounded.

The main theorem we are going to prove is the following.

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Theorem 1.1. Let $f(z) = z + \ldots$ be meromorphic in $\mathbb{C}$ (not necessarily transcendental), and let $f_\lambda(z) = \lambda f(z)$. Assume that $f$ has only one non-zero critical or asymptotic value $v_0$. Then there is a dense subset $S \subset \mathbb{R}$ such that $f_\lambda e^{2\pi i \alpha}$ has a smooth Siegel disc for $\alpha \in S$.

Remark. In the case where $f$ has a rotational symmetry $f(\omega z) = \omega f(z)$ with some root of unity $\omega$, it suffices to assume that $f$ has only one non-zero singular value up to multiplication by $\omega$. To prove this if $\omega$ is an $n$-th root of unity, consider $F(w) = f(w^{1/n})^n$. Then $F$ has at most one singular value, and $z \mapsto \lambda f(z)$ has a Siegel disc at $0$ if and only if $w \mapsto \lambda^n F(w)$ has one.

We can easily check that the assumption is fulfilled for some families and get the following immediate corollary.

Corollary 1.2. There are smooth Siegel discs in the following families:

$$P_{\lambda,d}(z) = \lambda \left( \left(1 + \frac{z}{d}\right)^d - 1 \right)$$
$$E_\lambda(z) = \lambda(e^z - 1)$$
$$f_\lambda(z) = \lambda z e^z$$
$$g_\lambda(z) = \lambda \sin z$$
$$h_\lambda(z) = \lambda \tan z$$

Note that the polynomials $P_{\lambda,d}$ are conjugate to $z \mapsto z^d + c$ for some $c = c(\lambda)$.

Recently Buff and Chéritat have refined the approximation technique used to produce smooth Siegel disk in order to get Siegel disks with any prescribed $C^\beta$ regularity [BC2].

2. Proof

We first recall some basic facts about linearization at attracting fixed points, for a reference see e.g. [CG]. For $0 < |\lambda| < 1$ the function $f_\lambda$ has an attracting fixed point at $0$, and by the Koenigs-Poincaré linearization theorem there exists a linearizing map $h_\lambda(z) = z + O(z^2)$ with $h_\lambda(f_\lambda(z)) = \lambda h_\lambda(z)$ near $0$. The functional equation allows to extend $h_\lambda$ to the basin of attraction of $0$ for $f_\lambda$. We define the Yoccoz function $w_\lambda$ of the family $f_\lambda$ as $w(\lambda) = h_\lambda(\lambda v)$. Notice that $\lambda v$ is the unique critical/asymptotic value of $f_\lambda$ with infinite forward orbit. The inverse map $h_\lambda^{-1}$ of $h_\lambda$ is defined in the disc $\{|w| < |w(\lambda)|/|\lambda|\}$ and does not extend to any larger disc centered at $0$. In [Yo], Yoccoz proved that the radial limit $R(\alpha) = \lim_{r \to 1} |w(re^{2\pi i \alpha})|$ exists for every $\alpha \in \mathbb{R}$. Furthermore, it coincides with the conformal radius of the
Siegel disc of $f_\lambda$ in the case of linearizability, and it is 0 otherwise. (Yoccoz proved this only for the case of the quadratic family, but his proof carries over in our case without any essential changes.) Apart from basic facts about conformal mappings, the only tool in Yoccoz’s proof is the Birkhoff Ergodic Theorem. By the Koebe 1/4-Theorem we get the estimate $|w(\lambda)| < 4|v|$ for all $\lambda \in \mathbb{D}$, and the Koebe distortion theorem also gives the asymptotic form $w(\lambda) = v\lambda + O(\lambda^2)$ at 0. Furthermore, $w(\lambda)/\lambda$ has no zeros in the unit disc, and it is still bounded by $4|v|$. Thus $u(\lambda) = \log |w(\lambda)/\lambda|$ is a harmonic function in the unit disc, bounded above by $M := \log 4 + \log |v|$. By Yoccoz’s result, the radial limit $\rho(\alpha) = \lim_{r \to 1} u(re^{2\pi i/\alpha}) \in [-\infty, M] = \log R(\alpha)$ exists for every $\alpha \in \mathbb{R}$.

**Proposition 2.1.** The function $\rho$ satisfies the following properties.

(a) $\rho(\alpha) = -\infty$ for every $\alpha \in \mathbb{Q}$.
(b) $\rho(\alpha)$ is finite for almost every $\alpha \in \mathbb{R}$.
(c) $\rho(\alpha) = \limsup_{\beta \searrow \alpha} \rho(\beta) = \limsup_{\beta \searrow \alpha} \rho(\beta)$ for every $\alpha$.
(d) $\rho$ satisfies the intermediate value property.

**Proof.** Claim (a) immediately follows from Yoccoz’s result. Claim (b) follows from Fatou’s Theorem for positive harmonic functions. Claim (c) implies (d) by the following argument: Let $\alpha < \beta$ and $\rho(\alpha) < r < \rho(\beta)$. Define $\gamma_0 := \inf\{\gamma \in [\alpha, \beta] : \rho(\gamma) \geq r\}$. Then there is a sequence $\gamma_n$ with $\rho(\gamma_n) \geq r$ and $\gamma_n \to \gamma_0$, thus $\rho(\gamma_0) \geq r$ by (c). On the other hand, $\rho(\gamma_0) = \limsup_{\gamma \searrow \gamma_0} \rho(\gamma) \leq r$, so $\rho(\gamma_0) = r$. The case $\rho(\alpha) > r > \rho(\beta)$ is treated similarly.

Proof of (c): Upper semicontinuity of $\rho$ is clear. For every $\alpha \in \mathbb{R}$ with $\rho(\alpha) > -\infty$ there is a conformal map $g_\alpha(w) = w + \ldots$ defined in $\{|w| < R(\alpha) = e^{\rho(\alpha)}\}$ with $f_\lambda(g_\alpha(w)) = g_\alpha(\lambda w)$, where we write $\lambda = e^{2\pi i/\alpha}$. If $\alpha_n \to \alpha$ with $\rho(\alpha_n) \to r > -\infty$, then the sequence of conformal maps $(g_{\alpha_n})$ is normal in $\{|w| < e^{r}\}$ and every subsequential limit $g(w) = w + \ldots$ is again a conformal map, satisfying $f_\lambda(g(w)) = g(\lambda w)$, so $f_\lambda$ has a Siegel disc of conformal radius $\geq e^r$, thus $\rho(\alpha) \geq r$.

The other direction is not quite as obvious, and the paper of Avila uses some deep results of Risler to show it. However, it also follows from the fact that $\rho$ can be represented as the boundary function of a harmonic function in the unit disc. Fix $\alpha \in \mathbb{R}$ and let $L := \limsup_{\beta \searrow \alpha} \rho(\beta)$ and $R := \limsup_{\beta \searrow \alpha} \rho(\beta)$. For every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho(\beta) \leq L + \epsilon$ for $\alpha - \delta \leq \beta < \alpha$ and $\rho(\beta) < R + \epsilon$ for $\alpha < \beta \leq \alpha + \delta$. Let $u_\epsilon$ denote the Poisson integral of the
Corollary 2.2. \[ \text{proof of the existence of smooth Siegel discs.} \]

One immediate corollary is the following curious property, central to the
terms which are still

corresponding Siegel disc of \( \rho \) with \( w \) and \( \alpha \) be arbitrary. We will find

\[ \text{integral formula.} \]

Proof. There exists a sequence of rational number \( (\beta_n) \) converging to \( \alpha \). We then have \( \rho(\beta_n) = -\infty < \rho(\alpha) \), and application of the intermediate value property yields the sequence \( \alpha_n \).

The rest of the proof essentially follows Avila’s presentation. Let \( F_r \) be the
space of holomorphic functions in \( \{|w| < r\} \), endowed with the compact-open topology. Define norms \( \|f\|_r := \sup_{|w| < r} \left| \frac{f^{(k)}(w)}{(k + 2) \log(k + 2)} \right|^k \)
for holomorphic functions in \( \{|w| < r\} \), and let \( E_r \) be the space of functions for
which this is finite. Then \( (E_r, \|\cdot\|_r) \) is a Banach space of holomorphic functions
which are still \( C^\infty \) in the closed disc \( \{|w| \leq r\} \). In fact, it is a class of quasi-analytic functions, i.e. if all derivatives of some \( f \in E_r \) vanish at a
point \( w \) with \( |w| = r \), then \( f \equiv 0 \). Furthermore, if \( r > s \), then the inclusion mapping \( F_r \hookrightarrow F_s \) is continuous, as can easily be seen from the Cauchy integral formula.

Now let \( \alpha_0 \in \mathbb{R} \) be a number with \( \rho_0 = \rho(\alpha_0) > \infty \), let \( \epsilon_0 > 0 \) and \( \rho_\infty < \rho_0 \)
be arbitrary. We will find \( \alpha_\infty \in (\alpha_0 - \epsilon_0, \alpha_0 + \epsilon_0) \), \( \rho(\alpha_\infty) = \rho_\infty \), and the

corresponding Siegel disc of \( f, \epsilon_{\alpha_\infty} \) having smooth boundary. Fix \( \delta > 0 \) such
that \( \|g - g_{\alpha_0}\|_{r_\infty} \leq 2\delta \) implies \( g' \neq 0 \) on the closed disk \( \{|w| \leq r_\infty\} \). Let \( (\rho_n) \) be a strictly decreasing sequence converging to \( \rho_\infty \). For convenience
we will adopt the notation \( r_n = e^{\rho_n} \). We recursively define sequences \( (\alpha_n) \)
and \( (\epsilon_n) \) as follows. Let \( \alpha_{n+1} \in (\alpha_n - \epsilon_n, \alpha_n + \epsilon_n) \) with \( \rho(\alpha_{n+1}) = \rho_{n+1} \)
and \( \|g_{\alpha_n} - g_{\alpha_{n+1}}\|_{r_\infty} \leq 2^{-n}\delta \). This is possible because by Corollary 2.2 there is a sequence \( (\alpha_{n,k}) \)
converging to \( (\alpha_n) \) with \( \rho(\alpha_{n,k}) = \rho_{n+1} \). The

corresponding linearizing maps \( g_{\alpha_{n,k}} \) converge locally uniformly on \( \{|w| < \)}
$\{g_{\alpha_n}\}$ to $g_{\alpha_\infty}$, so they converge in $\|\cdot\|_\infty$ because $r_\infty < r_{n+1}$. Thus we can pick $\alpha_{n+1}$ among the $(\alpha_{n,k})$ to satisfy our requirements. Now choose $\epsilon_{n+1}$ so that $[\alpha_{n+1} - \epsilon_{n+1}, \alpha_{n+1} + \epsilon_{n+1}] \subset [\alpha_n - \epsilon_n, \alpha_n + \epsilon_n]$ and $\rho(\alpha) < \rho_n$ for $|\alpha - \alpha_{n+1}| \leq \epsilon_{n+1}$. This is possible due to upper semicontinuity of $\rho$, implied by Proposition 2.1(c). Let $\alpha_\infty = \lim_{n \to \infty} \alpha_n$ the unique point in $\bigcap_n [\alpha_n - \epsilon_n, \alpha_n + \epsilon_n]$. Then by upper semicontinuity, $\rho(\alpha_\infty) \geq \lim_{n \to \infty} \rho(\alpha_n) = \rho_\infty$. On the other hand, by construction we have that $\rho(\alpha_\infty) < \rho_n$ for all $n$, so $\rho(\alpha_\infty) = \rho_\infty$. Furthermore, the sequence of linearizing maps $(g_{\alpha_n})$ is a Cauchy sequence in $\|\cdot\|_\infty$, so the limit $g_{\alpha_\infty}$ is in $E_{r_\infty}$, and $\|g_{\alpha_\infty} - g_{\alpha_0}\|_{r_\infty} \leq 2\delta$, so $g'_{\alpha_\infty} \neq 0$ on $\{|w| \leq r_\infty\}$. This implies that the Siegel disc of $f_{e^{2\pi i \alpha \infty}}$ is $S_\infty = g_{\infty}(\{|w| < r_\infty\})$ and that it is a domain with a $C^\infty$ boundary.

□

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