Abstract. Recent applications of large network models to machine learning, and to neural network suggest a need for a systematic study of the general correspondence, (i) discrete vs (ii) continuous. Even if the starting point is (i), limit considerations lead to (ii), or, more precisely, to a measure theoretic framework which we make precise. Our motivation derives from graph analysis, e.g., studies of (infinite) electrical networks of resistors, but our focus will be (ii), i.e., the measure theoretic setting. In electrical networks of resistors, one considers pairs (of typically countably infinite), sets $V$ (vertices), $E$ (edges) a suitable subset of $V \times V$, and prescribed positive symmetric functions $c$ on $E$. A conductance function $c$ is defined on $E$ (edges), or on $V \times V$, but with $E$ as its support. From an initial triple $(V, E, c)$, one gets graph-Laplacians, generalized Dirichlet spaces (also called energy Hilbert spaces), dipoles, relative reproducing kernel-theory, dissipation spaces, reversible Markov chains, and more.

Guided by applications to measurable equivalence relations, we extend earlier analyses to the non-discrete framework, and, with the use of spectral theory, we study correspondences. Our setting is that of standard Borel spaces $(M, \mathcal{B})$. Parallel to conductance functions in (i), we consider (in the measurable framework) a fixed positive, symmetric, and $\sigma$-finite measure $\rho$ on the product space $(M \times M, \mathcal{B} \times \mathcal{B})$. We study both the structures that arise as graph-limits, as well as the induced graph Laplacians, Dirichlet spaces, and reversible Markov processes, associated directly with the measurable framework.

Our main results include: spectral theory and Green’s functions for measure theoretic graph-Laplace operators; the theory of reproducing kernel Hilbert spaces related to Laplace operators; a rigorous analysis of the Laplacian on Borel equivalence relations; a new decomposition theory; irreducibility criteria; dynamical systems governed by endomorphisms and measurable fields; orbit equivalence criteria; and path-space measures and induced dissipation Hilbert spaces. We consider several applications of our results to other fields such as machine learning problems, reproducing kernel Hilbert spaces, Gaussian and determinantal processes, and joinings.

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1. **Introduction**

In the past decade, many researchers have studied diverse approaches to notions of limits of large networks. Hence there are two settings, (i) discrete vs (ii) continuous. More precisely, a starting point-setting often constitutes (i) a suitable class of network structures, each of independent interest. In this setting, an extensive (discrete) analysis has already been undertaken; see the references cited below. Now, the network analysis is typically undertaken before consideration of any kind of graph-limit, or limits. But our present focus will be a systematic study of (ii) suitable limit structures; the best known is perhaps the notion of “graphons”. In any case, the limit structures are continuous, or rather, they are studied in a measure theoretic framework. (Below, we include a brief technical summary of recent related studies; and our present opening comments here are meant merely to help non-expert readers with an orientation to the general setting.)
In summary, the setting before limits are taken is discrete: typically, a graph consisting of sets of vertices, and edges, as well as associated and specified classes of functions, and random processes. By contrast, the appropriate limit objects are “continuous”. More precisely, the category suitable for the limits is that of measure space, equipped with a definite structure which we make precise below. Prototypes of large networks (case (i)) include: the Internet, networks in chip designs, social networks, ecological networks, networks of proteins, the human brain, a network of neurons, statistical physics (large numbers of discrete particles, or spin-observables, realized on graphs, or electrical networks of resistors.

While each of these examples serves to illustrate our present analysis, we have chosen the latter (electrical networks of resistors) as a main reference for our general theory. And our focus will be (ii), i.e., the measure theoretic setting. To be more precise, by an electrical networks of resistors, we mean a pair (of typically countably infinite) sets $V$ (vertices), $E$ (edges) a suitable subset of $V \times V$, and a prescribed positive symmetric function $c$ on $E$ representing conductance. The conductance function $c$ is defined on $E$ (edges), or on $V \times V$, but with $E$ as its support. From an initial triple $(V, E, c)$, one is then lead to well defined graph-Laplacians, generalized Dirichlet spaces (also called energy Hilbert spaces), dipoles, relative reproducing kernel-theory, dissipation spaces, reversible Markov chains, and more. While this setting is of interest in its own right; see the papers cited below, our main focus here will be the structures which arise as limits, hence analogous structures on measure spaces; see Section 2 below for details. In summary, a measure space is a set $M$, and a prescribed sigma-algebra $\mathcal{B}$ of subsets of $M$. Our setting will be that of standard Borel spaces $(M, \mathcal{B})$. Parallel to the setting of conductance functions in the discrete case, our starting point for the “continuous” analysis will now be a fixed positive, symmetric, and sigma-finite measure $\rho$ on the product space $(M \times M, \mathcal{B} \times \mathcal{B})$. We shall study both the structures that arise as graph-limits, as well as the induced graph Laplacians, Dirichlet spaces, and reversible Markov processes that are associated directly with the measurable framework.

The setting in Section 2 below is motivated by a list of applications, detailed inside the paper. The list includes: a rigorous analysis of Borel equivalence relations, a new decomposition theory, irreducibility criteria, measurable fields, dynamical systems governed by endomorphisms, graph-Laplace operators in a measure theoretic framework (Sections 3 - 6). In our present general setting of measure spaces (precise details below), we present an explicit correspondence central to the questions in the paper; a correspondence between on the one hand (i) symmetric measures on product spaces, and on the other, (ii) reversible Markov transition processes. In our first results (Sections 2 and 3), we make precise the notion of equivalence for the setting of both (i) and (ii), and we prove that equivalence of two symmetric measures, leads to equivalence of the corresponding reversible Markov processes, and vice versa. Our general framework for these reversible Markov processes goes beyond earlier considerations in the literature; and hence allows us to attack questions from dynamics which were not previously accessible with existing tools.
With view to applications to spectral theory and potential theory, in Sections 3 - 7 for a given symmetric measure, we introduce a (measure) graph Laplacian $\Delta$, a finite energy space $H_E$, and a dissipation Hilbert space Diss, see Section 5. These are central notions, and their adaptation here is motivated by energy space for discrete weighted networks, such as electrical networks of resistors, and a host of areas studied extensively during last decades.

Our realization of the reversible Markov processes depends on a carefully designed infinite path space, and its associated path-space measures; both introduced in Section 5. These tools are used in turn in our study of orthogonality relations (Section 5).

In Section 8 we associate to every transient Markov processes a positive definite kernel and reproducing kernel Hilbert space (RKHS). We show that the latter is in turn a realization of the energy space. This then allows us to make precise a new Green’s function approach to our study general reversible Markov processes. Applications and examples are included.

Inside the paper, we shall include detailed citations, but to help readers with orientation, we give the following general pointers to the References.

Earlier work by the co-authors, related to the present paper, includes [BJ18a, BJ18b, Jor12, JT15, JT16, JT17].

For the theory of Dirichlet forms and generalized Laplacians, we refer to [GT17, Jor05, KZ12, Roz01]. Our definition of the finite energy Hilbert space and graph Laplacian is related to a special case of Dirichlet forms. The literature on Dirichlet forms is extensive; we recommend the following works: [AB05, AMR15, AFH11, Alb03, AKNT17, BG68, CF12, Kai92, LS88].

For the theory of reproducing kernels and their applications to Markov processes, see [AFMP94, AJ14, AJ15, Aro50, AS57, BTA04, SS16].

Throughout the paper, we shall make use of a number of tools from ergodic theory, Borel equivalence relations, orbit equivalence, and for this the reader may wish to consult [CPS82, FM77a, FM77b, GS00, Kec10, Roh49, Rev84].

The list of applications of our results includes models for online learning with Markov sampling. From the prior literature on this, we stress the following papers [AMP10, GFZ16, SY06, SJ07, SZ09, SZ09b].

We mention here also several adjacent areas where the ideas applied in this paper might be useful. First of all, the theory of electrical networks can regarded as a prototype for many of ours definitions and results. The reader can find the necessary information in various sources; for us the following books and articles are the most relevant [Kig01, JP10, JPT11, JP13, JP16, JP17, LP16, Woe00, Woe09].

The references to [FGJ+18a, FGJ+18b] deal with some aspects of the representation theory.

The reader can look at [AJL11, AJL12, AJL15, AJL17, AJL18] where Gaussian processes and path spaces are discussed.
2. PRELIMINARIES AND BASIC SETTING

Our aim in Section 2 below is to apply our present Hilbert space tools to a systematic study of measurable partitions and countable Borel equivalence relation in the context of standard (Borel) measure space, and their applications. Measurable partitions in turn serve as important tools in Borel dynamics, in ergodic theory, and in direct integral analysis in representation theory, to mention only a few. A sample of the relevant literature includes [AB05, AMR13, BJ18c, FM77b, FM77a, Gao09, JKL02, Kec10, Sim12].

2.1. Standard measure space and symmetric measures. Suppose \( V \) is a Polish space, i.e., \( V \) is a separable completely metrizable topological space. Let \( \mathcal{B} \) denote the \( \sigma \)-algebra of Borel sets generated by open sets of \( V \). Then \((V, \mathcal{B})\) is called a standard Borel space, see e.g., [Gao09, Kan08, Kec95, Kec10] and papers [Che89, Loe75] for detailed information about standard Borel spaces. We recall that all uncountable standard Borel spaces are Borel isomorphic, so that, without loss of generality, we can use any convenient realization of the space \((V, \mathcal{B})\).

If \( \mu \) is a positive non-atomic Borel measure on \((V, \mathcal{B})\), then \((V, \mathcal{B}, \mu)\) is called a standard measure space. Given \((V, \mathcal{B}, \mu)\), we will call \( \mu \) a measure for brevity. As a rule, we will deal with non-atomic \( \sigma \)-finite positive measures on \((V, \mathcal{B})\) which take values in the extended real line \( \overline{\mathbb{R}} \). We use the name of standard measure space for both probability (finite) and \( \sigma \)-finite measure spaces. Also the same notation, \( \mathcal{B} \), is applied for the \( \sigma \)-algebras of Borel sets and measurable sets of a standard measure space. Working with a measure space \((V, \mathcal{B}, \mu)\), we always assume that \( \mathcal{B} \) is complete with respect to \( \mu \). By \( \mathcal{F}(V, \mathcal{B}) \), we denote the space of real-valued bounded Borel functions on \((V, \mathcal{B})\). For \( f \in \mathcal{F}(V, \mathcal{B}) \) and a Borel measure \( \mu \) on \((V, \mathcal{B})\), we write

\[
\mu(f) = \int_V f \, d\mu.
\]

All objects, considered in the context of measure spaces (such as sets, functions, transformations, etc), are considered by modulo sets of zero measure. In most cases, we will implicitly use this mod 0 convention not mentioning the sets of zero measure explicitly.

For a \( \sigma \)-finite no-atomic measure \( \mu \) on a standard Borel space \((V, \mathcal{B})\), we denote by

\[
\mathcal{B}_{\text{fin}} = \mathcal{B}_{\text{fin}}(\mu) = \{ A \in \mathcal{B} : \mu(A) < \infty \} \tag{2.1}
\]

the algebra of Borel sets of finite measure \( \mu \). Clearly, any \( \sigma \)-finite measure \( \mu \) is uniquely determined by its values on \( \mathcal{B}_{\text{fin}}(\mu) \).

The linear space of simple function over sets from \( \mathcal{B}_{\text{fin}}(\mu) \) is denoted by

\[
\mathcal{D}_{\text{fin}}(\mu) : = \left\{ \sum_{i \in I} a_i \chi_{A_i} : A_i \in \mathcal{B}_{\text{fin}}(\mu), \; a_i \in \mathbb{R}, \; |I| < \infty \right\} = \operatorname{Span}\{\chi_A : A \in \mathcal{B}_{\text{fin}}(\mu)\} \, \tag{2.2}
\]
and it will play an important role in the further results since simple functions from $D_{fin}(\mu)$ form a norm dense subset in $L^p(\mu)$-space, $p \geq 1$.

**Definition 2.1.** Let $E$ be an uncountable Borel subset of the Cartesian product $(V \times V, B \times B)$ such that:

(i) $(x, y) \in E \iff (y, x) \in E$, i.e. $\theta(E) = E$ where $\theta(x, y) = (y, x)$ is the flip automorphism;

(ii) $E_x := \{y \in V : (x, y) \in E\} \neq \emptyset$, $\forall x \in X$;

(iii) for every $x \in V$, $(E_x, B_x)$ is a standard Borel space where $B_x$ is the $\sigma$-algebra of Borel sets induced on $E_x$ from $(V, B)$.

The set $E$ is called *symmetric subset* of $(V \times V, B \times B)$.

It follows from (ii) and (iii) of Definition 2.1 that the projection of a symmetric set $E$ on each margin of the product space $(V \times V, B \times B)$ is $V$. Moreover, condition (iii) assumes two cases: the Borel space $E_x$ can be either countable or uncountable. We will assume that $(E_x, B_x)$ is an uncountable Borel standard spaces.

There are several natural examples of symmetric sets related to dynamical systems. We mention here the case of a *Borel equivalence relation* $E$ on a standard Borel space $(V, B)$. By definition, $E$ is a Borel subset of $V \times V$ such that $(x, y) \in E$ for all $x \in V$, $(x, y)$ is in $E$ iff $(y, x)$ is in $E$, and $(x, y) \in E, (y, z) \in E$ implies that $(x, z) \in E$. Let $E_x = \{y \in V : (x, y) \in E\}$, then $E$ is partitioned into “vertical fibers” $E_x$. In particular, it can be the case when every $E_x$ is countable. Then $E$ is called a *countable Borel equivalence relation*.

As mentioned in Introduction, our approach is based on the study of *symmetric measures* defined on $(V \times V, B \times B)$, see Definition 2.4 below. We recall that every measure $\rho$ on the Cartesian product $(V \times V, B \times B)$ can be disintegrated with respect to a measurable partition. For this, denote by $\pi_1$ and $\pi_2$ the projections from $V \times V$ onto the first and second factor, respectively. Then $\{\pi_1^{-1}(x) : x \in V\}$ and $\{\pi_2^{-1}(y) : y \in V\}$ are the *measurable partitions* of $V \times V$ into vertical and horizontal fibers, see [Roh49] [CFS82] [BJ18a] for more information on properties of measurable partitions. The case of probability measures was studied by Rokhlin in [Roh49], whereas the disintegration of $\sigma$-finite measures is more delicate and has been considered somewhat recently, see for example [GS00]. We refer here to a result from [Sim12] whose formulation is adapted to our needs.

**Theorem 2.2** ([Sim12]). For a $\sigma$-finite measure space $(V, B, \mu)$, let $\rho$ be a $\sigma$-finite measure on $(V \times V, B \times B)$ such that $\rho \circ \pi_1^{-1} \ll \mu$. Then there exists a unique system of conditional $\sigma$-finite measures $(\tilde{\rho}_x)$ such that

$$\rho(f) = \int_V \tilde{\rho}_x(f) \, d\mu(x), \quad f \in F(V \times V, B \times B).$$

**Remark 2.3.** (1) The condition of Theorem 2.2 assumes that a measure $\mu$ is prescribed on the Borel space $(V, B)$. If one begins with a measure $\rho$ defined on $(V \times V, B \times B)$, then the measure $\mu$ can be taken as the projection of $\rho$ on $(V, B)$, $\rho \circ \pi_1^{-1} = \mu$. 
(2) Let $E$ be a Borel symmetric subset of $(V \times V, \mathcal{B} \times \mathcal{B})$, and let $\rho$ be a measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ satisfying the condition of Theorem 2.2. Then $E$ can be partitioned into the fibers $\{x\} \times E_x$. By Theorem 2.2, there exists a unique system of conditional measures $\tilde{\rho}_x$ such that, for any $\rho$-integrable function $f(x, y)$, we have

$$\int_{V \times V} f(x, y) \, d\rho(x, y) = \int_V \tilde{\rho}_x(f(x)) \, d\mu(x). \tag{2.3}$$

It is obvious that, for $\mu$-a.e. $x \in V$, $\text{supp}(\tilde{\rho}_x) = \{x\} \times E_x$ (up to a set of zero measure). To simplify the notation, we will write

$$\int_V f \, d\rho_x \quad \text{and} \quad \int_{V \times V} f \, d\rho$$

though the measures $\rho_x$ and $\rho$ have the supports $E_x$ and $E$, respectively.

(3) It follows from Theorem 2.2 that the measure $\rho$ determines the measurable field of sets $x \mapsto E_x \subset V$ and measurable field of $\sigma$-finite Borel measures $x \mapsto \rho_x$ on $(V, \mathcal{B})$, where the measures $\rho_x$ are defined by the relation

$$\tilde{\rho}_x = \delta_x \times \rho_x. \tag{2.4}$$

Hence, relation (2.3) can be also written in the following form, used in our subsequent computations,

$$\int_{V \times V} f(x, y) \, d\rho(x, y) = \int_V \left( \int_V f(x, y) \, d\rho_x(y) \right) \, d\mu(x). \tag{2.5}$$

In other words, we have a measurable family of measures $(x \mapsto \rho_x)$, and it defines a new measure $\nu$ on $(V, \mathcal{B})$ by setting

$$\nu(A) := \int_V \rho_x(A) \, d\mu(x), \quad A \in \mathcal{B}. \tag{2.6}$$

We note that the measure $\rho_x$ is defined on the subset $E_x$ of $(V, \mathcal{B})$, $x \in V$.

The next definition contains the key notion of a symmetric measure.

**Definition 2.4.** (1) Let $(V, \mathcal{B})$ be a standard Borel space. We say that a measure $\rho$ on $(V \times V, \mathcal{B} \times \mathcal{B})$ is symmetric if

$$\rho(A \times B) = \rho(B \times A), \quad \forall A, B \in \mathcal{B}.$$ 

In other words, $\rho$ is invariant with respect to the flip automorphism $\theta$.

**Assumption 1.** In this paper, we will consider the set of symmetric measures $\rho$ on $(V \times V, \mathcal{B} \times \mathcal{B})$ which satisfy the following property:

$$0 < c(x) := \rho_x(V) < \infty, \quad \mu\text{-a.e. } x \in V, \tag{2.7}$$

where $x \mapsto \rho_x$ is the measurable field of measures (the system of conditional measures) arising in Theorem 2.2.

Moreover, we will assume that symmetric measures satisfy the condition: $c(x) \in L^1_{\text{loc}}(\mu)$, i.e.,

$$\int_A c(x) \, d\mu(x) < \infty, \quad \forall A \in \mathcal{B}_{\text{fin}}(\mu). \tag{2.8}$$
Remark 2.5. It is natural to assume that the function \( c(x) \) satisfies properties (2.7) and (2.8). They correspond to local finiteness of discrete graphs in the theory of weighted networks. In several statements below, we need to use the requirement that \( c \in L^2_{\text{loc}}(\mu) \), i.e.,
\[
\int_A c^2 \, d\mu < \infty, \quad \forall A \in B_{\text{fin}}(\mu).
\]
We observe also that the case when the function \( c \) is bounded will lead to bounded Laplace operators and is not interesting for us.

Based on Assumption 1, we note that relation (2.6) defines the measure \( \nu \) which is equivalent to \( \mu \). As stated in Lemma 2.7, \( c(x) \) is the Radon-Nikodym derivative of \( \nu \) with respect to \( \mu \). If we want to reverse the definition and use \( \nu \) as a primary measure, then we need to require that the function \( c(x)^{-1} \) is locally integrable with respect to \( \nu \).

In the following remark, we discuss some natural properties of symmetric measures.

Remark 2.6. (1) If \( \rho \) is a symmetric measure on \((V \times V, B \times B)\), then the support of \( \rho \), the set \( E = E(\rho) \), is symmetric mod 0. Hence, without loss of generality, we can assume that any symmetric measure is supported by the sets \( E \) satisfying Definition 2.1. We will require that, for every \( x \in V \), the set \( E_x \subset E \) is an uncountable standard Borel space. The case when \( E_x \) is countable arises for a countable Borel equivalence relation \( E \) on \((V, B)\). The latter was considered in \([BJ18b]\). For countable sets \( E_x, x \in V \), we can take \( \rho_x \) as a finite measure which is equivalent to the counting measure, see, e.g. \([FM77a, FM77b, KM04]\) for details.

(2) In general, the notion of a symmetric measure is defined in the context of standard Borel spaces \((V, B)\) and \((V \times V, B \times B)\). But if a \( \sigma \)-finite measure \( \mu \) is given on \((V, B)\), then we have to include an additional relation between the measures \( \rho \) and \( \mu \). Let \( \pi_1 : V \times V \to V \) be the projection on the first coordinate. We require that \( \rho \circ \pi_1^{-1} \ll \mu \), see Theorem 2.2.

(3) The symmetry of the set \( E \) allows us to define a “mirror” image of the measure \( \rho \). Let \( E^y := \{ x \in V : (x, y) \in E \} \), and let \( (\tilde{\rho}^y) \) be the system of conditional measures with respect to the partition of \( E \) into the sets \( E^y \times \{y\} \). Then, for the measure
\[
\tilde{\rho} = \int_V \tilde{\rho}^y \, d\mu(y),
\]
the relation \( \rho = \tilde{\rho} \) holds.

(4) It is worth noting that, in general, when a measure \( \mu \) is defined on \((V, B)\), the set \( E(\rho) \) do not need to be a set of positive measure with respect to the product measure \( \mu \times \mu \). In other words, we admit both cases: (a) \( \rho \) is equivalent to \( \mu \times \mu \) on the set \( E \), and (b) \( \rho \) and \( \mu \times \mu \) are mutually singular.

The following (important for us) fact is deduced from the definition of symmetric measures. We emphasize that formula (2.9) will be used repeatedly in many proofs.
Lemma 2.7. (1) For a symmetric measure \( \rho \) and any bounded Borel function \( f \) on \((V \times V, \mathcal{B} \times \mathcal{B})\),
\[
\int_{V \times V} f(x,y) \, d\rho(x,y) = \int_{V \times V} f(y,x) \, d\rho(x,y).
\] (2.9)

Equality (2.9) is understood in the sense of the extended real line, i.e., the infinite value of the integral is allowed.

(2) Let \( \nu \) be defined as in (2.6). Then
\[
d\nu(x) = c(x) \, d\mu(x).
\]

We define below the key notion of an irreducible symmetric measure.

For a standard measure space \((V, \mathcal{B})\), a kernel \( k \) is a function \( k : V \times \mathcal{B} \to \mathbb{R}_+ = [0, \infty] \) such that
(i) \( x \mapsto k(x, A) \) is measurable for every \( A \in \mathcal{B} \);
(ii) for any \( x \in V \), \( k(x, \cdot) \) is a \( \sigma \)-finite measure on \((V, \mathcal{B})\).

A kernel \( k(x, A) \) is called finite if \( k(x, \cdot) \) is a finite measure on \((V, \mathcal{B})\) for every \( x \).
We will also use the notation \( k(x, dy) \) for the measure on \((V, \mathcal{B})\).

Given a kernel \( k \), one can construct inductively the sequence of kernels \((k^n : n \geq 1)\) by setting
\[
k^n(x, A) = \int_V k^{n-1}(y, A) \, k(x, dy), \quad n > 1.
\] (2.10)

It is said that a set \( A \in \mathcal{B} \) is attainable from \( x \in V \) if there exists \( n \geq 1 \) such that \( K^n(x, A) > 0 \); in symbols, we write \( x \to A \). A set \( A \in \mathcal{B} \) is called closed for
the kernel \( K \) if \( K(x, A^c) = 0 \) for all \( x \in A \). If \( A \) is closed, then it follows from (2.10) that \( K^n(x, A^c) = 0 \) for any \( n \in \mathbb{N} \) and \( x \in A \). Hence, \( A \) is closed if and only if \( x \to A^c \), see details in [Num84] [Rev84].

Every symmetric measure \( \rho \) defines a finite kernel \( K = K(\rho) \) where
\[
V \times \mathcal{B} \xrightarrow{K} [0, \infty) : K(x, A) = \rho_x(A),
\]
and \( x \to \rho_x(\cdot) \) is the measurable family of conditional measures from Theorem 2.2.

Definition 2.8. (1) A kernel \( x \to k(x, \cdot) \) is called irreducible with respect to a \( \sigma \)-finite measure \( \mu \) on \((V, \mathcal{B})\) (\( \mu \)-irreducible) if, for any set \( A \) of positive measure \( \mu \) and \( \mu \)-a.e. \( x \in V \), there exists some \( n \) such that \( k^n(x, A) > 0 \), i.e., any set \( A \) of positive measure is attainable from \( \mu \)-a.e. \( x \).

(2) A symmetric measure \( \rho \) on \((V \times V, \mathcal{B} \times \mathcal{B})\) is called irreducible if the corresponding kernel \( K(\rho) : (x, A) \to \rho_x(A) \) is \( \mu \)-irreducible where \( \mu \) is the projection of measure \( \rho \) onto \((V, \mathcal{B})\).

(3) A symmetric measure \( \rho \) (or the kernel \( x \to \rho_x(\cdot) \)) is called \( \mu \)-decomposable if there exists a Borel subset \( A \) of \( V \) of positive measure \( \mu \) such that
\[
E \subset (A \times A) \cup (A^c \times A^c)
\] (2.11)
where \( A^c = V \setminus A \) is also of positive measure. Otherwise, \( \rho \) is called indecomposable.
Let $\rho$ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. For any fixed $x \in V$, we define a sequence of subsets: $A_0(x) = \{x\}, A_1(x) = E_x$, 
$$ A_n(x) = \bigcup_{y \in A_{n-1}(x)} E_y, \quad n \geq 2. $$ Recall that $E_x$ is the support of the measure $\rho_x$, and $E_x$ can be identified with the vertical section of the symmetric set $E$. Note that $A_n(x) \in \mathcal{B}$ as $x \rightarrow E_x$ is a measurable field of sets.

**Lemma 2.9.** (1) Given $(V, \mathcal{B}, \mu)$, a symmetric measure $\rho$ is irreducible if and only if for $\mu$-a.e. $x \in V$ and any set $B \in \mathcal{B}$ of positive measure $\mu$ there exists $n \geq 1$ such that 
$$ \mu(A_n(x) \cap B) > 0. $$

(2) Let $K(x, A) = \rho_x(A)$. Suppose that the support of $\rho$, the set $E$, satisfies relation (2.11) where $\mu(A) > 0$ and $\mu(A^c) > 0$. Then the sets $A$ and $A^c$ are closed, and $x \mapsto \rho_x(A)$ is a $\mu$-reducible kernel. The converse statement also holds.

This lemma was proved in [BJ18b].

2.2. Symmetric measures and associated operators of Hilbert spaces. Let $(V, \mathcal{B}, \mu)$ be a $\sigma$-finite measure space, and let $\rho$ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ supported by a symmetric set $E$. Let $x \mapsto \rho_x$ be the measurable family of measures on $(V, \mathcal{B})$ that disintegrates $\rho$. Recall that, by Assumption 1, the function $c(x) = \rho_x(V)$ is finite for $\mu$-a.e. $x$. As discussed Subsection 2.1, the measure $\rho$ generates a finite kernel $K(\rho)$ which we use to define the following operators.

**Definition 2.10.** For a symmetric measure $\rho$ on $(V \times V, \mathcal{B} \times \mathcal{B})$, we define three linear operators $R, P$, and $\Delta$ acting on the space of bounded Borel functions $\mathcal{F}(V, \mathcal{B})$.

(i) The **symmetric operator** $R$:
$$ R(f)(x) := \int_V f(y) \, d\rho_x(y) = \rho_x(f). $$

(ii) The **Markov operator** $P$:
$$ P(f)(x) = \frac{1}{c(x)} R(f)(x) $$
or
$$ P(f)(x) := \frac{1}{c(x)} \int_V f(y) \, d\rho_x(y) = \int_V f(y) \, P(x, dy) $$
where $P(x, dy)$ is the probability measure obtained by normalization of $d\rho_x(y)$, i.e.
$$ P(x, dy) := \frac{1}{c(x)} d\rho_x(y). $$

(iii) The **graph Laplace operator** $\Delta$:
$$ \Delta(f)(x) := \int_V (f(x) - f(y)) \, d\rho_x(y) $$
(2.15)
or
\[ \Delta(f) = c(I - P)(f) = (cI - R)(f). \] (2.16)

Remark 2.11. We assemble in this remark several obvious properties of the defined operators.

1. The operator \( R \) (and therefore \( P \)) is positive, in the sense that \( f \geq 0 \) implies \( R(f) \geq 0 \). Moreover, \( R(1) = c(x) \) and therefore, using (2.7), we can write the operator \( \Delta \) in more symmetric form:
\[ \Delta(f) = R(1)f - R(f) \]
where \( 1 \) is a function on \( V \) identically equal to 1.

2. For the operator \( P \), we have \( P(1) = 1 \). It justifies the name of Markov operator used for \( P \). Conversely, any Markov operator \( P \) defines the probability kernel \( x \mapsto P(x,A) = P(\chi_A)(x) \) called also the transition probabilities.

3. It is worth noting that the Laplacian \( \Delta \) can be defined using a different approach. In Theorem 8.15, it is proved that \( \Delta(f) \) is the Radon-Nikodym derivative of a measure \( \mu_f \) with respect to \( \mu \).

4. Since every measure \( \rho \) on \( V \times V \) is uniquely determined by its values on a dense subset of functions, it suffices to define \( \rho \) on the set of the so-called “cylinder functions” \( (f \otimes g)(x,y) := f(x)g(y) \). This observation will be used below when we prove some relations for cylinder functions first.

5. In general, a positive operator \( R \) in \( \mathcal{F}(V,B) \) is called symmetric if it satisfies the relation:
\[ \int_V fR(g) \, d\mu = \int_V R(f)g \, d\mu, \] (2.17)
for any \( f,g \in \mathcal{F}(V,B) \). It can be easily seen that \( R \) is symmetric if and only if it defines a symmetric measure \( \rho \) by the formula
\[ \rho(A \times B) = \int_V \chi_A(x)R(\chi_B)(x) \, d\mu(x). \]

Remark 2.12. We do not discuss domains of the operators \( R, P, \) and \( \Delta \). They depend on the space where an operator is considered. In the current paper, we work with \( L^2 \)-spaces defined by the measures \( \mu, \nu, \) and \( \rho \). But the most intriguing is the case of the finite energy space Hilbert space \( h_E \) which is defined below.

We are interested in the following question. Let \( x \mapsto \rho_x \) be a measurable field of finite measures over a standard Borel space \( (V,B) \). How can one describe the set of Borel \( \sigma \)-finite measures \( \mu \) on \( (V,B) \) such that the measure \( \rho = \int_V \rho_x d\mu(x) \) would be a symmetric measure on \( (V \times V,B \times B) \). A partial answer was given in Remark...
if the operator $R : f \mapsto \int_V f \, d\rho_x$ satisfies (2.17), then the pair $\{x \mapsto \rho_x, \mu\}$ determines a symmetric measure. In the following proposition we clarify relations between symmetric measures $\rho$ and symmetric operators $R$ that were defined in Remark 2.11.

**Proposition 2.12.** Let $x \mapsto \rho_x$ be a measurable field of finite measures over a standard Borel space $(V, B)$ which defines the operator $R$. Suppose that $\mu$ is a measure on $(V, B)$ such that (2.17) holds. Let $p(x) \in L^1_{\text{loc}}(\mu)$ be a positive Borel function and $R'(f) := R(fp)$.

(1) The measure

$$\rho'(A \times B) := \int_V \chi_A R'(\chi_B) \, d\mu$$

is symmetric on $(V \times V, B \times B)$ if and only if $R \circ M_p = M_p \circ R$ where $M_p$ is the operator of multiplication by $p$.

(2) Given a positive Borel function $p \in L^1_{\text{loc}}(\mu)$ and the measure $d\beta(x) = p(x) d\mu(x)$, the measure $\rho_\beta = \int_V \rho_x d\beta(x)$ is symmetric on $(V \times V, B \times B)$ if and only if the operator $R'$ is symmetric with respect to the pair $\{(x \mapsto \rho_x), \beta\}$.

**Proof.** To show that (1) holds, we use the fact that the measure $\rho = \int_V \rho_x d\mu(x)$ is symmetric:

$$\rho'(A \times B) = \int_V \chi_A R'(\chi_B) \, d\mu$$

$$= \int_V \chi_A \left( \int_V p \chi_B \, d\rho_x \right) \, d\mu$$

$$= \int \int_{V \times V} \chi_A(x) p(y) \chi_B(y) \, d\rho(x, y)$$

$$= \int \int_{V \times V} \chi_A(y) p(x) \chi_B(x) \, d\rho(x, y)$$

$$= \int_V \chi_B p R(\chi_A) \, d\mu.$$

On the other hand,

$$\rho'(B \times A) = \int_V \chi_B R'(\chi_A) \, d\mu.$$

Then $\rho'(A \times B) = \rho'(B \times A)$ if and only if $R'(\chi_A) = p R(\chi_A)$. By linearity, the latter is extended to the relation $R \circ M_p = M_p \circ R$. 
(2) For this, we compute
\[
\int_V R'(f) g \, d\beta = \int_V R(fp) g \, d\beta \\
= \int_V R(fp) gp \, d\mu \\
= \int_V (fp) R(gp) \, d\mu \\
= \int_V f R'(g) \, d\beta
\]

\[\square\]

**Corollary 2.13.** Let \(\rho\) and \(\rho'\) be two symmetric measures on \((V \times V, \mathcal{B} \times \mathcal{B})\) defined by the pairs \(\{(x \mapsto \rho_x), \mu\}\) and \(\{(x \mapsto \rho'_x), \mu'\}\), respectively. Then the Laplace operators \(\Delta(\rho)\) and \(\Delta(\rho')\) coincide.

In the following statement we discuss properties of \(R, P, \) and \(\Delta\) as operators acting in \(L^2\)-spaces, see [BJ18a, BJ18b] for details.

**Theorem 2.14.** For a standard measure space \((V, \mathcal{B}, \mu)\), let \(\rho\) be a symmetric measure on \((V \times V, \mathcal{B} \times \mathcal{B})\) such that \(c(x) = \rho_x(V)\) satisfies Assumption 1. Let \(d\nu(x) = c(x) d\mu(x)\) be the \(\sigma\)-finite measure on \((V, \mathcal{B})\) equivalent to \(\mu\), and let the operators \(R, P, \) and \(\Delta\) be defined as in Definition 2.10.

1. Suppose that the function \(x \mapsto \rho_x(A) \in L^2(\mu)\) for every \(A \in \mathcal{B}_{fin}\). Then \(R\) is a symmetric unbounded operator in \(L^2(\mu)\), i.e.,
\[
\langle g, R(f) \rangle_{L^2(\mu)} = \langle R(g), f \rangle_{L^2(\mu)}.
\]

If \(c \in L^\infty(\mu)\), then \(R : L^2(\mu) \to L^2(\mu)\) is a bounded operator, and
\[
\|R\|_{L^2(\mu) \to L^2(\mu)} \leq \|c\|_{\infty}.
\]
Relation (2.18) is equivalent to the symmetry of the measure \(\rho\).

2. The operator \(R : L^1(\nu) \to L^1(\mu)\) is contractive, i.e.,
\[
\|R(f)\|_{L^1(\mu)} \leq \|f\|_{L^1(\nu)}, \quad f \in L^1(\nu).
\]
Moreover, for any function \(f \in L^1(\nu)\), the formula
\[
\int_V R(f) \, d\mu(x) = \int_V f(x)c(x) \, d\mu(x)
\]
holds. In other words, \(\nu = \mu R\), and
\[
\frac{d(\mu R)}{d\mu}(x) = c(x).
\]

3. The bounded operator \(P : L^2(\nu) \to L^2(\nu)\) is self-adjoint. Moreover, \(\nu P = \nu\).

\[\footnote{This means that the operator \(R\) is densely defined on functions from \(\mathcal{D}_{fin}(\mu)\); in particular, this property holds if \(c \in L^2(mu)\).}\]
The operator $P$, considered in the spaces $L^2(\nu)$ and $L^1(\nu)$, is contractive, i.e.,

$$||P(f)||_{L^2(\nu)} \leq ||f||_{L^2(\nu)}, \quad ||P(f)||_{L^1(\nu)} \leq ||f||_{L^1(\nu)}.$$  

(5) The spectrum of $P$ in $L^2(\nu)$ is a subset of $[-1, 1]$.

(6) The graph Laplace operator $\Delta : L^2(\mu) \to L^2(\mu)$ is a positive definite essentially self-adjoint operator with domain containing $D_{lin}(\mu)$. Moreover,

$$||f||^2_{H_E} = \int_V f \Delta(f) \, d\mu$$

when the integral in the right hand side exists.

Remark 2.15. Suppose that a non-symmetric measure $\rho$ is given on the space $(V \times V, \mathcal{B} \times \mathcal{B})$, i.e., $\rho(A \times B) \neq \rho(B \times A)$, in general. However, we will assume that $\rho$ is equivalent to $\rho \circ \theta$ where $\theta(x, y) = (y, x)$. Then we can define the following objects: margin measures $\mu_i := \rho \circ \pi_i^{-1}, i = 1, 2$, fiber measures $d\rho_x(\cdot)$ and $d\rho^x(\cdot)$ (see Remark 2.6), and functions $c_1(x) = \rho_x(V), c_2(x) = \rho^x(V)$.

Define now the symmetric measure $\rho^\#$ generated by $\rho$:

$$\rho^\# := \frac{1}{2}(\rho + \rho \circ \theta).$$

Then

$$\rho^\#(A \times B) = \frac{1}{2}(\rho(A \times B) + \rho(B \times A)).$$

Clearly, $\rho^\#$ is equivalent to $\rho$.

Let $E \subset V \times V$ be the support of $\rho$. Then $E^\# = E \cup \theta(E)$ is the support of the symmetric measure $\rho^\#$. The disintegration of $\rho = \int_V \rho_x \, d\mu_1(x)$ with respect to the partition $\{x\} \times E_x$ defines the disintegration of $\rho^\#$. For $\mu^\# := \frac{1}{2}(\mu_1 + \mu_2)$, we obtain that

$$\rho^\# = \int_V (\rho_x + \rho^x) \, d\mu^\#.$$

Having the symmetric measure $\rho^\#$ defined on $(V \times V, \mathcal{B} \times \mathcal{B})$, we can introduce the operators $R^\#$ and $P^\#$ as in (2.13) and (2.14). It turns out that, for $f \in \mathcal{F}(V, \mathcal{B})$,

$$R^\#(f)(x) = R_1(f)(x) + R_2(f)(x)$$

where

$$R_1(f) = \int_V f(y) \, d\rho_x(y), \quad R_2(f) = \int_V f(y) \, d\rho^x(y).$$

Similarly,

$$P^\#(f)(x) = \frac{1}{c^\#(x)} R^\#(f)(x)$$

where

$$c^\#(x) = \rho_x(V) + \rho^x(V).$$

Then we can define the measure $d\nu^\#(x) = c^\#(x) d\mu(x)$ such that the operator

$$P^\#(f)(x) = \int_V f(y) \frac{1}{c^\#(x)} \, d\rho^\#_x(y)$$
is self-adjoint in $L^2(\nu^\#)$. By Theorem 2.20 (see below), we obtain that the Markov process generated by $x \mapsto P^\#(x, \cdot)$ is reversible where $P^\#(x, A) = P^\#(\chi_A)(x)$.

Here we define an important class of functions that will be discussed throughout the paper.

**Definition 2.16.** A function $f \in \mathcal{F}(V, \mathcal{B})$ is called harmonic, if $Pf = f$. Equivalently, $f$ is harmonic if $\Delta f = 0$ or $R(f) = cf$. Similarly, $h$ is harmonic for a kernel $x \mapsto k(x, \cdot)$ if

$$\int_V h(y) k(x, dy) = h(x).$$

The set of harmonic functions will be denoted by $\mathcal{Harm}(P)$.

It turns out that $P$-harmonic functions cannot lie in the space $L^2(\nu)$ where $\nu$ is $P$-invariant.

**Theorem 2.17 ([BJ18a]).** Given a standard measure space $(V, \mathcal{B}, \nu)$, let $P$ be a Markov operator on $L^2(\nu)$ defined by transition probabilities as in (2.14). Suppose that $\nu P = \nu$. Then

$$L^2(\nu) \cap \mathcal{Harm}(P) = \begin{cases} 0, & \nu(V) = \infty \\ \mathbb{R}, & \nu(V) < \infty \end{cases}$$

Moreover, 1 does not belong to the point spectrum of the operator $P$ on the space $L^2(\nu)$.

In what follows, we discuss briefly the relationship between symmetric measures and the notion of reversible Markov processes.

Recall our setting. Let $(V, \mathcal{B})$ be a standard Borel space, $\rho$ a symmetric $\sigma$-finite measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ satisfying Assumption 1, $\mu$ the projection of $\rho$ on $(V, \mathcal{B})$, $c(x) = \rho_x(V)$ where $x \mapsto \rho_x$ is the system of conditional measures. Let $P$ denote the Markov operator defined by the family of transition probabilities $x \mapsto P(x, \cdot) = c(x)^{-1} d\rho_x(y)$:

$$P(f)(x) = \int_V f(y) P(x, dy) \quad (2.20)$$

Furthermore, we can use the kernel $x \mapsto P(x, \cdot) = P_t(x, \cdot)$ to define the sequence of probability kernels (transition probabilities) $(P_n(x, : n \in \mathbb{N})$ in accordance with (2.10):

$$P_{n+m}(x, A) = \int_V P_n(y, A) P_m(x, dy), \quad n, m \in \mathbb{N}.$$
Using the Markov operator $P$, we can define the sequence of measures $(\rho_n)$ (here $\rho_1 = \rho$) on $(V \times V, \mathcal{B} \times \mathcal{B})$ by setting
\[
\rho_n(A \times B) = \langle \chi_A, P^n(\chi_B) \rangle_{L^2(\nu)}, \quad n \in \mathbb{N}.
\] (2.21)

Then it can be easily seen that the following properties hold.

**Lemma 2.18.** For the measures $\rho_n$, defined by (2.21), we have:

(i) every measure $\rho_n, n \in \mathbb{N}$, is symmetric on $(V \times V, \mathcal{B} \times \mathcal{B})$, and $\rho_n$ is equivalent to $\rho$;

(ii) $\rho_x^{(n)}(V) = c(x)$;

(iii) $d\rho_n(x,y) = c(x)P_n(x,dy)d\mu(x) = P_n(x,dy)d\nu(x)$; (2.22)

(iv) $||P^n(\chi_A)||^2_{L^2(\nu)} = \rho_{2n}(A \times A)$;

(vi) $\rho_n(A \times B) = \langle \chi_A, RP^{n-1}(\chi_B) \rangle_{L^2(\nu)}$.

**Definition 2.19.** Suppose that $x \mapsto P(x,\cdot)$ is a measurable family of transition probabilities on the space $(V, \mathcal{B}, \mu)$, and let $P$ be the Markov operator determined by $x \mapsto P(x,\cdot)$. It is said that the corresponding Markov process is *reversible* with respect to a measurable functions $c : x \to (0, \infty)$ on $(V, \mathcal{B})$ if, for any sets $A, B \in \mathcal{B}$, the following relation holds:
\[
\int_B c(x)P(x,A)\ d\mu(x) = \int_A c(x)P(x,B)\ d\mu(x).
\] (2.23)

It turns out that the notion of reversibility is equivalent to the following properties.

**Theorem 2.20** ([BJ18a], [BJ18b]). Let $(V, \mathcal{B}, \mu)$ be a standard $\sigma$-finite measure space, $x \mapsto c(x) \in (0, \infty)$ a measurable function, $c \in L^1_{\text{loc}}(\mu)$. Suppose that $x \mapsto P(x,\cdot)$ is a probability kernel. The following are equivalent:

(i) $x \mapsto P(x,\cdot)$ is reversible (i.e., it satisfies (2.23));

(ii) $x \mapsto P_n(x,\cdot)$ is reversible for any $n \geq 1$;

(iii) the Markov operator $P$ defined by $x \mapsto P(x,\cdot)$ is self-adjoint on $L^2(\nu)$ and $\nu P = \nu$ where $d\nu(x) = c(x)d\mu(x)$;

(iv) $c(x)P(x,dy)d\mu(x) = c(y)P(y,dx)d\mu(y)$;

(v) the operator $R$ defined by the relation $R(f)(x) = c(x)P(f)(x)$ is symmetric;

(vi) the measure $\rho$ on $(V \times V, \mathcal{B} \times \mathcal{B})$ defined by
\[
\rho(A \times B) = \int_V c(x)R(\chi_A)\ d\mu = \int_V c(x)\chi_A P(\chi_B)\ d\mu
\]
is symmetric;

(vii) for every $n \in \mathbb{N}$, the measure $\rho_n$ defined by (2.27) is symmetric.
We finish this section discussing the following problem. We recall briefly our main setting. Let $\rho$ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. Then it generates the following objects: the marginal measure $\mu = \rho \circ \pi_1^{-1}$ such that $\rho = \int_V \rho_x d\mu$, function $c(x) = \rho_x(V)$, and the measure $\nu = c \mu$. The corresponding Markov operator $P$ is defined by the measurable field of transition probabilities $x \mapsto P(x, \cdot)$ where $c(x)P(x,dy) = d\rho_x(y)$, see (2.20). It was proved that the measure $\nu$ is $P$-invariant.

Suppose now we have two symmetric measures $\rho$ and $\rho'$ defined on $(V \times V, \mathcal{B} \times \mathcal{B})$ which are equivalent; let $d\rho'(x,y) = r(x,y)$. Then $r(x,y) = r(y,x)$, see [BJ18b] for details. Let the collection $(\mu', \rho'_x, c', \nu', P')$ be defined by the measure $\rho'$.

**Proposition 2.21.** Suppose that, for two equivalent symmetric measures $\rho$ and $\rho'$ on $(V \times V, \mathcal{B} \times \mathcal{B})$, the Markov operators $P$ and $P'$ coincide. Then $\nu'$ is a constant multiple of $\nu$.

**Proof.** Since $\rho \sim \rho'$, the margin measures $\mu$ and $\mu'$ are also equivalent, so that the Radon-Nikodym derivative

$$\frac{d\mu}{d\mu'}(x) = m(x)$$

is positive a.e. It can be seen that

$$\frac{d\rho_x'(y)}{d\rho_x}(y) = r_x(y)m(x) \quad (2.24)$$

where $r_x(\cdot) := r(x, \cdot)$. If $P = P'$, then $P(x,dy) = P'(x,dy)$ because $P(x,A) = P(\chi_A)(x)$. Hence,

$$\frac{d\rho_x'(y)}{d\rho_x}(y) = \frac{c'(x)}{c(x)},$$

and we obtain from (2.24) that

$$r_x(y) = \frac{c'(x)}{c(x)m(x)}.$$ 

This means that $r(x,y)$ depends on the variable $x$ only. But by symmetry of the function $r$, we conclude that $r(x,y)$ is a constant a.e. This proves the result. \qed

3. **Finite energy space: definitions and first results**

The finite energy space $\mathcal{H}_E$, see Definition 3.1 below, is one of the central notions of this article. It can be viewed as a generalization of the energy space for discrete weighted networks which have been extensively studied during last decades. We partially follow the paper [BJ18a] where this space was defined and studied.
3.1. Inner product and norm in $\mathcal{H}_E$.

**Definition 3.1.** Let $(V, \mathcal{B}, \mu)$ be a standard measure space with $\sigma$-finite measure $\mu$. Suppose that $\rho$ is a symmetric measure on the Cartesian product $(V \times V, \mathcal{B} \times \mathcal{B})$ whose projection on $V$ is $\mu$. We say that a Borel function $f : V \to \mathbb{R}$ belongs to the finite energy space $\mathcal{H}_E = \mathcal{H}_E(\rho)$ if

$$\int_{V \times V} (f(x) - f(y))^2 \, d\rho(x,y) < \infty. \quad (3.1)$$

To simplify the notation, we will also use the symbol $\mathcal{H}$ to denote the finite energy space $\mathcal{H}_E$.

**Remark 3.2.** (1) It follows from Definition 3.1 that $\mathcal{H}_E$ is a vector space containing all constant functions. We identify functions $f_1$ and $f_2$ such that $f_1 - f_2 = \text{const}$ and, with some abuse of notation, the quotient space is also denoted by $\mathcal{H}_E$. So that we will call elements $f$ of $\mathcal{H}_E$ functions assuming that a representative of the equivalence class $f$ is considered. It will be easily seen that our results do not depend on the choice of representatives.

(2) Definition 3.1 assumes that a symmetric irreducible measure $\rho$ is fixed on $(V \times V, \mathcal{B} \times \mathcal{B})$. This means that the space of functions $f$ on $(V, \mathcal{B})$ satisfying (3.1) depends on $\rho$, and, strictly speaking, this space must be denoted as $\mathcal{H}_E(\rho)$. As was mentioned above, it is a challenging problem to study relations between $\mathcal{H}_E(\rho)$ and $\mathcal{H}_E(\rho')$ for equivalent measures $\rho$ and $\rho'$, see [BJ18b, Theorem 4.11] for a discussion.

Define a bilinear form $\xi(f, g)$ in the space $\mathcal{H}$ by the formula

$$\xi(f, g) := \frac{1}{2} \int_{V \times V} (f(x) - f(y))(g(x) - g(y)) \, d\rho(x,y). \quad (3.2)$$

We denote $\xi(f) = \xi(f, f)$. Setting

$$\langle f, g \rangle_{\mathcal{H}} = \xi(f, g),$$

we define an inner product on the space $\mathcal{H}_E$. Then

$$||f||^2_{\mathcal{H}_E} := \frac{1}{2} \int_{V \times V} (f(x) - f(y))^2 \, d\rho(x,y), \quad f \in \mathcal{H}, \quad (3.3)$$

turns $\mathcal{H}_E$ in a normed vector space. As proved in [BJ18a], $\mathcal{H}_E$ is a Hilbert space with respect to the norm $|| \cdot ||_{\mathcal{H}_E}$.

We remark that the zero vector in $\mathcal{H}_E$ is represented by a constant function.

The definition of the Hilbert space $\mathcal{H}_E$ and the norm given in (3.3) allows us to define an embedding of the energy space into $L^2(\rho)$. The following lemma is based on the definition of the norm in $\mathcal{H}_E$.

**Lemma 3.3.** The map

$$\partial : f(x) \mapsto (\partial f)(x,y) := \frac{1}{\sqrt{2}} \, (f(x) - f(y))$$

is an isometric embedding of the space $\mathcal{H}_E$ into $L^2(\rho)$. 
We will use an assumption about local integrability of functions from the energy space. We will discuss the following assumption in detail below.

**Assumption 2.** Let a symmetric measure $\rho$ on $(V \times V, \mathcal{B} \times \mathcal{B})$ be chosen so that all functions from $\mathcal{H}_E = \mathcal{H}_E(\rho)$ are locally square integrable, i.e.,

$$\mathcal{H}_E \subset L^2_{\text{loc}}(\mu).$$

(3.4)

Since $L^2_{\text{loc}}(\mu) \subset L^1_{\text{loc}}(\mu)$, we can always assume that functions from the space $\mathcal{H}_E$ are locally integrable.

It is worth noting that this assumption is very mild. It holds automatically for the case of locally finite discrete weighted networks.

We consider first some immediate properties of functions from the space $\mathcal{H}_E$. These properties have been discussed in [BJ18a].

**Proposition 3.4.** (1) Let $\rho$ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $\mu = \rho \circ \pi_1^{-1}$. Suppose that $c(x) = \rho_x(V)$ is locally integrable with respect to $\mu$. Then:

$$\mathcal{D}_\text{fin}(\mu) \subset \mathcal{D}_\text{fin}(\nu) \subset \mathcal{H}_E$$

(3.5)

where $d\nu(x) = c(x)d\mu(x)$. Moreover, if $A \in \mathcal{B}_\text{fin}(\nu)$, then

$$||\chi_A||^2_{\mathcal{H}_E} = \rho(A \times A^c) \leq \int_A c(x) \, d\mu(x) = \nu(A),$$

(3.6)

where $A^c := V \setminus A$;

(2) for every $A \in \mathcal{B}$, the function $\chi_A$ belongs to $\mathcal{H}_E$ if and only if $\chi_{A^c}$ is in $\mathcal{H}_E$; moreover, $||\chi_A||_{\mathcal{H}_E} = ||\chi_{A^c}||_{\mathcal{H}_E}$ and

$$\langle \chi_A, \chi_{A^c} \rangle_{\mathcal{H}_E} = -\rho(A \times A^c).$$

In other words, the function $\chi_A$ is in $\mathcal{H}_E$ if and only if either $\mu(A) < \infty$ or $\mu(A^c) < \infty$:

$$\mathcal{B}_\text{fin}(\nu) \cap (\mathcal{B}_\text{fin}(\nu))^c \subset \mathcal{H}_E.$$

**Proof.** For (1), let $A \in \mathcal{B}_\text{fin}(\mu)$, then $\nu(A) = \int_A c \, d\mu < \infty$ because $c$ is a locally integrable function. Hence, $\mathcal{D}_\text{fin}(\mu) \subset \mathcal{D}_\text{fin}(\nu)$. To prove that $\mathcal{D}_\text{fin}(\nu) \subset \mathcal{H}_E$, we compute the norm of $\chi_A$:

$$||\chi_A||^2_{\mathcal{H}_E} := \frac{1}{2} \int_{V \times V} (\chi_A(x) - \chi_A(y))^2 \, d\rho(x, y) = \rho(A \times A^c).$$

Indeed, the function $(\chi_A(x) - \chi_A(y))^2$ takes value 1 if and only if either $x \in A$ and $y \in A^c$ or $y \in A$ and $x \in A^c$. Then the integral of $(\chi_A(x) - \chi_A(y))^2$ with respect to $\rho$ is $2\rho(A \times A^c)$ because $\rho$ is symmetric. To finish the proof of (3.6), we calculate

$$\rho(A \times A^c) = \int_A \rho_x(A^c) \, d\mu(x) \leq \int_A c(x) \, d\mu(x) = \nu(A).$$

This proves (3.5).

To see that (2) holds, we recall that the energy norm of any constant function is zero. Hence, the relation $\chi_A + \chi_{A^c} = 0$ is true when the characteristic functions are considered as elements of $\mathcal{H}_E$. It follows that the vectors $\chi_A$ and $\chi_{A^c}$ have the
same norm, and their inner product in \( \mathcal{H}_E \) is always non-positive. The criterion for \( \chi_A \in \mathcal{H}_E \) is a direct consequence of (1). \( \square \)

**Remark 3.5.** (1) As proved above, the set of functions from \( D_{\text{fin}}(\nu) \) (and therefore its subset \( D_{\text{fin}}(\mu) \)) belongs to \( \mathcal{H}_E \), and, for any set \( A \) of finite measure \( \nu \), we have

\[
||\chi_A||^2_{\mathcal{H}_E} = \rho(A \times A^c)
\]

\[
= \int_A \rho_x(A^c) \, d\mu(x)
\]

\[
= \int_A (c(x) - \rho_x(A)) \, d\mu(x).
\]

(2) \( \chi_A = 0 \) in \( \mathcal{H} \) \( \iff \) \( ||\chi_A||_{\mathcal{H}_E} = 0 \) \( \iff \) \( \rho_x(A) = c(x) \), \( \mu \)-a.e. \( x \in A \) \( \iff \) \( \rho(A \times A) = \rho(A \times V) \);

(3) It is useful to remember that \( D_{\text{fin}}(\mu) \subset D_{\text{fin}}(\nu) \) and \( D_{\text{fin}}(\mu) \subset L^2(\mu) \cap L^2(\nu) \cap \mathcal{H}_E \). But the set \( D_{\text{fin}}(\mu) \) is not dense in \( \mathcal{H}_E \), see details below.

(4) We observe that statements (1) and (2) of Proposition 3.4 give some criteria for a characteristic function \( \chi_A \) (and a function from \( D_{\text{fin}}(\mu) \)) to be in \( \mathcal{H}_E \). In particular, \( \chi_A \) is not in \( \mathcal{H}_E \) if and only if both \( A \) and \( A^c \) have infinite measure.

(5) Furthermore, since \( f = 0 \) in \( \mathcal{H}_E \) if and only if \( f \) is a constant a.e., and since two functions from \( \mathcal{H}_E \) are identified if they differ by a constant, we conclude that the equality \( \chi_A = -\chi_{A^c} \) holds when these functions are considered in \( \mathcal{H}_E \).

The description of the structure of the Hilbert space \( \mathcal{H}_E \) is a very intriguing problem. We formulate in the following statement several properties of vectors from \( \mathcal{H}_E \) related mostly to characteristic functions. Some of them have been proved in [BJ18a]. More statements will be added in Section 7.

**Theorem 3.6.** Let measures \( \rho \) and \( \mu \) be as in Proposition 3.4, and \( c(x) \in L^1_{\text{loc}}(\mu) \).

(1) If \( \chi_A, \chi_B \in \mathcal{H}_E \), then

\[
\langle \chi_A, \chi_B \rangle_{\mathcal{H}_E} = \rho((A \cap B) \times V) - \rho(A \times B) = \nu(A \cap B) - \rho(A \times B).
\]

(3.7)

(2) The following conditions are equivalent to the orthogonality of \( \chi_A \) and \( \chi_B \):

(i)

\[
\chi_A \perp \chi_B \iff \rho((A \setminus B) \times B) = \rho((A \cap B) \times B^c));
\]

(ii)

\[
\chi_A \perp \chi_B \iff \left( \int_{A \cap B} c(x) \, d\mu(x) = \int_{A} \rho_x(B) \, d\mu(x) \right);
\]

(iii) if \( A \subset B \) and \( \mu(A) > 0 \), then

\[
\chi_A \perp \chi_B \iff \rho_x(B^c) = 0 \quad \text{for a.e.} \; x \in A.
\]

(iv) if \( A \cap B = \emptyset \), then

\[
\chi_A \perp \chi_B \iff \rho(A \times B) = 0;
\]

and more generally,

\[
A \cap B = \emptyset \implies \langle \chi_A, \chi_B \rangle_{\mathcal{H}_E} \leq 0.
\]
Proof. (1) The computation is based on the definition, given in (3.3), and the property of symmetry for \( \rho \):

\[
\langle \chi_A, \chi_B \rangle_{\mathcal{H}_E} = \frac{1}{2} \int_{V \times V} (\chi_A(x) - \chi_A(y))(\chi_B(x) - \chi_B(y)) \, d\rho(x, y)
\]

\[
= \int_{V \times V} (\chi_A(x)\chi_B(x) - \chi_A(x)\chi_B(y)) \, d\rho(x, y)
\]

\[
= \int_V \int_V \chi_{(A \cap B) \times V}(x, y) \, d\rho(x, y) - \int_V \int_V \chi_{A \times B}(x, y) \, d\rho(x, y)
\]

\[
= \rho((A \cap B) \times V) - \rho(A \times B)
\]

\[
= \nu(A \cap B) - \rho(A \times B),
\]

because

\[
\rho((A \cap B) \times V) = \int_{A \cap B} c(x) \, d\mu(x)
\]

\[
= \nu(A \cap B).
\]

(2) For (i), we see from (3.7) that the vectors \( \chi_A \) and \( \chi_B \) are orthogonal if and only if \( \rho((A \cap B) \times V) = \rho(A \times C) \). Then \( \langle \chi_A, \chi_B \rangle_{\mathcal{H}_E} = 0 \) if and only if

\[
\rho((A \cap B) \times B^c) = \rho((A \cap B) \times V) - \rho((A \cap B) \times B)
\]

\[
= \rho(A \times B) - \rho((A \cap B) \times B)
\]

\[
= \rho((A \setminus B) \times B).
\]

For (ii), we use the first equality in (3.7) which is written in integrals.

If \( A \subset B \), then the condition \( \chi_A \perp \chi_B \) is equivalent to the property \( \rho(A \times B^c) = 0 \) what proves (iii).

The last equivalence in (2) is immediate from (3.7).

\( \square \)

3.2. Harmonic functions in \( \mathcal{H}_E \). Our goal is to study the properties of the Laplace operator \( \Delta = \Delta(\rho) \) considered acting on functions from the finite energy space \( \mathcal{H}_E \) according to formula (2.15). The next theorem is a key statement that has a number of important consequences. Its proof uses the made assumption that functions from \( \mathcal{H}_E \) belong to \( L^2_{\text{loc}}(\mu) \).

**Theorem 3.7.** Suppose that \( c \in L^2_{\text{loc}}(\mu) \). Let \( \varphi \in \mathcal{D}_{\text{fin}}(\mu) \) and \( f \in \mathcal{H}_E \). Then

\[
\langle \varphi, f \rangle_{\mathcal{H}_E} = \int_V \varphi \Delta(f) \, d\mu.
\]

**Proof.** Since \( \mathcal{D}_{\text{fin}}(\mu) \) is spanned by characteristic functions, it suffices to prove (3.8) for \( \varphi = \chi_A \) where \( \mu(A) < \infty \).

It follows from the condition of this theorem and Assumption 2 that, for any \( f \in \mathcal{H}_E \), the function \( fc \) belongs to \( L^1_{\text{loc}}(\mu) \). Indeed, since functions \( c \) and \( f \) are in
In the computation given below, we use the following facts: the formula for $R(f)$, see (2.13), the formula for $\Delta(f)$, see (2.16), the definition of the inner product in $H_E$, and the fact that the measure $\rho$ is symmetric, see (2.9) and (3.2).

\[
\langle \chi_A, f \rangle_{H_E} = \frac{1}{2} \int_{V \times V} (\chi_A(x) - \chi_A(y))(f(x) - f(y)) \, d\rho(x, y)
\]

\[
= \int_{V \times V} (\chi_A(x)f(x) - \chi_A(x)f(y)) \, d\rho(x, y)
\]

\[
= \int_V \left( \int_V (\chi_A(x)f(x) - \chi_A(x)f(y)) \, d\rho_x(y) \right) \, d\mu(x)
\]

\[
= \int_V (\chi_A(x)f(x)c(x) - \chi_A(x)R(f)(x)) \, d\mu(x)
\]

\[
= \int_V \chi_A(x)\Delta(f)(x) \, d\mu(x)
\]

□

**Remark 3.8.** (1) To justify the correctness of this computation, we note that in the relation

\[
\langle \chi_A, f \rangle_{H_E} = \int_V (\chi_A(x)f(x)c(x) - \chi_A(x)R(f)(x)) \, d\mu(x), \quad A \in B_{\text{fin}}(\mu),
\]

the integral $\int_V \chi_A f c \, d\mu$ is finite and therefore $\int_V \chi_A R(f) \, d\mu$ is finite too. We can state even more, namely, the function $R(f) \in L^1_{\text{loc}}(\mu)$.

Given $f \in H_E$, denote $f_+ = \max(f, 0), f_- = \min(f, 0)$; then $f_\pm \in H_E$. Indeed, since $|f_+ - f_-| \leq |f(x) - f(y)|$, we see that $\|f_\pm\|_{H_E} \leq \|f\|_{H_E}$.

Therefore, we see that $R(f_\pm)$ is locally integrable because $R$ is a positive operator, and the inequality

\[
|R(f)| \leq R(|f|) = R(f_+) + R(f_-)
\]

implies that $|R(f)|$ is locally integrable.

(2) Another simple consequence of the fact that $cf \in L^1_{\text{loc}}(\mu)$ is that $f \in L^1_{\text{loc}}(\nu)$.

**Corollary 3.9.** (1) In conditions of Theorem 3.7, the function $\Delta(f)$ is locally integrable for any $f \in H_E$.

(2)

\[
\left( \int_A \Delta(f) \, d\mu \right)^2 \leq \rho(A \times A^c)\|f\|_{H_E}^2, \quad A \in B_{\text{fin}}(\mu).
\]

(3) For every $f \in H_E$, the map

\[
A \mapsto \mu_f(A) := \int_A \Delta(f) \, d\mu
\]
determines a finite additive measure on \((V,B)\). The measure \(\mu_f(\cdot)\) is \(\sigma\)-additive if and only if the function \(\Delta(f)\) is integrable on \((V,B,\mu)\).

**Proof.** To see that (1) holds, we write \(\Delta(f) = cf - R(f)\) and use the proved facts that \(cf\) and \(R(f)\) are locally integrable.

The second statement is the Schwarz inequality where we uses the formula

\[||\chi_A||_{H_E}^2 = \rho(A \times A^c).\]

(3) is obvious. \(\Box\)

We denote by \(\text{Harm}_E\) the set of harmonic functions in \(\mathcal{H}_E\), i.e., a function \(h \in \mathcal{H}_E\) is harmonic if \(\Delta h = 0\). Equivalently, \(h\) is harmonic if \(P(h) = h\).

**Theorem 3.10.** The finite energy space \(\mathcal{H}_E\) admits the decomposition into the orthogonal sum

\[\mathcal{H}_E = D_{\text{fin}}(\mu) \oplus \text{Harm}_E\]  

(3.9)

where the closure of \(D_{\text{fin}}(\mu)\) is taken in the norm of the Hilbert space \(\mathcal{H}_E\).

**Proof.** It follows from Theorem 3.7 and (3.8) that if a function \(f \in \mathcal{H}_E\) is orthogonal to every characteristic function \(\chi_{A_i}, A_i \in B_{\text{fin}}(\mu)\), then

\[\int_V \chi_{A_i} \Delta(f) \, d\mu = 0.\]

Therefore, \(\Delta(f)(x) = 0\) for \(\mu\)-a.e. \(x \in V\). This means that \(f\) is harmonic in \(\mathcal{H}_E\).

Conversely, the same theorem implies that harmonic functions are orthogonal to \(D_{\text{fin}}(\mu)\) and therefore to the closure of \(D_{\text{fin}}(\mu)\). \(\Box\)

Formula (3.9) is an analogue of the so called Royden decomposition used in the theory of weighted networks.

**Remark 3.11.** Theorem 3.10 together with Theorem 3.6 show that the Hilbert space \(\mathcal{H}_E\) has no canonical orthonormal basis. Possible candidates \(\{\chi_{A_i}, A_i \in B_{\text{fin}}(\mu)\}\), where the sets \(\{A_i\}\) generate \(B\), are not orthogonal and their span is not even dense in \(\mathcal{H}_E\).

Another property of the Hilbert space \(\mathcal{H}_E\) that makes it nonstandard is the fact that the multiplication operator \(M_\varphi : f \mapsto \varphi f\) is not symmetric in \(\mathcal{H}_E\) when \(\varphi\) is nonzero. This result follows immediately from comparison the expressions for \(\langle \varphi f, g \rangle_{\mathcal{H}_E}\) and \(\langle f, \varphi g \rangle_{\mathcal{H}_E}\).

As seen from Remark 3.5 (5), pointwise identities should not be confused with Hilbert space identities in \(\mathcal{H}_E\). The point is that elements of \(\mathcal{H}_E\) are equivalence classes of functions which differ only by a constant. When working with representatives, we typically abuse notation and use the same symbol \(f\) to denote the equivalence class and the function.

**Proposition 3.12.** Let \(f \in \text{Harm}_E\) be a harmonic function for the Laplace operator \(\Delta = \Delta(\rho)\) acting in \(\mathcal{H}_E\) where \(\rho\) is a symmetric measure. Suppose that \(c \in L^2_{\text{loc}}(\mu)\) where \(c(x) = \rho_x(V)\). Then, for any function \(\varphi \in D_{\text{fin}}(\mu)\), we have

\[\int_V \Delta(\varphi) f \, d\mu = 0.\]
Proof. By Assumption 2, we can assume that the harmonic function $f$ is in $L^2_{\text{loc}}(\mu)$. The condition of the proposition means that the function $c(x)f(x)$ is locally integrable. Since $D_{\text{fin}}(\mu)$ is spanned by characteristic functions, it suffices to prove the result for $\varphi = \chi_A$. We have

$$\Delta(f)(x) = 0 \iff c(x)f(x) = R(f)(x),$$

where $R$ is the symmetric operator corresponding to $\rho$. Therefore, for any $A \in B_{\text{fin}}(\mu)$, one has

$$\int_V \chi_A(x)c(x)f(x) \, d\mu(x) = \int_V \chi_A(x)R(f)(x) \, d\mu(x) = \int_V R(\chi_A)(x)f(x) \, d\mu,$$

and this means that

$$\int_V (\chi_A c f - R(\chi_A)f) \, d\mu = 0$$

or equivalently

$$\int_V \Delta(\chi_A) f \, d\mu = 0.$$

□

Remark 3.13. (1) Clearly, the condition $c \in L^2_{\text{loc}}(\mu)$ can be replaced with $c \in L^1_{\text{loc}}(\nu)$.

(2) Fix a harmonic function $f$ and consider the set of all functions $g$ such that $\int_V gcf \, d\mu$ exists. Then we use the proof given above to conclude that

$$\int_V \Delta(g) f \, d\mu = 0.$$

Corollary 3.14. Let $f_1, f_2$ be any elements of the finite energy space $\mathcal{H}_E$ which is defined by a symmetric measure $\rho$. Then there are functions $\varphi_1, \varphi_2 \in D_{\text{fin}}(\mu)$ and $h_1, h_2 \in \mathcal{Harm}_E$ such that

$$\langle f_1, f_2 \rangle_{\mathcal{H}_E} = \int_V \varphi_1 \Delta(\varphi_2) \, d\mu + \langle h_1, h_2 \rangle_{\mathcal{H}_E} . \quad (3.10)$$

Moreover, if $f_1 \in D_{\text{fin}}(\mu)$, then

$$\langle f_1, f_2 \rangle_{\mathcal{H}_E} = \iint_{V \times V} f_1(x)(f_2(x) - f_2(y)) \, d\rho(x, y) . \quad (3.11)$$

Proof. By Theorem 3.10, every element $f \in \mathcal{H}_E$ is uniquely represented as $\varphi + h$ where $\varphi \in D_{\text{fin}}(\mu)$ and $h \in \mathcal{Harm}$. This property defines the functions $\varphi_i$ and $h_i$ for given $f_i, i = 1, 2$. 


To show that (3.10) holds, we use Theorem 3.7 and Proposition 3.12:

$$\langle f_1, f_2 \rangle_{H_E} = \langle \varphi_1, f_2 \rangle_{H_E} + \langle h_1, f_2 \rangle_{H_E}$$

$$= \int_V \varphi_1 \Delta(f_2) \, d\mu + \langle h_1, \varphi_2 + h_2 \rangle_{H_E}$$

$$= \int_V \varphi_1 \Delta(\varphi_2) \, d\mu + \langle h_1, h_2 \rangle_{H_E}. $$

Relation (3.11) follows directly from the proof of Theorem 3.7 and from formula (2.9) which characterizes symmetric measures. We leave details to the reader. □

Obviously, relation (3.10) can be written in the form

$$\langle f_1, f_2 \rangle_{H_E} = \langle \varphi_1, \Delta(\varphi_2) \rangle_{L^2(\mu)} + \langle h_1, h_2 \rangle_{H_E}. $$

Remark 3.15. (1) Suppose that the finite energy space consists of functions \( f \) that belong to \( L^2_{\text{loc}}(\mu) \cap L^2_{\text{loc}}(\nu) \). Then we claim that, for any \( \varphi \in \mathcal{D}_{\text{fin}}(\nu) \) and \( f \in H_E \),

$$\langle \varphi, f \rangle_{H_E} = \int_V \varphi \Delta(f) \, d\mu. $$

The proof repeats that of Theorem 3.7. The key point is that under the made assumption the function \( fc \) is in \( L^1_{\text{loc}}(\mu) \). Indeed, if \( \varphi = \chi_A \) where \( A \in \mathcal{B}_{\text{fin}}(\nu) \), then

$$\int_A fc \, d\mu = \int_A f \chi_A \, d\nu \leq \sqrt{\nu(A)} \left( \int_A f^2 \, d\nu \right)^{1/2} < \infty. $$

Having this result, we repeat word for word the computation of \( \langle \varphi, f \rangle_{H_E} \) from the proof of Theorem 3.7.

(2) Suppose that \( H_E \subset L^2_{\text{loc}}(\mu) \cap L^2_{\text{loc}}(\nu) \) as in (1). Then we can prove the following version of Proposition 3.12: for any \( \varphi \in \mathcal{D}_{\text{fin}}(\nu) \) and any harmonic function \( f \), we have

$$\int_V \Delta(\varphi) f \, d\mu = 0. $$

To prove this result, we again use the fact that \( fc \) is in \( L^1_{\text{loc}}(\mu) \) and then follow the proof of Proposition 3.12. We note that the assumption \( c \in L^2_{\text{loc}}(\mu) \) is not used in this version.

(3) Finally, we can deduce the result about the orthogonal decomposition of functions from \( H_E \). Assuming that \( H_E \subset L^2_{\text{loc}}(\mu) \cap L^2_{\text{loc}}(\nu) \), we can show that

$$\mathcal{D}_{\text{fin}}(\nu)^{H_E} = \mathcal{D}_{\text{fin}}(\mu)^{H_E}$$

and therefore

$$H_E = \mathcal{D}_{\text{fin}}(\nu) \oplus \text{Harm}_E. $$

We return to this property later in more general setting.
3.3. Six applications. In this subsection, we consider several types of symmetric measures. Each of these types corresponds to a possible direction for further applications of our approach based on symmetric measures. We give a few examples of symmetric measures $\rho$ and discuss how they can be used to define the objects we are interested in.

(I) Gaussian processes. This example is motivated by works on Gaussian processes where Cameron-Martin kernel $(x,y) \mapsto x \wedge y$ plays an important role.

Let $(V, \mathcal{B}, \mu)$ be a standard measure space with a $\sigma$-finite measure $\mu$, and let $c : V \to \mathbb{R}_+$ be a locally integrable Borel function. Define the measure $\nu$ on $(V, \mathcal{B}, \mu)$ by $d\nu(x) = c(x) d\mu(x)$. Set $\rho_\nu(A \times B) := \nu(A \cap B)$ where $A,B \in \mathcal{B}$. Then $\rho_\nu$ can be extended to a $\sigma$-finite symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. We note that, for $A \in \mathcal{B}$,

$$\rho_\nu \circ \pi^{-1}(A) = \rho(A \times V) = \nu(A) = \int_A c(x) \, d\mu(x).$$

(3.12)

Hence $\rho_\nu \circ \pi^{-1} \ll \mu$ and, by Theorem 2.2, the measure admits a decomposition $\rho = \int_V \rho_x \, d\mu(x)$. Since

$$\rho(A \times V) = \int_A \rho_x(V) \, d\mu(x),$$

we deduce from (3.12) that $\rho_x(V) = c(x)$.

Lemma 3.16. The measure $\rho_x$ is atomic for $\mu$-a.e. $x \in V$, and $d\rho_x(y) = c(x) \delta_x(y)$.

Proof. Indeed, we can write

$$\nu(A \cap B) = \int_V \chi_A(x) \chi_B(x) c(x) \, d\mu(x)$$

$$= \int_V \chi_A(x) \left( \int_V \chi_B(y) c(y) \delta_x(y) \right) \, d\mu(x)$$

$$= \rho(A \times B)$$

$$= \int_V \chi_A(x) \left( \int_V \chi_B(y) d\rho_x(y) \right) \, d\mu(x).$$

Since $A$ and $B$ are arbitrary sets, this proves that $\rho_x$ is the atomic measure supported at $(x,x)$ with weight $c(x)$.

Based on the Claim, we can easily determine the operators $R, P$, and $\Delta$ related to the measure $\rho_\nu$ as well as the finite energy space $\mathcal{H}_E(\rho_\nu)$. It turns out that the corresponding Markov process $(P_n)$ is deterministic since

$$P(x, A) = \delta_x(A), \quad A \in \mathcal{B},$$
and therefore $P$ is the identity operator. It follows that $R(f)(x) = c(x)f(x)$, and the Laplacian $\Delta = c(I - P) = 0$. By a similar argument, $\mathcal{H}_E(\rho_x) = \{0\}$.

**II Countable Borel equivalence relations.** For a measure space $(V, \mathcal{B}, \mu)$, let $c_{xy}$ be a symmetric positive function defined on a symmetric set $E \subset V \times V$. Consider a measurable field of finite Borel measures $x \mapsto \rho_x$ such that (i) the support of $\rho_x$ is the set $\{x\} \times E_x$, (ii) the measure

$$
\rho := \int_V c_{xy} \rho_x^c \, d\mu(x)
$$

is symmetric. In particular, the set $E$ can be of positive product measure $\mu \times \mu$. This case is discussed in Section 6. Another important example is the case of a countable Borel equivalence relation $E$.

By definition, a symmetric Borel subset $E \subset V \times V$ is a **countable Borel equivalence relation** if it satisfies the following properties:

(i) $(x, y), (y, z) \in E \implies (x, z) \in E$;
(ii) $E_x = \{y \in V : (x, y) \in E\}$ is countable for every $x$.

Countable Borel equivalence relations have been extensively studied during last decades in the context of the descriptive set theory, measurable and Borel dynamics, see e.g. [JKL02, KM04, Kan08, Gao09, Kec10, FM77a, FM77b] and references cited therein.

Let $|\cdot|$ be the counting measure on every $E_x$. Suppose that $c_{xy}$ is a symmetric function on $E$ such that, for every $x \in V$,

$$
c(x) = \sum_{y \in E_x} c_{xy} \in (0, \infty).
$$

Then we can define the atomic measure $\rho_x$ on $V$ by setting

$$
\rho_x(A) = \sum_{y \in E_x \cap A} c_{xy}.
$$

Finally, define the measure $\rho$ on $E$:

$$
\rho = \int_V \delta_x \times \rho_x \, d\mu(x). \quad (3.13)
$$

(We will identify measures $\rho_x$ and $\delta_x \times \rho_x$ as we did above.)

**Lemma 3.17.** The measure $\rho$ is a symmetric measure on $E$ which is singular with respect to $\mu \times \mu$. 
Proof. Since \((\mu \times \mu)(E) = 0\), the singularity of \(\rho\) is obvious. It follows from the symmetry of the function \(c_{xy}\) and (3.13) that, for \(A, B \in \mathcal{B}\),

\[
\rho(A \times B) = \int_A \sum_{y \in E_x \cap B} c_{xy} \, d\mu(x) \\
= \int_B \sum_{x \in E_y \cap A} c_{xy} \, d\mu(y) \\
= \rho(B \times A).
\]

□

Having the measure \(\rho\) defined, we apply the definitions given in Subsection 2.2 to construct the following operators:

\[
R(f)(x) = \int_V f(y) \, d\rho_x(y) = \sum_{y \in E_x} c_{xy} f(y),
\]

\[
P(f)(x) = \sum_{y \in E_x} \frac{c_{xy}}{c(x)} f(y) = \sum_{y \in E_x} p(x, y) f(y),
\]

and

\[
\Delta(f)(x) = c(x) f(x) - \sum_{y \in E_x} c_{xy} f(y).
\]

Functions \(f\) from the finite energy Hilbert space \(\mathcal{H}_E(\rho)\) are determined by the condition:

\[
\int_V \sum_{y \in E_x} c_{xy} (f(x) - f(y))^2 \, d\mu(x) < \infty.
\]

Definition 3.18. Let \(E\) be a countable Borel equivalence relation on a standard Borel space \((V, \mathcal{B})\). A symmetric subset \(G \subset E\) is called a graph if \((x, x) \not\in G, \forall x \in V\). A graphing of \(E\) is a graph \(G\) such that the connected components of \(G\) are exactly the \(E\)-equivalence classes. In other words, a graph \(G\) generates \(E\).

The notion of a graphing is useful for the construction of the path space \(\Omega\) related to a Markov process, see Section 5.

The following lemma can be easily proved.

Lemma 3.19. Let \(\rho\) be a countable equivalence relation on \((V, \mathcal{B})\), and let \(\rho\) be a symmetric measure on \(E\). Suppose \(G\) is a graphing of \(E\). Then \(\rho(G) > 0\).

We can use the notion of graphing to construct the path space \(\Omega\) and the family of probability measures \(x \mapsto P_x\) defined on the set of paths with starting point \(x\). This approach is realized in Section 5 in more general setting.

For more details regarding integral operators, and analysis of machine learning kernels, the reader may consult the following items [Atk75, CZ07, CWK17, Ho17, JT15] and the papers cited there.
(III) **Graphons.** Our approach in the study of symmetric measures, and the corresponding graph Laplace operators, is close to the basic setting of the theory of *graphons* and *graphon operators*. We refer to several basic works in this theory [APSS17, BCL+08, BCL+12, Jan13, Lov12]. Informally speaking, a graphon is the limit of a converging sequence of finite graphs with increasing number of vertices. Formally, a graphon is a symmetric measurable function \( W : (\mathcal{X}, m) \times (\mathcal{X}, m) \to [0, 1] \) where \((\mathcal{X}, m)\) is a probability measure space. The linear operator \( \mathbb{W} : L^2(\mathcal{X}, m) \to L^2(\mathcal{X}, m) \) acting by the formula

\[
\mathbb{W}(f)(x) = \int_\mathcal{X} W(x, y) f(y) \, dm(y)
\]

is called the graphon operator. The properties of \( \mathbb{W} \) have been extensively studied in many recent works, see e.g. [APSS17].

Below in Section 3, we consider a similar operator \( \tilde{R} \) defined by a symmetric measure \( \rho \). The principal difference is that we consider infinite measure spaces and symmetric functions which are not bounded, in general.

(IV) **Determinantal point processes.** One more application of our results can be used in the theory of determinantal measures and determinantal point processes, see e.g. [Lyo03, HKPV09, BQ15, BO17]. For example, the result of [Gho15, Proposition 4.1] gives the formula for the norm in the energy space for a specifically chosen symmetric measure \( \rho \). To make this statement more precise, we quote loosely the proposition proved in [Gho15]:

Let \( \Pi \) be a determinantal point process on a locally compact space \((X, \mu)\) with positive definite determinantal kernel \( K(\cdot, \cdot) \) such that \( K \) is an integral operator on \( L^2(\mu) \). Then, for every compactly supported function \( \psi \),

\[
\text{Var} \left[ \int_X \psi \, d[\Pi] \right] = \iint_{X \times X} |\psi(x) - \psi(y)|^2 |K(x, y)|^2 \, d\mu(x) d\mu(y).
\]

This formula is exactly the formula for the norm in the energy space when the symmetric measure \( \rho \) is defined by the symmetric function \( K(x, y) \): \( d\rho(x, y) = |K(x, y)|^2 d\mu(x) d\mu(y) \), see Section 7 below.

We refer to the following papers regarding the theory of positive definite kernels [Aro50, AFMP94, PR16]. The reader will find more references in the papers cited there. Various applications of positive definite kernels are discussed in [AJL11, AJV14, AJ14, AJK15, AJ15, AJLM15, AJL17]. More details and explicit constructions of reproducing kernel Hilbert spaces are considered in Section 8.

(V) **Dirichlet forms.** Another interesting application of symmetric measures and finite energy space is related to Dirichlet forms, see e.g., [AFH11, MR92, MR95]. We mention here the Beurling-Deny formula as given in [MR92]. It states that a symmetric Dirichlet form on \( L^2(U) \), where \( U \) is an open subset in \( \mathbb{R}^d \), can
be uniquely expressed as follows:

\[
\mathcal{E}(u, v) = \sum_{i,j=1}^{d} \int \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} d\nu_{ij}
\]

\[
+ \int_{(U \times U) \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y)) \ J(dx, dy)
\]

\[
+ \int uv \ dk.
\]

Here \( u, v \in C_{0}^{\infty}(U) \), \( k \) is a positive Radon measure on \( U \subset \mathbb{R}^{d} \), and \( J \) is a symmetric measure on \((U \times U) \setminus \text{diag}\). The first term on the right hand side in this formula is called the diffusion term, the second, the jump term, and the last, the killing term; a terminology deriving from their use in the study of general Levy processes [App09]. We see that the second term in this formula corresponds to the inner product in the finite energy space \( \mathcal{H}_{E} \) (details are in Section 7 below).

(VI) Joinings. The following application of symmetric measures is motivated by the theory of joinings developed in ergodic theory, see e.g., [Gla03, dlR12] and the literature cited therein.

Let \((V, \mathcal{B}, \mu)\) be a standard \(\sigma\)-finite measure space, and let \(S\) be a measure preserving surjective Borel endomorphism of \((V, \mathcal{B}, \mu)\), i.e., \(\mu \circ S^{-1} = \mu\). Define a measure \(\rho\) on \((V \times V, \mathcal{B} \times \mathcal{B})\) by setting

\[
\rho(A \times B) = \mu(A \cap S^{-1}(B)).
\]  

(3.14)

We note that the measure \(\rho\) is invariant with respect to \(S^{-1} \times S^{-1}\):

\[
\rho(S^{-1}(A) \times S^{-1}(B)) = \mu(S^{-1}(A) \cap S^{-1}[S^{-1}(B)]) = \mu(A \cap S^{-1}(B)) = \rho(A \times B).
\]

Moreover, the measure \(\rho\) defined in (3.14) is symmetric if and only if

\[
\mu(A \cap S^{-1}(B)) = \mu(S^{-1}(A) \cap B).
\]  

(3.15)

Lemma 3.20. Let the measure \(\rho\) be defined by (3.14) where \(\mu\) satisfies (3.15). Then:

(1) disintegration of \(\rho\) with respect to \(\mu\) defines the atomic fiber measures \(\rho_{x}\) such that \(d\rho_{x}(y) = \delta_{Sx}(y)\) for all \(x \in V\);

(2) the symmetric operator \(R = R(\rho)\) coincides with the Koopman operator \(f \mapsto f \circ S\) corresponding to the endomorphism \(S\).
Proof. (1) Indeed, setting \( d\rho_x(y) = \delta_{Sx}(y) \), we obtain that
\[
\int_A d\rho_x(B) \, d\mu(x) = \int_A \left( \int_V \chi_B(y) \, d\rho_x(y) \right) \, d\mu(x) \\
= \int_A \delta_{Sx}(B) \, d\mu(x) \\
= \int_A \chi_{S^{-1}(B)}(x) \, d\mu(x) \\
= \mu(A \cap S^{-1}(B)) \\
= \rho(A \times B).
\]
This proves that \( \rho = \int_V \delta_{Sx} d\mu(x) \).

(2) Since the field of measures \( x \mapsto \rho_x \) is determined, we can directly compute the operator \( R \):
\[
R(f)(x) = \int_V f(y) \, d\rho_x(y) = \delta_{Sx}(f)(y) = f(Sx).
\]
\[\square\]

Remark 3.21. To avoid a possible confusion, we mention that the Koopman operator \( R : f \mapsto f \circ S \) corresponds to a symmetric measure if it satisfies (2.17). Then
\[
\int_V \chi_A(\chi_B \circ S) \, d\mu = \int_V (\chi_A \circ S) \chi_B \, d\mu
\]
which is equivalent to the property (3.15).

It follows from Lemma 3.20 that the function \( c(x) = \rho_x(V) = 1 \) and the Markov operator \( P \) coincides with \( R \).

Corollary 3.22. The Laplace operator \( \Delta = \Delta(\rho) \), where the symmetric measure \( \rho \) is defined by (3.14) and (3.15), is the coboundary operator, i.e.,
\[
\Delta(f)(x) = f(x) - f(Sx).
\]

4. Embedding of \( \mathcal{D}_{\text{fin}}(\mu) \) and \( \mathcal{D}_{\text{fin}}(\nu) \) into \( \mathcal{H}_E \)

In this section, we focus on a description of subspaces in the finite energy Hilbert space \( \mathcal{H}_E \). We recall that this space is defined by a symmetric measure \( \rho \). It gives us the marginal measure \( \mu \), the function \( c \), and therefore one more measure \( \nu = c\mu \). We show below that the spaces of simple functions can be considered as subspaces of \( \mathcal{H}_E \) and describe their closures in \( \mathcal{H}_E \).

4.1. Locally integrable functions. We recall that the following chain of inclusions holds due to Assumptions 1 and 2, and the results proved above:
\[
\mathcal{D}_{\text{fin}}(\mu) \subset \mathcal{D}_{\text{fin}}(\nu) \subset \mathcal{H}_E \subset L^2_{\text{loc}}(\mu) \subset L^1_{\text{loc}}(\mu).
\]
In this section we will describe the closure of subspaces spanned by characteristic functions into the energy space \( \mathcal{H}_E \).
It is useful to have a criterion for local integrability of functions with respect to the measures $\mu$ and $\nu$ because, by the made assumption, all functions from the energy space $\mathcal{H}_E$ should be locally integrable.

**Lemma 4.1.** Let $f \in \mathcal{F}(V, \mathcal{B})$ and $\rho$ a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $c(x) = \rho_x(V)$. Then:

1. $f$ is locally integrable with respect to the measure $\mu$ on $(V, \mathcal{B})$ if and only if, for any $A \in \mathcal{B}_{\text{fin}}(\mu)$, the function
   \[ F(A, x) = \int_A \frac{f(y)}{c(y)} \, d\rho_x(y) \]
   is in $L^1(\mu)$;

2. $f$ is locally integrable with respect to the measure $\nu$ on $(V, \mathcal{B})$ if and only if, for any $A \in \mathcal{B}_{\text{fin}}(\nu)$, the function
   \[ F(A, x) = \frac{1}{c(x)} \int_A f(y) \, d\rho_x(y) \]
   is in $L^1(\nu)$.

**Proof.** We prove (1) only because the other statement is proved similarly. Let $A \in \mathcal{B}_{\text{fin}}(\mu)$; then we use $P$-invariance of $\nu$ and obtain
\[
\int_A f(x) \, d\mu(x) = \int_V f(x) / c(x) \, d\nu(x) = \int_V \chi_A(x) f(x) / c(x) \, d(\nu P)(x) = \int_V P \left( \frac{\chi_A f}{c} \right) (x) \, d\nu(x) = \int_V R \left( \frac{\chi_A f}{c} \right) (x) \, d\mu(x) = \int_V \left( \int_V \frac{\chi_A(y) f(y)}{c(y)} \, d\rho_x(y) \right) \, d\mu(x) = \int_V F(A, x) \, d\mu(x).
\]
If $f = f_+ - f_-$, where $f_+$ and $f_-$ are positive an negative parts of $f$, then the corresponding function $F(A, x)$ is represented as $F_+(A, x) - F_-(A, x)$. Hence the proved equality $\int_A f(x) \, d\mu(x) = \int_V F(A, x) \, d\mu(x)$ points out that $f \in L^1_{\text{loc}}(\mu)$ if and only $F(A, \cdot)$ is $\mu$-integrable for every $A \in \mathcal{B}_{\text{fin}}(\mu)$. \qed

We observe that condition (2) of Lemma 4.1 can be written in the following equivalent form:
\[
f \in L^1_{\text{loc}}(\nu) \iff [x \mapsto c(x) P(\chi_A f)(x)] \in L^1(\mu)
\]
for any $A \in \mathcal{B}_{\text{fin}}(\nu)$. 

4.2. **Embedding of** $D_{\mathbb{R}^n}(\nu)$ **into** $\mathcal{H}_E$. Let $f$ be a function from $L^2(\nu)$. We will show that this function determines an element of the finite energy space. In other words, the equivalence class generated by $f$ belongs to $\mathcal{H}_E$. In order to distinguish a function $f \in L^2(\nu)$ and the corresponding element of $\mathcal{H}_E$, we denote the latter by $\iota(f)$.

The following formula will be repeatedly used in further computations of the energy norm. This result extends Theorem 3.7.

**Lemma 4.2.** For $f \in \mathcal{H}_E \cap L^2(\nu)$, the following formula holds:

$$\|f\|_{\mathcal{H}_E}^2 = \iint_{V \times V} f(x)(f(x) - f(y)) \, d\rho(x, y) = \int_V f(x)\Delta(f)(x) \, d\mu(x). \quad (4.1)$$

**Proof.** We have

$$\|f\|_{\mathcal{H}_E}^2 = \frac{1}{2} \iint_{V \times V} (f(x) - f(y))^2 \, d\rho(x, y)$$

$$= \frac{1}{2} \iint_{V \times V} (f(x)^2 - 2f(x)f(y) + f(y)^2) \, d\rho(x, y)$$

$$= \frac{1}{2} \iint_{V \times V} (2f(x)^2 - 2f(x)f(y)) \, d\rho(x, y)$$

$$= \iint_{V \times V} f(x)(f(x) - f(y)) \, d\rho(x, y)$$

$$= \int_V f(x) \left( \int_V (f(x) - f(y)) \, d\rho_x(y) \right) \, d\mu(x)$$

$$= \int_V f(x)\Delta(f)(x) \, d\mu(x).$$

In the above computation, we used (2.9). \(\square\)

**Proposition 4.3.** Let $\rho$ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ with $c(x) = \rho_x(V)$. If $\mu$ is the projection of $\rho$ onto the margin $V$ and $d\nu = c d\mu$, then the map

$$L^2(\nu) \xrightarrow{\iota} \mathcal{H}_E : \iota(f) = f$$

is a well defined bounded linear operator such that

$$\|\iota\|_{L^2(\nu) \to \mathcal{H}_E} \leq \sqrt{2}.$$

Moreover the adjoint operator $\iota^* : \mathcal{H}_E \to L^2(\nu)$ acts by the formula:

$$\iota^*(g) = (I - P)(g), \quad g \in \mathcal{H}_E.$$
Proof. We first show that $\iota(f)$ is in $\mathcal{H}_E$. Indeed, the symmetry of $\rho$ implies that

$$||\iota(f)||^2_{\mathcal{H}_E} = \frac{1}{2} \int\int_{V \times V} (f(x) - f(y))^2 \, d\rho(x,y)$$

$$\leq \int\int_{V \times V} (f(x)^2 + f(y)^2) \, d\rho(x,y)$$

$$= 2 \int\int_{V \times V} f(x)^2 \, d\rho(x,y)$$

$$= 2 \int_V f(x)^2 c(x) \, d\mu(x)$$

$$= 2 \int_V f(x)^2 \, d\nu(x).$$

It follows from the last relation that the norm of the bounded linear operator $\iota$ is bounded by $\sqrt{2}$ what proves the statement.

To prove the formula for the adjoint operator $\iota^*$ we use Lemma 4.2. Then

$$\langle \iota(f), g \rangle_{\mathcal{H}_E} = \frac{1}{2} \int\int_{V \times V} (f(x) - f(y))(g(x) - g(y)) \, d\rho(x,y)$$

$$= \int\int_{V \times V} f(x)(g(x) - g(y))P(x,dy)d\nu(x)$$

$$= \int_V f(x)(I - P)(g)(x) \, d\nu(x)$$

$$= \langle f, \iota^*(g) \rangle_{L^2(\nu)},$$

and we obtain the formula for $\iota^*$.

$\square$

Corollary 4.4. For $\rho, \mu, \nu, \text{ and } c(x)$ as above in Proposition 4.3 we have:

(i) $$\overline{D_{\text{fin}}(\nu)}^{\mathcal{H}_E} = L^2(\nu)^{\mathcal{H}_E} = L^2(\nu),$$

where $L^2(\nu)$ is considered as a subspace of $\mathcal{H}_E$,

(ii) $$L^2(\nu) \cap \mathcal{H}_{\text{arm}_E} = \{0\}.$$

Proof. (i) Indeed, the embedding $\iota: L^2(\nu) \to \mathcal{H}_E$ is continuous by Proposition 4.3 so that $\iota(D_{\text{fin}}(\nu))$ is dense in $L^2(\nu)$ and the image of $L^2(\nu)$ in $\mathcal{H}_E$ is closed.

(ii) It was proved in [BJ18a], see Theorem 2.17 that $L^2(\nu)$ does not contain harmonic functions for a $\sigma$-finite measure $\nu$. $\square$

We deduce from the results proved in this subsection that the following fact holds.

Corollary 4.5. $$\overline{D_{\text{fin}}(\nu)}^{\mathcal{H}_E} = \overline{D_{\text{fin}}(\mu)^{\mathcal{H}_E}}.$$  

It is curious to compare this results with the proper inclusion

$$\overline{D_{\text{fin}}(\mu)^{L^2(\nu)}} \subset \overline{D_{\text{fin}}(\nu)^{L^2(\nu)}} = L^2(\nu).$$
4.3. **Embedding of $\mathcal{D}_{\text{fin}}(\mu)$ into $\mathcal{H}_E$.** It was proved in Lemma 3.5 that $\mathcal{D}_{\text{fin}}(\mu)$ can be viewed as a subset of $\mathcal{H}_E$. We define the map $J$ setting

$$
\mathcal{D}_{\text{fin}}(\mu) \ni \varphi \mapsto J \varphi \in \mathcal{H}_E.
$$

Here $\varphi$ is a linear combination of characteristic functions, so that the operator $J$ can be studied on $\chi_A$, $A \in \mathcal{B}_{\text{fin}}(\mu)$.

In the proof of Theorem 4.7 below, we need the density of the set

$$
D^* := \{ f \in \mathcal{H}_E : \Delta(f) \in L^2(\mu) \}.
$$

in the finite energy space $\mathcal{H}_E$.

**Lemma 4.6.** Suppose that $c \in L^2_{\text{loc}}(\mu)$. Then the set $D^*$ is a dense subset in $\mathcal{H}_E$.

**Proof.** Fix $A, B \in \mathcal{B}_{\text{fin}}(\mu)$. Let $\omega_{A,B}$ be an element of $\mathcal{H}_E$ such that

$$
\Delta(\omega_{A,B})(x) = c(x)(\chi_A(x) - \chi_B(x)).
$$

We call $\omega_{A,B}$ a $\nu$-dipole. It is proved, see Section 7.2 below, that $\{ \omega_{A,B} : A, B \in \mathcal{B}_{\text{fin}}(\mu) \}$ is a dense subset in $\mathcal{H}_E$. Hence, in order to prove the lemma, it suffices to show that

$$
\{ \omega_{A,B} : A, B \in \mathcal{B}_{\text{fin}}(\mu) \} \subset D^*.
$$

Indeed, we compute

$$
\int_V (\Delta(f))^2 \, d\mu = \int_V c^2(\chi_A - \chi_B) \, d\mu = \int_A c^2 \, d\mu - 2 \int_{A \cap B} c^2 \, d\mu + \int_B c^2 \, d\mu,
$$

and the result follows. □

**Theorem 4.7.** The operator $J : L^2(\mu) \to \mathcal{H}_E$ is closable, densely defined, and, in general, unbounded. The operator $JJ^*$ is a self-adjoint extension of the symmetric Laplace operator $\Delta$ acting in $\mathcal{H}_E$.

**Proof.** We first note that $J$ is a densely defined operator because $\mathcal{D}_{\text{fin}}(\mu)$ is dense in $L^2(\mu)$. Let $\varphi \in \mathcal{D}_{\text{fin}}(\mu)$ and $f \in \mathcal{H}_E$. We proved in Theorem 3.7 that

$$
\langle J(\varphi), f \rangle_{\mathcal{H}_E} = \int_V \varphi \Delta(f) \, d\mu.
$$

(4.2)

Suppose now that, in relation (4.2), the function $f$ belongs to $D^*$. Then (4.2) can be written as

$$
\langle J(\varphi), f \rangle_{\mathcal{H}_E} = \langle \varphi, \Delta(f) \rangle_{L^2(\mu)}.
$$

(4.3)

To define the adjoint operator $J^*$, we say that $f \in \text{Dom}(J^*)$ if there exists a finite constant $C_f$ such that

$$
(\langle J(\varphi), f \rangle_{\mathcal{H}_E})^2 \leq C_f \int_V \varphi^2 \, d\mu, \quad \varphi \in \mathcal{D}_{\text{fin}}(\mu).
$$

(4.4)
Then, by the Riesz representation theorem, there exists a unique element \( f^* \in L^2(\mu) \) such that
\[
\langle J(\varphi), f \rangle_{\mathcal{H}_E} = \langle \varphi, f^* \rangle_{L^2(\mu)}
\]
We note that, by the Schwarz inequality applied to the right hand side of (4.2), relation (4.4) holds for any \( f \in D^* \). This means that the domain of \( J^* \) contains \( D^* \), and we can set \( J^*(f) = f^* \). It follows from (4.2) and (4.3) that \( \Delta(f) = J^*(f) \), \( f \in D^* \) where \( \Delta \) is considered as an operator from \( \mathcal{H}_E \) to \( L^2(\mu) \).

Let \( H \) be a Hilbert space. It is well known that a linear operator \( T : \text{Dom}(T) \to H \) is closable if and only if the operator \( T^* \) is densely defined. As was shown in Lemma 4.6, \( D^* \) is dense in \( \mathcal{H}_E \), hence \( J^* \) is densely defined, and \( J \) is closable. Therefore \( J \) admits a closed extension that we denote again by \( J \). The operator \( JJ^* : \mathcal{H}_E \to \mathcal{H}_R \) is obviously self-adjoint, and as was proved above \( JJ^* \) is a self-adjoint extension of the operator \( \Delta \) viewed as operator acting in \( \mathcal{H}_E \), i.e.,
\[
\Delta(f) = JJ^*(f), \quad f \in D^*.
\]

It remains to observe that if \( J \) were a bounded operator, then \( ||J|| = ||J^*|| \) and the operator \( \Delta \) would be bounded too. It happens only in the case when the function \( c \) is essentially bounded. Our assumption about \( c \) does not require its boundness. \( \Box \)

The proved theorem can be used to show that the space \( \text{Harm}_E \) of harmonic functions from \( \mathcal{H}_E \) is nonempty, see Corollary 5.9. We use notations introduced in Lemma 4.6 and Theorem 4.7.

**Corollary 4.8.** The following equalities hold:
\[
\mathcal{H}_E \ominus J(D_{\text{fin}}(\mu)) = \{ f \in \text{Dom}(J^*) : J^*(f) = 0 \} = \text{Harm}_E.
\]

5. **Dissipation space**

5.1. **Path space and Markov chain.** Let \((V, \mathcal{B}, \mu)\) be a \( \sigma \)-finite standard measure space. Suppose that a transition probability kernel \( x \mapsto P(x, A) \) is defined on \((V, \mathcal{B})\), and let \( P \) be the Markov operator defined by the formula
\[
P(f)(x) = \int_V f(y) \, P(x, dy).
\]
In particular, the kernel \( x \mapsto P(x, A) \) and the operator \( P \) can be generated by a symmetric measure \( \rho = \int_V \rho_x d\mu(x) \) on \((V \times V, \mathcal{B} \times \mathcal{B})\) where
\[
P(x, dy) = \frac{1}{c(x)} d\rho_x(y), \quad c(x) = \rho_x(V).
\]
Our main interest will be focused on relations between the measure \( \rho \) and properties of \( P \).

Having the kernel \( P(x, A) \), we can define inductively the sequence of probability distributions \( (P_n) \) by setting \( P_0(x, A) = \chi_A(x), P_1(x, A) = P(x, A) \) and
\[
P_{n+1}(x, A) = \int_V P_n(y, A) \, P(x, dy), \quad n \in \mathbb{N}_0.
\]
It can be shown that
\[ P_n(x, A) = P^n(\chi_A)(x), \quad n \in \mathbb{N}_0. \]

We can define simultaneously the sequence of symmetric measures \((\rho_n)\) by the formula
\[ \rho_n(A \times B) = \int_A \chi_A(x) P_n(x, B) \, d\nu(x), \quad n \in \mathbb{N}, \]
see also [2.21] and [5.3].

For every Borel set \(A\), consider the series
\[ G(x, A) := \sum_{n \in \mathbb{N}_0} P_n(x, A) = \sum_{n \in \mathbb{N}_0} P^n(\chi_A)(x). \]

The Markov process \((P_n)\) is called **transient** if, for every set \(A\), \(G(x, A)\) is finite for a.e. on the space \((V, B, \mu)\). Then \(G(x, A)\) is called the **Green’s function**. We will discuss various properties of the Green’s function in Section 8.

We denote by \(\Omega\) the infinite Cartesian product \(V \times V \times \cdots = V^{\mathbb{N}_0}\). Let \((X_n(\omega) : n = 0, 1, ...)\) be the sequence of random variables \(X_n : \Omega \to V\) such that \(X_n(\omega) = \omega_n\). We call \(\Omega\) as the **path space** of the Markov process \((P_n)\).

Let \(\Omega_x, x \in V\), be the set of infinite paths beginning at \(x\):
\[ \Omega_x := \{ \omega \in \Omega : X_0(\omega) = x \}. \]

Clearly, \(\Omega = \bigsqcup_{x \in V} \Omega_x\).

The subset \(C(A_0, ..., A_k) := \{ \omega \in \Omega : X_0(\omega) \in A_0, ..., X_k(\omega) \in A_k \}\) of \(\Omega\) is called a **cylinder set** defined by Borel sets \(A_0, A_1, ..., A_k\) taken from \(B\), \(k \in \mathbb{N}_0\). The collection of cylinder sets generates the \(\sigma\)-algebra \(C\) of Borel subsets of \(\Omega\), and \((\Omega, C)\) is a standard Borel space. By definition of \(C\), he functions \(X_n : \Omega \to V\) are Borel.

Denote by \((\mathcal{F} \leq n)\) the increasing sequence of \(\sigma\)-subalgebras where \(\mathcal{F} \leq n\) is the smallest subalgebra for which the functions \(X_0, X_1, ..., X_n\) are Borel. By \(\mathcal{F}_n\), we denote the \(\sigma\)-subalgebra \(X_n^{-1}(B)\).

Define a probability measure \(\mathbb{P}_x\) on \(\Omega_x\). For a cylinder set \(C(A_1, ..., A_n)\) from \(\mathcal{F} \leq n\), we set
\[ \mathbb{P}_x(X_1 \in A_1, ..., X_n \in A_n) = \int_{A_1} \cdots \int_{A_n} P(y_{n-1}, A_n) P(y_{n-2}, dy_{n-1}) \cdots P(x, dy_1). \]

(5.1)

Then \(\mathbb{P}_x\) extends to the \(\sigma\)-algebra \(C\) of Borel sets on \(\Omega_x\) by the Kolmogorov extension theorem [Kol50].

The values of \(\mathbb{P}_x\) can be written as
\[ \mathbb{P}_x(X_1 \in A_1, ..., X_n \in A_n) = P(\chi_{A_1} P(\chi_{A_2} P(\cdots P(\chi_{A_{n-1}} P(\chi_{A_n}) \cdots ))))(x). \]

(5.2)

The joint distribution of the random variables \(X_i\) is given by
\[ d\mathbb{P}_x(X_1, ..., X_n)^{-1} = P(x, dy_1) P(y_1, dy_2) \cdots P(y_{n-1}, dy_n). \]

(5.3)

**Lemma 5.1.** The measure space \((\Omega_x, \mathbb{P}_x)\) is a standard probability measure space for every \(x \in V\).
5.2. Orthogonal functions in the dissipation space.

**Definition 5.2.** On the measurable space \((\Omega, \mathcal{C})\), define a \(\sigma\)-finite measure \(\lambda\) by
\[
\lambda := \int_V P_x \, d\nu(x) \tag{5.4}
\]
The Hilbert space
\[
\text{Diss} := \left\{ \frac{1}{\sqrt{2}} f : f \in L^2(\Omega, \mathcal{C}, \Lambda) \right\}
\]
is called the *dissipation space*.

Remark that: (i) the measure \(\lambda\) is infinite if and only if the measure \(\nu\) is infinite, and (ii) the dissipation Hilbert space \(\text{Diss}\) is formed, in fact, by functions from \(L^2(\Omega, \lambda)\) which are rescaled by the factor \(1/\sqrt{2}\), i.e.,
\[
\|f\|_{\mathcal{D}} = \frac{1}{\sqrt{2}} \|f\|_{L^2(\lambda)}, \quad f \in \text{Diss}.
\]

Because the partition of \(\Omega\) into \((\Omega_x : x \in V)\) is measurable, we have the decomposition into the direct integral of Hilbert spaces:
\[
L^2(\Omega, \lambda) = \int_V^\oplus L^2(\Omega_x, P_x) \, d\nu(x) \tag{5.5}
\]
As a consequence of (5.5), we obtain the following formula: for a measurable function over \((\Omega, \mathcal{C})\),
\[
\lambda(f) = \int_\Omega f(\omega) \, d\lambda(\omega) = \int_V \mathbb{E}_x(f) \, d\nu(x)
\]
where \(\mathbb{E}_x\) denotes the conditional expectation with respect to the measures \(P_x\),
\[
\mathbb{E}_x(f) = \int_{\Omega_x} f(\omega) \, dP_x(\omega).
\]

Then the inner product in the Hilbert space \(\text{Diss}\) is determined by the formula:
\[
(f, g)_\mathcal{D} = \frac{1}{2} \int_V \mathbb{E}_x(fg) \, d\nu(x) = \frac{1}{2} \int_V \int_{\Omega_x} f(\omega)g(\omega) \, dP_x(\omega) \, d\nu(x). \tag{5.6}
\]

Since \(X_n^{-1}(\mathcal{B})\) is a \(\sigma\)-subalgebra of \(\mathcal{C}\), there exists a projection
\[
E_n : L^2(\Omega, \mathcal{C}, \lambda) \to L^2(\Omega, X_n^{-1}(\mathcal{B}), \lambda).
\]
The projection \(E_n\) is called the *conditional expectation* with respect to \(X_n^{-1}(\mathcal{B})\) and satisfies the property:
\[
E_n(f \circ X_n) = f \circ X_n. \tag{5.7}
\]

The operator \(P\) and conditional expectations \(\mathbb{E}_x\) are related as follows: for any Borel functions \(f, h\), one has
\[
\mathbb{E}_x[(h \circ X_n) \, (f \circ X_{n+1})] = \mathbb{E}_x[(h \circ X_n) \, (P(f) \circ X_n)].
\]
In particular,
\[
P(f) \circ X_n = \mathbb{E}[f \circ X_{n+1} \mid \mathcal{F}_n] = E_n(f \circ X_{n+1}). \tag{5.8}
\]
Lemma 5.3. The Hilbert space $L^2(\nu)$ is isometrically embedded into the dissipation space $\text{Diss}$ by the map

$$W_n(f) = \sqrt{2}(f \circ X_n), \quad n \in \mathbb{N},$$

(5.9)

Moreover, the conditional expectation $E_n$ satisfies the property:

$$E_n = W_n W_n^*.$$ 

The lemma follows immediately from the definition of $W_n$ and (5.7).

We return here to the notion of reversible Markov process in connection with the dissipation space. We proved in [BJ18a] that the Markov process $P_n$ is irreducible if and only if the initial symmetric measure is irreducible.

Theorem 5.4 ([BJ18a, BJ18b]). Let $\rho$ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$, and let $A$ and $B$ be any two sets from $\mathcal{B}_{\text{fin}}(\mu)$. Then:

1. $\rho^n(A \times B) = \langle \chi_A, P^n(\chi_B) \rangle_{L^2(\nu)} = \lambda(X_0 \in A, X_n \in B), \quad n \in \mathbb{N},$ (5.10)

2. The Markov process $(P_n)$ is irreducible if and only if the measure $\rho$ is irreducible.

3. Let the measure $\lambda$ on $\Omega$ be defined by (5.4). The Markov operator $P$ is reversible if and only if

$$\lambda(X_0 \in A_0 \mid X_1 \in A_1) = \lambda(X_0 \in A_1 \mid X_1 \in A_0).$$

This theorem can be interpreted as follows. Relation (5.10) says that, for the Markov process $(P_n)$, the “probability” to get in $B$ for $n$ steps starting somewhere in $A$ is exactly $\rho^n(A \times B) > 0$. And the concept of reversible Markov processes can be reformulated in terms of the measure $\lambda$: roughly speaking $\lambda$ must be a symmetric distribution.

Corollary 5.5. Let $A_0, A_1, \ldots, A_n$ be a finite sequence of subsets from $\mathcal{B}_{\text{fin}}$. Then

$$\mathbb{P}_x(X_1 \in A_1, \ldots, X_n \in A_n \mid x \in A_0) > 0 \iff \rho(A_i-1 \times A_i) > 0$$

for $i = 1, \ldots, n$.

In what follows we formulate and prove a key property of functions from the dissipation space. We will use the fact that $L^2(\nu)$ can be seen as a subspace of $\text{Diss}$, see (5.9).

Theorem 5.6 (Orthogonal decomposition). (1) Let $g_1, g_2$ be functions from $L^2(\nu)$. Then

$$\langle g_1 \circ X_n, P(g_2) \circ X_n - g_2 \circ X_{n+1} \rangle_{\text{Diss}} = 0.$$ (5.11)

(2) For any function $f \in L^2(\nu)$ and any $n \in \mathbb{N}$, the functions

$$\langle (I - P)(f) \circ X_n - (P(f) \circ X_n - f \circ X_{n+1}) \rangle$$

(5.12)

in the dissipation space $\text{Diss}$. 


Proof. (1) It follows from (5.6) that the result would follow if we proved that the functions \( g_1 \circ X_n \) and \( P(g_2) \circ X_n - g_2 \circ X_{n+1} \) are orthogonal in \( L^2(\Omega_x, \mathbb{P}_x) \) for a.e. \( x \). We use here (5.8) and (5.7) to compute the inner product:

\[
\langle g_1 \circ X_n, P(g_2) \circ X_n - g_2 \circ X_{n+1} \rangle_{\mathbb{P}_x} = \mathbb{E}_x(E_n(g_1 \circ X_n) \cdot (P(g_2) \circ X_n - g_2 \circ X_{n+1}))
\]

\[
= \mathbb{E}_x((g_1 \circ X_n) \cdot E_n(P(g_2) \circ X_n - g_2 \circ X_{n+1}))
\]

\[
= \mathbb{E}_x((g_1 \circ X_n) \cdot (P(g_2) \circ X_n - E_n(g_2 \circ X_{n+1})))
\]

\[
= \mathbb{E}_x((g_1 \circ X_n) \cdot (P(g_2) \circ X_n - P(g_2) \circ X_n)) = 0
\]

(2) To prove (5.12), it suffices to show that

\[
(f \circ X_n) \perp (P(f) \circ X_n - f \circ X_{n+1}) \quad (5.13)
\]

and

\[
P(f) \circ X_n \perp (P(f) \circ X_n - f \circ X_{n+1}) \quad (5.14)
\]

Relation (5.13) has been proved in (1). It follows from (5.8) and the proof of statement (1) that, for \( \nu \)-a.e. \( x \in V \),

\[
\mathbb{E}_x((P(f) \circ X_n) \cdot (P(f) \circ X_n - f \circ X_{n+1}))
\]

\[
= \mathbb{E}_x((P(f) \circ X_n) \cdot (P(f) \circ X_n - f \circ X_{n+1}))
\]

\[
= \mathbb{E}_x((P(f) \circ X_n) \cdot (P(f) \circ X_n - E_n(f \circ X_{n+1})))
\]

\[
= 0.
\]

This proves (5.13) and we are done.

\[\square\]

### 5.3. Embedding of \( \mathcal{H}_E \) into Diss

The importance of Theorem 5.6 is explained by the fact that the finite energy space can be isometrically embedded into the dissipation space.

We remind first that, for a Markov operator \( P \) and the transition probabilities \( x \mapsto P(x, \cdot) \), we defined the path spaces \( (\Omega, \lambda) \) and \( (\Omega_x, \mathbb{P}_x), x \in V \), together with the sequence of random variables \( X_n(\omega) \) taking values in \( (V, \mathcal{B}) \), see Subsection 5.1. Then we have the following formulas for the conditional expectation \( \mathbb{E}_x \) with respect to the probability measure \( \mathbb{P}_x \):

\[
\mathbb{E}_x(f \circ X_0) = \int_{\Omega_x} f(X_0(\omega)) \, d\mathbb{P}_x(\omega) = \int_{\Omega_x} f(x) \, d\mathbb{P}_x(\omega) = f(x),
\]

\[
\mathbb{E}_x(f \circ X_1) = \int_{\Omega_x} f(X_1(\omega)) \, d\mathbb{P}_x(\omega) = \int_V f(y) \, P(x, dy) = P(f)(x)
\]

where \( y = X_1(\omega) \).

We define the operator \( \partial : \mathcal{H}_E \to Diss \) by setting

\[
\partial : f \mapsto f \circ X_1 - f \circ X_0.
\]

(5.17)
Similarly, we can set
\[ \partial_n : f \mapsto f \circ X_{n+1} - f \circ X_n. \]  
(5.18)

**Proposition 5.7.** (1) The operator \( \partial : \mathcal{H}_E \rightarrow \text{Diss} \) defined in (5.17) is an isometry.

(2) For \( f \in \mathcal{H}_E \) and \( \nu = c\mu \), we have
\[ \|f\|^2_{\mathcal{H}_E} = \frac{1}{2} \int_V \mathbb{E}_x[(f \circ X_1 - f \circ X_0)^2] \, d\nu(x). \]

In the next statements we strengthen the result of Proposition 5.7 (2) using the orthogonal decomposition given in Theorem 5.6. In Theorem 5.8 we give an explicit, canonical and orthogonal splitting of the energy form from Proposition 5.7 (2) into two terms, each one having a stochastic content, variation and dissipation.

**Theorem 5.8.** Let \( f \in \mathcal{H}_E \). Then
\[ \|f\|^2_{\mathcal{H}_E} = \frac{1}{2} \left( \int_V (P(f^2) - P(f)^2) \, d\nu + \int_V (P(f) - f)^2 \, d\nu \right) = \frac{1}{2} \left( \int_V (P(f^2) - P(f)^2) \, d\nu + \|f - P(f)\|^2_{L^2(\nu)} \right). \]  
(5.19)

In particular, the integrals in the right hand side of (5.19) are finite and non-negative. Moreover,
\[ \text{Var}_x(f \circ X_1) = P(f^2)(x) - P(f)^2(x) \geq 0 \]
and \( \text{Var}_x(f \circ X_1) \in L^1(\nu) \), for any \( f \in \mathcal{H}_E \); hence equivalently
\[ \|f\|^2_{\mathcal{H}_E} = \frac{1}{2} \left( \|\text{Var}_x(f \circ X_1)\|_{L^1(\nu)} + \mathbb{E}_x(f \circ X_0 - f \circ X_1)^2 \right). \]  
(5.20)

**Proof.** By Proposition 5.7 it suffices to prove that the right hand side of (5.19) equals \( \|\partial f\|^2_D \). Indeed, we can use the orthogonal decomposition given in Theorem 5.6 and write
\[ \|\partial f\|^2_D = \|f \circ X_0 - P(f) \circ X_0\|^2_D + \|P(f) \circ X_0 - f \circ X_1\|^2_D. \]

In the proof below, we use the following equality:
\[ \text{Var}_x(f \circ X_1) \]
\[ = \int_V (P(f)(x) - f(y))^2 P(x, dy) \]
\[ = P(f)^2(x) - 2P(f)(x) \int_V f(y) P(x, dy) + \int_V f(y)^2 P(x, dy) \]
\[ = P(f)^2(x) - 2P(f)^2(x) + P(f^2)(x) \]
\[ = P(f^2)(x) - P(f)^2(x). \]
Then the computation of $\| \partial f \|_D^2$ goes as follows:

$$
\| \partial f \|_D^2 = \frac{1}{2} \int_V E_x [(I - P)(f)^2 \circ X_0] \, d\nu(x)
+ \frac{1}{2} \int_V E_x [(P(f) \circ X_0 - f \circ X_1)^2] \, d\nu(x)
= \frac{1}{2} \int_V (f - P(f))^2(x) \, P(x, dy) d\nu(x)
+ \frac{1}{2} \int_V (P(f)(x) - f(y))^2 \, P(x, dy) d\nu(x)
= \frac{1}{2} \int_V (f - P(f))^2(x) d\nu(x)
+ \frac{1}{2} \int_V (P(f^2)(x) - P(f^2)(x)) \, d\nu(x).
$$

The proof is complete.

Theorem 5.8 allows us to deduce a number of important corollaries.

**Corollary 5.9.**

1. If $f \in \mathcal{H}_E$, then $f - P(f) \in L^2(\nu)$ and $P(f^2) - P(f)^2 \in L^1(\nu)$.

The operator

$$
I - P : f \mapsto f - P(f) : \mathcal{H}_E \to L^2(\nu)
$$

is contractive, i.e., $\| I - P \|_{\mathcal{H}_E \to L^2(\nu)} \leq 1$.

2. $\| f \|_{\mathcal{H}_E} = 0 \iff \begin{cases} P(f^2) = P(f)^2 \\ P(f) = f \end{cases}$ $\nu$ - a.e.
   $\iff$ both $f$ and $f^2$ are harmonic functions.

3. Let $f \in \mathcal{H}_E$, then

$$
\begin{align*}
& f \in \text{Harm}_E \iff \| f \|^2_{\mathcal{H}_E} = \frac{1}{2} \int_V (P(f^2)(x) - (P(f)^2)(x)) \, d\nu(x) \\
& \iff \| f \|^2_{\mathcal{H}_E} = \frac{1}{2} \int_V \text{Var}_x(f \circ X_1) \, d\nu(x).
\end{align*}
$$

4. $\int_V \text{Var}_x(f \circ X_1) \, d\nu(x) = \int_V \text{Var}_x(f \circ X_n) \, d\nu(x), \quad n \in \mathbb{N}$.

**Proof.** Statement (1) immediately follows from (5.19).

To see that (2) holds we use again (5.19). The right hand side is zero if and only if $P(f) = f$ and $P(f^2) = P(f)^2$ $\nu$ - a.e. (recall that, for any function $f$, $P(f^2) \geq P(f)^2$).

Since $f$ is harmonic, the latter means that $f^2$ is harmonic.

(3) This observation is a consequence of (5.19), Theorem 5.8. □
6. **Laplace operators in $\mathcal{H}_E$**

6.1. **Norm estimates for $\Delta$.** In this section we use the results of Sections 3 and 5 to establish several properties of the Laplace operator $\Delta$. This theme will be discussed in subsequent sections where Laplace operator $\Delta$ will be acting in the finite energy Hilbert space $\mathcal{H}_E$.

**Lemma 6.1.** For any function $f \in \mathcal{H}_E$, the function $\Delta(f)$ belongs to $L^2(c^{-1}\mu)$ and

$$
||f||^2_{\mathcal{H}_E} \geq \frac{1}{2} \int_V (\Delta(f))^2 c^{-1} \, d\mu = \frac{1}{2} ||c^{-1}\Delta(f)||^2_{L^2(\nu)}.
$$

**Proof.** Relation (5.19) implies that

$$
||f||^2_{\mathcal{H}_E} \geq \frac{1}{2} \int_V (f - P(f))^2 \, d\nu
= \frac{1}{2} \int_V [c(f - P(f))]^2 c^{-1} \, d\mu
= \frac{1}{2} \int_V (\Delta(f))^2 c^{-1} \, d\mu
= \frac{1}{2} ||\Delta(f)||^2_{L^2(c^{-1}\mu)}
= \frac{1}{2} ||c^{-1}\Delta(f)||^2_{L^2(\nu)}.
$$

□

The properties of the Laplace operator $\Delta$ depend on the spaces in which it acts, see Proposition 2.14 for details. In the next statement we show that, under some conditions, $\Delta$ can be even a bounded operator.

**Proposition 6.2.** Let the operator $\Delta$ be defined on Borel functions as in (2.15). Then if $\Delta$ is considered as an operator from $\mathcal{H}_E$ to $L^2(c^{-1}\mu)$, then $\Delta$ is bounded and

$$
||\Delta||_{\mathcal{H}_E \to L^2(c^{-1}\mu)} \leq \sqrt{2}.
$$

**Proof.** We compute, for $f \in \mathcal{H}_E$,

$$
||\Delta||^2_{\mathcal{H}_E \to L^2(c^{-1}\mu)} \leq \sqrt{2}.
$$
\[ ||\Delta(f)||_{L^2(c^{-1}\mu)}^2 = \int_V \left( \int_V (f(x) - f(y)) \, d\rho_x(y) \right)^2 c(x)^{-1} \, d\mu(x) \]
\[ = \int_V c(x)^2 \left( \int_V (f(x) - f(y)) \, P(x, dy) \right)^2 c(x)^{-1} \, d\mu(x) \]
\[ \leq \int_V c(x) \left( \int_V (f(x) - f(y))^2 \, P(x, dy) \right) \, d\mu(x) \]
\[ \leq \int_V \left( \int_V (f(x) - f(y))^2 \, d\rho_x(y) \right) \, d\mu(x) \]
\[ = \int_{V \times V} (f(x) - f(y))^2 \, d\rho(x, y) \]
\[ = 2||f||_{\mathcal{H}_E}^2. \]

It can be easily seen that the same proof as in Proposition 6.2 works to show that the following relation holds:

\[ ||\Delta(f)||_{L^2(c^{-1}\mu)}^2 \leq \int_V \int_V (f(x) - f(y))^2 \, d\rho_x(y) \, d\nu(x). \]

6.2. Example: the energy Hilbert space $\mathcal{H}_E$ over a probability space. In this subsection, we consider a particular case when the symmetric measure $\rho$ is the product of finite measure. It turns out that, in this case, the finite energy space can be identified with a $L^2$-space.

Our setting here are: $(V, \mathcal{B}, \mu)$ is a probability measure space, $\mu(V) = 1$, $r : V \to \mathbb{R}_+$ is a non-negative $\mu$-integrable Borel function, and $\mu_r$ is a measure on $(V, \mathcal{B})$ such that $d\mu_r(x) = r(x) \, d\mu(x)$. Since $r \in L^1(\mu)$, the measure $\mu_r$ is finite. To define a symmetric measure $\rho = \rho_r$ on $(V \times V, \mathcal{B} \times \mathcal{B})$, we set

\[ dp(x, y) = r(x)r(y) \, d\mu_r(x) \, d\mu_r(y), \quad (x, y) \in V \times V \]

Then $\rho = \mu_r \times \mu_r$, and the disintegration of $\rho = \int_V \rho_x \, d\mu(x)$ gives measures $\rho_x, x \in V$, such that $d\rho_x(y) = r(x)r(y) \, d\mu_r(y)$. Then the function $c(x) = \rho_x(V)$ is found by

\[ c(x) = \int_V d\rho_x(y) = r(x) \int_V r(y) \, d\mu_r(y) = E_\mu(r) r(x). \]

We note that $c \in L^1(\mu)$ because

\[ \int_V c(x) \, d\mu(x) = \mu_r(V)||r||_{L^1(\mu)}. \]

Having the measure $\rho$ on $(V \times V, \mathcal{B} \times \mathcal{B})$, we determine the operators $R, P$, and $\Delta$ acting on bounded Borel functions $\mathcal{F}(V, \mathcal{B})$ as follows

\[ R(f)(x) = \int_V f(y) \, d\rho_x(y) = \int_V f(y)r(x)r(y) \, d\mu_r(y) = E_{\mu_r}(f) r(x), \]

\[ P(f)(x) = \int_V f(y) \, d\rho_x(y) = \int_V f(y) \, d\mu_r(y). \]
This proves that $\alpha$ and $E$ identify here the number $E$.

Proof. To prove the theorem, we compute the norms $\|\cdot\|_{\mathcal{H}_E}$ and $\|\alpha(\cdot)\|_{L^2(\mu_r)}^2$. For $\|f\|_{\mathcal{H}_E}^2$:

$$\|f\|_{\mathcal{H}_E}^2 = \frac{1}{2} \int_{V \times V} (f(x) - f(y))^2 \, d\mu_r(x) \, d\mu_r(y) = \int_{V \times V} (f^2(x) - f(x)f(y)) \, d\mu_r(x) \, d\mu_r(y) = \mathbb{E}_{\mu_r}(f^2)\mu_r(V) - \left( \int_V f(x) \, d\mu_r(x) \right)^2 = \mathbb{E}_{\mu_r}(f^2)\mu_r(V) - \mathbb{E}_{\mu_r}(f)^2.
$$

On the other hand, we can find

$$\|\alpha(f)\|_{L^2(\mu_r)}^2 = \frac{1}{\mathbb{E}_{\mu_r}(f)^2} \int_V (\mathbb{E}_{\mu_r}(f) - \mathbb{E}_{\mu_r}(f))^2 \, d\mu_r(x) = \frac{1}{\mathbb{E}_{\mu_r}(f)^2} \int_V (\mathbb{E}_{\mu_r}(f)^2 - 2f(x)\mathbb{E}_{\mu_r}(f)\mathbb{E}_{\mu_r}(f) + \mathbb{E}_{\mu_r}(f)^2) \, d\mu_r(x) = \frac{1}{\mathbb{E}_{\mu_r}(f)^2} \left[ \mathbb{E}_{\mu_r}(f)^2 - 2\mathbb{E}_{\mu_r}(f)\mathbb{E}_{\mu_r}(f)^2 + \mathbb{E}_{\mu_r}(f)^2 \right] = \mathbb{E}_{\mu_r}(f^2) - \mathbb{E}_{\mu_r}(f)^2.
$$

This proves that $\alpha$ is an isometry. We observe that this proof shows that

$$\alpha(\mathcal{H}_E) = L^2_0(\mu_r),$$

where $L^2_0(\mu_r)$ is the subspace of functions from $L^2(\mu_r)$ with zero integral. $\square$
Remark 6.4. (1) It is clear that, in the case when $\rho_r = \mu_r \times \mu_r$, the space $\mathcal{H}_E(\rho_r)$ does not contain nontrivial harmonic functions because as follows from \[6.1\]

\[\Delta(f) = 0 \iff f(x) = \frac{E_{\mu_r}(f)}{E_{\mu}(r)} \iff f(x) = \text{const}.\]

(2) It follows from Theorem \[6.3\] that, in the case when $r = 1$,

\[\|f\|_{\mathcal{H}_E} = \|\Delta f\|_{L^2(\mu)}.\]

(3) A similar approach can be used to study symmetric measures $\rho_{r,q}$ on $(V \times V, \mathcal{B} \times \mathcal{B})$ which are defined by a pair of nonnegative integrable functions $r, q : V \to \mathbb{R}_+$:

\[d\rho_{r,q}(x, y) = (r(x)q(y) + r(y)q(x))d\mu(x)d\mu(y).\]

Suppose now that, for the probability measure space $(V, \mathcal{B}, \mu)$, the function $r(x) = 1$, and therefore $\rho = \mu \times \mu$. Then we can make more precise the formula for the inner product in $\mathcal{H}_E$.

**Corollary 6.5.** For any $f, g \in \mathcal{H}_E$,

\[\langle f, g \rangle_{\mathcal{H}_E} = \text{Cov}_\mu(f, g).\]

_in particular,

\[\langle \chi_A, \chi_B \rangle_{\mathcal{H}_E} = \mu(A \cap B) - \mu(A)\mu(B)\]

and $\langle \chi_A, \chi_B \rangle_{\mathcal{H}_E} = 0$ if and only if the sets $A$ and $B$ are independent.

**Proof.** By definition of the inner product in $\mathcal{H}_E$, we calculate

\[\langle f, g \rangle_{\mathcal{H}_E} = \frac{1}{2} \int \int_{V \times V} (f(x) - f(y))(g(x) - g(y)) \, d\mu(x)d\mu(y)\]

\[= \int \int_{V \times V} (f(x) - E_{\mu}(f))(g(x) - E_{\mu}(g)) \, d\mu(x)\]

\[= E_{\mu}(fg) - E_{\mu}(f)E_{\mu}(g)\]

\[= \text{Cov}_\mu(f, g).\]

The proof of the other formula in the lemma follows from Theorem \[3.6\](1)

\[\langle \chi_A, \chi_B \rangle_{\mathcal{H}_E} = \rho((A \cap B) \times V) - \rho(A \times B)\]

\[= \mu(A \cap B) - \mu(A)\mu(B).\]

Hence, the functions $\chi_A$ and $\chi_B$ are orthogonal if and only if $\mu(A \cap B) = \mu(A)\mu(B)$.

In what follows, we continue discussing energy spaces and Laplace operators on measure spaces with finite measures.

We recall that we consider symmetric measures $\rho = \int V \rho_x d\mu(x)$ on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that the function $c : x \mapsto \rho_x(V)$ belongs to $L^1_{\text{loc}}(\mu)$, see Assumption 1. Then, for the measure $d\nu(x) = c(x)d\mu(x)$, we have $D_{\text{fin}}(\mu) \subset L^1_{\text{loc}}(\mu) \cap L^1_{\text{loc}}(\nu)$. It follows from local integrability of $c$ that

\[B_{\text{fin}}(\rho) \supset \{ A \times B \in \mathcal{B} \times \mathcal{B} : A \in B_{\text{fin}}(\mu) \}. \tag{6.2}\]
Let $A \in \mathcal{B}_{\text{fin}}(\mu)$ be a Borel set of positive measure $\mu$. Consider the restriction $\rho_A$ of a symmetric measure $\rho$ on the subset $A \times A$, i.e.,

$$\rho_A(C \times D) = \rho((C \times D) \cap (A \times A)).$$

By [6.2], $\rho_A$ is a finite symmetric measure for every $A \in \mathcal{B}_{\text{fin}}(\mu)$.

For the finite measure space $(A, \mathcal{B}_A, \mu_A)$ and measure $\rho_A$, we can define the symmetric operator $R_A$ as follows:

$$R_A(f)(x) = \int_A f(y) \, d\rho^A_x(y),$$

where $x \mapsto \rho^A_x$ is the family of fiber measures arising in disintegration of $\rho_A$. We set $c_A(x) = \rho^A_x(V)$. When $R_A$ and $c_A$ are defined, we can construct the graph Laplacian

$$\Delta_A(f)(x) = c_A f(x) - R_A(f)(x).$$

Given a $\sigma$-finite measure space $(V, \mathcal{B}, \mu)$ and a symmetric measure $\rho$ on $(V \times V, \mathcal{B} \times \mathcal{B})$, we choose a sequence $(A_n)$ of Borel sets such that $A_n \subset A_{n+1}$, $\mu(A_n) < \infty$ and

$$\bigcup_{n=1}^{\infty} A_n = V. \quad (6.3)$$

For every $A_n$, we define the objects $\rho_n = \rho_{A_n}, \rho_x^{(n)} = \rho_x^{A_n}, c_n(x) = c_{A_n}(x), R_n = R_{A_n},$ and $\Delta_n = \Delta_{A_n}$ as it was done above.

**Lemma 6.6.** For the objects defined above, the following sequences converge:

1. for any set $C \in \mathcal{B} \times \mathcal{B}$, $\rho_n(C) \to \rho(C)$ and $\rho_x^{(n)}(C) \to \rho_x(C)$;
2. $c_n(x) \to c(x)$ a.e. $x \in V$;
3. for any function $f \in \mathcal{F}(V, \mathcal{B})$,

$$R_n(f)(x) \to R(f)(x), \quad \text{and} \quad \Delta_n(f)(x) \to \Delta(f)(x) \text{ a.e. } x \in V.$$

**Proof.** The proof follows directly from the definitions.

Suppose $A \in \mathcal{B}_{\text{fin}}(\mu)$ and $\rho_A$ is the restriction of a symmetric measure $\rho$ onto $A \times A$. Define the finite energy space $\mathcal{H}_E(\rho_A)$ as the space of functions on $(V, \mathcal{B})$ satisfying

$$||f||^2_{\mathcal{H}_E(\rho_A)} := \frac{1}{2} \iint_{A \times A} (f(x) - f(y))^2 \, d\rho_A(x,y) < \infty.$$

As usual, we denote by $\nu_A$ the measure $c_A \mu_A$ where the function $c_A$ is defined by $\rho_A$.

**Lemma 6.7.** The space $\mathcal{H}_E(\rho_A)$ is embedded into $L^2(A, \nu_A)$ and

$$||f||^2_{\mathcal{H}_E(\rho_A)} \leq 2||f||^2_{L^2(A, \nu_A)}.$$
Proof. We first remark that $\nu(A) < \infty$ since $c \in L^1_{\text{loc}}(\mu)$. We need to show that, for any $f \in L(A, \nu)$, the function $f$ belongs to $\mathcal{H}_E(\rho_A)$. Indeed, we have

$$||f||^2_{\mathcal{H}_E(\rho_A)} = \frac{1}{2} \int \int_{A \times A} (f(x) - f(y))^2 \, d\rho_A(x, y) \leq \int \int_{A \times A} (f(x)^2 + f(y)^2) \, d\rho_A(x, y) = 2 \int_A f(x)^2 c_A(x) \, d\mu_A(x) = 2 ||f||^2_{L^2(A, \nu_A)}.$$ 

□

Let $(A_n)$ be a sequence of Borel subsets of $V$ satisfying (6.3). For measures $\rho_{A_n} = \rho_n$, we can obviously define embedding of the Hilbert space $\mathcal{H}_E(\rho_n)$ into $\mathcal{H}_E(\rho_{n+1})$. It follows from Corollary 5.9 and Lemma 6.7 that every space $\mathcal{H}_E(\rho_n)$ does not contain harmonic functions. Therefore, there are functions in $\mathcal{H}_E(\rho)$ which are not in $\mathcal{H}_E(\rho_n)$.

7. Properties of functions from the finite energy space

In this section, our main object is the finite energy space $\mathcal{H}_E$, see Definition 3.1 and its properties. In the literature, some authors use the terms Dirichlet space and Dirichlet form for the inner product. We mention here several references that may be useful for the reader [GT17, Osh13, KZ12, Jon05, Roz01, Jor12].

7.1. Properties and structure of the energy space. We begin with a discussion of immediate properties of functions from the space $\mathcal{H}_E$. We first recall what results were proved in [BJ18a].

We recall that in Section 3 (see Subsections 3.1 and 3.2) some basic properties of functions from the energy space $\mathcal{H}_E$ have been already discussed. It is important for us to use the inclusions

$$\mathcal{D}_{\text{fin}}(\mu) \subset \mathcal{D}_{\text{fin}}(\nu) \subset \mathcal{H}_E, \quad \mathcal{D}_{\text{fin}}(\mu) \subset L^2(\mu) \cap L^2(\nu) \cap \mathcal{H}_E,$$

and the assumption (3.4) which states that

$$\mathcal{H}_E \subset L^2_{\text{loc}}(\mu).$$

Since $L^2_{\text{loc}}(\mu) \subset L^1_{\text{loc}}(\mu)$, this means that any function from the energy space is locally integrable.

In Section 4 we proved a number of statements about the closure of $\mathcal{D}_{\text{fin}}(\mu)$ and $\mathcal{D}_{\text{fin}}(\nu)$ in $\mathcal{H}_E$. In particular, the closure of $\mathcal{D}_{\text{fin}}(\mu)$ in $\mathcal{H}_E$ is the subspace which is orthogonal to the space of harmonic functions, see Theorem 3.6. Here we reprove the result for the closure of $\mathcal{D}_{\text{fin}}(\nu)$ in $\mathcal{H}_E$ using a different method based on the embedding of $\mathcal{H}_E$ into the dissipation space $\text{Diss}$ and Theorem 5.8.
Proposition 7.1. The following relation holds
\[ \overline{D_{\text{fin}}(\nu)}^\mathcal{H}_E = \overline{L^2(\nu)}^\mathcal{H}_E = L^2(\nu), \]
where \( L^2(\nu) \) is considered as a subspace of \( \mathcal{H}_E \).

Proof. We recall that because of the inequality \( ||\chi_A||_{\mathcal{H}_E}^2 \leq \nu(A) \), the linear space \( D_{\text{fin}}(\nu) \) is a subspace of the energy space \( \mathcal{H}_E \), hence \( D_{\text{fin}}(\nu) \subset \mathcal{H}_E \). On the other hand, for every \( f \in L^2(\nu) \) there exists a sequence \( (f_n) \subset D_{\text{fin}}(\nu) \) such that
\[ ||f - f_n||_{L^2(\nu)} \to 0, \quad n \to \infty. \]

It suffices to show that every function from \( L^2(\nu) \) can be approximated by functions from \( D_{\text{fin}}(\nu) \) in the norm of \( \mathcal{H}_E \).

We use relations (5.19) and (5.20), and find the norm of \( f - f_n \) in \( \mathcal{H}_E \):
\[ ||f - f_n||_{\mathcal{H}_E}^2 = \frac{1}{2} \left( \int_V [P((f - f_n)^2) - P(f - f_n)^2] \, d\nu + \|(f - f_n) - P(f - f_n)||_{L^2(\nu)}^2 \right). \]
By Proposition 2.14(4), the operator \( P \) considered in the space \( L^2(\nu) \) is contractive, and
\[ ||(f - f_n) - P(f - f_n)||_{L^2(\nu)}^2 \to 0, \quad n \to \infty \]
because \( f_n \to f \) in \( L^2(\nu) \).

Next, we note that
\[ \int_V P((f - f_n)^2) \, d\nu = \int_V (f - f_n)^2 \, d\nu \to 0, \]
because \( \nu \) is a \( P \)-invariant measure.

To prove that the remaining term in the formula for \( ||f - f_n||_{\mathcal{H}_E}^2 \) tends to zero, we represent it as inner product in \( L^2(\nu) \), and conclude that
\[ \int_V P(f - f_n)^2 \, d\nu = \langle P(f - f_n), P(f - f_n) \rangle_{L^2(\nu)} \to 0, \quad n \to \infty. \]
Therefore, \( (f_n) \) is a converging sequence of elements from \( \mathcal{H}_E \) and the limit, the function \( f \), belongs to \( \mathcal{H}_E \). \( \square \)

It follows from the results proved in Section 4 and Proposition 7.1 that the following corollary holds.

Corollary 7.2. The finite energy Hilbert space admits the orthogonal decomposition
\[ h_E = L^2(\nu) \oplus \text{Harm}. \]

Proof. Indeed, the statement follows from the relation
\[ \overline{D_{\text{fin}}(\nu)}^\mathcal{H}_E = \overline{D_{\text{fin}}(\mu)}^\mathcal{H}_E \perp \text{Harm} \]
Theorem 3.10 and Corollary 4.5. The proof of the fact that every harmonic function is orthogonal to \( \chi_A, A \in \mathcal{B}_{\text{fin}}(\nu) \), is similar to that of Proposition 3.12 \( \square \).
7.2. Dipoles in the energy Hilbert space. We recall that in the theory of electrical networks the notion of dipoles plays a crucial role for the study of properties and structure of the finite energy space. Let $(V, E, c)$ be an electrical network with conductance function $c = (c_{xy})$ and the Laplacian $\Delta$, see Introduction for details. Then one can show that, for any edge $(xy) \in E$, there exists a unique element $v_{xy}$ of $H_E$, called a dipole, such that

$$\Delta v_{xy} = \delta_x - \delta_y.$$ 

It turns out that, for any $f \in H_E$,

$$\langle f, v_{xy} \rangle_{H_E} = f(x) - f(y).$$

Our goal in this subsection is to formulate and prove similar results for the measurable analogue of $H_E$ and $\Delta$, discrete concept.

**Definition 7.3.** We say that the family of functions $\{v_{A,B} : A, B \in \mathcal{B}_{fin}(\mu)\}$ consists of dipoles (more precisely, $\mu$-dipoles) if they satisfy the equation

$$\Delta v_{A,B} = \chi_A - \chi_B. \quad (7.1)$$

Similarly, we define $\nu$-dipoles as functions $w_{A,B}$ such that

$$\Delta w_{A,B} = c(\chi_A - \chi_B). \quad (7.2)$$

**Proposition 7.4.** For any sets $A, B \in \mathcal{B}_{fin}(\mu)$,

$$\langle f, v_{A,B} \rangle_{H_E} = \int_A f \, d\mu - \int_B f \, d\mu, \quad (7.3)$$

and

$$\langle f, w_{A,B} \rangle_{H_E} = \int_A f \, d\nu - \int_B f \, d\nu. \quad (7.4)$$

**Proof.** The formulas follow immediately from our standing assumption that functions from $H_E$ are locally integrable. Indeed, we have

$$\langle f, v_{A,B} \rangle_{H_E} = \frac{1}{2} \int_{V \times V} (f(x) - f(y))(v_{A,B}(x) - v_{A,B}(y)) \, d\rho(x, y)$$

$$= \frac{1}{2} \int_{V \times V} [f(x)(v_{A,B}(x) - v_{A,B}(y)) - f(y)(v_{A,B}(x) - v_{A,B}(y))] \, d\rho_x(y) d\mu(x)$$

$$= \int_V f(x) \left( \int_V (v_{A,B}(x) - v_{A,B}(y)) \, d\rho_x(y) \right) \, d\mu(x)$$

$$= \int_V f(x) \Delta(v_{A,B})(x) \, d\mu(x)$$

$$= \int_V f(x)(\chi_A(x) - \chi_B(x)) \, d\mu(x).$$

Relation of (7.4) is proved similarly. \qed
Remark 7.5. We recall that the following formula follows from Theorem 3.7 for any function \( f \in \mathcal{H}_E \) and any set \( A \in \mathcal{B}_{\text{fin}}(\mu) \),
\[
\langle f, \chi_A \rangle_{\mathcal{H}_E} = \int_A \Delta f \, d\mu. \tag{7.5}
\]

It is important to remember that this formula is proved under our basic assumptions that the function \( c \in L^1_{\text{loc}}(\mu) \) and \( \mathcal{H}_E \subset L^2_{\text{loc}}(\nu) \).

We use (7.5) to define a new measure \( \mu_f(\cdot) \) on \((V, \mathcal{B})\) where \( f \in \mathcal{H}_E \). We observe that, in general, \( \mu_f \) is a finite additive measure. It is \( \sigma \)-additive when the function \( f \in \mathcal{H}_E \) satisfies the property \( \Delta(f) \in L^1(\mu) \). This measure \( \mu_f \) will be used in Section 8 to construct a reproducing kernel Hilbert space.

Corollary 7.6. The sets \( \mathcal{D}(w) := \text{Span}\{w_{A,B} : A, B \in \mathcal{B}_{\text{fin}}(\mu)\} \) and \( \mathcal{D}(v) := \text{Span}\{v_{A,B} : A, B \in \mathcal{B}_{\text{fin}}(\mu)\} \) are dense in the Hilbert space \( \mathcal{H}_E \).

Proof. Suppose, for contrary, that there exists a vector \( f \in \mathcal{H}_E \) which is orthogonal to \( \mathcal{D}(w) \). Then it follows from (7.4) that, for any sets \( A, B \in \mathcal{B}_{\text{fin}}(\mu) \),
\[
\int_A f \, d\nu = \int_B f \, d\nu.
\]

It is possible only when \( f = 0 \).

It remains to show that dipoles \( w_{A,B} \) always exist in the finite energy space \( \mathcal{H}_E \). To do this, we use the approach elaborated in the theory of electrical networks, see [JP11]. It is obvious that one set in \( w_{A,B} \) can be fixed because \( w_{A,B} = w_{A,A_0} - w_{A_0,B} \).

Lemma 7.7. Let \( A, A_0 \in \mathcal{B}_{\text{fin}}(\mu) \) and \( f \in \mathcal{H}_E \). Then
\[
f \mapsto \int_A f \, d\nu - \int_{A_0} f \, d\nu
\]
is a bounded linear functional on \( \mathcal{H}_E \).

Proof. The idea of the proof is similar to the case of discrete networks, see [JP10, JP11].

The situation with the family of \( \mu \)-dipoles is slightly different as shown in the following statement.

Proposition 7.8. For \( A, B \in \mathcal{B}_{\text{fin}}(\mu) \), the function \( v_{A,B} \) belongs to \( \mathcal{H}_E \) if and only if \( c^{-1} \in L^2_{\text{loc}}(\mu) \).

Proof. Without loss of generality, we can assume that, in the definition of \( v_{A,B} \) (see (7.3)), \( A \cap B = \emptyset \). Since for any \( f \in \mathcal{H}_E \), \( \Delta(f) = c(I - P)(f) \) and the operator \( I - P : \mathcal{H}_E \to L^2(\nu) \) is a contraction (see Corollary 5.9), we conclude that
\[
\Delta(f) \in cL^2(\nu) = L^2(c^{-1}\mu).
\]
Applying this fact to \( f = v_{A,B} \), we obtain that
\[
\infty > \int_V (\Delta(v_{A,B}))^2 c^{-1} \, d\mu \\
= \int_V (\chi_A - \chi_B)^2 c^{-1} \, d\mu \\
= \int_V (\chi_A + \chi_B) c^{-1} \, d\mu \\
= \int_A c^{-1} \, d\mu + \int_B c^{-1} \, d\mu
\]
which proves the proposition. \( \square \)

7.3. Application to machine learning problems. This subsection is devoted to an application of the graph Laplace operator considered in previous sections to the so-called learning problem. The problem we formulate below is an optimization problem with a penalty term, see [AMP10, GFZ16, PS03, SZ09a, SZ09b, SZ07, SY06] for more details.

We first recall the following results proved in [BJ18a]. Let the measure space \((V, \mathcal{B}, \mu)\) and the graph Laplace operator \(\Delta\) be as above, and let \(\mathcal{H}_E\) be the energy space. Then \(\Delta\) can be realized in \(L^2(\mu)\) and \(\mathcal{H}_E\), and denote by \(\Delta^2\) and \(\Delta_{\mathcal{H}_E}\) the corresponding operators in \(L^2(\mu)\) and \(\mathcal{H}_E\).

Let \(J\) and \(K\) be two densely defined operators that constitute a symmetric pair of operators:
\[
L^2(\mu) \xrightarrow{J} \mathcal{H}_E
\]
and
\[
\mathcal{H}_E \xrightarrow{K} L^2(\mu).
\]
For \(\varphi \in \mathcal{D}_Q, \psi \in \mathcal{C}\), where \(\mathcal{D}_Q\) and \(\mathcal{C}\) are dense subsets, we have
\[
(J\varphi, \psi)_{\mathcal{H}_E} = (\varphi, K\psi)_{L^2(\mu)},
\]
see details in [BJ18a, Lemma 8.4].

It follows that:
(1) \(J^* = K\) and \(K^* = J\),
(2) the operators \(J^*J\) and \(K^*K\) are self-adjoint in \(L^2(\mu)\) and \(\mathcal{H}_E\), respectively.

In [BJ18a, Theorem 8.5], we proved the following result.

**Theorem 7.9.** The Laplace operator \(\Delta\) admits its realizations in the Hilbert spaces \(L^2(\mu)\) and \(\mathcal{H}_E\) such that:
(i) \(\Delta_2 = J^*J\) is a positive definite essentially self-adjoint operator;
(ii) \(\Delta_{\mathcal{H}}\) is a positive definite and symmetric operator which is not self-adjoint, in general; a self-adjoint extension \(\Delta_{\mathcal{H}_E}\) of \(\Delta_{\mathcal{H}}\) is given by the operator \(JJ^* = K^*K\).

The goal of this subsection is to apply the above results to \(L^2\)-regulation for learning problem. We will show how to find the minimum of the function
\[
Q(h) = \|\psi - Kh\|^2_{L^2(\mu)} + \gamma \|h\|^2_{\mathcal{H}_E}, \quad h \in \mathcal{H}_E,
\]
where \( \psi \) is a fixed function from \( L^2(\mu) \), \( \gamma > 0 \), and \( K : H_E \to L^2(\mu) \) is defined in (7.7). This problem is interpreted as follows. Suppose \( \psi \) is a given function representing some data. Then the term \( \| \psi - Kh \|^2_{L^2(\mu)} \) corresponds to the least square approximation by functions \( h \) from a feature space, and \( \gamma \| h \|^2_{H_E} \) is the so-called penalty term, see [AMP10, GFZ16, PS03, SZ07, SZ09b, SZ09a, SY06] for more information.

**Proposition 7.10.** Let \( K, J \) be the symmetric pair of operators defined in (7.6) and (7.7), and let \( \bar{\Delta}_{H_E} = K^*K \) be the self-adjoint extension of \( \Delta_{H_E} \). Then, for a given function \( \psi \in L^2(\mu) \),

\[
\arg \min \{ Q(h) : h \in H_E \} = (\gamma I + \bar{\Delta}_{H_E})^{-1} J\psi.
\]

**Proof.** To minimize \( Q \), it suffices to find a function \( h \) such that

\[
\frac{d}{d\varepsilon} Q(h + \varepsilon k) \big|_{\varepsilon=0} = 0, \quad \forall k \in H_E.
\]

Clearly, we need to know only the linear term with respect to \( \varepsilon \) in \( Q(h + \varepsilon k) \) because other terms vanishes after differentiation and substitution \( \varepsilon = 0 \). We compute

\[
\frac{d}{d\varepsilon} Q(h + \varepsilon k) \big|_{\varepsilon=0} = -2\langle \psi, Kk \rangle_{L^2(\mu)} + 2\langle Kh, Kk \rangle_{L^2(\mu)} + 2\langle h, k \rangle_{H_E}
\]

\[
= -2\langle J\psi, k \rangle_{H_E} + 2\langle K^*Kh, k \rangle_{H_E} + 2\langle h, k \rangle_{H_E}
\]

\[
= 2\langle \bar{\Delta}_{H_E}h + \gamma h - J\psi, k \rangle_{H_E}.
\]

It follows that the function \( h \) must satisfy the property

\[
\bar{\Delta}_{H_E}h + \gamma h - J\psi = 0
\]

or

\[
h = (\gamma I + \bar{\Delta}_{H_E})^{-1} J\psi
\]

which is the desired conclusion. \( \square \)

We note that the operator

\[
(\gamma I + \bar{\Delta}_{H_E})^{-1} J = (\gamma I + JK)^{-1} J : L^2(\mu) \to H_E
\]

is bounded, contractive, and self-adjoint.

### 8. Reproducing Kernel Hilbert Spaces

In this section we will show that, for transient Markov processes, the energy space \( H_E \) can be realized as a reproducing kernel Hilbert space (RKHS) for a positive definite kernel. We give also two more reproducing kernel Hilbert spaces that are related to the symmetric measure \( \rho \) on \((V \times V, B \times B)\) and the measure \( \nu \) on \((V, B)\). The standard references for the theory of RKHS are [Aro50, AS57, AFMP94, PR16, SS16], see also more recent results and various applications in [AJ14, AJ15, JT15, JT16, BTA04].
8.1. Definition of RKHS. We begin with reminding the reader the definition of a RKHS.

Let $S$ be a set, and let $K : S \times S \rightarrow \mathbb{R}$ be a positive definite kernel, i.e., the function $K(s, t)$ has the property

$$\sum_{i,j=1}^{N} \alpha_i \alpha_j K(s_i, s_j) \geq 0$$

which holds for any $N \in \mathbb{N}$ and for any $s_i \in S, \alpha_i \in \mathbb{R}, i = 1, \ldots, N$. (For a complex-valued kernel $K$ some obvious changes must be made).

**Definition 8.1.** Fix $s \in S$ and denote by $K_s$ the function $K_s(t) = K(s, t)$ of one variable $t \in S$. Let $K := \text{span}\{K_s : s \in S\}$. The RKHS $H(K)$ is the Hilbert space obtained by completion of $K$ with respect to the inner product defined on $K$ by

$$\langle \sum_{i} \alpha_i K_s, \sum_{j} \beta_j K_s \rangle_{H(K)} := \sum_{i,j=1}^{N} \alpha_i \beta_j K(s_i, s_j)$$

It immediately follows from Definition 8.1 that

$$\langle K(\cdot, s), K(\cdot, t) \rangle_{H(K)} = K(s, t).$$

More generally, this result can be extended to the following property that characterizes functions from the RKHS $H(K)$. For any $f \in H(K)$ and any $s \in S$, one has

$$f(s) = \langle f(\cdot), K(\cdot, s) \rangle_{H(K)}. \quad (8.1)$$

It suffices to check that $(8.1)$ holds for any function from $K$ and then extend it by continuity.

One can check that the following property characterizes functions from the reproducing kernel Hilbert space $H(K)$ constructed by a positive definite kernel $K$ on the set $S$. We formulate it as a statement for further references.

**Lemma 8.2.** A function $f$ is in $H(K)$ if and only if there exists a constant $C = C(f)$ such that for any $n \in \mathbb{N}$, any $\{s_1, \ldots, s_n\} \subset S$, and any $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{R}$, one has

$$\left( \sum_{i=1}^{n} \alpha_i f(s_i) \right)^2 \leq C(f) \sum_{i,j=1}^{n} \alpha_i \alpha_j K(s_i, s_j). \quad (8.2)$$

We follow [JT17] in the following definition. Let $K(s, t)$ be a positive definite kernel as above. It is said that that a measure space $(X, \mathcal{A}, m)$ and functions $K_s^* : S \rightarrow L^2(m)$ define a realization of $K(s, t)$ if

$$K(s, t) = \langle K_s^*, K_t^* \rangle_{L^2(m)} = \int_{X} K_s^* K_t^* \, dm. \quad (8.3)$$

It is said that the realization is tight if the set of functions $\{K_s^*(\cdot) : s \in S\}$ is dense in $L^2(X, \mathcal{A}, m)$. 
We note that the converse approach can be also used. Namely, given a set of functions \( \{K^*_s\} \) from \( L^2(X,A,m) \), one can define a positive definite kernel \( K(s,t) \) by formula (8.3).

### 8.2. Reproducing kernel Hilbert space over \( B_{\text{fin}} \)

We use in this subsection our standard setting: a sigma-finite measure space \((V,B,\mu)\), a symmetric measure \( \rho \) on \( V \times V \), and the function \( c(x) = \rho_x(V) \). Then we define the sequence of transition probabilities \( P_n(x,A) \), the positive operator \( P \) acting by the formula

\[
P(f)(x) = \int_V f(y) P(x,dy) \quad \text{such that} \quad \nu P = \nu, \quad \text{where} \quad d\nu(x) = c(x)d\mu(x).
\]

Let also \( B_{\text{fin}} \) be the algebra of Borel sets of finite measure \( \mu \).

Recall that together with the symmetric measure \( \rho \) we have defined the sequence of symmetric measures \( (\rho_n) \) such that, for \( A,B \in B_{\text{fin}} \),

\[
\rho_n(A \times B) = \int_A P_n(x,B)\,d\nu(x) = \int_V \chi_A P^n(\chi_B)\,d\nu = \langle \chi_A, P^n(\chi_B) \rangle_{L^2(\nu)}.
\]

In particular, \( \rho_0(A \times B) = \nu(A \cap B) \) for \( A,B \in B_{\text{fin}}(\mu) \).

We will assume that the Markov process defined by \( (P_n) \) is transient. In other words, this assumption means that the Green’s function

\[
G(x,A) = \sum_{n=0}^{\infty} P_n(x,A)
\]

is well defined for any \( A \in B_{\text{fin}}(\mu) \). In order to emphasize that \( G(x,A) \) is a function in \( x \) for every fixed \( A \), we will use also the notation \( G_A(\cdot) \).

For every \( A \in B_{\text{fin}}(\mu) \) and \( n \in \mathbb{N}_0 \), the function \( P^n(\chi_A)(x) \) belongs to \( \mathcal{H}_E \), hence assuming the convergence of the series in (8.3), we note that the Green’s function \( G(x,A) \) can be viewed as an element of the energy space \( \mathcal{H}_E \). A direct computation gives the formula for the norm of \( G(x,A) \):

\[
||G(\cdot,A)||_{\mathcal{H}_E}^2 = \sum_{n=0}^{\infty} \rho_n(A \times A)
\]

(8.5)

(details are given in Theorem 8.3 below).

For the sake of completeness, we include the following theorem which was mostly proved in [BJ18b].

**Theorem 8.3.** Let \( (V,B,\mu), \rho_n, \mathcal{H}_E, \) and \( G(x,A) \) be the objects defined as above. Then the following properties hold.

1. For any sets \( A,B \in B_{\text{fin}} \), we have

\[
\langle G_A, G_B \rangle_{\mathcal{H}_E} = \sum_{n=0}^{\infty} \rho_n(A \times B);
\]

(8.6)

2. For any \( f \in \mathcal{H}_E \) and \( A \in B_{\text{fin}}(\mu) \),

\[
\langle f, G_A \rangle_{\mathcal{H}_E} = \int_A f\,d\nu.
\]
Furthermore, if
\[ G := \text{span}\{G_A(\cdot) : A \in B_{\text{fin}}\}, \] (8.7)
then \( G \) is dense in the energy space \( \mathcal{H}_E \).

(3) For \( A, B \in B_{\text{fin}}(\mu) \), we have
\[ \Delta G_A(x) = c(x) \chi_A(x), \]
and
\[ \Delta \omega_{A,B} = \Delta G_A - \Delta G_B = c(\chi_A - \chi_B) \]
is in \( L^2(\nu) \), where \( \omega_{A,B} \) is defined in Section 7.

We observe that the dipoles \( \omega_{A,B} \) can be determined using the formula \( \omega_{A,B} = G_A - G_B \), see also Lemma 7.7.

**Proof.** (1) Clearly, relation (8.6) follows from (8.5), so that it suffices to prove the formula for the norm of \( G_A \) in \( \mathcal{H}_E \). For this, one has

\[
\|G_A(x)\|_{\mathcal{H}_E}^2 = \int\int_{V \times V} (G_A(x) - P_A(y))^2 \, d\rho(x,y)
\]
\[
= \int\int_{V \times V} G_A(x)(G_A(x) - G_A(y)) \, d\rho(x,y)
\]
\[
= \int\int_{V \times V} G_A(x)(G_A(x) - P_A(y))c(x)P(x,dy)d\mu(x)
\]
\[
= \int V G_A(x)[G_A(x) - P(G_A)(x)]c(x) \, d\mu(x)
\]
\[
= \int V G_A(x) \left[ \sum_{n=0}^{\infty} P^n(\chi_A)(x) - \sum_{n=0}^{\infty} P^{n+1}(\chi_A)(x) \right] c(x) \, d\mu(x)
\]
\[
= \int \sum_{n=0}^{\infty} P^n(\chi_A)(x)\chi_A(x) \, d\nu(x)
\]
\[
= \sum_{n=0}^{\infty} \langle \chi_A, P^n(\chi_A) \rangle_{L^2(\nu)}
\]
\[
= \sum_{n=0}^{\infty} \rho_n(A \times A).
\]
For (2), we compute
\[
\langle f, G_A \rangle_{\mathcal{H}_E} = \frac{1}{2} \iiint_{V \times V} (f(x) - f(y))(G_A(x) - G_A(y)) \, d\rho(x, y)
\]
\[
= \iiint_{V \times V} (f(x)G_A(x) - f(x)G_A(y)) \, d\rho(x, y)
\]
\[
= \int_V \left[ f(x)G_A(x)c(x) - f(x) \left( \int_V G_A(y)P(x, dy) \right) c(x) \right] \, d\mu(x)
\]
\[
= \int_V f(x)c(x) \left[ \sum_{n=0}^{\infty} P^n(\chi_A)(x) - \sum_{n=0}^{\infty} P^{n+1}(\chi_A)(x) \right] \, d\mu(x)
\]
\[
= \int_V f(x)\chi_A(x) c(x) \, d\mu(x)
\]
\[
= \int_{A} f \, d\nu.
\]
It follows from the proved relation that if \( \langle f, G_A \rangle_{\mathcal{H}_E} = 0 \) for all \( A \in \mathcal{B}_{\text{fin}}(\mu) \), then \( f = 0 \), and \( \mathcal{G} \) is dense in \( \mathcal{H}_E \).

(3) We compute using the definition of Green’s function and the fact that the series \( \sum_n P_n(x, A) \) is convergent for all \( x \) and all \( A \in \mathcal{B}_{\text{fin}}(\mu) \):
\[
c(x)(I - P)G_A(x) = c(x)(I - P) \sum_{n=0}^{\infty} P_n(x, A)
\]
\[
= c(x) \sum_{n=0}^{\infty} P_n(x, A) - c(x) \sum_{n=1}^{\infty} P_n(x, A)
\]
\[
= c(x)\chi_A(x).
\]

\[\square\]

Corollary 8.4. (1) Assuming that, for every \( A \in \mathcal{B}_{\text{fin}}(\mu) \), the function \( G(\cdot, A) \) belongs to \( L^2(\nu) \), we have
\[
\langle \chi_A, G(\cdot, A) \rangle_{L^2(\nu)} = \sum_{n \in \mathbb{N}_0} \rho_n(A \times A).
\]

(2) If \( G(\cdot, A) \) belongs to \( L^1_{\text{loc}}(\nu) \), then \( \sum_{n \in \mathbb{N}_0} \rho_n(A \times A) < \infty \).

We use now the construction given in Subsection 8.1. Let \( S = \mathcal{B}_{\text{fin}}, \) and we set
\[
K(A, B) = \sum_{n \in \mathbb{N}_0} \rho_n(A \times B).
\]

(8.8)

We first observe that \( K(A, B) \) is a positive definite kernel. This fact follows from (8.6) of Theorem 8.3.

Moreover, one can point out a realization of the kernel \( K(A, B) \) in a \( L^2 \)-space.
Proposition 8.5. Let the Markov operator $P$ determine a transient Markov process. Then, for any $f \in L^2(\nu)$, the function $(I - P)^{-1/2}(f)$ belongs to $L^2(\nu)$. Moreover, $A \mapsto K_A^* (A \in \mathcal{B}_{\text{fin}})$ is a realization of the kernel $K$ on $L^2(\nu)$ where

$$K_A^*(\cdot) := (I - P)^{-1/2}(\chi_A)(\cdot).$$

Proof. We first need to show that $(I - P)^{-1/2} : L^2(\nu) \to L^2(\nu)$. From the spectral theorem for the self-adjoint operator $P$ acting on $L^2(\nu)$, we obtain that

$$\langle P(f), f \rangle_{L^2(\nu)} = \int_{-1}^{1} t \langle Q(dt)f, f \rangle_{L^2(\nu)},$$

where $Q(dt)$ is the projection valued measure for the operator $P$ in $L^2(\nu)$. For a Borel function $\varphi$, we have

$$\langle \varphi(P)f, f \rangle_{L^2(\nu)} = \int_{-1}^{1} \varphi(t) \langle Q(dt)f, f \rangle_{L^2(\nu)}.$$

Then

$$||(I - P)^{-1/2}(f)||_{L^2(\nu)} = \langle (I - P)^{-1}(f), f \rangle_{L^2(\nu)}$$

$$= \int_{-1}^{1} \frac{1}{1 - t} \langle Q(dt)f, f \rangle_{L^2(\nu)}$$

$$= \int_{-1}^{1} \sum_{n=0}^{\infty} t^n \langle Q(dt)f, f \rangle_{L^2(\nu)}$$

$$= \sum_{n=0}^{\infty} \langle P^n(f), f \rangle_{L^2(\nu)}.$$

Take $f = \chi_A, A \in \mathcal{B}_{\text{fin}}$. Then

$$||(I - P)^{-1/2}(\chi_A)||_{L^2(\nu)}^2 = \sum_{n=0}^{\infty} \langle P^n(\chi_A), \chi_A \rangle_{L^2(\nu)}$$

$$= \sum_{n=0}^{\infty} \rho_n(A \times A).$$

Since $(P_n)$ is a transient Markov process, we see that the $L^2$-norm of $(I - P)^{-1/2}(\chi_A)$ is finite. Then the result follows from the density of simple functions in $L^2(\nu)$.

A similar computation can be used in order to show that, for $A, B \in \mathcal{B}_{\text{fin}},$

$$\langle (I - P)^{-1/2}(\chi_A), (I - P)^{-1/2}(\chi_B) \rangle = \sum_{n=0}^{\infty} \rho_n(A \times B).$$

(8.9)

Since, by definition, $K(A, B) = \sum_{n=0}^{\infty} \rho_n(A \times B)$, this means that the functions $K_A^* = (I - P)^{-1/2}(\chi_A)$ define a realization of the kernel $K(A, B).$
Lemma 8.6. Let \((P_n)\) be a sequence of probability measures that defines a transient Markov process. Then, for any \(A \in \mathcal{B}_{\text{fin}}\), the function \(P_n(\cdot, A)\) belongs to \(\mathcal{H}_E\). Moreover, the following relation hold:

\[
\langle P_k(\cdot, A), P_l(\cdot, A) \rangle_{\mathcal{H}_E} = \rho_k(A \times A) - \rho_{k+l+1}(A \times A),
\]
(8.10)

\[
\langle P_k(\cdot, A), G(\cdot, A) \rangle_{\mathcal{H}_E} = \rho_k(A \times A).
\]

Proof. In [BJ18b], we proved that

\[
\|P_n(\cdot, A)\|_{\mathcal{H}_E}^2 = \rho_{2n}(A \times A) - \rho_{2n+1}(A \times A), \quad n \in \mathbb{N}.
\]

Using similar computation, one can generalize this result and prove that (8.10) holds. We leave details for the reader. □

Corollary 8.7. Let \((P_n)\) be a transient Markov process. Then the energy space \(\mathcal{H}_E\) consists of locally integrable functions.

This result formulated in this corollary follows from Theorem 8.3 and (8.8). We remark that Corollary 8.7 well agrees with Assumption 2 made in Section 3.

Proof. The proof follows from the following fact: for a function \(f \in \mathcal{H}_E\) and \(A \in \mathcal{B}_{\text{fin}}\), one has

\[
\langle f, G(\cdot, A) \rangle_{\mathcal{H}_E} = \int_A f \, d\nu.
\]
□

In the remaining part of this section, we will define and study isometries between the three Hilbert spaces: RKHS \(\mathcal{H}(K)\), energy space \(\mathcal{H}_E\), and \(L^2(\nu)\).

We define the operators \(I_1, I_2,\) and \(I_3\) by setting

\[
K(\cdot, A) \xrightarrow{I_1} G(\cdot, A)
\]

\[
G(\cdot, A) \xrightarrow{I_2} (I - P)^{-1/2}(\chi_A)
\]

\[
(I - P)^{-1/2}(\chi_A) \xrightarrow{I_3} K(\cdot, A).
\]

We recall that the family of functions \(\{G(\cdot, A) \mid A \in \mathcal{B}_{\text{fin}}\}\) is dense in \(\mathcal{H}_E\), so that \(I_2\) is a densely defined map. By linearity, the definition of \(I_1\) can be extended to a dense subset of functions from \(\mathcal{H}(K)\). One can also show that \(I_3\) is also densely defined operator.

Lemma 8.8. Let \(\mathcal{D}_{\text{fin}}(\mu)\) be the span of characteristic functions \(\chi_A, A \in \mathcal{B}_{\text{fin}}\). Then the set \((I - P)^{-1/2}(\mathcal{D}_{\text{fin}})\) is dense in \(L^2(\nu)\).

Proof. The proof is based on the fact that \(L^2(\nu)\) does not contain nontrivial harmonic functions. □

Corollary 8.9. The operators \(I_1, I_2\) and \(I_3\) implement isometric isomorphisms of the Hilbert spaces:

\[
I_1 : \mathcal{H}(K) \to \mathcal{H}_E, \quad I_2 : \mathcal{H}_E \to L^2(\nu), \quad I_3 : L^2(\nu) \to \mathcal{H}(K).
\]
Proof. The result follows immediately from the proved above formulas for the norm of functions in the Hilbert spaces $\mathcal{H}(K)$, $\mathcal{H}_E$, and $L^2(\nu)$. Indeed, we deduce from (8.5), (8.8), and (8.9) that the functions $K(\cdot, A)$, $G(\cdot, A)$, and $(I - P)^{-1/2}(\chi_A)$ have the same norm equal to $\sum_{n=0}^{\infty} \rho_n(A \times A)$.

□

Corollary 8.10. The energy space $\mathcal{H}_E$ is a RKHS.

8.3. Reproducing kernel Hilbert space generated by symmetric measures. We give here one more construction of a reproducing kernel Hilbert space defined by a symmetric measure $\rho$. We take the set $S$ to be the same as above, i.e., $S = \mathcal{B}_{\text{fin}}(\mu)$. Define a function $k = k_{\rho} : \mathcal{B}_{\text{fin}}(\mu) \times \mathcal{B}_{\text{fin}}(\mu) \to \mathbb{R}$ as follows:

$$k_{\rho} : (A, B) \to \rho((A \cap B) \times V) - \rho(A \times B) \quad (8.11)$$

It is difficult to determine whether the function $k_{\rho}(A, B)$ is positive definite using the definition (8.11) only. The next statement follows immediately from Theorem 3.6.

Lemma 8.11. The function $k_{\rho}$ is positive definite on the set $\mathcal{B}_{\text{fin}}$.

Proof. The proof is based on the formula

$$\langle \chi_A, \chi_B \rangle_{\mathcal{H}_E} = \rho((A \cap B) \times V) - \rho(A \times B)$$

that is proved in [BJ18a, Lemma 6.18]. Then, for any $A_1, \ldots, A_m \in \mathcal{B}_{\text{fin}}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$,

$$\sum_{i,j=1}^{m} \alpha_i \beta_j k_{\rho}(A_i, B_j) = \left( \sum_{i=1}^{m} \alpha_i \chi_{A_i}, \sum_{j=1}^{m} \beta_j \chi_{B_j} \right)_{\mathcal{H}_E}.$$

□

Let $\mathcal{H}(k_{\rho})$ be the reproducing kernel Hilbert space (RKHS) constructed by the positive definite function $k_{\rho}(\cdot, \cdot)$. By general theory, the Hilbert space $\mathcal{H}(k_{\rho})$ consists of functions on $\mathcal{B}_{\text{fin}}(\mu)$ such that, for any $f \in \mathcal{H}(k_{\rho})$,

$$f(B) = \langle f, k_{\rho}(\cdot, B) \rangle_{\mathcal{H}(k_{\rho})}.$$

In what follows, we will discuss relations between the two Hilbert spaces $\mathcal{H}(k_{\rho})$ and $\mathcal{H}_E$.

Lemma 8.12. Let $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where $A_i \in \mathcal{B}_{\text{fin}}(\mu)$. Then the map

$$i : f \mapsto f : \mathcal{D}_{\text{fin}}(\mu) \to \mathcal{H}_E$$

can be extended to an isometry $i$ from $\mathcal{H}(k_{\rho})$ to $\mathcal{H}_E$.

Proof. Since $\mathcal{H}(k_{\rho})$ is the closure of $\mathcal{D}_{\text{fin}}(\mu)$, it suffices to check that $i$ is an isometry on functions $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, from $\mathcal{D}_{\text{fin}}(\mu)$:

$$\|f\|_{\mathcal{H}(k_{\rho})} = \|f\|_{\mathcal{H}_E}.$$
Indeed, we have
\[
\left\| \sum_{i=1}^{n} \alpha_i \chi_{A_i} \right\|_{\mathcal{H}(k_\rho)}^2 = \sum_{i,j=1}^{n} \alpha_i \alpha_j k_\rho(A_i, A_j)
\]
\[
= \sum_{i,j=1}^{n} \alpha_i \alpha_j (\rho((A_i \cap A_j) \times V) - \rho(A_i \times A_j))
\]
\[
= \int \int_{V \times V} f(x)^2 \, d\rho(x, y) - \int \int_{V \times V} f(x)f(y) \, d\rho(x, y)
\]
\[
= \frac{1}{2} \int \int_{V \times V} (f(x) - f(y))^2 \, d\rho(x, y)
\]
\[
= ||f||^2_{\mathcal{H}_E}.
\]

\[\square\]

**Corollary 8.13.** The map \(i\) defined in Lemma 8.12 implements an isometric isomorphism between \(\mathcal{H}(k_\rho)\) and the subspace \(\mathcal{H}_E \ominus \text{Harm} = \overline{D_{\text{lin}}(\mu)}^{\mathcal{H}_E}\) of \(\mathcal{H}_E\).

For any \(f \in \mathcal{H}_E\), define a signed measure \(\mu_f\) on \((V, \mathcal{B})\) by setting
\[
\mu_f(A) = \langle \chi_A, f \rangle_{\mathcal{H}_E}. \tag{8.12}
\]

**Lemma 8.14.** The function \(A \mapsto \mu_f(A) \in \mathcal{H}(k_\rho)\) for any \(f \in \mathcal{H}_E\). Moreover, for any \(f \in \mathcal{H}_E \ominus \text{Harm}\), we have
\[
||\mu_f||^2_{\mathcal{H}(k_\rho)} = ||f||^2_{\mathcal{H}_E}.
\]

**Proof.** To prove this result, we use the criterion given in Lemma 8.2 for the function \(\mu_f(\cdot)\). It gives
\[
\left| \sum_{i=1}^{n} \alpha_i \mu_f(A_i) \right|^2 = \left| \sum_{i=1}^{n} \alpha_i \langle \chi_{A_i}, f \rangle_{\mathcal{H}_E} \right|^2
\]
\[
= \left| \left( \sum_{i=1}^{n} \alpha_i \chi_{A_i} \right)_{\mathcal{H}_E} \right|^2
\]
\[
\leq ||f||^2_{\mathcal{H}_E} \left| \sum_{i=1}^{n} \alpha_i \chi_{A_i} \right|_{\mathcal{H}_E}^2
\]
\[
= ||f||^2_{\mathcal{H}_E} \left\langle \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \sum_{i=1}^{n} \alpha_i \chi_{A_i} \right\rangle_{\mathcal{H}_E}
\]
\[
= ||f||^2_{\mathcal{H}_E} \sum_{i,j=1}^{n} \alpha_i \alpha_j \langle \chi_{A_i}, \chi_{A_j} \rangle_{\mathcal{H}_E}
\]
\[
= ||f||^2_{\mathcal{H}_E} \sum_{i,j=1}^{n} \alpha_i \alpha_j k_\rho(A_i, A_j).
\]
Hence, the result follows from (8.2).

For the second statement, it suffices to take \( f = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \) with \( A_i \in \mathcal{B}_{\text{fin}}(\mu) \).

Then
\[
\|\mu_f\|_{H(k_\rho)}^2 = \| \sum_{i=1}^{n} \alpha_i k_\rho(\cdot, A_i) \|_{H(k_\rho)}^2 = \sum_{i,j=1}^{n} \alpha_i \alpha_j k_\rho(A_i, A_j)
\]
\[
= \sum_{i,j=1}^{n} \alpha_i \alpha_j \langle \chi_{A_i}, \chi_{A_j} \rangle_{H_E}
\]
\[
= \|f\|_{H_E}^2.
\]

\[\square\]

**Theorem 8.15.** (1) For any \( f \in \mathcal{H}_E \), we have
\[
\mu_f(A) = \int_A \Delta f \, d\mu,
\]
(8.13)
i.e.,
\[
\frac{d\mu_f}{d\mu}(x) = \Delta(f)(x).
\]

(2) \( f \in \mathcal{H}_{\text{arm}} \iff \mu_f = 0 \).

**Proof.** (1) To prove (8.13), we use that functions from \( \mathcal{H}_E \) are locally integrable. Then one can compute
\[
\langle \chi_A, f \rangle_{\mathcal{H}_E} = \frac{1}{2} \int_{V \times V} (\chi_A(x) - \chi_A(y))(f(x) - f(y)) \, d\rho(x,y)
\]
\[
= \int_{V \times V} (\chi_A(x)f(x) - \chi_A(x)f(y)) \, d\rho(x,y)
\]
\[
= \int_{V} \chi_A(x) \left( \int_{V} (f(x) - f(y)) \, d\rho_x(y) \right) d\mu(x)
\]
\[
= \int_{A} \Delta(f)(x) \, d\mu(x).
\]

(2) We use (1) to prove the second assertion. Indeed, it follows from (8.13) that if \( f \in \mathcal{H}_{\text{arm}} \), then \( \Delta(f) = 0 \) and \( \mu_f(A) = 0 \) for any \( A \in \mathcal{B}_{\text{fin}}(\mu) \).

On the other hand, if \( \mu_f(A) = 0 \), then \( \int_{A} \Delta(f) \, d\mu = 0 \) for any \( A \in \mathcal{B}_{\text{fin}}(\mu) \). This means that \( \Delta(f) = 0 \) a.e. \[\square\]

It follows from Lemma 8.14 that the map
\[
\mathcal{H}_E \ni f \overset{W}{\mapsto} \mu_f \in \mathcal{H}(k_\rho)
\]
is well defined.
Corollary 8.16. Let \( f = v_{A,B} \) where \( A, B \in \mathcal{B}_{\text{fin}}(\mu) \). Then the function \( C \mapsto \mu_{v_{A,B}} \) satisfies the property
\[
\mu_{v_{A,B}}(C) = \mu(A \cap C) - \mu(B \cap C).
\]

Proof. The result follows from the following computation
\[
\mu_{v_{A,B}}(C) = \langle v_{A,B}, \chi C \rangle_{\mathcal{H}_E}
= \langle \Delta v_{A,B}, \chi C \rangle_{L^2(\mu)}
= \langle \chi_A - \chi_B, \chi C \rangle_{L^2(\mu)}
= \mu(A \cap C) - \mu(B \cap C).
\]

Corollary 8.17. The map \( W \) is a co-isometry and \( i^* = W \).

Proof. We will show that \( W(f) = \mu_f \) and \( i^*(f) \) coincide as functions on \( \mathcal{D}_{\text{fin}}(\mu) \).
Let \( A \) be any set from \( \mathcal{B}_{\text{fin}} \), then
\[
\mu_f(A) = \langle f, \chi_A \rangle_{\mathcal{H}_E}
= \langle f, i(\chi_A) \rangle_{\mathcal{H}_E}
= \langle i^*(f), \chi_A \rangle_{\mathcal{H}(k_\rho)}
= i^*(f)(A).
\]
The last equality follows from the reproducing property of \( \mathcal{H}(k_\rho) \). \( \square \)

8.4. Reproducing kernel Hilbert space generated by the measure \( \nu \). Let \((V, B, \nu)\) be a \( \sigma \)-finite measure space, and let \( A, B \) be any two elements of the set \( \mathcal{B}_{\text{fin}}(\nu) \). Define \( K_\nu : \mathcal{B}_{\text{fin}}(\nu) \times \mathcal{B}_{\text{fin}}(\nu) \to [0, \infty) \) as follows:
\[
K_\nu(A, B) := \nu(A \cap B).
\]
Then \( K_\nu \) is a positive definite kernel because \( K_\nu(A, B) = \langle \chi_A, \chi_B \rangle_{L^2(\nu)} \) and
\[
\sum_{i,j=1}^{n} \alpha_i \alpha_j K_\nu(A_i, A_j) = ||\sum_{i=1}^{n} \alpha_i \chi_{A_i}||_{L^2(\nu)}.
\]

Theorem 8.18. The kernel \( K_\nu(A, B) \) is positive definite, and, for the corresponding RKHS \( \mathcal{H}_\nu \), a function \( F \) on \( \mathcal{B}_{\text{fin}}(\nu) \) is in \( \mathcal{H}_\nu \) if and only if there exists a function \( f \in L^2_{\text{loc}}(\nu) \) such that
\[
F(A) = \int_A f \, d\nu, \quad A \in \mathcal{B}_{\text{fin}}(\nu). \tag{8.14}
\]
Moreover,
\[
||F||_{\mathcal{H}_\nu} = ||f||_{\mathcal{H}(k_\rho)}. \tag{8.15}
\]

Proof. Let \( F \) be a function on \( \mathcal{B}_{\text{fin}}(\nu) \) defined by (8.14). To show that \( F(\cdot) \) belongs to \( \mathcal{H}_\nu \), we use (8.2) of Lemma 8.2, i.e.,
\[
\left( \sum_{i=1}^{n} \xi_i F(A_i) \right)^2 \leq C_F \sum_{i,j=1}^{n} \xi_i \xi_j K_\nu(A_i, A_j), \tag{8.16}
\]
where \( \xi_i \in \mathbb{R}, A_i \in B_{\text{fin}}(\nu), i = 1, \ldots, n \), and the constant \( C_F \) depends on \( F \) only. Take the function
\[
\varphi(x) := \sum_{i=1}^{n} \xi_i \chi_{A_i}(x)
\]
which belongs to \( L^2(\nu) \) and find that
\[
||\varphi||^2_{L^2(\nu)} = \sum_{i,j=1}^{n} \xi_i \xi_j \nu(A_i \cap A_j) = \left| \sum_{i=1}^{n} \xi_i K(\cdot, A_i) \right|_{H^\nu}^2 .
\]
For \( F(\cdot) \) as in (8.14), we compute using the Schwarz inequality
\[
\left( \sum_{i=1}^{n} \xi_i F(A_i) \right)^2 = \left( \sum_{i=1}^{n} \xi_i \int_{A_i} f \, d\nu \right)^2 \\
\leq \left( \int_V f \left( \sum_{i=1}^{n} \xi_i \chi_{A_i} \right) \, d\nu \right)^2 \\
= ||f||^2_{L^2(B,\nu)} \left( \sum_{i,j=1}^{n} \xi_i \xi_j K(\cdot, A_i, A_j) \right)
\]
where \( B = \bigcup_i A_i \).

Relation (8.15) can be proved by using the corresponding definitions of the norms in \( H^\nu \) and \( H^E \). We leave the details for the reader. \( \Box \)

Remark 8.19. We note that Theorem 8.18 agrees with the definition of \( H^\nu \) because, for a fixed \( B \in B_{\text{fin}}(\nu) \), the function \( K(\cdot, B) \) is represented by (8.14) with \( f = \chi_B \).

If \( V = [0, \infty) \), \( \nu \) is the Lebesgue measure on \( V \), and \( K_\nu(A, B) = \nu(A \cap B) \), then
\[
K_\nu([0, s] \cap [0, t]) = s \wedge t, \quad s, t \in \mathbb{R}_+ .
\]
It follows that the RKHS \( H_\nu \) can be represented as
\[
H_\nu = \{ F : F(0) = 0, \quad F' \in L^2([0, \infty), \nu) \}
\]
with
\[
||F||^2_{H_\nu} = \int_0^\infty |F'|^2 \, d\nu .
\]

8.5. Conditionally negative definite kernel. In this subsection we discuss the notion of conditionally negative definite kernels.

Definition 8.20. Let \( X \) be arbitrary set. Then the map \( N : X \times X \to \mathbb{R} \) is called a conditionally negative definite kernel if for any \( n \in \mathbb{N} \), any finite set of points \( x_1, \ldots, x_n \), and any real numbers \( \lambda_1, \ldots, \lambda_n \), one has
\[
\sum_{i,j=1}^{n} \lambda_i \lambda_j N(x_i, x_j) \leq 0
\]
provided that $\sum_{i=1}^{n} \lambda_i = 0$.

Conditionally negative definite kernels were completely characterized in \cite{Sch38} where the following result was proved.

**Theorem 8.21.** Let $N : X \times X \to \mathbb{R}$ satisfy the following conditions: $N(x, y) = N(y, x) \geq 0$, and $N(x, x) = 0$. If $N(x, y)$ is a conditionally negative definite kernel, then there exists a real Hilbert space $\mathcal{H}(N)$ and a map $\alpha : X \to \mathcal{H}(N)$ such that

$$N(x, y) = ||\alpha(x) - \alpha(y)||^2_{\mathcal{H}(N)}.$$ 

It was also shown in \cite{J15} that, for any conditionally negative definite kernel $N(x, y)$, there exists a positive definite kernel $K(x, y)$ and a function $F : X \to \mathbb{R}$ such that

$$N(x, y) = -K(x, y) + F(x) + F(y).$$ 

Let now $\rho$ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$, and let $\mathcal{H}_E$ be the finite energy Hilbert space. For any sets $A, B \in \mathcal{B}_{\text{fin}}(\mu)$, we consider the dipoles $\omega_{A,B}$ defined in Section 7. We recall that these functions form a dense subset in $\mathcal{H}_E$ and satisfy the relation $\Delta(\omega_{A,B}) = c(\chi_A - \chi_B)$. As was mentioned in Section 7, we can fix a set $A_0 \in \mathcal{B}_{\text{fin}}(\mu)$ and represent $\omega_{A,B}$ as the difference $\omega_{A,A_0} - \omega_{B,A_0}$.

**Lemma 8.22.** Let

$$N_{\rho}(A, B) = ||\omega_{A,B}||^2_{\mathcal{H}_E}, \quad A, B \in \mathcal{B}_{\text{fin}}(\mu).$$

Then $N_{\rho}$ is a conditionally negative definite kernel.

The lemma follows directly from Theorem 8.21.

Applying Theorem 8.21 we can define a Hilbert space $\mathcal{H}(N)$ and a map $\alpha : \mathcal{B}_{\text{fin}}(\mu) \to \mathcal{H}(N)$ such that $N_{\rho}(A, B) = ||\alpha(A) - \alpha(B)||^2_{\mathcal{H}(N)}$.

**Theorem 8.23.** Let $\Lambda : \omega_{A,B} \mapsto \alpha(A) - \alpha(B)$ can be extended by linearity to an isometric isomorphism $\mathcal{H}_E \cong \mathcal{H}(N)$.

**Proof.** The proof is based on the given above definitions, Theorem 8.21 and Lemma 8.22. We leave the details to the reader. \hfill \square

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