Stabilization of Quantum Computer
Calculation Basis by Qubit Encoding in
Virtual Spin Representation

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Abstract

It is proposed to map the quantum information qubit not to individual spin 1/2 states, but to the collective spin states being eigenfunctions of the Hamiltonian including spin-spin interactions, which may be not small. Such an approach allows to introduce more stable calculation basis for quantum computer based on the solid state NMR systems.

Key words: quantum computer, basis, stability, gate, virtual spin, representation

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1 Introduction

NMR is excellent testing ground for approbation of the different ideas of quantum information science. Due to the combination of the developed theory and

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the refined experimental technique it becomes to be possible to realize some algorithms on a few qubit quantum computer being implemented on standard NMR spectrometers. Up to day the quantum information science achievements based on liquid state NMR systems far exceed other experimental realization of quantum calculations.

The success of the liquid state quantum information science connected partially with the fact that it was found the clear and convenient physical object, nuclear spin 1/2, for the representation of the quantum mechanical bit (qubit) conception. Two possible spin orientations in the dc magnetic field, which are described by the nuclear z-component eigenfunctions \(|m\rangle (m = \pm 1/2)\), are associated with two qubit states \(|0\rangle\) and \(|1\rangle\). The functions \(|m\rangle\) are also eigenfunctions of the spin-system main Hamiltonian. Two qubit calculation basis arranged on the states of two spins is

\[
|00\rangle = |m_1 = -1/2, m_2 = -1/2\rangle, |01\rangle = |m_1 = -1/2, m_2 = +1/2\rangle, \\
|10\rangle = |m_1 = +1/2, m_2 = -1/2\rangle, |11\rangle = |m_1 = +1/2, m_2 = +1/2\rangle. \tag{1}
\]

However the restrictions of liquid state NMR possibilities in quantum information science become evident yet now \([1,2]\). The next step in development of information science based on NMR will be connected probably with solid state NMR. The essential feature of solid state NMR is the existence in the Hamiltonian of the spin-spin interactions (of exchange or dipole-dipole type), which is not averaged by the thermal motion and contains not only spin Z-components, but also X- and Y- components.

For the last reason the orientation of individual spin in the dc magnetic field in solid state media becomes bad integral of motion, that is the stationary states of spin system do not correspond to a definite value of the individual spin.
Z-component, and functions (1) are unstable states. That is why the special efforts are required for damping the spin-spin interactions and for support the stability of calculation basis connected with individual spin orientation [2]. This can be reached by effecting every sort of spins in solid state by the pulse sequences developed in NMR, for example, by WAHUHA sequence [3] consisting of four resonance pulses.

Here we demonstrate that it is possible to achieve the increasing of the computational basis stability without damping the spin-spin coupling at the condition when spins Z-components do not commute with main Hamiltonian. The point is in the qubit mapping on a pair of the eigenstates of Hamiltonian, which include main Hamiltonian and spin-spin interaction, instead of eigenstates of the main Hamiltonian only, as it can be interpreted if functions (1) are used for encoding.

In terms of spin orientations it means that virtual spin orientation is mapped to the qubit [4]. In this case the qubits turn out to be delocalized and direct correspondence between the qubit and individual spin is lost.

To make clear the idea of stabilization let us take in consideration a quantum system with the Hamiltonian

\[ \mathcal{H} = \mathcal{H}_0 + V_1 + V_2 + V_3 + \ldots, \]

where \( \mathcal{H}_0 \) is a main Hamiltonian of non interacting particles and \( V_1 \gg V_2 \gg V_3 \gg \ldots \) is a hierarchy of other interactions. Let the eigenfunctions of the main Hamiltonian are (1).

In the case when qubits are encoded on the eigenstates \( |m_1, m_2\rangle \) of main
Hamiltonian, the qubit states remain stable only on the times \( t_1 \simeq h/V_1 \). Then they vary under the influence of the \( V_1 \) interaction. The calculation basis stability will increase and will hold during the time interval \( t_2 \simeq h/V_2 \), if the \( V_1 \) interaction is included in the main Hamiltonian and the qubits will be encoded on the eigenfunctions of the Hamiltonian \( H_0 + V_1 \). According to the definition of the interaction hierarchy it means: \( t_2 \gg t_1 \). In other words the including of more and more weaker Hamiltonians in the qubit definition increases the basis stability. Such including needs the usage of qubit encoding in the virtual spin representation.

2 A simple example: system of two interacting spins

The complete set of gates, which is sufficient for forming the algorithm of an arbitrary complexity, consists of one qubit rotations and two qubit controlled negation gate CNOT [6]. That is why for an example it is enough to consider system of two nonequivalent coupled spins \( I = 1/2 \) and \( S = 1/2 \). The Hamiltonian of such a system is

\[
H = \hbar \omega_0(I_z + S_z) + \hbar \delta/2(I_z - S_z) + \hbar J(IS) + \hbar V, \\
\omega_0 = (1/2)(\gamma_I + \gamma_S)H_0, \quad \delta = -(\gamma_I + \gamma_S)H_0, \quad (2)
\]

where \( \gamma_I \) and \( \gamma_S \) are the nuclear gyromagnetic ratio, \( J \) is the exchange integral, \( H_0 \) is static magnetic field, \( \hbar V \) is the spin-lattice or dipole-dipole interaction. Let the inequalities \( \omega_0 > J > V \) take place for the Hamiltonian parameters. The properties of this system are known for a long time [7].

This Hamiltonian eigenfunctions in case \( V = 0 \) are
\[ |\Psi_1\rangle = |++\rangle \equiv |m_I = +1/2, m_S = +1/2\rangle, |\Psi_2\rangle = |p|+\rangle + q|-\rangle, |\Psi_3\rangle = |p|-\rangle - q|+\rangle, |\Psi_3\rangle = |--\rangle, \quad (3) \]

where \( p = \cos(\phi/2), q = \sin(\phi/2) \). The eigenvalues

\[
E_1 \equiv \hbar \varepsilon_1 = \hbar \omega_0 + (1/4)\hbar J, \quad E_2 \equiv \hbar \varepsilon_2 = -(1/4)\hbar J + (1/2)\hbar \theta, \\
E_3 \equiv \hbar \varepsilon_3 = -(1/4)\hbar J - (1/2)\hbar \theta, \quad E_4 \equiv \hbar \varepsilon_4 = -\hbar \omega_0 + (1/4)\hbar J \quad (4)
\]

correspond to these eigenfunctions (3), where \( \theta^2 = J^2 + \delta^2 \). In such a system there are four allowed resonance transitions on the frequencies

\[
\varepsilon_{12} = \omega_0 + (1/4)J - (1/2)\theta, \quad \varepsilon_{13} = \omega_0 + (1/4)J + (1/2)\theta, \\
\varepsilon_{24} = \omega_0 - (1/4)J - (1/2)\theta, \quad \varepsilon_{34} = \omega_0 - (1/4)J - (1/2)\theta \quad (5)
\]

having the relative intensities

\[
A_{24} = A_{12} \propto |\langle \Psi_1 | I_x + S_x | \Psi_2 \rangle|^2 = 1 + \sin \phi, \\
A_{34} = A_{13} \propto |\langle \Psi_1 | I_x + S_x | \Psi_3 \rangle|^2 = 1 - \sin \phi. \quad (6)
\]

3 Calculation basis in virtual spin representation

Four-dimensional Hilbert space spanned on the functions (3) can be considered as a direct product of two two-dimensional Hilbert spaces of virtual spins \( I = 1/2 \) and \( S = 1/2 \) [8]. It means that the Hamiltonian (2) eigenfunctions (3) are taken as the calculation basis, that is

\[
|00\rangle = |\Psi_1\rangle, |01\rangle = |\Psi_2\rangle, \\
|10\rangle = |\Psi_3\rangle, |11\rangle = |\Psi_4\rangle, \quad (7)
\]

In such notations the separate virtual spin corresponds to the separate qubit,
that is

\[ |0\rangle = |m_Q = -1/2\rangle, |1\rangle = |m_Q = +1/2\rangle, |0\rangle = |m_R = -1/2\rangle, |1\rangle = |m_R = +1/2\rangle, \]

and coupled states of two qubits form the calculation basis

\[ |00\rangle = |m_Q = -1/2, m_R = -1/2\rangle, |01\rangle = |m_Q = -1/2, m_R = +1/2\rangle, |10\rangle = |m_Q = +1/2, m_R = -1/2\rangle, |11\rangle = |m_Q = +1/2, m_R = +1/2\rangle, \quad (8) \]

where \( m_Q \) and \( m_R \) are the "eigenvalues" of the virtual spins \( Q = 1/2 \) and \( R = 1/2 \). The functions (8) are connected to the Hamiltonian (2) eigenstates (3) by relations (7).

Now the resonance transition between two states of the real physical system attributed to a single qubit can be interpreted as a virtual spin reorientation. For example, the transition \( \langle \Psi_1 | \leftrightarrow | \Psi_2 \rangle \) can be interpreted in virtual spin representation as a virtual spin \( R \) rotation and so on. It can be shown (see the Table 1) using the results of papers [4,5] that universal gates being established in the two qubit system under consideration may be realized by suitable pulses of resonance radio frequency field.

These quantum gates in the two qubit quantum system can be realized by one double frequency pulse for qubit rotation and one single frequency pulse for controlled negation [5].
Table 1

Logic operations and implementation pulses

| Logic operation             | Excitation transitions                  |
|----------------------------|----------------------------------------|
| Virtual spin $Q$ rotation   | $\langle \Psi_1 | \leftrightarrow | \Psi_2 \rangle$ and $\langle \Psi_3 | \leftrightarrow | \Psi_3 \rangle$ |
| Virtual spin $R$ rotation   | $\langle \Psi_1 | \leftrightarrow | \Psi_3 \rangle$ and $\langle \Psi_2 | \leftrightarrow | \Psi_4 \rangle$ |
| Controlled negation of spin $Q$ | $\pi$-pulse on the $\langle \Psi_3 | \leftrightarrow | \Psi_4 \rangle$ transition |
| $\text{CNOT}_{R\rightarrow Q}$ |                                         |
| Controlled negation of spin $R$ | $\pi$-pulse on the $\langle \Psi_2 | \leftrightarrow | \Psi_4 \rangle$ transition |
| $\text{CNOT}_{Q\rightarrow R}$ |                                         |

4 Conclusion

As it was shown above the information encoding in the virtual spin representation permits to create rather simple the universal set of quantum gates in system of two interacting real spins located in the solid state. The preference of this type encoding lies in the possibility to use systems of two spins with large spin coupling and in the absence of necessity to subject the solid state to continuous pulse influence for supporting the calculation basis stability. The basis stability in system under consideration will hold during time interval $t_1 \simeq 1/J$, if qubit encoding is performed on the eigenstates (1) of Zeeman Hamiltonian. The stability spreads to a larger interval $t_2 \simeq 1/J$, if the encoding is established on the eigenstates (3) of the Hamiltonian (2) being considered without spin-lattice and dipole-dipole interaction. Besides the operation time of such a gate is in the whole experimentalists control and can be made sufficiently short, whereas the operation time when coding is fulfilled on
the real spins is determined by the exchange interaction value (by substance properties) [6] and, generally speaking, may be long.

The proposed approach is applicable to qubit encoding in the quantum information medium of an arbitrary nature, if there are suitable selection rules for external excitation, which is necessary for gate arrangement. In particular it may be the cluster of strong interacting particles or liquid state NMR systems in the case, where the exchange interaction is not averaged to the spin $z$-components. An example of virtual qubits encoding on the optical states of an individual atom is given in [8].

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