Application of Jain and Munczek’s bound–state approach to $\gamma\gamma$–processes of $\pi^0$, $\eta_c$ and $\eta_b$

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Abstract

We point out the problems affecting most quark–antiquark bound state approaches when they are faced with the electromagnetic processes dominated by Abelian axial anomaly. However, these problems are resolved in the consistently coupled Schwinger-Dyson and Bethe-Salpeter approach. Using one of the most successful variants of this approach, we find the dynamically dressed propagators of the light $u$ and $d$ quarks, as well as the heavy $c$ and $b$ quarks, and find the Bethe-Salpeter amplitudes for their bound states $\pi^0$, $\eta_c$ and $\eta_b$. Thanks to incorporating the dynamical chiral symmetry breaking, the pion simultaneously appears as the (pseudo)Goldstone boson. We give the theoretical predictions for the $\gamma\gamma$ decay widths of $\pi^0$, $\eta_c$ and $\eta_b$, and for the $\pi^0\gamma^* \rightarrow \gamma$ transition form factor, and compare them with experiment. In the chiral limit, the axial-anomaly result for $\pi^0 \rightarrow \gamma\gamma$ is reproduced analytically in the consistently coupled Schwinger-Dyson and Bethe–Salpeter approach, provided that the quark-photon vertex is dressed consistently with the quark propagator, so that the vector Ward-Takahashi identity of QED is obeyed. On the other hand, the present approach is also capable of quantitatively describing systems of heavy quarks, concretely $\eta_c$ and possibly $\eta_b$, and their $\gamma\gamma$-decays. We discuss the reasons for the broad phenomenological success of the bound–state approach of Jain and Munczek.

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1. INTRODUCTION

In Ref. [1], we calculated the two-photon decays $\pi^0, \eta_c, \eta_b \to \gamma\gamma$, and – in the case of the neutral pion – also the off-shell extension thereof, namely the transition form factor for the process $\gamma^*\pi^0 \to \gamma$. We used a kind of quark–antiquark ($q\bar{q}$) bound-state approach to mesons known as the coupled Schwinger-Dyson and Bethe-Salpeter (SD-BS) approach. More specifically, we used one of the phenomenologically most successful variant of this approach so far, namely the one of Jain and Munczek [2–4].

One of the motivations of this paper is to give the details of these calculations which could not be presented in the letter [1] for reasons of space, such as at least some of the solutions obtained with the gluon propagator Ansatz used by us, and the comparison of some other gluon propagator Ansätze with this gluon propagator used by us.

Another motivation is to present an updated comparison with the $\eta_c$ data, in order to show that the conclusions of [1] are still valid in spite of some of the recent developments regarding the measurements of $\eta_c \to \gamma\gamma$, which – superficially – may seem to have worked against the conclusions of [1] regarding $\eta_c$.

The third motivation lies in presenting the results for the $\gamma\gamma$-decay of the massive $\pi^0$, since in Ref. [1] the pion was treated in the chiral limit. Although the difference between the “massive” and the earlier, chiral-limit results for $\pi^0 \to \gamma\gamma$ are (predictably) small, we discuss them because of their importance as a test of validity of some recent attempts to take the SD-BS approach beyond ladder approximation. These results are therefore also tied to the last, fourth motivation.

Namely, the fourth motivation – which is the most important one – is to present more detailed explanations and discussions of why the coupled SD-BS approach can be so successful over such a surprisingly broad range of masses, and why this type of description of electromagnetic processes of mesons in the coupled SD-BS approach, represents a significant advance in understanding of mesons as $q\bar{q}$ bound states. In brief, this is because this approach incorporates the dynamical chiral symmetry breaking (D$\chi$SB) into the bound states consistently, so that the pion, although constructed as a $q\bar{q}$ bound state, appears as a Goldstone boson in the chiral limit. Precisely because of this feature, the coupled SD-BS approach correctly incorporates the Abelian axial (or Adler-Bell-Jackiw [5,6], ABJ) anomaly of QED, which other bound-state approaches – the ones without D$\chi$SB – simply cannot. These latter approaches consequently fail to provide adequate descriptions of the processes where the axial anomaly is important, such as the two-photon processes of light pseudoscalars, of which the $\pi^0 \to \gamma\gamma$ decay is the cleanest example of an anomalous electromagnetic decay. It is important to elucidate this issue in detail, because it is very often simply swept under the carpet in various model approaches to such processes – especially in the bound-state calculations. For example, model results for $\eta \to \gamma\gamma$ and $\eta' \to \gamma\gamma$ which may seem reasonable by themselves, cannot be really physically significant if the analogous calculation of the closely related decay $\pi^0 \to \gamma\gamma$ gives the result wrong by orders of magnitude. Similarly, if some approach yields good results for the normalized transition form factors for $\gamma^*\pi^0(\eta, \eta') \to \gamma$, they are only seemingly good if the results for the related on-shell processes $\pi^0(\eta, \eta') \to \gamma\gamma$ are wrong – since this of course implies equally wrong results for the absolute, un-normalized rates for $\gamma^*\pi^0(\eta, \eta') \to \gamma$ at any transferred momentum. (See Eq. (20) below.)
Therefore, in the next section we present a survey of the situation in bound-state approaches concerning the axial anomaly. In the third section we introduce the coupled SD-BS approach, and in the fourth, we sketch how the $q\bar{q}$ bound-state solutions are obtained. In the fifth section we explain how the quark–photon interactions are treated, leading to the correct results for the anomalous processes. In the sixth section we present the results and a discussion of them, whereas the seventh contains the concluding discussion on how the chosen model works.

2. BOUND-STATE MODELS AND AXIAL ANOMALY

It is not surprising that $q\bar{q}$ bound-state models have problems in accounting for the processes dominated by the axial anomaly, because their general philosophy is often such that the details of the internal dynamics and structure of bound-states are given precedence over the correct incorporation of the symmetries of the underlying theory which they would like to model. (The Nambu–Jona-Lasinio (NJL) model [7,8] applied to quarks – reviewed by, e.g., [9] – is a notable exception, but due to its low momentum cutoff, its usual versions have other problems with the axial anomaly, which we will comment on later.) On the other hand, the axial anomaly effects do not depend on the internal $q\bar{q}$ structure of pseudoscalar mesons at all, but appear only if the light pseudoscalars are (pseudo)Goldstone bosons of $D\chi_{SB}$. Therefore, the constituent quark bound state approaches which (unlike the consistently coupled SD-BS ones) do not incorporate $D\chi_{SB}$, cannot be consistent with the axial anomaly even if they successfully reproduce the magnitude of the $\pi^0 \rightarrow \gamma\gamma$ width, because they introduce strong structure dependence where it should remain small. Namely, anomalous amplitude is independent of the structure, while the non-anomalous effects on top of it should not exceed several percent [10] due to PCAC and Veltman–Sutherland theorem. The incorporation of the axial anomaly in the fashion of, e.g., Roberts [11], Frank et al. [12], Burden et al. [13], or Bando et al. [14], therefore represents a significant recent advance in the theory of bound states.

Approaches depicting the pion as a quark–antiquark bound state but not employing $D\chi_{SB}$, do not succeed in capturing the effect of the axial anomaly. Namely, except through the fine-tuning of model parameters (as in Ref. [15] for example), such modeling of the $q\bar{q}$ bound states has not provided even the numerical results for the $\pi^0 \rightarrow \gamma\gamma$ decay width that can quite favorably match the experiment and thus try to yield an explanation for the $\pi^0 \rightarrow \gamma\gamma$ width that would be based on hadronic structure, as an alternative to the physical understanding based on the Abelian axial anomaly. For example, Horbatsch and Koniuk [16] point out that even with relativistic corrections, the constituent quark approaches lead to $\pi^0 \rightarrow \gamma\gamma$ decay widths exceeding the experimental decay width

$$W_{exp}(\pi^0 \rightarrow \gamma\gamma) = 7.74 \pm 0.56 \text{ eV}$$

by three orders of magnitude, unless ad hoc correction factors are introduced. Although Ackleh and Barnes [17], for example, use them for multiplying the $P \rightarrow \gamma\gamma$-widths, they themselves stress that these factors, being difficult to justify physically, are largely arbitrary and primarily motivated by comparison with experimental data. On the other hand, the
authors of Ref. [16] reject this procedure, claiming that their improved approach gives acceptable $\eta, \eta'$ widths, the very large discrepancy of 3 orders of magnitude still affecting only the $\pi^0$-width. However, since the character of $\gamma\gamma$-decays of these pseudoscalars cannot be drastically different one from the other (as all should be in varying degrees influenced by the axial anomaly), their results on $\eta-\eta'$ can be regarded only as parameter fittings without much physical significance if the $\pi^0$-width is so very wrong at the same time.

Formulating calculations in the relativistically covariant manner solves some of the problems from which not-fully-relativistic approaches were suffering. However, improving the relativistic bound-state formalism in a manner which leaves the treatment of $\pi^0 \rightarrow \gamma\gamma$ structure- (model-)dependent also does not help decisively. E.g., in the Bethe–Salpeter formalism without $D\chi_{SB}$, Guiasu and Koniuk [18] did obtain a suppression with respect to the 3-orders-of-magnitude overestimate of the $\pi^0 \rightarrow \gamma\gamma$ width in the naive constituent model, but had to be satisfied with the correct order of magnitude, namely the width of roughly 4 eV; and the best (to our knowledge) result in those structure–dependent attempts was achieved by Münz et al. [15], who obtained — in one of their fits — the width of 7.6 eV in a Bethe–Salpeter approach without $D\chi_{SB}$, i.e., with constituent quarks simply postulated, but at the expense of fine–tuning the model parameters to very unlikely values which may contradict with other empirical quantities. Most notably, they are forced to suppose that the constituent $u(d)$-quark mass is quite light, $M_u = M_d = 170$ MeV, and such values of the constituent quark mass are certainly unsatisfactory if tried in some other applications, such as reproducing the spectrum of baryons. Indeed, the later and broader fits to the meson spectrum in practically the same approach [19], used more acceptable values of the parameters, but had to settle for $W(\pi^0 \rightarrow \gamma\gamma)$ between 3.81 eV and 5.07 eV. Münz et al. [15] see their achievement in providing a quark bound-state alternative to the Goldstone-plus-anomaly picture (see their page 429). However, considering which alternative is more favorable (or rather, not ruled out) on phenomenological grounds, does not seem very essential. Namely, it is not just that $D\chi_{SB}$ in QCD is favored on phenomenological (and other) grounds — it seems it is in fact obligatory, so there are probably no alternatives. This is especially clearly stated by 'tHooft in Ref. [20], according to which it is not possible that in some variant of QCD, chiral symmetry is not spontaneously (dynamically) broken. He stresses that in SU(N) binding theories, chiral symmetry must be broken spontaneously, so that in the chiral limit the anomaly must be reproduced by massless bound states which are Goldstone modes of this spontaneous (dynamical) chiral symmetry breaking. Therefore, one should reconcile the bound-state picture with the anomaly picture if one wants to have satisfactory understanding of the light pseudoscalar $q\bar{q}$ bound states and their interactions. Noting this need, already Hayne and Isgur [21] expressed the hope (in their concluding section) that there may exist a massive-constituent model which is physically equivalent to the chiral-symmetry picture.

The woe affecting $q\bar{q}$ bound states when axial anomaly is at work, makes one really appreciate the successes with anomalous processes achieved by the methods incorporating the symmetries and anomalies of QCD directly, like the chiral perturbation theory ($\chi$PT) with Wess–Zumino anomaly action. (See, e.g., [22].) However, quarks are nowhere present explicitly in $\chi$PT, so that comparing constituent quark approaches with such a very different, complementary approach, is not what is most frustrating. The until recently unsatisfactory situation in the bound-state approaches to $\pi^0 \rightarrow \gamma\gamma$ was especially bothersome if compared
with the simple free-quark loop (or even the old “baryon loop”) calculations of the triangle
diagram coupled to an “elementary”, structureless pion field, which readily reproduced the
anomaly amplitude $1/4\pi^2 f_\pi$ in the chiral limit if the Goldberger-Treiman relation is used.
(See, for example, [23] or [24] for expositions of the connection between the simple free-quark
loop calculation a la Steinberger and the $\sigma$-model anomaly analysis [25,26].)

Such calculations employing elementary pseudoscalar coupling to free quarks, thus seem
to indicate that the bound–state wave-functions that describe the internal $qq$ sub-structure,
are a superfluous element that can only spoil reproducing the axial anomaly effects when
calculating the triangle diagram. Does it, then, mean that the $\pi^0 \to \gamma \gamma$ decay can somehow
be the argument in favor of those (“the opponents of the quark model”, as Ref. [16] puts it)
who reject the view of the pion as a $q\bar{q}$ bound state in favor of the pion as a Goldstone boson
of $D\chi_{SB}$? Certainly not, unless one insists that one must choose: either a quark model,
or the Goldstone-plus-anomaly picture; our point is that in the literature there already is
at least one example [2–4] of a constituent quark model which is very successful over a
very wide range of masses, and which can also solve the problems with anomaly because it
incorporates the correct chiral symmetry behavior through the coupled SD-BS mechanism:
the constituent quarks come about through dynamical dressing in SD equations and the
light $qq$-solutions of the BS equation in the consistent approximation (here: rainbow-ladder)
are (pseudo)Goldstone bosons!

As we will explain in the following sections, constituent models equivalent to the chi-
ral symmetry picture are provided by the coupled SD-BS approach, because it simultaneously
describes light pseudoscalar mesons as both bound states of constituent quarks and
(pseudo)Goldstone bosons of $D\chi_{SB}$, which produces these constituent quarks in the first
place. This gives the present treatment of $\gamma\gamma$-decays of pseudoscalar mesons a fundamental
advantage over most of the other constituent quark approaches. For example, this enables us
(essentially in the fashion of Bando et al. [14] and Roberts and others [11,12] in the Ansatz
approach) to reproduce – even analytically and exactly – the famous, empirically successful
result of the Abelian axial anomaly for the $\gamma\gamma$ width of $\pi^0$, the lightest pseudoscalar,

$$W(\pi^0 \to \gamma\gamma) = \frac{\alpha_{em}^2 M_\pi^3}{64\pi^3 f_\pi^2},$$

while the same approach can arbitrarily depart from the chiral limit all the way to the
heaviest mesons.

Obtaining this form (2), i.e., this relationship between the width $W(\pi^0 \to \gamma\gamma)$ and the
pion leptonic decay constant $f_\pi$, is possible thanks to the correct chiral symmetry behavior
in the coupled SD-BS approach. However, this is not enough in the approaches which are so
predictive that, e.g., the pion leptonic decay constant $f_\pi$ is also calculable. What is needed
for obtaining the good value for the $\pi^0 \to \gamma\gamma$ width (2) consistently, is not just plugging in
the empirical value of $f_\pi$, but that the predicted $f_\pi$ is close to this empirical value. In the
light of that, Jain and Munczek’s model [2–4] is especially favorable variant of the coupled
SD-BS approach, because it is simultaneously a very successful constituent quark model in
the sense of reproducing many meson masses from very light to very heavy ones, as well as
the leptonic decay constants of pseudoscalar mesons. For the concrete model and parameters
[4] adopted in this work, we obtain $f_\pi = 93.2$ MeV. Let us note that the pion decay constant
is actually not very sensitive to parameter and model variations, as all Refs. [2–4] obtain $f_\pi$
close to that value (and to the experimental one, $f^{\exp}_\pi = 92.4$ MeV) although they differ in
details of modeling the infrared (IR) part of the gluon propagator and in parameter values.

At this point we can sketch one of the reasons why the present approach is more suitable
for treating the anomalous processes, than the NJL approach is. We make a more detailed
comparison with a variant [27] of the NJL model elsewhere [28], but here we should address
the most easily explainable reason, because one might in principle remark that recent coupled
SD-BS approaches were not the first to fulfill Hayne and Isgur's [21] hope for existence of a $q\bar{q}$
model equivalent to the chiral-symmetry picture, since already the NJL model has the
correct chiral symmetry behavior and incorporates the pion as the (pseudo-)Goldstone boson
of $D\chi_\text{SB}$. (In a sense, one can say that already the NJL model belongs to the class of the
coupled SD-BS approaches, but with a particularly schematic interaction.) In particular,
Eq. (2) seems – on the first thought – to be easily reproduced, because if in the chiral limit
the Goldberger-Treiman relation at the quark level holds in the NJL model, the $\pi^0 \to \gamma\gamma$
NJL-calculation (e.g., in [27]) effectively reduces to the simple free-quark loop calculation.
The second thought, however, reveals the following problem: in contradistinction to Jain
and Munczek's model, where the momentum cutoff is either not needed (in the chiral limit
[2]) or practically infinite [3,4] in comparison with the relevant hadronic scales, the Nambu–
Jona-Lasinio approach contains a low cutoff ($\Lambda_{\text{NJL}} < 1$ GeV). Nevertheless, its triangle
diagram calculation of $\pi^0 \to \gamma\gamma$ of course reproduces the anomaly result (2) only if there is
no cutoff! As Langfeld et al. [29] pointed out, the low cutoff in the NJL model would affect
the anomalous $\pi^0 \to \gamma\gamma$ width (2) by the factor $1/(1 + M^2/\Lambda_{\text{NJL}}^2)^2$. For the typical value of
the ratio of the constituent quark mass $M$ and the NJL-cutoff, which is $M/\Lambda_{\text{NJL}} \sim 0.5$, this
implies the reduction with respect to the empirically successful width (2) by 30% to 40%.

Although we perform a more concrete comparison with a specific NJL calculation [27]
in another paper [28], in this paper it will also – in the following three sections – gradually
become clear that the chosen bound-state approach [2–4] can be considered a NJL–inspired,
but more sophisticated approach to mesons. Correspondingly, our approach to two-photon
processes contains improvements (both in the conceptual consistency and in the quantitative
details) not only with respect to constituent models without $D\chi_\text{SB}$, but also with respect
to the NJL approach.

3. THE COUPLED SCHWINGER-DYSON AND BETHE-SALPETER
APPROACH, AND ABELIAN ANOMALY

The coupled Schwinger-Dyson and Bethe-Salpeter (SD–BS) approach is the approach in
which the Bethe–Salpeter (BS) equation employs the quark propagator obtained by solving
the Schwinger-Dyson (SD) equation, and both equations employ the same interaction be-
tween quarks, normally defined by a (partially) modeled gluon propagator and the ladder
approximation for the quark–gluon vertices. It is one of the most interesting applications
of SD and BS equations to the physics of hadrons. Refs. [30–33] provide good reviews of
the SD and BS equations in the hadronic context – including the coupled SD–BS approach,
which started developing in the eighties and the beginning of nineties; e.g., see [34–39] and
other similar references that can be found in [30–33].
So far, the most successful variant\(^a\) of the coupled SD–BS approach to the meson spectra and decay constants has been that of Jain and Munczek [2–4]. Hence, we chose their model to use it in Ref. [1] for the first time in the context of electromagnetic interactions (other than electromagnetic mass differences). While some aspects of the issues we discuss are model–dependent, so that the presented results are specific to the Jain–Munczek model, some are in fact common to all variants of the consistently coupled SD–BS approach, including some closely related approaches employing Ansätze for the dressed quark propagators. In particular, this is so for anomalous processes of chiral pions, as will become apparent below.

A crucial development for properly embedding the electromagnetic interactions in the context of the bound states composed of dynamically dressed quarks occurred when Bando et al. [14] and Roberts [11] demonstrated how the Adler-Bell-Jackiw axial anomaly can be incorporated in the framework of SD and BS equations, reproducing (in the chiral limit) the famous anomaly result for \(\pi^0 \rightarrow \gamma \gamma\) analytically. This was also extended [12] to the off–shell case \(\gamma^* \pi^0 \rightarrow \gamma\). However, these treatments are all restricted to the chiral limit (and its immediate vicinity), i.e., to quarks with vanishing (or almost vanishing) current masses \(m\) and their pseudoscalar meson composites – pions with zero (or almost zero) mass \(M_{\pi}\). In contradistinction to that, our application of Jain and Munczek’s model [2–4] to electromagnetic processes, is not subject to such limitations; while agreeing with [11,12,14] for \(m \rightarrow 0\), it can also be applied for large quark masses \(m\).

In the present paper we argue that in the light of the latest experimental results on \(\eta_c \rightarrow \gamma \gamma\) from CLEO [42], elements of these treatments [14,11,12] appear essential also for the understanding of the electromagnetic processes of mesons in a totally different regime, far away from the chiral limit.

Refs. [11,12] avoided solving the SD equation for the dressed quark propagator \(S\) by using an Ansatz quark propagator. Then, thanks to working in the chiral limit and the soft limit (where the momentum of the pion \(p \rightarrow 0\)), they also automatically obtained the solution of the BS equation. Namely, in this limit, when the chiral symmetry is not broken explicitly but spontaneously (dynamically), and when pions must consequently appear as Goldstone bosons, the solution for the pion bound-state vertex \(\Gamma_{\pi}\), corresponding to the Goldstone pion, is – to order \(\mathcal{O}(p^0)\) – determined (see, e.g., Ch. 9 in [31]) by the dressed quark propagator \(S(q)\):

\[
S^{-1}(q) = A(q^2)\phi - B(q^2) .
\]  

(3)

For the pion bound-state vertex \(\Gamma_{\pi}\), Ref. [11,12] concretely used the solution (given in Eq. (4) immediately below) that is of zeroth order in the pion momentum \(p\) (which is appropriate in the soft limit \(p^\mu \rightarrow 0\)). It is interesting that such \(\mathcal{O}(p^0)\) \(\Gamma_{\pi}\) fully saturates the Adler-Bell-Jackiw axial anomaly [14,11]. In the chiral limit, the pion decay constant \(f_{\pi}\) gives [43] the normalization of \(\Gamma_{\pi}\), whereas its \(\mathcal{O}(p^0)\) piece (again: in the chiral limit) is proportional to \(B(q^2)\) from Eq. (3):

\(^a\) Recently, similarly successful descriptions of phenomenology have been achieved employing separable interaction [40,41].
\[
\Gamma_\pi(q^2; p^2 = M_\pi^2 = 0) = 2 \frac{\gamma_8 B(q^2)_{m=0}}{f_\pi}, \tag{4}
\]
(where the flavor structure of pions has been suppressed). Thanks to using the chiral-and-
soft-limit solution (4) (and to satisfying the vector Ward-Takahashi identity of QED, on
which point we elaborate later), Bando et al. [14] and Roberts [11] analytically
reproduced the famous axial-anomaly result \textit{independently} of what precise \textit{Ansatz}
has been used [11] for \(S(q)\). Note that this implies the independence of this result also on the inter-quark
interactions, determining both the quark propagator \(S(q)\) and the internal pion structure.
Namely, even if one uses such an \textit{Ansatz} for \(S(q)\), one can in principle invert the SD equation
(Eq. (9) below) for the quark propagator, and find the effective gluon propagator \(G^{\mu\nu}(k)\)
that would lead to this \(S(q)\). Since the reproduction of the axial-anomaly amplitude is
independent of the \textit{Ansatz} for the quark propagator, it is consequently independent also of
what interaction, defined by \(G^{\mu\nu}(k)\), has formed the \(\pi^0\) bound state.

Whereas this \textit{Ansatz} scheme works beautifully in the chiral limit and very close to it,
one must obviously depart from it if one wants to consider heavier quarks. Already when
Strange (s) quarks are present, (4) can be regarded only as an “exploratory \textit{Ansatz}” [13].
For even heavier c– and b–quarks, the whole concept of the chiral limit is of course useless
even qualitatively. The two–photon processes of neutral pseudoscalar mesons that are much
heavier than \(\pi^0\) and cannot be described as Goldstone bosons, will depend strongly on their
internal hadronic structure. Thus, one needs to solve the pertinent bound-state equation,
which is determined by the interaction between quarks. On the other hand, one should also
have the axial anomaly incorporated correctly. This consistency requirement is satisfied by
the coupled SD-BS approach developed in a series of papers by Jain and Munczek [2–4],
because their treatment in the chiral limit yields pions as Goldstone bosons of dynamical
chiral symmetry breaking (D\(\chi\)SB). On the other hand, their treatment has also reproduced
[2–4] almost the whole spectrum of meson masses, including those in the heavy-quark regime,
and also the leptonic decay constants \((f_P)\) of pseudoscalar mesons \((P)\). We are therefore
motivated to use this so far successful approach for calculating other quantities. In this
paper we present the calculation of the \(\pi^0\), \(\eta_c\), and \(\eta_b \to \gamma \gamma\) decay widths, as well as of the \(\gamma^* \pi^0\)-to-\(\gamma\) transition form factor.

4. Obtaining \(q\bar{q}\) Bound-State Solutions

4.1. Effective gluon propagator

We define the interaction kernel for the SD and BS equations following Jain and Munczek
[2–4]: \textit{i.e.}, we use an effective, modeled Landau-gauge gluon propagator given by

\[
g_{st}^2 C_F G^{\mu\nu}(k) = G(-k^2)(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}), \tag{5}
\]
where we have indicated that our convention is such that not only the strong coupling
constant \(g_{st}\), but also \(C_F\), the second Casimir invariant of the quark representation, are
absorbed into the function \(G\). For the present case of SU(3)\(_c\), where the group generators
are \(\lambda^a/2\), namely the (halved) Gell-Mann matrices, \(C_F = \frac{2}{3}\).
It is essential that the effective propagator function $G$ is the sum of the perturbative contribution $G_{UV}$ and the nonperturbative contribution $G_{IR}$:

$$G(Q^2) = G_{UV}(Q^2) + G_{IR}(Q^2), \quad (Q^2 = -k^2).$$

(6)

The perturbative part $G_{UV}$ is required to reproduce correctly the ultraviolet (UV) asymptotic behavior that unambiguously follows from QCD in its high-energy, perturbative regime. Therefore, this part must essentially be given – up to the factor $1/Q^2$ – by the running coupling constant $\alpha_{st}(Q^2)$ which is well-known from perturbative QCD, so that $G_{UV}$ is in fact not modeled.

From the renormalization group, in the spacelike region $(Q^2 = -k^2)$,

$$G_{UV}(Q^2) = 4\pi C_F \frac{\alpha_{st}(Q^2)}{Q^2} \approx \frac{4\pi^2 C_F d}{Q^2 \ln(x_0 + \frac{Q^2}{\Lambda_{QCD}^2})} \left\{ 1 + b \frac{\ln[\ln(x_0 + \frac{Q^2}{\Lambda_{QCD}^2})]}{\ln(x_0 + \frac{Q^2}{\Lambda_{QCD}^2})} \right\},$$

(7)

where we employ the two–loop asymptotic expression for $\alpha_{st}(Q^2)$, and where $d = 12/(33 - 2N_f)$, $b = 2\beta_2/\beta_1^2 = 2(19N_f/12 - 51/4)/(N_f/3 - 11/2)^2$, and $N_f$ is the number of quark flavors. The parameter $x_0$ is the infrared cutoff, introduced to regulate the logarithmic behavior of $G_{UV}$ as the values of $Q^2$ approach $\Lambda_{QCD}^2$, the dimensional parameter of QCD. As in [4], we use $x_0 = 10$, but this is not really important since the results are only very weakly sensitive to the values of $x_0$, as was already pointed out by [4]. Following [4], we set $N_f = 5$ and $\Lambda_{QCD} = 228$ MeV. Although the top quark has meanwhile been found, its mass scale is far above the range of momenta relevant for bound state calculations, and even above the value of the UV cutoff needed in the massive version of our SD equations (see below). Therefore, there is no need to revise $G_{UV}$ (7) to include $N_f = 6$. (On the other hand, choosing $N_f$ below 5 would not be satisfactory because (i) the momentum range of the order of the $b$ quark mass still has non-negligible influence in our bound-state calculations, (ii) the $b$ quark mass is below the UV cutoff used in our “massive SD equations”, and (iii) sometimes we need the solutions for relatively high momenta, e.g., to be able to see the asymptotic behavior of the propagator functions $A(q^2)$ and $B(q^2)$ – see Figs. 3, 4 and 5.)

$G_{UV}(Q^2)$ is depicted by the dashed line in Fig. 1.

The non-perturbative part $G_{IR}$ should describe the infrared (IR) behavior. The infrared behavior of QCD is however still very far from being understood, so this non-perturbative part must be modeled. We choose $G_{IR}$ from Ref. [4]:

$$G_{IR}(Q^2) = 4\pi^2 C_F a Q^2 \exp(-\mu Q^2),$$

(8)

with their [4] parameters $a = (0.387 \text{ GeV})^{-4}$ and $\mu = (0.510 \text{ GeV})^{-2}$. This Ansatz for $G_{IR}$ is depicted by the dotted line in Fig. 1.

The whole effective propagator function $G$ that we use, is depicted by the solid curve in Fig. 1 and Fig. 2. It is interesting and indicative that the gluon propagator function $G(Q^2)$ (6)-(8) is reasonably similar to the effective gluon propagator functions obtained by

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\*We thank R. Cahill for pointing this out to us.
Cahill and Gunner in [44] (short–dashed line in Fig. 2) and [45] (dash–dotted line in Fig. 2) using a closely related approximation to QCD, called the Global Color Model [34], reviewed recently by Tandy [46]. One observes a somewhat less close agreement with the effective gluon propagator (the long–dashed line in Fig. 2) employed by Savkli and Tabakin [47] in another related approach, which, however, has not achieved such a broad fit to meson spectrum and decay constants as the approach of [4] with the propagator (5)–(8).

### 4.2. Generating constituent quarks through SD equations

In contradistinction to [11,12] for example, we do not use any Ansätze for the functions $A(q^2)$ and $B(q^2)$ in the dressed quark propagators (3). These propagators are obtained for various flavors by solving the SD equation in the ladder approximation (i.e., with bare quark-gluon vertices):

$$S^{-1}(p) = p - \tilde{m} - i g_m^2 C_F \int \frac{d^4 k}{(2\pi)^4} \gamma^\mu S(k) \gamma^\nu G_{\mu\nu}(p-k) ,$$

(9)

where $\tilde{m}$ is the bare mass term of the pertinent quark flavor, breaking the chiral symmetry explicitly.

The case $\tilde{m} = 0$ corresponds to the chiral limit where the current quark mass $m = 0$, and where the constituent quark mass $B(0)/A(0)$ stems exclusively from $D\chi$SB [2]. (Of course, calling “the constituent mass” the value of the “momentum-dependent constituent mass function” $B(q^2)/A(q^2)$ at exactly $q^2 = 0$ and not on some other low $q^2$, is a matter of a somewhat arbitrary choice. Another conventional choice (e.g., in [48,49]) is to call the solution of $-q^2 = B^2(q^2)/A^2(q^2)$ the Euclidean constituent-quark mass squared. However, since this is just a matter of choosing one’s terminology, we stick to that of Jain and Munczek [3].

With the assumption that $u$ and $d$ quarks are massless, which is an excellent approximation in the context of hadronic physics, solving of (9) yields the solutions for $A(q^2)$ and $B(q^2)$, displayed in respective Fig. 3 and Fig. 4 by the solid lines.

In these figures we also compare them with $A(q^2)$ and $B(q^2)$ corresponding to the dressed propagator Ansätze of the references [11,12], represented by the dashed lines. Our massless solutions lead to the constituent $u$ (and $d$) quark mass $B(0)/A(0) = 356$ MeV. The ratio $B(q^2)/A(q^2)$, namely our momentum-dependent mass function $B(q^2)/A(q^2)$ is depicted in Fig. 5 by the solid line, and the dashed line represents the analogous ratio formed from $A(q^2)$ and $B(q^2)$ corresponding to the Ansätze of Refs. [11,12].

Obviously, our chiral–limit solutions for $A(q^2)$ and $B(q^2)$ differ a lot from the Ansätze of [11,12]. We display and compare them for stressing one of the key points of the later sections, namely that in the chiral limit, what matters is the correct implementation of $D\chi$SB and Ward identities, rather than the concrete forms of the propagator and bound state solutions. Of course, the situation changes as the quark masses grow appreciably, and approaches the simple constituent picture. Nevertheless, for the small quark masses appropriate for the realistically light pion, the physical situation is still dominantly determined by $D\chi$SB and the resulting (pseudo)Goldstone character of the pion – and indeed, for the realistically
massive \( u \)- and \( d \)-quarks and their lightest bound state \( \pi \), the differences with respect to the chiral limit turn out to be small in the present approach.

When \( \tilde{m} \neq 0 \), the SD equation (9) must be regularized by a UV cutoff \( \Lambda \) [3,4], and the bare mass \( \tilde{m} \) is in fact a cutoff-dependent quantity. We adopted the parameters of [4], where (for \( \Lambda = 134 \) GeV) \( \tilde{m}(\Lambda^2) \) is 3.1 MeV for the isosymmetric \( u \)- and \( d \)-quarks, 73 MeV for \( s \)-quarks, 680 MeV for \( c \)-quarks, and 3.3 GeV for \( b \)-quarks. Solving of (9) then yields the solutions \( A(q^2) \) and \( B(q^2) \) for “slightly massive” \( u \)- and \( d \)-quarks, “intermediately massive” \( s \)-quarks, as well as the solutions for the heavy quarks \( c \) and \( b \). We essentially reproduce the results of Ref. [4] (within the accuracy permitted by numerical uncertainties). The \( A(q^2) \) and \( B(q^2) \) solutions for \( \tilde{m}(\Lambda^2) \neq 0 \) are displayed in Figs. 3 and 4 by dotted lines marked by \( u,d \) and \( s,c \) and \( b \), indicating which flavor a curve pertains to. For the lightest, \( u \)- and \( d \)-quarks (with \( \tilde{m} = 3.1 \) MeV), both \( A(q^2) \) and \( B(q^2) \) are only slightly above the curves representing our respective chiral-limit solutions. More precisely, the difference is then at most 1.4% (at \( q^2 = 0 \)) for \( A(q^2) \), while for \( B(q^2) \) the largest absolute value of the difference (again occurring at \( q^2 = 0 \)) amounts to an excess of 6.2% over our chiral-limit solution. The excess quickly becomes much smaller above \(-q^2 = 0.2 \) GeV. Admittedly, at \(-q^2 \) above 2 GeV, the relative difference between the “chiral” and “slightly massive” \( B(q^2) \)’s starts growing again because of the different asymptotic behaviors of these respective solutions. They are, respectively, \( B(q^2) \sim 1/\ln(-q^2/\Lambda_{QCD}^2) \) and \( B(q^2) \sim 1/\ln(-q^2/\Lambda_{QCD}^2) \), and are consistent with the asymptotic freedom of QCD [50,51]. (This in turn results in the asymptotic behavior of the momentum-dependent, dynamical mass functions \( B(q^2)/A(q^2) \), which is in accord with the behavior in perturbative QCD [2–4,51,52]). However, the absolute values of these \( B(q^2) \)’s (even for the “slightly massive” case) and of their difference are already very small at \(-q^2 > 2 \) GeV.

For the more massive flavors, the deep Euclidean asymptotic behavior \( B(q^2) \sim 1/\ln(-q^2/\Lambda_{QCD}^2) \) is also fulfilled, but of course with very different coefficients (which are essentially proportional to the current quark masses [3,4]). Also, \( A(q^2) \rightarrow 1 \) for all flavors as \(-q^2 \rightarrow \infty \). For low \(-q^2 \), however, \( A(q^2) \)’s belonging to different flavors exhibit interesting differences. The bump that characterizes the least massive (or chiral) \( u,d \)-quarks is absent already in \( A(q^2) \) of our “intermediately massive” \( s \)-quark, for which the fall-off is almost monotonical, as the increase (around \(-q^2 \sim 0.1 \) GeV) above the \( A(0) \)-value is practically imperceptibly small. Moreover, for even heavier \( c \)- and especially \( b \)-quarks, the \( A(q^2) \)-values for even lowest \(-q^2 \)’s, are below the corresponding values of the chiral-limit \( A(q^2) \).

Comparing the various \( A(q^2) \)- and \( B(q^2) \)-solutions illustrates well how the importance of the dynamical dressing decreases as one considers increasingly massive quark flavors.

These \( \tilde{m} \neq 0 \) solutions give us the constituent mass \( B(0)/A(0) \) of 375 MeV for the (isosymmetric) \( u \)- and \( d \)-quarks, 610 MeV for the \( s \)-quarks, 1.54 GeV for the \( c \)-quarks, and 4.77 GeV for the \( b \)-quarks. These are very reasonable values. Also, the momentum-dependent mass functions \( B(q^2)/A(q^2) \) – depicted in Fig. 5 – in the presently chosen variant of the coupled SD-BS approach [2–4] behave for all flavors in the way which correctly captures the differences between heavy and light quarks (stressed recently by Ref. [49]).
4.3. Bound states of dynamically dressed constituent quarks

In the chiral limit, Eqs. (3)–(4) reflect the fact that solving of (9) with $\bar{m} = 0$ is already sufficient to give us the Goldstone pion bound-state vertex (to $O(p^3)$) which saturates the anomalous $\pi^0 \to \gamma\gamma$ decay [11,14]. Of course, the usefulness of avoiding to solve the BS equation this way, diminishes with growing masses, and for heavier $q\bar{q}$ composites, such as presently interesting $\eta_c$ and $\eta_b$, this does not make sense even qualitatively. In addition, in this paper we are interested in the effects – however small – of the finite masses of the light quarks and of the pion. Therefore, the massive pseudoscalar ($P$) bound-state vertices $\Gamma_P$ for the massive pion, $\eta_c$, $\eta_b$, and also for the unphysical pseudoscalar $s\bar{s}$ bound state $\eta_s$, will be obtained in the same way as the bound-state vertex $\Gamma_M$ of any other meson $M$, by solving explicitly the homogeneous BS equation

$$\Gamma_M(q,p) = i g_\pi^2 C_F \int \frac{d^4q'}{(2\pi)^4} \gamma^\mu S(q' + \frac{p}{2}) \Gamma_M(q',p) S(q' - \frac{p}{2}) \gamma^\nu G_{\mu\nu}(q - q') . \quad (10)$$

Here we have written the BS equation again in the ladder approximation, consistently with (9). Note that $S$ is the quark propagator obtained by solving the SD equation (9) with the same gluon propagator $G_{\mu\nu}$.

For pseudoscalar mesons ($M = P$), the complete decomposition of the BS bound-state vertex $\Gamma_P$ in terms of the scalar functions $\Gamma_i^P$ is:

$$\Gamma_P(q,p) = \gamma_5 \left\{ \Gamma_0^P(q,p) + \not{p} \Gamma_1^P(q,p) + q \Gamma_2^P(q,p) + [\not{p}, \not{q}] \Gamma_3^P(q,p) \right\} . \quad (11)$$

(The flavor indices are again suppressed.) The BS equation (10) leads to a coupled set of integral equations for the functions $\Gamma_i^P$ ($i = 0,\ldots,3$), which can most easily be solved numerically in the Euclidean space.

The formulations of the relativistic bound–state problem through the “amputated” BS vertices $\Gamma_M(q,p)$ or the “unamputated” BS amplitudes $\chi_M(q,p)$, mutually related through

$$\Gamma_M(q,p) = S^{-1}(q + \frac{p}{2}) \chi_M(q,p) S^{-1}(q - \frac{p}{2}) , \quad (12)$$

are of course equivalent, but we find that it is technically somewhat more convenient to solve the BS equation for the BS amplitude,

$$S^{-1}(q + \frac{p}{2}) \chi_M(q,p) S^{-1}(q - \frac{p}{2}) = i g_\pi^2 C_F \int \frac{d^4q'}{(2\pi)^4} \gamma^\mu \chi_M(q',p) \gamma^\nu G_{\mu\nu}(q - q') , \quad (13)$$

as in [2–4]. For pseudoscalar mesons, the decomposition of the BS amplitudes in terms of the scalar functions $\chi_i^P$ proceeds in the same way as in (11) for the pseudoscalar bound-state vertices:

$$\chi_P(q,p) = \gamma_5 \left\{ \chi_0^P(q,p) + \not{p} \chi_1^P(q,p) + q \chi_2^P(q,p) + [\not{p}, \not{q}] \chi_3^P(q,p) \right\} . \quad (14)$$

Substitution of (14) into (13) results in a coupled set of integral equations for the functions $\chi_i^P$ ($i = 0,\ldots,3$), which we solve in the Euclidean space for the massive pion, $\eta_c$ and $\eta_b$. We also solve them for the $s\bar{s}$-pseudoscalar ($\eta_s$) even though it is unphysical, because it
is needed for constructing $\eta$ and $\eta'$ in Ref. [28]. ($\eta$ and $\eta'$ and their two-photon interactions are of similar interest to us as those of $\eta_c$ and $\eta_b$. However, obtaining the bound-state description for $\eta$ and $\eta'$ is much more complicated owing to their mixing, so we treat them separately in another paper [28].)

In our calculations – following Jain and Munczek [2–4] – we use the Chebyshev-polynomial-decomposition of the scalar functions $\chi^P_i$ ($i = 0, ..., 3$) appearing in Eq. (14). This way one avoids the angular integration not only in the integral equations for the functions $\chi^P_i$ themselves, but also in applications of these solutions, such as in our evaluation of the amplitudes for the $P\gamma\gamma$-processes addressed in the next section. While using the kernel and parameters of the Ref. [4] which often kept only the zeroth order Chebyshev moment because it was adequate for the spectrum of the pseudoscalars, we always retain the first Chebyshev moment too. The accuracy of our procedure has recently received an independent confirmation from Maris and Roberts [48]. In their study of $\pi$- and $K$-meson BS amplitudes (or, quite precisely, the BS-vertices (11)), they employed both the Chebyshev decomposition, and straightforward multidimensional integration. Comparison of these two techniques confirms the very quick convergence of the Chebyshev expansion [48]: in the case of equal quark and antiquark masses, such as in the pion, the zeroth and the first Chebyshev moment are enough for an accurate representation of the solution, while for the kaon, in spite of the difference in the masses of the constituents, just one more is needed.

We confirm the success in reproduction of many meson masses, especially of heavy ones, of the Jain–Munczek approach [2–4], some small differences being due to our independent numerical procedures. Solving for $\chi^P (P = \pi, \eta_s, \eta_c, \eta_b)$ using the parameter values fixed in Ref. [4], we obtain the following. i) $M_{\pi} = 140$ MeV – i.e., less than 2% away from the average over the empirical masses of the three pion isospin states, which value (138 MeV) is the most appropriate pion mass in the isospin limit. ii) $M_{\eta_s} = 721$ MeV, which value contributes to reasonable predictions for the masses of physical states $\eta$ and $\eta'$ in Ref. [28]. iii) Solving for $\chi^P_{\eta_c}$, we get $M_{\eta_c} = 2.875$ GeV, whereas the experimental $\eta_c$-mass is $M_{\eta_c}^{exp} = 2.979$ GeV. iv) For $\eta_b$, where there are no experimental results yet, we predict $M_{\eta_b} = 9.463$ GeV.

The amputated BS bound-state vertices $\Gamma_P$ are sometimes technically more convenient for describing some aspects of the presently interesting processes involving bound states, than the equivalent formulation through the BS amplitude $\chi_P$. The derivation of the closed-form expression (Eq. (24) below) for the amplitude of the ABJ anomaly–induced decay $\pi^0 \rightarrow \gamma\gamma$, is in the chiral limit therefore performed this way, employing the amputated vertex $\Gamma^\pi$. The reason why this is simpler is clear if we recall that [14,11] proved that the $O(p^0)$-piece of the amputated pion bound-state vertex, namely $\Gamma^\pi_0(q, 0)$ in (11), given in the chiral limit by (4), fully saturates the axial anomaly. In contrast to that, using the BS-amplitude here would introduce unnecessary complications since the connection between the functions $\Gamma^\pi_i$ and $\chi_i^\pi$ (defined by (12)) is such that $\Gamma^\pi_0(q, 0)$ is tied to all four functions $\chi^\pi_i(q, 0)$. On the other hand, we find the unamputated BS amplitudes $\chi_M$ generally more suitable for solving BS equations on the computer, since the number of quark propagator lines is reduced in the integral – see the BS equations (10) and (13).
5. \( \pi^0, \eta_s, \eta_c, \eta_b \rightarrow \gamma \gamma \), AND \( \gamma^* \pi^0 \rightarrow \gamma \) PROCESSES

The transition matrix element for all of the processes \( P \rightarrow \gamma \gamma \) \( (P = \pi^0, \eta_s, \eta_c, \eta_b) \) and \( \gamma^* \pi^0 \rightarrow \gamma \) has the form,

\[
S_{fi} = (2\pi)^4 \delta^{(4)}(p - k - k') \varepsilon^{\mu*}(k, \lambda) \varepsilon^{\nu*}(k', \lambda') i e^2 T_P^\mu\nu(k, k') ,
\]

where

\[
T_P^\mu\nu(k, k') = i \int d^4x \exp(ik \cdot x) \langle 0|T\{J^\mu(x)J^\nu(0)\}|P(p)\rangle ,
\]

and where \( J^\mu(x) \) is the electromagnetic current of quarks,

\[
J^\mu(x) = \bar{\psi}(x)\gamma^\mu Q\psi(x) ,
\]

and \( Q = \text{diag}(Q_u, Q_d, Q_s, Q_c, Q_b) = \text{diag}(+\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, +\frac{1}{3}, -\frac{1}{3}) \) is the quark charge operator in units of the proton charge \( e \). The photon momenta are \( k \) and \( k' \), while \( p = k + k' \) is the meson momentum.

For \( \pi^0, \eta_s, \eta_c \) and \( \eta_b \rightarrow \gamma \gamma \), the both photons \( (\gamma) \) are real, with \( k^2 = k'^2 = 0 \), and consequently \( \varepsilon^\mu(k, \lambda) \) and \( \varepsilon^\nu(k', \lambda') \) are both just the photon polarization vectors. In contrast, for \( \gamma^* \pi^0 \rightarrow \gamma \), only one photon is real \( (k^2 = 0) \), whereas the other is virtual, stemming for example from the electron current like in TJNAF. Only \( \varepsilon^\mu(k, \lambda) \) is the photon polarization vector, whereas \( \varepsilon^\nu(k', \lambda') \) should be understood as the product of the electron electromagnetic current and the photon propagator transferring the momentum \( k', k'^2 = -Q^2 \neq 0 \). In any case, the problem boils down to computing the tensor \( T^{\mu\nu}(k, k') \) defined by Eq. (16), i.e., the \( P \)-to–vacuum matrix element of electromagnetic quark currents. By symmetry arguments, \( T_P^{\mu\nu} \) can be written as [23]

\[
T_P^{\mu\nu}(k, k') = \varepsilon^{\alpha\beta\mu\nu}k_\alpha k'_\beta T_P(k^2, k'^2) ,
\]

where \( T_P(k^2, k'^2) \) is a scalar, a function of the scalar products of the momenta. For \( \pi^0 \rightarrow \gamma \gamma \) as well as \( \eta_c \rightarrow \gamma \gamma \) and \( \eta_b \rightarrow \gamma \gamma \), where all particles are real, i.e., on their mass shells, we have simply \( T_P(0, 0) = \text{const} \) in (18). After summing over polarizations and integration over the photon phase space we finally get the width

\[
W(P \rightarrow \gamma \gamma) = \frac{\pi\alpha_{em}^2M_P^3}{4} |T_P(0, 0)|^2 , \quad (P = \pi^0, \eta_c, \eta_b).
\]

On the other hand, for \( \gamma^* \pi^0 \rightarrow \gamma \) transition, only the final photon is necessarily on the mass shell, \( k'^2 = 0 \), whereas \( k^2 = -Q^2 \) is the square of the momentum transferred from the projectile beam. In general, the pion can be off–mass shell, with \( p^2 \neq M_{\pi}^2 \). Ref. [12] discusses how to handle also this more general possibility. In the present paper, we limit ourselves to the \( \gamma^* \pi^0 \rightarrow \gamma \) transition form factor \( F(Q^2) \) on the \( \pi^0 \) mass shell, defined by

\[
F(Q^2) = \frac{T_{\pi^0}(0, -Q^2)}{T_{\pi^0}(0, 0)} .
\]
How do we go about calculating $T_P$, or, equivalently, $T_{P}^{\mu\nu}$? We adopt the framework advocated by (for example) [14,11–13] in the context of electromagnetic interactions of BS bound states, and called the generalized impulse approximation (GIA) by [11–13]. This means that in the triangle graph we use the dressed quark propagator $S(q)$, Eq. (3), and the pseudoscalar BS bound–state vertex $\Gamma_{\mu}(q, p)$ instead of the bare $\gamma_5$ vertex. Another essential element of the GIA is to use an appropriately dressed electromagnetic vertex $\Gamma^{\mu}(q', q)$, which satisfies the vector Ward–Takahashi identity,

$$ (q' - q)_{\mu} \Gamma^{\mu}(q', q) = S^{-1}(q') - S^{-1}(q) . \quad (21) $$

Assuming that photons couple to quarks through the bare vertex $\gamma^\mu$ would be inconsistent with our quark propagator, which, dynamically dressed through Eq. (9), contains the momentum-dependent functions $A(q^2)$ and $B(q^2)$. The bare vertex $\gamma^\mu$ obviously violates (21), implying the non-conservation of the electromagnetic vector current and of the electric charge. Since no-one has yet satisfactorily solved the pertinent SD equation for the dressed quark-photon vertex $\Gamma^{\mu}$, it is customary to use realistic Ansätze, in the development of which a number of researchers invested much effort. Motivated by its earlier successful usage in some related work, such as [11] or [12], we choose the Ball–Chiu (BC) [53] vertex for $\Gamma^{\mu}(q', q)$:

$$ \Gamma^{\mu}(q', q) = A_{\pm}(q^2, q^2) \frac{\gamma^\mu}{2} + \frac{(q' + q)^\mu}{(q^2 - q^2)^2} \{ A_{\pm}(q^2, q^2) \frac{(q' + \not{q})}{2} - B_{\pm}(q^2, q^2) \} , \quad (22) $$

where $H_{\pm}(q^2, q^2) \equiv [H(q^2) \pm H(q^2)]$, for $H = A$ or $B$. Obviously, this Ansatz does not introduce any new parameters as it is completely determined by the quark propagator (3). Its four chief virtues, however, are (i) that it satisfies the Ward–Takahashi identity (21), (ii) that it reduces to the bare vertex in the free-field limit as must be in perturbation theory, (iii) that its transformation properties under Lorentz transformations and charge conjugation are the same as for the bare vertex, and (iv) it has no kinematic singularities.

It is important to note that the correct axial-anomaly result cannot be obtained [11] analytically in the chiral limit unless a quark–photon–quark ($qq\gamma$) vertex that satisfies the Ward-Takahashi identity is used (even if D$\chi$SB is employed and the pion does appear as a Goldstone boson, as comparison with [54,55] shows).

In the case of $\pi^0$, for example, the GIA yields the amplitude $T_{\pi^0}^{\mu\nu}(k, k')$:

$$ T_{\pi^0}^{\mu\nu}(k, k') = -N_c \frac{Q_u^2 - Q_d^2}{2} \int \frac{d^4q}{(2\pi)^4} \text{tr} \{ \Gamma^{\mu}(q - \frac{p}{2}, k + q - \frac{p}{2})S(k + q - \frac{p}{2}) \}

\times \Gamma^{\nu}(k + q - \frac{p}{2}, q + \frac{p}{2})S(q + \frac{p}{2})\Gamma_{\pi^0}(q, p)S(q - \frac{p}{2}) \} + (k \leftrightarrow k', \mu \leftrightarrow \nu) . \quad (23) $$

(The analogous expressions for $\eta_c$ and $\eta_b$, or any other neutral pseudoscalar, are straightforward.) The number of colors $N_c$ arose from the trace over the color indices. The $u$ and $d$ quark charges in units of $e$, $Q_u = 2/3$ and $Q_d = -1/3$, appeared from tracing the product of the the quark charge operators $Q$ and $\tau_3/2$, the flavor matrix appropriate for $\pi^0$: $\text{tr}(Q^2\tau_3/2) = (Q_u^2 - Q_d^2)/2$.}

$$ \text{(23)} $$
6. DISCUSSION OF THE RESULTS

In the chiral and soft limits, the present approach yields the form of the amplitude for \( \pi^0 \to \gamma \gamma \) which is completely independent of the pion structure. Namely, our framework is in this limit equivalent to \([14,11,12]\), as demonstrated by the fact that, in the chiral limit, with the \( \mathcal{O}(p^0) \) solution (4) but independently of its concrete shape \( B(q^2) \), we too can reproduce analytically, in the closed form, the famous “triangle” anomaly amplitude:

\[
T_{\pi^0}^{\text{chiral}}(0,0) = \frac{1}{4\pi^2 f_\pi}.
\]  

(24)

The decay width (19) is then given by Eq. (2), and its numerical value is in excellent agreement with experiment (see Table 1), if the pion decay constant \( f_\pi \) is also predicted correctly.

As already pointed out above, this result is independent of our concrete choice of the interaction kernel and the resulting hadronic structure of \( \pi^0 \); this is as it should be, because the axial anomaly, which dominates \( \pi^0 \to \gamma \gamma \), is of course independent of the structure.

It is then not surprising that the calculations for \( \pi^0 \to \gamma \gamma \) which rely on the details of the hadronic structure (be it in the context of the BS equation, nonrelativistic quarks, or otherwise) fail to describe this decay accurately even when the model parameters are fine-tuned for that purpose.

On the other hand, we should point out that Jain and Munczek’s \([2–4]\) approach to the bound states describes the processes such as the anomaly–dominated \( \pi^0 \to \gamma \gamma \) or \( \gamma^* \to \pi \pi \pi \), more consistently than some other approaches which also analytically obtain the correct expressions (24) and (2). The reason is that their [2–4] treatment of the all–important pion decay constant is exceptionally consistent. Namely, in the chiral limit, the axial-vector Ward–Takahashi identity

\[
(q' - q)\mu \Gamma_5^{\mu}(q', q) = \frac{1}{2} \lambda^a [S^{-1}(q')\gamma_5 + \gamma_5 S^{-1}(q)]
\]  

(25)

requires that the normalization\(^d\) \( \mathcal{N}_\pi \) of the BS pion bound state must be equal to the pion decay constant \([43]\), \( \mathcal{N}_\pi = f_\pi \). However, as pointed out recently by Maris, Roberts, and Tandy \([48,57]\), this equality is not obeyed in model studies to date (unless one unrealistically assumes \( A(q^2) \equiv 1 \)) because of neglecting the \( \mathcal{O}(p) \) terms in the decomposition of the BS amplitudes (14) or bound-state vertices (11). Nevertheless, the approach of Jain and Munczek is a notable exception, since it does satisfy this important consistency relation very accurately, within 2% to 3%. This was pointed out already in their first paper \([2]\) devoted

\(^c\)By the way, our approach in the chiral and soft limit also reproduces analytically the “box” anomaly result for \( \gamma^* \to \pi \pi \pi \), in the fashion of Alkofer and Roberts \([56]\) in the Ansatz approach.

\(^d\)As in \([28]\), the normalization is in this paper defined as in Eq. (2.8) of Ref. \([3]\), up to the factor of \( N_c \), since we do not adopt their \([3]\) conventional color factor of \( 1/\sqrt{N_c} \) in the definition of BS amplitudes.
precisely to the consistent calculation of $f_\pi$ in the SD–BS framework, where they stressed the importance of subleading Dirac components. This fact seems to have been largely overlooked, and we point it out especially because of its obvious favorable implications for the consistent description of $\gamma\gamma$-processes performed here.

The off-shell extension of $\pi^0 \to \gamma\gamma$, namely $\gamma^*\pi^0 \to \gamma$, is described by the transition form factor $F(Q^2)$ (20) which we calculated for space–like ($Q^2 > 0$) transferred momenta. As in [12], we have used the chiral-limit solution (4) for the pion bound state, but we have obtained $B(q^2)$ by solving the SD equation (9) with a specified kernel, concretely (5)–(8). We compare our $F(Q^2)$ in Fig. 7 with the CELLO data [58] and with the $F(Q^2)$ of [12], which is a quark-propagator-Ansatz approach but related to ours. We also display the Brodsky–Lepage interpolation [59] to the perturbative QCD factorization limit $Q^2 F(Q^2) \to 8\pi^2 f_\pi = 0.67$ GeV$^2$, and the curve resulting from the vector-meson-dominance type of approach of [60]. Although our solution for $B(q^2)$, parameterizing the pion structure in the chiral limit, is significantly different from the Ansatz used by [12] (see Fig. 4), our curve for $F(Q^2)$ is only marginally better than that of [12], and we conclude that for $Q^2 < 2.5$ GeV$^2$ these curves fit the CELLO experimental points$^*$ with comparable quality. The “interaction size”, defined by [12] via $\langle r^2_{\pi^0} \rangle = -6 F'(Q^2) Q^2 = 0$, discriminates even less between our respective $B(q^2)$’s; namely, our result of 0.46 fm is practically the same as $\langle r^2_{\pi^0} \rangle^{1/2} = 0.47$ fm of [12]. (The monopole fit to the CELLO data [58] yields 0.65 ± 0.03 fm.) It seems, therefore, that the differences in the pion structure play a relatively small role for the $\gamma^*\pi^0 \to \gamma$ transition form factor, at least in the chiral limit, although cannot be reduced to the dependence of $f_\pi$ as for the on-shell $\gamma\gamma$ amplitude of the (pseudo)Goldstone $\pi^0$.

Of course, the situation is very different for quark masses of the order of $\Lambda_{QCD}$ and higher: only the numerical evaluation of $T_P(0,0)$ is reliable in this regime. Moreover, the details of the chosen interaction kernel and the resulting propagator functions $A(q^2)$ and $B(q^2)$, as well as the bound state solutions, do matter in that regime (which gets further and further away from the domination of the axial anomaly with growing quark masses, as illustrated by the amplitude ratios in the last column of Table 1). This is naturally the case with the heavy-quark composites $\eta_c$ and $\eta_b$. Their two–photon widths are also given in Table 1. In the case of $\eta_b$, only predictions exist, as there are no experimental data yet, so that the calculated mass of $\eta_b$ had to be used in the phase-space factors, unlike $\pi^0$ and $\eta_c$.

The case of $\eta_c$ is very intriguing. The widths $W(\eta_c \to \gamma\gamma)$ resulting from the nonrelativistic constituent potential models had long seemed to be in good agreement with experiment, but have more recently been shown [63] to rise to far too large values of $11.8 \pm 0.8 \pm 0.6$ keV after the calculations have been improved by removing certain approximations. The estimates [63] of the relativistic corrections indicate that they are so large that they can reduce the width back down to around 8.8 keV. Such strong relativistic corrections corroborate the view that the relativistic approaches to bound states retain their importance on the mass scale of the $c$–quarks. Since the latest (1996) [64] Particle Data Group (PDG) average has also risen with respect to the earlier (1994) PDG average [65] up to $W^{PDG}_{96}(\eta_c \to \gamma\gamma) = 7.5 \pm 1.6$ keV,

$^*$Our comparison with the new CLEO data [61] at higher momenta up to $Q^2 \sim 8$ GeV$^2$, as well as the asymptotic behavior of the $\gamma^*\pi^0 \to \gamma$ form factor, is in preparation [62].
it can appear that the substantial relativistic corrections have indeed managed to solve the problem, bringing about the agreement with experiment. However, if this is so, then fully relativistic treatments through the BS equation should be able to yield results which are at least as good – i.e., in no worse agreement with the experimental $\eta_c \rightarrow \gamma\gamma$ width than the relativistically corrected width from the nonrelativistic constituent potential models [63]. Only if this is the case, we can say we understand $\eta_c \rightarrow \gamma\gamma$ simply through the relativistic effects in constituent models.

Nevertheless, the situation is not like that. If one surveys the theoretical treatments of the two–photon physics of mesonic $q\bar{q}$–composites attempted in the fully relativistic BS approach (but without $D\chi$SB and without a dressed $qq\gamma$ vertex), one finds that such BS calculations tend to yield the widths $W(\eta_c \rightarrow \gamma\gamma)$ typically below 4 keV – see for example Ref. [19], containing quite broad fits of $\gamma\gamma$-decays of numerous mesons including $\eta_c$, and references to other related BS calculations, which lead to similar results. Of course, many of such BS calculations could fit $\eta_c \rightarrow \gamma\gamma$ by fine–tuning of model parameters aimed especially at $\eta_c$, but simultaneous broad fits of many different quantities besides $W(\eta_c \rightarrow \gamma\gamma)$ performed by Ref. [19] and references therein, indicate that the fully relativistic bound-state approaches tend to reduce the $\eta_c \rightarrow \gamma\gamma$ width too far below its current experimental PDG average [64]. On the other hand, our coupled SD-BS approach, concretely Jain and Munczek model [4] plus GIA, is also fully relativistic, but predicts the somewhat higher value $W(\eta_c \rightarrow \gamma\gamma) = 5.3$ keV. This is thanks to its distinct characteristic – namely employing also in the heavy-quark sector, the quark–photon vertex dressed consistently with the dynamical dressing of the quarks (needed for $D\chi$SB essential in the light-quark sector) – which partially compensates the width reduction caused by the relativistic treatment. Although this is not enough to approach the current experimental average $W^{PDG}_{96}$ within one standard deviation, what is in our opinion more important, is that our result fits comfortably (and without any adjustment of model parameters whatsoever!) within the empirical error bars after the 1996 PDG average is supplemented with the 1995 CLEO [42] result of $(4.3 \pm 1.0 \pm 0.7 \pm 1.4)$ keV, resulting in $W^{exp}(\eta_c \rightarrow \gamma\gamma) = 6.2 \pm 1.2$ keV. In contrast to this, it seems – to the best of our knowledge – that unless they fine-tune their parameters for that purpose, most of the other theoretical approaches predict $\eta_c \rightarrow \gamma\gamma$ decay widths that will be either too large or too small after the inclusion of the 1995 CLEO [42] result in the PDG average. This is the reason we judge our SD-BS approach especially successful for $\eta_c \rightarrow \gamma\gamma$.

We stress that the 1995 CLEO [42] results on $\eta_c \rightarrow \gamma\gamma$ are reliable and should be taken into account [66]. Admittedly, they have not been published in a journal yet (and therefore – contrary to our expectations [1] – have not entered in the current [64] PDG average), but this is not due to the $\eta_c$ data being questionable. It has been due to some unresolved issues concerning the $c\bar{c}$ scalars $\chi_{c0}$ and $\chi_{c2}$. This has prevented the publication of all $c\bar{c}$ results of [42], even though the results for $\eta_c$ presented in [42] are definitive [66] and in fact, judging by their error bars, of higher quality than most other measurements contributing to the present average [64]. (The inclusion of the $\eta_c$-results presented in [42], would reduce the standard deviation of the 1996 PDG average [64] by some 20%.) While the formal reasons for these $\eta_c$-results not entering in compilations such as [64], are clear in the absence of their publication in a journal, it is also clear under the circumstances explained above, that (a) they should not be left out from discussion in research works, and (b) they should serve as a particularly strong reminder that new measurements and data re-analyses are still needed...
in order to make the empirical value of the $\eta_c \to \gamma\gamma$ width really trustable.

It is important to note that we have not done any fine-tuning of the parameters, or of the gluon propagator form (6)-(8) which we used; these are the propagator and the parameters of Ref. [4], which achieved a broad fit to the meson spectrum and pseudoscalar decay constants. Therefore, in the consistently applied generalized impulse approximation, the BS approach genuinely, without fitting, leads to the adequate amplitude strength, which is otherwise too low. In other words, the BS approach which is in accordance with the ideas of [14,11,12], seems to be able to describe electromagnetic processes well even in the heavy-quark sector without fine-tuning of model parameters. On the other hand, since $W(\eta_c \to \gamma\gamma)$ does depend on the interaction kernel and the parameters determining the internal structure of $\eta_c$, making the measurements of the processes such as $\eta_c \to \gamma\gamma$ more precise can, through our theoretical approach, contribute to determining the non-perturbative gluon propagator more accurately. In the first place, this pertains to the infrared part, which is at present poorly known, but is also of significance for the determination of the QCD running coupling $\alpha_{\text{st}}$ (contained in (7), the ultraviolet part of our gluon propagator), for example, at the scale $m_c$ naturally sampled by the $\eta_c$ system.

To close the circle, the calculations for $\eta_c$ and $\eta_b$ and their $\gamma\gamma$-decays bring us back to the pion, because the question arises: why treat the pion in the chiral limit only? Why not perform also for $\pi^0$ and its $\gamma\gamma$-decay the fully massive calculation? Well, until recently there was little motivation for doing this for the $\pi^0 \to \gamma\gamma$-decay, because it was clear that for this lightest pseudoscalar meson, the pure anomaly result (24) for $T^{\text{chiral}}(0,0)$ is an excellent approximation, and that one can get only very small corrections to (24) by discarding (4) and computing $\pi^0 \to \gamma\gamma$ beyond chiral limit. Recently, however, there appeared another treatment of $q\bar{q}$ bound states in the coupled SD-BS formalism beyond chiral limit [47] but also beyond ladder approximation, which obtained as much as 14% deviation from the pure anomaly amplitude (24). This is not only three standard deviations away from the experimental value, but also calls for some clarifications as to what corrections to the anomaly result can occur if the massive pion has the pseudo–Goldstone character consistent with PCAC.

For the massive pion, the amplitude cannot be obtained analytically any more. We have to evaluate $T_P(0,0)$ numerically, just like we did for the very massive $\eta_c$ and $\eta_b$ in [1]. The result for the massive pion is, however, little changed with respect to the purely anomalous result (24): numerically, $T_\pi(0,0) \approx 0.942 T^{\text{chiral}}(0,0)$. Rounded to two significant digits, our prediction for the massive $\pi^0 \to \gamma\gamma$ amplitude is thus just one standard deviation below the central experimental value – see Table I. In the case of the massive pion, there is of course some structure dependence, and since the largest influence on the choice of the model Ansatz for $G_{1R}$, as well as on the fixing of the model parameters in [4], was exercised by mesons much heavier than pions simply because they are most numerous, it is not surprising that the agreement with experiment is worsened with respect to the chiral limit where the decay amplitude is purely anomalous and thus structure-independent. What is important is that in spite of this, the deviation from the axial anomaly result is not larger than what is allowed by PCAC. Our amplitude for the massive pion deviates from the purely anomalous, structure-independent result in the chiral limit by 5.8%, which still permits the consistency of the used bound-state model [4] with the Goldstone character of the pion in the chiral limit and consequently with the PCAC. Namely, on top of the anomaly, Veltman-Sutherland theorem
allows only contributions which are at least of order $p^2$ higher than the anomaly result. (For a clear exposition thereof, see, e.g., [67]). This means that the corrections with respect to the anomaly result can be of the order of $M_\pi^2/\Lambda_H^2$ [10]. Here, $\Lambda_H$ is a typical hadronic scale. $M_\pi$ must be an order of magnitude smaller than $\Lambda_H$ in order that PCAC can make sense. Such a scale is usually taken to be roughly of the order of the $\rho$-meson mass $M_\rho \approx 770$ MeV (i.e., like 2 constituent quarks forming an ordinary, non-Goldstone meson), or the scale $\Lambda_\chi$ in the $\chi$PT expansions, given by (e.g., see [22] and [68])

$$\Lambda_\chi = \frac{4\pi f_\pi}{\sqrt{N_{lf}}}. \quad (26)$$

For the number of light flavors $N_{lf} = 3$ [meaning the number of quark flavors building up the octet of light pseudoscalar mesons that can be regarded as (pseudo)Goldstone bosons], its value is $\Lambda_\chi = 670$ MeV. This leads to the rough estimate that for the realistically massive pion, the deviation from the chiral-limit $\pi^0 \rightarrow \gamma \gamma$ amplitude (24) cannot exceed much the order of 4%. Our 5.8% deviation from the anomaly amplitude (24), effectively given by integrals over dressed propagators, the pion BS amplitude and dressed $q\gamma q$ Ball-Chiu vertices, is of this order and therefore consistent with PCAC and Veltman-Sutherland theorem. Since PCAC is really not a hypothesis any more, but is understood on the basis of the pseudo-Goldstone character of the pion, our massive $\pi^0 \rightarrow \gamma \gamma$ result is a check of the pion being – in the Jain-Munczek model [4] – both a $q\bar{q}$ bound state and the proper pseudo-Goldstone particle. In contradistinction to that, the 14%-deviation from the anomalous amplitude $T_{\chi}^{\text{chiral}}(0, 0)$, obtained by [47] for a massive pion, would come from $\Lambda_H = 360$ MeV = $2.6M_\pi$. This is larger than $M_\pi$, but not by an order of magnitude. Such a bound-state model therefore comes in the conflict with PCAC. This is an indication that their pion is not a proper pseudo-Goldstone particle. In fact, the approach of Ref. [47] is a specific attempt to go beyond the ladder approximation for the interaction used in the SD and BS equations. We take their result on $\pi^0 \rightarrow \gamma \gamma$ as an indication that their SD and BS equations are not coupled into a consistent approximation scheme any more. It is well-known that (rainbow-)ladder approximation – as used in [4] for example – is such a consistent approximation scheme, which preserves the Goldstone pseudoscalar in the chiral limit, and the pseudo-Goldstone one for quarks which are light in comparison with $\Lambda_{QCD}$. It is very desirable to go beyond this approximation, but it is obviously not yet clear how to do it and preserve successes of the coupled SD-BS regarding the axial anomaly and pseudo-Goldstone character of the pion.

7. CONCLUDING DISCUSSION OF THE MODEL

The concrete realization of the coupled SD–BS approach used in this work, namely the model of Jain and Munczek, is quite successful from the lightest to the heaviest meson masses. It should thus have wide future applicability, especially because this feature is combined with another one (common to coupled SD–BS approaches in general), namely that it is manifestly relativistically covariant. One can therefore relate its solutions – and thus also form factors, etc. – in different frames, for high recoil momenta, which becomes
increasingly important with the advent of ever more energetic facilities (for example TJNAF today, tomorrow ELFE, etc.).

Therefore, we would like to close this paper with a discussion of the model of Jain and Munczek, with the emphasis on clarifying how this model can – over such a broad range of masses – be so successful in describing the meson spectrum and pseudoscalar decay constants [2–4], and, as shown in [1] and this paper, also in describing the $\gamma\gamma$-interactions of pseudoscalars (including the anomalous $\pi^0 \to \gamma\gamma$), especially if one notes that all this was done with a small number of parameters.

This clarification is necessary, because it is known that the sector of light quark mesons is rather different from the one of heavy quark mesons, and at first is hard to understand how it is possible to treat them in such an unified manner as in the model of Munczek and Jain [4]. We especially want to clarify how is this possible in spite of employing: i) the ladder approximation, which has been almost universally used to make BS calculations tractable, but which is an uncontrolled approximation after all, and ii) a model assumption, namely the model Ansatz for $G_{IR}$, which has been justified only a posteriori, by its phenomenological success.

In the following, we will basically argue that good results over the wide mass range are indeed not accidental and are in fact understandable, because the way this model has been constructed suggests that it had the potential to combine (and hence in some things surpass) the successes of two reputable models: A.) Nambu–Jona-Lasinio (NJL) model which is successful in the light sector because it incorporates $D\chi_{SB}$, and B.) the constituent quark model which is certainly adequate for heavy quarks.

If one notes that Jain and Munczek’s approach includes the merits of both the NJL model and the constituent quark model, the success of Jain and Munczek’s model [4] in describing the meson spectrum from light to heavy is not so mysterious. Let us remember that various non-relativistic and relativistic constituent quark models work quite well also for almost entire spectrum of hadrons – not only mesons, but also baryons. However, they do not perform well for light pseudoscalar mesons even if we talk just about the spectrum. Besides the references already discussed in Section 2, let us recall another work on meson spectroscopy of Sommerer et al. [69] as an illustrative example. In this well-known paper, the spectrum of almost 50 meson states was fitted in a relativistic constituent quark model. Their interaction kernel, in the ladder approximation and same for all fitted mesons, consists of a one-gluon exchange interaction (roughly analogous to our $G_{UV}$) and a phenomenological, long range (i.e., roughly analogous to our $G_{IR}$) “string tension” potential. They do not have $D\chi_{SB}$, and constituent quark masses are simply parameters. Their so-called “reduction B” (their assumed prescription for reducing their BS equation from 4 to 3 dimensions) results in successful reproduction of even the very lightest, $\pi$–mass. However, they themselves note that there are symptoms that not all is well for small masses, because their other prescription for reducing their BS equation from 4 to 3 dimensions, “reduction A”, does poorly at representing these low–mass mesons, even though there is no apparent physical reason why one of those assumed prescriptions should be favored over the other.

Therefore, this is another interesting case showing that one can expect troubles for small masses if one treats them too similarly to the large ones; however, our point of view is that what is missing in the works of this kind is primarily $D\chi_{SB}$, which determines the properties of pseudoscalars in the light quark sector. When $D\chi_{SB}$ is not included, there will be a price
to be paid in the light sector: even if troubles with the light spectrum are avoided by fine-tuning model parameters and opting for most favorable reduction prescriptions, there is no escape from the trouble with the anomalous processes such as $\pi^0 \rightarrow \gamma\gamma$.

On the other hand, Jain and Munczek’s model is constructed to have the correct limiting behavior both in the non-relativistic limit (for heavy quarks and antiquarks of the same flavor), and in the chiral limit. Indeed, we will argue below that the success of Jain and Munczek’s approach in the light-quark sector is tied more to the incorporation of $D\chi_{SB}$ than to any of the specific interaction kernels used [2–4]. As for the heavy quark sector, even if their model cannot in all applications compete – in that sector – with a specialized scheme such as Heavy Quark Effective Theory (HQET [70], which explicitly takes advantage of simplifications available in the heavy-quark region), it is certainly capable of providing a bridge between these two limiting regions. As stressed by Burden [71], the understanding of nonperturbative dynamical self-dressing – which is the key feature both in QCD and in the SD-BS approach to QCD – is important also for the heavy quarks and for the proper understanding of the successes of HQET. The introductory section of Munczek and Jain’s second paper [3] is especially detailed in discussing their motivation (and to some extent, even justification) for resorting to the ladder approximation as a – for the time being – unavoidable but also acceptable modeling assumption, so here we just point out that the ladder BS equation is known ([72] and references therein) to reproduce the Schrödinger equation in the non-relativistic limit, namely when both quark masses become very large. In fact, Munczek and Jain [3,4] have even derived the non-relativistic potentials following from the gluon propagators assumed, and they are in reasonable agreement with the potentials used in non-relativistic models. (The limitations of the ladder approximation do lead to additional difficulties when one of the masses is much larger than the other. Later we will comment on this some more, but now let us remember that in the present paper we have dealt exclusively with quarks and antiquarks of the same flavor.)

So, the model of Munczek and Jain is certainly no worse than the non-relativistic quark model or the relativistic constituent quark model. In fact, for light quarks it is better than these other models because it satisfies chiral symmetry, enabling the correct chiral limit behavior, complete with the appearance of Goldstone bosons due to $D\chi_{SB}$, essentially in the Nambu–Jona-Lasinio fashion [7,8]. Thus all the results obtainable from pure symmetry arguments are automatically satisfied in this model. This is very satisfying and, as already stressed above, will happen only if the consistency between SD equation and BS equation is maintained. However, this consistency does not pertain just to the usage of the same interaction, that is the same gluon propagator; SD and BS equations must be treated in the consistent approximations. In particular, since the work of Nambu and Jona-Lasinio [7,8], the Goldstone boson solution of the BS equation was found to accompany the spontaneous chiral symmetry breaking appearing in the SD equation for the quark propagator in a variety of chirally invariant models; however, let us recall that until recently (see Ref. [73]), this result was established only when both SD and BS equations were in the ladder approximation. (In the case of SD equations, this approximation is also often called “the
rainbow approximation\(^f\)). Of course, it would be very desirable to go beyond the ladder approximation, and in fact in his paper on dynamical chiral symmetry breaking, Goldstone’s theorem, and the consistency of the SD and BS equations, (which we just quoted above) Munczek himself [73] discusses in which way the BS equation should change, after the SD equation has been taken beyond the ladder approximation, in order to preserve the appearance of Goldstone bosons and other dynamically broken chiral symmetry features. He has made progress, as have also some other authors [75], but the implementation in a concrete coupled SD-BS bound state model which would be elaborated analogously to the present Jain-Munczek model, and which would go beyond the ladder approximation, is still not an immediate possibility, and must be left for some later stage.

To summarize this issue:

i) The ladder approximation does give the correct non-relativistic limit. Jain and Munczek’s model therefore also has the correct non-relativistic limit, so that there are no problems for systems where both quarks are heavy. (As Ref. [3] cautioned, ladder approximation might cause problems when one of the quark masses becomes much larger – i.e., in heavy–light systems, but we do not treat them in the present paper. Also, see the comment in the last paragraph.)

ii) On the other hand, the applicability of the ladder approximation to the light systems is not really known, and the work of Munczek and Jain is an attempt to investigate that. Their results, along with some Miransky’s arguments (see especially the references of Miransky and his collaborators in [3]) and the recent results of Maris, Roberts and Tandy [48,57], are quite encouraging in this respect.

iii) Dynamical chiral symmetry breaking can be incorporated correctly (complete with the appearance of Goldstone pseudoscalars in the chiral limit) in the SD–BS framework using the ladder approximation. (In fact, until recently it was the only way one knew how to do it, and it may still remain one of the most practical ways for modeling even in the future. For example, Miransky [31] in his Chapter 12 judges that such models based on the improved ladder approximation are an improvement with respect to the widely used Extended Nambu–Jona-Lasinio model.)

Keeping in mind i) and iii), it is rather transparent how it was possible for Jain and Munczek to construct a model where the results for the light \(q\bar{q}\) ground-state masses were found to be in good qualitative and quantitative agreement with current algebra results, while for heavy quarks such as \(b\) and \(c\) the predicted masses of their bound states are proportional to the sum of their “constituent” masses in accordance with nonrelativistic limit expectations [4].

As for the point ii), the numerous empirical successes of Jain and Munczek’s model, especially for pseudoscalar and vector mesons (whereas the light scalars remain problematic), have provided a rather strong a posteriori indication that the ladder approximation is

\(^f\)In a more precise terminology of references such as [11,30,74], a framework such as the coupled SD-BS Jain–Munczek approach is called the “rainbow–ladder framework” or the “rainbow–ladder truncation of the quark DS equation and two–body BS equation”. However, in this paper, we follow somewhat less precise, but shorter Jain and Munczek’s terminology.
reasonably useful for the light quarkonia. In addition to that, recent investigations beyond the ladder approximation seem to provide new insight why it is so. It turns out \cite{75,76} that among the corrections beyond ladder approximation that have been computed so far, those computed for pseudoscalar, vector and axial $q\bar{q}$ bound states tend to roughly cancel out, whereas for scalar and tensor $q\bar{q}$ bound states they mostly add up. If this trend eventually turns out to continue also for higher corrections in the pseudoscalar, vector and/or axial channels, this will show that the ladder approximation is indeed reasonable for these channels. Even if one objects that so far this is still not a proof, and that Jain and Munczek \cite{4} maybe had just an accidental success in the above point \textit{ii)}, or, even if one insists that one has to go beyond the ladder approximation in the light sector to get a good description of the internal hadronic structure there, the point \textit{iii)} still permits Jain and Munczek’s model to unquestionably retain all good results in the light sector that follow just from symmetry and the correct incorporation of the dynamical chiral symmetry breaking.

In that connection, recall the following: of the $\gamma\gamma$-processes we calculated in the light sector, the chiral $\pi^0 \rightarrow \gamma\gamma$ is not affected at all, and $\gamma^* \pi^0 \rightarrow \gamma\gamma$ as well as the massive $\pi^0 \rightarrow \gamma\gamma$ are affected rather little by how Jain and Munczek’s model describes the internal pion structure, and here is why. We have found (along with, e.g., Bando \textit{et al.} \cite{14} and Roberts \cite{11}) that, in the chiral \textit{(and soft)} limit, our amplitude for $\pi^0 \rightarrow \gamma\gamma$ is completely independent of the bound-state vertex for the Goldstone boson case ($\propto \gamma_5 B(q^2)$) \textit{i.e.}, the pion structure falls out completely for the $\gamma\gamma$ decay of Goldstone pions, as it should, because there the axial anomaly dominates. In other words, for our treatment of the massless $\pi^0 \rightarrow \gamma\gamma$ the only important thing was the correct incorporation of the symmetry properties: the dynamical chiral symmetry breaking leading to Goldstone bosons in Jain and Munczek’s model, and the Ward-Takahashi identity of QED respected by GIA! It did not matter exactly what solution $B(q^2)$ we got, and from what interaction kernel. We analytically obtained exactly the same answer \textit{(the correct anomaly amplitude)} as Roberts \cite{11} with his \textit{Ansatz} for $B(q^2)$, even though it was very different from our chiral–limit solution for $B(q^2)$ – and as just said, it was because the internal structure did not matter – only obeying the Ward-Takahashi identity and the correct chiral limit behavior mattered. \textit{(A place where we can, and in fact do, detect some structure dependence is the off–shell extension $\gamma^* \pi^0 \rightarrow \gamma$. However, it seems that this dependence is rather weak, since we get results relatively similar to Frank \textit{et al.} \cite{12}, even though our solution for $B(q^2)$ is very different from their \textit{Ansatz} for $B(q^2)$. In $\pi^0 \rightarrow \gamma\gamma$ away from the chiral limit there is also some room for structure dependence, but again on the level of several percent, in keeping with PCAC and Veltman–Sutherland theorem. The correct incorporation of $D\chi\text{SB}$ obviously remains the dominant feature.) Therefore, even if a sceptical reader still doubts that Jain and Munczek’s model can simultaneously give realistic descriptions of the hadronic structure for both heavy and light systems, our simultaneously good results for both $\pi^0 \rightarrow \gamma\gamma$ and $\eta_c \rightarrow \gamma\gamma$ need not be suspect. It is in fact sufficient that: 1.) the GIA with the Ball–Chiu vertex is a good way to incorporate the interactions of dressed quarks with photons, and that 2.) we have a reasonable description of the hadronic structure for $\eta_c$ and $\eta_b$, because what matters for $\pi^0 \rightarrow \gamma\gamma$ is $D\chi\text{SB}$ and the (pseudo)Goldstone character of $\pi$, and not details of the internal pion structure.

With that we finish our detailed explanation of functioning of our chosen variant \cite{2–4} of the coupled SD-BS approach. We close the article with some additional comments on
the situations in the presence of heavy quarks where the present approach is useful and on the situations where, due to the limitations of the present approach such as the ladder approximation, more accurate results are expected from simpler approaches using the non-relativistic approximation.

Possible simplifications are always something valuable. In particular, it is clear that there is no way of beating the accuracy of non-relativistic approximation in the limit of very heavy quark mass. However, the question is what is the value of that quark mass. For top this is certainly true. Bottom is also treatable very well by non-relativistic approximation. In the case of $\eta_b$, we do not mean to challenge the accuracy of the nonrelativistic description; nevertheless, it would be instructive to see how well (or not so well) the coupled SD-BS approach does there once the experimental data on $\eta_b$ are obtained. However, for $\eta_c$ we believe that relativistic effects and effects of dynamical dressing on the quark-photon processes can be very important – especially in the light of recent experimental results from CLEO [42]. Besides already quoted Ref. [19], which advocates the view that relativistic effects are important for all $\gamma\gamma$ widths, including heavy quarkonia, let us also give the corroborating views of (i) Zöller et al. [77] who found that relativistic effects are non–negligible even for mesons involving only heavy quarks and in fact particularly important for $M1$ transitions in charmonium, then of (ii) Resag and Münz [78], who point out that in charmonium one still finds typical velocities of $v/c \approx 0.4$ [79], so that relativistic effects should become important especially for electroweak decay properties, as has been shown by Beyer et al. [80]. Also, we remark that for the hidden flavor systems, where constituents have equal masses, there were no problems in obtaining accurate solutions. On the other hand, in the cases such as $B$–mesons, where one quark is significantly heavier than the other, we did have problems in reproducing Jain and Munczek’s solutions as accurately as in the equal-mass cases. This is not surprising, because it is known that the ladder approximation is especially troublesome for such cases, as Jain and Munczek [3] themselves warned. Obviously, these are the cases where we may profit from making a systematic nonrelativistic reduction, say by expanding in powers of the inverse mass of the heavier quark and adopting elements of HQET, as well as utilizing the insights of SD–BS studies pertinent for heavy–light systems, such as [81,71,82]. Such considerations become really important when heavy–light systems are addressed. However, in the case at hand, for the systems built of a quark and its equally massive antiquark, the straightforward usage of accurately reproduced solutions of Ref. [4], is adequate in every respect.

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TABLE I. For the presently considered neutral pseudoscalars $P$ [$\pi^0$ both in the chiral limit and in the realistically massive case, the unphysical $ss$-pseudoscalar ($\eta_s$), $\eta_c$ and $\eta_b$], we compare their experimental masses, where applicable or available, with our respective theoretical predictions. The ratios $B(0)/A(0)$ roughly represent the constituent quark masses for the quark flavors [massless and massive (but isosymmetric) $u$ and $d$, as well as $s$, $c$, $b$] pertinent for these mesons $P$. The next two columns provide the comparison of the calculated $P \to \gamma\gamma$ decay amplitudes $T_P(0, 0)$ with their respective average experimental widths, where we have included the 1995 CLEO result [42] on $\eta_c \to \gamma\gamma$. At present, there are no experimental data on $\eta_b$. The masses [including the ratios $B(0)/A(0)$] are given in MeV, the decay amplitudes in MeV$^{-1}$, while the amplitude ratios are of course dimensionless. The model value of the pion decay constant we predict [and use in the expression (24) for $T_{\pi^0}^{\text{chiral}}$] is $f_\pi = 93.2$ MeV.

| $P$ | $M_P^{\text{exp}}$ | $M_P$ | $\frac{B(0)}{A(0)}$ | $T_P(0, 0)$ | $T_P^{\text{exp}}(0, 0)$ | $\frac{T_P(0, 0)}{T_{\pi^0}^{\text{chiral}}(0, 0)}$ |
|-----|-----------------|-------|----------------|-------------|----------------|----------------------------------|
| chiral $\pi^0$ | 0 | 356 | 0.272 · 10$^{-3}$ | | | 1 |
| $\pi^0$ | 135 | 140 | 375 | 0.256 · 10$^{-3}$ | | 0.942 |
| $\eta_s$ | 721 | 610 | 0.0798 · 10$^{-3}$ | | 0.294 |
| $\eta_c$ | 2979 | 2875 | 1540 | 0.0692 · 10$^{-3}$ | $(0.075 \pm 0.007) · 10^{-3}$ | 0.255 |
| $\eta_b$ | ? | 9463 | 4770 | 2.057 · 10$^{-6}$ | ? | 7.57 · 10$^{-3}$ |
FIGURE CAPTIONS

Fig. 1: The UV part (the dashed line) and the IR part (the dotted line) of the total presently used effective propagator function (the solid line).

Fig. 2: The comparison of the presently used effective propagator function (the solid line) with the effective gluon propagator functions of [44] (the short-dashed line), of [45] (the dash–dotted line) and of [47] (the long-dashed line).

Fig. 3: Our chiral-limit solution (the solid line) for the propagator function $A(q^2)$ is compared with our massive solutions for various $\bar{m}(\Lambda) \neq 0$ (the dotted lines marked by letters denoting the pertinent flavors). The dashed line denotes the $A(q^2)$-Ansatz (for $u,d$-quarks) of [11], and also of Frank et al. [12] who have such parameters that the difference with respect to the dashed line [11] cannot be seen on this figure.

Fig. 4: The comparison of our chiral-limit solution (the solid line) for the propagator function $B(q^2)$ with our massive solutions for various $\bar{m}(\Lambda) \neq 0$ represented by the dotted lines marked by letters denoting the pertinent flavors, and with the Ansatz (for $u,d$-quarks) for $B(q^2)$ employed by [11] (the dashed line), and that of [12], which cannot be distinguished from the dashed line in this plot.

Fig. 5: The solid line denotes our constituent quark mass function $B(q^2)/A(q^2)$ in the chiral limit, while the dotted lines (marked by letters indicating the pertinent flavors) denote our constituent quark mass functions for $\bar{m}(\Lambda) \neq 0$. The one following from the Ansätze of [11,12] is denoted by the dashed line.

Fig. 6: Diagram for $\pi^0, \eta_c, \eta_b \rightarrow \gamma \gamma$ decays, and for the $\gamma^* \pi^0 \rightarrow \gamma$ process if $k'^2 \neq 0$.

Fig. 7: $\gamma^* \pi^0 \rightarrow \gamma$ form factor. Experimental points are the results of the CELLO collaboration [58]. The solid line represents our results, while the dashed line represents those of Ref. [12]. The dash–dotted line is the Brodsky–Lepage interpolation [59], the line of open circles is the curve of [60], and open squares form the monopole curve corresponding to $\langle r^2_{\gamma \pi^0_\gamma} \rangle^{1/2} = 0.65$ fm.
\[ q + \frac{p}{2} \quad \Gamma_p \quad q - \frac{p}{2} \quad \Gamma^\mu \]

\[ k + q - \frac{p}{2} \quad k' \quad + (k \leftrightarrow k', \mu \leftrightarrow \nu) \]
