Robust Correction Procedure for Accurate Thin Shell Models via a Perturbation Technique

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Abstract—This research proposes a robust correction procedure to improve inaccuracies around edges and corners inherent to thin shell electromagnetic problems by means of perturbation technique. This proposal is developed with three processes: A classical thin shell approximation replaced with an impedance-type interface condition across a surface is first considered and then a volume correction is introduced to overcome the thin shell approximation. However, the volume correction is quite sensitive to cancellation errors, with dramatic effects in the calculation of the local fields near edges and corners. Therefore, a robust correction procedure is added to improve cancellation errors of the volume correction. Each step of the developed method is validated on the practical problem.

Keywords—thin shell approximation; magnetic field; eddy current; joule power loss; perturbation method; subproblem method

I. INTRODUCTION

Thin Shell (TS) models [1-3] are approximated by a priori known 1-D analytical distributions to avoid meshing the thin regions and lighten the mesh of surrounding regions with Interface Conditions (ICs). However, these ICs generally neglect end and border effects, which lead to inaccuracies in the calculation of local and global fields in the vicinity of geometrical discontinuities near borders and corners, increasing with thickness. In order to scope with this difficulty, a sub geometrical discontinuities near borders and corners, increasing

II. SEQUENCE OF THE PERTURBATION TECHNIQUE

A. Definition of Coupled Sub Problems

Based on the SPM strategy [3-5], the scenario of the perturbation method is herein considered in two steps: A problem attending with the stranded inductor and TS model is first solved on a simplified mesh. The inaccuracy on TS solution is then improved by the volume correction taken by a robust correction procedure in order to overcome the cancellation error [4-7]. The relationship between SPs is constrained by surface sources (SSs) or volume sources (VSs), where SSs show changes of ICs across surfaces from previous SPs, and VSs point out changes of material properties of thin volume regions. Each SP is directly performed on its own mesh without depending on other meshes or solving again a new full problem for each new set of parameters as a traditional finite element method [7, 8, 10].

B. Canonical Magnetodynamic Problem

In the SP scenario, a canonical magnetodynamic problem \( p \) is defined in a domain \( \Omega_p \), with boundary \( \partial \Omega_p = \Gamma_p = \Gamma_{h,p} \cup \Gamma_{e,p} \). It should be noted that the subscript \( p \) refers to the associated SP \( p \). The eddy current conducting part of \( \Omega_p \) is denoted \( \Omega_{c,p} \) and the non-conducting one \( \Omega_{c,p}^c \), with \( \Omega_p = \Omega_{c,p} \cup \Omega_{c,p}^c \). Stranded inductors belong to \( \Omega_{c,p}^c \). The equations, material relations, and boundary conditions (BCs) of the SP \( p \) are [4-7]:

\[
\begin{align*}
\text{curl } h_p &= j_{e,p} \text{ div } b_p = 0, \text{curl } e_p = -\partial_t b_p, \\
\text{curl } h &= \mu_p^{-1} b + h_{s,p} j_p = \sigma_p e_p + j_{s,p} \\
[\text{n} \times h_p]_{|\Gamma_{h,p}} &= k_{f_p} n \times h_p |_{|\Gamma_{h,p}} = 0
\end{align*}
\]

where \( h_p \) is the magnetic field, \( b_p \) is the magnetic flux density, \( e_p \) is the electric field, \( j_{s,p} \) is the electric current density, \( \mu_p \) is the magnetic permeability, \( \sigma_p \) is the electric conductivity and \( n \) is the unit normal exterior to \( \Omega_p \).

The source fields \( h_{s,p} \) and \( j_{s,p} \) in (2 a-b) are VSs. In the frame of SPM, the field \( h_{s,p} \) is usually considered as a remnant field in magnetic materials and expressed changes of permeability. The field \( j_{s,p} \) is an imposed electric current...
density in inductors and presented changes of conductivity. For
that, the changes from SP u (μu, and σu to SP p (μp, and σp),
hu,p, and jx,p) are defined [4-7]:

\[ h_{u,p} = (μ_p^{-1} - μ_u^{-1})b_u \quad j_{x,p} = (σ_p - σ_u)e_u \quad (4a-b) \]

The total fields to be related by the uploaded relations
\[ h = μ_p^{-1}(b_p + b_u) \quad j = σ_p(e_p + e_u) \quad [3, 4]. \]
The source field \( k_{f,p} \) in (3a) is SSS. In general, these SSS are defined as zero
for classical homogeneous BCs. ICs can define their
discontinuities through any interface \( γ_p \) (with sides \( γ_p^+ \) and \( γ_p^- \))
in \( Ω_p \), with the notation \( \{ \} \). On the other hand,
they can be considered as SSS for particular phenomena
appearing at the both sides (\( γ_p^+ \) and \( γ_p^- \)) of the thin regions [3, 
4].

C. Thin Shell Finite Element Model

The constraints between the TS model and volume
corrections are expressed via VSS and SSS. For the magnetic
vector potential formulation (b-conformal formulation) in
the TS model [3], these sources are defined via the BCs and ICs of
impedance-type boundary conditions (IBC) associated with
solutions from previous SP which can be a stranded inductor
alone or the inductors with the TS model. Therefore,
the constraint from the TS model to the volume correction
problems, or from the volume correction to the robust correction
procedure, can be obtained from [3, i.e.:

\[ \{ n \times h_{p} \} |_{γ_p} = -σ_0 \partial_γ \left( 2a_{c,p} + a_{d,p} \right) \quad (5) \]

\[ n \times h_{p} |_{γ_p} = \frac{1}{2} σ_0 \partial_γ \left( 2a_{c,p} + a_{d,p} \right) + \frac{1}{μ_0} a_{d,p} - n \times h_{u} |_{γ_p} \]

\[ = \frac{1}{2} σ_0 \partial_γ \left( 2a_{c,p} + a_{d,p} \right) + \frac{1}{μ_0} a_{d} - k_{f,p} \quad (6) \]

where \( a_{d,p} \) and \( a_{c,p} \) are respectively the discontinuous and
continuous components of the field \( a_p \). It should be noted that
\( a_{d,p} \) is considered as zero on the negative side \( Γ_{1,p} \) of the TS,
which cancels the magnetic flux entering there [3]. \( d \) (the thickness of the TS), \( δ \) (the skin depth) and \( β \) are also
given [3]. The discontinuity \( -n \times h_{u} |_{γ_p} \) in (6) is really an SS for
the volume correction.

III. SEQUENCE OF WEAK FORMULATIONS

A. b-Conformal Formulation

The weak \( b_p \)-conformal formulation for problem \( p \)
is achieved from the weak form of the Ampère’s law (1a), i.e. [4- 
7]:

\[ \left( μ_0^{-1} \text{curl} \ a_p, \text{curl} \ a_p' \right)_{Ω_p} + \left( σ_0 \partial_γ \left( 2a_{c,p} + a_{d,p} \right) \right)_{γ_p} + \left( μ_0^{-1} \text{grad} \ v_p, \text{curl} \ a_p' \right)_{Ω_p} + \left( h_{p}, \text{curl} \ a_p' \right)_{Ω_p} + \left( n \times h_{p}, a_{p}' \right)_{Γ_{1,p}} + \left( -n \times h_{u}, a_{p}' \right)_{Γ_p} = (f_p, a_p')_{Ω_p} \]

\[ \forall a_p' \in H^1_0(Ω_p) \quad (7) \]

where \( H^1_0(Ω_p) \) is a gauged curl-conform function space
presented in \( Ω_p \), containing the basis functions for \( a_p \), and for
the test function \( a_p' \). Factors \( (\cdot), Ω_p \) and \( (\cdot), Γ_1 \) are respectively
notations of a volume integral in \( Ω_i \) and a surface integral on \( Γ_i \)
of the product of their vector field arguments. The surface
integral term on \( Γ_{1,p} \) accounts for natural BCs of type (3 b),
which is usually zero. The term \( \left( -n \times h_{p} \right) a_p'_{Γ_p} \) in (9) is
the TS model and can be expressed as [3]:

\[ \left( n \times h_{p} \right) a_p'_{Γ_p} = \left( n \times h_{p} \right) a_p'_{Γ_p} + \left( n \times h_{p} \right) a_p'_{Γ_p} \]

\[ + \frac{1}{μ_0} a_p - k_{f,p} \quad (9) \]

B. Volume Correction and Robust Correction Procedure

Replacing TS Representation

The TS solution is obtained by combining (7), (8) and (9).
The solution is now improved by the volume correction (e.g. problem \( k \)) via VSS \( \left( j_{x,p}, \text{curl} \ a_p' \right)_{Ω_p} \ (h_{p}, \text{curl} \ a_p' \) \( a_{d,p} \) given by (4a) and (4b). This means that these fields need to be
transferred from the mesh of the TS problem \( p \) to the mesh of next
problem \( k \) (volume correction \( k \)) via a projection method
[9]. For that, the weak form for the problem \( k \) is written as

\[ \left( μ_0^{-1} \text{curl} \ a_p, \text{curl} \ a_p' \right)_{Ω_k} + \left( σ_0 \partial_γ \left( 2a_{c,p} + a_{d,p} \right) \right)_{Γ_{1,k}} + \left( μ_0^{-1} \text{grad} \ v_k, \text{curl} \ a_p' \right)_{Ω_k} + \left( h_{p}, \text{curl} \ a_p' \right)_{Ω_k} + \left( n \times h_{p}, a_{p}' \right)_{Γ_{1,k}} + \left( -n \times h_{u}, a_{p}' \right)_{Γ_k} = (f_p, a_p')_{Ω_k} \]

\[ ∀ a_p' \in H^1_0(Ω_k) \quad (10) \]

At the discrete level, the source field \( a_p \) solved in the mesh
of the TS problem \( p \) is projected to the mesh of problem \( k \) in
(10), with \( Ω_{k,l} \) limited to the volume correction, which thus
reduces the computational effort of the process of projection.
The volume correction in (10) is really sensitive to cancellation
errors, with dramatic effects on the calculation of the field \( a_p \)
(see Figure 5 in [6]). In order to avoid the cancellation error,
we need to combine a problem \( k-a \) with:

\[ h_{k-a} = μ_0^{-1} b_{k-a} + \left( μ_0^{-1} - μ_p^{-1} \right) b_p \quad (12) \]

Considering a perfect magnetic region in \( Ω_{k,a} \) (\( μ_{k-a} = 0 \)),
and a problem \( k-b \) with:

\[ h_{k-b} = μ_0^{-1} b_{k-b} + \left( μ_0^{-1} - μ_p^{-1} \right) b_{k-b} \quad (13) \]

presenting a change to the actual permeability (\( μ_{k-b} = μ_{actua} = μ_k \)). The problem \( k-a \) uses a SS only contributing on the positive side of \( Γ_{k,c} \) of \( Γ_{k,c} \), that is:
\[ \mathbf{n} \times \mathbf{h}_{k-a} = \mathbf{n} \times \mathbf{h}_{k-a} \big|_{\Gamma_{c,k}} = -\mathbf{n} \times \mathbf{h}_{p} \big|_{\Gamma_{c,k}} = J_{\text{surf},k-a} \tag{14} \]

with \( \mathbf{h}_{k-a} = 0 \) and \( \mathbf{b}_{k-a} \neq 0 \). The trace \( \mathbf{n} \times \mathbf{h}_{p} \big|_{\Gamma_{c,k}} \) is originally presented in (18) for the problem \( p \) limited to \( \Gamma_{c,k} = \Gamma_{c,k}^p \). It maybe also naturally presented via the volume in (10), that is:

\[
\langle \mathbf{n} \times \mathbf{h}_{k-a} \rangle_{\Gamma_{c,k}} = -\langle \mathbf{n} \times \mathbf{h}_{p} \rangle_{\Gamma_{c,k}} = (\mu_p^{-1} \mathbf{a}_p, \mathbf{curl} \mathbf{a}_p)_{\Omega_{\text{Tl}}} \tag{15}
\]

where \( \Omega_{\text{Tl}} \) is limited to one single layer of FE touching \( \Gamma_{c,k} \), because it involves only the associated trace \( \mathbf{n} \times \mathbf{a}_p \big|_{\Gamma_{c,k}} \) \( [5] \) \( \mathbf{a}_{k-a} = \mathbf{a}_{k-b} = \mathbf{a}_{k} \) because of the same property on the mesh. For the problem \( k-b \), the VS has:

\[
\mathbf{h}_{k-b} = (\mu_k^{-1} b_k - \mu_k^{-1} a_k) (b_{k-b} + \mathbf{b}_p) \tag{16}
\]

with \( \mu_{k-b} = \mu_{\text{actual}} \mu_k \) and \( \mu_{k-b}^{-1} = 0 \). Both problems \( k-a \) and \( k-b \) are being solved at the same time, with the SS \( j_{f,k} = \mathbf{n} \times \mathbf{h}_{p} \big|_{\Gamma_{c,k}} = j_{f,k-a} \) and the resulting relation is:

\[
\mathbf{h}_k = \mathbf{h}_{k-a} + \mathbf{h}_{k-b} = -\mu_p^{-1} \mathbf{b}_p + \mu_{k-b}^{-1} \mathbf{b}_{k-b} + \mu_{k-b} (b_{k-b} + \mathbf{b}_p) = \mu_k^{-1}(b_{k-a} + \mathbf{b}_p) + (\mu_k^{-1} - \mu_p^{-1}) \mathbf{b}_p \tag{17}
\]

This procedure needs the projection of the source field in the added magnetic region (for VSs) as well as in the layer of FE touching this region (for SSs).

C. Projections of Solutions between Meshes

In the SPM strategy, the TS is considered as an SS in a sub-model \( \Omega_{c,k} \) which is a subset of \( \Omega_k \). At the discrete level, the source field from the previous problem (e.g. \( \mathbf{a}_p \) in the previous mesh problem \( p \)) is projected in the mesh of the current (new) problem (e.g. problem \( k \)). This can be done via a projection method \([8]\) of its curl limited to \( \Omega_{c,k} \), i.e.:

\[
(\mathbf{curl} \mathbf{a}_{p,k-proj}, \mathbf{curl} \mathbf{a}_k)_{\Omega_{c,k}} = (\mathbf{curl} \mathbf{a}_p, \mathbf{curl} \mathbf{a}_k)_{\Omega_{c,k}}, \forall \mathbf{a}_k \in H^1_{\text{curl}}(\mathbf{curl}, \Omega_{c,k}) \tag{18}
\]

where \( H^1_{\text{curl}}(\mathbf{curl}, \Omega_{c,k}) \) is a gauged curl-conform function space for the \( k \)-projected source \( \mathbf{a}_{p,k-proj} \) and the test function \( \mathbf{a}_k \).

IV. Numerical Test

The numerical test is an actual problem including an inductor located below a shielding thin plate (Figure 1).

![Diagram](Figure 1. Flux geometry of inductor and thin plate (dimensions in mm))

The inductor carries a fixed current density (excitation current \( I = 1 \text{A} \), frequency \( f = 50 \text{Hz} \), number of turns \( N = 1000 \)).

The test problem is solved in three steps:

- **Step 1:** The stranded inductors with the TS model approximations (SP \( p \)) are first solved
- **Step 2:** The volume correction (SP \( k \)) is given to overcome the TS approximations at step 1
- **Step 3:** The robust correction procedure is considered to improve cancelation errors at step 2

The TS solution SP \( p \) on the magnetic vector potential at step 1 considered with the thin plate and the stranded inductor is shown in Figure 2(a). Next, a volume correction SP \( k \) then replaces the TS approximation with an actual volume covering the plates and their surroundings without including the stranded inductor anymore (Figure 2(b)). As presented, this volume correction is very sensitive to cancellation errors, with dramatic effects on the calculation, thus it is improved by a robust correction procedure (Figure 2(c)).

(a) Magnetic vector potential (Wb/m)

(b) Volume correction with cancellation error (Wb/m)

(c) Robust correction procedure (Wb/m)

![Flux lines (real part) on the magnetic vector potential for (a) the TS solution, (b) volume correction with cancellation errors, and (c) robust correction procedures (c) (d=10mm, p=50Hz, \( \mu_s=100 \) and \( s=59\text{MS/m} \)).](Figure 2. Flux lines (real part) on the magnetic vector potential for (a) the TS solution, (b) volume correction with cancellation errors, and (c) robust correction procedures (c) (d=10mm, p=50Hz, \( \mu_s=100 \) and \( s=59\text{MS/m} \)).)
The relative robust correction procedure of the longitudinal magnetic flux along the thin plate increases with the different thicknesses ($f = 50$ Hz, $\mu_r = 100$ and $\sigma = 59$ MS/m), as shown in Figure 3. For thickness $d = 10$ mm, it can reach several tens percent in the vicinity near edges and corners of the plate, i.e. 70%, with $f = 50$ Hz, $\mu_r = 100$, $\sigma = 59$ MS/m and $\delta = 1.977$ mm. For the lower thickness $d = 2.5$ mm, the error is lower than 15% from the middle to the end.

The eddy current density of the TS SP $p$ (Figure 4(a)) is corrected by the volume correction SP $k$ (Figure 4(b)) without taking the cancellation error into account. The robust correction procedure (Figure 4(c)) is proposed to improve the cancellation errors to overcome the volume correction at step 2. The error obtained is 32% at the plate end. The inaccuracy on the joule power loss density along the half plate of the TS solution approximation SP $p$, the volume correction SP $k$ and the robust correction procedure are pointed out in Figure 5, with $d = 10$ mm, $f = 50$ Hz, $\mu_r = 100$, $\sigma = 59$ MS/m. The significant errors approximately reach 58% between the TS solution SP $p$ and the volume correction SP $k$, or nearly 20% between the volume correction SP $k$ and the robust correction procedure. The robust correction procedure is then compared with the reference solution computed from the traditional Finite Element Method (FEM) [7, 8, 10]. This is an agreement to illustrate a very suitable validation of the robust correction procedure developed in the perturbation technique.

**V. DISCUSSION AND CONCLUSIONS**

Robust correction procedure with the magnetic vector potential formulation has been successfully developed via a perturbation technique in order to improve the cancellation errors on the local fields of magnetic vector potential, magnetic flux, eddy current loss, and Joule power loss density inherent to the TS approximation and the volume correction. The computed results of the extend method are close to the reference FEM calculation solution [7, 8, 10], which is a very good validation of the studied technique. The development has been successfully done with the linear case in the frequency domain. Extension of the method could be further performed in two-way coupling [12]. All the steps of the technique have been validated and applied to the practical problem.

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