Optimal designs for parameters of shifted
Ornstein-Uhlenbeck sheets measured on monotonic sets

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Abstract

Measurement on sets with a specific geometric shape can be of interest for many important applications (e.g. measurement along the isotherms in structural engineering). In the present paper the properties of optimal designs for estimating the parameters of shifted Ornstein-Uhlenbeck sheets, that is Gaussian two-variable random fields with exponential correlation structures, are investigated when the processes are observed on monotonic sets. Substantial differences are demonstrated between the cases when one is interested only in trend parameters and when the whole parameter set is of interest. The theoretical results are illustrated by computer experiments and simulated examples from the field of structure engineering. From the design point of view the most interesting finding of the paper is the loss of efficiency of the regular grid design compared to the optimal monotonic design.

Key words and phrases: D-optimality, efficiency, equidistant design, monotonic sets, optimal design, Ornstein-Uhlenbeck sheet

AMS 2010 subject classifications: Primary 62K05; Secondary 62M30

1 Introduction

Measurement on sets with a specific geometric shape is of interest for many important applications, e.g. measurement along the isotherms. Starting with the fundamental works of Hoel (1958, 1961), the central importance of equidistant designs for the estimation of parameters of correlated processes has been realized. Hoel (1958, 1961) compared the efficiencies of equally spaced designs for one dimensional polynomial models for several design regions and correlation structures. In this context by a design we mean a set \( \xi = \{x_1, x_2, \ldots, x_n\} \) of locations where the investigated process is observed. A comparison in a multi-dimensional setup including correlations can be found in Herzberg and Huda (1981). Later Kiseláčk and Stehlík (2008) proved that equidistant design is optimal for estimating the unknown trend parameter of an Ornstein-Uhlenbeck (OU) process. For other papers on exact designs we refer to the recent studies on stationary OU process with a constant mean by Zagoraiou and Baldi Antognini (2009) and Dette et al. (2008). For prediction of OU process the optimality of the equidistant design was proved by Baldi Antognini and Zagoraiou.
Further, for a process with a parametrized mean, it is often possible to find an asymptotic design that performs well for a large number of design points, see e.g. Sacks and Ylvisaker (1966, 1968). All above mentioned papers on optimal design for OU process considered design region to be an interval from real line. However, a one-dimensional interval is naturally a directed set induced by total ordering of real line. There is a big difference in geometry between plane and line and thus OU sheet sampled on a two dimensional interval provides much more delicate design strategies.

In the present work we derive the optimal exact designs for parameters of shifted OU sheet measured in the points constituting a monotonic set. A monotonic set can be defined in arbitrary Hilbert space $H$, with real or complex scalars. For $x,y \in H$, we denote by $\langle x,y \rangle$ the real part of the inner product. A set $E \subset H \times H$ is called monotonic (Minty, 1962, 1963) provided that for all $(x_1,y_1),(x_2,y_2) \in E$ we have $\langle x_1 - x_2,y_1 - y_2 \rangle \geq 0$. A practical example of such a set are measurements on isotherms of a stationary temperature field with several applications in thermal slab modelling (see e.g. Koizumi and Jin (2012) or Babiak et al. (2005)). Another important example in which monotonic measurements appear is motivated by measuring of methane adsorption (Lee and Weber, 1969) where keeping all measurements at isotherm decreases the problems with stability. Here we consider the following version of a monotonic set:

**Condition D** The potential design points $\{(s_1,t_1),(s_2,t_2),\ldots,(s_n,t_n)\} \subset X$, $n \in \mathbb{N}$, where $X$ denotes a compact design space, satisfy $0 < s_1 < s_2 < \ldots < s_n$ and $0 < t_1 < t_2 < \ldots < t_n$.

We remark that the same observation scheme is used in Baran et al. (2013) where the authors deal with prediction of OU sheets and derive optimal designs with respect to integrated mean square prediction error and entropy criteria.

The paper is organized as follows. In Section 2 we introduce the model to be studied and our notations. Section 3 deals with an example which motivates this study, namely a design experiment for measuring on isotherms of a stationary thermal field, while Sections 4, 5, and 6 deal with the optimal designs for the estimation of parameters in our model. We demonstrate the substantial differences between the cases when only trend parameters are of interests and when the whole parameter set is of interest. Finally, Section 7 contains some applications and we summarize our results in Section 8. To maintain the continuity of the explanation, the proofs are included in the Appendix.

## 2 Statistical Model

Consider the stationary process

$$Y(s,t) = \theta + \varepsilon(s,t) \quad (2.1)$$

with design points taken from a compact design space $X = [a_1,b_1] \times [a_2,b_2]$, where $b_1 > a_1$ and $b_2 > a_2$ and $\varepsilon(s,t), s,t \in \mathbb{R}$, is a stationary Ornstein-Uhlenbeck sheet, that is a zero mean Gaussian process with covariance structure

$$E \varepsilon(s_1,t_1)\varepsilon(s_2,t_2) = \frac{\sigma^2}{4\alpha\beta} \exp \left( -\alpha|t_1 - t_2| - \beta|s_1 - s_2| \right), \quad (2.2)$$

where $\alpha > 0$, $\beta > 0$, $\sigma > 0$. We remark that $\varepsilon(s,t)$ can also be represented as

$$\varepsilon(s,t) = \frac{\sigma}{2\sqrt{\alpha\beta}} e^{-\alpha|t| - \beta|s|} \mathcal{W}(e^{2\alpha|t|},e^{2\beta|s|}),$$

where $\mathcal{W}(s,t), s,t \in \mathbb{R}$, is a standard Brownian sheet (Baran et al., 2003; Baran and Sikolya, 2012). Covariance structure (2.2) implies that for $d = (d,\delta), d \geq 0$, $\delta \geq 0$, the variogram $2\gamma(d) := \text{Var}(\varepsilon(s+d,t+\delta) - \varepsilon(s,t))$ equals

$$2\gamma(d) = \frac{\sigma^2}{2\alpha\beta} \left( 1 - e^{-\alpha d - \beta \delta} \right).$$
and the correlation between two measurements depends on the distance through the semivariogram $\gamma(d)$.

In order to apply the usual approach for design in spatial modeling (Kiselák and Stehlík, 2008) we introduce $\sigma := \bar{\sigma}/(2\sqrt{\alpha\beta})$ and instead of (2.2) we investigate

$$E \epsilon(s_1, t_1)\epsilon(s_2, t_2) = \sigma^2 \exp(-\alpha|t_1 - t_2| - \beta|s_1 - s_2|),$$

where $\sigma$ is considered as a known parameter. This form of the covariance structure is more suitable for statistical applications, while (2.2) fits better to probabilistic modelling. Further, we require Condition D to be hold on the design points. Under Condition D we may use the construction of Kiselák and Stehlík (2008) to obtain the inverse of the covariance matrix of observations which is tridiagonal. Moreover, in case of an equidistant design the covariance matrix is Toeplitz.

An exact design allows the experimenter to plan where to measure the process to optimize a certain measure of variance of estimators, for optimal design in spatial case see Müller (2007). In the literature one can find applications of various criteria of design optimality for second-order models. Here we consider D-optimality, which corresponds to the maximization of objective function $\Phi(M) := \det(M)$, the determinant of the standard Fisher information matrix. This method, "plugged" from the widely developed uncorrelated setup, is offering considerable potential for automatic implementation, although further development is needed before it can be applied routinely in practice. Theoretical justifications for using Fisher information for D-optimal designing under correlation can be found in Abt and Welch (1998) and Pázmán (2007). Abt and Welch (1998) considered a design space $X = [0, 1]$ with the covariance structure $\text{Cov}(Y(x), Y(x + d)) = \sigma^2 e^{-rd}$ and showed that $\lim_{n \to +\infty} (M^{-1}(r, \sigma^2))_{1,1} = 0$ and $\lim_{n \to +\infty} n(M^{-1}(r, \sigma^2))_{1,1} = 2(r\sigma^2)^2$, where $(M^{-1}(r, \sigma^2))_{1,1}$ denotes the (1,1) entry of the inverse of the information matrix.

Zhu and Stein (2005) used simulations (under Gaussian random field and Matérn covariance structure) to study whether the inverse Fisher information matrix is a reasonable approximation of the covariance matrix of maximal likelihood (ML) estimators and a reasonable design criterion as well. For more references on the Fisher information as design criterion in the correlated setup see e.g. Stehlík (2007) where the structures of Fisher information matrices for stationary processes were studied. Stehlík (2007) showed that under mild conditions given on covariance structures the lower bound for the Fisher information is an increasing function of the distances between the design points. Particularly, this supports the idea of increasing domain asymptotics. If for a one-dimensional OU process only the trend parameters are of interest, then the designs covering uniformly the whole design space are very efficient. A similar observation is made in Dette et al. (2008) in a more general framework where the authors prove that if $r \to 0$, then any exact $n$-point D-optimal design in the linear regression model with exponential semivariogram converges to the equally spaced design on the one dimensional design interval. A recurring topic in the recent literature is that uniform or equispaced designs perform well in terms of model-robustness when a Bayesian approach is adopted, when the maximum bias is to be minimized or when the minimum power of the lack-of-fit test is to be maximized (Goos et al., 2005). However, an equidistant design is easy to construct in the case of a single experimental variable. When more than one variable is involved in an experiment and the number of observations available is small, it becomes much more difficult to construct these type of designs. Uniform design is a kind of space-filling design whose applications in industrial experiments, reliability testing and computer experiments is a novel endeavor. The concept of uniform designs was introduced by Fang (1978) and has now gained popularity and proved to be very successful in industrial applications (Pham, 2006, Chapter 13) and in computer experiments (Müller and Stehlík, 2006, Santner et al., 2003).

However, for an OU sheet the optimality of a monotonic set is a very interesting property. It has become standard practice to select the design points such as to cover the available space as uniformly as possible, e.g. to apply the so called space-filling designs. In higher dimensions there are several ways to produce such designs. The importance of the discussion whether space-filling designs are superior has been addressed in the literature recently, see e.g. Pronzato and Müller (2013). Therein the review of the circumstances under
which this superiority holds is given together with the clarification of the motives to go beyond space-filling. In this paper we illustrate that for the OU sheet the design satisfying monotonicity *Condition D* could be superior to the space filling grid design.

The idea of choosing a monotonic set is in particular motivated by Markovian properties of the OU sheet. We are not claiming that monotonic set designs should be used rigidly in engineering practice, but the aim of our paper is to show that for OU sheet in some scenarios a monotonic curve could provide better efficiency than the traditional grid designs. Therefore, the experimenter is advised to integrate monotonic set design into his candidate designs portfolio-especially in the cases when there is a strong intuition/justification of Markovianity of the process.

Being more particular, it is often overseen in practice, that information increases with the number of point only in the case of independence (or specific form of dependence). Thus general filling designs, generated without further caution, may increase variance instead of information (Smit, 1961). Further discussion on designing for correlated processes in the context of space filling and its limitations can be found in Müller and Stehlík (2009) and Pronzato and Müller (2012).

Many recent developments on optimal design strategies for estimation of parameters should admit that they are mostly a *benchmarks* in the more realistic setups for optimal design (like geometric progression ones discussed in in Zagoraiová and Baldi Antognini (2009) for one-dimensional design space, or designs for more complicated trends, see e.g. Rodríguez-Díaz et al. (2012)). These *benchmarks* should always be directly confronted with a subject science, e.g. with methane modelling in the case of modified Arrhenius model as in Rodríguez-Díaz et al. (2012). In the current paper we provide a monotonic design as a benchmark design for a Markovian stochastic process measured on rectangle (with continuous time) and a given subject science is taken from civil engineering where measurement of stationary thermal fields is an issue of interest (Minárová, 2005).

Finally, we should emphasize that we have not tried to find optimal among general design setups – the main aim of the paper is to concentrate entirely on *monotonic set designs*. However, we are working on finding the optimal regular grid designs for OU sheets where we are trying to make a comparison (at least for small sample sizes) with the globally optimal ones.
3 Motivating example: measurement of a stationary thermal field

Temperature distribution calculation during the process of designing a building is a necessary part of testing the critical places at the building envelope. The aim is to increase the minimal surface temperature, and to predict the possible thermal bridges which are possible locations of mould growth in the building. Figure 1 displays the composition of materials of the 2D section of a thermal bridge within the building construction, while on Figure 1 a net for a finite element method is drawn where the points of temperature computation are given. In this way the system investigated is a computer experiment (Sacks et al., 1989; Santner et al., 2003) modelling the real temperature distribution in the building.

Data are taken from Minárová (2005), where a finite element method for computation of the temperature field is applied using software package ANSYS. Figure 2a illustrates the isotherms of the thermal field which fit well to measurements in a monotonic set satisfying Condition D.

Data points in which we measure the temperature are plotted on Figure 2b. We assume that the covariance parameters $\alpha$ and $\beta$ are given and we are interested in the estimation of the trend parameter $\theta$ of model (2.1). Table 1 lists relative efficiency, information $M_\theta$ gained in the data points and the optimal information gain (max $M_\theta$) of the data from Figure 2b for three choices of known correlation parameters $\alpha, \beta$. Obviously, the relative efficiency of the given data points varies with these parameters.

| Correlation Parameters | $M_\theta$ | max $M_\theta$ | Efficiency ($M_\theta/ \text{max} M_\theta$) |
|------------------------|------------|----------------|------------------------------------------|
| $\alpha = \beta = 1$  | 1.481565   | 1.481695       | 0.99                                     |
| $\alpha = 1, \beta = 10$ | 4.97261    | 5.081253       | 0.978                                    |
| $\alpha = 10, \beta = 1$ | 2.212449   | 2.212854       | 0.999                                    |

Table 1: Efficiency depending on correlation parameters.
4 Estimation of trend parameter only

Assume first that parameters $\alpha$, $\beta$ and $\sigma$ of the covariance structure (2.3) of the OU sheet $\varepsilon$ are given and we are interested in estimation of the trend parameter $\theta$. In this case the Fisher information on $\theta$ based on observations $\{Y(s_i, t_i), \ i = 1, 2, \ldots, n\}$ equals $M_{\theta}(n) = 1_n^T C^{-1}(n, r) 1_n$, where $1_n$ is the column vector of ones of length $n$, $r = (\alpha, \beta)^T$, and $C(n, r)$ is the covariance matrix of the observations (Pázman, 2007; Xia et al., 2006). Further, let $d_i := s_{i+1} - s_i$ and $\delta_i := t_{i+1} - t_i$, $i = 1, 2, \ldots, n - 1$, be the distances between two adjacent design points. With the help of this representation one can prove the following theorem.

Theorem 1. Consider the model (2.1) with covariance structure (2.3) observed in points $\{(s_i, t_i), \ i = 1, 2, \ldots, n\}$ satisfying Condition D and assume that the only parameter of interest is the trend parameter $\theta$. In this case, the equidistant design of the form $\alpha d_i + \beta \delta_i$ to be maximal is optimal for estimation of $\theta$.

According to Theorem 1 the optimality holds for $\alpha d_i + \beta \delta_i = \frac{\lambda}{n-1}$, where $\lambda$ is the “skewed size” of the design region, i.e. $\lambda := \alpha \sum_{i=1}^{n-1} d_i + \beta \sum_{i=1}^{n-1} \delta_i$ and $\sum_{i=1}^{n-1} d_i < b_1 - a_1$, $\sum_{i=1}^{n-1} \delta_i < b_2 - a_2$. Several situations may appear in practice. As now we consider the covariance parameters $\alpha, \beta$ to be fixed and make inference only on unknown trend parameter $\theta$, from the proof of Theorem 1 we obtain

$$M_{\theta}(n) = 1 + \sum_{i=1}^{n-1} \frac{1 - q_i}{1 + q_i},$$

(4.1)

where $q_i := \exp(-\alpha d_i - \beta \delta_i)$. Thus, for an optimal design we have

$$M_{\theta}(n) = M_{\theta}(n; \lambda) = 1 + (n - 1) \frac{1 - \exp(-\lambda/(n - 1))}{1 + \exp(-\lambda/(n - 1))},$$

which is an increasing function of both the number of design points $n$ and the length $\lambda$. Further, $M_{\theta}(n; \lambda) \to \lambda/2 + 1$ as $n \to \infty$ and $M_{\theta}(n; \lambda) \to n$ as $\lambda \to \infty$, which values are bounds for information increase in experiments.

To illustrate the latter fact let us consider the design region $\mathcal{X} = [0, 1]^2$ and a four-point design, and assume that correlation parameters are $\alpha = \beta = 1$. We are comparing a regular grid design which puts the four points into the vertices of the rectangle $\mathcal{X}$ (this design does not satisfy Condition D). The information corresponding to this design is $M_{\theta} = 2.13$. Having the same design region we cannot reach such an efficiency, because $\lambda = 2$ and $M_{\theta}(n; \lambda) < \lambda/2 + 1$. Indeed, the maximal information gain can be $M_{\theta}(4; 2) = 1.965$ which gives us an efficiency of 0.919. If we allow the growth of the design region, e.g. $\mathcal{X} = [0, x]^2$, for a four-point design, under the above conditions we obtain $M_{\theta} = \frac{4}{1 + \exp(-2x) + \exp(-x)} \to 4$ for $x \to \infty$ at a regular grid design with vertices.

5 Estimation of covariance parameters only

Assume now that we are interested only in the estimation of the parameters $\alpha$ and $\beta$ of the OU sheet. According to the results of Pázman (2007) and Xia et al. (2006) the Fisher information matrix on $r = (\alpha, \beta)^T$ has the form

$$M_r(n) = \begin{bmatrix} M_{\alpha}(n) & M_{\alpha,\beta}(n) \\ M_{\alpha,\beta}(n) & M_{\beta}(n) \end{bmatrix},$$

(5.1)
where

\[ M_\alpha(n) := \frac{1}{2} \text{tr} \left\{ C^{-1}(n, r) \left( \frac{\partial C(n, r)}{\partial \alpha} C^{-1}(n, r) \frac{\partial C(n, r)}{\partial \alpha} \right) \right\}, \]

\[ M_\beta(n) := \frac{1}{2} \text{tr} \left\{ C^{-1}(n, r) \left( \frac{\partial C(n, r)}{\partial \beta} C^{-1}(n, r) \frac{\partial C(n, r)}{\partial \beta} \right) \right\}, \]

\[ M_{\alpha, \beta}(n) := \frac{1}{2} \text{tr} \left\{ C^{-1}(n, r) \left( \frac{\partial C(n, r)}{\partial \alpha} C^{-1}(n, r) \frac{\partial C(n, r)}{\partial \beta} \right) \right\}, \]

and \( C(n, r) \) is the covariance matrix of the observations \( \{Y(s_i, t_i), \ i = 1, 2, \ldots, n\} \). Note, that here \( M_\alpha(n) \) and \( M_\beta(n) \) are Fisher information on parameters \( \alpha \) and \( \beta \), respectively, taking the other parameter as a nuisance.

The following theorem gives the exact form of \( M_r(n) \) for the model (2.1).

**Theorem 2.** Consider the model (2.1) with covariance structure (2.3) observed in points \( \{(s_i, t_i), \ i = 1, 2, \ldots, n\} \) satisfying Condition D. Then

\[ M_\alpha(n) = \sum_{i=1}^{n-1} \frac{d_i^2 q_i^2 (1 + q_i^2)}{(1 - q_i^2)^2}, \quad M_\beta(n) = \sum_{i=1}^{n-1} \frac{\delta_i^2 q_i^2 (1 + q_i^2)}{(1 - q_i^2)^2}, \quad M_{\alpha, \beta}(n) = \sum_{i=1}^{n-1} \frac{d_i \delta_i q_i^2 (1 + q_i^2)}{(1 - q_i^2)^2}, \]

(5.2)

where \( d_i, \delta_i \) and \( q_i \) denote the same quantities as in the previous section, i.e. \( d_i := s_{i+1} - s_i, \ \delta_i := t_{i+1} - t_i \) and \( q_i := \exp(-\alpha d_i - \beta \delta_i), \ i = 1, 2, \ldots, n - 1. \)

Using Theorem 2 one can formulate the following statement on the optimal design for the parameters of the covariance structure of the OU sheet.

**Theorem 3.** The design which is optimal for estimation of the covariance parameters \( \alpha, \beta \) does not exist within the class of admissible designs.

### 6 Estimation of all parameters

Consider now the most general case, when both \( \alpha, \beta \) and \( \theta \) are unknown and the Fisher information matrix on these parameters equals

\[ M(n) = \begin{bmatrix} M_\theta(n) & 0 \\ 0 & M_r(n) \end{bmatrix}, \]

where \( M_\theta(n) \) and \( M_r(n) \) are Fisher information matrices on \( \theta \) and \( r = (\alpha, \beta)^\top \), respectively, see (4.1) and (5.1).

**Theorem 4.** The design which is optimal for estimation of the covariance parameters \( \alpha, \beta \) and of the trend parameter \( \theta \) does not exist within the class of admissible designs.

Loosely speaking, the optimal designs for the trend have the tendency to move the design points as far as possible, while the optimal designs for the covariance structure have the tendency to shrink the set of design points. However, we can choose a compromise between estimating the trend and correlation parameters. Therefore, similarly to Zagoraiou and Baldi Antognini (2009), we may consider the so-called geometric progression design, which is generated by the vectors of distances

\[ d_{n, r_1} := (k, kr_1, kr_1^2, \ldots, kr_1^{n-2}), \quad \delta_{n, r_2} := (\ell, \ell r_2, \ell r_2^2, \ldots, \ell r_2^{n-2}), \]
Figure 3: Total information corresponding to an OU process with parameters (a) $\alpha = 0.5, \beta = 0.8$; (b) $\alpha = 1, \beta = 1$; (c) $\alpha = 2.5, \beta = 1.5$; (d) $\alpha = 3, \beta = 3$.

where $0 < r_1, r_2 \leq 1$.

As $\sum_{i=1}^{n-1} d_i = 1$ and $\sum_{i=1}^{n-1} \delta_i = 1$, for $r_1 = 1$, $r_2 = 1$ both constants $k$ and $\ell$ are equal to $(n-1)^{-1}$, while for $r_1 < 1$ and $r_2 < 1$ we get $k = \frac{1-r_1}{1-r_1^2}$ and $\ell = \frac{1-r_2}{1-r_2^2}$, respectively. The tuning parameters $r_1$, $r_2$ can be varied according to the desired efficiency for the estimation of the trend or the correlation parameters.

Note, that case $r_1 = 1$, $r_2 = 1$ corresponds to the equidistant design, which we have proved to be optimal for estimation of the trend parameter, while for $r_1 \to 0$, $r_2 \to 0$, vectors $d_{n,r_1}$ and $\delta_{n,r_2}$ tend to the best design for the estimation of $\alpha$ and $\beta$.

**Theorem 5.** For any fixed $n > 2$, $\alpha > 0$, $\beta > 0$, the information $M_\theta(n)$ of the trend is increasing with respect to $r_1, r_2$, while the determinant of the Fisher information $M_r(n)$ of covariance parameters has a global minimum at $r_1 = r_2$.

Observe, that Theorem 5 obviously implies that the total information $\det(M(n))$ has the same behaviour as $\det(M_r(n))$, that is it has a global minimum at $r_1 = r_2$. This result is clearly illustrated on Figure 3 where for $n = 5$ the total information is plotted as a function of $r_1$ and $r_2$ for various combinations of covariance parameters.
Table 2: Efficiency depending on correlation parameters.

| Correlation Parameters | $M_\theta$   | $\max M_\theta$ | Efficiency ($M_\theta/\max M_\theta$) |
|------------------------|--------------|------------------|---------------------------------------|
| $\alpha = \beta = 1$  | 4.319177     | 4.374803         | 0.987285                              |
| $\alpha = 1, \beta = 10$ | 13.13952    | 17.85041         | 0.7360907                            |
| $\alpha = 10, \beta = 1$ | 14.1108     | 21.20754         | 0.6653671                            |

7 Applications to structural engineering

Deterioration of highways

Typically, engineers are using regular grids for estimation of a random field. Such an application is demonstrated in [Mohapl (1997)] on the data describing deterioration of a highway in New York state. Data were collected in four successive years at distances of 0.2 miles from each other and form a $4 \times 16$ table and based on these data the author estimated the parameters. What is the efficiency of such a design? The design region has the natural form $[0,4] \times [0,3.2]$ and the number of observed points is 64. In the case $\alpha = \beta = 1$ design satisfying Condition $D$ and having 64 points in such a region has $M_\theta(64,7.2) = 4.596$.

Now, let us have 16 time coordinates uniformly generated from time region $[0,4]$ and 16 place coordinates generated from space region $[0,3.2]$. Then for time points $1.35, 3.66, 1.86, 0.996, 0.89, 1.56, 3.37, 2.189, 0.5157, 2.58, 0.058, 0.32, 0.58, 1.4, 0.36, 1.82$ and lengths $0.64, 0.37, 1.2, 0.91, 1.34, 2.82, 2.56, 2.44, 0.257, 2.568, 2.223, 0.66, 2.298, 2.814, 2.75, 1.61$ we obtain $M_\theta = 5.2$ in the case both parameters $\alpha$ and $\beta$ are equal to 1. According to Section 4 the maximal information gain with Condition $D$ for $n = 64, \lambda = 5.12$ equals 3.558592, thus the relative efficiency is 0.6843446. However, there is an open question, how to estimate parameters in such a set of points, which is far not trivial. Since the observations form a Gaussian random vector, one can derive the likelihood function and find the ML estimates at least numerically. For a regular grid design [Ying (1993)] proved consistency and asymptotic normality of the ML estimators, but according to the authors’ best knowledge, this is the only result in this direction. The problem is that in the general case the dependence of the likelihood function on the parameters and design points is too complicated to find its asymptotic properties.

When one uses regular grids (for which [Mohapl (1997)] compared least squares, optimal estimation function and ML estimators) the following situation occurs: time is measured in 16 equispaced, optimal estimation moments starting from 0, until 3.75 by 0.25, while the deterioration of the highway is measured in 16 points (by 0.2 miles). Then $M_\theta = 4.319177$ (in the case $\alpha = \beta = 1$) with relative efficiency of 0.8275795. Table 2 is revealing an interesting fact, that regular grid design (with $256 = 16^2$ points) has a lost of efficiency with respect to the optimal design satisfying Condition $D$ with the same number of points in the same design region. This loss can be substantial, dependently on correlation parameters.

Bridge corrosion

As bridge infrastructures age throughout the world, more and more bridges are being classified as structurally deficient [Bhattacharya et al. (2006)]. Unfortunately, due to limited financial resources, bridge owners are not able to immediately repair or, if needed, replace all of the structurally deficient bridges in their inventory. As a result, methods for accurately assessing a bridge’s true load-carrying capacity are needed so that the limited resources can be spent wisely. Therefore, an efficient statistical modeling is needed to overcome the financial limitations via an efficient design.

[Bhattacharya et al. (2006)] proposed a new model for corrosion rate that incorporates a multiplicative noise term ensuring that the corrosion loss function $C(t)$ is non-decreasing in time, that is $dC'(t)/dt =
\(\beta(t-T_I)^\gamma \exp(\eta(t))\), for \(t > T_I\) and 0 for \(t \leq T_I\), where \(\beta\) and \(\gamma\) are parameters independent of time and \(\eta(t)\) is an Ornstein-Uhlenbeck process. Optimal designs for such a process are derived in Kiselák and Stehlík (2008) or Zagoraiou and Baldi Antognini (2009). However, several other corrosion sources can be available yielding a corrosion loss \(C\) field depending on two variables with an error term \(\eta\) forming a planar Ornstein-Uhlenbeck sheet. The design strategies studied in this paper might be of interest for practitioners in estimating the parameters of such a spatial random field.

8 Conclusions

We have constructed exact optimal designs for estimation of parameters of shifted Ornstein-Uhlenbeck sheets on monotonic sets. The central importance of equidistant designs is visible. Since the designs strategies for planar OU sheet are much more difficult than for univariate OU process, the possibility of efficient designing on a monotonic set is very interesting. We illustrated a possible loss of efficiency of the regular grid design with respect to the optimal design satisfying monotonicity Condition \(D\). A motivation example of isotherm measurement is given, simulated examples on highway deterioration are also presented.

In an uncorrelated model the parameter \(\sigma\) influences neither the estimation of the mean value parameters, nor the optimal design. In the present paper we assume \(\sigma\) to be known but a valuable direction for the future research will be the investigation of models with unknown nuisance parameter \(\sigma\).

Acknowledgment

Authors are grateful to Lenka Filová for her helpful comments during the preparation of the manuscript. We acknowledge Mária Minárová for providing us simulated data of thermal fields. This research was supported by the Hungarian Scientific Research Fund under Grants Nos OTKA T079128/2009 and OTKA NK101680/2012 and by the Hungarian –Austrian intergovernmental S&T cooperation program TÉT_10-1-2011-0712, and partially supported by the TÁMOP-4.2.2.C-11/1/KOV-2012-0001 project. The project was supported by the European Union, with co-financing from the European Social Fund. The second author acknowledges the support of the project DESIRE.

A Appendix

A.1 Proof of Theorem [1]

According to the notations of Sections 3 and 4 let \(d_i := t_{i+1} - t_i\), \(\delta_i := s_{i+1} - s_i\) and \(q_i := \exp(-\alpha d_i - \beta \delta_i)\). Similarly to the results of Kiselák and Stehlík (2008) we have

\[
C(n, r) = \begin{pmatrix}
1 & q_1 & q_1 q_2 & q_1 q_2 q_3 & \ldots & \ldots & \prod_{i=1}^{n-1} q_i \\
qu_1 & q_2 & q_2 q_3 & \ldots & \ldots & \prod_{i=2}^{n-1} q_i \\
q_1 q_2 & q_2 & q_3 & \ldots & \ldots & \prod_{i=3}^{n-1} q_i \\
q_1 q_2 q_3 & q_2 q_3 & q_3 & 1 & \ldots & \ldots & q_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & q_{n-1} \\
\prod_{i=1}^{n-1} q_i & \prod_{i=2}^{n-1} q_i & \prod_{i=3}^{n-1} q_i & \ldots & \ldots & q_{n-1} & 1
\end{pmatrix}
\]  

(A.1)
By symmetry it suffices to prove

A.2 Proof of Theorem 2

design is the D-optimal for the parameter \( \lambda / \alpha_d \)

\[
C^{-1}(n, r) = \begin{bmatrix}
\frac{1}{1-q_i} & \frac{q_1}{q_i q_i - 1} & 0 & 0 & \ldots & 0 \\
\frac{q_2}{q_i q_i - 1} & V_2 & \frac{q_2}{q_i q_i - 1} & 0 & \ldots & 0 \\
0 & \frac{q_3}{q_i q_i - 1} & V_3 & \frac{q_3}{q_i q_i - 1} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \frac{q_{n-1}}{q_{n-1} - 1} \\
\end{bmatrix},
\]

(A.2)

where \( V_k := \frac{1-q_k^2 q_{k-1}^2}{(q_k^2 - 1)(q_k^2 - 1)} = \frac{1}{1-q_k^2 - 1} + \frac{q_k^2}{1-q_k^2}, \) \( k = 2, \ldots, n-1. \) Hence, for \( M_\theta(n) = \mathbf{1}_n^T C^{-1}(n, r) \mathbf{1}_n \) we obtain

\[
M_\theta(n) = 1 - 2q_1 \frac{1}{1-q_1^2} + \frac{1}{1-q_{n-1}^2} + \sum_{i=2}^{n-1} \left( \frac{2q_i}{q_i^2 - 1} + \frac{1-q_i^2 q_i^2}{(q_i^2 - 1)(q_i^2 - 1)} \right) = 1 + \sum_{i=1}^{n-1} \frac{1-q_i}{1+q_i}. \quad (A.3)
\]

Now, consider reformulation

\[
M_\theta(n) = 1 + \sum_{i=1}^{n-1} g(\alpha d_i + \beta \delta_i), \quad \text{where} \quad g(x) := \frac{1 - \exp(-x)}{1 + \exp(-x)}.
\]

As \( g(x) \) is a concave function of \( x, \) by Proposition C1 of Marshall and Olkin (1979), \( M_\theta(n) \) is a Schur-concave function of \( \alpha d_i + \beta \delta_i, \) \( i = 1, 2, \ldots, n-1. \) In this way \( M_\theta(n) \) attains its maximum when \( \alpha d_i + \beta \delta_i = \lambda / (n-1), \) \( i = 1, 2, \ldots, n-1, \) where \( \lambda \) is the “skewed size” of the design rectangle. Hence, an equidistant design is the D-optimal for the parameter \( \theta. \) 

A.2 Proof of Theorem 2

By symmetry it suffices to prove

\[
M_\alpha(n) = \frac{1}{2} \text{tr} \left\{ C^{-1}(n, r) \frac{\partial C(n, r)}{\partial \alpha} C^{-1}(n, r) \frac{\partial C(n, r)}{\partial \alpha} \right\} = \sum_{i=1}^{n-1} \frac{d_i q_i q_i (1+q_i^2)}{(1-q_i^2)^2}. \quad (A.4)
\]

For \( n = 2 \) equation (A.4) holds trivially. Assume also that (A.4) is true for some \( n \) and we are going to show it for \( n+1. \) Let \( \mathbf{0}_{k, \ell} \) be the \( k \times \ell \) matrix of zeros and let

\[
\Delta(n) := \left( -(d_1 + d_2 + \ldots + d_n)q_1 q_2 \ldots q_n, -(d_2 + d_3 + \ldots + d_n)q_2 q_3 \ldots q_n, \ldots, -d_n q_n \right)^T.
\]

With the help of representation (A.1) one can easily see that

\[
\frac{\partial C(n + 1, r)}{\partial \alpha} = \begin{bmatrix}
\partial C(n, r) \\
\Delta(n) \\
\Delta^\top(n) \\
0
\end{bmatrix},
\]

while (A.2) implies

\[
C^{-1}(n + 1, r) = \begin{bmatrix}
C^{-1}(n, r) & \mathbf{0}_{n, 1} \\
\mathbf{0}_{1, n} & 0
\end{bmatrix} + \begin{bmatrix}
\Lambda_{1,1}(n) & \Lambda_{1,2}(n) \\
\Lambda_{2,1}(n) & (1-q_n^2)^{-1}
\end{bmatrix},
\]

\[11\]
where
\[ \Lambda_{1,1}(n) := \begin{bmatrix} 0_{n-1,n-1} & 0_{n-1,1} \\ 0_{1,n-1} & \frac{q_n^2}{1 - q_n^2} \end{bmatrix} \quad \text{and} \quad \Lambda_{1,2}(n) := \begin{bmatrix} 0_{n-1,1} \\ -\frac{q_n^2}{1 - q_n^2} \end{bmatrix}. \]

In this way
\[
C^{-1}(n + 1, r) \frac{\partial C(n + 1, r)}{\partial \alpha} = \begin{bmatrix} C^{-1}(n, r) \frac{\partial C(n, r)}{\partial \alpha} & C^{-1}(n, r) \Delta(n) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K_{1,1}(n) & K_{1,2}(n) \\ K_{2,1}(n) & K_{2,2}(n) \end{bmatrix},
\]
with
\[
K_{1,1}(n) := \begin{bmatrix} 0_{n-1,n} \\ -\frac{q_n}{1 - q_n^2} (\Delta^\top(n) - (q_n \Delta^\top(n - 1), 0)) \end{bmatrix}, \quad K_{1,2}(n) := \begin{bmatrix} 0_{n-1,1} \\ -\frac{q_n^2}{1 - q_n^2} \end{bmatrix},
\]
\[
K_{2,1}(n) := \frac{1}{1 - q_n^2} (\Delta^\top(n) - (q_n \Delta^\top(n - 1), 0)), \quad K_{2,2}(n) := \frac{\delta_n q_n^2}{1 - q_n^2}.
\]

Hence,
\[
M_\alpha(n + 1) = M_\alpha(n) + \text{tr} \left\{ C^{-1}(n, r) \frac{\partial C(n, r)}{\partial \alpha} K_{1,1}(n) \right\} + \text{tr} \left\{ C^{-1}(n, r) \Delta(n) K_{2,1}(n) \right\} + \frac{1}{2} \text{tr} \left\{ K_{1,1}^2(n) \right\} + K_{2,1}(n) K_{1,2}(n) + \frac{1}{2} K_{2,2}^2(n).
\]

After long but straightforward calculations one can get
\[
\text{tr} \left\{ C^{-1}(n, r) \frac{\partial C(n, r)}{\partial \alpha} K_{1,1}(n) \right\} = 0, \quad \text{tr} \left\{ C^{-1}(n, r) \Delta(n) K_{2,1}(n) \right\} = \frac{\delta_n q_n^2}{1 - q_n^2},
\]
\[
\text{tr} \left\{ K_{1,1}^2(n) \right\} = K_{2,1}(n) K_{1,2}(n) = K_{2,2}(n) = \frac{\delta_n^2 q_n^4}{(1 - q_n^2)^2},
\]
so \([A.5]\) implies
\[
M_\alpha(n + 1) = M_\alpha(n) + \frac{\delta_n^2 q_n^2(1 + q_n^2)}{(1 - q_n^2)^2},
\]
which completes the proof. \(\square\)

**A.3 Proof of Theorem 3**

Consider first the case when we are interested in the estimation of one of the parameters \(\alpha\) and \(\beta\) and other parameters are considered as nuisance. If \(\alpha\) is the parameter of interest then according to \([5,2]\) the Fisher information on \(\alpha\) equals
\[
M_\alpha(n) = \sum_{i=1}^{n-1} F(d_i, \delta_i),
\]
where
\[
F(d, \delta) := \frac{d^2 q^2 (1 + q^2)}{(1 - q^2)^2} \geq 0, \quad \text{with} \quad q := \exp(-\alpha d - \beta \delta).
\]

Due to the separation of the different data points in the expression of \(M_\alpha(n)\) it suffices to consider the properties of the function \(F(d, \delta)\) for \(d, \delta \geq 0, \ d \delta \neq 0\). Obviously,
\[
\frac{\partial F(d, \delta)}{\partial d} = \frac{2d q^2 ((1 - q^4) - \alpha d (1 + 3 q^2))}{(1 - q^2)^3} \quad \text{and} \quad \frac{\partial F(d, \delta)}{\partial \delta} = \frac{-2 \beta d^2 q^2 (1 + 3 q^2)}{(1 - q^2)^3}, \quad \text{for} \quad d, \delta \geq 0, \ d \delta \neq 0.
\]
so the critical points of \( F(d, \delta) \) are \((0, \delta)\), \( \delta > 0 \). However, at these points the determinant of the Hessian is zero and for \( \delta > 0 \) we have \( F(0, \delta) = 0 \). Moreover, short calculation shows that if \( d\delta \neq 0 \) then \( F(d, \delta) < 1/(2\alpha^2) \) and \( \lim_{d, \delta \to 0} F(d, \delta) = 1/(2\alpha^2) \). Hence, the supremum of \( F \) is reached at \( d = \delta = 0 \), but in our context, \( d_i \neq 0, \delta_i \neq 0 \) for \( i = 1, 2, \ldots, n - 1 \).

A similar result can be obtained in the case when \( \beta \) is the parameter of interest.

Now, consider the case when both \( \alpha \) and \( \beta \) are unknown. According to \((5.1)\) and \((5.2)\) the corresponding objective function to be maximized is

\[
\Phi(d_1, \ldots, d_{n-1}, \delta_1, \ldots, \delta_{n-1}) = \det(M_r(n)) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (d_i^2 \delta_j^2 - d_i \delta_i d_j \delta_j) \frac{q_i^2 (1 + q_i^2) q_j^2 (1 + q_j^2)}{(1 - q_i^2)^2 (1 - q_j^2)^2} \quad (A.7)
\]

\[
= \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} (d_i \delta_j - d_j \delta_i)^2 \frac{q_i^2 (1 + q_i^2) q_j^2 (1 + q_j^2)}{(1 - q_i^2)^2 (1 - q_j^2)^2} \geq 0.
\]

Obviously, for an equidistant design, where \( d_1 = \ldots = d_{n-1} \) and \( \delta_1 = \ldots = \delta_{n-1} \), the above function equals 0, that is this design cannot be optimal. Further,

\[
\frac{\partial \Phi}{\partial d_1} = \frac{2q_i^2 (1 + q_i^2)}{(1 - q_i^2)^2} (d_1 \tilde{M}_\beta(1) - \delta_1 \tilde{M}_\alpha(1) - \delta_1 M_\alpha(1) - 2d_1 \delta_1 \tilde{M}_\alpha(1)), \quad (A.8)
\]

\[
\frac{\partial \Phi}{\partial \delta_1} = \frac{2q_i^2 (1 + q_i^2)}{(1 - q_i^2)^2} (d_1 \tilde{M}_\alpha(1) - d_1 \tilde{M}_\alpha(1) - \delta_1 \tilde{M}_\beta(1) - 2d_1 \delta_1 \tilde{M}_\beta(1)), \quad (A.9)
\]

where \( \tilde{M}_\alpha(k), \tilde{M}_\beta(k) \) and \( \tilde{M}_\alpha(1), \tilde{M}_\beta(1) \), \( k = 1, 2, \ldots, n - 2 \), are the elements of the Fisher information matrix on \( r = (\alpha, \beta)^T \) corresponding to observations \( \{Y(s_i, \epsilon_i), i = k, k + 1, \ldots, n\} \) (see \((5.1)\)), that is

\[
\tilde{M}_\alpha(k) = \sum_{i=k+1}^{n} d_i^2 q_i^2 (1 + q_i^2) (1 - q_i^2)^2, \quad \tilde{M}_\beta(k) = \sum_{i=k+1}^{n} \delta_i^2 q_i^2 (1 + q_i^2) (1 - q_i^2)^2, \quad \tilde{M}_\alpha(1) = \sum_{i=k+1}^{n} d_i \delta_i q_i^2 (1 + q_i^2) (1 - q_i^2)^2,
\]

while for \( i = 2, 3, \ldots, n - 1 \) we have

\[
\frac{\partial \Phi}{\partial d_i} = \frac{2q_i^2 (1 + q_i^2)}{(1 - q_i^2)^2} (d_i \tilde{M}_\beta(i) - d_i \tilde{M}_\alpha(i) - \delta_i \tilde{M}_\alpha(i) - \delta_i M_\alpha(i) - 2d_i \delta_i M_\alpha(i)), \quad (A.9)
\]

\[
\frac{\partial \Phi}{\partial \delta_i} = \frac{2q_i^2 (1 + q_i^2)}{(1 - q_i^2)^2} (d_i \tilde{M}_\alpha(i) - d_i M_\alpha(i) - \delta_i \tilde{M}_\beta(i) - \delta_i \tilde{M}_\alpha(i) - 2d_i \delta_i M_\beta(i)).
\]

Solving recursively the equations \((A.9)\) under the assumption \( d_i \delta_i \neq 0, i = 1, 2, \ldots, n - 1 \), for the critical points of \( \Phi \) we obtain relations

\[
\frac{d_i}{d_1} = \frac{\delta_i}{\delta_1} =: c_i > 0, \quad \text{that is} \quad d_i = c_i d_1, \quad \delta_i = c_i \delta_1, \quad i = 1, 2, \ldots, n - 1. \quad (A.10)
\]

These solutions also solve \((A.8)\) and short calculations show that for all \( d_1, \delta_1, c_1, \ldots, c_{n-1} > 0 \) we have \( \Phi(d_1, c_1 d_1, \ldots, c_{n-1} d_1, \delta_1, c_1 \delta_1, \ldots, c_{n-1} \delta_1) = 0 \). Hence, critical points determined by \((A.10)\) are minimum points of \( \Phi \). Thus, the maximum of \( \Phi(d_1, \ldots, d_{n-1}, \delta_1, \ldots, \delta_{n-1}) \) can only be attained at the boundary points, but in our context, \( d_i \notin \{0, b_1 - a_1\} \) and \( \delta_i \notin \{0, b_2 - a_2\} \).

**A.4 Proof of Theorem 4**

As \( \det(M(n)) = M_\theta(n) \det(M_r(n)) = M_\theta(n) \Phi \), according to \((A.8)\) and \((A.7)\), for unknown parameters \( \alpha \), \( \beta \) and \( \theta \) the objective function to be maximized is

\[
\Psi(d_1, \ldots, d_{n-1}, \delta_1, \ldots, \delta_{n-1}) = \left( \frac{2}{1 + q_1} + \sum_{i=2}^{n-1} \frac{1 - q_i}{1 + q_i} \right) \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} (d_i \delta_j - d_j \delta_i)^2 \frac{q_i^2 (1 + q_i^2) q_j^2 (1 + q_j^2)}{(1 - q_i^2)^2 (1 - q_j^2)^2} \quad (A.11)
\]
For $d_1 = \ldots = d_{n-1}$ and $\delta_1 = \ldots = \delta_{n-1}$, we have $\Phi(d_1, \ldots, d_{n-1}, \delta_1, \ldots, \delta_{n-1}) = 0$, thus an equispaced design cannot be optimal.

Further, 
\[
\frac{\partial \Psi}{\partial d_i} = M_\theta(n) \frac{\partial \Phi}{\partial d_i} - \frac{2\alpha q_i}{(1 + q_i)^2} \Phi \quad \text{and} \quad \frac{\partial \Psi}{\partial \delta_i} = M_\theta(n) \frac{\partial \Phi}{\partial \delta_i} - \frac{2\beta q_i}{(1 + q_i)^2} \Phi,
\]
where the expressions for $\partial \Phi/\partial d_i$ and $\partial \Phi/\partial \delta_i$ are given by (A.10). Solving the above equations for the critical points of $\Psi$ we obtain the relations (A.10). However, $\Psi(d_1, c_1 d_1, \ldots, c_{n-1} d_1, \delta_1, c_1 \delta_1, \ldots, c_{n-1} \delta_1) = 0$, thus the function $\Psi$ attains its minimum at the points determined by (A.10).

\[ \square \]

A.5 Proof of Theorem 5

Consider first $M_\theta(n)$ and according to (4.1)
\[
M_\theta(n) = M_\theta(n; r_1, r_2) = 1 + \sum_{i=1}^{n-1} f(d_i(r_1), \delta_i(r_2)), \quad \text{where} \quad f(d, \delta) = \frac{e^{\alpha d + \beta \delta} - 1}{e^{\alpha d + \beta \delta} + 1}.
\]

Obviously, for $r_1 = 1$, $r_2 = 1$, the geometric progression design corresponds to the equidistant design, which is optimal for the estimation of the trend parameter. Let $0 < r_1, r_2 < 1$ and one has to prove that
\[
\frac{\partial M_\theta(n; r_1, r_2)}{\partial r_1} = \sum_{i=1}^{n-1} \frac{\partial f(d_i(r_1), \delta_i(r_2))}{\partial d} \frac{\partial d_i(r_1)}{\partial r_1} > 0 \quad \text{and} \quad \frac{\partial M_\theta(n; r_1, r_2)}{\partial r_2} = \sum_{i=1}^{n-1} \frac{\partial f(d_i(r_1), \delta_i(r_2))}{\partial \delta} \frac{\partial \delta_i(r_2)}{\partial r_2} > 0.
\]

Now,
\[
\frac{\partial f(d, \delta)}{\partial d} = \frac{2\alpha e^{\alpha d + \beta \delta}}{(e^{\alpha d + \beta \delta} + 1)^2} > 0,
\]
which, as a function of $\alpha d + \beta \delta$, is strictly decreasing. In this way we can use the arguments of Proof of Theorem 5.1 of Zagoraiou and Baldi Antognini (2009), where a one-dimensional OU process is investigated.

From $d_i(r) = \delta_i(r) = \frac{(1-r)^{i-1}}{1-r}, \quad i > 1$, we obtain $d_1(r_1) > \ldots > d_{n-1}(r_1)$ and $\delta_1(r_2) > \ldots > \delta_{n-1}(r_2)$ which implies
\[
0 < \frac{\partial f(d_1(r_1), \delta_1(r_2))}{\partial d} < \ldots < \frac{\partial f(d_{n-1}(r_1), \delta_{n-1}(r_2))}{\partial d}.
\]
Further,
\[
\frac{\partial d_i(r_1)}{\partial r_1} = \frac{r_1^i}{(r_1 - r_1^n)^2} \left( r_1^{n-1}(n-i) - r_1^n(n-i-1) + i - 1 - r_1 i \right), \quad i = 1, 2, \ldots, n-1,
\]
and due to $\sum_{i=1}^{n-1} d_i(r_1) = 1$, $0 < r_1 \leq 1$, we have $\sum_{i=1}^{n-1} \frac{\partial d_i(r_1)}{\partial r_1} = 0$. Now, let $j$ be the smallest integer such that
\[
\frac{\partial d_j(r_1)}{\partial r_1} \geq 0 \quad \text{for} \quad i = j, \ldots, n-1,
\]
and according to Zagoraiou and Baldi Antognini (2009) such integer exists. Then
\[
\sum_{i=1}^{n-1} \frac{\partial f(d_i(r_1), \delta_i(r_2))}{\partial d} \frac{\partial d_i(r_1)}{\partial r_1} = \sum_{i=1}^{j-1} \frac{\partial f(d_i(r_1), \delta_i(r_2))}{\partial d} \frac{\partial d_i(r_1)}{\partial r_1} + \sum_{j=1}^{n-1} \frac{\partial f(d_j(r_1), \delta_j(r_2))}{\partial d} \frac{\partial d_j(r_1)}{\partial r_1} > \frac{\partial f(d_j(r_1), \delta_j(r_2))}{\partial d} \sum_{i=1}^{n-1} \frac{\partial d_i(r_1)}{\partial r_1} = 0.
\]
The positivity of the other partial derivative of $M_\theta(n; r_1, r_2)$ can be proved exactly in the same way.

Finally, the second statement of the theorem is a direct consequence of (A.7), since if $r_1 = r_2$ then for all $i = 2, 3, \ldots, n-1$ and $j = 1, 2, \ldots, i-1$ we have $d_i(r_1)\delta_j(r_2) - d_j(r_1)\delta_i(r_2) = 0$. 

\[ \square \]
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