Precise charm to strange mass ratio and light quark masses from full lattice QCD

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By using a single formalism to handle charm, strange and light valence quarks in full lattice QCD for the first time, we are able to determine ratios of quark masses to 1%. For \( m_s/m_a \) we obtain 11.85(16), an order of magnitude more precise than the current PDG average. Combined with 1% determinations of the charm quark mass now possible this gives \( m_s(2\text{GeV}) = 92.4(1.5) \text{ MeV} \). The MILC result for \( m_s/m_l = 27.2(3) \) yields \( m_l(2\text{GeV}) = 3.46(7) \text{ MeV} \) for the average of \( u \) and \( d \) quark masses.

Introduction. – The masses of \( u \), \( d \) and \( s \) quarks are some of the least well-known parameters of the Standard Model. Even the most inaccurate lepton mass (that of the \( \tau \)) is known to better than 0.01% and yet errors on light quark masses of 30% are quoted in the Particle Data Tables [1]. The reason for the mismatch is the confinement property of the strong force that obscures the connection between the properties of the quark constituents and the hadron physics that is accessible to experiment. To make this connection requires accurate calculations in QCD and accurate experimental results for appropriate hadronic quantities. A method particularly well-suited to this is lattice QCD. Here we will demonstrate its use by determining \( m_c/m_s \) to 1% and obtaining as a result 1.5% errors for light quark masses, which brings them almost into line with those of heavy quarks.

Heavy quark masses, \( m_Q \), can be determined accurately because \( \alpha_s(m_Q) \) is relatively small. 1% errors for charm and bottom quark masses have recently become possible using \( \mathcal{O}(\alpha_s^3) \) calculations in QCD perturbation theory for the heavy quark vacuum ‘bubble’ [2] and therefore for the energy-derivative (or time) moments of correlation functions for a heavy quark-antiquark pair at zero momentum. Since the scale of \( \alpha_s \) is naturally related to the relevant heavy quark mass, the expressions can be evaluated accurately. To extract the quark mass the perturbative result is compared to a nonperturbative determination containing information from experiment. For a \( 1^- \) \( QQ \) configuration moments of the experimentally measured cross-section for \( (e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons}) \) can be used after isolating the heavy quark contribution and using dispersion relations [3]. Alternatively the time-moments for heavy quark current-current correlation functions of various \( J^{PC} \) can be directly determined in lattice QCD calculations that have been tuned so that a charmonium or bottomonium mass agrees with experiment [4, 5]. The time moments must be extrapolated to the zero lattice spacing (continuum) limit before the comparison to QCD perturbation theory. These two methods give results that agree, with 1% errors for \( m_c(3\text{GeV}) \) in the \( \overline{MS} \) scheme. The more traditional ‘direct’ lattice QCD method, although somewhat less accurate, also gives results in good agreement [6]. We can conclude from this that \( m_c \) is now accurately known.

The strange quark mass, \( m_s \), being much smaller, cannot be determined this way and is poorly known at present. Instead of a direct determination of \( m_s \), however, we can use the leverage of an accurate result for the ratio \( m_c/m_s \) combined with the accurate \( m_c \) above [2]. But simple ratios of hadron mass differences give unreliable estimates of \( m_c/m_s \). Two such estimates:

\[
\frac{m(B_c) - m(B_u)}{m(B_s) - m(B_d)} = 11; \frac{m(\Sigma_c) - m(N)}{m(\Sigma) - m(N)} = 6
\]

(1)
differ by almost a factor of 2. The ratio of \( m_s/m_l \) (where \( m_l = (m_u + m_d)/2 \)) is known to about 10% from ratios of squared masses of \( K \) and \( \pi \) mesons using SU(3) chiral perturbation theory [1]. Clearly neither ratio is determined well enough this way to provide the accuracy we need, because the relationship between hadron mass and well-defined running quark mass is more complicated than these simple ratios must assume.

Lattice QCD, on the other hand, can give very accurate results for the ratio of two quark masses but only if the same formalism is used for both quarks. This has already been used to give accurate results for \( m_s/m_l \), although neither \( m_s \) nor \( m_l \) is very well determined. Here, for the first time, we give an accurate result for \( m_c/m_s \) by using the same formalism for charm, strange and light quarks and this enables us to cascade the accuracy of the heavy quark mass down to the light quarks.

The Lattice QCD calculation. – Lattice QCD gives
direct access to quark masses through the lattice QCD Lagrangian. Tuning of the masses is done by calculating an appropriate hadron mass and adjusting the quark mass until the hadron mass agrees with experiment. Experimental measurements of appropriate hadron masses are extremely accurate in most cases, with errors at the level of tenths or hundredths of a percent. To make maximum use of this precision we need to calculate the hadron mass in lattice QCD with small statistical and systematic errors. In particular it requires the full effect of sea quarks in the hadron to be included. This is now possible in lattice QCD [8]. Fixing the four quark masses \((m_u, m_s, m_c, m_b)\) from four ‘gold-plated’ hadrons \((\pi, K, \eta_c, \Upsilon)\) enables other quantities to be calculated with errors of a few percent and agreement with experiment is obtained [8,9]. This is an important test that QCD, with only one scale parameter and one mass parameter per quark flavor, describes the full range of hadron physics consistently.

The lattice quark mass is a perfectly well-defined running quark mass. However, it is scheme-dependent and so varies with the discretisation of the Dirac equation used in the lattice calculation. For wider applicability it is more useful to convert the lattice quark mass to a standard continuum scheme such as \(\overline{M}\overline{S}\). This renormalization has been a major source of systematic error in previous determinations of light and strange quark masses. The best existing result for \(m_u(2\text{GeV})\), with a 7% error, uses the direct method of converting the tuned quark mass in the lattice QCD Lagrangian to the \(\overline{M}\overline{S}\) scheme using \(\alpha_s^2\) lattice QCD perturbation theory [10]. The error is dominated by the error in the renormalization and it is the error that we will remove here, by instead determining \(m_c/m_s\) accurately. The Highly Improved Staggered Quark action [11,12] allows us to use the same discretization of QCD for both charm and strange quarks because it is a fully relativistic ‘light quark’ action that can also be used for charm quarks. Then the mass renormalization factor cancels in the quark mass ratio.

We work with eight different ensembles of gluon field configurations provided by the MILC collaboration. These include the effect of \(u, d\) and \(s\) sea quarks using the improved staggered quark (asqtad) formalism using the fourth root ‘trick’. This procedure, although ‘ugly’, appears to be a valid discretization of QCD [13,16]. Tests include studies of the Dirac operator and comparisons to effective field theories. Configurations are available with large spatial volumes (> 2.4(fm)\(^3\)) at multiple values of the light sea masses (using \(m_u = m_d = m_l\)) and for a wide range of values of the lattice spacing, \(a\). We use configurations at five values of \(a\) between 0.15 fm and 0.05 fm with parameters as listed in Table [1].

On these configurations we have calculated quark propagators for charm quarks, strange quarks and light quarks (again \(m_u = m_d = m_l\)) using the HISQ action. The numerical speed of HISQ means that we have been able to use several nearby quark masses for charm and strange to allow accurate interpolation to the correct values. Table [1] gives masses for the goldstone pseudoscalars made from either a charm quark-antiquark pair or a strange one (the \(\eta_c\) and the \(\eta_s\), which are used for tuning. In the charm case, as well as the quark mass, we list the coefficient of the ‘Naik’ term in the HISQ action that corrects for discretisation errors through \((am_{0c})^4\). The quark propagators are generated from random wall sources and the goldstone mesons have good signal/noise properties so the meson masses can be determined to high precision using a standard multi-exponential fit [17].

The meson masses can be converted to physical units with a determination of the lattice spacing. On an en-

| Set \(\beta\) | \(r_1/a\) | \(a m_{0c}\) | \(a m_{0s}\) | \(L/a\) | \(T/a\) | \(N_{conf} \times N_t\) |
|---|---|---|---|---|---|---|
| 1 | 6.572 | 2.152(5) | 0.0097 | 0.0484 | 16 | 48 | 631 \times 2 |
| 2 | 6.586 | 2.138(4) | 0.0194 | 0.0484 | 16 | 48 | 631 \times 2 |
| 3 | 6.76 | 2.647(3) | 0.005 | 0.05 | 24 | 64 | 678 \times 2 |
| 4 | 6.76 | 2.618(3) | 0.01 | 0.05 | 20 | 64 | 595 \times 2 |
| 5 | 7.09 | 3.699(3) | 0.0062 | 0.031 | 28 | 96 | 566 \times 4 |
| 6 | 7.11 | 3.712(4) | 0.0124 | 0.031 | 28 | 96 | 265 \times 4 |
| 7 | 7.46 | 5.296(7) | 0.0036 | 0.018 | 48 | 144 | 201 \times 2 |
| 8 | 7.81 | 7.115(20) | 0.0028 | 0.014 | 64 | 192 | 208 \times 2 |

TABLE I: Ensembles (sets) of MILC configurations used, with gauge coupling \(\beta\), size \(L^2 \times T\) and sea masses \((x\,\text{ tadpole parameter} \, u_0)\) \(m_{0u}\) and \(m_{0d}\). The lattice spacing values in units of \(r_1\) after ‘smoothing’ are given in column 3 [14]. Column 8 gives the number of configurations and time sources per configuration used for calculating correlators.

| Set \(\beta\) | \(r_0/a\) | \(a m_{0u}\) | \(a m_{0d}\) | \(a m_{0s}\) |
|---|---|---|---|---|
| 1 | 0.59 | 0.865 | 0.295 | 0.975 |
| 2 | 0.75 | 0.865 | 0.306 | 0.975 |
| 3 | 0.92 | 0.85 | 0.306 | 0.975 |
| 4 | 0.28 | 0.95 | 0.306 | 0.975 |
| 5 | 0.24 | 1.0 | 0.306 | 0.975 |
| 6 | 0.19 | 1.05 | 0.306 | 0.975 |
| 7 | 0.15 | 1.1 | 0.306 | 0.975 |
| 8 | 0.12 | 1.15 | 0.306 | 0.975 |

TABLE II: Results for the masses in lattice units of the goldstone pseudoscalars made from valence HISQ charm or strange quarks on the different MILC ensembles enumerated in Table I. Columns 2 and 3 give the corresponding bare charm quark mass, and Naik coefficient respectively. Column 6 gives the bare strange quark mass \((\epsilon = 0 \text{ in that case}).
m/\pi \text{ must be determined by relating its properties to those of other flavor neutral combinations to show our extrapolation of the resulting tuned \( m_c/m_s \) to the continuum limit. Blue points with error bars are from a simple interpolation, separately for each ensemble, to the correct \( m_c/m_s \), and are shown for illustration.}

ensemble by ensemble basis this is taken from a parameter in the heavy quark potential called \( r_1 \). Values for \( r_1/a \) determined by the MILC collaboration \cite{14} are given in Table \( \ref{table:1} \). They have errors of 0.3-0.5%. The physical value for \( r_1 \) must then be obtained by comparing to experimentally known quantities and we use the value 0.3133(23) fm obtained from a set of four such quantities, tested for consistency in the continuum limit \cite{18, 19}.

Using the information about meson masses that we have on each ensemble we can interpolate to the correct ratio for \( am_{0c} \) and \( am_{0s} \) using appropriate continuum values for the masses of the \( \eta_c \) and \( \eta_s \). We correct the experimental value of \( m_{0c} \) of 2.9803 GeV to \( m_{0c,\text{phys}} = 2.9852(34) \) GeV. This allows for electromagnetic effects (2.4 MeV) \cite{18} and \( \eta_c \) annihilation to gluons (2.5MeV) \cite{11}, both of which are missing from our calculation, so increasing the \( \eta_c \) mass. We take a 50% error on each of these corrections and also increase the experimental error to 3 MeV to allow for the spread of results from different \( \eta_c \) production mechanisms \cite{1}. Since the total shift is only around 0.2% of the \( \eta_c \) mass it has a negligible effect as can be seen from our error budget below.

The \( \eta_c \) is not a physical particle in the real world because of mixing with other flavor neutral combinations to make the \( \eta \) and \( \eta' \). However, in lattice QCD, the particle calculated (as here) from only ‘connected’ quark propagators does not mix and is a well-defined meson. Its mass must be determined by relating its properties to those of mesons such as the \( \pi \) and \( K \) that do appear in experiment. From an analysis of the lattice spacing and \( m_l \)-dependence of the \( \pi \), \( K \), and \( \eta_c \) masses we conclude that the value of the \( \eta_c \) mass in the continuum and physical \( m_l \) limits is 0.6858(40) GeV \cite{18}.

The connection between the \( MS \) mass at a scale \( \mu \) and the lattice bare quark mass is given by \cite{10, 20}:

\[
\bar{m}(\mu) = \frac{am_0}{a} Z_m(\mu a, m_0 a),
\]

\[
Z_m = 1 + \alpha_s(-\frac{2}{\pi} \log(\mu a) + C + b(\mu a)^2 + \ldots) + \ldots.
\]

From these two equations it is clear that

\[
\frac{\bar{m}_c(\mu)}{\bar{m}_s(\mu)} = \frac{am_{0c}}{am_{0s}}|_{\text{phys}},
\]

where \( \text{phys} \) denotes extrapolation to the continuum limit and physical sea quark mass limit.

On each ensemble the ratios we have for \( am_{0c}/am_{0s} \) then differ from the physical value because of three effects: mistuning from the correct physical meson mass; finite \( a \) effects that need to be extrapolated away and effects because the sea light quark masses are not correct. We incorporate these into our fitting function:

\[
\frac{m_{0c}}{m_{0s}}|_{\text{lat}} = \frac{m_{0c}}{m_{0s}}|_{\text{phys}} \times \left( 1 + d_{\text{sea}} \delta m_{\text{sea}} \right) \times \left( 1 + \sum_{i,j,k,l} c_{ijkl} \delta_i \delta_j \left( \frac{am_{0s}}{2} \right)^{2k} (am_{0c})^{2l} \right).
\]

\[
\delta_c = \frac{m_{0c,MC} - m_{0c,\text{phys}}}{m_{0c,\text{phys}}}; \delta_s = \frac{m_{0s,MC} - m_{0s,\text{phys}}}{m_{0s,\text{phys}}}
\]

are the measures of mistuning, where \( MC \) denotes lattice values converted to physical units. The last bracket fits the finite lattice spacing effects as a power series in even powers of \( a \). These can either have a scale set by \( m_c \) (for which we use \( am_{0c}/2 \)) or by \( A_{QCD} \) (for which we use \( am_{0s} \)). \( i,j,k,l \) all start from zero and are varied in the ranges: \( i,j \leq 3, k \leq 6, l \leq 2 \) with \( i+j+k+l \leq 6 \). Doubling any of the upper limits has negligible effect on the result. The prior on \( c_{ijkl} \) is set to 0(1). \( \delta m_{\text{sea}} \) is the total difference between the sea-quark masses used in the simulation and the correct value for \( 2m_l + ms \) \cite{18}. This has a tiny effect and we simply use a linear term (adding higher orders has negligible effect). The prior for \( d_{\text{sea}} \) is 0.0(1). Figure \( \ref{fig:1} \) shows the results of the fit, giving \( m_c/m_s \) in the continuum limit as 11.85(16) (\( \chi^2/\text{dof} = 0.42 \)). The error budget is given in Table \( \ref{table:1} \). The ratio of \( m_s/m_l \) is known to 1% from lattice QCD as a byproduct of standard chiral extrapolations of \( m_s^2 \) and \( m_l^2 \) to the physical point \cite{21}. MILC quote 27.2(3) using asqtad quarks \cite{14}. Our HISQ analysis in \cite{12} gave a result in agreement at 27.8(3), using a Bayesian fit to a function including terms from chiral perturbation theory up to third order in \( m_l \) and allowing for discretisation errors up to and including \( a^4 \) and for mixed terms (i.e \( m_l \)-dependent discretisation errors). A full error budget is given in Table \( \ref{table:1} \).
Our $m_c/m_s$ can be used with any value for $m_c$ to give $m_s$. The best existing result \cite{20} (converted from $n_f=4$ to $3$) is $\bar{m}_{c}^{(3)}(2.0\text{GeV}) = 1.095(11) \text{ GeV}$ or $\bar{m}_{c}^{(3)}(3.0\text{GeV}) = 0.990(10) \text{ GeV}$. Dividing by $11.85(16)$ gives $\bar{m}_{s}^{(3)}(2.0\text{GeV}) = 92.4(1.5) \text{ MeV}$ and $\bar{m}_{s}^{(3)}(3.0\text{GeV}) = 83.5(1.4) \text{ MeV}.$

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline $m_c/m_s$ & $m_s/m_l$ & $m_s/m_l$ & $m_s/m_l$
\hline overall $r_1$ uncertainty & 0.4% & 0.1% & \\
r$_1/a$ uncertainties & 0.2 & & \\
continuum $M_{0_c}$ & 0.2 & & \\
continuum $M_{0_s}$ & 1.1 & & \\
Finite volume & - & 0.3 & \\
$a^2$ extrapolation, $m_q$ interpols & 0.4 & 0.8 & \\
sea-quark mass extrapolation & 0.0 & 0.2 & \\
statistical errors & 0.3 & 0.4 & \\
Total & 1.3% & 1.0% & \\
\hline
\end{tabular}
\caption{Error budgets for $m_c/m_s$ and $m_s/m_l$.}
\end{table}

Using the MILC values for $m_s/m_l$ and $m_{u}/m_d$ \cite{20}, we can then obtain: $\bar{m}_{l}^{(3)}(2.0\text{GeV}) = 3.40(7) \text{ MeV}$ and $\bar{m}_{l}^{(3)}(3.0\text{GeV}) = 3.07(6) \text{ MeV}$; $\bar{m}_{u}^{(3)}(2.0\text{GeV}) = 2.01(14) \text{ MeV}$ and $\bar{m}_{u}^{(3)}(2.0\text{GeV}) = 4.79(16) \text{ MeV}.$ The values for all four quark masses are plotted in Figure 2 in comparison to the current evaluations from the Particle Data Tables \cite{1}.

Thus our high accuracy on $m_c/m_s$ allows us to leverage 2% accurate values for $m_s$ and $m_l$ that are completely nonperturbative in lattice QCD, for the first time. Our $m_s$ mass is higher, by around 1σ, than our previous value of $\bar{m}_s(2\text{GeV}) = 87(6) \text{ MeV}$ which used 2-loop lattice QCD perturbation theory \cite{10}. Then the error was dominated by unknown $\alpha_s^3$ terms. Our new result, which does not have this limitation, has an error almost five times smaller. Our new error is almost an order of magnitude smaller than other lattice QCD results from full QCD \cite{22,23}. These use direct methods of converting the lattice mass to the $\overline{\text{MS}}$ mass, and have 10% errors.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Our results for the 4 lightest quark masses compared to the current PDG evaluations (shaded bands) \cite{1}. Each mass is quoted in the $\overline{\text{MS}}$ scheme at its conventional scale: 2 GeV for $u, d, s$ ($n_f=3$); $m_c$ for $c$ ($n_f=4$).}
\end{figure}

Note that \cite{20} determined $m_c$ using an earlier and less accurate value of $r_1(0.321(5) \text{ fm}).$ We have checked that using the new value does not change $m_s$ significantly.

The mass parameter $a_{0c}$ in equation \cite{3} should really be the perturbative pole mass at tree level \cite{11}, which differs from $a_{0c}$ by $\approx 0.04(a_{0c})^3$, but gives the same extrapolated $m_c/m_s$ ratio.

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