On a Lagrangian formulation of the 1D Green-Naghdi system

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Abstract

In this paper we consider the 1D Green-Naghdi system. This system describes the evolution of water waves over a flat bottom in the shallow water regime in terms of the surface height \( h \) and the horizontal velocity \( u \). We give a Lagrangian formulation of the 1D Green-Naghdi system on a Sobolev type diffeomorphism group. As an application of this formulation we prove local well-posedness for \( (h, u) \) in the Sobolev space \( (1 + H^s(\mathbb{R})) \times H^{s+1}(\mathbb{R}) \), \( s > 1/2 \). This improves the local well-posedness range for the 1D Green-Naghdi system.

1 Introduction

The 1D Green-Naghdi system is given by

\[
\begin{align*}
  u_t + uu_x + h_x &= \frac{1}{3h} \partial_x \left( h^3 (u_{tx} + uu_{xx} - u_x^2) \right), \quad t \geq 0, \ x \in \mathbb{R}, \\
  h_t + \partial_x (hu) &= 0, \quad t \geq 0, \ x \in \mathbb{R}, \\
  h(t = 0) &= h_0, \ u(t = 0) = u_0,
\end{align*}
\] (1)

where \( h(t, x) \in \mathbb{R} \) is the height of the upper free surface of the water wave over the flat bottom and \( u(t, x) \in \mathbb{R} \) its horizontal velocity. The system (1) describes the evolution of water waves over a flat bottom in the shallow water regime, i.e. when the typical wavelength is much larger than the typical water depth. The 2D version of (1) for a variable bottom is derived in a paper of Green and
Naghdi [1]. The name “Green-Naghdi system” originates from this paper. But the 1D version (1) appears already in a paper by Serre [5] and later in [6]. People refer to (1) sometimes also as the “Serre-Green-Naghdi system”.

Let us introduce the operator

\[ A_h : u \mapsto 3hu - \partial_x(h^3u_x). \] (2)

Then a simple calculation shows that the first equation in (1) is equivalent to

\[ A_h(u_t + uu_x) = -3hh_x - 2\partial_x(h^3u_x^2). \]

If \( A_h \) is invertible one can write (1) in non-local form (see also [4]) as

\[ u_t + uu_x = -A_h^{-1}(3hh_x + 2\partial_x(h^3u_x^2)), \quad t \geq 0, \ x \in \mathbb{R}, \]

\[ h_t + \partial_x(hu) = 0, \quad t \geq 0, \ x \in \mathbb{R}, \]

\[ h(t = 0) = h_0, \ u(t = 0) = u_0. \] (3)

The Green-Naghdi system in the form (3) is the starting point for the Lagrangian formulation. More precisely, we want to express (3) in the Lagrangian variable \( \varphi \), i.e. in terms of the flow map \( \varphi \) of \( u \). Recall that the flow map is defined as

\[ \varphi_t(t, x) = u(t, \varphi(t, x)), \ \varphi(0, x) = x, \ t \geq 0, \ x \in \mathbb{R}. \]

Note that this defines for each \( t \geq 0 \) a diffeomorphism \( \varphi(t) := \varphi(t, \cdot) \) of \( \mathbb{R} \). Using the second equation in (3) we easily see that

\[ \frac{d}{dt} (\varphi_x(t) \cdot h(t) \circ \varphi(t)) = 0. \]

In other words we can write the second equation in (3) in the Lagrangian variable \( \varphi \) as

\[ h(t) = \left( \frac{h_0}{\varphi_x(t)} \right) \circ \varphi(t)^{-1}, \ t \geq 0. \] (4)

To write the first equation of (3) in the Lagrangian variable \( \varphi \) consider

\[ \frac{d}{dt} \varphi_t(t) = \frac{d}{dt} u(t) \circ \varphi(t) = (u_t(t) + u(t)u_x(t)) \circ \varphi(t). \]
If we now replace \( u_t + uu_x \) by the corresponding expression from (3) and use theirin \( u(t) = \varphi(t) \circ \varphi(t)^{-1} \) and from (4) the identity \( h(t) = (u_0/\varphi_x(t)) \circ \varphi(t)^{-1} \) we end up with a second order initial value problem

\[
\varphi_{tt} = F(\varphi, \varphi_t, h_0), \quad t \geq 0, \quad \varphi(0) = \text{id}, \quad \varphi_t(0) = u_0,
\]

where \( \text{id} : \mathbb{R} \to \mathbb{R}, \ x \mapsto x \), is the identity map on \( \mathbb{R} \). To put (5) into a proper mathematical framework we need the right functional space for \( \varphi \) and the smoothness of \( F(\varphi, \varphi_t, h_0) \).

The functional space for the Green-Naghdi system (3) we have in mind in this paper are the Sobolev spaces. Recall that for \( s \geq 0 \) the Sobolev space \( H^s(\mathbb{R}) \) is defined as

\[
H^s(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) \mid \| f \|_{H^s} < \infty \},
\]

where the norm \( \| \cdot \|_{H^s} \) is given by

\[
\| f \|_{H^s} = \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}.
\]

Here we denote by \( \hat{f} \) the Fourier transform of \( f \). Sobolev spaces of negative order are defined as dual spaces

\[
H^{-s}(\mathbb{R}) = (H^s(\mathbb{R}))', \quad s \geq 0.
\]

Suppose now \( s > 1/2 \). The height \( h \) of the free upper surface is an \( H^s \) perturbation of the equilibrium height \( \bar{h} \equiv 1 \). We take as state space for \( h \)

\[
U^s = \{ h : \mathbb{R} \to \mathbb{R} \mid h - 1 \in H^s(\mathbb{R}), h(x) > 0 \text{ for all } x \in \mathbb{R} \}.
\]

By the Sobolev Imbedding Theorem we know that \( H^s(\mathbb{R}) \) can be embedded into \( C_0(\mathbb{R}) \), the space of continuous functions on \( \mathbb{R} \) vanishing at infinity. Thus for \( h \in U^s \) we conclude \( \inf_{x \in \mathbb{R}} h(x) > 0 \), which means that \( U^s - 1 \) is an open subset of \( H^s(\mathbb{R}) \). So \( U^s \) has naturally a differential structure. Thus to speak about

\[(h, u) \in C([0, T]; U^s \times H^{s+1}(\mathbb{R})), \quad T > 0,\]

makes sense. In [2] the authors studied for \( s' = s + 1 > 3/2 \) the functional space

\[
D^{s'}(\mathbb{R}) = \{ \varphi : \mathbb{R} \to \mathbb{R} \mid \varphi - \text{id} \in H^{s'}(\mathbb{R}), \varphi_x(x) > 0 \text{ for all } x \in \mathbb{R} \}.
\]
By the Sobolev Imbedding $H^{s'}(\mathbb{R}) \hookrightarrow C^1_0(\mathbb{R})$ one gets that $\mathcal{D}^{s'}(\mathbb{R})$ consists of $C^1$ diffeomorphisms of $\mathbb{R}$ and that $\mathcal{D}^{s'}(\mathbb{R}) - \text{id}$ is an open subset of $H^{s'}(\mathbb{R})$. So $\mathcal{D}^{s'}(\mathbb{R})$ has naturally a differential structure. Moreover, as was shown in [2], the maps

$$H^\sigma(\mathbb{R}) \times \mathcal{D}^{s'}(\mathbb{R}) \to H^\sigma(\mathbb{R}), \ (f, \varphi) \mapsto f \circ \varphi, \ 0 \leq \sigma \leq s',$$

and

$$\mathcal{D}^{s'}(\mathbb{R}) \to \mathcal{D}^{s'}(\mathbb{R}), \ \varphi \mapsto \varphi^{-1},$$

are continuous. In particular $\mathcal{D}^{s'}(\mathbb{R})$ is a topological group when the group operation is composition of maps. Now suppose that $(h, u)$ is a solution to the Green-Naghdi system (3) on $[0, T]$ with

$$(h, u) \in C([0, T]; U^s \times H^{s+1}(\mathbb{R})).$$

In [3] it was shown that for $s' = s + 1$ there is a unique

$$\varphi \in C^1([0, T]; \mathcal{D}^{s'}(\mathbb{R}))$$

satisfying $\varphi_t(t) = u(t) \circ \varphi(t), \ 0 \leq t \leq T, \ \varphi(0) = \text{id}$. Thus $\mathcal{D}^{s'}(\mathbb{R})$ is the right functional space for the Lagrangian variable $\varphi$. The first main result of this paper reads then as

**Theorem 1.1.** Let $s > 1/2$. Then

$$\mathcal{D}^{s+1}(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times U^s \to H^{s+1}(\mathbb{R}), \ (\varphi, v, h_0) \mapsto F(\varphi, v, h_0)$$

is real analytic. Here $F$ is the map from [5].

For the basics of analyticity in Banach spaces we refer to [3]. Using Theorem 1.1 and the Picard-Lindelöf Theorem we get for every $h_0 \in U^s$ and $u_0 \in H^{s+1}(\mathbb{R})$ a unique local solution $\varphi$ to [5] on some time interval $[0, T]$. By defining now

$$h(t) = \left(\frac{h_0}{\varphi_x(t)}\right) \circ \varphi(t)^{-1}, \ u(t) = \varphi_t(t) \circ \varphi(t)^{-1}, \ 0 \leq t \leq T,$$

we get a solution $(h, u) \in C([0, T]; U^s \times H^{s+1}(\mathbb{R}))$ to [3]. With this the second main result of the paper reads as

**Theorem 1.2.** Let $s > 1/2$. Then the 1D Green-Naghdi system [3] is locally well-posed for $(h, u)$ in $U^s \times H^{s+1}(\mathbb{R})$.

In [4] it was shown that [3] is locally well-posed in $U^s \times H^{s+1}(\mathbb{R}), \ s > 3/2$. Theorem 1.2 improves this result.
2 The operator $A_h$

The goal of this section is to prove that for $s > 1/2$ and $h \in U^s$ as in \cite{G} the operator $A_h$ in \cite{Z} is an isomorphism $A_h : H^{s+1}(\mathbb{R}) \to H^{s-1}(\mathbb{R})$, $u \mapsto 3hu - \partial_x (h^3u_x)$. To do that consider the following inner product on $H^1(\mathbb{R})$

$$\langle u, v \rangle_h = \int_{\mathbb{R}} 3huv + h^3u_x v_x \, dx.$$ 

Since $\|h\|_{L^\infty} < \infty$ and $\inf_{x \in \mathbb{R}} h(x) > 0$ we easily see that $\langle \cdot, \cdot \rangle_h$ is equivalent to the $H^1$ inner product

$$\langle u, v \rangle_{H^1} = \int_{\mathbb{R}} uv + u_x v_x \, dx.$$ 

Lemma 2.1. Let $s > 1/2$ and $h \in U^s$. Then

$$A_h : H^{s+1}(\mathbb{R}) \to H^{s-1}(\mathbb{R}), \quad u \mapsto 3hu - \partial_x (h^3u_x),$$

is an isomorphism.

Proof. In the following we will use $\langle \cdot, \cdot \rangle$ for the duality pairing between $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$. Suppose first $1/2 < s \leq 2$ and let $f \in H^{s-1}(\mathbb{R})$. We have $f \in H^{-1}(\mathbb{R}) = (H^1(\mathbb{R}))'$. By the Riesz Representation Theorem there is a unique $u \in H^1(\mathbb{R})$ s.t.

$$\langle u, \phi \rangle_h = \langle f, \phi \rangle$$

for all test functions $\phi \in C_0^\infty(\mathbb{R})$. We can write this in $H^{-1}(\mathbb{R})$ as

$$3hu - \partial_x (h^3u_x) = f.$$ 

Thus

$$\partial_x (h^3u_x) = 3hu - f.$$ 

Since we have by assumption $s - 1 \leq 1$ the right hand side is in $H^{s-1}(\mathbb{R})$. As $h^3u_x \in L^2(\mathbb{R})$ and $\partial_x (h^3u_x) \in H^{s-1}(\mathbb{R})$ we conclude $h^3u_x \in H^s(\mathbb{R})$. From \cite{Z} we know that dividing by $h^3$ is a bounded linear map $H^s(\mathbb{R}) \to H^s(\mathbb{R})$. We therefore have $u_x \in H^s(\mathbb{R})$ and with that $u \in H^{s+1}(\mathbb{R})$ satisfying $A_h(u) = f$. So we’ve proved that for $1/2 < s \leq 2$

$$A_h : H^{s+1}(\mathbb{R}) \to H^{s-1}(\mathbb{R})$$


suppose now $2 < s \leq 3$ and $f \in H^{s-1}(\mathbb{R})$. The previous step shows $u \in H^3(\mathbb{R})$ and hence $3hu - f \in H^{s-1}(\mathbb{R})$. Arguing as before we conclude $u \in H^{s+1}(\mathbb{R})$. Continuing like that for $3 < s \leq 4$, $4 < s \leq 5$, . . . shows that

$$A_h : H^{s+1}(\mathbb{R}) \to H^{s-1}(\mathbb{R}), \quad s > 1/2,$$

is an isomorphism.

\[\square\]

3 Lagrangian formulation

The goal of this section is to prove Theorem 1.1. Let us start by introducing some notation.

Let $s > 1/2$ and $0 \leq \sigma \leq s$. Then multiplication

$$H^s(\mathbb{R}) \times H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}), \quad (f, g) \mapsto f \cdot g,$$

is continuous – see [2]. For $1/2 < s < 1$ multiplication extends for $s - 1 \leq \sigma < 0$ to a continuous bilinear map

$$H^s(\mathbb{R}) \times H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}), \quad (f, g) \mapsto f \cdot g.$$

This follows from the fact that there is a constant $C > 0$ s.t.

$$\left| \int_\mathbb{R} f \cdot g \cdot \phi \, dx \right| \leq \|g\|_{\sigma} \|f \cdot \phi\|_{-\sigma} \leq C \|f\|_s \|g\|_{\sigma} \|\phi\|_{-\sigma},$$

for all $\phi \in C^\infty_c(\mathbb{R})$, where we used $-\sigma < s$. In particular we have for $\varphi \in D^{s+1}(\mathbb{R})$ a well-defined multiplication operator

$$M_{\varphi_x} : H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}), \quad f \mapsto \varphi_x \cdot f,$$

for $\min\{0, s - 1\} \leq \sigma \leq s$. Moreover

$$D^{s+1}(\mathbb{R}) \to L(H^\sigma(\mathbb{R}); H^\sigma(\mathbb{R})), \quad \varphi \mapsto M_{\varphi_x},$$

is affine linear and hence it is analytic. Here we denote by $L(X; Y)$ the space of bounded linear maps from $X$ to $Y$. From [2] we know that dividing by $\varphi_x$ is a bounded linear map $H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R})$. In other words $M_{\varphi_x}^{-1} \in L(H^\sigma(\mathbb{R}); H^\sigma(\mathbb{R}))$. Using Neumann series we see that inversion of linear maps is an analytic process, hence for $\min\{0, s - 1\} \leq \sigma \leq s$ the map

$$D^{s+1}(\mathbb{R}) \to L(H^\sigma(\mathbb{R}); H^\sigma(\mathbb{R})), \quad \varphi \mapsto M_{\varphi_x}^{-1}$$
is analytic. As an immediate consequence we get that the map
\[ \mathcal{D}^{s+1}(\mathbb{R}) \times U^s \to U^s, \ (\varphi, h_0) \mapsto \frac{h_0}{\varphi_x} = M_{\varphi_x}^{-1}h_0, \]
is analytic.

Let \( s > 1/2 \) and \( \varphi \in \mathcal{D}^{s+1}(\mathbb{R}) \). We denote by \( R_{\varphi} : f \mapsto f \circ \varphi \) composition with \( \varphi \) from the right. Note that \( R_{\varphi}^{-1} = R_{\varphi^{-1}} \). As mentioned in Section 1 we know from [2] that for \( 0 \leq \sigma \leq s + 1 \)
\[ R_{\varphi} : H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}), \ f \mapsto f \circ \varphi, \]
is a continuous linear map. If \( 1/2 < s < 1 \) then this extends for \( s - 1 \leq \sigma < 0 \) to a continuous linear map
\[ R_{\varphi} : H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}). \]
The reason is that there is a constant \( C > 0 \) such that we have for all test functions \( \phi \in C_\infty(\mathbb{R}) \)
\[ \left| \int_\mathbb{R} f \circ \varphi \cdot \phi \ dx \right| = \left| \int_\mathbb{R} f \cdot \frac{\phi \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \ dx \right| \leq C \| f \|_\sigma \| \phi \|_{-\sigma}. \]
This follows from \( -\sigma < s \) and the fact that division by \( \varphi_x \) and \( R_{\varphi}^{-1} \) are bounded linear maps \( H^{-\sigma}(\mathbb{R}) \to H^{-\sigma}(\mathbb{R}) \).

The composition map has poor regularity. It is not more than continuous. The reason is that to take the derivative with respect to \( \varphi \) in \( \varphi \mapsto f \circ \varphi \) we have to take the derivative of \( f \), which leads to a loss of derivative. But the conjugation with \( R_{\varphi}^{-1} \) turns out to be smooth.

**Lemma 3.1.** Let \( s > 1/2 \). Then
\[ \mathcal{D}^{s+1}(\mathbb{R}) \to L(H^{s+1}(\mathbb{R}); H^s(\mathbb{R})), \ \varphi \mapsto R_{\varphi} \partial_x R_{\varphi}^{-1} \]
and
\[ \mathcal{D}^{s+1}(\mathbb{R}) \to L(H^s(\mathbb{R}); H^{s-1}(\mathbb{R})), \ \varphi \mapsto R_{\varphi} \partial_x R_{\varphi}^{-1} \]
are analytic.

**Proof.** Using the chain rule we have
\[ (\partial_x (f \circ \varphi^{-1})) \circ \varphi = \frac{\partial_x f}{\varphi_x}. \]
Thus \( R_{\varphi} \partial_x R_{\varphi}^{-1} = M_{\varphi_x}^{-1}\partial_x \), which by the above considerations is analytic in \( \varphi. \)
Consider first the operator \( R \). Let \( \phi, v \) be analytic. We rewrite \( F(h, 0) \) as
\[
F(h, 0) = -R_{\varphi}A_{(h_0/\varphi_x)}^{-1}R_{\varphi}^{-1} \left( 3(h_0/\varphi_x) \circ \varphi^{-1} \cdot \partial_x ((h_0/\varphi_x) \circ \varphi^{-1}) + 2\partial_x \left( ((h_0/\varphi_x) \circ \varphi^{-1})^3 \cdot (\partial_x (v \circ \varphi^{-1})^2) \right) \right),
\]
is analytic. We rewrite \( F(h, 0) \) as
\[
F(h, 0) = -R_{\varphi}A_{(h_0/\varphi_x)}^{-1}R_{\varphi}^{-1} \left( 3M_{\varphi_x}^{-1}h_0 \cdot R_{\varphi} \partial_x R_{\varphi}^{-1}M_{\varphi_x}^{-1}h_0 + 2R_{\varphi} \partial_x R_{\varphi}^{-1} \left( (M_{\varphi_x}^{-1}h_0)^3 \cdot (R_{\varphi} \partial_x R_{\varphi}^{-1}v)^2 \right) \right).
\]
Consider first the operator \( R_{\varphi}A_{(h_0/\varphi_x)}^{-1}R_{\varphi}^{-1} \). We clearly have
\[
R_{\varphi}A_{(h_0/\varphi_x)}^{-1}R_{\varphi}^{-1} = (R_{\varphi}A_{(h_0/\varphi_x)}^{-1})^{-1}.
\]
We have for \( f \in H^{s+1}(\mathbb{R}) \)
\[
R_{\varphi}A_{(h_0/\varphi_x)}^{-1}R_{\varphi}^{-1}(f) = 3M_{\varphi_x}^{-1}h_0 \cdot f - R_{\varphi} \partial_x R_{\varphi}^{-1} \left( (M_{\varphi_x}^{-1}h_0)^3 \cdot R_{\varphi} \partial_x R_{\varphi}^{-1}f \right).
\]
Thus the map
\[
\mathcal{D}^{s+1}(\mathbb{R}) \times U^s \to L(H^{s+1}(\mathbb{R}); H^{s-1}(\mathbb{R})), \ (h_0, h_0) \mapsto R_{\varphi}A_{(h_0/\varphi_x)}^{-1}R_{\varphi}^{-1}(\cdot),
\]
is analytic. Since inversion of linear maps is an analytic process we get that
\[
\mathcal{D}^{s+1}(\mathbb{R}) \times U^s \to L(H^{s-1}(\mathbb{R}); H^{s+1}(\mathbb{R})), \ (h_0, h_0) \mapsto R_{\varphi}A_{(h_0/\varphi_x)}^{-1}R_{\varphi}^{-1}(\cdot),
\]
is analytic. We clearly have that
\[
\mathcal{D}^{s+1}(\mathbb{R}) \times U^s \to H^{s-1}(\mathbb{R}), \ (h_0, h_0) \mapsto 3M_{\varphi_x}^{-1}h_0 \cdot R_{\varphi} \partial_x R_{\varphi}^{-1}M_{\varphi_x}^{-1}h_0,
\]
and
\[
\mathcal{D}^{s+1}(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times U^s \to H^{s-1}(\mathbb{R}), \ (h_0, h_0) \mapsto 2R_{\varphi} \partial_x R_{\varphi}^{-1} \left( (M_{\varphi_x}^{-1}h_0)^3 \cdot (R_{\varphi} \partial_x R_{\varphi}^{-1}v)^2 \right),
\]
are analytic maps. So composing the latter two maps with \( R_{\varphi}A_{(h_0/\varphi_x)}^{-1}R_{\varphi}^{-1}(\cdot) \)
shows that \( F(h, 0) \) depends analytically on \( (h_0, 0) \). This finishes the proof. \( \square \)

Consequently we get by using Theorem 1.1 a Lagrangian formulation of the Green-Naghdi system (3) in the form of an analytic second order ODE on \( \mathcal{D}^{s+1}(\mathbb{R}) \) given by (3).
4 Local well-posedness of the Green-Naghdi system

The goal of this section is to prove the local well-posedness result stated in Theorem 1.2. We will prove this in two steps: local existence and uniqueness. But before we do that we prove the following technical lemma.

Lemma 4.1. Let $s > 1/2$ and $T > 0$. Suppose $g \in C^1([0,T];U^s)$ and $\varphi \in C^1([0,T];D^{s+1}(\mathbb{R}))$. Then $g \circ \varphi^{-1} \in C^1([0,T];(1 + H^{s-1}(\mathbb{R})))$ with

$$
\frac{d}{dt}g(t) \circ \varphi^{-1}(t) = g_t(t) \circ \varphi(t)^{-1} - \left( \frac{g_x(t)\varphi_t(t)}{\varphi_x(t)} \right) \circ \varphi(t)^{-1}, \quad 0 \leq t \leq T.
$$

Proof. Take a sequence $(g^{(k)})_{k \geq 1} \subset C^1([0,T];U^{s+1})$ s.t. $g^{(k)} \to g$ in $C^1([0,T];U^s)$ as $k \to \infty$. By the Sobolev imbedding $H^{s+1}(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$ we can differentiate $g^{(k)} \circ \varphi^{-1}$ pointwise in $t$

$$
\frac{d}{dt}g^{(k)}(t) \circ \varphi(t)^{-1} = g^{(k)}_t(t) \circ \varphi(t)^{-1} - \left( \frac{g^{(k)}_x(t)\varphi_t(t)}{\varphi_x(t)} \right) \circ \varphi(t)^{-1}, \quad 0 \leq t \leq T.
$$

By the Fundamental lemma of calculus we get pointwise

$$
g^{(k)}(t) \circ \varphi(t)^{-1} = g^{(k)}(0) \circ \varphi(0)^{-1} + \int_0^t g^{(k)}_t(s) \circ \varphi(s)^{-1} - \left( \frac{g^{(k)}_x(s)\varphi_t(s)}{\varphi_x(s)} \right) \circ \varphi(s)^{-1} \, ds.
$$

But this is an identity in $H^{s-1}(\mathbb{R})$ as well. Taking $k \to \infty$ shows the claim. 

Let us prove now the local existence of solutions to the Green-Naghdi system.

Lemma 4.2. Let $s > 1/2$ and $(h_0, u_0) \in U^s \times H^{s+1}(\mathbb{R})$. Then there is $T > 0$ and

$$(h, u) \in C([0,T];U^s \times H^{s+1}(\mathbb{R})) \cap C^1([0,T];(1 + H^{s-1}(\mathbb{R})) \times H^s(\mathbb{R}))$$

solving (3). Moreover the dependence of $(h, u)$ on $(h_0, u_0)$ is continuous.

We can take a uniform $T > 0$ in a neighborhood of $(h_0, u_0)$. Continuous dependence on $(h_0, u_0)$ means continuity in such a neighborhood with the same $T > 0$. 

9
Proof of Lemma 4.2. Using Theorem 1.1 and the Picard-Lindelöf Theorem we get a solution \( \varphi \in C^\infty([0, T]; D^{s+1}(\mathbb{R})) \) to
\[
\varphi_{tt} = F(\varphi, \varphi_t, h_0), \quad \varphi(0) = \text{id}, \quad \varphi_t(0) = u_0,
\]
on some time interval \([0, T]\) for some \( T > 0 \). For initial data in a neighborhood of \((h_0, u_0)\) we can take the same \( T \). We define
\[
h(t) := \left( \frac{h_0}{\varphi_x(t)} \right) \circ \varphi(t)^{-1}, \quad u(t) := \varphi_t(t) \circ \varphi(t)^{-1}, \quad 0 \leq t \leq T.
\]

So by the continuity properties of the composition map we see that
\[(h, u) \in C([0, T]; U^s \times H^{s+1}(\mathbb{R})).\]

By the Sobolev imbedding \( H^{s+1}(\mathbb{R}) \hookrightarrow C^1(\mathbb{R}) \) we know that \( u \in C^1([0, T] \times \mathbb{R}) \). Taking pointwise the \( t \) derivative in \( u \circ \varphi \) gives
\[
\varphi_{tt} = \frac{d}{dt} u \circ \varphi = (u_t + uu_x) \circ \varphi = F(\varphi, \varphi_t, u_0).
\]

Entangling the last equality leads to the pointwise identity
\[
u_t + uu_x = -A_h^{-1}(3hh_x + 2\partial_x(h^3u_x)).
\]

But this is an identity in \( H^s \) as well since \( U^s \rightarrow L(H^{s-1}(\mathbb{R}); H^{s+1}(\mathbb{R})) \), \( h \mapsto A_h^{-1} \) is continuous. Thus we have
\[u \in C^1([0, T]; H^s(\mathbb{R})).\]

and the first equation in (3) is satisfied. Using Lemma 4.1 one gets that
\[
h = \left( \frac{h_0}{\varphi_x} \right) \circ \varphi^{-1} \in C^1([0, T]; 1 + H^{s-1}(\mathbb{R}))
\]
and the second equation in (3) is satisfied. Continuous dependence on the initial data follows from the continuity properties of the composition map. This finishes the proof. \( \square \)

Now we show uniqueness of solutions to (3).
Lemma 4.3. Let $s > 1/2$ and $(h_0, u_0) \in U^s \times H^{s+1}(\mathbb{R})$. Suppose that 
$(h, u), (\tilde{h}, \tilde{u}) \in C([0,T]; U^s \times H^{s+1}(\mathbb{R})) \cap C^1([0,T]; (1 + H^{s-1}(\mathbb{R})) \times H^s(\mathbb{R}))$
are solutions to (3) on $[0,T]$ for some $T > 0$. Then $(h, u) \equiv (\tilde{h}, \tilde{u})$ on $[0,T]$.

Proof. From [3] there are $\varphi, \tilde{\varphi} \in C^1([0,T]; D^{s+1}(\mathbb{R}))$ satisfying 
$\varphi_t = u \circ \varphi, \tilde{\varphi}_t = \tilde{u} \circ \tilde{\varphi}, 0 \leq t \leq T, \varphi(0) = \tilde{\varphi}(0) = \text{id}.$

Taking the pointwise $t$ derivative in $u \circ \varphi$ gives 
$\varphi_{tt} = (u + uu_x) \circ \varphi = -R_{\varphi}A^{-1}_h(3hh_x + 2\partial_x(h^3u_x)),$
where in the last equality we used the first equation in (3). But this is an identity in $H^{s+1}(\mathbb{R})$ as well. So $\varphi$ solves the ODE (5) on $[0,T]$. A similar argument shows that $\tilde{\varphi}$ solves the same initial value problem on $[0,T]$. Thus by uniqueness of solutions to ODEs we get $\varphi \equiv \tilde{\varphi}$ on $[0,T]$, which implies $(h, u) \equiv (\tilde{h}, \tilde{u})$ on $[0,T]$. This finishes the proof. □

By combining Lemma 4.2 and Lemma 4.3 we can prove Theorem 1.2.

Proof of Theorem 1.2. The proof follows from Lemma 4.2 and Lemma 4.3.

□

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