Generating a Quadratic Forms from a Given Genus

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Abstract
Given a non-empty genus in $n$ dimensions with determinant $d$, we give a randomized algorithm that outputs a quadratic form from this genus. The time complexity of the algorithm is $\text{poly}(n, \log d)$; assuming Generalized Riemann Hypothesis (GRH).

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1 Introduction

Let $R$ be a commutative ring with unity and $R^\times$ be the set of units (i.e., invertible elements) of $R$. A quadratic form over the ring $R$ in $n$-formal variables $x_1, \cdots, x_n$ in an expression $\sum_{1 \leq i,j \leq n} a_{ij}x_ix_j$, where $a_{ij} = a_{ji} \in R$. A quadratic form can equivalently be represented by a symmetric matrix $Q^n = (a_{ij})$ such that $Q(x_1, \cdots, x_n) = (x_1, \cdots, x_n)^T Q(x_1, \cdots, x_n)$. The quadratic form is called integral if $R = \mathbb{Z}$ and the determinant of the quadratic form $Q$ is defined as $\det(Q)$. In this paper, we concern ourselves with integral quadratic forms, henceforth referred only as quadratic forms.

One of the classical problems in the study of quadratic forms is their classification into equivalence classes. Two quadratic forms $Q_1, Q_2$ are said to be equivalent over a ring $R$ if there exists a transformation $U \in \text{GL}_n(R)$ such that $Q_1 = U^T Q_2 U$. For example, $Q_1$ and $Q_2$ are $q$-equivalent, for an integer $q$ (denoted $Q_1 \sim_q Q_2$), if there exists a matrix $U \in \text{GL}_n(\mathbb{Z}/q\mathbb{Z})$ such that $Q_1 \equiv U^T Q_2 U \mod q$. Intuitively, $q$-equivalence means that there exists an invertible linear change of variables over $\mathbb{Z}/q\mathbb{Z}$ that transforms one form to the other. Gauss [Gau86] gives a complete classification of binary quadratic forms (i.e., $n = 2$).

Two quadratic forms are said to be in the same genus if they are equivalent over the reals $\mathbb{R}$ and also over $\mathbb{Z}/q\mathbb{Z}$ for all positive integers $q$. In this paper, we consider the following problem: given a description of a non-empty genus, produce a quadratic form from that genus. A discussion of the problem can be found in Conway and Sloane [CS99], page 403. The best algorithm for this problem is based on Minkowski Reduced forms and takes $O(d^{n^2})$ time for genus in dimension $n$ with determinant $d$.

The skeleton of our algorithm is similar to the algorithm given by Hartung [Har08]. His thesis uses an equivalent but different approach based on Cassels [Cas78]. Unfortunately, there are several gaps in his construction. There are also mistakes when dealing with prime 2. But, the most severe problem with the algorithm is that its time complexity is proportional to $n^n$ i.e., it is not polynomial. A discussion can be found in Section 8.3.2.

We mention here, a connection of our problem to lattices as studied in the Computer Science community. A full-rank lattice $L$ in $\mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$ which is the set of all integer linear combinations of $n$-linearly independent vectors, say $b_1, \ldots, b_n$ i.e., $L = \sum_{i=1}^n z_i b_i \mid z_1, \ldots, z_n \in \mathbb{Z}$. The matrix $B = [b_1, \ldots, b_n]$ is called the basis of the lattice and the matrix $Q = B^T B$ is called a Gram matrix of the lattice. A lattice is integral if its Gram matrix has only integer entries. It is not difficult to see that the Gram matrix of a lattice defines a positive definite quadratic form.

Two lattices are called isomorphic if one can be transformed into another by an orthogonal linear transformation. A fundamental question, called the Lattice Isomorphism Problem (LIP), is to decide if two given Gram matrices come from isomorphic lattices. In other words, given two Gram matrices $Q_1$ and $Q_2$ one has to decide if there exists a unimodular matrix $U$ such that $Q_2 = U^T Q_1 U$. For Gram matrices in dimension $n$ and determinant $d$, the problem can be solved using Minkowski Reduced Forms (see Section 10, Chapter 10 [CS99]) in time $O(d^{n^2})$. Other exhaustive search algorithms are known, see [Die03, Sie72]. Recently, Regev and Haviv [HR14] gave an algorithm with time complexity which is $n^{O(n)}$ times the size of the input.

The shortest vector problem (SVP) is the problem of finding the shortest non-zero vector in a given lattice. The current best known hardness for SVP is given by Regev-Haviv [HR07] and is based on tensoring lattice bases in the hope of amplifying the length of the shortest vector. This approach fails in general. For large enough dimension $n$, there are self-dual lattices with shortest vector $\Omega(\sqrt{n})$. The usual tensoring among these lattices fails to amplify the length of the shortest vector (Lemma 2.4, [HR07]). It is not known how one can construct self-dual lattice with shortest vector length $\Omega(\sqrt{n})$ but it can be shown that such lattices exist in large dimensions. The proof of existence (see page 48, [MH73]) uses the Smith-Minkowski-Siegel mass formula; which computes the average number vectors of a certain length in a genus. One way to generate a self-dual lattice with shortest vector $\Omega(\sqrt{n})$ is to sample a lattice according to a certain distribution from a specific genus (see Minkoff-Husenmoller [MH73]). Our result fails short in the following way. Given this specific genus, we can construct one lattice but we do not know how to sample according to the distribution specified in [MH73]. In this respect, our work can be seen as an important first step towards construction of self-dual lattices with shortest vector $\Omega(\sqrt{n})$. 

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Our Contributions. Let \( d \) be the determinant of a genus in dimension \( n \). We present a poly\((n, \log d)\) Las Vegas algorithm that outputs a quadratic form in the genus with constant probability. Our construction technique is inspired by the proof of Smith-Minkowski-Siegel mass formula given by Siegel [Sic35] and uses similar notations as Conway-Sloane [CS99].

A significant feature of our work is the simplification achieved by not using \( p \)-adic numbers, a staple in the analysis of integral quadratic forms [CS99, Kit99, Kne02, Sie35].

2 Preliminaries

Integers and ring elements are denoted by lowercase letters, vectors by bold lowercase letters and matrices by typewriter uppercase letters. The \( i \)th component of a vector \( \mathbf{v} \) is denoted by \( v_i \). We use the notation \((v_1, \ldots, v_n)\) for a column vector and the transpose of matrix \( A \) is denoted by \( A' \). The matrix \( A^n \) will denote a \( n \times n \) square matrix. The scalar product of two vectors will be denoted \( \mathbf{v}'\mathbf{w} \) and equals \( \sum v_iw_i \). The standard Euclidean norm of the vector \( \mathbf{v} \) is denoted by \( ||\mathbf{v}|| \) and equals \( \sqrt{\mathbf{v}'\mathbf{v}} \).

If \( Q_1, Q_2 \) are matrices, then the direct product of \( Q_1 \) and \( Q_2 \) is denoted by \( Q_1 \oplus Q_2 \) and is defined as \( \text{diag}(Q_1, Q_2) = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \). Given two matrices \( Q_1 \) and \( Q_2 \) with the same number of rows, \([Q_1, Q_2]\) is the matrix which is obtained by concatenating the two matrices columnwise. A matrix is called unimodular if it is an integer \( n \times n \) matrix with determinant \( \pm 1 \). If \( Q^\prime \) is a \( n \times n \) integer matrix and \( q \) is a positive integer then \( Q \mod q \) is defined as the matrix with all entries of \( Q \) reduced modulo \( q \).

Let \( R \) be a commutative ring with unity and \( R^\times \) be the set of units (i.e., invertible elements) of \( R \). If \( Q \in R^{n \times n} \) is a square matrix, the adjugate of \( Q \) is defined as the transpose of the cofactor matrix and is denoted by \( \text{adj}(Q) \). The matrix \( Q \) is invertible if and only if \( \det(Q) \) is a unit of \( R \). In this case, \( \text{adj}(Q) = \det(Q)Q^{-1} \). The set of invertible \( n \times n \) matrices over \( R \) is denoted by \( \text{GL}_n(R) \). The subset of matrices with determinant 1 will be denoted by \( \text{SL}_n(R) \).

Fact 1 A matrix \( U \) is in \( \text{GL}_n(R) \) iff \( \det(U) \in R^\times \).

The set of odd primes is denoted by \( \mathbb{P} \). We define \( \mathbb{Q}/(-1)\mathbb{Q} = \mathbb{Z}/(-1)\mathbb{Z} := \mathbb{R} \). For every prime \( p \) and positive integer \( k \), we define the ring \( \mathbb{Z}/p^k\mathbb{Z} = \{0, \ldots, p^k - 1\} \), where product and addition is defined modulo \( p^k \).

Let \( p \) be a prime, and \( a, b \) be integers. Then, \( \text{ord}_p(a) \) is the largest integer exponent of \( p \) such that \( p^{\text{ord}_p(a)} \) divides \( a \). We let \( \text{ord}_p(0) = \infty \). The \( p \)-coprime part of \( a \) is then \( \text{cpr}_p(a) = a^{\frac{1}{\text{ord}_p(a)}} \). Note that \( \text{cpr}_p(a) \) is, by definition, a unit of \( \mathbb{Z}/p\mathbb{Z} \). For a rational number, we define \( \text{ord}_p\left(\frac{a}{b}\right) = \text{ord}_p(a) - \text{ord}_p(b) \). The \( p \)-coprime part of \( \frac{a}{b} \) is denoted as \( \text{cpr}_p\left(\frac{a}{b}\right) \) and equals \( \frac{a^{\text{ord}_p(a)}}{b^{\text{ord}_p(b)}} \). For a positive integer \( q \), one writes \( a \equiv b \mod q \), if \( q \) divides \( a - b \). By \( x := a \mod q \), we mean that \( x \) is assigned the unique value \( b \in \{0, \ldots, q - 1\} \) such that \( b \equiv a \mod q \). An integer \( t \) is called a quadratic residue modulo \( q \) if \( \gcd(t, q) = 1 \) and \( x^2 \equiv t \mod q \) has a solution.

Definition 1 Let \( p \) be an odd prime, and \( t \) be a positive integer with \( \gcd(t, p) = 1 \). Then, the Legendre-symbol of \( t \) with respect to \( p \) is defined as follows.

\[
\left( \frac{t}{p} \right) = t^{(p-1)/2} \mod p = \begin{cases} 
1 & \text{if } t \text{ is a quadratic residue modulo } p \\
-1 & \text{otherwise.}
\end{cases}
\]

For the prime 2, there is an extension of Legendre symbol called the Kronecker symbol. It is defined for odd integers \( t \) and \( \left( \frac{t}{2} \right) \) equals 1 if \( t \equiv \pm 1 \mod 8 \), and \(-1\) if \( t \equiv \pm 3 \mod 8 \).

The Law of Quadratic Reciprocity, conjectured by Euler and Legendre and first proved by Gauss, says
Lemma 1 Let \( p \) be an odd prime. Then, there are \( \frac{p-1}{2} \) quadratic residues and \( \frac{p-1}{2} \) quadratic non-residues modulo \( p \). Also, every quadratic residue in \( \mathbb{Z}/p\mathbb{Z} \) can be written as a sum of two quadratic non-residues and every quadratic non-residue can be written as a sum of two quadratic residues.

An integer \( t \) is a square modulo \( q \) if there exists an integer \( x \) such that \( x^2 \equiv t \) (mod \( q \)). The integer \( x \) is called the square root of \( t \) modulo \( q \). If no such \( x \) exists, then \( t \) is a non-square modulo \( q \).

Definition 2 Let \( p \) be a prime and \( \frac{a}{b} \) be a rational number. Then, \( \frac{a}{b} \) can be uniquely written as \( \frac{a}{b} = p^\alpha \frac{a}{b'} \), where \( a, b \) are units of \( \mathbb{Z}/p\mathbb{Z} \). We say that \( \frac{a}{b} \) is a \( p \)-antisquare if \( \alpha \) is odd and \( \text{sgn}_p(a) \neq \text{sgn}_p(b) \).

The following lemma is folklore and gives the necessary and sufficient conditions for an integer \( t \) to be a square modulo \( p^k \). For completeness, a proof is provided in Appendix B.

Lemma 2 Let \( p \) be a prime, \( k \) be a positive integer and \( t \in \mathbb{Z}/p^k\mathbb{Z} \) be a non-zero integer. Then, \( t \) is a square modulo \( p^k \) if and only if \( \text{ord}_p(t) \) is even and \( \text{sgn}_p(t) = 1 \).

Definition 3 Let \( p^k \) be a prime power. A vector \( \mathbf{v} \in (\mathbb{Z}/p^k\mathbb{Z})^n \) is called primitive if there exists a component \( v_i, i \in [n] \), of \( \mathbf{v} \) such that \( \gcd(v_i, p) = 1 \). Otherwise, the vector \( \mathbf{v} \) is non-primitive.

Our definition of primitiveness of a vector is different but equivalent to the usual one in the literature. A vector \( \mathbf{v} \in (\mathbb{Z}/q\mathbb{Z})^n \) is called primitive over \( \mathbb{Z}/q\mathbb{Z} \) for a composite integer \( q \) if it is primitive modulo \( p^{\text{ord}_p(q)} \) for all primes that divide \( q \).

**Randomized Algorithms.** Our randomized algorithms are Las Vegas algorithms. They either fail and output nothing, or produce a correct answer. The probability of failure is bounded by a constant. Thus, for any \( \delta > 0 \), it is possible to repeat the algorithm \( O(\log \frac{1}{\delta}) \) times and succeed with probability at least \( 1 - \delta \). Henceforth, these algorithms will be called randomized algorithms.

Our algorithms perform two kinds of operations. Ring operations e.g., multiplication, additions, inversions over \( \mathbb{Z}/p^k\mathbb{Z} \) and operations over integers \( \mathbb{Z} \) e.g., multiplications, additions, divisions etc and operations over integers \( \mathbb{Z} \). The runtime for all these operations is treated as constant i.e., \( O(1) \) and the time complexity of the algorithms is measured in terms of ring operations. Note that the complexity cannot be assumed to be \( O(1) \) if the numbers are doubly exponential in \( n \). Thus, we make sure than the numbers generated during the algorithm are bounded by \( 2^{\text{poly}(n,d)} \). Sometimes, we also need to sample a uniform ring element from \( \mathbb{Z}/p^k\mathbb{Z} \). We adapt the convention that sampling a uniform ring elements also takes \( O(1) \) ring operations.

For example, the Legendre symbol of an integer \( a \) can be computed by fast exponentiation in \( O(\log p) \) ring operations over \( \mathbb{Z}/p\mathbb{Z} \) while \( \text{ord}_p(t) \) for \( t \in \mathbb{Z}/p^k\mathbb{Z} \) can be computed by fast exponentiation in \( O(\log k) \) ring operations over \( \mathbb{Z}/p^k\mathbb{Z} \).

Let \( \omega \) be the constant, such that multiplying two \( n \times n \) matrices over \( \mathbb{Z}/p^k\mathbb{Z} \) takes \( O(n^\omega) \) ring operations.
Dirichlet’s Theorem. Let \( a, q \) be positive integers such that \( \gcd(a, q) = 1 \). Dirichlet’s theorem states that there are infinitely many primes of the form \( a + zq \), where \( z \) is a non-negative integer. The following theorem gives a quantitative version of Dirichlet’s theorem using Generalized Riemann Hypothesis (GRH). A proof of the theorem can be found in any analytic number theory book, for example [IK04].

**Theorem 3** Let \( a, q \) be integers such that \( \gcd(a, q) = 1 \) and \( S \) be the set \( \{a + zq \mid z \in \mathbb{Z}, a + zq \leq q^3\} \). Then assuming GRH, there exists a constant \( c \) such that \( S \) has \( c \frac{|S|}{\log |S|} \) primes.

Another implication of GRH is that the smallest quadratic non-residue modulo \( p \), for odd prime \( p \); is a number less than \( 3(\log p)^2/2 \), see [Ank52, Wed01]. Thus, assuming GRH, a quadratic residue modulo \( p \) can be found deterministically in time \( O(\log^2 p) \) ring operations over \( \mathbb{Z}/p\mathbb{Z} \) by trying all integers \( \leq 3(\log p)^2/2 \).

**Quadratic Form.** An \( n \)-ary quadratic form over a ring \( R \) is a symmetric matrix \( Q \in R^{n \times n} \), interpreted as the following polynomial in \( n \) formal variables \( x_1, \ldots, x_n \) of uniform degree 2.

\[
\sum_{1 \leq i,j \leq n} Q_{ij} x_i x_j = Q_{11} x_1^2 + Q_{12} x_1 x_2 + \cdots = x^T Q x
\]

The quadratic form is called *integral* if it is defined over the ring \( \mathbb{Z} \). It is called positive definite if for all non-zero column vectors \( x \), \( x^T Q x > 0 \). This work deals with integral quadratic forms, henceforth called simply *quadratic forms*. The *determinant* of the quadratic form is defined as \( \det(Q) \). A quadratic form is called *diagonal* if \( Q \) is a diagonal matrix.

Given a set of formal variables \( x = (x_1 \cdots x_n)^T \) one can make a linear change of variables to \( y = (y_1 \cdots y_n)^T \) using a matrix \( U \in R^{n \times n} \) by setting \( y = Ux \). If additionally, \( U \) is invertible over \( R \) i.e., \( U \in \text{GL}_n(R) \), then this change of variables is reversible over the ring. We now define the equivalence of quadratic forms over the ring \( R \) (compare with Lattice Isomorphism).

**Definition 4** Let \( Q_1^n, Q_2^n \) be quadratic forms over a ring \( R \). They are called \( R \)-equivalent if there exists a \( U \in \text{GL}_n(R) \) such that \( Q_2 = U^T Q_1 U \).

If \( R = \mathbb{Z}/q\mathbb{Z} \), for some positive integer \( q \), then two integral quadratic forms \( Q_1^n \) and \( Q_2^n \) will be called \( q \)-equivalent (denoted, \( Q_1 \sim_q Q_2 \)) if there exists a matrix \( U \in \text{GL}_n(\mathbb{Z}/q\mathbb{Z}) \) such that \( Q_2 \equiv U^T Q_1 U \pmod{q} \). For a prime \( p \), they are \( p^\ast \)-equivalent (denoted, \( Q_1 \overset{p^\ast}{\sim} Q_2 \)) if they are \( p^k \)-equivalent for every positive integer \( k \). Additionally, \( (-1)^{\ast} \)-equivalence as well as \( (-1) \)-equivalence mean equivalence over the reals \( \mathbb{R} \).

Let \( Q^n \) be an \( n \)-ary integral quadratic form, and \( q, t \) be positive integers. If the equation \( x^T Q x \equiv t \pmod{q} \) has a solution then we say that \( t \) has a \( q \)-representation in \( Q \) (or \( t \) has a representation in \( Q \) over \( \mathbb{Z}/q\mathbb{Z} \)). Solutions \( x \in (\mathbb{Z}/q\mathbb{Z})^n \) to the equation called *\( q \)-representations* of \( t \) in \( Q \). We classify the representations into two categories: *primitive* and *non-primitive* (see Definition 3). The following lemma shows that a primitive representation can be extended to a invertible transformation.

**Lemma 4** Let \( p \) be a prime, \( k \) be a positive integer and \( x \in (\mathbb{Z}/p^k\mathbb{Z})^n \) be a primitive vector. Then, an \( A \) can be found in \( O(n^2) \) ring operation such that \( [x, A] \in \text{SL}_n(\mathbb{Z}/p^k\mathbb{Z}) \).

**Proof:** The column vector \( x = (x_1, \ldots, x_n) \) is primitive, hence there exists a \( x_i, i \in [n] \) such that \( x_i \) is invertible over \( \mathbb{Z}/p^k\mathbb{Z} \). It is easier to write the matrix \( U \), which equal \( [x, A] \) where the row \( i \) and 1 or \( [x, A] \) are swapped.

\[
U = \begin{pmatrix} x_i & 0 \\ x_{-i} & x_{i-1}^{-1} \mod p^k \otimes \mathbb{I}^{n-2} \end{pmatrix} \quad x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)
\]

The matrix \( U \) has determinant 1 modulo \( p^k \) and hence is invertible over \( \mathbb{Z}/p^k\mathbb{Z} \). The lemma now follows from the fact that the swapped matrix is invertible if the original matrix is invertible. \( \square \)

For the following result, see Theorem 2, [Jon50].
Theorem 5 An integral quadratic form $Q^n$ is equivalent to a quadratic form $q_1 \oplus \cdots \oplus q_a \oplus q_{a+1} \oplus \cdots \oplus q_n$ over the field of rationals $\mathbb{Q}$, where $a \in [n]$, $q_1, \cdots, q_n$ are positive rational numbers and $q_{a+1}, \cdots, q_n$ are negative rational numbers.

The signature (also, $(-1)$-signature) of the form $Q$ (denoted $\text{sig}(Q)$, also $\text{sig}_1(Q)$) is defined as the number $2a - n$, where $a$ is the integer in Theorem 5.

Each rational number $q_i$ in Theorem 5 can be written uniquely as $p^{\alpha_i}a_i$, where $\alpha_i = \text{ord}_p(q_i)$ and $a_i = \text{cpr}_p(q_i)$. Let $m$ be the number of $p$-antisquares among $q_1, \cdots, q_n$. Then, we define the $p$-signature of $Q$ as follows.

$$\text{sig}_p(Q) = \begin{cases} p^{\alpha_1} + p^{\alpha_2} + \cdots + p^{\alpha_n} + 4m & \text{mod } 8 \quad p \neq 2 \\ a_1 + a_2 + \cdots + a_n + 4m & \text{mod } 8 \quad p = 2 \end{cases}$$

(2)

The 2-signature is also known as the oddity and is denoted by $\text{odt}(Q)$. Even though there are different ways to diagonalize a quadratic form over $\mathbb{Q}$, the signatures are an invariant for the quadratic form.

For each $p \in \{-1, 2\} \cup \mathbb{P}$, we define the $p$-excess of $Q$ as follows.

$$\text{exs}_p(Q) = \begin{cases} \text{sig}_p(Q) - n & p \neq 2 \\ n - \text{sig}_2(Q) & p = 2 \end{cases}$$

(3)

Theorem 6 ([Cas78, page 76]; [Jon50, Theorem 29]) Let $Q^n$ be an integral quadratic form. Then,

$$\sum_{p \in \{-1, 2\} \cup \mathbb{P}} \text{exs}_p(Q) \equiv 0 \quad (\text{mod } 8) \quad \text{or equivalently,}$$

$$\text{sig}(Q) + \sum_{p \in \mathbb{P}} \text{exs}_p(Q) \equiv \text{odt}(Q) \quad (\text{mod } 8).$$

(4)

The Equation 4 is also referred to as the oddity formula in the literature.

Diagonalizing a Quadratic Form. For the ring $\mathbb{Z}/p^k\mathbb{Z}$ such that $p$ is odd, there always exists an equivalent quadratic form which is also diagonal (see [CS99], Theorem 2, page 369). Additionally, one can explicitly find the invertible change of variables that turns it into a diagonal quadratic form. The situation is tricky over the ring $\mathbb{Z}/2^k\mathbb{Z}$. Here, it might not be possible to eliminate all mixed terms, i.e., terms of the form $2a_{ij}x_ix_j$ with $i \neq j$. For example, consider the quadratic form $2xy$ i.e., \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}\] over $\mathbb{Z}/2^k\mathbb{Z}$. An invertible linear change of variables over $\mathbb{Z}/2^k\mathbb{Z}$ is of the following form.

$$\begin{align*}
x & \rightarrow a_1x_1 + a_2x_2 \\
y & \rightarrow b_1x_1 + b_2x_2
\end{align*}$$

The mixed term after this transformation is $2(a_1b_2 + a_2b_1)$. As $a_1b_2 + a_2b_1 \text{mod } 2$ is the same as the determinant of the change of variables above i.e., $a_1b_2 - a_2b_1 \text{modulo } 2$; it is not possible for a transformation in $\text{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$ to eliminate the mixed term. Instead, one can show that over $\mathbb{Z}/2^k\mathbb{Z}$ it is possible to get an equivalent form where the mixed terms are disjoint i.e., both $x_ix_j$ and $x_ix_k$ do not appear, where $i, j, k$ are pairwise distinct. One captures this form by the following definition.

Definition 5 A matrix $D^n$ over integers is in a block diagonal form if it is a direct sum of type I and type II forms; where type I form is an integer while type II is a matrix of the form \[
\begin{pmatrix}
2^{d+1}a & 2^d b \\
2^d b & 2^{d+1}c
\end{pmatrix}
\] with $b$ odd.

The following theorem is folklore and is also implicit in the proof of Theorem 2 on page 369 in [CS99]. For completeness, we provide a proof in Appendix A.

Theorem 7 Let $Q^n$ be an integral quadratic form, $p$ be a prime, and $k$ be a positive integer. Then, there is an algorithm that performs $O(n^{1+\epsilon} \log k)$ ring operations and produces a matrix $U \in \text{SL}_n(\mathbb{Z}/p^k\mathbb{Z})$ such that $U^TQU \equiv Q \pmod{p^k}$, is a diagonal matrix for odd primes $p$ and a block diagonal matrix (in the sense of Definition 5) for $p = 2$. 

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Canonical Forms. For a quadratic form $Q$ and prime $p$, the set $\{S^n \mid S \not\sim_p Q\}$ is the set of $p^*$-equivalent forms of $Q$ (also called $p^*$-equivalence class of $Q$). It is possible to define something called a “canonical” quadratic form for the $p^*$-equivalence class of a given quadratic form $Q$. In particular, we are interested in a function $\text{can}_p$ such that for all integral quadratic forms $Q$, $\text{can}_p(Q) \in \{S \mid S \not\sim_p Q\}$; with the property that if $Q_1 \not\sim_p Q_2$ then $\text{can}_p(Q_1) = \text{can}_p(Q_2)$. We also consider a related problem of coming up with a canonicalization procedure. In particular, we want a polynomial time algorithm that given $Q$, $p$ and a positive integer $k$, finds $U \in \text{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ such that $U^TQU \equiv \text{can}_p(Q) \mod p^k$.

It is not difficult to show the existence of a canonical form. For example, we can go over the $p^*$-equivalence class of $Q$ and output the form which is lexicographically the smallest one. But, this form gives us no meaningful information about $Q$ or the $p^*$-equivalence class of $Q$.

For odd prime $p$, the $p$-canonical form is implicit in Conway-Sloane $\text{CS99}$ and is also described explicitly by Hartung $\text{Har08}$, $\text{Cas78}$. The canonicalization algorithm in this case is not complicated and can be claimed to be implicit in Cassels $\text{Cas78}$.

The definition of canonical form for the case of prime 2 is quite involved and needs careful analysis.\footnote{Cassels (page 117, Section 4, $\text{Cas78}$), referring to the canonical forms for $p = 2$ observes that “only a masochist is invited to read the rest”.} Jones $\text{Jon14}$, Cassels $\text{CS99}$, Watson $\text{Wat60}$. Jones $\text{Jon14}$ presents the most complete description of the 2-canonical form. His method is to come up with a small 2-canonical forms and then showing that every quadratic form is $2^*$-equivalent to one of these. Unfortunately, a few of his transformations are existential i.e., he shows that a transformations with certain properties exists without explicitly finding them.

The following theorems appear in $\text{DH14a}$.

**Theorem 8** Let $Q^n$ be an integral quadratic form, $p$ be a prime and $k \geq \text{ord}_2(\det(Q)) + k_p$. Then, there is an algorithm (Las Vegas with constant probability of success for odd primes and deterministic for the prime 2) that given $(Q^n, p, k)$ performs $O(n^{1+\omega} \log k + nk^3 + n \log p + \log^3 p)$ ring operations over $\mathbb{Z}/p^k\mathbb{Z}$ and outputs $U \in \text{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ such that $U^TQU \equiv \text{can}_p(Q) \mod p^k$.

**Theorem 9** Let $Q^n$ be an integral quadratic form, $p$ be a prime and $k = \text{ord}_p(\det(Q)) + k_p$. If $D^n$ is a block diagonal form which is equivalent to $Q$ over $\mathbb{Z}/p^k\mathbb{Z}$, then $D \not\sim_p Q$.

Primitive Representations. The following theorem gives an algorithmic handle on the question of deciding if an integer $t$ has a primitive $p^*$-representation in $Q$. The theorem is implicit in Siegel $\text{Sie35}$ (a proof is provided in Appendix B for completeness).

**Theorem 10** Let $Q^n$ be an integral quadratic form, $t$ be an integer, $p$ be a prime and $k = \max\{\text{ord}_p(Q), \text{ord}_p(t)\} + k_p$. Then, if $t$ has a primitive $p^k$-representation in $Q$ then $t$ has a primitive $p^*$-representation in $\tilde{Q}$ for all $\tilde{Q} \not\sim_p Q$.

Next, we give several results from $\text{DH14b}$. This paper deals with the following problem. Given a quadratic form $Q$ in $n$-variables, a prime $p$, and integers $k, t$ find a solution of $x^TQx \equiv t \mod p^k$, if it exists. Note that it is easy (i.e., polynomial time tester exists) to test if $t$ has a $p^k$-representation in $Q$.

**Theorem 11** Let $Q^n$ be an integral quadratic form, $p$ be a prime, $k$ be a positive integer, $t$ be an element of $\mathbb{Z}/p^k\mathbb{Z}$. Then, there is a polynomial time algorithm (Las Vegas for odd primes and deterministic for the prime 2) that performs $O(n^{1+\omega} \log k + nk^3 + n \log p)$ ring operations over $\mathbb{Z}/p^k\mathbb{Z}$ and outputs a primitive $p^k$-representation of $t$ by $Q$, if such a representation exists. The time complexity can be improved for the following special cases.

| Type  | $p$ odd | $p = 2$ |
|-------|---------|---------|
| $I$   | $O(\log k + \log p)$ | $O(k)$ |
| $II$  | $O(k \log k)$ |         |
Next, we give necessary and sufficient conditions for a Type II block to represent an integer \( t \). A proof of this result can also be found in \[DH14b\].

**Lemma 12** Let \( Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \), \( b \) odd be a type II block, and \( t, k \) be positive integers. Then, \( Q \) represents \( t \) primitively over \( \mathbb{Z}/2^k\mathbb{Z} \) if \( \text{ord}_2(t) = 1 \).

**Additional Notation.** For convenience, we introduce the following notations, where \( q \) is a positive integer.

\[
k_p = \begin{cases} 
3 & \text{if } p = 2, \text{ and } \ p \text{ odd prime.} \\
1 & \text{if } p \text{ odd prime.}
\end{cases}
\]

\[
\Upsilon = q \prod_{p|2q} p^{k_p}
\]

\[
\mathcal{P}_q = \{ p \mid \text{ord}_p(2) > 0 \}
\]

\[
\text{SGN}^\times = \{1, 3, 5, 7\}
\]

\[
T^+ = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \quad T^- = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\]

### 3 Technical Overview

In this section, we give an overview of our algorithm. Note that this algorithm does not run in polynomial time. The final version of the algorithm, which is correct and runs in polynomial time will be presented in Section 6.

Before describing the algorithm, we need to describe the input to the algorithm. The question of succinct specification of a genus has several competing answers \[Wat76, Kit99, CS99, O’M73\]. In this work, we use the specification by Conway-Sloane, called the symbol; with the property that two quadratic forms are in the same genus iff they have the same symbol. An intuitive overview of the symbol follows (see Section 4.1 for the formal version).

It can be shown that two quadratic forms \( Q_1^* \) and \( Q_2^* \) are in the same genus iff:

(i) \( \text{det}(Q_1) = \text{det}(Q_2) \),

(ii) \( \text{sgn}(Q_1) = \text{sgn}(Q_2) \),

(iii) \( Q_1 \sim_\ast Q_2 \), for every prime \( p \) that divides \( 2 \text{det}(Q_1) \).

The set of primes \( \mathcal{P}_\Upsilon = \{ p \mid \text{ord}_p(2) > 0 \} \) is called the set of relevant primes for the genus \( \Upsilon \). The question now reduces to finding the necessary and sufficient condition for \( p^\ast \)-equivalence. It can be shown that two quadratic forms \( Q_1^* \) and \( Q_2^* \) are \( p^\ast \)-equivalent iff (a) \( \text{ord}_p(\text{det}(Q_1)) = \text{ord}_p(\text{det}(Q_2)) \), and (b) \( Q_1 \sim_\ast Q_2 \), for \( k = \text{ord}_p(\text{det}(Q_1)) + k_p \). These two conditions can be written in an equivalent way, using what is called the \( p \)-symbol of a quadratic form. The key property is that two quadratic forms are \( p^\ast \)-equivalent iff they have the same \( p \)-symbol. For now, we can think of \( \text{can}_p(Q) \) as the \( p \)-symbol of \( Q \). By definition of \( p \)-canonical forms, it follows that \( Q_1 \sim_\ast Q_2 \) iff \( \text{can}_p(Q_1) = \text{can}_p(Q_2) \).

**Symbol.** One can now give an informal description of the symbol. Intuitively, think of \( \text{sym}_p(Q) \) as the tuple \( (p, \text{can}_p(Q)) \). Then, by the definition of canonical forms, two quadratic form are \( p^\ast \)-equivalent iff they have the same \( p \)-symbol. The symbol of a quadratic form \( Q \) is the list of tuples \( (p, \text{can}_p(Q)) \), one for each prime that divides \( 2 \text{det}(Q) \) along with its signature i.e.,

\[
\text{sym}(Q) = \{(-1, \text{sgn}(Q))\} \bigcup \{(p, \text{can}_p(Q))\} \quad \text{for each } p|2 \text{det}(Q)
\]

Note that the determinant of \( Q \) is missing from the symbol. This is because it is possible to find the determinant from the symbol \( \text{sym}(Q) \). This is done by calculating \( \text{ord}_p(\text{det}(Q)) \) for each prime \( p \). The relevant primes of \( \text{sym}(Q) \) can be read from the first component in of the tuples in \( \text{sym}(Q) \). And, for a relevant prime \( p \) it can be shown that \( \text{ord}_p(\text{det}(Q)) = \text{ord}_p(\text{det}(\text{can}_p(Q))) \).
The notation $\Upsilon^n$ will denote both a genus and the symbol of the genus, depending on the context. The notation $\Upsilon_p$ will denote the $p$-symbol of the genus $\Upsilon$. The set of relevant primes for the symbol $\Upsilon$ is denoted by $\mathbb{P}_\Upsilon = \{p \mid \text{ord}_p(2 \det(\Upsilon)) > 0\}$. If $\text{sym}_p(\mathcal{Q})$ is the $p$-symbol of $\mathcal{Q}$ then $I_p(\mathcal{Q})$ denotes the set of $p$-scales of $\mathcal{Q}$. The size of the symbol of a genus $\Upsilon^n$ of determinant $d$ is $O(n|\mathbb{P}_\Upsilon| \log d)$.

**Local Form.** Given a symbol $\Upsilon^n$ and a positive integer $q$, it is possible to construct a quadratic form $S^n$ such that $S^n \sim Q$ for every $Q \in \Upsilon$; in $\text{poly}(n, \log \det(\Upsilon), \log q)$ time. Such a form is said to be locally equivalent to the genus $\Upsilon$ over $\mathbb{Z}/q\mathbb{Z}$ and is denoted by $S^n \sim \Upsilon$. The local form satisfies the following important property. For every prime $p$ that divides $q$, $\text{ord}_p(\det(S)) = \text{ord}_p(\det(\Upsilon))$. Note that $S$ does not need to have determinant $\det(\Upsilon)$ and may not be equivalent to $\Upsilon$ over $\mathbb{R}$. In particular, $S \not\in \Upsilon$.

**Primitive Representation.** If $t$ has a primitive $p^*$-representation in $\mathcal{Q}$ then, by Theorem 10, $t$ has a primitive $p^*$-representation in every quadratic form in the genus $\text{Gen}(\mathcal{Q})$. Hence, the primitive representativeness of an integer $t$ by a quadratic form $\mathcal{Q}$ only depends on the symbol $\text{sym}(\mathcal{Q})$.

**Definition 6** We say that an integer $t$ has a primitive representation in a genus $\Upsilon$ if $t$ has a primitive $p^*$-representation in $\Upsilon$ for all $p \in \{-1, 2\} \cup \mathbb{P}$.

**Simple Version of the Algorithm.** Let $\Upsilon^n$ be a symbol of non-empty genus. In this section, we give the simple version of the algorithm. The run time of the algorithm is not polynomial; mainly because the exponential blowup in the determinant after each recursive step. In Section 6, we show that by carefully selecting the embedding $x$ of $t$ and by simplifying the input before each recursive call, it is possible to avoid the blowup and show a polynomial bound on the runtime.

**GENSIMPLE** (input: $\Upsilon^n$) output: $Q^n \in \Upsilon$

1. If $n = 1$ then $Q = \det(\Upsilon)$.
2. Let $t$ be an integer such that $t$ has a primitive representation in $\Upsilon$.
3. Let $q = t^{n-1}\det(\Upsilon)$. Find $S$ such that $S \sim \Upsilon$.
4. Find a primitive $q$-representation $x$ such that $x'Sx \equiv t \mod q$.
5. Extend $x$ by $A$ so that $[x, A] \in \text{GL}_n(\mathbb{Z}/q\mathbb{Z})$.
6. Compute the following quantities.
   
   \[ d := x'SA \mod q \quad H := (tA'SA - d'd) \mod q \quad \text{mod } q \]  
   \[ (7) \]

7. Define the symbol $\tilde{\Upsilon}^{n-1}$ as follows.
   
   \[ \tilde{\Upsilon}_p = \begin{cases} 
   \text{sig}(t) (\text{sig}(\Upsilon) - \text{sig}(t)) & \text{if } p = -1 \\
   \text{sym}_p(H) & \text{if } p \text{ divides } q \\
   \text{sym}_p(\Upsilon^{n-2} \oplus t^{n-2} \det(\Upsilon)) & \text{otherwise}
   \end{cases} \]
   \[ (8) \]

8. Let $\tilde{H} = \text{GENSIMPLE}(\tilde{\Upsilon})$.
9. Find $\tilde{U} \in \text{GL}_{n-1}(\mathbb{Z}/q\mathbb{Z})$ such that $\tilde{H} \equiv \tilde{U}'\tilde{H}\tilde{U} \mod q$.
10. Output $Q = \left( \frac{t}{(d\tilde{U})'}, \frac{d\tilde{U}}{d'd\tilde{U}} \right)$, where the division in the lower right is (usual) rational division.
3.1 Intuitive Description of the Algorithm

For simplicity, we assume that \( \text{sig}_{-1}(\mathcal{Y}) = n \) i.e., the genus \( \mathcal{Y} \) is the genus of positive definite quadratic forms or Gram matrix of lattices.

In order to find a Gram matrix \( \mathcal{Q} \) in the genus \( \mathcal{Y} \), we start by finding a value \( t \) such that \( \mathcal{Q} \) has the following form.

\[
\mathcal{Q} = \begin{pmatrix}
    t & w \\
    w' & \tilde{Q}
\end{pmatrix}
\]  

(9)

It turns out that it suffices to find an integer \( t \) which has a primitive representation in the genus \( \mathcal{Y} \). The next step is best explained by thinking in terms of lattices.

The Gram matrix \( \mathcal{Q} \) equals \( \mathcal{B}' \mathcal{B} \), where \( \mathcal{B} = [b_1, \ldots, b_n] \) is a basis of a lattice with Gram matrix \( \mathcal{Q} \). Because \( \mathcal{Q} \) has the form given in Equation (9) \( b_1, b_1 = t \). Consider now the (possibly non-integral) lattice one obtains by projecting \([b_1, \ldots, b_n]\) onto the subspace orthogonal to \( b_1 \). It is possible to show that the Gram matrix of this lattice in \((n - 1)\)-dimensions is given by \( \mathcal{Q} - \frac{w'w}{t} \).

The matrix \( \mathcal{Q} \), and the matrix \( w'w \) are integral. Thus, \( t\mathcal{Q} - w'w \) is a Gram matrix of an integral lattice. To find \( \mathcal{Q} \), we therefore (a) find the symbol of the lattice \( t\mathcal{Q} - w'w \) and recursively find a corresponding lattice, and (b) find \( w \). To solve (a), the algorithm above constructs a locally equivalent quadratic form \( S \), then finds a representation \( x \) of \( t \) into \( S \), and transforms \( S \) with a transformation \([x, A] \in \text{GL}_n(\mathbb{Z}/q\mathbb{Z})\) which maps \( t \) into the top left

\[
[x, A]'[x, A] = \begin{pmatrix}
    t & d' \\
    d & \tilde{A}'SA
\end{pmatrix} \mod q
\]

(10)

To solve (b), it is possible to show that one can recover \( w \) from the vector \( d \).

Finally, one can show that because of the way we chose \( q \) in the algorithm, the expression \( \frac{t + 0'd'd0}{t} \) in the construction of \( \mathcal{Q} \) is actually integral.

3.2 Comparisons to Hartung’s Algorithm.

Our construction in Section 3 is similar to the algorithm given by Hartung [Har08]. This algorithm, as we will show, is not polynomial but \( O(n^n) \). There are other severe problems with Hartung’s work (i) several lemmas are incorrect because of insufficient care while handling prime 2, and (ii) the construction of \( t \), in case of dimension 2, is short but unfortunately incorrect. This construction takes us several pages (page 24-33).

A detailed discussion of the comparison follows.

The first non-trivial step is to construct an integer \( t \) which is primitively representatable in the genus \( \mathcal{Y} \). Hartung constructs a \( t \) such that \( t = 2s \), where \( s \) divides \( \text{det}(\mathcal{Y}) \) and \( \phi \) is a prime which does not divide \( \text{det}(\mathcal{Y}) \). This construction seems correct when \( n \geq 3 \) but incorrect for \( n = 2 \). One of the reasons is the treatment of the prime 2: which is not thorough. The prime 2 has been known to create problems, if not handled correctly [Pal65, Min10, Wat76].

For example, Lemma 3.3.1 [Har08] is incorrect for \( p = 2 \) because it is not possible to divide by 2 over the ring \( \mathbb{Z}/2^k\mathbb{Z} \) at the end of the proof. This leads to an easy counter example for Lemma 3.3.2, which claims that a quadratic form \( \mathcal{Q}^{n \geq 3} \) with \( \text{det}(\mathcal{Q}) \in (\mathbb{Z}/p\mathbb{Z})^* \) represents every integer \( t \) primitively over \( \mathbb{Z}/p^k\mathbb{Z} \) for all positive integers \( k \). A counter example is \( \mathcal{Q} = x^2 + y^2 + z^2, p = 2, k = 3, t = 7 \). By exhaustive search, it can be verified that \( x^2 + y^2 + z^2 \) does not represent 7 modulo 8. This mistake becomes more severe in the construction of \( t \) for \( \mathcal{Y}^{n = 2} \). The construction in this case is highly non-trivial and needs a separate treatment (page 24-33).

The construction of \( t \) for \( \mathcal{Y}^{n \geq 3} \) seems to be correct [Har08]. Our construction, though, gives smaller \( t \). Hartung needs a prime \( \phi \) which does not divide the determinant, each time he needs to find a primitively representable integer \( t \). In contrast, we need such a prime only once. Note that the construction of such a prime \( \phi \) takes polynomial time if ERH holds.
The most serious issue is that the algorithm by Hartung is not polynomial time. He argues that each step in the algorithm is polynomial and because each recursive step reduces the dimension by 1, overall the algorithm is polynomial. This is not true because after finding $t$ and reducing to one less dimension to a symbol $\tilde{\Upsilon}^{n-1}$, $\det(\tilde{\Upsilon}) = t^{n-2} \det(\Upsilon)$ (see Claim 1). The upper found on $t$ is $\det(\Upsilon)$ and so $\det(\tilde{\Upsilon})$ can be as large as $\det(\Upsilon)t$ leading to a blowup $\sim \det(\Upsilon)n$ if used $n$ times recursively. As it is, the time complexity of Hartung’s algorithm is proportional to $O(n^n)$. In contrast, our construction represents $t$ in a specific way and uses the property of this representation to show that the determinant blows up by $2^{n^2}$ at most; resulting in a polynomial time algorithm (see Section 8).

4 Formalizing the Input

This section describes the input to the algorithm which boils down to the question of a succinct representation of the genus.

4.1 Symbol of a Quadratic Form

There are several equivalent ways of giving a description of the $p^*$-equivalence [CS99, Kit99, O’M73, Cas78]. In this work, we go with a modified version of the Conway-Sloane description, called the $p$-symbol of a quadratic form. Our modification gets rid of the need to use the $p$-adic numbers. Note that $p$-adic numbers are a staple in this area and we are not aware of any work which does not use them [Kit99, O’M73, Sie35].

The $(-1)$-symbol of a quadratic form is equal to the signature of the quadratic form.

4.1.1 $p$-symbol, $p$ odd prime

Let $k = \text{ord}_p(\det(Q)) + 1$ and $D$ be the diagonal quadratic form which is $p^k$-equivalent to $Q$ (see Theorem 7). Then, $D$ can be written as follows.

$$D = D_0^n + pD_1^n + \ldots + p^kD_k^n + \ldots \quad i \leq \text{ord}_p(\det(Q)),$$

where $D_0, \ldots, D_{k-1}$ are diagonal quadratic forms, $\sum_i n_i = n$ and $p$ does not divide $\det(D_0) \ldots \det(D_{k-1})$. Let $I_p(Q)$ is the set of $p$-orders $i$ with non-zero $n_i$. Then, the $p$-symbol of $Q$ is defined as the set of scales $i$ occurring in Equation 11 with non-zero $n_i$, dimensions $n_i = \dim(D_i)$ and signs $\epsilon_i = \left(\frac{\det(D_i)}{p}\right)$.

$$\text{sym}_p(Q) = \left\{(p, i, \left(\frac{\det(D_i)}{p}\right), n_i) \mid i \in I_p(Q)\right\}$$

The following fundamental result follows from Theorem 9, page 379 [CS99] and Theorem 9.

**Theorem 13** For $p \in \{-1\} \cup \mathbb{P}$, two quadratic forms are $p^*$-equivalent iff they have the same $p$-symbol.

4.1.2 2-symbol

Let $k = \text{ord}_2(\det(Q)) + 3$ and $D$ be the block diagonal form which is $2^k$-equivalent to $Q$ (Theorem 7). Then, $D$ can be written as follows.

$$D = D_0^n + 2D_1^n + \ldots + 2^kD_k^n + \ldots \quad i \leq \text{ord}_2(\det(Q)),$$
where \( \det(D_0), \ldots, \det(D_i), \ldots \) are odd, \( \sum_i n_i = n \) and each \( D_i \) is in block diagonal form according to Definition 8. The 2-symbol of \( 'D_i \) are the following quantities.

\[
\begin{pmatrix}
  i \\
  n_i = \dim(D_i) \\
  \epsilon_i = \frac{\det(D_i)}{2} \\
  \text{type}_i = \text{I or II} \\
  \text{odt}_i \in \{0, \ldots, 7\}
\end{pmatrix}
\]

\[\text{scale of } D_i \]
\[\text{dimension of } D_i \]
\[\text{sign of } D_i \]
\[\text{type of } D_i \]
\[\text{odity of } D_i \]

Let the set of scales \( i \), with non-zero \( n_i \), be denoted \( \mathbb{I}_2(Q) \). Then, the 2-symbol of \( Q \) is written as follows.

\[
\text{sym}_2(Q) = \{(2, i, \epsilon_i, n_i, \text{type}_i, \text{odt}_i) \mid i \in \mathbb{I}_2(Q)\}
\]

In contrast to the \( p \in \{ -1 \} \cup \mathbb{P} \) case, two \( 2^* \)-equivalent quadratic forms may produce two different 2-symbols. These symbols are then said to be 2-equivalent.

Consider the following useful generalization of the function \( p \)-order.

**Definition 7** Let \( p \in \{ 2 \} \cup \mathbb{P} \) be a prime, and \( \Upsilon \) be a symbol with \( \Upsilon_p = \{(p, i, \epsilon_i, *, *, *)\}, \) where * is empty in case \( p \) is odd. Then, \( \text{ord}_p(\Upsilon) \) is defined as \( \arg \max_i \{i \in \mathbb{I}_p(\Upsilon)\} \).

### 4.2 Reduced Symbol

The set \( \bigcup_{p \in \{ -1, 2 \} \cup \mathbb{P}} \Upsilon_p \), where \( \Upsilon_p \) is a \( p \)-symbol, is a complete description of a genus and can be used as an input to the algorithm. Unfortunately, this description is too long because there are infinitely many primes. The following lemma helps us in giving a shorter description.

**Lemma 14** Let \( Q^n \) be a lattice with determinant \( d \) and \( p \) be an odd prime that does not divide \( d \). Then, \( Q \not\sim d \oplus \mathbb{I}^{n-1} \).

**Proof:** Let \( p \) be an odd prime that does not divide \( d \) and \( D = d_1 \oplus \cdots \oplus d_n \) be the diagonal matrix which is equivalent to \( Q \) over \( \mathbb{Z}/p\mathbb{Z} \). Then, \( p \) does not divide \( d_1 \cdots d_n \) and

\[
\text{sym}_p(Q) = \left\{ \left(p, 0, n, \left(\frac{\det(D)}{p}\right)\right) \right\}
\]

By the definition of the \( p \)-symbol, \( D \not\sim Q \). It follows that there is a \( U \in \text{GL}_n(\mathbb{Z}/p\mathbb{Z}) \) such that \( D \equiv U'QU \mod p \). But then, \( \det(D) \equiv \det(U)^2 \mod p \) and

\[
\left(\frac{\text{cpr}_p(\det(D))}{p}\right) \equiv \left(\frac{\text{cpr}_p(\det(Q) \det(U)^2)}{p}\right) = \left(\frac{\text{cpr}_p(d)}{p}\right).
\]

This implies that \( d \oplus \mathbb{I}^{n-1} \) has the same \( p \)-symbol as \( Q \), completing the proof (Theorem 13). \( \square \)

Our input to the algorithm is the symbol of a genus, defined as follows.

**Definition 8** Let \( Q \) be an integral quadratic form from a given genus. Then, the symbol the genus is defined as the set

\[
\bigcup_{p \in \{ -1 \} \cup \{ p \mid \text{ord}_p(2 \det(Q)) > 0 \}} \text{sym}_p(Q)
\]

12
From Theorem 13, it follows that two quadratic forms are in the same genus if they are $2^*$-equivalent and have the same $p$-symbol for each $p \in \{-1\} \cup \mathbb{P}$.

The following theorem is a direct implication of [Theorem 9, CS99, page 379], [Theorem 10, CS99, page 381], and the definition of the symbol.

**Theorem 15** Let $Q_1^n$ and $Q_2^n$ be two integral quadratic forms. Then, the following statements are equivalent.

(a) $Q_1 \in \text{Gen}(Q_2)$,  
(b) $\det(Q_1) = \det(Q_2)(= d)$, $Q_1 \underset{p}{\sim} Q_2$, and $Q_1 \overset{F}{\sim} Q_2$  
(c) $Q_1 \overset{F}{\sim} Q_2, \forall p \in \{-1\} \cup \{p \mid \text{ord}_p(2 \det(Q_1,Q_2)) > 0\}$.

Note that Theorem 15 implies that every quadratic form in a particular genus has the same determinant (for a proof see page 139, Lemma 4.1, Cas78). The determinant of a genus can be computed from its symbol.

We simplify the input description further by introducing the notion of reduced symbol.

**Definition 9** A symbol $\Upsilon$ is reduced if for every relevant prime $p \in \mathbb{P}_\Upsilon$ the $p$-scale 0 appears in $I_p(\Upsilon)$.

If $i_p = \min\{i \in I_p(\Upsilon)\}$ then $\Upsilon$ is $p^{i_p}$-equivalent to a matrix which is identically 0. Thus, if $D$ is a quadratic form with $D \overset{F}{\sim} \Upsilon$ then every entry of $D$ is divisible by $p^{i_p}$. Given a genus $\Upsilon$, we define the following quantity.

$$\gcd(\Upsilon) = \prod_{p \in \mathbb{P}_\Upsilon} p^{i_p}$$  

(16)

The reduced symbol corresponding to the symbol $\Upsilon$ can now be defined as follows.

$$\text{red}(\Upsilon) = \left\{ \text{sym}_p\left(\frac{D}{p^{\min\{i \in I_p(\Upsilon)\}}}ight) \mid p \in \mathbb{P}_\Upsilon, D \overset{F}{\sim} \Upsilon_p \right\} \cup \{\text{sym}_{(-1)}(\Upsilon)\}$$  

(17)

To find a quadratic form from a genus $\Upsilon$, it suffices to find a quadratic form in genus $\text{red}(\Upsilon)$, the proof of which is as follows.

**Lemma 16** Let $\Upsilon$ be a genus, and for each prime $p \in \mathbb{P}_\Upsilon$, $i_p$ be the integer $\min\{i \in I_p(\Upsilon)\}$. Then, $Q \in \text{red}(\Upsilon)$ iff $\gcd(\Upsilon)Q \in \Upsilon$.

**Proof:** Let $p \in \mathbb{P}_\Upsilon$ be a prime. If $S$ is a quadratic form such that $\text{sym}(S) = \Upsilon$ then every entry of $S$ is divisible by $p^{i_p}$. We define a quadratic form $Q$ as follows.

$$Q = \frac{S}{\prod_{p \in \mathbb{P}_\Upsilon} p^{i_p}}$$

By definition of $\text{red}(\Upsilon)$, $\text{sym}_p(Q) = \text{sym}_p(\text{red}(\Upsilon))$ for all $p \in \{-1\} \cup \mathbb{P}_\Upsilon$. Thus, $Q \in \text{red}(\Upsilon)$.

Conversely, if $Q \in \text{red}(\Upsilon)$ then $p^{i_p}Q$ has the same $p$-symbol as $\Upsilon_p$. Thus, $\left(\prod_{p \in \mathbb{P}_\Upsilon} p^{i_p}\right)Q$ has symbol $\Upsilon$.  

\[ \square \]

### 4.3 Valid Symbol

We now define the following three conditions on the symbol $\Upsilon$.

**Determinant Condition.** For every prime $p \in \{2\} \cup \mathbb{P}$ such that $\Upsilon_p = \left\{(p, i, \epsilon_i, n_i, *, *) \mid i \in I_p(\Upsilon)\right\}$, where $*$ is empty for odd primes;

$$\left(\frac{\text{cpr}_p(\det(\Upsilon))}{p}\right) = \prod_{i \in I_p(\Upsilon)} \epsilon_i$$  

(18)
Oddity Condition. The symbol $\Upsilon$ satisfies the oddity equation i.e.,
\[
sig(\Upsilon) + \sum_{p \in \mathcal{P}} \text{exs}_p(\Upsilon) \equiv \text{odt}(\Upsilon) \mod 8 \tag{19}
\]

Jordan Condition. Let $p$ be an odd prime and $\Upsilon_p = \{(p, i, \epsilon_i, n_i) \mid i \in \mathbb{I}_p(\Upsilon)\}$, then for each Jordan constituent $(p^i, \epsilon_i, n_i)$, we must have
\[
\text{if } n_i = 0 \text{ or } p = -1 \text{ then } \epsilon_i = + \tag{20}
\]

For $p = 2$, let $\text{sym}_2(\mathcal{Q}) = \{(2, i, \epsilon_i, n_i, \text{type}_i, \odt_i) \mid i \in \mathbb{I}_2(\Upsilon)\}$, then $\Upsilon$ satisfies the following conditions.
\[
\text{for } n_i = 0, \text{type}_i = \text{II and } \epsilon_i = + \Rightarrow s_i \equiv \pm 1 \mod 8
\]
\[
\text{for } n_i = 1, \left\{ \begin{array}{l}
\epsilon_i = + \Rightarrow s_i \equiv 0 \text{ or } \pm 2 \mod 8 \\
\epsilon_i = - \Rightarrow s_i \equiv 4 \text{ or } 2 \mod 8
\end{array} \right. \tag{21}
\]

The set of conditions are taken from [CS99, page 382-383]. A symbol $\Upsilon$ which satisfies these three conditions will be called valid.

5 \text{ q-equivalent forms, } q \text{ composite}

Given a valid symbol $\Upsilon^n$, it is useful to construct a quadratic form $\mathcal{Q}^n$ which is $q$-equivalent to $\Upsilon$ for a given positive integer $q$ (see Step 3 of GENSIMPLE).

The following is a helper lemma which shows how to construct a quadratic form $\mathcal{Q}$ such that $\mathcal{Q}^{p^*} \sim \Upsilon$.

**Lemma 17** There exists a randomized algorithm that takes a symbol $\Upsilon^n$ of determinant $d$, and a prime $p$ as input; performs $O(n + \log^3 p)$ ring operations over $\mathbb{Z}/p^{\text{ord}_p(d)+k}p\mathbb{Z}$; and outputs a block diagonal quadratic form $\mathcal{Q}^n$ such that $\mathcal{Q}^{p^*} \sim \Upsilon$.

**Proof:** There are three different constructions: for the prime 2, for relevant odd prime and for the odd prime that does not divide $\text{det}(\Upsilon)$.

(1.) The first and simplest construction deals with odd primes $p$ that do not divide $\text{det}(\Upsilon)$. By Lemma 14, $\Upsilon_p^{p^*} \sim \text{det}(\Upsilon) \oplus \mathbb{I}^{n-1}$. Hence, we set $\mathcal{Q} = \mathbb{I}^{n-1} \oplus \text{det}(\Upsilon)$.

(2.) The second type of primes are odd primes that divide $\text{det}(\Upsilon)$. Let $\Upsilon_p = \{(p, i, \epsilon_i, n_i) \mid i \in \mathbb{I}_p(\Upsilon)\}$. We use rejection sampling to find a quadratic non-residue modulo $p$, say $\tau_p$. Note that generating a random non-zero element from $\mathbb{Z}/p\mathbb{Z}$ yields a quadratic non-residue with probability $1/2$. The matrix $\mathcal{Q}$, in this case, is generated as follows.
\[
\mathcal{Q} = \bigoplus_{i \in \mathbb{I}_p(\Upsilon)} p^i \mathcal{D}_i \quad \mathcal{D}_i = \left\{ \begin{array}{l}
\mathbb{I}^{n_i} \quad \text{if } \epsilon_i = 1 \\
\mathbb{I}^{n_i-1} \oplus \tau_p \quad \text{otherwise}
\end{array} \right. \tag{22}
\]

(3.) The only remaining case is of the prime 2. Let $\Upsilon_2 = \{(2, i, \epsilon_i, n_i, \text{type}_i, \odt_i) \mid i \in \mathbb{I}_2(\Upsilon)\}$. Then, the
Table 1: Exhaustive List of Type I forms for $n = 2$

| Form          | $\epsilon$ | odt |
|---------------|------------|-----|
| $1 \oplus 7, 3 \oplus 5$ | +          | 0   |
| $1 \oplus 1, 5 \oplus 5$ | +          | 2   |
| $3 \oplus 3, 7 \oplus 7$ | +          | 6   |
| $3 \oplus 7$ | −          | 2   |
| $1 \oplus 3, 5 \oplus 7$ | −          | 4   |
| $1 \oplus 5$ | −          | 6   |

Table 2: List of Type I forms for $n = 3$

| $\epsilon$ | odt | Form          |
|------------|-----|---------------|
| +          | 1   | $1 \oplus 1 \oplus 7$ |
| 3          | 1   | $1 \oplus 1 \oplus 1$ |
| 5          | 7   | $7 \oplus 7 \oplus 7$ |
| 7          | 1   | $1 \oplus 7 \oplus 7$ |
| −          | 1   | $3 \oplus 3 \oplus 3$ |
| 3          | 3   | $3 \oplus 3 \oplus 5$ |
| 5          | 1   | $1 \oplus 1 \oplus 3$ |
| 7          | 1   | $1 \oplus 1 \oplus 5$ |

The quadratic form $Q$ is defined as $Q = \oplus_{i \in I} 2^i D_i$, where $D_i$ is defined as follows.

$$D_i^n = \begin{cases} T^+ \oplus \cdots \oplus T^+ \oplus T^- & \text{if } \epsilon_i = -1, \text{odt}_i = 2 \\ T^+ \oplus \cdots \oplus T^+ & \text{if } \epsilon_i = 1, \text{odt}_i = 2 \\ I^{n_i-3} \oplus D_i=3 & \text{odt}_i \in \{0, \cdots, 7\}, n_i > 3 \end{cases} \quad (23)$$

If $n_i = 1$ then $D_i$ has to be equal to odt$_i$. For $n_i = 2$, we exhaustively list all possible Type I forms in Table I. We observe that two situations are not possible: $\epsilon = +, \text{odt} = 4$ and $\epsilon = -, \text{odt} = 0$. For $n_i = 3$, we list forms for all possible choices of $\epsilon$ and odt.

The $D$ in Equation (23) is defined as follows. Suppose we are looking for a type I form in dimension $n_i > 3$ with odt$_i \in \{0, \cdots, 7\}$ and Legendre-Jacobi symbol $\epsilon_i$. In this case, we choose $D$ as the form in Table 2 with odt $= \text{odt}_i - (n_i - 3) \mod 8$ and $\epsilon = \epsilon_i$.

By construction $\text{sym}_2(Q) = \Upsilon_2$.

The algorithm needs to generate a quadratic non-residue modulo $p$ and hence performs $O(n + \log^3 p)$ ring operations. \hfill \Box

**Theorem 18** Let $T^n$ be a symbol and $q$ be a composite integer. Then, there is a randomized $\text{poly}(n, \log q)$ algorithm that takes $(\Upsilon, q)$, along with a factorization of $q$ as input; and produces a quadratic form $Q$ such that $Q \equalmod \Upsilon$.

**Proof:** For each $p \in P_\Upsilon$, we use Lemma 17 to generate $Q_p$ such that $Q_p \equalmod \Upsilon$. We now solve the following system of congruences using the Chinese Remainder Theorem.

$$Q \equiv Q_p \mod p^\text{ord}_p(q) \quad p \in P_q$$
6 Existence of a Quadratic Form with a given Symbol

In this section, we answer the following question. Given a symbol \( \Upsilon \), how does one verify that the genus corresponding to \( \Upsilon \) is non-empty i.e., there exists a quadratic form \( Q \) such that \( \text{sym}(Q) = \Upsilon \).

**Theorem 19** Let \( \Upsilon \) be a valid symbol (i.e., satisfies the determinant, oddity and the Jordan conditions); then there exists an integral quadratic form \( Q \) such that \( \text{sym}(Q) = \Upsilon \).

This is a well known theorem [Theorem 11, CS99, page 383]. A proof can also be found in [O’M73]. Our work, not only shows the existence, but also generates a form in polynomial time. In this section, we show that the algorithm GenSimple, if successful, generates a quadratic form from the correct genus.

**Proof:** (Theorem 19) Let \( \Upsilon \) be a valid input symbol. Run the GenSimple algorithm on \( \Upsilon \). We show that if the algorithm outputs a quadratic form then it must be from the genus \( \Upsilon \).

We prove several claims regarding the matrices constructed during the algorithm GenSimple.

**Claim 1** The determinant of the genus \( \tilde{\Upsilon} \) is \( t^{n-2} \det(\Upsilon) \). Also, \( t^{n-2} \det(\Upsilon) \) divides \( \det(H) \).

**Proof:** Recall that \( S \) is a quadratic form which is equivalent to \( \Upsilon \) over \( \mathbb{Z}/q\mathbb{Z} \). Let \( d \) be the row vector and \( H \) be the matrix defined in Equation (7). Note that all entries of these matrices i.e., \( d, H \) are integers. Define the matrix \( M = \begin{pmatrix} t & d \\ d' & (H + d'd)/t \end{pmatrix} \). By definition, \( H + d'd \equiv tA'SA \mod q \). The integer \( t \) divides \( q \). But then, each entry in the matrix \( H + d'd \) is divisible by \( t \) i.e., \((H + d'd)/t\) is a matrix over integers. Thus, \( M \) is a matrix over integers and the following equality implies that \( \det(H) = \det(M)/t^{n-2} \).

\[
M = \begin{pmatrix} t & d \\ d' & (H + d'd)/t \end{pmatrix} = \begin{pmatrix} 1 & d/t \\ 0 & I \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & H/t \end{pmatrix} \begin{pmatrix} 1 & d/t \\ 0 & I \end{pmatrix}
\]

From Equation (7) it follows that \((H + d'd)/t \equiv A'SA \mod q/t\). By definition, \( M \equiv (xA)'S(xA) \mod q/t \).

Hence,

\[
\det((xA)'S(xA)) \equiv \det(M) \mod q/t = \frac{\det(H)}{t^{n-2}} \mod q/t
\]

\[
t^{n-2} \det(S) \det(xA)^2 \equiv \det(H) \mod q/t^{n-3}
\]

Let \( p \) be a prime divisor of \( q \). Recall \((xA) \in GL_n(\mathbb{Z}/q\mathbb{Z})\). But then, \( p \) does not divide \( \det(xA) \). From the fact that \( S \) is equivalent to \( \Upsilon \) over \( \mathbb{Z}/q\mathbb{Z} \), it follows that \( \text{ord}_p(\det(S)) = \text{ord}_p(\det(\Upsilon)) \). By definition of \( q \), it follows that \( \text{ord}_p(q) > \text{ord}_p(\det(\Upsilon)) \). But then,

\[
\text{ord}_p(t^{n-2} \det(S) \det(xA)^2) = \text{ord}_p(t^{n-2}) + \text{ord}_p(\det(S))
\]

\[
= \text{ord}_p(t^{n-2} \det(\Upsilon)) < \text{ord}_p(q)
\]

From Equation (25) and Equation (26) we conclude that for all primes \( p \) that divide \( q \), \( \text{ord}_p(\det(H)) = \text{ord}_p(t^{n-2} \det(S)) = \text{ord}_p(t^{n-2} \det(\Upsilon)) \).

It also follows from the definition of the symbol that \( \text{ord}_p(\det(\tilde{\Upsilon})) = \text{ord}_p(\det(H)) \) for all relevant primes of symbol \( \tilde{\Upsilon} \). But we showed that \( \text{ord}_p(\det(H)) = \text{ord}_p(t^{n-2} \det(\Upsilon)) \) for every relevant prime \( p \) of \( \tilde{\Upsilon} \). Thus, \( \det(\tilde{\Upsilon}) = t^{n-2} \det(\Upsilon) \).

We now show that \( \tilde{\Upsilon} \) defined in Equation (8) is a valid symbol.
Theorem 20 Let $\Upsilon^{n>1}$ be a valid symbol, $t$ be a primitively representable integer in $\Upsilon$, $q = t^{n-1}\det(\Upsilon)$, $S^n$ be an integral quadratic form with $S \not\sim \Upsilon$, $x \in (\mathbb{Z}/q\mathbb{Z})^n$ be a primitive vector with $x'Sx \equiv t \mod q$, $A$ be such that $[x, A] \in \text{GL}_n(\mathbb{Z}/q\mathbb{Z})$, $d := x'SA \mod q$, $H^{n-1} := (tA'SA - d'd) \mod q$ and $\tilde{\Upsilon}$ be as defined in Equation 5. Then $\tilde{\Upsilon}$ is a valid symbol.

Proof: We divide the proof in three items, one for each condition.

(i). (Oddity Condition) Consider the matrix $M$ over integers (note that $t$ divides both $q$ and $H + d'd$ and so $(H + d'd)/t$ is integral).

$$M = \begin{pmatrix} t & d \\ d' & (H + d'd)/t \end{pmatrix} = \begin{pmatrix} 1 & d/t \\ 0 & I \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & H/t \end{pmatrix} \begin{pmatrix} 1 & d/t \\ 0 & I \end{pmatrix}$$

The matrix $V = \begin{pmatrix} 1 & d/t \\ 0 & I \end{pmatrix}$ is over rationals and has determinant 1. Thus, $M$ is equivalent to $\text{diag}(t, H/t)$ over rationals. By construction, $M \not\subset S \not\subset \Upsilon$. From [Theorem 3, [CS99, page 372]], it follows that there exists a (rational) quadratic form $X$ which is equivalent to $\tilde{\Upsilon}$ over rationals. Thus, $X$ satisfies the oddity condition (Theorem 6).

$$\text{exs}_p(\Upsilon) = \text{exs}_p(M) = \text{exs}_p(t \oplus H/t) = \text{exs}_p(t) + \text{exs}_p(H/t) \quad (27)$$

By the hypothesis of the theorem, the oddity condition holds for the symbol $\Upsilon$. And so,

$$0 \equiv \sum_{p \mid q \cup \{-1\}} \text{exs}_p(\Upsilon) \equiv \text{sig}(\Upsilon) - n + \sum_{p \mid q} \left( \text{exs}_p(t) + \text{exs}_p(H/t) \right)$$

$$\equiv \text{sig}(\Upsilon) - (n - 1) - \text{sig}(t) + (\text{sig}(t) - 1) + \sum_{p \mid q} \left( \text{exs}_p(t) + \text{exs}_p(H/t) \right)$$

$$\equiv \text{sig}(t)(1 + \text{sig}(\Upsilon)) - (n - 1) - \text{sig}(t) + \sum_{p \mid q} \text{exs}_p(H/t)$$

$$\equiv \text{sig}(t) \text{sig}(\Upsilon) - (n - 1) + \sum_{p \mid q} \text{exs}_p(tH)$$

$$\equiv \text{sig}(t) \text{sig}(\Upsilon) - (n - 1) + \sum_{p \mid q} \text{exs}_p(t\tilde{\Upsilon}) \quad (\text{mod } 8)$$

From [Theorem 5, [CS99, page 372]], it follows that there exists a (rational) quadratic form $X$ which is equivalent to $\tilde{\Upsilon}$ over rationals. This also implies that $\tilde{\Upsilon}$ satisfies the oddity condition (Theorem 6).

(ii). (Determinant Condition) By definition, $H \not\subset \tilde{\Upsilon}$. By item (i), $X$ is equivalent to $\tilde{\Upsilon}$ over rationals. Thus, $\frac{\text{det}(H)}{\text{det}(\Upsilon)}$ is a rational square modulo $q$ [Theorem 3, [CS99, page 372]]. From Claim 1 there exists an integer $x$ such that

$$\text{det}(H) \equiv t^{n-2} \text{det}(\Upsilon)x^2 \quad (\text{mod } q) \quad (28)$$

But then, for all primes $p$ that divide $q$,

$$\prod_{i \in \mathbb{I}_p(\tilde{\Upsilon})} \epsilon_i = \left( \frac{\text{cpr}_p(\text{det}(H))}{p} \right) = \left( \frac{\text{cpr}_p(\text{det}(\Upsilon)x^2)}{p} \right) = \left( \frac{\text{cpr}_p(\text{det}(\Upsilon))}{p} \right)$$

This equality show that the determinant condition holds for all primes $p$ that divide $q$. For all other primes, the determinant condition holds by construction.

(iii). (Jordan Condition) The Jordan constituents of the $H$ are the same as the Jordan constituents of $\tilde{\Upsilon}$.
This is because for all relevant primes of \( \det(\tilde{\Upsilon}) \). The quadratic form \( H \) is integral and so its Jordan constituents exist, proving that the Jordan Condition is satisfied for \( \tilde{\Upsilon} \).

\[ \Box \]

Claim 2  The matrix \( Q \) is an integral quadratic form with determinant \( \det(\Upsilon) \) and signature \( \text{sig}(\Upsilon) \).

Proof:  The matrix \( Q \) is symmetric by construction. By Claim 1, the determinant of \( \tilde{H} \) equals \( t^{n-2} \det(\Upsilon) \). Thus, the following equality implies that the determinant of \( Q \) equals \( \det(\Upsilon) \).

\[
Q = \left( \begin{array}{c}
\tilde{U}'d' \\
\tilde{U} + \tilde{U}'d'd\tilde{U}
\end{array} \right) = \left( \begin{array}{cc}
t & d\tilde{U} \\
0 & I
\end{array} \right) \left( \begin{array}{cc}
t & 0 \\
0 & t
\end{array} \right) \left( \begin{array}{cc}
1 & d\tilde{U} \\
0 & I
\end{array} \right)
\]

Let \( H \) be the integral quadratic form with symbol \( \Upsilon \), \( H = tA'SA - d'd \mod q \), and \( \tilde{U} \) be such that \( \tilde{U}'\tilde{H} \tilde{U} \equiv \tilde{H} \mod q \). Then,

\[
\left( \begin{array}{cc}
1 & 0 \\
0 & \tilde{U}
\end{array} \right)'(xA'SA')\left( \begin{array}{cc}
x & 0 \\
0 & \tilde{U}
\end{array} \right) \mod q = \left( \begin{array}{cc}
t & d\tilde{U} \\
0 & \tilde{U}'SA\tilde{U}
\end{array} \right) \mod q
\]

\[
t\tilde{U}'A'SA\tilde{U} - \tilde{U}'d'd\tilde{U} \equiv \tilde{U}'H\tilde{U} \equiv \tilde{H} \mod q
\]

The integer \( t \) divides \( q \). By Equation 31, \( \tilde{H} + \tilde{U}'d'd\tilde{U} \equiv 0 \mod t \). But then, \( t \) divides every entry of the matrix \( \tilde{H} + \tilde{U}'d'd\tilde{U} \) and so \( Q \) is an integral matrix.

Finally, by Equation 29, \( Q \sim t \oplus \tilde{H} \). Hence,

\[
\text{sig}(Q) = \text{sig}(t) + \text{sig}(\tilde{H})\text{sig}(t) = \text{sig}(t) + \text{sig}(\tilde{\Upsilon})\text{sig}(t) = \text{sig}(\Upsilon)
\]

\[ \Box \]

The proof of Theorem 19 now proceeds as follows.

(i). If the symbol \( \Upsilon \) is valid, then there exists an integer \( t \), which has a primitive representation in \( \Upsilon \) (see Theorem 22).

(ii). If \( \Upsilon^n \) satisfies the determinant, oddity and the Jordan conditions i.e., Equations 18-21; then so does \( \tilde{\Upsilon} \) (Theorem 20).

(iii). The symbol \( \tilde{\Upsilon} \) is well defined and has a short description. In particular, by Claim 1, \( \det(\tilde{\Upsilon}) = t^{n-2}\det(\Upsilon) \) and \( \tilde{\Upsilon} \) can equivalently be written as follows.

\[
\tilde{\Upsilon} = \left( \cup_{p \in \mathbb{P}_q} \text{sym}_p(H) \right) \cup \{ \text{sig}(t)(\text{sig}(\Upsilon) - \text{sig}(t)) \}
\]

(iv). The output matrix \( Q \) has determinant \( \det(\Upsilon) \). It also has the same signature as \( \text{sig}(\Upsilon) \). Thus, \( Q \sim \Upsilon \).

(v). If \( n = 1 \) then \( \det(\Upsilon) \) is the unique matrix with determinant equal to the determinant of the symbol \( \Upsilon \). This follows from the Determinant Condition.

(vi). If \( n > 1 \) then, it remains to show that for every relevant prime of \( \Upsilon \), \( \text{sym}_p(Q) = \Upsilon_p \). Consider the
following sequence of congruences.

\[
\begin{align*}
Q & = \left( \begin{array}{cc} t^2 & t \tilde{d} \\ t(\tilde{d}U)' & \tilde{H} + \tilde{U}'d' \tilde{d}U \end{array} \right) \mod q \\
& \equiv \left( \begin{array}{cc} t^2 & t \tilde{d} \\ t(\tilde{d}U)' & \tilde{H} + \tilde{U}'d' \tilde{d}U \end{array} \right) \mod q \\
& \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & \tilde{U} \end{array} \right) \left( \begin{array}{cc} t^2 & t \tilde{d} \\ t(\tilde{d}U)' & \tilde{H} + \tilde{U}'d' \tilde{d}U \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & \tilde{U} \end{array} \right) \mod q \\
& \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & \tilde{U} \end{array} \right) \left( \begin{array}{cc} t^2 & t \tilde{d} \\ t(\tilde{d}U)' & \tilde{H} + \tilde{U}'d' \tilde{d}U \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & \tilde{U} \end{array} \right) \mod q \\
& \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & \tilde{U} \end{array} \right) \left( \begin{array}{cc} t \tilde{S}x \tilde{A} & t \tilde{S}x \tilde{A} \\ t \tilde{S}x \tilde{A} & t \tilde{S}x \tilde{A} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & \tilde{U} \end{array} \right) \mod q \\
& \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & \tilde{U} \end{array} \right) \left( \begin{array}{cc} x \ A & t \tilde{S}x \ A \\ t \tilde{S}x \ A & t \tilde{S}x \ A \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & \tilde{U} \end{array} \right) \mod q/t \\
Q & \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & \tilde{U} \end{array} \right) \left( \begin{array}{cc} x \ A & t \tilde{S}x \ A \\ t \tilde{S}x \ A & t \tilde{S}x \ A \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & \tilde{U} \end{array} \right) \mod q/t 
\end{align*}
\]

Recall, \( \tilde{U} \in \text{GL}_{n-1}(\mathbb{Z}/q\mathbb{Z}) \) and \( (x \ A) \in \text{GL}_n(\mathbb{Z}/q\mathbb{Z}) \). This implies that \( U = [x, A](1 \oplus \tilde{U}) \in \text{GL}_n(\mathbb{Z}/q\mathbb{Z}) \).

But then, \( U \in \text{GL}_n(\mathbb{Z}/q\mathbb{Z}) \) and \( Q \sim S \). Note that for every prime \( p \) that divides \( 2 \det(\Upsilon) \) the following holds because \( \text{ord}_p(q/t) > \text{ord}_p(\det(Q)) + k_p \).

\[ \text{ord}_p(\det(\Upsilon)) = \text{ord}_p(\det(Q)) = \text{ord}_p(\det(S)) \]

Thus, by definition of \( p^* \)-equivalence one concludes that for every \( p \) that divides \( 2 \det(\Upsilon) \), \( Q \sim S \); completing the proof of Theorem 19.

\[ \square \]

7 Primitive Representation in a Genus

An important step in the algorithm \text{GENSIMPLE} is to find an integer \( t \) which has a primitive representation in the genus \( \Upsilon \).

Recall Definition 6. The following lemma shows that if \( n \geq 2 \) then \( t \) has a primitive \( p^* \)-representation in \( \Upsilon \) for all primes \( p \) such that \( p \) does not divide \( 2 \det(\Upsilon) \). A proof of this lemma can already be found in Siegel [Sie35], although in a different setting.

\textbf{Lemma 21} \( \Upsilon^n \geq 2 \) be a valid genus, \( t \) be an integer and \( p \) be an odd prime which does not divide \( t \det(\Upsilon) \). Then, \( t \) has a primitive \( p^* \)-representation in \( \Upsilon \).

\textbf{Proof:} Let \( p \) be an odd prime which does not divide \( t \det(\Upsilon) \). Then, by Lemma 14, \( \Upsilon \sim \text{diag}(\det(\Upsilon), 1, \cdots, 1) \). It suffices to show that \( t \) has a primitive representation in \( \text{diag}(\det(\Upsilon), 1) \) over \( \mathbb{Z}/p\mathbb{Z} \) (Theorem 10).

By assumption, \( \det(\Upsilon) \) and 1 are invertible modulo \( p \). If \( x^2 \equiv t \det(\Upsilon)^{-1} \mod p \) has a non-trivial solution. Otherwise, \( \det(\Upsilon) \) and 1 have the same Legendre symbol, different from \( t \). But then the result follows from Lemma 11. \[ \square \]
Theorem 22 Let $\mathcal{Y}^{n\geq 2}$ be a valid genus and $Q \in \mathcal{Y}$. A positive integer $t$ has a primitive representation in $\mathcal{Y}$ if $t$ has a primitive $p^K\mathcal{Y}$-representation in $Q$ for all $p$ that divides $2t \det(\mathcal{Y})$, where $K_p = \max\{\text{ord}_p(Q), \text{ord}_p(t)\} + k_p$.

**Proof:** Follows from Theorem 10, Lemma 21 and the definition of primitive representations in a genus. □

This simplifies our problem in the algorithmic sense. To find an integer $t$ which has a primitive representation in the genus $\mathcal{Y}^{n\geq 2}$, we only need to check all primes $p$ that divide $2t \det(\mathcal{Y})$ and only over the ring $\mathbb{Z}/p^K\mathbb{Z}$ for $K_p = \max\{\text{ord}_p(\mathcal{Y}), \text{ord}_p(t)\} + k_p$.

For $n > 3$, it is comparatively easy to find a $t$; in fact, it is possible to find a $t$ which divides $\det(\mathcal{Y})$. But for dimensions $n = 3$ and $n = 2$, the proof deteriorates to case analyses, especially for dimension 2. The proofs are constructive in the sense that it is also possible to find a representation $x$ such that $x'Sx \equiv t \mod q$ in time $\text{poly}(n, \log \det(\mathcal{Y}))$.

In this section, we prove the following theorem.

**Theorem 23** Let $\mathcal{Y}^{n\geq 2}$ be a valid reduced genus. Then, there exists a randomized algorithm that takes $\mathcal{Y}$ as input; runs in time $\text{poly}(|\mathbb{P}_\mathcal{Y}|, \log \det(\mathcal{Y}))$ and outputs an integer $t$ which has a primitive representation in the genus $\mathcal{Y}$.

Note that the run time of the algorithm does not depend on $n$. This is because for $n = 4$, we can already find a nice $t$ which divides $\det(\mathcal{Y})$ and we ignore the later dimensions. When we want to find an $x$ such that $x'Sx \equiv t \mod q$ then we use the same trick and only represents $t$ using at most $4 \times 4$ sub-form of $S$ (see Section 6).

But before starting the construction of $t$, we prove two lemmas which are going to be useful.

**Lemma 24** Let $t$ be an odd integer, and $2^{i_1}\tau_1 \oplus \cdots \oplus 2^{i_4}\tau_4$ be an integral quadratic form with $\tau_1, \cdots, \tau_4$ odd and $i_1 \leq \cdots \leq i_4$. Then, $2^it$ has a $2^k$-primitive representation in $D$. Additionally, for every positive integer $k$ there exists a primitive $2^k$-representation $(x_1, \cdots, x_4)$ such that $\text{ord}_2(x_4) = 0$ and $\text{ord}_2(2^j\tau_j x_j^2) \geq i_4$, for all $j \in [4]$.

**Proof:** Let $k = i_4 + 3$, then it suffices to show that the following has a primitive solution (see Theorem 10),

$$2^{i_1}\tau_1 x_1^2 + \cdots + 2^{i_4}\tau_4 x_4^2 \equiv 2^it \pmod{2^{i_4+4}}$$ (33)

We find a primitive solution where $x_4$ is odd. For $j \in [3]$, we set $x_j = 2^{[\frac{i_4-i_j}{2}]}y_j$ and divide the entire Equation 33 by $2^{i_4}$. The equation then reduces to the following.

$$\sum_{j \in [3]} 2^{(i_4-i_j) \mod 2} \tau_j y_j^2 + \tau_4 x_4^2 \equiv t \pmod{16}$$ (34)

An exhaustive search shows that for each possible choice of odd $t$ in $\mathbb{Z}/16\mathbb{Z}$, $\tau_1, \cdots, \tau_4 \in \{1, 3, 5, 7\}$ and $i_4 - i_j \pmod{2}$, the Equation 34 always has a solution, where $x_4$ is odd. □

**Lemma 25** Let $p$ be an odd prime, $D = \tau_1 \oplus p^i\tau_2$, where $\tau_1, \tau_2 \in (\mathbb{Z}/p\mathbb{Z})^\times$, $i$ even and $t \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that $\left(\frac{t}{p}\right) \neq \left(\frac{\tau_1}{p}\right)$ and $\left(\frac{t}{p}\right) \neq \left(\frac{\tau_2}{p}\right)$. Then, $p^it$ has a $p^k$-primitive representation in $D$.

**Proof:** If $\left(\frac{t}{p}\right) = \left(\frac{\tau_2}{p}\right)$, then $p^it$ has the same symbol as $p^i\tau_1$ and then $p^it$ has a primitive $p^k$-representation in $p^i\tau_2$. Otherwise, from the statement of the lemma, $\left(\frac{t}{p}\right) = \left(\frac{\tau_1}{p}\right) = \left(\frac{\tau_2}{p}\right)$. □
In this case, \( t \) can always be written as 
\[ t \equiv \tau_1 y_1^2 + \tau_2 y_2^2 \pmod{p}, \]
where both \( y_1 \) and \( y_2 \) are units of \( \mathbb{Z}/p\mathbb{Z} \). But then,
\[ p^i t \equiv p^i \tau_1 y_1^2 + p^i \tau_2 y_2^2 \pmod{p^{i+1}} \]
\[ \equiv \tau_1 (p^{i/2} y_1)^2 + p^i \tau_2 y_2^2 \pmod{p^{i+1}}. \]
It follows that \( p^i t \) has a primitive representation by \( \tau_1 \odot p^i \tau_2 \) over \( \mathbb{Z}/p^{i+1} \mathbb{Z} \). By Theorem 10, \( p^i t \) has a \( p^i \)-primitive representation in \( \mathbb{D} \). \( \square \)

7.1 Representation: \( n > 3 \)

As mentioned earlier, we construct an integer \( t \) such that \( t \) divides \( \det(\Upsilon) \) and \( t \) has a primitive representation in the input genus \( \Upsilon \). It turns out that when \( n > 3 \) we do not need to use the fact that the input symbol \( \Upsilon \) is reduced.

**Lemma 26** Let \( \Upsilon^{n>3} \) be a genus. Then, there exists an integer \( t \) such that \( t \) divides \( \det(\Upsilon) \) and \( t \) has a primitive representation in the genus \( \Upsilon \).

**Proof:** Let us suppose that \( p \) is an odd prime that divides \( \det(\Upsilon) \). In this case, we construct a diagonal form using Lemma 17 as follows.

\[ \Upsilon_p \sim p^{i_1} \tau_1 \oplus p^{i_2} \tau_2 \oplus \cdots \quad \tau_1, \tau_2, \cdots \in (\mathbb{Z}/p\mathbb{Z})^\times, i_1 \leq i_2 \leq \cdots \quad (35) \]

An integer can be equivalently written as \( \prod_{p \in \{-1, 2\} \cup \mathbb{P}} p^{e_p} \), where \( e_p \) is the \( p \)-order of the integer. The construction of the integer \( t \) is as follows.

(i). For every odd prime that does not divide \( \det(\Upsilon) \), \( e_p \) is identically 0. Also, if \( \text{sig}(\Upsilon) > -n \), then we set \( e_{-1} = 0 \). Otherwise, \( e_{-1} = 1 \).

(ii). For every odd prime \( p \) that divide \( \det(\Upsilon) \) our first step is to compute the value of \( e_p \mod 2 \). Consider a prime \( p \) that divides \( \det(\Upsilon) \). Consider the quadratic form constructed in Equation 35 for the prime \( p \). Then,
\[ e_p \mod 2 := \max(i_1 \mod 2, i_2 \mod 2, i_3 \mod 2) \quad (36) \]

(iii). Next we compute \( e_2 \). If \( \Upsilon_2 \) has a type II block of 2-order \( \ell \), then we set \( \text{ord}_2(t) = \ell + 1 \). Otherwise, \( \Upsilon_2 \) has only Type I blocks. Thus, \( \Upsilon \) is 2-equivalent to a diagonal matrix \( 2^{j_1} \tilde{\tau}_1 \oplus \cdots \), where \( \tilde{\tau}_1, \cdots \) are odd and \( j_1 \leq j_2 \leq \cdots \). The value of \( j_1, \cdots, j_4 \) can be read off the symbol \( \Upsilon_2 \) as the four smallest possible 2-orders in \( \Upsilon_2 \). We set \( e_2 = j_4 \).

(iv). Once the parity of all \( p \in \{-1, 2\} \cup \mathbb{P} \) is known (see item (i)-(iii)), we define an integer \( r \) as follows.
\[ r = \prod_{p \in \{-1, 2\} \cup \mathbb{P}} p^{e_p \mod 2} \quad (37) \]

(v). Finally, we compute \( e_p \) for all odd primes \( p \) which divide \( \det(\Upsilon) \). Consider the diagonal form constructed in Equation 35 for the prime \( p \). Out of \( (i_1, i_2, i_3) \), and \( (i_1, i_3) \); let \( (i_a, i_b) \), \( a < b \in \{1, 2, 3\} \) be the pair with the same parity. Then,
\[ e_p = \begin{cases} 
  i_a & \text{if } \left( \frac{e_{pr}(r)}{p} \right) = \left( \frac{i_a}{p} \right), \\
  i_b & \text{otherwise} \end{cases} \quad (38) \]
(vi). We now have $e_p$ for every $p \in \{-1, 2\} \cup P$. We define our integer $t$ as follows.

$$
t = \prod_{p \in \{-1, 2\} \cup P} p^{e_p} \quad (39)
$$

The next step is to show that $t$ has a primitive representation in the genus $\Upsilon$, or equivalently, $t$ has a $p^*$-primitive representation in $\Upsilon$ for all $p \in \{-1, 2\} \cup P$.

(i). $(p = -1)$ By construction, $t$ is negative if $\text{sig}(\Upsilon) = -n$. In this case $\Upsilon$ is a genus of negative definite matrices and hence must represent every negative integer over $\mathbb{R}$. Otherwise, $t$ is a positive integer and $\Upsilon$ is a genus of non-negative definite matrices i.e., $\Upsilon \sim 1 \oplus \cdots$. Hence, $\Upsilon$ must represent all positive integers over $\mathbb{R}$. In either case, the constructed $t$ has a primitive representation in $\Upsilon$ over $\mathbb{R}$.

(ii). $(p \text{ odd}, p \text{ does not divide } \det(\Upsilon))$ In this case, $p$ does not divide $t$. Hence, $t$ has a $p^*$-primitive representation in $\Upsilon$ (Lemma 24).

(iii). $(p = 2)$ If $\Upsilon_2$ has a Type II block then $\text{ord}_2(t) = \ell + 1$, where $\ell$ is the 2-order of one of the Type II blocks. Then, the Type II block represents every integer of 2-order $\ell + 1$ (by Lemma 12). The existence of a $2^*$-primitive representation now follows from Lemma 24. Otherwise, there are only Type I blocks in $\Upsilon_2$ and the existence of a $2^*$-primitive representation follows from Lemma 24 and Theorem 10.

(iv). $(p \text{ odd}, p \text{ divides } \det(\Upsilon))$ By construction, $e_p$ has the same parity as $i_a$ and $i_b$, see item (v) of the construction of $t$ and Equation 38. Thus, $\text{ord}_p(t) \equiv \text{ord}_p(r) \pmod{2}$, or

$$
\left(\frac{\text{cpr}_p(t)}{p}\right) = \left(\frac{\text{cpr}_p(r)}{p}\right)
$$

By Equation 38, $e_p = i_a$ if $\left(\frac{\text{cpr}_p(t)}{p}\right) = \left(\frac{r_a}{p}\right)$ and $e_p = i_b$, otherwise. If $\left(\frac{\text{cpr}_p(t)}{p}\right) = \left(\frac{r_a}{p}\right)$ then $t$ can be $p^*$-primitively represented by the diagonal entry $p^{i_a} \tau_a$ in Equation 35. Otherwise, $t$ can be $p^*$-primitively represented by $p^{i_a} \tau_a \oplus p^{i_b} \tau_b$ (see Lemma 25). In either case, $t$ has a $p^*$-primitive representation in $\Upsilon_p$.

\[\square\]

7.2 Representation: $n = 3$

In this case, we construct an integer $t$ with the following properties. If the input genus $\Upsilon_2$ has a Type II block then the constructed $t$ divides $\det(\Upsilon)$. Otherwise, $t$ is of the form $\varphi \ell$, where $\ell$ divides $\det(\Upsilon)$ and $\varphi$ is an odd prime that does not divide $\det(\Upsilon)$.

**Lemma 27** Let $\Upsilon^{n=3}$ be a genus. Then, there exists an integer $\varphi \ell$ such that $\ell$ divides $\det(\Upsilon)$, $\varphi \in P \setminus P_\Upsilon$ and $\varphi \ell$ has a primitive representation in the genus $\Upsilon$.

**Proof:** Let us suppose that $p$ is an odd prime that divides $\det(\Upsilon)$. In this case, we construct a diagonal form using Lemma 17 as follows.

$$
\Upsilon_p \cong p^{i_1} \tau_1 \oplus p^{i_2} \tau_2 \oplus p^{i_3} \tau_3 \quad \tau_1, \tau_2, \tau_3 \in (\mathbb{Z}/p\mathbb{Z})^\times, i_1 \leq i_2 \leq i_3 \quad (40)
$$

An integer can be equivalently written as $\prod_{p \in \{-1, 2\} \cup P} p^{e_p}$, where $e_p$ is the $p$-order of the integer. The construction of the integer $t$ is as follows.

(i). if $\text{sig}(\Upsilon) > -3$, then we set $e_{-1} = 0$. Otherwise, $e_{-1} = 1.$

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(ii). For every odd prime \( p \) that divides \( \det(\Upsilon) \), our first step is to compute the value of \( e_p \mod 2 \). Consider the quadratic form constructed in Equation 35 for the prime \( p \). Then,
\[
e_p \mod 2 := \text{maj}\{i_1 \mod 2, i_2 \mod 2, i_3 \mod 2\}
\]

(iii). If \( \Upsilon_2 \) has a type II block of 2-order \( \ell \), then we set \( \text{ord}_2(t) = \ell + 1 \). Also, for all odd primes \( p \notin \mathcal{P}_\Upsilon \), we set \( e_p = 0 \).

(iv). Otherwise, \( \Upsilon_2 \) has only Type I blocks. Thus,
\[
\Upsilon_2 \cong 2^{j_1} \hat{\tau}_1 \oplus 2^{j_2} \hat{\tau}_2 \oplus 2^{j_3} \hat{\tau}_3, \quad \hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3 \in \text{SGN}^\times, i_1 \leq i_2 \leq i_3
\]

The value of \( j_1 \) can be read off the symbol \( \Upsilon_2 \) as the smallest possible 2-orders in \( \Upsilon_2 \). We set \( e_2 = j_1 \).

We also pick an odd prime \( \wp \) not in \( \mathcal{P}_\Upsilon \) satisfying the following equation.
\[
\prod_{p \in \{-1, 2\} \cup \mathcal{P}_\Upsilon} p^{e_p \mod 2} \equiv \hat{\tau}_1 \pmod{8}
\]

Such a prime can be found by rejection sampling. A random odd prime satisfies the Equation 43 with probability \( 1/4 \). We set \( e_\wp = 1 \). Also, for all primes \( p \) that do not divide \( 2^\wp \det(\Upsilon) \), we set \( e_p = 0 \).

(v). Once the parity of all \( p \in \{-1, 2\} \cup \mathcal{P} \) is known (see item (i)-(iv) of the construction), we define an integer \( r \) as follows.
\[
r = \prod_{p \in \{-1, 2\} \cup \mathcal{P}} p^{e_p \mod 2}
\]

(vi). Finally, we compute \( e_p \) for all odd primes \( p \) which divide \( \det(\Upsilon) \). Consider the diagonal form constructed in Equation 35 for the prime \( p \). Out of \( (i_1, i_2), (i_2, i_3), \) and \( (i_1, i_3) \); let \( (i_a, i_b), a < b \in \{1, 2, 3\} \) be the pair which has the same parity. Then,
\[
e_p = \begin{cases} 
i_a & \text{if } \left( \text{cpr}_{(r)} \right) = \left( \tau_a \right), \\ i_b & \text{otherwise} \end{cases}
\]

(vii). We now have \( e_p \) for every \( p \in \{-1, 2\} \cup \mathcal{P} \). We define our integer \( t \) as follows.
\[
t = \prod_{p \in \{-1, 2\} \cup \mathcal{P}} p^{e_p}
\]

The next step is to show that \( t \) has a primitive representation in the genus \( \Upsilon \). Equivalently, it suffices to show that \( t \) has a \( p^* \)-primitive representation in \( \Upsilon \) for all \( p \in \{-1, 2\} \cup \mathcal{P} \). Note that if \( \Upsilon_2 \) has a Type II block then the construction of \( t \) in this case is the same as the construction of \( t \) in the case of Lemma 26. The correctness of the construction also follows from the same proof. In the rest, we assume that \( \Upsilon_2 \) has no Type II block.

(i). (\( p = -1 \)) By construction, \( t \) is negative iff \( \text{sig}(\Upsilon) = -3 \). In this case \( \Upsilon \) is a genus of negative definite matrices and hence must represent every negative integer over \( \mathbb{R} \). Otherwise, \( t \) is a positive integer and \( \Upsilon \) is a genus of non-negative definite matrices i.e., \( \Upsilon \cong 1 \oplus \cdots \). Hence, \( \Upsilon \) must represent all positive integers over \( \mathbb{R} \). In either case, the constructed \( t \) has a primitive representation in \( \Upsilon \) over \( \mathbb{R} \).

(ii). (\( p \) odd, \( p \) does not divide \( \wp \det(\Upsilon) \)) In this case, \( p \) does not divide \( t \). Hence, \( t \) has a \( p^* \)-primitive representation in \( \Upsilon \) (Lemma 21).
By assumption, there are only Type I blocks in Υ2. By construction of t, Sym2k(2j1τ) = Sym2k(t). But then, t has a 2*-primitive representation in 2j1τ (Lemma 2).

By construction of t, sym2k(t) = sym2k(τ1). But then, t has a 2*-primitive representation in 2j1τ (Lemma 2).

By construction, cp(t) has the same parity as ia and ib, see item (vi) of the construction of t and Equation 45. Thus, ordp(t) ≡ ordp(r) (mod 2), or

\[ cpr_p(t) \equiv cpr_p(r) \pmod{2} \]

By Equation 45, cp(t) = ia if \( cpr_p(t) \equiv \tau a_p \) and cp(t) = ib, otherwise. If \( cpr_p(t) \equiv \tau a_p \) then t can be p*-primitively represented by the diagonal entry piaτa (see Equation 40). Otherwise, t can be p*-primitively represented by piaτa ⊕ pibτb (see Lemma 25). In either case, t has a p*-primitive representation in Υp.

Finally, we show that t has a ℘*-primitive representation in Υ. The prime ℘ does not divide det(Υ) and hence by Lemma 14, Υ ⊕ ℘ − 1 ⊕ 1. Consider the following equation.

\[ \det(Υ)x_1^2 + x_2^2 + x_3^2 \equiv t \pmod{℘^2} \] \hspace{1cm} (47)

By Lemma 1, \( x_2^2 + x_3^2 \) represents \( t - \det(Υ) \) over \( \mathbb{Z}/℘\mathbb{Z} \). Also, \( \det(Υ)x_1^2 \) represents \( \det(Υ) \) primitively over \( \mathbb{Z}/℘^2\mathbb{Z} \). Thus, Equation 47 has a primitive solution. By Theorem 10, t has a ℘*-primitive representation in Υ.

\[ \square \]

7.3 Representation: \( n = 2 \), basics

Finding an integer representation in dimension 2 is the most difficult. As with dimension 3, we may need a prime ℘ but it needs to satisfy more stringent conditions. In this case, we strongly use the fact that the input symbol Υ is reduced and valid.

Recall the definition of a reduced genus. In dimension 2 a reduced genus Υ has the following form.

\[ Υ \{ a_p \oplus p\tau b_p \} \]

where \( a_p, b_p \in (\mathbb{Z}/p\mathbb{Z})^\times, p \text{ odd,} \)

\[ = \{ a_2 \oplus 2^ib_2, T^+, T^- \} \]

where \( a_2, b_2 \in \text{sgn}^\times \).

Note that Υ is a symbol in dimension 2 and hence \( \text{sig}(Υ) \in \{ 2, 0, -2 \} \). Define the quantities \( \epsilon, \rho, \) and the function \( \xi : \mathbb{Z} \to \{ 0, 1 \} \), as follows.

\[ \epsilon = \frac{\det(Υ)}{|\det(Υ)|} \]

\[ \rho = \begin{cases} 1 & \text{if } \text{sig}(Υ) \in \{ 0, 2 \} \\ -1 & \text{otherwise.} \end{cases} \]

\[ \xi(x) = \begin{cases} 1 & \text{if } \left( \frac{x}{2} \right) = \frac{x(x-1)}{2} \text{ is odd, and} \\ 0 & \text{otherwise.} \end{cases} \]

Then, the signature and the oddity of Υ can be computed as follows.

\[ \text{sig}(Υ) = \rho(1 + \epsilon) \]

\[ \text{odt}(Υ) = \begin{cases} 0 & \text{if } Υ \sim T^+ \text{ or } Υ \sim T^- \\ a_2 + b_2 \pmod{8} & \text{if } \left( \frac{a_2}{2} \right) = 1, \text{ or } i_2 \text{ even} \\ a_2 + b_2 + 4 \pmod{8} & \text{otherwise} \end{cases} \]

\[ (50) \]
For convenience, we define the set $S$ as the set of odd primes $p$ for which $i_p$ is odd i.e., $S = \{p \in \mathbb{P}_T \cap \mathbb{P} | i_p \text{ odd}\}$. Next, for each $d \in \{1, 3, 5, 7\}$ and $b \in \{-, +\}$ we define sets,

$$S_{db} = \left\{ p \in S \mid p \equiv d \mod 8, \left( \frac{a_p}{p} \right) = b \right\} \quad (51)$$

If we eliminate a subscript, it means a union of the sets with all possible values of the subscript. For example, $S_3 = S_{3+} \cup S_{3-}$. The calligraphic versions, as usual, will denote the size of the corresponding sets. For example, $S_{(3, 5)-}$ is $|S_{3-}| + |S_{5-}|$.

**Lemma 28** Let $\Upsilon^{n=2}$ be a valid reduced genus, $\epsilon = \frac{\det(\Upsilon)}{|\det(\Upsilon)|}$, and $m$ be the total number of antisquares in $\Upsilon$. Then,

$$\left( \sum_{p \in \mathbb{P}_T} \text{exs}_p(\Upsilon) \right) \equiv 2S_3 + 4S_5 + 6S_7 + 2(1 - (-1)^m) \pmod{8}$$

where,

$$(-1)^m = \begin{cases} (-1)^{S_+ + \xi(S_{(3, 7)})} c^{S_{(3, 7)}} & \text{if } \text{ord}_2(\det(\Upsilon)) \text{ is even} \\
(-1)^{S_+ + \xi(S_{(3, 7)}) + S_{(3, 5)}} c^{S_{(3, 7)}} & \text{otherwise.} \end{cases}$$

**Proof:** Let $p$ be an odd prime from $\mathbb{P}_T$. To compute the $p$-excess, we need to compute the number of $p$-antisquares in $\Upsilon_p$. By construction, $\Upsilon \overset{p}{\sim} a_p \oplus p^b b_p$ has a $p$-antisquare iff $i_p$ is odd and $\left( \frac{b_p}{p} \right) = -1$. But then, $\det(\Upsilon) \overset{p}{\sim} \det(a_p \oplus p^b b_p) = a_p b_p p^{i_p}$ and so $\left( \frac{b_p}{p} \right) = \left( \frac{cpr_p(\det(\Upsilon))}{p} \right)$. If $m$ is the total number of antisquares in $\Upsilon$ then,

$$(-1)^m = \prod_{p \in S} \left( \frac{b_p}{p} \right) = \prod_{p \in S} \left( \frac{\text{cpr}_p(\det(\Upsilon))}{p} \right) \prod_{p \in S} \left( \frac{a_p}{p} \right)$$

$$= \begin{cases} (-1)^{S_+} \prod_{p \in S} \left( \frac{\epsilon}{p} \right) \prod_{p, i_p \in S} \left( \frac{p}{p} \right) \left( \frac{p}{p} \right) & \text{if } \text{ord}_2(\det(\Upsilon)) \text{ is even} \\
(-1)^{S_+} \prod_{p \in S} \left( \frac{2}{p} \right) \prod_{p, i_p \in S} \left( \frac{p}{p} \right) \left( \frac{p}{p} \right) & \text{otherwise.} \end{cases}$$

Note that $\left( \frac{-1}{p} \right) = -1$ iff $p \equiv 3 \pmod{4}$ i.e., $\prod_{p \in S} \left( \frac{\epsilon}{p} \right) = \epsilon^{S_{(3, 7)}}$. Also, $\left( \frac{2}{p} \right) = -1$ iff $p \equiv 8 \pmod{25}$ i.e., $\prod_{p \in S} \left( \frac{2}{p} \right) = (-1)^{S_{(3, 5)}}$. By Quadratic Reciprocity, $\prod_{p, i_p \in S} \left( \frac{p}{p} \right) \left( \frac{p}{p} \right) = (-1)^{\xi(S_{(3, 7)})}$. Putting it together, we have,

$$(-1)^m = \begin{cases} (-1)^{S_+ + \xi(S_{(3, 7)})} c^{S_{(3, 7)}} & \text{if } \text{ord}_2(\det(\Upsilon)) \text{ is even} \\
(-1)^{S_+ + \xi(S_{(3, 7)}) + S_{(3, 5)}} c^{S_{(3, 7)}} & \text{otherwise.} \end{cases}$$

By definition, if $m$ is the total number of $p$-antisquares in $\Upsilon$ with $p$ odd then,

$$\sum_{p \in \mathbb{P}_T} \text{exs}_p(\Upsilon) = \sum_{p \in S} \text{exs}_p(\Upsilon) = 4m + \sum_{p \in S} (p - 1) \pmod{8}$$

$$= 2S_3 + 4S_5 + 6S_7 + 4m \pmod{8}$$

The expression $4m \pmod{8}$ evaluates to $4$ iff $m$ is odd. Equivalently, $4m \pmod{8}$ evaluates to $4$ iff $(-1)^m$ evaluates to $-1$; completing the proof. \[\square\]

**Lemma 29** Let $\Upsilon^{n=2}$ be a reduced genus, $r$ be an integer and $\psi$ be an odd prime that does not divide
Then, \( r \varphi \) has a \( \varphi^* \)-primitive representation in \( \Upsilon \) iff 
\[
\left( \frac{-\det(\Upsilon)}{\varphi} \right) = 1.
\]
If \( \text{ord}_2(\det(\Upsilon)) \) is even, then 
\[
\left( \frac{-\det(\Upsilon)}{\varphi} \right) = \begin{cases} 
\prod_{p \in S} \left( \frac{\varphi}{p} \right) & \text{if } \varphi \equiv 1 \mod 4 \\
( -1 )^{S_{\{3,7\}} + 1} \prod_{p \in S} \left( \frac{\varphi}{p} \right) & \text{otherwise}.
\end{cases}
\]
and if \( \text{ord}_2(\det(\Upsilon)) \) is odd then, 
\[
\left( \frac{-\det(\Upsilon)}{\varphi} \right) = \begin{cases} 
\prod_{p \in \{2\} \cup S} \left( \frac{\varphi}{p} \right) & \text{if } \varphi \equiv 1 \mod 4 \\
( -1 )^{S_{\{3,7\}} + 1} \prod_{p \in \{2\} \cup S} \left( \frac{\varphi}{p} \right) & \text{otherwise}.
\end{cases}
\]

**Proof:** By Lemma 14, \( \Upsilon \cong \text{diag}(\det(\Upsilon), 1) \). By Theorem 10, \( r \varphi \) has a \( \varphi^* \)-primitive representation in \( \Upsilon \) iff the following equation has a primitive solution:
\[
\varphi r \equiv \det(\Upsilon) x^2 + y^2 \pmod{\varphi^2}.
\]
Both \( x \) and \( y \) must be units of \( \mathbb{Z}/\varphi \mathbb{Z} \). But then,
\[
\left( \frac{-\det(\Upsilon)}{\varphi} \right) = \left( \frac{\varphi r - \det(\Upsilon) x^2}{\varphi} \right) = \left( \frac{y^2}{\varphi} \right) = 1.
\]

Conversely, if \( \left( \frac{-\det(\Upsilon)}{\varphi} \right) = 1 \) then, by Lemma 2, the following equation has a solution.
\[
x^2 \equiv -\det(\Upsilon) + \varphi r \pmod{\varphi^2}
\]
Thus, \( x^2 + \det(\Upsilon) \equiv \varphi r \pmod{\varphi^2} \) or \( \varphi r \) has a \( \varphi^* \)-primitive representation in \( 1 \oplus \det(\Upsilon) \).

Next, we write \( \left( \frac{-\det(\Upsilon)}{\varphi} \right) \) in terms of \( \epsilon, \rho, \) and \( S_{db} \) using the Law of Quadratic Reciprocity (Equation 1). If \( \text{ord}_2(\det(\Upsilon)) \) is even then,
\[
\left( \frac{-\det(\Upsilon)}{\varphi} \right) = \left( \frac{-\epsilon}{\varphi} \right) \prod_{p \in S} \left( \frac{\varphi}{p} \right)
\]
\[
= \left( \frac{-\epsilon}{\varphi} \right) \prod_{p \in S_{\{1,5\}}} \left( \frac{\varphi}{p} \right) \prod_{p \in S_{\{3,7\}}} \left( \frac{\varphi}{p} \right)
\]
\[
= \begin{cases} 
\prod_{p \in S} \left( \frac{\varphi}{p} \right) & \varphi \equiv 1 \mod 4, \\
( -1 )^{S_{\{3,7\}}} \left( \frac{\varphi}{\rho} \right) \prod_{p \in S} \left( \frac{\varphi}{p} \right) & \text{otherwise}.
\end{cases}
\]

On the other hand, if \( \text{ord}_2(\det(\Upsilon)) \) is odd then
\[
\left( \frac{-\det(\Upsilon)}{\varphi} \right) = \left( \frac{-2\epsilon}{\varphi} \right) \prod_{p \in S} \left( \frac{\varphi}{p} \right)
\]
\[
= \begin{cases} 
\left( \frac{\varphi}{\rho} \right) \prod_{p \in S} \left( \frac{\varphi}{p} \right) & \varphi \equiv 1 \mod 4, \\
( -1 )^{S_{\{3,7\}}} \left( \frac{\varphi}{\rho^2} \right) \prod_{p \in S} \left( \frac{\varphi}{p} \right) & \text{otherwise}.
\end{cases}
\]
7.4 Representation: $n = 2$, Type II

In this section, we construct an integer $t$ such that $t$ has a primitive representation in $\Upsilon$, where $\Upsilon_2 \cong \mathbb{T}^+$ or $\Upsilon_2 \cong \mathbb{T}^-$. Note that, in this situation, 2 does not divide $\det(\Upsilon)$.

**Lemma 30** Let $\Upsilon^{n=2}$ be a valid reduced genus with $\Upsilon_2 \cong \mathbb{T}^+$ or $\Upsilon_2 \cong \mathbb{T}^-$. Then, there exists an integer of the form $2\rho r^2$ with primitive representation in $\Upsilon$, where $\rho$ is an odd prime that does not divide $\det(\Upsilon)$ and $r^2$ is an integer that divides $\det(\Upsilon)$.

**Proof:** Recall the definition of $\epsilon, \rho \in \{-1, 1\}$ as in Equation 29. Let us define the following set of congruences.

$$
\varnothing \equiv 2\rho a_p \pmod{p} \quad \text{for all } p \in S
$$

1. \(\pmod{4}\) if $\rho = +, S_{(3,5)} + S_{2}$ even, or

2. \(\pmod{4}\) if $\rho = -, S_{(5,7)} + S_{2}$ even.

3. \(\pmod{4}\) if $\rho = +, \epsilon = +, S_{(5,7)} + S_{2}$ odd, or

4. \(\pmod{4}\) if $\rho = +, \epsilon = -, S_{(5,7)} + S_{2}$ even, or

5. \(\pmod{4}\) if $\rho = -, \epsilon = -, S_{(3,5)} + S_{2}$ even, or

6. \(\pmod{4}\) if $\rho = -, \epsilon = +, S_{(3,5)} + S_{2}$ odd.

Note that the set of possibilities under which we can write a modulo 4 congruence is not exhaustive. It is, as we show later, exhaustive for every valid symbol $\Upsilon$.

It is possible to solve the congruence in such a way that $\varnothing$ is a prime (Dirichlet’s Theorem). Consider an integer $r$ defined as follows.

$$
r = \prod_{p \in \mathbb{P}_\mathbb{R}} p^{r_p/2} \quad e_p = \begin{cases} 
0 & \text{if } p \in (\{2\} \cup S) \\
0 & \text{if } p \in \mathbb{P}_\mathbb{P} \setminus (\{2\} \cup S), \left(\frac{a_p}{p}\right) = \left(\frac{2\rho a_p}{p}\right) \\
i_p & \text{if } p \in \mathbb{P}_\mathbb{P} \setminus (\{2\} \cup S), \left(\frac{a_p}{p}\right) \neq \left(\frac{2\rho a_p}{p}\right)
\end{cases}
$$

Note that, if $p$ is an odd prime not in $S$ then $i_p$ is even. Thus, $r$ is an integer. Define $t = 2\rho r^2$. We next show that $t$ has a primitive representation in the genus $\Upsilon$. For this, it suffices to show that $t$ has a primitive $p^r$-representation in $\Upsilon$ for all $p \in \{p \mid \ord_p(2t \det(\Upsilon)) > 0\} \cup \{-1\}$ (see Lemma 21).

(i). ($p = -1$) By Equation 19, $\rho = -1$ if $\text{sig}(\Upsilon) = -2$. In this case, $\Upsilon$ is negative definite and represents all negative integers. Otherwise, $\Upsilon \cong \mathbb{R} \times x$ and $\rho = 1$. But then, $t$ is a positive integer and hence can be represented by $\Upsilon$ over $\mathbb{R}$. In either case, $t$ has a representation in $\Upsilon$ over $\mathbb{R}$.

(ii). ($p = 2$) By construction, 2 does not divide $\varnothing r^2$ and so $\ord_p(t) = 1$. By assumption, $\Upsilon_2 \cong \mathbb{T}^+$ or $\Upsilon_2 \cong \mathbb{T}^-$. In either case, by Lemma 12 and Theorem 10, $t$ has a $2^r$-primitive representation in $\Upsilon_2$.

(iii). ($p \in S$) By definition of $r$, $\ord_p(r) = 0$ for all primes $p \in S$. By construction in Equation 52, $\left(\frac{t}{p}\right) = \left(\frac{2\rho u}{p}\right) = \left(\frac{n_p}{p}\right)$, where $\Upsilon \cong a_p \oplus p^r b_p$. But then, by Lemma 24, $a_p$ (and hence, $\Upsilon$) represents $t$, $p^r$-primitively.

(iv). ($p$ odd, $p \in \mathbb{P}_\mathbb{P} \setminus S$) If $\Upsilon \cong a_p \oplus p^r b_p$ then, $i_p$ is even. If $\left(\frac{a_p}{p}\right) = \left(\frac{2\rho u}{p}\right)$ then, $p$ does not divide $2\rho r^2$ and $\left(\frac{a_p}{p}\right) = \left(\frac{t}{p}\right)$. Thus, $t$ has a $p^r$-primitive representation in $\Upsilon$ (Lemma 2 and Theorem 10). Otherwise, $\left(\frac{a_p}{p}\right) \neq \left(\frac{2\rho u}{p}\right)$. But then, $\ord_p(t) = \ord_p(r^2) = i_p$, and by Lemma 25 and Theorem 11, $t$ has a $p^r$-primitive representation in $a_p \oplus p^r b_p$. 

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(v). \((p = \varphi)\) Finally, it remains to show that \(t\) has a \(\varphi^*\)-primitive representation in \(\Upsilon\). By Lemma 29, one needs to show that \((-\det(\Upsilon)/\varphi) = 1\). Recall the Quadratic Reciprocity Laws in Equation 11. Also, note that \((-1/p) = 1\) iff \(p \equiv 1 \mod 4\). The computation of \((-\det(\Upsilon)/\varphi)\) can be done using Lemma 29 as follows.

\[
\prod_{p \in S} \left( \frac{\varphi}{p} \right) = \prod_{p \in S} \left( \frac{-\det(\Upsilon)}{p} \right) = (-1)^{S_{(3,7)}} \prod_{p \in S} \left( \frac{2}{p} \right) = (-1)^{S_{(3,7)}} \prod_{p \in S} \left( \frac{\varphi}{p} \right)
\]

It turns out that \(\varphi \mod 4\) was defined to satisfy exactly this equation (see Equation 52).

This completes the proof of the claim that \(t\) has a primitive representation in the genus \(\Upsilon\).

Finally, we show that the set of possibilities under the modulo 4 congruence in Equation 52 is exhaustive, if the input symbol \(\Upsilon\) is valid. The proof of this statement is computer assisted and the code can be found in Appendix C.

We design the test program as follows. For all possible choices of \(\epsilon, \rho \in \{1, -1\}\), and \(s_{db} \in \{0, 1, 2, 3\}\), we compute \(\text{sig}(\Upsilon), \text{odt}(\Upsilon)\) by Equation 54. We also compute \(\sum_{p \in P} \text{exs}_p(\Upsilon)\) by Lemma 28. Then, we check the oddity condition i.e.,

\[
\text{sig}(\Upsilon) + \sum_{p \in P} \text{exs}_p(\Upsilon) \equiv \text{odt}(\Upsilon) \pmod{8}
\]

If the oddity condition is satisfied then we check if at least one of these conditions hold.

\[
\begin{align*}
(p = +, S_{(3,5)} + S_{-} \text{ even}) & \quad (p = -, S_{(5,7)} + S_{-} \text{ even}) \\
(p = +, \epsilon = +, S_{(5,7)} + S_{-} \text{ odd}) & \quad (p = +, \epsilon = -, S_{(5,7)} + S_{-} \text{ even}) \\
(p = -, \epsilon = +, S_{(3,5)} + S_{-} \text{ even}) & \quad (p = -, \epsilon = +, S_{(3,5)} + S_{-} \text{ odd})
\end{align*}
\]

(54)

In each of these cases, a \(\varphi\) and hence \(t\) exists by Equation 52. The test program never finds itself in a situation when none of the conditions in Equation 54 are true. This completes the proof of existence of a primitively representable \(t\).

\( \square \)

7.5 Representation: \(n = 2\), Type I, Even

This section deals with the case when \(\Upsilon_2 \sim a_2 \oplus 2^{i_2} b_2\), where \(i_2\) is even, and \(a_2, b_2 \in \{1, 3, 5, 7\}\).

Lemma 31. Let \(\Upsilon^{n=2}\) be valid reduced genus with \(\Upsilon_2 \sim a_2 \oplus 2^{i_2} b_2\), where \(i_2\) is even and \(a_2, b_2 \in \text{SGN}^x\). Then, there exists an integer of the form \(\varphi r^2\) with primitive representation in \(\Upsilon\), where \(\varphi\) is an odd prime that does not divide \(\det(\Upsilon)\) and \(r^2\) is an integer that divides \(\det(\Upsilon)\).
Proof: Recall the definition of $\epsilon, \rho \in \{-1, 1\}$ as in Equation 49. Let us define the following set of congruences.

\[
\varphi \equiv \rho a_p \pmod{p} \quad \text{for all } p \in S
\]

\[
\varphi \equiv \begin{cases} 
 x \pmod{8} & \text{if } \rho = +, S_{n} \text{ even and } x \in X \cap \{1, 5\} \text{ or} \\
 y \pmod{8} & \text{if } \rho = -, S_{n} \text{ even and } y \in X \cap \{3, 7\} \text{ or} \\
 0 & \text{if } \rho = 2, \varphi \equiv a_2 \pmod{8} \\
i_2 & \text{if } p = 2, \varphi \equiv b_2 \pmod{8} \\
0 & \text{if } p \in \mathbb{P} \setminus \{(2) \cup S\}, \left(\frac{a_p}{p}\right) = \left(\frac{\rho a_p}{p}\right) \\
i_p & \text{if } p \in \mathbb{P} \setminus \{(2) \cup S\}, \left(\frac{a_p}{p}\right) \neq \left(\frac{\rho a_p}{p}\right)
\end{cases}
\]

(55)

where, $X = \{\rho a_2 \pmod{8}, \rho b_2 \pmod{8}\}$

A few words on notation. The modulo 8 congruences should be read as follows. Consider the first congruence “$\rho \equiv x \pmod{8}$, if $\rho = +, S_n$ even and $x \in X \cap \{1, 5\}$”. If $X \cap \{1, 5\}$ is empty then this statement is false. Otherwise, we pick any element $x$ from the intersection. Also, $x \equiv 1 \pmod{4}$ and $y \equiv 3 \pmod{4}$.

Note that the set of possibilities under which we can write a modulo 8 congruence is not exhaustive. It is, as we show later, exhaustive for every valid symbol $\Upsilon$.

It is possible to solve the congruence in such a way that $\varphi$ is a prime (Dirichlet’s Theorem). Consider an integer $r$ defined as follows.

\[
r = \prod_{p \in \mathbb{P}_r/2} p^{e_p/2}
\]

The exponent $e_p$ is always even and hence $r$ is an integer. Define $t = \rho p r^2$. We next show that $t$ has a primitive representation in the genus $\Upsilon$. For this, it suffices to show that $t$ has a primitive $p^r$-representation in $\Upsilon$ for all $p \in \{p \mid \text{ord}_p(2t \det(\Upsilon)) > 0\} \cup \{-1\}$ (see Lemma 21).

(i). $(p = -1)$ By Equation 49, $\rho = -1$ if $\text{sig}(\Upsilon) = -2$. In this case, $\Upsilon$ is negative definite and represents all negative integers. Otherwise, $\Upsilon \cong 1 \oplus x$ and $p = 1$. But then, $t$ is a positive integer and hence can be represented by $\Upsilon$ over $\mathbb{R}$. In either case, $t$ has a representation in $\Upsilon$ over $\mathbb{R}$.

(ii). $(p \in S)$ By definition of $r$, $\text{ord}_p(r) = 0$ for all primes $p \in S$. By construction in Equation 55, $\left(\frac{l}{p}\right) = \left(\frac{\rho a_p}{p}\right) = \left(\frac{a_p}{p}\right)$, where $\Upsilon \cong a_p \oplus p^{i_p} b_p$. But then, by Lemma 2 $a_p$ (and hence, $\Upsilon$) represents $t$, $p^r$-primitively.

(iii). $(p \text{ odd}, p \in \mathbb{P}_r \setminus S)$ If $\Upsilon \cong a_p \oplus p^i b_p$ then, $i_p$ is even. If $\left(\frac{a_p}{p}\right) = \left(\frac{\rho a_p}{p}\right)$ then, $p$ does not divide $\rho p r^2$ and $\left(\frac{a_p}{p}\right) = \left(\frac{l}{p}\right)$. Thus, $t$ has a $p^r$-primitive representation in $\Upsilon$ (Lemma 2 and Theorem 10). Otherwise, $\left(\frac{a_p}{p}\right) \neq \left(\frac{\rho a_p}{p}\right)$. But then, $\text{ord}_p(t) = \text{ord}_p(r^2) = i_p$, and by Lemma 29 and Theorem 10 $t$ has a $p^r$-primitive representation in $a_p \oplus p^{i_p} b_p$.

(iv). $(p = \varphi)$ Next, we show that $t$ has a $\varphi^r$-primitive representation in $\Upsilon$. By Lemma 29 one needs to show that $\left(\frac{-\det(\Upsilon)}{\varphi}\right) = 1$. Recall the Quadratic Reciprocity Laws in Equation 1. Also, note that $\left(\frac{l}{p}\right) = 1$
iff \( p \equiv 1 \mod 4 \). The computation of \( \left( -\frac{\det(\Upsilon)}{\varphi} \right) \) can be done using Lemma 29 as follows.

\[
\prod_{\varphi \in S} \left( \frac{\varphi}{p} \right) = \prod_{\varphi \in S} \left( \frac{\rho \alpha \varphi}{p} \right) = (-1)^{S_\varphi} \rho^{S_{(3,7)}}
\]

\[
\left( -\frac{\det(\Upsilon)}{\varphi} \right) = \begin{cases} 
(-1)^{S_-} \rho^{S_{(3,7)}} & \text{if } \varphi \equiv 1 \mod 4, \\
(-1)^{S_+} + S_- + 1 \rho^{S_{(3,7)}} & \text{otherwise.}
\end{cases}
\]

It turns out that \( \varphi \mod 4 \) was defined to satisfy exactly this equation (see Equation 52).

(v) (\( p = 2 \)) In this case, \( \Upsilon \sim 2^* \cong a_2 \oplus 2^i b_2 \), where \( a_2, b_2 \in \{1, 3, 5, 7\} \). From Equation 55, either \( a_2 \sim 2^* \) or \( 2^i b_2 \sim t \). In either case, \( t \) has a \( 2^* \)-primitive representation in \( \Upsilon \).

This completes the proof of the claim that \( t \) has a primitive representation in the genus \( \Upsilon \).

Finally, we show that the set of possibilities under the modulo 8 congruence in Equation 55 is exhaustive, if the input symbol \( \Upsilon \) is valid. The proof of this statement is computer assisted and the code can be found in Appendix [6].

We design the test program as follows. For all possible choices of \( \epsilon, \rho \in \{1, -1\}, a_2, b_2 \in \{1, 3, 5, 7\} \), and \( S_{db} \in \{0, 1, 2, 3\} \), we compute \( \text{sig}(\Upsilon), \text{odt}(\Upsilon) \) by Equation 50. We also compute \( \sum_{p \in \mathbb{P}} \text{exs}_{p}(\Upsilon) \) by Lemma 28. Then, we check the oddity condition i.e,

\[
\text{sig}(\Upsilon) + \sum_{p \in \mathbb{P}} \text{exs}_{p}(\Upsilon) \equiv \text{odt}(\Upsilon) \mod 8
\]

We next check the following determinant condition for \( \Upsilon_2 \),

\[
\left( \frac{a_2 b_2}{2} \right) = (-1)^{S_{(3,5)}}
\]

If either of these conditions is not satisfied then the symbol \( \Upsilon \) is not valid. For the others, we check if at least one of these condition holds.

\[
\begin{align*}
(\rho = +, S_- \text{ even}, |X \cap \{1, 5\}| > 0) \\
(\rho = -, S_- + S_{(3,7)} \text{ even}, |X \cap \{1, 5\}| > 0) \\
(\rho = +, \epsilon = +, S_{(3,7)} + S_- \text{ odd}, |X \cap \{3, 7\}| > 0) \\
(\rho = +, \epsilon = -, S_{(3,7)} + S_- \text{ even}, |X \cap \{3, 7\}| > 0) \\
(\rho = -, \epsilon = +, S_- \text{ even}, |X \cap \{3, 7\}| > 0) \\
(\rho = -, \epsilon = -, S_- \text{ odd}, |X \cap \{3, 7\}| > 0)
\end{align*}
\]

In each of these cases, \( \varphi \) and hence \( t \) exists by Equation 55. The test program never finds itself in a situation when none of the conditions in Equation 57 are true. This completes the proof of existence of a primitively representable \( t \).

\( \square \)
7.6 Representation: $n = 2$, Type I, Odd

This section deals with the case when $\mathcal{Y}_2 \cong a_2 \oplus 2^{i_2}b_2$, where $i_2$ is odd and $a_2, b_2 \in \{1, 3, 5, 7\}$.

**Lemma 32** Let $\mathcal{Y}^{n=2}$ be a valid reduced genus with $\mathcal{Y}_2 \cong a_2 \oplus 2^{i_2}b_2$, where $i_2$ is odd and $a_2, b_2 \in \text{SGN}^\times$. Then, there exists an integer of the form $\varphi^2$ or $2^{i_2} \varphi^2$ with primitive representation in $\mathcal{Y}$, where $\varphi$ is an odd prime that does not divide $\det(\mathcal{Y})$ and $r^2$ is an integer that divides $\det(\mathcal{Y})$.

**Proof:** By assumption $i_2$ is odd and hence an odd power of 2 divides $\det(\mathcal{Y})$.

Consider the following set of congruences, along with the construction of the candidate primitively representable integer $t$.

\[
\begin{align*}
\text{if } \left( \rho a_2 \equiv 1 \text{ mod } 4 \text{ and } (-1)^{S'} \rho^{S(3,7)} \left( \frac{a_2}{2} \right) = 1 \right) \text{ or } \\
\left( \rho a_2 \equiv 3 \text{ mod } 4 \text{ and } (-1)^{S'} + S(3,7) \rho^{S(3,7)} \left( \frac{a_2}{2} \right) = 1 \right) \text{ then } \\
\varphi \equiv \rho a_p \text{ mod } p \quad \text{for all } p \in S \\
\varphi \equiv \rho a_2 \text{ mod } 8 \\
\epsilon_p = \begin{cases} 0 & \text{if } p \in \{2\} \cup S \\
 0 & \text{if } p \in \mathbb{P} \setminus \{(2) \cup S\}, \left( \frac{a_2}{p} \right) = \left( \frac{2}{p} \right) \\
i_p & \text{if } p \in \mathbb{P} \setminus \{(2) \cup S\}, \left( \frac{a_2}{p} \right) \neq \left( \frac{2}{p} \right) \\
\end{cases} \\
\epsilon = \rho \varphi \left( \prod_{p \in \mathbb{P} \setminus \{2\} \cup S} p^{\epsilon_p} \right) \\
\end{align*}
\]

(58)

Note that the set of possibilities under which we can write the congruence for $\varphi$ is not exhaustive. It is, as we show later, exhaustive for every valid symbol $\mathcal{Y}$.

We show that $t$ has a primitive representation in $\mathcal{Y}$, or equivalently, $t$ has a $p'$-primitive representation in $\mathcal{Y}$ for all $p \in \{ -1, 2 \} \cup \mathbb{P}$. For this, it suffices to show that $t$ has a primitive $p'$-representation in $\mathcal{Y}$ for all $p \in \{ p \mid \text{ord}_p(2t \det(\mathcal{Y})) > 0 \} \cup \{-1\}$ (see Lemma 21).

(i). ($p = -1$) The value of $\rho = -1$ iff $\text{sig}(\mathcal{Y}) = -2$. Thus, $t$ has a representation over $\mathbb{R}$ in $\mathcal{Y}$.

(ii). (odd $p \in \mathbb{P} \setminus S$) In this case, $\mathcal{Y}_p \cong a_p \oplus p^{i_p}b_p$ and $\text{ord}_p(t) = i_p$, where $i_p$ is even. By Lemma 25, $t$ has a primitive representation in $a_p \oplus p^{i_p}b_p$.

(iii). ($p = 2$) By construction, either $t \cong a_2$ or $t \cong 2^{i_2}b_2$. In either case, $t$ has a primitive $2^*$-representation in $\mathcal{Y}$ (Lemma 2).

(iv). ($p = \varphi$) In this case, we need to show that $\left( \frac{-\det(\mathcal{Y})}{\varphi} \right) = 1$. We split the proof into two sub-cases.
(a). \( \varphi \equiv \rho a_2 \mod 8 \) We first compute the value of \( \prod_{p \in \{2\} \cup S} \left( \frac{\varphi}{p} \right) \).

\[
\prod_{p \in \{2\} \cup S} \left( \frac{\varphi}{p} \right) = \prod_{p \in \{2\} \cup S} \left( \frac{\rho a_2}{p} \right) = (-1)^{S^+} \rho^{S(3,7)} \left( \frac{a_2}{2} \right)
\]

And then, we insert it into the computation of \( \left( \frac{-\det(\Upsilon)}{\varphi} \right) \) in Lemma 29

\[
\left( \frac{-\det(\Upsilon)}{\varphi} \right) = \begin{cases} 
(-1)^{S^-} \rho^{S(3,7)} \left( \frac{a_2}{2} \right) & \text{if } \varphi \equiv 1 \mod 4 \\
(-1)^{S^-} \rho^{S(3,7)} \left( \frac{a_2}{2} \right) + 1 & \text{otherwise}
\end{cases}
\]

(b). \( \varphi \equiv \rho b_2 \mod 8 \) Similarly, we compute,

\[
\prod_{p \in \{2\} \cup S} \left( \frac{\varphi}{p} \right) = \prod_{p \in S} \left( \frac{2\rho a_2}{p} \right) = (-1)^{S^-} \rho^{S(3,7)} \left( \frac{b_2}{2} \right)
\]

And then, we insert it into the computation of \( \left( \frac{-\det(\Upsilon)}{\varphi} \right) \) in Lemma 29

\[
\left( \frac{-\det(\Upsilon)}{\varphi} \right) = \begin{cases} 
(-1)^{S^-} \rho^{S(3,7)} \left( \frac{b_2}{2} \right) & \text{if } \varphi \equiv 1 \mod 4 \\
(-1)^{S^-} \rho^{S(3,7)} \left( \frac{b_2}{2} \right) + 1 & \text{otherwise}
\end{cases}
\]

In either case, \( \left( \frac{-\det(\Upsilon)}{\varphi} \right) = 1 \), proving the primitive \( \varphi^* \)-representativeness.

This completes the proof of the claim that \( t \) has a primitive representation in the genus \( \Upsilon \).

Finally, we show that the set of possibilities when we can write a congruence (see Equation 58) is exhaustive, if the input symbol \( \Upsilon \) is valid. The proof of this statement is computer assisted and the code can be found in Appendix C

We design the test program as follows. For all possible choices of \( \epsilon, \rho \in \{1, -1\}, a_2, b_2 \in \{1, 3, 5, 7\} \) and \( S_{db} \in \{0, 1, 2, 3\} \), we compute \( \text{sig}(\Upsilon), \text{odt}(\Upsilon) \) by Equation 50. We also compute \( \sum_{p \in P} \text{exs}_p(\Upsilon) \) by Lemma 28. Then, we check the oddity condition i.e.,

\[
\text{sig}(\Upsilon) + \sum_{p \in P} \text{exs}_p(\Upsilon) \equiv \text{odt}(\Upsilon) \pmod{8}
\]

We next check the following determinant condition for \( \Upsilon_2 \).

\[
\left( \frac{a_2 b_2}{2} \right) = (-1)^{S(3,5)}
\]

If either of these conditions is not satisfied then the symbol \( \Upsilon \) is not valid. For the others, we check if at least one of these condition holds.

\[
\begin{align*}
(\rho a_2 &\equiv 1 \mod 4, (-1)^{S^-} \rho^{S(3,7)} \left( \frac{a_2}{2} \right) = 1) \\
(\rho a_2 &\equiv 3 \mod 4, (-1)^{S^-} \rho^{S(3,7)} \left( \frac{a_2}{2} \right) = 1) \\
(\rho b_2 &\equiv 1 \mod 4, (-1)^{S(3,5)} + S^- \rho^{S(3,7)} \left( \frac{b_2}{2} \right) = 1) \\
(\rho b_2 &\equiv 3 \mod 4, (-1)^{S(3,5)} + S^- \rho^{S(3,7)} \left( \frac{b_2}{2} \right) = 1)
\end{align*}
\]

(59)

In each of these cases, a \( \varphi \) and hence \( t \) exists as in Figure 5.1. The test program never finds itself in a situation when none of the conditions in Equation 59 are true. This completes the proof of existence of a primitively representable \( t \).

□

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7.7 Representation: putting it together

Proof: (Theorem 33) The construction follows from the constructive nature of Lemma 26, Lemma 27, and the constructions for the case of dimension 2. The only remaining task in case of dimension 2 is to find a $\varphi$ which satisfies the given set of congruence relations.

Assuming ERH, one can find $\sigma_p$, i.e., the smallest non-residue modulo $p$ in $O(\log^3 p)$ ring operations over $\mathbb{Z}/p\mathbb{Z}$. If done for every prime that divides $\det(\Upsilon)$, this takes $O(|p|\log^3 \det(\Upsilon))$ ring operations over $\mathbb{Z}/\det(\Upsilon)\mathbb{Z}$.

Let $p_1, \cdots, p_s$ be the primes which appear with odd parity in the symbol and $\alpha = 8p_1 \cdots p_s$. Then, we form the required set of congruent equations i.e.,

$$x \equiv \begin{cases} x_{p_i} \mod p_i & \text{if } \epsilon_i = -1, \text{ where } x_{p_i} \in (\mathbb{Z}/p\mathbb{Z})^x \\ \tau \mod 8 & \text{where } \tau \in \{1, 3, 5, 7\} \end{cases}$$

Solve this set of congruence using the Chinese Remainder and let $a$ be a solution. Pick a $b$ uniformly at random from the range $[0, \alpha^2]$. If $S = \{a + z\alpha \mid z \in \mathbb{Z}, \alpha \leq \alpha^2\}$, then $a + bo$ is a uniformly random element of $S$. By Theorem 3 with probability $\frac{1}{\log |S|}$ the number $a + bo$ is prime. One then sets $\varphi = a + bo$. If repeated $O(\log^2 |S|)$ times, one can find $\varphi$ with overwhelming probability. The time complexity of the algorithm follows from the fact that $|S| \leq \alpha^2 \leq \det(\Upsilon)^2$.

The next step is to devise an algorithm that given the local form $S$, positive integer $q$ and the generated $t$ finds a primitive $x$ such that $xSx \equiv t \mod q$. Instead, we find primitive representations $x_p$ for all $p$ that divides $q$ such that $x^*_p S_p x_p \equiv t \mod p^k$, where $S_p$ is the $p^*$-equivalent form, $k = \text{ord}_p(q)$, and then combining them using Chinese Remainder.

The construction of $t$ used at most 4 diagonal entries of $S_p$ and so to find $x_p$ we use Theorem 11 and construct $x$ by filling the rest of the dimensions with 0. The time taken by this algorithm does not depend on $n$ and is poly$(k, \log p)$, for each prime factor of $q$.

8 Polynomial Time Algorithm

In this section, we give the main contribution of this thesis.

Theorem 33 Let $\Upsilon^n$ be a valid genus. Then, there exists a randomized poly$(n, \log \det(\Upsilon))$ algorithm that ouputs a quadratic form $\mathcal{Q}^n \in \Upsilon$ with constant probability.

Proof: Recall definition of the reduced genus. By Lemma 16 it follows that finding a quadratic form in $\Upsilon^* = \text{red}(\Upsilon)$ suffices for generating a quadratic form in $\Upsilon$.

The algorithm described in Section 6 is correct; but is not polynomial as it is. The analysis of the time complexity will be done on a different algorithm, the correctness of which will follow from the proof of correctness of the algorithm in Section 6. We now describe the algorithm.

QFGenPoly (input: valid symbol $\Upsilon^n$) output: $\mathcal{Q}^n \in \Upsilon$

i. If $n < 4$ then return QFGen($\Upsilon$).

ii. Compute gcd($\Upsilon$) and let $\Upsilon^* = \Upsilon / \gcd(\Upsilon)$.

iii. Find $t$ which has a primitive representation in $\Upsilon^*$. Let $q = t^{n-1} \det(\Upsilon^*)$ and $K_p = \text{ord}_p(q)$.

iv. For every $p \in P\Upsilon$, we construct a block diagonal matrix $S_p$ and a matrix $[x_p, A_p] \in \mathbb{Z}/p^{K_p} \mathbb{Z}$ as follows.

Use Theorem 18 to find a block diagonal matrix $D_p \in \tilde{\Upsilon}^*$. Recall the construction of $t$ in Lemma 26.

If $p$ is odd then by construction, $t$ has a primitive representation in $\mathbb{Z}/p^{K_p} \mathbb{Z}$ of two different types: (a) $t$ has a primitive representation by the first entry of $D_p$. Let $x$ be the primitive representation. Then,
set $S_p = D_p$ and $[x_p, A_p] = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \oplus \mathbb{I}^{n-2}$, (b) otherwise, $t$ has a primitive representation by two of the first three entries of $D_p$, say $d_1, d_2$ where $\text{ord}_p(d_1) \geq \text{ord}_p(d_2)$. If $(x_1, x_2)$ is a primitive $p^K_p$ representation of $t$ then in this case $x_1$ is primitive. Let $(D_p)_{3+}$ be the rest of the blocks in $D_p$ then we set $S_p = d_1 \oplus d_2 \oplus (D_p)_{3+}$ and $[x_p, A_p] = \begin{pmatrix} x_1 & 0 \\ x_2 & x_1^{-1} \end{pmatrix} \oplus \mathbb{I}^{n-2}$.

On the other hand, when $p = 2$ then too $t$ has a primitive representation in $\mathbb{Z}/2^{K_p}\mathbb{Z}$ of two different kinds; (a) when $\Upsilon^*_2$ has a type II block then if $x_1, x_2$ be the primitive representation with $x_1$ odd then we set $S_2$ as the block diagonal form equivalent to $D_2$ where the first block is the type II block which was used to represent $t$. Then, we set $[x_2, A_2] = \begin{pmatrix} x_1 & 0 \\ x_2 & x_1^{-1} \end{pmatrix} \oplus \mathbb{I}^{n-2}$, (b) otherwise the first four Type I entries of $D_2$ were used to represent $t$ and $\text{ord}_2((D_2)_4) = \text{ord}_p(t)$, by construction. Also, if $x_1, x_2, x_3, x_4$ is the primitive representation of $t$ then $x_4$ is primitive. In this case, we set $S_2$ as $(D_2)_4 \oplus \cdots \oplus (D_2)_1 \oplus (D_2)_4+$, and $[x_2, A_2] = \begin{pmatrix} x_4 & 0 & 0 & 0 \\ x_3 & x_4^{-1} & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ x_1 & 0 & 0 & 1 \end{pmatrix} \oplus \mathbb{I}^{n-4}$.

**Property.** The construction satisfies the property that for each $p \in \Upsilon_p$, $[x_p, A_p] \in \text{GL}_n(\mathbb{Z}/p^{K_p}\mathbb{Z})$ and $x'_p S_p x_p \equiv t \mod p^K_p$.

v. For each $p \in \mathbb{P}_\Upsilon$, let $\hat{\Upsilon}_p = \text{sym}_p(\mathbb{H}_p)$, where $\mathbb{H}_p$ is defined as follows.

$$d_p = x'_p S_p A_p \mod p^{K_p} \quad \mathbb{H}_p = (tA'_p S_p A_p - d'_p d_p) \mod p^{K_p} \quad \text{(60)}$$

vi. Let $\hat{\Upsilon} = \{ \hat{\Upsilon}_p : p \in \mathbb{P}_\Upsilon \}$, and $\hat{\Upsilon}^* = \hat{\Upsilon} / \text{gcd}(\hat{\Upsilon})$.

vii. Call this algorithm recursively with input $\hat{\Upsilon}^*$. Let us suppose that the algorithm returns $\hat{h}^* \in \text{Gen}(\hat{\Upsilon}^*)$. Then, set $\hat{h} = \text{gcd}(\hat{\Upsilon}) \hat{h}^*$.

viii. Use Chinese Remaindering to compute $[x, A]$ from $\{[x_p, A_p] \mod p^{K_p} : p \in \mathbb{P}_\Upsilon \}$, $S$ from $\{S_p \mod p^{K_p} : p \in \mathbb{P}_\Upsilon \}$ and $\mathbb{H}$ from $\{\mathbb{H}_p \mod p^{K_p} : p \in \mathbb{P}_\Upsilon \}$.

ix. Canonicalize both $\mathbb{H}$ and $\hat{h}$ over $\mathbb{Z}/q\mathbb{Z}$, i.e., we find $\tilde{U} \in \text{GL}_{n-1}(\mathbb{Z}/q\mathbb{Z})$ such that $\hat{h} \equiv \tilde{U} \mathbb{H} \mod q$.

x. Output the following quadratic form.

$$Q = \text{gcd}(\hat{\Upsilon}) \begin{pmatrix} t \\ (\tilde{U})' \tilde{d} \tilde{d}' \tilde{U} \end{pmatrix} \quad \text{(61)}$$

The correctness of this algorithm follows from the proof before.

Let us compute the time complexity of this algorithm.

Our first step is to show that the recursions do not blow up the size of the symbol. Notice that to calculate the $(n-1)$-dimensional symbol, we multiply by $t$ in Equation (60) The analysis is done below, separately for odd primes and $p = 2$.

**Odd primes.** In this case, we show that $\text{ord}_p(\hat{\Upsilon}^*) = \text{ord}_p(\Upsilon^*)$.

For each odd $p \in \mathbb{P}_\Upsilon$, $x'_p S_p x_p \equiv t \mod p^{K_p}$ and $[x_p, A_p] \in \text{GL}_n(\mathbb{Z}/p^{K_p}\mathbb{Z})$. Recall the two cases discussed in the algorithm while constructing $S_p$ and $[x_p, A_p]$.

Case 1: Suppose $t$ is primitively representable by the first entry of $S_p$. Let $x$ be the primitive representation.

Note that $x$ is primitive and $\langle S_p \rangle$ has $p$-scale 0 because $\Upsilon^*$ is reduced. The computation for $\mathbb{H}_p$ (and
ord_p(Υ) is as follows.

\[
[x_p, A_p] \equiv \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \oplus I^{n-2} \mod p^{K_p} \quad S_p = s_1 + \cdots + s_n
\]

\[
[x_p, A_p]S_p[x_p, A_p] \equiv x^2 s_1 \oplus s_2 x^{-2} \text{diag}(s_3, \ldots) \mod p^{K_p}
\]

\[
H_p \equiv tA_pS_pA_p - d_p' d_p = s_1 s_2 \oplus x^2 s_1 \text{diag}(s_3, \ldots, s_n) \mod p^{K_p}
\]

ord_p(Υ) = ord_p(H_p) = ord_p(x^2 s_1 s_n) = ord_p(s_n) = ord_p(Υ^t)

Case 2: Otherwise, the first two entries of $S_p$ represent $t$. In this case, if $S_p = s_1 + \cdots + s_n$, then by construction, ord_p(s_1) \geq ord_p(s_2) and ord_p(x_1) = 0, where $x_1, x_2$ is the primitive representation of $t$ in $S_p$. Then,

\[
[x_p, A_p] = \begin{pmatrix} x_1 & 0 \\ x_2 & x_1^{-1} \end{pmatrix} \oplus I^{n-2} \quad S_p = s_1 + \cdots + s_n
\]

\[
[x_p, A_p]S_p[x_p, A_p] \equiv \begin{pmatrix} s_1 x_1^2 + s_2 x_2^2 \\ s_2 x_1^{-1} x_2 \\ s_2 x_1^{-2} \end{pmatrix} \oplus \text{diag}(s_3, \ldots, s_n) \mod p^{K_p}
\]

By construction, $s_1 x_1^2 + s_2 x_2^2 \equiv t \mod p^{K_p}$, $x_1$ is primitive and ord_p(s_2) = ord_p(t). Thus, each entry of $H_p$ is divisible by $p^{ord_p(t)}$. When we reduce $Υ$ to $Υ^t$, we have ord_p(Υ^t) = ord_p(s_n) = ord_p(Υ^t).

Thus, for odd prime $p$, in either case, ord_p(Υ^t) = ord_p(Υ^*)

**Prime $p = 2$.** Recall the two cases discussed in the algorithm. The first case is when $t$ has a primitive representation using a Type II block. By construction of $S_2$ in Theorem 13, the block is either $T^+$ or $T^-$. Let $i$ be the 2-order of the block, then

\[
[x_2, A_2] = \begin{pmatrix} x_2 & 0 \\ x_2 & x_1^{-1} \end{pmatrix} \oplus I^{n-2} \quad S_2 = 2^i \begin{pmatrix} 2 & 1 \\ 1 & 2c \end{pmatrix} \oplus (S_p)_{3+}
\]

\[
[x_2, A_2]S_2[x_2, A_2] \equiv 2^i \begin{pmatrix} 2x_1^2 + 2x_1 x_2 + 2c x_2^2 \\ 1 + 2c x_2 x_1^{-1} \\ 2c x_1^{-2} \end{pmatrix} \oplus (S_p)_{3+}
\]

\[
H_2 = 4^i(4c - 1) \oplus 2^i(2x_1^2 + 2x_1 x_2 + 2c x_2^2)(S_p)_{3+}
\]

Hence, ord_2(Υ^t) \leq 1 + ord_2(Υ^*). There could be at most $n$ Type II blocks in a quadratic form of dimension $n$, which can be generated during the recursion. Thus, the 2-order of the recursively generated reduced 2-symbols remain bounded by ord_p(Υ^t) + $n$.

Otherwise, $t$ has a primitive representation using the first four type I blocks of $S_2$. In this case, the
calculations are as follows.

\[
[x_2, A_2] = \begin{pmatrix}
  x_4 & 0 & 0 & 0 \\
  x_3 & x_4^{-1} & 0 & 0 \\
  x_2 & 0 & 1 & 0 \\
  x_1 & 0 & 0 & 1
\end{pmatrix} \oplus \mathbb{T}^{n-4} \quad S_2 = \text{diag}(d_4, \ldots, d_1, s_5, \ldots, s_n)
\]

\[
t \equiv d_4x_4^2 + \cdots + d_1x_1^2 \mod 2K^2
\]

\[
[x_2, A_2]S_2[x_2, A_2] = \begin{pmatrix}
  t & d_3x_3x_4^{-1} & d_2x_2 & d_1x_1 \\
  d_3x_3x_4^{-1} & d_3x_4^{-2} & 0 & 0 \\
  d_2x_2 & 0 & d_2 & 0 \\
  d_1x_1 & 0 & 0 & d_1
\end{pmatrix} \oplus (S_2)_{5+} \mod 2K^2
\]

\[
H_2 = \begin{pmatrix}
  d_3x_3^2 + d_2x_2^2 + d_1x_1^2 & -\frac{x_3d_2x_2d_1}{x_4} & -\frac{x_3d_3x_3d_1}{x_4} & (d_2x_2^2 + 3x_3^2 + d_4x_4^2)d_2 \\
  -\frac{x_3d_2x_2d_1}{x_4} & (d_1x_1^2 + d_3x_3^2 + d_4x_4^2)d_2 & -x_2d_2x_1d_1 & (d_2x_2^2 + 3x_3^2 + d_4x_4^2)d_1 \\
  -\frac{x_3d_3x_3d_1}{x_4} & -x_2d_2x_1d_1 & (d_2x_2^2 + 3x_3^2 + d_4x_4^2)d_1 & 0
\end{pmatrix} \oplus (S_2)_{5+} \mod 2K^2
\]

Recall Lemma [24] By construction of \(x_1, \ldots, x_4\) it follows that for each \(i \in [4]\), \(\text{ord}_2(d_ix_i^2) \geq \text{ord}_2(d_4) = \text{ord}_2(t)\), \(\text{ord}_2(d_4) \geq \cdots \geq \text{ord}_2(d_1)\) and \(\text{ord}_2(x_4) = 0\). This implies that \(\text{ord}_2(x_1) \geq \cdots \geq \text{ord}_2(x_4) = 0\) and by inspection, every entry in the first 3 x 3 submatrix of \(H_2\) is divisible by \(2^{\text{ord}_2(t)}\). Thus, \(\text{ord}_2(\tilde{\mathcal{T}}^*) = \text{ord}_2(\mathcal{T}^*)\).

To recapitulate, \(\text{ord}_p(\mathcal{T}^*)\) is equal to \(\text{ord}_p(\tilde{\mathcal{T}}^*)\) unless we use a Type II block to represent \(t\) modulo \(\mathbb{Z}/2K^2\mathbb{Z}\), in which case it increases by exactly 1.

The step by step calculation of the time taken by the algorithm is as follows.

(i.) After calculating the reduced symbol \(\mathcal{T}^*\), the algorithm starts by computing a positive integer \(t\) which is primitively representable in \(\mathcal{T}^*\). For \(n \geq 4\) such an integer can be found by looking at the first 4 dimensions of the symbol \(\mathcal{T}\), see Lemma [20] This takes time linear in the number of relevant primes of \(\mathcal{T}\) i.e., \(O(|\mathcal{T}| \log \det(\mathcal{T}^*))^2\).

(ii.) Next, we find a quadratic form \(S\) which is equivalent to \(\mathcal{T}^*\) over the ring \(\mathbb{Z}/q\mathbb{Z}\), for \(q = t^{n-1} \det(\mathcal{T}^*)\). By Lemma [26] the integer \(t\) has the property that \(t\) divides \(\det(\mathcal{T}^*)\). Thus, we do not introduce any new primes and for every prime \(p \in \mathbb{P}_T\);

\[
\text{ord}_p(q) \leq n \text{ord}_p(\det(\mathcal{T}^*)) + k_p.
\]

By Theorem [15] finding such an integral quadratic form \(S_p\) takes time \(\text{poly}(n, \log \det(\mathcal{T}^*), \log p)\). There are \(|\mathbb{P}_T|\) relevant primes and hence the total time in this step is.

\[
\text{poly}(|\mathbb{P}_T|, n, \log \det(\mathcal{T}^*))
\]

(iii.) Then, we find a primitive representation \(x_p\) of \(t\) in \(S_p\) over \(\mathbb{Z}/p^k\mathbb{Z}\), \(k = \text{ord}_p(q) \leq n \log \det(\mathcal{T}^*)\), for all \(p \in \mathbb{P}_T\). Note that the representation is done by Theorem [11] on a 4 x 4 submatrix, which takes time \(\text{poly}(k, \log p)\). By the bound on \(k\), we get the following expression.

\[
O(|\mathbb{P}_T|, n, \log \det(\mathcal{T}^*))
\]

(iv.) Then, we Chinese Remainder the matrices \([x_p, A_p], S_p\) and \(H_p\) entry-by-entry (\(\leq n^2\) entries in each matrix) to get \([x, A], S\) and \(H\), respectively. The modulus of the Chinese Remainder is \(q\). This takes time \(\text{poly}(|\mathbb{P}_T|, n, \log q)\).

(v.) Finally, we canonicalize both \(H\) and \(\tilde{H}\) modulo \(q\). This is again done by canonicalizing for each prime that divides \(q\) and then Chinese Remaindering the results. For each \(p\), \(\text{ord}_p(q) \leq n \log \det(\mathcal{T}^*)\). Thus,
the time taken for each $p$ is bounded by $\text{poly}(|\mathcal{P}_\Upsilon|, n, \log \det(\Upsilon^*))$.

The next step is to calculate the reduced form $\hat{\Upsilon}^*$ and recurse. By the discussion of the blowup above, it follows that $\det(\hat{\Upsilon}^*) \leq 2^{n-2} \det(\Upsilon^*)$. Or, $\log \det(\hat{\Upsilon}^*) \leq (n-2) \log 2 + \log \det(\Upsilon)$. Thus the total time complexity of the algorithm can be written recursively as

$$T(n, \det(\Upsilon^*)) = T(n-1, 2^{n-2} \det(\Upsilon^*)) + \text{poly}(|\mathcal{P}_\Upsilon|, n, \log \det(\Upsilon^*))$$

Although the blowup in the determinant is exponential, all our algorithms run in $\text{poly}(\log d, |\mathcal{P}_\Upsilon|)$, where $d$ is the determinant of the input genus. For $n \leq 3$, $t \leq \varphi d$, and $\varphi \leq d^2$. Thus, for any constant $\delta > 0$ the generation algorithm runs in time $\text{poly}(n, \log d, \log \frac{1}{\delta})$ and succeeds with probability at least $1 - \delta$.

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A Diagonalizing a Matrix

In this section, we provide a proof of Theorem 7.

Module. There are quadratic forms which have no associated lattice e.g., negative definite quadratic forms. To work with these, we define the concept of free modules (henceforth, called module) which behave as vector space but have no associated realization over the Euclidean space \( \mathbb{R}^n \).

If \( M \) is finitely generated \( \mathbb{R} \)-module with generating set \( x_1, \ldots, x_n \) then the elements \( x \in M \) can be represented as \( \sum_{i=1}^{n} r_i x_i \), such that \( r_i \in \mathbb{R} \) for every \( i \in [n] \). By construction, for all \( a, b \in \mathbb{R} \), and \( x, y \in M \):

\[
a(x + y) = ax + ay \quad (a + b)x = ax + bx \quad a(bx) = (ab)x \quad 1x = x
\]

Note that, if we replace \( \mathbb{R} \) by a field in the definition then we get a vector space (instead of a module). Any inner product \( \beta : M \times M \rightarrow \mathbb{R} \) gives rise to a quadratic form \( Q \in \mathbb{R}^{n \times n} \) as follows;

\[
Q_{ij} = \beta(x_i, x_j).
\]

Conversely, if \( R = \mathbb{Z} \) then by definition, every symmetric matrix \( Q \in \mathbb{Z}^{n \times n} \) gives rise to an inner product \( \beta \) over every \( \mathbb{Z} \)-module \( M \); as follows. Given \( n \)-ary integral quadratic form \( Q \) and a \( \mathbb{Z} \)-module \( M \) generated by the basis \( \{x_1, \ldots, x_n\} \) we define the corresponding inner product \( \beta : M \times M \rightarrow \mathbb{Z} \) as;

\[
\beta(x, y) = \sum_{i,j} c_i d_j Q_{ij} \quad \text{where,} \quad x = \sum_i c_i x_i \quad y = \sum_j d_j x_j.
\]

In particular, any integral quadratic form \( Q^n \) can be interpreted as describing an inner product over a free module of dimension \( n \).

For studying quadratic forms over \( \mathbb{Z}/p^k \mathbb{Z} \), where \( p \) is a prime and \( k \) is a positive integer; the first step is to find equivalent quadratic forms which have as few mixed terms as possible (mixed terms are terms like \( x_1 x_2 \)).

Proof(Theorem 7) The transformation of the matrix \( Q \) to a block diagonal form involves three different kinds of transformation. We first describe these transformations on \( Q \) with small dimensions (2 and 3).
Let \( Q \) be a \( 2 \times 2 \) integral quadratic form. Let us also assume that the entry with smallest \( p \)-order in \( Q \) is a diagonal entry, say \( Q_{11} \). Then, \( Q \) is of the following form; where \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are units of \( \mathbb{Z}/p\mathbb{Z} \).

\[
Q = \begin{pmatrix}
p^i \alpha_1 & p^j \alpha_2 \\
p^j \alpha_2 & p^k \alpha_3
\end{pmatrix} \quad i \leq j, s
\]

The corresponding \( U \in \text{SL}_2(\mathbb{Z}/p^k\mathbb{Z}) \), that diagonalizes \( Q \) is given below. The number \( \alpha_1 \) is a unit of \( \mathbb{Z}/p\mathbb{Z} \) and so \( \alpha_1 \) has an inverse in \( \mathbb{Z}/p^k\mathbb{Z} \).

\[
U = \begin{pmatrix} 1 & -p^{j-i} \alpha_2 \mod p^k \\ 0 & 1 \end{pmatrix} \quad U'QU \equiv \begin{pmatrix} p^i \alpha_1 & 0 \\
p^j \alpha_2 - p^{2j-i} \alpha_2^2 / \alpha_1 & p^k \alpha_3 \end{pmatrix} \pmod{p^k}
\]

(2) If \( Q^2 \) does not satisfy the condition of item (1) i.e., the off diagonal entry is the one with smallest \( p \)-order, then we start by the following transformation \( V \in \text{SL}_2(\mathbb{Z}/p^k\mathbb{Z}) \).

\[
V = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad V'QU = \begin{pmatrix} Q_{11} + 2Q_{12} + Q_{22} & Q_{12} + Q_{22} \\
Q_{12} + Q_{22} & Q_{22} \end{pmatrix}
\]

If \( p \) is an odd prime then \( \text{ord}_p(Q_{11} + 2Q_{12} + Q_{22}) = \text{ord}_p(Q_{12}) \), because \( \text{ord}_p(Q_{11}), \text{ord}_p(Q_{22}) > \text{ord}_p(Q_{12}) \). By definition, \( S = V'QU \) is equivalent to \( Q \) over the ring \( \mathbb{Z}/p^k\mathbb{Z} \). But now, \( S \) has the property that \( \text{ord}_p(S_{11}) = \text{ord}_p(S_{12}) \), and it can be diagonalized using the transformation in (1). The final transformation in this case is the product of \( V \) and the subsequent transformation from item (1). The product of two matrices from \( \text{SL}_2(\mathbb{Z}/p^k\mathbb{Z}) \) is also in \( \text{SL}_2(\mathbb{Z}/p^k\mathbb{Z}) \), completing the diagonalization in this case.

(3) If \( p = 2 \), then the transformation in item (2) fails. In this case, it is possible to subtract a linear combination of these two rows/columns to make everything else on the same row/column equal to zero over \( \mathbb{Z}/2^k\mathbb{Z} \). The simplest such transformation is in dimension 3. The situation is as follows. Let \( Q^3 \) be a quadratic form whose off diagonal entry has the lowest possible power of 2, say \( 2^r \) and all diagonal entries are divisible by at least \( 2^{\ell+1} \). In this case, the matrix \( Q \) is of the following form.

\[
Q = \begin{pmatrix} 2^{\ell+1}a & 2^r b & 2^d \\
2^r b & 2^\ell+1c & 2^e \\
2^d & 2^e & 2^{\ell+1}f 
\end{pmatrix} \quad b \text{ odd, } \ell \leq i, j
\]

In such a situation, we consider the matrix \( U \in \text{SL}_3(\mathbb{Z}/2^{k-\ell}\mathbb{Z}) \) of the form below such that if \( S = U'QU \pmod{2^k} \) then \( S_{13} = S_{23} = 0 \).

\[
U = \begin{pmatrix} 1 & 0 & -r \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
(U'QU)_{13} \equiv 0 \pmod{2^k} \quad \Rightarrow \quad r2a + sb \equiv 2^{i-\ell}d \pmod{2^{k-\ell}}
\]

\[
(U'QU)_{23} \equiv 0 \pmod{2^k} \quad \Rightarrow \quad rb + s2c \equiv 2^{j-\ell}e \pmod{2^{k-\ell}}
\]

For \( i, j \geq \ell \) and \( b \) odd, the solution \( r \) and \( s \) can be found by the Cramer’s rule, as below. The solutions exist because the matrix \( \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \) has determinant \( 4ac - b^2 \), which is odd and hence invertible over the ring \( \mathbb{Z}/2^{k-\ell}\mathbb{Z} \).

\[
\begin{pmatrix} r \\ s \end{pmatrix} = \frac{\det\begin{pmatrix} 2^{i-\ell}d & s \\ 2^{j-\ell}e & 2c \end{pmatrix}}{\det\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}} \pmod{2^{k-\ell}} \quad s = \frac{\det\begin{pmatrix} 2a & 2^{i-\ell}d \\ b & 2^{j-\ell}e \end{pmatrix}}{\det\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}} \pmod{2^{k-\ell}}
\]

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This completes the description of all the transformations we are going to use, albeit for \( n \)-dimensional \( Q \) they will be a bit technical. The full proof for the case of odd prime \( p \) follows.

Our proof will be a reduction of the problem of diagonalization from \( n \) dimensions to \((n-1)\)-dimensions, for the odd primes \( p \). We now describe the reduction.

Given the matrix \( Q^n \), let \( M \) be the corresponding \((\mathbb{Z}/p^k\mathbb{Z})\)-module with basis \( B = [b_1, \cdots, b_n] \) i.e., \( Q = B' \mathbb{B} \). We first find a matrix entry with the smallest \( p \)-order, say \( Q_{i', j'} \). The reduction has two cases: (i) there is a diagonal entry in \( Q \) with the smallest \( p \)-order, and (ii) the smallest \( p \)-order occurs on an off-diagonal entry.

We handle case (i) first. Suppose it is possible to pick \( Q_i \) as the entry with the smallest \( p \)-order. Our first transformation \( U_1 \in \text{SL}_n(\mathbb{Z}/p^k\mathbb{Z}) \) is the one which makes the following transformation i.e., swaps \( b_1 \) and \( b_i \),

\[
[b_1, \cdots, b_n] \rightarrow [b_i, b_2, \cdots, b_{i-1}, b_1, b_{i+1}, \cdots, b_n]
\]

(64)

Let us call the new set of elements \( B_1 = [v_1, \cdots, v_n] \) and the new quadratic form \( Q_1 = B_1' \mathbb{B} \mod p^k. \) Then, \( v_1' v_1 \) has the smallest \( p \)-order in \( Q_1 \) and \( U_1' Q U_1 \equiv Q_1 \mod p^k \). The next transformation \( U_2 \in \text{SL}_n(\mathbb{Z}/p^k\mathbb{Z}) \) is as follows.

\[
w_i = \begin{cases} 
    v_i & \text{if } i = 1 \\
    v_i - \frac{v_1' v_i}{p^{\text{ord}_p(Q_{11})}} \cdot \left( \frac{1}{\text{cpr}_p(Q_{11})} \mod p^k \right) \cdot v_1 & \text{otherwise}.
\end{cases}
\]

(65)

By assumption, \((Q_1)_{11}\) is the matrix entry with the smallest \( p \)-order and so \( p^{\text{ord}_p((Q_1)_{11})} \) divides \( v_1' v_1 \). Furthermore, \( \text{cpr}_p((Q_1)_{11}) \) is invertible modulo \( p^k \). Thus, the transformation in Equation (65) is well defined. Also note that it is a basis transformation, which maps one basis of \( B_1 = [v_1, \cdots, v_n] \) to another basis \( B_2 = [w_1, \cdots, w_n] \). Thus, the corresponding basis transformation \( U_2 \) is a unimodular matrix over integers, and so \( U_2 \in \text{SL}_n(\mathbb{Z}/p^k\mathbb{Z}) \). Let \( Q_2 = U_2' Q U_2 \mod p^k \). Then, we show that the non-diagonal entries in the entire first row and first column of \( Q_2 \) are 0.

\[
(Q_2)_{i}(\neq 1) = (Q_2)_{i1} = w_i' w_i \mod p^k \equiv v_i' v_i - \frac{v_1' v_i}{p^{\text{ord}_p((Q_2)_{11})}} \cdot \left( \frac{1}{\text{cpr}_p(Q_{11})} \mod p^k \right) \cdot v_1' v_1 \\
\equiv v_1' v_i - \frac{v_1' v_i}{p^{\text{ord}_p((Q_2)_{11})}} \cdot \left( \frac{1}{\text{cpr}_p(Q_{11})} \mod p^k \right) \cdot p^{\text{ord}_p((Q_2)_{11})} \cdot \text{cpr}_p((Q_1)_{11}) \\
\equiv 0 \mod p^k
\]

Thus, we have reduced the problem to \((n-1)\)-dimensions. We now recursively call this algorithm with the quadratic form \( S = [w_2, \cdots, w_n]/[w_2, \cdots, w_n] \mod p^k \) and let \( V \in \text{SL}_{n-1}(\mathbb{Z}/p^k\mathbb{Z}) \) be the output of the recursion. Then, \( V' S V \mod p^k \) is a diagonal matrix. Also, by construction \( Q_2 = \text{diag}((Q_2)_{11}), S \). Let \( U_3 = 1 \oplus V \), and \( U = U_1 U_2 U_3 \), then, by construction, \( U' Q U \mod p^k \) is a diagonal matrix; as follows.

\[
U' Q U = U_3' U_2' U_1' Q U_1 U_2 U_3 = U_3' Q U_3 = (1 \oplus V)' \text{diag}((Q_2)_{11})(1 \oplus V) \\
\equiv \text{diag}((Q_2)_{11}, V' S V) \mod p^k
\]

Otherwise, we are in case (ii) i.e., the entry with smallest \( p \)-order in \( Q \) is an off diagonal entry, say \( Q_{i', j'}, i* \neq j* \). Then, we make the following basis transformation from \([b_1, \cdots, b_n]\) to \([v_1, \cdots, v_n]\) as follows.

\[
v_i = \begin{cases} 
    b_i & \text{if } i = i* \\
    b_i + b_{j*} & \text{otherwise}.
\end{cases}
\]

(66)

The transformation matrix \( U_0 \) is from \( \text{SL}_n(\mathbb{Z}/p^k\mathbb{Z}) \). Recall, \( \text{ord}_p(Q_{i', j*}) < \text{ord}_p(Q_{i*, j'}) < \text{ord}_p(Q_{j', j*}) \), and so \( \text{ord}_p(v_1' v_i) = \text{ord}_p(b_{i*} b_{j*}) \). Furthermore, \( \text{ord}_p(v_1' v_j) \geq \text{ord}_p(b_{i*} b_{j*}) \), and so the minimum \( p \)-order does not change after the transformation in Equation (66). This transformation reduces the problem to the case.
when the matrix entry with minimum \( p \)-order appears on the diagonal. This completes the proof of the theorem for odd primes \( p \).

For \( p = 2 \), exactly the same set of transformations works, unless the situation in item (3) arises. In such a case, we use the type II block to eliminate all other entries on the same rows/columns as the type II block. Thus, in this case, the problem reduces to one in dimension \((n - 2)\).

The algorithm uses \( n \) iterations, reducing the dimension by 1 in each iteration. In each iteration, we have to find the minimum \( p \)-order, costing \( O(n^2 \log k) \) ring operations and then 3 matrix multiplications costing \( O(n^3) \) operations over \( \mathbb{Z}/p^k\mathbb{Z} \). Thus, the overall complexity is \( O(n^4 + n^3 \log k) \) or \( O(n^4 \log k) \) ring operations.

\( \square \)

## B Missing Proofs

**Proof:** (proof of Lemma 2) We split the proof in two parts: for odd primes \( p \) and for the prime 2.

**Odd Prime.** If \( 0 \neq t \in \mathbb{Z}/p^k\mathbb{Z} \) then \( \text{ord}_p(t) < k \). If \( t \) is a square modulo \( p^k \) then there exists a \( x \) such that \( x^2 \equiv t \mod p^k \). Thus, there exists an \( a \) such that \( x^2 = t + ap^i \). But then, \( 2 \text{ord}_p(x) = \text{ord}_p(t + ap^k) = \text{ord}_p(t) \). This implies that \( \text{ord}_p(t) \) is even and \( \text{ord}_p(x) = \text{ord}_p(t)/2 \). Substituting this into \( x^2 = t + ap^i \) and dividing the entire equation by \( p^{\text{ord}_p(t)} \) yields that \( \text{cpr}_p(t) \) is a quadratic residue modulo \( p \); as follows.

\[
\text{cpr}_p(x)^2 \equiv \text{cpr}_p(t) + ap^{k - \text{ord}_p(t)} \equiv \text{cpr}_p(t) \pmod{p}
\]

Conversely, if \( \text{cpr}_p(t) \) is a quadratic residue modulo \( p \) then there exists an \( u \in \mathbb{Z}/p^k\mathbb{Z} \) such that \( u^2 \equiv \text{cpr}_p(t) \mod p^k \), by Lemma 2. If \( \text{ord}_p(t) \) is even then \( x = p^{\text{ord}_p(t)/2}u \) is a solution to the equation \( x^2 \equiv t \mod p^k \).

**Prime 2.** If \( 0 \neq t \in \mathbb{Z}/2^k\mathbb{Z} \) then \( \text{ord}_2(t) < k \). If \( t \) is a square modulo \( 2^k \) then there exists an integer \( x \) such that \( x^2 \equiv t \mod 2^k \). Thus, there exists an integer \( a \) such that \( x^2 = t + a2^k \). But then, \( 2 \text{ord}_2(x) = \text{ord}_2(t + a2^k) = \text{ord}_2(t) \). This implies that \( \text{ord}_2(t) \) is even and \( \text{ord}_2(x) = \text{ord}_2(t)/2 \). Substituting this into the equation \( x^2 = t + a2^k \) and dividing the entire equation by \( 2^{\text{ord}_2(t)} \) yields,

\[
\text{cpr}_2(x)^2 = \text{cpr}_2(t) + a2^{k - \text{ord}_2(t)} \quad \text{cpr}_2(t) < 2^{k - \text{ord}_2(t)}.
\]

But \( \text{cpr}_2(x) \) is odd and hence \( \text{cpr}_2(x)^2 \equiv 1 \mod 8 \). If \( k - \text{ord}_2(t) > 2 \), then \( \text{cpr}_2(t) \equiv 1 \mod 8 \). Otherwise, if \( k - \text{ord}_2(t) \leq 2 \) then \( \text{cpr}_2(t) < 2^{k - \text{ord}_2(t)} \) implies that \( \text{cpr}_2(t) = 1 \).

Conversely, if \( \text{cpr}_2(t) \equiv 1 \mod 8 \) then there exists an \( u \in \mathbb{Z}/2^k\mathbb{Z} \) such that \( u^2 \equiv \text{cpr}_2(t) \mod 2^k \), by Lemma 2. If \( \text{ord}_2(t) \) is even then \( x = 2^{\text{ord}_2(t)/2}u \) is a solution to the equation \( x^2 \equiv t \mod 2^k \).

\( \square \)

**Proof:** (Theorem 10) We do the proof in two steps: (i) if \( t \) has a primitive \( p^k \)-representation in \( \mathbb{Q} \) then \( t \) has a primitive \( p^* \)-representation in \( \mathbb{Q} \), and (ii) if \( t \) has a primitive \( p^* \)-representation in \( \mathbb{Q} \) then \( t \) has a primitive \( p^* \)-representation in \( \overline{\mathbb{Q}} \) for all \( \overline{\mathbb{Q}} \) such that \( \overline{\mathbb{Q}} \cong \mathbb{Q} \).

The proof of (i) follows. By assumption, there exists a primitive \( x \in (\mathbb{Z}/p^k\mathbb{Z})^n \) such that \( x^t \mathbb{Q} x \equiv t \mod p^k \). Let \( a = x^t \mathbb{Q} x \) be a primitive unit, then by definition of symbols \( a \) and \( t \) have the same \( p^k \)-symbol. This implies that for all \( i \geq k \) there exists a unit \( u_i \in \mathbb{Z}/p^i \mathbb{Z} \) such that \( u_i^2 a \equiv t \mod p^i \). It follows that \( u_i \) is a primitive representation of \( t \) in \( \mathbb{Z}/p^i \mathbb{Z} \). But, if \( x \) is a primitive representation of \( t \) by \( \mathbb{Q} \) then \( x \) is also a primitive representation of \( t \) by \( \overline{\mathbb{Q}} \) over \( \mathbb{Z}/p^j \mathbb{Z} \), for all positive integers \( j \leq i \). This completes the proof of (i).

The proof of (ii) follows. Let \( K \) be an arbitrary positive integer and \( x \in (\mathbb{Z}/p^K\mathbb{Z})^n \) be a primitive vector such that \( x^t \mathbb{Q} x \equiv t \mod p^K \). As \( \overline{\mathbb{Q}} \cong \mathbb{Q} \), there exists \( u \in \text{GL}_n(\mathbb{Z}/p^K\mathbb{Z}) \) such that \( \mathbb{Q} \equiv u^t \mathbb{Q} u \mod p^K \). Thus,
\((\bar{u}x)^\tilde{Q}(\bar{u}x) \equiv t \mod p^K\) and \(\bar{u}x\) is a \(p^K\)-representation of \(t\) in \(\tilde{Q}\). If \(x\) is primitive then so is \(\bar{u}x\). As \(K\) is arbitrary, the proof of (ii) and hence the theorem is complete. \(\square\)

C Computer Assisted Proofs

In this section, we provide the Maple code for the computer Assisted proofs. The procedure \(f\xi\) computes the function \(\xi\) and the names of the other procedures are self-explanatory.

Following are the names of the variables that we use.

\[
\begin{align*}
rh &= \rho \\
a2 &= a_2 \\
sdp &= S_{d^+} \\
s37 &= S_{(3,7)} \\
sm37 &= S_+ + S_{(3,7)} \\
lega &= \left(\frac{\alpha}{\tilde{Q}}\right) \\
eps &= \epsilon \\
b2 &= b_2 \\
sm &= S_{d^+} \\
dm &= S_{d^+} \\
sd &= S_{(3,5)} \\
s35 &= S_{(3,5)} \\
sm35 &= S_+ + S_{(3,5)} \\
legb &= \left(\frac{\beta}{\tilde{Q}}\right)
\end{align*}
\]

When run on Maple, none of these codes output “FAIL!”.
fXi:=proc(s ::integer)::integer;
    f = 1;
    if s mod 4 = 1 or s mod 4 = 0 then f := 0; end if:
    return f:
end proc:

TypeIIBruteForce := proc();
    for rh in {−1,1} do:for eps in {−1,1} do:
        sig := rh · (1 + eps); odty := 0; pexs := (odty − sig) mod 8;
        for s1p in {0,1} do: for s1m in {0,1} do:
            for s5p in {0,1} do: for s5m in {0,1} do:
                for s3p in {0,1,2,3} do: for s3m in {0,1,2,3} do:
                    for s7p in {0,1,2,3} do: for s7m in {0,1,2,3} do:
                        s3 := s3p + s3m; s5 := s5p + s5m;
                        sm := s1m + s3m + s5m + s7m; s7 := s7p + s7m;
                        s37 := s3 + s7; sm35 := sm + s3 + s5; sm57 := sm + s5 + s7;
                        sx := 2 · s3 + 4 · s5 + 6 · s7;
                        if pexs = sx + 2 · (1 − eps)37 · (−1)((sm+fXi(s37))) mod 8 then
                            if not (rh = 1 and type(sm35,even)) and
                                not (rh = −1 and type(sm57,even)) and
                                not (rh = 1 and eps = 1 and type(sm57,odd)) and
                                not (rh = 1 and eps = −1 and type(sm57,even)) and
                                not (rh = −1 and eps = −1 and type(sm35,even)) and
                                not (rh = −1 and eps = 1 and type(sm35,odd)) then
                                print("FAIL!");
                            end if:
                        end if:
                    end do:
                end do:
            end do:
        end do:
    end do:
end proc:
**TypeIEvenBruteForce** := proc():

for rh in \{-1, 1\} do:
for eps in \{-1, 1\} do:
for a2 in \{1, 3, 5, 7\} do:
for b2 in \{1, 3, 5, 7\} do:

sig := rh \cdot (1 + eps); odty := a2 + b2 mod 8; pexs := (odty - sig) mod 8;
leg := numtheory[legendre](a2 \cdot b2, 2); X := \{rh \cdot a2 mod 8, rh \cdot b2 mod 8\};

for s1p in \{0, 1\} do:
for s1m in \{0, 1\} do:
for s5p in \{0, 1\} do:
for s5m in \{0, 1\} do:
for s3p in \{0, 1, 2, 3\} do:
for s3m in \{0, 1, 2, 3\} do:
for s7p in \{0, 1, 2, 3\} do:
for s7m in \{0, 1, 2, 3\} do:

sig := rh \cdot (1 + eps); odty := a2 + b2 mod 8; pexs := (odty - sig) mod 8;
leg := numtheory[legendre](a2 \cdot b2, 2); X := \{rh \cdot a2 mod 8, rh \cdot b2 mod 8\};

for s1p in \{0, 1\} do:
for s1m in \{0, 1\} do:

for s3p in \{0, 1, 2, 3\} do:
for s3m in \{0, 1, 2, 3\} do:

for s7p in \{0, 1, 2, 3\} do:
for s7m in \{0, 1, 2, 3\} do:

s3 := s3p + s3m; s5 := s5p + s5m; sx := 2 \cdot s3 + 4 \cdot s5 + 6 \cdot s7;
sm := s1m + s3m + s5m + s7m; s7 := s7p + s7m;
s37 := s3 + s7; sm37 := sm + s37;

if pexs = sx + 2 \cdot (1 - eps^{s37} \cdot (-1)^{sm + tx(s37)}) mod 8 and leg = (-1)^{s35} then

if not (rh = 1 and type(sm, even) and nops(X intersect \{1, 5\}) > 0) and
not (rh = -1 and type(sm37, even) and nops(X intersect \{1, 5\}) > 0) and
not (rh = 1 and eps = 1 and type(sm37, odd) and nops(X intersect \{3, 7\}) > 0) and
not (rh = -1 and eps = 1 and type(sm37, even) and nops(X intersect \{3, 7\}) > 0) and
not (rh = -1 and eps = -1 and type(sm, even) and nops(X intersect \{3, 7\}) > 0) and
not (rh = -1 and eps = -1 and type(sm, odd) and nops(X intersect \{3, 7\}) > 0) then
print("FAIL!");

end if:

end if:

end do: end do: end do: end do:

end do: end do: end do: end do:

end do: end do: end do: end do:

end proc:
PROCEDURE_TYPEIODODBRUTEFORCE := proc();

for rh in \{-1, 1\} do:
  for a2 in \{1, 3, 5, 7\} do:
    for b2 in \{1, 3, 5, 7\} do:
      legb := numtheory[legendre](b2, 2); lega := numtheory[legendre](a2, 2);
      if legb = -1 then odty := odty + 4 mod 8; end if;
      sig := rh \cdot (1 + eps); pexs := (odty - sig) mod 8;
      leg := numtheory[legendre](a2 \cdot b2, 2);
      for s1p in \{0, 1\} do:
        for s1m in \{0, 1\} do:
          for s5p in \{0, 1\} do:
            for s5m in \{0, 1\} do:
              for s7p in \{0, 1, 2, 3\} do:
                for s7m in \{0, 1, 2, 3\} do:
                  s3 := s3p + s3m; s5 := s5p + s5m; sx := 2 \cdot s3 + 4 \cdot s5 + 6 \cdot s7;
                  sm := s1m + s3m + s5m + s7m; s7 := s7p + s7m;
                  s37 := s3 + s7; s35 := s3 + s5; sm37 := sm + s37; sm35 := sm + s35;
                  if pexs = sx + 2 \cdot (1 - eps^{37} \cdot (1^{(s3m + 35x(s3}))}) mod 8 and
                    leg = \((-1)^{s35}\)
                    then
                    if not (rh \cdot a2 mod 4 = 1 and \((-1)^{sm} \cdot rh^{37} \cdot lega = 1\) and
                      not (rh \cdot a2 mod 4 = 3 and \((-1)^{(sm37+1)} \cdot rh^{37} \cdot eps \cdot lega = 1\) and
                      not (rh \cdot b2 mod 4 = 1 and \((-1)^{sm35} \cdot rh^{37} \cdot legb = 1\) and
                      not (rh \cdot b2 mod 4 = 3 and \((-1)^{(sm57+1)} \cdot rh^{37} \cdot eps \cdot legb = 1\) then
                      print(“FAIL!”);
                    end if:
                end if:
              end do:
            end do:
          end do:
        end do:
      end do:
    end do:
  end do:
end do:
end proc: