ON THE EXISTENCE OF RAMIFIED ABELIAN COVERS

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ABSTRACT. Given a normal complete variety $Y$ over an algebraically closed field $\mathbb{K}$, distinct irreducible effective Weil divisors $D_1, \ldots, D_n$ of $Y$ and positive integers $d_1, \ldots, d_n$, we spell out the conditions for the existence of an abelian cover $X \rightarrow Y$ branched with order $d_i$ on $D_i$ for $i = 1, \ldots, n$.

As an application, we prove that a cover of a normal complete toric variety branched on the torus-invariant divisors is itself a toric variety if $\text{char } \mathbb{K} = 0$ or if the cover is Galois of degree not divisible by $\text{char } \mathbb{K}$.

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Dedicated to Alberto Conte on his 70th birthday.

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1. Introduction

Given a projective variety $Y$ over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq 0$ and effective divisors $D_1, \ldots, D_n$ of $Y$, deciding whether there exists a Galois cover branched on $D_1, \ldots, D_n$ with given multiplicities is a very complicated question, which in the complex case is essentially equivalent to describing the finite quotients of the fundamental group of $Y \setminus (D_1 \cup \cdots \cup D_n)$.

In Section 2 of this paper we answer this question for a normal variety $Y$ in the case that the Galois group of the cover is abelian (Theorem 2.1) of order not divisible by $p$, using the theory developed in [Par91] and [AP12]. In particular, we prove that when the class group $\text{Cl}(Y)$ is torsion free, every abelian cover of $Y$ branched on $D_1, \ldots, D_n$ with given multiplicities not divisible by $p$ is the quotient of a maximal such cover, unique up to isomorphism.

In Section 3 we analyze the same question using toric geometry in the case when $Y$ is a normal complete toric variety and $D_1, \ldots, D_n$ are invariant divisors and we obtain results that parallel those in Section 2 (Theorem 3.5). Combining the two approaches we are able to show that any cover of a normal complete toric variety branched on the invariant divisors is toric if $p = 0$ or if the cover is Galois of order not divisible by $p$ (Theorem 3.7).

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was due to the fact that in the proof we made use of [Mi72, Prop. 1], where the assumption that the degree of the cover be prime to $p$ is made only implicitly.

**Notation.** $G$ always denotes a finite group, almost always abelian, and $G^* := \text{Hom}(G, \mathbb{K}^*)$ the group of characters; $o(g)$ is the order of the element $g \in G$ and $|H|$ is the cardinality of a subgroup $H < G$. We work over an algebraically closed field $\mathbb{K}$ whose characteristic $p$ does not divide the order of the finite abelian groups we consider.

If $A$ is an abelian group we write $A[d] := \{a \in A \mid da = 0\}$ ($d$ an integer), $A^Y := \text{Hom}(A, \mathbb{Z})$ and we denote by $\text{Tors}(A)$ the torsion subgroup of $A$.

The smooth part of a variety $Y$ is denoted by $Y_{\text{sm}}$. The symbol $\equiv$ denotes linear equivalence of divisors. If $Y$ is a normal variety we denote by $\text{Cl}(Y)$ the group of classes, namely the group of Weil divisors up to linear equivalence.

## 2. Abelian covers

### 2.1. The fundamental relations

We quickly recall the theory of abelian covers (cf. [Par91], [AP12], and also [PT95]) in the most suitable form for the applications considered here.

There are slightly different definitions of abelian covers in the literature (see, for instance, [AP12] that treats also the non-normal case). Here we restrict our attention to the case of normal varieties, but we do not require that the covering map be flat; hence we define a cover as a finite morphism $\pi: X \to Y$ of normal varieties and we say that $\pi$ is an abelian cover if it is a Galois morphism with abelian Galois group $G$ ($\pi$ is also called a “$G$-cover”).

Recall that, as already stated in the Notations, throughout all the paper we assume that $G$ has order not divisible by char $\mathbb{K}$.

To every component $D$ of the branch locus of $\pi$ we associate the pair $(H, \psi)$, where $H < G$ is the cyclic subgroup consisting of the elements of $G$ that fix the preimage of $D$ pointwise (the inertia subgroup of $D$) and $\psi$ is the element of the character group $H^*$ given by the natural representation of $H$ on the normal space to the preimage of $D$ at a general point (these definitions are well posed since $G$ is abelian). It can be shown that $\psi$ generates the group $H^*$.

If we fix a primitive $d$-th root $\zeta$ of 1, where $d$ is the exponent of the group $G$, then a pair $(H, \psi)$ as above is determined by the generator $g \in H$ such that $\psi(g) = \zeta^{d o(g)}$. We follow this convention and attach to every component $D_i$ of the branch locus of $\pi$ a nonzero element $g_i \in G$.

If $\pi$ is flat, which is always the case when $Y$ is smooth, the sheaf $\pi_* \mathcal{O}_X$ decomposes under the $G$-action as $\bigoplus_{\chi \in G^*} \mathcal{L}_\chi^{-1}$, where the $L_\chi$ are line bundles ($L_1 = \mathcal{O}_Y$) and $G$ acts on $L_\chi^{-1}$ via the character $\chi$.

Given $\chi \in G^*$ and $g \in G$, we denote by $\chi(g)$ the smallest non-negative integer $a$ such that $\chi(g) = \zeta^{a d o(g)}$. The main result of [Par91] is that the $L_\chi$, $D_1$ (the building data of $\pi$) satisfy the following **fundamental relations**:

\begin{equation}
L_\chi + L_{\chi'} \equiv L_{\chi + \chi'} + \sum_{i=1}^n \alpha_{\chi, \chi'}^i D_i \quad \forall \chi, \chi' \in G^*
\end{equation}
where $e^i_{\chi',\chi'} = |\frac{\chi'(g_i) + \chi(g_i)}{\deg(g_i)}|$. (Notice that the coefficients $e^i_{\chi,\chi'}$ are equal either to 0 or to 1). Conversely, distinct irreducible divisors $D_i$ and line bundles $L_\chi$ satisfying (2.1) are the building data of a flat (normal) $G$-cover $X \rightarrow Y$; in addition, if $h^0(\mathcal{O}_Y) = 1$ then $X \rightarrow Y$ is uniquely determined up to isomorphism of $G$-covers.

If we fix characters $\chi_1, \ldots, \chi_r \in G^*$ such that $G^*$ is the direct sum of the subgroups generated by the $\chi_j$, and we set $L_j := L_{\chi_j}$, $m_j := o(\chi_j)$, then the solutions of the fundamental relations (2.1) are in one-one correspondence with the solutions of the following reduced fundamental relations:

\[
\text{(2.2)} \quad m_j L_j \equiv \sum_{i=1}^{n} \frac{m_j \chi_j(g_i)}{d_i} D_i, \quad j = 1, \ldots, r
\]

As before, denote by $d$ the exponent of $G$; notice that if $\text{Pic}(Y)[d] = 0$, then for fixed $(D_i, g_i), i = 1, \ldots, n$, the solution of (2.2) is unique, hence the branch data $(D_i, g_i)$ determine the cover.

In order to deal with the case when $Y$ is normal but not smooth, we observe first that the cover $X \rightarrow Y$ can be recovered from its restriction $X' \rightarrow Y_{\text{sm}}$ to the smooth locus by taking the integral closure of $Y$ in the extension $\mathbb{K}(X') \supset \mathbb{K}(Y)$. Observe then that, since the complement $Y \setminus Y_{\text{sm}}$ of the smooth part has codimension $>1$, we have $h^0(\mathcal{O}_{Y_{\text{sm}}}) = h^0(\mathcal{O}_Y) = 1$, and thus the cover $X' \rightarrow Y_{\text{sm}}$ is determined by the building data $L_\chi, D_i$. Using the identification $\text{Pic}(Y_{\text{sm}}) = \text{Cl}(Y_{\text{sm}}) = \text{Cl}(Y)$, we can regard the $L_\chi$ as elements of $\text{Cl}(Y)$ and, taking the closure, the $D_i$ as Weil divisors on $Y$, and we can interpret the fundamental relations as equalities in $\text{Cl}(Y)$. In this sense, if $Y$ is a normal variety with $h^0(\mathcal{O}_Y) = 1$, then the $G$-covers $X \rightarrow Y$ are determined by the building data up to isomorphism.

We say that an abelian cover $\pi: X \rightarrow Y$ is totally ramified if the inertia subgroups of the divisorial components of the branch locus of $\pi$ generate $G$, or, equivalently, if $\pi$ does not factorize through a cover $X' \rightarrow Y$ that is étale over $Y_{\text{sm}}$. We observe that a totally ramified cover is necessarily connected; conversely, equations (2.2) imply that if $G$ is an abelian group of exponent $d$ and $Y$ is a variety such that $\text{Cl}(Y)[d] = 0$, then any connected $G$-cover of $Y$ is totally ramified.

2.2. The maximal cover. Let $Y$ be a complete normal variety, let $D_1, \ldots, D_n$ be distinct irreducible effective divisors of $Y$ and let $d_1, \ldots, d_n$ be positive integers (it is convenient to allow the possibility that $d_i = 1$ for some $i$). We set $d := \text{lcm}(d_1, \ldots, d_n)$.

We say that a Galois cover $\pi: X \rightarrow Y$ is branched on $D_1, \ldots, D_n$ with orders $d_1, \ldots, d_n$ if:

- the divisorial part of the branch locus of $\pi$ is contained in $\sum_i D_i$;
- the ramification order of $\pi$ over $D_i$ is equal to $d_i$.

Let $\eta: \widetilde{Y} \rightarrow Y$ be a resolution of the singularities and set $N(Y) := \text{Cl}(Y)/\eta_* \text{Pic}^0(\widetilde{Y})$. Since the map $\eta_*: \text{Pic}(\widetilde{Y}) = \text{Cl}(\widetilde{Y}) \rightarrow \text{Cl}(Y)$ is surjective, $N(Y)$ is a quotient of the Néron-Severi group $\text{NS}(\widetilde{Y})$, hence it is finitely generated. It follows that $\eta_* \text{Pic}^0(\widetilde{Y})$ is the largest divisible subgroup of $\text{Cl}(Y)$ and therefore $N(Y)$ does not depend on
the choice of the resolution of Y (this is easily checked also by a geometrical argument). The group $\text{Cl}(Y)^\vee$ coincides with $N(Y)^\vee$, hence it is a finitely generated free abelian group of rank equal to the rank of $N(Y)$.

Consider the map $\mathbb{Z}^n \to \text{Cl}(Y)$ that maps the $i$-th canonical generator to the class of $D_i$, let $\phi: \text{Cl}(Y)^\vee \to \bigoplus_{i=1}^n \mathbb{Z}d_i$ be the map obtained by composing the dual map $\text{Cl}(Y)^\vee \to (\mathbb{Z}^n)^\vee$ with $(\mathbb{Z}^n)^\vee = \mathbb{Z}^n \to \bigoplus_{i=1}^n \mathbb{Z}d_i$ and let $K_{\min}$ be the image of $\phi$. Let $G_{\max}$ be the abelian group defined by the exact sequence:

$$
0 \to K_{\min} \to \bigoplus_{i=1}^n \mathbb{Z}d_i \to G_{\max} \to 0.
$$

Then we have the following:

**Theorem 2.1.** Let $Y$ be a normal variety with $h^0(\mathcal{O}_Y) = 1$, let $D_1, \ldots, D_n$ be distinct irreducible effective divisors, let $d_1, \ldots, d_n$ be positive integers and set $d := \text{lcm}(d_1, \ldots, d_n)$. Then:

1. If $X \to Y$ is a totally ramified $G$-cover branched on $D_1, \ldots, D_n$ with orders $d_1, \ldots, d_n$, then:
   a. the map $\bigoplus_{i=1}^n \mathbb{Z}d_i \to G$ that maps $1 \in \mathbb{Z}d_i$ to $g_i$ descends to a surjection $G_{\max} \to G$;
   b. the map $\mathbb{Z}d_i \to G_{\max}$ is injective for every $i = 1, \ldots, n$.
2. If the map $\mathbb{Z}d_i \to G_{\max}$ is injective for $i = 1, \ldots, n$ and $N(Y)[d] = 0$, then there exists a maximal totally ramified abelian cover $X_{\max} \to Y$ branched on $D_1, \ldots, D_n$ with orders $d_1, \ldots, d_n$; the Galois group of $X_{\max} \to Y$ is equal to $G_{\max}$.
3. If the map $\mathbb{Z}d_i \to G_{\max}$ is injective for $i = 1, \ldots, n$ and $\text{Cl}(Y)[d] = 0$, then the cover $X_{\max} \to Y$ is unique up to isomorphism of $G_{\max}$-covers and every totally ramified abelian cover $X \to Y$ branched on $D_1, \ldots, D_n$ with orders $d_1, \ldots, d_n$ is a quotient of $X_{\max}$ by a subgroup of $G_{\max}$.

**Proof.** Let $H_1, \ldots, H_t \in N(Y)$ be elements whose classes are free generators of the abelian group $N(Y)/\text{Tors}(N(Y))$, and write:

$$
D_i = \sum_{j=1}^t a_{ij} H_j \mod \text{Tors}(N(Y)), \quad j = 1, \ldots, t
$$

Hence, the subgroup $K_{\min}$ of $\bigoplus_{i=1}^n \mathbb{Z}d_i$ is generated by the elements $z_j := (a_{1j}, \ldots, a_{nj})$, for $j = 1, \ldots, t$.

Let $X \to Y$ be a totally ramified $G$-cover branched on $D_1, \ldots, D_n$ with orders $d_1, \ldots, d_n$ and let $(D_i, g_i)$ be its branch data. Consider the map $\bigoplus_{i=1}^n \mathbb{Z}d_i \to G$ that maps $1 \in \mathbb{Z}d_i$ to $g_i$: this map is surjective, by the assumption that $X \to Y$ is totally ramified, and its restriction to $\mathbb{Z}d_i$ is injective for $i = 1, \ldots, n$, since the cover is branched on $D_i$ with order $d_i$. If we denote by $K$ the kernel of $\bigoplus_{i=1}^n \mathbb{Z}d_i \to G$, to prove (1) it suffices to show that $K \subseteq K_{\min}$. Dually, this is equivalent to showing that $G^* \subseteq K_{\min}^* \subseteq \bigoplus_{i=1}^n (\mathbb{Z}d_i)^*$. Let $\psi_i \in (\mathbb{Z}d_i)^*$ be the generator that maps $1 \in \mathbb{Z}d_i$ to $\frac{1}{d_i}$, and write $\chi \in G^*$ as $(\psi_1^{b_1}, \ldots, \psi_n^{b_n})$, with $0 \leq b_i < d_i$; if $o(\chi) = m$ then

$$
(2.2) \quad mL_\chi = \sum_{i=1}^n \frac{b_i a_{ij}}{d_i} D_i.
$$

Plugging (2.4) in this equation we obtain that

$$
(2.4) \quad D_i = \sum_{j=1}^t a_{ij} H_j \mod \text{Tors}(N(Y)), \quad j = 1, \ldots, t
$$

is an integer for $j = 1, \ldots, t$, namely $\chi \in K_{\min}^*$. 

2. Let $\chi_1, \ldots, \chi_r$ be a basis of $G_{\max}$ and, as above, for $s = 1, \ldots, r$ write $\chi_s = (\psi_1^{b_{s1}}, \ldots, \psi_n^{b_{sn}})$, with $0 \leq b_{si} < d_i$. Since by assumption $N(Y)[d] = 0$, by the proof of (1) the elements $\sum_{j=1}^t (\sum_{i=1}^n \frac{b_{si} a_{ij}}{d_i}) H_j$, $s = 1, \ldots, r$, can be lifted to solutions $\bar{\chi}_s \in \text{Tors}(N(Y))$.
There exists a $L$.

Lemma 3.3. The morphism $L$.

Proposition 3.4. $F$.

In this case the group $\text{Hom}(\mathbb{N},\mathbb{C})$ of the fan $\Sigma$. They are in a bijection with the $\text{Hom}(\mathbb{N},\mathbb{C})$-toric variety $Y$.

Example 2.1. Take $Y = \mathbb{P}^{n-1}$ and let $D_1, \ldots, D_n$ be the coordinate hyperplanes.

In general, $X_{\text{max}}$ is a weighted projective space $\mathbb{P}(\frac{d_1}{d_1}, \ldots, \frac{d_n}{d_n})$ and the cover is
given by $[x_1, \ldots, x_n] \mapsto [x_1^{d_1}, \ldots, x_n^{d_n}]$.

3. Toric covers

Notations 3.1. Here, we fix the notations which are standard in toric geometry.

A (complete normal) toric variety $Y$ corresponds to a fan $\Sigma$ living in the vector space $N \otimes \mathbb{R}$, where $N \cong \mathbb{Z}^s$. The dual lattice is $M = N^\vee$. The torus is $T = N \otimes \mathbb{C}^* = \text{Hom}(M, \mathbb{C}^*)$.

The integral vectors $r_i \in N$ will denote the integral generators of the rays $\sigma_i \in \Sigma(1)$ of the fan $\Sigma$. They are in a bijection with the $T$-invariant Weil divisors $D_i$ ($i = 1, \ldots, n$) on $Y$.

Definition 3.2. A toric cover $f: X \to Y$ is a finite morphism of toric varieties corresponding to the map of fans $F: (N', \Sigma') \to (N, \Sigma)$ such that:

(1) $N' \subseteq N$ is a sublattice of finite index, so that $N' \otimes \mathbb{R} = N \otimes \mathbb{R}$.

(2) $\Sigma' \subseteq \Sigma$.

The proof of the following lemma is immediate.

Lemma 3.3. The morphism $f$ has the following properties:

(1) It is equivariant with respect to the homomorphism of tori $T' \to T$.

(2) It is an abelian cover with Galois group $G = \ker(T' \to T) = N / N'$.

(3) It is ramified only along the boundary divisors $D_i$, with multiplicities $d_i \geq 1$
defined by the condition that the integral generator of $N' \cap \mathbb{R}_{\geq 0} r_i$ is $d_i r_i$.

Proposition 3.4. Let $Y$ be a complete toric variety such that $\text{Cl}(Y)$ is torsion free, and $X \to Y$ be a toric cover. Then, with notations as above, there exists the
following commutative diagram with exact rows and columns.

\[
\begin{array}{cccccc}
0 & 0 & \\
& \downarrow & \\
\Cl(Y)^\vee & \rightarrow & K & \\
& \downarrow & \\
\oplus_{i=1}^n \mathbb{Z}D_1^* & \rightarrow & \oplus_{i=1}^n \mathbb{Z}D_i^* & \rightarrow & \oplus_{i=1}^n \mathbb{Z}d_i & \rightarrow & 0 \\
& \downarrow & p' & \downarrow & p & \\
N' & \rightarrow & N' & \rightarrow & N & \rightarrow & G & \rightarrow & 0 \\
& & 0 & & 0 & & 0 & & \\
\end{array}
\]

(Here the $D_i^*$ are formal symbols denoting a basis of $\mathbb{Z}^n$). Moreover, each of the homomorphisms $\mathbb{Z}d_i \rightarrow G$ is an embedding.

**Proof.** The third row appeared in Lemma 3.3, and the second row is the obvious one.

It is well known that the boundary divisors on a complete normal toric variety span the group $\Cl(Y)$, and that there exists the following short exact sequence of lattices:

\[
0 \rightarrow M \rightarrow \oplus_{i=1}^n \mathbb{Z}D_i \rightarrow \Cl(Y) \rightarrow 0.
\]

Since $\Cl(Y)$ is torsion free by assumption, this sequence is split and dualizing it one obtains the central column. Since $\oplus_{i=1}^n \mathbb{Z}D_i^* \rightarrow N$ is surjective, then so is $\oplus_{i=1}^n \mathbb{Z}d_i \rightarrow G$. The group $K$ is defined as the kernel of this map.

Finally, the condition that $\mathbb{Z}d_i \rightarrow G$ is injective is equivalent to the condition that the integral generator of $N' \cap \mathbb{R}_{\geq 0}r_i$ is $d_ir_i$, which holds by Lemma 3.3. □

**Theorem 3.5.** Let $Y$ be a complete toric variety such that $\Cl(Y)$ is torsion free, let $d_1, \ldots, d_n$ be positive integers and let $K_{\min}$ and $G_{\max}$ be defined as in sequence (2.3). Then:

1. There exists a toric cover branched on $D_i$ of order $d_i$, $i = 1, \ldots, n$, iff the map $\mathbb{Z}d_i \rightarrow G_{\max}$ is injective for $i = 1, \ldots, n$.
2. If condition (1) is satisfied, then among all the toric covers of $Y$ ramified over the divisors $D_i$ with multiplicities $d_i$ there exists a maximal one $X_{\text{Tmax}} \rightarrow Y$, with Galois group $G_{\max}$, such that any other toric cover $X \rightarrow Y$ with the same branching orders is a quotient $X = X_{\text{Tmax}}/H$ by a subgroup $H < G_{\max}$.

**Proof.** Let $X \rightarrow Y$ be a toric cover branched on $D_1, \ldots, D_n$ with orders $d_1, \ldots, d_n$, let $N'$ be the corresponding sublattice of $N$ and $G = N/N'$ the Galois group. Let $N'_{\min}$ be the subgroup of $N$ generated by $d_ir_i$, $i = 1, \ldots, n$. By Lemma 3.3 one must have $N'_{\min} \subseteq N'$, hence the map $\mathbb{Z}d_i \rightarrow N/N'_{\min}$ is injective since $\mathbb{Z}d_i \rightarrow G = N/N'$ is injective by Proposition 3.4. We set $X_{\text{Tmax}} \rightarrow Y$ to be the cover for $N'_{\min}$. Clearly, the cover for the lattice $N'$ is a quotient of the cover for the lattice $N'_{\min}$ by the group $H = N'/N'_{\min}$.
Consider the second and third rows of the diagram of Proposition 3.4 as a short exact sequence of 2-step complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$. The associated long exact sequence of cohomologies gives

$$\text{Cl}(Y)^\vee \xrightarrow{q} K \xrightarrow{\text{coker}(p')} 0$$

For $N' = N'_{\text{min}}$, the map $p'$ is surjective, hence $\text{Cl}(Y)^\vee \to K$ is surjective too, and $K = K_{\text{min}}$, $N/N'_{\text{min}} = G_{\text{max}}$.

Vice versa, suppose that in the following commutative diagram with exact row and columns each of the maps $\mathbb{Z}_{d_i} \to G_{\text{max}}$ is injective.

$$\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\text{Cl}(Y)^\vee & q & K_{\text{min}} & 0 \\
\oplus_{i=1}^n \mathbb{Z}d_i D_i^\bullet & \oplus_{i=1}^n \mathbb{Z}D_i^\bullet & \oplus_{i=1}^n \mathbb{Z}d_i & 0 \\
N & G_{\text{max}} & 0 & 0
\end{array}$$

We complete the first row on the left by adding $\ker(q)$. We have an induced homomorphism $\ker(q) \to \oplus \mathbb{Z}d_i D_i^\bullet$, and we define $N'$ to be its cokernel.

Now consider the completed first and second rows as a short exact sequence of 2-step complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$. The associated long exact sequence of cohomologies says that $\ker(q) \to \oplus_{i=1}^n \mathbb{Z}d_i D_i^\bullet$ is injective, and the sequence

$$0 \to N' \to N \to G_{\text{max}} \to 0$$

is exact. It follows that $N' = N'_{\text{min}}$ and the toric morphism $(N'_{\text{min}}, \Sigma) \to (N, \Sigma)$ is then the searched-for maximal abelian toric cover.

**Remark 3.6.** Condition (1) in the statement of Theorem 3.5 can also be expressed by saying that for $i = 1, \ldots, n$ the element $d_ir_i \in N'_{\text{min}}$ is primitive, where $N'_{\text{min}} \subseteq N$ is the subgroup generated by all the $d_ir_i$.

We now combine the results of this section with those of §2 to obtain a structure result for Galois covers of toric varieties.

**Theorem 3.7.** Let $Y$ be a normal complete toric variety and let $f : X \to Y$ be a connected cover such that the divisorial part of the branch locus of $f$ is contained in the union of the invariant divisors $D_1, \ldots, D_n$.

If $\text{char} \mathbb{K} = 0$ or $f$ is Galois and $\text{char} \mathbb{K}$ does not divide $\deg f$, then $f : X \to Y$ is a toric cover.

**Proof.** Our first step is showing that $f$ is an abelian cover. Let $U \subseteq Y$ be the open orbit and let $X' \to U$ be the cover obtained by restricting $f$. Since $U$ is smooth, by the assumptions and by purity of the branch locus, $X' \to U$ is étale. If $f$ is Galois,
then $X' \to U$ is also Galois with the same Galois group, and if in addition $\text{char } K$ does not divide $\deg f$ then the Galois group is abelian by [Mi72, Thm. 1] (cf. also [BS13]).

Now drop the assumption that $f$ is Galois but assume that $\text{char } K = 0$. Let $X'' \to U$ be the Galois closure of $X' \to U$: the cover $X'' \to U$ is also étale, hence again by [Mi72, Thm. 1] it is, up to isomorphism, a homomorphism of tori. It follows that $X'' \to U$ is an abelian cover and so the intermediate cover $X' \to U$ is also abelian (actually $X' = X''$). The cover $f: X \to Y$ is abelian, too, since $X$ is the integral closure of $Y$ in $K(X')$. So we have proven that under our assumptions $f$ is an abelian cover. We denote by $G$ the Galois group of $f$ and by $d_1, \ldots, d_n$ the orders of ramification of $X \to Y$ on $D_1, \ldots, D_n$.

Assume first that $\text{Cl}(Y)$ has no torsion, so that every connected abelian cover of $Y$ is totally ramified (cf. §2). Then by Theorem 2.1 every connected abelian cover branched on $D_1, \ldots, D_n$ with orders $d_1, \ldots, d_n$ is a quotient of the maximal abelian cover $X_{\text{max}} \to Y$ by a subgroup $H < G_{\text{max}}$. In particular, this is true for the cover $X_{\text{Tmax}} \to Y$ of Theorem 3.5. Since $X_{\text{max}}$ and $X_{\text{Tmax}}$ have the same Galois group it follows that $X_{\text{max}} = X_{\text{Tmax}}$. Hence $X \to Y$, being a quotient of $X_{\text{Tmax}}$, is a toric cover.

Consider now the general case. Recall that the group $\text{Tors Cl}(Y)$ is finite, isomorphic to $N/\langle r_j \rangle$, and the cover $Y' \to Y$ corresponding to $\text{Tors Cl}(Y)$ is toric, and one has $\text{Tors Cl}(Y') = 0$. Indeed, on a toric variety the group $\text{Cl}(Y)$ is generated by the $T$-invariant Weil divisors $D_i$. Thus, $\text{Cl}(Y)$ is the quotient of the free abelian group $\bigoplus Z D_i$ of all $T$-invariant divisors modulo the subgroup $M$ of principal $T$-invariant divisors. Thus, $\text{Tors Cl}(Y) \simeq M'/M$, where $M' = \bigoplus Q D_i$ is the subgroup of $Q$-linear functions on $N$ taking integral values on the vectors $r_i$. Then $N' := M''$ is the subgroup of $N$ generated by the $r_i$, and the cover $Y' \to Y$ is the cover corresponding to the map of fans $(N', \Sigma) \to (N, \Sigma)$. On $Y'$ one has $N' = \langle r_i \rangle$, so $\text{Tors Cl}(Y') = 0$.

Let $X' \to Y'$ be a connected component of the pull back of $X \to Y$: it is an abelian cover branched on the invariant divisors of $Y'$, hence by the first part of the proof it is toric. The map $X' \to Y$ is toric, since it is a composition of toric morphisms, hence the intermediate cover $X \to Y'$ is also toric. \hfill \Box

**Remark 3.8.** The argument that shows that the map $f$ is an abelian cover in the proof of Theorem 3.7 was suggested to us by Angelo Vistoli. He also remarked that it is possible to prove Theorem 3.7 in a more conceptual way by showing that the torus action on the cover $X' \to U$ of the open orbit of $Y$ extends to $X$, in view of the properties of the integral closure. However our approach has the advantage of describing explicitly the fan/building data associated with the cover.

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