We study the statistical description of a small quantum system that interacts in a generic way with a large quantum environment, when the total system lies in an equilibrium state described by a microcanonical ensemble. The focus is on the difference between the reduced density matrix (RDM) of the central system in this interacting case and the RDM obtained in the uncoupled case. In the eigenbasis of the central system’s Hamiltonian, it is shown that the difference between diagonal elements is mainly confined by the ratio of the maximum width of the eigenfunctions of the total system in the uncoupled basis to the width of the microcanonical energy shell; meanwhile, the difference between offdiagonal elements is given by the ratio of certain property of the interaction Hamiltonian to the related level spacing of the central system. As an application, a condition is given under which the RDM has a canonical Gibbs form that includes certain averaged effect of the interaction. It is argued that, for a central system that interacts locally with a many-body quantum chaotic system, it is quite possible for the RDM to have a Gibbs form. We also study the RDM which is computed from a typical state of the total system within an energy shell.

I. INTRODUCTION

A. Motivation

In quantum statistical mechanics, one important topic is about the relationship between microcanonical(MC)-ensemble description and canonical-ensemble description. In particular, for a generic, isolated, and large quantum system that is described by a MC ensemble, the condition, under which the reduced density matrix (RDM) of an interacting small subsystem may have a canonical Gibbs form, is still a problem not completely solved. Unlike the corresponding problem in the classical statistical mechanics, which can be solved relatively easily (see, e.g., Ref.[1]), this problem is highly-nontrivial, due to the mathematical difficulty met when dealing with the total energy eigenstates under nonnegligible subsystem-environment interactions.

A related important topic is justification of the usage of a MC-ensemble description for the total system, in view of the fact that quantum mechanics in principle allows a pure-state description for the total system. This topic is also of relevance, in the effort of generalizing equilibrium-state statistical-mechanics principles to nonequilibrium processes. Modern studies show that a mathematical concept related to high-dimensional linear spaces plays an important role, namely typicality, an idea of which can be traced back to von Neumann’s original work[2]. Recently, by making use of the so-called Levy’s lemma[3, 4], a quantitative progress was reported in Ref.[5] in 2006, wherein an upper bound is derived for the distance between the RDM computed from an MC-ensemble description of the total system and that from a typical-state description of the total system[5]. The derived upper bound shows that the MC and typical-state descriptions for the total system are effectively identical in view of computing the RDM of a small subsystem, when the dimension of the effective environmental state space is sufficiently large.

In the same year of 2006, it was shown in Ref.[6] that the RDM of a small subsystem, which is computed from a typical state of the total system, is typically close to the Gibbs state when the subsystem-environment interaction is very weak, under the well-known assumption about the exponential shape of the density of states of the environment. However, the strength of interaction required in Ref.[6] is usually too weak for a macroscopic environment to satisfy, due to the exponential increase of its density of states with the particle number[5]. To solve this problem, weak (not necessarily extremely weak) interaction was studied in Ref.[7] in 2012, wherein an upper bound was given to the distance between two MC-ensemble-computed RDMs, which are obtained in the two cases with and without subsystem-environment interaction, respectively. The result shows closeness of the two RDMs and, as a consequence, to the Gibbs state, when the interaction is sufficiently weak.

Two problems remain open related to the approach of Ref.[5]. (i) The obtained results are for a generic environment, independent of whether it undergoes a complex motion or not. One interesting question is whether the upper bound given there may be significantly lowered for complex environments such as quantum chaotic systems. And, (ii) it is unclear how this approach may give a practically feasible method of finding the Hamiltonian that should be used in the Gibbs state, which may take into account some effects of the subsystem-environment interaction.

To the same problem of relatively-weak subsystem-environment interactions, the Gibbs state of a system that is described by a MC ensemble, is typically close to the RDM of a small subsystem, which is computed from a typical state of the total system within an energy shell.

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[1] Below, we refer to these two types of RDM as MC-ensemble-computed RDM and typical-state-computed RDM, respectively.
environment interaction, in the same year of 2012, a different approach was reported in Ref. [8], in which elements of the RDM in the eigenbasis of the central system's Hamiltonian are studied directly. In this approach, a more specific situation is considered, in which the subsystem is locally coupled to an environment as a many-body quantum chaotic system that satisfies the so-called eigenstate thermalization hypothesis (ETH) [9–12]. And, closeness is shown between the typical-state-computed RDM and the Gibbs state. This approach gives an explicit expression for the Hamiltonian that should be used in the Gibbs state, which takes into account certain averaged effect of the interaction.

Two problems remain open related to the approach of Ref. [8]. (a) No upper bound was derived explicitly for the difference between elements of the studied RDM and those of the Gibbs state. For this reason, although this approach and that of Ref. [7] reach the same qualitative conclusion of closeness of the studied RDM and the Gibbs state under sufficiently weak interactions, a quantitative comparison of their predictions for the condition and extent of the closeness is unavailable. And (b) an upper bound for the width of energy eigenfunctions (EFs) of the total system in the basis with zero subsystem-environment coupling was derived and made use of in Ref. [8], based on a first-order perturbation-theory treatment to long tails of the EFs. Although it was pointed out there that this perturbative treatment may be justified by a generalized Brillouin-Wigner perturbation theory [13, 14], a detailed analysis was not given.

More recently, a relationship was found among elements of the long-time averaged RDM of a qubit, which is locally coupled to a many-body quantum chaotic system that initially lies in a typical state within an energy shell [16]. This relation shows the existence of some nontrivial offdiagonal elements of RDM. It is unclear whether the above-discussed two approaches may accommodate this type of relationship among elements of RDM.

B. Problems to be studied and organization of the paper

In this paper, we are to derive upper bounds related to the first remaining problem of the second approach discussed above, but, for a generic environment and a generic type of interaction. We first study MC-ensemble-computed RDMs, then, study typical-state-computed RDMs.

Specifically, we are to consider a generic, isolated, and large quantum system, which is described by an MC ensemble or by a typical state within an energy shell. The isolated system is divided into a generic small subsystem and a large environment. We are to derive upper bounds for the difference between elements of two RDMs that are obtained with and without the subsystem-environment interaction. The upper bounds to be derived are expressed with properties of the systems involved, such as the width of the energy shell, level spacings of the subsystem, the maximum width of total EFs, and so on.

The paper is organized as follows. In Sec. III we describe the basic framework, within which we are to give our discussions, including notations to be used in later sections, basic properties of the systems to be studied, and basic properties of RDMs. In Sec. III we first derive an upper bound for the difference between diagonal elements of two MC-ensemble-computed RDMs, which are obtained in the two cases with and without the subsystem-environment interaction, respectively. Then, we discuss some application of the obtained result and compare it with a prediction of Ref. [7].

In Sec. IV we first derive an expression for the difference between offdiagonal elements of the above-mentioned RDMs, then, as an illustration, we discuss a simple example with a two-level system as the subsystem and a many-body quantum chaotic system as the environment. After that, we compare the obtained results and some predictions of Refs. [7, 8, 16]. In Sec. V as an application, we discuss some cases in which the RDM of an interacting subsystem may have a canonical Gibbs form, when the interaction is not very weak. In Sec. VI we discuss differences between elements of typical-state-computed and MC-ensemble-computed RDMs. Finally, conclusions and discussions are given in Sec. VII.

II. THE SETUP

A. Hamiltonians and their eigenstates

We consider a generic, isolated, and large quantum system, denoted by $T$, which is divided into a small subsystem denoted by $S$ and a large environment denoted by $E$. The Hilbert spaces of $S$ and $E$ are denoted by $\mathcal{H}^S$ and $\mathcal{H}^E$, respectively, with dimensions $d_S$ and $d_E$. The total Hamiltonian is written as

$$ H = H^S + H^I + H^E, $$(1)

where $H^S$ and $H^E$ are the self-Hamiltonians of $S$ and $E$, respectively, and $H^I$ represents the interaction. The interaction is of a generic type; there is only one restriction to it, that is, it is not very strong such that its influence in the density of states of the total system can be neglected. Note that, more precisely, say, $H^S$ on the right-hand side (rhs) of Eq. (1) should be written as $H^S \otimes I^E$, where $I^E$ represents the identity operator acting on $\mathcal{H}^E$, but, for brevity, we usually omit the identity operator.

Normalized eigenstates of $H$ are denoted by $|n\rangle$ with energies $E_n$ in the increasing-energy order,

$$ H|n\rangle = E_n|n\rangle. $$

(2)

[2] We are to give a further study for this problem in a different paper [15].
Normalized eigenstates of $H^S$ are denoted by $|\alpha\rangle$ with energies $e_\alpha^S$, and those of $H^E$ by $|i\rangle$ with energies $e_i$, both in the increasing-energy order,

$$H^S|\alpha\rangle = e_\alpha^S|\alpha\rangle, \quad H^E|i\rangle = e_i|i\rangle,$$

where for brevity we have omitted a superscript $\mathcal{E}$ for the environmental energy $e_i$. For the simplicity in discussion, we assume that the system $S$ has a nondegenerate spectrum.

We use $H^0$ to denote the uncoupled Hamiltonian of the total system, i.e.,

$$H^0 = H^S + H^E.$$

Eigenstates of $H^0$ with eigenenergies $E_{\alpha i}$ are written as $|\alpha i\rangle$, in short $|\alpha i\rangle$, satisfying

$$H^0|\alpha i\rangle = E_{\alpha i}|\alpha i\rangle, \quad E_{\alpha i} = e_\alpha^S + e_i.$$

In the energy order, the states $|\alpha i\rangle$ are indicated by $|E_r\rangle$ with one integer label $r$, which has a one-to-one correspondence to the pair $(\alpha, i)$, namely $r \leftrightarrow (\alpha, i)$, such that $E_r = E_{\alpha i}$ and

$$H^0|E_r\rangle = E_r|E_r\rangle, \quad E_r \leq E_{r+1}.$$

The expansions of the states $|n\rangle$ in the bases $|\alpha i\rangle$ and $|E_r\rangle$, with coefficients denoted by $C^0_{\alpha i}$ and $C^n_r$, respectively, are written as

$$|n\rangle = \sum_{\alpha, i} C^n_{\alpha i} |\alpha i\rangle = \sum_r C^n_r |E_r\rangle.$$

The coefficients $C^0_{\alpha i}$ and $C^n_r$ give the EFs.

Significant components $C^n_r$ of the EF of $|n\rangle$ usually occupy a restricted region in the unperturbed spectrum, say, for a region of $E_r$ with $r$ between $r_1^{(n)}$ and $r_2^{(n)}$. For brevity, we call such a region a “main-body” region of $|n\rangle$. To characterize a main-body region, one may employ a small positive parameter $\epsilon$, such that the population of $|n\rangle$ within this region is not smaller than $(1-\epsilon)$. Explicitly, we use $\Omega_n \equiv [r_1^{(n)}, r_2^{(n)}]$ to indicate such a region for which

$$\sum_{r \in \Omega_n} |\langle E_r |n\rangle|^2 \geq 1 - \epsilon,$$

where “$\geq$” means that the left-hand side is either equal to the rhs, or is just larger than the rhs, such that it become smaller than the rhs when $\Omega_n$ is enlarged by letting $r_1^{(n)} \rightarrow r_1^{(n)} - 1$ or $r_2^{(n)} \rightarrow r_2^{(n)} + 1$. We use $w_E$ to denote the maximum width of the energy region occupied by $\Omega_n$, i.e.,

$$w_E = \max \left\{ \left( E_{r_2^{(n)}} - E_{r_1^{(n)}} \right) \right\}$$

for those states $|n\rangle$ that lie in the energy region of the total system of relevance to our discussions to be given later.

The so-called local spectral density of states (LDOS), or strength function in nuclear physics, will also be used in our later discussions. They are the reverse of EFs, that is, the LDOS of an uncoupled state $|E_r\rangle$ is given by its expansion in the eigenbasis $\{|n\rangle\}$. We use $\Omega_r^L$ to denote a main-body region of $|E_r\rangle$, which is written as $\Omega_r^L = [n_1^{(r)}, n_2^{(r)}]$ for a region of the label $n$ between $n_1^{(r)}$ and $n_2^{(r)}$; it satisfies the following relation,

$$\sum_{n \in \Omega_r^L} |\langle E_r |n\rangle|^2 \geq 1 - \epsilon.$$

The maximum value of the energy width of $\Omega_r^L$, namely of $(E_{n_2^{(r)}} - E_{n_1^{(r)}})$, for those states $|E_r\rangle$ in the energy region of relevance, is denoted by $w_L$. We use $w_M$ to indicate the larger one of $w_E$ and $w_L$, namely,

$$w_M = \max\{w_E, w_L\}.$$

For a sufficiently small $\epsilon$, the value of $E_n$ lies within the main-body energy region of the EF of $|n\rangle$, meanwhile, $E_r$ lies within the main-body region of the LDOS of $|E_r\rangle$. It is not difficult to verify that these two properties imply the following relations, respectively,

$$E_n - w_M \leq E_{r_2^{(n)}} < E_{r_1^{(n)}} \leq E_n + w_M, \quad (13a)$$

$$E_r - w_M \leq E_{n_2^{(r)}} < E_{n_1^{(r)}} \leq E_r + w_M. \quad (13b)$$

That is, the main body of the EF of $|n\rangle$ lies within the region of $E_r \in [E_n - w_M, E_n + w_M]$, meanwhile, the main body of the LDOS of $|E_r\rangle$ lies within the region of $E_n \in [E_r - w_M, E_r + w_M]$.  

### B. Energy shell and RDM

We consider an MC description of the total system in an energy shell denoted by $\Gamma$, which starts at an energy denoted by $E_s$ and has a width $\Delta$, i.e., $\Gamma = [E_s, E_s + \Delta]$, with the subscript “$s$” standing for “starting of shell”. We use $\mathcal{H}_r$ to denote the subspace spanned by the eigenstates $|n\rangle$ with $E_n \in \Gamma$. The dimension of $\mathcal{H}_r$ is denoted by $d_r$, satisfying $d_r \gg 1$. We assume that $\Delta > 2w_M$. The microcanonical description of the total system for the energy shell $\Gamma$ is written as

$$\rho^T = \frac{1}{d_\Gamma} \sum_{E_n \in \Gamma} |n\rangle \langle n|.$$  

The RDM of the system $S$, denoted by $\rho^S$, is given by

$$\rho^S \equiv \text{Tr}_{\mathcal{E}} \left( \rho^T \right).$$

---

3 We neglect the trivial case of $[H^S + H^E, H^I] = 0$, in which the states $|n\rangle$ are equal to the uncoupled ones $|E_r\rangle$. 
Its elements are written as

$$\rho^S_{\alpha \beta} = \langle \alpha | \rho^S | \beta \rangle = d^{-1}_\Gamma \sum_{E_r \in \Gamma} \rho^S_{\alpha \beta}(n)$$  \hspace{1cm} (16)$$

where $\rho^S_{\alpha \beta}(n)$ indicates an element computed from a single eigenstate $|n\rangle$, i.e.,

$$\rho^S_{\alpha \beta}(n) = \langle \alpha | \mathrm{Tr}_\xi (|n\rangle \langle n|) | \beta \rangle = \sum_i C^\alpha_{\alpha i} C^\beta_{\beta i}.$$  \hspace{1cm} (17)$$

For the uncoupled system $H^0$, one may consider a similar energy shell denoted by $\Gamma^0$, with $\Gamma^0 = [E_s, E_s + \Delta]$. We use $d_{\Gamma^0}$ to denote the number of levels $E_r$ within $\Gamma^0$. The MC ensemble in the uncoupled case is described by

$$\rho^{T^0} = \frac{1}{d_{\Gamma^0}} \sum_{E_r \in \Gamma^0} |E_r\rangle \langle E_r|.$$  \hspace{1cm} (18)$$

This gives the RDM $\rho^{SO} = \mathrm{Tr}_\xi (\rho^{T^0})$, with elements $\rho^{SO}_{\alpha \beta} = \langle \alpha | \rho^{T^0} | \beta \rangle$. For a given state $|\alpha\rangle$ of the system $S$, we use $\Gamma^\alpha$ to denote the environmental energy shell, which contains those environmental levels $e_i$ that satisfy $E_{\alpha i} = E_r \in \Gamma^0$, i.e.,

$$\Gamma^\alpha = [E_s - e_{\alpha s}, E_s - e_{\alpha s} + \Delta].$$  \hspace{1cm} (19)$$

We use $H^\alpha$ to denote the subspace spanned by $|i\rangle \in \Gamma^\alpha$ and use $d^\alpha_{\Gamma^0}$ to indicate its dimension.

It is straightforward to find that

$$\rho^{SO}_{\alpha \beta} = \frac{1}{d_{\Gamma^0}} d^\alpha_{\Gamma^0}, \quad \forall \alpha,$$  \hspace{1cm} (20a)$$

$$\rho^{SO}_{\alpha \beta} = 0, \quad \forall \alpha \neq \beta.$$  \hspace{1cm} (20b)$$

Then, under the well-known assumption about an exponential shape of the density of states $[1]$, one gets that

$$\rho^{SO}_{\alpha \alpha} \simeq (\rho^S_G)_{\alpha \alpha}, \quad \forall \alpha,$$  \hspace{1cm} (21a)$$

$$\rho^{SO}_{\alpha \beta} = (\rho^S_G)_{\alpha \beta} = 0, \quad \forall \alpha \neq \beta.$$  \hspace{1cm} (21b)$$

where $\rho^S_G$ indicates the Gibbs state,

$$\rho^S_G = e^{-\beta H^S} / \mathrm{Tr} e^{-\beta H^S},$$  \hspace{1cm} (22)$$

with a parameter $\beta$ determined by the density of states of the environment. Hence, instead of studying the differences $|\rho^S_{\alpha \beta} - (\rho^S_G)_{\alpha \beta}|$, below we study $|\rho^S_{\alpha \beta} - \rho^{SO}_{\alpha \beta}|$.

**III. DIFFERENCE BETWEEN DIAGONAL ELEMENTS OF RDMS**

In this section, we first derive an upper bound for $|\rho^S_{\alpha \alpha} - \rho^{SO}_{\alpha \alpha}|$, then, discuss some applications of the obtained result.

### A. An upper bound for $|\rho^S_{\alpha \alpha} - \rho^{SO}_{\alpha \alpha}|$

In order to find an upper bound for the difference $|\rho^S_{\alpha \alpha} - \rho^{SO}_{\alpha \alpha}|$, we divide the environmental spectrum $\{e_i\}$ into several regions separated by the following parameters,

$$\epsilon_1 = E_s - e_{\alpha s} - w_M,$$  \hspace{1cm} (23a)$$

$$\epsilon_2 = \epsilon_1 + 2 w_M,$$  \hspace{1cm} (23b)$$

$$\epsilon_3 = \epsilon_1 + \Delta,$$  \hspace{1cm} (23c)$$

$$\epsilon_4 = \epsilon_2 + \Delta.$$  \hspace{1cm} (23d)$$

We use $R^E_{\kappa}$ with $\kappa = 0, 1, 2, 3$ to denote the following four regions of the spectrum separated by the above parameters, i.e.,

$$R^E_0 := [e_{\text{start}}, \epsilon_1] \cup (\epsilon_4, e_{\text{end}}],$$  \hspace{1cm} (24a)$$

$$R^E_1 := [\epsilon_1, \epsilon_2),$$  \hspace{1cm} (24b)$$

$$R^E_2 := [\epsilon_2, \epsilon_3),$$  \hspace{1cm} (24c)$$

$$R^E_3 := [\epsilon_3, \epsilon_4].$$  \hspace{1cm} (24d)$$

where $e_{\text{start}}$ and $e_{\text{end}}$ indicate the starting and ending levels of the environmental spectrum, respectively. Then, making use of Eqs. (10), (17), the diagonal element $\rho^S_{\alpha \alpha}$ is written as

$$\rho^S_{\alpha \alpha} = d^{-1}_\Gamma \sum_{\alpha i} \sum_{\alpha \alpha} |C^\alpha_{\alpha i}|^2,$$  \hspace{1cm} (25)$$

where

$$F_{\alpha \alpha} = \sum_{\epsilon_i \in R^E_0} \sum_{E_r \in \Gamma} |C^\alpha_{\alpha i}|^2.$$  \hspace{1cm} (26)$$

We use $N^E_{\alpha}$ to denote the number of the levels $e_i$ lying within a region $R^E_{\kappa}$.

Below, we discuss contributions from the four regions $R^E_{\kappa}$ separately. Firstly, we discuss the central region $R^E_2$, which usually gives the main contribution to $\rho^S_{\alpha \alpha}$. We write $F_{\alpha \alpha}$ in the following form,

$$F_{\alpha \alpha} = N^E_{\alpha} + \sum_{\epsilon_i \in R^E_2} (I_{\alpha i} - 1),$$  \hspace{1cm} (27)$$

where

$$I_{\alpha i} = \frac{1}{E_r \in \Gamma} |C^\alpha_{\alpha i}|^2.$$  \hspace{1cm} (28)$$

For a level $\epsilon_i \in R^E_2$, according to Eqs. (23b) - (23c) and Eq. (24c), the value of $E_r = E_{\alpha i}$ lies between $(E_s + w_M)$ and $(E_s + \Delta - w_M)$. Due to Eq. (23a), this implies that the main-body region of the LDOS of $|\alpha\rangle$ should lie within the energy shell $\Gamma$. Hence, $(1 - I_{\alpha i}) < \epsilon$ [see Eq. (11)]. As a result, $F_{\alpha \alpha}$ in Eq. (27) can be written as

$$F_{\alpha \alpha} = N^E_{\alpha} - a_2 \varepsilon N^E_{\alpha},$$  \hspace{1cm} (29)$$
where \(a_2\) is some undetermined parameter satisfying \(0 < a_2 < 1\).

Next, we discuss the two regions \(\mathcal{R}^{E_{\alpha}}\) of \(\kappa = 1\) and 3, each with a width \(2w_M\). The value of \(I_{\alpha}\) is close to 1 for some levels \(e_i\) in these two regions, while, is much smaller than 1 for some other levels. Note that the environmental density of states has an average value of \((d_{E_{\alpha}}^{E}/\Delta)\) within the energy shell \(\Gamma_{E_{\alpha}}^{E}\). Then, it is easy to see that \(F_{\alpha}\) of \(\kappa = 1, 3\) can be written in the following form,

\[
F_{\alpha} = 2a_1w_Md_{E_{\alpha}}^{E}/\Delta, \quad \kappa = 1, 3,
\]

where \(a_1\) and \(a_3\) are some undetermined parameters satisfying \(0 < a_{1(3)} < 1\). In most cases, the values of \(a_1\) and \(a_3\) are around 0.5 or smaller.

Finally, we discuss the region \(\mathcal{R}^{E_{\alpha}}_0\). For an energy level \(e_i\) lying in this region, the value of \(E_r = e_i + e_i^S\) is either smaller than \((E_s - w_M)\) or is larger than \((E_s + \Delta + w_M)\). This implies that \(E_r\) lies outside the main-body regions of all those states \(|n\rangle \in \Gamma\). Hence, according to Eq. (30), one has

\[
\sum_{e_i \in \mathcal{R}^{E_{\alpha}}_0} |C_{n|a}|^2 \ll \epsilon.
\]

This gives the following expression,

\[
F_{\alpha 0} = \sum_{E_n \in \Gamma} \sum_{e_i \in \mathcal{R}^{E_{\alpha}}_0} |C_{n|a}|^2 = a_0 \epsilon d_{\Gamma},
\]

with some undetermined parameter \(a_0\) satisfying \(0 < a_0 < 1\).

Putting the above results together, we get that

\[
\rho_{\alpha}^{S} = N_2^{E_{\alpha}} \frac{\omega_{BW}}{d_{\Gamma}} + \frac{2(a_1 + a_3)w_Md_{E_{\alpha}}^{E}}{\Delta} \frac{d_{E_{\alpha}}^{E}}{d_{\Gamma}} + (a_0 - a_2)N_2^{E_{\alpha}} \epsilon.
\]

Since the \(S-E\) interaction is not very strong such that the difference between the density of states of \(H_0\) and that of \(H\) can be neglected, one has \(d_{\Gamma} \approx d_{\Gamma}\). We consider the generic case that the energy shell \(\Gamma^{E_{\alpha}}\) is far from edges of the spectrum. In this generic case and for \(\Gamma^{E_{\alpha}}\) that is not wide, using that \(d_{\Gamma} = \sum_{\alpha} d_{E_{\alpha}}^{E}\), it is not difficult to verify that

\[
(a_1 + a_3)d_{E_{\alpha}}^{E}/d_{\Gamma} \lesssim 2/d_{S}.
\]

Then, one gets the following estimate,

\[
\rho_{\alpha}^{S} \lesssim N_2^{E_{\alpha}} \frac{\omega_{BW}}{d_{\Gamma}} + 4w_M \frac{d_{E_{\alpha}}^{E}}{\Delta} \frac{d_{E_{\alpha}}^{E}}{d_{\Gamma}} + \epsilon.
\]

To go further, we note that, when the value of \(\Delta\) is small, \(d_{E_{\alpha}}^{E} - N_2^{E_{\alpha}} \approx 2w_Md_{E_{\alpha}}^{E}/\Delta\). Then, making use of Eqs. (28) and (33), in the generic case that \(d_{S}d_{\Gamma}^{E} < 2d_{\Gamma}\), one gets that

\[
|\rho_{\alpha}^{S} - \rho_{\alpha}^{S0}| \lesssim \frac{2w_M}{d_{S} \Delta} \frac{d_{E_{\alpha}}^{E}}{d_{\Gamma}} + \epsilon.
\]

More concisely, we write

\[
|\rho_{\alpha}^{S} - \rho_{\alpha}^{S0}| \lesssim \frac{q_a w_M}{d_{S} \Delta} + \epsilon,
\]

where

\[
q_a = 4 - 2d_{S}d_{\Gamma}^{E}/d_{\Gamma}.
\]

In the case that the difference among \(d_{E_{\alpha}}^{E}\) of different \(\alpha\) is small compared with their own values, one has \(q_a \approx 2\).

Obviously, for sufficiently weak \(S-E\) interactions (unnecessarily extremely weak), the value of \((w_M/\Delta)\) is small for a fixed \(\Delta\). Hence, Eq. (37) shows that the differences \(|\rho_{\alpha}^{S} - \rho_{\alpha}^{S0}|\) are small for sufficiently weak interactions.

\section{Some further discussions}

In this section, we discuss some application of Eq. (37) and compare it with results of Refs. [7, 8].

\subsection{EFs with a Breit-Wigner form}

To apply Eq. (37), an important quantity is the width \(w_M\). As an illustration, we discuss a case that is often met in realistic models, in which the EFs and LDOS have on average a Breit-Wigner form [17, 18], described by the following Lorentz function \(f(E)\),

\[
f(E) = \frac{1}{2\pi} \frac{\omega_{BW}}{E^2 + (\omega_{BW}/2)^2},
\]

with a width \(\omega_{BW}\) given by

\[
\omega_{BW} \approx 2\pi |H_{rr}^E|^2 \rho_{dos},
\]

where \(\rho_{dos}\) indicates the density of states. Under this Breit-Wigner form, making use of Eq. (39), it is straightforward to find the following expression of \(w_E\) and \(w_L\),

\[
w_{E,L} = \frac{2\omega_{BW}}{\pi \epsilon}.
\]

It is sometimes convenient to introduce a parameter \(\lambda\) for characterizing the strength of the interaction, with \(||H|| \propto \lambda\). Then, Eq. (40) predicts that \(\omega_{BW} \propto \lambda^2\).

At first sight, it seems that the smallness of the parameter \(\epsilon\) in Eq. (41) may imply largeness of \(w_E\). However, this is not necessarily true, because for a large quantum chaotic environment it is possible for the rhs of Eq. (40) to be quite small such that \(w_E\) gets a small value at a given value of \(\epsilon\). (See the last paragraph of Sec. V A for a little more discussions.)
2. Comparison with results of Refs. [2, 3]  

Equation (37) can be regarded as a quantitative expression for some qualitative arguments used in Ref. [3] in order to derive main results given there.

In Ref. [3], by making use of a perturbation theorem about impact of perturbation in projection operators for eigen-subspaces of Hamiltonians, an upper bound is derived for the distance between the two MC descriptions \( \rho^T \) of the total system in the two cases with weak and none \( S-E \) interaction, respectively. This then gives the following upper bound for the distance between the corresponding RDMs, denoted by \( D(\rho^S, \rho^{S_0}) \), appearing as Eq. (2) in Ref. [3],

\[
D(\rho^S, \rho^{S_0}) \leq 4 \frac{\sqrt{H^I \| \infty}}{\Delta}, \tag{2)-[7]}
\]

where \( \| H^I \| \infty \) indicates the maximum absolute eigenvalue of \( H^I \). [19]

The two upper bounds given in Eq. (37) and Eq. ((2)-[7]), although not identical, are qualitatively consistent due to the fact that both \( w_M \) and \( \| H^I \| \infty \) are small for weak interactions. To compare them in a quantitative way, as one example, one may consider a special case, in which the width \( w_M \) is proportional to \( \lambda \). If the \( \epsilon \)-dependence of \( w_M \) is similar to that given in Eq. (11), then, \( w_M \propto \lambda/\epsilon \). As a result, the rhs of Eq. (37) has a minimum value proportional to \( \sqrt{\lambda/\Delta} \), at an appropriate value of the parameter \( \epsilon \). Then, since \( \| H^I \| \infty \propto \lambda \), the two upper bounds mentioned above show the same dependence of \( \sqrt{\lambda/\Delta} \).

As another example, one may consider a case, in which the EFs of \( |n_i \rangle \) in the uncoupled basis have on average a Breit-Wigner form. As discussed in the above subsection, \( w_M = \lambda^2 \). Substituting Eq. (11) into the rhs of Eq. (37) for \( w_M \), one gets the following expression for it,

\[
\frac{2q_{\alpha} \omega_{BW}}{\pi d \Delta} \frac{1}{\epsilon} + \epsilon, \tag{42}
\]

which has a minimum value given by \( 2\sqrt{2q_{\alpha} \omega_{BW}}/(\pi d \Delta) \). This minimum value has the same dependence on \( \Delta \) as the rhs of Eq. (2)-[7], but, the \( \lambda \)-dependence is different. Since \( \sqrt{\lambda} \) decreases slower than \( \lambda \) with decreasing \( \lambda \), one notes that the upper bound given here is smaller than that of Eq. (2)-[7] for sufficiently weak interactions.

IV. OFFDIAGONAL ELEMENTS OF \( \rho^S \)

In this section, we first derive a generic expression for offdiagonal elements \( \rho_{\alpha\beta}^S \) with \( \alpha \neq \beta \), then, illustrate the obtained result by an example with the central system \( S \) as a two-level system, and finally compare with results given in Refs. [7, 8, 16].

A. A generic expression of \( \rho_{\alpha\beta}^S \) with \( \alpha \neq \beta \)

In order to derive an expression for \( \rho_{\alpha\beta}^S \) with \( \alpha \neq \beta \), which is written with some property of the interaction Hamiltonian, we note that the Schrödinger equation (2) implies the following relation,

\[
\langle \alpha | H^I | n \rangle = (E_n - e_{\alpha} - e_{i}) C_{\alpha \alpha}^n. \tag{43}
\]

Multiplying both sides of Eq. (43) by \( C_{\beta \alpha}^{n*} \) and noting that the equality obtained also holds under the exchange of \( \alpha \leftrightarrow \beta \), one finds that

\[
C_{\beta \alpha}^{n*} \langle \alpha | H^I | n \rangle = C_{\beta \alpha}^{n*} (E_n - e_{\alpha} - e_{i}) C_{\alpha \alpha}^n, \tag{44a}
\]

\[
C_{\alpha \alpha}^n (\alpha | H^I | \beta i) = C_{\alpha \alpha}^n (E_n - e_{\beta}^S - e_{i}) C_{\beta \beta}^n. \tag{44b}
\]

This gives that

\[
C_{\beta \alpha}^{n*} C_{\alpha \alpha}^n = \frac{1}{\Delta_{\beta \alpha}} \left( C_{\beta \alpha}^{n*} \langle \alpha | H^I | n \rangle - C_{\alpha \alpha}^n (\alpha | H^I | \beta i) \right), \tag{45}
\]

where \( \Delta_{\beta \alpha} := e_{\beta}^S - e_{\alpha} \), being nonzero as assumed. Substituting Eq. (45) into Eq. (17) and writing \( C_{\beta \alpha}^{n*} \) and \( C_{\alpha \alpha}^n \) as \( \langle n | \beta i \rangle \) and \( \langle \alpha | n \rangle \), respectively, one finds that

\[
\rho_{\alpha\beta}^S = \frac{1}{\Delta_{\beta \alpha}} Q_{\beta \alpha}^n, \tag{46}
\]

where

\[
Q_{\beta \alpha}^n = \langle n | A_{\beta \alpha}, H^I | n \rangle. \tag{47}
\]

Here, \( A_{\beta \alpha} \) is an operator defined by

\[
A_{\beta \alpha} := \sum_i | \beta i \rangle \langle \alpha i | = | \beta \rangle \langle \alpha | I^E. \tag{48}
\]

From Eqs. (46) and (16), we get that

\[
\rho_{\alpha\beta}^S = \frac{1}{\Delta_{\beta \alpha}} Q_{\beta \alpha}, \tag{49}
\]

where

\[
Q_{\beta \alpha} := d_{\Gamma}^{-1} \sum_{E_n \in \Gamma} Q_{\beta \alpha}^n. \tag{50}
\]

Since \( \rho_{\alpha\alpha}^S = 0 \), one directly gets that

\[
| \rho_{\alpha\beta}^S - \rho_{\alpha\alpha}^S | = \frac{Q_{\beta \alpha}}{\Delta_{\beta \alpha}}. \tag{51}
\]

From Eqs. (37) and (51), it is seen that the RDM \( \rho^S \) is indeed close to the Gibbs state \( \rho_G^S \), when the interaction is sufficiently weak and the environmental density of states has an exponential shape.

To see more clearly the physical meaning of the commutator \([A_{\beta \alpha}, H^I]\), let us consider a special case in which \( H^I \) has a direct-product form, namely,

\[
H^I = H^I_S \otimes H^I_E, \tag{52}
\]
where $H^{IS}$ and $H^{IE}$ are operators acting on the two spaces $H^S$ and $H^E$, respectively. Elements of $H^{IS}$ and $H^{IE}$ in the bases of $|\alpha\rangle$ and of $|i\rangle$ are written as

$$H^{IS}_{\alpha\beta} = \langle \alpha | H^{IS} | \beta \rangle,$$

$$H^{IE}_{ij} = \langle i | H^{IE} | j \rangle. \tag{53a}$$

Writing

$$H^{IS} = \sum_{\alpha'\beta'} H^{IS}_{\alpha'\beta'} |\alpha'\rangle \langle \beta'|,$$

one finds that

$$|\beta\rangle \langle \alpha| H^I = \sum_{\beta'} H^{IS}_{\alpha\beta'} |\beta'\rangle \langle \beta' | \otimes H^{IE}, \tag{55a}$$

$$H^I |\beta\rangle \langle \alpha| = \sum_{\alpha'} H^{IS}_{\alpha'\beta} |\alpha'\rangle \langle \alpha | \otimes H^{IE}. \tag{55b}$$

This gives that

$$[A_{j\alpha}, H^I] = \sum_{\alpha'} (H^{IS}_{\alpha\alpha'} |\beta\rangle \langle \alpha' | - H^{IS}_{\alpha'\beta} |\alpha\rangle \langle \alpha|) \otimes H^{IE}, \tag{56}$$

or explicitly,

$$[A_{j\alpha}, H^I] = (H^{IS}_{\alpha\alpha} - H^{IS}_{\beta\beta}) |\beta\rangle \langle \alpha| \otimes H^{IE} + H^{IS}_{\alpha\beta} |\beta\rangle \langle \beta| - |\alpha\rangle \langle \alpha| \otimes H^{IE} + \sum_{\alpha' (\neq \alpha, \beta)} (H^{IS}_{\alpha\alpha'} |\beta\rangle \langle \alpha' | - H^{IS}_{\alpha'\beta} |\alpha'\rangle \langle \alpha|) \otimes H^{IE}. \tag{57}$$

The above expression shows that $[A_{j\alpha}, H^I]$ can be regarded as certain “rearranged” (non-Hermitian) interaction Hamiltonian.

### B. A model with a two-level system

In this section, as an illustration of Eq. (32), we discuss a model, in which the subsystem $S$ is a two-level system (a qubit) and the environment is a many-body quantum chaotic system to which the ETH ansatz is applicable. For the simplicity in discussion, we assume that the interaction Hamiltonian $H^I$ has the following properties: (i) It has a direct-product form given in Eq. (32), with $H^{IE}$ being a local operator and $H^{IS} = 0$ for both values of $\alpha$; and (ii) within the considered energy region, the function $h(e)$ to be used in Eq. (60) is a constant, denote by $h_0$.

Under the conditions stated above, Eq. (57) gives that

$$[A_{j\alpha}, H^I] = H^{IS}_{\alpha\beta} |\beta\rangle \langle \beta| - |\alpha\rangle \langle \alpha| \otimes H^{IE} \tag{58}$$

with $\beta \neq \alpha$. Substituting this result into Eq. (17) and making use of the expansion of $|n\rangle = \sum_{\alpha} C^n_{\alpha}|\alpha\rangle$, one gets that

$$Q_{\alpha\beta}^{n} = H^{IS}_{\alpha\beta} \sum_{i,j} (C^n_{\beta\delta} C^n_{\beta\delta} - C^n_{\alpha\delta} C^n_{\alpha\delta}) H^{IE}_{ij}. \tag{59}$$

For a local operator $H^{IE}$, the ETH ansatz predicts that

$$H^{IE}_{ij} = h(e_i) \delta_{ij} + e^{-S(e_i)/2} g(e_i, e_j) R_{ij}, \tag{60}$$

where $h(e)$ is a slowly-varying function of $e$, $S(e)$ is proportional to the particle number of $E$ and is related to the microcanonical entropy in a semiclassical treatment, $g(e_i, e_j)$ is some smooth function of its variables, and the quantity $R_{ij}$ has certain random feature with a normal distribution (zero mean and unit variance).

Due to the random feature of $R_{ij}$ and the smallness of the term $e^{-S(E)/2}$ at large $N$, the contribution from the second term on the rhs of Eq. (60) to $Q_{\alpha\beta}^{n}$ in Eq. (59) is usually negligible, when the environment is sufficiently large. Then, making use of Eqs. (16)–(17), one gets the following expression of $Q_{\alpha\beta}^{n}$ with $h(e)$ written as $h_0$,

$$Q_{\alpha\beta}^{n} = h_0 H^{IS}_{\alpha\beta} \beta_{\alpha\beta} - \rho_{\alpha\beta} S_{\alpha\beta}^{(n)}). \tag{61}$$

Substituting Eq. (61) into Eq. (60), one gets the following simple relation among the elements of the RDM of a single state $|n\rangle$,

$$\rho_{\alpha\beta}^{S\langle n\rangle} \simeq \frac{H^{IS}_{\alpha\beta} h_0}{\Delta_{\alpha\beta}} (\rho_{\alpha\beta}^{S} - \rho_{\alpha\beta}^{(S\langle n\rangle)}) \quad \text{for } \alpha \neq \beta. \tag{62}$$

This implies the following relation for elements of $\rho_{\alpha\beta}^{S}$,

$$\rho_{\alpha\beta}^{S} \simeq \frac{H^{IS}_{\alpha\beta} h_0}{\Delta_{\alpha\beta}} (\rho_{\alpha\beta}^{S} - \rho_{\alpha\beta}^{(S\langle n\rangle)}) \quad \text{for } \alpha \neq \beta. \tag{63}$$

### C. Comparison with results of Refs. [7, 8, 16]

In this section, we compare results given in the previous two sections and those given in Refs. [7, 8, 16] for offdiagonal elements $\rho_{\alpha\beta}^{S\langle n\rangle}$ with $\alpha \neq \beta$.

We first discuss Ref. [8]. There, only a specific situation was studied for offdiagonal elements $\rho_{\alpha\beta}^{S\langle n\rangle}$ (appendix C), in which quantities like $\langle H^{IS}_{\alpha\beta} h(e) / \Delta_{\alpha\beta}^{(n)} \rangle$ have very small values and the EFs are very narrow. It is shown there that the offdiagonal elements $\rho_{\alpha\beta}^{S\langle n\rangle}$ have small values, when the dimension of the effective environmental state space is large. This prediction is clearly in agreement with Eq. (63) for a two-level system. For a multi-level system $S$'s agreement can also be found by making use of Eq. (70) to be derived later.

Next, we compare with Ref. [7]. It is easy to see that predictions of Eq. (11) and of Eq. (2)–(11) (as Eq. (2) of Ref. [7]) can not be always consistent, because the latter contains a term $\Delta^{-1/2}$, while, the former shows no explicit dependence on $\Delta$. The difference between the two predictions is seen more clearly from Eq. (63) for a two-level system, which shows that the value of $|\rho_{\alpha\beta}^{S\langle n\rangle} - \rho_{\alpha\beta}^{S\langle 0\rangle}|$ does not necessarily decrease with increasing $\Delta$.

To be precise, let us consider a solvable example, in which the interaction Hamiltonian has the simple form of $H^I = H^{IS} \otimes I^E$ with $[H^{IS}, H^S] \neq 0$. Clearly, one...
may equivalently take \( \tilde{H}^S = H^S + H^{IS} \) as the self-Hamiltonian of \( S \), with the corresponding interaction Hamiltonian vanishing; in other words, the total Hamiltonian \( H \) can be reformulated as \( H = \tilde{H}^S + H^E \). Under this formulation of \( H \), following arguments similar to those leading to Eq. (20), one finds that the RDM \( \rho^S \) has the following elements in the eigenbasis of \( \tilde{H}^S \), denoted by \( |\tilde{\alpha}\rangle \),

\[
\rho^S_{\alpha\alpha} = \frac{1}{d_{\Gamma\alpha}} e^{\tilde{d}_{\Gamma\alpha}^S}, \quad \rho^S_{\alpha\beta} = 0 \quad (\tilde{\alpha} \neq \tilde{\beta}), \quad \tag{64}
\]

where \( d_{\Gamma\alpha}^S \) is similar to \( d_{\Gamma\alpha}^E \) but related to the state \( |\tilde{\alpha}\rangle \). Transforming from the basis \( \{ |\tilde{\alpha}\rangle \} \) to \( \{ |\alpha\rangle \} \), since \( [H^S, H^{IS}] \neq 0 \), \( \rho^S \) usually gets nonzero offdiagonal elements \( \rho^S_{\alpha\beta} \) (unless the values of \( d_{\Gamma\alpha}^S \) are independent of the label \( \tilde{\alpha} \)), which do not depend on the value of \( \Delta \).

It is not difficult to check that the above-discussed nonzero \( \rho^S_{\alpha\beta} \) obtained from Eq. (61) are consistent with Eq. (63) related to the formulation of \( H = \tilde{H}^S + H^I + H^E \). (See also discussions to be given later at the end of Sec. V)\(^4\) In contrast, the obtained nonzero \( \rho^S_{\alpha\beta} \) conflict with the prediction of Eq. (21) that they should decrease as \( \sqrt{1/\Delta} \) or faster with increasing \( \Delta \). This conflict suggests that Eq. (21) may work under a condition stricter than that given in Ref. \( \tilde{H}^S \).

Finally, a formula given in Ref. \( \tilde{H}^S \) for a long-time averaged RDM has a form similar to Eq. (63). In fact, that formula of Ref. \( \tilde{H}^S \) can be derived from Eq. (63) \( \tilde{H}^S \).

\section{V. Gibbs states with impact of interaction}

In this section, making use of results obtained in previous sections, we discuss some situations in which the RDM \( \rho^S \) may have a Gibbs form. As mentioned previously, when the system-environment interaction is sufficiently weak, \( \rho^S \) has approximately a Gibbs form \( \tilde{H}^S \) below. We are to discuss interactions that are not so weak.

\subsection{A. Renormalized Gibbs state — diagonal elements}

Equation (37) supplies a method of studying possible modification to the Gibbs state due to interaction. To study this, we note that Eq. (37) is still valid under the following reformulation of the total Hamiltonian in Eq. (1),

\[
H = \tilde{H}^S + \tilde{H}^I + \tilde{H}^E, \quad \tag{65}
\]

where

\[
\tilde{H}^S = H^S + O^S, \quad \tilde{H}^I = H^I - O^S \otimes I^E, \quad \tag{66}
\]

with an arbitrary operator \( O^S \) that acts on the state space of the system \( S \). Formally, one may regard \( \tilde{H}^S \) as a renormalized self-Hamiltonian of the system \( S \) and \( \tilde{H}^I \) as the corresponding renormalized interaction Hamiltonian. Hereafter, unless otherwise stated, we use tilde to indicate items related to \( \tilde{H}^S \), e.g., \( |\tilde{\alpha}\rangle \) as eigenstates of \( \tilde{H}^S \), \( \tilde{H}^S |\tilde{\alpha}\rangle = \tilde{E}_{\alpha} |\tilde{\alpha}\rangle \), and \( \tilde{w}_M \) as a width like \( w_M \) but related to the basis states \( |E^\Gamma\rangle \equiv |\tilde{\alpha}\rangle \).

Like the relations shown in Eq. (21), it is not difficult to verify that \( \tilde{\rho}^{S0} \), which is the RDM obtained in the case of \( \tilde{H}^I = 0 \), is close to the following operator,

\[
\tilde{\rho}^{S}_{G} := e^{-\tilde{\beta} \tilde{H}^S} / \text{Tr}e^{-\tilde{\beta} \tilde{H}^S}, \quad \tag{67}
\]

under an exponential environmental density of states. We call \( \tilde{\rho}^{S}_{G} \) a renormalized Gibbs state.

Clearly, \( \tilde{w}_M \) usually has different values, when different operators \( O^S \) are adopted. Suppose that \( \tilde{w}_M / \Delta \) has a small value, when \( O^S \) is taken as some operator denoted by \( O^\text{sw} \), with “sw” standing for “small width”. Then, according to Eq. (37), \( |\tilde{\alpha}\rangle \) have small values, when \( \tilde{H}^S \) is taken as \( H^S + O^\text{sw} \). As a result, in the renormalized basis \( \{ |\tilde{\alpha}\rangle \} \), the renormalized Gibbs state supplies an appropriate description for the diagonal elements of the RDM \( \rho^S \). In other words, \( O^\text{sw} \) gives a useful description for the influence of the \( S-E \) interaction in these diagonal elements of \( \rho^S \).

Finally, we recall a mechanism that may give rise to smallness of \( \tilde{w}_E \) for EFs, for an environment as a many-body quantum chaotic system to which the ETH ansatz is applicable. \(^6\) In fact, according to a result of Ref. \( \tilde{w}_E \), \( O^\text{sw} \) may be given by an average of \( H^I \), which is obtained by taking partial trace over certain effective environmental state space. This may make the width \( \tilde{w}_E \) scaling as \( 1/\Delta_E \), where \( \Delta_E \) represents the total energy scale of the environment \( E \). Clearly, \( \Delta_E \) can be very large for a sufficiently large environment, implying smallness of \( \tilde{w}_E \). Thus, chaotic motion of the environment may indeed considerably suppress the upper bound given in Eq. (37).

\subsection{B. Renormalized Gibbs state in a big class of systems}

In the case that an operator \( O^\text{sw} \) can be found, according to discussions given above, if the offdiagonal elements of \( \rho^S \) in the related renormalized basis \( \{ |\tilde{\alpha}\rangle \} \) are

\[^4\] Since the proof of Eq. (21) given in Ref. \( \tilde{H}^S \) is sketchy, it is difficult to give a more detailed comparison.

\[^5\] Note that the RDM \( \rho^S \) is independent of the above reformulation of the total Hamiltonian and, hence, there is no need to write a tilde above it.

\[^6\] One notes that narrowness of EFs usually implies narrowness of LDOS.
small, then, one has $\rho^S \approx \tilde{\rho}_G^S$, otherwise, the renormalized Gibbs state $\tilde{\rho}_G^S$ gives an appropriate description only for the diagonal part of $\rho^S$. For a generic total system, there is no reason to expect that the two requirements of smallness of $\tilde{w}_M$ and of approximate diagonalization of $\rho^S$ in the basis $\{|\tilde{\alpha}\rangle\}$ may be satisfied at the same time. This suggests that the RDM $\rho^S$ does not need to have a Gibbs form in a generic total system.

Below, we show that it is possible for $\rho^S$ to be close to $\tilde{\rho}_G^S$ in a big class of systems of realistic relevance. Specifically, we consider systems that satisfy the following requirements: (i) The environmental part of the interaction Hamiltonian is local; (ii) the ETH ansatz is applicable to the environment; and (iii) an operator $O_{sw}^S$ exists such that the EFs of $|n\rangle$ in the basis $\{|\tilde{E}_n\rangle\}$ are narrow, in particular, $\tilde{w}_M \ll \Delta$ and $\tilde{w}_M \ll |\Delta_{\alpha\beta}|$.

For the simplicity in discussion, we assume that $H^I$ has the direct-product form given in Eq. (62), with a local operator $H^{IS}$. (Extension of the discussions to be given below to the case of $H^I$ as a sum of product terms is straightforward.) Below in this section, except for the last paragraph, we consider only the renormalized self-Hamiltonian $\tilde{H}^S = H^S + O_{sw}^S$ and its eigenbasis, hence, for brevity, we omit the tildes until the last paragraph of this section.

Since all significant components of the EF of $|n\rangle$ are included in its main-body region, noting the relation in Eq. (65), one writes

$$ |n\rangle \approx \sum_\alpha \sum_{\epsilon_i \in \Upsilon^S_\alpha} C^n_{\alpha\epsilon_i} |\alpha\rangle, $$

where $\Upsilon^S_\alpha$ indicates the environmental energy region of $[E_n - e^S_\alpha - w_M, E_n - e^S_\alpha + w_M]$. The assumed smallness of $w_M$ implies that $\epsilon_i + e^S_\alpha \approx E_n$ for $\epsilon_i \in \Upsilon^S_\alpha$ and there is no overlap between $\Upsilon^S_\alpha$ of different $\alpha$.

Substituting Eq. (61) into Eq. (60) and inserting $\sum_i |i\rangle\langle i|$, one finds that, for $\alpha \neq \beta$,

$$ Q^S_{\beta\alpha} = \sum_{ij} (H^{IS}_{\alpha\alpha} - H^{IS}_{\beta\beta})(|n\rangle\langle \beta|\langle \alpha j|n\rangle H^{IE}_{ij} + \sum_{\alpha' \neq \alpha, \beta} (H^{IS}_{\alpha\alpha'} (|n\beta\rangle\langle \alpha' j|n\rangle - H^{IS}_{\alpha'\beta} (|n\alpha'\rangle\langle \alpha j|n\rangle)) H^{IE}_{ij} + H^{IS}_{\beta\beta} (|n\beta\rangle\langle \beta j|n\rangle - |\beta\rangle\langle \alpha j|n\rangle) H^{IE}_{ij}. $$

Now, we insert the ETH ansatz of Eq. (60) into Eq. (69). Note that, for the same reason as that discussed previously in Sec. IV B, the second term on the rhs of Eq. (60) can be neglected and hence we are left with the diagonal term of $h(\epsilon_i)\delta_{ij}$. Furthermore, since there is no overlap between $\Upsilon^S_\alpha$ of different $\alpha$ for a given state $|n\rangle$, all those terms on the rhs of Eq. (69), which has the form of $\langle n\beta\rangle\langle \alpha|n\rangle$ with $\alpha \neq \beta$, can be neglected. Then, we get that

$$ Q^S_{\beta\alpha} \approx \sum_i H^{IS}_{\beta\beta}(\langle n\beta\rangle^2 - |\langle \alpha|n\rangle|^2) h(\epsilon_i) \simeq H^{IS}_{\beta\beta} h(E_n - e^S_\beta) \rho^S(\alpha), $$

where $\rho^S(\alpha)$ is defined in Eq. (47) and slow variation of $h(\epsilon)$ has been used in the derivation of the second equality.

To go further, we note that the assumed smallness of $w_M$ requires some restriction to properties of the interaction. To see this point clearly, let us consider an arbitrary pair of basis states $|\alpha i\rangle$ and $|\beta i\rangle$ with the same label $i$ and $\alpha \neq \beta$. The level spacing of these two states is given by $\Delta_{\beta\alpha}$. Making use of the ETH ansatz, one finds that the coupling between these two states, i.e., the offdiagonal element $\langle \alpha i|H^{IS}|\beta i\rangle$ has the following form,

$$ \langle \alpha i|H^{IS}|\beta i\rangle \simeq H^{IS}_{\alpha\beta} h(\epsilon_i). $$

According to the perturbation theory, the assumed narrowness of the EF of $|n\rangle$ usually requires that $|H^{IS}_{\alpha\beta}h(\epsilon_i)/\Delta_{\beta\alpha}| \ll 1$, at least for those levels $\epsilon_i$ that are not far from the values of $(E_n - e^S_\alpha(\beta))$. Then, from Eq. (67), one finds that $|Q^S_{\alpha\beta}/\Delta_{\beta\alpha}| \ll 1$. According to Eqs. (69)-(70), this implies smallness of the offdiagonal elements $\rho^S_{\alpha\beta}$.

To summarize, resuming the tilde, for systems satisfying the requirements stated in the second paragraph of this section, one has $\rho^S \approx \tilde{\rho}_G^S$ with $\tilde{H}^S = H^S + O_{sw}^S$. Finally, to illustrate the above result, as an example, let us consider the model discussed in Sec. IV B. Making use of Eq. (73), it is straightforward to verify that $\tilde{\rho}^S$ has the following matrix form in the eigenbasis of $\tilde{H}^S$,

$$ [\tilde{\rho}^S] = \frac{\rho^S_{\alpha\beta} - \rho^S_{\beta\alpha}}{\Delta_{\beta\alpha}} \begin{pmatrix} 0 & H^{IS}_{\beta\alpha}h_0 & H^{IS}_{\alpha\beta}h_0 \end{pmatrix} + \rho^S_{\beta\alpha} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}. $$

The main result of Ref. [8] is applicable to this model, which shows that $\tilde{w}_E$ is small under the renormalized self-Hamiltonian $\tilde{H}^S = H^S + h_0 H^{IS}$, implying that $O_{sw}^S = h_0 H^{IS}$. In the basis of $|\alpha\rangle$, $\tilde{H}^S$ has the following matrix form,

$$ [\tilde{H}^S] = \begin{pmatrix} e^S_\alpha & H^{IS}_{\alpha\beta}h_0 & H^{IS}_{\alpha\beta}h_0 \\ e^S_\beta & H^{IS}_{\beta\alpha}h_0 & H^{IS}_{\beta\alpha}h_0 \\ e^S_\beta & H^{IS}_{\alpha\beta}h_0 & H^{IS}_{\alpha\beta}h_0 \end{pmatrix}. $$

It is then easy to see that $[\tilde{\rho}^S]$ and $[\tilde{H}^S]$ are diagonalized by almost a same transformation. Therefore, $\tilde{\rho}^S \approx \tilde{\rho}_G^S$, i.e., the RDM is close to the renormalized Gibbs state in Eq. (47) with $O_{sw}^S = h_0 H^{IS}$.

### VI. RDM COMPUTED FROM TYPICAL STATES OF THE TOTAL SYSTEM

In this section, we discuss the RDM of $S$, which is computed from a typical state of the total system in the
energy shell $\Gamma$. We denote it by $\rho_{S}^{\Gamma}$. In particular, we give estimates to the elements $(\rho_{S}^{\Gamma})_{\alpha\beta} = \langle \alpha | \rho_{S}^{\Gamma} | \beta \rangle$, which enable one to get details of the difference between $\rho_{S}^{\Gamma}$ and the previously-discussed RDM $\rho^{S}$. For brevity, we use $O(x)$ to indicate an undetermined quantity, whose order of magnitude is the same as that of a quantity $x$.

A. Basic properties of $\rho_{S}^{\Gamma}$

We use $|\Psi_{\Gamma}^{\Gamma}\rangle$ to denote a normalized typical vector in the subspace $\mathcal{H}_{\Gamma}$, written as

$$|\Psi_{\Gamma}^{\Gamma}\rangle = \mathcal{M}_{\Gamma}^{-1} \sum_{E_{n} \in \Gamma} D_{n} |n\rangle,$$

(74)

where the real and imaginary parts of $D_{n}$ are independent Gaussian random variables, with mean zero and variance $1/2$, and $\mathcal{M}_{\Gamma}$ is the normalization coefficient. The RDM $\rho_{S}^{\Gamma}$ is given by

$$\rho_{S}^{\Gamma} = \text{Tr}_{ \Gamma } (|\Psi_{\Gamma}^{\Gamma}\rangle \langle \Psi_{\Gamma}^{\Gamma}|).$$

(75)

As shown in Ref. [3], the averaged trace distance between $\rho^{S}$ and $\rho_{S}^{\Gamma}$ satisfies

$$\langle D(\rho^{S}, \rho_{S}^{\Gamma}) \rangle \leq \frac{1}{2} \sqrt{ \frac{d_{\Gamma}^{2}}{d_{\Gamma}} }.$$

(76)

To study $\rho_{S}^{\Gamma}$, we expand the typical state $|\Psi_{\Gamma}^{\Gamma}\rangle$ according to the system $S$’s states $|\alpha\rangle$, i.e.,

$$|\Psi_{\Gamma}^{\Gamma}\rangle = \mathcal{M}_{\Gamma}^{-1} \sum_{\alpha} |\alpha\rangle |\Phi_{\alpha}^{\Gamma}\rangle,$$

(77)

where

$$|\Phi_{\alpha}^{\Gamma}\rangle = \sum_{i} \left( \sum_{E_{n} \in \Gamma} D_{n} C_{\alpha i}^{n} \right) |i\rangle.$$

(78)

It is easy to verify that

$$\langle \rho_{S}^{\Gamma} \rangle_{\alpha\beta} = \mathcal{M}_{\Gamma}^{-2} \langle \Phi_{\beta}^{\Gamma} | \Phi_{\alpha}^{\Gamma} \rangle,$$

(79)

$$\langle \Phi_{\beta}^{\Gamma} | \Phi_{\alpha}^{\Gamma} \rangle = \sum_{i} \sum_{E_{n} \in \Gamma} \sum_{E_{n'} \in \Gamma} D_{n}^{*} D_{n'}^{*} C_{\beta i}^{n} C_{\alpha i}^{n'}.$$  

(80)

From Eq. (74) and the randomness of the coefficients $D_{n}$, one finds that

$$\mathcal{M}_{\Gamma}^{2} = \sum_{E_{n} \in \Gamma} |D_{n}|^{2} = d_{\Gamma} + O(\sqrt{d_{\Gamma}}),$$

(81)

and, as a result,

$$\langle \rho_{S}^{\Gamma} \rangle_{\alpha\beta} = \frac{1}{d_{\Gamma}} \left( 1 + O(\frac{1}{\sqrt{d_{\Gamma}}} ) \right) \langle \Phi_{\beta}^{\Gamma} | \Phi_{\alpha}^{\Gamma} \rangle.$$  

(82)

In the two sections following this one, we discuss properties of the diagonal part $\langle \Phi_{\alpha}^{\Gamma} | \Phi_{\alpha}^{\Gamma} \rangle$ and of the offdiagonal part $\langle \Phi_{\beta}^{\Gamma} | \Phi_{\alpha}^{\Gamma} \rangle$ with $\alpha \neq \beta$, separately. There, it proves convenient to divide the overlap $\langle \Phi_{\beta}^{\Gamma} | \Phi_{\alpha}^{\Gamma} \rangle$ into two parts,

$$\langle \Phi_{\beta}^{\Gamma} | \Phi_{\alpha}^{\Gamma} \rangle = K_{\beta\alpha}^{(1)} + K_{\beta\alpha}^{(2)},$$

(83)

where $K_{\beta\alpha}^{(1)}$ represents the diagonal contribution of the rhs of Eq. (80) with $n = n'$ and $K_{\beta\alpha}^{(2)}$ is for the offdiagonal contribution with $n \neq n'$.

B. Diagonal overlap — $\langle \Phi_{\beta}^{\Gamma} | \Phi_{\alpha}^{\Gamma} \rangle$

In this section, we discuss properties of $\langle \Phi_{\beta}^{\Gamma} | \Phi_{\alpha}^{\Gamma} \rangle$. We first discuss $K_{\alpha\alpha}^{(1)}$, which by definition is written as

$$K_{\alpha\alpha}^{(1)} = \sum_{E_{n} \in \Gamma} |D_{n}|^{2} B_{\alpha\alpha},$$

(84)

where

$$B_{\alpha\alpha} = \sum_{i} |C_{\alpha i}^{n}|^{2}. $$

(85)

We use $\overline{B}_{\alpha}$ to indicate the average value of $B_{\alpha\alpha}$ within the energy shell $\Gamma$, i.e.,

$$\overline{B}_{\alpha} := \frac{1}{d_{\Gamma}} \sum_{E_{n} \in \Gamma} B_{\alpha\alpha}. $$

(86)

It is straightforward to verify that the RDM element $\rho_{\alpha\alpha}^{S}$ [see Eq. (16)] is equal to $\overline{B}_{\alpha}$, i.e.,

$$\rho_{\alpha\alpha}^{S} = \overline{B}_{\alpha}. $$

(87)

Since the average of $|D_{n}|^{2}$ is equal to 1, the sum of $\sum_{E_{n} \in \Gamma} |D_{n}|^{2} - 1) B_{\alpha\alpha}$ has an absolute value that has the same order of magnitude as $\rho_{\alpha\alpha}^{S} \sqrt{d_{\Gamma}}$. Then, from Eq. (84) one gets that

$$K_{\alpha\alpha}^{(1)} = d_{\Gamma} \rho_{\alpha\alpha}^{S} + \rho_{\alpha\alpha}^{S} e^{i\varphi_{1}} O(\sqrt{d_{\Gamma}}), $$

(88)

where $\varphi_{1}$ represents some undetermined phase. Note that $\rho_{\alpha\alpha}^{S}$ are elements of $\rho^{S}$ discussed in previous sections.

Next, we discuss the term $K_{\alpha\alpha}^{(2)}$, which has the following expression,

$$K_{\alpha\alpha}^{(2)} = \sum_{i} J_{\alpha i}, $$

(89)

where

$$J_{\alpha i} = \sum_{E_{n} \in \Gamma} \sum_{E_{n'} \in \Gamma} D_{n}^{*} D_{n'}^{*} C_{\alpha i}^{n} C_{\alpha i}^{n'}. $$

(90)

We divide $K_{\alpha\alpha}^{(2)}$ into subparts according to the regions $R_{\kappa}^{s}$, like what was done in Sec. IIIA, that is,

$$K_{\alpha\alpha}^{(2)} = \sum_{\kappa=0}^{3} K_{\alpha\alpha,\kappa}^{(2)}, $$

(91)
where
\[ K^{(2)}_{\alpha_\alpha, \kappa} = \sum_{e_i \in \mathcal{R}_L^{\alpha}} J_{\alpha i}. \] (92)

It proves convenient to introduce the following quantity,
\[ \mathcal{I}_{\alpha i} = \sum_{e_n \in \Gamma} |D_n|^2 |C_{\alpha i}|^2. \] (93)

We note that, due to the randomness of the components \( D_n \), usually, \( J_{\alpha i} \) and \( \mathcal{I}_{\alpha i} \) have the following relation,
\[ J_{\alpha i} = \mathcal{I}_{\alpha i} e^{i\theta_i} O(1), \] (94)

with random phases \( \theta_i \).

Firstly, we discuss the term \( K^{(2)}_{\alpha_\alpha,2} \), the contribution coming from the central region \( \mathcal{R}_2^{\alpha} \). For \( e_i \) lying in this region, like \( \mathcal{I}_{\alpha i} \) in Eq. (28), \( \mathcal{I}_{\alpha i} \) also fluctuate around \((1 - a_2\epsilon)\). Then, making use of Eq. (94), one finds that
\[ K^{(2)}_{\alpha_\alpha,2} = \sqrt{N_2^{\alpha}} (1 - a_2\epsilon) e^{i\theta_2} O(1), \] (95)

with some undetermined phase \( \theta_2 \).

Secondly, we discuss \( K^{(2)}_{\alpha_\alpha,1} \), coming from the region \( \mathcal{R}_1^{\alpha} \). We use \( i_0 \) and \( i_f \) to indicate the starting and ending labels of \( \mathcal{R}_1^{\alpha} \). In this region, \( \mathcal{I}_{\alpha i} \) is close 1 for \( e_i \) close to the region \( \mathcal{R}_2^{\alpha} \), while, it is small for \( e_i \) close to the region \( \mathcal{R}_0^{\alpha} \). Loosely speaking, with the label \( i \) increasing from \( i_0 \) to \( i_f \), \( \mathcal{I}_{\alpha i} \) increases on average from some value close to 0 to some value close to 1. It is this difference in the values of \( \mathcal{I}_{\alpha i} \) that makes it uneasy to get an estimate to \( K^{(2)}_{\alpha_\alpha,1} \).

In order to circumvent the above-mentioned difficult, we construct new variables from \( J_{\alpha i} \). At the first step, we construct a series of variables, denoted by \( X^{(s)}_t \) with \( s = 0, 1, \ldots, s_f \), where \( s_f \) is given by the integer part of \((i_f - i_0)/2\). Specifically,
\[ X^{(1)}_s = J_{\alpha (i_0 + s)} + J_{\alpha (i_0 + s - 1)} \quad \text{for} \quad s = 0, 1, \ldots, s_f - 1; \] (96)
\[ X^{(1)}_{s_f} \] is given by Eq. (96) if \((i_f - i_0)\) is odd, otherwise, \( X^{(1)}_{s_f} = J_{\alpha (i_0 + s_f)} \). Clearly, the variance of \( X^{(1)}_t \) is equal to the sum of the variances of \( J_{\alpha (i_0 + s)} \) and \( J_{\alpha (i_0 + s - 1)} \) for \( s \leq s_f - 1 \). We proceed following the above procedure, until an \( L \)-th step is reached, at which most \( X^{(L)}_t \) have similar variances. It is easy to see that
\[ K^{(2)}_{\alpha_\alpha,1} = \sum_{t} X^{(L)}_t. \] (97)

We assume that \( N_1^{\alpha} \) is sufficiently large, such that \( N_1^{\alpha} \gg 2^L \). Note that the number of the variables \( X^{(L)}_t \) at the \( L \)-the step is about \( N_1^{\alpha}/2^L \).

According to the construction of \( X^{(L)}_t \), the sum of the variances of \( X^{(L)}_t \) over \( t \) is equal to that of \( J_{\alpha i} \) over \( i \) with \( e_i \in K^{(2)}_{\alpha_\alpha,1} \). This implies that the averaged variance of \( X^{(L)}_i \) is around \((\sigma^{2}_i 2^L)\), where \( \sigma^{2}_i \) is the averaged variance of these \( J_{\alpha i} \). Then, one finds that
\[ K^{(2)}_{\alpha_\alpha,1} = \sigma_i e^{i\theta_i} \sqrt{N_1^{\alpha}} O(1), \] (98)

with some undetermined phase \( \theta_i \). The above arguments are also applicable to \( K^{(2)}_{\alpha_\alpha,3} \). Thus, when the environmental density of states does not change much around the energy shell \( \Gamma_0 \), we find that
\[ K^{(2)}_{\alpha_\alpha,3} = \sigma_i e^{i\theta_i} \sqrt{2\omega_T d_{2L}^{\alpha}} O(1) \quad \text{for} \quad \kappa = 1, 3. \] (99)

It is easy to check that \( 0 < \sigma_{1,3} < 1 \).

Thirdly, we discuss \( K^{(2)}_{\alpha_\alpha,0} \), coming from \( e_i \) lying in the region \( \mathcal{R}_0^{\alpha} \). Making use of Eq. (94), we write it in the following form,
\[ K^{(2)}_{\alpha_\alpha,0} = \sum_{e_n \in \Gamma} |D_n|^2 \left( \sum_{e_i \in \mathcal{R}_0^{\alpha}} |C_{\alpha i}|^2 x_i \right), \] (100)

where \( x_i \) represents a complex random variable, whose variance is of the order of magnitude of 1. Noting Eq. (54), one sees that
\[ \sum_{e_n \in \mathcal{R}_0^{\alpha}} |C_{\alpha i}|^2 x_i = b_0 \epsilon, \] (101)

where \( b_0 \) is some undetermined parameter satisfying \( 0 < b_0 < 1 \) (usually \( b_0 \ll 1 \)). Then, one gets that
\[ K^{(2)}_{\alpha_\alpha,0} = b_0 \epsilon \sqrt{d_T} e^{i\theta_0} O(1), \] (102)

with some undetermined phase \( \theta_0 \).

To summarize, we find that \( K^{(2)}_{\alpha_\alpha} \) has the following form,
\[ K^{(2)}_{\alpha_\alpha} = \sqrt{N_2^{\alpha}} \epsilon^{i\theta_2} O(1) + (\sigma_1 e^{i\theta_1} + \sigma_3 e^{i\theta_3}) \sqrt{2\omega_T d_{2L}^{\alpha}} O(1) \]
\[ -a_2 \epsilon \sqrt{N_2^{\alpha}} \epsilon^{i\theta_2} O(1) + b_0 \epsilon \sqrt{d_T} e^{i\theta_0} O(1). \] (103)

It is seen that, for a large \( d_T \), usually \( K^{(2)}_{\alpha_\alpha} \) gives a small contribution to \( \rho^{\alpha}_{\gamma_\gamma, \alpha} \) in Eq. (52).

C. Nondiagonal terms

For an offdiagonal term \( \langle \Phi^{\alpha}_{\gamma_\gamma} | \Phi^{\beta}_{\beta} \rangle \) with \( \alpha \neq \beta \), its two parts are written in the following forms,
\[ K^{(1)}_{\beta \alpha} = \sum_{E_n \in \Gamma} \sum_{t} |D_n|^2 C^{*}_{\beta t} C_{\alpha i}, \] (104)
\[ K^{(2)}_{\beta \alpha} = \sum_{E_n \in \Gamma} \sum_{E_n \neq E_n} \sum_{t} D^{*}_{n} D_{n'} C^{*}_{\beta t} C_{\alpha i} \] (105)
The two quantities $K^{(1)}_{\alpha\alpha}$ and $K^{(2)}_{\alpha\alpha}$ can be studied by a method similar to that used in the previous section for diagonal terms, and qualitatively similar results can be obtained.

For example, to study $K^{(1)}_{\beta\alpha}$, we write it as

$$K^{(1)}_{\beta\alpha} = \sum_{E_n \in \Gamma} |D_n|^2 \rho^\alpha_{\beta\alpha}. \quad (106)$$

Similar to $K^{(1)}_{\alpha\alpha}$ discussed previously, one finds that

$$K^{(1)}_{\beta\alpha} = d\rho^\beta_{\beta\alpha} + \sqrt{d\rho^\beta_{\beta\alpha}} O(\sigma_{\beta\alpha}), \quad (107)$$

where $\theta_{\beta\alpha}$ is some undetermined phase and $\sigma_{\beta\alpha}^2$ represents the variance of $\rho^\beta_{\beta\alpha}$.

Finally, substituting results thus obtained for the offdiagonal contributions $\langle \phi^S | \phi^S \rangle$ and the previously-obtained results for diagonal elements in Eqs. (108) and (109) into Eq. (82), it is straightforward to get an estimate to $(\rho^S_{\alpha\beta} - (\rho^S_{\gamma\gamma})_{\alpha\beta})$.

VII. CONCLUSIONS AND DISCUSSIONS

In this paper, for a generic, isolated, and large quantum system which is described by an MC ensemble, we study the RDM $\rho^S$ of a small subsystem $S$ in the eigenbasis of its Hamiltonian. Upper bounds have been derived for the difference between elements of $\rho^S$ and those of $\rho^{S0}$, the latter of which is for the case of $S$ being uncoupled to the environment. The derived upper bound for diagonal elements is mainly confined by the ratio of the maximum width of the total EFs in the uncoupled basis to the width of the MC energy shell. The difference between the offdiagonal elements is given by the ratio of certain property of the interaction Hamiltonian to the related level spacing of the system $S$. Properties of the RDM $\rho^S_{\gamma\gamma}$, which is computed from a typical state of the total system within an energy shell, have also been studied.

By making use of the well-known relationship between $\rho^{S0}$ and the Gibbs state, the above-discussed results enable a comparison between the RDM $\rho^S$ and the Gibbs state. Under subsystem-environment interactions that are sufficiently weak, independent of the type of the environment (integrable, or chaotic, or some mixed type), the two ratios mentioned above are small and hence $\rho^S_{\alpha\beta}$ is close to the canonical Gibbs state, as usually expected and explicitly shown in Refs. [7, 8]. However, the situation is much more complex, when the interaction is not so weak. In this case, one may consider a renormalized Gibbs state given by a renormalized self-Hamiltonian of $S$, which includes certain averaged effect of the interaction.

When a renormalized self-Hamiltonian of $S$ can be found such that the total EFs in the uncoupled basis are sufficiently narrow, the renormalized Gibbs state supplies an appropriate description for diagonal elements of the RDM $\rho^S$ in the renormalized basis. While, this is unnecessarily true for the offdiagonal elements in a generic total system. However, it is argued that $\rho^S_{\alpha\beta}$ may be close to certain renormalized Gibbs state in the class of systems possessing the following properties: local interaction, environments as many-body quantum chaotic systems to which the ETH ansatz is applicable, and narrow total EFs in the uncoupled basis.

Results of this paper are also of relevance to the study of some aspects of decoherence, particularly the concept of preferred (pointer) basis. For example, if certain type of initial state of the total system may evolve at long times to states that can be regarded as typical states within an energy shell, then, the RDM of a subsystem $S$ may approach to $\rho^S_{\gamma\gamma}$ at long times. As being computed from a typical state, this operator $\rho^S_{\gamma\gamma}$ is independent of many details of the initial state of the total system. Hence, in the case that $\rho^S_{\alpha\beta} \approx \rho^S_{\gamma\gamma}$, the eigenbasis of $\rho^S$ is a good candidate for preferred basis. In particular, when $\rho^S_{\alpha\beta}$ is close to a renormalized Gibbs state, the eigenbasis of the renormalized self-Hamiltonian may give a preferred basis [21].

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