THE TOUSCHEK EFFECT
IN STRONG FOCUSING STORAGE RINGS

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Abstract

The lifetime of a stored beam due to the Touschek effect is calculated for arbitrary ratios of beam height to beam width. A variation of the beam envelopes is taken into account, i.e. the derivatives of the horizontal and vertical amplitude functions and dispersions are included. The calculation is done for arbitrary energies in the rest frame of the colliding particles.
1. Introduction

Coulomb scattering of charged particles in a stored beam causes an exchange of energies between the transverse and longitudinal motion. It changes, therefore, the betatron and synchrotron oscillation coordinates of the colliding particles. One consequence is the Touschek effect which is the transformation of a small transverse momentum into a large longitudinal momentum due to the scattering. Then both scattered particles are lost, one with too much and one with too little energy. The amplification of the momentum change is a relativistic effect so that the change of the longitudinal momentum is increased by the Lorentz factor $\gamma$.

The Touschek effect is different from intra-beam scattering which is also caused by Coulomb scattering. The intra-beam scattering, however, is a multiple scattering which leads to diffusion in all three directions and, primarily, changes the beam dimensions. The Touschek effect, on the other hand, is a single scattering effect which leads to the immediate loss of the colliding particles. Here only the energy transfer from the transverse to the longitudinal direction plays a role.

The Touschek effect was observed for the first time in the small electron storage ring ADA [1] and it turned out later that it is a serious problem in storage rings at low energies and in synchrotron light sources where it can reduce the lifetime considerably. Theoretical investigations were made in [1, 2] for the non-relativistic case, where the velocities of two colliding particles are non-relativistic in their center of mass system. The theory was then extended in [3] to the ultra-relativistic case, and in [4] to arbitrary energies. Dispersion was included in [5]. In all these calculations only the transfer of horizontal momentum into longitudinal momentum was considered whereas the transfer of the vertical momentum was neglected. In [6] a 100% coupling of horizontal and vertical betatron oscillations, i.e. the special case of a round beam, was considered for the non-relativistic case.

The main aim of the following calculation is to take into account both the horizontal and the vertical betatron oscillations. In that case one has a two-dimensional distribution of the transverse momenta which leads to a weaker dependence on the maximum stable energy deviation than a one-dimensional distribution. This can increase the Touschek lifetime by a factor of two as compared to the flat beam approximation at energies around 1 GeV, but the correction can be larger at lower energies.

At the same time a variation of the beam envelopes is taken into account. The derivatives of the amplitude functions $\beta_x$ and $\beta_z$ and of the dispersions $D_x$ and $D_z$ increase the angle between colliding particles. But the derivatives of the amplitude functions play a role only at positions where the dispersion does not vanish, or to be exact, where $\beta_{x,z} D'_{x,z} - \beta'_{x,z} D_{x,z}/2$ is different from zero. In that case they can increase the loss rate and decrease the Touschek lifetime.

We will use here the same method as for the calculation of the intra-beam scattering [7]. Thus we will consider the collision of two arbitrary particles in their center of mass system using the complete Møller scattering cross section, i.e. assuming arbitrary energies in the center of mass system. After transforming into the laboratory
system we will calculate the number of collisions with those momentum changes which lead to the loss of both particles. When averaging over the positions and angles of the colliding particles we will assume Gaussian distributions for all coordinates.

2. Change of the momenta of two colliding particles

The momenta $\vec{p}_1$ and $\vec{p}_2$ of the two colliding particles before the collision are given in the laboratory coordinate system $\{s, x, z\}$ by the two vectors

$$\vec{p}_{1,2} = \begin{pmatrix} p_{s1,2} \\ p_{x1,2} \\ p_{z1,2} \end{pmatrix}_{s,x,z}$$

where $s$ indicates the longitudinal, $x$ the horizontal, and $z$ the vertical direction. We define a coordinate system with the axes $j$, $k$, and $l$ which are parallel to $\vec{p}_1 + \vec{p}_2$, $\vec{p}_1 \times \vec{p}_2$, and $(\vec{p}_1 + \vec{p}_2) \times (\vec{p}_1 \times \vec{p}_2)$, respectively. The two momenta then take the form

$$\vec{p}_{1,2} = \begin{pmatrix} \cos \chi_{1,2} \\ 0 \\ \pm \sin \chi_{1,2} \end{pmatrix}_{j,k,l}$$

where $\chi_1$ and $\chi_2$ are the angles between the vector $\vec{p}_1 + \vec{p}_2$ and the vectors $\vec{p}_1$ and $\vec{p}_2$, respectively. They are given by the two relations

$$\vec{p}_1 \vec{p}_2 = p_1 p_2 \cos(\chi_1 + \chi_2)$$

and

$$p_1 \sin \chi_1 = p_2 \sin \chi_2$$

When we apply a Lorentz transformation parallel to $\vec{p}_1 + \vec{p}_2$ we obtain for the momenta $\vec{\bar{p}}_1$ and $\vec{\bar{p}}_2$ in the c. o. m. system the representation

$$\vec{\bar{p}}_{1,2} = \begin{pmatrix} \gamma_t (\cos \chi_{1,2} - \beta_t/\beta_{1,2}) \\ 0 \\ \pm \sin \chi_{1,2} \end{pmatrix}_{j,k,l}$$

The transformed energies $\bar{E}_1$ and $\bar{E}_2$ are given by (see App. A1)

$$\bar{E}_{1,2} = \gamma_t E_{1,2} (1 - \beta_t \beta_{1,2} \cos \chi_{1,2}) = (E_1 + E_2)/(2\gamma_t)$$

where $\beta_t$ is the relative velocity of the c. o. m. system, $\gamma_t$ is the Lorentz factor of the transformation, $\beta_{1,2}$ are the relative velocities of the two particles in the laboratory system, and the bars denote all quantities in the c. o. m. system. $\beta_t$ is determined by the condition that the sum of the two momenta vanishes in the c. o. m. system and is given by:

$$\beta_t = \frac{c|\vec{p}_1 + \vec{p}_2|}{E_1 + E_2} = \frac{\beta_1 \gamma_1 \cos \chi_1 + \beta_2 \gamma_2 \cos \chi_2}{\gamma_1 + \gamma_2}$$
We may now assume that the quantities \((p_1 - p_2)^2/p_{1,2}^2, p_{x,1,2}^2/p_{1,2}^2, \) and \(p_{z,1,2}^2/p_{1,2}^2,\) and therefore also \(\chi_{1,2}^2,\) are small as compared to one, and Eqs. (3) and (4) become, if we replace \(p_{s,1,2}\) by \(p_{1,2} - (p_{x,1,2}^2 + p_{z,1,2}^2)/(2p_{1,2}):\)

\[
(p_{x,1}/p_1 - p_{x,2}/p_2)^2 + (p_{z,1}/p_1 - p_{z,2}/p_2)^2 = (\chi_1 + \chi_2)^2
\]

and

\[p_1 \chi_1 = p_2 \chi_2\]

Finally one obtains with \(\chi_1 \approx \chi_2 \approx \chi\)

\[4\chi^2 = (p_{x,1} - p_{x,2})^2/p^2 + (p_{z,1} - p_{z,2})^2/p^2\]

where \(p\) is the mean value of all momenta in the bunch. Eqs. (5) and (6) then simplify to (see App. A1)

\[
\vec{p}_{1,2} = \pm p \begin{pmatrix} \xi \sqrt{1 + \gamma^2 \chi^2}/2 \\ 0 \\ \chi \end{pmatrix}_{\bar{j}, \bar{k}, \bar{l}}
\]

and

\[\bar{E}_{1,2} = E/\gamma_t\]

with

\[
\xi = \frac{p_1 - p_2}{\gamma p}, \quad \theta = \frac{p_{x,1} - p_{x,2}}{p}, \quad \zeta = \frac{p_{z,1} - p_{z,2}}{p}, \quad 4\chi^2 = \theta^2 + \zeta^2
\]

We now assume that the momenta are almost perpendicular to the \(\bar{j}\)-axis in the c. o. m. system, which means (Eq. (9))

\[\xi^2(1 + \gamma^2 \chi^2) \ll 4\chi^2 \quad \text{or} \quad \xi^2 \ll (4 - \gamma^2 \xi^2) \chi^2 \approx 4\chi^2
\]

which can be written as

\[(p_1 - p_2)^2 \ll \gamma^2 ((p_{x,1} - p_{x,2})^2 + (p_{z,1} - p_{z,2})^2)\]

This assumption gives

\[|\vec{p}_1| \approx |\vec{p}_2| \approx p\]

and one obtains

\[
\vec{p}_{1,2} = p \begin{pmatrix} 0 \\ 0 \\ \pm \chi \end{pmatrix}_{\bar{j}, \bar{k}, \bar{l}}
\]

The relative velocity \(\beta_t\) of the c. o. m. system simplifies to

\[\beta_t = \beta \cos \chi\]

where \(\beta\) is given by the mean value \(p\) of all momenta and \(\gamma_t\) is

\[\gamma_t^2 = \frac{1}{1 - \beta_t^2} = \frac{\gamma^2}{1 + \beta^2 \gamma^2 \chi^2}\]

4
After the collision the absolute values of the two momenta are not changed in the c. o. m. system and the new directions of the momenta can be described with help of the two angles \( \bar{\psi} \) and \( \bar{\phi} \), where \( \bar{\psi} \) is the angle between the momentum of the scattered particle \( \vec{p}_1' \) and the (longitudinal) \( \bar{j} \)-axis, and \( \bar{\phi} \) is the angle between the projection of \( \vec{p}_1' \) on the \( \bar{l}-k \)-plane and the \( k \)-axis. The momenta after the collision are then given by
\[
\vec{p}_{1,2}' = \pm \bar{p} \begin{pmatrix} \cos \bar{\psi} \\ \sin \bar{\psi} \cos \bar{\phi} \\ \sin \bar{\psi} \sin \bar{\phi} \end{pmatrix} = \pm p \sin \chi \begin{pmatrix} \cos \bar{\psi} \\ \sin \bar{\psi} \cos \bar{\phi} \\ \sin \bar{\psi} \sin \bar{\phi} \end{pmatrix}
\]
(15)
The inverse Lorentz transformation gives the two rotated momenta in the laboratory system:
\[
\vec{p}_{1,2}' = \begin{pmatrix} \gamma (\mp \bar{p}_j' + \beta \tilde{E}/c) \\ \pm \bar{p}_k' \\ \pm \bar{p}_l' \end{pmatrix} = \begin{pmatrix} \pm \gamma \sin \chi \cos \bar{\psi} + \cos \chi \\ \pm \sin \bar{\psi} \cos \bar{\phi} \\ \pm \sin \bar{\psi} \sin \bar{\phi} \end{pmatrix} \approx \begin{pmatrix} \gamma \sin \chi \cos \bar{\psi} + \cos \chi \\ \sin \bar{\psi} \cos \bar{\phi} \\ \sin \bar{\psi} \sin \bar{\phi} \end{pmatrix}
\]
(16)
The change of the momenta due to the collision is
\[
\vec{p}_{1,2}' - \vec{p}_{1,2} = \pm p \sin \chi \begin{pmatrix} \gamma \sin \bar{\psi} \\ \sin \bar{\psi} \cos \bar{\phi} \\ \sin \bar{\psi} \sin \bar{\phi} \end{pmatrix} \approx \pm p \sin \chi \begin{pmatrix} \gamma \sin \bar{\psi} \\ \sin \bar{\psi} \cos \bar{\phi} \\ \sin \bar{\psi} \sin \bar{\phi} \end{pmatrix}
\]
(17)
With \( p_{1,2}' = \sqrt{p_{j1,2}'^2 + p_{k1,2}'^2 + p_{l1,2}'^2} \approx p_{j1,2}' + (p_{k1,2}'^2 + p_{l1,2}'^2)/(2p_{j1,2}') \) we get finally
\[
\vec{p}_{1,2}' - \vec{p}_{1,2} \approx \vec{p}_{j1,2}' - p_{j1,2} \\ \approx \pm p \gamma \chi \cos \bar{\psi}
\]
(18)

3. Scattering cross section

The probability for scattering of one of the two colliding particles into the solid angle \( d\Omega \) is given by the Møller scattering cross-section in the c. o. m. system [8]:
\[
d\sigma = \frac{r_p^2}{4\bar{\gamma}^2} \left( 1 + \frac{1}{\beta^2} \right)^2 \left( \frac{4}{\sin^4 \Phi} - \frac{3}{\sin^2 \Phi} + \frac{4}{\sin^2 \Phi} + 1 \right) d\Omega
\]
(19)

\( r_p \) is the classical particle radius, \( \bar{\gamma} \) is \( \gamma / \gamma_t \), and \( \bar{\Phi} \) is the angle between the momenta before and after the collision, i.e. between the \( \bar{l} \)-axis and \( \vec{p}_1' \). In order to determine all collisions with an energy change larger than the maximum stable energy deviation we have to integrate over \( \bar{\psi} \) and \( \bar{\phi} \) with the conditions
\[
0 \leq \bar{\phi} \leq 2\pi, \quad 0 \leq \bar{\psi} \leq \bar{\psi}_m
\]
where \( \bar{\psi}_m \) is given by the maximum stable momentum deviation \( \Delta p_m \). If one of the two particles is scattered into the region \( \pi - \bar{\psi}_m \leq \bar{\psi} \leq \pi \) the other one is scattered into \( 0 \leq \bar{\psi} \leq \bar{\psi}_m \) so that both regions are included. With Eq.(18) we obtain for \( \Delta p_m \)
\[ \Delta p_m = |p_{1,2}' - p_{1,2}| = p \gamma t \cos \bar{\psi}_m \]

or

\[ \cos \bar{\psi}_m = \frac{\delta_m}{\gamma t \chi} = \frac{\delta_m \sqrt{1 + \beta^2 \gamma^2 \chi^2}}{\gamma \chi} \leq 1 \quad (20) \]

with

\[ \delta_m = \frac{\Delta p_m}{p} \]

Eq. (20) gives also the condition

\[ \chi^2 \geq \chi_m^2 = \frac{\delta_m^2}{\gamma^2 (1 - \beta^2 \delta_m^2)} \approx \frac{\delta_m^2}{\gamma^2} \quad (21) \]

With \( d\Omega = \sin \bar{\psi} d\bar{\phi} d\bar{\psi} \) one gets for the total cross section

\[ \bar{\sigma} = \frac{r_p^2}{4 \gamma^2} \int_0^{\bar{\psi}_m} \int_0^{2\pi} \left( (1 + \frac{1}{\beta^2})^2 \left( \frac{4}{\sin^4 \Phi} - \frac{3}{\sin^2 \Phi} \right) + \frac{4}{\sin^2 \Phi} + 1 \right) \sin \bar{\psi} d\bar{\phi} d\bar{\psi} \quad (22) \]

With the relation

\[ \bar{\rho}' = \bar{\rho} \cos \bar{\Phi} = \bar{\rho} \sin \bar{\psi} \cos \bar{\phi} \]

which follows from the definitions of the angles \( \bar{\Phi}, \bar{\psi}, \) and \( \bar{\phi}, \) one obtains

\[ \sin^2 \Phi = \sin^2 \bar{\phi} + \cos^2 \bar{\psi} \cos^2 \bar{\phi} \]

Substituting \( \tan \bar{\phi} = |\cos \bar{\psi}| \times \tan u \) and integrating with respect to \( u \) and \( \bar{\psi} \) gives

\[ \bar{\sigma} = \frac{\pi r_p^2}{2 \gamma^2} \int_0^{\bar{\psi}_m} \left( (1 + \frac{1}{\beta^2})^2 \left( \frac{2 (1 + \cos^2 \bar{\psi})}{\cos^3 \bar{\psi}} - \frac{3}{\cos \bar{\psi}} \right) + \frac{4}{\cos \bar{\psi}} + 1 \right) \sin \bar{\psi} d\bar{\psi} \]

\[ = \frac{\pi r_p^2 \gamma t}{2 \gamma^2} \left( 3 - \frac{2}{\beta^2} - \frac{1}{\beta^4} \right) \ln \frac{\gamma t \chi}{\delta_m} + \left( 1 + \frac{1}{\beta^2} \right) \frac{2 \gamma^2 \chi^2 - \delta_m^2}{\delta_m^2} + 1 - \frac{\delta_m}{\gamma t \chi} \right) \quad (23) \]

where \( \bar{\beta} \) is given by

\[ \bar{\beta}^2 = \frac{\bar{p}'^2 c^2}{E^2} = \gamma^2 \beta^2 \chi^2 = \frac{\beta^2 \gamma^2 \chi^2}{1 + \beta^2 \gamma^2 \chi^2} \approx \frac{\beta^2 \gamma^2 \chi^2}{1 + \gamma^2 \chi^2} \quad (24) \]

since \( \chi^2 \ll 1 \). The cross section \( \bar{\sigma} \) is parallel to \( \bar{p}_1 + \bar{p}_2 \), i.e. parallel to the \( \bar{\ell} \)-axis, and it is transformed into the laboratory system by:

\[ \sigma = \frac{\bar{\sigma}}{\gamma t} \quad (25) \]
4. Loss rate

We want to calculate the number of collisions per unit time which lead to the loss of both particles. We shall integrate, therefore, the cross section with respect to all positions and angles of the two colliding particles satisfying the condition Eq.(21). The distribution of the positions and angles of electrons and positrons is, in good approximation, a Gaussian distribution:

$$P_{\beta}(x_{\beta}, x'_{\beta}, z_{\beta}, z'_{\beta}) = \frac{\beta_{x_{\beta}}}{4\pi^2\sigma_{x_{\beta}}^2\sigma_{z_{\beta}}^2} \exp\left\{ -\frac{x_{\beta}^2 + (\alpha_{x_{\beta}}/\beta_{x_{\beta}})^2 + z_{\beta}^2 + (\alpha_{z_{\beta}}/\beta_{z_{\beta}})^2}{2\sigma_{x_{\beta}}^2} \right\}$$

with $\alpha_{x_{\beta}} = -\beta_{x_{\beta}}/2$. $\sigma_{x_{\beta}}$ and $\sigma_{z_{\beta}}$ are the standard deviations for the horizontal and vertical betatron distribution. The distribution of the synchrotron coordinates $\Delta s$ and $\Delta p$ is given by

$$P_s(\Delta s, \Delta p) = \frac{1}{2\pi\sigma_s\sigma_p} \exp\left\{ -\frac{\Delta s^2}{2\sigma_s^2} - \frac{\Delta p^2}{2\sigma_p^2} \right\}$$

where $\sigma_p$ and $\sigma_s$ determine the relative momentum spread and the bunch length, respectively.

The number of scattering events per unit time for a single particle moving with an angle of $2\chi$ with respect to the momentum of the opposing particle is $\beta_{rel}cP_o\sigma$ where the relative velocity $\beta_{rel}c$ between the two colliding particles is $2\beta c \sin \chi \approx 2\beta c \chi$. $P_o$ is the spatial density in the laboratory system. The total number of scattering events per unit time is obtained by integrating over all positions and angles of all particles with the condition Eq.(21):

$$R = 2\beta c \int Ps(\Delta s, \Delta p) \sigma \ dV$$

$P$ and $dV$ are given by

$$P = N_p^2 P_s(\Delta s_1, \Delta p_1) P_s(\Delta s_2, \Delta p_2) P_{\beta}(x_{\beta_1}, x'_{\beta_1}, z_{\beta_1}, z'_{\beta_1}) P_{\beta}(x_{\beta_2}, x'_{\beta_2}, z_{\beta_2}, z'_{\beta_2})$$

and

$$dV = d\Delta s_1 dx_{\beta_1} dz_{\beta_1} d\Delta p_1 d\Delta p_2 dx'_{\beta_1} dx'_{\beta_2} dz'_{\beta_1} dz'_{\beta_2}$$

$N_p$ is the number of particles per bunch. When averaging the position coordinates over the whole beam we assume that the two colliding particles have always the same position, i.e. we take into account the following conditions:

$$\Delta s_1 = \Delta s_2, \quad x_{\beta_1} + D_x \frac{\Delta p_1}{p} = x_{\beta_2} + D_x \frac{\Delta p_2}{p}, \quad z_{\beta_1} + D_z \frac{\Delta p_1}{p} = z_{\beta_2} + D_z \frac{\Delta p_2}{p}$$

The integration in Eq.(28) is carried out in Appendix A2 and yields ($\tau = \beta^2\gamma^2\chi^2$)

$$R = \frac{r_p^2 c^5 \beta_x \beta_z \sigma_h N_p^2}{8\sqrt{\pi} \beta^2 \gamma^4 \sigma_{x_{\beta}}^2 \sigma_{z_{\beta}}^2 \sigma_p^2} \int_{\tau_m}^\infty \left( (2 + \frac{1}{\tau})^2 \frac{\tau}{\tau_m} - 1 \right) - \frac{\sqrt{1 + \tau}}{\sqrt{\tau/\tau_m}}$$

$$- \frac{1}{2\tau} \left( 4 + \frac{1}{\tau} \right) \ln \left( \frac{\tau/\tau_m}{1 + \tau} \right) e^{-B_1\tau} I_o(B_2\tau) \frac{\sqrt{\tau} d\tau}{\sqrt{1 + \tau}}$$

(31)
$I_o$ is the modified Bessel function and the other quantities are given by

$$
\frac{1}{\sigma_h^2} = \frac{1}{\sigma_p^2} + \frac{D_x^2 + \tilde{D}_x^2}{\sigma_{x\beta}^2} + \frac{D_z^2 + \tilde{D}_z^2}{\sigma_{z\beta}^2}
$$

$$
= \frac{1}{\sigma_p^2 \sigma_{x\beta}^2 \sigma_{z\beta}^2}(\sigma_{x\beta}^2 \sigma_{x\beta}^2 + \sigma_{z\beta}^2 \sigma_{z\beta}^2 - \sigma_{x\beta}^2 \sigma_{z\beta}^2)
$$

$$
B_1 = \frac{\beta_x^2}{2\beta^2 \gamma^2 \sigma_{x\beta}^2}(1 - \frac{\sigma_h^2 \tilde{D}_x^2}{\sigma_{x\beta}^2}) + \frac{\beta_z^2}{2\beta^2 \gamma^2 \sigma_{z\beta}^2}(1 - \frac{\sigma_h^2 \tilde{D}_z^2}{\sigma_{z\beta}^2})
$$

$$
B_2 = \frac{1}{4\beta^4 \gamma^4} \left( \frac{\beta_x^2}{\sigma_{x\beta}^2} (1 - \frac{\sigma_h^2 \tilde{D}_x^2}{\sigma_{x\beta}^2}) - \frac{\beta_z^2}{\sigma_{z\beta}^2} (1 - \frac{\sigma_h^2 \tilde{D}_z^2}{\sigma_{z\beta}^2}) \right)^2 + \frac{\sigma_h^2 \beta_x^2 \beta_z^2 \tilde{D}_x^2 \tilde{D}_z^2}{\beta^4 \gamma^4 \sigma_{x\beta}^4 \sigma_{z\beta}^4}
$$

$$
\tau_m = \beta^2 \delta_m^2
$$

In order to simplify the representation we have introduced

$$
\tilde{D}_{x,z} = \alpha_{x,z} D_{x,z} + \beta_{x,z} D'_{x,z}
$$

and

$$
\tilde{\sigma}_{x,z}^2 = \sigma_{x,z}^2 + \sigma_p^2 \tilde{D}_{x,z}^2 = \sigma_{x\beta,z\beta}^2 + \sigma_p^2 (D_{x,z}^2 + \tilde{D}_{x,z}^2)
$$

6. Touschek lifetime

The number of particles lost per unit time is given by

$$
\frac{dN_p}{dt} = -R = -aN_p^2
$$

The fact that always two particles are lost is taken into account by averaging over both particles, 1 and 2, in Eqs.(28) to (30), so that there are always two contributions to the scattering rate which differ only by the indices 1 and 2. Integration of Eq.(38) with respect to $t$ gives

$$
-\frac{1}{N_p} = -at + \text{const.}
$$

or

$$
N_p = \frac{1}{at - \text{const.}} = \frac{N_o}{at + N_o at}
$$

where $N_o$ is the number of particles at the time $t = 0$. A lifetime $T_\ell$ can be defined by

$$
\frac{1}{T_\ell} = \langle a N_o \rangle = \langle \frac{R}{N_o} \rangle
$$
after that the number of particles drops to half the initial number. The brackets denote the average over the whole circumference of the storage ring. Using the same convention as other calculations we write

$$ \frac{1}{T_\ell} = \left\langle \frac{r_p e N_p}{8 \pi \gamma^2 \sigma_s \sqrt{\sigma_x^2 \sigma_z^2 - \sigma_p^4 D_x D_z}} F(\tau_m, B_1, B_2) \right\rangle $$ \tag{41}

with

$$ F(\tau_m, B_1, B_2) = \sqrt{\pi (B_1^2 - B_2^2)} \tau_m \int_{\tau_m}^{\infty} \left( \left( 2 + \frac{1}{\tau} \right)^2 \left( \frac{\tau}{\tau_m} - 1 \right) - 1 - \sqrt{1 + \tau} \right) - \frac{1}{2 \tau} \left( 4 + \frac{1}{\tau} \right) \ln \frac{\tau}{\tau_m} \frac{1}{1 + \tau} e^{-B_1 \tau} I_0(B_2 \tau) \sqrt{1 + \tau} d\tau \tag{42} $$

where $B_1^2 - B_2^2$ is given by Eq.(34). A faster numerical integration is achieved by substituting $\tau = \tan^2 \kappa$, $\tau_m = \tan^2 \kappa_m$:

$$ F(\tau_m, B_1, B_2) = 2 \sqrt{\pi (B_1^2 - B_2^2)} \tau_m \int_{\kappa_m}^{\pi/2} \left( (2 \tau + 1)^2 \left( \frac{\tau}{\tau_m} - 1 \right) - 1 - \sqrt{1 + \tau} \right) - \frac{1}{2 \tau} \left( 4 + \frac{1}{\tau} \right) \ln \frac{\tau}{\tau_m} \frac{1}{1 + \tau} e^{-B_1 \tau} I_0(B_2 \tau) \sqrt{1 + \tau} d\kappa $$

Eqs.(41) and (42) describe the most general case with respect to the horizontal and vertical betatron oscillation, the horizontal and vertical dispersion, and the derivatives of the amplitude functions and dispersions. Special cases with some simplifications will be discussed in the following sections.

Figures 1 to 4 show $F(\tau_m, B_1, B_2)$ as a function of $\tau_m = \beta^2 \delta_m^2 \approx \delta_m^2$ for different ratios of beam height to width and for different energies.

7. Special cases

7.1 Plane orbit

In case of a plane orbit and without coupling of the horizontal dispersion one obtains

$$ D_z = \tilde{D}_z = 0 $$ \tag{43}

and $\sigma_h$ is given by (Eq.(32))

$$ \frac{1}{\sigma_h^2} = \frac{1}{\sigma_p^2} + \frac{D_x^2 + \tilde{D}_x^2}{\sigma_x^2 \sigma_{x\beta}^2} = \frac{\sigma_x^2 + \tilde{\sigma}_x^2}{\sigma_p^2 \sigma_{x\beta}^2} $$

With Eqs.(33) and (34) and with $\sigma_{x\beta} = \sigma_x B_1$ and $B_2$ simplify to

$$ B_{1,2} = \frac{1}{2 \beta^2 \gamma^2} \left| \frac{\beta_x^2 \sigma_x^2}{\sigma_{x\beta}^2 \sigma_x^2} \pm \frac{\beta_x^2}{\sigma_{x\beta}^2} \right| $$ \tag{44}
and the lifetime is given by
\[
\frac{1}{T_{\ell}} = \left\langle \frac{r_p^2 c N_p}{8 \pi \gamma^2 \sigma_x \sigma_y} F(\tau_m, B_1, B_2) \right\rangle \tag{45}
\]

### 7.2 Flat beam

For flat beams we assume
\[
\frac{\sigma_{x,\beta}}{\beta_z^2} \gg \frac{\sigma_{z,\beta}^2}{\beta_z^2} \tag{46}
\]
and \(D_z = \tilde{D}_z = 0\). The last condition (plane orbit) is not necessary but simplifies the calculation. Thus we obtain with Eq.(44)
\[
B_{1,2} \approx \frac{\beta_z^2}{2 \beta^2 \gamma^2 \sigma_{x,\beta}^2}
\]
and
\[
B_1 - B_2 = \frac{\beta_z^2 \sigma_x^2}{\beta^2 \gamma^2 \sigma_{x,\beta}^2 \sigma_x^2}
\]
and with Eq.(34) and \(\sigma_h = \sigma_p \sigma_{x,\beta}/\tilde{\sigma}_x\)
\[
B_1^2 - B_2^2 = \frac{\beta_z^2 \beta_z^2}{\beta^2 \gamma^2 \sigma_{z,\beta}^2 \sigma_z^2}
\]
The argument of the Bessel function at the lower integration limit (Eq.(42)) is \((\chi_m = \delta_m/\gamma)\)
\[
\tau_m B_2 = \frac{\delta_m^2}{2 \gamma} \left| \frac{\beta_z^2 \sigma_x^2}{\sigma_{x,\beta}^2} - \frac{\beta_z^2}{\sigma_{z,\beta}^2} \right| \approx \frac{\delta_m^2 \beta_z^2}{2 \gamma \sigma_{z,\beta}^2}
\]
If this value is large as compared to one we have \(\gamma \sigma_{z,\beta}^2/\beta_z^2 \ll \delta_m^2\). This means that the change \(\gamma \sigma_{z,\beta}^2/\beta_z^2\) of the longitudinal momentum due to the initial vertical momentum \(\sigma_{z,\beta}/\beta_z\) is smaller than the maximum stable momentum deviation \(\delta_m\) so that the vertical momentum does not contribute to the loss rate. The Bessel function can then be written approximately as \([11]\)
\[
e^{-\tau B_1 I_0(\tau B_2)} \approx \frac{1}{\sqrt{2 \pi \tau B_2}} e^{-\tau(B_1 - B_2)} = \frac{\beta \gamma \sigma_{z,\beta}}{\sqrt{\pi \tau \beta_z}} \exp\left\{-\frac{\tau \beta_z^2 \sigma_x^2}{\beta^2 \gamma^2 \sigma_{z,\beta}^2 \sigma_x^2}\right\} \tag{47}
\]
The lifetime follows with Eqs.(31) and (47) to
\[
\frac{1}{T_{\ell}} = \left\langle \frac{r_p^2 c \beta_z N_p}{8 \pi \gamma \sigma_x} \right\rangle \int_{\tau_m}^{\infty} \left(\frac{2 - \frac{1}{\tau}}{1 + \tau} \right)^2 \left(\frac{\tau}{\tau_m} - 1\right) + 1 - \frac{\sqrt{1 + \tau}}{\tau / \tau_m} \right) e^{-\frac{\tau \epsilon_m}{\tau_m}} \exp\left\{-\frac{\tau \epsilon_m}{\tau_m} \right\} \frac{d\tau}{\sqrt{1 + \tau}} \tag{48}
\]
and \( F(\tau_m, B_1, B_2) \) (Eq.(42)) is given by

\[
F(\tau_m, B_1, B_2) = \sqrt{\frac{B_1^2 - B_2^2}{2B_2}} \tau_m \int_{\tau_m}^{\infty} \left( \left( 2 + \frac{1}{\tau} \right)^2 \left( \frac{\tau}{\tau_m} \right) - 1 \right) + 1 - \frac{1}{\tau/\tau_m} \\
- \frac{1}{2\tau} \left( 4 + \frac{1}{\tau} \right) \ln \frac{\tau/\tau_m}{1+\tau} e^{-\tau(B_1-B_2)} d\tau
\]

\[
= \sqrt{\epsilon_m} \tau_m \int_{\tau_m}^{\infty} \left( \left( 2 + \frac{1}{\tau} \right)^2 \left( \frac{\tau}{\tau_m} \right) - 1 \right) + 1 - \frac{1}{\tau/\tau_m} \\
- \frac{1}{2\tau} \left( 4 + \frac{1}{\tau} \right) \ln \frac{\tau/\tau_m}{1+\tau} \exp\left\{-\frac{\tau\epsilon_m}{\tau_m} \right\} d\tau
\]

where \( \epsilon_m \) is a generalized expression already used in other investigations:

\[
\epsilon_m = \frac{\delta_2 \beta^2 \sigma_x^2}{\gamma^2 \sigma_{x\beta}^2 \sigma_x^2} = \frac{\delta_2 \beta^2 \sigma_x^2}{\gamma^2 \sigma_{x\beta}^2 (\sigma_x^2 + \sigma_p^2 D_0^2)} \quad (50)
\]

7.3 Round beam

Here we assume again \( D_z = \tilde{D}_z = 0 \). Then we can define the round beam by

\[
\frac{\beta_x \sigma_x}{\sigma_{x\beta} \sigma_x} = \frac{\beta_x}{\sigma_{x\beta}} \quad (51)
\]

and \( B_1 \) and \( B_2 \) are given by

\[
B_1 = \frac{\beta_x^2 \sigma_x^2}{\beta^2 \gamma^2 \sigma_{x\beta}^2 \sigma_x^2} = \frac{\epsilon_m}{\tau_m}, \quad B_2 = 0
\]

\( F(\tau_m, B_1, B_2) \) simplifies to

\[
F(\tau_m, B_1, 0) = \sqrt{\pi} B_1 \tau_m \int_{\tau_m}^{\infty} \left( \left( 2 + \frac{1}{\tau} \right)^2 \left( \frac{\tau}{\tau_m} \right) - 1 \right) + 1 - \frac{1}{\tau/\tau_m} \\
- \frac{1}{2\tau} \left( 4 + \frac{1}{\tau} \right) \ln \frac{\tau/\tau_m}{1+\tau} \exp\left\{-\tau B_1 \right\} \frac{\sqrt{\tau} d\tau}{\sqrt{1+\tau}} \quad (52)
\]

With help of the integral [11]

\[
\int_{0}^{\infty} e^{-pr} \frac{d\tau}{\sqrt{\tau(\tau+a)}} = \exp\{-ap/2\} K_0(ap/2) \quad (53)
\]

where \( K_0 \) is a Bessel function, the integral in Eq.(52) can be solved exactly except for the terms with \( \ln(\tau/(1+\tau)) \). Then one gets for small \( \tau_m \):

\[
F(\tau_m, B_1, 0) = \sqrt{\pi} B_1 (2 + B_1) (K_1 - K_0)e^{B_1^2/2} + 0(\sqrt{\tau_m}) \quad (54)
\]
where $K_1$ and $K_o$ are Bessel functions with the argument $B_1/2$.

Thus for a round beam $F$ becomes constant with decreasing $\tau_m$. This is different in the case of a flat beam where $F$ increases continuously with decreasing $\tau_m$. The reason for the different behaviour is the different distribution of the transverse momentum. When the maximum stable energy deviation $\delta_m$ is decreased by $-d\delta_m$ more particles will be lost having the smaller transverse momentum difference $\Delta p_\perp-d\Delta p_\perp$. In a one-dimensional distribution the number of particles which can be scattered additionally is proportional to $2d\Delta p_\perp$, but in a two-dimensional distribution it is proportional to $2\pi\Delta p_\perp d\Delta p_\perp$ with $\Delta^2 p_\perp = \Delta^2 p_x + \Delta^2 p_z$. Thus with decreasing $\Delta p_\perp$ the increase in loss rate is larger in a one-dimensional distribution, i.e. in a flat beam.

### 7.4 Non-relativistic case

The non-relativistic case is defined by $\bar{\beta}^2 \ll 1$, i.e. (Eq.(24))

$$\beta^2 \gamma^2 \chi^2 \ll 1$$

which means also $\gamma^2 \chi^2 \ll 1$ since $\chi^2 \ll 1$. $\bar{\psi}_m$ is given by

$$\cos \bar{\psi}_m = \frac{\delta_m}{\gamma \chi} = \frac{\delta_m}{\gamma \chi}$$

since $\gamma_\ell = \gamma$ (Eq.(14)). One obtains for the cross sections with Eq.(23)

$$\bar{\sigma} = \frac{\pi r_p^2}{2 \beta^4} \left( \frac{1}{\cos^2 \bar{\psi}_m} - 1 + \ln \cos \bar{\psi}_m \right)$$

$$= \frac{\pi r_p^2}{2 \beta^4 \gamma^4 \chi^4} \left( \gamma^2 \chi^2 - 1 + \ln \frac{\delta_m}{\gamma \chi} \right)$$

(56)

and with $\chi^2 = \tau/(\beta^2 \gamma^2)$

$$\sigma = \frac{\pi r_p^2}{2 \gamma \tau^2} \left( \frac{\tau}{\beta^2 \delta_m^2} - 1 + \ln \frac{\beta \delta_m}{\sqrt{\tau}} \right)$$

(57)

With Eqs.(40), (55) and (A2.5) one gets for the lifetime

$$\frac{1}{T_\ell} = \left\langle \frac{c r_p^2 \beta_x \beta_z \sigma_h N_p}{8 \sqrt{\pi} \beta^2 \gamma^4 \sigma_x^2 \sigma_z^2 \sigma_s^2 \sigma_p^2} \int_{\tau_m}^\infty \left( \frac{\tau}{\tau_m} - 1 - \frac{1}{2} \ln \frac{\tau}{\tau_m} \right) e^{-\tau B_1} I_o(\tau B_2) \frac{d\tau}{\tau^{3/2}} \right\rangle$$

(58)

and

$$F(\tau_m, B_1, B_2) = \sqrt{\pi (B_1^2 - B_2^2)} \tau_m \int_{\tau_m}^\infty \left( \frac{\tau}{\tau_m} - 1 - \frac{1}{2} \ln \frac{\tau}{\tau_m} \right) e^{-\tau B_1} I_o(\tau B_2) \frac{d\tau}{\tau^{3/2}}$$

(59)

One can obtain this result also from Eq.(42) by neglecting $\tau$ as compared one. Using the same notation as [9] one can write with $D_z = D_z = 0$

$$\frac{1}{T_\ell} = \left\langle \frac{c r_p^2 \beta_x \beta_z \sigma_x N_p}{8 \sqrt{\pi} \beta^2 \gamma^5 \sigma_x^2 \sigma_z^2 \sigma_s^2 \sigma_p^2} \int_{\epsilon_m}^{\epsilon} \left( \frac{\epsilon}{\epsilon_m} - 1 + \frac{1}{2} \ln \frac{\epsilon_m}{\epsilon} \right) e^{-\epsilon G_1} I_o(\epsilon G_2) \frac{d\epsilon}{\epsilon^{3/2}} \right\rangle$$

(60)
with
\[ G_{1,2} = \frac{\beta^2 \gamma^2 \sigma_{x\beta}^2 \sigma_{z\beta}^2}{\beta_{x\beta}^2 \beta_{z\beta}^2} B_{1,2} = \frac{1}{2} \left| \frac{\sigma_{z\beta}^2}{\beta_{x\beta}^2} \pm \sigma_{x\beta}^2 \right| \]

### 7.5 Non-relativistic case for a flat beam

For the non-relativistic case one obtains with Eqs.(47) and (58) and with \( \bar{\beta}^2 = \gamma \), \( \bar{\gamma}^2 \chi^2 = \gamma / \beta^2 \), \( \chi = \sqrt{\gamma / (\beta \gamma)} \), and \( D_z = \bar{D}_z = 0 \)

\[
\frac{1}{T}\ell = \left\langle r^2 p c \frac{B_x N_p}{\beta^3 \gamma^3} \right\rangle \int_{\tau_m}^{\infty} \left( \frac{\tau}{\tau_m} - 1 + \frac{1}{2} \ln \frac{\tau_m}{\tau} \right) \exp \left\{ -\frac{\tau \epsilon_m}{\tau_m} \right\} \frac{d\tau}{\tau^2} \quad (61)
\]

and
\[
F(\tau_m, B_1, B_2) = \frac{\beta \gamma \sigma_{z\beta} \tau_m}{\beta_{z} z} \int_{\tau_m}^{\infty} \left( \frac{\tau}{\tau_m} - 1 - \frac{1}{2} \ln \frac{\tau_m}{\tau} \right) \exp \left\{ -\frac{\tau \epsilon_m}{\tau_m} \right\} \frac{d\tau}{\tau^2} \quad (62)
\]

Following existing representations (see f. e. [9, 10]) we can write \( \tau = \epsilon \tau_m / \epsilon_m, \epsilon_m / \tau_m = \beta_{x\beta}^2 \sigma_{z\beta}^2 / (\beta^2 \gamma^2 \sigma_{x\beta}^2 \sigma_{z\beta}^2) \)

\[
\frac{1}{T}\ell = \left\langle r^2 p c \frac{B_x N_p}{\beta^3 \gamma^3} \sigma_{x\beta}^2 \sigma_{z\beta}^2 \right\rangle \int_{\epsilon_m}^{\infty} \left( \frac{\epsilon}{\epsilon_m} - 1 - \frac{1}{2} \ln \frac{\epsilon}{\epsilon_m} \right) \exp \left\{ -\epsilon \right\} \frac{d\epsilon}{\epsilon^2} = \left\langle r^2 p c \frac{B_x N_p}{\beta^3 \gamma^3} \sigma_{x\beta}^2 \sigma_{z\beta}^2 \right\rangle C(\epsilon_m) \quad (63)
\]

with

\[
C(\epsilon_m) = \epsilon_m \int_{\epsilon_m}^{\infty} \left( \frac{\epsilon}{\epsilon_m} - 1 - \frac{1}{2} \ln \frac{\epsilon}{\epsilon_m} \right) e^{-\epsilon} \frac{d\epsilon}{\epsilon^2} = -3 e^{-\epsilon_m} + \int_{\epsilon_m}^{\infty} \left( 1 + \frac{3 \epsilon_m}{2} + \frac{\epsilon_m}{2} \ln \frac{\epsilon}{\epsilon_m} \right) e^{-\epsilon} \frac{d\epsilon}{\epsilon} \quad (64)
\]

or

\[
C(\epsilon_m) = \frac{\sqrt{\pi} \beta_{z} \sigma_{z\beta}}{\beta \gamma \sigma_{x\beta}} F(\tau_m, B_1, B_2)
\]

For \( \bar{D}_x = 0 \) and \( \bar{\sigma}_x = \sigma_x \) this representation was obtained in [9].

### 7.6 Ultra-relativistic case

The ultra-relativistic case is defined by

\[ \beta^2 \gamma^2 \chi^2 \gg 1 \]

which means \( \beta^2 \approx 1 \) since \( \chi^2 \ll 1 \). With Eqs.(14) and (20) one obtains

\[
\gamma_t = \frac{1}{\chi}, \quad \cos \bar{\psi}_m = \delta_m
\]
The cross section follows from Eq.(23) with \( \bar{\beta}^2 = 1 - 1/\bar{\gamma}^2 = 1 - 1/(\gamma^2 \chi^2) \approx 1 \) to

\[
\bar{\sigma} = \frac{\pi r_p^2}{2\gamma^2 \chi^2} \left( 4 \frac{1 - \delta_m^2}{\delta_m^2} + 1 - \delta_m + \frac{4}{\gamma^2 \chi^2} \ln \delta_m \right)
\]

\[
\approx \frac{2\pi r_p^2}{\gamma^2 \chi^2 \delta_m^2}
\]  

(65)

With Eqs.(25), (40) and (A2.5) and with \( \beta = 1 \) one obtains

\[
\frac{1}{T_e} = \left\langle \frac{c r_p^2 \beta_x \beta_z \sigma_h N_p}{2 \sqrt{\pi} \gamma^4 \sigma_x^2 \sigma_z^2 \delta_m^2} \left( \frac{1}{B_1^2 - B_2^2} \right) \int_{\delta_m^2}^{\infty} \exp\{-\tau B_1\} I_0(\tau B_2) d\tau \right\rangle
\]

(66)

Using a definite integral [11] we can write

\[
\frac{1}{T_e} = \left\langle \frac{c r_p^2 \beta_x \beta_z \sigma_h N_p}{2 \sqrt{\pi} \gamma^4 \sigma_x^2 \sigma_z^2 \delta_m^2} \left( \frac{1}{B_1^2 - B_2^2} \right) \int_0^{\delta_m^2} \exp\{-\tau B_1\} I_0(\tau B_2) d\tau \right\rangle
\]

\[
\approx \left\langle \frac{c r_p^2 N_p}{2 \sqrt{\pi} \gamma^2 \sigma_x \sqrt{\sigma_x^2 - \sigma_z^2 D_x D_z \delta_m^2}} \right\rangle
\]

(67)

and \( F(\tau_m, B_1, B_2) \) simplifies to

\[
F(\tau_m, B_1, B_2) = 4\sqrt{\pi}
\]

(68)

7.7 Ultra-relativistic case for a flat beam

A better approximation for a flat beam at high energy can be derived by using Eq.(49). Since \( \beta_x^2/(\gamma^2 \sigma_x^2) \ll 1 \) and \( \beta_z^2/(\gamma^2 \sigma_z^2) \gg 1 \) it follows that \( B_{1,2} \gg 1 \) but \( B_1 - B_2 \ll 1 \). Eq.(49) can then be written approximately as

\[
F(\tau_m, B_1, B_2) = \sqrt{B_1 - B_2} \int_{\tau_m}^{\infty} \left( (4 - 3 \tau_m - \tau^{3/2}/1+1/\tau) e^{-\tau(B_1 - B_2)} \right.
\]

\[
- \frac{4}{1+\tau} + \left( \frac{4}{\tau} + \frac{1}{\tau^2} \right) \left( \frac{\tau}{1+\tau} - \tau_m - \frac{\tau_m}{2} \ln \frac{\tau}{\tau_m} \right) \frac{d\tau}{\sqrt{1+\tau}}
\]

(69)

For small \( \tau_m \) and small \( B_1 - B_2 \) the function \( F \) simplifies to

\[
F(\tau_m, B_1, B_2) = \sqrt{\pi} \left( 4 + 2(B_1 - B_2) \right) + \sqrt{B_1 - B_2} \left( \ln \left( 4/\tau_m \right) - 11 \right)
\]

(70)

where \( B_1 - B_2 \) is given by \( \beta_x^2 \sigma^2 / (\gamma^2 \sigma_x^2 \bar{\sigma}^2) \). For \( \tau_m < 10^{-3} \) (\( \delta < 3.2\% \)) and \( B_1 - B_2 < 0.3 \) (\( \gamma > 2\beta_x/\sigma_x \)) the error is smaller than \( 8 \times 10^{-3} \).

Eq.(70) shows that for decreasing \( \tau_m \) \( F \) increases continuously, and that the gradient of the increase goes to zero with increasing energy.
7.8 Ultra-relativistic case for a round beam

In this case both, \( \beta^2_x/(\gamma^2 \sigma^2_{x\beta}) \) and \( \beta^2_z/(\gamma^2 \sigma^2_{z\beta}) \), and therefore also \( B_1 \) are small as compared to one (\( B_2 = 0 \)). With approximations for the Bessel functions \( K_0 \) and \( K_1 \) [11] and with Eqs.(52) and (54) one obtains

\[
F(\tau_m, B_1, 0) = \sqrt{\pi} \left( 4 + B_1 \left( 1.73 + 2 \ln(B_1) - 8\sqrt{\tau_m} \right) \right)
\]

(71)

where \( B_1 \) is given by \( \beta^2_x \sigma^2_{x\beta}/(\gamma^2 \sigma^2_{x\beta} \bar{\sigma}^2_x) \). For \( \tau_m < 10^{-3} \) (\( \delta < 3.2\% \)) and \( B_1 < 0.1 \) (\( \gamma > 3\beta_x/\sigma_{x\beta} \)) the error is smaller than \( 3 \times 10^{-3} \).

* * *

References

[1] C. Bernardini, G. F. Corazza, G. Di Giugno, G. Ghigo, J. Haissinski, P. Marin, R. Querzoli, and B. Touschek, Phys. Rev. Lett. 10, 407 (1963)

[2] J. Haissinski, Rapport technique interne No. 41-63, LAL Orsay (1963)

[3] B. Gittelmann, D. M. Ritson, HEPL-291, Stanford University, Stanford, 1963

[4] U. Voelkel, DESY 67/5 March 1965

[5] J. Le Duff, Proc. of the CERN Accelerator School, Berlin, 1987

[6] Y. Miyahara, Jap. Journal of Appl. Phys., Vol. 24, No. 9, p. L742(1985)

[7] A. Piwinski, Proc. 9th Int. Conf. on High Energy Accel., Stanford, 1974, p. 405

[8] W. Heitler, The Quantum Theory of Radiation, Oxford 1960

[9] H. Brueck, Accelerateurs Circulaires de Particules, PUF, Paris, 1966

[10] H. Wiedemann; Particle Accelerator Physics II, 1994

[11] I. S. Gradshteyn, I. M. Ryzhik; Tables of Integrals, Series, and Products
Appendix A1

Since in the c. o. m. system \( \bar{E}_1 = \bar{E}_2 \) Eq.(6) can be written as

\[
\bar{E}_{1,2} = \frac{(E_1 + E_2)}{2} = \gamma_t(E_1 + E_2 - \beta_t(\beta_1 E_1 \cos \chi_1 + \beta_2 E_2 \cos \chi_2))/2
\]

\[
= \gamma_t(E_1 + E_2 - \beta_t^2(1 + E_1 + E_2))/2
\]

\[
= (E_1 + E_2)/(2\gamma_t)
\]

\[
\approx E/\gamma_t
\]

(A1.1)

With Eq.(7) one gets

\[
\frac{1}{\gamma_t^2} = 1 - \frac{c^2(\bar{p}_1 + \bar{p}_2)^2}{(E_1 + E_2)^2}
\]

\[
= \frac{E_1^2 - c^2p_1^2 + E_2^2 - c^2p_2^2 + 2E_1E_2 - 2c^2p_1p_2\cos(\chi_1 + \chi_2)}{(E_1 + E_2)^2}
\]

\[
= 2 \frac{1 + \gamma_1\gamma_2 - \beta_1\beta_2\gamma_1\gamma_2 \cos(\chi_1 + \chi_2)}{\gamma_1^2 + \gamma_2^2}
\]

\[
\approx 1 + \gamma^2 - \beta^2\gamma^2(1 - 2\chi^2)
\]

\[
\approx \frac{1 + \beta^2\gamma^2\chi^2}{\gamma^2}
\]

(A1.2)

From the Lorentz transformation

\[
p_{j1,2} = \gamma_t(\bar{p}_{j1,2} + \beta_t\bar{E}_{1,2}/c)
\]

follows with Eqs.(7) and (A1.1)

\[
\bar{p}_{j1,2} = p_{j1,2}/\gamma_t - \beta_t\bar{E}_{1,2}/c
\]

\[
= p_{j1,2}/\gamma_t - \beta_t(E_1 + E_2)/(2c\gamma_t)
\]

\[
= (2p_{j1,2} - p_{j1} - p_{j2})/(2\gamma_t)
\]

\[
= \pm(p_{j1} - p_{j2})/(2\gamma_t)
\]

\[
= \pm\frac{p_{j1}^2 - p_{j2}^2}{2\gamma_t(p_{j1} + p_{j2})}
\]

With \( p_{\ell 1} = p_{\ell 2} \) (Eq.(4)) and with Eq.(A1.2) one gets

\[
\bar{p}_{j1,2} = \pm\frac{p_{j1}^2 - p_{j2}^2}{2\gamma_t(p_{j1} + p_{j2})}
\]

\[
= \pm\frac{\xi p(p_1 + p_2)}{2\gamma_t(p_{j1} + p_{j2})}
\]

\[
= \pm\frac{\xi(p_1 + p_2)}{2\gamma_t(p_{j1} + p_{j2})}\sqrt{1 + \beta^2\gamma^2\chi^2}
\]

\[
\approx \pm\xi p\sqrt{1 + \beta^2\gamma^2\chi^2}/2
\]

(A1.3)
Appendix A2

We replace the variables $x_{β,1,2}$, $x'_{β,1,2}$, $z_{β,1,2}$, $z'_{β,1,2}$, and $Δp_{1,2}$ by the variables $x_{β}$, $x'_{β}$, $z_{β}$, $z'_{β}$, $Δp$, $ξ$, $θ$, and $ζ$ with help of the relations

\[ x_{β,1,2} = x_{β} + D_{x}γξ/2, \quad z_{β,1,2} = z_{β} + D_{z}γξ/2 \]

\[ x'_{β,1,2} = x'_{β} ± θ/2 \mp D'_{x}γξ/2, \quad z'_{β,1,2} = z'_{β} ± θ/2 \mp D'_{z}γξ/2, \quad Δp_{1,2} = Δp ± pγξ/2 \]

and obtain

\[ R = \frac{2cββ_{2}β'_{2}N_{p}^{2}}{64π^{6}σ_{xβ}^{4}σ_{zβ}^{4}σ_{p}^{4}σ_{p}^{2}} \int \chiσ(χ) e^{-H} dV \]

\[ = \frac{cβγβ_{2}β'_{2}N_{p}^{2}}{32π^{6}σ_{xβ}^{4}σ_{zβ}^{4}σ_{p}^{4}σ_{p}^{2}} \int \chiσ(χ) e^{-H} dV^* \] (A2.1)

with the condition $χ^{2} \geq χ_{m}^{2} = δ^{2}_{m}/γ^{2}$ (Eq.(21)) and with

\[ H = -\frac{x_{β}^{2} + (αx_{β} + β_{x}x'_{β})^{2}}{σ_{xβ}^{2}} - \frac{z_{β}^{2} + (αz_{β} + β_{z}z'_{β})^{2}}{σ_{zβ}^{2}} - \frac{1}{Δ^{2}_{p}σ_{p}^{2}} - \frac{Δ^{2}_{s}}{σ_{s}^{2}} \]

\[ -\frac{γ^{2}ζ^{2}}{4σ_{p}^{2}} - D_{x}^{2}γζ^{2} + (D_{x}ζ - β_{ε}ζ^{2})^{2} - D_{z}^{2}γζ^{2} + (D_{z}ζ - β_{ε}ζ^{2})^{2} \]

and

\[ dV^* = dΔs_{1} dx_{β} dz_{β} dΔp dx'_{β} dz'_{β} dξ dθ dζ \]

The Jacobian of the transformation is $γp$. In Eq.(A2.1) six of the nine integrations can be performed immediately and one obtains

\[ R = \frac{cββ_{2}β'_{2}N_{p}^{2}}{32π^{6}σ_{xβ}^{4}σ_{zβ}^{4}σ_{p}^{4}σ_{p}^{2}} \int_{-∞}^{∞} \int_{-∞}^{∞} \int_{-∞}^{∞} \exp\left\{ \frac{(γζ - (β_{x}D_{x}ζ/σ_{xβ}^{2} + β_{z}D_{z}ζ/σ_{zβ}^{2})σ_{h}^{2})^{2}}{4σ_{h}^{2}} \right\} \chiσ(χ) dξ dθ dζ \] (A2.2)

with the condition $χ^{2} \geq χ_{m}^{2}$. Integration over $ξ$ gives

\[ R = \frac{cββ_{2}β'_{2}σ_{h}N_{p}^{2}}{16π^{5/2}σ_{xβ}^{2}σ_{zβ}^{2}σ_{p}^{2}σ_{p}} \int_{-∞}^{∞} \int_{-∞}^{∞} \chiσ(χ) \]

\[ \times \exp\left\{ \frac{σ_{h}^{2}}{4} (\frac{β_{x}D_{x}ζ/σ_{xβ}^{2} + β_{z}D_{z}ζ/σ_{zβ}^{2})^{2}} - \frac{β_{x}^{2}θ^{2}}{4σ_{xβ}^{2}} - \frac{β_{z}^{2}ζ^{2}}{4σ_{zβ}^{2}} \right\} dθ dζ \] (A2.3)

with the condition $θ^{2} + ζ^{2} = 4χ^{2} \geq 4χ_{m}^{2}$ for the double integral. $σ_{h}$ is given by Eq.(32). Substituting

\[ θ = \sqrt{p} cos ν, \quad ζ = \sqrt{p} sin ν, \quad dθ dζ = dρ dν/2 \]

one obtains ($χ = \sqrt{p}/2$)
With Eqs. (23) and (25) one gets (\ref{eq:31})

\[ \int_{A_2}^{A_3} \int_{\tilde{z}_2}^{\tilde{z}_1} \chi \sigma (\chi) \exp \left\{ -\rho (A_1 + A_3 \cos (2\nu) + A_4 \sin (2\nu)) \right\} d\nu d\rho \]

(A2.4)

with

\[ A_{1,3} = \frac{2}{3} \left( 1 - \frac{\sigma_h D^2}{\sigma_x^2} \right) + \frac{2}{3} \left( 1 - \frac{\sigma_h D^2}{\sigma_x^2} \right), \quad A_4 = \frac{\sigma_h^2 D^2}{4\sigma_x^2 \sigma_x^2} \]

With cos \( \phi_o = A_3/\sqrt{A_3^2 + A_4^2} \) and sin \( \phi_o = A_4/\sqrt{A_3^2 + A_4^2} \) and using [11]

\[ I_o (\rho A_2) = \frac{1}{2\pi} \int_0^{2\pi} e^{\pm \rho A_2 \cos \theta} d\theta = \frac{1}{4\pi} \int_0^{2\pi} e^{\pm \rho A_2 \cos (2\theta)} d\theta \]

where \( I_o \) is the modified Bessel function, the double integral simplifies to

\[ R = \frac{c_0 \beta s \sigma_h N_P^2}{2\pi \sqrt{2} \sigma_x^2 (1 - \sigma_h \sigma_x^2) \sigma_s \sigma_p} \int_{A_2}^{A_3} \int_{\tilde{z}_2}^{\tilde{z}_1} \chi \sigma (\chi) \exp \left\{ -\rho (A_1 - \rho A_2 \cos (2\nu + \phi_o)) \right\} d\nu d\rho \]

(A2.5)

with

\[ A_2^2 = A_3^2 + A_4^2 = \frac{1}{64} \left( \frac{\beta_x^2}{\sigma_x^2} \right) \left( 1 - \frac{\sigma_h D^2}{\sigma_x^2} \right) - \frac{\beta_x^2}{\sigma_x^2} \left( 1 - \frac{\sigma_h D^2}{\sigma_x^2} \right)^2 + \frac{\sigma_h^2 D^2}{16\sigma_x^2 \sigma_x^2} \]

(A2.6)

With Eqs. (23) and (25) one gets (\( \rho = 4\chi^2 \))

\[ R = \frac{r_0^2 c_0 \beta s \sigma_h N_P^2}{32 \sqrt{2} \gamma^2 (1 - \sigma_h \sigma_x^2 \sigma_s \sigma_p)} \int_{A_2}^{A_3} \int_{\tilde{z}_2}^{\tilde{z}_1} \left( 1 + \frac{1}{\beta_x^2} \right)^2 \left( \frac{\gamma^2 \chi^2}{\delta_m^2} - 1 \right) + 1 - \frac{\delta_m}{\gamma \chi} \]

(A2.7)

With \( \rho = 4\chi/(\beta^2 \gamma) \), \( B_{1,2} = 4A_{1,2}/(\beta^2 \gamma) \), and \( \tau_m = \beta^2 \delta_m \) one obtains finally Eq. (31).
$F(\tau_m, B_1, B_1)$

\[
\sigma_z \beta_x \sigma_x / (\sigma_x \beta_z \bar{\sigma_x}) = 0.0 \\
0.02 \\
0.05 \\
0.1 \\
0.2 \\
0.4 \\
0.6 \\
0.8 \\
1.0
\]

Figure 1: $F$ as a function of $\tau_m$ for different ratios of beam height to beam width.

$\gamma = 10^3$, $\sigma_x \beta_x \bar{\sigma_x} / (\beta_x \bar{\sigma_x}) = 10^{-3}$, $D_z = 0$, $D'_z = 0$
Figure 2: $F$ as a function of $\tau_m$ for different ratios of beam height to beam width.

$\left( \gamma = 10^4, \sigma_x \beta_x \bar{\sigma}_x / (\beta_x \sigma_x) = 10^{-3}, D_z = 0, D_z' = 0 \right)$
Figure 3: $F$ as a function of $\tau_m$ for a flat beam and different energies.

$\sigma_x \beta \bar{\chi} / (\beta_x \sigma_x) = 10^{-3}, D_z = 0, D'_z = 0$
Figure 4: \( F \) as a function of \( \tau_m \) for a round beam and different energies.

\[
\frac{\sigma_x \beta_x}{\beta_x \sigma_x} = 10^{-3}, \quad D_z = 0, \quad D'_z = 0
\]