Dirichlet and Neumann problems
for Klein-Gordon-Maxwell systems

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Abstract
This paper deals with the Klein-Gordon-Maxwell system in a bounded spatial domain. We study the existence of solutions having a specific form, namely standing waves in equilibrium with a purely electrostatic field. We prescribe Dirichlet boundary conditions on the matter field, and either Dirichlet or Neumann boundary conditions on the electric potential.

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1 Introduction
In this paper we pursue the investigation of the existence and multiplicity of solutions for a class of Klein-Gordon-Maxwell (KGM for short) systems in a bounded spatial domain.

Let us recall the general setting for KGM systems. We are concerned with a matter field $\psi$, whose free Lagrangian density is given by

$$ L_{KG} = \frac{1}{2} \left( |\partial_t \psi|^2 - |\nabla \psi|^2 - m^2 |\psi|^2 \right), $$

with $m > 0$. The field is charged and in equilibrium with its own electromagnetic field $(E, B)$, represented by means of the gauge potentials $(\phi, A)$,

$$ E = - (\nabla \phi + \partial_t A), \quad B = \nabla \times A. $$

Abelian gauge theories provide a model for the interaction: formally we replace the ordinary derivatives ($\partial_t, \nabla$) in (1) with the so-called gauge covariant derivatives

$$ (\partial_t + iq\phi, \nabla - iqA), $$

where $q$ is a nonzero coupling constant (see e.g. [10]). Since the electromagnetic field is not prescribed, the total Lagrangian density contains also the term

$$ L_M = \frac{1}{8\pi} \left( |E|^2 - |B|^2 \right). $$

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The KGM system is given by the Euler-Lagrange equations corresponding to the Lagrangian density

\[ L_{KGM} = L_{KG}(\psi, \phi, A) + L_M(\phi, A) = \]

\[ = \frac{1}{2} \left( |(\partial_t + iq\phi)\psi|^2 - |(\nabla - iqA)\psi|^2 - m^2 |\psi|^2 \right) + \]

\[ + \frac{1}{8\pi} \left( |\nabla \phi + \partial_t A|^2 - |\nabla \times A|^2 \right). \]

The study of KGM systems is usually carried out for special classes of solutions (and for some classes of lower-order nonlinear perturbations in \( L_{KG} \)). Here we consider

\[ \psi = u(x)e^{-i\omega t}, \quad \phi = \phi(x), \quad A = 0, \]

that is, a standing wave in equilibrium with a purely electrostatic field

\[ E = -\nabla \phi(x), \quad B = 0. \]

Under this ansatz, the KGM system reduces to

\[ \begin{cases} 
-\Delta u - (q\phi - \omega)^2 u + m^2 u = 0, \\
\Delta \phi = 4\pi q (q\phi - \omega) u^2
\end{cases} \quad \text{(S0)} \]

(see [1] where the set of equations in the general case has been derived).

We will study (S0) in a bounded domain \( \Omega \subset \mathbb{R}^3 \) with smooth boundary \( \partial\Omega \). The unknowns are the real functions \( u \) and \( \phi \) defined on \( \Omega \) and the frequency \( \omega \in \mathbb{R} \). We are interested in finding nontrivial solutions, that is, solutions such that \( u \neq 0 \). We shall consider two different boundary conditions, specifically,

- either Dirichlet boundary conditions
  \[ \begin{cases} 
u = h \\
\phi = \zeta
\end{cases} \quad \text{on } \partial\Omega, \quad \text{(D0)} \]

- or “mixed” boundary conditions
  \[ \begin{cases} 
u = h \\
\frac{\partial \phi}{\partial n} = \theta
\end{cases} \quad \text{on } \partial\Omega, \quad \text{(M0)} \]

that is, Dirichlet boundary conditions on \( u \) and Neumann boundary conditions on \( \phi \),

where \( h, \zeta \) and \( \theta \) are smooth functions defined on the boundary \( \partial\Omega \).

With \( q \neq 0 \), the change of variables

\[ \begin{cases} 
u_q = \sqrt{4\pi q} u, \\
\phi_q = q\phi - \omega
\end{cases} \quad \text{(2)} \]

transforms the system (S0) and the boundary conditions (D0) and (M0) into

\[ \begin{cases} 
-\Delta u_q - \phi_q^2 u_q + m^2 u_q = 0, \\
\Delta \phi_q = \phi_q u_q^2
\end{cases} \quad \text{(S1)} \]
\[
\begin{align*}
\begin{cases}
u_q = \sqrt{4\pi q} h \\
\phi_q = q \zeta - \omega
\end{cases} & \quad \text{on } \partial \Omega, \\
\begin{cases}
\nu_q = \sqrt{4\pi q} h \\
\frac{\partial \phi_q}{\partial n} = q \theta
\end{cases} & \quad \text{on } \partial \Omega,
\end{align*}
\]
respectively.

First we study problem (S1)-(D1). Let \( \{\lambda_k\} \) denote the eigenvalues of \( -\Delta \) with homogeneous Dirichlet boundary conditions.

**Theorem 1** Assume
\[
\|q \zeta - \omega\|_2^2 < m^2 + \lambda_1.
\] (3)

1. If \( h \neq 0 \), the problem (S1)-(D1) has a nontrivial solution.
2. If \( h = 0 \), the problem (S1)-(D1) has no nontrivial solutions.

It is immediately seen that (S1)-(D1) has a trivial solution if and only if \( h = 0 \). Hence from Theorem 1 we deduce the following

**Corollary 2** Assume (3). Then problem (S1)-(D1) has a solution. This solution is trivial if and only if \( h = 0 \).

**Remark 3** If we assume
\[
\omega^2 < m^2 + \lambda_1,
\] (4)
then (3) is satisfied whenever \( |q| \) is sufficiently small. Then it is interesting to study the limit case \( q = 0 \).

Since the change of variables (2) is not allowed for \( q = 0 \), we consider the “original” problem (S0)-(D0). Being uncoupled, it can be split into
\[
\begin{cases}
-\Delta u - (\omega^2 - m^2) u = 0, \\
u = h
\end{cases} \quad \text{on } \partial \Omega
\] (5)
and
\[
\begin{cases}
\Delta \phi = 0, \\
\phi = \zeta
\end{cases} \quad \text{on } \partial \Omega.
\] (6)

Problem (6) has a unique solution (independent of \( u \)). The existence and uniqueness of solutions of problem (5) depend on the value \( \omega^2 - m^2 \). If \( \omega^2 - m^2 \neq \lambda_k \) (in particular if (4) holds), then problem (5) has a unique solution; this solution is nontrivial if and only if \( h \neq 0 \). Hence, at least in the case (4), the existence of a (nontrivial) solution depends continuously on \( q \), as \( q \to 0 \).

Now we address problem (S0)-(M1).

First we notice that, after the change of variables (2), which is valid for \( q \neq 0 \), the system does not depend on the frequency \( \omega \). Hence, for any \( \omega \in \mathbb{R} \), the existence of a standing wave \( \psi = u(x)e^{-i\omega t} \) in equilibrium with a purely electrostatic field is equivalent to the existence of a static matter field \( \psi = u(x) \), in equilibrium with the same electric field.

**Theorem 4** If \( \|q\| \|\theta\|_{H^{1/2}(\partial \Omega)} \) is sufficiently small and \( h \neq 0 \), then there exists a nontrivial solution of (S0)-(M1).
It is easily seen that the problem (S1)-(M1) has infinitely many trivial solutions when $h = 0$ and $\int_{\partial \Omega} \theta \, d\sigma = 0$.

The case $h \neq 0$ is covered by Theorem 4.

In [9, Theorem 1.1] we have shown that, if $|q| \|\theta\|_{H^{1/2}(\partial \Omega)}$ is sufficiently small, $h = 0$ and $\int_{\partial \Omega} \theta \, d\sigma \neq 0$, then problem (S1)-(M1) has a nontrivial solution.

Taking into account the Neumann boundary condition on $\phi_q$, the second equation of (S1) gives the following necessary condition

$$\int_{\Omega} \phi_q u_q^2 \, dx = q \int_{\partial \Omega} \theta \, d\sigma.$$ 

Hence, if (S1)-(M1) has trivial solutions (i.e. $u_q = 0$), then we have necessarily $\int_{\partial \Omega} \theta \, d\sigma = 0$ and, of course, $h = 0$. Vice versa, in the same paper [9, Theorem 1.1] we have shown that, if $|q| \|\theta\|_{H^{1/2}(\partial \Omega)}$ is sufficiently small, the joint conditions $h = 0$ and $\int_{\partial \Omega} \theta \, d\sigma = 0$ force $u_q$ to be 0.

All these results are summarized in the following statement.

**Theorem 5** If $|q| \|\theta\|_{H^{1/2}(\partial \Omega)}$ is sufficiently small, then problem (S1)-(M1) has a solution. This solution is trivial if and only if $h = 0$ and $\int_{\partial \Omega} \theta \, d\sigma = 0$.

**Remark 6** Under the boundary conditions (M0), the existence of solutions (trivial or nontrivial) of (S0) is not continuous with respect to $q \to 0$ in the following sense. If we fix boundary data such that $\int_{\partial \Omega} \theta \, d\sigma \neq 0$, Theorem 4 gives (via (2)) a nontrivial solution of (S0)-(M0) for all $q \neq 0$ sufficiently small. However the limit problem has no solutions. Indeed, for $q = 0$, (S0) decouples into (5) and

$$\begin{align*}
\Delta \phi &= 0, \\
\frac{\partial \phi}{\partial n} &= \theta \quad \text{on } \partial \Omega,
\end{align*}$$

and the latter system has a solution if and only if $\int_{\partial \Omega} \theta \, d\sigma = 0$. Moreover, unlike the case $q \neq 0$, the limit problem depends on $\omega$.

From now on, for the sake of simplicity, we shall omit the subscript $q$.

As we said before, we can consider a nonlinear lower order term $g(x,u)$ in the first equation of (S1). Thus we obtain the system

$$\begin{align*}
-\Delta u - \phi^2 u + m^2 u - g(x,u) &= 0, \\
\Delta \phi &= \phi u^2.
\end{align*}$$

(S2)

We assume that $g$ behaves like $|u|^{p-2} u$ with $p \in (2,6)$. More precisely, $g \in C(\bar{\Omega} \times \mathbf{R}, \mathbf{R})$ satisfies the following well-known Ambrosetti-Rabinowitz conditions (see e.g. [12]):

(g1) there exist $a_1, a_2 \geq 0$ and $p \in (2,6)$ such that

$$|g(x,t)| \leq a_1 + a_2 |t|^{p-1};$$

(g2) $g(x,t) = o(|t|)$ as $t \to 0$ uniformly in $x$;

(g3) there exist $s \in (2,p]$ and $r \geq 0$ such that

$$0 < sG(x,t) \leq tg(x,t),$$
for every $|t| \geq r$, where

$$G(x, t) = \int_0^t g(x, \tau) \, d\tau.$$ 

**Theorem 7** Let us consider the system (S2) with boundary conditions

\[
\begin{align*}
  u &= 0 \\
  \phi &= q\zeta - \omega \\
\end{align*}
\]  

on $\partial \Omega$. \hspace{1cm} (D2)

1. If (3) holds, then there exists a nontrivial solution.

2. If $g$ is odd, for every $m, \omega, q$, there exist infinitely many solutions $(u_i, \phi_i) \in H_0^1(\Omega) \times H^1(\Omega)$ with $\|\nabla u_i\|_2 \to +\infty$ and $\{\phi_i\}$ bounded in $L^\infty(\Omega)$.

The system (S2) with mixed boundary conditions has been studied in [9, Theorem 1.3]. For the sake of completeness, we quote the statement.

**Theorem 8** Let us consider the system (S2) with boundary conditions

\[
\begin{align*}
  u &= 0 \\
  \frac{\partial \phi}{\partial n} &= q\theta \\
\end{align*}
\]  

on $\partial \Omega$. Assume that $\int_{\partial \Omega} \theta \, d\sigma = 0$.

1. If $q\|\theta\|_{H^{1/2}(\partial \Omega)}$ is sufficiently small, then there exists a nontrivial solution.

2. If $g$ is odd, for every $m, \omega, q$, there exist infinitely many solutions $(u_i, \phi_i) \in H_0^1(\Omega) \times H^1(\Omega)$, with $\|\nabla u_i\|_2 \to +\infty$ and $\{\phi_i\}$ bounded in $L^\infty(\Omega)$.

We conclude this section by recalling some results which have motivated our research. The application of global variational methods to the study of KGM systems started with the pioneering paper of Benci and Fortunato [1]. They studied the nonlinear KGM system

\[
\begin{align*}
  -\Delta u - (q\phi - \omega)^2 u + m^2 u &= |u|^{p-2} u, \\
  \Delta \phi &= 4\pi q (q\phi - \omega)^2 u^2 \\
\end{align*}
\]  

in the whole space $\mathbb{R}^3$. They proved the existence of infinitely many solutions if $p \in (4, 6)$ and $\omega^2 < m^2$. Then D’Aprile and Mugnai [6] proved two interesting nonexistence results: the “linear” system (S0) has no nontrivial finite energy solution in $\mathbb{R}^3$; the “nonlinear” system (7) has no nontrivial finite energy solutions in $\mathbb{R}^3$ if $p \notin [2, 6]$. The existence result in [1] has been generalized in [5] and then in [2]. In [2], Benci and Fortunato prove the existence of solutions of (7) when $p \in (2, 6)$ and

$$|\omega| < m \sqrt{\min \left(1, \frac{p-2}{2}\right)}.$$ 

Further results on this topic are contained in [4, 8].
On the other hand, the lower-order term $|u|^{p-2}u$ in (7) is not suitable for physical models since

$$W(u) = \frac{1}{2} m^2 u^2 - \frac{1}{p} |u|^p$$

is not positive and the conservation of the energy does not guarantee global existence for the initial value problems. Some recent papers (\cite{3, 11}) are concerned with systems

$$\begin{cases}
-\Delta u - (q\phi - \omega)^2 u + m^2 u = G'(u), \\
\Delta \phi = 4\pi q (q\phi - \omega) u^2,
\end{cases}$$

where

$$W(u) = \frac{1}{2} m^2 u^2 - G(u) \geq 0.$$  \hspace{1cm} (8)

In \cite{3} it is shown that there exist solutions of (8) if the coupling constant $q$ is sufficiently small. It is easy to see that our Theorems 1 and 4 (concerning the system (S0)) are consistent with this kind of results.

Lastly we recall that Benci and Fortunato have introduced a different class of solutions of the KGM system: three-dimensional vortices, i.e. solutions such that the matter field $\psi$ has nontrivial angular momentum and the corresponding electromagnetic field has nontrivial magnetic component (see \cite{7} and references therein).

## 2 Proof of Theorem 11

To get homogeneous boundary conditions, we change variables as follows

$$v = u - U, \quad \varphi = \phi - \Phi_D,$$

where $U$ and $\Phi_D$ are the solutions of

$$\begin{cases}
-\Delta U + m^2 U = 0, \\
U = \sqrt{4\pi qh} \quad \text{on} \ \partial \Omega, \\
\Delta \Phi_D = 0, \\
\Phi_D = q\zeta - \omega \quad \text{on} \ \partial \Omega.
\end{cases}$$

By (3) we have

$$\|\Phi_D\|_\infty^2 = \|q\zeta - \omega\|_\infty^2 < m^2 + \lambda_1.$$  \hspace{1cm} (10)

Then problem (S1)-(D1) can be written as

$$\begin{cases}
-\Delta v - (\varphi + \Phi_D)^2 (v + U) + m^2 v = 0, \\
\Delta \varphi = (\varphi + \Phi_D) (v + U)^2, \\
v = \varphi = 0 \quad \text{on} \ \partial \Omega.
\end{cases}$$

The solutions of (11) are the critical points of the functional

$$F(v, \varphi) = \frac{1}{2} \|\nabla v\|_2^2 - \frac{1}{2} \int_\Omega (v + U)^2 dx + \frac{m^2}{2} \|v\|_2^2 - \frac{1}{2} \|\nabla \varphi\|_2^2,$$

defined in $H_0^1(\Omega) \times H_0^1(\Omega)$. The functional $F$ is strongly indefinite, so we apply a well known reduction argument (see e.g. \cite{12}).
Let $\varphi_v \in H^1_0(\Omega)$ denote the unique solution of
\[
\begin{align*}
\Delta \varphi &= (\varphi + \Phi_D) (v + U)^2, \\
\varphi &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]  
Since the map $v \mapsto \varphi_v$ is $C^1$, we define the reduced $C^1$ functional $J(v) = F(v, \varphi_v)$. It is easy to see that the pair $(v, \varphi)$ is a solution of (11) if and only if $v$ is a critical point of $J$ and $\varphi = \varphi_v$. So we have to prove the existence of a critical point of $J$.

The function $\varphi_v$ satisfies
\[
\|\nabla \varphi_v\|_2^2 + \int_\Omega \varphi_v^2 (v + U)^2 \, dx = - \int_\Omega \varphi_v \Phi_D (v + U)^2 \, dx.
\]  
(12)
Then we obtain
\[
J(v) = \frac{1}{2} \|v\|_2^2 + \frac{m^2}{2} \|v\|_2^2 - \frac{1}{2} \int_\Omega \Phi_D (\varphi_v + \Phi_D) (v + U)^2 \, dx
\]  
and, for every $w \in H^1_0(\Omega)$,
\[
\langle J'(v), w \rangle = \int_\Omega \nabla v \nabla w \, dx + m^2 \int_\Omega v w \, dx - \int_\Omega (\varphi_v + \Phi_D)^2 (v + U) \, w \, dx.
\]

The following lemma shows that the functions $\{\varphi_v\}$ are uniformly bounded in $L^\infty(\Omega)$.

**Lemma 9** For every $v \in H^1_0(\Omega)$, the function $\varphi_v$ satisfies the following inequalities
\[
- \max \{0, \Phi_D\} = - \Phi_D^+ \leq \varphi_v \leq \Phi_D^- = \max \{0, - \Phi_D\}, \quad \text{a.e. in } \Omega.
\]  
(13)

Hence $\varphi_v \in L^\infty(\Omega)$ and satisfies
\[
\|\varphi_v\|_\infty \leq \|\Phi_D\|_\infty, \\
\|\varphi_v + \Phi_D\|_\infty \leq \|\Phi_D\|_\infty.
\]  
(14)

**Proof.** Fix $v \in H^1_0(\Omega)$. Let $\tilde{\varphi}_v$ be the unique solution of
\[
\begin{align*}
\Delta \varphi &= (\varphi - \Phi^-_D) (v + U)^2, \\
\varphi &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]  
We claim that
\[
0 \leq \tilde{\varphi}_v \leq \Phi^-_D, \quad \text{a.e. in } \Omega.
\]
Indeed, since $\tilde{\varphi}_v$ is the minimum of the functional
\[
f(\varphi) = \frac{1}{2} \int_\Omega |\nabla \varphi|^2 \, dx + \frac{1}{2} \int_\Omega \varphi^2 (v + U)^2 \, dx - \int_\Omega \varphi \Phi^-_D (v + U)^2 \, dx
\]
and
\[
f(\|\tilde{\varphi}_v\|) \leq f(\tilde{\varphi}_v),
\]
we have that $\tilde{\varphi}_v$ is positive.
On the other hand, suppose that

\[ 0 < \tilde{\varphi}_v - \Phi_D, \quad \text{a.e. in } \Omega_1 \subset \Omega. \tag{15} \]

Since \( \tilde{\varphi}_v - \Phi_D \) solves

\[
\begin{cases}
\Delta w = w (v + U)^2 & \text{in } \Omega_1, \\
w = 0 & \text{on } \partial \Omega_1, \\
w > 0 & \text{in } \Omega_1,
\end{cases}
\]

then

\[-\| \nabla (\tilde{\varphi}_v - \Phi_D) \|^2_2 = \int_{\Omega_1} (\tilde{\varphi}_v - \Phi_D)^2 (v + U)^2 \, dx.\]

This implies that \( \tilde{\varphi}_v - \Phi_D = 0 \) a.e. in \( \Omega_1 \), which contradicts (15).

Analogously, the unique solution \( \hat{\varphi}_v \) of

\[
\begin{cases}
\Delta \varphi = (\varphi - \Phi_D^+) (v + U)^2, \\
\varphi = 0 & \text{on } \partial \Omega,
\end{cases}
\]

satisfies

\[ 0 \leq \hat{\varphi}_v \leq \Phi_D^+, \quad \text{a.e. in } \Omega. \]

Thus, linearity and uniqueness imply that

\[ \tilde{\varphi}_v \equiv \varphi_v^+ \quad \text{and} \quad \hat{\varphi}_v \equiv \varphi_v^- \]

and estimate (13) is proved. ■

**Proposition 10** Under the assumption (10), the functional \( J \) is bounded from below, coercive and satisfies the Palais-Smale condition.

**Proof.** We have

\[
J(v) \geq \frac{1}{2} \| \nabla v \|^2_2 + \frac{m^2}{2} \| v \|^2_2 - \frac{\| \Phi_D \|^2_{\infty}}{2} \| v + U \|^2_2
\]

\[
\geq \frac{1}{2} \| \nabla v \|^2_2 - \frac{\| \Phi_D \|^2_{\infty} - m^2}{2} \| v \|^2_2 - \frac{\| \Phi_D \|^2_{\infty}}{2} \| U \|^2_2 - \frac{\| \Phi_D \|^2_{\infty}}{2} \| v \|^2_2 \| U \|^2_2
\]

\[
\geq \frac{\lambda_1 - \max \{ 0, \| \Phi_D \|^2_{\infty} - m^2 \}}{2\lambda_1} \| \nabla v \|^2_2 - \frac{\| \Phi_D \|^2_{\infty}}{2} \| U \|^2_2 - c \| \nabla v \|^2_2.
\]

Taking into account (10), we deduce that \( J \) is bounded from below and coercive.

Let \( \{ v_n \} \) be a P-S sequence for \( J \), that is a sequence such that \( \{ J(u_n) \} \) is bounded and \( \{ J'(v_n) \} \) tends to zero. Since \( J \) is coercive, the sequence \( \{ v_n \} \) is bounded in \( H^1_0(\Omega) \) and, up to subtracting a subsequence, it is weakly convergent. By (14), \( \{ (\varphi_{v_n} + \Phi_D)^2 \} \) is uniformly bounded in \( L^\infty(\Omega) \), hence the right hand side of

\[
\Delta v_n = m^2 v_n - (\varphi_{v_n} + \Phi_D)^2 (v_n + U) - J'(v_n)
\]

is bounded in \( H^{-1}(\Omega) \). The claim immediately follows. ■

By a standard argument, the functional \( J \) has a minimum and the first statement of Theorem 1 is thereby proved.
Now we prove the second statement. First we notice that, if $h = 0$, then $U = 0$. Problem (11) becomes

$$\begin{align*}
-\Delta v - (\varphi + \Phi_D)^2 v + m^2 v &= 0, \\
\Delta \varphi &= (\varphi + \Phi_D) v^2, \\
v &= \varphi = 0 \quad \text{on } \partial \Omega.
\end{align*}$$

If $(v, \varphi)$ is a solution of (16), by the first equation we have

$$\|\nabla v\|_2^2 - \int_\Omega (\varphi + \Phi_D)^2 v^2 \, dx + m^2 \|v\|_2^2 = 0. \quad (17)$$

Substituting (12) into (17) we get

$$\|\nabla v\|_2^2 + \int_\Omega \varphi^2 v^2 \, dx + 2 \|\nabla \varphi\|_2^2 + \int_\Omega (m^2 - \Phi_D^2) v^2 \, dx = 0.$$

Then

$$0 \geq \lambda_1 \|v\|_2^2 + \int_\Omega \varphi^2 v^2 \, dx + 2 \|\nabla \varphi\|_2^2 + \int_\Omega (m^2 - \Phi_D^2) v^2 \, dx \geq \left(\lambda_1 + m^2 - \|\Phi_D\|_\infty^2\right) \|v\|_2^2 + 2 \|\nabla \varphi\|_2^2$$

and, by (10), we conclude that $v = \varphi = 0$. Since $u = v$, the claim follows.

### 3 Proof of Theorem 4

We consider again the function $U \neq 0$ defined by (9).

On the other hand, let $\Phi_N$ denote the unique solution of

$$\begin{align*}
\Delta \Phi_N &= q\kappa, \\
\frac{\partial \Phi_N}{\partial n} &= q\theta \quad \text{on } \partial \Omega, \\
\int_\Omega \Phi_N \, dx &= 0.
\end{align*}$$

With

$$\kappa = \frac{1}{|\Omega|} \int_{\partial \Omega} \theta \, d\sigma.$$

It is well known that $\|\Phi_N\|_\infty \leq c|q|\|\theta\|_{H^{1/2}(\partial \Omega)}$; we choose $|q|\|\theta\|_{H^{1/2}(\partial \Omega)}$ small enough to get

$$m^2 - \Phi_N^2 \geq 0.$$

If we set

$$v = u - U, \quad \varphi = \phi - \Phi_N,$$

then, problem (S1)-(M1) becomes

$$\begin{align*}
-\Delta v - (\varphi + \Phi_N)^2 (v + U) + m^2 v &= 0, \\
\Delta \varphi &= (\varphi + \Phi_N) (v + U)^2 - q\kappa, \\
v &= 0 \quad \text{on } \partial \Omega, \\
\frac{\partial \varphi}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{align*}$$

(18)

The following lemma has been proved in [9, Lemma 2.3].
Lemma 11 For every \( w \in H^1(\Omega) \setminus \{0\} \) and \( \rho \in L^{6/5}(\Omega) \), the problem
\[
\begin{cases}
-\Delta \varphi + \varphi w^2 = \rho, \\
\frac{\partial \varphi}{\partial n} = 0
\end{cases}
\text{on } \partial \Omega
\]
has a unique solution in \( H^1(\Omega) \).

Since \( U = \sqrt{4\pi qh} \neq 0 \) on \( \partial \Omega \), for every \( v \in H^1_0(\Omega) \) we have \( v + U \neq 0 \).
Hence, by the previous lemma, the problem
\[
\begin{cases}
\Delta \varphi = (\varphi + \Phi_N) (v + U)^2 - q\kappa, \\
\frac{\partial \varphi}{\partial n} = 0
\end{cases}
\text{on } \partial \Omega
\]
has always a unique solution \( \varphi_v \in H^1(\Omega) \).

As well as in the previous section, we use a variational principle: we look for the critical points of a reduced functional \( J = J(v) \) defined in \( H^1_0(\Omega) \); then the solutions of (18) are the pairs \((v, \varphi_v)\).

The reduced functional has the form
\[
J(v) = \frac{1}{2} \| \nabla v \|_2^2 + \frac{1}{2} \int_\Omega (m^2 - \Phi_N^2) (v + U)^2 \, dx + \frac{1}{2} \int_\Omega \Phi_N (v + U)^2 \, dx + \frac{q\kappa}{2} \int_\Omega \varphi_v \, dx.
\]

With the aim of studying the functional \( J \), as in [9, Section 3], we consider
\[
\varphi_v = \xi_v + \eta_v,
\]
where \( \xi_v \) and \( \eta_v \) solve respectively
\[
\begin{cases}
\Delta \xi - (v + U)^2 \xi = (v + U)^2 \Phi_N, \\
\frac{\partial \xi}{\partial n} = 0
\end{cases}\text{on } \partial \Omega,
\]
\[
\begin{cases}
\Delta \eta - (v + U)^2 \eta = -q\kappa, \\
\frac{\partial \eta}{\partial n} = 0
\end{cases}\text{on } \partial \Omega.
\]

We obtain the following estimates
\[
\int_\Omega \xi_v \Phi_N (v + U)^2 \, dx \leq 0, \quad -\max \Phi_N \leq \xi_v \leq -\min \Phi_N, \quad q\kappa \eta_v \geq 0, \quad \| \nabla \eta_v \|_2 \leq c \| \eta_v \|_2 \| v + U \|_4^2,
\]
where \( c > 0 \) and \( \bar{\eta}_v \) is the average of \( \eta_v \).

Then we can show that the functional \( J \) is coercive, bounded from below and satisfies the PS condition. So we get the existence of a minimum of \( J \), which gives rise to a solution of (18).
4 Proof of Theorem 7

With the same change of variables of Section 2 (with $U = 0$, since $h = 0$), problem (S2)-(D2) can be written as

\[
\begin{align*}
-\Delta v &- (\varphi + \Phi_D)^2 v + m^2 v - g(x,v) = 0, \\
\Delta \varphi &- (\varphi + \Phi_D) v^2, \\
v &- \varphi = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

As in the previous sections, the solutions of (19) have the form $(v, \varphi_v)$, where $v$ is a critical point of the reduced functional

\[
J(v) = \frac{1}{2} \|\nabla v\|^2 + \frac{m^2}{2} \int_\Omega v^2 dx - \frac{1}{2} \int_\Omega \Phi_D (\varphi_v + \Phi_D) v^2 dx - \int_\Omega G(x,v) dx
\]

which is $C^1$ in $H_0^1(\Omega)$.

Let us recall that the well-known conditions $(g_1) - (g_3)$ imply that:

$(G_1)$ for every $\varepsilon > 0$ there exists $A \geq 0$ such that for every $t \in \mathbb{R}$

\[|G(x,t)| \leq \frac{\varepsilon}{2} t^2 + A |t|^p;\]

$(G_2)$ there exist two constants $b_1, b_2 > 0$ such that for every $t \in \mathbb{R}$

\[G(x,t) \geq b_1 |t|^s - b_2.\]

Now we can state some properties of $J$.

Lemma 12 The functional $J$ satisfies the Palais-Smale condition on $H_0^1(\Omega)$ and diverges negatively on every finite dimensional subspace of $H_0^1(\Omega)$.

Proof. Let $\{v_n\} \subset H_0^1(\Omega)$ such that

\[|J(v_n)| \leq c_1 \quad (20)\]

\[J'(v_n) \to 0. \quad (21)\]

As before, we set $\varphi_n = \varphi_{v_n}$ and we use $c_i$ to denote suitable positive constants. By (20), (g1) and (g3)

\[
\frac{1}{2} \|\nabla v_n\|^2 \leq c_1 + \int_\Omega G(x,v_n) dx + \frac{1}{2} \int_\Omega \Phi_D (\varphi_n + \Phi_D) v_n^2 dx \leq c_2 + \frac{1}{s} \int_{\{x \in \Omega: |v_n(x)| \geq r\}} g(x,v_n) v_n dx + \frac{1}{2} \|\Phi_D\|_\infty^2 \|v_n\|^2 \]

\[
\leq c_3 + \frac{1}{s} \int_\Omega g(x,v_n) v_n dx + \frac{1}{2} \|\Phi_D\|_\infty^2 \|v_n\|^2. \quad (22)
\]

On the other hand, by (21),

\[
\|\nabla v_n\|^2 + m^2 \|v_n\|^2 - \int_\Omega (\varphi_n + \Phi_D)^2 v_n^2 dx - \int_\Omega g(x,v_n) v_n dx = |\langle J'(v_n), v_n \rangle| \leq c_4 \|\nabla v_n\|^2,
\]
\[
\int_\Omega g(x,v_n) \, dx \leq c_4 \|\nabla v_n\|_2^2 + \|\nabla v_n\|_2^2 + m^2 \|v_n\|_2^2 - \int_\Omega (\varphi_n + \Phi_D)^2 \, v_n^2 \, dx
\]

\[
\leq c_4 \|\nabla v_n\|_2^2 + \|\nabla v_n\|_2^2 + m^2 \|v_n\|_2^2.
\] (23)

Therefore, substituting (23) into (22), we easily find

\[
\frac{s}{2} - \frac{2}{s} \|\nabla v_n\|_2^2 \leq c_3 + \frac{c_4}{s} \|\nabla v_n\|_2^2 + c_5 \|v_n\|_2^2
\]

or, equivalently,

\[
\|v_n\|_2^2 \geq c_6 \|\nabla v_n\|_2^2 - c_7 \|\nabla v_n\|_2^2 - c_8.
\] (24)

Now we claim that \(\{v_n\}\) is bounded in \(H_0^1(\Omega)\). Otherwise, by (24), up to a subsequence, we have

\[
\|v_n\|_2^2 \geq c_9 \|\nabla v_n\|_2^2 \to +\infty
\]

and then, by (G2),

\[
J(v_n) \leq \frac{1}{2} \|\nabla v_n\|_2^2 + \frac{m^2}{2}\|v_n\|_2^2 + \frac{\|\Phi_D\|_2^2}{2}\|v_n\|_2^2 - \int_\Omega G(x,v_n) \, dx
\]

\[
\leq c_{10}\|v_n\|_2^2 - b_1\|v_n\|_*^2 + b_2 |\Omega| \to -\infty,
\]

which contradicts (20).

Thus, we can assume that

\[
v_n \rightharpoonup v \text{ in } H_0^1(\Omega).
\]

This convergence is strong. Indeed, by (G1) and the same arguments used in the proof of Proposition 10 we get that the right hand side of

\[
\Delta v_n = m^2 v_n - (\varphi_n + \Phi_D)^2 v_n - g(x,v_n) - J'(v_n)
\]

is bounded in \(H^{-1}(\Omega)\).

Finally, by (G2),

\[
J(v) \leq \frac{1}{2} \|\nabla v\|_2^2 + c_1 \|v\|_2^2 - \int_\Omega G(x,v) \, dx
\]

\[
\leq \frac{1}{2} \|\nabla v\|_2^2 + c_1 \|v\|_2^2 - b_1 \|v\|_*^2 + b_2 |\Omega| \to -\infty
\]

when \(\|v\| \to +\infty\) on every finite dimensional subspace. ■

The first part of Theorem 7 follows from the classical Mountain Pass Theorem. Indeed from (G1) and Lemma 9 we deduce

\[
J(v) \geq \frac{1}{2} \|\nabla v\|_2^2 + \frac{m^2}{2} \|v\|_2^2 - \frac{\|\Phi_D\|_\infty^2}{2} \|v\|_2^2 - \frac{\|v\|_2^2}{2} \|v\|_2^2 - \frac{\|v\|_2^2}{2} - A \|v\|^p_p
\]

\[
\geq \frac{\lambda_1 - \max \left\{0, \frac{\|\Phi_D\|_\infty^2 - m^2}{2}\right\}}{2\lambda_1} \|\nabla v\|_2^2 - A' \|\nabla v\|_2^p, \] (25)

with \(A, A' > 0\) depending on \(\varepsilon\). Taking into account (10), if we choose \(\varepsilon\) sufficiently small, we have

\[
J(v) \geq c \|\nabla v\|_2^2 - A' \|\nabla v\|_2^p
\] (26)

with \(c > 0\). Hence \(J\) has a strict local minimum in 0. By Lemma 12 the claim immediately follows.
Remark 13 The proof of Lemma 13 does not use any assumption on the value \( \omega^2 - m^2 \). Hence, if \( \omega^2 - m^2 \neq \lambda_k \) (and \( q \) is sufficiently small) we conjecture that a critical point could be obtained by means of some variant of the Mountain Pass Theorem.

If \( g \) is odd, the functional \( J \) is even and we apply the following \( Z_2 \)-Mountain Pass Theorem (see [12]).

Theorem 14 Let \( E \) be an infinite dimensional Banach space and let \( I \in C^1 (E, \mathbb{R}) \) be even, satisfy the Palais-Smale condition and \( I (0) = 0 \). If \( E = V \oplus X \), where \( V \) is finite dimensional and \( I \) satisfies

1. there are constants \( \rho, \alpha > 0 \) such that \( I|_{\partial B_\rho \cap X} \geq \alpha \), and
2. for each finite dimensional subspace \( \tilde{E} \subset E \), there is an \( R = R(\tilde{E}) \) such that \( I \leq 0 \) on \( E \setminus B_R(\tilde{E}) \),

then \( I \) possesses an unbounded sequence of critical values.

We have only to prove the first geometrical condition. We distinguish two cases.

(a) If \( \| \Phi_D \|_\infty^2 - m^2 < \lambda_1 \), we proceed as in (25) and (26) and we obtain that \( J \) has a strict local minimum in 0. Applying Theorem 14 with \( V = \{0\} \) we infer the existence of infinitely many solutions \( v_i \).

(b) If \( \lambda_1 \leq \| \Phi_D \|_\infty^2 - m^2 \), denoted with \( \{\lambda_j\} \) the eigenvalues of \( -\Delta \) with Dirichlet boundary condition, \( M_j \) the (finite dimensional) corresponding eigenspaces and

\[
\kappa = \min \left\{ j \in \mathbb{N} : \| \Phi_D \|_\infty^2 - m^2 < \lambda_j \right\},
\]

we consider

\[
V = \bigoplus_{j=1}^{k-1} M_j, \quad X = V^\perp = \bigoplus_{j=k}^{+\infty} M_j.
\]

Since

\[
\lambda_k = \min \left\{ \frac{\| \nabla v \|_2^2}{\| v \|_2^2} : v \in X, v \neq 0 \right\},
\]

for every \( v \in X \) we have that

\[
J (v) \geq \frac{\lambda_k - (\Phi_D^2 - m^2)}{2\lambda_k} \| \nabla v \|_2^2 - \int_\Omega G(x, v) \, dx.
\]

Thus, arguing as before, \( J \) is strictly positive on a sphere in \( X \) and we obtain the existence of infinitely many finite energy solutions \( v_i \).

In both cases we have \( J(v_i) \to +\infty \). To complete the proof of Theorem 7 we simply notice that, by Lemma 10 and (G1),

\[
J (v_i) \geq \frac{1}{2} \| \nabla v_i \|_2^2 + \frac{m^2}{2} \| v_i \|_2^2 - \frac{1}{2} \int_\Omega \Phi_D (\phi_i + \Phi_D) v_i^2 \, dx - \int_\Omega G (x, v_i) \, dx \leq c_1 \| \nabla v_i \|_2^2 + c_3 \| \nabla v_i \|_2^p.
\]
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