The structure of graphs with forbidden induced $C_4, \overline{C}_4, C_5, S_3$, chair and co-chair

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Abstract

We find the structure of graphs that have no $C_4, \overline{C}_4, C_5, S_3$, chair and co-chair as induced subgraphs. Then we deduce the structure of the graphs having no induced $C_4, \overline{C}_4, S_3$, chair and co-chair and the structure of the graphs $G$ having no induced $C_4, \overline{C}_4$ and such that every induced $P_4$ of $G$ is contained in an induced $C_5$ of $G$.

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1. Introduction

In this paper, graphs are finite and simple. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$ respectively. Two edges of a graph $G$ are said to be adjacent if they have a common endpoint and two vertices $x$ and $y$ are said to be adjacent if $xy$ is an edge of $G$. The neighborhood of a vertex $v$ in a graph $G$, denoted by $N_G(v)$, is the set of all vertices adjacent to $v$ and its degree is $d_G(v) = |N_G(v)|$. We omit the subscript if the graph is clear from the context. For two set of vertices $U$ and $W$ of a graph $G$, let $E[U,W]$ denote the set of all edges in the graph $G$ that joins a vertex in $U$ to a vertex in $W$. A graph is empty if it has no edges. For $A \subseteq V(G)$, $G[A]$ denotes the sub-graph of $G$ induced by $A$. If $G[A]$ is an empty graph, then $A$ is called a...
stable set. While, if $G[A]$ is a complete graph, then $A$ is called a clique set, that is any two distinct vertices in $A$ are adjacent. The complement graph of $G$ is denoted by $\overline{G}$ and defined as follows: $V(G) = V(\overline{G})$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$. A graph $H$ is called a forbidden subgraph of $G$ if $H$ is not (isomorphic to) an induced subgraph of $G$.

A cycle on $n$ vertices is denoted by $C_n = v_1v_2...v_nv_1$ while a path on $n$ vertices is denoted by $P_n = v_1v_2...v_n$. A chair is any graph on 5 distinct vertices $x, y, z, t, v$ with exactly 5 edges $xy, yz, zt$ and $zv$. The co-chair or chair is the complement of a chair. $S_3$ is the graph on 6 vertices as indicated in Figure 1.

Many graphs encountered in the study of graph theory are characterized by configurations or subgraphs they contain. However, there are occasions where it is easier to characterize graphs by subgraphs or induced subgraphs they do not contain. For example, trees are the connected graph without (induced) cycles. Bipartite graphs are those without (induced) odd cycles ([1]). Split graphs are those without induced $C_4$, $\overline{C}_4$ and $C_5$. Line graphs are characterized by the absence of only nine particular graphs as induced sub-graph (see [2]). Perfect graphs are characterized by $C_{2n+1}$ and $\overline{C}_{2n+1}$ being forbidden, for all $n \geq 2$ (see [3]). The purpose of this paper is to find the structure of graphs such that $C_4, \overline{C}_4, C_5, S_3$ chair and co-chair are forbidden subgraphs. These graphs will be called generalized combs and they are generalization of generalized stars and generalization of combs (See [6, 8]). Seymour’s Second Neighborhood Conjecture (see [9]) is proved for orientation of graphs obtained from the complete graph by deleting the edges of a generalized star and for those obtained by deleting the edges of a comb [6, 8]. Generalized stars (also called threshold graphs) are the graphs with $C_4, \overline{C}_4, C_5$, $S_3$ chair and co-chair forbidden. Finding the structure of the generalized comb, might give a clearer vision for an attempt to prove Seymour’s conjecture for oriented graphs obtained from the complete graph by deleting the edges of a generalized comb.
2. Preliminary Definitions and Theorems

Definition 1. A graph $G$ is a called a split graph if its vertex set is the disjoint union of a stable set $S$ and a clique set $K$. In this case, $G$ is called an $\{S, K\}$-split graph.

If $G$ is an $\{S, K\}$-split graph and $\forall s \in S, \forall x \in K$ we have $sx \in E(G)$, then $G$ is called a complete split graph.

If $G$ is an $\{S, K\}$-split graph and $E[S, K]$ forms a perfect matching of $G$, then $G$ is called a perfect split graph.

Theorem 2.1. (Földes and Hammer [4]) $G$ is a split graph if and only if $C_4, \overline{C_4}$ and $C_5$ are forbidden subgraphs of $G$.

Definition 2. ([5]) A threshold graph $G$ can be defined as follows:

1) $V(G) = \bigcup_{i=1}^{n+1} (X_i \cup A_{i-1})$, where the $A_i$’s and $X_i$’s are pair-wisely disjoint sets.

2) $K := \bigcup_{i=1}^{n+1} X_i$ is a clique and the $X_i$’s are nonempty, except possibly $X_{n+1}$.

3) $S := \bigcup_{i=0}^{n} A_i$ is a stable set and the $A_i$’s are nonempty, except possibly $A_0$.

4) $\forall i, j \in [1, n]$ and $j \leq i$, $G[A_i \cup X_j]$ is a complete split graph.

5) The only edges of $G$ are the edges of the subgraphs mentioned above.

In this case, $G$ is called an $\{S, K\}$-threshold graph.

In fact, threshold graphs are exactly the generalized stars defined in [6].

Theorem 2.2. (Hammer and Chvátal [5]) $G$ is a threshold graph if and only if $C_4, \overline{C_4}$ and $P_4$ are forbidden subgraphs of $G$.

Theorem 2.3. ([7]) $C_4, \overline{C_4}$ are forbidden subgraphs of a graph $G$ if and only if $V(G)$ is disjoint union of three sets $S, K$ and $C$ such that:

1) $G[S \cup K]$ is an $\{S, K\}$-split graph;

2) $G[C]$ is empty or isomorphic to the cycle $C_5$;

3) every vertex in $C$ is adjacent to every vertex in $K$ but to no vertex in $S$. 

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3. Main Results

Lemma 3.1. Suppose that $C_4$, $\overline{C_4}$, $C_5$, chair and co-chair are forbidden subgraphs of $G$. If the path $mbb'm'$ is an induced subgraph of $G$, then:

$$N(m) - \{b\} = N(m') - \{b'\}$$

and

$$N(b) - \{m\} = N(b') - \{m'\}.$$  

Proof. Since $C_4$, $\overline{C_4}$, and $C_5$ are forbidden, then $G$ is an $\{S, K\}$-split graph for some stable set $S$ and a clique set $K$. Since $mbb'm'$ is an induced subgraph of $G$, then $m, m' \in S$ and $b, b' \in K$.

Assume that there is $x \in N(m) - \{b\}$ but $x \notin N(m') - \{b'\}$. Since $xm$ is an edge of $G$ and $S$ is stable, then we must have $x \in K$. But $K$ is a clique, then $x$ is adjacent to $b$ and $b'$. Thus $G'\{x, m, b, b', m'\}$ is a co-chair. Contradiction. So $N(m) - \{b\} \subseteq N(m') - \{b'\}$. By symmetry, $N(m') - \{b'\} \subseteq N(m) - \{b\}$. Thus $N(m) - \{b\} = N(m') - \{b'\}$.

Assume that there is $x \in N(b) - \{m\}$ but $x \notin N(b') - \{m'\}$. Suppose that $x \in S$. Then $G'\{x, m, b, b', m'\}$ is a chair. Contradiction. Thus $x \notin K$. But $K$ is a clique. Whence $x \in N(b') - \{m'\}$. Thus $N(b) - \{m\} \subseteq N(b') - \{m'\}$. By symmetry, $N(b') - \{m'\} \subseteq N(b) - \{m\}$. Therefore $N(b) - \{m\} = N(b') - \{m'\}$. □

Proposition 3.1. If $P_4$ is a forbidden subgraph of an $\{S, K\}$-split graph $G$, then $G$ is an $\{S, K\}$-threshold graph.

Proof. We prove this by induction on the number of vertices of $G$. This is clearly true for small graphs. Suppose that $P_4$ is a forbidden subgraph of an $\{S, K\}$-split graph $G$. It is clear that $G$ is a threshold graph. We have to prove that $G$ is an $\{S, K\}$-threshold graph. Let $x \in K$ be a vertex with minimum degree in $G$, that is $d_G(x) = \min \{d_G(y); y \in K\}$ and $G' := G - x$ be the graph induced by the vertices of $G$ except $x$ (if $K = \phi$, then the statement is true). Then $P_4$ is a forbidden subgraph of the $\{S, K - \{x\}\}$-split graph $G'$. By the induction hypothesis, $G'$ is an $\{S, K - \{x\}\}$-threshold graph. We follow the notations in Definition 2. Assume that $\exists a \in S - A_n$ such that $ax \in E(G)$. Let $x_n \in X_n$. Since $d(x_n) \geq d(x)$, then there is $a_n \in A_n$ such that $a_nx \in E(G)$ but $a_nx \notin E(G)$. Then $ax_n, a_n$ is an induced $P_4$ in $G$. Contradiction. Thus we may suppose that $N(x) \cap S \subseteq A_n$. If $N(x) \cap A_n = \phi$, then we add $x$ to $X_{n+1}$. If $N(x) \cap A_n = A_n$, then we add $x_n$ to $X_n$. Otherwise $\phi \subseteq N(x) \cap A_n \subseteq A_n$. In this case we do the following: remove from $A_n$ the element of $N(x) \cap A_n$, create $A_{n+1} = N(x) \cap A_n$, remove the elements of $X_{n+1}$ to the new set $X_{n+2}$ and add $x$ to $X_{n+1}$ (so that the new $X_{n+1} = \{x\}$). Then $G$ is $\{S, K\}$-threshold graph. □

Definition 3. A graph $G$ is called a generalized comb if:

1) $V(G)$ is disjoint union of sets $A_0, ..., A_n, M_1, ..., M_l, X_1, ..., X_{n+1}, Y_1, ..., Y_{l+2}$. Let $Y_1 = X_1$ (These sets are called the sets of the generalized comb $G$).

2) $S := A \cup M$ is a stable set, where $M = \bigcup_{i=1}^{l} M_i$ and $A = \bigcup_{i=0}^{n} A_i$. 

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3) $K := X \cup Y$ is a clique, where $X = \bigcup_{i=1}^{n+1} X_i$ and $Y = \bigcup_{i=1}^{l+2} Y_i$.

4) $\forall i, j \in [1, n]$ and $j \leq i$, $G[A_i \cup X_j]$ is a complete split graph.

5) $G[A \cup Y]$ is a complete split graph.

6) $\forall i \in [1, l], G[Y_i \cup M_i]$ is a perfect split graph or $M_i = \phi$.

7) $\forall i, j \in [1, l + 1]$ and $i < j$, $G[Y_j \cup M_i]$ is a complete split graph.

8) $X_{n+1}, Y_{l+2}, Y_{l+1}$ and $A_0$ are the only possibly empty sets among the $X_i's$, $Y_i's$, $A_i's$.

9) The only edges of $G$ are the edges of the subgraphs mentioned above.

In this case, we say that $G$ is an $\{S, K\}$-generalized comb. Note, that we may assume that no two consecutive sets $M_i$ and $M_{i+1}$ are both empty. We use this assumption in the rest.

It is clear that the comb defined in [8] is a particular case of the generalized comb (see Figure 3). Moreover, we have the following:

**Lemma 3.2.** Every $\{S, K\}$-threshold graph is an $\{S, K\}$-generalized comb.

**Proof.** Let $G$ be an $\{S, K\}$-threshold graph defined as in Definition 2. Following the notations in Definition 3, we take $l = 1$ and $M_l = Y_{l+1} = Y_{l+2} = \phi$. This shows that $G$ is an $\{S, K\}$-generalized comb. \[\square\]
Theorem 3.1. If $S_3$, chair and co-chair are forbidden subgraphs of an $\{S, K\}$-split graph $G$, then $G$ is an $\{S, K\}$-generalized comb.

Proof. We prove the statement by induction on the number of vertices. The statement is true for small graphs. Suppose that $S_3$, chair and co-chair are forbidden subgraphs of an $\{S, K\}$-split graph $G$. If $P_4$ is also a forbidden subgraph of $G$, then $G$ is an $\{S, K\}$-threshold graph, and hence, $G$ is an $\{S, K\}$-generalized comb. So we may suppose that $G$ contains at least one induced path of length four.

Suppose that $G$ has exactly one induced path of length four, say $mbb'm'$. Suppose $N(m) = \{b\}$. Then $N(m') = \{b'\}$. Let $H = G[K \cup S - \{m, m'\}]$. By induction hypothesis, we have $H$ is $\{S - \{m, m'\}, K\}$-generalized comb. But $H$ has no induced $P_4$, then $H$ is in fact $\{S - \{m, m'\}, K\}$-threshold graph. We use the nation in the definition of threshold graph, in what follows. Assume that $\exists i \geq 2$ such that $b \in X_i$. Let $x \in X_1$ and $a \in A_1$. Then $mbx$a is induced $P_4$ in $G$, a contradiction. So $b \in X_1$. Then also $b' \in X_1$, because $b$ and $b'$ have the same neighborhood in $H$. Define $Y_2 = \phi$, $M_1 = \{m, m'\}$, $Y_3 = X_1 - \{b, b'\}$ and the new $X_1$ is the $\{b, b'\}$. Then $G$ is an $\{S, K\}$-generalized comb with $l = 1$ and $Y_{i+1} = \phi$.

Otherwise, $G$ has at least two induced $P_4$. Let $m$ be a vertex of $G$ such that $d(m) = \min\{d(z); z \text{ is a leaf of an induced } P_4 \text{ in } G\}$ and let $P = mbb'm'$ be an induced $P_4$. Note that $d(m) = d(m')$. Let $Q = udd'u'$ be an induced $P_4$ distinct from $P$ (Note that $m, m', u, u' \in S$ while $b, b', d, d' \in K$). Either $m \notin \{u, u'\}$ or $m' \notin \{u, u'\}$, since $N(m) - \{b\} = N(m) - \{b'\}$ (Lemma 3.1). We may assume without loss of generality that $m \notin \{u, u'\}$ and let $H = G[(S - m') \cup (K - b')]$. By the induction hypothesis, $H$ is an $\{S - m', K - b'\}$-generalized comb.

Suppose first that $m' \in \{u, u'\}$ and assume without loss of generality that $m' = u'$. Assume that $b' \neq d'$. If $b = d$, then by using Lemma 3.1 repeatedly, we can prove easily that $G[\{m', m, u, b, b', d'\}]$ is an $S_3$, a contradiction. So $b \neq d$. Note that $b' \neq d$, because $u'b' = mb' \in E(G)$, while $u'd' \notin E(G)$. By applying Lemma 3.1 repeatedly, we have the following: Since $u'b' = m'b' \in E(G)$, then $ub' \in E(G)$, thus $ub \in E(G)$, whence $u'b \in E(G)$, therefore $m'b \in E(G)$, which is a contradiction. Therefore, $b' = d'$. Note that $b \neq d$, since otherwise, we
get \( u \in N(b) - \{m\} \), thus by Lemma 3.1, we get \( u \in N(b') - \{m'\} = N(d') - \{u'\} \), whence \( ud'u = udd'm' \) is an induced path of length four of \( G \), then by Lemma 3.1 also \( udbm \) is an induced path of \( G \) and thus of \( H \). Then, by the definition of the generalized comb \( H, \exists i; u, m \in M_i \) (We follow the notations of definition 3.). In this case we add \( m' \) to \( M_i \) and \( b' \) to \( Y_i \). This shows that \( G \) is an \( \{S, K\} \)-generalized comb.

Now, suppose that \( m' \notin \{u, u'\} \). Assume that \( m \in A \). By definition of the generalized comb \( H \) and since \( ud'u' \) is an induced \( P_4 \) of \( H \), we get that \( N_H(u) \subseteq N_H(m) \) and \( d' \in N_H(m) - N_H(u) \).

So \( d_H(u) < d_H(m) \). Assume that \( b \notin N_H(u) \). Then \( b \notin N(u) \) and thus by Lemma 3.1, we get \( b' \notin N(u) \).

Therefore, \( d_G(u) = d_H(u) < d_H(m) = d_G(m) \), which is a contradiction to the choice of \( m \). Hence, \( b \in N_H(u) \) and so, by Lemma 3.1, we get \( b, b' \in N(u) \cap N(u') \). Note that \( d, d' \in N(m) \) and hence \( d, d' \in N(m') \). Thus \( G[\{u, d', m', b, m, b'\}] \) is an induced \( S_3 \) in \( G \), a contradiction.

So \( m \in M \). Let \( l \) be the greatest such that \( M_l \neq \phi \). Suppose that \( m \notin M_l \). Let \( m'' \in M_l \) and \( b'' \in Y_j \) be its neighbor. \( \exists i < l \) such that \( m \in M_i \). Then \( b''m \in E(G) \) and \( N_H(m'') \subseteq N_H(m) \).

Let \( c \in Y_j \) be the neighbor of \( m \). Let \( k \) be the smallest such that \( k > i \) and \( M_k \neq \phi \) (Note that \( k \) exists and \( i < k \leq l \), moreover we may assume \( k = i + 1 \) or \( k = i + 2 \)).

Suppose \( b \in N(m'') \). Then also \( b' \in N(m'') \). If \( b \neq b'' \), then \( \exists j > k \) such that \( b \in Y_j \).

Then by using Lemma 3.1, we can observe that \( G[\{m, m', m'', b, b', c\}] \) is an induced \( S_3 \) in \( G \), a contradiction. However, if \( b = b' \), then also by using Lemma 3.1, we can observe that \( G[\{m, m', m'', b, b', c\}] \) is an induced \( S_3 \) in \( G \), a contradiction.

Suppose \( b \notin N(m'') \). Then \( b' \notin N_H(m) - N_H(m'') \), \( b \neq b'' \) and \( \exists i < j \leq k \) such that \( b \in Y_j \).

Thus \( d(m'') = d_H(m'') < d_H(m) = d_G(m) \), a contradiction is reached if \( m'' \) is a leaf of an induced \( P_4 \) of \( G \). So, we have \( m'' \) is not a leaf of an induced \( P_4 \) of \( G \) and thus \( k = \{m''\} \) and \( j < k \).

If \( c = b \), then we add \( b' \) to \( Y_i \) and \( m' \) to \( M_i \) and thus \( G \) is an \( \{S, K\} \)-generalized comb. So suppose \( c \neq b \). Assume there is \( mcm''b'' \) an induced \( P_4 \) in \( H \). Then \( m'' \in M_i \) and \( b'' \in Y_i \).

Then by using Lemma 3.1, we can observe that \( G[\{m, m', m'', b, b', c\}] \) is an induced \( S_3 \) in \( G \), a contradiction. Thus \( m \) is not a leaf of an induced \( P_4 \) of \( H \), that is \( M_i = \{m\} \). By definition of \( k \), we get \( M_i = \phi \). Thus \( j = i + 1 \) and \( k = i + 2 \). Now, to \( Y_{i+1} \) we add \( c \) and remove \( b' \), while to \( Y_i \) we add \( b \) and remove \( c \). Then, we can add \( b' \) to \( Y_i \) and \( m' \) to \( M_i \) to get that \( G \) is an \( \{S, K\} \)-generalized comb.

Therefore \( m \in M_i \). Let \( Y_i \cap N(m) = \{c\} \). If \( b = c \), then we add \( b' \) to \( Y_i \) and \( m' \) to \( M_i \) and thus \( G \) is an \( \{S, K\} \)-generalized comb. Now suppose that \( b \neq c \). Suppose that \( c \) is not the only vertex in \( Y_i \) and thus there is an induced path \( mce''m'' \) with \( c, e'' \in Y_i \) and \( m'' \in Y_i \).

By using Lemma 3.1, we can prove easily that \( G[\{b, b', c, m, m', m''\}] \) is an induced \( S_3 \) of \( G \) a contradiction. Therefore \( c \) is the only vertex in \( Y_i \). Since \( bm \in E(H) \), then \( e'' = Y_i \). We do the following: To \( Y_{i+1} \) add \( c \) and remove \( b' \) and to \( Y_i \) add \( b \) and remove \( c \). Then we add \( b' \) to \( Y_i \) and \( m' \) to \( M_i \) (as in the case \( b = c \)) and this shows that \( G \) is an \( \{S, K\} \)-generalized comb.

\[\square\]

**Corollary 3.1.** \( G \) is a generalized comb if and only if \( C_4, \overline{C_4}, C_5, S_3 \) chair and co-chair are forbidden subgraphs of \( G \).

**Proof.** The necessary condition is obvious by the definition of the generalized comb. For the sufficient condition it is enough to note that the statement \( C_4, \overline{C_4}, C_5, S_3, \) chair and co-chair are
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forbidden subgraphs of $G$ is equivalent to the statement that $G$ is a split graph and $S_3$, chair and co-chair are forbidden subgraphs of $G$.

**Corollary 3.2.** $G$ is a generalized comb if and only if every induced subgraph of $G$ is a generalized comb.

*Proof.* Let $G'$ be an induced subgraph of a generalized comb $G$. It is clear that $G'$ contains no induced $C_4, \overline{C}_4, C_5$, chair and co-chair. Thus $G'$ is a generalized comb. The sufficient condition is clear.

**Corollary 3.3.** $C_4, \overline{C}_4, S_3$, chair and co-chair are forbidden subgraphs of a graph $G$ if and only if $V(G)$ is disjoint union of three sets $S, K$ and $C$ such that:

1) $G[S \cup K]$ is an $\{S, K\}$-generalized comb;
2) $G[C]$ is empty or isomorphic to the cycle $C_5$;
3) every vertex in $C$ is adjacent to every vertex in $K$ but to no vertex in $S$.

*Proof.* The sufficient condition is clear by construction of $G$. We prove the necessary condition. Suppose that $C_4, \overline{C}_4, S_3$, chair and co-chair are forbidden subgraphs of a graph $G$. Then by Theorem 2.3, $V(G)$ is disjoint union of three sets $S, K$ and $C$ such that:

1) $G[S \cup K]$ is an $\{S, K\}$-threshold;
2) $G[C]$ is empty or isomorphic to the cycle $C_5$;
3) every vertex in $C$ is adjacent to every vertex in $K$ but to no vertex in $S$.

Then $C_4, \overline{C}_4, C_5, S_3$, chair and co-chair are forbidden subgraphs of $G[S \cup K]$. Thus $G[S \cup K]$ is an $\{S, K\}$-generalized comb.

**Corollary 3.4.** $C_4, \overline{C}_4$ are forbidden subgraphs of $G$ and every induced $P_4$ of $G$ is contained in an induced $C_5$ of $G$ if and only if $V(G)$ is disjoint union of three sets $S, K$ and $C$ such that:

1) $G[S \cup K]$ is an $\{S, K\}$-threshold;
2) $G[C]$ is empty or isomorphic to the cycle $C_5$;
3) every vertex in $C$ is adjacent to every vertex in $K$ but to no vertex in $S$.

*Proof.* The sufficient condition is clear by construction of $G$. We prove the necessary condition. Suppose that $C_4, \overline{C}_4$ are forbidden subgraphs of a graph $G$ and every induced $P_4$ of $G$ is contained in an induced $C_5$ of $G$. Then by Theorem 2.3, $V(G)$ is disjoint union of three sets $S, K$ and $C$ such that:

1) $G[S \cup K]$ is an $\{S, K\}$-threshold;
2) $G[C]$ is empty or isomorphic to the cycle $C_5$;
3) every vertex in $C$ is adjacent to every vertex in $K$ but to no vertex in $S$.

Then $G[C]$ is the unique induced $C_5$ of $G$ or $G$ has no induced $C_5$. Then $C_4, \overline{C}_4, P_4$ are forbidden subgraphs of $G[S \cup K]$. Thus $G[S \cup K]$ is an $\{S, K\}$-threshold graph.
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