A Few Questions on the
Topologization of the Ring of Fermat
Reals.

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Abstract

This work is developing, and we will include many additions in the near future. Our purpose here is to highlight that there is plenty of space for a topological development of the Fermat Real Line.

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1 Introduction

The idea of the ring of Fermat Reals \( \mathbb{R} \) has come as a possible alternative to Synthetic Differential Geometry (see e.g. [11, 12, 13]) and its main aim is the development of a new foundation of smooth differential geometry for finite and infinite-dimensional spaces. In addition, \( \mathbb{R} \) could play a role of a potential alternative in some certain problems in the field \( \mathbb{R} \) in Nonstandard Analysis (NSA), because the applications of NSA in differential geometry are very few. One of the “weak” points of \( \mathbb{R} \) at the moment is the lack of a natural topology, carrying the strong topological properties of the line.

P. Giordano and M. Kunzinger have recently done brave steps towards the topologization of the ring \( \mathbb{R} \) of Fermat Reals. In particular, they have constructed two topologies; the Fermat topology and the omega topology (see [3]). The Fermat topology is generated by a complete pseudo-metric and is linked to the differentiation of non-standard smooth functions. The omega topology is generated by a complete metric and is linked to the differentiation...
of smooth functions on infinitesimals. Although both topologies are very useful in developing infinitesimal instruments for smooth differential geometry, none of these two topologies aims to characterize the Fermat real line from an order-theoretic perspective. In fact, neither makes the space $T_1$, while an appropriate order-topology would equip the Fermat Real Line with the structure of a monotonically normal space, at least. The possibility to define a linear order relation on $\mathbb{R}$, so that it can be viewed as a LOTS (linearly ordered topological space) can be considered important, because $\mathbb{R}$ is an alternative mathematical model of the real line, having some features with respect to applications in smooth differential geometry and mathematical physics. It is therefore natural to ask whether for $\mathbb{R}$ peculiar characteristics of $\mathbb{R}$ hold or not.

In this paper we will focus in the order relation which is introduced in [4] (which is linear, but it generates the discrete topology on the space and also if considered minus the diagonal, i.e. as a strict order, the topology when restricted to the set of proper reals is again the discrete topology) and we will add properties to it, so that it will both extend the natural order of the real line and it will also give a stronger topology than the Fermat topology and the omega topology. We aim to do this using interlocking nests.

As we shall see in Definition 2.6 the idea of the formation of $\mathbb{R}$ starts with an equivalence relation in the little-oh polynomials, where $\mathbb{R}$ is the quotient space under this relation. This treatment permits us to view these little-oh polynomials as numbers.

2 Preliminaries.

2.1 LOTS and GO-spaces via Nests.

The notions nest and order are closely related. J. van Dalen and E. Wattel gave a complete characterization of LOTS (linearly ordered topological spaces) and of GO-spaces (generalized ordered spaces, i.e. subspaces of LOTS) using properties of nests (see [1]). In this paper we will use tools from [2], where the authors improved the techniques of van Dalen and Wattel in order to characterize ordinals in topological terms and from [5], where the author studies further properties of order relations via nests.

Definition 2.1. Let $X$ be a set.

1. A collection $\mathcal{L} \subset \mathcal{P}(X)$ of subsets of $X$ $T_0$-separates $X$, if and only if for every $x, y \in X$, such that $x \neq y$, there exists $L \in \mathcal{L}$, such that
2. A collection \( L \subset \mathcal{P}(X) \) of subsets of \( X \) \( T_1 \)-separates \( X \), if and only if for every \( x, y \in X \), such that \( x \neq y \), there exist \( L, L' \in L \), such that \( x \in L \) and \( y \notin L \) and also \( y \in L' \) and \( x \notin L' \).

**Definition 2.2.** Let \( X \) be a set and let \( L \) be a family of subsets of \( X \). \( L \) is a nest on \( X \) if, for every \( M, N \in L \), either \( M \subset N \) or \( N \subset M \).

**Definition 2.3.** Let \( X \) be a set and let \( x, y \in X \). We declare \( x \triangleleft_L y \), if and only if there exists \( L \in L \), such that \( x \in L \) and \( y \notin L \).

It follows that \( x \triangleleft_L y \), if and only if either \( x \triangleleft_L y \) or \( x = y \). One can easily see that \( \triangleleft_L \) is a linear order, if \( L \) is a \( T_0 \)-separating nest.

**Theorem 2.1** (See [2]). Let \( X \) be a set and suppose that \( L \) and \( R \) are two nests on \( X \). Then, \( L \cup R \) is \( T_1 \)-separating, if and only if \( L \) and \( R \) are both \( T_0 \)-separating and \( \triangleleft_L = \triangleleft_R \).

**Definition 2.4** (van Dalen & Wattel). Let \( X \) be a set and let \( L \subset \mathcal{P}(X) \). We say that \( L \) is interlocking if and only if, for each \( L \in L \), \( L = \bigcap\{N \in L : L \subset N\} \) implies \( L = \bigcup\{N \in L : N \subset L\} \).

**Theorem 2.2** (See [2]). Let \( X \) be a set and let \( L \) be a \( T_0 \)-separating nest on \( X \). The following are equivalent:

1. \( L \) is interlocking;
2. for each \( L \in L \), if \( L \) has a \( \triangleleft_L \)-maximal element, then \( X - L \) has a \( \triangleleft_L \)-minimal element;

**Theorem 2.3** (van Dalen & Wattel). Let \((X, \mathcal{T})\) be a topological space. Then:

1. If \( L \) and \( R \) are two nests of open sets, whose union is \( T_1 \)-separating, then every \( \triangleleft_L \)-order open set is open, in \( X \).
2. \( X \) is a GO-space, if and only if there are two nests \( L \) and \( R \) of open sets, whose union is \( T_1 \)-separating and forms a subbasis for \( \mathcal{T} \).
3. \( X \) is a LOTS, if and only if there are two interlocking nests \( L \) and \( R \) of open sets, whose union is \( T_1 \)-separating and forms a subbasis for \( \mathcal{T} \).
2.2 The Ring $\mathbb{R}$ of Fermat Reals.

The material in this subsection can be found in [7], [6] and also in [4].

**Definition 2.5.** A little-oh polynomial $x_t$ (or $x(t)$) is an ordinary set-theoretical function, defined as follows:

1. $x_t : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ and
2. $x_t = r + \sum_{i=1}^{k} \alpha_i t^{a_i} + o(t)$, as $t \to 0^+$, for suitable $k \in \mathbb{N}$, $r, \alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and $a_1, \ldots, a_k \in \mathbb{R}_{\geq 0}$.

The set of all little-oh polynomials is denoted by the symbol $\mathbb{R}_o[t]$. So, $x \in \mathbb{R}_o(t)$, if and only if $x$ is a polynomial function with real coefficients, of a real variable $t \geq 0$, with generic positive powers of $t$ and up to a little-oh function $o(t)$, as $t \to 0^+$.

**Definition 2.6.** Let $x, y \in \mathbb{R}_o[t]$. We declare $x \sim y$ (and we say $x = y$ in $\mathbb{R}$), if and only if $x(t) = y(t) + o(t)$, as $t \to 0^+$.

The relation $\sim$ in Definition 2.6 is an equivalence relation and $\mathbb{R} := \mathbb{R}_o[t]/\sim$.

A first attempt to define an order in $\bullet \mathbb{R}$ has come from Giordano.

**Definition 2.7** (Giordano). Let $x, y \in \mathbb{R}$. We declare $x \leq y$, if and only if there exists $z \in \mathbb{R}$, such that $z = 0$ in $\mathbb{R}$ (i.e. $\lim_{t \to 0^+} z_t/t = 0$) and for every $t \geq 0$ sufficiently small, $x_t \leq y_t + z_t$.

For simplicity, one does not use equivalence relation but works with an equivalent language of representatives. If one chooses to use the notations of [4], one has to note that Definition 2.7 does not depend on representatives.

As the author describes in [4], the order relation in NSA admits all formal properties among all the theories of (actual) infinitesimals, but there is no good dialectic of these properties with their informal interpretation. In particular, the order in $\mathbb{R}$ inherits by transfer all the first order properties but, on the other hand, in the quotient field $\mathbb{R}$ it is difficult to interpret these properties of the order relation as intuitive properties of the corresponding representatives. For example, a geometrical interpretation like that of $\mathbb{R}$ seems not possible for $\mathbb{R}$. Definition 2.7 provides a clear geometrical representation of the ring $\mathbb{R}$ (see, for instance, section 4.4 of [4]).
2.3 The Fermat Topology and the omega-topology on \(\mathbb{R}^n\).

A subset \(A \subset \mathbb{R}^n\) is open in the Fermat topology, if it can be written as \(A = \bigcup\{U \subset A : U \text{ is open in the natural topology in } \mathbb{R}^n\}\). Giordano and Kunzinger describe this topology as the best possible one for sets having a “sufficient amount of standard points”, for example \(\mathbb{U}\). They add that this connection between the Fermat topology and standard reals can be glimpsed by saying that the monad \(\mu(r) := \{x \in \mathbb{R} : \text{standard part of } x = r\}\) of a real \(r \in \mathbb{R}\) is the set of all points which are limits of sequences with respect to the Fermat topology. However it is obvious that in sets of infinitesimals there is a need for constructing a (pseudo-)metric generating a finer topology that the authors call the omega-topology (see [3]). Since neither the Fermat nor the omega-topology are Hausdorff when restricted to \(\mathbb{R}\) and since each of them describes sets having a “sufficient amount” of standard points or infinitesimals, respectively, there is a need for defining a natural topology on \(\mathbb{R}\) describing sufficiently all Fermat reals and carrying the best possible properties.

3 Interlocking Nests on \(\mathbb{R}\).

A first disadvantage of the construction in Definition 2.7 is that the order \(\leq\) in \(\mathbb{R}\) does not generate interlocking nests, missing points from the Fermat real line. In particular, the nest \(L\) consisting of sets \(L = \{k \in \mathbb{R} : k \leq l\}\), for some \(l \in \mathbb{R}\), has as maximal element the fermat real \(l\), but the complement of \(L\), i.e. \(L^c = \{k \in \mathbb{R} : k > l\}\), for some \(l \in \mathbb{R}\), does not have a minimal element. Thus, we first remark that the order of Definition 2.7 makes \(\mathbb{R}\) a GO-space, a subspace of a particular LOTS. So we will now have to construct an appropriate order in \(\mathbb{R}\) which makes it LOTS, by completing the missing minimal elements from complements of sets with maximal elements. Even the fact that \(\leq\) is linear, it generates the discrete topology on \(\mathbb{R}\) and, if considered as a strict order, the restriction of its topology in \(\mathbb{R}\) will be again the discrete topology.

3.1 Order Relations and an Order Topologies on \(\mathbb{R}\).

**Theorem 3.1.** The pair \((\mathbb{R}, <_F)\), where \(<_F\) is defined as follows:
let $y \in L$). So, according to Theorem 2.2, neither $L \in L$ nor $x \in R$ has a $\preceq_L$-maximal element (resp. $\preceq_R$-maximal element for $R$), such that $X - L$ has no $\preceq_L$-minimal element (resp. $X - R$ has no $\preceq_R$-minimal element). So, according to Theorem 2.2, neither $L$ nor $R$ are interlocking.

Now, for all $L = \{k \in \mathbb{R} : k \leq l\} \in \mathcal{L}$, some $l \in \mathbb{R}$, let $x_L$ denote the $\preceq_L$-maximal element of $L$ and for all $R = \{k \in \mathbb{R} : l \leq k\} \in \mathcal{R}$, some $l \in \mathbb{R}$, let $y_R$ denote the $\preceq_L$-minimal element of $R$.

Furthermore, for each $L \in \mathcal{L}$ choose $x_L^+ \in \mathbb{R}$ and for each $R \in \mathcal{R}$ choose $y_R^- \in \mathbb{R}$, where $x_L^+$ and $y_R^-$ are distinct points in $\mathbb{R}$, and define a map $p : \mathbb{R} \to \mathbb{R} - (\{x_L^+ : L \in \mathcal{L}\} \cup \{y_R^- : R \in \mathcal{R}\})$, as follows:

$$p(x) = \begin{cases} 
    x, & \text{if } x \in \mathbb{R} - (\{x_L^+ : L \in \mathcal{L}\} \cup \{y_R^- : R \in \mathcal{R}\}) \\
    x_L, & \text{if } x = x_L^+ \\
    y_R, & \text{if } x = y_R^-
\end{cases}$$

Now, define an order $<_F$ on $\mathbb{R}$, so that:

$$x <_F y \Leftrightarrow \begin{cases} 
    p(x) \preceq_L p(y) \\
    x = x_L \text{ and } y = x_L^+ \\
    x = y_R^- \text{ and } y = y_R
\end{cases}$$

Obviously, $<_F$ is a linear order and the restriction of $<_F$ to $\mathbb{R} - (\{x_L^+ : L \in \mathcal{L}\} \cup \{y_R^- : R \in \mathcal{R}\})$ equals $\preceq_L$, the order in Definition 2.7. In addition,
we can set $x^+_L = x_L + h$, where $h$ is not zero in $\mathbb{R}$ and $h > 0$, that is, $\lim_{t \to 0^+} h/t \neq 0$ and, respectively, we set $x^-_R = x_R - h$, and this completes the proof. 

**Theorem 3.2.** $\mathbb{R}$ equipped with the order topology from $<_F$ is a LOTS.

**Proof.** We will now show that the topology $T$ on $\mathbb{R} - \{x^+_L : L \in \mathcal{L}\} \cup \{y^-_R : R \in \mathcal{R}\}$ coincides with the subspace topology on $\mathbb{R} - \{x^+_L : L \in \mathcal{L}\} \cup \{y^-_R : R \in \mathcal{R}\}$ that is inherited from the $<_F$-order topology on $\mathbb{R}$.

But, since $\mathcal{L} \cup \mathcal{R}$ forms a subbasis for $T$, that consists of two nests, every set in $T$ can be written as a union of sets of the form $L \cap R$, where $L \in \mathcal{L}$ and $R \in \mathcal{R}$. It suffices therefore to show that every $L \in \mathcal{L}$ and $R \in \mathcal{R}$ can be written as the intersection of an order-open set with $\mathbb{R} - \{x^+_L : L \in \mathcal{L}\} \cup \{y^-_R : R \in \mathcal{R}\}$. But this is always true, since if $L \in \mathcal{L}$, with $\sqsupset L$-maximal element $x_L$, then $L = \mathbb{R} - \{x^+_L : L \in \mathcal{L}\} \cup \{y^-_R : R \in \mathcal{R}\} \cap \{x \in \mathbb{R} : x < F x^+_L\}$.

The argument for $R \in \mathcal{R}$ is similar, and this completes the proof. 

### 3.2 Remarks.

1. The order topology $T_{<_F}$ equals the topology $T_{\mathcal{L} \cup \mathcal{R} < _F}$, where $\mathcal{L} < _F = \{k \in \mathbb{R} : k <_F l\}$, some $l \in \mathbb{R}$ and $\mathcal{R} < _F = \{k \in \mathbb{R} : l <_F k\}$, some $l \in \mathbb{R}$. This is because $\mathcal{L} < _F \cup \mathcal{R} < _F$ $T_1$-separates $\mathbb{R}$ and both $\mathcal{L} < _F$ and $\mathcal{R} < _F$ are interlocking nests. So, unlike the GO-space topology $T_{\leq}$ on $\mathbb{R}$, where $T_{\leq} \subset T_{\mathcal{L} \cup \mathcal{R}}, <_F$ provides a natural extension of the natural linear order of the set of real numbers to the Fermat real line and the order topology from $<_F$ can be completely described via the nests $\mathcal{L} < _F$ and $\mathcal{R} < _F$.

2. Viewing the Fermat real line as a LOTS and working with nests $\mathcal{L} < _F$ and $\mathcal{R} < _F$, one can now define the product topology for $\mathbb{R}^n$, some positive integer $n$, or even more generally for $\Pi_{i \in I} \mathbb{R}_i$, some arbitrary indexing set $I$, in the usual way via the subbasis $\pi_{j_0}^{-1}(A_{j_0}) = \Pi_{i \in I} (\mathbb{R}_i : i \neq j_0) \times A_{j_0}$, where $A_{j_0}$ is an open subset in the coordinate space $\mathbb{R}_{j_0}$ in the order topology $T_{<_F}$ and $\pi_i : \Pi_{i \in I} \mathbb{R}_i \to \mathbb{R}_i$ the projection.

3. In this way one can define continuity for any function $f$ from a topological space $Y$ into the product space $\Pi_{i \in I} \mathbb{R}_i$ via the continuity of $\pi_i \circ f : Y \to \mathbb{R}_i$.

4. The height of $\mathbb{R}$ is 2 and the height of $\mathbb{R}^n = n + 1$ (see [10]). Using the product topology, as stated in Remark (2), we use four nests in...
order to define -for example- the topology in \( \mathbb{R}^2 \), but since the height of \( \mathbb{R}^2 \) is 3, one can define a topology using three nests exclusively.

### 3.3 Questions.

1. As a LOTS, \((\mathbb{R}, <_F)\) has rich topological properties. It is, for example, a monotone normal space. It would be interesting though to have an extensive study on the metrizability of this space. It is known that in a GO-space the terms metrizable, developable, semistratifiable, etc. are equivalent (see [8] and [9]). The real line (i.e. the set of all standard reals, from the point of view of \( \mathbb{R} \)) is a developable LOTS and this is equivalent to say that it is also a metrizable LOTS. Is \((\mathbb{R}, \mathcal{T}_{<_F})\) developable?

2. Which of the subspaces of \((\mathbb{R}, \mathcal{T}_{<_F})\) are developable?

Since any sequence \( x_1, x_2, \cdots \) of points in \( \Pi_{i \in I} \mathbb{R}_i \) will converge to a point \( x \in \Pi_{i \in I} \mathbb{R}_i \), iff for every projection \( \pi_i : \Pi_{i \in I} \mathbb{R}_i \to \mathbb{R}_i \) the sequence \( \pi_i(x_1), \pi_i(x_2), \cdots \) converges to \( \pi_i(x) \) in the coordinate space \( \mathbb{R}_i \), any answer to the above questions will be foundamental towards our understanding of convergence in the ring of Fermat Reals.
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