GROTHENDIECK-WITT GROUPS OF QUADRICS

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Abstract. Let $k$ be a commutative ring containing $\frac{1}{2}$. In this paper, we construct certain homotopy fibration sequences for Grothendieck-Witt spectra of smooth quadric hypersurfaces over $k$. As an application, we compute Witt groups of smooth quadric hypersurfaces over certain kinds of bases.

Contents

1. Introduction 1
2. Review: Grothendieck-Witt theory 4
3. Review: Semi-orthogonal decomposition on $\mathcal{S}^bQ$ 8
4. The canonical involution 11
5. Grothendieck-Witt spectra of odd dimensional quadrics 13
6. Witt groups of odd dimensional quadrics 18
7. An exact sequence 21
8. Grothendieck-Witt spectra of even dimensional quadrics 22
9. Witt groups of even dimensional quadrics 26
10. Witt groups of Quadrics defined by $m\langle -1 \rangle \perp n\langle 1 \rangle$ 29

Appendix A. Witt groups of Clifford algebras 32

References 34

1. Introduction

Let $(P, q)$ be a non-degenerate quadratic form over a commutative ring $k$ of rank $n = d + 2$. Let $Q_d \subset \mathbb{P}^{d+1}_k$ be the quadric hypersurface $\text{Proj } S(P^*)(q)$ where $S(P^*)$ is symmetric algebra of $P^*$ and $q$ is considered as an element in $S_2(P^*)$. We denote the connective $K$-theory spectrum and the Grothendieck-Witt theory spectrum by $K$ and $GW$ respectively; see Section 2 for a review and [26] for details. In this paper, we prove the following result.

Theorem 1.1. Let $k$ be a commutative ring with $\frac{1}{2} \in k$. If $d = 2m + 1$, then there is a stable equivalence of spectra

$$GW^{[i]}(Q_d) \approx m \bigoplus_{i=1} K(k) \oplus GW^{[i]}(\mathcal{A}).$$

If $d = 2m$, then there is a homotopy fibration sequence of spectra

$$\bigoplus_{j=1}^{m-1} K(k) \oplus GW^{[i]}(\mathcal{A}) \to GW^{[i]}(Q_d) \to GW^{[i-d]}(k, \det(P)).$$

Date: January 26, 2016.
where \( \det(P) = \Lambda^n P \) is the determinant of the projective module \( P \). In both cases, the spectrum \( GW^{[i]}(\mathcal{A}) \) fits into a homotopy fibration sequence

\[
GW^{[i]}(k) \rightarrow GW^{[i]}(\mathcal{A}) \rightarrow GW^{[i+1]}(C_0(q)_\sigma)
\]

where \( C_0(q) \) is the even part of the Clifford algebra of \( q \) and \( \sigma \) is the canonical involution of \( C_0(q) \). Moreover, this result also holds for \( GW \)-spectra.

Algebraic K-theory of quadric hypersurfaces was computed by Swan in the 1980s, cf. [28]. Swan proved that \( K(Q_d) \cong K(C_0(q)) \oplus K(k)^d \) for any commutative ring \( k \). Theorem 1.1 above may be viewed as the Grothendieck-Witt analogue to Swan’s result. Grothendieck-Witt theory (aka. Hermitian K-theory) as a refinement of algebraic K-theory was first studied by Bass, Karoubi and others in terms of algebras with involution, cf. [6]. The refinement stems from the data of duality. Loosely speaking, the data of duality allow us to study the category of non-degenerate quadratic forms which refines the category of finitely generated projective modules. Recently, Schlichting generalized Grothendieck-Witt theory of algebras with involution to that of exact categories and dg categories with duality, the results of which make Grothendieck-Witt theory bear a heavier burden, cf. [23], [24] and [26].

Grothendieck-Witt theory performs in Algebraic Geometry like the role of \( KO \)-theory in Topology (topological K-theory of real vector bundles), while algebraic K-theory is the algebraic analogue of \( KU \)-theory (topological K-theory of complex vector bundles). In topology, \( KO \)-theory usually refines \( KU \)-theory. Moreover, this refinement even helps to give the optimal solution. For example, the main cohomology theory involved in Adams’ solution to vector fields on spheres problem is just \( KO \)-theory, cf. [1].

In Section 2 we review the background of our framework, Grothendieck-Witt theory of dg categories [26]. To extend Swan’s result to Grothendieck-Witt theory, we need the semiorthogonal decomposition on the derived category \( D^b Q_d \) of bounded complexes of finitely generated vector bundles on \( Q_d \) which is reviewed in Section 3. We need a manipulation of the even part of the Clifford algebras with the canonical involution which is explored in Section 4. We also need to manipulate the duality on the dg categories of strictly perfect complexes on quadrics. This is done in Section 5 for the case of odd dimensional quadrics and in Section 8 for the case of even dimensional quadrics. In Section 6 we define a trace map

\[
tr : W^0(C_0(q)_\sigma) \rightarrow W^0(k)
\]

of Witt groups. Let \( p : Q_d \rightarrow \text{Spec} \ k \) be the structure map. As an application of Theorem 1.1 we get the following result in Section 6 and 9.

**Theorem 1.2.** Let \( k \) be a commutative semilocal ring with \( \frac{1}{2} \in k \). Assume that the odd indexed Witt groups \( W^{2i+1}(C_0(q)_\sigma) \) vanish (see Theorem A.3 for examples, e.g. \( k \) is a regular local ring containing a field of characteristic \( \neq 2 \)). Then,

\[
W^i(Q_d) = \begin{cases} 
\text{coker}(tr) & \text{if } d \neq 0 \text{ mod } 4 \text{ and if } i = 0 \text{ mod } 4 \\
W^2(C_0(q)_\sigma) & \text{if } d \neq 0 \text{ mod } 4 \text{ and if } i = 1 \text{ mod } 4 \\
0 & \text{if } d \neq 2 \text{ mod } 4 \text{ and if } i = 2 \text{ mod } 4 \\
\text{ker}(tr) & \text{if } d \neq 2 \text{ mod } 4 \text{ and if } i = 3 \text{ mod } 4
\end{cases}
\]
If \( d \equiv 2 \mod 4 \), then there is an exact sequence
\[
0 \rightarrow W^2(Q_d) \rightarrow W^0(k, \det(P)) \rightarrow \ker(tr) \rightarrow W^3(Q_d) \rightarrow 0.
\]

If \( d \equiv 0 \mod 4 \), then there is an exact sequence
\[
0 \rightarrow \ker(tr) \rightarrow W^0(Q_d) \rightarrow W^0(k, \det(P)) \rightarrow W^2(C_0(q)_\sigma) \rightarrow W^1(Q_d) \rightarrow 0.
\]

For the case of isotropic quadrics, we obtain the following result in Corollary 6.7 and Corollary 9.3.

**Corollary 1.3.** Let \( k \) be a field of characteristic \( \neq 2 \). Let \( Q_d \) be the quadric defined by an isotropic quadratic form \( q \) over \( k \). Then,
\[
W^i(Q_d) = \begin{cases} 
W^0(k) & \text{if } d \neq 0 \mod 4 \text{ and if } i = 0 \mod 4 \\
W^2(C_0(q)_\sigma) & \text{if } d \neq 0 \mod 4 \text{ and if } i = 1 \mod 4 \\
0 & \text{if } d \neq 2 \mod 4 \text{ and if } i = 2 \mod 4 \\
W^0(C_0(q)_\sigma) & \text{if } d \neq 2 \mod 4 \text{ and if } i = 3 \mod 4 \\
\end{cases}
\]

If \( d \equiv 2 \mod 4 \), then there is an exact sequence
\[
0 \rightarrow W^2(Q_d) \rightarrow W^0(k) \rightarrow W^0(C_0(q)_\sigma) \rightarrow W^3(Q_d) \rightarrow 0.
\]

If \( d \equiv 0 \mod 4 \), then there is an exact sequence
\[
0 \rightarrow W^0(k) \rightarrow W^0(Q_d) \rightarrow W^0(k) \rightarrow W^2(C_0(q)_\sigma) \rightarrow W^1(Q_d) \rightarrow 0
\]

where \( p^* \) is split injective.

Define the number \( \delta(n) := \#\{l \in \mathbb{Z} : 0 < l < n, l \equiv 0, 1, 2 \text{ or } 4 \mod 8\} \). This number is related to the vector fields on spheres problem; see [11]. For the case of quadric hypersurfaces defined by quadratic forms \( n(1) \), we prove the following result in Theorem 10.3

**Theorem 1.4.** Let \( k \) be a field in which \(-1\) is not a sum of two squares. Let \( Q_{0,n} \) be the smooth quadric associated to the quadratic form \( n(1) \) of dimension \( d = n - 2 \). If \( d \not\equiv 0 \mod 4 \), then \( W^0(Q_{0,n}) \approx W^0(k)/2^{\delta(n)}W^0(k) \). If \( d \equiv 0 \mod 4 \), then there is an exact sequence
\[
0 \rightarrow W^0(k)/2^{\delta(n)}W^0(k) \rightarrow W^0(Q_{0,n}) \rightarrow W^0(k).
\]

This exact sequence may not be split, as the following result shows. This result is proved in Corollary 10.2 and Corollary 10.4.

**Corollary 1.5.** Let \( Q_{0,n} \) be the quadric defined by the anisotropic quadratic form \( n(1) \) over \( \mathbb{R} \). Then,
\[
W^i(Q_{0,n}) = \begin{cases} 
\mathbb{Z}/2^{\delta(n)}\mathbb{Z} & \text{if } i = 0 \mod 4 \\
\mathbb{Z}/2\mathbb{Z} & \text{if } i = 1 \mod 4 \text{ and } n \equiv 3, 5 \mod 8 \\
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } i = 1 \mod 4 \text{ and } n \equiv 4 \mod 8 \\
0 & \text{otherwise}
\end{cases}
\]

**Remark 1.6.** The \( \frac{1}{2} \in k \) assumption is crucial for our computation, because we use the invariance theorem for \( GW \) [26, Theorem 6.5] which requires this assumption.
In Nenashev’s work [21], Witt groups of split quadrics were studied. In Zibrowski’s work [31], Grothendieck-Witt (and Witt) groups of quadrics over \( \mathbb{C} \) were computed. This paper focuses on the case of a general projective quadric over an arbitrary commutative ring \( k \) with \( \frac{1}{2} \in k \). Besides, Dell’Ambrogio and Fasel [7] studied Witt groups of certain kinds of quadrics by forgetting the 2 primary torsions. This paper determines the 2 primary torsion for certain cases. Charles Walter gave a talk on Witt groups of quadrics on a conference; cf. [29]. Presumably, there is some overlap between our work. However, Walter’s work has not been published. Calmès informed me that Walter’s approach (presented in [29]) involved the theory of spectral sequences, which is different from this paper.

**Preliminary.** Although some details are recalled, the author assumes that readers are familiar with the paper [26], especially Section 1, 3, 6, 8 and 9, and with the paper [28].

2. Review: Grothendieck-Witt theory

Let \( k \) be a commutative ring containing \( \frac{1}{2} \). Let \( C(k) \) denote the symmetric monoidal category of cochain complexes of \( k \)-modules. Let \( \mathcal{A} \) be a small dg category (that is a small category enriched in \( C(k) \)). The category \( Z^0\mathcal{A} \) of zero cycles (resp. \( H^0\mathcal{A} \) of zero cohomology) is called the underlying category (resp. the homotopy category) of the dg category \( \mathcal{A} \). A morphism in the dg category \( \mathcal{A} \) means a morphism in \( Z^0\mathcal{A} \). Every dg category \( \mathcal{A} \) has a pretriangulated-hull \( \mathcal{A}_{\text{ptr}} \) (which is pretriangulated) such that \( \mathcal{A} \) is a full dg subcategory of \( \mathcal{A}_{\text{ptr}} \). \( Z^0\mathcal{A}_{\text{ptr}} \) is Frobenius exact and \( H^0\mathcal{A}_{\text{ptr}} \) is triangulated, cf. Section 1 [26]. Moreover, \((-)^{\text{ptr}} \) is functorial.

A dg category with duality is a triplet \((\mathcal{A}, *, \text{can})\) consisting of a dg category \( \mathcal{A} \), a dg functor \( * : \mathcal{A}^{\text{op}} \to \mathcal{A} \) and a natural transformation \( \text{can} : 1 \to * \circ *^{\text{op}} \) such that \( \text{can}^* \circ \text{can}_{\mathcal{A}} = 1_{\mathcal{A}} \). A dg form functor \((\mathcal{A}, *, \text{can}_A) \to (\mathcal{B}, *, \text{can}_\mathcal{B})\) is a pair \((F, \varphi)\) consisting of a dg functor \( F : \mathcal{A} \to \mathcal{B} \) and a natural transformation \( \varphi : F \circ *_{\mathcal{A}} \to *_{\mathcal{B}} \circ F^{\text{op}} \) (called duality compatibility morphism) such that the diagram

\[
\begin{array}{ccc}
F \mathcal{A} & \xrightarrow{\text{can}_{F\mathcal{A}}} & F(A^* \circ A) \\
\downarrow{\text{can}_F} & & \downarrow{\varphi} \\
(F\mathcal{A})^{*\circ*} & & F(A^*A)^{*\circ*}
\end{array}
\]

commutes. A dg category with weak equivalences is a pair \((\mathcal{A}, w)\) consisting of the following data: a full dg subcategory \( \mathcal{A}^w \subset \mathcal{A} \) and a set of morphisms \( w \) in \( \mathcal{A}_{\text{ptr}} \) such that \( f \in w \) if and only if \( f \) is an isomorphism in the quotient triangulated category \( T(\mathcal{A}, w) := H^0\mathcal{A}_{\text{ptr}} / H^0(\mathcal{A}^w)_{\text{ptr}} \). An exact dg functor \( F : (\mathcal{A}, w) \to (\mathcal{C}, v) \) of dg categories with weak equivalences consists of a dg functor \( F : \mathcal{A} \to \mathcal{C} \) preserving weak equivalences, i.e. \( F(\mathcal{A}^w) \subset \mathcal{C}^v \). A quadruple \( \mathcal{A} = (\mathcal{A}, w, *, \text{can}) \) is called a dg category with weak equivalences and duality if \((\mathcal{A}, w)\) is a dg category with weak equivalences, if \((\mathcal{A}, *, \text{can})\) is a dg category with duality, if \( \mathcal{A}^w \) is invariant under the duality and if \( \text{can}_A \in w \) for all objects \( A \) in \( \mathcal{A} \). An exact dg form functor \((F, \varphi) : (\mathcal{A}, w, *, \text{can}) \to (\mathcal{C}, v, \#, \text{can})\) consists of a dg form functor \((F, \varphi) : (\mathcal{A}, *, \text{can}) \to (\mathcal{C}, \#, \text{can})\) such that the dg functor \( F : (\mathcal{A}, w) \to (\mathcal{C}, v) \) is exact. For every dg category \( \mathcal{A} \) with weak equivalences and duality, there is an
associated triangulated category $\mathcal{T} = (\mathcal{T}(\mathcal{A}, w), *, \text{can}, \lambda)$ with duality, cf. Section 3 [26]. Any exact dg form functor $\mathcal{A} \to \mathcal{C}$ gives a duality preserving functor $\mathcal{T} \mathcal{A} \to \mathcal{T} \mathcal{C}$ of triangulated categories with duality. Any dg category $\mathcal{A}$ with weak equivalences and duality gives a pretriangulated dg category $\mathcal{A}^{ptr}$ with weak equivalences and duality.

Let $\mathcal{A} = (\mathcal{A}, w, *, \text{can})$ be a dg category with weak equivalences and duality. Its Grothendieck-Witt group $GW(\mathcal{A})$ (resp. Witt group $W(\mathcal{A})$) is defined to be the Grothendieck-Witt group $GW(Z^0 \mathcal{A}^{ptr})$ (resp. the Witt group $W(Z^0 \mathcal{A}^{ptr})$) where $Z^0 \mathcal{A}^{ptr} := (Z^0 \mathcal{A}^{ptr}, w, *, \text{can})$ is the corresponding exact categories with weak equivalences and duality, cf. Section 1 [26]. The $n$-th shifted version $W^{[n]}(\mathcal{A})$ and $GW^{[n]}(\mathcal{A})$ are also defined in loc. cit. Moreover, $GW^{[n]}_0(\mathcal{A})$ (resp. $W^{[n]}(\mathcal{A})$) is isomorphic to $GW^n(\mathcal{T}\mathcal{A})$ (resp. $W^{[n]}(\mathcal{T}\mathcal{A})$) which is just isomorphic to Walter’s Grothendieck-Witt group (resp. Balmer’s Witt group) of a triangulated category with duality, cf. Section 3 [26]. Given a dg category $\mathcal{A}$ with weak equivalences and duality, one can define the $n$-th Grothendieck-Witt spectrum $GW^{[n]}(\mathcal{A})$ (cf. Section 5 [26]) and the $n$-th Karoubi-Grothendieck-Witt spectrum $GW^{[n]}(\mathcal{A})$ (cf. Section 8 [26]). Note that $GW^{[n]}(\mathcal{A})$ and $GW^{[n]}(\mathcal{A})$ are actually objects in the (large) category $Sp$ of spectra. For the following theorems, we consider them as objects in the stable homotopy category $\mathcal{SH}$ of spectra. By a stable equivalence of spectra, we mean an isomorphism in $\mathcal{SH}$. Actually, objects in $\mathcal{Sp}$ and $\mathcal{SH}$ are the same, but morphisms are different. One may find details in Appendix B2 [26].

**Theorem 2.1** (Agreement [26]). Assume that $\frac{1}{2} \in \mathcal{A}$. There are group isomorphisms $\pi_0GW^{[n]}(\mathcal{A}) \approx GW^{[n]}_0(\mathcal{A})$ and $\pi_iGW^{[n]}(\mathcal{A}) \approx W^{[n-i]}(\mathcal{A})$ for $i < 0$.

From now on, we shall use the notation

$$GW^{[n]}_i(\mathcal{A}) := \pi_iGW^{[n]}(\mathcal{A})$$

for any $i \in \mathbb{Z}$ (slightly abuse of the notation when $i = 0$).

Any exact dg form functor $\mathcal{A} \to \mathcal{C}$ gives maps of spectra $GW^{[n]}(\mathcal{A}) \to GW^{[n]}(\mathcal{C})$, $GW^{[n]}_0(\mathcal{A}) \to GW^{[n]}_0(\mathcal{C})$ and maps of groups $GW^{[n]}_0(\mathcal{A}) \to GW^{[n]}_0(\mathcal{C})$, $W^{[n]}(\mathcal{A}) \to W^{[n]}(\mathcal{C})$. The following theorem is proved in [26, Theorem 6.5].

**Theorem 2.2** (Invariance of $GW$). Assume $\frac{1}{2} \in \mathcal{A}, \mathcal{C}$. If $F : \mathcal{A} \to \mathcal{C}$ induces an equivalence $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{C}$ of triangulated categories, then $F$ gives an equivalence of spectra $GW^{[n]}(\mathcal{A}) \to GW^{[n]}(\mathcal{C})$.

Recall that a triangle functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ is cofinal if it is fully faithful and every object in $\mathcal{T}_2$ is a direct summand of an object in $\mathcal{T}_1$. The following is proved in [26, Theorem 8.9].

**Theorem 2.3** (Invariance of $GW$). Assume $\frac{1}{2} \in \mathcal{A}, \mathcal{C}$. If $F : \mathcal{A} \to \mathcal{C}$ induces a cofinal triangle functor $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{C}$ of triangulated categories, then $F$ gives a stable equivalence of spectra $GW^{[n]}(\mathcal{A}) \to GW^{[n]}(\mathcal{C})$.

A sequence $\mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3$ of triangulated categories is called exact if the composition is trivial, $\mathcal{T}_1 \to \mathcal{T}_2$ makes $\mathcal{T}_1$ into a full subcategory which is closed under direct factor, and the induced functor $\mathcal{T}_2/\mathcal{T}_1 \to \mathcal{T}_3$ is an equivalence where $\mathcal{T}_2/\mathcal{T}_1$ is
the Verdier quotient. A sequence \( \mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_3 \) of dg categories with weak equivalences and duality is \( \text{quasi-exact} \) if the induced sequence \( \mathcal{T} \mathcal{A}_1 \rightarrow \mathcal{T} \mathcal{A}_2 \rightarrow \mathcal{T} \mathcal{A}_3 \) is an exact sequence of triangulated categories. The following is proved in [26, Lemma 6.6].

**Theorem 2.4 (Localization for GW).** Let \( \frac{1}{2} \in \mathcal{A}_1 \). Assume that \( \mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_3 \) is quasi-exact. Then, there is a homotopy fibration of spectra

\[
GW^{[n]}(\mathcal{A}_1) \rightarrow GW^{[n]}(\mathcal{A}_2) \rightarrow GW^{[n]}(\mathcal{A}_3).
\]

Consequently, there is a long exact sequence of groups

\[
\cdots \rightarrow GW^{[n]}_{i+1}(\mathcal{A}_3) \rightarrow GW^{[n]}_i(\mathcal{A}_1) \rightarrow GW^{[n]}_i(\mathcal{A}_2) \rightarrow GW^{[n]}_i(\mathcal{A}_3) \rightarrow GW^{[n]}_{i-1}(\mathcal{A}_1) \rightarrow \cdots
\]

A sequence \( \mathcal{T}_1 \rightarrow \mathcal{T}_2 \rightarrow \mathcal{T}_3 \) of triangulated categories is called \( \text{exact up to factors} \) if the composition is trivial, \( \mathcal{T}_1 \rightarrow \mathcal{T}_2 \) is fully faithful, and the induced functor \( \mathcal{T}_2/\mathcal{T}_1 \rightarrow \mathcal{T}_3 \) is cofinal. A sequence \( \mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_3 \) of dg categories with weak equivalences and duality is \( \text{Morita exact} \) if the induced sequence \( \mathcal{T} \mathcal{A}_1 \rightarrow \mathcal{T} \mathcal{A}_2 \rightarrow \mathcal{T} \mathcal{A}_3 \) is exact up to factors. The following is proved in [26, Theorem 8.10].

**Theorem 2.5 (Localization for \( \mathbb{G} \mathcal{W} \)).** Let \( \frac{1}{2} \in \mathcal{A}_1 \). Assume that \( \mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_3 \) is Morita exact. Then, there is a homotopy fibration of spectra

\[
GW^{[n]}(\mathcal{A}_1) \rightarrow GW^{[n]}(\mathcal{A}_2) \rightarrow GW^{[n]}(\mathcal{A}_3).
\]

Consequently, there is a long exact sequence of groups

\[
\cdots \rightarrow GW^{[n]}_{i+1}(\mathcal{A}_3) \rightarrow GW^{[n]}_i(\mathcal{A}_1) \rightarrow GW^{[n]}_i(\mathcal{A}_2) \rightarrow GW^{[n]}_i(\mathcal{A}_3) \rightarrow GW^{[n]}_{i-1}(\mathcal{A}_1) \rightarrow \cdots
\]

Recall that a \( \text{semi-orthogonal decomposition} \) of a triangulated category \( \mathcal{T} \), denoted by \( \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n \), consists of the following data:

1. \( \mathcal{T}_i \) are full triangulated subcategories of \( \mathcal{T} \);
2. \( \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n \) generate \( \mathcal{T} \);
3. \( \text{Hom}(\mathcal{T}_i, \mathcal{T}_j) = 0 \) for all \( i < j \).

**Theorem 2.6 (Additivity [26]).** Assume \( \frac{1}{2} \in \mathcal{A} \). Let \( \mathcal{A} = (\mathcal{A}, w, *, \text{can}) \) be a pretriangulated dg category with weak equivalences and duality, and let \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \) be full dg subcategories of \( \mathcal{A} \) containing the full dg subcategory \( \mathcal{A}^w \) of \( w \)-acyclic objects. Assume that

1. \( \mathcal{A}_2 \) is fixed by the duality (that is \( * \mathcal{A}_2 = \mathcal{A}_2 \)),
2. \( \mathcal{A}_1 \) and \( \mathcal{A}_3 \) are exchanged by the duality, i.e. \( * \mathcal{A}_1 = \mathcal{A}_3 \) and \( * \mathcal{A}_3 = \mathcal{A}_1 \),
3. \( (\mathcal{T}(\mathcal{A}_1, w), \mathcal{T}(\mathcal{A}_2, w), \mathcal{T}(\mathcal{A}_3, w)) \) is a semi-orthogonal decomposition of \( \mathcal{T} \mathcal{A} \).

Then, there are stable equivalences of spectra (in \( SH \))

\[
K(\mathcal{A}_1) \oplus GW^{[n]}(\mathcal{A}_2) \xrightarrow{\sim} GW^{[n]}(\mathcal{A}) \quad \text{and} \quad K(\mathcal{A}_1) \oplus GW^{[n]}(\mathcal{A}_2) \xrightarrow{\sim} GW^{[n]}(\mathcal{A}_2)
\]

**Proof.** We prove this for \( GW \), and \( \mathbb{G}W \) is an analog. Let \( v \) be the set of morphisms that become isomorphisms in the Verdier quotient \( \mathcal{T}(\mathcal{A}, w)/\mathcal{T}(\mathcal{A}_2, w) \). Then, we have a quasi-exact sequence \( (\mathcal{A}_2, w) \rightarrow (\mathcal{A}, w) \rightarrow (\mathcal{A}, v) \). By localization, we see that this sequence induces a homotopy fibration of spectra

\[
GW^{[n]}(\mathcal{A}_2) \rightarrow GW^{[n]}(\mathcal{A}) \rightarrow GW^{[n]}(\mathcal{A}, v)
\]

Recall the hyperbolic dg category \( \mathcal{H} \mathcal{A}_1 \), cf. [26, Section 4.7]. Define a dg form functor \( \mathcal{H} \mathcal{A}_1 \rightarrow \mathcal{A}, (A, B) \rightarrow A \oplus B^* \). Claim that the composition

\[
\mathcal{H} \mathcal{A}_1 \rightarrow (\mathcal{A}, w) \rightarrow (\mathcal{A}, v)
\]
induces an equivalence $GW^{[n]}(\mathcal{H}_{\mathcal{A}_1}) \to GW^{[n]}(\mathcal{A}, v)$. Observe that $\mathcal{T}(\mathcal{A}, v) \cong (\mathcal{T}(\mathcal{A}_1, w), \mathcal{T}(\mathcal{A}_3, w))$. The functor $\mathcal{H}_{\mathcal{A}_1} \to (\mathcal{A}, v)$ gives a $K$-theory equivalence by the Invariance Theorem for $K$-theory. Moreover, the Witt groups of $\mathcal{H}_{\mathcal{A}_1}$ and $(\mathcal{A}, v)$ are both trivial, so that $\mathcal{H}_{\mathcal{A}_1} \to (\mathcal{A}, v)$ gives an isomorphism of Witt groups. Then, we can apply the Invariance Theorem for $GW$. One may find these details in the proof of Proposition 6.8 [26]. Thus, we have

$$GW^{[n]}(\mathcal{H}_{\mathcal{A}_1}) \oplus GW^{[n]}(\mathcal{A}_2) \xrightarrow{\sim} GW^{[n]}(\mathcal{A})$$

where $GW^{[n]}(\mathcal{H}_{\mathcal{A}_1})$ is equivalent to $GW(\mathcal{H}_{\mathcal{A}_1}) = K(\mathcal{A}_1)$, cf. [26, Section 6].

Let $(\mathcal{A}, *_{\mathcal{A}}, \text{can}_{\mathcal{A}})$ and $(\mathcal{B}, *_{\mathcal{B}}, \text{can}_{\mathcal{B}})$ be dg categories with duality. Recall the definition of tensor product dg category

$$(\mathcal{A}, *_{\mathcal{A}}, \text{can}_{\mathcal{A}}) \otimes (\mathcal{B}, *_{\mathcal{B}}, \text{can}_{\mathcal{B}}) := (\mathcal{A} \otimes \mathcal{B}, *_{\mathcal{A}} \otimes *_{\mathcal{B}}, \text{can}_{\mathcal{A}} \otimes \text{can}_{\mathcal{B}})$$

with duality in Section 1.8 [26].

**Lemma 2.7.** Assume that $(\mathcal{A}, *_{\mathcal{A}}, \text{can}_{\mathcal{A}}) \otimes (\mathcal{B}, *_{\mathcal{B}}, \text{can}_{\mathcal{B}}) \to (\mathcal{C}, *_{\mathcal{C}}, \text{can}_{\mathcal{C}})$ is a dg form functor. Then any symmetric form $(A, \varphi)$ on $(\mathcal{A}, *_{\mathcal{A}}, \text{can}_{\mathcal{A}})$ induces a dg form functor $(A, \varphi) \otimes ? : (\mathcal{B}, *_{\mathcal{B}}, \text{can}_{\mathcal{B}}) \to (\mathcal{C}, *_{\mathcal{C}}, \text{can}_{\mathcal{C}})$.

**Proof.** Recall that a dg form functor $(\mathcal{A}, *_{\mathcal{A}}, \text{can}_{\mathcal{A}}) \otimes (\mathcal{B}, *_{\mathcal{B}}, \text{can}_{\mathcal{B}}) \to (\mathcal{C}, *_{\mathcal{C}}, \text{can}_{\mathcal{C}})$ consists of the following data: a dg functor

$$F : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C} : (A, B) \to A \otimes B$$

and a duality compatibility natural transformation $\epsilon_{A, B} : A^* \otimes B^* \to (A \otimes B)^*$ such that the diagram

(1)

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{\varphi} & A^* \otimes B^* \\
\downarrow & & \downarrow \\
(A \otimes B)^* & \xrightarrow{\epsilon_{A, B}} & (A^* \otimes B^*)
\end{array}$$

commutes. Given a symmetric form $(A, \varphi)$ on $(\mathcal{A}, *_{\mathcal{A}}, \text{can}_{\mathcal{A}})$, we can define a dg form functor

$$(A, \varphi) \otimes ? : (\mathcal{B}, *_{\mathcal{B}}, \text{can}_{\mathcal{B}}) \to (\mathcal{C}, *_{\mathcal{C}}, \text{can}_{\mathcal{C}})$$

by the data, a dg functor $A \otimes ? : \mathcal{B} \to \mathcal{C} : B \mapsto A \otimes B$ and a duality compatibility map

$$\varphi : A \otimes B^* \xrightarrow{\varphi \otimes 1} A^* \otimes B^* \xrightarrow{\epsilon_{A, B}} (A \otimes B)^*$$

Consider the diagram
We conclude the commutativity of \(\Box_1\) by the symmetry of the form \(\varphi\), the commutativity of \(\Box_2\) by the commutative diagram (1) and the commutativity of \(\Box_3\) by the naturality of \(\epsilon\).

3. Review: Semi-orthogonal decomposition on \(\mathcal{D}^b\mathcal{Q}\)

Let \(k\) be a commutative ring. We do not assume \(\frac{1}{2} \in k\) in this section. Let \((P, q)\) be a non-degenerate quadratic form of rank \(n\) over \(k\). One has a homogeneous ring \(A = S(P^*)/(q)\) by regarding \(q\) as an element in \(S^2(P^*)\), cf. [28, Section 2]. Define \(Q\) (or \(Q_d\)) to be the projective variety \(\text{Proj} A\) which is smooth of relative dimensional \(d = n - 2\) over \(k\), cf. [28, Proposition 2.2].

Let \(\mathcal{D}^b\mathcal{Q}\) be the derived category of bounded chain complexes of finite rank locally free sheaves over \(Q\). It is well-known that Swan’s computation of \(K\)-theory of quadrics to a semi-orthogonal decomposition of \(\mathcal{D}^b\mathcal{Q}\). Meanwhile, \(\mathcal{D}^b\mathcal{Q}\) has been extensively studied, cf. [15], [18]. Since Swan’s version is mostly related to what we are doing, I will explain how to adapt Swan’s computation of \(K\)-theory of quadrics to a semi-orthogonal decomposition of \(\mathcal{D}^b\mathcal{Q}\). I thank Marco Schlichting for sharing with me his personal notes on this.

Define \(\mathcal{O}(i) := \overline{A(i)}\). Recall from [28] Section 8] that there is a pairing \(\mathcal{O}(i) \otimes P^* \rightarrow \mathcal{O}(i + 1)\) induced by the multiplication in the symmetric algebra. It follows that one has a map \(\mathcal{O}(i) \rightarrow \mathcal{O}(i + 1) \otimes P\) as the following composition

\[
\mathcal{O}(i) \rightarrow \text{Hom}_k(P^*, \mathcal{O}(i + 1)) \xrightarrow{\sim} \mathcal{O}(i + 1) \otimes \text{Hom}_k(P^*, k) \rightarrow \mathcal{O}(i + 1) \otimes P
\]

where the first map is induced by the pairing, where the middle map is the canonical isomorphism and where the last map is induced by the double dual identification.

If \((P, q)\) is a quadratic form over \(k\), the Clifford algebra \(C(q)\) is defined as \(T(P)/I(q)\) where \(T(P)\) is the tensor algebra of \(P\) and where \(I(q)\) is the two-sided ideal generated by \(v \otimes v - q(v)\) for all \(v \in P\). Moreover, the Clifford algebra \(C(q) = C_0(q) \oplus C_1(q)\) is \(\mathbb{Z}/2\mathbb{Z}\)-graded from the grading on \(T(P)\). Let \(\xi : P \rightarrow C(q)\) be the inclusion. The Clifford algebra \(C(q)\) has the universal property in the following sense: for any \(k\)-algebra \(B\), if \(\varphi : P \rightarrow B\) satisfies \(\varphi(v)^2 = q(v)\), then there exists a unique \(k\)-algebra homomorphism \(\tilde{\varphi} : C(q) \rightarrow B\) such that \(\tilde{\varphi} \circ \xi = \varphi\).

If \(M = M_0 \oplus M_1\) is a \(\mathbb{Z}/2\mathbb{Z}\)-graded left \(C(q)\)-module, then there are natural maps \(P \otimes M_j \rightarrow M_{j+1}\) which induce maps

\[
\ell = \ell_{i,j} : \mathcal{O}(i) \otimes M_j \rightarrow \mathcal{O}(i + 1) \otimes M_{j+1}
\]

by the composition

\[
\mathcal{O}(i) \otimes M_j \rightarrow \mathcal{O}(i + 1) \otimes P \otimes M_j \rightarrow \mathcal{O}(i + 1) \otimes M_{j+1}.
\]

It follows that there is a sequence \((s_i = i + d + 1 \in \mathbb{Z}/2\mathbb{Z})\)

\[
\cdots \rightarrow \mathcal{O}(-i - 1) \otimes M_{s_i + 1} \rightarrow \mathcal{O}(-i) \otimes M_{s_i} \rightarrow \mathcal{O}(-i + 1) \otimes M_{s_i - 1} \rightarrow \cdots
\]

which is called Clifford sequence in [28, Section 8]. Swan defines

\[
\mathcal{U}_i(M) := \text{coker}\left[ \mathcal{O}(-i - 2) \otimes M_{s_i + 2} \rightarrow \mathcal{O}(-i - 1) \otimes M_{s_i + 1} \right]
\]
and \( \mathcal{V}_i := \mathcal{V}(C(q)) \). Swan proves \( \text{End}(\mathcal{V}_i) \cong C_0(q) \), cf. [28] Corollary 8.8. Let \( \Lambda^n := \Lambda^n(P^*) \) be the \( n \)-th exterior power. Let \( \Lambda := \bigoplus_n \Lambda^n \) and \( \Lambda^{(i)} := \bigoplus_n \Lambda^{2n+i} \) for \( i \in \mathbb{Z}/2\mathbb{Z} \). Then, \( \Lambda = \Lambda^{(0)} \oplus \Lambda^{(1)} \) can be viewed as a \( \mathbb{Z}/2\mathbb{Z} \)-graded left \( C(q) \)-module, cf. [28] Corollary 8.8. The inclusion \( \text{det}(P^*) \subset \Lambda \) induces a \( \mathbb{Z}/2\mathbb{Z} \)-graded \( C(q) \)-module isomorphism \( C(q) \otimes \text{det} P^* \to \Lambda \), cf. [28] Lemma 8.3. Then, one sees that \( \mathcal{V}_n(\Lambda) = \mathcal{V}_n \otimes \text{det} P^* \) and that \( \text{End}(\mathcal{V}_n(\Lambda)) = C_0(q) \), cf. [28] Corollary 8.8. Taking \( \Lambda = \Lambda^{(0)} \oplus \Lambda^{(1)} \) to the sequence (2), we get

\[
\cdots \to \mathcal{O}(-i - 1) \otimes \Lambda^{(s_{i-1})} \to \mathcal{O}(-i) \otimes \Lambda^{(s_i)} \to \mathcal{O}(-i + 1) \otimes \Lambda^{(s_{i-1})} \to \cdots
\] (3)

Recall the Tate resolution

\[
\cdots \to \mathcal{O}(-2) \otimes (\Lambda^1 \otimes \Lambda^3 \otimes \Lambda^5) \to \mathcal{O}(-1) \otimes (\Lambda^0 \otimes \Lambda^2) \to \mathcal{O} \otimes \Lambda^1 \to \mathcal{O}(1),
\]

for \( \mathcal{O}(1) \), cf. [28] Section 7] or [28] Proof of Lemma 8.4. Precisely, the resolution (4) is given by

\[
T_{-i} := \mathcal{O}(-i) \otimes \left( \bigoplus_{d \geq 0} \Lambda^i \right)
\]

where \( \Lambda^i := 0 \) whenever \( i > n \) and \( i \geq 0 \). The differential

\[
\partial_i : \mathcal{O}(-i) \otimes \left( \bigoplus_{d \geq 0} \Lambda^i \right) \to \mathcal{O}(-i + 1) \otimes \left( \bigoplus_{d \geq 0} \Lambda^{i-2d} \right)
\]

is defined by \( \partial_i + \partial_i '' \) where

\[
\partial_i : f \otimes (p_1 \wedge \cdots \wedge p_k) \mapsto \sum (-1)^{s+1} f p_s \otimes (p_1 \wedge \cdots \wedge \hat{p}_s \wedge \cdots \wedge p_k)
\]

and

\[
\partial_i '' : f \otimes w \mapsto \gamma \wedge (f \otimes w) = \sum \xi_i \otimes (\beta_i \wedge w)
\]

where the element \( \gamma = \sum \xi_i \otimes \beta_i \in P^* \otimes P^* \) lifts \( q \in S_2(P^*) \) via the natural surjective map \( P^* \otimes P^* \to S_2(P^*) \).

The sequence (3) is exact, since Swan shows that, when \( i > d \), the sequence (3) and the Tate resolution (4) coincide and the maps in the sequence (3) have '2-periodicity', so the exactness of the sequence (3) when \( i < d \) is analogous to the case \( i \geq d \), cf. [28] Lemma 8.4.

**Proposition 3.1.** The functors

\[
\mathcal{O}(i) \otimes \mathcal{O} : \mathcal{D}^b k \to \mathcal{D}^b Q \text{ and } \mathcal{V}_i \otimes C_0(q) : \mathcal{D}^b C_0(q) \to \mathcal{D}^b Q
\]

are fully faithful, where \( \mathcal{D}^b C_0(q) \) is the derived category of bounded complexes of finitely generated projective left \( C_0(q) \)-modules.

**Proof.** Note that

\[
\text{Hom}_{\mathcal{D}^b Q}(\mathcal{O}(i), \mathcal{O}(i)[p]) = H^p(Q, \mathcal{O}) = \begin{cases} 0 & \text{if } p > 0 \\ k & \text{if } p = 0 \end{cases}
\]

by [28] Lemma 5.2 and that

\[
\text{Hom}_{\mathcal{D}^b Q}(\mathcal{V}_i, \mathcal{V}_i[p]) = \text{Ext}^p(\mathcal{V}_{d-1}, \mathcal{V}_{d-1}) = \begin{cases} 0 & \text{if } p > 0 \\ C_0(q) & \text{if } p = 0 \end{cases}
\]

by [28] Corollary 8.8 and [28] Lemma 6.1. The result follows. \( \square \)
Let \( \mathcal{U}_i \) (resp. \( \mathcal{A}_i \)) be the smallest full idempotent complete triangulated subcategory of \( \mathcal{D}^b \mathcal{Q} \) containing the bundle \( \mathcal{V}_i \) (resp. line bundle \( \mathcal{O}(i) \)), that is the essential image of the functor \( \mathcal{V}_i \otimes \mathcal{C}_{n_i} : \mathcal{D}^b \mathcal{Q}_0 \rightarrow \mathcal{D}^b \mathcal{Q} \) (resp. the essential image of the functor \( \mathcal{O}(i) \otimes \mathcal{?} : \mathcal{D}^b k \rightarrow \mathcal{D}^b \mathcal{Q} \)).

**Theorem 3.2.** There is a semi-orthogonal decomposition
\[
\mathcal{D}^b \mathcal{Q} = (\mathcal{U}_{d-1}, \mathcal{A}_{1-d}, \ldots, \mathcal{A}_{-1}, \mathcal{A}_0).
\]

**Proof.** Firstly, we show that the set
\[
\Sigma = \{ \mathcal{U}_{d-1}, \mathcal{O}(1-d), \ldots, \mathcal{O} \}
\]
generates \( \mathcal{D}^b \mathcal{Q} \) as an idempotent complete triangulated category. Then, \( \mathcal{D}^b \mathcal{Q} \) is generated by the idempotent complete full triangulated subcategories \( \mathcal{U}_{d-1}, \mathcal{A}_{1-d}, \ldots, \mathcal{A}_0 \).

Let \( \langle \Sigma \rangle \subset \mathcal{D}^b \mathcal{Q} \) denote the full triangulated subcategory generated by \( \Sigma \). Note that \( \mathcal{Q} \) is projective and thus a scheme with an ample line bundle \( \mathcal{O}(1) \). By [25 Lemma 3.5.2 or Lemma A.4.7], the triangulated category \( \mathcal{D}^b \mathcal{Q} \) is generated (as an idempotent complete triangulated category) by line bundles \( \mathcal{O}(i) \) for \( i \leq 0 \). Taking duals, we see \( \mathcal{D}^b \mathcal{Q} \) is generated (as an idempotent complete triangulated category) by \( \mathcal{O}(i) \) for \( i \geq 0 \). The Tate resolution [4] implies that \( \mathcal{O}(1) \) is in \( \langle \Sigma \rangle \). The canonical resolution (cf. [28, p. 126 Section 6]) shows that \( \mathcal{O}(i) \) is in \( \langle \Sigma \rangle \) for \( i \geq 2 \).

It is clear that
\[
\text{Hom}_{\mathcal{D}^b \mathcal{Q}}(\mathcal{O}(i), \mathcal{O}(j)[p]) = H^p(Q, \mathcal{O}(j-i)) = 0
\]
for \( 1-d \leq j < i \leq 0 \), cf. [28 Lemma 5.2]. Moreover, we have
\[
\text{Hom}_{\mathcal{D}^b \mathcal{Q}}(\mathcal{O}(i), \mathcal{U}_{d-1}[p]) = H^p(Q, \mathcal{U}_{d-1}(-i)) = 0
\]
for \( 1-d \leq i \leq 0 \), cf. [28 Proof of Lemma 9.3] for \( p > 0 \) and [28 Lemma 9.5] for \( p = 0 \). Thus, we conclude \( \text{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0 \) for \( j < i \) and \( \text{Hom}(\mathcal{A}_i, \mathcal{U}_{d-1}) = 0 \).

**Corollary 3.3.** There is a semi-orthogonal decomposition
\[
\mathcal{D}^b \mathcal{Q}_d = \begin{cases} 
(\mathcal{A}_{-m}, \ldots, \mathcal{A}_{1-d}, \mathcal{U}_0, \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_m) & \text{if } d = 2m+1; \\
(\mathcal{A}_{1-m}, \ldots, \mathcal{A}_{-1}, \mathcal{U}_0, \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{m-1}, \mathcal{A}_m) & \text{if } d = 2m.
\end{cases}
\]

**Proof.** Let \( d = 2m+1 \). By taking the tensor product \( \mathcal{O}(m) \otimes E \) for every element \( E \in \Sigma \), we get another set
\[
\{ \mathcal{V}_m, \mathcal{O}(-m), \ldots, \mathcal{O}, \mathcal{O}(m) \}.
\]
Clearly, this set generates \( \mathcal{D}^b \mathcal{Q} \) as an idempotent complete triangulated category. Note that
\[
\Sigma' = \{ \mathcal{O}(-m), \ldots, \mathcal{O}(1), \mathcal{U}_0, \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(m) \}
\]
also generates \( \mathcal{D}^b \mathcal{Q} \) as an idempotent complete triangulated category. It is enough to show \( \mathcal{V}_m \) is in \( \langle \Sigma' \rangle \). But this follows from the Clifford sequence and the definition of \( \mathcal{V}_m \) which gives the following exact sequence
\[
0 \rightarrow \mathcal{V}_m \rightarrow \mathcal{O}(-m) \otimes C_{s_m} \rightarrow \cdots \rightarrow \mathcal{O}(-1) \otimes C_{s_{-1}} \rightarrow \mathcal{U}_0 \rightarrow 0.
\]
Besides, we have \( H^p(Q, \mathcal{F} \otimes \mathcal{V}_m') = \text{Ext}^p(\mathcal{V}_0, \mathcal{F}) \). Thus, by Lemma 4.5 below,
\[
\text{Hom}(\mathcal{V}_0, \mathcal{O}(i)[p]) = H^p(Q, \mathcal{V}_0(i))
\]
is isomorphic to
\[
H^p(Q, \mathcal{U}_{d-1}(i)) = H^p(Q, \mathcal{U}_{d-1}(d+i))
\]
For $i \leq j$, let $\mathcal{A}_{i,j} \subset \mathcal{D}^B Q$ (resp. $\mathcal{A} \subset \mathcal{D}^B Q$) denote the full triangulated subcategory
$$\langle \mathcal{A}_i, \ldots, \mathcal{A}_j \rangle$$
that is the smallest triangulated subcategory of $\mathcal{D}^B Q$ containing $\mathcal{A}_i, \ldots, \mathcal{A}_j$ (resp. $\mathcal{U}_0, \mathcal{A}_0$).

Corollary 3.4. There is a semi-orthogonal decomposition
$$\mathcal{D}^B Q_d = \begin{cases} \langle \mathcal{A}_{[-m,-1]}, \mathcal{A}, \mathcal{A}_{[1,m]} \rangle & \text{if } d = 2m + 1; \\ \langle \mathcal{A}_{[-m,-1]}, \mathcal{A}, \mathcal{A}_{[1,m-1]}, \mathcal{A}_m \rangle & \text{if } d = 2m. \end{cases}$$

4. The canonical involution

Let $k$ be a commutative ring. Let $A$ be a $k$-algebra. Recall that an involution $\tau : A \to A^{op}$ is a $k$-algebra homomorphism such that $\tau^2 = id$. The inclusion $\varphi : P \to C(q)^{op}$ satisfies $\varphi(v)^2 = q(v)$, hence it provides a $k$-algebra homomorphism $\sigma : C(q) \to C(q)^{op}$, $x \mapsto \bar{x}$ which is an involution (called canonical involution). Certainly, the canonical involution preserves the $\mathbb{Z}/2\mathbb{Z}$-grading on $C(q)$, so it restricts to the even part $C_0(q)$.

Assume $(P, q)$ is free of rank $n$ over $k$. Let $\{e_1, \ldots, e_n\}$ be a basis of $P$. Then the Clifford algebra $C(q)$ is free of rank $2^n$ over $k$ with a basis $\{e^\Delta : \Delta \in \mathbb{F}_2^n\}$ where $e^\Delta := e_1^{b_1} \cdots e_n^{b_n}$ with $\Delta = (b_1, \ldots, b_n) \in \mathbb{F}_2^n$, cf. [16, Theorem IV.1.5.1]. In this case, one could explicitly define a “trace” form $tr : C(q) \to k$ by $tr(\alpha e^\delta) = a$ (for $a \in k$) and $tr(e^\Delta) = 0$ for any $\Delta \neq 0 \in \mathbb{F}_2^n$. Let $(k^n, q_h)$ be the free quadratic form given by
$$q_h := \begin{cases} \mathbb{H}^m & \text{if } n = 2m \\ \mathbb{H}^m \perp \{a\} & \text{if } n = 2m + 1 \end{cases}$$
where $\mathbb{H}^n := \bigoplus_{i=1}^m \langle 1,-1 \rangle$ is the hyperbolic form and $a \in k^\times$. Let $(P, q)$ be a non-degenerate quadratic form of rank $n$ over $k$. Then, there is an isomorphism
$$\alpha : C(q) \otimes_k S \to C(q_h)_S$$
where $S$ is some etale extension over $k$, cf. [28]. Then, one defines a (generalized) “trace” form $C(q) \to k$ by $a \mapsto tr(\alpha(a \otimes 1))$. The trace form restricts to the even part $C_0(q)$.

Lemma 4.1. Let $k$ be a commutative ring. Then, the trace form $tr : C(q) \to k$ has the property $tr(uv) = tr(vu)$.

Proof. Step I. Assume $(P, q)$ is free over $k$. It is enough to show that
$$tr(\sum a_\Delta e^\Delta \cdot \sum b_\Xi e^{\Xi}) = tr(\sum b_\Xi e^{\Xi} \cdot \sum a_\Delta e^\Delta)).$$
Observe that
$$tr(\sum a_\Delta e^\Delta \cdot \sum b_\Xi e^{\Xi}) = \sum a_\Delta b_\Delta (e^\Delta e^{\Delta})$$
and that
$$tr(\sum b_\Xi e^{\Xi} \cdot \sum a_\Delta e^\Delta) = \sum b_\Delta a_\Delta (e^\Delta e^{\Delta}).$$
Lemma 4.3. Let \( M \) be an opposite module, then \( A \) considered as a right \( A \)-quadratic form over \( C \). Let the map \( \vartheta \) be a right \( C \)-quadratic form over \( k \). Then the map \( \mathcal{E} \) serve that \( \vartheta \) preserves the \( C \)-quadratic form over \( k \). Then, we apply [28, Proposition 1.1. (a)] to reduce the problem to the case when the base \( k \) is a field. The form \( B \) is non-degenerate over a field of characteristic \( \neq 2 \), cf. [27, Exercise 3.14].

Proof. Since \( \frac{1}{2} \in k \), we can use [28, Corollary 1.2] to identify non-singularity and non-degeneracy. Then, we apply [28, Proposition 1.1. (a)] to reduce the problem to the case when the base \( k \) is a field. The form \( B \) is non-degenerate over a field of characteristic \( \neq 2 \), cf. [27, Exercise 3.14].

Let \( A \) be any \( k \)-algebra, and let \( A^\ast := \text{Hom}_k(A, k) \). If \( A \) is viewed as a right \( A \)-module, then \( A^\ast \) may be considered as a left \( A \)-module via the right multiplication. Let \( M \) be a right \( A \)-module. If we endow \( A \) with an involution \( a \mapsto \bar{a} \), then its opposite module \( \overline{M}^\ast \) equals \( M \) as a \( k \)-module and can be viewed as a left \( A \)-module: \( a \cdot \overline{m}^\ast = (\overline{ma})^\ast \) for any \( a \in A \). It follows that \( \text{Hom}_k(\overline{M}^\ast, k) \) can be considered as a right \( A \)-module.

Lemma 4.2. Let \( k \) be a commutative ring with \( \frac{1}{2} \). Assume \( (P, q) \) is non-degenerate. Then \( B \) is non-degenerate.

Proof. Since \( \frac{1}{2} \in k \), we can use [28, Corollary 1.2] to identify non-singularity and non-degeneracy. Then, we apply [28, Proposition 1.1. (a)] to reduce the problem to the case when the base \( k \) is a field. The form \( B \) is non-degenerate over a field of characteristic \( \neq 2 \), cf. [27, Exercise 3.14].

Let \( A \) be any \( k \)-algebra, and let \( A^\ast := \text{Hom}_k(A, k) \). If \( A \) is viewed as a right \( A \)-module, then \( A^\ast \) may be considered as a left \( A \)-module via the right multiplication. Let \( M \) be a right \( A \)-module. If we endow \( A \) with an involution \( a \mapsto \bar{a} \), then its opposite module \( \overline{M}^\ast \) equals \( M \) as a \( k \)-module and can be viewed as a left \( A \)-module: \( a \cdot \overline{m}^\ast = (\overline{ma})^\ast \) for any \( a \in A \). It follows that \( \text{Hom}_k(\overline{M}^\ast, k) \) can be considered as a right \( A \)-module.

Lemma 4.3. Let \( k \) be a commutative ring with \( \frac{1}{2} \). Consider \( C(q) \) as a right \( C(q) \)-module. Then the map

\[
\vartheta : C(q) \to \text{Hom}_k(C(q)^\ast, k) : x \mapsto (\vartheta(x) : y^\ast \mapsto B(x, y))
\]

is a right \( C(q) \)-module isomorphism such that \( \vartheta \circ \text{can} = \vartheta \) where \( \text{can} : C(q) \to C(q)^{\ast\ast} \) is the double dual identification.

Proof. The map \( \vartheta \) is clearly a \( k \)-linear isomorphism. The only less obvious thing to check is that \( \vartheta(xa)(y^\ast) = [\vartheta(x)a](y^\ast) \) for all \( a \in C(q) \) and \( y^\ast \in C(q)^\ast \). Observe that \( \vartheta(xa)(y^\ast) = tr(\overline{x}a \cdot y) = tr(\overline{a}x\overline{y}) \) and \( [\vartheta(x)a](y^\ast) = (\vartheta(x))(a \cdot y^\ast) = \vartheta(x)((y\overline{a})^\ast) = tr(\overline{x}y\overline{a}) \). Using the identity \( tr(uv) = tr(vu) \) for any \( u, v \in C(q) \), we conclude the result.

Remark 4.4. The map \( \vartheta \) preserves the \( \mathbb{Z}/2\mathbb{Z} \)-grading of \( C(q) \).

If \( \mathcal{E} \) is a vector bundle, we denote the sheaf hom \( \mathcal{H}om(F, \mathcal{O}) \) by \( F^\ast \). Let \( \text{can}_\mathcal{F} : \mathcal{F} \to \mathcal{F}^{\ast\ast} \) be the canonical double dual identification. Recall the notation in the previous section.

Lemma 4.5. Let \( k \) be a commutative ring with \( \frac{1}{2} \). Let \( (P, q) \) be a non-degenerate quadratic form over \( k \). Then, there are isomorphisms

\[
h_i : \mathcal{H}_i \to (\mathcal{H}_{i-1}^\ast)^\ast
d\text{of } \mathcal{O} \text{-modules and right } C_0(q) \text{-modules. Moreover, we have the following equalities}
\]

\[
h_{i-1} = (h_i)^\ast \circ \text{can}_\mathcal{H}_{i-1}.
\]
Proof. Let $C_s$ denote $C_s(q)$ for $s \in \mathbb{Z}/2\mathbb{Z}$ for simplicity. Define a map
\[
\eta_i : \mathcal{O}(-i) \otimes C_{s_i} \longrightarrow (\mathcal{O}(i) \otimes C_{s_i}^{\text{op}})^{\vee}
\]
by $\eta_i(f \otimes x)(g \otimes y^{\text{op}}) = f \cdot \text{tr}(\bar{xy})$. It is evident that $\eta_i$ is an isomorphism, because $\eta_i$ is the following composition
\[
\mathcal{O}(-i) \otimes C_{s_i} \xrightarrow{\psi \otimes \vartheta} \mathcal{O}(i) \otimes (C_{s_i}^{\text{op}})^{\ast} \longrightarrow (\mathcal{O}(i) \otimes C_{s_i}^{\text{op}})^{\vee}
\]
where $\psi : \mathcal{O}(i) \to \mathcal{O}(i)^{\vee}$ is the natural isomorphism, $\vartheta$ is defined in Lemma 4.3 and the last map is just the canonical isomorphism. Note that $C_s = C_{-s}$. By Lemma 4.3 and Remark 4.4 we see $\eta_i$ are well-defined right $C_0(q)$-homomorphisms.

The diagram labeled by the symbol $\ast$
\[
\begin{array}{ccc}
\mathcal{O}(-i-2) \otimes C_{s_{i+2}} & \xrightarrow{\ell} & \mathcal{O}(-i-1) \otimes C_{s_{i+1}} \\
\eta & & \eta \\
\mathcal{O}(i+2) \otimes C_{s_{i+2}}^{\text{op}} & \xrightarrow{\ell^\prime} & \mathcal{O}(i+1) \otimes C_{s_{i+1}}^{\text{op}}^{\vee}
\end{array}
\]
is commutative. This may be checked locally. Let $f \otimes m \in \mathcal{O}(-i-2) \otimes C_{s_{i+2}}$. Assume $k$ is a local ring (by localizing at a prime ideal), we see the map $\ell$ may be interpreted as $\ell(f \otimes m) = \sum \beta_i f \otimes \xi_i m$ for some $\beta_i \in P^\ast, \xi_i \in P$. Let $g \otimes n^{\text{op}} \in \mathcal{O}(i+1) \otimes C_{s_{i+1}}^{\text{op}}$. It reduces to check
\[
\eta(\sum \beta_i f \otimes \xi_i m)(g \otimes n^{\text{op}}) = \eta(f \otimes m)(\sum \beta_i g \otimes (\xi_i n)^{\text{op}})
\]
The left-hand side equals $\sum \beta_i fg \cdot \text{tr}(\bar{\xi_i mn}) = \sum \beta_i fg \cdot \text{tr}(\bar{\xi_i n})$, while the right-hand side equals $\sum \beta_i \cdot \text{tr}(\bar{\xi_i n})$. Note that $\bar{\xi} = \xi \in P$. This provides the equality.

Then, the dotted map $h_i$ is just given by the universal property of cokernels. Since $\eta_i$ are isomorphisms, so are the maps $h_i$. The last assertion is obtained by the symmetry of $\vartheta$ in Lemma 4.3.

5. Grothendieck-Witt Spectra of Odd Dimensional Quadrics

Let $k$ be a commutative ring with $\frac{1}{2}$ throughout this section. Let $\text{sPerf}(Q_d)$ be the dg category of strictly perfect complexes over $Q_d$, that is the dg category of bounded cochain complexes of finite rank locally free sheaves over $Q_d$. Let $\mathcal{L}[i]$ be any shift of a line bundle over $Q_d$. Let $#_{\mathcal{L}[i]}$ denote the dg functor $[-, \mathcal{L}[i]]$ and $\text{can}_{\mathcal{L}[i]}$ be the canonical double dual identification. Take quasi-isomorphisms as weak equivalences. The quadruple
\[(\text{sPerf}(Q_d), \text{quis}, #_{\mathcal{L}[i]}, \text{can}_{\mathcal{L}[i]})\]
is a dg category with weak equivalences and duality, cf. \[26\] Section 9. Then, the triangulated category $\mathcal{T}(\text{sPerf}(Q_d), \text{quis})$ is just $\mathcal{G}bQ_d$.

Set
\[GW[i](Q_d) := GW(\text{sPerf}(Q_d), \text{quis}, #_{\mathcal{L}[i]}, \text{can}_{\mathcal{L}[i]}).\]
Note that
\[GW[i](Q_d) \cong GW[i](\text{sPerf}(Q_d), \text{quis}, #_{\mathcal{O}}, \text{can}_{\mathcal{O}}).\]
One could write a similar notation for the case of $GW$-spectra.

We define $\mathcal{A}, \mathcal{A}_{[k,l]}$ and $\mathcal{A}$ to be the full dg categories of $sPerf(Q_d)$ corresponding to the full triangulated categories $\mathcal{A}_i, \mathcal{A}_{[k,l]}$ and $\mathcal{A}$ of $D^bQ_d$ respectively. Explicitly, $\mathcal{A}_i, \mathcal{A}_{[k,l]}, \mathcal{A}$ have objects which lie in $\mathcal{A}_i, \mathcal{A}_{[k,l]}, \mathcal{A}$ respectively. Moreover, $\mathcal{A}_i, \mathcal{A}_{[k,l]}$ and $\mathcal{A}$ are pretriangulated.

**Lemma 5.1.** $\mathcal{A} \subset sPerf(Q_d)$ is fixed by the duality $\#_\sigma$.

**Proof.** It is enough to show $\mathcal{A}$ is fixed by the duality $\#_\sigma$. By the definition of $\mathcal{A}_i$, we have an exact sequence

$$\mathcal{A}_0 \longrightarrow \mathcal{O} \otimes C_0(q) \longrightarrow \mathcal{A}_1$$

It follows that $\mathcal{A}_1 \in \mathcal{A}$. By Lemma 4.5 we see $\mathcal{A}_0'' \simeq \mathcal{A}_1$ in $\mathcal{A}$. $\square$

Thus we have a pretriangulated dg category with weak equivalences and duality $(\mathcal{A}, \text{quis}, \#_\sigma, \text{can}^\sigma)$.

Next, recall the hyperbolic dg category $\mathcal{H}_{\mathcal{A}_{[1,m]}}$ defined in [26, Section 4].

**Theorem 5.2.** There is a stable equivalence of Grothendieck-Witt spectra

$$GW(\mathcal{H}_{\mathcal{A}_{[1,m]}}) \oplus GW^{[i]}(\mathcal{A}) \longrightarrow GW^{[i]}(Q_d)$$

and a stable equivalence of Karoubi-Grothendieck-Witt spectra

$$GW(\mathcal{H}_{\mathcal{A}_{[1,m]}}) \oplus GW^{[i]}(\mathcal{A}) \longrightarrow GW^{[i]}(Q_d).$$

**Proof.** This result is a consequence of the additivity theorem (cf. Theorem 2.6) and Corollary 3.4. $\square$

Recall from [26] that we have

$$K(\mathcal{A}_{[1,m]}) = GW(\mathcal{H}_{\mathcal{A}_{[1,m]}}) \quad \text{and} \quad K(\mathcal{A}_{[1,m]}) = GW(\mathcal{H}_{\mathcal{A}_{[1,m]}}).$$

By additivity in $K$-theory, we conclude

$$K(\mathcal{A}_{[1,m]}) = \bigotimes_{i=1}^m K(\mathcal{A}_i) \quad \text{and} \quad K(\mathcal{A}_{[1,m]}) = \bigotimes_{i=1}^m K(\mathcal{A}_i).$$

Moreover, note that the exact dg functor

$$\mathcal{O}(i)\otimes : (sPerf(k), \text{quis}) \longrightarrow (\mathcal{A}_i, \text{quis})$$

induces an equivalence of associated triangulated categories

$$\mathcal{O}(i)\otimes : D^b(k) \longrightarrow \mathcal{A}_i.$$

Applying [25, Theorem 3.2.24] and [25, Theorem 3.2.29], we see

$$\mathcal{O}(i)\otimes : K(k) \longrightarrow K(\mathcal{A}_i) \quad \text{and} \quad \mathcal{O}(i)\otimes : K(k) \longrightarrow K(\mathcal{A}_i).$$

**Corollary 5.3.** There is a stable equivalence of Grothendieck-Witt spectra

$$(H\mathcal{O}(1)\otimes, \ldots, H\mathcal{O}(m)\otimes, ?) : \bigoplus_{i=1}^m K(k) \oplus GW^{[i]}(\mathcal{A}) \longrightarrow GW^{[i]}(Q_d)$$

and a stable equivalence of Karoubi-Grothendieck-Witt spectra

$$(H\mathcal{O}(1)\otimes, \ldots, H\mathcal{O}(m)\otimes, ?) : \bigoplus_{i=1}^m K(k) \oplus GW^{[i]}(\mathcal{A}) \longrightarrow GW^{[i]}(Q_d).$$
It remains to understand $GW^{[i]}(\mathcal{A})$ and $GW^{[i]}(\mathcal{A})$. Let $w$ be the set of morphisms in $\mathcal{A}$ that become isomorphisms in the Verdier quotient $\mathcal{T}(\mathcal{A}, \text{quis})/\mathcal{T}(\mathcal{A}_0, \text{quis})$.

This quotient category is equivalent to $\mathcal{T}(\mathcal{A}, w)$. Then, there is a quasi-exact sequence (hence Morita exact)

\[
(\mathcal{A}_0, \text{quis}) \to (\mathcal{A}, \text{quis}) \to (\mathcal{A}, w)
\]

in the sense of [26, Section 6] which provides localization sequences of $GW^{[i]}$-spectra

\[
GW^{[i]}(\mathcal{A}_0) \to GW^{[i]}(\mathcal{A}) \to GW^{[i]}(\mathcal{A}, w)
\]

by [26, Theorem 6.6], and localization sequences of Karoubi $GW^{[i]}$-spectra

\[
GW^{[i]}(\mathcal{A}_0) \to GW^{[i]}(\mathcal{A}) \to GW^{[i]}(\mathcal{A}, w)
\]

by [26, Theorem 8.9]. The exact dg form functor

\[
(\mathcal{O}, \text{id}) \otimes : (\text{sPerf}(k), \text{quis}, \#_k) \to (\mathcal{A}_0, \text{quis})
\]

gives an equivalence $\mathcal{O} \otimes : \mathcal{D}^b(k) \to \mathcal{A}_0$ of associated triangulated categories. By the invariance for $GW$ (cf. [26, Theorem 6.5]) and the invariance for $GW$ (cf. [26, Theorem 8.10]), one has equivalences of spectra

\[
GW^{[i]}(k) \to GW^{[i]}(\mathcal{A}_0) \quad \text{and} \quad GW^{[i]}(k) \to GW^{[i]}(\mathcal{A}_0).
\]

Thus, we deduce homotopy fibrations of $GW$-spectra

\[
GW^{[i]}(k) \to GW^{[i]}(\mathcal{A}) \to GW^{[i]}(\mathcal{A}, w)
\]

and of $GW$-spectra

\[
GW^{[i]}(k) \to GW^{[i]}(\mathcal{A}) \to GW^{[i]}(\mathcal{A}, w).
\]

Next, we study $GW^{[i]}(\mathcal{A}, w)$ and $GW^{[i]}(\mathcal{A}, w)$.

Recall the exact sequence $\mathcal{Z}_0 \to \mathcal{O} \otimes C_0(q) \to \mathcal{Z}_1$ obtained by the definition of $\mathcal{Z}_i$, and delete the component $\mathcal{Z}_{-1}$. We obtain a (cochain) complex concentrated in degree $[0, 1]$

\[
\cdots \to 0 \to \mathcal{Z}_0 \to \mathcal{O} \otimes C_0(q) \to 0 \to \cdots.
\]

We denote this complex by $Cl_{[0,1]}$.

**Lemma 5.4.** There is a symmetric space $(\mathcal{Z}_0, \mu)$ in the category with duality

\[
(\mathcal{T}(\mathcal{A}, w), \#_e[-1], \text{can}^\mathcal{O})
\]

where the form $\mu$ is represented by the following left roof

\[
\mu : \begin{array}{c}
\mathcal{Z}_0 \\
\downarrow t \\
\downarrow s \\
\end{array} \begin{array}{c}
Cl_{[0,1]} \\
\end{array} \begin{array}{c}
(\mathcal{Z}_0)^\vee[-1].
\end{array}
\]
The only non-trivial component in the morphism \( t : Cl_{[0,1]} \to \mathcal{U}_0 \) is the map id : \( \mathcal{U}_0 \to \mathcal{U}_0 \) in the degree 0, and the only non-trivial component of the map \( s : Cl_{[0,1]} \to (\mathcal{U}_0)^\vee[-1] \) is the composition \( \Theta \otimes C_0(q) \to \mathcal{U}_{-1} \xrightarrow{h_{-1}} (\mathcal{U}_0)^\vee \) in the degree 1.

**Proof.** It is clear that \( s \) is a quasi-isomorphism so that \( s \in w \). Moreover, we observe that cone\((t)\) is in \( T(\mathcal{A},w) \), so that \( t \in w \). Thus, \( \mu \) is an isomorphism in \( T(\mathcal{A},w) \). We show \( \mu \) is symmetric, i.e. \( \mu^\vee \circ \text{can}_{\mathcal{U}_0} = \mu \). Observe \( \mu^\vee \circ \text{can}_{\mathcal{U}_0} \) is represented by a right roof. The result can be obtained by noting that the following morphism of complexes is null-homotopic.

Here, all the maps are defined (recall Lemma 5.5 and its proof).

The proof of [26, Lemma 3.9] tells us that the form \((\mathcal{U}_0,\mu)\) (in Lemma 5.4) can be lifted to a symmetric form \((B_{\mathcal{U}_0},B_{\mu})\) in the dg category

\[(\mathcal{A},w,\#_{\Theta[-1]},\text{can})\]

with weak equivalences and duality, such that the morphism \( B_{\mu} \) is in \( w \) and that \((B_{\mathcal{U}_0},B_{\mu})\) is isometric to \((\mathcal{U}_0,\mu)\) in \((T(\mathcal{A},w),\#_{\Theta[-1]},\text{can},\lambda)\). The form \((B_{\mathcal{U}_0},B_{\mu})\) is displayed as follows.

Let \( A,B \) be dg \( k \)-algebras. Recall from [24, Section 7.2] the definition of dg \( A \)-modules. We denote by \( A\text{-dgMod} := A\text{-dgMod}-k \), \( A\text{-dgMod-B} := k\text{-dgMod-B} \) and \( A\text{-dgMod-B} \) the dg category of dg left \( A \)-modules, dg right \( B \)-modules, and of dg left \( A \)-modules and right \( B \)-modules. If \( A \) is a dg algebra with involution, then for any dg left \( A \)-module \( M \) we have a dg right \( A \)-module \( M^{op} \), cf. [24 Section 7.3].

Let \((I,i)\) denote an \( A \)-bimodule \( I \) together with an \( A \)-bimodule isomorphism \( i : I \to I^{op} \) such that \( i^{op} \circ i = \text{id} \). By abuse of the notation, we write \( I \) for \((I,i)\) if the isomorphism \( i \) is understood. In fact, \( I \) is called a duality coefficient in [24 Section 7.3]. There is a dg category with duality \((A\text{-dgMod-B},\#,\text{can}_I)\)

with \( \#_I : (A\text{-dgMod-B})^{op} \to A\text{-dgMod-B} \) by \( M^{\#_I} = [M^{op},I]_A \) and \( \text{can}_I : M \to M^{\#_I} \) by

\[\text{can}(x)(f^{op}) = (-1)^{|f||x|}i(f(x^{op})).\]

By abuse of notation, we write \((A\text{-dgMod-B},\#_I)\) for this dg category with duality. Let \( A,B \) be dg algebras with involution. Then, there is a dg form functor

\[(A\text{-dgMod-B},\#_I) \otimes (B\text{-dgMod},\#_B) \to (A\text{-dgMod},\#_I)\]
sending \((M, N)\) to \(M \otimes_B N\) with the duality compatibility map
\[
\gamma : [M^{op}, I]_A \otimes_B [N^{op}, B] \rightarrow [(M \otimes_B N)^{op}, I]_A
\]
defined by
\[
\gamma(f \otimes g)(n^{op}) = (-1)^{|m||n|} f(g(n^{op})m^{op}).
\]
Applying \(A = (\mathcal{O}, \text{id}), I = (\mathcal{O}[-1], \text{id})\) and \(B = (C_0(q)_{\sigma})\), we deduce the following result.

**Lemma 5.5.** There is a dg form functor
\[
(\mathcal{O}-\text{dgMod}-C_0(q), \#_{\mathcal{O}[-1]}) \otimes (C_0(q)-\text{dgMod}, \#_{C_0(q)}) \rightarrow (\mathcal{O}-\text{dgMod}, \#_{\mathcal{O}[-1]}).
\]

We have already seen \(B_{\mu} : C_{[0,1]} \rightarrow [C_{[0,1]}, \mathcal{O}[-1]]\) is a symmetric form in the dg category with duality \((\mathcal{O}-\text{dgMod}, \#_{\mathcal{O}[-1]})\). We further observe the following result.

**Lemma 5.6.** The map \(B_{\mu} : C_{[0,1]} \rightarrow [(C_{[0,1]})^{op}, \mathcal{O}[-1]]\) is a symmetric form in \((\mathcal{O}-\text{dgMod}-C_0(q), \#_{C_0(q)})\).

**Proof.** We only need to show \(B_{\mu}\) is a right \(C_0(q)\)-module map. This can be seen directly by Lemma 4.5 and its proof. \(\square\)

By Lemma 2.7, we obtain the following result.

**Corollary 5.7.** There is a dg form functor
\[
(B_{\mathcal{O}}, B_{\mu}) \otimes ? : (C_0(q)-\text{dgMod}, \#_{C_0(q)}) \rightarrow (\mathcal{O}-\text{dgMod}, \#_{C_0(q)})
\]

Let \(\text{sPerf}(C_0(q))\) be the dg category of strictly perfect complexes of finitely generated left projective \(C_0(q)\)-modules. Since \(\text{sPerf}(C_0(q)) \subset C_0(q)-\text{dgMod}\) and \(\mathcal{A} \subset \mathcal{O}-\text{dgMod}\) are full dg subcategories, we get the following result.

**Corollary 5.8.** There is a dg form functor
\[
(B_{\mathcal{O}}, B_{\mu}) \otimes ? : (\text{sPerf}(C_0(q)), \#_{C_0(q)}) \rightarrow (\mathcal{A}, \#_{C_0(q)})
\]

Taking weak equivalences into account, we get an exact dg form functor
\[
(B_{\mathcal{O}}, B_{\mu}) \otimes ? : (\text{sPerf}(C_0(q)), \text{quis}, \#_{C_0(q)}) \rightarrow (\mathcal{A}, \text{quis}, \#_{C_0(q)})
\]
which induces an equivalences of associated triangulated categories. Let
\[
GW^{[i]}(C_0(q)_\sigma) := GW^{[i]}(\text{sPerf}(C_0(q)), \text{quis}, \#_{C_0(q)}).
\]
By invariance for \(GW\) and \(GW\), we find stable equivalences
\[
(B_{\mathcal{O}}, B_{\mu}) \cup ? : GW^{[i+1]}(C_0(q)_\sigma) \rightarrow GW^{[i]}(\mathcal{A}, \text{quis}),
\]
and
\[
(B_{\mathcal{O}}, B_{\mu}) \cup ? : GW^{[i+1]}(C_0(q)_\sigma) \rightarrow GW^{[i]}(\mathcal{A}, \text{quis}).
\]

Summarizing the above discussion, we have proved the following.
Theorem 5.9. Let $k$ be a commutative ring containing $\frac{1}{2}$. Then there is a stable equivalence of spectra

$$(H\mathcal{O}(1)\otimes?, \cdots, H\mathcal{O}(m)\otimes?) : \bigoplus_{i=1}^{m} K(k) \oplus GW^{[i]}(\mathcal{A}) \sim GW^{[i]}(Q_d).$$

where $GW^{[i]}(\mathcal{A})$ fits into another homotopy fibration sequence

$$GW^{[i]}(k) \longrightarrow GW^{[i]}(\mathcal{A}) \longrightarrow GW^{[i+1]}(C_0(q)_\sigma).$$

Moreover, this result also holds for $GW$-spectra.

6. Witt groups of odd dimensional quadrics

In this section, we prove Theorem 1.2 when $d$ is odd. Note that Balmer’s Witt groups $W^i(Q_d)$ are just $W^i(Q_d)$ in Schlichting’s framework. In fact, $W^0(Q_d)$ is isomorphic to the classical Witt group $W^0(Q_d)$, cf. \cite{3}.

Lemma 6.1. Let $k$ be a commutative ring with $\frac{1}{2} \in k$. Let $d > 0$ be an odd integer. Then there is an isomorphism $W^i(\mathcal{A}) \approx W^i(Q_d)$.

Proof. Note that $K(k)$ is the connective $K$-theory in our framework. Taking negative homotopy groups over both sides of the equivalence

$$\bigoplus_{i=1}^{m} K(k) \oplus GW^{[i]}(\mathcal{A}) \sim GW^{[i]}(Q_d),$$

we see the result. \qed

Recall the trace map $tr : C_0(q) \rightarrow k$ in Section 4. Now, we introduce a map

$$tr : W^0(C_0(q)_\sigma) \rightarrow W^0(k).$$

Observe that the forgetful dg functor

$$F : C_0(q)\text{-dgMod-}k \longrightarrow \text{dgMod-}k$$

(by forgetting the left $C_0(q)$-module structure) induces a dg form functor

$$(F, tr) : (C_0(q)\text{-dgMod-}k, \#_{C_0(q)}) \longrightarrow (\text{dgMod-}k, \#_k)$$

with the duality compatibility map defined by the composition of $k$-module maps

$$[M, C_0(q)]_{C_0(q)} \stackrel{i}{\longrightarrow} [M, C_0(q)]_k \stackrel{[1, tr]}{\longrightarrow} [M, k]_k$$

where the map $i$ is the inclusion and where the map $tr$ is the trace map in Section 4. This dg form functor restricts to an exact dg form functor

$$(F, tr) : (\text{sPerf}(C_0(q)), \text{quis}, \#_{C_0(q)}) \longrightarrow (\text{sPerf}(k), \text{quis}, \#_k)$$

which provides the map $tr : W^0(C_0(q)_\sigma) \rightarrow W^0(k)$. We shall prove Theorem 1.2 for $d$ odd as follows.
Theorem 6.2. Let \( k \) be a commutative semilocal ring with \( \frac{1}{2} \in k \). Assume that the odd indexed Witt groups \( W^{2i+1}(C_0(q)_{\sigma}) = 0 \) (see Theorem A.3 for examples). If \( d \) is odd, then
\[
W^i(Q_d) = \begin{cases} 
\text{coker}(tr) & \text{if } i = 0 \mod 4 \\
W^2(C_0(q)_{\sigma}) & \text{if } i = 1 \mod 4 \\
0 & \text{if } i = 2 \mod 4 \\
\ker(tr) & \text{if } i = 3 \mod 4
\end{cases}
\]

To prove Theorem 6.2, we need the following lemma. Recall the exact sequence \( \mathcal{A}_0 \to \mathcal{A} \to \mathcal{A}/\mathcal{A}_0 \) of triangulated categories which gives a map \( \partial^3 : W^i(\mathcal{A}/\mathcal{A}_0) \to W^{i+1}(\mathcal{A}_0) \) called the “connecting homomorphism” in \([2]\). The map \( \partial^3 \) may be interpreted by the trace map \( tr \) in the following sense.

**Lemma 6.3.** Let \( k \) be a commutative ring with \( \frac{1}{2} \in k \). Then the diagram
\[
\begin{array}{ccc}
W^0(C_0(q)_{\sigma}) & \xrightarrow{tr} & W^0(k) \\
\downarrow U & & \downarrow G \\
W^{-1}(\mathcal{A}/\mathcal{A}_0) & \xrightarrow{\partial^3} & W^0(\mathcal{A}_0)
\end{array}
\]
is commutative, where \( U \) (resp. \( G \)) denotes the map \((\mathcal{Z}_0, \mu)\otimes? \) (resp. \((\mathcal{S}, \text{id})\otimes?)\).

**Proof.** Let \( b \) be an element in \( W^0(C_0(q)_{\sigma}) \) corresponding to a symmetric space
\[
b : M \to [M^{op}, C_0(q)]_{C_0(q)},
\]
where \( M \) is a finitely generated left projective \( C_0(q) \)-module. It is enough to check
\[
G \circ tr(b) = \partial^3 \circ U(b).
\]
Note that \( G \circ tr(b) \) is the symmetric space \( \psi : \mathcal{S} \otimes M \to [\mathcal{S} \otimes M, \mathcal{S}]_{\mathcal{S}} \) defined by \( \psi(f \otimes x)(y \otimes y) = fg \cdot tr(b(x)(y^{op})) \). Moreover, I claim \( \partial^3 \circ U \) sends \( b \) to the symmetric space \( \psi \). Firstly, observe that \( U(b) \) is the symmetric space \( \mathcal{Z}_0 \otimes_{C_0(q)} M \to (\mathcal{Z}_0 \otimes_{C_0(q)} M)^{[-1]} \) represented by the left roof \( t^{-1}s \)
\[
\begin{array}{c}
\mathcal{Z}_0 \otimes_{C_0(q)} M \\
\downarrow t \\
\mathcal{Z}_0 \otimes_{C_0(q)} M^{op} \otimes_{C_0(q)} M
\end{array}
\]
where \( t \) is the projection and where \( s \) consists of the composition
\[
e : (\mathcal{S} \otimes C_0(q)) \otimes_{C_0} M \xrightarrow{\ell \otimes 1} \mathcal{Z}_0 \otimes_{C_0} M \xrightarrow{h \otimes b} (\mathcal{Z}_0^{op})^{\mathcal{S}} \otimes_{C_0} [M^{op}, C_0(q)] \xrightarrow{\gamma} (\mathcal{Z}_0 \otimes_{C_0} M)^{\mathcal{S}}
\]
in the degree 1. Note that \( s \) is a quasi-isomorphism. Now, we use the procedure introduced in \([2]\) to investigate \( \partial^3(U(b)) \). Observe that there is an exact triangle
\[
(\mathcal{Z}_0 \otimes_{C_0} M)^{[-1]} \xrightarrow{s^{-1}t} \mathcal{Z}_0 \otimes_{C_0} M \xrightarrow{\ell \otimes 1} \mathcal{S} \otimes M \xrightarrow{e} (\mathcal{Z}_0 \otimes_{C_0} M)^{\mathcal{S}}
\]
in the triangulated category \( \mathcal{A} \), and a symmetric form of exact triangles
\[
\begin{array}{ccc}
(\mathcal{Z}_0 \otimes_{C_0} M)^{[-1]} & \xrightarrow{s^{-1}t} & \mathcal{Z}_0 \otimes_{C_0} M \\
\downarrow & & \downarrow \text{can} \\
(\mathcal{Z}_0 \otimes_{C_0} M)^{\mathcal{S}} & \xrightarrow{e} & (\mathcal{S} \otimes M)^{\mathcal{S}}
\end{array}
\]

\[
\begin{array}{ccc}
(\mathcal{Z}_0 \otimes_{C_0} M)^{[-1]} & \xrightarrow{s^{-1}t} & \mathcal{Z}_0 \otimes_{C_0} M \\
\downarrow & & \downarrow \psi \\
(\mathcal{Z}_0 \otimes_{C_0} M)^{\mathcal{S}} & \xrightarrow{e^s} & (\mathcal{S} \otimes M)^{\mathcal{S}} \xrightarrow{(\ell \otimes 1)^s}
\end{array}
\]
in the triangulated category \((\mathcal{A}, \#_{\sigma[-1]})\) with duality. Note that the map
\[
s^{-1} t : (\mathcal{U}_0 \otimes_{C_0} M)^{\vee}[-1] \rightarrow \mathcal{U}_0 \otimes_{C_0} M
\]
is a symmetric form in the triangulated category \((\mathcal{A}, \#_{\sigma[-1]})\) with duality. Moreover, it is isometric to the symmetric space
\[
U(b) = \tau^{-1} s : \mathcal{U}_0 \otimes_{C_0} M \rightarrow (\mathcal{U}_0 \otimes_{C_0} M)^{\vee}[-1]
\]
in the triangulated category \((\mathcal{A}/\mathcal{U}_0, \#_{\sigma[-1]})\) with duality.

**Proof of Theorem 6.2.** By four-periodicity, Lemma 6.1, Lemma 6.3 and Theorem 5.9 we find a 12-term long exact sequence (5)

Note that if \(k\) is a commutative semilocal ring containing \(\frac{1}{2}\), then \(W^i(k) = 0\) for \(i \neq 0 \mod 4\) by [4] and \(W^{2i+1}(C_0(q)_\sigma) = 0\) by the assumption.

**Proposition 6.4.** Let \(k\) be a field of characteristic \(\neq 2\). Let \(n = d + 2\) be an odd number. Let \(Q_d\) be the quadric defined by the quadratic form \((a_1, \ldots, a_n)\) with \(a_i \in k^x\). Then, the map \(p^* : W^0(k) \rightarrow W^0(Q_d)\) induces a well-defined surjective map

\[
W^0(k)/\langle a_1a_2, \ldots, a_1a_n \rangle W^0(k) \twoheadrightarrow W^0(Q_d)
\]

where \(\langle a_1a_2, \ldots, a_1a_n \rangle := \langle 1, a_1a_2 \rangle \otimes \cdots \otimes \langle 1, a_1a_n \rangle\) is the Pfister form.

**Proof.** There is always an element \(\beta\) in \(W^0(C_0(q)_\sigma)\) represented by the form

\[
\beta : C_0(q) \times C_0(q) \rightarrow C_0(q) : (x, y) \mapsto xy
\]

Let \(\{e_1, \ldots, e_n\}\) be an orthogonal basis of the quadratic form \(q\). The Clifford algebra \(C(q)\) has a basis \(\{e^\Delta : \Delta \in \mathbb{F}_2^n\}\) where \(e^\Delta := e_1^{b_1} \cdots e_n^{b_n}\) with \(\Delta = (b_1, \ldots, b_n)\). Let \(|\Delta| = \sum b_i\). Then, \(C_0(q)\) has a basis \(\{e^\Delta : \Delta \in \Omega\}\) where \(\Omega\) is the set \(\{\Delta \in \mathbb{F}_2^n : |\Delta|\text{ is even}\}\). Thus, we see \(\text{tr}(\beta) = \sum_{\Delta \in \Omega} q(e^\Delta) \in W^0(k)\) where \(q(e^\Delta) := q(e_1^{b_1} \cdots q(e_n)^{b_n} = a_1^{b_1} \cdots a_n^{b_n}\). Observe that \(\sum_{\Delta \in \Omega} q(e^\Delta) = \langle a_1a_2, \ldots, a_1a_n \rangle\) in \(W^0(k)\) (one may find that [27] Exercise 3.14] is helpful to see this).
Remark 6.5. Assume $C_0(q)$ is a division algebra. See [27, Exercise 3.16] for a discussion of Clifford division algebras. If $n = 3$, then the map

$$p^* : W^0(k) / \langle a_1a_2, \ldots, a_1a_n \rangle W^0(k) \to W^0(Q_d)$$

is an isomorphism. However, this map need not be an isomorphism for $n > 3$. This means in general we could have $tr(b) \notin \langle a_1a_2, \ldots, a_1a_n \rangle W^0(k)$ for some $b \in W^0(C_0(q)_*)$.

Lemma 6.6. Let $k$ be a field of characteristic $\neq 2$. Let $Q_d$ be the quadric defined by an isotropic quadratic form $q$ over $k$. Then, the map $p^* : W^0(k) \to W^0(Q_d)$ is split injective.

Proof. Since $q$ is isotropic, the quadric $Q_d$ has a rational point $s : \text{Spec} \ k \to Q_d$ which induces a map $s^* : W^0(Q_d) \to W^0(k)$. The map $s^*$ is a splitting of $p^*$. □

We immediately deduce the following result.

Corollary 6.7. Let $k$ be a field of characteristic $\neq 2$. Let $Q_d$ be the quadric defined by an isotropic quadratic form $q$ over $k$. If $d$ is odd, then

$$W^i(Q_d) = \begin{cases} W^0(k) & \text{if } i = 0 \mod 4 \\ W^2(C_0(q)_*) & \text{if } i = 1 \mod 4 \\ 0 & \text{if } i = 2 \mod 4 \\ W^0(C_0(q)_*) & \text{if } i = 3 \mod 4 \end{cases}$$

7. An exact sequence

Let $k$ be a commutative ring. Let $(F, q)$ be an $n$-dimensional non-degenerate quadratic form over $k$. Recall that $Q_d$ is the variety $\text{Proj}(S(P^*)/q)$ of dimension $d = n - 2$. Consider the following maps of exact sequences

$$0 \to 0 \to \ker(l) \xrightarrow{z} \ker(v) \to 0$$

$$\text{CL}_{d-1}(\Lambda) \to \text{CL}_d(\Lambda) \to \text{CL}_{d+1}(\Lambda) \to \mathcal{U}_{d-2}(\Lambda) \to 0$$

$$T_{d-1} \xrightarrow{=} \mathcal{U}_d \xrightarrow{=} T_{d+1} \xrightarrow{\varphi_d} \text{coker}(\varphi_d) \to 0$$

where the middle row is the exact sequence \([7\,\text{with } \text{CL}_i(\Lambda) := \mathcal{O}(-i) \otimes \Lambda^{(s_i)}, \text{ the bottom row is the Tate resolution, and where the map } l \text{ is the projection by taking the component } \mathcal{O}(1-d) \otimes \Lambda^n(P^*) \text{ to } 0. \text{ By the universal property of cokernels, we observe that there is a surjective map } v : \mathcal{U}_{d-2}(\Lambda) \to \text{coker}(\varphi_d) \text{ such that the lower-third square in the diagram } \([7\) \text{ is commutative. By taking kernels of maps between the bottom and the middle exact sequences of the diagram } \([7\), we obtain the exact sequence in the top row. In particular, we see that } \mathcal{O}(1-d) \otimes \det(P^*) = \ker(l) = \ker(v). \text{ Let } s \text{ denote the composition } \mathcal{O}(1-d) \otimes \det(P^*) \to \ker(l) \to \ker(v) \to \mathcal{U}_{d-2}(\Lambda).$$

Then, we have proved the following result.
Theorem 8.1. Let $\mathcal{P}, \mathcal{Q}$ be a dg category with weak equivalences and duality. Let $\mathcal{C}$ be the full dg subcategory of $s\text{Perf}(Q_d)$ associated to the triangulated category $\mathcal{A}'$. Let $\nu$ be the set of morphisms in $s\text{Perf}(Q_d)$ which become isomorphisms in the Verdier quotient $T(s\text{Perf}(Q_d), \text{quis})/T(\mathcal{C}', \text{quis})$. Note that $\mathcal{C}'$ is fixed by the duality $\#_{\mathcal{P}} = [-, \mathcal{P}]$.

Observe that the sequence of dg categories with weak equivalences and duality

$$(\mathcal{C}', \text{quis}) \rightarrow (s\text{Perf}(Q_d), \text{quis}) \rightarrow (s\text{Perf}(Q_d), \nu)$$

is quasi-exact (hence Morita exact). So, it induces localization sequences of $GW$-spectra

$$GW^{[i]}(\mathcal{C}') \rightarrow GW^{[i]}(Q_d) \rightarrow GW^{[i]}(Q_d, \nu).$$

Proposition 7.1. There is an exact sequence

$$0 \rightarrow \mathcal{O}(1 - d) \otimes \det(P^*) \rightarrow \mathcal{O}_{d-2}(\Lambda) \rightarrow T_{d-2} \rightarrow \cdots \rightarrow T_{-1} \rightarrow T_0 \rightarrow \mathcal{O}(1) \rightarrow 0$$

where the sequence of maps

$$T_{-d+2} \rightarrow \cdots \rightarrow T_{-1} \rightarrow T_0 \rightarrow \mathcal{O}(1) \rightarrow 0$$

is the Tate resolution truncated from $T_{-d+2}$.

Corollary 7.2. Let $d = 2m$ be an even number. Tensoring the exact sequence of Proposition 7.1 with the line bundle $\mathcal{O}(m - 1)$, we obtain another exact sequence (7)

$$0 \rightarrow \mathcal{O}(-m) \otimes \det(P^*) \rightarrow \mathcal{O}_{d-2}(\Lambda)(m-1) \rightarrow T_{-d+2}(m-1) \rightarrow \cdots \rightarrow T_0(m-1) \rightarrow \mathcal{O}(m) \rightarrow 0.$$

This exact sequence is of length $n = d + 2$. We denote it by $T_{[-m,m]}$.

8. Grothendieck-Witt spectra of even dimensional quadrics

Let $d = 2m$ be an even integer throughout this section. Let $Q_d$ be a smooth quadric hypersurface of dimension $d$ corresponding to a non-degenerate quadratic form $(P, q)$ of rank $n = d + 2$. This section is devoted to prove the following theorem.

Theorem 8.1. Let $d = 2m$ and let $k$ be a commutative ring containing $\frac{1}{2}$. Then there exists a homotopy fibration sequence of spectra

$$\bigoplus_{i=1}^{m-1} K(k) \oplus GW^{[i]}(\mathcal{O}) \rightarrow GW^{[i]}(Q_d) \rightarrow GW^{[i+1]}(C_0(q))$$

where $GW^{[i]}(k, \det(P)) := GW^{[i]}(s\text{Perf}(k), \text{quis}, \#_{\det(P)}, \text{can})$ and $GW^{[i]}(\mathcal{O})$ fits into another homotopy fibration sequence

$$GW^{[i]}(k) \rightarrow GW^{[i]}(\mathcal{O}) \rightarrow GW^{[i+1]}(C_0(q)).$$

Moreover, this result also holds for $GW$-spectra.

Proof. Recall from Corollary 7.2 the semi-decomposition

$$D^h Q_d = \{A_{[1-m,-1]}, A, A_{[1,m-1]} \}.$$

Define $A' := \{A_{[1-m,-1]}, A, A_{[1,m-1]} \}$. Note that $A'$ is fixed by the duality $\nu := \text{Hom}(-, \mathcal{O})$.

Recall that the quadruple

$$(s\text{Perf}(Q_d), \text{quis}, \#_{\mathcal{P}}, \text{can}_{\mathcal{P}})$$

is a dg category with weak equivalences and duality. Let $\mathcal{C}'$ be the full dg subcategory of $s\text{Perf}(Q_d)$ associated to the triangulated category $\mathcal{A}'$. Let $\nu$ be the set of morphisms in $s\text{Perf}(Q_d)$ which become isomorphisms in the Verdier quotient $T(s\text{Perf}(Q_d), \text{quis})/T(\mathcal{C}', \text{quis})$. Note that $\mathcal{C}'$ is fixed by the duality $\#_{\mathcal{P}} = [-, \mathcal{P}]$.
We shall compute \(GW^{[i]}(\mathcal{C}')\) in Lemma \[8.2\] and \(GW^{[i]}(Q_d, v)\) in Lemma \[8.3\] below. The result follows.

**Lemma 8.2.** Let \(k\) be a commutative ring containing \(\frac{1}{2}\). Then, there is a stable equivalence of spectra

\[
(H^{\mathcal{C}}(1)\otimes\cdots,H^{\mathcal{C}}(m)\otimes\cdot) : \bigoplus_{i=1}^m K(k) \oplus GW^{[i]}(\mathcal{A}) \rightarrow GW^{[i]}(\mathcal{C}')
\]

where \(GW^{[i]}(\mathcal{A})\) fits into another homotopy fibration sequence

\[
GW^{[i]}(k) \rightarrow GW^{[i]}(\mathcal{A}) \rightarrow GW^{[i+1]}(C_0(q)_\sigma).
\]

Moreover, this result also holds for \(GW\)-spectra.

**Proof.** Observe that \(GW^{[i]}(\mathcal{C}')\) and \(GW^{[i]}(\mathcal{A})\) may be manipulated similarly as in the case of odd dimensional quadrics. The results follow by the additivity theorem. \(\square\)

**Lemma 8.3.** Let \(k\) be a commutative ring containing \(\frac{1}{2}\). Then, there is an equivalence

\[
(B_{\mathcal{C}}(m), B_\psi) : GW^{[i]}(k,\det(P)) \rightarrow GW^{[i+d]}(Q, v)
\]

of spectra. A similar result holds for the case of \(GW\)-spectra.

Before proving Lemma \[8.3\] we provide the necessary background information. Consider the exact sequence \(T_{[-m,m]}\) in Corollary \[7.2\]. Delete the component \(\mathcal{O}(-m)\otimes\det(P^*)\) in \(T_{[-m,m]}\). We obtain a complex concentrated in degree \([-d,0]\)

\[
0 \rightarrow \mathcal{U}_{d-2}(\Lambda)(m-1) \rightarrow T_{-d+2}(m-1) \rightarrow \cdots \rightarrow T_0(m-1) \rightarrow \mathcal{O}(m) \rightarrow 0
\]

(8) Denote this new complex by \(T_{[1-m,m]}\). Note that

\[
T_{-d+2}(m-1) = \mathcal{O}(1-m) \otimes \left( \bigoplus_{j \geq 0} \Lambda^{(d-2)+1-2j}(P^*) \right).
\]

Let \((\mathcal{O}(m), \psi)\) denote the right roof

\[
\xymatrix{T_{[1-m,m]} \ar[dr]^s \ar[dl]_\epsilon \ar[rr] & & \mathcal{O}(m) \ar[rr] & & \mathcal{O}(m, \mathcal{O} \otimes \det(P^*)[d])_\sigma}
\]

The only non-trivial component of \(\epsilon : \mathcal{O}(m) \rightarrow T_{[1-m,m]}\) is the map \(\text{id} : \mathcal{O}(m) \rightarrow \mathcal{O}(m)\) in the degree 0. The only non-trivial component of

\[
s : [\mathcal{O}(m, \mathcal{O} \otimes \det(P^*)[d])_\sigma \rightarrow T_{[1-m,m]}]
\]

is the composition

\[
\mathcal{O}(m)^\vee \otimes \det(P^*) \rightarrow \mathcal{O}(-m) \otimes \det(P^*) \rightarrow \mathcal{U}_{d-2}(\Lambda)
\]

in the degree \(-d\).

**Lemma 8.4.** Let \(k\) be a commutative ring containing \(\frac{1}{2}\). Then, the left roof \((\mathcal{O}(m), \psi)\) is a symmetric space in the category with duality

\[
(\mathcal{T}(s\text{Perf}(Q), v), \#_{\det(P^*}, \epsilon, \text{can})
\]
for some $\epsilon \in k^\ast$ such that $\epsilon^2 = 1$ where $\#_{\text{det}(P^\ast)}$ is the duality given by the functor

$$[-, \mathcal{O} \otimes \text{det}(P^\ast)[d]] : \mathcal{T}(\text{sPerf}(Q), v)^{\text{op}} \to \mathcal{T}(\text{sPerf}(Q), v).$$

If $k$ is a local ring containing $\frac{1}{2}$, then $\epsilon = \pm 1$.

**Proof.** It is enough to show that $t$ and $s$ are both weak equivalences and that $\psi$ is $\epsilon$-symmetric. Indeed, $s$ is a quasi-isomorphism, hence also a weak equivalence, cf. Corollary 7.2. Furthermore, the cone of the morphism $t$ in $\mathcal{D}^bQ$ is already in the triangulated category $\mathcal{A}_{1-m,m-1}$. Thus, $t$ is a weak equivalence and we conclude $\psi$ is an isomorphism in

$$\text{Hom}_{\mathcal{T}(\text{sPerf}(Q), v)}(\mathcal{O}(m), \mathcal{O}(m)^{\#_{\text{det}(P^\ast)}}).$$

Define $\psi^t := \psi \#_{\text{can}}.\mathcal{O}(m)$. Note that $\psi^t = \psi$. Next, we show that $\psi = \epsilon \psi^t$ for some $\epsilon \in k^\ast$ such that $\epsilon^2 = 1$. In fact, the morphism $\psi$ can also induce an isomorphism

$$\text{Hom}_{\mathcal{T}(\text{sPerf}(Q), v)}(\mathcal{O}(m), \mathcal{O}(m)^{\#_{\text{det}(P^\ast)}}) \approx \text{Hom}_{\mathcal{T}(\text{sPerf}(Q), v)}(\mathcal{O}(m), \mathcal{O}(m)).$$

The right-hand side is isomorphic to

$$\text{Hom}_{\mathcal{D}^bQ}(\mathcal{O}(m), \mathcal{O}(m)) \approx k,$$

since $\mathcal{T}(\text{sPerf}(Q), v)$ is just the Verdier quotient $\mathcal{D}^bQ/\mathcal{A}'$. Thus, we conclude $\psi^t = \epsilon \psi$ for some $\epsilon \in k^\ast$. Observe that $\psi = \psi^t = (\epsilon \psi)^t = \epsilon^2 \psi$. This implies $\epsilon^2 = 1$, because $\psi$ is an isomorphism.

It is clear that $\epsilon^2 = 1$ implies $\epsilon = \pm 1$ when the base is $\mathbb{Z}[\frac{1}{2}]$. If $k$ is $\mathbb{Z}[\frac{1}{2}]$ or a local ring, then every projective module over $k$ is free. In particular, the projective module $P$ involved in the duality $\#_{\text{det}(P^\ast)}$ is free. Thus, we can conclude $\epsilon = \pm 1$ by using the base change via the canonical map $\mathbb{Z}[\frac{1}{2}] \to k$. □

The element $\epsilon \in k^\ast$ will be determined later. Consider the dg form functor

$$(\mathcal{O}^\ast, \text{dgMod}-k, \#_{\mathcal{O} \otimes \text{det}(P^\ast)[d]}) \otimes (k, \text{dgMod}, \#_{\mathcal{O}^d}) \to (\mathcal{O}^\ast, \text{dgMod}, \#_{\mathcal{O}^d})$$

with the duality compatibility map

$$\gamma : [M, \mathcal{O} \otimes \text{det}(P^\ast)[d]] \otimes [N, \text{det}(P)] \to [M \otimes N, \mathcal{O}^d]$$

given by $\gamma(f \otimes g)(m \otimes n) = (-1)^{|g|m}(f(m) \otimes g(n))$ where the map

$$\kappa : \mathcal{O} \otimes \text{det}(P^\ast) \otimes \text{det}(P) \to \mathcal{O}$$

defined by $a \otimes f \otimes p \mapsto a f(p)$ is an isomorphism. The proof of [26, Lemma 3.9] tells us that the form $(\mathcal{O}(m), \psi)$ (in Lemma 8.4) can be lifted to a symmetric form $(B_{\mathcal{O}(m), B_{\psi}})$ in the dg category

$$(\text{sPerf}(Q), v, \#_{\mathcal{O} \otimes \text{det}(P^\ast)[d]}, \epsilon \cdot \text{can})$$

with weak equivalences and duality, such that the morphism $B_{\psi}$ is in $v$ and that $(B_{\mathcal{O}(m), B_{\psi}})$ is isometric to $(\mathcal{O}(m), \psi)$ in $(\mathcal{T}(\text{sPerf}(Q), v), \#_{\mathcal{O} \otimes \text{det}(P^\ast)[d]}, \epsilon \cdot \text{can})$ for some $\epsilon \in k^\ast$ such that $\epsilon^2 = 1$.

**Lemma 8.5.** Let $k$ be a commutative ring containing $\frac{1}{2}$. Then, there is a dg form functor

$$(B_{\mathcal{O}(m), B_{\psi}}) \otimes ? : (k, \text{dgMod}, \#_{\text{det}(P^\ast)}) \to (\mathcal{O}^\ast, \text{dgMod}, \#_{\mathcal{O}^d}, \epsilon \cdot \text{can}^\mathcal{O})$$

for some $\epsilon \in k^\ast$ such that $\epsilon^2 = 1$. 

Proof: This is a consequence of Lemma [2.7] and the preceding discussion. \qed

Restricting our attention to the full dg subcategory sPerf(k) ⊂ k-dgMod and taking the class v of weak equivalences into account, we record the following result.

Lemma 8.6. Let k be a commutative ring containing \( \frac{1}{2} \). Then, there is an exact dg form functor

\[
(B_{\theta(m)}, B_{\psi}) \circled{\otimes} : (\text{sPerf}(k), \text{quis, } \#_{\det(P)}, \text{can}) \to (\text{sPerf}(Q_{d}), v, \#_{\varepsilon[d]}, \varepsilon \cdot \text{can})
\]

of dg categories with weak equivalences and duality for some \( \varepsilon \in k^* \) such that \( \varepsilon^2 = 1 \).

It is well-known that the composition of functors

\[
i : \mathcal{A}_{m} \to (\mathcal{A}', \mathcal{A}_{m}) \to (\mathcal{A}', \mathcal{A}_{m})/\mathcal{A}' \approx s\text{Perf}(Q), v)
\]

is an equivalence, where the first functor is the inclusion and the second one is the quotient. Furthermore, this equivalence induces an equivalence of categories

\[
\mathcal{O}^h k \to s\text{Perf}(Q), v), E \mapsto \mathcal{O}'(m) \otimes E.
\]

Thus, the functor \( B_{\theta(m)} \circled{\otimes} : \text{sPerf}(k) \to \text{sPerf}(Q) \) induces an equivalence of associated triangulated categories. By invariance of GW and GW, we obtain the following lemma.

Lemma 8.7. Let k be a commutative ring containing \( \frac{1}{2} \). Then, the map

\[
(B_{\theta(m)}, B_{\psi}) \circled{\otimes} : \text{GW}^{[i]}(k, \det(P)) \to ^{\epsilon} \text{GW}^{[i+d]}(Q, v)
\]

is a stable equivalence of spectra for some \( \epsilon \in k^* \) such that \( \epsilon^2 = 1 \).

Assume that k is a field of characteristic \( \neq 2 \) (Then, the duality given by \( \det(P) \cong k \) is the trivial duality). In this case, we can compute the element \( \epsilon \) in Lemma [8.7]. It is clear that \( \epsilon^2 = 1 \) implies \( \epsilon = \pm 1 \) if k is a field. Recall that \( d = 2m \). If k is a field containing \( \frac{1}{2} \), I claim \( \epsilon = -1 \) is impossible, hence \( \epsilon \) must equal 1. Note that by our notation \( _{-1} \text{GW}^{[i]}(Q, v) = \text{GW}^{[i+2]}(Q, v) \). If m is odd, then there is a homotopy fibration

\[
\text{GW}^{[2]}(\mathcal{O}') \to \text{GW}^{[2]}(Q) \to _{-1} \text{GW}^{[0]}(Q, v)
\]

of spectra by the localization theorem. If \( \epsilon = -1 \), we have \( _{-1} \text{GW}^{[0]}(Q, v) \cong \text{GW}^{[2]}(k) \) by Lemma [8.7] and the 4-periodicity. Taking the negative homotopy group \( \pi_{-4} \) to the fibration, one has an exact sequence

\[
W^2(\mathcal{A}') \to W^2(Q) \to W^2(k)
\]

By the base change via the obvious map \( k \to \bar{k} \), an exact sequence

\[
W^2(\mathcal{A}'_{\bar{k}}) \to W^2(Q_{\bar{k}}) \to W^2(\bar{k})
\]

comes to life. Note that \( W^2(\mathcal{A}'_{\bar{k}}) = 0 \) by the proof of the odd dimensional case in Section [6]. Besides, \( W^2(\bar{k}) = 0 \) is well-known. This implies \( W^2(Q_{\bar{k}}) = 0 \). This is a contradiction because \( W^2(Q_{\bar{k}}) \cong W^2(Q_{\bar{k}}) \) by [30] Corollary 4.1 and \( W^2(Q_{\bar{k}}) \cong \mathbb{Z}/2\mathbb{Z} \) by [31] Theorem 4.9. If m is even, then we have a homotopy fibration sequence

\[
\text{GW}^{[0]}(\mathcal{O}') \to \text{GW}^{[0]}(Q) \to _{-1} \text{GW}^{[2]}(Q, v)
\]
Assume \( \epsilon = -1 \). By Lemma \([8.7]\) and the 4-periodicity, we have \( -1GW^2(Q, v) \approx GW^2(k) \). These data provide an exact sequence

\[
W^0(A'_k) \rightarrow W^0(Q_k) \rightarrow W^2(k).
\]

By the proof of the odd dimensional case in Section 6, we deduce that we see the results.

\[\Box\]

Lemma 8.8. Let \( k \) be a field of characteristic \( \neq 2 \). Then, there is a stable equivalence

\[
(B_E(m), B_\psi) \otimes ? : GW^{[i]}(k, \det(P)) \rightarrow GW^{[i+d]}(Q, v)
\]

of spectra.

If \( k \) is a local ring containing \( \frac{1}{2} \), then \( \epsilon = \pm 1 \) by Lemma \([5.4]\).

Lemma 8.9. Let \( k \) be a local ring containing \( \frac{1}{2} \). Then, there is a stable equivalence

\[
(B_E(m), B_\psi) \otimes ? : GW^{[i]}(k, \det(P)) \rightarrow GW^{[i+d]}(Q, v)
\]

of spectra.

Proof. Let \( F \) be the residue field of \( k \). Using the base change via the quotient map \( k \rightarrow F \), we conclude the result. \( \Box \)

Now, we have enough information to prove Lemma \([8.3]\).

Proof of Lemma \([8.3]\). If \( k \) is a commutative ring containing \( \frac{1}{2} \), we need to compute the element \( \epsilon \in k^* \) in Lemma \([8.3]\). Let \( k_p \) be the localization of \( k \) at a prime ideal \( p \), and let \( \epsilon_p \) be the image of \( \epsilon \) under the map \( k \rightarrow k_p \). If \( d = 2m \), then we have \( \epsilon_p = 1 \) in \( k_p \) for any prime ideal \( p \) of \( k \) by Lemma \([8.3]\). This implies \( \epsilon = 1 \) in \( k \). Note that \( GW^{[i]}(Q_d) \) is just \( GW^{[i]}(Q_d) \). \( \Box \)

9. WITT GROUPS OF EVEN DIMENSIONAL QUADRICS

Let \( d = 2m \) be an even integer. In this section, we continue our study of Balmer’s Witt groups \( W^i(Q_d) \) in Section 6 by considering \( d = 2m \).

Lemma 9.1. Let \( k \) be a commutative ring with \( \frac{1}{2} \). Let \( d = 2m \). Then, there is an exact sequence

\[
\cdots \rightarrow W^{[i]}(\mathcal{A}) \rightarrow W^{[i]}(Q) \rightarrow W^{[i-d]}(k, \det(P)) \rightarrow W^{[i+1]}(\mathcal{A}) \rightarrow \cdots
\]

Proof. Taking the negative homotopy groups of the fibration sequence

\[
\bigoplus_{j=1}^{n-1} K(k) \oplus GW^{[i]}(\mathcal{A}) \rightarrow GW^{[i]}(Q_d) \rightarrow GW^{[i-d]}(k, \det(P)) \; ,
\]

we see the results. \( \Box \)

Having this lemma in hand, we are able to prove
Theorem 9.2. Let $k$ be a commutative semilocal ring with $\frac{1}{2} \in k$. Assume that the odd indexed Witt groups $W^{2l+1}(C_0(q)_\sigma) = 0$ (see Theorem A.3 for a vanishing result). If $d$ is even, then

$$W^i(Q_d) = \begin{cases} 
\text{coker}(\text{tr}) & \text{if } d \equiv 2 \mod 4 \text{ and if } i \equiv 0 \mod 4 \\
W^2(C_0(q)_\sigma) & \text{if } d \equiv 2 \mod 4 \text{ and if } i \equiv 1 \mod 4 \\
0 & \text{if } d \equiv 0 \mod 4 \text{ and if } i \equiv 2 \mod 4 \\
\ker(\text{tr}) & \text{if } d \equiv 0 \mod 4 \text{ and if } i \equiv 3 \mod 4 
\end{cases}$$

If $d \equiv 2 \mod 4$, then there is an exact sequence

$$0 \to W^2(Q_d) \to W^0(k, \det(P)) \to \ker(\text{tr}) \to W^3(Q_d) \to 0.$$

If $d \equiv 0 \mod 4$, then there is an exact sequence

$$0 \to \ker(\text{tr}) \to W^0(Q_d) \to W^0(k, \det(P)) \to W^2(C_0(q)_\sigma) \to W^1(Q_d) \to 0.$$

Proof. By Lemma 9.1, we are reduced to studying $W^i(A) = W^i(A)$. Similar to the odd dimension case in Section 6, we have the following 12-term long exact sequence

$$\begin{array}{ccccccc}
& & W^0(k) & \to & W^0(A) & \to & W^1(C_0(q)_\sigma) \\
& & W^0(C_0(q)_\sigma) & \downarrow & \downarrow & & W^1(k) \\
& W^3(A) & \downarrow & & \downarrow & & W^1(A) \\
W^3(k) & \downarrow & & \downarrow & & \downarrow \\
& W^3(C_0(q)_\sigma) & \leftarrow & \leftarrow & \leftarrow & \leftarrow & W^2(k) \\
& W^3(k) & \leftarrow & \leftarrow & \leftarrow & \leftarrow & W^2(C_0(q)_\sigma) \\
& W^3(k) & \leftarrow & \leftarrow & \leftarrow & \leftarrow & W^2(k) \\
\end{array}$$

Note that if $k$ is a commutative semilocal ring containing $\frac{1}{2}$, then $W^i(k) = 0$ for $i \not\equiv 0 \mod 4$ by [4] and $W^{2l+1}(C_0(q)_\sigma) = 0$ by the assumption. So, we have

$$W^i(A) = \begin{cases} 
\text{coker}(\text{tr}) & \text{if } i \equiv 0 \mod 4 \\
W^2(C_0(q)_\sigma) & \text{if } i \equiv 1 \mod 4 \\
0 & \text{if } i \equiv 2 \mod 4 \\
\ker(\text{tr}) & \text{if } i \equiv 3 \mod 4 
\end{cases}$$
According to Lemma 9.1, we obtain another 12-term long exact sequence
\[
\begin{array}{ccccccc}
\coker(tr) & \rightarrow & W^0(Q) & \rightarrow & W^{-d}(k, \det(P)) & \rightarrow & W^2(C_0(q)_\sigma) \\
0 & \rightarrow & W^3(Q) & \rightarrow & W^1(Q) & \rightarrow & 0 \\
\ker(tr) & \rightarrow & W^{2-d}(k, \det(P)) & \rightarrow & W^2(Q) & \rightarrow & 0
\end{array}
\]
where we use the fact that \(W^{2i+1}(k, \det(P)) = 0\) for any semi-local ring \(k\) (because any projective module over a connected semi-local ring is free, and \(W^i(R_1 \times R_2, L_1 \times L_2) = W^i(R_1, L_1) \oplus W^i(R_2, L_2)\)). □

Corollary 9.3. Let \(d\) be an even number. Let \(k\) be a field of characteristic \(\neq 2\). Let \(Q_d\) be the quadric defined by an isotropic quadratic form \(q\) over \(k\). Then,
\[
W^i(Q_d) = \begin{cases} 
W^0(k) & \text{if } d \equiv 0 \mod 4 \text{ and if } i = 0 \mod 4 \\
W^2(C_0(q)_\sigma) & \text{if } d \equiv 2 \mod 4 \text{ and if } i = 1 \mod 4 \\
0 & \text{if } d \equiv 0 \mod 4 \text{ and if } i = 2 \mod 4 \\
W^0(C_0(q)_\sigma) & \text{if } d \equiv 0 \mod 4 \text{ and if } i = 3 \mod 4 
\end{cases}
\]
If \(d \equiv 2 \mod 4\), then there is an exact sequence
\[
0 \rightarrow W^2(Q_d) \rightarrow W^0(k) \rightarrow W^0(C_0(q)_\sigma) \rightarrow W^3(Q_d) \rightarrow 0.
\]
If \(d \equiv 0 \mod 4\), then there is an exact sequence
\[
0 \rightarrow W^0(k) \xrightarrow{p^*} W^0(Q_d) \rightarrow W^0(k) \rightarrow W^2(C_0(q)_\sigma) \rightarrow W^1(Q_d) \rightarrow 0
\]
where \(p^*\) is split injective.

Proof. If \(q\) is isotropic, then the pull-back \(p^* : W^0(k) \rightarrow W^0(Q)\) is split injective, cf. Lemma 6.6. It is enough to prove the map \(tr : W^0(C_0(q)_\sigma) \rightarrow W^0(k)\) vanishes. Observe that there is a commutative diagram
\[
\begin{array}{ccc}
W^0(k) & \xrightarrow{p^*} & W^0(A) \\
\downarrow & & \downarrow \\
W^0(Q) & & W^0(Q)
\end{array}
\]
where all the maps are uniquely defined by the context above. Therefore, the map \(W^0(k) \rightarrow W^0(A)\) is injective. From the exact sequence
\[
W^0(C_0(q)_\sigma) \xrightarrow{tr} W^0(k) \rightarrow W^0(A),
\]
one can conclude the map \(tr : W^0(C_0(q)_\sigma) \rightarrow W^0(k)\) vanishes. □
10. Witt Groups of Quadrics Defined by $m(-1) \perp n(1)$

Let $k$ be a field in which $-1$ is not a sum of two squares. Let $Y$ denote the quaternion algebra $\left( \frac{-1,-1}{k} \right)$. Note that the algebra $Y$ is a division algebra. Let $Q_{m,n}$ be the quadric hypersurface defined by the quadratic form $q_{m,n} = m(-1) \perp n(1)$. Following Karoubi (cf. [13, Chapter III.3] and [20, Chapter V]), we write $C_{0,n} := C_0(q_{m,n})$ and $C_{m,n} := C(q_{m,n})$ for convenience. Assume $n + m \geq 2$ (otherwise $Q_{m,n} = \emptyset$).

There are, namely, three cases to be considered:

Case 1. $m \geq 1$ and $n \geq 1$. In this case, the quadric $Q_{m,n}$ is defined by an isotropic quadratic form. Then, Corollary [6.7] and Corollary [9.3] apply.

Case 2. $m = 0$ or $n = 0$, and $m + n$ is odd. We only need to consider $Q_{0,n}$, since $Q_{0,n} \cong Q_{n,0}$ as algebraic varieties. Note that $C_{0,n} \cong C_{n,0}$. We copy the table on [20, p. 125] for this case.

| $n$ | Type of $\sigma$ | $\sigma_0$ | $\sigma_5$ | $\sigma_9$ |
|-----|-----------------|-----------|-----------|-----------|
| 3   | symplectic       | $M_2(Y)$  | $M_8(k)$  | $M_{16}(k)$ |
| 5   | orthogonal       | $W^0(k)$  | $W^0(k)$  | $W^0(k)$  |
| 7   | orthogonal       | $W^0(k)$  | $W^0(k)$  | $W^0(k)$  |
| 9   | orthogonal       | $W^0(k)$  | $W^0(k)$  | $W^0(k)$  |

The type of the canonical involution is determined in [17] Proposition 8.4. The Clifford algebra is 8-periodic in the sense that $C^{n+8} \cong C^{n}$. By Morita equivalence, we see that

$$W^0(C_{0,n}) \cong W^0(k)/2^{\delta(n)}W^0(k)$$

where $\sigma$ is the involution which is the 1 on the center and $-1$ outside the center.

**Theorem 10.1.** Assume $k$ is a field in which $-1$ is not a sum of two squares. Then,

$$W^0(Q_{0,n}) \cong W^0(k)/2^{\delta(n)}W^0(k)$$

where $\delta(n)$ is the cardinality of the set $\{l \in \mathbb{Z}: 0 < l < n, l \equiv 1, 2 \text{ or } 4 \pmod{8}\}$.

**Proof.** It is enough to show $\text{Im}(tr) \cong \ker(\rho^*) \cong 2^{\delta(n)}W^0(k)$. Recall that the map $tr$ takes a symmetric space $M \times M \to C_{0,n}$ in $W^0(C_{0,n})$ to the symmetric space $M \times M \to C_{0,n}$ by forgetting the left $C_{0,n}$ structure. Since there is a surjective group homomorphism $\mathbb{Z}/[k^* / k^{2*}] \to W^0(C_{0,n})$ (via the Morita equivalence), it is enough to investigate where a symmetric space represented by the irreducible $C_{0,n}$-module goes under the map $tr$. Note that in this case $C_{0,n}$ is simple. Let $\Sigma(q)$ be the unique (up to isomorphism) irreducible left $C_0(q)$-module. By Table 1, $\Sigma(q)$ may be understood as a left ideal of $C_{0,n}$. As a $k$-vector space, $\Sigma(q)$ is of rank $2^{\delta(n)}$. Consider a space represented by the form $b : \Sigma(q) \times \Sigma(q) \to C_{0,n} \times C_{0,n}$. Note that $\beta(e,e) = e\bar{e} = 1$ for any $e$ in the set of basis $\{ e^\Delta : \Delta \in \mathbb{F}_2^2 \}$. And that $tr(e^\Delta) = 0$ if $\Delta \neq 0$. Let $A$ be a $k$-algebra and let $a \in A$. Denote $h_{s,t}(a)$ the matrix with $a \in A$ in the $(s,t)$-spot and 0 in other places. Then $C_{0,n}$ is isomorphic to either $M_{2^n}(Y)$ or $M_{2^n}(k)$ for some $n,m \geq 0$. If $C_{0,n}$ is isomorphic to $M_{2^n}(Y)$, we can choose the $k$-vector space basis $\{ h_{1,i}(a) : a \in \{1, i, j, k\} \}$ of $\Sigma(q)$. If $C_{0,n}$ is isomorphic to $M_{2^n}(k)$, we can choose the $k$-vector space basis $\{ h_{1,1}(1) \}$ of $\Sigma(q)$. In terms of this choice of basis, it is then an exercise to check that $tr$ sends $b : \Sigma(q) \times \Sigma(q) \to C_{0,n}$ to the diagonal form $2^{\delta(n)}(1)$ in $W^0(k)$ for $n = 3, 5, 7, 9$. 

Theorem 10.3. Let $-1$ outside the center. \hfill \Box

Proof. To obtain this result, we need to compute the cokernel of the trace map of Witt groups. The situation is similar as in the proof of Theorem 10.1. \hfill \Box

If $k = \mathbb{R}$, then $Y$ is just the Hamilton’s quaternion algebra $\mathbb{H}$.

Corollary 10.2. Let $n = d + 2$ be an odd number. Let $Q_{0,n}$ be the quadric defined by the anisotropic quadratic form $n(1)$ over $\mathbb{R}$. Then,

\[
W^i(Q_{0,n}) = \begin{cases} 
\mathbb{Z}/2\mathbb{Z} & \text{if } i = 0 \pmod{4} \\
\mathbb{Z}/2\mathbb{Z} & \text{if } i = 1 \pmod{4} \text{ and } n \equiv 3, 5 \pmod{8} \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. By [10], Chapter I (10.5), we have $W^0(\mathbb{R}) \approx W^0(\mathbb{H}_{\sigma}) \approx \mathbb{Z}$, $W^2(\mathbb{R}) = 0$ and $W^2(\mathbb{H}_{\sigma}) \approx \mathbb{Z}/2$. It follows that $W^0(Q_{0,n}) \approx \mathbb{Z}/2^{\delta(n)}\mathbb{Z}$ by Theorem 10.1. To compute $W^1(Q_{0,n})$, we need to study $W^2((C^0_{0,n})_{\sigma})$ by Theorem 6.2. By previous discussion, we conclude that

\[
W^2((C^0_{0,n})_{\sigma}) = \begin{cases} 
W^2(\mathbb{H}_{\sigma}) \approx \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 3, 5 \pmod{8} \\
W^2(\mathbb{R}) = 0 & \text{if } n \equiv 1, 7 \pmod{8}
\end{cases}
\]

If $n = d + 2$ is odd, then we have $W^0((C^0_{0,n})_{\sigma}) \approx \mathbb{Z}$ and a commutative diagram

\[
\begin{array}{ccc}
W^0((C^0_{0,n})_{\sigma}) & \xrightarrow{\text{tr}} & W^0(\mathbb{R}) \\
\downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{2^{\delta(n)}} & \mathbb{Z}
\end{array}
\]

for $n \geq 2$ by the proof of Theorem 10.1 so that $\ker(\text{tr}) = 0$. By Theorem 6.2, we conclude $W^2(Q_{0,n}) = W^3(Q_{0,n}) = 0$. \hfill \Box

Case 3. $m = 0$ or $n = 0$, and $m + n$ is even. Again, we only consider $Q_{0,n}$ and copy parts of the table on [21, p. 125] for this case.

| Table 2 | $n=2$ | $n=4$ | $n=6$ | $n=8$ |
|---------|------|------|------|------|
| $C^{n-1,0}$ | $X$ | $Y \times Y$ | $M_4(X)$ | $M_8(k) \times M_8(k)$ |
| Type of $\sigma$ | unitary | symplectic | unitary | orthogonal |

The type of the canonical involution is determined in [17, Proposition 8.4]. The Clifford algebras are also 8-periodic. By Morita equivalence, we see that

\[
W^0(C_0(q)_{\sigma}) = \begin{cases} 
W^0(X_{\sigma_{\sigma}}) & \text{if } n \equiv 2, 6 \pmod{8} \\
W^0(Y_{\sigma_{\sigma}}) \oplus W^0(Y_{\sigma_{\sigma}}) & \text{if } n \equiv 4 \pmod{8} \\
W^0(k) \oplus W^0(k) & \text{if } n \equiv 0 \pmod{8}
\end{cases}
\]

where $\sigma_{\sigma}$ is the unitary involution and $\sigma_{\sigma}$ is the involution which is identity on the center and $-1$ outside the center.

Theorem 10.3. Let $k$ be a field in which $-1$ is not a sum of two squares. If $d \equiv 2 \pmod{4}$, then $W^0(Q_d) \approx W^0(k)/2^{\delta(n)}W^0(k)$ and if $d \equiv 0 \pmod{4}$, then there is an exact sequence

\[
0 \to W^0(k)/2^{\delta(n)}W^0(k) \to W^0(Q_d) \to W^0(k).
\]

Proof. To obtain this result, we need to compute the cokernel of the trace map of Witt groups. The situation is similar as in the proof of Theorem 10.1 \hfill \Box
If \( k = \mathbb{R} \), then \( X \) is the field of complex numbers \( \mathbb{C} \) and \( Y \) is just the quaternion algebra \( \mathbb{H} \).

**Corollary 10.4.** Let \( n = d + 2 \) be an even number. Let \( Q_{0,n} \) be the quadric defined by the anisotropic quadratic form \( n(1) \) over \( \mathbb{R} \). Then,

\[
W^i(Q_{0,n}) \cong \begin{cases} 
\mathbb{Z}/2^i(n)\mathbb{Z} & \text{if } i = 0 \mod 4 \\
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } i \equiv 1 \mod 4 \text{ and } n \equiv 4 \mod 8 \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** By [16, Chapter I (10.5)] again, we have that \( W^0(\mathbb{R}) \cong W^0(\mathbb{H}_{\sigma_a}) \cong W^0(\mathbb{C}_{\sigma_a}) \cong W^2(\mathbb{C}_{\sigma_a}) \cong \mathbb{Z} \), \( W^2(\mathbb{R}) = 0 \) and \( W^2(\mathbb{H}_{\sigma_a}) \cong \mathbb{Z}/2\mathbb{Z} \) where \( \sigma_a \) is just the complex conjugation. According to previous discussion, we conclude that

\[
W^0((C_{0,n}^{0,n})_{\sigma}) \cong \begin{cases} 
W^0(\mathbb{C}_{\sigma_a}) \cong \mathbb{Z} & \text{if } n \equiv 2, 6 \mod 8 \\
W^0(\mathbb{H}_{\sigma_a}) \oplus W^0(\mathbb{H}_{\sigma_a}) \cong \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \equiv 4 \mod 8 \\
W^0(\mathbb{R}) \oplus W^0(\mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \equiv 0 \mod 8
\end{cases}
\]

and that

\[
W^2((C_{0,n}^{0,n})_{\sigma}) \cong \begin{cases} 
W^2(\mathbb{C}_{\sigma_a}) \cong \mathbb{Z} & \text{if } n \equiv 2, 6 \mod 8 \\
W^2(\mathbb{H}_{\sigma_a}) \oplus W^2(\mathbb{H}_{\sigma_a}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 4 \mod 8 \\
W^2(\mathbb{R}) \oplus W^2(\mathbb{R}) = 0 & \text{if } n \equiv 0 \mod 8.
\end{cases}
\]

If \( n \equiv 0 \mod 4 \), we have a commutative diagram

\[
\begin{array}{ccc}
W^0((C_{0,n}^{0,n})_{\sigma}) & \xrightarrow{tr} & W^0(\mathbb{R}) \\
\downarrow \cong & & \downarrow \cong \\
\mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(2^i(n), 2^i(n))} & \mathbb{Z} \\
\end{array}
\]

for \( n \geq 2 \), so that \( \ker(tr) \cong \mathbb{Z} \). If \( n \equiv 2 \mod 4 \), we get a commutative diagram

\[
\begin{array}{ccc}
W^0((C_{0,n}^{0,n})_{\sigma}) & \xrightarrow{tr} & W^0(\mathbb{R}) \\
\downarrow \cong & & \downarrow \cong \\
\mathbb{Z} & \xrightarrow{2^i(n)} & \mathbb{Z} \\
\end{array}
\]

for \( n \geq 2 \), so that \( \ker(tr) = 0 \). By [7, Proposition 3.1], we also note that \( W^i(Q_{0,n}) \) are all 2-primary torsion groups. Thus, if \( d \) is even, \( W^0(Q_{0,n}) \) is isomorphic to \( \mathbb{Z}/2^i(n)\mathbb{Z} \) by Theorem 10.3 (because \( \text{Hom}(\mathbb{Z}/2^c\mathbb{Z}, \mathbb{Z}) = 0 \) for any integer \( c > 0 \)). If \( n \equiv 2 \mod 4 \), then there is a commutative diagram of exact sequences

\[
0 \to \coker(tr) \to W^0(Q_{0,n}) \to W^0(\mathbb{R}) \to W^2((C_{0,n}^{0,n})_{\sigma}) \to W^1(Q_{0,n}) \to 0
\]

\[
0 \to \mathbb{Z}/2^i(n)\mathbb{Z} \xrightarrow{\cong} W^0(Q_{0,n}) \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \to W^1(Q_{0,n}) \to 0
\]
which allows us to deduce \( W^2(Q_{0,n}) = 0 \). If \( n \equiv 0 \) mod 4, then there is a commutative diagram of exact sequences

\[
\begin{array}{ccccccc}
0 & \to & W^2(Q_{0,n}) & \to & W^0(\mathbb{R}) & \to & \ker(tr) & \to & W^2(Q_{0,n}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W^2(Q_{0,n}) & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & W^2(Q_{0,n}) & \to & 0.
\end{array}
\]

which gives \( W^2(Q_{0,n}) = W^3(Q_{0,n}) = 0 \). Other cases can be simply observed from Theorem 9.2

\[\square\]

Appendix A. Witt groups of Clifford algebras

The purpose of this appendix is to give examples for which Theorem 1.2 can be applied. Let \( k \) be a commutative ring containing \( \frac{1}{2} \). Let \( A \) be a \( k \)-algebra with an involution \( \sigma \). Let \( W^i(A_\sigma) \) be Balmer’s Witt groups of finitely generated projective modules over \( A \), which are identified as \( W^i(A_\sigma - \text{proj}) \) in Balmer and Preedi’s work [4] and as

\[ W^i(A_\sigma) := W^i(\text{sPerf}(A), \text{quis}, \#_A^\sigma, \text{can}) \]

in Schlichting’s framework [29]. Note that \( W^0(A_\sigma) \) is isomorphic to the classical Witt group of the algebra \( A \) with the involution \( \sigma \). Let \( W^i(A_\sigma - \text{free}) \) be the Balmer’s Witt groups of finite free modules over \( A \) (see [4] for this notation). The following result may be compared to [8, Theorem 5.2].

**Theorem A.1.** Let \( k \) be a commutative ring with \( \frac{1}{2} \). Let \( A \) be a semilocal \( k \)-algebra with an involution \( \sigma \). Let \( J(A) \) be the Jacobson radical of \( A \) and \( \tilde{A} := A/J(A) \). If the natural map \( K_0(A) \to K_0(\tilde{A}) \) is an isomorphism, then \( W^{2i+1}(A_\sigma) = 0 \).

**Proof.** Let \( \tilde{\sigma} \) be the induced involution on \( \tilde{A} \). Consider the following diagram consisting of Hornbostel-Schlichting exact sequences (cf. [12, Appendix A])

\[
\begin{array}{ccccccc}
\hat{H}^{i-1}(C_2, \tilde{K}_0(A)) & \to & W^i(A_\sigma - \text{free}) & \to & W^i(A_\sigma) & \to & \hat{H}^i(C_2, \tilde{K}_0(A)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\hat{H}^{i-1}(C_2, \tilde{K}_0(\tilde{A})) & \to & W^i(\tilde{A}_\sigma - \text{free}) & \to & W^i(\tilde{A}_\sigma) & \to & \hat{H}^i(C_2, \tilde{K}_0(\tilde{A}))
\end{array}
\]

where \( \tilde{K}_0(A) := \text{coker}(K_0(A - \text{free}) \to K_0(A)) \). If \( K_0(A) \to K_0(\tilde{A}) \) is an isomorphism, then both the left and the right vertical maps are isomorphisms. The second vertical map is injective for \( i \) odd by [8, Theorem 5.2]. Since \( \tilde{A} \) is semisimple, we have \( W^i(\tilde{A}_\sigma) = 0 \) for \( i \) odd by [22, or 4 Theorem 4.2]. Finally, we conclude \( W^i(A_\sigma) = 0 \) by a diagram chasing.

Recall that a \( k \)-algebra \( A \) is called semiperfect if \( A \) is semilocal and idempotents of \( A/J(A) \) can be lifted to \( A \). For examples, see below.

**Corollary A.2.** Let \( k \) be a commutative ring containing \( \frac{1}{2} \). If \( A \) is a semiperfect \( k \)-algebra with an involution \( \sigma \), then \( W^{2i+1}(A_\sigma) = 0 \).

**Proof.** The proof of [5, Proposition 2.12 Chapter III on p. 90] can be applied to show that \( K_0(A) \to K_0(\tilde{A}) \) is an isomorphism when \( A \) is semiperfect.
Let \((P, q)\) be a nondegenerate quadratic form of rank \(n\) over \(k\). Let \(C(q)\) be the Clifford algebra of \((P, q)\) and let \(C_0(q)\) be its even part. Let \(\sigma\) be an involution of the first kind on \(C(q)\) (the one induces the identity on the center). Certainly, \(W^{2l+1}(C_0(q)_\sigma) = W^{2l+1}(C(q)_\sigma) = 0\) if the base \(k\) is a field of characteristic \(\neq 2\) by [22], since Clifford algebras are semisimple over a field. In this appendix, we go a little further. If \(k\) is local of characteristic \(\neq 2\), then we can assume \(P\) is free and we can find a basis \(\{e_1, \ldots, e_n\}\) of \((P, q)\) and write \(q \approx (\alpha_1, \ldots, \alpha_n)\) with \(\alpha_i = q(e_i) \in k^*\).

**Theorem A.3.** Let \(k\) be a commutative semilocal ring containing \(\frac{1}{2}\). Then

\[W^{2l+1}(C_0(q)_\sigma) = W^{2l+1}(C(q)_\sigma) = 0\]

if one of the following additional assumptions hold.

(i) \((k, J(k))\) is a Henselian pair,

(ii) \(k\) is a local ring containing a field \(F\) such that \(F\) is isomorphic to the residue field and \(q(e_i) \in F^*\),

(iii) \(k\) is a regular local ring containing a field \(F\).

To prove this theorem, we need the following lemmas.

**Lemma A.4.** Let \(k\) be a commutative ring. If \(A\) is a separable \(k\)-algebra, then \(J(A) = J(k)A\).

**Proof.** See [13, Lemma 1.1 (d)].

**Lemma A.5.** Let \(k\) be a commutative semilocal ring. Then finite \(k\)-algebras are semilocal.

**Proof.** See [19, Proposition 20.6 on p. 297]

**Lemma A.6.** Under the assumption (i) of Theorem A.3, separable \(k\)-algebras are semiperfect.

**Proof.** By Lemma A.5, we learn that separable algebras over commutative semilocal rings are semilocal. Under the assumption (i) of Theorem A.3 the idempotent lifting property with respect to the Jacobson radical holds by [9, Theorem 2.1] and Lemma A.4.

**Lemma A.7.** Let \(k\) be a commutative ring. Then both \(C(q)\) and \(C_0(q)\) are separable \(k\)-algebras.

**Proof.** See [11, Theorem 9.9 on p. 127].

**Lemma A.8.** Under the assumption (ii) of Theorem A.3, both \(C(q)\) and \(C_0(q)\) are semiperfect.

**Proof.** Let \(m\) be the unique maximal ideal of \(k\). Note that \(J(C(q)) = mC(q)\) by Lemma A.4 and A.7. It is enough to show that idempotents of \(C(q)/mC(q)\) can be lifted to \(C(q)\). Let \(\{e_i\}_{1 \leq i \leq n}\) be an orthogonal basis of \((P, q)\). If \(\Delta = (e_1, \ldots, e_n) \in F^*_2\), then we write \(e^\Delta := e_1^\Delta \cdots e_n^\Delta\). It is well-known that \(C(q)\) has a \(k\)-basis \(\{e^\Delta : \Delta \in F^*_2\}\).

Let \(\bar{e} = \sum \bar{a}_ie_i^\Delta\) (with \(\bar{a}_i \in k/m\)) be a non-zero idempotent in \(C(q)/mC(q)\). For each nonzero \(\bar{a}_i \in k/m\), we can find an element \(a_i \in F^*\) lifting \(\bar{a}_i\) via the isomorphism \(F \rightarrow k \rightarrow k/m\). Define \(e = \sum a_ie_i^\Delta\). Claim that \(e\) is an idempotent in \(C(q)\). If not, then \(e^2 - e = \sum b_\lambda e^\Delta\) for some \(b_\lambda \in F^*\) (because \(a_i\) and \(q(e_i)\) are all in \(F^*\)). It follows that \(\bar{e}^2 - \bar{e} = 0 \in C(q)/mC(q)\). This contradicts the assumption that \(\bar{e}\) is an
idempotent in \( k/\mathfrak{m} \). Using the \( k \)-basis \( \{ e^\Delta : \Delta = (c_1, \ldots, c_n) \in \mathbb{F}_2^n \text{ and } \sum_i c_i = 0 \in \mathbb{F}_2 \} \) for \( C_0(q) \), we see that the same strategy works for the idempotent lifting property of \( C_0(q) \) with respect to \( J(C_0(q)) \).

**Proof of Theorem A.3.** By Corollary A.2, Lemma A.6 and Lemma A.8, we obtain the result under the assumption (i) or (ii). The result under the assumption (iii) can be concluded by [10]. □

**Remark A.9.** Clifford algebras over local rings need not be semiperfect in general. For example, the quaternion algebra \( (\frac{1-p}{\mathbb{Z}(p)}) \) is not semiperfect for any prime \( p > 0 \) where \( \mathbb{Z}(p) \) is the localization of \( \mathbb{Z} \) at the multiplicative system \( \mathbb{Z} - (p) \). However, it still might be the case for \( W^{2l+1}(C_0(q)_{\sigma}) = W^{2l+1}(C(q)_{\sigma}) = 0 \).

**Acknowledgement.** This paper is part of my PhD studies. I would like to thank my supervisor Marco Schlichting for his supervision and enlightening ideas. I am grateful to Baptiste Calmès, Jean Fasel, Alexander Kuznetsov, Simon Markett and Dmitriy Rumynin for comments and discussion. Parts of the results were found during my stay at MPIM, Bonn. I thank MPIM for its hospitality.

**References**

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. 75 (1962), 603-632.
[2] P. Balmer, *Triangular Witt groups. Part I: The 12-term localization exact sequence*, K-Theory (2) 19 (2000), 311-363.
[3] P. Balmer, *Witt groups*, Handbook of K-Theory II, Springer-Verlag Berlin Heidelberg, 2005.
[4] P. Balmer and R. Preeti, *Shifted Witt groups of semi-local rings*, manucripta math., 117 (2005), 1-27.
[5] H. Bass, *Algebraic K-theory*, W. A. Benjamin, INC, New York, (1968).
[6] H. Bass, *Hermitian K-Theory and geometric applications*, Proceedings of the conference held at the Seattle research center of the Battelle memorial institute, from August 28 to September 8, (1972).
[7] I. Dell’Ambrogio and J. Fasel, *The Witt groups of the spheres away from two*, Journal of Pure and applied algebra 212 (2008), 1039-1045.
[8] J. Davis and A. Ranicki, *Semi-invariants in surgery*, K-Theory 1 (1987), 83-109.
[9] S. Greco, *Algebras over nonlocal Hensel rings*, J. algebra 8 (1968), 45-59.
[10] S. Gille, *On coherent hermitian Witt groups*, manucripta math. 141 (2013), 423-446.
[11] A. J. Hahn, *Quadratic algebras, Clifford algebras and arithmetic Witt groups*, Universitext. Springer-Verlag, New York, 1994.
[12] J. Hornbostel and M. Schlichting, *Localization in Hermitian K-Theory of Rings*, J. London Math. Soc. (2) 70 (2004), 77-124.
[13] E. C. Ingraham, *Inertial subalgebras of algebras over commutative rings*, Trans. Amer. Math. Soc. 124 (1966), 77-93.
[14] M. Karoubi, *K-theory: An introduction*, Grundlehren der mathematischen Wissenschaften (226), Springer-Verlag Berlin Heidelberg, 1978.
[15] M. M. Kapranov, *On derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. 92 (1988), 479-508.
[16] M. A. Knus, *Hermitian forms and quadratic forms*, Grun. der math. Wiss. 294, Springer-Verlag, 1991.
[17] M.-A. Knus, A. Merkurjev, M. Rost and J.-P. Tignol, *The book of involution*, AMS Colloquium Publications 44, 1998.
[18] A. Kuznetsov, *Derived categories of quadric fibrations and intersections of quadrics*, Advances in Mathematics 218 (2008), 1340-1369.
[19] T. Y. Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics 131. Springer-Verlag, New York, 1991.
[20] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate studies in mathematics 67. American Mathematical Society, Providence, RI, 2005.
[21] A. Nenashev, *On the Witt groups of projective bundles and split quadrics: Geometric reasoning*, J. K-theory 3 (2009), 533-546.
[22] Andrew Ranicki, *On the algebraic L-theory of semisimple rings*, J. Algebra 50 (1978), 242-243.
[23] M. Schlichting, *Hermitian K-theory of exact categories*, J. K-theory 5 (2010), no. 1, 105-165.
[24] M. Schlichting, *The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes*, Invent. Math. 179 (2010), no. 2, 349-433.
[25] M. Schlichting, *Higher Algebraic K-theory (After Quillen, Thomason and Others)*, Springer Lecture Notes in Math. 2008 (2011), 167-242.
[26] M. Schlichting, *Hermitian K-theory, derived equivalences, and Karoubi's fundamental theorem*, preprint, (2012).
[27] D. B. Shapiro, *Composition of quadratic forms*, De Gruyter Expositions in Mathematics 33, Walter de Gruyter, 2000.
[28] R. G. Swan, *K-theory of quadric hypersurfaces*, Ann. of Math., 122 (1985), 113-153.
[29] C. Walter, *Witt groups of quadrics*, A talk given in the conference at the IHP, Paris, (2004).
[30] H. Xie, *An application of Hermitian K-theory: Sums-of-squares formulas*, Documenta Math. 19 (2014), 195-208.
[31] M. Zibrowius, *Witt Groups of Complex Cellular Varieties*, Documenta Math. 16 (2011), 465-511.

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