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Subdifferential inclusions for stress formulations of unilateral contact problems

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Abstract
We consider two classes of inclusions involving subdifferential operators, both in the sense of Clarke and in the sense of convex analysis. An inclusion that belongs to the first class is stationary while an inclusion that belongs to the second class is history-dependent. For each class, we prove existence and uniqueness of the solution. The proofs are based on arguments of pseudomonotonicity and fixed points in reflexive Banach spaces. Then we consider two mathematical models that describe the frictionless unilateral contact of a deformable body with a foundation. The constitutive law of the material is expressed in terms of a subdifferential of a nonconvex potential function and, in the second model, involves a memory term. For each model, we list assumptions on the data and derive a variational formulation, expressed in terms of a multivalued variational inequality for the stress tensor. Then we use our abstract existence and uniqueness results on the subdifferential inclusions and prove the unique weak solvability of each contact model. We end this paper with some examples of one-dimensional constitutive laws for which our results can be applied.

Keywords
Clarke subdifferential, convex subdifferential, pseudomonotone operator, history-dependent operator, inclusion, unilateral contact condition, frictionless contact, weak solution

1. Introduction
Subdifferential inclusions play an important and challenging role in the study of nonlinear boundary value problems, which arise in mechanics, physics and engineering sciences. In particular, they represent a powerful instrument, which enables new and interesting results to be obtained in the study of various classes of variational and hemivariational inequalities. This is because variational inequalities are closely related to inclusions involving the subgradient of convex functions and, in turn, hemivariational inequalities are related to inclusions involving the Clarke subgradient of locally Lipschitz functions. Variational and hemivariational inequalities have been widely used in the study of mathematical models that describe the contact process of deformable bodies [1–13]. Various classes of stationary and history-dependent subdifferential inclusions have been studied [8, 14–17]. There, existence and uniqueness results were proved, through arguments of surjectivity for pseudomonotone
operators and the Banach fixed point theorem. These results have been used in the study of various classes of variational and hemivariational inequalities, for which continuous dependence of the solution on the data was shown, numerical methods were introduced, and convergence was established rigorously. Finally, results on the well-posedness and error estimation of numerical solutions were applied to inequalities arising in the study of new models of contact.

The aim of this paper is to study a new class of subdifferential inclusion and to apply the corresponding results in the analysis of contact problems with unilateral constraints. Unlike the inequalities considered in the papers mentioned in the previous paragraph, the inclusions considered here involve both the subdifferential of the indicator of a closed convex set and the Clarke subdifferential of a locally Lipschitz function. This represents a first trait of novelty of our paper, which allows us to obtain existence and uniqueness results for new classes of variational inequalities, with multivalued operators and unilateral constraints. We apply these results in the study of frictionless models of unilateral contact, both in the static and time-dependent case. The constitutive law of the material is expressed in terms of the Clarke subdifferential, whose argument is the stress function. Consideration of such models, in which the unilateral constraints are formulated in terms of displacement and the constitutive law is governed by a subdifferential in stress, leads to new and challenging mathematical problems. Their variational analysis represents the second trait of novelty of our paper.

The rest of the manuscript is structured as follows. In Section 2, we present some notation and preliminary material. In Section 3, we consider a class of stationary subdifferential inclusions in reflexive Banach spaces, for which we state and prove an abstract existence and uniqueness result, Theorem 2. We extend this result in Section 4 to a class of history-dependent subdifferential inclusions. There, we state and prove our second abstract existence and uniqueness result, Theorem 3. In Section 5, we consider a static frictionless model of contact with unilateral constraints. We list the assumptions on the data, then we derive a variational formulation of the problem, in terms of stress. Next, we use Theorem 2 to prove the unique solvability of the problem. In Section 6, we use Theorem 3 to extend these results to a history-dependent frictionless model of contact. Finally, in Section 7, we provide examples of one-dimensional constitutive laws for which our results work.

2. Notation and preliminaries

In this section we briefly present the notation and some preliminary material to be used later in this paper. More details on the material presented below can be found elsewhere [8, 9, 18–20].

First, we remark that all linear spaces used in this paper are assumed to be real. Unless stated otherwise, in this section we denote by $X$ a normed space and use the notation $\| \cdot \|_X$ and $0_X$ for the norm and the zero element of $X$, respectively. We denote by $X^*$ its topological dual, and $\langle \cdot , \cdot \rangle_{X^\ast \times X}$ will represent the duality pairing of $X$ and $X^*$. The symbol $2^{X^*}$ is used to represent the set of all subsets of $X^*$. We start with a definition of the subdifferential in the sense of Clarke.

**Definition 1.** Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function. The Clarke generalized directional derivative of $\varphi$ at the point $x \in X$ in the direction $v \in X$, is defined by

$$\varphi^0(x; v) = \limsup_{\lambda \downarrow 0} \frac{\varphi(x + \lambda v) - \varphi(x)}{\lambda}. $$

The Clarke subdifferential of $\varphi$ at $x$ is a subset of $X^*$ given by

$$\partial_{\text{Cl}} \varphi (x) = \left\{ \xi \in X^* : \varphi^0(x; v) \geq \langle \xi , v \rangle_{X^* \times X} \text{ for all } v \in X \right\}. $$

We recall the basic properties of the Clarke subdifferential (see Proposition 3.23 (iv) of Migórski et al. [8]).

**Proposition 1.** Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function. Then

1. For every $x \in X$, the set $\partial_{\text{Cl}} \varphi (x)$ is nonempty, convex and weakly* compact subset of $X^*$.
2. The graph of the Clarke subdifferential $\partial_{\text{Cl}} \varphi$ is closed in $X \times (w^* - X^*)$ topology, i.e., if $\{x_n\} \subset X$ and $\{\xi_n\} \subset X^*$ are sequences such that $\xi_n \in \partial_{\text{Cl}} \varphi (x_n)$ and $x_n \to x$ in $X$, $\xi_n \to \xi$ weakly* in $X^*$, then $\xi \in \partial_{\text{Cl}} \varphi (x)$.

For a convex function, we recall the definition of its subdifferential in the sense of convex analysis.
Definition 2. Let $X$ be a Banach space and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The subdifferential of $\Phi$ at $x \in X$ is then defined by

$$\partial \Phi(x) = \{ \eta \in X^* \mid \Phi(y) - \Phi(x) \geq \langle \eta, v - x \rangle_{X^* \times X} \text{ for all } v \in X \}.$$ 

Moreover, the effective domain of $\Phi$ is the set defined by

$$\text{dom } \Phi = \{ x \in X \mid \Phi(x) < +\infty \}.$$ 

We now recall the definition of pseudomonotonicity, for both single-valued and multivalued operators.

Definition 3. A single-valued operator $A : X \to X^*$ is called pseudomonotone if, for any sequence $\{v_n\}_{n=1}^{\infty} \subset X$, $v_n \to v$ weakly in $X$ and

$$\limsup_{n \to \infty} \langle Au_n, v_n - v \rangle_{X^* \times X} \leq 0$$

imply that

$$\langle Av, v - y \rangle_{X^* \times X} \leq \liminf_{n \to \infty} \langle Av_n, v_n - y \rangle_{X^* \times X},$$

for all $y \in X$.

Definition 4. A multivalued operator $A : X \to 2^{X^*}$ is called pseudomonotone if the following conditions hold:

1. $A$ has values that are nonempty, weakly compact and convex.
2. $A$ is upper semicontinuous from every finite dimensional subspace of $X$ into $X^*$ endowed with the weak topology.
3. For any sequence $\{v_n^*\}_{n=1}^{\infty} \subset X$ and any $v^* \in Av_n$, $v_n \to v$ weakly in $X$ and $\limsup_{n \to \infty} \langle v^*_n, v_n - v \rangle_{X^* \times X} \leq 0$ imply that for any $y \in X$ there exists $u(y) \in Av$, such that

$$\langle u(y), v - y \rangle_{X^* \times X} \leq \liminf_{n \to \infty} \langle v^*_n, v_n - y \rangle_{X^* \times X}.$$ 

The next proposition corresponds to Proposition 3.58 in Migórski et al. [8].

Proposition 2. Let $X$ be a real reflexive Banach space, and assume that $A : X \to 2^{X^*}$ satisfies the following conditions:

1. For each $v \in X$, $Av$ is a nonempty, closed and convex subset of $X^*$.
2. $A$ is bounded, i.e., it maps bounded sets into bounded ones.
3. If $v_n \to v$ weakly in $X$, $v^*_n \to v^*$ weakly in $X^*$ with $v^*_n \in Av_n$ and, moreover, $\limsup_{n \to \infty} \langle v^*_n, v_n - v \rangle_{X^* \times X} \leq 0$, then $v^* \in Av$ and $\langle v^*, v_n \rangle \to \langle v^*, v \rangle$.

Then the operator $A$ is pseudomonotone.

The following proposition deals with a multivalued operator that is a perturbation of the Clarke subdifferential of a locally Lipschitz functional. We acknowledge that the idea of the proof of Proposition 3 comes from Migórski et al. [21]. However, for the convenience of the reader, we provide a detailed proof.

Proposition 3. Let $X$ be a real reflexive Banach space and let the operator $A : X \to X^*$ and the functional $J : X \to \mathbb{R}$ be such that:

1. $A$ is demicontinuous, i.e., for every sequence $\{u_n\}_{n=1}^{\infty}$ if $u_n \to u$ in $X$ then $Au_n \to Au$ weakly in $X^*$.
2. $A$ is strongly monotone, i.e., there exists a constant $\alpha > 0$ such that, for all $u_1, u_2 \in X$ we have

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle_{X^* \times X} \geq \alpha \|u_1 - u_2\|_{X^*}^2.$$ 

3. $J$ is locally Lipschitz.
4. $\partial C_J J$ is bounded, i.e., if $U \subset X$ is a bounded set in $X$ then the set

$$\{ x^* \in X^* \mid \exists u \in U, \text{ such that } x^* \in \partial C_J J(u) \}$$

is bounded in $X^*$.
5. \( \partial_{\text{Cl}} J \) is relaxed monotone, i.e., there exists a constant \( m > 0 \) such that, for all \( u_1, u_2 \in X \), \( \eta_1, \eta_2 \in X^* \), if \( \eta_i \in \partial_{\text{Cl}} J(u_i), \ i = 1, 2 \), then
\[
\langle \eta_1 - \eta_2, u_1 - u_2 \rangle_{X^* \times X} \geq -m \| u_1 - u_2 \|^2_{X^*}.
\]

6. \( \alpha > m. \)

Then the operator \( T : X \to 2^{X^*} \) defined by \( Tu = Au + \partial_{\text{Cl}} J(u) \) for all \( u \in X \) is pseudomonotone.

**Proof.** To prove that operator \( T \) is pseudomonotone, we shall apply Proposition 2. To this end, we need to show that \( T \) satisfies the three conditions of Proposition 2. First, it follows from condition 3 of Proposition 3 and Proposition 1 that, for all \( u \in X \), the set \( Tu \) is nonempty, convex and closed and, therefore, condition 1 of Proposition 2 holds. Moreover, by condition 4 of Proposition 3, it follows that \( T \) is bounded, which shows that condition 2 of Proposition 2 holds, too. It remains to check assumption 3 of Proposition 2. To this end, let \( \{u_n\}_{n=1}^{\infty} \subset X \) and \( \{u_n^*\}_{n=1}^{\infty} \subset X^* \) be sequences such that \( u_n \to u \) weakly in \( X \), \( u_n^* \to u^* \) weakly* in \( X^* \), \( u_n^* \in Tu_n \) for \( n \in \mathbb{N} \) and
\[
\limsup_{n \to \infty} \langle u_n^*, u_n - u \rangle_{X^* \times X} \leq 0. \tag{1}
\]

Our goal is to show that \( u^* \in Tu \) and \( \langle u_n^*, u_n \rangle_{X^* \times X} \to \langle u^*, u \rangle_{X^* \times X} \) as \( n \to \infty \). Since \( u_n^* \in Tu_n \), it follows that, for all \( n \in \mathbb{N} \), there exists \( \eta_n \in X^* \) such that
\[
u_n^* = Au_n + \eta_n \tag{2}
\]
and
\[
\eta_n \in \partial_{\text{Cl}} J(u_n). \tag{3}
\]

Let us choose arbitrary \( \xi \in \partial_{\text{Cl}} J(u) \). Using conditions 2 and 5 of Proposition 3, we get
\[
(\alpha - m)\| u_n - u \|^2_{X^*} \leq \langle Au_n + \eta_n - Au - \xi, u_n - u \rangle_{X^* \times X}
= \langle u_n^*, u_n - u \rangle_{X^* \times X} - \langle Au + \xi, u_n - u \rangle_{X^* \times X}. \tag{4}
\]

Combining equations (4) with (1), using assumption 6 of Proposition 3 and the fact that \( u_n \to u \) weakly in \( X \), we conclude that
\[
u_n \to u \text{ in } X. \tag{5}
\]

Since the sequence \( \{u_n\}_{n=1}^{\infty} \) converges in \( X \), it follows that it is bounded in \( X \). Conversely, using assumption 4 of Proposition 3 and equation (3), we see that the sequence \( \{\eta_n\} \) is bounded in \( X^* \). Since the space \( X^* \) is reflexive, there exists \( \eta \in X^* \) such that for a subsequence, still denoted \( \eta_n \), we have
\[
\eta_n \to \eta \text{ weakly in } X^*. \tag{6}
\]

Using equations (3), (5) and (6) and applying condition 2 of Proposition 1, we see that
\[
\eta \in \partial_{\text{Cl}} J(u). \tag{7}
\]

Conversely, using equation (5) and assumption 1 of Proposition 3, we obtain that
\[
Au_n \to Au \text{ weakly in } X^*. \tag{8}
\]

Next, combining equations (2), (6) and (8), we get \( u^*_n \to Au + \eta \) weakly in \( X^* \). Thus, by the uniqueness of the weak limit, it follows that \( u^* = Au + \eta \). Combining this equality with equation (7) we obtain that \( u^* \in Tu \). Finally, from equation (5) and the fact that \( u^*_n \to u^* \) weakly in \( X^* \), it follows that \( \langle u_n^*, u_n \rangle_{X^* \times X} \to \langle u^*, u \rangle_{X^* \times X} \), which completes the proof of the proposition. \( \square \)

The next proposition deals with an existence result for an abstract elliptic inclusion and corresponds to Theorem 2.2. in Lea [22].
Theorem 1. Assume that \( X \) is a Banach space and \( \mathcal{S} \) is a given operator. Then for all \( v \in D(\mathcal{S}) \) and \( \xi \in \mathcal{S} \), we have
\[
\langle \xi + \eta - L, v - v_0 \rangle_{X^* \times X} > 0,
\]
for all \( v \in D(\mathcal{S}) \) with \( \|v\|_X = R \) and all \( \xi \in F(v), \eta \in G(v) \). Then there exists at least an element \( u \in D(\mathcal{S}) \), such that
\[
F(u) + G(u) \ni L.
\]

Note that in the statement of Proposition 4 we denote by \( D(F) \) and \( D(G) \) the effective domains of the operators \( F \) and \( G \), respectively, and that \( B_R(0_X) \) represents the sphere of radius \( R \) and centre \( 0_X \).

We end this section with some preliminaries useful in the study of history-dependent inclusions. Thus, for \( T > 0 \), we use the usual notation \( L^2(0, T; X) \) for the Bochner–Lebesgue space and we recall the following definition.

Definition 5. An operator \( \mathcal{S} : L^2(0, T; X) \to L^2(0, T; X) \) is called a history-dependent operator if the following condition holds:
\[
\left\{ \begin{array}{l}
\text{There exists } L_S > 0 \text{ such that} \\
\quad \| (\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t) \|_X \leq L_S \int_0^t \| u_1(s) - u_2(s) \|_X \, ds \\
\quad \forall u_1, u_2 \in L^2(0, T; X), \text{ a.e. } t \in (0, T).
\end{array} \right.
\]

We also recall the following fixed point result.

Theorem 1. Assume that \( X \) is a Banach space and \( \mathcal{S} : L^2(0, T; X) \to L^2(0, T; X) \) is a history-dependent operator. Then \( \mathcal{S} \) has a unique fixed point, i.e., there exists \( \eta^* \in L^2(0, T; X) \) such that \( \mathcal{S}\eta^* = \eta^* \).

A proof of Theorem 1 could be found, for instance, in Migórski et al. [8] or Sofonea and Matei [13].

3. Stationary subdifferential inclusions

Everywhere in this section, we assume that \( Q \) is a real reflexive Banach space. We denote by \( \| \cdot \|_Q \) its associated norm and by \( \langle \cdot, \cdot \rangle_Q \) the duality pairing between \( Q \) and its dual \( Q^* \). Let \( \Sigma \) be a subset of \( Q \), \( \Lambda : Q \to Q^* \) a given operator, \( J : Q \to \mathbb{R} \) a locally Lipschitz function and \( f \in Q^* \). We denote by \( \partial J \) the subdifferential of the function \( J \) in the sense of Clarke, by \( \psi_\Sigma : Q \to \mathbb{R} \cup \{+\infty\} \) the indicator function of the set \( \Sigma \) and by \( \partial \psi_\Sigma \) its subdifferential in the sense of convex analysis. Recall that
\[
\psi_\Sigma(\sigma) = \begin{cases} 
0 & \text{for } \sigma \in \Sigma, \\
+\infty & \text{for } \sigma \in Q \setminus \Sigma.
\end{cases}
\]
and, in addition,
\[
\partial \psi_\Sigma(\sigma) = \begin{cases} 
\{ \xi \in Q^* \mid \langle \xi, \tau - \sigma \rangle_Q \leq 0 \text{ for all } \tau \in \Sigma \} & \text{if } \sigma \in \Sigma, \\
\emptyset & \text{if } \sigma \in Q \setminus \Sigma.
\end{cases}
\]

With these data, we consider the following subdifferential inclusion.

Problem P. Find an element \( \sigma \in Q \), such that
\[
A\sigma + \partial J(\sigma) + \partial \psi_\Sigma(\sigma) \ni f.
\]

In the study of Problem P we consider the following hypotheses:

\( H(\Sigma) \) The set \( \Sigma \) is a convex, nonempty, closed subset of \( Q \).
H(A) The operator $A: Q \rightarrow Q^*$ is Lipschitz continuous, strongly monotone, i.e.:

1. There exists a constant $L_A > 0$, such that
   \[ ||A\sigma_1 - A\sigma_2||_{Q^*} \leq L_A ||\sigma_1 - \sigma_2||_Q \quad \text{ for all } \sigma_1, \sigma_2 \in Q. \]

2. There exists a constant $\alpha_A > 0$, such that
   \[ \langle A\sigma_1 - A\sigma_2, \sigma_1 - \sigma_2 \rangle_{Q^* \times Q} \geq \alpha_A ||\sigma_1 - \sigma_2||_Q^2 \quad \text{ for all } \sigma_1, \sigma_2 \in Q. \]

H(J) The function $J: Q \rightarrow \mathbb{R}$ is such that:

1. $J$ is locally Lipschitz.
2. There exists a constant $c_J > 0$, such that
   \[ ||\xi||_{Q^*} \leq c_J (1 + ||\sigma||_Q) \quad \text{ for all } \sigma \in Q, \text{ all } \xi \in \partial \mathcal{C}_J(\sigma). \]

3. There exists a constant $m_J > 0$, such that
   \[ \langle \xi_1 - \xi_2, \sigma_1 - \sigma_2 \rangle_{Q^* \times Q} \geq -m_J ||\sigma_1 - \sigma_2||_Q^2 \]
   for all $\sigma_1, \sigma_2 \in Q, \xi_1, \xi_2 \in Q^*$, with $\xi_i \in \partial \mathcal{C}_J(\sigma_i), i = 1, 2.$

H(f) $f \in Q^*.$

Finally, we consider the smallness assumption
\[ \alpha_A > \max \{m_J, c_J\} \quad (15) \]
where, recall, $\alpha_A, m_J$ and $c_J$ are the positive constants that appear in assumptions $H(A)$ and $H(J)$.

Our main existence and uniqueness result in this section is the following.

**Theorem 2.** Assume that $H(\Sigma), H(A), H(J), H(f)$ and (15) hold. Then Problem $P$ has a unique solution $\sigma \in \Sigma.$

**Proof.** We consider three multivalued operators $T_1, T_2$ and $T: Q \rightarrow 2^{Q^*}$, defined by
\[ T_1\sigma = \partial \psi_\Sigma(\sigma), \quad T_2\sigma = A\sigma + \partial \mathcal{C}_J(\sigma), \quad T\sigma = T_1\sigma + T_2\sigma, \quad \text{for all } \sigma \in Q. \]

We show that operator $T$ is surjective, i.e., for all $F \in Q^*$ there exists $\sigma \in Q$, such that
\[ T\sigma \ni F. \]

To this end, we apply Proposition 4. First, we note that assumption $H(\Sigma)$ implies that $\psi$ is a convex, proper, lower semicontinuous function. Then, it follows that operator $T_1$ is maximal monotone as a subdifferential of a convex, proper and lower semicontinuous function and, moreover, $D(T_1) = \Sigma$. Conversely, by Proposition 3, operator $T_2$ is pseudomonotone and $D(T_2) = Q$. Let $\tau_0 \in \Sigma$ be fixed, $F \in Q^*$ and, for simplicity, denote by $0$, the zero element of the space $Q$, i.e., $0 = 0_Q$. We define the constants
\[ \tilde{C} = \frac{5}{(\alpha_A - c_J)^2} \left( ||F||_{Q^*}^2 + c_J^2 + ||A0||_{Q^*}^2 + (L_A^2 + c_J^2)||\tau_0||_Q^2 \right) \]
\[ + \frac{2}{\alpha_A - c_J} \left( ||A0||_{Q^*} + c_J + ||F||_{Q^*} \right) ||\tau_0||_Q, \]
\[ R_1 = \sqrt{\tilde{C}}. \]

It follows from equation (15) that $\tilde{C} > 0$ and, therefore, $R_1$ is well defined. Let $\tau \in \Sigma, \xi \in T_1\tau$ and $\eta \in T_2\tau$. The last inclusion shows that there exists $\zeta \in \partial \mathcal{C}_J(\tau)$ such that $\eta = A\tau + \zeta$. Suppose that $||\tau||_Q \geq R_1$. Then, using equation (15), we have
\[ (\alpha_A - c_J - 5\tilde{C}) ||\tau||_Q^2 \]
\[ \geq \frac{1}{4\tilde{C}} \left( ||F||_{Q^*}^2 + c_J^2 + ||A0||_{Q^*}^2 + (L_A^2 + c_J^2)||\tau_0||_Q^2 \right) + (||A0||_{Q^*} + c_J + ||F||_{Q^*}) ||\tau_0||_Q. \quad (16) \]
where \( \bar{\epsilon} := (\alpha_A - c_J)/10 \). Conversely, using the Hölder inequality and \( H(J)(2) \), we have
\[
\langle F - \zeta, \tau \rangle_{Q^* \times Q} + \langle A\tau + \zeta - F, \tau_0 \rangle_{Q^* \times Q}
 \leq (c_J + 4\bar{\epsilon})\|\tau\|_{Q}^2 + \frac{1}{4\bar{\epsilon}}(\|F\|_{Q^*}^2 + c_J^2 + (L_A^2 + c_J^2)\|\tau_0\|_{Q}^2) + (\|A\|_{Q^*} + c_J + \|F\|_{Q^*})\|\tau_0\|_{Q}.
\] (17)
Moreover, from \( H(A) \), we find that
\[
\langle A\tau, \tau \rangle_{Q^* \times Q} = \langle A\tau - A0, \tau - 0 \rangle_{Q^* \times Q} + \langle A0, \tau \rangle_{Q^* \times Q}
 \geq \alpha_A\|\tau\|_{Q}^2 + \langle A0, \tau \rangle_{Q^* \times Q} \geq (\alpha_A - \bar{\epsilon})\|\tau\|_{Q}^2 - \frac{1}{4\bar{\epsilon}}\|A0\|_{Q^*}^2.
\] (18)
Combining equations (16) to (18), we get
\[
\langle A\tau, \tau \rangle_{Q^* \times Q} \geq (F - \zeta, \tau)_{Q^* \times Q} + \langle A\tau + \zeta - F, \tau_0 \rangle_{Q^* \times Q}
\]
In addition, it follows from equation (13) that
\[
\langle \xi, \tau_0 - \tau \rangle_{Q^* \times Q} \leq 0.
\]
Summing up the last two inequalities we obtain
\[
\langle \xi + \eta - F, \tau - \tau_0 \rangle_{Q^* \times Q} \geq 0.
\] (19)
Since \( D(T_1) = \Sigma \neq \emptyset \), there exists \( R_2 > 0 \), such that \( D(T_1) \cap B_{R_2}(0) \neq \emptyset \). Define \( R = \max \{R_1, R_2, \|\tau_0\|_{Q} \} \). Then, \( R \geq \|\tau_0\|_{Q} \) and \( D(T_1) \cap B_R(0) \neq \emptyset \). Moreover, for all \( \tau \in \Sigma \), such that \( \|\tau\|_{Q} = R \), and for all \( \xi \in T_1 \tau \), \( \eta \in T_2 \tau \), inequality (19) holds.

Thus, we are in a position to apply Proposition 4 to conclude that \( T \) is surjective, i.e., there exists \( \sigma \in Q \) such that equation (14) holds. Hence, \( \sigma \) is a solution of Problem \( P \). We now show that \( \sigma \in \Sigma \). It follows from equation (14) that there exists \( \xi \in \partial_{C\Sigma}(\sigma) \), such that
\[
-A\sigma - \xi + f \in \partial\psi_{\Sigma}(\sigma).
\]
The last inclusion implies that \( \partial\psi^{\Sigma}(\sigma) \neq \emptyset \) and, using equation (13), it follows that \( \sigma \in \Sigma \), which concludes the existence part of the theorem.

To prove the uniqueness part, we suppose that \( \sigma_1, \sigma_2 \in \Sigma \) are two solutions of Problem \( P \). Then, there exist \( \xi_1, \xi_2 \in Q \), such that
\[
\langle A\sigma_i + \xi_i - f, \tau - \sigma_i \rangle_{Q^* \times Q} \geq 0 \quad \text{for all} \quad \tau \in \Sigma, \quad \text{with} \quad \xi_i \in \partial_{C\Sigma}(\sigma_i), \quad i = 1, 2.
\] (20)
We add equation (20) for \( i = 1, 2 \), taking \( \tau = \sigma_2 \) for \( i = 1 \) and \( \tau = \sigma_1 \) for \( i = 2 \). Therefore, we obtain
\[
\langle A\sigma_1 + \xi_1 - A\sigma_2 - \xi_2, \sigma_1 - \sigma_2 \rangle_{Q^* \times Q} \leq 0.
\]
Now, from \( H(A)(2) \) and \( H(J)(3) \), we get
\[
(\alpha_A - m_J)\|\sigma_1 - \sigma_2\|_{Q}^2 \leq 0.
\]
Using the smallness assumption, equation (15), we conclude that \( \sigma_1 = \sigma_2 \), which completes the proof. \( \square \)

**Remark 1.** Using the definition of the subdifferential \( \partial\psi_{\Sigma} \), it is easy to see that Problem \( P \) can be formulated, equivalently, as follows.

**Problem \( P' \).** Find an element \( \sigma \in \Sigma \) with the property that there exists \( \xi \in Q^* \) such that
\[
\langle A\sigma, \tau - \sigma \rangle_{Q^* \times Q} + \langle \xi, \tau - \sigma \rangle_{Q^* \times Q} \geq \langle f, \tau - \sigma \rangle_{Q^* \times Q} \quad \text{for all} \quad \tau \in \Sigma, \quad \xi \in \partial_{C\Sigma}(\sigma).
\] (21)
Note that Theorem 2 provides an existence and uniqueness result for Problem \( P' \). Nevertheless, if \( \sigma \) is a solution of this problem, the element \( \xi \) that satisfies equation (22) cannot be uniquely determined.
4. History-dependent subdifferential inclusions

We now introduce a history-dependent version of Problem \( P \). To this end, we consider a time interval \((0, T)\), with \( T > 0 \), and we allow the set \( \Sigma \), the function \( J \) and the element \( f \) to depend on the time variable. More precisely, we assume in what follows that \( H(A) \) holds and we replace the assumptions \( H(\Sigma) \), \( H(f) \) and \( H(J) \) with the following assumptions.

\[ H_{\text{hd}}(\Sigma) \text{ The set } \Sigma(t) \text{ is a convex, nonempty, closed subset of } Q, \text{ for a.e. } t \in (0, T). \]

\[ H_{\text{hd}}(J) \text{ The function } J : (0, T) \times Q \rightarrow \mathbb{R} \text{ satisfies:} \]

1. \( J(\cdot, \sigma) \) is measurable on \((0, T)\) for all \( \sigma \in Q \).
2. \( J(t, \cdot) \) is locally Lipschitz for a.e. \( t \in (0, T) \).
3. There exists a function \( c_J : (0, T) \rightarrow \mathbb{R} \), such that \( c_J(t) > 0 \) a.e. \( t \in (0, T) \) and
   \[ \| \xi \|_{Q^*} \leq c_J(t)(1 + \| \sigma \|_Q) \text{ for all } \sigma \in Q, \text{ all } \xi \in \partial_{\text{C}} J(t, \sigma), \text{ a.e. } t \in (0, T). \]
4. There exists a function \( m_J : (0, T) \rightarrow \mathbb{R} \), such that \( m_J(t) > 0 \) a.e. \( t \in (0, T) \) and
   \[ \langle \xi_1 - \xi_2, \sigma_1 - \sigma_2 \rangle_{Q^* \times Q} \geq -m_J(t)\| \sigma_1 - \sigma_2 \|_Q^2 \]
   for all \( \sigma_1, \sigma_2 \in Q, \xi_1, \xi_2 \in Q \), with \( \xi_i \in \partial_{\text{C}} J(t, \sigma_i), i = 1, 2, \text{ a.e. } t \in (0, T) \).

\[ H_{\text{hd}}(f) \ f \in L^2(0, T; Q^*). \]

We also consider the following smallness assumption.

There exists \( \delta > 0 \) s.t. \( \alpha_4 > \max \{ m_J(t), c_J(t) \} + \delta \) a.e. \( t \in (0, T) \). \hspace{1cm} (23)

Finally, let \( S \) be an operator such that

\[ H_{\text{hd}}(S) \ S : L^2(0, T; Q) \rightarrow L^2(0, T; Q^*) \text{ is a history-dependent operator.} \]

We now consider the following subdifferential inclusion.

**Problem** \( P_{\text{hd}} \). Find a function \( \sigma : (0, T) \rightarrow Q \), such that

\[ A\sigma(t) + (S\sigma)(t) + \partial_{\text{C}} J(t, \sigma(t)) + \partial \psi_{\Sigma(t)}(\sigma(t)) \ni f(t), \text{ a.e. } t \in (0, T). \hspace{1cm} (24) \]

Our main existence and uniqueness result in this section is the following.

**Theorem 3.** Assume that \( H_{\text{hd}}(\Sigma), H(A), H_{\text{hd}}(J), H_{\text{hd}}(f), H_{\text{hd}}(S) \) and equation (23) hold. Then Problem \( P_{\text{hd}} \) has a unique solution \( \sigma \in L^2(0, T; Q) \). Moreover, \( \sigma(t) \in \Sigma(t) \) a.e. \( t \in (0, T) \).

The proof of this theorem will be carried out in several steps, which we present next. In the first step, we consider a given element \( \eta \in L^2(0, T; Q^*) \) together with the following intermediate problem.

**Problem** \( P_{\text{hd}}^{\eta} \). Find a function \( \sigma_{\eta} : (0, T) \rightarrow Q \), such that

\[ A\sigma_{\eta}(t) + \eta(t) + \partial_{\text{C}} J(t, \sigma_{\eta}(t)) + \partial \psi_{\Sigma(t)}(\sigma_{\eta}(t)) \ni f(t) \text{ a.e. } t \in (0, T). \hspace{1cm} (25) \]

We have the following existence and uniqueness result.

**Lemma 1.** Assume that \( H_{\text{hd}}(\Sigma), H(A), H_{\text{hd}}(J), H_{\text{hd}}(f) \) and equation (23) hold. Then Problem \( P_{\text{hd}}^{\eta} \) has a unique solution with regularity \( \sigma_{\eta} \in L^2(0, T; \Sigma) \).
Proof. The following equalities and inequalities hold for a.e. \( t \in (0, T) \), supposed to be fixed. We note that for such \( t \) the set \( \Sigma(t) \) satisfies assumption \( H(\Sigma) \). Moreover, the functions \( J(t, \cdot) \) and \( f(t) \) satisfy assumptions \( H(J) \) and \( H(f) \), respectively, and, in addition, equation (15) follows from equation (23). Therefore, using Theorem 2 we deduce that there exists a unique element \( \sigma_{\eta}(t) \in \Sigma(t) \), which solves equation (25) at time \( t \).

We now prove that the function \((0, T) \ni t \mapsto \sigma_{\eta}(t) \in Q\) belongs to the space \( L^{2}(0, T; Q) \). To this end, let \( \eta_{i} \in L^{2}(0, T; Q^{*}), f_{i} \in L^{2}(0, T; Q^{*}) \) be given and, for simplicity, denote \( \sigma_{\eta_{i}} = \sigma_{i}, i = 1, 2 \). Let \( t \in (0, T) \) be fixed. It follows from equation (25) that there exists \( \xi_{i}(t) \in \partial \psi_{\Sigma(t)}(\sigma_{i}(t)) \) such that

\[
-A\sigma_{i}(t) - \eta_{i}(t) - \xi_{i}(t) + f_{i}(t) \in \partial \psi_{\Sigma(t)}(\sigma_{i}(t))
\]

and, therefore,

\[
\left\langle A\sigma_{i}(t) + \eta_{i}(t) + \xi_{i}(t) - f_{i}(t), \tau - \sigma_{i}(t) \right\rangle_{Q^{*} \times Q} \geq 0 \quad \text{for all } \tau \in \Sigma(t), \ i = 1, 2. \tag{26}
\]

We add equation (26) for \( i = 1, 2 \), taking \( \tau = \sigma_{2}(t) \) for \( i = 1 \) and \( \tau = \sigma_{1}(t) \) for \( i = 2 \). As a result, we obtain

\[
\left\langle A\sigma_{1}(t) - A\sigma_{2}(t) + \xi_{1}(t) - \xi_{2}(t), \sigma_{1}(t) - \sigma_{2}(t) \right\rangle_{Q^{*} \times Q} + \left\langle \eta_{1}(t) + f_{2}(t) - \eta_{2}(t) - f_{1}(t), \sigma_{1}(t) - \sigma_{2}(t) \right\rangle_{Q^{*} \times Q} \leq 0.
\]

Now, using assumptions \( H(A)(2) \) and \( H_{\text{hd}}(J)(4) \), we find that

\[
(\alpha - m_{J}(t))\Vert \sigma_{1}(t) - \sigma_{2}(t) \Vert_{Q} \leq \Vert \eta_{1}(t) - \eta_{2}(t) \Vert_{Q^{*}} + \Vert f_{1}(t) - f_{2}(t) \Vert_{Q^{*}}
\]

and, therefore, equation (23) implies that

\[
\Vert \sigma_{1}(t) - \sigma_{2}(t) \Vert_{Q} \leq \frac{c}{\delta} \left( \Vert \eta_{1}(t) - \eta_{2}(t) \Vert_{Q^{*}} + \Vert f_{1}(t) - f_{2}(t) \Vert_{Q^{*}} \right). \tag{27}
\]

Inequality (27) provides the continuity of the function \( Q^{*} \times Q^{*} \ni (\eta, f) \mapsto \sigma \in Q \), where \( \sigma = \sigma_{\eta} \) is the unique solution of equation (25) corresponding to \( (\eta, f) \). Since, clearly, the functions \( \eta \) and \( f \) are measurable, we deduce that the function \((0, T) \ni t \mapsto \sigma_{\eta}(t) \in Q\) is measurable as a composition of continuous and measurable functions. In addition, equation (27) shows that \( \sigma_{\eta} \) satisfies an inequality of the form

\[
\Vert \sigma_{\eta}(t) \Vert_{Q} \leq c \left( 1 + \Vert \eta(t) \Vert_{Q^{*}} \right), \tag{28}
\]

where \( c \) denotes a positive constant that does not depend on \( t \). Now, since \( \eta \in L^{2}(0, T; Q^{*}) \) we deduce from equation (28) that \( \sigma_{\eta} \in L^{2}(0, T; Q) \). Conversely, recall that \( \sigma_{\eta}(t) \in \Sigma \) a.e. \( t \in (0, T) \). This concludes the existence part of the lemma. The uniqueness follows from the uniqueness of the solution of equation (25) for a.e. \( t \in (0, T) \), guaranteed by Theorem 2. \( \square \)

We now use Lemma 1 to define the operator \( \Lambda : L^{2}(0, T; Q^{*}) \rightarrow L^{2}(0, T; Q^{*}) \) by equality

\[
\Lambda \eta = S \sigma_{\eta} \quad \text{for all } \eta \in L^{2}(0, T; Q^{*}). \tag{29}
\]

We have the following fixed point result.

**Lemma 2.** Assume that \( H_{\text{hd}}(\Sigma), H(A), H_{\text{hd}}(S), H_{\text{hd}}(J), H_{\text{hd}}(f) \) and equation (23) hold. Then the operator \( \Lambda \) has a unique fixed point \( \eta^{*} \in L^{2}(0, T; Q^{*}) \).

**Proof.** Let \( \eta_{1}, \eta_{2} \) be two elements in the space \( L^{2}(0, T; Q^{*}) \). Then using inequality (27) we deduce that

\[
\Vert \sigma_{\eta_{1}}(t) - \sigma_{\eta_{2}}(t) \Vert_{Q} \leq \frac{1}{\delta} \Vert \eta_{1}(t) - \eta_{2}(t) \Vert_{Q^{*}} \quad \text{a.e. } t \in (0, T). \tag{30}
\]

We now combine inequality (30) with assumption \( H_{\text{hd}}(S) \) to see that \( \Lambda \) is a history-dependent operator. Then, we use Theorem 1 to conclude the proof. \( \square \)

We are now in a position to present the proof of Theorem 3.

**Proof.** We use Lemma 2. The solution of the auxiliary Problem \( P^{*}_{\text{hd}} \) represents the unique solution to Problem \( P_{\text{hd}} \). \( \square \)
Remark 2. Using the definition of the subdifferential $\partial \psi_\Sigma$, it is easy to see that Problem $P_{hd}$ can be formulated, equivalently, as follows.

Problem $P'_{hd}$. Find a function $\sigma : (0, T) \to Q$ with the property that $\sigma(t) \in \Sigma(t)$ a.e. $t \in (0, T)$ and there exists $\xi : (0, T) \to Q^*$, such that

$$
\langle A\sigma(t), \tau - \sigma(t) \rangle_{Q^* \times Q} + \langle (S\sigma)(t), \tau - \sigma(t) \rangle_{Q^* \times Q} + \langle \xi(t), \tau - \sigma(t) \rangle_{Q^* \times Q} \geq \langle f(t), \tau - \sigma(t) \rangle_{Q^* \times Q} \quad \text{for all } \tau \in \Sigma(t),
$$
a.e. $t \in (0, T)$.

As in the previous section, we note that Theorem 3 provides an existence and uniqueness result for Problem $P'_{hd}$. Nevertheless, if $\sigma$ is a solution of this problem, we have no information on the uniqueness and the regularity of the function $\xi$ that satisfies equation (31).

5. A static model of contact

Theorem 2 is useful in the study of various models of contact with deformable bodies. To provide an example, in this section we consider a frictionless contact problem for elastic bodies. To provide an example, in this section we consider a frictionless contact problem for elastic bodies. Let

$$
\Omega \subset \mathbb{R}^d \quad (d = 1, 2, 3)
$$
be the reference configuration of an elastic body, $\Gamma$ the boundary of $\Omega$ and $\Gamma_1, \Gamma_2, \Gamma_3$ a partition of $\Gamma$ such that $\text{meas}(\Gamma_i) > 0$. Here, and in the following, $\text{meas}(\Gamma_i)$ denotes the $d-1$ dimensional Lebesgue measure of the set $\Gamma_i$. We denote by $S^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order $d = 1, 2, 3$. The inner product and norm on $\mathbb{R}^d$ and $S^d$ are defined by

$$
\begin{align*}
\langle u, v \rangle &= \langle u, v \rangle_{\mathbb{R}^d} = u \cdot v, \\
\|\tau\| &:= \|\tau\|_{S^d} = \tau \cdot \tau,
\end{align*}
$$

for all $u, v \in \mathbb{R}^d$, $\tau \in S^d$.

Here, and in the following, the indices $i$ and $j$ run between 1 and $d$ and, unless stated otherwise, the summation convention over repeated indices is used.

We use the notation $x = (x_i)$ for a typical point in $\Omega \cup \Gamma$. For a vector field $v : \Omega \to \mathbb{R}^d$, we use the notation $v = (v_i)$, and a tensor field $\sigma : \Omega \to S^d$ will be denoted $\sigma = (\sigma_{ij})$. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable $x$, e.g., $u_{ij} = \partial u_i / \partial x_j$.

We use $\varepsilon$ and $\text{Div}$ for the deformation and divergence operators, respectively, i.e.,

$$
\varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2} (v_{ij} + v_{ji}), \quad \text{Div} \sigma = (\sigma_{ij}).
$$

Let $v = (v_i)$ be the outward unit normal at $\Gamma$. Given a vector field $v : \Gamma \to \mathbb{R}^d$, we define its normal and tangential components by equalities $v_n = v \cdot v$ and $v_t = v - v_n v$, respectively. Similarly, for a tensor field $\sigma : \Gamma \to S^d$, we define its normal and tangential components by $\sigma_n = (\sigma v) \cdot v$ and $\sigma_t = \sigma v - \sigma_n v$, respectively.

We use the standard notation for Sobolev and Lebesgue spaces associated with $\Omega$ and $\Gamma$. In addition, we consider the space

$$
Q = \left\{ \sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\},
$$

which is a real Hilbert spaces endowed with the canonical inner product given by

$$
\langle \sigma, \tau \rangle_Q = \int_{\Omega} \sigma_{ij} \tau_{ij} d\mathbf{x}.
$$

The associated norm is denoted by $\| \cdot \|_Q$. For an element $v \in H^1(\Omega)^d$, we still write $v$ for the trace of $v$. Recall also that for a regular stress function $\sigma$ the following Green’s formula holds:

$$
\int_{\Omega} \sigma \cdot \varepsilon(v) d\mathbf{x} + \int_{\Omega} \text{Div} \sigma \cdot v d\mathbf{x} = \int_{\Gamma} \sigma \cdot v d\mathbf{a} \quad \text{for all } v \in H^1(\Omega)^d.
$$

We consider the space

$$
V = \left\{ v \in H^1(\Omega)^d : v = 0 \text{ on } \Gamma_1 \right\}.
$$
It is well known that $V$ is a real Hilbert space endowed with the inner product

$$(u, v)_V = (e(u), e(v))_Q = \int_\Omega e(u) \cdot e(v) \, dx$$

and the associated norm $\| \cdot \|_V$. Completeness of the space $(V, \| \cdot \|_V)$ follows from the assumption $\text{meas}(\Gamma_1) > 0$, which allows the use of Korn’s inequality. We also recall that there exists $c_0 > 0$, which depends on $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq c_0 \|v\|_V \quad \text{for all } v \in V. \quad (33)$$

Inequality (33) represents a consequence of the Sobolev trace theorem.

With these preliminaries, the classical formulation of the unilateral frictionless contact problem that we study in this section is the following.

**Problem $P$.** Find a displacement field $u : \Omega \to \mathbb{R}^d$ and a stress field $\sigma : \Omega \to \mathbb{S}^d$, such that

$$e(u) \in \mathcal{A}\sigma + \partial_{CJ}(\sigma) \quad \text{in } \Omega, \quad (34)$$

$$\text{Div } \sigma + f_0 = 0 \quad \text{in } \Omega, \quad (35)$$

$$u = 0 \quad \text{on } \Gamma_1, \quad (36)$$

$$\sigma v = f_2 \quad \text{on } \Gamma_2, \quad (37)$$

$$u_v \leq g, \quad \sigma_v \leq 0, \quad \sigma_v(u_v - g) = 0 \quad \text{on } \Gamma_3, \quad (38)$$

$$\sigma_\tau = 0 \quad \text{on } \Gamma_3. \quad (39)$$

We recall that equation (34) represents the constitutive law in which $\mathcal{A}$ is the compliance operator and $j$ is a nonlinear potential. Examples of such laws will be provided in Section 7. Here we restrict ourselves to remark that equation (34) shows that the strain tensor has an additive decomposition into a single-valued part, $\mathcal{A}\sigma$, and a multivalued part, $\partial_{CJ}(\sigma)$. Equation (35) is the equilibrium equation in which $f_0$ denotes the density of body forces. We use it here, since we assume that the mechanical process is static. Conditions (36) and (37) are the displacement–traction boundary conditions, in which $f_2$ represents the density of traction on $\Gamma_2$. Condition (38) represents the Signorini contact condition in a form with a gap $g$. Finally, condition (39) represents the frictionless condition, which states that the tangential component of the stress, denoted $\sigma_\tau$, vanishes on the contact surface $\Gamma_3$.

The assumptions on the data of Problem $P$ are the following.

1. $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$.
2. There exists $L_\mathcal{A} > 0$ such that
   $$\|\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)\| \leq L_\mathcal{A} \|\varepsilon_1 - \varepsilon_2\|$$
   for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
3. There exists $\alpha_\mathcal{A} > 0$ such that
   $$\langle \mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2), (\varepsilon_1 - \varepsilon_2) \rangle \geq \alpha_\mathcal{A} \|\varepsilon_1 - \varepsilon_2\|^2$$
   for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
4. The mapping $x \mapsto \mathcal{A}(x, \varepsilon)$ is measurable on $\Omega$.
5. The mapping $x \mapsto \mathcal{A}(x, 0)$ belongs to $Q$.

1. $j : \mathbb{S}^d \to \mathbb{R}$.
2. $j$ is locally Lipschitz.
3. There exists $c_j > 0$ such that
   $$\|\xi\| \leq c_j(1 + \|\sigma\|) \quad \text{for all } \sigma \in \mathbb{S}^d, \, \xi \in \partial_{CJ}(\sigma).$$
4. There exists $m_j > 0$ such that
   $$\langle \xi_1 - \xi_2, (\sigma_1 - \sigma_2) \rangle \geq -m_j \|\sigma_1 - \sigma_2\|^2$$
   for all $\sigma_1, \sigma_2 \in \mathbb{S}^d$, $\xi_1, \xi_2 \in \mathbb{S}^d$, with $\xi_i \in \partial_{CJ}(\sigma_i)$, $i = 1, 2$.
5. $f_0 \in L^2(\Omega)^d$, $f_2 \in L^2(\Gamma_2)^d$. \quad (42)
\[ \alpha_A = \max \{ m_j, c_j, c_j^2 \sqrt{\text{meas}(\Omega)} \}, \]  

(43)

where \( \text{meas}(\Omega) \) represents the \( d \)-dimensional measure of \( \Omega \). Finally, we assume that

\[
\text{there exists an element } \tilde{g} \in V \text{ such that } \tilde{g}_v = g \text{ on } \Gamma_3,
\]

(44)

and we refer the reader to Kalita et al. [23] and Sofonea et al. [24] for examples and details of this condition. We use Riesz’s representation theorem to define the element \( \tilde{f} \in V \) by equality

\[
(\tilde{f}, v)_V = \int_{\Omega} f_0 \cdot v \, dx + \int_{\Gamma_2} f_2 \cdot v \, da \quad \text{for all } v \in V.
\]

(45)

Then, we introduce the set of admissible displacements \( U \) and the set of admissible stress fields \( \Sigma \) defined by

\[
U = \{ v \in V : v_v \leq g \text{ a.e. on } \Gamma_3 \},
\]

(46)

\[
\Sigma = \{ \tau \in Q : (\tau, \varepsilon(v) - \varepsilon(\tilde{g}))_Q \geq (\tilde{f}, v - \tilde{g})_V \text{ for all } v \in U \}.
\]

(47)

We now turn to the variational formulation of the contact Problem \( P \) and, to this end, we assume that \( u, \sigma \) are regular functions that satisfy equations (34) to (39). Then, multiplying equation (35) by \((v - u)\), where \( v \in U \) and using Green’s formula, we have

\[
\int_{\Omega} \sigma \cdot (\varepsilon(v) - \varepsilon(u)) \, dx + \int_{\Omega} \text{Div} \sigma \cdot (v - u) \, dx = \int_{\Gamma} \sigma v \cdot (v - u) \, da.
\]

We now split the surface integral on \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \); then we use the equilibrium equation (35), the boundary conditions (36), (37) and the definition (45) to deduce that

\[
\int_{\Omega} \sigma \cdot (\varepsilon(v) - \varepsilon(u)) \, dx = (\tilde{f}, v - u)_V + \int_{\Gamma_3} \sigma v \cdot (v - u) \, da.
\]

(48)

Next, using equations (38) and (39), it is easy to see that

\[
\sigma v \cdot (v - u) = \sigma_v (v_v - u_v) \geq 0 \text{ a.e. on } \Gamma_3
\]

and, therefore,

\[
\int_{\Gamma_3} \sigma v \cdot (v - u) \, da \geq 0.
\]

(49)

We now combine equations (48) and (49) to see that

\[
\int_{\Omega} \sigma \cdot (\varepsilon(v) - \varepsilon(u)) \, dx \geq (\tilde{f}, v - u)_V.
\]

(50)

Note that assumption (44) implies that \( 2u - \tilde{g} \in U \) and \( \tilde{g} \in U \). This allows us to test equation (50) with \( v = 2u - \tilde{g} \) and \( v = \tilde{g} \), to deduce that

\[
\int_{\Omega} \sigma \cdot (\varepsilon(u) - \varepsilon(\tilde{g})) \, dx = (\tilde{f}, u - \tilde{g})_V.
\]

(51)

Next, we add inequality (50) and equation (51) and use definition (47) to deduce that

\[
\sigma \in \Sigma.
\]

(52)

Consider now an arbitrary element \( \tau \in \Sigma \). Then using equations (47) and (51) it is easy to see that

\[
\int_{\Omega} (\tau - \sigma) \cdot (\varepsilon(u) - \varepsilon(\tilde{g})) \, dx \geq 0.
\]

(53)
Next, the constitutive law, equation (34), shows that there exists a function $\eta : \Omega \to S^d$ such that

$$
\varepsilon(u) = A\sigma + \eta, \quad \eta \in \partial Cl(\sigma) \quad \text{a.e. in } \Omega.
$$

(54)

We now gather relations (52) to (54) to deduce the following variational formulation of Problem $\mathcal{P}$, in terms of stress.

**Problem $\mathcal{P}^V$.** Find a stress field $\sigma \in \Sigma$ with the property that there exists $\eta : \Omega \to S^d$ such that

$$
\int_\Omega A\sigma \cdot (\tau - \sigma) \, dx + \int_\Omega \eta \cdot (\tau - \sigma) \, dx \geq \int_\Omega \varepsilon(\tilde{g}) \cdot (\tau - \sigma) \, dx \quad \text{for all } \tau \in \Sigma,
$$

(55)

$$
\eta \in \partial Cl(\sigma) \quad \text{a.e. in } \Omega.
$$

(56)

Our main result in this section is the following.

**Theorem 4.** Assume that equations (40) to (44) hold. Then, Problem $\mathcal{P}^V$ has a unique solution, which satisfies $\sigma \in \Sigma$.

To provide the proof of Theorem 4, we consider the dual of the space $Q$, denoted $Q^*$ and let $\iota : Q \to Q^*$ be the isometry provided by the Riesz representation theorem defined by

$$
\langle \iota \tau, \sigma \rangle_{Q^* \times Q} = \int_\Omega \tau \cdot \sigma \, dx \quad \text{for all } \tau \in Q.
$$

We also denote by $\iota^* : Q^* \to Q$, the inverse of $\iota$ and define the operators $A : Q \to Q^*$, $\varepsilon^* : V \to Q^*$ and the function $J : Q \to \mathbb{R}$ by

$$
\langle A\sigma, \tau \rangle_{Q^* \times Q} = \int_\Omega A\sigma \cdot \tau \, dx \quad \text{for all } \sigma, \tau \in Q,
$$

(57)

$$
\langle \varepsilon^* v, \tau \rangle_{Q^* \times Q} = \int_\Omega \varepsilon(v) \cdot \tau \, dx \quad \text{for all } \tau \in Q, \ v \in V,
$$

(58)

$$
J(\tau) = \int_\Omega j(\tau) \, dx \quad \text{for all } \tau \in Q.
$$

(59)

Then, we have the following result.

**Lemma 3.** Assume that equation (41) holds. Then, the function $J$ is well defined and satisfies assumption $H(J)$ on the space $Q = Q$ with the constants

$$
c_J = \max \left\{ c_J, c_J \sqrt{\text{meas} (\Omega)} \right\}, \quad m_J = m_J.
$$

(60)

In addition, for all $\tau \in Q$, the following implication holds:

$$
\xi \in \partial Cl(\tau) \implies \iota^* \xi \in \partial Cl(j(\tau)) \quad \text{a.e. in } \Omega.
$$

(61)

**Proof.** It follows from Theorem 3.47 of Migórski et al. [8] that the function $J$ satisfies conditions 1 and 2 of hypothesis $H(J)$. Moreover, the same theorem guarantees that equation (61) holds. Note that the value of the constant $c_J$ that appears in equation (60) follows from a direct calculations, based on equation (61). Moreover, the validity of condition 3 of hypothesis $H(J)$ with the constant $m_J = m_J$ follows straightforwardly from equations (61) and (41)(d), which concludes the proof.

We are now in a position to provide the proof of Theorem 4.

**Proof.** We start by considering the subdifferential inclusion

$$
A\sigma + \partial Cl(\sigma) + \partial \psi_S(\sigma) \ni \varepsilon^*(\tilde{g}),
$$

(62)
for which we apply Theorem 2 with \( Q = \mathcal{Q} \) and \( f = \mathbf{e}^\star(\mathbf{g}) \). To this end, we note that the set \( \Sigma \) given by equation (47) is nonempty since, for instance, it contains the element \( \epsilon(\mathbf{f}) \). Conversely, it is easy to check that \( \Sigma \) is a closed convex subset of \( \mathcal{Q} \) and, therefore, it satisfies assumption \( H(\Sigma) \). The assumption of equation (40) on the elasticity operator \( A \) implies that operator \( A \) defined by equation (57) is Lipschitz continuous and strongly monotone, i.e., it satisfies assumption \( H(A) \) with \( L_A = L_A \) and \( \alpha_A = \alpha_A \). We also note that, using Lemma 3, it follows that condition \( H(J) \) holds, too. Finally, the smallness assumption, equation (43), combined with equation (60), implies equation (15). Therefore, since the element \( f = \mathbf{e}^\star(\mathbf{g}) \) obviously satisfies assumption \( H(f) \), it follows from Theorem 2 that there exists a unique solution \( \mathbf{\sigma} \in \Sigma \) to equation (62). Using Remark 1, we obtain the existence of a unique element \( \mathbf{\eta} \in \mathcal{Q}^\circ \) such that

\[
\langle A\mathbf{\sigma}, \mathbf{\tau} - \mathbf{\sigma} \rangle_{\mathcal{Q}^\circ \times \mathcal{Q}} + \langle \mathbf{\eta}, \mathbf{\tau} - \mathbf{\sigma} \rangle_{\mathcal{Q}^\circ \times \mathcal{Q}} \geq \langle \mathbf{e}^\star(\mathbf{g}), \mathbf{\tau} - \mathbf{\sigma} \rangle_{\mathcal{Q}^\circ \times \mathcal{Q}} \quad \text{for all } \mathbf{\tau} \in \Sigma, \quad (63)
\]

\[
\mathbf{\eta} \in \partial_{CJ}(\mathbf{\sigma}). \quad (64)
\]

Let \( \mathbf{\eta} = \epsilon^\star\mathbf{\xi} \). Then, it follows that

\[
\langle \mathbf{\xi}, \mathbf{\tau} \rangle_{\mathcal{Q}^\circ \times \mathcal{Q}} = \int_\Omega \mathbf{\eta} \cdot \mathbf{\tau} \, dx \quad \forall \mathbf{\tau} \in \mathcal{Q}. \quad (65)
\]

We now use equations (57), (58) and (65) in equation (63) to see that inequality (55) holds. Conversely, (64) and (61) imply that equation (56) holds, too. We conclude from here that \( \mathbf{\sigma} \in \Sigma \) is a solution to Problem \( \mathcal{P}_{\mathcal{P}_1} \), which concludes the existence part of the proof.

The uniqueness part could be proved directly, by using arguments similar to those used in the proof of Theorem 2. For this reason, we skip the details. We restrict ourselves to note that it follows by using equations (55), (56), assumptions (40)(c), (41)(d) and the smallness assumption, equation (43). \( \square \)

### 6. A history-dependent model of contact

We now present a history-dependent version of Problem \( \mathcal{P} \). The classical formulation of the problem is the following.

**Problem** \( \mathcal{P}_{\text{hd}} \). Find a displacement field \( \mathbf{u} : \Omega \times (0, T) \to \mathbb{R}^d \) and a stress field \( \mathbf{\sigma} : \Omega \times (0, T) \to \mathbb{S}^d \), such that

\[
\epsilon(\mathbf{u}(t)) \in A\mathbf{\sigma}(t) + \partial_{CJ}(t, \mathbf{\sigma}(t)) + \int_0^t B(t - s)\mathbf{\sigma}(s) \, ds \quad \text{in } \Omega, \quad (66)
\]

\[
\text{Div } \mathbf{\sigma}(t) + f_0(t) = 0 \quad \text{in } \Omega, \quad (67)
\]

\[
\mathbf{u}(t) = 0 \quad \text{on } \Gamma_1, \quad (68)
\]

\[
\mathbf{\sigma}(t)\nu = f_2(t) \quad \text{on } \Gamma_2, \quad (69)
\]

\[
u_\nu(t) \leq g, \quad \sigma_\nu(t) \leq 0, \quad \sigma_\nu(t)(u_\nu(t) - g) = 0 \quad \text{on } \Gamma_3, \quad (70)
\]

\[
\mathbf{\sigma}_\nu(t) = 0 \quad \text{on } \Gamma_3, \quad (71)
\]

for all \( t \in (0, T) \).

These equations and boundary conditions are similar to those in Problem \( \mathcal{P} \). The difference arises in the fact that now the constitutive equation (34) is replaced with the history-dependent constitutive equation (66), in which \( B \) represents the relaxation tensor. Note that now the function \( j \) is assumed to depend on time, which makes the problem more general from a mathematical point of view. From a physical point of view, this dependence could model the dependence of \( j \) with respect to the temperature, which is considered as given. We assume that

\[
B \in L^2(0, T; \mathcal{Q}_\infty), \quad (72)
\]

where \( \mathcal{Q}_\infty \) represents the space of fourth-order tensor fields, given by

\[
\mathcal{Q}_\infty = \{ \mathcal{E} = (\mathcal{E}_{ijkl}) : \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \leq i, j, k, l \leq d \}. \quad (73)
\]
We recall that $Q_\infty$ is a real Banach space with the norm

$$\|E\|_{Q_\infty} = \max_{1 \leq i, l, k, l \leq d} \|E_{ijkl}\|_{L^\infty(\Omega)}.$$ 

Moreover, a simple calculation shows that

$$\|E\|_Q \leq d \|E\|_{Q_\infty} \|\tau\|_Q \quad \text{for all } E \in Q_\infty, \tau \in Q. \quad (73)$$

In the study of Problem $P_{nh}$ we keep assumptions (40) and (44) and use the notation of equation (46) for the set of admissible displacement fields. Nevertheless, we replace assumptions (41) to (43) with the following.

\[
\begin{aligned}
(a) & : (0, T) \times \mathbb{S}^d \rightarrow \mathbb{R}, \\
(b) & j(\cdot, \sigma) \text{ is measurable, for all } \sigma \in Q, \\
(c) & j(t, \cdot) \text{ is locally Lipschitz for a.e. } t \in (0, T). \\
(d) & \text{There exists a function } c_j : (0, T) \rightarrow \mathbb{R} \text{ such that } c_j(t) > 0 \\
& \text{for a.e. } t \in (0, T) \text{ and } \|\xi\| \leq c_j(t)(1 + \|\sigma\|) \text{ for all } \sigma \in \mathbb{S}^d, \\
& \text{a.e. } t \in (0, T). \quad (74) \\
(e) & \text{There exists a function } m_j : (0, T) \rightarrow \mathbb{R} \text{ such that } m_j(t) > 0 \\
& \text{for a.e. } t \in (0, T) \text{ and } (\xi_1 - \xi_2) \cdot (\sigma_1 - \sigma_2) \geq -m_j(t)\|\sigma_1 - \sigma_2\|^2 \\
& \text{for all } \sigma_1, \sigma_2, \xi_1, \xi_2 \in \mathbb{S}^d \text{ with } \xi_i \in \partial_{\mathcal{C}j}(t, \sigma_i) \text{ for a.e. } t \in (0, T), i = 1, 2. \\
\end{aligned}
\]

Next, we define the function $\tilde{f} : (0, T) \rightarrow V$ by

$$\tilde{f}(t, v) = \int_{\Omega} f_0(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot v \, da \quad \text{for all } v \in V, \text{ a.e. } t \in (0, T) \quad (77)$$

and, for a.e. $t \in (0, T)$, we define the set

$$\Sigma(t) = \{ \tau \in Q : (\tau, \varepsilon(v) - \varepsilon(\tilde{g}))_Q \geq (\tilde{f}(t), v - \tilde{g})_V \text{ for all } v \in U \}. \quad (78)$$

Then, the variational formulation of the contact Problem $P_{nh}$ is obtained by arguments similar to those used in the previous section and is as follows.

**Problem $P_{nh}^V$.** Find a stress field $\sigma : (0, T) \rightarrow Q$ with the property that $\sigma(t) \in \Sigma(t)$ a.e. $t \in (0, T)$ and there exists $\eta : (0, T) \rightarrow Q$, such that

$$\int_{\Omega} A\sigma(t) : (\tau - \sigma(t)) \, dx + \int_{\Omega} \left( \int_0^t \mathcal{B}(t - s)\sigma(s) \, ds \right) : (\tau - \sigma(t)) \, dx$$

$$+ \int_{\Omega} \eta(t) : (\tau - \sigma(t)) \, dx \geq \int_{\Omega} \varepsilon(\tilde{g}) : (\tau - \sigma) \, dx \quad \forall \tau \in \Sigma(t),$$

$$\eta(t) \in \partial_{\mathcal{C}j}(t, \sigma(t)) \quad \text{a.e. in } \Omega, \quad (79)$$

for a.e. $t \in (0, T)$.

Our main result in this section is the following.

**Theorem 5.** Assume that equations (40), (44), (72), (74), (75) and (76) hold. Then, Problem $P_{nh}^V$ has a unique solution, which satisfies $\sigma \in L^2(0, T; Q)$. 


To provide the proof of Theorem 5, we consider the operator \( S : L^2(0, T; \mathcal{Q}) \to L^2(0, T; \mathcal{Q}') \) and the functional \( J : (0, T) \times \mathcal{Q} \to \mathbb{R} \), defined by

\[
\langle (S\sigma)(t), \tau \rangle_{\mathcal{Q}' \times \mathcal{Q}} = \int_{\Omega} \left( \int_{0}^{t} B(t-s)\sigma(s) \, dx \right) \cdot \tau \, dx \quad \text{for all } \tau, \sigma \in L^2(0, T; \mathcal{Q}), \quad \text{a.e. } t \in (0, T).
\]

(81)

\[
J(t, \tau) = \int_{\Omega} j(t, \tau) \, dx \quad \text{for all } \tau \in \mathcal{Q}, \quad \text{a.e. } t \in (0, T).
\]

(82)

The next lemma deals with the properties of the function \( J \).

**Lemma 4.** Assume that equation (74) holds. Then, the functional \( J \) is well defined and satisfies assumption \( H_{\text{sup}(J)} \) on the space \( \mathcal{Q} = \mathcal{Q} \), with the functions

\[
c_j(t) = \max \left\{ c_j(t), c_j(t)\sqrt{\text{meas}(\Omega)} \right\}, \quad m_j(t) = m_j(t).
\]

(83)

In addition, the following implication holds:

\[
\xi \in \partial c_j J(t, \tau) \implies i^* \xi \in \partial c_j J(t, \tau) \quad \text{a.e. in } \Omega, \quad \text{for a.e. } t \in (0, T).
\]

(84)

**Proof.** We use Theorem 3.47 of Migórski et al. [8] to see that the function \( J \) satisfies conditions 1 to 3 of hypothesis \( H_{\text{sup}(J)} \) and, moreover, equation (84) holds. The value of functions \( c_j \), that appears in equation (83) follows from a direct calculation based on equation (84). In addition, the validity of condition 4 of hypothesis \( H_{\text{sup}(J)} \), with \( m_j(t) = m_j(t) \) follows straightforwardly from equations (84) and (74)(e).

We now pass to the proof of Theorem 5.

**Proof.** We consider the subdifferential inclusion

\[
A\sigma(t) + (S\sigma)(t) + \partial c_j J(t, \sigma(t)) + \partial \psi_{\Sigma(t)}(\sigma(t)) \ni \epsilon^*(\tilde{g}) \quad \text{a.e. } t \in (0, T),
\]

(85)

for which we apply Theorem 3 with \( \mathcal{Q} = \mathcal{Q} \) and \( f = \epsilon^*(\tilde{g}) \). To this end, we note that for a.e. \( t \in (0, T) \) the set \( \Sigma(t) \) given by equation (78) is a nonempty closed convex subset of \( \mathcal{Q} \) and, therefore, satisfies assumption \( H_{\text{sup}}(\Sigma) \). The assumption (40) on the elasticity operator \( A \) implies that operator \( A \) is Lipschitz continuous and strongly monotone and, therefore, satisfies assumption \( H(A) \) with \( A_\ell = L_A \) and \( \alpha_\ell = \alpha_A \). We also note that Lemma 4 and condition (74) on the function \( j \) shows that the assumption \( H_{\text{sup}(J)} \) holds. Next, the smallness assumption of equation (76) combined with equation (83) implies equation (23). We now use inequality (73) to see that the operator \( S \) satisfies condition (11). We conclude from here that \( S \) is a history-dependent operator, i.e., it satisfies condition \( H_{\text{sup}}(S) \). Therefore, since the element \( f = \epsilon^*(\tilde{g}) \) obviously satisfies the assumption \( H_{\text{sup}}(f) \), it follows now from Theorem 3 that there exists a unique function \( \sigma \in L^2(0, T; \mathcal{Q}) \), such that \( \sigma(t) \in \Sigma(t) \) a.e. \( t \in (0, T) \). Moreover, equation (85) holds, too. Using Remark 2, we obtain the existence of a unique function \( \sigma \in L^2(0, T; \mathcal{Q}) \) with the property that \( \sigma(t) \in \Sigma(t) \) a.e. \( t \in (0, T) \) and, moreover, there exists \( \xi : (0, T) \to \mathcal{Q}^* \) such that

\[
\langle A\sigma(t), \tau - \sigma(t) \rangle_{\mathcal{Q}' \times \mathcal{Q}} + \langle (S\sigma)(t), \tau - \sigma(t) \rangle_{\mathcal{Q}' \times \mathcal{Q}} + \langle \xi(t), \tau - \sigma(t) \rangle_{\mathcal{Q}' \times \mathcal{Q}} \geq \langle \epsilon^*(\tilde{g}), \tau - \sigma(t) \rangle_{\mathcal{Q}' \times \mathcal{Q}} \quad \text{for all } \tau \in \Sigma,
\]

(86)

\[
\xi(t) \in \partial c_j J(t, \sigma(t)) \quad \text{a.e. } t \in (0, T). \tag{87}
\]

(87)

a.e. \( t \in (0, T) \). Let \( \eta(t) = i^*\xi(t) \). Then, using arguments similar to those used in the proof of Theorem 4, it is easy to see that \( \sigma \) is a solution to Problem \( \mathcal{P}_{\text{sup}}^0 \), which concludes the existence part of the proof.

The uniqueness part can be proved directly, by using equations (79) and (80). It is based on the history-dependence of the operator \( (S) \), assumptions (40)(c), (74)(e) and the smallness assumption of equation (76).
7. One-dimensional examples

In this section we present some examples of potential functions \( j \) for which our results apply. For simplicity, we restrict ourselves to the one-dimensional time-independent case, i.e., we assume in what follows that \( d = 1 \) and \( j \) does not depend explicitly on time. Then, assumptions (41) and (74) are identical and can be formulated as follows.

\[
\begin{align*}
(a) & \quad j: \mathbb{R} \to \mathbb{R}, \\
(b) & \quad j \text{ is locally Lipschitz}, \\
(c) & \quad \text{There exists } \alpha > 0 \text{ such that } |\xi| \leq \alpha(1 + |\sigma|) \text{ for all } \sigma \in \mathbb{R}, \text{ all } \xi \in \partial_C j(\sigma). \\
(d) & \quad \text{There exists } m_j > 0 \text{ such that } \\
& \quad (\xi_1 - \xi_2)(\sigma_1 - \sigma_2) \geq -m_j|\sigma_1 - \sigma_2|^2 \\
& \quad \text{for all } \sigma_1, \sigma_2 \in \mathbb{R}, \xi_1, \xi_2 \in \mathbb{R}, \text{ with } \xi_i \in \partial_C j(\sigma_i), i = 1, 2.
\end{align*}
\]

(88)

**Example 1.** Let \( \alpha > 0, \sigma > 0 \) and let \( j: \mathbb{R} \to \mathbb{R} \) be the function defined by

\[
j(\sigma) = \begin{cases} 
\frac{1}{2} \alpha \sigma_0^2 - \alpha \sigma_0(\sigma + \sigma_0) & \text{if } \sigma < -\sigma_0, \\
\frac{1}{2} \alpha \sigma^2 & \text{if } |\sigma| \leq \sigma_0, \\
\frac{1}{2} \alpha \sigma_0^2 + \alpha \sigma_0(\sigma - \sigma_0) & \text{if } \sigma > \sigma_0.
\end{cases}
\]

Then, it is easy to see that \( j \) is a \( C^1 \) function and, therefore, condition (88)(b) is satisfied. Moreover,

\[
\partial_C j(\sigma) = \begin{cases} 
-\alpha \sigma_0 & \text{if } \sigma < -\sigma_0, \\
\alpha \sigma & \text{if } |\sigma| \leq \sigma_0, \\
\alpha \sigma_0 & \text{if } \sigma > \sigma_0.
\end{cases}
\]

This imply that \( |\partial_C j(\sigma)| \leq \alpha \sigma_0 \) for all \( \sigma \in \mathbb{R} \) and, hence, condition (88)(c) holds with \( c_j = \alpha \sigma_0 \). In addition, since \( \partial_C j \) is a monotone function, we deduce that condition (88)(d) holds with any \( m_j > 0 \).

**Example 2.** Let \( \alpha \in [0, 1) \) and let \( j: \mathbb{R} \to \mathbb{R} \) be the function defined by

\[
j(\sigma) = (\alpha - 1)e^{-|\sigma|} + \alpha|\sigma| \quad \forall \sigma \in \mathbb{R}.
\]

(90)

It is easy to see that \( j \) satisfies condition (88)(b). Moreover, using the definition of the Clarke subdifferential it follows that

\[
\partial_C j(\sigma) = \begin{cases} 
(\alpha - 1)e^\sigma - \alpha & \text{if } \sigma < 0, \\
[-1, 1] & \text{if } \sigma = 0, \\
(1 - \alpha)e^{-\sigma} + \alpha & \text{if } \sigma > 0.
\end{cases}
\]

In addition, it is easy to check that \( |\xi| \leq 1 \) for all \( \xi \in \partial_C j(\sigma) \) and \( \sigma \in \mathbb{R} \) and, therefore, condition (88)(c) holds with \( c_j = 1 \). Finally, a simple calculation shows that condition (88)(d) holds with \( m_j = 1 - \alpha \).

**Example 3.** Let \( \alpha \geq 0 \) and let \( j: \mathbb{R} \to \mathbb{R} \) be the function defined by

\[
j(\sigma) = \begin{cases} 
0 & \text{if } \sigma < 0, \\
e^{-\sigma} + \alpha \sigma + 1 & \text{if } \sigma \geq 0.
\end{cases}
\]

(91)

It is easy to see that \( j \) satisfies condition (88)(b). Moreover, using elementary computation it follows that

\[
\partial_C j(\sigma) = \begin{cases} 
0 & \text{if } \sigma < 0, \\
[0, 1 + \alpha] & \text{if } \sigma = 0, \\
e^{-\sigma} + \alpha & \text{if } \sigma > 0.
\end{cases}
\]

In addition, it is easy to check that \( |\xi| \leq 1 + \alpha \) for all \( \xi \in \partial_C j(\sigma) \) and \( \sigma \in \mathbb{R} \) and, therefore, condition (88)(c) holds with \( c_j = 1 + \alpha \). Finally, a simple calculation shows that condition (88)(d) holds with \( m_j = 1 \).
We conclude from Examples 1 to 3 that Theorem 2 could be applied in the study of contact problems with elastic constitutive laws of the form

$$\varepsilon \in a \sigma + \partial_c j(\sigma),$$  \hspace{1cm} (92)

where $a > 0$ is a given compliance coefficient and $j$ represents one of the functions of equations (89) to (91). Note that in the case of equation (89) the constitutive law of equation (92) is single-valued and is given by

$$\varepsilon = \begin{cases} 
    a \sigma - a \sigma_0 & \text{if } \sigma < -\sigma_0, \\
    (a + \alpha) \sigma & \text{if } |\sigma| \leq \sigma_0, \\
    a \sigma + a \sigma_0 & \text{if } \sigma > \sigma_0.
\end{cases}$$ \hspace{1cm} (93)

In contrast, in the case of equations (90) and (91), the constitutive law (92) is multivalued. In addition, Theorem 3 could be applied in the study of contact problems with viscoelastic constitutive laws of the form

$$\varepsilon(t) \in a \sigma(t) + \partial_c j(\sigma(t)) + \int_0^t c(t - s) \sigma(s) ds.$$

Here, $a > 0$ is a given compliance coefficient, $j$ represents one of equations (89) to (91) and $c \in L^2(0, T)$ is a relaxation function. Both theorems provide the existence of a unique solution, in terms of stress, to the corresponding frictionless unilateral contact problems.

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**References**

[1] Duvaut, G, and Lions, J-L. *Inequalities in mechanics and physics*. Berlin: Springer-Verlag, 1976.

[2] Eck, C, Jarušek, J, and Krbce, M. *Unilateral contact problems: variational methods and existence theorems* (Pure and Applied Mathematics, vol. 270). New York: Chapman/CRC Press, 2005.

[3] Han, H, Migórski, S, and Sofonea, M (eds). *Advances in variational and hemivariational inequalities: theory, numerical analysis and applications* (Advances in Mechanics and Mathematics, vol. 33). New York: Springer, 2015.

[4] Han, W, and Sofonea, M. *Quasistatic contact problems in viscoelasticity and viscoplasticity* (Studies in Advanced Mathematics, vol. 30). Providence, RI: American Mathematical Society, 2002.

[5] Haslinger, J, Hlaváček, I, and Nečas, J. *Numerical methods for unilateral problems in solid mechanics*. In: Ciarlet, PG and Lions, J-L (eds.) *Handbook of numerical analysis vol. IV*. Amsterdam: North-Holland, 1996, 313–485.

[6] Hlaváček, I, Haslinger, J, Necás, J, and Lovišek, J. *Solution of variational inequalities in mechanics*. New York: Springer-Verlag, New York, 1988.

[7] Haslinger, J, Miettinen, M, and Panagiotopoulos, PD *Finite element method for hemivariational inequalities: theory, methods and applications*. Boston: Kluwer Academic Publishers, 1999.

[8] Migórski, S, Ochal, A, and Sofonea, M. *Nonlinear inclusions and hemivariational inequalities: models and analysis of contact problems* (Advances in Mechanics and Mathematics, vol. 26). New York: Springer, 2013.

[9] Naniewicz, Z, and Panagiotopoulos, PD. *Mathematical theory of hemivariational inequalities and applications*. New York: Marcel Dekker, Inc., 1995.

[10] Panagiotopoulos, PD. *Inequality problems in mechanics and applications*. Boston, MA: Birkhäuser, 1985.

[11] Panagiotopoulos, PD. *Hemivariational inequalities: applications in mechanics and engineering*. Berlin: Springer-Verlag, 1993.

[12] Shillor, M, Sofonea, M, and Teleaga, JJ *Models and analysis of quasistatic contact* (Lecture Notes in Physics, vol. 655). Berlin: Springer, 2004.

[13] Sofonea, M, and Matei, A. *Mathematical models in contact mechanics* (London Mathematical Society Lecture Note Series, vol. 398). Cambridge: Cambridge University Press, 2012.

[14] Han, W, Migorski, S, and Sofonea, M. A class of variational-hemivariational inequalities with applications to frictional contact problems. *SIAM J Math Anal* 2014; 46: 3891–3912.

[15] Migórski, S, Ochal, A, and Sofonea, M. History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics. *Nonlinear Anal Real World Appl* 2011; 12: 3384–3396.

[16] Migórski, S, Ochal, A, and Sofonea, M. History-dependent variational-hemivariational inequalities in contact mechanics. *Nonlinear Anal Real World Appl* 2015; 22: 604–618.
[17] Sofonea, M, Han, W, and Migorski, S. Numerical analysis of history-dependent variational inequalities with applications to contact problems. *Eur J Appl Math* 2015; 26: 427–452.

[18] Clarke, FH. *Optimization and nonsmooth analysis*. New York: Wiley Interscience, 1983.

[19] Denkowski, Z, Migórski, S, and Papageorgiou, NS. *An introduction to nonlinear analysis: theory*. Boston: Kluwer Academic, 2003.

[20] Zeidler, E. *Nonlinear functional analysis and its applications, vol. II/B: nonlinear monotone operators*. New York: Springer-Verlag, 1990.

[21] Migórski, S, Ochal, A, and Sofonea, M. A class of variational-hemivariational inequalities in reflexive Banach spaces. *J Elast* 2017; 127: 151–178.

[22] Lea, VK. Range and existence theorem for pseudomonotone perturbations of maximal monotone operators. *Proc Am Math Soc* 2011; 139: 1645–1658.

[23] Kalita, P, Migorski, S, and Sofonea, M. A class of subdifferential inclusions for elastic unilateral contact problems. *Set-Valued Var Anal* 2016; 24: 355–379.

[24] Sofonea, M, Danan, D, and Zheng, C. Primal and dual variational formulation of a frictional contact problem. *Mediterr J Math* 2016; 13: 857–872.