Improbability of Wandering Orbits Passing Through a Sequence of Poincaré Surfaces of Decreasing Size

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Abstract

Given a volume preserving dynamical system with non-compact phase space, one is sometimes interested in special subsets of its wandering set. One example from celestial mechanics is the set of initial values leading to collision. Another one is the set of initial values of semi-orbits, whose asymptotic velocity does not exist as a limit. We introduce techniques that can be helpful in showing that these sets are of measure zero: by defining a sequence of hypersurfaces, that are eventually hit by each of those semi-orbits and whose total surface area decreases to zero.

1 Introduction and Main Result

Let \( P^d \) be a smooth manifold with a volume form \( \Omega \) and a \( C^1 \) vector field \( X : P \to TP \), so that the Lie derivative \( \mathcal{L}_X \Omega \) vanishes. By standard results of ordinary differential equations, the flow \( \Phi \) associated to the differential equation \( \dot{x} = X(x) \) uniquely exists on a maximal neighborhood \( D \subseteq \mathbb{R} \times P \) of \( \{0\} \times P \) in extended phase space, \( \Phi \in C^1(D, P) \), and \( \Phi \) preserves the volume form \( \Omega \).

The flow’s domain of definition is of the form

\[
D = \left\{ (t, x) \in \mathbb{R} \times P \mid T^-(x) < t < T^+(x) \right\}
\]  

with the so-called escape time \( T := T^+ : P \to (0, +\infty] \), which is a lower semi-continuous function. Similarly, \( T^- : P \to [-\infty, 0) \) is upper semi-continuous. By
\(\mathcal{O}(x) := \Phi((T^{-}(x),T^{+}(x)),x)\) we denote the orbit passing through \(x \in P\), and by \(\mathcal{O}^{+}(x) := \Phi([0,T^{+}(x)),x)\) the semi-orbit.

We consider the wandering set of \(\Phi\)

\[
\text{Wand} \equiv \text{Wand}_{\Phi} \subseteq P,
\]

consisting of those \(x \in P\) which have a neighborhood \(U_{x}\) so that for a suitable time \(t_{-} \in (0,T(x))\)

\[
U_{x} \cap \Phi\left(\{(t_{-},T(x)) \times U_{x}\} \cap D\right) = \emptyset.
\]

**Remark 1.1** (Wandering set)

1. In view of applications (see Example 1.6 below) we allowed for finite escape times \(T^{\pm}(x)\). So \(\Phi\) does not in general define an \(R\)-action on \(P\). In this sense our notion of ‘wandering set’ is a generalization of the usual definition.

2. As \(T\) is lower semi-continuous, the set of singular points

\[
\text{Sing} := \{x \in P \mid T(x) < \infty\},
\]

is a Borel set. It is wandering for the same reason, see Lemma 1.2 below.

3. Trivially, equilibrium points are nonwandering, so that

\[
\text{Wand} \subseteq \{x \in P \mid X(x) \neq 0\}.
\]

The latter is an open submanifold of \(P\) because of \(X \in C^{1}(P,TP)\). Thus, we assume the vector field \(X\) to be non-vanishing on \(P\) from the outset, without loss of generality. \(\diamond\)

**Lemma 1.2** \(\text{Sing} \subseteq \text{Wand}\).

**Proof:** Let \(x \in \text{Sing}\), so that \(T(x) \in (0, +\infty)\). For any small \(\epsilon_1 > 0\) there exists a flow-box chart (see also Lemma 4.2) \(\varphi : U_{1} \rightarrow (-\epsilon_1, \epsilon_1) \times W_{1} \subseteq R \times R^{2n-1}\) with \(\varphi(x) = (0,0)\) that is reentered only a finite number of times by \(\mathcal{O}^{+}(x)\).

Inside \(U_{1}\), for any small \(\epsilon_2 > 0\) there is a compact neighborhood \(U_{2}\) of \(x\) with \(\varphi(U_{2}) = [-\epsilon_2, \epsilon_2] \times W_{2}\) that is not reentered at all by \(\mathcal{O}^{+}(x)\). By a compactness argument, for any \(\epsilon_3 \in (0, \epsilon_2/2)\) there is a compact neighborhood \(U_{3} \subseteq U_{2}\) of \(x\) with \(\varphi(U_{3}) = [-\epsilon_3, \epsilon_3] \times W_{3}\) and \(\Phi(([2\epsilon_2,T(x)-\epsilon_3] \times U_{3}) \cap U_{2} = \emptyset\). Then also \(\Phi((\{t\} \times U_{3}) \cap D) \cap U_{3} = \emptyset\) for all \(t \in [2\epsilon_2, T(x)]\), so that \(x\) is wandering. \(\square\)

Now let

\[
\tau_{m} : \mathcal{H}_{m} \rightarrow P \quad (m \in \mathbb{N})
\]
be a sequence of (pairwise disjoint) codimension one closed ∂-submanifolds\(^1\) of \(P\), which we will call Poincaré surfaces for reasons explained below.

**Assumptions:**

1. The vector field \(X\) is transversal to their relative interior \(\iota_m : \mathcal{H}_m \to P\). Thus (\(\iota\) being the inner product) the \((d - 1)\)-form

\[\mathcal{V} := \iota_X \Omega\]

on \(P\) induces the volume forms \(\mathcal{V}_m := \iota^* \mathcal{V}\) on \(\mathcal{H}_m\).

2. We assume that the \(\mathcal{H}_m\) are of finite volume: \(\int_{\mathcal{H}_m} \mathcal{V}_m < \infty\) (\(m \in \mathbb{N}\)), and the volumes go to zero:

\[
\lim_{m \to \infty} \int_{\mathcal{H}_m} \mathcal{V}_m = 0. \tag{1.3}
\]

Then we denote the set of *transition points*, whose forward orbits eventually hit all of these surfaces by

\[
\text{Trans} \equiv \text{Trans}_\Phi := \{ x \in P \mid \exists m_0 \in \mathbb{N} \forall m \geq m_0 : \mathcal{O}^+(x) \cap \mathcal{H}_m \neq \emptyset \}. \tag{1.4}
\]

Our main result is the following.

**Theorem A** From the assumptions it follows that \(\Omega(\text{Trans} \cap \text{Wand}) = 0\).

**Remark 1.3 (Theorem A)**

1. A volume form on a manifold defines a positive linear functional on the vector space of continuous, compactly supported functions. Thus by the Riesz representation theorem, it induces a measure on its Borel sets, and like in Theorem A we do not in general make a distinction between these notions and use the same symbol. So instead of \(\int_{\mathcal{H}_m} \mathcal{V}\) we write \(\mathcal{V}(\mathcal{H}_m)\), etc.

2. The volume form also defines an orientation of the given (sub-)manifold.

3. In general \(\Omega(\text{Trans}) > 0\) (e.g. for every ergodic flow on a closed manifold \(P\)).

4. Clearly in general \(\Omega(\text{Wand})\) is positive, too. \(\diamond\)

**Remark 1.4 (Cases of symplectic and Kähler structures)**

To apply our result, we make use of additional structures of our volume preserving dynamical system:

\(^1\)We call (sub-)manifolds with boundary \(\partial\)-(sub)-manifolds.
1. An important subclass is the one of Hamiltonian systems $(P^{2n}, \omega, H)$, with
\[ \Omega := \frac{(-1)^{\lfloor n/2 \rfloor}}{n!} \omega ^{\wedge n}, \]
$H \in C^2(P, \mathbb{R})$ and $X \equiv X_H$ uniquely given by the equation $i_{X_H} \omega = dH$ (with the exterior derivative $d$).

In this case we obtain useful expressions for the volume forms $\mathcal{V}_m$ in terms of the symplectic form $\omega$.

2. As in Hamiltonian systems $H$ is a constant of the motion, we may restrict the maximal flow to the energy surfaces $\Sigma_E := H^{-1}(E)$ ($E \in \mathbb{R}$). By our assumption, $dH$ is non-vanishing. So the $i_E : \Sigma_E \rightarrow P$ are submanifolds. It is well-known, that there exists a $(2n-1)$-form $\sigma$ on $P$ with $dH \wedge \sigma = \Omega$. Although $\sigma$ is not unique, its pullbacks $\sigma_E := i_E^* \sigma$ to $\Sigma_E$ are uniquely defined volume forms [AM78, Theorem 3.4.12]. These are invariant under the restricted flow. We denote by $W_{\Sigma_E}$ the set of wandering points and by $Trans_{\Sigma_E}$ the set of transition points on $\Sigma_E$ w.r.t. a sequence of hypersurfaces $\mathcal{H}_m \in \Sigma_E$.

Applying Theorem A to $P = \Sigma_E$ hence yields $\sigma_E(W_{\Sigma_E} \cap Trans_{\Sigma_E}) = 0$, if $\lim_{m \rightarrow \infty} \int_{\mathcal{H}_m} i_{XH} \sigma_E = 0$.

3. Finally, the symplectic manifold $(P, \omega)$ may be Kähler, that is, equipped with a Riemannian metric $g$ and complex structure $J$, so that $\omega(X, Y) := g(JX, Y)$. This then allows to use simple estimates for the volume forms in terms of Riemannian volumes.

Considering motion on an energy surface $\Sigma_E \subseteq P$ of a Kähler manifold $P^{2n}$, we can modify Assumption 2.:

2'. We assume that the hypersurfaces $\mathcal{H}_m \subseteq \Sigma_E$ are of finite volume w.r.t. the Riemannian volume form $d\mathcal{H}_m$: $\int_{\mathcal{H}_m} d\mathcal{H}_m < \infty$ ($m \in \mathbb{N}$), and the volumes go to zero:
\[ \lim_{m \rightarrow \infty} \int_{\mathcal{H}_m} d\mathcal{H}_m = 0. \]

Theorem B On any energy surface $\Sigma_E \subseteq P$ in a Kähler manifold it follows from Assumptions 1. and 2'. for the Hamiltonian flow that $\sigma_E(Trans_{\Sigma_E} \cap W_{\Sigma_E}) = 0$.

Remark 1.5 (Comparison with Theorem A) In the case of energy surfaces $\Sigma_E \subseteq P$ of a symplectic manifold $(P^{2n}, \omega)$, the natural invariant volume form on a hypersurface $i : \mathcal{H} \rightarrow \Sigma_E$ is $i^* \omega^{\wedge n-1}$. This would be the $V$ entering in (1.3). Although Assumption 2', using the Riemannian volume form $d\mathcal{H}_m$, is stronger than Assumption 2., it may be easier to check.
We give a simple example.

**Example 1.6 (Collision of two particles is improbable for \( n \geq 2 \))**

By reduction we can model the motion by the Hamiltonian function

\[
H : T^*(\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}, \quad H(q, p) = \frac{1}{2} \|p\|^2 + V(q)
\]

with a potential \( V \in C^2(\mathbb{R}^n \setminus \{0\}, \mathbb{R}) \). We assume that for some \( \alpha \in (0, 2) \), \( c > 0 \)

\[
|V(q)| \leq \frac{c}{\|q\|^\alpha} \quad (\|q\| \leq 1)
\]

and, say \( \lim_{\|q\| \to \infty} V(q) = 0 \). Then for all \( x_0 \in \text{Sing} \) we have \( \lim_{t \to T(x_0)} q(t, x_0) = 0 \). So all singular points are collision points.

To show that \( \sigma_E(\text{Sing}_E) = 0 \) for all energy surfaces \( \Sigma_E \), we first tacitly delete the rest points \((q, p) \in \Sigma_E \) with \( V(q) = E, \nabla V(q) = 0 \) and \( p = 0 \), see Remark 1.1.3. Then we define the \( \partial \)-hypersurfaces in \( \Sigma_E \) by

\[
H_m := \{(q, p) \in \Sigma_E \mid \|q\| = 1/m, \langle p, q \rangle \leq 0\} \quad (m \in \mathbb{N}). \tag{1.5}
\]

It is clear that all singular points are in \( \text{Trans} \) and \( \sigma_E(\overline{H}_m) < \infty \). Also, the \( \overline{H}_m \) are transversal to the flow. For \( n = 1 \) dimension \( \overline{H}_m \) consists of two points and thus has \( m \)-independent volume, clearly violating Condition (1.3). To show (1.3) for \( n \geq 2 \), we note that on \( \overline{H}_m \) we have

\[
\|q\| = \frac{1}{m} \quad \text{and} \quad \|p\| = \sqrt{2(E - V(q))} \leq \sqrt{2(E + cm^\alpha)} \leq \sqrt{cm^{\alpha/2}}
\]

for all sufficiently large \( m \in \mathbb{N} \). The symplectic manifold \( (T^*(\mathbb{R}^n \setminus \{0\}), \omega_0) \) is Kähler. Using Theorem B, for \( \mathcal{F}_m := \{q \in \mathbb{R}^n \mid \|q\| = 1/m\} \)

\[
\int_{\overline{H}_m} d\overline{H}_m = \frac{1}{2} \text{vol}(S^{n-1}) \tag{1.6}
\]

\[
\int_{\mathcal{F}_m} (2(E - V(q))^{(n-2)/2} \sqrt{2(E - V(q))} + \|Q(q)\nabla V(q)\|^2 d\mathcal{F}_m(q),
\]

with the Riemannian volume element \( d\mathcal{F}_m \). Here \( Q(q) \) is the projection along the direction \( q/\|q\| \). The expression is derived as follows:

- **The factor** \( \text{vol}(S^{n-1}) \) **in (1.6)** **is the Riemannian volume of a unit sphere in momentum space.**

- **The factor** \( 1/2 \) **is due to the condition** \( \langle p, q \rangle \leq 0 \) **in the definition (1.5) of the surface** \( \overline{H}_m \).

- **At the point** \( q \) **on the sphere** \( \mathcal{F}_m \) **of radius** \( 1/m \), **the corresponding half-sphere in momentum space has radius** \( \sqrt{2(E - V(q))} \). **For** \( \nabla V(q) \) **parallel to** \( q \) **this would lead to a Riemannian volume element** \( (2(E - V(q)))^{(n-1)/2} d\mathcal{F}_m(q) \).
By Pythagoras, this is multiplied by the ratio
\[
\left(1 + \frac{\|Q(q)\nabla V(q)\|^2}{2(E - V(q))}\right)^{1/2}
\]
between hypotenuse and adjacent side for the direction of the gradient.

If we additionally assume that the force is radial, that is \(Q\nabla V(q) = 0\), then it follows that the singular set has measure zero.

On the other hand, it is well known that collision occurs with positive measure if \(V(q) = -\|q\|^{-\alpha}\) and \(\alpha \geq 2\). So in the example the criterion is optimal.

If \((P, \omega) = (T^*M, \omega_0)\), with the canonical symplectic form \(\omega_0\) on the cotangent bundle of a Riemannian manifold \((M, h)\) and if the Hamiltonian function is of the form 'kinetic+potential', then there is a useful integration formula, see Theorem C. This can be applied to our example.

**Example 1.7** If we instead use Theorem A, employing Theorem C, we obtain, without assuming that \(Q\nabla V(q) = 0\),
\[
\int_{H_m} \omega^{n-1} = \nu_{n-1} \int_{F_m} (2(E - V(q)))^{(n-1)/2} dF_m(q).
\]
The integral is bounded above by \(c_3m^{(n-2)(n-1)/2}\) and thus goes to zero as \(m \to \infty\). So the singular set has measure zero. ☐

**Remark 1.8** (Transversality)

1. It was important in the example to use closed \(\partial\)-submanifolds \(\overline{H}_m\) to ensure that every collision orbit belongs to \(\text{Trans}\). On the other hand, the proof of Theorem A will be based on the transversality of the flow w.r.t. the Poincaré surfaces \(H_m\). So it will be necessary to show in general that the orbits meeting \(\partial H_m\) do not contribute to \(\Omega(\text{Trans} \cap \text{Wand})\). This will be done in Section 3.

2. In Example 1.6 we defined for every energy surface \(\Sigma_E \subseteq P\) of the Hamiltonian flow \(\Phi : D \to P\) hypersurfaces \(H^E_m \subseteq \Sigma_E\) that are transversal to the flow.
   Instead, we could define from the outset hypersurfaces \(H_m \subseteq P\). This does not imply the aimed-for result for all energy surfaces intersected by the \(H_m\). However, by Sard’s Theorem one could conclude in this approach that \(\lambda^1\)-almost all of these \(\Sigma_E\) are met transversally, see Remark 3.2. ☐
The outline of the article is as follows: Section 2 abstractly considers a discretized version of the dynamics. In Section 3, we show that indeed only the surface areas of the transverse parts $H_m$ are of interest, since hitting the boundaries $\partial H_m$ of the surfaces will be shown to be improbable. In Section 4, we make a crucial step in the proof of Theorem A by contradiction: if the set of wandering transition points had positive measure, then in one Poincaré surface we would find a compact subset, whose intersection with the set of these points had positive area. The main tool in doing so is a version of the Flow-Box Theorem, which we will state at the beginning of that section. Then, in Section 5 we will take a look at the progression of those transition points by defining Poincaré maps between the surfaces (hence the name Poincaré surfaces), which are shown to be area preserving. Since the total area of the surfaces is assumed to decrease to zero, this contradicts an initial set to have positive area.

In Section 6, we show how the result can be restated if the symplectic manifold is Kähler. Finally, in Section 7 we indicate how we apply the scheme given here in forthcoming articles.

2 Discretization of the Problem

Here we consider a discrete dynamics. This will be used in Section 5 to model the dynamics on the Poincaré surfaces.

Let $T : M \to M$ be a continuous injective map of a topological space, preserving a Borel measure $\mu : \mathcal{M} \to [0, \infty]$ on the Borel $\sigma$-algebra $\mathcal{M}$.

The wandering set

$$Wand \equiv \text{Wand}_T \subseteq M$$

of the corresponding discrete dynamical system consists of those $x \in M$ which have a neighborhood $U_x$ so that $U_x \cap T^t(U_x) = \emptyset \ (t \in \mathbb{N})$.

We denote by $O^+(x) := \{T^t(x) \mid t \in \mathbb{N}_0\}$ the forward orbit of $x \in M$. The subsets

$$H_m \equiv H^T_m \in \mathcal{M} \quad (m \in \mathbb{N})$$

are assumed to have finite measures. The transition set is defined by

$$\text{Trans} \equiv \text{Trans}_T := \{x \in M \mid \exists m_0 \in \mathbb{N} \forall m \geq m_0 : O^+(x) \cap H^T_m \neq \emptyset\}. \ (2.1)$$

Lemma 2.1 Assuming $\lim_{m \to \infty} \mu(H_m) = 0$, then $\mu(\text{Trans} \cap \text{Wand}) = 0$.

Proof: Assuming the converse, we find $k \in \mathbb{N}$ and $x \in H_k \cap \text{Trans} \cap \text{Wand}$ such that $\mu(U_x \cap H_k \cap \text{Trans} \cap \text{Wand}) > 0$ for all neighborhoods $U_x$ of $x$. As
For such a $k$ and $U_x$ we set $\mathcal{K}_k := U_x \cap \text{Trans} \cap \text{Wand}$ and
\[
\mathcal{K}_{k,\ell} := \{ y \in \mathcal{K}_k \mid |O^+(y) \cap \mathcal{H}_k| = \ell \} \quad (\ell \in \mathbb{N} \cup \{\infty\}).
\]
As we have the disjoint union
\[
\mathcal{K}_k = \coprod_{\ell \in \mathbb{N} \cup \{\infty\}} \mathcal{K}_{k,\ell},
\]
by our assumption $\mu(\mathcal{K}_k) > 0$ there is an $\ell$ with $\mu(\mathcal{K}_{k,\ell}) > 0$. However:

- We notice that $\mu(\mathcal{K}_{k,\infty}) = 0$, since otherwise by (2.2)
\[
\mu\left( \bigcup_{t \in \mathbb{N}_0} T^t(\mathcal{K}_{k,\infty}) \cap \mathcal{H}_k \right) = \sum_{t \in \mathbb{N}_0} \mu\left( T^t(\mathcal{K}_{k,\infty}) \cap \mathcal{H}_k \right) = \infty,
\]
contradicting $\mu(\mathcal{H}_k) < \infty$.

- But also $\mu(\mathcal{K}_{k,\ell}) = 0$ for all $\ell \in \mathbb{N}$. For otherwise, using $\mathcal{K}_{k,\ell} \subseteq \text{Trans}$, with
\[
\mathcal{K}_{k,\ell,m_0} := \{ y \in \mathcal{K}_{k,\ell} \mid \forall m \geq m_0 : O^+(y) \cap \mathcal{H}_m \neq \emptyset \} \quad (m_0 > k)
\]
there exists a $m_0 > k$ with $\mu(\mathcal{K}_{k,\ell,m_0}) > 0$. But this is impossible: Choose $m \geq m_0$ so that $\mu(\mathcal{H}_m) < \mu(\mathcal{K}_{k,\ell,m_0})$, and let
\[
\tau(y) := \min\{t \in \mathbb{N} \mid T^t(y) \in \mathcal{H}_m\} \quad (y \in \mathcal{K}_{k,\ell,m_0}).
\]
Then the map
\[
\mathcal{H}_{k,\ell,m_0} \to \mathcal{H}_m, \quad y \mapsto T^{\tau(y)}(y)
\]
is one to one, and thus $\mu(\mathcal{H}_m) \geq \mu(\mathcal{H}_{k,\ell,m_0})$, as $T$ preserves $\mu$.

So we derived a contradicting to the assumption $\mu(\text{Trans} \cap \text{Wand}) > 0$. \hfill \Box

In Section 3 we will apply Lemma 2.1 to a Poincaré map for the flow $\Phi$, on a certain subset of $\cup_{m \in \mathbb{N}} \mathcal{H}_m$. 

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Figure 1: Transverse and tangent orbits: here $P = \mathbb{R}^2$, $X_H = e_1$ and $H = \text{Graph}(x \mapsto x^2 + 1)$, so the flow is transverse to $H \setminus \{(0, 1)\}$ and $T = \mathbb{R} \times \{1\}$.

3 Transversality

In the following sections, we need to move points along their orbits, until they hit one of the Poincaré surfaces. We usually need the orbits to be transverse to the surfaces to ensure that the corresponding (local) Poincaré maps are $C^1$. Also, we want to define a global continuous discrete dynamics for using Lemma 2.1. As we would like to exclude orbits hitting a boundary $\partial H_m$ from our considerations early on, we claim:

**Lemma 3.1** The measure $\Omega(T)$ vanishes for the set

$$T := \bigcup_{m \in \mathbb{N}} T_m \quad \text{with} \quad T_m := \{x \in P \mid O(x) \cap \partial H_m \neq \emptyset\} \quad (m \in \mathbb{N}).$$

**Remark 3.2** (Transversality)

1. Hence, if we want to prove Theorem A, i.e. show that $\Omega(\text{Trans} \cap \text{Wand}) = 0$, we only need to show $\Omega(\text{Trans} \cap \text{Wand} \cap T^C) = 0$, since $T$ is of measure zero.

2. $T$ contains all phase space points $x$, whose orbit $O(x)$ hits the submanifolds $\overline{H}_m$ tangentially. Applying Sard’s Theorem, we could even show the analog of Lemma 3.1 if we allowed the flow to be tangential to the relative interior $H_m$ of $\overline{H}_m$, see Figure 1.

**Proof of Lemma 3.1:** We must show that $\Omega(T_m) = 0$ for all $m \in \mathbb{N}$. But as $\partial H_m$ is a codimension two embedded submanifold of $P$ and $\Phi \in C^1(D, P)$,

$$T_m = \Phi(D \cap (\mathbb{R} \times \partial H_m))$$

is of measure zero.

**Proof of Theorem A:**
We now use Lemma 2.1 in the following way.
• $M := \bigcup_{m \in \mathbb{N}} \mathcal{H}^T_m$ with $\mathcal{H}^T_m := \mathcal{H}_m \cap \text{Set}$, is the union of the parts of the Poincaré surfaces, belonging to

$$\text{Set} := \text{Trans} \cap \text{Wand} \setminus \mathcal{T}.$$ 

As $\text{Set} \cap \mathcal{T} = \emptyset$, the flow $\Phi$ is transversal to $M$.

• $T : M \to M$ is defined as the next intersection with a Poincaré surface. As $\text{Set} \subseteq \text{Trans}$, this exists. So the \textit{return time}

$$\tau : M \to \mathbb{N} \cup \{\infty\}, \quad \tau(x) = \inf\{t \in (0, T(x)) | \Phi(t, x) \in M\}$$

is finite. Since $\Phi$ is transversal to $M$, it is positive, and one sets

$$T(x) := \Phi(\tau(x), x).$$

Notice that by transversality $T \in C(M, M)$. By definition $T$ is injective.

• $\mu := \iota_M^* \nu$ is the invariant measure, with $\iota_M : M \to P$. So $\mu(\mathcal{H}^T_m) = \mathcal{V}(\mathcal{H}_m) < \infty$, using Lemma 3.1. In particular, the assumption $\lim_{m \to \infty} \mu(\mathcal{H}_m) = 0$ of Lemma 2.1 is true.

Moreover, by Lemma 5.1 the map $T$ preserves the measure $\mu$.

So in this application the whole topological space $M$ is now the wandering set of the continuous transformation $T$ and also equals $\text{Trans} T$ as defined in (2.1). We apply Lemma 2.1 to show that $\mu(M) = 0$. By Lemma 4.1 below this implies that $\Omega(\text{Trans} \cap \text{Wand}) = 0$. 

\[\square\]

### 4 Localization to one Poincaré Surface

As pointed out in the introduction, one step of proving Theorem A is the following implication:

if $\text{Trans}_\Phi$ had positive measure, then there would be some compact subset of some Poincaré surface, whose intersection with $\text{Trans}_\Phi$ had positive area:

**Lemma 4.1** Suppose $\Omega(\text{Trans} \cap \text{Wand}) > 0$. Then there exist $m \in \mathbb{N}$ and a compact subset $\mathcal{K}_m \subseteq \mathcal{H}_m$ with

$$\mathcal{V}_m(\mathcal{K}_m \cap \text{Set}) > 0.$$ 

Our main tool in proving this is the following Flow-Box Theorem, which identifies the change of volume of sets like $\Phi((−t, t) \times \mathcal{K})$ (with an appropriate codimension one $\partial$-submanifold $\mathcal{K}$) in ongoing time with the surface area of the submanifold $\mathcal{K}$. To be more precise:
**Lemma 4.2 (Flow-Box Theorem)** Let $H \subseteq P$ be a submanifold of codimension one, such that the vector field $X$ is transverse to $H$. Then $x \in H$ has a compact $\partial$-submanifold $K \subseteq H$ as a neighborhood, $L_t := \Phi((-t, t) \times K)$ is a $\partial$-submanifold of $P$ for sufficiently small $t > 0$, and

$$\int_{L_t} F \Omega = 2t \int_K f \mathcal{V} \quad (4.1)$$

for all continuous functions $f$ on $K$; here, $F$ is the continuation of $f$ to $L_t$ that is constant along the flow lines, i.e.

$$F: L_t \to \mathbb{C}, \quad F(\Phi(s, x)) := f(x) \quad (s \in (-t, t), x \in K). \quad (4.2)$$

**Proof:** Let $x \in H$. As initially assumed (Remark 1.1.3), $X(x) \neq 0$. By the 'straightening out' Theorem 2.1.9 in [AM78], this means that on a suitable open neighborhood $U$ of $x$ in $P$ there exists a chart $\varphi: U \to I \times W \subseteq \mathbb{R} \times \mathbb{R}^{2n-1}$, where $I := (-\varepsilon, \varepsilon)$ for $\varepsilon > 0$, and $W \subseteq \mathbb{R}^{2n-1}$ is open, such that (see Figure 2)

1. $I \ni \lambda \mapsto \varphi^{-1}(\lambda, \eta)$ is an integral curve of $X$ for all $\eta \in W$.
2. $\varphi^{-1}(\{0\} \times W) = H \cap U$.
3. The local representative of the vector field $X$ at any point of the chart is $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$.

We choose a $\partial$-submanifold $K \subseteq H \cap U$ as a compact neighborhood of $x$. Then for $t \in (0, \varepsilon)$

$$\varphi(L_t) = (-t, t) \times \varphi(K).$$

So $L_t$ is a $\partial$-submanifold of $P$, and its closure $\overline{L_t}$ is a submanifold with corners. By Property 3. we can write the volume form $\Omega$ on $L_t$ as

$$\Omega = dt \wedge \mathcal{V},$$

using the time coordinate $t$ of the chart $\varphi$. Therefore

$$F \Omega = dt \wedge F \mathcal{V} = dt \wedge i_X F \Omega. \quad (4.3)$$

As the Lie derivative $L_X F \Omega = (L_X F) \Omega + F L_X \Omega$ vanishes by Def. (4.2), we have $dF \mathcal{V} = di_X F \Omega = 0$, too. So (4.3) implies $F \Omega = d(t i_X F \Omega) = d(t F \mathcal{V})$, and Stokes’ Theorem tells us that

$$\int_{L_t} F \Omega = \int_{\partial L_t} t F \mathcal{V} = \int_{\partial L_t} t F \mathcal{V} = 2t \int_K f \mathcal{V}.$$

In the last equation we used that $\mathcal{V} = i_X \Omega$ vanishes on $\Phi((-t, t), \partial K)$. \qed
With this, we can give the proof of the preceding lemma:

**Proof of Lemma 4.1:** To simplify notation, we use the flow invariant set

\[ \text{Set} = \text{Trans} \cap \text{Wand} \setminus \mathcal{T}. \]

By Lemma 3.1, \( \Omega(\text{Trans} \cap \text{Wand}) > 0 \) implies \( \Omega(\text{Set}) > 0 \) as well. In this case, there exists \( x_0 \in \text{Set} \) with

\[ \Omega(U \cap \text{Set}) > 0 \quad \text{for every neighborhood } U \subseteq P \text{ of } x_0. \quad (4.4) \]

By definition of the set \( \text{Trans} \), for a sufficiently large \( m \in \mathbb{N} \) there exists a time \( t_m \in (0, T^+(x_0)) \) with

\[ x_m := \Phi(t_m, x_0) \in \overline{\mathcal{H}}_m. \]

Specifically because of \( x_0 \in \mathcal{T}^C \), we have that \( x_m \in \mathcal{H}_m \). For a sufficiently small \( \delta > 0 \), we have that \( \delta \) the ball \( B_\delta(x_m) := \{ y \in P \mid d(y, x_0) < \epsilon \} \) is a subset of

\[ \{ x \in P \mid (-t_m, x) \in D \}, \]

since the latter is open (see [Lee03, Th. 17.8]) and contains \( x_m \); here, \( D \) is the domain of the flow \( \Phi \), see (1.1). Then

\[ \mathcal{K}_m := \overline{\mathcal{H}}_m \cap B_{\delta/2}(x_m) \]

is a compact subset of \( \overline{\mathcal{H}}_m \) with \( x_m \in \mathcal{K}_m \), which just like \( \mathcal{H}_m \) is of codimension one in \( P \). Then we find \( \epsilon > 0 \), such that for every point in \( \mathcal{K}_m \) the flow exists at least on the time interval \( (-\epsilon, \epsilon) \); then

\[ L_\epsilon := \Phi((-\epsilon, \epsilon) \times \mathcal{K}_m) \]
Figure 3: Localization on a Poincaré surface: we illustrate the points $x_0$ and $x_m \in \mathcal{H}_m$ as constructed in the proof, as well as the sets $B_\delta(x_m)$ and $\mathcal{K}_m$ plus $L_\varepsilon$ and $\Phi_{-t_m}(L_\varepsilon)$.

is well-defined, see Figure 3.

By a possible diminishment of $\varepsilon$, we have that $L_\varepsilon \subseteq B_\delta(x_m)$, hence $\Phi_{-t_m}(L_\varepsilon)$ is well-defined. This is a neighborhood of $x_0$, such that this point’s construction implies that

$$\Omega \left( \Phi_{-t_m}(L_\varepsilon) \cap \text{Set} \right) > 0,$$

see (4.4). Thus

$$\Omega \left( L_\varepsilon \cap \text{Set} \right) > 0,$$

(4.5)
since by flow invariance of $\text{Set}$

$$
\Phi_{-t_m}(L_\varepsilon) \cap \text{Set} = \Phi_{-t_m}(L_\varepsilon \cap \text{Set})
$$

and $\Phi_{-t_m}$ is volume preserving, see [AM78, Prop 3.3.4].

Now we show that $\mu_m(\mathcal{K}_m \cap \text{Set}) > 0$. This is an immediate consequence of the Flow-Box Theorem, see Lemma 4.2: equation (4.1) yields that

$$
\int_{L_\varepsilon} F \Omega = 2\varepsilon \int_{\mathcal{K}_m} f \mathcal{V}
$$

(4.6)

for all continuous functions $f$ on the compact set $\mathcal{K}_m$, where $\Omega$ is the canonical volume form on $P$; for $F$, we adopt the notation from Lemma 4.2 (continuation of $f$, which is constant along the flow lines).

\[3\] with respect to an arbitrary Riemannian metric $g$ on $P$ and the metric $d$ induced by $g
Now let \( f_n (n \in \mathbb{N}) \) be a sequence of smooth functions on \( \mathcal{H}_m \) with compact support in \( \mathcal{K}_m \), with \( L^1 \lim_{n \to \infty} f_n = 1_{\mathcal{K}_m \cap \text{Set}} \). Thus
\[
2\varepsilon \int_{\mathcal{H}_m} f_n \nu_m = \int_{L_\varepsilon} F_n \Omega \quad (n \in \mathbb{N}) \tag{4.7}
\]
where again \( F_n \) is the continuation of \( f_n \), that is constant along the flow lines. Hence, \( L^1 \lim_{n \to \infty} F_n = 1_{L_\varepsilon \cap \text{Set}} \), so taking the limit \( n \to \infty \) in (4.7) exactly yields
\[
2\varepsilon \cdot \nu_m (\mathcal{K}_m \cap \text{Set}) = \Omega (L_\varepsilon \cap \text{Set}).
\]
By (4.5), it follows that \( \nu_m (\mathcal{K}_m \cap \text{Set}) > 0 \). \( \square \)

5 Transit between the Poincaré Surfaces

Below, we will identify points on one surface with certain elements on their positive semi-orbit, lying on another surface. We will call such a mapping Poincaré map, and show that they are volume preserving.

Now we give the general notion of Poincaré maps as used within this paper. We first consider general (incomplete) flows and then specialize to the Hamiltonian flow \( \Phi \), and to its restriction to energy surfaces.

Lemma 5.1 (Poincaré Maps)

1. Let \( X \) be a vector field on a manifold \( M \) whose flow \( \Phi \in C^1(D, M) \) has maximal domain \( D \subseteq \mathbb{R} \times M \). Consider two codimension one submanifolds \( \mathcal{H}_0, \mathcal{H}_1 \subseteq M \), such that \( X \) is transverse to \( \mathcal{H}_i \) \((i = 0, 1)\). Fix a point \( x_0 \in \mathcal{H}_0 \) and assume there is a time \( t_0 > 0 \), such that \( x_1 := \Phi(t_0, x_0) \in \mathcal{H}_1 \).

Then there exist a neighborhood \( U_0 \subseteq \mathcal{H}_0 \) of \( x_0 \) and a so-called hitting time \( \tau \in C^1(U_0, \mathbb{R}) \) with \( \tau(x_0) = t_0 \), such that on \( U_0 \) the Poincaré map \( \psi \),
\[
\psi(x) := \Phi(\tau(x), x) \in \mathcal{H}_1 \quad (x \in U_0)
\]
is a diffeomorphism onto its image \( U_1 \).

2. Now consider the case of the flow \( \Phi \) on the manifold \( P \) generated by \( X \) and preserving the volume form \( \Omega \). Then the restrictions of \( \nu = i_X \Omega \) to \( \mathcal{H}_i \) are volume forms, and the Poincaré map, see Figure 4, is volume preserving.

3. Finally consider the restriction of a Hamiltonian flow \( \Phi \) to an energy surface \( \Sigma_E \). Then \( \mathcal{H}_i \subseteq \Sigma_E \) are symplectic submanifolds of \((P, \omega)\), and the Poincaré map is a symplectomorphism onto its image.

Proof:
1. The proof is analogous to [AM78, Theorem 7.1.2], using the lower continuity of the escape time.

2. Consider for the unit ball $B \subseteq \mathbb{R}^{2(n-1)}$ a (small) embedded $\partial$-submanifold $\iota_0 : B \to V_0 \subseteq U_0$ with $\iota_0(0) = x_0$ and its image under the Poincaré map $\psi$, the $\partial$-submanifold $\iota_1 : V_1 := \psi(V_0) \to U_1$. We extend these embeddings to the embedding

$$\iota : [0,1] \times B \to P \ , \ \iota(t,x) = \Phi(t \tau(x), x)$$

of a cylinder. Then $\int_{[0,1] \times B} \iota^* d\mathcal{V} = 0$, since $\mathcal{V}$ is closed:

$$d\mathcal{V} = di_X \Omega = (di_X + i_X d) \Omega = \mathcal{L}_X \Omega = 0.$$

Similarly $\int_{[0,1] \times \partial B} \iota^* \mathcal{V} = 0$, since $X$ is tangential to the $\partial$-submanifold $\iota([0,1] \times \partial B)$. Thus by Stokes’ theorem for manifolds with corners

$$0 = \int_{[0,1] \times B} \iota^* d\mathcal{V} = \int_{\partial([0,1] \times B)} \iota^* \mathcal{V} = \int_{(1) \times B} \iota^* \mathcal{V} - \int_{(0) \times B} \iota^* \mathcal{V} = \int_B \iota_1^* \mathcal{V} - \int_B \iota_0^* \mathcal{V}.$$

3. The proof goes along the lines of [MS95, Lemma 8.2].

\section{Submanifolds of Kähler Manifolds}

If there is an underlying metric on the manifold inducing the symplectic structure, then it may be more convenient from a technical point of view, to use the metric structure to estimate the symplectic volume of a codimension two submanifold.
Let \((P, g, J)\) be a 2n-dimensional Kähler manifold, with Riemannian metric \(g\) and complex structure \(J\), which induce a symplectic form \(\omega\) by

\[
\omega(X, Y) := g(JX, Y) \quad (X, Y \in \mathcal{X}(P)).
\]

Assume\(^4\) that \(E \in \mathbb{R}\) is a regular value of \(H\), such that \(\Sigma_E\) is a codimension-1 submanifold (if non-empty) of \(P\). Let \((\mathcal{H}^E_m)_{m \in \mathbb{N}}\) be a sequence of submanifolds of \(\Sigma_E\) of codimension one.

Let \(dP\) be the Riemannian volume form on \(P\) induced by \(g\); then \(dP\) equals the canonical volume form \(\Omega\) as defined above. Further, let \(d\Sigma_E\) and \(d\mathcal{H}^E_m\) be the Riemannian volume forms on \(\Sigma_E\) and \(\mathcal{H}^E_m\) respectively, induced by the restrictions \(g_{\Sigma_E}\) resp. \(g_{\mathcal{H}^E_m}\) of \(g\) to \(\Sigma_E\) resp. \(\mathcal{H}^E_m\).

Then it is sufficient to calculate the Riemannian surface areas of the submanifolds, in order to apply the technique from above.

The following two lemmas will be used in the proof of Theorem B.

**Lemma 6.1** Let \(i : N \rightarrow P\) be a submanifold of codimension two, being the preimage of a regular value of a smooth map \(F \equiv (F_1, F_2) : P \rightarrow \mathbb{R}^2\). Then

\[
\frac{i^* \omega^{n-1}}{(n - 1)!} = \frac{\omega(X_{F_1}, X_{F_2})}{\sqrt{\|\nabla F_1\|^2 \|\nabla F_2\|^2 - g(\nabla F_1, \nabla F_2)^2}} dN,
\]

\(dN\) being the Riemannian volume form on \(N\) induced by \(g\).

**Proof:** We denote the Hodge star operator by \(\star : \Omega^k(P) \rightarrow \Omega^{2n-k}(P)\). So for \(\phi, \psi \in \Omega^k(P)\) and the Riemannian volume form \(dP\) on \(P\) we have \(\phi \wedge \star \psi = \langle \phi, \psi \rangle dP\), with the bilinear extension of

\[
\langle \alpha_1 \wedge \ldots \wedge \alpha_k, \beta_1 \wedge \ldots \wedge \beta_k \rangle := \det \left( (\alpha_i, \beta_j)_{i,j=1}^k \right) \quad (\alpha_i, \beta_j \in \Omega^1(P))
\]

and

\[
\langle \alpha_i, \beta_j \rangle := g(\alpha_i^\#, \beta_j^\#) \quad (\text{with } g(\gamma^\#, X) := \gamma(X) \text{ for } \gamma \in \Omega^1(P) \text{ and all } X \in \mathcal{X}(P)).
\]

It is well known (see, e.g. [Ba06]) that \(\star \omega^{n-k}/(n - k)! = \omega^{\wedge k}/k!\). The formula follows by applying \(\star\) to both sides, since at any \(x \in N\) the denominator \(\sqrt{\|\nabla F_1\|^2 \|\nabla F_2\|^2 - g(\nabla F_1, \nabla F_2)^2}\) is the Riemannian volume in the normal bundle of \(N\), spanned by \(\nabla F_1(x), \nabla F_2(x) \in T_x^\perp N\), whereas \(\omega(X_{F_1}, X_{F_2})(x) = \omega(\nabla F_1(x), \nabla F_2(x))\).

\(\square\)

So the Riemannian volume form \(dN\) is multiplied by the normalized Poisson bracket \(\{F_1, F_2\}\).

\(^4\)As explained in Remark 1.1.3, this is possible without loss of generality.
Lemma 6.2 (Wirtinger’s inequality)
On an $\mathbb{R}$-vector space $V$ with scalar product $\langle \cdot , \cdot \rangle$, complex structure $J : V \to V$ and Kählerian symplectic form $\omega$ one has the inequality
$$
\omega(X,Y)^2 \leq \|X\|^2\|Y\|^2 - \langle X,Y \rangle^2 \quad (X,Y \in V).
$$

Proof: We assume without loss of generality that $\|X\| = 1$. Then the formula follows from Bessel’s inequality, as $\omega(X,Y) = \langle JX,Y \rangle$, and the pair $\{X,JX\} \subseteq V$ is an orthonormal system. $\blacksquare$

Now we use Lemma 6.1 and 6.2 to prove Theorem B from Section 1, which we cite for convenience:

Theorem B. On any energy surface $\Sigma_E \subseteq P$ in a Kähler manifold it follows from Assumptions 1. and 2' for the Hamiltonian flow that $\sigma(\text{Trans}_E \cap \text{Wand}_E) = 0$.

Proof. We show $(i_{X_H}\sigma)|_{\mathcal{H}^E_m} = \eta \, d\mathcal{H}^E_m$ with a function $\eta$, that (in modulus) is bounded above by 1. Then the assertion follows from Theorem A.

Locally, $\mathcal{H}^E_m$ is the preimage of the regular value $(E,0)$ of a smooth map $(H,F) : P \to \mathbb{R}^2$. With $Y := \|\nabla H\|^2\nabla H$ we may take
$$
\sigma := i_Y \Omega,
$$
since then $dH \wedge \sigma = dH \wedge i_Y \Omega = (i_Y dH) \wedge \Omega = \Omega$, using $i_{\nabla H} dH = g(\nabla H, \nabla H) = \|\nabla H\|^2$. With $s := (-1)^{[n/2]}$ we have
$$
i_{X_H} \sigma = -\frac{s}{(n-1)!} \omega^{\wedge n-1}, \quad (6.1)
$$
since
$$
i_{X_H} \sigma = \|\nabla H\|^{-2} i_{X_H} i_{\nabla H} \Omega = s\|\nabla H\|^{-2} i_{X_H} i_{\nabla H} \frac{\omega^{\wedge n}}{n!} = s\frac{n}{n!} \frac{\omega(\nabla H, X_H)}{\|\nabla H\|^2} \omega^{\wedge n-1}.
$$

By applying Lemma 6.1, we get $i_{X_H} \sigma|_{\mathcal{H}^E_m} = \eta \, d\mathcal{H}_m$ with
$$
\eta := \frac{\omega(\nabla H, \nabla F)}{\sqrt{\|\nabla H\|^2\|\nabla F\|^2 - g(\nabla H, \nabla F)^2}}. \quad (6.2)
$$

Then $|\eta| \leq 1$ follows from Lemma 6.2, applied to the tangent spaces $T_x P$ for $x \in \mathcal{H}^E_m$. $\blacksquare$

Finally we consider the metric $g$ on the cotangent bundle $\pi : T^*M \to M$ of a Riemannian manifold $(M^n, h)$, given at $x \in T^* M$ by
$$
g_x((Q_1, P_1),(Q_2, P_2)) := h_{\pi(x)}(Q_1, Q_2) + h^{-1}_{\pi(x)}(P_1, P_2) \quad ((Q_i, P_i) \in T_x T^* M).
$$
The tangent bundle $TT^*M$ of the phase space splits as

$$TT^*M = \text{ver} \oplus \text{hor}$$

with the vertical subspace $\text{ver}_x \subseteq T_xT^*M$ given by the kernel of $T_x\pi^*$, and the horizontal subspace $\text{hor}_x \subseteq T_xT^*M$ defined by the Riemannian connection. With the natural vector bundle isomorphisms

$$I_{\text{ver}} : \text{ver} \to TM \quad \text{and} \quad I_{\text{hor}} : \text{hor} \to TM$$

(induced by $T\pi|_{\text{hor}} : \text{hor} \to TM$, respectively by the restriction of the connection, see, e.g. Klingenberg [Kli95, Proposition 1.5.10]), the vector bundle isomorphism $J : TT^*M \to T^*M$, given by

$$J_x = \left( \begin{array}{cc} 0 & I_{x,\text{ver}}^{-1} \circ I_{x,\text{hor}} \\ -I_{x,\text{hor}}^{-1} \circ I_{x,\text{ver}} & 0 \end{array} \right) \quad (x \in T^*M).$$

is an almost complex structure.\footnote{i.e., a smooth (1,1) tensor field with $J^2 = -1$.} So the symplectic manifold $(T^*M, \omega_0)$ with $J$ and $g(u,v) = \omega_0(u,Jv)$ is an almost Kähler manifold.

For a Hamiltonian $H \in C^2(T^*M, \mathbb{R})$ of the form

$$H(q,p) = \frac{1}{2}h_q^{-1}(p,p) + V(q), \quad (6.3)$$

a regular value $E \in V(\mathbb{R})$ of $V$ and a hypersurface $\mathcal{F} \subseteq M$ of finite Riemannian volume, we obtain a hypersurface in $\Sigma_E = H^{-1}(E)$ given by

$$\mathcal{H}_E := \{(q,p) \in \Sigma_E \mid q \in \mathcal{F}\} = \{(q,p) \in T_q^*M \mid h_q^{-1}(p,p) = 2(E - V(q))\}.$$

So $\mathcal{H}_E$ is of codimension two in $T^*M$. By localization, if necessary we can assume that $\mathcal{F}$ is orientable and denote by $N : \mathcal{F} \to T\mathcal{F}$ a (continuous) unit normal vector field. Then (applying $p \in T_q^*M$ to $N \in T_qM$)

$$\mathcal{H}_E^\pm := \{(q,p) \in \mathcal{H}_E \mid \pm p(N(q)) > 0\}$$

both project diffeomorphically to their common image in $T^*\mathcal{F} \subseteq T^*M$, via

$$n^\pm : \mathcal{H}_E^\pm \to T^*\mathcal{F}, \quad (q,p) \mapsto \left(q, p - p(N(q))N^\flat(q)\right).$$

The cotangent bundle $T^*\mathcal{F}$ carries the canonical symplectic form $\omega_\mathcal{F}$.\footnote{As shown by Dombrowski in [Do62, Appendix III] (for the tangent bundle $TM$ instead of the cotangent bundle $T^*M$), $J$ defines a Kähler structure on $T^*M$, if $(M,h)$ is of vanishing curvature.}
Theorem C. With respect to the embeddings \( \imath_E^\pm : \mathcal{H}_E^\pm \to T^*M \) one has
\[
(\imath_E^\pm)^*\omega_0 = (n^\pm)^*\omega_F.
\] (6.4)

So for the Riemannian volume element \( d\mathcal{F} \) induced by \( h \) on \( \mathcal{F} \subseteq M \)
\[
\int_{\mathcal{H}_E^\pm} \omega_0^{\wedge n-1} (n-1)! = \nu_{n-1} \int_{\mathcal{F}} (2(E - V(q)))^{(n-1)/2} d\mathcal{F}(q),
\] (6.5)
with the Lebesgue measure \( \nu_k \) of the \( k \)-dimensional unit ball.

Proof: To prove (6.4), we represent \( \mathcal{F} \) as a zero set of a smooth function \( f \),
defined in a neighborhood \( U \subseteq M \) of a point \( q_0 \) of \( \mathcal{F} \). We additionally require that \( df(N(q)) = +1 \) \( q \in \mathcal{F} := \mathcal{F} \cap U \). Next we multiply the phase space function \( \pi^* f : T^*U \to \mathbb{R} \) with the functions (depending on the parameter \( E \))
\[
(q,p) \mapsto \mp \sqrt{2(E - V(q)) - \|p\|_{h^{-1}}^2 + p(N(q))^2}.
\]
The resulting functions \( g^\pm : T^*U \to \mathbb{R} \) define maximal Hamiltonian flows \( \Gamma^\pm : D \to T^*U \). As \( g^\pm \) vanishes on \( T^*_F M \), there the flow lines of \( \Gamma^\pm \) coincide with those of the Hamilton flow of \( f \). This means that \( \Gamma^\pm \) acts on the fibers of \( \pi : T^*\mathcal{F} \to \tilde{\mathcal{F}} \):
\[
\Gamma^\pm_t(q,p) = \left( q, p \mp \sqrt{2(E - V(q)) - \|p\|_{h^{-1}}^2 + p(N(q))^2}N^p(q)t \right) \quad (q \in \tilde{\mathcal{F}}).
\]

Note that \( \Gamma^\pm_t \) does not change the argument of the square root. For initial conditions \( (q,p) \in \mathcal{H}_E^\pm \cap T^*U \), this square root is the modulus of the (initial) momentum component \( p(N(q)) \) in the normal direction. So on \( T^*U \) the flow \( \Gamma^\pm_t \) maps \( \mathcal{H}_E^\pm \) into the cotangent bundle \( T^*\mathcal{F} \).

Formula (6.4) then follows like in Remark 1.4.3, using that \( \omega_\mathcal{F} = (\imath_T^\ast \mathcal{F})^*\omega_0 \) for the embedding \( \imath_T^\ast : T^*\mathcal{F} \to T^*M \).

In the left hand side of (6.5) we wrote for simplicity \( \omega_0 \) instead of \((\imath_E^\pm)^*\omega_0 \). When we apply (6.4), we obtain
\[
\int_{\mathcal{H}_E^\pm} \frac{((\imath_E^\pm)^*\omega_0)^{\wedge n-1}}{(n-1)!} = \int_{\mathcal{H}_E^\pm} \frac{((n^\pm)^*\omega_F)^{\wedge n-1}}{(n-1)!} = \int_{\mathcal{H}_E^\pm} \frac{\omega_F^{\wedge n-1}}{(n-1)!}.
\]

We write \( \omega_F^{\wedge n-1} \) in the form \( d\mathcal{F} \wedge d\mathcal{G} \), where on the fiber of \( \pi : T^*\tilde{\mathcal{F}} \to \tilde{\mathcal{F}} \) over \( q \), \( d\mathcal{G}_q \) denotes the Riemannian volume element with respect to \( h_q^{-1} \). Restricted to the fiber over \( q \in \tilde{\mathcal{F}} \) the image \( n^\pm(\mathcal{H}_E^\pm) \) is a ball of radius \( 2(E - V(q)) \) with respect to \( \| \cdot \|_{h_q^{-1}} \). Integrating out the fiber yields (6.5). \( \square \)
Assume now that there is a \( h \)-orthogonal decomposition \( M = M_1 \times M_2 \) of configuration space, that is, for \( Q = (Q_1, Q_2), \ Q' = (Q'_1, Q'_2) \in T_qM \)

\[
h_{q}(Q, Q') = h_{1,q}(Q_1, Q'_1) + h_{2,q}(Q_2, Q'_2).
\]

This then extends to a \( g \)-orthogonal decomposition \( T^*M = (T^*M_1) \times (T^*M_2) \) of the symplectic manifold \( (T^*M, \omega) \). So the Hamiltonian \((6.3)\) has the form

\[
H(q, p) = T_1(q, p_1) + T_2(q, p_2) + V(q_1, q_2)
\]

with \( T_i(q, p_i) := \frac{1}{2}h_{i,q}^{-1}(p_i, p_i) \). For the foot point maps \( \pi_i : T^*(M_i) \to M_i \), the volume form \( \Omega := \omega^{\otimes n}_{M_1} \otimes \omega^{\otimes n}_{M_2} \) on \( T^*M \) can be written as \( \Omega = \pi_1^*\Omega_1 \wedge \pi_2^*\Omega_2 \) w.r.t. the volume forms \( \Omega_i := \omega_i^{\otimes n_i} \) on \( T^*M_i \).

Set \( n_i := \dim(M_i) \) so that \( n = n_1 + n_2 \). We assume that a hypersurface \( \mathcal{F} \subseteq M \) has the property that both families

\[
\mathcal{F}_1^{q_2} := \{ q_1 \in M_1 \mid (q_1, q_2) \in \mathcal{F} \} \quad (q_2 \in M_2)
\]

and

\[
\mathcal{F}_2^{q_1} := \{ q_2 \in M_2 \mid (q_1, q_2) \in \mathcal{F} \} \quad (q_1 \in M_1)
\]

consist of hypersurfaces of \( M_1 \) respectively of \( M_2 \).

**Corollary 6.3** Then, assuming finiteness of the integrals, \( \int_{\mathcal{F}} \frac{\omega^{\otimes n-1}}{(n-1)!} \) equals

\[
v_{n_2-1} \int_{T^*(M_1)} \left( \int_{\mathcal{F}_2^{q_1}} (2(E - T_1(q_1, p_1) - V(q)))^{(n_2-1)/2} d\mathcal{F}_2^{q_1}(q_2) \right) \Omega_1(q_1, p_1) + v_{n_1-1} \int_{T^*(M_2)} \left( \int_{\mathcal{F}_1^{q_2}} (2(E - T_2(q_2, p_2) - V(q)))^{(n_1-1)/2} d\mathcal{F}_1^{q_2}(q_1) \right) \Omega_2(q_2, p_2).
\]

**Proof:** Using Theorem C, we have to prove that

\[
\omega^{\wedge n-1} = \frac{n_1-1}{(n-1)!} ! \omega_1^{\wedge n_1-1} \wedge \Omega_2 + \frac{n_2-1}{(n-1)!} \omega_2^{\wedge n_2-1} \wedge \Omega_1.
\]

But as \( \omega = \pi_1^*\omega_1 + \pi_2^*\omega_2 \) and \( n - 1 - k = n_2 + (n_1 - 1) - k \)

\[
\omega^{\wedge n-1} = \sum_{k=0}^{n} \binom{n-1}{k} \pi_1^*\omega_1^{\wedge k} \wedge \pi_2^*\omega_2^{\wedge n-1-k} = \sum_{k=n_1-1}^{n} \binom{n-1}{k} \pi_1^*\omega_1^{\wedge k} \wedge \pi_2^*\omega_2^{\wedge n-1-k}.
\]

Noting that

\[
\frac{n-1}{(n-1)!} = \frac{1}{(n_1-1)!n_2!), \quad \text{and} \quad \frac{n-1}{(n-1)!} = \frac{1}{(n_2-1)!n_1!},
\]

we have proven the corollary. \( \square \)

---

6Locally this is a generic property. If this is violated, then \( \mathcal{F} \) may still have this property outside a subset of measure zero.
7 Applications and Open Questions

In this paper, we have defined the set of transition points depending on a given sequence of Poincaré surfaces. In applications, the perspective is usually vice versa: one is interested in showing the improbability of a given subset of the wandering set. If the technique presented here is ought to be applied, the task then is to find an appropriate sequence of Poincaré surfaces, such that their total area decreases to zero and that the wandering orbits under considerations can be shown to be transition points to that sequence.

One important model, where these techniques can be implemented, is the \( N \)-body problem of celestial mechanics. A subset of singular orbits (1.2) which are of special interest are collisions: they are exactly those singular orbits, where every particle has a limit position in configuration space, as time approaches the singularity. In the upcoming paper [FK18], we show the improbability of collisions in the \( N \)-body problem that may include some fixed centers as well. More generally, the result holds for a wide class of two-body interactions, including the gravitational case of celestial mechanics, but also e.g. Coulomb fields in electrostatic motion. Due to the technique devised in the present article, these estimates are optimal, concerning the power law of the two-body interactions.

For the same family of two-body interactions, also non-collision singularities can be treated in certain situations: in the upcoming paper [Fle18a] we show the improbability of non-collision singularities in systems with two free particles and an arbitrary amount of fixed centers. In the upcoming paper [Fle18b] we show the improbability of non-collision singularities in the four-body system.

As pointed out, the technique presented here can only be used to show the improbability of certain subsets of wandering orbits. However, in principle the given result can be applied to show the improbability of certain sets of non-singular wandering orbits, e.g. those for which the asymptotic velocity does not exist as a limit.

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