Dejean’s conjecture holds for $n \geq 30$

James Currie* and Narad Rampersad†
Department of Mathematics and Statistics
University of Winnipeg
515 Portage Avenue
Winnipeg, Manitoba R3B 2E9 (Canada)

Abstract

We extend Carpi’s results by showing that Dejean’s conjecture holds for $n \geq 30$.

1 Introduction

Repetitions in words have been studied starting with Thue [12, 13] at the beginning of the previous century. Much study has also been given to repetitions with fractional exponent [1, 3, 4, 5, 6, 8]. If $n > 1$ is an integer, then an $n$-power is a non-empty word $x^n$, i.e., word $x$ repeated $n$ times in a row. For rational $r > 1$, a fractional $r$-power is a non-empty word $w = x[\lfloor r \rfloor]x'$ such that $x'$ is the prefix of $x$ of length $(r - \lfloor r \rfloor)|x|$. For example, 01010 is a 5/2-power. A basic problem is that of identifying the repetitive threshold for each alphabet size $n > 1$:

What is the infimum of $r$ such that an infinite sequence on $n$ letters exists, not containing any $r$-powers?

*The author is supported by an NSERC Discovery Grant.
†The author is supported by an NSERC Postdoctoral Fellowship.
We call this infimum the repetitive threshold of an \( n \)-letter alphabet, denoted by \( RT(n) \). Dejean’s conjecture \[4\] is that

\[
RT(n) = \begin{cases} 
7/4, & n = 3 \\
7/5, & n = 4 \\
n/(n-1) & n \neq 3, 4 
\end{cases}
\]

The values \( RT(2), RT(3), RT(4) \) were established by Thue, Dejean and Pansiot, respectively \[12, 4, 11\]. Moulin-Ollagnier \[10\] verified Dejean’s conjecture for \( 5 \leq n \leq 11 \), while Mohammad-Noori and Currie \[9\] proved the conjecture for \( 12 \leq n \leq 14 \).

An exciting new development has recently occurred with the work of Carpi \[3\], who showed that Dejean’s conjecture holds for \( n \geq 33 \). Verification of the conjecture is now only lacking for a finite number of values. In the present paper, we sharpen Carpi’s methods to show that Dejean’s conjecture holds for \( n \geq 30 \).

2 Preliminaries

The following definitions are from sections 8 and 9 of \[3\]: Fix \( n \geq 30 \). Let \( m = \lfloor (n-3)/6 \rfloor \). Let \( A_m = \{1, 2, \ldots, m\} \). Let \( \ker \psi = \{ v \in A_m^+ | \forall a \in A_m, 4 \text{ divides } |v_a| \} \). (In fact, this is not Carpi’s definition of \( \ker \psi \), but rather the assertion of his Lemma 9.1.) A word \( v \in A_m^+ \) is a \( \psi \)-kernel repetition if it has period \( q \) and a prefix \( v' \) of length \( q \) such that \( v' \in \ker \psi \), \((n-1)(|v|+1) \geq nq - 3 \).

It will be convenient to have the following new definition: If \( v \) has period \( q \) and its prefix \( v' \) of length \( q \) is in \( \ker \psi \), we say that \( q \) is a kernel period of \( v \).

As Carpi states at the beginning of section 9 of \[3\]:

By the results of the previous sections, at least in the case \( n \geq 30 \), in order to construct an infinite word on \( n \) letters avoiding factors of any exponent larger than \( n/(n-1) \), it is sufficient to find an infinite word on the alphabet \( A_m \) avoiding \( \psi \)-kernel repetitions.

For \( m = 5 \), Carpi produces such an infinite word, based on a paper-folding construction. He thus establishes Dejean’s conjecture for \( n \geq 33 \). In the present paper, we give an infinite word on the alphabet \( A_d \) avoiding \( \psi \)-kernel repetitions. We thus establish Dejean’s conjecture for \( n \geq 30 \).
**Definition 1.** Let $f : A^*_4 \rightarrow A^*_4$ be defined by $f(1) = 121$, $f(2) = 123$, $f(3) = 141$, $f(4) = 142$. Let $w$ be the fixed point of $f$.

It is useful to note that the frequency matrix of $f$, i.e.,

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

has an inverse modulo 4.

**Remark 1.** Let $q$ be a non-negative integer, $q \leq 1966$. Fix $n = 32$.

*R1:* Word $w$ contains no $\psi$-kernel repetition $v$ with kernel period $q$.

*R2:* Word $w$ contains no factor $v$ with kernel period $q$ such that $|v|/q \geq 35/34$.

Note that $\frac{32}{31} - \frac{34}{31q} = \frac{35}{31} - \frac{34}{31q}$ when $q = \frac{34}{3} = 385\frac{1}{3}$, so neither piece of the remark implies the other. Note also that the conditions of the remark become less stringent for $n = 30, 31$. One also verifies that

$$\frac{35}{34} + \frac{9}{2(1967)} \leq \frac{32}{31} - \frac{34}{31q}$$

for $q \geq 1967$. To show that $w$ contains no $\psi$-kernel repetitions for $n = 30, 31, 32$, it thus suffices to verify R1 and to show that word $w$ contains no factor $v$ with kernel period $q \geq 1967$ such that

$$|v|/q \geq 35/34 + 9/2(1967). \quad (1)$$

The remarks R1 and R2 are verified by computer search, so we will consider the second part of this attack. Fix $q \geq 1967$, and suppose that $v$ is a factor of $w$ with kernel period $q$, and $|v|/q \geq 35/34$. Without loss of generality, suppose that no extension of $v$ has period $q$. Write $v = sf(u)p$ where $s$ (resp. $p$) is a suffix (resp. prefix) of the image of a letter, and $|s|$ (resp. $|p|$) $\leq 2$.

If $|v| \leq q + 2$, then $35/34 \leq (q + 2)/q$ and $1/34 \leq 2/q$, forcing $q \leq 68$. This contradicts R2. We will therefore assume that $|v| \geq q + 3$.

Suppose $|s| = 2$. Since $|v| \geq q + 3$, write $v = s0zs0v'$, where $|s0z| = q$. Examining $f$, we see that the letter $a_s$ preceding any occurrence of $s0$ in $w$ is
uniquely determined if $|s| = 2$. It follows that $a_s v$ is a factor of $w$ with kernel period $q$, contradicting the maximality of $v$. We conclude that $|s| \leq 1$.

Again considering $f$, we see that if $t$ is any factor of $w$ of length 3, and $u_1 t$, $u_2 t$ are prefixes of $w$, then $|u_1| \equiv |u_2| \pmod{3}$. Since $|v| \geq q + 3$, we conclude that 3 divides $q$. Write $q = 3q_0$. Since $|s| \leq 1$, $|p| \leq 2$ and $|v| \geq q + 3$, we see that $|f(u)| \geq q$. Thus $f(u)$ has a prefix of length $q = 3q_0$ which is in ker $\psi$. As the frequency matrix of $f$ is invertible modulo 4, the prefix of $u$ of length $q_0$ is in ker $\psi$. We see that

$$\frac{|v|}{q} \leq \frac{3|u| + 3}{3q_0} = \frac{|u|}{q_0} + \frac{1}{q_0}.$$

Lemma 2. Let $s$ be a non-negative integer. If factor $v$ of $w$ has kernel period $q$, where $q \leq 1966(3^s)$, then

$$\frac{|v|}{q} < \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{s-1} 3^{-j}.$$

Proof: If $s = 0$, this is implied by R2. Suppose $t > 0$ and the result holds for $s < t$. Suppose that $1966(3^{t-1}) < q \leq 1966(3^t)$ and there is a factor $v$ of $w$ such that $v$ has kernel period $q$. Suppose that $|v|/q \geq 35/34$. Without loss of generality, suppose that no extension of $v$ has period $q$. We have seen that there is a factor $u$ of $w$ with kernel period $q_0 = q/3$, $1966(3^{t-2}) < q_0 \leq 1966(3^{t-1})$ such that

$$\frac{|v|}{q} \leq \frac{|u|}{q_0} + 1/q_0$$

$$< \left( \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{t-2} 3^{-j} \right) + \frac{1}{q_0} \text{ (by the induction hypothesis)}$$

$$< \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{t-2} 3^{-j} + \frac{1}{1966(3^{t-2})}$$

$$= \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{t-1} 3^{-j} + \frac{3}{1966(3^{t-1})}$$

$$= \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{t-1} 3^{-j} \square$$

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Theorem 3. Word \( w \) contains no factor \( v \) with kernel period \( q \) such that
\[
\frac{|v|}{q} \geq \frac{35}{34} + \frac{9}{2(1966)}.
\]

Proof: Suppose that factor \( v \) of \( w \) has kernel period \( q \) such that (1) holds. By Remark \( \Pi \), we have \( q \geq 1966 \). By the previous lemma, for some non-negative \( s \),
\[
\frac{|v|}{q} < \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{s-1} 3^{-j} < \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{\infty} 3^{-j} = \frac{35}{34} + \frac{9}{2(1966)}. \]

Corollary 4. Dejean’s conjecture holds for \( n = 30, 31, 32 \).

Appendix: Computer search

Suppose that some factor \( v \) of \( w \) has kernel period \( q \leq 1966 \) and either
\[
31(|v| + 1) \geq 32q - 3 \quad \text{or} \quad \frac{|v|}{q} \geq \frac{35}{34} + \frac{9}{2(1967)}.
\]
Without loss of generality, taking such a \( v \) as short as possible, we may assume that
\[
|v| \leq \left\lfloor \frac{32(1966) - 3}{31} - 1 \right\rfloor = 2029.
\]
(We also have \( \left\lfloor (1966) \left( \frac{35}{34} + \frac{9}{2(1967)} \right) \right\rfloor = 2029. \))

If \( |v| > 3 \), \( v \) is a factor of \( f(u) \) for some factor \( u \) of \( w \) with \( |u| \leq (|v| + 4)/3 \).

For a non-negative integer \( r \), let \( g(r) = \lfloor (r + 4)/3 \rfloor \). Since \( g^7(2029) = 2 < 3 \), (here the exponent denotes iterated function composition) word \( v \) must be a factor of \( f^7(u) \) for some factor \( u \) of \( w \), \( |u| = 2 \).

The word \( u_0 = 23141121142 \) contains all 8 factors of \( w \) which have length 2. To establish R1 and R2, one thus checks that they hold for the single word \( f^7(u_0) \) (which is of length 24,057). In fact, we performed this computer check on the word \( f^7(u_1) \), where \( u_1 = 11421231211231411 \) contains all 13 factors of \( w \) which have length 3.

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