Constructing a $c$-function for SUSY Gauge Theories

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Recently a non-perturbative formula for the RG flow between UV and IR fixed points of the coefficient in the trace of the energy momentum tensor of the Euler density has been obtained for $N = 1$ SUSY gauge theories by relating the trace and R-current anomalies. This result is compared here with an earlier perturbation theory analysis based on a naturally defined metric on the space of couplings for general renormalisable quantum field theories. This approach is specialised to $N = 1$ supersymmetric theories and extended, using consistency arguments, to obtain the Euler coefficient at fixed points to 4-loops. The result agrees completely, to this order, with the exact formula.

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The work of Seiberg [1] and many other authors has greatly clarified the non-perturbative structure of $N = 1$ supersymmetric theories. There is considerable evidence that there exist infrared attractive fixed points in many models, where beta functions vanish and which define non-trivial superconformal invariant theories. As a consequence there has been renewed interest in the quest for a four-dimensional version of the Zamolodchikov $c$-theorem, which defines a function of the couplings for two-dimensional field theories which decreases monotonically under RG flow to the infrared limit and coincides at fixed points with the Virasoro central charge of the associated conformal theory.

In a four-dimensional theory coupled to a background metric, the external trace anomaly of the energy-momentum tensor is

$$16\pi^2 T_{\alpha}{}^\alpha = \text{operator terms} + c C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} - a R^{\alpha\beta\gamma\delta} R^*_{\alpha\beta\gamma\delta}, \quad (1)$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor and $R^*_{\alpha\beta\gamma\delta} = \frac{1}{4} \epsilon_{\alpha\beta\epsilon\eta} \epsilon^{\gamma\delta\sigma\rho} R^{\epsilon\eta}_{\sigma\rho}$, so that $R^{\alpha\beta\gamma\delta} R^*_{\alpha\beta\gamma\delta}$ is the Euler density. Cardy [2] has suggested that the Euler anomaly coefficient $a(g)$ as a function of the running couplings $g^I(\mu)$ provides the desired $c$-function. The total flow $a_{\text{UV}} - a_{\text{IR}}$ between UV and IR fixed points has been calculated in both non-supersymmetric [3] and more recently in many supersymmetric models [4,5,6] and has been found to be positive in all models for flows from the trivial asymptotic freedom UV fixed point, while $c_{\text{UV}} - c_{\text{IR}}$ has no definite sign in the models studied.

In the analysis of [5,6] the values $a_{\text{UV}}$ and $c_{\text{UV}}$ were obtained from the free field content of the asymptotically free theories studied, while $a_{\text{IR}}$ and $c_{\text{IR}}$ were related to the R-current anomalies usually computed in studies of $N = 1$ duality. At a fixed point, where the renormalisation group beta functions vanish, by virtue of superconformal invariance the R-charges of chiral fields are related to their scale dimensions so that the result for the flow of $a$ can be simply written in terms of the anomalous dimensions $\gamma_r$, at the IR fixed point, of the chiral superfields in the theory. With $r$ denoting the gauge group representation $R_r$, of dimension $\text{dim} \, R_r$, the non-perturbative formula, for all models with a unique R-current, is

$$8(a_{\text{UV}} - a_{\text{IR}}) = \sum_r \text{dim} \, R_r \, \gamma_r^2 \left(1 - \frac{2}{3} \gamma_r \right), \quad (2)$$

(Analogous formulae for the flows of $c$ and of flavour current anomalies are given in [3] but they are not relevant here). The formula (2) is a non-perturbative result, but it is a desirable check of the methodology to compare it with perturbative calculations which is the purpose of the present note. For models with a perturbatively accessible fixed point, the $\gamma_r$ and also the coefficient $a$ in (1) can be expressed as a power series in the couplings so that comparison can be made.
We now discuss the perturbative formalism of [7], in which the usual RG equations describing the variation of an arbitrary renormalisation scale \( \mu \) were extended to local rescalings of the background space-time metric. In this approach \( a \) can be calculated to high orders of perturbation theory in a general renormalisable quantum field theory, thus generalising earlier results in special cases [8]. (The present notation for the anomalies in [7] is related to that of [7] by \( a = 16\pi^2\beta_b, c = -16\pi^2\beta_a \).) The initial study [7] used dimensional regularisation, but the results were also shown [8] to follow from consistency conditions related to the commutativity of local rescalings. In a renormalisable theory with couplings \( g^I \) and associated beta functions \( \beta^I \), it was shown how to define, to all orders in perturbation theory, a metric \( g_{IJ} = g_{JI} \) on the space of couplings and associated one-form \( W_I \) so that

\[
\partial_I \tilde{a} = t_{IJ} \beta^J, \quad t_{IJ} = g_{IJ} + \partial_I W_J - \partial_J W_I, \quad \tilde{a} = a + W_I \beta^I. \tag{3}
\]

\( g_{IJ} \) and also \( W_I \) may be calculated in terms of connected one particle irreducible vacuum graphs in the background geometry and (3) then permits calculations of the perturbative expansion for \( a \). The lowest order corrections to the one loop free field results occur in general at three loops and were given in [7]. If \( W_I \) is an exact one-form then (3) represents a gradient flow [10]. Clearly at a fixed point \( \tilde{a} \) is equal to \( a \) and under renormalisation flow (3) implies

\[
\frac{dt}{dt} g^I(t) = -\beta^I(g_t) \quad \Rightarrow \quad \frac{dt}{dt} \tilde{a}(g_t) = -g_{IJ}(g_t)\beta^I(g_t)\beta^J(g_t), \tag{4}
\]

Thus, if the metric is positive, \( \tilde{a}(g_t) \) monotonically decreases, as does the Zamolodchikov c-function in two dimensions. The positivity properties of the metric are not known in general but the leading perturbative contribution is positive, so one can establish the c-theorem at least in some region of weak coupling. It should be noted that it is the associated quantity \( \tilde{a} \) which is monotonic and not the anomaly \( a \) itself (which actually increases as one flows away from the UV fixed point in some models). Further the formula (2) was derived under the condition that \( \beta \)-functions vanish, so our comparison can only be made at an IR fixed point.

In this note we apply the method of [7,8] to general \( N = 1 \) supersymmetric theories, assuming a simple gauge group \( G \) with coupling \( g \) and chiral \( G \)-invariant couplings \( Y_{ijk}, \bar{Y}^{ijk} \) so that the superpotential is \( \frac{1}{6}Y_{ijk}\phi^i\phi^j\phi^k \) for \( \phi^i \) complex chiral superfields acting on which the generators of \( G \) are \( t_a = -t_a^\dagger \). In an appendix the 3-loop results of [7] are briefly reviewed and specialised to the case where gauge, Yukawa, and quartic scalar couplings are related by supersymmetry. This gives a straightforward check of the formula (2) to 3-loop order. However the integrability conditions following from (3) allow, without any new Feynman graph calculations, the expression for the metric to be extended to
higher order and then 4-loop results for $\tilde{a}$ to be found. To achieve this we use the NSVZ beta function \cite{[11]} for the gauge coupling $g$ to three loop order.

It is convenient to eliminate factors of $4\pi$ by writing $\hat{g} = g/4\pi$, $\hat{Y}_{ijk} = Y_{ijk}/4\pi$, $\hat{Y}^{ijk} = \bar{Y}^{ijk}/4\pi$. We then restrict the results of two loop calculations in \cite{[7]}, as given in (A.1), to a supersymmetric theory with $n_V$ vector superfields and $n_S$ chiral superfields. The metric and one-form become

\[
d s^2 = g_{IJ}(g) dg^I dg^J = 4 n_V \frac{1}{\hat{g}^2} (d\hat{g})^2 (1 + \sigma \hat{g}^2) + \frac{2}{3} d\hat{Y}^{ijk} d\hat{Y}_{ijk},
\]

\[
W_I(g) dg^I = 2 n_V \frac{1}{\hat{g}} d\hat{g} (1 + \frac{1}{2} \sigma \hat{g}^2) + \frac{1}{12} (\hat{Y}^{ijk} d\hat{Y}_{ijk} + d\hat{Y}^{ijk} \hat{Y}_{ijk}).
\]

The parameter $\sigma$, which is linear in $R$ and $C$ (where $\text{tr}(t_a t_b) = -R \delta_{ab}$ and $C$ is similarly defined for the adjoint representation, $C = N$ for $G = SU(N)$), is scheme dependent. To the order given in (5) the metric is clearly flat and the one-form is exact. The leading perturbative result is also manifestly positive which should therefore remain valid at least in some region of weak coupling. The corresponding 3-loop order result (A.5) for $a$, which is independent of $\sigma$ because of cancellation between the the $\tilde{a}$ and $W_I \beta^I$ terms of (3), becomes

\[
8a = 8a_0 + 2 n_V C (3C - R) \hat{g}^4 - 4 \text{tr}(t^2 t^2) \hat{g}^4 - \text{tr}(t^2(\hat{Y} \hat{Y})) \hat{g}^2, \quad (\hat{Y} \hat{Y})^i_j = \hat{Y}^{ilm} \hat{Y}_{jlm},
\]

with $a_0$ the result for free fields, $8a_0 = \frac{1}{6}(9n_V + n_S)$. For asymptotically free theories $a_0 = a_{UV}$, the value at the UV fixed point. Applying (3) to pure SSYM theory without chiral matter gives $8a = \frac{3}{2} n_V (1 + 4C^2 \hat{g}^4)$ which increases away from the free UV fixed point. This shows that $a$ itself does not define a good $c$-function describing monotonic RG flow between fixed points. The three loop corrections in both (A.5) and (3) are proportional to the two loop terms in the gauge beta function and are scheme independent.

In this paper we require expressions for the gauge beta function to three loops and for the Yukawa beta function to two loops. At this order they are dependent on the choice of renormalisation scheme. For the gauge $\beta$-function we use the NSVZ form, which expresses $\beta^g$ to all orders in terms of the anomalous dimension matrix $\gamma^i_j$ for the chiral fields and has been verified at three and four loops in \cite{[12]}. This is here written as

\[
\frac{1}{g} \beta^g(g) = \frac{\tilde{\beta}(\hat{g})}{1 - 2C \hat{g}^2}, \quad \tilde{\beta}(\hat{g}) = -(\beta_0 - 2 \overline{tr}(\gamma t^2)) \hat{g}^2, \quad \beta_0 = 3C - R,
\]

where $\overline{tr} = n_V^{-1} \text{tr}$. The Yukawa beta function is also in general directly expressible in terms of $\gamma$, as a consequence of the non-renormalisation of the superpotential,

\[
\beta^Y_{ijk} = 3 Y_{\ell(ij} \gamma^{\ell)k}, \quad \tilde{\beta}^{Yijk} = 3 \gamma_{\ell(ij} \hat{Y}^{\ell)k}.\]
To one and two loop order using dimensional reduction

\[
\gamma^{(1)}_{ij} = P_{ij} \equiv \frac{1}{2}(\hat{Y}\hat{Y})_{ij} + 2\hat{g}^2(t^2)_{ij},
\]

\[
\gamma^{(2)}_{ij} = -\hat{Y}^{ik\ell}\hat{Y}_{jkm}P^{m}_{\ell} + 2\hat{g}^2(\hat{P}t^2)_{ij} + 2\beta_0\hat{g}^4(t^2)_{ij}.
\] (9)

From (A.9) in the same dimensional reduction scheme

\[
\sigma = -2C + 5\beta_0.
\] (10)

We first show that (6) at a perturbative fixed point agrees with (2) to 3-loop order. Using \(\hat{Y} \cdot \beta \hat{Y} = 3\text{tr}((\hat{Y}\hat{Y})\gamma)\) and the one loop form for \(\gamma\) in (9) it is easy to see that

\[
8a = 8a_0 - \text{tr}(\gamma\gamma) + 2nV\beta_0\hat{g}^4 + \frac{1}{6}\hat{Y} \cdot \beta \hat{Y}.
\] (11)

To obtain a perturbative fixed point it is necessary to choose the gauge group \(G\) and representation content of other fields so that \(\beta_0\) is effectively small. This is possible in a large \(N\) limit, where as \(N \to \infty\), the gauge group is such that \(C = O(N), nV = O(N^2)\), and \(\beta_0 > 0\) is tuned to be \(O(1)\). In this case there is a IR fixed point with \(g^2 = O(N^{-2})\) and the other couplings are given in terms of \(g_*\) so that in the case of \(N = 1\) SUSY theories discussed above \(Y_*, Y_* = O(g_*)\). In these circumstances \(\gamma = O(N^{-1})\) and \(\text{tr}(\gamma\gamma) = O(1)\) while the last two terms in (11) are of higher order in \(1/N\). It is evident therefore that (11) is compatible with (2) to leading order. Equivalently in discussing a perturbative fixed point it is legitimate to assume that the lowest order beta functions which actually appear in (11) are ‘completed’ by higher order terms, \(-\beta_0\hat{g}^2 \to \tilde{\beta}(\hat{g})(1 + O(\hat{g}^2))\), which make them vanish. This allows the last two terms in (11) to be dropped and we then observe agreement with (2) to 3-loop order.

We now demonstrate that the formalism can be extended in a fairly simple way to obtain results for \(\bar{a}\) to four-loop order, which can then be compared with the exact formula (2). The main idea is to use the integrability conditions for (3) to generate the three-loop contributions to \(g_{IJ}\) and \(W_I\) in the scheme in which the beta functions are presented above. We begin with the the \(g_{Y\bar{Y}}\) sector and then extend results to the complete metric. The compatibility of these results with the NSVZ \(\beta\)-function with independent 3 loop calculations for \(\beta^g\), up to the freedom corresponding to a change of scheme, was shown in [12, 13]. The form assumed (8) does not itself remove possible scheme dependence since taking \(\delta Y_{ij} = 3Y_{ij}(h,h^T)\), \(\delta Y^{jk} = 3h^{ij}Y^{jk}\) and using the general result for the variation of the \(\beta\)-function, \(\delta \beta^I = \beta^I \partial_I \delta^I - \delta^I \partial_I \beta^I\), gives in (8) \(\delta \gamma = \beta \cdot \partial h - \delta g \cdot \partial \gamma + [\gamma, h]\). At a fixed point then \(\delta \gamma_* = [\gamma_*, h_*]\) so that the eigenvalues of \(\gamma_*\) are invariant. However this does not maintain the NSVZ form in (7) which appears to define the renormalisation scheme uniquely.
form of possible contributions are determined by vacuum diagrams, exhibited in Fig. 1 for the cubic interactions determined by the superpotential. Hence, for suitable coefficients $\alpha, \beta, \gamma, \epsilon$ (which in general are scheme dependent), it may be written as

$$dY \cdot g_{Y\bar{Y}} \cdot d\bar{Y} = \frac{2}{3} d\hat{Y}^{ijk} d\hat{Y}_{ijk} + \alpha \text{tr}((d\hat{Y} d\bar{Y})(\hat{Y} \bar{Y})) + \beta \text{tr}((d\hat{Y} \bar{Y})(\hat{Y} d\bar{Y})) + \gamma \text{tr}((\hat{Y} d\bar{Y})(\bar{Y} d\bar{Y}) + (d\hat{Y} \bar{Y})(d\hat{Y} d\bar{Y})) + \epsilon \hat{g}^2 \text{tr}((d\hat{Y} d\bar{Y})t^2). \tag{12}$$

In a similar fashion we may extend the result in (5) for $\bar{Y}$ provided by (8) and (9) then gives an equation determining the dependence of $\hat{a}$ on $\bar{Y}$,

$$d\bar{Y} \cdot \partial_{\bar{Y}} 8\hat{a} = \frac{1}{2} \text{tr}((d\hat{Y} \bar{Y})(\hat{Y} \bar{Y})) + 2\hat{g}^2 \text{tr}((d\hat{Y} \bar{Y})t^2)$$

$$+ \frac{1}{3} \alpha d\hat{Y}^{ikl} \hat{Y}_{jkm}(\hat{Y} \bar{Y})_{l}^{m}(\hat{Y} \bar{Y})_{j}^{i} + \frac{1}{2} (\beta + 2\gamma - 1) \hat{Y}^{ikl} \hat{Y}_{jkm}(d\hat{Y} \bar{Y})_{l}^{m}(\hat{Y} \bar{Y})_{j}^{i}$$

$$+ \frac{1}{3} (\alpha + \beta + 2\gamma) \text{tr}((d\hat{Y} \bar{Y})(\hat{Y} \bar{Y})(\hat{Y} \bar{Y})) + (2\alpha + \frac{1}{2} \epsilon) \hat{g}^2 d\hat{Y}^{ikl} \hat{Y}_{jkm}(\hat{Y} \bar{Y})_{l}^{m}(t^2)_{j}^{i}$$

$$+ 4\alpha \hat{g}^2 \hat{Y}^{ikl} \hat{Y}_{jkm}(d\hat{Y} \bar{Y})_{l}^{m}(t^2)_{j}^{i} + (3\alpha + \frac{1}{4} \epsilon + 2) \hat{g}^2 \text{tr}((d\hat{Y} \bar{Y})(\hat{Y} \bar{Y})t^2)$$

$$+ 2\epsilon \hat{g}^4 d\hat{Y}^{ikl} \hat{Y}_{jkm}(t^2)_{l}^{m}(t^2)_{j}^{i} + (\epsilon + 4) \hat{g}^4 \text{tr}((d\hat{Y} \bar{Y})t^2 t^2) + 2(1 - \eta) \beta_0 \hat{g}^4 \text{tr}((d\hat{Y} \bar{Y})t^2). \tag{14}$$

The first line here follows just from (5) and the one loop result in (5). Integrability of (14) imposes the conditions

$$\beta + 2\gamma - 1 = 2\alpha, \quad \epsilon = 4\alpha. \tag{15}$$

If, in terms of the previous footnote, we take $h = \lambda(\hat{Y} \bar{Y})$ then $\delta a = \delta \beta = -4\lambda$, $\delta \gamma = -2\lambda$, while $\delta \gamma^{(2)} = 4\lambda \hat{g}^2 ((\hat{Y} \bar{Y})t^2)_{j}^{i} + 8\lambda \hat{g}^2 \hat{Y}^{ikl} \hat{Y}_{jkm}(t^2)_{l}^{m}$. 

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**Fig. 1** Two, three and four loop vacuum diagrams for chiral scalar fields. Each diagram represents a potential contribution to $a$, although the results here contain no term corresponding to the non-planar diagram, while selecting one vertex determines the possible forms for the one form $W$ and pairs of vertices the metric $g$. Since there is just a single diagram at two and three loops $W$ is exact to this order while there are three possible terms in the metric at three loops.
and then (14) then gives

\[ 8\tilde{a} = 8a_0 + \frac{1}{4} \text{tr}
\left((\hat{Y}Y)(\hat{Y}Y)\right) + 2\hat{g}^2 \text{tr}
\left((\hat{Y}Y)t^2\right)
\left(1 + (1 - \eta)\beta_0\hat{g}^2\right) \\
+ 2\alpha \hat{Y}^i k t \hat{Y}^j m P^m_i P^j + \frac{1}{12} (3\alpha + 1) \text{tr}
\left((\hat{Y}Y)(\hat{Y}Y)(\hat{Y}Y)\right) \\
+ (2\alpha + 1)\hat{g}^2 \text{tr}
\left((\hat{Y}Y)(\hat{Y}Y)t^2\right) + 4(\alpha + 1)\hat{g}^4 \text{tr}
\left((\hat{Y}Y)t^2t^2\right) + \ldots . \] (16)

It remains to consider also the dependence on the gauge coupling. To this end we
extend the metric in (5) to the form

\[ g_{gg} \text{d}g^2 = 4nV \left(\frac{1}{\hat{g}^2} \left(\text{d}\hat{g}\right)^2
\left(1 + \sigma\hat{g}^2 + \tau\hat{g}^2 \text{tr}
\left((\hat{Y}Y)t^2\right) + O(\hat{g}^4)\right)\right) , \] (17)

as well as requiring

\[ \text{d}g t_g Y \cdot \text{d}Y + \text{d}g \bar{g} Y \cdot \text{d} \bar{Y} = 2\kappa \hat{g} \text{d} \hat{g} \text{tr}
\left(\text{d}(\hat{Y}Y)t^2\right) . \] (18)

Then (3), with (7), gives

\[ g \partial_g 8\tilde{a} = 4nV \hat{\beta}(\hat{g}) \left(1 + (2C + \sigma)\hat{g}^2 + \tau\hat{g}^2 \text{tr}
\left((\hat{Y}Y)t^2\right) + O(\hat{g}^4)\right) \\
+ 2\kappa \hat{g}^2 \text{tr}
\left((\hat{Y}Y)t^2 + (\hat{Y}\hat{Y})t^2\right) . \] (19)

It is of course crucial that the \( Y, \bar{Y} \) dependent terms obtained from integrating (19) are
consistent with those already found (16). This requires, after using (10),

\[ \kappa - 2\alpha = 1 , \quad \tau - 2\eta = 3 . \] (20)

We may then calculate the purely \( \hat{g} \) dependent terms in \( \tilde{a} \) from (19) to be

\[ 8\tilde{a} = -2nV \beta_0 \hat{g}^2 \left(1 + \frac{5}{2} \beta_0 \hat{g}^2\right) + 4\hat{g}^4 \text{tr}
\left(t^2t^2\right) + \frac{16}{3} \hat{g}^6 \text{tr}
\left(t^2t^2t^2\right) + O(\beta_0 \hat{g}^6) . \] (21)

Combining (16) and (21) we can express \( \tilde{a} \) to effectively 4 loop order in the form

\[ 8\tilde{a} = 8a_0 - 2nV \beta_0 \hat{g}^2 \left(1 + \frac{5}{2} \beta_0 \hat{g}^2\right) + \text{tr}(PP) \\
+ 2\alpha \hat{Y}^i k t \hat{Y}^j m P^m_i P^j + \frac{3}{2} (3\alpha + 1) \text{tr}(PPP) - 4\alpha \hat{g}^2 \text{tr}(Ptt^2) \\
+ O(\beta_0 \hat{g}^6) + O(\beta_0 \hat{g}^4) \text{tr}(Pt^2) . \] (22)

For finite theories with zero beta functions \( \tilde{a} = a_0 \) is independent of the couplings and this
is reflected in (22) since the corrections vanish if \( \beta_0 = 0, P = 0 \) which are sufficient for a
finite theory to two loop order.
To make the connection with the results in \[^8\] we use, to the required order,

\[
\begin{align*}
n_V \tilde{\beta}(\hat{g}) + \frac{1}{6} \hat{Y} \cdot \beta \hat{Y} &= -n_V \beta_0 \hat{g}^2 + \text{tr}(PP) - \hat{Y}_j \hat{Y}_m \hat{P}_i \hat{P}_j \text{tr}(PP) \\
&+ 2 \hat{g}^2 \text{tr}(PP_1^2) + 2 \beta_0 \hat{g}^2 \text{tr}(PP_2^2),
\end{align*}
\]

and hence (22) becomes consistent with

\[
8\tilde{a} = 8a_0 - \text{tr}(\gamma \gamma) + 2 \frac{1}{6} \hat{Y} \cdot \beta \hat{Y} + \alpha \text{tr}(\hat{Y} \beta \hat{Y} P). \tag{24}
\]

At a fixed point the second line, and hence the dependence on the undetermined parameter \(\alpha\), vanishes. The result (24) is in accord with (2) taking \(\gamma\) at the IR fixed point to have eigenvalues \(\gamma_r\). It is a tribute to the consistency of quantum field theory that two such different approaches give identical results— the non-perturbative method of \[^8\] effectively uses only 1-loop triangle anomalies while the second method strictly requires 4-loop calculations in curved space which were bypassed here by use of the consistency conditions associated with (3).

As a specific illustration we consider an example based on magnetic SQCD \[^1\], as described by Kogan \textit{et al} \[^15\], which for \(G = SU(N)\) contains chiral fields \(\varphi^i, \tilde{\varphi}_i\) belonging to \(N, \tilde{N}\) representations of \(SU(N)\), with \(i = 1, \ldots N_f\) as well as a \(SU(N)\) singlet \(M^i_j\). There are then Yukawa interactions preserving \(SU(N_f) \times SU(N_f)\) symmetry represented by the superpotential \(y \tilde{\varphi}_i M^i_j \varphi^j\) and the matrix \(P\) defined in (9) has the form, in a basis for the fields given by \((M, \varphi, \tilde{\varphi})\),

\[
P = \begin{pmatrix}
N \hat{y}^2 1_{N_f} & 0 & 0 \\
0 & (N_f \hat{y}^2 - \frac{N^2 - 1}{N} \hat{g}^2) 1_{N_f} & 0 \\
0 & 0 & (N_f \hat{y}^2 - \frac{N^2 - 1}{N} \hat{g}^2) 1_{N_f}
\end{pmatrix}.
\]

For this theory \(\beta_0 = 3N - N_f\) and the lowest order formula for \(\tilde{a}\) becomes

\[
8\tilde{a} = \frac{1}{6} \left(9(N^2 - 1) + 2NN_f\right) \\
- 2(N^2 - 1)(3N - N_f) \hat{g}^2 + N^2N_f^2 \hat{y}^4 + 2 \frac{N_f}{N} (NN_f \hat{y}^2 - (N^2 - 1) \hat{g}^2)^2. \tag{26}
\]

This satisfies, to lowest order since from (11) \(ds^2 = 4(N^2 - 1)d\hat{g}^2/\hat{g}^2 + 4NN_f d\hat{y}^2\),

\[
\hat{g} \frac{\partial}{\partial \hat{g}} 8\tilde{a} = 4(N^2 - 1)\tilde{\beta}^g(\hat{g}, \hat{y}), \quad \frac{\partial}{\partial \hat{y}} 8\tilde{a} = 4NN_f^2 \beta^y(\hat{g}, \hat{y}). \tag{27}
\]

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and has a minimum at the IR fixed point

\[ NN_f \hat{g}_*^2 = \frac{2N_f}{N + 2N_f} (N^2 - 1) \hat{g}_*^2 = 3N - N_f. \]  

It is easy to show that then

\[ 8(a_{UV} - a_{IR}) = \text{tr}(P^2) = \frac{N + 2N_f}{2N_f} (3N - N_f)^2. \]  

Of course these perturbative results are only valid in a large \( N \) limit with \( N_f \) tuned to ensure \( 3N - N_f = O(1) \).

We may also note the restriction to \( N = 2 \) SUSY theories which may be defined by taking the superpotential to be \( \sqrt{2}g \eta_a \chi^T T_a \xi \) where \( \eta, \xi \) and \( \chi \) are chiral superfields transforming according to the adjoint, \( R \) and \( R^* \) representations of the gauge group. In this case there is a single coupling \( g \) with the perturbative beta function non zero only at one loop with \( \beta_0 = 2(C - R) \) where \( \text{tr}(T_a T_b) = -R \delta_{ab} \). From (4) and (10) we have

\[ g_{gg} \text{d}g^2 = 4n_V \frac{1}{g^2} (\text{d}g)^2 \left( 1 + 4 \beta_0 \hat{g}^2 \right), \]  

and, with the analogous result for \( W \), \( a = a_0 \) to three loops, independent of \( g \). The finiteness of \( N = 2 \) theories beyond one loop, in particular the vanishing of \( \gamma \), suggest that this may be true to all orders, compatible with there being no perturbative fixed point.

Appendix A.

In [7] results were described for the non supersymmetric theories. In order to show the connection with above results for SUSY models we describe briefly the necessary transcription code. For \( n_{\phi} \) real scalars \( \phi^i \) and \( n_{\psi} \) Majorana fermions \( \psi, \bar{\psi} = \psi^T C, \ C = -C^T \), the essential dimensionless couplings for a general renormalisable four dimensional theory are \( g^I = (g, \Gamma_i, \lambda_{ijk\ell}) \), with \( g \) the gauge coupling, a Yukawa interaction \( \frac{1}{2} \bar{\psi} \Gamma_i \psi \phi^i \), where \( \Gamma_i \) is a matrix linear in \( \gamma_5 \) satisfying \( CT_i = -(CT_i)^T \) (\( T_i \) is also defined by \( \gamma_{\mu} \Gamma_i = \hat{T}_i \gamma_{\mu} \)), and a quartic scalar interaction \( \frac{1}{4!} \lambda_{ijk\ell} \phi^i \phi^j \phi^k \phi^\ell \). The gauge interactions are specified by the gauge group generators \( t^a, t^a_\psi, a = 1, \ldots n_V \) acting on \( \phi, \psi \) satisfying \( t^a = -t^a_\psi T, \ C t^a_\psi C^{-1} = -i t^a_\psi T \) (where \( \gamma_{\mu} t^a_\psi = i \delta^a_\psi \gamma_{\mu} \)). For such couplings the metric calculated in [7] can be written as

\[ ds^2 = 4n_V \frac{1}{g^2} (\text{d}g)^2 \left( 1 + A \frac{g^2}{16\pi^2} \right) + \frac{1}{16\pi^2} \frac{1}{6} \text{tr}(\text{d}\hat{\Gamma}_i \text{d}\Gamma_i) + \frac{1}{(16\pi^2)^2} \frac{1}{72} \text{d}\lambda_{ijk\ell} \text{d}\lambda_{ijk\ell}. \]  

(A.1)
The associated one form to the same order was also
\[ W_I(g) dg^I = 2 n_V \frac{1}{g} \frac{dg}{g} \left( 1 + \frac{1}{2} A \frac{g^2}{16\pi^2} \right) + \frac{1}{16\pi^2} \frac{1}{24} \text{tr} (\hat{\Gamma}_i d \hat{\Gamma}_i) + \frac{1}{(16\pi^2)^2} \frac{1}{216} \lambda_{ijkl} d\lambda_{ijkl} \,. \] (A.2)

The trace in the terms involving \( \Gamma_i \) and \( t^\psi \), also involves a sum over spinor indices. Using the results for the \( \beta \)-functions
\[ \frac{1}{g} \beta^g = - \beta_0 \frac{g^2}{16\pi^2} - \frac{g^2}{(16\pi^2)^2} \left( \beta_1 g^2 - \frac{1}{4} \text{tr} (t^\psi \hat{\Gamma}_i \Gamma_i) \right) , \]
\[ \beta^\Gamma_i = \frac{1}{16\pi^2} \left( \hat{\Gamma}_j \hat{\Gamma}_i \Gamma_j + \frac{1}{2} (\hat{\Gamma}_j \hat{\Gamma}_j + 6 t^\psi \hat{\psi}^2) \Gamma_i + \Gamma_i \frac{1}{2} (\hat{\Gamma}_j \hat{\Gamma}_j + 6 t^\psi \hat{\psi}^2) + \frac{1}{4} \text{tr} (\Gamma_i \hat{\Gamma}_j) \Gamma_j \right) , \] (A.3)

where
\[ \beta_0 = \frac{1}{3} (11 C - 2 R_\psi - \frac{1}{2} R_\phi) , \quad R_\psi = - \frac{1}{4} \text{tr} (t^\psi_\alpha t^\psi_\alpha) , \quad R_\phi = - \frac{1}{4} \text{tr} (t^\phi_\alpha t^\phi_\alpha) , \]
\[ \beta_1 = \frac{1}{3} C (34 C - 10 R_\psi - R_\phi) - \frac{1}{2} \frac{1}{4} \text{tr} (t^\psi \hat{\psi}^2) - 2 \frac{1}{4} \text{tr} (t^\phi \hat{\phi}^2) , \] (A.4)

then \( a \) can be found, from (3) and the results for free fields, to three loops, independent of \( A \), to be
\[ 8a = \frac{1}{90} (124 n_V + 11 n_\psi + 2 n_\phi) + \frac{1}{(16\pi^2)^2} \left( n_V \beta_1 g^4 - \frac{1}{4} \text{tr} (t^\psi \hat{\Gamma}_i \Gamma_i) g^2 \right) . \] (A.5)

Completing the calculation in [7] to include scalar fields, using dimensional regularisation, gives
\[ A_{\text{dim.reg.}} = 17 C - \frac{10}{3} R_\psi - \frac{7}{6} R_\phi . \] (A.6)

The reduction to a general SUSY theory, as described above when \( n_\psi = n_V + n_S \), \( n_\phi = 2 n_S \), may be obtained by taking
\[ t^\phi_\alpha \rightarrow \begin{pmatrix} t_\alpha & 0 \\ 0 & -t_\alpha^T \end{pmatrix} , \quad t^\psi_\alpha \rightarrow \begin{pmatrix} t_\alpha & 0 \\ 0 & t_\alpha \end{pmatrix} P_\pm + \begin{pmatrix} t_\alpha & 0 \\ 0 & -t_\alpha^T \end{pmatrix} P_\mp , \] (A.7)

with \( t_\alpha^\text{ad} \) the adjoint representation generators and \( P_\pm = \frac{1}{2} (1 \pm \gamma_5) \), and also
\[ \hat{\Gamma}_i \rightarrow \begin{pmatrix} 0 & -\sqrt{2} g(t_a^T)_i^j \\ -\sqrt{2} g(t_a)_i^j & Y_{ijk} \end{pmatrix} P_\pm + \begin{pmatrix} 0 & \sqrt{2} g(t_a^T)_i^j \\ -\sqrt{2} g(t_a)_i^j & Y_{ijk} \end{pmatrix} P_\mp . \] (A.8)

Thus \( R_\psi = C + R, \quad R_\phi = 2 R \) and (A.1), (A.2) reduce to (3) as well as (A.5) leading to (6). To apply (A.6) we transform to dimensional reduction, as appropriate for supersymmetric perturbative calculations, using [16] \( 16\pi^2/g_{\text{reg.}}^2 = 16\pi^2/g_{\text{red.}}^2 + \frac{1}{3} C \) which gives
\[ A_{\text{dim.red.}} = A_{\text{dim.reg.}} - \frac{2}{3} C = \frac{1}{3} (39 C - 17 R) , \quad \sigma = A_{\text{dim.red.}} + \frac{2}{3} R = 13 C - 5 R . \] (A.9)

**Acknowledgements**

The research of D.Z.F. is supported in part by NSF Grant No. PHY-97-22072. Both of us are very grateful to Ian Jack for several very helpful discussions.
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