IWASAWA MAIN CONJECTURE FOR $p$-ADIC FAMILIES OF ELLIPTIC MODULAR CUSPFORMS

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ABSTRACT. In this article, we discuss Iwasawa Main Conjecture for $p$-adic families of elliptic modular cuspforms. After an overview on the situation of the ordinary case of Hida family, we introduce the Coleman map for the non-ordinary case (Coleman family) which was obtained as a joint work with Filippo Nuccio [NO16] and we give some results on Iwasawa Theory for Coleman families as applications of [NO16].

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Among the contents of the paper, Section 1 to Section 4 are intended as a review of known results. However, even the proof of a half of the divisibility of the Cyclotomic Iwasawa Main Conjecture for ordinary cusp forms which are supposed to be given by Kato [Ka04] is missing if we would like to find the totality of the result. For example, the case of ordinary cusp forms whose level is divisible by $p$ is missing in [Ka04]. Also, there are some subtle points in the formulation of the Cyclotomic Iwasawa Main Conjecture for ordinary cusp forms such as the residually reducible case and the dependence of the $p$-adic $L$-function on the choice of complex periods. Such subtle points were sometimes forgotten in most of references. We believe that Section 1 to Section 4 give a reliable reference on Iwasawa Main Conjecture for modular forms even for this classical situation

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1His proof relies on the paper [Pe94] where we assume that Galois representations are cristalline.
from the above view point. Section 4 to Section 6 are also indispensable to follow the strategy of Section 5.

Section 6 gives some new results. First, we give a construction of a two-variable \( p \)-adic \( L \)-function for a Coleman family thanks to the ingredients given in [NO16] (Theorem 5.1). Combining this result and Coleman map obtained in [NO16], we also construct Beilinson-Kato Euler systems over a Coleman family (Thorem 5.4). Finally we formulate Iwasawa Main conjecture for a Coleman family and prove the half of Iwasawa Main Conjecture (Theorem 5.8).

1. Iwasawa Main Conjecture for a Cuspidal Form \( f \) (Review on Classical Results)

Let \( p > 2 \) be a fixed prime number. Throughout the paper, we fix embeddings

\[
\mathbb{Q} \hookrightarrow \mathbb{C}, \quad \mathbb{Q}_p \hookrightarrow \mathbb{C}_p
\]

of \( \mathbb{Q} \). For any field \( L \), we denote by \( G_L \) the absolute Galois group \( \text{Gal}(\overline{L}/L) \).

Let \( f \in S_k(\Gamma_1(M)) \) be a normalized eigen cuspform of weight \( k \geq 2 \) and of level \( \Gamma_1(M) \) where \( M \) is a natural number divisible by \( p \). We usually denote by \( f = \sum_{n=1}^{\infty} a_n(f)q^n \) the Fourier expansion of \( f \) at \( i\infty \). Thus \( f \) is \( p \)-stabilized\(^2\) in this paper. We denote by \( K = K_f \) the finite extension of \( \mathbb{Q}_p \) obtained by adjoining Fourier coefficients \( a_n(f) \) to \( \mathbb{Q}_p \).

By Deligne [Del71] and Shimura [Sh68], to each normalized eigen cuspform \( f \) as above, we can attach a modular \( p \)-adic Galois representation space \( V_f = K^{\otimes 2} \) equipped with a continuous irreducible representation \( \rho_f : G_{\mathbb{Q}} \to \text{Aut}_K(V_f) \) unramified outside primes dividing \( M \). The representation \( \rho_f \) is characterized by the property

\[
\text{Tr}(\rho_f(\text{Frob}_\ell)) = a_\ell(f)
\]

for every prime \( \ell \) not dividing \( M \).

The one–variable cyclotomic Iwasawa main conjecture for each modular \( p \)-adic Galois representation \( V_f \) has been formulated and was studied by various people including Greenberg, Mazur, Perrin-Riou, Kato and Skinner–Urban.

1.1. Selmer group and \( p \)-adic \( L \)-function. When \( f \) is ordinary at \( p \) in the sense that \( a_p(f) \) is a \( p \)-adic unit, Wiles proved that the representation \( V_f \) is reducible as \( G_{\mathbb{Q}_p} \)-module giving a canonical submodule \( F_p^+V_f \subset V_f \) stable by the action of \( G_{\mathbb{Q}_p} \). For any \( G_{\mathbb{Q}_p} \)-stable lattice \( T \subset V_f \) and for each number field \( F \), a Selmer group \( \text{Sel}_A(F) \) for \( A := V_f/T \) over \( F \) is defined as a subgroup of the Galois cohomology \( H^1(F,A) \) according to Greenberg [Gr89]:

\[
\text{Sel}_A(F) = \text{Ker} \left[ H^1(F,A) \to \prod_{\lambda \mid p} H^1(I_{\lambda}, A) \times \prod_{\ell \mid p} \frac{H^1(I_p,A)}{\text{image}(H^1(I_p,F_p^+A))} \right]
\]

\(^2\)An eigenform \( f \) is said to be \( p \)-stabilized either if \( f \) is new at \( p \) or if there exists an eigenform \( g \) of weight \( k \) whose level is prime to \( p \) such that \( f = g(g) - \beta g(q^p) \) where \( \beta \) is one of roots of \( X^2 - a_p(g)X + \psi_f(p)p^{k-1} \). Here \( \psi_f \) is the Neben character of \( f \). In the latter case, we have \( a_n(f) = a_n(g) \) for all natural number \( n \) prime to \( p \), which implies that the \( p \)-adic Galois representation \( V_g \) associated to \( g \) is isomorphic to the \( p \)-adic Galois representation \( V_f \) associated to \( f \). Thus the assumption of being \( p \)-stabilized is not restrictive at all.
where $I_\lambda$ (resp. $I_p$) means the inertia subgroup at each place $\lambda$ (resp. $p$) not dividing $p$ (resp. dividing $p$).

**Lemma 1.1.** When $F$ is a finite extension over $\mathbb{Q}$, the Pontrjagin dual $\text{Sel}_A(F)^\vee$ of $\text{Sel}_A(F)$ is a finitely generated $\mathbb{Z}_p$-module.

This lemma can be proved by reducing the problem to Hermite’s theorem on the finiteness of the number of the extensions of a given number fields whose ramified places and the degrees are fixed. Since the argument of the proof is well-known and found in [Gr94 §4] for example, we omit the proof.

When $F = \mathbb{Q}(\mu_{p^\infty})$, the Galois group $G_{\text{cyc}} := \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ acts on $\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))$ thanks to the functoriality of Galois cohomology. Since the Pontrjagin dual $\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee$ is a compact module with continuous action of $G_{\text{cyc}}$, the module $\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee$ naturally has a structure of a module over the semi-local algebra $\Lambda_{\text{cyc}} := \mathbb{Z}_p[[G_{\text{cyc}}]]$. We note that $\Lambda_{\text{cyc}}$ is a semi-local algebra which is isomorphic to $\mathbb{Z}_p[[X]] \times \cdots \times \mathbb{Z}_p[[X]]$ ($p - 1$ components).

We have also the following lemma.

**Lemma 1.2.** $\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee$ is finitely generated over $\Lambda_{\text{cyc}}$.

**Proof.** The proof for the fact that $\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee$ is finitely generated over $\Lambda_{\text{cyc}}$ is reduced to the fact that $\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee/(g - 1)\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee$ is finitely generated over $\mathbb{Z}_p = \Lambda_{\text{cyc}}/(g - 1)$ by using Nakayama’s lemma, where $g$ is a topological generator of $G_{\text{cyc}}$. Then, by the control theorem of Selmer group (see [Och01 Proposition 3.8]), the fact that $\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee/(g - 1)\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee$ is finitely generated over $\mathbb{Z}_p$ is reduced to the fact that $\text{Sel}_A(\mathbb{Q})^\vee$ is finitely generated over $\mathbb{Z}_p$, which is the conclusion of Lemma 1.1. This completes the proof. □

On the other hand, the following theorem relies essentially on the technique of Euler system.

**Theorem A (Torsion property of the Selmer group of $f$/Kato–Rubin–Rohrlich)**

Let $f$ be an ordinary eigen cuspform of weight $k \geq 2$. Assume that $p \geq 5$. Then the finitely generated $\Lambda_{\text{cyc}}$-module $\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee$ is torsion over $\Lambda_{\text{cyc}}$.

**Remark 1.3.** In ([1.2]), we required $f$ to be ordinary in order to define the Selmer group. However, there is another Selmer group $\text{Sel}_A^{\text{BK}}(\mathbb{Q}(\mu_{p^\infty}))$ defined by using the local condition $H_f^1$ of Bloch-Kato (see [BK90 Section 3]), which is almost equal to the above $\text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))$ and is also valid when $V_f$ is non-ordinary. When the form $f$ is non-ordinary, Lemma 1.2 is also applied also to see that $\text{Sel}_A^{\text{BK}}(\mathbb{Q}(\mu_{p^\infty}))^\vee$ is a finitely generated $\Lambda_{\text{cyc}}$-module. However, it is known that $\text{Sel}_A^{\text{BK}}(\mathbb{Q}(\mu_{p^\infty}))^\vee$ is not a torsion $\Lambda_{\text{cyc}}$-module. Let us

3We exclude $p = 3$ only because we did not find a complete reference for CM case.
4Note that the algebra $\Lambda_{\text{cyc}}$ is not local, but a product of $p - 1$ copies of local domain $\mathbb{Z}_p[[X]]$. A $\Lambda_{\text{cyc}}$-module $M$ is said to be torsion when the base extension of $M$ to each local component $\mathbb{Z}_p[[X]]$ is torsion.
recall the following exact sequence obtained by the global duality theorem of Poitou-Tate:

\[
\lim_{\nu} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)) \longrightarrow \text{Sel}_{A}^{BK} (\mathbb{Q}(\mu_{p^n}))^\vee \longrightarrow \lim_{\nu} H^2(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1))
\]

where we denote by \( T^* \) the \( \mathbb{Z}_p \)-linear dual of \( T \) and by \( T^*(1) \) the Tate twist of \( T^* \). Let \( K \) be a finite extension of \( \mathbb{Q}_p \) obtained by adjoining Fourier coefficients of \( f \) and \( \mathcal{O}_K \) the ring of integers in \( K \). By using the semi-global Euler–Poincaré characteristic formula of the Galois cohomology theory, the \( \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \)-module \( \lim_{\nu} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)) \) is generically of rank one over \( \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \). On the other hand, by a result of Berger [Ber05 Théorème A], we have

\[
\lim_{\nu} H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1)) \simeq \lim_{\nu} H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1)).
\]

By using the local Euler–Poincaré characteristic formula of the Galois cohomology theory, the \( \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \)-module \( \lim_{\nu} H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1)) \) is generically of two one over \( \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \). Hence, we conclude that \( \text{Sel}_{A}^{BK} (\mathbb{Q}(\mu_{p^n}))^\vee \) is not a torsion \( \Lambda_{\text{cyc}} \)-module by (1.3). We remark that, as we will discuss later in the proof of the inequality (1.10) in Theorem C, a result of Kato [Ka04] implies that the first map in (1.3) is injective and the last term \( \lim_{\nu} H^2(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)) \) in (1.3) is \( \Lambda_{\text{cyc}} \)-torsion. By (1.3), this implies that \( \text{Sel}_{A}^{BK} (\mathbb{Q}(\mu_{p^n}))^\vee \) is generically of two one over \( \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \) more precisely.

**Proof of Theorem A.** We remark that the proof is completely different depending on whether the cuspsform \( f \) has complex multiplication (in the sense of Ribet [Rib77]) or not.

When the cuspsform \( f \) has complex multiplication, the result is due to Rubin and Rohrlich. Rubin studied Euler systems of elliptic units which are constructed in the Galois cohomology of the Galois representations associated to cuspsforms with complex multiplication. An Euler system of elliptic units is related to a \( p \)-adic \( L \)-function of \( f \) stated below and this \( p \)-adic \( L \)-function of \( f \) is known to be nontrivial (nonzero) due to Rohrlich [Ro84]. Hence, Euler systems of elliptic units are nontrivial. On the other hand, Rubin proves that the size of the Selmer group is bounded by an Euler system of elliptic units. Thanks to the nontriviality of the Euler system, we conclude that the Pontrjagin dual of the Selmer group is \( \Lambda_{\text{cyc}} \)-torsion.

When the cuspsform \( f \) does not have complex multiplication, we can use an Euler system called Beilinson-Kato Euler system, which was constructed by Kato in the Galois cohomology of the Galois representations associated to any cuspsforms. By the same reasoning as the above, Euler systems of Beilinson-Kato are nontrivial thanks to Rohrlich. Under the condition that the cuspsform \( f \) does not have complex multiplication, Kato proves that the Pontrjagin dual of the Selmer group is torsion. \( \square \)

In order to introduce the \( p \)-adic \( L \)-function, we recall the following result which is due to Shimura [Sh76].

**Theorem (algebraicity of special values/Shimura)**
There exist complex numbers \( \Omega_{+}^{+}, \Omega_{-}^{-} \in \mathbb{C}^* \) (called **complex periods**) such that we
have

\[
\frac{L(f, \phi^{-1}, j)}{(2\pi)^j \Omega_{f, \infty}^{\text{sgn}(j, \phi)}} \in \mathbb{Q}_f[\phi]^\times
\]

for any integer \( j \) satisfying \( 1 \leq j \leq k - 1 \) and for any Dirichlet character \( \phi \), where \( \text{sgn}(j, \phi) \) denotes the sign of \((-1)^{(j-1)\phi(-1)}\), \( \mathbb{Q}_f \) means the Hecke field of \( f \) and \( \mathbb{Q}_f[\phi] \) is the field obtained by adjoining values of \( \phi \).

**Remark 1.4.**

1. When \( f \) is an eigen cuspidal form of level \( \Gamma_1(M) \). The periods \( \Omega_{f, \infty}^+ \) and \( \Omega_{f, \infty}^- \) can be obtained via “determinant of geometric comparison isomorphism”.

\[
\text{Fil}^k H_{\text{dr}}^1(X_1(M), \omega^{\otimes k_2-2})[f] \otimes_{\mathbb{Q}_f} \mathbb{C} \cong H_c^1(Y_1(M) \mathbb{C}, \mathcal{L}_{k_2-2}(\mathbb{Q}_f))^{\pm}[f] \otimes_{\mathbb{Q}_f} \mathbb{C}
\]

where \( \omega \) is a standard coherent sheaf on \( X_1(M) \), obtained from the push-forward of the sheaf of relative differential forms on the universal elliptic curve on \( Y_1(M) \) and \( \mathcal{L}_{k_2-2}(\mathbb{Q}_f) \) is the standard local system of rank \( k - 1 \) over \( Y_1(M) \mathbb{C} \). The symbol \([f]\) means cutting out the rank-one \( \mathbb{Q}_f \)-subspace of the cohomology on which the Hecke operators act via Fourier coefficients of \( f \). The period can be defined by:

\[
d_f \otimes 1 = \Omega_{f, \infty}^+ \cdot b_f^+ \otimes 1
\]

where \( d_f \) is a canonical basis of \( \text{Fil}^k H_{\text{dr}}^1(X_1(M), \omega^{\otimes k_2-2})[f] \) given by \( f \) and \( b_f^+ \) is a basis of a rank-one \( \mathbb{Q}_f \)-space \( H_c^1(Y_1(M) \mathbb{C}, \mathcal{L}_{k_2-2}(\mathbb{Q}_f))^{\pm}[f] \). Since there are no canonical choices of \( b_f^+ \), in general, \( \Omega_{f, \infty}^+ \) are defined only modulo multiplication by elements of \( \mathbb{Q}_f^\times \) they make sense as an element of \( \mathbb{C}^\times / \mathbb{Q}_f^\times \).

2. The algebraicity theorem gives a supporting example of Deligne conjecture (see [Del79]) on special values of the \( L \)-function associated to the motif of \( f \).

3. Let \( \mathbb{Z}_{f,(p)} \) be the localization of the ring of integers \( \mathbb{Z}_f \) of \( \mathbb{Q}_f \) at the prime over \( p \) of \( \mathbb{Z}_f \) induced by the fixed inclusion \( \mathbb{Q}_f \hookrightarrow \mathbb{Q}_p \). Then \( H_c^1(Y_1(M) \mathbb{C}, \mathcal{L}_{k_2-2}(\mathbb{Z}_{f,(p)})\hat{\otimes}[f] \) is a free module of rank one over the discrete valuation ring \( \mathbb{Z}_{f,(p)} \). When we choose \( b_f \) to be a \( \mathbb{Z}_{f,(p)} \)-basis of \( H_c^1(Y_1(M) \mathbb{C}, \mathcal{L}_{k_2-2}(\mathbb{Z}_{f,(p)})\hat{\otimes}[f] \), periods \( \Omega_{f, \infty}^+, \Omega_{f, \infty}^- \in \mathbb{C}^\times \) are called \( p \)-optimal complex periods. Such \( p \)-optimal complex periods \( \Omega_{f, \infty}^+, \Omega_{f, \infty}^- \) are unique modulo multiplication by elements in \( (\mathbb{Z}_{f,(p)})^\times \) and hence \( \Omega_{f, \infty}^+ \) and \( \Omega_{f, \infty}^- \) make sense as an element of \( \mathbb{C}^\times / (\mathbb{Z}_{f,(p)})^\times \).

The \( p \)-adic orders of the values \( \{\Omega_{f, \infty}^+\} \) are believed to match with the generalized Birch and Swinnerton-Dyer conjecture for a Galois stable lattice of \( V_f \) when we choose \( \Omega_{f, \infty}^+, \Omega_{f, \infty}^- \) to be \( p \)-optimal.

Recall that \( K \) is the completion of \( \mathbb{Q}_f \) in \( \mathbb{Q}_p \). Now we recall the \( p \)-adic \( L \)-function due to Amice-Vélu [AV75] and Visik [V76] following the modular symbol method developed by Manin and Mazur.

**Theorem B (Existence of one-variable ordinary \( p \)-adic \( L \)-function)**

Let \( f \) be an ordinary eigen cuspidal form of weight \( k \geq 2 \). Let us choose complex periods \( \Omega_{f, \infty}^+ \) and \( \Omega_{f, \infty}^- \). Then, there exists an analytic \( p \)-adic \( L \)-function \( L_p(\{\Omega_{f, \infty}^\pm\}) \in \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} K \)
such that we have the following interpolation formula.\footnote{The both sides of the equation are in \( \mathbb{Q} \) thanks to algebraicity theorem of Shimura stated above.}

\begin{equation}
\chi_{\text{cyc}}(L_p(\{\Omega_{f,\infty}^\pm\})) = (-1)^j(j - 1)! \times e_p(f, j, \phi) \times \tau(\phi) \times \frac{L(f, \phi^{-1}, j)}{(2\pi \sqrt{1})^j\Omega_{f,\infty}^{\text{reg}(j,\phi)}},
\end{equation}

for any integer \( j \) satisfying \( 1 \leq j \leq k - 1 \) and for any Dirichlet character \( \phi \) whose conductor \( \text{Cond}(\phi) \) is a power of \( p \), where \( \tau(\phi) \) means a Gauss sum \( \sum_{i=1}^{\text{Cond}(\phi)} \phi(i) \left( \exp \frac{2\pi \sqrt{-1}}{\text{Cond}(\phi)} \right)^i \) and the factor \( e_p(f, j, \phi) \) is as follows:

\[ e_p(f, j, \phi) = \begin{cases} 1 - \frac{p^{j-1}}{a_p(f)} & \text{when } \phi = 1, \\ \left( \frac{p^{j-1}}{a_p(f)} \right)^{\text{ord}_p(\text{Cond}(\phi))} & \text{when } \phi \neq 1. \end{cases} \]

**Remark 1.5.**

1. There are several constructions of \( L_p(\{\Omega_{f,\infty}^\pm\}) \) other than the method of modular symbol mentioned above. First, thanks to Hida, Panchishkin and Dabrowski, we should be able to construct \( L_p(\{\Omega_{f,\infty}^\pm\}) \) by Rankin-Selberg method using Eisenstein series. Also, \( L_p(\{\Omega_{f,\infty}^\pm\}) \) is constructed from a Beilinson-Kato Euler system via a generalized Coleman map (see \cite{Ka04} chap. IV). Finally, thanks to Coates-Wiles, de Shalit and Yager, when the cuspform \( f \) has complex multiplication, \( L_p(\{\Omega_{f,\infty}^\pm\}) \) is constructed from an Euler system of elliptic units via a generalized Coleman map.

2. We expect that, when the complex periods \( \Omega_{f,\infty}^+ \) and \( \Omega_{f,\infty}^- \) are \( p \)-optimal, the \( p \)-adic \( L \)-function \( L_p(\{\Omega_{f,\infty}^\pm\}) \) is integral in the sense that \( L_p(\{\Omega_{f,\infty}^\pm\}) \in \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \) where \( \mathcal{O}_K \) is the ring of integers in \( K \).

3. When the weight \( k \) of \( f \) is greater than 2, the value \( j = k - 1 \) among the critical values \( 1 \leq j \leq k - 1 \) is inside the convergence region \( \text{Re}(s) > \frac{k-1}{2} \) of the Euler product expansion of the \( L \)-function \( L(f, \phi^{-1}, s) \). By the definition of the convergence of infinite product, \( L_p(\{\Omega_{f,\infty}^\pm\}) \) is nonzero thanks to the interpolation property.\footnote{When \( k = 1 \), the value \( j = k - 1 \) is on the border of the convergence region. In this case, Jacquet-Shalika \cite{JS76} shows that \( L(f, \phi^{-1}, 2) \) is always nonzero for \( k = 3 \).} When \( k = 2 \), the only critical value \( L(f, \phi^{-1}, 1) \) can be sometimes zero. Hence, a priori, it is not clear if \( L_p(\{\Omega_{f,\infty}^\pm\}) \) is nonzero. However, Rohrlich \cite{Ro84} proves that, for any eigen cuspform \( f \) of weight 2, there exists a Dirichlet character \( \phi \) whose conductor \( \text{Cond}(\phi) \) is a power of \( p \) such that we have \( L(f, \phi^{-1}, 1) \neq 0 \). Hence, \( L_p(\{\Omega_{f,\infty}^\pm\}) \) is always nonzero as an element of \( \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} K \).

**1.2. Iwasawa Main Conjecture.** With the preparation above, we recall the one-variable cyclotomic Iwasawa Main conjecture for elliptic cuspforms:

**Conjecture (Cyclotomic Iwasawa Main Conjecture for \( f \))**
Let \( f \) be an ordinary eigen cuspform of weight \( k \geq 2 \). Let \( T \subset V_f \) be a Galois stable lattice and we choose complex periods \( \Omega_{f,\infty}^+, \Omega_{f,\infty}^- \) to be \( p \)-optimal with respect to the lattice \( T \). Then, we have the following equality of principal ideals in the ring \( \Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \):

\[
(1.8) \quad (L_p(\{\Omega_{f,\infty}^\pm\})) = \text{char}_{\Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K} \text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee
\]

where \( A := V_f/T \).

**Remark 1.6.** The right-hand side of (1.8) is nonzero thanks to Theorem A introduced above. Also, the left-hand side of (1.8) is nonzero thanks to Rohrlich’s result [Ro84] explained in Remark 1.5.

Before introducing the result on the above conjecture, we recall the relation between the ideals of the ring \( \Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \) and the ideals of the ring \( \Lambda_{cyc} \otimes_{\mathbb{Z}_p} K \). Since \( \Lambda_{cyc} \otimes_{\mathbb{Z}_p} K \) is a localization of \( \Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \) with respect to the multiplicative set \( \{\varpi^n\}_{n \in \mathbb{Z}} \) with \( \varpi \) a uniformizer of \( \mathcal{O}_K \), we have an injection

\[
(\text{prime ideals in } \Lambda_{cyc} \otimes_{\mathbb{Z}_p} K) \hookrightarrow (\text{prime ideals in } \Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K)
\]

and the image coincide with prime ideals which do not contain \( \varpi(\Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K) \). For a finitely generated torsion \( \Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \)-module \( M \), by the functor

\[
\text{char}_{\Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K} M \mapsto \text{char}_{\Lambda_{cyc} \otimes_{\mathbb{Z}_p} K} (M \otimes_{\mathcal{O}_K} K),
\]

we lose the information on prime factors containing \( \varpi(\Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K) \). The following result is known:

**Theorem C (Cyclotomic Iwasawa Main Conjecture for \( f \))**

Let \( f \) be an ordinary eigen cuspform of weight \( k \geq 2 \). We fix a Galois stable lattice \( T \subset V_f \) and we choose complex periods \( \Omega_{f,\infty}^+, \Omega_{f,\infty}^- \) to be \( p \)-optimal with respect to the lattice \( T \). Under certain assumptions (on the prime number \( p \) and the level of modular forms etc), we have the following equality of principal ideals in the ring \( \Lambda_{cyc} \otimes_{\mathbb{Z}_p} K \):

\[
(1.9) \quad (L_p(\{\Omega_{f,\infty}^\pm\})) = \text{char}_{\Lambda_{cyc} \otimes_{\mathbb{Z}_p} K} \text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee \otimes_{\mathcal{O}_K} K.
\]

**Remark 1.7.** In the known case of (1.9), the proof is completely different depending on whether the cuspform \( f \) has complex multiplication or not.

1. When \( f \) has complex multiplication by an imaginary quadratic field \( F \), the equality (1.9) is obtained by specializing the two-variable Iwasawa Main conjecture for \( F \) proved by Rubin [Ru91] at certain height-one prime of the two-variable Iwasawa algebra for \( F \). We refer the reader to [HO, Theorem A] for this "descent" from two-variable to one-variable.

The reason why we only have a weaker equality over \( \Lambda_{cyc} \otimes_{\mathbb{Z}_p} K \) (not over \( \Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \)) is the difficulty on analytic side due to the lack of knowledge on the periods. It is not known whether the ratio of a \( p \)-optimal complex period in the context of complex multiplication and a \( p \)-optical complex period in the above context becomes a \( p \)-adic unit (see [HO, Conj. 2.26]). Thus we can compare the
ideal of a $p$-adic $L$-function obtained in [HO, Theorem A] and the ideal of a $p$-adic $L$-function obtained in Theorem B above only modulo a power of $\varpi$.

(2) When $f$ does not have complex multiplication, Kato [Ka04] proved the following inclusion of principal ideals in the ring $\Lambda_{cyc} \otimes_{\mathbb{Z}_p} K$ using a Beilinson-Kato Euler system:

\[
(L_p(\{\Omega^+_{f,\infty}\})) \subset \text{char}_{\Lambda_{cyc} \otimes_{\mathbb{Z}_p} K} (\text{Sel}_A(\mathbb{Q}(\mu_{p^n}))^\vee \otimes_{\mathcal{O}_K} K).
\]

We note that, under the condition that the residual representation is full, Kato proves a stronger result than (1.10) and he obtains the following inclusion of principal ideals in the ring $\Lambda_{cyc} \otimes_{\mathbb{Z}_p} \mathcal{O}_K$:

\[
(L_p(\{\Omega^+_{f,\infty}\})) \subset \text{char}_{\Lambda_{cyc} \otimes_{\mathbb{Z}_p} K} (\text{Sel}_A(\mathbb{Q}(\mu_{p^n}))^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_K).
\]

On the other hand, Skinner-Urban [SU14] takes a completely different approach of Eisenstein ideal for $U(2, 2)$. Under certain conditions, they claim the following opposite inclusion of principal ideals in the ring $\Lambda_{cyc} \otimes_{\mathbb{Z}_p} K$:

\[
(L_p(\{\Omega^+_{f,\infty}\})) \supset \text{char}_{\Lambda_{cyc} \otimes_{\mathbb{Z}_p} K} (\text{Sel}_A(\mathbb{Q}(\mu_{p^n}))^\vee \otimes_{\mathcal{O}_K} K).
\]

We remark that cuspforms $f$ with complex multiplication are excluded in their result since they need to assume that there exists a prime number $l$ which divides the level of $f$ exactly.

We discuss a further generalization of the above result in later sections, especially a generalization of (1.10). For this purpose, it is important to look into the proof of (1.10) in Theorem C (In fact, we prove Theorem A at the same time under the setting of (1.10)).

**Brief idea of the proof of the inequality (1.10) in Theorem C.** Let $\Sigma$ be a finite set of primes of $\mathbb{Q}$ which contain $\{p\}, \{\infty\}$ and the places dividing the level of the cuspform $f$. Let $\mathbb{Q}_\Sigma$ be the maximal Galois extension of $\mathbb{Q}$ which is unramified outside $\Sigma$.

We denote by $\text{loc}_p$ the $\Lambda_{cyc}$-linear homomorphism:

\[
\lim_n H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)) \longrightarrow \lim_n H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1))
\]

and we denote by $\text{loc}_p^*$ the $\Lambda_{cyc}$-linear homomorphism obtained by composing the following $\Lambda_{cyc}$-linear homomorphism after $\text{loc}_p$:

\[
\lim_n H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1)) \longrightarrow \lim_n \frac{H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1))}{\text{Im}(H^1(\mathbb{Q}_p(\mu_{p^n}), F_p T^*(1)))}.
\]

Then we have the following sequence of $\Lambda_{cyc}$-modules:

\[
\begin{align*}
0 & \longrightarrow \lim_n H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)) \overset{\text{loc}_p}{\longrightarrow} \lim_n \frac{H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1))}{\text{Im}(H^1(\mathbb{Q}_p(\mu_{p^n}), F_p T^*(1)))} \\
& \quad \longrightarrow \text{Sel}_A(\mathbb{Q}(\mu_{p^n}))^\vee \\
& \quad \longrightarrow \lim_n H^2(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)) \longrightarrow 0
\end{align*}
\]

such that

(i) The sequence (1.12) $\otimes_{\mathcal{O}_K} K$ is exact.

(ii) The last two terms are torsion over (every local component of) $\Lambda_{cyc}$.

(iii) The first two terms are of rank one over (every local component of) $\Lambda_{cyc}$.
Here, the assertion (i) is nothing but the Poitou-Tate exact sequence of the Galois cohomology except that the injectivity of the map $\text{loc}_p$ is due to Kato's result on Beilinson-Kato Euler system in [Ka04]. Usually the Pontrjagin dual of $H^2(\mathbb{Q}(\mu_{p^n}), A)$ contributes to the kernel of $\text{loc}_p$. However Kato's result assures that this group is cotorsion $\Lambda_{\text{cyc}}$-module. Since the cohomological dimension of $G_{\mathbb{Q}(\mu_{p^n})}$ is two, the Pontrjagin dual of $H^2(\mathbb{Q}(\mu_{p^n}), A)$ is $p$-torsion free. This proves that $H^2(\mathbb{Q}(\mu_{p^n}), A) = 0$.

The assertion (ii) is also a consequence of deep result by Kato's work on Beilinson-Kato Euler system in [Ka04]. Usually the Pontrjagin dual of $\mathbb{Q}$ is equal to $\mathbb{Z}$. Here, the assertion (i) is nothing but the Poitou-Tate exact sequence of the Galois cohomology except that the injectivity of the map $\text{loc}_p$ is due to Kato's result on Beilinson-Kato Euler system in [Ka04].

For the assertion (iii), the rank of the first term is obtained as an application of the semi-global Euler-Poincare characteristic formula of Galois cohomology as well as the fact that $\lim_{n \to \infty} H^2(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1))$ is torsion, the rank of the second term is obtained as an application of the local Euler-Poincare characteristic formula of Galois cohomology.

Now, we take the (first layer of the) Beilinson–Kato Euler system

$$\lim_{n \to \infty} z_n \in \lim_{n \to \infty} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)).$$

Then the sequence (1.12) induces the following sequence of $\Lambda_{\text{cyc}}$-modules:

$$0 \longrightarrow \lim_{n \to \infty} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)) / \Lambda_{\text{cyc}} \lim_{n \to \infty} z_n \longrightarrow \lim_{n \to \infty} \frac{H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1))}{\text{loc}_p(z_n)} \longrightarrow \text{Sel}_A(\mathbb{Q}(\mu_{p^n}))^\vee \longrightarrow \lim_{n \to \infty} H^2(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)) \longrightarrow 0,$$

which becomes exact after the base extension $\otimes_{\mathcal{O}_K} K$. Now, the following theorem is a special case of general machinery of Euler system bound established by Kato [Ka99], Perrin-Riou [Pe98], and Rubin [Ru00] independently for Galois representations of arbitrary rank:

**Theorem 1.8** (Kato, Perrin-Riou, Rubin). *Let us assume the setting of Theorem C of §1. Suppose further that the following conditions are satisfied:

1. The first layer of a given Euler system $\lim_{n \to \infty} z_n \in \lim_{n \to \infty} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1))$ is not contained in the $\Lambda_{\text{cyc}}$-torsion part of $\lim_{n \to \infty} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1))$.
2. The Galois representation $G_{\mathbb{Q}} \to \text{Aut}_{\mathbb{Z}_p} \hat{V}_f \cong GL_2(\mathbb{Z}_p)$ contains a nontrivial unipotent element.

Then, $\lim_{n \to \infty} H^2(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1))$ is a torsion $\Lambda_{\text{cyc}}$-module and we have:

$$\lim_{n \to \infty} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)) / \Lambda_{\text{cyc}} \lim_{n \to \infty} z_n \otimes_{\mathcal{O}_K} K \subseteq \text{char}_{\Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} K} \lim_{n \to \infty} H^2(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1)) \otimes_{\mathcal{O}_K} K.$$

On the other hand, we have the following theory of Coleman map:

**Theorem 1.9** ([Och03], [Pe94]). *Let us assume the setting of Theorem C of §1.*

\footnote{This condition is equivalent to the condition that $f$ has no complex multiplication.}
(1) There is a $\Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} K$-linear homomorphism:

$$\text{Col} : \left( \lim_{\longrightarrow n} H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1)) \right) \otimes_{\mathcal{O}_K} K \longrightarrow \text{Fil}^0 D_{\text{dR}}(V_f^* (1)) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$$

such that we have the following commutative diagram for any positive integer $j$ and for any Dirichlet character $\phi$ whose conductor $\text{Cond}(\phi)$ is a power of $p$:

$$\begin{array}{ccc}
\left( \lim_{\longrightarrow n} H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1)) \right) \otimes_{\mathcal{O}_K} K & \xrightarrow{\text{Col}} & \text{Fil}^0 D_{\text{dR}}((V_f)^*(1)) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}} \\
\chi_{\text{cyc}} j \phi^{-1} & \downarrow & \chi_{\text{cyc}} j \phi^{-1} \\
H^1(\mathbb{Q}_p, (V_f \otimes \phi)^*(1 - j)) & \xrightarrow{\text{exp}^*} & \text{Fil}^0 D_{\text{dR}}((V_f \otimes \phi)^*(1 - j))
\end{array}$$

where $\text{exp}^*$ is the dual exponential map of Bloch-Kato and $e_p(f, j, \phi)$ is as follows:

$$e_p(f, j, \phi) = \begin{cases} 
(-1)^j (j - 1)! \left(1 - \frac{p^j - 1}{a_p(f)}\right) & \text{when } \phi = 1, \\
(-1)^j (j - 1)! \left(\frac{p^j - 1}{a_p(f)}\right)^n & \text{when } \phi \neq 1 \text{ (with conductor } p^n > 1). 
\end{cases}$$

(2) Recall that there is a natural homomorphism

$$P : \left( \lim_{\longrightarrow n} H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1)) \right) \otimes_{\mathcal{O}_K} K \longrightarrow \left( \lim_{\longrightarrow n} \frac{H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1))}{\text{Im}(\text{Fil}^0 D_{\text{dR}}((V_f)^*(1)))} \right) \otimes_{\mathcal{O}_K} K$$

Then, the map

$$\text{Col} : \left( \lim_{\longrightarrow n} H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1)) \right) \otimes_{\mathcal{O}_K} K \longrightarrow \text{Fil}^0 D_{\text{dR}}(V_f^* (1)) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$$

is factorized as $\overline{\text{Col}} \circ P$ where $\overline{\text{Col}}$ is given by

$$\overline{\text{Col}} : \left( \lim_{\longrightarrow n} \frac{H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1))}{\text{Im}(\text{Fil}^0 D_{\text{dR}}((V_f)^*(1)))} \right) \otimes_{\mathcal{O}_K} K \longrightarrow \text{Fil}^0 D_{\text{dR}}(V_f^* (1)) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}.$$ 

When $a_p(f) \neq 1$, the map $\overline{\text{Col}}$ is an isomorphism. When $a_p(f) = 1$ \footnote{This happens only when the weight of $f$ is equal to 2.}, $\text{Ker}(\overline{\text{Col}})$ and $\text{Coker}(\overline{\text{Col}})$ are both $K$-vector space of dimension one.

Finally, by combining the above results, we deduce the inequality (1.10). First, the exactness of the sequence (1.13) $\otimes_{\mathcal{O}_K} K$ and the inclusion (1.14), implies

$$\text{char}_{\Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} K} \left( \lim_{\longrightarrow n} \frac{H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1))}{\text{Im}(\text{Fil}^0 D_{\text{dR}}((V_f)^*(1)))} \right) / \Lambda_{\text{cyc}} \lim_{\longrightarrow z} \text{loc}_{p}(z) \otimes_{\mathcal{O}_K} K$$

$$\subset \text{char}_{\Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} K} \text{Sel}_A(\mathbb{Q}(\mu_{p^n}))' \otimes_{\mathcal{O}_K} K$$

Also, it is known that Beilinson-Kato Euler system is related to the special-value of the Hecke $L$-function $L(f, \phi^{-1}, s)$. 
Remark 1.10.

For example, let us denote by \( z_{1, \chi_\text{cyc} \phi^{-1}} \in H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T^*(1) \otimes \chi_\text{cyc}^{-1}) \) the image of twisted element

\[
(\lim_n z_n) \otimes \chi_\text{cyc}^{-1} \phi^{-1} \in \lim_n H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1) \otimes \chi_\text{cyc}^{-1})
\]

by the corestriction map to \( H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^n}), T^*(1) \otimes \chi_\text{cyc}^{-1}) \) Then the element \( z_{1, \chi_\text{cyc} \phi^{-1}} \) is related to the \( L \)-value in the following manner:

\[
\exp\left( \log_p(z_{1, \chi_\text{cyc} \phi^{-1}}) \right) = \tau(\phi) \times \frac{L(p)(f, \phi^{-1}, j)}{(2\pi \sqrt{-1})^j \Omega_{p,\phi}^{(j)}} \cdot T \otimes \phi^{-1}
\]

where \( L(p)(f, \phi^{-1}, s) \) is the Hecke \( L \)-function of \( f \) twisted by \( \phi^{-1} \) whose \( p \)-Euler factor is removed and \( T \) is a dual modular form of \( f \) whose \( q \)-expansion is given by \( T = \sum_{n=0}^{\infty} a_n(f)q^n \). We note that \( T \otimes \phi^{-1} \) defines a \( K \)-vector space \( \text{Fil}^0D_{dR}((V_f \otimes \phi)^*(1-j)) \) of dimension one.

Now if we apply the interpolation property of Theorem 1.9 to the Beilinson-Kato euler system, we obtain

\[
\text{char}_{\chi_\text{cyc} \otimes \mathbb{Z}_p K} \left( \lim_n \frac{H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1)) / \Lambda_\text{cyc} \lim_n \log_p(z_n)}{\text{Im}(H^1(\mathbb{Q}_p(\mu_{p^n}), F_p^{+}T^*(1)))} \right) \otimes_{\mathcal{O}_K} K
\]

The equality (1.17) combined with Rohrlich’s result on the non-triviality of \( L_p(\{\Omega_{j,\infty}^\pm\}) \) explained in Remark 1.5 implies that the left-hand side of (1.17) is non-zero. The final fact combined with (1.15) implies that \( \text{char}_{\chi_\text{cyc} \otimes \mathbb{Z}_p K} \text{Sel}_A(\mathbb{Q}(\mu_{p^n}))^{\vee} \otimes_{\mathcal{O}_K} K \) is non-zero, which means that \( \text{Sel}_A(\mathbb{Q}(\mu_{p^n}))^{\vee} \otimes_{\mathcal{O}_K} K \) is torsion over \( \Lambda_\text{cyc} \otimes \mathbb{Z}_p K \). Thus Theorem A is proved under the setting of (1.10). Finally, the proof of the desired inequality (1.10) follows simply by combining (1.15) and (1.17). \( \square \)

Here are several remarks on Theorem 1.9.

Remark 1.10.

(1) Perrin-Riou ([Pe94], [Pe01]) constructs Coleman maps similar to Theorem 1.9 for lattices \( T \) of more general \( p \)-adic representations \( V \) than \( V_f \) (In her work, \( V \) is of arbitrary rank and \( V \) is not necessarily ordinary). However, the paper [Pe94] requires an assumption that the representation \( V \) is crystalline. The result was later generalized in her paper [Pe01]. Though the condition gets milder in [Pe01], there still remains an assumption of \( V \) being semi-stable.

(2) Unfortunately, the representation \( V_f \) associated to an ordinary cuspidal form \( f \) is not semi-stable when the conductor of Neben character of \( f \) is divisible by \( p \) and the results of [Pe94] and [Pe01] cannot be applied to lattices of \( V_f \) in such cases. The author [Och03] constructs Coleman map as in Theorem 1.9 in a different manner as [Pe94] (see also Remark 3.8) and [Pe01] and the result in [Och03] is also valid for ordinary eigen cuspidal forms \( f \) whose associated to Galois representations \( V_f \) are not semi-stable.

(3) We remark that the results of Perrin-Riou ([Pe94], [Pe01]) are the interpolation of exponential maps of Bloch-Kato rather than the interpolation of dual exponential maps of Bloch-Kato as we formulated above. However, once the interpolation of
exponential maps are constructed in a suitable manner, we can recover the above result by taking "Kummer dual" of the above sequence.

We finish this section with an another version of Iwasawa Main Conjecture proposed by Kato.

**Conjecture (another version of Cyclotomic Iwasawa Main Conjecture for \( f \))**

(Recall that) we take \( f \) to be an ordinary eigen cuspform of weight \( k \geq 2 \). Let \( T \subset V_f \) be a Galois stable lattice and we choose complex periods \( \Omega_{\pm\infty} \) to be \( p \)-optimal with respect to the lattice \( T \). Then, we have the following equality of principal ideals in the ring \( \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \):

\[
\text{char}_{\Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathcal{O}_K} \left( \lim_{\longleftarrow n} H^1(\overline{\mathbb{Q}}_\Sigma/\mathbb{Q}((\mu_{p^n})), \mathbb{Q}^1) / \Lambda_{\text{cyc}} \lim_{\longleftarrow n} \text{col}(z_n) \right) = \text{char}_{\Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathcal{O}_K} \left( \lim_{\longleftarrow n} H^2(\overline{\mathbb{Q}}_\Sigma/\mathbb{Q}((\mu_{p^n})), \mathbb{Q}^1) \right).
\]

The formulation of this version of Cyclotomic Iwasawa Main Conjecture is due to Kato \[\text{[Ka93]}\]. We have several remarks on the above conjecture.

**Remark 1.11.**

1. Thanks to the sequence \((\mathbf{1.13}) \otimes_{\mathcal{O}_K} \mathcal{O}_K\), the equality \((\mathbf{1.18})\) in the above Iwasawa Main Conjecture is equivalent to the equality \((\mathbf{1.8})\) in the previously stated Iwasawa Main Conjecture modulo multiplication by \( (\varpi^r) \), where \( \varpi \) is a uniformizer of \( \mathcal{O}_K \) and \( r \) is an integer. By discussing more carefully, it is also possible to eliminate the ambiguity of multiplication by \( (\varpi^r) \).

   The latter Iwasawa Main conjecture makes sense also when the cuspform \( f \) is not ordinary.

2. There are various choices involved in the definition of the Beilinson-Kato Euler system and this poses an ambiguity in the left-hand side of \((\mathbf{1.18})\), by multiplication of a power of \( \varpi \). Since the right-hand side of \((\mathbf{1.18})\) has nothing to do with an Euler system, the latter version of Iwasawa Main Conjecture seems to have an ambiguity of multiplication by a power of \( \varpi \) as it is presented.

   As a solution, we propose to characterize \( \{z_n\} \) to be the one which satisfies \( \text{Col}(\lim_{\longleftarrow n} \text{loc}_p(z_n)) = L_p(\{\Omega^2_{f,\infty}\}) \) where \( \text{Col} \) is the Coleman map given in Theorem \[\text{[1.9]}\] and we choose the complex periods \( \Omega^{\text{sgn}(j,\phi)}_{f,\infty} \) in the interpolation property of the \( p \)-adic \( L \)-function \( L_p(\{\Omega^2_{f,\infty}\}) \) to be \( p \)-optimized with respect to the lattice \( T \) in an appropriate sense.

3. Sometimes, we call the latter Iwasawa Main Conjecture as "Iwasawa Main Conjecture without \( p \)-adic \( L \)-function". However, from what we discussed in (2), it is important to note that, even for this version of the conjecture, we need a \( p \)-adic \( L \)-function implicitly to normalize \( \{z_n\} \) precisely. We would like to stress this point, especially because the ambiguity becomes a more serious problem if we consider Iwasawa Main Conjecture for a family of cuspforms \( f \), which is the theme of this article.
2. Setting on Iwasawa Main Theory for a Family of Cuspsfoms \( f \)

In the last section, we briefly explained the formalism and the known results of the one-variable cyclotomic Iwasawa Main Conjecture for \( f \). On the other hand, the theory of \( p \)-adic families of cuspforms are much developed since mid 1980’s. First, when \( f \) is an ordinary eigen cusp form of weight \( \geq 2 \), there is a family of ordinary eigen cuspforms which contain \( f \). This family called a Hida family. Below, we will recall the results of Hida theory briefly and without proof and fix the setting at the same time.

2.1. Review of Hida family. Before stating the results, we prepare some notations:

**Definition 2.1.**  
(1) Let \( \kappa : \mathbb{Z}_p^\times \rightarrow (\mathbb{Q}_p)^\times \) be a continuous character. When there is an open subgroup \( U \subset \mathbb{Z}_p^\times \) such that the restriction \( \kappa|_U \) of \( \kappa \) to \( U \subset \mathbb{Z}_p^\times \) coincides with a character \( x \mapsto x^{w(\kappa)} \) \( (x \in U) \) with \( w = w(\kappa) \in \mathbb{Z} \), we call \( \kappa \) an arithmetic character of weight \( w \).

(2) If we have a continuous character \( \kappa : \mathbb{Z}_p^\times \rightarrow (\mathbb{Q}_p)^\times \), it induces a specialization map of algebra \( \kappa : \mathbb{Z}_p[[\mathbb{Z}_p^\times ]] \rightarrow \mathbb{Q}_p \), which we denote by \( \kappa \) by abuse of notation.
When \( \kappa \) is an arithmetic character of weight \( w \), we call \( \kappa : \mathbb{Z}_p[[\mathbb{Z}_p^\times ]] \rightarrow \mathbb{Q}_p \) an arithmetic specialization of weight \( w \).

Let \( R \) be an algebra which is finite over \( \mathbb{Z}_p[[\mathbb{Z}_p^\times ]] \). Then, a continuous specialization map \( \kappa : R \rightarrow \mathbb{Q}_p \) is an arithmetic specialization on \( R \) of weight \( w \) if \( \kappa|_{\mathbb{Z}_p[[\mathbb{Z}_p^\times ]]} \) is an arithmetic specialization of weight \( w \) in the sense stated above.

The following theorem is due to Hida (See [Hi86a] and [Hi86b] for two different proofs of the following theorem):

**Theorem 2.2** (Hida). Let \( f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_{k_0}(\Gamma_1(M)) \) be an ordinary normalized \( p \)-stabilized eigen cuspform of weight \( k_0 \geq 2 \) and of level \( \Gamma_1(M) \). We denote by \( K = K_f \) the finite extension of \( \mathbb{Q}_p \) obtained by adjoining Fourier coefficients \( a_n(f) \) to \( \mathbb{Q}_p \).

Then there are a local domain \( \mathbb{I} \) which is finite over \( O_K[[1 + p\mathbb{Z}_p]] \) and a formal \( q \)-expansion \( \mathbb{F} = \sum_{n=1}^{\infty} A_n(\mathbb{F})q^n \in \mathbb{I}[[q]] \) such that the following properties hold:

1. For each arithmetic specialization \( \kappa \) on \( \mathbb{I} \) of weight \( w(\kappa) \geq 0 \), \( f_\kappa := \kappa(\mathbb{F}) \in \kappa(\mathbb{I})[[q]] \) is the \( q \)-expansion of a classical ordinary eigen cuspform of weight \( w(\kappa) + 2 \).
2. There exists an arithmetic specialization \( \kappa_0 \) on \( \mathbb{I} \) of weight \( k_0 - 2 \) such that \( f_{\kappa_0} = f \).

This result gives a deformation \( \mathbb{F} \) of a given ordinary form \( f \). We call a family of ordinary cuspforms \( \mathbb{F} \) given in the above theorem a Hida family.

Hida [Hi86b] and Wiles [Wi88] prove the following result.

**Theorem 2.3** (Hida, Wiles). Let \( \mathbb{F} \) be a Hida family in the sense of Theorem 2.2. Then we have a Galois deformation \( \mathbb{V}_\mathbb{F} \cong \text{Frac}(\mathbb{I})[2] \) over the fraction field \( \text{Frac}(\mathbb{I}) \) of \( \mathbb{I} \) equipped with a continuous irreducible representation \( \rho_\mathbb{F} : G_\mathbb{Q} \rightarrow \text{Aut}_{\text{Frac}(\mathbb{I})}(\mathbb{V}_\mathbb{F}) \) unramified outside primes dividing \( M \) such that the equality

\[ \text{Tr}(\rho_\mathbb{F}(\text{Frob}_\ell)) = A_\ell(\mathbb{F}) \]

holds for every prime \( \ell \) not dividing \( M \).
Remark 2.4. The field \( \text{Frac}(\mathcal{I}) \) is not locally compact with respect to the topology induced by the maximal ideal of \( \mathcal{I} \). Hence, it is not clear (and maybe not known) if \( V_F \) always has a free Galois stable lattice over \( \mathcal{I} \).

The following local property at \( p \) of the representation \( V_F \) is also important.

**Theorem 2.5 (Wiles).** Let \( V_F \cong \text{Frac}(\mathcal{I})^{\oplus 2} \) be a representation associated to a Hida family \( \mathcal{F} \) given in Theorem 2.3. Then, the representation \( \rho_F|_{G_{Q_p}} \) obtained by restricting \( \rho_F \) to the decomposition group at \( p \) admits the following local filtration on \( V_F \):

\[
0 \to F^+V_F \to V_F \to V_F/F^+V_F \to 0
\]

where the submodule \( F^+V_F \) is of dimension one over \( \text{Frac}(\mathcal{I}) \), hence \( V_F/F^+V_F \) is also of dimension one over \( \text{Frac}(\mathcal{I}) \). The rank-one subspace \( F^+V_F \) is characterized by the property that the action of \( G_{Q_p} \) on \( F^+V_F \) is unramified.

We refer the reader to the article [Wi88] for the proof of this theorem.

In order to avoid technical complications of working over non-free lattices, we will consider the following conditions:

(F) We have a free \( \mathcal{I} \)-submodule \( T \subset V_F \) of rank two which is stable under the action of \( G_{Q_p} \). Further the representation \( \rho_F|_{G_{Q_p}} \) obtained by restricting \( \rho_F \) to the decomposition group at \( p \) admits the following local filtration on \( T \):

\[
0 \to F^+T \to T \to T/F^+T \to 0
\]

such that the sequence is stable under the action of \( G_{Q_p} \), graded pieces \( F^+T \) and \( T/F^+T \) are free of rank one over \( \mathcal{I} \) and the sequence \((2.2)\) recovers \((2.1)\) by applying the base extension \( \otimes_{\mathcal{I}} \text{Frac}(\mathcal{I}) \).

**Remark 2.6.**

(1) The first half of the statement of condition (F) holds when the residual representation \( G_{Q_p} \to \text{Aut}_{/\mathcal{M}_I}(T/\mathcal{M}_IT) \cong GL_2(\mathbb{F}_q) \) is irreducible where \( \mathcal{M}_I \) is the maximal ideal of \( \mathcal{I} \). As another example for the first half of the condition (F), it suffices that \( \mathcal{I} \) is a regular local ring. In fact, when we are given a Galois stable lattice \( L \subset V_F \) which is not necessarily free over \( \mathcal{I} \), we define \( T \) to be the double \( \mathcal{I} \)-linear dual \( L^{**} \) of \( L \). The lattice \( T \) is reflexive module over \( \mathcal{I} \). Since \( \mathcal{I} \) is of Krull dimension two by definition and is regular by assumption, any reflexive module over \( \mathcal{I} \) is free over \( \mathcal{I} \).

(2) As a caution, we remark that Theorem 2.5 and the first statement of the condition (F) might not be sufficient to imply the second statement of the condition (F). For example, if the residual representation \( T/\mathcal{M}_IT \) decomposed into two different characters as \( G_{Q_p} \)-module, Theorem 2.5 and the first statement of the condition (F) imply the second statement of the condition (F) (See [FO12, Rem. 2.13] for the proof of this fact).

---

9 Note that Galois representations with values in \( p \)-adic field have always a Galois stable lattice over the ring of integers (See [Se89, Chap. I §1] for example).

10 We take this occasion to correct a typo in [FO12]. The sequence obtained by \( \otimes_{\mathcal{R}} \mathcal{R}/\mathcal{M}_\mathcal{R} \) in the third line of the proof of [FO12, Rem. 2.13] is only right exact and the phrase "Note that the sequence is left-exact since \( T_F \) is a torsion-free \( \mathcal{R} \)-module." is not correct. However, the argument below this phrase uses only the right exactness of the sequence and the argument remains correct except this point.
2.2. Review of Coleman family. About ten years later from Hida theory, Coleman ([Co96], [Co97]) proved a family of modular forms for non-ordinary eigen cuspforms. The way of the construction of Coleman family is more "analytic" than Hida family and it is obtained as a rigid analytic local section over the weight space. Thus, Coleman family exists only locally over the weight space while Hida family exists over the whole weight spaces. Below, we will recall the results of Coleman theory briefly and without proof and fix the setting at the same time.

Let \( W \) be weight space over \( \mathbb{Q}_p \) whose \( K \)-valued points \( W(K) \) is identified with the set of continuous characters \( \mathbb{Z}_p^\times \to K^\times \). We have an embedding \( \mathbb{Z} \hookrightarrow W(\mathbb{Q}_p) \) by sending \( k \in \mathbb{Z} \) to the character \( \mathbb{Z}_p^\times \to \mathbb{Q}_p^\times \), \( a \mapsto a^k \). The space \( W \) is a \( p-1 \) copies of open discs of radius 1 and the ring of power bounded rigid analytic functions on \( W \) is isomorphic to \( \mathbb{Z}_p[[Z_p^\times]] \).

Now as we mentioned earlier, we need to work locally on \( W \). So we prepare some definitions and notations.

**Definition 2.7.** Let \( k_0 \) be an integer and \( r \) a natural number.

1. We denote by \( B(k_0; r) \) a rigid analytic space whose rational points are identified with an open disc of radius \( \frac{1}{p^r} \) centered at \( k_0 \in W(\mathbb{Q}_p) \) with the above mentioned embedding of \( \mathbb{Z} \) in \( W(\mathbb{Q}_p) \).

2. The ring of power bounded rigid analytic functions on \( B(k_0; r) \) is noncanonically isomorphic to \( \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \) and we denote it by \( \Lambda_{(k_0; r)} \) in this article. When \( \mathcal{O} \) is the ring of the integers of a finite extension of \( \mathbb{Q}_p \), we denote \( \Lambda_{(k_0; r; \mathcal{O})} \) by \( \Lambda_{(k_0; r; \mathcal{O})} \) for short.

3. A character \( \eta : \mathbb{Z}_p^\times \to \mathbb{Q}_p^\times \) identified with a point \( B(k_0; r) \subset W(\mathbb{Q}_p) \), is called an arithmetic character if it lies in \( \mathbb{Z} \cap B(k_0; r) \) and the corresponding integer \( k \) is called the weight of \( \eta \).

4. As explained in Definition 2.7, any continuous character \( \eta : \mathbb{Z}_p^\times \to \mathbb{Q}_p^\times \) extends to a specialization map \( \mathbb{Z}_p[[Z_p^\times]] \to \mathbb{Z}_p \). Further, if \( \eta \) is in \( B(k_0; r) \subset W(\mathbb{Q}_p) \), the map extends to \( \eta : \Lambda_{(k_0; r; \mathcal{O})} \to \mathcal{O} \). Here, we note that the ring \( \Lambda_{(k_0; r; \mathcal{O})} \) contains one of the local components \( \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \) of \( \mathbb{Z}_p[[Z_p^\times]] \) as the ring of power bounded rigid analytic functions on \( W \) since \( B(k_0; r) \subset W \). A specialization \( \eta : \Lambda_{(k_0; r; \mathcal{O})} \to \mathcal{O} \) thus obtained is called an arithmetic specialization if the corresponding character \( \mathbb{Z}_p^\times \to \mathbb{Q}_p^\times \) is arithmetic.

5. The set of arithmetic specializations is identified with \( \mathbb{Z} \cap B(k_0; r) \). Hence, we denote it by an integer \( k \in \mathbb{Z} \cap B(k_0; r) \) rather than \( \eta \) and we call it as an arithmetic point.

**Theorem 2.8 (Coleman).** Let \( f = \sum_{n=1}^{\infty} a_n(f) q^n \in S_{k_0}(\Gamma_1(M)) \) be a normalized \( p \)-stabilized eigen cuspform of weight \( k_0 \geq 2 \) and of level \( \Gamma_1(M) \). Assume that \( a_p(f) \neq 0 \) and put \( s \in \mathbb{Q}_{\geq 0} \) to be \( s = \text{val}_p(a_p(f)) \) where \( \text{val}_p : (\mathbb{Q}_p)^\times \to \mathbb{Q}_{\geq 0} \) is a valuation map normalized by \( \text{val}_p(p) = 1 \). When \( f \) is a \( p \)-stabilization of a newform \( f_0 \) whose level is prime to \( p \), we assume that the \( p \)-th Hecke polynomial \( X^2 - a_p(f_0)X + \psi_{f_0}(p)p^{k_0-1} \) does not have double root.\(^{11}\) We denote by \( K = K_f \) the finite extension of \( \mathbb{Q}_p \) obtained by adjoining Fourier coefficients \( a_n(f) \) to \( \mathbb{Q}_p \).

\(^{11}\)In fact, this assumption is conjectured to be always true and the conjecture is already proved for \( k = 2 \) by Coleman-Edixhoven [CE98].
Then there exists a natural number $r$ and a formal $q$-expansion $F = \sum_{n=1}^{\infty} A_n(F) q^n \in \Lambda_{(k_0, r), \mathcal{O}_K[[q]]}$ such that the following properties hold:

(1) At each arithmetic point $k \in \mathbb{Z} \cap B(k_0; r)$ larger than $s + 1$, $f_k := F(k) \in \mathcal{O}_K[[q]]$ is the $q$-expansion of a classical ordinary eigen cuspform of weight $k$.

(2) For each arithmetic point $k \in \mathbb{Z} \cap B(k_0; r)$, we have $\text{val}_p(a_p(f_k)) = s$.

(3) At the arithmetic point $k_0 \in \mathbb{Z} \cap B(k_0; r)$, we have $f_{k_0} = f$.

We call the number $s$ which appeared in the theorem the **slope** of a given Coleman family $F$.

**Remark 2.9.**

(1) In Definition 2.1, the arithmetic characters remain to be arithmetic after twisting by Dirichlet characters of $p$-power conductor. On the other hand, the arithmetic characters of Definition 2.7 do not remain to be arithmetic after twisting by Dirichlet characters of $p$-power conductor. The construction of Coleman family does not allow “infinitesimal deformation” (by finite order characters of $p$-power conductor) as in Hida family.

(2) In [Co96], [Co97], Coleman works rigid analytically over affinoid spaces. Hence, we refer the reader to [NO16, §2.2] for the proof of a reformulation of Coleman family over a local complete algebra as presented in Theorem 2.8. Since an affinoid algebra is not complete, Beilinson-Kato elements and Coleman maps do not seem to be glued over an affinoid algebra. Hence, the existence of Coleman family over a formal base ring as presented in Theorem 2.8 plays important roles in the paper [NO16].

As an application of the existence of Coleman family over a formal base as presented in Theorem 2.8, we can apply Wiles’ method of pseudo-representation to obtain the following consequence:

**Theorem 2.10.** Let $F$ be a Coleman family in the sense of Theorem 2.8. Then we have a Galois deformation $T \cong \Lambda^{\otimes 2}_{(k_0, r), \mathcal{O}_K}$ equipped with a continuous irreducible representation $\rho_F : G_\mathbb{Q} \to \text{Aut}_{\Lambda_{(k_0, r), \mathcal{O}_K}(T)} \text{ unramified outside primes dividing } M$ such that the equality

$$\text{Tr}(\rho_F(\text{Frob}_\ell)) = A_\ell(F)$$

holds for every prime $\ell$ not dividing $M$.

We refer the reader to [NO16, §2.3] for the proof of Theorem 2.10.

Our motivation is to study the two-variable Iwasawa theory (Iwasawa Main Conjecture) for a given $p$-adic families of $V_f$ which are as universal as possible. In each of the ordinary case and the non-ordinary case presented above, there is a complete Noetherian local ring $\mathcal{R}$ of characteristic 0 with finite residue field of characteristic $p$ and a family of Galois representation $T \cong \mathcal{R}^{\otimes d}$ on which the absolute Galois group $G_\mathbb{Q}$ acts continuously. Then there should be Iwasawa Main Conjecture over $T$ as an equality of ideals in $\mathcal{R}^{\otimes \mathbb{Z}_p} \Lambda_{\text{cyc}}$

\textit{Note that a typical example of affinoid space is a $p$-adic closed discs corresponding to the Tate algebra $\mathbb{Z}_p(X) := \lim_{\leftarrow n} (\mathbb{Z}/p^n\mathbb{Z})[X]$ and a $p$-adic open disc is not an affinoid space.}
relating an algebraic object and an analytic object.}

3. Iwasa-Main Conjecture for an Ordinary Family of Cuspforms $f$

3.1. Statements of known results. As for our motivation stated in the end of the previous section, we have already some conditional results in the ordinary setting given in 3.1. In this section, we shall introduce the results and the proof. Then we will make our motivation more precise explaining new difficulties and new phenomena on this project of “Iwasawa theory for deformation spaces”.

We fix the setting of “(Review of Hida family)” introduced above. We take a Hida family $F = \sum_{n=1}^{\infty} A_n(F) q^n \in \mathbb{Q}(q)$. We choose and fix an integral Galois representation $T \cong \mathbb{T} \oplus \mathbb{Z}_p^2$ associated to $F$ by Theorem 3.3 which satisfies the condition (F). Similarly as in 3.1 for each number field $K$, we define the Greenberg type Selmer group

$$\text{Sel}_G(K) = \text{Ker} \left( H^1(K, \overline{A}) \rightarrow \prod_{\lambda \nmid p} H^1(I_\lambda, \overline{A}) \times \prod_{\lambda \mid p} \frac{H^1(I_p, \overline{A})}{\text{Image}(H^1(I_p, F_p^+) \otimes \mathbb{Z}_p^\alpha)} \right)$$

where $\overline{A}$ is a discrete Galois representation defined to be $\overline{A} = \mathbb{T} \otimes \mathbb{Z}_p^\alpha$ and where $I_\lambda$ (resp. $I_p$) means the inertia subgroup at each place $\lambda$ (resp. $p$) not dividing $p$ (resp. dividing $p$). We deduce that the Pontrjagin dual $\text{Sel}(\mathbb{Q}(\mu_p^\infty))^{\vee}$ of $\text{Sel}(\mathbb{Q}(\mu_p^\infty))$ is finitely generated over $\mathbb{I} \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}$ by a similar argument as the argument of Lemma 1.1 and Lemma 1.2.

The following theorem follows from Theorem A of 3.1 combined with the control theorem of Selmer group specializing from two-variable to one-variable (see [Och01] and [FO12]).

**Theorem A'** (Torsion property of the Selmer group for a Hida family $F$)

Let $F$, $T$ and $A$ as above. Assume that $p \geq 3$. Then the finitely generated $\mathbb{I} \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}$-module $\text{Sel}_A(\mathbb{Q}(\mu_p^\infty))^{\vee}$ is torsion over $\mathbb{I} \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}$.

In order to generalize Theorem B of 3.1 from a cuspform $f$ to a family of cuspforms $f_\kappa$, it is important to understand “periods”. In fact, a complex period $\Omega_{\text{sgn} f_\kappa, \kappa}$, not canonical at all and it is defined only modulo multiplication by elements of $(\mathbb{Q}(f_\kappa))^\times$ at each $\kappa$. Hence, it does not make sense to formulate a family of $p$-adic $L$-functions for cuspforms $f_\kappa$, without “regularizing a choice of periods $\Omega_{f,\kappa}^{\text{reg}}$”.

Greenberg-Stevens [GS93] and Kitagawa [Kit94] constructed an $\mathbb{I}$-module called “$\mathbb{I}$-adic modular symbols” to overcome the problem of “regularization of periods” and they constructed two-variable $p$-adic $L$-functions for given Hida families. Though the constructions of [GS93] and [Kit94] are similar, the former interpolates parabolic cohomologies and the construction is only local over $\mathbb{I}$. The latter interpolates compactly supported cohomologies and the construction is global over the whole $\mathbb{I}$.

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13We outlined the setting of $p$-adic deformations of modular forms very quickly for our later use later in this article. The setting of eigencurves (cf. [CM98]) which unifies the ordinary case and the non-ordinary case is also known. Formulating and proving the results which will be satet in later sections of this paper over eigencurves should be also an interesting problem.

14We exclude $p = 3$ only because we did not find a complete reference for CM case.
Hence, we follow more closely the construction of \[\text{Kit94}\] and denote the module of $\mathbb{M}S(\mathbb{I})^\pm$. Though we do not recall the definition of $\mathbb{M}S(\mathbb{I})^\pm$, we remark that $\mathbb{M}S(\mathbb{I})^\pm$ is the same as the one we denoted by $\mathcal{M}S(\mathbb{I})^\pm[\lambda]$ in \[\text{Kit94}\]. As an important property of $\mathbb{M}S(\mathbb{I})^\pm$, for each arithmetic specialization $\kappa$ of non-negative weight $w(\kappa) \geq 0$, $\mathbb{M}S(\mathbb{I})^\pm \otimes_{\mathbb{I}} \kappa(\mathbb{I})$ is canonically identified with $H^1_c(Y_1(M)_\mathbb{C}, \mathcal{L}_{k_{j-2}}(\mathbb{Q}_f))^\pm[\kappa_\mathbb{I}]$.

**Definition 3.1.** Suppose that $\mathbb{M}S(\mathbb{I})^\pm$ is free of rank one over $\mathbb{I}$ for each of the signs $\pm$. Let us fix a basis $\Xi^\pm$ of $\mathbb{M}S(\mathbb{I})^\pm$ over $\mathbb{I}$ for each of the signs $\pm$. Then, for each arithmetic specialization $\kappa$ of non-negative weight $w(\kappa) \geq 0$, the specialization $\mathbb{M}S(\mathbb{I})^\pm \otimes_{\mathbb{I}} \kappa(\mathbb{I})$ is naturally identified with a lattice of $H^1_c(Y_1(N_{f_\kappa})_\mathbb{C}, \mathcal{L}_{w(\kappa)}(\mathbb{Q}_{f_\kappa}))^\pm[\kappa_\mathbb{I}] \otimes_{\mathbb{Q}_f} \text{Frac}(\kappa(\mathbb{I}))$. We define a $p$-adic period to be an error term given by:

$$\kappa(\Xi^\pm) = C^\pm_{f_\kappa,p} \cdot b^\pm_{f_\kappa} \otimes 1.$$  

**Remark 3.2.** As was cautioned earlier, $C^\pm_{f_\kappa,\infty}$ and $\Omega^\pm_{f_\kappa,\infty}$ depend on the choice of a $\mathbb{Q}_{f_\kappa}$-basis $b^\pm_{f_\kappa}$ on $H^1_c(Y_1(M)_\mathbb{C}, \mathcal{L}_{k_{j-2}}(\mathbb{Q}_f))^\pm[\kappa_\mathbb{I}]$. However, the “ratio” is independent of the choice of $b^\pm_{f_\kappa}$. If we denote the $p$-adic period and the complex period obtained by another $\mathbb{Q}_{f_\kappa}$-basis $(b^\pm_{f_\kappa})'$ on $H^1_c(Y_1(M)_\mathbb{C}, \mathcal{L}_{k_{j-2}}(\mathbb{Q}_f))^\pm[\kappa_\mathbb{I}]$ by $(C^\pm_{f_\kappa,\infty})'$ and $(\Omega^\pm_{f_\kappa,\infty})'$, we have

$$\frac{C^\pm_{f_\kappa,p}}{(C^\pm_{f_\kappa,\infty})'} = \frac{\Omega^\pm_{f_\kappa,\infty}}{(\Omega^\pm_{f_\kappa,\infty})'}. $$

Thus the interpolation property of two-variable ordinary $p$-adic $L$-functions stated in Theorem B below is well-defined and makes sense.

In order to state the result on two-variable ordinary $p$-adic $L$-function, we prepare the following condition.

*(M)* The $\mathbb{I}$-module $\mathbb{M}S(\mathbb{I})^\pm$ are free of rank one over $\mathbb{I}$.

Here is the main result of \[\text{Kit94}\].

**Theorem B' (Existence of two-variable ordinary $p$-adic $L$-function)**

Let $\mathbb{F}$, $\mathbb{T}$ and $\mathbb{A}$ as above. Assume that $p \geq 5$. We assume the condition \(\text{(M)}\) and fix an $\mathbb{I}$-basis $\Xi^\pm$ of $\mathbb{M}S(\mathbb{I})^\pm$ respectively.

Then, there exists an analytic $p$-adic $L$-function $L_p(\Xi^\pm) \in \mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that we have the following equality:

$$\frac{(\chi_{\text{cyc}} \phi \times \kappa)(L_p(\Xi^\pm))}{C^\pm_{f_\kappa,p}} = (-1)^j(j-1)! \times e_p(f_\kappa,j,\phi) \times \tau(\phi) \times \frac{L(f_\kappa,\phi^{-1},j)}{(2\pi \sqrt{-1})^{\frac{1}{2}} \Omega^\pm_{f_\kappa,\infty}},$$

for any arithmetic specialization of $\mathbb{I}$ with $w(\kappa) \geq 0$, for any integer $j$ satisfying $1 \leq j \leq w(\kappa) + 1$ and for any Dirichlet character $\phi$ whose conductor $\text{Cond}(\phi)$ is a power of $p$, where $\tau(\phi)$ and $e_p(f,j,\phi)$ is the same as Theorem B.

**Remark 3.3.**

\[\text{Some sufficient conditions so that } \mathbb{M}S(\mathbb{I}) \text{ becomes free of rank one over } \mathbb{I} \text{ is listed in } \text{Kit94}. \text{ For example, this holds if } \mathbb{I} \text{ is a UFD.}\]
The $p$-adic $L$-function $L_p\left(\{\Xi^{\pm}\}\right)$ is expected to be in $\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}} \subset \mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ when the complex period is $p$-optimal way. By lack of appropriate reference for this integrality, we gave $L_p\left(\{\Xi^{\pm}\}\right)$ as an element of $\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. When the residual Galois representation associated to the Hida family $\mathbb{F}$ is irreducible, we can check that $L_p\left(\{\Xi^{\pm}\}\right) \in \mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$. In this case, if we choose a complex period $\Omega^{\text{sgn}(j, \phi)}_{f_e, \infty}$ to be $p$-optimal, the special value $\frac{L(\omega, \phi^{-1}, j)}{(2\pi)^{j-1} \Omega^{\text{sgn}(j, \phi)}_{f_e, \infty}}$ is $p$-integral for every $\kappa$ and $p$-adic period $C^{\text{sgn}(j, \phi)}_{f_e, p}$ is a $p$-adic unit for every $\kappa$. Thus $L_p\left(\{\Xi^{\pm}\}\right)$ has to be integral since its specializations are $p$-integral.

The following result realizes our motivation which was proposed in the end of the previous section:

**Theorem C’** (Two-variable Iwasawa Main Conjecture for a Hida family $\mathbb{F}$)

Under certain assumptions (on the prime number $p$, the tame conductor of the Hida family $\mathbb{F}$, the fullness of the residual representation, the condition (M), the regularity of the local ring $\mathbb{I}$ etc), we have the following equality of principal ideals in the ring $\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$:

\[(L_p\left(\{\Xi^{\pm}\}\right)) = \text{char}_{\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}} \text{Sel}_{h}(\mathbb{Q}(\mu_{p^\infty}))^\vee.\]

### Proof of known results

By Euler system approach (cf. [Och03], [Och05] and [Och06]), we prove an inclusion

\[(L_p\left(\{\Xi^{\pm}\}\right)) \subset \text{char}_{\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}} \text{Sel}_{h}(\mathbb{Q}(\mu_{p^\infty}))^\vee.\]

Once we prove this inclusion for two-variable situation, we specialize at an arithmetic specialization $\kappa$ on $\mathbb{I}$. Since we have already an equality Theorem C under appropriate conditions, the inclusion may be upgraded to an equality (see [Och06] §7 for the detail of the proof of this part). We stress that this strategy implicitly relies on Skinner-Urban [SU14] since the equality in Theorem C is due to the opposite inclusion obtained in [SU14] by means of the Eisenstein ideal for $U(2, 2)$. Note that Skinner-Urban [SU14] also proves the opposite inequality of (3.6) for the situation of two-variable.
We will discuss a bit the proof with Euler system approach ([Och03], [Och05] and [Och06]) for Theorem C', which should be also an opportunity to explain new difficulties of Theorem C' compared to Theorem C.

From now on, we concentrate on the proof of (3.6). The proof is composed of two independent steps (the Euler system bound and the existence of a Coleman map) as in the proof of (1.10) explained after Remark 1.7. However, we need some preparations.

First, we have to note that a general machinery of Euler system bound obtained in Theorem 1.8 is only for usual Galois representation with coefficients in the discrete valuation rings with finite residue field. In fact, the characteristic ideal of a torsion in Theorem 1.8 is only for usual Galois representation with coefficients in the discrete valuation ring with finite residue field. In fact, the characteristic ideal of a torsion in the proof of (1.10) explained after Remark 1.7 is based on the counting. However, we need some preparations.

Since we need to generalize Theorem 1.8 to Galois representations with coefficients in larger deformation rings, we need essentially new ideas. Below, we will give the absolute Galois group $G$.

Definition 3.4 (Def. 2.1 of [Och05]). Let $T \cong R^\oplus 2$ be a Galois representation of the absolute Galois group $G_Q$ over a complete Noetherian local ring $R$ of characteristic 0 with finite residue field of characteristic $p$. Assume that the action of $G_Q$ on $T$ is continuous and unramified outside a finite set of primes $\Sigma$ of $Q$ which contain $\{p\}, \{\infty\}$ and the ramified primes of the representation $T$.

We denote by $S$ the set of all square-free natural numbers which are prime to $\Sigma$. An Euler system for $T$ is a collection of cohomology classes

$$\{Z(r) \in H^1(Q(\mu_v), (\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^\sharp(1)))\}_{r \in S}$$

with the following properties:

1. The element $Z(r)$ is unramified outside $\Sigma \cup \{r\}$ for each $r \in S$.
2. The image of the norm $\text{Norm}_{Q(\mu_v)}(Z(r))$ of $Z(r)$ is equal to $P_q(\text{Frob}_q)Z(r)$, where $P_q(X) \in R \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}[X]$ is the polynomial $\text{det}(1 - \text{Frob}_qX; T \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}})$ and $\text{Frob}_q$ is a (conjugacy class of) geometric Frobenius element at $q$ in the Galois group $\text{Gal}(Q(\mu_v)/Q)$.

Here, $\Lambda_{\text{cyc}}^\sharp$ is a free $\Lambda_{\text{cyc}}$-module of rank one on which the absolute Galois group acts via the character

$$G_Q \to \text{Gal}(Q(\mu_v)/Q) \hookrightarrow \Lambda_{\text{cyc}}^\times.$$  

The following theorem is a deformation theoretic generalization of Theorem 1.8 which gave an Euler system bound for cyclotomic twists of usual $p$-adic representations. In order to apply the result also to non-ordinary situations, we do not restrict ourselves to Hida families and state the result with a more general setting.

Theorem 3.5 (Thm 2.4 of [Och05]). Let us assume the setting of Definition 3.4. Assume further that the following conditions are satisfied:

The cohomology $H^1(Q(\mu_v), (\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^\sharp)^*(1))$ is isomorphic to $\lim_{\leftarrow n} H^1(Q(\mu_{p^n}), T^*(1))$ by Shapiro’s lemma on Galois cohomology.
Remark 3.6.

(i) The residual Galois representation of $\mathbb{T}$ at the maximal ideal of $\mathcal{R}$ is absolutely irreducible as a representation of $G_{\mathbb{Q}}$.

(ii) The first layer of a given Euler system $Z(1) \in H^1(\mathbb{Q}, (\hat{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i)^*(1))$ is not contained in the $\mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i$-torsion part of $H^1(\mathbb{Q}, (\hat{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i)^*(1))$ at each local component $\mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i$. Here, $\omega$ denotes the Teichmüller character and we recall that $\Lambda_{\text{cyc}}$ is decomposed into a product of local rings as $\Lambda_{\text{cyc}} = \prod_{i=0}^{p-2} \Lambda_{\text{cyc}}^i$ by character decomposition.

(iii) The $\mathcal{R}$-module $\mathbb{T}$ splits into eigenspaces: $\mathbb{T} = \mathbb{T}^+ \oplus \mathbb{T}^-$ with respect to the complex conjugation in $G_{\mathbb{Q}}$, and $\mathbb{T}^+$ (resp. $\mathbb{T}^-$) is free $\mathcal{R}$-module of rank one.

(iv) There exist $\sigma_1 \in G_{\mathbb{Q}(\mu_{p\infty})}$ and $\sigma_2 \in G_{\mathbb{Q}}$ such that $\rho(\sigma_1)$ is conjugate to $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in GL_2(\mathcal{R})$ for a unit $u \in \mathcal{R}^\times$ and $\sigma_2$ acts on $\mathbb{T}$ as multiplication by $-1$ for the Galois representation $\rho : G_{\mathbb{Q}} \rightarrow \text{Aut}_{\mathcal{R}}(\mathbb{T}) \cong GL_2(\mathcal{R})$.

(v) Every local component $\mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i$ of the semi-local ring $\mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$ is isomorphic to a ring of two-variable formal power series $\mathcal{O}[[X, Y]]$ where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$.

Then, $H^2(\mathcal{O}_{\Sigma}/\mathbb{Q}, (\hat{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i)^*(1))$ is a torsion $\Lambda_{\text{cyc}}$-module and we have:

\[
\text{char}_{\mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}} \left( H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q}, (\hat{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i)^*(1)) / \mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}} \cdot Z(1) \right) \subset \text{char}_{\mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}} \left( H^2(\mathcal{O}_{\Sigma}/\mathbb{Q}, (\hat{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i)^*(1)) \right).
\]

Remark 3.6.

(1) The paper [Och05] considers Euler systems with coefficients in the ring of power series in $n$ variables $\mathcal{O}[[X_1, \ldots, X_n]]$ where $\mathcal{O}$ is the ring of integers of a $p$-adic field. To overcome an essential difficulty to generalize Theorem 1.8 explained before Definition 3.3, the proof of the paper [Och05] uses an approach called “specialization method”.

When $n = 1$, the proof of Euler system bound in [Och05] is based on a method to recover the characteristic ideal of a torsion module over $\mathcal{O}[[X]]$ from the size of various specializations of the modules (see [Och05], Prop. 3.7). When $n$ is greater than one, [Och05] requires a method to recover the characteristic ideal of a torsion module over $\mathcal{O}[[X_1, \ldots, X_n]]$ from the characteristic ideal of various specializations of the modules to situations of $n - 1$ variables (see [Och05], Prop. 3.6) and the proof proceeds by the induction with respect to the number of variables $n$.

(2) At the same time as [Och05], Mazur-Rubin [MR04] also considered Euler systems for Galois deformations according to their own motivation. They also considered Euler systems with coefficients in $\mathcal{O}[[X]]$ which are not necessary the cyclotomic deformation of a usual $p$-adic representation. Since their result is limited to the case $n = 1$, Theorem 3.5 does not follow from [MR04]. The method of proof of

\[17\] This condition excludes the case where $F$ has complex multiplication.
Euler system bound in \[\text{[MR04]}\] is based on the same principle as \[\text{[Och05]}\ Prop. 3.7].

(3) In order that “specialization method” works well, the assumption (v) was essential. A joint work with Shimomoto (\[\text{[SO15]}\] and \[\text{[SO17]}\]) relaxed the assumption by replacing the condition (v) by a weaker assumption (v′) as follows:

\(\text{(v')}\) Every local component \(\mathcal{R} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i\) of the semi-local ring \(\mathcal{R} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda_{\text{cyc}}\) is an integral domain which is integrally closed in its field of fractions. Here, \(\omega\) denotes the Teichmüller character and we recall that \(\Lambda_{\text{cyc}}\) is decomposed into

\[
a \text{product of local rings as } \Lambda_{\text{cyc}} = \prod_{i=0}^{p-2} \Lambda_{\text{cyc}}^i \text{ by character decomposition.}
\]

and assuming an extra condition on vanishing of \(H^2(\mathbb{Q}_v, T \widehat{\otimes}_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i)\) for every \(v \in \Sigma \setminus \{\infty\}\).

As in the proof of (1.9), we go on to the theory of Coleman map. Though we stated the above result for Euler system bound in a general setting, the result for Coleman map is restricted to the case of Hida family. In fact, the construction of Coleman map is much harder in non-ordinary case and the construction in non-ordinary case (Coleman family) is a joint work with Filippo Nuccio \[\text{[NO16]}\] which is explained in the next section.

Let \(\mathcal{F}\) be a Hida family over \(\mathbb{I}\). By \[\text{[MW83]}\ Prop. 1\], we have a \(G_{\mathbb{Q}}\)-stable lattice \(T \subset \mathbb{V}_\mathcal{F}\) and the following \(G_{\mathbb{Q}_p}\)-stable exact sequence

\[
0 \longrightarrow F^+ T \longrightarrow T \longrightarrow T/F^+ T \longrightarrow 0
\]

such that \(F^+ T\) is free of rank one over \(\mathbb{I}\) and the action of \(G_{\mathbb{Q}_p}\) on \(F^+ T\) is unramified.

\(T\) a lattice and \(A\) the discrete Galois representation \(T \otimes_\mathbb{I} \mathbb{V}'\). Let us define

\[
D_{\text{ord}} = (F^+ T \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p_{\text{ur}})^{G_{\mathbb{Q}_p}}
\]

Since \(F^+ T\) is unramified, the \(\mathbb{I}\)-module \(D_{\text{ord}}\) is free of rank one by definition. Also, the specialization \(D_{\text{ord}} \otimes_\mathbb{I} \kappa(\mathbb{I})\) is naturally identified with a \(\kappa(\mathbb{I})\)-lattice of \(D_{\text{dR}}(V_f, V_{f^0})\) for any arithmetic specialization \(\kappa\) on \(\mathbb{I}\) of non-negative weight.

We denote by \((D_{\text{ord}})^* (1)\) the \(\mathbb{I}\)-module \(\text{Hom}_{\mathbb{I}}(D_{\text{ord}}, \mathbb{I}) \otimes_{\mathbb{Z}_p} D_{\text{crys}}(\mathbb{Z}_p(1))\). Here, \(D_{\text{crys}}(\mathbb{Z}_p(1))\) is a \(\mathbb{Z}_p\)-lattice of \(D_{\text{crys}}(\mathbb{Q}_p(1)) = (\mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_{\mathbb{Q}_p}}\) generated by \(\zeta_{p^\infty} \otimes t^{-1}\) with \(\zeta_{p^\infty}\) a generator of \(\mathbb{Z}_p\)-module \(\mathbb{Z}_p(1)\) and \(t \in B_{\text{crys}}\) is a standard element of Fontaine on which \(G_{\mathbb{Q}_p}\) acts by the \(p\)-adic cyclotomic character. The specialization \((D_{\text{ord}})^* (1) \otimes_\mathbb{I} \kappa(\mathbb{I})\) is naturally identified with a \(\kappa(\mathbb{I})\)-lattice of \(\text{Fil}^0 D_{\text{dR}}(V_f, V_{f^0}(1))\) for any arithmetic specialization \(\kappa\) on \(\mathbb{I}\) of non-negative weight.

**Theorem 3.7** ([Thm 3.13 of \[\text{Och03}\]). Let us assume the setting of Theorem \(C'\).

(1) There is a \(\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}\)-linear homomorphism:

\[
\text{Col} : H^1(\mathbb{Q}_p, (T \widehat{\otimes}_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^i)^*(1)) \longrightarrow (D_{\text{ord}})^* (1) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}
\]

such that we have the following commutative diagram for any arithmetic specialization \(\kappa\) on \(\mathbb{I}\) of non-negative weight, any positive integer \(j\) and for any Dirichlet
character $\phi$ whose conductor $\text{Cond}(\phi)$ is a power of $p$:

$$H^1(\mathbb{Q}_p, (T \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^\iota)^*(1)) \xrightarrow{\text{Col}} (D^{\text{ord}})^*(1) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$$

$$H^1(\mathbb{Q}_p, (V_f \otimes \phi)^*(1-j)) \xrightarrow{e_p(f, j, \phi) \times \text{exp}^*} \text{Fil}^0 D_{dR}((V_f \otimes \phi)^*(1-j))$$

where $\text{exp}^*$ is the dual exponential map of Bloch-Kato and the factor $e_p(f, j, \phi)$ is given in Theorem C.

(2) Recall that there is a natural homomorphism

$$P : H^1(\mathbb{Q}_p, (T \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^\iota)^*(1)) \rightarrow H^1(\mathbb{Q}_p, (T \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^\iota)^*(1))$$

$$\text{Im}(H^1(\mathbb{Q}_p, F^+ (T \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^\iota)^*(1)))$$

Then, the map

$$\text{Col} : \lim_n H^1(\mathbb{Q}_p(\mu_{p^n}), T^*(1)) \rightarrow (D^{\text{ord}})^*(1) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$$

is factorized as $\overline{\text{Col}} \circ P$ where $\overline{\text{Col}}$ is given by

$$\overline{\text{Col}} : \frac{H^1(\mathbb{Q}_p, (T \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^\iota)^*(1))}{\text{Im}(H^1(\mathbb{Q}_p, F^+ (T \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^\iota)^*(1)))} \rightarrow (D^{\text{ord}})^*(1) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}.$$ 

Further, $\overline{\text{Col}}$ is an $\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$-linear injective homomorphism whose cokernel is a pseudo-null $\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$-module.

**Remark 3.8.** The proof of Theorem 3.7 relies very much on the fact that the Galois representation $V_F$ associated to a Hida family $F$ is obtained as an extension of a rank-one Galois representation by a rank-one Galois representation. Though $V_F$ is of rank two, we reduce the proof of Theorem 3.7 to a Coleman map of rank-one Galois representation which was classically known and was proved by Coleman power series of local units (see [Och03] for the detail of the proof).

According to the construction of Hida family, it is not difficult to see that Kato’s work [Ka01] implies that an Euler system for T in the sense of Definition 3.4 exists and nontrivial as follows (see Remark 3.10 (1) below).

**Theorem 3.9.** For natural numbers $c, d$ which does not divide the tame level of the Hida family nor $6p$, we have an Euler system:

$$\{c, d \mathcal{Z}_\kappa(r) \in H^1(\mathbb{Q}(\mu_r), (T \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^\iota)^*(1))\}_{r \in S}$$

such that $\text{Col}((\text{loc}_p(c, d \mathcal{Z}_\kappa(r))))$ has the following relation to the special value for any arithmetic specialization $\kappa$ on $\mathbb{T}$ of non-negative weight, any integer $j$ satisfying $1 \leq j \leq w(\kappa) + 1$ and for any Dirichlet character $\phi$ whose conductor $\text{Cond}(\phi)$ is a power of $p$:

$$\text{exp}^*((\kappa, \chi^\iota_{\text{cyc}} \phi^{-1})(c, d \mathcal{Z}_\kappa(r)))) = \tau(\phi) \times \frac{L_{(pr)}(f, \phi^{-1}, j)}{(2\pi \sqrt{-1})^j \Omega_{f_{(pr)}(c, d \mathcal{Z}_\kappa), \infty}^{\text{sgn}(j, \phi)} \cdot f_{(pr)} \otimes \phi^{-1}},$$

where $L_{(pr)}(f, \phi^{-1}, s)$ is the Hecke $L$-function of $f$ twisted by $\phi^{-1}$ whose Euler factors are removed at primes dividing $pr$. The element $\Omega_{f_{(pr)}(c, d \mathcal{Z}_\kappa), \infty}^{\text{sgn}(j, \phi)}$ is a complex period which satisfies Algebraicity Theorem of Shimura presented in [7].
Remark 3.10.

(1) Let $N$ be the tame conductor of Hida family. The Euler system of Theorem 3.9 is obtained by taking the inverse limit of Beilinson-Kato Euler systems on modular curves $Y(Np^n)$ constructed by Kato [Ka04] with respect to norm maps $Y(Np^{n+1}) \to Y(Np^n)$. Kato [Ka04] does not discuss the situation of Hida family at all and it only provides an Euler system for each cuspform $f$. However, he proves the Beilinson-Kato Euler system on modular curves $Y(Np^n)$ are norm compatible in [Ka04]. As seen in [Hi86b], Hida families are constructed by the inverse limit of cohomologies of $Y(Np^n)$ and “cutting out the ordinary part by Hida’s $e$-operator”. Hence we deduce Theorem 3.9 by taking inverse limit of Beilinson-Kato elements in [Ka04].

(2) The parameters $c$, $d$ and $\xi$ appear necessarily through the construction in [Ka04]. Since these parameters are of no importance to understand the rest of the paper we do not explain how these elements $c, d, \xi$ depend on these parameters $c, d, \xi$. See [Ka04] for the detail.

Remark 3.11. If we consider Iwasawa theory for a family of modular forms, there appears an “essential problem on periods” as explained below. The period $\Omega_{f_{\kappa, \infty}}^{\text{p-ann}(j, \phi)}$ in (3.10) is based on Rankin-Selberg method. In order that the ideal $\text{Col}(\text{loc}_{p(c, d; \xi}(1)))$ defines the same principal ideal as $L_p(\{\Xi^\pm\})$ given in Theorem B of this section, it is necessary that, for any arithmetic specialization $\kappa$ of non-negative weight on $I$, the ratio

$$\frac{\Omega_{f_{\kappa, \infty}}^{\text{p-ann}(j, \phi)}}{\Omega_{f_{\kappa, (c, d, \xi), \infty}}^{\text{p-ann}(j, \phi)}} \in \mathbb{Q}$$

is a $p$-adic unit for a complex period $\Omega_{f_{\kappa, \infty}}^{\text{p-ann}(j, \phi)}$ which is $p$-optimal in the sense of Remark 1.4 We are not sure if the Euler system $\{c, d; \xi(r)\}_{r \in S}$ (1.16) satisfies this condition for certain choice of the parameters $c, d, \xi$.

In fact, Kato [Ka04] was already faced with the problem of the comparison of the complex periods as discussed in Remark 3.11. In [Ka04], we only consider a cuspform. Hence he manages to prove that a linear combination of Euler systems for some parameters $(c, d, \xi)$ (see [Ka04] §16) provides a $p$-optimized period.

If we consider a family of cuspforms, we have to optimize the Euler system at infinitely many arithmetic specializations at the same time. Taking a certain finite linear combination like in [Ka04] will not work. Finally, we obtain the following optimized Euler system with help of two-variable Coleman map (Theorem 3.7).

Theorem 3.12. Let us assume the setting of Theorem C’ and let $\mathbb{F}$ and $\mathbb{T}$ as above. Assume that $p \geq 5$. We assume that the image of the residual representation $\overline{\rho} : G_\mathbb{Q} \to \text{GL}_2(\mathbb{F})$ is non-solvable and fix an $I$-basis $\Xi^\pm$ of $\text{MS}(I)^\pm$ respectively.

Then, we have an Euler system (depending on the choice of $\Xi^\pm$):

$$\{Z(r) \in H^1(\mathbb{Q}(\mu_r), (\mathbb{T} \otimes \mathbb{Z}_\mathbb{p} \Lambda_c^\text{cyc})^r(1))\}_{r \in S}$$

such that $\text{Col}(\text{loc}_{p}(Z(r)))$ has the following relation to the special value for any arithmetic specialization $\kappa$ on $I$ of non-negative weight, any integer $j$ satisfying $1 \leq j \leq w(\kappa) + 1$

\[18\] It seems that the condition is necessary but not sufficient.

\[19\] We remark that the irreducibility of the residual representation implies the validity of the condition (M).
and for any Dirichlet character $\phi$ whose conductor $\text{Cond}(\phi)$ is a power of $p$:

\[
(3.11) \quad \frac{\exp^*(\text{loc}_p((\kappa, \chi_{\text{cyc}}^j\phi^{-1}))(\mathcal{Z}(r))))}{\Omega^\text{sgn}(j,\phi)_{f_s,p}} = \tau(\phi) \times \frac{L(p)(f_\kappa, \phi^{-1}, j)}{(2\pi)^{2j-1}} \Omega_f^{\text{sgn}(j,\phi)_\infty} \cdot L_f(\kappa, \phi^{-1}),
\]

where $\Omega^{\text{sgn}(j,\phi)}_{f_s,\infty}$ is a complex period which satisfy Algebraicity Theorem of Shimura presented in in [7] and $\Lambda_{f_s,p}^{\text{sgn}(j,\phi)}$ is a $p$-adic period defined in [72]20.

In particular, we have the equality:

\[
(3.12) \quad \text{Col}^0(\text{loc}_p(\mathcal{Z}(1))) = L_p(\{\Xi^\pm\})
\]
in $\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$.

Though the proof of Theorem 3.12 is found in [Och06, §5.3], we will prove it below since the same method will be used later in a similar but slightly different context of the construction of Euler systems of Coleman families. We prepare a lemma before going into the proof of Theorem 3.12.

**Lemma 3.13.** Let us assume the setting of Theorem $C'$ and let $F$, $T$ and $\mathbb{A}$ as above. Assume that $p \geq 5$. We assume that the image of the residual representation is nonsolvable. Let $r$ be a natural number and put $J, J' \in \mathbb{A}$. Then, we have the exact sequence:

\[
(3.13) \quad 0 \rightarrow H^1((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J \cap J'}) \rightarrow H^1((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J}) \oplus H^1((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J'}) \rightarrow H^1((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J+J'}).\]

Here we denote $(\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J}$ by $(\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J}$ and the cohomology $H^1(\mathbb{Q}(\mu_r) \otimes / \mathbb{Q}(\mu_r), (\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J})$ by $H^1(\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))$ for any ideal $J$ of $\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$.

The first map of (3.13) sends each element $x_{J \cap J'}$ to $(x_{J \cap J'}) \oplus (x_{J \cap J'})$. The second map (3.13) sends each element $x_{J} \oplus y_{J'}$ to $(x_{J} \mod J + J') - (y_{J'} \mod J + J')$.

**Proof.** Let us consider the short exact sequence of $G_{\mathbb{Q}}$-module:

\[
(3.14) \quad 0 \rightarrow ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J \cap J'} \rightarrow ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J} \oplus ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J'}) \rightarrow ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J+J'} \rightarrow 0.
\]

By the assumption that the image of the residual representation $\overline{\rho}$ restricted to $G_{\mathbb{Q}(\mu_r)}$ is irreducible for any natural number $r$. Thus, $H^0(G_{\mathbb{Q}(\mu_r)}, ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^* (1))_{J+J'}) = 0$ for any natural number $r$. Thus by taking the Galois cohomology of the long exact sequence of $\text{Gal}(\mathbb{Q}(\mu_r) \otimes \mathbb{Q}(\mu_r))$-module associated to the short exact sequence (3.14), we obtain (3.13). This completes the proof.

Now let us go into the proof of Theorem 3.12.

**Proof of Theorem 3.12.** Let us define the following set

\[
(3.15) \quad \mathcal{S} = \{I = \text{Ker}(\kappa) \mid \kappa \text{ is an arithmetic specialization on } \mathbb{I} \text{ of non-negative weight}\}.
\]

See Remark 3.2 for the fact that the interpolation property is well-defined.
We denote by $\mathfrak{A}$ a subset of the set of height one ideals of $\widehat{\mathbb{Z}}_p \Lambda_{\text{cyc}}$ as follows:

\begin{equation}
\mathfrak{A} = \left\{ J = \bigcap_{i \in S} I \cdot \widehat{\mathbb{Z}}_p \Lambda_{\text{cyc}} \mid S \subset \mathcal{S}, zS < \infty \right\}.
\end{equation}

Note that $J \cap J' \in \mathfrak{A}$ for any $J, J' \in \mathfrak{A}$ and that the intersection $\bigcap J$ for infinitely many $J \in \mathfrak{A}$ is zero. For any natural number $r$, we define $\Sigma_r$ to be $\Sigma_r = \Sigma \cup \{ \text{primes } q \text{ dividing } r \}$. We denote by $\mathbb{Q}_{\Sigma_r}$ the maximal unramified extension of $\mathbb{Q}$ unramified outside $\Sigma_r$.

Let us consider the following morphism:

\begin{equation}
H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^t)^*(1))(1)) \xrightarrow{\text{loc}} H^1(\mathbb{Q}_p, ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^t)^*(1))(1))
\end{equation}

\begin{equation}
\xrightarrow{(d_J \otimes 1, \circ \text{Col})} (\mathbb{I}/J) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}
\end{equation}

where $d_J$ is a basis of free $\mathbb{I}/J$-module $\mathbb{D}^\text{ord}/J\mathbb{D}^\text{ord}$ of rank one which coincide with $j_\kappa$ for any arithmetic specialization $\kappa$ on $\mathbb{I}$ and $(\cdot, \cdot)$ is the paring:

\begin{equation}
(\mathbb{D}^\text{ord}/J\mathbb{D}^\text{ord}) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}} \times ((\mathbb{D}^\text{ord})^*/J(\mathbb{D}^\text{ord})^*) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}} \rightarrow (\mathbb{I}/J) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}
\end{equation}

Now we take $I = \text{Ker}(\kappa)$ and $I' = \text{Ker}(\kappa')$ from $\mathcal{S}^\prime$ given in (3.15). Set $J = I \cdot \widehat{\mathbb{Z}}_p \Lambda_{\text{cyc}}$ and $J' = I' \cdot \widehat{\mathbb{Z}}_p \Lambda_{\text{cyc}}$.

As explained before Theorem 3.12 the result of Kato for the cyclotomic deformation of a cuspform assures the existence of the Euler system:

\begin{equation}
\{ Z_J(r) \in H^1(\mathbb{Q}(\mu_r), ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^t)^*(1))(1)) \}_{r \in \mathcal{S}}
\end{equation}

\begin{equation}
(\text{resp. } \{ Z_{J'}(r) \in H^1(\mathbb{Q}(\mu_r), ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^t)^*(1))(1)) \}_{r \in \mathcal{S}})
\end{equation}

such that we have

\begin{equation}
\text{Col}(\text{loc}_p(Z_J(1))) = L_p(\{\Xi^\pm\}) \text{ mod } J \text{ in } (\mathbb{I}/J) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}
\end{equation}

\begin{equation}
(\text{resp. } \text{Col}(\text{loc}_p(Z_{J'}(1))) = L_p(\{\Xi^\pm\}) \text{ mod } J' \text{ in } (\mathbb{I}/J') \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}})
\end{equation}

Note that, since $L_p(\{\Xi^\pm\})$ is an element of $\mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$, $L_p(\{\Xi^\pm\}) \text{ mod } J$ and $L_p(\{\Xi^\pm\}) \text{ mod } J'$ can be glued together to have an element of $(\mathbb{I}/(\mathbb{J} \cap \mathbb{J})) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$. By the injectivity of the map $(d_J \otimes 1, \circ \text{Col})$ for $(\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^t)^*(1))(1))$ and by Lemma 3.13 we have the first layer $Z_{J \cap J'}(1) \in H^1(\mathbb{Q}(\mu_r), ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^t)^*(1))(1) J \cap J')$ which glues $Z_{J'}(1)$ and $Z_{J'}(1)$. Similarly for each $r \in \mathcal{S}$, we can use the Coleman map $\text{Col}(r)$ over $\mathbb{Q}_p \otimes \mathbb{Q}(\mu_r)$ and the fact that we have

\begin{equation}
\text{Col}(r)(\text{loc}_p(Z_J(r))) = L_p(r)(\{\Xi^\pm\}) \text{ mod } J \text{ in } (\mathbb{I}/J) \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}
\end{equation}

\begin{equation}
(\text{resp. } \text{Col}(r)(\text{loc}_p(Z_{J'}(r))) = L_p(r)(\{\Xi^\pm\}) \text{ mod } J' \text{ in } (\mathbb{I}/J') \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}})
\end{equation}

where $L_p(r)(\{\Xi^\pm\}) \in \mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$ is the $p$-adic $L$-function obtained by removing Euler factors at primes dividing $r$ from $L_p(\{\Xi^\pm\})$. Thus we obtain a unique Euler system

\begin{equation}
\{ Z_{J \cap J'}(r) \in H^1(\mathbb{Q}(\mu_r), ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^t)^*(1))(1) J \cap J') \}_{r \in \mathcal{S}}
\end{equation}

which glues Euler systems $\{ Z_J(r) \}_{r \in \mathcal{S}}$ and $\{ Z_{J'}(r) \}_{r \in \mathcal{S}}$.

By repeating the induction on the numbers of the ideals of the form $I \cdot \mathbb{I} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$ with $I \in \mathcal{S}$ containing a given ideal $J \in \mathfrak{A}$, we construct for each $J \in \mathfrak{A}$ an Euler system $\{ Z_J(r) \in H^1(\mathbb{Q}(\mu_r), ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^t)^*(1))(1)) \}_{r \in \mathcal{S}}$ such that it provides an Euler system for
each cuspform corresponding to the representation mod $I \cdot \Xi_{Z_p} \Lambda_{cyc}$ with $I \in \mathcal{S}$ such that $J \subset I \cdot \Xi_{Z_p} \Lambda_{cyc}$. The Euler systems \{ $Z_J(r)$ \}$_{r \in \mathcal{S}}$ form an inverse system for $J \subset \mathfrak{A}$. Since we have
$$
\lim_{J \to \mathfrak{A}} H^1(\mathbb{Q}(\mu_r), ((T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)), 1) \cong H^1(\mathbb{Q}(\mu_r), ((T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)) )
$$
we obtain the desired Euler system \{ $Z(r)$ \}$_{r \in \mathcal{S}}$ by putting $Z(r) = \lim_{J \to \mathfrak{A}} Z_J(r)$. Here, we remark that the topology defined by \{ $J \in \mathfrak{A}$ \} is equivalent to the topology defined by the maximal ideals of $\Xi_{Z_p} \Lambda_{cyc}$ by the fact that $\Xi_{Z_p} \Lambda_{cyc}$ is complete semi-local and by well-known Chevalley’s theorem (cf. [Ch43, §II, Lemma 7]).

After we constructed a $p$-optimized Euler system, the argument goes similarly as the one-variable cyclotomic Iwasawa theory explained in [11]. We go back to the proof of (3.6) in Theorem C’.

**Proof of (3.6) in Theorem C’**. By the Poitou-Tate exact sequence of the Galois cohomology, Euler-Poincare characteristic formula of Galois cohomology and the existence of nontrivial Euler system explained above, we have the following sequence of $\Xi_{Z_p} \Lambda_{cyc}$-modules:

\[
(3.17) \quad 0 \rightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, (T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)) \rightarrow \frac{H^1(q_p, (T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1))}{\text{Im}(H^1(q_p, F_p^+(T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)))} \rightarrow \text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee \rightarrow H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, (T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)) \rightarrow 0
\]

such that

(i) The sequence (3.17) is exact modulo zero by pseudo-null $\Xi_{Z_p} \Lambda_{cyc}$-modules.

(ii) The last two terms are torsion over every local component of $\Xi_{Z_p} \Lambda_{cyc}$.

(iii) The first two terms are of rank one over every local component of $\Xi_{Z_p} \Lambda_{cyc}$.

The reason of these statements (i), (ii) and (iii) is similar to the reason of similar statements (ii), (ii) and (iii) explained in the proof of Theorem C.

Now, we take the first layer of the modified Beilinson–Kato Euler system:

\[
Z(1) \in H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, (T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)).
\]

obtained in Theorem 3.12 and we denote by $\overline{\text{loc}}_p$ the $\Xi_{Z_p} \Lambda_{cyc}$-linear homomorphism:

\[
H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, (T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)) \rightarrow \frac{H^1(q_p, (T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1))}{\text{Im}(H^1(q_p, F_p^+(T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)))}.
\]

Then the sequence (3.17) induces the following sequence of torsion $\Xi_{Z_p} \Lambda_{cyc}$-modules:

\[
(3.18) \quad 0 \rightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, (T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)) / \Xi_{Z_p} \Lambda_{cyc} \cdot Z(1)
\]

\[
\rightarrow \frac{H^1(q_p, (T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1))}{\text{Im}(H^1(q_p, F_p^+(T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)))} / \Xi_{Z_p} \Lambda_{cyc} \cdot \overline{\text{loc}}_p(Z(1))
\]

\[
\rightarrow \text{Sel}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee \rightarrow H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, (T \otimes_{Z_p} \Lambda_{cyc}^\sharp)^*(1)) \rightarrow 0
\]
which is exact modulo error by pseudo-null $I \otimes_{\Z_p} \Lambda^2_{\text{cyc}}$-modules. Finally, by applying Theorem 3.5 for $R = I$, we obtain

\[ (3.19) \quad \text{char}_{I \otimes_{\Z_p} \Lambda_{\text{cyc}}} \left( H^1(Q_{\Sigma}/Q, (T \otimes_{\Z_p} \Lambda^2_{\text{cyc}}) \cdot Z(1)) \right) \subset \text{char}_{I \otimes_{\Z_p} \Lambda_{\text{cyc}}} \left( H^2(Q_{\Sigma}/Q, (T \otimes_{\Z_p} \Lambda^2_{\text{cyc}}) \cdot Z(1)) \right). \]

We recall that we have the following inclusion thanks to (3.12):

\[ (3.20) \quad \text{char}_{I \otimes_{\Z_p} \Lambda_{\text{cyc}}} \left( \frac{H^1(Q_p, (T \otimes_{\Z_p} \Lambda^2_{\text{cyc}}) \cdot Z(1))}{\text{Im}(H^1(Q_p, F^+(T \otimes_{\Z_p} \Lambda^2_{\text{cyc}}) \cdot Z(1)) \otimes \Lambda_{\text{cyc}}} \right) = \left( L_p(\{e\})\right). \]

By combining (3.18), (3.19) and (3.20), we obtain (3.6). \qed

4. COLEMAN MAP FOR COLEMAN FAMILY

In the non-ordinary situation, the evaluations of the $p$-adic $L$-functions might have denominators which are divisible by higher powers of $p$ depending on the power of $p$ which divides the conductor of evaluating characters. Hence, $p$-adic $L$-functions in non-ordinary case might need to be formulated as elements in a space of distributions which allows unbounded denominators. Though distributions with unbounded denominators might not be well-behaved as in the case of bounded measures, the space of distributions $H_{h,\text{cyc}}$ introduced by Amice–Vélu is quite well-behaved and has nice properties to characterize and recover a function in $H_{h,\text{cyc}}$ from the values of its evaluations (see Proposition 4.2). Thus it is quite important to work with $H_{h,\text{cyc}}$.

Definition 4.1. (1) For $i \in \Z_{\geq 0}$, we set $\ell(0) = 0$ and we define $\ell(i)$ to be \[ \left\lceil \frac{\ln(i)}{\ln(p)} \right\rceil + 1 \] if $i \geq 1$.

(2) For $h \geq 0$, we define

\[ H_h = \left\{ \sum_{i=0}^{+\infty} a_i X^i \in Q_p[[X]] \mid \inf \{ \text{ord}(a_i) + h\ell(i) \mid i \in \Z_{\geq 0} \} > -\infty \right\} \]

and call the elements of $H_h$ the power series of logarithmic order $h$.

We have a Banach norm on $H_h$ and we have an integral structure $H^+_h \subset H_h$ such that $H_h \cong H^+_h \otimes_{\Z_p} Q_p$ by means of this norm (see [NO16, §4.1]). We do not give a precise definition of the Banach-module structure on $H_h$. But we only remark that $H^+_h$ is equal to $Z_p[[X]]$.

For every $j \in \Z_{\geq 0}$, we set

\[ \omega_n^{[j]} = \omega_n^{[j]}(X) = (1 + X)^{p^n} - (1 + p)^{p^n}. \]

For a fixed $n$ and for different $j \neq j'$ the polynomials $\omega_n^{[j]}(X)$ and $\omega_n^{[j']}(X)$ have no common factor in $Z_p[[X]]$. We will denote $\omega_n^{[0]}(X)$ by $\omega_n(X)$. Finally, we recall that $A_{\text{cyc}}$ is a semi-local ring isomorphic to $Z_p[\text{Gal}(Q(\mu_p)/Q)] \otimes Z_p[\text{Gal}(Q(\mu_{p^n})/Q(\mu_p))]$ which is non-canonically isomorphic to $Z_p[\text{Gal}(Q(\mu_{p^n})/Q(\mu_p))]$ induced by the Iwasawa-Serre isomorphism $Z_p[\text{Gal}(Q(\mu_{p^n})/Q(\mu_p))] \cong Z_p[[X]] = H^+_0$ sending a chosen topological generator $\gamma$.
of \( \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p)) \) to \( 1 + X \). We define \( \mathcal{H}_{h,\text{cyc}} \) (resp. \( \mathcal{H}^+_{h,\text{cyc}} \)) to be

\[
\mathcal{H}_{h,\text{cyc}} = \mathbb{Z}_p[\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))] \otimes_{\mathcal{H}_0^+} \mathcal{H}_h
\]

(resp. \( \mathcal{H}^+_{h,\text{cyc}} = \mathbb{Z}_p[\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))] \otimes_{\mathcal{H}_0^+} \mathcal{H}^+_h \)).

We have the following important results of Amice–Vélu, which we can find also in §1.2 and §1.3 of [Pe94]:

**Proposition 4.2.** Let \( F \in \mathcal{H}_{h,\text{cyc}} \) and let us choose \( l, l' \in \mathbb{Z} \) such that \( h = l' - l \). Suppose that \( F \) is contained in \( Q\mathcal{H}_{h,\text{cyc}} \) for every element of \( Q[l,l'] \). Then we have \( F = 0 \).

**Proposition 4.3.** Let us choose \( l, l' \in \mathbb{Z} \) with \( l \leq l' \) and put \( h = l' - l \). Let \( \{G_{n,j} \in K[X]\}_{n \in \mathbb{Z}_{\geq 1}, l \leq j \leq l'} \) be a sequence of polynomials satisfying the following conditions:

1. For each \( j \) satisfying \( l \leq j \leq l' \), \( \|p^n G_{n,j}\| \) is bounded when \( n \) varies.
2. For each \( j \) satisfying \( l \leq j \leq l' \) and for each \( n \in \mathbb{Z}_{\geq 1} \), \( G_{n+1,j} - G_{n,j} \equiv 0 \) modulo \( \omega_n(X) K[X] \).
3. For each \( j \) satisfying \( l \leq j \leq l' \),

\[
\left\| p^{(h-j)} \sum_{k=l}^{j} (-1)^{j-k} \binom{j}{k} G_{n,l+k}((1 + X) - (1 + p)^k) \right\|
\]

is bounded when \( n \) varies.

Then there exists a unique element \( F \in \mathcal{H}_h \) such that \( F \equiv G_{n,j} \mod \omega_n^{[j]} \mathcal{H}_h \) for every \( n \in \mathbb{Z}_{\geq 1} \) and for every \( j \) satisfying \( l \leq j \leq l' \).

Let us define a \( \Lambda_{(k_0:r)} \mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \)-module \( \mathbb{D} \) by

\[
\mathbb{D} := (\mathbb{T} \otimes_{\mathbb{Z}_p} B_{\text{cris}})^{G_{\mathbb{Q}_{\mathbb{F}}(\mathbb{F}_p)}}_{\mathbb{Z}}(\mathbb{A}_p(\mathbb{F})).
\]

**Lemma 4.4.** The \( \Lambda_{(k_0:r)} \mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \)-module \( \mathbb{D} \) is free of rank one over \( \Lambda_{(k_0:r)} \mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) and the specialization of \( \mathbb{D} \) at each arithmetic point \( k \in \mathbb{Z} \cap B(k_0;r) \) is canonically identified with \( D_k := D_{\text{cris}}^{\mathbb{Z}}(\mathbb{V}_k) \).

The lemma is essentially due to Kisin’s result (See [Kis03 Corollary 5.6]). However the paper [Kis03] does not really prove exactly the same result as in Lemma 4.4. For the proof of Lemma 4.4 we refer the reader to [NO16, Lemma 3.4].

After this preparation, we can now recall the main result of the paper [NO16], namely the existence of the Coleman maps for Coleman families.

**Theorem 4.5** (Nuccio-Ochiai). Let \( s \in \mathbb{Q}_{\geq 0} \) be the slope of the Coleman family \( \mathbb{F} \) and \( h \) an integer satisfying \( h \geq s \). Assume that the residual representation \( p : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F}_p) \) associated to the family is irreducible when restricted to \( G_{\mathbb{Q}_{\mathbb{F}}(\mathbb{F}_p)} \).

Then, we have a unique \( \Lambda_{(k_0:r)} \mathcal{O}_K \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}} \)-linear big exponential map

\[
\text{Col} : H^1(\mathbb{Q}_p, (\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}})^*(1)) \to \mathbb{D}^*(1) \otimes_{\mathbb{Z}_p} \mathcal{H}_{h,\text{cyc}}
\]
such that, for each arithmetic point $k \in \mathbb{Z} \cap B(k_0; r)$ and for each arithmetic character $\chi_{\text{cyc}} \phi : G_{\text{cyc}} \to \hat{\mathbb{Q}}_p^\times$ with $j$ a positive integer, we have the following commutative diagram:

$$
\begin{array}{ccc}
H^1\left(q_p, (T \hat{\otimes}_p \Lambda^j_{\text{cyc}})\right)^{\star}(1) & \xrightarrow{\text{Col}} & \mathbb{D}^\star(1) \hat{\otimes}_p \mathcal{H}_{h, \text{cyc}} \\
(k, \chi_{\text{cyc}} \phi^{-1}) & & \downarrow \text{exp}^{\star} \\
H^1\left(q_p, (V_{f_k} \otimes \phi)^{\ast} \otimes \chi_{\text{cyc}}^{-1} \right) & \xrightarrow{e_p(f_k, j, \phi) \times \exp^{\star}} & (D_k)^{\star}(1) \otimes D_{\text{dr}}(\chi_{\text{cyc}}^{-1} \phi^{-1})
\end{array}
$$

where $e_p(f_k, j, \phi)$ is as given in Theorem C.

**Remark 4.6.** We give some remarks on the assumption of Theorem 4.5 on the irreducibility of the residual representation $\pi : G_Q \to \text{GL}_2(\mathbb{F}_p)$ restricted to $G_{Q_p(\mu_p)}$.

1. Thanks to the description of the mod $p$ modular Galois representation $G_Q \to \text{GL}_2(\mathbb{F}_p)$ of Serre Conjecture, $\pi$ restricted to $G_{Q_p(\mu_p)}$ is irreducible if and only if $\pi$ restricted to $G_{Q_p}$ is irreducible.

2. This assumption implies that $H^1\left(q_p, (T \hat{\otimes}_p \Lambda^j_{\text{cyc}})\right)^{\star}(1)$ is free of rank two over each local component of $\Lambda_{(k_0, r), \mathcal{O}_K} \hat{\otimes}_p \Lambda_{\text{cyc}}$. By this assumption, we also see that $H^1\left(q_p, (T \hat{\otimes}_p \Lambda^j_{\text{cyc}})\right)^{\star}(1)$ is exactly controlled under the specialization maps to all arithmetic points. These properties help us on the proof of the above theorem in [NO16].

To be precise, Main Theorem of [NO16] gives a $\Lambda_{(k_0, r), \mathcal{O}_K} \hat{\otimes}_p \Lambda_{\text{cyc}}$-linear map

$$
\text{EXP} : \mathbb{D} \hat{\otimes}_p \Lambda_{\text{cyc}} \to H^1\left(q_p, T \hat{\otimes}_p \Lambda^j_{\text{cyc}}\right) \hat{\otimes}_{\Lambda_{\text{cyc}}} \mathcal{H}_{h, \text{cyc}}
$$

which interpolates exponential maps (but not dual exponential maps as stated in Theorem 4.5) for each arithmetic point $k \in \mathbb{Z} \cap B(k_0; r)$ and for each arithmetic character $\chi_{\text{cyc}} \phi : G_{\text{cyc}} \to \hat{\mathbb{Q}}_p^\times$ with $j$ a positive integer. However, once we obtain an interpolation of exponential maps, it is straightforward to obtain an interpolation of dual exponential maps by taking its Kummer dual. In fact, we can extend the scalars to the ring $\mathcal{H}_{\infty, \text{cyc}} = \bigcup_{h=0}^{\infty} \mathcal{H}_{h, \text{cyc}}$ and we obtain a $\Lambda_{(k_0, r), \mathcal{O}_K} \hat{\otimes}_p \Lambda_{\text{cyc}}$-linear map

$$
\text{EXP} \otimes \mathcal{H}_{\infty, \text{cyc}} : \mathbb{D} \hat{\otimes}_p \mathcal{H}_{\infty, \text{cyc}} \to H^1\left(q_p, T \hat{\otimes}_p \Lambda^j_{\text{cyc}}\right) \hat{\otimes}_{\Lambda_{\text{cyc}}} \mathcal{H}_{\infty, \text{cyc}}
$$

By Tate local duality of Galois cohomology, we have an canonical isomorphism

$$
\text{Hom}_{\Lambda_{\text{cyc}}}(H^1\left(q_p, T \hat{\otimes}_p \Lambda^j_{\text{cyc}}\right), \Lambda_{\text{cyc}}) \cong H^1\left(q_p, (T \hat{\otimes}_p \Lambda^j_{\text{cyc}})^{\ast}(1)\right)
$$

Hence, by taking the $\mathcal{H}_{\infty, \text{cyc}}$-linear dual of the map (1.2), we obtain a $\Lambda_{(k_0, r), \mathcal{O}_K} \hat{\otimes}_p \Lambda_{\text{cyc}}$-linear map

$$
H^1\left(q_p, (T \hat{\otimes}_p \Lambda^j_{\text{cyc}})^{\ast}(1)\right) \hat{\otimes}_{\Lambda_{\text{cyc}}} \mathcal{H}_{\infty, \text{cyc}} \to \mathbb{D}^{\ast}(1) \hat{\otimes}_p \mathcal{H}_{\infty, \text{cyc}}
$$

which have exactly the same interpolation property as Theorem 4.5. The map $\text{Col}$ is defined to be a $\Lambda_{(k_0, r), \mathcal{O}_K} \hat{\otimes}_p \Lambda_{\text{cyc}}$-linear map obtained by restricting the map (1.3) to $H^1\left(q_p, (T \hat{\otimes}_p \Lambda^j_{\text{cyc}})^{\ast}(1)\right) \subset H^1\left(q_p, (T \hat{\otimes}_p \Lambda^j_{\text{cyc}})^{\ast}(1)\right) \hat{\otimes}_{\Lambda_{\text{cyc}}} \mathcal{H}_{\infty, \text{cyc}}$. By the growth of the denominators of map $\text{Col}$ thus defined, it is clear that the image of $\text{Col}$ falls in $\mathbb{D}^{\ast}(1) \hat{\otimes}_p \mathcal{H}_{h, \text{cyc}} \subset \mathbb{D}^{\ast}(1) \hat{\otimes}_p \mathcal{H}_{\infty, \text{cyc}}$.

Below, we note some important ingredients that go into the proof of Theorem 4.5.
(1) Theorem 3.7 was proved by reducing to Coleman map for a rank-one Galois deformation thanks to Theorem 2.5 which says that the Galois deformation associated to a Hida deformation is locally an extension of rank-one deformations (see Remark 1.10 and Remark 3.8). Since the analogue of Theorem 2.5 does not hold for Coleman families, the proof of Theorem 4.5 is rather close to original techniques of Perrin-Riou in [Pe94] and [Pe01], which consists of gluing argument of exponential maps based on explicit calculation of the growth of denominators.

(2) As for the technical difference of [NO16] from [Pe94] and [Pe01], [NO16] takes care of the integral structure $\mathcal{H}_h^+$ of the Banach module $\mathcal{H}_h$, which did not appear in [Pe94] and [Pe01]. In [NO16], we need to calculate some of constants which appeared in [Pe94] and make them explicit. Since we work on a family, we need to assure that all these constants are universally bounded with respect to the integral structure $\mathcal{H}_h^+$ in a Coleman family.

5. APPLICATION TO TWO-VARIABLE IWASAWA MAIN CONJECTURE

In this section, we apply Theorem 4.5 and other techniques explained in earlier sections to obtain a partial result on the two-variable Iwasawa Main Conjecture for Coleman families.

In the ordinary case, the construction of an Euler system as in Theorem 3.9 was quite automatic as inverse limit with respect to the $p$-power of the level of the modular curves $Y(Np^n)$ as explained in Remark 3.10 (1). In the non-ordinary case, we do not have such a construction of an Euler system over the family and the construction of an Euler system over Coleman family is possible only under strong conditions by which the gluing of Euler systems works with help of Coleman map and $p$-adic $L$-function (see Theorem 5.4).

Let us fix the setting of Coleman family as in §2. We have a certain open disc of radius $p^r$ centered at some weight $k_0$ and a Coleman family $F = \sum_{n=1}^{\infty} A_n(F) q^n \in \Lambda(k_0,r,\mathcal{O}_K[[q]])$ over $\Lambda(k_0,r,\mathcal{O}_K)$. Let $s \in \mathbb{Q}_{\geq 0}$ be the slope of the family. By Theorem 2.10, we associate a Galois representation $T \cong (\Lambda(k_0,r,\mathcal{O}_K))^\otimes 2$ on which $G_\mathbb{Q}$ acts continuously. By Kato’s result [Ka04] discussed in §1, we already know that Beilinson-Kato Euler systems exist in a point-wise manner at arithmetic points $k$ on $\Lambda(k_0,r,\mathcal{O}_K)$. We need to construct an Euler system over the whole $\Lambda(k_0,r,\mathcal{O}_K)$ which amounts to an analogue of Theorem 3.12 of ordinary case. In the non-ordinary case, the situation is different from the ordinary situation. We will show that we can glue Beilinson-Kato Euler systems over arithmetic points to obtain an Euler system over $\Lambda(k_0,r,\mathcal{O}_K)$ under some assumptions. The construction of an Euler system is realized with help of Coleman map for Coleman family which was given by Theorem 4.5 and two-variable $p$-adic $L$-functions in $\Lambda(k_0,r,\mathcal{O}_K) \otimes \mathbb{Z}_p \mathcal{H}_h^{cyc}$ which will be given by Theorem 5.1 where $h$ is an integer satisfying $h \geq s$.

5.1. Two-variable $p$-adic $L$-function on Coleman family. The construction of the two-variable $p$-adic $L$-function in the non-ordinary case proceeds in the same manner as the ordinary case (Theorem B in §3) except that the interpolated values of the $p$-adic $L$-function has denominators and we have to extend the coefficient to $\mathcal{H}_h^{cyc} \supset \Lambda_{cyc}$. Following the construction of [Kit94] in the ordinary case (see Theorem B in §3), we need to introduce the module of $\Lambda_{k_0,r,\mathcal{O}_K}$-adic modular symbols $"\mathcal{MS}(\Lambda(k_0,r,\mathcal{O}_K)"^\pm$ which serve as an analogue of the module of $\mathbb{I}$-adic modular symbols $\mathcal{MS}(\mathbb{I})^{\pm}$ in §3.
We have the following theorem for the existence of two-variable non-ordinary $p$-adic $L$-function which is an non-ordinary analogue of \cite{Kit94} as follows.

**Theorem 5.1.** Let $\mathbb{F}$ be a Coleman family over $\Lambda_{(k_0,r)}$ with the slope $s \in \mathbb{Q}_{\geq 0}$. Let $h$ be an integer satisfying $h \geq s$. Let $\mathbb{T} \cong (\Lambda_{(k_0,r)})^{\otimes 2}$ be the Galois representation associated to $\mathbb{F}$. Assume that the residual representation $\overline{\pi} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F}_p)$ associated to the family is irreducible. Then

1. There exists a $\Lambda_{(k_0,r),\mathcal{O}_K}$-module of $\Lambda_{(k_0,r),\mathcal{O}_K}$-adic modular symbols $\overline{\text{MS}}(\Lambda_{(k_0,r)},\mathcal{O}_K)\pm$ which is free of rank one over $\Lambda_{(k_0,r),\mathcal{O}_K}$ and interpolates the $f_k$-part of classical modular symbols with coefficients in $\mathcal{O}_K$ for arithmetic points $k \in \mathbb{Z}_{\geq s+1} \cap \mathcal{B}(k_0; r)$.

2. Let us fix a $\Lambda_{(k_0,r)}$-basis $\Xi \pm$ of $\overline{\text{MS}}(\Lambda_{(k_0,r)})\pm$ respectively. Then, there exists an analytic $p$-adic $L$-function $L_p(\{\Xi \pm\}) \in \Lambda_{(k_0,r)} \hat{\otimes}_{\mathbb{Z}_p} \mathcal{H}_{h,\text{cyc}}^+\pm$ such that we have the following interpolation formula:

\[
\frac{(k \times \chi^j \phi)(L_p(\{\Xi \pm\}))}{C_{s,J}(j,\phi)} = (-1)^j(j-1)! \times \epsilon_p(f_k, j, \phi) \times \tau(\phi) \times \frac{L(f_k, \phi^{-1}, j)}{(2\pi \sqrt{-1})^j} \xi_{f_k,\infty},
\]

for any arithmetic point $k$ of $\Lambda_{(k_0,r)}$ larger than the slope of the family, for any integer $j$ satisfying $1 \leq j \leq k-1$ and for any Dirichlet character $\phi$ whose conductor $\text{Cond}(\phi)$ is a power of $p$, where $\tau(\phi)$ and $\epsilon_p(f, j, \phi)$ is the same as Theorem B of [7]. Here, for each $k \in \mathbb{Z}_{\geq s+1} \cap \mathcal{B}(k_0; r)$, the specialization of $\overline{\text{MS}}(\Lambda_{(k_0,r),\mathcal{O}_K})\pm$ at $k$ is naturally identified with a lattice of $H^1_c(Y_1(M)_{\mathbb{C}}, \mathcal{L}_{k-2}(\mathbb{Q}_{f_k}))\pm[f_k] \otimes_{\mathbb{Q}_{f_k}} K$ and $C_{f_k,p} \in K$ is a $p$-adic period which is defined to be an error term given by $[21]:$

\[
\Xi^\pm(k) = C_{f_k,p} \cdot b_{f_k}^\pm \otimes 1.
\]

**Remark 5.2.** We remark that the existence of two-variable non-ordinary $p$-adic L-function is already known. There are several similar results. Glenn Stevens announced such a result early 90’s by developing method of families of distributions-valued modular symbols over the weight space. Though it was never published, Joël Bellaïche [Hel12] publishes a result of two variable $p$-adic $L$-functions over eigencurves along Stevens method (see [Hel12] Theorem 3).

Panchishkin [Pa03] also published a result on a similar results to Theorem 5.1. His construction relies on Rankin-Selberg method and there appears no $p$-adic periods (analogue of $C_{s,J}(f_k)\pm$ of the left-hand side of (5.1)) in the interpolation of his two-variable $p$-adic $L$-function.

The following Remark is parallel to Remark 3.2 of the ordinary case and is an important remark which assures that the interpolation property of Theorem 5.1 is well-defined and makes sense.

**Remark 5.3.** As was cautioned also in the ordinary case, $C_{f_k,p}^\pm$ and $\Omega_{f_k,\infty}^\pm$ depend on the choice of a $\mathbb{Q}_{f_k}$-basis $b_{f_k}^\pm$ on $H^1_c(Y_1(M)_{\mathbb{C}}, \mathcal{L}_{k-2}(\mathbb{Q}_{f_k}))\pm[f_k]$ where $M$ is the level of $f_k$. However, the “ratio” is independent of the choice of $b_{f_k}^\pm$. If we denote the $p$-adic period and the complex period obtained by another $\mathbb{Q}_{f_k}$-basis $(b_{f_k}^\pm)'$ on $H^1_c(Y_1(M)_{\mathbb{C}}, \mathcal{L}_{k-2}(\mathbb{Q}_{f_k}))\pm[f_k]$.

\textsuperscript{21}Recall that the $p$-adic completion of the Hecke field $\mathbb{Q}_f$ is contained in the fraction field $K$ of $\mathcal{O}_K$. 

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\textsuperscript{21}Recall that the $p$-adic completion of the Hecke field $\mathbb{Q}_f$ is contained in the fraction field $K$ of $\mathcal{O}_K$. 

---
by \((C_{f,k}^\pm,p)^{e}\) and \((\Omega_{f,k}\)∞,∞)\(^{e}\), we have
\[
\frac{C_{f,k}^\pm}{(C_{f,k}^\pm,p)^{e}} = \frac{\Omega_{f,k}\)∞,∞}{}^{e}.
\]

**Proof of Theorem 5.1.** Recall that, when \(k\) varies in \(\mathbb{Z}_{>s+1} \cap B(k_0; r)\), the compactly supported cohomology \(H^1_c(Y_1(M)_{\mathbb{C}}, \mathcal{L}_{k-2}(\mathcal{O}_K))^\pm[f_k]\) is naturally identified with the modules of modular symbols with coefficient in \(\mathcal{O}_K\) (To understand the identification, see [Kit94] §3.2 for example).

Since we assume that the residual representation \(G_{\mathbb{Q}} \rightarrow \text{Aut}_{\Lambda(k_0,r),\mathcal{O}_K/\mathfrak{m}(\mathbb{T}/\mathcal{M})}\) associated to the Coleman family \(\mathcal{F}\) is irreducible, we have a natural isomorphism
\[
H^1_{\text{par}}(Y_1(M)_{\mathbb{C}}, \mathcal{L}_{k-2}(\mathcal{O}_K))^\pm[f_k] \rightarrow H^1_c(Y_1(M)_{\mathbb{C}}, \mathcal{L}_{k-2}(\mathcal{O}_K))^\pm[f_k]
\]
from the parabolic cohomology to the compactly supported cohomology at any arithmetic point \(k \in \mathbb{Z} \cap B(k_0; r)\) larger than \(s + 1\). On the other hand, by the comparison of the Betti cohomology and the étale cohomology, we have an isomorphism as follows
\[
H^1_{\text{par}}(Y_1(M)_{\mathbb{C}}, \mathcal{L}_{k-2}(\mathcal{O}_K))^\pm[f_k] \cong H^1_{\text{par,\text{ét}}}(Y_1(M)_{\overline{\mathbb{Q}}}, \mathcal{L}_{k-2}(\mathcal{O}_K))[[f_k]],
\]

Recall that \(H^1_{\text{par,\text{ét}}}(Y_1(M)_{\overline{\mathbb{Q}}}, \mathcal{L}_{k-2}(\mathcal{O}_K))[[f_k]]\) which appears in the right-hand side of (5.4) is naturally equipped with continuous Galois action of \(G_{\mathbb{Q}}\) by the functoriality of étale cohomology and it is nothing but an integral lattice of the Galois representation for \(f_k\) as proved in [Del71]. Hence \(\mathbb{T}^\pm\) interpolates \(H^1_{\text{par,\text{ét}}}(Y_1(M)_{\overline{\mathbb{Q}}}, \mathcal{L}_{k-2}(\mathcal{O}_K))[[f_k]]\) when \(k\) varies in the arithmetic points \(k \in \mathbb{Z} \cap B(k_0; r)\) larger than \(s + 1\). Thanks to the above identification, we define the \(\Lambda(k_0,r),\mathcal{O}_K\)-module of \(\Lambda(k_0,r),\mathcal{O}_K\)-adic modular symbols \(M\)\(\mathcal{S}(\Lambda(k_0,r),\mathcal{O}_K)^\pm\) to be
\[
M\mathcal{S}(\Lambda(k_0,r),\mathcal{O}_K)^\pm = \mathbb{T}^\pm
\]
each of the signs \(\pm\).

Once the space of modular symbols are interpolated and we have a module of \(\Lambda\)-adic modular symbols \(M\mathcal{S}(\Lambda(k_0,r),\mathcal{O}_K)^\pm\), the construction of the two-variable \(p\)-adic \(L\)-function \(L_p(\{\Xi^\pm\})\) as stated in Theorem [5.1] is parallel to the construction of the two-variable \(p\)-adic \(L\)-function done by [Kit94] out of the existence of modules of \(\Lambda\)-adic modular symbols in the ordinary case, except that we need to extend the coefficient to \(\mathcal{H}_{h,\text{cyc}}\) because of denominators. For each \(k \in \mathbb{Z} \cap B(k_0; r)\), we denote by \(I_k\) the principal prime ideal of \(\Lambda(k_0,r),\mathcal{O}_K\), which corresponds to the closed point \(k\) of \(\cap B(k_0; r)\). Let Let us define the following set
\[
\mathcal{G} = \{I_k \mid k \in \mathbb{Z}_{>s+1} \cap B(k_0; r)\}.
\]
We denote by \(\mathfrak{A}\) a subset of the set of height one ideals of \(\Lambda(k_0,r),\mathcal{O}_K\) as follows:
\[
\mathfrak{A} = \left\{ J = \bigcap_{I \in S} I \right\} \mid S \subset \mathcal{G}, \# S < \infty \right\}.
\]
Note that \(J \cap J' \in \mathfrak{A}\) for any \(J, J' \in \mathfrak{A}\) and that the intersection \(\bigcap J\) of infinitely many elements \(J\) of \(\mathfrak{A}\) is zero. Thanks to the existence of the interpolation of the space of modular symbols \(M\mathcal{S}(\Lambda(k_0,r),\mathcal{O}_K)^\pm\), we have a unique element
\[
L_p(\{\Xi^\pm\})_J \in (\Lambda(k_0,r),\mathcal{O}_K/J) \otimes_{\mathbb{Z}_p} \mathcal{H}_{h,\text{cyc}}^+.
\]
such that we have the following interpolation formula:

$$\frac{(k \times \chi_{\text{cyc}}^j \phi)(L_p(\{\Xi^\pm\}, j))}{C_{f_k, p}^{\text{sgn}(j, \phi)}} = (-1)^{(j - 1)!} \times e_p(f_k, j, \phi) \times \tau(\phi) \times \frac{L(f_k, j, \phi^{-1}, j)}{(2\pi \sqrt{-1})^j \Omega_{f_k, \infty}^{\text{sgn}(j, \phi)}},$$

for any arithmetic point $p \in \mathbb{Z}_{>0} \cap B(k_0; r)$ such that $J \subset I_k$, for any integer $j$ satisfying $l \leq j \leq l'$ and for any Dirichlet character $\phi$ whose conductor $\text{Cond}(\phi)$ divides $p^n$. The set of elements $\{L_p(\{\Xi^\pm\}, j)\}$ provides a projective system with respect to $p \in \mathfrak{A}$ and $n \in \mathbb{N}$. Thanks to Proposition 4.3, the projective limit $L_p(\{\Xi^\pm\}, j)$ defines a unique element of $\Lambda_{(k_0, r)} \otimes_{\mathbb{Z}_p} \mathcal{H}_{k, \text{cyc}}^{-}$ satisfying the desired interpolation property (5.1) for any arithmetic point $p \in \mathcal{S}$, for any integer $j$ satisfying $1 \leq j \leq k - 1$ and for any Dirichlet character $\phi$ whose conductor $\text{Cond}(\phi)$ is a power of $p$. This completes the proof. □

5.2. Construction of Beilinson-Kato Euler system on Coleman family. As application of Theorem 5.4, we have a theorem on the existence of an Euler system associated to a family of Galois representation $\mathcal{T}$ for a given Colaman family $\mathcal{F}$.

Theorem 5.4. Let us assume the setting of Theorem 5.4. Let us fix a $\Lambda_{(k_0, r), \mathcal{O}_K}$-basis $\Xi^\pm$ of $\mathcal{MS}(\Lambda_{(k_0, r), \mathcal{O}_K})^\pm$ respectively. Assume that the image of the residual representation $\overline{\rho} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F}_p)$ associated to the family is non-solvable and that $\overline{\rho}$ is irreducible when restricted to $G_{\mathbb{Q}_p(\mu_p)}$. Then, the following statements hold:

1. We have an Euler system (depending on the choice of $\Xi^\pm$):

$$\{ Z(r) \in H^1(\mathbb{Q}(\mu_r), (\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^r)^{\ast}(1)) \}_{r \in \mathcal{S}}$$

which satisfies the axiom of Euler system given Definition 3.4 for $\mathcal{R} = \Lambda_{(k_0, r), \mathcal{O}_K}$.

2. The elements $Z(r) \in H^1(\mathbb{Q}(\mu_r), (\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^r)^{\ast}(1))$ have the following relation to the special value for any $k \in \mathbb{Z}_{=1} \cap B(k_0; r)$, any integer $j$ satisfying $1 \leq j \leq k - 1$ and for any Dirichlet character $\phi$ whose conductor $\text{Cond}(\phi)$ is a power of $p$:

$$\frac{\exp^\ast\left(\text{loc}_p\left(k, \chi_{\text{cyc}}^j \phi^{-1}, \left(Z(r)\right)\right)\right)}{C_{f_k, p}^{\text{sgn}(j, \phi)}} = \tau(\phi) \times \frac{L(\mu_p, f_k, \phi^{-1}, j)}{(2\pi \sqrt{-1})^j \Omega_{f_k, \infty}^{\text{sgn}(j, \phi)}} \cdot \mathcal{T}_k \otimes \phi^{-1},$$

where $\Omega_{f_k, \infty}^{\text{sgn}(j, \phi)}$ is a complex period which satisfies the algebraicity which was shown in Theorem of Shimura presented of §1.1 and $C_{f_k, p}^{\text{sgn}(j, \phi)}$ is a $p$-adic period defined in §3.7.

In particular, we have the equality

$$\text{Col}(\text{loc}_p\left(Z(1)\right)) = L_p(\{\Xi^\pm\})$$

in $\Lambda_{(k_0, r), \mathcal{O}_K} \otimes_{\mathbb{Z}_p} \mathcal{H}_{k, \text{cyc}}$.

We prove Theorem 5.4 by using Theorem 4.5.

Proof of Theorem 5.4. The proof proceeds in the same manner as the proof of Theorem 3.12 which is an analogue of Theorem 5.4 in the ordinary case.\[22\]

\[22\]See Remark 3.2 for the fact that the interpolation property is well-defined.

\[23\]In the ordinary case, Theorem 5.12 played only a role of a modification of an Euler system which already existed in Theorem 3.9. Note that we have no analogue of Theorem 5.9 in the non-ordinary case and the method of the proof of Theorem 5.12 plays a more essential role in the non-ordinary case.
Similarly as in the proof of Theorem 5.4.1 we define the following set
\[(5.10) \quad \mathcal{S} = \{ I_k \mid k \in \mathbb{Z}_{\geq 0} \cap B(k_0; r) \}.\]
We denote by \(\mathfrak{A}\) the following subset of height one ideals of \(\Lambda_{(k_0;r),\mathcal{O}_K} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}:\)
\[(5.11) \quad \mathfrak{A} = \left\{ J = \bigcap_{I \in \mathcal{S}} I \cdot \Lambda_{(k_0;r),\mathcal{O}_K} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}} \mid S \subset \mathcal{S}, \#S < \infty \right\}.\]
Note again that \(J \cap J' \in \mathfrak{A}\) for any \(J, J' \in \mathfrak{A}\) and that the intersection \(\bigcap J\) for infinitely many \(J \in \mathfrak{A}\) is zero. We remark that \(\Lambda_{(k_0;r),\mathcal{O}_K} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}\) is a semi-local ring whose local components are all isomorphic to the ring of power series in two variables \(\mathcal{O}_K[[X_1, X_2]]\).
In particular, local components are regular local rings.
Recall that we defined \(\Sigma_r\) to be \(\Sigma_r = \Sigma \cup \{ \text{primes } q \text{ dividing } r \}\) for any natural number \(r\). We denote by \(\mathbb{Q}_{\Sigma}\) the maximal unramified extension of \(\mathbb{Q}\) unramified outside \(\Sigma_r\). We have the following lemma which is the non-ordinary analogue of Lemma 3.13.

**Lemma 5.5.** Let us assume the setting of Theorem 5.4. Let \(r\) be a natural number and \(J, J' \in \mathfrak{A}\). Then, we have the exact sequence:
\[(5.12) \quad 0 \rightarrow H^1((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1))_{J \cap J'} \rightarrow H^1((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1))_{J} \oplus H^1((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1))_{J'} \rightarrow H^1((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1))_{J + J'}.\]
Here we denote \((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1)/\mathbb{T}(\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1)\) by \((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1), J\) and the cohomology \(H^1(\mathbb{Q}(\mu_r), \Sigma_r, \mathbb{Q}(\mu_r), ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1), J)\) by \(H^1((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1), J)\) for any ideal \(J\) of \(\Lambda_{(k_0;r),\mathcal{O}_K} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}\).

The first map of (5.12) sends each element \(x_{J \cap J'}\) to \((x_{J \cap J'} \mod J) \oplus (x_{J \cap J'} \mod J')\). The second map (5.14) sends each element \(x_J \oplus y_J\) to \((x_J \mod J + J') - (y_J \mod J + J')\).

Let us denote by \(\text{Col}_J\) the Coleman map for \(((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1), J)\), which is the cyclotomic deformation of a usual \(p\)-adic Galois representation \(\mathbb{T}/\mathbb{F}_p\). The map \(\text{Col}_J\) is obtained by taking the \(\mathcal{H}_{\infty, \text{cyc}}\)-linear dual of Perrin-Riou’s big exponential map obtained in [Pe94]. By observing the denominators appearing in the interpolation, we easily see the image of \(\text{Col}\) falls in the submodule \((\mathbb{D}^*(1)/\mathbb{J}\mathbb{D}^*(1)) \otimes_{\mathbb{Z}_p} \mathcal{H}_{h, \text{cyc}} \subset (\mathbb{D}^*(1)/\mathbb{J}\mathbb{D}^*(1)) \otimes_{\mathbb{Z}_p} \mathcal{H}_{\infty, \text{cyc}}\). We consider the following composed morphism:
\[(5.13) \quad H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q}_I, ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1), J)) \xrightarrow{\text{loc}} H^1(\mathbb{Q}_p, ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1), J)) \xrightarrow{\text{Col}_J} (\mathbb{D}^*(1)/\mathbb{J}\mathbb{D}^*(1)) \otimes_{\mathbb{Z}_p} \mathcal{H}_{h, \text{cyc}} \xrightarrow{(d_J \otimes 1)} (\Lambda_{(k_0;r),\mathcal{O}_K}/J) \otimes_{\mathbb{Z}_p} \mathcal{H}_{h, \text{cyc}}\]
where \(d_J\) is a basis of a free \(\Lambda_{(k_0;r),\mathcal{O}_K}/J\)-module \(\mathbb{D}/\mathbb{J}\mathbb{D}\) of rank one which coincides with \(\mathfrak{J}_k\) for any \(k \in \mathbb{Z}_{\geq 0} \cap B(k_0; r)\) where \(\mathbb{D}\) is a free \(\Lambda_{(k_0;r),\mathcal{O}_K}\)-module of rank one constructed in Lemma 5.4. The symbol \(\langle , , \rangle\) represents the paring:
\[(\mathbb{D}/\mathbb{J}\mathbb{D}) \otimes_{\mathbb{Z}_p} \mathcal{H}_{\infty, \text{cyc}} \times (\mathbb{D}^*(1)/\mathbb{J}\mathbb{D}^*(1)) \otimes_{\mathbb{Z}_p} \mathcal{H}_{\infty, \text{cyc}} \longrightarrow (\Lambda_{(k_0;r),\mathcal{O}_K}/J) \otimes_{\mathbb{Z}_p} \mathcal{H}_{\infty, \text{cyc}}\]
and we note that the image of the composition map \((d_J \otimes 1, \ ) \circ \text{Col}_J\) falls in the submodule \((\Lambda_{(k_0;r),\mathcal{O}_K}/J) \otimes_{\mathbb{Z}_p} \mathcal{H}_{h, \text{cyc}} \subset (\Lambda_{(k_0;r),\mathcal{O}_K}/J) \otimes_{\mathbb{Z}_p} \mathcal{H}_{\infty, \text{cyc}}\). The following lemma will be used in the following arguments.

**Lemma 5.6.** Let us assume the setting of Theorem 5.4. Then the map
\[(d_J \otimes 1, \ ) \circ \text{Col}_J \circ \text{loc} : H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q}_I, ((\mathbb{T} \otimes_{\mathbb{Z}_p} \Lambda_{\text{cyc}}^e)^*(1), J)) \longrightarrow (\Lambda_{(k_0;r),\mathcal{O}_K}/J) \otimes_{\mathbb{Z}_p} \mathcal{H}_{h, \text{cyc}}\]
is injective for every \( J \in \mathfrak{A} \). Similarly, for any natural number \( r \), the composed morphism

\[
H^1(Q_{\Sigma r}/Q(\mu_r), ((T \otimes_{Z_p} \Lambda_{\text{cyc}}^r)^*(1)), J) \xrightarrow{\text{loc}} H^1(Q_p \otimes_Q Q(\mu_r), ((T \otimes_{Z_p} \Lambda_{\text{cyc}}^r)^*(1)), J)
\]

is injective for every \( J \in \mathfrak{A} \) where \( \text{Col}_J^{(r)} \) is a Coleman map over \( Q_p \otimes_Q Q(\mu_r) \) which is similar as \( \text{Col}_J \) above.

Though we do not go into the proof of the lemma, we remark that the lemma follows from the fact that \( H^1(Q_{\Sigma r}/Q(\mu_r), ((T \otimes_{Z_p} \Lambda_{\text{cyc}}^r)^*(1)), J) \) is not torsion over any local component of \( \Lambda_{\text{cyc}} \) thanks to the assumption of Theorem 5.4 saying that the image of the residual representation is non-solvable.

Now we take \( I, I' \) in the set \( S \) given in (5.10). Let us set \( J = I \cdot \Lambda_{(k_0,r), \mathcal{O}_K} \otimes_{Z_p} \Lambda_{\text{cyc}} \) and \( J' = I' \cdot \Lambda_{(k_0,r), \mathcal{O}_K} \otimes_{Z_p} \Lambda_{\text{cyc}} \).

Recall that we denote by \( S \) the set of all square-free natural numbers which are prime to \( \Sigma \). As explained before Theorem 5.12, the result of Kato for the cyclotomic deformation of a cuspform assures the existence of the Euler system:

\[
\{ Z_J(r) \in H^1(Q(\mu_r), ((T \otimes_{Z_p} \Lambda_{\text{cyc}}^r)^*(1)), J) \}_{r \in S}
\]

such that we have

\[
\text{Col}_J^{(r)}(\text{loc}_p(Z_J(r))) = L_p^{(r)}(\{ \Xi^\pm \}) \mod J \in (\Lambda_{(k_0,r), \mathcal{O}_K}/J) \otimes_{Z_p} \mathcal{H}^+_{h,cyc}
\]

\[
\text{resp. } \text{Col}_J^{(r)}(\text{loc}_p(Z_{J'}(r))) = L_p^{(r)}(\{ \Xi^\pm \}) \mod J' \in (\Lambda_{(k_0,r), \mathcal{O}_K}/J') \otimes_{Z_p} \mathcal{H}^+_{h,cyc}
\]

where \( L_p^{(r)}(\{ \Xi^\pm \}) \in \Lambda_{(k_0,r), \mathcal{O}_K} \otimes_{Z_p} \mathcal{H}^+_{h,cyc} \) is the \( p \)-adic \( L \)-function obtained by removing Euler factors at primes dividing \( r \) from \( L_p(\{ \Xi^\pm \}) \). Note that, since \( L_p^{(r)}(\{ \Xi^\pm \}) \) is an element of \( \Lambda_{(k_0,r), \mathcal{O}_K} \otimes_{Z_p} \mathcal{H}^+_{h,cyc} \), \( L_p^{(r)}(\{ \Xi^\pm \}) \mod J \) and \( L_p^{(r)}(\{ \Xi^\pm \}) \mod J' \) can be glued together to have a unique element \( L_p^{(r)}(\{ \Xi^\pm \}) \mod J \cap J' \) of \( (I/(J \cap J')) \otimes_{Z_p} \mathcal{H}^+_{h,cyc} \).

By applying Lemma 5.6 to the ideal \( J \cap J' \in \mathfrak{A} \), we have a unique element \( Z_{J \cap J'}(r) \in H^1(Q(\mu_r), ((T \otimes_{Z_p} \Lambda_{\text{cyc}}^r)^*(1)), J \cap J') \) such that \( ((d_j \otimes 1, \circ \text{Col}_J^{(r)} \circ \text{loc})(Z_{J \cap J'}(r)) \) is equal to \( L_p^{(r)}(\{ \Xi^\pm \}) \mod J \cap J' \). Thus, we obtain a unique Euler system

\[
\{ Z_{J \cap J'}(r) \in H^1(Q(\mu_r), ((T \otimes_{Z_p} \Lambda_{\text{cyc}}^r)^*(1)), J \cap J') \}_{r \in S}
\]

which glues Euler systems \( \{ Z_{J'}(r) \}_{r \in S} \) and \( \{ Z_{J'}(r) \}_{r \in S} \). By Lemma 5.3 and by repeating the induction on the numbers of the ideals of the form \( I \cdot \Lambda_{(k_0,r), \mathcal{O}_K} \otimes_{Z_p} \Lambda_{\text{cyc}} \) with \( I \in \mathfrak{S} \) containing a given ideal \( J \in \mathfrak{A} \), we construct for every \( J \in \mathfrak{A} \) a unique Euler system

\[
\{ Z_J(r) \in H^1(Q(\mu_r), ((T \otimes_{Z_p} \Lambda_{\text{cyc}}^r)^*(1)), J) \}_{r \in S}
\]

which provides an Euler system for each cuspform corresponding to the representation mod \( I \cdot \Lambda_{(k_0,r), \mathcal{O}_K} \otimes_{Z_p} \Lambda_{\text{cyc}} \) with \( I \in \mathfrak{S} \) such that \( J \subset I \cdot \Lambda_{(k_0,r), \mathcal{O}_K} \otimes_{Z_p} \Lambda_{\text{cyc}} \).

The Euler systems \( \{ Z_J(r) \}_{r \in S} \) form an inverse system for \( J \subset \mathfrak{A} \). By Chevalley’s theorem stated in the proof of Theorem 5.12 we have

\[
H^1(Q(\mu_r), ((T \otimes_{Z_p} \Lambda_{\text{cyc}}^r)^*(1))) \cong \lim_{\overline{J} \in \mathfrak{A}} H^1(Q(\mu_r), ((T \otimes_{Z_p} \Lambda_{\text{cyc}}^r)^*(1)), J).
\]
Hence, we obtain the desired Euler system \( \{ Z(r) \in H^1(\mathbb{Q}(\mu_r), ((\hat{T} \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^1)^*(1))) \}_{r \in S} \) by putting \( Z(r) = \lim_{\rightarrow J \in \mathbb{R}} Z_J(r) \) and this completes the proof. \( \square \)

5.3. Two-variable Iwasawa Main Conjecture on Coleman family. Thanks to Theorem 5.4, we can now state two-variable Iwasawa Main Conjecture.

Conjecture 5.7 (Two-variable Iwasawa Main Conjecture for a Coleman family). Let \( \mathbb{F} \) be a Coleman family over \( \Lambda_{(k_0; r_0)} \) with the slope \( s \in \mathbb{Q}_{\geq 0} \). Let \( h \) be an integer satisfying \( h \geq s \). Let \( \mathbb{T} \cong (\Lambda_{(k_0; r_0)})^{\oplus 2} \) be the Galois representation associated to \( \mathbb{F} \). Let us fix an \( \Lambda_{(k_0; r_0), \mathbb{O}_K} \) basis \( \Xi^\pm \) of \( \mathrm{MS}(\Lambda_{(k_0; r_0), \mathbb{O}_K})^\pm \) respectively. Assume that the residual representation \( \overline{\rho} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{F}_p) \) associated to the family is irreducible when restricted to \( G_{\mathbb{Q}_p(\mu_r)} \).

Then, the following statements hold:

1. The finitely generated \( \Lambda_{(k_0; r_0), \mathbb{O}_K} \hat{\otimes} \mathbb{Z}_p \Lambda_{\text{cyc}} \) module \( H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, (\hat{T} \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^1)^*(1)) \) is torsion.

2. We should have the following equality of principal ideals in \( \Lambda_{(k_0; r_0), \mathbb{O}_K} \hat{\otimes} \mathbb{Z}_p \Lambda_{\text{cyc}} \):

\[
(5.14) \quad \text{char} \Lambda_{(k_0; r_0), \mathbb{O}_K} \hat{\otimes} \mathbb{Z}_p \Lambda_{\text{cyc}} \left( H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, (\hat{T} \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^1)^*(1))/\Lambda_{(k_0; r_0), \mathbb{O}_K} \hat{\otimes} \mathbb{Z}_p \Lambda_{\text{cyc}} \cdot Z(1) \right) = \text{char} \Lambda_{(k_0; r_0), \mathbb{O}_K} \hat{\otimes} \mathbb{Z}_p \Lambda_{\text{cyc}} \left( H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, (\hat{T} \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^1)^*(1)) \right)
\]

where \( Z(1) \) is the first layer of the Euler system obtained by Theorem 5.4 and \( \mathbb{Q}_\Sigma \) is the maximal extension of \( \mathbb{Q} \) unramified outside the finite set of places \( \Sigma \) of \( \mathbb{Q} \) which consists of ramified places of the Galois representation \( \mathbb{T} \) and \( \{ \infty \} \).

The main result is as follows:

Theorem 5.8. Let us assume the setting of Conjecture 5.7. We also assume the following condition:

(SL) The image of Galois representation \( G_{\mathbb{Q}} \rightarrow \text{Aut}_{\Lambda_{(k_0; r_0), \mathbb{O}_K}}(\mathbb{T}) \cong \mathrm{GL}_2(\Lambda_{(k_0; r_0), \mathbb{O}_K}) \) contains \( \mathrm{SL}_2(\Lambda_{(k_0; r_0), \mathbb{O}_K}) \).

Then, \( H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, (\hat{T} \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^1)^*(1)) \) is a torsion \( \Lambda_{\text{cyc}} \) module and we have:

\[
(5.15) \quad \text{char} \Lambda_{(k_0; r_0), \mathbb{O}_K} \hat{\otimes} \mathbb{Z}_p \Lambda_{\text{cyc}} \left( H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, (\hat{T} \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^1)^*(1))/\Lambda_{(k_0; r_0), \mathbb{O}_K} \hat{\otimes} \mathbb{Z}_p \Lambda_{\text{cyc}} \cdot Z(1) \right) \subset \text{char} \Lambda_{(k_0; r_0), \mathbb{O}_K} \hat{\otimes} \mathbb{Z}_p \Lambda_{\text{cyc}} \left( H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, (\hat{T} \otimes \mathbb{Z}_p \Lambda_{\text{cyc}}^1)^*(1)) \right).
\]

Remark 5.9. We note that the condition (SL) above is studied by Conti–Iovita–Tilouine [CIT16]. For example, they prove that the Lie algebra of Galois representation is full under certain regularity conditions (see Thm 6.2 and Cor. 7.2 of [CIT16]).

Proof. We apply a general machinery Theorem 3.5 to the Euler system obtained in Theorem 5.4. By our setting and the hypothesis (SL) of the theorem, all the conditions (i) to (v) of Theorem 3.5 is checked to be true. Thus the desired statement follows immediately from Theorem 3.5. \( \square \)

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