CARTEL STABILITY UNDER QUALITY DIFFERENTIATION

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Abstract. This note considers cartel stability when the cartelized products are vertically differentiated. If market shares are maintained at pre-collusive levels, then the firm with the lowest competitive price-cost margin has the strongest incentive to deviate from the collusive agreement. The lowest-quality supplier has the tightest incentive constraint when the difference in unit production costs is sufficiently small.

Keywords: Cartel Stability, Collusion, Vertical Differentiation.

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1. Introduction

In this note, we examine cartel stability when the cartelized products are vertically differentiated. Goods or services are differentiated vertically when there is consensus among consumers about how to rank them quality-wise; comparing products A and B, all agree A to have a higher (perceived) value than B or vice versa. There might, however, still be a demand for lower-quality goods when buyers face budget constraints or differ in their willingness to pay for quality. This creates an incentive for suppliers to compete through offering different price-quality combinations.

One implication of this price-quality dispersion is that firms that consider colluding typically face heterogeneous incentive constraints. The fact that firms are induced to charge different prices, for example, affects both collusive and noncollusive profits. From a supply-side perspective, there commonly exists a positive relationship between the quality of a good and its production costs. This, too, impacts both sides of the constraint. It is therefore a priori unclear how quality differentiation impacts the sustainability of collusion.

The scarce literature on this topic provides mixed results and, moreover, does not consider the potential impact of cost heterogeneity. Assuming identical costs, Häckner (1994) and Symeonidis (1999) both analyze an infinitely repeated vertically differentiated duopoly game. Häckner (1994) studies a variation of the setting in Gabszewicz and Thisse (1979)
and Shaked and Sutton (1982) and finds that it is the high-quality supplier who has the strongest incentive to deviate. By contrast, Symeonidis (1999) considers a representative consumer model with horizontal and vertical product differentiation and establishes that it is the low-quality seller who is most eager to leave the cartel.

In the following, we analyze an \( n \)-firm infinitely repeated game version of the classic vertical differentiation model of Mussa and Rosen (1978) where production costs are assumed to be increasing in quality.\(^3\) Under the assumption that colluding firms maintain their pre-collusive market shares, we find that it is the competitive mark-up rather than the quality of the product that drives the incentive to deviate. Specifically, it is the supplier with the lowest noncooperative price-cost margin who has the strongest incentive to chisel on the cartel. Moreover, our analysis confirms the above-mentioned conclusion by Symeonidis (1999) when the difference in unit costs is sufficiently small.

The next section presents the model. Section 3 contains the main finding. Section 4 concludes.

2. Model

There is a given set of suppliers, denoted \( N = \{1, \ldots, n\} \), who repeatedly interact over an infinite, discrete time horizon. In every period \( t \in \mathbb{N} \), they simultaneously make price decisions with the aim to maximize the expected discounted sum of their profit stream. Firms face a common discount factor \( \delta \in (0, 1) \) and all prices set up until \( t-1 \) are assumed public knowledge.

Each firm \( i \in N \) sells a single variant of the product with quality \( v_i \). We assume \( \infty > v_n > v_{n-1} > \ldots > v_1 > 0 \) and refer to firm \( n \) as the top firm, firm 1 as the bottom firm and all others as intermediate firms. Unit production costs of firm \( i \in N \) are given by the constant \( c_i \) and we suppose these costs to be positive and (weakly) increasing in quality, i.e., \( c_n \geq c_{n-1} \geq \ldots \geq c_1 > 0 \).

Consumers have a valuation for the various product types of \( \theta \), which is uniformly distributed on \( [\theta, \bar{\theta}] \subset \mathbb{R}^+ \) with mass normalized to one. A higher \( \theta \) corresponds to a higher gross utility when consuming variant \( v_i \). Buyers purchase no more than one item so that someone 'located' at \( \theta \) obtains the following utility

\[
U(\theta) = \begin{cases} 
\theta v_i - p_i & \text{when buying from firm } i \\
0 & \text{when not buying,}
\end{cases}
\]

where \( p_i \in [0, \bar{\theta} v_n] \) is the price set by firm \( i \).\(^4\) Using (2.1), it can be easily verified that a consumer at \( \theta_i \in [\theta, \bar{\theta}] \) is indifferent between buying from, say, firm \( i+1 \) and firm \( i \) when

\[
\theta_i(p_i, p_{i+1}) = \frac{p_{i+1} - p_i}{v_{i+1} - v_i},
\]

for every \( i = 1, 2, ..., n-1 \). In the ensuing analysis, we further assume that the market is and remains covered (i.e., all consumers buy a product).\(^5\)

\(^3\)Like the model in H"ackner (1994), this is a model with heterogeneous customers. Apart from the number of firms and cost heterogeneity, it differs in terms of consumers' utility specification.

\(^4\)Note that none of the buyers would buy at prices in excess of \( \bar{\theta} v_n \).

\(^5\)This is a common assumption in contributions that employ this type of spatial setting. See, for example, Tirole (1988, pp.296-298) and Ecchia and Lambertini (1997). We discuss some implications of this assumption.
Current profit of the bottom firm \((i = 1)\) is therefore given by
\[(2.3)\]
\[\pi_1 (p_1, p_2) = (p_1 - c_1) \cdot (\theta_1 - \theta),\]
where \(\theta_1 = \theta(p_1, p_2)\) is as specified by \((2.2)\). For each intermediate firm \((i = 2, 3, ..., n - 1)\) profit is
\[(2.4)\]
\[\pi_i (p_{i-1}, p_i, p_{i+1}) = (p_i - c_i) \cdot (\theta_i - \theta_{i-1}),\]
and for the top firm \((i = n)\) it is
\[(2.5)\]
\[\pi_n (p_{n-1}, p_n) = (p_n - c_n) \cdot (\theta - \theta_{n-1}).\]

Before analyzing the infinitely repeated version of the above game, let us first consider the one-shot case in more detail. In this setting, each firm simultaneously picks a price to maximize its profit as specified in \((2.3)-(2.5)\). Following the first-order conditions, this yields three types of best-response functions:
\[(2.6)\]
\[\hat{p}_1 (p_2) = \frac{1}{2} (p_2 + c_1 - \theta (v_2 - v_1))\]
for the bottom firm \((i = 1)\). For each intermediate firm \((i = 2, 3, ..., n - 1)\), the best-reply is given by
\[(2.7)\]
\[\hat{p}_i (p_{i-1}, p_i, p_{i+1}) = \frac{1}{2} \frac{p_{i-1} (v_{i+1} - v_i) + p_{i+1} (v_i - v_{i-1})}{(v_{i+1} - v_{i-1})} + \frac{1}{2} c_i.\]
The best-response function of the top firm \((i = n)\) is
\[(2.8)\]
\[\hat{p}_n (p_{n-1}) = \frac{1}{2} (p_{n-1} + c_n + \theta (v_n - v_{n-1})).\]

Since the action sets are compact and convex and the above best-reply functions are contractions, there exists a unique static Nash equilibrium price vector \(p^*\) for any finite number of firms.\(^6\) Finally, we impose two more conditions to ensure that the equilibrium solution is interior (i.e., all firms have a positive output at \(p^*\)) and that the market is indeed covered at the single-shot Nash equilibrium:
\[(2.9)\]
\[\theta > \theta_{n-1} > \theta_{n-2} > \ldots > \theta^*_i > \ldots > \theta^*_1 > \theta > \frac{p_1^*}{v_1} > 0,\]
where \(\theta^*_i = \theta_i (p_i^*, p_{i+1}^*)\) and \(p_i^* \geq c_i\), for all \(i \in N\).

at the end of Section 3 and consider the possibility of an uncovered collusive market in an online appendix to this paper.

\(^6\) See, for instance, Friedman (1991, p.84). A sufficient condition for the contraction property to hold is (see, for example, Vives 2000, p.47):
\[\frac{\partial^2 \pi_i}{\partial (p_i)^2} + \sum_{j \neq i} \left| \frac{\partial^2 \pi_i}{\partial p_i \partial p_j} \right| < 0,\]
which, using \((2.4)\) for all intermediate firms \(i = 2, ..., n - 1\), becomes
\[\frac{v_{i-1} - v_{i+1}}{(v_{i+1} - v_i) (v_i - v_{i-1})} < 0,\]
which holds. The same applies for the top and the bottom firm.
3. Sustainability of Collusion

Within the above framework, we now study the sustainability of collusion assuming a standard grim-trigger punishment strategy. The incentive compatibility constraint (ICC) of a firm $i \in N$ is then given by:

$$\Omega_i \equiv \pi^c_i - (1 - \delta) \cdot \pi^d_i - \delta \cdot \pi^*_i \geq 0,$$

where $\pi^c_i = \pi_i(p_{c_{i-1}}, p_{c_i}, p_{c_{i+1}})$ is its collusive payoff, $\pi^d_i = \pi_i(p_{d_{i-1}}, p_{d_i}, p_{c_{i+1}})$, with $p_{d_i} = \hat{p}_i(p_{c_{i-1}}, p_{c_{i+1}})$, is its deviation payoff and $\pi^*_i = \pi_i(p^*_i, p^*_i, p^*_i)$ is its Nash equilibrium payoff.

A cartel comprising the entire industry is thus sustainable only when $\Omega_i \geq 0$ for all $i \in N$.

In principle, this set-up allows for a plethora of sustainable collusive contracts. In the following, we limit ourselves to what is perhaps the simplest possible agreement. Specifically, we consider the maximization of total cartel profits without side payments under the assumption that firms maintain their market shares at pre-collusive levels. Such an agreement is appealing for several reasons. First, it seems a natural focal point in the issue of how to divide the market. Second, there have been quite a few cartels that employed such (or similar) market-sharing scheme. Third, it is arguably one of the most subtle arrangements in that firm behavior maintains a competitive appearance, thereby minimizing the possibility of cartel detection.

Let us now address the question of what collusive price vector such an all-inclusive cartel would pick. As an initial observation, notice that the fixed market share assumption implies that the price ranking should stay the same (i.e., collusive prices are strictly increasing in quality). Moreover, as market size is given, the lowest-valuation buyer should still be willing to buy the product. This means that

$$\theta v_1 - p^c_1 \geq 0.$$

Next, note that the fixed market share rule in combination with the covered market assumption implies that each ‘marginal consumer’s location’ remains unaffected. Specifically, the consumer who was indifferent between firm 1 and 2 absent collusion has now the following utility when buying from firm 1:

$$U(\theta^*_1) = \left( \frac{p^*_2 - p^*_1}{v_2 - v_1} \right) v_1 - p^c_1,$$

which in turn determines the collusive price for the product of firm 2:

$$\left( \frac{p^*_2 - p^*_1}{v_2 - v_1} \right) v_1 - p^c_1 = \left( \frac{p^*_2 - p^*_1}{v_2 - v_1} \right) v_2 - p^c_2.$$

Rearranging gives,

$$p^c_2 = p^*_2 + (p^c_1 - p^*_1).$$

A higher collusive price by firm 2 would mean that the customer on the boundary prefers firm 1, which contradicts market shares being fixed. Likewise, a lower price implies a decrease in sales for firm 1 and therefore cannot occur either. The collusive prices for all other firms

\[^7\text{See, for example, Harrington (2006).}\]
can be determined in a similar fashion. In general, the collusive price of firm \( i \in N \backslash \{1\} \) is equal to its Nash price plus the price increase by the lowest-quality firm:

\[
(3.2) \quad p_i^c = p_i^* + (p_1^* - p_1^c) .
\]

Both cartel profits and the incentive constraints are therefore effectively a function of \( p_1^c \) alone.

Since prices are strategic complements, it is clear that the cartel would like to set \( p_i^c = \theta v_1 \). The collusive contract with \( p_i^c = \theta v_1 \) might not be sustainable, however, because one or more ICC’s may be binding. The next result shows that it is the supplier with the lowest noncollusive profit margin who has the tightest incentive constraint.

**Proposition 1.** For any \( i, j \in N \) and \( j \neq i \), if \( p_i^* - c_i > p_j^* - c_j \), then \( \Omega_i > \Omega_j \).

**Proof.** Consider the ICC of an intermediate firm \( i = 2, 3, ..., n - 1 \):

\[
\Omega_i \equiv \pi_i^c - (1 - \delta) \cdot \pi_i^d - \delta \cdot \pi_i^c \geq 0 \Leftrightarrow \delta \geq \delta_i \equiv \frac{\pi_i^d - \pi_i^c}{\pi_i^d - \pi_i^c}.
\]

To evaluate the critical discount factor \( \delta_i \) of every intermediate firm, let us focus on \( \pi_i^d, \pi_i^c \) and \( \pi_i^c \) in turn. Every firm’s \( i = 2, 3, ..., n - 1 \) deviating profit is given by \( \pi_i^d = (p_i^d - c_i) (\theta_i^d - \theta_{i-1}^d) \), which, using best-replies (2.7), yields

\[
p_i^d - c_i = \frac{p_{i-1}^c (v_{i+1} - v_i) + p_{i+1}^c (v_i - v_{i-1}) - (v_{i+1} - v_{i-1}) c_i}{2(v_{i+1} - v_{i-1})}
\]

or

\[
(3.3) \quad 2(v_{i+1} - v_{i-1}) (p_i^d - c_i) = p_{i-1}^c (v_{i+1} - v_i) + p_{i+1}^c (v_i - v_{i-1}) - (v_{i+1} - v_{i-1}) c_i.
\]

Moreover,

\[
\theta_i^d - \theta_{i-1}^d = \frac{(v_i - v_{i-1}) p_{i+1}^c + (v_{i+1} - v_i) p_{i-1}^c - (v_{i+1} - v_{i-1}) c_i}{2(v_{i+1} - v_i) (v_i - v_{i-1})},
\]

from which

\[
(3.4) \quad 2(v_{i+1} - v_i) (v_i - v_{i-1}) (\theta_i^d - \theta_{i-1}^d) = (v_i - v_{i-1}) p_{i+1}^c + (v_{i+1} - v_i) p_{i-1}^c - (v_{i+1} - v_{i-1}) c_i.
\]

Combining (3.3)-(3.4) yields

\[
2(v_{i+1} - v_{i-1}) (p_i^d - c_i) = 2(v_{i+1} - v_i) (v_i - v_{i-1}) (\theta_i^d - \theta_{i-1}^d)
\]

and therefore

\[
(\theta_i^d - \theta_{i-1}^d) = \frac{(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i) (v_i - v_{i-1})} (p_i^d - c_i).
\]

Hence, deviating profits can be written as

\[
(3.5) \quad \pi_i^d = (p_i^d - c_i)^2 \left( \frac{v_{i+1} - v_{i-1}}{(v_{i+1} - v_i) (v_i - v_{i-1})} \right).
\]

In a similar vein, intermediate firms’ Nash profit can be written as
Thus, by (5.1) and (5.2), the critical discount factor of every intermediate firm is given by

\[ \theta_i^* - \theta_{i-1}^* = \theta_i^* - \theta_{i-1}^* \]

and

\[ p_i^* = p_i^* + p_i^* - p_i^* . \]

Hence, every intermediate firm’s collusive profit can be written as

\[ \pi_i^c = (p_i^c - c_i) (\theta_i^c - \theta_{i-1}^c) = (p_i^* + p_i^* - p_i^* - c_i) \left( \frac{(v_{i+1} - v_i)}{(v_{i+1} - v_i) (v_i - v_{i-1})} (p_i^* - c_i) \right) . \]

Note further that

\[ \pi_i^d = \frac{(v_i - v_{i-1}) p_i^n + (v_{i+1} - v_i) p_i^n - (v_{i+1} - v_i) c_i}{2 (v_{i+1} - v_i)} = \frac{(v_i - v_{i-1}) (p_i^* + p_i^* - p_i^* + p_i^* + p_i^* - p_i^* - p_i^* - c_i)}{2 (v_{i+1} - v_i)} = p_i^* + \frac{1}{2} (p_i^* - p_i^*) . \]

Combining (3.5)-(3.7) yields

\[ \pi_i^d - \pi_i^c = \left( \frac{(v_i - v_{i-1})}{(v_{i+1} - v_i) (v_i - v_{i-1})} \right) \left[ \frac{1}{4} (p_i^* - p_i^*)^2 \right] , \]

and

\[ \pi_i^d - \pi_i^* = \left( \frac{(v_i - v_{i-1})}{(v_{i+1} - v_i) (v_i - v_{i-1})} \right) (p_i^* - p_i^*) \left[ \frac{1}{4} (p_i^* - p_i^*) + (p_i^* - c_i) \right] . \]

Thus, by (5.1) and (5.2), the critical discount factor of every intermediate firm \( i = 2, 3, \ldots, n - 1 \) is given by

\[ \delta_i \equiv \frac{\pi_i^d - \pi_i^c}{\pi_i^d - \pi_i^*} = \frac{1}{4} (p_i^* - p_i^*) (p_i^* - p_i^*) + (p_i^* - c_i) , \]

which is decreasing in the noncollusive price-cost margin.

Turning to the top-quality firm, by following the above steps, it can be verified that

\[ \pi_n^d = \left( \frac{p_n^d - c_n}{v_n - v_{n-1}} \right) , \quad \pi_n^* = \left( \frac{p_n^* - c_n}{v_n - v_{n-1}} \right) \]

and

\[ \pi_n^c = (p_1^c - p_1^* + p_n^* - c_n) \left( \frac{\theta (v_n - v_{n-1}) - (p_n^* - p_n^* - c_n)}{v_n - v_{n-1}} \right) , \]

yielding

\[ \delta_n \equiv \frac{\pi_n^d - \pi_n^c}{\pi_n^d - \pi_n^*} = \frac{1}{4} (p_1^* - p_1^*) (p_1^* - p_1^*) + (p_n^* - c_n) . \]
Hence, it is the supplier with the smaller noncooperative price-cost margin (between firm \( n \) and firm \( n - 1 \)) who has the tighter incentive constraint.

Finally, in a similar fashion, it can be shown that

\[
\pi^1_i = \frac{(p^d_i - c_1)^2}{(v_2 - v_1)}, \quad \pi^1_* = \frac{(p^*_i - c_1)^2}{(v_2 - v_1)} \quad \text{and} \quad
\]

\[
\pi^2_i = (p^*_{ij} - c_1) \left( \frac{p^*_2 - p^*_1 - (v_2 - v_1) \theta}{v_2 - v_1} \right) = \frac{(p^*_i - c_1) (p^*_i - c_1)}{(v_2 - v_1)},
\]

yielding

\[
\delta_1 \equiv \frac{\pi^1_d - \pi^1_*}{\pi^1_d - \pi^1_1} = \frac{1}{4} \frac{(p^*_i - p^*_1)}{\pi^1_d - \pi^1_1} + \frac{(p^*_i - c_1)}{(v_2 - v_1)}.
\]

Hence, \( p^*_2 - c_2 > p^*_i - c_1 \) implies \( \Omega_2 > \Omega_1 \), whereas \( p^*_i - c_1 > p^*_2 - c_2 \) implies \( \Omega_1 > \Omega_2 \). We thus conclude that, if \( p^*_i - c_i > p^*_j - c_j \), then \( \Omega_i > \Omega_j \) for all \( i, j \in N, j \neq i \).

The incentive to deviate from the collusive agreement is determined by the short-term gain of defection \((\pi^d_i - \pi^*_i)\) and the severity of the resulting punishment \((\pi^d_i - \pi^*_i = \pi^d_i - \pi^*_i + \pi^c_i - \pi^*_i)\). The proof of Proposition 1 reveals that the ‘extra profit effect’ is the same across firms and that the heterogeneity in incentive constraints is exclusively driven by differences in the punishment impact \((\pi^c_i - \pi^*_i)\). Specifically, there is a positive relation between a firm’s market share and its noncollusive price-cost margin. Since market shares are fixed at pre-collusive levels, members with a higher competitive mark-up are hit relatively harder by a cartel breakdown and this creates a stronger incentive to abide by the agreement.

The next result follows immediately. In stating this result, let \( \Delta c_{ij} = c_i - c_j \) for any firm \( i, j \in N \) and \( j \neq i \).

**Corollary 1.** For any firm \( i, j \in N \) and \( j \neq i \), \( \exists \mu \in \mathbb{R}_{++} \) such that if \( \Delta c_{ij} < \mu \) and \( v_i > v_j \), then \( \Omega_i > \Omega_j \).

Hence, if the difference in unit production costs is sufficiently small, then it is the lowest-quality supplier who has the tightest incentive constraint. Our analysis thus confirms the above-mentioned conclusion by Symeonidis (1999) in case quality heterogeneity is primarily driven by (sunk) fixed costs rather than variable costs.\(^8\)

Let us conclude this section with a remark on the covered market assumption. Note that since each ICC is strictly concave in the own cartel price, a sufficient condition for the cartel to keep market size constant is that at \( p^*_i = \theta v_1 \), \( \Omega_i \leq 0 \) and \( \frac{\partial \Omega_i}{\partial p^*_i} < 0 \) for all \( i \in N \). Endogenizing the size of the market would therefore not affect the above findings when the discount factor is sufficiently low. If its members are patient enough, however, then the cartel would like to uncover the market. This case is generally far less tractable analytically, but we show in an online appendix that this paper’s results also hold when the number of low-valuation buyers leaving the market is sufficiently small.

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\(^8\)This result can also be shown to hold in an \( n \)-firm variation of the model in Häckner (1994). The analysis is available upon request.
Many markets are characterized by some degree of quality differentiation with corresponding firm heterogeneity in cost and demand. One implication of such differences is that colluding firms typically face non-identical incentive constraints. Existing literature on this topic focuses on demand differences, while ignoring the potential impact of cost heterogeneity. In this note, we considered how cartel stability is affected when unit costs are increasing in product quality. Under the assumption that colluding firms maintain their pre-collusive market shares, we found that the incentive to deviate from the collusive agreement is monotonic in the noncollusive price-cost margin. Specifically, the supplier with the lowest competitive mark-up is *ceteris paribus* most inclined to leave the cartel. Moreover, it is the lowest-quality seller who has the tightest incentive constraint when differences in unit costs are sufficiently small.

5. Appendix 1: Uncovered Market Case

In the note we show that the incentive to deviate from a cartel agreement is monotonic in the noncollusive price-cost margin and that the firm with the lowest profit margin has the tightest incentive constraint (Proposition 1). This result is derived under the assumption of a fixed market size. In this appendix, we consider the possibility that the cartel ‘uncovers the market’ by setting collusive prices at a level where some of the lowest-valuation customers prefer to no longer buy the product. In the following, we show that the result of Proposition 1 also holds when the number of buyers leaving the market is sufficiently small.

To begin, consider the incentive compatibility constraint (ICC) of an intermediate firm $i \in N$:

$$
\Omega_i \equiv (p_i^c - c_i) \left( \theta_i^c - \theta_{i-1}^c \right) - (1 - \delta) \left( p_i^d - c_i \right) \left( \theta_i^d - \theta_{i-1}^d \right) - \delta \left( p_i^s - c_i \right) \left( \theta_i^s - \theta_{i-1}^s \right) \geq 0.
$$

As the cartel uses a fixed market share rule, it holds that:

$$
\frac{\theta_i^c - \theta_{i-1}^c}{\bar{\theta} - \frac{v_i}{p_i^c}} = \frac{\theta_i^s - \theta_{i-1}^s}{\bar{\theta} - \bar{\theta}}.
$$

The ICC can thus be written as:

$$
\Omega_i \equiv (p_i^c - c_i) \left( \theta_i^s - \theta_{i-1}^s \right) \frac{\left( \bar{\theta} - \frac{v_i}{p_i^c} \right)}{\left( \bar{\theta} - \bar{\theta} \right)} - (1 - \delta) \left( p_i^d - c_i \right) \left( \theta_i^d - \theta_{i-1}^d \right) - \delta \left( p_i^s - c_i \right) \left( \theta_i^s - \theta_{i-1}^s \right) \geq 0,
$$

which is equivalent to

$$
\delta \geq \delta_i \equiv \frac{\pi_i^d - \pi_i^c}{\pi_i^d - \pi_i^s} = \frac{(p_i^d - c_i) \left( \theta_i^d - \theta_{i-1}^d \right) - (p_i^c - c_i) \left( \theta_i^s - \theta_{i-1}^s \right)}{(p_i^d - c_i) \left( \theta_i^d - \theta_{i-1}^d \right) - (p_i^s - c_i) \left( \theta_i^s - \theta_{i-1}^s \right)} \frac{\left( \pi_i^c - \pi_i^s \right)}{\left( \pi_i^d - \pi_i^s \right)}.
$$

Let us now specify $\pi_i^d$, $\pi_i^s$ and $\pi_i^c$. Following the proof of Proposition 1, deviating and Nash profits are respectively given by

$$
\pi_i^d = (p_i^d - c_i)^2 \left( \frac{v_i + v_{i-1}}{v_i (v_i - v_{i-1})} \right),
$$

$$
\pi_i^s = (p_i^s - c_i)^2 \left( \frac{v_i + v_{i-1}}{v_i (v_i - v_{i-1})} \right),
$$

$$
\pi_i^c = (p_i^c - c_i)^2 \left( \frac{v_i + v_{i-1}}{v_i (v_i - v_{i-1})} \right).
$$
and

\[ \pi^*_i = (p^*_i - c_i)^2 \left( \frac{(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} \right). \]

As to collusive profits, note that uncovering the market in combination with the fixed market share rule implies that the lowest quality firm has the smallest price increase and that the price increase is rising in quality. In other words, cartel prices must be chosen such that each marginal consumer’s location ‘shifts upwards’ in order to maintain market shares at pre-collusive levels. Rather than adding \( p^*_i - p^*_1 \) to its Nash price \( p^*_i \) (as in the covered market case), intermediate firm \( i \) should therefore raise its price by more. Let this additional amount be indicated by \( x_i > 0 \) so that its collusive price is given by \( p^c_i = p^*_i + p^c_i - p^*_1 + x_i \). Collusive profits are then

\[ \pi^c_i = (p^c_i - c_i) (\theta_i^c - \theta_{i-1}^c) = (p^*_i + p^c_i - p^*_1 + x_i - c_i) \left( \frac{(v_{i+1} - v_{i-1})(p^*_i - c_i)}{(v_{i+1} - v_i)(v_i - v_{i-1})} \right) \left( \frac{\theta - \theta^c_i}{\theta^c - \theta} \right). \]

Focusing on the numerator of the critical discount factor first, we have

\[ \pi^d_i - \pi^c_i = \frac{(p^d_i - c_i)^2}{(v_{i+1} - v_i)(v_i - v_{i-1})} \left[ (p^*_i + p^c_i - p^*_1 + x_i - c_i) \frac{(p^*_i - c_i)(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} \right] \left( \frac{\theta - \theta^c_i}{\theta^c - \theta} \right) = \frac{(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} \left[ (p^*_i - c_i) (\theta^c_i - \theta^c_{i-1}) \right] \left( \frac{\theta - \theta^c_i}{\theta^c - \theta} \right). \]

In this case, the deviating price is given by

\[ p^d_i = \frac{(v_i - v_{i-1})}{2(v_{i+1} - v_i)} \left( p^c_{i+1} + (v_{i+1} - v_i) p^c_{i-1} + (v_{i+1} - v_{i-1}) c_i \right) = \frac{(v_i - v_{i-1})}{2(v_{i+1} - v_i)} \left( p^*_i + p^c_i - p^*_1 + x_i + (v_{i+1} - v_i) (p^c_{i-1} + p^c_i - p^*_1 + x_{i-1}) + (v_{i+1} - v_{i-1}) c_i \right) = p^*_i + \frac{1}{2}(p^*_i - p^*_1) + \frac{(v_i - v_{i-1})}{2(v_{i+1} - v_i)} x_i + \frac{(v_{i+1} - v_i)}{2(v_{i+1} - v_i)} x_{i-1}. \]

To facilitate the presentation of the analysis, let us denote

\[ y_i = \frac{(v_i - v_{i-1})}{2(v_{i+1} - v_i)} x_i + \frac{(v_{i+1} - v_i)}{2(v_{i+1} - v_i)} x_{i-1}, \]

and \( s = \frac{\theta - \theta^c_i}{\theta^c - \theta} \).

Substituting in the above equation gives

\[ \pi^d_i - \pi^c_i = \left( \frac{(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} \right) \left[ (p^*_i + \frac{1}{2}(p^*_i - p^*_1) + y_i - c_i) \right] \left( \frac{\theta - \theta^c_i}{\theta^c - \theta} \right) = \left( \frac{(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} \right) \left[ \frac{1}{4}(p^c_i - p^*_i)^2 + (1 - s) (p^*_i - c_i) \right] \left( \frac{\theta - \theta^c_i}{\theta^c - \theta} \right) \]

or

\[ \pi^d_i - \pi^c_i = \left( \frac{(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} \right) \left[ \frac{1}{4}(p^c_i - p^*_i)^2 + z \right], \]

(5.1)
where

\[ z \equiv (1 - s) (p_i^* - c_i) [(p_i^* - c_i) + (p_i^* - p_i^*) + x_i] + (p_i^* - c_i) (2y_i - x_i) + y_i ((p_i^* - p_i^*) + y_i). \]

Note that with a covered market it holds that \( s = 1, x_i = 0 \) and \( y_i = 0 \), in which case \( z = 0 \) and the numerator reduces to the corresponding value in the proof of Proposition 1. Turning to the denominator of the critical discount factor, we have

\[
\pi_i^d - \pi_i^* = (p_i^d - c_i)^2 \frac{(v_i+1 - v_i-1)}{(v_i+1 - v_i)(v_i - v_i-1)} - (p_i^* - c_i)^2 \frac{(v_i+1 - v_i-1)}{(v_i+1 - v_i)(v_i - v_i-1)}
\]

\[
= \frac{(v_i+1 - v_i-1)}{(v_i+1 - v_i)(v_i - v_i-1)} \left[ (p_i^d - c_i)^2 - (p_i^* - c_i)^2 \right]
\]

\[
= \frac{(v_i+1 - v_i-1)}{(v_i+1 - v_i)(v_i - v_i-1)} \left[ (p_i^* + \frac{1}{2} (p_i^c - p_i^*) + y_i - c_i)^2 - (p_i^* - c_i)^2 \right]
\]

\[
= \frac{(v_i+1 - v_i-1)}{(v_i+1 - v_i)(v_i - v_i-1)} \left[ \frac{1}{4} (p_i^c - p_i^*)^2 + (p_i^* - c_i) (p_i^c - p_i^* + 2y_i) + y_i ((p_i^* - p_i^*) + y_i) \right]
\]

or

\[
\pi_i^d - \pi_i^* = \frac{(v_i+1 - v_i-1)}{(v_i+1 - v_i)(v_i - v_i-1)} \left[ \frac{1}{4} (p_i^c - p_i^*)^2 + r \right],
\]

where

\[ r \equiv (p_i^* - c_i) (p_i^c - p_i^* + 2y_i) + y_i ((p_i^* - p_i^*) + y_i). \]

Note that in case of a covered market \( y_i = 0 \) so that \( r = (p_i^c - p_i^*) (p_i^* - c_i) \) as in the proof of Proposition 1. Combining both terms \([5.1]\) and \([5.2]\) gives the critical discount factor:

\[
\delta \geq \tilde{\delta}_i \equiv \frac{\frac{1}{4} (p_i^c - p_i^*)^2 + (1 - s) (p_i^* - c_i) [(p_i^* - c_i) + (p_i^c - p_i^*) + x_i] + (p_i^* - c_i) (2y_i - x_i) + y_i ((p_i^* - p_i^*) + y_i)}{\frac{1}{4} (p_i^c - p_i^*)^2 + (p_i^* - c_i) (p_i^c - p_i^* + 2y_i) + y_i ((p_i^* - p_i^*) + y_i)}. \]

As a final step, let us evaluate this critical discount factor with respect to the price-cost margin \((p_i^* - c_i)\). Taking the first-derivative with respect to the noncollusive profit margin of firm \(i\) yields:

\[
\frac{\partial \tilde{\delta}_i}{\partial ((p_i^* - c_i))} = \left\{ \begin{array}{ll}
\frac{1}{4} (p_i^c - p_i^*)^2 + (p_i^* - c_i) (p_i^c - p_i^* + 2y_i) + y_i ((p_i^* - p_i^*) + y_i) \\
(1 - s) [2 (p_i^c - c_i) + (p_i^c - p_i^*) + x_i] \\
+ (2y_i - x_i)
\end{array} \right\} \left[ \frac{1}{4} (p_i^c - p_i^*)^2 + (p_i^* - c_i) (p_i^c - p_i^* + 2y_i) + y_i ((p_i^* - p_i^*) + y_i) \right]^2
\]

\[
- \left\{ \begin{array}{ll}
\frac{1}{4} (p_i^c - p_i^*)^2 + (1 - s) (p_i^* - c_i) [(p_i^* - c_i) + (p_i^c - p_i^*) + x_i] \\
+ (p_i^* - c_i) (2y_i - x_i) + y_i ((p_i^* - p_i^*) + y_i)
\end{array} \right\} (p_i^c - p_i^* + 2y_i)
\]

\[
\frac{1}{4} (p_i^c - p_i^*)^2 + (p_i^* - c_i) (p_i^c - p_i^* + 2y_i) + y_i ((p_i^* - p_i^*) + y_i) \right)^2,
\]
which is negative when
\[(1 - s) (p_i^* - c_i) \left\{ 2y_i ((p_i^* - p_1^*) + y_i) + (p_i^* - c_i) (p_i^* - p_1^* + 2y_i) + \frac{1}{2} (p_i^* - p_1^*)^2 \right\}
- s \left\{ (p_i^* - p_1^*) + x_i \right\} y_i ((p_i^* - p_1^*) + y_i) + \frac{1}{4} (p_i^* - p_1^*)^2 < 0.\]

Note that this condition holds for \( s \to 1 \). In a similar fashion, this can be shown to be true for the bottom and top quality firm cases. Thus, we conclude that the result of Proposition 1 also holds when the cartel uncovers the market and the fraction of buyers no longer buying the product is sufficiently small.

6. Appendix 2: Example with Nonuniform Distribution

Let us present a simple two-firm example with a non-uniform distribution of \( \theta \). The results of the paper are also shown to hold in this case. Though of course far from being a proof, this suggests that our findings may apply for a wider class of customer distributions.

Consider the model of the paper, but with two firms and a simple ‘two-step uniform’ distribution function where a subset of consumers (of mass \( s \)) is uniformly distributed over a given interval \([\tilde{\theta}, \theta]\), with \( \tilde{\theta} \in (\theta, \tilde{\theta}) \) and another subset (of mass \((1 - s) \neq s\)) is uniformly distributed over \((\tilde{\theta}, \tilde{\theta})\) (higher willingness to pay consumers). The density function takes the form:

\[f(\theta) = \begin{cases} \frac{s}{(\theta - \tilde{\theta})} & \text{for } \theta \in \left[\tilde{\theta}, \theta\right] \\ \frac{1-s}{(\theta - \tilde{\theta})} & \text{for } \theta \in \left(\tilde{\theta}, \tilde{\theta}\right) \end{cases} \]

In the following, suppose that \( \theta_1(p_1, p_2) = \frac{p_2 - p_1}{v_2 - v_1} \leq \tilde{\theta} \) so that the profit functions are given by:

\[\pi_1 = (p_1 - c_1) \left( \frac{p_2 - p_1}{v_2 - v_1} - \tilde{\theta} \right) \frac{s}{(\theta - \tilde{\theta})}, \quad \text{and} \]

\[\pi_2 = (p_2 - c_2) \left( \tilde{\theta} - \frac{p_2 - p_1}{v_2 - v_1} \right) \frac{s}{(\theta - \tilde{\theta})} + (p_2 - c_2) \left( \tilde{\theta} - \tilde{\theta} \right) \frac{1-s}{(\tilde{\theta} - \tilde{\theta})} \]

\[= (p_2 - c_2) \left( \frac{\tilde{\theta} (v_2 - v_1) s - sp_2 + sp_1 + (v_2 - v_1) (\tilde{\theta} - \tilde{\theta}) (1 - s)}{(v_2 - v_1) (\tilde{\theta} - \tilde{\theta})} \right).\]

The first-order conditions give the best response functions:

\[\hat{p}_1 = \frac{1}{2} \left( p_2 - (v_2 - v_1) \tilde{\theta} + c_1 \right)\]

\[\hat{p}_2 = \frac{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} (1 - s) \right) + sp_1 + sc_2}{2s}\]
Combining gives the Nash prices and corresponding profits:

\[ p_1^* = \frac{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} (1 + s) \right) + 2 sc_1 + sc_2}{3s} \]

\[ p_2^* = \frac{(v_2 - v_1) \left( 2 \left( \tilde{\theta} - \tilde{\theta} \right) + \theta s \right) + 2 sc_2 + sc_1}{3s} \]

\[ \pi_1^* = \frac{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} (1 + s) \right) - sc_1 + sc_2}{3s} \left( \frac{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} (1 + s) \right) - sc_1 + sc_2}{3 (v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \right) \]

\[ = \frac{(p_1^* - c_1)^2}{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \frac{s}{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \]

\[ \pi_2^* = \frac{(v_2 - v_1) \left( 2 \left( \tilde{\theta} - \tilde{\theta} \right) + \theta s \right) - sc_2 + sc_1}{3s} \left( \frac{(v_2 - v_1) \left[ 2 \left( \tilde{\theta} - \tilde{\theta} \right) + \theta s \right] - sc_2 + sc_1}{3 (v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \right) \]

\[ = \frac{(p_2^* - c_2)^2}{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \frac{s}{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \]

Under collusion, prices and profits are respectively given by:

\[ p_1^c = \frac{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} (1 + s) \right) + 2 sc_1 + sc_2}{3s} \]

and

\[ p_2^c = p_2^* + (p_1^c - p_1^*) = \frac{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} + 2\theta s \right) + sc_2 - sc_1 + 3sp_1^c}{3s} \]

\[ \pi_1^c = (p_1^c - c_1) \left( \frac{p_2^c - p_1^c}{v_2 - v_1} - \frac{s}{\tilde{\theta} - \tilde{\theta}} \right) = (p_1^c - c_1) \left( \frac{p_2^c - p_1^c}{v_2 - v_1} - \frac{s}{\tilde{\theta} - \tilde{\theta}} \right) \]

\[ = (p_1^c - c_1) \left( \frac{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} (1 + s) \right) + sc_2 - sc_1}{3 (v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \right) = (p_1^c - c_1) \left( \frac{p_1^c - c_1}{v_2 - v_1} \right) \frac{s}{\tilde{\theta} - \tilde{\theta}} \]

\[ \pi_2^c = (p_2^c - c_2) \left( \frac{\tilde{\theta} (v_2 - v_1) s - s (p_2^c - p_1^c) + (v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right) (1 - s)}{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \right) \]

\[ = (p_2^c - c_2) \left( \frac{\tilde{\theta} (v_2 - v_1) s - s (p_2^c - p_1^c) + (v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right) (1 - s)}{(v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \right) \]

\[ = (p_2^c - c_2) \left( \frac{3\tilde{\theta} (v_2 - v_1) s - (v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} + 2\theta s \right) - sc_2 + sc_1 + 3 (v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right) (1 - s)}{3 (v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \right) \]

\[ = (p_2^c - c_2) \left( \frac{(v_2 - v_1) \left[ 2 \left( \tilde{\theta} - \tilde{\theta} \right) + \theta s \right] - sc_2 + sc_1}{3 (v_2 - v_1) \left( \tilde{\theta} - \tilde{\theta} \right)} \right) = (p_2^c - c_2) \left( \frac{p_2^c - c_2}{v_2 - v_1} \right) \frac{s}{\tilde{\theta} - \tilde{\theta}} \]
Finally, deviating price and profits are:

\[ p^d_1 = \frac{1}{2} (p^c_2 - (v_2 - v_1) \tilde{\theta} + c_1) \]

\[ p^d_2 = \frac{(v_2 - v_1) \left( \tilde{\theta} - \frac{\theta}{(1 - s)} \right) + sp^c_1 + sc_2}{2s} \]

and

\[ \pi^d_1 = (p^d_1 - c_1) \left( \frac{p^c_2 - p^d_1}{v_2 - v_1} - \tilde{\theta} \right) \frac{s}{(\tilde{\theta} - \theta)} = \]

\[ = \frac{1}{2} (p^c_2 - (v_2 - v_1) \tilde{\theta} - c_1) \left( \frac{p^c_2 - (v_2 - v_1) \tilde{\theta} - c_1}{2 (v_2 - v_1)} \right) \frac{s}{(\tilde{\theta} - \theta)} \]

\[ = \frac{(p^d_1 - c_1)^2}{(v_2 - v_1) (\tilde{\theta} - \theta)} s \]

and,

\[ \pi^d_2 = (p^d_2 - c_2) \left( \frac{\tilde{\theta} (v_2 - v_1) s - s \left( p^c_2 - p^d_1 \right) + (v_2 - v_1) \left( \tilde{\theta} - \theta \right) (1 - s)}{(v_2 - v_1) (\tilde{\theta} - \theta)} \right) \]

\[ = \left( \frac{(v_2 - v_1) \left( \tilde{\theta} - \frac{\theta}{(1 - s)} \right) + sp^c_1 - sc_2}{2s} \right) \left( \frac{(v_2 - v_1) \left[ \tilde{\theta} - \theta (1 - s) \right] + sp^c_1 - sc_2}{2 (v_2 - v_1) (\tilde{\theta} - \theta)} \right) \]

\[ = \frac{(p^d_2 - c_2)^2}{(v_2 - v_1) (\tilde{\theta} - \theta)} s \]

Combining to obtain the critical discount factor of each firm gives:

\[ \tilde{\delta}_1 = \frac{\pi^d_1 - \pi^c_1}{\pi^c_1 - \pi^*_1} = \frac{(p^d_1 - c_1)^2}{(v_2 - v_1) (\tilde{\theta} - \theta)} \frac{s}{(p^c_1 - c_1) (p^*_1 - c_1)} - \frac{(p^c_1 - c_1) (p^*_1 - c_1)}{(v_2 - v_1) (\tilde{\theta} - \theta)} \]

\[ = \frac{(p^d_1 - c_1)^2 - (p^c_1 - c_1) (p^*_1 - c_1)}{(p^d_1 - c_1)^2 - (p^*_1 - c_1)^2} = \frac{1}{4} \frac{(p^c_1 - p^*_1)^2}{(p^c_1 - p^*_1) \left( \frac{1}{4} (p^c_1 - p^*_1) + (p^*_1 - c_1) \right)} \]

\[ = \frac{1}{4} \frac{(p^c_1 - p^*_1)}{(p^c_1 - p^*_1) + (p^*_1 - c_1)} \]
and

$$\overline{\delta}_2 = \frac{\pi_2^d - \pi_2^c}{\pi_2^d - \pi_2^c} - \frac{(p_2^d - c_2)^2}{(v_2 - v_1)} \frac{s}{(\hat{\theta} - \theta)} - \frac{(p_2^c - c_2) (p_2^* - c_2)}{(v_2 - v_1)} \frac{s}{(\hat{\theta} - \theta)}$$

$$= \frac{(p_2^d - c_2)^2 - (p_2^c - c_2) (p_2^* - c_2)}{(p_2^d - c_2)^2 - (p_2^c - c_2)^2} = \frac{\frac{1}{4} (p_1^c - p_1^*)^2}{(p_1^c - p_1^*) (\frac{1}{4} (p_1^c - p_1^*) + (p_2^* - c_2))}$$

$$= \frac{\frac{1}{4} (p_1^c - p_1^*)}{\frac{1}{4} (p_1^c - p_1^*) + (p_2^* - c_2)}.$$ 

7. Appendix 3: Comparisons with Other Models

Symeonidis (1999) analyzes a representative consumer model, which differs from ours in many ways. Meaningful comparisons are therefore difficult to make. Häckner’s (1994) model is also different, but in many ways comparable. Specifically, he is using a utility specification of the form $U(\theta) = v_i (\theta - p_i)$, whereas we follow Mussa and Rosen (1978) which uses $U(\theta) = v_i \theta - p_i$. To address this issue, we have tried to clarify the differences between the different settings in the introduction. Moreover, we have performed the same analysis in Häckner’s (1994) setting. This analysis is presented below, but let us first summarize the main conclusion. The main finding of Proposition 1 may not generally hold in Häckner’s (1994) model. In particular, conclusions may be different when the highest quality firms have the lowest profit margins and vice versa. Yet, the results of Proposition 1 do hold when the noncollusive profit margin is increasing in quality. Moreover, if differences in unit costs are sufficiently small, then it is indeed the lowest-quality seller who is most inclined to deviate, all else unchanged. Consequently, the result of the Corollary also holds when taking Häckner’s (1994) approach. We have added a footnote highlighting this point (footnote 8). Consider an intermediate firm in an n-firm variant of Häckner’s (1994) model with cost heterogeneity. A consumer located at $\theta_i$ is indifferent between buying from firm $i$ and $i + 1$ when:

$$\theta_i v_{i+1} - v_{i+1} p_{i+1} = \theta_i v_i - v_i p_i \text{ or } \theta_i = \frac{v_{i+1} p_{i+1} - v_i p_i}{v_{i+1} - v_i}.$$ 

The profit function is then given by

$$\pi_i = (p_i - c_i) \left( \frac{v_{i+1} (v_i - v_{i-1}) p_{i+1} + v_{i-1} (v_{i+1} - v_i) p_{i-1} - v_i (v_{i+1} - v_{i-1}) p_i}{(v_{i+1} - v_i) (v_i - v_{i-1})} \right),$$ 

which yields the following best-response function:

$$\hat{p}_i (p_{i-1}, p_{i+1}) = \frac{v_{i+1} (v_i - v_{i-1}) p_{i+1} + v_{i-1} (v_{i+1} - v_i) p_{i-1} + v_i (v_{i+1} - v_{i-1}) c_i}{2v_i (v_{i+1} - v_{i-1})}.$$
Following the same steps in the proof of Proposition 1, this gives demand:

$$\theta_i^* - \theta_{i-1}^* = \frac{v_i (v_{i+1} - v_{i-1})}{(v_{i+1} - v_i) (v_i - v_{i-1})} (p_i^* - c_i)$$

and Nash profits

$$\pi_i^* = (p_i^* - c_i)^2 \left( \frac{v_i (v_{i+1} - v_{i-1})}{(v_{i+1} - v_i) (v_i - v_{i-1})} \right).$$

In a similar fashion, it can be shown that:

$$\pi_i^d = (p_i^d - c_i)^2 \left( \frac{v_i (v_{i+1} - v_{i-1})}{(v_{i+1} - v_i) (v_i - v_{i-1})} \right).$$

As to collusive profits, we know that (due to the fixed market share rule):

$$\theta_i^* v_{i+1} - v_{i+1} p_{i+1}^c = \theta_i^* v_i - v_i p_i^c \quad \text{or} \quad p_{i+1}^c = p_i^* + \frac{v_i}{v_{i+1}} (p_i^c - p_i^*).$$

In general, the collusive price is then:

$$p_i^c = p_i^* + \frac{v_i}{v_{i+1}} (p_i^c - p_i^*).$$

Following the covered market assumption, we know that $\theta_i^c - \theta_{i-1}^c = \theta_i^* - \theta_{i-1}^*$ and therefore,

$$\pi_i^c = (p_i^c - c_i) (\theta_i^c - \theta_{i-1}^c) = \left( p_i^* + \frac{v_i}{v_{i+1}} (p_i^c - p_i^*) - c_i \right) \left( \frac{v_i (v_{i+1} - v_{i-1})}{(v_{i+1} - v_i) (v_i - v_{i-1})} (p_i^* - c_i) \right).$$

Moreover,

$$p_i^d = \frac{v_i (v_{i+1} - v_{i-1}) p_{i+1}^c + v_{i-1} (v_{i+1} - v_i) p_{i-1}^c + v_i (v_{i+1} - v_{i-1}) c_i}{2 v_i (v_{i+1} - v_{i-1})}$$

$$= \frac{v_i (v_{i+1} - v_i) \left( p_{i+1}^* + \frac{v_i}{v_{i+1}} (p_i^c - p_i^*) \right) + v_{i-1} (v_{i+1} - v_i) \left( p_{i-1}^* + \frac{v_i}{v_{i-1}} (p_i^c - p_i^*) \right) + v_i (v_{i+1} - v_{i-1}) c_i}{2 v_i (v_{i+1} - v_{i-1})}$$

$$= \frac{v_i (v_{i+1} - v_{i-1}) p_i^* + v_{i-1} (v_{i+1} - v_i) p_i^* + v_i (v_{i+1} - v_{i-1}) c_i}{2 v_i (v_{i+1} - v_{i-1})}$$

$$= p_i^* + \frac{1}{2} \frac{v_i}{v_{i+1}} (p_i^c - p_i^*).$$

Combining the above profit specifications gives

$$\pi_i^d - \pi_i^c = \frac{v_i (v_{i+1} - v_{i-1}) (p_i^d - c_i)^2}{(v_{i+1} - v_i) (v_i - v_{i-1})} - \frac{v_i (v_{i+1} - v_{i-1}) (p_i^* - c_i) \left( p_i^* + \frac{v_i}{v_{i+1}} (p_i^c - p_i^*) - c_i \right) (v_{i+1} - v_i) (v_i - v_{i-1})}{(v_{i+1} - v_i) (v_i - v_{i-1})}$$

$$= \frac{v_i (v_{i+1} - v_{i-1}) \left[ \left( p_i^* + \frac{v_i}{2 v_{i+1}} (p_i^c - p_i^*) - c_i \right)^2 - (p_i^* - c_i)^2 - \frac{v_i}{v_{i+1}} (p_i^c - p_i^*) (p_i^* - c_i) \right]}{(v_{i+1} - v_i) (v_i - v_{i-1})}$$

$$= \frac{v_i (v_{i+1} - v_{i-1}) \left[ \frac{1}{4} \left( \frac{v_i}{v_{i+1}} \right)^2 (p_i^c - p_i^*)^2 \right]}{(v_{i+1} - v_i) (v_i - v_{i-1})}.$$
\[ \pi_i^d - \pi_i^e = \frac{v_i(v_{i+1} - v_{i-1})(p_i^d - c_i)^2}{(v_{i+1} - v_i)(v_i - v_{i-1})} - \frac{v_i(v_{i+1} - v_{i-1})(p_i^e - c_i)^2}{(v_{i+1} - v_i)(v_i - v_{i-1})} \\
= \frac{v_i(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} \left( \frac{1}{4} \left( \frac{v_i}{v_{i+1}} \right)^2 (p_i^e - p_i^e)^2 + \frac{v_i}{v_{i+1}} (p_i^e - p_i^e) (p_i^e - c_i) \right) \\
= \frac{v_i(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} \left( \frac{1}{4} \left( \frac{v_i}{v_{i+1}} \right)^2 (p_i^e - p_i^e)^2 + \frac{v_i}{v_{i+1}} (p_i^e - p_i^e) (p_i^e - c_i) \right) \\
= \frac{1}{4} \left( \frac{v_i}{v_{i+1}} \right) (p_i^e - p_i^e) + (p_i^e - c_i) \\
= \frac{1}{4} v_1 (p_i^e - p_i^e) + v_i (p_i^e - c_i). \]

Thus, the critical discount factor of an intermediate firm \( i \) in Hättner’s (1994) setting with cost heterogeneity and \( n \) firms is:

\[ \delta \geq \delta_i = \frac{\pi_i^d - \pi_i^e}{\pi_i^d - \pi_i^e} = \frac{\frac{v_i(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} \left[ \frac{1}{4} \left( \frac{v_i}{v_{i+1}} \right)^2 (p_i^e - p_i^e)^2 \right]}{\frac{1}{4} \left( \frac{v_i}{v_{i+1}} \right) (p_i^e - p_i^e) + (p_i^e - c_i)} \]

\[ = \frac{1}{4} \left( \frac{v_i}{v_{i+1}} \right) (p_i^e - p_i^e) + (p_i^e - c_i) = \frac{1}{4} v_1 (p_i^e - p_i^e) + v_i (p_i^e - c_i). \]

Results for the highest and lowest quality firm can be derived in a similar way (see the proof of Proposition 1).

Recall that the corresponding critical discount factor in Proposition 1 is given by

\[ \delta \geq \delta_i = \frac{\pi_i^d - \pi_i^e}{\pi_i^d - \pi_i^e} = \frac{1}{4} \left( p_i^e - p_i^e \right) \]

Observe that, compared to the result of Proposition 1, it is now possible that a firm with a higher competitive profit margin is more eager to leave the cartel when it is of sufficiently low quality (i.e., a high \( p_i^e - c_i \) may be more than neutralized by a low \( v_i \)) and vice versa. The result of Proposition 1 does apply, however, when the noncollusive price-cost margin is increasing in quality. Moreover, and in contrast to Hättner, it is indeed the lowest-quality supplier who has the tightest incentive constraint when differences in unit costs are sufficiently small (in accordance with the result of the Corollary). As to the latter, note that Hättner (1994) considers a duopoly without costs and does not impose a fixed market share rule, which is an important driver of our finding.

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