Dark Energy From Fifth Dimensional Brans–Dicke Theory

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June 4, 2013

Abstract

Following the approach of the induced–matter theory, we investigate the cosmological implications of a five–dimensional Brans–Dicke theory, and propose to explain the acceleration of the universe. After inducing in a four–dimensional hypersurface, we classify the energy–momentum tensor into two parts in a way that, one part represents all kind of the matter (the baryonic and dark) and the other one contains every extra terms emerging from the scale factor of the fifth dimension and the scalar field, which we consider as the energy–momentum tensor of dark energy. We also separate the energy–momentum conservation equation into two conservation equations, one for matter and the other for dark energy. We perform this procedure for different cases, without interacting term and with two particular (suitable) interacting terms between the two parts. By assuming the parameter of the state equation for dark energy to be constant, the equations of the model admit the power–law solutions. Though, the non–interacting case does not give any accelerated universe, but the interacting cases give both decelerated and accelerated universes. For the interacting cases, we figure out analytically the acceptable ranges of some parameters of the model, and also investigate the data analysis to test the model parameter values consistency with the observational data of the distance modulus of 580 SNe Ia compiled in Union2.1. For one of these interacting cases, the best fitted values suggest that the Brans–Dicke coupling constant ($\omega$) is $\approx -7.75$, however, it also gives the state parameter of dark energy ($w_x$) equal to $\approx -0.67$. In addition, the model gives the Hubble and deceleration parameters at the present time to be $H_0 \approx 69.4$ (km/s)/Mpc and $q_0 \approx -0.38$ (within their confidence intervals), where the scale factor of the fifth dimension shrinks with the time.

PACS number: 04.50. – h ; 04.50.Kd ; 95.36. + x ; 98.80.Es

Keywords: Brans–Dicke Theory; Induced–Matter Theory; FRW Cosmology; Dark Energy; Data Analysis of Observational Cosmology

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1 Introduction

While improving and searching for more suitable gravitational theory, attempts for a geometrical unification of gravity with other interactions has begun by employing extra dimensions beyond the conventional four–dimensional (4D) space–time. At first, Nordstrøm [1] built a unified theory based on extra dimensions, then Kaluza [2] and Klein [3] established a 5D version of general relativity (GR) in which electrodynamics rises from an extra fifth dimension. Till now, an intensive amount of researches have been performed on a similar idea either via different mechanism of compactification of extra dimension or generalizing it to non–compact scenarios [4] such as the Brane–World theories [5], the space–time–matter or induced–matter (IM) theories [6, 7], and higher dimensional cosmology, e.g. Ref. [8]. The latter theories are grounded on the Campbell–Magaard theorem that asserts every analytical N–dimensional Riemannian space can locally be embedded in an (N + 1)–dimensional vacuum one [9]–[13]. The importance of this theorem is that the matter sources of 4D space–times can be viewed as a manifestation of extra dimensions. In another word, 4D field equations with matter sources can be achieved by inducing 5D field equations without matter sources. This idea has been the core of the IM theory using GR as the underlying theory.

On the other hand, Jordan [14] conceived a new kind of gravitational theory, known as the scalar–tensor theory, by embedding a curved 4D space–time in a flat 5D one. Following his idea, Brans and Dicke [15, 16] introduced a version of the scalar–tensor gravitational theory, as an alternative to GR, with a non–minimally scalar field coupled to the curvature, in which the weak equivalence principle is preserved and makes it to be more Machian than GR. Unfortunately, its weakness is mismatching with the solar system observations [17, 18], which is a generic difficulty of the scalar–tensor theories in the solar system constraint [19, 20]. However, it does not necessarily denote that the evolution of the universe, at all scales, should be close to GR, in which there are some debates on its tests on the cosmic scales as well [21, 22].

The measurements of anisotropies in the cosmic microwave background suggest that the ordinary 4D universe is very close to a spatially flat universe [23]–[28], and the observations of type Ia–supernovae indicate that the expansion of the universe presently is accelerating [29]–[33]. Hence, in one school of thought, the main component of the universe should be consist of what usually has been called dark energy [34, 35]. Since then, a considerable amount of work has been performed in the literature to explain the acceleration of the universe. Most of the dark energy models, such as the quintessence [36]–[39], the chaplygin gas [40, 41] and the k-essence [42, 43] models, involve minimally coupled scalar fields with different potentials. These scalar fields or the potentials have been added in priori by hand, and hence, their origins are not understood. Nevertheless, in the recent decades, explaining the accelerated expansion of the universe via fundamental theories has been a great challenge.

Recently, a 5D vacuum Brans–Dicke (BD) theory based on the idea of IM theory has been investigated [44], in which the role of GR has been replaced by the BD theory of gravitation as the fundamental underlying theory. It has been illustrated that 5D vacuum BD equations, when reduced to four dimensions, give a modified version of the 4D BD theory with an induced potential. Whereas in the literature, to obtain accelerating universes, such potentials has been added in priori by hand. A few applications of this approach have also been performed in Refs. [45]–[48].

Although one of the aims of higher dimensional theories is to obtain 4D matter from pure geometry, but it is sometimes desirable to have a higher dimensional energy–momentum tensor or a scalar field, for example in compactification of the extra curved dimensions [49]. Indeed, the reduction procedure of a 5D analogue of the BD theory with matter content, on every hypersurface orthogonal to an extra cyclic dimension (that recovers the modified BD theory described by a 4–metric coupled to two scalar fields) has previously been accomplished in the literature [50, 51]. Besides, the main idea of the brane models is the existence of a higher dimensional bulk in which the universe is sitting as a hypersurface and all the matter fields, except gravity, are confined to this brane [5]. Cosmological implications in the context of brane worlds have also been studied, e.g. it has been shown that the
recent accelerated expansion of the universe can be explained by a geometrical originated dark energy caused by an addition of a brane curvature scalar term in the action [52, 53].

Similar to the idea of brane scenarios, but based on the IM theory with underlying the BD theory of gravitation, in this work, we employ a generalized Friedmann–Lemaître–Robertson–Walker (FLRW) type solution for a 5D BD theory and investigate its cosmological implications. For this purpose, in the next section, we give a brief review of a 5D BD theory following the idea of IM theory. Then, in a 4D hypersurface, we gather the extra terms emerging from the scale factor of the fifth dimension and the scalar field as dark energy component of an energy–momentum tensor in addition to the usual energy–momentum tensor of all (the baryonic and dark) matter. In Section 3, we consider a generalized FLRW metric in a 5D space–time, and derive the FLRW cosmological equations. Then, we manage and obtain the total energy conservation equation as separated into two energy conservation equations for all matter and dark energy, with non–interacting and two simple interacting cases. However, as the cosmological equations are strictly non–linear and do not have exact solution, in order to proceed, we assume some simplifications. In Section 4, we apply the observational constraints for the interacting cases of the model, and use the Markov Chain Monte Carlo (MCMC) method to fit the free parameters of the model with the SCP Union2.1 SN Ia compilation [54, 55] based on the Bayesian statistics [56]. A few tables and figures are also provided for a better view of the acceptable range and the best fitted values of parameters. Finally, conclusions are presented in the last section.

2 Five–Dimensional Brans–Dicke Theory

Following the idea of IM theories, one can consider the BD theory of gravitation as the underlying theory instead of GR. In this respect, the action of 5D BD theory in the Jordan frame can analogously be written as

$$\mathcal{S}[g_{AB}, \phi] = \int \sqrt{|\mathcal{g}|} \left( \phi \mathcal{R}^{(5)} - \frac{\omega}{\phi} g^{AB} \phi_{,A} \phi_{,B} + 16\pi L_m \right) d^5 x ,$$

where \(c = 1\), the capital Latin indices run from zero to four, \(\mathcal{R}^{(5)}\) is 5D Ricci scalar, \(\mathcal{g}\) is the determinant of 5D metric \(g_{AB}\), \(\phi\) is a positive scalar field that describes the gravitational coupling in five dimensions, \(L_m\) represents the matter Lagrangian and \(\omega\) is a dimensionless coupling constant. Hence, the field equations obtained from action (1) are

$$\mathcal{G}^{(5)}_{AB} = \frac{8\pi}{\phi} \mathcal{T}_{AB} + \frac{\omega}{\phi^2} \left( \phi_{,A} \phi_{,B} - \frac{1}{2} g_{AB} \phi_{,C} \phi_{,C} \right) + \frac{1}{\phi} \left( \phi_{,AB} - g_{AB} \mathcal{G}^{(5)}_{\Box} \right)$$

(2)

and

$$\mathcal{G}^{(5)}_{\Box} = \frac{8\pi}{4 + 3\omega} \mathcal{T} ,$$

(3)

where \(\mathcal{G}^{(5)}_{AB}\) is 5D Einstein tensor, \(\mathcal{T}_{AB}\) is 5D energy–momentum tensor, \(\mathcal{T} \equiv \mathcal{T}^{(5)}_{C} \mathcal{C}\) and \(\Box \equiv \Box^{(5)}_{A} \mathcal{A}\). In order to have a non–ghost scalar field in the conformally related Einstein frame, i.e. a field with a positive kinetic energy term in that frame, the BD coupling constant must be \(\omega > -4/3\) [57]–[59]. Though recently, some new ghost dark energy models have been suggested to explain the observed acceleration of the universe [60].

As a plausible assumption, we consider that \(\mathcal{T}_{AB}\) exactly represents the same baryonic and dark matter sources of a 4D hypersurface, i.e. \(\mathcal{T}_{\alpha\beta}^{(M)}\). Hence, in this case \(\mathcal{T}^{(5)}_{AB} = \text{diag}(\rho_M, -p_M, -p_M, -p_M, 0)\), where \(\rho_M\) and \(p_M\) are the energy density and the pressure of the matter, and the Greek indices run from zero to three. Also, cosmological purposes usually restrict attentions to 5D metrics of the simple form

$$d\mathcal{S}^2 = g_{AB}(x^C) dx^A dx^B = (\mathcal{g}_{\mu\nu}(x^C) dx^\mu dx^\nu + g_{4i}(x^C) dy^2 \equiv (\mathcal{g}_{\mu\nu}(x^C) dx^\mu dx^\nu + e^2(x^C)dy^2$$

(4)
in local coordinates \( x^A = (x^\mu, y) \), where \( y \) represents the fifth coordinate and \( \epsilon^2 = 1 \). By assuming the 5D space–time is foliated by a family of hypersurfaces, say \( \Sigma \), that are defined by fixed values of \( y \), then, one can obtain the metric intrinsic to every generic hypersurface, e.g. \( \Sigma_0(y = y_0) \), by restricting the line element (4) to displacements confined to it. Thus, the induced metric on the hypersurface \( \Sigma_0 \) becomes

\[
    ds^2 = (g_{\mu\nu}(x^\alpha, y_0))dx^\mu dx^\nu \equiv g_{\mu\nu}dx^\mu dx^\nu,
\]

in such a way that the usual 4D space–time metric, \( g_{\mu\nu} \), can be recovered.

Therefore, after some manipulations, equation (2) on the hypersurface \( \Sigma_0 \) can be written as

\[
    G_{\alpha\beta} = \frac{8\pi}{\phi} (T^{(M)}_{\alpha\beta} + T^{(X)}_{\alpha\beta}),
\]

where we consider \( T^{(X)}_{\alpha\beta} \) as dark energy component of the energy–momentum tensor that is defined by

\[
    T^{(X)}_{\alpha\beta} \equiv T^{(IM)}_{\alpha\beta} + T^{(e)}_{\alpha\beta} + \frac{\omega}{8\pi\phi} (\phi,_{\alpha}\phi,_{\beta} - \frac{1}{2}g_{\alpha\beta}\phi,^\sigma\phi,^\sigma) + \frac{1}{8\pi} (\phi,_{\alpha\beta} - g_{\alpha\beta}\Box \phi),
\]

in which, analogous to the IM theory [47],

\[
    T^{(IM)}_{\alpha\beta} \equiv \frac{\phi}{8\pi} \left\{ \frac{b_{,\alpha}}{b} - \Box b - \frac{\epsilon}{2b^2} \left[ \frac{b'}{b} g_{\alpha\beta}' - g_{\alpha\beta}' - g_{\alpha\sigma} g_{\beta\mu} g_{\mu\sigma}' - \frac{1}{2} g_{\alpha\mu} g_{\beta\nu} g_{\mu\nu}' \right] \right\}
\]

and

\[
    T^{(e)}_{\alpha\beta} \equiv -\frac{\epsilon}{8\pi b^2} \left\{ g_{\alpha\beta} \left[ \phi,\phi, + \left( \frac{1}{2} g_{\mu\nu} g_{\mu\nu}' - \frac{b'}{b} + \frac{\omega}{2\phi} \phi' \right) \phi' + \epsilon b_{,\alpha\mu} g_{\mu\nu} \right] - \frac{1}{2} g_{\alpha\beta}' \phi' \right\}.
\]

The prime denotes derivative with respect to the fifth coordinate.

In the following section, we consider a generalized FLRW metric in a 5D universe and investigate its cosmological properties.

3 Generalized FLRW Cosmology

For a 5D universe with an extra space–like dimension in addition to the three usual spatially homogenous and isotropic ones, metric (4), as a generalized FLRW solution, can be written as

\[
    ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] + b^2(t)dy^2.
\]

Generally, the scalar field \( \phi \) and the scale factors \( a \) and \( b \) should be functions of \( t \) and \( y \). However, for physical plausibility and simplicity, we assume that the hypersurface–orthogonal space–like is a Killing vector field in the underlying 5D space–time [50, 51], i.e. the extra dimension to be a cyclic coordinate. Besides, the functionality of the scale factor \( b \) on \( y \) could be eliminated either by transforming to a new extra coordinate (if \( b \) be a separable function) or making no changes in the consequent equations if \( b \) is the only field that depends on \( y \). In another word, in the compactified extra dimension scenarios, all fields can be Fourier–expanded around the fixed value \( y_0 \), and hence, one can get the observable terms independent of \( y \), i.e. physics would be effectively independent of the compactified fifth dimension, see, e.g. Ref. [4]. With this solution, we will show that the universe can accept both accelerating and decelerating expansion eras.

By considering metric (10), the BD equations (2) reduce as follows. The time component \( A = 0 = B \) gives

\[
    H^2 = \frac{8\pi}{3\phi} \rho_M + \frac{\omega}{6} T^2 - HB + \frac{\phi,}{3\phi} - \frac{k}{a^2} \equiv \frac{8\pi}{3\phi} (\rho_M + \rho_X) - \frac{k}{a^2},
\]

(11)
the spatial components $A = B = 1, 2, 3$ provide
\[
\frac{\ddot{a}}{a} = -\frac{4\pi}{\phi} p_M - \frac{1}{2} H^2 - \frac{\omega}{4} F^2 + \frac{1}{2} HF - HB - \frac{\ddot{b}}{2b} - \frac{k}{2a^2} \equiv -\frac{4\pi}{3\phi} \left( (\rho_M + \rho_X) + 3(p_M + p_X) \right)
\] (12)
and the $A = 4 = B$ component yields
\[
\frac{\ddot{a}}{a} = -H^2 - \frac{\omega}{6} F^2 + \frac{1}{3} BF - \frac{k}{a^2} .
\] (13)
Also, the scalar field equation (3) becomes
\[
\frac{\ddot{\phi}}{\phi} + 3HF = \frac{8\pi(\rho_M - 3p_M)}{\phi(4 + 3\omega)} - BF ,
\] (14)
where $H \equiv \dot{a}/a$, $B \equiv \dot{b}/b$ and $F \equiv \dot{\phi}/\phi$.

In the last part of equations (11) and (12), we have defined the energy density and pressure of dark energy as
\[
\rho_X \equiv -T^{(x)}_t = \frac{\phi}{8\pi} \left( \frac{\omega}{2} F^2 - 3HB + \frac{\ddot{\phi}}{\phi} \right)
\] (15)
and
\[
p_X \equiv T^{(x)}_i = \frac{\phi}{8\pi} \left( \frac{\omega}{2} F^2 - HF + 2HB + \frac{\ddot{b}}{b} \right).
\] (16)
Hence, the state equation of dark energy obviously yields
\[
w_X = \frac{p_X}{\rho_X} = \frac{\omega F^2/2 - HF + 2HB + \frac{\ddot{b}}{b}}{\omega F^2/2 - 3HB + \frac{\ddot{\phi}}{\phi}} .
\] (17)

In order to find the functionality of $w_X$ with the time, equations (11)–(14) must be exactly solved. However, as these equations are coupled non–linearly, one should apply numerical methods. As an alternative, in the following, by considering a few simplifications, we proceed to fit the parameters of the model by observational data analysis, and discuss the properties of the model.

First of all, let us obtain the energy conservation equations. In this respect, one can take the time derivative of equation (11) and substitutes equation (12) into it, and finally gets
\[
\dot{\rho} + 3H(\rho + \rho_M) = F\ddot{\rho} ,
\] (18)
where $\rho \equiv \rho_M + \rho_X$ and $\ddot{\rho} \equiv p_M + p_X$. As the nature of dark energy is unknown, the detailed coupling form among it and matter is unclear. Hence, one can expect that their conservation equations should not be independent. In this case, let us apply a plausible simplification by separating equation (18) into two distinguished continuity equations for $\rho_X$ and $\rho_M$ as
\[
\dot{\rho}_M + 3H(\rho_M + p_M) = F\rho_M + Q
\] (19)
and
\[
\dot{\rho}_X + 3H(\rho_X + p_X) = F\rho_X - Q ,
\] (20)
where the $Q$ term stands for interacting terms among matter and dark energy. Such a similar interacting term has also been used in the literature, e.g. Ref. [61].

To proceed further, we take $w_M = 0$ as dust (the baryonic and dark) matter, and assume $w_X$ to be a constant. Hence, relation (17) imposes the power–law solutions
\[
a(t) = a_0 \left( \frac{t}{t_0} \right)^{\alpha} \quad \text{with} \quad H = \frac{\alpha}{t} ,
\] (21)
\[ b(t) = b_o \left( \frac{t}{t_o} \right)^\beta \quad \text{with} \quad B = \frac{\beta}{t} \]  

(22)

and

\[ \phi(t) = \phi_o \left( \frac{t}{t_o} \right)^\gamma \quad \text{with} \quad F = \frac{\gamma}{t}. \]  

(23)

If one assumes the power–law solutions (21)–(23), equations (11)–(13) will restrict either the geometry to be spatially flat or \( \alpha \) to be one, i.e. a free expanding universe. As the latter choice is not interested, we consider \( k = 0 \) in this work. This choice is also consistent with the measurements of anisotropies in the cosmic microwave background radiation that indicate the universe must be very close to spatially flat one [23]–[28].

Note that, there are four independent equations (11)–(14) that, in general, determine \( a, b, \phi \) (or equivalently \( \alpha, \beta, \gamma \) in the power–law solutions) and \( \rho_M \). These unknowns also depend on the BD parameter \( \omega \). In order that our ansätze do not lead to over–constraining equations, we manage the values of the parameters of the model including \( \omega \) to be, by using the data analysis method, consistent with the observations.

In the next two subsections, we find the relations of \( \rho_X \) and \( \rho_M \) with the scale factors and the scalar field for both the non–interacting and interacting cases. For interacting cases, we consider two simple and reasonable choices of \( Q \equiv \Gamma \rho_M \) and \( Q \equiv \Gamma \rho_X \), where

\[ \Gamma \equiv \frac{\dot{\psi}}{\psi}. \]  

(24)

The symbol \( \psi \) is denoted as a general notation that represents either the scale factor of the ordinary spatial dimensions, \( a \), or the scale factor of the fifth dimension, \( b \), or the scalar field, \( \phi \). That is, \( \Gamma \) can be either \( H \) or \( B \) or \( F \), however, by matching the model with the observations, we will figure out the best choice of \( \psi \).

### 3.1 Non–Interacting Case \( Q = 0 \)

Now, for the matter to be a dust one and the dark energy parameter of state to be time independent, if there is no interacting term, then equations (19) and (20) lead to

\[ \frac{\rho_M}{\rho_{Mo}} = \frac{\phi}{\phi_o} \left( \frac{a}{a_o} \right)^{-3} \]  

(25)

and

\[ \frac{\rho_X}{\rho_{Xo}} = \frac{\phi}{\phi_o} \left( \frac{a}{a_o} \right)^{-3(1+w_X)}, \]  

(26)

which for the power–law solutions (21)–(23) become

\[ \frac{\rho_M}{\rho_{Mo}} = \left( \frac{t}{t_o} \right)^{\gamma-3\alpha} \]  

(27)

and

\[ \frac{\rho_X}{\rho_{Xo}} = \left( \frac{t}{t_o} \right)^{\gamma-3\alpha(1+w_X)}. \]  

(28)

where \( \rho_{Mo}, \rho_{Xo}, a_o \) and \( \phi_o \) are the energy density of matter, the energy density of dark energy, the scale factor and the scalar field at the present time, respectively.

Substituting the power–law solutions in equation (11) (and also equation (14)) imposes \( \alpha = 2/3 \) and \( w_X = 0 \), which does not give an accelerated universe. Hence, in the following, we investigate different interacting cases to figure out any possible accelerating solutions. However, when \( w_X \) is not constant (as a more general case), then the non–interacting choice may also provide some interesting results.
3.2 Interacting Cases $Q \equiv \Gamma \rho_M$ and $Q \equiv \Gamma \rho_X$

With the same assumptions as in the non–interacting cases, to find out analytically the ranges of the parameters of our model for interacting cases $Q \equiv \Gamma \rho_M$ and $Q \equiv \Gamma \rho_X$, we consider the power–law solutions (21)–(23). Thus, we assume

$$\psi(t) = \psi_o \left(\frac{t}{t_o}\right)^{\lambda} \quad \text{with} \quad \Gamma = \frac{\lambda}{t}. \quad \text{(29)}$$

Now, for the chosen interacting cases, we will show, in the following two parts, that the model can accept both acceleration and deceleration for the universe expansion eras.

A. Interacting Case $Q \equiv \Gamma \rho_M$

Substituting $Q \equiv \Gamma \rho_M$ into equation (19) gives

$$\frac{\rho_M}{\rho_{M_o}} = \frac{\phi}{\phi_o} \left(\frac{a}{a_o}\right)^{-3}, \quad \text{(30)}$$

that for the power–law solutions (21)–(23), becomes

$$\frac{\rho_M}{\rho_{M_o}} = \left(\frac{t}{t_o}\right)^{\gamma + \lambda - 3\alpha}. \quad \text{(31)}$$

Thus, an acceptable solution of (20), which satisfies equations (11)–(14), is

$$\frac{\rho_X}{\rho_{X_o}} = \left(\frac{t}{t_o}\right)^{\gamma + \lambda - 3\alpha} \quad \text{with} \quad \rho_{X_o} = -\frac{\lambda \rho_{M_o}}{\lambda + 3\alpha w_X}. \quad \text{(32)}$$

Solutions (31) and (32) reveals that both the matter and dark energy evolutions are the same in this model. Substituting $\rho_M$ and $\rho_X$ into equation (11) or (12), imposes

$$\lambda = 3\alpha - 2, \quad \text{(33)}$$

and then, using it into the second part of (32), gives

$$w_X = -\frac{(u_o + 1)(3\alpha - 2)}{3\alpha}, \quad \text{(34)}$$

where $u_o \equiv \rho_{M_o}/\rho_{X_o}$. The acceptable ranges of $w_X$, $\alpha$, and $\lambda$, for an expanding universe, are given in Table 1 and are depicted in Fig. 1.

For this interacting case, equation (11) can also be presented as

$$H^2 = H_o^2 \left[\Omega_{M_o} + \Omega_{X_o}\right] (1 + z)^2 \frac{\alpha}{\pi}, \quad \text{(35)}$$

where $\Omega_{M_o} + \Omega_{X_o} = 1$. Relation (35) is the same as the result obtained in Ref. [62] for a 4D BD model with a priori assumed (i.e. added–by–hand) potential term. However, in the next section, we also match the model consistency with the observational data of SN Ia, and search the best fitted parameters of the model. If one sets $\psi \equiv a$, i.e. $\Gamma \equiv H$ and $\lambda \equiv \alpha$, then from equation (33) it is obvious that $\alpha = 1$, which gives a free expanding universe, thus, in the data analyzing of equation (35), we only consider the other two (more interesting) cases $\lambda = \beta$ and $\lambda = \gamma$.

B. Interacting Case $Q \equiv \Gamma \rho_X$
Another suitable choice for non–interacting case is to take \( Q \equiv \Gamma \rho_X \). Thus, by substituting it into equation (20), one gets
\[
\frac{\rho_X}{\rho_{X_0}} = \frac{\phi}{\phi_0} \left( \frac{\psi}{\psi_0} \right)^{-1} \left( \frac{a}{a_0} \right)^{-3(1+w_X)},
\]
that for the power–law solutions (21)–(23), yields
\[
\frac{\rho_X}{\rho_{X_0}} = \left( \frac{t}{t_0} \right)^{\gamma-\lambda-3\alpha(1+w_X)}.
\]
Therefore, an acceptable solution of (19) is
\[
\frac{\rho_M}{\rho_{M_0}} = \left( \frac{t}{t_0} \right)^{\gamma-\lambda-3\alpha(1+w_X)} \quad \text{with} \quad \rho_{M_0} = -\frac{\lambda \rho_{X_0}}{\lambda + 3\alpha w_X}.
\]

Once again, the matter and dark energy evolve in the same manner in this model. Solutions (37) and (38) must also satisfy equations (11)–(14), that give
\[
\lambda = u_0(3\alpha - 2)
\]
and
\[
w_X = -\frac{(u_0 + 1)(3\alpha - 2)}{3\alpha}.
\]
It is notable that equations (40) and (34) are exactly the same. Finally, one can rewrite equation (11) as
\[
H^2 = H_0^2 \left[ \Omega_{M_0} + \Omega_{X_0} \right] (1+z)^{3(1-u_0)+\frac{2u_0}{\alpha}},
\]
where again \( \Omega_{M_0} + \Omega_{X_0} = 1 \). In the following, we highlight the conditions that correspond to a decelerating and an accelerating universe for both of the interacting cases.

I. Deceleration and Free Expansion

The observational data reveals that when the universe was in the radiation or dust dominated phases, it was in a decelerating regime for a long time [63]. In our model, decelerating and free expanding universe can be obtained from equation (12) for \( w_X \geq -\frac{1+u}{3} \) (where \( u \equiv \rho_M/\rho_X \)) and \( 0 < \alpha \leq 1 \) in the power–law solutions. The acceptable domains of \( w_X, \alpha \) and \( \lambda \), in accord with this situation, are given in Table 1 and drown in Fig. 1.

II. Acceleration

Recent observations illustrate that the universe is in an accelerating phase at the present epoch [29]–[33]. An accelerating universe from equation (12) makes \( w_X < -\frac{1+u}{3} \) and \( \alpha > 1 \) for the power–law solutions. The acceptable values of \( w_X, \alpha \) and \( \lambda \) corresponding to this condition are also given in Table 1 and Fig. 1.

Table 1 and Fig. 1 surprisingly show that both the decelerating and the accelerating solutions are acceptable in the range \( w_X < -1/3 \). However, a fixed value for the parameter \( \alpha \) for all epochs implies that the universe is always either decelerating or accelerating [62], which is the common property of the power–law solutions. In this respect, one should consider the model only for the late time acceleration of the universe.

To analyze the nature of the acceleration part of the model, it is instructive to investigate the model consistency with the observational data. For this purpose, in the next section, we employ the MCMC method to fit the free parameters of the model with the SCP Union2.1 SN Ia compilation [54, 55] based on the Bayesian statistics [56].
Universe Phase | $\alpha$ | $\lambda$ for case A | $\lambda$ for case B | $w_X$
---|---|---|---|---
Expansion | $0 < \alpha \leq \frac{2}{3}$ | $-2 < \lambda \leq 0$ | $\lambda \leq 0$ | $w_X \geq \frac{(2-3\alpha)}{3\alpha}$
| $\alpha > \frac{2}{3}$ | $\lambda > 0$ | $\lambda > 0$ | $w_X < \frac{(2-3\alpha)}{3\alpha}$
Deceleration and Free Expansion | $0 \leq \alpha \leq \frac{2}{3}$ | $-2 < \lambda \leq 0$ | $\lambda \leq 0$ | $w_X \geq \frac{(2-3\alpha)}{3\alpha}$
| $\frac{2}{3} < \alpha \leq 1$ | $0 < \lambda \leq 1$ | $\lambda \geq 0$ | $w_X \leq \frac{(2-3\alpha)}{3\alpha}$
| $\alpha > 1$ | $\lambda > 1$ | $\lambda > 0$ | $w_X \leq \frac{(2-3\alpha)}{3\alpha}$

Table 1: The ranges of $w_X$, $\alpha$ and $\lambda$ for the expanding, decelerating and accelerating power-law solutions of interacting cases A and B.

Figure 1: The domains of $w_X$ and $\alpha$ correspond to Table 1. The dashed line corresponds to the border value $\alpha = 1$, that separates the acceleration (the upper) and the deceleration (the lower) regions.
4 Observational Constraints

The various data information from different observations are used to constrain the cosmological models. Among them, the SN Ia distance modulus declares the accelerated expansion of the universe [29–33]. We employ the Bayesian statistics to investigate the model consistency with the SCP Union2.1 SN Ia compilation that is an update of the Union2 compilation [64]. This compilation consists of nineteen data sets from 833 supernovae. However, it has been indicated [54, 55] that only 580 of these 833 supernovae can pass the usability cuts, which contain new data from the HST Cluster Survey. The HST Cluster Survey supplies the latest and the most complete data set for SN Ia observations till now.

The most popular techniques for the parameter estimation in cosmology is the MCMC method. These methods were first employed in astrophysics [65], and since then, it has also been used in cosmology. We have used our own package, however, the standard packages for the cosmological parameter estimation are also publicly available [66, 67].

The likelihood function in the Bayesian statistics is defined to be proportional to $\exp\left(-\chi^2/2\right)$. Hence, the best fitted values for the parameters of the model are obtained by minimizing the $\chi^2$, which for the SN Ia is defined as

$$\chi^2_{SN} = \sum_{i=1}^{580} \frac{\left[\mu_{th}(z; \{l\}) - \mu_{ob}(z_i)\right]^2}{\sigma_i^2}. \quad (42)$$

In equation (42), $\{l\}$ refers to the parameters of the model which can be estimated by the data analysis process and $\sigma_i$ stands for the $1\sigma$ uncertainty associated to the $i$th data point. The symbol $\mu_{ob}$ is the observed distance modulus and $\mu_{th}$ is the theoretical distance modulus that is defined as

$$\mu_{th}(z; \{l\}) \equiv m - M = 5\log D_L(z; \{l\}) + 5\log \left(\frac{c}{1\text{Mpc}}\right) + 25, \quad (43)$$

where $m$ is the apparent magnitude, $M$ is the absolute magnitude and

$$D_L(z; \{l\}) \equiv (1 + z) \int_0^z \frac{H_0dz'}{H(z'; \{l\})}, \quad (44)$$

is the luminosity distance.

We constrain the parameters of the model with the observational data from the supernovae type Ia, for both interacting cases A and B of the previous section. The analysis reveals that for the interacting case B, the parameters $\alpha$ and $\lambda$ are strictly related and any change for each of them in the MCMC process yields different non–compatible results. By increasing the steps of the MCMC process, almost the whole parameter space will be covered. In other words, there are no preferred values for the parameters of the model in case B regarding to the SNe Ia observations, hence, we ignore this case.

The result of the data analysis corresponding to the interacting case A is as follow. The best fitted values for $\alpha$ and $H_0$ are listed in Table 2, that suggest $\alpha \simeq 1.61$ and $H_0 \simeq 69.4 \text{ (km/s)/Mpc}$ within their own confidence intervals. Fig. 2 illustrates the likelihood functions for them. The confidence intervals are also represented in Fig. 3 down. Also, the best fitted value for the deceleration parameter, $q = -\ddot{a}/a^2$, within its confidence intervals is given in Table 2. The Hubble diagram (the distance modulus in terms of the redshift) simulated based on this case shows agreement between the model prediction and the observed data, Fig. 3 up. The Hubble time corresponding to the best fitted value of $H_0$ is $t_{H_0} \simeq 14.1 \times 10^9 \text{yr}$.

Now, we can find the other parameters of the model with the best fitted value of $\alpha$, namely $\alpha = 1.61$. First of all, from relation (33), one gets $\lambda = 2.83$, thus, there are two choices, either $\lambda = \beta$ or $\lambda = \gamma$ (where for the latter choice, there are two solutions). Substituting $\alpha$ and $\beta$ (or $\gamma$) into equations (12) and (13) gives the best values of $\omega$ and $\gamma$ (or $\beta$) and then, from relation (17), one
Table 2: The joint likelihood analysis results of $\alpha$ (with the prior used from 1.00 to infinity), $q_0$ and $H_0$ up to $3\sigma$ confidence levels for the interacting case A.

| Parameter      | Best fit | 1$\sigma$ | 2$\sigma$ | 3$\sigma$ |
|----------------|----------|-----------|-----------|-----------|
| $\alpha$      | 1.61     | $+0.16$   | $+0.29$   | $+0.38$   |
|                |          | $-0.12$   | $-0.20$   | $-0.25$   |
| $q_0$          | $-0.37$  | $\pm0.05$ | $\pm0.09$ | $\pm0.12$ |
| $H_0$(km/s)/Mpc| 69.4     | $+0.5$    | $+0.7$    | $+1.13$   |
|                |          | $-0.6$    | $-0.9$    | $-1.12$   |

Table 3: The best values of the parameters of the model corresponding to the best fitted value $\alpha = 1.61$ for the interacting case A. The consistent values of the parameters with the condition $w_X \leq (2 - 3\alpha)/3\alpha = -0.58$ are those given in the last column.

| Parameter | $\lambda \equiv \beta$ | $\lambda \equiv \gamma$ | $\lambda \equiv \gamma$ |
|-----------|-------------------------|-------------------------|-------------------------|
|           | Solution I | Solution II | Solution II |
| $\beta$   | 2.83         | 1.74         | $-6.78$         |
| $\gamma$  | $-5.74$      | 2.83         | 2.83          |
| $\omega$  | $-1.69$      | $-1.72$      | $-7.75$       |
| $w_X$     | 1.52         | 0.45         | $-0.67$       |

gets the best value for $w_X$. The given value of $w_X$ from this procedure must fulfil the acceptable ranges of $w_X$ according to Table 1, which for $\alpha = 1.61$, it reveals that one must have $w_X \leq -0.58$. Table 3 shows all the three groups of values for the parameters of the model. The results illustrate that the consistent values of the parameters of the model are those listed in the last column, which corresponds to $\lambda \equiv \gamma$ case. This imposes that the best interacting term consists of the energy density of the matter multiplied by the ratio of the time derivative of the scalar field to itself.

The consistent values of the parameters indicate that the scale factor of the fifth dimension shrinks with the time and the model has a ghost scalar field with $\omega \approx -7.75$. Note that, as described in the Introduction, this mismatching value of $\omega$ with the values obtained from the solar system observations [17, 18] is expected [19]–[22]. Indeed, this value of the BD coupling constant actually corresponds to an imaginary conformal scalar field in the Einstein frame, where the issue of which conformal frame, the Jordan or the Einstein one, is physical is a matter of debate [68, 69]. In addition, in this model, the obtained dark energy state parameter, $w_X$, is also $\simeq -0.67 > -1$ which does not belong to a ghost one and the corresponding dark energy does not lead to big rip singularities.

5 Conclusions

It is a general belief that the universe is in an accelerated expanding phase. Thus, the main content of it should be consisted of what usually has been called dark energy. Hence, a considerable amount of work has been performed to explain the acceleration of the universe, but until now, the origin and the nature of dark energy is unknown. Most of the dark energy models, such as the quintessence, the chaplygin gas and the k-essence models, involve minimally coupled scalar fields with different potentials which have been added by hand, and nothing has been asserted about their origins. Also in the recent decade, explaining the accelerated expansion of the universe by alternative theories of gravitation has been a great challenge.

In this work, following the approach of the induced–matter theory, we have investigated the cosmological implications of a 5D BD theory, in order to explain the acceleration of the universe.
Figure 2: The likelihood functions of the model free parameters $H_o$ and $\alpha$ for the interacting case A.
Figure 3: Upper figure: the Hubble diagram with simulation based for the interacting case A and the SCP Union2.1 compilation of the SNe Ia observations. Lower figure: the joint likelihood analysis of $\alpha$ and $H_0$ for the interacting case A up to 1$\sigma$, 2$\sigma$ and 3$\sigma$ levels.
After inducing in a 4D hypersurface, we have classified the energy–momentum tensor into two parts. One part represents all kind of the matter (the baryonic and dark), and the other one contains every extra terms emerging from the scale factor of the fifth dimension and the scalar field, which we have considered as the energy–momentum tensor of dark energy.

As the cosmological equations of the model are extremely non–linearly coupled, we have assumed some simplifications in order to proceed the properties of the model. The total energy conservation has been separated into two equations, one for the matter conservation and the other for dark energy one. Of course, such a procedure has been performed without interacting term and with two particular interacting terms between the two parts. We consider the parameter of the state equation of dark energy to be constant. Hence, the equations of the model admit the power–law solutions which impose a spatially flat geometry.

The non–interacting case gives decelerated universes, though, the interacting cases give both decelerated and accelerated universes. For the latter case, we have figured out analytically the acceptable ranges of parameters of the model, and have illustrated the results in a few tables and figures.

Then after, for these interacting cases, we have employed the MCMC method based on the Bayesian statistics to investigate the consistency of the model parameters with the observational data from supernovae type Ia. For this purpose, we have used the data of the SCP Union2.1 SN Ia compilation. The data analysis process for the case, which its interacting term between the matter and dark energy is proportional to the energy density of dark energy, reveals that the involved parameters are strictly related and any change for each of them in the MCMC process yields different non–compatible results. By increasing the steps of the MCMC process, almost the whole parameter space will be covered. In other words, there are no preferred values for the parameters of the model in this interacting case. But for the other interacting case, which the interacting term is proportional to the energy density of the matter, the best fitted values suggest that $H_0 \simeq 69.4$ (km/s)/Mpc, $q_0 \simeq -0.38$ within their confidence intervals. The Hubble time corresponding to the best fitted value of $H_0$ is $t_{H_0} \simeq 14.1 \times 10^9$ yr. The best fitted values of the parameters of the model are also listed in a table, and the results suggest that the energy density of the matter multiplied by the ratio of the time derivative of the scalar field to itself plays the role of the best interacting term between the matter and dark energy. Also, the consistent values of the parameters indicate that the model has a ghost scalar field with $\omega \simeq -7.75$, while the scale factor of the fifth dimension shrinks with the time. Although the model has a ghost scalar field with this value for the BD coupling constant, the consistent values of the parameters reveals a non–ghost dark energy with the state parameter $w_X \simeq -0.67$ as well. That is, with this value of the dark energy state parameter, there should not be big rip singularities and ghost instabilities.

Acknowledgements

We would like to thank Dr. M.S. Movahed for useful comments. M.F. also thanks the Research Office of Shahid Beheshti University G.C. for financial support.

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