ON A TWO-VARIABLE $p$-ADIC $l_q$-FUNCTION

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Abstract. We prove that a two-variable $p$-adic $l_q$-function has the series expansion

$$l_{p,q}(s, t, \chi) = \frac{[2]_{q^F}}{[2]_{q^F}} \sum_{a=1}^{P} (-1)^a \frac{\chi(a)q^a}{(a + pt)^s} \sum_{m=0}^{\infty} \left( \frac{F}{(a + pt)^m} \right) m E_{m,q}^*$$

which interpolates a linear combinations of terms of the generalized $q$-Euler polynomials at non positive integers. The proof of this original construction is due to Kubota and Leopoldt in 1964, although the method given this note is due to Washington.

1. Introduction

The ordinary Euler polynomials $E_n(t)$ are defined by the equation

$$\frac{2e^{tx}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(t) \frac{x^n}{n!}.$$ Setting $t = 1/2$ and normalizing by $2^n$ gives the ordinary Euler numbers $E_n = 2^n E_n \left( \frac{1}{2} \right)$.

The ordinary Euler polynomials appear in many classical results (see [1]). In [2], the values of these polynomials at rational arguments were expressed in term of the Hurwitz zeta function. Congruences for Euler numbers have also received much attention from the point of view of $p$-adic interpolation. In [9], Kim et al. recently defined the natural $q$-extension of ordinary Euler numbers and polynomials by $p$-adic integral representation and proved properties generalizing those satisfied by $E_n$ and $E_n(t)$. They also constructed the one-variable $p$-adic $q$-l-function $l_{p,q}(s, \chi)$ for Dirichlet characters $\chi$ and $s \in \mathbb{C}_p$ with $|s|_p < p^{1-p^{1-p}}$, with the property that

$$l_{p,q}(-n, \chi) = E_{n,\chi \omega^{-n},q}^* - [2]_{q^P}^{-1} \omega^{-n}(p)E_{n,\chi \omega^{-n},q}^*$$

for $n = 0, 1, \ldots$, where $E_{n,\chi \omega^{-n},q}^*$ is a generalized $q$-Euler numbers attached to the Dirichlet characters $\chi \omega^{-n}$ (see Section 2 for definitions).

In the present paper, we shall construct a specific two-variable $p$-adic $l_q$-function $l_{p,q}(s, t, \chi)$ by means of a method provided in [13, 8]. We also prove that $l_{p,q}(s, t, \chi)$ is analytic in $s$ and $t$ for $s \in \mathbb{C}_p$ with $|s|_p < p^{1-p^{1-p}}$ and $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, which interpolates a linear combinations of terms of the generalized $q$-Euler polynomials at non positive integers. This two-variable function is a generalization.
of the one-variable $p$-adic $q$-I-function, which is the function obtained by putting $t = 0$ in $I_p,q(s,t,\chi)$ (cf. \[3, 5, 6, 7, 8, 9, 11, 12, 13\]).

Thought this paper $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of integers, the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and completion of the algebraic closure of $\mathbb{Q}_p$, respectively. We will use $\mathbb{Z}^+$ for the set of non positive integers. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = \frac{1}{p}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $|1 - q|_p < 1$. If $q \in \mathbb{C}$, then we assume that $|q| < 1$. Also we use the following notations:

$$[x]_q = \frac{1 - q^x}{1 - q} \quad \text{and} \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}, \quad \text{cf. \[5, 9\].}$$

Let $d$ be a fixed integer, and let

$$X = X_d = \lim_{N \to \infty} (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{0 \leq a < dp} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-integral was defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(a) d\mu_q(a) = \int_X f(a) d\mu_q(a)$$

$$= \lim_{N \to \infty} \frac{1}{dp^N} \sum_{a=0}^{dp^N-1} f(a) q^a \quad \text{for } |1 - q|_p < 1.$$  

In \[6\], the bosonic integral was considered from a more physical point of view to the bosonic limit $q \to 1$ as follows:

$$I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(a) d\mu_1(a) = \lim_{N \to \infty} \frac{1}{dp^N} \sum_{a=0}^{dp^N-1} f(a).$$

Furthermore, we can consider the fermionic integral in contrast to the conventional “bosonic.” That is, \(I_{-1}(f) = \int_{\mathbb{Z}_p} f(a) d\mu_{-1}(a) \) (see \[7\]). From this, we derive \(I_{-1}(f_1) + I_{-1}(f) = 2f(0)\), where \(f_1(a) = f(a + 1)\). Also we have

$$I_{-1}(f_n) + (-1)^{n-1}I_{-1}(f) = 2 \sum_{a=0}^{n-1} (-1)^{n-1-a} f(a),$$

where \(f_n(a) = f(a + n)\) and \(n \in \mathbb{Z}^+ \) (see \[7\]). For $|1 - q|_p < 1$, we consider fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ which is the $q$-extension of $I_{-1}(f)$ as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(a) d\mu_{-q}(a) = \lim_{N \to \infty} \frac{1}{dp^N} \sum_{a=0}^{dp^N-1} f(a)(-q)^a \quad \text{cf. \[9\].}$$

2. $q$-Euler numbers and polynomials

In this section, we review some notations and facts in \[9\].

From \[1, 4\], we can derive the following formula:

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0),$$

where \([2]_q f(0)\) denotes the elementary symmetric polynomial of degree 2 in \(f(0)\).
where \( f_1(a) \) is translation with \( f_1(a) = f(a + 1) \). If we take \( f(a) = e^{ax} \), then we have \( f_1(a) = e^{(a+1)x} = e^{ax}e^x \). From (2.11), we derive \((qe^x + 1)I_{e^x(e^{ax})} = [2]_q\). Hence we obtain

\[
(2.2) \quad I_{e^x(e^{ax})} = \int_{\mathbb{Z}_p} e^{ax}d\mu_{e^x}(a) = \frac{[2]_q}{qe^x + 1}.
\]

We now set

\[
(2.3) \quad \frac{[2]_q}{qe^x + 1} = \sum_{n=0}^{\infty} E_{n, q}^* \frac{x^n}{n!}.
\]

\( E_{n, q}^* \) is called \( q \)-Euler numbers. By (2.2) and (2.3), we see that \( \int_{\mathbb{Z}_p} a^n d\mu_{e^x}(a) = E_{n, q}^* \). From (2.2), we also note that

\[
(2.4) \quad \int_{\mathbb{Z}_p} e^{t(a)x}d\mu_{e^x}(a) = \frac{[2]_q}{qe^x + 1}e^{tx}.
\]

In view of (2.3) and (2.4), we can consider \( q \)-Euler polynomials associated to \( t \) as follows:

\[
(2.5) \quad \frac{[2]_q}{qe^x + 1}e^{tx} = \sum_{n=0}^{\infty} E_{n, q}^* (t) \frac{x^n}{n!} \quad \text{and} \quad \int_{\mathbb{Z}_p} (t + a)^n d\mu_{e^x}(a) = E_{n, q}^*(t).
\]

Put \( \lim_{q \to 1} E_{n, q}^* = E_n^* \) and \( \lim_{q \to 1} E_{n, q}^*(t) = E_n^*(t) \). Then we have \( E_n(t) = E_n^*(t) \) and

\[
E_n = \sum_{m=0}^{n} 2^m \binom{n}{m} E_m^*.
\]

where \( E_n \) and \( E_n(t) \) are the ordinary Euler numbers and polynomials. By (2.3) and (2.4), we easily see that \( E_{n, q}^*(t) = \sum_{m=0}^{n} \binom{n}{m} t^{n-m} E_{m, q}^* \). For \( d \in \mathbb{Z}_p\), let \( f_d(a) = f(a + d) \). Then we have

\[
(2.6) \quad q^d I_{-e^x}(-1)^{d-1}I_{e^x} = [2]_q \sum_{a=0}^{d-1} (-1)^d a^{d-a-1}q^n f(a), \quad \text{see [9].}
\]

If \( d \) is odd positive integer, we have

\[
(2.7) \quad q^d I_{-e^x}(fd) + (-1)^{d-1}I_{e^x} = [2]_q \sum_{a=0}^{d-1} (-1)^d a^d f(a).
\]

Let \( \chi \) be a Dirichlet character with conductor \( d = d_\chi(=\text{odd}) \in \mathbb{Z}_p\). If take \( f(a) = \chi(a)e^{(t+a)x} \), then we have \( f_d(a) = f(a + d) = \chi(a)e^{dx}e^{(t+a)x} \). From (1.1) and (2.7), we derive

\[
(2.8) \quad \int_{X} \chi(a)e^{(t+a)x} d\mu_{e^x}(a) = \frac{\sum_{a=1}^{d} (-1)^a q^a \chi(a)e^{(t+a)x}}{q^d e^{dx} + 1}.
\]

In view of (2.8), we also consider the generalized \( q \)-Euler polynomials attached to \( \chi \) as follows:

\[
(2.9) \quad F_{\chi, q}(x, t) = \sum_{a=0}^{d} (-1)^a q^a \chi(a)e^{(t+a)x} = \sum_{n=0}^{\infty} E_{n, \chi, q}^*(t) \frac{x^n}{n!}.
\]

From (2.8) and (2.9), we derive the following

\[
(2.10) \quad \int_{X} \chi(a)(t + a)^n d\mu_{e^x}(a) = E_{n, \chi, q}^*(t)
\]
for $n \geq 0$. Put $\lim_{q \to 1} E^*_{n, \chi, q}(t) = E^*_{n, \chi}(t)$. On the other hand, the generalized $q$-Euler polynomials attached to $\chi$ are easily expressed as the $q$-Euler polynomials:

$$E^*_{n, \chi, q}(t) = d^n \frac{[2]_q}{[2]_q} \sum_{a=1}^{d} (-1)^a q^a \chi(a) E^*_{n, q^a} \left( \frac{a + t}{d} \right), \quad n \geq 0. \tag{2.11}$$

Let $\chi$ be a Dirichlet character with conductor $d = d_\chi \in \mathbb{Z}^+$. It is well known (see \cite{4} \cite{12}) that, for positive integers $m$ and $n$,

$$\sum_{a=1}^{dn} \chi(a) a^m = \frac{1}{m + 1} (B_{m+1, \chi}(dn) - B_{m+1, \chi}(0)), \tag{2.12}$$

where $B_{m+1, \chi}(t)$ is the generalized Bernoulli polynomials. When $d = d_\chi (=\text{odd}) \in \mathbb{Z}^+$, note that

$$\frac{[2]_q}{[2]_q} \sum_{a=1}^{d} \sum_{t=0}^{n-1} (-1)^a q^a \chi(a) e^{(t+a)x} (1 - (-q^d e^{dx})^n) \quad \frac{1}{1 - (-q^d e^{dx})}$$

$$= \frac{[2]_q}{[2]_q} \sum_{a=1}^{d} \sum_{t=0}^{n-1} (-1)^a q^a dl q^{a+dt} \chi(a + dt) e^{x(t+a+dt)}$$

$$= \frac{[2]_q}{[2]_q} \sum_{a=1}^{dn} (-1)^a q^a \chi(a) e^{x(t+a)}$$

$$= \sum_{m=0}^{\infty} \left( \frac{[2]_q}{[2]_q} \sum_{a=1}^{dn} (-1)^a q^a \chi(a)(t + a)^m \right) \frac{x^m}{m!}. \tag{2.13}$$

By (2.9), the relation (2.13) can be rewritten as

$$\frac{[2]_q}{[2]_q} \sum_{a=1}^{d} \sum_{t=0}^{n-1} (-1)^a q^a \chi(a) e^{(t+a)x} (1 - (-q^d e^{dx})^n) \quad \frac{1}{1 - (-q^d e^{dx})}$$

$$= \sum_{m=0}^{\infty} \left( E^*_{m, \chi, q}(t) + (-1)^{n+1} q^{dn} E^*_{m, \chi, q}(t + dn) \right) \frac{x^m}{m!}. \tag{2.14}$$

Now, we give the $q$-analogue of (2.12) for the generalized Euler polynomials. From (2.13) and (2.14), it is easy to see that

$$\sum_{a=1}^{dn} (-1)^a q^a \chi(a)(t + a)^m = \frac{1}{[2]_q} \left( E^*_{m, \chi, q}(t) + (-1)^{n+1} q^{dn} E^*_{m, \chi, q}(t + dn) \right) \frac{x^m}{m!}. \tag{2.15}$$

for positive integers $m$ and $n$. In particular, replacing $q$ by 1 in (2.15), if $\chi = \chi^0$, the principal character ($d_\chi = 1$), and $t = 0$, then

$$\sum_{a=1}^{n-1} (-1)^a a^m = \frac{1}{2} \left( E_m(0) + (-1)^{n+1} E_m(n) \right).$$

Definition 2.1. Let $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. Let $\chi$ be a primitive Dirichlet character with conductor $d = d_\chi (=\text{odd}) \in \mathbb{Z}^+$. We set

$$l_q(s, t, \chi) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n \chi(n)}{(t + n)^s}, \quad 0 < t \leq 1.$$
Remark 2.2. We assume that \( q \in \mathbb{C} \) with \(|q| < 1\). Let \( \chi \) be a primitive Dirichlet character with conductor \( d = d_q(=\text{odd}) \in \mathbb{Z}^+ \). From (2.9), we consider the below integral which known the Mellin transformation of \( F_{\chi,q}(x, t) \) (cf. [10]).

\[
\frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} F_{\chi,q}(-x, t) dx = [2]^q \sum_{a=1}^d (-1)^n q^n \chi(a) \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \frac{e^{-(t+a)x}}{1 - (-q^a)e^{-dx}} dx
\]

\[
= [2]^q \sum_{a=1}^d (-1)^n q^n \chi(a + dl) \int_0^\infty (-1)^{dl \chi(a + dl + t)} (a + dl + t)^s.
\]

We write \( n = a + dl \), where \( n = 1, 2, \ldots \), and obtain

\[
\frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} F_{\chi,q}(-x, t) dx = [2]^q \sum_{n=0}^\infty (-1)^n q^n \chi(n)(t+n)^s = l_q(s, t, \chi).
\]

Note that \( l_q(s, t, \chi) \) is an analytic function in the whole complex space.

By using a geometric series in (2.9), we obtain

\[
[2]^q e^{tx} \sum_{n=0}^\infty (-1)^n q^n \chi(n)e^{nx} = 2 \sum_{n=0}^\infty E_{n,\chi,q}^*(t) \frac{x^n}{n!}.
\]

We also note that

\[
E_{n,\chi,q}^*(t) = \left. \left( \frac{d}{dx} \right)^k [2]^q e^{tx} \sum_{n=0}^\infty (-1)^n q^n \chi(n)e^{nx} \right|_{x=0}.
\]

By Definition 2.1 and (2.16), we obtain the following theorem.

Proposition 2.3. For \( n \in \mathbb{Z}^+ \), we have \( l_q(-n, t, \chi) = E_{n,\chi,q}^*(t) \).

These values of \( l_q(s, t, \chi) \) at negative integers are algebraic, hence may be regarded as being in an extension of \( \mathbb{Q}_p \). We therefore look for a \( p \)-adic function which agrees with \( l_q(s, t, \chi) \) at the negative integers in Section 3.

3. A \( p \)-ADIC \( l_q \)-FUNCTION

We shall consider the \( p \)-adic analogue of the \( l_q \)-functions which are introduced in the previous section (see Definition 2.1). Throughout this section we assume that \( p \) is an odd prime. Note that there exists \( \varphi(p) \) distinct solutions, modulo \( p \), to the equation \( x^{\varphi(p)} = 1 \), and each solution must be congruent to one of the values \( a \in \mathbb{Z} \), where \( 1 \leq a < p \). Given \( a \in \mathbb{Z} \), let \( (a, p) = 1 \), there exists a unique \( \omega(a) \in \mathbb{Z}_p \), where \( \omega(a)^{\varphi(p)} = 1 \), such that \( \omega(a) \equiv a \mod p\mathbb{Z}_p \). Letting \( \omega(a) = 0 \) for \( a \in \mathbb{Z} \) such that \( (a, p) \neq 1 \), it can be seen that \( \omega \) is actually a Dirichlet character having conductor \( d_\omega = p \), called the Teichmüller character. Let \( \langle a \rangle = \omega^{-1}(a) \). Then \( \langle a \rangle \equiv 1 \mod p\mathbb{Z}_p \). For the context in the sequel, an extension of the definition of the Teichmüller character is needed. We denote a particular subring of \( \mathbb{C}_p \) as

\[
R = \{ a \in \mathbb{C}_p \mid |a|_p \leq 1 \}.
\]

If \( t \in \mathbb{C}_p \) such that \(|t|_p \leq 1\), then for any \( a \in \mathbb{Z}, a + pt = a \mod pR \). Thus, for \( t \in \mathbb{C}_p, |t|_p \leq 1 \), \( \omega(a + pt) = \omega(a) \). Also, for these values of \( t \), let \( \langle a + pt \rangle = \omega^{-1}(a)(a + pt) \).

Let \( \chi \) be the Dirichlet character of conductor \( d = d_\chi \). For \( n \geq 1 \), we define \( \chi_n \) to be the primitive character associated to the character \( \chi_n : (\mathbb{Z}/lcm(d, p)\mathbb{Z})^\times \to \mathbb{C}^\times \) defined by \( \chi_n(a) = \chi(a)\omega^{-n}(a) \).
**Definition 3.1.** Let $\chi$ be the Dirichlet character with conductor $d = d_{\chi}$ (odd) and let $F$ be a positive integral multiple of $p$ and $d$. Now, we define the two-variable $p$-adic $l_{q}$-functions as follows:

$$l_{p,q}(s,t,\chi) = \frac{[2]_{q}^{q}}{[2]_{q^{F}}} \sum_{a=1}^{F} (-1)^{a}\chi(a)q^{a}\langle a + pt \rangle^{-s} \sum_{m=0}^{\infty} \left( \frac{-s}{m} \right) \left( \frac{F}{a+pt} \right)^{m} E_{m,q,F}^{*}.$$ 

Let $D = \{ s \in \mathbb{C} \mid |s| < p^{-1} \}$ and let $a \in \mathbb{Z}, (a, p) = 1$. For $t \in \mathbb{C}_{p}, |t|_{p} \leq 1$, the same argument as that given in the proof of the main theorem of [3, 13] can be the functions $\sum_{m=0}^{\infty} (\chi\langle a + pt \rangle^{m} E_{m,q,F}^{*}$ and $\langle a + pt \rangle^{s} = \sum_{m=0}^{\infty} \langle m \rangle(\langle a + pt \rangle - 1)^{m}$ is analytic for $s \in D$. According to this method, we see that the function $\sum_{m=0}^{\infty} (\chi\langle a + pt \rangle^{m} E_{m,q,F}^{*}$ is analytic for $t \in \mathbb{C}_{p}, |t|_{p} \leq 1$, whenever $s \in D$. It readily follows that $\langle a + pt \rangle^{s} = \langle a \rangle^{s} \sum_{m=0}^{\infty} (\langle m \rangle(a^{-1}pt)^{m}$ is analytic for $t \in \mathbb{C}_{p}, |t|_{p} \leq 1$, when $s \in D$. Therefore, $l_{q}(s,t,\chi)$ is analytic for $t \in \mathbb{C}_{p}, |t|_{p} \leq 1$, provided $s \in D$ (see [3]).

We set

$$(3.1) \quad h_{p,q}(s,t,a|F) = (-1)^{a}q^{a}\langle a + pt \rangle^{-s} \frac{[2]_{q}^{q}}{[2]_{q^{F}}} \sum_{m=0}^{\infty} \left( \frac{-s}{m} \right) \left( \frac{F}{a+pt} \right)^{m} E_{m,q,F}^{*}.$$ 

Thus, we note that

$$(3.2) \quad h_{p,q}(-n, t, a|F) = \omega^{-n}(a)(-1)^{a}q^{a}F^{n} \frac{[2]_{q}^{q}}{[2]_{q^{F}}} E_{n,q,F}^{*} \left( \frac{a + pt}{F} \right)$$

for $n \in \mathbb{Z}^{+}$. We also consider the two-variable $p$-adic $l_{q}$-functions which interpolate the generalized $q$-Euler polynomials at negative integers as follows:

$$(3.3) \quad l_{p,q}(s,t,\chi) = \sum_{a=1}^{F} \chi(a)h_{p,q}(s,t,a|F).$$

We will in the process derive an explicit formula for this function. Before we begin this derivation, we need the following result concerning generalized $q$-Euler polynomials:

**Lemma 3.2.** Let $F$ be a positive integral multiple of $d = d_{\chi}$. Then for each $n \in \mathbb{Z}, n \geq 0$,

$$E_{n,\chi,q}^{*}(t) = F^{n} \frac{[2]_{q}^{q}}{[2]_{q^{F}}} \sum_{a=1}^{F} (-1)^{a}q^{a}\chi(a)E_{n,q,F}^{*} \left( \frac{a + t}{F} \right).$$

We can derive by a manipulation of an appropriate generating functions. Set $\chi_{n} = \chi\omega^{-n}$. From (3.2) and (3.3), we obtain

$$l_{p,q}(-n, t, \chi) = F^{n} \frac{[2]_{q}^{q}}{[2]_{q^{F}}} \sum_{a=1}^{F} \chi_{n}(a)(-1)^{a}q^{a}E_{n,q,F}^{*} \left( \frac{a + pt}{F} \right)$$

$$(3.4) \quad = F^{n} \frac{[2]_{q}^{q}}{[2]_{q^{F}}} \sum_{a=1}^{F} \chi_{n}(a)(-1)^{a}q^{a}E_{n,q,F}^{*} \left( \frac{a + pt}{F} \right)$$

$$- F^{n} \frac{[2]_{q}^{q}}{[2]_{q^{F}}} \sum_{a=1}^{F} \chi_{n}(pa)(-1)^{pa}q^{pa}E_{n,q,F}^{*} \left( \frac{pa + pt}{F} \right).$$
for \( n \in \mathbb{Z}^+ \). From Lemma 3.2, we see that

\[
E_{n, \chi_n, q}(pt) = F^n \frac{[2]}{[q]^{q^F}} \sum_{a=1}^{F} (-1)^{a} q^{a} \chi_n(a) E_{n, \chi_n, q}^{*} \left( \frac{a + pt}{F} \right)
\]

and

\[
E_{n, \chi_n, q}^{*}(t) = \left( \frac{F}{p} \right)^n \frac{[2]}{[q]^{q^F}} \sum_{a=1}^{F} (-1)^{a} (q^{a}) \chi_n(a) E_{n, \chi_n, q}^{*} \left( \frac{a + t}{p} \right).
\]

From (3.4), (3.5), and (3.6), we obtain the following theorem:

**Theorem 3.3.** Let \( F(=\text{odd}) \) be a positive integral multiple of \( p \) and \( d = (d_{\chi}) \). Then the two-variable \( p \)-adic \( l_{q} \)-functions

\[
l_{p,q}(s, t, \chi) = \frac{[2]}{[q]^{q^F}} \sum_{a=1}^{F} \left( -1 \right)^{a} \chi(a) q^{a} \langle a + pt \rangle^{-s} \sum_{m=0}^{\infty} \left( -s \right) \left( \frac{F}{\langle a + pt \rangle} \right)^m E_{m, q}^{*}
\]

admits an analytic function for \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \) and \( s \in D \), and satisfies the relation

\[
l_{p,q}(-n, t, \chi) = E_{n, \chi_n, q}(pt) - p^n \chi_n(p) \frac{[2]}{[q]^{q^F}} E_{n, \chi_n, q}^{*}(t)
\]

for \( n \in \mathbb{Z}^+ \) and \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \).

From (3.3) and Theorem 3.3, it follows that \( h_{p,q}(s, t, a|F) \) is analytic for \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \) and \( s \in D \).

**Remark 3.4.** Let \( \langle a + pt \rangle = \omega^{-1}(a(a + pt)) \), and let \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \) and \( s \in D \). Then the two-variable \( p \)-adic \( l_{q} \)-functions defined above is redefined by

\[
l_{p,q}(s, t, \chi) = \int_X \chi(a) \langle a + pt \rangle^{-s} d\mu_{-q}(a), \quad \text{cf. \cite{9, 11}}.
\]

Then we have

\[
l_{p,q}(-n, t, \chi) = \int_X \chi_n(a) \langle a + pt \rangle^n d\mu_{-q}(a) - \int_X \chi_n(pa) \langle pa + pt \rangle^n d\mu_{-q}(pa)
\]

\[
= E_{n, \chi_n, q}(pt) - p^n \chi_n(p) \frac{[2]}{[q]^{q^F}} E_{n, \chi_n, q}^{*}(t),
\]

since \( X^* = X - pX \) and \( [2]_{q} d\mu_{-q}(pa) = [2]_{q} d\mu_{-q}(a) \).

**Corollary 3.5.** Let \( F(=\text{odd}) \) be a positive integral multiple of \( p \) and \( d = (d_{\chi}) \), and let the two-variable \( p \)-adic \( l \)-functions

\[
l_{p}(s, t, \chi) = \sum_{a=1}^{F} \left( -1 \right)^{a} \chi(a) \langle a + pt \rangle^{-s} \sum_{m=0}^{\infty} \left( -s \right) \left( \frac{F}{\langle a + pt \rangle} \right)^m E_{m}^{*}
\]

Then

1. \( l_{p}(s, t, \chi) \) is analytic for \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \) and \( s \in D \).
2. \( l_{p}(-n, t, \chi) = E_{n, \chi_n}(pt) - p^n \chi_n(p) E_{n, \chi_n}(t) \) for \( n \in \mathbb{Z}^+ \).
3. \( l_{p}(s, t, \chi) = \int_X \chi(a) \langle a + pt \rangle^{-s} d\mu_{-1}(a) \) for \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \) and \( s \in D \).
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