The Fourier Transform and its application in solving the equation

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Abstract. Fourier transform is a linear transform that applicants in the field of engineering and physics. In this research study, we summarize the fundamental properties of the Fourier transform and its application in solving the wave equation. First, we review the definition of Fourier transform and its inverse form. Then, we show Schwartz space is the invariant subspace of the Fourier transform. We also prove the Plancherel theorem. At last, we apply the Fourier transform to solve the wave equation.

1. Introduction

Fourier transform was first proposed by a French mathematician physicist Jean Baptiste Joseph Fourier [1-4]. He published a paper in the French Academy of Sciences in 1822, which contains an important content that a sine curve can be used to describe the temperature distribution [5]. The Fourier transform mainly calculates the frequency and amplitude of different sinusoidal signals by using the way of summarizing the original signal, which has been directly measured [1][5]. Although the Fourier transform was originally intended as a tool for the analytical analysis of thermal processes, it also gives an idea in mathematics. Any function could be decomposed to a linear combination of sine functions. One of the important applications of Fourier transform is to solve the partial differential equations that originated from many mathematical physics.

In order to construct a Fourier transform on a space which is as big as possible, we will use a duality principle. The process is the following: if \( T \) is a continuous linear application from \( E \) into \( E' \) then \( T'^* \) is continuous from \( E' \) into \( E'' \). Moreover if \( E \subset L^1(\mathbb{R}^d) \), then \( L^1(\mathbb{R}^d) \subset E'' \). So to extend the definition of the Fourier transform to a space larger than \( L^1(\mathbb{R}^d) \), we will try to define it as the adjoint of an isomorphism of a space \( E \) included in \( L^1(\mathbb{R}^d) \). Indeed, It is not be able to work in the scope of...
Banach spaces. It should to work in the framework of Fréchet spaces.
In the first section, we first give notations of $|x|$ and $x \cdot y$ for Euclidean norm and standard inner product. Then we define the expression of a multi-index differential operator.
In the second section, we talk about the Fourier transform of a Schwartz function. In the third section, we talk about the solution of the Cauchy Problem of the wave equation.

2. Preliminaries
In this paper, we use the notation $|x|$ to represent the usual Euclidean norm on $\mathbb{R}^d$

$$|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}},$$

which $x = (x_1, \ldots, x_n) \in \mathbb{R}^d$.

We use the notation $x \cdot y$ as the $\mathbb{R}^d$ to be the standard inner product,

$$x \cdot y = x_1y_1 + \cdots + x_ny_n.$$

Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each $\alpha_i$ is a positive integer, we use the symbol $x^\alpha$ to be monomial

$$x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n};$$

We define a multi-index differential operator in the following way:

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

**Definition 2.1.1 (Translation)** We say a mapping $\mathcal{A}$ from one function space $F_1$ to another function space $F_2$ is a translation, if such a mapping is in the form of the following:

$$\mathcal{A}: F_1 \to F_2 \quad f(x) \to \mathcal{A}(f) = f(x + h).$$

**Definition 2.1.2 (Dilations)**

$$\mathcal{A}: F_1 \to F_2 \quad f(x) \to \mathcal{A}(f) = f(\lambda x)$$

**Definition 2.1.3 (Rotation)**
A rotation is a linear transformation $\mathcal{R}: \mathbb{R}^d \to \mathbb{R}^d$, which remains the inner product.

Remark: for all $x, y \in \mathbb{R}^d$, we have

1) $\mathcal{R}(ax + by) = a\mathcal{R}(x) + b\mathcal{R}(y)$, for $a, b \in \mathbb{R}$.
2) $\mathcal{R}(x \cdot y) = x \cdot y$.
3) $\mathcal{R}^T = \mathcal{R}^{-1}$.

To make integrations on $\mathbb{R}^d$, we shall first define what a rapidly decreasing function is.

**Definition 2.1.4 (Rapidly decreasing functions)** A rapidly decreasing function is a continuous complex-valued function on a Cartesian space.
Rapidly decreasing means for every multi-index \( \alpha \), \( |x^\alpha f(x)| \) is bounded on \( \mathbb{R}^d \). That is,

\[
\sup_{x \in \mathbb{R}^d} |x^\alpha f(x)| < \infty
\]

for every \( k = 0, 1, 2 \ldots \)

If \( f(x) \) is of rapid decreasing, the integration defined as

\[
\int_{\mathbb{R}^d} f(x) dx = \lim_{N \to \infty} \int_{Q_N} f(x) dx
\]

is well-defined, where \( Q_N \) is

\[
Q_N = x \in \mathbb{R}^d : |x_i| \leq \frac{N}{2}, \; i = 1, 2, 3, \ldots
\]

The integral on \( Q_N \) is a multiple integral in the usual Riemann integration.

As, \( \int_{Q_N} f(x) dx = \int_{N/2}^N \ldots \int_{N/2}^N f(x_1, x_2, \ldots, x_d) \; dx_1 \ldots dx_d \)

The limit above exists because \( l_N = \int_{Q_N} f(x) dx \) forms a Cauchy sequence.

**Theorem 2.1.5** The sequence \( \{l_N = \int_{Q_N} f(x) dx\} \) is a Cauchy sequence.

**Proof.** From \( \sup_{x \in \mathbb{R}^d} |x^\alpha f(x)| < \infty \), for every \( k = 0, 1, 2, \ldots \), take \( k > d \)

We can have

\[
\left| \int_{-\infty}^{\infty} f(x) dx \right| < \int_{-\infty}^{\infty} |f(x)| dx.
\]

Taking limits on both sides, we have \( \int_{-\infty}^{\infty} f(x) dx \to 0 \).

Similarly, we have

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{N} f(x) dx \to 0.
\]

While \( d = 1 \), \( I_N = \int_{Q_N} f(x) dx = \int_{N/2}^{2N} f(x) dx \)

Assume when \( N > N_0 \), then,

\[
\max \left\{ \int_{N/2}^{\infty} f(x) dx, \int_{-\infty}^{N/2} f(x) dx \right\} < \epsilon
\]

Take \( M > N > N_0 \), then we have

\[
|I_M - I_N| = \left| \int_{N/2}^{M/2} f(x) dx + \int_{M/2}^{\infty} f(x) dx \right|
\]

\[
< \left| \int_{N/2}^{\infty} f(x) dx - \int_{M/2}^{\infty} f(x) dx \right| + \left| \int_{-\infty}^{N/2} f(x) dx - \int_{-\infty}^{M/2} f(x) dx \right|
\]

\[
< 2 \epsilon + 2 \epsilon = 4 \epsilon
\]

With \( \epsilon \to 0 \), \( I_N \) form a Cauchy sequence when \( d = 1 \).

Assume \( l_{N,d-1} \) form a Cauchy sequence at \( d-1 \), and \( N > N_0 \), we have

\[
\max \left\{ \int_{N/2}^{\infty} l_{N,d-1}(x) dx, \int_{-\infty}^{N/2} l_{N,d-1}(x) dx \right\} < \epsilon
\]

Then when \( M > N > N_0 \)
\[
|I_{M,d} - I_{N,d}| = \left| \int_{-M}^{M} I_{M,d-1}(x)dx - \int_{-N}^{N} I_{N,d-1}(x)dx \right|
\]

\[
= \left| \int_{-M}^{M} I_{M,d-1}(x)dx - \int_{-M}^{M} I_{N,d-1}(x)dx + \int_{-M}^{M} I_{N,d-1}(x)dx - \int_{-N}^{N} I_{N,d-1}(x)dx \right|
\]

\[
< \int_{-M}^{M} I_{M,d-1}(x)dx - \int_{-M}^{M} I_{N,d-1}(x)dx + \int_{-M}^{M} I_{N,d-1}(x)dx - \int_{-N}^{N} I_{N,d-1}(x)dx
\]

\[
< M \varepsilon + \int_{-N}^{N} I_{N,d-1}(x)dx + \int_{-M}^{M} I_{N,d-1}(x)dx
\]

\[
< (M + 2) \varepsilon \to 0
\]

So, \( I_{N,d} \) form a Cauchy sequence at any \( d \).

3. Main works

3.1 (Schwartz space)

The Schwartz space \( \mathcal{S}(\mathbb{R}^d) \) is composed of all infinitely differentiable functions \( f \) on \( \mathbb{R}^d \), we can know

\[
\text{Sup} \left| x^\alpha \left( \frac{\partial}{\partial x} \right)^\beta f(x) \right| < \infty,
\]

for each \( \alpha \) and \( \beta \). In other words, \( f \) and all its derivatives need to decrease quickly.

Theorem 3.1.1 Schwartz Space \( \mathcal{S}(\mathbb{R}^d) \) is a linear space, for \( f \in \mathcal{S}, \ g \in \mathcal{S} \).

We have,

\[
\sup_{x \in \mathbb{R}^d} \left| x^\alpha \left( \frac{\partial}{\partial x} \right)^\beta f(x) \right| < \infty
\]

and

\[
\sup_{x \in \mathbb{R}^d} \left| x^\alpha \left( \frac{\partial}{\partial x} \right)^\beta g(x) \right| < \infty
\]

Which means for every \( x_n \in \mathbb{R}^d \), we can find \( N_n \), s.t.,

\[
x_n^\alpha \left( \frac{\partial}{\partial x_n} \right)^\beta f(x_n) < N_n
\]

And for every \( x_m \in \mathbb{R}^d \), we can find \( M_n \), s.t.,

\[
|x_m^\alpha \left( \frac{\partial}{\partial x_m} \right)^\beta g(x_m)| < M_n
\]

Then for every \( x_k \in \mathbb{R}^d \), we can find \( N_k \), s.t.\( x_k^\alpha \left( \frac{\partial}{\partial x_k} \right)^\beta f(x_k) | < N_k \);
and $M_k$, s.t. $|x_k^\alpha \left(\frac{\partial}{\partial x_k}\right)^\beta g(x_k)| < M_k$.

So $|x_k^\alpha \left(\frac{\partial}{\partial x_k}\right)^\beta [f(x_k) + g(x_k)]| < N_k + M_k \equiv K_k$

Therefore, we get $f(x) + g(x) \in \mathcal{S}$.

We also notice that $|x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta [mf(x)]| = m |x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta f(x)|$ when $m \in \mathbb{R}$.

Thus, we have finished proving that Schwartz Space $\mathcal{S}(\mathbb{R}^d)$ is a linear space.

### 3.2 Fourier transform

**Definition 3.2.2** We define the Fourier transform of a Schwartz function $f$ is

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} \, dx, \quad \text{for } \xi \in \mathbb{R}^d.$$

**Proposition 3.2.2** Let $f \in \mathcal{S}(\mathbb{R}^d)$.

1) If we assume $g(x) = f(x + h)$, then $\hat{g}(\xi) = \hat{f}(\xi)e^{-2\pi i x \cdot h}$, whenever $h \in \mathbb{R}^d$.

   **Proof.**

   $$\hat{g}(\xi) = \hat{f}(\xi + h) = \int_{\mathbb{R}^d} f(x + h)e^{-2\pi i x \cdot \xi} \, dx = \int_{\mathbb{R}^d} f(x)e^{-2\pi i (x - h) \cdot \xi} \, dx$$

   $$= e^{-2\pi i h \cdot \xi} \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} \, dx = \hat{f}(\xi)e^{-2\pi i h \cdot \xi}.$$

2) If we assume $g(x) = f(x)e^{-2\pi i x \cdot h}$, then $\hat{g}(\xi) = \hat{f}(\xi + h)$, whenever $h \in \mathbb{R}^d$.

   **Proof.**

   $$\hat{g}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot h} \cdot e^{-2\pi i x \cdot \xi} \, dx = \int_{\mathbb{R}^d} f(x)e^{-2\pi i (x + h) \cdot \xi} \, dx$$

   $$= \hat{f}(\xi + h).$$

3) If we assume $g(x) = f(\delta x)$, then $\hat{g}(\xi) = \delta^{-d} \hat{f}(\delta^{-1} \xi)$, whenever $\delta > 0$.

   **Proof.**

   $$\hat{g}(\xi) = \int_{\mathbb{R}^d} f(\delta x)e^{-2\pi i x \cdot \xi} \, dx = \int_{\mathbb{R}^d} f(x)e^{-2\pi i \delta^{-1} x \cdot \xi} \, \frac{1}{\delta^d} = \delta^{-d} \hat{f}(\delta^{-1} \xi).$$

4) If we assume $g(x) = \left(\frac{\partial}{\partial x}\right)^\alpha f(x)$, then $\hat{g}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$.

   **Proof.** We first prove the equation at $d=1$, then $g(x) = \left(\frac{\partial}{\partial x}\right)^\alpha f(x)$, So

   $$\hat{g}(\xi) = \int_{\mathbb{R}} \left(\frac{\partial}{\partial x}\right)^\alpha f(x)e^{-2\pi i x \cdot \xi} \, dx$$

   $$= (2\pi i \xi)^\alpha \int_{\mathbb{R}} f(x)e^{-2\pi i x \cdot \xi} \, dx$$

   $$= (2\pi i \xi)^\alpha \hat{f}(\xi).$$

   Let’s assume the equation true at $1, 2, \ldots, d - 1$, then at $d$, we have

   $$\hat{g}(\xi) = \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial x}\right)^\alpha f(x)e^{-2\pi i x \cdot \xi} \, dx$$

   $$= \int_{\mathbb{R}^{d-1}} e^{-2\pi i x_d \cdot \xi_d} \left(\frac{\partial}{\partial x_d}\right)^\alpha \int_{\mathbb{R}} \left(\frac{\partial}{\partial x}\right)^{\alpha/d} f(x)e^{-2\pi i x_1,\ldots,d-1 \cdot \xi_1,\ldots,d-1 \cdot \xi_d} \, dx_1 \cdots dx_d$$

   $$= (2\pi i \xi)^\alpha \hat{f}(\xi).$$

5) If we assume $g(x) = (-2\pi i)^\alpha f(x)$, then $\hat{g}(\xi) = \left(\frac{\partial}{\partial \xi}\right)^\alpha \hat{f}(\xi)$.

   $$\hat{g}(\xi) = \int_{\mathbb{R}^d} (-2\pi i)^\alpha f(x)e^{-2\pi i x \cdot \xi} \, dx$$
\begin{align*}
& = \int_{\mathbb{R}^d} f(x) \cdot \prod_{k=1}^{d} (-2\pi i x_k)^{a_k} \cdot e^{-2\pi i x_k \xi_k} \, dx \\
& = \int_{\mathbb{R}^d} f(x) \cdot \prod_{k=1}^{d} \left( \frac{\partial}{\partial \xi_k} \right)^{a_k} \cdot e^{-2\pi i x_k \xi} \, dx \\
& = \int_{\mathbb{R}^d} f(x) \left( \frac{\partial}{\partial \xi} \right)^{a} \cdot e^{-2\pi i x \xi} \, dx \\
& = \left( \frac{\partial}{\partial \xi} \right)^{a} \hat{f} (\xi).
\end{align*}

**Definition 3.2.3** If a function \( f(x) \) can be viewed as a composition of two functions, \( f_0(u), u(x) \), where \( u(x) = |x| \), then we say \( f(x) \) is radial.

**Corollary 3.2.4** The Fourier transform of a radial function is radial.

Proof. This follows at once from property (vi) in Proposition 3.2.2. Indeed, the condition \( f(Rx) = f(x) \) for all \( R \) implies that \( \hat{f}(R\xi) = \hat{f}(\xi) \) for all \( R \), thus \( \hat{f} \) is radial whenever \( f \) is.

An example of a radial function in \( \mathbb{R}^d \) is the Gaussian \( e^{-\pi |x|^2} \). Also, we observe that when \( d = 1 \), the radial functions are precisely the even functions, that is, those for which \( f(x) = f(-x) \).

**Theorem 3.2.5** Suppose \( f \in S(\mathbb{R}^d) \). Then
\[
 f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi.
\]

Moreover
\[
\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, d\xi = \int_{\mathbb{R}^d} |f(x)|^2 \, dx.
\]

The proof is separated into several steps.

**Step 1:** The Fourier transform of \( e^{-\pi |x|^2} \) is \( e^{-\pi |\xi|^2} \). To prove this, notice that the properties of the exponential functions imply that \( e^{-\pi |x|^2} = e^{-\pi x_1^2} \ldots e^{-\pi x_d^2} \) and \( e^{-2\pi i x \xi} = e^{-2\pi i \xi_1} \ldots e^{-2\pi i \xi_d} \).

So that the integrand in the Fourier transform is a product of \( d \) functions, each depending on the variable \( x_j (1 < j < d) \) only. Thus, the assertion follows by writing the integral over \( \mathbb{R}^d \) as a series of repeated integrals, each taken over \( \mathbb{R} \). For example, when \( d = 2 \),
\[
\int_{\mathbb{R}^2} e^{-\pi |x|^2} e^{2\pi i x \xi} \, dx = \int_{\mathbb{R}^2} e^{-\pi x_1^2} e^{2\pi i x_1 \xi_1} \left( \int_{\mathbb{R}} e^{-\pi x_2^2} e^{2\pi i x_2 \xi_2} \, dx_2 \right) \, dx_1 \]
\[
= e^{-\pi |\xi|^2}.
\]

**Step 2:** The family \( K_0(x) = \delta^{-\frac{d}{2}} e^{-\frac{|x|^2}{\delta}} \) is a family of good kernels in \( \mathbb{R}^d \). By this we mean that:
\[
1. \int_{\mathbb{R}^d} K_0(x) \, dx = 1
\]
\[
\int_{\mathbb{R}^d} K_0(x) \, dx = \int_{\mathbb{R}^d} \delta^{-\frac{d}{2}} e^{-\frac{|x|^2}{\delta}} \, dx
\]
\[
= \delta^{-\frac{d}{2}} \int_{-\infty}^{+\infty} e^{-\frac{\pi x_1^2}{\delta}} \, dx_1 \times \cdots \times \int_{-\infty}^{+\infty} e^{-\frac{\pi x_d^2}{\delta}} \, dx_d
\]
\[
= \delta^{-\frac{d}{2}} \left( \int_{-\infty}^{+\infty} e^{-\frac{\pi y^2}{\delta}} \, dy \right)^d
\]

Let \( z = \sqrt{\frac{\pi}{\delta}} y \).
2. $\int_{\mathbb{R}^d} |K_\delta(x)| \, dx < M$

Because $K_\delta(x) > 0$ is always true for every $x$, so it is obvious according to $\int_{\mathbb{R}^d} K_\delta(x) \, dx = 1$.

3. For every $\mu > 0, \int_{|x|\geq \mu} |K_\delta(x)| \, dx \to 0$ as $\delta \to 0$

\[
\begin{align*}
\int_{|x|\geq \mu} |K_\delta(x)| \, dx &= \int_{|x|\geq \mu} \left| \delta^{-\frac{d}{2}} e^{-\frac{|x|^2}{\delta}} \right| \, dx \\
&= \int_{|x|\geq \mu} \frac{e^{-\frac{|x|^2}{\delta}}}{\delta^{\frac{d}{2}}} \, dx \\
&\leq \int_{|x|\geq \mu} \frac{e^{-\frac{|x|^2}{\delta}}}{\delta^{\frac{d}{2}}} \, dx
\end{align*}
\]

Consider $\lim_{\delta \to 0} \frac{e^{-\frac{|x|^2}{\delta}}}{\delta^{\frac{d}{2}}} = \lim_{\delta \to 0} \frac{e^{-\frac{|x|^2}{\delta}}}{\delta^{\frac{d}{2}}} = \lim_{\delta \to 0} e^{-\frac{|x|^2}{\delta}} = 0$

So $\int_{|x|\geq \mu} |K_\delta(x)| \, dx \to 0$ as $\delta \to 0$

Step 3. As we know that if $f, g \in S(\mathbb{R}^d)$ then

\[
\int_{\mathbb{R}^d} f(x)g(x) \, dx = \int_{\mathbb{R}^d} f(x)g(x) \, dy
\]

This is also true for $f, g \in S(\mathbb{R}^d)$

Then we have the Fourier transform $\mathcal{F}$ for $g(x)$

\[
\mathcal{F}(x) = \int_{\mathbb{R}^d} g(\xi)e^{i2\pi ix\xi} \, d\xi
\]

And $\mathcal{F}$ for $f(x)$

\[
\mathcal{F}(f(x)) = \int_{\mathbb{R}^d} f(x)e^{-i2\pi ix\xi} \, dx
\]

Step 4: Combining them we get $f \ast g = g \ast f$ and $f \ast g(\xi) = \mathcal{F}(f(\xi))g(\xi)$. Also, notice that the Fourier transforms for $f(x)$ and $g(x)$ have the result $\mathcal{F}(f(x)) \ast \mathcal{F}(g(x)) = 1$, identity map, so we can conclude that $\mathcal{F}(f(x))$ has inverse $\mathcal{F}(g(x))$. So $\mathcal{F}$ is bijective.

3.3 The wave equation in $\mathbb{R}^d \times \mathbb{R}$

In this section, we will apply the method of Fourier transform to solve the wave equation.

The one-dimensional wave equation is noted as the following,

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
\]

A natural generalization of this equation to $d$ space variables is also called the d-dimensional wave equation (with $c =$ speed of light),

\[
\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
\]
which determine the behavior of electromagnetic waves in a vacuum.

The Laplacian operator in d dimensions is defined as \( \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \).

Then the wave equation can be written as \( \Delta u = \frac{\partial^2 u}{\partial t^2} \).

To find the solution of the function, we should first give initial conditions

\[ u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x), \]

where \( f, g \in S(\mathbb{R}^d) \). This is called the Cauchy problem for the wave equation.

**Theorem 3.3.1** A solution of the Cauchy problem for wave equation

\[ u(x, t) = \int_{\mathbb{R}^d} \left[ f(\xi) \cos(2\pi|\xi|t) + g(\xi) \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right] e^{2\pi i \xi} \, d\xi \]

We give a proof of this equation:

To prove this, we need to verify that \( u \) satisfy the conditions of the Cauchy problem, which are

\[ u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = g(x). \]

First, we prove that \( u(x, 0) = f(x) \),

By letting \( t = 0 \), we get

\[ u(x, 0) = \int_{\mathbb{R}^d} \bar{f}(\xi) e^{2\pi i \xi} \, d\xi. \]

As we have proved,

\[ \int_{\mathbb{R}^d} \bar{f}(\xi) e^{2\pi i \xi} \, d\xi = f(x). \]

So, we get \( u(x, 0) = f(x) \), the first condition of Cauchy problem.

Then, taking partial differentiate to both sides of our solution

\[ u(x, t) = \int_{\mathbb{R}^d} \left[ f(\xi) \cos(2\pi|\xi|t) + g(\xi) \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right] e^{2\pi i \xi} \, d\xi \]

with respect to \( t \), we get

\[ \frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{R}^d} \left[ 2\pi|\xi| \bar{f}(\xi) \sin(2\pi|\xi|t) + \bar{g}(\xi) \frac{2\pi|\xi| \sin(2\pi|\xi|t)}{2\pi|\xi|} \right] e^{2\pi i \xi} \, d\xi \]

And let \( t = 0 \), we get

\[ \frac{\partial u}{\partial t}(x, 0) = \int_{\mathbb{R}^d} \bar{g}(\xi) e^{2\pi i \xi} \, d\xi \]

Using Fourier inversion we have proved, we get our final result

\[ \frac{\partial u}{\partial t}(x, 0) = \int_{\mathbb{R}^d} \bar{g}(\xi) e^{2\pi i \xi} \, d\xi = g(x) \]

So, we also proved that this solution satisfies second condition of Cauchy problem and we are done.

**4. Conclusion**

The Fourier transform is a bijection mapping from the Schwartz space to itself. In other words, Schwartz space is the invariant subspace of the Fourier transform. After giving a detailed proof of the Plancherel theorem, it is easy to see that e Fourier transform map is an isometry with respect to the \( L^2 \) norm and this isometry in fact is a unitary map. The Fourier transformation has a lot of applications in other branches of mathematics and in the field of engineering. We apply the Fourier transformation to solve the Cauchy problem of the wave equation in high dimension. Although the Fourier transform is well-defined in the framework of Schwartz functions as well as \( L^2 \), the Fourier transform is well-behaved in the sense of \( L^1 \) norm. In the future, we will report the behaviour of the Fourier transformation on the tempered distribution space. Tempered distributions provide a larger framework in which Fourier transform is well-behaved. Also, we will give a self-contained review to microlocal analysis, a branch
of modern analysis utilized in many mathematical research fields, from the research of partial differential equations to dynamical systems. We will give complete proofs of several main theorems in these fields, such as the continuity of pseudodifferential operators on Sobolev spaces.

Appendix

Polar coordinates

In \( \mathbb{R}^2 \), use \((r, \theta)\) with \( r > 0 \) and \( 0 \leq \theta < 2\pi \), then

\[
\int_{\mathbb{R}^2} f(x)\,dx = \int_0^{2\pi} \int_0^\infty f(r\cos\theta, r\sin\theta) r\,dr\,d\theta
\]

Choose a point on the unit circle \( S^1 \) as \( \gamma = (\cos\theta, \sin\theta) \)

Then for any function \( g \) on \( S^1 \), we can have

\[
\int_{S^1} g(\gamma)\,d\sigma(\gamma) = \int_0^{2\pi} g(\cos\theta, \sin\theta)\,d\theta
\]

So that we get

\[
\int_{\mathbb{R}^2} f(x)\,dx = \int_{S^1} \int_0^\infty f(r\gamma) r\,dr\,d\sigma(\gamma)
\]

Example: In \( \mathbb{R}^3 \), \( x_1 = r\sin\theta\cos\varphi, x_2 = r\sin\theta\sin\varphi, x_3 = r\cos\theta \), where \( r > 0, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi \).

Thus, we can have

\[
\int_{\mathbb{R}^3} f(x)\,dx = \int_0^{2\pi} \int_0^\pi \int_0^\infty f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) r^2\,dr\,\sin\theta\,d\theta\,d\varphi
\]

For unit sphere \( S^2 \), \( \gamma = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \)

Similarly,

\[
\int_{\mathbb{R}^3} f(x)\,dx = \int_{S^2} \int_0^\infty f(r\gamma) r^2\,dr\,d\sigma(\gamma)
\]

Generally, we can write any point in \( \mathbb{R}^d - \{0\} \) uniquely as \( x = r\gamma \) lies on the unit \( S^{d-1} \). We can take \( r = |x| \) and \( \gamma = x/|x| \)

Let \( g \) be any function on the unit sphere \( S^2 = \{x \in \mathbb{R}^3 : |x| = 1\} \), if \( \gamma \in \mathbb{R}^3 \), define \( \gamma \) in spherical coordinates, let \( \gamma = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \).

Since the surface element \( d\sigma(\gamma) \) can be defined by

\[
\int_{S^2} g(\gamma)\,d\sigma(\gamma) = \int_0^{2\pi} \int_0^\pi g(\gamma) \sin\theta\,d\theta\,d\varphi.
\]

We can get

\[
\int_{\mathbb{R}^3} f(x)\,dx = \int_{S^2} \int_0^\infty f(r\gamma) r^2\,dr\,d\sigma(\gamma).
\]

In general, any point in \( \mathbb{R}^d - \{0\} \) can be written as \( x = r\gamma \), for \( \gamma \in S^{d-1} \subset \mathbb{R}^d \) and \( r > 0 \).

To define spherical coordinates, we proceed when \( d = 2 \) or \( d = 3 \). Take \( r = x \) and \( \gamma = x/|x| \). We use the formula

\[
\int_{\mathbb{R}^d} f(x)\,dx = \int_{S^{d-1}} \int_0^\infty f(r\gamma) r^{d-1}\,dr\,d\sigma(\gamma).
\]

Here \( f \) decrease steadily and \( d\sigma(\gamma) \) is the surface element of the sphere \( S^{d-1} \) obtained from spherical coordinates.
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