Magnetohydrodynamic simulations of self-consistent rotating neutron stars with mixed poloidal and toroidal magnetic fields

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We perform the first magnetohydrodynamic simulations in full general relativity of self-consistent rotating neutron stars (NSs) with ultrastrong mixed poloidal and toroidal magnetic fields. The initial uniformly rotating NS models are computed assuming perfect conductivity, stationarity, and axisymmetry. Although the specific geometry of the mixed field configuration can delay or accelerate the development of various instabilities known from analytic perturbative studies, all our models finally succumb to them. Differential rotation is developed spontaneously in the cores of our magnetars which, after sufficient time, is converted back to uniform rotation. The rapidly rotating magnetars show a significant amount of ejecta, which can be responsible for transient kilonova signatures. However no highly collimated, helical magnetic fields or incipient jets, which are necessary for gamma-ray bursts, arise at the poles of these magnetars by the time our simulations are terminated.

Introduction.—Neutron stars are not only the densest objects in the Universe, but in some cases they possess a magnetic field billions of times larger than the strongest magnet on Earth. These so called magnetars [1–3] have magnetic fields that surpass the quantum electrodynamic threshold of $4 \times 10^{13}$ G and are responsible for many exotic phenomena, such as vacuum birefringence, photon splitting, and the distortion of atoms [4] for a review. They are invoked in order to explain the large bursts of gamma-rays and X-rays in soft-gamma repeaters [5] or the related anomalous X-ray pulsars [6].

Magnetars are also naturally produced after the merger of two NSs via the instigation of various magnetic instabilities such as the Kelvin-Helmholtz instability [7–11], the magnetorotational instability (MRI) [10, 12–14], or magnetic wind- ing [9, 15, 16]. Even if the two NSs that compose the binary system have magnetic fields of the order of $\sim 10^{13}$ G, when they merge because of the aforementioned mechanisms the magnetic field reaches magnetar strengths and beyond on a dynamical timescale. This was first demonstrated with the very high resolution studies in [9, 10] where an initial magnetic field of $10^{13}$ G was amplified to $\gtrsim 10^{15}$ G, with local values reaching $\sim 10^{17}$ G, 5 ms following merger. Similar results have been reported more recently in [17] where an even larger amplification was achieved. The existence of this ultrastrong magnetic field is one of the most crucial factors for the realization of multimessenger astronomy. According to our current understanding, the merger of the two NSs in event GW170817 [18] produced such a magnetar that was instrumental for the creation of the sGRB [19] (possibly following its delayed collapse), and the kilonova [20, 21] that followed.

Despite the large amount of research in analytical and perturbative magnetohydrodynamics, self-consistent general relativistic solutions of the Einstein-Maxwell-Euler system are still in their infancy. In [22] self-consistent equilibria have been obtained with only poloidal magnetic field, while a different formalism was employed in [23, 24] to obtain self-consistent equilibria with only toroidal magnetic fields. Other authors have explored such solutions in great detail [25–29] while in [30–32] solutions with mixed poloidal and toroidal magnetic fields were obtained with the price of greatly reducing the number of Einstein equations solved. On the other hand equilibrium solutions are not necessarily stable, and indeed, the first general relativistic MHD simulations with either purely toroidal magnetic fields [33] or purely poloidal magnetic fields [34–37] confirmed the unstable nature of these solutions predicted decades ago [38–43]. In [34–37] the initial conditions were based on the self-consistent poloidal solutions of [22]. In all cases the Cowling approximation was used, i.e. the Einstein field equations were not evolved but only the MHD equations on the fixed initial background. In [33] the initial toroidal conditions were those of [23] and an axisymmetric GRMHD simulation was employed.

The stability of a pulsar magnetic field, as well as the recent results by NICER [44, 45], demand a more sophisticated modelling of a NS magnetic field. As a first step we go beyond the previous works above by constructing rotating, magnetized equilibria with mixed ultrastrong poloidal and toroidal components and evolve them in full GRMHD in order to assess their evolutionary fate. Our initial data are constructed with the magnetized, rotating NS libraries of the COCAL code [46, 47], where the whole set of Einstein-Maxwell-Euler system is solved to construct models under the assumptions of perfect conductivity, stationarity, and axisymmetry. These models are then evolved using the Illinois GRMHD code [48] in full general relativity. The salient characteristics of our simulations are summarized in the Results section below, while details on the construction of the self-consistent models, as well as our choices for the simulations, are provided in the Supplemental Material. Here and throughout we adopt units of $G = c = M_\odot = 1$, unless otherwise noted.

Initial data.—We construct the initial magnetized equilibria, models A1-A7 in Table I by solving the Einstein equations, Maxwell’s equations, and ideal MHD equations self-consistently under the assumptions of stationarity and axisymmetry. All of our models use a polytropic equation of state

\[ P = \frac{\gamma}{\gamma - 1} \rho \varepsilon, \]

where $\rho$ is the mass density, $\varepsilon$ is the internal energy, and $P$ is the pressure. The polytropic index $\gamma$ is set to 5/3, which corresponds to the gamma-ray burst (GRB) model of [4]. The gravitational collapse starts from a uniform distribution of mass and energy, with a constant density of $\rho = 10^{-5} M_\odot$ and a constant internal energy of $\varepsilon = 10^{-6} M_\odot$. The initial velocity of the NS is $v = 0$, and the initial magnetic field is $B = 0$. The initial conditions are imposed on a spherical radius of $r = 10^{13}$ cm, and the simulation is evolved until a final time of $t = 10^{13}$ s.

The initial conditions are obtained by solving the equations of ideal MHD in the presence of a uniform magnetic field, with a constant density and internal energy. The initial magnetic field is $B = 0$, and the initial velocity of the NS is $v = 0$. The initial conditions are imposed on a spherical radius of $r = 10^{13}$ cm, and the simulation is evolved until a final time of $t = 10^{13}$ s.

Results.—We construct the initial magnetized equilibria, models A1-A7 in Table I by solving the Einstein equations, Maxwell’s equations, and ideal MHD equations self-consistently under the assumptions of stationarity and axisymmetry. All of our models use a polytropic equation of state

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TABLE I. The magnetar models evolved in this work. Columns are: the model name, the polytropic index, the central rest-mass density in $g/cm^3$, the gravitational mass, the period, the ratio of polar to equatorial radii, the rotational kinetic energy, the total magnetic energy, the toroidal magnetic energy, the poloidal magnetic energy, the dynamical timescale ($1/\sqrt{M}$), and the Alfvén timescale. $|W|$ denotes the gravitational energy, while the * denotes that this model collapses to a black hole.

| Case | $\Gamma$ | $\rho_{oc}$ ($\times 10^{15}$) | $M$ | $P/M$ | $R_p/R_e$ | $\mathcal{T}/|W|$ ($\times 10^{-2}$) | $\mathcal{M}/|W|$ ($\times 10^{-2}$) | $\mathcal{M}_{\text{tor}}/|W|$ ($\times 10^{-4}$) | $\mathcal{M}_{\text{pol}}/|W|$ ($\times 10^{-2}$) | $t_d/M$ | $t_A/M$ |
|------|----------|-----------------|------|------|---------|-----------------|-----------------|-----------------|-----------------|-----------|-----------|
| A1   | 2        | 1.072           | 1.385| 173.0| 0.7     | 4.531            | 3.026            | 0               | 2.970           | 17        | 56        |
| A2   | 2        | 1.072           | 1.366| 169.3| 0.7     | 4.871            | 1.632            | 7.863           | 1.525           | 18        | 70        |
| A3   | 2        | 1.072           | 1.362| 172.0| 0.7     | 4.730            | 1.794            | 8.876           | 1.669           | 18        | 61        |
| A4   | 2        | 1.072           | 1.359| 175.3| 0.7     | 4.568            | 1.983            | 8.707           | 1.852           | 18        | 47        |
| A5   | 2        | 1.072           | 1.197| 769.3| 0.925   | 0.2612           | 1.709            | 7.492           | 1.632           | 20        | 45        |
| A6*  | 2        | 2.225           | 1.586| 90.77 | 0.6     | 5.055            | 0.1624           | 0.5361          | 0.1504          | 11        | 126       |
| A7   | 3        | 1.225           | 1.592| 119.1| 0.7     | 4.043            | 4.399            | 17.81           | 4.134           | 14        | 18        |

FIG. 1. Initial magnetic field strength along the x and z axes for all magnetars A1-A7, where z is the rotational axis. Vertical dashed lines show the corresponding NS radii. The toroidal magnetic field ($B^\parallel$) is concentrated in a region below the NS surface. Note that the lines (radii) for A1-A3 closely overlap.

with $\Gamma = 2$, except the last model that has $\Gamma = 3$. Model A6 is supramassive [49], while all others are normal NSs. Models A1-A4 are rapidly rotating NSs with the same central density $\rho_{oc}$ and polar to equatorial radius deformation $R_p/R_e$, but with the ratio of toroidal to poloidal B-field energies $\mathcal{M}_{\text{tor}}/\mathcal{M}_{\text{pol}}$ changed systematically. Model A5 is a slowly rotating NS whose parameters that determine the B-fields are the same as in model A4. Model A6 is close to the mass-shedding limit curve and the maximum mass of unmagnetized, uniformly rotating equilibrium (to the left). Finally model A7 is a moderately rotating normal NS. All magnetars have magnetic energy which is at most 4.4% of their gravitational potential energy. Most of this energy is poloidal in nature (expressions of how these energies are computed are given in [47]) since the toroidal magnetic field is confined to a region below the surface of the NS, therefore its volume is much smaller than the corresponding one for the poloidal field, which extends to infinity. However, the maximum values of the toroidal and the poloidal magnetic fields are of the same order. This can be seen in Fig. 1 where we plot various components of the magnetic field (toroidal is $B^\parallel$) across the x and z axes. Vertical dashed lines signify the magnetar radii. In the last two columns of Table I we show the dynamical and Alfvén timescales ($t_A = R_e\sqrt{4\pi \rho_{oc}}/B$, where $B$ is the value of the magnetic field at the NS center) of our models. A more precise estimate of the Alfvén timescale based on the relativistic formula is given in the Supplemental Material, and broadly agrees with the timescales of Table I. All models have been evolved for $10^{-20}$ Alfvén times, with the longest being magnetar A7 ($\sim 20t_A$).

Evolutions.—Magnetars A1-A7 are evolved using the Illinois GRMHD moving-mesh-adaptive-refinement code (see e.g. [48]) using the settings described in [50], and summarized in the Supplemental Material. In all our simulations we use high resolution, with the finest refinement level having $\Delta x_{\min} \approx 87 \text{ m}$ for the $\Gamma = 2$ models, having radii $10.5 - 12.3$ km, and $\Delta x_{\min} \approx 72 \text{ m}$ for the $\Gamma = 3$ model, which has a radius of $8.7 \text{ km}$.

Results.—The behavior of our magnetized neutron stars can be broadly described by the following characteristics: (i) Large radial density oscillations, especially for the rapidly rotating magnetars. (ii) Uniform rotation is destroyed in the core of the stars, which at instances becomes counterrotating. (iii) The NSs remain axisymmetric to a large degree. (iv) The toroidal magnetic field is the first to become unstable. (v) The timescale of the instability is longer for smaller toroidal magnetic field strengths, although the strength is comparable in all cases. Models with the strongest toroidal B-field exhibit a density dip inside the star (as described in [47]), are the most unstable and develop a “gear”-like shape at the NS surface. (vi) All our models develop the “varicose” and “kink” instabilities [39–41].

In Fig. 2 we show 3D renderings of model A2 (top row)
FIG. 2. Three-dimensional renderings of model A2 (top row) and A7 (bottom row) at four different instances of time. White lines show the poloidal field lines while green lines show a poloidal + toroidal one. The tighter the coil of the helix, the smaller the toroidal magnetic field.

FIG. 3. Rest-mass density and magnetic field lines for model A2 on the equatorial and meridional planes at four time instances. Dark field lines signify a stronger magnetic field.

which has proven to be the most stable, and model A7 (bottom row) which has the strongest toroidal magnetic field at 4 different instances. Also in Fig. 3 along with the density profile we show the poloidal and toroidal field lines on the meridional and equatorial planes respectively at 4 instances for model A2. After approximately $\sim 10t_A$ model A2 preserves broadly both its shape as well as the geometry of its toroidal and poloidal magnetic fields. In Fig. 2 green lines signify the combined poloidal and toroidal magnetic field while in Fig. 3 black field lines signify regions of strong magnetic field. On the other hand model A7 after $\sim 10t_A$ exhibits turbulent motion on its surface, together with large density oscillations close to the surface and at $\pm 45^\circ$ from the equatorial plane. By that time the toroidal geometry of the magnetic field is lost and the kink instability is fully developed. It has been suggested that the instability can trigger giant magnetar flares [3] and may be accompanied by a change in the mass quadrupole moment that can potentially lead to detectable gravitational waves [51, 52]. In our simulations our stars preserve their general (hydrostatic) axisymmetry and we
did not observe any significant nonaxisymmetric mode growth that can lead to appreciable gravitational wave emission, in accordance with [36, 37, 53].

Figure 4 shows the azimuthally averaged angular velocity \( \bar{\Omega}(r) = 1/(2\pi) \int_0^{2\pi} \bar{\Omega} \bar{u}/d\phi \) in the equator, plotted along the x-axis, for models A2 (top panel) and model A7 (bottom panel) at 6 different instances. Although all our models initially are uniformly rotating, in a couple of dynamical timescales differential rotation arises in their core at distances within half their radii. One broad characteristic of this differential rotation is that it resembles the one found after the merger of two NSs [54–57]. In our simulations this behavior has developed spontaneously and is probably associated with the strong poloidal magnetic field. If the strength of the poloidal magnetic field is indeed responsible for the angular velocity drop at the center of the merger remnant (before uniform rotation is restored) that can have consequences in its evolution. A second characteristic is that the angular velocity in the core at various instances drops to zero and even takes negatives values, which is reminiscent of the behavior found in analytical models [16]. Differential rotation generates toroidal Alfvén waves that convert kinetic energy into magnetic field and thermal energy. Eventually we expect that the differential rotation will be washed out and the star will come back to uniform rotation at the end due to the effective turbulent viscosity induced by MRI, as can be seen for model A7 in Fig. 4.

Models A3, A4, and A5 exhibit a dip in the density profile just below the NS surface due to the strength of the toroidal magnetic field. This phenomenon was first found in [47] in mixed poloidal and toroidal configurations. In our case it is more pronounced in model A4 in which the pressure becomes zero inside that magnetar, creating a toroidal electro-vacuum region. These equilibria turn out to be the most unstable since the radial oscillations in conjuction with the “varicose” and “kink” instabilities destroy this electro-vacuum on Alfvén timescales and create turbulent-like phenomena on the NS surface. The density profile of model A2, which was the most stable magnetar, had essentially no such density dip and therefore no hydrostatic pressure depletion below its surface.

The normalized maximum density of our magnetar solutions is plotted in Fig. 5 (top panel) where oscillations of 10 – 30% are visible. Note that the largest oscillations are present for the rapidly rotating magnetars A1-A4 and A6. However, the very slowly rotating model A5 and the moderately rotating model A7 exhibit oscillations of \( \sim 10\% \). The large oscillations of model A6 are responsible for its collapse to a black hole, since the close proximity of this magnetar to the maximum supramassive limit [49] drives it to the unstable branch [59] (we have performed a resolution study to confirm this conclusion). In the scenario of a binary NS merger we expect that supramassive remnants close to the maximum mass limit will be similarly unstable. The timescale of the oscillations in Fig. 5 are of the order of an Alfvén timescale and are not present in the absence of the magnetic field. Indeed, if for the collapsing model A6 we reset the magnetic field to zero, it stays in a quasiequilibrium state and no collapse is triggered. In the bottom panel of Fig. 5 we plot the ejected material from our simulations. Note that detectable, transient kilonova signatures powered by radioactive decay of unstable elements formed by neutron-rich material are expected for ejecta masses \( \gtrsim 10^{-3}M_\odot \) [60, 61]. Using the ejecta of models \{A1, A2, A3, A4\}, and the fitting formulae provided in [62], we infer the peak time \( \tau_{\text{peak}} \) of the kilonova emission is \( \{5.4, 4.6, 3.7, 5.2\} \) days, the peak luminosity \( L_{\text{kil}}\) of the ejecta is \( \{1.3, 1.2, 2.1, 2.6\} \times 10^{47} \) ergs/s, and the effective temperature \( T_{\text{peak}} \) at the peak is \( \{2.4, 2.6, 2.3, 2.1\} \times 10^{5} \) K, respectively. Significant ejecta are being created only from our rapidly rotating magnetars A1-A4, consistent with the suggestion that the magnetic field lines of a rotating compact object may accelerate fluid elements due to a magneto-centrifugal mechanism [63]. However no highly collimated, helical magnetic fields or incipient jets, which are necessary for gamma-ray bursts, arise at the poles of these magnetars by the time our simulations are terminated.
Discussion.—In this work we presented our latest GRMHD simulations of self-consistent, ultramagnetized equilibria. We constructed and evolved a diverse set of magnetars from slowly to rapidly rotating, most having mixed poloidal and toroidal magnetic fields of comparable strength. Although the specific geometry of the mixed field configuration can delay or accelerate the development of various well-known instabilities [39–41] all our models finally succumb to them. In addition, our initially uniformly rotating models develop differential rotation in their cores on a dynamical timescale, similar to that found in binary NS mergers. To establish exactly how differential rotation is spontaneously created requires further analysis. Our models (especially the rapidly rotating ones) exhibit large radial oscillations in the NS’s F-mode, and produce significant amounts of ejecta that can power a kilonova. Our equilibria can be explored further with improved numerical evolution schemes that will address effects from the artificial atmosphere typically found in all ideal MHD simulations. In that direction one would employ the equations of resistive MHD in full GR [11, 64–66] or a scheme that reliably matches GRMHD to its force-free limit [67]. Finally the recent results by NICER [44, 45] are calling for the development of solutions beyond the large scale dipolar magnetic field configurations. One important question is how do observed pulsars with their magnetic fields reach and maintain uniform rotation, which is believed necessary for pulsars to serve as very precise clocks? All of these matters will be the subject of our future explorations.

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Supplemental Material

Initial data.—Our initial data (blue circles in Fig. 6) are constructed with the magnetized rotating neutron star libraries of the COCAL code [46, 47]. These are stationary and axisymmetric equilibrium solutions for the simultaneous system of Einstein’s field equations, Maxwell’s equations, and the ideal magnetohydrodynamic (MHD) equations.

In the 3+1 decomposition, the spacetime metric is written in terms of the lapse $\alpha$, shift $\beta^i$, and the spatial metric $\gamma_{ij}$, which is further decomposed as $\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$, with $\tilde{\gamma}_{ij} = f_{ij} + h_{ij}$. Here $\psi$ is the conformal factor, $f_{ij}$ the flat metric, and $h_{ij}$ the nonflat part of the conformal geometry. The introduction of an additional degree of freedom (from 6 components of $\gamma_{ij}$ we now have 7 components for $\psi$ and $h_{ij}$) results to a constraint for the conformal metric, which we set to be $\tilde{\gamma} = 1$. Through combinations of the Einstein equations, elliptic (Poisson-type) equations are derived for the 11 components $\{\alpha, \beta^i, \psi, h_{ij}\}$.

For the gauge conditions we use maximal slicing $K = 0$, and the Dirac gauge $\bar{D}_a \tilde{\gamma}^{ab} = 0$, where $\bar{D}_a$ is the covariant spatial derivative with respect to the flat metric $f_{ij}$. A method to impose the latter is described in [47, 68] and results in an additional 3 elliptic equations. Although as shown in [69] a waveless condition $\partial_t \tilde{\gamma}^{ij} = O(r^{-3})$ is sufficient for the metric potentials to fall off as Coulomb fields, here we use the stronger condition $\partial_t \tilde{\gamma}^{ij} = 0$. Similarly, for the time derivative on the tracefree part of the extrinsic curvature $\bar{A}_{ij}$, as well as on the matter fields, we use the condition of stationarity.

As shown in [70], Maxwell’s and the ideal MHD equations reduce to a single elliptic equation for a master potential called the relativistic master transfield equation associated with a system of first integrals on a fluid support. The transfield equation in the absence of a meridional flow field becomes the well-known Grad-Shafranov equation [71–73]. In our formulation, we do not reduce the system of Maxwell’s and ideal MHD equations to the transfield equation, but instead, we use the 3+1 decomposition of Maxwell’s equations and solve 4 elliptic equations for the projections of the electromagnetic 1-form $A_i$, subject to the Coulomb gauge $\bar{D}^i A_i = 0$, similar
to the Dirac gauge for \( \tilde{\gamma}_{ij} \). The ideal MHD condition implies that surfaces of constant \( A_t \) and \( A_\phi \) coincide, and therefore these variables are functions of a single master potential, which is taken to be \( A_\phi \) itself. Then, the first integrals of the MHD-Euler equations and the ideal MHD conditions become relations to determine the specific enthalpy \( h \), the components of 4-velocity \( u^\mu \) and \( u_\phi \), and the components of the current \( j^\alpha \). The latter involves the following arbitrary functions of the potential \( A_\phi \), which are chosen to be

\[
\begin{align*}
\Lambda(A_\phi) &= -\Lambda_0 \Xi(A_\phi) - \Lambda_1 A_\phi - \mathcal{E}, \\
A_t(A_\phi) &= -\Omega_c A_\phi + C_c, \\
\sqrt{-g} \Lambda_\Phi(A_\phi) &= \Lambda_{00} \Xi(A_\phi). 
\end{align*}
\]

In Eqs. (1)-(3) \( \Lambda_0, \Lambda_1, \) and \( \Lambda_{00} \) are input parameters that control the poloidal and toroidal magnetic field strength, while constants \( \mathcal{E} \) and \( \Omega_c \) are determined during the iteration procedure. The former signifies the injection energy [74], while the latter is the constant angular velocity of the magnetar. Constant \( C_c \) controls the net charge of the star, which in our case is zero. \( \Xi(A_\phi) \) is an integral of the “sigmoid” function [47] which is used where \( A_\phi \) varies on the fluid support, and its derivative is written

\[
\Xi'(A_\phi) = \frac{1}{2} \left[ \tanh \left( \frac{1}{b} \frac{A_\phi - A_{\phi,S}^{\max}}{A_{\phi,S}^{\max} - c} \right) + 1 \right],
\]

where \( A_{\phi,S}^{\max} \) is the maximum value of \( A_\phi \), and \( A_{\phi}^{\max} \) is the maximum value of \( A_\phi \) at the stellar surface. Parameters \( b, c \in [0, 1] \) control the width and the position of the sigmoid. Therefore \( \Xi'(A_\phi) \) vary from zero to one in the interval \( A_\phi \in [A_{\phi,S}^{\max}, A_{\phi}^{\max}] \). This guarantees that the current and the toroidal B field are confined in the star, and the components of electromagnetic fields extend continuously into the exterior vacuum region. Together with \( \Lambda_0, \Lambda_1, \) and \( \Lambda_{00} \), the parameters \( b \) and \( c \) are reported in Table II for the seven models A1-A7 presented here.

**Evolutions.**—We evolve the initial data above using the Illinois GRMHD moving-mesh-adaptive-refinement code (see e.g. [48]), which employs the Baumgarte-Shapiro-Shibata-Nakamuru formulation of the Einstein’s equations [75, 76] with puncture gauge conditions (see Eq. (2)-(4) in [77]). The MHD equations are solved in conservation-law form adopting high-resolution shock-capturing methods. Imposition of

\[
\nabla \cdot \vec{B} = 0 \quad \text{during evolution is achieved by integrating the magnetic induction equation using a vector potential} \, A^\mu [48].
\]

The generalized Lorenz gauge [78] is employed to avoid the appearance of spurious magnetic fields [79]. We employ a \( \Gamma \)-law equation of state (EOS), \( P = (\Gamma - 1) \rho_0 \epsilon \), with \( \epsilon \) the specific internal energy, and allow for shock heating. In models A1-A6 we set \( \Gamma = 2 \), and \( \Gamma = 3 \) in A7. To capture one of the properties of the force-free conditions \( B^2/(8\pi \rho_0) > 1 \) that likely characterize the neutron star exterior, we set a variable- and low-density magnetosphere outside the star such that the magnetic-to-gas pressure ratio is \( \beta = P_{\text{mag}}/P_{\text{gas}} = 100 \) everywhere [80]. This one-time reset of the low-density magnetosphere increases the total rest-mass on the entire grid by less than 1%, consistent with the values reported previously (see e.g. [81, 82]). The ideal GRMHD equations are then integrated everywhere, imposing on top of the magnetosphere a density floor in the low density regions similar to [50, 81, 82].

**Magnetic field strength.**—In Fig. 7 (top row) we show the magnetic field strength for the model A2 at the initial and final (end of our simulations) moments. Although the broad structure of the poloidal and toroidal magnetic field is preserved, it is apparent from the figure that the centers of the toroidal configuration have moved slightly relative to the equatorial axis as a result of the kink instability. In the bottom panels we show the Alfvén timescale \( t_A \) on the meridional plane based on the relativistic formula \( v_A = \sqrt{b^2/(b^2 + \rho_0)} \), where \( v_A \) is the Alfvén velocity, \( b^4 = B^\mu/\sqrt{4\pi} \), and \( t_A = R_e/v_A \). At the end of our simulation the profile of the Alfvén timescale follows closely the initial one.

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**TABLE II.** Parameters of functions in the integrability conditions Eqs. (1)-(3) and Eq. (4).

| Models | \( \Lambda_0 \) | \( \Lambda_1 \) | \( \Lambda_{00} \) | \( b \) | \( c \) | Descriptions |
|--------|----------------|----------------|----------------|------|------|----------------|
| A1     | 1.6            | 0.3            | 0.0            | 0.2  | 0.5  | Systematic change in \( B_{\text{tor}}^{\max}/P_{\text{pol}}^{\max} \) |
| A2     | -0.6           | 0.3            | 1.1            | 0.2  | 0.5  | Slow rotation of A4 |
| A3     | -1.8           | 0.3            | 1.7            | 0.2  | 0.5  | Supramassive model |
| A4     | -3.0           | 0.3            | 2.3            | 0.2  | 0.5  | \( \Gamma = 3 \) model |
| A5     | -3.0           | 0.3            | 2.3            | 0.2  | 0.5  | \( \Gamma = 3 \) model |
| A6     | -1.7           | 0.1            | 1.7            | 0.2  | 0.5  | \( \Gamma = 3 \) model |
| A7     | -0.2           | 0.3            | 1.0            | 0.2  | 0.5  | \( \Gamma = 3 \) model |

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**FIG. 7.** Magnetic field strength and Alfvén timescale for model A2 at two different instances.
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