SOLVING THE HEAT EQUATION IN PIECEWISE-HOMOGENEOUS ANISOTROPIC MEDIA USING THE MULTIDIMENSIONAL FOURIER TRANSFORMS

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Abstract. Multidimensional integral transformations with non-separated variables for problems with discontinuous coefficients are constructed in this work. The coefficient discontinuities focused on the of parallel hyperplanes. In this work explicit formulas for the kernels in the case of ideal coupling conditions are obtained; the basic identity of the integral transform is proved; technique of integral transforms is developed.

Keywords: integral transformations, non-separated variables, coupling conditions

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1. Introduction

Integral transforms arise in a natural way through the principle of linear superposition in constructing integral representations of solutions of linear differential equations. First note that the structure of integral transforms with the relevant variables are determined by the type of differential equation and the kind of media in which the problem is considered. Therefore decision of integral transforms are the problem for mathematical physics piecewise-homogeneous (heterogeneous) media. It is clear this method is an effective for obtaining the exact solution of boundary-value problems for piecewise-homogeneous structures mathematical physics. Integral transforms and their applications are appeared in the mathematical monographs of Uflyand Y.S. [1,2], Lenuk M.P. [3,4]; Nayda L.S. [4], Protsenko V.S. [5], etc. The author together with I.I.Bavrin has proposed integral transforms with non-separate variables for solving multidimensional problems in the work [7].

Let $V$ from $R^{n+1}$ be the half-space

$$V = \{(y_1, ..., y_n, x) \in R^{n+1} : x > 0\},$$

then solution of the Dirichlet’s problem for the half-space is expressed by Poisson formula takes the form:

$$u(x, y) = \Gamma \left( \frac{n+1}{2} \right) \pi^{-\frac{n+1}{2}} \int_{y=0}^{\infty} \frac{x}{[(y-\eta)^2 + x^2]^{\frac{n+1}{2}}} f(\eta) d\eta.$$
Obviously Poisson’s kernel is the form of integral Laplace transform and therefore expansion of the function \( f(y) \) for the eigenfunctions of the Laplace operator \( \Delta \) is obtained from the reproduce properties of the Poisson kernel:

\[
f(y) = \lim_{\tau \to 0} \int_0^\infty \lambda^2 e^{-\lambda \tau} \left( \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{J_{n-2} \left( \lambda |y - \eta| \right)}{|y - \eta|^{n-2}} f(\eta) \, d\eta \right) \, d\lambda,
\]

here \( J_\nu \) is Bessel’s function of order \( \nu \) [8]. We may assume that integral transforms with non-separate variables are defined as follows [7] on the basis of this expansion: direct integral Fourier transform has the form

\[
F[f](y, \lambda) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{J_{n-2} \left( \lambda |y - \eta| \right)}{|y - \eta|^{n-2}} f(\eta) \, d\eta \equiv \hat{f}(y, \lambda),
\]

inverse Fourier integral transform has the form

\[
F^{-1}[\hat{f}](y) = \lim_{\tau \to 0} \int_0^\infty \lambda^2 e^{-\lambda \tau} \hat{f}(y; \lambda) d\lambda \equiv f(y).
\]

In our case the construction of multi-dimensional analogues for integral transforms (1)-(2) with discontinuous coefficients is the purpose of this research.

2. ONE-DIMENSIONAL INTEGRAL TRANSFORMS WITH DISCONTINUOUS COEFFICIENTS

In this paper integral transforms with discontinuous coefficients are constructed in accordance with author’s work [10]. Let \( \varphi(x, \lambda) \) and \( \varphi^*(x, \lambda) \) be eigenfunctions of primal and dual problems Sturm-Liouville for Fourier operator on sectionally homogeneous axis \( I_n \),

\[
I_n = \left\{ x : x \in \bigcup_{j=1}^{n+1} (l_{j-1}, l_j), \quad l_0 = -\infty, \quad l_{n+1} = \infty, \quad l_j < l_{j+1}, \quad j = 1, n \right\}.
\]

Let us remark that eigenfunction \( \varphi(x, \lambda) \),

\[
\varphi(x, \lambda) = \sum_{k=2}^{n} \theta(x - l_{k-1}) \theta(l_k - x) \varphi_k(x, \lambda) + \theta(l_1 - x) \varphi_1(x, \lambda) + \theta(x - l_n) \varphi_{n+1}(x, \lambda)
\]

is the solution of separated differential equations system

\[
\left( a_m \frac{d^2}{dx^2} + \lambda^2 \right) \varphi_m(x, \lambda) = 0, \quad x \in (l_m, l_{m+1}); \quad m = 1, \ldots, n + 1,
\]

by the coupling conditions

\[
\left[ \alpha_m^k \frac{d}{dx} + \beta_m^k \right] \varphi_k = \left[ \alpha_m^k \frac{d}{dx} + \beta_m^k \right] \varphi_{k+1},
\]

on the boundary conditions

\[
\varphi_1|_{x=-\infty} = 0, \quad \varphi_{n+1}|_{x=\infty} = 0.
\]

Similarly, the eigenfunction \( \varphi^*(x, \lambda) \),

\[
\varphi^*(\xi, \lambda) = \sum_{k=2}^{n} \theta(\xi - l_{k-1}) \theta(l_k - \xi) \varphi_k^*(\xi, \lambda) + \theta(\xi - l_n) \varphi_{n+1}^*(\xi, \lambda)
\]
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\[ + \theta (l_1 - \xi) \varphi_1^* (\xi, \lambda) + \theta (\xi - l_n) \varphi_{n+1}^* (\xi, \lambda) \]

is the solution of separate differential equations system

\[ \left( a_m^2 \frac{d^2}{dx^2} + \lambda^2 \right) \varphi_m^* (x, \lambda) = 0, \quad x \in (l_m, l_{m+1}); \quad m = 1, \ldots, n + 1, \]

by the coupling conditions

\[ \frac{1}{\Delta_{1,k}} \left[ \alpha_{m1}^k \frac{d}{dx} + \beta_{m1}^k \varphi_k^* = \frac{1}{\Delta_{2,k}} \left[ \alpha_{m2}^k \frac{d}{dx} + \beta_{m2}^k \right] \varphi_{k+1}^*, \quad x = l_k, \]

where

\[ \Delta_{i,k} = \det \left( \begin{array}{cc} \alpha_{1i}^k & \beta_{1i}^k \\ \alpha_{2i}^k & \beta_{2i}^k \end{array} \right) \]

on the boundary conditions

\[ \varphi_1|_{x=-\infty} = 0, \quad \varphi_{n+1}|_{x=\infty} = 0. \]

Further normalization eigenfunctions is accepted by the following:

\[ \varphi_{n+1}^* (x, \lambda) = e^{ia_{n+1} x \lambda}, \quad \varphi_{n+1}^* (x, \lambda) = e^{-ia_{n+1} x \lambda}. \]

Let direct \( F_n \) and inverse \( F_n^{-1} \) Fourier transforms on the Cartesian axis with \( n \) division points be defined by the rules in [10]:

\[ F_n [f] (\lambda) = \sum_{m=0}^{n+1} \int_{l_m}^{l_{m-1}} u_m^* (\xi, \lambda) f_m (\xi) d\xi \equiv \hat{f} (\lambda), \]

\[ f_k (x) = \frac{1}{\pi i} \int_0^{\infty} u_k (x, \lambda) \hat{f} (\lambda) \lambda d\lambda. \]

3. Heat conduction in piecewise-homogeneous anisotropic media

Here the method of delta- functions [4] is the foundation for integral transforms. Namely kernels of the integral solutions mixed Cauchy’s problem are the delta-functions.

This means that we fined the solution of the separated matrix systems \((n + 1)\) parabolic equations:

\[ \left( \frac{\partial}{\partial t} - A_j \frac{\partial^2}{\partial x^2} - \Delta_y \right) U_j (t, x, y) = 0, \quad (t, x, y) \in D_+ \times \mathbb{R}^m, \quad j = 1, n + 1 \]

bounded on the set \( D \times \mathbb{R}^m, D_+ = (0, \infty) \times I_n, \)

where

\[ I_n = \left\{ x : x \in \bigcup_{j=1}^{n+1} (l_j - l_j), \quad l_0 = -\infty, \quad l_{n+1} = \infty, \quad l_j < l_{j+1}, \quad j = 1, n \right\} \]

\[ \Delta_y = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_m^2}, \]

\[ A_j = \left( a_{ik}^j \right) \text{ positive definite matrix } r \times r, \]

to initial data

\[ U_j (t, x, y) |_{t=0} = g_j (x, y), \quad x \in I_n, y \in \mathbb{R}^m \]
by edge condition
\[ U_1 |_{x=-\infty} = 0, \quad U_{n+1} |_{x=\infty} = 0 \]

by the coupling condition
\[ \left[ \alpha^k_{m1} \frac{\partial}{\partial x} + \beta^k_{m1} \right] U_k = \left[ \alpha^k_{m2} \frac{\partial}{\partial x} + \beta^k_{m2} \right] U_{k+1}, \quad x = l_k, \quad k = 1, n; \quad m = 1, 2, \]

here \( U_j(t, x, y) \) is unknown vector-function, \( g_j(x, y) \) given vector-function, \( \alpha^k_{mi}, \beta^k_{mi}, \gamma^k_{mi}, \delta^k_{mi} \) are matrix’s \( r \times r \).

Fourier integral with discontinuous coefficients section 2 and the Fourier integral with non-separated variable (1) - (2) give the idea of solving the problem (3) - (6).

This solution takes the form:
\[
U_k(t, x, y) = \frac{1}{\pi i} \frac{1}{(\sqrt{2\pi})^m} \int_{R^m} \sum_{j=1}^{l_j} \lim_{\tau \to 0} \left( \int_0^\infty \frac{J_{m/2} (\lambda |y - \eta|)}{|y - \eta|^{m/2}} e^{-\lambda \tau} \lambda^{m/2} d\lambda \right) \int_{-\infty}^{\infty} e^{-\beta t} \varphi_k(x, \beta) \varphi^*_j(\xi, \beta) d\beta \]

\[
\cdot f_j(\xi, \eta) \, d\xi d\eta, \quad k = 1, n + 1, \]

where \( \varphi_k(x, \beta), \varphi^*_j(\xi, \beta) \) are the eigenfunctions of the direct and dual Sturm – Liouville problems, respectively.

Suppose iterated integral
\[
\int_0^\infty \frac{J_{m/2} (\lambda |y - \eta|)}{|y - \eta|^{m/2}} e^{-\lambda \tau} \lambda^{m/2} d\lambda \int_{-\infty}^{\infty} e^{-\beta t} \varphi_k(x, \beta) \varphi^*_j(\xi, \beta) d\beta
\]
is considered as a double then pass to the polar coordinates
\[
\lambda = \rho \sin \varphi, \quad \beta = \rho \cos \varphi; \quad 0 \leq \rho < \infty, \quad 0 \leq \varphi \leq \pi.
\]

Then we prove:
\[
\int_0^\pi \rho^{m/2} \rho d\rho \int_0^{\pi \rho^2 \cos^2 \alpha} \sin \frac{\pi \rho^2}{\alpha} \frac{J_{m/2} (\rho \sin \alpha |y - \eta|)}{|y - \eta|^{m/2}} e^{-\rho \tau \sin \alpha} \cdot \varphi_k(x, \rho \cos \alpha) \varphi^*_j(\xi, \rho \cos \alpha) d\alpha.
\]

If perform to the limit as \( \tau \to 0 \) in the formula (9), then we obtain
\[
u_k(t, x, y) = \frac{1}{\pi i} \frac{1}{(\sqrt{2\pi})^m} \int_{R^m} \sum_{j=1}^{l_j} \int_0^\infty e^{-\lambda^2 t} \rho^{m/2} \rho d\rho \int_0^{\pi \rho^2 \cos^2 \alpha} \frac{J_{m/2} (\rho \sin \alpha |y - \eta|)}{|y - \eta|^{m/2}} \cdot \varphi_k(x, \rho \cos \alpha) \varphi^*_j(\xi, \rho \cos \alpha) \, d\alpha \int f_j(\xi, \eta) \, d\xi d\eta.
\]
Here we use the notation
\[ \varphi_{k,j} = \varphi_{k,j}(\rho, x, \xi, |y - \eta|) = \pi \int_0^\pi \sin^{\frac{m}{2}} \alpha J^{\frac{m}{2}} - \frac{1}{|y - \eta|^{\frac{m}{2}}} \cdot \varphi_k(x, \rho \cos \alpha) \varphi_j^*(\xi, \rho \cos \alpha) \, d\alpha \]

It is clear formula (10) be written as
\[ u_k(t, x, y) = \frac{1}{\pi} \int_0^\infty e^{-\rho^2 t} \frac{1}{(\sqrt{2\pi})^m} \int_{R^m} \int_{l_{ij} - 1}^{l_i} \varphi_{k,j}(\xi, \eta) d\xi d\eta. \]

4. THE MULTIDIMENSIONAL FOURIER TRANSFORMS WITH DISCONTINUOUS ON PLANES

If perform to the limit as \( t \to 0 \) in the formula (11), then we obtain
\[ f_k(x, y) = \frac{1}{\pi} \int_0^\infty \rho^{\frac{m}{2}} \rho^{\frac{m+1}{2}} d\rho \frac{1}{(\sqrt{2\pi})^m} \int_{R^m} \int_{l_{ij} - 1}^{l_i} \varphi_{k,j}(\xi, \eta) d\xi d\eta. \]

Note that direct and inverse multidimensional Fourier transforms with discontinuous on planes \( x = l_k \) are determined due to the integral identity (12):
\[ F_n[f](x, y, \lambda) = \frac{1}{(\sqrt{2\pi})^m} \int_{l_{ij} - 1}^{l_i} \varphi_{k,j}(\lambda, x, \xi, |y - \eta|) f_j(\xi, \eta) d\xi d\eta, \]
\[ f(x, y) = \int_0^\infty \lambda^{\frac{m}{2} + 1} F_n[f](x, y, \lambda) d\lambda, \]
respectively.

Finally we prove the basic integral identity for differential operator
\[ B = \theta (l_1 - x t) \left( A_1^2 \frac{d^2}{dx^2} + \Delta_y \right) + \sum_{k=1}^n \theta (x - l_{k-1}) \theta (l_k - x) \left( A_k^2 \frac{d^2}{dx^2} + \Delta_y \right) + \theta (x - l_n) \left( A_{n+1}^2 \frac{d^2}{dx^2} + \Delta_y \right). \]

**Theorem 1.** Suppose twice continuously differentiable on \( D_+ \times R^m \) vector-function
\[ f(x, y) = \sum_{k=1}^{n+1} \theta (x - l_{k-1}) \theta (l_k - x) f_k(x, y) + \theta (x - l_n) f_{n+1}(x, y), \]
satisfies the conditions at infinity:
\[ f_{n+1}(x, y), \frac{\partial f_{n+1}(x, y)}{\partial x} \]
limits to zero as \( x \to +\infty, y \) -is fixed,
limits to zero as \( x \to -\infty, y \)-is fixed,

\[ f_i(x, y), \quad \frac{\partial f_i(x, y)}{\partial x} \]

limits to zero as \( y_j \to \pm \infty; x, y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m \)-are fixed,

the coupling conditions (8) are performed

then hold true:

\[ F_n[B(f)] = -\lambda^2 F_n[f]. \]

Proof. Let us twice integrate by parts with respect to each of the variables in the left part. Further outside the integral terms are disappeared by the conditions at infinity, connection summands are disappeared by the coupling conditions. Therefore the operator \( B \) is placed as the kernel:

\[ F_n[B(f)](x, y, \lambda) = \frac{1}{(\sqrt{2\pi})^m} \int_{\mathbb{R}^m} \sum_{j=1}^{n+1} \int B_j[\varphi_{k,j}(\lambda, x, |y - \eta|)] f_j(\xi, \eta) \, d\xi \, d\eta. \]

Hence the equality \( B_j[\varphi_{k,j}] = -\lambda^2 \varphi_{k,j} \) is proved. We shall prove that —

\[ B_j[\varphi_{k,j}] = \int_0^{\pi} \sin \frac{\pi}{m} \alpha \Delta_\eta \left( \frac{J_{m-2} \left( \rho \sin \alpha |y - \eta| \right)}{|y - \eta|^{\frac{m-2}{2}}} \right) \cdot \varphi_k(x, \rho \cos \alpha) \left( \varphi_j^*(\xi, \rho \cos \alpha) \right) \, d\alpha + \]

\[ + \int_0^{\pi} \sin \frac{\pi}{m} \alpha \left( \frac{J_{m-2} \left( \rho \sin \alpha |y - \eta| \right)}{|y - \eta|^{\frac{m-2}{2}}} \right) \cdot \varphi_k(x, \rho \cos \alpha) \left( \varphi_j^*(\xi, \rho \cos \alpha) \right) \, d\alpha = \]

\[ = -\rho^2 \sin^2 \alpha \int_0^{\pi} \sin \frac{\pi}{m} \alpha \left( \frac{J_{m-2} \left( \rho \sin \alpha |y - \eta| \right)}{|y - \eta|^{\frac{m-2}{2}}} \right) \cdot \varphi_k(x, \rho \cos \alpha) \left( \varphi_j^*(\xi, \rho \cos \alpha) \right) \, d\alpha + \]

\[ -\rho^2 \cos^2 \alpha \int_0^{\pi} \sin \frac{\pi}{m} \alpha \left( \frac{J_{m-2} \left( \rho \sin \alpha |y - \eta| \right)}{|y - \eta|^{\frac{m-2}{2}}} \right) \cdot \varphi_k(x, \rho \cos \alpha) \left( \varphi_j^*(\xi, \rho \cos \alpha) \right) \, d\alpha = -\rho^2 \varphi_{k,j}. \]

We take into consideration that in this proof \( \varphi_j^*(\xi, \rho \cos \alpha) \) are dual Sturm-Liouville problems eigenfunctions.

\[ \Delta_\eta \left( \frac{J_{m-2} \left( \rho \sin \alpha |y - \eta| \right)}{|y - \eta|^{\frac{m-2}{2}}} \right) = -\rho^2 \sin^2 \alpha \left( \frac{J_{m-2} \left( \rho \sin \alpha |y - \eta| \right)}{|y - \eta|^{\frac{m-2}{2}}} \right), \]
By basis identity [9] we conclude
\[
\frac{\rho^{\frac{m-1}{2}} J_{\frac{m-1}{2}}(\rho |y|)}{|y|^{\frac{m-1}{2}}} = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{S_\rho} e^{i<y, \xi>} dS_\rho.
\]
\[
\square
\]

This completes the proof.
Specifically the formulas for the direct and inverse Fourier transforms with non-separated variables are significantly simplified in the case of ideal coupling conditions on one surface. This case is the most distributed in engineering practice. As an example the scalar case is considered. Suppose the ideal coupling conditions are in the plane \(x = 0\)

\[
\varphi_1 (x, y) = \varphi_2 (x, y), x = 0, y \in \mathbb{R}^m ;
\]

\[
\varphi_{1x} (x, y) = \nu \varphi_{2x} (x, y), x = 0, y \in \mathbb{R}^m ; \nu = \frac{\lambda_2}{\lambda_1}
\]

then the analytical expressions for the one-dimensional components eigenfunctions are in [4]:

\[
\varphi_1 (x, \lambda) = \left( \cos \frac{x}{\alpha_1} + i \frac{1}{\sqrt{\delta_0}} \sin \frac{x}{\alpha_1} \right) (1 + \delta_0) ;
\]

\[
\varphi_2 (x, \lambda) = \left( \cos \frac{x}{\alpha_2} + i \sqrt{\delta_0} \sin \frac{x}{\alpha_2} \right) (1 + \delta_0) ;
\]

\[
\varphi_k (x, \lambda) = r_k \varphi_k (x, \lambda), k = 1, 2, r_1 = \frac{\alpha_2}{\nu_0 \alpha_1}, r_2 = \frac{1}{\alpha_2}, \delta_0 = \frac{\alpha_2}{\nu_0 \alpha_1}.
\]

It is clear that the expressions for the multidimensional components of eigenfunctions with non-separated variables \(\varphi_{kj}\) have the form:

\[
\varphi_{11} = \frac{1 + \delta_0}{\alpha_1} J_{\frac{m-1}{2}} \left( \rho \sqrt{\left( \frac{x-\xi}{\alpha_1} \right)^2 + |y-\eta|^2} \right) - \left( 1 - \delta_0 \right) \frac{\rho}{\alpha_1} \frac{J_{\frac{m-1}{2}} \left( \rho \sqrt{\left( \frac{x+\xi}{\alpha_1} \right)^2 + |y-\eta|^2} \right)}{\left( \frac{x-\xi}{\alpha_1} \right)^2 + |y-\eta|^2}^{\frac{m-1}{2}}.
\]

\[
\varphi_{12} = \frac{1 + \delta_0}{\alpha_2 \sqrt{\delta_0}} J_{\frac{m-1}{2}} \left( \rho \sqrt{\left( \frac{x-\xi}{\alpha_2} \right)^2 + |y-\eta|^2} \right) + \left( \frac{x}{\alpha_2} - \frac{\xi}{\alpha_1} \right) \left( \frac{x}{\alpha_2} - \frac{\xi}{\alpha_1} \right)^2 + |y-\eta|^2 \right)^{\frac{m-1}{2}}.
\]

\[
\varphi_{21} = \sqrt{\delta_0} \frac{1 + \delta_0}{\alpha_1} J_{\frac{m-1}{2}} \left( \rho \sqrt{\left( \frac{x-\xi}{\alpha_1} \right)^2 + |y-\eta|^2} \right) + \left( \frac{x}{\alpha_1} - \frac{\xi}{\alpha_2} \right) \left( \frac{x}{\alpha_1} - \frac{\xi}{\alpha_2} \right)^2 + |y-\eta|^2 \right)^{\frac{m-1}{2}}.
\]
This shows that integral transforms by formulas (12) - (13) are constructed.

5. Conclusion

Let us remark that integral transforms (12) - (13) are used in solving problems of mathematical physics by the standard algorithm. If we find the solution in the images then return to the original. We stress if one spectral parameter involved in the final formula then the practical profit is achieved. At the same time the integral transforms with separate variables contain $m$ parameters.

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