A SHORT ELEMENTARY PROOF OF REVERSED
BRUNN–MINKOWSKI INEQUALITY FOR COCONVEX BODIES

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Abstract. The theory of coconvex bodies was formalized by A. Khovanskii and V. Timorin in [KT14]. It has fascinating relations with the classical theory of convex bodies, as well as applications to Lorentzian geometry. In a recent preprint [Sch17], R. Schneider proved a result that implies a reversed Brunn–Minkowski inequality for coconvex bodies, with description of equality case. In this note we show that this latter result is an immediate consequence of a more general result, namely that the volume of coconvex bodies is strictly convex. This result itself follows from a classical elementary result about the concavity of the volume of convex bodies inscribed in the same cylinder.

Let $C$ be a closed convex cone in $\mathbb{R}^n$, with non empty interior, and not containing an entire line. A $C$-cococonvex body $K$ is a non-empty closed bounded proper subset of $C$ such that $C \setminus K$ is convex. The set of $C$-cococonvex bodies is stable under positive homotheties. It is also stable for the $\oplus$ operation, defined as $K_1 \oplus K_2 = C \setminus (C \setminus K_1 + C \setminus K_2)$, where $+$ is the Minkowski sum. The following reversed Brunn–Minkowski theorem is proved in [Sch17] (see [KT14] for a partial result). We denote by $V_n$ the volume in $\mathbb{R}^n$.

Theorem 1. Let $K_1, K_2$ be $C$-cococonvex bodies, and $\lambda \in (0, 1)$. Then

$$V_n((1 - \lambda)K_1 \oplus \lambda K_2)^{1/n} \leq (1 - \lambda)V_n(K_1)^{1/n} + \lambda V_n(K_2)^{1/n},$$

and equality holds if and only if $K_1 = \alpha K_2$ for some $\alpha > 0$.

Remark 2. What is actually proved in [Sch17] in the analogous of Theorem 1 for $C$-cococonvex sets instead of $C$-cococonvex bodies: the set is not required to be bounded but only to have finite Lebesgue measure. So the result of [Sch17] requires a more involved proof than the one presented here.

Actually, we will see that the following result holds.

Theorem 3. The volume is strictly convex on the set of $C$-cococonvex bodies. More precisely, if $K_1, K_2$ are $C$-cococonvex bodies, and $\lambda \in (0, 1)$, then

$$V_n((1 - \lambda)K_1 \oplus \lambda K_2) \leq (1 - \lambda)V_n(K_1) + \lambda V_n(K_2).$$

Moreover, equality holds if and only if $K_1 = K_2$.

The following elementary lemma, together with the fact that $V_n$ is positively homogeneous of degree $n$ (i.e. $V_n(tA) = t^nV_n(A)$ for $t > 0$), shows that Theorem 3 implies Theorem 1.

Lemma 4. Let $f$ be a positive convex function, positively homogeneous of degree $n$. Then $f^{1/n}$ is convex.

Suppose moreover that $f$ is strictly convex. If there is $\lambda \in (0, 1)$ such that $f^{1/n}((1-\lambda)x + \lambda y)$ equals $(1 - \lambda)f^{1/n}(x) + \lambda f^{1/n}(y)$, then there is $\alpha > 0$ with $x = \alpha y$.

Proof. For $\lambda \in [0, 1]$ and any $x, y$, we have $f((1 - \lambda)f^{1/n}(x) + \lambda f^{1/n}(y)) \leq 1$, and the result follows by taking, for any $\lambda \in (0, 1)$, $\tilde{\lambda} = \lambda f(y)^{1/n} / ((1 - \lambda)f(x)^{1/n} + \lambda f(y)^{1/n})$.

Let us prove Theorem 3.

Let $H$ be an affine hyperplane of $\mathbb{R}^n$ with the following properties: it has an orthogonal direction in the interior of $C$, $K_1$, $K_2$ and the origin are contained in the same half-space $H^+$ bounded by $H$, and $H \cap C = B$ is compact. For $\lambda \in [0, 1]$, let $K_\lambda = (1 - \lambda)K_1 \oplus \lambda K_2$, which is also contained in $H^+$, and let $\text{cap}_H(K_\lambda) = H^+ \cap (C \setminus K_\lambda)$, see Figure 1.

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Also, the quantity \( V_n(K_\lambda) + V_n(\text{cap}_H(K_\lambda)) \) does not depend on \( \lambda \), as it is equal to \( V_n(C \cap H^+) \). Hence Theorem 3 is equivalent to

\[
V_n(\text{cap}_H(K_\lambda)) \geq (1 - \lambda)V_n(\text{cap}_H(K_1)) + \lambda V_n(\text{cap}_H(K_2))
\]

for \( \lambda \in (0, 1) \), with equality if and only if \( K_1 = K_2 \).

This last result itself follows from the following elementary result. Here “elementary” means that the most involved technique in its proof is Fubini theorem (see Chapter 50 in [BF87] or Lemma 3.30 in [BF17]).

**Lemma 5.** Let \( A_0 \) and \( A_1 \) be two convex bodies in \( \mathbb{R}^n \) contained in \( H^+ \), such that their orthogonal projection onto \( H \) is \( B \). Then, for \( \lambda \in [0, 1] \),

\[
V_n((1 - \lambda)A_0 + \lambda A_1) \geq (1 - \lambda)V_n(A_0) + \lambda V_n(A_1).
\]

Equality holds if and only if either \( A_0 = A_1 + U \) or \( A_1 = A_0 + U \), where \( U \) is some segment whose direction is orthogonal to \( H \).

In our case, if \( K \) is a \( C \)-coconvex body, then \( K \oplus U \) is a \( C \)-coconvex body if and only if \( U = \{0\} \).

**Remark 6.** In the classical convex bodies case, the Brunn–Minkowski inequality (saying that the \( n \)th-root of the volume of convex bodies is concave) follows from the more general result that the volume of convex bodies is log-concave. This is the genuine analogue of our situation, due to the following implications:

\[
f \text{concave} \implies f \text{ log-concave}
\]

\[
f \text{log convex} \implies f \text{ convex}
\]

If moreover \( f \) is positively homogenous of degree \( n \), we have:

\[
f \text{ log-concave} \implies f^{1/n} \text{ concave}
\]

\[
f \text{ convex} \implies f^{1/n} \text{ convex}
\]

**Remark 7.** Actually we didn’t use the fact that the convex set \( C \) is a cone, as the only thing that really matters is the stability of \( C \)-coconvex bodies under convex combinations. See e.g. [BF17] for an application to this more general situation. If \( C \) is a cone, the \( C \)-coconvex bodies are furthermore stable under positive homotheties and \( \oplus \), that allows to develop a mixed-volume theory for \( C \)-coconvex sets, see [Fil13, KT14, Sch17].

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