Some Cosmological Solutions of Einstein Equations

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Abstract. Contribution of Kurt Gödel to the development of general relativity along with its consequences is discussed. Particular attention is devoted to contemporary research which has been stimulated by Gödel’s paper on stationary dust–filled rotating universe. Contribution of Brno relativity group to the research is mentioned as well.

1. The Gödel solution

1.1. History
Kurt Gödel, during his staying at the Princeton university, grew together with A. Einstein, who inspired his interests in general relativity. In the period 1949 to 1952 Gödel published two papers from this field. In particular he was attracted by the notion of time and rotation in cosmology. The first paper was published in 1949, in occasion of Einstein 70–th [1]. The paper is very interesting from a number of reasons which will be discussed in next, and it has become a gold mine for many known relativists. Gödel interests in cosmology continued and in 1952 he published the other paper [2].

1.2. Basic properties
The Gödel solution describes a homogeneous and cylindrically symmetric spacetime, which represents an universe filled with a dust and with a negative cosmological constant. The solution turns out to be homogeneous, which means that it looks like identically at any spacetime point. Therefore the dust has in every point the same density. Moreover, the solution exhibits cylindrical symmetry.

1.3. Rotation of the universe
Gödel solution provides one of the first examples of rotating universe. The physical content is constituted by dust particles. They in fact stay, i.e. move only in time direction. But if one computes vorticity of these particles one finds that they are in an uniform rotation around the symmetry axis. Their trajectories are tilted. Since the whole universe is entirely filled with the dust particles, it can be said that it rotates. This means that there is a preferred direction. Because of its homogeneity, the universe rotates around each spacetime point. This provides an example that the Mach principle is not incorporated into general relativity. According to Mach principle, the inertial properties of the spacetime are determined by distant bodies. In Gödel solution, because the matter is homogeneous everywhere, it would mean that there should be no preferred direction which contradicts to the fact that the universe rigidly rotates.
1.4. Geodesics in Gödel spacetime
Let us briefly mention the geodesic behavior of the Gödel solution. Let us remind that a geodesic is roughly speaking a line in a curved manifold. As it is known two points in the Euclidean space can be joined by a line, which minimizes their distance. In a curved spacetime, on the other hand, the minimal distance of two causally connected points is always zero, but a geodesic in a curved spacetime maximize the distance of two points [3].

Among the geodesic, null geodesics are of particular interest, i.e. the trajectories of photons, lightrays. The behavior of the null geodesics in the Gödel solution is rather interesting. The null geodesics emitted from a point first expand until they reach certain radius. Then they begin to refocus [4].
1.5. **Closed timelike curves**

Probably the most interesting fact, which made it so famous, is that the solution admits possibility of time travel. Imagine a circle lying in the plane of constant time coordinate. If one looks at behavior of the light cones, one finds out that they tip over in the rotation direction. The larger is their distance from the rotation axis, the more significant is their slope. With increasing distance from the rotation axis they tip over until they reach the plane of constant constant time. Because the surface of the light cone is generated by light rays, we can see that this circle is closed null curve [4]. If the distance is even larger, one has a closed timelike curve.

If this curve is slightly deformed as it is depicted in the figure, then we still have timelike curve. Thus we may spiral against time into the past. In general for every point there exists a CTC which passes through it.
1.6. Predictability

Another thing is connected with appearance of CTC’s – predictability. We define the Cauchy development of a spacetime set as a region which uniquely determines the time evolution of the set. The boundary of Cauchy development is called Cauchy horizon [5, 6, 7].

It follows that Cauchy development in Gödel solution is empty. If we impose an initial data then it may happen that an information may appear which is not controlled by the initial data. Thus the Cauchy problem is not well posed. This means that we are unable to predict evolution of a system.

1.7. Reality of the solution

However, despite of many interesting properties, the solution from a number of reasons cannot represent our universe. First of all the infinite source does not correspond to what we observe. Furthermore the solution does not exhibit the red shift, which we observe, corresponding to the
universe expansion. And also the fact that our world is not crowded by groups of visitors from the future tells us that CTC plausibly are not present at least in our part of our universe.

2. Impact of the solution
The Gödel solution stimulated a great effort in clarifying causal relationships between various concepts of the geometry and topology of a spacetime. A number of results that was direct consequences are connected with names of R. Penrose, R. Geroch, S. Hawking and others. Of course I am not saying that all of these concepts would not be formulated without Gödel contribution, but certainly the Gödel work had pronounced influence on this. Let us briefly look at some of the most important.

2.1. CTC
Closed timelike curves (CTC’s) are something what we do not like in physics. But Gödel solution provides an example that the chronology violation is admitted by GR. Well, but Gödel solution, as we know, is not a good candidate for a description of our universe. It suffers from a number of
disasters which make it not realistic. Could the causality be violated in more realistic situations? Several topological concepts have been defined for classifying the level of the causality violation. For instance, it is not enough to forbidden just only the closed causal curves. We must also demand the causality of the spacetime to be stable against small metric perturbations, since we know that in our real universe the metric fulfills the uncertainty relations. So that it must not happen that the spacetime does not possess a CTC, but contains almost closed timelike curve. Altogether a few basic concepts have been developed, mainly in 60–th and 70–th, with increasing power, from the most mild (the spacetime does not contain any CTC) up to the most severe. This is the notion of global hyperbolicity, which means that Cauchy development coincides with the entire spacetime. In a globally hyperbolic spacetime the time evolution can be entirely determined by imposing initial data on the so called Cauchy surface.

But is the violation also permitted by the laws of nature? We do not know the definite answer yet, although we believe that the causality is prohibited. If not, than we have to encounter with causal pathologies which could be in principle resolved, but with a deep logical and philosophical impact. Therefore there has been a considerable effort to find such a mechanism that would enforce the causality at least in parts of the universe, which are governed by the currently known physical laws.

In 90’s the possibility of time travel was widely reexamined. First, in 1988 the well known theoretical physicist Kip Thorne with his collaborators published a wormhole solution that enables time travel [8, 9]. This work stimulated a series of papers treating this subject from different viewpoints. In 1992 Stephen Hawking suggested the Chronology protection conjecture, which states that the chronology can not be violated under any circumstances [10]. In attempts to prove the conjecture a considerable effort has been released. In particular this conjecture has been tested within framework of quantum fields extending in classical backgrounds, of course in the Gödel universe too [11].

But this is not end of the story. Gödel solution turned out to be extremely helpful in formulation of another possible chronology protection mechanism. It has to do with so called holographic principle, according to which the number of states available to an observer is given by its holographic screen. An holographic screen can be roughly understood like a horizon which bounds physically accessible degrees of freedom to the observer [12]. The idea was firstly put forward and demonstrated in the classical as well as higher–dimensional Gödel–type spacetimes. Petr Hořava, who by the way studied at the MU, found that the CTC’s are shielded by a holographic screen. There is still many questions remaining, but the principle itself seems to be underlying structure of our physical laws [13].

Figure 9. Holographic principle
2.2. Kinematical quantities

The Gödel solution also inspired relativists in a systematic study of so-called kinematical quantities. Every timelike or null congruence of curves is characterized by its tangent vector field and its derivatives projected onto orthogonal spacelike hypersurfaces. We can describe a congruence by its velocity, acceleration, twist, expansion, and shear [3, 4, 14].

Figure 10. Congruences

These quantities are related by several constraints, involving a curvature of the spacetime. In his original paper, Gödel had employed some basic ideas which then were developed by himself and others to a special branch – relativistic kinematics and geometry of congruences. Moreover, in the context of null geodesics, the subject was widely used in singularity theorems of Penrose and Hawking [3, 4].

2.3. Horizons

The previous ideas were successively extended. An extremely important role play boundaries between regions which are fully predictable and the ones which are not. We are led to the notion of the horizon [5, 6, 7]. Here we just comment on that although Gödel solution contains no horizons, it stimulated the research which led to formulation of Cauchy horizons.

3. Further generalization

We have seen that the Gödel solution have become a gold mine for many issues connected geometry and spacetime structure. Many scientists tried to generalize it in order to obtain answers to various questions. Particularly, the homogeneity of the solution and its chronological behavior have attracted attention over decades up to now.

3.1. Gödel–type spacetimes

The high degree of symmetry of the solution led to formulation of the Gödel–type spacetimes. If we demand a spacetime to be homogeneous and cylindrically symmetric, then the class of admissible metrics is reduced to very special form [15]. The Gödel solution is one of them. But the whole class is called Gödel–type. The Gödel–type metrics are direct product of a real line \( \mathbb{R} \) and a three–dimensional timelike manifold. If one employ the usual cylindrical coordinates, in which the symmetry of the solution is manifested, the general form of the metric can be written down as

\[
\begin{align*}
\text{d}s^2 = \text{d}t^2 + \frac{8\Omega}{m^2}\text{sh}^2\left(\frac{mr}{2}\right)\text{d}t\text{d}\varphi + \frac{4}{m^2}\text{sh}^2\left(\frac{mr}{2}\right)\left[\frac{4\Omega^2}{m^2} - 1\right] \text{sh}^2\left(\frac{mr}{2}\right) - 1 \text{d}\varphi^2 + \text{d}z^2 + \text{d}r^2,
\end{align*}
\]
with constants $m$ and $\Omega$. $\Omega$ corresponds to rotation of the spacetime, so called vorticity. By switching off the vorticity one gets the ordinary Minkowski spacetime. In fact just $\Omega$ is what responds for appearance of CTC’s [16]. The bigger is $\Omega$, the smaller is the radius necessary for CTC’s. On the other hand the constant $m$ corresponds to physical content of the spacetime. Namely it is proportional to the density.

3.2. Various additional matter fields
In order to get some new results the Gödel solution have been generalized in a number of ways. The first attempts concerned with adding various matter fields [17]. Naturally one may ask how the metric changes if the perfect fluid becomes charged. Most attempts was connected with the case, when the Lorentz force acting on the fluid particles, which carry the charge, vanishes [18, 19, 20]. From this and the symmetry consideration it follows that there is purely magnetic field, parallel to the rotation axis. It turns out that resulting solution again takes the Gödel–type form, but with $2\Omega^2 = m^2 + 2B^2$, i.e. the vorticity is shifted by the magnetic field $B$. Since $\Omega$ is increased, the chronology violation is even worse.

Furthermore a scalar field $\phi$ was added [17, 19, 21]. It could depend also on the longitudinal direction. This results in Gödel–type metric with modified vorticity, $2\Omega^2 = m^2 + 2B^2 - \alpha^2$, where $\alpha$ is constant, gradient of the scalar field along the rotation direction. Note, that now the vorticity is decreased by the scalar field, which in a sense compensate the electromagnetic field.

Generalizations were found when the source is constituted by a spinor field [22].

3.3. Higher order correction terms, higher dimensions
A number of attempts have been performed to get Gödel solution within framework of semiclassical quantum gravity. The motivation arises from the fact that the Lagrangian is enriched by a contribution of terms which are quadratic in the curvature. Let us remind that in classical relativity the Lagrangian coming from the gravitational field essentially equals to the curvature. Semiclassical theories contain in addition correction terms which are negligible in the classical regime, i.e. at low energies, but they become significant when the energies approach the Planck scale. And what is the role of Gödel solution? There was a hope that these terms will be simplified in the case of Gödel solution due to its homogeneity. Indeed this is the case and a few new solutions of the Gödel–type have been obtained [23, 24].

Another possible direction of generalization relies in searching for higher dimensional solutions with the same symmetries. Indeed, it is possible to find a solution within framework [13, 17].

4. Brno contribution to the Gödel heritage
4.1. Classical relativity
The research activities of members of the relativistic group at the Masaryk university in Brno include extension of the Gödel ideas to some more general situations. In fact we examined spacetimes with only three symmetries. These are cylindrical symmetry and stationarity. As the matter content we considered composition of the perfect fluid, a massless scalar field and an electromagnetic field. Thus we attempted to find a solution which is more general than the Gödel one, but which should reduce to it in a sufficient limit [20, 21, 25]. The goal have been accomplished. The solution obtained is an inhomogeneous generalization of the Gödel spacetime. Let us mention that the Lorentz force acting on the fluid particles generally does not vanish [18, 26]. One may ask whether the chronology violation is removed by the generalization. It turns out that it is true within certain range of the metric functions.

4.2. Quantum gravity
Quantum gravity is a dream of all theoretical physicists. Unfortunately we are still on the beginning of the journey to the complete quantum theory of gravitation. An efforts of the Brno
relativity group in connection with the Gödel solution consists of searching for exact solutions for modified field equations (which are generalizations of the Einstein equations). Namely we looked for solutions that follow from the so called Einstein–Maxwell–dilaton (EMD) gravity. EMD gravity is a low energy limit of string cosmology [27]. Currently the most promising candidate for complete quantum theory of gravitation is superstring theory. In contrast with former theories, superstring theory supposes that the particles are not point–like, but rather one–dimensional objects – strings with the lengths about ten to minus thirty four. We can construct cosmology which follows from superstring theory [28]. The task is extremely difficult, but partial steps have been carried out. One can obtain an effective theory which contains some additional fields and correction terms. For instance it contains correction terms quadratic in the curvature as we have already mentioned [29]. As well it has massless scalar field modes etc.

Following our approach connected with generalizations of Gödel solution to the inhomogeneous case, we have succeeded in finding a solution, which persists in so called string–inspired theory of gravity [30, 31]. Furthermore we intend to investigate the holographic principle within this solution in order to get an answer whether the holographic principle, which is a chronology protection mechanism, is valid or not. In any case, the Gödel solution again led us to new and interesting results.

To conclude the talk, I have to mention that the Gödel heritage will certainly give us a number of others important and unexpected results and maybe it will lead us as a gold thread to truth about our universe and the laws of nature.

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