CERTAIN UNIFIED INTEGRATION FORMULAS ASSOCIATED WITH GENERALIZED $k$-BESSEL FUNCTION

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Abstract. Our purpose in this present paper is to investigate generalized integration formulas containing the generalized $k$-Bessel function $W_{v,c}^k(z)$ to obtain the results in representation of Wright-type function. Also, we establish certain special cases of our main result.

1. INTRODUCTION

The generalized $k$-Bessel function defined in [11] as:

$$W_{v,c}^k(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk+v+k)n!} \left( \frac{z}{2} \right)^{2n+k},$$

(1.1)

where $k > 0$, $v > -1$, and $c \in \mathbb{R}$ and $\Gamma_k(z)$ is the $k$-gamma function defined in [5] as:

$$\Gamma_k(z) = \int_0^\infty t^{z-1}e^{-\frac{t}{k}} dt, z \in \mathbb{C}.$$  

(1.2)

By inspection the following relation holds:

$$\Gamma_k(z + k) = z\Gamma_k(z)$$  

(1.3)

and

$$\Gamma_k(z) = k^{\frac{k}{k}-1} \Gamma\left( \frac{z}{k} \right).$$  

(1.4)

In the same paper, the researchers also defined Pochhammer $k$-symbols which is defined as:

$$(x)_n = x(x+k) \cdots (x+(n-1)k), n \neq 0, n \in \mathbb{N}, (x)_0 = 1.$$  

The relation between Pochhammer $k$-symbols and $k$-gamma function is defined as:

$$(x)_n = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}.$$  

If $k \to 1$ and $c = 1$, then the generalized $k$-Bessel function defined in [21] reduces to the well known classical Bessel function $J_v$ defined in [7]. For further detail.
about \(k\)-Bessel function and its properties (see [8]-[10]).

The generalized hypergeometric function \(p \mathbf{F}_q(z)\) is defined in [6] as:

\[
p \mathbf{F}_q(z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!},
\]

where \(\alpha_i, \beta_j \in \mathbb{C}; i = 1, 2, \ldots, p, j = 1, 2, \ldots, q\) and \(b_j \neq 0, -1, -2, \ldots\) and \((z)_n\) is the Pochhammer symbols. The gamma function is defined as:

\[
\Gamma(\mu) = \int_0^\infty t^{\mu-1} e^{-t} dt, \mu \in \mathbb{C}, (1.6)
\]

\[
\Gamma(z + n) = z \Gamma(z), z \in \mathbb{C}, (1.7)
\]

and beta function is defined as:

\[
B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.
\]

The Wright type hypergeometric function is defined (see [16]-[18]) by the following series as:

\[
p \Psi_q(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n) n!} \frac{z^n}{n!} (1.9)
\]

where \(\beta_r\) and \(\mu_s\) are real positive numbers such that

\[
1 + \sum_{s=1}^{q} \beta_s - \sum_{r=1}^{p} \alpha_r > 0.
\]

Equation (3.1) differs from the generalized hypergeometric function \(p \mathbf{F}_q(z)\) defined (2.2) only by a constant multiplier. The generalized hypergeometric function \(p \mathbf{F}_q(z)\) is a special case of \(p \Psi_q(z)\) for \(A_i = B_j = 1\), where \(i = 1, 2, \ldots, p\) and \(j = 1, 2, \ldots, q\):

\[
\frac{1}{\prod_{j=1}^{q} \Gamma(\beta_j)} p \mathbf{F}_q \left[ \begin{array}{c} (\alpha_1), \cdots (\alpha_p) \\ (\beta_1), \cdots (\beta_q) \end{array} ; z \right] = \frac{1}{\prod_{i=1}^{p} \Gamma(\alpha_i)} p \Psi_q \left[ \begin{array}{c} (\alpha_i, 1)_{1,p} \\ (\beta_j, 1)_{1,q} \end{array} ; z \right].
\]

In this paper, we define a class of integral formulas which containing the generalized \(k\)-Bessel function as defined in (1.1). Also, we investigate some special cases as the
corollaries. For this continuation of our study, we recall the following result of Lavoie and Trottier [12].

\[
\int_0^1 z^{\alpha-1}(1-z)^{2\beta-1}(1-\frac{z}{3})^{2\alpha-1}(1-\frac{z}{4})^{\beta-1}dz = \left( \frac{2}{3} \right)^{2\alpha} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
\Re(\alpha) > 0, \Re(\beta) > 0. \tag{1.11}
\]

For various other investigations containing special function, the reader may refer to the recent work of researchers (see [3], [4], [13], [14], [15]).

2. Main Result

In this section, we establish two generalized integral formulas containing \(k\)-Bessel function defined in (1.1), which represented in terms of Wright-type function defined in (1.9) by inserting with the suitable argument defined in (1.11).

**Theorem 2.1.** For \(\lambda, \rho, v, c \in \mathbb{C}\) with \(\Re(\frac{v}{k}) > -1, \Re(\lambda + \rho) > 0, \Re(\lambda + \frac{v}{k}) > 0\) and \(z > 0\), then the following result holds:

\[
\int_0^1 z^{\lambda+\rho-1}(1-z)^{2\lambda-1}(1-\frac{z}{3})^{2(\lambda+\rho)-1}(1-\frac{z}{4})^{\lambda-1}W_{v,c}^k \left( \frac{\nu(1-\frac{v}{k})}{2} \left( 1-\frac{z}{4} \right)^{2} \right) dz
\]

\[= \left( \frac{\nu}{2} \right)^{\frac{v}{k}} \Gamma(\lambda + \rho) \left( \frac{\nu}{2} \right)^{2(\lambda+\rho)} \]

\[\times \left( k \right)^{\frac{v}{k}} \Psi_2 \left( \begin{array}{c} (\lambda + \frac{v}{k}, 2); \\
(\frac{v}{k} + 1, 1), (2\lambda + \frac{v}{k} + \rho, 2) \\
\end{array} \right) \left| -\frac{cy^2}{4k} \right|. \tag{2.1}
\]

**Proof.** Let \(S\) be the left hand side of (2.1) and applying (1.1) to the integrand of (2.1), we have

\[
S = \int_0^1 z^{\lambda+\rho-1}(1-z)^{2\lambda-1}(1-\frac{z}{3})^{2(\lambda+\rho)-1}(1-\frac{z}{4})^{\lambda-1} \]

\[\times \sum_{n=0}^\infty \frac{(-c)^n}{\Gamma(nk+v+k)n!} \left( \frac{\nu(1-\frac{v}{k})}{2} \left( 1-\frac{z}{4} \right)^{2} \right)^{2n+\frac{v}{k}} dz
\]

By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of theorem 2.1, we have

\[
S = \sum_{n=0}^\infty \frac{(-c)^n}{\Gamma(nk+v+k)n!} \left( \frac{\nu}{2} \right)^{2n+\frac{v}{k}} \]

\[\times \int_0^1 z^{\lambda+\rho-1}(1-z)^{2(\lambda+\frac{v}{k}+2n)-1}(1-\frac{z}{3})^{2(\lambda+\rho)-1}(1-\frac{z}{4})^{\lambda+\frac{v}{k}+2n-1} dz.
\]
By considering the assumption given in theorem 2.1 since \( \Re(\frac{\lambda}{4}) > 0, \Re(\lambda + \frac{\rho}{4} + 2n) > \Re(\lambda + \frac{\rho}{4}) > 0, \Re(\lambda + \rho) > 0, k > 0 \) and applying (1.11), we obtain
\[
S = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + \nu + k)n!} \left( \frac{y}{2} \right)^{2n + \frac{\nu}{2}} \frac{2^{(\lambda + \rho)} \Gamma(\lambda + \rho) \Gamma(\lambda + \frac{\nu}{4} + 2n)}{\Gamma(2\lambda + \rho + \frac{\nu}{4} + 2n)}.
\]
Using (1.4), we get
\[
S = \frac{\left( \frac{\nu}{4} \right)^{\frac{\nu}{4} + 2} \Gamma(\lambda + \rho) \Gamma(\lambda + \frac{\nu}{4} + 2n)}{k^{\frac{\nu}{4}}} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + \nu + k)n!} \left( \frac{y}{2} \right)^{2n} \frac{\Gamma(\lambda + \frac{\nu}{4} + 2n)}{\Gamma(2\lambda + \rho + \frac{\nu}{4} + 2n)}
\]
which upon using (1.3), we get the required result.

**Theorem 2.2.** For \( \lambda, \rho, v, c \in \mathbb{C} \) with \( \Re(\frac{\lambda}{4}) > -1, \Re(\lambda + \rho) > 0, \Re(\lambda + \frac{\rho}{4}) > 0 \) and \( z > 0 \), then the following result holds:
\[
\int_{0}^{1} z^{\lambda - 1} (1 - z)^{\nu(\lambda + \rho)^{-1} - 1} \left( 1 - \frac{z}{3} \right)^{2\lambda - 1} \left( 1 - \frac{z}{4} \right)^{(\lambda + \rho)^{-1}} dz
\]
\[
= \frac{\left( \frac{\nu}{4} \right)^{\frac{\nu}{4} + 2} \Gamma(\lambda + \rho) \Gamma(\lambda + \frac{\nu}{4} + 2n)}{k^{\frac{\nu}{4}}} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + \nu + k)n!} \left( \frac{y}{2} \right)^{2n} \left( \frac{v z (1 - \frac{\nu}{4})}{2} \right)^{2n + \frac{\nu}{2}}.
\]

**Proof.** Let \( \mathfrak{L} \) be the left hand side of (2.2) and applying (1.1) to the integrand of (2.1), we have
\[
\mathfrak{L} = \int_{0}^{1} z^{\lambda - 1} (1 - z)^{\nu(\lambda + \rho)^{-1} - 1} \left( 1 - \frac{z}{3} \right)^{2\lambda - 1} \left( 1 - \frac{z}{4} \right)^{(\lambda + \rho)^{-1}} dz
\]
\[
\times \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + \nu + k)n!} \left( \frac{y z (1 - \frac{\nu}{4})}{2} \right)^{2n + \frac{\nu}{2}}.
\]
By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of theorem 2.2, we have
\[
\mathfrak{L} = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + \nu + k)n!} \left( \frac{y}{2} \right)^{2n + \frac{\nu}{2}} \int_{0}^{1} z^{\lambda + \frac{\nu}{4} + 2n - 1} (1 - z)^{\nu(\lambda + \rho)^{-1} - 1} \left( 1 - \frac{z}{3} \right)^{2\lambda - 1} \left( 1 - \frac{z}{4} \right)^{(\lambda + \rho)^{-1}} dz.
\]
By considering the assumption given in theorem 2.2 since \( \Re(\frac{\lambda}{4}) > 0, \Re(\lambda + \frac{\rho}{4} + 2n) > \Re(\lambda + \frac{\rho}{4}) > 0, \Re(\lambda + \rho) > 0, k > 0 \) and applying (1.11), we obtain
\[
\mathfrak{L} = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + \nu + k)n!} \left( \frac{y}{2} \right)^{2n + \frac{\nu}{2}} \frac{2^{(\lambda + \rho) + 2n)} \Gamma(\lambda + \rho) \Gamma(\lambda + \frac{\nu}{4} + 2n)}{\Gamma(2\lambda + \rho + \frac{\nu}{4} + 2n)}.
\]
Using (1.4), we get
\[ S = \frac{(y^2)^{\nu} \Gamma(\nu + \rho + \frac{\nu}{2} + 2\nu)}{k^2} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\nu + 1 + n)n!} \frac{y^{2n}}{4^n k^n} \frac{\Gamma(\nu + \frac{\nu}{2} + 2n)}{\Gamma(2\nu + \rho + \frac{\nu}{2} + 2n)} \]
which upon using (1.9), we get the required result. □

3. Special Cases

In this section, we present the generalized form of classical and modified Bessel functions which are the special cases of $k$-Bessel function defined (1.1). Also, we prove two corollaries which are the special cases of obtained theorems in Section 2.

Case 1. If we set $c = 1$ in (1.1), then we get another definition of $k$-Bessel function. We call it the classical $k$-Bessel function
\[ J_v^k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{\nu})^{\nu + 2n}}{\Gamma(v + nk + k)n!} \] (3.1)

Case 2. If we set $c = -1$ in (1.1), then we get another definition of $k$-Bessel function. We call it the modified $k$-Bessel function
\[ I_v^k(z) = \sum_{n=0}^{\infty} \frac{(z)^{\nu + 2n}}{\Gamma(v + nk + k)n!} \] (3.2)

Corollary 3.1. Assume that the conditions of Theorem 2.1 are satisfied. Then the following integral formula holds:
\[
\int_0^1 z^{\lambda + \rho - 1} (1 - z)^{2\lambda - 1} (1 - \frac{z}{3})^{2(\lambda + \rho) - 1} (1 - \frac{z}{4})^{\lambda - 1} J_v^k \left( \frac{y}{4} \left( 1 - \frac{z}{4} \right) \left( 1 - z \right)^2 \right) \, dz
\]
\[ = \frac{(y^2)^{\nu} \Gamma(\lambda + \rho + \frac{\nu}{2} + 2\nu)}{k^2} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\nu + 1 + n)n!} \frac{y^{2n}}{4^n k^n} \frac{\Gamma(\nu + \frac{\nu}{2} + 2n)}{\Gamma(2\lambda + \rho + \frac{\nu}{2} + 2n)} \times 1_\Psi_2 \left[ \begin{array}{c} (\lambda + \frac{\nu}{2}, 2); \\ (\frac{\nu}{2} + 1, 1), (2\lambda + \frac{\nu}{2} + \rho, 2) \\ | - \frac{y^2}{4\pi} \end{array} \right]. \] (3.3)

Corollary 3.2. Assume that the conditions of Theorem 2.1 are satisfied. Then the following integral formula holds:
\[
\int_0^1 z^{\lambda + \rho - 1} (1 - z)^{2\lambda - 1} (1 - \frac{z}{3})^{2(\lambda + \rho) - 1} (1 - \frac{z}{4})^{\lambda - 1} J_v^k \left( \frac{y}{4} \left( 1 - \frac{z}{4} \right) \left( 1 - z \right)^2 \right) \, dz
\]
\[ = \frac{(y^2)^{\nu} \Gamma(\lambda + \rho + \frac{\nu}{2} + 2\nu)}{k^2} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\nu + 1 + n)n!} \frac{y^{2n}}{4^n k^n} \frac{\Gamma(\nu + \frac{\nu}{2} + 2n)}{\Gamma(2\lambda + \rho + \frac{\nu}{2} + 2n)} \times 1_\Psi_2 \left[ \begin{array}{c} (\lambda + \frac{\nu}{2}, 2); \\ (\frac{\nu}{2} + 1, 1), (2\lambda + \frac{\nu}{2} + \rho, 2) \\ | - \frac{y^2}{4\pi} \end{array} \right]. \] (3.4)
Corollary 3.3. Assume that the conditions of Theorem 2.2 are satisfied. Then the following integral formula holds:

\[
\int_0^1 z^{\lambda-1}(1-z)^{2(\lambda+\rho)-1}\left(1 - \frac{z}{3}\right)^{2\lambda-1}\left(1 - \frac{z}{4}\right)^{(\lambda+\rho)-1}J_v^k\left(\frac{yz\left(1 - \frac{z}{3}\right)^2}{2}\right)\,dz
\]

\[
= \frac{\left(\frac{y}{2}\right)^2 \Gamma(\lambda + \rho)(\frac{z}{3})^{2(\lambda+\rho)}}{k\pi} \times _1\Psi_2\begin{bmatrix}
\lambda + \frac{v}{k}, 2; \\
(v + 1, 1), (2\lambda + \frac{v}{k} + \rho, 2)
\end{bmatrix} | - \frac{4y^2}{\pi k} \end{align}

(3.5)

Corollary 3.4. Assume that the conditions of Theorem 2.2 are satisfied. Then the following integral formula holds:

\[
\int_0^1 z^{\lambda-1}(1-z)^{2(\lambda+\rho)-1}\left(1 - \frac{z}{3}\right)^{2\lambda-1}\left(1 - \frac{z}{4}\right)^{(\lambda+\rho)-1}I_v^k\left(\frac{yz\left(1 - \frac{z}{3}\right)^2}{2}\right)\,dz
\]

\[
= \frac{\left(\frac{y}{2}\right)^2 \Gamma(\lambda + \rho)(\frac{z}{3})^{2(\lambda+\rho)}}{k\pi} \times _1\Psi_2\begin{bmatrix}
\lambda + \frac{v}{k}, 2; \\
(v + 1, 1), (2\lambda + \frac{v}{k} + \rho, 2)
\end{bmatrix} | - \frac{4y^2}{\pi k} \end{align}

(3.6)

Remark. If we set \(k = 1\) in (3.1) to (3.6), then we get the well known result for case 1 (see [1]) and some new result for the familiar function defined in [11, 2, 19].

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