On the anyon description of the Laughlin hole states

Heidi Kjønsberg†‡ and Jon Magne Leinaas⋆

Centre for Advanced Study
at the Norwegian Academy of Science and Letters,
P.O. Box 7606 Skillebekk, N-0205 Oslo, Norway
and
Department of Physics, University of Oslo
P.O. Box 1048 Blindern, N-0316 Oslo, Norway

ABSTRACT

We examine the anyon representation of the Laughlin quasi-holes, in particular the one-dimensional, algebraic aspects of the representation. For the cases of one and two quasi-holes an explicit mapping to anyon systems is given, and the connection between the hole-states and coherent states of the fundamental algebras of observables is examined. The quasi-electron case is discussed more briefly, and some remaining questions are pointed out.
1 Introduction

In Laughlin’s theory of the fractional quantum Hall effect [1] the quasi-holes and quasi-electrons satisfy fractional statistics [2]. They are anyons in a strong magnetic field. The fractional statistics property of these excitations was first suggested by Haldane [3] and Halperin [4] in the construction of the hierarchy of Hall states. It was demonstrated more directly by Arovas, Schrieffer and Wilczek, who considered a pair of quasi-holes encircling each other and interpreted the two-particle contribution to the Berry phase as a statistics phase [5]. Laughlin later suggested an explicit particle representation of the quasi-holes and quasi-electrons where these have anyonic properties [6, 7].

Due to the strong magnetic field, the quasi-holes and quasi-electrons effectively behave like particles in one dimension. The relevant coordinates are the particle coordinates projected to the lowest Landau level, which are the guiding center coordinates of the particles. These do not commute, and the physical plane gets the character of a two-dimensional phase space. In fact the quasi-holes and quasi-electrons are more strictly one-dimensional than the electrons of the system, since excitations of these to higher Landau levels are missing. This suggests that the hole and particle excitations may be characterized by one-dimensional fractional statistics as an alternative to the two-dimensional anyon statistics referred to above.

An algebraic approach to fractional statistics in one dimension was introduced in ref.[8]. For a single particle the Hilbert space defines an irreducible representation of a Heisenberg-Weyl (HW) algebra of observables, the algebra defined by the coordinate and momentum. For two identical particles the fundamental algebra of observables does not consist of two independent HW algebras. This is due to the symmetrization postulate for the observables. Instead the algebra may be viewed as consisting of a HW algebra for the center-of-mass coordinates and an $su(1, 1)$ algebra for the relative coordinates. The latter is quadratic in the coordinates. The irreducible representations of the $su(1, 1)$ algebra are characterized by a real parameter which is interpreted as the one-dimensional statistics parameter.

In ref.[9] this one-dimensional algebraic approach was applied to a two-anyon system in the lowest Landau level. It was shown explicitly how to construct the raising and lowering operators of the $su(1, 1)$ algebra and a linear relation was found between the one-dimensional statistics parameter and the (two-dimensional) anyon parameter. For bosons and fermions the generators of the $su(1, 1)$ were quadratic in anyon coordinates and derivatives, but in the general case the expressions were more complicated.

The purpose of this paper is to examine the anyon representation of the quasi-holes, in particular the one-dimensional aspects, somewhat closer. We restrict the discussion mainly to one and two quasi-holes, although with some comments also about the quasi-electron case. We follow the ideas of Laughlin and construct an explicit anyon representation of the quasi-holes. We do it in a rather detailed way and focus special attention on the algebraic properties of the one-dimensional system. A particular question is whether the quasi-holes can be viewed as coherent states of the fundamental (one-dimensional) algebra of observables. The discussion is closely related to that of ref.[9], but now with application...
to quasi-holes of the quantum Hall system rather than to a general anyon system.

2 One particle — one quasi-hole.

In this section we consider the case of a single quasi-hole and its relation to the single particle in a strong magnetic field. We first have a brief discussion of the one particle case and in the next subsection consider the explicit mapping between the quasi-hole of the Laughlin theory and the single particle system.

2.1 The one particle case.

For a single, planar particle in a magnetic field, the Hamiltonian and conserved angular momentum are

\[
H = \frac{m}{2} \mathbf{v}^2, \quad L = m (\mathbf{r} \times \mathbf{v})_z + \frac{1}{2} m \omega r^2; \quad m \mathbf{v} = \mathbf{p} - \frac{q}{e} \mathbf{A}.
\] (1)

The charge \( q \) is assumed to be positive, and \( \mathbf{B} = B \mathbf{e}_z \) with \( B > 0 \), so the cyclotron frequency \( \omega = \frac{qB}{mc} \) is positive and the magnetic length is \( l_B = (\frac{hc}{qB})^{1/2} \). In terms of the guiding center coordinate \( \mathbf{R} \) and the relative coordinate \( \mathbf{r}_{rel} \) (using the notation \( \mathbf{v}^* = (-v_y, v_x) \))

\[
\mathbf{R} = \mathbf{r} - \frac{1}{\omega} \mathbf{v}^*, \quad \mathbf{r}_{rel} = -\frac{1}{\omega} \mathbf{v}^*,
\] (2)

a convenient set of variables is defined by the dimensionless operators

\[
a = \frac{1}{\sqrt{2l_B}} (X - iY), \quad b = \frac{1}{\sqrt{2l_B}} (x_{rel} + iy_{rel}),
\] (3)

and their adjoints. The Hamiltonian and angular momentum are then given by the well known gauge independent expressions

\[
H = \hbar \omega (b^\dagger b + \frac{1}{2}), \quad L = \hbar (a^\dagger a - b^\dagger b).
\] (4)

The only non-vanishing commutation relations are

\[
[b, b^\dagger] = [a, a^\dagger] = 1,
\] (5)

and the system thus consists of two uncoupled one-dimensional harmonic oscillators. The harmonic oscillator states \( |l, n\rangle \), \( n, l = 0, 1, ..., \) correspond to energies \( E_n = \hbar \omega (n + \frac{1}{2}) \) and angular momentum eigenvalues \( L = (l - n) \) (measured in units of \( \hbar \)), and the variable \( n \) then specifies a given Landau level.

In each specific energy level (Landau level) the fast relative coordinates are frozen out, and the resulting system has one-dimensional dynamics. The corresponding state space
yields (in the unbounded case) a representation of the Heisenberg-Weyl algebra \{a, a^\dagger, 1\}. The coherent states of this algebra are defined by

\[ a \mid \alpha^*, n \rangle = \alpha^* \mid \alpha^*, n \rangle. \quad (6) \]

The state can be written as

\[ \mid \alpha^*, n \rangle = e^{-\frac{1}{2} \alpha^*} \sum_{l=0}^{\infty} \frac{(\alpha^*)^l}{\sqrt{l!}} \mid l, n \rangle = e^{\frac{1}{2} \alpha^* + \alpha^* a^\dagger} \mid 0, n \rangle. \quad (7) \]

One can show that for each specific Landau level \( n \) the uncertainty in position generally must obey \( \Delta x \Delta y \geq \frac{l_B^2}{B} (n + 1) \). The coherent states satisfy the equality sign in this relation, and in this sense they are maximally localized in the physical plane.

Introducing the complex particle coordinate \( z = \frac{1}{\sqrt{2l_B}} (x + iy) \), the coordinate representation (using symmetric gauge \( A = \frac{1}{2} B \times r \)) of the angular momentum states of the lowest Landau level (LLL) is

\[ \langle z, z^* \mid l \rangle \equiv \langle z, z^* \mid l, n = 0 \rangle = \frac{1}{\sqrt{\pi l!}} z^r e^{-\frac{1}{2} zz^*}. \quad (8) \]

Comparing with (7), we see that the coherent states can be identified, up to a normalization factor, with the projected position eigenstates of the full system,

\[ \mid z^*, n = 0 \rangle = \sqrt{\pi} P \mid z^*, z \rangle, \quad P = \sum_l \mid l \rangle \langle l \mid. \quad (9) \]

Thus, for a general wave function of LLL, which has the form

\[ \langle z, z^* \mid \psi \rangle = \psi(z) e^{-\frac{1}{2} zz^*} \quad (10) \]

with \( \psi(z) \) as an analytic function of \( z \), the variable \( z \) can be identified either as the position variable projected on LLL or, equivalently, as the coherent state variable of the HW algebra.

When acting only on the analytic part of the wave functions, the guiding center operators \( a, a^\dagger \) are in the LLL represented by

\[ a^\dagger = z, \quad a = \frac{d}{dz}, \quad (11) \]

and a Fock-Bargmann representation \([10, 11]\) is obtained.

As emphasized in ref.\([12]\), restriction to the lowest Landau level involves a projection for the observables with an associated operator ordering rule. This is because although the operators \( z \) and \( z^* \) commute, the same is not true for the projected operators. In fact, the operator ordering rule is easily obtained by an algebraic approach. According to (2) and (3), the particle coordinate operators are given by

\[ z = a^\dagger - b, \quad z^* = a - b^\dagger. \quad (12) \]
Since $b$ annihilates any state in the LLL, while $b^\dagger$ lifts the state up to the next Landau level we have

$$bP = Pb^\dagger = 0 \quad (13)$$

with $P$ as the projection on LLL. $P$ also commutes with $a$ and $a^\dagger$, and this implies for the projection of a product of powers of $z$ and $z^*$,

$$Pz^k(z^*)^lP = P(a - b^\dagger)(a^\dagger - b)^kP = a^l(a^\dagger)^k. \quad (14)$$

The operator ordering is unique, and the rule found in ref.\[12\] is the coordinate representation of this expression.

So far we have implicitly assumed that the particle is allowed to move in a plane with no boundary, in which case each Landau level is infinitely degenerated and yields a representation of the Heisenberg-Weyl algebra. For the comparison with the quasi-hole case it may be of interest to consider also a case where there is a boundary to the system. The simplest case is a circular boundary, which can be obtained by introducing a limit to the angular momentum number, $l \leq l_{\text{max}} = \frac{\Phi}{\Phi_0}$, where $\Phi$ is the total magnetic flux and $\Phi_0 = \frac{\hbar e}{q}$ is the flux quantum. This implies that the raising operator $a^\dagger$ no longer leaves the state space invariant. However, an operator annihilating the uppermost state may be defined as

$$d_+ = \sqrt{l_{\text{max}} - L + 1} a^\dagger. \quad (15)$$

With the adjoint operator $d_- \equiv d_+^\dagger$ and $d \equiv L - \frac{l_{\text{max}}}{2}$ we then obtain an operator set \{d_+, d_-, d\} spanning an $su(2)$ algebra. Thus a consequence of limiting the area is that the state space no longer defines an irreducible representation of the HW algebra, but instead an irreducible representation of this $su(2)$ algebra. The dimension of the representation is determined by the area of the system. The finite value of $l_{\text{max}}$ also implies that the coherent states must be modified. An obvious modification is to use the spin coherent states of the $su(2)$ algebra. Following ref.\[10\] these states are given by

$$|\eta^*\rangle_{sc} = (1 + |\eta|^2)^{-\frac{l_{\text{max}}}{2}} e^{\eta^*d_+} |l = 0\rangle = (1 + |\eta|^2)^{-\frac{l_{\text{max}}}{2}} \sum_{l=0}^{l_{\text{max}}} (\eta^*)^l \sqrt{\frac{l_{\text{max}}!}{l!(l_{\text{max}} - l)!}} |l\rangle, \quad (16)$$

and satisfy the equation

$$(d_- + 2\eta^*d - (\eta^*)^2d_+) |\eta^*\rangle_{sc} = 0. \quad (17)$$

Now choose $\alpha = \sqrt{l_{\text{max}}} \eta$, and hold $\alpha$ fixed while the area of the system grows. Then the spin coherent state $|\eta^*\rangle_{sc}$ evolves into the HW-algebra coherent state $|\alpha^*\rangle$ as the area goes
We also have

$$\lim_{l_{\text{max}} \to \infty} \frac{1}{\sqrt{l_{\text{max}}}} d_+ |l\rangle = \sqrt{l+1} |l+1\rangle,$$

$$\lim_{l_{\text{max}} \to \infty} \frac{1}{\sqrt{l_{\text{max}}}} d_- |l\rangle = \sqrt{l} |l-1\rangle,$$

$$\lim_{l_{\text{max}} \to \infty} \frac{1}{l_{\text{max}}^2} d |l\rangle = |l\rangle.$$ (18)

This shows explicitly how the Heisenberg-Weyl algebra is reestablished for the infinite system \([10]\). The correspondence also implies that the localization property of the HW-algebra coherent state is taken well care of by the \(su(2)\) algebra spin coherent state as long as the particle is not close to the boundary of the system, \(|\eta| \ll 1\).

One should note that the coherent states defined in this way no longer are identical to the position eigenstates projected to the restricted area of the lowest Landau level. Such a projection is obtained from the expression of the HW coherent states (7) by limiting the sum to \(l \leq l_{\text{max}}\).

### 2.2 One quasi-hole.

Consider \(N\) electrons in a magnetic field \(B = -Be_z\) \((B > 0)\), and assume that the system contains a single quasi-hole. The filling factor is \(\nu = 1/m\) where \(m\) is odd. The symmetric gauge is used. Laughlin’s wave function for this system is \([4]\)

$$\Psi_{m,z_0}(z_1, \ldots, z_N) = \psi_{z_0}(z_1, \ldots, z_N) \phi_m(z_1, \ldots, z_N),$$ (19)

where

$$\phi_m(z_1, \ldots, z_N) = e^{-\frac{1}{2} \sum_{i=1}^N z_i z_i^*} \prod_{i<j} (z_i - z_j)^m$$ (20)

is the ground state wave function and

$$\psi_{z_0}(z_1, \ldots, z_N) = \prod_{i=1}^N (z_i - z_0)$$ (21)

is the factor which pushes the electrons away from a small area around the point \(z_0\). The electron charge is by convention \(-e\). The ground state \(\phi_m\) corresponds, for finite \(N\), to an electron ”droplet” of constant density \(1/(2\pi ml_B^2)\) and of circular shape around the origin. The radius is \(l_B \sqrt{2mN}\).

For the discussion of the one hole-states it is sufficient to consider only the \(\psi_{z_0}\)-part of the wave function. The ground state factor \(\phi_m\) does not depend on the position \(z_0\) of the quasi-hole and can be absorbed in the integration measure. Notice that \(\psi_{z_0}\) does not depend on the complex conjugate coordinates, it is an analytic function of each of the \(z_i\)’s.

\(\psi_{z_0}\) will now be expanded in terms of eigenfunctions of the angular momentum. These states define a complete set of orthogonal states for the state space of the \(N\)-electron
system with a single quasi-hole. When the angular momentum operator is pulled through the ground state $\phi_m$ it picks up a term $m$ from each pair of electrons, and since $\psi_{z_0}$ is independent of the complex conjugates the angular momentum operator now is

$$L = \sum_{i=1}^{N} z_i \partial_{z_i} + \frac{m}{2} N(N - 1).$$  \tag{22}$$

The expansion is done by use of the so-called elementary symmetric polynomials $S_k$. In fact, $\psi_{z_0}$ can be viewed as defining these functions as follows [13]:

$$\psi_{z_0}(z_1, \ldots, z_N) = \prod_{i=1}^{N} (z_i - z_0) = \sum_{k=0}^{N} (-z_0)^N S_k(z_1, \ldots, z_N).$$  \tag{23}$$

But this expression also is an expansion of the wave function $\psi_{z_0}$ in terms of angular momentum eigenstates. This is realized by noting that the elementary symmetric polynomials are homogeneous of degree $k$ in $z_1, \ldots, z_N$. Hence,

$$LS_k(z_1, \ldots, z_N) = \left( \sum_{i=1}^{N} z_i \partial_{z_i} + \frac{m}{2} N(N - 1) \right) S_k(z_1, \ldots, z_N)$$

$$= \left( k + \frac{m}{2} N(N - 1) \right) S_k(z_1, \ldots, z_N).$$  \tag{24}$$

The state space for the system of $N$ electrons with a single quasi-hole may then be defined as being the function space spanned by the set of $N + 1$ states.

$$\{ S_k(z_1, \ldots, z_N) \}_{k=0}^{N}. \tag{25}$$

(Note that $S_0$ is identical to the ground state of a system with $N$ electrons.)

The functions $S_k$ are orthogonal since they all have different eigenvalues for the total angular momentum. However, they are not normalized. The scalar product is

$$(S_k, S_{k'}) = \int d\mu(z) \left( S_k(z_1, \ldots, z_N) \right)^* S_{k'}(z_1, \ldots, z_N),$$  \tag{26}$$

where the integration measure is

$$d\mu(z) = d^2z_1 \cdots d^2z_N \prod_{i<j}^{N} |z_i - z_j|^{2m} e^{-\sum_{i=1}^{N} |z_i|^2}.$$  \tag{27}$$

For arbitrary $m$ this scalar product indeed is troublesome because of the product of all pairs, $\prod |z_i - z_j|^{2m}$. In fact, only for $m = 1$ we are able to calculate the norm of the angular momentum eigenfunctions $S_k$ exactly. This case corresponds to fermionic quasi-holes. On the other hand, for arbitrary values of $m$ it is in the limit of large $N$ possible to use the plasma analogy to find approximate solutions for the relative normalization.
Let us briefly review how the plasma analogy [6] is used to determine the normalization factor $N_{z_0}$ for the wave function $\Psi_{m,z_0}(z_1,\ldots,z_N^*)$ (19). The normalization integral then is interpreted as the classical probability distribution for a charge (the quasi-hole) in a plasma of $N$ other charges (the electrons). The integrand is written in the form $|N_{z_0}\Psi_{m,z_0}|^2 \equiv e^{-\beta U(z,z^*,z_0,z_0^*)}$ where $\beta$ is some constant (the inverse temperature) and $U$ is interpreted as the potential energy of the classical system of charges. The potential $U$ is dictated by the form of the hole state (19). The $N$ charges have values $q = \sqrt{m/\beta}$ and interact with a logarithmic Coulomb potential. They also have a Coulomb interaction with a uniform background charge density $\sigma = -1/(\pi \sqrt{m\beta})$. The additional charge, of value $\bar{q} = q/m$ (the hole), has a logarithmic Coulomb interaction with the $N$ other charges. If also this charge has a Coulomb interaction with the background charge, the probability distribution becomes independent of the position of the charge due to screening by the $N$ free charges. This is interpreted as the condition for the normalization integral to be independent of $z_0$.

Thus, the normalization factor is determined by the form of this interaction and is (up to a constant factor)

$$N_{z_0} = e^{-\frac{1}{2m}|z_0|^2}. \quad (28)$$

The validity of this method to determine the normalization factor rests upon the assumption that the number $N$ of electrons is a large number and the main contribution to the integral is due to the classical configuration where the $N$ identical particles form a droplet of uniform charge density. One generally expects corrections when $\bar{q}$ is near the boundary. Now, holding both the filling fraction and the magnetic field at constant values, an increase in the number of electrons must be accompanied by an increase in area. Hence, the limit $N \to \infty$ effectively means that boundary effects in the quantum Hall system are neglected.

With the normalization factor determined we can use the expansion

$$\int d^2z_1 \cdots d^2z_N \ |N_{z_0}\Psi_{m,z_0}|^2 = e^{-\frac{1}{2m}|z_0|^2} \sum_{k=0}^{N} (z_0 z_0^*)^k (S_{N-k}, S_{N-k}),$$

to find the relative norm of the angular momentum eigenfunctions. The result is

$$\frac{(S_{k+1}, S_{k+1})}{(S_{k}, S_{k})} = m \ (N - k) \quad (N \gg 1). \quad (29)$$

It is of interest to compare this result with the exact result which can be obtained for $m = 1$. The state $S_k$ then is an $N$-electron state with all angular momentum states $l = 0, 1, \ldots, N$ filled except for a hole in the state $l = N - k$. The normalization integral then is simply equal to the product of the normalization integrals of the single-particle states of the filled levels. They are of the form

$$\int d^2z \ e^{-z z^*} (z z^*)^l = \pi l!.$$
The relative norm of two elementary symmetric polynomials \((26)\) is given by the ratio between two such integrals. In particular we have

\[
\frac{(S_{k+1}, S_{k+1})}{(S_k, S_k)} = (N - k) \quad (m = 1).
\]

(30)

This shows that the approximate result \((29)\) derived from the plasma analogy is in fact exact for \(m = 1\).

Before proceeding we introduce an abstract notation for the hole wave functions and let the normalized ket \(|k\rangle\) correspond to the angular momentum wave function \(S_{N-k}\). Thus, the electron coordinate representation of the state is \((20, 23, 26)\)

\[
\langle z_1, \ldots, z_N^* |k\rangle = \frac{1}{(S_{N-k}, S_{N-k})^{1/2}} S_{N-k}(z_1, \ldots, z_N) \phi_m(z_1, \ldots, z_N^*). \tag{31}\]

(Note that our notation for the states emphasizes the role of the quasi-hole. The quantum number \(k\) of the ket \(|k\rangle\) in the case \(m = 1\) corresponds to a hole in the single-electron state \(z_k^* e^{-\frac{1}{2} |z|^2}\). When we later on let \(N \to \infty\) while holding \(k\) fixed, this means that the hole is held at a fixed position in the plane.) Similarly, the quasi-hole wave function \((19)\) is represented by the abstract state \(|z_0\rangle_{nn}\)

\[
\langle z_1, \ldots, z_N^* |z_0\rangle_{nn} = \Psi_{z_0,m}(z_1, \ldots, z_N^*). \tag{32}\]

The subscript ‘nn’ indicates that the state is not normalized, and in this abstract notation we have

\[
|z_0\rangle_{nn} = \sum_{k=0}^{N} (-z_0)^k (S_{N-k}, S_{N-k})^{1/2} |k\rangle. \tag{33}\]

A unitary transformation \(U\) from the quasi-hole system onto the single particle system can now be defined. Motivated by the physical pictures for \(m = 1\) we define for arbitrary \(m\)

\[
U |k\rangle_{qh} \equiv |k\rangle_{sp}, \tag{34}\]

where we have introduced the subscript \(qh\) for the hole states and \(sp\) for the single-particle states of the previous subsection. Transforming the normalized quasi-hole state \(|z_0\rangle\) we then obtain

\[
U |z_0\rangle = \frac{1}{nn\langle z_0 | z_0\rangle_{nn}^{1/2}} U |z_0\rangle_{nn}
\]

\[
= \frac{1}{nn\langle z_0 | z_0\rangle_{nn}^{1/2}} \sum_{k=0}^{N} (-z_0)^k (S_{N-k}, S_{N-k})^{1/2} |k\rangle_{sp}
\]

\[
= \left( \sum_{k=0}^{N} (z_0 z_0^*)^k \frac{(S_{N-k}, S_{N-k})}{(S_N, S_N)} \right)^{-1/2} \sum_{k=0}^{N} (-z_0)^k \frac{(S_{N-k}, S_{N-k})^{1/2}}{(S_N, S_N)^{1/2}} |k\rangle_{sp}
\]

9
\[
\approx \left( \sum_{k=0}^{N} \frac{(z_0 z_0^*)^k}{m^{k} k!} \right)^{-1/2} \sum_{k=0}^{N} \frac{(-\frac{z_0}{\sqrt{m}})^k}{\sqrt{k!}} |k\rangle_{sp} \quad (N \gg 1)
\]
\[
\rightarrow |\alpha^* = -\frac{z_0}{\sqrt{m}}\rangle_{sp}, \quad (N \to \infty). \tag{35}
\]

This shows that when boundary effects in the Hall system are neglected and the normalization found by the plasma analogy is applied, the quasi-hole state \(|z_0\rangle\) is mapped into a HW algebra coherent state of the single particle system. In this sense the system of \(N\) electrons with one (quasi)hole is effectively a one-dimensional single particle system \([9]\).

The relation \(\alpha^* = -\frac{z_0}{\sqrt{m}}\) between the coordinates can be understood if we take into mind the physical picture of the two systems. The change of sign and complex conjugation is related to the change in sign of the charge of the hole relative to that of the electrons, and the factor \(1/\sqrt{m}\), which means an effectively larger magnetic length for the holes, determines the size of the hole charge relative to that of the electrons.

Returning to the case of large, but finite \(N\), we can also find the expressions for the \(su(2)\)-operators of the one-hole system. They are related to the \(su(2)\) operators of the single-particle case by the unitary transformation \(U\). It is convenient to introduce the following raising and lowering operators which leave the \((N+1)\)-dimensional one-hole space invariant,

\[
D'_+ = \sum_{i=1}^{N} \partial_{z_i}, \quad D'_- = \sum_{i=1}^{N} (z_i - z_i^2 \partial_{z_i}). \tag{36}
\]

Together with the shifted angular momentum operator

\[
D' = -L + \frac{m}{2} N(N - 1) + \frac{N}{2}
\]
\[
= -\sum_{i=1}^{N} z_i \partial_{z_i} + \frac{N}{2}, \tag{37}
\]

they define an \(su(2)\) algebra. However, this is a non-hermitian algebra. The correctly normalized operators, which correspond to the operators \(d_{\pm},d\) in the single-particle case are

\[
D = D', \quad D_- = \frac{1}{\sqrt{m}} D'_- \left( \frac{N}{2} - D' + 1 \right)^{-1/2},
\]
\[
D_+ \equiv D'_+ = \sqrt{m} \left( \frac{N}{2} - D' + 1 \right)^{1/2} D'_+ \tag{38}
\]

and these define a standard representation of the \(su(2)\) generators.

If the number of electrons is increased while we consider the action of \(D_+, D_-\) and \(D\) on some state \(|k\rangle\) with fixed value \(k\), we obtain

\[
\lim_{N \to \infty} \frac{1}{\sqrt{N}} D_- |k\rangle = \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sqrt{k(N-k+1)} |k-1\rangle
\]
\[
\lim_{N \to \infty} \frac{1}{\sqrt{N}} D_+ |k\rangle = \sqrt{k+1} |k+1\rangle,
\]
\[
\lim_{N \to \infty} \frac{2}{N} D |k\rangle = - |k\rangle.
\] (39)

This shows how the HW algebra is obtained from the \(su(2)\) algebra when \(N \to \infty\).

The raising and lowering operators have a rather complicated form when expressed as electron operators. However, a hole-coordinate representation can be defined which is identical to a coherent state representation of the HW algebra for \(N \to \infty\). For an arbitrary state \(|\psi\rangle = \sum c_k |k\rangle\) this representation is defined by \(\psi(-\frac{z_0^*}{\sqrt{m}}) \equiv e^{\frac{1}{2m}|z_0|^2} \langle z_0 | \psi \rangle\). In this representation the lowering operator \(A = \lim_{N \to \infty} \frac{1}{\sqrt{N}} D_+\) and its hermitian conjugate gets a simple form
\[
\langle z_0 | A | \psi \rangle = e^{-\frac{1}{2m}|z_0|^2} \frac{d}{d(-\frac{z_0^*}{\sqrt{m}})} \psi(-\frac{z_0^*}{\sqrt{m}}),
\]
\[
\langle z_0 | A^\dagger | \psi \rangle = e^{-\frac{1}{2m}|z_0|^2} (-\frac{z_0^*}{\sqrt{m}}) \psi(-\frac{z_0^*}{\sqrt{m}}).
\] (40)

This is the Fock-Bargmann representation \[11\] for the system of \(N\) electrons with a quasi-hole.

The mapping (35) between the quasi-hole states and the coherent states of the single particle system was based on the (approximate) form of the norm of the elementary symmetric polynomials obtained by use of the plasma analogy. Also the form of the hermitian \(su(2)\) operators (38) was derived by use of this expression for the norm. However, we can now check directly the action of these operators on the quasi-hole states. The lowering operator of the HW algebra, when applied to the quasi-hole state gives
\[
\frac{1}{\sqrt{N}} D_+ \sum_{k=0}^{N} (-z_0)^k S_{N-k} = - \frac{z_0}{\sqrt{m}} \sum_{k=0}^{N-1} (-z_0)^k \sqrt{\frac{N-k}{N}} S_{N-k}.
\] (41)

There is a correction factor \(\sqrt{\frac{N-k}{N}}\) in this expression, but this tends to 1 for large \(N\) (if the hole stays at finite distance from the origin when \(N\) increases). Thus, the interpretation of the quasi-hole states as coherent states of a HW algebra is confirmed without the use of the detailed form of the norm of the polynomials \(S_k\). However, to check the hermitian properties of the algebra the scalar product is needed, and for this the plasma analogy has to be applied.
3 Two particles.

We now turn to the case of two quasi-holes. It is useful to start by recalling some properties of the two-anyon system, where we also include some comments on the distinction between the anyon coordinate and the coherent state coordinate. This distinction will be important when we in the next subsection discuss the two-hole system and its relation to the anyon system. An explicit mapping is performed.

3.1 The two anyon system.

Consider two anyons in a magnetic field. For a homogeneous field the center of mass motion can be separated out, and the hamiltonian for the relative coordinate of the two particles is identical to the hamiltonian \( H \). The parameters \( m \) and \( q \) then are reduced mass and charge, respectively. Denoting the anyon parameter by \( \nu \), the wave functions now should satisfy the symmetry condition \( \psi(e^{i\pi}z) = e^{i\nu\pi}\psi(z) \), where \( z \) is the complex, dimensionless, relative coordinate. The solution of the energy problem then shows that there is a (generalized) lowest Landau level where the energy is independent of \( \nu \), and the function space is spanned by the orthonormal angular momentum functions

\[
\psi_{l,\nu}(z, z^*) = \frac{1}{\sqrt{\pi \Gamma(2l + \nu + 1)}} z^{2l+\nu} e^{-\frac{1}{2}zz^*}.
\]  

(42)

It is convenient to restrict \( \nu \) to the interval \( 0 \leq \nu < 2 \), which implies \( l = 0, 1, \ldots \). Due to the anyonic properties, there is now a shift in the eigenvalues of the (relative) angular momentum,

\[
L\psi_{l,\nu}(z, z^*) = (2l + \nu)\psi_{l,\nu}(z, z^*).
\]  

(43)

In the single-particle case the projection on the lowest Landau level can be viewed as a dimensional reduction of the system. Thus, the hermitian combinations \( a + a^\dagger \) and \( i(a - a^\dagger) \) are canonically conjugate and can be viewed as the phase space variables for a particle moving in one dimension. The Heisenberg-Weyl algebra generated by these operators is the fundamental algebra of observables of the system.

For two particles in one dimension, the relative coordinate and momentum are not true observables since they do not respect the symmetry of the system. However, the quadratic combinations do, and they span the \( su(1,1) \) algebra, which can now be viewed as the fundamental algebra of observables. The irreducible representations are labelled by a real parameter which is interpreted as the one-dimensional statistics parameter. In ref. [9] it was shown how to identify this \( su(1,1) \) algebra for two anyons in the lowest Landau level. The angular momentum operator \( L \) is, up to a constant, identical to the compact generator of the algebra, and the shift of the eigenvalues is identified with the one-dimensional statistics parameter. In this way the algebraic approach to the dimensional reduction can be carried out also for the two-anyon system in the lowest Landau level. The result is a simple linear relation between the one-dimensional statistics parameter and the (two-dimensional) anyon parameter.
We let the set \( \{B, B_+, B_-\} \) span the \( su(1,1) \) algebra, with \( B \) as the compact generator, \( B_+ \) and \( B_- \) as the (hermitian conjugate) raising and lowering operators. Represented in anyon coordinates the operators are given by ref.\[9\],

\[
B_+ = \frac{1}{2}z^2 M, \quad B_- = \frac{1}{2}M \frac{d^2}{dz^2}, \quad B = \frac{1}{2}(L + \frac{1}{2}),
\]

where \( M \) is the square root of \( M^2 \);

\[
M^2 = 1 - \frac{\nu(\nu - 1)}{(L + 1)(L + 2)}, \quad L = z \frac{d}{dz}.
\]

These operators are assumed to act on the non-exponential part of the wave functions \( \psi_{l,\nu} \). The eigenvalues of the operator \( B \) in this specific representation are given by

\[
b_k = k + \frac{1}{2}(\nu + 1)
\]

and with the one-dimensional statistics parameter identified with the shift in the spectrum, this exhibits the simple linear relation between the one-dimensional parameter and the 2-dimensional anyon parameter \( \nu \) referred to above. Note however the restriction \( 0 \leq \nu < 2 \) which applies to the anyon parameter. For general hermitian representations of the \( su(1,1) \)-algebra one has a less restrictive condition, \( -1 < \nu \). Values of \( \nu \geq 2 \) can in fact be included also in the anyon picture, but only when a repulsive interaction is introduced, which effectively excludes the states with the lowest relative angular momenta from the space of states. In the same way the values \( -1 < \nu < 0 \) can be interpreted within the anyon picture as due to a short range attractive interaction which makes the wave functions singular (but still normalizable) when the relative coordinate tends to zero.

In the case of fermions and bosons \( M \) equals the identity, which yields \( B_+ = \frac{1}{2}(a^\dagger)^2 \), \( B_- = \frac{1}{2}a^2 \) and \( B = \frac{1}{4}(a^\dagger a + aa^\dagger) \) \([14]\). Therefore, coherent states of the \( su(1,1) \) algebra, defined as eigenstates of \( B_- \) \([14]\), will in the case of general \( \nu \) be generalizations of the symmetric/antisymmetric combinations of the maximally localized HW-algebra coherent states. Using an abstract bra-ket notation where \( |l,\nu\rangle \) corresponds to the wave function \( \psi_{l,\nu} \), these coherent states are given by

\[
|\beta,\nu\rangle_{cs} = N_{\beta}^{(1)} \sum_{l=0}^{\infty} \frac{(\beta^*)^{2l}}{\sqrt{l!\Gamma(l + \nu + \frac{1}{2})}} |l,\nu\rangle
\]

and satisfy

\[
B_- |\beta,\nu\rangle_{cs} = \frac{1}{2}(\beta^*)^2 |\beta,\nu\rangle_{cs}.
\]

The Berry phase corresponding to an interchange of the two anyons has been calculated for the state \( |\beta,\nu\rangle_{cs} \), and the asymptotic form has been shown to have a term equal to (minus) the anyon parameter \([9]\). We would now like to give some comments on questions
related to coherent states, localization and the Berry phase. The considerations will be of importance when we later on shall discuss the quantum Hall system. In the single-particle case there was in the lowest Landau level an identification of the particle coordinate and the coherent state coordinate. In the present two-particle case such an identification holds only for fermions and bosons. This is readily noted by observing that a coherent state representation of \( B_+ \) now gives \( B_+ = \frac{1}{2} \beta^2 \), since \( B_+ = B_+^\dagger \). The anyon coordinate \( z \) rather corresponds to the state

\[
|z, \nu\rangle = N_z^{(2)} \sum_{l=0}^{\infty} \frac{(z^*)^{2l}}{\sqrt{\pi \Gamma(2l + 1 + \nu)}} |l, \nu\rangle.
\]

This state, which is the projection to LLL of a position eigenvector of the full anyon system, is an eigenstate of the operator \( M^{-1} B_- \);

\[
M^{-1} B_- |z, \nu\rangle = \frac{1}{2} (z^*)^2 |z, \nu\rangle.
\]

This operator is the adjoint of the operator \( \frac{1}{2} z^2 \) when the system is restricted to the lowest Landau level, hence \( M^{-1} B_- \) is the projection onto the LLL of the operator \( \frac{1}{2} (z^*)^2 \).

The distinction between \( |z, \nu\rangle \) and \( |\beta, \nu\rangle_{cs} \) will not affect the asymptotic behaviour of localization properties or geometric phase. However, the leading corrections to the asymptotic form of the Berry phase are different. To demonstrate this, let us first consider a general state of the form

\[
|\eta, \eta^*\rangle = N(\eta \eta^*) \sum_{l=0}^{\infty} (\eta^*)^{2l} a_l |l, \nu\rangle,
\]

where \( a_l \) are some expansion coefficients. We parameterize \( \eta \) along a circle around the origin, \( \eta = \eta_0 e^{i \theta} \). The Berry connection then is

\[
i \langle \eta, \eta^* | \partial_\theta | \eta, \eta^* \rangle = - \langle \eta, \eta^* | (\eta \partial_\eta - \eta^* \partial_\eta^*) | \eta, \eta^* \rangle = -\eta\eta^* \frac{d}{d(\eta^*)} \ln |N(\eta^*)|^2.
\]

We note that the normalization factor \( N(\eta^*) \) on one hand determines the Berry phase and on the other hand uniquely determines the expansion coefficients \( a_l \) and therefore the state \( |\eta, \eta^*\rangle \).

We specialize this to the case of the \( su(1, 1) \) algebra coherent states. The normalization factor is given in terms of a modified Bessel function, so the asymptotic behaviour is

\[
|N_{\beta}^{(1)}|^2 = \frac{\left(\beta \beta^*\right)^{\nu-\frac{1}{2}}}{I_{\nu-\frac{1}{2}}(\beta \beta^*)} \sim \sqrt{\pi} 2^{1-\nu} (\beta \beta^*)^\nu e^{-\beta \beta^*} (1 + \frac{\nu(\nu - 1)}{2\beta \beta^*}).
\]

Hence,

\[
i \langle \beta, \nu | \partial_\theta | \beta, \nu \rangle_{cs} \sim \beta \beta^* - \nu + \frac{\nu(\nu - 1)}{2\beta \beta^*}
\]

14
for $|\beta| \gg 1$. For the anyon coordinate state the normalization factor is more complicated. However, it equals a symmetrized degenerate hypergeometric function, and using relations between these and the Whittaker functions \[15\] we find the asymptotic behaviour

$$|\mathcal{N}_z^{(2)}|^{-2} = \sum_{l=0}^{\infty} \frac{(zz^*)^{2l}}{\pi \Gamma(2l + \nu + 1)}$$

$$\sim \frac{1}{2\pi} \left( e^{zz^*} (zz^*)^{-\nu} + e^{-zz^*} \cos(\pi \nu) (zz^*)^{-\nu} \right)$$

$$- \frac{\nu}{\pi \Gamma(\nu + 1)} \sum_{n=0}^{\infty} \frac{(\nu - 1)(\nu - 3) \cdots (\nu - 2n - 1)}{(zz^*)^{2n+2}}$$

$$\approx \frac{1}{2\pi} e^{zz^*} (zz^*)^{-\nu} \left( 1 - \frac{2\nu(\nu - 1)}{\Gamma(\nu + 1)} e^{-zz^*} (zz^*)^{-\nu-2} \right).$$

This yields

$$i \langle z, \nu | \partial_{\theta} | z, \nu \rangle \sim zz^* - \nu + \frac{2\nu(\nu - 1)}{\Gamma(\nu + 1)} e^{-zz^*} (zz^*)^{-\nu-1}. \quad (56)$$

As advertized, the anyon coordinate state and the $su(1,1)$-coherent state are equivalent as far as the asymptotic behaviour of the Berry phase is concerned. The difference shows up in the leading correction to the asymptotic form. For fermions and bosons the difference disappears, but in the general anyon case the state $|z, \nu\rangle$ reaches the asymptotic form $zz^* - \nu$ faster than the coherent state.

For the later discussion it is of interest to note that the specific way in which the statistics parameter $\nu$ appears in the Berry phase \[54\] (and in \[56\]) depends on some conventions chosen in the anyon description. To be more precise, let us for the moment denote the one-dimensional statistics parameter by $\nu_1$. This is the parameter which defines the shift in the spectrum of the operator $B$ of the $su(1,1)$ algebra, and it is the same parameter as we read from the Berry phase \[54\]. It can furthermore be related to the asymptotic behavior of the wave functions $|\psi\rangle \approx |z|^{v_1+2m}, m = 0, 1, ...$, when the relative coordinate $z$ tends to zero. (It is also the same as the exclusion statistics parameter for anyons in the lowest Landau level \[16, 17, 18\]). With the conventions chosen here it is up to a factor $1/\pi$ also identical to the phase angle obtained by the interchange of the two anyons,

$$\nu_1 = \frac{\theta}{\pi} \equiv \nu_2, \quad 0 \leq \theta < 2\pi. \quad (57)$$

The convention implies that the orientation of the loop which interchanges the two particles is fixed relative to the vector $q\vec{B}$, which depends on the charge $q$ of the particles. However, the opposite orientation may be chosen. The relation then is changed to

$$\nu_1 = 2 - \frac{\theta}{\pi} = 2 - \nu_2, \quad 0 \leq \theta < 2\pi. \quad (58)$$
This implies that if the statistics angle is kept fixed, relative to a fixed orientation of the interchange loop, but the sign of the charge is changed, then the relation between the (two-dimensional) statistics angle and the one-dimensional statistics parameter is changed from (57) to (58).

3.2 Two quasi-holes.

The Laughlin wave function for two quasi-holes at positions \(z_{0_1}\) and \(z_{0_2}\) is

\[
\Psi_{m,z_{0_1},z_{0_2}}(z_1, \ldots, z_N^*) = \prod_{i=1}^N (z_i - z_{0_1}) \prod_{j=1}^N (z_j - z_{0_2}) \phi_m(z_1, \ldots, z_N),
\]

(59)

where \(\phi_m\) is the Laughlin ground state (20). In the general two-hole case the total angular momentum is degenerate. However, since we shall be mainly concerned with the anyonic properties of the system, which are related to the relative motion of the two quasi-holes, we choose to investigate the restricted case with two holes located symmetrically around the origin. We then have \(z_{0_1} = -z_{0_2} \equiv Z/2\), and the degeneracy in the total angular momentum is removed. The hole-part of the wave function is

\[
\psi_Z(z_1, z_2, \ldots, z_N) \equiv \prod_{i=1}^N (z_i - \frac{Z}{2}) \prod_{j=1}^N (z_j + \frac{Z}{2})
\]

\[
= \prod_{i=1}^N (z_i^2 - (\frac{Z}{2})^2)
\]

\[
= \sum_{k=0}^N (\frac{Z}{2})^{2N-2k} (-1)^{N-k} S_k(z_1^2, z_2^2, \ldots, z_N^2).
\]

(60)

The angular momentum eigenstates are represented by the elementary symmetric polynomials \(S_k\) which now are functions of the squared electron coordinates. Defining \(\tilde{S}_k(z_1, \ldots, z_N) \equiv S_k(z_1^2, z_2^2, \ldots, z_N^2)\) and using the homogeneity of \(S_k\) we find for the angular momentum eigenvalues

\[
L \tilde{S}_k(z_1, z_2, \ldots, z_N) = \left( \sum_{i=1}^N z_i \partial_{z_i} + \frac{m}{2} N(N-1) \right) \tilde{S}_k(z_1, z_2, \ldots, z_N)
\]

\[
= \left( 2k + \frac{m}{2} N(N-1) \right) \tilde{S}_k(z_1, z_2, \ldots, z_N).
\]

(61)

Since there is no degeneracy, the angular momentum states \(\tilde{S}_k\) are orthogonal, but also in this case they are not normalized.

We introduce an abstract notation for the states of the two-hole system. Let the ket \(|k\rangle\) be defined by

\[
\langle z_1, \ldots, z_N^* | k \rangle = \frac{1}{\sqrt{c_k}} \tilde{S}_{N-k}(z_1, \ldots, z_N) \phi_m(z_1, \ldots, z_N^*),
\]

(62)
where $\phi_m$ is the ground state wave function (20), and let $c_k$ be the norm (26) of the angular momentum eigenfunction $\tilde{S}_{N-k}$:

$$c_k = (\tilde{S}_{N-k}, \tilde{S}_{N-k}).$$

(63)

Similarly, let $|Z\rangle$ be the normalized state of which the coordinate representation is (apart from the normalization factor) given by $\psi_Z \phi_m$ (24, 30), that is

$$|Z\rangle = \mathcal{N}(ZZ^*) \sum_{k=0}^{N} \left(\frac{Z}{2}\right)^{2k} (-1)^k \sqrt{c_k} |k\rangle$$

(64)

where

$$|\mathcal{N}(ZZ^*)|^2 = \sum_{k=0}^{N} (ZZ^*)^{2k} \frac{c_k}{4^{2k}}.$$

(65)

To determine the normalization factor $\mathcal{N}(ZZ^*)$ Laughlin’s plasma analogy is again used. The normalization factor then represents the interaction potential between two (external) charges of strength $\bar{q} = \sqrt{\frac{1}{m\beta}}$ (the quasi-holes) and also the interaction between these charges and the uniform background charge density $\sigma = -\frac{1}{\pi \sqrt{m\beta}}$. The interactions with the free charges of the plasma are determined by the $z_i$-dependent part of the wave function, and they have the form of two-dimensional (logarithmic) Coulomb potentials. Therefore, if $\mathcal{N}(ZZ^*)$ is assumed to have a form determined by Coulomb interactions between the external charges and between these charges and the background charge, then the plasma analogy asserts that the normalization integral will be (almost) independent of $ZZ^*$ due to charge screening. For small $ZZ^*$ the screening will not be complete, but this effect is expected to vanish exponentially. There will also be a correction for values of $ZZ^*$ which brings the two charges close to the boundary of the plasma. Motivated by these considerations we write the normalization factor as

$$|\mathcal{N}(ZZ^*)|^2 = Re^{-\frac{1}{4m}(ZZ^*)^2} (1 + \Delta V(ZZ^*)).$$

(66)

The first two $Z$-dependent factors correspond to the Coulomb interactions and $\Delta V$ represents the interaction potential which survives the screening. This potential is expected to vanish exponentially as the charges are moved away from each other. $R$ is an $N$- and $m$-dependent factor related to the ground state energy of the system.

We now return to the discussion of the localized two-anyon states. The Berry connections of the coherent state (54) and the anyon coordinate state (56) were shown to have the same asymptotic behaviour, but with different corrections for finite distance between the anyons. These results we will compare with the corresponding result for the quasi-hole system. Using (52) and (66), with $Z = |Z| e^{-i\phi}$, we find for the Berry connection of the two-quasi-hole system

$$i\langle Z | \partial_\phi | Z \rangle = \frac{1}{2m} ZZ^* - \frac{1}{m} - ZZ^* \frac{d}{d(ZZ^*)} \Delta V(ZZ^*).$$

(67)
The two first terms correspond to the results of ref. [5], and hence show that the main assumption from the plasma analogy, namely the Coulomb interaction, is in accordance with the independent calculation of the Berry phase. The two leading terms also agree with the corresponding terms of the anyon system (both for the anyon coordinate state and the \(su(1,1)\)-coherent state), provided we identify the statistics parameter \(\nu\) with \(1/m\) and \(zz^*\) with \(ZZ^*/2m\), with \(z\) as the anyon coordinate. However, the form of the third term of (67), which is also determined by the plasma analogy, seems to agree only with the result for the anyon coordinate state, but not with the coherent state. The coherent state has a correction term with a slower, non-exponential fall-off with distance. In the plasma picture it would correspond to a surviving dipole-dipole potential, which falls off like \(1/r^2\). To the best of our knowledge such a dipole interaction is not expected to be present in the two-dimensional plasma. For this reason the Laughlin quasi-hole state corresponds more closely to the anyon coordinate state than to the \(su(1,1)\)-coherent state.

Based on the correspondence with the anyon system we have the following expression for the normalization factor \(N\)

\[
|N(ZZ^*)|^{-2} = R \sum_{k=0}^{N} \frac{(ZZ^*)^{2k}}{\pi \Gamma(2k + \frac{1}{m} + 1)}.
\]

(68)

It gives the correct behaviour of the equivalent plasma as well as the expected form of the Berry phase (see (55) and (66)). Thus, for a large electron number \(N\) the expression (68) approximates the exact one with high accuracy when \(ZZ^*\) is not too small (of order the magnetic length squared). The deviation from the exact result tends exponentially fast to zero when \(ZZ^*\) increases.

The formal definition of the mapping from the quasi-hole system onto the anyon system can now be given. We introduce the subscripts \(qh\) for the quasi-hole states and \(a\) for the anyon states to distinguish between the states of this and the previous subsection. The mapping is

\[
U |k\rangle_{qh} = (-1)^k |k, \nu = \frac{1}{m}\rangle_a.
\]

(69)

\(U\) is a unitary operator, and we have identified the anyon parameter of the quasi-holes as \(\nu = \frac{1}{m}\) in order to have accordance between (67) and (66). Then

\[
U |Z\rangle_{qh} = N(ZZ^*) \sum_{l=0}^{N} \frac{Z^{2l}}{\sqrt{c_l}} |l, \frac{1}{m}\rangle_a
\]

\[
\approx \sqrt{R} N(ZZ^*) \sum_{l=0}^{N} \frac{(\sqrt{2m})^{2l}}{\pi \Gamma(2l + \frac{1}{m} + 1)} |l, \frac{1}{m}\rangle_a \quad (N \gg 1)
\]

\[
\rightarrow |z = -\frac{Z^*}{\sqrt{2m}} \frac{1}{m}\rangle_a \quad (N \to \infty).
\]

(70)

This gives the explicit correspondence between the two-hole system and the anyon system with statistics parameter \(\nu = 1/m\). The complex conjugation and the rescaling of the


|     | \( N = 2 \) | \( N = 3 \) | \( N = 4 \) | \( N = 5 \) |
|-----|-------------|-------------|-------------|-------------|
| \( k = 1 \) | 0.42        | 0.91        | 1.08        | 1.07        |
| \( k = 2 \) | 0.28        | 0.49        | 0.78        | 1.16        |
| \( k = 3 \) | 0.31        | 0.52        | 0.72        |             |
| \( k = 4 \) |             | 0.34        | 0.54        |             |
| \( k = 5 \) |             |             | 0.36        |             |

Table 1: Expansion coefficients of the hole states for \( m = 3 \). The table gives a comparison between results from the plasma analogy and numerical results (exact) for a small number of electrons. The entries are \( \frac{c_k/c_0}{c_k^{\text{exact}}/c_0^{\text{exact}}} \), and there is a tendency to obtain the correct \( k \)-dependency as \( N \) grows. The exact results are taken from ref.[19].

coordinates is related to the change in sign and rescaling of the charge, as has already been discussed in the one-hole case. Note however the additional factor \( 1/\sqrt{2} \) that is due to the reduced charge which appears in the two-anyon system.

Our mapping is based on an approximate expression for the normalization factor \( N^1 \), and thereby approximate expressions for the norms \( c_k \) of the angular momentum eigenstates;

\[
c_k = \frac{R}{\pi} \frac{2^{2k}}{m^{2k} \Gamma(2k + \frac{1}{m} + 1)}.
\]

In the case \( m = 1 \) the latter can be calculated exactly. The result is

\[
c_k^{\text{exact}} = \begin{cases} 
N!\pi^N (N + 1)! & 2^{2k} \frac{2k}{(2k+1)!} 
(N = 0, 1, \ldots) \\
N!\pi^N (N + 1)! & 2^{2k} \frac{2k}{(2k+1)!} - N!\pi^N (N + 1)! \sum_{l=0}^{2k-N-1} \frac{1}{l!(2k+1-l)!} 
(N = \frac{N}{2}, \ldots, N) 
\end{cases}
\]

(72)

where we have used the notation \((N + 1)! = (N + 1)!N!(N - 1)! \cdots 1\). In this case the approximate expression is in fact exact for \( k \leq N/2 \), which means that in the limit \( N \to \infty \) it has the correct \( k \)-dependency for all finite values of \( k \). The \( N \)-dependence of the factor \( R \) is also specified by the exact result. For each specific value of \( k \) we have investigated how the \( N \) electrons occupy the single-electron states in the lowest Landau level. We find that \( k > \frac{N}{2} \) always corresponds to having one of the quasi-holes outside the boundary of the system, that is it is actually a single-hole state. On the other hand, if \( k \leq \frac{N}{2} \) both holes are inside the system. Hence, in the case \( m = 1 \) the approximative results equal the correct results for all true two-hole states.

For \( m = 3 \) we do not have exact analytical results for \( c_k \), but numerical values are found for up to five electrons by use of the results of ref.[19]. For each specific value of \( N \) the ratio \( \frac{c_k/c_0}{c_k^{\text{exact}}/c_0^{\text{exact}}} \) has been evaluated and the results are listed in Table 1. Although the number of electrons in these examples is small, there seems to be a tendency to reach the correct \( k \)-dependency for \( c_k \), i.e. the value 1 for the ratio tabulated, as \( N \) grows.

For the finite system we can, similarly to the case with one quasi-hole, find ladder operators leaving the space spanned by \( \{ \tilde{S}_k(z_1, z_2, \ldots, z_N) \}_{k=0}^{N} \) invariant. These are given by

\[
C_- = \sum_{i=1}^{N} \left( z_i^2 - \frac{1}{2} z_i^4 \partial^2_{z_i} \right), \\
C_+ = \sum_{i=1}^{N} \frac{1}{2} \partial^2_{z_i},
\]
\[ C = -\frac{1}{2} \sum_{i=1}^{N} z_i \partial_{z_i} + \frac{N}{2}, \] (73)

and span a non-hermitian \( su(2) \) algebra.

The unitary mapping between the \((N \to \infty)\) two-hole state space and the state space for two anyons in the lowest Landau level, determines the form of the generators of the \( su(1,1) \) algebra in the quasi-hole case. Expressed in terms of the operators \( C_\pm \) and \( K \equiv C + N/2 \) they have the form

\[
\tilde{B}_- = \frac{1}{m} C_-(K - N - 1)^{-1} \sqrt{\frac{K(K + 1/m - 1/2)}{(K + 1/2m)(K + 1/2m - 1/2)}},
\]

\[
\tilde{B}_+ = -m C_+(K + 1)^{-1} \sqrt{(K + 1/2m + 1)(K + 1/2m + 1/2)(K + 1)(K + 1/m + 1/2)} . \quad (74)
\]

The expressions fully demonstrates that the electron coordinate representation of the ladder operators \( \tilde{B}_+ \) and \( \tilde{B}_- \), which are mutually adjoint in the limit \( N \to \infty \), is rather complicated. However, as already discussed, the quasi-hole state \( |Z\rangle \) is not identified as a coherent state of this algebra, but rather as an eigenvector of the operator \( U^\dagger M^{-1} B_- U \) \((44, 45)\). For this operator we have the slightly simpler form

\[
U^\dagger M^{-1} B_- U = \frac{1}{m} C_-(K - N - 1)^{-1}
\]

\[
= -\frac{2}{m} \sum_{i=1}^{N} (z_i^2 - \frac{1}{2} z_i^4 \partial_{z_i}^2) \left( \sum_{i=1}^{N} z_i \partial_{z_i} + 2 \right)^{-1} . \quad (75)
\]
4 Quasi-electrons.

In the case of Laughlin quasi-electrons an anyon representation is not easily obtained. This was noted also in ref.[7]. We will in this section present some of the difficulties and make some additional comments to the problem.

We start out by considering a single quasi-electron, for which the variational wave function

$$\langle z_1, \ldots, z_N | \tilde{z}_0 \rangle = \Psi_{m,z_0}^{-} (z_1, \ldots, z_N) = \tilde{N}(|z_0|) e^{-\frac{1}{2} \sum_{i=1}^{N} z_i z_i^*} \prod_{i=1}^{N} (\partial_{z_i} - z_0^*) \prod_{i<j} (z_i - z_j)^m$$

(76)

has been proposed [1]. The normalization integral for this state is

$$1 \equiv \langle \tilde{z}_0 | \tilde{z}_0 \rangle = \int d^2 z_1 e^{-\sum_{j=1}^{N} z_j z_j^*} \prod_{k<l} |z_k - z_l|^2 \prod_{i=1}^{N} (|z_i - z_0|^2 - 1) |\tilde{N}(|z_0|)|^2.$$  (77)

In the language of the plasma analogy, there are now corrections to the pure (logarithmic) Coulomb interaction between the external charge (the quasi-electron) and the free charges (the electrons). If these corrections are neglected one simply has the normalization integral of the single quasi-hole, and $\tilde{N}(|z_0|)$ equals (28). Using (52) (which still is valid since (76) can be expanded in terms of orthogonal angular momentum states), with the parameterization $z_0 = |z_0| e^{-i\phi}$, one then obtains a quasi-electron charge $q_{qe} = -e/m$. This equals the result proposed in ref.[5]. In the large $N$ limit it is then possible to map the quasi-electron state $| \tilde{z}_0 \rangle$ onto a (HW algebra) coherent state of an ordinary single-particle system, the coherent state coordinate now being $\alpha^* = z_0^*/\sqrt{m}$ (see (55) for comparison).

There is however an objection to this approximation: For the full Landau level ($m = 1$) it gives the wrong expression both for the normalization factor and for the Berry phase. Since $\Psi_{m,z_0}^{-}$ (76) is entirely in the lowest Landau level, it describes a situation with no excess charge anywhere for $m = 1$. The wave function is simply that of the ground state, and the Berry phase is zero.

We will now show that the approximation mentioned above is equivalent to neglecting some of the terms in the Berry connection. The latter can, when assuming the normalization factor depends only on $|z_0|$, be written

$$\frac{d\gamma^{-}}{d\phi} \equiv i \langle \tilde{z}_0 | \frac{\partial}{\partial \phi} | \tilde{z}_0 \rangle$$

$$= i |\tilde{N}(|z_0|)|^2 \int d^2 z_1 e^{-\sum_{j=1}^{N} z_j z_j^*} \left( \prod_{j=1}^{N} (\partial_{z_j}^* - z_0^*) \prod_{k<l} (z_k^* - z_l^*) \right)^m$$

$$\sum_{i=1}^{N} (-\frac{dz_i}{d\phi})(\partial_{z_i} - z_0^*)^{-1} \prod_{j=1}^{N} (\partial_{z_j} - z_0^*) \prod_{k<l} (z_k - z_l)^m.$$  (78)
The operator \((\partial_z - z^*_o)^{-1}\) is not well defined in single particle spaces of infinite dimension. However, we may define the operator

\[
A \equiv -\frac{1}{z^*_o} \sum_{l=0}^{\infty} \frac{1}{(z^*_o)^l} \partial_z^l,
\]

which for any finite value of \(k\) satisfies

\[
A(\partial_z - z^*_o)z^k = (\partial_z - z^*_o)Az^k = z^k.
\]

Laughlin’s quasi-electron wave function contains only finite polynomials, hence \(A\) may be used to find the Berry connection. The details of the calculation are shown in an appendix, and the following expression appears:

\[
\frac{d\gamma^-}{d\phi} = i \int d^2z \rho^-_o(z, z^*) \frac{dz^*_o}{d\phi} \frac{1}{z^*_o - z^*} + i\pi \frac{1}{z^*_o} \sum_{l=0}^{L} \frac{1}{(z^*_o)^l} \left((\partial_z + z^*)^l \rho^-_o(z, z^*)\right)_{z=z_o, z^*=z_o^*},
\]

where \(\rho^-_o(z, z^*)\) is the single particle density \(N\int d^2z_2 \cdots d^2z_N |\Psi_{m,z_0}|^2\). The upper limit of the summation index is \(L \equiv m(N-1)\) and corresponds to the highest power in the polynomials arising. The expression (81) is valid for any filling fraction \(1/m\). The first term corresponds to the results of ref.\([5]\). Assuming \(\rho^-_o\) equals \(1/m\pi\), which is the ground state density for \(N\) large, this term gives the Berry phase \(-2\pi |z_0|^2 /m\), which in turn gives the charge and normalization factor mentioned above. The second term gives an additional contribution to the Berry phase, and hence corrections to the normalization factor. We conclude (for arbitrary \(m\)) that to neglect this term is equivalent to use the approximation \(|z_i - z_0|^2 - 1 \to |z_i - z_0|^2\) in (77).

Using the exact expression for \(\rho^-_{z_o}\) in the case \(m = 1\), we have calculated the sum in (81). Taking the large \(N\) limit we then obtain \(2\pi |z_0|^2 - 2\pi N\) for this contribution to the Berry phase. Hence, we obtain the correct phase and also the correct \(|z_0|\)-dependence of the normalization factor\([4]\) but at the same time we have explicitly demonstrated that the additional term does not vanish when \(N \to \infty\). One might argue that \(m = 1\) is a special case, but we see no obvious reason why the corrections can be overlooked for other values of \(m\), while they are important for \(m = 1\). This implies some uncertainty in the derivation of the charge of the quasi-electrons from the Berry phase.

Proceeding to the case of two quasi-electrons at \(z_a\) and \(z_b\), the normalization integral is written on the same form as (77) but with the substitution

\[
|z_i - z_0|^2 - 1 \to |z_i - z_0|^2 |z_i - z_b|^2 - 4 |z_i - \frac{1}{2}(z_a + z_b)|^2 + 2.
\]

\(^{1}\)There is a subtlety here. Eq.(81) rests upon the assumption \(\mathcal{N} = \mathcal{N}(|z_0|)\). For \(m = 1\) this implies a nonphysical singularity \((\frac{z_i}{|z_0|^2})^{N}\) in the wave function. This singularity is responsible for the artificial \(-2\pi N\) in the Berry phase. Such singularities are not present for other filling fractions.
Using the plasma analogy language, there are again corrections to the pure Coulomb interaction, which are also reflected in the Berry connection. With \( z_a = |z_a| e^{-i\phi} \), \( z_b \) fixed, the latter is given by (81) when substituting

\[ \rho^-_{z_a, z_b}(z, z^*) \]

\( \rho^-_{z_a, z_b} \) being the single particle density when two quasi-electrons are present. We write \( \rho^-_{z_a, z_b} = \rho^-_{z_a} + \delta \rho^-_{z_b} \), where \( \delta \rho^-_{z_b} = \rho^-_{z_b} - \rho_0 \) is the deviation from the ground state density due to the quasi-electron at \( z_b \). For sufficiently large separation between the quasi-electrons we assume that \( \delta \rho^-_{z_b}(z, z^*) \) as well as all its derivatives are vanishingly small at the point \( z = z_a \), and derive the following approximate expression for the Berry phase,

\[
\begin{align*}
\gamma^- &= i \int_0^{2\pi} d\phi \int d^2z \rho^-_{z_a}(z, z^*) \frac{dz^*}{d\phi} \frac{1}{z^*_a - z^*} \\
&\quad + i\pi \int_0^{2\pi} d\phi \sum_{l=0}^{L} \frac{1}{(z^*_a)^l} \left( (\partial_{z^*} + z^*)^l \rho^-_{z_a}(z, z^*) \right)_{z=z_a, z^*=z^*_a} \\
&\quad - 2\pi \int_{|z|<|z_a|} d^2z \delta \rho^-_{z_b},
\end{align*}
\]

where Cauchy’s residue theorem has been used on the last term. We notice that no new types of correction terms arise as compared to those of (81). The two first terms determine the Aharonov-Bohm charge \( q_{qe} \) of the quasi-electrons and give no information about interchange phases. We assume, as usual, this charge to be equal to the charge determined by \( \delta \rho^-_{z_b} \). This implies that the statistics phase is

\[ \nu_{qe} = \frac{q_{qe}}{-e}, \]

where \(-e\) is the electron charge, and we note that this is the same relation between charge and statistics as for the quasi-holes [5]. When the complications referred to above in the derivation of the quasi-electron charge is neglected this gives \( \nu_{qe} = -\frac{1}{m} \).

One should note that the negative value of \( \nu_{qe} \), as discussed in section 3.1, implies that if the quasi-electrons are interpreted as anyons in a magnetic field, they are not free anyons, but anyons with a special short-range attraction which effectively gives a negative statistics parameter.

It is of interest to compare the result (85) with numerical simulations which have been performed for a small number of interacting electrons in a strong magnetic field on a sphere [20]. By state counting the one-dimensional statistics parameter (exclusion statistics parameter), for \( m = 3 \), has there been determined to be \( \nu_1 = 2 - \frac{1}{m} \). The fractional part agrees with the value cited above, but not the integer part. (To be certain that this part is correctly accounted for one has to check that the Berry phase is well behaved also for \( z \to 0 \).) A possible explanation for this is that the correction terms we have neglected are important for the (integer part of the) statistics parameter. However, numerical calculation of the Berry phase indicates that the result (85) may in fact be correct for the Laughlin
quasi-electron state \cite{21}. Therefore, a more likely explanation is that the Laughlin state does not reproduce correctly this aspect of the physical quasi-electron state of the quantum Hall system.

5 Conclusion.

In summary, we have examined the anyon representation of Laughlin’s quasi-particle states with emphasis on one-dimensional, algebraic aspects. The motivation has been to give a detailed account of the correspondence to the one-dimensional description of anyons in the lowest Landau level.

In particular, we have formulated explicit mappings from the state spaces of one and two quasi-holes to the state spaces of a single particle and two anyons in the lowest Landau level. We have further examined the question whether the Laughlin states can be viewed as coherent states of the fundamental algebra of observables for these systems. In the case of a single hole the Laughlin hole-state was found to correspond to a coherent state for the one-dimensional Heisenberg-Weyl algebra in the limit where the electron number \( N \) tends to infinity. In the case of two quasi-holes we showed that the two-hole state corresponds more closely to the anyon position eigenstate projected onto the lowest Landau level than to the coherent state of the underlying \( su(1,1) \) algebra. The two-dimensional parameter was identified as \( 1/m \) and the charge of the anyons as \(-1/m\) times the electron charge.

However, one should note that the difference between the projected anyon position eigenstate and the coherent state only shows up in the corrections to the asymptotic form of the Berry phase for particle interchange, and the corrections tend to zero for large separation. The physical relevance of these correction terms is not so clear, and in particular it is not clear whether the Laughlin hole state gives a good representation of the physical hole state of the interacting electron system in this respect.

For both the one-hole and the two-hole systems we have considered the (edge) effects of a finite electron number \( N \). The state spaces then have finite dimensions and define hermitian representations of the compact \( su(2) \) algebra rather than of the non-compact Heisenberg-Weyl or \( su(1,1) \) algebra which are relevant for the infinite dimensional cases. The unitary mapping between the hole system and the single particle/anyon system then determines the form of the generators. The operators have a quite complicated form when written in electron coordinates. However, in the hole-representation the operator expressions take a simple form.

We have compared the approximative results (found from plasma analogy and Berry phase calculations) for the norm of the angular momentum eigenstates to known, exact results. For \( m = 1 \) analytic expressions can be found and these show exact agreement for all single-hole states as well as for all true two-hole states. In the case of two quasi-holes we have made a comparison to numerical results previously obtained for \( m = 3 \) and a small number of electrons. We find a reasonably good agreement when this number is increased.

For quasi-electrons explicit single-particle and two-anyon representations are not so
easily obtained. We have pointed out that to perform a specific approximation in the plasma analogy calculation is equivalent to neglect an additional contribution to the Berry phase. We have also shown that this extra term is important for the filling fraction \( m = 1 \). Whether it is important for other values of \( m \) is unsettled.

**Acknowledgements.**

We would like to thank G.S. Canright, J. Myrheim and D. Arovas for several useful discussions.

### A Berry phase calculation for quasi-electrons.

In this appendix we show the detailed derivation of the Berry connection (81) for Laughlin quasi-electrons. We start from (78) and (79),

\[
\frac{d\gamma}{d\phi} = iN \frac{dz_0^*}{z_0^*} \frac{dz}{d\phi} \int \prod_{i=1}^N d^2 z_i \left| \tilde{N}(z_0) \right|^2 e^{-\sum_{i=1}^N z_i z_i^*} \left( \prod_{j=1}^N (\partial_{z_j^*} - z_0) \prod_{k<n} (z_k^* - z_n^*)^m \right) \sum_{l=0}^{\infty} \frac{1}{(z_0^*)^l} \prod_{j=1}^N \left( \partial_{z_j} - z_0^* \right) \prod_{k<n} (z_k - z_n)^m. \tag{86}
\]

In the expression \( \prod_{j=1}^N (\partial_{z_j} - z_0^*) \prod_{k<n} (z_k - z_n)^m \) the highest power of any single electron coordinate is \( m(N - 1) \equiv L \). Thus the infinite sum over \( l \) is cut off, and integrating by parts we obtain

\[
\frac{d\gamma}{d\phi} = iN \frac{dz_0^*}{z_0^*} \frac{dz}{d\phi} \int d^2 z_1 \sum_{l=0}^L \frac{z_1^*}{z_0^*} \int \prod_{i=2}^N d^2 z_i \left| \Psi_{m,z_0}^- \right|^2 \left( \prod_{j=1}^N (\partial_{z_j} - z_0) \prod_{k<n} (z_k - z_n)^m \right) \sum_{l=0}^{L+1} \frac{1}{(z_0^*)^l} \prod_{j=1}^N \left( \partial_{z_j} - z_0^* \right) \prod_{k<n} (z_k - z_n)^m \left( z_0^* \right)^{L+1}. \tag{87}
\]

Here \( \rho_{m,z_0}^-(z_1, z_1^*) = N \int \prod_{i=2}^N d^2 z_i \left| \Psi_{m,z_0}^- \right|^2 \) is the single electron density. The first integral corresponds to the result of ref. [3], but we have also obtained an additional contribution to the Berry connection. To further investigate this extra term we use the notation
\[ p(z_1, z_1^*) \equiv (z_1^*)^{L+1} \int \prod_{i=2}^{N} d^2 z_i e^{-\sum_{i=2}^{N} z_i z_i^*} \left( \prod_{j=1}^{N} (\partial_{z_j} - z_o) \prod_{k<n} (z_k - z_n)^m \right) \]

\[ \prod_{j=2}^{N} (\partial_{z_j} - z_o^*) \prod_{k<n} (z_k^* - z_n)^m \), \quad (88) \]

which is a polynomial in \( z_1 \) and \( z_1^* \). It is important to realize that the highest power of \( z_1 \) is \( m(N-1) = L \) whereas the lowest power of \( z_1^* \) is \( L + 1 \). Using the expression for the single electron density we find

\[ \int d^2 z \rho(z, z_0^*) (z_1^*)^{L+1} \frac{1}{z_0 - z_1^*} p(z, z_1^*) = N |\tilde{N}|^2 \int d^2 z e^{-zz^*} \frac{1}{z_0 - z_0^*} p(z, z_1^*) \]

\[ = N |\tilde{N}|^2 \int d^2 z \partial_z \left( e^{-zz^*} \frac{1}{z_0 - z_0^*} p(z, z_1^*) \right) \]

\[ + N |\tilde{N}|^2 \int d^2 z e^{-zz^*} p(z, z_1^*). \quad (89) \]

Integration by parts has been used to arrive at the last equality. The last integral vanishes because there exist no matching powers of \( z \) and \( z_1^* \) in the polynomial \( p(z, z^*) \). However, the surface integral is nonvanishing due to the pole at the position of the quasi-electron. We remove a small area \( A \) around \( z_0 \). Integration over the remaining part of the plane is performed by using Stokes theorem to rewrite the integral to a line integral over the boundary of \( A \). We then find

\[ \lim_{A \to 0} \int d^2 z \partial_z \left( e^{-zz^*} \frac{1}{z_0 - z_0^*} p(z, z_1^*) \right) = -\pi e^{-z_0 z_0^*} p(z_0, z_0^*). \quad (90) \]

The polynomial \( p(z, z_1^*) \) is related to the single electron density by

\[ N |\tilde{N}|^2 p(z, z_1^*) = -(z_1^*)^{L+1} e^{-zz^*} \sum_{l=0}^{L} \frac{1}{(z_0^*)^l} (\partial_z + z^*)^l \rho_{z_0}^-(z, z_1^*), \quad (91) \]

so the final result for the Berry connection then is

\[ \frac{d\gamma^-}{d\phi} = i \int d^2 z \rho_{z_0}^-(z, z_1^*) \frac{dz_0^*}{d\phi} \frac{1}{z_0 - z_1^*} \]

\[ + i\pi \frac{1}{z_0^*} \frac{dz_0^*}{d\phi} \sum_{l=0}^{L} \frac{1}{(z_0^*)^l} \left( (\partial_z + z^*)^l \rho_{z_0}^-(z, z_1^*) \right) \bigg|_{z=z_0, z_1^*=z_0^*}. \quad (92) \]
References

[1] R.B. Laughlin, Phys.Rev.Lett. 50 (1983) 1395
[2] J.M. Leinaas and J. Myrheim, Nuovo Cimento 37 (1977) 1
[3] F.D.M. Haldane, Phys.Rev.Lett. 51 (1983) 605
[4] B.I. Halperin, Phys.Rev.Lett. 52 (1984) 1583
[5] D. Arovas, R. Schrieffer and F. Wilczek, Phys.Rev.Lett. 53 (1984) 722
[6] R.B. Laughlin, in The Quantum Hall Effect eds. S.M. Girvin and R.E. Prange, Springer Verlag, (1987)
[7] R.B. Laughlin, in Fractional Statistics and Anyon Superconductivity ed. F. Wilczek, World Scientific, (1990)
[8] J.M. Leinaas and J. Myrheim, Phys.Rev. B37 (1988) 9286
[9] T.H. Hansson, J.M. Leinaas and J. Myrheim, Nucl.Phys. B 384 (1992) 559
[10] A. Perelomov, Generalized Coherent States and Their Applications, Springer-Verlag (1986)
[11] V. Bargmann, Communications on pure and applied mathematics XIV (1961) 187
[12] S.M. Girvin and T. Jach, Phys.Rev. B29 (1984) 5617
[13] S. Lang, Algebra, Addison-Wesley Publ. Comp. (1977)
[14] A.O. Barut and L. Girardello, Commun.Math.Phys. 21 (1971) 41
[15] I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series, and products, Academic Press, Inc. (1980)
[16] F.D.M. Haldane, Phys.Rev.Lett. 67 (1991) 937
[17] G.S. Canright and M.D. Johnsen, J.Phys.A:Math.Gen. 27 (1994) 3579
[18] Y.-S. Wu, Phys.Rev.Lett. 73 (1994) 922
[19] J. Myrheim, Anyons, Notes for the Course on Geometrical Phases ICTP, Trieste, 6-17 September 1993 (Theoretical Physics Seminar in Trondheim, Norwegian Institute of Technology, No 13 1993)
[20] M.D. Johnson and G.S. Canright, Phys.Rev. B49 (1994) 2947
[21] J. Myrheim, priv. com.