Probabilistic Dense Coding Using a Non-symmetric Multipartite Quantum Channel

Qiu-Bo Fan and Shou Zhang∗

Department of Physics, College of Science,
Yanbian University, Yanji, Jilin, 133002, PR China

We investigate probabilistic dense coding in non-symmetric Hilbert spaces of the sender’s and the receiver’s particles. The sender and the receiver share the multipartite non-maximally quantum channel. We also discuss the average information.

PACS numbers: 03.67.-a, 89.70.+c

Keywords: Probabilistic dense coding; Non-symmetric Hilbert space; Quantum channel

Quantum dense coding [1] is one of the important applications of quantum entangled state [2, 3] in quantum communication. Some people have proposed some schemes [4, 5, 6, 7, 8] for quantum dense coding using mixed state entanglement, Greenberger-Horne-Zeilinger (GHZ) state, multi-level entangled state, and multipartite entangled state. Other people have proposed a number of schemes about dense coding [9, 10] using the interaction between atoms and cavity quantum electrodynamics (QED). However, the quantum channels of these schemes are all symmetrically and maximally entangled states. And then Yan et al. [11] and Fan et al. [12] have discussed dense coding using two-particle and multi-particle entangled states as quantum channels, respectively. In their schemes the quantum channels are all non-symmetric and maximal. Wang et al. [13] have proposed another scheme for probabilistic dense coding using a non-maximally and symmetrically entangled pair. And Pati et al. [14] have also proposed a scheme for probabilistic super dense coding with non-maximally and symmetrically entangled states as a resource, and they generalized the scheme to higher dimension and more entanglement. There are other people proposed distributed quantum dense coding [15], i.e., the generalization of quantum dense coding to more than one sender and more than one receiver. But in our scheme, we investigate probabilistic dense coding using two non-symmetrically and non-maximally entangled pairs as quantum channel and

∗ E-mail: szhang@ybu.edu.cn
generalize it to $N$ non-symmetrically and non-maximally entangled pairs, that it to say, our quantum channel is a non-symmetric and multi-particle state. Our scheme only has one sender and one receiver.

Now we discuss our scheme in detail. For clarity, we first use four particles $1, 2, 1', 2'$ to realize the probabilistic dense coding. We suppose particles 1 and 2, both in 3-dimension Hilbert space, belong to Alice; particles $1', 2'$, both in 2-dimension Hilbert space, belong to Bob. The initial state which they share is as follows:

$$|\Psi\rangle_{121'2'} = (\alpha_{01}|00\rangle + \alpha_{11}|11\rangle)_{11'} \otimes (\alpha_{02}|00\rangle + \alpha_{12}|11\rangle)_{22'},$$

where the subscripts 1 and 2 of $\alpha_0$ and $\alpha_1$ indicate the 1, 2-th two-particle entangled state, respectively; $\alpha_{01}, \alpha_{11}, \alpha_{02}, \alpha_{12}$ are real numbers; and $|\alpha_{01}|^2 + |\alpha_{11}|^2 = 1, |\alpha_{02}|^2 + |\alpha_{12}|^2 = 1$. Without loss of generality, we suppose that $|\alpha_{01}| \leq |\alpha_{11}|$ and $|\alpha_{02}| \leq |\alpha_{12}|$.

Firstly, Alice introduces two auxiliary two-level particles in the quantum state $|00\rangle_{a_1a_2}$, So the total state of the system is

$$|\Psi\rangle_T = (\alpha_{01}|00\rangle + \alpha_{11}|11\rangle)_{11'} \otimes (\alpha_{02}|00\rangle + \alpha_{12}|11\rangle)_{22'} \otimes |00\rangle_{a_1a_2}.$$

Alice performs two unitary operations $U_{1a_1}$ and $U_{2a_2}$ on her particles $(1, a_1)$ and $(2, a_2)$, under the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle, |20\rangle, |21\rangle\}_{1a_1(2a_2)}$, respectively. We can write the two unitary operations as one $U = U_{1a_1} \otimes U_{2a_2}$ as follows:

$$U = U_{1a_1} \otimes U_{2a_2} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & A_1 & B_1 & 0 & 0 \\
0 & 0 & B_1 & -A_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \otimes \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & A_2 & B_2 & 0 & 0 \\
0 & 0 & B_2 & -A_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

where $A_i = \alpha_{0i}/\alpha_{1i}, B_i = \sqrt{(1 - \alpha_{0i}^2/\alpha_{1i}^2)}(i = 1, 2)$. The unitary operation will transform $|\Psi\rangle_T$ into the corresponding state:

$$|\Psi'\rangle_T = \left\{ \alpha_{01}(|00\rangle + |11\rangle)_{11'} |0\rangle_{a_1} + \sqrt{\alpha_{11}^2 - \alpha_{01}^2} |11\rangle_{11'} |1\rangle_{a_1} \right\}$$

$$\otimes \left\{ \alpha_{02}(|00\rangle + |11\rangle)_{22'} |0\rangle_{a_2} + \sqrt{\alpha_{12}^2 - \alpha_{02}^2} |11\rangle_{22'} |1\rangle_{a_2} \right\}$$
\[
\begin{align*}
= \left\{ \sqrt{2\alpha_{01}} \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right)_{11'} \langle 0 |_{a_1} + \sqrt{\alpha_{11}^2 - \alpha_{01}^2} |11\rangle_{11'} |1 |_{a_1} \right\} \\
\otimes \left\{ \sqrt{2\alpha_{02}} \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right)_{22'} \langle 0 |_{a_2} + \sqrt{\alpha_{12}^2 - \alpha_{02}^2} |11\rangle_{22'} |1 |_{a_2} \right\} \\
= \left\{ \sqrt{2\alpha_{01}} |\Psi_{00}^0 \rangle_{11'} \langle 0 |_{a_1} + \sqrt{\alpha_{11}^2 - \alpha_{01}^2} |\Psi_{00}^1 \rangle_{11'} |1 |_{a_1} \right\} \\
\otimes \left\{ \sqrt{2\alpha_{02}} |\Psi_{00}^0 \rangle_{22'} \langle 0 |_{a_2} + \sqrt{\alpha_{12}^2 - \alpha_{02}^2} |\Psi_{00}^1 \rangle_{22'} |1 |_{a_2} \right\}.
\end{align*}
\]

where \( |\Psi_{00}^0 \rangle_{11'(22')} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{11'(22')}, |\Psi_{00}^1 \rangle_{11'(22')} = |11\rangle_{11'(22')} \). Then Alice makes orthogonal measurement on the auxiliary particles. If she gets the result \( |0 \rangle_{a_1} |0 \rangle_{a_2} \), she ensures that the four particles 1, 1', 2, and 2' are in the product state of the two maximally entangled pairs, i.e., \( |\Psi_{00}^0 \rangle_{11'} \otimes |\Psi_{00}^0 \rangle_{22'}, \) and the probability of obtaining \( |0 \rangle_{a_1} |0 \rangle_{a_2} \) is \( 4\alpha_{01}^2 \alpha_{02}^2 \) according to Eq. (4); if she gets the result \( |0 \rangle_{a_1} |1 \rangle_{a_2} \), she ensures that the four particles are in the state \( |\Psi_{00}^0 \rangle_{11'} \otimes |\Psi_{00}^1 \rangle_{22'}, \) and the probability of this result is \( 2\alpha_{01}^2 (\alpha_{12}^2 - \alpha_{02}^2) \); if she gets the result \( |1 \rangle_{a_1} |0 \rangle_{a_2} \), she ensures that the four particles are in the state \( |\Psi_{00}^0 \rangle_{11'} \otimes |\Psi_{00}^0 \rangle_{22'}, \) and the probability is \( 2\alpha_{01}^2 (\alpha_{12}^2 - \alpha_{02}^2) \); if she gets the result \( |1 \rangle_{a_1} |1 \rangle_{a_2} \), she ensures the four particles are in the state \( |\Psi_{00}^1 \rangle_{11'} \otimes |\Psi_{00}^1 \rangle_{22'}, \) and the probability is \( (\alpha_{11}^2 - \alpha_{01}^2) (\alpha_{12}^2 - \alpha_{02}^2) \).

Secondly, Alice encodes classical information by a unitary transformation on her particles 1 and 2. If particles 1 and 1'(2 and 2') are in the state \( |\Psi_{00}^0 \rangle_{11'(22')} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{11'(22')}, \) she can perform six single-particle operators on particle 1(2):

\[
U_{00}^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
U_{01}^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
U_{10}^0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
U_{11}^0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},
U_{20}^0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},
U_{21}^0 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.
\]
The state $|\Psi_{00}^{0}\rangle_{11',(22')}$ will be transformed into the corresponding states:

$$U_{00}^{\dagger} |\Psi_{00}^{0}\rangle_{11',(22')} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{11',(22')} = |\Psi_{00}^{0}\rangle_{11',(22')}$$

$$U_{01}^{\dagger} |\Psi_{00}^{0}\rangle_{11',(22')} = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)_{11',(22')} = |\Psi_{01}^{0}\rangle_{11',(22')}$$

$$U_{10}^{\dagger} |\Psi_{00}^{0}\rangle_{11',(22')} = \frac{1}{\sqrt{2}} (|10\rangle + |21\rangle)_{11',(22')} = |\Psi_{10}^{0}\rangle_{11',(22')}$$

$$U_{11}^{\dagger} |\Psi_{00}^{0}\rangle_{11',(22')} = \frac{1}{\sqrt{2}} (|10\rangle - |21\rangle)_{11',(22')} = |\Psi_{11}^{0}\rangle_{11',(22')}$$

$$U_{20}^{\dagger} |\Psi_{00}^{0}\rangle_{11',(22')} = \frac{1}{\sqrt{2}} (|20\rangle + |01\rangle)_{11',(22')} = |\Psi_{20}^{0}\rangle_{11',(22')}$$

$$U_{21}^{\dagger} |\Psi_{00}^{0}\rangle_{11',(22')} = \frac{1}{\sqrt{2}} (|20\rangle - |01\rangle)_{11',(22')} = |\Psi_{21}^{0}\rangle_{11',(22')}.$$  \(6\)

The above states are orthogonal mutually.

If particles 1 and 1' (2 and 2') are in the product state $|\Psi_{00}^{1}\rangle_{11',(22')} = |11\rangle_{11',(22')}$, Alice can perform three single-particle operators on particle 1(2):

$$U_{00}^{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U_{10}^{1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U_{20}^{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$  \(7\)

The state $|\Psi_{00}^{1}\rangle_{11',(22')}$ will be transformed into the corresponding states:

$$U_{00}^{1\dagger} |\Psi_{00}^{1}\rangle_{11',(22')} = |11\rangle_{11',(22')} = |\Psi_{00}^{1}\rangle_{11',(22')}$$

$$U_{10}^{1\dagger} |\Psi_{00}^{1}\rangle_{11',(22')} = |01\rangle_{11',(22')} = |\Psi_{10}^{1}\rangle_{11',(22')}$$

$$U_{20}^{1\dagger} |\Psi_{00}^{1}\rangle_{11',(22')} = |21\rangle_{11',(22')} = |\Psi_{20}^{1}\rangle_{11',(22')}.$$  \(8\)

These states are also orthogonal mutually.

In all, there are four cases Alice can encode classical information on her particles 1 and 2 according to the measurement results of the auxiliary particles $a_1$ and $a_2$, which have different probabilities. TABLE I shows the four cases (where the superscripts 1 and 2 of $U_{mam}^{0}, U_{m1n1}^{1}$...
The states which Alice makes ensures the states of the corresponding particles. The unitary operators indicate the 1, 2-th particles; the subscripts 0 and 1 of each $\Psi$ and $U$ correspond to the results of each auxiliary particle; $m_0, m_1 = 0, 1, 2; n_0 = 0, 1; n_1 = 0$).

Thirdly, after performing one of these unitary operators on her particles 1 and 2, Alice sends her particles to Bob and tells Bob her measurement result of particles $a_1$ and $a_2$.

Finally, Bob receives the particles, and makes measurements on the four particles 1, 1’, 2, and 2’. The measurement basis is selected according to Alice’s measurement result of particles $a_1$ and $a_2$. After that, Bob will obtain the classical information that Alice has encoded.

From the above procedure, we can calculate the average information transformation:

$$I_{\text{ave}} = 4\alpha_0^2\alpha_0^2 \log_2 36 + 2\alpha_0^2(\alpha_1^2 - \alpha_0^2) \log_2 18$$

$$+ 2\alpha_0^2(\alpha_1^2 - \alpha_0^2) \log_2 18 + (\alpha_1^2 - \alpha_0^2)(\alpha_2^2 - \alpha_0^2) \log_2 9.$$  

(9)

In addition, the above scheme needs $\log_2 4$ bits of classical information for Alice to tell Bob her measurement result on the two auxiliary particles.

We can also generalize the above scheme to arbitrarily different dimensions Hilbert space for $N$ non-maximally entangled pairs. Alice and Bob need to share $N$ non-maximally entangled pairs. One particle of each entangled pair in $p$ dimension belongs to Alice, and the other in $q$ dimension belongs to Bob (that it to say, particles 1, 2, \ldots, $N$ belong to Alice, and particles 1’, 2’, \ldots, $N'$ belong to Bob), where $p \neq q$, i.e., the Hilbert space of Alice’s particles is non-symmetric with that of Bob’s particles. Without loss of generality, we choose $p > q$.

| The results of particles $a_1$ and $a_2$ | The states which Alice ensures | The unitary operators which Alice makes | The corresponding states | The number of the states |
|---------------------------------|---------------------------------|------------------------------------------|------------------------|-------------------------|
| $|0\rangle a_1|0\rangle a_2$ | $|\Psi^0_{00}\rangle_{11'} \otimes |\Psi^0_{00}\rangle_{22'}$ | $(U^0_{\text{monog}})^1 \otimes (U^0_{\text{monog}})^2$ | $|\Psi^0_{\text{monog}}\rangle_{11'} \otimes |\Psi^0_{\text{monog}}\rangle_{22'}$ | 36 |
| $|0\rangle a_1|1\rangle a_2$ | $|\Psi^0_{00}\rangle_{11'} \otimes |\Psi^1_{00}\rangle_{22'}$ | $(U^0_{\text{monog}})^1 \otimes (U^1_{\text{monog}})^2$ | $|\Psi^0_{\text{monog}}\rangle_{11'} \otimes |\Psi^1_{\text{monog}}\rangle_{22'}$ | 18 |
| $|1\rangle a_1|0\rangle a_2$ | $|\Psi^1_{00}\rangle_{11'} \otimes |\Psi^0_{00}\rangle_{22'}$ | $(U^1_{\text{monog}})^1 \otimes (U^0_{\text{monog}})^2$ | $|\Psi^1_{\text{monog}}\rangle_{11'} \otimes |\Psi^0_{\text{monog}}\rangle_{22'}$ | 18 |
| $|1\rangle a_1|1\rangle a_2$ | $|\Psi^1_{00}\rangle_{11'} \otimes |\Psi^1_{00}\rangle_{22'}$ | $(U^1_{\text{monog}})^1 \otimes (U^1_{\text{monog}})^2$ | $|\Psi^1_{\text{monog}}\rangle_{11'} \otimes |\Psi^1_{\text{monog}}\rangle_{22'}$ | 9 |
The total state which Alice and Bob share is

$$|\Psi_T\rangle = \bigotimes_{k=1}^{N} (\alpha_{0k}|00\rangle + \alpha_{1k}|11\rangle + \cdots + \alpha_{(q-1)k}|q-1q-1\rangle)_{kk'},$$  \(\text{(10)}\)

where \(\alpha_{0k}, \alpha_{1k}, \cdots, \alpha_{(q-1)k}\) are real numbers and satisfy \(|\alpha_{0k}| \leq |\alpha_{1k}| \leq \cdots \leq |\alpha_{(q-1)k}|\),

Similarly, the scheme of probabilistic dense coding can be realized by the following steps.

Firstly, Alice introduces \(N\) auxiliary \(q\)-level particles in the quantum state \(\bigotimes_{k=1}^{N} |0\rangle_{ak}\). Then she performs a proper unitary transformation on her particles and the auxiliary particles. The unitary transformation \(U = \bigotimes_{k=1}^{N} U_{ak}\) transforms the state \(|\Psi_T\rangle \otimes_{k=1}^{N} |0\rangle_{ak}\) into the state

$$|\Psi_T\rangle = \bigotimes_{k=1}^{N} \{\alpha_{0k}(|00\rangle + |11\rangle + \cdots + |q-1q-1\rangle)_{kk'}|0\rangle_{ak}$$

$$+ \sqrt{\alpha_{1k}^2 - \alpha_{0k}^2} (|11\rangle + \cdots + |q-1q-1\rangle)_{kk'}|1\rangle_{ak}$$

$$+ \cdots$$

$$+ \sqrt{\alpha_{(q-1)k}^2 - \alpha_{(q-2)k}^2} (q-1q-1\rangle)_{kk'}|q-1\rangle_{ak}$$

$$= \bigotimes_{k=1}^{N} \left\{\sqrt{q\alpha_{0k}} \frac{1}{\sqrt{q}} (|00\rangle + |11\rangle + \cdots + |q-1q-1\rangle)_{kk'}|0\rangle_{ak}$$

$$+ \sqrt{q-1} \alpha_{1k} \frac{1}{\sqrt{q-1}} \sqrt{\alpha_{0k}^2 - \alpha_{1k}^2} (|11\rangle + \cdots + |q-1q-1\rangle)_{kk'}|1\rangle_{ak}$$

$$+ \cdots$$

$$+ \sqrt{\alpha_{(q-1)k}^2 - \alpha_{(q-2)k}^2} (q-1q-1\rangle)_{kk'}|q-1\rangle_{ak}\right\}.  \(\text{(11)}\)

Then Alice makes orthogonal measurement on the auxiliary particles. She ensures that the quantum channel will be in the states \(\bigotimes_{k=1}^{N} \frac{1}{\sqrt{q}} (|00\rangle + |11\rangle + \cdots + |q-1q-1\rangle)_{kk'}, \bigotimes_{k=1}^{N} \frac{1}{\sqrt{q}} (|00\rangle + |11\rangle + \cdots + |q-1q-1\rangle)_{kk'} \bigotimes_{k=1}^{N} \frac{1}{\sqrt{q-1}} (|11\rangle + \cdots + |q-1q-1\rangle)_{NN'}, \cdots, \) or \(\bigotimes_{k=1}^{N} |q-1q-1\rangle_{kk'}\) corresponding to the results of auxiliary particles \(\bigotimes_{k=1}^{N} |0\rangle_{ak}, \bigotimes_{k=1}^{N} |0\rangle_{ak} |1\rangle_{aN}, \cdots, \) or \(\bigotimes_{k=1}^{N} |q-1\rangle_{ak}.\) The corresponding probabilities of the results of auxiliary particles are

$$q^N \prod_{k=1}^{N} \alpha_{0k}^2, q^{N-1}(q-1) \prod_{k=1}^{N-1} \alpha_{0k}^2 (\alpha_{1N}^2 - \alpha_{0N}^2), \cdots, \prod_{k=1}^{N} (\alpha_{(q-1)k}^2 - \alpha_{(q-2)k}^2).$$

Secondly, Alice encodes classical information by making a unitary transformation on her particles 1, 2, \cdots, and \(N\). According to the results of the auxiliary particles \(\bigotimes_{k=1}^{N} |0\rangle_{ak},\)
| particles according to Alice’s measurement result on the auxiliary particles. Then Bob can
| Her particles are \( \bigotimes_{k=1}^{N} |q - 1\rangle_{a_k} \), \( \bigotimes_{k=1}^{N} (U_{m_0n_0})^k \), \( \bigotimes_{k=1}^{N} (U_{m_1n_1})^k \), \( \bigotimes_{k=1}^{N} (U_{m_2n_2})^k \). The corresponding states are \( \bigotimes_{k=1}^{N} |\Psi_{m_0n_0}\rangle_{kk'} \), \( \bigotimes_{k=1}^{N} |\Psi_{m_1n_1}\rangle_{kk'} \), \( \bigotimes_{k=1}^{N} |\Psi_{m_2n_2}\rangle_{kk'} \). The unitary operations \( U_{m_0n_0} \), \( U_{m_1n_1} \), \( U_{m_2n_2} \) are showed as follows:

\[
U_{m_0n_0} = \sum_{j=0}^{q-1} e^{2\pi ij_0/q} |(j \oplus m_0)\rangle \langle j|,
\]

\[
U_{m_1n_1} = \sum_{j=0}^{q-1} e^{2\pi ij_1/q} |(j \oplus m_1)\rangle \langle j|,
\]

\[
\ldots
\]

\[
U_{m_{q-1}n_{q-1}} = \sum_{j=0}^{q-1} e^{2\pi ij_{q-1}/q} |(j \oplus m_{q-1})\rangle \langle j|;
\]

the states \( |\Psi_{m_0n_0}\rangle_1 |\Psi_{m_1n_1}\rangle_1 \), \( \ldots \), \( |\Psi_{m_{q-1}n_{q-1}}\rangle_1 \) are

\[
|\Psi_{m_0n_0}\rangle = \sum_{j=0}^{q-1} e^{2\pi ij_0/q} |(j \oplus m_0)\rangle \langle j|/\sqrt{q},
\]

\[
|\Psi_{m_1n_1}\rangle = \sum_{j=1}^{q-1} e^{2\pi ij_1/q} |(j \oplus m_1)\rangle \langle j|/\sqrt{q-1},
\]

\[
\ldots
\]

\[
|\Psi_{m_{q-1}n_{q-1}}\rangle = \sum_{j=q-1}^{q-1} e^{2\pi ij_{q-1}/q} |(j \oplus m_{q-1})\rangle \langle j|;
\]

where \( m_0, m_1, \ldots, m_{q-1} = 0, 1, \ldots, p - 1; n_0 = 0, 1, \ldots, q - 1; n_1 = 0, 1, \ldots, q - 2; n_2 = 0, 1, \ldots, q - 3; \ldots; n_{q-1} = 0. \)

Thirdly, Alice sends her particles to Bob, and tells him her measurement result of the auxiliary particles.

Finally, Bob receives Alice’s particles 1, 2, ⋅⋅⋅, and \( N \), and makes measurement on all his particles according to Alice’s measurement result on the auxiliary particles. Then Bob can obtain the classical information that Alice has encoded on her particles via his measurement.

Obviously, the average information Bob can obtain is

\[
I_{ave} = q^N \prod_{k=1}^{N} \alpha_{0_k}^2 \log_2 \left[ (p \times q)^N \right] + q^{N-1}(q - 1) \prod_{k=1}^{N-1} \alpha_{0_k}^2 (\alpha_{1k}^2 - \alpha_{0k}^2) \log_2 \left[ (p \times q)^{N-1}p(q - 1) \right]
\]
FIG. 1: The relationship of $\alpha_{01}, \alpha_{02}$ and $I_{\text{ave}}$

\[ + q^{N-1} (q-2) \prod_{k=1}^{N-1} \left( \alpha_{0_k}^2 \right) \left( \alpha_{2_N}^2 - \alpha_{1_N}^2 \right) \log_2 \left[ (p \times q)^{N-1} p (q-2) \right] \]

\[ + \cdots + \prod_{k=1}^{N} \left( \alpha_{(q-1)_k}^2 - \alpha_{(q-2)_k}^2 \right) \log_2 \left[ p^N \right]. \]  

(14)

In addition, this scheme consumes $N \log_2 q$ bits of classical information to transmit Alice’s measurement results of the auxiliary particles.

We discuss the average information transformation $I_{\text{ave}}$. When the quantum channel is composed of two entangled pairs, we can draw a figure about $\alpha_{01}, \alpha_{02}$ and $I_{\text{ave}}$. So from FIG. 1 we can see $I_{\text{ave}}$ clearly. When $\alpha_{01}^2 = 0.5$ and $\alpha_{02}^2 = 0.5$, $I_{\text{ave}}$ is more than 5, i.e., when the two entangled pairs are in the maximally entangled states, $I_{\text{ave}}$ has the maximal value $\log_2 36$, more than 5 bits information. Similarly, when we generalize this scheme to $N$ entangled pairs, if all the entangled pairs are in maximally entangled states, $I_{\text{ave}}$ has the maximal value $N \log_2 (p \times q)$.

In this scheme, we discuss probabilistic dense coding via two non-symmetrically and non-maximally entangled pairs as quantum channel and generalize it to $N$ non-symmetrically and non-maximally entangled pairs. We also consider the maximal value of the average information transformation. The average information has maximal value when all entangled pairs are in the maximally entangled states.

Comparing with Ref [12] (not probabilistic and with multipartite quantum channel) which operates with the same physical resource, our scheme is more general, because the maximally
entangled state is not only difficult for permanently because of decoherence, but also is hard to be prepared in the experiment. Therefore, we chose the non-maximally entangled state as quantum channel. A comparison of the symmetric quantum channel \cite{13} with our non-symmetric multipartite quantum channel shows our scheme can increase the efficiency of information transmission. Comparing with Ref. \cite{14}, we use different quantum channel and different path to realize probabilistic dense coding. In their scheme, the quantum channel is composed of two particles and in the same dimensions, and they performed Positive Operator Valued Measurements (POVMs) on the qubit states to distinguish these non-orthogonal states. They find that the success probability of performing super dense coding is exactly the same as the success probability of distinguishing a set of non-orthogonal. But in our scheme, we use non-symmetric multipartite state as quantum channel. The non-symmetric multipartite state can be converted into orthogonal states by introducing a set of auxiliary particles and making some unitary operations. And these orthogonal states can be distinguished only by some simple measurements. The probability depends on the measurement results of auxiliary particles. In conclusion, we have proposed a general efficiency scheme for dense coding.

\begin{itemize}
  \item \cite{1} C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. \textbf{69}, 2881 (1992).
  \item \cite{2} A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. \textbf{47}, 777 (1935).
  \item \cite{3} E. Schrödinger, Proc. Cambridge Philos. Soc. \textbf{31}, 555 (1935).
  \item \cite{4} T. Hiroshima, J. Phys. A \textbf{34}, 6907 (2001).
  \item \cite{5} V. N. Gorbachev, A. I. Trubilko, A. I. Zhiliba, and E. S. Yakovleva, eprint quant-ph/0011124.
  \item \cite{6} J. L. Cereceda, eprint quant-ph/0105096.
  \item \cite{7} X. S. Liu, G. L. Long, D. M. Tong, and F. Li, Phys. Rev. A \textbf{65}, 022304 (2002).
  \item \cite{8} G. Rigolin, eprint quant-ph/0407193.
  \item \cite{9} X. M. Lin, Z. W. Zhou, P. Xue, Y. J. Gu, and G. C. Guo, Phys. Lett. A \textbf{313}, 351 (2003).
  \item \cite{10} L. Ye and G. C. Guo, Phys. Rev. A \textbf{71}, 034304 (2005).
  \item \cite{11} F. L. Yan and M. Y. Wang, Chin. Phys. Lett. \textbf{21}, 1195 (2004).
  \item \cite{12} Q. B. Fan, L. L. Sun, and S. Zhang, J Kor. Phys. Soc. \textbf{46}, 769 (2005).
  \item \cite{13} M. Y. Wang, L. G. Yang, and F. L. Yan, Chin. Phys. Lett. \textbf{22}, 1053 (2005).
\end{itemize}
[14] A. K. Pati, P. Parashar, and P. Agrawal, eprint quant-ph/0412039.

[15] D. Bruß, G. M. D’Ariano, M. Lewenstein, C. Macchiavello, A. Sen(De), and U. Sen, Phys. Rev. Lett. 93, 210501 (2004).