Existence of solution for Mean-field Reflected Discontinuous Backward Doubly Stochastic Differential Equation

Mostapha Abdelouahab Saouli †

Laboratory of Applied Mathematics, University of Biskra, POB 145, Algeria
Department of Mathematics, University of Kasdi Merbah Ouargla, Algeria

Abstract

In this paper we prove the existence of a solution for mean-field reflected backward doubly stochastic differential equations (MF-RBDSDEs) with one continuous barrier and discontinuous generator (left-continuous). By a comparison theorem establish here for MF-RBDSDEs, we provide a minimal or a maximal solution to MF-RBDSDEs.

Keywords: Reflected Backward doubly stochastic differential equations; Mean-field; Minimal solution; Comparison theorem; Discontinuous generator

AMS 2010 codes: 60H10, 60H05.

1 Introduction

The theory of nonlinear backward stochastic differential equations (BSDEs in short) have been first introduced by Pardoux and Peng [6] (1990). They proved the existence and uniqueness of the adapted processes \((Y,Z)\), solution of the following equation:

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s, \quad 0 \leq t \leq T, \]

where the terminal value \(\xi\) is square integrable and the coefficient \(f\) is uniformly Lipschitz in \((y,z)\), several authors interested in weakening this assumption; In [5] (1997), the authors prove the existence of a solution for one dimensional backward stochastic differential equations where the coefficient is continuous and it has a linear growth, they also obtain the existence of a minimal solution. In [3] (2008) the author prove the existence of the solution to BSDEs whose coefficient may be discontinuous in \(y\) and continuous in \(z\).

†Corresponding author.
Email address: saoulimoustapha@yahoo.fr saoulimoustapha@yahoo.fr
A new kind of backward stochastic differential equations was introduced by Pardoux and Peng [7] in (1994) which is a class of backward doubly stochastic differential equation (BDSDE for short) of the form:

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \]

with two different directions of stochastic integrals, i.e., the equation involves both a standard (forward) stochastic integral \(dW_t\) and a backward stochastic integral \(dB_t\) and \(\xi\) is a random variable termed the terminal condition.

After the authors have proved an existence and unique solution when \(f\) and \(g\) are uniform Lipschitz, several authors interested to weakening this assumption, see [4]. In [9](2005) the authors obtained the existence of the solution of BDSDE under continuous assumption and gave the comparison theorem for one dimensional BDSDE.

On the other hand Bahlali et al [1] (2009) introduced a special class of reflected BDSDEs (RBDSDEs in short) which is a BDSDE but the solution is forced to stay above a lower barrier. In particular, a solution of such equation is a triplet of processes \((Y, Z, K)\) satisfying

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) dB_s + \int_t^T dK_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \]

and \(Y_t \geq S_t\) a.s. for any \(t \in [0, T]\). The role of the nondecreasing continuous process \((K_t)_{t \in [0, T]}\) is to push upward the process \(Y\) in order to keep it above \(S\), it satisfies the skorohod condition

\[ \int_0^T (Y_s - S_s) dK_s = 0. \]

In this paper, motivated by the above results and by the result introduced by Xu, R. (2012) [10], we establish the existence of the a minimal solution to the following reflected MF-BDSDE,

\[ Y_t = \xi + \int_t^T E \left( f(s, \omega, \omega', Y_s, (Y_s)'(Z_s)'(Z_s)'(Z_s)'(Z_s)') \right) ds + \int_t^T dK_s + \int_t^T E \left( g(s, \omega, \omega', Y_s, (Y_s)'(Z_s)'(Z_s)'(Z_s)'(Z_s)') \right) dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \]

whose coefficient may be discontinuous in \(y\) and continuous in \(z\).

- **In Section 2**, we give some preliminaries about MF-BDSDE with one continuous barrier.
- **In Section 3**, under certain assumptions, we obtain the existence for a minimal solution to the Mean-field backward doubly stochastic differential equation with one continuous barrier and discontinuous generator (left-continuous).

2 Framework

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. For \(T > 0\), let \(\{W_t, 0 \leq t \leq T\}\) and \(\{B_t, 0 \leq t \leq T\}\) be two independent standard Brownian motion defined on \((\Omega, \mathcal{F}, P)\) with values in \(\mathbb{R}^d\) and \(\mathbb{R}\), respectively. Let \(\mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t)\), and \(\mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \leq s \leq T)\), completed with \(P\)-null sets. We put,

\[ \mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B. \]

It should be noted that \((\mathcal{F}_t)\) is not an increasing family of sub \(\sigma\)-fields, and hence it is not a filtration.

Let \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) = (\Omega \times \Omega, \bar{\mathcal{F}} \otimes \bar{\mathcal{F}}, P \otimes P)\) be the (non-completed) product of \((\Omega, \mathcal{F}, P)\) with itself. We denote the filtration of this product space by \(\bar{\mathcal{F}} = \{\mathcal{F}_t = \mathcal{F}_t \otimes \mathcal{F}_t, 0 \leq t \leq T\}\).
Remark \( K \equiv 2 \) (assumptions on the data \( K \)). The terminal value \( \bar{\xi} \) is defined by \( \bar{\xi}(\omega) = \xi(\omega) |_{T} \) and \( \bar{\xi} \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d) \). The state process \( \Delta Y \) is a continuous progressively measurable real valued process satisfying \( E \left( \sup_{0 \leq t \leq T} |Y_t|^2 \right) < \infty \). Its expectation by \( \bar{\xi} \):

\[
\bar{\xi}(\omega) = \xi(\omega) |_{T} = \bar{\xi}(\omega) \in \bar{\Omega} = \Omega \times \Omega.
\]

Remark for every \( \theta \in L^1(\bar{\Omega}, \mathcal{F}, P) \), the variable \( \theta(\cdot, \omega) : \Omega \rightarrow \mathbb{R} \) belongs to \( L^1(\bar{\Omega}, \mathcal{F}, P) \). We denote its expectation by \( \bar{E}(\theta(\cdot, \omega)) = \int_{\Omega} \theta(\omega, \omega) P(d\omega) \).

Notice that

\[
\begin{align*}
\bar{E}(\theta) &= \bar{E}(\theta(\cdot, \omega)) \\
\bar{E}(\theta) &= \int_{\Omega} \theta(\omega, \omega) P(d\omega) = E(\bar{E}(\theta)).
\end{align*}
\]

We consider the following spaces of processes:

- \( \mathcal{M}^2(0, T, \mathbb{R}^d) \) denote the set of \( d \)-dimensional \( \mathcal{F}_t \)-progressively measurable processes \( \{\varphi_t; t \in [0, T]\} \), such that \( \mathbb{E} \int_0^T |\varphi_t|^2 dt < \infty \).

- We denote by \( \mathcal{S}^2(0, T, \mathbb{R}^d) \), the set of \( \mathcal{F}_t \)-adapted càdlàg processes \( \{\varphi_t; t \in [0, T]\} \), which satisfy \( \mathbb{E}(\sup_{0 \leq t \leq T} |\varphi_t|^2) < \infty \).

- \( \mathcal{S}^2 \) set of continuous, increasing. \( \mathcal{F}_t \)-adapted process \( K : [0, T] \times \Omega \rightarrow [0, +\infty) \) with \( K_0 = 0 \) and \( \mathbb{E}(K_T)^2 < +\infty \).

- \( \mathbb{L}^2 \) set of \( \mathcal{F}_T \) measurable random variables \( \xi : \Omega \rightarrow \mathbb{R} \) with \( \mathbb{E} |\xi|^2 < +\infty \).

**Definition 1.** A solution of equation (2) is a triple \( (Y, Z, K) \) which belongs to the space \( \mathcal{S}^2(0, T, \mathbb{R}^d) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{S}^2 \) and satisfies (2) such that:

\[
\left\{ \begin{array}{l}
S_t \leq Y_t, \quad 0 \leq t \leq T, \\
\int_0^T (Y_t - L_s) dK_s = 0.
\end{array} \right.
\]

**Remark 1.** In the case where \( S = -\infty \) (i.e., MF-BDSDEs without lower barrier), the process \( K \) has no effect i.e., \( K \equiv 0 \).

**Remark 2.** In the setup of system (2) the process \( S(\cdot) \) play the role of reflecting barrier.

**Remark 3.** The state process \( Y(\cdot) \) is forced to stay above the lower barrier \( S(\cdot) \), thanks to the action of the increasing reflection process \( K(\cdot) \).

The coefficient of mean-field Reflected BDSDE is a function. We assume that \( f \) and \( g \) satisfy the following assumptions on the data \((\xi, f, g, S)\):

- \( \textbf{(H.1)} \) The terminal value \( \xi \) be a given random variable in \( \mathbb{L}^2 \).

- \( \textbf{(H.2)} \)  \( S_t \geq 0 \), is a continuous progressively measurable real valued process satisfying

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left( S_t^+ \right)^2 \right) < +\infty, \quad \text{where} \quad S_t^+ := \max(S_t, 0).
\]

- \( \textbf{(H.3)} \) For \( t \in [0, T] \), \( S_T \leq \xi \), \( \mathbb{P} \)-almost surely.

- \( \textbf{(H.4)} \) \( f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \); \( g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) be jointly measurable such that for any \( (y, y', z, z') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \),

\[
\left\{ \begin{array}{l}
f(\cdot, \omega, y, y', z, z') \in \mathcal{M}^2(0, T, \mathbb{R}^d), \\
g(\cdot, \omega, y, y', z, z') \in \mathcal{M}^2(0, T, \mathbb{R}^d).
\end{array} \right.
\]
Mostapha Abdelouahab Saouli. (2020). Applied Mathematics and Nonlinear Sciences, 5(2020), 205–216.

Following assumptions:

(2) Under the assumptions (H.1), (H.2), (H.3), and (H.4),

We recall the following existence results.

In this section we are interested in weakening the conditions on \( f \) and \( g \).

- **(H.6)** (i) For a.e \( (t, \omega) \) the mapping \( (y, y', z, f) \rightarrow f(t, y, y', z, f) \) is continuous.
- (ii) There exist constants \( C \geq 0 \) and \( 0 \leq \alpha \leq \frac{1}{2} \) such that for every \((\omega, t) \in \Omega \times [0, T] \) and \((y, y') \in \mathbb{R}^2 \times \mathbb{R}^d \),

\[
\left| f(t, y, y', z, f) \right| \leq C \left( 1 + |y| + |y'| + |z| + |f| \right),
\]

\( g \) satisfies (H.2) (ii).

We recall the following existence results.

**Proposition 1.** ([2] (2014)). Under the assumptions (H.1)-(H.5) the reflected BDSDE (2) has a unique solution \((Y, Z, K) \in \mathcal{S}^2(0, T, \mathbb{R}^d) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{A}^2\).

### 3 Existence result

In this section we are interested in weakening the conditions on \( f \). We assume that \( f \) and \( g \) satisfy the following assumptions:

- **(H.7)** Linear growth: There exists a nonnegative process \( f \in \mathcal{M}^2(0, T, \mathbb{R}^d) \) such that

\[
\forall (t, y, y', z) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^d, \quad \left| f(t, y, y', z) \right| \leq f(\omega) + C \left( |y| + |y'| + |z| \right).
\]

- **(H.8)** \( f(t, \cdot, \cdot, z) : \mathbb{R} \rightarrow \mathbb{R} \) is a left continuous function and \( f(t, y, \cdot, z) \) is a continuous function.

- **(H.9)** There exists a continuous function \( \pi : [0, T] \times \mathbb{R}^2 \times \mathbb{R}^d \) satisfying for \( y_1 \geq y_2, \quad (y_1', y_2') \in \mathbb{R}^2 \), \((z_1, z_2) \in \mathbb{R}^d\)

\[
\left\{ \begin{array}{l}
\left| \pi(t, y, y', z) \right| \leq C \left( |y| + |y'| + |z| \right), \\
\left| f(t, \omega, y_1, y_1', z_1) - f(t, \omega, y_2, y_2', z_2) \right| \geq \pi(t, y_1 - y_2, y_1' - y_2', z_1 - z_2)
\end{array} \right.
\]

- **(H.10)** Monotonicity in \( y' \): \( \forall (y, y', z), \ f(t, y, y', z) \) is increasing in \( y' \).

- **(H.11)** \( g \) satisfies (H.5) (ii) and \( g(t, 0, 0, 0) \equiv 0 \).
Hence, we only consider the following type of Mean-field reflected BDSDE:

\[
Y_t = \xi + \int_t^T E^\prime \left( f(s, \omega^t, Y_s, (\tilde{Y}_s)^{\prime}, Z_s) \right) ds + \int_t^T dK_s \\
+ \int_t^T E^\prime \left( g(s, \omega^t, Y_s, (\tilde{Y}_s)^{\prime}, Z_s) \right) d\tilde{B}_s - \int_t^T Z_s dW_s, \ 0 \leq t \leq T.
\] (3)

**Proposition 2.** [2] (2014). Under the assumption (H.1)-(H.4) and (H.6), and for any random variable \( \xi \in \mathbb{L}^2 \), the mean-field RDSDSE (3) a has an adapted solution \((Y, Z, K) \in \mathcal{S}^2(0, T, \mathbb{R}) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{S}^2 \), which is a minimal one, in the sense that, if \((Y^*, Z^*, K^*)\) is any other solution we \( Y \leq Y^*, \ P - a.s. \)

Now we prove a technical Lemma before we introduce the main theorem.

**Lemma 3.** Let \( \pi(t, y, y^t, z) \) satisfies (H.9), \( g \) satisfies (H.11) and \( h \) belongs in \( \mathcal{M}^2(0, T, \mathbb{R}^d) \). For a continuous function of finite variation \( \hat{K} \) belong in \( \mathcal{S}^2 \) we consider the processes \((\tilde{Y}, \tilde{Z}) \in \mathcal{S}^2(0, T, \mathbb{R}) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \) such that:

\[
\begin{align*}
(i) \quad \tilde{Y}_t = \xi + \int_t^T E^\prime \left( \pi(s, \omega^t, \tilde{Y}_s, (\tilde{Y}_s)^{\prime}, \tilde{Z}_s) \right) ds + \int_t^T d\hat{K}_s \\
+ \int_t^T E^\prime \left( g(s, \omega^t, \tilde{Y}_s, (\tilde{Y}_s)^{\prime}, \tilde{Z}_s) \right) d\tilde{B}_s - \int_t^T \tilde{Z}_s dW_s, \ 0 \leq t \leq T, \\
(ii) \quad \int_0^T \tilde{Y}_t d\hat{K}_t \geq 0.
\end{align*}
\] (4)

Then we have

(i) The MF-RBDSDE (4) has at least one solution \((\tilde{Y}, \tilde{Z}, \tilde{K}) \in \mathcal{S}^2(0, T, \mathbb{R}) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{S}^2 \)

(ii) if \( h(t) \geq 0 \) and \( \xi \geq 0 \), we have \( \tilde{Y}_t \geq 0, \ d\mathbb{P} \times dt - a.s. \)

**Proof.** (i) See [2], (2014). (ii) Applying Tanaka’s formula to \( |\tilde{Y}_t^-|^2 \), we have

\[
\mathbb{E} |\tilde{Y}_t^-|^2 \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^t < 0\}} |\tilde{Z}_s|^2 ds = \mathbb{E} |\xi^-|^2 - 2\mathbb{E} \int_t^T \tilde{Y}_s^- E^\prime \left( \pi(s, \omega^t, (\tilde{Y}_s)^{\prime}, \tilde{Z}_s) \right) ds \\
- 2\mathbb{E} \int_t^T \tilde{Y}_s^- d\hat{K}_s + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^t < 0\}} \left| E^\prime \left( g(s, \omega^t, (\tilde{Y}_s)^{\prime}, \tilde{Z}_s) \right) \right|^2 ds.
\]

Since \( -2\mathbb{E} \int_t^T \tilde{Y}_s^- d\hat{K}_s \leq 0, \ h(s) \geq 0 \) and \( \xi \geq 0 \), we get

\[
\mathbb{E} |\tilde{Y}_t^-|^2 + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^t < 0\}} |\tilde{Z}_s|^2 ds \leq -2\mathbb{E} \int_t^T \tilde{Y}_s^- E^\prime \left( \pi(s, \omega^t, (\tilde{Y}_s)^{\prime}, \tilde{Z}_s) \right) ds \\
+ \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^t < 0\}} \left| E^\prime \left( g(s, \omega^t, (\tilde{Y}_s)^{\prime}, \tilde{Z}_s) \right) \right|^2 ds.
\]

By (H.9), we get \( \pi(s, \tilde{Y}_s, (\tilde{Y}_s)^{\prime}, \tilde{Z}_s) \leq C \left( |\tilde{Y}_s| + |(\tilde{Y}_s)^{\prime}| + |\tilde{Z}_s| \right) \) and by assumption (H.11) for \( g \), we have

\[
\mathbb{E} |\tilde{Y}_t^-|^2 + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^t < 0\}} |\tilde{Z}_s|^2 ds \leq \left( 4C^2 + \frac{C^2}{P} + 2C \right) \mathbb{E} \int_t^T |\tilde{Y}_s^-|^2 ds + (\alpha + \beta) \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^t < 0\}} |\tilde{Z}_s|^2 ds.
\]

Therefore, choosing \( 0 \leq \beta \leq 1 - \alpha \) and using Gronwall inequality, we have \( \tilde{Y}_t^- = 0, \ \mathbb{P} - a.s., \ \forall t \in [0, T] \), which implies that \( \tilde{Y}_t \geq 0, \ \mathbb{P} - a.s., \ \forall t \in [0, T] \). \( \square \)
Before we prove the main result, we construct a sequence of MF-RBDSDEs as follows:

\[
\begin{align*}
\bar{Y}_t^n &= \xi + \int_t^T E' \left( -C \left( |\bar{Y}_s^n| + |\bar{Z}_s^n| \right) - f_s \right) ds \\
&\quad + \int_t^T E' \left( g \left( s, \bar{Y}_s^n, \bar{Z}_s^n \right) \right) d\bar{B}_s + \int_t^T d\bar{K}_s^n - \int_t^T \bar{Z}_s^n dW_s, \quad 0 \leq t \leq T,
\end{align*}
\]

(ii) \( \bar{Y}_0^n \geq S_t \),

(iii) \( \int_0^T (\bar{Y}_s^n - S_s) d\bar{K}_s^n = 0. \)

(5)

\[
\begin{align*}
\bar{Y}_t^n &= \xi + \int_t^T E' \left( f \left( s, \bar{Y}_s^n, \bar{Z}_s^n \right) + \pi \left( s, \bar{Y}_s^n, \bar{Z}_s^n \right) \right) ds \\
&\quad + \int_t^T E' \left( g \left( s, \bar{Y}_s^n, \bar{Z}_s^n \right) \right) d\bar{B}_s - \int_t^T \bar{Z}_s^n dW_s, \quad 0 \leq t \leq T,
\end{align*}
\]

(ii) \( \bar{Y}_0^n \geq S_t \),

(iii) \( \int_0^T (\bar{Y}_s^n - S_s) d\bar{K}_s^n = 0. \)

(6)

(7)

For these solutions above, we get some properties as follows:

**Lemma 4.** Under the assumptions (H.1) - (H.4) and (H.7) - (H.11), we have for any \( n \geq 1 \) and \( t \in [0, T] \)

\[ \bar{Y}_t^n \leq \bar{Y}_t^{n+1} \leq \bar{Y}_t^0. \]

**Proof.** We will prove \( \bar{Y}_t^0 \leq \bar{Y}_t^n \) at first. By Eqs. (5), and (6), we have

\[
\begin{align*}
\bar{Y}_t^1 - \bar{Y}_t^0 &= \int_t^T E' \left( \pi \left( s, \delta \bar{Y}_s^1, \delta (\bar{Y}_s^1) \right), \delta \bar{Z}_s^1 \right) + \Lambda_t^1 \right) ds \\
&\quad + \int_t^T E' \left( g \left( s, \bar{Y}_s^1, \bar{Z}_s^1 \right) \right) d\bar{B}_s - \int_t^T \delta \bar{Z}_s^1 dW_s,
\end{align*}
\]

where \( \Lambda_t^1 = f \left( s, \bar{Y}_s^0, \bar{Z}_s^0 \right) + C \left( |\bar{Y}_s^0| + |\bar{Z}_s^0| \right) + f_s \). By hypothesis (H.7) we have \( \Lambda_t^1 \geq 0 \), because \((\bar{Y}_t^0, \bar{Z}_t^0)\) is the solution of Eq. (5), we get \( \Lambda_t^1 \in \mathcal{M}^2 (0, T, \mathbb{R}^d) \). Therefore, from Lemma 3 we get \( \bar{Y}_t^1 \geq \bar{Y}_t^0 \). Now we want to prove \( \bar{Y}_t^n \leq \bar{Y}_t^{n+1} \), for any \( n \geq 0 \). We set

\[
\begin{align*}
\delta \rho_t^n + \rho_t^{n+1} - \rho_t^n, \\
\Delta \psi^n + \delta \bar{Y}_t^{n+1} + \delta (\bar{Y}_t^{n+1}) + \delta \bar{Z}_t^{n+1} + \Lambda_t^{n+1} = \psi \left( s, \delta \bar{Y}_s^n, \delta (\bar{Y}_s^n) + \delta \bar{Z}_s^n \right) - \psi \left( s, \bar{Y}_s^n, \bar{Z}_s^n \right).
\end{align*}
\]
Using Eq. (6), we have

\[
\delta Y_t^{n+1} = \int_t^T E' \left(\pi \left( s, \delta Y_s^{n+1}, \delta (\bar{Y}_s^{n+1})', \delta Z_s^{n+1} \right) + \theta_s^{n+1} \right) \, ds - \int_t^T \delta Z_s^{n+1} \, dW_s \\
+ \int_t^T E' \left( \Delta g^{n+1}(s, \delta Y_t^{n+1}, \delta (\bar{Y}_s^{n+1})', \delta Z_s^{n+1}) \right) \, d\tilde{B}_s \\
+ \int_t^T d(\delta \bar{K}_s^{n+1}),
\]

where \( \theta_s^{n+1} = \Delta f^n(s, \delta Y_s^n, \delta (\bar{Y}_s^n)', \delta Z_s^n) - \pi \left( s, \delta Y_s^n, \delta (\bar{Y}_s^n)' \right) \) and \( \theta_0^n = \Lambda_1^n \), \( \forall n \geq 0 \). According to its definition, one can show that \( \theta_0^n \) and \( \Delta g^{n+1}, \forall n \geq 0 \) satisfy all assumption of Lemma 3. Moreover, since \( \bar{K}_s^n \) is a continuous and increasing process, for all \( n \geq 0 \), \( \delta \bar{K}_s^{n+1} \) is a continuous process of finite variation and, using the same argument as one appear in [2], on can show that

\[
\int_0^T (\bar{Y}_s^n - \bar{Y}_s) \, d(\delta \bar{K}_s^{n+1}) = \int_0^T (\bar{Y}_s^{n+1} - \bar{Y}_s^n) \, d\bar{K}_s^n - \int_0^T (\bar{Y}_s^n - \bar{Y}_s) \, d\bar{K}_s^n \\
= \int_0^T (\bar{Y}_s^{n+1} - \bar{Y}_s^n) \, d\bar{K}_s^n + \int_0^T (\bar{Y}_s^n - \bar{Y}_s) \, d\bar{K}_s^n \geq 0,
\]

by Lemma 3, we deduce that \( \delta Y_t^{n+1} \geq 0 \), i.e. \( \bar{Y}_t^{n+1} \geq \bar{Y}_t^n \forall t \in [0, T] \), we have

\[
\bar{Y}_t^{n+1} \geq \bar{Y}_t^n \geq \bar{Y}_t^0.
\]

Now we shall prove that \( \bar{Y}_t^{n+1} \leq \bar{Y}_t^n \forall n \geq 0 \), by Eqs. (3) and (7)

\[
Y_0^n - Y_t^{n+1} = \int_t^T E' \left( -C \left( |Y_0^n - Y_s^{n+1}| + |(Y_t^n)' - (\bar{Y}_s^{n+1})'| + |Z_s^n - Z_s^{n+1}| \right) + \Lambda_s^{n+1} \right) \, ds \\
+ \int_t^T E' \left( g(s, Y_0^n, (Y_t^n)', Z_s^n) - g(s, s, \bar{Y}_s^n, (\bar{Y}_s^n)', Z_s^n) \right) \, d\tilde{B}_s \\
+ \int_t^T (d\bar{K}_s^0 - d\bar{K}_s^{n+1}) + \int_t^T (Z_s^n - Z_s^{n+1}) \, dW_s,
\]

where

\[
\Lambda_s^{n+1} = C \left( |Y_0^n - \bar{Y}_s^n| + |(Y_t^n)' - (\bar{Y}_s^{n+1})'| + |Z_s^n - Z_s^{n+1}| + |Y_t^n| + (Y_t^n)' + |Z_0^n| \right) \\
+ f_s - f(s, \bar{Y}_s^n, (\bar{Y}_s^n)', Z_s^n) + \pi \left( s, \delta \bar{Y}_s^n, \delta (\bar{Y}_s^{n+1})', \delta Z_s^{n+1} \right).
\]

By Lemma 3, we deduce that \( Y_t^n - \bar{Y}_t^{n+1} \geq 0 \), i.e. \( Y_t^n \geq \bar{Y}_t^{n+1} \), for all \( t \in [0, T] \). Thus we have for all \( n \geq 0 \)

\[
Y_t^n \geq \bar{Y}_t^{n+1} \geq \bar{Y}_t^n \geq \bar{Y}_t^0, \quad \forall t \in [0, T].
\]

The proof of Lemma 4 is complete. \( \square \)

**Theorem 5.** Let \( \xi \in L^2(\mathcal{F}_T, \mathbb{R}) \) and \( t \in [0, T] \). Under assumption (H.1) - (H.4) and (H.7) - (H.11), the reflected MF-BDSDEs (2) has a minimal solution

\[
(Y_t, Z_t, K_t)_{0 \leq t \leq T} \in \mathcal{S}^2(0, T, \mathbb{R}) \times \mathcal{M}^2 \left( 0, T, \mathbb{R}^d \right) \times \mathcal{S}^2.
\]

**Proof.** From Lemma 4, we know \( (\bar{Y}_t^n)_{n \geq 0} \) is increasing and bounded in \( \mathcal{M}^2 \left( 0, T, \mathbb{R}^d \right) \). Since \( |\bar{Y}_t^n| \leq \max \{ |\bar{Y}_t^n|, |\bar{Y}_t^0| \} \leq |\bar{Y}_t^n| + |\bar{Y}_t^0| \) for all \( t \in [0, T] \), we have

\[
\sup_n \mathbb{E} \left( \sup_{0 \leq t \leq T} |\bar{Y}_t^n|^2 \right) \leq \mathbb{E} \left( \sup_{0 \leq t \leq T} |\bar{Y}_t^0|^2 \right) + \mathbb{E} \left( \sup_{0 \leq t \leq T} |\bar{Y}_t^0|^2 \right) < \infty.
\]
then according to the Lebesgue’s dominated convergence theorem, we deduce that \((\bar{Y}_n^T)_{n \geq 0}\) converges in \(\mathcal{S}^2(0, T, \mathbb{R})\). We denote by \(\bar{Y}\) the limit of \((\bar{Y}_n^T)_{n \geq 0}\).

On the other hand from Eq. (6), we deduce that

\[
\bar{Y}^{n+1}_0 = \bar{Y}^{n+1}_T + \int_0^T E' \left( f \left( s, \bar{Y}_n^s, (\bar{Y}_n^s)' \right), \bar{Z}^n_s \right) + \pi \left( s, \delta \bar{Y}^{n+1}_s, \delta (\bar{Y}^{n+1}_s)', \delta \bar{Z}^{n+1}_s \right) \right) ds + \int_0^T E' \left( g \left( s, \bar{Y}_n^s, (\bar{Y}_n^s)'; \bar{Z}^n_s \right), \bar{K}^n_s \right) d\bar{B}_s + \int_0^T d\bar{K}^{n+1}_s - \int_0^T \bar{Z}^{n+1}_s dW_s.
\]

Applying Itō’s formula, we obtain

\[
\mathbb{E}\left| \bar{Y}^{n+1}_0 \right|^2 + \mathbb{E} \int_0^T |\bar{Z}^{n+1}_s|^2 ds \leq \mathbb{E}\left| \bar{Y}^{n+1}_T \right|^2 + 2\mathbb{E} \int_0^T \bar{Y}^{n+1}_s d\bar{K}^{n+1}_s + \mathbb{E} \int_0^T \left| E' \left( g(s, \bar{Y}_n^s, (\bar{Y}_n^s)', \bar{Z}^n_s) \right) \right|^2 ds + 2\mathbb{E} \int_0^T \bar{Y}^{n+1}_s E' \left( f(s, \bar{Y}_n^s, (\bar{Y}_n^s)', \bar{Z}^n_s) \right) + \pi \left( s, \delta \bar{Y}^{n+1}_s, \delta (\bar{Y}^{n+1}_s)', \delta \bar{Z}^{n+1}_s \right) \right) ds.
\]

From assumption \((H.7)\) and Young’s inequality, we get

\[
2\mathbb{E} \int_0^T \bar{Y}^{n+1}_s E' \left( f(s, \bar{Y}_n^s, (\bar{Y}_n^s)', \bar{Z}^n_s) \right) ds \leq 2\mathbb{E} \int_0^T \bar{Y}^{n+1}_s E' \left( f_s(\omega) + C \left( 1 + |\bar{Y}^n_s| + |(\bar{Y}^n_s)'| \right) \right) ds,
\]

\[
\leq 4C^2 \mathbb{E} \int_0^T |\bar{Y}^{n+1}_s|^2 ds + (4C^2 + 1) \mathbb{E} \int_0^T |\bar{Y}^{n+1}_s|^2 ds + \mathbb{E} \int_0^T |\bar{Y}^n_s|^2 ds + 16C^2 \mathbb{E} \int_0^T |\bar{Z}^n_s|^2 ds + \mathbb{E} \int_0^T |f_s(\omega)|^2 ds,
\]

\[
(24C^2 + 1) \mathbb{E} \int_0^T |\bar{Y}^{n+1}_s|^2 ds + \mathbb{E} \int_0^T |\bar{Y}^n_s|^2 ds + \frac{1}{16} \mathbb{E} \int_0^T |\bar{Z}^n_s|^2 ds + \mathbb{E} \int_0^T |f_s(\omega)|^2 ds,
\]

and from hypothesis\((H.9)\) we get

\[
2\mathbb{E} \int_0^T \bar{Y}^{n+1}_s E' \left( \pi \left( s, \delta \bar{Y}^{n+1}_s, \delta (\bar{Y}^{n+1}_s)', \delta \bar{Z}^{n+1}_s \right) \right) ds \leq 2\mathbb{E} \int_0^T \bar{Y}^{n+1}_s E' \left( C \left( |\delta \bar{Y}^{n+1}_s| + \left| (\delta \bar{Y}^{n+1}_s)' \right| + |\delta \bar{Z}^{n+1}_s| \right) \right) ds,
\]

\[
\leq 4C \mathbb{E} \int_0^T |\bar{Y}^{n+1}_s|^2 ds + 4C^2 \mathbb{E} \int_0^T |\bar{Y}^{n+1}_s|^2 ds + \mathbb{E} \int_0^T |\bar{Y}^n_s|^2 ds + 8C^2 \mathbb{E} \int_0^T |\bar{Y}^n_s|^2 ds + 16C^2 \mathbb{E} \int_0^T |\bar{Z}^n_s|^2 ds,
\]

\[
(4C + 8C^2) \mathbb{E} \int_0^T |\bar{Y}^{n+1}_s|^2 ds + \mathbb{E} \int_0^T |\bar{Y}^n_s|^2 ds + \frac{1}{8} \mathbb{E} \int_0^T |\bar{Z}^n_s|^2 ds + \frac{1}{16} \mathbb{E} \int_0^T |\bar{Z}^n_s|^2 ds.
\]

Using the two inequalities \((8)\) and \((9)\), we obtain

\[
2\mathbb{E} \int_0^T \bar{Y}^{n+1}_s E' \left( f(s, \bar{Y}_n^s, (\bar{Y}_n^s)', \bar{Z}^n_s) \right) + \pi \left( s, \delta \bar{Y}^{n+1}_s, \delta (\bar{Y}^{n+1}_s)', \delta \bar{Z}^{n+1}_s \right) \right) ds \leq (52C^2 + 4C + 1) \mathbb{E} \int_0^T |\bar{Y}^{n+1}_s|^2 ds + 2\mathbb{E} \int_0^T |\bar{Y}^n_s|^2 ds
\]

\[
+ \frac{1}{8} \mathbb{E} \int_0^T \left( |\bar{Z}^{n+1}_s|^2 + |\bar{Z}^n_s|^2 \right) ds + \mathbb{E} \int_0^T |f_s(\omega)|^2 ds.
\]
Using Young’s inequality, we obtain
\[ C \text{ two constants by the Hölder inequality and } \]
Therefore, there exists a constant \( C \) such that
\[ (g(s, Y_{s}^{n+1}), Z_{s}^{n+1})) ds \]
Applying hypothesis (H.11), we have
\[ E \int_{0}^{T} |\tilde{Y}_{s}^{n+1}|^{2} ds + 2\alpha E \int_{0}^{T} |\tilde{Z}_{s}^{n+1}|^{2} ds + 2E \int_{0}^{T} ||g(s, 0, 0)||^{2} ds. \]
Using Young’s inequality, we obtain
\[ 2E \int_{0}^{T} \tilde{Y}_{s}^{n+1} d\kappa_{s}^{n+1} \leq 2E \int_{0}^{T} S_{s} d\kappa_{s}^{n+1} + \frac{1}{\theta} E \left( \sup_{0 \leq s \leq T} |S_{s}|^{2} \right) + \theta E |\kappa_{T}^{n+1}|^{2}. \]
Therefore, there exists a constant \( C^{\theta} \) depending on \( \alpha, \xi, C \) and \( \theta \), we derive
\[ E \int_{0}^{T} |\tilde{Z}_{s}^{n+1}|^{2} ds \leq C^{\theta} + \left( \frac{1}{8} + 2\alpha \right) E \int_{0}^{T} |\tilde{Z}_{s}^{n+1}|^{2} ds + \frac{1}{8} E \int_{0}^{T} |\tilde{Z}_{s}^{n}|^{2} ds + \theta E |\kappa_{T}^{n+1}|^{2}, \] (10)
where \( C^{\theta} = C + E|\xi|^{2} + 4C \int_{0}^{T} |\tilde{Y}_{s}^{n+1}|^{2} ds + \frac{1}{\theta} E \left( \sup_{0 \leq s \leq T} |S_{s}|^{2} \right) + 2E \int_{0}^{T} ||g(s, 0, 0)||^{2} ds. \]
Choosing \( \alpha \) such that \( 0 < \frac{1}{8} + 2\alpha < 1 \), we obtain
\[ E \int_{0}^{T} |\tilde{Z}_{s}^{n+1}|^{2} ds \leq C^{\theta} + \frac{1}{8} E \int_{0}^{T} |\tilde{Z}_{s}^{n}|^{2} ds + \theta E |\kappa_{T}^{n+1}|^{2}. \]
Moreover, since
\[ \kappa_{T}^{n+1} = \tilde{Y}_{0}^{n+1} - \xi - \int_{0}^{T} E' \left( f \left( s, \tilde{Y}_{s}^{n}, (\tilde{Y}_{s}^{n})', \tilde{Z}_{s}^{n} \right) + \pi \left( s, \delta \tilde{Y}_{s}^{n+1}, \delta (\tilde{Y}_{s}^{n+1})', \delta \tilde{Z}_{s}^{n+1} \right) \right) ds \]
- \[ \int_{0}^{T} E' \left( g \left( s, \tilde{Y}_{s}^{n+1}, (\tilde{Y}_{s}^{n+1})', \tilde{Z}_{s}^{n+1} \right) \right) d\tilde{b}_{s} + \int_{0}^{T} \tilde{Z}_{s}^{n+1} dW_{s}, \]
by the Hölder inequality and B-D-G inequality, \( E(X)^{2} \leq E(X^{2}) \) and the properties on \( f, g, \pi \) that there exists two constants \( C_{1} \) and \( C_{2} \) depending on \( \alpha, \xi \) and \( C \) of \( n \) such that
\[ E |\kappa_{T}^{n+1}|^{2} \leq C_{1} + C_{2} \left( E \int_{0}^{T} |\tilde{Z}_{s}^{n+1}|^{2} ds \right). \]
Return to inequality (10), we get
\[ E \int_{0}^{T} |\tilde{Z}_{s}^{n+1}|^{2} ds \leq C^{\theta} + \theta C_{1} + \left( \frac{1}{8} + \theta C_{2} \right) E \int_{0}^{T} |\tilde{Z}_{s}^{n}|^{2} ds + \theta C_{2} E \int_{0}^{T} |\tilde{Z}_{s}^{n+1}|^{2} ds, \]
we choosing \( \theta \), such that \( \theta C_{2} \leq 1 \), we have
\[ E \int_{0}^{T} |\tilde{Z}_{s}^{n+1}|^{2} ds \leq C^{\theta} + \theta C_{1} + \left( \frac{1}{8} + \theta C_{2} \right) E \int_{0}^{T} |\tilde{Z}_{s}^{n}|^{2} ds \]
\[ \leq \left( C^{\theta} + \theta C_{1} \right) \sum_{i=0}^{n-1} \left( \frac{1}{8} + \theta C_{2} \right)^{i} + \left( \frac{1}{8} + \theta C_{2} \right)^{n} E \int_{0}^{T} |\tilde{Z}_{s}^{0}|^{2} ds. \]
Now choosing $\theta$ such that $\frac{1}{8} + \theta C_2 < 1$ and noting $E \int_0^T |\bar{Z}_s^n| \, ds < \infty$. Obtain
\[
\sup_{n \in \mathbb{N}} E \int_0^T |\bar{Z}_s^{n+1}|^2 \, ds < \infty,
\]
consequently, we deduce
\[
E |\bar{K}_T^{n+1}|^2 < \infty.
\]
Now we shall prove that $(\bar{Z}^n, \bar{K}^n)$ is a Cauchy sequence in $\mathcal{M}^2 (0, T, \mathbb{R}^d) \times \mathcal{A}^2$.

Applying Itô’s formula to $|\delta \bar{Y}_{s,m}^n|^2 = |\bar{Y}_s^n - \bar{Y}_s^m|^2$, we have
\[
E |\bar{Y}_s^n - \bar{Y}_s^m|^2 + E \int_0^T |\bar{Z}_s^n - \bar{Z}_s^m|^2 \, ds = 2E \int_0^T (\bar{Y}_s^n - \bar{Y}_s^m) (\Gamma_s^n - \Gamma_s^m) \, ds + 2 \int_0^T \bar{Y}_s^{n+1} (d \bar{K}_s^n - d \bar{K}_s^m)
+ \int_0^T \left| E' \left( g(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n) - g(s, \bar{Y}_s^m, (\bar{Y}_s^m)', \bar{Z}_s^m) \right) \right|^2 \, ds.
\]
where $\Gamma_s^n = f(s, \bar{Y}_s^{n-1}, (\bar{Y}_s^{n-1})', \bar{Z}_s^{n-1}) + \pi \left(s, \delta \bar{Y}_s^n, \delta (\bar{Y}_s^n)', \delta \bar{Z}_s^n\right)$. Since $\int_0^T \bar{Y}_s^{n+1} (d \bar{K}_s^n - d \bar{K}_s^m) \leq 0$, we obtain
\[
E \int_0^T |\bar{Z}_s^n - \bar{Z}_s^m|^2 \, ds \leq 2E \int_0^T (\bar{Y}_s^n - \bar{Y}_s^m) (\Gamma_s^n - \Gamma_s^m) \, ds + 2 \int_0^T \bar{Y}_s^n (d \bar{K}_s^n - d \bar{K}_s^m)
+ \int_0^T \left| E' \left( g(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n) - g(s, \bar{Y}_s^m, (\bar{Y}_s^m)', \bar{Z}_s^m) \right) \right|^2 \, ds.
\]
By the Hölder inequality and hypothesis (H.11), we deduce that
\[
(1 - \alpha) E \int_0^T |\bar{Z}_s^n - \bar{Z}_s^m|^2 \, ds \leq 2E \left( \int_0^T |\bar{Y}_s^n - \bar{Y}_s^m|^2 \, ds \right)^{\frac{1}{2}} E \left( \int_0^T \left| E' (\Gamma_s^n - \Gamma_s^m) \right|^2 \, ds \right)^{\frac{1}{2}}
+ 2CE \int_0^T |\bar{Y}_s^n - \bar{Y}_s^m|^2 \, ds.
\]
The boundedness of the sequence $(\bar{Y}_s^n, \bar{Z}_s^n, \bar{K}_s^n)$, we deduce that $\Lambda = \sup_{n \in \mathbb{N}} \left[ E \int_0^T |\Gamma_s^n|^2 \, ds \right] < \infty$, this yields that
\[
(1 - \alpha) E \int_0^T |\bar{Z}_s^n - \bar{Z}_s^m|^2 \, ds \leq 4 \Lambda E \left( \int_0^T |\bar{Y}_s^n - \bar{Y}_s^m|^2 \, ds \right)^{\frac{1}{2}} + 2CE \int_0^T |\bar{Y}_s^n - \bar{Y}_s^m|^2 \, ds,
\]
which yields that $(\bar{Z}^n)_{n \geq 0}$ is a Cauchy sequence in $\mathcal{M}^2 (0, T, \mathbb{R}^d)$. Then there exists $Z \in \mathcal{M}^2 (\mathbb{R}^d)$ such that
\[
E \int_0^T |\bar{Z}_s^n - Z_s|^2 \, ds \to 0 \text{ as } n \to \infty.
\]
On the other hand, by Burkholder-Davis-Gundy inequality, we get
\[
\begin{align*}
E \sup_{0 \leq t \leq T} \left| \int_0^T \bar{Y}_s^n \, dW_s - \int_0^T Z_s \, dW_s \right|^2 & \leq E \int_0^T |\bar{Z}_s^n - Z_s|^2 \, ds \to 0, \text{ as } n \to \infty, \\
E \sup_{0 \leq t \leq T} \left| \int_0^T E \left( g(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n) - E \left( g(s, \bar{Y}_s^n, (\bar{Y}_s^n)', Z_s) \right) \right) \, ds \right|^2 & \leq 2CE \int_0^T |\bar{Y}_s^n - Z_s|^2 \, ds + \alpha E \int_0^T |\bar{Z}_s^n - Z_s|^2 \, ds \to 0, \text{ as } n \to \infty.
\end{align*}
\]
Therefore, from the properties of $f$ and $\pi$
\[
\Gamma_s^n = f(s, \bar{Y}_s^{n-1}, (\bar{Y}_s^{n-1})', \bar{Z}_s^{n-1}) + \pi \left(s, \delta \bar{Y}_s^n, \delta (\bar{Y}_s^n)', \delta \bar{Z}_s^n\right) \to f(s, Y_s, (Y_s)', Z_s),
\]
\[ \hat{p} - a.s., \text{ for all } t \in [0, T] \text{ as } n \to \infty. \] Then follows by Lebesgue's dominated convergence theorem that

\[ \mathbb{E} \left[ \int_0^T \left| E' \left( \Gamma^n_y - f(s, Y_s, (Y_s')', Z_s) \right) \right|^2 ds \right] \to 0, \ n \to \infty \]

Since \( (\hat{Y}_s, \hat{Z}_s, \Gamma^n_s) \) converges in \( \mathcal{S}^2 \left( (0, T, \mathbb{R}) \times \mathcal{M}^2 \left( (0, T, \mathbb{R}^d) \right) \times \mathcal{M}^2 \left( (0, T, \mathbb{R}^d) \right) \right) \) and

\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} |R^n_t - R^m_t|^2 \right) \leq \mathbb{E} |S^n_0 - S^m_0|^2 + \mathbb{E} \sup_{0 \leq t \leq T} |S^n_t - S^m_t|^2 + \mathbb{E} \int_0^T \left[ E' \left( \Gamma^n_y - \Gamma^m_y \right) \right]^2 ds \]

\[ + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t E' \left( g(s, \hat{Y}^n_s, (\hat{Y}^n_s')', \hat{Z}^n_s) - g(s, \hat{Y}^m_s, (\hat{Y}^m_s')', \hat{Z}^m_s) \right) d\hat{B}_s \right|^2 \]

\[ + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t (\hat{Z}^n_s - \hat{Z}^m_s) dW_s \right|^2 \]

for any \( n \geq 0 \), we deduce from Bukhölder-Davis-Gundy inequality that

\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} |R^n_t - R^m_t|^2 \right) \to 0, \]

as \( n \to \infty \). Consequently, there exists a \( \mathcal{F}_t \)-measurable process \( K \) which value in \( \mathbb{R} \) such that

\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} |\hat{K}^n_t - K_t|^2 \right) \to 0, \]

as \( n \to \infty \). Obviously, \( K_0 = 0 \) and \( \{K_t; 0 \leq t \leq T\} \) is a increasing and continuous process. From Eq. (6), we have for all \( n \geq 0, \hat{Y}^n_s \geq S_t, \forall t \in [0, T] \), then \( Y_t \geq S_t, \forall t \in [0, T] \). On the other hand, from the result of Saisho [8] (in 1987, p. 465), we have

\[ \int_0^T (\hat{Y}^n_s - S_s) d\hat{K}^n_s \to \int_0^T (Y_s - S_s) dK_s, \]

\[ \hat{p} - a.s., \text{ as } n \to \infty. \] Using the identity \( \int_0^T (\hat{Y}^n_s - S_s) d\hat{K}^n_s = 0 \), for all \( n \geq 0 \) we conclude that \( \int_0^T (Y_s - S_s) dK_s = 0 \). Letting \( n \to +\infty \) in Eq. (3), we prove that \( (Y, Z, K) \) is solution to Eq. (3). Let \( (Y^*, Z^*, K^*) \) be any solution of the MF-RBDSDE (3), we have \( \hat{Y}^n \leq Y^* \), for all \( n \geq 0 \) and therefore, \( Y \leq Y^* \) i.e., \( Y \) is the minimal solution. \( \square \)

References

[1] Bahlali, K. Hassanli, M. Mansouri, B. Mrhardy, N. (2009), One barrier reflected backward doubly stochastic differential equations with continuous generator, Comptes Rendus Mathematique, Volume 347, Issue 19, Pages 1201-1206. http://doi.org/10.1016/j.crma.2009.08.001.

[2] Chaouchkhouan, N. Labed. Boubakeur & Badreddine. Mansouri, (2014), Mean-field Reflected Backward Doubly Stochastic DE With Continuous Coefficients*, Journal of Numerical Mathematics and Stochastics,. 6. 62-72.

[3] Jia, G. (2008), A class of backward stochastic differential equations with discontinuous coefficients, Statist. Probab. Lett. 78, 231–237. http://doi.org/10.1016/j.spl.2007.05.028.

[4] Lin, Q. (2011), Backward doubly stochastic differential equations with weak assumptions on the coefficients, Applied Mathematics and Computation, 217, 9322–9333. https://doi.org/10.1016/j.amc.2011.04.016.

[5] Lepeltier, J.P., San Martin, J. (1997), Backward stochastic differential equations with continuous coefficients, Stat. Probab. Lett. 32, 425–430. https://doi.org/10.1016/0167-6911(96)00103-4.

[6] Pardoux, E. Peng, S. (1990), Adapted solution of a backward stochastic differential equation, Systems Control Lett. 4, 55–61. https://doi.org/10.1016/0167-6911(90)90082-6.

[7] Pardoux, E. Peng, S. (1994), Backward doubly stochastic differential equations and systems of quasili near SPDEs, Probability Theory and Related Fields, 98, 209-227. https://doi.org/10.1007/BF01192514.

[8] Saisho, Y. (1987), Stochastic differential equations for multi-dimensional domain with reflecting boundary, Probability Theory and Related Fields, 74(3), 455-477. https://doi.org/10.1007/BF00699100.
[9] Shi, Y., Gu, Y., Liu, K. (2005), Comparison theorems of backward doubly stochastic differential equations and applications, Stochastic Analysis and Application, 23, 97–110. https://doi.org/10.1081/SAP-200044444.

[10] Xu, R. (2012), Mean-field backward doubly stochastic differential equations and related SPDEs, Boundary Value Problems, 2012(1), 114. https://doi.org/10.1186/1687-2770-2012-114.