Abstract
This article proposes a generalisation of the delete-$d$ jackknife to solve hyperparameter selection problems for time series. I call it artificial delete-$d$ jackknife to stress that this approach substitutes the classic removal step with a fictitious deletion, wherein observed datapoints are replaced with artificial missing values. This procedure keeps the data order intact and allows plain compatibility with time series. This manuscript justifies the use of this approach asymptotically and shows its finite-sample advantages through simulation studies. Besides, this article describes its real-world advantages by regulating high-dimensional forecasting models for foreign exchange rates.

Keywords: Jackknife, Hyperparameter optimisation, Model selection, Time series.

1. Introduction
Using large datasets with standard predictive models is not straightforward. There is often a proliferation of parameters, high estimation uncertainty and the tendency of over-fitting in-sample, but performing poorly out-of-sample. This so-called curse of dimensionality is often handled regularising statistical models with a collection of tuning parameters. Since the latter are often determined before the estimation process takes place, they are denoted as hyperparameters. This paper proposes a systematic approach for selecting them in the case of time-series data.

There is a large number of techniques for high-dimensional prediction problems. Classical methods include ridge (Hoerl and Kennard, 1970), LASSO (Tibshirani, 1996) and elastic-net (Zou and Hastie, 2005) regressions. They make the estimation feasible for linear regressions by penalising the magnitude of the coefficients to downweight the variables that do not help in predicting. The strength of the penalties is tuned with

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a vector of hyperparameters. Regression trees (Morgan and Sonquist, 1963; Breiman et al., 1984; Quinlan, 1986) are a classical example from the machine learning literature for exploring high-dimensional datasets. These techniques can handle non-linearities and complex data generating processes. However, they must be regulated via a range of penalties and stopping rules to perform well out-of-sample. This is again achieved using hyperparameters. Large datasets are also commonly handled with Bayesian methods. In this literature, hyperparameters are often necessary to define prior distributions (Gelman et al., 2014) and obtain parsimonious models with shrinkage techniques similar or equivalent to ridge and LASSO (Giannone et al., 2017), and the elastic-net (Li and Lin, 2010). Hyper-parameters are also crucial for low-dimensional problems. For example, hyperparameters such as the number of lags for autoregressive models are fundamental for structuring forecasting exercises.

Cross-validation (Stone, 1974) is among the most well-known approaches for selecting hyperparameters in independent data settings. It is a statistical method to estimate the expected accuracy of a model on unseen data. Its basic formulation is straightforward: data is split into complementary partitions, and the resulting subsamples are used for estimating and validating a predictive method. The performance within the validation samples is used as an estimate of the prediction error on unseen datapoints and the hyperparameters are generally selected to minimise this measure.

Cross-validation is challenging for time series since data is ordered and autocorrelated. Several authors have proposed generalisations to handle these complexities. One of the first contributions came from Snijders (1988). The latter used insights from Brown et al. (1975) and Ljung and Söderström (1983) to propose a cross-validatory method based on realised pseudo out-of-sample errors. Indeed, it suggested to split the observed data into complementary partitions and then use the first as an estimation sample, and the remaining observations to measure the realised pseudo out-of-sample error. The hyperparameters are selected to minimise this error measure.

While this approach is very intuitive and consistent with the structure of the data, it is not necessarily robust, since it uses only a single estimation and validation set. Kunst (2008) proposed overcoming this downside by applying standard pseudo out-of-sample evaluations to random subsamples. However, the results are relatively difficult to interpret since the algorithm used for generating these partitions is initialised with in-sample regression parameters.

Burman et al. (1994) introduced a different way to address this problem: the so-called $h$-block cross-validation. This methodology, based on Györfi et al. (1989) and Burman and Nolan (1992), uses blocking techniques to generate validation samples independent from the data used for estimation. Indeed, Burman et al. (1994) proposed creating a set of estimation samples by removing, in turn, each block of dimension
$2h + 1$ (for a given $h$) from the data. $h$-block cross-validation then uses the median item of this block as a one-dimensional validation sample. Even though this approach has interesting properties, keeping a fixed distance between the partitions is costly, given that a large share of observations is lost in the process. This is especially severe when there are not so many observations, because the number of validation samples available is small.

Most recently, Bergmeir et al. (2018) proposed cross-validating autoregressive models with uncorrelated errors with techniques for i.i.d. data. This approach makes good use of all available observations, but its properties do not hold for models with correlated errors and it disregards the order in the data.

Jackknife (Quenouille, 1956; Tukey, 1958) and bootstrap (Efron, 1979a,b, 1981) can be used as alternative approaches to estimate the prediction error on unseen data-points and thus select hyperparameters. These techniques are typically more efficient than cross-validation (Efron, 1979a; Efron and Gong, 1983) since they measure the accuracy of a model on the average prediction error committed over a large range of data subsamples. Bootstrap builds these partitions sampling with replacement from the data. Instead, jackknife constructs subsamples by removing sets of observations from the observables. In particular, the delete-$d$ jackknife (Wu, 1986; Shao and Wu, 1989) generates a sequence of partitions by removing, in turn, all the combinations of $d > 0$ observations from the data.

Jackknife and bootstrap require modifications to be compatible with time series, since the subsampling schemes do not take the data order into account. Kunsch (1989) extended these methodologies to stationary series. Indeed, building on Carlstein (1986), Kunsch (1989) proposed developing block-wise subsampling schemes. Let $c$ be an integer lower or equal to the total number of observed time periods. The block jackknife generates the partitions by removing or down-weighting, in turn, all the $c$-dimensional blocks of consecutive observations from the data. Instead, the block bootstrap draws with replacement a fixed number of $c$-dimensional blocks of observations from the data. Politis and Romano (1992, 1994) developed this technique further proposing the so-called stationary bootstrap. This approach wraps the data “in a circle”, so that the first observation follows the last, and generates the bootstrap samples drawing and merging blocks of random length. Differently than the block bootstrap and its variations, the block jackknife does not impact the data order when constructing subsamples.

This paper introduces a version of the standard delete-$d$ jackknife compatible with time series. In this version of the jackknife, the data removal step is replaced with a fictitious deletion that consists in imposing (artificial) patterns of missing observations on the data. I call this new approach artificial delete-$d$ jackknife (or artificial jackknife) to emphasise that $d$ observations are artificially removed from the original data to
Figure 1: Subsampling schemes for dependent data. (b) Blocks of random (stationary bootstrap) or fixed (block bootstrap) length are drawn with replacement from the data. (c) Subsamples are constructed down-weighting, in turn, all the $c$-dimensional blocks of consecutive observations from the data. As in section 2.2, the down-weighting scheme is operated by turning blocks of consecutive observations into missing values. (d) Subsamples are constructed imposing (artificial) patterns of missing data to the original sample. This is a generalisation of the delete-$d$ jackknife.

generate each subsample. This article proposes using this new methodology to compute a robust measure of the forecast error (or, artificial jackknife error) as a means for selecting hyperparameters. The advantages of this approach depend on the finite-sample properties of the artificial jackknife. In fact, all errors based on pseudo out-of-sample evaluations converge in probability to the true error with the same rate (as shown in section 6). However, the artificial jackknife error has a smaller finite-sample variance than the pseudo out-of-sample error and the block jackknife (for most configurations of $c$ and $d$). This is crucial for stability and to select hyperparameters when the number of observations (i.e., time periods) is limited.

The artificial delete-$d$ jackknife is compatible with forecasting models able to handle missing observations. Within the scope of this paper, this is not a strong restriction. Most predictive problems with missing observations in the measurements can be written in state-space form and estimated via a large number of methods, as surveyed in Shumway and Stoffer (2011, ch. 6) and Särkkä (2013, ch. 12).

As an illustration, this article employs the artificial jackknife for tuning vector autoregressive moving average (VARMA) models regulated via an elastic-net penalty (Zou and Hastie, 2005). These models are estimated on a high-dimensional dataset of weekly
exchange rate returns. In order to provide full compatibility with the artificial jackknife, this article proposes to estimate the VARMA.s with an Expectation-Conditional Maximisation (ECM) algorithm (Meng and Rubin, 1993) able to handle incomplete time series. This estimation method is a secondary contribution of the paper given that, to my best knowledge, the literature has not proposed a way for handling missing observations in the measurements with similar settings.\textsuperscript{1}

2. Methodology

One of the main objectives of time series is to predict the future. This article aims to select optimal vectors of hyperparameters consistently with this maxim and thus in a way that minimises the expected forecast error.

2.1. Foundations

This subsection sets out the foundations for the hyperparameter selection process and delimits the scope of the article to a broad family of forecasting methods that encompasses common techniques such as ARMA, ADL and VARMA models.\textsuperscript{2}

**Assumption 1** (Data). Let $n, T \in \mathbb{N}$ and $n_Z \in \mathbb{N}_0$. Assume that $Y_{i,t}$ and $Z_{j,t}$ are finite realisations of some real-valued stochastic processes observed at time periods in the sets $\mathcal{T}_i, \mathcal{T}_j \subseteq \{t : t \in \mathbb{Z}, 1 \leq t \leq T\}$ for $i = 1, \ldots, n$ and $j = n + 1, \ldots, n + n_Z$.

**Assumption 2** (Lags). Define $q, r \in \mathbb{N}_0$ to be such that $p := \max(q, r)$ and $0 < p \ll T − 1$.

**Assumption 3** (Predictors). Let $X_t := (Y'_{t} \ldots Y'_{t−q+1} Z'_{t} \ldots Z'_{t−r+1})'$ be $m \times 1$ and defined at any point in time $t \in \mathbb{Z}$.

**Assumption 4** (Model structure). Finally, assume that

$$Y_{t+1} = f(X_t, \Psi) + V_{t+1},$$

where $f$ is a finite function, $\Psi$ is a matrix of finite coefficients, $V_{t+1} \overset{i.i.d.}{\sim} (0_n, \Sigma)$ with $\Sigma$ being a positive definite matrix\textsuperscript{3} and $\mathbb{E}(V_{t+1} | X_t) = 0$, for any integer $t$.\textsuperscript{4}

\textsuperscript{1}The replication code for this empirical application is available on GitHub.
\textsuperscript{2}Please note that this subsection and section 2 in its entirety do not limit the manuscript by looking at a specific forecasting model. Hence, the theoretical results are widely applicable.
\textsuperscript{3}This part of assumption 4 could be relaxed following an approach similar to the one employed in Barigozzi and Luciani (2020). However, this is outside the scope of the paper.
\textsuperscript{4}Under assumptions 1–3, $X_t$ is allowed to include $Y_t, \ldots, Y_{t−q+1}$ and $V_t, \ldots, V_{t−r+1}$ (for some $0 \leq q \leq p$ and $0 \leq r \leq p$), and more explanatory variables referring up to time $t − p + 1$. 

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Remark. The dependence on the sample size is highlighted in the notation only when strictly necessary, in order to ease the reading experience. Also, this article uses the same symbols to indicate the realisations at some integer point in time \( t \) and their general value in the underlying process. This is again for simplifying the notation and it should be clear from the context whether the manuscript is referring to the first or second category.

Knowing the data generating process in assumption 4, one could use it for obtaining the most accurate prediction (true forecast) for \( Y_{t+1} \) given \( X_t \) at any point in time \( t \).

Definition 1 (True error). Under a weighted square loss, the expected error associated with the true forecast is

\[
 err := \sum_{i=1}^{n} w_i \mathbb{E} \left[ |Y_{i,t+1} - f_i(X_t, \Psi)|^2 \right] = \sum_{i=1}^{n} w_i \mathbb{E}(V_{i,t+1}^2) = \sum_{i=1}^{n} w_i \Sigma_{i,i},
\]

with \( w_i \geq 0 \) for \( 1 \leq i \leq n \). This article refers to \( err \) as the true error.

In most practical applications, the data generating process is unknown and forecasters’ objective can be then reduced to approximating the true forecast.

Assumption 5 (Information set). Formally, at any time period \( p \leq s \leq T \), forecasters have an information set \( \mathcal{I}(s) \) containing the data observed up to that point and their expectation for \( Y_{t+1} \) conditional on \( \mathcal{I}(s) \) is

\[
 \hat{Y}_{t+1|s}(\gamma) := \mathbb{E} \left[ Y_{t+1} | X_t, \hat{\theta}_s(\gamma) \right] = g(X_t, \hat{\theta}_s(\gamma)),
\]

where \( g(X_t, \hat{\theta}_s(\gamma)) \) is a finite function whose coefficients \( \hat{\theta}_s(\gamma) \) are also finite and estimated on the basis of the data in \( \mathcal{I}(s) \), and given a vector of hyperparameters \( \gamma \).

Remark. The forecast function is further specified in section 2.3 with assumption 8. Note that forecasters’ may consider different predictors than those in the true forecast function when constructing their approximation. The article does not explicitly consider this case to simplify notation, but allowing for it would not change the results.

The predictions generated as in assumption 5 are clearly less accurate than the corresponding true forecasts. However, as shown in the following subsections, empirical error estimators converge in probability to the true error for a wide class of forecast functions. Therefore, the use of these approximations can be justified asymptotically.

2.2 Error estimators

This subsection expands on these empirical error estimators. It starts by describing the most well-known, broadens the discussion with the block jackknife and introduces the artificial delete-\( d \) jackknife error.
Before getting into details, I need to define a loss function for measuring the forecast error at each point in time.

**Definition 2 (Loss).** Consistently with **definition 1**, this paper uses

\[
L(Y_{t+1}, \hat{Y}_{t+1|s}(\gamma)) := \sum_{i \in \mathcal{D}(t+1)} w_i \left[ Y_{i,t+1} - \hat{Y}_{i,t+1|s}(\gamma) \right]^2,
\]

where \( \mathcal{D}(t+1) := \{ i : 1 \leq i \leq n \text{ and } Y_{i,t+1} \neq \text{NA} \} \), \( \text{NA} \) denotes a generic missing value, for any \( p \leq t \leq T - 1 \) and \( p \leq s \leq T \).

The next thing to consider is the conceptual relation between the forecast and its conditioning set. As a general point, the difficulty in obtaining an accurate \( \hat{Y}_{t+1|s}(\gamma) \) changes depending on whether \( \mathcal{S}(s) \) includes information about the future. This is what leads to the distinction between the two most common categories of forecast error estimators: in-sample and pseudo out-of-sample.

**Definition 3 (In-sample error).** The in-sample error

\[
\text{err}(\gamma) := \frac{1}{T-p} \sum_{t=p+1}^{T} L(Y_t, \hat{Y}_{t|T}(\gamma))
\]

is a measure of the average loss between the data and predictions generated conditioning on the full information set.

Estimating the coefficients once and on the full information set is beneficial for very short time-series problems, as there may not be enough observations to compute more sophisticated estimators. However, this approach tends to overstate the forecast accuracy since the information set is (at least partially) aware of the future. Indeed, in a realistic environment, forecasters would only have information about the past when computing their predictions.

**Definition 4 (Pseudo out-of-sample error).** The pseudo out-of-sample error

\[
\text{err}(\gamma) := \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} L(Y_{t+1}, \hat{Y}_{t+1|t}(\gamma)),
\]

overcomes this limitation by using forecasts generated on the basis of an expanding and backward-looking information set, starting from \( p \leq t_0 \leq T - 1 \).

**Remark.** The pseudo out-of-sample error can be extended to forecast horizons larger than one, but this is not further explored in the manuscript.\(^5\)

\(^5\)When long run predictions are calculated iteratively from the one step ahead forecast, it is not
Unfortunately, the pseudo out-of-sample error can be either over or under confident depending on the time periods used for estimating and validating the model. This can be overcome using estimators based on the average of pseudo out-of-sample errors computed on a series of data subsamples. The article generates these partitions using time-series generalisations of the jackknife (Quenouille, 1956; Tukey, 1958). The generic jackknife error in definition 5 is an estimator that can accommodate for different jackknife partitioning algorithms.

**Definition 5** (Generic jackknife error). Let $\mathcal{J}$ be an indexed family of sets such that each element contains ordered pairs $(i,t)$ with $1 \leq i \leq n$ and $0 \leq t \leq T$. The generic jackknife pseudo out-of-sample error

$$\tilde{\text{err}}(\mathcal{J}, \gamma) := \frac{1}{|\mathcal{J}|} \cdot \frac{(T - t_0)}{T} \sum_{j=1}^{T-1} \sum_{t=t_0}^{T-1} L(Y_{t+1}^{-j}, \hat{Y}_{t+1|t}(\gamma)),$$

where $Y^{−j}$ is the $n \times T$ matrix such that

$$Y_{i,t}^{−j} := \begin{cases} Y_{i,t}, & \text{if } (i,t) \notin \mathcal{J}, \\ NA, & \text{if } (i,t) \in \mathcal{J}, \end{cases}$$

$\hat{Y}_{t+1|t}(\gamma)$ is analogous to $\hat{Y}_{t+1|t}(\gamma)$, but the autoregressive data component of $X_t$ is now based on $\hat{Y}^{−j}$. As for definition 4, $p \leq t_0 \leq T - 1$.

**Remark.** Allowing the ordered pairs $(i,t)$ to have $t = 0$, permits to write the pseudo out-of-sample error as a banal case of jackknife error in which $\mathcal{J}$ contains only one element external to the sample. For instance via $\mathcal{J} = \{(1,0)\}$. Indeed, the actual data has observations referring to the points in time between 1 and $T$ (included). Therefore, any $t = 0$ is to be considered external.

The most well-known approach to generate jackknife subsamples for dependent data is the block jackknife (Kunsch, 1989). This technique partitions the data into block jackknife samples by removing or down-weighting, in turn, all the unique non-interrupted blocks of $1 \leq c \leq T$ observations.

necessary to generalise definition 4 to handle longer horizons. The latter would need to be modified only in the case of direct forecast. It is important to stress that when the model is correctly specified, producing iterative forecasts is more efficient than computing horizon-specific ones. However, the latter are more robust to misspecification (Marcellino et al., 2006). For simplicity, this paper focusses only on the one-step ahead forecast, wherein iterative and direct forecasts are identical. Implicitly, this approach is also consistent with iterative forecast methods targeting longer horizons.
**Definition 6** (Block jackknife error). This paper denotes the block jackknife error as

\[ \tilde{err}^{BJK}(c, \gamma) \equiv \tilde{err}(\mathcal{B}(c), \gamma), \]  

where \( \mathcal{B}(c) \) is the family of sets

\[ \mathcal{B}(c) := \{ \mathcal{B}(1, c), \ldots, \mathcal{B}(T - c + 1, c) \} \]

and

\[ \mathcal{B}(j, c) := \{(i, t) : 1 \leq i \leq n \text{ and } j \leq t \leq j + c - 1\}. \]

**Remark.** In other words, this article constructs the individual blocks by replacing, in turn, all the unique non-interrupted blocks of \( c \) observations with missing values. This is compatible with Kunsch (1989) since imposing blocks of NAs can be interpreted as fully down-weighting groups of observations. Furthermore, it simplifies the use of the block jackknife to estimate hyperparameters in forecasting settings. In fact, by processing the data via filtering and smoothing techniques compatible with missing observations, it is easier to estimate forecasting models without pre-processing the measurements to remove breaks introduced in the subsampling process.

The main issue with this estimator is that the number of partitions that can be generated from the data is generally small. Thus, the overall improvement over the standard pseudo out-of-sample error is somewhat limited. Also, for those partitions wherein a huge chunk of observations are removed after \( t_0 \), dividing for a factor of \( T - t_0 \) may produce inaccurate estimates of the expected error. This is especially true in small-sample problems where \( c \) is large relative to \( T - t_0 \). A simple way for reducing this issue in a finite-sample problem consists in adjusting the \( \tilde{err}^{BJK}(c, \gamma) \) multiplying it by

\[ \frac{T - t_0}{|\mathcal{B}(c)|} \sum_{j=1}^{\left|\mathcal{B}(c)\right|} \frac{1}{|\{(i, t) \in \mathcal{B}(j, c) : t > t_0\}|}. \]

However, this is difficult to justify asymptotically.

This paper proposes to surpass these problems using an error estimator based on a generalisation of delete-\( d \) jackknife (Wu, 1986; Shao and Wu, 1989) compatible with time-series problems: the artificial delete-\( d \) jackknife. The classical delete-\( d \) jackknife for i.i.d. data (Wu, 1986; Shao and Wu, 1989) generates subsamples by removing, in turn, all the combinations of \( d > 0 \) observations from the data. This is clearly incompatible with dependent data, since the autocorrelation structure would break
during the subsampling process. The artificial jackknife overcomes this complexity by generating the partitions replacing, in turn, all the combinations of \(d\) observations with (artificial) missing values. This allows to handle dependent data, as the resulting partitions keep the original ordering and the autocorrelation structure is not altered. Moreover, this approach permits to generate a much larger number of subsamples than block jackknife.\(^6\)

**Definition 7** (Artificial delete-\(d\) jackknife error). Let

\[
P := \{i \in \mathbb{Z} : 1 \leq i \leq n\} \times \{t \in \mathbb{Z} : 1 \leq t \leq T\}
\]

be the set of all data pairs. Hence, define \(A(d)\) as a family of sets with cardinality

\[
|A(d)| = \frac{(nT)!}{d!(nT - d)!}
\]

such that each element is a \(d\)-dimensional combination of \(P\). Next, let

\[
\tilde{err}^{AJK}(d, \gamma) \equiv \tilde{err}(A(d), \gamma).
\]

This is the artificial delete-\(d\) jackknife error.

The higher reliability of this error estimator is given by the large number of partitions that the artificial delete-\(d\) jackknife is able to generate and their heterogeneity. This can be formalised in terms of efficiency as follows.

**Assumption 6** (Finite-sample variance). Assume, for simplicity of notation, that the constituent pseudo out-of-sample errors in definition 5 follow a common finite-sample distribution with variance \(\sigma^2(T - t_0, \gamma)\).

**Proposition 1.** Under assumption 6, it follows that, in finite-sample problems,

\[
\begin{align*}
\text{var} \left[ \tilde{err}^{AJK}(d, \gamma) \right] & \leq \text{var}(\tilde{err}(\gamma)), \\
\text{var} \left[ \tilde{err}^{BJK}(c, \gamma) \right] & \leq \text{var}(\tilde{err}(\gamma)).
\end{align*}
\]

**Proof.** The proof is reported in section 5.A. \(\Box\)

**Remark.** Assumption 6 can be released without impacting the structure of the proof. However, the notation becomes quite convoluted and hard to read.

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\(^6\)It is interesting to notice that the block jackknife in equation 2 is a special case of the artificial delete-\(d\) jackknife, in which blocks of consecutive datapoints are replaced with missing values.
Section 5.B compares the variance of the block and artificial jackknife errors through a simulation exercise. This exercise shows that the artificial jackknife outperforms the block jackknife especially in small-sample problems.

When \( nT \) is large and \( \sqrt{nT} < d < nT \), the cardinality \(|\mathcal{A}(d)|\) can be large and it might not be computationally feasible to calculate equation 3 evaluating all combinations. Following common practice (Efron and Tibshirani, 1994, p. 149), this computational issue is handled with an approximation. Define \( \tilde{\mathcal{A}}(d) \subset \mathcal{A}(d) \) as a family of sets constructed by drawing at random, without replacement, for a sufficiently large number of times from \( \mathcal{A}(d) \). Hence, use this newly defined subset to compute \( \tilde{\varepsilon}(\tilde{\mathcal{A}}(d), \gamma) \), an approximation of the artificial delete-\(d\) jackknife error. Clearly, the accuracy of the approximation

\[
\tilde{\varepsilon}^{\text{AJK}}(d, \gamma) \approx \tilde{\varepsilon}(\tilde{\mathcal{A}}(d), \gamma)
\]  

depends on how close \(|\tilde{\mathcal{A}}(d)|\) is to \(|\mathcal{A}(d)|\).

In most empirical problems, the artificial jackknife will likely be truncated. Thus, this article proposes a simple heuristics for selecting its number of artificial missing observations. As detailed in section 5 and with the simplified notation in assumption 6, the artificial jackknife error variance depends on two factors: \( \sigma^2(T - t_0, \gamma) \) and the heterogeneity across jackknife subsamples. The latter is controlled by \( d \) and, ceteribus paribus, \( \text{var}(\tilde{\varepsilon}^{\text{AJK}}(d, \gamma)) \) is at its minimum when the subsamples are the most diverse. The exact functional form of this variance is unknown and, in the case of the truncated artificial jackknife, one would need to choose a value for \( d \) that guarantees a large pool of combinations. Besides, it would be ideal to exclude from \( \tilde{\mathcal{A}}(d) \) the combinations that are the most similar to the block jackknife. In other words, those where all series are missing for one or more periods.

**Conjecture** (Rule of thumb for selecting \( d \)). As a result, this paper proposes selecting \( d \) for the truncated artificial jackknife error to be

\[
\hat{d} = \arg \max_d \left( \frac{nT}{d} - \mathbb{I}_{d \geq n} \left( \frac{nT}{d} - n \right) T - \frac{\lfloor d/n \rfloor}{\sum_{i=2}^{\lfloor d/n \rfloor} (-1)^{i-1} \left( \frac{T}{i} \right) \left( \frac{nT}{d} - in \right)} \right),
\]  

\[\text{(5)}\]

where

\[
\mathbb{I}_{d \geq n} \left( \frac{nT}{d} - n \right) T + \frac{\lfloor d/n \rfloor}{\sum_{i=2}^{\lfloor d/n \rfloor} (-1)^{i-1} \left( \frac{T}{i} \right) \left( \frac{nT}{d} - in \right)},
\]

is the amount of subsamples with points in time where all series are artificially missing.\(^7\)

\(^7\)To further reduce the effect of those combinations where all series are missing in one or more points.
Remark. The maximisation is trivial since the objective function is particularly fast to compute for each admissible \( d \), that is every integer \( d \in [1, nT] \).

2.3. Asymptotic properties for estimators based on pseudo out-of-sample evaluations

This subsection provides the asymptotic justification needed for using the approximation in assumption 5 to forecast the target data. It starts by describing the underlying assumptions and continues by proving that pseudo out-of-sample evaluations are consistent, even in the presence of missing observations. The proofs are reported in section 6.

Assumption 7 (Absolute summability). For any finite \( n > 0 \),

\[
\sum_{i=1}^{n} w_i \leq M_1, \\
\sum_{i=1}^{n} \sum_{j=1}^{n} |\text{cov}(V_{i,t}, V_{j,t})| \leq M_2, \\
\sum_{i=1}^{n} \sum_{j=1}^{n} |\text{cov}(V_{i,t}^2, V_{j,t}^2)| \leq M_3,
\]

where \( M_1, M_2, M_3 \in (0, \infty) \) are non-negative finite constants.

Remark. Recall that the elements of \( w \) are non-negative. Thus, for any finite \( n > 0 \),

\[
\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} |w_i|
\]

by definition.

Assumption 8 (Mean squared error of the forecast). For any \( t > 0 \),

\[
\mathbb{E} \left( \|f(X_t, \Psi) - g(X_t, \hat{\theta}_t(\gamma))\|_2^2 \right) \leq M_4/t,
\]

where \( M_4 \in (0, \infty) \) is a positive finite constant.

Remark. Note that

\[
\sup_t \|f(X_t, \Psi) - g(X_t, \hat{\theta}_t(\gamma))\|_2^2 \leq \sup_t \|f(X_t, \Psi)\| + |g(X_t, \hat{\theta}_t(\gamma))|\|_2^2.
\]

Since under assumptions 4–5 both the true forecast and its approximation are always finite, the assumption holds within the context of this paper. However, this bound can

\[\tilde{A}(d)\.\]
be loose as it is a function on the problem at hand and it depends on the true forecast and all modelling choices.

**Assumption 9** (Limiting size of the presample). Assume that

$$
\lim_{T \to \infty} \frac{t_0}{T} = 0.
$$

**Assumption 10** (Limiting number of missing observations). Denote with $0 \leq t_{NA} < T - t_0$ the number of periods between $t_0 + 1$ and $T$ (included) where the data contains missing observations, and assume that

$$
\lim_{T \to \infty} \frac{t_{NA}}{T} = 0.
$$

**Remark.** Note that assumption 9 serves a crucial purpose: making sure that as $T$ approaches infinity, the pseudo out-of-sample period increases. Similarly, assumption 10 limits the number of periods with missing observations, as $T$ approaches infinity. This implies that as $T$ increases the information set expands, because the number of observed datapoints increases. Without assumption 10, the total number of missing values could become predominant, relative to the amount of observed datapoints.

**Proposition 2.** Denote with $\text{err}_T(\gamma)$ the pseudo out-of-sample error for a dataset with $T$ periods. Under assumptions 1–5 and assumptions 7–9, and with complete data it holds that

$$
\lim_{T \to \infty} \frac{T}{\ln T} \mathbb{E} \left[ |\text{err}_T(\gamma) - \text{err}| \right] \leq M_1 M_4.
$$

This proposition shows that with complete data, pseudo out-of-sample errors are consistent estimators of the true error. This is a first stepping stone to prove convergence in probability for the generic jackknife errors. Proposition 3 bridges further the gap by extending these results to estimators based on potentially incomplete data.

**Proposition 3.** Under assumptions 1–5 and assumptions 7–10, and with potentially incomplete data it holds that

$$
\lim_{T \to \infty} \frac{T}{\ln T} \mathbb{E} \left[ |\text{err}_T(\gamma) - \text{err}| \right] \leq M_1 M_4.
$$

**Remark.** Under assumption 10 the rate of convergence in proposition 2 is preserved with potentially incomplete data.
The following corollary of proposition 3 extends its conclusions to the generic jackknife pseudo-out-of-sample error estimators described in section 2.2. Clearly, this includes the artificial delete-\(d\) jackknife error.

**Corollary 3.1.** Let \(\tilde{\text{err}}_{T}(J, \gamma)\) be a generic jackknife pseudo out-of-sample error based on a dataset with \(T\) time periods. Under the assumptions of proposition 3, it holds that

\[
\lim_{T \to \infty} \frac{T}{\ln T} \mathbb{E} \left[ |\tilde{\text{err}}_{T}(J, \gamma) - \text{err}| \right] \leq M_1 M_4.
\]

2.4. **Hyperparameter selection**

Having justified asymptotically the use of the forecasters’ approximation in assumption 5, this subsection shows how to optimise its accuracy through hyperparameter selection. It does so by exploring a grid of candidate hyperparameters to find the minimiser for a pseudo out-of-sample error estimator of choice between those reported in section 2.2.\(^8\)

Prior to entering into details, let me formalise the hyperparameter selection problem in general terms.

**Definition 8** (Search region and optimal hyperparameters). Let \(\mathcal{H}\) be a compact set of ordered tuples that defines the region of existence of the vector of hyperparameters of interest. Hence, the optimal hyperparameters are

\[
\begin{align*}
\hat{\gamma}(\mathcal{H}) &:= \arg \min_{\gamma \in \mathcal{H}} \tilde{\text{err}}(\gamma), & \text{when using the estimator in definition 4,} \\
\hat{\gamma}_{BJK}(c, \mathcal{H}) &:= \arg \min_{\gamma \in \mathcal{H}} \tilde{\text{err}}_{BJK}(c, \gamma), & \text{when using the estimator in definition 6,} \\
\hat{\gamma}_{AJK}(d, \mathcal{H}) &:= \arg \min_{\gamma \in \mathcal{H}} \tilde{\text{err}}_{AJK}(d, \gamma), & \text{when using the estimator in definition 7.}
\end{align*}
\]

The simplest way to explore a region of interest is via a grid search.

**Definition 9** (Grid search). Let \(\mathcal{H}^{GS} \subseteq \mathcal{H}\) be a finite set of candidate vectors of hyperparameters. Grid search considers every candidate in \(\mathcal{H}^{GS}\) and computes

\[
\begin{align*}
\hat{\gamma}(\mathcal{H}^{GS}), & \quad \text{when using the estimator in definition 4,} \\
\hat{\gamma}_{BJK}(c, \mathcal{H}^{GS}), & \quad \text{when using the estimator in definition 6,} \\
\hat{\gamma}_{AJK}(d, \mathcal{H}^{GS}), & \quad \text{when using the estimator in definition 7,}
\end{align*}
\]

via a naive brute-force optimisation.

\(^8\)This is in line with classical empirical risk minimisation (see, for instance, Elliott and Timmermann, 2016, ch. 3 for a complete survey).
This approach explores a small to medium finite grid of candidates and evaluates the relevant error estimator for each one of them. It then returns the candidate vector of hyperparameters associated to the smaller error. The set $\mathcal{H}^{GS}$ is generally constructed to include combinations of hyperparameters within some predetermined ranges. This can be done agnostically (e.g., specifying a rule to take candidates lying in some broad range) or via user expertise (e.g., selecting a few candidates of interest according to a judgmental component). In both cases, there is a strong risk of excluding valid candidates, since it is unfeasible to explore large search regions by using a brute force approach. A simple solution for this problem is given by a random search.

**Definition 10** (Random search). Define $\mathcal{H}^{RS} \subseteq \mathcal{H}$ as a set of candidate vectors of hyperparameters constructed via means of independent and uniform draws without replacement from $\mathcal{H}$. A random search considers every candidate in $\mathcal{H}^{RS}$ and computes

$$
\begin{align*}
\hat{\gamma}(\mathcal{H}^{RS}), & \quad \text{when using the estimator in definition 4,} \\
\tilde{\gamma}_{BJK}(c, \mathcal{H}^{RS}), & \quad \text{when using the estimator in definition 6,} \\
\tilde{\gamma}_{AJK}(d, \mathcal{H}^{RS}), & \quad \text{when using the estimator in definition 7,}
\end{align*}
$$

with the same approach employed for grid search.

In its most naive implementation, it is a grid search based on a region of interest constructed by taking random candidates from $\mathcal{H}$. This operation allows to keep the computational advantages of grid search, while exploring a more heterogeneous section of $\mathcal{H}$. This is especially relevant if $\gamma$ is high-dimensional, since it is difficult to generate a proper set of candidates on the basis of some deterministic or subjective rule.

This formulation for the random search is rather naive. Nonetheless, Bergstra and Bengio (2012) showed that it is (at least) as good as more advanced versions of random search. Further details on these algorithms can be found in Solis and Wets (1981) and Andradóttir (2015). Random search tends to be less effective for cases where the number of hyperparameters to tune is very large. For these cases, alternative and more powerful techniques (e.g., simulated annealing, particle swarm optimization) surveyed in Weise (2009) could help. However, since they would inevitably increase the computational burden and the complexity of the hyperparameter optimisation, they are left for future research.

A final point that should be taken into account is that while definition 8 is intuitive, it is also prone to errors in some circumstances. Indeed, when the expected error surface is flat, it is hard to pick one candidate in particular. In these cases, it is often more

---

9The latter is also called manual search (Bergstra and Bengio, 2012).
sensible to evaluate the whole grid of interest and use the threshold where the surface starts flattening as optimal hyperparameters.

3. Empirical application

This section illustrates the functionality of the artificial delete- \( d \) jackknife by tuning the hyperparameters of penalised VARMAs on weekly exchange rate returns.

The exchange rates complexities serve as a good empirical example to benchmark different techniques for selecting hyperparameters.\(^{10}\) Starting with the contribution of Meese and Rogoff (1983), a large body of empirical economic research has found that forecasting models for exchange rates based on macroeconomic data or informed by economic theory are often outperformed by simple univariate techniques and parsimonious multivariate methods usually difficult to tune.

3.1. Penalised VARMA

This subsection describes the case in which forecasters form their predictions using high-dimensional elastic-net VARMA(\( q, r \)) models.\(^{11}\) Clearly, the following assumptions and definitions affect only the empirical example in section 3.

Assumption 11 (VARMA model). Within section 3, forecasters form their expectations assuming that

\[
Y_{t+1} = \Pi_1 Y_t + \ldots + \Pi_q Y_{t-q+1} + \Xi_1 V_t + \ldots + \Xi_r V_{t-r+1} + V_{t+1},
\]

where \( V_{t+1} \sim \mathcal{N}(0_{n \times 1}, \Sigma) \) with \( \Sigma \) being positive definite, \( t \in \mathbb{Z} \).\(^{12}\) The autoregressive and moving average coefficients are \( n \times n \) matrices for which the VARMA is causal and invertible (Brockwell et al., 1991, pp. 418-420).

For simplicity of notation, let

\[
\Pi := \begin{pmatrix} \Pi_1 & \ldots & \Pi_q \end{pmatrix},
\Xi := \begin{pmatrix} \Xi_1 & \ldots & \Xi_r \end{pmatrix}.
\]

Moreover, consider only parametrisations where \( \min(q, r) = 0 \).\(^{13}\)

---

\(^{10}\)It is important to remark that this manuscript does not intend to find the best model (among a class of techniques) for predicting exchange rates, but rather it aims to show that the artificial jackknife is a valid approach for tuning the models in this example.

\(^{11}\)It is important to stress that the VARMA model encompasses common univariate (i.e., AR, MA, ARIMA) and multivariate (i.e., VAR, VMA, VARIMA) forecasting methods.

\(^{12}\)The data is assumed to have zero mean and unit standard deviation for simplicity of notation.

\(^{13}\)Under assumption 2, \( \max(q, r) > 0 \). Therefore, letting \( \min(q, r) = 0 \) does not exclude the white noise case, since the model could be parametrised to have autoregressive and moving average coefficients equal to zero. This point is purely to simplify the notation in section 7.
**Definition 11** (Penalised maximum likelihood estimation). Forecasters use penalised maximum likelihood estimation to estimate the estimated VARMA coefficients. With complete data, this implies

\[
\hat{\theta}_s(\gamma) := \arg \max_{\theta \in \mathcal{R}} \mathcal{L}(\theta \mid Y_{1:s}) - \mathcal{P}(\theta, \gamma),
\]

where \(\mathcal{P}(\mathcal{R}, \gamma)\) is the region of interest for the parameters implicitly defined in assumption 11, \(\mathcal{L}(\theta \mid Y_{1:s}) \simeq -\frac{s}{2} \ln \|\Sigma\| - \frac{1}{2} \text{Tr} \left[ \sum_{t=1}^{s} \Sigma^{-1}_t V_t(\theta)V_t(\theta)' \right]\) denotes the log-likelihood of the VARMA model (Lütkepohl, 2005, ch. 11) and \(\mathcal{P}(\theta, \gamma)\) is a penalty function, for \(\max(q, r) \leq s \leq T\). By extension, \(\Pi, \Xi\) and \(\Sigma\) are the VARMA coefficients built from \(\theta\).

Performing penalised maximum likelihood with incomplete data is non-trivial. In order to overcome the related complexities, this article uses an Expectation-Conditional Maximisation (ECM) algorithm (Meng and Rubin, 1993). The details of this iterative estimation procedure are described in section 7.

The penalty function of interest for this empirical application builds on the elastic-net literature (Zou and Hastie, 2005; Zou and Zhang, 2009).

**Definition 12** (Generalised elastic-net penalty). For any \(p \in \mathbb{N}\), let

\[
\Gamma(\gamma, p) := \lambda \begin{pmatrix}
I_n & 0_{n \times n} & \ldots & 0_{n \times n} \\
0_{n \times n} & \beta \cdot I_n & \ldots & 0_{n \times n} \\
\vdots & \ddots & \ddots & \vdots \\
0_{n \times n} & \ldots & \ldots & \beta^{p-1} \cdot I_n
\end{pmatrix}
\]

where \(\gamma := (q \ r \ \lambda \ \alpha \ \beta)'\) is a given vector of hyperparameters with \(\lambda \geq 0, 0 \leq \alpha \leq 1\) and \(\beta \geq 1\). Building on that, this manuscript uses the penalty

\[
\mathcal{P}(\theta, \gamma) := \begin{cases}
\frac{1-\alpha}{2} \left\| \Pi \Gamma(\gamma, q) \right\|_F^2 + \frac{\alpha}{2} \left\| \Pi \Gamma(\gamma, q) \right\|_{1,1} & \text{if } q > 0 \text{ and } r = 0, \\
\frac{1-\alpha}{2} \left\| \Xi \Gamma(\gamma, r) \right\|_F^2 + \frac{\alpha}{2} \left\| \Xi \Gamma(\gamma, r) \right\|_{1,1} & \text{if } q = 0 \text{ and } r > 0.
\end{cases}
\]

**Remark.** Note that when the penalty is active (i.e., \(\lambda > 0\)), \(\Gamma(\gamma, q)\) and \(\Gamma(\gamma, r)\) are 

\[\text{The innovations } V_t(\theta) \equiv V_t \text{ in equation 6. This notation is used for stressing its dependence from the coefficients in } \theta \text{ and obtain a compact formula.}\]
diagonal and positive definite matrices, and thus

\[
P(\theta, \gamma) = \begin{cases} 
\sum_{i=1}^{n} \sum_{j=1}^{nq} \frac{1-\alpha}{2} \Pi_{i,j}^2 |\Gamma(\gamma, q)|_{i,j} + \frac{\alpha}{2} |\Pi_{i,j}| |\Gamma(\gamma, q)|_{j,j} & \text{if } q > 0 \text{ and } r = 0, \\
\sum_{i=1}^{n} \sum_{j=1}^{nr} \frac{1-\alpha}{2} \Xi_{i,j}^2 |\Gamma(\gamma, r)|_{i,j} + \frac{\alpha}{2} |\Xi_{i,j}| |\Gamma(\gamma, r)|_{j,j} & \text{if } q = 0 \text{ and } r > 0.
\end{cases}
\]

The penalty \(P(\theta, \gamma)\) is a generalisation of the elastic-net that allows to penalise more autoregressive and moving average coefficients referring to distant points in time. As for its standard implementation, when \(\alpha = 1\) and \(\alpha = 0\) the function is equivalent to the LASSO (Tibshirani, 1996) and ridge (Hoerl and Kennard, 1970) penalties. These penalties perform differently depending on the empirical setting in which they are employed, as extensively described in Zou and Hastie (2005).\(^{15}\) For \(0 < \alpha < 1\) the model allows for a sparse model and benefits from the co-movement of correlated predictors. With respect to the standard elastic-net, the penalty function in equation 7 includes \(\beta\), an additional hyperparameter. If \(\beta > 1\), then \(P(\theta, \gamma)\) penalises more coefficients referring to distant points in time.\(^{16}\)

### 3.2. Results

The time series for the exchange rates are collected from the Federal Reserve Board H.10 and include regular weekly (Friday, EOP) observations from January 1999 to the end of December 2020, for a set of major economies reported in table 2. This dataset contains a total of 1,148 weeks and 21,812 observations. These are all the exchange rates in the Federal Reserve Board H.10 that did not have a fixed or pegged rate with the dollar in the sample. Moreover, these exchange rates are not taken in levels, but they are transformed in weekly log-returns instead.

The sample is divided into three blocks: a presample (January 1999 to December 1999), a selection sample (January 2000 to December 2001) and a test sample (January 2002 to December 2020). The presample is used for computing \(w\) and then discarded. Each entry in this vector is equal to 1 over the variance of the corresponding series in the presample. This is done to equally weight each series, regardless of its volatility. Next, the grid of candidate hyperparameters \(H = H_p \times H_\lambda \times H_\alpha \times H_\beta\) is explored on the selection sample following section 2.4 to compute the associated expected errors. This is done settings \(H_p := \{4\}, H_\lambda := [10^{-2}, 2.5], H_\alpha := [0, 1]\) and \(H_\beta := [1, 2]\) and

---

\(^{15}\)LASSO gives a sparse representation of the model and thus a simple regression in few predictors. Ridge does not select subsets of regressors, but it shrinks all of them jointly. The ability of LASSO in selecting the same covariates over time is rather poor when some of them are highly correlated.\(^{16}\)This idea is commonly used in time series and a simple parallel can be made by looking at Bayesian VARs with Minnesota priors (Doan et al., 1984; Litterman, 1986). Indeed, in stationary settings, this set of priors shrinks the vector autoregression toward a white noise (i.e., it shrinks the coefficients to zero) and penalises more distant lags. The penalty in equation 7 is similar in spirit, but it allows for a sparse representation of the model and for the use of moving average coefficients.
Figure 2: Expected error for the candidate hyperparameters in $\mathcal{H}$.

Notes: For each model, the first row describes the expected error in absolute terms, while the second one shows it in relative terms (per subsampling method). The scalar $\lambda \beta^3$ denotes the shrinkage associated to the farthest lag. The block jackknife output is adjusted to reduce the finite-sample methodological defects as described in section 2.2.

letting the methods based on pseudo out-of-sample criteria defining $t_0$ to be such that the part of the selection sample used for estimation purposes ends in December 2000. The set $\mathcal{H}_p$ fixes the number of lags to 4: a value considered large enough to forecast the weekly financial returns.\(^{17}\) The sets referring to the remaining hyperparameters

\(^{17}\)Since the vector moving averages in this manuscript are constrained to be invertible, they account for a higher persistence than the vector autoregressions of the same order. Indeed, any invertible VMA can be equivalently thought as a VAR($\infty$).
allow to control the overall shrinkage level and kill superfluous lags, if needed. Finally, the expected error associated to each hyperparameter is assessed by computing the realised pseudo out-of-sample error over the test sample, having estimated the relevant model (i.e., vector autoregression or vector moving average) once on the full selection sample.

Figure 2 describes the random search output obtained with the error estimators described in section 2.2. The vector autoregression results show a series of important features. First, it is evident that the expected error decreases when the shrinkage level increases, no matter the error estimator. Second, the area with the lowest expected error is where $\lambda_3 \geq 15$. This location seems independent from $\alpha$ and it is again found regardless of the error estimator of choice. Third, there are strong differences in the scale of the expected error obtained via different estimators. Indeed, the artificial jackknife estimates are the most conservative, since the expected error is higher in scale across all candidate hyperparameters. The pseudo out-of-sample gives similar expected errors for any configuration with $\lambda_3 < 15$, but a more pronounced fall before the common flattening. The block jackknife measures are the least conservative. The picture observed through the lenses of the vector moving average results is quite different. Indeed, while the expected error still decreases when the shrinkage level increases, it does so at a different rate for the case of the artificial jackknife. Indeed, the artificial jackknife expected error decreases sharply already at $\lambda_3 \approx 1$, whereas it takes about a fivefold figure to start decreasing when estimated with the pseudo out-of-sample and block jackknife. Furthermore, the artificial jackknife expected error starts flattening at a much smaller shrinkage level compared to the benchmarks.

| Error estimator | Selection RMSE |
|-----------------|----------------|
|                 | Vector autoregression | Vector moving average |
| In-sample error | 7.28 | 1.50 |
| Pseudo out-of-sample error | 1.00 | 1.00 |
| Block jackknife, $c/T = 0.1$ | 1.68 | 1.20 |
| Block jackknife, $c/T = 0.2$ | 2.47 | 1.30 |
| Block jackknife, $c/T = 0.1$ (adjusted) | 1.17 | 1.14 |
| Block jackknife, $c/T = 0.2$ (adjusted) | 1.28 | 1.20 |
| Artificial delete-$d$ jackknife | 0.93 | 0.89 |

Table 1: Selection relative mean squared error.

Notes: The selection MSE is computed by averaging the squared error between the realised and expected error associated to each candidate hyperparameter. The realised error is the pseudo out-of-sample error computed in the test sample. The selection RMSE is rescaled so that values lower than one indicate a better performance compared to the pseudo-out-of-sample selection.

These expected errors are then compared with the pseudo out-of-sample error re-

\footnote{Figure 4 shows the same output using an alternative graphical representation.}
alised in test sample. Table 1 summarises the quality of each selection method for both vector autoregressions and moving averages. These results show that the artificial jackknife gives the best estimate of the expected error across all candidate hyperparameters. This is evident both for vector autoregressions and vector moving averages. A further interesting result is that the pseudo out-of-sample is better than the block jackknife (both raw and adjusted). This is likely due to the small number of partitions that the block jackknife is able to generate in this empirical application. Finally, it follows from table 1 and figure 2 that the best configuration for vector moving averages requires a much smaller shrinkage level compared to vector autoregressions.

4. Concluding comments

This article proposes a new approach for selecting hyperparameters in time series denoted as artificial delete-$d$ jackknife: a generalisation of the delete-$d$ jackknife.

By contrast with existing approaches, the artificial delete-$d$ jackknife can partition dependent data into a large set of unique partitions, even when $T$ is relatively small. These partitions are used for constructing a robust forecast error estimator, based on pseudo out-of-sample evaluations. The artificial delete-$d$ jackknife has strong finite-sample advantages and converges in probability to the true error. Empirical results on weekly exchange rate returns are also promising.

While the theory developed in this paper is based on a weighted mean square loss, the artificial jackknife error could be extended to other loss functions for prediction and classification problems. Also, it could be expanded to compute the uncertainty around sample statistics in time series. These and a few other points are not fully developed in this article and they are left for future research.

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5. Finite sample results

5.A. Proposition 1

PROOF OF PROPOSITION 1. Let

\[ \overline{\text{err}}^{-j}(\gamma) := \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} L(Y_{t+1}^{-j}, \hat{Y}_{t+1}^{-j}(\gamma)), \]

for any \( j = 1, \ldots, |J| \). Hence,

\[ \overline{\text{err}}(J, \gamma) = \frac{1}{|J|} \left( \frac{|J|}{T-t_0} \sum_{j=1}^{T-1} L(Y_{t+1}^{-j}, \hat{Y}_{t+1}^{-j}(\gamma)) \right) = \frac{1}{|J|} \sum_{j=1}^{|J|} \overline{\text{err}}^{-j}(\gamma). \]

It follows that

\[
\begin{align*}
\text{var}(\overline{\text{err}}(J, \gamma)) & = \frac{1}{|J|^2} \left[ \sum_{j=1}^{|J|} \overline{\text{err}}^{-j}(\gamma) \right]^2 \\
& = \frac{1}{|J|^2} \sum_{i=1}^{|J|} \sum_{j=1}^{|J|} \text{cov} \left( \overline{\text{err}}^{-i}(\gamma), \overline{\text{err}}^{-j}(\gamma) \right) \\
& = \frac{1}{|J|^2} \left\{ \sum_{i=1}^{|J|} \text{var} \left( \overline{\text{err}}^{-i}(\gamma) \right) + \sum_{i=1}^{|J|} \sum_{j \neq i} \text{cov} \left( \overline{\text{err}}^{-i}(\gamma), \overline{\text{err}}^{-j}(\gamma) \right) \right\}.
\end{align*}
\]

Under assumption 6,

\[
\text{var}(\overline{\text{err}}(J, \gamma)) = \frac{\sigma^2(T-t_0, \gamma)}{|J|} + \frac{1}{|J|^2} \sum_{i=1}^{|J|} \sum_{j \neq i} \text{cov} \left( \overline{\text{err}}^{-i}(\gamma), \overline{\text{err}}^{-j}(\gamma) \right).
\]

For any \( J \) with cardinality equal to one, including the pseudo out-of-sample error (cf. definition 5 remark),

\[ \text{var}(\overline{\text{err}}(J, \gamma)) = \sigma^2(T-t_0, \gamma). \]

Instead, for \( J \) with a larger cardinality and heterogeneous partitions, the \( \text{var}(\overline{\text{err}}(J, \gamma)) \) is lower or equal to \( \sigma^2(T-t_0, \gamma) \). Among others, this is the case of the block and artificial delete-\( d \) jackknife errors. Hence,

\[
\begin{align*}
\text{var} \left( \overline{\text{err}}^{AJK}(d, \gamma) \right) & \leq \text{var}(\overline{\text{err}}(\gamma)), \\
\text{var} \left( \overline{\text{err}}^{BJK}(c, \gamma) \right) & \leq \text{var}(\overline{\text{err}}(\gamma)).
\end{align*}
\]
5.B. Simulation results

Determining which one between the block and artificial jackknife error estimators has the smaller variance is a little more complicated, since it requires to study the covariances.

Note that

\[
\frac{1}{|J|^2} \sum_{i=1}^{|J|} \sum_{\substack{j=1 \atop j \neq i}}^{|J|} \text{cov} \left[ \tilde{e}_{\gamma}^{-i}(\gamma), \tilde{e}_{\gamma}^{-j}(\gamma) \right]
\]

is inversely proportional to the heterogeneity across subsamples. One way to measure it is through the expected value of the Jaccard similarity coefficient (denoted as sim) of a pair of subsamples \(S_1\) and \(S_2\) selected at random among those in the family of sets \(J\). The lower it is, the more diverse the partitions. Formally, the expected value of this Jaccard index is denoted as \(E[\text{sim}(S_1, S_2)]\). Besides, for all partitioning methods in which the elements of \(J\) have the same cardinality,

\[
E[\text{sim}(S_1, S_2)] = E \left[ \frac{|S_1 \cap S_2|}{|S_1 \cup S_2|} \right] = \sum_{k=0}^{|S_1|} \frac{k}{2|S_1|} \text{Pr}(|S_1 \cap S_2| = k).
\]

In the case of the artificial delete-\(d\) jackknife, the random subsamples are selected from those in \(A(d)\), they are dependent on \(d\) and such that \(|S_1(d)| = |S_2(d)| = d\). Therefore,

\[
E[\text{sim}(S_1(d), S_2(d))] = \sum_{k=1}^{d-1} \frac{k}{2d-k} \frac{d \binom{nT-d}{d-k}}{\binom{nT}{d}}.
\]

With the block jackknife, the random subsamples are selected from those in \(B(c)\), they are dependent on \(c\) and such that \(|S_1(c)| = |S_2(c)| = c\). In this case, the partitions are non-interrupted blocks of consecutive observations. Considering each \(i\)-th subsample as a different case and employing the law of total probability allows to determine that

\[
\text{Pr}(|S_1(c) \cap S_2(c)| = k) = \frac{1}{(T-c+1)^2} \sum_{i=1}^{T-c+1} \left( \mathbb{I}_{T-c+1-i \geq c-k} + \mathbb{I}_{i \geq c-k} \right),
\]

for \(1 \leq k < c\). Hence, the expected Jaccard index for the block jackknife is

\[
E[\text{sim}(S_1(c), S_2(c))] = \frac{1}{(T-c+1)^2} \sum_{k=1}^{c-1} \sum_{i=1}^{T-c+1} \frac{k}{2c-k} \left( \mathbb{I}_{T-c+1-i \geq c-k} + \mathbb{I}_{i \geq c-k} \right).
\]

These expectations are compared within deterministic computer simulations to understand whether the artificial jackknife produces more heterogeneous partitions than
the block jackknife. In order to put $d$ and $c$ onto the same scale, the former is set to be equal to $nc$ in this simulation exercise.\footnote{Indeed, these versions of the jackknife place the same number of artificial missing observations when $d = nc$, since $c$ refers to a number of time periods.}

**Figure 3:** Difference between the expected Jaccard similarity associated to the block and artificial jackknifes for a broad set of configurations.

**Notes:** A positive difference indicates a configuration in which the artificial jackknife outperforms the block version. The $n$-axis ranges from 5 to 50 with a step size of 5. The $T$-axis ranges from 50 to 500 with a step size of 50. Hence, the surface comprises 100 points.

Results in figure 3 show that the artificial jackknife significantly outperforms the block jackknife especially in small samples. However, its relative advantage decreases as the sample size increases. This is consistent for different $c$ and compatible with the asymptotic results in section 6.
6. Asymptotic results

6.A. Proposition 2

Recall that with complete data
\[
\text{err}_T(\gamma) = + \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 + \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i=1}^{n} w_i \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma)) \right]^2
\]
\[
+ \frac{2}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1} \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma)) \right],
\]
where \( f_i(X_t, \Psi) \equiv [f(X_t, \Psi)]_i \), \( g_i(X_t, \hat{\theta}_t(\gamma)) \equiv [g(X_t, \hat{\theta}_t(\gamma))]_i \) for \( i = 1, \ldots, n \) and any integer point in time \( t \). The proof of proposition 2 relies on the following three lemmas, and it is reported hereinafter. Each term on the RHS of equation 8 is linked to one of these lemmas.

Lemma 1. Under assumptions 1–5, assumption 7, assumption 9 and with complete data, it holds that
\[
\lim_{T \to \infty} T \mathbb{E} \left( \left| \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 - \text{err} \right|^2 \right) \leq M_1^2 M_3.
\]

Proof. Recall that the vector of weights \( w \) is made of \( n \) given non-negative finite scalars, \( V_{t+1} \overset{i.i.d.}{\sim} (0_{n \times 1}, \Sigma) \), the model parameters are finite, \( \Sigma \) is a positive definite matrix and \( \text{err} = \sum_{i=1}^{n} w_i \Sigma_{i,i} \). Using the bias-variance decomposition
\[
\mathbb{E} \left( \left| \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 - \sum_{i=1}^{n} w_i \Sigma_{i,i} \right|^2 \right)
\]
can be written in the equivalent form
\[
\left[ \text{bias} \left( \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 \right) \right]^2 + \text{var} \left( \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 \right).
\]
The bias is equal to zero since
\[
\text{bias} \left( \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 \right) = \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i=1}^{n} \mathbb{E} \left( w_i V_{i,t+1}^2 \right) - \sum_{i=1}^{n} w_i \Sigma_{i,i} = 0.
\]
The variance is

\[
\text{var} \left( \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 \right) \\
= \frac{1}{(T-t_0)^2} \sum_{t=t_0}^{T-1} \sum_{s=t_0}^{T-1} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \left[ \mathbb{E} \left( V_{i,t+1}^2 V_{j,s+1}^2 \right) - \mathbb{E} \left( V_{i,t+1}^2 \right) \mathbb{E} \left( V_{j,s+1}^2 \right) \right] \\
= \frac{1}{(T-t_0)^2} \sum_{t=t_0}^{T-1} \sum_{s=t_0}^{T-1} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{cov} \left( V_{i,t+1}^2, V_{j,s+1}^2 \right).
\]

Given that \( V_{t+1} \) is i.i.d.

\[
\text{var} \left( \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 \right) = \frac{1}{(T-t_0)^2} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i w_j \text{cov} \left( V_{i,t+1}^2, V_{j,t+1}^2 \right).
\]

Since all weights are non-negative and under assumption 7,

\[
\mathbb{E} \left( \left| \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 - \sum_{i=1}^{n} w_i \Sigma_{n,i} \right|^2 \right) \\
\leq \frac{M_1^2}{(T-t_0)^2} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov} \left( V_{i,t+1}^2, V_{j,t+1}^2 \right) \\
\leq \frac{M_1^2 M_3}{T(1-t_0/T)}.
\]

Hence, under assumption 9,

\[
\lim_{T \to \infty} T \mathbb{E} \left( \left| \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 - \text{err} \right|^2 \right) \leq M_1^2 M_3.
\]

\[\square\]

**Lemma 2.** Under assumptions 1–5, assumptions 7–9 and with complete data

\[
\lim_{T \to \infty} \frac{T}{\ln T} \mathbb{E} \left\{ \left| \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \left[ f_i(X_t; \Psi) - g_i(X_t; \hat{\theta}_t(\gamma)) \right] \right|^2 \right\} \leq M_4.
\]

**Proof.** Note that, since all weights are non-negative,

\[
\mathbb{E} \left\{ \left| \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \left[ f_i(X_t; \Psi) - g_i(X_t; \hat{\theta}_t(\gamma)) \right] \right|^2 \right\} \\
= \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \mathbb{E} \left[ \left| f_i(X_t; \Psi) - g_i(X_t; \hat{\theta}_t(\gamma)) \right|^2 \right].
\]

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Under assumptions 7–8,

\[
\frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \mathbb{E} \left[ |f_i(\mathbf{X}_t, \Psi) - g_i(\mathbf{X}_t, \hat{\theta}_t(\gamma))|^2 \right] \\
\leq \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \mathbb{E} \left[ \|f(\mathbf{X}_t, \Psi) - g(\mathbf{X}_t, \hat{\theta}_t(\gamma))\|_2^2 \right] \\
\leq \frac{M_4}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \\
\leq \frac{M_1 M_4}{T-t_0} \sum_{t=t_0}^{T-1} \frac{1}{t}.
\]

Since

\[
\sum_{t=t_0}^{T-1} \frac{1}{t} = \sum_{s=1}^{T-t_0} \frac{1}{s} \leq \sum_{s=1}^{T} \frac{1}{s} - t_0 - 1 \leq \sum_{s=1}^{T} \frac{1}{s} \leq \int_{1}^{T} \frac{1}{s} \, ds \leq \ln T + 1,
\]

it follows that

\[
\mathbb{E} \left\{ \left| \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \left[ f_i(\mathbf{X}_t, \Psi) - g_i(\mathbf{X}_t, \hat{\theta}_t(\gamma)) \right]^2 \right| \right\} \leq \frac{(\ln T + 1)(M_1 M_4)}{T(1 - t_0/T)}.
\]

Therefore, under assumption 9, it holds that

\[
\lim_{T \to \infty} \frac{T}{\ln T} \mathbb{E} \left\{ \left| \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \left[ f_i(\mathbf{X}_t, \Psi) - g_i(\mathbf{X}_t, \hat{\theta}_t(\gamma)) \right]^2 \right| \right\} \leq M_1 M_4.
\]

\[
\square
\]

**Lemma 3.** Under assumptions 1–5, assumptions 7–9 and with complete data

\[
\lim_{T \to \infty} \sqrt{T} \mathbb{E} \left\{ \left| \frac{2}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1} \left[ f_i(\mathbf{X}_t, \Psi) - g_i(\mathbf{X}_t, \hat{\theta}_t(\gamma)) \right] \right| \right\} \leq 4M_1 \sqrt{M_2} \sqrt{M_4}.
\]

**PROOF.** Since all weights are non-negative, it follows from the Cauchy-Schwarz inequality that

\[
\mathbb{E} \left\{ \left| \frac{2}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1} \left[ f_i(\mathbf{X}_t, \Psi) - g_i(\mathbf{X}_t, \hat{\theta}_t(\gamma)) \right] \right| \right\} \\
\leq \frac{2}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \mathbb{E} \left[ \left| V_{i,t+1} \right| \left| f_i(\mathbf{X}_t, \Psi) - g_i(\mathbf{X}_t, \hat{\theta}_t(\gamma)) \right| \right] \\
\leq \frac{2}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \sqrt{\mathbb{E} \left[ \left| V_{i,t+1} \right|^2 \right]} \sqrt{\mathbb{E} \left[ \left| f_i(\mathbf{X}_t, \Psi) - g_i(\mathbf{X}_t, \hat{\theta}_t(\gamma)) \right|^2 \right]}.
\]
Under assumptions 7–8,

\[
\frac{2}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \sqrt{\mathbb{E} \left[ |V_{i,t+1}|^2 \right]} \sqrt{\mathbb{E} \left[ |f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma))|^2 \right]} \leq \frac{2 \sqrt{M_2}}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \sqrt{\mathbb{E} \left[ |f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma))|^2 \right]} \leq \frac{2 \sqrt{M_2} \sqrt{M_4}}{T-t_0} \sum_{t=t_0}^{T-1} \frac{1}{\sqrt{t}} \sum_{i=1}^{n} w_i \leq \frac{2 M_1 \sqrt{M_2} \sqrt{M_4} (2 \sqrt{T} - 1)}{T(1-t_0/T)}
\]

Note that

\[
\sum_{t=t_0}^{T-1} \frac{1}{\sqrt{t}} = \sum_{s=1}^{T-t_0} \frac{1}{\sqrt{s+t_0-1}} \leq \sum_{s=1}^{T} \frac{1}{\sqrt{s}} \leq \sum_{s=1}^{T} \frac{1}{\sqrt{s}} \leq \int_{1}^{T} \frac{1}{\sqrt{s}} ds \leq 2 \sqrt{T} - 1.
\]

Thus,

\[
\mathbb{E} \left\{ \left| \frac{2}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1} \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma)) \right] \right| \right\} \leq \frac{2 M_1 \sqrt{M_2} \sqrt{M_4} (2 \sqrt{T} - 1)}{T(1-t_0/T)}
\]

and, under assumption 9,

\[
\lim_{T \to \infty} \sqrt{T} \mathbb{E} \left\{ \left| \frac{2}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1} \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma)) \right] \right| \right\} \leq 4 M_1 \sqrt{M_2} \sqrt{M_4}.
\]

\[
\boxed{\text{PROOF OF PROPOSITION 2.} \text{ From equation 8 it follows that}}
\]

\[
\mathbb{E} \left[ \left| \text{err}_{T}(\gamma) - \text{err} \right| \right] \leq \mathbb{E} \left\{ \left| \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1}^2 - \sum_{i=1}^{n} w_i \Sigma_{i,i} \right| \right\}
\]

\[
+ \mathbb{E} \left\{ \left| \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma)) \right]^2 \right| \right\}
\]

\[
+ \mathbb{E} \left\{ \left| \frac{2}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i=1}^{n} w_i V_{i,t+1} \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma)) \right] \right| \right\}.
\]

By lemmas 1–3 the second term has the slowest rate of convergence.
Therefore, under assumption 9, it holds that
\[
\lim_{T \to \infty} \frac{T}{\ln T} \mathbb{E} \left[ |\tilde{e}_{\mathcal{R}}(\gamma) - err| \right] \leq M_1 M_4.
\]

\[\square\]

6.B. Proposition 3

Recall that with potentially incomplete data
\[
\tilde{e}_{\mathcal{R}}(\gamma) = + \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in \mathcal{D}(t+1)} w_i V_{i,t+1}^2
\]
\[
+ \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in \mathcal{D}(t+1)} w_i \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_i(\gamma)) \right]^2
\]
\[
+ \frac{2}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in \mathcal{D}(t+1)} w_i V_{i,t+1} \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_i(\gamma)) \right].
\]

Lemma 4 analyses the first term of equation 9. This is then used for structuring the proof of proposition 3 (reported hereinafter).

Lemma 4. Under assumptions 1–5, assumption 7, assumptions 9–10 and with potentially incomplete data, it holds that
\[
\lim_{T \to \infty} T \mathbb{E} \left( \left| \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in \mathcal{D}(t+1)} w_i V_{i,t+1}^2 - err \right|^2 \right) \leq M_1^2 M_3.
\]

Proof. Similarly to lemma 1, using the bias-variance decomposition,
\[
\mathbb{E} \left( \left| \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in \mathcal{D}(t+1)} w_i V_{i,t+1}^2 - \sum_{i=1}^{n} w_i \Sigma_{i,i} \right|^2 \right)
\]
can be re-written in the equivalent form
\[
\left[ \text{bias} \left( \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in \mathcal{D}(t+1)} w_i V_{i,t+1}^2 \right) \right]^2 + \text{var} \left( \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in \mathcal{D}(t+1)} w_i V_{i,t+1}^2 \right).
\]

Consider that
\[
\frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in \mathcal{D}(t+1)} w_i V_{i,t+1}^2 = \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \left( \sum_{i=1}^{n} w_i V_{i,t+1}^2 + \sum_{i \in \mathcal{D}(t+1) \setminus \mathcal{D}(t+1) \cap \{ \gamma \}} w_i V_{i,t+1}^2 \right).
\]
It follows that the bias is

\[
\begin{align*}
\text{bias} & \left( \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i \in \mathcal{X}(t+1)} w_i V_{i,t+1}^2 \right) \\
& = \frac{T-t_0-t_{\text{NA}}}{T-t_0} \sum_{i=1}^{n} w_i \Sigma_{i,i} + \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i \in \mathcal{X}(t+1) \neq n} w_i \Sigma_{i,i} - \sum_{i=1}^{n} w_i \Sigma_{i,i} \\
& \leq \frac{1-t_0/T - t_{\text{NA}}}{1-t_0/T} \sum_{i=1}^{n} w_i \Sigma_{i,i} + \frac{t_{\text{NA}}/T}{1-t_0/T} M_5 - \sum_{i=1}^{n} w_i \Sigma_{i,i},
\end{align*}
\]

where

\[
M_5 := \sup_t \left( \sum_{i \in \mathcal{X}(t+1) \neq n} w_i \Sigma_{i,i} \right)
\]

and thus \(0 \leq M_5 < \sum_{i=1}^{n} w_i \Sigma_{i,i}\). Furthermore, following an approach analogous to lemma 1 (but allowing for potentially incomplete data), it holds that

\[
\begin{align*}
\text{var} \left( \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i \in \mathcal{X}(t+1)} w_i V_{i,t+1}^2 \right) & = \\
& + \frac{1}{(T-t_0)^2} \sum_{t=t_0}^{T-1} \sum_{i \in \mathcal{X}(t+1) = n} w_i w_j \text{cov} \left( V_{i,t+1}^2, V_{j,t+1}^2 \right) \\
& + \frac{1}{(T-t_0)^2} \sum_{t=t_0}^{T-1} \sum_{i \in \mathcal{X}(t+1) \neq n} \sum_{j \in \mathcal{X}(t+1)} w_i w_j \text{cov} \left( V_{i,t+1}^2, V_{j,t+1}^2 \right).
\end{align*}
\]

Under assumption 7,

\[
\begin{align*}
\text{var} \left( \frac{1}{T-t_0} \sum_{t=t_0}^{T-1} \sum_{i \in \mathcal{X}(t+1)} w_i V_{i,t+1}^2 \right) & \leq \frac{(T-t_0-t_{\text{NA}})}{(T-t_0)^2} (M_1^2 M_3) + \frac{t_{\text{NA}} M_1^2 M_6}{T-t_0} \\
& \leq \frac{(1-t_0/T - t_{\text{NA}}/T)}{T(1-t_0/T)^2} (M_1^2 M_3) + \frac{(t_{\text{NA}}/T) (M_1^2 M_6)}{T(1-t_0/T)^2},
\end{align*}
\]

where

\[
M_6 := \sup_t \left[ \sum_{i \in \mathcal{X}(t+1) \neq n} \sum_{j \in \mathcal{X}(t+1)} \left| \text{cov} (V_{i,t+1}^2, V_{j,t+1}^2) \right| \right]
\]

and thus \(0 \leq M_6 < M_3\). 

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Under assumptions 9–10,

\[ \lim_{T \to \infty} \text{bias} \left( \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in D(t+1)} w_i V_{i,t+1}^2 \right) = 0, \]

\[ \lim_{T \to \infty} T \text{var} \left( \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in D(t+1)} w_i V_{i,t+1}^2 \right) \leq M_1^2 M_3. \]

As a result,

\[ \lim_{T \to \infty} T \mathbb{E} \left( \left| \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in D(t+1)} w_i V_{i,t+1}^2 - \text{err} \right|^2 \right) \leq M_1^2 M_3. \]

\[ \square \]

**Remark** (Upper bounds in lemma 4). Defining \( M_5 \) and \( M_6 \) to be equal to \( \sum_{i=1}^{n} w_i \Sigma_{i,i} \) and \( M_3 \) would give loose upper bounds for the bias and variance. This would mask the relevancy of assumption 10, and it would not be ideal especially for the case in which \( t_{N/A} \) is large and all series are always jointly missing.

**Proof of Proposition 3.** In the presence of potentially incomplete data,

\[ \mathbb{E} \left[ |\text{err}_T(\gamma) - \text{err}| \right] \leq + \mathbb{E} \left( \left| \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in D(t+1)} w_i V_{i,t+1}^2 - \sum_{i=1}^{n} w_i \Sigma_{i,i} \right| \right) 

+ \mathbb{E} \left\{ \left| \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in D(t+1)} w_i \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma)) \right] \right|^2 \right\} 

+ \mathbb{E} \left\{ \left| \frac{2}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in D(t+1)} w_i V_{i,t+1} \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma)) \right] \right| \right\}. \]

Note that

\[ \mathbb{E} \left[ |\text{err}_T(\gamma) - \text{err}| \right] \leq + \mathbb{E} \left( \left| \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i \in D(t+1)} w_i V_{i,t+1}^2 - \sum_{i=1}^{n} w_i \Sigma_{i,i} \right| \right) 

+ \mathbb{E} \left\{ \left| \frac{1}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i = 1}^{n} w_i \left[ f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma)) \right] \right|^2 \right\} 

+ \frac{2}{T - t_0} \sum_{t = t_0}^{T-1} \sum_{i = 1}^{n} w_i \mathbb{E} \left[ |V_{i,t+1}||f_i(X_t, \Psi) - g_i(X_t, \hat{\theta}_t(\gamma))| \right]. \]

By lemmas 2–4 the second term has the slowest rate of convergence. Thus, under
assumptions 9–10,

\[ \lim_{T \to \infty} \frac{T}{\ln T} \mathbb{E} \left[ \left| \tilde{e}_{\text{err}}(T) - err \right| \right] \leq M_1 M_4. \]

even with potentially incomplete data. \qed

**Proof of Corollary 3.1.** It follows from the argument in section 5 that \( \tilde{e}_{\text{err}}(\mathcal{I}, \gamma) \) is the average of a number of i.d. pseudo out-of-sample errors. Trivially, this implies that proposition 3 is also valid for this case. \qed

7. Estimation of penalised VARMA models for incomplete data

Traditional estimation methods for VARMA\((q, r)\) models are unable to handle incomplete time series. This is especially problematic for the scope of the manuscript since the artificial jackknife introduces missing values into the data. This appendix proposes overcoming the issue with an ECM algorithm (Meng and Rubin, 1993).

7.A. State-space representation

The ECM algorithm developed in this appendix is structured similarly to the EM algorithm in Shumway and Stoffer (1982) and Watson and Engle (1983), and thus starting from a model representation in state-space form.

**Definition 13** (State-space). Recall that \( \min(q, r) = 0 \) and \( \max(q, r) \geq 1 \), and let \( m := nq + nr + n \mathbb{I}_{q=0} \). The representation chosen for the VARMA\((q, r)\) is such that, for any integer \( t \),

\begin{align*}
Y_t &= BX_t + \epsilon_t, \\
X_t &= CX_{t-1} + D\tilde{V}_t,
\end{align*}

where \( X_t \) denotes a vector of \( m \) latent states, \( \epsilon_t \overset{\text{w.n.}}{\sim} N(0_{n \times 1}, R) \), \( \tilde{V}_t \overset{\text{w.n.}}{\sim} N(0_{n \times 1}, \Sigma) \),

\[
B := \left( \begin{array}{c|c}
I_n & B_*
\end{array} \right), \\
R := \varepsilon \cdot I_n, \\
D := \left( \begin{array}{c}
I_n \\
0_{m-n \times n}
\end{array} \right),
\]

\( \varepsilon \) is a small positive real number and \( \Sigma \) is a \( n \times n \) positive definite covariance matrix.\(^{20}\)

\(^{20}\)In the empirical implementation reported in this manuscript \( \varepsilon = 10^{-4} \).
The structure of $B_*$ and $C$ is described in definitions 14–15, differentiating between the VAR and VMA cases. It follows from these definitions that $V_t \approx \tilde{V}_t$ and $\Sigma \approx \tilde{\Sigma}$. The precision of these approximations is inversely proportional to $\varepsilon$.

**Definition 14** (Direct state-space representation: VAR). If $q > 0$ and $r = 0$,

$$B_* := 0_{n \times m - n}$$

$$C := \begin{pmatrix}
\hat{\Pi}_1 & \hat{\Pi}_2 & \ldots & \hat{\Pi}_{q-1} & \hat{\Pi}_q \\
I_n & 0_{n \times n} & \ldots & 0_{n \times n} & \\
0_{n \times n} & \ddots & \ddots & \vdots & \\
\vdots & \ddots & \ddots & \vdots & \\
0_{n \times n} & \ldots & \ldots & I_n & 0_{n \times n}
\end{pmatrix},$$

where $\hat{\Pi}_i$ denotes a $n \times n$ matrix, for any integer $1 \leq i \leq q$. With this representation, the vectors of latent states can be partitioned as

$$X_t = (\hat{Y}_t' \ldots \hat{Y}_{t-q+1}')',$$

where $Y_t \approx \hat{Y}_t$. The precision of this approximation is inversely proportional to $\varepsilon$.

**Definition 15** (Direct state-space representation: VMA). If $q = 0$ and $r > 0$,

$$B_* := \begin{pmatrix}
\hat{\Xi}_1 & \ldots & \hat{\Xi}_r
\end{pmatrix}$$

$$C := \begin{pmatrix}
0_{n \times n} & 0_{n \times n} & \ldots & 0_{n \times n} & 0_{n \times n} \\
I_n & 0_{n \times n} & \ldots & 0_{n \times n} & \\
0_{n \times n} & \ddots & \ddots & \vdots & \\
\vdots & \ddots & \ddots & \vdots & \\
0_{n \times n} & \ldots & \ldots & I_n & 0_{n \times n}
\end{pmatrix},$$

where $\hat{\Xi}_i$ denotes a $n \times n$ matrix, for any integer $1 \leq i \leq r$. With this representation, the vectors of latent states can be partitioned as

$$X_t = (\hat{V}_t' \ldots \hat{V}_{t-r}')'.$$

**Assumption 12** (Initial conditions). Consistently with assumption 1, it is assumed that the first observation for the measurements refers to $t = 1$. Therefore, the state-space representation is initialised such that $X_0 \sim N(\mu_0, \Omega_0)$, where $\mu_0$ and $\Omega_0$ denote a generic $m \times 1$ vector and a $m \times m$ positive definite covariance matrix.

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Assumption 13 (Causality and invertibility of the state-space model). In all cases, the state-space parameters are assumed to be such that the model is causal and invertible. The region in which the parameters must lie is indicated with $\mathcal{R}$ (as in section 3.1).

Example 1 (Vector autoregression of order 2). For this example, let $q = 2$ and $r = 0$. Under definition 14, the state-space representation in equations 10–11 becomes

$$ Y_t = \begin{pmatrix} I_n & 0_{n \times n} \end{pmatrix} \begin{pmatrix} \tilde{Y}_t \\ \tilde{Y}_{t-1} \end{pmatrix} + \epsilon_t, $$

$$ \begin{pmatrix} \tilde{Y}_t \\ \tilde{Y}_{t-1} \end{pmatrix} = \begin{pmatrix} \tilde{\Pi}_1 & \tilde{\Pi}_2 \\ I_n & 0_{n \times n} \end{pmatrix} \begin{pmatrix} \tilde{Y}_{t-1} \\ \tilde{Y}_{t-2} \end{pmatrix} + \begin{pmatrix} I_n \\ 0_{n \times n} \end{pmatrix} \tilde{V}_t. $$

Example 2 (Vector moving average of order 2). For this example, let $q = 0$ and $r = 2$. Under definition 15, the state-space representation in equations 10–11 becomes

$$ Y_t = \begin{pmatrix} I_n & \tilde{\Xi}_1 & \tilde{\Xi}_2 \end{pmatrix} \begin{pmatrix} \tilde{V}_t \\ \tilde{V}_{t-1} \\ \tilde{V}_{t-2} \end{pmatrix} + \epsilon_t, $$

$$ \begin{pmatrix} \tilde{V}_t \\ \tilde{V}_{t-1} \\ \tilde{V}_{t-2} \end{pmatrix} = \begin{pmatrix} 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \\ I_n & 0_{n \times n} & 0_{n \times n} \\ 0_{n \times n} & I_n & 0_{n \times n} \end{pmatrix} \begin{pmatrix} \tilde{V}_{t-1} \\ \tilde{V}_{t-2} \\ \tilde{V}_{t-3} \end{pmatrix} + \begin{pmatrix} I_n \\ 0_{n \times n} \\ 0_{n \times n} \end{pmatrix} \tilde{V}_t. $$

7.B. The Expectation-Conditional Maximisation algorithm

The ECM algorithm builds on the penalised maximum likelihood problem described in definition 11 and proposes an iterative estimation method compatible with missing observations.

Let

$$ \vartheta := \left( \text{vec}(B)' \quad \text{vech}(R)' \quad \text{vec}(C)' \quad \text{vech}(\hat{\Sigma})' \quad \text{vec}(\mu_0)' \quad \text{vech}(\Omega_0)' \right)'.$$

Assumption 14. Assume that the ECM algorithm is initialised as in section 7.C and denote with $\hat{\vartheta}_0^0(\gamma)$ the corresponding initial vector of VARMA coefficients.

The ECM algorithm proceeds from the initial value assigned under assumption 14 and repeats the process described in definition 16 until it converges.

Assumption 15 (Convergence). The ECM algorithm is said to be converged when the conditions described in algorithm 1 are reached.
Definition 16 (Estimation routine). At any iteration $k + 1 > 0$, the ECM algorithm computes the vector of coefficients

$$
\hat{\vartheta}^{k+1}(\gamma) := \arg \max_{\vartheta \in \mathcal{R}} \mathbb{E} \left[ \mathcal{L}(\vartheta | Y_{1:s}, X_{1:s}) | \mathcal{Y}(s), \hat{\vartheta}^k(\gamma) \right] - \mathbb{E} \left[ \mathcal{P}(\vartheta, \gamma) | \mathcal{Y}(s), \hat{\vartheta}^k(\gamma) \right],
$$

where $\mathcal{Y}(s)$ is the information set available at time $t = s$.

$$
\mathcal{L}(\vartheta | Y_{1:s}, X_{1:s}) \simeq -\frac{1}{2} \ln |\Omega_0| - \frac{1}{2} \text{Tr} \left[ \Omega_0^{-1}(X_0 - \mu_0)(X_0 - \mu_0)' \right] - \frac{s}{2} \ln |\Sigma| - \frac{1}{2} \text{Tr} \left[ \sum_{t=1}^{s} \Sigma^{-1}(X_{1:n,t} - C_*, X_{t-1})(X_{1:n,t} - C_*, X_{t-1})' \right] - \frac{s}{2} \ln |R| - \frac{1}{2} \text{Tr} \left[ \sum_{t=1}^{s} R^{-1}(Y_t - BX_t)(Y_t - BX_t)' \right].
$$

$C_* \equiv C_{1:n,1:m}$ and the underlined coefficients denote the state-space parameters within $\hat{\vartheta}$. The function $\mathcal{L}(\vartheta | Y_{1:s}, X_{1:s})$ is known as complete-data (i.e., as if the states were known and the data was fully observed) log-likelihood. Finally,

$$
\mathcal{P}(\vartheta, \gamma) := \begin{cases} 
\frac{1-\alpha}{2} \| C_* \Gamma(\gamma, q) \|_F^2 + \frac{\alpha}{2} \| C_* \Gamma(\gamma, q) \|_{1,1} & \text{if } q > 0 \text{ and } r = 0, \\
\frac{1-\alpha}{2} \| B_* \Gamma(\gamma, r) \|_F^2 + \frac{\alpha}{2} \| B_* \Gamma(\gamma, r) \|_{1,1} & \text{if } q = 0 \text{ and } r > 0.
\end{cases}
$$

This function is a compact version of the elastic-net penalty in equation 7 for the state-space representations illustrated in this appendix.

Every $\hat{\vartheta}^{k+1}(\gamma)$ is estimated via the so-called E-step and CM-step. The E-step corresponds to the operation of computing the complete-data penalised log-likelihood expectation, conditional on the parameters estimated at the $k$-th iteration and $\mathcal{Y}(s)$. The CM-step estimates $\hat{\vartheta}^{k+1}(\gamma)$ to conditionally maximise the resulting expected penalised log-likelihood.

In order to formalise the E-step for the complete-data log-likelihood, it is convenient to clarify which measurements are observed at each point in time.

Definition 17 (Observed measurements). Let

$$
\mathcal{F} := \bigcup_{i=1}^{n} \mathcal{F}_i,
$$

$$
\mathcal{F}(s) := \{ t : t \in \mathcal{F}, 1 \leq t \leq s \},
$$

describe two sets representing the points in time (either over the full sample or up to

---

1. Direct maximisation of the complete-data log-likelihood is not feasible since there are missing observations in the measurements.
time \( s \) in which at least one measurement is observed, for \( 1 \leq s \leq T \). Thus, let

\[
Y_{t}^{\text{obs}} := \left( Y_{i,t} \right)_{i \in \mathcal{D}(t)} \\
B_{t}^{\text{obs}} := A_{t}B
\]

be the vector of observed measurements at time \( t \) and the corresponding \(|\mathcal{D}(t)| \times m\) matrix of coefficients, for \( t \in \mathcal{F} \). Every \( A_{t} \) is indeed a selection matrix constituted by ones and zeros that permits to retrieve the appropriate rows of \( B \) for every \( t \in \mathcal{F} \).

Building on Shumway and Stoffer (1982) and Watson and Engle (1983), it is also handy to formalise the E-step by using the Kalman smoother output described in definition 18. Indeed, lemmas 5–6 build on that to compute a series of conditional expectations. Finally, proposition 4 uses these results to design the E-step for the complete-data log-likelihood.

**Definition 18** (Kalman smoother output). The Kalman smoother output used for formalising the E-step is

\[
\hat{X}_{t} := \mathbb{E} \left[ X_{t} \mid \mathcal{Y}(s), \hat{\vartheta}_{s}^{k}(\gamma) \right], \\
\hat{P}_{t,t-j} := \text{Cov} \left[ X_{t}, X_{t-j} \mid \mathcal{Y}(s), \hat{\vartheta}_{s}^{k}(\gamma) \right],
\]

for any \( k \geq 0, 0 \leq j \leq t \) and \( t \geq 0 \). Let also \( \hat{P}_{t} \equiv \hat{P}_{t,t} \).

**Remark.** These estimates can be computed using a range of different recursions. This article follows the approach in Durbin and Koopman (2012) for \( \hat{X}_{t} \) and \( \hat{P}_{t} \), and the one in Watson and Engle (1983) for \( \hat{P}_{t,t-1} \).

**Lemma 5.** Building on definition 18, it follows that

\[
\mathbb{E} \left[ X_{t}X'_{t-j} \mid \mathcal{Y}(s), \hat{\vartheta}_{s}^{k}(\gamma) \right] = \hat{X}_{t}\hat{X}'_{t-j} + \hat{P}_{t,t-j},
\]

for any \( k \geq 0, 0 \leq j \leq t \) and \( t \geq 0 \).

**Proof.** Note that \( X_{t} = \hat{X}_{t} + (X_{t} - \hat{X}_{t}) \). Thus,

\[
\mathbb{E} \left[ X_{t}X'_{t-j} \mid \mathcal{Y}(s), \hat{\vartheta}_{s}^{k}(\gamma) \right] \\
= \mathbb{E} \left\{ X_{t} \left[ \hat{X}_{t-j} + (X_{t-j} - \hat{X}_{t-j}) \right]' \mid \mathcal{Y}(s), \hat{\vartheta}_{s}^{k}(\gamma) \right\} \\
= \mathbb{E} \left[ X_{t}\hat{X}'_{t-j} \mid \mathcal{Y}(s), \hat{\vartheta}_{s}^{k}(\gamma) \right] + \mathbb{E} \left[ X_{t}X'_{t-j} \mid \mathcal{Y}(s), \hat{\vartheta}_{s}^{k}(\gamma) \right] - \mathbb{E} \left[ X_{t}\hat{X}'_{t-j} \mid \mathcal{Y}(s), \hat{\vartheta}_{s}^{k}(\gamma) \right].
\]
Since
\[
E \left[ X_t X'_{t-j} \mid \mathcal{Y}(s), \hat{\theta}_s^k(\gamma) \right] = E \left[ X_t \mid \mathcal{Y}(s), \hat{\theta}_s^k(\gamma) \right] E \left[ X_{t-j} \mid \mathcal{Y}(s), \hat{\theta}_s^k(\gamma) \right]' = \hat{X}_t \hat{X}'_{t-j},
\]
it holds that
\[
E \left[ X, X'_{t-j} \mid \mathcal{Y}(s), \hat{\theta}_s^k(\gamma) \right] = \hat{X}_t \hat{X}'_{t-j} + \hat{P}_{t,t-j}.
\]

**Lemma 6.** Building on definition 13 and definition 18, it follows that
\[
E \left[ X_{i_1:i_2} X'_{i_3:i_4,t-j} \mid \mathcal{Y}(s), \hat{\theta}_s^k(\gamma) \right] = \left[ \hat{X}_t \hat{X}'_{t-j} + \hat{P}_{t,t-j} \right]_{i_1:i_2,i_3:i_4},
\]
for any \( k \geq 0, 1 \leq i_1 \leq i_2 \leq m, 1 \leq i_3 \leq i_4 \leq m \) and \( 0 \leq j \leq t \).

**Proof.** Note that
\[
X_{i_1:i_2,t} X'_{i_3:i_4,t-j} = \left[ X_t X'_{t-j} \right]_{i_1:i_2,i_3:i_4}.
\]
It then follows from the same logic employed in the proof of lemma 5 that
\[
E \left[ X_{i_1:i_2} X'_{i_3:i_4,t-j} \mid \mathcal{Y}(s), \hat{\theta}_s^k(\gamma) \right] = \left[ \hat{X}_t \hat{X}'_{t-j} + \hat{P}_{t,t-j} \right]_{i_1:i_2,i_3:i_4}.
\]

**Proposition 4.** Let
\[
\mathcal{L}_e \left[ \hat{\theta} \mid \mathcal{Y}(s), \hat{\theta}_s^k(\gamma) \right] \equiv E \left[ \mathcal{L} \left( \hat{\theta} \mid Y_{1:s}, X_{1:s} \right) \mid \mathcal{Y}(s), \hat{\theta}_s^k(\gamma) \right].
\]

Building on definition 13 and definition 18, it follows that
\[
\mathcal{L}_e \left[ \hat{\theta} \mid \mathcal{Y}(s), \hat{\theta}_s^k(\gamma) \right] \simeq -\frac{1}{2} \ln |\Sigma_0| - \frac{1}{2} \text{Tr} \left[ \Sigma_0^{-1} (\hat{\mathcal{E}} - \hat{X}_0 \hat{\mu}_0' - \mu_0 \hat{X}_0' + \mu_0 \hat{\mu}_0') + \frac{s}{2} \ln |\Sigma| - \frac{1}{2} \text{Tr} \left[ \hat{\Sigma}_s^{-1} (\hat{F}_s - \hat{C}_s \hat{C}_s' - \hat{C}_s \hat{G}_s' + \hat{C}_s \hat{H}_s \hat{C}_s') \right] \right] - \frac{1}{2} \text{Tr} \left\{ \sum_{t \in \mathcal{Y}(s)} \left[ (Y_{t}^{\text{obs}} - \hat{B}_{t}^{\text{obs}} \hat{X}_t) (Y_{t}^{\text{obs}} - \hat{B}_{t}^{\text{obs}} \hat{X}_t)' + \hat{B}_{t}^{\text{obs}} \hat{P}_t \hat{B}_{t}^{\text{obs}} \right] \right\}.
\]
where

\[ \hat{E} := \mathbb{E}\left[ X_0X'_0 \mid \mathcal{Y}(s), \hat{\theta}^k_s(\gamma) \right] = \hat{X}_0\hat{X}'_0 + \hat{P}_0, \]

\[ \hat{F}_s := \sum_{t=1}^{s} \mathbb{E}\left[ X_{1:n,t}X'_{1:n,t} \mid \mathcal{Y}(s), \hat{\theta}^k_s(\gamma) \right] = \sum_{t=1}^{s} \left( \hat{X}_t\hat{X}'_t + \hat{P}_{t,t} \right)_{1:n,1:n}, \]

\[ \hat{G}_s := \sum_{t=1}^{s} \mathbb{E}\left[ X_{1:n,t}X'_{t-1} \mid \mathcal{Y}(s), \hat{\theta}^k_s(\gamma) \right] = \sum_{t=1}^{s} \left( \hat{X}_t\hat{X}'_{t-1} + \hat{P}_{t,t-1} \right)_{1:n,1:m}, \]

\[ \hat{H}_s := \sum_{t=1}^{s} \mathbb{E}\left[ X_{t-1}X'_{t-1} \mid \mathcal{Y}(s), \hat{\theta}^k_s(\gamma) \right] = \sum_{t=1}^{s} \left( \hat{X}_{t-1}\hat{X}'_{t-1} + \hat{P}_{t-1} \right). \]

**Proof.** Note that

\[ -\frac{s}{2} \ln |R| - \frac{1}{2} \text{Tr} \left[ \sum_{t=1}^{s} R^{-1}(Y_t - BX_t)(Y_t - BX_t)' \right] \simeq -\frac{1}{2\varepsilon} \text{Tr} \left[ \sum_{t=1}^{s} (Y_t - BX_t)(Y_t - BX_t)' \right], \]

since the covariance matrix \( R = \varepsilon \cdot I_n \). Thus, the complete-data log-likelihood

\[ \mathcal{L}(\hat{\theta} \mid Y_{1:s}, X_{1:s}) \simeq -\frac{1}{2} \ln |\Omega_0| - \frac{1}{2} \text{Tr} \left[ \Omega_0^{-1}(X_0X'_0 - X_0\mu_0' - \mu_0X'_0 + \mu_0\mu_0') \right] \]

\[ -\frac{s}{2} \ln |\Sigma| - \frac{1}{2} \text{Tr} \left[ \sum_{t=1}^{s} \Sigma^{-1}(X_{1:n,t}X'_{1:n,t} - X_{1:n,t}X'_{t-1}C_s') \right] \]

\[ -\frac{1}{2} \text{Tr} \left[ \sum_{t=1}^{s} \Sigma^{-1}(-C_XX_{t-1}X'_{1:n,t} + C_XX_{t-1}X'_{t-1}C_s') \right] \]

\[ -\frac{1}{2} \varepsilon \text{Tr} \left[ \sum_{t=1}^{s} (Y_t - BX_t)(Y_t - BX_t)' \right]. \]

It follows from definition 18 and lemma 5 that the expectation of the terms in the first row of equation 12, conditional on the information set \( \mathcal{Y}(s) \) and \( \hat{\theta}^k_s(\gamma) \), is

\[ -\frac{1}{2} \ln |\Omega_0| - \frac{1}{2} \text{Tr} \left[ \Omega_0^{-1}(\hat{E} - \hat{X}_0\mu_0' - \mu_0\hat{X}'_0 + \mu_0\mu_0') \right]. \]

The following terms are a bit harder to handle. It follows from lemma 6 that

\[ \mathbb{E}\left[ X_{1:n,t}X'_{1:n,t} \mid \mathcal{Y}(s), \hat{\theta}^k_s(\gamma) \right] = \left( \hat{X}_t\hat{X}'_t + \hat{P}_{t,t} \right)_{1:n,1:n}, \]

\[ \mathbb{E}\left[ X_{1:n,t}X'_{t-1} \mid \mathcal{Y}(s), \hat{\theta}^k_s(\gamma) \right] = \left( \hat{X}_t\hat{X}'_{t-1} + \hat{P}_{t,t-1} \right)_{1:n,1:m}. \]

Building on that, it holds that the expectation of the terms in the second and third row of equation 12, conditional on the information set \( \mathcal{Y}(s) \) and \( \hat{\theta}^k_s(\gamma) \), is

\[ -\frac{s}{2} \ln |\Sigma| - \frac{1}{2} \text{Tr} \left[ \Sigma^{-1}(\hat{F}_s - \hat{G}_sC_s' - C_s\hat{G}'_s + C_s\hat{H}_sC_s') \right]. \]

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Finally, it follows directly from Shumway and Stoffer (1982, Section 3) that

\[- \frac{1}{2\varepsilon} \text{Tr} \left\{ \mathbb{E} \left[ \sum_{t=1}^{s} (Y_t - B_t X_t)(Y_t - B_t X_t)' \mid \mathcal{Y}(s), \hat{\vartheta}_s^k(\gamma) \right] \right\} \]

\[\simeq - \frac{1}{2\varepsilon} \text{Tr} \left\{ \sum_{t \in \mathcal{T}(s)} \left[ (Y_{t \text{obs}} - B_{t \text{obs}} \hat{X}_t) (Y_{t \text{obs}} - B_{t \text{obs}} \hat{X}_t)' + B_{t \text{obs}} \hat{P}_t B_{t \text{obs}}' \right] \right\}.
\]

\[\square\]

**Lemma 7.** Building on definition 13 and definitions 16–18, it follows that

\[\mathbb{E} \left[ \mathcal{P}(\vartheta, \gamma) \mid \mathcal{Y}(s), \hat{\vartheta}_s^k(\gamma) \right] = \mathcal{P}(\vartheta, \gamma).\]

**Proof.** A formal proof is not reported since it is immediate. Indeed, the penalty function in this ECM algorithm depends on the current coefficients (i.e., \(C_\ast\) or \(B_\ast\)) and hyperparameters only.

**Proposition 4** and **Lemma 7** give the structure of the expected penalised log-likelihood

\[\mathcal{M}_e \left[ \vartheta, \gamma \mid \mathcal{Y}(s), \hat{\vartheta}_s^k(\gamma) \right] := \mathcal{L}_e \left[ \vartheta \mid \mathcal{Y}(s), \hat{\vartheta}_s^k(\gamma) \right] - \mathcal{P}(\vartheta, \gamma).\]

(13)

The CM-step conditionally maximises \(\mathcal{M}_e \left[ \vartheta, \gamma \mid \mathcal{Y}(s), \hat{\vartheta}_s^k(\gamma) \right]\) to estimate the state-space coefficients. **Lemmas 8–11** detail the estimation procedure. For internal consistency, the estimated coefficients are denoted with the same naming used in definitions 14–15, a “\(\text{‘hat’}\)” symbol on top, an \(s\) in the subscript to highlight the sample size and a superscript denoting the reference to the ECM iteration.

**Lemma 8.** Building on definition 13 and definitions 16–18, it follows that the ECM estimators at a generic iteration \(k + 1 > 0\) for \(\mu_0\) and \(\Omega_0\) are

\[\hat{\mu}_{0,s}^{k+1}(\gamma) = \hat{X}_0,\]

\[\hat{\Omega}_{0,s}^{k+1}(\gamma) = \hat{P}_0.\]

**Proof.** The derivative of equation 13 with respect to \(\mu_0\) is

\[\frac{\partial \mathcal{M}_e \left[ \vartheta, \gamma \mid \mathcal{Y}(s), \hat{\vartheta}_s^k(\gamma) \right]}{\partial \mu_0} = -\frac{1}{2} \Omega_0^{-1} (-2\hat{X}_0 + 2\mu_0).\]

It follows that the maximiser for the expected penalised log-likelihood is

\[\hat{\mu}_{0,s}^{k+1}(\gamma) = \hat{X}_0.\]
The derivative of equation 13 with respect to \( \Omega_0 \) and fixing \( \mu_0 = \hat{\mu}_{0,s}^{k+1}(\gamma) \) is

\[
-\frac{1}{2} \Omega_0^{-1} + \frac{1}{2} \Omega_0^{-1} \left[ \hat{E} - \hat{X}_0 \hat{p}_{0,s}^{k+1}(\gamma) - \hat{p}_{0,s}^{k+1}(\gamma) \hat{X}_0' + \hat{p}_{0,s}^{k+1}(\gamma) \hat{X}_0' \right] \Omega_0^{-1},
\]

or,

\[
-\frac{1}{2} \Omega_0^{-1} + \frac{1}{2} \Omega_0^{-1} \left[ \hat{E} - \hat{X}_0 \hat{X}_0' - \hat{X}_0 \hat{X}_0' + \hat{X}_0 \hat{X}_0' \right] \Omega_0^{-1}.
\]

Thus, due to the structure of \( \hat{E} \),

\[
\hat{\Omega}_{0,s}^{k+1}(\gamma) = \hat{P}_0.
\]

\[\square\]

**Lemma 9.** Partition the output of the following Cartesian as

\[
\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\} = \{\mathcal{E}(i, j), (i, j), \mathcal{E}''(i, j)\},
\]

and let

\[
\mathcal{F}(i, j) := \{\mathcal{E}(i, j), \mathcal{E}''(i, j)\},
\]

for any integer \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). Let also \( S(a, b) := \text{sign}(a) \max(|a| - b, 0) \) be the soft-thresholding operator, for any \( a, b \in \mathbb{R} \). Building on definition 13 and definitions 16–18, it follows that, if \( q > 0 \) and \( r = 0 \), the ECM estimators at a generic iteration \( k + 1 > 0 \) for \( C \) is such that

\[
\hat{C}_{s,s}^{k+1}(\gamma) = \begin{pmatrix}
\hat{C}_{1,1,s}^{k+1}(\gamma) & \cdots & \hat{C}_{1,m,s}^{k+1}(\gamma) \\
\vdots & \ddots & \vdots \\
\hat{C}_{n,1,s}^{k+1}(\gamma) & \cdots & \hat{C}_{n,m,s}^{k+1}(\gamma)
\end{pmatrix},
\]

where

\[
\hat{C}_{i,j,s}^{k+1}(\gamma) = S \left[ \sum_{l_1=1}^{n} \hat{\Sigma}_{i,l_1,s}^{k-1}(\gamma) \hat{C}_{l_1,j,s} - \sum_{(l_1,l_2) \in \mathcal{F}(i,j)} \hat{\Sigma}_{i,l_1,s}^{k-1}(\gamma) \hat{C}_{l_1,l_2,s}^{k+1}(\gamma) \hat{H}_{l_2,j,s} + \frac{\nu}{2} \Gamma_{j,j}(\gamma, q) \right]
\]

for any integer \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), and the remaining entries are constant and specified according to the prescriptions in definition 14. If \( q = 0 \) and \( r > 0 \), it follows from definition 15 that \( \hat{C}_{s,s}^{k+1}(\gamma) = 0_{n \times m} \).

**Proof.** This proof starts with the case in which \( q > 0 \) and \( r = 0 \). Given that the absolute value function in the penalty is not differentiable at zero, this part of the
ECM algorithm estimates the free entries of $C$ one-by-one starting from the $C_{1,1}$, in a column-major order and conditioning on a series of coefficients estimated in previous rounds of the same algorithm. Namely, it estimates every $C_{i,j}$ by fixing $\hat{\Sigma} = \hat{\Sigma}_s^k(\gamma)$ and any other free entry of $C$ to the latest estimate available, with $1 \leq i \leq n$ and $1 \leq j \leq m$. In other words, the derivative of equation 13 with respect to $C_{i,j}$ is taken having fixed the parameters as described in the last sentence. If $C_{i,j} \neq 0$, this is

$$\begin{align*}
+ \sum_{l_1=1}^{n} \hat{\Sigma}_{i,l_1,s}^{k-1}(\gamma) \hat{G}_{i,j,s} - \hat{\Sigma}_{i,i,s}^{k-1}(\gamma) C_{i,j} \hat{H}_{j,j,s} - \sum_{(l_1,l_2) \in \mathcal{F}(i,j)} \hat{\Sigma}_{i,l_1,s}^{k-1}(\gamma) \hat{C}_{l_1,l_2,s}^{k+1}(\gamma) \hat{H}_{l_2,j,s} \\
- \left(1 - \alpha\right) C_{i,j} \Gamma_{j,j}(\gamma, q) - \frac{\alpha}{2} \Gamma_{j,j}(\gamma, q) \text{sign}(C_{i,j}).
\end{align*}$$

It follows that

$$\hat{\Sigma}_{i,j,s}^{k+1}(\gamma) = \frac{S \left[ \sum_{l_1=1}^{n} \hat{\Sigma}_{i,l_1,s}^{k-1}(\gamma) \hat{G}_{i,j,s} - \sum_{(l_1,l_2) \in \mathcal{F}(i,j)} \hat{\Sigma}_{i,l_1,s}^{k-1}(\gamma) \hat{C}_{l_1,l_2,s}^{k+1}(\gamma) \hat{H}_{l_2,j,s}, \frac{\alpha}{2} \Gamma_{j,j}(\gamma, q) \right]}{\hat{\Sigma}_{i,i,s}^{k-1}(\gamma) \hat{H}_{j,j,s} + \left(1 - \alpha\right) \Gamma_{j,j}(\gamma, q)}. $$

When $q = 0$ and $r > 0$ the coefficients of interest for this proof are not free parameters and fixed to zero as described in definition 15. □

**Lemma 10.** Building on definition 13 and definitions 16–18, it follows that the ECM estimators at a generic iteration $k + 1 > 0$ for $\hat{\Sigma}$ is

$$\hat{\Sigma}_s^{k+1}(\gamma) = \frac{1}{s} \left[ \hat{F}_s - \hat{G}_s \hat{C}_s^{k+1}(\gamma) - \hat{C}_s^{k+1}(\gamma) \hat{G}_s' + \hat{C}_s^{k+1}(\gamma) \hat{H}_s \hat{C}_s^{k+1}(\gamma) \right].$$

**PROOF.** The derivative of equation 13 with respect to $\hat{\Sigma}$ and fixing $\hat{C}_s = \hat{C}_s^{k+1}(\gamma)$ is

$$\begin{align*}
- \frac{s}{2} \hat{\Sigma}^{-1} - \frac{1}{2} \hat{\Sigma}^{-1} \left[ \hat{F}_s - \hat{G}_s \hat{C}_s^{k+1}(\gamma) - \hat{C}_s^{k+1}(\gamma) \hat{G}_s' + \hat{C}_s^{k+1}(\gamma) \hat{H}_s \hat{C}_s^{k+1}(\gamma) \right] \hat{\Sigma}^{-1}.
\end{align*}$$

It follows that

$$\hat{\Sigma}_s^{k+1}(\gamma) = \frac{1}{s} \left[ \hat{F}_s - \hat{G}_s \hat{C}_s^{k+1}(\gamma) - \hat{C}_s^{k+1}(\gamma) \hat{G}_s' + \hat{C}_s^{k+1}(\gamma) \hat{H}_s \hat{C}_s^{k+1}(\gamma) \right].$$

□

**Remark** (Vector moving average case). When $q = 0$ and $r > 0$, $\hat{C}_s^{k+1}(\gamma) = 0_{n \times m}$. Thus, it follows that $\hat{\Sigma}_s(\gamma) = \frac{1}{s} \hat{F}_s$.

---

22 This approach is similar, in spirit, to Friedman et al. (2010).
Lemma 11. Let

\[ \hat{M}_s := \sum_{t \in T(s)} A_t'Y_{t,obs} X_t', \]

\[ \hat{N}_t := A_t' A_t, \]

\[ \hat{O}_t := \hat{X}_t \hat{X}_t' + \hat{P}_t. \]

Building on definition 13 and definitions 16–18, it follows that, if \( q = 0 \) and \( r > 0 \), the ECM estimators at a generic iteration \( k + 1 > 0 \) for \( B \) is such that

\[ \hat{B}_{s,s}^{k+1}(\gamma) = \begin{pmatrix} \hat{B}_{1,1,s}^{k+1}(\gamma) & \cdots & \hat{B}_{1,nr,s}^{k+1}(\gamma) \\ \vdots & \ddots & \vdots \\ \hat{B}_{n,1,s}^{k+1}(\gamma) & \cdots & \hat{B}_{n,nr,s}^{k+1}(\gamma) \end{pmatrix}, \]

where

\[ \hat{B}_{i,j,s}^{k+1}(\gamma) = S \left[ \hat{M}_{i,j,s} - \sum_{t \in T(s)} \sum_{(l_1,l_2) \in \mathcal{T}(i,j)} \hat{N}_{i,l_1,t} \hat{B}_{l_1,l_2}^{k+1}(\gamma) \hat{O}_{l_2,j,t} + (1 - \alpha) \hat{P}_{l_2} \right] \frac{\alpha \epsilon_{ij} \Gamma_j \cdot \Gamma_{j-n-j-n}(\gamma,r)}{\sum_{t \in T(s)} \hat{N}_{i,l_1,t} \hat{O}_{l_2,j,t}} \]

for any integer \( 1 \leq i \leq n \) and \( n + 1 \leq j \leq m \), and the remaining entries are constant and specified according to the prescriptions in definition 15. If \( q > 0 \) and \( r = 0 \), it follows from definition 14 that \( \hat{B}_{s,s}^{k+1}(\gamma) = 0_{n \times m - n} \).

PROOF. This proof starts with the case in which \( q = 0 \) and \( r > 0 \). Note that

\[ \sum_{t \in T(s)} \left[ (Y_{t,obs} - B_{t,obs} \hat{X}_t) (Y_{t,obs} - B_{t,obs} \hat{X}_t)' + B_{t,obs} \hat{P}_t B_{t,obs}' \right] = \sum_{t \in T(s)} \left[ (Y_{t,obs} - A_t B \hat{X}_t) (Y_{t,obs} - A_t B \hat{X}_t)' + A_t B \hat{P}_t B' A_t' \right] \]

\[ = \sum_{t \in T(s)} \left[ Y_{t,obs}' Y_{t,obs} - Y_{t,obs}' \hat{X}_t B' A_t' - A_t B \hat{X}_t Y_{t,obs}' + A_t B \hat{X}_t Y_{t,obs}' + A_t B (\hat{X}_t \hat{X}_t' + \hat{P}_t) B' A_t' \right]. \]

Since the absolute value function in the penalty is not differentiable at zero, this part of the ECM algorithm estimates the free entries of \( B \) one-by-one starting from the \( B_{1,n+1} \), in a column-major order and conditioning on a series of coefficients estimated in previous rounds of the same algorithm. Indeed, as in lemma 9, the derivative of equation 13 with respect to \( B_{i,j} \) is taken having fixed any other free entry of \( B \) to the
latest estimate available, for $1 \leq i \leq n$ and $n + 1 \leq j \leq m$. If $B_{i,j} \neq 0$, this is

$$+ \varepsilon^{-1} \hat{M}_{i,j,s} - B_{i,j} \sum_{t \in \mathcal{F}(s)} \varepsilon^{-1} \hat{N}_{i,i,t} \hat{O}_{j,j,t} - \sum_{t \in \mathcal{F}(s)} \sum_{(l_1,l_2) \in \mathcal{F}(i,j)} \varepsilon^{-1} \hat{N}_{i,l_1,t} \hat{B}_{l_1,l_2,s}^{k+1}(\gamma) \hat{O}_{l_2,j,t}$$

$$- (1 - \alpha) B_{i,j} \Gamma_{j-n,j-n}(\gamma, r) - \frac{\alpha}{2} \Gamma_{j-n,j-n}(\gamma, r) \text{sign}(B_{i,j}).$$

It follows that

$$\hat{B}_{i,j,s}^{k+1}(\gamma) = \mathcal{S} \left[ \frac{\hat{M}_{i,j,s} - \sum_{t \in \mathcal{F}(s)} \sum_{(l_1,l_2) \in \mathcal{F}(i,j)} \hat{N}_{i,l_1,t} \hat{B}_{l_1,l_2,s}^{k+1}(\gamma) \hat{O}_{l_2,j,t} + (1 - \alpha) \varepsilon \Gamma_{j-n,j-n}(\gamma, r)}{\sum_{t \in \mathcal{F}(s)} \hat{N}_{i,i,t} \hat{O}_{j,j,t}} \right].$$

When $q > 0$ and $r = 0$ the coefficients of interest for this proof are not free parameters and fixed to zero as described in definition 14. \(\square\)

7.C. Initialisation of the ECM algorithm

In small-data settings, the ECM algorithm is initialised interpolating missing observations (if any) with sample average of the observed datapoints. If $q > 0$ and $r = 0$, the model is initialised via OLS (in small data settings) or ridge (in high-dimensional settings). If $q = 0$ and $r > 0$, an estimate of the VMA innovations is computed by taking the sample residuals of a VAR with $\lfloor \sqrt{T} \rfloor$ lags. Indeed, these residuals can be interpreted as those of a truncated VAR(\(\infty\)) resulting from an invertible VMA. The VMA coefficients are then initialised regressing the data on the estimated residuals (either with OLS or ridge, depending on the problem dimensionality).

In both cases, the approach in section 7.D is used for making sure that the estimated coefficients are within the feasible region $\mathcal{R}$.

7.D. Enforcing causality and invertibility

The ECM algorithm makes sure that the autoregressive and moving average coefficients are causal and invertible. If, at any iteration $k + 1 > 1$, the matrix of autoregressive coefficients $\hat{\Pi}_{s}^{k+1}(\gamma)$ needs to be adjusted, it is replaced by the causal

$$\eta^{k+1} \hat{\Pi}_{s}^{k+1}(\gamma) + (1 - \eta^{k+1}) \hat{\Pi}_{s}^{k}(\gamma)$$

associated to the largest feasible $\eta^{k+1} \in \{0, 0.1, 0.2, \ldots, 0.9\}$. An analogous procedure is followed to adjust $\hat{\Omega}_{s}^{k+1}(\gamma)$ when necessary. This approach can be thought as a slowing mechanism that restricted the CM-step to the feasible region $\mathcal{R}$.
7.E. Estimation algorithm summary

Algorithm 1: VARMA with elastic-net penalty

Initialization
The ECM algorithm is initialised as described in section 7.C.

Estimation
for $k \leftarrow 1$ to max_iter do
  for $j \leftarrow 1$ to $m$ do
    Run the Kalman filter and smoother using $\hat{\vartheta}^{k-1}_s(\gamma)$;
    if converged then
      Store the parameters and stop the loop.
    end
    Estimate $\hat{\mu}^k_{s,0}(\gamma)$ and $\hat{\Omega}^k_{s,0}(\gamma)$ as in lemma 8;
    Estimate $\hat{C}^k_s(\gamma)$, $\hat{\Sigma}^k_s(\gamma)$ and $\hat{B}^k_s(\gamma)$ as in lemmas 9–11;
    Build $\hat{\vartheta}^k_s(\gamma)$;
  end
end

Notes
• The results are computed fixing max_iter to 1000. This is a conservative number, since the
  algorithm generally requires substantially less iterations to converge.
• The ECM algorithm is considered to be converged when the estimated coefficients (all
  relevant parameters in lemmas 9–11) do not significantly change in two subsequent iterations.
  This is done by computing the absolute relative change per parameters and comparing at the
  same time the median and 95th quantile respectively with a fixed tolerance of $10^{-3}$ and $10^{-2}$.
  Intuitively, when the coefficients do not change much, the expected log-likelihood and the
  parameters in lemma 8 should also be stable.
• The scalar $\epsilon$ is summed to the denominator of each relative change in order to ensure
  numerical stability.

The replication code for this paper is available on GitHub.
8. Tables and charts

| Mnemonic     | Description                               | Transformation             |
|--------------|-------------------------------------------|----------------------------|
| australia    | Foreign exchange rate: Australia / USA    | Week-on-week (log-returns) |
| brazil       | Foreign exchange rate: Brazil / USA       | Week-on-week (log-returns) |
| canada       | Foreign exchange rate: Canada / USA       | Week-on-week (log-returns) |
| denmark      | Foreign exchange rate: Denmark / USA      | Week-on-week (log-returns) |
| ea           | Foreign exchange rate: EA / USA           | Week-on-week (log-returns) |
| india        | Foreign exchange rate: India / USA        | Week-on-week (log-returns) |
| japan        | Foreign exchange rate: Japan / USA        | Week-on-week (log-returns) |
| mexico       | Foreign exchange rate: Mexico / USA       | Week-on-week (log-returns) |
| new_zealand  | Foreign exchange rate: New Zealand / USA  | Week-on-week (log-returns) |
| norway       | Foreign exchange rate: Norway / USA       | Week-on-week (log-returns) |
| singapore    | Foreign exchange rate: Singapore / USA    | Week-on-week (log-returns) |
| south_africa | Foreign exchange rate: South Africa / USA  | Week-on-week (log-returns) |
| south_korea  | Foreign exchange rate: South Korea / USA  | Week-on-week (log-returns) |
| sweden       | Foreign exchange rate: Sweden / USA       | Week-on-week (log-returns) |
| switzerland  | Foreign exchange rate: Switzerland / USA  | Week-on-week (log-returns) |
| taiwan       | Foreign exchange rate: Taiwan / USA       | Week-on-week (log-returns) |
| thailand     | Foreign exchange rate: Thailand / USA     | Week-on-week (log-returns) |
| uk           | Foreign exchange rate: UK / USA           | Week-on-week (log-returns) |

Table 2: Foreign exchange rates used for the empirical application in section 3.

Notes: The time series are collected from the Federal Reserve Board H.10 and include regular weekly (Friday, EOP) observations from January 1999 to the end of December 2020. This dataset contains a total of 1,148 weeks and 21,812 observations.

Source: Federal Reserve Board.

Figure 4: Expected error for the candidate hyperparameters in $\mathcal{H}$.

Notes: Alternative graphical representation for the VAR (first row) and VMA (second row) selection. The colormap highlights the relative scale of the expected error for each subsampling method. The scalar $\lambda^3\beta^3$ denotes the shrinkage associated to the farthest lag. The block jackknife output is adjusted to reduce the finite-sample methodological defects as described in section 2.2.