Exact solutions to relativistic singular fractional power potentials

Davids Agboola and Yao-Zhong Zhang

School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia

E-mail: d.agboola@maths.uq.edu.au and yzz@maths.uq.edu.au

Received 3 July 2013, in final form 16 October 2013
Published 26 November 2013
Online at stacks.iop.org/JPhysA/46/505301

Abstract

We present (exact) solutions of the Dirac equation with equally mixed interactions for a single fermion bounded by the family of fractional power singular potentials. Closed-form expressions as well as numerical values for the energies were obtained. The wave functions and the allowed values of the potential parameters for the first two members of the family are obtained in terms of a set of algebraic equations. The non-relativistic limit is also discussed and using the Hellmann–Feynmann theorem, some useful expectation values are obtained.

PACS numbers: 03.65.-w, 03.65.Fd, 03.65.Pm, 03.65.Ge, 02.30.Ik

(Some figures may appear in colour only in the online journal)

1. Introduction

The family of singular potentials appears to be very significant in many aspects of modern physics as extensive literature has been developed on the subject (see for example [1–20] and the references therein). One of the early works that generated much interest in the study of singular potentials was presented in [3] where it was argued that real world interactions were likely to be highly singular and thus the study of singular potentials rather than regular potentials might be more relevant physically. This was thereafter followed by applications of singular potentials to the study of gaseous ions moving through a gas [2] and also in the investigation of the elastic differential cross sections for high energy scattering [4–7], magnetic resonances between massless and massive spin-\(\frac{1}{2}\) particles [8], decay rate in mesons [9], interatomic or intermolecular diatomic molecules [10, 11] and non-polar molecules [12, 13].

One particular class of the power-law potentials is the singular fractional power potentials, which are power-law potentials with rational powers. This class of potentials has recently found applications in models exhibiting shape resonance behaviour [9] and in the description of quark–antiquark interactions [14]. Bound state solutions of relativistic and non-relativistic models with such potentials cannot be analytically obtained, as the models are not exactly...
solvable. Thus, these models have been discussed via various approximation and computational methods [15, 21–23]. The methods used in most previous works were ill motivated, without a well-defined solution structure. As a result, closed-form representation expressions for the general solutions have not been previously obtained.

Recently, the functional Bethe ansatz method [24] has been used to obtain the (exact) solutions to the non-relativistic quantum mechanical model with integer power singular potentials proposed in [25, 26]. The aim of this paper is to solve the Dirac equation with the singular fractional power-law potentials of the form

$$V_N(r) = \sum_{p=1}^{2N-1} \frac{a_p}{r^{p/N}}, \quad 2 \leq N \in \mathbb{N},$$

where $r \in (0, \infty)$ and $a_p$ are real parameters. Obviously, $V_1(r)$ is the well-known Coulomb interaction, which will not be considered in our further discussion except as a limiting case. In particular, we obtain exact, closed-form solutions of the Dirac equation for the $N = 2$ and $N = 3$ cases using the Bethe ansatz method.

This work is organized as follows. Section 2 deals with the transformation of the Dirac equation with the singular fractional power-law potentials into a QES second-order differential equation. Sections 3 and 4 present exact closed-form polynomial solutions to the square-root singular potential ($N = 2$) and the third-order singular potential ($N = 3$) respectively. With appropriate limits, the non-relativistic spectrum and wavefunction for the square-root singular potential ($N = 2$) are obtained in section 5. In addition, using the Hellmann–Feynmann theorem (HFT), we worked out some expectation values are for the model. Finally, in section 6 we provide some concluding remarks.

2. Dirac equation with fractional singular potentials

The Dirac equation for a single fermion with mass $\mu$, moving in spherically symmetric central scalar $S(r)$ and vector $V(r)$ potentials can be written in natural units $\hbar = c = 1$ as [27]

$$H \Psi(r) = E_{\text{rel}} \Psi(r), \quad \text{where} \quad H = \sum_{j=1}^{3} \hat{\alpha}_j p_j + \hat{\beta} [\mu + S(r)] + V(r)$$

and $E_{\text{rel}}$ is the relativistic energy, $\{\hat{\alpha}_j\}$ and $\hat{\beta}$ are Dirac matrices defined as

$$\hat{\alpha}_j = \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $\hat{\sigma}_j$ are the Pauli $2 \times 2$ matrices and $\mathbb{1}$ is a $2 \times 2$ unit matrix, which satisfy the anti-commutation relations

$$\hat{\alpha}_j \hat{\alpha}_k + \hat{\alpha}_k \hat{\alpha}_j = 2 \delta_{jk} \mathbb{1},$$

$$\hat{\alpha}_j \hat{\beta} + \hat{\beta} \hat{\alpha}_j = 0,$$

$$\hat{\alpha}_j^2 = \hat{\beta}^2 = \mathbb{1},$$

and $p_j$ is the three momentum which can be written as

$$p_j = -i \partial_j = -i \frac{\partial}{\partial x_j}, \quad 1 \leq j \leq 3.$$
The orbital angular momentum operators $L_{jk}$, the spinor operators $S_{jk}$ and the total angular momentum operators $J_{jk}$ can be defined as follows:

$$L_{jk} = -L_{jk} = i\kappa_j \frac{\partial}{\partial x_k} - i\kappa_k \frac{\partial}{\partial x_j}, \quad S_{jk} = -S_{jk} = i\hat{\alpha}_j \hat{\alpha}_k/2, \quad J_{jk} = L_{jk} + S_{jk},$$

$$\begin{align*}
L^2 &= \sum_{j<k} L_{jk}^2, \quad S^2 = \sum_{j<k} S_{jk}^2, \quad J^2 = \sum_{j<k} J_{jk}^2, \quad 1 \leq j < k \leq 3. \tag{6}
\end{align*}$$

For a spherically symmetric potential, the total angular momentum operator $J_{jk}$ and the spin-orbit operator $K = -\beta(J^2 - L^2 - S^2 + 1/2)$ commute with the Dirac Hamiltonian. For a given total angular momentum $j$, the eigenvalues of $K$ are $\kappa = \pm (j + 1/2)$; $\kappa = -(j + 1/2)$ for aligned spin $j = \ell + \frac{1}{2}$ and $\kappa = (j + 1/2)$ for unaligned spin $j = \ell - \frac{1}{2}$. Moreover, the spin-orbital quantum number $\kappa$ is related to the orbital angular number $\ell$ and the pseudo-orbital angular number $\tilde{\ell}$ by the expressions $\kappa(\kappa + 1) = \ell(\ell + 1)$ and $\kappa(\kappa - 1) = \tilde{\ell}(\tilde{\ell} + 1)$ respectively for $\kappa = \pm 1, \pm 2, \ldots$. The spinor wave functions can be classified according to the radial quantum number $n_r$ and the spin-orbital quantum number $\kappa$ and can be written using the Dirac–Pauli representation

$$\Psi_{n,\kappa}(r) = \frac{1}{r} \begin{pmatrix} F_{n,\kappa}(r) Y_{jm}(\theta, \phi) \\ iG_{n,\kappa}(r) Y_{jm}(\theta, \phi) \end{pmatrix}, \tag{7}$$

where $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ are the radial wave functions of the upper- and the lower-spinor components respectively, $Y_{jm}(\theta, \phi)$ and $Y_{jm}(\theta, \phi)$ are the spinor spherical harmonic functions coupled with the total angular momentum $j$. The orbital and the pseudo-orbital angular momentum quantum numbers for spin symmetry $\ell$ and pseudospin symmetry $\tilde{\ell}$ refer to the upper- and lower-component respectively. Substituting equation (7) into equation (2), and separating the variables we obtain the following coupled radial Dirac equation for the spinor components:

$$\left( \frac{d}{dr} + \frac{\kappa}{r} \right) F_{n,\kappa}(r) = [\mu + E_{n,\kappa} - \Delta(r)] G_{n,\kappa}(r), \tag{8}$$

$$\left( \frac{d}{dr} - \frac{\kappa}{r} \right) G_{n,\kappa}(r) = [\mu - E_{n,\kappa} + \Sigma(r)] F_{n,\kappa}(r), \tag{9}$$

where $\Sigma(r) = V(r) + S(r)$, $\Delta(r) = V(r) - S(r)$ and $n_r$ is the radial quantum number. For an equally mixed interaction, $\Delta(r) = 0$, thus, if we differentiate equation (8), followed by the insertion of equations (9) and (1), we arrive at the following Schrödinger-like second-order differential equation

$$F''_{n,\kappa}(r) + \left[ \left( E_{n,\kappa}^2 - \mu^2 \right) - V_{\text{eff}}(r) \right] F_{n,\kappa}(r) = 0, \tag{10}$$

where the effective potential $V_{\text{eff}}(r)$ is given as

$$V_{\text{eff}}(r) = \frac{\kappa(\kappa + 1)}{r^2} + 2(\mu + E_{n,\kappa}) \sum_{j=1}^{2N-1} \frac{a_j}{r^{j/N}}$$

$$= \frac{\kappa(\kappa + 1)}{r^2} + 2(\mu + E_{n,\kappa}) \left[ \frac{N}{r^{2/N}} \sum_{j=1}^{2N-1} \frac{a_j}{r^{j/N}} + \frac{N}{r^{2N-2}} \sum_{j=1}^{N} \frac{a_j}{r^{j/N}} \right]. \tag{11}$$

If all $a_j$ are positive, then the effective potential $V_{\text{eff}}(r)$ becomes monotonously decreasing, which leads to the no bound state [28–30]. However, by alternating the signs before the first $N \text{ terms}$, the potential becomes bounded (as shown in figure 1) and thus, we may determine the possible bound states. Moreover, for numerical computation, the parameters $a_j$ can be made free while $a_q$ must be constrained for the systems to possess exact solutions.
With the transformation
\[ F_{n,\kappa}(r) = x^{N(k+1)} e^{w(x)} y(x), \quad w(x) = \sum_{p=1}^{N} b_p x^p, \quad x = r^{1/N}, \]
\[ (12) \]
equation (10) becomes
\[ \left\{ \frac{x}{N^2} \frac{d^2}{dx^2} + \left[ \frac{1 - N}{N^2} + \frac{2xw'}{N^2} + \frac{2(k + 1)}{N} \right] \frac{d}{dx} + \left[ \left( E_{n,\kappa}^2 - \mu^2 \right)x^{2N-1} \right. \\
-2(\mu + E_{n,\kappa}) \sum_{p=1}^{N} \alpha_p x^{2N-1-p} + \frac{(1 - N)w'}{N^2} + \left. \left( \frac{\mu}{N} \right)^2 \right) x^{2N-1} \right\} y(x) = 0. \]
\[ (13) \]
By suitably choosing the parameter \( b_p \), which appears in the exponential prefactor, one can obtain exact solutions to equation (13), provided, the potential parameters satisfy certain conditions. This is an essential feature of a quasi-exactly solvable system which can also be attributed to the quantization procedure of the system [33]. In what follows, we examine the \( N = 2 \) and \( N = 3 \) cases using the Bethe ansatz method.
3. Square-root singular potential \((N = 2)\)

In this case, the interaction becomes

\[
V_2(r) = -\frac{a_1}{\sqrt{r}} + \frac{a_2}{r} + \frac{a_3}{r^2}, \quad a_1, a_2 > 0, \tag{14}
\]

and with an appropriate choice of parameter \(b_p\), equations (12) and (13) reduce to

\[
F_{n_r}(r) = x^{2(k+1)} e^{w(x)} y(x), \quad w(x) = -x^2 \left(\mu^2 - E_{n_r}^2\right)^{\frac{1}{2}} + \frac{2a_1 x (\mu + E_{n_r})}{(\mu^2 - E_{n_r}^2)^{\frac{1}{2}}}, \quad x = r^\frac{1}{\kappa}, \tag{15}
\]

and

\[
x y''(x) + \left[-4x^2 \left(\mu^2 - E_{n_r}^2\right)^{\frac{1}{2}} + \frac{4x a_1 (\mu + E_{n_r})}{(\mu^2 - E_{n_r}^2)^{\frac{1}{2}}} + (4\kappa + 3)\right] y'(x) + 4x a_2 (\mu + E_{n_r}) \left[\frac{a_2^2 (\mu + E_{n_r})^2}{(\mu^2 - E_{n_r}^2)^{\frac{1}{2}}} - 2(\kappa + 1) (\mu^2 - E_{n_r}^2)^{\frac{1}{2}} - 2a_2 (\mu + E_{n_r})\right] y(x)
\]

\[
= 8(\mu + E_{n_r}) \left[a_3 - \frac{(\kappa + \frac{3}{2}) a_1}{(\mu^2 - E_{n_r}^2)^{\frac{1}{2}}}\right] y(x), \tag{16}
\]

respectively. Although, the Klein–Gordon case for the above model has been discussed in [22] using the \(s\)-algebraization, in what follows, we show that equation (16) is quasi-exactly solvable and therefore possesses polynomial solutions of degree \(n_r \geq 0\), which we write in the form

\[
y(x) = \prod_{i=1}^{n_r} (x - x_i), \quad y(x) \equiv 1 \quad \text{for} \quad n_r = 0, \tag{17}
\]

where \(\{x_i\}\) are the roots of the polynomials to be determined. To solve equation (16), we apply the functional Bethe ansatz method. Substituting (17) into (16), we obtain

\[
\sum_{i=1}^{n_r} \frac{x}{x - x_i} \sum_{j=1}^{n_r} \frac{2}{x_i - x_j} + \left[-4x^2 \left(\mu^2 - E_{n_r}^2\right)^{\frac{1}{2}} + \frac{4x a_1 (\mu + E_{n_r})}{(\mu^2 - E_{n_r}^2)^{\frac{1}{2}}} + (4\kappa + 3)\right]
\]

\[
\times \sum_{i=1}^{n_r} \frac{1}{x - x_i} + 4x a_2 (\mu + E_{n_r}) \left[\frac{a_2^2 (\mu + E_{n_r})^2}{(\mu^2 - E_{n_r}^2)^{\frac{1}{2}}} - 2(\kappa + 1) (\mu^2 - E_{n_r}^2)^{\frac{1}{2}} - 2a_2 (\mu + E_{n_r})\right]
\]

\[
= 8(\mu + E_{n_r}) \left[a_3 - \frac{(\kappa + \frac{3}{2}) a_1}{(\mu^2 - E_{n_r}^2)^{\frac{1}{2}}}\right]. \tag{18}
\]

The right-hand side of this equation is a constant, while the left-hand side is a meromorphic function with simple poles \(x = x_i\) and singularity at \(x = \infty\). For this equation to be valid, the left-hand side must also be a constant. We thus demand that the coefficients of the powers of \(x\) as well as the residues at the simple poles of the left-hand side be zero. Following Liouville’s theorem, this is the necessary and sufficient condition for the left-hand side of (18) to be a constant.

Executing this demand, we have the following transcendental energy equation

\[
[n_r + 2(\kappa + 1)] (\mu^2 - E_{n_r}^2)^{\frac{1}{2}} - \frac{a_2^2 (\mu + E_{n_r})^2}{(\mu^2 - E_{n_r}^2)^{\frac{1}{2}}} + 2a_2 (\mu + E_{n_r}) = 0, \tag{19}
\]
and the corresponding spinor wavefunction

\[ F_{n,s}(r) \sim x^{3(n+1)} \left[ \prod_{i=1}^{n} (x - x_i) \right] \exp \left[ -x^2 \left( \mu^2 - E_{0,n,s}^2 \right)^{1/2} + \frac{2a_1 x (\mu + E_{n,s})}{(\mu^2 - E_{0,n,s}^2)^{1/2}} \right], \quad x = r^2 \]

\[ G_{n,s}(r) \sim x^{2n} \left[ \prod_{i=1}^{n} (x - x_i) \right] \exp \left[ -x^2 \left( \mu^2 - E_{n,s}^2 \right)^{1/2} + \frac{2a_1 x (\mu + E_{n,s})}{(\mu^2 - E_{0,n,s}^2)^{1/2}} \right] \times \frac{1}{\mu + E_{n,s}} \left\{ -x^2 \left( \mu^2 - E_{0,n,s}^2 \right)^{1/2} + x \left[ \frac{a_1 (\mu + E_{n,s})}{(\mu^2 - E_{0,n,s}^2)^{1/2}} + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{x - x_i} \right] + 2 \kappa + 1 \right\}, \quad (20) \]

subject to the constraint

\[ 8(\mu + E_{n,s}) \left[ a_3 - \frac{(\kappa + \frac{3}{2}) a_1}{(\mu^2 - E_{0,n,s}^2)^{1/2}} \right] = -4(\mu^2 - E_{n,s}^2)^{1/2} \sum_{i=1}^{n} x_i + n_r(4\kappa + 3), \quad (21) \]

where the roots \([x_i]\) satisfy the Bethe ansatz equations

\[ \sum_{j \neq i} \frac{1}{x_i - x_j} = 2x_i \left( \mu^2 - E_{n,s}^2 \right)^{1/2} - \frac{2a_1 (\mu + E_{n,s})}{(\mu^2 - E_{0,n,s}^2)^{1/2}} - \frac{(4\kappa + 3)}{2x_i}, \quad i = 1, \ldots, n_r. \quad (22) \]

It is easy to see that \(y(x) = 1\) is a solution of (16) for certain values of the potential parameters. This solution corresponds to the \(n_r = 0\) case in the general expressions above. Thus from equations (19) and (20) we have the ground state solutions

\[ \begin{align*}
2(\kappa + 1) \left( \mu^2 - E_{0,s}^2 \right)^{1/2} - \frac{a_1^2 (\mu + E_{0,s})^2}{(\mu^2 - E_{0,s}^2)^{1/2}} + 2a_2 (\mu + E_{0,s}) &= 0, \quad \kappa = 1, 2, \ldots \\
\end{align*} \quad (23) \]

and wavefunctions

\[ F_{0,s}(r) \sim x^{2(n+1)} \exp \left[ -x^2 \left( \mu^2 - E_{0,s}^2 \right)^{1/2} + \frac{2a_1 x (\mu + E_{0,s})}{(\mu^2 - E_{0,s}^2)^{1/2}} \right], \quad x = r^2 \]

\[ G_{0,s}(r) \sim x^{2n} \exp \left[ -x^2 \left( \mu^2 - E_{0,s}^2 \right)^{1/2} + \frac{2a_1 x (\mu + E_{0,s})}{(\mu^2 - E_{0,s}^2)^{1/2}} \right] \times \frac{1}{\mu + E_{0,s}} \left\{ -x^2 \left( \mu^2 - E_{0,s}^2 \right)^{1/2} + \frac{a_1 x (\mu + E_{0,s})}{(\mu^2 - E_{0,s}^2)^{1/2}} + 2 \kappa + 1 \right\}, \quad (24) \]

where

\[ a_3 = \frac{(\kappa + \frac{3}{2}) a_1}{(\mu^2 - E_{0,s}^2)^{1/2}}. \quad (25) \]

Similarly, the first excited state solutions corresponding to \(n_r = 1\) are given as

\[ \begin{align*}
(2\kappa + 3) \left( \mu^2 - E_{1,s}^2 \right)^{1/2} - \frac{a_1^2 (\mu + E_{1,s})^2}{(\mu^2 - E_{1,s}^2)^{1/2}} + 2a_2 (\mu + E_{1,s}) &= 0, \quad \kappa = 1, 2, \ldots \\
\end{align*} \quad (26) \]
and the corresponding wavefunction

$$\begin{align*}
F_{1,x}(r) & \sim x^{2(n+1)}(x-x_1) \exp \left[ -x^2 \left( \mu^2 - E_{1,x}^2 \right)^{\frac{1}{2}} + \frac{2a_1x(\mu + E_{1,x})}{(\mu^2 - E_{1,x}^2)^{\frac{1}{2}}} \right], \\
\quad x = r^\frac{1}{7} \\

G_{1,x}(r) & \sim x^{2n}(x-x_1) \exp \left[ -x^2 \left( \mu^2 - E_{1,x}^2 \right)^{\frac{1}{2}} + \frac{2a_1x(\mu + E_{1,x})}{(\mu^2 - E_{1,x}^2)^{\frac{1}{2}}} \right] \\
& \times \frac{1}{\mu + E_{1,x}} \left\{ -x^2 \left( \mu^2 - E_{1,x}^2 \right)^{\frac{1}{2}} + \frac{a_1(\mu + E_{1,x})}{(\mu^2 - E_{1,x}^2)^{\frac{1}{2}}} + \frac{1}{2} \frac{1}{x-x_1} \right\} \\
& + 2\kappa + 1, \quad (27)
\end{align*}$$

subject to the constraint

$$8(\mu + E_{1,x}) \left[ a_3 - \frac{(\kappa + \frac{3}{2})a_1}{(\mu^2 - E_{1,x}^2)^{\frac{1}{2}}} \right] = -4(\mu^2 - E_{1,x}^2)^{\frac{1}{2}} x_1 + 4\kappa + 3, \quad (28)$$

where the roots \{x_i\} satisfy the equation

$$x_i^2 \left( \mu^2 - E_{1,x}^2 \right)^{\frac{1}{2}} - \frac{a_1x_1(\mu + E_{1,x})}{(\mu^2 - E_{1,x}^2)^{\frac{1}{2}}} = -(\kappa + 3/4) = 0. \quad (29)$$

It is important to note that a careful selection of \(a_1\) and \(a_2\) is crucial for the existence of bound state. Moreover, it is obvious from equation (26) that the energy can either be positive or negative depending on the values of parameters \(a_1\) and \(a_2\). For a given value of \(\mu\), the energy moves from positive to negative states as \(a_1\) and \(a_2\) increase while for any given \(\mu\), \(a_1\) and \(a_2\), the energy moves from negative to positive states as the quantum number \(n_r + 2(\kappa + 1)\) increases. Thus it is interesting to investigate the condition for zero-point energy state. To do this, we set \(E_{n_r,x} = 0\) in equation (19) and obtain the critical condition

$$n_r = n_r + 2(\kappa + 1). \quad (30)$$

As an example, we set \(\mu = 1\), \(a_1 = a_2 = 1\) and solve equations (19) and (21) simultaneously to explicitly obtain the energy \(E_{n_r,x}\) and allowed values of the parameter \(a_{3}\) for \(n_r = 0, 1, 2\) as shown in Table 1. We note the degenerate states \(|n_r, \kappa + 1\rangle\) and \(|n_r + 2, \kappa\rangle\).

### 4. Third-root singular potential (N = 3)

We now consider the potential of the form

$$V_3(r) = -\frac{a_1}{r^2} + \frac{a_2}{r} - \frac{a_3}{r} + \frac{a_4}{r^2} + \frac{a_5}{r^3}, a_1, a_2, a_3 > 0 \quad (31)$$

thus, taking the transformation

$$F_{n,x}(r) = x^{3(n+1)}e^{b_0r^2+b_2r^4+b_1r^6}u(x), \quad x = r^\frac{1}{7}, \quad (32)$$

and using equation (13), we have

$$\begin{align*}
xu''(x) + 2[3b_3x^3 + 2b_2x^2 + b_1x + 3\kappa + 2]u'(x) & + \left[ 9(E_{n,x}^2 - \mu^2 + b_3^2) \right] x^5 \\
+ [12b_3b_1 + 18(\mu + E_{n,x})a_1]x^2 & + \left[ 4b_3^2 + 6b_1^2b_3 - 18(\mu + E_{n,x})a_2 \right] x^3 \\
+ [18b_3(\kappa + 1) + 4b_1b_2 + 18(\mu + E_{n,x})a_3]x^2 & + [b_3^2 + 2b_2(6\kappa + 5) - 18(\mu + E_{n,x})a_4]x + 2b_1(3\kappa + 2) \\
& - 18(\mu + E_{n,x})a_5 \right] x = 0. \quad (33)
\end{align*}$$
In this case, we seek the polynomial solution of the form
\[ x u''(x) + 2[3b_3 x^3 + 2b_2 x^2 + b_1 x + 3\kappa + 2] u'(x) + \left\{ (18b_3(\kappa + 1) + 4b_1 b_2 - 18(\mu + E_{n,k}) a_3 \right\} x + 2b_1 (3\kappa + 2) - 18(\mu + E_{n,k}) a_3 \left\{ \mu u(x) = 0. \right\}

Equation (33) is quasi-exactly solvable provided coefficients of the terms \( x^5, x^4, x^3 \) are zero. This yields the values of the parameters
\[ b_3 = -\left( \mu^2 - E_{n,k}^2 \right)^\frac{1}{2}, \quad b_2 = \frac{3(\mu + E_{n,k}) a_1}{(\mu^2 - E_{n,k}^2)^2}, \quad b_1 = -\frac{3(\mu + E_{n,k}) a_2}{(\mu^2 - E_{n,k}^2)^2} + \frac{3(\mu + E_{n,k})^2 a_3^2}{(\mu^2 - E_{n,k}^2)^2}, \]

and reduces equation (33) to
\[ x u''(x) + 2[3b_3 x^3 + 2b_2 x^2 + b_1 x + 3\kappa + 2] u'(x) + \left\{ (18b_3(\kappa + 1) + 4b_1 b_2 - 18(\mu + E_{n,k}) a_3 \right\} x + 2b_1 (3\kappa + 2) - 18(\mu + E_{n,k}) a_3 \left\{ \mu u(x) = 0. \right\}

In this case, we seek the polynomial solution of the form
\[ u(x) = \prod_{m=1}^{n_r} (x - x_m), \quad u(x) \equiv 1 \quad \text{for} \quad n_r = 0, \]

where \( \{x_m\} \) are the roots of the polynomials to be determined, followed by the Bethe ansatz procedure used in section 2, and we obtain the energy equation
\[ [n_r + 3(\kappa + 1)](\mu^2 - E_{n,k}^2)^\frac{1}{2} = \frac{3a_1^2 (\mu + E_{n,k})^3}{2(\mu^2 - E_{n,k}^2)^2} + \frac{3a_1 a_2 (\mu + E_{n,k})^2}{\mu^2 - E_{n,k}^2} - 3(\mu + E_{n,k}) a_3 = 0, \]
and the corresponding spinor wavefunction

\[ F_{n,κ} (r) = x^{3κ+1} \left[ \prod_{m=1}^{n} (x - x_m) \right] \exp \left\{ - x^3 (\mu^2 - E_{n,κ}^2)^{1/2} + \frac{3a_1 x^2 (\mu + E_{n,κ})}{2(\mu^2 - E_{n,κ}^2)^{3/2}} \right\} \]

\[ = \frac{3(μ + E_{n,κ})a_2}{(μ^2 - E_{n,κ}^2)^{3/2}} \cdot \frac{3(μ + E_{n,κ})^2a_1^2}{2(μ^2 - E_{n,κ}^2)^{5/2}} x \quad \text{for } x = r^\dagger \]

\[ G_{n,κ} (r) = x^{3κ} \left[ \prod_{m=1}^{n} (x - x_m) \right] \exp \left\{ - x^3 (μ^2 - E_{n,κ}^2)^{1/2} + \frac{3a_1 x^2 (μ + E_{n,κ})}{2(μ^2 - E_{n,κ}^2)^{3/2}} \right\} \]

\[ = \left[ \frac{3(μ + E_{n,κ})a_2}{(μ^2 - E_{n,κ}^2)^{3/2}} \right] x \quad \text{for } x = r^\dagger \]

\[ \times \frac{1}{μ + E_{n,κ}} \left( - x^3 (μ^2 - E_{n,κ}^2)^{1/2} + \frac{a_1 x^2 (μ + E_{n,κ})}{(μ^2 - E_{n,κ}^2)^{3/2}} \right) \]

\[ \times \frac{(μ + E_{n,κ})^2a_1^2}{2(μ^2 - E_{n,κ}^2)^{5/2}} x \quad + \frac{1}{3} \sum_{m=1}^{n} \frac{x - x_m + 2κ + 1}{x} \quad \text{for } x = r^\dagger \]

provided \( a_4 \) and \( a_5 \) take the values

\[ a_4 = \frac{1}{(μ + E_{n,κ})} \left[ \frac{b_2^2}{18} + \frac{b_2}{9} (2n_r + 6κ + 5) + \frac{b_3}{3} \sum_{m=1}^{n} x_m \right], \]

\[ a_5 = \frac{1}{(μ + E_{n,κ})} \left[ \frac{b_1}{9} (n_r + 3κ + 2) + \frac{b_1}{3} \sum_{m=1}^{n} x_m^2 - \frac{2b_2}{9} \sum_{m=1}^{n} x_m \right], \]

\[ Figure 2. \text{ The wavefunction } F_{n,κ} (r) \text{ for the ground and first excited states of the square-root singular potential } (N = 2) \text{ with parameters } a_1 = a_2 = 1 \text{ and } μ = 1 \text{ and } κ = 1 - 3. \]
as the roots \( a_3 \) and solutions \( x \) take the values \( \mu + E_{0,a}^2 \) and \( \mu + E_{0,a}^2 \), provided, \( a_4 \) and \( a_5 \) take the values

\[
a_4 = \frac{1}{\mu + E_{0,a}^2} \left[ \frac{b_1^2}{18} + \frac{b_2}{9} (6\kappa + 5) + \frac{b_3}{3} \right],
\]

\[
a_5 = -\frac{1}{\mu + E_{0,a}^2} \left[ \frac{b_1}{9} (3\kappa + 2) \right],
\]

respectively, with \( b_1 \) and \( b_2 \) given by equation (34). In a similar fashion, the first excited state corresponds to \( n^* = 1 \), with solutions

\[
(3\kappa + 4)(\mu^2 - E_{1,\kappa}^2)^{\frac{3}{2}} - \frac{3a_1^3 (\mu + E_{1,\kappa})^3}{2(\mu^2 - E_{1,\kappa}^2)^2} + \frac{3a_1 a_2 (\mu + E_{1,\kappa})^2}{\mu^2 - E_{1,\kappa}^2} - 3(\mu + E_{1,\kappa})a_3 = 0,
\]

As examples of the general solution, the ground state solutions, which correspond to \( n^* = 0 \), are given by

\[
F_{0,a}(r) = x^{3(n^*+1)} \exp \left\{ -x^3 (\mu^2 - E_{0,a}^2)^{\frac{3}{2}} + \frac{3a_1 x^2 (\mu + E_{0,a})}{2(\mu^2 - E_{0,a}^2)^2} \right\},
\]

\[
G_{0,a}(r) = x^3 \exp \left\{ -x^3 (\mu^2 - E_{0,a}^2)^{\frac{3}{2}} + \frac{3a_1 x^2 (\mu + E_{0,a})}{2(\mu^2 - E_{0,a}^2)^2} \right\},
\]

\[
F_{1,a}(r) = x^{3(n^*+1)}(x - x_1) \exp \left\{ -x^3 (\mu^2 - E_{1,\kappa}^2)^{\frac{3}{2}} + \frac{3a_1 x^2 (\mu + E_{1,\kappa})}{2(\mu^2 - E_{1,\kappa}^2)^2} \right\},
\]

\[
G_{1,a}(r) = x^3 (x - x_1) \exp \left\{ -x^3 (\mu^2 - E_{1,\kappa}^2)^{\frac{3}{2}} + \frac{3a_1 x^2 (\mu + E_{1,\kappa})}{2(\mu^2 - E_{1,\kappa}^2)^2} \right\},
\]

respectively, where \( b_1, b_2 \) and \( b_3 \) are given in equation (34), with the roots \( \{x_m\} \) satisfying the algebraic equations

\[
\sum_{x \neq x_m} \frac{3}{x_m - x} + 3b_3 x_m^2 + 2b_2 + b_1 + \frac{3}{x_m} = 0, \quad m = 1, \ldots, n_r.
\]

As examples of the general solution, the ground state solutions, which correspond to \( n^* = 0 \), are given by

\[
3(\kappa + 1)(\mu^2 - E_{0,a}^2)^{\frac{3}{2}} - \frac{3a_1^3 (\mu + E_{0,a})^3}{2(\mu^2 - E_{0,a}^2)^2} + \frac{3a_1 a_2 (\mu + E_{0,a})^2}{\mu^2 - E_{0,a}^2} - 3(\mu + E_{0,a})a_3 = 0,
\]
The wavefunction $F_{n\kappa}(r)$ for the ground and first excited states of the third-root singular potential ($N = 3$) with parameters $a_1 = a_2 = a_3 = 1$ and $\mu = 1$ and $\kappa = 1 - 3$.

\[
\begin{align*}
F_{0,1}(r) & = \frac{1}{\mu + E_{1,1}} \left[ \frac{3(\mu + E_{1,1})a_2}{ \left( \mu^2 - E_{1,1}^2 \right)^{\frac{3}{2}}} - \frac{3(\mu + E_{1,1})^2 a_2^2}{2 \left( \mu^2 - E_{1,1}^2 \right)^{\frac{5}{2}}} \right] \times \frac{1}{\mu + E_{1,1}} \left( -x^3(\mu^2 - E_{1,1}^2)^{\frac{3}{2}} + \frac{a_1 x^2(\mu + E_{1,1})}{(\mu^2 - E_{1,1}^2)^{\frac{5}{2}}} \right) - \frac{(\mu + E_{1,1})a_2}{\left( \mu^2 - E_{1,1}^2 \right)^{\frac{3}{2}}} \times \frac{1}{\mu + E_{1,1}} \left[ \frac{\mu + E_{1,1}}{2 \left( \mu^2 - E_{1,1}^2 \right)^{\frac{5}{2}}} \right] x + \frac{x/3}{x - x_1} + 2\kappa + 1 \right),
\end{align*}
\]

provided $a_4$ and $a_5$ take the values

\[
\begin{align*}
a_4 & = \frac{1}{\mu + E_{1,1}} \left[ \frac{b_2^2}{18} + \frac{b_2}{9}(6\kappa + 7) + \frac{b_3 x_1}{3} \right], \\
a_5 & = \frac{1}{\mu + E_{1,1}} \left[ \frac{b_1}{3}(\kappa + 1) + \frac{b_3 x_1^2}{3} - \frac{2b_3 x_1}{9} \right],
\end{align*}
\]

with $x_1$ satisfying the equation

\[
3b_3 x_1^3 + 2b_2 x_1 + b_1 + \frac{3\kappa + 2}{x_1} = 0.
\]

The fact that $a_4$ and $a_5$ are obtained in terms of other parameters is as a result of the quasi-exact solvability of the system. While these restrictions are necessary for the quantization of the spectrum, we note that they do not affect the generality of the model. In other words, these constraints simply indicate that $a_4$ and $a_5$ must have some specific values (depending on $b_i$, $\kappa$ and roots $x_i$) for the equation (33) to be exactly solvable. Moreover, the values of $a_1$, $a_2$, and $a_3$ are important for the bound state to exist.
Thus, a careful selection of $a_{1} = a_{2} = a_{3} = 2$ and $\mu = 1$ yields the numerical values for the energies for the generalized third-root singular model, as shown in table 2, with energy degeneracies within states $|n_{r}, \kappa + 1\rangle$ and $|n_{r} + 3, \kappa\rangle$. In a similar fashion, the critical condition for the zero-energy state ($E_{n_{r}, \kappa} = 0$) can be easily obtained from equation (37) as

$$[(n^{2} - 3a_{1}\mu - 3a_{1}a_{2})\mu = 3a_{1}^{2}, \quad n^{2} = n_{r} + 3(\kappa + 1). \quad (48)$$

5. Non-relativistic bound states and expectation values

Interestingly, within the limits $\kappa \rightarrow \ell, \mu \rightarrow E_{n_{r}, \kappa} \rightarrow 1/2$ and $\mu^{2} - E_{n_{r}, \kappa}^{2} \rightarrow -E_{n_{r}, \ell}^{\prime}$, equation (10) can be written as

$$H_{n_{r}}^{\prime} \phi(r) = E_{n_{r}, \ell}^{\prime} \phi(r) \quad (49)$$

where $H_{n_{r}}^{\prime}$ is the non-relativistic effective Hamiltonian given by $(2\mu = \hbar = 1)$

$$H_{n_{r}}^{\prime} = -\frac{d^{2}}{dr^{2}} + \frac{\ell(\ell + 1)}{r^{2}} + \sum_{p=1}^{2N-1} \frac{a_{p}}{r^{p}}, \quad (50)$$

and $\phi(r)$ and $E_{n_{r}, \ell}^{\prime}$ are the wavefunction and energy, respectively. In particular, for the $N = 2$ case, the non-relativistic energies are obtainable as

$$[n_{r} + 2\ell + 2]\sqrt{-E_{n_{r}, \ell}^{\prime}} - \frac{a_{1}^{2}}{4\sqrt{-E_{n_{r}, \ell}^{\prime}}} + a_{2} = 0 \Rightarrow E_{n_{r}, \ell}^{\prime} = -\frac{1}{4n^{2}}\left[\frac{a_{2}^{2}}{n_{r} + \ell + 1} + 2a_{2} + 2a_{2}\sqrt{a_{2}^{2} + a_{1}^{2}}\right]. \quad (51)$$

where we have defined $n = n_{r} + 2\ell + 2$. We note from equation (50), that if $N = 1$, we have the Coulomb interaction whose energy levels $E_{n}^{\prime} = -a_{2}^{2}/(n_{r} + \ell + 1)^{2}$ are obtainable from equation (51) when $a_{1} = a_{3} = 0$. Moreover, the corresponding wavefunctions are given as

$$\phi_{n_{r}, \ell}(r) \sim r^{2(\ell + 1)} \prod_{i=1}^{n_{r}} (x - x_{i}) \exp \left[-\frac{x^{2}}{2}\sqrt{-E_{n_{r}, \ell}^{\prime}} - \frac{a_{1}x}{\sqrt{-E_{n_{r}, \ell}^{\prime}}} \right], \quad x = r^{2} \quad (52)$$
subject to the constraint

$$a_3 - \frac{(\ell + \frac{3}{4})}{\sqrt{E_{n,\ell}}\ell} = -\frac{E_{n,\ell}}{\ell} \sum_{i=1}^{n_r} x_i + n_r \left( \ell + \frac{3}{4} \right),$$

where the roots \( \{x_i\} \) satisfy the Bethe ansatz equations

$$\sum_{j \neq i}^{n_r} \frac{1}{x_i - x_j} = 2x_i \sqrt{-E_{n,\ell} \ell} - \frac{a_1}{\sqrt{-E_{n,\ell} \ell}} - \frac{(4\ell + 3)}{2x_i}, \quad i = 1, \ldots, n_r. \quad (53)$$

We note that equation (51) is in agreement with equation (9) of [23]; however, no closed-form expression for the wavefunction and constraint were obtained in the previous work.

Furthermore, since the Hamiltonian \( H'_N \) for the system is a function of some parameters \( q \) (say) then by the HFT \([31, 32, 34, 35]\) we have

$$\frac{\partial E_{n,\ell}'}{\partial q} = \langle \phi(q) | \frac{\partial H'_N(q)}{\partial q} | \phi(q) \rangle. \quad (55)$$

The effective Hamiltonian for case \( N = 2 \) is given by

$$H'_2 = -\frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} - a_1 \frac{a_2}{r} + a_3 \frac{a_3}{r^2},$$

thus with the choice of \( q = a_1 \), we have

$$\langle \phi(a_1) | \frac{\partial H'_2(a_1)}{\partial a_1} | \phi(a_1) \rangle = -\langle r^{-1} \rangle$$

and

$$\frac{\partial E_{n,\ell}'}{\partial a_1} = -\frac{a_1}{2n} \left[ 1 + \frac{a_2}{\sqrt{a_2^2 + a_1^2 n}} \right]. \quad (58)$$

Thus by HFT,

$$\langle r^{-1} \rangle = \frac{a_1}{2n} \left[ 1 + \frac{a_2}{\sqrt{a_2^2 + a_1^2 n}} \right]. \quad (59)$$

Similarly, for \( q = a_2 \), we obtain

$$\langle r^{-1} \rangle = -\frac{1}{2n^2} \left[ 2a_2 + \frac{a_2^2}{\sqrt{a_2^2 + a_1^2 n}} + \sqrt{a_2^2 + a_1^2 n} \right], \quad (60)$$

and for \( q = \ell \),

$$\langle r^{-2} \rangle = -\frac{1}{n(2\ell + 1)} \left[ 2E_{n,\ell} + \frac{a_1^2}{4n} \left( 1 + \frac{a_2}{\sqrt{a_2^2 + a_1^2 n}} \right) \right]. \quad (61)$$

Finally, we note that a similar procedure can be followed to obtain the expectation values for the generalized third-root singular potential.
6. Concluding remarks

In this paper, we have presented the exact (Bethe ansatz) solutions to the Dirac equation for a class of singular fractional power potentials. For the first two members of this class of potentials, we showed that the Dirac equation is reducible to a quasi-exactly solvable differential equation which has exact solutions, provided the parameters satisfy certain constraints. We obtained closed-form expressions for the energies and eigenfunctions in terms of the roots of Bethe ansatz equations. Similar degeneracy between the states $|n_r, \kappa + 1\rangle$ and $|n_r + N, \kappa\rangle$ was observed for any member potential $V_N(r)$.

By taking appropriate limits, we discussed some non-relativistic properties of the models, and using the Hellmann–Feynmann theorem, explicit expressions for some expectation values were obtained. We hope that the present findings will lead to new applications for this class of fractional singular potentials.

Acknowledgments

This work was supported by Australian IPRS, the University of Queensland Centennial Scholarship and the University of Queensland GSIT Award. YZZ was partly supported by the ARC through the discovery project DP110103434. DA is indebted to Father J and Agboola B for their support during the preparation of the manuscript and also acknowledges the warm hospitality of Professor R L Hall (Concordia University, Montreal) and Professor N Saad (University of Prince Edward Island, PEI) during his recent research visit.

References

[1] Case K M 1950 Phys. Rev. 80 797
[2] Vogt E and Wannier G 1954 Phys. Rev. 95 1190
[3] Predazzi E and Regge T 1962 Nuovo Cimento 24 518
[4] Brander O 1981 J. Math. Phys. 22 1229
[5] Esposito G 1998 J. Phys. A: Math. Gen. 31 9493
[6] Pais A and Wu T T 1964 J. Math. Phys. 5 799
   Pais A and Wu T T 1964 Phys. Rev. 134 B1303
[7] Dolinskiy T 1990 Nucl. Phys. A 338 495
[8] Barut A O 1980 J. Math. Phys. 21 568
[9] Song X 1991 J. Phys. G: Nucl. Part. Phys. 17 49
[10] Ikhdair S and Sever R 2007 J. Math. Chem. 42 461
[11] Erkoc S and Sever R 1984 Phys. Rev. D 30 2117
[12] Dutt R and Varshni Y P 1987 J. Phys. B: At. Mol. Phys. 20 2437
[13] Silva F R and Filho E D 2010 Mod. Phys. Lett. A 25 641
[14] Stillinger F H 1979 J. Math. Phys. 20 1891
[15] Bose S K and Schulze-Halberg A 2000 Mod. Phys. Lett. A 15 1583
[16] Frank W M, Land D J and Spector R M 1971 Rev. Mod. Phys. 43 36
[17] Miller H G 1994 J. Math. Phys. 35 2229
[18] Aguilera-Navarro V C and Coelho A L 1994 Phys. Rev. A 49 1477
[19] Das A and Pernice S A 1999 Nucl. Phys. B 561 357
[20] de Castro A S and Alberto P 2012 Phys. Rev. A 86 032122
[21] Schulze-Halberg A 2002 Phys. Scr. 65 373
[22] Panahi H and Baradaran M 2012 Mod. Phys. Lett. A 27 1250176
[23] Brihaye Y, Devaux N and Kosinski P 1995 Int. J. Mod. Phys. A 10 4633
[24] Zhang Y-Z 2012 J. Phys. A: Math. Theor. 45 065206
[25] Agboola D and Zhang Y-Z 2013 Ann. Phys., NY 330 246
[26] Agboola D and Zhang Y-Z 2012 J. Math. Phys. 53 042101
[27] Greiner W 1981 Relativistic Quantum Mechanics (Berlin: Springer)
[28] Long C and Robson D 1983 Phys. Rev. D 27 644
[29] Su R K and Zhang Y 1984 J. Phys. A: Math. Gen. 17 851
[30] Hall R L 1987 J. Math. Phys. 28 457
[31] Agboola D 2010 Chin. Phys. Lett. 27 040301
[32] Agboola D 2009 Phys. Scr. 80 065304
[33] Castro L B 2012 Phys. Rev. C 86 052201
[34] Hellmann G 1937 Einführung in die Quantenchemie (Vienna: Denticke)
[35] Feynman R P 1939 Phys. Rev. 56 340