On the Need for Large Quantum Depth

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Abstract

Near-term quantum computers are likely to have small depths due to short coherence time and noisy gates, and thus a potential way to leverage these quantum devices is using a hybrid scheme that interleaves them with classical computers. For example, the quantum Fourier transform can be implemented by a hybrid of logarithmic-depth quantum circuits and a classical polynomial-time algorithm. Along the line, it seems possible that a general quantum computer may only be “polynomially” faster than a hybrid quantum-classical computer. Jozsa raised the question of whether $\text{BQP} = \text{BPP}^{\text{BQNC}}$ and conjectured that they are equal, where $\text{BQNC}$ means poly log-depth quantum circuits. Nevertheless, Aaronson conjectured an oracle separation for these two classes, and gave a candidate. In this work, we prove Aaronson’s conjecture for a different but related oracle problem. Our result also proves that Jozsa’s conjecture fails relative to an oracle.

1 Introduction

Circuit depth may become an essential consideration when designing algorithms on near-term quantum computers. Quantum computers with more than 50 qubits have been realized recently [IBM17, Kel18]; both the quantity and quality of the qubits are continually improving. These small quantum computers may be beyond the capabilities of the most powerful existing supercomputers in terms of brute-force simulation [Pre18]. However, with noisy gates and limited coherence time, these quantum computers are only able to operate for a short period. Hence the effective circuit depths of these quantum computers are limited, and this seems to be an essential bottleneck for quantum technologies.

On the other hand, it is worth noting that the capability of constant-depth quantum computers is limited. Specifically, consider the standard setting where the composing gate set only includes one- and two-qubit gates. It is obvious that a small-depth quantum circuit cannot solve any classically intractable decision problem or even some classically easy problems, e.g., computing a parity
function. This is because that each output qubit depends on only $O(1)$ input qubits. Although involving unbounded fan-out gates allows a quantum circuit to conduct many operations in small depth, such as parity, mod[q], AND, OR, majority, threshold[t], exact[t], counting, arithmetic operations, phase estimations, and the quantum Fourier transform \cite{HS05}, it seems that unbounded fan-out gates are hard to implement in practice and thus is rarely considered for near-term quantum device.\footnote{Our oracle separation results hold even if the unbounded fan-out gates are allowed.}

A natural idea to exploit the power of small depth quantum computers is to use hybrid quantum-classical computers. In this scheme, we allow a classical algorithm, which itself can have an arbitrarily polynomial depth, to access some small-depth quantum circuits during the computation.

Many of the known quantum algorithms require only small-depth in the quantum part. Most notably, Cleve and Watrous showed that the quantum Fourier transform can be parallelized to have only logarithmic depth \cite{CW00}, which implies that quantum algorithms for hidden subgroup problems, such as Shor’s factoring algorithm, can be implemented on a hybrid quantum-classical computer with quantum circuit depth logarithmic in the input size. Moreover, we also observe that several sampling problems \cite{BJS10, BFK18}, which are proven to be hard for classical simulation, are efficiently solvable when access to a constant-depth quantum circuit is allowed. In the presence of near-term quantum device, Quantum Approximate Optimization Algorithm (QAOA) \cite{FGG14} and Variational Quantum Eigensolver (VQE) \cite{MRBAG16} were proposed for solving some practical problems in specific areas, including many-body physics and chemistry. Farhi and Harrow showed that QAOA can sample distributions that are classically intractable unless PH collapses \cite{FH16}. Therefore, “quantum poly log depth is as powerful as quantum poly depth in the presence of classical computation” seems to be a live possibility!

Regarding to this possibility, Jozsa \cite{Joz05} conjectured that

> “Any polynomial time quantum algorithm can be implemented with only $O(\log n)$ quantum depth interspersed with polynomial-time classical computations.”

If we only consider decision problems, Jozsa’s conjecture can be rephrased to $\text{BQP} = \text{BPP}^{\text{BQNC}}$. However, the community does not have consensus; Aaronson \cite{Aar11, Aar19} conjectured that

> “There exists an oracle $\mathcal{O}$ such that $\text{BQP}^\mathcal{O} \neq (\text{BPP}^{\text{BQNC}})^\mathcal{O}$. Furthermore, this $\mathcal{O}$ can be obtained by composing a problem with exponential quantum speedup, like Simon’s problem, with another problem requiring large sequential depth.”

Note that an efficient classical algorithm cannot solve classically hard problem, and a low-depth quantum circuit can only do limited-depth sequential search. Our idea is to construct a new problem that combines both harness problems for these two computational models. If such an oracle exists, then proving or disproving $\text{BQP} = \text{BPP}^{\text{BQNC}}$ must require non-relativizing techniques.

In this work, we prove Aaronson’s conjecture. Namely, we show that there exists an oracle $\mathcal{O}$, relative to which, there exists a language $\mathcal{L}^{\mathcal{O}}$ such that $\mathcal{L}^{\mathcal{O}}$ is in $\text{BQP}^{\mathcal{O}}$ but $\mathcal{L}^{\mathcal{O}}$ is not in $(\text{BPP}^{\text{BQNC}})^{\mathcal{O}}$.\footnote{Independent to our work, we learned recently that Coudron and Menda got a similar result showing that Jozsa’s conjecture failed relative to a different oracle \cite{CM19}. We have coordinated with them to post our work on Arxiv on the same day.}

Another scheme related to hybrid quantum-classical computation is small-depth measurement-based quantum computation (MBQC). In MBQC, we prepare a set of qubits on a grid in advance. For each layer, we apply disjoint unitaries to the qubits, measure a subset of the qubits, and then decide the subsequent unitaries according to the measurement outcomes. Note that the last step...
may require additional classical computational resources. MBQC has been shown to be equivalent to the quantum circuit model [RB00]. Furthermore, Broadbent and Kashefi [BK09] proved that MBQC scheme computes the parity function in constant depth, whereas there is no constant-depth quantum circuit for computing the same function. Due to the limited coherence time and the noise in the grid, the depth of MBQC may still be limited in the near future. In this work, we also prove that relative to the same oracle, $BQP^O \neq (BQNC^{BPP})^O$, where $BQNC^{BPP}$ can be viewed as the set of languages decided by $d$-depth MBQC.

1.1 Main results

We first define two hybrid models, which interleave $d$-depth quantum circuits and classical computers. The first scheme is the $d$-depth quantum-classical scheme ($d$-QC scheme), which is a generalized model for small-depth MBQC, and the second scheme is the $d$-depth classical-quantum scheme ($d$-CQ scheme), which characterizes the hybrid quantum-classical computation. Briefly, the $d$-QC scheme is based on a $d$-depth quantum circuit and can access some classical computational resources after each depth. The $d$-CQ scheme is based on a classical computer with access to some $d$-depth quantum circuits. We define $BQNC_d^{BPP}$ to be the set of languages decided by $d$-QC schemes, and $BPP^{BQNC_d}$ to be the set of languages decided by $d$-CQ schemes. (Formal definitions will be given in Section 3.)

The oracle problem is a variant of Simon’s problem. Given a function $f : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^n$ with the promise that there exists $s \in \mathbb{Z}_2^n$ such that $f(x) = f(x \oplus s)$ for $x \in \mathbb{Z}_2^n$, Simon’s problem is to find $s$ [Sim94]. Simon’s problem is easy for quantum-polynomial-time (QPT) algorithms but hard for all probabilistic-polynomial-time (PPT) algorithms; however, Simon’s problem can be solved by a constant-depth quantum circuit with classical postprocessing. To devise a harder problem, we provide access to Simon’s function only as a composition of random one-to-one functions $f_0, \ldots, f_{d-1}$ and a two-to-one function $f_d$ such that $f = f_d \circ \cdots \circ f_0$ as shown in Fig. 1. If a quantum circuit must query $f_0, \ldots, f_d$ in sequence to evaluate $f$, then a $d$-depth quantum circuit cannot solve this variant of Simon’s problem via the standard approach, since it can only make at most $d$ sequential queries but we have $d + 1$ random functions. Nevertheless, a cleverer approach that does not query the functions in sequence might exist.

![Figure 1: Composing $d$ random one-to-one function $f_0, \ldots, f_{d-1}$ and a two-to-one function $f_d$ such that $f = f_d \circ \cdots \circ f_0$.](image)

To rule out such an approach, we make it infeasible for the algorithm to access the domain of $f_i$ before the $(i + 1)$-th parallel queries. Specifically, we enlarge the domains of the random one-to-one functions $f_0, \ldots, f_{d-1}, f_d$ such that the original domains of these functions are hidden in larger domains. Therefore, to evaluate $f_i$, one must find the original domain from the
larger domain, and the success probability could be negligible. For clarity, we denote the original domain of \( f_j \) as \( S_j \), the new larger domain as \( S_j^{(0)} \), and \(|S_j|/|S_j^{(0)}|\) is negligible for \( j \in [d] \). The oracle we now consider is as follows: Let \( f \) be a random Simon’s function. In defining the oracle, we first choose a sequence of random one-to-one functions \( f_0, \ldots, f_{d-1} \) defined on much larger domains \( S_0^{(0)}, \ldots, S_d^{(0)} \), and then sets \( f_d \) to be the function such that \( f_d \circ \cdots \circ f_0(x) = f(x) \) for \( x \in S_0 \). An algorithm which has access to this oracle can access \( f_0, \ldots, f_d \). We illustrate the oracle in Fig. 2.

It is worth noting that the algorithm can learn \( S_i \) after the \( i \)-th query with probability one, and thus is able to evaluate \( f \) after \( d+1 \) sequential queries. However, our goal is to make it impossible for the algorithm to evaluate \( f \) by using at most \( d \) sequential queries.

We call this oracle the **shuffling oracle** of \( f \). The **d-Shuffling Simon’s Problem** (**d-SSP**) is a decision problem defined as follows:

**Definition 1.1 (d-SSP (Informal)).** Let \( f \) be a random one-to-one function or a random Simon function. Given oracle access to the shuffling oracle of \( f \) (as in Fig. 2), the problem is to decide whether \( f \) is a Simon function or not.

Then, our main theorem is as follows.

**Theorem 1.2 (Informal).** \( d \)-SSP \( \in \text{BQP} \), but \( d \)-SSP \( \not\in \text{BPP}^{\text{BQNC}} \cup \text{BQNC}^{\text{BPP}} \) for \( d = \Theta(n) \).

1.2 Discussion and open problems

There are several open directions to study. One central question is: can we instantiate our oracle? For example, under some cryptographic primitives, \( \text{BQP} \neq \text{BPP}^{\text{BQNC}} \) in the real world. Additionally, many computational models seem to be intermediate between classical and general quantum computers, such as IQP \([BJS10]\) and unitary conjugate Clifford circuit (U-CCC) \([BFK18]\). Is it possible to show the existence of a problem that is in \( \text{BQP} \) but is intractable by interleaving a classical computer with these computational models? Finally, in this work, we actually show an oracle \( \mathcal{O} \) such that \( \text{BPP}^{\text{BQNC}_d \mathcal{O}} \neq \text{BPP}^{\text{BQNC}_{d+1} \mathcal{O}} \). Can we make the separation sharper? For example, showing that \( \text{BPP}^{\text{BQNC}_d} \neq \text{BPP}^{\text{BQNC}_{d+k}} \) for some \( k \) a constant less than \( d \)?
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2 Proof overview

In this section, we give an overview of the proofs. For ease of presentation, some notations and descriptions will be informal here. Formal definitions and proofs will be given in subsequent sections.

The main idea is to show that given access to the shuffling oracle, an easier task: distinguishing whether \( f \) is a Simon’s function or some dummy function is already hard. To be more specific, let \( \mathcal{F} \) be the shuffling oracle of \( f \). We consider some oracle \( \mathcal{G} := (g_0, \ldots, g_d) \) which agrees with \( \mathcal{F} \) on a set of elements out of \( S_0, \ldots, S_d \), and maps the rest (including \( S_0, \ldots, S_d \)) to \( \bot \). We call the set where these two oracles are different the hidden set. We can think of \( \mathcal{G} \) as a shadow of \( \mathcal{F} \), which has the function values in the hidden set be blocked but has the same function values as \( \mathcal{F} \) on non-blocked elements. Obviously, one given access to the shadow \( \mathcal{G} \) cannot do better than random guessing. Therefore, if we can show that distinguishing \( \mathcal{F} \) from \( \mathcal{G} \) is hard, then computing the hidden shift with \( \mathcal{F} \) is also hard.

To show the indistinguishability of \( \mathcal{F} \) and its shadow \( \mathcal{G} \), we generalize a technique proposed by Ambainis et al. \cite{AHU18} called the Oneway-to-Hiding (O2H) lemma to the setting of the shuffling oracle. Briefly, the O2H Lemma shows that given any two random functions which have same function values on most elements except for some hidden subset, we can bound the probability of distinguishing these two functions by the probability that the queries are in the hidden set, which we call the finding probability. This probability will be negligible if the subset is small enough. We generalize the O2H Lemma to a sequence of random functions. We show that there exists a sequence of shadows \( \mathcal{G}_1, \ldots, \mathcal{G}_d \) such that the \( d \)-depth quantum circuit which makes the \( i \)-th quantum parallel queries to \( \mathcal{G}_i \) is indistinguishable from making queries to \( \mathcal{F} \). Here, \( \mathcal{G}_1, \ldots, \mathcal{G}_d \) are built depending on the scheme we consider. In the following, we briefly describe how to construct the shadows for \( \text{QNC}_d \), \( d \)-QC, and \( d \)-CQ.

2.1 QNC\(_d\) circuit

Consider the first parallel queries: Let \( \mathcal{S}^{(0)} := (S_0^{(0)}, \ldots, S_d^{(0)}) \) be the sequence of the domains of \( \mathcal{F} := (f_0, \ldots, f_d) \), we choose \( S_1^{(1)} \) uniformly randomly from \( S_1^{(0)} \) with the promise that \( |S_1^{(1)}|/|S_1^{(0)}| = 1/2^n \) and \( S_1 \subset S_1^{(1)} \), and then choose \( S_i^{(1)} := f_{i-1} \circ \cdots \circ f_1(S_1^{(1)}) \) for \( i \in [d] \). It is not hard to show that \( \mathcal{S}^{(1)} := (S_1^{(1)}, \ldots, S_d^{(1)}) \) is a uniformly random subset in \( \mathcal{S}^{(0)} \), i.e., for any element \( x \in \mathcal{S}^{(0)} \), \( \Pr[ x \in \mathcal{S}^{(1)} ] = 1/2^n \). Let \( \mathcal{S}^{(1)} \) be the hidden set where \( \mathcal{F} \) and the shadow \( \mathcal{G}_1 \) are different. Then, \( \mathcal{F} \) and \( \mathcal{G}_1 \) are indistinguishable since \( \mathcal{S}^{(1)} \) is a small subset in \( \mathcal{S}^{(0)} \). It is worth noting that we have ignored the domain \( S_0^{(0)} \) of \( f_0 \), which is because the quantum circuit has already known elements \( S_{i-1} \) before making the \( i \)-th parallel queries. Furthermore, since \( \mathcal{G}_1 \) and \( \mathcal{F} \) are consistent with elements out of \( \mathcal{S}^{(1)} \), we assume the algorithm learns \( \mathcal{F} \) on elements out of \( \mathcal{S}^{(1)} \) without loss of generality after the first parallel queries. We then suppose, after the \( i-1 \)-th parallel queries, the
algorithm learns $\mathcal{F}$ on $\overline{S}^{(i-2)} \setminus S^{(i-1)}$. We inductively choose $S^{(i)}_i$ to be a random subset in $S^{(i-1)}_i$ with the same promise, set $S^{(i)}_j := f_{j-1} \circ \cdots \circ f_1(S^{(i)}_i)$ for $j > i$ and $\overline{S}^{(i)} := (S^{(i)}_1, \ldots, S^{(i)}_d)$, and let $G_i$ be the shadow which maps elements in $S^{(i)}$ to $\bot$. Finally, we can show that the outputs of $U_{d+1} \circ \mathcal{F} \circ U_d \circ \cdots \circ \mathcal{F} \circ U_1$ and $U_{d+1} \circ G_d \circ U_d \circ \cdots \circ G_1 \circ U_1$ are indistinguishable via the hybrid argument, and $G_1, \ldots, G_d$ contain no information about $f_d$ on $S_d$. This fact implies that $U_{d+1} \circ G_d \circ U_d \circ \cdots \circ G_1 \circ U_1$ can do no better than guessing and so is $U_{d+1} \circ \mathcal{F} \circ U_d \circ \cdots \circ \mathcal{F} \circ U_1$.

2.2 $d$-QC scheme

We can use the above approach to handle $d$-QC scheme by further removing coordinates that has been queried by the PPT algorithm from the domain. The observation is that a $d$-QC scheme has a PPT algorithm before applying each single-depth unitary as in Fig. 5, which PPT algorithm can learn $x, f_1(x), f_2 \circ f_1(x), \ldots, f(x)$ for any $x$. If we choose $\overline{S}^{(0)}, \ldots, \overline{S}^{(d)}$ as above, the probability that the circuit finds these hidden sets is actually 1. However, except for elements queried by $\mathcal{A}_c$, the rest elements are still uniformly random, and $\mathcal{A}_c$ can only make poly($n$) queries. Regarding this observation, we choose $\overline{S}^{(0)}, \ldots, \overline{S}^{(d)}$ as below.

Consider the first quantum parallel queries. Let $T^{(1)}_j \subset S^{(0)}$ for $j \in [f]$ be the set of elements queried by $\mathcal{A}_c$ before applying $U_1$ in Fig. 5. We assume $f_j(T^{(1)}_j) = T^{(1)}_{j+1}$ for all $j$ without loss of generality. Now, we set $S^{(0)} := \overline{S}^{(0)} \setminus T^{(1)}$ and choose $S^{(1)}_1$ uniformly randomly with the promise that $|S^{(1)}_1|/|S^{(1)}_0| = 1/2^n$ and $(S^{(1)}_1 \setminus T^{(1)}) \subset S^{(1)}_1$. We then choose $S^{(1)}_2, \ldots, S^{(1)}_d$ according to $f_1, \ldots, f_{d-1}$, and it is not hard to show that $\overline{S}^{(1)} := (\overline{S}^{(1)}_1, \ldots, \overline{S}^{(1)}_d)$ is uniformly random in $\overline{S}^{(0)}$. Following this procedure, we construct $\overline{S}^{(i)}$ in sequence for all $i \in [d]$. Finally, via the same hybrid argument for analyzing $d$-depth quantum circuit, we show that no $d$-QC scheme can solve the $d$-SSP with high probability.

2.3 $d$-CQ scheme

Proving that $d$-SSP is not in $\text{BPP}^{Q\text{C}_d}$ seems to be harder. The main obstacle is that the measurement outcome of the quantum circuit can fail the uniformity of the shuffling oracle when considering the following procedures. More precisely, the measurement outcome will be a short advice that is correlated to the whole shuffling oracle. Conditioned on this short advice, the shuffling can be far from uniform distribution, and thus our previous analysis fails.

To deal with this difficulty, we show that the shuffling oracle is still “almost” uniform when conditioning a short classical advice string. To show this, we use the presampling argument proposed by Coretti et al. [CDGS18]. Informally, the argument states that a random function conditioned on a short classical advice string is classically computationally indistinguishable from a convex combination of random functions which are fixed on a few elements. We generalize this argument to the setting of the shuffling oracle, i.e., we show that conditioned on a short classical advice string, a uniform shuffling oracle is indistinguishable from a convex combination of shuffling oracles with a few elements fixed and the rest are almost-uniform. For the ease of illustrating the idea, we just assume the uniform shuffling oracle is close to a shuffling oracle that has a few elements be fixed and has the rest elements be uniformly random.\footnote{In the formal proof, we still consider a convex combination of shuffling oracles which are almost uniform. Whether we can replace the almost-uniform oracle by uniform oracle in the quantum setting is an open question for the presampling argument.}
For the first time \( \mathcal{A}_c \) accesses the quantum circuit, the analysis is the same as \( d \)-QC scheme since there is no advice correlated to the whole shuffling oracle. We can replace \( \mathcal{F} \) by a sequence of shadows \( G_1, \ldots, G_d \). Consider the second time the \( d \)-depth quantum circuit is called, an advice string \( \bar{a} \) correlated to \( U_1 \rightarrow G_1 \rightarrow \cdots G_d \rightarrow U_{d+1} \) is given. This advice is uncorrelated to \( f \) since \( G_1, \ldots, G_d \) has no information about \( f \). Therefore, by the presampling argument, we can replace \( \mathcal{F} \) conditioned on \( \bar{a} \) by a shuffling oracle \( G'_1 \) that has a few elements be fixed and has the rest elements be uniform. Let \( T^{(1)}_1 \) be elements fixed in \( G'_1 \) and by \( \mathcal{A}_c \). We let \( S^{(0)}_1 := S^{(0)}_1 \setminus (T^{(1)}_1) \), choose \( S^{(1)}_1 \) uniformly randomly from \( S^{(0)}_1 \), and set \( S^{(1)}_2, \ldots, S^{(1)}_d \) according to \( G'_1 \) and \( S^{(1)}_1 \). Then, we show that the probability of finding \( S^{(1)} \) is still negligible. The rest of the analysis follows from mathematical induction on the quantum circuit depth.

3 Preliminaries

In this section, we first introduce the distance measures of quantum states. Then, we give formal definitions of the \( d \)-CQ and \( d \)-QC schemes.

3.1 State distance

**Definition 3.1.** Let \( \mathcal{H} \) be a Hilbert space. For any two pure states \( |\psi\rangle, |\phi\rangle \in \mathcal{H} \), we define

- (Fidelity) \( F(|\psi\rangle, |\phi\rangle) := |\langle \psi | \phi \rangle| \);
- (Two-norm distance) \( |||\psi\rangle - |\phi\rangle|| \).

Then, we define distance measures between mixed states.

**Definition 3.2.** Let \( \mathcal{H} \) be a Hilbert space. For any two mixed states \( \rho, \rho' \in \mathcal{H} \),

- (Fidelity) \( F(\rho, \rho') := \text{tr}(\sqrt{\sqrt{\rho} \rho' \sqrt{\rho}}) \);
- (Trace distance) \( TD(\rho, \rho') := \frac{1}{2} \text{tr} |\rho - \rho'| \);
- (Bures distance) \( B(\rho, \rho') := \sqrt{2 - 2F(\rho, \rho')} \).

The probability for a quantum procedure to distinguish two states can be bounded by the Bures distance between the two states.

**Claim 3.3.** For any two mixed states \( \rho \) and \( \rho' \), for any quantum algorithm \( \mathcal{A} \) and for any classical string \( s \),

\[
\left| \Pr[\mathcal{A}(\rho) = s] - \Pr[\mathcal{A}(\rho') = s] \right| \leq B(\rho, \rho').
\]

**Proof.** It is well-known that \( \left| \Pr[\mathcal{A}(\rho) = s] - \Pr[\mathcal{A}(\rho') = s] \right| \leq \frac{1}{2} \text{tr} |\rho - \rho'|. \) Then,

\[
TD(\rho, \rho') \leq \sqrt{1 - F(\rho, \rho')^2} = \sqrt{\frac{1 + F(\rho, \rho')}{2}} \sqrt{2 - 2F(\rho, \rho')} \leq B(\rho, \rho').
\]

The state distance and Claim 3.3 will be used shortly in the following sections.
3.2 Computational Model

We here define two schemes which interleave low-depth quantum circuits and classical computers. The first scheme is called \(d\)-depth quantum-classical scheme (\(d\)-QC scheme) and the second scheme is \(d\)-depth classical-quantum scheme (\(d\)-CQ scheme).

We say a set of gates forms a layer if all the gates in the set operate on disjointed qubits. Gates in the same layer can be parallelly applied. We define the number of layers in a circuit as the depth of the circuit. In the following, we define circuit families which has circuit depth \(d\) as in \cite{TD02, MN98}.

**Definition 3.4** \((d\text{-depth quantum circuit } QNC_d)\). A \(QNC_d\) quantum circuit family \(\{C_n : n > 0\}\) is defined as below:

- There exists a polynomial \(p\) such that for all \(n > 0\), \(C_n\) operates on \(n\) input qubits and \(p(n)\) ancilla qubits;
- for \(n > 0\), \(C_n\) has the initial state \(|0^{n+p(n)}\rangle\), consists of \(d\) layers of one- and two-qubit gates, and measures all qubits after the last layer.

We can illustrate a \(QNC_d\) as in Fig. 3, where \(U_i\) for \(i \in [d]\) is a unitary which can be implemented by one layer of one- and two-qubit gates, and the last computational unit is a qubit-wise measurement in the standard basis.

In some studies, \(QNC\) is also used to refer the set of languages decided by the quantum circuits. For clarity, we define the set of languages decided by \(QNC_d\) as \(BQNC_d\) as follows:

**Definition 3.5** \((BQNC_d)\). The set of languages \(L = \{L_n : n > 0\}\) for which there exists a circuit family \(\{C_n : n > 0\} \in QNC_d\) such that for \(n > 0\), for any \(x\) where \(|x| = n\),

- if \(x \in L_n\), then \(\Pr[C_n(x) = 1] \geq 2/3\);
- otherwise, \(\Pr[C_n(x) = 1] \leq 1/3\).

Then we define the quantum analogue of Nick’s class

**Definition 3.6** \((BQNC^k)\). The set of languages \(L = \{L_n : n > 0\}\) for which there exists a circuit family \(\{C_n : n > 0\} \in QNC_d\) for \(d = O(\log^kn)\) such that for \(n > 0\), for any \(x\) where \(|x| = n\),

- if \(x \in L_n\), then \(\Pr[C_n(x) = 1] \geq 2/3\);
- otherwise, \(\Pr[C_n(x) = 1] \leq 1/3\).
For simplicity, we define \( \text{BQNC} \) as the set of languages which can be decided by poly log-depth quantum circuit that is \( \text{BQNC} := \bigcap_k \text{BQNC}^k \).

A \( d \)-depth quantum circuit with oracle access to some classical function \( f \) is a sequence of operations

\[
U_1 \xrightarrow{d} f \xrightarrow{d} U_2 \xrightarrow{d} f \xrightarrow{d} \cdots U_d \xrightarrow{d} f \xrightarrow{d} U_{d+1},
\]

where \( U_i \)'s are unitaries as we have defined in \[3.4\] The transition \( \xrightarrow{d} \) implies that \( U_i \) can send and receive quantum messages from \( f \). We add an additional layer of unitary \( U_{d+1} \) to the computational model to process the information from the last call of \( f \).

In the quantum query model, we usually consider \( f \) as an unitary operator \( U_f \)

\[
U_f|x\rangle|0\rangle = |x\rangle|f(x)\rangle.
\]

We represent the process in Eq. 1 as follows: let \( |0,0\rangle_{QA}|0\rangle_W \) be the initial state, where registers \( Q \) and \( A \) consist of qubits the algorithm sends to or receives from the oracle and register \( W \) consists of the rest of the qubits the algorithm holds as working space. Let

\[
U_1|0,0\rangle_{QA}|0\rangle_W = \sum_{x,y} c(x,y)|x,y\rangle_{QA}|z(x,y)\rangle_W
\]

be the quantum message before applying the first \( U_f \). After applying \( U_f \),

\[
U_f \sum_{x,y} c(x,y)|x,y\rangle_{QA}|z(x,y)\rangle_W = \sum_{x,y} c(x,y)|x,y \oplus f(x)\rangle_{QA}|z(x,y)\rangle_W.
\]

We represent Eq. 1 as a sequence of unitaries \( U_{d+1}U_fU_d \cdots U_fU_1 \) in this way. For simplicity, we rewrite Eq. 1 as \( U_{d+1}fU_d \cdots fU_1 \) in the rest of the paper.

### 3.2.1 The \( d \)-QC scheme and \( \text{BQNC}^\text{BPP}_d \)

The first scheme we consider is the \( d \)-QC scheme, which is a generalized model for \( d \)-depth MBQC and can be represented as the following sequence

\[
\mathcal{A}_c \xrightarrow{c/q} (\Pi_{0/1} \otimes I) \circ U^{(1)} \xrightarrow{c/q} \mathcal{A}_c \xrightarrow{c/q} \cdots \xrightarrow{c/q} (\Pi_{0/1} \otimes I) \circ U^{(d)} \xrightarrow{c/q} \mathcal{A}_c,
\]

where \( \mathcal{A}_c \) is a randomized algorithm, \( U^{(i)} \) is a single-depth quantum circuit, and \( \Pi_{0/1} \) is a projective measurement in the standard basis on a subset of the qubits. The arrows \( \xrightarrow{c/q} \) indicate the classical and quantum messages transmitted. We illustrate the scheme as in Fig. 4.

In this model, after applying \( U^{(i)} \), one measures a subset of qubits, uses the measurement outcome to decide \( U^{(i+1)} \) by the classical polynomial algorithm \( \mathcal{A}_c \), and then apply \( U^{(i+1)} \) to the rest of the qubits. Let \( L^{(i)} \) be the procedure \( \mathcal{A}_c \rightarrow c(\Pi_{0/1} \otimes I) \circ U^{(i)} \). Then, we rewrite Eq. 2 as

\[
L^{(1)} \xrightarrow{c/q} \cdots \xrightarrow{c/q} L^{(d)} \xrightarrow{c/q} \mathcal{A}_c.
\]

Then, we define the languages which can be decided by \( d \)-QC scheme.

**Definition 3.7 (\( \text{BQNC}^\text{BPP}_d \)).** The set of languages \( \mathcal{L} = \{ \mathcal{L}_n : n > 0 \} \) for which there exists a family of \( d \)-QC schemes \( \{ \mathcal{A}_n : n > 0 \} \) such that for \( n > 0 \), for any \( x \) where \( |x| = n \),

- if \( x \in \mathcal{L}_n \), then \( \Pr[\mathcal{A}_n(x) = 1] \geq 2/3; \)
Figure 4: The $d$-depth quantum-classical ($d$-QC) scheme: The red line stands for the quantum wire, and the black one is for classical wire.

Figure 5: The $d$-QC scheme with access to an oracle $O$.

- otherwise, $\Pr[A_n(x) = 1] \leq 1/3$.

Let $A$ be a $d$-QC scheme with access to some oracle $O$. We represent $A^O$ as a sequence of operators:

$$(L^{(1)})^O \xrightarrow{c/q} \cdots \xrightarrow{c/q} (L^{(d)})^O \xrightarrow{c} A^O_c,$$

where $(L^{(i)})^O := A^O_c \xrightarrow{c} (\Pi_{0/1} \otimes I) \circ OU^{(i)}$. We illustrate the scheme as in Fig. 5.

**Definition 3.8 $((BQNC_d^{BPP})^O)$.** The set of languages $L^O := \{L^O_n : n > 0\}$ for which there exists a family of $d$-QC schemes $\{A^O_n : n > 0\}$ such that for $n > 0$, for any $x$ where $|x| = n$,

- if $x \in L^O_n$, then $\Pr[A^O_n(x) = 1] \geq 2/3$;
- otherwise, $\Pr[A^O_n(x) = 1] \leq 1/3$.

Similar to the definition of $BQNC$, we define $BQNC^{BPP}$ as a set of languages which can be decided by a family of $d$-QC schemes with $d = O(\text{poly log } n)$.

### 3.2.2 The $d$-CQ scheme and $BPP^{BQNC_d}$

Here we define the $d$-CQ scheme that is a classical algorithm which has access to a $\text{QNC}_d$ circuit during the computation. We represent the scheme as follows:

$$A_{c,1} \xrightarrow{c} \Pi_{0/1} \circ C \xrightarrow{c} \cdots \xrightarrow{c} A_{c,m} \xrightarrow{c} \Pi_{0/1} \circ C \xrightarrow{c} A_{c,m+1}.$$

where \( m = \text{poly}(n) \), \( A_{c,i} \) is a randomized algorithm, \( C \) is a \( d \)-depth quantum circuit, and \( \Pi_{0/1} \) is the standard-basis measurement. We illustrate the scheme as in Fig. 6.

In this scheme, the classical algorithm can perform queries to the \( \text{QNC}_d \) circuit, and then use the measurement outcomes from the circuit as part of the input to the following procedures. We let \( L^{(i)} := A_{c,i} \xrightarrow{\xi} \Pi_{0/1} \circ C \) and rewrite Eq. 5 as

\[
L^{(1)} \xrightarrow{\xi} L^{(2)} \xrightarrow{\xi} \cdots \xrightarrow{\xi} L^{(m)} \xrightarrow{\xi} A_{c,m+1}.
\]

**Definition 3.9 (\( \text{BPP}^{\text{BQNC}}_d \)).** The set of languages \( L = \{ L_n : n > 0 \} \) for which there exists a family of \( d \)-CQ schemes \( \{ A_n : n > 0 \} \) such that for \( n > 0 \), for any \( x \) where \( |x| = n \),

- if \( x \in L_n \), then \( \Pr[A_n(x) = 1] \geq 2/3 \);
- otherwise, \( \Pr[A_n(x) = 1] \leq 1/3 \).

Let \( A \) be a \( d \)-CQ scheme with access to some oracle \( O \). We represent \( A^O \) as

\[
(L^{(1)})^O \xrightarrow{\xi} (L^{(2)})^O \xrightarrow{\xi} \cdots \xrightarrow{\xi} (L^{(m)})^O \xrightarrow{\xi} (A_{c,m+1})^O
\]

where \( (L^{(i)})^O := (A_{c,i})^O \xrightarrow{\xi} \Pi_{0/1} \circ (U^{(d)}O U^{(d-1)} \cdots O U^{(1)}) \). We illustrate the scheme as in Fig. 7.

**Definition 3.10 ((\( \text{BPP}^{\text{BQNC}}_d \))^O)).** The set of languages \( L^O = \{ L_n^O : n > 0 \} \) for which there exists a family of \( d \)-CQ schemes \( \{ A_{c,i}^O : n > 0 \} \) such that for \( n > 0 \), for any \( x \) where \( |x| = n \),

- if \( x \in L_n^O \), then \( \Pr[A_n^O(x) = 1] \geq 2/3 \);
- otherwise, \( \Pr[A_n^O(x) = 1] \leq 1/3 \).

We define \( \text{BPP}^{\text{BQNC}}_d \) as a set of languages which can be decided by a family of \( d \)-CQ schemes with \( d = O(\text{poly log } n) \).

The main differences between \( d \)-CQ and \( d \)-QC schemes are that 1) the \( d \)-QC scheme can transmit quantum messages from one layer to the next, but the \( d \)-CQ scheme can only send classical messages, and 2) a \( d \)-QC scheme has at most \( d \) layers, but a \( d \)-CQ scheme may have \( m \times d \) layers. According to these observations, these two schemes seem to be incomparable.

### 4 The \( d \)-shuffling Simon’s problem (\( d \)-SSP)

In this section, we define the oracle and the corresponding oracle problem which separates the general quantum algorithms from \( d \)-QC and \( d \)-CQ schemes. We call the oracle the shuffled oracle and the corresponding problem as the \( d \)-shuffling Simon’s problem (\( d \)-SSP).
4.1 The shuffling oracle

For $X$ and $Y$ any two sets, we let $P(X, Y)$ be the set of one-to-one functions from $X$ to $Y$. For example, $P(Z^n_2, Z^n_2)$ is the set of all permutations over $Z^n_2$. Let $f : Z^n_2 \rightarrow Z^n_2$ be an arbitrary function and $d \in \mathbb{N}$. We define the $d$-depth shuffling of $f$ as below.

**Definition 4.1** ($(d, f)$-Shuffling). A $(d, f)$-shuffling of $f$ is defined by $F := (f_0, \ldots, f_{d-1})$, where $f_0, \ldots, f_{d-1} \in P(Z^{(d+2)n}_2, Z^{(d+2)n}_2)$ are chosen randomly, and then we choose $f_d$ be the function satisfying the following properties: Let $S_d := \{f_{d-1} \circ \cdots \circ f_0(x') : x' = 0, \ldots, 2^n - 1\}$.

- For $x \in S_d$, let $f_{d-1} \circ \cdots \circ f_0(x') = x$, and we choose the function $f_d : S_d \rightarrow [0, 2^n - 1]$ satisfying that $f_d \circ f_{d-1} \circ \cdots \circ f_0(x') = f(x')$.
- For $x / \in S_d$, we let $f_d(x) = \perp$.

We let $\text{SHUF}(d, f)$ be the set of all $(d, f)$-shuffling functions of $f$. For simplicity, we denote $f_d$ on the subdomain $S_d$ as $f_d^*$.

In this paper, we consider the case that the $(d, f)$-shuffling is given randomly. One of the most natural way is sampling a shuffling function $F$ uniformly at random from $\text{SHUF}(d, f)$. We describe the sampling procedure as below.

**Definition 4.2** $(D(f, d))$. Draw $f_0, \ldots, f_{d-1}$ uniformly at random from $P(Z^{(d+2)n}_2, Z^{(d+2)n}_2)$ and then choose $f_d^*$ such that $f_d^* \circ \cdots \circ f_0(x) = f(x)$ for $x \in Z^n_2$.

Fix $d \in \mathbb{N}$ and a function $f : Z^n_2 \rightarrow Z^n_2$, we can define a random oracle which is a $(d, f)$-shuffling chosen uniformly randomly from $\text{SHUF}(d, f)$.

**Definition 4.3** (Shuffling oracle $O^{f, d}_{\text{uni}}$). Let $f$ be an arbitrary function from $Z^n_2$ to $Z^n_2$. Let $d \in \mathbb{N}$. We define the shuffling oracle $O^{f, d}_{\text{uni}}$ as a $(d, f)$-shuffling $F$ chosen from $\text{SHUF}(d, f)$ according to $D(f, d)$.
Applying \( O \) algorithm; the state \(|R\rangle\) oracle, while the remaining local working qubits in the register \( Q \) are unchanged and hold by the algorithm; the state \(|i, X_i\rangle\) denotes the set of parallel queries to function \( f_i \).

Now, we describe the oracle access to the shuffling oracle \( O_{\text{unif}} \). Let \(|\phi\rangle\) be the input state to \( O_{\text{unif}} \), which we represent in the form

\[
|\phi\rangle := \sum_{X_0, \ldots, X_d} c(X_0, \ldots, X_d) \left( \bigotimes_{i=0}^{d} |i, X_i\rangle \right) \otimes |0\rangle_{R_Q} \otimes |w(X_0, \ldots, X_d)\rangle_{R_W},
\]

(7)

where \(|w(X_0, \ldots, X_d)\rangle\)'s are some arbitrary states and \( X_i \) is a set of elements in the domain of \( f_i \). The queries in the register \( R_Q \) and the ancillary qubits in the register \( R_N \) will be processed by the oracle, while the remaining local working qubits in the register \( R_W \) are unchanged and hold by the algorithm; the state \(|i, X_i\rangle\) denotes the set of parallel queries to function \( f_i \).

We let \( F \in \text{SHUF}(d, f) \) be sampled according to \( D(f, d) \), then

\[
F|\phi\rangle := \left( \bigotimes_{i=0}^{d} |i, X_i\rangle |f_i(X_i)\rangle \right)_{R_Q, R_N} \otimes |w(X_0, \ldots, X_d)\rangle_{R_W}.
\]

Applying \( O_{\text{unif}} \) on \(|\phi\rangle\) gives the mixed state

\[
O_{\text{unif}}^{f, d}(|\phi\rangle \langle \phi|) := \sum_{F \in \text{SHUF}(d, f)} \frac{1}{|\text{SHUF}(d, f)|} F|\phi\rangle \langle \phi|F.
\]

### 4.2 Shuffling Simon’s problem

We recall the definitions of Simon’s function and Simon’s problem.

**Definition 4.5 (Simon’s function).** A two-to-one function \( f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n \) for \( n \in \mathbb{N} \) is a Simon’s function if there exists an \( s \in \mathbb{Z}_2^n \) such that \( f(x) = f(x + s) \) for \( x \in \mathbb{Z}_2^n \).

**Definition 4.6 (Simon’s problem).** Let \( n \in \mathbb{N} \). Let \( F \) be the set of all Simon’s functions from \( \mathbb{Z}_2^n \) to \( \mathbb{Z}_2^n \). Choose \( f \) from \( F \) uniformly at random, the problem is to find the hidden shift \( s \) of \( f \).

The decision version of Simon’s problem is as follows:

**Definition 4.7 (Decision Simon’s problem).** Let \( n \in \mathbb{N} \). Let \( F \) be the set of all Simon’s functions from \( \mathbb{Z}_2^n \) to \( \mathbb{Z}_2^n \). Choose \( f \) to be either a random Simon’s function from \( F \) or a random one-to-one function from \( \mathbb{Z}_2^n \) to \( \mathbb{Z}_2^n \) with equal probability, the problem is to decide which case \( f \) is.

Both problems have been shown to be hard classically and can be solved in quantum polynomial time. We define the \( (d, f) \)-SSP by combining Simon’s problem and the shuffling oracle.

**Definition 4.8 (\( (d, f) \)-Shuffling Simon’s Problem (d-SSP)).** Let \( d \in \mathbb{N} \). Let \( f : \{0, 1\}^n \to \{0, 1\}^n \) be a random Simon’s function or a random one-to-one function with equal probability. Given access to the \( (d, f) \)-shuffling oracle \( O_{\text{unif}}^{f, d} \) of \( f \), the problem is to decide which case \( f \) is.
The search version of the $d$-SSP is as follows:

**Definition 4.9** (Search $d$-SSP). Let $d \in \mathbb{N}$. Let $f : \{0,1\}^n \rightarrow \{0,1\}^n$ be a random Simon’s function. Given access to the $(d,f)$-shuffling oracle $O_{\text{uni}}^{f,d}$ of $f$, the problem is to find the hidden shift of $f$.

Then, we define an oracle $O$ and a language $\mathcal{L}(O)$ corresponding to the $d$-SSP.

**Definition 4.10.** Let $\{f_i : \mathbb{Z}_2^d \rightarrow \mathbb{Z}_2^d; i > 0\}$ be the set of functions satisfying the promise of the decision Simon’s problem. Let $O := \{O_{\text{uni}}^{f_i,d(i)} : i > 0\}$, where $O_{\text{uni}}^{f_i,d(i)}$ is the $(d(i), f_i)$-shuffling oracle for the function $f_i$ and $d(i) = i$. The language is defined as follows:

$$\mathcal{L}(O) := \{1^n : f_n \text{ is a Simon’s function.}\}$$

**Remark 1.** If the $d$-SSP is intractable for any $d$-CQ or $d$-QC scheme for any $d$, then

$$\mathcal{L}(O) \not\in (\text{BPPBQNC})^O \cup (\text{BQNCBPP})^O.$$

This is because $d(n) = \Theta(n)$ is asymptotically greater than $\log^k n$ for any constant $k$. We can prove it by contradiction. If there is a $d$-CQ or $d$-QC algorithm to solve the $d$-SSP, then we can use that algorithm to solve the $d$-SSP.

We can show that the $d$-SSP can be solved with a $2d + 1$-depth quantum circuit. The idea is using Simon’s algorithm and erasing the queries on the path.

**Theorem 4.11.** The $d$-SSP and the search $d$-SSP can be solved by a QNC$_{2d+1}$ circuit with classical post-processing.

**Proof.** The $d$-SSP can be solved via Simon’s algorithm. We show the proof here.

$$\sum_{x \in \mathbb{Z}_2^n} |x\rangle|0,\ldots,0\rangle$$

$f_0 \sim D_f \quad \sum_{x \in \mathbb{Z}_2^n} |x\rangle f_0(x),\ldots,0\rangle$

$f_1 \sim D_f \quad \sum_{x \in \mathbb{Z}_2^n} |x\rangle f_0(x), f_1(f_0(x)),\ldots,0\rangle$

$f_d \sim D_f \quad \sum_{x \in \mathbb{Z}_2^n} |x\rangle f_0(x), f_1(f_0(x)),\ldots,f(x)\rangle$

Measure $\frac{1}{\sqrt{2}}(|x\rangle f_0(x),\ldots,f_{d-1}(f_0(x)))$

$+|x+s\rangle f_0(x+s),\ldots,f_{d-1}(f_0(x+s))\rangle f(x)\rangle$

uncompute $f_0,\ldots,f_{d-1} \quad \frac{1}{\sqrt{2}}(|x\rangle + |x+s\rangle) f(x)\rangle$

$QFT \quad \frac{1}{\sqrt{2^n}} \sum_{j \in \mathbb{Z}_2^n} ((-1)^{x \cdot j} + (-1)^{(x+s)j}) |j\rangle.$

When $s \cdot j = 0$, measuring the first register outputs $j$ with non-zero probability. Other other hand, when $s \cdot j = 1$, the probability that measurement outputs $j$ is zero. Therefore, by sampling $O(n)$ copies of $j$’s, one can find $s$ which is orthogonal to all $j$’s with high probability. □

**Theorem 4.11** directly implies that the language defined in Def. 4.10 is in BQP relative to $O$. 

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5 Analyzing the shuffling oracle

In this section, we are going to prove some properties related to the shuffling oracle. We first define as sequence of subsets which are in the domains of $f_0, \ldots, f_d$.

**Definition 5.1 ($S$).** Let $S_0 = \{0, \ldots, 2^n - 1\}$. For $j = 1, \ldots, d$, let $S_{j+1} = f_j \circ f_{j-1} \circ \cdots \circ f_0(S_0)$. We define $S := (S_0, \ldots, S_d)$.

We define a sequence of hidden sets $S$ corresponding to the shuffling oracle.

**Definition 5.2 (The hidden sets $S$).** Let $d, n \in \mathbb{N}$, $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$, and $F$ be a $(d, f)$-shuffling of $f$. Then, we define the sequence of hidden sets $S = (\bar{S}(0), \ldots, \bar{S}(d))$ as follows.

1. Let $S_j^{(0)} = \mathbb{Z}_2^{(d+2)n}$ for $j = 0, \ldots, d$. We define $S_j^{(0)} = (S_j^{(0)}, \ldots, S_d^{(0)})$.

2. For $\ell = 1, \ldots, d$, for $j = \ell, \ldots, d$, we choose $S_j^{(\ell)} \subseteq S_j^{(\ell-1)}$ randomly satisfying that $\frac{|S_j^{(\ell)}|}{|S_j^{(\ell-1)}|} \leq 1 \frac{1}{2^\ell}$. Let $f_j(S_{j-1}^{(\ell)}) = S_j^{(\ell)}$, and $S_j \subseteq S_j^{(\ell)}$. We define $S_j^{(\ell)} = (S_j^{(\ell)}, \ldots, S_d^{(\ell)})$.

Note that $S$ is a concept which we will use to show that a $d$-depth quantum circuit cannot successfully evaluate $f_d^*$ with high probability. Hence, we will choose $S$ in the ways such that some properties are satisfied depending on the computational models we are considering. We will see how to construct $S$ in the following sections.

With the concept of $S$, we can introduce the notation $\text{Shadow}$ which we will use to analyze the shuffling oracle.

**Definition 5.3 (Shadow function).** Let $F := (f_0, \ldots, f_d)$ be a $(d, f)$-shuffling of $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$. Fix the hidden sets $S := (\bar{S}(0), \ldots, \bar{S}(d))$. The shadow $G$ of $F$ in $\bar{S}(\ell) = (S_\ell^{(\ell)}, \ldots, S_d^{(\ell)})$ is as follows: For $j = \ell, \ldots, d$, let $g_j$ be the function such that if $x \in S_j^{(\ell)}$, $g_j(x) = \bot$; otherwise, $g_j(x) = f_j(x)$. We let $G := (f_0, \ldots, f_{\ell-1}, g_{\ell}, \ldots, g_d)$.

We can also represent a $(d, f)$-shuffling $F$ in terms of mappings corresponding to elements in $\bar{S}(\ell)$ of $S$.

**Definition 5.4 ($F^{(\ell)}$ and $\hat{F}^{(\ell)}$).** For $\ell = 1, \ldots, d$, we let $f_j^{(\ell)}$ be $f_j$ on $S_j^{(\ell-1)} \setminus S_j^{(\ell)}$ and $\hat{f}_j^{(\ell)}$ be $f_j$ on $S_j^{(\ell)}$. Then, we define

$$
F^{(1)} := (f_0, f_1^{(1)}, \ldots, f_d^{(1)}) \quad F^{(d+1)} := (\hat{f}_d^{(d)}), \text{ and }
$$

$$
F^{(\ell)} := (f_{\ell-1}^{(\ell)}, f_{\ell}^{(\ell)}, \ldots, f_d^{(\ell)})
$$

for $\ell = 2, \ldots, d$. Also, we define

$$
\hat{F}^{(0)} := (f_0, \ldots, f_d) \quad \text{and} \quad \hat{F}^{(\ell)} := (\hat{f}_{\ell}^{(\ell)}, \ldots, \hat{f}_d^{(\ell)})
$$

for $\ell = 1, \ldots, d$.

We can say that $\hat{F}^{(\ell)}$ is the mapping of elements in $\bar{S}(\ell)$, and $F^{(1)}, \ldots, F^{(\ell)}$ are the mappings out of $\bar{S}(\ell)$ for $\ell \in [d]$.  

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We rewrite $F$ in the following form

$$F := (F^{(1)}, \ldots, F^{(d+1)}).$$

This representation will be convenient for our analysis. There are several facts corresponding to this representation.

**Observation 1.** Let $F$ be an uniform $(d, f)$-shuffling of some function $f$.

- The key function $f^*_d$ is in $F^{(d+1)} := (f^{(d)}_d)$.
- Let $\mathcal{G} := (f_1, \ldots, f_{d-1}, g_{\ell}, \ldots, g_d)$ be the shadow of $F$ in $\mathcal{S}^{(\ell)}$, $\mathcal{G}$ must be consistent with $F$ on $F^{(1)}, \ldots, F^{(\ell)}$, and maps the elements in $\mathcal{S}^{(\ell)}$ to $\perp$. In other words, $\hat{\mathcal{F}}^{(\ell)}$ is totally blocked by $\perp$ when given $\mathcal{G}$.
- For $\ell = 1, \ldots, d$, conditioned on $F^{(1)}, \ldots, F^{(\ell)}$, the function $f^{(\ell)}_j$ is still drawn uniformly randomly from $P(\mathcal{S}^{(j)}_j, \mathcal{S}^{(j)}_{j+1})$ for $j = \ell, \ldots, d-1$ according to the definition of $D(f, d)$.

**Remark 2.** In this work, we will say a quantum state $\rho$ or a classical bit string $\bar{s}$ is **uncorrelated** to $F^{(\ell)}$. This means that if we replace $F^{(\ell)}$ by any other function, the process which outputs $\rho$ or $\bar{s}$ will not change the output distribution. On the other hand, if a quantum state $\rho$ or a bit string $\bar{s}$ is correlated to $F^{(\ell)}$, then we will assume a process which is given $\rho$ (or $\bar{s}$) knows everything about $F^{(\ell)}$ without loss of generality.

### 5.1 Semi-classical shuffling oracle

In this section, we are going to combine the concepts of “semi-classical” oracle introduced in [AHU18] and the shuffling oracle together.

**Definition 5.5** $(U^{(\mathcal{S}^{(\ell)})})$. Let $f$ be an arbitrary function and $F$ be a random $(d, f)$-shuffling of $f$. Let $S := (\mathcal{S}^{(0)}, \ldots, \mathcal{S}^{(d)})$ be a sequence of hidden sets. Let $U$ be single-depth quantum circuit. For $\ell \in [d]$, we define $U^{(\mathcal{S}^{(\ell)})}$ to be an unitary operating on registers $(R, I)$ where $I$ is a single-qubit register. $U^{(\mathcal{S}^{(\ell)})}$ simulates $F_U$ and that:

Before applying $F$, $U^{(\mathcal{S}^{(\ell)})}$ first applies $U_{\mathcal{S}^{(\ell)}}$ on $(R, I)$ and then performs $F$. Here $U_{\mathcal{S}^{(\ell)}}$ is defined by:

$$U_{\mathcal{S}^{(\ell)}}(\ell, X_1, \ldots, (d, X_d))_R|b\rangle_I := \begin{cases} U_{\mathcal{S}^{(\ell)}}(\ell, X_1, \ldots, (d, X_d))|b\rangle & \text{if every } X_i \cap S_i^{(\ell)} = \phi, \\ U_{\mathcal{S}^{(\ell)}}(\ell, X_1, \ldots, (d, X_d))|b + 1 \pmod{2}\rangle & \text{otherwise.} \end{cases}$$

In other words, for any state $|\psi\rangle_{R, I}$:

$$U^{(\mathcal{S}^{(\ell)})}|\psi\rangle := FU_{\mathcal{S}^{(\ell)}}U|\psi\rangle$$

In the following, we define a quantity which is the probability that the parallel queries are in a particular hidden set $\mathcal{S}^{(\ell)}$.

**Definition 5.6** $(\Pr(f \text{ find } \mathcal{S}^{(k+1)} : U^{(\mathcal{S}^{(k+1)})}, \rho))$. Let $k, d \in \mathbb{N}$ and $k+1 < d$. Let $U$ be a single-depth quantum circuit and $\rho$ be any input state. We define

$$\Pr(f \text{ find } \mathcal{S}^{(k+1)} : U^{(\mathcal{S}^{(k+1)})}, \rho) := \mathbb{E} \left[ \mathbb{E} \left[ \log \left( I_{R} \otimes (I - |0\rangle\langle 0|) \right) \circ U^{(\mathcal{S}^{(k+1)})} \circ \rho \right] \right].$$
Following Def. 5.6 we let $|\psi\rangle$ be a pure state and $U$ be a single-depth quantum circuit. Then,

$$U^{F\backslash S^{(t)}}|\psi\rangle_R|0\rangle_I := |\phi_0\rangle_R|0\rangle_I + |\phi_1\rangle_R|1\rangle_I$$

and $\Pr[\text{find } S^{(t)} : U^{F\backslash S^{(t)}}, |\psi\rangle] = E[||\phi_1\rangle_R||^2]$. Note that $|\phi_0\rangle$ and $|\phi_1\rangle$ are orthogonal by the fact that $|\phi_0\rangle$ involves no query to $S^{(t)}$ but $|\psi\rangle$ does. Therefore,

$$\mathcal{F}U|\psi\rangle = |\phi_0\rangle + |\phi_1\rangle.$$ 

5.2 Oneway-to-hiding (O2H) lemma for the shuffling oracle

Here, we extend the Oneway-to-hiding lemma (O2H lemma) by Ambainis et al. [AHU18] to the shuffling oracle. Briefly, the O2H lemma shows that for any two functions $g$ and $h$, the probability for a quantum algorithm to distinguish them is bounded by the probability that the quantum algorithm ever "find" an element in the input domain that $g$ and $h$ disagree with each other times the depth of the quantum algorithm.

**Lemma 5.7** (O2H lemma for the shuffling oracle). Let $k, d \in \mathbb{N}$ satisfying that $k < d$. Let $U$ be any single depth quantum circuit and $\rho$ be any input state. Let $f$ be any function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^n$. Let $F$ be a random $(d, f)$-shuffling of $f$. Let $S := (\overline{S}^{(0)}, \ldots, \overline{S}^{(d)})$ be a sequence of random hidden sets as defined in Def. 5.2. Let $G$ be the shadow of $F$ in $\overline{S}^{(k)}$. Then, for any binary string $t$,

$$|\Pr[\Pi_{0/1} \circ \mathcal{F}U(\rho) = t] - \Pr[\Pi_{0/1} \circ \mathcal{G}U(\rho) = t]| \leq B(\mathcal{F}U(\rho), \mathcal{G}U(\rho)) \leq \sqrt{2 \Pr[\text{find } \overline{S}^{(k)} : U^{F\backslash \overline{S}^{(k)}}, \rho]},$$

where $\Pi_{0/1}$ is the measurement in the standard basis. Here the probability is over $F$, $S$, and the randomness of the quantum mechanism.

**Proof.** We will prove the case where the initial state is a pure state and then the general case directly follows from the concavity of the mixed state. For simplicity, we denote $\Pr[\text{find } \overline{S}^{(k)} : U^{F\backslash \overline{S}^{(k)}}, \rho]$ as $P_{\text{find}}$.

Fix $F$ and $\overline{S}^{(k)}$. We let $|\psi\rangle$ be any initial state and

$$\mathcal{F}U_{\overline{S}^{(k)}}U|\psi\rangle_R|0\rangle_I := |\phi_0\rangle_R|0\rangle_I + |\phi_1\rangle_R|1\rangle_I,$$

where $|\phi_0\rangle$ and $|\phi_1\rangle$ are two unnormalized states. $|\phi_0\rangle$ and $|\phi_1\rangle$ are orthogonal due to the fact that all queries $|\phi_0\rangle$ consists of are not in $\overline{S}^{(k)}$, while the queries $|\phi_1\rangle$ performs are elements in $U_{\overline{S}^{(k)}}$. This, therefore, implies that $|\psi_f\rangle := \mathcal{F}U|\psi\rangle = |\phi_0\rangle + |\phi_1\rangle$ and

Similarly, we let

$$\mathcal{G}U_{\overline{S}^{(k)}}|\psi\rangle_R|0\rangle_I := |\phi_0\rangle_R|0\rangle_I + |\phi_1^\perp\rangle_R|1\rangle_I,$$

and due to the same fact that $|\phi_0\rangle$ and $|\phi_1^\perp\rangle$ are orthogonal, we have that $|\psi_g\rangle := \mathcal{G}U|\psi\rangle = |\phi_0\rangle + |\phi_1^\perp\rangle$.

Here, $|\phi_1^\perp\rangle$ and $|\phi_1\rangle$ are orthogonal since $G$ maps all elements in $\overline{S}^{(k)}$ to $\perp$. 

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Let $P_{\text{find}}(\mathcal{F}, \overline{\mathcal{S}}^{(k)})$ be the probability that a standard-basis measurement in the register $I$ of $\mathcal{F}U_{\overline{\mathcal{S}}^{(k)}}U|\psi\rangle_R|0⟩_I$ returns 1, which is equal to $\|\phi_1\|^2$. Consider the two-norm distance between $|ψ_f⟩$ and $|ψ_g⟩$,

$$\|ψ_f⟩ − |ψ_g⟩\|_2^2 = \|φ_1⟩ − |φ_⊥^1⟩\|_2^2 = \|φ_1⟩\|^2 + \|φ_⊥^1⟩\|^2 \leq 2\|φ_1⟩\|^2 = 2P_{\text{find}}(\mathcal{F}, \overline{\mathcal{S}}^{(k)}).$$

The second equality follows from the fact that $|φ_1⟩$ and $|φ_⊥^1⟩$ are orthogonal. The inequality is because that $\|φ_1⟩\|^2 = \|φ_⊥^1⟩\|^2 = 1 − \|φ_0⟩\|^2$.

Then, consider the case that $\mathcal{S}^{(k)}$ and $\mathcal{F}$ are random. The output states of $\mathcal{F}U$ and $\mathcal{G}U$ becomes

$$\rho_f := \sum_{\mathcal{F}, \mathcal{S}^{(k)}} \Pr[\mathcal{F} \wedge \overline{\mathcal{S}}^{(k)}] |ψ_f⟩⟨ψ_f|,$$

$$\rho_g := \sum_{\mathcal{F}, \mathcal{S}^{(k)}} \Pr[\mathcal{F} \wedge \overline{\mathcal{S}}^{(k)}] |ψ_g⟩⟨ψ_g|.$$

Consider the fidelity of these two mixed states.

$$F(\rho_f, \rho_g) = \frac{1}{2} \sum_{\mathcal{F}, \mathcal{S}^{(k)}} \Pr[\mathcal{F} \wedge \overline{\mathcal{S}}^{(k)}] |ψ_f⟩⟨ψ_f| + \Pr[\mathcal{F} \wedge \overline{\mathcal{S}}^{(k)}] |ψ_g⟩⟨ψ_g| \geq 1 - \frac{1}{2} \sum_{\mathcal{F}, \mathcal{S}^{(k)}} \Pr[\mathcal{F} \wedge \overline{\mathcal{S}}^{(k)}] \|ψ_f⟩ − |ψ_g⟩\|^2 \geq 1 - \frac{1}{2} \sum_{\mathcal{F}, \mathcal{S}^{(k)}} \Pr[\mathcal{F} \wedge \overline{\mathcal{S}}^{(k)}] 2P_{\text{find}}(\mathcal{F}, \overline{\mathcal{S}}^{(k)}).$$

Then, $B(\rho_f, \rho_g) \leq \sqrt{2 − 2F(\rho_f, \rho_g)} \leq \sqrt{2 − 2(1 − P_{\text{find}})} = \sqrt{2P_{\text{find}}}.$

Finally, by Lemma 3.3

$$|\Pr[\Pi_{0/1} \circ \mathcal{F}U(\rho) = t] − \Pr[\Pi_{0/1} \circ \mathcal{G}U(\rho) = t]| \leq \sqrt{2P_{\text{find}}}.$$

5.3 Bounding the finding probability

As we have just shown that the probability of distinguishing $\mathcal{F}$ and its shadow can be bounded by the probability of finding the “shadow”. Then, we would like to show how to bound the finding probability.
Follow the previous section, we let $\mathcal{F}$ be a random $(d, f)$-shuffling of $f$ and $S := (S^{(0)}, \ldots, S^{(d)})$ be a sequence of random hidden sets as defined in Definition 5.2 (which could be chosen according to arbitrary distribution). We show that the finding probability of $S^{(k)}$ is bounded.

**Lemma 5.8.** Suppose $\Pr[x \in S_i^{(k)} | x \in S_i^{(k-1)}] \leq p$ for $i = k, \ldots, d$. Then for any single-depth quantum circuit $U$ and initial state $\rho$, which are promised to be uncorrelated to $\mathcal{F}^{(k-1)}$ and $S^{(k)}$,

$$\Pr[\text{find } S^{(k)} : U \mathcal{F} \setminus S^{(k)}, \rho] \leq q \cdot p,$$

where $q$ is the number of queries $U$ performs.

**Proof.** It is sufficient to prove the case where $\rho$ is a pure state. Let $|\psi\rangle$ be the initial state and be promised to be uncorrelated to $\mathcal{F}^{(k-1)}$. We represent $\mathcal{F} U_S^{(k)} U |\psi\rangle$ as

$$\sum_{X_0, \ldots, X_d} c(X_0, \ldots, X_d) \left( \bigotimes_{i=0}^{d} |i, X_i, f_i(X_i)\rangle \bigotimes w(X_0, \ldots, X_d) |b(X_1, \ldots, X_d)\rangle \right)_{1}$$

where $b(X_1, \ldots, X_d) = 1$ if there exists $i \in [\ell, d]$ such that $X_i \cap S_i^{(k)} \neq \emptyset$; otherwise, $b(X_1, \ldots, X_d) = 0$. We can assume all queries are in $S^{(k-1)}$ without loss of generality. Since both $U$ and $|\psi\rangle$ are uncorrelated to $S^{(k)}$, the probability that $b(X_1, \ldots, X_d) = 1$ is at most $p \cdot (\sum_{i=k}^{d} |X_i|)$ for all $X_1, \ldots, X_d$ by union bound. Therefore,

$$\Pr[\text{find } S^{(k)} : U \mathcal{F} \setminus S^{(k)}, |\psi\rangle] = E \left[ \left| \sum_{X_0, \ldots, X_d : b(X_0, \ldots, X_d) = 1} c(X_0, \ldots, X_d) \left( \bigotimes_{i=0}^{d} |i, X_i, f_i(X_i)\rangle \bigotimes w(X_0, \ldots, X_d) |b(X_1, \ldots, X_d)\rangle \right) \right|^2 \right]$$

$$= \sum_{X_0, \ldots, X_d : b(X_0, \ldots, X_d) = 1} |c(X_0, \ldots, X_d)|^2 \cdot \Pr[\bigvee_{i=k}^{d} (X_i \cap S_i^{(k)} \neq \emptyset)]$$

$$\leq q \cdot p$$

for $q$ the number of queries $U$ performs. The second equality follows from the fact that for different set of queries, $|i, X_i, f_i(X_i)\rangle$’s are orthogonal. The last inequality follows from the union bound. \hfill \qed

### 6 The $d$-SSP is hard for QNC$_d$

We start from showing that the $d$-SSP is intractable for any QNC$_d$ circuit as a warm-up. We first prove the main theorem in this section.

**Theorem 6.1.** Let $n, d \in \mathbb{N}$. Let $(\mathcal{A}, \rho)$ be any $d$-depth quantum circuit and initial state. Let $f$ be a random Simon’s function from $\mathbb{Z}_2^d$ to $\mathbb{Z}_2^n$ with hidden shift $s$. Give $\mathcal{A}$ the access to the shuffling oracle $\mathcal{O}_\text{unit}^{f, d}$. Let $\mathcal{F}$ be the $(d, f)$-shuffling sampled from $\mathcal{O}_\text{unit}^{f, d}$, then

$$\Pr[\mathcal{A}^{\mathcal{F}}(\rho) = s] \leq d \cdot \sqrt{\frac{\text{poly}(n)}{2^n} + \frac{1}{2^n}}.$$

Here, $\rho$ and $U$ can be arbitrarily correlated to $\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(k-1)}$, and $\mathcal{F}$ and $S^{(k)}$ can be sampled arbitrarily.
Proof. We choose \( S = (S^{(0)}, \ldots, S^{(d)}) \) according to Procedure 1 and represent \( F \) in form \( (F^{(1)}, \ldots, F^{(d+1)}) \) regarding to \( S \). Let \( G_\ell \) be the shadow function of \( F \) in \( S^{(\ell)} \) for \( \ell \in [d] \). We define
\[
A^F := U_{d+1} G_d U_d \cdots G_1 U_1.
\]
Then, for any initial state \( \rho_0 \) which is uncorrelated to \( F \),
\[
\left| \Pr[A^F(\rho_0) = s] - \Pr[A^G(\rho_0) = s]\right| \\
= \left| \Pr[U_{d+1} F U_d \cdots F U_1(\rho_0) = s] - \Pr[U_{d+1} G_d U_d \cdots G_1 U_1(\rho_0) = s]\right| \\
\leq \left| \Pr[U_{d+1} F U_d \cdots F U_2 F U_1(\rho_0) = s] - \Pr[U_{d+1} G_d U_d \cdots G_2 G_1 U_1(\rho_0) = s]\right| \\
+ \left| \Pr[U_{d+1} F U_d \cdots F U_2 F U_1(\rho_0) = s] - \Pr[U_{d+1} G_d U_d \cdots G_2 G_1 U_1(\rho_0) = s]\right| \\
\leq \sum_{i=1}^{d} B(F U_i(\rho_{i-1}), \rho_i) \\
\leq \sum_{i=1}^{d} \sqrt{2 \Pr[find S^{(i)} : U_i^\top \rho_i]} \\
\] where \( \rho_i := G_i U_i \rho_{i-1} U_i^\top G_i^\top \) for \( i \geq 1 \).

**Procedure 1** The hidden sets for QNC

Let \( d, n \in \mathbb{N} \) and \( f \) a random Simon’s problem from \( \mathbb{Z}_2^n \to \mathbb{Z}_2^n \). Consider \( F \sim D(f, d) \), we construct \( S \) as follows:
- Let \( S^{(0)} := (S_0^{(0)}, \ldots, S_d^{(0)}) \), where \( S_j^{(0)} := \mathbb{Z}_2^{(d+2)n} \) for \( j = 0, \ldots, d \).
- For \( \ell = 1, \ldots, d \),
  1. let \( S_\ell^{(\ell)} \) be a subset chosen uniformly at random with the promise that \( |S_\ell^{(\ell)}|/|S_\ell^{(\ell-1)}| = \frac{1}{2^n} \) and \( S_\ell \subset S_\ell^{(\ell)} \);
  2. for \( j = \ell + 1, \ldots, d \), let \( S_j^{(\ell)} := \{f_{j-1} \circ \cdots \circ f_\ell(S_\ell^{(\ell)})\} \);
  3. let \( S^{(\ell)} := (S_\ell^{(\ell)}, \ldots, S_d^{(\ell)}) \).
- \( S := (S^{(0)}, \ldots, S^{(d)}) \).

It is not hard to see that \( \Pr[A^G(\rho_0) = s] \) is at most \( \frac{1}{2^n} \). This follows from the fact that \( G_1, \ldots, G_d \) does not contain information of \( f_d^* \) and therefore \( A^G(\rho_0) \) can do no better than guess. The rest to show is that \( \Pr[find S^{(i)} : U_i^\top \rho_i] \) is at most \( \frac{\text{poly}(n)}{2^n} \) for all \( i \in [d] \). To prove it, we show that \( \Pr[x \in S_j^{(\ell)} \mid x \in S_j^{(\ell-1)}] = \frac{1}{2^n} \) for \( \ell = 1, \ldots, d \) and \( j = \ell, \ldots, d \). We prove it by induction on \( \ell \).

For the base case \( \ell = 1 \), for all \( j \in [d] \), and \( x \in S_j^{(0)} \),
\[
\Pr[x \in S_j^{(1)}] = \Pr[x \in S_j] \Pr[x \in S_j^{(1)} \mid x \in S_j] + \Pr[x \notin S_j] \Pr[x \in S_j^{(1)} \mid x \notin S_j] \\
= \Pr[x \in S_j] + (1 - \Pr[x \in S_j]) \Pr[x \in S_j^{(1)} \mid x \notin S_j] \\
= \frac{1}{2^{(d+1)n}} + (1 - \frac{1}{2^{(d+1)n}}) \frac{2^{(d+1)n} - 2^n}{2^{(d+2)n} - 2^n} = \frac{1}{2^n}. \\
\]
Then, given $F$ and therefore, we assume $j$ the fact that $\rho$

Proof. We also consider the same shadow $G$ about $f$

Then, the rest to check is that $A$

Theorem 6.2. The $d$-SSP cannot be decided by any $\text{QNC}_d$ circuit with probability greater than $\frac{1}{2} + d \cdot \sqrt{\frac{\text{poly}(n)}{2^n}}$.

Proof. We also consider the same shadow $G$ in the proof of Theorem 6.1. Following that proof, for any $\rho$ and $A$,

Then, the rest to check is that $A^G$ cannot solve the $d$-SSP with high probability. Similar to the case of the search $d_0$-SSP, since $G_1, \ldots, G_d$ have the core function $f^d_1$ be blocked, $A^G$ has no information about $f$ and thus cannot do better than guess. This implies that $\Pr[A^F(\rho_0) = 0] \leq \frac{1}{2} + d \cdot \sqrt{\frac{\text{poly}(n)}{2^n}} < \frac{2}{3}$.

Therefore, the language defined in Def. 4.10 is also hard for $\text{QNC}$ circuit.

Corollary 6.3. Let $O$ and $\mathcal{L}(O)$ be defined as in Def. 4.10, $\mathcal{L}(O) \notin \text{BQNC}$

The second equality is because that $x \in S_j$ implies $x \in S_j^{(1)}$ and the third inequality follows from the fact that $f_0, \ldots, f_{d-1}$ are uniformly random one-to-one functions.

Finally, $U_i$ and $\rho_{i-1}$ are uncorrelated to $\hat{F}^{(i)}$. By Lemma 5.8, $\Pr[\text{find } S^{(i)} : U_i^{\hat{F} \setminus S^{(i)}}, \rho_{i-1}]$ is at most $q_i \cdot \frac{1}{2^n}$ where $q_i$ is the number of queries $U_i$ performs. Therefore,

Theorem 6.1 shows that the search $d$-SSP is hard for any $\text{QNC}_d$ circuit. By following the same proof, we can show that for any $\text{QNC}_d$ circuit, the $d$-SSP is also hard.
7  The \(d\)-SSP is hard for \(d\)-QC scheme

The main theorem we are going to show in this section is that the search \(d\)-SSP is hard for all \(d\)-QC scheme.

**Theorem 7.1.** Let \(d, n \in \mathbb{N}\). For any \(d\)-QC scheme \(A\) and initial state \(\rho\), let \(f\) be a random Simon’s function from \(\mathbb{Z}_2^n\) to \(\mathbb{Z}_2^n\) with hidden shift \(s\), and \(F \sim D(f, d)\), then

\[
\Pr[A^F(\rho) = s] \leq d \cdot \sqrt{\frac{\text{poly}(n)}{2^n}}.
\]

Before proving Theorem 7.1, we first recall the classical lower bound for the Simon’s problem. However, for the purpose of

**Lemma 7.2.** Let \(A_c\) be any PPT algorithm. Let \(f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n\) be a uniformly random Simon function. Let \(q \in \mathbb{N}\) be the number of queries \(A_c\) performs, and \(S \subset \mathbb{Z}_n\) be the set where \(f(x)\) is known for \(x \in S\) and \(f(x) \neq f(x')\) for \(x \neq x'\). Then the probability that \(A_c^f(S, f(S))\) outputs the hidden shift correctly is at most

\[
\frac{(q + 1 + |S|)^2}{2^{n+1} - (q + 1 + |S|)^2},
\]

which is \(O(\text{poly}(n)/2^n)\) when \(q\) and \(|S|\) are polynomial in \(n\).

**Proof.** We only need to consider the case where \(A_c\) is a deterministic algorithm. A probabilistic algorithm can be seen as a convex combination of deterministic algorithm; therefore, the success probability of a probabilistic algorithm must be an average over deterministic algorithms.

Let \(S \subset \mathbb{Z}_2^n\) and \(f(x) \neq f(y)\) for \(x, y \in S\) and \(x \neq y\). The probability that \(A_c\) finds a collision is

\[
\Pr[\text{collision} : A_c^f(S)] \leq \sum_{i=1}^{q} \frac{i + |S| - 1}{2^n - (i + |S|)^2/2} \leq \frac{(q + |S|)^2}{2^{n+1} - (q + |S|)^2}.
\]

For any algorithm which can find \(s\) with probability \(p\) by performing \(q\) queries, it can find a collision with the same probability by performing \(q+1\) queries. The probability of finding a collision by using \(q + 1\) queries is at most \(\frac{(q+|S|+1)^2}{2^{n+1}-(q+|S|+1)^2}\). Therefore,

\[
\Pr[A_c^f(S) = s] \leq \frac{(q + |S| + 1)^2}{2^{n+1} - (q + |S| + 1)^2}.
\]

\(\square\)

7.1  Proof of Theorem 7.1

Recall that we can represent a \(d\)-QC scheme \(A\) with access to \(F \sim D(f, d)\) as

\[
A^F \circ (\Pi_{0/1} \circ FU_d \circ A^F_{c}) \circ \cdots \circ (\Pi_{0/1} \circ FU_1 \circ A^F_{c}).
\]

We denote \(\Pi_{0/1} \circ FU_i \circ A^F_{c}\) as \(L_i^F\) for \(i = 1, \ldots, d\) and rewrite the representation above as \(A^F \circ L_{d}^F \circ \cdots \circ L_{1}^F\). We let \(q_i\) be the number of quantum queries and \(r_i\) be the number of classical queries the algorithm performs in \(L_i\). We let \(q := \sum_{i=1}^{d} q_i\) and \(r := \sum_{i=1}^{d} r_i\).

For the ease of the analysis, we allow \(A_c\) to learn the whole path from \(f_0\) to \(f_d\) by just one query, which we called the “path query”. It is worth noting that \(A_c\) that can make path queries can be
simulated by the original model. This follows from the fact that the original model can achieve the same thing by using \( d \) times as many queries as \( A_c \).

To prove the theorem, we need to define an \( S \) which has property that \( \Pr [ x \in S_j^{(\ell)} ] \leq p \) as described in Lemma 5.8 for some \( p \) that is small enough. In the following, we show that \( S = (S^{(1)}, \ldots, S^{(d)}) \) in Procedure 2 satisfies this property with \( p = \frac{1}{2^n} \).

**Claim 7.3.** Let \( d, n \in \mathbb{N} \). Let \( \ell \in [d] \). Let \( A_c \) be any randomized algorithm. Let \( f : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^n \) be any function. Let \( \mathcal{F} \) be defined as in Procedure 2 regarding to \( A_c \). Given \( (\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(\ell-1)}) \) and \( (\overline{T}^{(1)}, \ldots, \overline{T}^{(\ell)}) \), then

\[
\Pr_{\overline{F},S}[x \in S_j^{(\ell)} | x \in S_j^{(\ell-1)} \setminus T_j^{(\ell)}] = \frac{1}{2^n} \text{ for } j = \ell, \ldots, d.
\]

**Proof.** We prove it via induction on the depth of \( \mathcal{F} \). For the base case where \( \ell = 1 \), given \( T_j^{(1)} \) and \( f_j \) on \( T_j^{(1)} \) for \( j = 1, \ldots, d \), for all \( i \in [d] \) and \( x_i \in S_i^{(0)} \setminus T_i^{(1)} \),

\[
\Pr [ x_i \in S_i^{(1)} ] = \Pr [ x_i \in S_i \setminus T_i^{(1)} ] + \Pr [ x_i \in S_i^{(1)} \setminus S_i ]
\]

\[
= \frac{2^n - |T_i^{(1)}|}{2^{(d+2)n} - |T_i^{(1)}|} + \frac{|S_i^{(1)}| - 2^n + |T_i^{(1)}|}{2^{(d+2)n} - |T_i^{(1)}|}
\]

\[
= \frac{|S_i^{(1)}|}{2^{(d+2)n} - |T_i^{(1)}|} = \frac{1}{2^n}.
\]

We now suppose that when \( \ell = k - 1 \), given \( \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(k-2)} \) and \( \overline{T}^{(1)}, \ldots, \overline{T}^{(k-1)} \),

\[
\Pr [ x_i \in S_i^{(k-1)} | x_i \in S_i^{(k-2)} \setminus T_i^{(k-1)} ] = \frac{1}{2^n}
\]

for \( i = k - 1, \ldots, d \).

Then, for \( \ell = k \), given \( \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(k-1)} \) and \( \overline{T}^{(1)}, \ldots, \overline{T}^{(k)} \), for \( i = k, \ldots, d \) and \( x \in S_i^{(k-1)} \setminus T_i^{(k)} \),

\[
\Pr [ x \in S_i^{(k)} ] = \Pr [ x \in S_i \setminus ((\cup_{m=1}^k (T_i^{(m)}))] + \Pr [ x \in S_i^{(k)} \setminus S_i ]
\]

\[
= \frac{2^n - \sum_{m=1}^k |T_i^{(m)}|}{|S_i^{(k-1)}| - |T_i^{(k)}|} + \frac{|S_i^{(k)}| - 2^n + \sum_{m=1}^k |T_i^{(m)}|}{|S_i^{(k-1)}| - |T_i^{(k)}|}
\]

\[
= \frac{1}{2^n}.
\]

Now, we are ready to prove Theorem 7.1.

**Proof of Theorem 7.1.** We choose the hidden set \( S = (S^{(0)}, \ldots, S^{(d)}) \) according to Procedure 2. In the procedure, we choose \( S^{(\ell)} \) after the \( A_c \) in \( L_\ell \) has performed. We represent \( \mathcal{F} \) as \( (\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(d+1)}) \) according to \( S \). Let \( \mathcal{G}_\ell \) be the shadow of \( \mathcal{F} \) in \( S^{(\ell)} \) for \( \ell \in [d] \). We define

\[
\mathcal{A}^\mathcal{G} := A_c^\mathcal{G} \circ (\Pi_{0/1} \circ \mathcal{G}_{d} U_d \circ \mathcal{A}_c^\mathcal{F}) \circ \cdots \circ (\Pi_{0/1} \circ \mathcal{G}_{1} U_1 \circ \mathcal{A}_c^\mathcal{F})
\]

\[
:= A_c^\mathcal{F} \circ L_d^\mathcal{G} \circ \cdots \circ L_1^\mathcal{G}.
\]

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Procedure 2 The hidden sets for $d$-QC scheme

Let $d, n \in \mathbb{N}$ and $f$ a random Simon’s problem from $\mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^n$. Given $\mathcal{F} \sim \mathcal{D}(f, d)$ and $\mathcal{A}$ a $d$-QC scheme, We construct $\mathcal{S}$ as follows:

- Let $\bar{S}^{(0)} := (S_0^{(0)}, \ldots, S_d^{(0)})$, where $S_j^{(0)} := \mathbb{Z}_2^{(d+2)n}$ for $j = 0, \ldots, d$.
- For $\ell = 1, \ldots, d$:
  1. After the $\ell$-th $\mathcal{A}_F^\mathcal{F}$ is applied, let $\overline{T}^{(\ell)} = (T^{(\ell)}_1, \ldots, T^{(\ell)}_d)$ be the set of points the $\ell$-th $\mathcal{A}_c^\mathcal{F}$ queried. As we have mentioned before, we allow $\mathcal{A}_c$ to query the whole path by one query. Hence, $f_j(T^{(\ell)}_j) = T^{(\ell)}_{j+1}$ for $j = \ell, \ldots, d$.
  2. Let $W^{(\ell-1)}_\ell := S^{(\ell-1)}_\ell \setminus T^{(\ell)}_\ell$. Then, we choose $S^{(\ell)}_\ell$ uniformly randomly from $W^{(\ell-1)}_\ell$ with the promise that $|S^{(\ell)}_\ell|/|W^{(\ell-1)}_\ell| = 1/2^n$ and $S^{(\ell)}_\ell \setminus (T^{(1)}_\ell \cup \cdots \cup T^{(\ell)}_\ell) \subset S^{(\ell)}_\ell$.
  3. For $j = \ell + 1, \ldots, d$, let $S^{(\ell)}_j := \{ f_{j-1} \circ \cdots \circ f_\ell(S^{(\ell)}_\ell) \}$.
  4. Let $\bar{S}^{(\ell)} := (S^{(\ell)}_1, \ldots, S^{(\ell)}_d)$.

- $\mathcal{S} := (\bar{S}^{(0)}, \ldots, \bar{S}^{(d)})$

$\mathcal{A}^\mathcal{F}$ succeeds to output the hidden shift with probability at most $\frac{(r+1)^2}{2^n-\binom{r+1}{2}^2}$, where $r$ is the number of queries the classical algorithms perform. Note that the outputs of $U_1, \ldots, U_d$ are uncorrelated to $\mathcal{F}(d)$. This fact implies that given the measurement outcomes of the $i$-th layer quantum unitaries, $\mathcal{A}_c$ can only learn information about $\mathcal{F}(1), \ldots, \mathcal{F}(d)$. This does not give $\mathcal{A}_c$ more information about $f$. Therefore, $\mathcal{A}_c$ at $L_i$ succeeds with probability at most $\frac{(\sum_{j=1}^{r_i+1}(r_i+1)^2)}{2^n-(\sum_{j=1}^{r_i+1}(r_i+1))^2}$ and $\mathcal{A}^\mathcal{G}$ succeeds with probability at most $\frac{\text{poly}(n)}{2^n}$.

Let $\rho_0$ be the initial state and $\rho_i$ be the output state of $(L_i^n \circ \cdots \circ L_1^n)(\rho_0)$ for $i = 1, \ldots, d$, we can show that

$$
\begin{align*}
|\Pr[\mathcal{A}(\rho_0) = s] - \Pr[\mathcal{A}^\mathcal{G}(\rho_0) = s]| \\
\leq |\Pr[\mathcal{A}(\rho_0) = s] - \Pr[(L_d \circ \cdots \circ L_2)^{\mathcal{F}}(L_1^n(\rho_0)) = s]| + \\
|\Pr[(L_d \circ \cdots \circ L_2)^{\mathcal{G}}(L_1^n(\rho_0)) = s] - \Pr[(L_d \circ \cdots \circ L_2)^{\mathcal{F}}(L_1^n(\rho_0)) = s]| \\
\leq B(L_1^n(\rho_0), L_2^n(\rho_0)) + \\
|\Pr[(L_d \circ \cdots \circ L_2)^{\mathcal{F}}(\rho_1) = s] - \Pr[(L_d \circ \cdots \circ L_2)^{\mathcal{G}}(\rho_1) = s]| \\
\leq \sum_{i=1}^{d} B(\rho_i, L_i^n(\rho_{i-1})). \\
\leq \sum_{i=1}^{d} \sqrt{2} \Pr[\text{find } \mathcal{S}^{(i)} : U_i^{\mathcal{F} \setminus \mathcal{S}^{(i)}}(\rho_{i-1}). \\
\end{align*}
$$

Eq. 8 is by the hybrid argument and Eq. 9 is from Lemma 5.7

Then, by Lemma 5.8 and Claim 7.3

$$
\Pr[\text{find } \mathcal{S}^{(i)} : U_i^{\mathcal{F} \setminus \mathcal{S}^{(i)}}(\rho_{i-1})] \leq \frac{q_i}{2^n}.
$$

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This implies that
\[
\Pr[\mathcal{A}^F(\rho_0) = s] \leq \frac{\text{poly}(n)}{2^n} + d \cdot \sqrt{\frac{\text{poly}(n)}{2^n}} \leq \sqrt{\frac{\text{poly}(n)}{2^n}}.
\]

7.2 On separating the depth hierarchy of $d$-QC scheme

By following the same proof of Theorem 7.1, we can show that for any $d$-QC scheme, the $d$-SSP is also hard.

**Theorem 7.4.** The $d$-SSP cannot be decided by any $d$-QC scheme with probability greater than \( \frac{1}{2} + \sqrt{\frac{\text{poly}(n)}{2^n}} \).

**Proof.** We consider the same shadow $\mathcal{G}$ in the proof of Theorem 7.1. Following that proof, for any $\rho_0$ and $A$,
\[
|\Pr[\mathcal{A}^G(\rho_0) = 1] - \Pr[\mathcal{A}^G(\rho_0) = 1]| \leq d \cdot \sqrt{\frac{\text{poly}(n)}{2^n}}.
\]
Then, the rest to check is that $\mathcal{A}^G$ cannot solve the $d$-SSP with high probability. In case that $f$ is a random Simon’s function, $\mathcal{A}^G$ finds $s$ with probability at most $\frac{\text{poly}(n)}{2^n}$. Therefore, $\Pr[\mathcal{A}^G(\rho_0) = 1]$ is at most $1/2 + \text{poly}(n)/2^n$. This implies that $\Pr[\mathcal{A}^F(\rho_0) = 1] \leq 1/2 + d \cdot \sqrt{\frac{\text{poly}(n)}{2^n}} + \text{poly}(n)/2^n < 2/3$.

**Corollary 7.5.** For any $d \in \mathbb{N}$, there is a $(2d + 1)$-QC scheme which can solve the $d$-SSP with high probability, but there is no $d$-QC scheme which can solve the $d$-SSP.

**Proof.** This corollary follows from Theorem 7.4 and Theorem 4.11 directly.

Finally, we can conclude that

**Corollary 7.6.** Let $\mathcal{O}$ and $\mathcal{L}(\mathcal{O})$ be defined as in Def. 4.10. $\mathcal{L}(\mathcal{O}) \in \text{BQP}^\mathcal{O}$ and $\mathcal{L}(\mathcal{O}) \notin (\text{BQNC}^\text{BPP})^\mathcal{O}$.

**Proof.** Note that for each $n \in \mathbb{N}$, $\mathcal{O}_{\text{uni}}^{f_n,d(n)} \in \mathcal{O}$ has depth equal to the input size. A quantum circuit with depth $\text{poly}(n)$ can decide if $1^n$ is in $\mathcal{L}(\mathcal{O})$ by solving the $d$-SSP by Theorem 4.11. However, for $d$-QC scheme which only has quantum depth $d = \text{polylog} n$, it cannot decide the language by Theorem 7.4.

8 The $d$-SSP is hard for $d$-CQ scheme

The main result we are going to show here is the following theorem.

**Theorem 8.1.** Let $d, n \in \mathbb{N}$. Let $\mathcal{A}$ be any $d$-CQ scheme. Let $f$ be a random Simon’s function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^n$ with hidden shift $s$. Let $\mathcal{F} \sim D(f,d)$. Then
\[
\Pr[\mathcal{A}^F() = s] \leq d \cdot \sqrt{\frac{\text{poly}(n)}{2^n}}.
\]
Recall that we can represent a d-CQ scheme $A$ as
\[ A_{c,m}^{\mathcal{F}} \circ (\Pi_{0/1} \circ U_{d+1} \mathcal{F} \cdots \mathcal{F} U_1 \circ A_{c,m}^{\mathcal{F}}) \circ \cdots \circ (\Pi_{0/1} \circ U_{d+1} \mathcal{F} \cdots \mathcal{F} U_1 \circ A_{c,1}^{\mathcal{F}}) := A_{c,m}^{\mathcal{F}} \circ L_{m} \circ \cdots \circ L_{1}. \]

The main difficulty for proving Theorem 8.1 is that $L_{i}$ can send some short classical advice to the proceeding processes, which advice can be correlated to all mappings in $\mathcal{F}$. Therefore, conditioned on the short advice, the distribution of the shuffling oracle may not be uniform enough for us to follow the same proofs above. To deal with this difficulty, we show that given a short classical advice, by fixing the shuffling function on a few paths, the rest paths of the shuffling oracle are still almost-uniform.

### 8.1 The presampling argument for the shuffling oracle

Here, we are going to show that for $\mathcal{F} \sim \mathcal{D}(f,d)$ and a short classical string correlated to $\mathcal{F}$, we can approximate $\mathcal{F}|\overline{a}$ ($\mathcal{F}$ given $\overline{a}$) by a convex combination of $(p, 1+\delta)$-uniform shuffling functions. In the following, we first define $(p, 1+\delta)$-uniform shuffling functions and then prove the statement.

Let $X$ and $Y$ be two sets of elements such that $|X| = |Y|$. Recall that $\mathcal{P}(X,Y)$ is the set of all one-to-one functions from $X$ to $Y$.

**Definition 8.2 (Random variable $\mathcal{H}_{\overline{X}}$).** Let $k, N \in \mathbb{N}$ and $\overline{X} := (X_1, \ldots, X_{k+1})$ be a set of sets with size $N$. Let $h_i$ be random variables distributed in $\mathcal{P}(X_i, X_{i+1})$ for $i = 1, \ldots, k$. Then, we define
\[ \mathcal{H}_{\overline{X}} := (h_1, \ldots, h_k). \]

$\mathcal{H}_{\overline{X}}$ is a sequence of random one-to-one functions which distribution could be arbitrary. In the following, we introduce the distributions we will use shortly.

**Definition 8.3 (Almost-uniform Shuffling).** Let $k, N, p' \in \mathbb{N}$ and $0 < \delta < 1$. Let $\overline{X} = (X_1, \ldots, X_{k+1})$ be a set of sets with size $N$. Consider $\mathcal{H}_{\overline{X}} = (h_1, \ldots, h_k)$.

- $\mathcal{H}_{\overline{X}}$ is $(1+\delta)$-uniform if for all subset of paths $P = (\overline{P}_1, \ldots, \overline{P}_m)$ where
  \[ \overline{P}_i = (x_{i,1}, \ldots, x_{i,k+1}) \]
  satisfying $h_j(x_{i,j}) = x_{i,j+1}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, k$,
  \[ \Pr[P \text{ is in } \mathcal{H}_{\overline{X}}] \leq (1+\delta)^m \left( \frac{(N-m)!}{N!} \right)^k. \]

- $\mathcal{H}_{\overline{X}}$ is $(p', 1+\delta)$-uniform if there exist a set of paths $P'$ with size at most $p'$ such that $\mathcal{H}_{\overline{X}}|P'$ is $(1+\delta)$-uniform, i.e., let $P'$ be fixed, then for all subset of unfixed paths $P = (\overline{P}_1, \ldots, \overline{P}_m)$,
  \[ \Pr[P \text{ is in } \mathcal{H}_{\overline{X}}|P' \text{ is in } \mathcal{H}_{\overline{X}}] \leq (1+\delta)^m \left( \frac{N-m-p'}{N-p'} \right)! \left( \frac{N-m-p'}{(N-p)!} \right)^k. \]

A convex combination of $(p', 1+\delta)$-uniform shuffling functions can be defined by the following formula:
\[ \mathcal{H} := \sum_{t=1}^{T} p_t \mathcal{H}_t, \]

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where $\mathcal{H}_1, \ldots, \mathcal{H}_T$ are $(p', 1+\delta)$-uniform shuffling functions and $p_1, \ldots, p_T$ are the probabilities for how much each $\mathcal{H}_t$ contributes to $\mathcal{H}$. In out context, we will only consider convex combination of finitely objects, i.e., $T$ is finite. We will show in Claim 8.4 that $\mathcal{F}$ conditioned on some classical advice $\bar{a}$ is close to a convex combination of finitely many $(p', 1+\delta)$-uniform shufflings.

**Claim 8.4.** Let $0 < \gamma, \delta < 1$. Let $p, k, N \in \mathbb{N}$. Let $\bar{X} = (X_1, \ldots, X_{k+1})$ be a set of sets with size $N$. Let $\mathcal{H}_{\bar{X}} = (h_1, \ldots, h_k)$ be distributed uniformly. Let $P$ be a set of fixed paths on $h_1, \ldots, h_k$ and $\bar{a}$ be a $p$-bit advice. Let $\mathcal{H}_{\bar{X}}(P, \bar{a})$ be $\mathcal{H}_{\bar{X}}$ conditioned on $(P, \bar{a})$. Then, there exists a convex combination $\mathcal{H}(P, \bar{a})$ of $(p', 1+\delta)$-uniform shufflings such that

$$\mathcal{H}_{\bar{X}}(P, \bar{a}) = \mathcal{H}_{\bar{X}}(P, \bar{a}) + \gamma' \mathcal{H}'.$$

where $p' \leq \frac{p+\log(1/\gamma)}{\log(1+\delta)} + |P|$, $\gamma' \leq \gamma$ and $\mathcal{H}'$ is an arbitrary random shuffling.

**Proof.** We let $\mathcal{H}_{\bar{X}}(P, \bar{a})$ be the random variable conditioned on $P$ and $\bar{a}$. It is obvious that conditioned on $P$, $\mathcal{H}_{\bar{X}}$ is uniform on the rest. We let $P_1$ be the maximal set of paths satisfying that

$$\Pr[P_1 \text{ in } \mathcal{H}_{\bar{X}}(P, \bar{a})] \geq (1+\delta)^{|P_1|} \left( \frac{(N - |P_1| - |P|)!}{(N - |P|)!} \right)^d.$$

Conditioned on $P_1$, we show that $\mathcal{H}_{\bar{X}}$ is $(1+\delta)$-uniform by contradiction. Suppose there exists another set of paths $P'$ such that

$$\Pr[P' \text{ in } \mathcal{H}_{\bar{X}}(P_1, P, \bar{a})] \geq (1+\delta)^{|P'|} \left( \frac{(N - |P_1| - |P'| - |P|)!}{(N - |P|)!} \right)^d.$$

Then,

$$\Pr[P_1 \cup P' \text{ in } \mathcal{H}_{\bar{X}}(P, \bar{a})] \geq (1+\delta)^{|P'| + |P_1|} \left( \frac{(N - |P_1| - |P'| - |P|)!}{(N - |P|)!} \right)^d$$

which contradicts the maximallity of $P_1$. This proves that conditioned on $P_1$, $\mathcal{H}_{\bar{X}}(P, \bar{a})$ is $1+\delta$-uniform.

The size of $P_1$ is bounded as follows: Since $\bar{a}$ is a $p$-bit advice,

$$\Pr[P_1 \text{ in } \mathcal{H}_{\bar{X}}(P, \bar{a})] \leq 2^p \left( \frac{(N - |P| - |P_1|)!}{(N - |P|)!} \right)^d$$

which implies that $|P_1| \leq \frac{p}{\log(1+\delta)}$.

Now, we can decompose $\mathcal{H}_{\bar{X}}(P, \bar{a})$ as

$$\Pr[P_1 \text{ in } \mathcal{H}_{\bar{X}}(P, \bar{a}) \cdot \mathcal{H}_{\bar{X}}(P, P_1, \bar{a}) + \Pr[\neg P_1 \text{ in } \mathcal{H}_{\bar{X}}(P, \bar{a}) \cdot \mathcal{H}_{\bar{X}}(P, \neg P_1, \bar{a})].$$

Note that $\mathcal{H}_{\bar{X}}(P, \neg P_1, \bar{a})$ may not be $(p', 1+\delta)$-uniform. In case that it is not and

$$\Pr[\neg P_1 \text{ in } \mathcal{H}_{\bar{X}}(P, \bar{a})] \geq \gamma,$$

we keep decomposing $\mathcal{H}_{\bar{X}}(P, \neg P_1, \bar{a})$.

We then find another maximal set of paths $P_2$ satisfying that

$$\Pr[P_1 \text{ in } \mathcal{H}_{\bar{X}}(P, \neg P_1, \bar{a})] \geq (1+\delta)^{|P_2|} \left( \frac{(N - |P_2| - |P|)! - |P_1|}{(N - |P|)!} \right)^d.$$
It is obvious that conditioned on $P_2$, $H_{\vec{X}}|(P, -P_1, \vec{a})$ is $(1+\delta)$-uniform following the same calculation. Furthermore, the size of $P_2$ can be bounded by the inequality
\[
(1 + \delta)|P_2| \left( \frac{(N - |P_2| - |P|)! - |P_1|}{(N - |P|)! - |P_1|} \right)^d \leq 2\gamma \cdot \left( \frac{(N - |P_2| - |P|)! - |P_1|}{(N - |P|)! - |P_1|} \right)^d.
\]
This implies that $|P_2| \leq \frac{p + \log(1/\gamma)}{\log(1+\delta)}$.

We recursively decompose $H_{\vec{X}}|(P, \vec{a})$ until the probability that the rest is less than $\gamma$. Then, $H_{\vec{X}}|(\vec{a}, P) = \sum_{i=1}^q \Pr[P_i|\neg P_1, \ldots, \neg P_{i-1}] \cdot H_{\vec{X}}|(P, -P_1, \ldots, -P_{i-1}, P_i, \vec{a})$.

An algorithm $A$ which has access to a convex combination of shuffling functions, e.g., $H := \sum_t p_t H_t$ can be represented as
\[
A^H = \sum_t p_t A^H_t.
\]

In the following, we show that if the shuffling is $(p', 1 - \delta)$-uniform, the probability to find the hidden sets in the shuffling is still bounded as we need for Lemma 5.8.

**Claim 8.5.** Let $p, k, N, N' \in \mathbb{N}$ and $0 < \delta < 1$. Let $\vec{X} = (X_1, \ldots, X_{k+1})$ be a set of sets with each size $N$. Let $H_{\vec{X}} = (h_1, \ldots, h_k)$ be $(p, 1 + \delta)$-uniform as defined in Def. 8.3 and $P$ be the set of $p$ paths fixed in $H_{\vec{X}}$. We choose a $N'$-element set $Y_1$ uniformly randomly from $X_1$, and let $Y_i := h(Y_{i-1})$ for $i = 2, \ldots, k$. Then, for $j \in [k]$, for $x_j \in X_j$,
\[
\Pr[x_j \in Y_j] \leq (1 + \delta) \cdot \frac{N'}{N - p}.
\]

**Proof.** It is obvious that $\Pr[x_1 \in Y_1] = \frac{N'}{N - p}$ since $Y_1$ is chosen randomly uniformly from $X_1$. For $i = 2, \ldots, k$, $\Pr[x_i \in Y_i]$ can be calculated as follows:
\[
\Pr[x_i \in Y_i] = \Pr[ \bigcup_{Y_1 \subset X_1} (x_i \in (h_{i-1} \circ \cdots \circ h_1(Y_1)) \land (Y_1 \text{ is chosen})] \\
\leq \sum_{Y_1 \subset X_1} \Pr[Y_1 \text{ is chosen}] \cdot \Pr[x_i \in (h_{i-1} \circ \cdots \circ h_1(Y_1))|(Y_1 \text{ is chosen})] \\
\leq \sum_{Y_1 \subset X_1} \Pr[Y_1 \text{ is chosen}] \sum_{y \in Y_1} \Pr[h_{i-1} \circ \cdots \circ h_1(y) = x_i|(Y_1 \text{ is chosen})] \\
\leq (1 + \delta) \frac{N'}{N - p}.
\]
The first two inequalities follow from the union bound, and the last inequality follows from the fact that $H_{\vec{X}}$ is $(p, 1 + \delta)$-uniform.

\[\square\]
8.2 Proof of Theorem 8.1

Following the similar idea in previous sections, we want to show that there exists a sequence of shadows which is indistinguishable from $F$. However, to prove Theorem 8.1, we actually show that there exist a “convex combinatio” of finitely many shadows, which are indistinguishable from $F$.

Specifically, we show that there exist a convex combination $\sum_{t=1}^{T} p_t (G_t^{(1)}, \ldots, G_t^{(m)})$ such that

$$|\Pr[L_m^F \circ \cdots \circ L_1^F () = s] - \sum_{t=1}^{T} p_t \cdot \Pr[(L_m^{G_t^{(m)}} \circ \cdots \circ L_1^{G_t^{(1)}}) () = s]| \leq md \cdot \sqrt{\frac{\text{poly}(n)}{2n}}.$$  

We will give the details of the shadows later in Lemma 8.9.

We denote a convex combination of bit strings as $ar{z} := \sum_{t=1}^{T} p_t \bar{z}_t,$ where $\bar{z}_1, \ldots, \bar{z}_T$ are bit string, $p_1, \ldots, p_T$ are the probability that $\bar{z}_t$ is sampled, and $T$ is finite in our context.

Let $f$ be a random Simon’s function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^n$. Let $F$ be the random $(d, f)$-shuffling of $f$. In this section, $s$ will always be in the form $(P, \bar{a})$, where $P$ is a set of paths in $F$, and $\bar{a}$ is some bit string correlated to $F$. For example, $\bar{a}$ could be the statement “$f(0) \oplus f(1) = 1$” and so on.

We say $\bar{a}$ is uncorrelated to $f$ conditioned on $P$ if the procedure producing $\bar{a}$ will not change the output when all mappings in $f$ except for mappings in $P$ are erased by $\perp$.

In the following, we define two kinds of advice, which are ideal and semi-ideal for our analysis.

**Definition 8.6 (Ideal advice).** $(P, \bar{a})$ is ideal if

- $P$ does not have a collision in $f$,
- $|P| = \text{poly}(n)$,
- and the bit string $\bar{a}$ is uncorrelated to $f$ conditioned on $P$.

**Definition 8.7 (Semi-ideal advice).** $(P, \bar{a})$ is semi-ideal if

- $|P| = \text{poly}(n)$,
- and the bit string $\bar{a}$ is uncorrelated to $f$ conditioned on $P$.

Note that the only difference between ideal and semi-ideal advice is that the paths fixed in an ideal advice is promised to have no collision to reveal $s$, while the paths in a semi-ideal advice may have a collision.

**Claim 8.8.** Let $A$ be any algorithm. Let $\bar{s} := (P, \bar{a})$ be an ideal advice of $F$. Then,

$$\Pr[A(\bar{s}) = s] \leq \text{poly}(n)/2^n.$$  

**Proof.** If $\bar{s}$ is ideal, then there is no collision in $P$, and $\bar{a}$ is uncorrelated to $f$ conditioned on $P$. Given $(P, \bar{a})$, $A$ cannot distinguish whether $f$ is a one-to-one function or a Simon function. Therefore,

$$\Pr[A(\bar{s}) = s] \leq \frac{|P| + 1}{2^n} \leq \frac{\text{poly}(n)}{2^n}.$$  

\[ \square \]
We consider
\[ \mathcal{B} := \Pi_{0/1} \circ U_{d+1} \mathcal{F} \ldots \mathcal{F} U_1 \circ \mathcal{A}_c, \] (10)
where \( U_1, \ldots, U_d \) are single depth quantum circuit and \( \mathcal{A}_c \) is a PPT algorithm. As we have mentioned earlier, the output of \( \mathcal{B}^\mathcal{F} \) can be represented as \((P, \tilde{a})\), where \( P \) is a set of paths fixed by \( \mathcal{A}_c \) and \( \tilde{a} \) is corresponding to the measurement outcome of the quantum circuit.

**Lemma 8.9.** Let \( \mathcal{F} \) be a \((d, f)\)-shuffling sampled from \( \mathcal{D}_{f,d} \). Let \( \tilde{s} := \sum_{t=1}^{T} p^{(t)} \tilde{s}^{(u)} \) be a convex combination of semi-ideal advice strings. For any \( \mathcal{B} \) in Eq. (10), there exist \( \{G_{\tilde{s}^{(1)}}, \ldots, G_{\tilde{s}^{(T)}}\} \), which are convex combinations of sequences of shadows corresponding to \( \tilde{s}^{(1)}, \ldots, \tilde{s}^{(T)} \) such that for all bit string \( \tilde{s}' \), for \( u \in [T] \),
\[ |\Pr[\mathcal{B}^\mathcal{F}(\tilde{s}^{(u)}) = \tilde{s}'] - \Pr[\mathcal{B}^\mathcal{G}_{\tilde{s}^{(u)}}(\tilde{s}^{(u)}) = \tilde{s}']| \leq d \cdot \sqrt{\frac{\text{poly}(n)}{2^n}}, \]
and the output of \( \mathcal{B}^\mathcal{G}_{\tilde{s}^{(u)}}(\tilde{s}^{(u)}) \) is semi-ideal. Moreover, if \( \tilde{s} \) is ideal, then the output of \( \mathcal{B}^\mathcal{G}_{\tilde{s}^{(u)}}(\tilde{s}^{(u)}) \) is also ideal with probability at least \( 1 - \frac{\text{poly}(n)}{2^n} \).

**Proof of Lemma 8.9.** For the ease of the analysis, we allow the classical algorithm \( \mathcal{A}_c \) to make path queries. We prove the lemma by math induction on the quantum circuit depth.

Given \( \tilde{s} := (P, \tilde{a}) \) which is semi-ideal, we let \( P_0 \) be \( P \) and the set of paths queried by \( \mathcal{A}_c \), let \( \tilde{s}^{(0)} := (P_0, \tilde{a}) \), and let \( p := |\tilde{a}| \). Note that the main difficulty which fails the previous proofs is that \( \mathcal{F} \) may not be uniform conditioned on \( \tilde{a} \). By applying Lemma 8.4, \( \mathcal{F}((\tilde{s}^{(0)})) \) is \( \gamma \)-close to a convex combination of \((p', 1+\delta)\)-uniform shuffling functions
\[ \mathcal{F}|(\tilde{s}^{(0)})) = H_{\tilde{s}^{(0)}}|(\tilde{s}^{(0)}) + \gamma' \mathcal{H}', \]
where \( H_{\tilde{s}^{(0)}}|(\tilde{s}^{(0)}) \) is a convex combination of \((p', 1+\delta)\)-uniform shuffling functions, \( \mathcal{H}' \) is an arbitrary random shuffling, and \( \gamma' \leq \gamma \). According to claim 8.4, \( p' \leq \frac{p + \log(1/\gamma)}{\log(1+\delta)} + |P_0| \). We will set the parameters \( p' \), \( \gamma' \), and \( \delta \) shortly.

We let
\[ H_{\tilde{s}^{(0)}}|(\tilde{s}^{(0)}) := \sum_{t_1=1}^{T_1} p_{t_1} \cdot \mathcal{H}_{t_1}^{(1)}, \]
and \( \rho^{(0)} \) is the initial state. Then,
\[ \mathcal{F} U_1(\rho^{(0)}, \tilde{s}^{(0)}) = \sum_{t_1=1}^{T_1} p_{t_1} \cdot \mathcal{H}_{t_1}^{(1)} U_1(\rho^{(0)}, \tilde{s}^{(0)}) + \gamma' \mathcal{H}' U_1(\rho^{(0)}, \tilde{s}^{(0)}), \] (11)
where $\mathcal{H}_1^{(1)}$, \ldots, $\mathcal{H}_{T_1}^{(1)}$ are ($p', 1 + \delta$)-uniform shuffling functions. Moreover, since $\bar{a}$ is uncorrelated to $f$ conditioned on $P_0$, the additional paths fixed in $\mathcal{H}_t^{(1)}$ is uncorrelated to $f$ given $P_0$.

Eq. 11 implies that for all $\bar{z}$

$$
| \Pr[\Pi_{j_0/1} \circ U_{d+1} \mathcal{F} \cdots \mathcal{F} U_1(\rho^{(0)}, \bar{s}^{(0)}) = \bar{z}] 
- \Pr[\Pi_{j_0/1} \circ U_{d+1} \mathcal{F} \cdots \mathcal{F} U_2\left(\mathcal{H}_{S_1^{(0)}} (\bar{s}^{(0)}) \right) U_1(\rho^{(0)}, \bar{s}^{(0)}) = \bar{z}] | \leq \gamma,
$$

where

$$
U_{d+1} \mathcal{F} \cdots \mathcal{F} U_2\left(\mathcal{H}_{S_1^{(0)}} (\bar{s}^{(0)}) \right) U_1(\rho^{(0)}, \bar{s}^{(0)}) = \sum_{t_1=1}^{T_1} p_{t_1} \cdot U_{d+1} \mathcal{F} \cdots \mathcal{F} U_2\left(\mathcal{H}_{t_1}^{(1)} \right) U_1(\rho^{(0)}, \bar{s}^{(0)}).
$$

Then, we construct shadows for each $\mathcal{H}_t^{(1)}$ as follows: Let $P_{t_1}^{(1)}$ be $P_0$ and the set of paths $\mathcal{H}_t^{(1)}$ is fixed on in addition to $P_0$. We construct the hidden set $S_{t_1}^{(1)}$ based on $\mathcal{H}_t^{(1)}$, $P_{t_1}^{(1)}$, and $S_0^{(0)}$ as in Procedure 3. Let $G_{t_1}^{(1)}$ be the shadow of $\mathcal{H}_{t_1}^{(1)}$ in $S_{t_1}^{(1)}$. Then,

$$
B(\mathcal{H}_{t_1}^{(1)} U_1(\rho^{(0)}, \bar{s}^{(0)}), G_{t_1}^{(1)} U_1(\rho^{(0)}, \bar{s}^{(0)}))
\leq \sqrt{2 \Pr[\text{find } S_{t_1}^{(1)} : \mathcal{H}_{t_1}^{(1)} S_{t_1}^{(1)}, \rho^{(0)}]}
\leq \sqrt{(1 + \delta) \frac{2q_1}{2n}}
$$

(13)

where $q_1$ is the number of queries $U_1$ performs. The first inequality follows from Lemma 5.7 and the last inequality follows from Claim 8.5 and Lemma 5.8. The output of $G_{t_1}^{(1)} U_1(\rho^{(0)}, \bar{s}^{(0)})$ is uncorrelated to $\mathcal{F}$ in $S_{t_1}^{(1)}$ by following the definition of $G_{t_1}^{(1)}$.

**Procedure 3** The hidden sets for $d$-CQ scheme

Given $j \in \mathbb{N}$, $S_{j}^{(j-1)} := (S_{j-1}^{(j-1)}, \ldots, S_d^{(j-1)})$, $\mathcal{H}_{k,j} = (h_j, \ldots, h_d)$, and $P$ a set of fixed paths.

1. Let $S_j^{(j-1)}$ be $S_j^{(j-1)}$ except for elements on $P$.

2. Let $S_j^{(j)}$ be a subset chosen uniformly at random with the promise that $|S_j^{(j)}|/|S_j^{(j-1)}| = 1/2^k$, and $S_j^{(j)}$ includes all elements in $S_j$ except for elements on $P$.

3. For $\ell = j + 1, \ldots, d$, let $S_{\ell}^{(j)} := \{h_{\ell-1} \circ \cdots \circ h_j(S_j^{(j)})\}$.

4. We let $S^{(j)} = (S_j^{(j)}, \ldots, S_d^{(j)})$. 

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By combining Eq. 12 and Eq. 13 we have proven that for all \( \bar{z} \)

\[
| \Pr[\Pi_{0/1} \circ U_{d+1} \cdots \mathcal{F} U_1(\rho^{(0)}, \tilde{s}^{(0)}) = \bar{z}] 
- \sum_{t_1=1}^{T_1} p_{t_1} \cdot \Pr[\Pi_{0/1} \circ U_{d+1} \cdots \mathcal{F} U_2 \mathcal{G}^{(1)}_{t_1} \mathcal{U}_1(\rho^{(0)}, \tilde{s}^{(0)}) = \bar{z}] |
\leq \gamma + \sqrt{(1 + \delta) \frac{q_1}{2^n}},
\]

The output state of \( \sum_{t_1=1}^{T_1} p_{t_1} \cdot \mathcal{G}^{(1)}_{t_1} \mathcal{U}_1(\rho^{(0)}, \tilde{s}^{(0)}) \) is

\[
\rho^{(1)} := \sum_{t_1=1}^{T_1} p_{t_1} \cdot \mathcal{G}^{(1)}_{t_1} \mathcal{U}_1(\rho^{(0)}, \tilde{s}^{(0)})
:= \sum_{t_1=1}^{T_1} p_{t_1} \cdot \rho_{t_1}^{(1)},
\]

and we let \( \tilde{s}^{(1)} := \sum_{t_1=1}^{T_1} p_{t_1} \tilde{s}_{t_1}^{(1)} \), where \( \tilde{s}_{t_1}^{(1)} := (\rho_{t_1}^{(1)}, \tilde{a}) \).

\( \tilde{s}_{t_1}^{(1)} \) must be semi-ideal. First, \( \tilde{a} \) is still uncorrelated to \( f \) conditioned on \( P_{t_1}^{(1)} \) because \( P \subseteq P_{t_1}^{(1)} \).

By Claim 8.4 \( p' \leq \frac{p' + \log 1/\gamma}{\log(1+\delta)} \). The size \( |P_{t_1}^{(1)}| = |P_0| + p' \) is at most \( \text{poly}(n) \) by setting \( \gamma = 1/\text{poly}(n) \) and \( \delta = O(1) \).

We then show that if \( \tilde{s} \) is ideal, \( \tilde{s}_{t_1}^{(1)} \) for \( t_1 \in [T_1] \) is ideal with high probability at least \( 1 - \frac{\text{poly}(n)}{2^n} \).

Note that \( \tilde{a} \) is uncorrelated to \( f \) conditioned on \( P \). This implies that for all \( \mathcal{H}_{t_1}^{(1)} \in \mathcal{H}_{S^{(0)}}^{(0)} |\tilde{s}^{(0)}, \mathcal{H}_{t_1}^{(1)} \) can only have at most \( p' \) paths be fixed in \( \mathcal{F} \), and these paths are uncorrelated to \( f \) conditioned on \( P_0 \). Since these paths are uncorrelated to \( f \) conditioned on \( P_0 \), the probability that \( P_{t_1}^{(1)} \) gives \( s \) is at most \( (|P_{t_1}^{(1)}| + 1)^2 \). Therefore, the probability that \( \tilde{s}^{(1)} \) is ideal is at least 

\[
1 - \frac{\text{poly}(n)}{2^n}.
\]

Then, we consider

\[
\Pi_{0/1} \circ U_{d+1} \mathcal{F} \cdots \mathcal{F} U_2(\rho^{(1)}, \tilde{s}^{(1)}).
\]

The formula above can be decomposed as

\[
\sum_{t_1=1}^{T_1} p_{t_1} \cdot \Pi_{0/1} \circ U_{d+1} \mathcal{F} \cdots \mathcal{F} U_2(\rho_{t_1}^{(1)}, \tilde{s}_{t_1}^{(1)}).
\]

Note that \( \rho_{t_1}^{(1)} \) is uncorrelated to mappings in \( S_{t_1}^{(1)} \) since \( \mathcal{G}^{(1)}_{t_1} \) has mappings in \( S_{t_1}^{(1)} \) be blocked. This implies that conditioned on \( \rho_{t_1}^{(1)} \), the mappings in \( S_{t_1}^{(1)} \) are still uniformly random. Therefore, for each input \( (\rho_{t_1}^{(1)}, \tilde{s}_{t_1}^{(1)}) \), we can apply the presampling argument in Claim 8.4 again and get the convex combination \( \mathcal{H}_{S_{t_1}^{(1)}} (|\tilde{s}_{t_1}^{(1)}|) \) satisfying that for all \( \bar{z} \),

\[
| \Pr[\Pi_{0/1} \circ U_{d+1} \cup U_d \cdots \mathcal{F} U_2(\rho_{t_1}^{(1)}, \tilde{s}_{t_1}^{(1)}) = \bar{z}] 
- \Pr[\Pi_{0/1} \circ U_{d+1} \mathcal{F} \cdots \mathcal{F} U_2(\mathcal{H}_{S_{t_1}^{(1)}} (|\tilde{s}_{t_1}^{(1)}|), \mathcal{U}_1(\rho_{t_1}^{(1)}, \tilde{s}_{t_1}^{(1)}) = \bar{z})] | \leq \gamma.
\] (14)
Here, we represent the convex combination as following formula

$$H_{S_{t_1}}((s^{(1)}_{t_1}) := \sum_{t_2=1}^{T_2} p_{t_1,t_2} H_{t_1,t_2}^{(2)}$$

and then construct the shadow $G_{t_1,t_2}^{(2)}$ and the hidden set $S_{t_1,t_2}^{(2)}$ for each $H_{t_1,t_2}^{(2)}$ according to Procedure 3. It satisfies that

$$B(H_{t_1,t_2}^{(2)} U_2(\rho_{t_1}^{(1)}, s^{(1)}_{t_1}), G_{t_1,t_2}^{(2)} U_2(\rho_{t_1}^{(1)}, s^{(1)}_{t_1})) \leq \sqrt{2 \Pr[\text{find } S_{t_1,t_2}^{(2)} : U_1^{H_{t_1,t_2}^{(2)} \setminus S_{t_1,t_2}^{(2)}}, \rho(0)]} \leq \sqrt{(1 + \delta) \frac{q_2}{2n}}, \quad (15)$$

where $q_2$ is the number of queries $U_2$ performs. We let $P_{t_1,t_2}^{(2)}$ be $P_{t_1}^{(1)}$ and the additional set of paths fixed in $H_{t_1,t_2}^{(2)}$. We let $\bar{s}_{t_1,t_2}^{(2)} := (\bar{a}, P_{t_1,t_2}^{(2)})$.

By Eq. 14 and Eq. 15 for all $\bar{z}$,

$$| \sum_{t_1=1}^{T_1} p_{t_1} \cdot \Pr[\Pi_{0/1} \circ U_{d+1} F \circ U_{d} \circ \cdots \circ U_2(\rho_{t_1}^{(1)}, s^{(1)}_{t_1}) = \bar{z}] - \sum_{t_1=1}^{T_1} \sum_{t_2=1}^{T_2} p_{t_1,t_2} \cdot \Pr[\Pi_{0/1} \circ U_{d+1} F \circ \cdots G_{t_1,t_2}^{(2)} U_2(\rho_{t_1}^{(1)}, s^{(1)}_{t_1}) = \bar{z}] | \leq \gamma + \sqrt{(1 + \delta) \frac{q_2}{2n}}.$$

Again, we let the output state of $\sum_{t_1=1}^{T_1} \sum_{t_2=1}^{T_2} p_{t_1,t_2} \cdot G_{t_1,t_2}^{(2)} U_2 G_{t_1}^{(1)} U_1$ be

$$\rho^{(2)} := \sum_{t_1=1}^{T_1} \sum_{t_2=1}^{T_2} p_{t_1,t_2} G_{t_1,t_2}^{(2)} U_2 G_{t_1}^{(1)} U_1(\rho^{(0)}, \bar{s}^{(0)})$$

$$:= \sum_{t_1=1}^{T_1} \sum_{t_2=1}^{T_2} p_{t_1,t_2} \rho_{t_1,t_2}^{(2)},$$

which satisfies that $\rho_{t_1,t_2}^{(2)}$ is uncorrelated the mappings in $S_{t_1,t_2}^{(2)}$. We let

$$\bar{s}^{(2)} := \sum_{t_1=1}^{T_1} \sum_{t_2=1}^{T_2} p_{t_1,t_2} \bar{s}_{t_1,t_2}^{(2)}.$$

Here, in case that $\bar{s}_{t_1}^{(1)}$ is ideal, $\bar{s}_{t_1,t_2}^{(2)}$ is also ideal with probability at least $1 - \frac{\text{poly}(n)}{2n}$ via the same analysis. Therefore, the probability that $\bar{s}_{t_1,t_2}^{(2)}$ is ideal is at least $1 - \frac{\text{poly}(n)}{2n}$. $\bar{s}_{t_1,t_2}^{(2)}$ must be semi-ideal since $\bar{s}_{t_1}^{(1)}$ is semi-ideal via the same analysis.
Now, we can suppose when the \( k \)-th parallel queries are applied, for all \( \bar{z} \), there exist \( \{G^{(k)}_{t_1, \ldots, t_k}\} \) and the corresponding hidden sets \( \{S^{(k)}_{t_1, \ldots, t_k}\} \) such that

\[
| \sum_{t_1, \ldots, t_{k-1}} p_{t_1} \cdots p_{t_{k-1}} \cdot \Pr[\Pi_{0/1} \circ U_{d+1} \cdots U_{k} (\rho^{(k-1)}_{t_1, \ldots, t_{k-1}}, \bar{s}^{(k-1)}_{t_1, \ldots, t_{k-1}}) = \bar{z}] - \sum_{t_1, \ldots, t_k} p_{t_1} \cdots p_{t_k} \Pr[\Pi_{0/1} \circ U_{d+1} \cdots G^{(k)}_{t_1, \ldots, t_k} U_{k} (\rho^{(k-1)}_{t_1, \ldots, t_{k-1}}, \bar{s}^{(k-1)}_{t_1, \ldots, t_{k-1}}) = \bar{z}] | 
\leq \gamma + \sqrt{(1 + \delta) \frac{q_k}{2^n}}.
\]

(16)

Here, \( \rho^{(k-1)}_{t_1, \ldots, t_{k-1}} \) is the output of \( G^{(k-1)}_{t_1, \ldots, t_{k-1}} U_{k-1} \cdots G^{(1)}_{t_1, t_k} U_1 (\rho^{(0)}, \bar{s}^{(0)}) \) and

\[
\bar{s}^{(k-1)}_{t_1, \ldots, t_{k-1}} := (\bar{a}, \bar{P}^{(k-1)}_{t_1, \ldots, t_{k-1}}),
\]

where \( \bar{s}^{(k-1)}_{t_1, \ldots, t_{k-1}} \) is ideal with probability at least \( 1 - \frac{\text{poly}(n)}{2^n} \), and \( \rho^{(k-1)}_{t_1, \ldots, t_{k-1}} \) is uncorrelated to mappings in \( S^{(k-1)}_{t_1, \ldots, t_{k-1}} \).

The output of the scheme with access to \( \{G^{(k)}_{t_1, \ldots, t_k}\} \) in Eq. 16 is

\[
\rho^{(k)} := \sum_{t_1, \ldots, t_k} p_{t_1} \cdots p_{t_k} \cdot \rho^{(k)}_{t_1, \ldots, t_k}
\]

which satisfies that \( \rho^{(k)}_{t_1, \ldots, t_k} \) is uncorrelated to the mappings in \( S^{(k)}_{t_1, \ldots, t_k} \). We let

\[
\bar{s}^{(k)} := \sum_{t_1, \ldots, t_k} (p_{t_1} \cdots p_{t_k} \bar{s}^{(k)}_{t_1, \ldots, t_k}),
\]

where \( \bar{s}^{(k)}_{t_1, \ldots, t_k} := \bar{P}^{(k)}_{t_1, \ldots, t_k}, \bar{a} \) is ideal with probability at least \( 1 - \frac{\text{poly}(n)}{2^n} \).

Consider the \( (k + 1) \)-th quantum parallel queries,

\[
\Pi_{0/1} \circ U_{d+1} \cdots U_{k+1} (\rho^{(k)}, \bar{s}^{(k)}) = \sum_{t_1, \ldots, t_k} p_{t_1} \cdots p_{t_k} \cdot \Pi_{0/1} \circ U_{d+1} \cdots U_{k} (\rho^{(k)}_{t_1, \ldots, t_k}, \bar{s}^{(k)}_{t_1, \ldots, t_k}).
\]

Following the fact that \( \rho^{(k)}_{t_1, \ldots, t_k} \) is only correlated to mappings out of \( S^{(k)}_{t_1, \ldots, t_k} \), we apply the presampling argument in Claim 8.4 and there exists a convex combination

\[
H^{(k+1)}_{\bar{s}^{(k)}_{t_1, \ldots, t_k}} \mid (\bar{s}^{(k)}_{t_1, \ldots, t_k}) := \sum_{t_{k+1}} p_{t_1, \ldots, t_{k+1}} \cdot H^{(k+1)}_{t_1, \ldots, t_{k+1}}
\]

such that for all \( \bar{z} \),

\[
| \sum_{t_1, \ldots, t_k} p_{t_1} \cdots p_{t_k} \cdot \Pr[\Pi_{0/1} \circ U_{d+1} \cdots U_{k+1} (\rho^{(k)}_{t_1, \ldots, t_k}, \bar{s}^{(k)}_{t_1, \ldots, t_k}) = \bar{z}] - \sum_{t_1, \ldots, t_{k+1}} p_{t_1} \cdots p_{t_k} \Pr[\Pi_{0/1} \circ U_{d+1} \cdots H^{(k+1)}_{t_1, \ldots, t_{k+1}} \circ U_{k+1} (\rho^{(k)}_{t_1, \ldots, t_k}, \bar{s}^{(k)}_{t_1, \ldots, t_k}) = \bar{z}] | 
\leq \gamma.
\]
We then construct $G_{t_1,\ldots, t_{k+1}}^{(k+1)}$ and $S_{t_1,\ldots, t_{k+1}}^{(k+1)}$ for each $H_{t_1,\ldots, t_{k+1}}^{(k+1)}$. Following the same proof, we can show that for all $\bar{z}$

$$
\left| \sum_{t_1,\ldots, t_k} p_{t_1} \cdots p_{t_{k+1}} \cdot \text{Pr}[\Pi_0/1 \circ U_{d+1} \mathcal{F} \cdots \mathcal{F} U_{k+1}(\rho_{t_1,\ldots, t_k}, \tilde{s}_{t_1,\ldots, t_k}) = \bar{z}] - \sum_{t_1,\ldots, t_{k+1}} p_{t_1} \cdots p_{t_{k+1}} \text{Pr}[\Pi_0/1 \circ U_{d+1} \mathcal{F} \cdots \mathcal{F} U_{k+1}(\rho_{t_1,\ldots, t_{k+1}}, \tilde{s}_{t_1,\ldots, t_{k+1}}^{(k+1)}) = \bar{z}] \right|
\leq \gamma + \sqrt{(1 + \delta) \frac{q_{k+1}}{2^n}}.
$$

Moreover, the output of the scheme with access to $\{G_{t_1,\ldots, t_{k+1}}^{(k+1)}\}$ is

$$\rho^{(k+1)} := \sum_{t_1,\ldots, t_{k+1}} p_{t_1} \cdots p_{t_{k+1}} \cdot \rho_{t_1,\ldots, t_{k+1}}^{(k+1)}$$

which satisfies that $\rho_{t_1,\ldots, t_{k+1}}^{(k+1)}$ is uncorrelated to the mappings in $S_{t_1,\ldots, t_{k+1}}^{(k+1)}$. We let

$$\tilde{s}^{(k+1)} := \sum_{t_1,\ldots, t_{k+1}} (p_{t_1} \cdots p_{t_{k+1}} \tilde{s}_{t_1,\ldots, t_{k+1}}^{(k+1)}),$$

where $\tilde{s}_{t_1,\ldots, t_{k+1}}^{(k+1)} := (P_{t_1,\ldots, t_{k+1}}, \tilde{a})$ is ideal with probability $1 - \frac{\text{poly}(n)}{2^n}$. Then, for all $\bar{z}$

$$
\left| \text{Pr}[\mathcal{B}^{\tilde{s}^{(u)}}(\tilde{s}^{(u)}) = \bar{z}] - \text{Pr}[\mathcal{B}^{G_{\tilde{s}^{(u)}}}(\tilde{s}^{(u)}) = \bar{z}] \right|
\leq \sum_{t_1,\ldots, t_d} p_{t_1} \cdots p_{t_d} \text{Pr}[\Pi_0/1 \circ U_{d+1} \mathcal{F} \cdots \mathcal{F} U_{1}(\rho^{(0)}, \tilde{s}^{(0)}) = \bar{z}]
\leq d \gamma + \sum_{i=1}^d \sqrt{(1 + \delta) \frac{q_i}{2^n}}.
$$

Eq. [17] follows from the hybrid argument and the indistinguishability we have just proven by math induction.

Finally, we need to show that the output of $\mathcal{B}^{G_{\tilde{s}^{(u)}}}(\tilde{s}^{(u)})$ is ideal with probability $1 - \frac{\text{poly}(n)}{2^n}$. The output can be represented by

$$\mathcal{B}^{G_{\tilde{s}^{(u)}}}(\tilde{s}^{(u)}) := \sum_{t_1,\ldots, t_d} p_{t_1} \cdots p_{t_d} \tilde{s}_{t_1,\ldots, t_d}^{(d)}$$

and

$$\tilde{s}_{t_1,\ldots, t_d}^{(d)} := (\tilde{a}', P_{t_1,\ldots, t_d}^{(d)}).$$

First, we show that $|P_{t_1,\ldots, t_d}^{(d)}|$ is at most a polynomial in $n$. The number of fixed paths by the sequence of shadows is at most $d \cdot \frac{p' + \log 1/\gamma}{\log 1 + \delta}$. We set $\gamma = 1/\text{poly}(n)$ and $\delta = O(1)$ such that $d \gamma + \sum_{i=1}^d \sqrt{(1 + \delta) \frac{q_i}{2^n}} \leq d \sqrt{\frac{\text{poly}(n)}{2^n}}$ and $|P_{t_1,\ldots, t_d}^{(d)}|$ is at most $\text{poly}(n)$. Moreover, since $\tilde{a}$ is uncorrelated to $f$ given $P$, the additional paths fixed in $G_{t_1,\ldots, t_d}^{(1)}$, $G_{t_1,\ldots, t_d}^{(d)}$ is also uncorrelated to $f$ given $P_0$. This implies that the probability that $P_{t_1,\ldots, t_d}^{(d)}$ gives the hidden shift $s$ is at most
we can follow Lemma 8.9, which also gives the same conclusion.

Proof. Following the proof for Theorem 8.1, there exist

\[ L_{t_1, \ldots, t_d}^{(d)} \]

and the output of

\[ L_{t_1, \ldots, t_d}^{(d)} \]

which is correlated to all

\[ F \]

This completes the proof.

\[ \mathrm{Claim 8.8} \]

\[ L_{t_1, \ldots, t_d} \]

Finally, by combining Eq. 18 and Eq. 19,

\[ \Pr[(L_m \circ \cdots \circ L_1)^{F}() = s] \leq (dm \sqrt{\frac{\text{poly}(n)}{2^n}} + \frac{\text{poly}(n)}{2^n}) \]

This completes the proof.

8.3 On separating the depth hierarchy of \( d \)-CQ scheme

By using the same proof for Theorem 8.1, we can show that the \( d \)-SSP is also hard for any \( d \)-CQ scheme.

\[ \textbf{Theorem 8.10.} \text{ The } d \text{-SSP cannot be decided by any } d \text{-CQ scheme with probability greater than } \frac{1}{2} + \sqrt{\frac{\text{poly}(n)}{2^n}}. \]

Proof. Following the proof for Theorem 8.1, there exist \( G^{(1)}, G^{(2)}, \ldots, G^{(m)} \) such that \( \mathcal{A} \) cannot distinguish \( F \) from \( G^{(1)}, G^{(2)}, \ldots, G^{(m)} \). Moreover, in case that \( f \) is a random Simon function, \( \mathcal{A} \) with access to \( G^{(1)}, \ldots, G^{(m)} \) cannot find \( s \). Therefore, \( \Pr[\mathcal{A}^F() = 1] \leq 1/2 + md \cdot \sqrt{\frac{\text{poly}(n)}{2^n}} \).
**Corollary 8.11.** For any \( d \in \mathbb{N} \), there is a \((2d + 1)\)-CQ scheme which can solve the \( d \)-SSP with high probability, but there is no \( d \)-CQ scheme which can solve the \( d \)-SSP.

*Proof.* This corollary follows from Theorem 8.10 and Theorem 4.11 directly. \(\square\)

Finally, we can conclude that

**Corollary 8.12.** Let \( O \) and \( \mathcal{L}(O) \) be defined as in Def. 4.10. \( \mathcal{L}(O) \in \text{BQP}^O \) and \( \mathcal{L}(O) \notin (\text{BPP}^{\text{BQNC}})^O \).

*Proof.* Note that each \( n \in \mathbb{N}, O_{\text{uni}}^{f_n, d(n)} \in O \) has depth equal to the input size. A quantum circuit with depth \( \text{poly}(n) \) can decide whether \( f_n \) is a Simon’s function by Theorem 4.11 and thus decides if \( 1^n \) is in \( \mathcal{L}(O) \). However, for \( d \)-CQ scheme which only has quantum depth \( d = \text{poly log } n \), it cannot decide the language by Theorem 8.10. \(\square\)

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