Iwasawa invariants for symmetric square representations

Anwesh Ray, R. Sujatha and Vinayak Vatsal*

Abstract

Let \( p \geq 5 \) be a prime, and \( p \) a prime of \( \overline{\mathbb{Q}} \) above \( p \). Let \( g_1 \) and \( g_2 \) be \( p \)-ordinary, \( p \)-distinguished and \( p \)-stabilized cuspidal newforms of nebentype characters \( \epsilon_1, \epsilon_2 \), respectively, and even weight \( k \geq 2 \), whose associated newforms have level prime to \( p \). Assume that the residual representations at \( p \) associated to \( g_1 \) and \( g_2 \) are absolutely irreducible and isomorphic. Then, the imprimitive \( p \)-adic \( L \)-functions associated with the symmetric square representations are shown to exhibit a congruence modulo \( p \).

Furthermore, the analytic and algebraic Iwasawa invariants associated to these representations of the \( g_i \) are shown to be related. Along the way, we give a complete proof of the integrality of the \( p \)-adic \( L \)-function, normalized with Hida’s canonical period. This fills a gap in the literature, since, despite the result being widely accepted, no complete proof seems to ever have been written down. On the algebraic side, we establish the corresponding congruence for Greenberg’s Selmer groups and verify that the Iwasawa main conjectures for the twisted symmetric square representations for \( g_1 \) and \( g_2 \) are compatible with the congruences.

Keywords: Iwasawa theory, Symmetric square representations, Congruences, \( p \)-adic \( L \)-functions

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1 Introduction

Let \( g_1 \) and \( g_2 \) be eigencuspforms of the same even weight \( k \geq 2 \) and levels \( M_1 \) and \( M_2 \), respectively, normalized so that \( a(1, g_i) = 1, i = 1, 2 \), where \( a(1, g_i) \) denotes the \( 1 \)-st Fourier coefficient of \( g_i \). Throughout, we fix a prime \( p \geq 5 \), and \( p \) a prime in the ring of integers \( \mathcal{O}_L \) of a suitably large number field \( L \) containing the Hecke eigenvalues of \( g_1 \) and \( g_2 \), as well as the values of the nebentype characters \( \epsilon_i \) of the \( g_i \). Let \( \mathcal{O} \) denote the completion of \( \mathcal{O}_L \) at \( p \) and write \( K \) for the fraction field of \( \mathcal{O} \). We also fix an isomorphism of \( \mathcal{C}_p \) with \( \mathbb{C} \), where \( \mathcal{C}_p \) is the completion of \( \overline{\mathbb{Q}} \) at a prime above \( p \). Assume that all but finitely many of the Hecke eigenvalues at primes \( \ell \) of the \( g_i \) are congruent modulo \( p \). Then, we say that the newforms \( g_i \) are \( p \)-congruent, or simply \( p \)-congruent, since \( p \) is fixed. In the situation of \( p \)-congruent newforms, there is a general philosophy that the critical values of any \( L \)-functions functorially associated to \( g_1 \) and \( g_2 \) will also be congruent, in a suitable sense. Furthermore, one expects the \( p \)-adic \( L \)-functions of \( g_1 \) and \( g_2 \), if they exist,
to also be congruent. Finally, the corresponding $p$-primary Selmer groups defined over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ should also be related.

One case where these expectations can be verified was given by Greenberg and the third author of the present work in [8], where they studied the main conjecture of Iwasawa theory for the standard two-dimensional representations associated to modular forms. In the two-dimensional case, the $p$-adic $L$-functions of ordinary $p$-congruent modular forms are the well-known $p$-adic $L$-functions arising from modular symbols; the Selmer groups are Greenberg’s $p$-ordinary Selmer groups; and the general expectation of the previous paragraph takes the following precise shape: the Iwasawa invariants of the Selmer groups and $p$-adic $L$-functions, associated to congruent Hecke eigencuspforms, are related by an explicit formula. The authors of [8] combine their formula with deep results of Kato [14] to deduce certain cases of the Main Conjecture of Iwasawa theory.

The primary goal of the present paper is to generalize the explicit relationship between the Iwasawa invariants of the congruent forms $g_1, g_2$ from the case of the standard representation of dimension 2 to the case of the symmetric square representation, which has dimension 3. A secondary accomplishment in this paper is the complete proof of the integrality for the $p$-adic $L$-functions of degree 3, since this result (although apparently known to experts) seems not to be found in the literature. Implicit in this discussion of integrality is a careful normalization of the periods appearing in the definition of the $p$-adic $L$-function. This is a subtle point: it turns out to be quite difficult to show that the normalization which gives rise to congruences coincides with the canonical normalization given by Hida. We show that Hida’s period gives the correct congruences if a certain variant of Ihara’s lemma holds. This lemma is unknown in weight $k > 2$ in the generality which we require, although cases of it are known weight 2. We remark also that all the results in this paper are much easier to prove in the case of weight 2 and the main novelty lies in the results for higher weight.

To state our results, we require some hypotheses and notation. We work basically under the same hypotheses as in Loeffler-Zerbes [15] and Schmidt [26], which we now describe. Let $a(n, g_i)$ denote the $n$-th Fourier coefficient of the eigenform $g_i$. In this paper, we make the following running assumptions.

- The forms $g_i$ are $p$-ordinary, i.e., that $a(p, g_i)$ is a $p$-adic unit for $i = 1, 2$;
- The $g_i$ are $p$-stabilized newforms, meaning that the level $M_i$ of $g_i$ is divisible by precisely the first power of $p$; that each $g_i$ is an eigenvector for the $p$-th Hecke operator $U_p$ (with unit eigenvalue at $p$), and that the level of the newform $g_{0,i}$ associated to $g_i$ has level $M_i$ or $M_i/p$.
- The nebentype characters $\epsilon_i$ are trivial on $(\mathbb{Z}/p\mathbb{Z})^*$, and
- the common two-dimensional Galois representation with values in $\text{GL}_2(\overline{\mathbb{F}}_p)$ associated to the $g_i$ and $p$ is irreducible and $p$-distinguished (in the sense of [32], page 481).

Remark 1.1 The cases above cover situations where the representations $\rho_{g_i}$ are either crystalline or semistable. We have excluded the case of potentially good reduction, namely ordinary eigenforms of level $M_0p^r$, $(M_0, p) = 1$, such the character $\epsilon_i$ of $g_i$ is nontrivial on $(\mathbb{Z}/p^r\mathbb{Z})^*$. The exclusion is only to keep the calculations simple, and the paper relatively short, as all the explicit formulae take on a different shape in the potentially good case.
In particular, there are additional Euler-like factors in the interpolation formula for the $p$-adic $L$-function (see [11], equation (0.2) on page 97), and the definition of the canonical integral period must also be adapted. Our method below applies to ordinary $p$-new forms, as well as to $\Lambda$-adic forms, but the calculations are repetitive, and do not add anything new beyond different cases of the formulae. Note that Loeffler-Zerbes [15] and Schmidt [26] do not treat the case of ordinary semistable reduction. Our treatment applies to this case without change.

**Remark 1.2** It is well known that the symmetric square of a CM form is reducible, and that many statements about the symmetric square that hold in the non-CM case are no longer true. However, we do not exclude the case that the $g_i$ are CM forms. We assume only that the Selmer groups defined below are cotorsion (which is known in the non-CM case [15] under some hypotheses, and also known in the CM case whenever it is expected to be true [16], [24]) and that the auxiliary characters $\psi$ under consideration are suitably restricted to avoid the problematic cases. The restriction on $\psi$ could be removed by following the analysis in [11], but we have not pursued this.

To continue, we require more notation. Let the $g_i$ be as above. Let $\mathbb{Q}_{\text{cyc}}$ denote the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, i.e., the unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$ contained in $\mathbb{Q}(\mu_{p^{\infty}})$, where $\mu_{p^{\infty}}$ denotes the $p$-power roots of unity. Let $\Lambda = \mathcal{O}[[\text{Gal}(\mathbb{Q}_{\text{cyc}}/\mathbb{Q})]]$ denote the usual Iwasawa algebra. The group $\text{Gal}(\mathbb{Q}(\mu_{p^{\infty}})/\mathbb{Q})$ is canonically identified with $\mathbb{Z}_p^\times$ via the action on the $p$-power roots of unity. For a prime $q \neq p$, we normalize our Frobenius elements so that $\text{Frob}(q)$ acts via $\zeta \mapsto \zeta^q$, where $\zeta$ is a $p$-power root of unity. Then we may write $\text{Frob}(q) \in \text{Gal}(\mathbb{Q}(\mu_{p^{\infty}})/\mathbb{Q}) \cong \mathbb{Z}_p^\times$ as

$$\text{Frob}(q) = q = \eta_1(q)q_w$$

where $\eta_1$ is the Teichmüller character taking values in the $(p-1)$-th roots of unity, and $q_w$ is the projection of $q$ to the wild part, namely projection to the pro-$p$ subgroup $1 + p\mathbb{Z}_p$.

The reason for the subscript 1 is to label characters of $\mathbb{Z}_p^\times$; any character $\eta$ is a product of a wild part $\eta_w$, and a tame part $\eta_t = \eta_1^t$. Eventually, we will use this labelling of characters to write the interpolation formula for the various power series giving rise to the $p$-adic $L$-functions.

If we fix a rank-2 Galois-stable $\mathcal{O}$-lattice $T_{g_i}$ in the $p$-adic representation associated to $g_i$ (normalized so that $\text{Frob}(q)$ has trace equal to the $q$-Hecke eigenvalue of $g_i$, for almost all $q$), we may view the representations $\rho_{g_i}$ as taking values in $\mathcal{O}$. For $i = 1, 2$, let

$$\rho_{g_i} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathcal{O})$$

be the associated Galois representation. The residual representation is denoted by $\bar{\rho}_{g_i} := \rho_{g_i} \mod p$. Since the modular forms $g_1$ and $g_2$ are $p$-congruent, and since we have assumed that $\bar{\rho}_{g_1}$ and $\bar{\rho}_{g_2}$ are absolutely irreducible, we find that $\bar{\rho}_{g_1}$ and $\bar{\rho}_{g_2}$ are isomorphic. We will simply write $\pi$ to denote the common residual representation. Since the residual representation is assumed to be irreducible, the lattices $T_{g_i}$ are unique up to homothety.

Let $\psi$ denote a Dirichlet character of conductor $c_\psi$, where $(c_\psi, p) = 1$. In this paper we will assume that

- $\psi$ is even;
- the reduction of $(\psi \epsilon_i)^2 \neq 1 \mod p$, for the characters $\epsilon_i$ of the $g_i$; and
- the coefficient field $L$ contains the values of $\psi$. 

We may view $\psi$ as a Galois character via $\psi(\text{Frob}(q)) = \psi(q)$. Conversely, we may view a Galois character as a Dirichlet character, simply by inverting this convention. In particular, the Teichmüller character $\eta_1$ mentioned above may be viewed as a Dirichlet character mod $p$, with values in $\mu_{p-1}$. The second condition above is inserted to rule out the problematic dihedral cases.

For each even integer $t$, we let $\psi_t$ denote the Dirichlet character $\psi\eta_t^1$. Let $r_{gi} = \text{Sym}^2(\mu_{gi})$ denote the symmetric square representation for $g_i$, with $i = 1, 2$, viewed as taking values in the symmetric square of the lattice $T_{gi}$. In this setting the representations $r_{gi} \otimes \psi : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_3(\mathcal{O})$ are residually isomorphic. Let $A_{t,gi}$ be the $p$-primary representation associated to the underlying Galois-stable $\mathcal{O}$-lattice for $r_{gi} \otimes \psi_t$. Note that since $g_i$ is $p$-ordinary, so is $r_{gi} \otimes \psi$. For each value of $t$ as above, we work with the primitive Selmer group $\text{Sel}_{p^\infty}(A_{t,gi}/\mathbb{Q}_{cyc})$ as defined by Greenberg in [7]. We are assuming that $\text{Sel}_{p^\infty}(A_{t,gi}/\mathbb{Q}_{cyc})$ is $\Lambda$-cotorsion, and this allows us to define a nonzero algebraic $p$-adic $L$-function $L^{\text{alg}}(r_{gi} \otimes \psi_t) \in \Lambda$ as a generator of the characteristic ideal of the Pontrjagin dual of the Selmer group. It is well-defined up to a unit factor in $\Lambda^X$. We also point out that the existence of $L^{\text{alg}}(r_{gi} \otimes \psi_t)$ and part of the main conjecture is proven, in some cases, in unpublished work of Urban [30]. The arguments in this paper do not assume the main conjecture. By the Weierstrass preparation theorem,

$$L^{\text{alg}}(r_{gi} \otimes \psi_t) = p^\mu a(T)u(T),$$

where $a(T)$ is a distinguished polynomial and $u(T)$ is a unit in $\Lambda$ (each depending on $i$ and $t$). The $\mu$-invariant $\mu(t_{gi} \otimes \psi_t)$ is the number $\mu$ in the above factorization, and the $\lambda$-invariant $\lambda(t_{gi} \otimes \psi_t)$ is the degree of $a(T)$.

Next, we define a primitive $p$-adic $L$-function $L^{\text{an}}(r_{gi} \otimes \psi_t)$, for $i = 1, 2$. This is essentially done in old work of Schmidt and others; see [26] for the basic source, and the discussion in [15] for an account of the various refinements. In this paper, we shall normalize and label our $p$-adic $L$-functions as in [7], to facilitate comparison with the Selmer groups defined there.

Under the present hypotheses, work of Schmidt, Hida, and Dabrowski-Delbourgo proves the existence of an element $L^{\text{an}}(r_{gi} \otimes \psi_t) \in \Lambda \otimes \mathbb{Q}$ satisfying a certain interpolation property with respect to special values of the complex symmetric square $L$-function. The interpolation property defining the $p$-adic $L$-function is given in formula (1.2) below.

It is important to remark that the definition of $L^{\text{an}}(r_{gi} \otimes \psi_t)$ presupposes the choice of a certain transcendental period. We discuss our normalization and our choice of period in further detail in Sect. 2 of this paper and summarize the key points later in this introduction (see equation (1.5)). According to our normalization, the main conjecture predicts that the two power series $L^{\text{alg}}(r_{gi} \otimes \psi_t)$ and $L^{\text{an}}(r_{gi} \otimes \psi_t)$ are equal up to a unit factor in $\Lambda$.

Once the integrality of the $p$-adic $L$-function is established, we have well-defined Iwasawa invariants $\lambda^{\text{an}}(r_{gi} \otimes \psi_t), \mu^{\text{an}}(r_{gi} \otimes \psi_t) \in \mathbb{Z}_{\geq 0}$ on the analytic side as well. The main conjecture implies that

$$\lambda^{\text{an}}(r_{gi} \otimes \psi_t) = \lambda^{\text{alg}}(r_{gi} \otimes \psi_t)$$

and

$$\mu^{\text{an}}(r_{gi} \otimes \psi_t) = \mu^{\text{alg}}(r_{gi} \otimes \psi_t)$$

for $i = 1, 2$. 


Our goal is to relate the invariants for the congruent forms \( g_1 \) and \( g_2 \). However, the primitive invariants as defined above are not equal; it is quite possible for the primitive invariants to be trivial in one case yet highly nontrivial in the other. The correct relationship, as discovered in [8], is between imprimitive Iwasawa invariants, which we now proceed to define. Thus, let \( S_0 \) denote any finite set of primes \( q \neq p \). For each \( i \), we have an imprimitive Selmer group, obtained by relaxing the local conditions at all the primes \( q \in S_0 \). It is shown in [8] that the imprimitive Selmer group is cotorsion if and only if the primitive Selmer group is so. Thus, we have imprimitive invariants \( \lambda_{S_0}^{\text{alg}}(r_{gi} \otimes \psi_t) \) and \( \mu_{S_0}^{\text{alg}}(r_{gi} \otimes \psi_t) \). For each prime \( q \), we can define an integer \( \sigma_i(q)(\psi_t) \) to be the degree of a certain polynomial coming from applying the Weierstrass preparation theorem to the annihilator of a local cohomology group at \( q \), which is known unconditionally to be torsion, and whose annihilator can be described explicitly in terms of Euler factors. Then the basic result is the following.

**Proposition 1.3** Assume that the Selmer groups \( \text{Sel}_{p^\infty}(A_i, \psi_t/\mathbb{Q}_{\text{cycl}}) \) are cotorsion as \( \Lambda \)-modules. The following statements hold:

- \( \mu_{S_0}^{\text{alg}}(r_{gi} \otimes \psi_t) = \mu_{S_0}^{\text{alg}}(r_{gi} \otimes \psi_t) \), and
- \( \lambda_{S_0}^{\text{alg}}(r_{gi} \otimes \psi_t) = \lambda_{S_0}^{\text{alg}}(r_{gi} \otimes \psi_t) + \sum_{q \in S_0} \sigma_i(q)(\psi_t) \).

Most of this proposition is carried through from [8], where an analysis of the local conditions and cohomology groups is made under very mild hypotheses.

Similar considerations apply on the analytic side. By virtue of a construction due to Coates, Hida, and Schmidt, there exists an imprimitive \( p \)-adic \( L \)-function \( \mathcal{L}^{\text{can}}_{S_0}(r_{gi} \otimes \psi_t) \in \Lambda \otimes \mathbb{Q} \). Once again, \( S_0 \) denotes a finite set of primes \( q \neq p \).

We want to study congruences and integrality, but the results of Hida, Coates, and Schmidt only produce elements of \( \Lambda \otimes \mathbb{Q} \). Thus, we have to show that the \( p \)-adic \( L \)-functions actually lie in \( \Lambda \). This integrality property is a folklore result for which a complete proof seems not to have ever been written down. It is stated as Proposition 2.3.5 in [15], where it is attributed to Hida, but no reference is given. Despite searching many papers of Hida, we were unable to find a complete proof, except for the sketch in [13], which is only complete in weight 2, and subject to various complicated restrictions. Therefore, our first task is to provide a through discussion of integrality, and give an integral construction valid for all weights. The proof turns out to be somewhat delicate, once the weight is large compared to \( p \). And, as we have remarked above, the choice of periods is important and must be specified precisely. For now, we simply record that Hida’s theory leads to a choice of period \( \Omega_{g_i}^{\text{can}} \) that is determined up to a \( p \)-adic unit, and that it is this choice that one would like to use. The precise definition of \( \Omega_{g_i}^{\text{can}} \) is given in equation (1.5).

To state the result, let us fix \( g = g_i, M = M_i, \epsilon = \epsilon_i \). In this theorem, \( \psi \) will denote an even character of conductor \( c_\psi \) coprime to \( p \) such that \((\psi \epsilon_i)^2 \neq 1 \mod p \), and \( S_0 \) is any finite set of primes \( q \neq p \). Recall that we have set \( \psi_t = \psi^t \eta_1 \), for any even \( t \).

**Theorem 1.4** Let the notation be as above. Then the primitive \( L \)-function \( \mathcal{L}^{\text{can}}_{S_0}(r_{gi} \otimes \psi_t) \in \Lambda \) is integral, normalized with the integral period \( \Omega_{g_i}^{\text{can}} \). The imprimitive \( p \)-adic \( L \)-function \( \mathcal{L}^{\text{can}}_{S_0}(r_{gi} \otimes \psi_t) \in \Lambda \) is integral as well, for the same choice of period.
With the integrality theorem in hand, our main result in Sect. 2 is the following. As stated here, the result is dependent on the validity of a certain variant of Ihara’s lemma (see Hypothesis 2.18). One can make an unconditional statement at the cost of introducing a certain ambiguity in the choice of periods, but the statement is somewhat clumsy, so we avoid it. Let \( \lambda_{S_0}^{an}(r_g \otimes \psi_t) \) and \( \mu_{S_0}^{an}(r_g \otimes \psi_t) \) denote the Iwasawa invariants of \( L_{S_0}^{an}(r_g \otimes \psi_t) \).

**Proposition 1.5** Let the notation and assumptions be as above. Assume further that Hypothesis 2.18 holds. Then the following statements hold.

- \( \mu_{S_0}^{an}(r_g \otimes \psi_t) = \mu^{an}(r_g \otimes \psi_t) \), and
- \( \lambda_{S_0}^{an}(r_g \otimes \psi_t) = \lambda^{an}(r_g \otimes \psi_t) + \sum_{q \in S_0} \sigma_{i}^{(q)}(\psi_t) \)

Here the integers \( \sigma_{i}^{(q)}(\psi_t) \) are the same as the ones occurring in the algebraic case, since, the Euler factors in the algebraic and analytic sides are exactly the same. This is a deep fact: the equality between the Galois-theoretic Euler factors in the algebraic case and the complex Euler factors in the analytic \( L \)-function is the local Langlands correspondence for the three-dimensional representations \( r_g \otimes \psi_t \); see [5], or [26].

With regard to the Iwasawa invariants for congruent forms, our result is as follows. Again, we include the Ihara Lemma as a hypothesis, since formulating a general result without it would give a clumsy statement. We remark however that the Ihara hypothesis may be proved to be true in weight 2. We will say more about this point in Sect. 2.19. Consider the pair \( g_1, g_2 \) of \( p \)-congruent forms, of level \( M_1, M_2 \), respectively. In the situation where we consider the congruences, we have to delete the Euler factors at all bad primes, just as in [8]. We also have to take care of the prime \( q = 2 \), to apply the results of Shimura. Thus, we let \( S \) denote any set of primes \( q \) containing \( q = 2 \) and all primes dividing \( M_1, M_2 \). Let \( S_0 = S \setminus \{p\} \).

**Theorem 1.6** Let the notation be as above. Assume Hypothesis 2.18, and that the Selmer groups \( \text{Sel}_{p}^{\infty}(A, \psi_t/\mathbb{Q}_{cyc}) \) are cotorsion as \( \Lambda \)-modules. Then the following statements hold.

1. If \( \mu_{S_0}^{an}(r_{g_1} \otimes \psi_t) = 0 \), we have \( \mu_{S_0}^{an}(r_{g_2} \otimes \psi_t) = 0 \), and \( \lambda_{S_0}^{an}(r_{g_1} \otimes \psi_t) = \lambda_{S_0}^{an}(r_{g_2} \otimes \psi_t) \).
2. If \( \mu_{S_0}^{al}(r_{g_1} \otimes \psi_t) = 0 \), we have \( \mu_{S_0}^{al}(r_{g_2} \otimes \psi_t) = 0 \), and \( \lambda_{S_0}^{al}(r_{g_1} \otimes \psi_t) = \lambda_{S_0}^{al}(r_{g_2} \otimes \psi_t) \).
3. If \( \mu_{S_0}^{an}(r_{g_1} \otimes \psi_t) = \mu_{S_0}^{al}(r_{g_1} \otimes \psi_t) = 0 \), and \( \lambda_{S_0}^{an}(r_{g_1} \otimes \psi_t) = \lambda_{S_0}^{al}(r_{g_1} \otimes \psi_t) \), then \( \mu_{S_0}^{an}(r_{g_2} \otimes \psi_t) = \mu_{S_0}^{al}(r_{g_2} \otimes \psi_t) = 0 \), and \( \lambda_{S_0}^{an}(r_{g_2} \otimes \psi_t) = \lambda_{S_0}^{al}(r_{g_2} \otimes \psi_t) \).

It is clear from the relationships given in Propositions 1.3 and 1.5, that the third statement follows from the first two. Furthermore, it follows from the third statement that if one knows the main conjecture and vanishing of the \( \mu \)-invariants for \( g_1 \), that the same conclusions follow for \( g_2 \). Examples where the main conjecture is known for a particular form may be found in [15].

**Remark 1.7** Our theorems come in two flavors—some (e.g., Theorem 1.4, and Proposition 1.5) pertain to properties of a single form \( g = g_1 \), whilst others (Theorem 1.6) pertain to two congruent forms \( g_1 \) and \( g_2 \). In the theorems dealing with a single form, the set \( S_0 \) is allowed to be an arbitrary set of primes \( q \neq p \), while in the case of congruences, we must assume that \( S_0 \) contains all primes dividing the levels \( M_1, M_2 \), together with the prime 2.
There is no simple description of the polynomials described above. The proof of the congruences of \( p \)-adic L-functions turns out to be more delicate than in the case of \([8]\) and relies crucially on knowledge of the possible non-minimal deformations of \( \mathfrak{p} \) (Lemma 2.26 below). Indeed, to get the required results, one has to redo the Coates–Hida–Schmidt construction of the \( p \)-adic L-function, which goes back more than 30 years, and apply various refinements which were not available at that time.

We start by writing down the Euler products and L-functions that are the subject of this paper. The story is a bit complicated, because our results concern the primitive and imprimitive automorphic L-functions of \( GL_3 \) but the proofs are concerned almost exclusively with certain related degree 3 Euler products defined by Shimura, which are almost—but not quite—the same. Since the distinction is important in understanding why the objects in the sketch of the proofs do not appear in the statements of the theorems, we spell out the definitions in this introduction.

For notational simplicity, assume that \( g = g_i \) is a \( p \)-stabilized newform for some fixed value \( i \in \{1, 2\} \) and set \( M := M_i \). Then \( M \) is divisible by precisely the first power of \( p \). If \( g = \sum a(n, g)q^n \), then the Dirichlet series \( \sum_n a(n, g)n^{-s} \) has an Euler product of the form

\[
\prod_q (1 - \alpha_q q^{-s})^{-1}(1 - \beta_q q^{-s})^{-1}
\]

with certain parameters \( \alpha_q, \beta_q \) at each prime \( q \) (including \( p \)). If \( q \) divides \( M \), then one or both of these parameters may be zero. Whether or not these factors may be succinctly described as follows.

Let \( T \cong \mathcal{O} \oplus \mathcal{O} \) denote a realization of the Galois representation \( \rho_g = \rho_{g_i} \), as in the previous discussion above. If \( q \neq p \) is a prime, we shall say that \( \rho_g \) is ordinary at \( q \) if the submodule \( T^{(q)} \) of invariants under an inertia group \( I_q \) has \( \mathcal{O} \)-rank 1. This terminology is taken from [6]. Analogously, we say that \( q \) is unramified if \( T^{(q)} = T \) has rank 2, and that \( T \) is depleted if \( T^{(q)} = 0 \). It is then known that precisely one of \( \alpha_q, \beta_q \) is non-zero when \( q \) is ordinary; that both are zero if \( q \) is depleted; and that both are non-zero when \( q \) is unramified. As for \( q = p \), we are in the \( p \)-stabilized case, so \( \alpha_p \) is a unit, and \( \beta_p = 0 \).

Now let \( \eta \) be an even finite order Dirichlet character of conductor \( c_\eta = p^\nu \), and set \( \chi = \psi \eta \). Thus, \( \chi \) is an even Dirichlet character of conductor \( c_\chi = p^{-\nu} \). We may write \( \eta = \eta_1 \nu_\eta \), where \( \eta_1 = \eta_1^t \), for the Teichmüller character \( \eta_1 \), and a character \( \nu_\eta \) of \( p \)-power order, so that \( \chi = \psi_\nu s \eta \). We will use this notation throughout the paper.

Let \( S_0 \) denote any set of prime numbers such that \( p \notin S_0 \). We allow that \( S_0 \) could be empty. Then the \( S_0 \)-imprimitive symmetric square L-function (twisted by \( \chi \)) is defined as follows:

\[
L_{S_0}(r_g \otimes \chi, s) = \prod_{q \notin S_0} P_q(q^{-s})^{-1}
\]

(1.1)

where \( P_q(X) \) is a certain polynomial with algebraic integer coefficients (see Sect. 2.11). There is no simple description of the polynomials \( P_q \), beyond the fact that \( P_q(q^{-s}) = (1 - \chi(q)\alpha_q\beta_q q^{-s})(1 - \chi(q)\beta_q q^{-s})(1 - \chi(q)\alpha_q^2 q^{-s}) \) for almost all \( q \), where \( \alpha_q \) and \( \beta_q \) are...
determined from the standard degree 2 Euler product given above. The exact form of the finitely many remaining factors \( P_q \) may be found in [26], Section 1; we do not need the prescription here. The primitive symmetric square L-function \( L(r_g \otimes \chi, s) \) corresponds to the case \( S = \emptyset \); it is the L-function corresponding to an automorphic representation of \( \text{GL}_3 \).

Next we want to define Shimura’s Euler products for the modular form \( g \) and a certain depletion \( f \) of \( g \). Given the \( p \)-stabilized modular form \( g \) of level \( M \), define

\[
D_g(\chi, s) = \prod_p \left( (1 - \chi(q) \alpha_q \beta_q q^{-s})(1 - \chi(q) \beta_q^2 q^{-s})(1 - \chi(q) \alpha_q^2 q^{-s}) \right)^{-1},
\]

where the product is taken over all primes \( q \). This function is written as \( D(s, g, \chi) \) by Shimura in [28]. We note that Shimura’s results don’t require that \( g \) be a newform; his results apply to any cuspform that is an eigenform for all the Hecke operators. It is not necessarily true that \( D_g(\chi, s) \) coincides with the automorphic L-function \( L(r_g \otimes \chi, s) \).

The analogue of the imprimitive L-function \( L_{S_0}(r_g \otimes \chi, s) \) is given by \( D_f(\chi, s) \), where \( f \) is the depletion of \( g \) at the set of primes in \( S_0 \), defined as follows. If \( g = \sum_n a(n, g) e^{2\pi i n z} \), then \( f = \sum_n a(n, g) e^{2\pi i n z} \) where the second sum is taken over integers \( n \) such that \( (n, q) = 1 \) for all \( q \in S_0 \). The modular form \( f \) is an eigenform of some level \( N \), and the standard L-function of \( f \) admits a degree 2 Euler product. We get parameters \( \alpha_q', \beta_q' \) associated to \( f \) just as before, so that \( \alpha_q = \alpha_q' \beta_q = \beta_q' \) if \( q \notin S_0 \), and \( \alpha_q = \beta_q = 0 \) if not. Then we can follow Shimura and define a degree three Euler product \( D_f(\chi, s) \) for \( f \) just as above, with \( \alpha_q', \beta_q' \) instead of \( \alpha_q, \beta_q \). Again, it is not necessarily true in general that \( D_f(\chi, s) = L_{S_0}(r_g \otimes \chi, s) \).

This fact does become true when \( S_0 \) is sufficiently large and (since \( p \notin S_0 \)) when \( \chi \) is ramified at \( p \). For example, it suffices to assume that \( S_0 \) includes all the primes \( q \neq p \) dividing the level \( M \) of \( g \) and the conductor \( c_\chi \) of \( \chi \).

With these preliminaries on complex L-functions in hand, we can now describe the \( p \)-adic L-functions we consider. Let \( g \) be a \( p \)-stabilized newform as above; let \( S_0 \) denote any sufficiently large finite set of primes \( q \neq p \), and let \( f \) denote the depletion of \( g \) at \( S_0 \).

Let \( \psi \) be a fixed even character of conductor relatively prime to \( p \). Set \( \chi = \psi \eta = \psi_1 \eta_w \), for \( \eta \) of \( p \)-power conductor. Then the imprimitive \( p \)-adic L-function \( \mathcal{L}^\text{an}_{S_0}(r_g \otimes \psi_1) \) is defined via interpolation of \( D_f(\psi_1 \eta_w, s) \) at critical values of \( s \), and \( \eta_w \) varying over cyclotomic characters of \( p \)-power order and \( S_0 \) is chosen sufficiently large to ensure that \( D_f(\chi, s) = L_{S_0}(r_g \otimes \chi, s) \). To write an interpolation formula, we view \( \mathcal{L}^\text{an}_{S_0}(r_g \otimes \psi_1) \) as an element of \( \Lambda = \mathcal{O}[[\text{Gal}(\mathbb{Q}_{\text{cycl}}/\mathbb{Q})]] \). Given a finite order character \( \eta_w \) of \( \text{Gal}(\mathbb{Q}_{\text{cycl}}/\mathbb{Q}) \simeq 1 + p\mathbb{Z}_p \), and a rational integer \( n \), let us write \( \eta_{w,n} \) for the character of \( \text{Gal}(\mathbb{Q}_{\text{cycl}}/\mathbb{Q}) \) defined by

\[
\eta_{w,n}(u) = \eta_w(u) \cdot u^{1-n}.
\]

Thus, we have

\[
\eta_{w,n}(\text{Frob}(q)) = (\eta_w \eta_1^{n-1})(\text{Frob}(q)) q^{1-n},
\]

for all primes \( q \neq p \).

The \( p \)-adic L-function is characterized (assuming it exists) by its specialization at any infinite set of characters. We consider characters of the form \( \eta_{w,n} \), where \( \eta_w \) is non-trivial, hence of conductor \( p^r, r > 1 \), and odd \( n \) in the range \( 1 \leq n < k \). Let

\[
E_p(n, \eta_w) = E_p(n, \psi, \eta_w) = (p^{n-1} \psi(p)^{-1} \alpha_p^{-2})^r.
\]

Here \( \alpha_p \) denotes the \( U_p \) eigenvalue of \( g \), as always. The number \( \psi(p) \) is nonzero because \( \psi \) is assumed to be unramified at \( p \). If \( \chi \) is any Dirichlet character, we let \( G(\chi) \) denote the
usual Gauss sum of $\chi$. Fix an even $t$ with $0 \leq t \leq p - 2$. Then the characterizing interpolation formula is stated as follows: there exists a complex period $\Omega_{g_0}^{\text{can}}$ associated to the newform $g_0$, determined up to a $p$-adic unit, and independent of $S_0$ and the corresponding depletion $f$ of $g_0$, such that

$$
\eta_{w,n}(L_{S_0}^\text{can}(r_g \otimes \psi_t)) = \Gamma(n)E_p(n, \eta_w) \cdot G(\eta_{1-n-t}^{-1}) \frac{L_{S_0}(\psi_{t+n-1}^{-1} \eta_w, n)}{\pi n \Omega_{g}^{\text{can}}}.
$$

(1.2)

**Remark 1.8** This formula is adapted from [15], Theorem 2.3.2, which deals with the primitive $L$-function in the case of good reduction. We have transposed the formula to the imprimitive and semistable case (the formula is the same), and suppressed various conventionsof[7], since we are working with Greenberg’s Selmer groups.

We remark also that the congruence ideal which appears in the statement of the Main Conjecture in Section 2 of [15] does not play a role for us, since our definition of the period will subsume this factor. Finally, we point out that in the convention of [15], $\Lambda$ refers to the completed group algebra of $\text{Gal} (\mathbb{Q}(\mu_{p^\infty}) / \mathbb{Q}) \cong \mathbb{Z}_p^\times$, whereas we have taken $\Lambda$ to be the group algebra of $\text{Gal} (\mathbb{Q}_{\text{cyc}} / \mathbb{Q}) \cong \mathbb{Z}_p \cong 1 + p\mathbb{Z}_p$, since the latter formulation is more convenient for congruences. However, the case of nontrivial even tame cyclotomic twists is incorporated by choosing a tame twist $\psi_t$, for each even $t$.

The term “imprimitive” is used in [15] to refer to $D_g(\chi, s)$. Our $D_f(\chi, s)$ is even less primitive than the already defective $D_g(\chi, s)$. The function $D_f(\chi, s)$ does not appear in [15].

For the present, we fix $S_0$ sufficiently large, and focus on $L_{S_0}(r_g \otimes \chi, s) = D_g(\chi, s)$, for $\chi = \psi \eta = \psi_t \eta_w$, with $\psi, \eta$ even, $\eta$ of $p$-power conductor, and $\psi$ of conductor prime to $p$, and describe how to prove congruences for the quantity appearing on the right hand side of (1.2). The starting point is Shimura’s formula expressing $D_g(\chi, s)$ in terms of the Petersson inner product of a theta series $\theta$ and an Eisenstein series $\Phi^\epsilon$:

$$
(4\pi)^{-s/2} \Gamma(s/2)D_g(\chi, s) = \int_{B_0(N_\chi)} f(z, \overline{\Theta}(z)\Phi^\epsilon(z, \overline{z}, s)) \, dxdy
$$

(1.3)

where $z = x + iy$ runs over a fundamental domain $B_0(N_\chi)$ for the group $\Gamma_0(N_\chi)$. Here $N_\chi$ is the least common multiple of $4, N$, and the conductor of $\chi$. The superscript $\epsilon$ on the Eisenstein series indicates a dependence on the character $\epsilon$ of the cusp form $f$. If $n$ is an odd integer in the range $1 \leq n \leq k - 1$, Shimura in [28] has shown that $\theta(\chi)\Phi^\epsilon(z, \overline{z}, n)$ is a nearly holomorphic modular form of level $N_\chi$, weight $k$, and character $\epsilon$. For the precise definitions of the theta function and Eisenstein series, see Sect. 2.4 below.

**Remark 1.9** The integral in Shimura’s formula is taken over a fundamental domain for $\Gamma_0(N_\chi)$. The modular forms $f$ and $\overline{\Theta}(z)\Phi(z, \overline{z}, s)$ both have nontrivial nebentype character, but the integrand as a whole is invariant under $\Gamma_0(N_\chi)$. 


Our first job is to prove that the values on the right hand side of Shimura’s formula are algebraic and integral, when divided by the canonical period. The algebraicity is well-known and was basically proven by Shimura himself: it follows from the fact that the Fourier coefficients of \( \theta(z) \Phi(z, \chi, n) \) can be calculated explicitly, and turn out to be algebraic. Then one has to replace the nearly holomorphic form \( \theta(z) \Phi(z, \chi, n) \) with its holomorphic projection \( \theta(z) \Phi(z, \chi, n)^{\text{hol}} \). This projection will have algebraic Fourier coefficients, so Shimura’s method shows that the Petersson product \( \langle f, \theta(z) \Phi(z, \chi, n)^{\text{hol}} \rangle \) is equal to the Petersson inner product of \( f \) with itself, up to an algebraic factor. This procedure is elaborated by Schmidt in [25] and [26], in the case that \( S_0 \) is the empty set, so that \( f = g \) and \( N = M \). His work leads to the construction of the \( p \)-adic L-functions described above. The period that emerges is simply the Petersson inner product of the \( p \)-stabilized newform \( g \) with itself.

Once one has a \( p \)-adic L-function for \( S_0 = \emptyset \), it is a straightforward matter to obtain the \( p \)-adic L-function for general \( S_0 \): one has only to multiply by the finitely many Euler factors. However, this approach is useless for studying congruences, since one has no way to compare the constructions for forms of different minimal levels. One could simply multiply each one by an arbitrary unit factor, and the resulting objects would not be congruent. Thus, one has to construct both the L-functions simultaneously, at some fixed common level where suitably depleted forms co-exist. In other words, one has to verify that Schmidt’s construction works for \( D_f(\chi, s) \), when \( S_0 \neq \emptyset \); this is carried out in Section 2 below. Here, the problem is that we have to work with the depleted form \( f \), rather than the stabilized newform \( g \), so multiplicity one is not automatic. Furthermore, we have to make sure our construction works over an integer ring, rather over a field.

We describe the solution to the integrality problem first. The main difficulty is that the holomorphic projection in Shimura and Schmidt introduces denominators dividing \( k! \). To get around this, we have to change tactics, and use methods from \( p \)-adic modular forms—we replace the holomorphic projection with the ordinary projection, which is denominator-free. This is enough for our purposes, since we are dealing with \( f \) ordinary.

Furthermore, one has to give an integral definition for the periods, rather than just the Petersson product. The key idea (due to Hida) is that the Petersson inner product is related to a certain algebraic, and even integral, inner product, up to canonical scalar multiple. Let \( M \) denote the level of the \( p \)-stabilized newform \( g \). Let \( S_k(M, \mathcal{O}) \) be the space of cusp forms of weight \( k \) on the group \( \Gamma_1(M) \), with coefficients in \( \mathcal{O} \), and let \( T \) be the ring generated by Hecke operators acting on \( S_k(M, \mathcal{O}) \). Let \( P_f \) be the kernel of the map \( T \rightarrow \mathcal{O} \) associated to \( f \) and \( m \) the unique maximal ideal of \( T \) generated by \( P_f \) and \( p \). We have assumed that the residual representation associated to \( f \) is absolutely irreducible, ordinary, and \( p \)-distinguished, so it follows that \( T_m \) is Gorenstein. This induces an algebraic and integral duality pairing

\[
(\cdot, \cdot)_N : S_k(M, \mathcal{O})_m \times S_k(M, \mathcal{O})_m \rightarrow \mathcal{O},
\]

see (2.17) and the discussion preceding it for more details. Following Hida, we use the algebraic pairing defined to replace the usual Petersson inner product. To compare the two, we define a modified Petersson product on \( S_k(M, \mathcal{C}) \) by setting

\[
[v, w]_M = \langle v | W_M, w^c \rangle_M
\]

where the pairing on the right is the Petersson product. The superscript \( c \) denotes complex conjugation on the Fourier coefficients, and \( W_M \) is the Atkin–Lehner–Li involution. It
is then shown that the two pairings are essentially scalar multiples of each other, thus 
\[ \langle g, g \rangle_M = \Omega_M \langle g, g \rangle_M \], where \( \Omega_M = \frac{\langle g_0, g_0 \rangle_M}{\langle g, g \rangle_M} \) is well-defined up to unit factor. The number 
\( \langle g, g \rangle_M \) is the so-called congruence number for \( f \). We shall compute the number \( \Omega_M \) more explicitly in Sect. 2 below, and the computation leads to the following definition of the 
canonical period: we put
\[
\Omega^\text{can}_{g_0} = \frac{\langle g_0, g_0 \rangle_{M_0}}{\langle g_0, g_0 \rangle_{M_0}} = \text{unit} \cdot p^{k/2 - 1} \cdot \Omega_M
\]  
(1.5)
where \( g_0 \) is the newform of level \( M_0 \) associated to \( g \), so that \( M_0 = M \) or \( M/p \), and 
\[
\langle g_0, g_0 \rangle_{M_0} = \int_{B_1(M_0)} |g_0(z)|^2 y^{k-2} \, dx \, dy,
\]
for \( z = x + iy \) running over a fundamental domain \( B_1(M_0) \) for \( \Gamma_1(M_0) \). The number 
\( \langle g_0, g_0 \rangle_{M_0} \) is a certain congruence number associated to the newform \( g_0 \). The period \( \Omega^\text{can}_{g} \) is 
defined up to a \( p \)-adic unit factor, coming from the choice of the Gorenstein isomorphism, 
which is fixed once and for all.

It remains to explain how to modify Schmidt’s construction to construct \( p \)-adic \( L \)-
functions from depleted forms rather than newforms. Some care is required, since Schmidt 
relies crucially on some kind of semi-simplicity and multiplicity one result, in order to 
compute certain Petersson products which arise. In the depleted case, the Hecke algebra 
is not semi-simple, and it is unclear how to compute the Petersson products, or even to 
show that they are nonzero. We are able to resolve this problem because depletion at 
\( q \) increases the level by at most \( q^2 \), and in fact the level doesn’t increase at all unless \( g \) 
is ordinary or unramified. This fact allows us to restrict attention to a one-dimensional 
spacemotivated by degeneracy maps. Thus, there is a unique \( g \)-isotypic line at level \( Mq^2 \) 
where \( U_q = 0 \), and the depleted form \( f \) is chosen to lie in exactly this one-dimensional space.

To deal with congruences, we actually need a bit more. Indeed, for the purposes of 
relating the Iwasawa invariants of the \( p \)-congruent forms \( g_1 \) and \( g_2 \), we must show that 
we can simultaneously add primes to the level to obtain imprimitive modular forms \( f_1 \) 
and \( f_2 \) of the same level for which all Fourier coefficients are \( p \)-congruent, and for which 
all \( U_q \) eigenvalues are zero, and for which we can retain some version of semisimplicity 
for the Hecke action. The fact that the level so obtained satisfies all our conditions is an 
application of a result due to Gouvea, (Lemma 2.26 below). Once we have semisimplicity, 
and a perfect integral pairing is shown to exist at a fixed common level, it is a simple matter 
to show that normalized special values of \( p \)-adic \( L \)-functions associated to \( r_{g_1} \otimes \psi_1 \) 
and \( r_{g_2} \otimes \psi_1 \) are \( p \)-congruent, see Theorem 2.27. The hard part of the theorem is showing that 
a suitable algebraic pairing exists at some common choice of level, and for this Gouvea’s 
result is indispensible.

Finally, we mention a further—and more stubborn—point that arises in dealing with 
imprimitive forms. One would like to compare the periods of the depleted form \( f \) at level 
\( N \) with the canonical periods of the newform \( g_0 \) at level \( M_0 \), since it is the imprimitive 
forms and imprimitive periods that show up when dealing with congruences. Relating 
these periods requires us to assume that a certain version of Ihara’s lemma is satisfied, 
see Hypothesis 2.18. Cases of the result are known unconditionally when \( k = 2 \), but it is 
not yet resolved in all the higher weight cases. We are therefore required to carry around
Hypothesis 2.18. If the reader is willing to allow the periods to depend on the level, then our results become unconditional. We mention also that the results in this paper could be generalized to cover the remaining critical values, and odd characters $\psi$, at the cost of additional calculation of Fourier expansions of theta functions, sometimes of weight $3/2$ and of slightly different Eisenstein series.

On the algebraic side, there is little difficulty. The results on Galois cohomology in [8] are quite general, and apply to the situation treated here, so require little more than translation. The key condition is that the common reduction modulo $p$ of the representations $r_{g_i} \otimes \psi_t$ does not contain any non-zero submodule on which the action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is trivial. This hypothesis is ensured by the condition that $(\psi \epsilon_i)^2 \neq 1 \pmod{p}$.

In conclusion, we mention that the methods of this paper have been used by Delbourgo to prove congruences for Rankin-Selberg $L$-functions. We refer the reader to his forthcoming article for an interesting application.

2 Congruences for symmetric square $L$-functions

2.1 Definitions and normalizations

Petersson inner product

Consider modular forms $v, w$ of weight $k \geq 2$ on $\Gamma_1(m)$ (at least one of which is cuspidal). Then the Petersson inner product of $v$ and $w$ is defined via

$$\langle v, w \rangle_m = \int_{B_1(m)} v(z)\overline{w(z)} y^{k-2} dxdy,$$

(2.1)

where $B_1(m)$ is a fundamental domain for $\Gamma_1(m)$. Since we will be comparing to the work of Schmidt and Shimura, it is important to record that their normalizations are different: they write the Petersson product as an integral of $v^*w$ on a fundamental domain for $\Gamma_0(m)$.

Slash operator and Hecke operators

We follow the classical conventions of [17], Chapter 4, when speaking of Hecke operators acting on modular forms. We refer also to [1]. Let $m$ be any positive integer, which may be divisible by our fixed prime $p$. For a prime $q$ $\nmid m$ (resp. $q \mid m$, including possibly $q = p$) the Hecke operator $q^{1-k/2}T_q$ (resp. $q^{1-k/2}U_q$) of level $m$ corresponds to the right action of the double coset $\Gamma_1(m) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_1(m)$, acting via the slash operator, normalized so that

$$(f|_{k} \gamma)(z) = \text{det}((\gamma)^{k/2}(cz+d)^{-k}f((az+b)/(cz+d))), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The Hecke operators $T_q$ are normal with respect to the Petersson inner-product on $\Gamma_1(m)$, but the operators $U_q$ are not. Write $W_m$ for the operator on modular forms of level $m$ induced by the action of the matrix $\begin{pmatrix} 0 & 1 \\ -m & 0 \end{pmatrix}$. Note that the matrix $W_m$ normalizes $\Gamma_1(m)$, and recall that $W^2_m$ acts via the scalar $(-1)^k$. Furthermore, the adjoint of $U_q$ acting on cuspforms of level $m$ is the operator $U_q^*$ given by $W_m^{-1}U_qW_m$. Observe that the adjoint of $U_q$ depends on the level, although $U_q$ itself does not.

Since we are working on $\Gamma_1(m)$, we must also specify additional operators giving the character: if $(n, m) = 1$, then we have an operator $S_n$ defined by $f|S_n = f|_{k} \sigma_n$, and $\sigma_n \equiv \begin{pmatrix} n^{-1} \star \\ 0 & n \end{pmatrix} \pmod{m}$. We have not used the usual diamond notation for these operators, since that notation is reserved for the Petersson product below.
2.1.1 Atkin–Lehner–Li operators

Let \( m \) denote any positive integer, and let \( m = QQ' \) with \( Q, Q' \geq 1 \), be a factorization such that \((Q, Q') = 1\). We let \( W_Q \) and \( W_{Q'} \) denote the Atkin–Lehner–Li operators defined by Atkin–Li in [1], top of page 223, acting on the modular forms for the group \( \Gamma_1(m) \).

Explicitly, the operator \( W_Q \) is given by the action of any matrix \( \gamma = \left( \begin{array}{cc} Qx & y \\ Nz & Qw \end{array} \right) \), where \( x, y, z, w \in \mathbb{Z} \), and \( y \equiv 1 \) (mod \( Q \)), \( x \equiv 1 \) (mod \( Q' \)), and \( \det(\gamma) = Q \). The operator \( W_m \) defined above corresponds to the factorization \( m = m \cdot 1 \). Suppose that \( F \) has nebentype character \( \epsilon \) mod \( m \), and write \( \epsilon = \epsilon_Q \cdot \epsilon_Q' \), where \( \epsilon_Q \) (resp. \( \epsilon_Q' \)) is a character defined mod \( Q \) (resp. defined mod \( Q' \)). Then by [1], Proposition 1.4, we have \( W_m = \epsilon_Q'(Q) \cdot W_Q \circ W_{Q'} \).

In particular, \( W_Q \) may not commute with \( W_{Q'} \).

The operator \( W_Q \) depends on the level \( m \), not just on the index \( Q \), although it is not customary to put that in the notation. We will put some indication of the level if it is required. However, for \( Q, Q' \) as above, the matrix written above which defines \( W_Q \) at level \( m \) also satisfies the definition of the matrix giving \( W_Q \) at level \( Q \) corresponding to \( Q = Q \cdot 1 \). Thus, if we view the forms of level \( Q \) as a subspace of the forms of level \( m \), via the natural inclusion of functions on the upper half plane, the action of the operator \( W_Q \) of level \( Q \) coincides with the action of \( W_Q \) at level \( m \). We will use this fact later. Note also that the eigenvalues of the \( W \) operators on a given form \( f \) may not be contained in the ring generated by the Fourier coefficients of \( f \); thus, we enlarge the coefficient ring to contain these eigenvalues if required.

2.2 Modular forms and their depletions

With the definitions in hand, we return to the situation of \( p \)-stabilized newforms, as in the introduction. In this section, we will be dealing exclusively with the construction and integrality of \( p \)-adic \( L \)-functions for a single modular form, so we simply fix some \( g = g_i \).

Write \( M \) for the level of \( g \), so that \( M = M_0p \), where \( p \nmid M_0 \). Let \( \epsilon \) denote the character of \( g \), and denote the newform associated to \( g \) by \( g_0 \), so that \( g_0 \) has level \( M_0 \) or \( M_0p \).

If \( z \) denotes a variable in the upper half plane, write the Fourier expansion of \( g \) as \( g(z) = \sum a(n, g)e^{2\pi inz} \). Let \( S_0 \) be a finite set of primes \( q \neq p \). Define the modular form

\[
f(z) := \sum_{(n, S_0) = 1} a(n, g)e^{2\pi inz},
\]

where the sum is restricted to indices \( n \) that are indivisible by each prime in \( S_0 \). Then the \( L \)-function \( L(f, s) = \sum a(n, f)n^{-s} \) of \( f \) has the formal Euler product expansion

\[
L(f, s) = \prod_q (1 - \alpha_q q^{-s})^{-1}(1 - \beta^*_q q^{-s})^{-1},
\]

where the product is taken over all prime numbers \( q \), and \( \alpha_q, \beta^*_q \) are certain complex numbers. The basic properties of \( f \) are well-known, and we summarize them as a lemma.

**Lemma 2.1** The modular form \( f \) has level \( N \), where \( \ord_q(N) = \ord_q(M) \) for all \( q \neq S_0 \). If \( q \in S_0 \) then \( \ord_q(N) = \ord_q(M) + 1 \) if \( q \) is ordinary for \( g \). If \( q \in S_0 \) is unramified for \( g \), then \( \ord_q(N) = \ord_q(M) + 2 \). Finally, if \( q \in S_0 \) is depleted for \( g \), then \( \ord_q(N) = \ord_q(M) \).

Furthermore, the following statements hold:

1. \( N = pN_0 \), where \( (N_0, p) = 1 \);
2. \( 4|N \) if \( 2 \in S_0 \);
3. \( f \) is an eigenvector of the Hecke operators \( T_q \) for \( q \nmid N \) and for \( U_q \) when \( q | N \);
4. the eigenvalue \( \alpha_p \) of the \( U_p \) operator on \( f \) is a \( p \)-adic unit;
5. the eigenvalue of \( U_q \) on \( f \) is zero, for \( q | N, q \neq p \); and
6. \( \alpha'_q = \beta'_q = 0 \) for all \( q | N, q \neq p \).

2.3 The naive symmetric square, and the Petersson product formula

We keep the notations for the modular form \( g \) and its depletion \( f \), as in the previous subsection. Consider a fixed character \( \psi \), of conductor \( c_\psi \) relatively prime to \( p \). Let \( \chi \) be a Dirichlet character of the form \( \psi \eta \), where \( \eta \) has \( p \)-power conductor. Then the naive \( \chi \)-twisted symmetric square \( L \)-function of \( f \) is given by:

\[
D_f(\chi, s) = \prod_q \left( (1 - \chi(q)\alpha'_q \beta'_q q^{-s})(1 - \chi(q)\beta'^2 q^{-s})(1 - \chi(q)\alpha'^2 q^{-s}) \right)^{-1}.
\]  
(2.2)

The product is taken over all primes \( q \), with \( \chi(q) = 0 \) for \( q | c_\chi = c_\psi p^r \), for \( r \geq 0 \). It converges in a suitable half-plane, and admits a meromorphic continuation to all \( s \). Let \( G(\chi) \) denote the Gauss sum of \( \chi \). Then the quantity

\[
D_f(\chi, s)^{\text{alg}} = \frac{D_f(\chi, s)}{\pi^{k-1} (\langle f, f \rangle)_N} \cdot \frac{G(\chi)}{(2\pi)^{s-k+1}}
\]  
(2.3)

is algebraic when \( s = n \) is an integer in the range \( 1 \leq n \leq k - 1 \) satisfying \((-1)^n = -\chi(-1)\). This is well-known, see [15, Theorem 2.2.3], for the present formulation; the result goes back to Shimura, whose method was elaborated by Schmidt [25], [26], and Sturm [29].

In this situation we say that \( n \) is critical. We remark that the functional equation for the primitive symmetric square \( L \)-function leads to similar algebraicity results for \( D_f(\chi, s)^{\text{alg}} \) for integer values of \( s \) in the range \( k \leq s \leq 2k - 2 \); we will not need these results here. Note that the algebraic quantities above are not necessarily integral. We make the following assumption, to keep the notation and book-keeping simple:

- we have \( \chi = \psi \eta \), where \( \eta \) is a nontrivial even character of \( p \)-power conductor, and
- \( s = n \) is an odd integer with \( 1 \leq n \leq k - 1 \) in the algebraicity formula above.

Remark 2.2 The first assumption above would be problematic if one were trying to construct a \( p \)-adic \( L \)-function, since in that case one must consider the trivial character. However, we are simply trying to prove congruences for a \( p \)-adic \( L \)-function that is known to exist, so it suffices to demonstrate the congruences for almost all characters.

2.4 Explicit formulae for \( L \)-values

The explicit formulae we will need for congruences originate in [28]. We have elected to follow the treatment in [25], with some improvements, since it is relatively easy to compare the formulae given there to those originally given by Shimura, whose work remains the basic reference. But before giving Shimura’s formula, we need to clear up an important technical point. Shimura considers the convolution of \( f \) with a modular form of half-integral weight, whose level is therefore divisible by 4. It will be convenient for the computations to insist that the level \( N \) of \( f \) is also divisible by 4, and we can achieve this by requiring \( 2 \in S_0 \). Thus, until further notice, we assume that

- The set \( S_0 \) contains \( q = 2 \).
We will remove this condition on $S_0$ at the very end of the argument, when we give the proofs of the theorems stated in the introduction. Thus, define

$$\theta_\chi(z) = \frac{1}{2} \sum_{j \in \mathbb{Z}} \chi(j) \exp(2\pi ij^2z), \quad (2.4)$$

which is a modular form of weight $1/2$ and level $4c_\chi^2$. Further, let

$$\Phi^\epsilon(z, \chi, s) = L_{N_{\chi}}(\chi^2, 2s + 2 - 2k)E(z, s + 2 - 2k, 1 - 2k, \omega)$$

denote the Eisenstein series as defined in [25, p.210]. We reproduce the formula here:

$$E(z, s, \chi, \omega) = y^{s/2} \sum_{\gamma} \omega(d_\gamma) j(\gamma, z)^2 |j(\gamma, z)|^{-2s},$$

where $z, s, \chi, \omega \in \mathbb{C}$, and $\omega$ is the Dirichlet character given by

$$\omega(d) = \epsilon(d) \chi(d) \left( \frac{-1}{d} \right)^k \quad (2.5)$$

for $(d, 4N_{\chi}) = 1$. The sum is taken over $\gamma \in \Gamma_\infty \backslash \Gamma_0(N_{\chi})$, where $N_{\chi} := \text{lcm}(N, 4c_\chi^2)$.

The subgroup $\Gamma_\infty$ is the set of matrices of the form $\pm \left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right), n \in \mathbb{Z}$, and $d_\gamma$ denotes the bottom right entry of the matrix $\gamma$.

The Eisenstein series is a real analytic modular form of weight $k - 1/2$, and $\theta_\chi(z)\Phi^\epsilon(z, \chi, s)$ is a (non-holomorphic) modular form of weight $k$, character $\epsilon$, and level $N_{\chi}$. Recall that the level $N$ of $f$ is divisible by precisely the first power of $p$, since $f$ is assumed to be $p$-stabilized. We shall write $c_\chi = p^{m_\chi} = p^{m_\psi}$ for the $p$-part of the level of $\chi = \psi n$. Since $n \neq 1$ we have $m_\chi \neq 0$ and $N_{\chi} = \text{lcm}(N, 4c_\chi^2p^{2m_\chi}) = p^{2m_\chi} \text{lcm}(N, 4c_\psi^2)$.

Consider an odd integer $n$ in the range $1 \leq n \leq k - 1$. Shimura has shown that $\theta_{\chi}(z)\Phi^\epsilon(z, \chi, n)$ is a nearly holomorphic modular form of level $N_{\chi}$, weight $k$, and trivial character. In fact, one has the following formula (see equation (1.5) in [28]):

$$(4\pi)^{-s/2} \Gamma(s/2) D_f(\chi, s) = \frac{1}{\phi(N_{\chi})} \langle f, \theta_{\chi}(z)\Phi^\epsilon(z, \chi, s) \rangle_{N_{\chi}}. \quad (2.6)$$

Here the inner product is taken as an integral over a fundamental domain for $\Gamma_1(N_{\chi})$. The factor of $\frac{1}{\phi(N_{\chi})}$ is inserted to match Shimura’s integral over a fundamental domain for $\Gamma_0(N_{\chi})$.

### 2.5 Trace computations and the Petersson product formula at level $N$

For the purpose of proving congruences, it is vital to work at the fixed level $N$. However, observe that the form $\theta_{\chi}(z)\Phi^\epsilon(z, \chi, n)$ satisfies a transformation property with respect to the smaller group $\Gamma_0(N_{\chi})$, which varies with $\chi$. Our goal is to bring $\theta_{\chi}(z)\Phi^\epsilon(z, \chi, n)$ down to level $N$ by taking a trace, and eventually verify that we can retain control of integrality of the Fourier coefficients. In this section we define and compute the trace, and prepare for the discussion of holomorphicity and integrality in the next section.

Let $N_\psi = \text{lcm}(N, 4c_\psi^2)$. The level $N$ of $f$ and the level $N_{\chi} = p^{2m_\chi}N_\psi$ of $\theta_{\chi}(z)\Phi^\epsilon(z, \chi, n)$ differ only by a power of $p$, and other primes $q | c_\psi$. It may happen that the primes dividing $c_\psi$ also divide $N$.

We start by dealing with the powers of $p$. In this we reproduce the method of Schmidt [25, 26], with a few improvements. Let $T_\chi$ denote the trace operator that takes modular forms on $\Gamma_1(N_{\chi})$ down to $\Gamma_1(N_\psi)$. The formula for the trace on $\Gamma_1$ may be deduced
from the formulae on pages 68-69 of Ohta [18], especially (2.3.1) on the top of page 69, where coset representatives for \( \Gamma_1(N \psi p^{-1}) \) to \( \Gamma_1(N \psi p^r) \) are computed, for any \( r \geq 1 \). Ohta’s notation is different from ours—he’s \( N \) is a number prime to \( p \), which corresponds to our \( N \psi/p \). Let \( F \) denote a modular form on \( \Gamma_1(N \chi) \). Assume that the character of \( F \) has conductor dividing \( N \psi \). In the following pages, we will use the symbol \( \circ \) to denote the action of operators on the right (instead of the usual slash) because the formulae are somewhat messy, and \( \circ \) renders the various compositions of operators more readable.

With this convention, Ohta’s formulae (2.3.2) and (2.3.3) op. cit. yield
\[
p^{k/2-2} F \circ \mathrm{Tr}_{\Gamma_1(N \psi p^r)} = F \circ W_{N \psi p^r} \circ U_p \circ W_{N \psi p^r}^{-1},
\]
Here we have used the fact that the nebentype character is defined modulo \( N \psi \) to conclude that the \( p \) matrices denoted by \( \sigma_\chi \) in Ohta act trivially, together with the definition of \( U_p \). Iterating this relationship yields the following formula, which is valid for any \( F \) as above:
\[
p^{2m_N-1}(k/2-2) F \circ T_\chi \circ W_{N \psi} = F \circ W_{N \psi} \circ U_p^{2m_N-1}. \tag{2.7}
\]
We remark that \( T_\chi \) is defined purely in terms of matrices, and can be applied to the non-holomorphic form \( \theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \), and that the latter has character \( \epsilon \), which is defined modulo \( N \).

Thus, we may apply our formula to \( F = \theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \). As a result, we find that \( \theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \circ W_{N \psi} \circ U_p^{2m_N-1} \) is of level \( N \psi \). It follows further that for any \( m \geq m_\chi \), that \( \theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \circ W_{N \psi} \circ U_p^{2m_N-1} \) is also a modular form of level \( N \psi \), since the \( U_p \) operator at level \( N \chi \) is given by the same matrices as \( U_p \) at level \( N \psi \), and \( U_p \) stabilizes the space of forms of level \( N \psi \). Since
\[
\theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \circ W_{N \psi} \circ U_p^{2m_N-1} = \theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \circ W_{N \psi} \circ U_p^{2m_N-1} \circ U_p^{2(m-m_N)},
\]
the equation (2.6) gives
\[
\Gamma(n/2)D_f(\chi, n) \frac{\phi(N \chi)}{(4\pi)^{n/2}} = \frac{1}{\phi(N \chi)} \left( f, \theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \right)_N \chi = \frac{1}{\phi(N \chi)} \left( f, \theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \circ T_\chi \right)_N \psi = \frac{1}{\phi(N \chi)} \left( f \circ W_{N \psi}, \theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \circ T_\chi \circ W_{N \psi} \right)_N = \frac{p^{-(2m_N-1)(k/2-1)}}{\phi(N \psi)} \left( f \circ W_{N \psi}, \theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \circ W_{N \psi} \circ U_p^{2m_N-1} \right)_N.
\]
Here we have used (2.7), as well as the factorization \( \phi(N \chi) = \phi(N \psi)p^{2m_N-1} \).

Computing a bit further, and setting \( H = \theta_{\chi}(z) \Phi^\xi(z, \overline{\chi}, n) \) for brevity, we consider some \( m \geq m_\chi \), and we find that
\[
\left( f \circ W_{N \psi}, H \circ W_{N \psi} \circ U_p^{2m_N-1} \right)_{N \psi} = \left( f \circ W_{N \psi}, H \circ W_{N \psi} \circ U_p^{2m_N-1} \circ U_p^{2(m-m_N)} \right)_{N \psi} = \left( f \circ W_{N \psi} \circ (U_p)^{2(m-m_N)}, H \circ W_{N \psi} \circ U_p^{2m_N-1} \right)_{N \psi} = \alpha_p^{2(m-m_N)} \left( f \circ W_{N \psi}, H \circ W_{N \psi} \circ U_p^{2m_N-1} \right)_{N \psi}.
\]
In the second calculation above, we have used the fact that adjoint of $U_p$ at level $N_\psi$ is given by $U_p^*=W_{N_\psi}^{-1} \circ U_p \circ W_{N_\psi}$. Putting together the strings of equalities above, we conclude that
\[
\phi(N_\psi)\frac{\Gamma(n/2)}{(4\pi)^{n/2}}p^{(2m-1)(k/2-1)}D_f(\chi, n)
\]
\[
=\epsilon_q^{2(m-1)}(f \circ W_{N_\psi}, \theta_\psi(z) \Phi(\chi, n) \circ W_{N_\psi} \circ U_p^{2m-1})_{N_\psi}
\]
for any $m \geq m_\chi$.

**Remark 2.3** The factor $\phi(N_\psi)$ in the formula above depends on $\psi$, and may be divisible by additional powers of $p$, coming from primes $q \equiv 1 \pmod{p}$.

We want to replace the inner product in the formula above by an inner product at the fixed level $N$. Thus, we have to take another trace. We warn the reader that the calculation is a bit messy, although more or less elementary. But we need to do the calculation in detail, as it turns out to be what we need to get rid of the extra powers of $p$ in the quantity $\phi(N_\psi)$ that come from primes $q$ with $q \mid c_\psi$, $q \not \equiv 1 \pmod{p}$. Let $N = \prod_{i=1}^t q_i^{e_i}$ be the prime factorization of $N$. Reordering the factors if necessary, suppose that $q_i \not \equiv 2c_\psi \iff i < t$. If $i < t$, then clearly $\text{ord}_{q_i}(N_\psi) = \text{ord}_{q_i}(N)$, because $N_\psi = \text{lcm}(N, 4c_\psi^2)$. The level $N_\psi = \text{lcm}(N, c_\psi^2)$ differs from $N$ only at the primes $q_i$ with $i \geq t$, where $f_i = \text{ord}_{q_i}(N_\psi) \geq e_i = \text{ord}_{q_i}(N) \geq 1$, or at primes which divide $4c_\psi$, which do not divide $N$ at all.

To start with, we want to get rid of the extra powers of the $q_i$, $i \geq t$, so we define $N_{\psi, t} = N_\psi / (\prod_{i>t} q_i^{e_i})$ and set
\[
U_{\psi, t} = U_{\psi, t}^{e_t} \circ \ldots \circ U_{\psi, t}^{e_1}.
\]

If we set $N_{t} = N_\psi / q_t^{e_t}$, then the same calculation as in the case $q = p$ treated above shows that, if $F$ is a form of level $N_\psi$ whose nebentype character has conductor dividing $N$, we have
\[
F \circ \text{Tr}_{q_t} \circ W_{N_{t}} = \epsilon_{q_t} F \circ W_{N_{\psi}} \circ U_{t}^{e_t}.
\]
Here $\epsilon_{q_t}$ is a power of $q_t$, and $\text{Tr}_{q_t}$ is the trace to level $\Gamma_1(N_{q_t})$. By construction, we have $\text{ord}_{q_t}(N_{q_t}) = \text{ord}_{q_t}(N)$. Repeating this procedure, we conclude that $F \circ U_1^{\psi}$ has level $N_{\psi, t}$. Furthermore, it is clear, by induction, that if $G$ is a cusps form of level $N_{\psi, t}$, and $F$ has character with conductor dividing $N_{\psi, t}$, that we have
\[
(G, F \circ W_{N_{\psi}} \circ U_1^{\psi} \circ W_{N_{\psi}}^{-1})_{N_{\psi, t}} = a_\psi (G, F)_{N_{\psi}}.
\]
where $a_\psi$ is a $p$-adic unit. Let $Q = \prod_{q|N} q^{\text{ord}_{q}(N_{\psi, t})}$, and set $Q' = N_{\psi, t}/Q$. Then the following lemma holds by construction:

**Lemma 2.4** We have $Q = N$, and a factorization $N_{\psi, t} = QQ'$ where $(Q', N) = 1$.

Let $W_Q$ and $W_Q'$ be the Atkin–Lehner–Li operators at level $N_{\psi, t}$. As we have already recorded in Sect. 2.1.1, we have $W_N \circ W_Q = W_Q \circ W_Q' = \epsilon_Q(Q)W_QW_Q = W_{N_{\psi, t}}$, since $\epsilon$ is defined modulo $Q = N$, on forms with character $\epsilon$ and level $N_{\psi, t} = QQ'$. In the situation at hand where $Q = N$, the operator $W_Q$ acting on forms of level $N_{\psi, t}$ agrees with $W_N$ defined in the same way on the subspace of forms of level $N$. 

Now, let $U_{Q}^{2}$ denote the trace operator from level $N_{\psi}'$ to level $N$. We define $\tilde{T}_{\psi} : S_{k}(N_{\psi}, K) \to S_{k}(N, K)$ by

$$\tilde{T}_{\psi} : h \mapsto h \circ U_{1}^{\psi} \circ W_{Q}^{-1} \circ U_{2}^{\psi}.$$ 

The reason for the factor $W_{Q}^{-1}$ will become clear very shortly. The key lemma is the following.

**Lemma 2.5** Suppose $F$ is a holomorphic modular form of level $N_{\psi}$, character $\epsilon$ modulo $N$, and weight $k$, with $p$-integral Fourier coefficients. Assume that $\epsilon$ has trivial restriction to $(\mathbb{Z}/p\mathbb{Z})^\times$, namely, that $\Gamma$ is modular with respect to $\Gamma_1(a, p) = \Gamma_1(a) \cap \Gamma_0(p)$, for $a = N_{\psi}/p$.

Then the following statements hold:

1. $F \circ \tilde{T}_{\psi}$ has $p$-integral Fourier coefficients, and
2. The Fourier coefficients of $F \circ \tilde{T}_{\psi}$ are divisible by $\prod_{q}(q - 1)$, where the product is taken over all primes $q$ dividing $c_{\psi}$ which do not divide $N$.

**Proof** We start with the first statement. The fact that $U_{1}^{\psi}$ preserves integrality is clear from the definition, virtue of the standard expression for the action of the $U_{q}$ operators on the Fourier coefficients. The remainder of the proof boils down to more or less well-known facts about the special fiber of the modular curve $X_{1}(a, p)$, so we merely sketch the details. For $W_{Q}$, this is treated in the case of modular forms with trivial central character by Theorem A.1 in the appendix by Conrad to [20], since $(Q', p) = 1$. Conrad only treats the case of $\Gamma_0$ level structure. However, an examination of Conrad’s argument show that it works equally well for modular forms on $\Gamma_1(a) \cap \Gamma_0(p)$. The key point is that the special fiber of the moduli stack of $X_{0}(a p)$ in characteristic $p$ has a distinguished irreducible component, and that this component contains both the cusp at infinity as well as its image under the operator $W_{Q}$ (because $p \nmid Q'$). Conrad then applies the Katz $q$-expansion principle on this component.

The existence of the distinguished component is unaffected by the auxiliary level structure at primes away from $p$. Indeed, it is obtained as the image of a section $X_{1}(a) \to X_{1}(a, p)$ over the ordinary locus, which enhances an ordinary elliptic curve, plus any given level structure at $a$, with the level structure at $p$ coming from the unique subgroup order $a$ isomorphic to $\mu_{p}$ (which exists because of ordinarity). We remark also that Conrad’s geometric definition of the $W_{Q}$ operator at the bottom of page 60, also carries through to our situation, since $(Q', p) = 1$ as in Conrad, and the $p$-part of our moduli problem is exactly the same. The prime-to-$p$ part of the level causes no problems, since we are working over a ring in which $a$ is invertible. Thus, we obtain the our result for $W_{Q}$ by applying the $q$-expansion principle on the distinguished component: if a modular form vanishes identically in a formal neighborhood of one cusp, it must vanish on the whole component, and hence, the expansion is zero about the other cusp as well.

As for $U_{Q}^{2}$, the proof is implicit in Hida’s work (see [10], IV, page 11) and is in fact very similar to the above. The trace is given by $F \mapsto \sum_{\gamma} F \circ \gamma$, where $\gamma$ runs over a set of coset representatives of $\Gamma_{1}(N_{\psi}') \backslash \Gamma_{1}(N)$. By definition of $N$ we have $\Gamma_{1}(N) \subset \Gamma_{1}(p)$. Thus, the cusp $s = \gamma(\infty)$ is one of Hida’s ‘unramified at $p$’ cusps (see [9], Section 5) and it is well-known that if $F$ has integral $q$-expansion at $\infty$ then it has integral $q$-expansion at $s$. Again, the key point is that both $\infty$ and $s$ reduce modulo $p$ to points on the same component of the special fiber, just as in Conrad’s argument. Thus, $t_{\psi}$ preserves integrality as well.
To prove the second statement in the lemma, let \( N_q' \) denote the prime-to-\( q \) part of the level \( N_q' \), for \( q \) as in the statement of the Lemma. Since the character \( \epsilon \) has conductor \( N \), and \( q \) does not divide \( N \), the form \( F \) is invariant under the group \( \Gamma_q = \Gamma_0(q^\infty) \cap \Gamma_1(N_q') \). Since we have \( \Gamma_1(N_q') \supset \Gamma_q \supset \Gamma_1(N_q') \), and the latter index is \((q-1)q^{e_q-1}\), the trace to \( \Gamma_1(N_q') \) is divisible by \( q-1 \). The claim now follows by iterating this process with \( N_q' \) instead of \( N \), and any other prime \( q \) appearing in the statement of the Lemma. \( \Box \)

Let \( b_\psi = \prod_q (q-1) \), where the product is taken over the same primes \( q \) as in the statement of the lemma, and define the operator \( T_\psi \) by \( T_\psi = \frac{1}{b_\psi} T_\psi \). The operator \( T_\psi \) will be our tame trace operator. The lemma above only applies to holomorphic forms, but the operator \( T_\psi \) can be applied even in the nearly holomorphic case, since it just involves matrices. We compute the result for later use. If \( F \) is a cuspform of level \( N \), and \( G \) is a nearly holomorphic form with character defined modulo \( N \), and \( m \geq m_\chi \), then plugging in the definitions, and using Lemma 2.4 and the remarks that follow, shows that we have

\[
\begin{align*}
\psi(F \circ W_N, G \circ U_p^{2m-1} \circ T_\psi)_N &= \langle F \circ W_N, G \circ U_p^{2m-1} \circ U_q^{-1} \circ U_\psi \rangle_N^i \\
&= \langle F \circ W_N, G \circ U_p^{2m-1} \circ U_q^{-1} \circ U_\psi \rangle_{N_q'} \\
&= \langle F \circ W_N, G \circ U_p^{2m-1} \circ U_1 \circ W_{N_q'}^{-1} \circ T_\psi \rangle_{N_q'} \\
&= \langle F \circ W_N, G \circ U_p^{2m-1} \circ W_{N_q'}^{-1} \circ T_\psi \rangle_{N_q'}. 
\end{align*}
\]

Taking \( G = \theta_\pi(z) \Phi^{\xi}(z, \chi, n) \circ W_{N_q} \) and \( F = f \) in the above, using the definition of \( b_\psi \), and comparing with (2.8), we obtain the desired inner product formula at level \( N \):

\[
\begin{align*}
\phi(N) \frac{\Gamma(n/2)}{(4\pi)^{n/2} p^{(2m_\chi-1)(k/2-1)}} D_f(\chi, n) &= \text{unit} \cdot a_p^{2(m_\chi-m)} f \circ W_N, \theta_\pi(z) \Phi^{\xi}(z, \chi, n) \circ W_{N_q} \circ U_p^{2m-1} \circ T_\psi \rangle_N \\
&= \text{specify exactly, but we will not need it.}
\end{align*}
\]

Finally, we observe that the number \( \phi(N) \) may still be divisible by \( p \), in the event that \( N \) itself is divisible by primes \( q \) with \( q \equiv 1 \pmod{p} \). To get rid of these, let \( U \) denote the \( p \)-Sylow subgroup of \((\mathbb{Z}/N\mathbb{Z})^\times \). Since \( N \) is divisible only by the first power of \( p \), we see that \( U \) is the direct product of the \( p \)-Sylow subgroups of \((\mathbb{Z}/q_i\mathbb{Z})^\times \), where \( q_i \) runs over primes dividing \( N \) such that \( q \equiv 1 \pmod{p} \). In particular, there is no contribution from \( q = p \). If \( N_\chi = \prod q_i^{e_i} \) is the prime factorization of \( N_\chi \), we have \((\mathbb{Z}/N_\chi\mathbb{Z})^\times \cong \prod (\mathbb{Z}/q_i^{e_i}\mathbb{Z})^\times \). Therefore, we may identify \( U \) with a certain direct summand of the \( p \)-Sylow subgroup of \((\mathbb{Z}/N_\chi\mathbb{Z})^\times \). With this in mind, let \( \epsilon' \) denote a character of \((\mathbb{Z}/N_\chi\mathbb{Z})^\times \) of \( p \)-power order. We may identify \( \epsilon' \) with a character of \( U \) in the obvious way, and hence as a character of \((\mathbb{Z}/N_\chi\mathbb{Z})^\times \). For each fixed character \( \epsilon \) of \((\mathbb{Z}/N\mathbb{Z})^\times \), and for an auxiliary character \( \chi' = \psi \eta \) and an odd critical integer \( n \) as above, each of which is also fixed, consider the function

\[
H_\chi(n) = \frac{1}{|U|} \sum_{\epsilon'} \theta_\pi(z) \Phi^{\epsilon'}(z, \chi', n)
\]

where we sum over all characters \( \epsilon' \) of \( U \). Then clearly we have \( \phi(n) = |U| \cdot \text{unit} \), so that if \( f \) is a cuspidal eigenform of level \( N \) with fixed character \( \epsilon \), then we have

\[
\begin{align*}
\frac{\Gamma(n/2)}{(4\pi)^{n/2} p^{(2m_\chi-1)(k/2-1)}} D_f(\chi, n) &= \text{unit} \cdot a_p^{2(m_\chi-m)} f \circ W_N, H_\chi(N) \circ W_{N_\chi} \circ U_p^{2m-1} \circ T_\psi \rangle_N
\end{align*}
\]
2.6 Holomorphic and ordinary projectors

In this section, we analyze the form \( H_X(n) \circ W_{N^*_X} \circ U^2_{f^m-1} \) defined above and show how 
to replace it with something holomorphic and integral. The classical method of going from a nearly holomorphic form to something holomorphic, and which is adopted in 
[2], [25], [26], is to pass from \( H_X \) to its so-called holomorphic projection. This is a bit complicated, since the formulae giving the holomorphic projection of a nearly holomorphic form involve factorials and binomial coefficients, and one cannot easily control the denominators. This is why the results of [25], [26] are only stated up to some unspecified rational constant.

One of the main contributions of this paper is a solution to this problem, using \( p \)-adic methods: the form \( f \) is ordinary, so we can replace the nearly holomorphic form with a certain ordinary projection, without losing any information. This has the significant advantage that the ordinary projector is denominator-free. In view of Hida’s control theorems for ordinary forms, the ordinary projection is automatically holomorphic. We shall follow this alternative path, but to complete the journey, we have to compute the Fourier expansion of \( H_X(n) \circ W_{N^*_X} \). Since \( H_X(n) \) is a linear combination of products of an Eisenstein series and a theta series, we work out the expansions of these two first, starting with the Eisenstein series.

Following [25], page 213, and Shimura, [28], Section 3, page 86, let us fix a character \( \epsilon \) modulo \( N \), and write

\[
\Phi^\epsilon \left( \frac{-1}{N^*}, \chi, n \right) \cdot (\sqrt{N^*_X}z)^{1/2-k} = \sum_{j=0}^{(n-1)/2} \sum_{\nu=0}^{\infty} (4\pi y)^{-j} d_{j,\nu} q^\nu.
\]

**Lemma 2.6** Let \( \chi = \psi \chi \), as before, and that \( (\psi \epsilon)^2 \not\equiv 1 \pmod{p} \). If \( \chi \) is ramified at \( p \), the quantities \( \frac{\Gamma((n+1)/2)}{\pi(n+1/2)} p^{m_X} (3-2k+2n)^2 d_{j,\nu} \) are algebraic. Furthermore, the quantities \( \frac{\Gamma((n+1)/2)}{\pi(n+1/2)} p^{m_X} (3-2k+2n)^2 d_{j,\nu} \) are \( p \)-integral if \( \nu > 0 \).

**Proof** The formulae for the \( d_{j,\nu} \) may be deduced from those on pages 212-213 of [25], whose \( n \) is our \( \nu \), and whose \( m \) is our \( n \).

For \( \nu > 0 \), one obtains

\[
d_{j,\nu} = (-2)^{k-1/2} \cdot \pi^{n+1} \cdot v^{n-1} \cdot B_j \cdot \left( \frac{n+1}{2} \right) \cdot N_X^{(2k-2n-3)/4} \cdot L_{N^*}(n+1-k, \omega_\nu) \cdot \beta(v, n+2-2k),
\]

where \( B_j = \frac{\Gamma((n+1)/2)}{\Gamma((n+1)/2+1-\beta)} \in \mathbb{Q} \). The definition of \( \beta \) may be found on page 212 of [25], or in Proposition 1 of [28]. Note also that \( \beta \) depends on \( \epsilon \) and \( \chi \). The character \( \omega_\nu \) is defined in (2.5), and \( \omega_\nu = \left( \frac{-1}{\nu} \right)^{k+1} \left( \frac{\nu}{N^*_X} \right) \omega = \epsilon_X \left( \frac{-\nu N^*_X}{4} \right) \). The power of \( p \) dividing \( N_X \) is \( p^{2m_X} \), and hence is a perfect square, so the character \( \omega_\nu \) is unramified at \( p \) if \((\nu, p) = 1 \).

As for the constant term, one has \( d_{j,0} = 0 \) unless \( j = (n-1)/2 \), in which case

\[
d_{(n-1)/2,0} = (-2)^{(k-1)/2} \cdot \pi^{n+1} \cdot B_j \cdot N_X^{(2k-2n-3)/4} \cdot L_{N^*}(2n+1-2k, \omega^2).
\]

It is clear from the formula for \( B_j \) that \( \Gamma((n+1/2))B_j \) is a rational integer, and one knows from properties of Kubota-Leopoldt \( p \)-adic \( L \)-functions that their special values \( L(\omega_\nu, n+1-k) \) are \( p \)-integral, except in the case \( \omega_\nu \) is one of certain even powers of the

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1We remark that there is a typo in Schmidt, where \( L_{N^*_X}(2n+2-2k, \omega^2) \) is written.
Teichmüller character $\eta_1$, and $n + 1 - k$ is one of certain odd negative integers. These bad cases correspond to special values of the $p$-adic Riemann zeta function, which is the only $p$-adic Dirichlet L-function with a pole. We have assumed that $(\psi_\epsilon)^2 \not\equiv 1 \mod p$, so that $\omega_\epsilon$ is ramified at some prime other than $p$, and is never a power of the Teichmüller character, for any $\nu > 0$. But this means that $L(\omega_\epsilon, n + 1 - k)$ occurs as a value of the $p$-adic zeta function of a nontrivial character, and it is therefore $p$-integral. Then the result for $\nu > 0$ follows upon clearing the powers of $p$ coming from $N\chi$.

Remark 2.7 The integrality properties of Kubota-Leopoldt L-functions are summarized in the paper [21]. As for the constant terms with $\nu = 0$ in the Lemma above, we cannot exclude the possibility that the $L$-values arise in interpolation of the Riemann zeta function without further hypothesis. We do not pursue this point, since it is not needed.

One has next to compute the Fourier expansion of the quantity $\theta_\chi(-1/N\chi z) \cdot (\sqrt{N\chi z})^{-1/2}$. Here the exponent comes from the fact that $\theta_\chi$ is a form of weight $1/2$.

The requisite formula may be found in [27], Proposition 2.1, and we record the result here. Recall the notations: $N = N_0p$ is an integer divisible by precisely the first power of $p$, and $\chi$ is a character of conductor $\psi \chi$. Furthermore, $N$ is divisible by 4. We let $N' = N\chi/4\chi^2$. Thus, $(N', p) = 1$, if $\chi$ is ramified.

Lemma 2.8 Suppose that $\chi$ is ramified. We have

$$\theta_\chi(-1/N\chi z) \cdot (\sqrt{N\chi z})^{-1/2} = \theta_\chi(N'z) \cdot \frac{\Gamma(z)}{\sqrt{c_\psi p^m \chi}} \cdot i^{3/2} \cdot (N')^{1/4}.$$

Proof See [27], Proposition 2.2. \hfill \Box

Corollary 2.9 Suppose that $\eta$ is ramified, and let $\epsilon$ be any character modulo $N$ such that $(\psi_\epsilon)^2 \not\equiv 1 \mod p$. Then

$$\tilde{H}_\chi(n)^\epsilon = \frac{1}{|U|} \sum_{\epsilon'} \tilde{H}_\chi(n)^\epsilon \epsilon'.$$

Proof The first statement is obvious, from the formulae for the coefficients of the theta function and the Eisenstein series. As the for the second statement, the only issue is the possible denominators coming from the constant terms of the Eisenstein series. However, $\theta_\chi = \sum b_nq^n = \sum \chi(m)q^{m^2}$ has $b_n = 0$ once $p | n$, and the second claim follows by multiplying out the Fourier expansions. \hfill \Box

Remark 2.10 We have not made any analysis of the case where $\eta$ is unramified, although it may be done just as above. The exact formulae are slightly different, and we will not need them. As we have already remarked, it suffices, for the purpose of integrality, to show that almost all the values are integral (since we are assuming the existence of imprimitive $p$-adic L-functions).

Now, we follow (2.12) and define (for fixed $\epsilon, \psi, n$)

$$\tilde{H}_\chi(n) = \frac{1}{|U|} \sum_{\epsilon'} \tilde{H}_\chi(n)^\epsilon \epsilon'. \quad (2.14)$$
Corollary 2.11 Suppose that $(\psi \epsilon)^2 \not\equiv 1 \pmod{p}$. If $\eta$ is ramified, then $\tilde{H}_\chi(n)$ is a nearly holomorphic form of level $N_\chi$ whose Fourier coefficients $c_{j,\nu}$ are algebraic. Furthermore, the coefficients $c_{j,\nu}$ are $p$-integral for all $\nu > 0$ such that $p|\nu$.

Proof We need the analogue of Lemma 2.6, which dealt with a single fixed character. With the notation of that Lemma, we have to show that
$$\frac{1}{p^k} \sum_{\nu} L_{N_\chi}(n+1-k, \omega_\nu') \beta'(\nu, n+2k)$$
is integral, for each $\nu > 0$. Here $\omega_\nu'$ and $\beta'(\nu, n+2k)$ are given by the same prescription as $\omega_\nu$ and $\beta(\nu, n+2-2k)$, but with $\epsilon' \epsilon$ instead of $\epsilon$. The quantity $\beta'(\nu, n+2k)$ is an integral linear combination of terms of the form $c_a \epsilon' (a)$, for certain integers $a$, with integral coefficients $c_a$, which depend on $\psi, \epsilon, \nu$. Observe that $\epsilon' \epsilon$ has $p$-power order, and hence $\epsilon' \epsilon \psi$ is never quadratic and $\omega_\nu'$ is never a power of the Teichmüller character. Furthermore, if $a \in (\mathbb{Z}/N_\chi \mathbb{Z})^\times$, then
$$\frac{1}{p^k} \sum_{\nu} \epsilon' (a)$$
is $p$-integral, by orthogonality of characters of $U$. Then the corollary follows from the fact that Kubota-Leopoldt L-functions are given by an integral pseudo-measure with a pole only at the trivial character, as described in [21].

Now we want to pass from $\tilde{H}_\chi(n)$ to something holomorphic, while preserving integrality. Let $M_k(N_\chi, \mathcal{O})$ denote the space of all modular forms of level $N_\chi$ with coefficients in $\mathcal{O}$.

Proposition 2.12 Let e denote Hida’s ordinary projection operator, acting on $M_k(N_\chi, \mathcal{O}) \otimes \hat{\mathbb{Q}}$. Then $\tilde{H}_\chi(n)^{\text{hol}} \circ e \in M_k(N_\psi) \otimes \hat{\mathbb{Q}}$ has $p$-integral Fourier coefficients and level $N_\psi$. Here $\tilde{H}_\chi(n)^{\text{hol}}$ denotes the holomorphic projection of $\tilde{H}_\chi(n)$.

Proof The easiest way to prove the Proposition is to use the geometric theory of nearly ordinary and nearly holomorphic modular forms, as developed by Urban [31]. A resumé of Urban’s work adapted to this setting may be found in [23]. However, we give a proof along classical lines, for the convenience of the reader. Write $\tilde{H}_\chi(n) = \tilde{H}_\chi(n)^{\text{hol}} + \sum h_i$, where each $h_i$ is in the image of a Maass-Shimura differential operator. It follows from the explicit formulæ for the differential operators and the formulæ for the Fourier coefficients of $\tilde{H}_\chi(n)$ that $\tilde{H}_\chi(n)^{\text{hol}}$ and each $h_i$ has algebraic Fourier coefficients. Let $m$ denote a positive integer, divisible by $p - 1$, and consider $\tilde{H}_\chi(n) \circ U_p^m$. Then one has
$$\tilde{H}_\chi(n) \circ U_p^m = \tilde{H}_\chi(n)^{\text{hol}} \circ U_p^m + \sum h_i \circ U_p^m.$$
It is well known that $U_p$ multiplies each $h_i$ by a power of $p$ (For example, see [23]), formula above Lemma 2.3) Thus, the quantity on the right converges to $\tilde{H}_\chi(n)^{\text{hol}} \circ e$, as $m$ increases, the convergence being assured by the existence of the ordinary projector. On the other hand, the quantity on the left is $p$-integral, by Corollary 2.9. This proves the proposition. The fact that the level comes down to $N_\psi$ arises from the fact that $U_p^m$ already brings the level down to level $N_\psi$ for any $m$ sufficiently large, as noted in the course of proving (2.11).

Finally, we bring the $p$-integral and holomorphic form $\tilde{H}_\chi(n)^{\text{hol}} \circ e$ all the way down to level $N$. Thus, we define
$$\mathcal{H}_\chi(n) = \tilde{H}_\chi(n)^{\text{hol}} \circ e \circ T_\psi \in M_k(N, \mathcal{O}) \otimes \hat{\mathbb{Q}}$$
(2.15)
and
$$\mathcal{H}_\psi^m(n) = \tilde{H}_\chi(n)^{\text{hol}} \circ U_p^{2m-1} \circ T_\psi \in M_k(N, \mathcal{O}) \otimes \hat{\mathbb{Q}}.$$ (2.16)
Then Lemma 2.5, and the proof of Proposition 2.12, gives
The modular form $\mathcal{H}_T(n)$ has $p$-integral Fourier coefficients. The modular forms $\mathcal{H}_T^m(n)$ have $p$-integral Fourier coefficients for all $m$ sufficiently large.

In the next section, we shall see how to compute the inner product of $f$ with $\mathcal{H}_T(n)$ and derive integrality properties for the special values of $D_f(\chi, s)$.

2.7 Algebraic and analytic inner products

Our goal is to use (2.11) to describe the imprimitive $p$-adic $L$-function, and show that it behaves well with respect to congruences. We accomplish this by combining the inner product formula with a certain algebraic incarnation of the Petersson inner product formula due to Hida.

Let $S_k(N, \mathbb{Z})$ denote the space of modular forms of weight $k$ with rational integral Fourier coefficients. If $R$ is any ring, set $S_k(N, R) = S_k(\Gamma_1(N), \mathbb{Z}) \otimes R$. Let $T$ denote the ring generated by the Hecke operators $T_q, U_q, U_p, S_n$ acting on $S_k(N, \mathcal{O})$, and set $T(R) = T \otimes R$. Recall that our convention is that Hecke operators act on the right.

Then the eigenform $f$ determines a ring homomorphism $T \rightarrow \mathcal{O}$, sending a Hecke operator $T \in T$ to the $T$-eigenvalue of $f$. Define $\mathcal{P}_f$ to be the kernel of this homomorphism. There is a unique maximal ideal $m$ of $T$ that contains $\mathcal{P}_f$ and the maximal ideal of $\mathcal{O}$.

There is a canonical duality of $T_m$-modules between $S_k(N, \mathcal{O})_m$ and $T_m$, which we view as a pairing

$$S_k(N, \mathcal{O})_m \times T_m \rightarrow \mathcal{O}.$$ 

This pairing is explicitly given by $(s, t) \mapsto a(t, st) \in \mathcal{O}$, where $t \in T_m$ and $s \in S_k(N, \mathcal{O})$ is a $\mathcal{O}$-linear combination of elements in $S_k(N, \mathbb{Z})$, given by the Fourier expansion $s = \sum a(n, s)q^n$, with $a(n, s) \in \mathcal{O}$.

2.8 The definition of periods and the integrality of special values.

To proceed further, we assume that the Galois representation associated to $f$ at $m$ is irreducible, ordinary and $p$-distinguished. It is well-known that under these conditions that $T_m$ is Gorenstein, and isomorphic as a left $T_m$-module to $\text{Hom}(T_m, \mathcal{O})$. A proof may be found in [32, Theorem 2.1 and Corollary 2 (p. 482)]. We fix, once and for all, an isomorphism

$$T_m \sim \text{Hom}(T_m, \mathcal{O})$$

of $T_m$-modules. This choice is determined up to a unit factor in $T_m$. Composing with the canonical isomorphism above leads to an isomorphism $T_m \cong S_k(N, \mathcal{O})_m$, as modules over $T_m$, determined up to multiplication by a unit.

Thus, the space $S_k(N, \mathcal{O})_m$ is equipped with both a left and right action of Hecke operators; one coming from the classically defined slash action of Hecke operators on modular forms, and the other the abstract left action obtained from the choice of Gorenstein isomorphism in the previous paragraph. In fact, the left action of $T_m$ on $\text{Hom}(T_m, \mathcal{O})$ coincides with the usual right action of $T_m$ on $S_k(N, \mathcal{O})_m$. Indeed, the isomorphism $T_m \sim \text{Hom}(T_m, \mathcal{O})$ is an isomorphism of $T_m$-modules, and both are free of rank 1, so the two actions are seen to coincide. One deduces that there is a duality pairing

$$(\cdot, \cdot)_N : S_k(N, \mathcal{O})_m \times S_k(N, \mathcal{O})_m \rightarrow \mathcal{O},$$

which satisfies the equivariance condition $a(t, f_1 \cdot t) = (f_1, f_2)$. This is Hida’s algebraic inner product (see [12], Chapters 7 and 8). Unlike the usual Petersson product, it is linear in both
variables, and the Hecke operators are self-adjoint. Note that this pairing is dependent on
the choice of isomorphism $T_m \sim \text{Hom}(T_m, \mathcal{O}) \equiv S_k(N, \mathcal{O})_m$.

With this machinery in hand, we return to the case of interest, as set out in Sect. 2.2.
Thus, we consider a fixed $p$-ordinary, $p$-distinguished, and $p$-stabilized newform $g$ of level
$M$, and a finite set $S_0$ of primes $q \neq p$, including $q = 2$. We let $f$ denote the depletion of $g$
at the primes in $S_0$, and write $N$ for the level of $g$, as determined in Lemma 2.1. We have
a maximal ideal $m$ in the Hecke algebra $T_m(N, \mathcal{O})$ corresponding to $f$ and the fixed prime
$p \mid p$, to which we may apply the considerations above.

Consider the function

$$\varphi_f : v \mapsto (f, v)_N,$$

(2.18)

for $v \in S_k(N, \mathcal{O})_m$. Let $f^\perp \subset S_k(N, \mathcal{O})_m$ denote the kernel of $\varphi_f$, and let

$$\eta_f = (f, f)_N = \varphi_f(f).$$

The quantity $\eta_f = (f, f)_N$ is the famous congruence number of Wiles [32]. Indeed, it follows
from the calculations above that $\eta_f = \varphi_f(\text{Ann}(\text{ker}(\varphi_f)))$, which is one of the equivalent
definitions given by Wiles.

We would like to say that $\eta_f$ is nonzero.

**Lemma 2.14** Let $f, g, S_0, N$ be as in Sect. 2.2, so that $f$ is the depletion of $g$ at some finite set
$S_0$ of primes $q \neq p$ (not necessarily containing $q = 2$). Let $\mathcal{P} = \mathcal{P}_f$ denote the kernel of the
homomorphism $T_m(O) \rightarrow O$ associated to $f$. Then, there is an isomorphism $T_m(O)|_{\mathcal{P}} \simeq K$.

**Proof** The proof is elementary, but since the result is important, we write out the details.
Setting $S := T_m(O) \otimes \mathbb{Q}_p$, we note that $S$ is a finite-dimensional algebra over $K$.
Hence, $S$ is the product of local rings $R_i$, each of which is finite-dimensional over $K$.
The ring $R_i$ corresponds to a height one prime ideal of $T_m(O)$. The localization of $T_m(O)$
at $\mathcal{P}$ is equal to one of the rings $R = R_i$. Thus, we are to show that $R$ is a field. The subalgebra
$T'$ of $T_m(O) \otimes \mathbb{Q}$ generated by the Hecke operators prime to the level is semisimple,
and hence the product of fields $K_i$, each corresponding to a newform of level $N$. Thus,
$K \subset R$ is the image of $T'$ in $R$. Note that the summands $K_i$ are not necessarily in bijection
with the summands $R_j$, since $K_j$ could potentially be contained in multiple $R_j$s. We have a
homomorphism $T' \rightarrow K \hookrightarrow R$ with kernel $\mathcal{P}'$. By construction, the ideal $\mathcal{P}$ lies above $\mathcal{P}'$.

Consider the subspace of forms in $S_k(N, O)_m \otimes \mathbb{Q}_p$ annihilated by the ideal $\mathcal{P}'$. By duality,
it suffices to show that this subspace is one-dimensional over $K$. To achieve this, recall
that the newform associated to $f$ is the form $g_0$ at level $M_0$, which differs from $N$ only at
the prime $p$, and at primes $q \in S = S_0 \cup \{p\}$ at which $q$ is either unramified or ordinary.
Note that if $q \in S_0, q \neq p$, then $U_q f = 0$.

The argument follows by adding one prime at a time to the level. Let $q \neq p \in S$, and let
$N_q = M_0 q^{\ell_q}$, where $\ell_q = 0, 1, 2$ depending on whether or not $q$ is depleted, ordinary, or
unramified, respectively. If $q = p$, let $N_p = M = M_0 p$. Then consider the space $S_q$ given
as follows.

- If $q = p$, and $g_0$ has level divisible by $p$, then $S_q$ is generated by $g_0 = g$.
- If $q = p$, and $g_0$ has level prime to $p$, then $S_p$ is spanned by $g_0(z)$ and $g_0(pz)$.
- If $q \neq p$ is ordinary, then $S_q$ is spanned by the $g_0(z), g_0(qz)$.
- If $q \neq p$ and $q$ is unramified, $S_q$ is spanned by

$$\{g_0(z), g_0(qz), g_0(q^2 z)\}.$$
Each of these spaces is stable under the Hecke operator $U_q$, and is annihilated by $P'$. The eigenvalues of $U_q$ are given as follows (see [1], or [17]).

- In the first case, $U_p$ has the eigenvalue $\alpha_p$, which is a $p$-adic unit.
- In the second, the eigenvalues are $\alpha_p$, $\beta_p$, and $\beta_p$ is a non-unit.
- In the third, the eigenvalues are $\alpha_q = \sqrt{\epsilon(q)q^{k-2}}$ and 0.
- In the fourth, we have $\alpha_q$, $\beta_q$, 0, and $\alpha_q\beta_q = q^{k-1}$, so both these numbers are nonzero (and in fact units).

We claim that, in each case, the localization of $S_q$ at $m$ has dimension 1. In the case when $q = p$, the localization of $S_q$ at $m$ is one-dimensional. This is because $U_p \not\in m$ and one of the two eigenvalues $\alpha_p$ and $\beta_p$ is a $p$-adic unit and the other is not. On the other hand, in the case when $q \neq p$, $U_q \in m$. Since, $U_q$ has a unique non-unit eigenvalue ($q \neq p$) it follows that the localization of $S_q$ at $m$ has dimension 1.

An iteration of this argument over the primes $q$, using the fact that the level raising operators commute with Hecke operators away from the level, and replacing the form $g_0$ with the one-dimensional space produced in the previous step, implies that our space is one-dimensional. In greater detail, express $S_0$ as $\{q_1, \ldots, q_r\}$ and for $m \leq r$, set $S_m := \{q_1, \ldots, q_m\}$ and $N_m$ be the largest divisor of $N$ which is divisible by $M$ and the primes in $S_m$. Assume that the subspace of $S_k(N_m, O)_m \otimes \mathbb{Q}_p$ which is annihilated by $P'$ is one-dimensional over $K$, and let $g_m(z)$ be a generator of this one-dimensional space. Then, apply the same argument as above to $g_m(z)$ in place of $g_0(z)$, to prove that the subspace of $S_k(N_m+1, O)_m \otimes \mathbb{Q}_p$ which is annihilated by $P'$ is one-dimensional over $K$. This inductive argument completes the proof.

**Corollary 2.15** Let $f, g, S_0, N$ be as in Sect. 2.2, so that $f$ is the depletion of $g$ at some finite set $S_0$ of primes $q \neq p$, including $q = 2$. Then the quantity $\eta_f = (f, f)_N$ is nonzero.

**Proof** It suffices to prove the corollary upon extending the pairing to $S_k(N, O)_m \otimes \mathbb{Q} \cong \oplus R_i$. Let $(\cdot, \cdot)_{ij}$ be the restriction of the pairing $(\cdot, \cdot)_N$ to $R_i \times R_j$. It follows from Hecke-equivariance that this pairing $(\cdot, \cdot)_{ij}$ is zero if $i \neq j$. Since the pairing $(\cdot, \cdot)_N$ is non-degenerate, it follows that

$$(\cdot, \cdot)_{ii} : R_i \times R_i \rightarrow K$$

is non-zero. On the other hand, since Lemma 2.14 gives $R_i \cong K$ where $i$ is the local component of interest, it follows from $K$-linearity that $(x, x) \neq 0$ for all $x \in R_i$ such that $x \neq 0$. Note that since $f$ is an eigenform, $f$ is nonzero, and hence, we have that $(f, f)_N \neq 0$.

To continue, let $a$ denote a fixed generator of the rank-1 $T_m$-module $S_k(N, O)_m$, and consider the number $\varphi(f, a) = (f, a) \in O$. Then any element of $S_k(N, O)_m$ is of the form $t \cdot a$ for $t \in T_m$, so the Hecke equivalence of the pairing shows that

$$(f, t \cdot a)_N = (f|_t, a)_N = a(1, f|_t)(f, a)_N.$$  

In particular, it follows that $f^\perp$ is the submodule $P S_k(N, O)_m$, and

$$S_k(N, O)_m/f^\perp \cong (T_m \otimes O)/P(T_m \otimes O) \cong O,$$

where $P$ is the kernel of the canonical homomorphism $T_m(O) \rightarrow O$ associated to $f$. Since $(f, f)_N$ is nonzero, we find that $f \not\in P S_k(N, O)_m$, and that the function $\varphi_f$ is determined by the nonzero number $\eta_f = (f, f)_N \in O$.
Next we need to compare the algebraic pairing defined above to the usual Petersson inner product. Thus, given a modular form \( v(z) = \sum a_n q^n \in S_k(N, \mathbb{C}) \), define \( v^\ell(z) = \sum \overline{a}_n q^n \), where the bar denotes complex conjugation. We define a modified Petersson product on \( S_k(N, \mathbb{C}) \) by setting
\[
\{ v, w \}_N = \langle v | W_N, wc \rangle_N \tag{2.19}
\]
where the pairing on the right is the Petersson product (with our chosen normalization).

One sees from the definition that \( \{ \cdot, \cdot \}_N \) is \( \mathbb{C} \)-linear in both variables, and that it satisfies
\[
\{ v | t, w \}_N = \{ v, w | t \}_N,
\]
for any Hecke operator \( t \), just like the algebraic pairing defined above.

Recall that \( \mathcal{O} \) is the completion of the ring of integers of a number field at a prime \( p \) corresponding to an embedding in to \( \mathbb{C} \). Since we have identified \( \mathbb{C} \) with \( \mathbb{C}_p \) at the outset, we find that the space \( S_k(N, \mathcal{O})_m \) is equipped with two \( \mathbb{C}_p \)-valued pairings \( (\cdot, \cdot)_N \) and \( \{ \cdot, \cdot \}_N \). Each pairing is bilinear, and renders the Hecke operators self-adjoint. Just as in the algebraic case, we have a function \( \varphi_\infty : S_k(N, \mathcal{O})_m \to \mathbb{C}_p \) defined by \( v \mapsto \{ f, v \}_N \), and the adjointness implies that the kernel of \( \varphi_\infty \) is the submodule \( \mathcal{P}S_k(N, \mathcal{O})_m \). Thus, we have two different \( \mathbb{C}_p \)-valued functions on the rank 1 \( \mathcal{O} \)-module \( S_k(N, \mathcal{O})_m / \mathcal{P}S_k(N, \mathcal{O})_m \), and to compare them, it suffices to evaluate on any given element, say on \( f \) itself. One is therefore led to consider \( \{ f, f \}_N = \langle f| W_N, f^c \rangle_N \). It is not clear from the definition that this number is nonzero; that it is so follows from the same argument that was used in the algebraic case above.

**Definition 2.16** Define a period associated to \( f \) and the level \( N \) via \( \Omega_N = \frac{\langle f | f \rangle_N}{\langle f | f \rangle_N} \).

**Remark 2.17** As defined, the quotient is a ratio of a complex number with a \( p \)-adic number. One could remedy this by making the congruence number in the denominator algebraic, by following Hida’s original approach from the 1980s where the congruence number is defined in terms of a certain cup product pairing on cohomology, but that would require a substantial digression. Our definition presupposes an identification of \( \mathbb{C}_p \) with \( \mathbb{C} \), and that the period depends (up to unit) upon the choice of isomorphism \( T_m \sim \text{Hom}(T_m, \mathcal{O}) \). However, once this choice is made, the period is defined for all Hecke eigenforms of level \( N \) whose localization at \( m \) is nonzero.

A more serious issue is that this definition depends on the level \( N \). We would like to claim that in fact \( \Omega_N \) is independent of \( N \), and depends only on the \( p \)-stabilized newform \( g \), up to a \( p \)-adic unit. More precisely, we would like to assert that
\[
\Omega_N = \text{unit} \cdot \Omega_M
\]
where \( M = M_0p \). Here \( \Omega_M \) is defined by the same prescription as before:
\[
\Omega_M = \frac{\langle g, g \rangle_M}{\langle g, g \rangle_M}
\]
where the pairings at level \( M \) are derived from the Gorenstein condition and the modified Petersson product at level \( M \). The construction is easier in this case, since multiplicity one at level \( M \) is automatic.

Unfortunately, we cannot quite prove this claim for general weight \( k \). The case of weight 2 is known, at least under some hypotheses—this is due to Diamond, see [4, Theorem 4.2], and relies on Ihara’s lemma. While there are various versions of Ihara’s lemma known for weight \( k > 2 \), the specific version needed here does not seem to be available.
Thus, we will state the precise variant of Ihara’s lemma that we need, and make some remarks about what is known and what is required. We will then prove the independence of the period from the auxiliary level under the assumption that a suitable Ihara-type lemma holds.

To set the framework, fix a prime $p \geq 5$, and consider integers $A, B$ (the levels), together with an auxiliary odd prime $q \neq p$. We assume that $A | B$, and that one of the two following conditions holds:

1. $B = q^2A$, and $(A, q) = 1$ (the unramified case), or
2. $B = qA$, and $q | A$ (the ordinary case).

Let $T_A$ and $T_B$ denote the Hecke rings generated by all the Hecke operators, including $U_q$ as well as $T_q$, at levels $A$ and $B$, respectively. We work with the groups $\Gamma_1(A)$ and $\Gamma_1(B)$. Let $S(A), S(B)$ denote the lattice of cuspforms of levels $A, B$, respectively, whose Fourier coefficients are in $O$. Let $S(A, C), S(B, C)$ denote the corresponding complex vector spaces. We have Hecke-equivariant and $C$-bilinear and perfect analytic pairings $\langle \cdot, \cdot \rangle_A : S(A, C) \times S(A, C) \to C$ and $\langle \cdot, \cdot \rangle_B : S(B, C) \times S(B, C) \to C$, defined as above. Then let $L_A, L_B$ denote the $O$-dual to $S_A, S_B$, respectively. Namely, we have $x \in L_A$ if and only if $\{x, s\} \in O$, for all $s \in S_A$, and similarly for $S_B, L_B$. Then we have $L_A \cong T_A$ as a $T_A$-module, and similarly $L_B \cong T_B$ over $T_B$.

Next, we define a map $\tau : S(A, C) \to S(B, C)$, as follows. Let $h = h(z) \in S(A, C)$, where $z$ denotes a variable in the upper half plane. We define $\tau$ via

$$
\tau(h(z)) = \begin{cases} h(z) - (U_q h)(qz) & \text{if } B = Aq, \\ h(z) - (T_q h)(qz) + S_q q^{-1} h(q^2 z) & \text{if } B = A q^2. \end{cases}
$$

It is clear that $\tau(S(A)) \subset S(B)$, and that the image is stable under $T_B$. To check the stability under $U_q$, one can calculate explicitly that $U_q = 0$ on the image. Thus, $\tau$ is a map that removes the Euler factor at $q$.

Now let $h_A \in S(A)$ be a modular form that is an eigenvector for every element $t \in T_A$. Let $P_A$ denote the kernel of the homomorphism $\phi_A : T_A \to O$ associated to $h_A$. Let $m_A$ be the maximal ideal of $T_A$ corresponding to the inverse image of the maximal ideal of $O$, under $\phi_A$. Then, $h_B = \tau(h_A)$ is an eigenvector for $T_B$ (in fact, with $U_q h_B = 0$). We may repeat the constructions above for $h_B$, and obtain a height one prime $P_B$ and a maximal ideal $m_B$ inside $T_B$. The ideals $P_A, P_B, m_A, m_B$ are required to satisfy additional properties, which we record below:

- The localizations of $T_A, T_B$ at $m_A, m_B$, respectively, are both Gorenstein, and
- The localizations of $T_A, T_B$ at $P_A, P_B$ are fields isomorphic to the fraction field $K$ of $O$.

We will be applying these considerations to the case that $A$ divides $B$ and $B$ divides $N$, where $N$ is a level derived via depletion at $S_0$ of a $p$-stabilized newform $g$, as in Sect. 2.2. Thus, the maximal ideals we obtain at levels $m_A$ and $m_B$ are such that the residual representations at $m_A$ and $m_B$ are absolutely irreducible and $p$-distinguished (by assumption on $g$). It follows from Lemma 2.14 that the second condition will also be satisfied in the case of interest.

With these assumptions in place, we can now state the Ihara-type results that we need.
Hypothesis 2.18 With the conditions and notations above, and any choice of \( A, B, q \) as above, we have

- (Ihara-1) We have \( \tau(L_{A,m_A}) \subset L_{B,m_B} \), and
- (Ihara-2) \( L_{B,m_B}/\tau(L_{A,m_A}) \) is \( \mathcal{O} \)-torsion-free.

Remark 2.19 The lattices \( S(A, C), S(B, C) \) are, by definition, \( \mathcal{O} \)-dual to the lattices of cuspforms of level \( A, B \), respectively. It is well-known, in weight 2, that the dual lattices to the space of integral cuspforms occur in the cohomology of the modular curves \( X_1(A), X_1(B) \). The map \( \tau : S(A, C) \to S(B, C) \) is completely explicit in terms of the usual degeneracy maps of modular curves. Translating the statements we have written to the language of cohomology gives the familiar Ihara lemma, which was proven by Wiles in [32], Chapter 2. Wiles’s results were used by Diamond to prove the Ihara hypothesis as stated here, for the case \( B = q^2 A \) (see Theorem of 4.2 [4]).

The case of weight \( k > 2 \) is a bit more complicated. The cohomology of a modular curve with coefficients in the Eichler-Shimura module of coefficients may have torsion, and it is not true in clear how to identify the dual lattice with any simply described submodule of the cohomology. Furthermore, the duality pairing on cohomology is only defined with rational coefficients, and does not give any integral duality. However, one can solve these problems in the ordinary case by using Hida’s control theorems to reduce to the case of weight 2. For the details of the computation, we refer to the forthcoming thesis of Maletto.

Now we return to the situation of Definition 2.16. Consider a \( p \)-stabilized newform \( g \) of level \( M \), and the oldform \( f \) of level \( N \) associated to a choice of \( S_0 \) as before. We want to show that the periods at level \( N \) and \( M \) are equal up to a unit. This turns out to be a simple inductive argument, once Ihara’s lemma is known.

Lemma 2.20 Suppose that the Hypotheses Ihara-1 and Ihara-2 hold, for any \( A, B, q \), with \( A | B | N \). Then \( \Omega_N = u \Omega_M \) for some \( p \)-adic unit \( u \).

Proof We start at level \( M \), and work our way upwards, adding one prime at time. To spell out the induction, we start with a modular form \( h_A \) at level \( A \), and we move up to level \( B = Aq \) or \( Aq^2 \), and replace \( h_A \) with the \( q \)-depleted form \( h_B \). In this situation, we are required to show that the periods of \( h_A \) and \( h_B \) are equal up to a unit.

It is clear from the definition of the periods that \( \Omega_A \) at level \( A \) is characterized up to a unit by the properties:

- \( \delta_A := \Omega_A^{-1} \cdot h_A \) is contained in \( L_{A,m_A} \),
- \( L_{A,m_A}/\mathcal{O}\delta_A \) is torsion-free.

Similarly, the period \( \Omega_B \) at level \( B \) is characterized by:

- \( \delta_B := \Omega_B^{-1} \cdot h_B \) is contained in \( L_{B,m_B} \),
- \( L_{B,m_B}/\mathcal{O}\delta_B \) is torsion-free.

Since \( \tau(h_A) = h_B \) by definition, Ihara-1 shows that \( \delta_B := \tau(\delta_A) = \tau(\Omega_A^{-1} h_A) \) is contained in \( L_{B,m_B} \). Let \( u \) be such that \( \delta_B = u \delta_B' \). We show that \( u \) is a \( p \)-adic unit. We have that \( \delta_B' \in L_{B,m_B} \) and \( L_{B,m_B}/\mathcal{O}\delta_B \) is torsion-free. Hence, \( u^{-1} \) is contained in \( \mathcal{O} \). According to Ihara-2, \( L_{B,m_B}/\tau(L_{A,m_A}) \) is torsion-free. We have that \( u^{-1} \delta_B = \delta_B' = \tau(\delta_A) \in \tau(L_{A,m_A}) \). Hence, \( \delta_B \) is contained in \( \tau(L_{A,m_A}) \), and we write \( \delta_B = \tau(\eta_A) \). Since the map \( \tau \) is injective, it
follows that $u^{-1} \eta_A = \delta_A$. Since $L_{A,m_A}/\mathcal{O}\delta_A$ is torsion-free, it follows that $u \in \mathcal{O}$. We have shown that $u \in \mathcal{O}$ and $u^{-1} \in \mathcal{O}$, so we have deduced that $u \in \mathcal{O}^\times$. Therefore, $\Omega_A = u \Omega_B$ for some unit $u$.

\begin{remark}
To get a nice formula at the end, and to verify that our final formulae agree with those in [15], we need to further calculate further. We have shown above that the ratio of the algebraic and analytic pairings at level $M$ and level $N$ are the same. It remains to express everything in terms of the newform $g_0$ associated to $f$ and $g$. In other words, we have to bring everything down to level $M_0$. There are two cases to consider, depending on whether or not the $p$-stabilized form is new or old at $p$ (so $M = M_0p$ or $M = M_0$).

Start with the case that $g_0$ is old at $p$. Then a further calculation (see Lemma 27 of [19]) shows that $(g,g)_M = (p - 1) \cdot E_p \cdot (g_0, g_0)_{M_0}$, where $E_p = \pm p^{1-k/2} \alpha_p (1 - \frac{p^{k-2}}{a_p^2})(1 - \frac{p^{k-1}}{a_p^2})$, and $g_0$ is the newform of level $M_0$ associated to $f$, and $\alpha_p$ is the unit root of the Hecke polynomial. The factor of $p - 1$ comes from the fact that the Petersson inner product in [19] is defined as an integral over a fundamental domain for $\Gamma_0$, not $\Gamma_1$. The factors $1 - \frac{p^{k-2}}{a_p^2}$ and $1 - \frac{p^{k-1}}{a_p^2}$ are units for $k > 2$. When $k = 2$, the term $1 - 1/\alpha_p^2$ may be a non-unit; this is so precisely when $g_0$ is congruent to a $p$-new form of level $pM_0$. The number $1 - 1/\alpha_p^2$ is the relative congruence number of Ribet [22].

In the $p$-old situation (and all weights), we define

$$ (g_0, g_0) = \frac{(g,g)_M}{1 - 1/\alpha_p^2}. $$

Note that $\alpha_p \neq \pm 1$, by the Weil bounds. We remark that it can be shown that in fact $(g_0, g_0)$ as defined above coincides with the pairing of $g_0$ with itself defined via a Gorenstein pairing at level $M_0$ (as opposed to the $p$-stabilized level $M = M_0p$). We don’t need this result, but mention it simply to justify the notation. We refer the reader to [32], Chapter 2, Section 2, for a full discussion of relative congruence numbers in weight 2.

If $f$ is new at $p$ (and hence of weight 2), then of course $(g,g)_M = E_p \cdot (g_0, g_0)_{M_0}$ where $E_p = \pm 1$ is the eigenvalue of the Fricke involution.

The number $\Omega_M$ is almost the canonical period, but not quite: it depends not only on $g_0$, but on the choice of the prime $p$, and the stabilization of $g_0$ at the unit root $\alpha_p$ of the Hecke polynomial. We can get rid of this dependency as follows.

If $f$ is old at $p$, we have

$$ \Omega_M = \frac{(g,g)_M}{(g,g)_M} = \text{unit} \cdot E_p \cdot \frac{(g_0, g_0)_{M_0}}{(g,g)_M} = \text{unit} \cdot p^{1-k/2} \frac{(g_0, g_0)_{M_0}}{(g_0, g_0)_{M_0}}. $$

Note that we have used Proposition 2.4 of [32] to account for the non-unit congruence number for $g_0$ in the last step of equalities above.

If $f$ is new at $p$, so that $g = g_0$ and $M = M_0$, and we are in weight 2, we have

$$ \Omega_M = \frac{(g,g)_M}{(g,g)_M} = E_p \cdot \frac{(g_0, g_0)_{M_0}}{(g,g)_M} = \text{unit} \cdot \frac{(g,g)_M}{(g,g)_M}. $$

A common way of expressing the above formulae is by defining

$$ \Omega_M = \text{unit} \cdot p^{1-k/2} \cdot \frac{(g_0, g_0)_{M_0}}{(g_0, g_0)_{M_0}}. \tag{2.20} $$

With this formulation, the only dependence on $p$ or $g$ is absorbed in the unit factor. Thus, we may simply set

$$ \Omega_M = \text{unit} \cdot p^{1-k/2} \cdot \frac{(g_0, g_0)_{M_0}}{(g_0, g_0)_{M_0}}. $$
\[ \Omega_{\varphi_0}^{\text{can}} = \frac{\langle \varphi_0, \varphi_0 \rangle_M}{\langle \varphi_0, \varphi_0 \rangle_M}, \]  
(2.21)

as stated in the introduction.

Remark 2.22 The factor \( p^{1-k/2} \) which shows up in the comparison with \( \Omega_M \), is important—it shows up in the formulae (2.11), and those of Schmidt in [25], [26], where it is simply carried around. As we shall see, it exactly cancels unwanted powers of \( p \) arising from (2.11).

Now, if \( h \in S_k(N, \mathcal{O})_m \) is arbitrary, then we have \( \frac{\langle f, h \rangle_N}{\langle f \rangle_N} = \frac{\langle f, h \rangle_N}{\langle f \rangle_N} = \frac{\langle f, h \rangle_N}{\langle f \rangle_N} \). In view of the independence of the period on the level, we get the following key evaluation formula, valid for any \( h \in S_k(N, \mathcal{O})_m \):

Proposition 2.23 Assume that the Ihara hypotheses are valid. Then we have the equalities
\( \langle f, h \rangle_N = \frac{\langle f \rangle_N}{\langle f \rangle_N} = \text{unit} \cdot \frac{\langle f \rangle_N}{\langle f \rangle_N} \). Further, the quantity
\[ \frac{\langle f, h \rangle_N}{\Omega_M} = \text{unit} \cdot p^{k/2-1} \cdot \frac{\langle f, h \rangle_N}{\langle \varphi_0, \varphi_0 \rangle_M} \]
is \( p \)-integral.

Remark 2.24 Our next task will be to apply the machinery developed above to the case where \( h = \mathcal{H}_f(n) \) is derived from a product of a theta series and Eisenstein series. However, there are two problems. First, this product is unlikely to be cuspidal, and second, it is not an element of \( S_k(N, \mathcal{O})_m \). Some care is therefore required. The number \( \langle f, h \rangle_N = \langle f \circ W_N, h^c \rangle_N \) makes sense for any \( h \in M_k(N, \mathcal{O}) \otimes \hat{\mathbb{Q}} \), since \( f \) is cuspidal. Since the maximal ideal \( m \) corresponding to \( f \) is residually irreducible, we find that if \( e_m \) is the idempotent in the Hecke algebra \( T = \oplus T_m \), corresponding to the maximal ideal \( m \), then \( h \circ e_m \) is cuspidal for any modular form \( h \) of level \( N \) and weight \( k \). But now \( f \circ e_m = f \), and the Hecke operators are self-adjoint under the modified pairing, hence \( \langle f, h \circ e_m \rangle_N = \langle f, h \rangle_N \).

Thus, we may replace \( h \) with \( h \circ e_m \), and define \( \langle f, h \rangle_N = \langle f, h \circ e_m \rangle_N \) for any \( h \in M_k(N, \mathcal{O}) \otimes \hat{\mathbb{Q}} \). Then the same formalism as above applies. In particular, we still have \( \frac{\langle f \rangle_N}{\langle f \rangle_N} = \frac{\langle f \rangle_N}{\langle f \rangle_N} = \frac{\langle f \rangle_N}{\langle f \rangle_N} \), and the conclusion of Proposition 2.23 applies without change.

2.9 Integrality
In view of the considerations above, we are led to compute the algebraic pairings \( \langle f, \mathcal{H}_f(n) \rangle_N \) and \( \langle f, \mathcal{H}^m_f(n) \rangle_N \), with \( \mathcal{H}_f(n) \) and \( \mathcal{H}^m_f(n) \) being as defined in (2.15) and (2.16).

To use our previous formulae, we need to assume \( 2 \in S_0 \). Then, we know already that \( \mathcal{H}_f(n) \) and \( \mathcal{H}^m_f(n) \) are integral, hence the corresponding pairings are integral as well. It remains only to relate them to special values of \( L \)-functions. The starting point is Proposition 2.23, which reduces the calculation to that of analytic pairings \( \langle f, \mathcal{H}_f(n) \rangle_N \) and \( \langle f, \mathcal{H}^m_f(n) \rangle_N \). Pick any \( m \geq m_f \). The analytic pairing is computed in equation (2.13), which states that
\[ \frac{\Gamma(n/2)}{(4\pi)^{n/2}} p^{(2m_f - 1)(k/2 - 1)} D_f(\chi, n) \]
\[ = \text{unit} \cdot \alpha^2_p(m_f - m) f \circ W_N, H_f(n) \circ W_{N_f} \circ U_{p^{2m_f - 1}} \circ T_p \rangle_N. \]

Using this formula, and plugging in all the definitions, we obtain
Corollary 2.25 Suppose that $\eta$ is ramified and $m \geq m_\chi$ is any integer. There exists a $p$-adic unit $u$ depending on $n$ and $k$ and $\epsilon$ and $\psi$ such that we have

$$u \cdot \frac{p^{1-k/2}}{\pi^n} \left( \frac{p^{n-1}}{\psi(p)} \right)^{m_\chi} \left( \frac{1}{\alpha_{p}^{2}} \right)^{m_\chi - m} \cdot \frac{\eta_0}{G(\eta)} \cdot D_{\psi}(\chi, n) = \Omega_{\chi}^{\eta}(n).$$

where $\Omega_{\chi}^{\eta}(n) = H_{\chi}(n)^{\eta} \cdot U_{p}^{2m-1} \cdot T_{\psi} = H_{\chi}(n)^{\eta} \cdot U_{p}^{2m-1}$. 

**Proof** This is a direct computation, using Lemmas 2.6 and 2.8 and applying the doubling formula for the $\Gamma$-function in the formula for the coefficients $d_{i,v}$; see also Lemma 4.2 of [25]. One has also to use the factorization $g(\chi) = \psi(p^{n_{x}}) \cdot \eta_{0} \cdot G(\eta)$, as well as the normalization of the modified pairing (2.19). The constant $u$ collects up all the various powers of $2$, $i$, and other quantities prime to $p$.

Observe that the formula above contains the nuisance factor $p^{k/2-1}$, which also appears in our period. Assuming Hypothesis 2.18, so that $\Omega_{\chi}^{\eta} = \text{unit} \cdot p^{k/2-1} \cdot \Omega_{\chi}$, where $N$ is the level of the depleted form $f$, and plugging in (2.20), we find that

$$\langle f, \Omega_{\chi}^{\eta}(n) \rangle = u' \cdot \frac{p^{1-k/2}}{\pi^n} \left( \frac{p^{n-1}}{\psi(p)} \right)^{m_\chi} \left( \frac{1}{\alpha_{p}^{2}} \right)^{m_\chi - m} \cdot \Gamma(n) \cdot \frac{D_{\psi}(\chi, n)}{\pi^n \Omega_{\chi}}$$

$$= u' \cdot \frac{p^{n-1}}{\psi(p)} \left( \frac{1}{\alpha_{p}^{2}} \right)^{m_\chi - m} \cdot \Gamma(n) \cdot \frac{D_{\psi}(\chi, n)}{\Omega_{\chi}^{\eta}}$$

$$= u' \cdot \frac{p^{n-1}}{\psi(p)} \left( \frac{1}{\alpha_{p}^{2}} \right)^{m_\chi - m} \cdot \Gamma(n) \cdot \frac{D_{\psi}(\chi, n)}{\Omega_{\chi}^{\eta}}.$$  

(2.24)

Here $u'$ is some other unit, independent of $\chi$.

Finally, we have to deal with $\Omega_{\chi}(n) \circ e$. It is not hard to see that the twisted trace operator $T_{\psi}$, which goes from level $N_{\psi}$ to level $N$, commutes with the Hecke operator $U_{p}$, since $N_{\psi} \parallel N$ has no common factor with $p$. There does not seem to be any particularly pleasant way to deduce this fact from a classical perspective where the trace operator is given by matrices with rational integer entries, but it is more or less obvious from the point of view of representation theory, since the local trace involves primes away from $p$, while $U_{p}$ is concentrated at $p$. It is also evident it one considers modular forms as functions on test objects on moduli spaces of enhanced elliptic curves-$U_{p}$ is a sum over certain subgroups of order $p$, while the trace from level $N_{\psi}$ involves subgroups of order prime to $p$. Anyway, we take this fact for granted, so that if $m$ is large, we have

$$\Omega_{\chi}^{\eta}(n) = H_{\chi}(n)^{\eta} \cdot U_{p}^{2m-1} \cdot T_{\psi} = H_{\chi}(n)^{\eta} \cdot U_{p}^{2m-1}.$$ 

We may then consider an suitable increasing sequence of integers $m$, divisible by $p - 1$, so that the forms $\Omega_{\chi}^{\eta}(n) \circ U_{p}$ converge to $\Omega_{\chi}(n) \circ e$. The algebraic inner product is linear, and we conclude that

$$\langle f, \Omega_{\chi}^{\eta}(n) \circ e \rangle = \text{unit} \cdot \left( \frac{p^{n-1}}{\psi(p)\alpha_{p}^{2}} \right)^{m_\chi} \cdot \Gamma(n) \cdot \frac{D_{\psi}(\chi, n)}{\pi^n \Omega_{\chi}^{\eta} \cdot D_{\psi}(\chi, n).}$$

(2.25)

In particular, the right-hand side is integral. Observe that the quantity on the right is a constant multiple of the one appearing in the definition of the $\psi$-twisted $p$-adic L-function given in (1.2). The Euler factor disappears because $\eta$ is ramified at $p$. 
2.10 Level raising and congruences

In view of the construction given above, it is more or less clear that that the $p$-adic $L$-functions satisfy good congruences. To state the result, consider two $p$-ordinary and $p$-stabilized newforms $g_1$, $g_2$, which are such that the residual representations $\overline{\rho}_{g_1}$ and $\overline{\rho}_{g_2}$ are isomorphic to some fixed representation $\overline{\rho}$. We assume, as always, that $\overline{\rho}$ is irreducible, ordinary, and distinguished. We write $M_i$ for the level of $g_i$, and $\epsilon_i$ for the corresponding nebentype character. The considerations in the previous paragraphs give an integral construction of imprimitive $p$-adic $L$-functions for each $g_i$, associated to some choice of a set $S_0$ of primes $q \neq p, 2 \in S_0$, which depends on $i$. We now show that we can choose a set for $S_0$ such that depletion of $g_1$ and $g_2$ yields a common level $N$ where semisimplicity is retained and we can apply our construction.

Recall that $\overline{\rho}$ denotes the common two-dimensional residual representation for the $g_i$. Let $q$ denote a prime number, $q \neq p$. In line with our previous convention, we shall say that $\overline{\rho}$ is ordinary at $q$ if the subspace $\overline{\rho}^{\psi_q}$ of invariants in $\overline{\rho}$ under an inertia group $I_q$ at $q$ has dimension 1. If $\overline{\rho}^{\psi_q} = 0$, we say that $\overline{\rho}$ is depleted at $q$, and if $\overline{\rho}^{\psi_q} = \overline{\rho}$, we say that $\overline{\rho}$ is unramified. Since each $g_i$ is a lift of $\overline{\rho}$, it follows that each $\rho_{g_i}$ is depleted at $q$ if the common representation $\overline{\rho}$ is so. If $\overline{\rho}$ is ordinary, then each $\rho_{g_i}$ is either depleted or ordinary. Let $\overline{M}$ denote the conductor of $\overline{\rho}$, as defined on page 104 of [6].

Our first task is to determine the possible difference between the levels $M_1$ and $M_2$. The answer is given by Proposition 6 of [6], which we paraphrase as follows:

**Lemma 2.26** For each prime $q$ we have $\text{ord}_q(M_1) = \text{ord}_q(M_2) = \text{ord}_q(\overline{M})$, unless one of the following occurs:

- Either $\overline{\rho}$ is unramified, in which case, $\text{ord}_q(M_i) \leq 2$, for $i = 1, 2$; or
- $\overline{\rho}$ is ordinary, in which case $\text{ord}_q(M_i) \leq \text{ord}_q(\overline{M}) + 1$, for $i = 1, 2$. If $\text{ord}_q(M_i) = \text{ord}_q(\overline{M}) + 1$, then $\rho_{g_i}$ is depleted at $q$. If $\text{ord}_q(M_i) = \text{ord}_q(\overline{M})$, then $\rho_{g_i}$ is ordinary.

It is therefore clear that if we take $S_0$ to be any finite set primes containing 2, as well as all primes dividing $M_1 M_2$, plus any finite set of primes away from the levels, then the considerations of the previous paragraphs apply. The corresponding depletions have the same level $N$, where $\text{ord}_q(N) = \text{ord}_q(\overline{M})$, except at primes $q \in S_0$ where $\overline{M}$ is unramified, in which case $\text{ord}_q(N) = 2$, and primes $q \in S_0$ where $\overline{M}$ is ordinary, in which case $\text{ord}_q(N) = \text{ord}_q(\overline{M}) + 1$.

We can now state the theorems around congruences for the imprimitive symmetric square $L$-function, but we need to recall all the hypotheses and notation. Thus, suppose that $p$ is a prime of $\bar{\mathbb{Q}}$ with residue characteristic $p \geq 5$, and that $g_1, g_2$ are $p$-stabilized and $p$-ordinary newforms of weight $k$, and level $M_1, M_2$, respectively, such that the Fourier coefficients $a(q, g_i)$ satisfy the congruence $a(q, g_1) \equiv a(q, g_2) \pmod{p}$, for each prime $q \mid M_1 M_2 p$. Assume that the corresponding residual representations are absolutely irreducible and $p$-distinguished, and the nebentype characters are trivial on $(\mathbb{Z}/p\mathbb{Z})^*$. Let $\alpha_{l,p}$ denote the eigenvalue under $U_p$ of $g_i$. Let $\psi$ denote an even character of conductor prime to $p$, such that $(\psi \epsilon_i)^2 \not\equiv 1 \pmod{p}$, and let $\eta$ denote a nontrivial Dirichlet character of $p$-power conductor, and set $\chi = \psi \eta$. Let $\Omega_1, \Omega_2$ denote the canonical periods associated to $g_1, g_2$ and the prime $p$, as above. Thus, $\Omega_i = p k/2 - 1 \langle g_0, g_i \rangle M_0 g_0 \langle g_i, g_0 \rangle M_0$, where $g_0$ is the newform of level $M_0$, corresponding to $g_i$. Let $n$ denote an odd integer in the range $1 \leq n \leq k$. 
Finally, fix a finite set $S_0$ of primes $q \neq p$ containing 2 and all primes dividing $M_1M_2$. Let $f_1, f_2$ denote the depleted forms of level $N$, associated to the forms $g_1, g_2$, and the set $S_0$.

**Theorem 2.27** Let the hypotheses and notation be as above. Then there exist units $u_i$, independent of $\chi$, such that we have the congruence

$$
u_1 \cdot \left( \frac{p^{n-1}}{\psi(p)\omega^2_{L_p}} \right)^{m_x} \Gamma(n)G(n) \cdot \frac{D_{f_1}(\chi, n)}{\pi^n \Omega_1} \equiv u_2 \cdot \left( \frac{p^{n-1}}{\psi(p)\omega^2_{L_p}} \right)^{m_x} \Gamma(n)G(n) \cdot \frac{D_{f_2}(\chi, n)}{\pi^n \Omega_2} \pmod{p}. \quad (2.26)$$

**Proof** This follows from the linearity of the functional $S_k(N, \mathcal{O}) \to \mathcal{O}$ given by $x \mapsto (x, \mathcal{H}_\chi(n) \circ e)_N$. \hfill \Box

### 2.11 The primitive L-function and p-adic interpolation

We now write down the relationships between the primitive and variously imprimitive $L$-functions, and the interpolation properties that characterize the $p$-adic $L$-functions. We also have to remove the hypothesis $2 \in S_0$ which we imposed for the purposes of our calculation of special values. For notational simplicity, let us fix the newform $g_0$, and write the level of $g_0$ as $M_0$. The corresponding $p$-stabilized newform will be denoted by $g$ and its level shall be denoted by $M$. For each prime $q$, we have a complex representation $\pi_q$ of $GL_2(\mathbb{Q}_q)$ associated to $g$. The first task is to work out the Euler factors of the symmetric square lift $\Pi_q$ of $\pi_q$ to $GL_3$. This is all contained in [5], and is recapitulated in Section 1 of [26], especially Lemmas 1.5 and 1.6, but some translation is required. We notice first of all that the representations $\Pi$ and $\Sigma$ considered by Schmidt are not exactly the symmetric square of [15], and that his normalization introduces an inverse when comparing with the Euler product of Shimura considered here. The exact relationship is given in the last line of page 603 of [26]. For us, the point is that the Euler factors of our $D_g(\chi_0, s)$ coincide with Schmidt’s $L(s - k + 1, \Sigma \otimes \chi^{-1})$ for any primitive (in our case even) Dirichlet character $\chi_0$ with corresponding idle class character $\chi$, at almost all primes. With this normalization in mind, one can read off the Euler factors for the automorphic representation $\Pi$ from Lemmas 1.5 and 1.6 of [26].

To state the result, let us write

$$L(r_g \otimes \chi, s) = \prod_q P_q(\chi, q^{-s})^{-1}$$

to denote the complex $L$-function associated to the Galois representation $r_g \otimes \chi$, as in the introduction. Now let $S_0$ be any finite set of primes $q \neq p$. We no longer require $2 \in S_0$. Then the imprimitive $L$-function is defined by the same formula, except with the product taken over primes $q \notin S_0$. Recall also that we have are identifying Galois characters and Dirichlet characters via $\chi(\text{Frob}(q)) = \chi(q)$, and $\text{Frob}(q)$ is normalized so that $\text{Frob}(q)$ lifts $x \mapsto x^q$.

For all but finitely primes $q$, we have

$$P_q(\chi, q^{-s}) = \left( 1 - \chi(q)\omega_q \beta_q q^{-s} (1 - \chi(q)\beta_2^2 q^{-s}) (1 - \chi(q)\omega_2^2 q^{-s}) \right).$$

If the set $S_0$ is sufficiently large, and $\chi$ is ramified at $p$, then $D_f(\chi, s) = L_{S_0}(r_g \otimes \chi, s)$, and so we get

$$D_f(\chi, s) = D_g(\chi, s) \cdot \prod_q P'_q(\chi, q^{-s}) = L(r_g \otimes \chi, s) \cdot \prod_q P_q(\chi, q^{-s}) = L_{S_0}(r_g \otimes \chi, s).$$
where once again the $P'_q$ and $P_q$ are polynomials in the variables $X = q^{-s}$ (compare [15], Proposition 2.1.5). The polynomials $P_q$ are those given by Schmidt, whereas the $P'_q$ are the ones given by the Euler factor at $q \in S_0$ in Shimura’s Euler product. Each of the products is taken over the finite set of primes in $S_0$, together with the primes of ramification of $r_g \otimes \chi$, with the understanding that some of the factors may be trivial.

Furthermore, we have

\[ \psi \eta \]

for any infinite collection of characters of the form $\psi \eta \in \psi \eta$, where $\eta$ is our fixed even character of conductor prime to $p$. We remark that, under the identification of $\mathcal{O}[\text{Gal}(\mathbb{Q}_{\text{cyc}}/\mathbb{Q})]$ with the subgroup of $1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, we have $\text{Frob}(q) = q_\omega = \eta_\omega^{-1}(q) q$, as in the introduction.

In the setup of $p$-adic L-functions, we have $\chi = \psi \eta$, where $\eta$ has $p$-power conductor. Furthermore, we have $\psi \eta = \psi \eta_\omega$, for some even $t$, $0 \leq t \leq p - 2$, and some character $\eta_\omega$ of $p$-power order. A glance at the formulae on pages 604–605 in [26] shows that in fact one has

\[ \eta_{w,n}(P_q(\psi t)) = P_q(\psi_{t+n-1} \eta_\omega q^{-n}), \]

consistent with the basic formula (2.12).

Following [15], the primitive $p$-adic L-function associated to the newform $g_0$ of level $M$ and the representation $r_g \otimes \psi_1$ is an element $L^{\text{an}}(r_g \otimes \psi_1)$ of $\Lambda$ characterized by

\[ \eta_{w,n}(L^{\text{an}}(r_g \otimes \psi_1)) = \Gamma(n) \cdot E_p(n,\chi) \cdot G(\eta_{1-t-n} \eta_\omega^{-1}) \cdot \frac{L(r_g \otimes \psi_{t+n-1} \eta_\omega, n)}{\pi^{n} \Omega_{g0}^{\text{can}}}, \]

for $n$ odd, $1 \leq n \leq k - 1$. The Euler factor $E_p(n,\chi)$ is given by

\[ E_p(n,\chi) = (p^{n-1} \psi(p)^{-1} \alpha_p^{-2})^{m_\chi}, \]

if $\eta$ is nontrivial and has conductor $p^m > 1$. If $\eta$ is trivial and $g_0$ has level prime to $p$, then

\[ E_p(n,\eta) = (1 - p^{n-1} \psi(p)^{-1} \alpha_p^{-2})(1 - \psi(p)p^{k-1-\eta})(1 - \psi(p)p^{k-1-\eta}), \]

A similar formula holds when $k = 2$ and $g$ has level divisible by the first power of $p$; we omit it here, as we do not need it. We remark also that the construction of this paper does not prove that the $p$-adic L-function exists.

Observe that our formula above is the same as that in [15], except that

- We have used the canonical period rather than the Petersson product.
- We have suppressed unit factors of 2 and $i$ that depend only on the weight.
- We have used our parity assumptions to get rid of various minus signs.
- We have adjusted the action of the Iwasawa algebra to match the Selmer group defined by Greenberg.

It is clear that if such a function exists, then it is characterized by the validity of the formula above, for any infinite collection of characters of the form $\eta_{w,n}$. If $S_0$ denote any finite set of prime numbers $q \neq p$ (which may not contain $q = 2$), then $L^{\text{an}}(r_g \otimes \psi_1)$ (assuming it exists) is an element of $\Lambda$ characterized by the analogue of (2.28).

The existence of an L-function interpolating the values of $D_g(\chi, s)$ was proven by Schmidt [25], [26]. He states in his work that very similar results were obtained by Hida, but never published. He (Schmidt) subsequently proved the existence of the primitive L-function $L^{\text{an}}(r_g \otimes \psi_1)$ under some hypotheses, which were later removed by Hida [11] and Dabrowski-Delbourgo [3]. Schmidt did not construct an interpolation of the
imprimitive $D_f(\chi, s)$, or the imprimitive $L_{S_0}^\text{an}(r_g \otimes \psi_t)$, but in fact those follow easily: one simply multiplies the existing $L$-function by the appropriate Euler factors, each of which is represented by an element $P_1(X)$ or $P'_q(X)$ of $\Lambda$.

As we have already remarked, this construction of imprimitive $p$-adic $L$-functions does not help in proving congruences, since we cannot compare the $L$-functions for different forms in any way: each one is obtained from a construction at a different minimal level, which are not related in any simple way.

In any case, we see from equation (2.27) that the relationship between the primitive and imprimitive $p$-adic $L$-functions is given by

$$L_{S_0}^\text{an}(r_g \otimes \psi) = \prod_{q \in S_0} P_q(\psi_t) \cdot L_{S_0}^\text{an}(r_g \otimes \psi_t).$$

(2.29)

A priori, both $L$-functions above are elements of $\Lambda \otimes \mathbb{Q}$. A similar formula of course holds for the $p$-adic $L$-function interpolating $D_f(\chi, s)$ and $D_g(\chi, s)$.

We can now give the proof of the various remaining results on $p$-adic $L$-functions stated in the introduction. We start with a simple lemma.

**Lemma 2.28** For any prime $q \in S_0$, the elements $P_q, P'_q \in \Lambda$ have $\mu$-invariant zero.

**Proof** The statement for $P_q$ follows from the explicit formulae for the Euler factors in [26], or the observation there on page 605 that the polynomials $P_q$ all satisfy $P(0) = 1$. The same argument works for $P'_q$, since these are given explicitly in Shimura’s defining Euler product. \qed

**Corollary 2.29** We have $\mu_{S_0}^\text{an} = 0 \iff \mu^\text{an} = 0$.

This lemma implies Proposition 1.5, simply by taking $g = g_i$, and $\sigma_i^{(q)}$ to be the degree of the polynomial $P_q$ associated above to $g_i$ at $q$, and using the fact that the $\mu$-invariant of $P_q$ is zero in the formula (2.29).

Next we deal with integrality properties, as stated in Theorem 1.4. We claim that in fact both $L_{S_0}^\text{an}(r_g \otimes \psi)$ and $L_{S_0}^\text{an}(r_g \otimes \psi)$ lie in $\Lambda$ and are integral, with the same canonical period. First we apply our construction to the $p$-stabilized newform $g$ and large set $S_0$ containing 2 and all the bad primes. Then since $D_f(\chi, s) = L_{S_0}(r_g \otimes \chi, s)$ for sufficiently large $S_0$, it follows from the formulae (2.22) and (2.25) that the quantity in the interpolation formula (1.2) is integral, for almost all characters $\eta$. The Weierstrass preparation theorem, applied to $L_{S_0}^\text{an}(r_g \otimes \psi)$, shows that the latter is an element of $\Lambda$. But now we have $L_{S_0}^\text{an}(r_g \otimes \psi) = L_{S_0}^\text{an}(r_g \otimes \psi) \cdot \prod P_q$, for integral polynomials $P_q$ with $\mu$-invariant zero, so $L_{S_0}^\text{an}(r_g \otimes \psi)$ is integral because the ring of power series with coefficients in a field is an integral domain. We may now obtain the result for any given set $S_0$, simply by multiplying by the factors $P_q(X), q \in S_0$. This proves Theorem 1.4.

Finally, we have to deal with congruences. Let $g_1, g_2$ be $p$-congruent newforms satisfying our running conditions. Let $S$ denote any set of primes including 2, and the set of primes dividing $M_1 M_2$, and let $S_0 = S\setminus \{p\}$ be as above. Furthermore, we have to assume that the Ihara hypotheses hold. We claim that we have

**Proposition 2.30** Let the notation be as above. Then we have $L_{S_0}^\text{an}(r_{g_1} \otimes \psi_t) \equiv u L_{S_0}^\text{an}(r_{g_2} \otimes \psi_t) \pmod{p}$, where $u$ is a $p$-adic unit and the congruence is that of elements in the completed group algebra $\mathcal{O}[[\mathbb{Z}_p]]$. 
Proof. This follows from the congruence of special values in Theorem 2.27, and the Weierstrass preparation theorem. □

We remark that the result in Theorem 2.27, and the statement of the theorem above, remain valid without the Ihara hypothesis, if replaces the canonical periods with the periods $\Omega_{\mu,N}$. However, $N$ depends on the set $S_0$.

Finally, we observe that the analytic part of Theorem 1.6 follows immediately, since two congruent Iwasawa functions with $\mu$-invariant zero necessarily have the same $\lambda$-invariant, and that if one has $\mu$-invariant zero, then so does the other.

3 Imprimitive Iwasawa Invariants: the algebraic side

We start by recalling the notation. From the previous sections. Throughout, let $p \geq 5$ be a fixed prime and $g$ be a normalized Hecke-eigencuspform of weight $k \geq 2$ on the congruence group $\Gamma_0(M)$.

Denote the number field generated by the field of Fourier coefficients of $g$ by $L$. For each prime $q$, choose an embedding $\iota_q : \bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_q$. Let $p|q$ be the prime of $L$ such that the inclusion of $L$ in $\mathcal{O}_q$ is compatible with $\iota_q$. Denote by $K$ the completion of $L$ at $p$, and $\mathcal{O}$ the valuation ring of $K$ with uniformizer $\varpi$. Associated with $g$ is the continuous Galois representation $\rho_g : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathcal{O})$. Let $V_g \simeq K^2$ be the underlying two-dimensional vector space on which $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts via $K$-linear automorphisms. Fix a Galois stable $\mathcal{O}$-lattice $T_g$ inside $V_g$. Let $F$ be the residue field of $\mathcal{O}$.

The mod-$p$ reduction of $\rho_g$ is denoted by $\bar{\rho}_g : G_Q \rightarrow \text{GL}_2(F)$, and it follows from the Brauer-Nesbitt theorem that the semi-simplification of $\bar{\rho}_g$ is independent of the choice of lattice $T_g$. Throughout, we make the following assumptions on $g$:

1. $g$ is ordinary and $p$-distinguished at $p$,
2. $\bar{\rho}_g$ is absolutely irreducible.

Since $\bar{\rho}_g$ is absolutely irreducible, the choice of Galois stable lattice $T_g$ is unique. Letting $G_q$ denote $\text{Gal}((\bar{\mathbb{Q}}_q/\mathbb{Q}_q)$, we note that the choice of embedding $\iota_q$ gives an inclusion of $G_q$ into $G_{\mathcal{O}}$. Let $X_{\text{cyc}} : G_p \rightarrow \mathcal{O}^\times$ denote the $p$-adic cyclotomic character. Since $g$ is ordinary at $p$, there is a short exact sequence

$$0 \rightarrow T_g^+ \rightarrow T_g \rightarrow T_g^- \rightarrow 0$$

of $G_p$-stable $\mathcal{O}$-lattices such that there are unramified characters $\gamma_1, \gamma_2 : G_p \rightarrow \mathcal{O}^\times$ for which

$$T_g^+ \simeq \mathcal{O}(X_{\text{cyc}}^{k-1}\gamma_1) \text{ and } T_g^- \simeq \mathcal{O}(\gamma_2).$$

Fix a finite order even character $\psi$ of conductor $c_\psi$ coprime $Mp$. Let $t$ be an even integer in the range $0 \leq t \leq p - 2$ and recall that $\psi_t$ denotes the Dirichlet character $\psi^{t_1}$, defined in the introduction. Consider the lattice $T_g := \text{Sym}^2 T_g$ and the symmetric square representation

$$r_g \otimes \psi_t := \text{Sym}^2(\rho_g) \otimes \psi_t : \text{Gal}((\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_3(\mathcal{O}).$$

Set $V_g := T_g \otimes \mathbb{Q}_p$ and $A_g := V_g/T_g$. The representation $T_g$ is $p$-ordinary, i.e., is equipped with a filtration of $G_p$-modules

$$T_g = F^0(T_g) \supset F^1(T_g) \supset F^2(T_g) \supset F^3(T_g) = 0.$$

...
For $j = 1, 2$, and unramified characters $\delta_j$, we have that
\[
\text{gr}_0(T_{g_j}) \simeq \mathcal{O}(\chi_{\text{cyc}}^{k-2} \delta_0),
\]
\[
\text{gr}_1(T_{g_j}) \simeq \mathcal{O}(\chi_{\text{cyc}}^{k-1} \delta_1),
\]
\[
\text{gr}_2(T_{g_j}) \simeq \mathcal{O}(\delta_2).
\]

With this notation in place, we consider Hecke $p$-stabilized eigencusps $g_j$ of weight $k \geq 2$, level $M_i$, and character $\epsilon_i$, as in the introduction. Setting $L$ to be the number field generated by the Fourier coefficients of $g_1$ and $g_2$, let $p$ be the prime of $L$ above $p$ corresponding to the choice of $\iota_p$. Let $\psi$ denote a Dirichlet character of conductor coprime to $p$ such that $(\psi \epsilon_i)^2 \equiv 1 \pmod{p}$. Assume that $\rho_{g_j}$ is absolutely irreducible and that the following equivalent conditions are satisfied.

1. The residual representations are isomorphic: $\bar{\rho}_{g_1} \simeq \bar{\rho}_{g_2}$.
2. For all primes $q \neq p$ coprime to the level of $g_1$ and $g_2$, the Fourier coefficients satisfy the congruence
   \[
a(q, g_1) \equiv a(q, g_2) \mod \mathfrak{m}.
   \]

Note that $T_{g_j}$ fits into a short exact sequence
\[
0 \rightarrow T_{g_j}^+ \rightarrow T_{g_j} \rightarrow \ker \to 0,
\]
where $T_{g_j}^+ = F^1(T_{g_j})$ and $T_{g_j}^\pm = T_{g_j}/T_{g_j}^\pm$. Set $A_i$ (resp. $A_i^\pm$) to denote the $p$-divisible Galois module $T_{g_j} \otimes \mathbb{Q}_p/\mathbb{Z}_p$ (resp. $T_{g_j}^\pm \otimes \mathbb{Q}_p/\mathbb{Z}_p$). Note that $A_i \simeq (K/\mathcal{O})^d$, where $d = 3$. Let $d^\pm$ be the dimensions (over $K$) of the $\pm$ eigenspaces for complex conjugation on $V_{g_j}$, we have that $d^+ = 2$ and $d^- = 1$ and that $A_i^\pm \simeq (K/\mathcal{O})^{d^\pm}$.

Let $Q_n$ be the subfield of $\mathbb{Q}(\mu_{p^{n+1}})$ degree $p^n$ and set $Q_{\text{cyc}} := \bigcup_{n \geq 0} Q_n$. Letting $\Gamma := \text{Gal}(Q_{\text{cyc}}/\mathbb{Q})$, we fix an isomorphism $\text{Gal}(Q_{\text{cyc}}/\mathbb{Q}) \simeq \mathbb{Z}_p$. The extension $Q_{\text{cyc}}$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. The Iwasawa algebra $L$ is defined as the following inverse limit $L := \lim_n \mathbb{Z}_p[\text{Gal}(Q_n/\mathbb{Q})]$, and is isomorphic to the formal power series ring $\mathbb{Z}_p[[T]]$. We fix a finite set $S$ of primes $q$ including all those that divide $c_\psi M_1 M_2 p$, and we let $Q_S$ denote the maximal algebraic extension of $\mathbb{Q}$ unramified outside the set of primes $S$ and the infinite places. Further, define $A_i \psi_t := A_i \otimes \psi_t$ for an even integer $t$. Then the $p$-primary Selmer group $\text{Sel}_{p^\infty}(A_i \psi_t/Q_{\text{cyc}})$ is defined as the kernel of the following restriction maps
\[
\lambda_i : H^1(Q_S/Q_{\text{cyc}}, A_i \psi_t) \rightarrow \bigoplus_{q \in S} \mathcal{H}_q(A_i \psi_t/Q_{\text{cyc}}).
\]

Here for each prime $q \neq p$, the local term is defined as follows
\[
\mathcal{H}_q(A_i \psi_t/Q_{\text{cyc}}) = \bigoplus_{\eta \mid q} H^1(Q_{\text{cyc}, \eta}/A_i \psi_t),
\]
where $Q_{\text{cyc}, \eta}$ is the union of all completions of number fields contained in $Q_{\text{cyc}}$ at the prime $\eta$. Note that since all primes are finitely decomposed in $Q_{\text{cyc}}$, the above direct sum is finite. The definition at the prime $q = p$ is more subtle, set
\[
\mathcal{H}_p(A_i \psi_t/Q_{\text{cyc}}) = H^1(Q_{\text{cyc, } \eta_p}/A_i \psi_t)/L_{\eta_p}
\]
with
\[
L_{\eta_p} = \ker \left( H^1(Q_{\text{cyc, } \eta_p}/A_i \psi_t) \rightarrow H^1(I_{\eta_p}, A_i \psi_t^-) \right).
\]
Here $\eta_p$ is the unique prime of $Q_{\text{cyc}}$ above $p$, and $I_{\eta_p}$ denotes the inertia group at $\eta_p$. Then we have the following conjecture of Coates-Schmidt and Greenberg.
**Conjecture 3.1** ([2], [7]) Assume that $\psi_t$ is even. Then $\text{Sel}_{p^\infty}(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}})$ is a cotorsion $\Lambda$-module.

This conjecture has been settled by Loeffler and Zerbes in many cases (cf. [15]).

**Lemma 3.2** Let $W = A_{i,\psi_t}[p]$ and let $W^* = \text{Hom}(W, \mu_{p^\infty})$ denote the Tate dual. Then we have $H^0(\mathbb{Q}, W) = H^0(\mathbb{Q}, W^*) = 0$.

*Proof* This follows from our assumption that $(\psi_t)^2 \not\equiv 1 \pmod{p}$. Indeed, since $V = \mathbb{F}_p^i$ is irreducible, we have $V \otimes \det^{-1} \cong \text{Hom}(V, \mathbb{Q}_p/\mathbb{Z}_p)$ and $\text{Ad}(V) = \text{Hom}(V, V) \cong V \otimes \text{Hom}(V, \mathbb{Q}_p/\mathbb{Z}_p) \cong V \otimes V \otimes \det^{-1}$, so that $\text{Ad}^0(V) \cong \text{Sym}^2(V) \otimes \det^{-1} \cong W \otimes \psi^{-1}_t \epsilon^{-1}_i \eta^{-1}_{1-k}$. So we find that if $H^0(\mathbb{Q}, W) \neq 0$, then $\text{Ad}^0(V)$ contains a line on which $\text{Gal}((\mathbb{Q}/\mathbb{Q})$ acts via the character $\mu = (\psi_t \epsilon_i \eta_{k-1+t})^{-1}$. Since $V$ is irreducible, it is easy to see that that $V \otimes \mu \cong V$. By taking determinants of both sides, we conclude that $\mu = \psi_t \epsilon_i \eta_{k-1+t}$ must be quadratic. But each of $\psi$ and $\epsilon$ is unramified at $p$, whilst the Teichmüller character is unramified outside $p$, so we find that $(\psi \epsilon_t)^2 \equiv 1 \pmod{p}$, which is a contradiction. The proof for $W^*$ is analogous, using the fact that $(\text{Ad}^0)^* \cong \text{Ad}^0(1)$. □

**Proposition 3.3** Let $A_{i,\psi_t}$ be as above. Assume that $\psi_t$ is even. Then the localization map $\lambda_i$ is surjective.

*Proof* We let $T_{i,\psi_t}^* := \text{Hom}(T_{i,\psi_t}, \mu_{p^\infty})$. Then we have that $H^0(\mathbb{Q}, T_{i,\psi_t}^*) = 0$. Since $\text{Gal}(\mathbb{Q}_{\text{cyc}}/\mathbb{Q})$ is pro-$p$, it follows that $H^0(\mathbb{Q}_{\text{cyc}}, T_{i,\psi_t}^*) = 0$ as well and thus in particular, is finite. The result follows from [8, Proposition 2.1]. □

Denote by $S_0 := S^i \{p\}$ and introduce the $S_0$-imprimitive Selmer group to be the Selmer group obtained by imposing conditions only at $p$.

**Definition 3.4** The imprimitive Selmer group is defined by:

$$\text{Sel}_{p^\infty}(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}}) = \ker \left( H^1 \left( \mathbb{Q}_S/\mathbb{Q}_{\text{cyc}}, A_{i,\psi_t} \right) \twoheadrightarrow \mathcal{H}_p(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}}) \right).$$

Since the map defining the Selmer group is surjective, it follows that

$$\text{Sel}_{p^\infty}(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}})/\text{Sel}_{p^\infty}(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}}) = \bigoplus_{q \in S_0} \mathcal{H}_q(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}}).$$

(3.1)

**Lemma 3.5** If $q \neq p$, then $\mathcal{H}_q(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}})$ is a cofinitely generated and cotorsion $\Lambda$-module with $\mu$-invariant equal to 0.

*Proof* It suffices to show that $\mathcal{H}_q(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}})$ is a cofinitely generated as a $\mathbb{Z}_p$-module, or equivalently, the $p$-torsion subgroup $\mathcal{H}_q(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}})[p]$ is finite. Consider the short exact sequence

$$0 \rightarrow \bigoplus_{\eta \neq q} H^0(\mathbb{Q}_{\text{cyc}}, \psi_{i,\psi_t}) \rightarrow H^1(\mathbb{Q}_{\text{cyc}}, \psi_{i,\psi_t}) \rightarrow \bigoplus_{\eta \neq q} H^1(\mathbb{Q}_{\text{cyc}}, \psi_{i,\psi_t})[p] \rightarrow 0.$$ 

The set of primes $\eta \neq q$ of $\mathbb{Q}_{\text{cyc}}$ is finite and so is $H^1(\mathbb{Q}_{\text{cyc}}, A_{i,\psi_t}[p])$. The result follows. □

Let $\sigma_{i,q}^\lambda$ denote the $\mathbb{Z}_p$-cokernel of $\mathcal{H}_q(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}})$ for $q \in S_0$. Set $\lambda^S_{S_0}(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}})$ to be the $\lambda$-invariant of $\text{Sel}_{p^\infty}(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}})$. It follows from the structure theory of $\Lambda$-modules that

$$\lambda^S_{S_0}(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}}) = \text{coker}_{\mathbb{Z}_p} \left( \text{Sel}_{p^\infty}(A_{i,\psi_t}/\mathbb{Q}_{\text{cyc}}) \right).$$
It follows from (3.1) that the following relation is satisfied:
\[ \lambda^{S_0}\left(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}\right) = \lambda\left(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}\right) + \sum_{q \in S_0} \alpha_i^{(q)} . \]  

(3.2)

Analogous to the classical and imprimitive Selmer group, we also define the reduced classical and imprimitive Selmer groups which we denote by \( \text{Sel}_{p^\infty}(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) \) and \( \text{Sel}_{p^\infty}^{S_0}(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) \), respectively.

For \( p \) \in \( S_0 \) set
\[ \mathcal{H}_p(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) := \prod_{q \mid p} H^1\left(\mathbb{Q}_{\text{cyc}}/q, \mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}\right) , \]

and for \( p = q \), set
\[ \mathcal{H}_q(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) := H^1(\mathbb{Q}_{\text{cyc}}, \mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) / \overline{\mathcal{Z}}_{np} , \]

where
\[ \overline{\mathcal{Z}}_{np} := \ker\left( H^1(\mathbb{Q}_{\text{cyc}}, \mathbf{A}_{k,\psi_1}) \rightarrow H^1(I_{np}, \mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) \right) . \]

**Definition 3.6** The reduced imprimitive Selmer group is defined as follows
\[ \text{Sel}_{p^\infty}^{S_0}(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) := \ker\left( H^1(\mathbb{Q}_{\text{cyc}}, \mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) \rightarrow \mathcal{H}_p(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) \right) . \]

**Proposition 3.7** For \( i = 1, 2 \), we have a natural isomorphism
\[ \text{Sel}_{p^\infty}^{S_0}(\mathbf{A}_{k,\psi_i}/\mathbb{Q}_{\text{cyc}}) \simeq \text{Sel}_{p^\infty}^{S_0}(\mathbf{A}_{k,\psi_i}/\mathbb{Q}_{\text{cyc}}) . \]

**Proof** We consider the diagram relating the two Selmer groups
\[
\begin{array}{ccc}
0 & \to & \text{Sel}_{p^\infty}^{S_0}(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) \\
& \downarrow f & \downarrow g \\
0 & \to & H^1(\mathbb{Q}_{\text{cyc}}, \mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) \\
& \downarrow h & \\
0 & \to & \text{im} \ \theta_0 \\
\end{array}
\]

where the vertical maps are induced by the Kummer sequence. Additionally, we have
\[ H^0(\mathbb{Q}, \mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) = H^0(\mathbb{Q}_{\text{cyc}}, \mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) . \]

Since \( \mathbb{Q}_{\text{cyc}}/\mathbb{Q} \) is a pro-\( p \) extension, we deduce that \( H^0(\mathbb{Q}_{\text{cyc}}, \mathbf{A}_{k,\psi_1}) = 0 \) and therefore \( g \) is injective. On the other hand, it clear that \( g \) is surjective.

It only remains to show that \( h \) is injective. For \( q = p \), denote by \( \iota \) the natural map
\[ \iota : \mathcal{H}_q(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) \to \mathcal{H}_q(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) \]

Consider the commutative square with injective horizontal maps
\[
\begin{array}{ccc}
\mathcal{H}_q(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) & \to & H^1\left(I_{np}, \mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}\right) \\
& \downarrow \iota & \downarrow f \\
\mathcal{H}_q(\mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}) & \to & H^1\left(I_{np}, \mathbf{A}_{k,\psi_1}/\mathbb{Q}_{\text{cyc}}\right) [p] .
\end{array}
\]
Since $A_{i,\psi}$ is unramified at $p$, it follows that $H^0(I_{\eta_p}, A_{i,\psi}^-) = A_{i,\psi}^-$ is divisible. It is easy to see that if $t \neq 0$, then $H^0(I_{\eta_p}, A_{i,\psi}^-) = 0$. The kernel of the map

$$i : H^1(I_{\eta_p}, A_{i,\psi}^-[p]) \rightarrow H^1(I_{\eta_p}, A_{i,\psi}^-)[p]$$

is $H^0(I_{\eta_p}, A_{i,\psi}^-)/p = 0$. □

**Lemma 3.8** The isomorphism $A_{1,\psi}[p] \simeq A_{2,\psi}[p]$ of Galois modules induces an isomorphism of residual Selmer groups

$$\text{Sel}^0_{\mu}(A_{1,\psi}[p]/\mathbb{Q}_{\text{cyc}}) \simeq \text{Sel}^0_{\mu}(A_{2,\psi}[p]/\mathbb{Q}_{\text{cyc}}).$$

**Proof** Note that the $G_p$-action on $A_{i,\psi}^+[p]$ is ramified and that on $A_{i,\psi}^-[p]$ is via an unramified character. Let $\Phi : A_{1,\psi}[p] \rightarrow A_{2,\psi}[p]$ be a choice of isomorphism of Galois modules, in which case it is easy to see that $\Phi$ induces an isomorphism

$$\Phi : A_{1,\psi}^+[p] \rightarrow A_{2,\psi}^+[p].$$

As a result, we have an isomorphism of $G_p$-modules $A_{1,\psi}^+[p] \simeq A_{2,\psi}^+[p]$. Clearly, $\Phi$ induces an isomorphism $H^1(QS/Q_{\text{cyc}}, A_{1,\psi}[p]) \rightarrow H^1(QS/Q_{\text{cyc}}, A_{2,\psi}[p])$. It suffices to show that for $q \in S$, the isomorphism $\Phi : A_{1,\psi}[p] \rightarrow A_{2,\psi}[p]$ induces an isomorphism

$$\mathcal{H}_q(QS, A_{1,\psi}[p]) \rightarrow \mathcal{H}_q(QS, A_{2,\psi}[p]).$$

This is clear for $q \neq p$. For $q = p$, this follows from the fact that $\Phi$ induces an isomorphism $A_{1,\psi}^-[p] \rightarrow A_{2,\psi}^-[p]$. □

**Corollary 3.9** The isomorphism $A_{1,\psi}[p] \simeq A_{2,\psi}[p]$ of Galois modules induces an isomorphism

$$\text{Sel}^0_{\mu}(A_{1,\psi}/\mathbb{Q}_{\text{cyc}})[p] \simeq \text{Sel}^0_{\mu}(A_{2,\psi}/\mathbb{Q}_{\text{cyc}})[p].$$

**Lemma 3.10** The $\mu$-invariant of the Selmer group $\text{Sel}_{\mu}(A_{i,\psi}/\mathbb{Q}_{\text{cyc}})$ coincides with that of $\text{Sel}^0_{\mu}(A_{i,\psi}/\mathbb{Q}_{\text{cyc}})$, i.e.,

$$\mu(A_{i,\psi}/\mathbb{Q}_{\text{cyc}}) = \mu^0(A_{i,\psi}/\mathbb{Q}_{\text{cyc}}).$$

**Proof** The result follows from Lemma 3.5, which states that $H_q(A_{i,\psi}/\mathbb{Q}_{\text{cyc}})$ has $\mu = 0$ for $q \neq p$. □

**Proposition 3.11** Let the notation be as above. Then, we have that

$$\mu(A_{1,\psi}/\mathbb{Q}_{\text{cyc}}) = 0 \iff \mu(A_{2,\psi}/\mathbb{Q}_{\text{cyc}}) = 0.$$

Moreover, if these $\mu$-invariants are 0, then the imprimitive $\lambda$-invariants coincide, i.e.,

$$\lambda^0(A_{1,\psi}/\mathbb{Q}_{\text{cyc}}) = \lambda^0(A_{2,\psi}/\mathbb{Q}_{\text{cyc}}).$$

Furthermore, we have

$$\lambda(A_{2,\psi}/\mathbb{Q}_{\text{cyc}}) - \lambda(A_{1,\psi}/\mathbb{Q}_{\text{cyc}}) = \sum_{q \in S_0} \left( \sigma_q^{(1)} - \sigma_q^{(2)} \right),$$

where $\sigma_q^{(i)}$ is the $\mathbb{Z}_p$-corank of $H_q(A_{i,\psi}/\mathbb{Q}_{\text{cyc}})$. 
Lemma 3.10 asserts that the $\mu$-invariant of $X_i$ coincides with $\mu(A_{1, \psi_1}/\mathbb{Q}_{\text{cycl}})$. Therefore, $\mu(A_{1, \psi_1}/\mathbb{Q}_{\text{cycl}}) = 0$ if and only if $X_i$ is cotorsion generated as a $\mathbb{Z}_p$-module. Note that $X_i$ is cotorsion generated as a $\mathbb{Z}_p$-module if and only if $X_i[p]$ has finite cardinality. Corollary 3.9 asserts that $X_1[p] \simeq X_2[p]$; thus,

$$
\mu(A_{1, \psi_1}/\mathbb{Q}_{\text{cycl}}) = 0 \iff \mu(A_{2, \psi_1}/\mathbb{Q}_{\text{cycl}}) = 0.
$$

Assume that $X_1[p]$ (or equivalently $X_2[p]$) is finite. It follows from [8, Proposition 2.5] that $X_i$ has no proper $\Lambda$-submodules of finite index. It is an easy exercise to show that $X_i$ is a free $\mathcal{O}$-module. Therefore, $X_i \simeq (\mathbb{K}/\mathcal{O})^{\lambda(X_i)}$, where $\lambda(X_i) := \frac{\lambda^S(X_i)}{[\mathbb{K} : \mathbb{Q}_p]}$.

As a result,

$$
\lambda(X_i) = \dim_{\mathbb{F}_p} X_i[p],
$$

and the isomorphism $X_1[p] \simeq X_2[p]$ implies that $\lambda(X_1) = \lambda(X_2)$. As a result, it follows that

$$
\lambda^S(A_{1, \psi_1}/\mathbb{Q}_{\text{cycl}}) = \lambda^S(A_{2, \psi_1}/\mathbb{Q}_{\text{cycl}}).
$$

The result then follows from (3.2).

Let $g_1$ and $g_2$ be $p$-congruent Hecke eigencuspsforms satisfying the conditions stated in the introduction. We denote by $\mu_{S_0}^\text{alg}(r_{g_1} \otimes \psi_1)$ and $\lambda_{S_0}^\text{alg}(r_{g_1} \otimes \psi_1)$ to denote the Iwasawa invariants of the imprimitve Selmer group $\text{Sel}_{p_{\infty}}^\text{alg}(A_{1, \psi_1}/\mathbb{Q}_{\text{cycl}})$ obtained by dropping conditions at the primes $q \in S_0$. Denote by $\mu_{S_0}^\text{an}(r_{g_1} \otimes \psi_1)$ and $\lambda_{S_0}^\text{an}(r_{g_1} \otimes \psi_1)$ the Iwasawa invariants of the $S_0$-imprimitive $p$-adic $\mathbb{Q}^\text{cycl}$-L-function $L_{S_0}^\text{an}(r_{g_1} \otimes \psi_1)$ obtained by dropping Euler factors at primes $q \not\in S_0$.

**Proposition 3.12** Let $g_1$ and $g_2$ be as above and $* \in \{\text{an, alg}\}$. Then, $\mu^*(r_{g_1} \otimes \psi_1) = \mu_{S_0}^*(r_{g_1} \otimes \psi_1)$ for $i = 1, 2$ and

$$
\lambda_{S_0}^*(r_{g_1} \otimes \psi_1) = \lambda_{S_0}^*(r_{g_1} \otimes \psi_1) + \sum_{q \in S_0} \sigma_i(q).
$$

**Proof** When $* = \text{alg}$, the result follows from Lemma 3.5 and (3.2). On the other hand, when $* = \text{an}$, the result follows from (2.29). The equality of the local factors $\sigma_i(q)$ comes from the local Langlands correspondence of [5] for $GL(3)$.

**Theorem 3.13** Let $g_1$ and $g_2$ satisfy the conditions stated in the introduction. Suppose the relations $\mu_{S_0}^\text{alg}(r_{g_1} \otimes \psi_1) = \mu_{S_0}^\text{an}(r_{g_1} \otimes \psi_1) = 0$ and $\lambda_{S_0}^\text{alg}(r_{g_1} \otimes \psi_1) = \lambda_{S_0}^\text{an}(r_{g_1} \otimes \psi_1)$ hold. Then, we have further equalities $\mu_{S_0}^\text{alg}(r_{g_1} \otimes \psi_1) = \mu_{S_0}^\text{an}(r_{g_1} \otimes \psi_1) = 0$ and $\lambda_{S_0}^\text{alg}(r_{g_1} \otimes \psi_1) = \lambda_{S_0}^\text{an}(r_{g_1} \otimes \psi_1)$.

**Proof** Let $* \in \{\text{an, alg}\}$, it follows from Proposition 3.12 that the equalities $\mu_{S_0}^\text{alg}(r_{g_1} \otimes \psi_1) = \mu_{S_0}^\text{an}(r_{g_1} \otimes \psi_1) = 0$ and $\lambda_{S_0}^\text{alg}(r_{g_1} \otimes \psi_1) = \lambda_{S_0}^\text{an}(r_{g_1} \otimes \psi_1)$ hold if and only if the relations $\mu_{S_0}^\text{alg}(r_{g_1} \otimes \psi_1) = \mu_{S_0}^\text{an}(r_{g_2} \otimes \psi_1) = 0$ and $\lambda_{S_0}^\text{alg}(r_{g_1} \otimes \psi_1) = \lambda_{S_0}^\text{an}(r_{g_2} \otimes \psi_1)$ hold. Therefore, by assumption, these relations hold for $i = 1$, and we are to deduce them for $i = 2$. Proposition 2.30 asserts that there is a $p$-adic unit $u$ such that $L_{S_0}^\text{an}(r_{g_1} \otimes \psi_1) \equiv u L_{S_0}^\text{an}(r_{g_2} \otimes \psi_1) \mod p$.

From the above congruence, we find that

$$
\mu_{S_0}^\text{an}(r_{g_1} \otimes \psi_1) = 0 \iff \mu_{S_0}^\text{an}(r_{g_2} \otimes \psi_1) = 0
$$

and if these $\mu$-invariants vanish, then

$$
\lambda_{S_0}^\text{an}(r_{g_1} \otimes \psi_1) = \lambda_{S_0}^\text{an}(r_{g_2} \otimes \psi_1).
$$
On the other hand, the same assertion for algebraic L-functions holds by Proposition 3.11. Therefore, the result follows. □

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