The Ising Transition in the double-frequency sine-Gordon model

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Abstract

In the present paper we utilize the renormalization group (RG) technique to analyse the Ising critical behavior in the double frequency sine-Gordon model. The one-loop RG equations obtained show unambiguously that there exist two Ising critical points besides the trivial Gaussian fixed point. The topology of the RG flows is obtained as well.
The sine-Gordon (SG) model plays an important role in the condensed matter physics since many one-dimensional physical system can be mapping onto it. The SG hamiltonian can be written as a Gaussian model of a scalar field \( \phi \) perturbed by a vertex operator \( \cos \beta \phi \), which has the form

\[
\mathcal{H}_{SG} = \mathcal{H}_{Gauss} + g' \cos \beta \phi
\]

(1)

\[
\mathcal{H}_{Gauss} = \frac{1}{2} \pi^2 + \frac{1}{2} (\partial_x \phi)^2
\]

(2)

with coupling constant \( g' \). As well known, for \( \beta^2 > 8\pi \), the perturbation is irrelevant and the system is gapless, and for \( 0 < \beta^2 < 8\pi \), the vertex operator becomes relevant and drive the system to a strong-coupling massive phase with a mass gap \( m \sim g^{4\pi/(8\pi - \beta^2)} \).

Recently, the double-frequency sine-Gordon (DSG) model has received much attention which is the SG model subjected to another vertex operator

\[
\mathcal{H}_{DSG} = \mathcal{H}_{Gauss} + g' \cos \beta \phi + \lambda' \cos \frac{\beta}{2} \phi.
\]

(3)

This model was found to display an Ising criticality with central charge \( c = 1/2 \) on a quantum critical line \( \lambda' = \lambda'_c(g) \) (quasi-classically \( \lambda'_c(g) = 4g' \)) by Delfino and Mussardo \[1\]. They also argued that this Ising transition is a universal property of the DSG model \[1\] as long as \( \beta^2 < 8\pi \). Fabrizio et al. investigated in detail the critical properties of this transition by mapping the DSG model onto the deformed quantum Ashkin-Teller (DAT) model. Under a new representation of the DAT model, they succeeded in identifying those degrees of freedom that become critical, and they also discussed the application of the Ising transition in some physical realization of the DSG model.

In this letter, we shall give the renormalization group (RG) analysis of the Ising criticality in the DSG model. In the following, we assume that \( 32\pi/9 < \beta^2 < 8\pi \), so that both the vertex operators in Eq.(3) are relevant and no other relevant operators are generated upon renormalization. It is clear that both these two vertex operators will make the theory fully massive, if acting alone. However if they coexist, it will become gapless on some critical line.
The RG equations obtained explore this feature in an explicit way and give the RG flows in the coupling parameters plane, in which there exist three fixed points to the RG equations including the trivial Gaussian fixed point and two non-trivial Ising fixed points.

To construct the one loop RG equations, we employ the formalism in Refs. [4,5] which is based on the operator product expansion (OPE) in real space. Firstly, we replace the coupling constants $g'$ and $\lambda'$ by the dimensionless bare coupling constants $g = a^{\Delta_g - 2} g'$ and $\lambda = a^{\Delta_\lambda - 2} \lambda'$ with $\Delta_g = 4 \Delta_\lambda = \beta^2 /(4\pi)$ and $a$ being the microscopic short distance cut-off.

In our case, the coefficients of the following OPEs are needed to construct the RG equations:

$$
: \cos \beta \phi(z, \bar{z}) :: \cos \beta \phi(w, \bar{w}) : \sim \frac{1}{2} |z - w|^\frac{\beta^2}{8} \cos \beta \phi(w, \bar{w}) : - \frac{\beta^2}{8} |z - w|^2 \frac{\beta^2}{16\pi} : \partial_w \phi \partial_{\bar{w}} \phi : + \cdots , \quad (4)
$$

$$
: \cos \beta \phi(z, \bar{z}) :: \cos \beta \phi(w, \bar{w}) : \sim \frac{1}{2} |z - w|^\frac{\beta^2}{8} \cos 2 \beta \phi(w, \bar{w}) : - \frac{\beta^2}{2} |z - w|^2 \frac{\beta^2}{16\pi} : \partial_w \phi \partial_{\bar{w}} \phi : + \cdots , \quad (5)
$$

$$
: \cos \beta \phi(z, \bar{z}) :: \cos \beta \phi(w, \bar{w}) : \sim \frac{1}{2} |z - w|^\frac{\beta^2}{8} \cos \frac{\beta}{2} \phi(w, \bar{w}) : + \cdots . \quad (6)
$$

It should be noted that in Eqs.(4,5,6) we only pick up the operators which have contribution to the RG equation.

Through Eqs.(4,5,6) one can obtain the one-loop RG equations [4,5]

$$
\frac{dg}{d\ln l} = (2 - \Delta_g) g - \frac{\pi}{2} \lambda^2 , \quad (7)
$$

$$
\frac{d\lambda}{d\ln l} = (2 - \Delta_\lambda) \lambda - \pi g \lambda \quad (8)
$$

under the scaling transformation $a \sim e^l a$. The second term in Eqs.(4,5) leads to the renormalization of $\beta^2$

$$
\frac{d\beta^2}{d\ln l} = - \frac{\pi}{4} \beta^4 (g^2 + \lambda^2) , \quad (9)
$$

which results in the modification of $\Delta_g$ and $\Delta_\lambda$ in Eqs.(4,5,6). However, this higher order correction can be neglected since it will not change our conclusion qualitatively for small $g$ and $\lambda$.

The zeros of the RG equations (4,5,6) give the fixed points of the system, which include two Ising critical points $(g_c, \pm \lambda_c)$ with
\[ g_c = \frac{1}{\pi} (2 - \Delta \lambda) \]  
(10)

\[ \lambda_c = \frac{1}{\pi} \sqrt{2(2 - \Delta g)(2 - \Delta \lambda)} \]  
(11)

and the trivial Gaussian fixed point \( g = \lambda = 0 \). The topology of the RG flows implied by Eqs.\((7,8)\) is plotted in Fig.\((1)\).

In the vicinity of the Gaussian fixed point which is unstable with respect to the parameters \( g \) and \( \lambda \), the RG trajectories have the form \( \lambda \sim g^\mu \) with \( \mu = (2 - \Delta \lambda)/(2 - \Delta g) \) and flow outward.

In order to extract the asymptotic behavior of the RG flows near the Ising critical points, one can substitute \( g = g_c + \tilde{g} \) and \( \lambda = \lambda_c + \tilde{\lambda} \) into the RG equations and neglect the quadratic terms of \( \tilde{g} \) and \( \tilde{\lambda} \), which leads to

\[ \frac{d\tilde{g}}{d\ln l} = \frac{\pi \lambda_c^2}{2g_c} \tilde{g} - \pi \lambda_c \tilde{\lambda}, \]  
(12)

\[ \frac{d\tilde{\lambda}}{d\ln l} = -\pi \lambda_c \tilde{g}. \]  
(13)

These two equations \((12,13)\) can be rewritten in a decoupled form with the new variables \( \xi_\pm = \tilde{g} + a_\pm \tilde{\lambda} \) as

\[ \frac{d\xi_\pm}{d\ln l} = \pi \lambda_c a_\pm \xi_\pm, \]  
(14)

where \( a_\pm = (\lambda_c/g_c \pm \sqrt{16 + \lambda_c^2/g_c^2})/4 \).

Clearly Eq.\((14)\) implies that \( \xi_- \) is relevant and will opens a mass gap proportional to \( \xi_-^{1/\pi \lambda_c a_+} \) and \( \xi_+ \) is irrelevant. As seen from Fig.\((1)\), the Ising fixed point \( (g_c, \lambda_c) \) is unstable with respect to \( \xi_- \) and stable with respect to \( \xi_+ \). The lines \( \xi_+ = 0 \) and \( \xi_- = 0 \) act as separatrix dividing the region around the Ising fixed point into four parts. Fig.\((1)\) also indicates there exist a critical line connecting the Gaussian fixed point and the Ising fixed point, any point on which will flow toward the Ising fixed point and the central charge of the system changes from \( c = 1 \) to \( c = 1/2 \) according to the Zamolodchikov’s C-theorem \([4,3]\). Thus, all the initial hamiltonian on this line are in the same university class. This critical
trajectory has the form $\lambda = \zeta g^\mu$ with some fixed parameter $\zeta > 0$ near the Gaussian fixed point and has a linear asymptotic form $\xi_-=0$, namely,

$$g + a_- \lambda - g_c - a_+ \lambda_c = 0$$

around the Ising fixed point $(g_c, \lambda_c)$.

As for the other Ising fixed point $(g_c, -\lambda_c)$, the similar analysis gives that this fixed point is unstable with respect to $\xi_+$ and stable with respect to $\xi_-$. The critical line connecting this Ising fixed point and the Gaussian fixed point also has the form $\lambda = \zeta g^\mu$ but the parameter $\zeta < 0$ near the Gaussian fixed point, and in the vicinity of the fixed point $(g_c, -\lambda_c)$ it becomes $\xi_+ = 0$, namely

$$g + a_+ \lambda - g_c + a_- \lambda_c = 0.$$  \hfill (16)

As a conclusion, we employ the RG approach to study the Ising criticality occurring in the DSG model which finds some interesting applications in the 1D strongly correlated electron systems. Two Ising fixed points are found besides the trivial Gaussian fixed point. The quantum critical line $\lambda = \lambda_c(g)$, on which the system displays Ising criticality with central charge $c = 1/2$, is obtained near the fixed points together with the RG flows.

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Fig.(1) The RG trajectories $\lambda$ vs. $g$