Twisting and $\kappa$-Poincaré

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Abstract
We demonstrate that the coproduct of $D = 2$ and $D = 4$ quantum $\kappa$-Poincaré algebras in a classical algebra basis cannot be obtained by a cochain twist depending only on Poincaré algebra generators. We also argue that the nonexistence of such a twist does not imply the nonexistence of a universal $R$-matrix.

Keywords: quantum groups, twist deformation, $\kappa$-Poincaré algebra

1. Introduction

In this note we wish to present the result that the quantum $\kappa$-deformation of the Poincaré–Hopf algebra [1–3] written in an undeformed Poincaré algebra basis [4] cannot be obtained by twisting of the classical Poincaré–Hopf algebra even if we relax the coassociativity condition. We shall consider below the quantum deformations in the category of triangular quasi-Hopf (tqH) algebras [5–7] with nontrivial coassociator $\phi \neq 1 \otimes 1 \otimes 1$. Our arguments are based on the existence of an explicit deformation map called a quantum map [4, 8–10] which connects the generators of $\kappa$-deformed (e.g. in a bicrossproduct basis [3]) and the classical (undeformed) Poincaré algebras. In such a classical basis, the $\kappa$-deformed quantum Poincaré algebra can be described by using classical Poincaré generators with quite complicated coproducts. Our aim is to show that one cannot obtain such $\kappa$-deformed coproducts via a general cochain twist transformation of primitive (classical) Poincaré coproducts. One can relate this result with the fact that until now the universal $R$-matrix for the $\kappa$-deformed Poincaré algebra has not been known, at least in compact form. We argue however (see the Appendix) that for $D = 2$ $\kappa$-deformation, the existence of a universal $R$-matrix does not imply the existence of a cochain twist defining $\kappa$-deformed coproducts.

It should be added that several authors [11–17] have tried to obtain $\kappa$-deformed Poincaré algebra in the framework of Hopf algebras by proposing nonstandard twists which were embedded in some enlargements of Poincaré algebra, but they never reproduced in an
algebraic way either the complete $\kappa$-Poincaré coalgebra sector or the correct universal $R$-matrix. We also mention here that in the recent paper [18], the twists were studied in the Hopf algebroid framework (see e.g. [19, 20]) with the use of the Hopf algebroid structure of deformed phase space, but such an approach falls outside of the standard (quasi-)Hopf algebra scheme used in this paper.

The description of quantum deformation using a twist provides the universal $R$-matrix as well as an explicit formula for the star-product realization of quantum algebra (see e.g. [21, 22]) consistent with the Hopf-algebraic actions. Unfortunately the standard Poincaré twist depending only on Poincaré generators and satisfying the two-cocycle condition (i.e. with $\phi = 1 \otimes 1 \otimes 1$) cannot describe the quantum $\kappa$-deformation of Poincaré algebra. This property can be deduced also from the formula for the classical $\kappa$-Poincaré $r$-matrix (see e.g. [3]; $\in \otimes \in \otimes$):

$$\kappa = \wedge_{ri}^{1} \wedge_{N_{i}}^{1} \wedge_{1} (1)$$

which satisfies the modified Yang–Baxter equation\(^1\).

It is known that in the case of deformed semisimple Lie algebras the canonical Drinfeld–Jimbo quantum deformation can be obtained by a so-called Drinfeld twist in the category of quasi-Hopf algebras [5, 6]. This generalization of standard quantum groups introduces the category of quantum quasi-groups which is characterized by the universal $R$-matrix describing the flip operation ($a \otimes b)^T = b \otimes a; a, b \in A$) on the coproducts

$$\Delta^T (a) = R\Delta(a)R^{-1},$$

as well as by the non-unital coassociator $\phi = A \otimes A \otimes A$ modifying the quasi-triangularity relations for the universal $R$-matrix as follows:

$$(\Delta \otimes \text{id}) (R) = \phi_{312}^{1} R_{13}^{1} \phi_{123}^{-1} R_{23}^{1} \phi_{123}^{-1},$$

$$(\text{id} \otimes \Delta) (R) = \phi_{231}^{-1} R_{23}^{1} \phi_{213}^{1} R_{12}^{1} \phi_{123}^{-1}.$$\(^2\)

Quasi-triangular quasi-Hopf algebras $\tilde{H} = (A, \Delta, S, c; R, \phi, \alpha, \beta)$ generalize the notion of quasi-triangular Hopf algebras $H = (A, \Delta, S, c; R)$ and are characterized additionally by a coassociator $\phi$ generalizing the coassociativity condition as follows:\(^2\):

$$((\text{id} \otimes \Delta) O \Delta (a)) \phi = \phi ((\Delta \otimes \text{id}) O \Delta (a))$$

where two special elements $\alpha, \beta \in A$ are linking the antipode $S$ with the coproducts $\Delta$ (see e.g. [5]).

We recall that Poincaré algebra is not semisimple, but can be obtained by Wigner–Inonu contraction of simple AdS or dS algebras. Interestingly, one can introduce a suitable quantum modification of Wigner–Inonu contraction which when applied to the Drinfeld–Jimbo deformation $U_q(\mathfrak{o}(3, 2))$ of (A)dS algebra provides in the limit $R \to \infty$ ($R$ is the AdS radius) the $\kappa$-deformed quantum Poincaré–Hopf algebra [1, 23]. It has been shown however by Young and Zegers [24, 25] (see also [18, 26]) that there exists as well a quantum contraction of the universal Drinfeld twist for $U_q(O(3, 2))$ (or $U_q(O(4, 1)))$\(^3\) which permits one to

\(^1\) We denote the 3-momenta by $P_i$, and the $N_i$ are the boost generators.

\(^2\) The condition (5) describes quasi-coassociativity, introducing coassociativity up to the similarity transformation.

\(^3\) Universal Drinfeld twists in the category of quasi-Hopf algebras are given for quantum semisimple Lie algebras in [5]. They were defined as inner automorphisms which map via a twist the coproducts of quantum and undeformed enveloping algebras. It should be stressed that Drinfeld did prove the existence of a universal twist by using cohomological methods, which are not constructive.
introduce \(\kappa\)-deformed Poincaré–Hopf algebra as belonging to the category of triangular quasi-Hopf algebras. In this paper we shall perform, for a particular choice of \(\kappa\)-Poincaré coproducts, simple calculations demonstrating that a twist generating the \(\kappa\)-deformed Poincaré algebra from classical (undeformed) Poincaré–Hopf algebra does not exist even if one considers general cochain twists allowing noncoassociativity \((\phi \neq 1 \otimes 1 \otimes 1)^4\).

Firstly, in section 2 we shall consider a simpler case of standard \(\kappa\)-deformed Poincaré algebra for \(D = 2\). Further on, in section 3 we shall consider \(D = 4\) \(\kappa\)-deformation. Using a perturbation expansion in the \(1/\kappa\) deformation parameter, it will be shown that the description of coproducts in terms of twist already breaks down in the second order of the \(1/\kappa\) power expansion. In order to clarify that our result does not contradict the presence of a universal \(R\)-matrix for the \(\kappa\)-Poincaré case, which has been calculated perturbatively in [24], we show in the Appendix that in the classical basis the universal \(R\)-matrix up to second order of \(1/\kappa\) exists (for simplicity we consider the \(D = 2\) case).

### 2. The \(\kappa\)-Poincaré from twisting: \(D = 2\) case

#### 2.1. General considerations

We deal with nontrivial deformations of the enveloping algebra \((U(\hat{g}) \rightarrow U_\kappa(\hat{g}))\) only if we consider \(U_\kappa(\hat{g})\) as a Hopf algebra (or at least as a bialgebra). If we do not consider the coalgebraic sector, there always exists an isomorphism which maps the algebraic sector \(U(\hat{g})\) into \(U_\kappa(\hat{g})\). The quantum deformation, described infinitesimally by a classical \(r\)-matrix, modifies necessarily only the coalgebraic sector. Concluding, the basis of the algebraic sector \(U_\kappa(\hat{g})\) of quantum algebra can be chosen classical, but for any choice of algebra generators the coalgebra will be deformed and described by some nonsymmetric, nonprimitive coproducts \((\Delta \neq \Delta^T)\).

For the case of the \(\kappa\)-deformed Poincaré enveloping algebra \(U_\kappa(\hat{g})\) \((\hat{g} = (P_\mu, M_{\mu\nu}) \in T^{3,1})\), one can also show explicitly the existence of a quantum map by presenting the formula transforming the undeformed algebra \(U(\hat{g})\) into the deformed one \(U_\kappa(\hat{g})\). First calculations in [8, 9] were provided for an inverse quantum map from the \(\kappa\)-deformed basis of Poincaré algebra to the classical one, but an explicit calculation of \(\kappa\)-deformed coproducts of Poincaré generators in a classical basis has not been provided. The explicit formulas for such coproducts were given later in [4, 10].

In this paper, we shall discuss firstly the two-dimensional case \((D = 2)\) and then, further on, the four-dimensional case \((D = 4)\). We shall assume that the coproducts of \(\kappa\)-deformed quantum algebra in a classical algebra basis are generated by some twist \(F \in U(\hat{g}) \otimes U(\hat{g})\). We expand the logarithm of the twist into a power series in \(1/\kappa\) as follows:

\[
F = \exp\left(\frac{1}{\kappa} f_1 + \frac{1}{\kappa^2} f_2 + O\left(\frac{1}{\kappa^3}\right)\right) = 1 \otimes 1 + \frac{1}{\kappa} F_1 + \frac{1}{\kappa^2} F_2 + O\left(\frac{1}{\kappa^3}\right)
\]

(6)

where \(F_1 = f_1, F_2 = \frac{1}{2} f_1^2 + f_2, \) etc. This implies the following perturbative formula for the \(\kappa\)-deformed coproducts \(\Delta \in U(\hat{g}) \otimes U(\hat{g})[[\frac{1}{\kappa}]]\):

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4 We stress that the two-cocycle condition is not assumed for the cochain twists.
\[ \Delta = F \Delta_0 F^{-1} = \Delta_0 + \frac{1}{\kappa} \Delta_1 + \frac{1}{\kappa^2} \Delta_2 + O \left( \frac{1}{\kappa^3} \right) = \Delta_0 + \frac{1}{\kappa} \left[ f_1, \Delta_0 \right] + \frac{1}{2 \kappa^2} \left[ f_1, \left[ f_1, \Delta_0 \right] \right] + \frac{1}{\kappa^2} \left[ f_2, \Delta_0 \right] + O \left( \frac{1}{\kappa^3} \right) \] (7)

In the general case we obtain from \( F \) the formula for the universal \( R \)-matrix:
\[ R = F^T F^{-1}, \] (8)
as well as the nontrivial coassociator \( \phi \in U(\hat{\mathfrak{g}}) \otimes U(\hat{\mathfrak{g}}) \otimes U(\hat{\mathfrak{g}}) \) [\( 1/\kappa \)] (see e.g. [26]):
\[ \phi = (1 \otimes F)(id \otimes \Delta)(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1). \] (9)

One can show that the universal \( R \)-matrix describing the quasi-triangular quasi-Hopf algebra (see (4, 3)) satisfies the following modified quantum Yang–Baxter equation:
\[ R_{12} \phi_{312} R_{13} \phi_{132} R_{23} \phi_{231} R_{12} = \phi_{321} R_{23} \phi_{213} R_{13} \phi_{123} R_{12}. \] (10)

2.2. \( D = 2 \kappa \)-Poincaré algebra in a bicrossproduct basis

(i) Algebra.
\[ [P_0, P_1] = 0, \quad [N, P_0] = i P_1, \quad [N, P_1] = \frac{i}{\kappa} \left( 1 - \exp \left( -\frac{2 P_0}{\kappa} \right) \right) - \frac{i}{2 \kappa} P_1^2. \] (11)

(ii) Coalgebra.
\[ \Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \] (12)
\[ \Delta(P_1) = P_1 \otimes 1 + \exp \left( -\frac{P_0}{\kappa} \right) \otimes P_1, \] (13)
\[ \Delta(N) = N \otimes 1 + \exp \left( -\frac{P_0}{\kappa} \right) \otimes N. \] (14)

2.3. \( D = 2 \kappa \)-Poincaré–Hopf algebra in a classical basis

(i) Algebra.
The \( D = 2 \) classical Poincaré algebra described by 2-momentum generators \( P_0 = (R_0, R) \) and a boost generator \( N \):
\[ [R_0, R] = 0, \quad [N, R_0] = i P_1, \quad [N, R] = i R_0 \] (15)
is derived from the \( \kappa \)-deformed Poincaré algebra (11) by using the following inverse quantum map:
\[ P_0 = \frac{\kappa}{2} \left( \exp \left( \frac{P_0}{\kappa} \right) - \exp \left( -\frac{P_0}{\kappa} \right) \left( 1 - \frac{1}{\kappa^2} P_1^2 \right) \right), \quad R = P_1 \exp \left( \frac{P_0}{\kappa} \right). \] (16)
The quantum map which is inverse to (16) has the form
\[ P_0 = \kappa \ln \Pi_0, \quad P_1 = P_1 \Pi_0^{-1}, \] (17)
where

\[ \Pi_0 = \frac{1}{k} P_0 + \sqrt{1 - \frac{1}{k^2} C_0}, \quad \Pi_0^{-1} = \frac{1 - \frac{1}{k^2} C_0 - \frac{1}{k^2} P_0}{1 - \frac{1}{k^2} P_1^2}, \]

with \( C_0 \) describing the standard undeformed mass Casimir

\[ C_0 = P^0 P_0 + P^1 P_1 = -P_0^2 + P_1^2. \]

(ii) Coalgebra.

\[ \Delta(\Pi_0) = \Pi_0 \otimes \Pi_0 + \Pi_0^{-1} \otimes P_0 + \frac{1}{k} P_1 \Pi_0^{-1} \otimes P_1, \]

\[ \Delta(P_1) = P_1 \otimes \Pi_0 + 1 \otimes P_1, \]

\[ \Delta(N) = N \otimes 1 + \Pi_0^{-1} \otimes N. \]

The \( \kappa \)-deformed mass Casimir is the following:

\[ C = \kappa^2 \left( \Pi_0 + \Pi_0^{-1} - 2 - \frac{1}{k^2} P_1^2 \Pi_0^{-1} \right). \]

It can be checked by explicit calculation that the relations (20)–(22) satisfy the \( D = 2 \) classical Poincaré algebra (15).

2.4. The no-go theorem for \( D = 2 \)

One can expand the coproducts (20)–(22) in powers of \( \frac{1}{k} \) using

\[ \Pi_0 = 1 + \frac{1}{k} P_0 - \frac{1}{2k^2} C_0 + O \left( \frac{1}{k^3} \right), \quad \Pi_0^{-1} = 1 - \frac{1}{k} P_0 + \frac{1}{k^2} \left( P_0^2 + \frac{1}{2} C_0 \right) + O \left( \frac{1}{k^3} \right). \]

One gets

\[ \Delta(\Pi_0) = P_0 \otimes 1 + 1 \otimes P_0 + \frac{1}{k} P_1 \otimes P_1 \]

\[ + \frac{1}{k^2} \left( P_0^2 \otimes P_0 + \frac{1}{2} C_0 \otimes P_0 - \frac{1}{2} P_0 \otimes C_0 - P_0 \otimes P_0 \right) + O \left( \frac{1}{k^3} \right); \]

\[ \Delta(P_1) = P_1 \otimes 1 + 1 \otimes P_1 + \frac{1}{k} P_1 \otimes P_1 - \frac{1}{2k^2} P_1^2 \otimes C_0 + O \left( \frac{1}{k^3} \right); \]

\[ \Delta(N) = N \otimes 1 + 1 \otimes N - \frac{1}{k} P_0 \otimes N + \frac{1}{k^2} \left( P_0^2 \otimes N + \frac{1}{2} C_0 \otimes N \right) + O \left( \frac{1}{k^3} \right). \]

From \( \Delta_1 = \left[ f_1, \Delta_0 \right] \) and (27), one can easily calculate that\(^5\)

\[ f_1 = -i P_1 \otimes N, \]

i.e. we get ‘half’ of the classical \( r \)-matrix because \( r = f_1 - f_1^T \). The equation determining the term \( f_2 \) in (6) looks as follows:

\(^5\) In fact, \( f_1 \) is defined up to a term \( f_1^{(0)} \) for which \([f_1^{(0)}, \Delta_0] = 0\); such a term does not modify our result.
\[ \Delta_2 = \frac{1}{2} [f_1, [f_1, \Delta_0]] + [f_2, \Delta_0], \]  
where \( \Delta_2 \) is given explicitly by formulas (25)–(27). We shall show below that no such \( f_2 \) exists which provides \( \frac{1}{\kappa} \) terms in the coproducts (25)–(27).

Let us note that due to (28), we get
\[ \Delta_2 = f N, (0), \]  
and we see from (27) that \( \Delta_2(0) \) contains in the left factors of the tensor product the terms quadratic in \( P \) and in the right ones the terms linear in \( N \). Such a property, due to (29), implies that if \( f_2 = A \otimes B \), then the factors \( A \) have to be quadratic in momenta and the factors \( B \) have to be linear in \( N \).

In such circumstances the most general ansatz for \( f_2 \) is the following:
\[ f_2 = \alpha P_0^2 \otimes N + \beta P_1^2 \otimes N + \gamma P_0 P_1 \otimes N + f_2^{(0)} \]  
where \( [f_2^{(0)}, \Delta_0(0)] = 0 \). We get
\[ \Delta_2(N) = \alpha \left( P_0^2, N \right) \otimes N + \beta \left( P_1^2, N \right) \otimes N + \gamma (P_0 P_1, N) \otimes N = -i(C_0 R_0 R_0) \otimes N. \]  
Comparing this result with the coproduct (27), we obtain \(-i\gamma = \frac{1}{2}; \alpha + \beta = 0\), which implies that \( f_2 = \frac{1}{2} R_0 R_0 \otimes N + \beta C_0 \otimes N + f_2^{(0)} \). Further it is easy to see that for such a choice of \( f_2 \) the terms \( R_0 \otimes C_0 \) in (25) and \( R_1 \otimes C_0 \) in (26) cannot be obtained from the formula (29) with any choice of \( f_2^{(0)} \). In particular, the term \( R_0 \otimes P_0^2 \) can never be obtained from the commutator \([f_2, R_0 \otimes 1 + 1 \otimes R_0]\) for any element \( f_2 \in U(\hat{g}) \otimes U(\hat{g}) \). Concluding, we see that the coproducts (20)–(22) cannot be obtained by twist and formula (7).

3. The \( \kappa \)-Poincaré from twisting: \( D = 4 \) case

3.1. \( D = 4 \) \( \kappa \)-Poincaré algebra in a bicrossproduct basis
(i) Algebra with Lorentz generators \( M_{\mu\nu} = \left( M_k = \frac{1}{2} \epsilon_{ijk} M_{ij}; N_i = M_{0i} \right) \) and 4-momenta \( P_\mu = (\rho_0, P_j) \).
\[ [M_i, M_j] = i \epsilon_{ijk} M_k, \quad [M_i, N_j] = i \epsilon_{ijk} N_k, \quad [N_i, N_j] = -i \epsilon_{ijk} M_k \]  
\[ [M_i, P_j] = i \epsilon_{ijk} P_k, \quad [M_i, P_0] = 0, \quad [N_i, P_0] = i P_j \]  
\[ [N_i, P_j] = \frac{i}{2} \delta_{ij} \left( \frac{1}{\kappa} - e^{-\frac{2\rho_0}{\kappa}} \right) + \frac{i}{\kappa} P_j. \]  
(ii) Coalgebra.

The coproducts satisfying the relations (32)–(34) have the form
\[ \Delta(P_0) = 1 \otimes P_0 + P_0 \otimes 1, \quad \Delta(M_i) = 1 \otimes M_i + M_i \otimes 1 \]  
6 The necessity of adding to (30) the nontrivial term \( f_2^{(0)} \) was pointed out by V N Tolstoy. However, it can be shown that such terms do not modify the conclusion obtained if \( f_2^{(0)} = 0 \). The most important terms belonging to \( f_2^{(0)} \) are given by the formula \( f_2^{(0)} = \delta (R \otimes NR - R \otimes NR) + \rho (P \otimes NR - R \otimes NR) \).
7 A similar argument will be used later for the case \( D = 4 \).
\[ \Delta(P_k) = e^{-\frac{\kappa}{2}} \otimes P_k + P_k \otimes 1, \]  
(36)

\[ \Delta(N_i) = N_i \otimes 1 + e^{-\frac{\kappa}{2}} \otimes N_i - \frac{1}{\kappa} P_j \otimes \epsilon_{ijk} M_k. \]  
(37)

### 3.2. $D = 4 \kappa$-Poincaré algebra in a classical basis

(i) Algebra.

The relations (32)–(34) are mapped into classical Poincaré algebra:

\[ [M_i, M_j] = i\epsilon_{ijk} M_k, \quad [M_i, N_j] = i\epsilon_{ijk} N_k, \quad [N_i, N_j] = -i\epsilon_{ijk} M_k \]  
(38)

\[ [M_i, P_k] = i\epsilon_{ijk} P_j, \quad [M_i, P_0] = 0, \quad [N_i, P_0] = i\epsilon_{ijk} P_j, \quad [N_i, P_j] = i\delta_{ij} P_0 \]  
(39)

with the use of the inverse quantum map:

\[ P_0 = \kappa \ln \Pi_0, \quad P_i = P_i \Pi_0^{-1} \]  
(40)

where $\Pi_0$ and $\Pi_0^{-1}$ are given by the formulas (18) with the four-dimensional classical mass Casimir $C_0 = P^0 P_0 + P^i P_i = -P_0^2 + \vec{p}^2$.

(ii) Coalgebra.

\[ \Delta(R_0) = P_0 \otimes \Pi_0 + \Pi_0^{-1} \otimes P_0 + \frac{1}{\kappa} P_i \Pi_0^{-1} \otimes P_i \]  
(41)

\[ \Delta(R_i) = P_i \otimes \Pi_0 + 1 \otimes P_i \]  
(42)

\[ \Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i \]  
(43)

\[ \Delta(N_i) = N_i \otimes 1 + \Pi_0^{-1} \otimes N_i - \frac{1}{\kappa} \epsilon_{ijk} P_i \Pi_0^{-1} \otimes M_j \]  
(44)

where $\Pi_0$ and $\Pi_0^{-1}$ are expanded in $\frac{1}{\kappa}$ power series by means of the formula (17), with the four-dimensional classical mass Casimir $C_0$. By expanding (39), (40), (42) in $\frac{1}{\kappa}$ we get

\[ \Delta(R_0) = P_0 \otimes 1 + 1 \otimes P_0 + \frac{1}{\kappa} P_i \otimes P_i \]  
(45)

\[ + \frac{1}{\kappa^2} \left( P_0^2 \otimes P_0 + \frac{1}{2} C_0 \otimes P_0 - P_0 P_0 \otimes P_0 - \frac{1}{2} P_0 \otimes C_0 \right) + O \left( \frac{1}{\kappa^3} \right) \]  
(46)

\[ \Delta(R_i) = P_i \otimes 1 + 1 \otimes P_i + \frac{1}{\kappa} P_i \otimes P_0 - \frac{1}{2\kappa^2} P_i \otimes C_0 + O \left( \frac{1}{\kappa^3} \right) \]  
(47)

\[ \Delta(N_i) = N_i \otimes 1 + 1 \otimes N_i - \frac{1}{\kappa} \left( \epsilon_{ijk} P_i \otimes M_j + P_0 \otimes N_i \right) \]  
(48)

\[ + \frac{1}{\kappa^2} \left( P_0^2 + \frac{1}{2} C_0 \right) \otimes N_i + \epsilon_{ijk} P_0 P_0 \otimes M_j \]  
(49)

Following (28) one can postulate the following formula for $D = 4 \kappa$-deformation:

\[ f_i = -iP_i \otimes N_i. \]  
(50)
One can check that in accordance with formula (7), one gets correctly the linear terms in the coproducts (45)–(47).

3.3. The no-go theorem for $D = 4$

After using formulas (29) and (45), we present the term $\Delta_2(R_\alpha)$ in two ways:

$$
\Delta_2(R_\alpha) = \frac{1}{2} \left[ P_\alpha \otimes N_\alpha \left( P_\alpha \otimes N_\alpha, R_\alpha \otimes 1 + 1 \otimes R_\alpha \right) \right] + \left[ f_2, R_\alpha \otimes 1 + 1 \otimes R_\alpha \right]
$$

$$
= \frac{1}{2} P_\alpha^2 \otimes R_\alpha + \left[ f_2, R_\alpha \otimes 1 + 1 \otimes R_\alpha \right]
$$

$$
= \frac{1}{2} \left( P_\alpha \otimes P_\alpha^2 + P_\alpha^2 \otimes R_\alpha + R_\alpha^2 \otimes R_\alpha - R_\alpha \otimes R_\alpha^2 \right) - P_\alpha P_\alpha \otimes P_\alpha.
$$

(49)

We shall show the impossibility of finding such an $f_2 = A_\alpha \otimes B_\alpha$ that leads to the validity of the last equality in (49). For that purpose, let us consider just the derivation by twist of the term $R_\alpha \otimes P_\alpha^2$. One can write the general formula

$$
\left[ f_2, \Delta_0(R_\alpha) \right] = [A_\alpha, R_\alpha] \otimes R_\alpha + A_\alpha \otimes [B_\alpha, R_\alpha].
$$

(50)

Because $A_\alpha$ and $B_\alpha$ depend only on Poincaré generators, the commutators $[A_\alpha, R_\alpha]$ and $[B_\alpha, R_\alpha]$ will necessarily generate a term different from $R_\alpha$; in fact, because $[P_\alpha, R_\alpha] = [M_\alpha, R_\alpha] = 0$, the two commutators on the right-hand side of (50) are different from zero only if $A_\alpha$ and/or $B_\alpha$ depends on the boost generators $N_\alpha$. Because the $O(3)$-covariance of $\kappa$-deformed algebra and coalgebra relations is their basic feature [1–3], we assumed that the term $f_2$ is $O(3)$-invariant, i.e. generators $N_\alpha$ in $f_2$ are always accompanied by $P_\alpha$, on the same side or on the other side of the tensor product $\otimes$. By writing down all possible $O(3)$-invariant terms in $f_2$ linear in Lorentz generators (and quadratic in momenta for dimensional reasons), it can be shown that the term $R_\alpha \otimes P_\alpha^2$ can never be obtained by twist from the formula (50).

The only way of satisfying the equality (49) would be to consider the elements $A_\alpha$ and $B_\alpha$ containing some generators $X$ which satisfy the commutation relation $[X, R_\alpha] \sim R_\alpha$. Such a generator $X$ can be provided only if we enlarge the Poincaré symmetries, in particular through scale transformations with $X$ identified with the dilatation generator $D$. In such a way, the twist $F$ will be introduced as spanned by the generators of the eleven-dimensional extension $(P_\alpha^4, M_\alpha, D)$ of the $D = 4$ Poincaré algebra called also the $D = 4$ Weyl algebra. It can be mentioned that such enlarged 'outer' Poincaré twists depending on the dilatation generators $D$ have been considered before (see e.g. [11, 15, 18]), and were used in order to describe the star-product providing $\kappa$-Minkowski noncommutative spacetime. In particular, in [15] the twist used for the construction of the universal $R$-matrix depends on the phase space realization of the dilatation generator.

4. Discussion and outlook

The existence of a universal Drinfeld twist generating quantum $\kappa$-Poincaré algebra from an undeformed one by twist [5–7] in the category of quasi-Hopf algebras has been proven in [25] by the use of cohomological arguments. The proof was based on the analogy with the quantum (depending on the deformation parameter) contraction $R \rightarrow \infty$ ($R$ is the AdS radius) of $U_q(su(3, 2))$ to a $\kappa$-deformed Poincaré–Hopf algebra [1]. But the following should be mentioned:
(i) Although the explicit formula for the universal $R$-matrix for $\mathcal{U}_q(\mathfrak{so}(3, 2))$ is known (see e.g. [27]), due to the appearance of divergent terms in the quantum contraction limit which we are not able to compensate it was not possible to obtain the universal $R$-matrix for the $\kappa$-Poincaré case by quantum contraction.

(ii) Although the cohomological arguments imply the existence of a universal Drinfeld twist for the $D = 4\, \kappa$-Poincaré case [25], we do not have an explicit formula even for the Drinfeld twist describing the $\mathcal{U}_q(\mathfrak{o}(3, 2))$ deformation\(^8\) before quantum $\kappa$-contraction.

It should be recalled, however, that the universal $R$-matrix has been calculated perturbatively (in the standard basis [24] up to the fifth order in $\kappa$) as a quasi-Hopf algebra with nontrivial coassociator. In order to demonstrate that nonexistence of a cochain twist defining $\kappa$-deformed coproducts is not in contradiction with the existence of a universal $R$-matrix, we did calculate for $D = 2$ in the classical algebra basis (see the Appendix) the universal $R$-matrix up to the second order in $\kappa$. If we define the universal $R$-matrix by perturbative expansion of its logarithm:

$$R = \exp \left( \frac{1}{\kappa} \eta + \frac{1}{\kappa^2} \nu + \cdots \right)$$ \hspace{1cm} (51)

and calculate $\eta, \nu$ using relation (2) and the perturbative expansions (25)–(27), we get the following formula (see (A.8)):

$$\eta = iP_1 \wedge N$$ \hspace{1cm} (52)

$$\nu = \frac{i}{2} (N \wedge P_1 P_0 + NP_0 \wedge R).$$ \hspace{1cm} (53)

We see that the nonexistence of a twist satisfying relation (7) does not prohibit the existence of a universal $R$-matrix. In fact, if we express the $R$-matrix (51)–(53) using the formula (8), the twist factor $F$ will not satisfy the basic relation (7).

We would like to point out that in order to consider a twist as belonging to the tensor product of classical Poincaré enveloping algebras $\mathcal{U}(\mathfrak{P}^{3,1}) \otimes \mathcal{U}(\mathfrak{P}^{3,1})[[\frac{1}{\kappa}]]$, we performed the calculation in the particular classical $\kappa$-Poincaré basis defined by maps (16) and (17). In order to conclude that the $\kappa$-Poincaré case is not endowed with triangular quasi-Hopf algebra structure, i.e. to contest the results of [25], the statement of the nonexistence of a twist should be shown to hold for all isomorphisms between the $\kappa$-Poincaré algebra basis and the undeformed one. Indeed our formulas (16) correspond to a particular fixed choice of such an isomorphism, but we mention that these formulas were generalized in [9] using a form containing two arbitrary parameters. The generalization of our discussion here to a more general choice of the quantization map, e.g. that given in [9], will be considered in the near future.

Concluding, we add also that a problem which can be studied further is the consideration of generalized $\kappa$-deformations (see e.g. [30, 31]) with the extension of the standard $\kappa$-deformed Minkowski space relations by the introduction of an arbitrary constant 4-vector $\tau^\mu$:

$$[\hat{\epsilon}^\mu, \hat{\epsilon}^\nu] = i \frac{1}{\kappa} (\tau^\rho \hat{\epsilon}^\nu - \tau^\nu \hat{\epsilon}^\rho).$$ \hspace{1cm} (54)

The standard $\kappa$-deformed Poincaré algebras considered in sections 2 and 3 correspond to the choice $\tau^\mu = (1, 0) \, (D = 2)$ and $\tau^\mu = (1, 0, 0, 0) \, (D = 4)$. The general $\kappa$-deformed Poincaré algebras which lead to the relations (54) can be split into two cases:

\(^8\) Partial explicit results for Drinfeld twists are known only for $\mathcal{U}_q(\mathfrak{sl}(2; 2))$ quantum algebra and its real forms (see e.g. [28, 29]).
(i) $\tau^\mu \tau_\mu = 0$ (a light-like constant 4-vector determining a light-cone $\kappa$-deformation). In such a case, following old results from the 1990s [32, 33], one can show that the twist $F$ providing the coproducts via formula (7) does exist and satisfies also the two-cocycle condition (i.e. generates through formula (9) the trivial coassociator $\phi = 1 \otimes 1 \otimes 1$).

(ii) $\tau^\mu \tau_\mu \neq 0$. Here one can demonstrate that the result of section 3.3 concerning the nonexistence of a twist remains valid for any choice of time-like $\tau_\mu$ (equivalent to the standard $\kappa$-deformation) and for any space-like $\tau_\mu$ (tachyonic $\kappa$-deformation).

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Appendix: Perturbative expansion of the universal $R$-matrix for the $D = 2 \kappa$-Poincaré case in a classical basis.

The universal $R$-matrix describes the flip operation $((a \otimes b)^T = b \otimes a; a, b \in A)$ on the coproducts (see also (2)):

$$\Delta^T (a) = R \Delta (a) R^{-1}.$$  (A.1)

We assume (see also [24]) that the universal $R$-matrix can be expanded as in formula (51), and one gets a $\frac{1}{\kappa}$ expansion of the transposed coproducts and

$$\Delta^T (a) = R \Delta (a) R^{-1} = \Delta + [r, \Delta] + \frac{1}{2} [r, [r, \Delta]] + \cdots = \\
= \Delta + \frac{1}{\kappa} [\eta, \Delta] + \frac{1}{2 \kappa^2} [\eta, [\eta, \Delta]] + \frac{1}{\kappa^2} [r_2, \Delta] + O \left( \frac{1}{\kappa^3} \right)$$  (A.2)

where

$$\Delta = \Delta_0 + \frac{1}{\kappa} \Delta_1 + \frac{1}{\kappa^2} \Delta_2 + O \left( \frac{1}{\kappa^3} \right).$$  (A.3)

We obtain

$$\Delta^T (a) = \Delta_0 + \frac{1}{\kappa} \Delta_1 + \frac{1}{\kappa^2} \Delta_2 + \frac{1}{\kappa} [\eta, \Delta_0] + \frac{1}{\kappa^2} [\eta, \Delta_1] + \frac{1}{\kappa} [\eta, \Delta_0] + \frac{1}{\kappa^2} [\eta, \Delta_1] + \frac{1}{2} [\eta, [\eta, \Delta_0]] + \frac{1}{\kappa} [r_2, \Delta_0] + O \left( \frac{1}{\kappa^3} \right)$$

$$= \Delta_0 + \frac{1}{\kappa} (\Delta_1 + [\eta, \Delta_0]) + \frac{1}{\kappa^2} \left( [\eta, \Delta_1] + [r_2, \Delta_0] + \frac{1}{2} [\eta, [\eta, \Delta_0]] \right) + O \left( \frac{1}{\kappa^3} \right).$$  (A.4)
Comparing order by order, we get the following equations:
\[ \Delta_0^T = \Delta_0 \quad (A.5a) \]
\[ \Delta_1^T = \Delta_1 + [\eta, \Delta_0] \quad (A.5b) \]
\[ \Delta_2^T = \Delta_2 + [\eta, \Delta_1] + [\eta, \Delta_0] + \frac{1}{2} [\eta, [\eta, \Delta_0]] . \quad (A.5c) \]
where \( \Delta^T = \Delta_0^T + \frac{1}{\kappa} \Delta_1^T + \frac{1}{\kappa^2} \Delta_2^T + O \left( \frac{1}{\kappa^3} \right) \).

The expansions of the opposite coproducts in the classical \( D = 2 \kappa \)-Poincaré basis are the following:
\[ \Delta^\text{op} (R_0) = R_0 \otimes 1 + 1 \otimes R_0 + \frac{1}{\kappa} R_1 \otimes R_1 \]
\[ + \frac{1}{\kappa^2} \left( R_0 \otimes P_0^2 + \frac{1}{2} P_0 \otimes C_0 - \frac{1}{2} C_0 \otimes P_0 - P_1 \otimes R_1 P_0 \right) + O \left( \frac{1}{\kappa^3} \right) ; \quad (A.6a) \]
\[ \Delta^\text{op} (P_1) = P_1 \otimes 1 + 1 \otimes R_1 + \frac{1}{\kappa} R_0 \otimes R_1 - \frac{1}{2 \kappa^2} C_0 \otimes P_1 + O \left( \frac{1}{\kappa^3} \right) ; \quad (A.6b) \]
\[ \Delta^\text{op} (N) = N \otimes 1 + 1 \otimes N - \frac{1}{\kappa} N \otimes R_0 + \frac{1}{\kappa^2} \left( N \otimes P_0^2 + \frac{1}{2} N \otimes C_0 \right) + O \left( \frac{1}{\kappa^3} \right) . \quad (A.6c) \]

One can check easily that \( \eta = i R_1 \wedge N \) satisfies equation \((A.5b)\) for all generators: \( N, R_0, P_1 \).

Equation \((A.5c)\) applied to the \( D = 2 \kappa \)-Poincaré case gives
\[ \left[ \eta, \Delta_0 (R_1) \right] = P_1 R_0 \otimes P_1 - P_1 \otimes R_1 P_0 \]
\[ + \frac{1}{2} \left( P_0^2 \otimes P_1 - P_1 \otimes P_0^2 - R_0 \otimes R_1 P_0 + R_1 \otimes R_0 \right) \quad (A.7a) \]
\[ \left[ \eta, \Delta_0 (P_1) \right] = \frac{1}{2} \left( P_0^2 \otimes R_1 - P_1 \otimes P_0^2 - R_0 \otimes R_1 P_0 + R_1 \otimes R_0 \right) \quad (A.7b) \]
\[ \left[ \eta, \Delta_0 (N) \right] = N \otimes P_0^2 + \frac{1}{2} N \otimes C_0 - R_0^2 \otimes N - \frac{1}{2} C_0 \otimes N - \frac{1}{2} N R_0 \otimes R_0 \]
\[ + \frac{1}{2} R_0 \otimes NR_1 + \frac{1}{2} R_0 \otimes R_0 N - \frac{1}{2} R_0 N \otimes R_1 . \quad (A.7c) \]

One can show that on taking
\[ r_2 = \frac{i}{2} (N \wedge R_1 R_0 - N R_0 \wedge R_1) , \quad (A.8) \]
the equations \((A.7a)\)–\((A.7c)\) are satisfied, since
\[ \left[ N \otimes R_1 P_0 + N R_0 \otimes P_1 - P_1 R_0 \otimes N - R_1 \otimes N R_0, N \otimes 1 + 1 \otimes N \right] \]
\[ = N R_1 \otimes R_1 - P_0^2 \otimes N - P_1^2 \otimes N - R_0 \otimes N R_0 + N \otimes (P_0^2 + P_1^2) \]
\[ + N R_0 \otimes R_0 - R_1 \otimes N R_1 . \quad (A.9) \]
The perturbative calculations for \( D = 2 \) can be extended also to the \( D = 4 \) case.

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