Kronecker factorization theorems for the exceptional Malcev algebra

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Abstract

We prove that a Malcev algebra $\mathcal{M}$ containing the 7-dimensional exceptional Malcev algebra $\mathcal{M}$ is isomorphic to a direct sum $(\mathcal{M} \otimes F A) \oplus N$, where $A$ is a commutative associative algebra. Also, we prove that a Malcev superalgebra $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$ whose even part $\mathcal{M}_0$ contains $\mathcal{M}$ is isomorphic to a direct sum $(\mathcal{M} \otimes F A) \oplus N'$, where $A$ is a supercommutative associative superalgebra.

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1 Introduction

A Malcev algebra is a vector space $\mathcal{M}$ with a bilinear binary operation $(x, y) \mapsto xy$ satisfying the following identities:

$$x^2 = 0, \quad J(x, y, xz) = J(x, y, z)x,$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ is the Jacobian of the elements $x, y, z \in \mathcal{M}$. Denote by $\mathcal{J}(\mathcal{M}, \mathcal{M}, \mathcal{M})$ the subspace spanned by the Jacobians, which is an ideal of $\mathcal{M}$ (see [15]). The set $N_\mathcal{M} = \{x \in \mathcal{M} : \mathcal{J}(x, \mathcal{M}, \mathcal{M}) = 0\}$ is also an ideal of $\mathcal{M}$ called the Lie nucleus of $\mathcal{M}$. A Malcev algebra $\mathcal{M}$ is a Lie algebra if and only if $\mathcal{J}(\mathcal{M}, \mathcal{M}, \mathcal{M}) = 0$ if and only if $N_\mathcal{M} = \mathcal{M}$. In some way, the Malcev algebras with trivial Lie nucleus, $N_\mathcal{M} = 0$, are the most distant from being Lie algebras.
All Lie algebras are clearly Malcev algebras because the Jacobian of any three elements vanish. The tangent space $T(L)$ of an analytic Moufang loop $L$ is another example of Malcev algebra. Let $\mathcal{A}$ be an alternative algebra, if we introduce a new product by means of a commutator $[x, y] = xy - yx$ into $\mathcal{A}$, we obtain a new algebra that will be denoted by $\mathcal{A}^{(-)}$. It is easy to verify that the algebra $\mathcal{A}^{(-)}$ is a Malcev algebra. All Malcev algebras obtained in this form are called special. The classic example of a non-Lie Malcev algebra $\mathcal{M}$ is formed by traceless elements of the Cayley-Dickson algebra with the commutator. The algebra $\mathcal{M}$ is of dimension 7 and is one of most cornerstone examples.

H. M. Wedderburn proved that if a unital associative algebra $\mathcal{A}$ contains a central simple subalgebra of finite dimension $\mathcal{B}$, where $\mathcal{A}$ is isomorphic to a Kronecker product $S \otimes F \mathcal{B}$, where $S$ is the subalgebra of the elements that commute with each $b \in \mathcal{B}$. In particular, if $\mathcal{A}$ contains $M_n(F)$ as subalgebra with the same identity element, we have $\mathcal{A} \cong S \otimes F M_n(F)$ where $S$ is the subalgebra of the elements that commute with each unitary matrix $e_{ij}$. Kaplansky extended (see [7], Theorem 2) Wedderburn result to alternative algebras and the split Cayley algebra. Jacobson uses his classification of completely reducible alternative bimodules over fields of characteristic different of 2 to give a new proof of Kaplansky result in [5], Theorem 1. The author proved that the result is valid for any characteristic (see [10]). Jacobson used his above mentioned result in [5] to prove a Kronecker Factorization Theorem for the simple exceptional Jordan algebra of $3 \times 3$ Hermitian matrices over the Cayley algebra. In the case of right alternative algebras, S. Pchelintsev, O. Shashkov and I. Shestakov [13] proved that every unital right alternative bimodule over a Cayley algebra (over an algebraically closed field of characteristic not 2) is alternative and they used that result to prove a coordinatization theorem for unital right alternative algebras containing a Cayley subalgebra with the same unit.

In the case of superalgebras, M. López-Díaz and I. Shestakov [9, 8] studied the representations of simple alternative and exceptional Jordan superalgebras in characteristic 3 and through these representations, they obtained some analogues of the Kronecker Factorization Theorem for these superalgebras. Also, the author [10] obtained some analogues of the Kronecker Factorization Theorem for central simple alternative superalgebras $M_{(1|1)}(F)$, $\mathcal{O}(4|4)$ and $\mathcal{O}[u]$, where in particular the Kronecker Factorization Theorem for $M_{(1|1)}(F)$ answers the analogue for superalgebras of the Jacobson’s problem [5]. We state that Jacobson’s problem [5] was recently solved by the author and I. Shestakov [11] in the split case. Similarly, C. Martinez and E. Zelmanov [12] obtained a Kronecker Factorization Theorem for the exceptional ten dimensional Kac superalgebra $K_{10}$. Also, Y. Popov [14] studied the representations of simple finite-dimensional noncommutative Jordan superalgebras and proved some analogues of the Kronecker factorization theorem for such superalgebras.

The aim of this paper is to determine the structure of Malcev (super)algebras containing the 7-dimensional exceptional Malcev algebra $\mathcal{M}$. So the paper is organized as follows: In Section 2, we provided definitions about Malcev (super)algebras and their representations. Also we describe some special subvariety $\mathcal{H}$ of Malcev (super)algebras and their representations. In Section 3, we prove analogs of the Kronecker Factorization Theorems for Malcev
(super)algebras that contain $\mathbb{M}$. In Section 4, we establish certain equivalences of categories. Finally, in section 5, we study (super)algebras and modules with (super)involutions.

2 Preliminaries

In this section we provide background material that is used along the way and some preliminary results.

For arbitrary elements $x, y, z, t, u$ in a Malcev algebra $\mathcal{M}$ we consider the following functions that play an important role in the theory of Malcev algebras:

\[
J(x, y, z) = (xy)z + (yz)x + (zx)y, \text{ the Jacobian of } x, y, z,
\]

\[
[x, y, z] = (xy)z + x(yz), \text{ the antiassociator,}
\]

\[
\{x, y, z\} = (xy)z - (xz)y + 2x(yz) = J(x, y, z) + 3x(yz) = [x, y, z] - [x, z, y],
\]

\[
h(y, z, t, x, u) = \{yz, t, u\}x + \{yz, t, x\}u + \{yx, z, u\}t + \{yu, z, x\}t.
\]

Throughout this paper $F$ will be a field of characteristic different of 2 and 3. Then expanding the Jacobian, the Malcev identities (1) can be rewritten as follows

\[
xy = -yx,
\]

\[
(xz)(yt) = ((xy)z)t + ((yz)t)x + ((zt)x)y + ((tx)y)z.
\]

And the following identity hold in $\mathcal{M}$ which was proved in [15]

\[
J(tx, y, z) = tJ(x, y, z) + J(t, y, z)x - 2J(t, x, yz).
\]

2.1 Seven dimensional exceptional Malcev algebra

The most important example of a non-Lie Malcev algebra is the algebra $\mathbb{M}$ of dimension 7 which we are going to describe now. First, we consider the central simple Malcev algebra $\mathbb{M} = \mathcal{M}_7(F)$ obtained by means of the commutator $[x, y] = xy - yx$ introduced in the split Cayley-Dickson matrix algebra $C(F)$.

As in [15], we write the elements of the algebra $C(F)$ in the form

\[
\begin{bmatrix}
a & \alpha \\
\beta & b
\end{bmatrix},
\]

where $a, b$ are elements of the field $F$ and $\alpha, \beta$ are 3-dimensional vectors $(a_1, a_2, a_3)$ over $F$.

The sum is defined as

\[
\begin{bmatrix}
a & \alpha \\
\beta & b
\end{bmatrix} + \begin{bmatrix}
c & \gamma \\
\delta & d
\end{bmatrix} = \begin{bmatrix}
a + c & \alpha + \gamma \\
\beta + \delta & b + d
\end{bmatrix}.
\]
We use the notations $\bullet$ and $\times$ for the scalar and product of 3-dimensional vectors and define the multiplication by

$$\begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix} \cdot \begin{bmatrix} c & \gamma \\ \delta & d \end{bmatrix} = \begin{bmatrix} ac + \alpha \bullet \delta & a\gamma + d\alpha - \beta \times \delta \\ c\beta + b\delta + \alpha \times \gamma & \beta \bullet \gamma + bd \end{bmatrix}.$$  

In the split Cayley-Dickson algebra $C(F)$ we define a new multiplication $[,]$ by $[A, B] = AB - BA$, or

$$[A, B] = \begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix} \cdot \begin{bmatrix} c & \gamma \\ \delta & d \end{bmatrix} - \begin{bmatrix} c & \gamma \\ \delta & d \end{bmatrix} \cdot \begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix} = \begin{bmatrix} \alpha \bullet \delta - \beta \bullet \gamma & (a - b)\gamma + (d - c)\alpha - 2\beta \times \delta \\ (c - d)\beta + (b - a)\delta + 2\alpha \times \gamma & -(\alpha \bullet \delta - \beta \bullet \gamma) \end{bmatrix}.$$  

The algebra $C(F)^{(-)}$, with multiplication $[,]$ defined above, is a Malcev algebra. We chose a basis for $C(F)^{(-)}$ the elements

$$e_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ (0, 1, 0) \\ 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ (0, 0, 1) \\ 0 \end{bmatrix}, \quad e_5 = \begin{bmatrix} 0 \\ 0 \\ (1, 0, 0) \\ 0 \end{bmatrix},$$

$$e_6 = \begin{bmatrix} 0 \\ 0 \\ (0, 1, 0) \\ 0 \end{bmatrix}, \quad e_7 = \begin{bmatrix} 0 \\ 0 \\ (0, 0, 1) \\ 0 \end{bmatrix}.$$  

For all $x$ in $C(F)^{(-)}$, we have $[e_0, x] = 0$ and so let us consider the algebra

$$\mathcal{M}_7(F) = C(F)^{(-)}/F \cdot e_0.$$  

We identify $e_i$ in $C(F)^{(-)}$ with the coset $e_i + F \cdot e_0$ in $\mathcal{M}_7(F)$ and we denote the multiplication $[,]$ by juxtaposition, that is, $xy = [x, y]$. So we have the following multiplication table of $\mathcal{M}_7(F)$

|    | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|----|------|------|------|------|------|------|------|
| $e_1$ | 0    | 2$e_2$ | 2$e_3$ | 2$e_4$ | $-2e_5$ | $-2e_6$ | $-2e_7$ |
| $e_2$ | $-2e_2$ | 0    | 2$e_7$ | $-2e_6$ | $e_1$ | 0    | 0    |
| $e_3$ | $-2e_3$ | $-2e_7$ | 0    | 2$e_5$ | 0    | $e_1$ | 0    |
| $e_4$ | $-2e_4$ | 2$e_6$ | $-2e_5$ | 0    | 0    | 0    | $e_1$ |
| $e_5$ | 2$e_5$ | $-e_1$ | 0    | 0    | 0    | $-2e_4$ | 2$e_3$ |
| $e_6$ | 2$e_6$ | 0    | $-e_1$ | 0    | 2$e_4$ | 0    | $-2e_2$ |
| $e_7$ | 2$e_7$ | 0    | 0    | $-e_1$ | $-2e_3$ | 2$e_2$ | 0    |
and we know that the algebra $\mathcal{M}_7(F)$ is simple.

Now we consider the Malcev algebra $\mathcal{M} = \mathcal{M}_7$ obtained by means of the commutator $[x, y] = xy - yx$ introduced in the division Cayley-Dickson algebra $O$. Remember that $O = \mathbb{H} \oplus v\mathbb{H}$, $v^2 = \gamma \neq 0$ ($\gamma \in F$) has the $\mathbb{Z}_2$-grading induced by the Cayley-Dickson process applied to the division quaternion algebra $\mathbb{H}$. So, the odd part $v\mathbb{H}$ is a bimodule over $\mathbb{H}$ and it is a Cayley bimodule because satisfy the relation

$$aw = w\bar{a},$$

where $a \in \mathbb{H}$, $w \in v\mathbb{H}$ and $a \rightarrow \bar{a}$ is the canonical involution in $\mathbb{H}$. Also, we have

$$(ab)v = b(av),$$

$$v(ab) = (vb)a,$$

$$(va)(vb) = \gamma b\bar{a},$$

for all $a, b \in \mathbb{H}$.

The algebra $O^{(-)}$, with multiplication $[x, y] = xy - yx$, is a Malcev algebra. As the characteristic of the base field is different of 2, let $\{1, i, j, k, v, vi, vj, vk\}$ be a basis of $O^{(-)}$, where

$$i^2 = j^2 = k^2 = -1.$$

For all $x$ in $O^{(-)}$, we have $[1, x] = 0$ and so let us consider the algebra

$$\mathcal{M}_7 = O^{(-)}/F \cdot 1.$$

We identify any the base elements $x \in \{1, i, j, k, v, vi, vj, vk\}$ in $O^{(-)}$ with the coset $x + F \cdot 1$ in $\mathcal{M}_7$ and again we denote the multiplication $[,]$ by juxtaposition, that is, $xy = [x, y]$. Then by (4) and (5) we have the following multiplication table of $\mathcal{M}_7$

|     | $i$ | $j$ | $k$ | $v$ | $vi$ | $vj$ | $vk$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $i$ | 0   | $2k$| $-2j$| $-2vi$| $2v$ | $-2vk$| $2vj$ |
| $j$ | $-2k$| 0   | $2i$ | $-2vj$| $2vk$| $2v$  | $-2vi$|
| $k$ | $2j$ | $-2i$| 0   | $-2vk$| $-2vj$| $2vi$ | $2v$  |
| $v$ | $2vi$| $2vj$| $-2vk$| 0   | $2\gamma i$| $2\gamma j$| $2\gamma k$|
| $vi$| $-2v$| $-2vk$| $-2vj$| $-2\gamma i$| 0   | $2\gamma k$| $-2\gamma j$|
| $vj$| $2vk$| $-2v$| $-2vi$| $-2\gamma j$| $-2\gamma k$| 0   | $2\gamma i$|
| $vk$| $-2vj$| $2vi$| $-2v$| $-2\gamma k$| $2\gamma j$| $-2\gamma i$| 0   |

and also we know that the algebra $\mathcal{M}_7$ is simple.
2.2 Representation of Malcev superalgebras

In this subsection we provide some definitions about the representation theory of Malcev superalgebras.

Let $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$ be a superalgebra (this is, $\mathbb{Z}_2$-graded algebra) and let $G = G_0 \oplus G_1$ be the Grassmann algebra generated by the elements $1, e_1, \ldots, e_n, \ldots$ over a field $F$. The Grassmann envelope of $\mathcal{M}$ is defined to be $G(\mathcal{M}) := G_0 \otimes \mathcal{M}_0 + G_1 \otimes \mathcal{M}_1$. Then $\mathcal{M}$ is said to be Malcev superalgebra if $G(\mathcal{M})$ is a Malcev algebra. So from this definition and by (2), $\mathcal{M}$ satisfies the following superidentities

$$xy = -(-1)^{|x||y|}yx,$$

$$(-1)^{|y||z|}(xz)(yt) = ((xy)z)t + (-1)^{|z||y|+|z||t|}((yz)t)x$$

$$+ (-1)^{(|x||y|+|z||t|)}((zt)x)y$$

$$+ (-1)^{|z||y|+|y||t|}((tx)y)z.$$

where $x, y, z \in \mathcal{M}_0 \cup \mathcal{M}_1$ and $|x|$ denotes the parity index of a homogeneous element $x$ of $\mathcal{M}$: $|x| = i$ if $x \in \mathcal{M}_i$.

In any Malcev superalgebra $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$, the identities for Malcev algebras are easily superized to obtain the analogous graded identities. Thus, for homogeneous elements $x, y, z, t, u \in \mathcal{M}$ we define the following functions

$$\tilde{J}(x, y, z) = (xy)z - x(yz) - (-1)^{|y||z|}(xz)y,$$

$(super Jacobian)$

$$\{x, y, z\} = (xy)z - (-1)^{|y||z|}(xz)y + 2x(yz),$$

$$\tilde{h}(y, z, t, x, u) = \{yz, t, u\}x + (-1)^{|x||u|}\{yz, t, x\}u$$

$$+ (-1)^{|x||y|+|t||u|+|u||t|}\{xy, z, u\}t + (-1)^{|u||y|+|t||u|+|x||t|}\{yu, z, x\}t.$$
A regular module, $\text{Reg}\mathcal{M}$, for superalgebra $\mathcal{M}$, is defined on the vector space $\mathcal{M}$ with the action of $\mathcal{M}$ coinciding with the multiplication in $\mathcal{M}$.

Let $V$ be a module for the Malcev superalgebra $\mathcal{M}$ and let $E = \mathcal{M} \oplus V$ be the corresponding split null extension. Let us consider
\[ \tilde{\Gamma}_i = \{ \alpha \in \text{End}_F(E)_i : (xy)\alpha = x(y\alpha) = (-1)^{|y|} (x\alpha)y \quad \forall \ x, y \in E_0 \cup E_1 \}, \quad i = 0, 1, \]
\[ \tilde{\Gamma} = \tilde{\Gamma}(E) = \tilde{\Gamma}_0 \oplus \tilde{\Gamma}_1, \text{ the supercentroid of } E, \]
\[ Z = Z(E) = \{ \alpha \in \tilde{\Gamma} : V\alpha \subseteq V \text{ and } \mathcal{M}\alpha \subseteq \mathcal{M} \}, \]
\[ K_i = K_i(V) = \{ \varphi \in \text{End}_F(V)_i : \varphi \rho x = (-1)^{|\varphi|} \rho \varphi x \quad \forall \ x \in \mathcal{M}_0 \cup \mathcal{M}_1 \}, \quad i = 0, 1, \]
\[ K = K(V) = K_0 \oplus K_1, \text{ the supercentralizer of } V. \]

The following proposition provides some basic properties of the subsuperalgebras $Z$ and $K$ when we consider irreducible almost faithful modules.

**Proposition 2.1** ([2], Proposition 4). Assume that $V$ is an irreducible almost faithful module for $\mathcal{M}$. Then

(i) $Z_1 = 0$ and $Z = Z_0$ is an integral domain which acts without torsion on $\mathcal{M}$.

(ii) $K_0$ is a skew field and any nonzero homogeneous element in $K$ acts bijectively on $V$.

(iii) The restriction homomorphism $\phi : Z \longrightarrow K_0 (\alpha \mapsto \alpha|_V)$ is one-to-one.

### 2.3 (Super)algebras and modules in $\mathcal{H}$

In this subsection we define the subvariety $\mathcal{H}$ of the Malcev algebras which was introduced by Filippov and considered in several papers (see for example, [3, 4]) and in the case of superalgebras by Elduque and Shestakov [1, 2, 16].

Let $\mathcal{M}$ be a Malcev algebra and consider the subspace $H(\mathcal{M})$ generated by the elements $h(y, z, t, u, x)$; $H(\mathcal{M})$ is an ideal of $\mathcal{M}$ (see [3]). The variety $\mathcal{H}$ (over $F$) is defined as the class of Malcev algebras $\mathcal{M}$ over $F$ that satisfy the identity $h(y, z, t, u, x) = 0$, that is, $H(\mathcal{M}) = 0$.

Some elements of the variety $\mathcal{H}$ are the 3-dimensional simple Lie algebra $\mathfrak{sl}(2, F)$ and the 7-dimensional exceptional Malcev algebra $\mathfrak{M}$ over its centroid $\Gamma$, which is a field.

If we consider in $\mathcal{M}$ the function $[4]$
\[ p(x, y, z, t) = -\{zt, x, y\} - \{yt, z, x\} + \{xt, y, z\}. \]

We have the following result.
Lemma 2.2 ([16, 2], Lemma 2). Let $\mathcal{M}$ be a Malcev algebra and assume that $H(\mathcal{M}) = 0$. Then for any $x, y, z, t, u \in \mathcal{M}$:

$$p(x, y, z, t)u = p(xu, y, z, t),$$

(7)

$$p(x, y, z, ut) = p(x, u, t, yz).$$

(8)

If $\mathcal{M}$ is a Malcev superalgebra. Similarly, we define $\tilde{p}(x, y, z, t)$. Consider the subspace $\tilde{H}(\mathcal{M})$ generated by the elements $\tilde{p}(x, y, z, t)$. Without generating confusion we denote again by $H$ the variety of the class of Malcev superalgebras in which $\tilde{h} = 0$, that is, $\tilde{H}(\mathcal{M}) = 0$.

Let $\mathcal{M}$ be a Malcev superalgebra in $\mathcal{H}$ and let $V = V_0 \oplus V_1$ be a module for $\mathcal{M}$. The module $V$ is called a module for $\mathcal{M}$ in the variety $\mathcal{H}$ or $\mathcal{H}$-module if the split extension $E = \mathcal{M} \oplus V$ is again a Malcev superalgebra in $\mathcal{H}$ with a multiplicación (6).

Elduque studied the subvariety $\mathcal{H}$ of Malcev superalgebras and their representations. He obtained the following results:

Theorem 2.3 ([1], Theorem 5). Let $\mathcal{M}$ be a simple superalgebra in the variety $\mathcal{H}$ and let $V$ be an $\mathcal{H}$-module for $\mathcal{M}$. Then $V$ is completely reducible.

As a consequence of the proof of the theorem 2.3 the author deduced:

Corollary 2.4. Let $\mathcal{M}$ be a simple algebra in $\mathcal{H}$, $V$ an $\mathcal{H}$-module for $\mathcal{M}$, and $\Gamma$ the centroid of $\mathcal{M}$. Then $V = N_V \oplus J_V$ where the submodules $N_V$ and $J_V$ are given by $N_V = \{v \in V : J(v, \mathcal{M}, \mathcal{M}) = 0\}$ and $J_V = J(V, \mathcal{M}, \mathcal{M})$. Moreover,

(i) If $\dim_{\Gamma} \mathcal{M} = 3$, then $N_V = Z_V \oplus \tilde{N}_V$, with $Z_V = \{v \in V : v\mathcal{M} = 0\}$, $\tilde{N}_V = V\mathcal{M} \cap N_V$. Besides, $\tilde{N}_V$ is a direct sum of copies of the adjoint module for $\mathcal{M}$ and $J_V$ is a direct sum of copies of the unique irreducible non-Lie module for $\mathcal{M}$.

(ii) If $\dim_{\Gamma} \mathcal{M} = 7$, then $N_V\mathcal{M} = 0$ and $J_V$ is a direct sum of copies of the adjoint module for $\mathcal{M}$.

Theorem 2.3 can be easily extended to the case in which $\mathcal{M}$ is a finite direct sum of simple algebras in $\mathcal{H}$.

Proposition 2.5. Let $\mathcal{M}$ be a finite direct sum of simple algebras in $\mathcal{H}$, $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$, with $\mathcal{M}_i$ simple in $\mathcal{H}$, and let $V$ be an $\mathcal{H}$-module for $\mathcal{M}$. Then $V = V_0 \oplus V_1 \oplus \cdots \oplus V_n$, where $V_i = V_i\mathcal{M}_i = V_i\mathcal{M}_i$ for $i = 1, \ldots, n$ and $V_i\mathcal{M}_j = 0$ if $i \neq j$.

From simple non-Lie Malcev algebras we can suppress the hypothesis of the module being considered in the variety $\mathcal{H}$.

Theorem 2.6 ([1], Theorem 8). Let $\mathcal{M}$ be a finite direct sum of simple non-Lie Malcev algebras. Then any module for $\mathcal{M}$ (in the variety of Malcev algebras) is a $\mathcal{H}$-module.
Combining Theorems 2.3 and 2.6, the following result establishes the complete reducibility of Malcev modules over direct sum of simple non-Lie Malcev algebras.

**Corollary 2.7** ([1], Corollary 9). Let $\mathcal{M}$ be a finite direct sum of simple non-Lie Malcev algebras. Then any representation of $\mathcal{M}$ is completely reducible.

As for any variety algebras, the study of the irreducible representations of Malcev (super)algebras is a key point in its investigation. Thus in this direction, Elduque andlesh
takov [2] described the irreducible Non-Lie modules for Malcev superalgebras.

To finish this subsection we will mention some of the results obtained in [2] that will be necessary to obtain our main results.

Consider an irreducible $H$-module $V$ for $\mathcal{M}$ (in the variety of Malcev superalgebras), then $E = \mathcal{M} \oplus V$ is a Malcev superalgebra in $H$ that is $\tilde{H}(E) = 0$. So, by Lemma 2.2 we have

\begin{align*}
\tilde{p}(x, y, z, t) & = (-1)^{|x||y||z|}|zt| \tilde{p}(x, y, z, t), \quad (9) \\
\tilde{p}(x, y, z, ut) & = (-1)^{|x|(|y|+|z|+|t|)} \tilde{p}(x, u, t, yz) \quad (10)
\end{align*}

for any homogeneous $x, y, z, t, u \in E$.

Let us define the operator $\tilde{\alpha}(y, z, t) \in \text{End}_F(E)$ by

\[ x \tilde{\alpha}(y, z, t) = \tilde{p}(x, y, z, t). \]

Then $\tilde{\alpha}$ is super-skewsymmetric on its arguments and it follows from (9) that $\tilde{\alpha}(y, z, t)$ belongs to $\tilde{\Gamma}(E)$.

In the following results, the operator $\tilde{\alpha}(y, z, t)$ plays an important role. So the theorem shows that the only irreducible almost faithful non-Lie module is the adjoint module for $\mathcal{M}$.

**Theorem 2.8** ([2], Theorem 2). Let $V$ be an irreducible almost faithful non-Lie module for the Malcev superalgebra $\mathcal{M}$ and let $E = \mathcal{M} \oplus V$. Assume $\tilde{H}(E) = 0 \neq \tilde{\alpha}(V, \mathcal{M}, \mathcal{M})$. Then, $\mathcal{M}_1 = 0$, $\mathcal{M} = \mathcal{M}_0$ is a simple non-Lie Malcev algebra, so seven-dimensional over its centroid, and $V$ is the regular (adjoint) module.

To describe the structure of every irreducible non-Lie Malcev module $V$ over $\mathcal{M}$, we will do the following: let $v \in V$ and $x, y \in \mathcal{M}$ be with $\tilde{\alpha}(v, x, y) \neq 0$, this induces the isomorphism of modules that was proved in the Theorem 2.8.

\[ \alpha: \mathcal{M} \rightarrow V \]

given by $\alpha(z) = z \tilde{\alpha}(v, x, y)$. Then $V \simeq \mathcal{M} \alpha$ as modules.

**Proposition 2.9** ([2], Proposition 8). Let $V$ be an irreducible almost faithful non-Lie module for the Malcev superalgebra $\mathcal{M}$ and let $E = V \oplus \mathcal{M}$. If $\tilde{H}(E) = 0 = \tilde{\alpha}(V, \mathcal{M}, \mathcal{M})$. Then, $\mathcal{M}_1 = 0$, $\mathcal{M} = \mathcal{M}_0$ is a Lie algebra and the map $\varphi: \mathcal{M} \rightarrow \text{End}_F(V)$, $x \mapsto -\frac{1}{2} \rho_x$, is a Lie representation of $\mathcal{M}$. 

9
3 Factorization Theorems

In this section we consider arbitrary Malcev algebras \( \mathcal{M} \) containing the 7-dimensional exceptional Malcev algebra \( \mathbb{M} \) such that \( m\mathbb{M} \neq 0 \) for any \( m \neq 0 \) from \( \mathcal{M} \). Also, we consider arbitrary Malcev superalgebra \( \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \) whose even part \( \mathcal{M}_0 \) contains \( \mathbb{M} \) with \( m\mathbb{M} \neq 0 \) for any homogeneous element \( 0 \neq m \in \mathcal{M}_0 \cup \mathcal{M}_1 \). In the general context, we can drop this assumption and so we describe the Malcev (super)algebras containing \( \mathbb{M} \).

3.1 Factorization Theorem for Malcev algebras

The objective of this subsection is to prove an analogy of the Kronecker Factorization Theorem for Malcev algebras that contain \( \mathbb{M} \).

Theorem 3.1. Let \( \mathcal{M} \) be a Malcev algebra such that \( \mathcal{M} \) contains \( \mathbb{M} \), with \( m\mathbb{M} \neq 0 \) for any \( m \neq 0 \) from \( \mathcal{M} \). Then \( \mathcal{M} \cong \mathbb{M} \otimes \mathcal{U} \) for a certain commutative associative algebra \( \mathcal{U} \).

Proof: Consider \( \mathcal{M} \) as a Malcev \( \mathbb{M} \)-module. Then by Corollary 2.7, \( \mathcal{M} \) is completely reducible, that is, \( \mathcal{M} = \sum_i \oplus V_i \), where \( V_i \) is an irreducible almost faithful non-Lie Malcev module over \( \mathbb{M} \). By Proposition 2.9 we have \( 0 \neq \alpha_i(V_i, \mathbb{M}, \mathbb{M}) \) because \( \mathbb{M} \) is a non-Lie Malcev algebra. Let us take elements \( v_i \in V_i \) and \( a_i, b_i \in \mathbb{M} \) such that

\[
\alpha_i = \alpha_i(v_i, a_i, b_i) \neq 0.
\]

Thus

\[
\mathcal{M} = \sum_i \oplus \mathbb{M} \alpha_i
\]

where for each \( i \) we have \( V_i \cong \mathbb{M} \alpha_i \). Also by Theorem 2.6 \( \mathcal{M} \) is an \( \mathcal{H} \)-module for \( \mathbb{M} \), then \( E = \mathbb{M} \oplus \mathcal{M} \) is a Malcev algebra in the variety \( \mathcal{H} \); this implies that \( \mathcal{M} \) satisfies the identity \( h = 0 \). So, by Lemma 2.2 for all \( x, y, z \in \mathcal{M} \)

\[
p(x, y, a_i, b_i) = p(xy, x_i, a_i, b_i),
\]

\((x\alpha_i)y = (xy)\alpha_i; \) thus \( \alpha_i \in \Gamma(\mathcal{M}) \), where \( \Gamma(\mathcal{M}) \) is the centroid of \( \mathcal{M} \).

Let \( \mathcal{U} = \sum_i F \alpha_i \) denote the span of all \( \alpha_i \). Then \( \mathcal{M} = \mathbb{M} \mathcal{U} \). So let’s go to prove that \( \mathcal{U} \) is a subalgebra of \( \Gamma(\mathcal{M}) \).

First we consider \( \mathbb{M} = \mathcal{M}_7(F) \), so we fix arbitrary elements \( \alpha, \beta \in \mathcal{U} \) and \( a, b \in \mathcal{M}_7(F) \). So \( a\alpha\beta \in \mathcal{M} \), then \( a\alpha\beta = \sum_i a_i \alpha_i \) for some \( a_i \in \mathcal{M}_7(F) \) and \( \alpha_i \in \mathcal{U} \). We denote

\[
w = a\alpha\beta - \sum_i a_i \alpha_i = 0.
\]

(11)

where \( a = \sum_{t=1}^{7} \lambda_t e_t, a_i = \sum_{t=1}^{7} \lambda_{ti} e_t \) and \( 0 \neq \lambda_t, \lambda_{ti} \in F, t = 1, 2, \ldots, 7 \). As \( \alpha\beta \in \Gamma(\mathcal{M}) \) and using the multiplication table of \( \mathcal{M}_7(F) \), we have

\[
0 = e_2 w = (-2\lambda_1 e_2 + 2\lambda_3 e_7 - 2\lambda_4 e_6 + \lambda_5 e_1)\alpha\beta - \sum_i (-2\lambda_{1i} e_2 + 2\lambda_{3i} e_7 - 2\lambda_{4i} e_6 + \lambda_{5i} e_1)\alpha_i.
\]
\[ 0 = e_2(e_2w) = -2(\lambda_5 e_2)\alpha\beta + 2 \sum_i (\lambda_5 e_2)\alpha_i, \]

which implies, \( e_2\alpha\beta = e_2\bar{\alpha}, \) where \( \bar{\alpha} = \sum_i (\lambda_5^{-1} \lambda_5 i)\alpha_i \in U. \) Hence, it is easy to see \( e_t(\alpha\beta - \bar{\alpha}) = 0 \) for \( t = 1, 2, \ldots, 7; \) so

\[ \mathcal{M}_7(F)(\alpha\beta - \bar{\alpha}) = 0 \]

and \( \alpha\beta - \bar{\alpha} = 0. \) Therefore \( \alpha\beta = \bar{\alpha} \in U; \mathcal{U} \subseteq U. \)

Now let \( m \in \mathcal{M} \) and \( a \in \mathcal{M}_7(F) \) be, then

\[ ((m\alpha)\beta)a = (m\alpha)(a\beta) = ((m\alpha)a)\beta = (m\beta)(aa)\alpha = ((m\beta)a)\alpha = (m\beta)\alpha a. \]

If \( [\alpha, \beta] = \alpha\beta - \beta\alpha, \) we have

\[ (\mathcal{M}[\alpha, \beta])\mathcal{M}_7(F) = 0, \]

so \( \mathcal{M}[\alpha, \beta] = 0 \) and \( [\alpha, \beta]|_{\mathcal{M}} = 0. \) In particular, \( [\alpha, \beta]|_{V_i} = 0 \) for any irreducible component \( V_i \) of \( \mathcal{M}, \) then by Proposition \( \text{(2.1 iii)} \) \( [\alpha, \beta] = 0 \) because \( \phi : Z \rightarrow K(\alpha \mapsto \alpha|_{V_i}) \) is one-to-one. Therefore \( [\mathcal{U}, \mathcal{U}] = 0; \) hence \( \mathcal{U} \) is a commutative and associative algebra.

Also

\[ (a\alpha)(b\beta) = (a(b\beta))\alpha = ((ab)\beta)\alpha = (ab)(\beta\alpha) = (ab)(a\beta). \]

Let \( v = \sum_{i=1}^{7} e_i\alpha_i = 0 \) be, where \( \alpha_i \in U. \) Then \( 0 = e_4v = -2e_4\alpha_1 + 2e_6\alpha_2 - 2e_5\alpha_3 \) and \( 0 = e_5(e_4v) = -4e_4\alpha_2. \) Hence \( e_4\alpha_2 = 0. \) Also

\[ 0 = e_2(e_4\alpha_2) = (e_2e_4)\alpha_2 = -2e_6\alpha_2, \]
\[ 0 = e_3(e_4\alpha_2) = (e_3e_4)\alpha_2 = 2e_5\alpha_2, \]
\[ 0 = e_7(e_4\alpha_2) = (e_7e_4)\alpha_2 = -e_1\alpha_2, \]

so \( e_6\alpha_2 = e_5\alpha_2 = e_1\alpha_2 = 0, \) and

\[ 0 = e_2(e_1\alpha_2) = (e_2e_1)\alpha_2 = -2e_2\alpha_2, \]
\[ 0 = e_3(e_1\alpha_2) = (e_3e_1)\alpha_2 = -2e_3\alpha_2, \]
\[ 0 = e_7(e_1\alpha_2) = (e_7e_1)\alpha_2 = 2e_7\alpha_2, \]

so \( e_2\alpha_2 = e_3\alpha_2 = e_7\alpha_2 = 0. \) Thus \( \mathcal{M}_7(F)\alpha_2 = 0 \) and \( \alpha_2 = 0. \) Similarly \( \alpha_i = 0 \) for \( i = 1, 3, \ldots, 7. \) Therefore \( \mathcal{M} \cong \mathcal{M}_7(F) \otimes_F \mathcal{U}. \)

If \( \mathbb{M} = \mathcal{M}_7. \) Without generating confusion we denote again by \( \mathcal{U} \) the vector space generated by all the isomorphisms \( \alpha_t \) between the irreducible \( \mathbb{M}_7 \)-modules and \( \text{Reg} \mathbb{M}_7. \)

As in above, we will prove that \( \mathcal{U} \) is a subalgebra of \( \Gamma(\mathcal{M}). \) We fix arbitrary elements \( \alpha, \beta \in \mathcal{U} \) and \( a, b \in \mathbb{M}_7. \) So \( a\alpha\beta \in \mathcal{M}, \) then \( a\alpha\beta = \sum_t a_t\alpha_t \) for some \( a_t \in \mathbb{M}_7 \) and \( \alpha_t \in \mathcal{U}. \)
Denote \( w = a\alpha\beta - \sum_t a_t\alpha_t = 0 \), that is
\[
  w = (\lambda_1 i + \lambda_2 j + \lambda_3 k + \lambda_4 v + \lambda_5 vi + \lambda_6 vj + \lambda_7 vk)\alpha\beta \\
  - \sum_t (\lambda_{1t} i + \lambda_{2t} j + \lambda_{3t} k + \lambda_{4t} v + \lambda_{5t} vi + \lambda_{6t} vj + \lambda_{7t} vk)\alpha_t = 0,
\]
where \( a = \lambda_1 i + \lambda_2 j + \lambda_3 k + \lambda_4 v + \lambda_5 vi + \lambda_6 vj + \lambda_7 vk \), \( a_t = \lambda_{1t} i + \lambda_{2t} j + \lambda_{3t} k + \lambda_{4t} v + \lambda_{5t} vi + \lambda_{6t} vj + \lambda_{7t} vk \) and \( 0 \neq \lambda_s, \lambda_{st} \in F, s = 1, \ldots, 7 \). So, using the multiplication table of \( \mathbb{M}_7 \)
\[
0 = iw = (2\lambda_2 k - 2\lambda_3 j - 2\lambda_4 v - 2\lambda_5 v + 2\lambda_6 vj + 2\lambda_7 vk)\alpha\beta \\
  - \sum_t (2\lambda_{2t} k - 2\lambda_{3t} j - 2\lambda_{4t} v - 2\lambda_{5t} v + 2\lambda_{6t} vj + 2\lambda_{7t} vk)\alpha_t,
\]
\[
0 = k(iw) = (4\lambda_3 i + 4\lambda_4 v - 4\lambda_5 v + 4\lambda_6 v - 4\lambda_7 v)\alpha\beta \\
  - \sum_t (4\lambda_{3t} i + 4\lambda_{4t} v - 4\lambda_{5t} v + 4\lambda_{6t} v - 4\lambda_{7t} v)\alpha_t,
\]
\[
0 = i(k(iw)) = (16\lambda_5 \gamma i + 16\lambda_6 \gamma j - 16\lambda_7 \gamma k)\alpha\beta \\
  - \sum_t (16\lambda_{5t} \gamma i + 16\lambda_{6t} \gamma j - 16\lambda_{7t} \gamma k)\alpha_t,
\]
\[
0 = (vk)(i(k(iw)))) = (32\lambda_6 \gamma k + 32\lambda_7 \gamma j)\alpha\beta \\
  - \sum_t (32\lambda_{6t} \gamma k + 32\lambda_{7t} \gamma j)\alpha_t,
\]
\[
0 = k(i((vk)(i(k(iw)))))) = (-64\lambda_7 \gamma i)\alpha\beta - \sum_t (-64\lambda_{7t} \gamma i)\alpha_t.
\]
As \( \lambda_7, \gamma \neq 0 \), we have
\[
i(\alpha\beta - \hat{\alpha}) = 0
\]
where \( \hat{\alpha} = \sum_t (\lambda_t^{-1}\lambda_{7t})\alpha_t \in \mathcal{U} \). Hence, it is easy to see \( x(\alpha\beta - \hat{\alpha}) = 0 \) for any base elements \( x \in \{ i, j, k, v, vi, vj, vk \} \); hence
\[
\mathbb{M}_7(\alpha\beta - \hat{\alpha}) = 0
\]
and \( \alpha\beta - \hat{\alpha} = 0 \). Thus \( \alpha\beta = \hat{\alpha} \in \mathcal{U} \); \( \mathcal{U} \mathcal{U} \subseteq \mathcal{U} \).

Also, following the same ideas of the case \( \mathcal{M}_7(F) \), we can prove that \([\mathcal{U}, \mathcal{U}] = 0\); thus the algebra \( \mathcal{U} \) is associative and commutative. Also \( (aa)(b\beta) = (ab)(\alpha\beta) \) for any \( \alpha, \beta \in \mathcal{U} \) and \( a, b \in \mathbb{M}_7 \).

Finally, we shall prove that \( \mathcal{U} \) is free over \( \mathbb{M}_7 \). We consider \( \bar{w} = i\alpha_1 + j\alpha_2 + k\alpha_3 + v\alpha_4 + (vi)\alpha_5 + (vj)\alpha_6 + (vk)\alpha_7 = 0 \), where \( \alpha_i \in \mathcal{U}, i = 1, 2, \ldots, 7 \). Then, by the multiplication table of \( \mathbb{M}_7 \)
0 = i\hat{w} = 2k\alpha_2 - 2j\alpha_3 - 2(vi)\alpha_4 + 2v\alpha_5 - 2(vk)\alpha_6 + 2(vj)\alpha_7,

0 = k(i\hat{w}) = 4i\alpha_3 + 4(vj)\alpha_4 - 4(vk)\alpha_5 - 4v\alpha_6 + 4(vi)\alpha_7,

0 = i(k(i\hat{w})) = -8(vk)\alpha_4 - 8(vj)\alpha_5 + 8(vi)\alpha_6 + 8v\alpha_7,

0 = (vk)(i(k(i\hat{w}))) = 16\gamma\alpha_5 + 16\gamma j\alpha_6 - 16\gamma k\alpha_7,

0 = i((vk)(i(k(i\hat{w})))) = 32\gamma k\alpha_6 + 32\gamma j\alpha_7,

0 = k(i((vk)(i(k(i\hat{w})))))) = -64\gamma i\alpha_7.

As \gamma \neq 0, of the last equality we have 0 = i\alpha_7. Hence, it is easy to see x\alpha_7 = 0 for any base elements \(x \in \{i, j, k, v, vi, vj, vk\}\); hence

\[M_7\alpha_7 = 0\]

and \(\alpha_7 = 0\). Similarly \(\alpha_i = 0\) for \(i = 1, 2, 3, \ldots, 6\). Therefore \(M \cong M_7 \otimes_F U\).

The theorem is proved.

Q.E.D.

Using Theorem 3.1 we can easily drop the assumption that \(mM \neq 0\) for any \(m \neq 0\) from \(M\).

**Theorem 3.2.** Let \(M\) be a Malcev algebra containing \(M\). Then \(M\) is isomorphic to a direct sum \((M \otimes_F A) \oplus N\), where \(A\) is a certain commutative associative algebra.

**Proof:** Consider \(M\) as a Malcev \(M\)-module. Then by the proof of Theorem 3.1 we have the first decomposition \(M = \sum \oplus M\alpha_i\), where \(\alpha_i\) belong to \(\Gamma(M)\) (the centroid of \(M\)). From this decomposition it follows that \(M = MU\), where \(U = \sum_i F\alpha_i\) denote the span of all \(\alpha_i\).

On the other hand, from Corollary 2.3 \(M = N_M \oplus J_M\), where the submodules \(N_M\) and \(J_M\) are given by \(N_M = \{x \in M : J(x, M, M) = 0\}\) and \(J_M = J(M, M, M)\). Moreover, by (ii) of Corollary 2.4 \(N_M M = 0\) and \(J_M\) is a direct sum of the copies of the adjoint module of \(M\), that is, \(J_M = \sum_j \oplus \text{Reg}_j M\), where \(\text{Reg}_j M \cong \text{Reg}_M\) for all \(j\). So \(J_M = \sum_j \oplus M\alpha_j\), where \(\alpha_j \in \Gamma(M)\) and they are some of the elements that appear in the first decomposition of \(M\). Denote \(A = \sum_j F\alpha_j\) the span of all \(\alpha_j\). Then \(J_M = MA\).

For arbitrary elements \(x \in N_M, a, b, c \in M, \alpha \in U\), the equality \(M = MU\) and using (3) we have

\[J(x(a\alpha), b, c) = J(xa, b, c)\alpha = (xJ(a, b, c) + J(x, b, c)a - 2J(x, a, bc))\alpha = 0;\]

so \(N_M\) is an ideal of \(M\). Now as \(A^2 \subseteq \Gamma(M)\) and \(M A^2 \subseteq N_M + J_M\) we obtain
\[ J^2_M = (M A)(M A) = M^2 A^2 = (M A^2)M \subseteq N_M M + J_M M \subseteq J_M, \]

and \( J^2_M \subseteq J_M \); so \( J_M \) is a subalgebra of \( M \). Then by Theorem 3.1 \( J_M \cong M \otimes_F A \), where \( A \) is a commutative associative algebra. Denote \( N := N_M \triangleleft M \). Thus

\[ M \cong (M \otimes_F A) \oplus N. \]

The theorem is proved.

Q.E.D.

### 3.2 Factorization Theorem for Malcev superalgebras

The objective of this subsection is to prove an analogy of the Kronecker Factorization Theorem for Malcev superalgebras whose even part contains the 7-dimensional exceptional Malcev algebra \( M \).

**Theorem 3.3.** Let \( M = M_0 \oplus M_1 \) be a Malcev superalgebra such that \( M_0 \) contains \( M \), with \( m M \neq 0 \) for any homogeneous element \( 0 \neq m \in M_0 \cup M_1 \). Then \( M \cong M \otimes_F U \) for a certain supercommutative associative superalgebra \( U \).

**Proof:** Consider \( M_0 \) as a Malcev \( M \)-module. Then, as in the Theorem 3.1, every irreducible non-Lie Malcev \( M \)-module is almost faithful and so the regular module \( \text{Reg} M \).

As \( M_0 \) is a Malcev module over \( M \), by Corollary 2.7, \( M_0 \) is completely reducible, that is, \( M_0 = \sum_i \oplus V_i \), where \( V_i \) is an irreducible almost faithful non-Lie Malcev \( M \)-module. Again by Proposition 2.9 we have \( 0 \neq \alpha_i(V_i, M, M) \) because \( M \) is a non-Lie Malcev algebra. Let us take elements \( v_i \in V_i \) and \( a_i, b_i \in M \) such that

\[ \alpha_i = \alpha_i(v_i, a_i, b_i) \neq 0 \]

since \( M \) is a non-Lie Malcev algebra. Therefore

\[ M_0 = \sum_i \oplus M \alpha_i \]

because for each \( i \) we have \( V_i \simeq M \alpha_i \). As in the Theorem 3.1 \( \alpha_i \in \tilde{\Gamma}_0(M) \).

Let \( U_0 = \sum_i F \alpha_i \) denote the span of all \( \alpha_i \). Then \( M_0 = M U_0 \).

Also we can consider \( M_1 \) as a Malcev module over \( M \), by Corollary 2.7, \( M_1 \) is completely reducible, that is, \( M_1 = \sum_j \oplus W_j \), where \( W_j \) is an irreducible almost faithful non-Lie Malcev \( M \)-module. Thus

\[ M_1 = \sum_j \oplus M \beta_j \]

because for each \( j \) we have \( W_j \simeq M \beta_j \), with

\[ \beta_j = \beta_j(w_j, a_j, b_j) \neq 0 \]
for some $w_j \in W_j$ and $a_j, b_j \in \mathcal{M}$ because by Proposition 2.9 we have $0 \neq \beta_j(W_j, \mathcal{M}, \mathcal{M})$ since $\mathcal{M}$ is a non-Lie Malcev algebra. Also $\beta_j \in \bar{\Gamma}_1(\mathcal{M})$.

We denote by $U_1 = \sum_j F\beta_j$ the span of all $\beta_j$. Then $\mathcal{M}_1 = \mathcal{M}U_1$.

We have

$$\mathcal{M} = \mathcal{M}U_0 \oplus \mathcal{M}U_1.$$  

Let’s go to prove that $U = U_0 \oplus U_1$ is a subsuperalgebra of $\bar{\Gamma}(\mathcal{M})$. For all $a, b \in \mathcal{M}$ we have

$$(a\alpha)(b\beta) = (-1)^{|a||b|+|\beta|}(a(b\beta))\alpha = (-1)^{|a||b|+|\beta|}((ab)\beta)\alpha = (-1)^{|a||b|+|\beta|}(ab)(\beta\alpha). \quad (12)$$

So by (12) and using the relationships

$$M_iM_j \subseteq M_{(i+j)\mod 2}, \quad i, j = 0, 1$$

of the $\mathbb{Z}_2$-grading of $\mathcal{M}$ and the proof of the Theorem 3.1 we get

$$U_iU_j \subseteq U_{(i+j)\mod 2}, \quad i, j = 0, 1.$$  

Hence $U = U_0 \oplus U_1$ is a subsuperalgebra.

We fix arbitrary homogeneous elements $\alpha, \beta \in U$, $m \in \mathcal{M}$ and $a \in \mathcal{M}$, then

$$((m\alpha)\beta)a = (-1)^{|a||\beta|}(m\alpha)(a\beta)$$

$$= (-1)^{|a||\beta|+|\alpha|+|\beta|}(m(a\alpha))\beta$$

$$= (-1)^{|a||\beta|}((m\beta)a)\alpha$$

$$= (-1)^{|a||\beta|}((m\beta)\alpha)a.$$  

If $[\alpha, \beta] = \alpha\beta - (-1)^{|a||\beta|}\beta\alpha$, we have

$$(\mathcal{M}[\alpha, \beta])\mathcal{M} = 0,$$

so $[\alpha, \beta]|_{\mathcal{M}} = 0$. In particular, $[\alpha, \beta]|_{V_i} = 0$ or $[\alpha, \beta]|_{W_j} = 0$ for any irreducible components $V_i$, $W_j$ of $\mathcal{M}$, then by Proposition 2.1(iii) $[\alpha, \beta] = 0$ because $\phi : Z \rightarrow U(\alpha \mapsto \alpha|_{V_i})$ or $\phi : Z \rightarrow U(\beta \mapsto \beta|_{W_j})$ are one-to-one. Therefore $[U, U] = 0$; hence $U$ is a supercommutative and associative superalgebra.

Also, by (12) and $[U, U] = 0$

$$(a\alpha)(b\beta) = (-1)^{|a||b|+|\beta|}(ab)(\beta\alpha) = (-1)^{|a||b|+|\beta|}(-1)^{|\alpha||\beta|}(ab)(\alpha\beta) = (-1)^{|a||b|}(ab)(\alpha\beta).$$

for all $a, b \in \mathcal{M}$.

As in the Theorem 3.1 using the multiplication table of $\mathcal{M}_7$ and $\mathcal{M}_7(F)$, it is easy to see that the superalgebra $U$ is free over $\mathcal{M}$. Therefore $\mathcal{M} \cong \mathcal{M} \otimes_{F} U$.

The theorem is proved.

Q.E.D.
Using Theorem 3.3 we can drop the assumption that $mM \neq 0$ for any homogeneous element $0 \neq m \in M_0 \cup M_1$. The proof of the Theorem 3.1 mimics the Theorem 3.2 proof of the corresponding result for Malcev algebras that contains the 7-dimensional exceptional Malcev algebra.

**Theorem 3.4.** Let $\mathcal{M} = M_0 \oplus M_1$ be a Malcev superalgebra such that $\mathcal{M}_0$ contains $M$. Then $\mathcal{M}$ is isomorphic to a direct sum $(\mathcal{M} \otimes_F A) \oplus N'$, where $A$ is a certain supercommutative associative superalgebra.

**Proof:** Consider $\mathcal{M}$ as a Malcev $\mathcal{M}$-module. Then by the proof of Theorem 3.3, we have the first decomposition $\mathcal{M} = (\sum_i \mathcal{M} \alpha_i) \oplus (\sum_j \mathcal{M} \beta_j)$, where $\alpha_i \in \tilde{\Gamma}_0(\mathcal{M})$ and $\beta_j \in \tilde{\Gamma}_1(\mathcal{M})$. From this decomposition it follows that $\mathcal{M} = \mathcal{M}U$, where $U = U_0 \oplus U_1 = \sum_i F \alpha_i + \sum_j F \beta_j$ is a subsuperalgebra of $\tilde{\Gamma}(\mathcal{M})$ (the supercentroid of $\mathcal{M}$) spanned by all $\alpha_i$ and $\beta_j$.

Also from Corollary 2.4 we have the decomposition $\mathcal{M} = N_\mathcal{M} \oplus J_\mathcal{M}$, where the submodules $N_\mathcal{M}$ and $J_\mathcal{M}$ are given by $N_\mathcal{M} = \{ x \in \mathcal{M} : J(x, \mathcal{M}, \mathcal{M}) = 0 \}$ and $J_\mathcal{M} = J(\mathcal{M}, \mathcal{M}, \mathcal{M})$. Moreover, by (ii) of Corollary 2.4, $N_\mathcal{M} \mathcal{M} = 0$ and $J_\mathcal{M}$ is a direct sum of the copies of the adjoint module of $\mathcal{M}$, that is,$J_\mathcal{M} = (\sum_r \mathcal{M} \alpha_r) \oplus (\sum_s \mathcal{M} \beta_s)$, where $\mathcal{M} \alpha_r \cong \mathcal{M} \beta_s \cong \mathcal{M}$ for all $r, s$. So $J_\mathcal{M} = (\sum_r \mathcal{M} \alpha_r) \oplus (\sum_s \mathcal{M} \beta_s)$, where $\alpha_r \in \tilde{\Gamma}_0(\mathcal{M})$ and $\beta_s \in \tilde{\Gamma}_1(\mathcal{M})$ and they are some of the elements that appear in the first decomposition of $\mathcal{M}$. Denote $A = \sum_r F \alpha_r + \sum_s F \beta_s$ the span of all $\alpha_r$ and $\beta_s$. Then $J_\mathcal{M} = \mathcal{M}A$.

For arbitrary elements $x \in N_\mathcal{M}, a, b, c \in \mathcal{M}, \alpha \in U_0 \cup U_1$, the equality $\mathcal{M} = \mathcal{M}U$ and the superlinearization of (3) imply
\[
J(x(aa), b, c) = \pm J(xa, b, c)\alpha = \pm (\pm xJ(a, b, c) \pm J(x, b, c)a \pm 2J(x, a, bc))\alpha = 0;
\]
so $N_\mathcal{M}$ is a superideal of $\mathcal{M}$. Also as $A^2 \subseteq \tilde{\Gamma}(\mathcal{M})$ and $\mathcal{M}A^2 \subseteq N_\mathcal{M} + J_\mathcal{M}$ we obtain
\[
J_\mathcal{M}^2 = (\mathcal{M}A)(\mathcal{M}A) = \mathcal{M}^2A^2 = (\mathcal{M}A^2)\mathcal{M} \subseteq N_\mathcal{M}\mathcal{M} + J_\mathcal{M}\mathcal{M} \subseteq J_\mathcal{M},
\]
and $J_\mathcal{M}^2 \subseteq J_\mathcal{M}$, then $J_\mathcal{M}$ is a subsuperalgebra of $\mathcal{M}$ and by Theorem 3.3, $J_\mathcal{M} \cong \mathcal{M} \otimes_F A$, where $A$ is a supercommutative associative superalgebra. Denote $N' := N_\mathcal{M} \triangleleft \mathcal{M}$. Thus
\[
\mathcal{M} \cong (\mathcal{M} \otimes_F A) \oplus N'.
\]

The theorem is proved.

**Q.E.D.**

## 4 Some equivalence of categories

In this section, we provide some equivalence of categories. In the case of Malcev algebras, observe that when $\mathcal{M} = \mathcal{M}_7(F)$, the Theorem 3.1 states that the Malcev algebra $\mathcal{M}$ is coordinated by $U$, that is, $\mathcal{M} \cong \mathcal{M}_7(U)$. Thus the following result establishes that in this case $\mathcal{M}$ is Morita equivalent to $U$. 

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Let $\mathcal{U}$ be a unital associative commutative algebra, and let $C(\mathcal{U})$ be the Cayley-Dickson matrix algebra over $\mathcal{U}$. So introducing the commutator $[A, B] = AB - BA$ in $C(\mathcal{U})$ we obtain the Malcev algebra $C(\mathcal{U})^{(-)}$ and the non-Lie Malcev algebra $\mathcal{M}_7(\mathcal{U}) = C(\mathcal{U})^{(-)}/F \cdot e_0$, where $e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. It’s clear that $\mathcal{M}_7(\mathcal{U})$ contains the 7-dimensional exceptional Malcev algebra $\mathcal{M}_7(F)$.

Then consider the following categories:

- let $\text{Mod}_{\text{Malcev}} \mathcal{M}_7(\mathcal{U})$ denote the category of Malcev modules $V$ over $\mathcal{M}_7(\mathcal{U})$ such that $v\mathcal{M}_7(F) \neq 0$ for any $0 \neq v \in V$.
- let $\text{Mod}_{\text{Ass-Com}} \mathcal{U}$ denote the category of unital associative and commutative modules $W$ over $\mathcal{U}$ such that $w\mathcal{U} \neq 0$ for any $0 \neq w \in W$.

We define a functor $T$ from the category $\text{Mod}_{\text{Ass-Com}} \mathcal{U}$ into the category $\text{Mod}_{\text{Malcev}} \mathcal{M}_7(\mathcal{U})$.

Let $W \in \text{Obj}(\text{Mod}_{\text{Ass-Com}} \mathcal{U})$ and $E = \mathcal{U} \oplus W$ be the split null extension of $\mathcal{U}$ by $W$ such that $w\mathcal{U} \neq 0$ for any $0 \neq w \in W$, $E$ is a unital associative and commutative algebra. Consider the Cayley-Dickson matrix algebra $C(E)$ that contains $C(\mathcal{U})$ as a subalgebra. Also $C(E)$ contains the ideal $C(W)$ which is the set of matrices of $C(E)$ whose entries are in the ideal $W$ of $E$. So introducing the commutator in $C(E)$ we obtain the non-Lie Malcev algebra $\mathcal{M}_7(E) = C(E)^{(-)}/F \cdot e_0$.

That contains the non-Lie Malcev algebra $\mathcal{M}_7(\mathcal{U}) = C(\mathcal{U})^{(-)}/F \cdot e_0$. Moreover $\mathcal{M}_7(E)$ contains the ideal $V = \mathcal{M}_7(W)$. Then $V = \mathcal{M}_7(W)$ is a Malcev module over $\mathcal{M}_7(\mathcal{U})$ relative to the multiplication defined in $\mathcal{M}_7(E)$. Note that $v\mathcal{M}_7(F) \neq 0$ for any $0 \neq v \in V$ because $W$ is unital. We call to $V$ the $\mathcal{M}_7(\mathcal{U})$-module associated with the given module $W$ of $\mathcal{U}$ and denote $V = T(W)$.

As $E = \mathcal{U} \oplus W$ we have $\mathcal{M}_7(E) = \mathcal{M}_7(\mathcal{U}) \oplus V$. Also $W^2 = 0$ in $E$ which implies that $V^2 = 0$ in $\mathcal{M}_7(E)$, thus $\mathcal{M}_7(E)$ is the split null extension of $\mathcal{M}_7(\mathcal{U})$ by its module $V$.

We can easily verify that

$$T : \text{Mod}_{\text{Ass-Com}} \mathcal{U} \longrightarrow \text{Mod}_{\text{Malcev}} \mathcal{M}_7(\mathcal{U})$$

is really a functor. In addition each pair of objects $W$ and $W'$ of $\text{Mod}_{\text{Ass-Com}} \mathcal{U}$, the following equality is valid:

$$T(\text{Hom}(W, W')) = \text{Hom}(T(W), T(W')).$$

Thus $W$ and $W'$ are isomorphic if and only if $T(W)$ and $T(W')$ are too.

Similarly the functor $T$ gives a lattice isomorphism of the lattice of submodules of $W$ relative to $\mathcal{U}$ onto the lattice of submodules of $V = T(W)$ relative to $\mathcal{M}_7(\mathcal{U})$. 

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To complete our reduction of the theory of modules for \( \mathcal{M}_7(\mathcal{U}) \) to that of modules for \( \mathcal{U} \) we shall now show that every \( \mathcal{M}_7(\mathcal{U}) \)-module is isomorphic to a module associated with a module for \( \mathcal{U} \).

**Corollary 4.1.** Let \( V \) be a Malcev module for \( \mathcal{M}_7(\mathcal{U}) \) such that \( v\mathcal{M}_7(F) \neq 0 \) for any \( 0 \neq v \in V \). Then there exists a unital commutative and associative module \( W \) for \( \mathcal{U} \) such that \( V \) is isomorphic to \( T(W) \).

**Proof:** The fact that \( V \) is a Malcev module over \( \mathcal{M}_7(\mathcal{U}) \) such that \( v\mathcal{M}_7(F) \neq 0 \) for any \( 0 \neq v \in V \), implies that in particular \( v\mathcal{M}_7(\mathcal{U}) \neq 0 \) for any \( 0 \neq v \in V \). Let \( \mathcal{M} = \mathcal{M}_7(\mathcal{U}) \oplus V \) be the split null extension of \( \mathcal{M}_7(\mathcal{U}) \) by \( V \). So \( \mathcal{M} \) is a Malcev algebra containing the 7-dimensional simple non-Lie Malcev algebra \( \mathcal{M}_7(F) \) as a subalgebra such that \( m\mathcal{M}_7(F) \neq 0 \) for any \( 0 \neq m \in \mathcal{M} \), then by Theorem 3.3 there is a certain commutative and associative algebra \( D \) such that \( \mathcal{M} = \mathcal{M}_7(D) \). Let \( W \) be the set of the elements of \( D \) that appear in the matrix entries of \( V \). Thus \( V = \mathcal{M}_7(W) \) where \( W \triangleleft D \) and \( W^2 = 0 \) in \( D \), because \( V \triangleleft \mathcal{M} \) and \( V^2 = 0 \) in \( \mathcal{M} \); so \( D = \mathcal{U} \oplus W \). Then \( D \) is a split null extension of \( \mathcal{U} \) by its module \( W \), hence \( W \) is a commutative and associative module over \( \mathcal{U} \) such that \( w\mathcal{U} \neq 0 \) for any \( 0 \neq w \in W \). Therefore \( T(W) = V \).

Q.E.D.

Evidently, \( T(\text{Reg} \mathcal{U}) = \text{Reg}(\mathcal{M}_7(\mathcal{U})) \). Is straightforward to proof that \( T \) is faithful and full, and using Corollary 4.1 we have the desired equivalence of categories.

**Theorem 4.2.** The categories \( \text{Mod}_{\text{Ass-Com}} \mathcal{U} \) and \( \text{Mod}_{\text{Malcev}} \mathcal{M}_7(\mathcal{U}) \) are Morita equivalent.

Now in the case of Malcev superalgebras, the Theorem 3.3 implies that if \( \mathcal{M} = \mathcal{M}_7(F) \), then the Malcev superalgebra \( \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \) is coordinated by \( U = U_0 \oplus U_1 \), that is, \( \mathcal{M} \cong \mathcal{M}_7(U) \). Thus the following result establishes that in this case \( \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \) is Morita equivalent to \( U = U_0 \oplus U_1 \).

Let \( U \) be a unital associative supercommutative superalgebra, and let \( C(U) \) be the Cayley-Dickson matrix superalgebra over \( U \). So introducing the supercommutator \( [A, B] = AB - (-1)^{|A||B|}BA \) in \( C(U) \) we obtain the Malcev superalgebra \( C(U)^{(-)} \) and the non-Lie Malcev superalgebra \( \mathcal{M}_7(U) = C(U)^{(-)}/F \cdot e_0 \). Note that \( \mathcal{M}_7(U) \) contains the 7-dimensional simple non-Lie Malcev algebra \( \mathcal{M}_7(F) \).

Let’s consider the following categories:

- let \( \tilde{\text{Mod}}_{\text{Malcev}} \mathcal{M}_7(U) \) denote the category of Malcev modules \( V = V_0 \oplus V_1 \) over \( \mathcal{M}_7(U) \) such that \( v\mathcal{M}_7(F) \neq 0 \) for any homogeneous element \( 0 \neq v \in V_0 \cup V_1 \).
- let \( \tilde{\text{Mod}}_{\text{Ass-Com}} U \) denote the category of unital associative supercommutative supermodules \( W = W_0 \oplus W_1 \) over \( U \) such that \( wU \neq 0 \) for any homogeneous element \( 0 \neq w \in W_0 \cup W_1 \).
Thus analogous to the case of Malcev algebras we can define a functor

$$S : \tilde{\text{Mod}}_{\text{Ass-Com}} U \rightarrow \tilde{\text{Mod}}_{\text{Malcev}} \mathcal{M}_7(U).$$

We can get a result analogous to the Corollary and from there $S(\text{Reg } U) = \text{Reg}(\mathcal{M}_7(U))$. Also, is straightforward to proof that $S$ is faithful and full. Then we have equivalence of categories.

**Theorem 4.3.** The categories $\tilde{\text{Mod}}_{\text{Ass-Com}} U$ and $\tilde{\text{Mod}}_{\text{Malcev}} \mathcal{M}_7(U)$ are Morita equivalent.

5 (Super)algebras and Modules with (super)involution

Consider $\mathcal{A}$ an arbitrary algebra. Recall that a linear mapping $\ast : \mathcal{A} \rightarrow \mathcal{A}$ is called an *involution* of an algebra $\mathcal{A}$, if it satisfies the conditions

$$(a^\ast)^\ast = a, \ (ab)^\ast = b^\ast a^\ast$$

for any elements $a, b \in \mathcal{A}$.

If $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is an arbitrary superalgebra. A linear even mapping $\ast : \mathcal{A} \rightarrow \mathcal{A}$ is called a *superinvolution* of a superalgebra $\mathcal{A}$, if it satisfies the conditions

$$(a^\ast)^\ast = a, \ (ab)^\ast = (-1)^{|a||b|} b^\ast a^\ast$$

for any homogeneous elements $a, b \in \mathcal{A}_0 \cup \mathcal{A}_1$.

Now, let $V$ be a module over an algebra $(\mathcal{A}, \ast)$ with involution. We will call $V$ an *$\mathcal{A}$-module with involution*, if there exists a linear mapping $- : V \rightarrow V$ such that the mapping

$$v + a \mapsto \overline{v} + a^\ast$$

is an involution of the split null extension algebra $E = V + \mathcal{A}$. Evidently, for an algebra with involution $\mathcal{A}$, the bimodules $\text{Reg } \mathcal{A}$ and $(\text{Reg } \mathcal{A})^{op}$ have the involutions induced by that of $\mathcal{A}$.

Let $\mathcal{O}$ be the Cayley-Dickson algebra over $F$. Then we know that $\mathcal{O} = F \oplus \mathcal{M}$, where $\mathcal{M} = \{x \in \mathcal{O} : t(x) = 0\}$ and multiplication in $\mathcal{O}$, for elements $a, b \in \mathcal{M}$ is defined by the following relation:

$$a \cdot b = -(a, b) + a \times b$$

where $(\cdot)$ is a non-singular symmetric bilinear form on $\mathcal{M}$ and $\times$ is anticommutative multiplication on $\mathcal{M}$. As $t(x) = x + \overline{x}$, then

$$\overline{a} = -a$$
for all \( a \in M \). So \( \overline{a} = a \) and \( ab = -ab \Rightarrow ba = (-b)(-a) = b\overline{a} \); hence \( a \mapsto \overline{a} = -a \) is an involution in \( \overline{M} \).

So in this section we assume that in every Malcev (super)algebra that contains \( M \) is defined a non-singular symmetric bilinear form and an (super)involution that extend that of \( \overline{M} \).

Let \( V \) be a Malcev module for \( M \) and let \( E = V + \overline{M} \) be the split null extension. So \( E \) is a Malcev algebra that contains \( M \). Let

\[
 f : E \times E \longrightarrow F
\]

denote the non-singular symmetric bilinear form defined in \( E \). Then consider the associative algebra with involution \( \mathcal{L}(E) \) where

\[
 \mathcal{L}(E) = \{ \alpha \in \text{End}_F(E) : \exists \alpha^* \in \text{End}_F(E) \text{ such that } f(x\alpha, y) = f(x, y\alpha^*) \forall x, y \in E \}. 
\]

Note that \( \alpha^* \) is unique by the non-singularity of \( f \), which gives the involution in \( \mathcal{L}(E) \).

Now we will study the structure of modules with involution over \( M \). Our objective is to prove that every module with involution over this algebra is completely reducible and every irreducible module with involution is of the type \( \text{Reg}_M \). In fact, we will consider modules with involution that satisfy the additional condition of so-called \( J \)-admissibility (see [6]). A module with involution \( (V, -) \) over an algebra with involution \( (M, \ast) \) is called \( J \)-admissible if all the symmetric elements of the algebra with involution \( E = V + M \) lie in the associative center (the nucleus) of \( E \).

**Theorem 5.1.** Let \( V \) be a \( J \)-admissible Malcev module over \( \overline{M} \) and let \( E = V \oplus \overline{M} \) be the corresponding split extension. If \( f : E \times E \longrightarrow F \) is a non-singular symmetric bilinear form defined in \( E \). Then \( V \) is completely reducible and is a direct sum of irreducible modules with involution isomorphic to \( \text{Reg}_M \), that is, isomorphic to \( M\alpha_i \) for a certain symmetric element \( \alpha \) of the centroid \( \Gamma(E) \) with involution.

**Proof:** Let \( V \) be a module under consideration, with a involution \( v \mapsto \overline{v} \). By Corollary [2.7] \( V \) is completely reducible, that is, \( V = \sum_i \oplus V_i \), where \( V_i \) is an irreducible almost faithful non-Lie Malcev \( \overline{M} \)-module. Also, by Theorem [2.8] every irreducible almost faithful non-Lie Malcev \( \overline{M} \)-module is the regular module \( \text{Reg}\overline{M} \). So we have \( V_i = M\alpha_i \) as modules, where \( \alpha_i : \overline{M} \longrightarrow V_i \) is the isomorphism of modules that was proved in the Theorem [2.8] given by \( \alpha_i(z) = z\alpha_i(v_i, a_i, b_i) \) for some \( v_i \in V_i \) and \( a_i, b_i \in \overline{M} \) with \( \alpha_i(v_i, a_i, b_i) \neq 0 \). The condition \( 0 \neq \alpha_i(v_i, a_i, b_i) \) is true since by Proposition [2.9]

\[
 0 \neq \alpha_i(V_i, \overline{M}, \overline{M}) 
\]

because \( \overline{M} \) is a non-Lie Malcev algebra. So \( V = \sum_i \oplus M\alpha_i \).

Denote by \( U = \sum_i F\alpha_i \) the span of all elements \( \alpha_i \).
By Theorem 2.6, $E$ satisfies $h = 0$, then by Lemma 2.2 for all $x, y \in E$
\[ p(x, v_i, a_i, b_i) y = p(xy, v_i, a_i, b_i), \]
$(x\alpha_i)y = (xy)\alpha_i$; thus $\alpha_i \in \Gamma(E)$, where $\Gamma(E)$ is the centroid of $E$.

We can associate the involution $x \mapsto \overline{x}$ of $E$ with the involution of $\mathcal{L}(E)$ by
\[ \overline{x\alpha} = \overline{x} \alpha^* \]  
for all $x \in E$ and $\alpha \in \mathcal{L}(E)$.

Fix arbitrary $a, b \in \mathbb{M}$, $v \in V$ and $\alpha \in \Gamma(E)$. Then there exist unique $\alpha^* \in \mathcal{L}(E)$ and so by (13)
\begin{align*}
(ab)\alpha^* &= (\overline{b} \overline{a}) \alpha^* = (\overline{ba}) \alpha^* = (\overline{ba}) \alpha = \overline{\alpha(\alpha a)} = a(b \alpha^*), \\
(ab)\alpha^* &= (\overline{a} \overline{b}) \alpha^* = (\overline{ba}) \alpha^* = (\overline{ba}) \alpha = \overline{b(\alpha a)} = \overline{b(\alpha b)} = a(\alpha^* b), \\
(va)\alpha^* &= -(\overline{a} \overline{v}) \alpha^* = -(\overline{av}) \alpha = -(\overline{av}) \overline{\alpha} = -\overline{(av)} \overline{\alpha} = -(\overline{a}) \overline{v} = -\overline{(av)} \overline{\alpha} = v(\alpha^*) \alpha \\
\end{align*}
and
\begin{align*}
(va)\alpha^* &= -(\overline{a} \overline{v}) \alpha^* = -(\overline{av}) \alpha = -\overline{(av)} \overline{\alpha} = -(\overline{a}) \overline{v} = -(\overline{a}) \overline{v} \overline{\alpha} = (va) \alpha; \end{align*}
thus $\Gamma(E)^*_E \subseteq \Gamma(E)$ and $\Gamma(E)^* = \Gamma(E)$. So $\Gamma(E)$ is a subalgebra with involution of $\mathcal{L}(E)$; hence
\[ \Gamma(E) = \text{Sym} \Gamma(E) \oplus \text{Skew} \Gamma(E). \]
Assume that there exists $0 \neq \alpha \in \Gamma(E)$ such that $\alpha^* = -\alpha$. Then, if $\mathbb{M} = \mathcal{M}_7(F)$ we denote $a = e_1 \alpha$, then $\overline{a} = a$; so $a$ is a symmetric element; hence, by $J$-admissibility of $V$ and using the multiplication table of $\mathcal{M}_7(F)$
\[ 0 = (e_2, e_1 \alpha, e_3) = (e_2(e_1 \alpha))e_3 - e_2((e_1 \alpha)e_3) = -8e_7 \alpha \]
which implies $e_i \alpha = 0$ for all $i = 1, \ldots, 7$. Hence $\mathcal{M}_7(F)\alpha = 0$ and $\alpha = 0$, a contradiction. Similarly in the case $\mathbb{M} = \mathcal{M}_7$, the element $a = i \alpha$ is symmetric, then by $J$-admissibility of $V$ and using the multiplication table of $\mathcal{M}_7$, we have $0 = (j, i \alpha, k) = 4(\nu k)\alpha$; so $\mathbb{M}_7\alpha = 0$. Hence $\alpha = 0$, a contradiction. Thus $\Gamma(E) = \text{Sym} \Gamma(E)$.

Now, if $V$ is irreducible then, let us take elements $v \in V$ and $a, b \in \mathbb{M}$ with $\alpha = \alpha(v, a, b) \neq 0$, where $\alpha \in \mathcal{U}$ and $V = \mathbb{M} \alpha$, which is isomorphic to $\text{Reg} \mathbb{M}$, under the isomorphism $\alpha : z \mapsto z \alpha(v, a, b)$.

In the general case, it suffices to notice that every $\alpha_i$ generates an irreducible module which is invariant under the involution and is isomorphic to $\text{Reg} \mathbb{M}$.

The theorem is proved.

Q.E.D.
The following results affirm that the commutative associative algebra \( \mathcal{U} \) of the theorem 3.1 and the supercommutative associative superalgebra \( U \) of the theorem 3.3 are generated by symmetric elements.

Let \( \mathcal{M} \) be an arbitrary Malcev algebra and let \( f : \mathcal{M} \times \mathcal{M} \rightarrow F \) denote the non-singular symmetric bilinear form defined in \( \mathcal{M} \). As in the previous paragraphs, consider the associative algebra with involution \( \mathcal{L}(\mathcal{M}) \) with

\[ \mathcal{L}(\mathcal{M}) = \{ \alpha \in \text{End}_F(\mathcal{M}) : \exists \alpha^* \in \text{End}_F(\mathcal{M}) \text{ such that } f(x\alpha, y) = f(x, y\alpha^*) \forall x, y \in \mathcal{M} \}. \]

Also \( \alpha^* \) is unique by the non-singularity of \( f \), which gives the involution in \( \mathcal{L}(\mathcal{M}) \).

**Corollary 5.2.** Let \( \mathcal{M} \) be a Malcev algebra with \( J \)-admissible involution (that is, every symmetric element lies in the nucleus of \( \mathcal{M} \)) such that \( \mathcal{M} \) contains \( \mathbb{M} \) as a subalgebra with \( m\mathbb{M} \neq 0 \) for any \( m \neq 0 \) from \( \mathcal{M} \). If \( f : \mathcal{M} \times \mathcal{M} \rightarrow F \) is a non-singular symmetric bilinear form defined in \( \mathcal{M} \). Then \( \mathcal{M} \cong \mathbb{M} \otimes_F \mathcal{U} \) for a certain commutative associative algebra \( \mathcal{U} \) of symmetric elements.

**Proof:** By Theorem 3.1 we have \( \mathcal{M} \cong \mathbb{M} \otimes_F \mathcal{U} \), where \( \mathcal{U} \) is a certain commutative subalgebra of \( \Gamma(\mathcal{M}) \). Using the same arguments from Theorem 5.1, we obtain that \( \Gamma(\mathcal{M}) \) is a subalgebra with involution of \( (\mathcal{L}(\mathcal{M}), \ast) \) and is formed only by symmetric elements. Thus \( \mathcal{U} \) is a subalgebra with involution of \( \Gamma(\mathcal{M}) \) whose elements are symmetric.

The corollary is proved.

Q.E.D.

Now let \( \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \) be an arbitrary Malcev superalgebra and let \( h : \mathcal{M} \times \mathcal{M} \rightarrow F \) denote the non-singular symmetric bilinear form defined in \( \mathcal{M} \). Consider the associative superalgebra with superinvolution \( \mathcal{L}(\mathcal{M}) \) with

\[ \mathcal{L}(\mathcal{M})_i = \{ \alpha \in \text{End}_F(\mathcal{M})_i : \exists \alpha^* \in \text{End}_F(\mathcal{M}) \text{ such that } h(x\alpha, y) = (-1)^{|\alpha||y|}h(x, y\alpha^*) \forall x, y \in \mathcal{M}_0 \cup \mathcal{M}_1 \}. \]

We observe that \( \alpha^* \) is unique by the non-singularity of \( h \), which gives the superinvolution in \( \mathcal{L}(\mathcal{M}) \).

The following result is a generalization of the Corollary 5.2 for superalgebras.

**Corollary 5.3.** Let \( \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \) be a Malcev superalgebra with \( J \)-admissible superinvolution (that is, every symmetric element lies in the nucleus of \( \mathcal{M} \)) such that \( \mathcal{M}_0 \) contains \( \mathbb{M} \), with \( m\mathbb{M} \neq 0 \) for any \( m \neq 0 \) from \( \mathcal{M}_0 \cup \mathcal{M}_1 \). If \( h : \mathcal{M} \times \mathcal{M} \rightarrow F \) is a non-singular symmetric bilinear form defined in \( \mathcal{M} \). Then \( \mathcal{M} \cong \mathbb{M} \otimes_F \mathcal{U} \) for a certain supercommutative and associative superalgebra \( \mathcal{U} \) of symmetric elements.
**Proof:** By Theorem 3.3 we have $\mathcal{M} \cong M \otimes_F U$, where $U = U_0 + U_1$ is a certain supercommutative subsuperalgebra of $\tilde{\Gamma}(\mathcal{M})$.

Consider the superalgebra $\mathcal{L}(\mathcal{M})$ with superinvolution $\ast$. We can associate the superinvolution $a \mapsto \bar{a}$ of $\mathcal{M}$ with the superinvolution of $\mathcal{L}(\mathcal{M})$ by

$$\bar{a} = \bar{a}^\ast$$

for all $a \in \mathcal{M}$ and $\alpha \in \mathcal{L}(\mathcal{M})$.

Remember that as $\tilde{\Gamma}(\mathcal{M})$ is a subsuperalgebra of $\text{End}_F(\mathcal{M})$ we have $\mathcal{M}_j\mathcal{M}_j \subseteq \mathcal{M}_{i+j}$ for any homogeneous elements $\alpha_j$ of $\tilde{\Gamma}(\mathcal{M})$. Fix arbitrary $x, y \in \mathcal{M}$ and $\alpha \in \tilde{\Gamma}(\mathcal{M})$. Then there exist unique $\alpha^\ast \in \mathcal{L}(\mathcal{M})$ such that

$$(xy)\alpha^\ast = (xy)\bar{\alpha} = (-1)^{|x||y|}(\overline{yx})\alpha$$

$$= (-1)^{|x||y|}(\overline{yx})\alpha = (-1)^{|x||y|}(-1)^{|\alpha|}(\overline{yx})\alpha$$

$$= (-1)^{|x||y|}(-1)^{|\alpha|}(\overline{yx})\alpha$$

and

$$= x(\bar{y}\alpha^\ast)$$

Then $(xy)\alpha^\ast = x(\bar{y}\alpha^\ast) = (-1)^{|\alpha||y|}(x\alpha^\ast)y$; thus $\tilde{\Gamma}(\mathcal{M})^* \subseteq \tilde{\Gamma}(\mathcal{M})$ and $\tilde{\Gamma}(\mathcal{M})^* = \tilde{\Gamma}(\mathcal{M})$. So $\tilde{\Gamma}(\mathcal{M})$ is a subsuperalgebra with superinvolution of $\mathcal{L}(\mathcal{M})$ and hence it is clear that

$$\tilde{\Gamma}(\mathcal{M}) = \text{Sym}\tilde{\Gamma}(\mathcal{M}) \oplus \text{Skew}\tilde{\Gamma}(\mathcal{M})$$

Assume that there exists $0 \neq \alpha \in \tilde{\Gamma}(\mathcal{M})$ such that $\alpha^\ast = -\alpha$. So as in the Theorem 5.1 if $\tilde{M} = \tilde{M}_7(F)$ we denote $a = e_2\alpha$ and $\overline{a} = a$; then $a$ is a symmetric element and by $J$-admissibility of $\mathcal{M}$ and using the mutiplication table of $\mathcal{M}_7(F)$ we have $0 = (e_4,e_2\alpha,e_3) = (e_4(e_2\alpha))e_3 - e_4((e_2\alpha)e_3) = -4e_1\alpha$ which implies $e_i\alpha = 0$ for all $i = 1, \ldots, 7$. Hence $\mathcal{M}_7(F)\alpha = 0$ and $\alpha = 0$, a contradiction. Similarly in the case $\tilde{M} = \tilde{M}_7$ we get a contradiction. Therefore $\tilde{\Gamma}(\mathcal{M}) = \text{Sym}\tilde{\Gamma}(\mathcal{M})$.

It is easy to see that $U$ is invariant under the superinvolution of $\tilde{\Gamma}(\mathcal{M})$; thus $U$ is generated by symmetric elements.

The corollary is proved.

Q.E.D.
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7 Declarations

Conflict of Interests The author declares that they have no conflict of interest.

Availability of data and materials The author declares that data supporting the findings of this study are available within the article.

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