ELEMENTARY EVALUATION OF $\int_0^\infty \frac{\sin^p t}{t^q} dt$

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Abstract. Let $p$ and $q$ be two positive integers. The goal of this note is to demonstrate, in a very simple and elementary way and without using advanced tools, a formula that expresses the value of the integral $\int_0^\infty \frac{\sin^p t}{t^q} dt$ when it converges.

1. Introduction

For positive integers $p$ and $q$ we consider the family of integrals

$I(p, q) = \int_0^\infty \frac{\sin^p t}{t^q} dt$

It seems that these integrals were first considered by N. I. Lobachevskii [4]. An explicit evaluation of $I(p, q)$ when $q - p$ is even was given by T. Hayashi [7]. The formula given in [7] shows that in this case $I(p, q)$ is a rational multiple of $\pi$ and this was precisely the object of Problem 11423 proposed to the American Mathematical Monthly [5]. A detailed evaluation of these integrals can be found in the literature, for example an evaluation using distribution theory and Fourier transforms can be found in [3]. Other methods using contour integration can also be applied to derive these formulas.

The aim of this note is to present elementary proofs for the formulas for $I(p, q)$ when the integral converges.

2. The main results

Our basic tool in the proof is the sequence of polynomials $(P_n)_{n \geq 0}$ defined as follows:

$$P_{2n}(X) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} X^{2k}$$

$$P_{2n+1}(X) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} X^{2k+1}$$

and the associated sequence of functions $(f_n)_{n \geq 1}$ given by:

$$\forall n \geq 0, \quad f_{2n+1} : \mathbb{R} \rightarrow \mathbb{R}, f_{2n+1}(t) = (-1)^n (\sin t - P_{2n-1}(t))$$

$$f_{2n+2} : \mathbb{R} \rightarrow \mathbb{R}, f_{2n+2}(t) = (-1)^{n+1} (\cos t - P_{2n}(t))$$

with the convention $P_{-1} = 0$.

The evaluation of the integral $I(p, q)$ is based upon the next proposition which summarizes some properties of the functions $(f_n)_{n \geq 1}$:

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Proposition 1. The sequence of functions \((f_n)_{n \geq 1}\) satisfies the following properties:

i. For every \(n \in \mathbb{N}^*\) we have \(f'_{n+1} = f_n\).

ii. For every \(n \in \mathbb{N}^*\) we have \(\lim_{t \to 0} \frac{f_n(t)}{t^n} = \frac{1}{n!}\).

iii. For every \(n \in \mathbb{N}^*\) the improper integral \(J_n = \int_0^\infty f_n(t) \, dt\) converges, and
\[
J_n = \frac{1}{(n-1)!} J_1 = \frac{1}{(n-1)!} \frac{\pi}{2}
\]

iv. For every \(n \geq 2\) the improper integral \(K_n = \int_1^\infty \frac{f_{n-1}(t)}{t^n} \, dt\) converges, and for all \(\lambda > 0\) we have:
\[
\lim_{X \to \infty} \int_X^{\lambda X} \frac{f_{n+1}(t)}{t^n} \, dt = \frac{\ln \lambda}{(n-1)!} \quad \text{and} \quad \lim_{X \to \infty} \int_1^X \frac{f_{n+1}(t)}{t^n} \, dt = +\infty
\]

v. For every \((m, q) \in \mathbb{N}^2\) such that \(1 \leq q \leq m\) we have
\[
\forall t \in \mathbb{R}, \quad (\sin t)^{2m} = \frac{1}{2^{2m-1}} \sum_{k=1}^m \left( \frac{2m}{m-k} \right) (-1)^{k+q} f_{2q}(2kt)
\]

vi. For every \((m, q) \in \mathbb{N}^2\) such that \(0 \leq q \leq m\) we have
\[
\forall t \in \mathbb{R}, \quad (\sin t)^{2m+1} = \frac{1}{2^{2m}} \sum_{k=0}^m \left( \frac{2m+1}{m-k} \right) (-1)^{k+q} f_{2q+1}(2k+1)t
\]

Proof. i. The verification of the first property is straightforward.

ii. The second property follows from the power series expansions:
\[
\cos t = P_{2n}(t) + \sum_{k=n+1}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k} \quad \text{and} \quad \sin t = P_{2n-1}(t) + \sum_{k=n}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1}
\]

which are valid for all \(n \in \mathbb{N}\) and all \(t \in \mathbb{R}\).

iii. It is well-known that the integral \(J_1 = \int_0^\infty \frac{\sin t}{t} \, dt\) converges and that \(J_1 = \frac{\pi}{2}\). In the case \(n \geq 2\) we have \(\frac{|f_n(t)|}{t^n} = O \left( \frac{1}{t^2} \right) \) and this, with ii, proves the convergence of the integral \(J_n\). Now, for \(n \geq 2\) and \(X > 0\) we have
\[
\int_0^X \frac{f_n(t)}{t^n} \, dt = \left[ -\frac{f_n(t)}{(n-1) t^{n-1}} \right]_0^X + \frac{1}{n-1} \int_0^X \frac{f'_n(t)}{t^{n-1}} \, dt
\]
\[
= \frac{1}{n-1} \int_0^X \frac{f_{n-1}(t)}{t^{n-1}} \, dt + O \left( \frac{1}{X} \right)
\]

Letting \(X\) tend to infinity we find that \(J_n = \frac{1}{n-1} J_{n-1}\) and this proves iii by induction.

iv. Considering two cases according to the parity of \(n\) we see immediately that:
\[
\forall t > 0, \quad \frac{f_{n+1}(t)}{t^{n+1}} = \frac{1}{(n-1)!} \frac{1}{t} + \frac{f_{n-1}(t)}{t^n}
\]

But, for \(n \geq 2\), we have \(f_{n-1}(t) = O \left( t^{\max(n-3,0)} \right) \) in the neighborhood of \(+\infty\), so \(\frac{|f_{n-1}(t)|}{t^n} = O \left( \frac{1}{t^2} \right)\), and this proves the convergence of \(K_n = \int_1^\infty \frac{f_{n-1}(t)}{t^n} \, dt\). Now, using \(\text{iii}\) we conclude that
\[
\lim_{X \to \infty} \int_{1}^{X} \frac{f_{n+1}(t)}{t^n} \; dt = +\infty \quad \text{and that}
\]
\[
\int_{X}^{\lambda X} \frac{f_{n+1}(t)}{t^n} = \frac{\ln \lambda}{(n-1)!} - \int_{X}^{\lambda X} \frac{f_{n-1}(t)}{t^n}
\]

The convergence of \( K_n \) proves that
\[
\lim_{X \to \infty} \int_{X}^{\lambda X} \frac{f_{n-1}(t)}{t^n} = 0
\]
so that
\[
\lim_{X \to \infty} \int_{X}^{\lambda X} \frac{f_{n+1}(t)}{t^n} = \frac{\ln \lambda}{(n-1)!}
\]

v. Let us start with a well-known and standard calculation. From Euler’s formula: \( \sin t = \frac{e^{it} - e^{-it}}{2i} \) and using the binomial theorem we can write the following:

\[
(sin t)^{2m} = \frac{(-1)^m}{2^{2m}} \left( \sum_{k=0}^{m} \binom{2m}{k} (-1)^k e^{ikt} e^{-(2m-k)t} \right) = \frac{(-1)^m}{2^{2m}} \left( \sum_{k=0}^{2m} \binom{2m}{k} (-1)^k e^{2i(k-m)t} \right)
\]

\[
= \frac{(-1)^m}{2^{2m}} \left( \sum_{k=0}^{m-1} \binom{2m}{k} (-1)^k e^{2i(k-m)t} + \binom{2m}{m} + \sum_{k=m+1}^{2m} \binom{2m}{k} (-1)^k e^{2i(k-m)t} \right)
\]

\[
= \frac{(-1)^m}{2^{2m}} \left( \sum_{k=0}^{m-1} \binom{2m}{k} (-1)^k e^{2i(k-m)t} + \binom{2m}{m} \frac{1}{2^{2m-1}} \sum_{k=1}^{m} \binom{2m}{m-k} (-1)^k \cos(2kt) \right)
\]

Using that \( \cos u = (-1)^q f_{2q}(u) + P_{2q-2}(u) \) for all \( u \in \mathbb{R} \), we obtain

\[
\forall t \in \mathbb{R}, \quad (sin t)^{2m} = Q_{m,q}^0(t) + \frac{1}{2^{2m-1}} \sum_{k=1}^{m} \binom{2m}{m-k} (-1)^{k+q} f_{2q}(2kt)
\]

with \( Q_{m,q}^0(t) = \frac{(2m)}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=1}^{m} \binom{2m}{m-k} (-1)^k P_{2q-2}(2kt) \).

Since, in a neighborhood of 0, we have, \( \forall k \in \{1, 2, \ldots, m\} \), \( f_{2q}(2kt) = O(t^{2q}) \) and \( (sin t)^{2m} = O(t^{2m}) = O(t^{2q}) \) (this results from the assumption \( q \leq m \)), we conclude that the polynomial \( Q_{m,q}^0(t) \) which is of degree at most \( 2q-2 \), satisfies: \( Q_{m,q}^0(t) = O(t^{2q}) \) in a neighborhood of 0. This proves that \( Q_{m,q}^0 = 0 \), and establishes \([1]\).
vi. In a similar way, starting from Euler’s formula and using the binomial theorem we can write:

\[
\sin^2 t^{2m+1} = \frac{(-1)^{m+1}}{2^{2m+1}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} (-1)^k e^{ikt} e^{-i(2m+1-k)t}
\]

\[
= \frac{(-1)^{m+1}}{2^{2m+1}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} (-1)^k e^{i(2(k-m)-1)t}
\]

\[
= \frac{(-1)^{m+1}}{2^{2m+1}} \sum_{k=0}^{m} \binom{2m+1}{k} (-1)^k e^{i(2(k-m)-1)t} + \sum_{k=m+1}^{2m+1} \binom{2m+1}{k} (-1)^k e^{i(2(k-m)-1)t}
\]

\[
= \frac{(-1)^{m+1}}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{k} (-1)^k \sin((2m-k)+1)t
\]

\[
= \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{m-k} (-1)^k \sin((2k+1)t)
\]

But \( \sin u = (-1)^q f_{2q+1}(u) + P_{2q-1}(u) \) for all \( u \in \mathbb{R} \), hence

\[
\forall t \in \mathbb{R}, \quad (\sin t)^{2m+1} = Q_{m,q}^1(t) + \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{m-k} (-1)^k f_{2q+1}((2k+1)t)
\]

with \( Q_{m,q}^1(t) = \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{m-k} (-1)^k P_{2q-1}((2k+1)t) \).

Since, in a neighborhood of 0, we have, \( \forall k \in \{1, \ldots, m\}, f_{2q+1}((2k+1)t) = O(t^{2q+1}) \) and \( (\sin t)^{2m+1} = O(t^{2m+1}) = O(t^{2q+1}) \) (this results from the assumption \( q \leq m \)), we conclude that the polynomial \( Q_{m,q}^1 \), which is of degree at most \( 2q-1 \), satisfies : \( Q_{m,q}^1(t) = O(t^{2q+1}) \) in a neighborhood of 0. This proves that \( Q_{m,q} = 0 \), and establishes (2). \( \square \)

**Theorem 2.** Consider \((q, m) \in \mathbb{N} \).

i. If \( 1 \leq q \leq m \) then

\[
\int_0^\infty \frac{\sin^{2m} t}{t^{2q+1}} dt = \frac{\pi}{2^{2m}} \sum_{k=1}^{m} (-1)^{k+q} \binom{2m}{m-k} (2k)^{2q-1} (2q-1)!.
\]

ii. If \( 0 \leq q \leq m \) then

\[
\int_0^\infty \frac{\sin^{2m+1} t}{t^{2q+1}} dt = \frac{\pi}{2^{2m+1}} \sum_{k=0}^{m} (-1)^{k+q} \binom{2m+1}{m-k} (2k+1)^{2q}. (2q)!.
\]

iii. If \( 2 \leq q \leq m \) then

\[
\int_0^\infty \frac{\sin^{2m} t}{t^{2q-1}} dt = \frac{1}{2^{2m-1}} \sum_{k=1}^{m} (-1)^{k+q} \binom{2m}{m-k} (2k)^{2q-2} (2q-2)! \ln k.
\]

iv. If \( 1 \leq q \leq m \) then

\[
\int_0^\infty \frac{\sin^{2m+1} t}{t^{2q-1}} dt = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{k+q} \binom{2m+1}{m-k} (2k+1)^{2q-1} (2q-1)! \ln(2k+1).
\]
Proof.  

i. Using formula (1) from Proposition 1 we conclude that
\[
\int_0^\infty \frac{\sin^{2m} t}{t^{2q}} \, dt = \frac{1}{2^{2m-1}} \sum_{k=1}^{m} (-1)^{k+q} \binom{2m}{m-k} \int_0^\infty \frac{f_{2q}(2kt)}{t^{2q}} \, dt
\]
\[
= \frac{1}{2^{2m-1}} \sum_{k=1}^{m} (-1)^{k+q} \binom{2m}{m-k} (2k)^{2q-1} \int_0^\infty \frac{f_{2q}(u)}{u^{2q}} \, du
\]
\[
= \frac{1}{2^{2m-1}} \sum_{k=1}^{m} (-1)^{k+q} \binom{2m}{m-k} (2k)^{2q-1} J_{2q}
\]

Now, using iii from Proposition 1 we conclude that
\[
\int_0^\infty \frac{\sin^{2m} t}{t^{2q}} \, dt = \frac{\pi}{2^{2m}} \sum_{k=1}^{m} (-1)^{k+q} \left( \frac{2m}{m-k} \right) \frac{(2k)^{2q-1}}{(2q-1)!}
\]

ii. Using formula (2) from Proposition 1 we conclude that
\[
\int_0^\infty \frac{\sin^{2m+1} t}{t^{2q+1}} \, dt = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{k+q} \binom{2m+1}{m-k} \int_0^\infty \frac{f_{2q+1}(2k+1)t}{t^{2q+1}} \, dt
\]
\[
= \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{k+q} \binom{2m+1}{m-k} (2k+1)^{2q} \int_0^\infty \frac{f_{2q+1}(u)}{u^{2q+1}} \, du
\]
\[
= \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{k+q} \binom{2m+1}{m-k} (2k+1)^{2q} J_{2q+1}
\]

Now, using iii from Proposition 1 we conclude that once more that
\[
\int_0^\infty \frac{\sin^{2m+1} t}{t^{2q+1}} \, dt = \frac{\pi}{2^{2m+1}} \sum_{k=0}^{m} (-1)^{k+q} \left( \frac{2m+1}{m-k} \right) \frac{(2k+1)^{2q}}{(2q)!}
\]

iii. Using formula (1) from Proposition 1 we conclude that, for \( X > 0 \) we have:
\[
\int_0^X \frac{\sin^{2m} t}{t^{2q-1}} \, dt = \frac{1}{2^{2m-1}} \sum_{k=1}^{m} (-1)^{k+q} \binom{2m}{m-k} \int_0^X \frac{f_{2q}(2kt)}{t^{2q-1}} \, dt
\]
\[
= \frac{1}{2^{2m-1}} \sum_{k=1}^{m} (-1)^{k+q} \binom{2m}{m-k} (2k)^{2q-2} \int_0^{2kX} \frac{f_{2q}(u)}{u^{2q-1}} \, du
\]
\[
= \frac{1}{2^{2m-1}} \sum_{k=1}^{m} (-1)^{k+q} \binom{2m}{m-k} (2k)^{2q-2} \int_0^{2kX} \frac{f_{2q}(u)}{u^{2q-1}} \, du + C_{m,q} \int_0^{2kX} \frac{f_{2q}(u)}{u^{2q-1}} \, du
\]

with \( C_{m,q} = \frac{1}{2^{2m-1}} \sum_{k=1}^{m} (-1)^{k+q} \binom{2m}{m-k} (2k)^{2q-2} \). But, for \( 2 \leq q \leq m \) the integral \( \int_0^\infty \frac{\sin^{2m} t}{t^{2q-1}} \, dt \) converges, so that using iv from Proposition 1 we conclude that we must have \( C_{m,q} = 0 \), and consequently
\[
\int_0^\infty \frac{\sin^{2m} t}{t^{2q-1}} \, dt = \frac{1}{2^{2m-1}} \sum_{k=1}^{m} (-1)^{k+q} \binom{2m}{m-k} \frac{(2k)^{2q-2}}{(2q-2)!} \ln k
\]
iv. Using \( \Box \) we conclude that, for \( X > 0 \), we have:
\[
\int_0^X \frac{\sin^{2m+1} t}{t^{2q}} dt = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{k+q} \binom{2m+1}{m-k} \int_0^X \frac{f_{2q+1}(u)}{u^{2q}} du
\]
\[
= \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{k+q} \binom{2m+1}{m-k} (2k+1)^{2q-1} \int_0^{(2k+1)X} \frac{f_{2q+1}(u)}{u^{2q}} du
\]
\[
= \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{k+q} \binom{2m+1}{m-k} (2k+1)^{2q-1} \int_0^X \frac{f_{2q+1}(u)}{u^{2q}} du + C_{m,q} \int_0^X \frac{f_{2q+1}(u)}{u^{2q}} du
\]
— with \( C_{m,q} = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{k+q} \binom{2m+1}{m-k} (2k+1)^{2q-1} \). But, for \( 1 \leq q \leq m \) the integral \( \int_0^\infty \frac{\sin^{2m+1} t}{t^{2q}} dt \) converges, so that using \( \Box \) from Proposition 1 we conclude that we must have \( C_{m,q} = 0 \), and consequently
\[
\int_0^\infty \frac{\sin^{2m+1} t}{t^{2q}} dt = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{k+q} \binom{2m+1}{m-k} (2k+1)^{2q-1} \int_0^X \frac{f_{2q+1}(u)}{u^{2q}} du
\]
This concludes the proof of the theorem.

**Corollary 3.** If \( q \) is a positive integer and \( k \) is a nonzero integer then \( \int_0^\infty \frac{\sin^{2q+1} t}{t^{2q}} dt \) is a rational multiple of \( \pi \).

**Example 4.** Here are some numerical examples for even values of the difference \( p - q \):

| \( p = q \) | \( k = q + 2 \) | \( k = q + 4 \) | \( k = q + 6 \) |
|---|---|---|---|
| 0 | \( \frac{\pi}{2} \) | \( \frac{\pi}{4} \) | \( \frac{3\pi}{16} \) | \( \frac{5\pi}{32} \) |
| 0 | \( \frac{\pi}{2} \) | \( \frac{\pi}{4} \) | \( \frac{3\pi}{16} \) | \( \frac{5\pi}{32} \) |
| 0 | \( \frac{3\pi}{7} \) | \( \frac{5\pi}{17} \) | \( \frac{7\pi}{63} \) | \( \frac{45\pi}{312} \) |
| 0 | \( \frac{3\pi}{8} \) | \( \frac{3\pi}{14} \) | \( \frac{7\pi}{12} \) | \( \frac{25\pi}{384} \) |

**Example 5.** Here are some numerical examples for odd values of the difference \( p - q \), (note that the considered integral diverges for \( q = 1 \) in this case):

| \( p = q + 1 \) | \( p = q + 3 \) | \( p = q + 5 \) |
|---|---|---|
| \( q = 2 \) | \( \frac{3}{4} \log 3 \) | \( \frac{15}{16} \log 3 - \frac{5}{16} \log 5 \) | \( \frac{63}{64} \log 3 - \frac{35}{64} \log 5 + \frac{7}{64} \log 7 \) |
| \( q = 3 \) | \( \log 2 \) | \( \frac{3}{2} \log 2 - \frac{9}{16} \log 3 \) | \( \frac{9}{7} \log 2 - \frac{9}{8} \log 3 \) |
| \( q = 4 \) | \( -\frac{45}{42} \log 3 + \frac{125}{96} \log 5 \) | \( -\frac{189}{128} \log 3 + \frac{875}{384} \log 5 - \frac{341}{384} \log 7 \) | \( -\frac{270}{572} \log 3 + \frac{1500}{512} \log 5 - \frac{1029}{512} \log 7 \) |
| \( q = 5 \) | \( -2 \log 2 + \frac{27}{16} \log 3 \) | \( -5 \log 2 + \frac{27}{8} \log 3 \) | \( -\frac{55}{8} \log 2 + \frac{1215}{208} \log 3 - \frac{625}{208} \log 5 \) |
Remark 1. It is interesting to note that these integrals do not appear in [1], but only particular cases are given in paragraphs 3.82–3.83.

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