Coherent State Representation of Semiclassical Quantum Gravity

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Abstract

We elaborate the recently introduced asymptotically exact semiclassical quantum gravity derived from the Wheeler-DeWitt equation by finding a particular coherent state representation of a quantum scalar field in which the back-reaction of the scalar field Hamiltonian exactly gives rise to the classical one. In this coherent state representation classical spacetime emerges naturally from semiclassical quantum gravity.

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Canonical quantum gravity based on the Wheeler-DeWitt (WDW) equation has been used as one of its applications to provide a self-consistent theory of quantum fields in a curved spacetime (for review and references, see [1]). Most of the methods in this direction rely on the Born-Oppenheimer idea that the larger mass scale out of several mass scales of a quantum system becomes first semiclassical and the relatively smaller mass scales retain the quantum mechanical nature. In this approach to quantum field theory in the curved spacetime the classical spacetime emerges from the quantum domain but the matter fields (typically scalar fields) keep their quantum mechanical properties satisfying the time-dependent functional Schrödinger equation. The advantage of this approach is that one may treat semiclassical theory relatively simply, including some parts of quantum gravitational corrections to the matter fields and replacing the energy-momentum tensor by its quantum mechanical expectation values. The disadvantage is that the quantum corrections of gravity to matter fields may not be achieved fully and the renormalization problem of wave functions may not be resolved in this approach to semiclassical quantum gravity. This approach to semiclassical quantum gravity, despite its shortcomings, proves quite useful when one considers the quantum gravity effects semiclassically and especially in the context of cosmology.

However, there has been one step not completely resolved in deriving classical gravity from the WDW equation. One usually assumes that classical gravity could be directly obtained from the WDW equation as the Hamilton-Jacobi equation. It could, of course, presumably sound physical in a certain domain, but the logical steps are not fulfilled, since considering the different mass scales of the gravitational fields and matter fields it would be more correct for the gravity to emerge first from the quantum domain and then the larger mass scaled matter fields. The scheme of deriving semiclassical quantum gravity

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1 We distinguish this approach so-called semiclassical quantum gravity from conventional quantum field theory in curved spacetimes. We refer to [2] for review and references of the latter approach.
$G_{\mu\nu} = 8\pi \langle \hat{T}_{\mu\nu} \rangle$ from canonical quantum gravity $\hat{G}_{\mu\nu} = 8\pi \hat{T}_{\mu\nu}$ and then classical gravity $G_{\mu\nu} = 8\pi T_{\mu\nu}$ from semiclassical quantum gravity needs a rigorous foundation.

In the previous papers [3] we developed an asymptotic method to derive the quantum field theory of matter fields in a curved spacetime from canonical quantum gravity based the WDW equation. In the case of the quantum Friedmann-Robertson-Walker (FRW) cosmology minimally coupled to a massive scalar field we were able to construct a particular Fock space on which the expectation value of the scalar field Hamiltonian operator has the same form in terms of a complex classical solution as that of classical one [4].

In this paper, we elaborate further that scheme of deriving the semiclassical quantum and classical gravity from canonical quantum gravity based on the WDW equation. It is found that the coherent states constructed on the particular Fock space [4] make matter fields emerge as classical variables. In this scheme one is able to show how semiclassical quantum gravity, the quantum field theory of matter fields, can derive from canonical quantum gravity and also how classical gravity theory, the gravity coupled to classical matter fields, can emerge from semiclassical quantum theory of gravity. For simplicity we shall consider the quantum FRW cosmology minimally coupled to a massive scalar field. One of the reasons for specifying the scalar field is that the coherent states of the massive scalar field in a curved spacetime can be constructed explicitly as will be shown in this paper.

As a simple quantum cosmological model, we consider a FRW cosmology with the metric

$$ds^2 = -N^2 dt^2 + a^2 d\sigma_k^2. \quad (1)$$

The action for the FRW cosmology minimally coupled to a homogeneous and isotropic massive scalar field is

$$I = \int dt \left[ -\frac{3m_P^2}{8\pi}a^3 \left( \frac{1}{N} \left( \frac{\dot{a}}{a} \right)^2 + N \frac{k}{a^2} \right) + a^3 \left( \frac{\dot{\phi}^2}{2N} - \frac{Nm^2 \phi^2}{2} \right) \right], \quad (2)$$

where $k = 1, 0, -1$ corresponds to a closed, flat and open universe, respectively. In the above action we dropped the surface term. We used the units system such that $c = \hbar = 1$ and $\frac{1}{c} = m_P^2$. The conjugate momenta are
\[ \pi_a = -\frac{3m_p^2}{4\pi} a \dot{a}, \quad \pi_\phi = \frac{a^3 \dot{\phi}}{N}. \] (3)

From the super-Hamiltonian constraint of the ADM formulation
\[ \mathcal{H} = -\frac{2\pi}{3m_p} \frac{1}{a} \pi_a^2 + \frac{3m_p^2}{8\pi} \dot{k}a + \frac{1}{2a^3} \pi_\phi^2 + \frac{a^3 m_p^2}{2} \phi^2 = 0 \] (4)

one obtains the WDW equation
\[
\left[ \frac{2\pi}{3m_p^2 a} \frac{\partial^2}{\partial a^2} - \frac{3m_p^2}{8\pi} k a - \frac{1}{2a^3} \frac{\partial^2}{\partial \phi^2} + \frac{a^3 m_p^2}{2} \right] \Psi(a, \phi) = 0. \] (5)

One of the questions closely related with the correspondence between quantum and classical theory in general is now to understand how and when one may recover classical gravity theory
\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi}{3m_p^2} \left( \frac{\dot{\phi}^2}{2} + \frac{m^2}{2} \phi^2 \right), \] (6)

which is obtained by varying the action (2) with respect to the lapse function \( N \) and fixing the temporal gauge \( N = 1 \). The other scalar field equation that constitutes classical gravity is obtained from the variation of the action with respect to \( \phi \)
\[ \ddot{\phi} + \frac{3}{a} \dot{a} \dot{\phi} + m^2 \phi = 0. \] (7)

One has frequently used the WKB wave function \( \Psi(a, \phi) = F(a, \phi) e^{iS(a, \phi)} \), where \( F \) is a slowly varying function. The rapidly changing phase factor satisfies the Hamilton-Jacobi equation
\[
-\frac{2\pi}{3m_p^2 a} \left( \frac{\partial S}{\partial a} \right)^2 - \frac{3m_p^2}{8\pi} k a + \frac{1}{2a^3} \left( \frac{\partial S}{\partial \phi} \right)^2 + \frac{a^3 m_p^2}{2} \phi^2 = 0. \] (8)

By identifying \( \frac{\partial S}{\partial a} = \pi_a \) and \( \frac{\partial S}{\partial \phi} = \pi_\phi \) in (3), we recover the classical equation (6). But in this approach to classical gravity, there remains one problem unexplained that the large mass scale difference between the gravitational field and matter field in a later stage of cosmological evolution makes the gravitational field classical but keeps the matter field quantum mechanical following the Born-Oppenheimer idea. This is the main conceptual idea behind semiclassical quantum gravity theory.
Below we shall develop an alternative to it, in which classical gravity can be derived from semiclassical quantum gravity which in turn can be derived from quantum gravity based on the WDW equation. It was shown \[3\] that the semiclassical quantum gravity derived from the WDW equation consists of the gravitational field equation

\[
\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3m_P^2 a^3} \langle \hat{H}_m \rangle,
\]

and time-dependent Schrödinger equation of scalar field

\[
i \frac{\partial}{\partial t} \Phi(\phi, t) = \hat{H}_m \Phi(\phi, t),
\]

where

\[
\hat{H}_m = \frac{1}{2a^3} \hat{\pi}_\phi^2 + \frac{a^3 m^2}{2} \hat{\phi}^2
\]

is the scalar field Hamiltonian. As was shown explicitly and fully in \[3\] the semiclassical quantum gravity equations are asymptotically exact in the sense of \( \frac{1}{m_P^2} \to 0 \), i.e \( O\left(\frac{1}{m_P^2}\right) \), provided that one chooses the quantum states of the scalar field as the eigenstates of invariant \( \hat{I}_m \) satisfying \[3\]

\[
\frac{\partial \hat{I}_m}{\partial t} - i[\hat{I}_m, \hat{H}_m] = 0.
\]

The equivalence between different approaches to semiclassical quantum gravity was recently shown in \[4\]. We shall, however, use the approach of \[3\] in which it is relatively easier to construct the coherent state representation compared with the others.

Of many invariants it was found that two particular invariants are very useful and convenient in constructing the Fock space \[4\]:

\[
\hat{A}^\dagger(t) = \phi_c(t) \hat{\pi}_\phi - a^3(t) \dot{\phi}_c(t) \hat{\phi},
\]

\[
\hat{A}(t) = \phi^*_c(t) \hat{\pi}_\phi - a^3(t) \dot{\phi}^*_c(t) \hat{\phi},
\]

where \( \phi_c \) is a complex solution of \[7\] with the boundary conditions\[4\]

\[2\]The sign of the second condition is corrected from \[4\]
\[
a^3(t) \left( \dot{\phi}_c(t) \dot{\phi}_c^*(t) - \dot{\phi}_c^*(t) \dot{\phi}_c(t) \right) = i,
\]
\[
\text{Im} \left( \frac{\dot{\phi}_c(t)}{\phi_c(t)} \right) > 0.
\]

In fact \( \hat{A}^\dagger(t) \) acts as the creation operator and \( \hat{A}(t) \) as the annihilation operator on the Fock space of number states
\[
\hat{A}^\dagger(t) \hat{A}(t) |n, t> = n |n, t>.
\]

The exact quantum states are given by
\[
\Phi_n(\phi, t) = e^{-i \omega_n(t)} |n, t>
\]
where
\[
\omega_n(t) = \int < n, t | \hat{H}_m - i \frac{\partial}{\partial t} | n, t >,
\]
is a time-dependent phase factor.

We may find the Bogoliubov transformation between two different times
\[
\hat{A}^\dagger(t) = u(t) \hat{A}^\dagger(t_0) + v(t) \hat{A}(t_0),
\]
\[
\hat{A}(t) = v^*(t) \hat{A}^\dagger(t_0) + u^*(t) \hat{A}(t_0),
\]
where
\[
u(t, t_0) = \frac{i a^3}{2} \left( \dot{\phi}_c(t) \dot{\phi}_c^*(t_0) - \dot{\phi}_c^*(t) \dot{\phi}_c(t_0) \right),
v(t, t_0) = \frac{i a^3}{2} \left( \dot{\phi}_c(t) \dot{\phi}_c(t_0) - \dot{\phi}_c(t) \dot{\phi}_c(t_0) \right).
\]

The relation
\[
|u(t, t_0)|^2 - |v(t, t_0)|^2 = 1
\]
can be shown by direct substitution. The above relation can be parameterized as
\[
u(t, t_0) = \cosh \nu e^{-i \theta_u},
v(t, t_0) = \sinh \nu e^{-i \theta_v}.
\]
Then we find a unitary transformation of the creation operators between two different times

\[ \hat{A}^\dagger(t) = \hat{S}^\dagger(t, t_0) \hat{A}^\dagger(t_0) \hat{S}(t, t_0), \quad (22) \]

in terms of the squeeze operator

\[ \hat{S}(t, t_0) = \exp \left( i \theta_u \hat{A}^\dagger(t_0) \hat{A}(t_0) \right) \exp \left( \frac{\nu e^{-i(\theta_u - \theta_v)} \hat{A}^{\dagger 2}(t_0)}{2} - \text{h.c.} \right). \quad (23) \]

The unitary transformation of the annihilation operators can also be found similarly by taking the hermitian conjugate of that of the creation operators. Note that the squeeze operator is a unitary operator. This implies the unitary evolution of the operators on the Fock space. This does not always mean that the Fock representations are unitarily equivalent. In the case of a homogeneous but not isotropic scalar field in the flat universe which has an infinite volume, it is known that the vacua at any two different times are mutually orthogonal and therefore there can be an infinitely many unitarily inequivalent Fock representations [7,8].

We introduce the coherent states on the Fock space of the eigenstates of the invariant. At an arbitrary initial time we define the coherent state by [8]

\[ \hat{A}(t_0) \alpha, t_0 \rangle = \alpha \alpha, t_0 \rangle \quad (24) \]

where \( \alpha \) is a complex number. In terms of the creation operator acting on the vacuum state at that time they read that

\[ \alpha, t_0 \rangle = e^{-\frac{|\alpha|^2}{2}} \sum_0^\infty \frac{\alpha^n}{\sqrt{n!}} |n, t_0 \rangle. \quad (25) \]

It can be shown that the coherent states also transform unitarily

\[ \alpha, t \rangle = \hat{S}^\dagger(t, t_0) \alpha, t_0 \rangle. \quad (26) \]

Thus it follows that

\[ \hat{A}(t) \alpha, t \rangle = \alpha \alpha, t \rangle. \quad (27) \]
The action of the creation operator is the hermitian conjugate of the annihilation operator \( < \alpha, t | \hat{\mathcal{A}}^\dagger (t) = \alpha^* < \alpha, t | \). From the relations
\[
\hat{\phi} = (-i) (\phi_c \hat{\mathcal{A}}^\dagger (t) - \phi_c^* \hat{\mathcal{A}} (t)),
\]
\[
\hat{\pi}_\phi = (-ia^3) (\dot{\phi}_c \hat{\mathcal{A}}^\dagger (t) - \dot{\phi}_c^* \hat{\mathcal{A}} (t)),
\]
(28)
we can show that \( |\alpha, t \rangle \) really describes the classical trajectory
\[< \alpha, t | \hat{\phi}|\alpha, t > = \varphi_c (t), \]
(29)
where
\[
\varphi_c = \frac{\alpha^* \phi_c - \alpha \phi_c^*}{i},
\]
(30)
is a real classical solution. The expectation value of the position operator gives indeed the
real classical orbit. The expectation value of the scalar field Hamiltonian taken with respect
to the coherent state has a simple form
\[
< \alpha, t | \hat{H}_m |\alpha, t > = \frac{a^3}{2} (\dot{\varphi}_c^2 (t) + m^2 \varphi_c^2 (t)) + \frac{a^3}{2} (\dot{\phi}_c^* (t) \dot{\phi}_c (t) + m^2 \phi_c^* (t) \phi_c (t)).
\]
(31)
Note that the last two terms of the expectation value come from the quantum fluctuation
of vacuum which are removed by the normal ordering of the operators
\[
< \alpha, t | : \hat{H}_m : |\alpha, t > = \frac{a^3}{2} (\dot{\varphi}_c^2 (t) + m^2 \varphi_c^2 (t)).
\]
(32)
Remembering that the coherent state is a superposition of the eigenstates of the partic-
ular invariant, we see that the decoupling theorem of Lewis and Riesenfeld between
off-diagonal terms still holds and the semiclassical quantum gravity in the coherent state
representation is asymptotically exact.

In summary we elaborated the previous scheme in which canonical quantum gravity based
on the Wheeler-DeWitt equation leads to semiclassical quantum and classical gravity.
The coherent state representation is found to make the expectation value of the quantum
energy-momentum tensor reduce to classical one. Even though we showed the coherent state
representation for the quantum Friedmann-Robertson-Walker cosmology minimally coupled to a free massive scalar field, we put forth a conjecture that there may exist the coherent state representations of semiclassical quantum gravity for a generic geometry coupled to fundamental scalar fields such as scalar fields and fermionic fields.

The result of this paper may have an important implication and application to cosmology. Assuming that the Wheeler-DeWitt equation is valid just below the Planck scale, we can investigate the condition under which semiclassical quantum and classical gravity coupled to inflatons and some other quantum fluctuations necessary for inflation and reheating emerge and hold true.

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