GLOBAL SOLVABILITY OF THE NAVIER-STOKES EQUATIONS WITH A FREE SURFACE IN THE MAXIMAL $L_p$-$L_q$ REGULARITY CLASS

HIROKAZU SAITO

Abstract. We consider the motion of incompressible viscous fluids bounded above by a free surface and below by a solid surface in the N-dimensional Euclidean space for $N \geq 2$. The aim of this paper is to show the global solvability of the Navier-Stokes equations with a free surface, describing the above-mentioned motion, in the maximal $L_p$-$L_q$ regularity class. Our approach is based on the maximal $L_p$-$L_q$ regularity with exponential stability for the linearized equations, and also it is proved that solutions to the original nonlinear problem are exponentially stable.

1. Introduction

This paper is concerned with the global solvability of the Navier-Stokes equations with a free surface, describing the motion of incompressible viscous fluids bounded above by a free surface and below by a solid surface in the $N$-dimensional Euclidean space for $N \geq 2$, in the maximal $L_p$-$L_q$ regularity class (cf. [36] for the class). Such equations were mathematically treated by Beale [6] for the first time. He proved, in an $L_2$-in-time and $L_2$-in-space setting with the gravity, the local solvability for large initial data in [6], whereas we prove in the maximal $L_p$-$L_q$ regularity class the global solvability for small initial data in the case where the gravity is not taken into account in the present paper.

The problem is stated as follows: We are given an initial domain $\Omega \subset \mathbb{R}^N$, occupied by an incompressible viscous fluid, such that

$$\Omega = \{ \xi = (\xi', \xi_N) \mid \xi' = (\xi_1, \ldots, \xi_{N-1}) \in \mathbb{R}^{N-1}, 0 < \xi_N < d \} \quad (d > 0),$$

as well as an initial velocity field $a = a(\xi) = (a_1(\xi), \ldots, a_N(\xi))^T$ of the fluid on $\Omega$. The symbols $\Gamma, S$ denote the boundaries of $\Omega$ such that

$$\Gamma = \{ \xi = (\xi', \xi_N) \mid \xi' = (\xi_1, \ldots, \xi_{N-1}) \in \mathbb{R}^{N-1}, \xi_N = d \},$$

$$S = \{ \xi = (\xi', \xi_N) \mid \xi' = (\xi_1, \ldots, \xi_{N-1}) \in \mathbb{R}^{N-1}, \xi_N = 0 \}.$$

We wish to find for each $t \in (0, \infty)$ a transformation $\Theta = \Theta(\cdot, t) : \Omega \ni \xi \mapsto x = \Theta(\xi, t) \in \mathbb{R}^N$, a velocity field $v = v(x, t) = (v_1(x, t), \ldots, v_N(x, t))^T$ of the fluid, and a pressure field $\pi = \pi(x, t)$ of the fluid so that

$$\partial_t \Theta = v \circ \Theta, \quad \Theta(\xi, 0) = \xi, \quad \xi \in \Omega,$$

$$\Omega(t) = \Theta(\Omega, t), \quad \Gamma(t) = \Theta(\Gamma, t), \quad S = \Theta(S, t).$$

2010 Mathematics Subject Classification. Primary: 35Q30; Secondary: 76D05.

Key words and phrases. Global solvability, Navier-Stokes equations, Free surfaces, Maximal regularity, $L_p$-$L_q$ framework, Exponential stability, Infinite layers.

$^1$M$^T$ denotes the transposed M.
\[ (1.3) \quad \partial_t v + (v \cdot \nabla)v = \text{Div}\, T(v, \pi), \quad x \in \Omega(t), \]
\[ (1.4) \quad \text{div } v = 0, \quad x \in \Omega(t), \]
\[ (1.5) \quad T(v, \pi)n = -\pi_0 n, \quad x \in \Gamma(t), \]
\[ (1.6) \quad v = 0, \quad x \in S, \]
\[ (1.7) \quad v|_{t=0} = a, \quad \xi \in \Omega, \]

where \( v \circ \Theta = (v \circ \Theta)(\xi, t) = v(\Theta(\xi, t), t). \)

Here the density of the fluid have been set to 1; \( n \) is the unit outward normal to \( \Gamma(t); \) the constant \( \pi_0 \) is the atmospheric pressure, and it is assumed in this paper that \( \pi_0 = 0 \) without loss of generality. The stress tensor \( T(v, \pi) \) is given by \( T(v, \pi) = \mu D(v) - \pi I, \) where \( \mu \) is a positive constant and denotes the viscosity coefficient of the fluid; \( I \) is the \( N \times N \) identity matrix; \( D(v) = \nabla v + (\nabla v)^T \) is the doubled strain tensor. Here and subsequently, we use the following notation to \( \Gamma(\xi,t) \): the constant \( \pi_0 = 0 \) without loss of generality. The stress tensor \( T(v, \pi) \) is given by \( T(v, \pi) = \mu D(v) - \pi I, \) where \( \mu \) is a positive constant and denotes the viscosity coefficient of the fluid; \( I \) is the \( N \times N \) identity matrix; \( D(v) = \nabla v + (\nabla v)^T \) is the doubled strain tensor. Here and subsequently, we use the following notation for differentiations: Let \( f = f(x), \) \( g = (g_1(x), ..., g_N(x))^T, \) and \( M = (M_{ij}(x)) \) be a scalar-, a vector-, and an \( N \times N \) matrix-valued function on a domain of \( \mathbb{R}^N, \) respectively, and then for \( \partial_j = \partial/\partial x_j \)

\[ \nabla f = (\partial_1f, \ldots, \partial_Nf)^T, \quad \Delta f = \sum_{j=1}^N \partial_j^2 f, \quad \Delta g = (\Delta g_1, \ldots, \Delta g_N)^T, \]

\[ \text{div } g = \sum_{j=1}^N \partial_j g_j, \quad \nabla^2 g = \{ \partial_i \partial_j g_k \mid i, j, k = 1, \ldots, N \}, \]

\[ \nabla g = \begin{pmatrix} \partial_1 g_1 & \cdots & \partial_N g_1 \\ \vdots & \ddots & \vdots \\ \partial_1 g_N & \cdots & \partial_N g_N \end{pmatrix}, \quad (g \cdot \nabla)g = \begin{pmatrix} \sum_{j=1}^N g_j \partial_j g_1, \ldots, \sum_{j=1}^N g_j \partial_j g_N \end{pmatrix}^T, \]

\[ \text{Div } M = \begin{pmatrix} \sum_{j=1}^N \partial_j M_{1j}, \ldots, \sum_{j=1}^N \partial_j M_{Nj} \end{pmatrix}^T. \]

Let \( u(\xi, t) = (v \circ \Theta)(\xi, t), \) which is the so-called Lagrangian velocity, and then the solution \( \Theta \) to (1.1) is represented as

\[ (1.8) \quad \Theta(\xi, t) = \xi + \int_0^t u(\xi, s) \, ds \quad (\xi \in \Omega, \ t > 0). \]

We now write the equations (1.3)-(1.7) in the Lagrangian formulation. Thus our unknowns will be the Lagrangian velocity \( u(\xi, t) = v(\Theta(\xi, t), t) \) and pressure \( p(\xi, t) = \pi(\Theta(\xi, t), t) \) for \( (\xi, t) \in \Omega \times (0, \infty). \) If we substitute the new unknowns \( u = u(\xi, t) \) and \( p = p(\xi, t) \) in (1.3)-(1.7), then the equations turn into\(^2\)

\[ (1.9) \quad \partial_t u - \text{Div } T(u, p) = F(u) \quad \text{in } \Omega, \ t > 0, \]
\[ (1.10) \quad \text{div } u = G(u) = \text{div } G(u) \quad \text{in } \Omega, \ t > 0, \]
\[ (1.11) \quad T(u, p)e_N = H(u)e_N \quad \text{on } \Gamma, \ t > 0, \]
\[ (1.12) \quad u = 0 \quad \text{on } S, \ t > 0, \]
\[ (1.13) \quad u|_{t=0} = a \quad \text{in } \Omega, \]

\(^2\)The derivation of (1.9)-(1.11) is discussed in the appendix.
where $e_N = (0, \ldots, 0, 1)^T$. Here $F(u)$, $G(u)$, $G(u)$, and $H(u)$ are nonlinear terms, with respect to $u$, of the forms:

\begin{equation}
F(u) = U_1 \left( \int_0^t \nabla u(\xi, s) \, ds \right) \partial_t u + V \left( \int_0^t \nabla u(\xi, s) \, ds \right) \nabla^2 u
+ \left[ W \left( \int_0^t \nabla u(\xi, s) \, ds \right) \int_0^t \nabla^2 u(\xi, s) \, ds \right] \nabla u,
\end{equation}

\begin{equation}
G(u) = U_2 \left( \int_0^t \nabla u(\xi, s) \, ds \right) : \nabla u, \quad G(u) = U_3 \left( \int_0^t \nabla u(\xi, s) \, ds \right) u,
\end{equation}

\begin{equation}
H(u) = D(u) U_4 \left( \int_0^t \nabla u(\xi, s) \, ds \right) + (\nabla u)^T U_5 \left( \int_0^t \nabla u(\xi, s) \, ds \right)
+ U_6 \left( \int_0^t \nabla u(\xi, s) \, ds \right) (\nabla u)^T \left( I + U_7 \left( \int_0^t \nabla u(\xi, s) \, ds \right) \right),
\end{equation}

where $U_i : R^{N \times N} \rightarrow R^{N \times N}$ ($i = 1, \ldots, 7$),

\begin{equation}
V(\cdot) \nabla^2 u = \left( \sum_{i,j,k=1}^N V_{ijk}^1(\cdot) \partial_i \partial_j u_k, \ldots, \sum_{i,j,k=1}^N V_{ijk}^N(\cdot) \partial_i \partial_j u_k \right)^T,
\end{equation}

\begin{equation}
\left[ W(\cdot) \int_0^t \nabla^2 u \, ds \right] \nabla v = \left( \sum_{i,j,k,l,m=1}^N W_{ijklm}^1(\cdot) \int_0^t \partial_i \partial_j u_k \, ds \partial_l v_m, \ldots, \sum_{i,j,k,l,m=1}^N W_{ijklm}^N(\cdot) \int_0^t \partial_i \partial_j u_k \, ds \partial_l v_m \right)^T,
\end{equation}

for some $V_{ijk}^1, \ldots, V_{ijk}^N, W_{ijklm}^1, \ldots, W_{ijklm}^N : R^{N \times N} \rightarrow R$ and for any $N$-vectors $u = (u_1, \ldots, u_N)^T$, $v = (v_1, \ldots, v_N)^T$. Note that, for $N \times N$ matrices $A = (A_{ij})$, $B = (B_{ij})$, we have set

\begin{equation}
A : B = \sum_{i,j=1}^N A_{ij} B_{ij}.
\end{equation}

One has the following information about $U_i$, $V$, $W$: Let $X = (X_{mn})$ be $N \times N$ matrices. Then all the components of $U_i(X)$ ($i = 1, \ldots, 7$), $V(X)$, and $W(X)$ are polynomials with respect to $X_{mn}$ for $m, n = 1, \ldots, N$. Furthermore,

\begin{equation}
U_i(O) = O \quad (i = 1, \ldots, 7), \quad V_{ijk}^l(O) = 0 \quad (i, j, k, l = 1, \ldots, N),
\end{equation}

where $O$ denotes the $N \times N$ zero matrix.

Let us introduce historical remarks and key ideas of the present paper at this point. If we consider free boundary problems, then we first usually transform them to nonlinear problems on given domains, independent of time $t$, by using a suitable transformation. Roughly speaking, such transformations are divided into

- Lagrangian transformation;
- Eulerian approaches (e.g. Beale’s transformation in [7], [8]; Hanzawa’s transformation in [19]).

Lagrangian transformation denotes the transformation $\Theta$ of (1.8). It is quite useful to show the local solvability for a lot of situations. In fact, by using the Lagrangian transformation, Shibata [36] and Enomoto et al. [17] proved, respectively, the local solvability of the Navier-Stokes equations with a free surface for
incompressible viscous fluids and for compressible viscous fluids in the case where
the initial domain $\Omega$ has uniform $W^{2-1/q}_q$ regularity (cf. their papers for the defi-
tion). Here half-spaces, bent half-spaces, layer-like domains, cylinder-like domains,
bounded domains, and exterior domains are typical examples of unif orm $W^{2-1/q}_q$ do-
mains. As for the local solvability with the Lagrangian transformation, we also refer
e.g. to the following papers: Solonnikov [42, 43] and references therein, Mogilevskii
and Solonnikov [28], Mucha and Zajączkowski [29], Shibata and Shimizu [38] for
incompressible viscous fluids in bounded domains; Beale [6], Allain [4], Tani [50],
Abels [1] for incompressible viscous fluids in layer-like domains; Tani [48], Solon-
nikov and Tani [44, 45], Secchi and Valli [35], Secchi [32, 33, 34], Zajączkowski
[55, 56, 57], Zadzieńska and Zajączkowski [52, 53, 54], Strömmer and Zajączkowski
[46], Denisova and Solonnikov [15] for compressible viscous fluids in bounded do-
mains; Tanaka and Tani [47] for compressible viscous fluids in layer-like domains;
Tani [49], Denisova [9, 10, 11, 12, 13], Denisova and Solonnikov [14] for the motion
of two fluids separated by a closed free surface.

The advantage of the Lagrangian transformation for the local solvability is that
\[ \int_0^t \nabla u(\xi,s) \, ds \]
appears in nonlinear terms (cf. (1.14)). By choosing $T > 0$ small
enough, we can see $\int_0^t \nabla u(\xi,s) \, ds$ ($t \in (0,T)$) as a small coefficient, and thus the
nonlinear terms would be small with suitable norms. This enables us to show
the local solvability by using the contraction mapping theorem.

In order to prove the global solvability, we need the integrability of $\nabla u(\xi,t)$
with respect to time $t \in (0,\infty)$ because of $\int_0^t \nabla u(\xi,s) \, ds$. If we are in an $L_p$-in-
time and $L_q$-in-space setting, then a key idea to guarantee the time integrability
is exponential stability of solutions for the linearized equations as was seen e.g. in
Shibata-Shimizu [39], Shibata [36]. These two papers tell us that the exponential
stability can be proved for bounded domains by some abstract approach. However,
for unbounded domains containing $\Omega$ of the present paper, it is not true in general
that the exponential stability holds. This is one of main difficulties to prove the
global solvability in our situation.

Eulerian approaches are useful to show the large-time behavior of solutions to the
Navier-Stokes equations with a free surface, because $\int_0^t u(\xi,s) \, ds$ as above does not
appear. We here introduce e.g. the following papers as references of the Eulerian
approaches: Beale and Nishida [8] (cf. also [20] for the detailed proof), Hataya and
Kawashima [21] proved polynomial decay of solutions for layer-like domains in an
$L_2$-in-time and $L_2$-in-space setting; Köhne, Prüss, and Wilke [22], Solonnikov [41]
proved exponential stability of solutions for bounded domains in an $L_p$-in-time and
$L_p$-in-space setting; Saito and Shibata [31] showed $L_p$-$L_r$ estimates of the Stokes
semigroup with surface tension and gravity on the half-space.

The key idea of this paper is to prove the maximal $L_p$-$L_q$ regularity with expo-
nential stability for the linearized equations of (1.9)-(1.13). As mentioned above,
we can prove such an exponential stability for bounded domains, but the technique
can not be applied to our situation because our domain $\Omega$ is unbounded. To over-
come this difficulty, we make use of Abels [2] and Saito [30] in this paper. We then
apply the maximal $L_p$-$L_q$ regularity with exponential stability and the contraction
mapping theorem to (1.9)-(1.13) in order to prove their global solvability.

This paper is organized as follows: The next section first introduce notation and
function spaces that are used throughout this paper. Secondly, we state main
results of the global solvability for (1.9)-(1.13) and the maximal $L_p$-$L_q$ regularity
with exponential stability for the linearized equations. Section 3 introduces some results concerning a time-shifted problem, a variational problem, and the Helmholtz decomposition on $\Omega$. Section 4 shows the generation of an analytic $C_0$-semigroup associated with the linearized equations. In Section 5, we prove the maximal $L_p$-$L_q$ regularity with exponential stability for the linearized equations by means of results introduced in Sections 3, 4. Section 6 proves the global solvability of (1.9)-(1.13) based on the contraction mapping theorem together with the maximal $L_p$-$L_q$ regularity with exponential stability proved in Section 5. In Section 7, we show the global existence and uniqueness of solutions for the original problem (1.1)-(1.7), and also their exponential stability.

2. Notation and main results

In this section, we first introduce notation and function spaces that are used throughout this paper. Next our main results are stated.

2.1. Notation. The set of all natural numbers, real numbers, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $m, n \in \mathbb{N}$ and $G$ be a domain of $\mathbb{R}^n$. We set $a \cdot b = \sum_{j=1}^m a_j b_j$ for $m$-vectors $a = (a_1, \ldots, a_m)^T$ and $b = (b_1, \ldots, b_m)^T$, while we set $(f, g)_G = \int_G f(x) \cdot g(x) \, dx = \sum_{j=1}^m \int_G f_j(x) g_j(x) \, dx$ for $m$-vector functions $f = (f_1(x), \ldots, f_m(x))^T$, $g = (g_1(x), \ldots, g_m(x))^T$ on $G$. In addition, $C_0^\infty(G)$ denotes the set of all $C^\infty$-functions on $\mathbb{R}^n$ whose supports are compact and contained in $G$.

Let $X$ and $Y$ be Banach spaces, and let $1 \leq p \leq \infty$. The Banach space of all bounded linear operators from $X$ to $Y$ is denoted by $\mathcal{L}(X, Y)$, and $\mathcal{L}(X) = \mathcal{L}(X, X)$. $L_p(G, X)$ and $W^m_p(G, X)$ denote, respectively, the standard $X$-valued Lebesgue spaces on $G$ and the standard $X$-valued Sobolev spaces on $G$, and $W^0_p(G, X) = L_p(G, X)$. If $X = \mathbb{R}$ or $X = \mathbb{C}$, then $L_p(G, X)$, $W^m_p(G, X)$, and $W^0_p(G, X)$ are denoted by $L_p(G)$, $W^m_p(G)$, and $W^0_p(G)$, respectively, for short.

The symbol $C(G, X)$ stands for the set of all $X$-valued continuous function on $G$, while $C^m(G, X)$ is the set of all $m$-times continuously differentiable functions on $G$ with values in $X$. Let $BUC(G, X)$ be the Banach space of all $X$-valued uniformly continuous and bounded functions on $G$. In addition,

$$BUC^m(G, X) = \{ f \in C^m(G, X) \mid \partial^\alpha f \in BUC(G, X) \text{ for } |\alpha| = 0, 1, \ldots, m\},$$

where we have set $\partial^\alpha f = \partial^{\alpha_1}_1 \cdots \partial^{\alpha_n}_n f$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0$. Here $C(G)$, $C^m(G)$, $BUC(G)$, and $BUC^m(G)$ are defined similarly as above, and also $G$ can be replaced by the closure $\overline{G}$ of $G$.

Let $\gamma \in \mathbb{R}$. We then define functions spaces with exponential weights as

$$L_{p, \gamma}(\mathbb{R}, X) = \{ f \in L_{p, \text{loc}}(\mathbb{R}, X) \mid e^{-\gamma t} f(t) \in L_p(\mathbb{R}, X) \},$$

$$W^k_{1, \gamma}(\mathbb{R}, X) = \{ f \in W^k_{p, \text{loc}}(\mathbb{R}, X) \mid e^{-\gamma t} \partial_t^k f(t) \in L_p(\mathbb{R}, X) \text{ for } k = 0, 1\}.$$

On the other hand, one sets for $\mathbb{R} = (0, \infty)$

$$W^1_p(\mathbb{R}_+, X) = \{ f \in W^1_p(\mathbb{R}_+, X) \mid f|_{t=0} = 0\}.$$

In order to define Bessel potential spaces of order $1/2$, we introduce the Fourier transform and its inverse transform as follows: Let $f = f(t)$ and $g = g(\tau)$ be
functions defined on \( \mathbb{R} \), and then
\[
\mathcal{F}[f](\tau) = \int_{\mathbb{R}} e^{-it\tau} f(t) \, dt, \quad \mathcal{F}^{-1}_\tau[g](t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\tau} g(\tau) \, d\tau.
\]
The Bessel potential spaces are given by
\[
H^{1/2}_p(\mathbb{R}, X) = \{ f \in L_p(\mathbb{R}, X) \mid \|f\|_{H^{1/2}_p(\mathbb{R}, X)} < \infty \},
\]
\[
\|f\|_{H^{1/2}_p(\mathbb{R}, X)} = \|\mathcal{F}^{-1}_\tau[(1 + \tau^2)^{1/4} \mathcal{F}[f]](\tau)\|_{L_p(\mathbb{R}, X)},
\]
\[
H^{1/2}_{p,\gamma}(\mathbb{R}, X) = \{ f \in L_{p,\gamma}(\mathbb{R}, X) \mid \|e^{-\gamma t} f\|_{H^{1/2}_p(\mathbb{R}, X)} < \infty \},
\]
and furthermore,
\[
H^{1/2}_{p,\gamma}(\mathbb{R}_+, X) = [L_p(\mathbb{R}_+, X), W^1_{q,\gamma}(\mathbb{R}_+, X)];_{1/2},
\]
\[
oH^{1/2}_{p,\gamma}(\mathbb{R}_+, X) = [L_p(\mathbb{R}_+, X), 0W^1_{q,\gamma}(\mathbb{R}_+, X)];_{1/2},
\]
where \([\cdot, \cdot]\) is the complex interpolation functor with \(0 < \gamma < 1\). For notational convenience, we set for \(1 < p, q < \infty\) and \(Z \in \{H_{0,H}\}
\]
\[
H^{1,1/2}_{p,q}(\Omega \times \mathbb{R}) = H^{1/2}_{p,q}(\mathbb{R}, L_q(\Omega)) \cap L_{p,\gamma}(\mathbb{R}, W^1_{q,\gamma}(\Omega)),
\]
\[
H^{1,1/2}_{p,q}(\Omega \times \mathbb{R}) = H^{1/2}_{p,q}(\mathbb{R}, L_q(\Omega)) \cap L_{p,\gamma}(\mathbb{R}, W^1_{q,\gamma}(\Omega)),
\]
\[
Z^{1/2}_{q,p}(\Omega \times \mathbb{R}_+) = H^{1/2}_{p,q}(\mathbb{R}_+, L_q(\Omega)) \cap L_p(\mathbb{R}_+, W^1_{q,\gamma}(\Omega)),
\]
\[
W^{2,1}_{q,p}(\Omega \times \mathbb{R}) = W^1_{q,p}(\mathbb{R}, L_q(\Omega)) \cap L_{p,\gamma}(\mathbb{R}, W^2_{q,\gamma}(\Omega)),
\]
\[
W^{2,3}_{q,p}(\Omega \times \mathbb{R}_+) = W^1_{p,\gamma}(\mathbb{R}_+, L_q(\Omega)) \cap L_p(\mathbb{R}_+, W^2_{q,\gamma}(\Omega)).
\]

Let \(1 < q < \infty\) and \(q' = q/(q-1)\). A closed subspace \(W^{1,1}_{q,\gamma}(\Omega)\) of \(W^1_{q,\gamma}(\Omega)\) is defined as \(W^{1,1}_{q,\gamma}(\Omega) = \{ f \in W^1_{q,\gamma}(\Omega) \mid f = 0 \text{ on } \Gamma \}. \) Then the solenoidal space \(J_q(\Omega)\) is given by
\[
J_q(\Omega) = \{ f \in L_q(\Omega)^N \mid (f, \nabla \varphi)_\Omega = 0 \text{ for all } \varphi \in W^1_{q',\gamma}(\Omega) \}.
\]
Here we set \(D_{q,p}(\Omega) = (J_q(\Omega), D(A_q))_{1-1/p, p}^{3)}, \) where \((\cdot, \cdot)_{\theta,p}\) is the real interpolation functor with \(0 < \theta < 1\).

**Remark 2.1.** The interpolation space \(D_{q,p}(\Omega)\) is characterized as follows\(^3\):
\[
D_{q,p}(\Omega) = \begin{cases} 
\{ u \in J_q(\Omega) \cap B^{2-2/p}_{q,p}(\Omega)^N \mid (\mu D(\mathbf{u}) e_N)_\gamma = 0 \text{ on } \Gamma, \mathbf{u} = 0 \text{ on } S \} \\
\text{when } 2 - 2/p > 1 + 1/q, \\
\{ u \in J_q(\Omega) \cap B^{2-2/p}_{q,p}(\Omega)^N \mid u = 0 \text{ on } S \} \\
\text{when } 1/q < 2 - 2/p < 1 + 1/q, \\
J_q(\Omega) \cap B^{2-2/p}_{q,p}(\Omega)^N \text{ when } 2 - 2/p < 1/q,
\end{cases}
\]
where we have set \(B^{2-2/p}_{q,p}(\Omega) = (L_q(\Omega), W^2_{q,\gamma}(\Omega))_{1-1/p, p}\) and \(v_\tau = v - e_N(e_N \cdot v)\) for any \(N\)-vector \(v\).

Throughout this paper, the letter \(C\) denotes generic constants and \(C_{a,b,c,...}\) means that the constant depends on the quantities \(a, b, c, \ldots\). The values of constants \(C\) and \(C_{a,b,c,...}\) may change from line to line.\(^4\)

\(^3\) \(D(A_q)\) is the domain of the Stokes operator \(A_q\) associated with the linearized equations of (1.9)-(1.13). They are discussed in Section 4 below in more detail, especially in (4.5).

\(^4\) We refer e.g. to [36, page 415].
2.2. Main results. This subsection introduces our main results of this paper.

First the global solvability of (1.9)-(1.13) is stated as follows:

Theorem 2.2. Let $p$ and $q$ be exponents satisfying

$$2 < p < \infty, \quad N < q < \infty, \quad \frac{2}{p} + \frac{N}{q} < 1.$$  

Then there exist positive constants $\gamma_0$, $\delta_0$, and $\varepsilon_0$ such that, for any $a \in D_{q,p}(\Omega)$ with $\|a\|_{D_{q,p}(\Omega)} \leq \varepsilon_0$, the equations (1.9)-(1.13) admit a unique solution

$$(u, p) \in W^{2,1}_{q,p}(\Omega \times \mathbb{R}^+) \times L_p(\mathbb{R}^+, W^1_q(\Omega)),$$

with $\lim_{t \to 0^+} \|u(t) - a\|_{B_{q,p}^{2,1}(\Omega)} = 0$, satisfying the estimate:

$$\|e^{\gamma_0 t}(\partial_t u, u, \nabla u, \nabla^2 u)\|_{L_p(\mathbb{R}^+, L_q(\Omega))} + \|e^{\gamma_0 t}g\|_{L_p(\mathbb{R}^+, W^1_q(\Omega))} \leq \delta_0.$$  

Remark 2.3. We discuss the equations (1.1)-(1.7) in Section 7 below.

Next we introduce the maximal $L_p$-$L_q$ regularity with exponential stability for the following linearized system associated with (1.9)-(1.13):

$$\begin{aligned}
\partial_t u - \text{Div } T(u, p) &= f \quad \text{in } \Omega, \ t > 0, \\
\text{div } u &= g \quad \text{in } \Omega, \ t > 0, \\
T(u, p)e_N &= h \quad \text{on } \Gamma, \ t > 0, \\
u &= 0 \quad \text{on } S, \ t > 0, \\
u|_{t=0} &= a \quad \text{in } \Omega.
\end{aligned}$$  

To this end, following [25], we introduce some function spaces related to the solvability of the divergence equation $\text{div } u = g$ in $\Omega$ with boundary condition $u \cdot (-e_N) = 0$ on $S$. Let $1 < q < \infty$ and $q' = q/(q - 1)$. Noting [2, Lemma 2.3], we can regard $W^1_{q', s}(\Omega)$ as a Banach space with norm $\|\nabla \cdot \|_{L_{q'}(\Omega)}$, which is denoted by $W^1_{q', s}(\Omega)$. Assume that $g \in L_q(\Omega)$, and then one has

$$\|(g, \varphi)\| \leq C\|g\|_{L_q(\Omega)}\|\nabla \varphi\|_{L_{q'}(\Omega)}$$

for any $\varphi \in \hat{W}^1_{q', s}(\Omega)$, with some positive constant $C$ independent of $g$ and $\varphi$. This inequality implies that $g$ is an element of $\hat{W}^{-1}_{q', s}(\Omega)$, where $\hat{W}^{-1}_{q', s}(\Omega)$ is the dual space of $\hat{W}^1_{q', s}(\Omega)$. Here we see $g$ as a functional on $\{\nabla \varphi \mid \varphi \in \hat{W}^1_{q', s}(\Omega)\} \subset L_q(\Omega)^N$, which, combined with Hahn-Banach's theorem, furnishes that there is a $G \in L_q(\Omega)^N$ such that

$$\|g\|_{\hat{W}^{-1}_{q', s}(\Omega)} = \|G\|_{L_q(\Omega)}, \quad (g, \varphi) = -(G, \nabla \varphi) \quad \text{for all } \varphi \in \hat{W}^1_{q', s}(\Omega).$$

Let $|G| = \{G + f \mid f \in J_q(\Omega)\} \in L_q(\Omega)^N / J_q(\Omega)$. Then $L_q(\Omega) \ni g \mapsto |G| \in L_q(\Omega)^N / J_q(\Omega)$ is well-defined, so that we denote $|G|$ by $G(g)$. One especially notes that $\|G(g)\|_{L_q(\Omega)^N / J_q(\Omega)} = \|g\|_{\hat{W}^{-1}_{q', s}(\Omega)}$. Thus, for any $g \in L_q(\Omega)$ and any representative $g$ of $G(g)$ regular enough, we have

$$(\text{div } g, \varphi)_{\Omega} - (g \cdot (-e_N), \varphi)_S = (g, \varphi)_{\Omega} \quad \text{for all } \varphi \in \hat{W}^1_{q', s}(\Omega),$$

which implies that $u = g$ solves the divergence equation mentioned above.

Now we state the main result for (2.3) as follows:
Theorem 2.4. Let $1 < p, q < \infty$ with $2/p + 1/q \neq 1$ and $2/p + 1/q \neq 2$. Then there exists a positive constant $\sigma_0$ such that, for every $f$, $g$, $h$, and $a$ satisfying

$$e^{\sigma_0 t}f \in L_p(R_+, L_q(\Omega))^N, \quad e^{\sigma_0 t}g \in 0 H^{1,1/2}_{q,p}(R_+ \times \Omega) \cap \partial W^1_p(R_+, \hat{W}^{−1}_{q,1}(\Omega)), \quad e^{\sigma_0 t}h \in 0 H^{1,1/2}_{q,p}(R_+ \times \Omega)^N, \quad a \in D_{q,p}(\Omega),$$

the system (2.3) admits a unique solution $(u, p) \in W^{2,1}_{q,p}(\Omega) \times L_p(R_+, W^1_q(\Omega))$ with $\lim_{t \to 0+} \|u(t) - a\|_{B^{1/2-p}_{q,p}(\Omega)} = 0$. In addition, the solution $(u, p)$ satisfies

$$\|e^{\sigma_0 t}(\partial_t u, u, \nabla u, \nabla^2 u)\|_{L_p(R_+, L_q(\Omega))} + \|e^{\sigma_0 t}q\|_{L_p(R_+, W^1_q(\Omega))} \leq c_0 \left(\|e^{\sigma_0 t}f\|_{L_p(R_+, L_q(\Omega))} + \|e^{\sigma_0 t}g\|_{W^2_p(R_+, \hat{W}^{−1}_{q,1}(\Omega))} + \|e^{\sigma_0 t}(g, h)\|_{0 H^{1,1/2}_{q,p}(\Omega) \times R_+} + \|a\|_{D_{q,p}(\Omega)}\right),$$

with some positive constant $c_0 \geq 1$ depending only on $N$, $d$, $p$, $q$, $\mu$, and $\sigma_0$.

3. Preliminaries

This section introduces some results concerning a time-shifted problem for (2.3), a variational problem, and the Helmholtz decomposition on $\Omega$.

3.1. A time-shifted problem. We consider in this subsection the following time-shifted linear system:

$$\begin{aligned}
\partial_t u + 2\delta u - \text{Div} T(u, p) &= f \quad \text{in} \: \Omega, \: t \in R, \\
\text{div} u &= g \quad \text{in} \: \Omega, \: t \in R, \\
T(u,p)e_N &= h \quad \text{on} \: \Gamma, \: t \in R, \\
u &= 0 \quad \text{on} \: S, \: t \in R,
\end{aligned}
$$

(3.1)

where $\delta$ is a positive constant. More precisely, we prove

Proposition 3.1. Let $1 < p, q < \infty$ and $\delta > 0$. Then, for every

$$f \in L_{p,-\delta}(R, L_q(\Omega))^N, \quad g \in H^{1,1/2}_{q,p,-\delta}(\Omega \times R) \cap W^1_{p,-\delta}(R, \hat{W}^{−1}_{q,1}(\Omega)),$$

$$h \in H^{1,1/2}_{q,p,-\delta}(\Omega \times R)^N,$$

the system (3.1) admits a unique solution $(u, p)$ with

$$u \in W^{2,1}_{q,p,-\delta}(\Omega \times R)^N, \quad p \in L_{p,-\delta}(R, W^1_q(\Omega)).$$

In addition, the following assertions hold true.

1. The solution $(u, p)$ satisfies the estimate:

$$\|e^{\delta t}(\partial_t u, u, \nabla u, \nabla^2 u)\|_{L_p(R, L_q(\Omega))} + \|e^{\delta t}q\|_{L_p(R, W^1_q(\Omega))} \leq C \left(\|e^{\delta t}f\|_{L_p(R, L_q(\Omega))} + \|e^{\delta t}(\partial_t g, g)\|_{L_p(R, \hat{W}^{−1}_{q,1}(\Omega))} + \|e^{\delta t}(g, h)\|_{H^{1,1/2}_{q,p}(\Omega \times R)}\right),$$

with some positive constant $C = C_{N,d,p,q,\delta,\mu}$.

2. If $f$, $g$, and $h$ vanish for $t < 0$, then $u$ also vanishes for $t < 0$.

Remark 3.2. Proposition 3.1 (2) implies $u|_{t=0} = 0 \in L_q(\Omega)^N$, provided that $f$, $g$, and $h$ vanish for $t < 0$. 

To prove Proposition 3.1, we start with

\[
\begin{align*}
\partial_t U - \text{Div} \mathbf{T}(U, P) &= F \quad \text{in } \Omega, \ t \in \mathbb{R}, \\
\text{div} \mathbf{U} &= G \quad \text{in } \Omega, \ t \in \mathbb{R}, \\
\mathbf{T}(U, P)e_N &= H \quad \text{on } \Gamma, \ t \in \mathbb{R}, \\
\mathbf{U} &= 0 \quad \text{on } S, \ t \in \mathbb{R}.
\end{align*}
\]

(3.2)

Concerning this system, we have

**Lemma 3.3.** Let \(1 < p, q < \infty\) and \(\gamma > 0\). Then, for every

\[
\begin{align*}
\mathbf{F} &\in L_{p,\gamma}(\mathbb{R}, L_q(\Omega))^N, \\
\mathbf{G} &\in H^{1,1/2}_{q,p,\gamma}(\Omega \times \mathbb{R}) \cap W^{1}_{p,\gamma}(\mathbb{R}, \widetilde{W}^{-1}_{q,1}(\Omega)), \\
\mathbf{H} &\in H^{1,1/2}_{q,p,\gamma}(\Omega \times \mathbb{R})^N,
\end{align*}
\]

the system (3.2) admits a unique solution \((\mathbf{U}, P)\) with

\[
\mathbf{U} \in W^{2,1}_{q,p,\gamma}(\Omega \times \mathbb{R})^N, \quad P \in L_{p,\gamma}(\mathbb{R}, W^1_q(\Omega)).
\]

In addition, the following assertions hold true.

1. The solution \((\mathbf{U}, P)\) satisfies the estimate:

\[
\begin{align*}
&\|e^{-\gamma t}(\partial_t \mathbf{U}, \mathbf{U}, \nabla \mathbf{U}, \nabla \mathbf{U}^2)\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t} P\|_{L_p(\mathbb{R}, W^1_q(\Omega))} \\
&\leq C \left( \|e^{-\gamma t} \mathbf{F}\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t}(\partial_t G, G)\|_{L_p(\mathbb{R}, \widetilde{W}^{-1}_{q,1}(\Omega))} \\
&\quad + \|e^{-\gamma t}(G, \mathbf{H})\|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R})} \right),
\end{align*}
\]

with some positive constant \(C = C_{N, \delta, \rho, \gamma, \mu} \).

2. If \(\mathbf{F}, \mathbf{G},\) and \(\mathbf{H}\) vanish for \(t < 0\), then \(\mathbf{U}\) also vanishes for \(t < 0\).

**Proof.** This lemma was proved in [30, Theorem 2.1]. \(\square\)

**Proof of Proposition 3.1.** In order to apply Lemma 3.3 with \(\gamma = \delta\) to (3.1), we set

\[
\begin{align*}
\mathbf{F} &= e^{2\delta t} \mathbf{F}, \quad G = e^{2\delta t} \mathbf{G}, \\
\mathbf{H} &= e^{2\delta t} \mathbf{H}.
\end{align*}
\]

It then is clear that

\[
\begin{align*}
\mathbf{F} &\in L_{p,\delta}(\mathbb{R}, L_q(\Omega))^N, \quad G \in H^{1,1/2}_{q,p,\delta}(\Omega \times \mathbb{R}) \cap W^{1}_{p,\delta}(\mathbb{R}, \widetilde{W}^{-1}_{q,1}(\Omega)), \\
\mathbf{H} &\in H^{1,1/2}_{q,p,\delta}(\Omega \times \mathbb{R})^N.
\end{align*}
\]

Thus, by Lemma 3.3, we have a solution \((\mathbf{U}, P) \in W^{2,1}_{q,p,\delta}(\Omega \times \mathbb{R})^N \times L_{p,\delta}(\mathbb{R}, W^1_q(\Omega))\) to (3.2), which satisfies

\[
\begin{align*}
&\|e^{-\delta t}(\partial_t \mathbf{U}, \mathbf{U}, \nabla \mathbf{U}, \nabla \mathbf{U}^2)\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\delta t} P\|_{L_p(\mathbb{R}, W^1_q(\Omega))} \\
&\leq C \left( \|e^{\delta t} \mathbf{F}\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{\delta t}(\partial_t \mathbf{G}, \mathbf{G})\|_{L_p(\mathbb{R}, \widetilde{W}^{-1}_{q,1}(\Omega))} + \|e^{\delta t}(\mathbf{G}, \mathbf{H})\|_{L^2(\Omega \times \mathbb{R})} \right)
\]

for some positive constant \(C = C_{N, \delta, \rho, \gamma, \mu}\). Let \((\mathbf{u}, p) = (e^{-2\delta t} \mathbf{U}, e^{-2\delta t} P)\), and then \((\mathbf{u}, p)\) solves the system (3.1) and satisfies the required estimate of Proposition 3.1 (1) by the last inequality. The other assertions immediately follow from Lemma 3.3, which completes the proof of Proposition 3.1. \(\square\)
3.2. A variational problem. Let \( 1 < q < \infty \) and \( q' = q/(q - 1) \). This subsection is concerned with the following variational problem:

\[
(\nabla u, \nabla \varphi)_\Omega = (f, \nabla \varphi)_\Omega \quad \text{for all } \varphi \in W^1_{q',q}(\Omega),
\]

which is the so-called weak Dirichlet-Neumann problem. Our aim in this subsection is to prove

**Proposition 3.4.** Let \( 1 < q < \infty \) and \( q' = q/(q - 1) \). Then, for every \( f \in L_q(\Omega)^N \), the variational problem (3.3) admits a unique solution \( u \in W^1_{q',q}(\Omega) \), and also \( \|u\|_{W^1_{q',q}(\Omega)} \leq C_{N,d,q}\|f\|_{L_q(\Omega)} \) for a positive constant \( C_{N,d,q} \). In this case, we define the solution operator \( Q_q \) from \( L_q(\Omega)^N \) to \( W^1_{q',q}(\Omega) \) by \( Q_q f = u \).

**Proof.** Since \( C^\infty_0(\Omega) \) is dense in \( L_q(\Omega) \), it suffices to consider the case where \( f = (f_1, \ldots, f_N)^T \in C^\infty_0(\Omega)^N \).

We first consider a strong problem associated with (3.3) as follows:

\[
\begin{aligned}
\Delta u &= \text{div } f \quad \text{in } \Omega, \\
0 &= \text{on } \Gamma, \\
\partial_N u &= 0 \quad \text{on } S.
\end{aligned}
\]

For \( g = g(x', x_N) \) defined on \( \mathbb{R}^N_+ \), let \( g^o \) and \( g^e \) be the odd extension of \( g \) and the even extension of \( g \), respectively, i.e.

\[
g^o = \begin{cases} g(x', x_N) & (x_N > 0), \\
-g(x', -x_N) & (x_N < 0),
\end{cases} \quad g^e = \begin{cases} g(x', x_N) & (x_N > 0), \\
-g(x', -x_N) & (x_N < 0).
\end{cases}
\]

Regarding \( f \) as an element of \( C^\infty_0(\mathbb{R}^N_+)^N \), we set

\[
(3.5) \quad F = (F_1, \ldots, F_N)^T = (f_1^e, \ldots, f_N^e, f_N^o)^T.
\]

Let \( \mathcal{F}[g](\xi) \) be the Fourier transform of \( g = g(x) \) and \( \mathcal{F}_\xi^{-1}[h](x) \) the inverse Fourier transform of \( h = h(\xi) \), i.e.

\[
\mathcal{F}[g](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} g(x) \, dx, \quad \mathcal{F}_\xi^{-1}[h](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} h(\xi) \, d\xi.
\]

Now we define a function \( v = v(x) \) on \( \mathbb{R}^N \) by

\[
(3.6) \quad v = -\mathcal{F}_\xi^{-1} \left[ \frac{1}{|\xi|^2} \mathcal{F}[\text{div } F](\xi) \right] (x) = -\sum_{j=1}^N \mathcal{F}_\xi^{-1} \left[ \frac{i\xi_j}{|\xi|^2} \mathcal{F}[F_j](\xi) \right] (x).
\]

It then holds that \( \Delta v = \text{div } F \) in \( \mathbb{R}^N \), which implies that \( \Delta v = \text{div } f \) in \( \Omega \). On the other hand, one has for \( k, l = 1, \ldots, N \)

\[
\partial_k v = \sum_{j=1}^N \mathcal{F}_\xi^{-1} \left[ \frac{\xi_j \xi_k}{|\xi|^2} \mathcal{F}[F_j](\xi) \right] (x), \quad \partial_k \partial_l v = \mathcal{F}_\xi^{-1} \left[ \frac{\xi_k \xi_l}{|\xi|^2} \mathcal{F}[\text{div } F](\xi) \right] (x),
\]

which, combined with the Fourier multiplier theorem of Mikhlin (cf. [27, Appendix Theorem 2]), furnishes that

\[
(3.7) \quad \|\nabla v\|_{L_q(\mathbb{R}^N)} \leq C_{N,d}\|f\|_{L_q(\Omega)}, \quad \|\nabla^2 v\|_{L_q(\mathbb{R}^N)} \leq C_{N,d}\|\nabla f\|_{L_q(\Omega)},
\]

for some positive constant \( C_{N,d} \).
Next we estimate \( \|v\|_{L^2(\Omega)} \). Let \( \hat{g}(\xi', x_N) \) be the partial Fourier transform of \( g = g(x', x_N) \) with respect to \( x' \) and \( \mathcal{F}_{\xi'}^{-1}[h(\xi', x_N)](x') \) the inverse partial Fourier transform of \( h = h(\xi', x_N) \) with respect to \( \xi' \), i.e.,

\[
\hat{g}(\xi', x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix' \cdot \xi'} g(x', x_N) dx',
\]

\[
\mathcal{F}_{\xi'}^{-1}[h(\xi', x_N)](x') = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{ix' \cdot \xi'} h(\xi', x_N) d\xi'.
\]

We then observe that, for \( j = 1, \ldots, N - 1 \),

\[
\mathcal{F}[F_j](\xi) = \int_0^d \left( e^{-iy_N \xi_N} + e^{iy_N \xi_N} \right) \hat{f}_j(\xi', y_N) dy_N,
\]

and that

\[
\mathcal{F}[F_N](\xi) = \int_0^d \left( e^{-iy_N \xi_N} - e^{iy_N \xi_N} \right) \hat{f}_N(\xi', y_N) dy_N.
\]

Inserting these formulas into (3.6) yields

\[
v = v(x', x_N) = -\sum_{j=1}^{N-1} \int_0^d \mathcal{F}_{\xi'}^{-1} \left[ \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(x_N-y_N)\xi_N} + e^{i(x_N+y_N)\xi_N}}{\xi^2} d\xi \right) i\xi_j \hat{f}_j(\xi', y_N) \right] (x') dy_N
\]

\[
- \int_0^d \mathcal{F}_{\xi'}^{-1} \left[ \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\xi_N(e^{i(x_N-y_N)\xi_N} + e^{i(x_N+y_N)\xi_N})}{\xi^2} d\xi \right) \hat{f}_N(\xi', y_N) \right] (x') dy_N.
\]

On the other hand, it holds by the residue theorem that for \( a \in \mathbb{R} \setminus \{0\} \)

\[
\int_{-\infty}^{\infty} e^{ia\xi_N} d\xi_N = \frac{\pi}{|\xi'|}, \quad \int_{-\infty}^{\infty} i\xi_N e^{ia\xi_N} d\xi_N = -\pi e^{-a|\xi'|} \text{sign}(a),
\]

where \( \text{sign}(a) = 1 \) when \( a > 0 \) and \( \text{sign}(a) = -1 \) when \( a < 0 \). We combine (3.8) with the above formula in order to obtain

\[
v = -\frac{1}{2} \sum_{j=1}^{N-1} \int_0^d \mathcal{F}_{\xi'}^{-1} \left[ \frac{i\xi_j}{|\xi'|} \left( e^{-|x_N-y_N|\xi'} + e^{-(x_N+y_N)\xi'} \right) \hat{f}_j(\xi', y_N) \right] (x') dy_N
\]

\[
+ \frac{1}{2} \int_0^d \mathcal{F}_{\xi'}^{-1} \left[ \left( \text{sign}(x_N-y_N)e^{-|x_N-y_N|\xi'} + e^{-(x_N+y_N)\xi'} \right) \hat{f}_N(\xi', y_N) \right] (x') dy_N,
\]

which implies that \( \|v\|_{L^2(\Omega)} \leq C_{N,d,q} \|f\|_{L^2(\Omega)} \) in the same manner as in the proof of [30, pages 1897-1898]. Hence, together with (3.7), one has

\[
\|v\|_{W^{1,q}_0(\Omega)} \leq C_{N,d,q} \|f\|_{L^2(\Omega)}, \quad \|v\|_{W^{1,q}_0(\Omega)} \leq C_{N,d,q} \|f\|_{W^{1,q}_0(\Omega)}.
\]

We here prove that \( \partial_N v = 0 \) on \( \mathbb{R}_0^N \). Noting \( \text{div} F = (\text{div} f)^* \) by the definition (3.5) and setting \( z = \text{div} f \), we have

\[
\mathcal{F}[\text{div} F](\xi) = \mathcal{F}[z^*](\xi) = \int_0^d \left( e^{-iy_N \xi_N} + e^{iy_N \xi_N} \right) \bar{z}(\xi', y_N) dy_N.
\]

By this formula and (3.6), one obtains

\[
\partial_N v = (\partial_N v)(x', x_N) =
\]

\[
- \int_0^d \mathcal{F}_{\xi'}^{-1} \left[ \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i\xi_N(e^{i(x_N-y_N)\xi_N} + e^{i(x_N+y_N)\xi_N})}{\xi^2} d\xi \right) \bar{z}(\xi', y_N) \right] (x') dy_N,
\]
which, combined with (3.8), furnishes \((\partial_N v)(x',0) = 0\). Let \(u = v + w\) in (3.4), and thus (3.4) is reduced to

\[
\begin{aligned}
\Delta w &= 0 \quad \text{in } \Omega, \\
w &= -v \quad \text{on } \Gamma, \\
\partial_N w &= 0 \quad \text{on } S.
\end{aligned}
\]

(3.10)

From now on, we solve the system (3.10). Applying the partial Fourier transform to (3.10) yields

\[
\begin{aligned}
(\partial_N^2 - |\xi'|^2)\hat{w}(\xi',x_N) &= 0 \quad (0 < x_N < d), \\
\hat{w}(\xi',d) &= -\hat{v}(\xi',d), \\
\partial_N \hat{w}(\xi',0) &= 0.
\end{aligned}
\]

One then solves this system as ordinary differential equations with respect to \(x_N\) in order to obtain

\[
w = w(x',x_N) = -\sum_{k=1}^{2} \mathcal{F}^{-1}_{\xi'} \left[ e^{-\frac{|\xi'|d}{1 + e^{-2|\xi'|d}}} \hat{v}(\xi',y_N) \right] (x').
\]

Let \(\varphi = \varphi(s)\) be a function in \(C^\infty(\mathbb{R})\) such that \(0 \leq \varphi \leq 1\) and

\[
\varphi(s) = \begin{cases} 
1 & \text{for } s \geq 2d/3, \\
0 & \text{for } s \leq d/3,
\end{cases}
\]

and note that for functions \(f(x_N), g(x_N)\) and for \(k = 1, 2\)

\[
f(x_N)g(d) = \int_{0}^{d} \frac{d}{dy_N} \left\{ \varphi(y_N) f(x_N + (-1)^k(y_N - d))g(y_N) \right\} dy_N.
\]

Combining these identities with the above formula of \(w\), we obtain

\[
w = -\sum_{k=1}^{2} \int_{0}^{d} \mathcal{F}^{-1}_{\xi'} \left[ \varphi(y_N) e^{-\frac{\xi' d}{1 + e^{-2\xi'|d}}} \hat{v}(\xi',y_N) \right] (x') dy_N
\]

\[
+ \sum_{k=1}^{2} \int_{0}^{d} \mathcal{F}^{-1}_{\xi'} \left[ \varphi(y_N) \xi' e^{-\frac{\xi' (y_N + (-1)^k x_N)}{1 + e^{-2\xi'|d}}} \hat{v}(\xi',y_N) \right] (x') dy_N
\]

\[- \sum_{k=1}^{2} \int_{0}^{d} \mathcal{F}^{-1}_{\xi'} \left[ \varphi(y_N) e^{-\frac{\xi' (y_N + (-1)^k x_N)}{1 + e^{-2\xi'|d}}} \partial_N \hat{v}(\xi',y_N) \right] (x') dy_N
\]

\[=: I_1 + I_2 + I_3,
\]

where \(\varphi(y_N) = (d\varphi/dy_N)(y_N)\). Noting \(|\xi'| = |\xi'|^2/|\xi'| = -\sum_{j=1}^{N-1} (i\xi_j)^2/|\xi'|\), we can write \(I_2\) as

\[
I_2 = -\sum_{j=1}^{N-1} \sum_{k=1}^{2} \int_{0}^{d} \mathcal{F}^{-1}_{\xi'} \left[ \varphi(y_N) i\xi_j e^{-\frac{\xi' (y_N + (-1)^k x_N)}{1 + e^{-2\xi'|d}}} \partial_j \hat{v}(\xi',y_N) \right] (x') dy_N.
\]

The following lemma was essentially proved in Lemma [30, Lemma 5.5].

**Lemma 3.5.** Let \(1 < q < \infty\). Assume that \(m(\xi')\) satisfies

\[
|\partial_{\xi'}^\alpha m(\xi')| \leq C_\alpha |\xi'|^{-|\alpha'|} \quad (\xi' \in \mathbb{R}^{N-1} \setminus \{0\})
\]
for any multi-index \( \alpha' \in \mathbb{N}_0^{N-1} \), and set for \( k = 1, 2 \)
\[
[L_k f](x) = \int_0^d \mathcal{F}_q^{-1} \left[ \hat{f}(y_N) m(\xi') e^{-|\xi'|((y_N + (-1)^k x_N))} \right] (x') dy_N,
\]
\[
[M_k f](x) = \int_0^d \mathcal{F}_q^{-1} \left[ \hat{f}(y_N) m(\xi') e^{-|\xi'|((y_N + (-1)^k x_N))} \right] (x') dy_N.
\]

Then, for any \( f \in L_q(\Omega) \), we have
\[
\| (L_k f, M_k f) \|_{W^2_q(\Omega)} \leq C_{N,d,q} \| f \|_{L_q(\Omega)} \quad (k = 1, 2).
\]

It is known by e.g. [40, Section 5] that for any multi-index \( \alpha' \in \mathbb{N}_0^{N-1} \)
\[
| \partial_{\xi'}^\alpha (i \xi | \xi'|^{-1}) | + | \partial_{\xi'}^\alpha e^{-a|\xi'|} | \leq C_{\alpha'} |\xi'|^{-|\alpha'|} \quad (\xi' \in \mathbb{R}^{N-1} \setminus \{0\}),
\]
where \( j = 1, \ldots, N-1 \) and \( a \geq 0 \), with some positive constant \( C_{\alpha'} \) independent of \( \xi' \) and \( a \). In order to estimate \( (1 + e^{-2|\xi'|^d})^{-1} \), we introduce Bell’s formula for derivatives of the composite function of \( f(t) \) and \( t = g(\xi') \) as follows: For any multi-index \( \alpha' \in \mathbb{N}_0^{N-1} \),
\[
\partial_{\xi'}^\alpha f(g(\xi')) = \sum_{l=1}^{\alpha'} f^{(l)}(g(\xi')) \sum_{\alpha'_1 + \cdots + \alpha'_l = \alpha'} \Gamma_{\alpha'_1, \ldots, \alpha'_l}^\alpha \partial_{\xi'}^{\alpha'_1} g(\xi') \cdots \partial_{\xi'}^{\alpha'_l} g(\xi'),
\]
with suitable coefficients \( \Gamma_{\alpha'_1, \ldots, \alpha'_l}^\alpha \), where \( f^{(l)}(t) \) is the \( l \)th derivative of \( f(t) \). By Bell’s formula with \( f(t) = t^{-1} \) and \( t = g(\xi') = 1 + e^{-2|\xi'|^d} \) and by (3.11),
\[
| \partial_{\xi'}^\alpha \left(1 + e^{-2|\xi'|^d} \right)^{-1} | \leq C_{\alpha'} \sum_{l=1}^{\alpha'} |1 + e^{-2|\xi'|^d} |^{-l+1} |\xi'|^{-|\alpha'|} \leq C_{\alpha'} |\xi'|^{-|\alpha'|}
\]
for any \( \xi' \in \mathbb{R}^{N-1} \setminus \{0\} \) and any multi-index \( \alpha' \in \mathbb{N}_0^{N-1} \). One thus obtains \( \| (I_1, I_2) \|_{W^1_q(\Omega)} \leq C_{N,d,q} \| v \|_{W^1_q(\Omega)} \) by Lemma 3.5 and (3.12). In addition, it holds by Leibniz’s rule, (3.11), and (3.12) that
\[
| \partial_{\xi'}^\alpha \left( i \xi_j | \xi'| \left(1 + e^{-2|\xi'|^d} \right) \right) | \leq C_{\alpha'} |\xi'|^{-|\alpha'|} \quad (\xi' \in \mathbb{R}^{N-1} \setminus \{0\}),
\]
which, combined with Lemma 3.5, furnishes \( \| I_2 \|_{W^1_q(\Omega)} \leq C_{N,d,q} \| v \|_{W^1_q(\Omega)} \). Analogously, we can prove that \( \| \nabla (I_1, I_2, I_3) \|_{W^1_q(\Omega)} \leq C_{N,d,q} \| \nabla v \|_{W^1_q(\Omega)} \). Summing up these inequalities for \( I_1, I_2, \) and \( I_3, \) one has by (3.9)
\[
\| u \|_{W^1_q(\Omega)} \leq C_{N,d,q} \| f \|_{L_q(\Omega)}, \quad \| w \|_{W^2_q(\Omega)} \leq C_{N,d,q} \| f \|_{L_q(\Omega)}.
\]

It now holds that \( u = v + w \) solves (3.4) and satisfies \( \| u \|_{W^1_q(\Omega)} \leq C_{N,d,q} \| f \|_{L_q(\Omega)} \) by (3.9) and (3.13). Clearly, such an \( u \) is a solution to (3.3).

Finally, we prove the uniqueness of solutions to (3.3). Let \( u \in W^1_{q,1}(\Omega) \) be a solution to (3.3) with \( f = 0 \), and let \( \Phi \in C_0^\infty(\Omega)^N \). Since \( \Phi \in L_q(\Omega)^N \), there is a \( v \in W^1_{q,1}(\Omega) \) such that
\[
(\nabla v, \nabla \psi)_{\Omega} = (\Phi, \nabla \psi)_{\Omega} \quad \text{for all} \ \psi \in W^1_{q,1}(\Omega).
\]
In this equation, we set \( \psi = u \) in order to obtain
\[
(\Phi, \nabla u)_{\Omega} = (\nabla v, \nabla u)_{\Omega} = 0,
\]
which implies that $u$ is a constant. Hence, we have $u = 0$, because $u = 0$ on $\Gamma$. This completes the proof of the Proposition 3.4. \hfill \Box

3.3. Helmholtz decomposition on $\Omega$. We here introduce the Helmholtz decomposition on $\Omega$. Let $G_q(\Omega) = \{\nabla \theta \mid \theta \in W_{q,1}^1(\Omega)\}$, and then one has

Proposition 3.6. Let $1 < q < \infty$. Then the following assertions hold true.

1. $L_q(\Omega)^N = J_q(\Omega) \oplus G_q(\Omega)$.
2. Let $P_q$ be the projection from $L_q(\Omega)^N$ to $J_q(\Omega)$, and let $Q_q$ be the solution operator of Proposition 3.4 from $L_q(\Omega)^N$ to $W_{q,1}^1(\Omega)$. Then, for any $f \in L_q(\Omega)^N$, we have $f = P_q f + \nabla Q_q f \in J_q(\Omega) \oplus G_q(\Omega)$, and

\begin{equation}
\|P_q f\|_{L_q(\Omega)} + \|Q_q f\|_{W_{q,1}^1(\Omega)} \leq C_{N,d,q} \|f\|_{L_q(\Omega)}
\end{equation}

for some positive constant $C_{N,d,q}$.

Proof. (1). Let $f \in L_q(\Omega)^N$. By Proposition 3.4, we have a unique solution $u = Q_q f \in W_{q,1}^1(\Omega)$ to the variational problem:

$$(\nabla u, \nabla \varphi)_{\Omega} = (f, \nabla \varphi)_{\Omega} \quad \text{for all } \varphi \in W_{q',1}^1(\Omega),$$

where $q' = q/(q-1)$. Setting $g = f - \nabla Q_q f$, we observe that $g \in J_q(\Omega)$ and $f = g + \nabla Q_q f \in J_q(\Omega) + G_q(\Omega)$.

Next we prove that $L_q(\Omega)^N = J_q(\Omega) \oplus G_q(\Omega)$. To this end, let $f \in J_q(\Omega) \cap G_q(\Omega)$. We then have

$$(\nabla Q_q f, \nabla \varphi)_{\Omega} = (f, \nabla \varphi)_{\Omega} = 0 \quad \text{for all } \varphi \in W_{q',1}^1(\Omega),$$

which, combined with the uniqueness of Proposition 3.4, furnishes $Q_q f = 0$. On the other hand, since there is a $\theta \in W_{q',1}^1(\Omega)$ such that $f = \nabla \theta$, we have

$$(\nabla Q_q f, \nabla \varphi)_{\Omega} = (f, \nabla \varphi)_{\Omega} = (\nabla \theta, \nabla \varphi)_{\Omega} \quad \text{for all } \varphi \in W_{q',1}^1(\Omega).$$

By the uniqueness of Proposition 3.4 again, it holds that $\theta = Q_q f = 0$. Thus $f = 0$, which implies $L_q(\Omega)^N = J_q(\Omega) \oplus G_q(\Omega)$.

(2). For any $f \in L_q(\Omega)^N$, the projection $P_q$ is given by $P_q f = f - \nabla Q_q f \in J_q(\Omega)$ as was seen in the proof of (1). Thus the first assertion clearly holds true, and also the second assertion (3.14) follows from Proposition 3.4 immediately. This completes the proof of Proposition 3.6. \hfill \Box

4. Generation of the Stokes semigroup

Our aim in this section is to construct an analytic $C_0$-semigroup associated with

\begin{equation}
\begin{aligned}
\partial_t u - \text{Div} \, T(u, p) &= 0 \quad \text{in } \Omega, \ t > 0, \\
\text{div} \, u &= 0 \quad \text{in } \Omega, \ t > 0, \\
T(u, p) e_N &= 0 \quad \text{on } \Gamma, \ t > 0, \\
\mathbf{u} &= 0 \quad \text{on } S, \ t > 0, \\
\mathbf{u}|_{t=0} &= \mathbf{a} \quad \text{in } \Omega.
\end{aligned}
\end{equation}

(4.1)

To this end, we start with the following resolvent problem:

\begin{equation}
\begin{aligned}
\lambda v - \text{Div} \, T(v, q) &= f \quad \text{in } \Omega, \\
\text{div} \, v &= 0 \quad \text{in } \Omega, \\
T(v, q) e_N &= 0 \quad \text{on } \Gamma, \\
v &= 0 \quad \text{on } S,
\end{aligned}
\end{equation}

(4.2)
with the resolvent parameter $\lambda \in \Sigma_\varepsilon = \{ \omega \in \mathbb{C} \setminus \{0\} \mid |\arg \omega| < \pi - \varepsilon \}$ for $0 < \varepsilon < \pi/2$. The following lemma was proved in [2, Theorem 1.1].

**Lemma 4.1.** Let $f, \omega \in \mathbb{C}$ such that, for every $\lambda \in \Sigma_\varepsilon \cup \{ \omega \in \mathbb{C} \mid |\omega| < \omega_1 \}$ and $f \in L_q(\Omega)^N$, there is a unique solution $\tilde{q} \in W^1_\eta(\Omega)$ to (4.2). In addition,

$$\|\lambda\|_{L_q(\Omega)} + |\lambda|^{1/2} \|\nabla \tilde{q}\|_{L_q(\Omega)} + \|\tilde{q}\|_{W^1_\eta(\Omega)} \leq C \|f\|_{L_q(\Omega)}$$

for any $\lambda \in \Sigma_\varepsilon \cup \{ \omega \in \mathbb{C} \mid |\omega| < \omega_1 \}$ with a positive constant $C = C_{N,d,q,\varepsilon,\mu,\omega_1}$.

Let $1 < q < \infty$ and $q' = q/(q-1)$. By Proposition 3.4, we see that, for any $f \in L_q(\Omega)^N$ and $q \in W^{1-1/q}(\Gamma)$, there exists a unique solution $\tilde{q} \in W^1_{q',r}(\Omega)$ to the variational problem:

$$(\nabla \tilde{q}, \varphi)_{\Omega} = (f - \nabla \tilde{g}, \nabla \varphi)_{\Omega} \quad \text{for all } \varphi \in W^1_{q',r}(\Omega),$$

where $\tilde{g}$ is an extension of $g$ satisfying $\|\tilde{g}\|_{W^1_\eta(\Omega)} \leq C \|g\|_{W^{1-1/q}(\Gamma)}$ for some positive constant $C$ independent of $g, \tilde{g}$. Furthermore, the solution $\tilde{q}$ satisfies

$$\|\tilde{q}\|_{W^1_\eta(\Omega)} \leq C_{N,d,q}(\|f\|_{L_q(\Omega)} + \|\nabla \tilde{g}\|_{L_q(\Omega)})$$

$$\leq C_{N,d,q}(\|f\|_{L_q(\Omega)} + \|g\|_{W^{1-1/q}(\Gamma)}).$$

Setting $q = \tilde{q} + \tilde{g} \in W^1_{q',r}(\Omega) + W^1_{q'}(\Omega)$ yields

$$(\nabla q, \nabla \varphi)_{\Omega} = (f, \nabla \varphi)_{\Omega} \quad \text{for all } \varphi \in W^1_{q',r}(\Omega), \quad q = g \quad \text{on } \Gamma,$$

and also $\|q\|_{W^1_\eta(\Omega)} \leq C_{N,d,q}(\|f\|_{L_q(\Omega)} + \|g\|_{W^{1-1/q}(\Gamma)})$. From this viewpoint, we define an operator $K : W^1_{q'}(\Omega)^N \ni v \mapsto K(v) \in W^1_{q',r}(\Omega) + W^1_{q'}(\Omega)$ as follows:

$$(\nabla K(v), \nabla \varphi)_{\Omega} = (\text{Div}(\mu D(v)) - \nabla \text{div } v, \nabla \varphi)_{\Omega} \quad \text{for all } \varphi \in W^1_{q',r}(\Omega),$$

$$K(v) = e_N \cdot (\mu D(v)e_N) - \text{div } v \quad \text{on } \Gamma.$$

Then $K(v)$ satisfies $\|K(v)\|_{W^1_\eta(\Omega)} \leq C_{N,d,q,\mu}\|v\|_{W^2_\eta(\Omega)}$ with some positive constant $C_{N,d,q,\mu}$ independent of $v$ and $\varphi$.

At this point, we introduce a result concerning the weak Dirichlet-Neumann problem with resolvent parameter $\lambda$.

**Proposition 4.2.** Let $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Then there exists a positive number $\omega_2$ such that, for every $\lambda \in \Sigma_\varepsilon \cup \{ \omega \in \mathbb{C} \mid |\omega| < \omega_2 \}$ and $f \in L_q(\Omega)^N$, there is a unique solution $u \in W^1_{q',r}(\Omega)$ to the variational problem:

$$(\lambda u, \varphi)_{\Omega} + (\nabla u, \nabla \varphi)_{\Omega} = (f, \nabla \varphi)_{\Omega} \quad \text{for all } \varphi \in W^1_{q',r}(\Omega),$$

where $q' = q/(q-1)$.

**Proof.** The case $\lambda = 0$ was already proved in Proposition 3.4. Then, by a small perturbation method, we can prove that there exists a positive constant $\omega_2$ such that, for every $\lambda \in \{ \omega \in \mathbb{C} \mid |\omega| < \omega_2 \}$ and $f \in L_q(\Omega)^N$, (4.3) admits a unique solution $u \in W^1_{q',r}(\Omega)$. In the case $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \geq \omega_2/2$, we consider the strong problem with resolvent parameter $\lambda$ for (3.4). One can construct solutions to the strong problem in a similar way to Proposition 3.4 (cf. also [30]). The uniqueness follows from the existence of solutions for a dual problem as was seen in the proof of Proposition 3.4. This completes the proof of the proposition. \qed
We now consider the reduced resolvent problem:

\[
\begin{aligned}
\lambda v - \text{Div} (v, K(v)) &= f & \text{in } \Omega, \\
T(v, K(v))e_N &= 0 & \text{on } \Gamma, \\
v &= 0 & \text{on } S.
\end{aligned}
\]

Then the following proposition holds.

**Proposition 4.3.** Let \(1 < q < \infty\) and \(0 < \varepsilon < \pi/2\). Assume that \(\omega_1\) and \(\omega_2\) are, respectively, the same positive constants as in Lemma 4.1 and Proposition 4.2, and set \(\omega_0 = \min(\omega_1, \omega_2)\). Then (4.2) is equivalent to (4.4) for every \(\lambda \in \Sigma_\varepsilon \cup \{ \omega \in \mathbb{C} \mid |\omega| < \omega_0 \}\) and \(f \in \mathcal{J}_q(\Omega)\), which means that the following assertions hold true:

\[(v, q) = (v, K(v)) \in W^2_q(\Omega)^N \times W^1_q(\Omega)\]

is a unique solution to (4.2) if \(v \in W^2_q(\Omega)^N\)

is a solution to (4.4), and conversely, \(v \in W^2_q(\Omega)^N\) is a unique solution to (4.4) if \((v, q) \in W^2_q(\Omega)^N \times W^1_q(\Omega)\) is a solution to (4.2).

**Proof.** Suppose that \(v \in W^2_q(\Omega)^N\) is a solution to (4.4). Let \(\varphi \in W^1_{q', \Gamma}(\Omega)\) with \(1/q + 1/q' = 1\). We then see, by the definition of \(K\), that \(\text{div } v = 0\) on \(\Gamma\) and

\[
0 = -(f, \nabla \varphi)_\Omega = -(\lambda v - \text{Div} (v, K(v)), \nabla \varphi)_\Omega = -(\lambda v - \nabla \text{div } v, \nabla \varphi)_\Omega
\]

\[
= (\lambda \text{div } v, \varphi)_\Omega + (\nabla \text{div } v, \nabla \varphi)_\Omega.
\]

Hence, \(\text{div } v = 0\) by Proposition 4.2 when \(\lambda \in \Sigma_\varepsilon \cup \{ \omega \in \mathbb{C} \mid |\omega| < \omega_0 \}\), and therefore setting \(q = K(v)\) implies that \((v, q) \in W^2_q(\Omega)^N \times W^1_q(\Omega)\) solves (4.2).

The uniqueness of solutions to (4.2) follows from Lemma 4.1.

Next we show the opposite direction. Suppose that \((v, q) \in W^2_q(\Omega)^N \times W^1_q(\Omega)\) is a solution to (4.2). Let \(\varphi \in W^1_{q', \Gamma}(\Omega)\), and then we see, by the definition of \(K\) and \(\text{div } v = 0\) in \(\Omega\), that

\[
0 = (f, \varphi)_\Omega = (\nabla q - \text{Div}(\mu D(v)), \varphi)_\Omega = (\nabla (q - K(v)), \varphi)_\Omega,
\]

\[
0 = q - K(v) \quad \text{on } \Gamma,
\]

where we have used \((\lambda v, \nabla \varphi)_\Omega = 0\) by \(\text{div } v = 0\) in \(\Omega\) and \(v = 0\) on \(S\). Combining these two equations with the uniqueness of Proposition 3.4 implies \(q = K(v)\). Thus \(v\) is a solution to (4.4). The uniqueness of solutions to (4.4) follows from the first half of this proof, which completes the proof of the proposition. \(\square\)

In view of (4.4), we set the Stokes operator \(A_q\) as \(A_q v = \text{Div} (v, K(v))\) with the domain \(D(A_q)\):

\[
\begin{aligned}
D(A_q) &= W^2_{q,B}(\Omega) \cap J_q(\Omega), \\
W^2_{q,B}(\Omega) &= \{ v \in W^2_q(\Omega)^N \mid (\mu D(v)e_N)_{|\tau} = 0 \text{ on } \Gamma, \ v = 0 \text{ on } S \}.
\end{aligned}
\]

The system (4.4) then can be written as \(\lambda v - A_q v = f\), and one has

**Lemma 4.4.** Let \(1 < q < \infty\) and \(0 < \varepsilon < \pi/2\). Suppose that \(\omega_0\) is the same positive constant as in Proposition 4.3. Then there exists a positive constant \(C = C_{N,d,q,r,\varepsilon,\mu,\omega_1,\omega_2}\) such that, for every \(\lambda \in \Sigma_\varepsilon \cup \{ \omega \in \mathbb{C} \mid |\omega| < \omega_0 \}\),

\[
\| (\lambda - A_q)^{-1} \|_{\mathcal{L}(J_q(\Omega))} \leq \frac{C}{1 + |\lambda|}.
\]

In addition, \(A_q\) is a densely defined closed operator on \(J_q(\Omega)\).
Proof. The required estimate follows from Lemma 4.1 and Proposition 4.3.

Let \( v \in D(A_q) \). Then \((\operatorname{div} v, \varphi)_{\Omega} = -(v, \nabla \varphi)_{\Omega} = 0 \) for any \( \varphi \in C_0^\infty(\Omega) \), because \( v \in J_q(\Omega) \) and \( C_0^\infty(\Omega) \subset W^{1,q}_{0}(\Omega) \) with \( q' = q/(q-1) \). This implies \( \operatorname{div} v = 0 \) in \( \Omega \). It thus holds by the definition of \( K \) that

\[
(A_q v, \nabla \varphi)_{\Omega} = (\nabla \operatorname{div} v, \nabla \varphi)_{\Omega} = 0 \quad \text{for all } \varphi \in W^{1,q}_{0}(\Omega),
\]

which furnishes \( A_q v \in J_q(\Omega) \). Then, following the proof of [39, Lemma 3.7], we can show the last assertion of Lemma 4.4. This completes the proof of the lemma. \( \square \)

The following proposition follows from Lemma 4.4 and the standard theory of analytic \( C_0 \)-semigroups.

**Proposition 4.5.** Let \( 1 < q < \infty \). Then \( A_q \) generates an analytic \( C_0 \)-semigroup \( \{e^{A_qt}\}_{t \geq 0} \) on \( J_q(\Omega) \). Furthermore, there exist positive constants \( \sigma_0 \) and \( C = C_{N,d,p,q,\mu,\sigma_0} \) such that for any \( t > 0 \)

\[
\begin{align*}
\|e^{A_qt}a\|_{J_q(\Omega)} &\leq Ce^{-2\sigma_0 t}\|a\|_{J_q(\Omega)} \quad (a \in J_q(\Omega)), \\
\|\partial_t e^{A_qt}a\|_{J_q(\Omega)} &\leq Ct^{-1}e^{-2\sigma_0 t}\|a\|_{J_q(\Omega)} \quad (a \in J_q(\Omega)), \\
\|\partial_t e^{A_qt}a\|_{J_q(\Omega)} &\leq Ce^{-2\sigma_0 t}\|a\|_{D(A_q)} \quad (a \in D(A_q)),
\end{align*}
\]

where \( \|a\|_{D(A_q)} = \|a\|_{J_q(\Omega)} + \|A_qa\|_{J_q(\Omega)} \).

Recall \( D_{q,p}(\Omega) = (J_{q}(\Omega), D(A_q))_{\frac{1}{p}-1/p, \frac{1}{p}} \) for \( 1 < p, q < \infty \). We have a corollary of Proposition 4.5 that can be proved in the same manner as in [39, Theorem 3.9].

**Corollary 4.6.** Let \( 1 < p, q < \infty \) and \( \sigma_0 \) be the same positive constant as in Proposition 4.5. Then, for every \( a \in D_{q,p}(\Omega) \), \( (u,p) = (e^{A_qt}a, K(e^{A_qt}a)) \) is a unique solution to (4.1), and also

\[
\|e^{\sigma_0 t}(\partial_t u, u, \nabla u, \nabla^2 u)\|_{L_p(\mathbb{R} \times L_q(\Omega))} + \|e^{\sigma_0 t}p\|_{L_p(\mathbb{R} \times W^{1,q}_{0}(\Omega))} \leq C\|a\|_{D_{q,p}(\Omega)}
\]

for a positive constant \( C = C_{N,d,p,q,\mu,\sigma_0} \).

**Remark 4.7.** By [24, Propositions 2.22, 2.28], it holds that \( \lim_{t \to 0+}\|e^{A_qt}a - a\|_{D_{q,p}(\Omega)} = 0 \) for any \( a \in D_{q,p}(\Omega) \). Then, under the conditions \( 2/p + 1/q \neq 1 \) and \( 2/p + 1/q \neq 2 \), we observe by Remark 2.1 that \( \lim_{t \to 0+}\|e^{A_qt}a - a\|_{B^{2-2/q}_q(x^2, \Omega)} = 0 \) for any \( a \in D_{q,p}(\Omega) \).

5. Proof of Theorem 2.4

This section proves Theorem 2.4. Assume that \( \sigma_0 \) is the same positive constant as in Proposition 4.5 in what follows.

**Step 1.** The aim of this step is to decompose \( (u,p) \) of (2.3). Let

\[
u = u_1 + u_2 + \bar{u}, \quad p = p_1 + p_2 + \bar{p},
\]

where each term on the right-hand sides satisfies the following systems:

\[
\begin{align*}
\begin{cases}
\partial_t u_1 - \operatorname{Div} T(u_1, p_1) = 0 \quad \text{in } \Omega, \quad t > 0, \\
\operatorname{div} u_1 = 0 \quad \text{in } \Omega, \quad t > 0,
\end{cases}
\end{align*}
\]

(5.1)

\[
\begin{align*}
T(u_1, p_1)e_N = 0 \quad \text{on } \Gamma, \quad t > 0, \\
u_1 = 0 \quad \text{on } S, \quad t > 0, \\
u_1|_{t=0} = a \quad \text{in } \Omega,
\end{align*}
\]
\[
\begin{align*}
\partial_t u_2 + 2\sigma_0 u_2 - \text{Div} T(u_2, p_2) &= 0 \quad \text{in } \Omega, \ t > 0, \\
\text{div } u_2 &= g \quad \text{in } \Omega, \ t > 0, \\
T(u_2, p_2)e_N &= h \quad \text{on } \Gamma, \ t > 0, \\
u_2 &= 0 \quad \text{on } S, \ t > 0, \\
u_2|_{t=0} &= 0 \quad \text{in } \Omega,
\end{align*}
\]  
(5.2)

Where we have used the fact that \( P_q \) and \( Q_q \) be the operators studied in Subsection 3.3. We then have

\[ f + 2\sigma_0 u_2 = P_q(f + 2\sigma_0 u_2) + \nabla Q_q(f + 2\sigma_0 u_2), \]

which gives further decompositions of \((\tilde{u}, \tilde{p})\) as follows:

\[ \tilde{u} = u_3 + u_4, \quad \tilde{p} = p_3 + p_4, \]

where

\[
\begin{align*}
\partial_t u_3 + 2\sigma_0 u_3 - \text{Div} T(u_3, p_3) &= \nabla Q_q(f + 2\sigma_0 u_2) \quad \text{in } \Omega, \ t > 0, \\
\text{div } u_3 &= 0 \quad \text{in } \Omega, \ t > 0, \\
T(u_3, p_3)e_N &= 0 \quad \text{on } \Gamma, \ t > 0, \\
u_3 &= 0 \quad \text{on } S, \ t > 0, \\
u_3|_{t=0} &= 0 \quad \text{in } \Omega,
\end{align*}
\]  
(5.3)

\[
\begin{align*}
\partial_t u_4 - \text{Div} T(u_4, p_4) &= P_q(f + 2\sigma_0 u_2) + 2\sigma_0 u_3 \quad \text{in } \Omega, \ t > 0, \\
\text{div } u_4 &= 0 \quad \text{in } \Omega, \ t > 0, \\
T(u_4, p_4)e_N &= 0 \quad \text{on } \Gamma, \ t > 0, \\
u_4 &= 0 \quad \text{on } S, \ t > 0, \\
u_4|_{t=0} &= 0 \quad \text{in } \Omega.
\end{align*}
\]  
(5.4)

By Corollary 4.6 and Remark 4.7, there exists a solution \((u_1, p_1)\) of (5.1) such that \( \lim_{t \to +0} \|u_1(t) - a\|_{B^{2-p}_{q,p}((\Omega))} = 0 \) and

\[
\begin{align*}
\|e^{\sigma t}(\partial_t u_1, u_1, \nabla u_1, \nabla^2 u_1)\|_{L_p(\Omega)} + \|e^{\sigma t} p_1\|_{L_p(\Omega)} &\leq C_{N,d,p,q,\sigma,0}\|a\|_{L^{q,p}(\Omega)}. \\
\end{align*}
\]  
(5.5)

**Step 2.** We consider (5.2) in this step. To this end, one extends \( g \) and \( h \) to functions defined on the whole line with respect to time \( t \). Let \( E_0 \) be the zero extension operator and \( X \) be a Banach space. Then,

\[
E_0 \in L(L_p(\mathbb{R}^+, X), L_p(\mathbb{R}, X)) \cap L(0W^1_p(\mathbb{R}^+, X), W^1_p(\mathbb{R}, X)).
\]  
(6.6)

Especially, by (6.6) with \( X = L_q(\Omega) \) and the complex interpolation method,

\[
E_0 \in L(0H^{1/2}_p(\mathbb{R}^+, L_q(\Omega)), H^{1/2}_p(\mathbb{R}, L_q(\Omega))),
\]  
(5.7)

where we have used the fact that

\[ [L_p(\mathbb{R}, L_q(\Omega)), W^1_p(\mathbb{R}, L_q(\Omega))]_{1/2} = H^{1/2}_p(\mathbb{R}, L_q(\Omega)). \]
(cf. e.g. [16, Theorem 1.56]). By (5.6) and (5.7), one has
\[ E_0(e^{\sigma_0 t}g) \in H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}) \cap W^{1,1}_{p,0}(\mathbb{R}, \mathbb{W}^{-1,q}_{1,0}(\Omega)), \quad E_0(e^{\sigma_0 t}h) \in H^{1,1/2}_{q,p}(\Omega \times \mathbb{R})^N, \]

together with the estimates:
\[
\| E_0(e^{\sigma_0 t}g) \|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R})} \leq C_{p,q} \| e^{\sigma_0 t}g \|_{0,H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}^+)},
\]
\[
\| E_0(e^{\sigma_0 t}g) \|_{W^{1,1}_{p,0}(\mathbb{R}, \mathbb{W}^{-1,q}_{1,0}(\Omega))} \leq C_{p,q} \| e^{\sigma_0 t}g \|_{W^{1,1}_{p,0}(\mathbb{R}, \mathbb{W}^{-1,q}_{1,0}(\Omega))},
\]
\[
\| E_0(e^{\sigma_0 t}h) \|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R})} \leq C_{p,q} \| e^{\sigma_0 t}h \|_{0,H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}^+)}.\]

Thus, setting \( G = e^{-\sigma_0 t}E_0(e^{\sigma_0 t}g) \) and \( H = e^{-\sigma_0 t}E_0(e^{\sigma_0 t}h) \) yields
\[
G \in H^{1,1/2}_{q,p,-\sigma_0}(\Omega \times \mathbb{R}) \cap W^{1,1}_{p,-\sigma_0}(\mathbb{R}, \mathbb{W}^{-1,q}_{1,0}(\Omega)), \quad H \in H^{1,1/2}_{q,p,-\sigma_0}(\Omega \times \mathbb{R})^N,
\]

and also
\[
\| e^{\sigma_0 t}G \|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R})} \leq C_{p,q} \| e^{\sigma_0 t}g \|_{0,H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}^+)},
\]
\[
\| e^{\sigma_0 t}(\partial_t G, G) \|_{L^p(\mathbb{R}, \mathbb{W}^{-1,q}_{1,0}(\Omega))} \leq C_{p,q} \| e^{\sigma_0 t}g \|_{W^{1,1}_{p,0}(\mathbb{R}, \mathbb{W}^{-1,q}_{1,0}(\Omega))},
\]
\[
\| e^{\sigma_0 t}H \|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R})} \leq C_{p,q} \| e^{\sigma_0 t}h \|_{0,H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}^+)}.\]

Combining these properties with Proposition 3.1 for \( \delta = \sigma_0 \) furnishes that there is a solution \((U, P) \in W^{1,1}_{q,p,-\sigma_0}(\Omega \times \mathbb{R})^N \times L^p_{\sigma_0}(\mathbb{R}, W^1_q(\Omega))\) to
\[
\begin{cases}
\partial_t U + 2\sigma_0 U - \text{Div} T(U, P) = 0 & \text{in } \Omega, \ t \in \mathbb{R}, \\
\text{div } U = G & \text{in } \Omega, \ t \in \mathbb{R}, \\
T(U, P)e_N = H & \text{on } \Gamma, \ t \in \mathbb{R}, \\
U = 0 & \text{on } S, \ t \in \mathbb{R},
\end{cases}
\]

and furthermore, \( U \) vanishes for \( t < 0 \) and
\[
\| e^{\sigma_0 t}(\partial_t U, U, \nabla U, \nabla^2 U) \|_{L^p(\mathbb{R}, L^q(\Omega))} + \| e^{\sigma_0 t}P \|_{L^p(\mathbb{R}, W^1_q(\Omega))} 
\leq C_{N,d,p,q,\mu,\sigma_0} \left( \| e^{\sigma_0 t}g \|_{W^{1,1}_{p,0}(\mathbb{R}, \mathbb{W}^{-1,q}_{1,0}(\Omega))} + \| e^{\sigma_0 t}(g, h) \|_{0,H^{1,1/2}_{q,p}(\Omega \times \mathbb{R})} \right).
\]

Therefore, \((u_2, p_2) = (U, P)\) solves (5.2) and satisfies
\[
(5.8) \quad \| e^{\sigma_0 t}(\partial_t u_2, u_2, \nabla u_2, \nabla^2 u_2) \|_{L^p(\mathbb{R}, L^q(\Omega))} + \| e^{\sigma_0 t}p_2 \|_{L^p(\mathbb{R}, L^q(\Omega))} 
\leq C_{N,d,p,q,\mu,\sigma_0} \left( \| e^{\sigma_0 t}f \|_{0,H^{1,1/2}_{q,p}(\Omega \times \mathbb{R})} + \| e^{\sigma_0 t}(f, h) \|_{0,H^{1,1/2}_{q,p}(\Omega \times \mathbb{R})} \right).
\]

**Step 3.** We consider (5.3) in this step. By Proposition 3.6 and (5.8),
\[
\| e^{\sigma_0 t}E_0 \nabla Q_0(f + 2\sigma_0 u_2) \|_{L^p(\mathbb{R}, L^q(\Omega))} \leq C \| e^{\sigma_0 t}(f + 2\sigma_0 u_2) \|_{L^p(\mathbb{R}, L^q(\Omega))} 
\leq C_{N,d,p,q,\mu,\sigma_0} \left( \| e^{\sigma_0 t}f \|_{L^p(\mathbb{R}, L^q(\Omega))} + \| e^{\sigma_0 t}g \|_{W^{1,1}_{p,0}(\mathbb{R}, \mathbb{W}^{-1,q}_{1,0}(\Omega))} 
+ \| e^{\sigma_0 t}(g, h) \|_{0,H^{1,1/2}_{q,p}(\Omega \times \mathbb{R})^N} \right),
\]
which implies \( E_0 \nabla Q_0(f + 2\sigma_0 u_2) \in L^p_{\sigma_0}(\mathbb{R}, L^q(\Omega))^N \). One thus observes by Proposition 3.1 with \( \delta = \sigma_0 \) that there is a solution \((V, Q) \in W^{2,1}_{q,p,-\sigma_0}(\Omega \times \mathbb{R})^N \times \mathbb{R} \).
$L_{p,-\sigma_0}(\mathbf{R}, W^1_q(\Omega))$ to

$$
\begin{align*}
\partial_t \mathbf{V} + 2\sigma_0 \mathbf{V} - \text{Div } \mathbf{T}(\mathbf{V}, Q) &= E_0 \nabla Q_q(\mathbf{f} + 2\sigma_0 \mathbf{u}_2) &\text{in } \Omega, \ t \in \mathbf{R}, \\
\text{div } \mathbf{V} &= 0 &\text{in } \Omega, \ t \in \mathbf{R}, \\
\mathbf{T}(\mathbf{V}, Q)\mathbf{e}_N &= 0 &\text{on } \Gamma, \ t \in \mathbf{R}, \\
\mathbf{V} &= 0 &\text{on } S, \ t \in \mathbf{R},
\end{align*}
$$

and furthermore, $\mathbf{V}$ vanishes for $t < 0$ and

$$
\begin{align*}
&\|e^{\sigma t}(\partial_t \mathbf{V}, \nabla \mathbf{V}, \nabla^2 \mathbf{V})\|_{L_p(\mathbf{R}, L_q(\Omega))} + \|e^{\sigma t}Q\|_{L_p(\mathbf{R}, W^1_q(\Omega))} \\
&\leq C_{N,d,p,q,\mu,\sigma_0}\left(\|e^{\sigma t}\mathbf{f}\|_{L_p(\mathbf{R}^+), L_q(\Omega))} + \|e^{\sigma t}g\|_{W^{1}_q(\mathbf{R}^+, \widehat{W}^{-1}_q(\Omega))} + \|e^{\sigma t}(g, h)\|_{H^{1/2}_{q,p}(\mathbf{R}^+)\times \mathbf{R}^+}\right).
\end{align*}
$$

It is then clear that $(\mathbf{u}_3, p_3) = (\mathbf{V}, Q)$ solves (5.3) and satisfies

$$
\begin{align*}
&\|e^{\sigma t}(\partial_t \mathbf{u}_3, \nabla \mathbf{u}_3, \nabla^2 \mathbf{u}_3)\|_{L_p(\mathbf{R}^+, L_q(\Omega))} + \|e^{\sigma t}p_3\|_{L_p(\mathbf{R}^+, W^1_q(\Omega))} \\
&\leq C_{N,d,p,q,\mu,\sigma_0}\left(\|e^{\sigma t}\mathbf{f}\|_{L_p(\mathbf{R}^+, L_q(\Omega))} + \|e^{\sigma t}g\|_{W^{1}_q(\mathbf{R}^+, \widehat{W}^{-1}_q(\Omega))} + \|e^{\sigma t}(g, h)\|_{H^{1/2}_{q,p}(\mathbf{R}^+)\times \mathbf{R}^+)\right).
\end{align*}
$$

**Step 4.** We consider (5.4) in this step. Let $\mathbf{F} = P_q(\mathbf{f} + 2\sigma_0 \mathbf{u}_2 + 2\sigma_0 \mathbf{u}_3)$. Then, by Proposition 3.6, (5.8), and (5.9), one has $e^{\sigma t}\mathbf{F} \in L_p(\mathbf{R}^+, J_q(\Omega))$ with

$$
\begin{align*}
&\|e^{\sigma t}\mathbf{F}\|_{L_p(\mathbf{R}^+, L_q(\Omega))} \\
&\leq C_{N,d,p,q,\mu,\sigma_0}\left(\|e^{\sigma t}\mathbf{f}\|_{L_p(\mathbf{R}^+, L_q(\Omega))} + \|e^{\sigma t}g\|_{W^{1}_q(\mathbf{R}^+, \widehat{W}^{-1}_q(\Omega))} + \|e^{\sigma t}(g, h)\|_{H^{1/2}_{q,p}(\mathbf{R}^+)\times \mathbf{R}^+)\right).
\end{align*}
$$

Such an $\mathbf{F}$ can be approximated by an element of $C_0^\infty(\mathbf{R}^+, J_q(\Omega))$ under the norm $\|e^{\sigma t} . \|_{L_p(\mathbf{R}^+, L_q(\Omega))}$, so that it suffices to consider the case $\mathbf{F} \in C_0^\infty(\mathbf{R}^+, J_q(\Omega))$.

As the first step, we estimate the solution $\mathbf{u}_4$ to (5.4) given by Duhamel’s formula:

$$
\mathbf{u}_4(t) = \int_0^t e^{A_q(0-s)}\mathbf{F}(s) \, ds \quad (t > 0),
$$

which is also written as

$$
e^{\sigma t}\mathbf{u}_4(t) = \int_0^t e^{\sigma(0-s)}e^{A_q(0-s)}(e^{\sigma_0 s}\mathbf{F}(s)) \, ds.
$$

Let $\chi_A$ be the characteristic function of $A \subset \mathbf{R}$, and then by Proposition 4.5

$$
\begin{align*}
\|e^{\sigma t}\mathbf{u}_4(t)\|_{L_q(\Omega)} &\leq C_{N,d,q,\mu,\sigma_0} \int_0^t e^{\sigma_0(0-s)}\|e^{\sigma_0 s}\mathbf{F}(s)\|_{L_q(\Omega)} \, ds \\
&= C_{N,d,q,\mu,\sigma_0} \int_{-\infty}^t e^{\sigma_0(0-s)}\|e^{\sigma_0 s}\mathbf{F}(s)\|_{L_q(\Omega)} \, ds \\
&= C_{N,d,q,\mu,\sigma_0} \int_{-\infty}^\infty \chi((0,(0)) (t-s) e^{\sigma_0(0-s)}\|e^{\sigma_0 s}\mathbf{F}(s)\|_{L_q(\Omega)} \, ds \\
&= C_{N,d,q,\mu,\sigma_0} \chi((0,(0)) (\cdot) e^{\sigma_0 \cdot} + (e^{\sigma_0 \cdot} \mathbf{F}(\cdot)))(t),
\end{align*}
$$

where $(f * g)(t) = \int_\mathbf{R} f(t-s)g(s) \, ds$. Combining this inequality with Young’s inequality $\|f \ast g\|_{L_p(\mathbf{R})} \leq \|f\|_{L_q(\mathbf{R})}\|g\|_{L_p(\mathbf{R})}$ furnishes that

$$
\|e^{\sigma t}\mathbf{u}_4\|_{L_p(\mathbf{R}^+, L_q(\Omega))} \leq C_{N,d,q,\mu,\sigma_0} \chi((0,(0)) (\cdot) e^{\sigma_0 \cdot} \|\mathbf{F}\|_{L_p(\mathbf{R}, L_q(\Omega))}
$$

(5.11)
which implies that
\[ W(5.14) \]
Then, similarly to Step 3, one observes by (5.11) that
\[ a \]
which, combined with (5.10), furnishes
\[ (1) \]
proof of Theorem 2.4.

solutions for a dual problem (cf. e.g. [30, Subsection 7.2]), which completes the
solution to (2.3) and satisfies the required estimate of Theorem 2.4 by (5.5), (5.8),
two Banach spaces \( X \) and \( Y \), i.e. there is a positive constant \( C \) such that
\[ \| u(t) - a \|_{B^{2-2/p}_{q,p}(\Omega)} = 0. \]
The uniqueness of solutions to (2.3) follows from the existence of

Lemma 6.1. Let \( 1 < p, q < \infty \). Then the following assertions hold true.

(1) For any \( f \in W^{1}_{\infty}(R_{+}, L_{\infty}(\Omega)) \) and \( g \in H^{1/2}_{p}(R_{+}, L_{q}(\Omega)) \),
\[ \| fg \|_{H^{1/2}_{p}(R_{+}, L_{q}(\Omega))} \leq \| f \|_{W^{1}_{\infty}(R_{+}, L_{\infty}(\Omega))} \| g \|_{H^{1/2}_{p}(R_{+}, L_{q}(\Omega))}. \]
(2) Let \( q > N \). Then there exists a positive constant \( C_{N,q} \) such that for any \( f, g \in B_q(\Omega \times \mathbb{R}_+) \) and \( h \in H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}_+) \),
\[
\|f g\|_{B_q(\Omega \times \mathbb{R}_+)} \leq C_{N,q} \|f\|_{B_q(\Omega \times \mathbb{R}_+)} \|g\|_{B_q(\Omega \times \mathbb{R}_+)},
\]
\[
\|f h\|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}_+)} \leq C_{N,q} \|f\|_{B_q(\Omega \times \mathbb{R}_+)} \|h\|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}_+)}.\]
(3) For any \( f \in H^{1/2}_p(\mathbb{R}_+, W^1_q(\Omega)) \),
\[
\|\nabla f\|_{H^{1/2}_p(\mathbb{R}_+, L^q(\Omega))} \leq \|f\|_{H^{1/2}_p(\mathbb{R}_+, W^1_q(\Omega))}.
\]

Proof. (1). It clearly holds that
\[
\|f g\|_{L^p(\mathbb{R}_+, L^q(\Omega))} \leq \|f\|_{W^{1,q}_p(\mathbb{R}_+, L^\infty(\Omega))} \|g\|_{L^p(\mathbb{R}_+, L^q(\Omega))},
\]
\[
\|f g\|_{W^{1,q}_p(\mathbb{R}_+, L^q(\Omega))} \leq \|f\|_{W^{1,q}_p(\mathbb{R}_+, L^\infty(\Omega))} \|g\|_{W^{1,q}_p(\mathbb{R}_+, L^q(\Omega))}.
\]
Let \( T_f g = fg \), and then \( T_f \in \mathcal{L}(L^p(\mathbb{R}_+, L^q(\Omega))) \cap \mathcal{L}(W^{1}_p(\mathbb{R}_+, L^q(\Omega))) \) with
\[
\|T_f\|_{\mathcal{L}(L^p(\mathbb{R}_+, L^q(\Omega)))} \leq \|f\|_{W^{1,q}_p(\mathbb{R}_+, L^\infty(\Omega))},
\]
\[
\|T_f\|_{\mathcal{L}(W^{1}_p(\mathbb{R}_+, L^q(\Omega)))} \leq \|f\|_{W^{1,q}_p(\mathbb{R}_+, L^\infty(\Omega))}.
\]
Combining these properties with the complex interpolation method furnishes that
\[
\|T_f\|_{[L^p(\mathbb{R}_+, L^q(\Omega)), W^{1}_p(\mathbb{R}_+, L^q(\Omega))]}_{1/2} \leq \|f\|_{[W^{1,q}_p(\mathbb{R}_+, L^\infty(\Omega)), W^{1}_p(\mathbb{R}_+, L^q(\Omega))]}_{1/2}.
\]
Noting \([L^p(\mathbb{R}_+, L^q(\Omega)), W^{1}_p(\mathbb{R}_+, L^q(\Omega))]}_{1/2} = H^{1/2}_p(\mathbb{R}_+, L^q(\Omega))\) by the definition, we complete the proof of Lemma 6.1 (1).
(2). Since \( W^1_q(\Omega) \) is a Banach algebra for \( q > N \), we have
\[
\|f g\|_{L^\infty(\mathbb{R}_+, W^1_q(\Omega))} \leq C_{N,q} \|f\|_{L^\infty(\mathbb{R}_+, W^1_q(\Omega))} \|g\|_{L^\infty(\mathbb{R}_+, W^1_q(\Omega))},
\]
\[
\|f h\|_{L^p(\mathbb{R}_+, W^1_q(\Omega))} \leq C_{N,q} \|f\|_{L^\infty(\mathbb{R}_+, W^1_q(\Omega))} \|h\|_{L^p(\mathbb{R}_+, W^1_q(\Omega))}.
\]
It is clear that \( \|f g\|_{W^1_q(\mathbb{R}_+, L^\infty(\Omega))} \leq \|f\|_{W^1_q(\mathbb{R}_+, L^\infty(\Omega))} \|g\|_{W^1_q(\mathbb{R}_+, L^\infty(\Omega))} \), so that one has, together with the first inequality of (6.1),
\[
\|f g\|_{B_q(\Omega \times \mathbb{R}_+)} \leq C_{N,q} \|f\|_{B_q(\Omega \times \mathbb{R}_+)} \|g\|_{B_q(\Omega \times \mathbb{R}_+)}.
\]
The other estimate of Lemma 6.1 (2) follows from Lemma 6.1 (1) and the second inequality of (6.1).

(3). The required inequality follows from the complex interpolation method immediately, which completes the proof of Lemma 6.1 (3).

At this point, we introduce some embedding properties.

Lemma 6.2. Let \( p, q \) satisfy (2.1). Then there exists a positive constant \( M_1 \geq 1 \) such that the following assertions hold true.
(1) \( \|f\|_{BUC^1(\Omega)} \leq M_1 \|f\|_{W^1_q(\Omega)} \) for any \( f \in W^1_q(\Omega) \).
(2) \( \|f\|_{BUC^0([0,\infty), BUC^1(\Omega))} \leq M_1 \|f\|_{W^{2,1}_{q,p}(\Omega \times \mathbb{R}_+)} \) for any \( f \in W^{2,1}_{q,p}(\Omega \times \mathbb{R}_+) \).
(3) \( \|f\|_{H^{1/2}_{q,p}(\mathbb{R}_+, W^1_q(\Omega))} \leq M_1 \|f\|_{W^{2,1}_{q,p}(\Omega \times \mathbb{R}_+)} \) for any \( f \in W^{2,1}_{q,p}(\Omega \times \mathbb{R}_+) \).

Proof. (1). See e.g. [3, Theorem 4.12].
(2). It follows from (5.14) and \( B^{q-2/p}_{q,p}(\Omega) \hookrightarrow BUC^1(\overline{\Omega}) \) under the condition (2.1) (cf. [51, Theorem 4.6.1]).
(3). See e.g. [26, Proposition 3.2, Remark 3.3], [37].
We prove Theorem 2.2 in the remaining part of this section by using Theorem 2.4 and the contraction mapping theorem. Let $\gamma_0 = \sigma_0$, where $\sigma_0$ is the same positive constant as in Theorem 2.4. We here define a closed set $X_{q,p}^{\gamma_0}(R)$ by

$$X_{q,p}^{\gamma_0}(R) = \{(u, p) \in W^{2,1}_q(\Omega \times R_+) \times L_p(R_+, W^{1}_q(\Omega)) \mid \|u\|_{X_{q,p}^{\gamma_0}} \leq R, \quad u = 0 \text{ on } S, \quad \lim_{t \to 0^+} \|u - a\|_{B_{q,p}^{2,2/p}(\Omega)} = 0 \},$$

where $0 < R < 1$ is a positive constant and $\|u\|_{X_{q,p}^{\gamma_0}} = \|u\|_{Y_{q,p}^{\gamma_0}} + \|p\|_{Z_{q,p}^{\gamma_0}}$ with

$$\|u\|_{Y_{q,p}^{\gamma_0}} = \|e^{\gamma_0 t}(\partial_t u, u, \nabla u, \nabla^2 u)\|_{L_p(R_+, L_q(\Omega))}, \quad \|p\|_{Z_{q,p}^{\gamma_0}} = \|e^{\gamma_0 t}p\|_{L_p(R_+, L_q(\Omega))}.$$

In addition, by e.g. [23], we know the following characterization:

$$(6.2) \quad \Omega^{1/2}(\Omega \times R_+) = \{f \in H^{1/2}_q(\Omega \times R_+) \mid f|_{t=0} = 0 \text{ in } L_q(\Omega) \}$$

under the condition (2.1).

Let $(v, q) \in X_{q,p}^{\gamma_0}(R)$, and one considers

$$\begin{align*}
\partial_t u - \text{Div } T(u, p) &= F(v) \quad \text{in } \Omega, \quad t > 0, \\
\text{Div } u &= G(u) = \text{Div } G(v) \quad \text{in } \Omega, \quad t > 0, \\
T(u, p)e_N &= H(v)e_N \quad \text{on } \Gamma, \quad t > 0, \\
u &= 0 \quad \text{on } S, \quad t > 0, \\
u|_{t=0} &= a \quad \text{in } \Omega.
\end{align*}$$

Here we prove

**Lemma 6.3.** Let $p$, $q$ satisfy (2.1), and let $\gamma_0$ be as above. Then there exists positive constants $M_2 \geq 1$ such that the following assertions hold true.

1. For any $v_r$ with $\|v_r\|_{Y_{q,p}^{\gamma_0}} \leq 1$ ($r = 1, 2$), we have

$$\left\| U_1 \left( \int_0^t \nabla v_2(\xi, s) \, ds \right) - U_1 \left( \int_0^t \nabla v_1(\xi, s) \, ds \right) \right\|_{L_\infty(R_+, L_\infty(\Omega))} \leq M_2 \|v_2 - v_1\|_{Y_{q,p}^{\gamma_0}},$$

$$\sum_{i,j,k,l=1}^N \left\| V_{ijk} \left( \int_0^t \nabla v_2(\xi, s) \, ds \right) - V_{ijk} \left( \int_0^t \nabla v_1(\xi, s) \, ds \right) \right\|_{L_\infty(R_+, L_\infty(\Omega))} \leq M_2 \|v_2 - v_1\|_{Y_{q,p}^{\gamma_0}},$$

$$\sum_{i,j,k,l,m,n=1}^N \left\| W_{ijklm} \left( \int_0^t \nabla v_2(\xi, s) \, ds \right) - W_{ijklm} \left( \int_0^t \nabla v_1(\xi, s) \, ds \right) \right\|_{L_\infty(R_+, L_\infty(\Omega))} \leq M_2 \|v_2 - v_1\|_{Y_{q,p}^{\gamma_0}},$$

$$\left\| U_r \left( \int_0^t \nabla v_2(\xi, s) \, ds \right) - U_r \left( \int_0^t \nabla v_1(\xi, s) \, ds \right) \right\|_{B_q(\Omega \times R_+)} \leq M_2 \|v_2 - v_1\|_{Y_{q,p}^{\gamma_0}} \quad (r = 2, \ldots, 7).$$

2. For any $v$ with $\|v\|_{Y_{q,p}^{\gamma_0}} \leq 1$, we have

$$\left\| U_1 \left( \int_0^t \nabla v(\xi, s) \, ds \right) \right\|_{L_\infty(R_+, L_\infty(\Omega))} \leq M_2 \|v\|_{Y_{q,p}^{\gamma_0}}.$$
Proof. For any \( p \) where (6.4) and that by Lemma 6.2 (1)

\[
\max \left\{ \sum_{(\alpha, \beta) \in N} \left| \partial_{(\alpha, \beta)} U_{r_1}(X) \right| \left| |X| \leq 2M_1(p')^{-1/p'} \right\} \leq m_2,
\]

\[
\max \left\{ \sum_{(\alpha, \beta) \in N} \sum_{i,j,k,l} \left| \partial_{(\alpha, \beta)} V_{ij}^{l}(X) \right| \left| |X| \leq 2M_1(p')^{-1/p'} \right\} \leq m_2,
\]

\[
\max \left\{ \sum_{(\alpha, \beta) \in N} \sum_{i,j,k,l,m,n=1} \left| \partial_{(\alpha, \beta)} W_{ij klmn}^{n}(X) \right| \left| |X| \leq 2M_1(p')^{-1/p'} \right\} \leq m_2,
\]

where \( p' = p/(p-1) \) and \( M_1 \) is the same positive constant as in Lemma 6.2.

(1) First, we consider \( U_1 \). Let us write \( v_r = (v_{r1}, \ldots, v_{rN})^T \) for \( r = 1, 2 \). It then holds that

\[
(6.4) \quad U_1 \left( \int_0^t \nabla v_2(\xi, s) \, ds \right) - U_1 \left( \int_0^t \nabla v_1(\xi, s) \, ds \right)
\]

\[
= \sum_{(\alpha, \beta) \in N} \int_0^1 \left( \theta \int_0^t \nabla v_2(\xi, s) \, ds + (1 - \theta) \int_0^t \nabla v_1(\xi, s) \, ds \right) \, d\theta
\]

\[
\cdot \left( \int_0^t \partial_{\beta} v_{1\alpha}(\xi, s) \, ds - \int_0^t \partial_{\beta} v_{2\alpha}(\xi, s) \, ds \right)
\]

and that by Lemma 6.2 (1)

\[
(6.5) \quad \left\| \theta \int_0^t \nabla v_2(\xi, s) \, ds + (1 - \theta) \int_0^t \nabla v_1(\xi, s) \, ds \right\|_{L_\infty(R_+, L_\infty(\Omega))}
\]

\[
\leq \sum_{r=1}^2 \int_0^\infty \| v_r(\cdot, s) \|_{L_\infty(\Omega)} \, ds \leq M_1 \sum_{r=1}^2 \int_0^\infty \| v_r(\cdot, s) \|_{W_2^2(\Omega)} \, ds
\]

\[
\leq M_1 \sum_{r=1}^2 \left( \int_0^\infty e^{-p'\gamma_0 s} \, ds \right)^{1/p'} \| v_r \|_{Y_{q,p}^\gamma} \leq 2M_1(p')^{-1/p'}.
\]
Thus we choose $M_2$ large enough so that $M_2 \geq m_2 M_1 (p' \gamma_0)^{-1/p'}$ in order to obtain the required estimate for $U_1$. Analogously, one can prove that the required estimates hold true for $V_{i,j,k}$ and $W_{i,j,k,l,m}$. Next, we consider $U_r$ for $r = 2, \ldots, 7$. Following (6.4), we observe that for $\partial \in \{ \partial_t, \partial_1, \ldots, \partial_N \}$

\[
\partial \left\{ U_r \left( \int_0^t \nabla v_2(\xi, s) \, ds \right) - U_r \left( \int_0^t \nabla v_1(\xi, s) \, ds \right) \right\} = \sum_{(\alpha', \beta') \in N} \sum_{(\alpha, \beta) \in N} \int_0^1 (\partial_{(\alpha', \beta')} \partial_{(\alpha, \beta)} U_r) \left( \theta \int_0^t \nabla v_2(\xi, s) \, ds + (1 - \theta) \int_0^t \nabla v_1(\xi, s) \, ds \right) \cdot \partial \left( \theta \int_0^t \partial_{\beta'} v_{2\alpha'}(\xi, s) \, ds + (1 - \theta) \int_0^t \partial_{\beta'} v_{1\alpha'}(\xi, s) \, ds \right) \, d\theta \\
- \sum_{(\alpha, \beta) \in N} \int_0^1 (\partial_{(\alpha, \beta)} U_r) \left( \theta \int_0^t \nabla v_2(\xi, s) \, ds + (1 - \theta) \int_0^t \nabla v_1(\xi, s) \, ds \right) \, d\theta \\
+ \int_0^1 (\partial_{(\alpha, \beta)} U_r) \left( \theta \int_0^t \nabla v_2(\xi, s) \, ds + (1 - \theta) \int_0^t \nabla v_1(\xi, s) \, ds \right) \, d\theta.
\]

By (6.5), (6.6), (6.7), and Lemma 6.2 (2), it holds that

\[
\left\| \partial_t \left\{ U_r \left( \int_0^t \nabla v_2(\xi, s) \, ds \right) - U_r \left( \int_0^t \nabla v_1(\xi, s) \, ds \right) \right\} \right\|_{L^\infty(R^+, L^\infty(\Omega))} \leq \left( 2m_2 M_2^2 (p' \gamma_0)^{-1/p'} + m_2 M_1 \right) \| v_2 - v_1 \|_{Y^{\gamma_0}_{p', \beta}}, \\
\left\| \partial_j \left\{ U_r \left( \int_0^t \nabla v_2(\xi, s) \, ds \right) - U_r \left( \int_0^t \nabla v_1(\xi, s) \, ds \right) \right\} \right\|_{L^\infty(R^+, L^4(\Omega))} \leq \left( 2m_2 M_1 (p' \gamma_0)^{-2/p'} + m_2 (p' \gamma_0)^{-1/p'} \right) \| v_2 - v_1 \|_{Y^{\gamma_0}_{p', \beta}}.
\]
One thus obtains the required estimates for $U_r$ (for $j = 2, \ldots, 7$) by choosing a larger $M_2 \geq 1$ if necessary. This completes the proof of Lemma 6.3 (1).

(2). The estimates follows from (1.15) and (1) with $(v_2, v_1) = (v, 0)$ immediately. This completes the proof of Lemma 6.3 (2).

(3). One sets $(v_2, v_1) = (v, 0)$ in the inequality proved in (1) in order to obtain

$$\sum_{i,j,k,l,m,n=1}^N W_{ijklm}^n \left( \int_0^t \nabla v(\xi, s) ds \right) - W_{ijklm}^n (O) \leq M_2 \|v\|_{Y_{q,p}^{\gamma_0}} \leq M_2,$$

which implies that

$$\sum_{i,j,k,l,m,n=1}^N \left\| W_{ijklm}^n \left( \int_0^t \nabla v(\xi, s) ds \right) \right\|_{L_\infty(R^+, L_\infty(\Omega))} \leq M_2 + \sum_{i,j,k,l,m,n=1}^N \left\| W_{ijklm}^n (O) \right\|_{L_\infty(R^+, L_\infty(\Omega))}.$$

Noting $W_{ijklm}^n (O) \in L_\infty(R^+, L_\infty(\Omega))$, we complete the proof of Lemma 6.3 (3) with $M_3 = M_2 + \sum_{i,j,k,l,m,n=1}^N \|W_{ijklm}^n (O)\|_{L_\infty(R^+, L_\infty(\Omega))}$. \qed

Let $M_1$, $M_2$, and $M_3$ be the same positive constants as in Lemmas 6.2, Lemma 6.3, and also assume that $(v, q), (v_1, q_1), (v_2, q_2) \in X_{q,p}^{\gamma_0}(1)$ in what follows.

**Estimates of $F(v)$**. One observes that

$$e^{\gamma_0 t}(F(v_2) - F(v_1))$$

$$= \left\{ U_1 \left( \int_0^t \nabla v_2 ds \right) - U_1 \left( \int_0^t \nabla v_1 ds \right) \right\} e^{\gamma_0 t} \partial_t v_2$$

$$+ U_1 \left( \int_0^t \nabla v_1 ds \right) e^{\gamma_0 t} \partial_t (v_2 - v_1)$$

$$+ \left\{ V \left( \int_0^t \nabla v_2 ds \right) - V \left( \int_0^t \nabla v_1 ds \right) \right\} e^{\gamma_0 t} \nabla^2 v_2$$

$$+ V \left( \int_0^t \nabla v_1 ds \right) e^{\gamma_0 t} \nabla^2 (v_2 - v_1)$$

$$+ \left[ \left\{ W \left( \int_0^t \nabla v_2 ds \right) - W \left( \int_0^t \nabla v_1 ds \right) \right\} \int_0^t \nabla^2 v_2 ds \right] e^{\gamma_0 t} \nabla v_2$$

$$+ \left[ W \left( \int_0^t \nabla v_1 ds \right) \int_0^t \nabla^2 (v_2 - v_1) ds \right] e^{\gamma_0 t} \nabla v_2$$
especially, setting \( (v) \).

Summing up the above estimates for \( j = 1, \ldots, 4 \).

On the other hand, by Lemma 6.2 (1)
\[
\|e^{\gamma t} \nabla v_2\|_{L_p(\mathbb{R}^n, L_q(\Omega))} \leq M_1 \|e^{\gamma t} \nabla v_2\|_{L_p(\mathbb{R}^n, W^1_2(\Omega))} \leq M_1 \|v_2\|_{Y_{q,p}^0} \leq M_1,
\]
which, combined with Lemma 6.3 and (6.7), furnishes
\[
\|I_5\|_{L_p(\mathbb{R}^n, L_q(\Omega))} \leq \left\| \mathbf{W} \left( \int_0^t \nabla v_2 \, ds \right) - \mathbf{W} \left( \int_0^t \nabla v_1 \, ds \right) \right\|_{L_{\infty}(\mathbb{R}^n, L_\infty(\Omega))} \cdot \int_0^t \nabla^2 v_2 \, ds \|e^{\gamma t} \nabla v_2\|_{L_p(\mathbb{R}^n, L_q(\Omega))}
\]
\[
\leq (p'\gamma_0)^{-1/p'} M_1 M_2 \left( \|v_2\|_{Y_{q,p}^0} + \|v_1\|_{Y_{q,p}^0} \right) \|v_2 - v_1\|_{Y_{q,p}^0}.
\]

Analogously, it holds that for \( j = 6, 7 \)
\[
\|I_j\|_{L_p(\mathbb{R}^n, L_q(\Omega))} \leq (p'\gamma_0)^{-1/p'} M_1 M_3 \left( \|v_2\|_{Y_{q,p}^0} + \|v_1\|_{Y_{q,p}^0} \right) \|v_2 - v_1\|_{Y_{q,p}^0}.
\]

Summing up the above estimates for \( I_1, \ldots, I_7 \), we have achieved
\[
e^{\gamma t}(F(v_2) - F(v_1))\|_{L_p(\mathbb{R}^n, L_q(\Omega))}\]
\[
\leq \left( 4M_2 + (p'\gamma_0)^{-1/p'} M_1 M_2 + 2(p'\gamma_0)^{-1/p'} M_1 M_3 \right) \left( \|v_2\|_{Y_{q,p}^0} + \|v_1\|_{Y_{q,p}^0} \right) \|v_2 - v_1\|_{Y_{q,p}^0}.
\]

Especially, setting \((v_2, v_1) = (v, 0)\) in (6.8) yields
\[
e^{\gamma t} F(v)\|_{L_p(\mathbb{R}^n, L_q(\Omega))}\]
\[
\leq \left( 4M_2 + (p'\gamma_0)^{-1/p'} M_1 M_2 + 2(p'\gamma_0)^{-1/p'} M_1 M_3 \right) \|v\|_{Y_{q,p}^0}.
\]

Estimates of \( G(v) \), \( G(v) \). One observes that
\[
e^{\gamma t}(G(v_2) - G(v_1)) = \left\{ U_2 \left( \int_0^t \nabla v_2 \, ds \right) - U_2 \left( \int_0^t \nabla v_1 \, ds \right) \right\} \cdot e^{\gamma t} \nabla v_2
\]
\[
+ U_2 \left( \int_0^t \nabla v_1 \, ds \right) : e^{\gamma t} (\nabla v_2 - \nabla v_1) =: I_8 + I_9.
\]

By Lemmas 6.1, 6.3, we have
\[
\|I_8\|_{H^{1/2}_{p}(\Omega \times \mathbb{R}^n)}
\]
\[
\leq \left\| U_2 \left( \int_0^t \nabla v_2 \, ds \right) - U_2 \left( \int_0^t \nabla v_1 \, ds \right) \right\|_{H^{1/2}_{p}(\Omega \times \mathbb{R}^n)} \cdot \left\| e^{\gamma t} \nabla v_2 \right\|_{H^{1/2}_{p}(\Omega \times \mathbb{R}^n)}
\]
\[
\leq M_2 \|v_2 - v_1\|_{Y_{q,p}^0} \left( \|e^{\gamma t} v_2\|_{H^{1/2}_{p}(\mathbb{R}^n, W^1_2(\Omega))} + \|e^{\gamma t} v_2\|_{L_p(\mathbb{R}^n, W^2_2(\Omega))} \right).
\]

Since it holds by Lemma 6.2 (3) that
\[
\|e^{\gamma t} v_2\|_{H^{1/2}_{p}(\mathbb{R}^n, W^1_2(\Omega))} \leq M_1 \|e^{\gamma t} v_2\|_{W^{2,1}_{q,p}(\Omega \times \mathbb{R}^n)}
\]
the above inequality for $I_8$ yields
\[ \|I_8\|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}^+)} \leq 2M_1M_2(\gamma_0 + 1)\|v_2\|_{Y_{q,p}^{\gamma_0}}\|v_2 - v_1\|_{Y_{q,p}^{\gamma_0}}. \]

Analogously, we have
\[ \|I_9\|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}^+)} \leq 2M_1M_2(\gamma_0 + 1)\|v_1\|_{Y_{q,p}^{\gamma_0}}\|v_2 - v_1\|_{Y_{q,p}^{\gamma_0}}, \]
and therefore
\[ \|e^{\gamma_0 t}(G(v_2) - G(v_1))\|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}^+)} \leq 2M_1M_2(\gamma_0 + 1)\|v_2 - v_1\|_{Y_{q,p}^{\gamma_0}}. \]

In addition, it is clear that $e^{\gamma_0 t}(G(v_2) - G(v_1))|_{t=0} = 0$ in $L_q(\Omega)$, which, combined with (6.2) and (6.10), furnishes
\[ e^{\gamma_0 t}(G(v_2) - G(v_1)) \in 0H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}^+). \]

Especially, setting $(v_2, v_1) = (v, 0)$ in (6.11) and (6.10) yields, respectively,
\[ e^{\gamma_0 t}G(v) \in 0H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}^+), \]
\[ \|e^{\gamma_0 t}G(v)\|_{H^{1,1/2}_{q,p}(\Omega \times \mathbb{R}^+)} \leq 2M_1M_2(\gamma_0 + 1)\|v\|_{Y_{q,p}^{\gamma_0}}^2. \]

Next we show $e^{\gamma_0 t}G(v) \in 0W^1_{q'(1)}(\Omega)$ and its estimate. For any $\varphi \in W^1_{q'(1)}(\Omega)$ with $q' = q/(q - 1)$, we have for $k = 0, 1$
\[ (\partial^k_t(G(v_2) - G(v_1)), \varphi)_{\Omega} = (\partial^k_t \text{div}(G(v_2) - G(v_1)), \varphi)_{\Omega} = -\langle \partial^k_t(G(v_2) - G(v_1)), \nabla \varphi \rangle_{\Omega}, \]
where we have used $G(v_1) = 0$ on $S$ for $l = 1, 2$. Thus $\partial^k_t(G(v_2) - G(v_1)) \in \mathcal{G}(\partial^k_t(G(v_2) - G(v_1)))$ as was discussed in Subsection 2.2, and also
\[ \|\partial^k_t(G(v_2) - G(v_1))\|_{\mathcal{W}^{-1}_{q,-1}(\Omega)} \leq \|\partial^k_t(G(v_2) - G(v_1))\|_{L_q(\Omega)}. \]

This inequality, together with $e^{\gamma_0 t}(G(v_2) - G(v_1))|_{t=0} = 0$ in $\mathcal{W}^{-1}_{q,-1}(\Omega)$, implies that
\[ e^{\gamma_0 t}(G(v_2) - G(v_1))|_{t=0} = 0 \text{ in } \mathcal{W}^{-1}_{q,-1}(\Omega), \]
\[ \|e^{\gamma_0 t}\partial^k_t(G(v_2) - G(v_1))\|_{L_q(R^+ \times \mathcal{W}^{-1}_{q,-1}(\Omega))} \leq \|e^{\gamma_0 t}\partial^k_t(G(v_2) - G(v_1))\|_{L_{q'}(R^+ \times L_q(\Omega))} (k = 0, 1). \]

From now on, we estimate the right-hand side of the inequality in (6.13). To this end, we set
\[ e^{\gamma_0 t}(G(v_2) - G(v_1)) = \left( U_3 \left( \int_0^t \nabla v_2 ds \right) - U_3 \left( \int_0^t \nabla v_1 ds \right) \right) e^{\gamma_0 t}v_2 \]
\[ + U_3 \left( \int_0^t \nabla v_1 ds \right) e^{\gamma_0 t}(v_2 - v_1) =: I_{10} + I_{11}, \]
\[ e^{\gamma_0 t}\partial_t(G(v_2) - G(v_1)) = \left\{ \partial_t \left( U_3 \left( \int_0^t \nabla v_2 ds \right) - U_3 \left( \int_0^t \nabla v_1 ds \right) \right) \right\} e^{\gamma_0 t}v_2 \]
\[ + \left( U_3 \left( \int_0^t \nabla v_2 ds \right) - U_3 \left( \int_0^t \nabla v_1 ds \right) \right) e^{\gamma_0 t}\partial_t v_2 \]
\[
+ \left\{ \partial_t U_3 \left( \int_0^t \nabla v_1 \, ds \right) \right\} e^{\gamma_0 t} (v_2 - v_1)
+ U_3 \left( \int_0^t \nabla v_1 \, ds \right) e^{\gamma_0 t} \partial_t (v_2 - v_1) =: I_{12} + I_{13} + I_{14} + I_{15}.
\]

By Lemma 6.3, we see that for \( j = 10, \ldots, 15 \)
\[
\| I_j \|_{L_p (\mathbb{R}^+ \setminus \partial \Omega)} \leq M_2 \left( \| v_2 \|_{Y_{q,p}^0} + \| v_1 \|_{Y_{q,p}^0} \right) \| v_2 - v_1 \|_{Y_{q,p}^0}.
\]

It thus holds that
\[
\sum_{k=0}^1 \| e^{\gamma_0 t} \partial_t^k (G(v_2) - G(v_1)) \|_{L_p (\mathbb{R}^+ \setminus \partial \Omega)}
\leq 6M_2 \left( \| v_2 \|_{Y_{q,p}^0} + \| v_1 \|_{Y_{q,p}^0} \right) \| v_2 - v_1 \|_{Y_{q,p}^0},
\]

which, combined with (6.13), furnishes
\[
\| e^{\gamma_0 t} (G(v_2) - G(v_1)) \|_{L_p (\mathbb{R}^+ \setminus \partial \Omega)}
\leq 6M_2 (\gamma_0 + 1) \left( \| v_2 \|_{Y_{q,p}^0} + \| v_1 \|_{Y_{q,p}^0} \right) \| v_2 - v_1 \|_{Y_{q,p}^0}.
\]

Especially, setting \( (v_2, v_1) = (v, 0) \) in (6.14) yields
\[
\| e^{\gamma_0 t} G(v) \|_{L_p (\mathbb{R}^+ \setminus \partial \Omega)}
\leq 6M_2 (\gamma_0 + 1) \| v \|_{Y_{q,p}^0}^2.
\]

**Estimates of \( H(v) \).** In the same manner as in the case \( G(v) \), we have
\[
e^{\gamma_0 t} \left( H(v_2) e_N - H(v_1) e_N, e^{\gamma_0 t} H(v) e_N \right) \in \partial H^{1,1/2}_q (\Omega \times \mathbb{R}_+)^N,
\]
\[
\| e^{\gamma_0 t} \left( H(v_2) e_N - H(v_1) e_N \right) \|_{H^{1,1/2}_q (\Omega \times \mathbb{R}_+)}
\leq (8M_1M_2(\gamma_0 + 1) + 6M_1M_2^2(\gamma_0 + 1)) \cdot \left( \| v_2 \|_{Y_{q,p}^0} + \| v_2 \|_{Y_{q,p}^0} \right) \| v_2 - v_1 \|_{Y_{q,p}^0},
\]
\[
\| e^{\gamma_0 t} H(v) e_N \|_{H^{1,1/2}_q (\Omega \times \mathbb{R}_+)}
\leq (8M_1M_2(\gamma_0 + 1) + 6M_1M_2^2(\gamma_0 + 1)) \| v \|_{Y_{q,p}^0}^2.
\]

Let \( M_4 \geq 1 \) be a positive constant defined as
\[
M_4 = 4M_2 + (p'\gamma_0)^{-1/p'} M_1M_2 + 2(p'\gamma_0)^{-1/p'} M_1M_3 + 2M_1M_2(\gamma_0 + 1)
+ 6M_2(\gamma_0 + 1) + 8M_1M_2(\gamma_0 + 1) + 6M_1M_2^2(\gamma_0 + 1).
\]

Summing up (6.8)-(6.12) and (6.14)-(6.16), we have obtained
\[
e^{\gamma_0 t} (F(v_2) - F(v_1)) e_N \in L_p (\mathbb{R}^+ \setminus \partial \Omega)^N,
\]
\[
e^{\gamma_0 t} (G(v_2) - G(v_1)) e_N \in \partial H^{1,1/2}_q (\Omega \times \mathbb{R}_+)^n \cap \partial W^{1,1}_q (\mathbb{R}^+ \setminus \partial \Omega)^N,
\]
\[
e^{\gamma_0 t} (H(v_2) e_N - H(v_1) e_N) e_N \in H^{1,1/2}_q (\Omega \times \mathbb{R}_+)^N,
\]

\[
\| e^{\gamma_0 t} (F(v_2) - F(v_1)) \|_{L_p (\mathbb{R}^+ \setminus \partial \Omega)}
\]
Next we show the global existence and uniqueness of such solutions, and problem (1.1)-(1.7). We first introduce the definition of proof of Theorem 2.2.

One here chooses $X_t$ and $R_t$ such that enable us to define the operator:

\[ \Phi : X_{q,p}^0(\delta_0) \ni (v, q) \mapsto \Phi(v, q) = (u, p) \in X_{q,p}^0(\delta_0). \]

Let $(v_i, q_i) \in X_{q,p}^0(\delta_0)$ and $(u_i, p_i) = \Phi(v_i, q_i)$ for $i = 1, 2$. Setting $\bar{u} = u_2 - u_1$ and $\bar{p} = p_2 - p_1$, we observe that

\[ \begin{align*}
\partial_t \bar{u} - \text{Div}(\bar{T}(\bar{u}, \bar{p})) &= F(v_2) - F(v_1) \quad \text{in } \Omega, \ t > 0, \\
\text{div } \bar{u} &= G(v_2) - G(v_1) = \text{div}(G(v_2) - G(v_1)) \quad \text{in } \Omega, \ t > 0, \\
\bar{T}(\bar{u}, \bar{p})e_N &= (H(v_2) - H(v_1))e_N \quad \text{on } \Gamma, \ t > 0, \\
\bar{u} &= 0 \quad \text{on } S, \ t > 0, \\
\bar{u}|_{t=0} &= 0 \quad \text{in } \Omega.
\end{align*} \]

Together with (6.18), (6.19), and (6.21), one has by Thorem 2.4

\[ \|\bar{u}, \bar{p}\|_{X_{q,p}^0} \leq 4c_0 M_4 \delta_0^2 \leq \frac{\delta_0}{2}, \]

which enables us to define the operator:

\[ \Phi : X_{q,p}^0(\delta_0) \ni (v, q) \mapsto \Phi(v, q) = (u, p) \in X_{q,p}^0(\delta_0). \]

7. Original nonlinear problem

This section is concerned with the global solvability of the original nonlinear problem (1.1)-(1.7). We first introduce the definition of $L_p-L_q$ solutions to (1.1)-(1.7). Next we show the global existence and uniqueness of such solutions, and also their exponential stability. Let $M_1, M_2, M_3,$ and $M_4$ be the same positive constants as in Section 6, and let $c_0$ be the positive constant given by Theorem 2.4.
7.1. Definition of $L_p$-$L_q$ solutions. Following [18], we introduce the definition of $L_p$-$L_q$ solutions for (1.1)-(1.7) in this subsection.

One first recalls the definition of $L_p$-$L_q$ solutions for the equations (1.9)-(1.13).

**Definition 7.1.** We call a pair $(u, p)$ an $L_p$-$L_q$ solution global in time to (1.9)-(1.13) if $(u, p) \in W^{2,1}_{q,p}(\Omega \times \mathbb{R}^+)^N \times L_q(R^+, W^1_q(\Omega))$ and if $(u, p)$ satisfies (1.9)-(1.13) in the $L_p$-$L_q$ sense for some $1 < p, q < \infty$ and $a \in B^{2-2/p}(\Omega)^N$.

**Remark 7.2.** Due to [36], the maximal $L_p$-$L_q$ regularity class means the function space of $(u, p)$ in Definition 7.1.

Then we can define $L_p$-$L_q$ solutions to (1.1)-(1.7) as follows:

**Definition 7.3.** We call a triplet $(\Theta, v, \pi)$, where for $\Omega(t) = \Theta(\Omega, t)$

$$v : \bigcup_{t \in (0, \infty)} (\Omega(t) \times \{t\}) \to \mathbb{R}^N, \quad \pi : \bigcup_{t \in (0, \infty)} (\Omega(t) \times \{t\}) \to \mathbb{R},$$

an $L_p$-$L_q$ solution global in time to (1.1)-(1.7) if the following assertions hold true for some $1 < p, q < \infty$ and $a \in B^{2-2/p}(\Omega)^N$:

1. $\Theta = \Theta(\xi, t)$ is a solution to (1.1) in the classical sense.
2. $\Theta(\cdot, t)$ is a $C^1$-diffeomorphism from $\Omega$ onto $\Omega(t)$ for each $t > 0$.
3. $(u, p) = (v \circ \Theta, \pi \circ \Theta)$ is an $L_p$-$L_q$ solution global in time to (1.9)-(1.13).

7.2. Global solvability and exponential stability. We here prove

**Theorem 7.4.** Let $p, q$ satisfy (2.1). Suppose that $\varepsilon_0$ is the same positive number as in Theorem 2.2 and that $a \in D_{q,p}(\Omega)$ with $\|a\|_{D_{q,p}(\Omega)} \leq \varepsilon_0$. Then there exists an $L_p$-$L_q$ solution $(\Theta, v, \pi)$ global in time to (1.1)-(1.7), which is unique. In addition, $\|v(t)\|_{L_q(\Omega(t))} = O(e^{-\gamma_0 t})$ as $t \to \infty$, where $\gamma_0$ is the same positive constant as in Theorem 2.2.

**Proof.** By Theorem 2.2, we have an $L_p$-$L_q$ solution $(u, p)$ to (1.9)-(1.13). Let us define for $t > 0$

$$\Theta(\xi, t) = \xi + \int_0^t u(\xi, s) \, ds \quad (\xi \in \Omega), \quad \Omega(t) = \Theta(\Omega, t),$$

where we note that $u \in BUC([0, \infty), BUC^1(\overline{\Omega}))$ by Lemma 6.2 (2).

**Step 1.** In this step, we prove the existence of $L_p$-$L_q$ solutions global in time to the equations (1.1)-(1.7).

Let $x_1, x_2 \in \Omega(t)$ with $x_1 = x_2$ for

$$x_i = \xi_i + \int_0^t u(\xi, s) \, ds \quad (i = 1, 2),$$

where $\xi_1, \xi_2 \in \Omega$. Since it holds that by Lemma 6.2 (1), (6.17), and (6.21)

$$\int_0^t \|\nabla u(\xi, s)\|_{BUC(\overline{\Omega})} \, ds \leq \left(\int_0^\infty e^{-p' \gamma_0 s} \, ds\right)^{1/p'} \|e^{\gamma_0 t} \nabla u\|_{L_p(\mathbb{R}^+, BUC(\overline{\Omega}))}$$

$$\leq (p' \gamma_0)^{-1/p'} M_1 \|e^{\gamma_0 t} \nabla u\|_{L_p(\mathbb{R}^+, W^1_q(\Omega))}$$

$$\leq (p' \gamma_0)^{-1/p'} M_1 \delta_0 \leq (p' \gamma_0)^{-1/p'} M_1 M_2 \delta_0 \leq 4c_0 M_4 \delta_0 \leq \frac{1}{4}.$$
for any \( t > 0 \), one observes that

\[
0 = |x_1 - x_2| \geq |\xi_1 - \xi_2| - \int_{0}^{t} |u(\xi_1, s) - u(\xi_2, s)| \, ds \\
\geq |\xi_1 - \xi_2| - |\xi_1 - \xi_2| \int_{0}^{t} \|\nabla u(\cdot, s)\|_{BUC(\Omega)} \, ds \\
\geq \frac{3}{4}|\xi_1 - \xi_2|.
\]

This inequality implies \( \xi_1 = \xi_2 \), and thus \( \Theta(\cdot, t) \) is bijective from \( \Omega \) onto \( \Omega(t) \) for each \( t > 0 \). We denote inverse mapping of \( \Theta(\cdot, t) \) by \( \Theta^{-1}(\cdot, t) \) in what follows. Here \( \Theta(\cdot, t) : \Omega \to \Omega(t) \) is a \( C^1 \) function, so that we see by the inverse function theorem that \( \Theta^{-1}(\cdot, t) : \Omega(t) \to \Omega \) is also a \( C^1 \) function. Hence, \( \Theta(\cdot, t) \) satisfies Definition 7.3 (3) and \( \Theta \) is a solution to (1.1).

**Step 2.** Let \( J_{\Theta} \) be the Jacobian matrix of \( \Theta \). Since \( J_{\Theta} = I + \int_{0}^{t} \nabla u(\xi, s) \, ds \), one has \( |J_{\Theta}| \leq c_1 \) by (7.1) for some positive constant \( c_1 \). It then holds by (7.2) that

\[
\|v(t)\|_{L_q(\Omega(t))} = \left( \int_{\Omega} |v(\Theta(x, t), t)|^q |J_{\Theta}| \, dx \right)^{1/q} \\
\leq (c_1)^{1/q} e^{-\gamma_0 t} \|\Theta(t)\|_{BUC((0, \infty), L_q(\Omega))} \\
\leq (c_1)^{1/q} e^{-\gamma_0 t} \|\Theta(t)\|_{BUC((0, \infty), \mathcal{B}_{q-p}^{2, \mathbb{R}^2})}.
\]

This inequality, together with (2.2) and (5.14), furnishes the exponential stability of the solution \( v \).

**Step 3.** We prove the uniqueness of solutions in this step. Let \( i = 1, 2 \) and \( (\Theta_i, v_i, \pi_i) \) be solutions to (1.1)-(1.7) with \( a \in D_{q,p}(\Omega) \) satisfying \( \|a\|_{D_{q,p}(\Omega)} \leq \varepsilon_0 \). Then \( (u_1, p_1) = (v_1 \circ \Theta_1, \pi_1 \circ \Theta_1) \) are \( L^p-L^q \) solutions global in time to (1.9)-(1.13). By Theorem 2.2, we see that \( u_1 = u_2 \) and \( p_1 = p_2 \). One integrates (1.1) with respect to time \( t \) in order to obtain

\[
\Theta_1(\xi, t) = \xi + \int_{0}^{t} v_1(\Theta_1(\xi, s), s) \, ds = \xi + \int_{0}^{t} u_1(\xi, s) \, ds \\
\Theta_2(\xi, t) = \xi + \int_{0}^{t} v_2(\Theta_2(\xi, s), s) \, ds = \Theta_2(\xi, t).
\]

Furthermore, for \( t > 0 \) and \( x \in \Theta_1(\Omega, t) = \Theta_2(\Omega, t) \), we observe that

\[
v_1(x, t) = v_1(\Theta_1(\xi, t), t) = u_1(\xi, t) = u_2(\xi, t) = v_2(\Theta_2(\xi, t), t) = v_2(\xi, t)
\]

and that \( \pi_1(x, t) = \pi_2(x, t) \). This completes the proof of Theorem 7.4.

**Remark 7.5.** One can prove similarly further properties of the solution \( (\Theta, v, \pi) \) as follows:

1. Let \( \Gamma(t) = \Theta(\Gamma, t) \) for \( t > 0 \). Then \( \Theta(\cdot, t) \) is a \( C^1 \)-diffeomorphism from \( \Gamma \) onto \( \Gamma(t) \) for each \( t > 0 \). In addition, it holds that \( \Theta(\Omega, t) = \Omega(t) \).
(2) Let \( \widetilde{\Theta}(\xi, t) = (\Theta(\xi, t), t) \) for \( (\xi, t) \in \Omega \times \mathbb{R}_+ \). Then \( \widetilde{\Theta} \) is a \( C^1 \)-diffeomorphism from \( \Omega \times \mathbb{R}_+ \) onto \( \bigcup_{t \in (0, \infty)} (\Omega(t) \times \{t\}) \).

(3) \( \| \nabla v(t) \|_{L_p(\Omega(t))} = O(e^{-\gamma_0 t}) \) as \( t \to \infty \).

Acknowledgments. This research was partly supported by JSPS Grant-in-aid for Young Scientists (B) \#17K14224, JSPS Japanese-German Graduate Externship at Waseda University, and Waseda University Grant for Special Research Projects (Project number: 2017K-176).

A.

Following [38, Appendix], we derive the equations (1.9)-(1.11) in this appendix. To this end, we assume that the equations (1.1)-(1.7) have a sufficiently regular solution \( (\Theta, v, \pi) \) in the following argumentation. Let \( x = \Theta(\xi, t) \) with (1.8), and recall that \( u(\xi, t) = v(\Theta(\xi, t), t) = v(x, t) \) and \( p(\xi, t) = \pi(\Theta(\xi, t), t) = \pi(x, t) \). If we write \( M = (M_{ij}) \), then \( M \) is an \( N \times N \) matrix whose \( (i, j) \)-component is \( M_{ij} \). In addition, \( \delta_{ij} \) denotes Kronecker’s delta defined by the formula: \( \delta_{ij} = 1 \) when \( i = j \) and \( \delta_{ij} = 0 \) when \( i \neq j \).

Case (1.9). It is clear that

\[
(A.1) \quad \frac{\partial}{\partial t} u(\xi, t) = \frac{\partial}{\partial t} v(x, t) + \sum_{j=1}^{N} \frac{\partial x_j}{\partial t} \frac{\partial}{\partial x_j} v(x, t) = \frac{\partial}{\partial t} v + (v \cdot \nabla)v.
\]

We here set

\[
A = \left( \frac{\partial x_i}{\partial \xi_j} \right) = I + B, \quad B = (B_{ij}), \quad B_{ij} = \int_0^t \frac{\partial u_i(\xi, s)}{\partial \xi_j}(s) \, ds.
\]

Let \( A \) be the cofactor matrix of \( A \), i.e. \( A^{-1} = (\det A)^{-1} A \). Then \( A = A^{-1} \) because \( \det A = 1 \) (cf. e.g. [38, page 271]), and also one observes by \( \delta_{ij} = \partial \xi_i/\partial \xi_j = \sum_{k=1}^{N} \frac{\partial \xi_i}{\partial \xi_k} \frac{\partial x_k}{\partial \xi_j} \), that

\[
A = A^{-1} = \left( \begin{array}{ccc}
\frac{\partial \xi_1}{\partial \xi_1} & \cdots & \frac{\partial \xi_1}{\partial \xi_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial \xi_N}{\partial \xi_1} & \cdots & \frac{\partial \xi_N}{\partial \xi_N}
\end{array} \right)
\]

In addition,

\[
A = I + B \quad \text{for some} \quad N \times N \text{ matrix } B = (B_{ij}) = B \left( \int_0^t \nabla u(\xi, s) \, ds \right),
\]

where \( B : \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N} \) with \( B(O) = O \) and the \( (i, j) \)-component \( B_{ij}(X) \) of \( B(X) \), \( X = (X_{kl}) \), are polynomials with respect to \( X_{kl} \) for \( k, l = 1, \ldots, N \). Now it holds, by direct calculations and by [14, page 771], that

\[
(A.2) \quad \nabla_x = A^T \nabla_\xi, \quad \text{div}_x = (A^T : \nabla_\xi) = \text{div}_\xi (A \cdot), \quad \nabla_x \text{div}_x = A^T \nabla_\xi \text{div}_\xi + A^T \nabla_\xi (B^T : \nabla_\xi),
\]

where the subscripts \( x \) and \( \xi \) denote their variables, and also

\[
(A.3) \quad \Delta_x = \text{div}_x \nabla_x = (\text{div}_\xi + B^T : \nabla_\xi)(I + B^T) \nabla_\xi = \Delta_\xi + \text{div}_\xi B^T \nabla_\xi + B^T : \nabla_\xi \nabla_\xi + B^T : \nabla_\xi B^T \nabla_\xi
\]
\[ = \Delta \zeta + \sum_{i,j,k=1}^{N} B_{ji} (2\delta_{ik} + B_{ki}) \frac{\partial^2}{\partial \xi_j \partial \xi_k} + \sum_{i,j,k=1}^{N} (\delta_{ij} + B_{ji}) \left( \frac{\partial B_{ki}}{\partial \xi_j} \right) \frac{\partial}{\partial \xi_k}. \]

Then we can write
\[
(A.4) \quad \sum_{i,j,k=1}^{N} B_{ji} (2\delta_{ik} + B_{ki}) \frac{\partial^2}{\partial \xi_j \partial \xi_k} = \sum_{i,j,k=1}^{N} \nu_{ijk} \left( \int_{0}^{t} \nabla u(\xi, s) \, ds \right) \frac{\partial^2}{\partial \xi_j \partial \xi_k},
\]
\[
= \sum_{i,j,k=1}^{N} \left[ \nu_{ijk} \left( \int_{0}^{t} \nabla u(\xi, s) \, ds \right) \int_{0}^{t} \nabla v(\xi, s) \, ds \right] \frac{\partial}{\partial \xi_k}
\]

by \( \nu_{ijk}(\cdot) : \mathbb{R} \to \mathbb{R} \) with \( \nu_{ijk}(0) = 0 \) and by \( \nu_{ijk}(\cdot) : \mathbb{R}^N \to \mathbb{R} \). Note that both \( \nu_{ijk}(\cdot) \) and \( \nu_{ijk}(\cdot) \), \( X = (X_{lm}) \), are polynomials with respect to \( X_{lm} \) for \( l, m = 1, \ldots, N \). Since \( \text{Div} T(v, \pi) = \mu(\Delta v + \nabla \text{div} v) - \nabla \pi \), we insert (1.1) and (A.2)-(A.4) into (1.3) in order to obtain
\[
\partial_t u - \mu (\Delta u + A^T \nabla \text{div} u + A^T \nabla (B^T : \nabla u)) + A^T \nabla p = \mu \left( \sum_{i,j,k=1}^{N} \nu_{ijk} \left( \int_{0}^{t} \nabla u(\xi, s) \, ds \right) \frac{\partial^2}{\partial \xi_j \partial \xi_k} - u \right)
\]
\[
= \mu \left( \sum_{i,j,k=1}^{N} \nu_{ijk} \left( \int_{0}^{t} \nabla u(\xi, s) \, ds \right) \int_{0}^{t} \nabla v(\xi, s) \, ds \right] \frac{\partial}{\partial \xi_k} u.
\]

Let \( A^{-T} = (A^T)^{-1} = (A^{-1})^T \), and multiply the last equation by \( A^{-T} = (I + B)^T \) from the left-hand side. Thus,
\[
\partial_t u - \mu(\Delta u + \nabla \text{div} u) + \nabla p = -B^T \partial_t u + \mu B^T \Delta u + \mu \nabla (B^T : \nabla u)
\]
\[
+ \mu \left( I + B^T \right) \sum_{i,j,k=1}^{N} \nu_{ijk} \left( \int_{0}^{t} \nabla u(\xi, s) \, ds \right) \frac{\partial^2}{\partial \xi_j \partial \xi_k} u
\]
\[
+ \mu \left( I + B^T \right) \sum_{i,j,k=1}^{N} \left[ \nu_{ijk} \left( \int_{0}^{t} \nabla u(\xi, s) \, ds \right) \int_{0}^{t} \nabla v(\xi, s) \, ds \right] \frac{\partial}{\partial \xi_k} u,
\]

which completes Case (1.9).

**Case** (1.10). The equation (1.10) follows from (1.4) and the second relation of (A.2) immediately. This completes Case (1.10).

**Case** (1.11). Let \( F(\xi) = \xi_N - d \), and then \( \Gamma = \{ \xi \in \mathbb{R}^N \mid F(\xi) = 0 \} \). We can regard \( \xi \in \Gamma \) as \( \xi = \Theta^{-1}(x, t) \) for \( x \in \Gamma(t) \), where \( \Theta^{-1}(\cdot, t) \) is the inverse mapping of \( \Theta(\cdot, t) \). Let \( G(x, t) = F(\xi(x, t)) \). Since \( \Gamma(t) \) is defined by \( G(x, t) = 0 \), we see that the unit outward normal \( u \) to \( \Gamma(t) \) is given by
\[
(A.5) \quad n = \frac{\nabla G(x, t)}{|\nabla G(x, t)|}.
\]
Now it holds that
\[ \nabla G(x, t) = \left( \sum_{j=1}^{N} \frac{\partial \xi_j}{\partial x_1} \frac{\partial F}{\partial \xi_j}, \ldots, \sum_{j=1}^{N} \frac{\partial \xi_j}{\partial x_N} \frac{\partial F}{\partial \xi_j} \right)^T = A^T e_N, \]
and thus we have by (A.5) the relation:
\[ (A.6) \quad n = \frac{A^T e_N}{|A^T e_N|}. \]

On the other hand, we see by (A.2) that
\[ T(v, \pi) = -p I + \mu (A^T \nabla u + (\nabla u)^T A), \]
which, combined with (A.6) and inserted into (1.11), furnishes
\[ -p \frac{A^T e_N}{|A^T e_N|} + \mu \frac{(A^T \nabla u + (\nabla u)^T A) A^T e_N}{|A^T e_N|} = 0. \]
The last equation is multiplied by \(|A^T e_N| A^{-T}\) from the left-hand side to give
\[ -p e_N + \mu (\nabla u + A^{-T}(\nabla u)^T A) A^T e_N = 0, \]
which, combined with \(A^{-T} = I + B^T\) and \(A = I + B\), implies
\[ T(u, p)e_N = -\mu (D(u)B^T + \{(\nabla u)^T B + B^T(\nabla u)^T (I + B)\} (I + B^T)) e_N. \]

This completes Case (1.11).

**References**

[1] H. Abels. The initial-value problem for the Navier-Stokes equations with a free surface in
\(L_q\)-Sobolev spaces. *Adv. Differential Equations*, 10(1):45–64, 2005.

[2] H. Abels. Generalized Stokes resolvent equations in an infinite layer with mixed boundary
conditions. *Math. Nachr.*, 279(4):351–367, 2006.

[3] R.A. Adams and J.J.F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics*.
Elsevier/Academic Press, Amsterdam, 2nd edition, 2003.

[4] G. Allain. Small-time existence for the Navier-Stokes equations with a free surface. *Appl.
Math. Optim.*, 16(1):37–50, 1987.

[5] H. Amann. *Linear and quasilinear parabolic problems. Vol. I. Abstract linear theory*,
volume 89 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1995.

[6] J.T. Beale. The initial value problem for the Navier-Stokes equations with a free surface.
*Comm. Pure Appl. Math.*, 34(3):359–392, 1981.

[7] J.T. Beale. Large-time regularity of viscous surface waves. *Arch. Rational Mech. Anal.*,
84(4):307–352, 1983/84.

[8] J.T. Beale and T. Nishida. Large-time behavior of viscous surface waves. In *Recent Topics
in Nonlinear PDE II*, volume 128 of *North-Holland Math. Stud.*, pages 1–14. North-Holland,
Amsterdam, 1985.

[9] I.V. Denisova. Problem of the motion of two viscous incompressible fluids separated by a
closed free interface. *Acta Appl. Math.*, 37(1-2):31–40, 1994.

[10] I.V. Denisova. Problem of the motion of two compressible fluids separated by a closed free
surface. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 243:61–86;
English transl.: *J. Math. Sci. (N. Y.)* 99 (2000), no. 1, 837–853, 1997.

[11] I.V. Denisova. Evolution of compressible and incompressible fluids separated by a closed
interface. *Interfaces Free Bound.*, 2(3):283–312, 2000.

[12] I.V. Denisova. Evolution of a closed interface between two liquids of different types. In *European
Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages
263–272. Birkhäuser, Basel, 2001.

[13] I.V. Denisova. Solvability in weighted Hölder spaces for a problem governing the evolution
of two compressible fluids. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*,
295:57–89; English transl.: *J. Math. Sci. (N. Y.)* 127 (2005), no. 2, 1849–1868, 2003.
[14] I.V. Denisova and V.A. Solonnikov. Classical solvability of the problem of the motion of two viscous incompressible fluids. *Algebra i Analiz*, 7(5):101–142; English transl.: St. Petersburg Math. J. 7 (1996), no. 5, 755–786, 1995.

[15] I.V. Denisova and V.A. Solonnikov. Classical solvability of the problem of the motion of an isolated mass of compressible fluid. *Algebra i Analiz*, 14(1):71–98; English transl.: St. Petersburg Math. J. 14 (2002), no. 1, 53–74, 2002.

[16] R. Denk and M. Kaip. *General Parabolic Mixed Order Systems in $L^p$ and Applications*. Oper. Theory Adv. Appl. 239. Birkhäuser/Springer, Cham, 2013.

[17] Y. Enomoto, L. von Below, and Y. Shibata. On some free boundary problem for a compressible barotropic viscous fluid flow. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 60(1):55–89, 2014.

[18] J. Escher, J. Prüss, and G. Simonett. Analytic solutions for a Stefan problem with Gibbs-Thomson correction. *J. Reine Angew. Math.*, 563:1–52, 2003.

[19] E. Hanzawa. Classical solutions of the stefan problem. *Tohoku Math. J. (2)*, 33(3):297–335, 1981.

[20] Y. Hataya. A remark on Beale-Nishida’s paper. *Bull. Inst. Math. Acad. Sin. (N. S.)*, 6(3):293–303, 2011.

[21] Y. Hataya and S. Kawashima. Decaying solution of the Navier-Stokes flow of infinite volume without surface tension. *Nonlinear Anal.*, 71(12):e2535–e2539, 2009.

[22] M. Köhne, J. Prüss, and M. Wilke. Qualitative behaviour of solutions for the two-phase navier-stokes equations with surface tension. *Math. Ann.*, 2(356):737–792, 2013.

[23] N. Lindemulder, M. Meyries, and M. Veraar. Complex interpolation with Dirichlet boundary conditions on the half line. 2017. preprint (arXiv:1705.11054).

[24] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*, volume 16 of Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Verlag, Basel, 1995.

[25] S. Maryani and H. Saito. On the $R$-boundedness of solution operator families for two-phase Stokes resolvent equations. *Differential Integral Equations*, 30(1-2):1–52, 2017.

[26] M. Meyries and R. Schnaubelt. Interpolation, embeddings and traces of anisotropic fractional Sobolev spaces with temporal weights. *J. Funct. Anal.*, 262(3):1200–1229, 2012.

[27] S.G. Mikhlin. *Multidimensional Singular Integrals and Integral Equations*. Pure and Applied Mathematics Monograph. Pergamon Press, Oxford-New York-Paris, 1965.

[28] I.S. Mogilevskii and V.A. Solonnikov. On the solvability of an evolution free boundary problem for the Navier-Stokes equations in Hölder spaces of functions. In *Mathematical problems relating to the Navier-Stokes equation*, volume 11 of Ser. Adv. Math. Appl. Sci., pages 105–181. World Sci. Publ., River Edge, NJ, 1992.

[29] P.B. Mucha and W.M. Zajączkowski. On local existence of solutions of the free boundary problem for an incompressible viscous self-gravitating fluid motion. *Appl. Math. (Warsaw)*, 27(3):319–333, 2000.

[30] Y. Shibata. On the $R$-boundedness of solution operator families of the generalized stokes resolvent problem in an infinite layer. *Math. Methods Appl. Sci.*, 38(9):1888–1925, 2015.

[31] Y. Shibata and Y. Shibata. On decay properties of solutions to the Stokes equations with surface tension and gravity in the half space. *J. Math. Soc. Japan*, 68(4):1559–1614, 2016.

[32] P. Secchi. On the motion of gaseous stars in the presence of radiation. *Comm. Partial Differential Equations*, 15(2):185–204, 1990.

[33] P. Secchi. On the uniqueness of motion of viscous gaseous stars. *Math. Methods Appl. Sci.*, 13(5):391–404, 1990.

[34] P. Secchi. On the evolution equations of viscous gaseous stars. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 18(2):295–318, 1991.

[35] P. Secchi and A. Valli. A free boundary problem for compressible viscous fluids. *J. Reine Angew. Math.*, 341:1–31, 1983.

[36] Y. Shibata. On some free boundary problem of the Navier-Stokes equations in the maximal $L^p-L^q$ regularity class. *J. Differential Equations*, 258(12):4127–4155, 2015.

[37] Y. Shibata. On the local wellposedness of free boundary problem for the Navier-Stokes equations in an exterior domain. 2017. preprint.

[38] Y. Shibata and S. Shimizu. On a free boundary value problem for the Navier-Stokes equations. *Differential Integral Equations*, 20(3):241–276, 2007.

[39] Y. Shibata and S. Shimizu. On the $L^p-L^q$ maximal regularity of the Neumann problem for the Stokes equations in a bounded domain. *J. Reine Angew. Math.*, 615:157–209, 2008.
[40] Y. Shibata and S. Shimizu. On the maximal $L_p$-$L_q$ regularity of the Stokes problem with first order boundary condition; model problems. *J. Math. Soc. Japan*, 64(2):561–626, 2012.

[41] V. A. Solonnikov. $l_p$-theory of the problem of motion of two incompressible capillary fluids in a container. *J. Math. Sci. (N.Y.)*, 198(6):761–827, 2014.

[42] V.A. Solonnikov. Lectures on Evolution Free Boundary Problems: Classical Solutions. In *Mathematical Aspects of Evolving Interfaces*, volume 1812 of *Lecture Notes in Math*. Springer, Berlin, 2003.

[43] V.A. Solonnikov. $L_q$-estimates for a solution to the problem about the evolution of an isolated amount of a fluid. *J. Math. Sci. (N. Y.)*, 117(3):4237–4259, 2003.

[44] V.A. Solonnikov and A. Tani. A problem with a free boundary for navier-stokes equations for a compressible fluid in the presence of surface tension. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 182:142–148, 1990. English transl.: *J. Soviet Math.*, 62(3):2814–2818, 1992.

[45] V.A. Solonnikov and A. Tani. Free boundary problem for a viscous compressible flow with a surface tension. In *Constantin Carathéodory: An international tribute*, Vol. II, pages 1270–1303. World Sci. Publ., Teaneck, NJ, 1991.

[46] G. Ströhmer and W.M. Zajączkowski. Local existence of solutions of the free boundary problem for the equations of compressible barotropic viscous self-gravitating fluids. *Appl. Math. (Warsaw)*, 26(1):1–31, 1999.

[47] N. Tanaka and A. Tani. Surface waves for a compressible viscous fluid. *J. Math. Fluid Mech.*, 5(4):303–363, 2003.

[48] A. Tani. On the free boundary value problem for compressible viscous fluid motion. *J. Math. Kyoto Univ.*, 21(4):839–859, 1981.

[49] A. Tani. Two-phase free boundary problem for compressible viscous fluid motion. *J. Math. Kyoto Univ.*, 24(2):243–267, 1984.

[50] A. Tani. Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface. *Arch. Rationl Mech. Anal.*, 133(4):299–331, 1996.

[51] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[52] E. Zadrzyńska and W.M. Zajączkowski. On local motion of a general compressible viscous heat conducting fluid bounded by a free surface. *Ann. Polon. Math.*, 59(2):133–170, 1994.

[53] E. Zadrzyńska and W.M. Zajączkowski. Local existence of solutions of a free boundary problem for equations of compressible viscous heat-conducting fluids. *Appl. Math. (Warsaw)*, 25(2):179–220, 1998.

[54] E. Zadrzyńska and W.M. Zajączkowski. Local existence of solutions of a free boundary problem for equations of compressible viscous heat-conducting capillary fluids. *Ann. Polon. Math.*, 60(3):255–287, 1995.

Department of Mathematics, Faculty of Science and Engineering, Waseda University, Okubo 3-4-1, Shinjuku-ku, Tokyo 169-8555, Japan

E-mail address: hsaito@aoni.waseda.jp