Symmetry groups of the planar 3-body problem and action–minimizing trajectories

Vivina Barutello∗, Davide L. Ferrario∗ and Susanna Terracini∗

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Abstract

We consider periodic and quasi-periodic solutions of the three-body problem with homogeneous potential from the point of view of equivariant calculus of variations. First, we show that symmetry groups of the Lagrangian action functional can be reduced to groups in a finite explicitly given list, after a suitable change of coordinates. Then, we show that local symmetric minimizers are always collisionless, without any assumption on the group other than the fact that collisions are not forced by the group itself. Moreover, we describe some properties of the resulting symmetric collisionless minimizers (Lagrange, Euler, Hill-type orbits and Chenciner–Montgomery figure-eight).

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1 Introduction and main results

Among all periodic solutions of the planar 3-body problem, relative equilibrium motions – the equilateral Lagrange and the collinear Euler-Moulton solutions – are definitely the simplest and most known. They are both endowed with an evident symmetry (SO(2) and O(2) respectively), that is, they are equivariant with respect to the symmetry group of dimension 1 acting as SO(2) (resp. O(2)) on the time circle and on the plane, and trivially on the set of indexes \{1, 2, 3\}. In fact, they are minimizers of the Lagrangian action functional in the space of all loops having their same symmetry group. Hence, G-equivariant minimizers for the action functional
(given a symmetry group $G$) can be thought as the natural generalization of relative equilibrium motions. The purpose of the paper is to apply systematically the $G$-equivariant calculus of variations to the planar 3-body problem – with homogeneous interaction potential. We will develop a theory and describe all the main aspects involved in the $G$-equivariant minimization: in particular, the classification of all possible symmetry groups, the description of the main properties of their minimizers and the proof of the fact that 3-body minimizers are always collisionless.

In the past few years some variational methods have been exploited by several authors in the search of new periodic solutions as symmetric minimizers for the $n$-body problem (see e.g. [12, 17, 23, 28, 30] for some recent results and references, [1, 7, 10, 15, 20, 22, 24, 29, 32] for more details on the three-body problem or variational methods, and [2, 4, 6, 13, 14, 19, 33, 34] for a similar approach dating back to the early nineties). A major problem in the equivariant minimization is that minimizers might be a priori colliding trajectories. To exclude this possibility several arguments have been introduced and used in recent literature. In [17] the authors proposed a class of symmetry groups with the property that all local minimizers are collisionless (groups with the rotating circle property), for the general $\alpha$-homogeneous $n$-body problem in dimension $d \geq 2$. The key-step is a generalization of the averaging technique introduced by Marchal and exposed in [10]. In spite of its generality, the main theorem of [17] cannot be applied to some relevant symmetry groups, such as the symmetry group of the Chenciner–Montgomery eight-shape orbit [12].

Besides that of collisions, the major problems occurring in the variational approach are the following: first, the minimum has to be achieved (this requires the condition of coercivity of the action functional on the space of symmetric loops; see (2.26) below). This depends on the group $G$ and possibly on the angular velocity $\omega$ of the rotating frame (when we consider these coordinates). Roughly speaking, a group for which the action functional is not coercive for all angular velocities will be termed fully uncoercive (see (3.11) below for the precise definition).

As we already pointed out, one has to prove that the minimizer is collisionless; in particular, this obviously excludes the case when collisions are forced by symmetries. A group will be termed bound to collisions if every equivariant loop has a collision (see (2.18) below). Finally, it makes sense to investigate whether the minimizer is necessarily a homographic solution, and therefore we would like to exclude those groups for which all equivariant loops are rotating configurations, as $SO(2)$ and $O(2)$ above. Such groups will be termed homographic (see (2.15) below).

Since fully uncoercive, bound to collisions or homographic groups are unsuitable to this minimization approach, we are interested in the classification of all the other groups. Our main results are the following theorems:

**Theorem A.** Let $G$ be a symmetry group of the Lagrangian action functional in the 3-body problem. Then $G$ is either bound to collisions, fully uncoercive, homographic, or, up to a change of rotating frame, it is conjugated to one of the symmetry groups listed in table [7] (RCP stands for Rotating Circle Property and HGM for Homographic Global Minimizer).

**Theorem B.** Let $G$ a symmetry group of the Lagrangian in the 3-body problem (in a rotating frame or not). If $G$ is not bound to collisions, then any local minimizer is
Table 1: Planar symmetry groups with trivial core

| Name             | $|G|$ | type | action type | trans. dec. | RCP | HGM |
|------------------|-----|------|-------------|-------------|-----|-----|
| Trivial          | 1   | yes  |             | 1 + 1 + 1   | yes | yes |
| Line             | 2   | yes  | brake       | 1 + 1 + 1   | (no) no     |
| 2-1-choreography | 2   | yes  | cyclic      | 2 + 1       | yes | no  |
| Isosceles        | 2   | yes  | brake       | 2 + 1       | no  | yes |
| Hill             | 4   | yes  | dihedral    | 2 + 1       | no  | no  |
| 3-choreography   | 3   | yes  | cyclic      | 3           | yes | yes |
| Lagrange         | 6   | yes  | dihedral    | 3           | no  | yes |
| $C_6$            | 6   | no   | cyclic      | 3           | yes | no  |
| $D_6$            | 6   | no   | dihedral    | 3           | yes | no  |
| $D_{12}$         | 12  | no   | dihedral    | 3           | no  | no  |

Figure 1: The poset of symmetry groups for the planar 3-body problem

The sentence “up to a change of rotating frame” in Theorem A requires a few words of explanation. Consider a symmetry group $G$ for the Lagrangian functional: if, for every angular velocity, $G$ is a symmetry group for the Lagrangian functional in the rotating frame, then we will say that $G$ is of type R (see the equivalent definition (3.1) below). This is a fundamental property for symmetry groups. In fact, if $G$ is not of type R, it turns out (see (3.10)) that the angular momentum of all $G$-equivariant trajectories vanishes (and thus, that no homographic solution can occur). For example, the dihedral symmetry group used by Chenciner and Montgomery in [12] is not of type R and thus, as already remarked in their paper, the figure-eight solution has zero angular momentum. Only three groups are not of type R, for the planar 3-body problem. We will give a detailed description of each of them in section 4.8 together with their minimizers; in this way we will partially
answer to the open question (posed by Chenciner) whether their minimizers coincide or not: for two of them ($D_6$ and $D_{12}$) the answer is yes (under certain conditions, see [4.15] below).

On the other hand, if $G$ is of type $R$, then a suitable choice of the angular velocity of the rotating frame allows us to consider only the minimal groups, listed in table [1]. Furthermore, given such a minimal group $G$ and a rational $\omega$, $G$-symmetric trajectories (minimizers in the $\omega$-rotating frame) yield $G'$-symmetric trajectories for a different (generally bigger) group $G'$, which are minimizers in the inertial frame (see remark [3.6]). Of course, when the group is of type $R$ a key point is to understand whether the minimizer is homographic or not. This indeed happens every time the rotating Lagrange configuration is $G$-equivariant. In the opposite cases, whenever the Lagrange motion is not $G$-equivariant, it is always possible to find suitable choices of masses $m_i$ and angular velocity $\omega$ such that the minimizer is not homographic; it turns out that these are Hill-like solutions. We emphasize that our approach to exclude collisions is purely local, as in [33, 17]. We do not need action estimates on colliding trajectories: in contrast we shall exhibit local variations around parabolic ejection-collision solutions (which, as shown in [17], are the blow-ups of possible colliding minimizers). This approach brings two advantages: first, it allows the extension of the result in [17] to a unified treatment of all symmetry groups, and also it works with all $\alpha$-homogeneous potentials – actually, this approach can be extended to a larger class of potentials, under only reasonable local assumptions at singularities. Moreover, it allows one to prove existence of collisionless minimizers without any level estimate on the minimal actions of colliding trajectories, that in the literature have been often obtained numerically (see for example [29, 10, 12, 9, 8]). On the other hand, our local variations are not suitable to be used in a context where constraints are of a different nature, such as homotopy/homology/topology constraints [28].

2 Preliminaries

2.1 Settings and notation

Let $E = \mathbb{R}^2 \cong \mathbb{C}$ the 2-dimensional Euclidean space and $0 \in \mathbb{R}^2$ its origin. Let $m_1, m_2, m_3$ be positive real numbers and $\mathcal{X}$ the configuration space of three point particles with masses $m_i$ respectively with center of mass in $0$, then

\begin{equation}
\mathcal{X} = \left\{ x = (x_1, x_2, x_3) \in E^3 : \sum_{i=1}^{3} m_i x_i = 0 \right\}.
\end{equation}

We will denote by $\Delta_{i,j} = \{ x \in \mathcal{X} : x_i = x_j \}$ the collision set of the $i$-th and the $j$-th particle and with $\Delta = \bigcup_{i,j} \Delta_{i,j}$, the collision set in $\mathcal{X}$, i.e.

\begin{equation}
\Delta = \{ x \in \mathcal{X} : \exists i, j \text{ such that } x_i = x_j \}.
\end{equation}

Let $k$ be a subset of the index set $n = \{1, 2, 3\}$ and let $k'$ denote its complement in $n$, then $\Delta_{k,k'} = \bigcup_{i \in k, j \in k'} \Delta_{i,j}$ and $\Delta_k = \bigcap_{i,j \in k} \Delta_{i,j}$. 

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Let $\alpha$ be a given positive real number. We consider the potential function (opposite to the potential energy) and the kinetic energy defined respectively on $\mathcal{X}$ and on the tangent bundle of $\mathcal{X}$ as

\begin{align}
U(x) &= \frac{m_1 m_2}{|x_1 - x_2|^\alpha} + \frac{m_1 m_3}{|x_1 - x_3|^\alpha} + \frac{m_2 m_3}{|x_2 - x_3|^\alpha}, \\
K(x, \dot{x}) &= \frac{1}{2} \left( m_1 |\dot{x}_1|^2 + m_2 |\dot{x}_2|^2 + m_3 |\dot{x}_3|^2 \right).
\end{align}

(2.3) (2.4)

Associated to $U$ and $K$ there are the Newton equations

\begin{equation}
m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}.
\end{equation}

When we work in a uniform rotating frame the kinetic energy needs to be changed into the corresponding form

\begin{equation}
K_\omega(x, \dot{x}) = \sum_{i=1}^3 \frac{1}{2} m_i |\dot{x}_i - J \omega x_i|^2.
\end{equation}

(2.5)

where $J$ is the complex unit and $\omega$ is the angular velocity. We suppose that the origin (which coincide with the center of mass) is the point of the plane fixed by the rotation. The Lagrangian is

\begin{equation}
L_\omega(x, \dot{x}) = L_\omega = K_\omega + U.
\end{equation}

(2.6)

Let $T \subset \mathbb{R}^2$ denote a circle in $\mathbb{R}^2$ of length $T = |T|$. Moreover, let $\Lambda = H^1(T, \mathcal{X})$ be the Sobolev space of $L^2$ loops $T \to \mathcal{X}$ with $L^2$ derivative. It is an Hilbert space with the scalar product

\begin{equation}
x \cdot y = \int_T (x(t)y(t) + \dot{x}(t)\dot{y}(t))dt
\end{equation}

and the corresponding norm denoted by $\| \cdot \|$.

An equivalent norm is given by

\begin{equation}
\|x\|'' = \left( \sum_{i=1}^3 \left( \int_T \dot{x}_i^2(t)dt + [x_i]^2 \right) \right)^{\frac{1}{2}},
\end{equation}

where $[x]$ is the average $[x] = \int_T x(t)dt$. For every $x = x(t) \in \Lambda$ we denote $x^{-1}\Delta \subset T$ the set of collision times. The point $t_0 \in T$ is an isolated collision for $x \in \Lambda$ when $t_0 \in x^{-1}\Delta$ and there exists a neighborhood of $t_0$, $(t_0 - \epsilon, t_0 + \epsilon) \cap x^{-1}\Delta = \{t_0\}$.

A colliding cluster of particles at $t = t_0$ is a subset $k \subseteq \{1, 2, 3\}$ such that $x(t_0) \in \Delta_k$ and $x(t_0) \notin \Delta_{k', k}$.

The action functional is the positive-definite function $\mathcal{A}_\omega : \Lambda \to \mathbb{R} \cup \infty$ defined by

\begin{equation}
\mathcal{A}_\omega(x) = \int_T L_\omega(x(t), \dot{x}(t))dt, \quad \omega \in \mathbb{R}.
\end{equation}

(2.7)
The action functional \( \mathcal{A}_\omega \) is of class \( C^1 \) on the subspace of collision-free loops of \( \Lambda \); hence collisionless critical points of \( \mathcal{A}_\omega \) in \( \Lambda \) are \( T \)-periodic \( C^2 \)-solutions of the associated Euler–Lagrange equations

\[
(2.8) \quad m_i \ddot{x}_i = 2J \omega \dot{x}_i + \omega^2 x_i + \frac{\partial U}{\partial x_i};
\]

these solutions will be called classical solutions, and, in the non-rotating frame, they satisfy Newton equations (2.1).

(2.9) Remark. In (2.1) we assumed that configurations have center of mass in 0. This constraint requires a few words of comment. Indeed, while the action functional \( \mathcal{A}_0 \) is invariant under translations, the same is not true for \( \mathcal{A}_\omega \) with \( \omega \neq 0 \). This means that the center of mass constraint is in general not a natural reduction. However, the convexity property of the kinetic quadratic form with respect to the center of mass implies that all local minima of \( \mathcal{A}_\omega \) have center of mass in 0.

### 2.2 Symmetry constraints

Let \( G \) be a finite group, acting on a space \( X \), i.e. there exists a map \( G \times X \to X \), \( (g,x) \mapsto gx \), such that \( (g_1,g_2x) \mapsto (g_2 \cdot g_1)x \), where \( \cdot \) is the internal operations of \( G \). The space \( X \) is then called \( G \)-equivariant. Let \( x \in X \), the fixer or the isotropy group of \( x \) in \( G \) is the set \( G_x = \{ g \in G : gx = x \} \). When \( H \) is a subgroup of \( G \), then the space \( X^H \subset X \) consists of all points \( x \in X \) that are fixed by \( H \), \( X^H = \{ x \in X : G_x \supset H \} \).

Consider a finite group \( G \) and the 2-dimensional orthogonal representations \( \tau, \rho : G \to O(2) \) of \( G \). By \( \tau \) and \( \rho \), \( G \) acts on the time circle \( \mathbb{T} \subset \mathbb{R}^2 \) and on the Euclidean space \( E \), respectively. Moreover, we can consider the group homomorphism \( \sigma : G \to \Sigma_3 \), where \( \Sigma_3 \) is the group of all permutations of 3 elements, and we endow the set of indexes \( n = \{1, 2, 3\} \) with a \( G \) action. The homomorphism \( \sigma \) satisfies the property

\[
(2.10) \quad \forall g \in G : (\sigma(g)(i) = j \Rightarrow m_i = m_j).\]

Given \( \rho \) and \( \sigma \) with property (2.10), \( G \) acts orthogonally on the configuration space \( \mathcal{X} \) by

\[
(2.11) \quad \forall g \in G, \quad g \cdot (x_1, x_2, x_3) = (\rho(g)x_{\sigma(g^{-1})(1)}, \rho(g)x_{\sigma(g^{-1})(2)}, \rho(g)x_{\sigma(g^{-1})(3)}).\]

for every \( g \in G \). Understanding the action of \( G \) as a space linear transformation or a permutation on the index set, we can write

\[
(2.12) \quad \forall i \in n, \quad (gx)_i = gx_{g^{-1}i}.\]

Furthermore, we can use the representation \( \tau \) and the action of \( G \) on \( \mathcal{X} \) given by (2.11) to define the action of \( G \) on the set of loops \( \Lambda \) as follows

\[
(2.13) \quad \forall g \in G, \forall t \in \mathbb{T}, \forall x \in \mathcal{X}, \quad (g \cdot x)(t) = gx(g^{-1}t).\]

The loops in \( \Lambda^G \) are called equivariant loops and are the loops \( \mathbb{T} \to \mathcal{X} \) fixed by \( G \); \( \Lambda^G \) is a closed linear subspace of \( \Lambda \).
(2.14) Definition. Consider the action of $G$ on the index set $n = \{1, 2, 3\}$. The transitive decomposition of $n$ is the decomposition of $n$ into its $G$-orbits or, with an abuse of notation, their lengths. For example, if $G$ acts transitively on $n$ then $n$ is decomposed into a unique orbit of length 3. If $G$ acts trivially, then the decomposition is of type $1 + 1 + 1$, since $n$ is decomposed into its subsets of length 1.

In the sequel, if not explicitly stated, we will assume the following additional hypotheses on the group action:

(i) $\ker \tau \cap \ker \rho \cap \ker \sigma = 1$;
(ii) There exists no proper linear subspace $E' \subsetneq E$ such that $\forall i \in n, \forall x \in \Lambda^G, \forall t \in T, x_i(t) \in E' \subsetneq E$;
(iii) There is not an integer $k \neq \pm 1$ such that $\forall x \in \Lambda^G, \exists y \in \Lambda : \forall t \in T, x(t) = y(kt)$.

In fact, it is easy to show that there is a canonical isomorphism $\Lambda^G \cong \Lambda \hat{G}$ where $\hat{G} = G/\ker \tau \cap \ker \rho \cap \ker \sigma$, and so without loss of generality (i) can be assumed to be true. Furthermore, if (ii) holds, then it is possible to consider the $n$-body problem on the subspace $E' \subset E$. Finally, condition (iii) simply means that all loops are periodic of a period that is sub-multiple of $2\pi$, and therefore up to rescaling time it is possible to define another (smaller) group $G$. We will see later in section 3 how to use this idea to reduce the number of groups in the classification (condition (iii) is actually equivalent to definition (3.5) below).

(2.15) Definition. A symmetry group is termed homographic if all equivariant loops are constant up to similarity.

(2.16) Example. Consider a homographic group $G$ and a loop $x$ similar to a certain configuration $\xi$. Without loss of generality we can assume that $\sum m_i \xi_i^2 = 1$. Thus, elements of $\Lambda$ have the form $x(t) = z(t)\xi$, for a complex valued function $z(t)$ and a configuration $\xi$; the action functional $A$ is actually a function only of $z$ and $\xi$. It is easy to see that if $x(t) = z(t)\xi$ is a critical point of $A$ then $z(t)$ solves the Kepler problem and $\xi$ is a central configuration. We recall here the definition of central configuration (further details can be found in [26]).

(2.17) Definition. A configuration $\xi$ is said a central configuration if $\xi$ is a critical point for the potential $U$ on the ellipsoid $\sum m_i \xi_i^2 = 1$.

Central configurations for the 3-body problem are well-known. Up to similarity there is only one non-collinear central configuration, termed the Lagrange configuration (it is an equilateral triangle), which minimizes $U(x)$ in the ellipsoid $\sum m_i \xi_i^2 = 1$. On the other hand, there are 3 collinear central configurations, termed Euler-Moulton configurations (or simply Euler configurations), which are saddle points for $U$.

Associated with a planar central configuration there is a relative equilibrium motion, as we already pointed out in the introduction. Now we prove that they are
equivariant critical points for the action. Consider first the group \( G = SO(2) \) acting canonically on the time circle \( \mathbb{T} \) and the Euclidean plane \( \mathbb{R}^2 \cong \mathbb{C} \), and trivially on the index set \( \{1, 2, 3\} \). The equivariant constraints are the following: for every \( \theta \in \mathbb{R} \) and every \( t \in \mathbb{R} \) one has \( x(t + \theta) = e^{i\theta} x(t) \). Thus the group is homographic and one can easily verify that the action is an affine function of \( U(\xi) \); therefore critical points of \( A^G \) are the rotating central configurations above (for every choice of masses). The minimal action is attained on the Lagrange motion (since the Lagrange configuration minimizes the potential \( U \)).

Furthermore, consider the group \( G = O(2) \) acting again canonically. In this case the equivariant constraints are as follows: for every \( \theta \in \mathbb{R} \) and every \( t \in \mathbb{R} \) one has \( x(t + \theta) = e^{i\theta} x(t) \), and, additionally, it must be that \( x(-t) = \overline{x}(t) \) for every \( t \in \mathbb{R} \) (where \( \overline{x} \) is the complex conjugate of \( x \in \mathbb{C} \)). The latter constraint implies that \( x(0) \) belongs to the real line, that is, the rotating configuration must be collinear. Hence, the only critical points are rotating Euler-Moulton configurations. In the paper we will deal with finite groups of symmetries since, as shown above, infinite symmetry groups are always homographic.

From our point of view homographic groups are not interesting, since they yield critical points which are already well-known. Another class of groups that are unsuitable to our approach are groups for which symmetry constraints force the occurrence of collisions.

**Definition.** Let \( G \) be a finite group acting on the set \( \Lambda \), we say that \( G \) is bound to collisions if every \( G \)-equivariant loop has at least a collision time

\[ \forall x \in \Lambda^G, x^{-1} \Delta \neq \emptyset \]

**Example.** Consider the group \( G \) of order 2 acting on \( \mathbb{T} \) as a reflection along a line, trivially on \( \mathbb{R}^2 \) and by permuting 1 and 2 in the index set \( \{1, 2, 3\} \). Thus the equivariant constraints are:

\[ x_1(-t) = x_2(t), \quad x_2(-t) = x_1(t), \quad x_3(-t) = x_3(t). \]

This implies that, for \( t = 0 \) or \( t = \pi \) one has \( x_1(t) = x_2(t) \), and hence all equivariant loops have collisions at times 0 and \( \pi \).

**Example.** Consider the dihedral group \( K = D_6 \) of order 6 with generators \( g, h \) of order 3 and 2 respectively, acting on \( \mathbb{T}, \mathbb{R}^2 \cong \mathbb{C} \) and \( \{1, 2, 3\} \) as follows:

\[
\begin{align*}
t \in \mathbb{T} \implies g(t) &= t, \quad h(t) = t \\
v \in \mathbb{C} \implies g(v) &= e^{2\pi i v}, \quad h(v) = \overline{v} \\
g(1) &= 2, \quad g(2) = 3, \quad g(3) = 1, \quad h(1) = 2, \quad h(2) = 1, \quad h(3) = 3.
\end{align*}
\]

As a consequence, if \( x(t) \) is an equivariant loop in \( \Lambda^K \), then, for each instant \( t \in \mathbb{T} \), the configuration \( x(t) \) is an (oriented) equilateral triangle symmetric with respect to the real line. In other words, the configuration space has dimension 1, with a subspace of dimension 0 consisting of triple collisions. Notice that \( K \) acts trivially on the time circle \( \mathbb{T} \). Now extend \( K \) by an (central) element \( r \) of order 2, which acts on \( \mathbb{T} \) by \( r(t) = t + \pi \), on \( \mathbb{R}^2 \) by \( r(v) = -v \) and trivially on \( \{1, 2, 3\} \). It is easy to see that for the group \( G = K \times C_2 \) all loops in \( \Lambda^G \) have collisions at some time \( t \in \mathbb{T} \).
Consider the normal subgroup \( \ker \tau \triangleleft G \) (which will be termed \textit{core} of \( G \) – if it is trivial the group will be said \textit{with trivial core}) and the group \( \bar{G} = G/\ker \tau \) acting on \( T \).

\section{Definition}
If the group \( \bar{G} \) acts trivially on the orientation of \( T \) then \( \bar{G} \) is cyclic and we say that the action type of \( G \) on \( \Lambda \) is of \textit{cyclic type}. If the group \( G \) consists of a single reflection on \( T \), then we say that the action type of \( G \) on \( \Lambda \) is of \textit{brake type}. Otherwise we say that the action of \( G \) on \( \Lambda \) is of \textit{dihedral type}.

\section{Definition}
The \( T \)-\textit{isotropy subgroups} of \( G \) are the isotropy subgroups of the action of \( G \) on \( T \) (induced by \( \tau \)).

\section{Definition}
Let \( I \subset T \) be the closure of a \textit{fundamental domain} for the action of \( \bar{G} \) on \( T \) defined as follows. When the action type is cyclic then \( I \) is a closed interval connecting the time \( t = t_0 \) with its image \( zt \) with a cyclic generator \( z \) in \( \bar{G} \); in this case \( I \) can be chosen among infinitely many intervals. Otherwise, \( I = [t_0, t_1] \subset T \) where \( t_0, t_1 \) are distinct and have non minimal \( T \)-isotropy subgroups in \( G \); moreover no other point in \((t_0, t_1)\) has non-minimal isotropy. There are \( |\bar{G}| \) such intervals.

Let \( H_0 \) and \( H_1 \) be the isotropy subgroups of \( t_0, t_1 \) respectively. If the action type is brake then \( H_0 = H_1 = G \); while if it is dihedral, then \( H_0 \) and \( H_1 \) are distinct and proper subgroups of \( G \). Further more the length of \( I \) is always \( \frac{t_1 - t_0}{|\bar{G}|} \), since \( T = \bigcup_{g \in \bar{G}} gI \) and the interiors of the terms in the sum are disjoint.

Let now consider the restriction of the Lagrangian action to the \( G \)-equivariant loops

\begin{equation}
\mathcal{A}_G^G : \Lambda^G \rightarrow \mathbb{R} \cup \infty.
\end{equation}

\begin{equation}
\mathcal{A}_G^G(x) = \int_T L_\omega(x(t), \dot{x}(t)) dt = |\bar{G}| \int_I L_\omega(x(t), \dot{x}(t)) dt.
\end{equation}

The action functional \( (2.7) \) is termed \( G \)-invariant when \( A_\omega(x) = A_\omega(gx) \); then the following result holds:

\section{Palais Principle of symmetric criticality}
If \( A_\omega \) is \( G \)-equivariant then a collisionless critical point of \( A_\omega^G \) is a critical point of \( A_\omega \).

\subsection{Coercivity and generalized solutions}

The action functional \( A_\omega^G : \Lambda^G \rightarrow \mathbb{R} \cup \infty \) is termed \textit{coercive} in \( \Lambda^G \) if \( A_\omega^G(x) \) diverges to infinity as the \( H^1 \)-norm of \( x \) goes to infinity in \( \Lambda^G \). This is a fundamental property, for it guarantees the existence of minimizers on the set \( \Lambda^G \) for the functional \( A_\omega^G \). Remark that this definition is equivalent to

\section{Definition}
The action functional \( A_\omega^G(x) \) is coercive on \( \Lambda^G \) if and only if

\begin{equation}
\inf_{x \in \Lambda^G, \|x\|_{L^2} = 1} K_\omega > 0.
\end{equation}

One can easily deduce the following corollary.
**Corollary.** When $\omega \notin \mathbb{Z}$ (and the period $T = 2\pi$), for every finite group $G$ acting on the loop space the action functional $A^G_\omega$ is coercive on $\Lambda^G$.

From now on we take the period $T$ to be $2\pi$, in order to keep the statements simple, unless otherwise explicitly stated.

The next proposition immediately follows from Proposition (4.1) in [17].

\textbf{(2.28)} When $\omega = 0$, then $A^G_\omega = A^G_0$ is coercive on $\Lambda^G$ if and only if $X^G = 0$.

When the functional $A^G_\omega$ is coercive on the subspace $\Lambda^G$ and its minima are collisionless, the latter are classical solutions of the Newton equations (2.8). To deal with the possible occurrence of collisions some authors introduced the notion of generalized solutions. A generalized solution (of (2.8)) is a $H^1$ path $x(t)$ defined on an interval $(T_0, T_1)$ such that in $(T_0, T_1) \setminus x^{-1}(\Delta)$ is a classical $C^2$-solution, the Lagrangian action is finite on $(T_0, T_1)$ and partial energies are preserved after collisions:

$$\forall t \in [t_0, t_1] \subset (T_0, T_1), \forall k \subset n : x(t) \notin \Delta_{k,k'} \implies E_k \in H^1((t_0, t_1), \mathcal{X}),$$

where $E_k$ is the partial energy of the cluster $k$.

We can then conclude the study of the existence of minimizers of the restricted action functional $A^G_\omega$ with the following proposition ((4.12) in [17]).

\textbf{(2.29) Proposition.} When $A^G_\omega$ is $G$-invariant and coercive on $\Lambda^G \subset \Lambda$, then there exists at least a minimum of the Lagrangian action $A^G_\omega$ which yields a generalized solution of (2.8) in $\Lambda^G$.

Sometimes we shall refer the property of being coercive directly to the symmetry group $G$.

\textbf{(2.30) Definition.} A symmetry group $G$ is said to be coercive if the action functional $A^G_0$, obtained with $\omega = 0$, is coercive in $\Lambda^G$.

### 3 Symmetry groups of type R and rotating frames

The determinant of the homomorphism $\rho: G \to O(2)$ is defined by composition with $\det: O(2) \to \{+1, -1\}$ (the same for $\tau$).

\textbf{(3.1) Definition.} A symmetry group $G$ is said of type $R$ if the determinant homomorphisms $\det(\rho), \det(\tau): G \to \{+1, -1\}$ coincide.

\textbf{(3.2) For every $\omega \in \mathbb{R}$, the Lagrangian action functional is invariant with respect to the action of symmetry groups of type $R$.}

\textit{Proof.} The proof is straightforward. $q.e.d.$

Now, consider the following change of coordinates in $H^1(\mathbb{R}, \mathcal{X})$ (rotating frame with angular velocity $\omega$): $(\forall i \in n) x_i(t) = e^{i\omega t} q_i(t)$. A path $x(t)$ in $\Lambda$ is $G$-equivariant if an only if $x(gt) = gx(t)$ for every $g \in G$ and every $t \in \mathbb{T}$. That means that for
every $g \in G$ there exists a real number $\delta = \delta(g)$ such that for the corresponding path $q(t)$ in the rotating frame

$$e^{i\omega(t+\delta(g))}q(gt) = ge^{i\omega t}q(t)$$

for every $t \in \mathbb{R}$ if $g$ preserves the orientation in $\mathbb{T}$, while if $g$ reverses the orientation

$$e^{i\omega(-t+\delta(g))}q(gt) = ge^{i\omega t}q(t)$$

If we assume $G$ to be of type $\mathbb{R}$, then $g$ reverses the time orientation if and only if it reverses the space orientation, and therefore in both cases

$$(3.3) \quad q(gt) = e^{-i\omega \delta(g)}gq(t).$$

In order to let a single change of coordinates be enough to reduce as much as possible the group of symmetries, one chooses $g$ as an element in $\ker \det \tau \setminus \ker \tau$ rotating the time circle with minimal (non-zero) angle. Being a generator of the (cyclic) quotient $(\ker \det \tau) / \ker \tau$, such an element and all its powers will act trivially on the plane, once a suitable angular velocity $\omega$ is chosen. One can hence prove the following useful lemma:

**Lemma 3.4** Up to a suitable change of coordinates (in a rotating frame) a symmetry group of type $\mathbb{R}$ has the following property: every symmetry $g$ acting as a non-trivial shift on the time line acts as the identity on the euclidean space. In other words, for every element $g \in \ker \det \tau \setminus \ker \tau$ (equivalently, every element $g \in G$ such that $\tau(g)$ is a non-trivial translation on the time line), the image $\rho(g)$ is trivial: $(\ker \det \tau \setminus \ker \tau) \subset \ker \rho$.

Such a change of coordinates in rotating frames is useful for simplifying orbits with many symmetries. For example, consider the orbits in figure 2(a) and 2(b) (these orbits are equivariant minimizers for the group described in Remark 4.17).

In the inertial (non-rotating) frame the orbit enjoys a symmetry group of order 60, with many rotations; the synchronization of the rotating frame with the symmetry rotations allows the symmetry group to be reduced to a much simpler form, in rotating coordinates (practically, by canceling out the rotation part in the symmetry group). This is just part of the wider notion of redundant symmetries.

![Figure 2](image-url)
Definition. An element $g$ of a symmetry group $G$ is termed redundant if $g \neq 1$, $\tau(g) \neq 1$, $\det(\tau(g)) = 1$ and $\sigma(g) = \rho(g) = 1$. A group with redundant elements is termed a redundant group.

Hence a redundant symmetry is an element $g$ which not only is an element $g \in G$ such that $\tau(g)$ is a non-trivial translation on the time line and $\rho(g)$ is the identity, but also such that $\sigma(g)$ is the identical permutation. Clearly redundant elements do not give any true constraint on the loops, and they can be modded out. Given a symmetry group we will implicitly assume that the maximal redundant subgroup (i.e. the subgroup consisting of all redundant symmetries) has been modded out, so that, unless otherwise stated, the group has no redundant elements. On the other hand, a change in rotating coordinates given as seen above can be used to create a redundant part that once mod out gives a smaller symmetry group. Summarizing, there are two ways of considering a group $G$ equivalent to another group $G'$: either $G'$ is equal to $G/H$, where $H$ is a subgroup of redundant symmetries, or $G'$ is equal to $G$ after a change in the rotating angular velocity (for groups of type R). One might want to define an equivalence relation in symmetry groups, and to consider always the reduced minimal group, that is the group with the minimum number of elements. Since sometimes it is easier to work with non-minimal groups, we leave skip this notion of equivalence and will treat the groups on a case-by-case fashion.

Remark. A symmetry group of type R such that all its elements acting as a non-trivial shifts on the time line act as the identity on the euclidean space (that is, a group fulfilling the conclusion of (3.4)) of course can be redundant, in a frame rotating with suitable angular velocity $\omega$. To see how $\omega$ changes when we eliminate the redundant symmetries of the group (and hence the period has to be rescaled) we proceed in the following way. Let $g$ be an element in $\ker(\det(\tau)) \subset G$ which rotates the time circle $T$ of a minimal angle (i.e. the elements inducing one of the two standard cyclic generators in $\ker(\det(\tau)) / \ker \tau$). Let $c$ denote its order. If the group $G$ is of type R, then $\rho(g)$ is a rotation of angle $2\pi b/c$, for some integer $b$ with $|b| < c$. According to its definition, $\delta(g) = 2\pi/c$, and hence $\omega$ will be chosen in a way to satisfy the equation $\omega^{2\pi c} = \frac{2\pi b}{c} \mod 2\pi$, that is $\omega = b \mod c$. On the other hand the true period of the paths in the rotating frame will be a suitable multiple of $2\pi/c$ instead of $2\pi$ (depending on the period $s$ of the permutation $\sigma(g)$), hence, after making the action not redundant, the corresponding $\omega$ will be simply $\omega' = \frac{2\pi b}{c}$, and thus chosen so that

$$\frac{\omega'}{s} = \frac{b}{c} \mod 1.$$  

Therefore a symmetry group of type $R$, not redundant and with $\omega' \neq 0$, in this method yields back a symmetry group with $\omega = 0$ provided that equation (3.7) can be solved for $b, c$ integers. Furthermore, the symmetry group found is unique up to redundancies.

For an equivariant path $x(t)$ let $J(t)$ denote its angular momentum

$$J(t) = \sum_{i \in \mathbb{R}} m_i x_i \times \dot{x}_i$$

(since we are dealing with planar paths, $J(t)$ belongs to a copy of $\mathbb{R}$ orthogonal to the plane $E$).
(3.9) For every equivariant $x(t) \in \Lambda^G$,

$$J(gt) = \det(\rho(g)) \det(\tau(g))J(t) \in \mathbb{R}. $$

Proof. We have the chain of equalities:

$$J(gt) = \sum_{i \in \mathbb{N}} m_i x_i(gt) \times \dot{x}_i(gt)$$

$$= \sum_{i \in \mathbb{N}} m_i [(gx_{g^{-1}i}(t)) \times (\det(\tau(g))g\dot{x}_{g^{-1}i}(t))]$$

$$= \det(\tau(g)) \sum_{i \in \mathbb{N}} m_{g^{-1}i} [(gx_{g^{-1}i}(t)) \times (g\dot{x}_{g^{-1}i}(t))]$$

$$= \det(\tau(g)) \sum_{i \in \mathbb{N}} m_{g^{-1}i} \det(\rho(g)) [x_{g^{-1}i}(t) \times \dot{x}_{g^{-1}i}(t)]$$

$$= \det(\tau(g)) \det(\rho(g)) J(t).$$

q.e.d.

(3.10) If the symmetry group $G$ is not of type R, then every $G$-equivariant path $x(t)$ has zero angular momentum.

Proof. Since $J$ is constant, if there is $g \in G$ such that $\det(\rho(g)) \neq \det(\tau(g))$ then

$$J = \det(\rho(g)) \neq \det(\tau(g))J = -J$$

which means that $J = 0$ if $G$ is not of type R. q.e.d.

(3.11) Definition. A symmetry group $G$ is termed fully uncoercive if it is neither of type R nor coercive (actually, this implies that it is not possible to find a suitable $\omega$ such that the action is coercive – on the converse, if the action is of type R is always possible to find an $\omega$ such that the action functional is coercive).

4 Planar symmetry groups

In this section we will list the principal symmetry groups for the planar 3-body problem.

4.1 The trivial symmetry

Let $G$ be the trivial subgroup of order 1. It is clear that it is of type R, it has the rotating circle property. It yields a coercive functional on $\Lambda^G = \Lambda$ only for $\omega \neq 0 \mod 1$ (see [2.27] and [2.28]), that is, if and only if $\omega$ is not an integer. If $\omega = \frac{1}{2} \mod 1$ then the minimizers are minimizers for the anti-symmetric symmetry group (which is also termed Italian symmetry by some authors) $x(t + \pi) = -x(t)$. It is worth noting that the masses can be different.

(4.1) For every $\omega \notin \mathbb{Z}$ and every choice of masses the minimum for the trivial symmetry occurs in the relative equilibrium motion associated to the Lagrange central configuration.
Proof. The idea of the proof comes from the proof of a similar proposition in [11]. Consider the action functional defined in (2.7) and the energy and potential functionals

\[ K_\omega = \frac{1}{2} \sum_{i=1}^{3} m_i \int_0^{2\pi} |\dot{x}_i - J \omega x_i|^2, \quad U = \int_0^{2\pi} U(x). \]

The following estimate holds

\[ U \geq U_0 \int_0^{2\pi} \frac{1}{I^{\alpha/2}}, \]

where \( I = \sum_i m_i |x_i|^2 \) is the momentum of inertia of the system and \( U_0 \) is the minimal value of the normalized potential \( \tilde{U}(x) = \frac{U(x)}{I^{\alpha/2}} \).

The equality \( U = U_0 \int_0^{2\pi} \frac{1}{I^{\alpha/2}} \) is reached if and only if at every instant \( t \) the configuration is Lagrangian. Consider the Fourier representation of the trajectories \( x_i, x_i(t) = \sum_{n \in \mathbb{Z}} c_{i,n} e^{i n t}, c_{i,-n} = \bar{c}_{i,n}, i = 1,2,3 \); the kinetic part on the action functional can be estimated as

\[ K_\omega = \frac{1}{2} \sum_{i=1}^{3} m_i \int_0^{2\pi} \sum_{n \in \mathbb{Z}} (n - \omega)^2 |c_{i,n}|^2 \geq \frac{1}{2} \min_{n \in \mathbb{Z}} (n - \omega)^2 \int_0^{2\pi} I = \frac{c(\omega)}{2} \int_0^{2\pi} I \]

where \( c(\omega) = (k - \omega)^2 \) and \( k \) is the integer closest to \( \omega \). The equality is attained if and only if \( x_i(t) = \tilde{a}_i \sin(kt) + J\tilde{b}_i \cos(kt) \), where \( \tilde{a}_{i,k}, \tilde{b}_{i,k} \in \mathbb{C}, i = 1,2,3 \). We can conclude that

\[ A_\omega(x) \geq \frac{2\pi}{2} \frac{c(\omega)}{I^{\alpha/2}} \frac{U_0}{I^{\alpha/2}} \geq 2\pi \left( \frac{c(\omega)}{2} I_{\min} + \frac{U_0}{I_{\min}^{\alpha/2}} \right) \]

for every loop \( x \in \Lambda^G = \Lambda \). The equality is reached if and only if all the following conditions are verified:

(i) the momentum of inertia is constantly equal to \( I_{\min} = \left( \frac{2U_0}{c(\omega)} \right)^{1/2} \);

(ii) at every instant the configuration is Lagrangian;

(iii) there exist \( \tilde{a}_i, \tilde{b}_i \in \mathbb{R}^2 \), such that \( x_i(t) = \tilde{a}_i \sin(kt) + J\tilde{b}_i \cos(kt), i = 1,2,3 \), where \( k \) is the integer closest to \( \omega \).

From (iii) we deduce the existence of a time-dependent complex-valued function \( \lambda(t) \) such that \( x_i(t) = \lambda(t) \xi_i \), where \( (\xi_i) \) is a Lagrange central configuration. But the momentum of inertia is fixed and, up to reflection there is a unique central configuration with \( I = I_{\min} \). The complex function \( \lambda \) is then a rotation fixing the center of mass of the system and the trajectories \( x_i(t) \) are circles.

q.e.d.

4.2 The line symmetry

Another case of symmetry group of type R with arbitrary masses is the line symmetry: the group is a group of order 2 acting by a reflection on the time circle \( \mathbb{T} \), by
a reflection on the plane $E$, and trivially on the set of indexes $n$. That means, at time 0 and $\pi$ the masses are collinear, on a fixed line $l \subset E$. By (2.27) and (2.28) it is coercive only when $\omega \not\in \mathbb{Z}$. In this case the Lagrangian solution cannot be a minimum.

Consider the homographic motions from the Euler configuration (with the third body in the center)

$$x_1(t) = Re^{jkt}, x_2(t) = -Re^{jkt}, x_3(t) = 0 \quad (\text{for } t \in [0, 2\pi], R > 0, k \in \mathbb{Z}).$$

These trajectories are the Euler orbits. In a rotating frame with angular velocity $\omega$ (when the masses are all equal to 1) the action functional can be computed as

$$\mathcal{A}_E(R, \omega, \alpha) = R^2(k - \omega)^2 + \frac{2}{R^\alpha} + \frac{1}{(2R)^\alpha}.$$

Thus, for $\alpha = 1$, the minimal value of $\mathcal{A}_E$ is attained on $R_{\omega} = \sqrt[3]{\frac{5}{4(k-\omega)^2}}$ and

$$\min_{R>0} \mathcal{A}_E(R, \omega, 1) = 2\pi \frac{3}{2} \frac{\sqrt[3]{25(k-\omega)^2}}{\sqrt[3]{2}}.$$

In particular when $\omega = \frac{1}{2}$,

$$\min_{R>0} \mathcal{A}_E(R, \frac{1}{2}, 1) = \mathcal{A}_E(\sqrt[3]{5}, \frac{1}{2}, 1) = 2\pi \frac{3}{4} \sqrt[3]{25}.$$

Actually, there are two minima, one for $k = 0$ and one for $k = 1$ (the action levels of these two Euler orbits can be found in figure 3 under the names Euler1 and Euler2).

![Figure 3: Action levels for the line symmetry](image-url)
Our aim now is to prove the existence of a motion which is not homographic, whose action level is less than the minimum on the Euler orbits, for $\alpha = 1$ and $m_i = 1$ for $i = 1, 2, 3$. Consider the following family of test paths

$$x_1(t) = d + Re^{ikt}, x_2(t) = d - Re^{ikt}, x_3(t) = -2d,$$

for $t \in [0, 2\pi], 0 < R < 3d, k \in \mathbb{Z}$. The center of mass lies in the origin.

The action functional is

$$\frac{A_H(R, d, \omega)}{2\pi} = \frac{1}{2\pi} \int_0^{2\pi} \left( |\dot{x}_1 - J\omega x_1|^2 + \frac{4\omega^2 d^2}{2} + \frac{1}{2R} + \frac{2}{|3d + Re^{ikt}|} \right) dt$$

$$= 3\omega^2 d^2 + R^2 (k - \omega)^2 + \frac{1}{2R} + \frac{2}{3d} \int_0^{2\pi} \frac{dt}{|1 + \frac{2}{3d} e^{ikt}|}.$$

Thus when $\omega = \frac{1}{2}$, by Lemma (8.7) and (8.13) in [17],

$$\frac{A_H(R, d, \frac{1}{2})}{2\pi} \leq \frac{3}{4} d^2 + \frac{1}{4} R^2 + \frac{1}{2R} + \frac{2}{3d} \left( 1 - \frac{1}{2} \log \left( 1 - \frac{R^2}{9d^2} \right) \right).$$

Now observe that by setting $R = 1$ and $d = \frac{4}{5}$ one obtains

$$\frac{A_H(1, \frac{4}{5}, \frac{1}{2})}{2\pi} \leq \frac{3}{4} \left( \frac{4^2}{5^2} + \frac{1}{4} + \frac{1}{2} + \frac{5}{6} \left( 1 - \frac{1}{2} \log \left( 1 - \frac{5^2}{9 \cdot 4^2} \right) \right) \right)$$

$$= \frac{619}{300} - \frac{5}{12} \log \left( \frac{119}{144} \right).$$

Hence, by (4.4) the claim follows once it is shown that

$$\frac{619}{300} - \frac{5}{12} \log \left( \frac{119}{144} \right) < \frac{3}{4} \sqrt{25}.$$

But one has $\frac{144}{119} = 1 + \frac{25}{119}$ and hence

$$\frac{619}{300} + \frac{5}{12} \log \left( \frac{144}{119} \right) < \frac{619}{300} + \frac{5}{12} \log \left( \frac{144}{119} \right) = \frac{38393}{17850}$$

which can be easily (and rigorously) seen to be less than $\frac{3}{4} \sqrt{25}$.

In figure 3, besides the action levels of two Euler orbits, it is shown the graphs of the action functional of two minimizer orbits (computed numerically): the curve there named Hill1 corresponds to a test path with $k = 1$ and the curve named Hill2 corresponds to the value $k = 0$. By continuity of $A_H$ with respect to the masses $m_i \in \mathbb{R}_+^*$, $i = 1, 2, 3$, and to $\omega$, the following result holds.

(4.8) There exists an open set containing $\frac{1}{2} + \mathbb{Z} \subset \mathbb{R}$ such that if the masses are approximately equal then the minimum of the action functional restricted to the equivariant orbits with respect to a line symmetry group is not homographic (i.e. not the relative equilibrium motion associated to the Euler configuration).

A possible candidate for such minimizer is depicted in figure 4(a) (in rotating coordinates) and in figure 4(b) (inertial coordinates). It is a well-known orbit, which was found numerically for instance in [18, 31]. In particular, Chen in [8] propose a proof for this orbit in which numerical methods play a very minor role and are carefully justified. The reader is referred to [16] (which is also available on www.arxiv.org or from the web-site of the MPI) and to [8, 9] for other orbits and symmetries of this type. Numerical action level estimates of retrograde test paths can be found also in section 4 of [9] and section 7 of [16].
4.3 The 2-1-choreography symmetry

Consider the group of order 2 acting as follows: $\rho(g) = 1$, $\tau(g) = -1$ (that is, the translation of half-period) and $\sigma(g) = (1, 2)$ (that is, $\sigma(g)(1) = 2$, $\sigma(g)(2) = 1$, and $\sigma(g)(3) = 3$. That is, it is a half-period choreography for the bodies 1 and 2. It is of type R, and coercive for a suitable choice of $\omega$. By (2.28) and (2.27), the choice is $\omega \notin \mathbb{Z}$. The first two masses have to be equal $m_1 = m_2$. Consider as above the case of equal masses. Euler orbits (defined in (4.2)) and the test paths (defined in (4.5)) are $G$-equivariant if and only if $k = 1 \mod 2$ (this, incidentally, implies that $A^G_\omega$ is coercive if and only if $\omega \notin \mathbb{Z}$). Roughly speaking, this implies that the branch termed Euler1 in figure 3 (which is a Euler path with $k = 0$) does not appear in a corresponding graph for 2-1-choreographies, while both Hill1 and Hill2 do. We are going to prove that actually this is the case, and that therefore under such a constraint the minimizer is not homographic for a rather bigger interval of angular speeds $\omega$.

(4.9) There is an open set $A$ containing $[-\frac{1}{2}, \frac{1}{2}]$ such that if $\omega \in A + 2\mathbb{Z}$ then minimizers under the 2-1-choreography symmetry group are not homographic. (i.e. not the relative equilibrium motion associated to the Euler configuration).

Proof. Consider $\omega \in [0, \frac{1}{2}]$; the action $A_E$ of an Euler minimizer is given by (4.3) with $k = 1$, since for $k = 0$ the orbit is not equivariant. This is a decreasing function of $\omega \in [0, \frac{1}{2}]$. On the other hand, the action functional $A_H$ evaluated at the test path (4.5) as in (4.6) (with $k = 1$) is a (convex) function of $\omega$. It has already been shown in (4.8) that, for $\omega = \frac{1}{2}$, $A_H < A_E$. Deriving the difference $A_H - A_E$ after substituting $d = \frac{4}{5}$ and $R = 1$ as above one obtains a function

$$\frac{d}{d\omega}(A_H - A_E) = 2\pi \left( \frac{146}{25}\omega - 2 + \left(\frac{25}{2}\right)^{1/3}(1 - \omega)^{-1/3} \right),$$

which can be easily seen to be increasing in $[0, \frac{1}{2}]$ and positive in 0. Thus the difference $A_H - A_E$ is increasing in $[0, \frac{1}{2}]$, and hence everywhere negative since it is so in $\frac{1}{2}$. The graphs of the action levels are shown in figure 5, under the labels Hill1 and Euler2. q.e.d.
Remark. We give a few words on figure 5 which represents the action levels (computed numerically). There is a third branch of action levels, labeled Hill3, which corresponds to a non-retrograde Hill-like orbit. In fact, for \( \omega = 0 \), it is easy to see that the two action levels correspond to a Kepler 2-body problem (where, as \( \omega \) tends to 0, the third particle escapes to infinity – the functional is not coercive) and a rotating Euler solution with the third particle in 0. More precisely, for \( \omega = 0 \) there are two 2-body Kepler solutions (one rotating clockwise with \( k = -1 \) and the other counter-clockwise with \( k = 1 \)) and two Euler solutions (one clockwise with \( k = -1 \) and the other counter-clockwise with \( k = 1 \)) as limit cases. As \( \omega \) increases (thus considering a frame rotating counter-clockwise), the action of the Euler solution will decrease if \( k = 1 \) and will increase if \( k = -1 \) (the graph of the latter Euler solution does not appear in figure 5). On the other hand, both the Hill-like solutions will have an increased action, since there will appear the interaction of the third body, but the action of the orbit with \( k = 1 \) turns out to be less than that with \( k = -1 \), as one would expect. A possible trajectory for the minimizer (with \( \omega = \frac{2}{5} \) and equal masses) in the inertial frame is shown in figure 6. Equivalently, this orbit can be found as a minimizer of \( \mathcal{A}_0^G \), where the redundant symmetry group \( G \) is cyclic of order 10 (hence the action is of cyclic type) and generated by an element \( g \) which acts as follows: \( \tau(g) \) is a cyclic rotation of \( \pi/5 \), \( \rho(g) \) is a rotation of angle \( 2\pi/5 \) and \( \sigma(g) \) is the permutation \((1,2)\). More generally, choosing \( \rho(g) \) to be a rotation of angle \( \theta \cdot 2\pi \), with \( \theta \) rational, yields a non-homographic periodic symmetric orbit provided that \( \theta \in (0, \frac{1}{4}] \).

4.4 The isosceles symmetry

The isosceles symmetry can be obtained as follows: the group is of order 2, generated by \( h \); \( \tau(h) \) is a reflection in the time circle \( \mathbb{T} \), \( \rho(h) \) is a reflection along a line \( l \) in
Figure 6: A 2-1-choreography with $\omega = \frac{2}{3}

E$, and $\sigma(h) = (1,2)$ as above. The constraint is therefore that at time 0 and $\pi$ the 3-body configuration is an isosceles triangle with one vertex on $l$ (the third). By (2.27) and (2.28) it is coercive if and only if $\omega \not\in \mathbb{Z}$.

**4.11** For every $\omega \not\in \mathbb{Z}$ and every choice of masses (compatible with the isosceles constraint) the minimum for the isosceles symmetry occurs in the relative equilibrium motion associated to the Lagrange configuration.

Proof. It follows directly from (4.1). q.e.d.

### 4.5 The Euler–Hill symmetry

Now consider the symmetry group with a cyclic generator $r$ of order 2 (i.e. $\tau(r) = -1$) and a time reflection $h$ (i.e. $\tau(h)$ is a reflection of $\mathbb{T}$) given by $\rho(r) = 1$, $\sigma(r) = (1,2)$, $\rho(h)$ is a reflection and $\sigma(h) = ()$. It contains the 2-1-choreography (as the subgroup ker $\det(\tau)$), the isosceles symmetry (as the isotropy of $\pi/2 \in \mathbb{T}$) and the line symmetry (as the isotropy of $0 \in \mathbb{T}$) as subgroups.

Since Euler orbits (defined in (4.2)) and the Hill-like test paths (defined in (4.5)) are equivariant if $k = 1$, one can then draw the following immediate consequence (see (4.8) and (4.9)).

**4.12** The minimum of the Euler–Hill symmetry is not homographic for all values of $\omega$ in an open set containing $[-\frac{1}{2}, \frac{1}{2}] + 2\mathbb{Z}$, provided that the masses are approximately equal and two of them equal.

### 4.6 The choreography symmetry

The choreography symmetry is given by the group $C_3$ of order 3 acting trivially on the plane $E$, by a rotation of order 3 in the time circle $\mathbb{T}$ and by the cyclic permutation $(1,2,3)$ in $n$. The proof of (4.1) can be modified in a few points in order to give the following result, which has been proved in [5] – Remark 4. It is easy to see that the action functional is coercive if and only if $\omega \neq \pm 1/3 \mod 1$.

**4.13** For every $\omega \neq \pm 1/3 \mod 1$ the minimal choreography of the three-body problem is a rotating Lagrange configuration.
4.7 The Lagrange symmetry

The Lagrange symmetry group is the extension of the choreography symmetry group by the isosceles symmetry group. Thus, it is a dihedral group of order 6 and the action is of type R. Hence, the relative equilibrium motions associated to the Lagrange configuration are admissible motions for this symmetry and the following result follows from both \([4.1]\) and \([4.13]\). As for the choreography symmetry, the action functional is coercive if and only if \(\omega \neq \pm 1/3 \mod 1\).

\((4.14)\) For every \(\omega \neq \pm 1/3 \mod 1\) and every choice of masses the minimum for the Lagrange symmetry occurs in the relative equilibrium motion associated to the Lagrange configuration.

4.8 The Chenciner–Montgomery symmetry group and the eights

In this section we will describe three symmetry groups (up to change of coordinates) that might yield the Chenciner–Montgomery figure eight orbit as a minimizer: they are the only symmetry groups not of type R for \(n \leq 3\) bodies in the plane.

First, we consider the only symmetry group of cyclic action type, not of type R and coercive (under these hypotheses the group is automatically not bound to collisions and transitive, as we will see later). The group \(C_6\) (the cyclic eight) has order 6, acts cyclically on \(\mathbb{T}\) (i.e. by a rotation of angle \(\pi/3\)), by a reflection in the plane \(E\), and by the cyclic permutation \((1, 2, 3)\) in the index set. The second group, which we denote by \(D_{12}\), is the group of order 12 obtained by extending \(C_6\) with the element \(h\) defined as follows: \(\tau(h)\) is a reflection in \(\mathbb{T}\), \(\rho(h)\) is the antipodal map in \(E\) (thus, the rotation of angle \(\pi\)), and \(\sigma(h)\) is the permutation \((1, 2)\). This is the symmetry group used in \([12]\).

The third group is the subgroup of \(D_{12}\) generated by \(h\) and the subgroup \(C_3\) of order 3 of \(C_6 \subset D_{12}\). We denote this group \(D_6\) (since it is a dihedral group of order 6). The symmetry groups \(D_{12}\) and \(D_6\) are of dihedral type. The choreography group \(C_3\) is a subgroup of all the three groups, thus they are coercive. Since they are not of type R, the minimum is not a homographic solution (since it has trivial \(J\)).

\((4.15)\) Proposition. If \(\alpha > 2\), then the minimum in \(\Lambda^{D_{12}}\) (the Chenciner–Montgomery eight) coincides with the minimum in \(\Lambda^{D_6}\): Any local minimizer symmetric with respect to the group \(D_6\) is symmetric with respect to the group \(D_{12}\).

Proof. Let \(H_0, H_1 \subset D_6\) be the subgroups generated respectively by the elements \(h_0\) and \(h_1\) given by \(\rho(h_0) = \rho(h_1) = -1\) and \(\sigma(h_0) = (1, 2), \sigma(h_1) = (2, 3)\). If \(x(t)\) is a \(D_6\)-equivariant, then \(t \in T^{H_0} \implies x(t) \in X^{H_0}\) and \(t \in T^{H_1} \implies x(t) \in X^{H_1}\). Thus, without loss of generality we can assume that at time \(t = t_0 = 0\) one has \(x(t_0) \in X^{H_0}\), while at time \(t_1 = \frac{2\pi}{\alpha}\) one has \(x(t_1) \in X^{H_1}\). Since \([t_0, t_1]\) is a fundamental domain for the action of \(D_6\) in \(T\), if \(x(t)\) is a \(D_6\)-equivariant (local) minimum of the action functional \(A_\omega\), then it minimizes the action restricted to the interval \([t_0, t_1]\) together with the constraints \(x(t_0) \in X^{H_0}\) and \(x(t_1) \in X^{H_1}\). Now consider the function \(f: X \to \mathbb{R}\) defined by \(f(x_1, x_2, x_3) = (x_1 - x_3)x_2\). If a configuration (without collisions) \((\xi_1, \xi_2, \xi_3)\) belongs to \(X^{H_0}\), then \(\xi_1 = -\xi_2\) and \(\xi_3 = 0\), therefore
Let $L$ be the element of $O(\mathbb{T}) \times O(d) \times \Sigma_3$ given by: $\rho(g) = l$, $\sigma(g) = (1, 3)$ and $\tau(g)$ is the time reflection fixing the time $\frac{\pi}{6}$. It is easy to see that $g\mathcal{X}^{\sigma_0} = \mathcal{X}^{H_1}$, $g\mathcal{X}^{H_1} = \mathcal{X}^{H_0}$ and $gx(t') = x(t')$. Now, the restriction of $x(t)$ to the interval $[t_0, t']$ is not necessarily a (local) minimizer in the class of paths defined in $[0, t']$ with constraints $x(0) \in \mathcal{X}^{H_0}$ and $x(t') \in \mathcal{X}^g$, but sure its action is bounded below by such a minimal value. The same holds for the restriction of $x(t)$ to the interval $[t', t_1]$, with respect to paths with constraints $x(t') \in \mathcal{X}^g$ and $x(t_1) \in \mathcal{X}^{H_1}$. Now, if $\mathcal{A}_1$ denotes the corresponding local minimum of the action on paths defined in $[0, 1]$ with boundary constraints $x(0) \in \mathcal{X}^{H_0}$ and $x(1) \in \mathcal{X}^g$, then the sum of the actions over $[t_0, t']$ and $[t', t_1]$ is equal to

$$\left(\frac{2\pi}{6} + \frac{2\pi}{12}\right)\mathcal{A}_1,$$

where $T_0 = t'$ and $T_1 = t_1 - t'$. Such a sum is minimal for $T_0 = T_1$, hence the action of $x(t)$ (the minimal path with constraints $x(t_0) \in \mathcal{X}^{H_0}$ and $x(t_1) \in \mathcal{X}^{H_1}$) in $[t_0, t_1]$ is greater than twice the action of the path minimizing with constraints $x(t') \in \mathcal{X}^g$ and $x(t_0) \in \mathcal{X}^{H_0}$ where $t' = \frac{\pi}{6} = \frac{1}{2}(t_1 - t_0)$. But this minimizer, say $y(t)$, can be extended to a path defined on $[t_0, t_1]$ by the action of $g$:

$$y(t' + t) = gy(t' - t).$$

Hence $x(t)$ has to be equal to the symmetric path $y(t)$ on $[t_0, t_1]$. Now, the group generated by $D_6$ and $g$ is conjugated to $D_{12}$, and this implies that any $D_6$-symmetric local minimizer has the full $D_{12}$ symmetry group.

(4.16) Remark. Numerical experiments suggest that also the minimum in $\Lambda^{D_6}$ might coincide with the minimum in $\Lambda^{C_6}$ (and therefore that the occurring eight solution is unique), and that the restriction on the exponent is not necessary. Furthermore, it is not difficult to generalize (3.9) to orbits in 3-dimensional space, and thus to find symmetry groups for which all minimizers have zero angular momentum. Since 3-body orbits whose angular momentum vanishes are planar, in this paper actually we provide a complete list of symmetry groups (in the space as in the plane) not of type $R$, and hence that even in space eight-type orbits are at most two (one for the $C_6$ and one for $D_{12}$). It is interesting to find (if possible) a proof of the fact that actually it is unique.

(4.17) Remark. As it will be proved in the next section, we have described all possible symmetry groups for the planar three-body problem (in the sense of section 2.2). It is interesting that, even when the global minimum is proved to be homographic, such symmetry groups can be used to find (at the moment only numerically, but after estimates on the action levels of colliding trajectories rigorous results might follow) symmetric periodic orbits which are just local minimizers, such as the orbit of figure 2(b). This orbit is a choreography obtained as local minimum for the Lagrange symmetry group with $\omega = \frac{\pi}{10}$, which corresponds to a dihedral symmetry group $G$ of order 60 acting as follows: if $g$ denotes the generator of the maximal cyclic
subgroup of $G$ and $h$ one of the elements of order 2 not in the center, then $\rho(g)$ is a rotation of angle $\frac{\pi}{5}$, $\sigma(g) = (1, 2, 3)$, $\rho(h)$ is a reflection along a line, $\sigma(h) = (1, 2)$, $\tau(g)$ is a rotation of angle $\frac{\pi}{10}$ and $\tau(h)$ a time-reflection. In fact, for $\omega = 0$ the figure eight orbit seems to be a local minimum for the Lagrange symmetry group, and this minimum can be continued as $\omega$ grows to obtain a family of quasi-periodic orbits, which are choreographies in the rotating frame. It is easy to see that there is a dense subset $D$ of $(0, 1)$ such that for $\omega \in D$ the corresponding orbit is not only a choreography in the rotating frame, but also a choreography in the inertial frame.

5 The classification

In this section we will prove Theorem A. It will be an immediate consequence of (5.2) and (5.4).

(5.1) If $G$ is a symmetry group with trivial core for the 3-body problem in the plane, and the action is not redundant, then (up to a rotating frame) $\rho(g^2) = 1$ for every $g \in G$.

Proof. Let $r \in G$ denote the generator of $\ker \det(\tau)$ and $h$ a time reflection in $G$ (if it exists). It must be that $hrh^{-1} = r^{-1}$ and $h^2 = 1$. Thus the conclusion is equivalent to $\rho(r^2) = 1$. If the action is of type R and not redundant, then by (3.4) one has $\rho(r) = 1$. So assume that the action is not of type R. If the action type is cyclic then $\rho(r)$ is a reflection, so that $\rho(r^2) = 1$. If the action type is brake, then one has $\rho(r) = 1$ so the same follows. If the action type is dihedral and $\rho(r^2) \neq 1$, then $r \in \ker \det(\rho)$ (since elements not in $\ker \det(\rho)$ have order at most 2). Thus, being the action not of type R, $\rho(h)$ is orientation-preserving and of order at most 2, i.e. $\rho(h)$ is either trivial or the antipodal map and hence commutes with $\rho(r)$. Thus, by the equation $hrh^{-1} = r^{-1}$ it follows that $\rho(r) = \rho(r^{-1})$ and therefore that $\rho(r^2) = 1$ as claimed.

q.e.d.

(5.2) Proposition. For the planar 3-body problem with trivial core the symmetry groups not bound to collisions, not redundant and not fully uncoercive are those described in section 4 and summarized in table 1.

Proof. We can begin with symmetry groups acting trivially on the index set. First we consider groups of type R: any group of cyclic action type and not redundant acts trivially on $E$, so we have to assume $G$ to be of brake or dihedral type, if it is not trivial (see 4.1). But for the same reason it cannot be of dihedral type (since the cyclic part would be trivial), hence only brake. Let $h$ denote the time reflection. Since $G$ is of type R, then $\rho(h)$ has to be a reflection in $E$, and hence the group is the symmetry group of the line symmetry (see 4.2). Now, consider a group not of type R. We want to show that is always either redundant or bound to collisions. Let $r$ and $h$ be its cyclic generator (i.e. the generator of $\ker \det(\tau)$) and one of its time reflections (i.e. one of the elements not in $\ker \det(\tau)$). If the action type is cyclic, then $\rho(r)$ is a reflection, and $G$ is not coercive. If the action type is brake, then $\rho(h)$ is either trivial (and therefore $G$ is not coercive) or the antipodal map (and therefore bound to collisions). On the other hand if the action type is dihedral, then
\( \rho(r) \) is a reflection along a line \( l \) in \( E \) (otherwise would be trivial or redundant). As a consequence, if \( \rho(h) \) is trivial or a reflection along the same line \( l \), then \( G \) is not coercive. The other possibility is that \( \rho(h) \) is a reflection along a line \( l' \) orthogonal to \( l \), but in this case \( G \) would result to be bound to collisions. Thus there are no symmetry groups with the required properties and not of type \( R \).

Now we come to symmetry groups with transitive decomposition 2 + 1. Let \( r \) be the cyclic generator of \( \ker \det(\tau) \) and, if the action type is not cyclic, \( h \) an element of \( G \setminus \ker \det(\tau) \) (a time reflection). First consider groups of type \( R \). If the action type is cyclic, then, to avoid being redundant, the only choice is \( \rho(h) = 1 \) and \( \sigma(r) = (1, 2) \), thus the 2-1-choreography (see 4.3). If the action type is brake, then \( \rho(h) \) and \( \sigma(r) \) are trivial and thus \( \rho(h) \) is a reflection (since the group is of type \( R \)), while \( \sigma(h) = (1, 2) \) (up to a permutation of the indexes). This is the isosceles symmetry group (see 4.4).

As a third case, assume that the action type is dihedral. Being not redundant, it must be that \( \rho(r) = 1 \), so that \( \sigma(r) = (1, 2) \) (otherwise it would be brake); as above \( \rho(h) \) is a reflection while \( \sigma(h) = () \) or \( \sigma(h) = (1, 2) \). Both cases yield the same symmetry group (the two groups are conjugated in \( O(\mathbb{T}) \times O(2) \times \Sigma_3 \), which is the Hill symmetry group (see 4.5). It can be generated as the union of any two of the three listed symmetry groups of order 2.

Then, we deal with groups not of type \( R \). An enumeration of all possible cases shows that:

(5.3) There are no symmetry groups not of type \( R \) which are coercive, with transitive decomposition of type 2 + 1 and not bound to collisions.

Proof of (5.3) It suffices to prove it for non-redundant symmetry groups. The only symmetry group of cyclic action type is defined by \( \sigma(r) = (1, 2) \) while \( \rho(r) \) is a reflection, which is not coercive. The same happens for brake action type: \( \rho(r) \) and \( \sigma(r) \) are trivial, and \( \sigma(h) = (1, 2) \). Now, \( \rho(h) \) can be the antipodal map or the identity (the orientation-preserving orthogonal maps of the plane of order less than 2): in both cases the resulting symmetry group is not coercive.

Now we list all the symmetry groups of dihedral type. Up to a change in the generators, there are only two possibilities for \( \sigma \): either \( \sigma(r) = () \) and \( \sigma(h) = (1, 2) \), or \( \sigma(r) = (1, 2) \) and \( \sigma(h) = () \) (in fact, it cannot be \( \sigma(r) = \sigma(h) = () \) because of the transitive decomposition 2 + 1, and the case \( \sigma(r) = (1, 2) = \sigma(h) \) can be transformed into \( \sigma(r) = (1, 2) \), \( \sigma(h') = () \) by choosing \( h' = rh \). Let 1, a and \( l \) denote respectively the identity, the antipodal map and a reflection along a line in \( E \). Let \( l' \) denote the reflection along a line orthogonal to \( l \). The pair \( (\rho(r), \rho(h)) \), by (5.1) belongs to the following list (since the action is not of type \( R \)): \((1, 1), (1, a), (l, l), (l, l'), (l, a), (a, 1), (a, a) \). Hence there are 16 possible symmetry groups.

When the action on the index set is defined by \( \sigma(r) = () \) and \( \sigma(h) = (1, 2) \), then there are the following cases: if \( (\rho(r), \rho(h)) \) is in the set \( \{(1, 1), (1, a), (l, l')\} \), then the generated group is bound to collisions; if \( (\rho(r), \rho(h)) \) is in the set \( \{(1, a), (l, a), (l, l')\} \), then it is not coercive; if \( (\rho(r), \rho(h)) \) is in the set \( \{(a, 1), (a, a)\} \), then it is redundant. On the other hand, when \( \sigma(r) = (1, 2) \) and \( \sigma(h) = (1, 2) \), if the pair \( (\rho(r), \rho(h)) \) is in the set \( \{(1, 1), (1, a), (l, l), (l, l'), (l, a), (a, a)\} \) then the action is bound to collisions; if \( (\rho(r), \rho(h)) \) is in the set \( \{(l, 1), (a, 1)\} \) then the generated group is not coercive. This completes the proof of (5.3). q.e.d.
At last, we now consider the symmetry groups acting transitively on \{1, 2, 3\}. If the action is of cyclic type, then it has to be \(\sigma(r) = (1, 2, 3)\). If \(\rho(r) = 1\), then it is the choreography symmetry group (see 4.6). If \(\rho(r) = a\), then it is redundant (just the choreography counted twice). If \(\rho(r) = l\), then it is the symmetry group \(C_6\) (cyclic eight – see 4.8). Furthermore, the group cannot be of brake type, since the only groups acting transitively and effectively on a set of three elements are the cyclic group of order 3 and the dihedral group of order 6. Thus, if the action is of dihedral type, up to a permutation in the index set one has \(\sigma(r) = (1, 2, 3)\) and \(\sigma(h) = (1, 2)\). The only symmetry group which is of type R and not redundant is therefore the Lagrange symmetry group (see 4.7). We want to show that the groups which are not redundant, not bound to collisions and not of type R are \(C_6, D_6\) and \(D_{12}\) as listed in 4.8. By (5.1), \(\rho(r)\) has order at most 2. First consider the case \(\rho(r) = a\). Since \(\sigma(r)\) has order 3, it follows that \(\rho(r^3) = a\) and \(\sigma(r^3) = ()\), i.e. that the action is redundant. Thus \(\rho(r)\) belongs to the set \(\{1, l\}\), where as above 1 is the identity and \(l\) a reflection. If \(\rho(r) = 1\), then the element \(\rho(h)\) can be 1 or \(a\). In the first case the resulting action is bound to collisions and in the second case it is \(D_6\) (see 4.8). On the other hand, if \(\rho(r) = l\), then \(\rho(h)\) belongs to the set \(\{1, -1, l, l'\}\), where \(l'\) is the reflection along a line orthogonal to the line of the reflection \(l\) (it has to be an element in \(O(2)\) commuting with \(l\)). Since if \(\rho(h) = 1\) or \(l\) then the action is bound to collisions, the choices left are \(-1\) and \(l'\). In both cases the resulting symmetry group is \(D_{12}\) (see 4.8).

\(\text{q.e.d.}\)

(5.4) For the 3-body problem in the plane, paths constrained under symmetry groups with non-trivial core, not bound collisions and not fully uncoercive are always homographic.

Proof. We recall that the core of a symmetry group \(G\) is \(K = \ker \tau \subset G\). The subgroup \(K\) acts effectively on the index set, hence it is isomorphic to a subgroup of \(\Sigma_3\): either its order is 2 or 3, or it is the dihedral group of order 6 (the full permutation group on three elements). If its order is 2, then it is easy to see that to avoid being bound to collisions or not coercive it has to act on the plane by the antipodal map \(a\). If its order is 3, then necessarily its planar representation is given by a rotation of order 3. If it is a dihedral group of order 6, then its representation in the plane is the standard representation of a dihedral group. In the first case (\(|K| = 2\)) the configuration space \(X^K\) has dimension 2 and its elements are configurations \((x_1, x_2, x_3)\) where \(x_1 = -x_2\) and \(x_3 = 0\). In the second case (\(|K| = 3\)) the configuration space \(X^K\) has dimension 2 and its elements are all equilateral triangles. In the third case (\(|K| = 6\)), the configuration space \(X^K\) has dimension 1 and its elements are scalar multiples of a given equilateral triangle. Now, as above, it is possible to add elements to \(G\) which act on the time circle, but it is not difficult to see that in the first two cases minimizers will be, if not bound to collisions, rotating central configurations, while in the third case the only way to gain coercivity is to consider an action bound to collisions (since it is not of type R).

\(\text{q.e.d.}\)

(5.5) For the 3-body problem in the plane, if the symmetry group \(G\) has non-trivial core, is not bound collisions and not fully uncoercive, then local minimizers are collisionless homographic solutions.
Proof. As in the proof of (5.4), one is reduced to consider only the three cases \(|K| = 2, 3, 6\). For \(|K| = 6\) either \(G\) is bound to collisions or it is fully uncoercive; for \(|K| = 2, 3\) the equivariant 3-body problem results to be equivalent to an equivariant one-center planar \(\alpha\)-homogeneous Kepler problem, and the thesis follows since local minima of the equivariant Kepler problem are collisionless (see remark (5.6) below).

q.e.d.

(5.6) Remark. Consider the one-center Kepler problem in the plane and a symmetry group \(G\) of the action functional. In this case \(X = E \cong \mathbb{R}^2\), and obviously the action on the index set \(\{1\}\) is trivial. If \(G\) has non-trivial core then \(X^{\ker \tau}\) has dimension 0 or 1, and hence either \(G\) is bound to collisions or fully uncoercive. Thus, \(\ker \tau = 1\) and so in order to show that local minimizers are collisionless, one needs only to exclude boundary collisions (see (7.4) below). Let \(\rho(h)\) be the image under \(\rho\) of the generator \(h\) of the \(\mathbb{T}\)-isotropy in question. If \(\rho(h)\) is a rotation of angle \(\pi\), then \(G\) is bound to collisions. Otherwise, \(\rho(h)\) is a reflection along a line, and therefore the claim follows by theorem (7.5) in section 7.

As a consequence of theorem A one can easily proof the following proposition.

(5.7) Let \(G\) be a symmetry group such that \(A_G^\omega\) is coercive. If \(\Lambda_G^\omega\) contains a Lagrange relative equilibrium motion \(x\), then the minimum of the action functional \(A_G^\omega\) is attained on \(x\) (and hence the minimizer is homographic).

(5.8) Remark. It follows from (4.8), (4.9) and (3.10) that a sort of converse of (5.7) holds, at least in the case of almost equal masses: That is, for every \(G\) if the masses are approximately equal (subject to the symmetry constraints) and \(\Lambda_G^\omega\) does not contain Lagrange equilibrium motions, then there always exists a suitable choice of angular speed \(\omega\) such that the minimum of \(A_G^\omega\) is not homographic – that is, it is not a rotating collinear Euler configuration. It is of some interest to understand whether this is true for all values of masses.

6 Parabolic collisions with isosceles symmetry

In this section we will study a special class of colliding trajectories, constrained under a simple symmetry. This symmetry, which we term isosceles, is the only one not fulfilling the rotating circle property. Our aim is to prove that also for this symmetry there exists a local variation which implies that minimizers are collisionless. In the following we can assume \(\alpha \in (0, 2)\), \(k \supseteq \{1, 2\}\) and \(m_1 = m_2\).

(6.1) Definition. A parabolic collision trajectory for the cluster \(k \subseteq n\) is the path

\[ q_i(t) = |t|^{2/(2+\alpha)} \xi_i, \quad i \in k, \ t \in \mathbb{R} \]

where \(\xi = (\xi_i)_{i \in k}\) is a central configuration with \(k\) bodies.

(6.2) Definition. An escaping path for the cluster \(k \subseteq n\) is a path of type \(y = q + \varphi\), where \(q\) is a parabolic collision trajectory for the cluster \(k\) and \(\varphi \in H_0^1(\mathbb{R})\).

We now define the action of \(g_0\) on the time line \(\mathbb{R}\), on the space \(E\) and on the index set as follows: \(g_0(t) = -t\) for \(t \in \mathbb{T}\), \(g_0\) acts as a reflection along a line in \(E\) and \(\sigma(g_0) = (1, 2)\).
**Definition.** Given $g_0$, we say that an escaping path is $g_0$-equivariant if

$$y(g_0 t) = g_0 y(t).$$

Notice that Definition (6.3) covers both the cases of binary and triple clusters ($k = 2, 3$).

**Remark.** Let $l$ the line fixed by $\rho(g_0)$; then $y_1(0)$ and $y_2(0)$ are symmetric with respect to $l$ while $y_3(0)$ (if $n > 2$) belongs to $l$. A $g_0$-equivariant escaping path is determined by its restriction on the half line $[0, +\infty)$ provided that $y(0) \in X^{g_0}$.

We term $L_k$ the partial Lagrangian function, when $x = (x_i)_{i \in k}$, $L_k(x)$ is the Lagrangian restricted on the bodies of the cluster $k$.

**Definition.** We say that a parabolic collision trajectory, $q = (q_i)_{i \in k}$, defined in (6.1), is a $g_0$-equivariant minimizing parabolic collision trajectory if for every $g_0$-equivariant escaping path $y = q + \varphi$ the integral of the variation is positive:

$$\int_{-\infty}^{+\infty} [L_k(q + \varphi) - L_k(q)] dt \geq 0.$$

Let $\delta \in (\mathbb{R}^2)^k$, $k = 2, 3$, be a vector of norm $|\delta| = (\sum_{i=1}^{k} \delta_i^2)^{1/2}$ sufficiently small and $T > 0$ a real number.

**Definition.** The standard variation associated to $\delta$ and $T$ is defined as

$$v_\delta(t) = \begin{cases} 
\delta & \text{if } 0 \leq |t| \leq T - |\delta| \\
(T - t) \frac{\delta}{|\delta|} & \text{if } T - |\delta| \leq |t| \leq T \\
0 & \text{if } |t| \geq T.
\end{cases}$$

**Remark.** Let $q$ be a $g_0$-equivariant minimizing parabolic collision trajectory and $v_\delta(t)$ a standard variation. Then the path $y(t) = q(t) + v_\delta(t)$ (with $t \in \mathbb{R}$) is $g_0$-equivariant if and only if $v_\delta$ is $g_0$-equivariant, and hence $\delta = g_0 \delta$. Thus, in particular, any $\delta$ fixed by $g_0$ yields a $g_0$-equivariant standard variation $v_\delta(t)$.

In the next theorem we will prove that $g_0$-equivariant parabolic collision trajectories cannot be local minimizers (see definition (6.5)).

**Theorem.** Let $q$ be a $g_0$-equivariant parabolic collision trajectory. Then there exists a $g_0$-equivariant standard variation $v_\delta$ such that the path $q + v_\delta$ does not have a collision at $t = 0$ and

$$\Delta A = \int_{-\infty}^{+\infty} [L_k(q + v_\delta) - L_k(q)] dt < 0.$$

The proof of this result requires several intermediate steps. To begin with, consider the function

$$S(\xi, \delta) = \int_{0}^{+\infty} \left( \frac{1}{|\xi t^{2/(2+\alpha)} - \delta|^\alpha} - \frac{1}{|\xi t^{2/(2+\alpha)}|^\alpha} \right) dt$$

where $\xi, \delta \in \mathbb{R}^2$.

The next result allows to estimate the action differential involved in a standard variation.
Let \( q = \{ q_i \} = \{ t^{2/(2+\alpha)} \xi_i \}, \ i = 1, \ldots, k \) be a parabolic collision trajectory and \( v^g \) a \( g_0 \)-equivariant standard variation. Then for \( \delta \to 0 \)

\[
\Delta A = 2|\delta|^{1-\alpha/2} \sum_{i<j} m_i m_j S(\xi_i - \xi_j, \delta_i - \delta_j) + O(|\delta|).
\]

To prove Theorem \((6.8)\) we have to provide a suitable \( g_0 \)-equivariant standard variation such that the right end side of \((6.10)\) is negative. This will depend on the particular central configuration \( \xi \) drawn by the parabolic collision trajectory. The proof of \((6.10)\) can be found in \cite{[17]}, Lemma (9.2). Let \( \vartheta \in [0, 2\pi] \) such that

\[
\cos \vartheta = \langle \xi / |\xi|, \delta \rangle
\]

then

\[
(6.11) \quad S(\xi, \delta) = |\xi|^{-1-\alpha/2} \int_0^{+\infty} \frac{1}{\left( t^{\frac{4}{\pi+2}} - 2 \cos \vartheta t^{\frac{2}{\pi+2}} + 1 \right)^{\alpha/2}} \frac{1}{t^{\frac{2\alpha}{\pi+2}}} \, dt.
\]

\[
(6.12) \quad \Phi_\alpha(\vartheta) = \int_0^{+\infty} \frac{1}{\left( t^{\frac{4}{\pi+2}} - 2 \cos \vartheta t^{\frac{2}{\pi+2}} + 1 \right)^{\alpha/2}} \frac{1}{t^{\frac{2\alpha}{\pi+2}}} \, dt, \quad \alpha \in (0, 2)
\]

is defined and continuous on the interval \((0, 2\pi)\) and satisfies the following properties

(i) \( \Phi_\alpha(\vartheta) = \Phi_\alpha(2\pi - \vartheta) \), for every \( \vartheta \in (0, 2\pi) \) and \( \alpha \in (0, 2) \), i.e. its plot is symmetric with respect to \( \vartheta = \pi \);

(ii) is decreasing on \((0, \pi]\) and increasing on \([\pi, 2\pi)\), i.e. it achieves its minimal value on \( \vartheta = \pi \);

(iii) when \( \alpha < 1 \) then \( \Phi_\alpha(0) = \Phi_\alpha(2\pi) \) is finite, while when \( \alpha \geq 1 \)

\[
\lim_{\vartheta \to 0^+} \Phi_\alpha(\vartheta) = \lim_{\vartheta \to 2\pi^-} \Phi_\alpha(\vartheta) = +\infty.
\]

Proof. Properties (i) and (ii) are obvious. To prove (iii), we study the integral of the function

\[
\phi_\alpha(t, \vartheta) = \frac{1}{\left( t^{\frac{4}{\pi+2}} - 2 \cos \vartheta t^{\frac{2}{\pi+2}} + 1 \right)^{\alpha/2}} - \frac{1}{t^{\frac{2\alpha}{\pi+2}}},
\]

on the interval \((0, +\infty)\). For \( \vartheta \in (0, 2\pi) \), the function \( \phi_\alpha \) is a continuous with respect to \( t \) in \((0, +\infty)\) and we have to take into account the convergence of its integral at 0 and at \(+\infty\).

When \( t \to 0^+ \), the integral of \( \phi_\alpha(\cdot, \vartheta) \) in a right neighborhood of 0 is finite, being \( 2\alpha/(2 + \alpha) < 1, \forall \alpha \in (0, 2) \). About the integrability at infinity, we have

\[
\phi_\alpha(t, \vartheta) \approx \frac{\alpha}{2} \left( \frac{1}{t^2} - 2 \frac{\cos \vartheta}{t^2 \vartheta} \right)
\]
as $t \to +\infty$; since for every $\alpha \in (0, 2)$, $2(\alpha + 1)/(\alpha + 2) > 1$ and $\Phi_\alpha$ is integrable on $[2, +\infty)$.

When $\vartheta = 0$, the function $\phi_\alpha(\cdot, 0) = \phi_\alpha(\cdot, 2\pi)$ has a singularity of order $\alpha$ in $t = 1$. Therefore the integral function $\Phi_\alpha$ tends to $+\infty$ as $\vartheta$ tends to $0$ if and only if $\alpha \geq 1$.

(6.14) Corollary. There exists $\bar{\vartheta} = \bar{\vartheta}(\alpha) \in (0, \pi/2)$ such that $\Phi_\alpha(\bar{\vartheta}) = 0$ and $\Phi_\alpha(\vartheta) < 0$, for every $\bar{\vartheta} < \vartheta < 2\pi - \bar{\vartheta}$.

**Binary collisions**

The simplest case is when only two bodies are involved in the collision; thus $\xi_1 = -\xi_2$ and we have to take into account the angle between $\xi_1$ and the line $l$ fixed by $g_0$. Consider a vector $\delta$ orthogonal to $l$ with norm $|\delta| = 1/2$, in such a way that the angle $\vartheta = \arccos \left( \frac{\xi_1 - \xi_2}{|\xi_1 - \xi_2|}, \delta \right)$ lies in the interval $[\pi/2, \pi]$ (see Figure 7). To obtain a negative variation of the interaction energy, we add to the position of the first body a vector $-\delta$ and to the second a vector $\delta$.

**Figure 7: Binary collisions**

**Triple collisions**

A triple collision can take place in two different ways: from collinear configurations or from the Lagrange configuration. We study separately these two cases. Our aim is to show that is always possible to reduce a triple collision to a binary one.

**Triple collisions from collinear central configurations**

Two different situations occur whether the second or the third body lies between the other two (respectively we will have $\xi_1 = -\lambda \xi_3$ or $\xi_1 = -\lambda \xi_2$, $\lambda > 0$). In the first case we refer to Figure 8(a) and we move the trajectory of the third body of a vector $\delta$ parallel to $l$ whether $\xi_2 = \mu \xi_1$, or $\xi_2 = \mu \xi_3$, $\mu \in [0, 1)$.

When the third body lies in the middle, let $\delta$ be orthogonal to $l$, $|\delta| = 1/2$, such that $\vartheta = \arccos \left( \frac{\xi_1 - \xi_2}{|\xi_1 - \xi_2|}, \delta \right)$ is in the interval $[\pi/2, \pi]$, see Figure 8(b). If we shift the second body of $\delta$ and the first of $-\delta$, the interaction between these two bodies decreases. Since $\vartheta > \pi/2$ and $\mu < 1$, also the sum of the variations of the interactions of the third body with the other two is negative.
Consider now the 3 particles at the vertices of a regular triangle moving on a parabolic collision trajectory that makes them collide at \( t = 0 \) in their center of mass. Since \( m_1 = m_2 \), the center of mass of the cluster lies on the perpendicular line, \( h \), from the third body to the line joining the other two. Our aim is to show that there always exists a vector \( \delta \in S^1 \), such that, when replacing \( \xi_3 \) with \( \xi_3 + \delta \), the interaction potential decreases (the sum of the two variations is negative even thought they are not necessarily both negative) and therefore \( \Delta A \) of (6.10) is negative. In the following we refer to figure 9 and we call \( \gamma \) the angle in \([0, \pi/2]\) with edges the line \( h \) and the subspace \( l \). We will prove the following result.

\textbf{(6.15) Proposition.} For every \( \alpha \in (0, 2) \) the following inequality holds

\begin{equation}
\Phi_\alpha(\frac{2\pi}{3} + \gamma) + \Phi_\alpha(\frac{2\pi}{3} - \gamma) < 0, \quad \forall \gamma \in \left[0, \frac{\pi}{2}\right].
\end{equation}

Before proving Proposition (6.15) we need two preliminary Lemmata.

\textbf{(6.17)} For every \( \alpha \in (0, 2) \) and \( \vartheta \in (0, 2\pi) \)

\begin{equation}
\frac{2}{\alpha(\alpha + 2)} \Phi_\alpha(\vartheta) = \frac{1}{\alpha - 2} \beta\left(\frac{\alpha + 2}{4}, \frac{\alpha + 2}{4}\right) + \\
+ \frac{1}{\alpha} \sum_{k=1}^{\infty} \left(-\frac{\alpha}{2}/k\right)(-1)^k 2^k (\cos \vartheta)^k \frac{\alpha + 2k}{\alpha + 2k - 2} \beta\left(\frac{\alpha + 2}{4}, \frac{k}{2}, \frac{\alpha + 2}{4} + \frac{k}{2}\right).
\end{equation}
where the function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is defined as

$$
(6.19) \quad \beta(z, w) = \int_0^1 t^{z-1}(1 - t)^{w-1} dt.
$$

(6.20) Remark. It is well known that the function $\beta$ satisfies the following properties

(i) $\beta(z, w) = \beta(w, z)$, for every $z, w \in \mathbb{R}$;

(ii) $\beta(z + 1, w) = \frac{z}{z+w} \beta(z, w)$, for every $z, w \in \mathbb{R}$;

(iii) $\beta(x + n, x + n) = \frac{1}{4^n} \left( \frac{(-1)^n}{n!} \right) \beta(x, x)$, for every $x \in \mathbb{R}, n \in \mathbb{N}$.

Proof of Lemma (6.17). Consider the function $\Phi_\alpha$ as in (6.13) with $t = s^{2\alpha}$: then

$$
(6.21) \quad \Phi_\alpha(\vartheta) = \frac{2 + \alpha}{2 - \alpha} \int_0^{+\infty} \left[ \frac{s^{2\alpha}}{(s^{2\alpha} + 1)^{\alpha/2}} - 1 \right] ds.
$$

Since $x \leq \frac{1}{2}(x^2 + 1)$ and, for $|z| < 1$ there holds $(1 + z)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} z^k$, we can write

$$
(6.22) \quad \Phi_\alpha(\vartheta) = \frac{2 + \alpha}{2 - \alpha} \left\{ I_0 + \sum_{k=1}^{+\infty} \binom{\alpha/2}{k} (-1)^k 2^k \frac{s^{2k}}{(s^{2\alpha} + 1)^k} \right\} ds,
$$

or, equivalently,

$$
(6.23) \quad I_0 = \int_0^{+\infty} \left[ \frac{s^{2\alpha}}{(s^{2\alpha} + 1)^{\alpha/2}} - 1 \right] ds,
$$

$$
(6.24) \quad I_k = \int_0^{+\infty} \frac{s^{2k}}{(s^{2\alpha} + 1)^{\alpha/2}} \frac{s^{2k}}{(s^{2\alpha} + 1)^k} ds, \quad k \in \mathbb{N}^*.
$$

We now compute the expressions of $I_0$ and $I_k$ separately to obtain (6.18). Integrating by parts, (6.22) becomes

$$
I_0 = -\frac{2\alpha}{2 - \alpha} \int_0^{+\infty} \frac{s^{2\alpha}}{(s^{2\alpha} + 1)^{\alpha/2} + 1} ds;
$$

we then substitute

$$
(6.24) \quad s = \left( \frac{1 - t}{t} \right)^{2\alpha}, \quad \frac{ds}{dt} = \frac{2 - \alpha}{4} \left( \frac{t}{1 - t} \right)^{2\alpha} \left( -\frac{1}{t^2} \right).
$$
to obtain
\[(6.25)\quad I_0 = -\frac{\alpha}{2} \int_0^1 t^{\frac{\alpha - 2}{2}} (1 - t)^{\frac{\alpha - 2}{4}} dt = -\frac{\alpha}{2} \beta \left(\frac{\alpha + 2}{4}, \frac{\alpha + 2}{4}\right).\]

From (6.22), by (6.24), one deduces that
\[I_k = \frac{\alpha}{4} \int_0^1 t^{k + \frac{\alpha}{4} - \frac{3}{4}} (1 - t)^{k + \frac{\alpha}{4} - \frac{3}{4}} dt\]
\[(6.26)\]
By replacing (6.25) and (6.26) in (6.21) it follows that
\[
\Phi_\alpha(\vartheta) = \alpha \left(\frac{\alpha + 2}{2}\right) \left\{ 1 - \frac{\alpha + 2}{\alpha - 2} \beta \left(\frac{\alpha + 2}{4}, \frac{\alpha + 2}{4}\right) + \frac{1}{\alpha} \sum_{k=1}^{+\infty} \left(\frac{-\alpha/2}{k}\right) (-1)^k 2^{k-1} (\cos \vartheta)^k k \beta \left(\frac{\alpha}{4} + 1, \frac{\alpha}{4} + 1\right) \right\}.\]

Since \(\beta(z - 1, z) = \frac{2z - 1}{z - 1} \beta(z, z)\) (see (iv) in Remark (6.20)), the claim (6.18) is proved.
\[q.e.d.\]

(6.27) For every \(x \in (1/2, 1)\), the following inequality holds
\[(6.28)\quad \frac{1}{x - 1} + \sum_{k=1}^{+\infty} \left(\frac{3}{4}\right)^k \frac{1}{x + k} - \frac{4^k (k!)^2}{(2k)!} < 0.\]

**Proof.** Define the function \(f_k\) as \(f_k(x) = (-x)^2 \frac{1}{x + k - 1} \frac{4^k (k!)^2}{(2k)!}\) for every \(x \in (1/2, 1)\) and \(k \in \mathbb{N}\). For a fixed \(x \in (1/2, 1)\), \(f_k(x)\) is monotonically decreasing in \(k\), while, for a fixed \(k\), \(f_k(x)\) is increasing in \(x\) on \((1/2, 1)\). We first prove that the following inequality holds
\[(6.29)\quad \sum_{k=5}^{+\infty} f_k(x) \left(\frac{3}{4}\right)^k < \frac{27}{35}, \text{ for all } x \in (1/2, 1).\]

Indeed, by monotonicity of \(f_k(x)\), inequality (6.29) is implied by the fact that
\[f_5(1) \sum_{k=5}^{+\infty} \left(\frac{3}{4}\right)^k = f_5(1) \left(\frac{3}{4}\right)^5 = \frac{4^5 (5!)^2}{5^5 (10)!} = \frac{27}{35}.\]
Hence (6.28) would follow once it is proved that, for every \(x \in [1/2, 1]\)
\[(6.30)\quad \frac{1}{x - 1} + \sum_{k=1}^{4} f_k(x) \left(\frac{3}{4}\right)^k + \frac{27}{35} < 0.\]

Expanding the expression above we can write
\[
\frac{1}{x - 1} + \frac{3}{2} x + \frac{3}{8} (x + 1) x^2 + \frac{3}{80} (x + 2)(x + 1)^2 x^2 + \frac{9}{4480} (x + 3)(x + 2)^2(x + 1)^2 x^2 + \frac{27}{35} < 0
\]
or, equivalently, (6.28) holds if for every \( x \in (1/2, 1) \)

(6.31) \( p(x) = 9x^8 + 72x^7 + 366x^6 + 684x^5 + \\ + 1749x^4 - 756x^3 + 4596x^2 - 3264x + 1024 > 0; \)

one way of proving (6.31) is just to show that the polynomial \( p(x) \) has a Taylor expansion centered in 1/2 with positive coefficients; alternatively, since in \((1/2, 1)\)

\[ p(x) = 1024 - 3264x + 4596x^2 - 756x^3 > 1024 - 3264x + 4596x^2 - 756x^3 > 1024 - 3264x + 4596x^2 - 756x^3 \]

the claim follows from the fact that the minimum of the cubic polynomial \( 1 - \frac{10}{3}x + 4x^2 - x^3 \) in \((1/2, 1)\) is attained at \( x_0 = \frac{4 - \sqrt{6}}{4} \), with value \( \frac{35}{4} - \frac{4}{4} \sqrt{6} > 0. \)

q.e.d.

(6.32) For every \( \alpha \in (0, 2) \) the following inequality holds

(6.33) \( \Phi_\alpha \left( \frac{\pi}{6} \right) + \Phi_\alpha \left( \frac{7\pi}{6} \right) < 0. \)

Proof. We replace \( \gamma = \frac{\pi}{2} \) in (6.16) and we use the result proved in Lemma (6.17) to obtain

(6.34) \( \Phi_\alpha \left( \frac{\pi}{6} \right) + \Phi_\alpha \left( \frac{7\pi}{6} \right) = \frac{1}{\alpha - 2} \beta \left( \frac{\alpha + 2}{4}, \frac{\alpha + 2}{4} \right) + \\ + \frac{1}{\alpha} \sum_{k=1}^{+\infty} \left( -\frac{\alpha/2}{2k} \right) \left( \frac{3}{4} \right)^k 2^{2k} \alpha + 4k \alpha + 4k - 2 \beta \left( \frac{\alpha + 2}{4} + k, \frac{\alpha + 2}{4} + k \right). \)

By (iii) of Remark (6.20) one obtains

\[ \beta \left( \frac{\alpha + 2}{4} + k, \frac{\alpha + 2}{4} + k \right) = \frac{1}{4^k} \left( -\frac{\alpha+2}{k} \right)^{1/2} \beta \left( \frac{\alpha + 2}{4}, \frac{\alpha + 2}{4} \right) \]

and then (6.33) is implied by

\[ \frac{1}{\alpha - 2} + \frac{1}{\alpha} \sum_{k=1}^{+\infty} \left( -\frac{\alpha/2}{2k} \right) \left( \frac{3}{4} \right)^k \alpha + 4k \alpha + 4k - 2 \left( -\frac{\alpha+2}{k} \right)^{1/2} < 0. \]

Finally, this inequality is equivalent to

\[ \frac{1}{\alpha - 2} + \sum_{k=1}^{+\infty} \left( \frac{3}{4} \right)^k \left( -\frac{\alpha+2}{k} \right)^2 \frac{1}{\alpha + 4k - 2} \left( 2k \right)! < 0. \]

To conclude the proof we use lemma (6.27) with \( x = (\alpha + 2)/4. \) q.e.d.

Proof of Proposition (6.15). At first we remark that, when \( \gamma \in [0, \pi/6], \) then \( 2\pi/3 \pm \gamma \in [\pi/2, \pi] \) and therefore, since both \( \Phi_\alpha (2\pi/3 \pm \gamma) \) are negative, the thesis follows. Well then, let \( \gamma \in (\pi/6, \pi/2] \) and

\[ \delta = \frac{2\pi}{3} - \gamma, \quad \delta' = \frac{2\pi}{3} + \gamma \]
Then $\delta \in [\pi/6, \pi/2]$ and $\delta' \in (5\pi/6, 7\pi/6]$. Since $\Phi_\alpha$ is decreasing on $(0, \pi)$ and is symmetric with respect to $\vartheta = \pi$ we have that

$$\Phi_\alpha(\delta) \leq \Phi_\alpha\left(\frac{\pi}{6}\right), \forall \delta \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \quad \text{and} \quad \Phi_\alpha(\delta') \leq \Phi_\alpha\left(\frac{7\pi}{6}\right), \forall \delta' \in \left(\frac{5\pi}{6}, \frac{7\pi}{6}\right].$$

We conclude using the result we proved in Lemma (6.32) indeed

$$\Phi_\alpha(\delta) + \Phi_\alpha(\delta') \leq \Phi_\alpha\left(\frac{\pi}{6}\right) + \Phi_\alpha\left(\frac{7\pi}{6}\right) < 0.$$  
q.e.d.

### 7 Proof of Theorem B

We have already described (Proposition (2.29)) a criterion that guarantees the existence of minimizers, we now rule out the occurrence of collisions. Following [17], we introduce the key property of the action of a finite group $G$ on the loop space $\Lambda$ (we give here the definition specialized for the 3-body problem in the plane).

#### (7.1) Definition.** We say that a finite group $K$ acts on $\mathbf{n} = \{1, 2, 3\}$ (resp. $\{1, 2\}$) and $E$ with the rotating circle property (RCP) if the following conditions are true:

(i) for every $g \in K$ the determinant $\det(\rho(g)) = 1$;

(ii) there exist at least two different indices $i_1, i_2 \in \mathbf{n}$ such that $\forall g \in K, (gi_1 = i_1 \lor gi_2 = i_2) \implies \rho(g) = 1$ (resp. there exists at least one index $i_1 \in \{1, 2\}$ such that $\forall g \in K : gi_1 = i_1 \implies \rho(g) = 1$).

#### (7.2) Remark.** A group $G$ has the RCP if all its $T$-isotropy subgroups have the RCP, and a trivial group has the RCP. The underlying idea is the following: given a trajectory with a colliding cluster $k \subset \mathbf{n}$, since at least two indices in $\mathbf{n}$ satisfy property (ii), after a blow-up one is left with a parabolic $K$-equivariant collision-ejection trajectory, where $K$ is a suitable subgroup of $G$. Now, if $G$ has the RCP, then it follows that $K$ has the RCP. The first attempt would be to move away one of the particles, say $x_{i_1}$, and to let it rotate in a circle. Thanks to the averaging estimate [17] it would be possible to prove that the action decreases. But, this would yield a non-symmetric path, in general. So, it is necessary to move at the same time all those particles $x_{gi_1}$, for $g \in K$, which are related by the $K$-action to the $i_1$-particle.

#### (7.3) Definition.** We say that $x \in \Lambda^G$ has an interior collision at $t$ if $t \in (I \setminus \partial I) \cap \Delta^{-1}x$. We say that $x$ has a boundary collision at $t$ if $t \in \partial I \cap \Delta^{-1}x$.

The RCP yields to the following results (see [17], Theorems (10.7) and (10.10)):

#### (7.4) Theorem. Let $G$ be a finite group acting on $\Lambda$ such that $\ker \tau$ has the Rotating Circle Property. Then local minima of $\mathcal{A}^G_\omega$ in $\Lambda^G$ do not have any interior collision.

#### (7.5) Theorem. Let $G$ be a finite group acting on $\Lambda$ such that every maximal $T$-isotropy subgroup either has the RCP or acts trivially on $\mathbf{n}$, then any local minimizer of $\mathcal{A}^G_\omega$ in $\Lambda^G$ yields a collision-free periodic solution of the Newton equation (2.8).
Proof of Theorem 13. We assume that $x(t)$ is a $G$-equivariant local minimizer. If $\alpha \geq 2$ then $x(t)$ is collisionless, since the action level of colliding trajectories is infinite. When $\ker \tau$ is not trivial, Proposition (5.5) ensures that $x(t)$ is collisionless. When $\ker \tau$ is not trivial, Proposition (5.5) ensures that $x(t)$ is collisionless.

From now on we shall assume that $\ker \tau$ is trivial and therefore $G$ is isomorphic to its image $\tau(G) \subset O(2)$. On the other hand, Theorem (7.4) says that there are not interior collisions. This concludes the proof in the cyclic case.

Let us now assume, by the sake of contradiction that there is a collision instant at the boundary time $t = 0$ and let $G_0$ be its $T$-isotropy subgroup, which is a group of order two (since $G \cong \tau(G)$) generated by $g_0$. If $\sigma(g_0)$ acts trivially on the index set then Theorem (7.5) leads to a contradiction. Without loss of generality we can then suppose that $\sigma(g_0) = (1, 2)$. As to $\rho(g_0)$, three cases are possible: either $\rho(g_0)$ is the identity, the antipodal map or a reflection with respect to a line $l$. In the first two cases the rotating circle property holds and so we are left with the latter case.

Let $k \subset n = \{1, 2, 3\}$ be the colliding cluster; it is easy to see that either $k = \{1, 2\}$ or $k = n$. Let $q(t) = x(t) - x_0(t)$, where $x_0(t)$ is the center of mass of the bodies in $k$, and let $q^\lambda$ be defined by

$$q^\lambda(t) = \lambda^{-2(\alpha+1)/\alpha} q(\lambda t).$$

As shown in Section 7 of [17], there are sequences $\lambda_n \to 0$ such that $q^\lambda_n(t)$ converges to a $g_0$-equivariant parabolic minimizing collision trajectory, and this contradicts Theorem (6.8). The proof is now complete.

q.e.d.

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Glossary

Here we provide a list of terms together with the page at which the term is introduced or described the first time.

**Bound to collisions:** definition (2.18) at page 8

**Coercive:** definition (2.30) at page 10. See also page 9

**Cyclic, Brake or Dihedral type:** definition (2.21) at page 9

**Fully uncoercive:** definition (3.11) at page 13

**Homographic:** definition (2.15) at page 7

**Interior or boundary collision:** definition (7.3) at page 33

**Redundant:** definition (3.5) at page 12

**Rotating circle property:** definition (7.1) at page 33

**T-isotropy:** definition (2.22) at page 9

**Trivial core:** page 9

**Type R:** definition (3.1) at page 10