Examples of fourth-order scattering-type operators with embedded eigenvalues in their continuous spectra

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Abstract

We give examples of fourth-order scattering-type operators, acting on $L^2(\mathbb{R})$, which have eigenvalues embedded in their continuous spectra.

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Before we start, let us mention that an excellent reference for the general spectral theory of linear differential operators, which the reader may want to consult, is Naimark's monograph [1].

In the first two examples the fourth-order operator $L$ has the form $L = \frac{d^4}{dx^4} + q(x)$ where the “potential” $q(x)$ has compact support.

1 An example with a singular potential

Consider the function

$$\theta(x) = \begin{cases} 
\sin x, & 0 \leq x < 3\pi/4; \\
Ae^{-x}, & 3\pi/4 \leq x.
\end{cases}$$

(1)

We choose

$$A = \frac{1}{\sqrt{2}}e^{3\pi/4},$$

(2)
so that \( \theta(x) \in C^1[0, \infty) \). In particular,
\[
\theta\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \theta'\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}},
\] (3)
while \( \theta''(x) \) has a jump discontinuity at \( x = 3\pi/4 \).

Differentiating \( \theta(x) \) four times we get
\[
\theta'''(x) - \sqrt{2} \delta'(x - \frac{3\pi}{4}) + \sqrt{2} \delta\left(x - \frac{3\pi}{4}\right) = \theta(x),
\] (4)
where \( \delta(\cdot) \) is the Dirac delta function.

From the basic distribution theory we have that:

(i) \((x - b) \delta'(x - b) = -\delta(x - b)\)

and

(ii) if \( f, g \in C^1(\mathbb{R}) \), then \( f(b) = g(b) \) implies the distributional equality
\[
f(x) \delta(x - b) = g(x) \delta(x - b),
\]
while the two equalities \( f(b) = g(b) \) and \( f'(b) = g'(b) \) imply the distributional equality
\[
f(x) \delta'(x - b) = g(x) \delta'(x - b).
\]

Using the above properties in (4) we obtain
\[
\theta'''(x) - 2 \delta'(x - \frac{3\pi}{4}) \theta(x) + 4 \delta\left(x - \frac{3\pi}{4}\right) \theta(x) = \theta(x). \tag{5}
\]

Now, let \( f(x) \) be the odd extension of \( \theta(x) \) on \( \mathbb{R} \). Clearly \( f(x) \in C^1(\mathbb{R}) \cap L_2(\mathbb{R}) \) and (5) yields
\[
f'''(x) + q_s(x)f(x) = f(x), \quad x \in \mathbb{R}, \tag{6}
\]
where
\[
q_s(x) = Q_s(x) - Q_s(-x), \tag{7}
\]
with
\[
Q_s(x) = -2 \delta'(x - \frac{3\pi}{4}) + 4 \delta\left(x - \frac{3\pi}{4}\right), \quad x \in \mathbb{R}. \tag{8}
\]

Therefore, \( f(x) \) is an \( L_2(\mathbb{R}) \)-eigenfunction of the scattering-type operator
\[
Lu = u''' + q_s(x)u \tag{9}
\]
with eigenvalue \( \lambda = 1 \). Clearly, \( \sigma_c(L) = [0, \infty) \), thus the eigenvalue \( \lambda = 1 \) is embedded in the continuous spectrum of \( L \).

**Remark 1.** A similar example can be constructed if, instead of \( \theta(x) \) of (1), we start with the function
\[
\tilde{\theta}(x) = \begin{cases} 
\cos x, & 0 \leq x < \pi/4; \\
(e^{\pi/4}/\sqrt{2})e^{-x}, & \pi/4 \leq x,
\end{cases}
\]
and then, in place of \( f(x) \) above, we use the even extension, say \( \tilde{f}(x) \), of \( \tilde{\theta}(x) \) on \( \mathbb{R} \).

2
2 Piecewise constant potentials

Here we show how to construct piecewise constant compactly supported potentials \( q(x) \) with positive eigenvalues. The construction is based on the following proposition.

**Proposition 1.** Let

\[
Q_0(x) = \begin{cases} 
A, & a \leq x < b; \\
-B, & b \leq x,
\end{cases}
\]

We consider the solution \( u(x) \) of the equation

\[
u''''(x) + Q_0(x)u(x) = 0, \quad x > a,
\]

satisfying the initial conditions

\[
u(a) = \alpha_0 > 0, \quad \nu'(a) = \alpha_1 > 0, \quad \nu''(a) = \alpha_2 > 0, \quad \nu'''(a) = \alpha_3 > 0,
\]

where \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) are fixed. Then, given \( a \), we can choose \( b > a, A > 0, \) and \( B > 0 \) so that

\[
u'('ζ) = \nu'''('ζ) = 0,
\]

for some \( ζ \in (b, ∞) \).

Before giving the proof of Proposition 1, let us see how it helps to construct our desired example.

Assume without loss of generality (by shifting to the left by \( ζ \)) that \( ζ = 0 \) (thus, \( a < b < 0 \)) and consider the equation

\[
u''''(x) + q(x)u(x) = k_0^4 u(x), \quad x \in \mathbb{R},
\]

where \( k_0 > 0 \) is fixed and

\[
q(x) = \begin{cases} 
0, & x < a; \\
Q_0(x) + k_0^4, & a \leq x < 0; \\
q(-x), & 0 \leq x,
\end{cases}
\]

\( Q_0(x) \) being as in Proposition 1. Next, we choose the initial conditions of \( 12 \) to be

\[
α_0 = e^{k_0 a}, \quad α_1 = k_0 e^{k_0 a}, \quad α_2 = k_0^2 e^{k_0 a}, \quad \text{and} \quad α_3 = k_0^3 e^{k_0 a}
\]

(notice that they are all positive) and consider the (unique) solution \( g(x) \) of \( 14 \) satisfying

\[
g(a) = e^{k_0 a}, \quad g'(a) = k_0 e^{k_0 a}, \quad g''(a) = k_0^2 e^{k_0 a}, \quad \text{and} \quad g'''(a) = k_0^3 e^{k_0 a}.
\]

In other words, \( g(x) \) is the unique solution of \( 14 \) such that

\[
g(x) = e^{k_0 x} \quad \text{for} \quad x < a.
\]
Then, by Proposition 1 we know that $g(x)$ satisfies (13) (with $\zeta = 0$), i.e.

$$g'(0) = g'''(0) = 0,$$

while from (15) we see that our $q(x)$ is even. Therefore, by (19) we get that $g(x)$, too, is even and, finally, by (18) we can conclude that $g(x)$ is an $L_2(\mathbb{R})$-eigenfunction of the operator

$$Lu = u''' + q(x)u$$

with eigenvalue $\lambda = k_4^2 > 0$.

2.1 Proof of Proposition 1

We start with an observation.

Observation 1. Suppose that for some constant $A > 0$ we have

$$u'''(x) = -Au(x) \quad x \in \mathbb{R},$$

with

$$u(a) > 0, \quad u'(a) > 0, \quad u''(a) > 0, \quad \text{and} \quad u'''(a) > 0,$$

for some fixed $a \in \mathbb{R}$. If for $j = 0, 1, 2, 3$ we denote by $x_j$ the smallest zero of $u^{(j)}(x)$ in the interval $(a, \infty)$, then it is not hard to see (since for $u^{(j)}(x)$ to become negative, $j = 0, 1, 2$, we need first $u^{(j+1)}(x)$ to become negative) that

$$a < x_3 < x_2 < x_1 < x_0 < \infty.$$  

Let us, next, consider the initial value problem

$$u'''(x) = Bu(x), \quad x \in \mathbb{R},$$

$$u(0) = \gamma_0, \quad u'(0) = \gamma_1, \quad u''(0) = \gamma_2, \quad u'''(0) = \gamma_3,$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are fixed real numbers and $B$ is a (moving) parameter.

For the purpose of our analysis we need to derive a formula for the (partial) derivative of the solution $u(x) = u(x; B)$ of the above problem with respect to the parameter $B$.

Differentiating (24) and (25) with respect to $B$ yields

$$\partial_B u'''(x; B) = B \partial_B u(x; B) + u(x; B), \quad x \in \mathbb{R},$$

$$\partial_B u(0; B) = \partial_B u'(0; B) = \partial_B u''(0; B) = \partial_B u'''(0; B) = 0,$$

where primes will always denote derivatives with respect to $x$.

It is not hard to check that the solution $\partial_B u(x; B)$ of the initial value problem (26)–(27) is

$$\partial_B u(x; B) = \int_0^x U(x - \xi; B) u(\xi; B) \, d\xi,$$
where \( U(x) = U(x; B) \) is the solution of the initial value problem
\[
U''''(x) = B U(x), \quad x \in \mathbb{R},
\]
namely
\[
U(x; B) = \frac{1}{2B^{3/4}} \sinh \left( \frac{B^{1/4}}{4} x \right) - \frac{1}{2B^{3/4}} \sin \left( \frac{B^{1/4}}{4} x \right).
\]
Notice that \( U(x; B) \) is entire in \( B \) and positive for \( x, B > 0 \), and the same is true for its \( x \)-derivatives
\[
U''(x; B) = \frac{1}{2B^{1/4}} \cosh \left( \frac{B^{1/4}}{4} x \right) - \frac{1}{2B^{1/4}} \cos \left( \frac{B^{1/4}}{4} x \right),
\]
and
\[
U'''(x; B) = \frac{1}{2} \cosh \left( \frac{B^{1/4}}{4} x \right) + \frac{1}{2} \cos \left( \frac{B^{1/4}}{4} x \right).
\]
We are now ready for the main ingredient of the proof of Proposition 1.

**Lemma 1.** Consider the equation
\[
u''''(x) = Bu(x), \quad x \in \mathbb{R},
\]
where \( B > 0 \) is a parameter and let \( u(x) \) be the solution of (35) satisfying
\[
u(b) = \gamma_0 > 0, \quad u'(b) = \gamma_1 > 0, \quad u''(b) = \gamma_2 < 0, \quad u'''(b) = \gamma_3 < 0,
\]
where the numbers \( b, \gamma_0, \gamma_1, \gamma_2, \gamma_3 \) are fixed, hence independent of \( B \). If for \( j = 0, 1, 2, 3 \) we denote by \( z_j \) the smallest zero of \( u^{(j)}(x) \) in the interval \((b, \infty)\), with the convention that \( z_1 = \infty \) in the case where \( u^{(j)}(x) \) does not vanish in \((b, \infty)\), then we can choose \( B \) so that
\[
z_1 = z_3.
\]

**Proof.** Before we start, let us notice that without loss of generality we can take
\[
b = 0.
\]

The proof is divided into three parts: In the first part we determine the behavior of \( u^{(j)}(x) \), \( j = 0, 1, 2, 3, \) on \( x \in [0, \infty) \), in the case where \( B \) takes large values, while in the second part we determine the behavior of the same quantities in the case where \( B \) is close to 0. Finally, in the third part of the proof we show that as \( B \) moves (continuously) from larger to smaller values, it hits a value for which (37) is achieved.

1. The assumption \( u'(0) = \gamma_1 > 0 \) implies that \( u(x) \) is increasing on \([0, z_1)\); in particular,
\[
u(x) \geq u(0) = \gamma_0, \quad x \in [0, z_1).
\]
If we integrate (35) repeatedly (with respect to $x$) and use (36) and (38), we obtain

$$u'''(x) \geq \gamma_3 + B\gamma_0 x, \quad x \in [0, z_1),$$

(39)

$$u''(x) \geq \gamma_2 + \gamma_3 x + \frac{B\gamma_0}{2} x^2, \quad x \in [0, z_1),$$

(40)

and

$$u'(x) \geq \gamma_1 + \gamma_2 x + \frac{\gamma_3}{2} x^2 + \frac{B\gamma_0}{6} x^3, \quad x \in [0, z_1).$$

(41)

Since $\gamma_0, \gamma_1 > 0$, it is clear that there is $\delta > 0$ and a $B^\# > 0$ such that the right-hand side of (41) is $\geq \delta$ for every $B > B^\#$. In particular, if $z_1 < \infty$, then we would have from (41) that $u'(z_1) \geq \delta$, which is impossible since $u'(z_1) = 0$. Therefore $z_1 = \infty$ for every $B > B^\#$, i.e.

$$\text{if } B > B^\#, \text{ then } u'(x) > 0 \text{ for all } x \in [0, \infty).$$

(42)

From (42) we get that $u(x)$ is increasing and, hence $u(x) \geq \gamma_0 > 0$ on $[0, \infty)$. Thus (35) implies that $u'''(x)$ is (strictly) increasing and eventually positive on $[0, \infty)$. Consequently, $u''(x)$ has a unique zero $z_3 \in (0, \infty)$. Finally, $u''(x)$ has a unique minimum on $[0, \infty)$ attained at $x = z_3$; as for $x > z_3$, the function $u''(x)$ is (strictly) increasing and eventually positive, thus $z_3 < z_2 < \infty$.

II. The solution $u(x) = u(x; B)$ of (35) depends analytically in the parameter $B$; actually, it is entire in $B$. Thus, for every $x \in \mathbb{R}$ and every $B \in \mathbb{C}$ we have

$$u(x; B) = \sum_{n=0}^{\infty} u_n(x) B^n,$$

(43)

where

$$u_n(x) = \frac{\partial^n u(x; B)}{\partial B^n} \bigg|_{B=0},$$

(44)

in particular

$$u_0(x) = u(x; 0) = \gamma_0 + \gamma_1 x + \frac{\gamma_2}{2} x^2 + \frac{\gamma_3}{6} x^3.$$  

(45)

Differentiating (45) $n$ times with respect to $B$ and then setting $B = 0$ yields

$$u_n''''(x) = n u_{n-1}(x) \quad x \in \mathbb{R}, \quad n = 1, 2, \ldots$$

(46)

(as usual, primes denote derivatives with respect to $x$), with initial conditions (due to (36))

$$u_n(0) = u_n'(0) = u_n''(0) = u_n'''(0) = 0, \quad n = 1, 2, \ldots.$$  

(47)

By using (35) in (45) we obtain

$$u''''(x; B) = B u(x; B) = \sum_{n=0}^{\infty} u_n(x) B^{n+1},$$

(48)
thus, by integrating (13) with respect to $x$ repeatedly, we can obtain expansions in powers of $B$, similar to (13) for the $x$-derivatives $u'(x; B)$, $u''(x; B)$, and $u'''(x; B)$ (which are, therefore, entire in $B$ too).

From the analytic dependence and the form of $u(x; 0)$, as given in (14), it is not hard to see that there is a $B^0 > 0$ such that if $0 < B < B^0$, then: (i) $u'''(x; B)$ is negative for $x \in [0, \infty)$, with $\lim_{x \to \infty} u'''(x; B) = -\infty$; (ii) $u''(x; B)$ is negative and decreasing in $x$ on $[0, \infty)$, with $\lim_{x \to \infty} u''(x; B) = -\infty$; (iii) $u'(x; B)$ is decreasing in $x$ on $[0, \infty)$, with $\lim_{x \to \infty} u'(x; B) = -\infty$, thus $z_1 < \infty$; and (iv) $u(x; B)$ has a unique maximum in $[0, \infty)$ (attained at $x = z_1$), with $\lim_{x \to \infty} u(x; B) = -\infty$, hence $z_1 < z_0 < \infty$.

III. Suppose we start with a value of $B$ larger than $B^1$ and we decrease it towards 0 in a continuous motion. As we have seen in Parts I and II, $u'(x)$ has a unique minimum in $[0, \infty)$ attained at $x = z_2$. Initially $u'(z_2) > 0$, but as $B$ decreases it will reach a value, say $B_1$ for which $z_1 = z_2$, so that $u'(z_1) = u''(z_1) = u'''(z_1) = 0$. In addition, we must have that $u'''(z_2) > 0$ since $u''(x)$ is negative and decreasing near $x = 0$ and then starts increasing and becomes 0 at $x = z_2$; thus $u''(x)$ attains its minimum at $x = z_3 < z_2$, which implies that $u'''(z_2) > 0$.

Therefore, if $B$ becomes slightly smaller than $B_1$, then we will have $z_3 < z_1 < z_2 < \infty$. Let us now examine the quantities $dz_1/dB$ and $dz_3/dB$. Since $u'(z_1; B) = 0$ and $u'''(z_3; B) = 0$, by implicit differentiation we get

$$u''(z_1; B) \frac{dz_1}{dB} = -\partial_B u'(z_1; B)$$

(49)

and

$$u'''(z_3; B) \frac{dz_3}{dB} = -\partial_B u'''(z_3; B) \quad \text{or} \quad Bu(z_3; B) \frac{dz_1}{dB} = -\partial_B u'''(z_3; B)$$

(50)

(since $u''' = Bu$). Now

$$u''(z_1; B) < 0 \quad \text{and} \quad u(z_3; B) > 0.$$  

(51)

Also, by differentiating formula (25) and using (30) we get

$$\partial_B u'(z_1; B) = \int_0^{z_1} U'(z_1 - \xi; B) u(\xi; B) \, d\xi$$

(52)

and

$$\partial_B u'''(z_3; B) = \int_0^{z_3} U'''(z_3 - \xi; B) u(\xi; B) \, d\xi$$

(53)

In view (32) and (34) the quantity $U'(z_1 - \xi; B)$ is (strictly) positive for $\xi \in (0, z_1)$ and $U'''(z_3 - \xi; B)$ is (strictly) positive for $\xi \in (0, z_3)$. Also, $u(\xi; B) > 0$ for $\xi \in (0, z_1)$. Hence, (32) and (34) imply that

$$\partial_B u'(z_1; B) > 0 \quad \text{and} \quad \partial_B u'''(z_3; B) > 0.$$  

(54)
Using (51) and (54) in (49) and (50) we can conclude that

\[
\frac{dz_1}{dB} > 0, \quad \text{while} \quad \frac{dz_3}{dB} < 0,
\]

which tells us that \(z_1\) decreases, while \(z_3\) increases, as \(B\) decreases. As we have seen, for \(B\) slightly smaller than \(B_1\) we have that \(z_3 < z_1\). Also, as \(B\) decreases \(z_3\), remains smaller than \(z_0\) as long as it exists as a real number, and ceases to exist at the moment when \(z_3 = z_0\) (since \(z_0\) is the smallest positive zero of \(u''' = Bu\)). At the moment, though, when \(z_3 = z_0\) we must have \(z_1 < z_0\). Therefore there must be a \(B > 0\) for which \(z_1(B) = z_3(B)\).

We are now ready to finish the proof of Proposition 1.

**Proof of Proposition 1.** Given \(a\) we fix an \(A > 0\). Then we choose a \(b \in (x_2, x_1)\), where \(x_2, x_1\) are as in (23) of Observation 1. Clearly, \(b > a\) and for this \(b\) it is easy to see that (36) is satisfied. Finally, from Lemma 1 we obtain the existence of a \(B > 0\) for which \(z_1(B) = z_3(B)\).

\[\square\]

3 The square of a Schrödinger operator

Finally, let us mention a somehow trivial example where the fourth-order operator \(L\) is not of the form \(d^4/dx^4 + q(x)\).

Let \(H = -d^2/dx^2 + V(x)\) be a Schrödinger operator, acting on \(L_2(\mathbb{R})\), whose potential \(V(x)\) is smooth, say \(C^r, r \geq 2\). Suppose that \(V(x) \to 0\) as \(x \to \pm \infty\) and that \(H\) has bound states \(\kappa_1 < \kappa_2 < \cdots < 0\). Then, the operator

\[
L = H^2 = \frac{d^4}{dx^4} - V(x)\frac{d^2}{dx^2} - 2V'(x)\frac{d}{dx} + V(x)^2 - V''(x)
\]

has continuous spectrum \(\sigma_c(L) = [0, \infty)\) and eigenvalues \(\kappa_1^2 > \kappa_2^2 > \cdots > 0\).

**References**

[1] M. A. Naimark, *Linear Differential Operators: Two Volumes Bound as One*, Dover Publications Inc., New York, 2012.