THE LIST-CHROMATIC NUMBER AND THE COLORING NUMBER OF UNCOUNTABLE GRAPHS

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Abstract. We study the list-chromatic number and the coloring number of graphs, especially uncountable graphs. We show that the coloring number of a graph coincides with its list-chromatic number provided that the diamond principle holds. Under the GCH assumption, we prove the singular compactness theorem for the list-chromatic number. We also investigate reflection principles for the list-chromatic number and the coloring number of graphs.

1. Introduction

Throughout this paper, a graph means a non-directed simple graph, that is, a graph $X$ is a pair $\langle V_X, E_X \rangle$ where $V_X$ is the set of vertexes and $E_X \subseteq [V_X]^2$ the set of edges. We frequently identify $V_X$ with the graph $X$, and if $X$ is clear from the context, $E_X$ is denoted as $E$ for simplicity.

The cardinality of the graph $X$, denoted by $|X|$, is the cardinality of the vertex set $V_X$. For a graph $X = \langle X, E \rangle$ and $x, y \in X$, when $\{x, y\} \in E$ we write $x E y$ or $y E x$. For $x \in X$, let $E^x = \{y \in X \mid y E x\}$.

Definition 1.1. Let $X = \langle X, E \rangle$ be a graph.

1. A coloring of $X$ is a function on $X$. A good coloring of $X$ is a coloring $f : X \to \text{ON}$ such that whenever $x E y$, we have $f(x) \neq f(y)$.
2. For a finite or an infinite cardinal $\kappa$, a $\kappa$-assignment of $X$ is a function $F : X \to [\text{ON}]^\kappa$.
3. The list-chromatic number of $X$, List($X$), is the minimal finite or infinite cardinal $\kappa$ such that for every $\kappa$-assignment $F : X \to [\text{ON}]^\kappa$, there is a good coloring $f$ of $X$ with $f(x) \in F(x)$.
4. The coloring number of $X$, Col($X$), is the minimal finite or infinite cardinal $\kappa$ such that $X$ admits a well-ordering $\prec$ such that for every $x \in X$, we have $|\{y \in E^x \mid y \prec x\}| < \kappa$.

We know that List($X$) $\leq$ Col($X$) $\leq |X|$. The coloring number was introduced by Erdős-Hajnal [3]. The list-chromatic number was done in Vizing [19] and Erdős-Rubin-Taylor [4] independently, and Komjáth [12] studied the list-chromatic number of infinite graphs extensively.
See also Komjáth’s survey [11] for these numbers. In this paper, we study further combinatorial properties of the list-chromatic number and the coloring number of uncountable graphs, and we also investigate reflection principles for these numbers.

By Komjáth’s work, it turned out that the difference between the list-chromatic number and the coloring number of infinite graphs is sensitive. While it is consistent that there is a graph $X$ with $\text{Col}(X) > \text{List}(X) \geq \omega$, Komjáth [12] constructed a model of ZFC in which $\text{Col}(X) = \text{List}(X)$ holds for every graph $X$ with infinite coloring number. We show that this situation follows from the diamond principle, which gives another simple proof of Komjáth’s result.

Theorem 1.2. Suppose that for every regular uncountable cardinal $\kappa$ and stationary $S \subseteq \kappa$, $\diamondsuit(S)$ holds (e.g., assume $V = L$). Then for every graph $X$, if $\text{Col}(X)$ is infinite then $\text{Col}(X) = \text{List}(X)$.

A graph $Y$ is called a subgraph of the graph $X$ if $\mathcal{V}_Y \subseteq \mathcal{V}_X$ and $\mathcal{E}_Y \subseteq \mathcal{E}_X$. A subgraph $Y$ of $X$ is an induced subgraph if $\mathcal{E}_Y = [\mathcal{V}_Y]^2 \cap \mathcal{E}_X$. It is clear that if $Y$ is a subgraph of $X$ then $\text{Col}(Y) \leq \text{Col}(X)$ and $\text{List}(Y) \leq \text{List}(X)$.

The coloring number has many useful properties, one of these is the singular compactness. Shelah [15] showed that if $|X|$ is singular and $\text{Col}(Y) \leq \lambda$ for every subgraph $Y$ of size $< |X|$, then $\text{Col}(X) \leq \lambda$. Unlike the coloring number, one can prove that the singular compactness does not hold for the list-chromatic number in general (see Section 2 below). However, we prove that it is valid under the GCH assumption.

Theorem 1.3. Suppose $\kappa$ is a singular cardinal such that the set $\{\mu < \kappa \mid 2^\mu = \mu^+\}$ contains a club in $\kappa$. For every graph $X$ of size $\kappa$ and infinite cardinal $\lambda$, if $\text{List}(Y) \leq \lambda$ for every subgraph $Y$ of size $< \kappa$, then $\text{List}(X) \leq \lambda$.

In Sections 4–6, we investigate reflection principles for the coloring number and the list-chromatic number.

Definition 1.4. For an infinite cardinal $\lambda$, let $\text{RP}(\text{List}, \lambda)$ be the assertion that for every graph $X$ of size $\leq \lambda$, if $\text{List}(X) > \omega$ then $X$ has a subgraph $Y$ of size $\omega_1$ with $\text{List}(Y) > \omega$. This is equivalent to uncountable compactness for the list-chromatic number: For a graph $X$ of size $\leq \lambda$, if $\text{List}(Y) \leq \omega$ for every subgraph $Y$ of $X$ with size $\leq \omega_1$, then $\text{List}(X) \leq \omega$. $\text{RP}(\text{List})$ is the assertion that $\text{RP}(\text{List}, \lambda)$ holds for every cardinal $\lambda$. We define $\text{RP}(\text{Col}, \lambda)$ and $\text{RP}(\text{Col})$ by replacing the list-chromatic number with the coloring number.

Such reflection principles and uncountable compactness are studied in various fields, e.g., Balogh [1], Fleissner [5], Fuchino et al. [6], Fuchino-Rinot [7], Fuchino-Sakai-Soukup-Usuba [9], and Todorčević [17, 18].
Fuchino et al. [6] introduced the Fodor-type Reflection Principle FRP, which is a combinatorial principle and is consistent modulo large cardinal axiom. FRP implies various reflection principles ([6], [7]). Fuchino-Sakai-Soukup-Usuba [9] proved that FRP implies RP(Col), actually FRP is equivalent to RP(Col). Other notable principle in this context is Rado’s conjecture, which is a reflection principle for the chromatic number of intersection graphs, and is consistent modulo large cardinal axiom ([17]). The reflection RP(Col) is strictly weaker than Rado’s conjecture since Rado’s conjecture implies RP(Col) but the converse does not hold (Fuchino-Sakai-Torres-Perez-Usuba [10]).

For RP(List), Fuchino-Sakai [8] showed that RP(List) holds after collapsing a supercompact cardinal to $\omega_2$, hence RP(List) is also consistent modulo large cardinal axiom.

It is known that RP(Col), or even the local reflection RP(Col, $\lambda$) for some $\lambda > \omega_1$ is a large cardinal property; If $\lambda > \omega_1$ is regular and RP(Col, $\lambda$) holds then every stationary subset of $\lambda \cap \text{Cof}(\omega)$ is reflecting (see Fact 2.13). In contrast with this result, we prove that the local reflection RP(List, $\lambda$) for some fixed $\lambda$ is not a large cardinal property.

Theorem 1.5. Suppose GCH. Let $\lambda > \omega_1$ be a cardinal, and suppose $\text{AP}_{\mu}$ holds (see Definition 3.10) for every $\mu < \lambda$ with countable cofinality. Then there is a poset $P$ which is $\sigma$-Baire, satisfies $\omega_2$-c.c., and forces that “RP(List, $\lambda$) holds and $2^{\omega_1} > \lambda$”.

This theorem shows that the local reflection RP(List, $\lambda$) does not imply RP(Col, $\lambda$). On the other hand, under the assumption that List($X$) = Col($X$) for every graph $X$ of size $\omega_1$ with infinite coloring number, we have that RP(Col, $\lambda$) implies RP(List, $\lambda$) for every $\lambda$, in particular RP(Col) implies RP(List). This observation suggests a natural question:

Question 1.6. Does the global reflection RP(Col) imply RP(List)? How is the converse?

For this question, we show that RP(Col) and RP(List) can be separated one from the other. More precisely, we show that the global reflection does not imply the reflection at $\omega_2$ for the other number.

Theorem 1.7. If $\text{ZFC+}$ “there exists a supercompact cardinal” is consistent, then the following theories are consistent as well:

1. $\text{ZFC+}$ RP(List) holds but RP(Col, $\omega_2$) fails.
2. $\text{ZFC+}$ RP(Col) holds but RP(List, $\omega_2$) fails.

Notice that, in [8], they already showed the consistency of the theory that $\text{ZFC+}$ “RP(Col) holds but RP(List, $\omega_3$) fails”.

This paper is organized as follows. In Section 2, we present some basic definitions, facts, and easy observations. We study combinatorial properties about the list-chromatic number and the coloring number in
2. Preliminaries

We present some definitions, facts, and easy observations, which will be used later.

\( \kappa, \lambda, \mu \) will denote infinite cardinals unless otherwise specified. \( \text{ON} \) is the class of all ordinals. For an ordinal \( \alpha \), let \( \text{Cof}(\alpha) \) be the class of ordinals with cofinality \( \alpha \).

For a regular uncountable cardinal \( \theta \), let \( \mathcal{H}_\theta \) be the set of all sets with hereditary cardinality \( < \theta \).

Let \( \kappa \) be a regular uncountable cardinal. A stationary set \( S \subseteq \kappa \) is reflecting if there is some \( \alpha < \kappa \) such that \( S \cap \alpha \) is stationary in \( \alpha \). If \( S \) is not reflecting, then \( S \) is non-reflecting.

For a regular uncountable cardinal \( \kappa \) and a stationary set \( S \subseteq \kappa \), a sequence \( \langle d_\alpha \mid \alpha \in S \rangle \) is a ♦\((S)\)-sequence if for every \( A \subseteq \kappa \), the set \( \{ \alpha \in S \mid d_\alpha = A \cap \alpha \} \) is stationary in \( \kappa \). Let us say that ♦\((S)\) holds if there is a ♦\((S)\)-sequence.

**Fact 2.1** (Shelah [16]). Let \( \kappa \) be an infinite cardinal. Suppose \( 2^\kappa = \kappa^+ \). Then for every stationary subset \( S \subseteq \kappa^+ \setminus \text{Cof}(\text{cf}(\kappa)) \), ♦\((S)\) holds.

**Definition 2.2.** For an infinite set \( A \) and a limit ordinal \( \delta \), a filtration of \( A \) is a \( \subseteq \)-increasing continuous sequence \( \langle A_\alpha \mid \alpha < \delta \rangle \) such that \( |A_\alpha| < |A| \) for \( \alpha < \delta \) and \( \bigcup_{\alpha < \delta} A_\alpha = A \).

The following characterization of the coloring number is very useful.

**Fact 2.3** (Erdős-Hajnal [3]). Let \( X \) be a graph and \( \kappa \) an infinite cardinal. Then the following are equivalent:

1. \( \text{Col}(X) \leq \kappa \).
2. There is a function \( f : X \to [X]^{< \kappa} \) such that for every \( x, y \in X \), if \( x \in E y \) then either \( x \in f(y) \) or \( y \in f(x) \).
3. There is a filtration \( \langle X_i \mid i < \delta \rangle \) of \( X \) such that for every \( i < \delta \), identifying \( X_i \) with a induced subgraph of \( X \), \( \text{Col}(X_i) \leq \kappa \), and, for every \( i < \delta \) and \( x \in X \setminus X_i \), we have \( |X_i \cap \mathcal{E}^x| < \kappa \).
4. There is a 1-1 enumeration \( \langle x_i \mid i < |X| \rangle \) of \( X \) such that for every \( i < |X| \), we have \( |\{ x_j \mid j < i, x_j \in \mathcal{E} x_i \}| < \kappa \).

The following fact is immediate from the result in Komjáth [12], and we present the proof for completeness.

**Fact 2.4.** Let \( X = \langle X, \mathcal{E} \rangle \) be a graph and \( \lambda \) and \( \mu \) infinite cardinals with \( \mu \leq \lambda \). If there are \( Y_0, Y_1 \subseteq X \) such that \( |Y_0| = \lambda \), \( |Y_1| \geq 2^\lambda \), and \( |\mathcal{E}^z \cap Y_0| \geq \mu \) for every \( z \in Y_1 \), then \( \text{List}(X) > \mu \).
Proof. We may assume that \( Y_0 \cap Y_1 = \emptyset \). It suffices to show that some subgraph of \( X \) has list-chromatic number \( > \mu \). By removing edges of \( X \), we may assume that \( |E^z \cap Y_0| = \mu \) for every \( z \in Y_1 \). Fix a pairwise disjoint family \( \{ A_y \mid y \in Y_0 \} \) with \( A_y \subseteq [\lambda]^\mu \). Since \( |Y_1| \geq 2^\lambda \), we can take an enumeration \( (z_z \mid z \in Y_1) \) of \( \prod_{y \in Y_0} A_y \), possibly with repetitions. Let \( Y \) be the induced subgraph \( Y_0 \cup Y_1 \). Define the \( \mu \)-

assignment \( F \) of \( Y \) as follows. For \( z \in Y_1 \), let \( F(z) = \{ g_z(y) \mid y \in Y_0 \} \). For \( y \in Y_0 \), let \( F(y) = A_y \). We show that there is no good coloring \( f \) of \( Y \) with \( f(y) \in F(y) \). For a coloring \( f \) of \( Y \) with \( f(x) \in F(x) \), there must be \( z \in Y_1 \) with \( f \mid Y_0 = g_z \). Then \( \{ f(y) \mid y \in E_y, z \wedge y \in Y_0 \} = \{ g_z(y) \mid y \in E_y, z \wedge y \in Y_0 \} = F(z) \) and \( f(z) \in F(z) \), so \( f \) is never good. \( \square \)

For non-empty sets \( X \) and \( Y \), let \( K_{X,Y} \) be the complete bipartite graph on the bipartition classes \( X \) and \( Y \). Fact 2.4 yields the following fact.

**Fact 2.5** (Lemma 6 in [12]). The complete bipartite graph \( K_{\omega,2^\omega} \) has uncountable list-chromatic number.

**Fact 2.6** (Lemma 9 in [12]). If \( \lambda < 2^\omega \), then the complete bipartite graph \( K_{\omega,\lambda} \) has countable list-chromatic number. In particular, every subgraph of \( K_{\omega,2^\omega} \) with size \( < 2^\omega \) has countable list-chromatic number.

The following is also due to Komjáth.

**Fact 2.7** ([12]). Let \( X = (\kappa, E) \) be a graph on the regular uncountable cardinal \( \kappa \) and \( \lambda < \kappa \) an infinite cardinal. Suppose \( \text{Col}(Y) \leq \lambda \) for every subgraph \( Y \) of \( X \) with size \( < \kappa \). Then the following are equivalent:

1. The set \( S = \{ \alpha \in \kappa \cap \text{Col}(\text{cf}(\lambda)) \mid \exists \beta \geq \alpha (|\alpha \cap E^\beta| \geq \lambda) \} \) is stationary in \( \kappa \).

2. The set \( T = \{ \alpha \in \kappa \mid \exists \beta \geq \alpha (|\alpha \cap E^\beta| \geq \lambda) \} \) is stationary in \( \kappa \).

3. \( \text{Col}(X) > \lambda \).

Proof. (1) \( \Rightarrow \) (2) is trivial.

(2) \( \Rightarrow \) (3). Suppose \( T \) is stationary but \( \text{Col}(X) \leq \lambda \). Then by Fact 2.3, there is \( f : X \to [X]^{<\lambda} \) such that whenever \( \alpha, \beta \in X \) with \( \alpha \in E \beta \), we have \( \alpha \in f(\beta) \) or \( \beta \in f(\alpha) \). Since \( T \) is stationary, we can find \( \alpha \in T \) such that \( f(\gamma) \subseteq \alpha \) for every \( \gamma < \alpha \). Fix \( \beta \geq \alpha \) with \( |E^\beta \cap \alpha| \geq \lambda \).

Because \( |f(\beta)| < \lambda \), there is \( \gamma \in E^\beta \cap \alpha \) with \( \gamma \notin f(\beta) \). \( \beta \) is jointed with \( \gamma \) but \( \gamma \notin f(\beta) \), so \( \beta \notin f(\gamma) \subseteq \alpha \). This is a contradiction.

(3) \( \Rightarrow \) (2). Suppose \( T \) is non-stationary, and then we can deduce \( \text{Col}(X) \leq \lambda \) as follows. Fix a club \( D \) in \( \kappa \) disjointing from the set \( \{ \alpha \in \kappa \mid \exists \beta \geq \alpha (|\alpha \cap E^\beta| \geq \lambda) \} \). The sequence \( \{ \alpha \mid \alpha \in D \} \) is a filtration of \( X \). Moreover \( \text{Col}(\alpha) \leq \lambda \) and \( |\alpha \cap E^\beta| < \lambda \) for every \( \alpha \in D \) and \( \beta \in \kappa \setminus \alpha \). Applying Fact 2.3, we have \( \text{Col}(X) \leq \lambda \).
\( (2) \Rightarrow (1). \) Suppose \( S \) is non-stationary. Let \( T' = \{ \alpha \in \kappa \mid \text{Cof}(\text{cf}(\lambda)) \mid \exists \beta \geq \alpha (|\alpha \cap \mathcal{E}^\beta| \geq \lambda) \}. \) We will show that \( T' \) is non-stationary, then we can conclude that \( T \) is non-stationary. Suppose to the contrary that \( T' \) is stationary. For \( \alpha \in T', \) fix \( \beta \geq \alpha \) with \( |\mathcal{E}^\beta \cap \alpha| \geq \lambda. \) Since \( \text{cf}(\alpha) \neq \text{cf}(\lambda), \) there is \( g(\alpha) < \alpha \) with \( |\mathcal{E}^\beta \cap g(\alpha)| \geq \lambda. \) By Fodor’s lemma, there is \( \gamma < \kappa \) such that the set \( \{ \alpha \in T' \mid g(\alpha) = \gamma \} \) is stationary. Now take an arbitrary \( \alpha \) with \( \text{cf}(\alpha) = \text{cf}(\lambda) \) and \( \alpha > \gamma. \) Then we can take \( \alpha' \in T' \) with \( \alpha' > \alpha \) and \( g(\alpha') = \gamma. \) Then \( |\mathcal{E}^\beta \cap \gamma| \geq \lambda \) for some \( \beta \geq \alpha' > \alpha. \) This means that \( \alpha \in S, \) so \( S \) is stationary. This is a contradiction. \( \square \)

The next fact is a consequence of Shelah’s singular compactness theorem \([15]\).

**Fact 2.8** (Shelah \([15]\)). Let \( X \) be a graph, and suppose that \( |X| \) is a singular cardinal. For an infinite cardinal \( \lambda, \) if \( \text{Col}(Y) \leq \lambda \) for every subgraph \( Y \) of \( X \) with size \( < |X|, \) then \( \text{Col}(X) \leq \lambda. \)

The singular compactness for the list-chromatic number does not hold in general; Komjáth proved that \( \text{List}(K_{\omega,2^\omega}) > \omega \) but if \( H \) is a subgraph of \( K_{\omega,2^\omega} \) and \( |H| < 2^{\omega}, \) then \( \text{List}(H) \leq \omega. \) Hence if \( 2^\omega \) is singular then \( K_{\omega,2^\omega} \) exemplifies the failure of Shelah’s singular compactness with respect to the list-chromatic number. On the other hand, in the next section, we will show that the singular compactness for the list-chromatic number holds under GCH.

The singular compactness immediately yields the following, which we will use frequently.

**Corollary 2.9.** Let \( \lambda \) be an infinite cardinal, and \( X \) a graph with \( \text{Col}(X) > \lambda. \) Then \( X \) has a subgraph \( Y \) such that \( |Y| \) is regular uncountable, \( \text{Col}(Y) > \lambda, \) and \( \text{Col}(Z) \leq \lambda \) for every subgraph \( Z \) of \( Y \) with size \( < |Y|. \)

*Proof.* Let \( \kappa = \min\{|Y| \mid Y \text{ is a subgraph of } X, \text{Col}(Y) > \lambda\}. \) Clearly \( \kappa \) is uncountable. Pick a subgraph \( Y \) of \( X \) with size \( \kappa \) and \( \text{Col}(Y) > \lambda. \) By the minimality of \( \kappa, \) every subgraph of \( Y \) with size \( < \kappa \) has coloring number \( \leq \lambda. \) Hence \( \kappa \) must be regular by Fact 2.8. \( \square \)

We will use the forcing method, so we fix some basic notations and definitions. For a poset \( P \) and \( p, q \in P, \) if \( p \leq q \) then \( p \) is an extension of \( q. \) \( p \) and \( q \) are compatible if there is \( r \in P \) which is a common extension of \( p \) and \( q. \)

For a cardinal \( \kappa, \) a poset \( P \) is \( \kappa\text{-Baire} \) if for every family \( \mathcal{F} \) of open dense subsets of \( P \) with \( |\mathcal{F}| < \kappa, \) the intersection \( \bigcap \mathcal{F} \) is dense in \( P. \) A poset \( P \) is \( \kappa\text{-Baire} \) if and only if the forcing with \( P \) does not add new \( < \kappa \)-sequences. \( \sigma\text{-Baire} \) means \( \omega_1\text{-Baire}. \)

For an uncountable cardinal \( \kappa, \) and non-empty sets \( X \) and \( Y, \) let \( \text{Fn}(X,Y, < \kappa) \) be the poset of all partial functions from \( X \) to \( Y \) with size \( < \kappa. \) The ordering is the reverse inclusion.
For a regular cardinal $\lambda$ and a set $X$ of ordinals, let $\text{Coll}(\lambda, X)$ be the poset of all functions $p$ with size $< \lambda$ such that $\text{dom}(p) \subseteq \lambda \times X$ and $p(\alpha, \beta) \in \beta$ for every $\langle \alpha, \beta \rangle \in \text{dom}(p)$. The ordering is given by the reverse inclusion. $\text{Coll}(\lambda, X)$ is $\lambda$-closed, and forcing with $\text{Coll}(\lambda, X)$ adds a surjection from $\lambda$ onto $\beta$ for every $\beta \in X$. If $X$ is a regular cardinal $\kappa$, $\text{Coll}(\lambda, \kappa)$ is denoted as $\text{Coll}(\lambda, < \kappa)$, and if $\kappa$ is inaccessible, then $\text{Coll}(\lambda, < \kappa)$ satisfies the $\kappa$-c.c. and forces $\kappa = \lambda^+$.

A poset $\mathbb{P}$ is $\omega_1$-stationary preserving if for every stationary set $S \subseteq \omega_1$, $\mathbb{P}$ forces that “$S$ remains stationary in $\omega_1$”.

**Lemma 2.10.** Let $X$ be a graph.

1. If $|X| = \omega_1$ and $\text{Col}(X) > \omega$, then every $\omega_1$-stationary preserving poset forces that $\text{Col}(X) > \omega$.
2. Suppose $\text{Col}(X) \geq \omega_2$. Then every $\omega_2$-c.c. forcing notion forces that $\text{Col}(X) \geq \omega_2$.

**Proof.** (1) is immediate from Fact 2.7 and the $\omega_1$-stationary preservingness of $\mathbb{P}$.

For (2), take a subgraph $Y$ of $X$ such that $|Y|$ is regular uncountable, $\text{Col}(Y) > \omega_1$, and $\text{Col}(Z) \leq \omega_1$ for every subgraph $Z$ of $Y$ with size $< |Y|$. Let $\kappa = |Y|$. Clearly $\kappa \geq \omega_2$. We may assume $Y$ is of the form $\langle \kappa, E \rangle$. By Fact 2.7, the set $S = \{\alpha < \kappa \mid \exists \beta \geq \alpha (|E^\beta \cap \alpha| \geq \omega_1)\}$ is stationary in $\kappa$. Since $\mathbb{P}$ satisfies the $\omega_2$-c.c., we know that $S$ is stationary in $V^\mathbb{P}$. Then the stationarity of $S$ witnesses that $\text{Col}(Y) \geq \omega_2$, hence so does $\text{Col}(X) \geq \omega_2$ in $V^\mathbb{P}$.

We will also use the Fodor-type reflection principle $\text{FRP}$, which was introduced in Fuchino et al. [6].

**Definition 2.11** ([6]). For a regular $\kappa \geq \omega_2$, $\text{FRP}(\kappa)$ is the assertion that for every stationary $E \subseteq \kappa \cap \text{Cof}(\omega)$ and $g : E \rightarrow [\kappa]^\omega$ with $g(\alpha) \in [\alpha]^\omega$, there is $I \subseteq [\kappa]^\omega$ such that $\sup(I) \notin I$, $\text{cf}(\sup(I)) = \omega_1$, and the set $\{x \in [I]^\omega \mid \sup(x) \in E, g(\sup(x)) \subseteq x\}$ is stationary in $[I]^\omega$. $\text{FRP}$ is the assertion that $\text{FRP}(\kappa)$ holds for every regular $\kappa \geq \omega_2$.

**Fact 2.12** ([6], [9]).

1. $\text{RP}(\text{Col})$ holds if and only if $\text{FRP}$ holds.
2. $\text{FRP}$ is preserved by any c.c.c. forcing.
3. If $\kappa$ is supercompact, then $\text{Coll}(\omega_1, < \kappa)$ forces FRP, hence also RP(Col).

See also Corollary 4.8, which provides the proof of $\text{FRP} \Rightarrow \text{RP(\text{Col})}$. For completeness, let us sketch the proof of (3).

**Proof.** Take a $(V, \text{Coll}(\omega_1, < \kappa))$-generic $G$. In $V[G]$, fix a regular cardinal $\lambda \geq \kappa$. We show that $\text{FRP}(\lambda)$ holds in $V[G]$. Take a stationary $E \subseteq \lambda \cap \text{Cof}(\omega)$, and $g : E \rightarrow [\lambda]^\omega$ with $g(\alpha) \subseteq \alpha$. Let $D = \{x \in [\lambda]^\omega \mid \sup(x) \in E, g(\sup(x)) \subseteq x\}$. $D$ is stationary in $[\lambda]^\omega$.

In $V$, take a $\lambda$-supercompact elementary embedding $j : V \rightarrow M$ with critical point $\kappa$, that is, $\lambda^M \subseteq M$ and $\lambda < j(\kappa)$. Then take a
(V[G], Coll(ω₁, [κ, j(κ)]))-generic G_{tail}. We can take a (V, Coll(ω₁, j(κ)))-generic j(G) with j(G) ∩ Coll(ω₁, < κ) = G and V[j(G)] = V[G][G_{tail}]. Since Coll(ω₁, [κ, j(κ)]) is σ-closed in V[G], we have that M[j(G)] is closed under ω-sequences and D remains stationary in V[j(G)]. In addition one can check that j "D = \{x ∈ [j "\lambda]ω | sup(x) ∈ j(E), j(g)(sup(x)) ⊆ x\}, j "D ∈ M[j(G)], [j "\lambda] = ω₁, and j "D is stationary in [j "\lambda]ω in M[j(G)]. Hence in M[j(G)], j "D and j "D witness the statement that “there is I ∈ [j(\lambda)]ω such that sup(I) \notin I, cf(sup(I)) = ω₁, and \{x ∈ [I]ω | sup(x) ∈ j(E), j(g)(sup(x)) ⊆ x\} is stationary”. By the elementarity of j, it holds in V[G] that “there is I ∈ [\lambda]ω such that sup(I) \notin I, cf(sup(I)) = ω₁, and \{x ∈ [I]ω | sup(x) ∈ E, g(sup(x)) ⊆ x\} is stationary”. □

The following may be a kind of folklore, and the author found the proof of it in [8].

**Fact 2.13.** Let κ be a regular uncountable cardinal, and suppose S ⊆ κ ∩ Col(ω) is a non-reflecting stationary set. Then there is a graph X of size κ such that Col(X) > ω but Col(Y) ≤ ω for every subgraph Y of X with size < κ. In particular, if RP(\text{Col}, κ) holds for some regular \text{κ} ≥ ω₂, then every stationary subset of κ ∩ Col(ω) is reflecting.

**Proof.** For α ∈ S, take a cofinal set c_α ⊆ α in α with order type ω and c_α ∩ S = ∅. The vertex set of the graph X is S ∪ \{c_α | α ∈ S\}, and the edge set E is defined by β E α ⇔ α ∈ S and β ∈ c_α. We check that the graph X is as required.

For proving Col(X) > ω, suppose to the contrary that Col(X) ≤ ω. Then by Fact 2.3, there is f : X → [X]ω such that whenever β E α, we have α ∈ f(β) or β ∈ f(α). For α ∈ S, since f(α) is finite but c_α is infinite, we can choose γ_α ∈ c_α \ f(α). By Fodor’s lemma, there is γ < κ such that the set \{α ∈ S | γ = γ_α\} is stationary. However then \{α ∈ S | γ = γ_α\} ⊆ f(γ), this is impossible.

Next, take δ < κ such that X ∩ δ = (S ∩ δ) ∪ \{c_α | α ∈ S ∩ δ\}. We show that Col(X ∩ δ) ≤ ω. Since S ∩ δ is non-stationary in δ, we can find a function g on S ∩ δ such that g(α) < α and \{c_α \ g(α) | α ∈ S ∩ δ\} is a pairwise disjoint family (e.g., see Lemma 2.12 in Eisworth [2]). Note that g(α) ∩ c_α is finite. Define f : X ∩ δ → [X ∩ δ]ω as follows: For α ∈ X ∩ δ, if α ∈ S ∩ δ then f(α) = g(α) ∩ c_α. If α ∉ S ∩ δ and α ∈ g(α') ∩ c_α for some α' ∈ S ∩ δ, then f(α) = g(α). If α ∉ S ∩ δ but α ∉ g(α') ∩ c_α for every α' ∈ S ∩ δ, there is a unique α* ∈ S ∩ δ with α ∈ c_α \ g(α*). Then let f(α) = \{α*\}. One can check that α E β ⇒ α ∈ f(β) or β ∈ f(α), hence Col(X ∩ δ) ≤ ω by Fact 2.3. □

3. Combinatorial results

In this section, we present some combinatorial results about the list-chromatic number and the coloring number.
By the forcing method, Komjáth [12] showed the consistency of the statement that for every graph $X$, if $\text{Col}(X)$ is infinite then $\text{List}(X) = \text{Col}(X)$. We give another proof of Komjáth’s result, in fact the diamond principle is sufficient to obtain it.

**Proposition 3.1.** Let $\kappa$ be a regular uncountable cardinal and $\lambda < \kappa$ an infinite cardinal. Suppose $\Diamond(S)$ holds for every stationary $S \subseteq \kappa \cap \text{CoF}(\text{cf}(\kappa))$. Then for every graph $X = \langle \kappa, E \rangle$, if $\text{Col}(X) > \lambda$ but $\text{Col}(Y) \leq \lambda$ for every subgraph $Y$ of $X$ with size $< \kappa$, then $\text{List}(X) > \lambda$.

**Proof.** Let $S = \{ \alpha \in \kappa \cap \text{CoF}(\text{cf}(\kappa)) \mid \exists \beta \geq \alpha (|\alpha \cap \mathcal{E}^\beta| \geq \lambda) \}$. $S$ is stationary in $\kappa$ by Fact 2.7. For each $\alpha \in S$, take $\beta(\alpha) \geq \alpha$ with $|\alpha \cap \mathcal{E}^{\beta(\alpha)}| \geq \lambda$. If necessary, by shrinking $S$ we may assume that for every $\alpha, \alpha' \in S$, if $\alpha < \alpha'$ then $\beta(\alpha) < \beta(\alpha')$.

By our assumption, $\Diamond(S)$ holds. Then by a standard coding argument, there is a sequence $\langle d_\alpha, e_\alpha \mid \alpha \in S \rangle$ such that $d_\alpha, e_\alpha : \alpha \rightarrow \alpha$ and for every $f, g : \kappa \rightarrow \kappa$, the set $\{ \alpha \in S \mid f \upharpoonright \alpha = d_\alpha, g \upharpoonright \alpha = e_\alpha \}$ is stationary in $\kappa$.

Fix a pairwise disjoint sequence $\langle A_\beta \mid \beta < \kappa \rangle$ with $A_\beta \in [\kappa]^{\lambda}$. For each $\alpha \in S$, fix a set $x_\alpha \subseteq \alpha \cap \mathcal{E}^{\beta(\alpha)}$ with $|x_\alpha| = \lambda$. Now we define two $\lambda$-assignments $F, G$ in the following manner. For $\beta < \kappa$,

1. If $\beta \neq \beta(\alpha)$ for every $\alpha \in S$, then $F(\beta) = G(\beta) = A_\beta$.
2. Suppose $\beta = \beta(\alpha)$ for some (unique) $\alpha \in S$.
   
   a. If $|d_\alpha``x_\alpha| = \lambda$, then $F(\beta) = d_\alpha``x_\alpha$ and $G(\beta) = A_\beta$.
   
   b. If $|d_\alpha``x_\alpha| < \lambda$ and $|e_\alpha``x_\alpha| = \lambda$, then $F(\beta) = A_\beta$ and $G(\beta) = e_\alpha``x_\alpha$.
   
   c. If $|d_\alpha``x_\alpha|, |e_\alpha``x_\alpha| < \lambda$, let $F(\beta) = G(\beta) = A_\beta$.

Note that if $F(\beta) \neq A_\beta$, then the condition (2)(a) must be applied to $\beta$, so $G(\beta)$ is $A_\beta$.

Take arbitrary colorings $f, g : \kappa \rightarrow \kappa$ with $f(\beta) \in F(\beta)$ and $g(\beta) \in G(\beta)$. We show that if $f$ is good then $g$ is not good.

Now suppose $f$ is good. Take $\alpha \in S$ with $f \upharpoonright \alpha = d_\alpha$ and $g \upharpoonright \alpha = e_\alpha$. We know $f \upharpoonright x_\alpha = d_\alpha \upharpoonright x_\alpha$ and $g \upharpoonright x_\alpha = e_\alpha \upharpoonright x_\alpha$. Let $\beta = \beta(\alpha)$. If $|d_\alpha``x_\alpha| = \lambda$, then $F(\beta) = d_\alpha``x_\alpha = f``x_\alpha$. Since $f(\beta) \in F(\beta)$, there is $\eta \in x_\alpha$ with $f(\eta) = f(\beta)$, this is a contradiction because $\eta \in x_\alpha \subseteq \mathcal{E}^\beta$. Hence we have $|d_\alpha``x_\alpha| < \lambda$. In addition, since the family $\langle A_\beta \mid \beta < \kappa \rangle$ is pairwise disjoint, we know that the function $f \upharpoonright \{ \eta \in x_\alpha \mid F(\eta) = A_\eta \}$ is injective. Thus we have that $|\{ \eta \in x_\alpha \mid F(\eta) = A_\eta \}| < \lambda$, otherwise we have $|d_\alpha``x_\alpha| = \lambda$. Hence the set $\{ \eta \in x_\alpha \mid F(\eta) \neq A_\eta \}$ has cardinality $\lambda$. For $\eta \in x_\alpha$ with $F(\eta) \neq A_\eta$, we know that $G(\eta) = A_\eta$. By the same reason before, the map $g \upharpoonright \{ \eta \in x_\alpha \mid F(\eta) \neq A_\eta \}$ is injective, and we have $|e_\alpha``x_\alpha| = |g``x_\alpha| = \lambda$. Then $G(\beta) = e_\alpha``x_\alpha$, and we can find $\eta \in x_\alpha$ with $g(\eta) = g(\beta)$. Therefore $g$ is not good.

**Corollary 3.2.** Let $\kappa$ be an uncountable cardinal and $\lambda < \kappa$ an infinite cardinal. Suppose that for every regular uncountable $\mu < \kappa$ and every
stationary $S \subseteq \mu \cap \text{Col}(\text{cf}(\lambda))$, $\Diamond(S)$ holds. Then for every graph $X$ of size $< \kappa$, $\text{Col}(X) > \lambda$ if and only if $\text{List}(X) > \lambda$.

**Proof.** Take a graph $X$ of size $< \kappa$ and $\text{Col}(X) > \lambda$. We shall prove that $\text{List}(X) > \lambda$. By Corollary 2.9, there is a subgraph $Y$ of $X$ such that $|Y|$ is regular uncountable, $\text{Col}(Y) > \lambda$, and $\text{Col}(Z) \leq \lambda$ for every subgraph $Z$ of $Y$ with $|Z| < |Y|$. Then, by Proposition 3.1, we have $\text{List}(Y) > \lambda$, hence $\text{List}(X) > \lambda$. $\square$

Now we have Theorem 1.2.

**Corollary 3.3.** Suppose that for every regular uncountable $\kappa$ and stationary $S \subseteq \kappa$, $\Diamond(S)$ holds. Then for every graph $X$, if $\text{Col}(X)$ is infinite then $\text{Col}(X) = \text{List}(X)$.

**Proof.** Komjáth [12] proved that if $\text{Col}(X) = \omega$, then $\text{List}(X) = \omega$. The case $\text{Col}(X) > \omega$ follows from Corollary 3.2. $\square$

**Corollary 3.4.** If $\Diamond(S)$ holds for every stationary $S \subseteq \omega_1$, then every graph of size $\omega_1$ with uncountable coloring number has uncountable list-chromatic number.

Next we prove the list-chromatic version of (3) => (1) in Fact 2.3.

**Lemma 3.5.** Let $X$ be a graph and $\lambda$ an infinite cardinal. Suppose there is a filtration $\langle X_\alpha \mid \alpha < \delta \rangle$ of $X$ such that for every $\alpha < \delta$ and $x \in X \setminus X_\alpha$, we have $\text{List}(X_\alpha) \leq \lambda$ and $|E^\sharp \cap X_\alpha| < \lambda$. Then $\text{List}(X) \leq \lambda$.

**Proof.** Fix a $\lambda$-assignment $F$ of $X$. We construct a good coloring $f$ of $X$ with $f(x) \in F(x)$. We do this by induction on $\alpha < \delta$. Let $\alpha < \delta$ and suppose $f \upharpoonright X_\beta$ is defined to be a good coloring with $f(x) \in F(x)$ for every $\beta < \alpha$. If $\alpha$ is limit, then let $f \upharpoonright X_\alpha = \bigcup_{\beta < \alpha} f \upharpoonright X_\beta$. Suppose $\alpha = \gamma + 1$. Consider the induced subgraph $Y = X_\alpha \setminus X_\gamma$. Note that $\text{List}(Y) \leq \text{List}(X_\alpha) \leq \lambda$. By our assumption, for every $x \in Y$, we have that $E^\sharp \cap X_\gamma$ has cardinality $< \lambda$. Hence $F'(x) = F(x) \setminus (f^\sharp(E^\sharp \cap X_\gamma))$ has cardinality $\lambda$, and $F'$ is a $\lambda$-assignment of $Y$. Since $\text{List}(Y) \leq \lambda$, there is a good coloring $f'$ of $Y$ with $f'(x) \in F'(x)$. Now let $f \upharpoonright X_\alpha = (f \upharpoonright X_\gamma) \cup f'$. Finally, $f = \bigcup_{\alpha < \delta} f \upharpoonright X_\alpha$ is a good coloring of $X$ with $f(x) \in F(x)$. $\square$

As stated before, the singular compactness for the list-chromatic number does not hold in general. On the other hand, the singular compactness can hold under a certain cardinal arithmetic assumption. The following proposition is Theorem 1.3.

**Proposition 3.6.** Let $\kappa$ be a strong limit singular cardinal such that there is a club $C$ in $\kappa$ such that $2^\mu = \mu^+$ for every $\mu \in C$. Let $\lambda$ be an infinite cardinal with $\lambda < \kappa$. For every graph $X$ of size $\kappa$, if $\text{List}(Y) \leq \lambda$ for every subgraph $Y$ of size $< \kappa$, then $\text{List}(X) \leq \lambda$. 

10
Proof. Fix an increasing continuous sequence \( \langle \kappa_i \mid i < \operatorname{cf}(\kappa) \rangle \) with limit \( \kappa \) such that \( \operatorname{cf}(\kappa) + \lambda < \kappa_0 \) and \( 2^{\kappa_i} = \kappa_i^+ \) for \( i < \operatorname{cf}(\kappa) \).

First we claim:

**Claim 3.7.** For every \( i < \operatorname{cf}(\kappa) \) and subset \( Y \) of \( X \) with size \( \kappa_i \), the set \( \{ x \in X \mid |\mathcal{E}^x \cap Y| \geq \lambda \} \) has cardinality at most \( \kappa_i \).

Proof. Suppose to the contrary that the set \( \{ x \in X \mid |\mathcal{E}^x \cap Y| \geq \lambda \} \) has cardinality \( \geq \kappa_i \). There is \( Z \in [X]^{|\kappa_i|^+} \) such that \(|\mathcal{E}^x \cap Y| \geq \lambda \) for every \( x \in Z \). However then the induced subgraph \( Y \cup Z \) has cardinality \( < \kappa \) but \( \operatorname{List}(Y \cup Z) \geq \lambda \) by Fact 2.4 and the assumption that \( 2^{\kappa_i} = \kappa_i^+ \), this is a contradiction. \( \square \)

Fix a sufficiently large regular cardinal \( \theta \). We can take a sequence \( \langle M_i^\alpha \mid i < \operatorname{cf}(\kappa), \alpha < \lambda^+ \rangle \) such that:

1. \( M_i^\alpha \subseteq \mathcal{H}_\theta, |M_i^\alpha| = \kappa_i \subseteq M_i^0 \), and \( M_i^\alpha \) contains all relevant objects.
2. For every \( \alpha < \lambda^+ \), \( \langle M_i^\alpha \mid i < \operatorname{cf}(\kappa) \rangle \) is \( \subseteq \)-increasing and continuous.
3. For every \( \alpha < \lambda^+ \) and \( i < \operatorname{cf}(\kappa) \), if \( \alpha \) is limit then we have \( M_i^\alpha = \bigcup_{\beta < \alpha} M_i^\beta \).
4. For every \( \alpha < \lambda^+ \), \( \langle M_i^\alpha \mid i < \operatorname{cf}(\kappa) \rangle \in M_0^{\alpha+1} \).

Note that for every \( \alpha < \beta < \lambda^+ \) and \( i, j < \operatorname{cf}(\kappa) \), we have that \( M_i^\alpha \in M_j^\beta \).

For \( i < \operatorname{cf}(\kappa) \), let \( M_i = \bigcup_{\alpha < \lambda^+} M_i^\alpha \). By the choice of the \( M_i^\alpha \)'s, we know that \( \langle M_i \mid i < \operatorname{cf}(\kappa) \rangle \) is \( \subseteq \)-increasing, continuous, \( |M_i| = \kappa_i \subseteq M_i \), and \( X \subseteq \bigcup_{i < \operatorname{cf}(\kappa)} M_i \). We also know that \( M_i^\alpha \in M_i \) for every \( \alpha < \lambda^+ \) and \( i < \operatorname{cf}(\kappa) \). Let \( X_i = X \cap M_i \). The sequence \( \langle X_i \mid i < \operatorname{cf}(\kappa) \rangle \) is a filtration of \( X \). Now we show that \(|\mathcal{E}^x \cap X_i| < \lambda \) for every \( i < \operatorname{cf}(\kappa) \) and \( x \in X \setminus X_i \), and then the assertion follows from Lemma 3.5. Suppose to the contrary that \(|\mathcal{E}^x \cap X_i| \geq \lambda \) for some \( i < \operatorname{cf}(\kappa) \) and \( x \in X \setminus X_i \). Since \( X_i = \bigcup_{\alpha < \lambda^+} X \cap M_i^\alpha \), there is some \( \alpha < \lambda^+ \) with \(|\mathcal{E}^x \cap M_i^\alpha| \geq \lambda \). We know \( M_i^\alpha \in M_i \), so \( \{ y \in X \mid |\mathcal{E}^y \cap M_i^\alpha| \geq \lambda \} \in M_i \). By the above claim, the set \( \{ y \in X \mid |\mathcal{E}^y \cap M_i^\alpha| \geq \lambda \} \) has cardinality at most \( \kappa_i \). Since \( \kappa_i \subseteq M_i \), we have \( \{ y \in X \mid |\mathcal{E}^y \cap M_i^\alpha| \geq \lambda \} \subseteq M_i \). Now \( x \in \{ y \in X \mid |\mathcal{E}^y \cap M_i^\alpha| \geq \lambda \} \), so \( x \in M_i \cap X = X_i \), this is a contradiction. \( \square \)

**Question 3.8.** Can we weaken the assumption in Proposition 3.6? For instance, does the conclusion of Proposition 3.6 hold if \( \kappa \) is singular and \( \mu^\lambda < \kappa \) for every \( \mu < \kappa \), or \( \kappa \) is singular and \( 2^\lambda < \kappa \)?

The following is a partial answer to this question.

**Proposition 3.9.** Let \( \kappa \) be a singular cardinal with countable cofinality, and \( \lambda \) an infinite cardinal with \( \lambda < \kappa \). Suppose \( \mu^\lambda < \kappa \) for every \( \mu < \kappa \). For every graph \( X \) of size \( \kappa \), if \( \operatorname{List}(Y) \leq \lambda \) for every subgraph \( Y \) of size \( < \kappa \), then \( \operatorname{List}(X) \leq \lambda \).
Proof. Fix an increasing sequence \( \langle \kappa_n \mid n < \omega \rangle \) with limit \( \kappa \) such that \( \kappa^\lambda = \kappa_n \). This is possible by our assumption. Fix a sufficiently large regular cardinal \( \theta \). Take a \( \subseteq \)-increasing sequence of elementary submodels \( \langle M_n \mid n < \omega \rangle \) such that \( X \subseteq M_n \prec H_\theta \), \( |M_n| = \kappa_n \subseteq M_n \), and \( [M_n]^{\lambda \leq} \subseteq M_n \). Let \( X_n = M_n \cap X \). We know that \( \langle X_n \mid n < \omega \rangle \) is a filtration of \( X \). It is sufficient to show that for every \( n < \omega \) and \( x \in X \setminus X_n \), we have \( |E_x \cap X_n| < \lambda \). Suppose not. Take a set \( a \subseteq E_x \cap X_n \) with size \( \lambda \). By the choice of \( M_n \), we have \( a \in M_n \). The set \( A = \{ y \in X \mid a \subseteq E^y \} \) is in \( M_n \), and \( x \in A \). If the set \( A \) has cardinality \( \geq 2^\lambda \), by Fact 2.4 \( X \) has a subgraph with size \( < \kappa \) but list-chromatic number \( > \lambda \), this is impossible. Hence \( |A| < 2^\lambda \leq \kappa^\lambda = \kappa_n \), and \( A \subseteq M_n \) since \( \kappa_n \subseteq M \). Then \( x \in A \subseteq M_n \), this is a contradiction. \( \square \)

Erdős and Hajnal [3] showed that if \( \text{Col}(X) \) is uncountable, then for every \( n < \omega \), \( X \) contains a copy of \( K_{\omega_1 \omega_1} \) as a subgraph. We prove a variant of their result under some additional assumptions.

Definition 3.10 (Shelah). Let \( \lambda \) be a cardinal. \( \text{AP}_\lambda \) is the principle which asserts that there is a sequence \( \langle c_\xi \mid \xi < \lambda^+ \rangle \) such that:

1. \( c_\xi \subseteq \xi \).
2. There is a club \( C \) in \( \lambda^+ \) such that for every \( \alpha \in C \),
   a. \( c_\alpha \) is unbounded in \( \alpha \) and \( \text{ot}(c_\alpha) = \text{cf}(\alpha) \).
   b. \( \{ c_\alpha \cap \xi \mid \xi < \alpha \} \subseteq \{ c_\xi \mid \xi < \alpha \} \).

See Eisworth [2] about \( \text{AP}_\lambda \). We point out that \( \text{AP}_\lambda \) follows from the weak square principle at \( \lambda \).

Proposition 3.11. Let \( X \) be a graph with \( \text{Col}(X) > \omega_1 \). Suppose \( X^\omega = \lambda \) for every regular uncountable \( \lambda < |X| \) (in particular \( 2^\omega = \omega_1 \)), and \( \text{AP}_\lambda \) holds for every singular cardinal \( \lambda < |X| \) of countable cofinality. Then \( X \) contains a subgraph which is isomorphic to \( K_{\omega_1 \omega_1} \), in particular \( X \) contains a subgraph of size \( \omega_1 \) which has uncountable list-chromatic number.

Proof. Choose a subgraph \( Y \) of \( X \) such that \( |Y| \) is regular uncountable, \( \text{Col}(Y) > \omega_1 \), and \( \text{Col}(Z) \leq \omega_1 \) for every subgraph \( Z \) of \( Y \) with size \( < |Y| \). Let \( \kappa = |Y| \). \( \kappa \) is strictly greater than \( \omega_1 \). We may assume \( Y = \langle \kappa, E \rangle \). Since \( \text{Col}(Y) > \omega_1 \), the set \( S = \{ \alpha < \kappa \mid \exists \beta \geq \alpha (|E^\beta \cap \alpha| > \omega) \} \) is stationary in \( \kappa \) by Fact 2.7. Fix a sufficiently large regular cardinal \( \theta \).

Case 1: \( \kappa \) is not the successor of a singular cardinal of countable cofinality. Note that \( \gamma^\omega < \kappa \) for every \( \gamma < \kappa \) in this case.

Take \( M \prec H_\theta \) which contains all relevant objects and \( M \cap \kappa \subseteq S \). Fix \( \beta_0 \geq M \cap \kappa \) such that \( E^{\beta_0} \cap (M \cap \kappa) \) is uncountable. Then there is \( \gamma < M \cap \kappa \) such that \( E^{\beta_0} \cap \gamma \) is infinite. We know \( \gamma^\omega < \kappa \), hence \( \gamma^\omega < M \cap \kappa \) and \( [\gamma]^\omega \subseteq M \). Take \( Y_0 \in [\gamma]^\omega \) such that \( Y_0 \subseteq E^{\beta_0} \). By the elementarity of \( M \), the set \( \{ \beta < \kappa \mid Y_0 \subseteq E^\beta \} \) is unbounded in \( \kappa \).
Hence we can find \( Z \in [\kappa]^{\omega_1} \) such that \( Y_0 \cap Z = \emptyset \) and \( Y_0 \subseteq E^\beta \) for every \( \beta \in Z \). Then the induced subgraph \( Y_0 \cup Z \) contains a copy of \( K_{\omega_0,\omega_1} \).

Case 2: \( \kappa \) is the successor of a singular cardinal of countable cofinality, say \( \kappa = \lambda^+ \) with \( \text{cf}(\lambda) = \omega \). We have that \( \text{AP}_\lambda \) holds and \( \gamma^\omega < \lambda \) for every \( \gamma < \lambda \).

Take \( M \prec H_\theta \) which contains all relevant objects and \( M \cap \kappa \in S \). Take a sequence \( \langle \xi \mid \xi < \kappa \rangle \in M \) witnessing \( \text{AP}_\lambda \). Let \( \alpha = M \cap \kappa \). Then \( \sup(c_\alpha) = \alpha \) and \( \text{ot}(c_\alpha) < \lambda \). We also know \( c_\alpha \cap \gamma \in M \) for every \( \gamma < \alpha \), because \( \{ c_\alpha \cap \gamma \mid \gamma < \alpha \} \subseteq \{ c_\gamma \mid \gamma < \alpha \} \subseteq M \). Take a sequence \( \langle \pi_\xi \mid \xi < \kappa \rangle \in M \) such that each \( \pi_\xi \) is a surjection from \( \lambda \) to \( \xi \). Take also an increasing sequence \( \langle \lambda_n \mid n < \omega \rangle \in M \) with limit \( \lambda \). For \( n < \omega \), let \( A_n = \bigcup \{ \pi_\xi^\alpha \lambda_n \mid \xi \in c_\alpha \cap \gamma \} \). We have that \( |A_n| < \lambda \) and \( \bigcup_{n<\omega} A_n = \alpha \).

Fix \( \beta_0 \geq \alpha \) with \( |E^{\beta_0} \cap \alpha| > \omega \). Then there is \( n_0 < \omega \) such that \( A_{n_0} \cap E^{\beta_0} \) is uncountable. For \( \gamma < \alpha \), let \( B_\gamma = \bigcup \{ \pi_\xi^\alpha \lambda_{n_0} \mid \xi \in c_\alpha \cap \gamma \} \). The sequence \( \langle B_\gamma \mid \gamma < \alpha \rangle \) is \( \subseteq \)-increasing and \( \bigcup_{\gamma<\alpha} B_\gamma = A_{n_0} \). Thus there is some \( \delta < \alpha \) such that \( E^{\beta_0} \cap B_\delta \) is infinite. Since \( c_\alpha \cap \delta \in M \), we have that \( B_\delta \in M \). \( |B_\delta| < \lambda \), so we have that \( [B_\delta]^\omega \subseteq M \), and there is \( Y_1 \subseteq [B_\delta]^\omega \) such that \( Y_1 \in M \) and \( Y_1 \subseteq E^{\beta_0} \). The rest is the same as Case 1.

**Remark 3.12.** Under the assumption of Proposition 3.11, a graph \( X \) with \( \text{Col}(X) > \omega_1 \) actually contains a copy of \( K_{\omega_0,\omega_1} \). However we do not use this result in this paper.

In the proof of the previous proposition, we used the cardinal arithmetic assumption and the principle \( \text{AP}_\lambda \).

**Question 3.13.** Are the assumptions in Proposition 3.11 necessary?

4. Reflections for the List-Chromatic Number and the Coloring Number

In this section, we consider the reflection principles \( \text{RP}(\text{List}) \) and \( \text{RP}(\text{Col}) \), and, we prove some results about reflections. First we prove that \( \text{RP}(\text{List}) \) implies the Continuum Hypothesis.

**Lemma 4.1.** \( \text{RP}(\text{List}, 2^\omega) \) implies that \( 2^\omega = \omega_1 \).

**Proof.** The complete bipartite graph \( K_{\omega,2^\omega} \) has uncountable list-chromatic number by Fact 2.5. However, if \( 2^\omega > \omega_1 \), then every subgraph of \( K_{\omega,2^\omega} \) of size \( \omega_1 \) has countable list-chromatic number by Fact 2.6. This contradicts the principle \( \text{RP}(\text{List}, 2^\omega) \).

\( \text{FRP} \) and \( \text{RP}(\text{Col}) \) follow from Martin’s Maximum ([6]). However this lemma shows that \( \text{RP}(\text{List}) \) does not follow from Martin’s Maximum and other forcing axioms which imply \( 2^\omega > \omega_1 \).

**Question 4.2.** Does \( \text{RP}(\text{List}, \omega_2) \) imply \( 2^\omega = \omega_1 \)?
Now we have the following consistency result, which is (2) of Theorem 1.7.

**Corollary 4.3.** It is consistent that $RP(\text{Col})$ holds but $RP(\text{List}, \omega_2)$ fails.

**Proof.** Let $\kappa$ be a supercompact cardinal, and take a $(V, \text{Coll}(\omega_1, < \kappa))$-generic $G$. In $V[G]$, FRP holds by Fact 2.12. Now add $\omega_2$ many Cohen (or any other) reals by c.c.c. forcing. FRP is preserved by c.c.c. forcing, so FRP still holds in the extension, and we have $RP(\text{Col})$. On the other hand, since $2^\omega = \omega_2$ in the extension, we have that $RP(\text{List}, \omega_2)$ fails. □

On the other hand, by using Corollary 3.4, we have the following implication between $RP(\text{Col})$ and $RP(\text{List})$.

**Corollary 4.4.** Suppose that $\Diamond(S)$ holds for every stationary $S \subseteq \omega_1$. If $RP(\text{Col})$ holds, then $RP(\text{List})$ holds as well.

**Proof.** Let $X$ be a graph with $\text{List}(X) > \omega$. Then $\text{Col}(X) \geq \text{List}(X) > \omega$, hence $X$ has a subgraph $Y$ of size $\omega_1$ with uncountable coloring number. Now we have $\text{List}(Y) > \omega$ by Corollary 3.4. □

**Corollary 4.5 ([8]).** Suppose $\kappa$ is supercompact. Then $\text{Coll}(\omega_1, < \kappa)$ forces $RP(\text{List})$.

**Proof.** In $V^{\text{Coll}(\omega_1, < \kappa)}$, FRP holds and it is known that $\Diamond(S)$ holds for all stationary subsets $S$ in $\omega_1$. Then $RP(\text{List})$ holds as well by the previous corollary. □

Next we turn to the consistency strength of $RP(\text{List})$.

**Proposition 4.6.** Suppose $RP(\text{List})$. Then for every cardinal $\lambda \geq \omega_1$ of uncountable cofinality, either:

1. $2^\lambda > \lambda^+$,
2. Every stationary subset of $\lambda^+ \cap \text{Cof}(\omega)$ is reflecting.

**Proof.** Let $\lambda \geq \omega_1$ be a cardinal of uncountable cofinality. If $2^\lambda = \lambda^+$, then $\Diamond(S)$ holds for every stationary $S \subseteq \lambda^+ \cap \text{Cof}(\omega)$ by Fact 2.1. If $\lambda^+ \cap \text{Cof}(\omega)$ has a non-reflecting stationary subset, by Fact 2.13 there is a graph $X$ of size $\lambda^+$ such that $\text{Col}(X) > \omega$ but $\text{Col}(Y) \leq \omega$ for every subgraph $Y$ of size $< \lambda^+$. Then $\text{List}(X) > \omega$ but $\text{List}(Y) \leq \omega$ for every $Y \in [X]^{< \lambda^+}$ by Proposition 3.1, this is a contradiction. □

This proposition means that the global reflection $RP(\text{List})$ has a large cardinal strength: The singular cardinal hypothesis fails, or $\Box_\lambda$ fails at every singular cardinal $\lambda$ of uncountable cofinality.

We will use the following proposition later.

**Proposition 4.7.** Suppose $\Diamond(S)$ holds for every stationary $S \subseteq \omega_1$. Let $\kappa \geq \omega_2$ be regular and suppose $\text{FRP}(\kappa)$ holds. Then for every graph
$X$ of size $\kappa$, if $\text{List}(Y) \leq \omega$ for every subgraph $Y$ of size $< \kappa$, then $\text{List}(X) \leq \omega$.

Proof. We may assume that the graph $X$ is of the form $(\kappa, \mathcal{E})$. Let $S = \{ \alpha \in \kappa \cap \text{Cof}(\omega) \mid \exists \beta \geq \alpha (\mathcal{E}^{\beta(\alpha)} \cap \alpha$ is infinite)$. If $S$ is non-stationary, then we can deduce $\text{List}(X) \leq \text{Col}(X) \leq \omega$ by Fact 2.7.

Now we show that $S$ is non-stationary in $\kappa$. Suppose to the contrary that $S$ is stationary. For each $\alpha \in S$, fix $\beta(\alpha) \geq \alpha$ such that $\mathcal{E}^{\beta(\alpha)} \cap \alpha$ is infinite. Let $C = \{ \alpha < \kappa \mid \beta(\alpha') < \alpha \text{ for every } \alpha' \in S \cap \alpha \}$. $C$ is a club.

Let $S^* = S \cap C$. $S^*$ is also stationary. Note that $\beta(\alpha) \neq \beta(\alpha')$ for every distinct $\alpha, \alpha' \in S^*$. Take $g : S^* \rightarrow [\kappa]^\omega$ such that $g(\alpha) \in [\mathcal{E}^{\beta(\alpha)} \cap \alpha]^\omega$.

By FRP$(\kappa)$, there is $I \in [\kappa]^{|\omega|}$ such that $\text{sup}(I) \notin I$, $\text{cf}(\text{sup}(I)) = \omega_1$, and the set $A = \{ x \in [I]^\omega \mid \text{sup}(x) \in S^*, g(\text{sup}(x)) \subseteq x \}$ is stationary in $[I]^\omega$. Let $Y$ be the induced subgraph $I \cup \{ \beta(\alpha) \mid \alpha \in I \cap S^* \}$. $|Y| = \omega_1$, hence $\text{List}(Y) \leq \omega$ by our assumption, and $\text{Col}(Y) \leq \omega$ by Corollary 3.4. By Corollary 2.3, we can find $f : Y \rightarrow [Y]^{<\omega}$ such that for every distinct $\alpha, \alpha' \in Y$, if $\alpha \in \mathcal{E} \alpha'$ then either $\alpha \in f(\alpha')$ or $\alpha' \in f(\alpha)$. For each $\alpha \in Y$, $g(\alpha)$ is infinite but $f(\beta(\alpha))$ is finite. Thus we can take a function $h$ on $A$ so that $h(x) \in g(\text{sup}(x)) \setminus f(\beta(\text{sup}(x)))$. Then, we can find $\gamma \in \bigcup A$ such that $A' = \{ x \in A \mid h(x) = \gamma \}$ is stationary in $[I]^\omega$.

For $x \in A'$, since $\gamma = h(x) \in g(\text{sup}(x)) \subseteq \mathcal{E}^{\beta(\text{sup}(x))} \cap \text{sup}(x)$ but $\gamma \notin f(\beta(\text{sup}(x)))$, we have $\beta(\text{sup}(x)) \in f(\gamma)$. However this is impossible since $\{ \beta(\text{sup}(x)) \mid x \in A' \}$ is infinite but $f(\gamma)$ is finite. □

The proof of Proposition 4.7 yields the following:

Corollary 4.8 ([9]). If FRP holds, then RP(Col) holds as well.

Proof. By induction on size of graphs. Let $X$ be a graph of size $\geq \omega_2$, and suppose every subgraph of size $\omega_1$ has countable coloring number. By the induction hypothesis, every subgraph of size $< |X|$ has countable coloring number. If $|X|$ is regular, argue as in the proof of Proposition 4.7. If $|X|$ is singular, we can apply Fact 2.8. □

As mentioned before, in fact FRP is equivalent to RP(Col) ([9]).

5. Forcing notion adding a good coloring

In this section we define a forcing notion which adds a good coloring of a given graph. We will use this forcing notion for the proofs of Theorems 1.5 and 1.7 (1).

First we recall some basic definitions. Let $\mathbb{P}$ be a poset. Every set $x$ has the canonical name $\check{x}$ defined by $\check{x} = \{ \langle \check{y}, 1 \rangle \mid y \in x \}$, where 1 is the maximum element of the poset. We frequently omit the check of $\check{x}$, and simply write $x$.

Definition 5.1. Let $\mathbb{P}$ be a poset and $\theta$ a sufficiently large regular cardinal. Let $M \prec \mathcal{H}_\theta$ be a countable model with $\mathbb{P} \in M$. 

15
(1) A condition \( p \in \mathbb{P} \) is an \((M, \mathbb{P})\)-generic condition if for every dense open set \( D \in M \in \mathbb{P} \) and \( q \leq p \), there is \( r \in D \cap M \) which is compatible with \( q \).

(2) A condition \( p \in \mathbb{P} \) is a strong \((M, \mathbb{P})\)-generic condition if for every dense open set \( D \in M \in \mathbb{P} \), there is some \( q \in D \cap M \) with \( p \leq q \).

(3) A descending sequence \( \langle p_n \mid n < \omega \rangle \) in \( \mathbb{P} \) is an \((M, \mathbb{P})\)-generic sequence if \( p_n \in M \) for \( n < \omega \), and for every dense open set \( D \in M \in \mathbb{P} \), there is \( n < \omega \) with \( p_n \in D \cap M \).

Every strong \((M, \mathbb{P})\)-generic condition is an \((M, \mathbb{P})\)-generic condition. If an \((M, \mathbb{P})\)-generic sequence \( \langle p_n \mid n < \omega \rangle \) has a lower bound \( p \), then \( p \) is a strong \((M, \mathbb{P})\)-generic condition.

Let \( M \prec \mathcal{H}_\theta \) be a model with \( \mathbb{P} \in M \), and \( G \) be \((V, \mathbb{P})\)-generic. Let \( M[G] = \{ \dot{x}_G \mid \dot{x} \in M \} \), where \( \dot{x}_G \) is the interpretation of \( \dot{x} \) by \( G \). The following are known:

1. \( M[G] \prec \mathcal{H}_\theta^V[G] \).
2. If \( G \) contains an \((M, \mathbb{P})\)-generic condition, then \( M \cap \text{ON} = M[G] \cap \text{ON} \).

**Definition 5.2.** Let \( \mathbb{P}, \mathbb{Q} \) be posets, and suppose \( \mathbb{P} \) is a suborder of \( \mathbb{Q} \), that is, \( \mathbb{P} \subseteq \mathbb{Q} \) and for \( p_0, p_1 \in \mathbb{P} \), \( p_0 \leq p_1 \in \mathbb{P} \) if and only if \( p_0 \leq p_1 \) in \( \mathbb{Q} \).

1. For \( q \in \mathbb{Q} \), a condition \( p \in \mathbb{P} \) is a reduction of \( q \) if for every \( r \leq p \) in \( \mathbb{P} \), \( r \) is compatible with \( q \) in \( \mathbb{Q} \).
2. \( \mathbb{P} \) is a complete suborder of \( \mathbb{Q} \) if (i) \( p \perp q \) in \( \mathbb{P} \) then so does in \( \mathbb{Q} \), and (ii) every \( q \in \mathbb{Q} \) has a reduction \( p \in \mathbb{P} \). (ii) is equivalent to the property that every maximal antichain in \( \mathbb{P} \) is maximal in \( \mathbb{Q} \).
3. For a \((V, \mathbb{P})\)-generic \( G \), the quotient poset \( \mathbb{Q}/G \) is the suborder \( \{ q \in \mathbb{Q} \mid q \text{ is compatible with any } p \in G \} \). When \( G \) is clear from the context, \( \mathbb{Q}/G \) is denoted by \( \mathbb{Q}/\mathbb{P} \).

**Fact 5.3.** Let \( \mathbb{Q} \) be a poset and \( \mathbb{P} \) a complete suborder of \( \mathbb{Q} \).

1. If \( G \) is \((V, \mathbb{P})\)-generic and \( H \) is \((V[G], \mathbb{Q}/G)\)-generic, then \( H \) is \((V, \mathbb{Q})\)-generic and \( V[G][H] = V[H] \).
2. If \( H \) is \((V, \mathbb{Q})\)-generic, then \( G = H \cap \mathbb{P} \) is \((V, \mathbb{P})\)-generic, \( H \) is \((V[G], \mathbb{Q}/G)\)-generic, and \( V[H] = V[G][H] \).
3. Suppose \( q \in \mathbb{Q} \) has the greatest reduction \( p \in \mathbb{P} \). Then for every \((V, \mathbb{P})\)-generic \( G \), \( q \in \mathbb{Q}/G \) if and only if \( p \in G \).

In order to define our forcing notion, we need more definitions and lemmas.

**Definition 5.4.** Let \( \kappa \geq \omega_2 \) be a cardinal. We say that a graph \( X \) is \( \kappa \)-nice if \( X \) satisfies the following conditions:

1. \( X \) is of the form \( \langle \kappa, E \rangle \).
(2) Col(X) ≤ \omega_1.
(3) Col(Y) ≤ \omega for every subgraph Y of size \omega_1.
(4) For every \alpha < \kappa, |\mathcal{E}^\alpha \cap \alpha| \leq \omega.

Notice that for a given graph X of size \kappa, if Col(X) ≤ \omega_1, then there is an enumeration \langle x_i \mid i < \kappa \rangle of X with \mathcal{E}^{x_i} \cap \{x_j \mid j < i\} countable for every i < \kappa by Fact 2.3. Hence for every graph X of size \kappa, if X satisfies the conditions (2) and (3) then there is a \kappa-nice graph which is isomorphic to X.

Fix a \kappa-nice graph X and an \omega-assignment \( F : \kappa \rightarrow [\kappa]^\omega \). Under CH, we shall define a forcing notion \( \mathbb{P} \) such that \( \mathbb{P} \) satisfies the \omega_2-c.c., \sigma-Baire, and adds a good coloring \( f \) of X with \( f(\alpha) \in F(\alpha) \). Throughout this section, we assume CH. Let \( \theta \) be a sufficiently large regular cardinal.

**Lemma 5.5.** For every \( x \in [\kappa]^\omega \), the set \( \{ \beta < \kappa \mid \mathcal{E}^\beta \cap x \text{ is infinite} \} \) is at most countable.

**Proof.** Otherwise, we can find \( Z \in [\kappa]^\omega \) such that \( |x \cap \mathcal{E}^\beta| \geq \omega \) for every \( \beta \in Z \). By CH and Fact 2.4, we have \( \omega < \text{List}(x \cup Z) \leq \text{Col}(x \cup Z) \), this is a contradiction. \( \square \)

**Definition 5.6.** A set \( x \subseteq \kappa \) is said to be \( \langle X, F \rangle \)-complete (or simply complete) if the following hold:

1. \( \mathcal{E}^\alpha \cap \alpha \subseteq x \) and \( F(\alpha) \subseteq x \) for every \( \alpha \in x \).
2. For every \( \beta < \kappa \), if \( \mathcal{E}^\beta \cap x \) is infinite then \( \beta \in x \).

Note that if \( x \) and \( y \) are complete, then both \( x \cap y \) and \( x \cup y \) are complete as well.

We say that a set \( A \) is \( \sigma \)-closed if \( [A]^\omega \subseteq A \). By CH, for each \( x \in \mathcal{H}_\theta \) there is a \( \sigma \)-closed \( \mathcal{N} \in \mathcal{H}_\theta \) of size \( \omega_1 \) containing \( x \).

**Lemma 5.7.** Let \( \mathcal{N} \in \mathcal{H}_\theta \) be \( \sigma \)-closed with \( X, F \in \mathcal{N} \) and \( |\mathcal{N}| = \omega_1 \). Then \( \mathcal{N} \cap \kappa \) is complete.

**Proof.** It is clear that \( \mathcal{E}^\alpha \cap \alpha, F(\alpha) \subseteq \mathcal{N} \cap \kappa \) for every \( \alpha \in \mathcal{N} \cap \kappa \). Take \( \beta < \kappa \), and suppose \( \mathcal{E}^\beta \cap (\mathcal{N} \cap \kappa) \) is infinite. Take a countable subset \( a \subseteq \mathcal{E}^\beta \cap \mathcal{N} \cap \kappa \). By the \( \sigma \)-closure of \( \mathcal{N} \), we have that \( a \in \mathcal{N} \).

The set \( \{ \gamma < \kappa \mid \mathcal{E}^\gamma \cap a \text{ is infinite} \} \) is in \( \mathcal{N} \) and at most countable by Lemma 5.5. Hence \( \{ \gamma < \kappa \mid \mathcal{E}^\gamma \cap a \text{ is infinite} \} \subseteq \mathcal{N} \cap \kappa \), and we have \( \beta \in \{ \gamma < \kappa \mid \mathcal{E}^\gamma \cap a \text{ is infinite} \} \subseteq \mathcal{N} \cap \kappa \). \( \square \)

**Lemma 5.8.** Let \( Y \subseteq \kappa \) be a complete set of size \( \omega_1 \), and \( \mathcal{M} \in \mathcal{H}_\theta \) a countable model with \( X, F, Y \in \mathcal{M} \). Then \( \mathcal{M} \cap Y \) is complete. In particular, the set of all countable complete subsets of \( \kappa \) is stationary in \( [\kappa]^\omega \).

**Proof.** By the assumption, we have Col(Y) ≤ \omega. Hence we can find a function \( f : Y \rightarrow [Y]^\omega \) in \( \mathcal{M} \) such that for every \( \alpha, \beta \in Y \), if \( \alpha \in \mathcal{E}^\beta \), the \( \alpha \in f(\beta) \) or \( \beta \in f(\alpha) \) by Fact 2.3.
To show that $M \cap Y$ is complete, we only show that for every $\beta < \kappa$, if $M \cap Y \cap E^\beta$ is infinite then $\beta \in M \cap Y$. Because $Y$ is complete, we have $\beta \in Y$. On the other hand, since $M \cap Y \cap E^\beta$ is infinite but $f(\beta)$ is finite, we can take $\alpha \in (M \cap Y \cap E^\beta) \setminus f(\beta)$. Then $\beta \in f(\alpha) \subseteq M$, so $\beta \in M \cap Y$ as required. \hfill \Box

Now we are ready to define our forcing notion.

**Definition 5.9.** $\mathbb{P}(X,F)$ is the poset which consists of all countable functions $p$ such that:

1. $p$ is a good coloring of the induced subgraph $\text{dom}(p) \in [\kappa]^\omega$ with $p(\alpha) \in F(\alpha)$.
2. $\text{dom}(p)$ is complete.

Define $p \leq q$ if $p \supseteq q$.

For simplicity, we omit the parameters $X$ and $F$ in $\mathbb{P}(X,F)$ and just write $\mathbb{P}$.

**Lemma 5.10.**

1. For every $p \in \mathbb{P}$ and complete set $x \in [\kappa]^\omega$, if $x \supseteq \text{dom}(p)$ then there is $q \in \mathbb{P}$ such that $q \leq p$ and $\text{dom}(q) = x$.
2. For every $x \in [\kappa]^\omega$, the set $\{p \in \mathbb{P} \mid x \subseteq \text{dom}(p)\}$ is dense in $\mathbb{P}$.

**Proof.** (1) Take $p \in \mathbb{P}$. Let $\langle \alpha_n \mid n < \omega \rangle$ be an enumeration of $x \setminus \text{dom}(p)$. Note that $E^{\alpha_n} \cap \text{dom}(p)$ is finite for every $n < \omega$. Thus we can take a function $f$ on $\{\alpha_n \mid n < \omega\}$ such that $f(\alpha_n) \in F(\alpha_n) \setminus (p)(E^{\alpha_n} \cap \text{dom}(p)) \cup f(\alpha_n) \setminus \{\alpha_m \mid m < n\})$. Let $q = p \cup f$. It is easy to check that $q \in \mathbb{P}$ and $q \leq p$.

(2) follows from (1) and Lemma 5.8. \hfill \Box

**Lemma 5.11.**

1. For $p,q \in \mathbb{P}$, if $p \cup q$ is a function then $p \cup q$ is the greatest lower bound of $p$ and $q$.
2. $\mathbb{P}$ satisfies the $\omega_2$-c.c.

**Proof.** (1). Suppose $r = p \cup q$ is a function. The set $\text{dom}(r) = \text{dom}(p) \cup \text{dom}(q)$ is complete and $r(\alpha) \in F(\alpha)$ for every $\alpha \in \text{dom}(r)$ since $p$ and $q$ are conditions. Thus it is enough to check that $r = p \cup q$ is a good coloring of $\text{dom}(p) \cup \text{dom}(q)$. Take $\alpha, \beta \in \text{dom}(p) \cup \text{dom}(q)$ with $\alpha \in E^\beta$. We may assume $\alpha < \beta$. If $\beta \in \text{dom}(p)$, then $\alpha \in E^\beta \cap \beta \subseteq \text{dom}(p)$. Hence $r(\alpha) = p(\alpha) \neq p(\beta) = r(\beta)$. The case $\beta \in \text{dom}(q)$ follows from the same argument.

(2). For a given $\{p_i \mid i < \omega_2\} \subseteq \mathbb{P}$, by the $\Delta$-system lemma, there are $D \in [\omega_2]^\omega$ and $d$ such that $\text{dom}(p_i) \cap \text{dom}(p_j) = d$ for every distinct $i,j \in D$. For $\alpha \in d$ and $i \in D$, we have $p_i(\alpha) \in F(\alpha) \in [\kappa]^\omega$. Thus, by a standard pigeonhole argument, there is $D' \in [D]^\omega$ such that $p_i \upharpoonright d = p_j \upharpoonright d$ for every $i,j \in D'$. Then for every $i,j \in D'$, $p_i \cup p_j$ is a common extension of $p_i$ and $p_j$ by (1). \hfill \Box

The following lemmas are straightforward.
Lemma 5.12. For a descending sequence \( \langle p_n \mid n < \omega \rangle \) in \( P \), if the set \( \bigcup_{n<\omega} \text{dom}(p_n) \) is complete then \( \bigcup_{n<\omega} p_n \in P \). In particular, for every countable \( M < \mathcal{H}_\theta \) and every \((M, P)\)-generic sequence \( \langle p_n \mid n < \omega \rangle \), if \( M \cap \kappa \) is complete then the union \( \bigcup_{n<\omega} p_n \) is a strong \((M, P)\)-generic condition with domain \( M \cap \kappa \). Hence \( P \) is \( \sigma \)-Baire.

Lemma 5.13.  
1. \( P \) preserves all cofinalities.
2. Let \( G \) be a \((Y, P)\)-generic filter. Then \( f = \bigcup G \) is a good coloring of \( X \) with \( f(\alpha) \in F(\alpha) \) for every \( \alpha < \kappa \).

We do not know if the poset \( P \) is proper or even semiproper.

Question 5.14. Is \( P \) proper or semiproper?

Next let us consider complete suborders of \( P \).

Definition 5.15. For a subset \( Y \subseteq \kappa \), let \( P \upharpoonright Y = \{ p \in P \mid \text{dom}(p) \subseteq Y \} \). We identify \( P \upharpoonright Y \) as a suborder of \( P \).

Lemma 5.16. Let \( Y \subseteq \kappa \) be a complete set of size \( \omega_1 \).

1. The poset \( P \upharpoonright Y \) is a complete suborder of \( P \). Moreover, for each \( p \in P \), the function \( p \upharpoonright Y \) is in \( P \upharpoonright Y \) and is the greatest reduction of \( p \).
2. Let \( M < \mathcal{H}_\theta \) be a countable model with \( P, Y \in M \). For every \((M, P \upharpoonright Y)\)-generic sequence \( \langle p_n \mid n < \omega \rangle \), the union \( \bigcup_{n<\omega} p_n \) is a strong \((M, P \upharpoonright Y)\)-generic condition.

Proof. (1). For \( p, q \in P \upharpoonright Y \), if \( p \) is compatible with \( q \) in \( P \), then \( p \cup q \) is a common extension of \( p \) and \( q \), and \( \text{dom}(p \cup q) \subseteq Y \). Hence \( p \cup q \in P \upharpoonright Y \), and \( p \) is compatible with \( q \) in \( P \upharpoonright Y \).

Next take \( p \in P \). The sets \( \text{dom}(p) \) and \( Y \) are complete, hence \( \text{dom}(p) \cap Y \) is complete as well. Because \( \text{dom}(p \upharpoonright Y) = \text{dom}(p) \cap Y \), we have \( p \upharpoonright Y \in P \upharpoonright Y \). We show \( p \upharpoonright Y \) is a reduction of \( p \), that is, for every \( r \in P \upharpoonright Y \), if \( r \leq p \upharpoonright Y \) then \( r \) is compatible with \( p \) in \( P \), and this is immediate from Lemma 5.11, and it is straightforward to check that \( p \upharpoonright Y \) is the greatest reduction of \( p \).

(2). By Lemma 5.8, \( M \cap Y \) is complete. Since \( \text{dom}(\bigcup_{n<\omega} p_n) = M \cap Y \), the union \( \bigcup_{n<\omega} p_n \) is a condition in \( P \upharpoonright Y \). \( \square \)

Lemma 5.17. Let \( N < \mathcal{H}_\theta \) be a \( \sigma \)-closed model with \( |N| = \omega_1 \) and \( P \in N \). Then \( P \upharpoonright (N \cap \kappa) = P \cap N \) and \( P \cap N \) is a complete suborder of \( P \).

Proof. The inclusion \( P \cap N \subseteq P \upharpoonright (N \cap \kappa) \) is easy. For the converse, let \( p \in P \upharpoonright (N \cap \kappa) \). We know \( \text{dom}(p) \subseteq N \cap \kappa \). Because \( p \subseteq \text{dom}(p) \times \{ F(\alpha) \mid \alpha \in \text{dom}(p) \} \subseteq N \), we have \( p \in N \) by the \( \sigma \)-closure of \( N \).

The set \( N \cap \kappa \) is complete by Lemma 5.7, hence \( P \upharpoonright (N \cap \kappa) = P \cap N \) is a complete suborder of \( P \). \( \square \)
Let us say that a poset $Q$ is $\omega_1$-diamond preserving if for every stationary $S \subseteq \omega_1$ and $\diamond(S)$-sequence $\langle d_\alpha \mid \alpha \in S \rangle$, $Q$ forces “$\langle d_\alpha \mid \alpha \in S \rangle$ remains a $\diamond(S)$-sequence”.

**Lemma 5.18.**

1. $\mathbb{P}$ is $\omega_1$-stationary preserving and $\omega_1$-diamond preserving.

2. For every complete set $Y \subseteq \kappa$, the quotient $\mathbb{P}/(\mathbb{P} \upharpoonright Y)$ is $\omega_1$-stationary preserving.

**Proof.** First we note the following: Let $A \in \mathbb{V}^\mathbb{P}$ be a subset of $\omega_1$, and $\dot{A}$ be a name for $A$. Since $\mathbb{P}$ has the $\omega_2$-c.c., we can take a complete set $Y \subseteq \kappa$ with size $\omega_1$ such that $\dot{A}$ is a $\mathbb{P} \upharpoonright Y$-name. Thus, in order to show that $\mathbb{P}$ satisfies (1), it is enough to prove that for every complete set $Y \subseteq \kappa$ with size $\omega_1$, the complete suborder $\mathbb{P} \upharpoonright Y$ satisfies (1). We only show that $\mathbb{P} \upharpoonright Y$ is $\omega_1$-diamond preserving, the $\omega_1$-stationary preservingness follows from the same argument.

Fix a stationary $S \subseteq \omega_1$ and a $\diamond(S)$-sequence $\langle d_\alpha \mid \alpha \in S \rangle$. Take a $\mathbb{P} \upharpoonright Y$-name $\dot{A}$ for a subset of $\omega_1$. Then take an internally approachable sequence $\langle M_i \mid i < \omega_1 \rangle$ of countable elementary submodels of $\mathcal{H}_\theta$ containing all relevant objects, that is, $\langle M_i \mid i \leq j \rangle \in M_{j+1}$ for every $j < \omega_1$, and $M_j = \bigcup_{i < j} M_i$ if $j$ is limit. By Lemma 5.16, we can construct a descending sequence $\langle p_i \mid i < \omega_1 \rangle$ in $\mathbb{P}$ such that each $p_i$ is a strong $(M_i, \mathbb{P} \upharpoonright Y)$-generic condition, and $\langle p_i \mid i \leq j \rangle \in M_{j+1}$. Let $A = \{ \gamma < \omega_1 \mid \exists i < \omega_1 (p_i \forces_{\mathbb{P}^Y} \gamma \in \dot{A}) \}$. Since $\langle d_\alpha \mid \alpha \in S \rangle$ is a $\diamond(S)$-sequence, there is some $\alpha \in S$ such that $d_\alpha = A \cap \alpha$ and $M_\alpha \cap \omega_1 = \alpha$. We show that $p_\alpha \forces_{\mathbb{P}^Y} "d_\alpha = A \cap \alpha"$. Let $\gamma \in d_\alpha$. Since $A \cap \alpha = d_\alpha$, there is some $i < \omega_1$ such that $p_i \forces_{\mathbb{P}^Y} \gamma \in \dot{A}$. Pick $j < \alpha$ with $\gamma \in M_j$. Note that $p_\alpha \leq p_j$. Since $p_j$ is a strong $(M_j, \mathbb{P} \upharpoonright Y)$-generic condition, we know $p_j \forces_{\mathbb{P}^Y} \gamma \in \dot{A}$, or $p_j \forces_{\mathbb{P}^Y} \gamma \notin \dot{A}$. $p_j$ is compatible with $p_j$, so we have $p_j \forces_{\mathbb{P}^Y} \gamma \in \dot{A}$, and $p_\alpha \forces_{\mathbb{P}^Y} \gamma \in \dot{A}$. This means that $p_\alpha \forces_{\mathbb{P}^Y} "d_\alpha \subseteq A \cap \alpha"$. For the converse, let $\gamma < \alpha$ and suppose $\gamma \notin d_\alpha$. Pick $j < \alpha$ with $\gamma \in M_j$. Since $\gamma \notin \dot{d_\alpha} = A \cap \alpha$, $p_j$ does not force $\gamma \in \dot{A}$. Again, since $p_j$ is strong $(M_j, \mathbb{P} \upharpoonright Y)$-generic, we have that $p_j$ forces $\gamma \notin \dot{A}$, and $p_\alpha \forces_{\mathbb{P}^Y} "A \cap \alpha \subseteq d_\alpha"$.

For (2), take another complete set $Z \subseteq \kappa$ with size $\omega_1$ and $Y \subseteq Z$. $\mathbb{P} \upharpoonright Y$ is a complete suborder of $\mathbb{P} \upharpoonright Z$. By the same reason as before, it is enough to show that $(\mathbb{P} \upharpoonright Z)/\mathbb{P} \upharpoonright Y$ is $\omega_1$-stationary preserving.

Take a $(V, \mathbb{P} \upharpoonright Y)$-generic $G$, and we work in $V[G]$. Fix a stationary set $S \subseteq \omega_1$. Take a countable elementary submodel $M' < \mathcal{H}_\theta^{V[G]}$ such that $M' \cap \omega_1 \in S$, and an $(M', (\mathbb{P} \upharpoonright Z)/G)$-generic sequence $\langle p_n \mid n < \omega \rangle$. It is enough to prove that this sequence has a lower bound in $(\mathbb{P} \upharpoonright Z)/G$.

Here note that $p_n \forces Y \in G$ for every $n < \omega$. Let $M = M' \cap \mathcal{H}_\theta^V$, which is an elementary submodel of $\mathcal{H}_\theta^V$ in $V$, and $\langle p_n \mid n < \omega \rangle$ is an $(M, \mathbb{P} \upharpoonright Z)$-generic sequence belongs to $V$. By Lemma 5.16 again, the
union \( p = \bigcup_n p_n \) is in \( \mathbb{P} \upharpoonright Z \). Moreover \( p \upharpoonright Y = \bigcup_n (p_n \upharpoonright Y) \), which belongs to \( G \). Hence \( p \in (\mathbb{P} \upharpoonright Z)/G \), as required. \( \square \)

If \( X \) is a trivial graph on \( \kappa \) (that is, \( X = (\kappa, \emptyset) \) ) and \( F(\alpha) = \omega \), then it is clear that \( \mathbb{P}(X,F) \) is isomorphic to \( \text{Fn}(\kappa, \omega, < \omega) \). It is known that for every stationary \( S \subseteq \omega_1 \), \( \text{Fn}(\kappa, \omega, < \omega_1) \) forces \( \diamond(S) \). Thus we have:

**Lemma 5.19.** If \( X \) is a trivial graph and \( F(\alpha) = \omega \) for \( \alpha < \kappa \), then for every stationary \( S \subseteq \omega_1 \), \( \mathbb{P}(X,F) \) forces \( \diamond(S) \).

Now we consider an iteration, this is the hard part of this section. Let \( l \) be an ordinal and \( \langle \mathbb{P}_\xi, \overset{\frown}{\mathcal{Q}}_\eta \mid \eta \leq \xi < l \rangle \) a countable support iteration satisfying the following induction hypotheses:

1. \( \mathbb{P}_\xi \) satisfies the \( \omega_2 \)-c.c. and is \( \sigma \)-Baire.
2. \( \mathbb{P}_\xi \) is \( \omega_1 \)-stationary preserving and \( \omega_1 \)-diamond preserving.
3. For \( \xi < l \), there are \( \mathbb{P}_\xi \)-names \( \dot{X}_\xi \) and \( \dot{F}_\xi \) such that \( \| p_\xi \|^\mathbb{P}_\xi \) “\( \dot{X}_\xi \) is a \( \kappa \)-nice graph, \( \dot{F}_\xi : \kappa \to [\kappa]^\omega \), and \( \dot{\mathcal{Q}}_\xi = \mathbb{P}(\dot{X}_\xi, \dot{F}_\xi) \)”
4. For \( \xi < l \), let \( D_\xi \) be the set of all \( p \in \mathbb{P}_\xi \) such that for every \( \eta \in \text{supp}(p) \) there is \( r_\eta \) such that \( p(\eta) \) is the canonical name of \( r_\eta \). Then \( D_\xi \) is dense in \( \mathbb{P}_\xi \).

Let \( \xi < l \) and \( N \ll \mathcal{H}_\theta \) be a \( \sigma \)-closed model containing all relevant objects and \( |N| = \omega_1 \). Let \( p \in D_\xi \). For \( \eta \in \text{supp}(p) \), let \( r_\eta \) be the function such that \( p(\eta) \) is the canonical name of \( r_\eta \). Let \( p^N \) be the function defined by \( \text{dom}(p^N) = \xi, \text{supp}(p^N) = \text{supp}(p) \cap N \), and for \( \eta \in \text{supp}(p^N) \), let \( p^N(\eta) \) be the canonical name of \( r_\eta \upharpoonright (N \cap \kappa) \).

Let \( \langle p_n \mid n < \omega \rangle \) be a descending sequence in \( D_\xi \). For \( n < \omega \) and \( \eta \in \text{supp}(p_n) \), let \( r_{n,\eta} \) be the function such that \( p_n(\eta) \) is the canonical name of \( r_{n,\eta} \). The **canonical limit** of \( \langle p_n \mid n < \omega \rangle \) is the function \( p \) defined by \( \text{dom}(p) = \xi, \text{supp}(p) = \bigcup_n \text{supp}(p_n) \), and for \( \eta \in \text{supp}(p) \), let \( p(\eta) \) be the canonical name of \( \bigcup \{ r_{n,\eta} \mid n < \omega, \eta \in \text{supp}(p_n) \} \). Notice that the canonical limit is not necessarily a condition. Now we also require the following for the induction hypotheses:

5. Let \( \xi < l \) and \( N \ll \mathcal{H}_\theta \) be a \( \sigma \)-closed model containing all relevant objects and \( |N| = \omega_1 \).
   (a) \( \mathbb{P}_\xi \cap N \) is a complete suborder of \( \mathbb{P}_\xi \). In addition, for \( p \in D_\xi \), \( p^N \in \mathbb{P}_\xi \cap N \) and is the greatest reduction of \( p \).
   (b) The quotient \( \mathbb{P}_\xi / (\mathbb{P}_\xi \cap N) \) is \( \omega_1 \)-stationary preserving.
   (c) Let \( M \ll \mathcal{H}_\theta \) be countable with \( \mathbb{P}_\xi, N, \ldots \in M \). Then every \( (M, \mathbb{P}_\xi \cap N) \)-generic sequence in \( D_\xi \cap N \) has a lower bound. More precisely, let \( \langle p_n \mid n < \omega \rangle \) be an \( (M, \mathbb{P}_\xi \cap N) \)-generic sequence with \( p_n \in D_\xi \cap N \). Then the canonical limit is a condition in \( \mathbb{P}_\xi \cap N \), hence is a lower bound of \( \langle p_n \mid n < \omega \rangle \).

We define \( \mathbb{P}_l \) as intended. Now we verify that \( \mathbb{P}_l \) satisfies the induction hypotheses.
Case 1: \( l \) is successor, say \( l = k + 1 \). (1)–(4) follow from the induction hypotheses and lemmas above.

(5). Fix a \( \sigma \)-closed \( N \prec \mathcal{H}_\omega \) containing all relevant objects and \(|N| = \omega_1\). As usual, we can identify \( P_l \) with \( P_k \ast \dot{Q}_k \). By (4), \( D_l \) is dense in \( P_l \).

Before rendering the proof of (5), we give some observations: Let \( G \) be \((V, P_k)\)-generic, and \( G^* = G \cap N \) which is \((V, P_k \cap N)\)-generic. Let \( X_k \) and \( F_k \) be the interpretations of \( \dot{X}_k \) and \( \dot{F}_k \) by \( G \) respectively. We know \( N[G] \prec \mathcal{H}_\omega^{V[G]} \). Since \( P_k \) satisfies the \( \omega_2 \)-c.c. and \( N \) is \( \sigma \)-closed, we have that \( N[G] \) remains \( \sigma \)-closed and \( N[G] \cap \text{ON} = N \cap \text{ON} \). Hence \( N \cap \kappa \) is \( \langle X_k, F_k \rangle \)-complete by Lemma 5.7. For each \( \alpha, \beta \in Y = N \cap \kappa \), there is a maximal antichain \( I \in N \) in \( P_k \) which decides whether \( \alpha, \beta \) or not. \( I \subseteq P_k \cap N \), hence \( I \) is a maximal antichain in \( P_k \cap N \). Thus we have that the induced subgraph \( Y \) of \( X_k \) lies in \( V[G^*] \). Similarly, we have \( F_k \upharpoonright Y \subseteq V[G^*] \) and \( F_k(\alpha), E_k \cap \alpha \subseteq Y \) for \( \alpha \in Y \). Moreover, for each countable \( y \subseteq Y \), we know \( y \in N[G] \), and the set \( \{ \gamma < \kappa \mid \kappa, y \in \mathbb{E}_k \cap \gamma \} \) is in \( N[G] \). Hence it is a subset of \( N[G] \cap \kappa = N \cap \kappa = Y \) by Lemma 5.5. Thus, for each \( y \in [Y]^{\omega_1} \), \( y \) is \( (X_k, F_k) \)-complete in \( V[G] \) if and only if \( y \in (Y, F_k \upharpoonright Y) \)-complete in \( V[G^*] \), i.e., \( E^* \cap \alpha, F_k(\alpha) \subseteq y \) for \( \alpha \in y \), and for every \( \beta \in Y \), if \( E^*_k \cap y \) is infinite then \( \beta \notin y \). In addition, since \( \text{Col}(Y) \leq \omega \in V[G] \) and \( P_k/(P_k \cap N) \) is \( \omega_1 \)-stationary preserving, we have that \( \text{Col}(Y) \leq \omega \) in \( V[G^*] \) by Lemma 2.10.

(a). Let \( p_0, p_1 \in P_l \cap N \), and suppose \( p_0 \) is incompatible with \( p_1 \) in \( P_l \cap N \). We may assume that \( p_0, p_1 \in D_l \), so, for \( i < 2, p_i \) is of the form \( \langle q_i, r_i \rangle \).

If \( q_0 \) is incompatible with \( q_1 \) in \( P_k \cap N \), then these are incompatible in \( P_k \) by the induction hypotheses, and so \( \langle q_0, r_0 \rangle \) and \( \langle q_1, r_1 \rangle \) are incompatible in \( P_l \). Suppose \( q_0 \) and \( q_1 \) are compatible. Let \( q \in P_k \cap N \) be a common extension. Now, if \( r_0 \cup r_1 \) is a function, then we have that \( \langle q, (r_0 \cup r_1) \rangle \in P_l \cap N \) and is a common extension of \( \langle q_0, r_0 \rangle \) and \( \langle q_1, r_1 \rangle \). Thus \( \langle q_0, r_0 \rangle \) and \( \langle q_1, r_1 \rangle \) are compatible in \( P_l \cap N \), this is a contradiction. Hence \( r_0 \cup r_1 \) is not a function, and we have that \( p_0 \) and \( p_1 \) are incompatible in \( P_l \).

Next take \( p = \langle q, r \rangle \in D_l \). We shall prove that \( p^N \in N \) and is the greatest reduction of \( p \). Now, we identify \( p^N \) with \( \langle q^N, r \upharpoonright (N \cap \kappa) \rangle \). We have \( q^N \in N \) by the induction hypotheses, and \( \text{dom}(r \upharpoonright (N \cap \kappa)) \in N \) by the \( \sigma \)-closure of \( N \). Since \( P_k \) satisfies the \( \omega_2 \)-c.c., we can find a set \( A \subseteq \kappa \) such that \( A \in N \), \( |A| \leq \omega_1 \), and \( \models_{\mathcal{P}_k} \text{“} \dot{F}_k(\text{dom}(r \upharpoonright (N \cap \kappa))) \subseteq A \text{”} \). We know \( A \subseteq N \), and \( r \upharpoonright (N \cap \kappa) \subseteq (\text{dom}(r) \cap N \cap \kappa) \times A \) because \( q \models_{\mathcal{P}_k} r \in \dot{Q}_k \). Thus we have \( r \upharpoonright (N \cap \kappa) \in N \).

Next we show that \( q^N \models_{\mathcal{P}_k} \text{“} r \upharpoonright (\kappa) \in \dot{Q}_k \text{”} \). For this, first we show that \( q^N \models_{\mathcal{P}_k} \text{“} \text{dom}(r \upharpoonright (N \cap \kappa)) \text{ is } (X_k, F_k) \text{-complete} \text{”} \). If not, by the elementarity of \( N \), there is \( q' \leq q^N \) such that \( q' \in P_k \cap N \) and \( q' \models_{\mathcal{P}_k} \text{“} \text{dom}(r \upharpoonright (N \cap \kappa)) \text{ is not } (X_k, F_k) \text{-complete} \text{”} \). \( q' \) is compatible with
On the other hand, we know $q \models_{\mathbb{P}_k} \text{"dom}(r) \text{ is complete"}$ since $q$ forces $r \in \check{Q}_k$, and, $\models_{\mathbb{P}_k} \text{"}N \cap \kappa \text{ is complete"}$ because of the observation before. Hence $q \models_{\mathbb{P}_k} \text{"dom}(r \upharpoonright (N \cap \kappa)) = \text{dom}(r) \cap N \cap \kappa \text{ is complete"}$, this is a contradiction. The same argument shows that $q^N \models_{\mathbb{P}_k} \text{"}r \upharpoonright (N \cap \kappa) \text{ is a good coloring"}$, thus we have $q^N \models_{\mathbb{P}_k} \text{"}r \upharpoonright (N \cap \kappa) \in \check{Q}_k \text{"}$. Finally we must show that $\langle q^N, r \upharpoonright (N \cap \kappa) \rangle$ is the greatest reduction, but this can be verified by a standard argument.

(c). Take a countable $M \prec H_\theta$ containing all relevant objects. First we prove the following claim:

**Claim 5.20.** For every strong $(M, \mathbb{P}_k \cap N)$-generic condition $p$, $p \models_{\mathbb{P}_k} \text{"}M \cap N \cap \kappa \text{ is \langle \check{X}_k, \check{F}_k \rangle-complete\"}$.

**Proof of Claim.** Let $p$ be a strong $(M, \mathbb{P}_k \cap N)$-generic condition. Take a $(V, \mathbb{P}_k)$-generic $G$ with $p \in G$, and let $G^*$ be $(V, \mathbb{P}_k \cap N)$-generic induced by $G$.

Let $X_k = \langle \kappa, \mathcal{E}_k \rangle$ and $F_k$ be the interpretations of $\check{X}_k$ and $\check{F}_k$ by $G$ respectively. We know that $M[G^*] \cap ON = M \cap ON$ (but $M[G] \cap ON \neq M \cap ON$ may be possible). Let $Y = N \cap \kappa$. Now we show that $M \cap Y$ is $(\langle X_k, F_k \rangle)$-complete in $V[G]$. By the observation before, it is enough to show that $M \cap Y$ is $(\langle Y, F_k \upharpoonright Y \rangle)$-complete in $V[G^*]$. We argue as in the proof of Lemma 5.8. We work in $V[G^*]$. We know that $N[G] \cap \kappa = N[G^*] \cap \kappa = N \cap \kappa = Y$ and $Y$ is $(\langle Y, F_k \upharpoonright Y \rangle)$-complete in $V[G^*]$. We also know $\text{Col}(Y) \leq \omega$ in $V[G^*]$. In addition we have $Y, F_k \uparrow Y \in M[G^*]$. Since $M[G^*] \prec H_\theta^{V[G^*]}$ and $M \cap \kappa = M[G^*] \cap \kappa$, it is enough to check that for every $\beta \in Y$, if $M[G^*] \cap Y \cap \mathcal{E}_k^\beta$ is infinite then $\beta \in M[G^*] \cap Y$. Since $\text{Col}(Y) \leq \omega$, there is $f : Y \to [Y]^{<\omega}$ in $M[G^*]$ such that for every $\alpha, \alpha' \in Y$, if $\alpha \leq \mathcal{E}_k^\beta \alpha'$ then $\alpha \in f(\alpha')$ or $\alpha' \in f(\alpha)$. $M[G^*] \cap Y \cap \mathcal{E}_k^\beta$ is infinite but $f(\beta)$ is finite, so there is $\alpha \in (M[G^*] \cap Y \cap \mathcal{E}_k^\beta) \setminus f(\beta)$. Then $\beta \in f(\alpha) \subseteq M[G^*]$. \hfill \Box

Now we prove (c). Take an $(M, \mathbb{P}_l \cap N)$-generic sequence $\langle p_n \mid n < \omega \rangle$ in $D_l \cap N$. We can identify $p_n$ as $\langle q_n, r_n \rangle$ where $q_n \in D_k \cap N, r_n \in N$, and $q_n \models_{\mathbb{P}_k} r_n \in \check{Q}_k$. The sequence $\langle q_n \mid n < \omega \rangle$ is an $(M, \mathbb{P}_k \cap N)$-generic sequence. By the induction hypotheses, the canonical limit $q$ of this sequence is a condition in $\mathbb{P}_k \cap N$. By the claim above, we know $q \models_{\mathbb{P}_k} \text{"}M \cap N \cap \kappa = (\check{X}_k, \check{F}_k)\text{-complete\"}$. Since $\text{dom}(\bigcup_n r_n) = M \cap N \cap \kappa$, one can check that $q \models_{\mathbb{P}_k} \text{"}R \cap (\bigcup_n r_n) \in \check{Q}_k\text{"}$, thus $\langle q, (\bigcup_n r_n) \rangle$ is a lower bound of $\langle p_n \mid n < \omega \rangle$ in $\mathbb{P}_l \cap N$, and is the canonical limit of $\langle p_n \mid n < \omega \rangle$.

(b). Take another $\sigma$-closed model $N' \prec H_\theta$ with $|N'| = \omega_1$ and $N, \ldots \in N'$. As in the proof of Lemma 5.18, it is sufficient to prove that $(\mathbb{P}_l \cap N')/(\mathbb{P}_l \cap N)$ is $\omega_1$-stationary preserving.

Take a $(V, \mathbb{P}_l \cap N)$-generic $G^*$ and work in $V[G^*]$. Fix a stationary set $S \subseteq \omega_1$. Take a countable model $M' \prec H_\theta^{V[G^*]}$ such that $M'$ contains all relevant objects and $M' \cap \omega_1 \in S$. It is enough to prove that there is
an \( (M', (\mathbb{P}_l \cap N')/G^*) \)-generic condition. Take an \( (M', (\mathbb{P}_l \cap N')/G^*) \)-
 generic sequence \( \langle p_n \mid n < \omega \rangle \) with \( p_n \in D_l \cap N \). As before, we may
assume that each \( p_n \) is of the form \( \langle q_n, r_n \rangle \). Note that \( \langle p_n \mid n < \omega \rangle \in V \).

Let \( M = M' \cap V \), which is a countable elementary submodel of \( \mathcal{H}^V \)
and \( M \in V \). We know \( M[G^*] = M' \). In addition, \( \langle p_n \mid n < \omega \rangle \)
is an \( (M, \mathbb{P}_l \cap N') \)-generic sequence, and \( \langle q_n \mid n < \omega \rangle \) is \( (M, \mathbb{P}_k \cap N') \)
generic. By the induction hypotheses, the canonical limit \( q \) of the sequence \( \langle q_n \mid n < \omega \rangle \)
is a condition of \( \mathbb{P}_k \cap N' \). Let \( r = \bigcup_n r_n \in N' \).

By (c) and the claim above, \( q \Vdash \text{dom}(r) = M \cap N' \cap \kappa \) is \( (\dot{X}_k, \dot{F}_k) \)-
complete", hence \( q \Vdash \text{dom}(r) \in \dot{Q}_k \)" and \( \langle q, r \rangle \in \mathbb{P}_l \cap N' \). Now
consider the reduction \( q^N \) of \( q_n \). We have \( q^N \in G^* \cap \mathbb{P}_k \). By the
construction of \( q \), the reduction \( q^N \) of \( q \) is also in \( G^* \cap \mathbb{P}_k \). In addition,
\( q^N \) is a strong \( (M, \mathbb{P}_k \cap N) \)-generic condition. By the claim,
we have \( q^N \Vdash \text{dom}(r^N) = M \cap N \cap \kappa \) is \( (\dot{X}_k, \dot{F}_k) \)-complete", hence
\( q^N \Vdash \text{dom}(r^N) \in \dot{Q}_k \)" and \( \langle q, r \rangle^N = \langle q^N, r^N \rangle \in G^* \). Combining these
arguments, we have that \( \langle q, r \rangle \in (\mathbb{P}_l \cap N')/G^* \).

Case 2: \( l \) is limit. (3) is trivial. For (1), the chain condition of \( \mathbb{P}_l \)
follows from (4) and the standard \( \Delta \)-system argument. The \( \sigma \)-Baireness
follows from (5). We show that (2), (4), and (5) hold.

(4). Take \( p \in \mathbb{P}_l \). Now fix a \( \sigma \)-closed model \( N \prec \mathcal{H}_\theta \) of size \( \omega_1 \)
and \( p, \mathbb{P}_x, \ldots \in N \). Here notice that we do not know yet that \( \mathbb{P}_l \cap N \) is a
complete suborder of \( \mathbb{P}_l \), but this does not cause a problem. Take a
countable \( M \prec \mathcal{H}_\theta \) with \( N, p, \ldots \in M \). Take an \( (M, \mathbb{P}_l \cap N) \)-
generic sequence \( \langle p_n \mid n < \omega \rangle \) with \( p_0 \leq p \). Note that \( \text{supp}(p_n) \subseteq M \cap N \cap l \)
for every \( n < \omega \). Fix an increasing sequence \( \langle l_m \mid m < \omega \rangle \) with \( \text{sup}(M \cap N \cap l) \)
and \( l_m \in M \cap N \cap l \). For each \( m < \omega \), the sequence
\( \langle p_n \upharpoonright l_m \mid n < \omega \rangle \) is an \( (M, \mathbb{P}_l \cap N) \)-generic sequence. By the induction
hypotheses, the canonical limit \( q_m \) of \( \langle p_n \upharpoonright l_m \mid n < \omega \rangle \) is a condition
and in \( D_{l_m} \cap N \). Clearly we have \( q_m = q_n \upharpoonright l_m \) for \( m < n \leq \omega \). Let
\( q = \bigcup_{m < \omega} q_m \). We have that \( q \in D_l \) and \( q \leq p \).

(5). Fix a \( \sigma \)-closed \( N \prec \mathcal{H}_\theta \) of size \( \omega_1 \) containing all relevant objects.

For (a), let \( p \in \mathbb{P}_l \). Then for each \( \xi \in \text{supp}(p) \cap N \), \( (p \upharpoonright \xi)^N \in N \) is
a reduction of \( p \upharpoonright \xi \). Then clearly \( p^N = \bigcup_{\xi \in \text{supp}(p) \cap N} (p \upharpoonright \xi)^N \in N \)
and is the greatest reduction of \( p \).

For (c), take a countable \( M \prec \mathcal{H}_\theta \) with \( N, \ldots \in M \). Fix an
\( (M, \mathbb{P}_l \cap N) \)-generic sequence \( \langle p_n \mid n < \omega \rangle \). Take an increasing sequence
\( \langle l_m \mid m < \omega \rangle \) with \( \text{sup}(N \cap M \cap l) \) and \( l_m \in N \cap M \cap l \). For \( m < \omega \),
the sequence \( \langle p_n \upharpoonright l_m \mid n < \omega \rangle \) is an \( (M, \mathbb{P}_l \cap N) \)-generic sequence, so
the canonical limit \( q_m \) of the sequence is a condition in \( \mathbb{P}_l \cap N \) by
the induction hypotheses. Then \( q = \bigcup_{m < \omega} q_m \) is the canonical limit of
\( \langle p_n \mid n < \omega \rangle \) and a condition in \( \mathbb{P}_l \cap N \).

Finally we have to verify the conditions (5)(b) and (2). This can be done by the same argument used in the proof of Lemma 5.18 with the
condition (5)(c), so we show only (2). For the \( \omega_1 \)-stationary preserving
property of \(\mathbb{P}_t\), fix a stationary set \(S \subseteq \omega_1, p \in \mathbb{P}_t\), and a \(\mathbb{P}_t\)-name \(\dot{C}\) for a club in \(\omega_1\). Take a \(\sigma\)-closed \(N \prec \mathcal{H}_\theta\) with size \(\omega_1\) and \(S, \dot{C}, \ldots \in N\). Since \(\mathbb{P}_t\) satisfies the \(\omega_2\)-c.c., we may assume that \(\dot{C}\) is a \(\mathbb{P}_t \cap N\)-name. Take a countable model \(M \prec \mathcal{H}_\theta\) containing all relevant objects such that \(M \cap \omega_1 \in S\). By (5)(c), we can take a strong \((M, \mathbb{P}_t \cap N)\)-generic condition \(q \leq p\). We have \(q \Vdash_{\mathbb{P}_t \cap N} M \cap \omega_1 \in \dot{C}\), and since \(\mathbb{P}_t \cap N\) is a complete suborder, we also have \(q \Vdash_{\mathbb{P}_t} M \cap \omega_1 \in \dot{C}\). For the \(\omega_1\)-diamond preserving property, let \(\langle d_\alpha \mid \alpha \in S \rangle\) be a \(\Diamond(S)\)-sequence, and \(\dot{A}\) a \(\mathbb{P}_t\)-name for a subset of \(\omega_1\). Take a \(\sigma\)-closed \(N \prec \mathcal{H}_\theta\) of size \(\omega_1\) and \(\langle d_\alpha \mid \alpha \in S \rangle, \dot{A}, \ldots \in N\). As before, we may assume that \(\dot{A}\) is a \(\mathbb{P}_t \cap N\)-name. Take an internally approachable sequence \((M_i \mid i < \omega_1)\) of countable elementary submodels of \(\mathcal{H}_\theta\) containing all relevant objects. By (5)(c), we can take a descending sequence \(\langle p_i \mid i < \omega_1 \rangle\) in \(\mathbb{P}_t \cap N\) such that for each \(i < \omega_1, p_i\) is a strong \((M_i, \mathbb{P}_t \cap N)\)-generic condition, and \(\langle p_i \mid i \leq j \rangle \in M_{j+1}\). Let \(A = \{ \gamma < \omega_1 \mid \exists i < \omega_1 (p_i \Vdash_{\mathbb{P}_t \cap N} \gamma \in A) \}\), and take \(\alpha \in S\) such that \(d_\alpha = A \cap \alpha\). Then, as in Lemma 5.18, we can show \(p_\alpha \Vdash_{\mathbb{P}_t \cap N} \text{"}d_\alpha = \dot{A} \cap \alpha\text{"}\), and so \(p_\alpha \Vdash_{\mathbb{P}_t} \text{"}d_\alpha = \dot{A} \cap \alpha\text{"}\).

This completes the proof that \(\mathbb{P}_t\) satisfies the induction hypotheses (1)–(5).

For \(\xi < l\), let \(G\) be \((V, \mathbb{P}_\xi)\)-generic. In \(V[G]\), we can consider the tail poset \(\mathbb{P}_{\xi,l} = \mathbb{P}_t/G\). Since \(\mathbb{P}_\xi\) is \(\sigma\)-Baire and \(\omega_2\)-c.c., it is clear that the tail poset \(\mathbb{P}_{\xi,l}\) is forcing equivalent to an \((l-\xi)\)-stage countable support iteration. In particular the tail poset \(\mathbb{P}_{\xi,l}\) is \(\omega_1\)-stationary preserving and \(\omega_1\)-diamond preserving.

**Lemma 5.21.** (1) For \(\xi < l\), if \(Y \in V^{\mathbb{P}_\xi}\) is a \(\kappa\)-nice graph, then \(Y\) remains \(\kappa\)-nice in \(V^{\mathbb{P}_t}\).

(2) Suppose \(\text{cf}(l) > \kappa\). Let \(X \in V^{\mathbb{P}_t}\) be a graph of size \(\kappa\) and, in \(V^{\mathbb{P}_t}\), suppose that there is a \(\kappa\)-nice graph \(Y\) which is isomorphic to \(X\). Then there is some \(\xi < l\) such that \(Y \in \mathbb{P}_\xi\) and \(Y\) is \(\kappa\)-nice in \(V^{\mathbb{P}_\xi}\).

**Proof.** In \(V^{\mathbb{P}_\xi}\), the tail poset \(\mathbb{P}_{\xi,l}\) satisfies the \(\omega_2\)-c.c. and is \(\omega_1\)-stationary preserving.

(1). Let \(Y\) be a \(\kappa\)-nice graph in \(V^{\mathbb{P}_\xi}\). We check that \(Y\) is \(\kappa\)-nice in \(V^{\mathbb{P}_t}\), and it is enough to show that \(\text{Col}(Y) \leq \omega_1\), and \(\text{Col}(Z) \leq \omega\) for every \(Z \subseteq Y\) with size \(\omega_1\) in \(V^{\mathbb{P}_t}\).

Since \(\text{Col}(Y) \leq \omega_1\) in \(V^{\mathbb{P}_\xi}\), it is clear that \(\text{Col}(Y) \leq \omega_1\) in \(V^{\mathbb{P}_t}\). If \(Z \in V^{\mathbb{P}_t}\) is a subset of \(Y\) with size \(\omega_1\), by the \(\omega_2\)-c.c. we can find \(Z' \in V^{\mathbb{P}_\xi}\) such that \(Z \subseteq Z'\) and \(|Z'| \leq \omega_1\) in \(V^{\mathbb{P}_\xi}\). Since \(Y\) is \(\kappa\)-nice in \(V^{\mathbb{P}_\xi}\), we have that \(\text{Col}(Z') \leq \omega\) in \(V^{\mathbb{P}_\xi}\). Then we have that \(\text{Col}(Z) \leq \text{Col}(Z') \leq \omega\) in \(V^{\mathbb{P}_\xi}\).

(2). Let \(Y\) be a graph in \(V^{\mathbb{P}_t}\) which is \(\kappa\)-nice and is isomorphic to \(X\). Since \(\text{cf}(l) > \kappa\) and the chain condition of \(\mathbb{P}_t\), there is some \(\xi < l\) with \(Y \in V^{\mathbb{P}_\xi}\). We have to check that \(Y\) is \(\kappa\)-nice in \(V^{\mathbb{P}_\xi}\), and, again, it
is enough to show that \( \text{Col}(Y) \leq \omega_1 \), and \( \text{Col}(Z) \leq \omega \) for every \( Z \subseteq Y \) with size \( \leq \omega_1 \) in \( V^{P_\xi} \). We know \( \text{Col}(Y) \leq \omega_1 \) in \( V^{P_\xi} \). By Lemma 2.10 (2), we know \( \text{Col}(Y) \leq \omega_1 \) in \( V^{P_\xi} \). Next take \( Z \in [Y]^{\omega_1} \) in \( V^{P_\xi} \). Since \( \text{Col}(Z) \leq \omega \) in \( V^{P_\xi} \), we have that \( \text{Col}(Z) \leq \omega \) in \( V^{P_\xi} \) by Lemma 2.10 (1).

Suppose \( 2^\kappa = \kappa^+ \) and consider a \( \kappa^+ \)-stage iteration \( P_{\kappa^+} \). Using the standard book-keeping method and Lemma 2.10, we have:

**Proposition 5.22.** Suppose CH, \( \kappa \geq \omega_2 \), and \( 2^\kappa = \kappa^+ \). Then we can construct a poset \( P \) which is \( \sigma \)-Baire, satisfies the \( \omega_2 \)-c.c., and forces the following:

1. \( \diamond(S) \) holds for every stationary \( S \subseteq \omega_1 \).
2. For every graph \( X \) of size \( \leq \kappa \), if \( \text{Col}(X) \leq \omega_1 \) and \( \text{Col}(Y) \leq \omega \) for every \( Y \in [X]^{\omega_1} \), then \( \text{List}(X) \leq \omega \).

6. **Proofs of Theorem 1.5 and 1.7 (1)**

In this section, using the forcing notion constructed in Section 5, we give proofs of Theorem 1.5 and 1.7 (1).

First we give the proof of Theorem 1.5. Recall that:

**Theorem 1.5.** Suppose GCH. Let \( \lambda > \omega_1 \) be a cardinal, and suppose \( \text{AP}_\mu \) holds for every \( \mu < \lambda \) with countable cofinality. Then there is a poset \( P \) which is \( \sigma \)-Baire, satisfies the \( \omega_2 \)-c.c., and forces that “\( \text{RP}(\text{List}, \lambda) \) holds and \( 2^{\omega_1} > \lambda \)”.

**Proof of Theorem 1.5.** Suppose GCH. Let \( \kappa \geq \omega_2 \) be a cardinal. Suppose \( \text{AP}_\lambda \) holds for every singular \( \lambda < \kappa \) of countable cofinality. By Proposition 5.22, we can take an \( \omega_2 \)-c.c., \( \sigma \)-Baire poset which forces the following:

1. \( \diamond(S) \) holds for every stationary \( S \subseteq \omega_1 \).
2. \( \lambda^\omega = \lambda \) for every regular uncountable \( \lambda < \kappa \).
3. \( \text{AP}_\lambda \) holds for every singular \( \lambda < \kappa \) of countable cofinality.
4. For every graph \( X \) of size \( \leq \kappa \), if \( \text{Col}(X) \leq \omega_1 \) and \( \text{Col}(Y) \leq \omega \) for every \( Y \in [X]^{\omega_1} \), then \( \text{List}(X) \leq \omega \).

Then \( V^P \) is the required model; Take a graph \( X \) with \( \omega_2 \leq |X| \leq \kappa \) and \( \text{List}(X) > \omega \). By (4), we have \( \text{Col}(X) \geq \omega_2 \), or \( \text{Col}(Y) > \omega \) for some \( Y \in [X]^{\omega_1} \). If \( \text{Col}(X) \geq \omega_2 \), then there is \( Y \in [X]^{\omega_1} \) with \( \text{List}(Y) > \omega \) by (2), (3), and Proposition 3.11. If \( \text{Col}(Y) > \omega \) for some \( Y \in [X]^{\omega_1} \), then \( \text{List}(Y) > \omega \) by (1) and Corollary 3.4.

Before starting the proof of Theorem 1.7 (1), we introduce the following notion. For a poset \( P \) and an ordinal \( \alpha \), let \( \Gamma_\alpha(P) \) denote the following two players game of length \( \alpha \): At each inning, Players I and II choose conditions of \( P \) alternately with \( p_0 \geq q_0 \geq p_1 \geq q_1 \geq \cdots \), but if \( \beta < \alpha \) is limit, at the \( \beta \)-th inning, Player I does not move and only
Player II chooses a condition \(q_\beta\) which is a lower bound of the partial play \(\langle p_\xi, q_\zeta \mid \xi, \zeta < \beta, \xi = 0 \text{ or successor} \rangle\) (if it is possible):

\[
\begin{array}{ccccccc}
I & 0 & 1 & \cdots & \omega & \omega + 1 & \cdots \\
\hline
p_0 & p_1 & \cdots & p_{\omega+1} & \cdots \\
\hline
q_0 & q_1 & \cdots & q_\omega & q_{\omega+1} & \cdots \\
\end{array}
\]

Notice that Player I can always choose \(p_{i+1} = q_i\), so the only question is whether Player II can choose a condition. We shall say that Player II wins if II can choose a condition at each inning, and Player I wins otherwise.

\(\mathbb{P}\) is \(\alpha\)-strategically closed if Player II has a winning strategy in the game \(\Gamma_\alpha(\mathbb{P})\). If \(\kappa\) is a cardinal and \(\mathbb{P}\) is \(\kappa\)-strategically closed, then it is easy to show that \(\mathbb{P}\) is \(\kappa\)-Baire.

**Definition 6.1.** Let \(\kappa\) be a regular uncountable cardinal. \(S_\kappa\) is the poset consists of all non-reflecting bounded subsets of \(\kappa \cap \text{Cof}(\omega)\), that is, \(p \in S_\kappa \iff p\) is a bounded subset of \(\kappa \cap \text{Cof}(\omega)\) and \(p \cap \alpha\) is non-stationary in \(\alpha\) for every \(\alpha < \kappa\). Define \(p \leq q\) if \(p\) is an end-extension of \(q\).

Clearly \(|S_\kappa| = 2^{<\kappa}\), hence has the \((2^{<\kappa})^+\)-c.c. The following is well-known:

**Lemma 6.2.**

1. \(S_\kappa\) is \(\kappa\)-strategically closed.
2. Let \(G\) be \((V,S_\kappa)\)-generic. Then \(\bigcup G\) is a non-reflecting stationary set in \(\kappa\).

**Sketch of the proof.** (1). For a limit \(\beta < \kappa\) and a partial play \(\langle p_\xi, q_\zeta \mid \xi, \zeta < \beta, \xi = 0 \text{ or successor} \rangle\), suppose \(q = \bigcup_{\zeta < \beta} q_\zeta \in S_\kappa\), and let \(\gamma = \sup(q)\). Then Player II takes \(q \cup \{\gamma + \omega\}\) as his move. This is a winning strategy of Player II.

(2). To show that \(\bigcup G\) is stationary, take a name \(\dot{C}\) for a club in \(\kappa\). Take a descending sequence \(\langle p_n \mid n < \omega\rangle\) such that for every \(n < \omega\), there is \(\alpha_n\) such that \(\alpha_n < \sup(p_n) < \alpha_{n+1}\) and \(p_{n+1} \Vdash \alpha_n \in \dot{C}\). Let \(\alpha = \sup_n \alpha_n\), and \(p = \bigcup_n p_n \cup \{\alpha\}\). It is easy to check that \(p \in S_\kappa\) and \(p \Vdash \alpha \in \dot{C} \cap \bigcup \dot{G}\). \(\square\)

Now we start the proof of Theorem 1.7 (1):

**Theorem 1.7.** If \(\text{ZFC} + \text{"there exists a supercompact cardinal" is consistent, then the following theories are consistent as well:}

1. \(\text{ZFC} + \text{RP(List)}\) holds but \(\text{RP(Col, } \omega_2)\) fails.
2. \(\text{ZFC} + \text{RP(Col)}\) holds but \(\text{RP(List, } \omega_2)\) fails.

**Proof of Theorem 1.7 (1).** Suppose GCH. Let \(\kappa\) be a supercompact cardinal. First, force with the poset \(\text{Coll}(\omega_1, < \kappa)\). Let \(G\) be \((V, \text{Coll}(\omega_1, < \kappa))\)-generic. We know \(\kappa = \omega_2 = 2^{\omega_1}\) in \(V[G]\). Let \(S = S_\kappa\), and take a \((V[G], S)\)-generic \(H\). We work in \(V[G][H]\). In \(V[G][H]\), \(\kappa = \omega_2\), GCH holds, and \(S^* = \bigcup H\) is a non-reflecting stationary subset of
\( \omega_2 \cap \text{Cof}(\omega) \). Let \( \mathbb{P}_{\omega_3} \) be an \( \omega_3 \)-stage countable support iteration of the \( \mathbb{P}(X, F) \)'s which forces:

1. \( \triangleleft(S) \) holds for every stationary \( S \subseteq \omega_1 \).
2. For every graph \( X \) of size \( \omega_2 \), if \( \text{Col}(X) \leq \omega_1 \) and \( \text{Col}(Y) \leq \omega \) for every \( Y \in [X]^\omega_1 \), then \( \text{List}(X) \leq \omega \).
3. \( 2^\omega = \omega_1 \) and \( 2^\lambda = \lambda^+ \) for every \( \lambda \geq \omega_2 \).

As in the proof of Theorem 1.5, \( \mathbb{P}_{\omega_3} \) forces that:

4. For every graph \( X \) of size \( \omega_2 \), if \( \text{List}(Y) \leq \omega \) for every \( Y \in [X]^\omega_1 \) then \( \text{List}(X) \leq \omega \).

Since \( \mathbb{P}_{\omega_3} \) satisfies the \( \omega_2 \)-c.c., it also forces that:

5. There is a non-reflecting stationary subset of \( \omega_2 \cap \text{Cof}(\omega) \), hence \( \text{RP}(\text{Col}, \omega_2) \) fails.

Now we prove that \( \mathbb{P}_{\omega_3} \) also forces that:

6. \( \text{FRP}(\lambda) \) holds for every regular \( \lambda > \omega_2 \).

Then we can deduce that \( \text{RP}(\text{List}) \) holds in the generic extension as follows. We do this by induction on size of graphs. The \( \omega_2 \) case follows from (4). Let \( X \) be a graph with size \( > \omega_2 \), and suppose every \( Y \in [X]^\omega_1 \) has countable list-chromatic number. By the induction hypothesis, we have that every subgraph of \( X \) with size \( < |X| \) has countable list-chromatic number. If \( |X| \) is singular, then we have \( \text{List}(X) \leq \omega \) by (3) and Proposition 3.6. If \( |X| \) is regular, then we are done by (1), (6) and Proposition 4.7.

To show (6), consider the following poset \( S' \) defined in \( V[G][H] \). \( S' \) is the poset consists of all closed bounded subsets \( p \in \kappa \) with \( p \cap S^* = \emptyset \). Define \( p \leq q \) if \( p \) is an end-extension of \( q \). Clearly \( |S'| = \omega_2 \), so \( S' \) satisfies the \( \omega_2 \)-c.c. Moreover \( S' \) forces that “\( S^* \) is non-stationary”.

The following is straightforward:

**Lemma 6.3.** In \( V[G] \), let \( D = \{ (p, q) \in S * S' \mid p \Vdash_S \text{"sup}(p) = \max(q)^\# \} \). Then \( D \) is dense in \( S * S' \) and is \( \omega_2 \)-closed. In particular, \( S' \) is \( \omega_2 \)-Baire in \( V[G][H] \).

Now take a \( (V[G][H], \mathbb{P}_{\omega_3}) \)-generic \( G^* \). \( S * \mathbb{P}_{\omega_3} * S' \) is forcing equivalent to \( S * S' * \mathbb{P}_{\omega_3} \). Since \( S * S' \) has a \( \sigma \)-closed dense subset, and \( \mathbb{P}_{\omega_3} \) is \( \sigma \)-Baire in \( V[G][S*S'] \), we know that \( S * \mathbb{P}_{\omega_3} * S' \) is \( \sigma \)-Baire. Thus \( S' \) remains \( \sigma \)-Baire in \( V[G][H][G^*] \).

Fix a regular cardinal \( \lambda > \omega_2 \). We also fix a sufficiently large regular cardinal \( \theta > \omega_3 + \lambda \), and let \( N = H_\theta^{V[G][H]} \). To show that \( \text{FRP}(\lambda) \) holds in \( V[G][H][G^*] \), take a stationary \( E \subseteq \lambda \cap \text{Cof}(\omega) \) and \( g : E \rightarrow [\lambda]^{\omega} \) with \( g(\alpha) \in [\alpha]^{\omega} \). Let \( D = \{ x \in [\lambda]^{\omega} \mid \sup(x) \in E, g(\sup(x)) \subseteq x \} \). \( D \) is stationary in \( [\lambda]^{\omega} \). Take a \( (V[G][H][G^*], S') \)-generic \( H' \), and let \( V^* = V[G][H][G^*][H'] \). \( S^* \) is non-stationary in \( \omega_2 \) in \( V^* \). Since \( S' \) satisfies the \( \omega_3 \)-c.c., we have that \( E \) is stationary in \( V^* \). Moreover, since \( S' \) is \( \sigma \)-Baire, it is easy to show that \( D \) remains stationary in \( V^* \).
In $V$, take a $\theta$-supercompact embedding $j : V \to M$. We note $j''N \in M$. Since $S \ast S'$ has a $\sigma$-closed dense subset, we know that $\text{Coll}(\omega_1, < j(\kappa))$ is forcing equivalent to $\text{Coll}(\omega_1, < \kappa) \ast S \ast S' \ast \text{Coll}(\omega_1, [\kappa, j(\kappa))]$. Take a $(V^*, \text{Coll}(\omega_1, [\kappa, j(\kappa)]))$-generic $G_{\text{tail}}$. Note that $G^*$ is generic over $V[G][H][H'][G_{\text{tail}}]$.  

In $V^*[G_{\text{tail}}]$, we can construct a $(V, \text{Coll}(\omega_1, < j(\kappa)))$-generic $j(G)$ such that $j(G) \cap \text{Coll}(\omega_1, < \kappa) = G$ and $V[G][H][H'][G_{\text{tail}}] = V[j(G)]$. Then $j : V \to M$ can be extended to $j : V[G] \to M[j(G)]$ in $V^*[G_{\text{tail}}]$ (actually in $V[G][H][H'][G_{\text{tail}}]$). $M[j(G)]$ is still closed under $\omega$-sequences in $V^*[G_{\text{tail}}]$.

Since $\text{Coll}(\omega_1, [\kappa, j(\kappa)])$ is $\sigma$-closed in $V^*$, $D$ remains stationary in $[\lambda]^{< \omega}$ in $V^*[G_{\text{tail}}]$. We know that $S^* \subseteq M[j(G)]$ and is non-stationary in $\kappa$ since $H' \in M[j(G)]$. So $S^*$ is a non-reflecting bounded subset of $j(\kappa)$ in $M[j(G)]$, and $S^*$ is a condition in $j(S)$. Take a $(V^*[G_{\text{tail}}], j(S))$-generic $j(H)$ with $S^* \subseteq j(H)$. In $V^*[G_{\text{tail}}][j(H)]$, $j$ can be extended to $j : V[G][H] \to M[j(G)][j(H)]$. $j(S)$ is $j(\kappa)$-strategically closed in $M[j(G)]$ and $M[j(G)]$ is closed under $\omega$-sequences in $V^*[G_{\text{tail}}]$. Hence $j(S)$ is $\omega_1$-strategically closed in $V^*[G_{\text{tail}}]$, and $D$ remains stationary in $V^*[G_{\text{tail}}][j(H)]$. Again, because $M[j(G)] \subseteq V[G][H][H'][G_{\text{tail}}]$, we have that $G^*$ is generic over $V[G][H][H'][G_{\text{tail}}][j(H)]$.

We know that $j''N \in M \subseteq M[j(G)][j(H)], [j''N] = \omega_1 \in M[j(G)][j(H)]$, and $j''N$ is a $\sigma$-closed elementary submodel of $j(H)^{V[G][H]} \in M[j(G)][j(H)]$. Hence $j(P_{\omega_1}) \cap j''N$ is a complete suborder of $j(P_{\omega_1})$. The following is straightforward:

**Claim 6.4.** $j \upharpoonright P_{\omega_1}$ is a dense embedding from $P_{\omega_1}$ to $j(P_{\omega_1}) \cap j''N$.

Now $M[j(G)][j(H)] \subseteq V[G][H][H'][G_{\text{tail}}][j(H)]$ and $G^*$ is generic over $V[G][H][H'][G_{\text{tail}}][j(H)]$. Hence $j$ and $G^*$ induce the filter $G_0$ on $j(P_{\omega_1}) \cap j''N$ which is generic over $V[G][H][H'][G_{\text{tail}}][j(H)]$. In $M[j(G)][j(H)][G_0]$, we can consider the quotient $j(P_{\omega_1})/G_0$. Since $j \upharpoonright P_{\omega_1} \in M[j(G)][j(H)]$, we know that $G^* \in M[j(G)][j(H)][G_0]$. Using this observation, we can check that $j''D \in M[j(G)][j(H)][G_0], j''D$ is stationary in $[j''\lambda]^{\omega_1}$, and $[j''\lambda] = \omega_1$ in $M[j(G)][j(H)][G_0]$.

Take a $(V^*[G_{\text{tail}}][j(H)], j(P_{\omega_1})/G_0)$-generic $j(G^*)$. We can canonically extend $j : V[G][H] \to M[j(G)][j(H)]$ to $j : V[G][H][G^*] \to M[j(G)][j(H)][G^*]$. $M[j(G)][j(H)][G_0]$ thinks that $j(P_{\omega_1})/G_0$ is $\omega_1$-stationary preserving, thus we have that $j''D$ remains stationary in $[j''\lambda]^{\omega_1}$ in $M[j(G)][j(H)]$. Hence in $M[j(G)][j(H)][j(G^*)], j''\lambda$ and $j''D$ witness the statement that “there is $I \in [j(\lambda)]^{\omega_1}$ such that $\sup(I) \notin I, cf(\sup(I)) = \omega_1$, and $\{x \in [I]^{\omega_1} | sup(x) \in j(\omega), j(g)(sup(x)) \subseteq x\}$ is stationary in $[I]^{\omega_1}$”. By the elementarity of $j$, it holds in $V[G][H][G^*]$ that “there is $I \in [\lambda]^{\omega_1}$ such that $\sup(I) \notin I, cf(\sup(I)) = \omega_1$, and $\{x \in [I]^{\omega_1} | sup(x) \in E, g(sup(x)) \subseteq x\}$ is stationary in $[I]^{\omega_1}$”. This completes the proof of the condition (6). □
Finally let us pose some questions about RP(List).

**Question 6.5.** Does RP(List) imply some strong or interesting consequences?

At the moment, we know only Lemma 4.1 and Proposition 4.6. The following is a test question: It is known that the singular cardinal hypothesis follows from RP(Col) (Fuchino-Rinot [7]), hence we would like to ask:

**Question 6.6.** Does RP(List) imply the singular cardinal hypothesis?

If this question has a positive answer, we can improve the lower bound of the consistency strength of RP(List).

By Theorem 1.5, RP(List, λ) does not have a large cardinal strength. However $2^{ω_1} > λ$ in the resulting model of Theorem 1.5.

**Question 6.7.** Does RP(List, $2^{ω_1}$) have a large cardinal strength?

In the proofs involving RP(List), we always assumed the diamond principle.

**Question 6.8.** Does RP(List) imply $◊(ω_1)$, or $◊(S)$ for every stationary $S \subseteq ω_1$?

Concerning this question, Sakai ([14]) told us that the Game Reflection Principle GRP introduced by König [13] implies both FRP and $◊(S)$ for every stationary $S \subseteq ω_1$, hence by Proposition 4.7 it implies RP(List).

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