SOME RESULTS ON INFINITE DIMENSIONAL RIEMANNIAN GEOMETRY

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Abstract. In this paper we will investigate the global properties of complete Hilbert manifolds with upper and lower bounded sectional curvature. We shall prove the Focal Index lemma that will allow us to extend some classical results of finite dimensional Riemannian geometry as Rauch and Berger theorems and the Topogonov theorem in the class of manifolds in which the Hopf-Rinow theorem holds.

Key words: Riemannian geometry, Hilbert manifold.

1. Introduction

In infinite dimensional geometry, the most of the local results follow from general arguments analogous to those in the finite dimensional case (see [16] or [18]). The investigation of global properties is quite hard as finite dimensional case and the theorem of Hopf-Rinow is generic satisfy on complete Hilbert manifolds (see [11]). Moreover, the exponential map may not be surjective also when the manifold is a complete Hilbert manifold. In section 1 we briefly discuss the relationship between completeness and geodesically completeness (at some point) and we note that this facts are equivalent either when a manifold has constant sectional curvature or no positive sectional curvature. We conclude this section to prove that a group of bijective isometry coincide with the set of the maps that preserve the distance.

The fundamental tools to studying the geometry and topology of the finite dimensional manifolds are the Rauch and Berger theorems. These theorems allow us to understand the distribution of conjugate and focal points along geodesic and the geometry of the complete Hilbert manifolds with bounded sectional curvature. We recall briefly the notion of focal point: let $N$ be a submanifold of a riemannian manifold $M$. The exponential map of $M$ is defined on an open subset $W \subset TM$ and we restrict its on $W \cap T^\perp N$, that we will denote by $\text{Exp}^\perp$. A focal point is a singularity of $\text{Exp}^\perp : W \cap T^\perp N \to M$. In the infinite dimensional
manifolds two species of singularity can be appear: when the differential of $\exp^\perp$ fails to be injective (monofocal) or when the differential of $\exp^\perp$ fails to be surjective (epifocal). Clearly, when $N = p$ we have exactly the notion of conjugate points. In section 2, we shall study the singularity of the exponential map of differential point of view, as in [15], and we shall prove that always monofocal point implies epifocal but not conversely and the distribution of epifocal points and monofocal points can have cluster points. Moreover we will deduce a weak form of the Rauch theorem.

In section 4 we prove the fundamental tool to prove the main results: the Focal Index lemma. Firstly, we will prove the same version of the finite dimensional theory; then we note that we shall prove its in the case when we have only a finite number of epifocal points which are not monofocal (pathological points). Using the above results we will prove the Rauch and Berger theorems in infinite dimensional geometry, when we have at most a finite number of pathological points along a finite geodesic. The main applications will appear in the last section and the main result is the Topogonov Theorem in the class of the complete Hilbert manifolds on which the Hopf-Rinow theorem holds. The prove is almost the same as in [9] because Rauch, Berger and Hopf-Rinow theorems hold. Moreover, we shall prove the Maximal Diameter Theorem, and two version of sphere theorems, with the strong assumption on injectivity radius, with pinching $\sim \frac{\pi}{4}$ and $\frac{4}{9}$, in the class of Hopf-Rinow manifolds. Other simple applications are two results like Berger-Topogonov theorem, one using Rauch Theorem and one using Topogonov Theorem, and a results about the image of the exponential map of a complete manifold with upper bounded sectional curvature. Some basic references for infinite dimensional geometry are [18] and [16].

2. Preliminaries

In this section we give some general results of infinite dimensional Riemannian geometry and we briefly discuss some of the differences from the finite dimensional case. We begin recalling some basic facts and establishing our notation.

Let $(M, g)$ be an Hilbert manifold modeled on a infinite dimensional Hilbert space $\mathbb{H}$. Throughout this paper we shall assume that $M$ is a connected, paracompact and Hausdorff space. Any tangent space $T_pM$ has a scalar product $g(p)$, depending smoothly on $p$, and defining on $T_pM$ a norm equivalent to the original norm on $\mathbb{H}$. Using $g$, we can define the length of piecewise differential curve and it easy to check
that for any two points of $M$ there exists a piecewise differential curve joining them. Hence, we can introduce a metric $d$, defining $d(x, y)$ the infimum of the lengths of all differential paths joining $x, y$ and one can prove that $(M, d)$ is a metric space (see [21]) and $d$ induces the same topology of $M$. As in the finite dimensional case, $M$ admits a unique Levi Civita connection $\nabla$ (see [18]) defined by the Koszul’s formula. We recall that the criterion of tensoriality in infinite dimensional geometry doesn’t hold (see [7]) so we must deduce all properties of $\nabla$ by its local expression.

Let $c : [a, b] \to M$ be a smooth curve. For any $v \in T_{c(a)}M$ there exists a unique vector field $V(t)$ along $c$ such that $\nabla_{\dot{c}(t)}V(t) = 0$. Moreover, the Levi Civita connection satisfies $\nabla_X Y(p) = \frac{d}{dt} |_{t=0} \tau_t^0(Y)$, where $\tau_t^0$ is a parallel transport along any curve $\gamma$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X(p)$.

A geodesic in $M$ is a smooth curve $\gamma$ which satisfies $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Using the theorems of existence, uniqueness and smooth dependence on the initial data, we may prove the existence, at any point $p$, of the exponential map $\exp_p$, that is defined in a neighborhood of the origin in $T_pM$ by setting $\exp_p(v) = \gamma(1)$, where $\gamma$ is the geodesic in $M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. This map is smooth and is a local diffeomorphism, $d(\exp_p)_0 = id$, in a neighborhood of the origin in $T_pM$ by the inverse function theorem. Moreover, there exists an open neighborhood $W$ of $\{0_p \in T_pM : p \in M\}$ in the tangent bundle $TM$, such that the application $\exp_p(X) = \exp_p(X)$ is defined and differentiable.

Generally, the local theory works as in finite dimensional geometry and results as the Gauss lemma and existence of convex neighborhoods hold in infinite dimensional Riemannian geometry and the curvature tensor is defined as follows: let $x, y, z \in T_pM$, we extend them to vector fields $X, Y, Z$ and define $R(x, y)z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$. It is easy to check that $R$ doesn’t depend on the extension and it is antisymmetric in $x, y$ and satisfies the first Bianchi identity $R(x, y)z + R(z, x)y + R(y, z)x = 0$. Given any plane $\sigma$ in $T_pM$ and let $v, w \in \sigma$ be linearly independent. We define the sectional curvature $K(\sigma)$ to be

$$K(\sigma) = \frac{g(R(v, w)w, v)}{g(v, v)g(w, w) - g(v, w)^2}.$$

It is easy to check that $K$ doesn’t depends on the choice of the spanning vectors and the curvature tensor $R$ is completely determined by the sectional curvature. Moreover, as in the finite dimensional case, we shall prove the Cartan theorem, see [16] page 114, in which the existence of a local isometry is characterized by a certain property of the tensor curvature.
The global theory on Hilbert manifolds is more difficult: for example the Hopf-Rinow theorem fails. We recall that a Hilbert manifold is called complete if \((M, d)\) is complete as a metric space. On the other hand, we say that a manifold \(M\) is geodesically complete at a point \(p\) if the exponential map is defined in \(T_p M\) and \(M\) is geodesically complete if it is geodesically complete for all point \(q \in M\). It is easy to check the following implication:

\[ M \text{ complete} \Rightarrow M \text{ is geodesically complete} \Rightarrow M \text{ is geodesically complete at some point}. \]

If \(M\) is a finite dimensional manifold, the above sentences are equivalent, thanks to the Hopf-Rinow theorem. Grossman, see [14] constructs a simply connected complete Hilbert manifold on which there exist two points which cannot be connected with a minimal geodesic but the exponential map is surjective. On the other hand Ekeland ([11]) proved that the Hopf-Rinow theorem is generically satisfied, i.e. if one takes a point \(p \in M\), in a complete Hilbert manifold \(M\), the set of points \(q \in M\) such that there exists a unique minimal geodesic joining \(p\) and \(q\) is a \(G_\delta\) set and in particular it is a dense subset of \(M\). The Hopf-Rinow theorem implies also that the exponential map must be surjective on complete finite dimensional Riemannian manifolds. Atkin (see [4]) showed that there exists a complete Hilbert manifold \(M\) on which, at some point \(p \in M\), \(\exp_p(T_p M)\) is not surjective. Let \(q \in M - (\exp_p(T_p M))\). Then \(M - \{q\}\) is not complete as metric space, but, clearly, is geodesically complete at \(p\). In particular: in \textit{infinite dimensional Riemannian geometry}, \textit{geodesically complete at some point doesn’t imply completeness}. Moreover, Atkin (see [3]) constructed some infinite dimensional Hilbert manifolds in which the induced metric is incomplete, but geodesically complete and any two point may be joined by minimizing geodesic. The above discussion justify the following definition.

\textbf{Definition 1.} A complete Hilbert manifold \(M\) is called Hopf-Rinow if for every \(p, q \in M\) there exists at least a minimal geodesic joining \(p\) and \(q\).

In [12] Elísason showed that the Sobolev manifolds, i.e. the spaces of the Sobolev sections of a vector bundle on a compact manifold, are Hopf-Rinow. Other class of manifolds which are Hopf-Rinow are the simply connected complete Hilbert manifolds such that their sectional curvature is non positive; indeed the Cartan-Hadamard theorem holds, see [14] and [19]. Furthermore, see [18], we may prove that an Hilbert manifold \(M\) with non positive sectional curvature is a complete Hilbert manifold.
manifold if and only if \( M \) is geodesically complete at some point. It is easy to check that the same holds for a manifolds with constant sectional curvature: indeed, using the same arguments used in the classification of the simply connected complete Hilbert manifolds with constant sectional curvature, we may prove our claim.

The Bonnet theorem was proved by Anderson, see [2]; however we cannot conclude any information about fundamental group since we may prove, see [6], that there exist infinite groups acting isometrically and properly discontinuously on the infinite unit sphere. In particular, Weinstein theorem fails: the following example gives an isometry of the unit sphere of a separable Hilbert space \( S(l^2) \) without fixed points, such that \( \inf \{d(x, f(x)) : x \in S(l^2)\} = 0 \).

\[
\begin{align*}
f(\sum_{i=1}^{\infty} x_i e_i) &= \sum_{i=1}^{\infty} (\cos(\frac{1}{i})x_{2i-1} + \sin(\frac{1}{i})x_{2i}) e_{2i-1} \\
&\quad + \sum_{i=1}^{\infty} (\cos(\frac{1}{i})x_{2i}) - \sin(\frac{1}{i})x_{2i-1}) e_{2i}.
\end{align*}
\]

We conclude this section proving that the group of bijective isometry coincide with the set of applications that preserve the distance.

**Proposition 2.** Let \( F : (M, g) \to (N, h) \) be a surjective map. Then \( F \) is an isometry if and only if \( F \) preserves the distance, i.e. if and only if \( d(F(x), F(y)) = d(x, y) \).

The same proof of the finite dimensional case holds, ([17]) once we prove the following result

\[
\lim_{s \to 0} \frac{d(\exp_p(sX), \exp_p(sY))}{\|sX - sY\|} = 1.
\]

**Proof:** let \( r > 0 \) such that \( \exp_p : B_r(0_p) \to B_r(p) \), between the balls with their respectively metrics, is a onto diffeomorphism. Now recall that \( d(\exp_p)_0 = id \), hence there exists \( \epsilon > 0 \) and \( 0 < \eta \leq r \) such that

\[
1 - \epsilon \leq \|d(\exp_p)_q\| \leq 1 + \epsilon,
\]

for every \( q \in B_\eta(0_p) \). Let \( m, a \in B_{\frac{r}{4}}(p) \). By assumption, we shall calculate the distance from \( m \) to \( a \), restricting ourself to the curve \( c \) on \( B_\eta(p) \). Then \( c(t) = \exp_p(\xi(t)) \) and

\[
(1 - \epsilon) \int_0^1 \|\dot{\xi}(t)\|_p \, dt \leq \int_0^1 \|\dot{c}(t)\|_p \, dt \leq (1 + \epsilon) \int_0^1 \|\dot{\xi}(t)\|_p \, dt.
\]

That is: for every \( \epsilon > 0 \) there exists \( s_o = \frac{\eta}{4} \) such that, for every \( s < s_o \) we have

\[
(1 - \epsilon)(\|sX - sY\|_p) \leq d(\exp_p(sX), \exp_p(sY)) \leq (1 + \epsilon)(\|sX - sY\|),
\]

that implies our result. QED
3. Jacobi Flow

The linearized version of the geodesic equation is the famous Jacobi equation. In this section, we shall study the Jacobi field from the differential point of view getting some informations of the distribution of singular points of exponential map. Throughout this section, all estimates are formulated in terms of unit speed geodesics because one can easily reparametrize: if $J$ is a Jacobi fields along $c$, then

$$J_r(t) = J(rt)$$

is a Jacobi field along $c_r(t) = c(rt)$; with $J_r(0) = J(0)$ and $J'_r(0) = rJ'(0), c'_r(0) = rc'(0)$.

**Definition 3.** Let $c : [0, a] \rightarrow M$ be a geodesic. A vector field along $c$ is called Jacobi field if it satisfies the Jacobi differential equation

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} J(t) + R(J(t), \dot{c}(t))\dot{c}(t) = 0,$$

where $\nabla_{\frac{\partial}{\partial t}}$ denotes the covariant derivation along $c$.

The Jacobi equation is a second order differential equation and, by theorems of differential equation in Banach space, see [18], the solutions are defined in the whole domain of definition and the set of Jacobi fields along $c$ is a vector space isomorphic to $T_{c(0)}M \times T_{c(0)}M$ under the map $J \mapsto (J(0), \nabla_{\frac{\partial}{\partial t}} J(0))$. We recall also that the Jacobi fields are characterized as infinitesimal variation of $c$ by geodesic. We first give a lemma due to Ambrose (see [18] page 243).

**Lemma 4.** Let $c : [0, a] \rightarrow M$ be a geodesic and let $J$ and $Y$ be two Jacobi vector fields along $c$. Then

$$\langle \nabla_{\frac{\partial}{\partial t}} J(t), Y(t) \rangle - \langle \nabla_{\frac{\partial}{\partial t}} Y(t), J(t) \rangle = \text{constant} C.$$

Now, let $c : [0, b] \rightarrow M$ be a geodesic. We will denote by $p = c(0)$ and by $\tau^s_{\dot{c}(s)}$ the parallel transport along $c$ between the points $c(s)$ and $c(t)$. We define

$$R_s : T_{c(0)}M \rightarrow T_{c(0)}M, \quad R_s(X) = \tau^0_s(R(\tau^s_0(X), \dot{c}(s))\dot{c}(s))$$

that is a family of symmetric operators in $T_pM$. Take $H_o$ a closed subspace of $T_pM$ and let $A : H_o \rightarrow H_o$ be a continuous and symmetric linear operator. Clearly $T_pM = H_o \oplus H_o^\perp$ and we will study the solutions of the following linear differential equation

$$\begin{cases}
T''(s) + R_s(T(s)) = 0; \\
T(0)(v, w) = (v, 0), \quad T'(0)(v, w) = (-A(v), w),
\end{cases}$$
that we call *Jacobi flow* of $c$. Firstly, note that the family of bilinear applications

$$T_pM \times T_pM \xrightarrow{\Phi(t)} \mathbb{R}$$

$$(u, v) \mapsto \langle T(t)(u), T'(t)(v) \rangle$$

is symmetric; indeed is symmetric in $t = 0$, because $A$ is a symmetric operator, and

$$\langle \langle T(t)(u), T'(t)(w) \rangle - \langle T(t)(w), T'(t)(u) \rangle \rangle'$$

is zero. The solutions of the Jacobi flow are exactly the Jacobi fields.

**Proposition 5.** Let $(v, w) \in T_pM = \mathbb{H}_o \oplus \mathbb{H}_o^\perp$. Then the Jacobi field with $J(0) = v \in \mathbb{H}_o$ and $\nabla_{\partial t} J(0) + A(J(0)) = w \in \mathbb{H}_o^\perp$, is given by $\tau^t_0(T(t)(v, w))$.

**Proof:** Let $Z(t)$ be a parallel transport of $u \in T_pM$ along $c$. We indicate by $Y(t) = \tau^t_0(T(t)(v, w))$. Hence

$$\langle \nabla_{\partial t} \nabla_{\partial t} Y(t), Z(t) \rangle = \langle Y(t), Z(t) \rangle''$$

$$= \langle T(t)(v, w), u \rangle''$$

$$= -\langle R(Y(t), \dot{c}(t))\dot{c}(t), Z(t) \rangle.$$ QED

Our aim is to study the distribution of singular points of the exponential map along a geodesic $c$: then it is very useful compute the adjoint operator of $T(b)$. Let $u \in T_pM$ and let $J$ be the Jacobi field along the geodesic $c$ such that $J(b) = 0$, $\nabla_{\partial t} J(b) = \tau^t_0(u)$. By the Ambrose lemma we have

$$(1) \quad \langle T(b)(v, w), u \rangle = \langle T(0)(v, w), \nabla_{\partial t} J(0) \rangle - \langle T'(0)(v, w), J(0) \rangle.$$  

We denote by $\overline{c}(t) = c(b - t)$ and let

$$\begin{cases} \overline{T}'(s) + R_s(\overline{T}(s)) = 0; \\ \overline{T}(0) = 0, \overline{T}'(0) = id, \end{cases}$$

be the Jacobi flow of $\overline{c} : [0, b] \rightarrow M$. It is easy to check that $J$ is a vector field along $c$, then $\overline{J}(t) = J(b - t)$ is the Jacobi filed along $\overline{c}$ such that $\nabla_{\partial t} \overline{J}(b) = -\nabla_{\partial t} J(0)$. Then (1) becomes

$$\langle T(b)(v, w), u \rangle = \langle (v, o), \tau^0_0(\overline{T}'(b)(-\tau^b_0(u))) \rangle$$

$$- \langle (-A(v), w), \tau^0_0(\overline{T}(b)(-\tau^b_0(u))) \rangle$$

and the adjoint operator is given by

$$\langle T^*(b)(u), (v, 0) \rangle = -\langle \tau^b_0(\overline{T}'(b)(\tau^0_0(u))), A(\tau^0_0(\overline{T}(b)(\tau^b_0(u)))) \rangle(v, 0), \langle T^*(b)(u), (0, w) \rangle = \langle \tau^b_0(\overline{T}(b)(\tau^b_0(u))), (0, w) \rangle.$$ 

where $p_t$ is the component along $\mathbb{H}_o$. 
Proposition 6. There exists a bijective correspondence between the kernel of \( T(b) \) and the kernel of \( T^*(b) \).

Proof: let \( w \in T_p M \) such that \( T(b)(w) = 0 \). Then the Jacobi field

\[
Y(t) = \tau_0^t(T(t)(w)) = \tau_0^t(\dot{T}(t)(\overline{w}))
\]

and \( \overline{w} \in T_{c(b)} M \) is unique. Using the boundary condition on \( Y(t) \) we have \( T^*(b)(\tau_0^t(\overline{w})) = 0 \). Vice-versa, if \( T^*(b)(w) = 0 \) it is easy to check that \( \tau_0^t(\dot{T}(t)(\tau_0^t(w))) = \tau_0^t(\dot{T}(t)(\overline{w})) \) for some \( \overline{w} \in T_p M \). Clearly \( T(b)(\overline{w}) = 0 \) and \( w \) can be obtained, starting from \( \overline{w} \), with the above arguments. QED

We recall that \( \text{Ker} T^*(b) = \overline{\text{Im} T^+} \), hence we have proved also the following result

Proposition 7. If \( T(b) \) is injective then \( \text{Im} T(b) \) is a dense subspace.

We will study the behaviour of the Jacobi flow either when \( \mathbb{H}_o = 0 \) or \( \mathbb{H}_o = T_p M \) and \( A = 0 \). We will denote by \( f_\Delta \) a solution of the differential equation \( f''(t) + \Delta(s)f(s) = 0 \) with \( f(0) = 0 \), \( f'(0) = 1 \), when \( \mathbb{H}_o = 0 \), or with \( f(0) = 1 \), \( f'(0) = 0 \), when \( \mathbb{H}_o = T_p M \) and \( A = 0 \). Firstly, we note that in [15] it is proved the following results that holds in our context.

Proposition 8. Let \( \delta(s) \leq \langle R(u, \dot{c}(s))\dot{c}(s), u \rangle \leq \Delta(s) \) be an upper and lower curvature along \( c \). Hence

1. If \( T(s) \) is invertible then

\[
\langle T'T^{-1}u, u \rangle \leq -(\Delta(s) + \langle T'T^{-1}u, T'T^{-1}u \rangle^2);
\]

2. \( \| T(s)u \| \geq f_\Delta(s)\langle u, u \rangle^{\frac{1}{2}} \), \( f_\Delta(s) \) being positive in \( 0 < s \leq s_o \);

3. \( (T'u, Tu)f_\Delta \geq (Tu, Tu)f_\Delta', f_\Delta(s) \) being positive in \( 0 < s \leq s_o \);

4. \( (T'u, Tu)f_\delta \leq (Tu, Tu)f_\delta', \; 0 < s \leq s_o \) if \( T(s) \) is invertible in \( 0 < s \leq s_1 \) \((s_1 \geq s_o)\);

5. If \( T(s) \) is invertible in \( 0 < s \leq s_1 \) then

\[
\| T(s) \| f_\delta(t) \geq \| T(t) \| f_\delta(s), \; 0 \leq s \leq t \leq s_1;
\]

6. \( \| T(s) \| \leq f_\delta(s) < u, u >^\frac{1}{2}, \; 0 \leq s \leq s_1 \).

Corollary 9. Let

\[
\begin{cases}
T''(s) + R_s(T(s)) = 0; \\
T(0)(v, w) = (v, o), \; T'(0)(v, w) = (-A(v), w),
\end{cases}
\]

be the Jacobi flow. We suppose that we have a upper bound sectional curvature, i.e. \( K \leq H \). Then:
(a) if $H_0 = 0$ then the Jacobi flow is a topological isomorphism for $t \geq 0$, if $H \leq 0$, or for $0 \leq t < \frac{\pi}{\sqrt{H}}$, if $H > 0$;
(b) if $H_0 = T_p M$ and $A = 0$ then the Jacobi flow is a topological isomorphism for $t \geq 0$, if $H \leq 0$, or for $0 \leq t < \frac{\pi}{2\sqrt{H}}$, if $H > 0$.

**Proof:** using property (2) of proposition 8 and proposition 6 we have our claim.

Another interesting corollary is a weak form of the Rauch theorem.

**Theorem 10. (Rauch weak)** Let $c : [0, b] \to M$ be a unit speed geodesic. Suppose that we have a lower and upper bound of the sectional curvature, i.e. $L \leq K(c(t), v) \leq H$, for any $t$ and $v \in T_{c(t)} M$ such that $\langle v, \dot{c}(t) \rangle = 0$, $\langle v, v \rangle = 1$. Then $d(\exp_{c(0)})$ is a topological isomorphism for every $t \geq 0$, if $H \leq 0$ and for every $0 \leq t < \frac{\pi}{\sqrt{H}}$ if $H > 0$. Moreover, in the above neighborhoods, we have

$$\frac{f_H(t)}{t} \| v \| \leq \| d(\exp_{c(0)})(v) \| \leq \frac{f_L(t)}{t} \| v \| .$$

4. **Focal Index lemma and the Rauch and Berger theorems**

Let $N$ be a submanifold of an Hilbert manifold $(M, g)$, i.e. $N$ is a Hilbert manifold and the inclusion $i : N \hookrightarrow M$ is an embedding. Let $\gamma : [a, b] \to M$ be a geodesic such that

1. $\gamma(a) = p \in N$;
2. $\dot{\gamma}(a) = \xi \in T_p N^\perp$.

**Definition 11.** A Jacobi vector field along $\gamma$ is called $N$-Jacobi if it satisfies the following boundary conditions:

$$Y(a) \in T_p N, \nabla_{\dot{\gamma}} Y(a) + A_\xi(Y(a)) \in T^\perp_p N,$$

where $A_\xi$ is the operator of Weingartner relative to $N$.

The Jacobi flow along $\gamma$ on which $A = A_\xi$ is called Jacobi flow along $\gamma$ of $N$. Let $W$ be the open subset of $TM$ on which exp is defined. We denote by $\text{Exp}^\perp : T^\perp N \cap W \to M$, $\text{Exp}^\perp(X) = \text{exp}^M(X)$. It is easy to prove that $\text{Ker}(d(\text{Exp}^\perp)_{t, \xi})$ is isomorphic to the subspace of the $N$-Jacobi field along the geodesic $\gamma(t) = \text{Exp}^\perp(t\xi)$, $t \in [0, t_0]$, which is zero in $\gamma(t_0)$. On the other hand, in infinite dimensional manifolds, two species of singular points can appear; so the following definition is justified.

**Definition 12.** A element $q = \gamma(t_o)$ along $\gamma$ is called:
(1) monofocal if \( \text{d}(\text{Exp}^\perp)_{t_0\xi} \) fails to be injective;
(2) epifocal if \( \text{d}(\text{Exp}^\perp)_{t_0\xi} \) fails to be surjective.

By proposition 6 and by the formula of the adjoint of the Jacobi flow, we have the following results.

**Proposition 13.** Let \( N \) be a submanifold of \( M \) and let \( \gamma : [0, b] \rightarrow M \) such that \( \gamma(0) = p \in N \) and \( \xi = \dot{\gamma}(0) \in T_pN^\perp \). Then

1. if \( \gamma(t_0) \) isn’t monofocal then the image of \( \text{d}(\text{Exp}^\perp)_{t_0\xi} \) is a dense subspace;
2. \( \gamma(t_0) \) is monofocal then \( \gamma(t_0) \) is epifocal;
3. \( q = \gamma(t_0) \) is monofocal along \( \gamma \) if and only if \( p \) is monofocal along \( \gamma(t_0 - t) \);
4. \( q = \gamma(t_0) \) is epiconjugate and the image of \( \text{d}(\text{Exp}^\perp)_{t_0\xi} \) is closed then \( p \) is monofocal along \( \gamma(t_0 - t) \);

In the degenerate case, i.e. \( N = p \), then we call \( q = \gamma(t_0) \) either *monoconjugate* or *epiconjugate* along \( \gamma \). If there exist neither monofocal (monoconjugate) nor epifocal (epiconjugate) points then we will say that there aren’t focal (conjugate) points along \( \gamma \). The distribution of singular points of the exponential map along a finite geodesic is different from the finite dimensional case. Indeed, Grossman showed how the distribution of monoconjugate points be able to have cluster points. The same pathology appears in the case of focal points and we shall give a pathological example of the distribution of monofocal and epifocal points along a finite geodesic.

**Example 1.** Let \( M = \{ x \in l_2 : x_1^2 + x_2^2 + \sum_{i=3}^{\infty} a_i x_i^2 = 1 \} \), where \((a_i)_{i\in\mathbb{N}} \) is a positive sequence of real number. It is easy to check that

\[
\gamma(s) = \sin(s)e_1 + \cos(s)e_2
\]

is a geodesic and \( T_{\gamma(s)}M = \langle \dot{\gamma}(s), e_3, e_4, \ldots \rangle \). Let \( N \) be a geodesic submanifold defined by \( \dot{\gamma}(0) \). We shall restrict ourself to the normal Jacobi fields. We note that for \( k \geq 3 \)

\[
E_k := \{ x_1^2 + x_2^2 + a_k x_k^2 = 1 \} \hookrightarrow M
\]

is totally geodesic; then \( K(\dot{\gamma}(s), e_k) = a_k \) and the Jacobi fields, with boundary conditions \( J_k(0) = e_k \), \( \nabla_{\partial_t} J_k(0) = 0 \), are given by \( J_k(t) = \cos(\sqrt{a_k} t)e_k \). Hence

\[
\text{d}(\text{Exp}^\perp)_{s\dot{\gamma}(0)} (\sum_{k=3}^{\infty} b_k e_k) = \sum_{k=3}^{\infty} b_k \cos(\sqrt{a_k s})e_k.
\]
Clearly, the points $\gamma(r_m^k)$, $r_m^k = \frac{m\pi}{2\sqrt{a_k}}$ are monofocal. Specifically, let $a_k = (1 - \frac{1}{k})^2$. The points $\gamma(s_k)$, $s_k = \frac{k\pi}{2(1-k)}$ are monofocal, $s_k \to \frac{\pi}{2}$ and

$$d(\text{Exp}^\perp)_{\gamma(\frac{\pi}{2})} \left(\sum_{k=3}^{\infty} b_k e_k\right) = \sum_{k=3}^{\infty} b_k \cos\left(\frac{k-1}{k}\frac{\pi}{2}\right)e_k.$$ 

In particular, $\gamma(\frac{\pi}{2})$ is not monofocal along $\gamma$. On the other hand if $\sum_{k=3}^{\infty} \frac{1}{k} e_k = d(\text{Exp}^\perp)_{\gamma(\frac{\pi}{2})} \left(\sum_{k=3}^{\infty} b_k e_k\right)$ then $\sin\left(\frac{\pi}{2k}\right)b_k = \frac{1}{k}$ hence

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{\pi}{2k} \frac{1}{\sin\left(\frac{\pi}{2k}\right)} = \frac{2}{\pi}.$$ 

This means that $\gamma(\frac{\pi}{2})$ is epifocal.

This example shows that there exist epifocal points which are not monofocal. We call them pathological points. If the exponential map of a Hilbert manifold has only a finite number of pathological points we will say that the exponential map is almost non singular. Clearly, if the exponential map were Fredholm, and this one must be of zero index, monoconjugate points and epiconjugate points along geodesics would coincide. This holds for the Hilbert manifold $\Omega(M)$, the free loop space of a compact manifold (see [20]). Moreover, any finite geodesic in $\Omega(M)$ contains at most finitely many points which are conjugate.

Now we shall prove the Index lemma. This lemma allows us to extend Rauch and Berger theorems in infinite dimensional context.

Let $X : [0, 1] \to T_pM$, such that $X(0) \in T_pN$. We define the focal index of $X$ as follows:

$$I^N(X, X) = \int_0^1 \langle \ddot{X}(t), \dot{X}(t) \rangle - \langle R_t(X(t)), X(t) \rangle dt - \langle A_t(X(0)), X(0) \rangle.$$ 

If $N = p$, the Focal Index is called Index and we will denote it as $I(X, X)$. We note that any vector fields along $\gamma$ is a parallel transport of a unique application $X : [0, b] \to T_pM$; we will denote by $\overline{X}(t) = \tau^t_0(X)$ the vector field along $\gamma$ starting from $X$.

**Lemma 14.** $I^N(X, X) = D^2E(\gamma)(\overline{X}, \overline{X})$, where $D^2E(\gamma)$ is the index form of $B = N \times M \hookrightarrow M \times M$.

**Proof:** we recall that

$$D^2E(\gamma)(\overline{X}, \overline{X}) = \int_a^b \| \nabla_{\frac{\partial}{\partial t}} \overline{X}(t) \|^2 - \langle \overline{X}(t), R(\overline{X}(t), \dot{\gamma}(t))\dot{\gamma}(t) \rangle dt - \langle A_{\dot{\gamma}(a)}, -\dot{\gamma}(b) \rangle(\overline{X}(a), \overline{X}(b)), (\overline{X}(a), \overline{X}(b)) > >,$$

see [24], where $A$ is the Weingarter operator of $N \times M \hookrightarrow M \times M$. By the above expression, it is enough to prove that $\nabla_{\frac{\partial}{\partial t}} \overline{X}(t) = \tau^t_0(\dot{X}(t))$. 

Let $Z(t)$ be a parallel transport of a vector $Z \in T_pM$. Then
\[
\langle \nabla_{\partial \partial t} X(t), Z(t) \rangle = \langle X(t), Z(t) \rangle' \\
= \langle \dot{X}(t), Z \rangle \\
= \langle \tau^\prime_0(\dot{X}(t)), Z(t) \rangle.
\]
QED

We here give the Focal Index lemma formulated as in the finite-dimensional Riemannian geometry.

**Lemma 15.** Let $X : [0, b] \to T_pM$ be a piecewise differential application with $X(0) \in T_pN$. Suppose that $T(t)$ is invertible in $(0, a)$. Hence
\[
I^N(X, X) \geq I^N(J, J),
\]
where $J(t) = T(t)(u)$ with $X(b) = T(b)u$. The equality holds if and only if $X = T(t)(u)$. Hence, if there aren’t any focal points along $\gamma$, the index of a vector fields $Y$ along $\gamma$ is bigger than the focal index of the $N$-Jacobi field $J$ along $\gamma$ such that $W(a) = J(a)$.

**Proof:** $T(t)$ is invertible, then there exists a piecewise differential application $Y : [0, b] \to T_pM$ such that $Y(0) = X(0) \in T_pN$ and $X(t) = T(t)(Y(t))$. Hence
\[
\dot{X}(t) = T'(t)((Y(t)) + T(t)(\dot{Y}(t)) = A(t) + B(t).
\]
The focal index of $X$ is given by
\[
I(X, X) = \int_0^b \langle A(t), A(t) \rangle + 2\langle A(t), B(t) \rangle + \langle B(t), B(t) \rangle dt \\
- \int_0^b \langle R_t(T(t)(Y(t)), T(t)(Y(t)))dt - \langle A_\xi(X(0), X(0)) \rangle.
\]
A straightforward computation show that
\[
\langle A(t), A(t) \rangle = \langle T(t)(Y(t)), T'(t)(Y(t)) \rangle' \\
- \langle B(t), A(t) \rangle \\
+ \langle T(t)(Y(t)), R_t(T(t)(Y(t))) \rangle \\
- \langle T(t)(\dot{Y}(t)), T'(t)(Y(t)) \rangle = \langle A(t), B(t) \rangle,
\]
where the last equality depends on the fact that $\Phi(t)$ is a family of symmetric bilinear form. Hence, the focal index of $X$ is given by
\[
I^N(X, X) = \langle T(1)(u), T'(1)(u) \rangle + \int_0^b \| T(t)(\dot{Y}(t)) \|^2 dt.
\]
thus proving our lemma. QED

**Corollary 16.** Let $(M, g)$ be a Riemannian manifold and let $S$ and $\Sigma$ be submanifolds of codimension 1. We denote by $N$ and $\overline{N}$ the normal
vector fields respectively in $S$ and $\Sigma$. Suppose that in some point $p \in S \cap \Sigma$ we have $N_p = \overline{N}_p$ and
\[
g(\nabla_X N, X) < g(\nabla_X \overline{N}, X),
\]
for every $X \in T_p \Sigma = T_p S$. Then, if the Jacobi flow $T$ of $S$ is invertible in $(0, b)$ then the Jacobi flow of $\Sigma$ must be injective in $(0, b)$. Moreover, if $A - \overline{A}$ is invertible, where $A$ and $\overline{A}$ are the Weingarten operators in $p$ respectively of $S$ and $\Sigma$, then the Jacobi flow of $\Sigma$ is also invertible in $(0, b)$.

**Proof:** let $Y(t)$ be $\Sigma$-Jacobi field. Since $T(t)$ is invertible then there exists a piecewise application $X : [0, s] \rightarrow T_p M$ with $X(0) \in T_p S$ such that $Y(t) = T(t)(X(t))$. Hence
\[
\begin{align*}
Y(0) &= T(0)(X(0)), \\
\dot{Y}(0) &= T'(0)(X(0)) + T(0)(\dot{X}(0)) \\
(-A(Y(0)), \dot{Y}(0)^n) &= (-A(X(0)) + \dot{X}(0)^t, 0)).
\end{align*}
\]
Hence, $Y(0) = X(0)$ and the tangent component of $\dot{X}(0)$ is given by $(A - \overline{A})(X(0))$. Then,
\[
g(Y(s), \nabla_{\frac{d}{ds}} Y(s)) = I^S(Y, Y) = g((A - \overline{A})(X(0)), X(0)) + \int_0^s \| T(t)(\dot{X}(t)) \|^2 \, dt > 0.
\]
In particular, the Jacobi flow of $\Sigma$ is injective in $(0, b)$. Moreover, if $A - \overline{A}$ is invertible, using the above formula, it is easy to prove that the image of the Jacobi flow relative to $\Sigma$ is a closed subspace for every $t \in (0, b)$ and using (i) of proposition 13 we get our claim. QED

Now, assume that there exists a pathological point on the interior of $\gamma$; this means that the Jacobi flow is an isomorphism for every $t \neq t_o$ in $(0, b)$ and in $t_o$, $T(t_o)$ is a linear operator whose image is a dense subspace. Let $X : [0, b] \rightarrow T_p M$ be a piecewise application with $X(0) \in T_p N$. Given $\epsilon > 0$, there exist $X_{n,}^\epsilon$, $n = 1, 2$ such that
\[
\begin{align*}
\| T(t_o)(X_1^n) - X(t_o) \| &\leq \frac{\epsilon}{4}, \\
\| T(t_o)(X_2^n) - \dot{X}(t_o) \| &\leq \frac{\epsilon}{4}.
\end{align*}
\]
Choose $Y^\epsilon$ such that
\[
\| T(t_o)(Y^\epsilon) - T'(t_o)(X_1^n) \| \leq \frac{\epsilon}{4}.
\]
Hence there exists $\eta(\epsilon) \leq \frac{\epsilon}{2}$ such that for $t \in (\eta(\epsilon) - t_o, \eta(\epsilon) + t_o)$ we have
\[
\begin{align*}
(1) &\quad \| T(t)(X_1^n + (t - t_o)(X_2^n - Y^\epsilon)) - X(t) \| \leq \epsilon, \\
(2) &\quad \| \frac{d}{dt}(T(t)(X_1^n + (t - t_o)(X_2^n - Y^\epsilon))) - \dot{X}(t) \| \leq \epsilon.
\end{align*}
\]
We denote by $X^\epsilon$ the application

$$X^\epsilon(t) = \begin{cases} 
X(t) & \text{if } 0 \leq t \leq t_o - \eta(\epsilon); \\
T(t)(X_1^t + (t - t_o)(X_2^t - Y^t)) & \text{if } t_o - \eta(\epsilon) \leq t < t_o + \eta(\epsilon); \\
X(t) & \text{if } t_o - \eta(\epsilon) \leq t \leq b.
\end{cases}$$

Clearly, $X^\epsilon = T(t)(Y(t))$, where $Y(t)$ is again a piecewise application, at least at the points $t = t_o + (\eta(\epsilon)$ and $t = t_o - \eta(\epsilon)$, and using the same arguments in Lemma 15, we shall prove

$$I^N(X^\epsilon, X^\epsilon) \geq I^N(T(t)(u), T(t)(u)).$$

On the other hand, the Focal Index of $X$ is given by

$$I(X, X) = I(X^\epsilon, X^\epsilon) - \int_{t_o - \eta(\epsilon)}^{t_o + \eta(\epsilon)} \langle \dot{X}^\epsilon(t), \dot{X}^\epsilon(t) \rangle dt - \langle R(X^\epsilon(t), \dot{\gamma}(t))\gamma(t), X^\epsilon(t) \rangle dt + \int_{t_o - \eta(\epsilon)}^{t_o + \eta(\epsilon)} \langle \dot{X}(t), \dot{\gamma}(t) \rangle dt - \langle R(X(t), \dot{\gamma}(t))\dot{\gamma}(t), X(t) \rangle dt.$$

Now, using (1) and (2) it is easy to check that

$$\lim_{\epsilon \to 0} I^N(X^\epsilon, X^\epsilon) = I^N(X, X) \geq I^N(J, J),$$

where $J(t) = T(t)(u)$. This proves the Focal Index lemma when there is a pathological point. Clearly, the same proof can be generalized to a finite number of pathological points.

**Lemma 17. (Focal Index lemma)** Let $\gamma : [0, b] \to M$ be a geodesic with a finite number of pathological points. Then for every vector fields $X$ along $\gamma$ with $X(0) \in T_0 N$, the index form of $X$ relative to the submanifold $N \times M \hookrightarrow M \times M$, satisfies $D^2 E(\gamma)(X, X) \geq D^2 E(\gamma)(J, J)$, where $J$ is the $N$-Jacobi field such that $J(b) = X(b)$.

In particular, if $\gamma$ has a finite number of pathological points, then $\gamma$ is a local minimum and the Index form $D^2 E(\gamma)$ is non negative defined.

**Corollary 18.** Let $\gamma : [0, \infty) \to M$ be a geodesic, and let $\gamma(t_o)$ be a monoconjugate point. Then $\gamma : [0, t] \to M$ is not minimal for $t > t_o$.

Now, we shall prove the Rauch and Berger theorem and several corollaries. However, the proof are almost the same as the finite dimensional case, since these can be proved using Focal Index lemma: then we will give only a brief proof of the Rauch theorem.

**Theorem 19. (Rauch)** Let $(M, \langle , \rangle)$, $(N, \langle , \rangle^*)$ be Hilbert manifolds modeled on $\mathbb{H}_1$ and $\mathbb{H}_2$ respectively, with $\mathbb{H}_1$ isometric to a closed subspace of $\mathbb{H}_2$. Let

$$c : [0, a] \to M, \quad c^* : [0, a] \to N$$
be two geodesics of the same length. We assume that $c$ has at most a finite number of pathological points in its interior. Suppose also that for every $t \in [0, a]$ and for every $X \in T_0 M$, $X_0 \in T_{c^* (t)} N$ we have

$$K^N (X_0, c^* (t)) \geq K^M (X, c (t)).$$

Let $J$ and $J^*$ be Jacobi fields along $c$ and $c^*$ such that $J(0)$ and $J^*(0)$ are tangent to $c$ and $c^*$ respectively and

1. $\| J(0) \| = \| J^*(0) \| ;$
2. $\langle \dot{c}(0), \nabla_{\dot{c}} J(0) \rangle = \langle \dot{c}^*(0), \nabla_{\dot{c}^*} J^*(0) \rangle ;$
3. $\| \nabla_{\dot{c}} J(0) \| = \| \nabla_{\dot{c}^*} J^*(0) \|. $

Hence, for every $t \in [0, a]$

$$\| J(t) \| \geq \| J^*(t) \| .$$

**Proof:** It is easy to check that we will restrict ourself to the case in which the Jacobi fields satisfy the following condition:

$$0 = \| J(0) \| = \langle \dot{c}(0), \nabla_{\dot{c}} J(0) \rangle = \| J^*(0) \| = \langle \dot{c}^*(0), \nabla_{\dot{c}^*} J^*(0) \rangle .$$

Note, by assumption, $J^*(t) \neq 0$. Let $t_0 \in [0, a]$ be an isometry

$$F : T_{c(0)} M \longrightarrow T_{c^*(0)} N,$$

$$F(\dot{c}(0)) = \dot{c}^*(0),$$

$$F(\tau_{t_0}^0 (J(t_0))) = \chi_{t_0}^0 (J^*(t_0)) \frac{\| J(t_0) \|}{\| J^*(t_0) \|} .$$

We denote by

$$i_t : T_{c(t)} M \longrightarrow T_{c^*(t)} N,$$

$$i_t = \chi_{t_0}^0 \circ F \circ \tau_{t}^0 ,$$

a family of isometries for $0 \leq t \leq t_0$; it is easy to check that $i_t$ commute with the Levi Civita connection. Let $W(t) = i_t (J(t))$. Then

$$D^2 E(c^*)(W, W) = \int_0^{t_0} \| \nabla_{\dot{c}^*} W(t) \|^2 - \langle R^N (\dot{c}^*(t), W(t)) \dot{c}^*(t), W(t) \rangle dt$$

$$\leq \int_0^{t_0} \| \nabla_{\dot{c}^*} J(t) \|^2 - \langle R^M (\dot{c}(t), J(t)) \dot{c}(t), J(t) \rangle dt$$

$$= D^2 E(\gamma)(J, J) .$$

On the other hand,

$$\frac{1}{2} \frac{d}{dt} \bigg|_{t = t_0} \langle J(t), J(t) \rangle = \langle J(t_0), \nabla_{\dot{c}^*} J(t_0) \rangle$$

$$= \frac{d}{dt} \| J(t_0) \|^2$$

$$\geq \frac{d}{dt} \bigg|_{t = t_0} \langle J^*(t), J^*(t) \rangle \ast \frac{\| J(t_0) \|^2}{\| J^*(t_0) \|^2} .$$
where the last before inequality follows from the Focal Index lemma. Now, let $\epsilon > 0$. For every $t \geq \epsilon$ we have

$$\frac{d}{dt} \log(\| J(t) \|^2) \geq \frac{d}{dt} \log(\| J^*(t) \|^2),$$

that implies

$$\frac{\| J(t) \|^2}{\| J(\epsilon) \|^2} \geq \frac{\| J^*(t) \|^2}{\| J^*(\epsilon) \|^2}.$$ 

By $\| \nabla_{\frac{\partial}{\partial t}} J(0) \| = \| \nabla_{\frac{\partial}{\partial t}} J^*(0) \|$, when $\epsilon \to 0$ we get our claim. QED

Misiolek, see [20], proved that in $\Omega(M)$ the index of any finite geodesic if finite. By Rauch theorem, we have that the sectional curvature of $\Omega(M)$, with the $H^1$ metric, cannot be positive along any geodesic. Indeed, if $K \geq K_\alpha > 0$, we are able to compare $\Omega(M)$ with the sphere of radius $\frac{1}{\sqrt{K_\alpha}}$ and we may prove that the index along any geodesic of length bigger than $t = \frac{\pi}{\sqrt{K_\alpha}}$ is infinite.

**Corollary 20.** Let $M, N$ be Hilbert manifolds modeled on $\mathbb{H}_1$ and $\mathbb{H}_2$ where $\mathbb{H}_2$ is isometric to a closed subspace of $\mathbb{H}_1$. Assume that for every $m \in M$ and $n \in N$ and for every $\eta \in T_p M \in T_n N$ 2-subspaces we have

$$K^M(\eta) \geq K^N(\beta).$$

Let $i : T_n N \to T_p M$ be an isometry and let $r > 0$ such that

$$\exp_n : B_r(0_n) \to B_r(m) \text{ is a diffeomorphism}$$

$$\exp_m : B_r(0_m) \to B_r(n) \text{ is almost non singular.}$$

Let $c : [a, b] \to B_r(0_n)$ be a piecewise curve. Then

$$L(\exp_n(c)) \geq L(\exp_m(i \circ c)).$$

**Theorem 21. (Berger)** Let $(M, g)$ and $(N, h)$ be an Hilbert manifolds modeled on $\mathbb{H}_1 \in \mathbb{H}_2$, where $\mathbb{H}_1$ is isometric to a closed subspace of $\mathbb{H}_2$. Let $\gamma_1 : [0, b] \to M$ and $\gamma_2 : [0, b] \to N$ be two geodesics with the same length. Assume that for every $X_1 \in T_{\gamma_1(t)} M$ and $X_2 \in T_{\gamma_2(t)} N$ we have

$$K^N(X_2, \gamma_1(t)) \geq K^M(X_1, \gamma_2(t)), \langle X_1, \gamma_1(t) \rangle = \langle X_2, \gamma_2(t) \rangle = 0.$$ 

Assume furthermore that $\gamma_2$ has at most a finite number of pathological points, on its interior, of the geodesic submanifold $N$ defined by $\gamma_2(0)$. Let $J$ and $J^*$ Jacobi fields along $\gamma_1$ and $\gamma_2$ satisfying $\nabla_{\frac{\partial}{\partial t}} J(0)$ and $\nabla_{\frac{\partial}{\partial t}} J^*(0)$ are tangent to $\gamma_1$ and $\gamma_2$ and

1. $\| \nabla_{\frac{\partial}{\partial t}} J(0) \| = \| \nabla_{\frac{\partial}{\partial t}} J^*(0) \|,$
2. $\langle \gamma_1(0), J(0) \rangle = \langle \gamma_2(0), J^*(0) \rangle, \| J(0) \| = \| J^*(0) \|.$
Then
\[ \| J(t) \| \geq \| J^*(t) \|, \]
for every \( t \in [0, b] \).

**Corollary 22.** Let \( (M, g) \) and \( (N, h) \) be an Hilbert manifolds modeled on \( \mathbb{H}_1 \) and \( \mathbb{H}_2 \) respectively, where \( \mathbb{H}_1 \) is isometric to a closed subspace of \( \mathbb{H}_2 \). Let \( \gamma_1 : [0, b] \rightarrow M \) and \( \gamma_2 : [0, b] \rightarrow N \) be two geodesics with the same length. Let \( V_1(t) \) and \( V_2(t) \) be parallel unit vectors along \( \gamma_1 \) and \( \gamma_2 \) which are everywhere perpendicular to the tangent vectors of \( \gamma_1 \) and \( \gamma_2 \). Let \( f : I \rightarrow \mathbb{R} \) be a positive function and let
\[ b(t) = \exp_{\gamma_1(t)}(f(t)V_1(t)), \]
\[ b^*(t) = \exp_{\gamma_2(t)}(f(t)V_2(t)), \]
two curves. Assume that \( K_N \geq K_M \) and for any \( t \in I \) the geodesics
\[ \eta_0(s) = \exp_{\eta_2(t)}(sf(t)V_2(t)), \quad 0 \leq s \leq 1 \]
contains no focal points of the geodesic submanifold defined by \( \dot{\eta}(0) \). Then \( L(b) \geq L(b^*) \).

**Corollary 23.** Let \( (M, g) \) be an Hilbert manifold such that \( H \leq K \leq L, H > 0 \) and let \( \gamma : [0, b] \rightarrow M \) be a unit speed geodesic. Then

1. the distance \( d \), along \( \gamma \), from \( \gamma(0) \) to the first monoconjugate or epiconejugate satisfies the following inequality
\[ \frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}; \]

2. the distance \( d \), along \( \gamma \), from \( \gamma(0) \) to first monofocal or epifocal point, of the geodesic submanifold defined by \( \dot{\gamma}(0) \), satisfies the following inequality
\[ \frac{\pi}{2\sqrt{H}} \leq d \leq \frac{\pi}{2\sqrt{L}}. \]

### 5. Hilbert Manifolds: A Global Theory

The Rauch and Berger theorems are very important to understand the geometry of the complete manifolds with upper or lower curvature bounded. Indeed, we can compare these manifolds with the complete Hilbert manifolds with constant curvature and the geometry of these is well known. We saw that in a complete Hilbert manifold the exponential map may not be surjective. When the curvature is upper bounded by a constant we have the following result.

**Proposition 24.** Let \( (M, g) \) be a complete Hilbert manifold such that \( K \leq H \). If \( c : [0, 1] \rightarrow M \) is a piecewise differential curve, with \( L(c) < \)
\[
\pi \sqrt{H} \quad \text{if } H > 0, \text{ then there exists a unique piecewise differential curve } \overline{\tau}: [0, 1] \to B_{L(c)}(0_{c(0)}) \text{ such that } \exp_{c(0)}(c(t)) = c(t). \text{ In particular, }
\]
\[
\exp_p(B_r(0_r)) = B_r(p)
\]
for every \( p \in M \) and \( r \geq 0 \) if \( H \leq 0 \) or \( r < \frac{\pi}{\sqrt{H}} \text{ if } H > 0 \).

**Proof:** we will give the proof only when \( H > 0 \); the other case is similar. Take
\[
t_o = \sup \{ t \in [0, 1] : \exists! \overline{\tau}: [0, t] \to B_{L(c)}(0_{c(0)}), \text{ with } \exp_p(\overline{\tau}(t)) = c(t) \};
\]
\( t_o \) is positive by Rauch theorem, and we shall prove that \( t_o \) is in fact 1. Let
\[
\overline{\tau}: [0, t_o) \to B_{L(c)}(0_{c(0)})
\]
the unique lift of \( c \); using Rauch weak theorem we have
\[
\| \dot{c}(t) \| = \| d(\exp_{c(0)}(\overline{\tau}(t))) \| \dot{\overline{\tau}}(t) \|
\geq \sin(\| \overline{\tau}(t) \| \sqrt{H}) \| \dot{\overline{\tau}}(t) \|
\geq \sin(L(c)\sqrt{H}) \| \dot{\overline{\tau}}(t) \|
\]
so we get
\[
\lim_{t \to t_o} \int_0^{t_o} \| \dot{\overline{\tau}}(t) \| \, dt < \infty.
\]
However, \( B_{L(c)}(0_{c(0)}) \) is a complete metric space so \( \lim_{t \to t_o} \overline{\tau}(t) = q \) and by Gauss lemma and the definition of \( t_o \) we get \( t_o = 1 \). QED

**Corollary 25.** Let \((M, g)\) be a complete Hilbert manifold such \( K \leq H \). Let \( p, q \in M \), such that \( d(p, q) < \frac{\pi}{\sqrt{H}} \text{ if } H > 0 \). Hence, at least one of the following facts holds:

1. there exists a unique minimal geodesic between \( p, q \);
2. there exists a sequence \( \gamma_n \) of geodesics from \( p \) to \( q \) such that \( L(\gamma_n) > L(\gamma_{n+1}) \) and \( L(\gamma_n) \to d(p, q) \).

Next we claim a very useful lemma that we will use in the following proofs.

**Lemma 26.** Let \((M, g)\) be an Hilbert manifold such that \( K \geq L > 0 \). Suppose there exists a point \( p \in M \) on which \( \exp_p \) is almost non singular in \( B_r(0_p) \). Let \( \delta(s) \) be a curve joining two antipodal points on the sphere of radius \( s \) in \( T_pM \). Let \( \Delta \) be the Image, via \( \exp_{p'} \), of the curve \( \delta(s) \). Then
\[
L(\Delta) \leq \frac{\pi}{\sqrt{L}} \sin(s\sqrt{L}),
\]
for \( s < r \).
Proof: Let $S_{\frac{1}{\sqrt{L}}}(T_pM \times \mathbb{R})$ be the sphere of radius $\frac{1}{\sqrt{L}}$ and let $N = (0, \frac{1}{\sqrt{L}}) \in S_{\frac{1}{\sqrt{L}}}(T_pM \times \mathbb{R})$; it is easy to check that

$$\exp_N(v) = \frac{1}{\sqrt{L}}\left(\cos(\|v\|\sqrt{L})N + \sin(\|v\|\sqrt{L})\frac{v}{\|v\|}\right).$$

Let $v, w \in T_pM$ be such that $\langle v, w \rangle = 0$, $\langle v, v \rangle = \langle w, w \rangle = 1$; any meridian on the sphere of radius $s$ can be parametrized as follows:

$$c_s(t) = s(\cos(s)v + \sin(s)w)$$

and $L(\exp_N(c_s)) = \frac{\pi}{\sqrt{L}}\sin(s\sqrt{L})$. By corollary 20, we have

$$L(\Delta) \leq \frac{\pi}{\sqrt{L}}\sin(s\sqrt{L}).$$

QED

**Proposition 27.** Let $(M, g)$ be an Hilbert manifold such that $K \geq 1$. Suppose there exists a point $p \in M$ such that $\exp_p$ is almost non singular in $B_\pi(0_p)$. Then $M$ has constant curvature $K = 1$ and is covered by the unit sphere $S(T_pM \times \mathbb{R})$. Furthermore, $M$ is a complete Hilbert manifold.

Proof: let $S_r(T_pM)$, $r < \pi$ be the sphere of radius $r$ in $T_pM$; using lemma 26 we get that the diameter of $\exp_p(S_r(T_pM)) \to 0$ when $r \to \pi$. Hence $\exp_p(S_\pi(T_pM)) = q$. Let $N = (0, 1) \in S(T_pM \times \mathbb{R})$. We define

$$\phi(m) = \begin{cases} 
\exp_p(\exp_{N^{-1}}(m)) & \text{se } m \neq -N; \\
q & \text{se } m = -N.
\end{cases}$$

We claim that $\phi$ is a local isometry. Firstly, note that any geodesic which starts at $p$ get to $q$. Hence any Jacobi field along a geodesic which starts in $p$ is zero in $q$. Moreover the Index form of any geodesic $\gamma : [0, t] \to M$, $t \leq \pi$ and $\gamma(0) = p$, is non negative definite because $\exp_p$ is almost non singular in $B_\pi(0_p)$. Let $\gamma : [0, \pi] \to M$ be a geodesic such that $\gamma(0) = p$. Let $W(t)$ be a parallel transport along $\gamma$ of a unitary and perpendicular vector to $\dot{\gamma}(0)$. The index form of $W$ along $\gamma$ is given by

$$0 \leq \int_0^\pi \langle Y(t), Y(t) \rangle - \langle R(Y(t), \dot{\gamma}(t))\dot{\gamma}(t), Y(t) \rangle dt \leq \int_0^\pi (\cos^2 t - \sin^2 t) dt = 0.$$

Hence $K(W(t), \dot{\gamma}(t)) = 1$ and $Y(t)$ is a Jabobi field. Now, using Cartan theorem, about local isometry, and proposition 6.9 pag. 222 in [18], we get our result. QED

Next, we claim the Berger-Topogonov theorem of maximal diameter.
Theorem 28. (Berger-Topogonov) Let \((M, g)\) be a complete Hilbert manifold such that \(\delta \leq K \leq 1\). Suppose that there exist two points \(p, q\) with \(d(p, q) = \frac{\pi}{\sqrt{\delta}}\) and at least a minimal geodesic from \(p\) to \(q\). Then \(M\) is isometric to a sphere \(S_{\frac{1}{\sqrt{\delta}}}(T_pM \times \mathbb{R})\).

Proof: By corollary 23, the distance from the first focal point along any geodesic is at least \(\frac{\pi}{2}\); furthermore by Bonnet theorem, \(d(M) = \frac{\pi}{\sqrt{\delta}}\).

Let \(\gamma : [0, \frac{\pi}{\sqrt{\delta}}] \to M\) be a minimal geodesic from \(p\) to \(q\). Take the following vector field along \(\gamma\)

\[
Y(t) = \frac{1}{\sqrt{\delta}} \sin(t\sqrt{\delta}) W(t),
\]

where \(W(t)\) is the parallel transport along \(\gamma\) perpendicular to \(\dot{\gamma}(t)\). We define the following variation of \(\gamma\) to be

\[
\Omega(s, t) = \exp_{\gamma(t)}(sY(t)), \quad 0 \leq s \leq 1, \quad 0 \leq t \leq \frac{\pi}{\sqrt{\delta}}.
\]

Any curve \(\Omega(s, \cdot)\) joins \(p, q\) and by corollary 20 we get

\[
L[\Omega(s, \cdot)] \leq \frac{\pi}{\sqrt{\delta}}.
\]

Hence any curve is a minimal geodesic and \(Y(t)\) is a Jacobi field. Furthermore \(\Omega(\cdot, \cdot)\) is a totally geodesic submanifold. Now, it easy to check that \(\exp_p\) is non singular and injective in \(B_{\frac{\pi}{\sqrt{\delta}}}(0_p)\), \(\exp_p(S_{\frac{\pi}{\sqrt{\delta}}}(T_pM)) = q\) and the application

\[
\Phi : S_{\frac{1}{\sqrt{\delta}}}(T_pM \times \mathbb{R}) \to M
\]

defined by

\[
\begin{cases}
\Phi(m) = \exp_p \circ \exp_{N^{-1}}(m) & \text{se } m \neq -N; \\
\Phi(-N) = q & \text{se } m = -N;
\end{cases}
\]

where \(N = (0, \frac{1}{\sqrt{\delta}})\), is an isometry. QED

The Sphere theorem is one of the most beautiful theorem in classic Riemannian geometry. Unfortunately, we haven’t found yet a proof in infinite dimensional case. We will show two soft versions of Sphere theorem: one is the Sphere theorem due in finite dimensional Riemannian geometry by Rauch (see [23]), with the strong assumption on the injectivity radius, with pinching \(\sim \frac{3}{4}\) and the other is the Sphere theorem in the class of Hopf-Rinow manifolds, on which we shall prove that the Topogonov theorem holds, with pinching \(\frac{4}{9}\). In the first case the fundamental lemma is the following.
Lemma 29. Let \((M, g)\) be an Hilbert manifold such that \(K \leq H, H > 0\). Let \(\phi : S(\mathbb{H}) \to M\), where \(S(\mathbb{H})\) is an unit sphere in an Hilbert space, be a local homeomorphism onto the image such that: \(\phi(N) = p\) and the image of every meridian is a curve of length \(r \leq r_0 < \frac{\pi}{\sqrt{H}}\). Then there exists a locally homeomorphism

\[
\overline{\phi} : S(\mathbb{H}) \to B_{r_0}(0_p),
\]

such that \(\exp_p \circ \overline{\phi} = \phi\).

Proof: We apply the proposition 24 to each meridian and we get

\[
\overline{\phi} : S(\mathbb{H}) - \{-N\} \to B_{r_0}(0_p),
\]

with \(\exp_p \circ \overline{\phi} = \phi\). We claim that we can extend \(\overline{\phi}\) to \(-N\). Let \(\xi(t) = \exp_N(tv)\) be a meridian starting from \(N\). Let \(\gamma(t) = \phi(\xi(t))\) and let \(\overline{\gamma}(t)\) the lift of \(\gamma\). By assumption, for every \(t \in [0, \pi]\) there exists an open subset \(W(t)\) of \(\overline{\gamma}(t)\) and an open subset \(U(t)\) of \(\gamma(t)\) such that

\[
\exp_p : W(t) \to U(t)
\]

is an onto diffeomorphism. Now, \(\phi\) is a local homeomorphism then there exists an open subset \(V(t)\) of \(\xi(t)\) such that \(\phi(V(t)) \subset U(t)\) and \(\phi\) on \(V(t)\) is an homeomorphism. The closed interval is compact, then there exists a partition \(0 = t_0 \leq t_1 \leq \ldots \leq t_{n-1} \leq t_n = \pi\) such that

\[
\begin{align*}
\xi([0, \pi]) \subseteq V(t_0) \cup \cdots \cup V(t_n) = V; \\
\gamma([0, \pi]) \subseteq U(t_0) \cup \cdots \cup U(t_n) = U; \\
\overline{\gamma}([0, \pi]) \subseteq W(t_0) \cup \cdots \cup W(t_n) = W;
\end{align*}
\]

and another partition \(0 < s_1 < \cdots < s_n < s_{n+1} = \pi\) such that \(t_i < s_{i+1} < t_{i+1}\) and \(\xi(s_{i+1}) \in V(t_i) \cap V(t_{i+1})\) for \(0 \leq i \leq n - 1\). Now it is easy to see that

\[
C := \{w \in T_N S(\mathbb{H}) : \overline{\exp_N(tw)(\pi)} = \overline{\xi}(\pi)\}
\]

where \(\overline{\xi}\) is the unique lift of \(\xi\), is open and closed. Hence \(\overline{\phi}\) can be extended in \(-N\) and \(\phi\) is a local homeomorphism. QED

Now we claim that a manifold with pinching \(\sim \frac{3}{4}\) can be covered by two geodesic balls.

Let \((M, g)\) be a complete Hilbert manifold. Suppose that the sectional curvature satisfies the inequality \(0 < h \leq K \leq 1\), where \(h\) is a solution of the equation

\[
\sin \sqrt{\pi h} = \frac{\sqrt{h}}{2} (h \sim \frac{3}{4}).
\]
Let $p \in M$. By Rauch theorem we get that on the geodesic ball of radius $\pi$ there aren’t conjugate points. We denote by $\Delta$, the meridian of the sphere in $T_p M$ of radius $\pi$. Then
\[ L[\Delta] \leq \frac{\pi}{\sqrt{h}} \sin \pi \sqrt{h} \leq \frac{\pi}{2}. \]
In particular, there exists $\epsilon > 0$ such that the image of any meridian in the sphere of radius $\pi - \epsilon$ is a curve with length $r \leq r_1 < r_0 < \pi - \epsilon$. Furthermore, $\epsilon$ does not depend from $p$.

**Lemma 30.** Let $p \in M$ and let $q \in \exp_p(S_{\pi-\epsilon}(T_p M))$. Then
\[ M = \overline{B_{\pi-\epsilon}(p)} \cup \overline{B_{\pi-\epsilon}(q)}. \]

**Proof:** take $m \in M$ and let $c : [0,1] \longrightarrow M$ be a piecewise curve joining $p$ and $m$. Take
\[ t_o = \sup \{ t \in [0,1] : \exists \overline{\tau} : [0,t] \longrightarrow \overline{B_{\pi-\epsilon}(0_p)} : \exp_p(\overline{\tau}(s)) = c(s) \}. \]
As in proposition 24, $\overline{\tau}$ is defined in $t_o$ and $L(c_{[0,t_o]}) \geq \pi - \epsilon$. If $t_o = 1$ we get our claim; otherwise $c(t_o) \in \exp_q(B_{r_1}(0_q))$. Now, we define
\[ t_1 = \sup \{ t \geq t_o : \exists \overline{\tau} : [t_o,t] \longrightarrow \overline{B_{\pi-\epsilon}(0_q)} \}. \]
Now, $r_1 < \pi - \epsilon$ so $t_1 > 0$ and, as before, $\overline{\tau}(t_1)$ is well-defined. If $t_1 = 1$ we get our result. Otherwise $\overline{\tau}(t_1) \in \exp_p(B_{r_1}(0_p))$ and by Gauss lemma we get
\[ L[c_{[t_o, t_1]}] \geq r_1 - r_0. \]
However, the curve $c$ has finite length, then after a finite numbers of steps we get that $m$ either belongs to $\overline{B_{\pi-\epsilon}(p)}$ or to $\overline{B_{\pi-\epsilon}(q)}$. QED

Before proving the Sphere Rauch theorem, we recall that the injectivity radius of a complete Hilbert manifold is defined by
\[ i(M) = \sup \{ r > 0 : \exp_p : B_r(0_p) \longrightarrow B_r(p) \text{ is a diffeo. onto } \forall p \in M \}. \]

**Theorem 31. (Sphere Rauch theorem)** Let $(M, g)$ be an Hilbert complete manifold modeled on $\mathbb{H}$ such that $0 < h \leq K \leq 1$, where $h$ is the solution of the equation $\sin(\pi \sqrt{h}) = \frac{\sqrt{K}}{2}$. Assume also that the injectivity radius $i(M) \geq \pi$. Then $M$ is contractible. Furthermore, if $\mathbb{H}$ is a separable Hilbert space then $M$ is diffeomorphic to $S(l_2)$.

**Proof:** we recall that an infinite dimensional sphere is a deformation retract of the unit closed disk, because by Bessega theorem, see [5], there exists a diffeomorphism from $\mathbb{H}$ to $\mathbb{H} - \{0\}$ which is the identity outside the unit disk. When an infinite dimensional manifold $M$ is modeled on Banach space, $M$ is contractible if and only if $\pi_k(M) = 0$, for
every $k \in \mathbb{N}$ (see [22]). Let $f : S^k \to M$ be a continuous application and let

$$H : \overline{B}_{\pi-\frac{\pi}{4}}(p) \times [0,1] \to \overline{B}_{\pi-\frac{\pi}{4}}(p)$$

be the homotopy from the identity map and the retraction on the boundary. We can extend a map on $M$, that we denote by $\tilde{H}$, fixing the complementary of $\overline{B}_{\pi-\frac{\pi}{4}}(p)$. Then

$$F : S^k \times [0,1] \to M,$$

$$F(x,t) = \tilde{H}(f(x),t).$$

is a homotopy between $f$ and an application $\tilde{f} : S^k \to \overline{B}_{\pi-\epsilon}(q)$. Then $f$ is nullhomotopic. If $\mathbb{H}$ is separable, using the Kuiper-Burghelea theorem (see [8]), homotopy classifies the Hilbert manifolds, up to a diffeomorphism, so $M$ is diffeomorphic to the sphere. QED

Next we claim another version of the Sphere theorem in the class of Hopf-Rinow manifolds. However, the main result in this class of Hilbert manifolds is the Topogonov theorem. From our results appeared in section 4, we shall prove it using the same idea in [9] page 42. First of all, we start with the following result that we may prove as in [16] 2.7.11 Proposition page 224.

**Lemma 32.** Let $(M, g)$ be a Hilbert manifold with bounded sectional curvature $H \leq K \leq \Delta$, where $H, \Delta$ are constant. Let $(\gamma_1, \gamma_2, \alpha)$ be a geodesic segment in $M$ such that $\gamma_1(0) = \gamma_2(0)$ and $\alpha = (-\dot{\gamma}_1(l_1), \dot{\gamma}_2(0))$. We call such configuration hinge. Suppose that $\gamma_1$ and $\gamma_2$ are minimal geodesic with perimeter $P = l_1 + l_2 + d(\gamma_1(0), \gamma_2(l_2)) \leq \frac{\pi \sqrt{H}}{2} - 4\epsilon$, $\epsilon > 0$ if $H > 0$. In addition,

(i) if $H \leq 0$ then $l_2 \leq \frac{\pi}{2\sqrt{\Delta}}$;

(ii) if $H > 0$ then

$$l_2 \leq \inf(\epsilon, \frac{\sin \sqrt{H} \epsilon}{\sqrt{H}}, \frac{\pi \sqrt{H}}{4\Delta}, \frac{\pi}{2\sqrt{\Delta}}).$$

Let $(\gamma_1, \gamma_2, \alpha)$ be a hinge $M^H$ such that $L[\gamma_i] = L[\overline{\gamma_i}], i=1,2$. Then

$$d(\gamma_1(0), \gamma_2(l_2)) \leq d(\overline{\gamma_1}(0), \overline{\gamma_2}(l_2)).$$

**Theorem 33. (Topogonov)** Let $(M, g)$ be a Hopf-Rinow manifold such that $H \leq K \leq \Delta$. Then

(A) let $(\gamma_1, \gamma_2, \gamma_3)$ be a geodesic triangle in $M$. Assume $\gamma_1, \gamma_3$ are minimal geodesic and if $H > 0$, $l_2 \leq \frac{\pi}{\sqrt{H}}$. Then in $M^H$, simply connected 2-dimensional manifold of constant curvature $H$, there exists a geodesic triangle $(\overline{\gamma_1}, \overline{\gamma_2}, \overline{\gamma_3})$ such that $l_i = \overline{l_i}$ and
\( \alpha_1 \leq \alpha_1, \, \alpha_2 \leq \alpha_2 \). Except in case \( H > 0 \) and \( l_2 = \frac{\pi}{\sqrt{H}} \), the triangle is uniquely determined.

**(B)** Let \( (\gamma_1, \gamma_2, \alpha) \) be a hinge in \( M \). Let \( \gamma_1 \) be a minimal geodesic, and if \( H > 0 \), \( l_2 \leq \frac{\pi}{\sqrt{H}} \). Let \( (\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha}) \) be a hinge in \( M^H \) such that \( l_i = \overline{l}_i, \, i = 1, 2 \) e \( \alpha = \overline{\alpha} \). Then

\[
d(\gamma_1(0), \gamma_2(0)) \leq d(\overline{\gamma}_1(0), \overline{\gamma}_2(0)).
\]

**Proof:** the proof consists of numbers of steps as in [9] page 43. First of all, we recall briefly some facts of the proof in [9]. Let \( (\gamma_1, \gamma_2, \alpha) \) be a hinge. We call this hinge small if \( \frac{1}{2} r = \max L[\gamma_i], \, i = 1, 2 \) and exp\(_{\gamma_2(0)} \) is an embedding on \( B_r(0) \). Let \( (\gamma_1, \gamma_2, \gamma_3) \) be a geodesic triangle. We call \( (\gamma_1, \gamma_2, \gamma_3) \) a small triangle if any hinge of \( (\gamma_1, \gamma_2, \gamma_3) \) is small. Let \( (\gamma_1, \gamma_2, \gamma_3) \) as in *(A)*. We say that \( (\gamma_1, \gamma_2, \gamma_3) \) is thin if \( (\gamma_1, \gamma_2, \alpha_3) \) and \( (\gamma_3, \gamma_2, \alpha_1) \) are thin hinges, i.e. thin right hinge or thin obtuse hinge or thin acute hinge. We briefly describe the above terminology.

A thin right hinge is a hinge \( (\gamma_1, \gamma_2, \frac{\pi}{2}) \) if the hypothesis of corollary 22 hold.

Let \( (\gamma_1, \gamma_2, \alpha) \) be a hinge with \( \alpha > \frac{\pi}{2} \). Let \( (\overline{\gamma}_1, \overline{\gamma}_2, \frac{\pi}{2}) \) be the corresponding hinge in \( M^{H-\epsilon} \) with \( L[\overline{\gamma}_i] = L[\gamma_i], \, i = 1, 2 \). Let \( \overline{\gamma}_3 \) be a minimal geodesic from \( \overline{\gamma}_2(l_2) \) to \( \overline{\gamma}_1(0) \). Let \( \overline{\sigma} : [0, l] \rightarrow M^{H-\epsilon} \) be a geodesic starting from \( \overline{\gamma}_2(0) \) such that

\[
(\overline{\sigma}(0), \overline{\gamma}_2(0)) = 0, \, \overline{\sigma}(0) = \lambda_1 \overline{\gamma}_1(l_1) + \lambda_2 \overline{\gamma}_2(0), \, \lambda_i > 0
\]

and let \( \overline{\sigma}(l) \) be the first point of \( \overline{\sigma} \) which lies on \( \overline{\gamma}_3 \). Let \( \sigma \) be a geodesic in \( M \) starting from \( \gamma_2(0) \) with the same properties of \( \overline{\sigma} \). We call \( (\gamma_1, \gamma_2, \alpha) \) is a thin obtuse hinge if \( (\gamma_1, \sigma, \alpha - \frac{\pi}{2}) \) is a small hinge and \( (\sigma, \gamma_2, \frac{\pi}{2}) \) is a thin right hinge.

Let \( (\gamma_1, \gamma_2, \alpha) \) be a hinge with \( \alpha < \frac{\pi}{2} \) and let \( \gamma_2(l) \) be a point closest to \( \gamma_1(0) \). Let

\[
\tau = \gamma_2 : [0, l] \rightarrow M, \\
\theta = \gamma_2 : [l, l_2] \rightarrow M
\]

and let \( \sigma : [0, k] \rightarrow M \) be a minimal geodesic from \( \gamma_1(0) \) to \( \gamma_2(l_2) \). We call \( (\gamma_1, \gamma_2, \alpha) \) a thin acute hinge if \( (\gamma_1, \tau, \sigma) \) is a small triangle and \( (\sigma, \theta, \frac{\pi}{2}) \) is a small right hinge.

From step (1) to step (7), in [9], they essentially prove that *(B)* holds for thin right hinges, thin obtuse hinge and thin acute hinge. The same proofs work in our context since in any steps they use Rauch and Berger theorems, and the main corollaries, and the existence of at least a minimal geodesic joining any two points.
Now, we will prove it in general. Given an arbitrary hinge \((\gamma_1, \gamma_2, \alpha)\) as in (B), fix \(N\) and let
\[
\tau_{k,l} = \gamma_2 : \left[ \frac{kl_2}{N}, \frac{(k+l)l_2}{N} \right] \rightarrow M,
\]
where \(k, l\) are integers with \(0 \leq k, l \leq N\). Let \(\sigma_k\) be the minimal geodesic from \(\gamma_1(0)\) to \(\gamma_2(\frac{kl_2}{N})\). As in [9] page 48 we shall prove that any triangle \(T_{k,l} = (\sigma_k, \tau_{k,l}, \sigma_{k+l})\) is a geodesic triangle. If we prove that there exists \(N\) such that any \(T_{k,1}\) is thin we may continue the proof as in [9], proving our aim. If both \(H \leq 0\) and \(\Delta\) are non positive the result follows easily while if \(H \leq 0\) and \(\Delta > 0\) then it is enough to choose \(N\) such that (i) in lemma 32 holds. Then we shall assume \(H > 0\) and by Berger-Topogonov theorem we may suppose also that \(d(\gamma_1(0), \gamma_2(t)) < \frac{\pi}{\sqrt{H}}\). Indeed, if \(d(\gamma_1(0), \gamma_2(t)) = \frac{\pi}{\sqrt{H}}\) for some \(t\) then the manifold must be isometric to a sphere concluding our proof. Using the compactness of \(\gamma_2\) there exists \(\eta_{\epsilon} > 0\) such that for every \(\epsilon \leq \eta_{\epsilon}\) there exists \(t \in [0, t_2]\) we get
\[
d(\gamma_1(0), \gamma_2(t)) + d(\gamma_1(0), \gamma_2(s)) \leq \frac{2\pi}{\sqrt{H} - \epsilon} - 5\eta(\epsilon).
\]
Hence, for every \(\epsilon \leq \eta_{\epsilon}\), we choose \(N\) such that \(L[\tau_{k,1}] \leq \eta(\epsilon)\) and once apply lemma 32, comparing \(M\) with \(M^{H-\epsilon}\), on \((\sigma_k, \tau_{k,1}, \alpha_k)\) and \((\sigma_{k+l}, \tau_{k,i}, \beta_k)\). Moreover, using again the compactness of \(\gamma_2\) there then exists \(r_o\) such that
\[
\exp_{\gamma_2(t)} : B_{2r_o}(\gamma_2(t)) \rightarrow B_{2r_o}(\gamma_2(t))
\]
is a diffeomorphism onto. Now it is easy to see that any geodesic triangle \(T_{k,l}\) is thin. QED

**Corollary 34.** Let \((\gamma_1, \gamma_2, \gamma_3)\) be a geodesic triangle in a Hopf-Rinow manifold such that \(0 < H \leq K \leq \Delta\). Then the perimeter of \((\gamma_1, \gamma_2, \gamma_3)\) is at most \(\frac{2\pi}{\sqrt{H}}\).

**Corollary 35.** Let \((M, g)\) be a Hopf-Rinow manifold with \(0 < H \leq K \leq \Delta\). Assume that \(d(M) = \frac{\pi}{\sqrt{H}}\) and that there exists a point \(p\) such that the image of the function \(q \rightarrow d(p, q)\) has \([0, \frac{\pi}{\sqrt{H}}]\) as subset. Then \(M\) is isometric to \(S_{1,\frac{\pi}{\sqrt{H}}}(T_p M \times \mathbb{R})\).

**Proof:** by Ekeland theorem ([11]) there exists a sequence \(q_n\), in \(M\) such that
\[
d(p, q_n) \rightarrow \frac{\pi}{\sqrt{H}}.
\]
and there exists a unique geodesic \(\gamma_n\), that we shall assume parameterized in \([0, 1]\), from \(p\) to \(q_n\). Take the hinges \((\gamma_n, \gamma_m, \alpha_{n,m})\). Using
Topogonov theorem there exists a hinge \( (\gamma_n, \gamma_m, \alpha_{n,m}) \) in \( M^H \) such that
\[
d(q_n, q_m) \leq d(\gamma_n(1), \gamma_m(1)),
\]
Now, \( \gamma_n(1) \) converges then \( q_n \) is a Cauchy sequence in \( M \). In particular there exists the limit \( q \) of the sequence \( q_n \) that it satisfies \( d(p, q) = d(M) \). Using Berger-Topogonov theorem, \( M \) is isometric to the sphere \( S_{\sqrt{K}}(T_p M \times \mathbb{R}) \). QED

**Theorem 36. Sphere theorem** Let \( (M, g) \) be a Hopf-Rinow manifold such that \( \frac{4}{9} < \delta \leq K \leq 1 \). Assume that \( i(M) \geq \pi \). Then \( M \) is contractible and if \( \mathbb{H} \) is separable then \( M \) is diffeomorphic to \( S(\ell_2) \).

**Proof:** since \( \frac{4}{9} < \delta \) there exists \( \epsilon > 0 \) such that
\[
\frac{\pi}{\sqrt{\delta}} = \frac{3}{2}(\pi - \epsilon).
\]
Using the fact \( i(M) \geq \pi \), there exist two points \( p, q \in M \) such that \( d(p, q) = \pi - \epsilon \). We claim that \( M \) is covered by the following geodesic balls
\[
M = \overline{B_{\pi - \epsilon}(p)} \cup \overline{B_{\pi - \epsilon}(q)}.
\]
Let \( r \in M \) such that \( d(p, r) \geq \pi - \epsilon \). Using corollary 34, we have
\[
d(q, r) \leq 2(\frac{3}{2}(\pi - \epsilon)) - 2(\pi - \epsilon) = \pi - \epsilon.
\]
Now, we shall conclude our proof as in the Sphere Rauch theorem. QED

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