Finite non-solvable groups whose real degrees are prime-powers

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Abstract. We present a description of non-solvable groups in which all real irreducible character degrees are prime-power numbers.

1 Introduction

Let $G$ be a finite group. It is well known that $\text{cd}(G)$, the set of the degrees of all irreducible characters, has great impact on the structure of $G$. Manz in [11, 12] described the solvable and non-solvable groups in which all the real irreducible characters have prime-power degrees. In this paper, we give a structural description of non-solvable groups $G$ such that $\text{cd}_{rv}(G)$, the set of the degrees of all real irreducible characters, consists of prime-power numbers. In the following, $\text{Rad}(G)$ is the radical subgroup and $G^{(\infty)}$ is the last term of the derived series.

Theorem A. Let $G$ be a finite non-solvable group, and suppose that $\text{cd}_{rv}(G)$ consists of prime-power numbers. Then $\text{Rad}(G) = H \times O$ for a group $O$ of odd order and a 2-group $H$ of Chillag–Mann type. Furthermore, if $K = G^{(\infty)}$, then one of the following holds:

(i) $G = K \times \text{Rad}(G)$, and $K$ is isomorphic to $A_5$ or $L_2(8)$;

(ii) $G = (KH) \times O$, where $K \simeq \text{SL}_2(5)$, $HK = H \gamma K$, $K \cap H = Z(K) < H$.

About (ii), we remark that if $G$ is the SmallGroup(240, 93), then $K \simeq \text{SL}_2(5)$, $|H| = 4$ and $K \cap H = Z(K)$.

As a corollary, we get control on the set of real character degrees. We recall that $\text{cd}_{rv,2'}(G)$ is the set of odd real character degrees of a finite group $G$.

Theorem B. Let $G$ a non-solvable group such that $\text{cd}_{rv}(G)$ consists of prime-power numbers. Then either

(i) $\text{cd}_{rv}(G) = \text{cd}_{rv}(L_2(8))$ or

(ii) $\text{cd}_{rv,2'}(G) = \text{cd}_{rv,2'}(A_5)$.
2 Preliminary results and lemmas

Chillag and Mann are among the first authors that studied $\text{cd}_{rv}(G)$. They characterized the groups $G$ such that $\text{cd}_{rv}(G) = \{1\}$, namely where all real irreducible characters are linear. Now these groups are commonly known as groups of Chillag–Mann type.

**Theorem 2.1** ([2, Theorem 1.1]). Let $G$ be a finite group of Chillag–Mann type. Then $G = O \times T$, where $O$ is a group of odd order and $T$ is a 2-group of Chillag–Mann type.

One other important contribution was given by Dolfi, Navarro and Tiep in [3]. In that paper, there appears a version for real characters of the celebrated Ito–Michler Theorem for the prime $p = 2$. Recall that $\text{Irr}_{rv}(G)$ denotes the set of irreducible real-valued character of $G$.

**Theorem 2.2** ([3, Theorem A]). Let $G$ be a finite group and $T \in \text{Syl}_2(G)$. Then $2 \nmid \chi(1)$ for every non-linear $\chi \in \text{Irr}_{rv}(G)$ if and only if $T \leq G$ and $T$ is of Chillag–Mann type.

The corresponding condition for an odd prime $p$ was studied by Tiep in [15] and Isaacs and Navarro in [9]. Though a partial result, the techniques involved are deep. This confirms the special role of the prime 2 in the study of real character degrees.

**Theorem 2.3** ([15, Theorem A]). Let $G$ be a finite group and $p$ a prime. Suppose that $p \nmid \chi(1)$ for every $\chi \in \text{Irr}_{rv}(G)$ with Schur–Frobenius indicator 1. Then $O^p(G)$ is solvable; in particular, $G$ is $p$-solvable.

Navarro, Sanus and Tiep gave a version for real characters of Thompson’s Theorem for the prime 2 in [13]. Their work includes also a characterization of groups in which the real character degrees are powers of 2.

**Theorem 2.4** ([13, Theorem A]). Let $G$ be a finite group, and suppose that 2 divides $\chi(1)$ for all every real non-linear irreducible character of $G$. Then $G$ has a normal 2-complement.

The next two lemmas appear in [4].

**Lemma 2.5.** Let $N$ be a normal subgroup of $G$ and $\chi \in \text{Irr}_{rv}(G)$. The following hold:

(i) if $\chi(1)$ is odd, then $N \leq \ker(\chi)$;

(ii) if $|N|$ is odd and $N$ centralizes a Sylow 2-subgroup of $G$, then $N \leq \ker(\chi)$. 

Proof. Point (ii) is [4, Lemma 1.4]. Point (i) follows from the discussion before [4, Lemma 1.4], keeping in mind that a group of odd order does not have any real non-trivial character.

Let $N$ be a normal subgroup of $G$ and $\theta \in \text{Irr}_{rv}(N)$. The next lemmas provide some sufficient conditions for the existence of a real character of $G$ above $\theta$.

**Lemma 2.6** ([14, Lemma 2.1 and Corollary 2.2]). Let $N$ be a normal subgroup of a group $G$ and $\theta \in \text{Irr}_{rv}(N)$. If $[G : N]$ is odd, then $\theta$ allows a unique real-valued extension to $I_G(\theta)$. Furthermore, there exists a unique real-valued character $\chi \in \text{Irr}_{rv}(G \mid \theta)$.

**Lemma 2.7** ([14, Thorem 2.3]). Let $G$ be a finite group and $N \unlhd G$. Suppose that there is some $\theta \in \text{Irr}_{rv}(G)$ such that $\theta(1)$ is odd and $o(\theta) = 1$. Then $\theta$ extends to a character $\phi \in \text{Irr}_{rv}(I_G(\theta))$ and $\chi = \phi^G \in \text{Irr}_{rv}(G \mid \theta)$.

**Lemma 2.8.** Suppose that $N$ is a minimal normal subgroup of a group $G$ with $N = S_1 \times \cdots \times S_n$, where $S \simeq S$ is a non-abelian simple group. Let $\sigma \in \text{Irr}_{rv}(S)$, and suppose that $\sigma$ extends to a real character of $\text{Aut}(S)$. Then $\sigma \times \cdots \times \sigma$ extends to a real character of $G$.

**Proof.** The extension $\chi$ is constructed in [1, Lemma 5]. We see that if $\sigma$ takes real values, then so does $\chi$.

The technique used in the proof of Lemma 2.8 is known as tensor induction; for further details, see [7, Section 4].

**Lemma 2.9** ([4, Lemma 1.6]). Let $G$ be a finite group that acts by automorphisms on the group $M$. For every involution $x \in C_G(M) \in G/C_G(M)$, there exists a non-trivial character $\mu \in \text{Irr}(M)$ such that $\mu^x = \tilde{\mu}$.

3 Proofs

In the following, we call an integer a composite number if it is divisible by more than one prime. If $p$ is a prime, we denote by $p^*$ a general positive integer that is a power of $p$. Moreover, $\text{Rad}(G)$ is the solvable radical of $G$, namely the largest solvable normal subgroup of $G$.

**Theorem 3.1.** Let $G$ be a finite non-solvable group such that $\text{cd}_{rv}(G)$ consists of prime-power numbers. If $\text{Rad}(G) = 1$, then $G$ is isomorphic to $A_5$ or $\text{PSL}_2(8)$. 

Proof. Let $M$ be a minimal normal subgroup of $G$. Then $M = S_1 \times \cdots \times S_n$ is the product of simple groups, which are all isomorphic to a simple group $S$. Since $\text{Rad}(G) = 1$, the group $S$ is non-abelian.

Step 1: $S$ is isomorphic to one of the following groups:

$$A_5, A_6, \text{PSL}_2(8), \text{PSL}_3(3), \text{PSp}_4(3), \text{PSL}_2(7), \text{PSU}_3(3), \text{PSL}_2(17).$$

Let $p \in \pi(M)$. Since $M$ is minimal normal in $G$, we have $M \leq O^p'(G)$, so $O^p'(G)$ is non-solvable. By Theorem 2.3, there is a real irreducible character $\chi$ of $G$ such that $p \mid \chi(1)$. By the hypothesis, $\chi(1) = p^* > 1$. This means that, for every prime $p \in \pi(M)$, there is $\chi \in \text{Irr}_{rv}(S)$ such that $\chi(1) = p^* > 1$. By [3, Theorem B], if $\Delta_{rv}(G)$ is the prime graph on real character degrees of $G$, then the number of connected components of $\Delta_{rv}(G)$ is at most three. In our hypotheses, $\Delta_{rv}(G)$ consists of isolated vertices, and hence the number of primes that appear as divisors of the degree of some real irreducible character is at most 3. It follows that $M$, and hence $S$, is divisible by exactly 3 primes. Now, by [16, Lemma 2.1], the simple groups having order divided by exactly 3 distinct primes are those stated.

Step 2: $S$ is isomorphic to one of the following groups: $A_5, \text{PSL}_2(8), A_6$.

If $S \in \{\text{PSp}_4(3), \text{PSL}_3(3), \text{PSU}_3(3)\}$, there is a non-linear character $\sigma \in \text{Irr}_{rv}(S)$ such that $\sigma(1)$ is an odd composite number. Let $\theta = \sigma \times \cdots \times \sigma \in \text{Irr}_{rv}(M)$. Then we have $2 \nmid \theta(1)$ and $o(\theta) = 1$ since $M$ is perfect. So, by Lemma 2.7, there is $\chi \in \text{Irr}_{rv}(G \mid \theta)$. As $\theta(1)$ divides $\chi(1)$, the degree of $\chi$ is a composite number, contrary to the hypothesis. Suppose that $S \in \{\text{PSL}_2(7), \text{PSL}_2(17)\}$. There is a real character $\sigma \in \text{Irr}_{rv}(S)$ such that $\sigma(1)$ is a composite number and $\sigma$ extends to a real character of $A = \text{Aut}(S)$. By tensor induction (Lemma 2.8), the character $\theta = \sigma \times \cdots \times \sigma$ extends to a real character $\chi \in \text{Irr}_{rv}(G)$. Again, we get that $\chi(1) = \theta(1) = \sigma(1)^n$ is a composite number.

Step 3: $n = 1$ and $M$ is a simple group.

The only remaining possibilities are $S \in \{A_5, \text{PSL}_2(8), A_6\}$. Checking the character table of these groups, there are two non-linear characters $\sigma, \rho \in \text{Irr}_{rv}(S)$ such that $\sigma(1) = p^* > 1$ and $\rho(1) = q^* > 1$ for $p, q$ odd distinct primes. Let $\theta = \sigma \times \cdots \times 1 \in \text{Irr}_{rv}(M)$. Since $o(\theta) = 1$ and $\theta(1)$ is odd, the character $\theta$ extends to a character $\phi \in \text{Irr}_{rv}(I_G(\theta))$ by Lemma 2.7, and $\chi = \phi^G$ has degree $p^*$; hence $[G : I_G(\theta)] = p^* > 1$. Since $I_G(\theta) \leq N_G(S_1)$, we have that

$$n = [G : N_G(S_1)] \mid [G : I_G(\theta)] = p^* > 1,$$

so $n = p^*$. By the same argument with $\rho$ in place of $\sigma$, we get that $n = q^*$ and $n \mid (p^*, q^*) = 1$. 


Step 4: $C_G(M) = 1$.

Suppose, by contradiction, that $C_G(M) > 1$, and take $N$ to be a minimal normal subgroup of $G$ contained in $C_G(M)$. By the same arguments as used on $M$, we have that $N$ is isomorphic to a simple group of the following: $A_5, \text{PSL}_2(8), A_6$. As before, take $\sigma \in \text{Irr}_{\text{rv}}(M)$ with $\sigma(1) = p^*$ and $\rho \in \text{Irr}_{\text{rv}}(N)$ with $\rho(1) = q^*$ for $p, q$ odd distinct primes. Note that $[M, N] \leq M \cap N \leq M \cap C_G(M) = 1$ since $M$ is simple and non-abelian. So $MN = M \times N$ is perfect normal in $G$, and $\theta = \sigma \times \rho \in \text{Irr}_{\text{rv}}(MN)$. Note that $o(\theta) = 1$ and $2 \nmid \theta(1)$. By Lemma 2.7, there is $\chi \in \text{Irr}_{\text{rv}}(G | \theta)$, and this is impossible since $\chi(1)$ is not a composite number.

Conclusion: We have proved, so far, that $S \leq G \leq \text{Aut}(S)$ and that $S$ is isomorphic to either $A_5, A_6$ or $\text{PSL}_2(8)$. Now, $S$ cannot be the alternating group $A_6$ because each of the 5 subgroups between $S$ and $\text{Aut}(S)$ has a rational irreducible character of degree 10 (it is possible to check this with the software GAP), so $S \notin \{A_5, \text{PSL}_2(8)\}$. In any of these cases, $[\text{Aut}(S) : S]$ is a prime number and there is only one subgroup strictly above $S$, namely $\text{Aut}(S)$ itself. But both $\text{Aut}(A_5)$ and $\text{Aut}(\text{PSL}_2(8))$ have a real irreducible character with composite degree. Hence $G = A_5$ or $G = \text{PSL}_2(8)$. 

\[ \square \]

**Proposition 3.2.** Let $G$ be a finite non-solvable group such that $\text{cd}_{\text{rv}}(G)$ consists of prime-power numbers. Then $G = KR$ with $R = \text{Rad}(G)$ and $K = G^{(\infty)}$. Moreover, $K \cap R = L$ is a 2-group, and $K/L$ is isomorphic to $A_5$ or $\text{PSL}_2(8)$.

**Proof.** Let $K = G^{(\infty)}$ be the last term of the derived series of $G$, and define $\bar{G} = G/K \cap R$. Observe that quotients preserve the hypotheses. Hence, by Theorem 3.1, $G/R$ is a simple group. Since $1 < KR/R \leq G/R$, we have that $G = KR$ and $\bar{K} \simeq G/R$ is isomorphic to $A_5$ or $\text{PSL}_2(8)$. Moreover, $\bar{G} = \bar{K} \times \bar{R}$ because $[K, R] \leq L$.

Suppose by contradiction that there is some $\theta \in \text{Irr}_{\text{rv}}(\bar{R})$ of non-trivial degree. By Theorems 2.4 and 2.2, there are two non-linear characters $\phi_1, \phi_2 \in \text{Irr}_{\text{rv}}(\bar{K})$ such that $\phi_1(1)$ is even and $\phi_2(1)$ is odd. If $\theta(1)$ is odd, consider $\chi = \theta \phi_1$, and if $\theta(1)$ is even, consider $\chi = \theta \phi_2$. In any case, $\chi$ is a composite number, but this is impossible. It follows that every real character of $R/L$ is linear, and by Theorem 2.1, $\bar{R} = \bar{O} \times \bar{H}$, where $O \in \text{Hall}_2(R)$ and $H \in \text{Syl}_2(R)$. Write $G_0$ for the preimage in $G$ of $\bar{K} \bar{H}$; note that $G_0$ is a normal subgroup of odd index in $G$. Note that $G_0 = LKH = KH$. By Lemma 2.6, $\text{cd}_{\text{rv}}(G_0)$ consists of prime-power numbers. Moreover, $K = G_0^{(\infty)}$ and $\text{Rad}(G_0) \cap K = L$. Hence we can assume that $G = G_0$. This implies that $O \leq L$.

Suppose, working by contradiction, that $O > 1$, namely $L$ is not a 2-group. Consider $M/M_0$, the first term (from above) of a principal series of $G$ such that $M, M_0 \leq L$ and $M/M_0$ is not a 2-group. Hence $M/M_0$ is an elementary abelian
$p$-group for $p$ odd, and $L/M$ is a 2-group. Possibly replacing $G$ with $G/M_0$, we can assume that $M_0 = 1$ and $M$ is a minimal normal subgroup of $G$.

Since $K/L$ is simple, $C_K(M) = K$ or $C_K(M) \leq L$. If $C_K(M) = K$, then $M$ has a direct complement $N$ in $L$, and we consider $\overline{K} = K/N$. Note that

$$1 < \overline{M} \leq Z(\overline{K}) \cap \overline{K}$$

since $K = K'$ is perfect, and hence $|M|$ divides $|M(G)|$ by [6, Corollary 11.20], where $M(G)$ denotes the Schur multiplier of $G$. But this is impossible since $|M(A_5)| = 2$ and $M(PSL_2(8)) = 1$.

Hence $C_K(M) \leq L$, and the action of $K$ on $M$ is non-trivial. Moreover, $K/L$ has even order, so by Lemma 2.9, there are an element $\lambda \in \overline{M}$ and $x \in K$ such that $\lambda^x = \overline{\lambda}$. Let $I = I_G(\lambda)$, and note that $x \in N_G(I) \setminus I$, so $2$ divides $[G : I]$.

Let $\overline{I} = I/\ker(\lambda)$ (we remark that the “bar” notation here is not the same as in first part of the proof), and observe that $\overline{M} \leq Z(\overline{I})$. Take $P \in \text{Syl}_p(I)$; since the index of $K$ in $G$ is a 2-power, every subgroup of $G$ with odd order is contained in $K$; hence $P \leq K$. Moreover, $\overline{M} \leq Z(\overline{P})$, $\overline{P} \in \text{Syl}_p(\overline{I})$, and $PL/L$ is a $p$-subgroup of the simple group $K/L$ that is isomorphic to $A_5$ or $PSL_2(8)$. Now, if $p$ is an odd prime, every Sylow $p$-subgroup of $A_5$ or $PSL_2(8)$ is cyclic (see Tables 1 and 2). Hence $P/M \simeq \overline{P}/\overline{M} \simeq PL/L$ is cyclic and $\overline{P}$ is abelian.

Since $\overline{M} \leq Z(\overline{I})$, we have that $\overline{M} \nleq \overline{I}'$ by [8, Theorem 5.3]. In addition, we have $\overline{M} \cap \overline{I}' = 1$ because $\overline{M}$ has order $p$. Write

$$\overline{I}/\overline{I}' = Q \times B,$$

where $B \in \text{Hall}_p'(\overline{I}/\overline{I}')$ and $Q \in \text{Syl}_p(\overline{I}/\overline{I}')$.

Note that $Q$ and $B$ are $x$-invariant as $x$ normalizes $I$. By abuse of notation, we write $M \leq Q$ in the place of $\overline{M} \overline{I}'/\overline{I}' \leq Q$. In this notation, $M$ is a group of order $p$, and $\lambda$ is a faithful character of $\overline{M}$. The 2-group $\langle x \rangle$ acts on the abelian group $Q$; hence by Maschke’s Theorem [10, Theorem 8.4.6], there is an $\langle x \rangle$-invariant complement $T$ for the $\langle x \rangle$-invariant subgroup $M$, so $Q = M \times T$. Letting $\hat{\lambda} = \lambda \times 1_T \in \text{Irr}(Q)$ and $\delta = \hat{\lambda} \times 1_B \in \text{Irr}(\overline{I}/\overline{I}')$, we have that

$$\delta^x = \hat{\lambda}^x \times 1_B^x = (\lambda^x \times 1_T^x) \times 1_B = (\hat{\lambda} \times 1_T) \times 1_B = \delta.$$

We return to the previous notation, so $\delta$ lifts to a character of $I$ that we also call $\delta$. Note that $I < G$ as 2 divides $[G : I]$.

If $IH < G$, then $IH/H$ is a proper subgroup of $G/H$ that is a simple group isomorphic to $A_5$ or $PSL_2(8)$. The maximal subgroups of these two groups are known as well as their indexers; see Tables 1 and 2. In particular, there always is an odd prime $q$ such that $q$ divides $[G : IH]$, and hence $2q$ divides $[G : I]$. Note that $\delta \in \text{Irr}(I \mid \lambda)$, so $\chi = \delta^G \in \text{Irr}(G)$. Moreover,

$$\overline{\chi} = (\overline{\delta})^G = (\delta^x)^G = \delta^G = \chi.$$
Hence $\chi$ is a real character of $G$, and $2q \mid \chi(1)$ since $2q \mid [G : I]$, and this is impossible.

Suppose now $IH = G$. In this case, $I/I \cap H \simeq G/H$, which is isomorphic to $A_5$ or $PSL_2(8)$. These groups have a unique rational character $\phi$ of odd degree. The element $x$ stabilizes the section $I/I \cap H$; hence by uniqueness, $\phi^x = \phi$. By Gallagher’s Theorem [6, Theorem 6.17], $\phi \delta \in \text{Irr}(I \mid \lambda)$, and by the Clifford correspondence, $\chi = (\phi \delta)^G \in \text{Irr}(G)$. Since $\phi$ is a real $x$-invariant character and $\delta^x = \delta$, we have that $(\phi \delta)^x = \phi \delta$. Hence, as before, $\chi$ is a real irreducible character. Now $\theta(1) \mid \chi(1)$, and there is an odd prime $q$ such that $q$ divides $\chi(1)$. Moreover, $2 \mid \chi(1)$ since $2$ divides $[G : I]$. So $\chi(1)$ is a composite number, and this is impossible. 

We give the list of maximal subgroups of $A_5$ and $PSL_2(8)$ and their indices.

| $A_4$ | $D_{10}$ | $S_3$ | $F_{56}$ | $D_{18}$ | $D_{14}$ |
|-------|------|-------|-------|-------|-------|
| 12    | 10   | 6     | 56    | 18    | 14    |
| 5     | 6    | 10    | 9     | 28    | 72    |

Table 1. Maximal subgroups of $A_5$. Table 2. Maximal subgroups of $PSL_2(8)$.

**Lemma 3.3.** Let $K$ be a perfect group and $M$ a minimal normal subgroup of $K$ that is an elementary abelian 2-group. Suppose that $M$ is non-central in $K$ and $K/M$ is isomorphic to $L_2(8)$ or $A_5$. Then $K$ has an irreducible non-linear real character with odd composite degree.

**Proof.** Since $G/M$ is simple, we have that $C_G(M) = M$. Suppose that $K/M$ is isomorphic to $A_5$. There are two non-isomorphic irreducible $A_5$-modules $W_1, W_2$ of $A_5$ over $GF(2)$. Both have dimension 4, and $H^2(A_5, W_1) = H^2(A_5, W_2) = 0$. Hence $M$ has a complement $S$ in $K$. It is easy to construct these groups, and we see that $K = M \rtimes S = W_i \rtimes A_5$ has a real irreducible character of degree 15. Suppose now that $K/M \simeq L_2(8)$. Let $W_1, W_2, W_3$ be the non-isomorphic irreducible $L_2(8)$-modules over $GF(2)$, where $\dim(W_1) = 6$, $\dim(W_2) = 8$, $\dim(W_3) = 12$. If $M \simeq M_i$ with $i = 2, 3$, then $H^2(L_2(8), W_i) = 0$, and hence $M_i$ has a complement $S$ in $K$. Then, as before, we conclude by observing that $W_i \rtimes L_2(8)$ has a real irreducible character of degree 63. Suppose that $M \simeq W_1$. Then we have $\dim H^2(L_2(8), W_1) = 3$. Nevertheless, there are just two perfect groups of order $2^6 \cdot \lvert L_2(8) \rvert$. Both these groups have an irreducible real character of degree 63. 

In the previous lemma, dimensions of cohomology groups and all the perfect groups of a given order is information that is accessible with the GAP functions cohomolo and PerfectGroup.
Proposition 3.4. Let $G$ be a finite non-solvable group, and suppose that $cd_{rv}(G)$ consists of prime-power numbers. Let $K = G^{(\infty)}$ and $R = \text{Rad}(G)$. Then we have $|K \cap R| \leq 2$, and if equality holds, then $K \simeq \text{SL}_2(5)$.

Proof. By Proposition 3.2, we have that $N = K \cap R$ is a 2-group. We prove that if $N > 1$, then $|N| = 2$ and $K$ is isomorphic to $\text{SL}_2(5)$. Let $V = N/\Phi(N)$; then $V$ is a normal elementary abelian 2-subgroup of $G/\Phi(N)$. Let $V > V_1 > \cdots > V_n$ be a $K$-principal series of $V$. Let $N > N_1 > \cdots > N_n$ be such that $N_i$ is the preimage in $N$ of $V_i$. Then $N/N_1$ is an irreducible $K/N$-module, and $K/N$ is isomorphic $A_5$ or $L_2(8)$ by Proposition 3.2. By Lemmas 3.3 and 2.7, $N/N_1$ is central in $K/N_1$. Since $K$ is perfect, we have that $N/N_1$ is isomorphic to a subgroup of the Schur multiplier $M(K/N)$. The only possibility is $|N/N_1| = 2$ and $K/N_1 \simeq \text{SL}_2(5)$, the Schur covering of $A_5$. Suppose by contradiction that $N_1/N_2 > 1$; write $\tilde{K} = K/N_2$. Since $M(\text{SL}_2(5)) = 1$, $\tilde{N}_1$ cannot be central in $\tilde{K}$. Let $t \in K$ be a 2-element such that $\langle tN_1 \rangle = Z(K/N_1)$, namely the unique central involution in $\text{SL}_2(5)$ and $\langle tN_1 \rangle = O_2(K/N_1)$. Since $N_1$ is an irreducible module over $\text{GF}(2)$, we have that $t$ acts trivially on $\tilde{N}_1$. Suppose that $\tilde{t}^2 \neq 1$; then $\langle \tilde{t}^2 \rangle$ would be a proper, non-trivial submodule of $\tilde{N}_1$, against irreducibility. This means that $\tilde{t}^2 = 1$ and hence $\langle \tilde{t} \rangle$, which centralizes $\tilde{N}_2$, is a minimal normal subgroup of $\tilde{N}_2$. Observe that $\tilde{K}/(t)$ is a quotient of $K$ that satisfies the hypotheses of Lemma 3.3. Hence, by Lemma 2.7, we derive a contradiction. \qed

We now prove Theorem A, which we restate for convenience of the reader.

Theorem 3.5. Let $G$ be a finite non-solvable group, and suppose that $cd_{rv}(G)$ consists of prime-power numbers. Then $\text{Rad}(G) = H \times O$ for a group $O$ of odd order and a 2-group $H$ of Chillag–Mann type. Furthermore, if $K = G^{(\infty)}$, then one of the following holds:

(i) $G = K \times R$, and $K$ is isomorphic to $A_5$ or $L_2(8)$;

(ii) $G = (KH) \times O$, where $K \simeq \text{SL}_2(5)$, $HK = H \gamma K$, $K \cap H = Z(K) < H$.

Proof. By Proposition 3.4 and Proposition 3.2, if $K = G^{(\infty)}$ and $R = \text{Rad}(G)$, then $G = KR$, and either $K \cap R = 1$ and $K$ is simple isomorphic to $A_5$ or $L_2(8)$ or $K \simeq \text{SL}_2(5)$ and $K \cap R = Z(K)$. In the first case, point (i) follows. Suppose $K = \text{SL}_2(5)$ and $K \cap R = Z(K) = Z$. Note that $Z$ is a normal subgroup of order 2, hence is central in $R$. Consider $\tilde{G} = G/Z$. Then $\tilde{G} = \tilde{K} \times \tilde{R}$, and hence $\tilde{R}$ is a group of Chillag–Mann type since $\tilde{K}$ is simple and has irreducible real non-linear characters of both odd and even degree. This means that $\tilde{R} = \tilde{O} \times \tilde{H}$ for $O \in \text{Hall}_2(R)$ and $H \in \text{Syl}_2(R)$; note that $\tilde{H}$ is of Chillag–Mann type. We have
that $R$ is 2-closed, and hence $R = H \rtimes O$. Clearly, $O$ acts trivially on $H/Z$, so

$$H = C_H(O)Z \leq C_H(O)Z(R) \cap H.$$  

It follows that $O$ centralizes $H$ and $R = H \times O$. By the Dedekind modular law, $HK \cap O \leq HK \cap R \leq H(K \cap R) \leq H$, and hence $HK \cap O \leq H \cap O = 1$. This means that $G = (KH) \times O$ and $K \cap H = Z$, which has order 2. If $H$ and $K$ commute, then $KH = K \gamma H$. Suppose by contradiction that $[H, K] = Z$; hence there is a $Z \in H/Z$ that acts non-trivially by conjugation on $K/Z$. But this is impossible since $KH/Z = \tilde{K} \times \tilde{H}$. We now prove that $H$ is of Chillag–Mann type. Suppose that this is not the case, so there is $\theta \in \text{Irr}_r(H)$ such that $\theta(1) > 1$. Since $H$ is of Chillag–Mann type, we have that $Z \notin \text{ker} \phi$, so $\phi_Z = \phi(1)\lambda$, with $\lambda \neq 1_Z$. On the other hand, if $\theta$ is the unique character of $K$ of degree 6, then $Z \notin \text{ker} \theta$ and $\theta Z = \theta(1)\lambda$. Now, $KH = K \times H/N$, where $N = \{(z, z) \mid z \in Z\}$ (see [5, I9.10]) and $\psi = \theta \times \phi \in \text{Irr}_r(K \times H)$. Moreover,

$$\psi_N = \phi(1)\theta(1)\lambda^2 = \phi(1)\theta(1)1_N,$$

so it follows that $N \leq \ker \psi$ and $\psi \in \text{Irr}_r(KH)$. If $\chi = \psi \times 1_O$, then $\chi \in \text{Irr}(G)$ takes real values and has composite degree, which is impossible. Since $\text{SL}_2(5)$ does not satisfy the hypotheses, we have that $Z < H$. Point (ii) follows.

We remark that, in [6, Problem 4.4], we can find a stronger version of the argument used in the proof above.

As a consequence, we get Theorem B.

**Corollary 3.6.** Let $G$ be a non-solvable group, and suppose that $cd_{rv}(G)$ consists of prime-power numbers. Then either

$$cd_{rv}(G) = cd_{rv}(L_2(8)) \quad \text{or} \quad cd_{rv,2'}(G) = cd_{rv,2'}(A_5).$$

**Proof.** Apply Theorem 3.5. In case (i), there is nothing to prove. Suppose (ii). Then we have that $G = (KH) \times O$ with $O$ of odd order, $K = G^{(\infty)}$ and $H$ a normal 2-subgroup. Let $S$ denote the simple section $KH/H$; hence $S \simeq A_5$. Take $\chi \in \text{Irr}_r(G)$ to be a real non-linear character of odd degree. Hence $\chi(1) = p^n$ with $p$ odd and $\chi$ is a character of $HK$ since, by Lemma 2.5, $O \leq \ker(\chi)$. The degree of every irreducible constituent of $\chi_H$ divides $|H|$, $\chi(1) = 1$, and hence $\chi_H = e \sum_i \lambda_i$ for $\lambda_i \in \text{Lin}(H)$. By hypothesis, we have that $\chi(1) = p^* > 1$ for an odd prime $p$, and by [6, Corollary 11.29], $\chi(1)/\lambda(1)$ divides $[HK : H] = |S|$, where $S \simeq A_5$. Hence $p \leq \chi(1) \leq |S|_p$, the $p$-part of the number $|S|$, which is equal to $p$ if $p$ is an odd prime. It follows that $\chi(1) = p$. We have proved that $cd_{rv,2'}(G) \subseteq \{3, 5\} = cd_{rv,2'}(A_5)$. The right-to-left inclusion follows by observing that $A_5$ is a quotient of $G$. 

\[ \square \]
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