Pneumatic Artificial Muscle Actuator under Parametric and Hard Excitations

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Abstract: In this work, a single degree of freedom system consisting of a mass and a Pneumatic Artificial Muscle (PAM) subjected to time varying pressure inside the muscle is considered. The system is subjected to hard excitation and the governing equation of motion is found to be that of a nonlinear forced and parametrically excited system under super- and sub-harmonic resonance conditions. The solution of the nonlinear governing equation of motion is obtained using the method of multiple scales (MMS). The time and frequency response, phase portraits and basin of attraction have been plotted to study the system response along with the stability and bifurcations. Further, the different muscle parameters have been evaluated by performing experiments which are further used for numerically evaluating the system response using the theoretically obtained closed form equations. The responses obtained from the experiments are found to be in good agreement with those obtained from the method of multiple scales. With the help of examples, the procedure to obtain the safe operating range of different system parameters have been illustrated.

Keywords: pneumatic artificial muscle, parametrically excited, hard excitation, method of multiple scales, stability, bifurcations.

1. Introduction

Pneumatic Artificial Muscle (PAM) is an actuator that converts the pneumatic pressure attains from the air pressure to a linear pulling force. PAM offers a number of actuator features which are advantageous compare to the other conventional pneumatic actuators. The main advantages of these muscle actuators are high force to mass ratio, soft and flexible structure, lightweight and cheaper, large pulling forces with the least amount of compressed-air consumption [1, 2]. Due to these advantages, PAMs have been widely used in the field of bio-robotic as well as in industrial applications for its easy installation with safer human interactions. There are different types of

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artificial muscles available based on the use of rubber or similar types of elastic materials, viz., McKibben muscle [1-3], Rubbertuator made by Bridgestone company [4, 5], Air Muscle made by Shadow Robot company [6], PAM made by Festo company [7, 8], Pleated PAM developed by Vrije University of Brussel [9], ROMAC (RObotic Muscle ACtuator), Yarlott, Kukolj and many more [10]. Basically, these muscle actuators consist of a thin rubber tube (bladder) which is covered or embedded with a braided mesh shell. Both the ends of the muscle are closed where one end is used as air inlet and another is connected to the load. When the compressed air is supplied to the muscle through the inlet port, the internal bladder tends to increase its volume against the braided mesh shell. But the non-extensibility of the braided mesh threads causes the actuator to shorten and produce pulling forces if it is connected with the load. However, due to the presence of compressibility of air and natural properties of viscoelastic material, a high nonlinear characteristic is found in the PAM which makes them difficult to model and control. These qualities and drawbacks make the PAM as an attractive topic for many researchers and industry to study the behavior in various environments.

There are different models that have been described in the literature to understand the nonlinear nature present in the PAM. Chou and Hannaford [1], along with Tondu and Lopez [3] used virtual work for modeling of muscle force output and their models are based on conservation of energy in the system. Zhang et al. [11] proposed a load-dependent dynamic hysteresis model for PAM with experimental validation to predict the hysteresis behaviors with both rate-dependent and load-dependent effects. Wickramatunge and Leephakpreeda [8] derived an empirical model experimentally to understand the mechanical behaviour of the PAMs. Another experimental model is obtained by Li et al. [12], which establish a relation between the operating air pressures, muscle force along with the contraction of the PAM. The dynamic model of the PAM is very difficult for practical control and use because of the presence of high nonlinearity. Zhao et al. [13] modeled a nonlinear extended state observer (NESO) for double-joint manipulator systems which is actuated by artificial muscles with a trajectory tracking control strategy. Further, a motion mechanism of PAM based on NESO is studied by considering dead-zones of the muscles with an adaptive control method [14]. Chen et al. [15] proposed a neuroadaptive control method to handle the system nonlinearities and employed it on an experimental test-bed attached with the PAM to confirm its efficiency and robustness. Moreover, the major parameters of the PAM like pulling force, material properties, length and diameter are also added more nonlinearity to the dynamic behavior of the
muscle [16]. Kalita and Dwivedy [17-20] studied different PAM actuated single degree of system with weak, hard, forced and parametrically excited systems with different resonance conditions. Here, the experimentally obtained different models of system parameters have been considered.

Many researchers studied the response behavior of various dynamical systems at super- and sub-harmonic resonance conditions. Krishnan [21] studied the super- and sub-harmonic resonance conditions for strongly excited non-linear systems with the help of the difference equations. The super- and sub-harmonic resonance conditions along with the combination resonances, have been studied by Elnaggar and El-Basyouny [22] for the additive type of self-excited two coupled-second order systems subjected to multi-frequency excitation. These resonance conditions for the effect of fast harmonic base displacement with periodically time varying stiffness on Vibro impact dynamics of a forced single-sided Hertzian contact oscillator have been investigated by Bichri and Belhaq [23]. Davis and Rajan [24, 25] discussed the random sub and superharmonic resonance conditions extensively with the help of a Duffing oscillator. Cai and Yang [26] and Zhang and Yang [27] studied the bifurcations and chaos in Duffing Equation for the super- and sub-harmonic resonance conditions. An extensive study has been done for Duffing's equation in the literature for these resonance conditions to understand the bifurcations and stability [28-31] of the systems. Further, super- and sub-harmonic resonance conditions have been studied for some applications like in spur gear [32], cracked beams [33, 34], piezoelectric sandwich beams [35], elastic cables [36], Cartesian manipulator [37], wind turbine blades [38] and marine applications [39]. The theoretical and analytical results are investigated with the help of the method of multiple scales (MMS). A detail procedure of the method of multiple scales (MMS) can be found in the book of Nayfeh and Mook [40] and Nayfeh [41], which is a perturbation technique.

The muscle dynamics under simultaneous resonance condition has been explained by Kalita and Dwivedy [17] where the external excitation frequency of the system is nearly equal to the natural frequency of the system. In another work [18], they studied the nonlinear dynamics of the PAM under principal parametric resonance condition when the external excitation frequency of the system is nearly twice the natural frequency of the system. In the experimental work of Kalita and Dwivedy [19], they investigated a system attached with a nonlinear PAM and a spring at a constant pressure inside the muscle, which is subjected to simple resonance condition. It may also be noted that in most of the literature, the forcing is considered to be weak. So, a similar system with hard
excitation has been studied by Kalita and Dwivedy [20] where a constant pressure inside the muscle gives rise to super- and sub-harmonic resonance conditions. In the present work, an attempt has been made to study the hard excitation of a parametrically excited pneumatically actuated artificial muscle system where a time varying muscle force is considered which gives rise to the super- and sub-harmonic resonance conditions. It may be noted that as the superposition rule cannot be applied to the nonlinear systems, hence, it is not possible to extend the work considered in [17-20] to obtain the results reported in this work.

From these literature and to the best of the authors’ knowledge, it has been observed that no study has been made to investigate the artificial muscle under parametric and hard excitation considering super- and sub-harmonic resonance conditions. Hence, in this work, an attempt has been made to study the response of a single degree of freedom system with a PAM having periodically time varying pressure leading to a forced and parametrically excited system with super- and sub-harmonic resonance conditions. As mentioned earlier, it may be noted that the resonance conditions considered in this work are completely different from the earlier works of the authors [17-20]. Here, a second order nonlinear governing equation of motion with hard excitation is derived, which is solved by using the method of multiple scales to obtain the response of the system. This work will find several mechanical and bio-mechanical applications where the system is actuated through PAM, which include, the movement of upper arm or lower arm for elderly persons or persons with semi-paralytic disabilities [42-44] and for robotic manipulation in industrial applications [45, 46]. Also, for PAM actuated medical instrumentation for laparoscopic surgery [12] or keyhole surgery [47], the present analysis will be useful, where more precision is required to control the muscle actuator. The mathematical modeling along with the analytical solutions, have been discussed in the following sections.

2. Mathematical model of the system

Figure 1(a) shows a mass $m$ connected to a PAM with control valve circuit. The equivalent single degree of freedom system is shown in Fig. 1(b) where the PAM is modeled as a nonlinear spring and a damper, subjected to a muscle force $F_{mus}$. 
The expression for $F_{\text{mus}}$ is given by the following equation which is similar to the work of Li et al. [12] and Kalita and Dwivedy [17, 18].

$$F_{\text{mus}}(u, P) = \left( c_1 + c_2 P + c_3 P^2 \right) \left( \frac{u}{l_{\text{max}}} \right) + (d_1 + d_2 P) + Ku^3 \quad (1)$$

where $c_1$, $c_2$, $c_3$, $d_1$, $d_2$ and $K$ are the constants which can be obtained from the experiments. $P$ is the operating pressure in the muscle to actuate $u$ is the displacement with respect to the unstressed position of the system and $l_{\text{max}}$ is the maximum possible length that can be attained by the muscle. In this case, $P = P_m + P_0 \sin \omega t$ is considered, where $P_m$ and $P_0$ are static and dynamic pressure inside the muscle, respectively along with the frequency of the dynamic pressure is $\omega$.

Figure 1(c) shows the free-body diagram to derive the equation of motion, which can be given by the following equation.

$$m\ddot{u} + c\dot{u} + \left( c_1 + c_2 P + c_3 P^2 \right) \left( \frac{u}{l_{\text{max}}} \right) + (d_1 + d_2 P) + Ku^3 - mg = 0 \quad (2)$$

The overdot signifies the differentiation with respect to time $t$. Eq. (2) can be rearranged as follows.
\[ \ddot{u} + \frac{c}{m} \dot{u} + \left[ \frac{(c_1 + c_2 P + c_3 P^2)}{m l_{\text{max}}} \right] u + \frac{K}{m} u^3 = \left[ g - \frac{(d_1 + d_2 P)}{m} \right] \]  

(3)

Now, considering a non-dimensional time \( \tau = \omega_0 t \), where the fundamental natural frequency \( \omega_0 \) of the system can be written as below.

\[ \omega_0 = \sqrt{\frac{c_1 + c_2 P_m + c_3 \left( \frac{P^2}{2} + \frac{P_0^2}{2} \right)}{m l_{\text{max}}}} \]

(4)

The nondimensional displacement can be written as \( u = r x \) by using a scaling factor \( r \) and the following non-dimensional parameters have been used for formulation.

\[
\begin{align*}
\Omega & = \frac{\omega}{\omega_0}, \quad \mu = \frac{c}{2 \varepsilon m \omega_0}, \quad p_1 = \frac{c_2 P_0 + 2 c_3 P_0 P_m}{\varepsilon m \omega_0^2 l_{\text{max}}}, \quad p_2 = -\frac{c_3 P_0^2}{2 \varepsilon m \omega_0 L_{\text{max}}}, \\
\dot{f}_1 & = \frac{d_1 P_m + d_1 - mg}{m \varepsilon \omega_0^2}, \quad \dot{f}_2 = \frac{d_1 P_0}{m \varepsilon \omega_0}, \quad \alpha = \frac{r^2 K}{\varepsilon \omega_0^2}
\end{align*}
\]

(5)

Now, Eq. (3) can be simplified to the temporal equation of motion as follows.

\[
\frac{d^2 x}{d \tau^2} + 2 \varepsilon \mu \frac{dx}{d \tau} + x + \varepsilon \left( p_1 \sin \Omega \tau + p_2 \cos 2\Omega \tau \right) x + \varepsilon \alpha x^3 = \left( f_1 + f_2 \sin \Omega \tau \right)
\]

(6)

In Eq. (6), the book keeping parameter \( \varepsilon \) is less than 1 and \( \mu \) is the non-dimensional damping parameter. It may be noted that the nondimensional parameters \( p_1, p_2 \) and \( f_2 \) are the function of \( P_0 \); \( p_1 \) and \( f_1 \) are the function of \( P_m \). From the fourth term on the left-hand side of Eq. (6), it may clearly be observed that the coefficients of the response \( x \) contain time-varying terms with frequencies, \( \Omega \) and \( 2\Omega \). Hence, this is a parametrically excited system with multi frequency excitation. Finally, the last term of the left hand side contains a cubic nonlinear term with the coefficient \( \alpha \). In addition to this, the system is also a forced vibration system having a sinusoidally varying force of amplitude \( f_2 \) and frequency \( \Omega \). So, in this work, the governing equation Eq. (6) with cubic nonlinearity is subjected to hard excitation, where the magnitude of the forcing term is larger than that of the coefficient of the linear term. It may be noted that in the authors previous work [17, 18], the forcing is considered to be weak and the forcing parameter \( f_1 \) and \( f_2 \) are the function of the static pressure \( P_m \) and dynamic pressure \( P_0 \), respectively. Also, in some other works
by Kalita and Dwivedy [19, 20], the authors investigated the nonlinear characteristics of PAM with weak [19] and hard [20] excitation by considering constant pressure $P$ inside the muscle.

Due to the presence of various nonlinear terms in Eq. (6), it is very difficult to find the exact solution. So, there is a need to use the perturbation technique to find the approximate solution. Hence, the method of multiple scales [40, 41] is used to find out the solution of the equation of motion Eq. (6) where the displacement $x$ can be written as

$$x(t; \varepsilon) = x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) + O(\varepsilon^3)$$

in terms of different time scales $(T_0, T_1, \ldots)$ where $T_n = \varepsilon^n t, n = 0, 1$ with a book keeping parameter $\varepsilon$. Following standard procedure of method of multiple scales, the governing equation Eq. (6) can now be written as follows.

$$D_0^2 x_0 + \varepsilon D_0^2 x_1 + 2\varepsilon D_0 D_1 x_0 + 2\varepsilon \mu D_0 x_0 + \varepsilon x_1 + \varepsilon x_0 \left( p_1 \sin \Omega \tau + p_2 \cos 2\Omega \tau \right)$$

$$+ \varepsilon \alpha x_0^3 = f_1 + f_2 \sin \Omega \tau$$  \hspace{1cm} (7)

Equating the coefficient of $\varepsilon^0$ and $\varepsilon^1$ from Eq. (7), the following equations are found respectively.

$$D_0^2 x_0 + x_0 = f_1 + f_2 \sin \Omega \tau$$ \hspace{1cm} (8)

$$D_0^2 x_1 + x_1 = -2D_0 D_1 x_0 - 2\mu D_0 x_0 - x_0 p_1 \sin \Omega T_0 - x_0 p_2 \cos 2\Omega T_0 - \alpha x_0^3$$ \hspace{1cm} (9)

The general solution of Eq. (8) can be expressed as follows.

$$x_0 = \frac{f_1}{2} + A e^{i\tau_0} + \Lambda e^{i\Omega \tau_0} + \frac{f_1}{2} + \bar{A} e^{-i\tau_0} + \Lambda e^{-i\Omega \tau_0}$$ \hspace{1cm} (10)

where, $\Lambda = \frac{f_2}{2(1-\Omega^2)}$. $A$ is a complex number and $\bar{A}$ is the complex conjugate of $A$. Substituting the value of $x_0$ from Eq. (10) in Eq. (9) leads to the following equation.

$$D_0^2 x_1 + x_1 = -2iD_1 A e^{i\tau_0} - 2i\Omega D_1 \Lambda e^{i\Omega \tau_0} - 2i\mu A e^{i\tau_0} - 2i\Omega \mu \Lambda e^{i\Omega \tau_0} + \frac{i\Omega p_1}{2} \left[ \frac{f_1}{2} e^{i\Omega \tau_0} - \frac{f_1}{2} e^{-i\Omega \tau_0} \right]$$

$$+ A e^{(1+i)\tau_0} - A e^{(1-i)\tau_0} + \Lambda e^{i2\Omega \tau_0} - \Lambda \right] - \frac{p_2}{2} \left[ \frac{f_1}{2} e^{i2\Omega \tau_0} + \frac{f_1}{2} e^{-i2\Omega \tau_0} + A e^{i(1+2i)\tau_0} + A e^{i(1-2i)\tau_0} \right]$$

$$+ \Lambda e^{i\Omega \tau_0} + \Lambda e^{-i\Omega \tau_0} \right] - \alpha \left[ 3A^2 e^{i\Omega \tau_0} + A^3 e^{i3\Omega \tau_0} + 3A^3 e^{i3\Omega \tau_0} + \frac{1}{2} f_1^2 + 3f_1 A^2 + 3A f_1 \Lambda^2 + 3A f_1 e^{2i\Omega \tau_0} + 6A A \Lambda e^{i\Omega \tau_0} + 3A^2 e^{i\Omega \tau_0} e^{-2i\Omega \tau_0}$$

$$+ 3A^2 e^{i\Omega \tau_0} e^{-2i\Omega \tau_0} + 3A^2 \Lambda e^{i2\Omega \tau_0} e^{i\Omega \tau_0} + 3A^2 \Lambda e^{i2\Omega \tau_0} e^{-i\Omega \tau_0} + 6A f_1 \Lambda e^{i\Omega \tau_0} e^{i\Omega \tau_0} + 6A f_1 \Lambda e^{i\Omega \tau_0} e^{-i\Omega \tau_0} \right] + CC$$
where $CC$ stands for the complex conjugate of the proceeding terms. It can be observed from the Eq. (11), that it will contain secular or nearly secular terms when $\Omega \approx 1$, $\Omega \approx \frac{1}{3}$ or $\Omega \approx 3$. The particular solution of these terms will be unbounded and therefore should be eliminated. From Eq. (11) it may be noted that when $\Omega \approx 1$, the simple resonance condition may occur and but as $\Lambda \sim 0\infty$, the system will exhibit a response similar to that of a linear system. Hence, it is not explored in this work. The other two resonance conditions $\Omega \approx \frac{1}{3}$ and $\Omega \approx 3$ are considered in this work, which are known as super- and sub-harmonic resonance conditions, respectively. It may be noted that, the terms, i.e., $\frac{p_2}{2} \Lambda e^{i3\pi T_0}$ and $\alpha \Lambda^3 e^{3\pi T_0}$ will contribute to superharmonic resonance condition and the complex conjugate of the term $3\alpha A^2 \Lambda e^{i(2-\Omega)T_0}$ will lead to the subharmonic resonance condition. These two resonance cases have been studied in the following subsections.

2.1 Superharmonic resonance condition

In this case, the external excitation frequency $\Omega$ is nearly equal to the one third of the natural frequency of the system, which is known to be superharmonic resonance condition. Here, one may introduce the detuning parameter $\sigma$ as $\Omega = \frac{1}{3} + \varepsilon \sigma$ and substituting in Eq. (11), the following equation can be obtained.

$$D_i^2 x_i + x_i = -2iD_iA e^{i\pi_0} - 2i\Omega D_i\Lambda e^{i\pi_0} + 2i\mu A e^{i\pi_0} - 2i\Omega \mu \Lambda e^{i\pi_0} + \frac{i\varepsilon}{2} \left[ f_1 e^{i\frac{1}{3}e\pi_0} + f_2 e^{-i\frac{1}{3}e\pi_0} \right] - \Lambda \left[ -\frac{p_2}{2} \left[ f_1 e^{i\frac{1}{3}e\pi_0} + f_2 e^{-i\frac{1}{3}e\pi_0} \right] e^{i\pi_0} + A e^{i\frac{1}{3}e\pi_0} e^{i\pi_0} + A e^{-i\frac{1}{3}e\pi_0} e^{i\pi_0} \right] - \alpha \left[ 3A^3 e^{i\frac{1}{3}e\pi_0} e^{i\pi_0} + A e^{i\frac{1}{3}e\pi_0} e^{i\pi_0} + 3A^2 e^{i3\pi_0} e^{i\pi_0} + A e^{i\frac{1}{3}e\pi_0} e^{i\pi_0} \right] - 3\Lambda^2 e^{i\frac{1}{3}e\pi_0} e^{i\pi_0} + 3\Lambda^2 e^{i\frac{1}{3}e\pi_0} e^{i\pi_0} + 3A^2 \Lambda e^{i\pi_0} + 3A^2 e^{i\pi_0} e^{i2\pi_0} + 3A^2 e^{i\frac{1}{3}e\pi_0} e^{i\pi_0} + 6A \Lambda e^{i\pi_0} + 6A \Lambda e^{i\pi_0} e^{i\pi_0} + 3A e^{i\frac{1}{3}e\pi_0} e^{i\pi_0} + 3A^2 e^{i\frac{1}{3}e\pi_0} e^{i\pi_0} + 3A^2 e^{i2\pi_0} e^{i\pi_0} + 6A \Lambda e^{i\pi_0} e^{i\pi_0}$$

$$+6A f_1 \Lambda e^{i\pi_0} e^{i\pi_0} + 6A f_1 \Lambda e^{i\pi_0} e^{i\pi_0} + 6A^2 \Lambda e^{i\pi_0} e^{i\pi_0} + 6A f_1 \Lambda e^{i\pi_0} e^{i\pi_0} + 6A f_1 \Lambda e^{i\pi_0} e^{i\pi_0} + 6A f_1 \Lambda e^{i\pi_0} e^{i\pi_0} + CC$$

(12)
The secular and nearly secular terms in Eq. (12), which should be eliminated to have bounded solution are given below.

\[-2iD_t A - 2i \mu A - \Lambda e^{3i\sigma T_0} - \alpha \Lambda^2 e^{3i\sigma T_0} - 3\alpha A^2 A - 3\alpha A f_1^2 - 6\alpha A \Lambda^2 = 0\]  

(13)

\[D_t A = -\mu A + i/2 \Lambda e^{3i\sigma T_0} + i/2 \alpha \Lambda^3 e^{3i\sigma T_0} + 3i/2 \alpha A^2 A + 3i/2 \alpha A f_1^2 + 3i\alpha A \Lambda^2\]  

(14)

Again,

\[A' = \varepsilon D_t A = \varepsilon \left( -\mu A + i/2 \Lambda e^{3i\sigma T_0} + i/2 \alpha \Lambda^3 e^{3i\sigma T_0} + 3i/2 \alpha A^2 A + 3i/2 \alpha A f_1^2 + 3i\alpha A \Lambda^2 \right)\]  

(15)

Here, \(A\) is function of \(T_i\) and considering the polar form i.e., \(A = 1/2 ae^{i\beta}\) where \(a\) and \(\beta\) represents the amplitude and phase of the response. Now, by taking \(\gamma = 3\varepsilon \sigma T_0 - \beta\) in Eq. (15) and separating the real and imaginary parts, one may find a set of following reduced equations.

\[a' = \varepsilon \left( -\mu a - \Lambda \sin \gamma - \alpha \Lambda^3 \sin \gamma \right)\]  

(16)

\[a\gamma' = 3\varepsilon \sigma - \varepsilon \left( \Lambda \cos \gamma + \alpha \Lambda^3 \cos \gamma + 3/8 \alpha a^3 + 3/2 \alpha a f_1^2 + 3\alpha a \Lambda^2 \right)\]  

(17)

Now, for steady state response \((a_0, \gamma_0)\), a set of two nonlinear equation can be achieved by putting \(a' = 0\) and \(\gamma' = 0\) in Eq. (16) and Eq. (17) as follows.

\[\left(\Lambda + \alpha \Lambda^3\right) \sin \gamma = -\mu a\]  

(18)

\[\left(\Lambda + \alpha \Lambda^3\right) \cos \gamma = 3\alpha \sigma - 3/8 \alpha a^3 - 3/2 \alpha a f_1^2 - 3\alpha a \Lambda^2\]  

(19)

Squaring and adding these equations Eq. (18) and Eq. (19) leads to the following frequency response equation of the system.

\[\left(\Lambda + \alpha \Lambda^3\right)^2 = \left[\mu^2 + \left(3\sigma - 3/8 \alpha a^2 - 3/2 \alpha f_1^2 - 3\alpha \Lambda^2\right)^2\right] a^2\]  

(20)

From Eq. (20), one may noticed that \(a = 0\) is not a solution. Hence, there is no trivial state response exist in the case of superharmonic resonance condition. But, one may get the nontrivial state response for the system by solving the Eq. (16) and Eq. (17), simultaneously. Hence, substituting \(a = a_0 + a_1\) and \(\gamma = \gamma_0 + \gamma_1\) with \(a_0\) and \(\gamma_0\) are the equilibrium points in Eq. (16) and Eq. (17), the
stability of the steady state response can be found by determining the eigenvalues of the Jacobian matrix \((J)\). The Jacobian matrix \((J)\) is as follows.

\[
J = \begin{pmatrix}
-\mu & -3a_0\sigma + \frac{3}{8}a_0^3 + \frac{3}{2}\alpha a_0 f_1^2 + 3\alpha a_0 \Lambda^2 \\
\frac{3\sigma}{a_0} - \frac{9}{8}a_0 & -\frac{3}{2}a_0 \alpha f_1^2 & -\mu
\end{pmatrix}
\]

(21)

The system will be stable for the superharmonic resonance condition, if the real parts of all the eigenvalues of the Jacobian matrix \((J)\) in Eq. (21) are negative.

\[\text{2.2 Subharmonic resonance condition}\]

In this subharmonic resonance condition, the external excitation frequency \(\Omega\) is nearly equal to the thrice of the natural frequency of the system. Here, considering \(\Omega = 3 + \varepsilon\sigma\), Eq. (11) can be rewritten as follows.

\[
D_0^3 x_i + x_i = -2iD_1 A e^{iT_0} - 2i\Omega D_1 A e^{i(3+\varepsilon\sigma)T_0} - 2i\mu A e^{iT_0} - 2i\Omega\mu\Lambda e^{i(3+\varepsilon\sigma)T_0} + \frac{i\eta}{2} \left[ f_1 e^{i(3+\varepsilon\sigma)T_0} - f_1 e^{-i(3+\varepsilon\sigma)T_0} \right]
\]

(22)

Now, by eliminating the secular and nearly secular terms in Eq. (22), one may obtain the following expression.

\[
-2iD_1 A - 2i\mu A - 3\alpha A^2 \bar{A} - 3\alpha A f_1^2 - 6\alpha \Lambda^2 - 3\alpha A^2 \Lambda e^{i\varepsilon\sigma T_0} = 0
\]

(23)

\[
D_1 A = -\mu A + \frac{3i}{2} \alpha A^2 \bar{A} + \frac{3i}{2} \alpha A f_1^2 + 3i\alpha A \Lambda^2 + \frac{3i}{2} \alpha \bar{A}^2 \Lambda e^{i\varepsilon\sigma T_0}
\]

(24)

Here,

\[
A' = \varepsilon D_1 A = \varepsilon \left( -\mu A + \frac{3i}{2} \alpha A^2 \bar{A} + \frac{3i}{2} \alpha A f_1^2 + 3i\alpha A \Lambda^2 + \frac{3i}{2} \alpha \bar{A}^2 \Lambda e^{i\varepsilon\sigma T_0} \right)
\]

(25)
Now, substituting \( A = \frac{1}{2} a e^{i\beta} \) and \( \gamma = \varepsilon \sigma T_0 - 3\beta \) in Eq. (25) and separating the real and imaginary parts, one may get the following reduced equations.

\[
a' = \varepsilon \left(-\mu a - \frac{3}{4} \alpha a^2 \Lambda \sin \gamma\right) \tag{26}
\]

\[
a\gamma' = a\varepsilon \sigma - 3\varepsilon \left(\frac{3}{8} \alpha a^3 + \frac{3}{2} \alpha a f_i^2 + 3\alpha a \Lambda^2 + \frac{3}{4} \alpha a^2 \Lambda \cos \gamma\right) \tag{27}
\]

For steady state response \((a_0, \gamma_0)\), following similar procedure as that in superharmonic resonance condition, one can obtain the following frequency response equation.

\[
\frac{9}{16} \alpha^2 \Lambda^2 a^4 = \left(\mu^2 + \left(\frac{\sigma}{3} - \frac{3}{8} \alpha a^2 - \frac{3}{2} \alpha f_i^2 - 3\alpha \Lambda^2\right)^2\right) a^2 \tag{28}
\]

From Eq. (28), it may be noted that for subharmonic resonance condition, one may obtain both trivial state (i.e., \( a = 0 \)) and nontrivial state (i.e., \( a \neq 0 \)) responses. Hence, to obtain the nontrivial state response for the system consider \( a = a_0 + a_i \) and \( \gamma = \gamma_0 + \gamma_i \) with \( a_0 \) and \( \gamma_0 \) are the equilibrium points in Eq. (26) and Eq. (27) and the stability of the steady state response can be found by determining the eigenvalues of the Jacobian matrix \((J)\). The Jacobian matrix \((J)\) is as follows.

\[
J = \varepsilon \begin{pmatrix}
\mu & -\frac{a_0 \sigma}{3} + \frac{3}{2} \alpha a_0 f_i^2 + 3\alpha a_0 \Lambda^2 \\
-\frac{9}{4} \alpha a_0 - \frac{3}{8} \alpha a_0 + \frac{9}{2} a_0 \alpha f_i^2 + \frac{9}{8} \alpha a_0 \Lambda^2 & -3\mu
\end{pmatrix} \tag{29}
\]

The system will be stable for the subharmonic resonance condition, if the real parts of all the eigenvalues of the Jacobian matrix \((J)\) in Eq. (29) are negative.

### 2.2.1 Stability of the trivial steady state response

The polar form of modulation Eq. (27) contains term like \( a\gamma' \) but for finding the stability of the trivial state the linearized equation will not contain the perturbation \( \gamma_i' \). So, the stability of the trivial solutions may be found by converting the Eq. (26) and Eq. (27) to their Cartesian form of
modulations by introducing the transformation $p = a \cos \gamma$ and $q = a \sin \gamma$ [40, 41]. Now one may obtain from Eq. (26) and Eq. (27), the resulting Cartesian modulation equations are as follows.

\[ p' = \varepsilon \left( -\mu p + \frac{3}{2} \alpha \lambda p q + \frac{9}{8} \alpha q \left( p^2 + q^2 \right) + \frac{9}{2} \alpha f_i^2 q + 9 \alpha \lambda^2 q \right) - \varepsilon \sigma q \]  

(30)

\[ q' = \varepsilon \left( -\mu q - \frac{3}{4} \alpha \lambda \left( 3 p^2 + q^2 \right) - \frac{9}{8} \alpha p \left( p^2 + q^2 \right) - \frac{9}{2} \alpha f_i^2 p - 9 \alpha \lambda^2 p \right) + \varepsilon \sigma p \]  

(31)

Similarly, the stability of the steady state response $(p_0, q_0)$ for this case, can be achieved by examining the eigenvalues of the Jacobian matrix $(J)$ found by perturbing Eq. (30) and Eq. (31). The Jacobian matrix can be written as follows.

\[ J = \varepsilon \begin{pmatrix} 
-\mu + \frac{3}{2} \alpha \lambda q_0 + \frac{9}{4} \alpha p_0 q_0 & \left( \frac{3}{2} \alpha \lambda p_0 + \frac{9}{8} \alpha p_0^2 + \frac{27}{8} \alpha q_0^2 + \frac{9}{2} \alpha f_i^2 \right) \\
-\frac{9}{2} \alpha \lambda p_0 - \frac{27}{8} \alpha p_0^2 - \frac{9}{8} \alpha q_0^2 - \frac{9}{2} \alpha f_i^2 & +9 \alpha \lambda^2 \\
-9 \alpha \lambda^2 & \left( -\mu - \frac{3}{2} \alpha \lambda q_0 - \frac{9}{4} \alpha p_0 q_0 \right)
\end{pmatrix} \]  

(32)

Here, the system will be stable for trivial state responses when all the real parts the eigenvalues of the Jacobian matrix $(J)$ in Eq. (32) are negative.

### 3. Results and discussions

In this section, the time and frequency responses for the system as shown in Fig.1 have been plotted by taking different set of system parameters values for super- and sub-harmonic resonance conditions. The value of the different system parameters are given in Table 1 which is similar to the work of Li et al. [12] and Kalita and Dwivedy [17, 18]. Here, the steady state responses can be obtained from Eq. (21) for superharmonic resonance condition and Eq. (32) for subharmonic resonance condition from the method of multiple scales. In the following subsections results for super- and sub-harmonic resonance conditions have been discussed with the variation of system parameters. In all the frequency response plots red color depicts the unstable solution and blue color shows the stable solutions.
Table 1. System parameters used for numerical simulation.

| Parameter | Numerical Value | Parameter | Numerical Value | Parameter | Numerical Value |
|-----------|----------------|-----------|----------------|-----------|----------------|
| $I_{\text{max}}$ | 74 mm | $d_1$ | -100 N | $P_0$ | 58 kPa |
| $c_1$ | -234.25 N | $d_2$ | 1 N/kPa | $\varepsilon$ | 0.1 |
| $c_2$ | 1.96 N/kPa | $m$ | 6 N | $\mu$ | 0.01 |
| $c_3$ | -0.003 N/kPa$^2$ | $P_m$ | 300 kPa | $\alpha$ | 150 |

3.1 Superharmonic Resonance Condition

In this section, the responses are plotted for the superharmonic resonance condition, where the external excitation frequency is one third of the natural frequency of the system. A typical frequency response plot is shown Fig. 2. The stability of the steady state response of the system is examined with the help of the Jacobian matrix Eq. (21).

![Frequency response at superharmonic resonance condition](image)

**Fig. 2** Frequency response at superharmonic resonance condition taking system parameters as Table 1.

One may observe two distinct saddle node bifurcation points at $\sigma = 1.718$ (point P) and $\sigma = 3.758$ (point R) in the frequency response plot in Fig. 2. The jump up phenomenon can be noticed at $\sigma = 1.718$, where the unstable solution with amplitude $a = 0.0683$ at point P has always a tendency to reach the stable solution with $a = 0.1337$ at point Q in the stable branch of the nontrivial steady
state response. The maximum response amplitude $a = 0.3499$ for the system can be achieved at point R. Here, it may be noted that to obtain the frequency response using the conventional way of solving equation Eq. (6) numerically by using Runge-Kutta method, for each detuning parameter requires a huge time and memory space. But by using the closed form equation Eq. (20), one can obtain the same results within a fraction of seconds and less memory space. Also, the unstable equilibrium points cannot be obtained by the conventional method.

In Fig. 3, the time response and phase portraits have been plotted with the help of the reduced equations Eq. (16) and Eq. (17) near the mentioned saddle node bifurcation points. The responses are found to be periodic containing multi frequency components.

![Fig. 3 Time response and phase portraits corresponding to Fig. 2 at (a, b) point P (c, d) point R.](image)

In Fig. 4, the validation has been done for the frequency response plot in Fig. 2 by plotting the basin of attraction in the $a - \gamma$ plane with two different values of $\sigma$. At point M ($\sigma = 1$) in Fig. 2, the system has only a stable solution with amplitude $a = 0.0246$, which can be observed from the basin of attraction in Fig. 4(a). For the point N ($\sigma = 2$) marked in Fig. 2, the basin of attraction is shown in Fig. 4(b), where the system exhibits bi-stable nontrivial state where two
stable solutions can be obtained at amplitude $a = 0.022$ and $a = 0.1779$ along with an unstable solution at $a = 0.1586$. From these frequency and basin of attraction plots, it can be noticed that up to the point P ($\sigma = 1.718$) the system has single stable state, between both the saddle node bifurcation points viz., point P and point R the system exhibits bi-stable state and finally after point R ($\sigma = 3.758$) the system has single stable state.

![Fig. 4 Basin of attraction at (a) point M ($\sigma = 1$) (b) point N ($\sigma = 2$) in Fig. 2.](image)

In Fig. 5 (a) the maximum response amplitude of the system increases with increase in the value of the dynamic pressure in the muscle $P_0$ and decreases with decrease in the value of $P_0$. It can be observed that by reducing the value of $P_0$ into half from 58 kPa to 29 kPa, the maximum response amplitude $a = 0.1509$ of the system can be achieved at detuning parameter $\sigma = 1.664$, which is 56.9% decrease. But, doubling in the value of $P_0 = 116$ kPa in Fig. 5(a), the maximum response amplitude $a = 1.417$ at higher value of $\sigma = 40.54$, which is increased by around three times as that in Fig. 2.

Figure 5(b) shows the frequency response plots for two different values of the static pressure in the muscle $P_m$. Here, one may observed that either increase or decrease in the value of $P_m$, the maximum response amplitude is increased. The maximum response amplitude $a = 0.4509$ can be attained at $\sigma = 9.727$ which is 28.9% increase in comparison to Fig. 2 by increasing in the value of $P_m$ from 300 kPa to 400 kPa. With decrease in the value of $P_m = 200$ kPa, the maximum
response amplitude $a = 1.341$ which is also increased by almost three times and can be achieved at higher value of $\sigma = 35.21$.

From Fig. 5(c) with increase in the value of damping parameter $\mu$, the maximum response amplitude is decreased and vice-versa. At $\mu = 0.03$, which is increased as compared to Fig. 2 where $\mu = 0.01$, the maximum response amplitude $a = 0.1165$ at $\sigma = 1.723$ is decreased by 66.7%. But with decrease in the value of $\mu = 0.005$, the maximum response amplitude $a = 0.6999$ at $\sigma = 10.65$ which is increased by double as compared to Fig. 2.

From Fig. 2 and Fig. 5(d), it can be observed that by increasing in the value of nonlinearity $\alpha$, the maximum response amplitude can be achieved at higher value of detuning parameter $\sigma$. A similar observation can be noticed by decreasing the value of $\alpha$ in Fig. 5(d). By doubling the value of $\alpha$ from 150 to 300 in Fig. 5(d), the maximum response amplitude $a = 0.3932$ at $\sigma = 8.725$, which is increased by 12.4% in comparison with Fig. 2. Again by reducing the nonlinearity $\alpha$ to half i.e., $\alpha = 75$, the maximum response amplitude decreased by 6.2% as compared to the saddle node point R in Fig. 2.

Figure 5(e) depicts the frequency response plots for the muscle parameter $c_1$, where the maximum response amplitude decreases with increase or decrease in the value of $c_1$ as compared to Fig. 2. With decrease in the value of $c_1$ from $-234.25$ N to $-468.5$ N, the maximum response amplitude $a = 0.1607$ at $\sigma = 0.858$ which is decreased by 54% as compared to that in Fig. 2. Also, with increase in the value of $c_1 = -117.12$ N, the maximum response amplitude $a = 0.1259$ which is decreased by 64% as compared to Fig. 2 and can be achieved at saddle node bifurcation point at $\sigma = 0.535$ in Fig. 5(e). But from Fig. 5(e), it can be noticed that the maximum response amplitude at saddle node bifurcation point occurs at lower value of $\sigma$ as compared to point R in Fig. 2.

In Fig. 5(f), the frequency response curves have been plotted for the two different values of muscle parameter $c_2$. The maximum response amplitude of the system can be achieved with $a = 0.064$ at $\sigma = 0.142$ and $a = 0.111$ at $\sigma = 0.417$ for the muscle parameter value $c_2 = 2.96$ N/kPa and $c_2 = 0.96$ N/kPa, respectively. Like in the previous case for muscle parameter $c_1$, the maximum
response amplitude is decreased by 81.7% and 68.3% with increase and decrease in the value of $c_2$, respectively, as compared to the Fig. 2, where $c_2=1.96$ N/kPa.

**Fig. 5** Frequency responses under superharmonic resonance condition with variation in the system parameters (a) $P_0$ (b) $P_m$ (c) $\mu$ (d) $\alpha$ (e) $c_1$ (f) $c_2$ (g) $c_3$ (h) $d_1$ (i) $d_2$. 

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Like the previous cases for the muscle parameters $c_1$ and $c_2$, the maximum response amplitude decreases with either increase or decrease in the value of another muscle parameter $c_3$. The maximum response amplitude can be achieved $a = 0.0931$ at $\sigma = 0.297$ in Fig. 5(g) with increase in the value of $c_3 = -0.001 \text{ N/kPa}^2$ which is 73.4% decrease in comparison with Fig. 2. With decrease in the value of $c_3 = -0.006 \text{ N/kPa}^2$, the maximum response amplitude $a = 0.1256$ which is decreased by 64% and can be attained at $\sigma = 0.533$. In this case also like in Fig. 5(e) and Fig. 5(f), the saddle node bifurcation point, where the maximum response amplitude occurs, can be noticed at lower value of $\sigma$ as compared to Fig. 2.

From Fig. 5(h) and Fig. 2, it can be observed that the effect of the muscle parameter $d_1$ is not prominent. The maximum response amplitudes in Fig. 5(h) for both the values of $d_1 = -50 \text{ N}$ and $d_1 = -150 \text{ N}$ are same as that in Fig. 2. But, the value of detuning parameter $\sigma$ is slightly different, i.e., $\sigma = 4.862$ for $d_1 = -50 \text{ N}$ and $\sigma = 2.989$ for $d_1 = -150 \text{ N}$. 

In Fig. 5(i), the maximum response amplitude increases with increase in the value of another muscle parameter $d_2$ and decreases with decrease in the value of $d_2$ same as that in the cases for $P_0$ in Fig. 5(a) and $\alpha$ in Fig. 5(d). For $d_2 = 0.5 \text{ N/kPa}$, which is half of that value in Fig. 2, the maximum response amplitude $a = 0.1586$ at $\sigma = 0.514$ is obtained, which decreased by 54.7%. Now, by doubling the value of $d_2$ in Fig. 5(i), i.e., $d_2 = 2 \text{ N/kPa}$, the maximum response amplitude $a = 0.9597$ at $\sigma = 0.514$ can be achieved, which increased around 174%. Hence, to design a system for a particular application to achieve a definite response amplitude, one may find the optimum value of the muscle parameters $c_1, c_2, c_3, d_1$ and $d_2$. In all these cases, different PAMs have to be used.

To illustrate the application of the above developed results one may consider Table 2, where to achieve an amplitude $a = 0.1$, 17 different cases have been presented with different set of system parameters. It may be noted that the same response can be obtained by changing the parameter $P_0$, $P_m$ and $\sigma$ actively or the other parameters like $\mu$, $\alpha$, $c_1$, $c_2$, $c_3$, $d_1$ and $d_2$ passively. This will help the designer to choose the system parameters according to the particular application. From Table 2, it can be noticed that in Case 3 with low value of $P_0$ and $P_m$, one may obtain the same
displacement as compared to the other cases. Also, Case 17 provides the desired amplitude at a low frequency of the system. So, one may use the different set of system parameters to effectively move the PAM for the exact application.

Table 2. System parameters to attain an amplitude of $a = 0.1$.

| Case | $P_0$ | $P_m$ | $\mu$ | $\alpha$ | $c_1$ | $c_2$ | $c_3$ | $d_1$ | $d_2$ | $\sigma$ | Ref. Figure No. |
|------|-------|-------|-------|----------|-------|-------|-------|-------|-------|---------|----------------|
| 1    | 58    | 300   | 0.01  | 150      | -234.25 | 1.96  | -0.003 | -100  | 1     | 1.539   | Fig.2         |
| 2    | 116   | 300   | 0.01  | 150      | -234.25 | 1.96  | -0.003 | -100  | 1     | 2.608, 3.551 | Fig. 5(a)     |
| 3    | 29    | 300   | 0.01  | 150      | -234.25 | 1.96  | -0.003 | -100  | 1     | 1.385   | Fig. 5(a)     |
| 4    | 58    | 400   | 0.01  | 150      | -234.25 | 1.96  | -0.003 | -100  | 1     | 5.957   | Fig. 5(b)     |
| 5    | 58    | 200   | 0.01  | 150      | -234.25 | 1.96  | -0.003 | -100  | 1     | 1.211, 2.103 | Fig. 5(b)     |
| 6    | 58    | 300   | 0.03  | 150      | -234.25 | 1.96  | -0.003 | -100  | 1     | 1.590   | Fig. 5(c)     |
| 7    | 58    | 300   | 0.005 | 150      | -234.25 | 1.96  | -0.003 | -100  | 1     | 1.535   | Fig. 5(c)     |
| 8    | 58    | 300   | 0.01  | 300      | -234.25 | 1.96  | -0.003 | -100  | 1     | 3.174   | Fig. 5(d)     |
| 9    | 58    | 300   | 0.01  | 75       | -234.25 | 1.96  | -0.003 | -100  | 1     | 0.721   | Fig. 5(d)     |
| 10   | 58    | 300   | 0.01  | 150      | -117.12 | 1.96  | -0.003 | -100  | 1     | 0.398   | Fig. 5(e)     |
| 11   | 58    | 300   | 0.01  | 150      | -468.50 | 1.96  | -0.003 | -100  | 1     | 0.520   | Fig. 5(e)     |
| 12   | 58    | 300   | 0.01  | 150      | -234.25 | 0.96  | -0.003 | -100  | 1     | 0.357   | Fig. 5(f)     |
| 13   | 58    | 300   | 0.01  | 150      | -234.25 | 1.96  | -0.006 | -100  | 1     | 0.397   | Fig. 5(g)     |
| 14   | 58    | 300   | 0.01  | 150      | -234.25 | 1.96  | -0.003 | -100  | 1     | 2.641   | Fig. 5(h)     |
| 15   | 58    | 300   | 0.01  | 150      | -234.25 | 1.96  | -0.003 | -100  | 1     | 0.768   | Fig. 5(h)     |
| 16   | 58    | 300   | 0.01  | 150      | -234.25 | 1.96  | -0.003 | -100  | 2     | 13.34   | Fig. 5(i)     |
| 17   | 58    | 300   | 0.01  | 150      | -234.25 | 1.96  | -0.003 | -100  | 0.5   | 0.187   | Fig. 5(i)     |

Table 3 provides the variation of saddle node bifurcation point, where the system has a tendency to jump up while sweeping down the frequency for a wide range of system parameters. It may be noted that one may achieve the same amplitude either by sweeping up the frequency or by sweeping down the frequency. But, while sweeping down one must take care of the jump up phenomenon, which may lead to catastrophic failure of the system. In this table, the amplitude $(a)$ for both stable and unstable point are mentioned where jump up phenomenon will take place.
for different system parameters. It may be noted that in Case 12, the jump length is lowest and in
Case 2 the jump length is highest as compared to the other cases.

Table 3. Variation in the jump up phenomenon with different system parameters
while sweeping down the frequency.

| Case | Ref. Figure No. | Variation of System parameters | $\sigma$ | $a$ | Jump length |
|------|----------------|-------------------------------|---------|-----|-------------|
|      |                |                               |         |     |             |
| 1    | Fig. 2         | As per Table 1                |         |     |             |
| 2    | Fig. 5(a)      | $P_0 = 116$                   | 1.718   | 0.0683 | 0.1337 | 0.0654 |
| 3    | Fig. 5(a)      | $P_0 = 29$                    | 1.377   | 0.0524 | 0.0982 | 0.0458 |
| 4    | Fig. 5(b)      | $P_m = 400$                   | 6.219   | 0.0741 | 0.1461 | 0.0720 |
| 5    | Fig. 5(b)      | $P_m = 200$                   | 2.100   | 0.1062 | 0.2116 | 0.1054 |
| 6    | Fig. 5(c)      | $\mu = 0.03$                 | 1.688   | 0.0738 | 0.1144 | 0.0406 |
| 7    | Fig. 5(c)      | $\mu = 0.005$                | 1.721   | 0.0679 | 0.1352 | 0.0673 |
| 8    | Fig. 5(d)      | $\alpha = 300$               | 3.275   | 0.0562 | 0.1110 | 0.0548 |
| 9    | Fig. 5(d)      | $\alpha = 75$                | 0.924   | 0.0846 | 0.1634 | 0.0788 |
| 10   | Fig. 5(e)      | $c_1 = -117.12$              | 0.360   | 0.0496 | 0.0912 | 0.0416 |
| 11   | Fig. 5(e)      | $c_1 = -468.50$              | 0.523   | 0.0534 | 0.1006 | 0.0472 |
| 12   | Fig. 5(f)      | $c_2 = 2.96$                 | 0.135   | 0.0424 | 0.0636 | 0.0212 |
| 13   | Fig. 5(f)      | $c_2 = 0.96$                 | 0.298   | 0.0479 | 0.0863 | 0.0384 |
| 14   | Fig. 5(g)      | $c_3 = -0.001$               | 0.230   | 0.0457 | 0.0796 | 0.0339 |
| 15   | Fig. 5(g)      | $c_3 = -0.006$               | 0.359   | 0.0496 | 0.0911 | 0.0415 |
| 16   | Fig. 5(h)      | $d_1 = -50$                  | 2.820   | 0.0683 | 0.1337 | 0.0654 |
| 17   | Fig. 5(h)      | $d_1 = -150$                 | 0.948   | 0.0683 | 0.1337 | 0.0654 |
| 18   | Fig. 5(i)      | $d_2 = 2$                    | 13.980  | 0.0951 | 0.1898 | 0.0947 |
| 19   | Fig. 5(i)      | $d_2 = 0.5$                  | 0.187   | 0.0532 | 0.1001 | 0.0469 |

To find the response amplitude of the muscle in dimensional form an example has been illustrated here. The dimensional value of operating frequency can be calculated using the relation $\Omega = 1/3 + \varepsilon \sigma$ and $\omega = \Omega \omega_0$ from Eq. (5). Now, corresponding to nondimensional amplitude ($a$), nondimensional ($x$) and dimensional ($u$) displacement, $u = rx$ can be obtained. Using Fig. 2, at detuning parameter $\sigma = 3$ ($\omega = 8.4$ Hz), the nondimensional amplitude $a = 0.2884$ and the corresponding dimensional response amplitude is found to be 2.9 cm.
3.2 Subharmonic Resonance Condition

In this section, the numerical results for subharmonic resonance condition, where the external excitation frequency is three times of the natural frequency of the system have been explored. The frequency response have been plotted in Fig. 6 by taking the system parameters values as given in Table 1 except the value of static pressure is considered as $P_m = 200$ kPa. The system exhibits both trivial and nontrivial branches in the frequency plots for this resonance condition, which can be observed from Eq. (36), where $a = 0$ is a solution. Here, the stability of the steady state response on the nontrivial and trivial branch have been studied with the help Jacobian Matrix Eq. (32).

![Fig. 6 Frequency response at subharmonic resonance condition taking system parameters as Table 1.](image)

In the nontrivial branch in Fig. 6, the jump up phenomenon can be noticed at $\sigma = 12$, where the unstable solution with amplitude $a = 0.1827$ at point A has always a tendency to jump to the solution in the stable branch with amplitude $a = 0.1962$ at point B. The saddle node bifurcation point C can be observed in the nontrivial branch at detuning parameter $\sigma = 7.928$ with amplitude $a = 0.1087$ and the trivial branch in the frequency response plot is stable for a wide range of $\sigma$. The time response and phase portraits have been plotted in Fig. 7 with the help of the reduced equations Eq. (32) and Eq. (33) near the mentioned saddle node bifurcation point C ($\sigma = 7.928$) and the point A ($\sigma = 12$) where the responses are found to be periodic.
From Fig. 6, it can be observed that the system exhibits single stable solution in the trivial branch up to the saddle node bifurcation point C (\(\sigma = 7.928\)) and after that the nontrivial branch arises with a stable and an unstable solution, i.e., the system shows bistable state. So, to validate the frequency response curve in Fig. 6, the basin of attraction have been plotted in \(a \sim \gamma\) plane for two different values of \(\sigma\) in Fig. 8. For the point S (\(\sigma = 7.5\)) in Fig. 6, the system has only one stable solution, i.e., \(a = 0\) in the trivial branch of the frequency response curve, which can be verified from the Fig. 8(a). From Fig. 8(b), it can be observed that the system has two stable solution with \(a = 0\) and \(a = 0.1795\) in the trivial and nontrivial branch of the frequency response curve in Fig. 6 for \(\sigma = 11\). The system has also an unstable solution with \(a = 0.1668\) at \(\sigma = 11\) in the nontrivial branch of the frequency response, which always shows an affinity to jump to the nearest stable solution with \(a = 0.1795\). The frequency responses have been plotted for different set of system parameters and the trivial branch is found to be stable in all the cases for a wide range of \(\sigma\). Hence, the nature of the nontrivial branch of the frequency response plots has been discussed extensively.
In Fig. 9(a) the effect of the dynamic pressure in the muscle $P_0$ on the stability of the nontrivial branch has been studied with two different values of $P_0$. With decrease in $P_0 = 29$ kPa in Fig. 9(a), which is half to that in Fig. 6 ($P_0 = 58$ kPa), the saddle node bifurcation point occurs at detuning parameter $\sigma = 14.6$ with an amplitude $a = 0.2419$. But by making double, i.e., $P_0 = 116$ kPa, the saddle node bifurcation point is observed at higher value of detuning parameter $\sigma = 21.58$ with $a = 0.0422$. From Fig. 6 and Fig. 9(a), it can be noticed that the saddle node bifurcation point occurs at higher value of $\sigma$ with either increase or decrease in the value of $P_0$.

Figure 9(b) shows the frequency response plots for two different values of the static pressure in the muscle $P_m$. Here, one may observed that with increase in the value of $P_m = 300$ kPa as compared to Fig. 6, the saddle node bifurcation point in the nontrivial branch can be observed at $\sigma = 23.39$ with $a = 0.2608$. With decrease in the value of $P_m = 100$ kPa, the saddle node bifurcation point is at $\sigma = 12.35$ with $a = 0.2429$. Hence, from Fig. 6, Fig. 9(a) and Fig. 9(b), it can be observed that an optimum value of $P_0$ and $P_m$ should be chosen to operate the system at lower value of $\sigma$.

From Fig. 9(c) with decrease in the value of damping parameter $\mu = 0.005$ as compared to Fig. 6, the saddle node bifurcation point occurs at $\sigma = 6.449$ with $a = 0.0542$. Here, the response
amplitude is $a = 0.1628$ at $\sigma = 10$, which is increased by 1% from Fig. 6. With increase in the value of $\mu = 0.03$, the saddle node bifurcation point can be noticed at higher value $\sigma = 23.78$ with $a = 0.3252$ in the nontrivial branch of the frequency response curve.

From Fig. 6 and Fig. 9(d), it can be observed that by increasing in the value of nonlinearity $\alpha$, the saddle node bifurcation point can be observed at higher value of $\sigma$ and decreasing $\alpha$, the same can be attained at lower value of $\sigma$. By making double the nonlinearity $\alpha = 300$ and reducing to half $\alpha = 75$ as compared to Fig. 6, the saddle node bifurcation point occurs at $\sigma = 12.88$ with $a = 0.0545$ and $\sigma = 6.936$ with $a = 0.2169$, respectively. Hence, the response amplitude increases with decrease in the value of $\alpha$ and vice versa.

Figure 9(e) depicts that the saddle node bifurcation point occurs at higher value of $\sigma = 75.44$ with $a = 0.6679$ as compared to Fig. 6, by decreasing the value of muscle parameter $c_1$ from -234.25 to -468.5 N. But, with increase in the value of $c_1 = -117.12$ N, the saddle node bifurcation point in the nontrivial branch can be noticed at $\sigma = 41.88$ with $a = 0.4965$. From Fig. 6 and Fig. 9(e), the saddle node bifurcation point can be observed at higher value of $\sigma$ and the response amplitude increases with either increase or decrease in the value $c_1$.

From Fig. 9(f), the saddle node bifurcation point is observed at higher value of $\sigma = 52.1$ with $a = 0.5544$ as compared to Fig. 6, by decreasing the value of muscle parameter $c_2$ from 1 N/kPa to 0.96 N/kPa. With increase in the value of $c_2 = 2.96$ N/kPa, the saddle node bifurcation point occurs at $\sigma = 100.5$ with $a = 0.7712$ in the nontrivial branch. Like the previous case in Fig. 9(e), the response amplitude in Fig. 9(f) is increased with increase or decrease in the value of $c_2$ in comparison to Fig. 6.

In Fig. 9(g), the frequency response curves have been plotted for the two different values of another muscle parameter $c_3$. For muscle parameter $c_3 = -0.001$ N/kPa, which is increased to Fig. 6 ($c_3 = -0.003$ N/kPa), the saddle node bifurcation is noticed at higher value of $\sigma = 25.45$ with $a = 0.3847$. By decreasing the value of $c_3 = -0.006$ N/kPa, the saddle node bifurcation point occurs at $\sigma = 16.55$ with $a = 0.306$ in the nontrivial branch of the frequency response curve. In
this case also the response amplitude is increased at a higher value of $\sigma$ in comparison to Fig. 6, whether the value of $c_3$ is increased or decreased.

**Fig. 9** Frequency responses under subharmonic resonance condition with variation in the system parameters (a) $P_0$ (b) $P_m$ (c) $\mu$ (d) $\alpha$ (e) $c_1$ (f) $c_2$ (g) $c_3$ (h) $d_1$ (i) $d_2$. 
Like the case in Fig. 5(h) for superharmonic resonance condition, from Fig. 9(h) and Fig. 6, it can be observed that the effect of the muscle parameter \( d_1 \) is not distinct in case of subharmonic resonance condition. The response amplitude in Fig. 9(h) at the saddle node bifurcation point for both the values of \( d_1 = -50 \) N and \( d_1 = -150 \) N is same with Fig. 7. But the value of detuning parameter \( \sigma \) is different for the two values of \( d_1 = -50 \) N and \( d_1 = -150 \) N which is \( \sigma = 30.81 \) and \( \sigma = 2.329 \), respectively. Hence, it can be noted from Fig. 9(h) that with decrease in the value of \( d_1 \) as compared to Fig. 6 (\( d_1 = -100 \) N), the saddle node bifurcation point can be observed at lower value of \( \sigma \) and vice versa.

In Fig. 9(i), the saddle node bifurcation point occurs at a very high value \( \sigma = 201.9 \) with low response amplitude \( a = 0.0542 \) making the value of the muscle parameter \( d_2 = 2 \) N/kPa double in comparison with Fig. 6, where \( d_2 = 1 \) N/kPa. But decreasing the value of \( d_2 = 0.5 \) N/kPa which is half to that in Fig. 6, the saddle node bifurcation point is observed at \( \sigma = 19.91 \) with \( a = 0.2168 \). Hence, like the superharmonic resonance condition, the operator or designer should choose the muscle parameters \( c_1, c_2, c_3, d_1 \) and \( d_2 \) value optimally to obtain the required amplitude at lower frequency for subharmonic resonance condition.

4. Experimental Verification of the PAM model

Figure 10 (a) shows the schematic diagram and Fig. 10(b) shows the actual experimental setup used to conduct free vibration analysis. Further, it is used to determine the different muscle parameters in Eq. (2) in the quasi-static empirical PAM model. The most commonly used PAM, i.e., a McKibben Muscle has been considered which is composed of a gas-tight elastic tube or bladder surrounded by a nylon braided sleeve. Both the ends of the inner tube and braid of this muscle is closed with the end fitting. The McKibben Muscle is hung freely from a rigid support with a dead weight of 3 kg at its free end. This ensures that the rubber tube along with the outer braid of the muscle is at its maximum possible length and the characteristics of PAM (McKibben type) are studied from this position. The changes in the contraction of the muscle has been measured by varying the internal pressure inside the muscle with the help of a mechanical control valve. The pressure inside the muscle is increased from 50 kPa to the safe pressure limit of 200
kPa in discrete steps of 50 kPa. At each pressure step, the PAM contraction is measured and noted as given in Table 4. The above steps are repeated for different loads in the range 0-3 kg (0–30 N).

![Diagram](image)

**Fig. 10** Quasi-static PAM model identification (a) schematic diagram (b) actual experiment set-up.

| **Muscle Force** | **Pressure** | **50 kPa** | **100 kPa** | **150 kPa** | **200 kPa** |
|------------------|--------------|------------|-------------|-------------|-------------|
| 5 N              | 7.0          | 12.0       | 19.0        | 22.0        |
| 10 N             | 5.8          | 10.6       | 17.0        | 19.8        |
| 15 N             | 4.2          | 8.8        | 15.2        | 17.8        |
| 20 N             | 2.8          | 7.2        | 13.8        | 15.8        |
| 25 N             | 1.8          | 5.7        | 12.2        | 14.2        |
| 30 N             | 1.0          | 4.5        | 11.0        | 12.0        |

Table 4. Variation of contraction of the muscle in mm ($u$) with muscle force (applied load) and air pressure.

Now, taking $y = u/l_{\text{max}}$ and the data from Table 4, one may use the least square method to find the coefficients of Eq. (2). In this method the error can be written as follows.

$$E(c_1, c_2, c_3, d_1, d_2, K_n) = \sum_{i=1}^{24} \left[ F_{\text{max}} - \left\{ \left( c_1 + c_2 P_i + c_3 P_i^2 \right) y_i + \left( d_1 + d_2 P_i + K_n y_i^3 \right) \right\} \right]^2$$  \hspace{1cm} (33)

where, $K_n = K_{\text{max}}^3$. Hence, the error can be minimized by finding the first derivative with respect to $c_1$, $c_2$, $c_3$, $d_1$, $d_2$ and $K_n$ and equating it to zero as follows.
By solving the above equation Eq. (34) the values of the coefficients can be found out. The values of the muscle parameters in the present case are found to be 
\[ c_1 = -6.871 \times 10^3 \text{ N}, \quad c_2 = 34.6 \text{ N/kPa}, \]
\[ c_3 = 0.028 \text{ N/kPa}^2, \quad d_1 = -79.83 \text{ N} \quad \text{and} \quad d_2 = 0.579 \text{ N/kPa}. \]
The value of nonlinear parameter, \( K_n \) is \( 3.291 \times 10^{-9} \text{ N/mm}^3 \) and the value of the maximum possible length of the particular muscle used in this experiments \( l_{\text{max}} = 136 \text{ mm} \). Hence, the value of \( K \) in Eq. (2) is found to be \( 1.308 \times 10^{-15} \text{ N/mm}^6 \).

The dynamic characteristics of the PAM have been analyzing with time response over a range of external forced excitation. Figure 11(a) and Fig. 11(b) show the schematic diagram and actual experimental setup respectively. The load pan of weight 3.5 N is used, which will be considered as dead-weight. The pan has been attached to the free end of the muscle first to measure the dynamic characteristics. The system is then excited vertically as shown in Fig. 11 with the help of a shaker (Make: B&K, Model: 4824) with a sweep sinusoidal input force for three different external excitation frequencies, namely 
\[ f = 8 \text{ Hz} \ (\Omega = 50.27 \text{ rad/s}), \quad f = 10 \text{ Hz} \ (\Omega = 62.83 \text{ rad/s}) \]
and \( f = 12 \text{ Hz} \ (\Omega = 75.40 \text{ rad/s}) \). The acceleration-time history of the vibrating muscle is recorded using a Bruel and Kjaer made accelerometer (Type: 4396, Serial No.: 2247754) mounted on the load pan. A five-channel dynamic signal acquisition module (NI USB-4432) is connected to the accelerometer for making high-accuracy measurements from the accelerometer signals and NI LabVIEW 2009 software is used for post-processing the signals.
The response function is then calculated from the Fast-Fourier transformed force and acceleration data. This experiment is repeated by varying the loads in the load pan with different input pressure inside the muscle. In Fig. 12 the time responses (Acceleration data) are shown for the muscle for lifting a load of 2.5 kg with 200 kPa input pressure into the muscle for above mentioned three external excitation frequencies.

**Fig. 11** Dynamic PAM model identification using forced excitation (a) schematic diagram (b) actual experiment set-up.

**Fig. 12** Time response for the system with a load of 2.5 kg at 200 kPa

(a) $f = 8$ Hz  (b) $f = 10$ Hz  (c) $f = 12$ Hz.
The maximum displacement from experiments can be found out from Fig. 12 using the formula, 
\[ \text{Displacement} = \frac{\text{Acceleration Amplitude}}{(2 \times \pi \times f)^2}. \]

In Fig. 12(a), the acceleration amplitude is observed 13.5 m/s² for \( f = 8 \text{ Hz} \) and the corresponding maximum displacement is found to be 5.3 mm. Similarly, the acceleration amplitude is 21 m/s² in Fig. 12(b) for \( f = 10 \text{ Hz} \), where the maximum displacement is obtained 5.3 mm. Finally, for \( f = 12 \text{ Hz} \) in Fig. 12(c), the acceleration amplitude is 29 m/s² and the corresponding maximum displacement is noticed 5.1 mm.

By considering the value of the system parameters obtained from the experiments, the time response comparisons have been for the case studied in Fig. 12 (2.5 kg with 200 kPa). These comparisons are shown in Fig. 13 and Fig. 14 between the solution obtained by solving the reduced equations from the method of multiple scales (marked ‘AS’) and the original governing equation (marked ‘NS’) for super- and sub-harmonic resonance condition, respectively. The reduced equations Eq. (16) and Eq. (17) for superharmonic resonance condition and Eq. (26) and Eq. (27) for subharmonic resonance condition are used to find the responses. Further these results have been compared by solving the original equation Eq. (6) numerically using Runge-Kutta method.

For superharmonic resonance condition with \( f = 8 \text{ Hz} \) in Fig. 13(a), the maximum response of the solution obtained by solving the reduced equations is increased by 0.2% that from the original equation. From Fig. 13(b) with \( f = 10 \text{ Hz} \) it can be noticed that the response of the solution found from the reduced equations is increased around 0.17% that from the original equation. Similarly, it can be observed from Fig. 13(c) for \( f = 12 \text{ Hz} \) that the maximum response of the solution obtained from reduced equation also increased around 0.11% to that of the original equation. Hence, it can be noticed that the responses found by using method of multiple scales gives a conservative result in all the cases with same mean response amplitude. Again, in the case of subharmonic resonance condition in Fig. 14, the maximum response of the solution obtained by solving the reduced equations is in good agreement with that from the original equation. The responses have been plotted for \( f = 8 \text{ Hz} \) in Fig. 14(a), for \( f = 10 \text{ Hz} \) in Fig. 14(b) and for \( f = 12 \text{ Hz} \) in Fig. 14(c).
In Table 5 for the above mentioned three external excitation frequencies, the experimental displacements of the McKibben Muscle from Fig. 12 are presented. Further, these results have been compared with the theoretical displacement from the solution obtained by solving the reduced equations and original equation for super- and sub-harmonic resonance condition which can be observed from Fig. 13 and Fig. 14. Hence, in both the resonance conditions, the responses are
found to be in good agreement with the experimental results and the reduced equations from the method of multiple scales can be used to study system behaviour, which will reduce the computational time and memory.

**Table 5.** Comparison of experimental and theoretical response of the used McKibben Muscle.

| External Excitation frequency | Experimental Displacement | Theoretical Displacement (Superharmonic Resonance condition) Using governing equations Eq. (6) | Theoretical Displacement (Subharmonic Resonance condition) Using governing equations Eq. (6) Using reduced equations Eq. (16) and Eq. (17) | Theoretical Displacement (Subharmonic Resonance condition) Using governing equations Eq. (6) Using reduced equations Eq. (26) and Eq. (27) |
|-------------------------------|---------------------------|-------------------------------------------------------------------------------------------------|-----------------------------------------------------------------|-----------------------------------------------------------------|
| $f = 8 \text{ Hz}$           | 5.3 mm                    | 5.410 mm                                                                                     | 5.421 mm                                                       | 5.410 mm                                                       | 5.412 mm                                                       |
| $f = 10 \text{ Hz}$          | 5.3 mm                    | 5.408 mm                                                                                     | 5.417 mm                                                       | 5.408 mm                                                       | 5.409 mm                                                       |
| $f = 12 \text{ Hz}$          | 5.1 mm                    | 5.408 mm                                                                                     | 5.414 mm                                                       | 5.408 mm                                                       | 5.408 mm                                                       |

**5. Conclusion**

In the present work, the nonlinear dynamics of a pneumatic artificial muscle is studied under hard excitation for super- and sub-harmonic resonance conditions. Using the method of multiple scales, the governing equation of motion has been solved and a set of reduced equations are obtained for each resonance condition. So with the help of frequency response plots, the safe operating range of the different system parameters can be determined. Basin of attractions also have been plotted to verify the actually realized solutions from initial conditions. The frequency responses show the jump-up and jump-down phenomenon along with the saddle node bifurcations. The periodic responses are observed from the time responses and phase portraits in both the resonance conditions. But in the case of subharmonic resonance condition, both trivial and nontrivial responses have been found from the frequency response plots. While the trivial state is always stable, the nontrivial state undergoes transition from stable to unstable region by saddle node bifurcation. An experimental setup is developed to conduct a series of experiments which gives conservative results to validate the mathematical work. The different unknown coefficients present in the muscle force are found by using the least square method. The maximum response of the system obtained from the time response plots are in good agreement. Hence, the developed mathematical work and the reduced equations can be used for further analysis to know the behavior of various kind of PAM. This study can help to understand the nonlinear dynamics present in the PAM with a small amount of memory and less computational time.
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Declarations

Conflict of Interest: The authors declare that there is no conflict of interest.

References

[1] C. P. Chou, B. Hannaford, Measurement and modeling of McKibben pneumatic artificial muscles, IEEE Trans. Rob. Autom. 12(1) (1996) 90-102. DOI: 10.1109/70.481753

[2] G. K. Klute, J. M. Czerniecki, B. Hannaford, McKibben artificial muscles: pneumatic actuators with biomechanical intelligence, in: IEEE/ASME Int. Conf. Advanced Intelligent Mechatronics, Atlanta, GA, USA, 1999, pp. 221-226. DOI: 10.1109/AIM.1999.803170

[3] B. Tondu, P. Lopez, Modeling and control of McKibben artificial muscle robot actuators, IEEE Contr. Syst. Mag. 20 (2000) 15-38. DOI: 10.1109/37.833638

[4] R.T. Pack, J.L. Christopher Jr., K. Kawamura, A Rubbertuator- Based Structure-Climbing Inspection Robot, in: Proc. of the IEEE Int. Conf. on Robotics and Autom., Albuquerque, NM, USA, 1997, vol. 3, pp. 1869-1874. DOI: 10.1109/ROBOT.1997.619060

[5] K. Inoue, Rubbertuators and applications for robotics, in: Proc. of the 4th Int. Symp. on Robotics Research, 1988, pp. 57-63.

[6] Shadow Robot Company, UK, https://www.shadowrobot.com/.

[7] D. C. Thanh, K. K. Ahn, Nonlinear PID control to improve the control performance of 2 axes pneumatic artificial muscle manipulator using neural network, Mechatronics 16(9) (2006) 577-587. https://doi.org/10.1016/j.mechatronics.2006.03.011

[8] K.C. Wickramatunge, T. Leephakpreeda, Study on mechanical behaviors of pneumatic artificial muscle, Int. J. Eng. Sci. 48(2) (2010) 188-198. https://doi.org/10.1016/j.ijengsci.2009.08.001

[9] F. Daerden, D. Lefeber, The concept and design of pleated pneumatic artificial muscles, Int. J. Fluid Power, 2(3) (2001) 41-50. https://doi.org/10.1080/14399776.2001.10781119

[10] F. Daerden, D. Lefeber, Pneumatic Artificial Muscles: actuators for robotics and automation, Eur. J. Mech. Environ. Eng. 47 (2002) 10-21.
[11] Y. Zhang, H. Liu, T. Ma, L. Hao, Z. Li, A comprehensive dynamic model for pneumatic artificial muscles considering different input frequencies and mechanical loads. Mech. Syst. Signal Process. 148 (2021) 107133. https://doi.org/10.1016/j.ymssp.2020.107133

[12] H. Li, K. Kawashima, K. Tadano, S. Ganguly, S. Nakano, Achieving haptic perception in forceps’ manipulator using pneumatic artificial muscle, IEEE/ASME Trans. on Mechatronics, 18(1) (2011) 74-85. DOI: 10.1109/TMECH.2011.2163415

[13] L. Zhao, X. Liu, T. Wang, Trajectory tracking control for double-joint manipulator systems driven by pneumatic artificial muscles based on a nonlinear extended state observer. Mech. Syst. Signal Process. 122 (2019) 307-320. https://doi.org/10.1016/j.ymssp.2018.12.016

[14] L. Zhao, H. Cheng, J. Zhang, Y. Xia, Adaptive control for a motion mechanism with pneumatic artificial muscles subject to dead-zones. Mech. Syst. Signal Process. 148 (2021) 107155. https://doi.org/10.1016/j.ymssp.2020.107155

[15] Y. Chen, N. Sun, D. Liang, Y. Qin, Y. Fang, A neuroadaptive control method for pneumatic artificial muscle systems with hardware experiments. Mech. Syst. Signal Process. 146 (2021) 106976. https://doi.org/10.1016/j.ymssp.2020.106976

[16] M. D. Doumit, S. Pardoel, Dynamic contraction behaviour of pneumatic artificial muscle. Mech. Syst. Signal Process. 91 (2017) 93-110. https://doi.org/10.1016/j.ymssp.2017.01.001

[17] B. Kalita, S. K. Dwivedy, Nonlinear dynamics of a parametrically excited pneumatic artificial muscle (PAM) actuator with simultaneous resonance condition, Mech. Mach. Theory 135 (2019) 281-297. https://doi.org/10.1016/j.mechmachtheory.2019.01.031

[18] B. Kalita, S. K. Dwivedy, Dynamic analysis of pneumatic artificial muscle (PAM) actuator for rehabilitation with principal parametric resonance condition, Nonlinear Dyn. 97(4) (2019) 2271-2289. https://doi.org/10.1007/s11071-019-05122-2

[19] B. Kalita, S. K. Dwivedy, Nonlinear dynamic response of pneumatic artificial muscle: A theoretical and experimental study. Int. J. Nonlin. Mech. 125 (2020) 103544. doi: https://doi.org/10.1016/j.ijnonlinmec.2020.103544

[20] B. Kalita, S. K. Dwivedy, Numerical Investigation of Nonlinear Dynamics of a Pneumatic Artificial Muscle with Hard Excitation, ASME J. Comput. Nonlin. Dyn. 15(4) (2020), p.041003. https://doi.org/10.1115/1.4046246

[21] A. Krishnan, Difference equation analysis of non-linear subharmonic and superharmonic oscillations, J. Sound Vib. 79(1) (1981) 121-131. https://doi.org/10.1016/0022-460X(81)90332-1

[22] A. M. Elnaggar, A. F. El-Basyouny, Harmonic, subharmonic, superharmonic, simultaneous sub/super harmonic and combination resonances of self-excited two coupled second order systems to multi-frequency excitation, Acta Mech. Sinica 9(1) (1993) 61-71. https://doi.org/10.1007/BF02489163
[23] A. Bichri, M. Belhaq, Control of a forced impacting hertzian contact oscillator near sub-and superharmonic resonances of order 2, ASME J. Comput. Nonlin. Dyn. 7(1) (2012), p.011003. doi:10.1115/1.4004309

[24] H. G. Davies, S. Rajan, Random superharmonic response of a Duffing oscillator, J. Sound Vib. 111(1) (1986) 61-70. https://doi.org/10.1016/S0022-460X(86)81423-7

[25] H. G. Davies, S. Rajan, Random superharmonic and subharmonic response: multiple time scaling of a Duffing oscillator, J. Sound Vib. 126(2) (1988) 195-208. https://doi.org/10.1006/jsvi.1994.1192

[26] M. X. Cai, J. P. Yang, Bifurcation of periodic orbits and chaos in Duffing equation, Acta Math. Appl. Sin.-E 22(3) (2006) 495-508. https://doi.org/10.1007/s10255-006-0325-4

[27] M. Zhang, J. P. Yang, Bifurcations and chaos in Duffing equation, Acta Math. Appl. Sin.-E 23(4) (2007) 665-684. https://doi.org/10.1007/s10255-007-0404

[28] A. Hassan, On the Third Superharmonic Resonance in the Duffing Oscillator, J. Sound Vib. 172(4) (1994) 513-526. https://doi.org/10.1006/jsvi.1994.1192

[29] C. Holmes, P. Holmes, Second order averaging and bifurcations to subharmonics in Duffing's equation, J. Sound Vib. 78(2) (1981) 161-174. https://doi.org/10.1016/S0022-460X(81)80030-2

[30] M. Cai, H. Cao, Bifurcations of periodic orbits in Duffing equation with periodic damping and external excitations, Nonlinear Dyn. 70(1) (2012) 453-462. https://doi.org/10.1007/s11071-012-0467-2

[31] K. Yagasaki, Second-order averaging and Melnikov analyses for forced non-linear oscillators, J. Sound Vib. 190(4) (1996) 587-609. https://doi.org/10.1006/jsvi.1996.0080

[32] H. Moradi, H. Salarieh, Analysis of nonlinear oscillations in spur gear pairs with approximated modelling of backlash nonlinearity, Mech. Mach. Theory 51 (2012) 14-31. https://doi.org/10.1016/j.mechmachtheory.2011.12.005

[33] A. P. Bovsunovsky, O. Bovsunovsky, Crack detection in beams by means of the driving force parameters variation at non-linear resonance vibrations, Key Eng. Mater. 347 (2007) 413-420. https://doi.org/10.4028/www.scientific.net/KEM.347.413

[34] A. P. Bovsunovsky, C. Surace, Considerations regarding superharmonic vibrations of a cracked beam and the variation in damping caused by the presence of the crack, J. Sound Vib. 288(4-5) (2005) 865-886. https://doi.org/10.1016/j.jsv.2005.01.038

[35] M. S. Rechdaoui, L. Azrar, Active control of secondary resonances piezoelectric sandwich beams, Appl. Math.Comput. 216(11) (2010) 3283-3302. https://doi.org/10.1016/j.amc.2010.04.055
[36] F. Benedettini, G. Rega, Planar non-linear oscillations of elastic cables under superharmonic resonance conditions, J. Sound Vib. 132(3) (1989) 353-366. https://doi.org/10.1016/0022-460X(89)90630-5

[37] B. Pratiher, S. K. Dwivedy, Non-linear vibration of a single link viscoelastic Cartesian manipulator, Int. J. Nonlin. Mech. 43(8) (2008) 683-696. https://doi.org/10.1016/j.ijnonlinmech.2008.03.002

[38] V. Ramakrishnan, B. F. Feeny, Resonances of a forced Mathieu equation with reference to wind turbine blades, ASME J. Vib. Acoust. 134(6) (2012), p.064501. https://doi.org/10.1115/1.4006183

[39] D. Zou, Z. Rao, N. Ta, Coupled longitudinal-transverse dynamics of a marine propulsion shafting under superharmonic resonances, J. Sound Vib. 346 (2015) 248-264. https://doi.org/10.1016/j.jsv.2015.02.035

[40] A.H. Nayfeh, D.T. Mook, Nonlinear Oscillations, John Wiley & Sons, New Jersey, 2008.

[41] A.H. Nayfeh, Perturbation methods, John Wiley & Sons, New Jersey, 2008.

[42] S. Balasubramanian, R. Wei, M. Perez, B. Shepard, E. Koeneman, J. Koeneman, J. He, RUPERT: An exoskeleton robot for assisting rehabilitation of arm functions, IEEE Virtual Rehabilitation (2008) 163-167. DOI: 10.1109/ICVR.2008.4625154

[43] B.G. Do Nascimento, C.B.S. Vimieiro, D.A.P. Nagem, M. Pinotti, Hip orthosis powered by pneumatic artificial muscle: Voluntary activation in absence of myoelectrical signal, Artif. Organs 32(4) (2008) 317-322. doi: 10.1111/j.1525-1594.2008.00549.x.

[44] J.A. Norris, K.P. Granata, M.R. Mitros, E.M. Byrne, A.P. Marsh, Effect of augmented plantarflexion power on preferred walking speed and economy in young and older adults, Gait Posture 25(4) (2007) 620-627. doi: 10.1016/j.gaitpost.2006.07.002

[45] K. Kawashima, Development of robot using pneumatic artificial rubber muscles to operate construction machinery, J. Robot. Mechatron. 16 (2004) 8-16. doi: 10.20965/jrm.2004.p0008

[46] M. Van Damme, R. Van Ham, B. Vanderborght, F. Daerden, D. Lefeber, Design of a “soft” 2-DOF planar pneumatic manipulator, Climbing and Walking Robots (2006) 559-566, Springer, Berlin, Heidelberg. https://doi.org/10.1007/3-540-26415-9_67

[47] M. De Volder, A.J.M. Moers, D. Reynaerts, Fabrication and control of miniature McKibben actuators, Sens. Actuators, A 166(1) (2011) 111-116. https://doi.org/10.1016/j.sna.2011.01.002