On the Stability Analysis of the Stochastic Age-structured Infectious Model

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Abstract
This paper is concerned with the stability analysis of the stochastic age-structured infectious model. We have been faced with a threat of the infectious disease even in the modern society with a high medical technology. Hence, infection prevention and control of epidemic are cited as one of the important social problems. Until now, interesting and impactful research of infectious diseases has been performed using various kinds of mathematical models. However, most of the research is based on the deterministic model and the age-structure has not been taken into account. In the realistic spread of the infectious disease, environmental change and individual difference cause some kinds of random fluctuations in the infection, the recovery rates and the vaccination effect. Moreover, there exists the difference in the transmission intensity of infection by age. Taking these facts into consideration, we propose the stochastic age-structured infectious model. Since the spread of infection has reference to the stability of the disease-free steady state (DFS) of the stochastic infectious models, we analyze the stability of the DFS by using the stochastic Lyapunov theorem.

1 Introduction
The World Health Organization (WHO) warns of threat of infectious diseases on the home page. The WHO reports that nearly 50,000 men, women and children are dying every day from infectious diseases and we are faced with a global crisis in infectious diseases. In this way, a menace from the various kinds of infectious diseases including three major infectious diseases, HIV, tuberculosis, and malaria has been growing. The infectious disease prevention and control is a thorny problem to solve in the modern society with a high medical technology [1]-[4].

Mathematical models have become important tools in analyzing the infectious spread because an experiment on a human body is difficult from the standpoint of ethics in the study of the infectious disease. Hence, until now, interesting and impactful research of infectious diseases has been performed using various kinds of mathematical models [5]-[8] including the SIR (susceptible-infected-recovered) and the SIRV (susceptible-infected-recovered-vaccinated) models. However, most of the research is based on the deterministic model and the age structure has not been taken into account. In other words, previous studies have been performed under the deterministic environment and the assumption of homogeneous populations. Although the infectious models in [9]-[14] is stochastic ones, age structure is not considered.

In the realistic spread of the infectious disease, environmental change and individual difference cause some kinds of random fluctuations in the infection, the recovery rates and the vaccination effect. Moreover, there exists the difference in the transmission intensity of infection by age. Taking these facts into consideration, we propose the stochastic age-structured infectious model. Since the spread of infection has reference to the stability of the disease-free steady state (DFS) of the stochastic age-structured infectious model, we analyze the stability of the DFS by using the stochastic Lyapunov theorem. The DFS means the equilibrium solution with zero infected individuals.

In Section 2, firstly, as an example of the conventional infectious model, the deterministic SIR model without age structure is explained. We define some important demographic indexes including the net reproduction rate $R$. The value of the index $R$ determine whether the population increases or not as described in Section 2. Secondly, we introduce age structure for the SIR model and show the infectious model with age structure is given by the partial differential equation of the first order. And we explain the stable population theory for the age-structured infectious model. In Section 3, for more practical analysis of the spread of infectious diseases, introducing the random fluctuation in the recovery rate, we propose the stochastic SIR model with age structure. Since the behavior of the stochastic age-structured infectious model is determined by the stability of the DFS, we study the stability of the DFS in Section 4 using the stochastic Lyapunov theorem. We derive the sufficient conditions for the DFS to be stable. In the case where the DFS is unstable, by controlling the vaccination rate so as to satisfy the stability condition derived in this paper, the control of the infectious disease is possible.
2 Age-structured Infectious Model

2.1 Stable Population Theory

Denoting the population density of the susceptible, the infected and the recovered at time $t$ be $S(t)$, $I(t)$ and $R(t)$, the conventional SIR infectious model without the age structure is given by

$$
\dot{S}(t) = -\phi S(t) - \mu S(t) - \lambda(t)S(t),
\dot{I}(t) = \lambda(t)S(t) - (\mu + \gamma)I(t),
\dot{R}(t) = \gamma I(t) - \mu R(t),
$$

where the constant $\phi$ denotes the percentage of susceptible individuals being vaccinated per unit of time, $\mu$ the death rate, $b$ the birth rate, $\gamma$ the recovery rate and $\lambda(t)$ denotes the force of infection.

In the past, several function forms of $\lambda(t)$ have been proposed and the linear form below is typical:

$$
\lambda(t) = \beta I(t), \quad (\beta \text{ is a positive constant}).
$$

The initial conditions of (1) to (3) are given by

$$
S(0) = S_0 > 0, \quad I(0) = I_0 > 0, \quad R(0) = R_0 > 0.
$$

The interaction among each population density in (1) to (3) is described by Fig. 1.

![Diagram showing interaction between each population](image)

**Fig. 1: Interaction between each population**

Next, we introduce the age structure in (1) to (3). Firstly, letting $S(t,a)$, $I(t,a)$ and $R(t,a)$ be the density of the susceptible, the infected and the recovered at time $t$ and age $a$, we define the total population density $p(t,a)$ at time $t$ in such a way that

$$
p(t,a) = S(t,a) + I(t,a) + R(t,a), \quad (t,a) \in \Theta \times G,
$$

where $\Theta \times G = (0,T) \times (0,L)$ and where $T$ and $L$ are positive constants and especially $L$ denotes the maximum attainable age.

Then, the total population $P(t)$ at time $t$ is given by

$$
P(t) = \int_0^L p(t,x)dx.
$$

Denoting the age-dependent death and the birth rates by $\mu(a)$ and $b(a)$, the total birth, the survival and the net reproduction rates $B(t)$, $\Pi(a)$ and $\mathcal{R}$ are given by

$$
B(t) = \int_0^L b(x)p(t,x)dx,
\Pi(a) = \exp\{-\int_0^a \mu(x)dx\},
\mathcal{R} = \int_0^L b(x)\Pi(x)dx.
$$

Since the net reproduction rate $\mathcal{R}$ denotes the average number of live births of girls by a woman during her whole life, if this value is above (below) 1, the population increases (decreases) [15].

It should be noted that $\Pi(L) = 0$ in (9) holds because $L$ is the maximum attainable age.

Denoting the population under age $a$ at time $t$ by $N(t,a)$, we have

$$
N(t,a) = \int_0^a p(t,x)dx.
$$

It follows from (8) and (11) that

$$
N(t+h,a+h) - N(t,a) = \int_t^{t+h} B(s)ds - \int_0^h \int_0^a \mu(x)p(t+s,x)dxds.
$$

Differentiating (12) with respect to $h$ and taking $h = 0$ and using (11), we have

$$
\int_0^a \frac{\partial p(t,x)}{\partial t}dx + p(t,a) = B(t) - \int_0^a \mu(x)p(t,x)dx
$$

The relation (13) yields

$$
\frac{\partial p(t,a)}{\partial t} + \frac{\partial p(t,a)}{\partial a} = -\mu(a)p(t,a), \quad \text{in } \Theta \times G.
$$

We set the initial condition as below,

$$
p(0,a) = p_0(a), \quad \text{on } G.
$$

Taking $a = 0$ in (13), we have the boundary condition:

$$
p(t,0) = B(t), \quad \text{on } \Theta.
$$

Define the age profile $\theta(t,a)$ by

$$
\theta(t,a) = \frac{p(t,a)}{P(t)},
$$

then it is easily shown that $\theta(t,a)$ satisfies

$$
\frac{\partial \theta(t,a)}{\partial t} + \frac{\partial \theta(t,a)}{\partial a} = -\mu(a)\theta(t,a)
$$

$$
-\theta(t,a)\int_0^L (b(x) - \mu(x))\theta(t,x)dx, \quad \text{in } \Theta \times G,
$$

$$
\int_0^L \theta(t,x)dx = 1.
$$

The initial and the boundary conditions of (18) are given by

$$
\theta(0,a) = \frac{p_0(a)}{P(0)}.
$$


\[ \theta(t, 0) = \int_0^L b(x)\theta(t, x)dx. \]

Moreover, from (7), (8), (14), (16), (17) and \( p(t, L) = 0 \), we have

\[ \frac{dP(t)}{dt} = \alpha(t)P(t), \]

where \( \alpha(t) \) is defined by

\[ \alpha(t) = \int_0^L (b(x) - \mu(x))\theta(t, x)dx. \]

Let the stationary solution of (18) be \( \theta_\ast(a) \) which satisfies (19) and (21). Setting as

\[ \lambda = \int_0^L (b(x) - \mu(x))\theta_\ast(x)dx, \]

then noting (19), it follows from (18) that

\[ \theta_\ast(a) = \frac{e^{-\lambda a}\Pi(a)}{\int_0^L e^{-\lambda x}\Pi(x)dx}. \]

Since \( \theta_\ast(a) \) satisfies (21), from (25), we have

\[ \int_0^L e^{-\lambda x}b(x)\Pi(x)dx = 1. \]

It is known that (26) has a unique real solution \( \lambda = \alpha_\ast \)

and if the initial datum satisfies \( b(t + a)p_0(a) \neq 0 \), the following relation holds [15]

\[ \lim_{t \to \infty} \alpha(t) = \alpha_\ast. \]

From the discussion above, under the assumption below:

(A-1) The stable age profile is already attained,

we have

\[ \frac{dP(t)}{dt} = \alpha_\ast P(t). \]

Equation (28) means that the total population size \( P(t) \)

is exponentially changing but its age profile \( \theta(t, a) \)

defined by (17) has the stable age profile \( \theta_\ast(a) \). Moreover, we assume that

(A-2) The net reproduction rate \( R = 1 \).

Then, since we have \( \lambda = \alpha_\ast = 0 \) from (10) and (26),

it follows from (28) that the total population size \( P(t) \) is constant. Hence, we have

\[ P(t) = P(0) = \int_0^L p_0(x)dx = N(0, L) \equiv N_\ast. \]

Taking \( \lambda(= \alpha_\ast) = 0 \) in (25), from (17) and (29), we have

\[ p(t, a) \equiv p_\infty(a) = P(0)\theta_\ast(a) = N_\ast\theta_\ast(a) = \kappa\Pi(a), \]

where \( \kappa = N_\ast/M \) and where \( M \) denotes the life expectancy defined by \( \int_0^L \Pi(x)dx \).

In the same manner as the derivation of (14), we have the age-structured SIR mode instead of (1) to (3):

\[ \frac{\partial S(t, a)}{\partial t} + \frac{\partial S(t, a)}{\partial a} = - (\phi(a) + \mu(a))S(t, a) - \lambda(t, a)S(t, a), \]

\[ \frac{\partial I(t, a)}{\partial t} + \frac{\partial I(t, a)}{\partial a} = (\mu(a) + \gamma(a))I(t, a), \]

\[ \frac{\partial R(t, a)}{\partial t} + \frac{\partial R(t, a)}{\partial a} = \phi(a)S(t, a) + \gamma(a)I(t, a) - \mu(a)R(t, a), \]

where \( \phi(a) \) and \( \gamma(a) \) are age-dependent vaccination and recovery rates respectively.

It should be noted that (31) to (33) yield (14).

Letting \( \beta(a, \sigma) \) be the transmission rate between the susceptible aged \( a \) and the infective aged \( \sigma \), the force of infection \( \lambda(t, a) \) is given by

\[ \lambda(t, a) = \int_0^L \beta(a, \sigma)I(t, \sigma)d\sigma. \]

In order to clarify the \( I \)-dependency of \( \lambda(t, a) \), we write \( \lambda(t, a; I) \) for \( \lambda(t, a) \) in the sequel.

The initial and the boundary conditions are given by

(I. C.): \( S(0, a) = S_0(a) > 0, I(0, a) = I_0(a) > 0, \)

\[ R(0, a) = R_0(a) > 0, 0 < a < L, \]

(B. C.): \( S(t, 0) = S(t, a) = B(t) \equiv \int_0^L b(a)p(t, a)da > 0, \)

\[ I(t, 0) = 0, R(t, 0) = 0, 0 < t < T, \]

In the age-structured model (31) to (33), the newborn population is given by the boundary condition (36) because the age of the new born is zero.

### 3 Stochastic Age-structured Model

In this section, we propose the stochastic age-structured infectious model. First, we consider a temporal Wiener process \( w(t, a) \) with an incremental covariance \( Q \). We introduce the random fluctuation in the recovery rate \( \gamma(a) \). Modeling the random fluctuation in the recovery rate \( \gamma(a) \) as the white Gaussian noise \( \partial w(t, a)/\partial t \) with respect to time, we replace \( \gamma(a) \) by \( \gamma(a) + \varepsilon \partial w(t, a)/\partial t, (\varepsilon \) is constant), Eqs.(31) to (33) imply the stochastic age-structured model:

\[ dS(t, a) + \frac{\partial S(t, a)}{\partial a} dt = - (\phi(a) + \mu(a))S(t, a) dt \]

\[ - \lambda(t, a; I)S(t, a) dt, \text{ in } \Theta \times G, \]  

\[ dI(t, a) + \frac{\partial I(t, a)}{\partial a} dt = (\mu(a) + \gamma(a))I(t, a) dt - \varepsilon I(t, a)dw(t, a), \]

\[ \text{in } \Theta \times G, \]

\[ dR(t, a) + \frac{\partial R(t, a)}{\partial a} dt = \phi(a)S(t, a) dt + \gamma(a)I(t, a) dt \]
\[ -\mu(a)R(t,a)dt + \varepsilon I(t,a)dw(t,a), \quad \text{in } \Theta \times G, \quad (39) \]

The initial and the boundary conditions are given by (35) and (36).

It should be noted that the relation (14) holds for (37) to (39). Hence, it follows from (A-1), (A-2) and (30) that

\[ p(t,a) = S(t,a) + I(t,a) + R(t,a) \equiv p_{\infty}(a) = \kappa \Pi(a). \quad (40) \]

In order to normalize the variables, we define

\[ u(t,a) = \frac{S(t,a)}{p_{\infty}(a)}, \quad v(t,a) = \frac{I(t,a)}{p_{\infty}(a)}, \quad r(t,a) = \frac{R(t,a)}{p_{\infty}(a)}. \quad (41) \]

Then, noting that \( p_{\infty}(a) \) is the stationary solution of (14), it follows from (14) and (37) to (39) that

\begin{align*}
\frac{du(t,a)}{\partial a}dt & = -\phi(a)u(t,a)dt - \lambda(t,a;v)u(t,a)dt, \\
\frac{dv(t,a)}{\partial a}dt & = \lambda(t,a;v)u(t,a)dt - \gamma(a)v(t,a)dt - \varepsilon v(t,a)dw(t,a), \\
\frac{dr(t,a)}{\partial a}dt & = \phi(a)u(t,a)dt + \gamma(a)v(t,a)dt + \varepsilon v(t,a)dw(t,a),
\end{align*}

where

\[ \lambda(t,a;v) = \int_{0}^{L} \beta(a,\sigma)p_{\infty}(\sigma)v(t,\sigma)d\sigma. \quad (45) \]

Obviously, we have

\[ u(t,a) + v(t,a) + r(t,a) = 1. \quad (46) \]

From (46), \( u(t,a) \) is determined by \( v(t,a) \) and \( r(t,a) \), hence, we consider (42) and (43) in the following section.

### 4 Stability Analysis

In this section, we consider the stability of the disease-free steady state (DFS) of the stochastic age-structured infectious model proposed in Section 3. The DFS means the equilibrium solution with zero infected individuals. Firstly, we assume that there exist positive constants \( \phi_i, \beta_i \) and \( \gamma_i, \sigma_i \) for any \( a, \sigma \in G \) such that

\begin{align*}
\text{(A-3)} & : 0 < \phi_1 \leq \phi(a) \leq \phi_2, \quad 0 < \beta_1 \leq \beta(a,\sigma) \leq \beta_2, \\
\text{(A-4)} & : 0 < \mu_1 \leq \mu(a) \leq \mu_2, \quad 0 < \gamma_1 \leq \gamma(a) \leq \gamma_2.
\end{align*}

We summarize some definitions of the stochastic stability of the \( n \)-dimensional stochastic system below:

\begin{align*}
dx(t) & = f(x(t))dt + g(x(t))dw(t), \\
f(0) & = g(0) = 0,
\end{align*}

where \( w(t) \) is a Wiener process.

(Definition 1) (Mean Square Stable): The zero solution of (47) is called mean square stable if for any \( \varepsilon > 0 \), there exists a \( \delta(\varepsilon) > 0 \) such that

\[ ||\varphi|| < \delta(\varepsilon) \rightarrow E(||x(t)||^2) < \varepsilon, \quad \forall t \geq 0, \quad (49) \]

where \( || \cdot || \) is the norm in \( L^2(G) \).

(Definition 2) (Asymptotically Mean Square Stable): The zero solution of (47) is called asymptotically mean square stable if it is mean square stable and there exists \( \delta > 0 \) such that

\[ ||\varphi|| < \delta \rightarrow \lim_{t \to \infty} E(||x(t)||^2) = 0. \quad (50) \]

(Definition 3) (Stable in Probability): The zero solution of (47) is called stable in probability if for any \( \varepsilon_1 > 0, \varepsilon_2 > 0 \), there exists \( \delta(\varepsilon) \equiv \delta(\varepsilon_1,\varepsilon_2) \) such that

\[ P(||\varphi|| < \delta(\varepsilon)) = 1 \rightarrow P(\sup_{t \geq 0} ||x(t)|| > \varepsilon_1) < \varepsilon_2, \quad (51) \]

Using (46), (43) and (44) can be rewritten by

\begin{align*}
dv(t,a) + \frac{\partial v(t,a)}{\partial a}dt & = \{ \lambda(t,a)(1-v(t,a)) - \gamma(a)v(t,a))dt - \varepsilon v(t,a)dw(t,a), \\
dr(t,a) + \frac{\partial r(t,a)}{\partial a}dt & = \{ \phi(a)(1-v(t,a)) - \gamma(a)v(t,a))dt + \varepsilon v(t,a)dw(t,a),
\end{align*}

where

\begin{align*}
l(t,a,v) & = \int_{0}^{L} \beta(a,\sigma)p_{\infty}(\sigma)v(t,\sigma)d\sigma. \quad (45) \]

Obviously, we have

\[ u(t,a) + v(t,a) + r(t,a) = 1. \quad (46) \]

From (46), \( u(t,a) \) is determined by \( v(t,a) \) and \( r(t,a) \), hence, we consider (42) and (43) in the following section.

It is easily shown that the disease-free steady (DFS) state \((v_f(a), r_f(a))\) of (52) and (53) is given by

\[ (v_f(a), r_f(a)) = \left(0,1 - \exp\left(-\int_{0}^{a} \phi(x)dx\right)\right). \quad (56) \]

It should be noted that the steady state \( u_f(a) \) of \( u(t,a) \) becomes

\[ u_f(a) = 1 - r_f(a) = \exp\left(-\int_{0}^{a} \phi(x)dx\right). \quad (57) \]

Setting as

\[ x_1(t,a) = v(t,a), \quad x_2(t,a) = r(t,a) - r_f(a), \quad (58) \]

and using \( u(t,a) = 1 - v(t,a) - r(t,a) \) and \( dr_f(a)/da = \phi(a)(1-r_f(a)) = \phi(a)u_f(a) \), we have

\begin{align*}
dx_1(t,a) + \frac{\partial x_1(t,a)}{\partial a}dt & = \{ -\lambda(t,a;x_1(t,a))x_1(t,a) + x_2(t,a) \\
-u_f(a) - \gamma(a)x_1(t,a))dt - \varepsilon x_1(t,a)dw(t,a), \\
dx_2(t,a) + \frac{\partial x_2(t,a)}{\partial a}dt & = -\phi(a)(x_1(t,a) + x_2(t,a))dt
\end{align*}

\[ \text{where} \quad x_1(t,a) = v(t,a), \quad x_2(t,a) = r(t,a) - r_f(a). \quad (59) \]

\[ dx_1(t,a) + \frac{\partial x_1(t,a)}{\partial a}dt = \{ -\lambda(t,a;x_1(t,a))x_1(t,a) + x_2(t,a) \\
-u_f(a) - \gamma(a)x_1(t,a))dt - \varepsilon x_1(t,a)dw(t,a), \\
dx_2(t,a) + \frac{\partial x_2(t,a)}{\partial a}dt = -\phi(a)(x_1(t,a) + x_2(t,a))dt
\]
where
\[ \lambda(t, a; x_1) = \int_0^L \beta(a, \sigma) p_\infty(\sigma) x_1(t, \sigma) d\sigma. \] (61)

The linear parts of (59) and (60) become
\[ dx_1(t, a) + \frac{\partial x_1(t, a)}{\partial a} dt = \{\lambda(t, a; x_1) u_f(a) - \gamma(a) x_1(t, a)\} dt - \varepsilon x_1(t, a) dw(t, a), \] (62)
\[ dx_2(t, a) + \frac{\partial x_2(t, a)}{\partial a} dt = -\phi(a) (x_1(t, a) + x_2(t, a)) dt + \gamma(a) x_1(t, a) dt + \varepsilon x_1(t, a) dw(t, a). \] (63)

**Theorem 1** Under the assumptions (A-3), (A-4) and the following conditions
\[ \gamma_1 - \frac{1}{2}(1 + N^2 \beta_2^2) = \frac{1}{2} \varepsilon^2 \text{Tr}(Q) > 0, \] (64)
the origin of the linear system (62) and (63) is asymptotically mean square stable.

(Outline of Proof): Define \( V(x) \) in such a way that
\[ V(x(t)) = x_1^2(t) + Ax_2^2(t), \] (65)
where \( x = [x_1 \ x_2]' \) and \( A \) is given by
\[ A = \frac{\gamma_1 - \frac{1}{2}(1 + N^2 \beta_2^2) - \frac{1}{2} \varepsilon^2 \text{Tr}(Q)}{\varepsilon^2 \text{Tr}(Q) + \frac{2k}{\varepsilon}(\phi_2^2 + \gamma_2^2)}. \] (66)

Then, it suffices to show that there exist constants \( k_i > 0, (i = 0, 1, 2) \) such that
\[ k_0 ||x(t)||^2 \leq V(x(t)) \leq k_1 ||x(t)||^2, \] (67)
\[ L_1 V(x(t)) \leq -k_2 ||x(t)||^2, \] (68)
where the operator \( L_1(\cdot) \) in (68) is a generating operator of (62) and (63) given by
\[ L_1(\cdot) = \left\{ \frac{\partial(\cdot)}{\partial x} \right\}' F + \frac{1}{2} \text{tr} \left\{ \left( \frac{\partial(\cdot)}{\partial x} \right)' G GG' \right\}, \] (69)
and where \( F = [f_1 f_2]' \), \( G = [g_1 g_2]' \) and \( f_i, g_i, (i = 1, 2) \) are defined by
\[ f_1 = -\frac{\partial x_1}{\partial a} + \lambda u_f - \gamma x_1, \] (70)
\[ f_2 = -\frac{\partial x_2}{\partial a} - \phi(x_1 + x_2) + \gamma x_1, \] (71)
\[ g_1 = -\varepsilon x_1, \quad g_2 = \varepsilon x_1. \] (72)

For the detailed proof of Theorem 1, see Appendix A.

**Theorem 2** Under the same condition as Theorem 1, the origin of (59) and (60) is stable in probability.

(Outline of Proof): For any \( \delta > 0 \), considering \( x(t) \) satisfies
\[ P\{||x(t)|| < \delta\} = 1. \] (73)

By proving that the function \( V(x) \) is the stochastic Lyapunov function, i.e., which satisfies
\[ L_2 V(x(t)) \leq 0, \] (74)
the proof of Theorem 2 is performed.

In (74), the operator \( L_2(\cdot) \) is defined by replacing \( F = [f_1 f_2]' \) in (69), (70) and (71) by
\[ f_1 = -\frac{\partial x_1}{\partial a} - \lambda x_1(x_1 + x_2 - u_f) - \gamma x_1, \] (75)
\[ f_2 = -\frac{\partial x_2}{\partial a} - \phi(x_1 + x_2) + \gamma x_1. \] (76)

The function \( G \) in \( L_2(\cdot) \) is given by (72).

Since Theorem 2 is easily proved using the similar approach of Theorem 1 and the relation (73), the detailed proof is omitted.

### 5 Conclusions

In this paper, we have proposed the stochastic age-structured infectious model based on the conventional SIR (susceptible-infected-recovered) infectious model. The conventional infectious models are given by a system of the ordinary differential equations, whereas the age-structured model is described by a system of the partial differential equations.

Most of the previous studies for the infectious diseases have been performed under the deterministic framework and the assumption of the homogeneous population. However, in the realistic infectious diseases, the infection, the recovery rates and so on randomly change because of the environmental change and the individual difference. Moreover, the transmission of infection varies by age. Hence, the stochastic age-structured infectious model proposed in this paper plays an important role to analyze the real infectious diseases. Using the proposed model, we have derived the sufficient condition for the disease free steady state (DFS) to be stable by the stochastic Lyapunov theorem.

We are able to know the possibility of the epidemic spread by whether the DFS is stable or not. If the DFS is stable, the infectious disease has been eventually stamped out. In contrast, if the DFS is unstable, the disease prevails. In the case where the DFS is unstable, by controlling the vaccination rate so as to satisfy the stability condition given by Theorems 1 and 2, the control of the infectious disease is possible.

Although age structure is considered in the infectious model proposed in this paper, the spatial movement of individuals is not introduced. Hence, the study of the infectious disease by the stochastic diffusive infectious model with age structure is a future problem.

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A Proof of Theorem 1

Since it is clear that the function $V(x)$ satisfies (67), it suffices for the proof of Theorem 1 to prove (68). It follows from (62), (63) and (65) that

$$L_1 V(x) = -2 \left\{ \left( x_1, \frac{\partial x_1}{\partial u} + x_1 \lambda(x_1)u_f + \gamma x_1 \right) \right\}$$

$$\text{and}$$

$$-2A \left\{ \left( x_2, \frac{\partial x_2}{\partial u} + x_2(\phi - \gamma)x_1 + \phi x_2 \right) \right\} \epsilon^2 (1 + A)(Q x_1, x_1). \quad (77)$$

It should be noted that the following relations hold,

$$V_x(\infty) \leq \frac{N_v}{L} \quad |u_f(\infty)| \leq 1, \quad (78)$$

$$(Q x_1, x_1) \leq tr(Q)||x_1||^2, \quad (79)$$

$$||\lambda(x_1)|| \leq N_g^2 \beta_2 ||x_1||^2, \quad (80)$$

$$-2 \left( x_1, \frac{\partial x_1}{\partial u} \right) = -|x_1(L)|^2 + |x_1(0)|^2 \quad (81)$$

$$\text{and}$$

$$2|\lambda(x_1) u_f| \leq (1 + N_g^2 \beta_2)||x_1||^2, \quad (82)$$

$$2|\lambda(x_1) u_f| \leq (1 + N_g^2 \beta_2)||x_1||^2, \quad (83)$$

It follows from (77) to (83) that

$$L_2 V(x) \leq \left\{ \left( 2 \gamma_1 - 1 + N_g^2 \beta_2 - \epsilon^2 \text{tr}(Q) \right) + \left( \frac{2L}{\phi_1} \phi_2 + \gamma_2^2 + \epsilon^2 \text{tr}(Q) \right) A \right\} ||x_1||^2 - A \phi_1 ||x_2||^2 \quad (85)$$

(Substituting (66) into the 1st term of (84), we have)

$$= \left( \gamma_1 - \frac{1}{2} (1 + N_g^2 \beta_2) - \frac{1}{2} \epsilon^2 \text{tr}(Q) \right) ||x_1||^2 - A \phi_1 ||x_2||^2 \leq -C(||x_1||^2 + ||x_2||^2), \quad (86)$$

where $C$ is given by

$$C = \min \left\{ \epsilon^2 \text{tr}(Q) + \frac{2L}{\phi_1} \phi_2 + \gamma_2^2, \phi_1 \right\}. \quad (87)$$

Hence, since the function $V(x)$ satisfies (68), the origin of the linear system (62) and (63) is asymptotically mean square stable.