LABELLED COSPAN CATEGORIES AND PROPERADS

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Abstract. We prove Steinebrunner’s conjecture on the biequivalence between (colored) properads and labelled cospan categories. The main part of the work is to establish a 1-categorical, strict version of the conjecture, showing that the category of properads is equivalent to a category of strict labelled cospan categories via the symmetric monoidal envelope functor.

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One way to encode the data of a cobordism is as a cospan \(M_1 \to P \leftarrow M_2\), where \(P\) is a manifold and \(M_1\) and \(M_2\) are its “left” and “right” boundaries, respectively. By taking connected components, each such cospan gives a cospan of finite sets. This

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functor from cobordisms to cospans of finite sets precisely relates decompositions of objects and morphisms between the two categories. An abstract version of this appears under the name labelled cospan category in [Ste]. Therein, Steinebrunner develops the general theory of labelled cospan categories and uses it to show, among other things, that the classifying space of the category of 2-dimensional cobordisms is rationally equivalent to $S^1$.

The cobordism category is a prop in the sense of Adams and Mac Lane. It is in fact the free prop on a properad of connected cobordisms. Properads, introduced in [Val07] and independently under the name compact symmetric polycategory in [Dun06], are like props but without horizontal composition. Operations may have multiple inputs and outputs, and can be composed by attaching some (nonzero number) of inputs of one to the outputs of another. We do not give the precise definition (for that, see [HRY15, Chapter 3] or [YJ15, 11.7]), but rather an illustration in Figure 1 where the letters represent colors in the properad. In the example of cobordisms, the connected cobordisms form a properad and then become a prop when we allow disjoint unions. Steinebrunner conjectured that a similar connection between properads and labelled cospan categories holds in generality. Our main result is the following:

**Theorem A.** The 2-category of properads is biequivalent to the 2-category of labelled cospan categories.

This appears as Corollary 6.16 below, and establishes the first part of Conjecture 2.31 of [Ste] (we do not address the second part of the conjecture which concerns $\infty$-properads). The underlying functor of 1-categories from properads to labelled cospan categories is not an equivalence, so 2-categorical structure is essential. It is also the case (see Proposition 6.13) that if a symmetric monoidal category admits the structure of a labelled cospan category, then this structure is unique up to equivalence. Thus another interpretation of Theorem A is that it provides a faithful inclusion of properads into symmetric monoidal categories and identifies its image.

The main effort of this paper consists of a careful analysis of the symmetric monoidal envelope, which takes properads to symmetric monoidal categories; this functor extends the classical envelope of an operad [LV12, 5.4.1], which goes back to Boardman–Vogt. The envelope of the terminal properad is the category of cospans of finite sets, and applying the envelope to the unique map from a properad to the terminal properad yields a labelled cospan category. This construction actually produces a more rigid kind of object, which we have called a strict labelled cospan category. In the last section we establish a biequivalence (Theorem 6.15) between
labelled cospan categories and strict labelled cospan categories, which we combine with the following to obtain the main result.

**Theorem B** (Corollary 5.9 and Theorem 6.7). *There is a strict 2-equivalence between the 2-category of properads and the 2-category of strict labelled cospan categories.*

In particular, this yields an equivalence of categories between the underlying 1-categories. Though our focus in this introduction has been on 2-categories, it is this 1-categorical equivalence (Corollary 5.9) that is the core of the result.

A key ingredient in this paper is an efficient description of the symmetric monoidal envelope of a properad $P$. We use an alternative description of properads as Segal presheaves on a category of level graphs $\mathbf{L}$. That is, we use an equivalence of categories $\mathbf{Ppd} \simeq \text{Seg}(\mathbf{L})$: which appears below as Proposition 1.15 (the hard work of establishing this equivalence was already done in [HRY15] and [CH22]). The upshot is that a Segal $\mathbf{L}$-presheaf looks much more like a symmetric monoidal category than a properad does (in particular, it is important that the category $\mathbf{L}$ contains disconnected graphs, as these account for the monoidal structure). To form the envelope is then a relatively straightforward quotienting process, though managing symmetry isomorphisms still requires some care.

It is much more involved to go back in the other direction. Namely, given a strict labelled cospan category $C$, we would like to produce a Segal $\mathbf{L}$-presheaf that it comes from. There is an active-inert factorization system on $\mathbf{L}$, and there is essentially only one possible choice of an $\mathbf{L}_{\text{int}}$-presheaf that could be the restriction of our desired $\mathbf{L}$-presheaf. We must then extend the $\mathbf{L}_{\text{int}}$-presheaf by defining its action on active maps. Here we make use of additional simplicial structure: by using heights of level graphs, $\mathbf{L}$ is fibered over the simplicial category $\Delta$. The main thing that is missing are the inner face and degeneracy operators arising from this additional simplicial structure, and coskeletality arguments allow us to extend from low simplicial degree (using the composition and identities in $C$) to arbitrary simplicial degree.

**Remark** (Related work). As mentioned above, we can describe a properad by giving its envelope, along with map to the envelope of the terminal properad. The same thing can be done for operads, though we are not aware of any classical literature which pursues this idea. However, this is precisely the approach that is taken by recent work of Haugseng–Kock [HK] for $\infty$-operads (see also [BHS]). Following the initial public version of this paper, the preprints [KM] and [BS] appeared and offered related theorems to ours. The work by Barkan–Steinebrunner on $\infty$-properads is related to the Haugseng–Kock approach for $\infty$-operads. Some important results of [BS] are $\infty$-categorical versions our theorems, and by restricting to discrete objects they recover the $(2,1)$-categorical version of Theorem A (which is also an immediate consequence of Theorem A). Their elegant theory relies on the idea of ‘equifibered maps of $E_\infty$-monoids,’ and we highly recommend [BS] for an important alternative perspective. They also give information concerning the relationship with the ‘hereditary unique factorization categories’ of Kaufmann–Monaco [KM].

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1. Background

The category $\text{FinSet}$ is the category of finite sets and arbitrary functions between them. Let $F \subseteq \text{FinSet}$ be the full subcategory whose objects are the ordered sets $k = \{1, 2, \ldots, k\}$ for $k \geq 0$. We will write $\Delta$ for the (topologist’s) simplicial category, whose objects are the ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$ and morphisms preserve the $\leq$ relation. Note the shift in cardinality if you regard $\Delta$ as a subcategory of $F$.

The category $\text{Csp}$ has objects the same as $\text{FinSet}$, and morphisms from $A$ to $B$ are equivalence classes of cospans $A \to C \leftarrow B$, where two are identified if there is an isomorphism on the middle term:

\[
\begin{array}{ccc}
& C & \\
A & \cong & B \\
& \downarrow & \\
& C' & \\
\end{array}
\]

Composition of morphisms in $\text{Csp}$ is given by pushout

\[
\begin{array}{ccc}
& C & \\
A & \leftarrow & B \\
& \leftarrow & \downarrow & \leftarrow \\
& D & \leftarrow & E \\
\end{array}
\]

Choices of coproducts of finite sets yields a monoidal structure on $\text{Csp}$ with $\emptyset$ as the monoidal unit. This is not a cocartesian monoidal category, however.

1.1. Labelled cospan categories. This work is concerned with Steinebrunner’s notion of labelled cospan categories from [Ste], which are certain symmetric monoidal categories living over $\text{Csp}$. We briefly recall some definitions.

Definition 1.1. Let $\pi: C \to \text{Csp}$ be a symmetric monoidal functor. For the moment, it is convenient to write

\[
\begin{array}{ccc}
\pi_c & f & \pi_d \\
\downarrow & \downarrow & \downarrow \\
\pi_c & m & \pi_d \\
\end{array}
\]

for $\pi(f: c \to d)$.

- An object $c \in C$ is connected if $\pi(c)$ has cardinality one.
- A morphism $f: c \to d$ is connected if $m(f)$ has cardinality one.
- A morphism $f: c \to d$ is reduced if the cospan $\pi(f)$ is jointly surjective.
- $\text{hom}^c(c, d) \subseteq \text{hom}(c, d)$ denotes the connected morphisms and $\text{hom}^r(c, d) \subseteq \text{hom}(c, d)$ denotes the reduced morphisms.

By examining the diagrams

\[
\begin{array}{cc}
0 & 0 \\
\downarrow & \downarrow \\
1 & 1 \\
\downarrow & \downarrow \\
2 & 1 \\
\end{array}
\quad \& \quad
\begin{array}{cc}
0 & 1 \\
\downarrow & \downarrow \\
1 & 1 \\
\downarrow & \downarrow \\
1 & 1 \\
\end{array}
\]
whose diamonds are pushouts, we see that the sets of connected morphisms and reduced morphisms are not closed under composition.

If \( A \to X \leftarrow B \) represents an isomorphism in \( \mathbf{Csp} \), then both legs of the cospan are bijections. Thus every isomorphism in \( C \) is reduced, and every isomorphism involving a connected object is connected.

**Definition 1.2** (Steinebrunner). A labelled cospan category is a symmetric monoidal functor \( \pi : C \to \mathbf{Csp} \) satisfying the following:

1. If \( \pi(c) \) has cardinality \( n \), then we can find \( n \) objects \( c_1, \ldots, c_n \) which are connected so that \( c \) is isomorphic to \( c_1 \otimes \cdots \otimes c_n \).
2. If \( 1 \) is the tensor unit of \( C \), then the abelian monoid \( \text{hom}(1, 1) \) is freely generated by the set \( \text{hom}^r(1, 1) \) of connected morphisms.
3. The map
   \[
   \text{hom}^r(c, d) \times \text{hom}^r(1, 1) \longrightarrow \text{hom}(c, d)
   \]
   is a bijection.
4. For each four objects \( c, d, c', d' \) in \( C \), the following square is cartesian.
   \[
   \begin{array}{ccc}
   \text{hom}^r(c, d) \times \text{hom}^r(c', d') & \longrightarrow & \text{hom}^r(c \otimes c', d \otimes d') \\
   \downarrow \pi & & \downarrow \pi \\
   \text{hom}^r_{\mathbf{Csp}}(\pi c, \pi d) \times \text{hom}^r_{\mathbf{Csp}}(\pi c', \pi d') & \longrightarrow & \text{hom}^r_{\mathbf{Csp}}(\pi c \otimes \pi c', \pi d \otimes \pi d')
   \end{array}
   \]

A map of labelled cospan categories is a symmetric monoidal functor and a choice of monoidal natural isomorphism making the triangle over \( \mathbf{Csp} \) commute.

**Remark 1.3.** For a fixed symmetric monoidal functor \( f : C \to C' \), there is at most one monoidal natural isomorphism as displayed above; in other words, the forgetful functor from labelled cospan categories to symmetric monoidal categories is faithful. Uniqueness of the monoidal natural isomorphism follows from Definition 1.2(1) and the fact that there are unique isomorphisms in \( \mathbf{Csp} \) between objects of cardinality zero or one. Given two such natural isomorphisms, the following diagram of isomorphisms commutes for each of them, where \( c \cong c_1 \otimes \cdots \otimes c_n \) is the decomposition into connected objects guaranteed by (1).

\[
\begin{array}{ccc}
\pi'fc_1 \otimes \cdots \otimes \pi'fc_n & \longrightarrow & \pi c_1 \otimes \cdots \otimes \pi c_n \\
\downarrow \cong & & \downarrow \cong \\
\pi'(c_1 \otimes \cdots \otimes c_n) & \longrightarrow & \pi'f(c_1 \otimes \cdots \otimes c_n) \\
\downarrow \cong & & \downarrow \cong \\
\pi'fc & \longrightarrow & \pi fc
\end{array}
\]

Since the top map is the same for both natural isomorphisms, so too is the bottom map.
Remark 1.4 (2-categorical structure). We can regard the collection of labelled cospan categories as a 2-category, where a map between morphisms \((f, \alpha)\) and \((g, \beta)\) having the same source and target

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\alpha \cong & \swarrow & \nearrow \\
\Csp & & \Csp
\end{array}
\quad \quad
\begin{array}{ccc}
C & \xrightarrow{g} & C' \\
\beta \cong & \swarrow & \nearrow \\
\Csp & & \Csp
\end{array}
\]

is a monoidal natural transformation \(\gamma: f \Rightarrow g\) so that the composite natural transformation

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\gamma, \psi & \swarrow & \nearrow \\
\Csp & & \Csp
\end{array}
\quad \quad
\begin{array}{ccc}
C & \xrightarrow{g} & C' \\
\beta \cong & \swarrow & \nearrow \\
\Csp & & \Csp
\end{array}
\]

is an isomorphism. By Remark 1.3, we then have this composite is \(\alpha\). Steinebrunner only needs this when \(\gamma\) is an isomorphism, that is, he considers labelled cospan categories as forming a \((2,1)\)-category.

We will return to this 2-categorical structure of labelled cospan categories in Section 6.

1.2. The category of level graphs. In this section we recall the category of level graphs \(L\) from [CH22, §2.1]. As mentioned below [CH22, Lemma 2.1.9], this category is closely related to the double category of cospans, an enhancement of \(\Csp\). In Section 2 will see how to rederive \(\Csp\) (up to equivalence) from \(L\).

The category \(\mathcal{L}^n\) has, as its objects, pairs \((i,j)\) with \(0 \leq i \leq j \leq n\), and as morphisms, unique maps \((i,j) \rightarrow (k,\ell)\) whenever \(0 \leq k \leq i \leq j \leq \ell \leq n\). This presentation is as in [CH22, Definition 2.16], though it is also true that \(\mathcal{L}^n\) is isomorphic to the twisted arrow category of \([n] = \{0 \rightarrow 1 \rightarrow \cdots \rightarrow n\}\). Note that every square in \(\mathcal{L}^n\) commutes and is a pushout. Here is \(\mathcal{L}^3\):

\[
\begin{array}{cccc}
(0,0) & (1,1) & (2,2) & (3,3) \\
\uparrow & \uparrow & \uparrow & \uparrow \\
(0,1) & (1,2) & (2,3) & \\
\downarrow & \downarrow & \downarrow & \\
(0,2) & (1,3) & \\
\uparrow & \uparrow \\
(0,3)
\end{array}
\]

A level graph of height \(n\) is a functor \(G: \mathcal{L}^n \rightarrow \mathbf{F}\) that sends every square to a pushout.\footnote{This is slightly different from [CH22], where the target category was all finite sets and the functor only concerned the top layers \(\mathcal{L}_0^n \rightarrow \mathbf{FinSet}\). This makes no substantial difference, and we arrive at an equivalent category of level graphs below; compare with [CH22, 2.1.19].} In particular, level graphs of height 0 are just objects of \(\mathbf{F}\), and level graphs of height 1 are just cospans in \(\mathbf{F}\). We write \(G_{ij}\) or \(G_{i,j}\) for the value of \(G\) on
the object \((i, j)\) and do not label the structural maps \(G_{ij} \rightarrow G_{k\ell}\). Figure 2 gives a pictorial example of a particular height 2 level graph the following shape:

\[
\begin{array}{ccc}
6 & \check{\searrow} & 6 \\
4 & \check{\searrow} & 3 \\
2 & \check{\searrow}&
\end{array}
\]

Figure 2. A level graph of height 2

We consider the subcategory \(L_n\) of \(\text{Fun}(\mathcal{L}^n, \textbf{F})\) whose objects are the level graphs and whose morphisms are the natural transformations \(G \Rightarrow H\) satisfying the following two properties for each \(0 \leq i \leq j \leq n\):

1. The map \(G_{ij} \rightarrow H_{ij}\) is a monomorphism.
2. The naturality square

\[
\begin{array}{ccc}
G_{ij} & \rightarrow & H_{ij} \\
\downarrow & & \downarrow \\
G_{0n} & \rightarrow & H_{0n}
\end{array}
\]

is cartesian.

By the pasting law for pullbacks, the second condition is equivalent to the natural transformation being cartesian, as was required in [CH22, Definition 2.1.16]. This defines a functor \(L_* : \Delta^{op} \rightarrow \textbf{Cat}\), and we write \(L \rightarrow \Delta\) for the associated Grothendieck fibration. (The version of \(L\) appearing in [CH22] is a skeleton of this one.)

Some remarks on the simplicial category \(L_*\) will make later proofs easier to understand (see [CH22] for complete details). First we note that, in fact, the categories \(\mathcal{L}^n\) assemble into a cosimplicial category. Given a morphism \(\alpha : [m] \rightarrow [n]\) in \(\Delta\) the functor \(\alpha^* : \mathcal{L}^m \rightarrow \mathcal{L}^n\) takes the pair \((i, j)\) to the \((\alpha i, \alpha j)\). For instance all of the induced functors \(\mathcal{L}^0 \rightarrow \mathcal{L}^n\) take the unique object \((0, 0)\) to some object \((i, i)\) in \(\mathcal{L}^n\) (and never objects of the form \((i, j)\) with \(i \neq j\)). One can also consider the three coface morphisms \(\mathcal{L}^1 \rightarrow \mathcal{L}^2\). These can be visualized with color as follows:
These induce functors $L^2 \to L^1$ that, respectively, truncate level 2 of a height 2 level graph, contract level 1 of a height 2 level graph, and truncate level 0 of a height 2 level graph. More generally, on a height $n$ level graph $G: \mathcal{L}^n \to \mathcal{F}$, a morphism $\alpha: [m] \to [n]$ in $\Delta$ induces a functor $L^n \to L^m$ which precomposes with $\mathcal{L}^m \to \mathcal{L}^n$, i.e. $(\alpha^*G)_{ij} = G_{\alpha_i,\alpha_j}$ for each $(i,j) \in \mathcal{L}^m$.

If $G \to H$ is a map in $L_n$, then it is uniquely determined by $\{G_{ii} \to H_{ii}\}_{0 \leq i \leq n}$ and $\{G_{i-1,i} \to H_{i-1,i}\}_{1 \leq i \leq n}$, since the functors $G$ and $H$ are pushout-preserving. This implies that $L_n \to L_k \times_{L_0} L_{n-k}$ is fully faithful. Further, this functor is surjective on objects, hence an equivalence of categories. It is not injective on objects (unless $k = 0$ or $n$).

1.3. Properads as Segal $L$-presheaves. If $G$ is a level graph of height $m$ and $H$ is a level graph of height $n$, then an active map $G \to H$ is one whose image in $\Delta$ is active (preserves the top and bottom elements) and where $G_{0m} \to H_{0n}$ is a bijection [CH22, Remark 2.1.24]. Inert maps $G \to H$ are precisely those whose image in $\Delta$ is inert (distance preserving).

**Definition 1.5** (Elementary objects). The category $L$ contains several elementary objects. These consist of the edge $\epsilon \in L_0$, which corresponds to $1$ using $ob(L_0) = ob(F)$, and a collection of $m,n$-corollas $c_{m,n} \in L_1$ for non-negative integers $m$ and $n$. The graph $c_{m,n}$ is the following:

$$
\begin{array}{c}
\vdash \\
\downarrow \\
\downarrow \\
\downarrow \\
\vdash \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\begin{array}{c}
m \\
1 \\
1 \\
1 \\
n \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{array}
$$

**Definition 1.6.** The above allows to define three subcategories of $L$:

- Write $L_{\mathrm{int}}$ for the wide subcategory of $L$ on the inert morphisms.
- Write $L_{\mathrm{el}}$ for the full subcategory of $L_{\mathrm{int}}$ spanned by the elementary graphs of Definition 1.5.
• Write $\mathbf{L}_{\text{act}}$ for the wide subcategory of $\mathbf{L}$ on the active morphisms.

**Remark 1.7.** The three subcategories of Definition 1.6 make $\mathbf{L}^{\text{op}}$ into an algebraic pattern in the sense of [CH21, Definition 2.1].

**Definition 1.8 (Segal objects).** Let $\mathbf{L}^{el}_{/G}$ denote the category whose objects are inert maps $E \to G$ with $E$ elementary, and whose morphisms are commutative triangles with all maps inert. A presheaf $X \in \text{Psh}(\mathbf{L})$ is said to be Segal if

$$X_G \to \lim_{E \in (\mathbf{L}^{el}_{/G})^{\text{op}}} X_E$$

is a bijection for all $G \in \mathbf{L}$. We write $\text{Seg}(\mathbf{L}) \subseteq \text{Psh}(\mathbf{L})$ for the full subcategory consisting of the Segal presheaves.

**Definition 1.9 (Segal core).** Given $G \in \mathbf{L}$, the Segal core is the following colimit in $\text{Psh}(\mathbf{L})$

$$\text{Sc}(G) := \text{colim}_{E \in \mathbf{L}^{el}_{/G}} \mathcal{J}(E) \to \mathcal{J}(G)$$

which comes equipped with a map to the representable object $\mathcal{J}(G) = \text{hom}_{\mathbf{L}}(-, G)$.

A presheaf $X \in \text{Psh}(\mathbf{L})$ is Segal if and only if it is local with respect to the Segal core inclusions, that is, if and only if

$$\text{hom}(\mathcal{J}(G), X) \to \text{hom}(\text{Sc}(G), X)$$

is a bijection for all graphs $G$.

**Remark 1.10.** Suppose $X \in \text{Psh}(\mathbf{L})$ is Segal. We list explicit consequences. The first two are related to the ‘simplicial’ direction of $\mathbf{L}$; the second of these will play a role in the proof of Proposition 2.15. The third of these will be used repeatedly, and is about the behavior in the fibers.

• If $G \in \mathbf{L}_n$ and $0 < k < n$, let $\alpha : [k] \to [n]$ be the inert inclusion into the first part of the interval ($\alpha(t) = t$), $\beta : [n - k] \to [n]$ the inert inclusion into the last part of the interval ($\beta(t) = t + k$), and $k : [0] \to [n]$ pick out $k$. Then

$$X_G \to X_{\alpha^*(G)} \times_{X_{\kappa^*(G)}} X_{\beta^*(G)}$$

is an isomorphism. At the level of presheaves, we have a commutative square

$$\begin{array}{ccc}
\text{Sc}(\alpha^*(G)) \amalg_{\text{Sc}(k^*(G))} \text{Sc}(\beta^*(G)) & \cong & \mathcal{J}(\alpha^*(G)) \amalg_{\mathcal{J}(k^*(G))} \mathcal{J}(\beta^*(G)) \\
\downarrow \cong & & \downarrow \\
\text{Sc}(G) & & \mathcal{J}(G)
\end{array}$$

whose left edge is an isomorphism. The claimed statement follows by applying $\text{hom}(-, X)$ (which transforms pushouts to pullbacks) and using 2-of-3 for isomorphisms.

• We can iterate the previous observation. Suppose $G \in \mathbf{L}$ is a height $n$ level graph and for $1 \leq i \leq n$ that $\rho_i : [1] \to [n]$ in $\Delta$ is the inert map picking out $i - 1, i$, and for $0 \leq i \leq n$ the map $\kappa_i : [0] \to [n]$ picks out $i$. If $e \to G$ is an inert map in $\mathbf{L}$, then it factors uniquely through $\rho_i^* G$ (which has height 1) for some $i$. Likewise, any $e \to G$ factors uniquely through some $\kappa_i^* G$ (which has height 0). Further,

$$X_G \to X_{\rho_1^* G} \times_{X_{\kappa_1^* G}} X_{\rho_2^* G} \times_{X_{\kappa_2^* G}} \cdots \times_{X_{\kappa_n^* G}} X_{\rho_n^* G}$$

is a bijection. See [CH22, Proposition 3.2.9].
Suppose $G, H \in \mathbf{L}_n$ are two height $n$ level graphs. We can define a new height $n$ level graph $G + H$ with $(G + H)_{ij} = G_{ij} + H_{ij}$, which is a coproduct in the fiber $\mathbf{L}_n$ (see Section 2.3 where this will be used more). Then $\mathbf{L}_{(G+H)}^e = \mathbf{L}_G^e \amalg \mathbf{L}_H^e$, so it follows that the left hand map in the following square is an isomorphism:

\[
\begin{array}{ccc}
\text{Sc}(G) & \amalg & \text{Sc}(H) \\
\downarrow \cong & & \downarrow \\
\text{Sc}(G + H) & \longrightarrow & \text{Sc}(G + H) \\
\end{array}
\]

Applying hom($-\!, X$) and using 2-of-3 for isomorphisms, we see that $X_{G+H} \to X_G \times X_H$ is a bijection:

\[
\begin{array}{ccc}
X_{G+H} & \longrightarrow & X_G \times X_H \\
\downarrow \cong & & \downarrow \cong \\
\lim_{E \in (\mathbf{L}_{(G+H)}^e)^{op}} X_E & \cong & \left( \lim_{E \in (\mathbf{L}_G^e)^{op}} X_E \right) \times \left( \lim_{E \in (\mathbf{L}_H^e)^{op}} X_E \right) \\
\end{array}
\]

We also have the full subcategory $\mathbf{L}_c \subset \mathbf{L}$ whose objects are the connected level graphs (a level graph $G$ of height $m$ is connected just when $G_0^m$ is a point). This subcategory inherits an active-inert factorization system from the larger category $\mathbf{L}$. The elementary objects in $\mathbf{L}_c$ are precisely the elementary objects of $\mathbf{L}$ and there is a corresponding notion of Segal $\mathbf{L}_c$-presheaf. Finally, there is the graphical category defined in [HRY15] whose objects are certain connected graphs (see [HRY15, Definition 6.46] or [CH22, Definition 2.2.11]). Following [CH22], we call this category $\mathbf{G}$, and it has its own notion of Segal presheaf. We will not need any low-level details about this category or its algebraic pattern structure.

There is a zig-zag of functors over $\mathbf{FinSet}_{*}^{op}$

\[
\begin{array}{ccc}
\mathbf{L} & \leftarrow & \mathbf{L}_c \\
\downarrow & & \downarrow \\
\mathbf{FinSet}_{*}^{op} & \leftarrow & \mathbf{G} \\
\end{array}
\]

where the downward functors take sets of vertices.

**Proposition 1.11.** The functors $\iota^* : \text{Psh}(\mathbf{L}) \to \text{Psh}(\mathbf{L}_c)$ and $\tau^* : \text{Psh}(\mathbf{G}) \to \text{Psh}(\mathbf{L}_c)$ restrict to functors $\text{Seg}(\mathbf{L}) \to \text{Seg}(\mathbf{L}_c) \leftarrow \text{Seg}(\mathbf{G})$ which are equivalences of categories.

One can prove that $\iota^*$ is an equivalence on Segal presheaves by imitating the proof of Proposition 3.2.29 of [CH22], which is an $\infty$-categorical analogue. Similarly, one can imitate the proof of Theorem 5.1.4 of [CH22] to show that $\tau^* : \text{Seg}(\mathbf{G}) \to \text{Seg}(\mathbf{L}_c)$ is an equivalence. However, the latter proof is very involved. Instead, we deduce both of these results by recognizing that $\text{Seg}(\mathbf{X})$ is equivalent to the subcategory of discrete objects of the $\infty$-categorical version $\text{Seg}^\infty(\mathbf{X}) \subseteq \text{Psh}^\infty(\mathbf{X}) := \text{Fun}(\mathbf{X}^{op}, \mathbf{S})$ when $\mathbf{X} \in \{ \mathbf{L}, \mathbf{L}_c, \mathbf{G} \}$. Recall from [Lur09, 5.5.6.2] that an object $x$ in a quasi-category is called discrete if, for every object $y$, the Kan complex $\text{Map}(y, x)$ is equivalent to a set. We utilize the more general notion of algebraic pattern from [CH21, Definition 2.1]; in that source $\text{Seg}^\infty(\mathbf{X})$ would be denoted by $\text{Seg}^\infty_{\mathbf{X}^{op}}(\mathbf{S})$ and $\text{Seg}(\mathbf{X})$ would be denoted by $\text{Seg}_{\mathbf{X}^{op}}(\text{Set})$. 
Lemma 1.12. Let $\mathbf{X}$ be a category equipped with an active-inert factorization system and a class of elementary objects so that $\mathbf{X}^{\text{op}}$, together with this structure, is an algebraic pattern. There is a fully faithful functor $\text{Seg}(\mathbf{X}) \hookrightarrow \text{Seg}^\infty(\mathbf{X})$ whose essential image is the full subcategory of discrete objects.

Proof. Let $\text{Set} \hookrightarrow \mathcal{S}$ be the fully-faithful inclusion of the $\infty$-category of sets into that of spaces. As postcomposition with a fully faithful functor is fully faithful (see Remark 1.13 below), there is a fully faithful composite $\text{Seg}(\mathbf{X}) \hookrightarrow \text{Psh}(\mathbf{X}) \hookrightarrow \text{Psh}^\infty(\mathbf{X})$. This functor factors through $\text{Seg}^\infty(\mathbf{X})$ since $\text{Set} \hookrightarrow \mathcal{S}$ preserves limits [Lur09, 5.5.6.18]. It is automatic that $\text{Seg}(\mathbf{X}) \hookrightarrow \text{Seg}^\infty(\mathbf{X})$ is fully faithful. The essential image of the functor consists of those Segal objects $F: \mathbf{X}^{\text{op}} \to \mathcal{S}$ so that $F(x)$ is discrete for all $x \in \mathbf{X}$. We now show this is the same thing as $F \in \text{Seg}^\infty(\mathbf{X})$ being a discrete object.

Suppose $G: \mathbf{X}^{\text{op}} \to \mathcal{S}$ is an arbitrary presheaf, and write $G \simeq \text{colim}_a \chi(x_a)$, where $\chi: \mathbf{X} \to \text{Psh}^\infty(\mathbf{X})$ is the Yoneda embedding. Then there are equivalences

$$\lim \chi(x_a) \simeq \lim \text{Map}(\chi(x_a), F) \simeq \text{Map}(G, F) \simeq \text{Map}(LG, F)$$

where $L: \text{Psh}^\infty(\mathbf{X}) \to \text{Seg}^\infty(\mathbf{X})$ is the localization functor guaranteed by [CH21, Lemma 2.11(iii)]. If $F(x)$ is discrete for all $x \in \mathbf{X}$, then since $\text{Set} \hookrightarrow \mathcal{S}$ is closed under limits, we see that $\text{Map}(G, F)$ is discrete hence $F \in \text{Seg}^\infty(\mathbf{X})$ is discrete. Conversely, if $F$ is a discrete object of $\text{Seg}^\infty(\mathbf{X})$ and $x \in \mathbf{X}$ is arbitrary, taking $G = \chi(x)$ in the string of equivalences above implies that $F(x)$ is discrete. We have thus identified the essential image of $\text{Seg}(\mathbf{X}) \hookrightarrow \text{Seg}^\infty(\mathbf{X})$ with the discrete objects of $\text{Seg}^\infty(\mathbf{X})$. □

Remark 1.13. It is well known that if $\mathcal{C} \to \mathcal{D}$ is a fully faithful functor of quasi-categories, then the postcomposition functor $\text{Fun}(\mathcal{B}, \mathcal{C}) \to \text{Fun}(\mathcal{B}, \mathcal{D})$ is also fully faithful. One way to verify involves first observing that [RV20, 5.6] and [RV22, 3.5.6(iv)] together imply that being fully faithful as a functor of quasi-categories is equivalent to being fully faithful as a functor in the $\infty$-cosmos of quasi-categories. Since this $\infty$-cosmos is cartesian closed, the assertion follows from the characterization given in [RV22, 3.5.6(iii)].

Proof of Proposition 1.11. As a result of [Lur09, 5.5.6.28], an equivalence of quasi-categories preserves and reflects discrete objects. Therefore it restricts to an equivalence of categories between the full subcategories of its domain and codomain. Proposition 3.2.29 and Theorem 5.1.4 of [CH22] show that $\iota$ and $\tau$ induce equivalences $\text{Seg}^\infty(\mathbf{L}) \to \text{Seg}^\infty(\mathbf{L}_c) \leftarrow \text{Seg}^\infty(\mathbf{G})$. The result now follows from Lemma 1.12. □

Let $\chi_3: \mathbf{G} \to \mathbf{Ppd}$ be the functor which takes a graph to a properad freely generated by it; see [HRY15, §5.1.2]. We let $\chi_2 = \chi_3 \circ \tau$, and now describe $\chi_1$ appearing in the following diagram.

$$\begin{array}{ccc}
\mathbf{L} & \xleftarrow{\iota} & \mathbf{L}_c \\
\downarrow{\chi_1} & & \downarrow{\chi_2} \\
\mathbf{Ppd} & \xrightarrow{\chi_3} & \mathbf{G}
\end{array}$$

If $G$ is a height $m$ level graph, we can decompose $G$ into connected components $G \cong \coprod_{x \in G_{\text{om}}} G_x$ (in Definition 4.6 we will fix a particular such isomorphism). The functor $\chi_1: \mathbf{L} \to \mathbf{Ppd}$ is defined by sending $G$ to $\coprod \chi_2(G_x) = \coprod \chi_3 \tau(G_x)$.

\text{Let us describe $\chi_1$ applied to a morphism $f: \mathbf{G} \to \mathbf{H} \cong \coprod_{y \in H_{\text{om}}} H_y$ lying over $\alpha: [m] \to [n]$ in $\Delta$. For $x \in G_{\text{om}}$ write $\tilde{x}$ for the image of $x$ under the composite $G_{\text{om}} \to H_{\alpha(0)(\alpha(m))} \to H_{\text{om}}$. Then}
We obtain from these functors the following commutative diagram:

\[
Psh(L) \xrightarrow{\iota^*} Psh(L_c) \xrightarrow{\iota^*} Psh(G) \\
\downarrow N_1 \downarrow \downarrow N_2 \downarrow N_3 \downarrow \\
Ppd
\]

with \(N_1(P)_G = \text{hom}(\chi_1(G), P)\); this commutes since if \(G\) is a connected level graph, then

\[
\iota^*N_1(P)_G = N_1(P)_{\iota G} = \text{hom}(\chi_1(\iota G), P) = \text{hom}(\chi_2(G), P) = N_2(P)_G
\]

and likewise for the other triangle. By [HRY15, Lemma 7.38], \(N_3\) takes values in the subcategory of Segal objects.

Since we know that \(\tau^*\) takes Segal objects to Segal objects and \(N_2 = \tau^* \circ N_3\), we have that \(N_2\) lands in the subcategory of Segal objects. This same argument does not apply to \(N_1\), so we must check the following:

**Lemma 1.14.** If \(P\) is a properad, then \(N_1(P)\) is Segal.

**Proof.** Let \(G \in L\) be a height \(m\) level graph, and for \(x \in G_0\) let \(G_x \in L_c\) be the corresponding connected level graph with \(\coprod x G_x = G\) in the fiber \(L_m\). Notice that the canonical map \(N_1(P)_G \rightarrow \coprod x N_1(P)_{G_x}\) is an isomorphism, using our description of \(\chi_1:\)

\[
N_1(P)_G = \text{hom}(\chi_1(G), P) = \text{hom} \left( \coprod x \chi_2(G_x), P \right) = \coprod x N_1(P)_{G_x}.
\]

The map \(\coprod x \text{Sc}(G_x) \rightarrow \text{Sc}(G)\) is an isomorphism of presheaves, where \(\text{Sc}(G)\) is the Segal core of \(G\). In the commutative square

\[
\begin{array}{ccc}
\coprod x \text{Sc}(G_x) & \xrightarrow{=} & \text{Sc}(G) \\
\downarrow & & \downarrow \\
\coprod x \mathbb{k}(G_x) & \xrightarrow{=} & \mathbb{k}(G)
\end{array}
\]

we know that the bottom map becomes an isomorphism after applying \(\text{hom}(\mathbb{Z}, \cdot)\). Hence to see that \(N_1(P)\) is Segal it is enough to check that \(\text{hom}(\mathbb{k}(H), N_1(P)) \rightarrow \text{hom}(\text{Sc} H, N_1(P))\) is an isomorphism for connected graphs \(H\). But this is true: if \(H \in L_c\) is connected, then \(\iota_! \mathbb{k}(H) = \mathbb{k}(\iota H)\) and \(\iota_! \text{Sc}(H) = \text{Sc}(\iota H)\), and by Segality of \(N_2(P) = \iota^* N_1(P)\) the bottom map in the following square

\[
\begin{array}{ccc}
\text{hom}(\iota_! \mathbb{k} H, N_1 P) & \rightarrow & \text{hom}(\iota_! \text{Sc} H, N_1 P) \\
\downarrow & = & \downarrow \\
\text{hom}(\mathbb{k} H, \iota^* N_1 P) & \rightarrow & \text{hom}(\text{Sc} H, \iota^* N_1 P)
\end{array}
\]

is an isomorphism. \(\square\)

The composite \(G_x \rightarrow G \rightarrow H\) factors through a map between connected level graphs \(f_x: G_x \rightarrow H_x\), and \(\chi_1(f): \coprod x \chi_2(G_x) \rightarrow \coprod x \chi_2(H_x)\) is induced from the properad maps \(\chi_2(f_x)\).
Proposition 1.15. The functors $N_1$ and $N_2$ induces equivalences of categories

\[ N_1 : \text{Ppd} \simeq \text{Seg}(L) \]
\[ N_2 : \text{Ppd} \simeq \text{Seg}(L_c) \]

Proof. Each $N_i$ lands in the full subcategory of Segal objects (using Lemma 1.14 for $N_1$), hence we have the commutative diagram

\[
\begin{array}{ccc}
\text{Seg}(L) & \xrightarrow{\iota^*} & \text{Seg}(L_c) \\
\downarrow{N_1} & & \uparrow{N_2} \\
\text{Ppd} & \xrightarrow{\tau^*} & \text{Seg}(G)
\end{array}
\]

By Proposition 1.11, $\iota^*$ and $\tau^*$ are equivalences, and $N_3$ is an equivalence by [HRY15]. By 2-of-3, $N_1$ and $N_2$ are equivalences as well. □

We will have no further need for $L_c$ or $G$ in this paper.

2. The symmetric monoidal envelope of a properad

The envelope of a properad is the prop freely generated by it. In this section we give a detailed description of this structure, in a way that will make our later comparisons more transparent. We take as an input a Segal $L$-presheaf, relying on the equivalence of categories from Proposition 1.15.

Remark 2.1. In [HRY17, §4.A], a left adjoint to the forgetful functor from props to properads is exhibited. We caution the reader that this is not quite the same as what we are doing here, since the kind of prop used in that paper is slightly weaker than the original version [Mac65, HR15]. See [BB17, Remark 10.5] and [HR17, Remark 3.5] for details.

2.1. Congruences of level graphs.

Definition 2.2. A congruence in $L_n$ is an isomorphism $G \to H$ so that for $0 \leq i \leq n$, the map $G_{ii} \to H_{ii}$ is an identity. We denote such a congruence by

\[ G \equiv H. \]

We say that $G$ and $H$ are congruent, denoted $G \equiv H$, if there is a congruence between them.

If $G \in L_0$, then the only congruence involving $G$ is the identity. The same is true for the corollas $c_{n,m} \in L_1$.

Lemma 2.3. If $\alpha : [m] \to [n]$ is a map in $\Delta$ and $G \to H$ is a congruence in $L_n$, then $\alpha^*G \to \alpha^*H$ is a congruence in $L_m$.

Proof. If $0 \leq i \leq j \leq m$, then $(\alpha^*G)_{ij} \to (\alpha^*H)_{ij}$ is equal to $G_{\alpha_i \alpha_j} \to H_{\alpha_i \alpha_j}$. When $i = j$ this map is an identity by definition of congruence. □

Remark 2.4. Suppose $G \in L_1$ is a graph where every vertex has an input or output. Then the only congruence $G \equiv G$ is the identity, as the hypothesis tells us the horizontal morphisms in the following diagram are epimorphisms

\[
\begin{array}{ccc}
G_{00} + G_{11} & \longrightarrow & G_{01} \\
\downarrow{=} & & \downarrow{=} \\
G_{00} + G_{11} & \longrightarrow & G_{01}
\end{array}
\]
which implies that $G_{01} \to G_{01}$ is the identity as well. This is not to say that $G$ is not involved in any congruence at all, only that if one permutes the elements of $G_{01}$ then one must also alter at least one of the functions $G_{00} \to G_{01}$ and $G_{11} \to G_{01}$. Indeed, if $G_{01} = n$, then there are $n!$ different congruences with domain $G$.

**Definition 2.5** (Congruence category). Let $K_n \subseteq L_n$ denote the wide subcategory consisting of the congruences. This defines a functor $K_* : \Delta^{op} \to \text{Gpd} \subseteq \text{Cat}$, and we write $K \to \Delta$ for the associated (right) fibration.

It will be useful to collapse further:

**Definition 2.6.** Let $W \to \Delta$ be the discrete fibration associated to the composite

$$
\Delta^{op} \xrightarrow{K_*} \text{Gpd} \xrightarrow{\pi_0} \text{Set}.
$$

That is, $W_n = \pi_0(K_n)$ is the set of height $n$ level graphs modulo congruence. Note that $W_n$ is a discrete category, but $W$ itself is not.

**Remark 2.7.** The simplicial set $W_*$ turns out to be isomorphic to the nerve of a skeleton of $\text{Csp}$. Elements of $W_0$ can be identified with the set $\{k\}$ for $k \in \mathbb{N}$. Elements of $W_1$ are congruence classes of height 1 level graphs, and this relation means these are exactly the same thing as morphisms in $\text{Csp}$ between the sets appearing in $\{k\}_{k \in \mathbb{N}}$. In Corollary 2.9 below we will formally show that $W_*$ is Segal, and by inspection one can see that the compositions coincide in $W_*$ and $\text{NCsp}$. See Example 2.17 and Example 2.34 for more details.

The zig-zags $L_n \leftarrow K_n \to \pi_0(K_n) = W_n$ as $n$ varies constitute natural transformations

$$
\begin{array}{ccc}
\Delta^{op} & \xrightarrow{K_*} & \text{Gpd} \\
\uparrow & & \downarrow \\
W_* & \xrightarrow{\pi_0} & \text{Set}
\end{array}
$$

(considering $\text{Set} \subseteq \text{Gpd} \subseteq \text{Cat}$). This yields the following commutative diagram of fibrations over $\Delta$.

$$
\begin{array}{ccc}
W & \xleftarrow{\pi} & K \\
\downarrow{q} & & \downarrow{p} \\
\Delta & \xrightarrow{i} & L
\end{array}
$$

We saw above that $L_* : \Delta^{op} \to \text{Cat}$ is Segal, meaning that the Segal morphisms $L_n \to L_1 \times L_0 \cdots \times L_0 L_1$ are all equivalences of categories. The same holds for $K_*$.  

**Proposition 2.8.** The simplicial category $K_*$ is Segal. Moreover, the Segal morphisms are surjective on objects.

**Proof.** Let $n > 1, 1 < k < n$, and consider the pushout square in $\Delta_{int}$ whose vertical maps preserve last elements and horizontal maps preserve first elements.

$$
\begin{array}{ccc}
[0] & \xrightarrow{0} & [n - k] \\
\downarrow{k} & & \downarrow{r} \\
[k] & \xrightarrow{\alpha} & [n]
\end{array}
$$

We must show

$$
\alpha^* \times \beta^* : K_n \to K_k \times K_0 K_{n-k}
$$
is an equivalence of categories. We already know that the map is surjective on objects since \( L_n \to L_k \times L_0 \times L_{n-k} \) is. Since \( L_n \to L_k \times L_0 \times L_{n-k} \) is fully faithful, it is enough to show that if \( G, H \in K_n \) are height \( n \) level graphs and \( \alpha^* G \to \alpha^* H \) and \( \beta^* G \to \beta^* H \) are congruences, then they define a congruence \( G \to H \). But this is clear, since

\[
(G_{ii} \to H_{ii}) = \begin{cases} 
(\alpha^* G)_{ii} \to (\alpha^* H)_{ii} & \text{if } 0 \leq i \leq k \\
(\beta^* G)_{i-k,i-k} \to (\beta^* H)_{i-k,i-k} & \text{if } k \leq i \leq n
\end{cases}
\]

which are identities. \( \square \)

Though \( \pi_0 : Gpd \to Set \) does not preserve pullbacks in general, we still have the following result.

**Corollary 2.9.** The simplicial set \( W_* \) is Segal.

**Proof.** We use the same notation as in the previous proof. We are to show that the bottom map in the following square is a bijection.

\[
\begin{array}{ccc}
K_n & \xrightarrow{\sim} & K_k \times K_0 \times K_{n-k} \\
\downarrow & & \downarrow \\
\pi_0(K_n) & \xrightarrow{\alpha^* \times \beta^*} & \pi_0(K_k) \times \pi_0(K_0) \times \pi_0(K_{n-k})
\end{array}
\]

The bottom map is surjective, since the top-right composite is surjective on objects. Suppose \( w, v \in W_n \) map to the same element in \( W_k \times W_0 \times W_{n-k} \). Let \( G, H \in K_n \) be graphs in these path components. The assumption is that \( \alpha^* G \equiv \alpha^* H \) and \( \beta^* G \equiv \beta^* H \). Choose congruences witnessing these relations, giving a morphism \( (\alpha^* G, \alpha^* H) \to (\alpha^* H, \beta^* H) \in K_k \times K_0 \times K_{n-k} \). Since the top map in the square above is an equivalence, this comes from a congruence in \( K_n \). Hence \( G \equiv H \), that is, \( w = v \). \( \square \)

### 2.2. The category \( CX \).

Our next task is to describe the envelope of a properad. For now, we content ourselves in producing this as a category, rather than as a symmetric monoidal category. More specifically, we produce a functor \( S \) as in the following diagram by using restriction along \( i \) and left Kan extension along \( q \).

\[
\begin{array}{ccc}
\text{Seg}(L) & \xrightarrow{\sim} & \text{Seg}(\Delta) \\
\downarrow & & \downarrow \\
\text{Psh}(L) & \xrightarrow{\pi^*} & \text{Psh}(K) \xrightarrow{\pi_0} \text{Psh}(\Delta)
\end{array}
\]

and show that it takes Segal presheaves to Segal presheaves. We write \( C : \text{Seg}(L) \to \text{Cat} \) for the composite of \( S \) with the left adjoint of the nerve functor \( N \).

**Lemma 2.10.** Given a commutative triangle of small categories

\[
\begin{array}{ccc}
E & \xrightarrow{p} & B \\
\downarrow{q} & & \downarrow{r} \\
A & & 
\end{array}
\]

If \( q \) is a fibration and \( r \) is a discrete fibration, then \( p \) is a fibration.
Proof. If $b \in B$ is any object, then $B_{/b} \to A_{/rb}$ is an isomorphism. Indeed, it is bijective on objects since any $a \to rb$ has a unique lift $\tilde{a} \to b$. It is also fully faithful since a string $a_0 \to a_1 \to rb$ has a unique lift $\tilde{a}_0 \to \tilde{a}_1 \to b$ in $B$; but the composite $\tilde{a}_0 \to b$ is the unique lift of $a_0 \to rb$.

To show that $p$ is a fibration it is enough to exhibit a right adjoint right inverse to $E_{/e} \to B_{/pe}$ for every object $e$ (see [Gra66, Theorem 2.10] or [LR20, Theorem 2.2.2]). But since $q$ is a fibration, for each $e$ the functor $E_{/e} \to A_{/qe}$ admits a right adjoint right inverse. So choose one and this defines the desired functor $B_{/pe} \cong A_{/rpe} \to E_{/e}$. □

Since $q\pi : K \to \Delta$ is a fibration and $q$ is a discrete fibration, we have:

Lemma 2.11. The functor $\pi : K \to W$ is a Grothendieck fibration. □

It is classical that left Kan extension along an opfibration may be computed by taking the colimit over the fibers (see [na]), which gives the following.

Lemma 2.12. If $A$ is a $K$-presheaf, then its left Kan extension $\pi!A$ is given by

$$(\pi!A)_w = \colim_{G \in K^w} A_G.$$ where $K_w \subseteq K$ is the relevant connected component of $K_{q(w)}$. Likewise, if $B$ is a $W$-presheaf, then

$$(q_B)_n = \sum_{w \in W_n} B_w.$$ Definition 2.13. Let $S$ denote the composition

$$\text{Psh}(L) \xrightarrow{i^*} \text{Psh}(K) \xrightarrow{q^*\pi} \text{Psh}(\Delta).$$

That is, if $X \in \text{Psh}(L)$, define $SX \in \text{Psh}(\Delta)$ as

$$SX = (q\pi)i^*X.$$ In light of Lemma 2.12, this means that

$$SX_n = \sum_{w \in W_n} \colim_{G \in K^w} X_G.$$ We write $X_G$ for $\colim_{G \in K^w} X_G$, which is a quotient of $X_G$ by the action of self-congruences of $G$. We occasionally write $X_w$ for this set, where $w = |G| = \pi(G) \in W$.

Remark 2.14. If every vertex of $G \in L_1$ has an input or output, then the canonical map $X_G \to X_G = \colim_{G \in K^w} X_G$ is an isomorphism by Remark 2.4.

Proposition 2.15. If $X \in \text{Seg}(L)$, then $SX \in \text{Seg}(\Delta)$.

Proof. Suppose we have a pushout square of non-identity maps in $\Delta_{\text{int}}$ whose vertical maps preserve last elements and horizontal maps preserve first elements.

$$\begin{array}{ccc}
[0] & \xrightarrow{0} & [\ell] \\
\downarrow^k & \ & \downarrow^\beta \\
[k] & \xrightarrow{\alpha} & [n]
\end{array}$$

It is enough to show that $SX$ sends pushouts of this form to pullbacks.
We use the description of $SX_n$ from (1) above. Suppose we have two elements $s, s'$ of $SX_n$, represented by $(w, G, x)$ and $(w', G', x')$, which are sent to the same element of $SX_k \times_{SX_t} SX_t$ (Here $G$ is a graph in the path component $w$ of $K_n$ and $x$ is an element of $X_G$ with $s$ the image of $x$ under $X_G \to \text{colim } X_H \subseteq SX_n$). Then $w = w'$ by Corollary 2.9. Write $\alpha^* x \in X_{\alpha'^* G}$, $\beta^* x \in X_{\beta'^* G}$, $\alpha^* x' \in X_{\alpha'^* G'}$, and $\beta^* x' \in X_{\beta'^* G'}$ for the images of $x$ and $x'$. Since $\alpha^* x$ and $\alpha^* x'$ represent the same element in the colimit

$$\text{colim}_{H \in K_{n(w)}} X_H$$

there is a congruence $\gamma: \alpha^* G \to \alpha^* G'$ with $\gamma^*(\alpha^* x') = \alpha^* x$. Likewise, there is a congruence $\delta: \beta^* G \to \beta^* G'$ with $\delta^*(\beta^* x') = \beta^* x$. By Proposition 2.8, there is a unique congruence $\chi: G \to G'$ with $\alpha^*(\chi) = \gamma$ and $\beta^*(\chi) = \delta$. This implies, in particular, that $G_{kk} = G'_{kk}$, and we write $m$ for this common value. The following diagram commutes and consists of bijections since $X$ is a Segal $L$-presheaf (see Remark 1.10).

$$\begin{array}{ccc}
X_{G'} & \longrightarrow & X_{\alpha'^* G'} \times_{X_{m}} X_{\beta'^* G'} \\
x' \downarrow & & \downarrow \\
X_G & \longrightarrow & X_{\alpha^* G} \times_{X_{m}} X_{\beta^* G}
\end{array}$$

It follows that $\chi^* x' = x$. Since $x \in X_G$ and $x' \in X_{G'}$ represent the same element in the colimit $\text{colim}_{H \in K_{n(w)}} X_H$, it follows that $s = s' \in SX_n$. We have thus shown the Segal map is injective.

We now turn to surjectivity. Suppose $(s, s') \in SX_k \times_{SX_t} SX_t$, with $s$ represented by $(w, G, x)$ and $s'$ represented by $(w', G', x')$. We have $G_{kk} = G'_{00} = m$. Choose a height $n$ level graph $H$ with $\alpha^* H = G$ and $\beta^* H = G'$. Since $X$ is a Segal $L$-presheaf, there is a (unique) element $y \in X_H$ which maps to $(x, x')$ under

$$\alpha^* \times \beta^*: X_H \xrightarrow{=} X_G \times_{X_{m}} X_{G'}.$$ 

Now the element of $SX_n$ represented by $(\pi(H), H, y)$ is sent to $(s, s')$ by the Segal map. □

**Definition 2.16.** If $X \in \text{Seg}(L)$, we write $CX = \tau_1(SX) \in \text{Cat}$ for the associated category (where $\tau_1: \text{sSet} \to \text{Cat}$ is left adjoint to $N$). This defines a functor $C: \text{Seg}(L) \to \text{Cat}$ with $NCX = SX$.

The following example has previously been discussed in Remark 2.7, but it is extremely important for the rest of the paper and there is no harm in revisiting it. We will return to the monoidal structure in Example 2.34 below.

**Example 2.17** (Canonical inclusion). If $\ast \in \text{Psh}(L)$ is the terminal presheaf, then there is an injective-on-objects equivalence $C(\ast) \to \text{Csp}$. From the formula following Definition 2.13, we see $S(\ast)$ is isomorphic to the simplicial set $W_{\ast}$. Then the objects of $C(\ast)$ are identified with the height 0 level graphs, that is, with $\text{ob}(F) = \{0, 1, 2, \ldots \}$. The morphisms of $C(\ast)$ are the elements of $W_1$, which under the identification on objects are precisely the same as the morphisms between these objects in $\text{Csp}$. Finally, compositions in both categories are given by pushouts of equivalence classes of cospans.
2.3. Monoidal structure. In this section, we explain why \( C_X \) is a strict monoidal category. It turns out to also be symmetric monoidal (see Theorem 2.33), but this is more difficult to show since it requires working at several simplicial levels of \( SX \). The main result of this section is that if \( X \) is Segal, then \( SX \) is a (strict) monoid in \( sSet \). This means (see Proposition 2.22) that \( C_X \) is a monoid in \( \mathbf{Cat} \), that is, a strict monoidal category.

For each \( n \geq 0 \), the category \( L_n \subseteq \text{Fun}(\mathcal{L}^n, \mathbf{F}) \) is monoidal, using the ordinal sum in \( \mathbf{F} \). More specifically, if \( G \) and \( H \) are height \( n \) level graphs, then \( G + H \) is the height \( n \) level graph with \( (G + H)_{ij} = G_{ij} + H_{ij} \). The maps \( G_{ij} \hookrightarrow G_{ij} + H_{ij} \leftarrow H_{ij} \) are order-preserving, and every element in \( G_{ij} \) appears before every element in \( H_{ij} \). In fact, \( L_n \) is actually a permutative category since \( \mathbf{F} \) is (i.e. a symmetric strict monoidal category [May74, Definition 4.1]). From Remark 1.10 we know that a Segal presheaf \( X \) takes sums of height \( n \) level graphs to products of sets, which we use in the following.

**Definition 2.18.** Suppose \( X \in \text{Seg}(\mathbf{L}) \), \( G \) and \( H \) are height \( n \) level graphs, and let \( i: G \rightarrow G + H \) and \( j: H \rightarrow G + H \) be the inclusions. Then define 
\[- \otimes - : X_G \times X_H \rightarrow X_{G+H}\]
to be the inverse of the bijection \( i^* \times j^*: X_{G+H} \rightarrow X_G \times X_H \).

We will make use of the following lemma in Section 2.4.

**Lemma 2.19.** Let \( G, H \in L_n \) and let \( \sigma: H + G \rightarrow G + H \) be the flip map. If \( x \in X_G \) and \( y \in X_H \), then \( \sigma^*(x \otimes y) = y \otimes x \).

**Proof.** The diagram
\[
\begin{array}{c}
H \xrightarrow{i_L} H + G \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
G + H \xleftarrow{i_R} G
\end{array}
\]
commutes in \( L_n \), which implies the diagram
\[
\begin{array}{c}
X_G \times X_H \xrightarrow{\tau^*} X_{G+H} \\
\downarrow \quad \downarrow \\
X_H \times X_G \xrightarrow{\tau^*} X_{H+G}
\end{array}
\]
commutes, where \( \tau \) is the flip map. The result follows. \( \Box \)

**Lemma 2.20.** Let \( X \in \text{Seg}(\mathbf{L}) \) and suppose that \( G, H, K \in L_n \). If \( x \in X_G \), \( y \in X_H \), and \( z \in X_K \), then
\( (x \otimes y) \otimes z = x \otimes (y \otimes z) \)
in \( X_{(G+H)+K} = X_{G+(H+K)} \). If \( \emptyset_n \) is the empty height \( n \) level graph and \( * \) is the unique element of \( X_{\emptyset_n} \), then
\( * \otimes x = x = x \otimes * \)
in \( X_{\emptyset_n+G} = X_G = X_{G+\emptyset_n} \).
Proof. Associativity follows from commutativity of the following rectangle.

\[
\begin{array}{ccc}
(X_G \times X_H) \times X_K & \cong & X_{G+H} \times X_K \\
\downarrow & & \downarrow \\
X_G \times (X_H \times X_K) & \cong & X_G \times X_{H+K} & \cong & X_{G+(H+K)}
\end{array}
\]

This uses that the monoidal structure on \(L_n\) is strict, so \((G+H)+K = G+(H+K)\).

For the second statement, the Segal map at \(\emptyset_n\)

\[
X_{\emptyset_n} \cong \lim_{K \in (L_{op})_{\emptyset_n}} X_K
\]

identifies \(X_{\emptyset_n}\) with a limit over the empty category. Hence there is indeed a unique element \(\ast \) in \(X_{\emptyset_n}\). The Segal maps are

\[
X_G = X_{\emptyset_n+G} \longrightarrow X_{\emptyset_n} \times X_G \\
x \longmapsto (\ast, x) \\
X_G = X_{G+\emptyset_n} \longrightarrow X_G \times X_{\emptyset_n} \\
x \longmapsto (x, \ast)
\]

which establishes the result. \(\square\)

Lemma 2.21. Suppose \(X \in \text{Seg}(L)\) and \(n \geq 0\). The collection of functions \(\otimes\) from Definition 2.18 descend to a function

\[- \otimes - : SX_n \times SX_n \to SX_n.
\]

Proof. Recall the description of \(SX_n\) from (1) on page 16. Let \(\bar{x} \in \overline{X}_w \subseteq SX_n\) and \(\bar{x}' \in \overline{X}_{w'} \subseteq SX_n\). Choose a representative \(x \in X_G\) and \(x' \in X_{G'}\) for these elements and define \(\bar{x} \otimes \bar{x}' \in SX_n\) to be the image of \((x, x')\) under the following composite:

\[
X_G \times X_{G'} \overset{\cong}{\longrightarrow} X_{G+G'} \longrightarrow \overline{X}_{G+G'} \longrightarrow SX_n.
\]

The leftward map is a bijection since \(X\) is Segal (Remark 1.10). This definition does not depend on our choice of representatives: indeed, suppose we have \((y, y') \in X_H \times X_{H'}\) along with congruences \(f : H \to G\) and \(g : H' \to G'\) so that \(f^*(x) = y\) and \(g^*(x') = y'\). We then have a commutative diagram

\[
\begin{array}{ccc}
X_G \times X_{G'} & \cong & X_{G+G'} \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\
X_{H} \times X_{H'} & \cong & X_{H+H'} & \cong & SX_n
\end{array}
\]

\[
\begin{array}{ccc}
(f^* \otimes g^*) & \cong & (f+g)^* \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\
(f+g)^* & \cong & \overline{X}_{H+H'} & \longrightarrow SX_n
\end{array}
\]

which establishes that \(\bar{x} \otimes \bar{x}'\) does not depend on choice of representative. \(\square\)

Proposition 2.22. If \(X \in \text{Seg}(L)\), then \(CX\) is a strict monoidal category.

Proof. The assignment \(\otimes\) on objects and morphisms from Lemma 2.21 of \(CX\) is a functor \(CX \times CX \to CX\). This follows because the simplicial structure maps \(\alpha^* : L_m \to L_n\) preserve +. The tensor unit is the image of the unique element of \(X_{\emptyset_0}\) in \(SX_0 = \text{ob} CX\), and the tensor is strictly associative and unital by Lemma 2.20.

Essentially by definition of the monoidal structure, if \(X \to Y\) is a map in \(\text{Seg}(L)\) then \(CX \to CY\) is a strict monoidal functor.
2.4. The symmetry. In order to most efficiently produce the symmetry isomorphism and prove its properties, we first produce a more general twisted tensor of any two morphisms in $CX$, whose codomain is the opposite from the normal tensor. That is, if $f : x \to x'$ and $g : y \to y'$ are two morphisms, then we construct $f \otimes g : x \otimes y \to y' \otimes x'$. The symmetry isomorphism is then given in Definition 2.31 as $\text{id}_x \otimes \text{id}_y : x \otimes y \to y \otimes x$.

As an auxiliary useful construction, we first define twisted sums of height $n$ graphs. Though these are defined in general, we only need them for $n \leq 2$ which can be illustrated directly. For $G, H \in \mathbb{L}_n$ height $n$ level graphs and an integer $t$, define a new graph $G +_t H \in \mathbb{L}_n$ by

$$(G +_t H)_{ij} = \begin{cases} G_{ij} + H_{ij} & i < t \\ H_{ij} + G_{ij} & i \geq t. \end{cases}$$

The structure maps are constructed as sums and flips of sums of those appearing in $G$ and $H$.

**Example 2.23.** Special cases include $G + H = H + G$ and $G +_0 H = G +_n H$. This covers all cases when $n = 0$; when $n = 1$ there is one interesting case $G +_1 H$.

$$
\begin{array}{ccc}
G_{00} + H_{00} & \rightarrow & H_{11} + G_{11} \\
\downarrow & & \downarrow \\
G_{01} + H_{01} & \rightarrow & G_{01} + H_{01} \\
\end{array}
$$

while when $n = 2$ we have $G +_1 H$.

$$
\begin{array}{ccc}
G_{00} + H_{00} & \rightarrow & H_{11} + G_{11} & \rightarrow & H_{22} + G_{22} \\
\downarrow & & \downarrow & & \downarrow \\
G_{01} + H_{01} & \rightarrow & H_{12} + G_{12} & \rightarrow & G_{02} + H_{02} \\
\end{array}
$$

and $G +_2 H$.

$$
\begin{array}{ccc}
G_{00} + H_{00} & \rightarrow & G_{11} + H_{11} & \rightarrow & H_{22} + G_{22} \\
\downarrow & & \downarrow & & \downarrow \\
G_{01} + H_{01} & \rightarrow & G_{12} + H_{12} & \rightarrow & G_{02} + H_{02} \\
\end{array}
$$

Define the isomorphism

$$
G +_t H \xrightarrow{\sigma_t \cong \\ \text{ij}} G + H
$$

which is the identity in the $ij$ component for $i < t$ and is the flip map for $i \geq t$. We denote the inclusions by $i_t$ and $j_t$, and the following diagram commutes in $\mathbb{L}_n$.

$$
\begin{array}{ccc}
G & \xrightarrow{i_t} & G +_t H & \xleftarrow{j_t} & H \\
\downarrow & \cong & \downarrow & \cong & \downarrow \\
G + H & \cong & G + H & \cong & G + H \\
\end{array}
$$
The following is an essential lemma about the behavior of this structure with respect to coface maps.

**Lemma 2.24.** If \( k < t \), then \( d_k(G + H) = \frac{d_k(G) + (d_k H)}{t} \) and the following diagram in \( L \) commutes.

\[
\begin{array}{ccc}
(d_k G) + (d_k H) & \xrightarrow{\sigma_{t-1}} & d_k(G + H) \\
\downarrow d_k(\sigma_t) & & \downarrow d_k(\sigma_t) \\
d_k G + d_k H & \xrightarrow{\sigma_t} & d_k(G + H)
\end{array}
\]

If \( k \geq t \), then \( d_k(G + H) = (d_k G) + (d_k H) \) and the following diagram commutes.

\[
\begin{array}{ccc}
(d_k G) + (d_k H) & \xrightarrow{\sigma_1} & d_k(G + H) \\
\downarrow d_k(\sigma_t) & & \downarrow d_k(\sigma_t) \\
d_k G + d_k H & \xrightarrow{\sigma_t} & d_k(G + H)
\end{array}
\]

More concisely,

\[
\sigma_k d_k = \begin{cases} 
  d_k \sigma_{t-1} & \text{if } k < t \\
  d_k \sigma_t & \text{if } k \geq t.
\end{cases}
\]

**Proof.** This is a direct verification from the definitions. \( \square \)

Next is a special case which will be used in Lemma 2.29.

**Lemma 2.25.** If \( G, H \in L_n \) for \( n \geq 1 \), then the following diagram commutes.

\[
\begin{array}{ccc}
G + H & \xrightarrow{d^1} & G + H \\
\downarrow \sigma_1 & & \downarrow \sigma_2 \\
G + H & \xrightarrow{d^1} & G + H
\end{array}
\]

The dashed map is the isomorphism \( \sigma_2^{-1} \sigma_1 \).

**Proof.** The first part of Lemma 2.24 with \( k = 1 < 2 = t \) tells us that \( d^1 \sigma_1 = \sigma_2 d^1 \), while the second part with \( t = k = 1 \) tells us that \( d^1 \sigma_1 = \sigma_1 d^1 \), and both maps having domain \( d_1 G + d_1 H \). \( \square \)

**Definition 2.26.** Suppose \( G, H \in L_1, x \in X_G, \) and \( y \in X_H \). Let \( x \tilde{\otimes} y \in X_{G+H} \) be the element defined by the following diagram

\[
\begin{array}{ccc}
X_G \times X_H & \xleftarrow{\cong} & X_{G+H} \\
\downarrow & & \downarrow \sigma_i^* \\
X_G \times X_H & \xleftarrow{\cong} & X_{G+H}
\end{array}
\]

That is, \( x \tilde{\otimes} y = (i_1^* \times j_1^*)^{-1}(x, y) = \sigma_i^*(x \otimes y) \).
Proposition 2.27. The preceding definition descends to a well-defined function
\[- \circ \circ - : \text{mor} \mathcal{C}_X \times \text{mor} \mathcal{C}_X \to \text{mor} \mathcal{C}_X.\]

Proof. If \( G' \xrightarrow{\sim} G \) and \( H' \xrightarrow{\sim} H \) are congruences in \( \mathcal{K}_1 \), then there is an associated congruence \( G' + H' \xrightarrow{\sim} G + H \) as well, and the following diagram commutes.

Thus we may define \([x] \circ \circ [y] := [x \circ y]\) with no ambiguity about choice of representatives.

□

Lemma 2.28. If \( f : x_0 \to x_1 \) and \( g : y_0 \to y_1 \) are morphisms in \( \mathcal{C}_X \), then \( f \circ \circ g \) is a map \( x_0 \circ y_0 \to y_1 \circ x_1 \).

Proof. We may prove the result by working with representatives. The diagram

commutes, using \( d_0(G + H) = d_0G + d_0H = d_0H + d_0G \) along with Lemma 2.19 and Lemma 2.24. The two outer maps \( X_G \times X_H \to SX_0 \) are equal, implying that \( \text{cod}[x \circ y] = (\text{cod}[y]) \circ (\text{cod}[x]) \). A similar, but simpler diagram can be constructed for \( d_1 \) to see \( \text{dom}[x \circ y] = (\text{dom}[x]) \circ (\text{dom}[y]) \); it is simpler since \( d_1(G + H) = d_1G + d_1H = d_1G + d_1H \).

□

Lemma 2.29. Given morphisms

in \( \mathcal{C}_X \), we have

\[(f_2 \circ \circ g_2) \circ (f_1 \circ g_1) = (f_2f_1) \circ (g_2g_1) = (g_2 \circ f_2) \circ (f_1 \circ g_1).\]

Proof. The composable pairs given are elements in \( SX_2 \), and hence we can find height 2 level graphs \( G \) and \( H \) and elements \( x \in X_G \) and \( y \in X_H \) so that \( d_2x \in X_{d_2G} \) represents \( f_1 \) and \( d_0x \in X_{d_0G} \) represents \( f_2 \), and likewise for \( y \) and \( g_1, g_2 \). We have \( d_1(G + H) = (d_1G) + (d_1G) = d_1(G + H) \). Let \( \sigma_1 : G + H \to G + H \) and
\( \sigma_2 : G + H \rightarrow G + H \) be the partial flipping morphisms from above. By Lemma 2.25 we have

\[
d_1 \sigma_1^*(x \otimes y) = d_1 \sigma_2^*(x \otimes y) \in X_{d_1(G + H)} = X_{d_1(G + H)}.
\]

We claim that this element represents all three maps; we use freely the identities from Lemma 2.24. First, the equalities

\[
d_1 \sigma_1^*(x \otimes y) = \sigma_1^* d_1(x \otimes y) = \sigma_1^*((d_1 x) \otimes (d_1 y))
\]

show this element represents \((f_2 f_1) \otimes (g_2 g_1)\). The equations (the first of which uses \(\sigma_0 = \sigma\) and Lemma 2.19)

\[
d_0 \sigma_1^*(x \otimes y) = \sigma_0^* d_0(x \otimes y) = \sigma_0^*(d_0 x \otimes d_0 y) = d_0 y \otimes d_0 x
\]

\[
d_2 \sigma_2^*(x \otimes y) = \sigma_1^* d_2(x \otimes y) = \sigma_1^*(d_2 x \otimes d_2 y) = d_2 x \otimes d_2 y
\]

tell us our element represents \((g_2 \otimes f_2) \circ (f_1 \otimes g_1)\). Finally, we have

\[
d_0 \sigma_2^*(x \otimes y) = \sigma_1^* d_0(x \otimes y) = \sigma_1^*(d_0 x \otimes d_0 y) = d_0 x \otimes d_0 y
\]

\[
d_2 \sigma_2^*(x \otimes y) = \sigma_1^* d_2(x \otimes y) = \sigma_1^*(d_2 x \otimes d_2 y) = d_2 x \otimes d_2 y
\]

so the element represents \((f_2 \otimes g_2) \circ (f_1 \otimes g_1)\).

\[ \Box \]

**Lemma 2.30.** Given morphisms

\[
\begin{array}{c}
x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \\
y_0 \xrightarrow{g_1} y_1 \xrightarrow{g_2} y_2
\end{array}
\]

in \( CX \), we have

\[
(g_2 \otimes f_2) \circ (f_1 \otimes g_1) = (f_2 f_1) \otimes (g_2 g_1).
\]

**Proof.** Use the same notation from the beginning of the previous proof. We introduce a new graph \( G + H \) as displayed:

\[
\begin{array}{ccc}
G_{00} + H_{00} & \xleftarrow{G_{01} + H_{01}} & H_{11} + G_{11} & \xleftarrow{H_{12} + G_{12}} & G_{22} + H_{22}
\end{array}
\]

\[
\begin{array}{c}
G_{02} + H_{02}
\end{array}
\]

Define \( \tilde{\sigma} : G + H \rightarrow G + H \) which is the flip map in components 11 and 12, and the identity elsewhere. Our strategy is to show that

\[
d_1 \tilde{\sigma}^*(x \otimes y) \in X_{d_1(G + H)}
\]

represents both maps in the statement of the lemma.

Notice that we have \( d_0(G + H) = (d_0 H) + (d_0 G), d_1(G + H) = d_1 G + d_1 H, \) and \( d_2(G + H) = (d_2 G) + (d_2 H) \). Furthermore, the following diagrams commute:

\[
\begin{array}{ccc}
(d_0 H) + 1 d_0 G & \xrightarrow{\sigma_1} & d_0 H + d_0 G & \xrightarrow{\sigma} & d_0 G + d_0 H
\end{array}
\]

\[
\begin{array}{c}
\downarrow^{d^0}
\end{array}
\]

\[
\begin{array}{ccc}
G + H & \xrightarrow{\tilde{\sigma}} & G + H
\end{array}
\]

\[
\begin{array}{c}
\downarrow^{d^0}
\end{array}
\]
Theorem 2.33. The functor $C : \text{Seg}(L) \to \text{Cat}$ factors through the category $\text{Perm}$ of small permutative categories and strict symmetric monoidal functors. More precisely, this uses the monoidal structure on $CX$ from Proposition 2.22 and the symmetry isomorphism from Definition 2.31.

Proof. By Proposition 2.22 and Proposition 2.32, $CX$ is a permutative category for every $X \in \text{Seg}(L)$. But the definitions imply that both the monoidal structure and the symmetry are natural in $X$. □

**Example 2.34.** The equivalence $C(*) \to \text{Csp}$ from Example 2.17 is a symmetric monoidal functor. To see this, recall that objects of $C(*)$ have been identified with objects of $F$ and morphisms are equivalence classes of cospans in $F$. Direct inspection shows that the symmetry isomorphisms in $C(*)$ and $F$ coincide. The monoidal constraint for $C(*) \to \text{Csp}$ is then inherited from that for $F \to \text{FinSet}$, and we conclude that $C(*) \to \text{Csp}$ is a symmetric monoidal functor. That $\text{Csp}$ is equivalent to a skeletal permutative category is well known; see [Lac04, 5.4].
Lack shows that $C(*)$ is the prop for special Frobenius monoids \cite[Proposition 6.1]{CF17}, also known as commutative separable algebras \cite{Car91}. The introduction of \cite{RSW06} contains a nice overview of these structures.

It seems to be a folklore result that special Frobenius monoids are also the algebras for the terminal properad. This is consistent with the fact that $C$ gives the free prop generated by a properad.

3. PROPERADS GIVE LABELLED COSPAN CATEGORIES

In this section, we show that the envelope of a properad is a labelled cospan category. More precisely, if $P$ is a properad then the composite

$$C(N_1(P)) \to C(*) \cong \text{Csp}$$

is a labelled cospan category. Since $N_1$ is an equivalence of categories between $\text{Ppd}$ and $\text{Seg}(\mathbf{L})$, it is enough to prove the following. Below, we will verify the axioms from Definition 1.2 in a series of lemmas.

**Proposition 3.1.** If $X \in \text{Seg}(\mathbf{L})$, then $C(X) \to \text{Csp}$ is a labelled cospan category.

**Proof.** Combine Lemma 3.2, Lemma 3.4, Lemma 3.7, and Lemma 3.8. \qed

Suppose $X \in \text{Seg}(\mathbf{L})$, and consider functor $CX \to C(*)$. The set

$$Xc = X_1 \subseteq \bigoplus_{k \in \mathbb{N}} X_k = \bigoplus_{w \in \mathbf{W}_0} X_w = \text{ob} CX$$

is the set of connected objects of $CX$ in the sense of Definition 1.1. Likewise, the set

$$\bigoplus_{n,m \geq 0} X_{c_{n,m}} \subseteq \bigoplus_{w \in \mathbf{W}_1} X_w = \text{mor} CX$$

is the set of connected morphisms. These displays use that the component of $k \in \mathbf{K}_0$ and the component of $c_{n,m} \in \mathbf{K}_1$ are discrete. The reduced morphisms are those on (congruence classes) of height 1 level graphs so that each vertex has either an input or an output. Alternatively, these graphs are those without a $c_{00}$-summand.

**Lemma 3.2.** Let $c$ be an object of $CX$ so that $\pi(c) = \underline{n}$. Then $c = c_1 \otimes \cdots \otimes c_n$ for some connected objects $c_1, \ldots, c_n$.

**Proof.** Recall that $CX$ is a strict monoidal category, and on objects the $n$-fold tensor product is given as the inverse of the Segal map

$$X_n \xrightarrow{\text{Segal}} X_1 \times \cdots \times X_1.$$ 

Thus given an object $c$ with $\pi(c) = \underline{n}$, we see that $c$ is the tensor product of $n$ connected objects $c_1, \ldots, c_n$. \qed

**Remark 3.3** (Uniqueness of decomposition). We emphasize that as the $n$-fold tensor product is given by the inverse of the Segal map $X_n \xrightarrow{\text{Segal}} X_1 \times \cdots \times X_1$ there is actually a unique list of connected objects $c_1, \ldots, c_n$ with $c = c_1 \otimes \cdots \otimes c_n$ in the previous lemma.

See also Definition 4.2 below.

**Lemma 3.4.** The abelian monoid $\text{hom}(1, 1)$ is freely generated by the set $\text{hom}^c(1, 1)$ of connected morphisms.
Proof. The set of objects of $L_1$ with empty input and empty output is in bijection with $\mathbb{N}$, and we write temporarily abbreviate the object corresponding with $n$ by

$$n\varepsilon_0 := \varepsilon_0 + \cdots + \varepsilon_0.$$ 

This object is not isomorphic, hence not congruent, to any other object in $L_1$. Every automorphism of $n\varepsilon_0$ is a congruence, and the group of automorphisms is the symmetric group $\Sigma_n$. For $\gamma \in \Sigma_n$, the left diagram below commutes

$$\xymatrix{n\varepsilon_0 \ar[r]^-k \ar[d]_{\gamma} & n\varepsilon_0 \ar[d]_{\gamma(k)} \\
\varepsilon_0 \ar[r]_-\gamma & \varepsilon_0}$$

hence so too does the right diagram, where $\pi_t$ is the $t$th projection. This implies that the following commutes

$$\xymatrix{X_{n\varepsilon_0} \ar[r]^-{=} \ar[d]_{\gamma^*} & \prod_{i=1}^n X_{\varepsilon_0} \ar[d]_{\pi_k} \\
X_{n\varepsilon_0} \ar[r]^-{=} & \prod_{i=1}^n X_{\varepsilon_0}}$$

where the right-hand map is the standard right action on lists, that is $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n) \cdot \gamma = (x_{\gamma(1)}, \ldots, x_{\gamma(n)})$. Further, the monoidal structure comes from the bottom right isomorphism in the following diagram.

$$\xymatrix{X_{(n+m)\varepsilon_0} \ar[r]^-{=} \ar[d]_{=} & \prod_{i=1}^{n+m} X_{\varepsilon_0} \ar[d] \\
X_{n\varepsilon_0+m\varepsilon_0} \ar[r]^-{=} & \prod_{i=1}^n X_{\varepsilon_0} \times \prod_{i=1}^m X_{\varepsilon_0}}$$

Thus we see that $\text{hom}(1, 1)$ as a monoid is isomorphic to

$$\sum_{n \geq 0} \left( \prod_{i=1}^n X_{\varepsilon_0} \right) / \Sigma_n \text{op}$$

which is the free abelian monoid on $X_{\varepsilon_0} = \text{hom}^r(1, 1)$.

Let $R = \text{mor}^r C(*) \subseteq W_1$ be the congruence classes of graphs where each vertex has an input or output. The following lemma, concerning the connected components of the $q$-fibers in the congruence category $K$ appearing in Lemma 2.12, will be used in the proofs of Lemma 3.7 and Proposition 3.9. If $r \in R$, then by Remark 2.4 the category $K_r$ is a preorder, and $\overline{X}_r \cong X_G$ if $[G] = r$ (see Remark 2.14).
Lemma 3.5. If \( r \in R \subseteq W_1 \) is reduced and \( w \in W_1 \) is arbitrary, then
\[
K_r \times K_w \to K_{r \otimes w}
\]
sending \((G, H)\) to \(G + H\) is an equivalence of categories. It follows that if \( X \in \text{Seg}(L) \), then \( X_r \times X_w \to X_{r \otimes w} \) is a bijection.

Proof. By the definition in Lemma 2.21, the class \( r \otimes w \in W_1 \) contains some graph \( G + H \) where \([G] = r\) and \([H] = w\). Thus (2) is essentially surjective. It is also clearly faithful. We now prove that it is full. We must show that if \( G, G' \in r \) and \( H, H' \in w \) then any congruence \( f : G + H \equiv G' + H' \) restricts to congruences \( f^L : G \equiv G' \) and \( f^R : H \equiv H' \) with \( f = f^L + f^R \). For \( i = 0, 1 \) we know that \( \text{id}_{G_{ii}} : G_{ii} \to G'_{ii} \) and \( \text{id}_{H_{ii}} : H_{ii} \to H'_{ii} \) sum to \( f_{ii} = \text{id}_{(G + H)_{ii}} : (G + H)_{ii} \to (G + H')_{ii} \), so this part restricts. Since every vertex of \( G \) or \( G' \) has either an input or an output, the first horizontal maps in the following commutative rectangle are surjective.

\[
\begin{array}{ccc}
G_{00} + G_{11} & \longrightarrow & G_{01} \\
\downarrow f_{01}^L + f_{11}^R & & \downarrow f_{01} \\
G_{00} + G'_{11} & \longrightarrow & G'_{01} + H_{01}
\end{array}
\]

It follows that \( f_{01} \) restricts to a function \( f_{01}^L : G_{01} \to G'_{01} \). We must still check that \( f^L \) is a congruence, and for this simply note that \((f^{-1})_{01} \) restricts to a function \( G'_{01} \to G_{01} \) by the same argument, providing an inverse to \( f_{01}^L \). Since \( f^L \) is an isomorphism, it is a congruence.

Now the composite \( H \to G + H \to G' + H' \) must land in \( H' \), since \( f \) is a monomorphism and \( f^L : G \to G' \) is an epimorphism. Thus \( f \) restricts to a map \( f^R : H \to H' \), which must be an epimorphism since \( f \) is. We conclude that (2) is full, hence an equivalence.

For the conclusion, suppose \( X \) is Segal. In the following diagram, we use the result to deduce the isomorphism at the bottom of the triangle; the others come from Segal.

\[
\begin{array}{ccc}
X_G \times X_H & \cong & X_{G + H} \\
\downarrow \text{colim} & & \downarrow \text{colim} \\
\text{colim}_{G \in K^{op}_r} X_G \times \text{colim}_{H \in K^{op}_r} X_H & \cong & \text{colim}_{(G, H) \in K^{op}_r \times K^{op}_w} X_{G + H} \\
\downarrow \cong & & \downarrow \cong \\
\overline{X}_r \times \overline{X}_w & \longrightarrow & \overline{X}_{r \otimes w} \\
\downarrow \cong & & \downarrow \cong \\
SX_1 \times SX_1 & \longrightarrow & SX_1
\end{array}
\]

We conclude that the dotted arrow is an isomorphism. \( \square \)

Remark 3.6. The proof of Lemma 3.5 truly depended on one of the elements being reduced. The simplest issue was apparent in Lemma 3.4 and its proof, since
\[
X_{c_{00}} \times X_{c_{00}} = \overline{X}_{c_{00}} \times \overline{X}_{c_{00}} = \overline{X}_{c_{00} + c_{00}} = X_{c_{00} + c_{00}} / \Sigma^\text{op}_2 \cong (X_{c_{00}} \times X_{c_{00}}) / \Sigma^\text{op}_2
\]
is not typically a bijection. But this also occurs for other congruence classes. For example, if \( G = c_{10} + c_{00} \) and \( H = c_{01} + c_{00} \), then the non-trivial endo-congruence
on \( G + H = c_{10} + c_{00} + c_{01} + c_{00} \) does not restrict in the way of the proof. Further, the automorphism group of \( G + H \) in \( K_1 \) is \( \Sigma_2 \), whereas \( G \) and \( H \) have trivial automorphism groups.

**Lemma 3.7.** The map

\[
\text{hom}^r(c,d) \times \text{hom}(1,1) \rightarrow \text{hom}(c,d)
\]

is a bijection.

**Proof.** Let \( R = \text{mor}^r C(*) \subseteq W_1 \) be the congruence classes of graphs where each vertex has an input or output, and \( T = \{ nc_{00} \mid n \in \mathbb{N} \} \subseteq W_1 \) be the set of graphs with empty inputs and outputs (see proof of Lemma 3.4). Every element \( w \in W_1 \) decomposes uniquely as \( w = r \otimes t \) where \( r \in R \) and \( t \in T \). Indeed, to get this decomposition, if \( G \) is an arbitrary height 1 level graph, then we can form new graphs \( G^r \) and \( G^t \) and morphisms in \( L_1 \)

\[
G^r \xrightarrow{f} G \xleftarrow{g} G^t
\]

so that every vertex of \( G^r \) has an input or output, \( f_{ii} : G^r_{ii} \to G_{ii} \) is an identity, \( g_{ii} : 0 \to G_{ii} \) is the unique map, and \( f_{01}, g_{01} \) are order-preserving. See Figure 3 for an illustration. There is a unique congruence \( G^r + G^t \equiv G \) taking \( f \) and \( g \) to the usual inclusions into the sum. If \( H^r + H^t \equiv G \) is some other congruence with \( [H^r] \in R \) and \( [H^t] \in T \), then the composite \( H^r + H^t \to G \) decomposes uniquely as \( w = r \otimes t \) where \( r \in R \) and \( t \in T \). Indeed, to get this decomposition, if \( G \) is an arbitrary height 1 level graph, then we can form new graphs \( G^r \) and \( G^t \) and morphisms in \( L_1 \)

\[
\text{mor}^r C(X) \times \text{hom}(1,1) = \sum_{R \times T} X_r \times X_t = \sum_{R \times T} X_r \times X_t \xrightarrow{\cong} \sum_{W_1} X_w = \text{mor} C(X)
\]

is a bijection.

The large rectangle and bottom square in the following are pullbacks,

\[
\begin{array}{ccc}
\text{hom}^r(c,d) \times \text{hom}(1,1) & \longrightarrow & \text{mor}^r C(X) \times \text{hom}(1,1) \\
\text{hom}(c,d) & \longrightarrow & \text{mor} C(X) \\
\{ (c,d) \} & \longrightarrow & \text{ob} C(X) \times \text{ob} C(X)
\end{array}
\]

hence so is the upper square. The result follows. \( \square \)
Lemma 3.8. For each four objects $c, d, c', d'$ in $C$, the following square is cartesian.

$$
\begin{array}{ccc}
\text{hom}^r(c,d) \times \text{hom}^r(c',d') & \xrightarrow{\otimes} & \text{hom}^r(c \otimes c', d \otimes d') \\
\downarrow \pi & & \downarrow \pi \\
\text{hom}_\text{Csp}(\pi c, \pi d) \times \text{hom}_\text{Csp}(\pi c', \pi d') & \xrightarrow{\otimes} & \text{hom}_\text{Csp}(\pi c \otimes \pi c', \pi d \otimes \pi d')
\end{array}
$$

This follows from a different result.

Proposition 3.9. If $X \to Y$ is a morphism in $\text{Seg}(\mathbb{L})$, then

$$
\begin{array}{ccc}
\text{mor}^r C(X) \times \text{mor}^r C(X) & \longrightarrow & \text{mor}^r C(X) \\
\downarrow & & \downarrow \\
\text{mor}^r C(Y) \times \text{mor}^r C(Y) & \longrightarrow & \text{mor}^r C(Y)
\end{array}
$$

is cartesian.

Proof. Let $R = \text{mor}^r C(*) \subseteq W_1$ be the congruence classes of height 1 level graphs where each vertex has an input or output, so that

$$
\text{mor}^r C(X) = \sum_{r \in R} X_r.
$$

By Lemma 3.5, there is a canonical bijection

$$
\text{mor}^r C(X) \times \text{mor}^r C(X) = \sum_{r \in R} X_r \times \sum_{r' \in R} X_{r'} \cong \sum_{(r, r') \in R \times R} X_r \times X_{r'} \cong \sum_{(r, r') \in R \times R} X_{r \otimes r'}.
$$

Thus it suffices to observe that the following square,

$$
\begin{array}{ccc}
\sum_{r, r' \in R} X_{r \otimes r'} & \longrightarrow & \sum_{s \in R} X_s \\
\downarrow & & \downarrow \\
\sum_{r, r' \in R} Y_{r \otimes r'} & \longrightarrow & \sum_{s \in R} Y_s
\end{array}
$$

whose horizontal legs are induced by $\otimes: R \times R \to R$, is cartesian.

Proof of Lemma 3.8. Apply the previous lemma in the case when $Y = *$, using that $C(*) \to \text{Csp}$ is fully faithful. Suppose $f: \pi c \to \pi d$, $g: \pi c' \to \pi d'$, and $h: c \otimes c' \to d \otimes d'$ are reduced morphisms with $\pi(h) = f \otimes g$. By the previous lemma, we know there are unique maps $\hat{f}: a \to b$ and $\hat{g}: a' \to b'$ with $\hat{f} \otimes \hat{g} = h$, $\pi(\hat{f}) = f$, and $\pi(\hat{g}) = g$. The only thing to verify is that $a = c$, $b = d$, $a' = c'$ and $b' = d'$, but this follows from the uniqueness of decompositions of objects of $C X$ into connected objects (as was observed in Remark 3.3).

Proposition 3.10. The functor $C: \text{Seg}(\mathbb{L}) \to \text{Cat}$ is faithful.

Proof. If $X, Y \in \text{Seg}(\mathbb{L})$, then a map $X \to Y$ is uniquely determined by its action on elementary objects $X_e \to Y_e$ and $X_{e_{p,q}} \to Y_{e_{p,q}}$ (as $p$ and $q$ vary). If $f_0, f_1: X \to Y$
are two maps in Seg(L), then the following diagrams commute for \( i = 0, 1 \).

\[
\begin{align*}
X_{p,q} & \xrightarrow{f_i} Y_{p,q} \\
\text{mor}^e C(X) & \xrightarrow{C(f_i)} \text{mor}^e C(Y) \quad \text{ob} C(X) & \xrightarrow{C(f_i)} \text{ob} C(Y)
\end{align*}
\]

Thus if \( C(f_0) = C(f_1) \), then \( f_0 = f_1 \). \( \square \)

This proposition implies that the functors from Seg(L) to permutative categories and labelled cospan categories are also faithful.

4. **Strict labelled cospan categories**

In this section we identify a special kind of labelled cospan category, and show that to each we may associate a Segal L-presheaf (and hence a properad). This will provide an inverse to our previous construction: the colors of this properad will be the connected objects, and the operations in the associated properad will be the connected morphisms. Unfortunately, actually proving this gives a properad comes with substantial challenges.

We begin with an explanatory example.

**Example 4.1.** From the category \( C(*) \), create a new category \( C' \) with a single additional object \( 1' \) which is isomorphic to \( 1 \in C(*) \). As a permutative category, it will satisfy the equation \( 1' \otimes n = 1 \otimes n \) for \( n \geq 1 \) and the equation \( 1' \otimes 1' = 2 \).

There is an evident functor \( C \to C' \to Csp \) sending the new object to \( 1 \). By Remark 3.3 this labelled cospan category is not \( C(X) \) for any \( X \in \text{Seg}(L) \), since \( 2 = 1 \otimes 1 = 1' \otimes 1 = 1 \otimes 1' = 1' \otimes 1' \) admits several distinct decompositions into connected objects.

Thus not all labelled cospan categories arise from properads (though they do up to equivalence). We now isolate the image of the functor \( C \). Recall that a strict map of permutative categories is a strict monoidal functor which is compatible with the symmetries.

**Definition 4.2.** A **strict labelled cospan category** is a strict map of permutative categories \( \pi: C \to C(*) \) satisfying the conditions of Definition 1.2 along with

(1') If \( \pi(c) = n \), then there are unique connected objects \( c_1, \ldots, c_n \) with \( c = c_1 \otimes \cdots \otimes c_n \).

Maps are strict maps of permutative categories so that the triangle

\[
\begin{array}{ccc}
C & \xrightarrow{id} & C' \\
\downarrow & & \downarrow \\
C(*) & \xrightarrow{\pi} & C(*')
\end{array}
\]

commutes. We write SLCC for the associated category.

Alternatively, (1') can be rephrased as follows, which means that each strict labelled cospan category is a (colored) prop of a special type [HR15].

(1'') The monoid \( \text{ob}(C) \) is the free monoid on the set of connected objects \( \text{ob}^e(C) \).
One can show (see Proposition 6.13) that being a strict labelled cospan category is a \textit{property} of a permutative category, rather than a \textit{structure}. Indeed, if $C$ is a permutative category, there is at most one functor $\pi: C \to C(\ast)$ making $C$ into a strict labelled cospan category. Nevertheless, we will often require the map $\pi$, so we do not attempt a definition free from it.

Corollary 5.9 states that the category of properads is equivalent to $\text{SLCC}$. We know that $P_{\text{pd}} \simeq \text{Seg}(L)$ by Proposition 1.15, and we have already exhibited a functor $\text{Seg}(L) \to \text{SLCC}$ by Proposition 3.1 and Remark 3.3. Proposition 3.10 implies that this functor is faithful. It remains to show this functor is full and essentially surjective.

Suppose $\pi: C \to C(\ast)$ is in $\text{SLCC}$. We will define a presheaf $P = P(\pi) \in \text{Psh}(L)$ from this data, and later show that $CP \to C(\ast)$ is isomorphic to $C \to C(\ast)$, proving that $\text{Seg}(L) \to \text{SLCC}$ is essentially surjective.

4.1. The inert part. We first define the restriction of the presheaf to $L_{el}$; it is slightly more convenient to define this also on arbitrary height zero level graphs. We define $P_n$ and $P_{c_{m,n}}$ as subsets of the objects of $C$ and the connected morphisms of $C$ so that the following diagrams are pullbacks.

\[
\begin{array}{ccc}
P_n & \rightarrow & \text{ob } C \\
\downarrow & & \downarrow \\
\{n\} & \rightarrow & \text{ob } C(\ast)
\end{array}
\]

\[
\begin{array}{ccc}
P_{c_{m,n}} & \rightarrow & \text{mor}^F C \\
\downarrow & & \downarrow_{s \times t} \\
P_m \times P_n & \rightarrow & \text{ob } C \times \text{ob } C
\end{array}
\]

\[
\begin{array}{ccc}
\{ (m, n) \} & \rightarrow & \text{ob } C(\ast) \times \text{ob } C(\ast)
\end{array}
\]

Notice that $P_1 = P_{\perp}$ is precisely the set of connected objects in $C$, and there is a bijection

\[
P_n \xrightarrow{\simeq} P_{\perp} \times \cdots \times P_{\perp}
\]

that takes an object to the list of connected objects from (1') in Definition 4.2.

We must still define the action of our restriction on inert morphisms. Between the elementary objects of $L$, there are the following inert morphisms:

- For $1 \leq i \leq m$ the following composite
  \[
  \begin{array}{c}
  1 \xrightarrow{\gamma_i} m = d_1(\epsilon_{m,n}) \xrightarrow{d^1} \epsilon_{m,n}
  \end{array}
  \]
  which we call $\ell_i$.

- For $1 \leq j \leq n$ the following composite
  \[
  \begin{array}{c}
  1 \xrightarrow{\gamma_j} n = d_0(\epsilon_{m,n}) \xrightarrow{d^0} \epsilon_{m,n}
  \end{array}
  \]
  which we call $r_j$.

- Automorphisms $f = (\gamma, \chi): \epsilon_{m,n} \to \epsilon_{m,n}$, where $\gamma = d_1(f): m \to m$ and $\chi = d_0(f): n \to n$ are bijections.

The only relations among the above these morphisms are the following:

- $(\gamma, \chi) \circ \ell_i = \ell_{\gamma(i)}$
- $(\gamma, \chi) \circ r_j = r_{\chi(j)}$
- $(\gamma, \chi) \circ (\gamma', \chi') = (\gamma \gamma', \chi \chi')$. 

Notation 4.3. If \( \sigma : \bar{k} \to \bar{k} \) is a bijection, and \( c_1, \ldots, c_k \) are connected objects of \( C \) then we write \( \hat{\sigma} : c_1 \otimes \cdots \otimes c_k \to c_{\sigma^{-1}(1)} \otimes \cdots \otimes c_{\sigma^{-1}(k)} \) for the map determined by the symmetry isomorphism. Then \( \hat{\sigma} \circ \hat{\sigma} = \hat{\sigma} \).

Definition 4.4. Suppose that \( p : c_1 \otimes \cdots \otimes c_m \to b_1 \otimes \cdots \otimes b_n \) is in \( P_{m,n} \subseteq \text{mor}^c C \). We define \( \ell^*_p = c_i \in \mathcal{P} \) and \( r^*_p = b_j \in \mathcal{P} \). If \( f = (\gamma, \chi) : c_{m,n} \to c_{m,n} \) is an isomorphism, we define \( f^*p = (\gamma, \chi)^*(p) \) to be the following composite of \( p \) with symmetry isomorphisms.

\[
\begin{array}{ccc}
c_{\gamma(1)} \otimes \cdots \otimes c_{\gamma(m)} & \xrightarrow{\hat{\gamma}} & b_{\chi(1)} \otimes \cdots \otimes b_{\chi(n)} \\
c_1 \otimes \cdots \otimes c_m & \xrightarrow{p} & b_1 \otimes \cdots \otimes b_n
\end{array}
\]

that is,

\[
f^*p = (\gamma, \chi)^*(p) = \hat{\chi}^{-1} \circ p \circ \hat{\gamma} = d_0(f)^{-1} \circ p \circ d_1(f).
\]

We then have

\[
(\gamma', \chi')^*(\gamma, \chi)^*(p) = (\gamma', \chi')^*(\hat{\chi}^{-1}p\hat{\gamma}) = (\hat{\chi}')^{-1}\hat{\chi}'^{-1}p\hat{\gamma}' = \hat{\chi'}^{-1}p\hat{\gamma}'
\]

which is \((\gamma'\gamma', \chi'\chi')^*(p)\), as desired. Likewise,

\[
\ell^*_i(\gamma, \chi)^*(p) = c_{\gamma(i)} = \ell^*_i(\gamma)(p)
\]

\[
r^*_j(\gamma, \chi)^*(p) = b_{\chi(j)} = r^*_j(\chi)(p)
\]

so we conclude that \( P|_{\mathcal{L}_{\text{el}}} : \mathcal{L}_{\text{el}}^{\text{op}} \to \text{Set} \) is indeed a functor.

Notice that our definition of \( \ell^*_i \) actually factors through a map \( d_1 : P_{m,n} \to P_{m,2} \) taking \( p \) to \( c_1 \otimes \cdots \otimes c_m \) which appeared in (3), and then following by the \( i \)th projection from (4). A similar situation occurs for \( r^*_j \). We cannot do the same for \((\gamma, \chi)^*\).

Definition 4.5. We define \( P|_{\mathcal{L}_{\text{int}}} \) to be the right Kan extension of \( P|_{\mathcal{L}_{\text{el}}} \) along the fully faithful inclusion \( \mathcal{L}_{\text{el}}^{\text{op}} \hookrightarrow \mathcal{L}_{\text{int}}^{\text{op}} \).

Since the functor is fully faithful, we can take the restriction of \( P|_{\mathcal{L}_{\text{int}}} \) to be exactly equal to \( P|_{\mathcal{L}_{\text{el}}} \). Further, by (4) we can arrange things so our previously-defined \( P_2 \) agrees with the new one. By [CH21, Lemma 2.9], if we can show that \( P|_{\mathcal{L}_{\text{int}}} \) is the restriction of some \( \mathcal{L} \)-presheaf \( P \), then \( P \) will automatically be Segal.

4.2. The active part. We have defined \( P|_{\mathcal{L}_{\text{int}}} \) in Definition 4.5. In particular, this values of this presheaf are defined on every level graph \( G \) and on every isomorphism. Since \( (\mathcal{L}_{\text{act}}, \mathcal{L}_{\text{int}}) \) is an orthogonal factorization system on \( \mathcal{L} \), it remains to define \( P \) on every active map. But each active map is isomorphic to \( \alpha^*G \to G \) for \( \alpha : [m] \to [n] \) an active map of \( \Delta \), so we will only extend to maps of this form.

Further if \( G \) is a height \( n \) level graph, we can focus on the maps \( d^i : d_iG \to G \) for \( 0 < i < n \) and \( s^i : s_iG \to G \) for \( 0 \leq i \leq n \), since \( \Delta_{\text{act}} \) is generated by such maps.

In this section it will be helpful to have the following explicit version of splitting a graph \( G \) into its connected components. A looser version of this was used in Section 1.3 in proving Lemma 1.14.

Definition 4.6 (Canonical splitting of \( G \)). If \( G \) is a height \( m \) level graph with \( G_{0m} = \bar{k} \), then for \( x \in G_{0m} \) we define \( G_x \) to be the connected height \( m \) level graph
where for each \( i,j \) the following is a pullback whose top map \( \iota_x \) is order-preserving.

\[
\begin{array}{ccc}
(G_x)_{ij} & \xleftarrow{\iota_x} & G_{ij} \\
\downarrow & & \downarrow \\
1 & \xleftarrow{x} & G_{0m}
\end{array}
\]

Taking the sum of the \( \iota_x \), we have an isomorphism \( \iota : \sum_{x=1}^{k} G_x \to G \) in \( L_m \).

**Construction 4.7.** For each height 1 level graph \( G \in L_1 \), we define a function \( \mu = \mu_G : P_G \to \text{mor} C \) so that the following diagram commutes.

\[
P_{d_1 G} \xleftarrow{d_1} P_G \xrightarrow{d_0} P_{d_0 G}
\]

\[
\begin{array}{ccc}
\text{ob} C & \xleftarrow{s} & \text{mor} C \\
\downarrow & & \downarrow \\
\text{ob} C & \xleftarrow{t} & \text{ob} C.
\end{array}
\]

When \( G = \epsilon \) is a corolla, \( \mu \) is just the standard inclusion \( P_\epsilon \subseteq \text{mor}^C C \subseteq \text{mor} C \) from the beginning of Section 4.1. More generally, suppose that \( G \) has \( n \) input-edges \( (d_1 G = G_{00} = n) \), \( m \) output-edges \( (d_0 G = G_{11} = m) \), and \( k \) vertices \( (G_{01} = k) \). If \( p \in P_G \), then we must define \( \mu_G(p) \) to be a map

\[
d_1(p) = c_1 \otimes \cdots \otimes c_n \to e_1 \otimes \cdots \otimes e_m = d_0(p)
\]

where \( c_i \) and \( e_i \) are connected objects. Recall the canonical splitting

\[
\iota : \sum_{x=1}^{k} G_x \cong \to G
\]

from Definition 4.6. We write \( \iota^0 = d_0(\iota) = \iota_{11} \) and \( \iota^1 = d_1(\iota) = \iota_{00} \). If \( p \in P_G \), write \( p_x := \iota^*_x(p) \in P_{G_x} \subseteq \text{mor}^C C \), which is a map

\[
p_x : c_{i_1} \otimes \cdots \otimes c_{i_r} \to e_{j_1} \otimes \cdots \otimes e_{j_r},
\]

where \( \{i_1, \ldots, i_r\} = \iota_x((G_x)_{00}) \) and \( \{j_1, \ldots, j_r\} = \iota_x((G_x)_{11}) \). Tensoring these maps together, we have

\[
p_1 \otimes \cdots \otimes p_k : c_{i_1} \otimes \cdots \otimes c_{i_r(n)} \to e_{i_1^0(1)} \otimes \cdots \otimes e_{i_r^0(m)}.
\]

Then \( \mu_G(p) \) is defined as the following composite:

\[
\begin{array}{ccc}
c_{i_1^0(1)} \otimes \cdots \otimes c_{i_r^0(m)} & \xrightarrow{p_1 \otimes \cdots \otimes p_k} & e_{i_1^0(1)} \otimes \cdots \otimes e_{i_r^0(m)} \\
(\beta^*)^{-1} & & \beta
\end{array}
\]

\[
\begin{array}{ccc}
c_1 \otimes \cdots \otimes c_n & \xrightarrow{\mu_G(p)} & e_1 \otimes \cdots \otimes e_m
\end{array}
\]

As a special case, we have the following.

**Lemma 4.8.** Let \( G \in L_1 \) be a graph with both legs isomorphisms, as follows.

\[
\begin{array}{ccc}
\n \xrightarrow{\alpha} \cong \xrightarrow{\beta} \n
\end{array}
\]
Suppose \( c_1, \ldots, c_n \) are connected objects of \( C \) and \( p \in P_G \) is the element which goes to \((\text{id}_{c_1}, \text{id}_{c_2}, \ldots, \text{id}_{c_n})\) under the Segal map \( P_G \to \prod_{i=1}^n P_{c_i} \). Then \( \mu_G(p) \) is the following composite

\[
c_{\alpha^{-1}(1)} \otimes \cdots \otimes c_{\alpha^{-1}(n)} \xrightarrow{\hat{\alpha}} c_1 \otimes \cdots \otimes c_n \xrightarrow{\beta^{-1}} c_{\beta^{-1}(1)} \otimes \cdots \otimes c_{\beta^{-1}(n)}. \tag{\*}
\]

\[\square\]

**Lemma 4.9.** Suppose that \( G \) and \( G' \) are in \( L_1 \) and \( i: G \to G+G' \), \( j: G' \to G+G' \) are the inclusions. If \( p \in P_{G+G'} \), then \( \mu_{G+G'}(p) = \mu_G(i^*p) \otimes \mu_{G'}(j^*p) \).

**Proof.** The canonical decomposition for \( H = G + G' \)

\[
\sum_{z=1}^{k+\ell} H_z = \sum_{x=1}^{k} G_x + \sum_{y=1}^{\ell} G'_y \xrightarrow{i=\alpha+\beta} G + G' = H
\]

is the sum of the canonical decompositions (denoted by \( \alpha \) and \( \beta \)) for \( G \) and \( G' \). Write \( \iota^1 := d_1(\iota) \) and \( \iota^0 := d_0(\iota) \) and likewise for \( \alpha \) and \( \beta \). Then if \( p_z = \iota_z^* p \), we have

\[
\mu_H(p) = (\iota^1)^{-1} \circ (p_1 \otimes \cdots \otimes p_k \otimes p_{k+1} \otimes \cdots \otimes p_{k+\ell}) \circ \hat{\iota}^0 = (\alpha^1 \otimes \beta^1)^{-1} \circ (p_1 \otimes \cdots \otimes p_k \otimes p_{k+1} \otimes \cdots \otimes p_{k+\ell}) \circ (\alpha^0 \otimes \beta^0) \]

\[
= ((\alpha^1)^{-1} \otimes (\beta^1)^{-1}) \circ (\alpha_1^* p \otimes \cdots \otimes \alpha_k^* p \otimes \cdots \beta_{k+1}^* p) \otimes (\alpha^0 \otimes \beta^0) \]

\[
= ((\alpha^1)^{-1} \circ (\alpha_1^* p \otimes \cdots \otimes \alpha_k^* p) \otimes \alpha^0) \otimes ((\beta^1)^{-1} \circ (\beta_{k+1}^* p) \otimes \cdots \beta_{\ell+1}^* p) \otimes \beta^0) \]

\[
= \mu_G(i^*p) \otimes \mu_{G'}(j^*p). \tag{\*}
\]

\[\square\]

**Lemma 4.10.** Suppose \( G \) and \( G' \) are isomorphic height 1 level graphs, and let \( \mu_G \) and \( \mu_{G'} \) be as in Construction 4.7. Let \( f: G' \to G \) be an isomorphism. If \( p \in P_G \), then

\[
\mu_{G'} f^*(p) = d_0(f)^{-1} \circ \mu_G(p) \circ d_1(f).
\]

In particular, if \( f \) is a congruence then \( \mu_{G'} f^*(p) = \mu_G(p) \), since \( d_0(f) \) and \( d_1(f) \) are identities.

**Proof.** If an isomorphism \( f \) satisfies the lemma, then so does \( f^{-1} \). Further, if two composable isomorphisms \( f, f' \) satisfy the lemma, then so does their composition. We use these facts to reduce the proof to several special cases.

If \( G \) and \( G' \) are connected, that is, if they are both a corolla \( c \), then \( \mu \) is the canonical inclusion into \( \text{mor}^c C \) and the equation already appeared in Definition 4.4. We can boost this up to the case where we have isomorphisms of connected graphs \( f^i: H^i \to K^i \) for \( 1 \leq i \leq m \), and we consider their sum

\[
f: G' = \sum_{i=1}^m H^i \to \sum_{i=1}^m K^i = G.
\]

(Note that \( H^i = K^i \) since these are isomorphic corollas.) Letting \( i'_i: H^i \to G' \) and \( i_i: K^i \to G \) be the inclusions and using Lemma 4.9 and the result for corollas, we
have
\[
\mu_{G'}(f^*(p)) = (\iota'_1)^*(f^*(p)) \otimes \cdots \otimes (\iota'_m)^*(f^*(p)) \\
= (f_1^*)^*(\iota'_1(p)) \otimes \cdots \otimes (f_m^*)^*(\iota'_m(p)) \\
= [(d_0f_1)^{-1} \circ \iota'_1(p) \circ \hat{d}_1f_1] \otimes \cdots \otimes [(d_0f_m)^{-1} \circ \iota'_m(p) \circ \hat{d}_1f_m] \\
= d_0f^{-1} \circ (\iota'_1(p) \otimes \cdots \otimes \iota'_m(p)) \circ \hat{d}_1f \\
= d_0f^{-1} \circ \mu_G(p) \circ \hat{d}_1f.
\]

On the other hand, suppose we have

\[ f : G' = \sum_{i=1}^m G'_i \to \sum_{j=1}^m G_j = G. \]

with \( f \) acting as an automorphism on \( m \) but as the identity on each component, that is, \( f \circ \iota'_i = \iota_{f_i} : G_{f_i} = G'_i \to G' \to G \). Then

\[
\mu_{G'}(f^*(p)) = (\iota'_1)^* f^*(p) \otimes \cdots \otimes (\iota'_m)^* f^*(p) \\
= \iota'_{f_1}(p) \otimes \cdots \otimes \iota'_{f_m}(p) \\
= d_0f^{-1} \circ (\iota'_1(p) \otimes \cdots \otimes \iota'_m(p)) \circ \hat{d}_1f
\]

where these are now block permutations.

In the special case when \( f \) is the canonical decomposition from Definition 4.6

\[ \iota : G' = \sum G_x \to G, \]

the required formula follows from the definition of \( \mu_G \) in Construction 4.7. That is, if \( p_k = \iota'_k p \) then we have

\[
\mu_G(p) = d_0\iota \circ (p_1 \otimes \cdots \otimes p_k) \circ \hat{d}_1\iota^{-1} = d_0\iota \circ (\mu_{G'}(\iota^*(p))) \circ \hat{d}_1\iota^{-1}.
\]

We can combine all of these cases as follows

\[
\begin{array}{cccc}
\sum_{x=1}^m G'_x & \longrightarrow & \sum_{y=1}^m G'_{f_y} & \longrightarrow & \sum_{y=1}^m G_y \\
\downarrow & & \downarrow & & \downarrow \\
G' & \longrightarrow & f & \longrightarrow & G.
\end{array}
\]

The vertical maps are canonical decompositions, the first map on the top rearranges factors, and the second map on top is a sum of isomorphisms between corollas. The result now follows for \( f \) by the facts in the first paragraph of the proof. \( \square \)

**Proposition 4.11.** Let \( G \in \mathbf{L}_1 \) and \( \pi : C \to C(*) \). Then \( \pi(\mu_G(p)) \in \mathbf{W}_1 = \text{mor} C(*) \) is the congruence class \([G] \) of \( G \).

**Proof.** In the connected case, that is, when \( G = \varepsilon_{n,m} \) is a corolla, this is true by definition of \( \mu_{\varepsilon_{n,m}} \). If \( G \) is empty, then \( \mu_{G}(p) = 1 \), so the result follows. If the result is true for \( G \) and \( G' \), then applying Lemma 4.9 we have

\[
\pi_{\mu_{G+G'}(p)} = \pi(\mu_G(i^*p) \otimes \mu_{G'}(j^*p)) = \pi\mu_G(i^*p) \otimes \pi\mu_{G'}(j^*p) = [G] \otimes [G'] = [G+G'].
\]
Finally, suppose the result is true for $G'$ and that $f: G' \to G$ is an isomorphism. Let $H \in L_3$ denote the following height 3 level graph

\[
\begin{array}{ccc}
G_{00} & \cong & G'_{00} \\
\cong & & \cong \\
G_{01} & \cong & G'_{01} \\
\cong & & \cong \\
G_{11} & = & G_{11}
\end{array}
\]

where the unlabelled maps are the structure maps. This graph has the property that $d_1d_2H = G$ and $d_0d_3H = G'$, while $d_2d_3G$ and $d_0d_0G$ are of the form in Lemma 4.8. Let $p \in P_G$ be an element with $d_1(p) = c_1 \otimes \cdots \otimes c_n$ and $d_0(p) = e_1 \otimes \cdots \otimes e_m$. Using the Segal condition, define an element $q$ of $P_H$ so that $P_H \overset{d_2d_3}{\longrightarrow} P_{d_2d_3H}$ sends $q$ to $(\text{id}_{c_1}, \ldots, \text{id}_{c_n})$, $P_H \overset{d_0d_3}{\longrightarrow} P_{d_0d_3H}$ sends $q$ to $(\text{id}_{e_1}, \ldots, \text{id}_{e_m})$, and $d_0d_3: P_H \to P_{G'}$ sends $q$ to $f^*(p)$. By Lemma 4.8,

$$
\mu_{d_2d_3}(d_2d_3q) = d_1(f)^{-1} \quad \text{and} \quad \mu_{d_0d_3}(d_0d_0q) = d_0(f),
$$

and notice that $\pi$ sends these to $[d_2d_3H]$ and $[d_0d_0H]$, respectively. By Lemma 4.10 we also have

$$
\mu_G(p) = d_0(f) \circ \mu_G; f^*(p) \circ d_1(f)^{-1},
$$

which is sent to $[d_0d_0H] \circ [d_0d_3H] \circ [d_2d_3H]$ by $\pi$, since our assumption is $\pi(\mu_G; f^*(p)) = [G']$. But this element is equal to $[d_1d_2H] = [G]$. Thus $\pi(\mu_G(p)) = [G]$.

Since every graph is isomorphic to a finite sum of connected graphs, the general result follows.

**Definition 4.12** (Inner face in lowest dimension). Suppose $G$ is a height 2 level graph. If $G$ is connected, then $d_1G$ is a corolla. In this case we get a map $P_G \to P_{d_1G}$ by using composition in $C$. 

\[
\begin{array}{c}
P_G \cong P_{d_2G} \times P_{d_0d_2G} P_{d_0G} \to \text{mor } C \times_{\text{ob } C} \text{mor } C \\
\downarrow \downarrow \\
P_{d_1G} \to \text{mor } C \leftarrow \text{mor } C \\
\downarrow \downarrow \\
\{d_1G\} \leftarrow \{e_{p,q}\}_{p,q} \to \text{mor } C(*)
\end{array}
\]
For an arbitrary height 2 level graph, define $d_1 : P_G \to P_{d_1 G}$ by examining the following square.

$$
\begin{array}{ccc}
\sum_{x \in G_{x_2}} d_1 G_x & \longrightarrow & d_1 G \\
\downarrow d^1 & & \downarrow d^1 \\
\sum_{x \in G_{x_2}} G_x & \longrightarrow & G
\end{array}
$$

Here, each $G_x$ is connected and the bottom map is an isomorphism. Then applying $P$ we see $d_1$ is the dashed map in the following.

$$
P_G \xrightarrow{\sim} \prod_x P_{G_x} \\
\bigg\downarrow \bigg\downarrow \\
P_{d_1 G} \xrightarrow{\sim} \prod_x P_{d_1 G_x}
$$

**Lemma 4.13.** Suppose $f : G' \to G$ is an isomorphism between connected graphs in $\mathcal{L}_2$, and $d_1(f) = g : d_1 G' \to d_1 G$. Then the diagram

$$
P_G \xrightarrow{f^*} P_{G'} \\
\downarrow d_1 \downarrow d_1 \\
P_{d_1 G} \xrightarrow{g^*} P_{d_1 G'}
$$

commutes.

**Proof.** Write

$$
\alpha := d_2(f) : H' = d_2 G' \to d_2 G = H \\
\beta := d_0(f) : K' = d_0 G' \to d_0 G = K
$$

for the induced isomorphisms between height 1 level graphs. For $i = 0, 1$, we also write $\alpha_i := d_i(\alpha)$ and $\beta_i := d_i(\beta)$. If $p \in P_G$, then by definition

$$
d_1(p) = \mu_K(d_0 p) \circ \mu_H(d_2 p)
$$

and

$$
d_1(f^* p) = \mu_{K'}(d_0 f^* p) \circ \mu_{H'}(d_2 f^* p) = \mu_{K'}(\beta^* d_0 p) \circ \mu_{H'}(\alpha^* d_2 p).
$$

By Lemma 4.10 this means

$$
d_1(f^* p) = \hat{\beta}_0^{-1} \circ \mu_K(d_0 p) \circ \hat{\beta}_1 \circ \hat{\alpha}_0^{-1} \circ \mu_H(d_2 p) \circ \hat{\alpha}_1.
$$

Now $\alpha_0 = d_0 d_2(f) = d_1 d_0(f) = \beta_1$, so

$$
d_1(f^* p) = \hat{\beta}_0^{-1} \circ \mu_K(d_0 p) \circ \mu_H(d_2 p) \circ \hat{\alpha}_1 = \hat{\beta}_0^{-1} \circ d_1(p) \circ \hat{\alpha}_1.
$$

The result now follows from Definition 4.4 since $\alpha_1 = d_1 d_2(f) = d_1 d_1(f) = d_1(g)$ and $\beta_0 = d_0 d_0(f) = d_0 d_1(f) = d_0(g)$, implying $\hat{\beta}_0^{-1} \circ d_1(p) \circ \hat{\alpha}_1 = g^* d_1(p)$. □

**Proposition 4.14.** If $G \in \mathcal{L}_2$, then

1. $d_0 d_1 : P_G \to P_{d_1 G} \to P_{d_0 d_1 G}$ is equal to $d_0 d_0 : P_G \to P_{d_0 G} \to P_{d_0 d_0 G}$.
2. $d_1 d_1 : P_G \to P_{d_1 G} \to P_{d_1 d_1 G}$ is equal to $d_1 d_2 : P_G \to P_{d_2 G} \to P_{d_2 d_1 G}$. 
Proof. We prove only the first equality, as the second is dual. Suppose $G$ is connected.

The above diagram commutes, where the two upward maps are projections onto the second factor, so $d_0d_1 = d_0d_0$ in this case. The general case follows from the connected case:

Now that we have defined $d^1 : P_G \to P_{d_1G}$ for each $G \in \mathcal{L}_2$ and shown it is compatible at the ends, we can use the Segal condition to define inner faces at arbitrary height graphs. It is convenient to use the following uniform notation for the generating outer face maps.

Convention 4.15. If $X$ is a simplicial set, we write $d_\bot : X_n \to X_{n-1}$ for the bottom face map $d_0$ and $d^\top : X_n \to X_{n-1}$ for the top face map $d_n$. We use the same symbols when $G$ is a height $n$ level graph to write $d_\bot = d_0 : P_G \to P_{d_1G} = P_{d_0G}$ and $d^\top = d_n : P_G \to P_{d^\top G} = P_{d_nG}$.

If $G$ is a height $n$ level graph, then for $0 < i < n$ we can define $d_i$ by means of the following diagram, which is well-formed by Proposition 4.14.

\[
\begin{align*}
P_G & \xrightarrow{\cong} P_{d_iG} \\
& \xrightarrow{d_i} P_{d_{i+1}G} \times P_{d_{i-1}G} \times P_{d_{i}G} \\
& \xrightarrow{id \times d_i \times id} P_{d_iG} \times P_{d_{i+1}G} \times P_{d_{i-1}G} \\
\end{align*}
\]
In other words, inner face maps are defined as follows.

**Definition 4.16** (Inner face operators). If \( G \in L_n \), \( 0 < i < n \), and \( x \in P_G \), then \( d_i(x) \in P_{d_iG} \) is the unique element so that

- \( d_i^{n-1}(d_{i+1}) = d_i^{n-1+1}(x) \)
- \( d_i^{n-1}d_{i+1}^{n-1}(d_{i+1}) = d_i^1(d_{i+1}^{n-1}d_{i+1}^{n-1}x) \), and
- \( d_i^1(d_{i+1}x) = d_i^{n+1}(x) \).

In order to utilize this definition effectively, we now examine how an inner face \( d_i \) interacts with arbitrary inert simplicial operators. The identity in the following lemma also holds in an arbitrary simplicial object.

**Lemma 4.17.** Suppose \( G \) is a height \( n \) level graph and \( x \in P_G \). If \( k + \ell \leq n - 1 \), then

\[
d_{\ell+1}^k d_{\ell+1}^{k+1} d_{i+1}^k d_{i+1}^{k+1} d_{i+1}^k d_{i+1}^{k+1} \frac{\partial}{\partial x} = d_{\ell+1}^k d_{\ell+1}^{k+1} d_{i+1}^k d_{i+1}^{k+1} d_{i+1}^k d_{i+1}^{k+1} \frac{\partial}{\partial x}.
\]

**Proof.** If \( i \geq n - \ell \), then by Definition 4.16

\[
d_i^k d_i(x) = d_i^{k-n+i} d_i^{n-i} d_i(x) = d_i^{k-n+i} d_i^{n-i+1}(x) = d_i^{k+1}(x),
\]

while if \( i \leq k \) then

\[
d_i^k d_i(x) = d_i^{k-i} d_i^k d_i(x) = d_i^{k-i} d_i^{k+1}(x) = d_i^{k+1}(x),
\]

so the first and last options hold. The interesting case is the second option. Notice that \( 0 < i - k < s \), and we use the definition of the inner face map \( d_{i-k} \) from Definition 4.16. We make the following three computations

\[
d_i^{n-k-\ell}(i-k)^{n-k-\ell}(d_{i+1}^k d_{i+1}^{k+1} d_{i+1}^k d_{i+1}^{k+1} d_{i+1}^k d_{i+1}^{k+1} \frac{\partial}{\partial x}) = d_i^{n-k-\ell}(i-k)^{n-k-\ell}(d_{i+1}^k d_{i+1}^{k+1} d_{i+1}^k d_{i+1}^{k+1} d_{i+1}^k d_{i+1}^{k+1} \frac{\partial}{\partial x})
\]

and conclude that the element \( d_i^k d_i^{k+1} d_i x \) in \( P_{d_i^k d_i^{k+1} d_i^k d_i^{k+1} d_i^k d_i^{k+1} G} = P_{d_i^k d_i^{k+1} d_i^k d_i^{k+1} G} \) is equal to \( d_i^k d_i^{k+1} d_i^k d_i^{k+1} G \). \( \square \)

**Proposition 4.18.** Suppose \( f: G' \rightarrow G \) is a map in \( L_n \), and \( 0 < i < n \). Let \( d_i(f) = g: d_i G' \rightarrow d_i G \). Then the diagram

\[
\begin{array}{ccc}
P_G & \xrightarrow{f^*} & P_{G'} \\
\downarrow{d_i} & & \downarrow{d_i} \\
P_{d_i G} & \xrightarrow{g^*} & P_{d_i G'}
\end{array}
\]

commutes.
Proof. By Definition 4.16, since \( f \) and \( g \) are inert, it suffices to prove the result when \( n = 2 \) and \( i = 1 \). Further, the diagram commutes if and only if the outer rectangle in

\[
\begin{array}{ccc}
P_G & \xrightarrow{f^*} & P_{G'} \\
\downarrow{d_1} & & \downarrow{d_1} \\
P_{d_1G} & \xrightarrow{g^*} & P_{d_1G'}
\end{array}
\]

commutes for every \( x \in G'_{02} = (d_1G')_{01} \), since \( d_1 \) is defined as a product over the connected pieces. Since the cube

\[
\begin{array}{ccc}
d_1G' \xrightarrow{d_1G} & \xrightarrow{G'} & d_1G \\
\downarrow{d_1G} & \downarrow{d_1G} & \downarrow{d_1G} \\
G' & \xrightarrow{G'} & G
\end{array}
\]

commutes, it is enough to show that

\[
\begin{array}{ccc}
P_G & \xrightarrow{P_G} & P_{G'} \\
\downarrow{d_1} & & \downarrow{d_1} \\
P_{d_1G} & \xrightarrow{P_{d_1G}} & P_{d_1G'}
\end{array}
\]

commutes. The left square commutes by definition of \( d_1 \) on disconnected graphs, and \( G'_x \rightarrow G_{fx} \) is an isomorphism since any map in \( L_n \) between connected graphs is an isomorphism. Thus it suffices to establish the result only in the case when \( f: G' \rightarrow G \) is an isomorphism between height 2 connected graphs, which we already did in Lemma 4.13. \( \square \)

**Lemma 4.19.** If \( G \in L_2 \), then

\[
\begin{array}{ccc}
P_G & \xrightarrow{\cong} & P_{d_1G} \times P_{d_1G} \times P_{d_1G} \times P_{d_1G} \\
\downarrow{d_1} & & \downarrow{d_1} \\
P_{d_1G} & \xrightarrow{\mu_{d_1G} \times \mu_{d_1G} \times \mu_{d_1G} \times \mu_{d_1G}} & \mor C \times \mor C \times \mor C \times \mor C = NC_2
\end{array}
\]

commutes.

Proof. The result holds when \( G \) is connected; in fact, it is the definition of \( d_1 \). In a moment we will induct on the number of connected components of \( G \). First, suppose the result is true for \( G' \) and that \( f: G' \rightarrow G \) is an isomorphism. Then

\[
\mu_{d_1G'}((d_1f)^*d_1p) = \mu_{d_1G'}((d_1f)^*d_1p) \circ \mu_{d_1G'}((d_1f)^*d_1p).
\]

By Proposition 4.18 on the left and the fact that \( P \) is a presheaf on \( L_{\text{int}} \), the preceding equation can be written as

\[
\mu_{d_1G'}((d_1f)^*d_1p) = \mu_{d_1G'}((d_1f)^*d_1p) \circ \mu_{d_1G'}((d_1f)^*d_1p).
\]

We now apply Lemma 4.10 to see that

\[
\mu_{d_1G'}((d_1f)^*d_1p) = \mu_{d_1G'}(d_1p) \circ \mu_{d_1G'}(d_1f).
\]
If rectangles commute by Lemma 4.19.

\[
\mu_{d \downarrow G}((d \downarrow f)^* d \downarrow p) \circ \mu_{d \uparrow G}((d \uparrow f)^* d \uparrow p)
\]
\[
= d_0 d_\uparrow f^{-1} \circ \mu_{d \downarrow G}(d \downarrow p) \circ d_\uparrow d_\downarrow f \circ d_0 d_\uparrow f^{-1} \circ \mu_{d \uparrow G}(d \uparrow p) \circ d_\downarrow d_\uparrow f,
\]
using that \(d_0 d_\uparrow = d_0 d_1 = d_1 d_0 = d_1 d_\downarrow\). But also \(d_0 d_\uparrow = d_0 d_1\) and \(d_1 d_\uparrow = d_1 d_2 = d_1 d_1\), so we see that \(\mu_{d_1 G}(d_1 p) = \mu_{d_1 G}(d_\downarrow p) \circ \mu_{d_\uparrow G}(d_\uparrow p)\).

Now suppose the result is known for graphs with strictly fewer than \(n\) connected components, and let \(H + K \rightarrow G\) be an isomorphism where \(H\) and \(K\) are non-empty graphs. By the preceding paragraph, it is enough to prove the result for the graph \(H + K\). Let \(h : H \rightarrow H + K\) and \(k : K \rightarrow H + K\) be the inclusions. Write \(h_1 = d_1(h)\) and \(k_1 = d_1(k)\). Then using the induction hypothesis and Lemma 4.9, we have

\[
\mu_{d_1 (H + K)}(d_1 p)
\]
\[
= \mu_{d_1 H}(d_1^* d_1 p) \otimes \mu_{d_1 K}(k_1^* d_1 p)
\]
\[
= \mu_{d_1 H}(d_1 h^* p) \otimes \mu_{d_1 K}(d_1 k^* p)
\]
\[
= [(\mu_{d_1 H}(d_0 h^* p)) \circ (\mu_{d_2 H}(d_2 h^* p))] \otimes [(\mu_{d_1 K}(d_0 k^* p)) \circ (\mu_{d_2 K}(d_2 k^* p))]
\]
\[
= [(\mu_{d_1 H}(d_0 h^* p)) \circ (\mu_{d_1 K}(d_0 k^* p))] \otimes [(\mu_{d_1 H}(d_0 k^* p)) \circ (\mu_{d_1 K}(d_0 k^* p))]
\]
\[
= (\mu_{d_1 H}(h_0^* d_0 p)) \otimes (\mu_{d_1 K}(k_0^* d_0 p)) \otimes (\mu_{d_1 H}(h_2^* d_2 p)) \otimes (\mu_{d_1 K}(k_2^* d_2 p))
\]
\[
= (\mu_{d_1 H + K}(d_0 p)) \circ (\mu_{d_1 H + K}(d_2 p)).
\] \(\square\)

Lemma 4.20. If \(G\) is a height 3 level graph and \(x \in P_G\), then \(d_1 d_2(x) = d_1 d_1(x)\).

Proof. For space reasons, in the following two diagrams we do not notate the fiber products, nor do we label the \(\mu\). We use the notation \(NC_3 = \text{mor } C \times_{obC} \text{mor } C \times_{obC} \text{mor } C\) so that \(d_2 = (id \times \circ)\) and \(d_1 = (\circ \times id)\). In each diagram the two rectangles commute by Lemma 4.19.

If \(G\) is connected, then so are \(d_1 d_2 G\) and \(d_1 d_3 G\), and the bottom map of both diagrams is an inclusion, so the result holds in the connected case. The general case follows from the connected case and Proposition 4.18. \(\square\)
Proposition 4.21. If $G$ is a height $n$ level graph, $x \in P_G$, and $0 \leq i < j \leq n$, then $d_id_j(x) = d_{j-1}d_i(x)$.

Proof. If $i \leq j - 2$, then using Lemma 4.17 one can compute that $d_i^m d_{m-3}^n d_id_j$ and $d_{i+1}^m d_{i+1}^n d_{j-1}d_i$ are both equal to

$$\begin{cases} 
  d_i^m d_{i-1}^n & m \leq i - 2 \\
  d_{i+1}^m d_{i-2}^n & m = i - 1 \\
  d_i^{m+1} d_{i-2}^n & i \leq m \leq j - 3 \\
  d_i^{m+1} d_{i-3}^n & m = j - 2 \\
  d_i^{m+2} d_{i-3}^n & m \geq j - 1.
\end{cases}$$

By the Segal condition, this shows that $d_id_j(x) = d_{j-1}d_i(x)$ in this case.

It remains to consider the case when $i = j - 1$. In this case we still have for $m \leq i - 2$ that

$$d_i^m d_{m-3}^n d_id_{i+1} = d_i^m d_{m-1}^n = d_i^m d_{m-3}^n d_id_i$$

and for $m \geq i$ that

$$d_i^m d_{m-3}^n d_id_{i+1} = d_i^{m+2} d_{m-3}^n = d_i^m d_{m-3}^n d_id_i.$$

For the remaining value $m = i - 1 = j - 2$, we must verify that $d_i^{i-1} d_{i-2}^{i-2} d_i d_{i+1} = d_i^{i-1} d_{i-2}^{i-2} d_i d_i$. By Lemma 4.17 we have $d_i^{i-1} d_{i-2}^{i-2} d_i d_{i+1} = d_i d_i d_{i-1} d_{i-2}^{i-2}$ and $d_i^{i-1} d_{i-2}^{i-2} d_i d_i = d_i d_i d_{i-1} d_{i-2}^{i-2}$. These are equal by Lemma 4.20. \(\square\)

Definition 4.22 (Degeneracy in lowest dimension). If $n$ is height 0 level graph, define $s_0 : P_n \to P_{s_0(n)}$ by declaring that

$$P_n \to P_{s_0(n)} \cong \prod_{i=1}^n P_{s_1}$$

takes $c_1 \otimes \ldots \otimes c_n$ to $(id_{c_1}, \ldots, id_{c_n})$.

Lemma 4.23. If $x \in P_n$, then $d_0s_0(x) = x = d_1s_0(x)$.

Proof. For $i = 0, 1$ the full composite of the diagram

$$P_n \xrightarrow{s_0} P_{s_0(n)} \cong \prod_{i=1}^n P_{s_1} \xrightarrow{d_i} \prod_{i=1}^n P_{s_i}$$

takes $c_1 \otimes \ldots \otimes c_n$ to $(c_1, \ldots, c_n)$. Hence $d_is_0 = id$. \(\square\)

Lemma 4.24. If $f : n \to m$ is a map in $L_n$ and $g = s_0(f)$, then

$$s_0f^* = g^*s_0 : P_m \to P_{s_0(n)}.$$

Proof. By Segality, it is enough to show that the the equality holds after postcomposing with $P_{s_0(n)} \to P_{s_1}$ for each of the $n$ inert maps $s_{11} \to s_0(n)$. Since the two
squares on the right in the following diagram commute,

\[
\begin{array}{c}
P_m \xrightarrow{f^*} P_n \xrightarrow{\prod_{i=1}^n P_i} P_\ell \\
\downarrow s_0 \quad \downarrow s_0 \quad \downarrow \\
P_{s_0(m)} \xrightarrow{g^*} P_{s_0(n)} \xrightarrow{\prod_{i=1}^n P_{\ell_1}} P_{\ell_1}
\end{array}
\]

it thus enough to prove the result for \( n = 1 \). But if \( f: \frac{1}{m} \rightarrow n \) hits an element \( k \), then \( s_0 f^*(c_1 \otimes \cdots \otimes c_m) = s_0(c_k) = id_{c_k} \), and by Definition 4.22 we have \( g^* s_0(c_1 \otimes \cdots \otimes c_m) = id_{c_k} \).

**Definition 4.25** (Degeneracy operators). If \( G \) is a height \( n \) level graph, then \( s_j: P_G \rightarrow P_{s_j G} \) is defined so that \( s_j(x) \) is the unique element satisfying

- \( d_T^{n+1-j}(s_j x) = d_T^{n-j}(x) \)
- \( d_1^j d_T^{n-j}(s_j x) = s_0(d_1^j d_T^{n-j} x) \), and
- \( d_1^{j+1}(s_j x) = d_1^j(x) \).

In other words, \( s_j \) is defined so the following diagram commutes.

\[
\begin{array}{ccc}
P_G & \xrightarrow{\cong} & P_{d_T^{n-j}G} \times P_{d_1^j d_T^{n-j}G} \times P_{d_T^{n-j+1}G} \times P_{d_1^j G} \\
\cong & \downarrow s_j & \downarrow \text{id} \times s_0 \times \text{id} \\
\cong & P_{d_T^{n-1}G} \times P_{d_1^1 d_T^{n-1}G} \times P_{d_T^{n-1}G} \times P_{d_1^1 G}
\end{array}
\]

In order to utilize this definition effectively, we now examine how a degeneracy \( s_j \) interacts with arbitrary inert simplicial operators. The identity in the following lemma also holds in an arbitrary simplicial object.

**Lemma 4.26.** Suppose \( G \) is a height \( n \) level graph and \( x \in P_G \). If \( k + \ell \leq n + 1 \), then

\[
d_1^k d_T^\ell s_j(x) = \begin{cases} d_1^k d_T^\ell(x) & n - \ell + 1 \leq j \\ s_{j-k} d_1^k d_T^\ell(x) & k \leq j \leq n - \ell \\ d_1^{j-1} d_1^k d_T(x) & j \leq k - 1. \end{cases}
\]

**Proof.** If \( j \geq n - \ell + 1 \), then by Definition 4.25

\[
d_T^\ell s_j(x) = d_T^{n-j} d_T^{n+1-j} s_j(x) = d_T^{n-j+1} d_T^{n-j} = d_T^{n-j}(x),
\]

while if \( j \leq k - 1 \) then

\[
d_1^k s_j(x) = d_1^{k-j} d_1^j s_j(x) = d_1^{k-j} d_1^j(x) = d_1^{k-1} d_1^j(x),
\]

so the first and last options hold. We now use the definition of degeneracy operators from Definition 4.25. We make the following three computations

\[
d_T^{n-k-\ell+1} d_T^{(j-k)} (d_1^k d_T^\ell s_j x) = d_1^k d_T^{n-j+1} s_j(x) = d_1^k d_T^{n-j} x = d_T^{(n-\ell)-(j-k)} (d_1^k d_T^\ell(x)).
\]
\[ d_{-1}^{j-k} d_{1}^{(n-k-\ell)-(j-k)} (d_{1}^{k} d_{-1}^{\ell} s_{j} x) = d_{1}^{j-k} d_{-1}^{(n-j)} s_{j} x = s_{0}(d_{1}^{j-k} d_{-1}^{(n-j)} x) = s_{0}(d_{1}^{j-k} d_{1}^{(n-k-\ell)-(j-k)} (d_{1}^{k} d_{-1}^{\ell} (x)) \]

and conclude that the element \( d_{1}^{k} d_{-1}^{\ell} s_{j} x \) in \( P_{d_{1}^{k} d_{-1}^{\ell} s_{j} G} = P_{s_{j-k} d_{1}^{k} d_{-1}^{\ell} G} \) is equal to \( s_{j-k}(d_{1}^{k} d_{-1}^{\ell} x) \).

**Proposition 4.27.** Suppose \( f : G' \to G \) is a map in \( L_{n} \), and \( 0 \leq j \leq n \). Let \( s_{j}(f) = g : s_{j} G' \to s_{j} G \). Then the diagram

\[
\begin{array}{ccc}
P_{G} & \xrightarrow{f^{*}} & P_{G'} \\
\downarrow{s_{j}} & & \downarrow{s_{j}} \\
P_{s_{j} G'} & \xrightarrow{g^{*}} & P_{s_{j} G'}
\end{array}
\]

commutes.

**Proof.** By Segality it is enough to show that \( d_{-1}^{m} d_{1}^{m-m} s_{j} f^{*}(x) = d_{-1}^{m} d_{1}^{m-m} g^{*} s_{j}(x) \) for all \( x \in P_{G} \) and each \( 0 \leq m \leq n \). Set

\[
h = d_{-1}^{m} d_{1}^{m-m} (g) = d_{-1}^{m} d_{1}^{m-m} s_{j} (f) = \begin{cases} 
  d_{1}^{m} d_{1}^{m-m-1}(f) & m + 1 \leq j \\
  s_{0}d_{1}^{m} d_{1}^{m-m}(f) & j = m \\
  d_{-1}^{m-1} d_{1}^{m-m}(f) & j \leq m - 1.
\end{cases}
\]

By Lemma 4.26, for \( x \in P_{G} \) we have

\[
d_{-1}^{m} d_{1}^{m-m} s_{j} f^{*} x = \begin{cases} 
  d_{1}^{m} d_{1}^{m-m-1} f^{*} x & m + 1 \leq j \\
  s_{0}d_{1}^{m} d_{1}^{m-m} f^{*} x & j = m \\
  d_{-1}^{m-1} d_{1}^{m-m} f^{*} x & j \leq m - 1.
\end{cases}
\]

Since \( P|_{L_{m}} \) is a presheaf we have the first equality below, while the second follows from Lemma 4.26.

\[
d_{-1}^{m} d_{1}^{m-m} g^{*} s_{j} x = h^{*} d_{-1}^{m} d_{1}^{m-m} s_{j} x = \begin{cases} 
  h^{*}d_{1}^{m} d_{1}^{m-m-1} x & m + 1 \leq j \\
  h^{*} s_{0}d_{1}^{m} d_{1}^{m-m} x & j = m \\
  h^{*} d_{-1}^{m-1} d_{1}^{m-m} x & j \leq m - 1
\end{cases}
\]

Again using that \( P|_{L_{m}} \) is a presheaf, we have the desired equality for \( m \neq j \). When \( m = j \), we have

\[
s_{0}d_{1}^{m} d_{1}^{m-m} f^{*} x = s_{0}(d_{1}^{m} d_{1}^{m-m}(f))^{*} d_{-1}^{m} d_{1}^{m-m}(x).
\]

But

\[
s_{0}(d_{1}^{m} d_{1}^{m-m}(f))^{*} = (s_{0}d_{1}^{m} d_{1}^{m-m}(f))^{*} s_{0} = h^{*} s_{0}
\]

by Lemma 4.24, so

\[
d_{1}^{j} d_{1}^{n-j} s_{j} f^{*}(x) = d_{1}^{j} d_{1}^{n-j} g^{*} s_{j}(x)
\]

holds as well. \(\square\)
Proposition 4.28. Suppose that $G$ is a height $n$ level graph and $x \in P_G$. If $i \leq j$, then $s_is_j(x) = s_{j+1}s_i(x)$.

Proof. We have $s_is_jG = s_{j+1}s_iG$ in $L_{n+2}$. By Segality, it is enough to verify, for $0 \leq m \leq n + 1$ and $x \in P_G$, that the operator $d^m: d^m_{m+1-m}$ takes the two elements $s_is_j(x)$ and $s_{j+1}s_i(x)$ to the same element. Applying Lemma 4.26 several times, and also using Lemma 4.23, one computes that $d^m_{m+1-m}d_is_j(x)$ and $d^m_{m+1-m}d_{j+1}s_i(x)$ are both equal to

\[
\begin{align*}
&\left\{ \begin{array}{ll}
d^m_{m+1-m}d_i & m \leq i - 1 \\
s_0d^m_{m+1-m} & m = i \\
d^m_{m+1-m}d_i & i + 1 \leq m \leq j \\
s_0d^m_{m+1-m} & m = j + 1 \\
d^m_{m+1-m}d_i & m \geq j + 2.
\end{array} \right.
\]

Thus $s_is_j(x) = s_{j+1}s_i(x)$.

Proposition 4.29. If $G$ is a height 1 level graph and $x \in P_G$, then

\[d_1s_0x = x = d_1s_1x.\]

Proof. If we can prove the relation for a corolla $\varsigma$, then it will follow for an arbitrary $G \in L_1$ using Proposition 4.18, Proposition 4.27, and Segality. Indeed, if $\iota : \varsigma \rightarrow G$ is a component of $G$, then for $x \in P_G$ and $k = 0, 1$, we have

\[\iota^*d_1s_kx = d_1(s_k\iota)^*s_kx = d_1s_k\iota^*x = \iota^*x\]

using the corolla case, so by Segality $d_1s_kx = x$.

Let $x \in P_\varsigma$ where $\varsigma$ is a corolla. By Definition 4.12, $d_1$ applied to the element $s_0x \in P_{s_0\varsigma}$ is

\[\mu_{d_0s_0\varsigma}(d_0s_0x) \circ \mu_{d_2s_0\varsigma}(d_2s_0x)\]

which is equal to

\[\mu_{d_0s_0\varsigma}(x) \circ \mu_{d_2s_0\varsigma}(s_0d_1x) = x \circ \text{id} = x\]

by Definition 4.25. Likewise, $d_1s_1x$ is

\[\mu_{d_0s_1\varsigma}(d_0s_1x) \circ \mu_{d_2s_1\varsigma}(d_2s_1x) = \mu_{d_0s_1\varsigma}(s_0d_0x) \circ \mu_{d_2s_1\varsigma}(x) = \text{id} \circ x = x.\]

The proof of the following lemma uses the same idea as the proofs of Proposition 4.21 and Proposition 4.28.

Proposition 4.30. If $G$ is a height $n$ level graph and $x \in P_G$, then

\[d_is_jx = \begin{cases} 
  s_{j-1}d_i & i < j \\
  x & i = j, j + 1 \\
  sjd_{i-1} & i > j + 1.
\end{cases}\]

Proof. We will apply the operator $d^m_{m+1-m}$ for $0 \leq m \leq n - 1$ to both sides of the equation, and show that our answers coincide. Since $P$ is Segal, that implies that equation holds.

Using Lemma 4.17, Lemma 4.26, and Proposition 4.29, one computes that

\[d^m_{m+1-m}d_is_jx = d^m_{m+1-m}x\]
whenever \( m \leq \min(i-2, j-1) \), \( m \geq \max(i, j) \), \( m = j = i - 1 \), or \( m = i - 1 = j - 1 \). If \( i = j \) or \( i = j + 1 \), this covers all possible values of \( m \), so the formula holds in these cases. Otherwise, one computes using Lemma 4.17 and Lemma 4.26 that

\[
\begin{align*}
d^m_{11}d^m_{m-1}d_s j x &= \begin{cases} 
    d^m_{11}d^m_{m-2}x & i \leq m \leq j - 2 \\
    d^m_{11}d^m_{m-2}x & i - 1 = m \leq j - 2 \\
    s_0d^m_{11}d^m_{m-1}x & i \leq m = j - 1 \\
    d^m_{11}d^m_{m-1}x & j + 1 \leq m \leq i - 2 \\
    s_0d^m_{11}d^m_{m-1}x & j + 1 = m = i - 1 \\
    s_0d^m_{11}d^m_{m-1}x & m = j \leq i - 2.
\end{cases}
\end{align*}
\]

When \( i < j \), one computes

\[
\begin{align*}
d^m_{11}d^m_{m-1}s_j - 1 d s j x &= \begin{cases} 
    d^m_{11}d^m_{m-2}x & m \leq i - 2 \text{ or } j \leq m \\
    d^m_{11}d^m_{m-2}x & m = i - 1 \\
    d^m_{11}d^m_{m-2}x & \leq j \leq m \\
    s_0d^m_{11}d^m_{m-1}x & m = j \\
    s_0d^m_{11}d^m_{m-1}x & m = j - 1 \leq 1 \text{ or } m \leq i - 2 \\
    s_0d^m_{11}d^m_{m-1}x & m = i - 1,
\end{cases}
\end{align*}
\]

which implies \( d_i s_j x = s_j - 1 d_i x \) by the previous calculation. When \( i > j + 1 \), one computes

\[
\begin{align*}
d^m_{11}d^m_{m-1}s_j d_{i-1} x &= \begin{cases} 
    d^m_{11}d^m_{m-2}x & m \leq j - 1 \text{ or } i \leq m \\
    s_0d^m_{11}d^m_{m-1}x & m = j \\
    s_0d^m_{11}d^m_{m-1}x & m = j - 1 \leq j \leq m \\
    d^m_{11}d^m_{m-1}x & j + 1 \leq m \leq i - 2 \\
    d^m_{11}d^m_{m-1}x & m = i - 1,
\end{cases}
\end{align*}
\]

implying \( d_i s_j x = s_j d_{i-1} x \).

\( \square \)

**Theorem 4.31.** The \( \text{L}_{\text{int}} \)-presheaf \( P|_{\text{L}_{\text{int}}} \in \text{Seg}(\text{L}_{\text{int}}) \) extends to an \( \text{L} \)-presheaf \( P \in \text{Seg}(\text{L}) \) with inner face maps and degeneracies defined as in Definition 4.16 and Definition 4.25. Further, the assignment \( C \rightsquigarrow P \) is a functor \( \text{SLCC} \to \text{Seg}(\text{L}) \).

**Proof.** Every map \( G \to H \) in \( \text{L} \) factors uniquely as follows:

\[
G \xrightarrow{f} \alpha^* H \xrightarrow{\alpha} H
\]

where \( \alpha \) is a map in \( \Delta \) and \( f \in \text{L}_n \) where \( n \) is the height of \( G \). We have an existing map \( f^* : P_\alpha^* H \to P_G \) using the \( \text{L}_{\text{int}} \)-presheaf structure, and we define \( \alpha^* : P_H \to P_\alpha^* H \) by decomposing into face and degeneracy operators; by Proposition 4.21, Proposition 4.28, and Proposition 4.30 the result does not depend on any choices of how to do this decomposition. On such a composite, we define \( (\alpha \circ f)^* : P_H \to P_G \) to be the composite \( P_H \to P_\alpha^* H \to P_G \). To see that composition is preserved, consider a pair of composable morphisms in \( \text{L} \) presented in this way:

\[
G \xrightarrow{f} \alpha^* H \xrightarrow{\alpha} H \xrightarrow{g} \beta^* K \xrightarrow{\beta} K
\]

Then setting \( h := \alpha^*(g) : \alpha^* H \to \alpha^* \beta^* K \), the composite is presented as in the dashed maps. Then

\[
(\alpha \circ f)^*(\beta \circ g)^* = f^* \alpha^* g^* \beta^* = f^* h^* \alpha^* \beta^* .
\]
where the last equality uses Proposition 4.18, Proposition 4.27, and that \( P \) is an \( \mathbf{L}_{\text{int}} \)-presheaf. But then \( f^*h^* = (hf)^* \) since \( P \) is an \( \mathbf{L}_{\text{int}} \)-presheaf and \( \alpha^*\beta^* = (\beta\alpha)^* \) by Propositions 4.21, 4.28 and 4.30. Thus \((\alpha \circ f)^*(\beta \circ g)^*) = ((\beta \circ g) \circ (\alpha \circ f))^*\), so \( P \) is an \( \mathbf{L} \)-presheaf.

All constructions from this section were functorial on maps in \( \mathbf{SLCC} \): the composite of the \( \mathbf{L}_{\text{el}} \)-presheaf, the right Kan extension \( \text{Psh}(\mathbf{L}_{\text{el}}) \to \text{Psh}(\mathbf{L}_{\text{int}}) \), and finally the definition of the inner face operators (Definitions 4.12 and 4.16) and degeneracy operators (Definitions 4.22 and 4.25). Thus we have constructed a functor \( \mathbf{SLCC} \to \text{Seg}(\mathbf{L}) \).

\[ \square \]

5. Comparison of constructions

We now have two constructions: the first is the functor \( \text{Seg}(\mathbf{L}) \to \mathbf{SLCC} \) that takes a presheaf \( X \) to the image of the unique map \( X \to * \) under the envelope functor \( \mathbf{C} \), and the second is the functor \( \mathbf{SLCC} \to \text{Seg}(\mathbf{L}) \) that takes a strict labelled cospan category \( \pi : \mathbf{C} \to \mathbf{C}(*) \) to the presheaf \( P = P(\pi) \). In this section, we will show that these are part of an equivalence of categories \( \text{Seg}(\mathbf{L}) \simeq \mathbf{SLCC} \) (Theorem 5.8).

First we show that the composite \( \mathbf{SLCC} \to \text{Seg}(\mathbf{L}) \to \mathbf{SLCC} \) is isomorphic to the identity.

Let \( \pi : \mathbf{C} \to \mathbf{C}(*) \) be a strict labelled cospan category, and \( P = P(\pi) \) be the associated Segal \( \mathbf{L} \)-presheaf. Our goal is to compare \( C \) with the envelope \( \mathbf{CP} \) of \( P \).

The set of connected objects of \( \mathbf{CP} \) is \( P_\epsilon \), which is exactly the set of connected objects of \( \mathbf{C} \). (See the beginning of Section 3 and the beginning of Section 4.1). Since these are both strict labelled cospan categories, the set of objects of each is the free monoid on this set, hence the sets of objects can be identified. Under this identification the tensor units coincide. Moreover, the set of connected morphisms of \( \mathbf{CP} \) is \( \sum P_{\alpha_{n,m}} \), which is naturally in bijection with the set of connected morphisms of \( \mathbf{C} \) (each \( P_{\alpha_{n,m}} \) is a subset of \( \text{mor}^\text{el} \mathbf{C} \), see (3) on page 31). The abelian monoid \( \text{hom}(\mathbf{1}, \mathbf{1}) \) is the same in both categories, that is, it is freely generated by the identical sets \( \text{hom}^\text{el}_C(\mathbf{1}, \mathbf{1}) = P_{\alpha_0} = \text{hom}^\text{el}_C(\mathbf{1}, \mathbf{1}) \) of connected endomorphisms (see Definition 1.2(2)).

In Construction 4.7, we defined, for each \( G \in \mathbf{L}_1 \), a function \( \mu_G : P_G \to \text{mor} \mathbf{C} \). We now identify the image of this function.

**Lemma 5.1.** If \( G \in \mathbf{L}_1 \), then \( \mu_G : P_G \to \text{mor} \mathbf{C} \) factors through \( \mathbf{T}_G \) and the map \( \mathbf{T}_G \to \text{mor} \mathbf{C} \) is injective.

**Proof.** If \( f : H \equiv G \) is a congruence between height 1 graphs, then by Lemma 4.10 we have \( \mu_H f^*(p) = \mu_G(p) \) for all \( p \in P_G \). This implies that \( \mu_G \) factors through \( \mathbf{T}_G = \text{colim}_{H \in \mathbf{W}_{G}(\mathbf{L}_1)} P_H \). For injectivity, we may assume that \( G \) has the property that all vertices without inputs or outputs come at the end; that is, we utilize a congruence

\[
H = \left( G' + \sum_{i=1}^{k-j} e_{00} \right) \equiv G
\]

where \( G' \) has \( j \) vertices and is reduced, and \( G' \to G \) is order-preserving at each level. Now \( \mathbf{T}_G \to \text{mor} \mathbf{C} \) is equal to \( \mathbf{T}_H \to \text{mor} \mathbf{C} \), so we replace \( G \) with \( H \).

Let \( p, p' \in P_G \), where \( G \in \mathbf{L}_1 \) has \( j \) vertices which have an input or output and \( k-j \) vertices which have no input or output, and the former come before the latter in the order on \( G_{01} \). Assume \( \mu_G(p) = \mu_G(p') \); we wish to show that \( p \) and \( p' \)
represent the same element of $P_G$. We use similar notation as in Construction 4.7; specifically, we let

$$\iota = \sum t_x : \sum_{x=1}^k G_x \xrightarrow{\sim} G$$

be the canonical splitting from Definition 4.6, $p_x = \iota_x^* p$, $p'_x = \iota_x^* p'$, $\iota^0 = d_0(\iota)$, and $\iota^1 = d_1(\iota)$. Then

$$\mu(p) = \hat{\iota}^0 \circ (p_1 \otimes \cdots \otimes p_k) \circ (\hat{\iota})^{-1}$$

$$\mu(p') = \hat{\iota}^0 \circ (p'_1 \otimes \cdots \otimes p'_k) \circ (\hat{\iota})^{-1}.$$

Our assumption was $\mu(p) = \mu(p')$, hence

$$p_1 \otimes \cdots \otimes p_k = p'_1 \otimes \cdots \otimes p'_k$$

with all $p_x, p'_x$ connected morphisms of $C$. By Definition 1.2(3) we have $p_1 \otimes \cdots \otimes p_j = p'_1 \otimes \cdots \otimes p'_j$ and $p_{j+1} \otimes \cdots \otimes p_k = p'_{j+1} \otimes \cdots \otimes p'_k$. By (4) of Definition 1.2, $p_x = p'_x$ for $0 \leq x \leq j$. In light of Definition 1.2(2), there is an automorphism $\gamma'$ of $\mathbb{k}$ by letting $\gamma(x) = x$ for $1 \leq x \leq j$, and this comprises a congruence $f : G \xrightarrow{\sim} G$ with $f_{01} = \gamma$. We next observe that $f^*(p') = p$. By Segality, it is enough to show $\iota_x^* f^*(p') = \iota_x^*(p)$ for $1 \leq x \leq k$. Notice that $\iota_x^* f^*(p') = \iota_x^*(p')$.

For $1 \leq x \leq j$, we have $\iota_x^* \gamma_x(p') = \iota_x^*(p') = p'_x = p_x = \iota_x^*(p)$. For $j + 1 \leq x \leq k$, we have $\iota_x^* \gamma_x(p') = \iota_x^*(p') = p'_x = p_x = \iota_x^*(p)$.

We have shown that if $p, p' \in P_G$ are two elements with $\mu(p) = \mu(p')$, then they are identified in $P_G$. Since $P_G \to P_G$ is surjective, this proves that $P_G \to \text{mor} C$ is injective. \hfill \Box

As we saw at the beginning of the preceding proof, the functions $\mu_G : P_G \to \text{mor} C$ give well defined functions $P_w \to \text{mor} C$ where $w \in W_1 = \text{mor} C(*).$ Thus we can make the following definition.

**Definition 5.2** (Comparison of morphisms). Let

$$\bar{\mu} : \text{mor} CP = SP_1 = \sum_{w \in W_1} P_w \to \text{mor} C$$

be the function induced from the collection of functions $\mu_G$ (for $G \in L_1$).

**Lemma 5.3.** The function $\bar{\mu} : \text{mor} CP \to \text{mor} C$ is compatible with sources and targets of morphisms.

**Proof.** For each graph $G \in L_1$, the outer square and all inner chambers except the inner rectangle of the following diagram are known to commute (including using
Construction 4.7).

$$\begin{array}{c}
P_G \\
\mu \\
\downarrow \\
\text{mor } C \\
\downarrow \\
\text{ob } C \\
\downarrow \\
P_{d_1 G} \\
\end{array}$$

$$\begin{array}{c}
P_G \\
\downarrow \\
\text{mor } CP \\
\downarrow \\
\text{ob } CP \\
\downarrow \\
SP_0 \\
\end{array}$$

It follows from surjectivity of $P_G \to P_G$ that the inner rectangle commutes as well, so the function $\text{mor } CP \to \text{mor } C$ preserves sources of maps. Compatibility with targets is the evident modification of this argument. □

The next proof uses that $\bar{\mu}$ preserves identities between connected objects, which is immediate from Definition 4.22, since $s_0: P_1 \to P_{s_0(1)} = P_{c_1}$ takes $c$ to $\text{id}_c \in P_{c_1} \subseteq \text{mor } C$.

**Lemma 5.4.** The function $\bar{\mu}: \text{mor } CP \to \text{mor } C$ strictly preserves the monoidal product of morphisms, and also preserves the symmetry isomorphisms.

**Proof.** Recall from Lemma 2.21 and Definition 2.18 that the tensor product of morphisms in $CP$ is induced as in the left rectangle in the following diagram, while the right square commutes by Lemma 4.9.

$$\begin{array}{c}
SP_1 \times SP_1 \\
\downarrow \circ \\
SP_1 \\
\end{array} \leftrightarrow \begin{array}{c}
P_G \times P_H \\
\mu \times \mu \\
\downarrow \approx \\
\mu \\
\end{array} \leftrightarrow \begin{array}{c}
P_G \times P_H \\
\downarrow \approx \\
\text{mor } C \times \text{mor } C \\
\downarrow \approx \\
\text{mor } C \\
\end{array}$$

Thus $\bar{\mu}$ preserves monoidal product.

To see that $\bar{\mu}$ preserves the symmetry, it is enough to check that it is true for a pair of connected objects $c,d$. Let $G$ be the following graph

$$\begin{array}{c}
\begin{array}{c}
2 \\
\approx \tau \approx 2 \\
\gamma \\
2 \\
\end{array} \\
\end{array}$$

(using the notation of Section 2.4, this means $G = c_1 + c_1$). Let $p \in P_G$ be the unique element which is sent to $(\text{id}_c, \text{id}_d)$ under the isomorphism $P_G \to P_{c_1} \times P_{c_1}$. By Definition 2.26 and Definition 2.31, the symmetry $\tau_{c,d}: c \otimes d \to d \otimes c$ of $CP$ is the image of $p$ under $P_G \to P_G \subseteq \text{mor } CP$. The definition of $\bar{\mu}$ implies that $\bar{\mu}(\tau_{c,d}) = \mu_G(p)$. By Lemma 4.8, $\mu_G(p): c \otimes d \to d \otimes c$ is the symmetry isomorphism of $C$. Hence $\bar{\mu}$ preserves the symmetry isomorphism. □

**Lemma 5.5.** The function $\bar{\mu}: \text{mor } CP \to \text{mor } C$ is bijective.
The function $\bar{\mu} : \text{mor} \, CP \to \text{mor} \, C$ gives an isomorphism of strict labelled cospan categories $CP \cong C$.

Proof. By Lemma 4.19, $\bar{\mu}$ preserves composition. Lemma 5.3 and Lemma 5.5 imply that $\text{hom}_{C(P)}(x,y) \to \text{hom}_C(x,y)$ is a bijection for all $x,y$. Preservation of identities follows from surjectivity of these functions and the fact that $\bar{\mu}$ preserves composition. Hence $CP \to C$ is a functor. Since it is bijective on objects and fully-faithful, it is an isomorphism of categories. By Lemma 5.4 this is a map of permutative categories, hence an isomorphism of such. By Proposition 4.11 this is a map over $C(*)$, hence an isomorphism in $\text{SLCC}$.\[\square\]
permutative functor \( f \) lives over \( C(\ast) \), we have induced functions \( f_k: X_k \to Y_k \) and \( f_{\varepsilon_{m,n}}: X_{\varepsilon_{m,n}} \to Y_{\varepsilon_{m,n}} \); we will first show these give a map \( X_{[\mathcal{L}_0]} \to Y_{[\mathcal{L}_0]} \) of \( \mathcal{L}_0 \)-presheaves. Three types of maps are indicated in \( \S 4.1 \) that we must contend with, namely \( \ell_i, r_j: 1 \to \varepsilon_{m,n} \) and \( (\gamma, \chi): \varepsilon_{m,n} \to \varepsilon_{m,n} \). The diagram for \( \ell_i \) is listed just below, where the top composite is \( \ell_i^*: X_{\varepsilon_{m,n}} \to X_1 \) and similarly for the bottom composite. The two left squares commute since \( f \) preserves domains and the monoidal product.

\[
\begin{array}{cccc}
X_{\varepsilon_{m,n}} & \xrightarrow{d_1} & X_m & \xleftarrow{X_m^r} X_1 \\
\downarrow & & \downarrow & \downarrow \\
Y_{\varepsilon_{m,n}} & \xrightarrow{d_1} & Y_m & \xleftarrow{Y_m^r} Y_1 
\end{array}
\]

We similarly have \( f_1 r_j^* = r_j^* f_{\varepsilon_{m,n}} \). For the bipermutations \( (\gamma, \chi) \) acting on \( \varepsilon_{m,n} \) it is enough to consider the cases where one of \( \gamma \) or \( \chi \) is an adjacent transposition and the other is an identity. Let \( H \) be the height 2 level graph

\[
\begin{array}{c}
m \\
\tau \\
\id \\
1
\end{array}
\]

where the leftmost bijection \( \tau \) interchanges \( i \) and \( i + 1 \) and acts as the identity otherwise. There is an isomorphism \( H \cong s_0 \varepsilon_{m,n} \), which is \( \tau: H_{00} \to (s_0 \varepsilon_{m,n})_{00} \), and the identity elsewhere. Then the composite

\[
s_0 \varepsilon_{m,n} \xrightarrow{d^i} H \xrightarrow{\cong} s_0 \varepsilon_{m,n} \xrightarrow{s^0} \epsilon_{m,n}
\]

is the automorphism of \( \varepsilon_{m,n} \) which interchanges the \( i \)th and \((i + 1)\)st inputs. There is a congruence as to the left of the following commutative square (see Section 2.4).

\[
\begin{array}{cccc}
d_2 H & \cong & d_2 s_0 \varepsilon_{m,n} \\
\equiv & & \equiv \\
s_0(i - 1) + (c_{11} + \chi_1) + s_0(n - i - 1) & \xrightarrow{id_1 + \sigma_1 + \id_0} & s_0 m
\end{array}
\]

so that \( X_{s_0\Omega} \to X_{d_2 H} \) sends the element representing \( \id_{x_1 \otimes \cdots \otimes x_m} \) to the element representing \( \id_{x_1 \otimes \cdots \otimes x_{i-1} \otimes \tau_{x_i, x_{i+1}} \otimes \id_{x_{i+2} \otimes \cdots \otimes x_m}} \) in \( CX \). Commutativity of the following diagram then gives that a connected morphism \( \phi: x_1 \otimes \cdots \otimes x_m \to x'_1 \otimes \cdots \otimes x'_n \) is sent to \( \phi \circ (\id_{x_1 \otimes \cdots \otimes x_{i-1} \otimes \tau_{x_i, x_{i+1}} \otimes \id_{x_{i+2} \otimes \cdots \otimes x_m}}) \).

\[
\begin{array}{cccc}
X_{\varepsilon_{m,n}} & \xleftarrow{d_1} & X_H & \xleftarrow{s_0} X_{\varepsilon_{m,n}} \\
\equiv & & \equiv & \equiv \\
X_{d_2 H} \times X_{d_0 H} & \xleftarrow{X_{s_0 m} \times X_{\varepsilon_{m,n}}} & X_{s_0 m} \times X_{\varepsilon_{m,n}} & \xleftarrow{s_0 d_1 \times \id}
\end{array}
\]

All of this structure is compatible with \( f: CX \to CY \), so we conclude that

\[
\begin{array}{cccc}
X_{\varepsilon_{m,n}} & \xleftarrow{X_{\varepsilon_{m,n}}} & X_{\varepsilon_{m,n}} \\
\downarrow & & \downarrow \\
Y_{\varepsilon_{m,n}} & \xleftarrow{Y_{\varepsilon_{m,n}}} & Y_{\varepsilon_{m,n}}
\end{array}
\]
commutes. The case when an adjacent transposition acts on an output is similar. It follows that $f$ defines a map $X|_{L_{\text{int}}} \to Y|_{L_{\text{int}}}$.

Since $Y$ is Segal, $Y|_{L_{\text{int}}}$ is the right Kan extension of $Y|_{L_2}$, hence we obtain a unique extension $X|_{L_{\text{int}}} \to Y|_{L_{\text{int}}}$. In particular, we have functions $f_G: X_G \to Y_G$ for every level graph $G$. It remains only to check that these maps are compatible with degeneracies and inner faces. By (5) above Definition 4.16, we only need to check compatibility with the inner faces $d_1: X_G \to X_{d_1 G}$, where $G \in L_2$, and by (6) below Definition 4.25 we only need to check compatibility with the degeneracies $s_0: X_n \to X_{s_0(n)}$. Again by Segality we may further reduce to the cases where $G$ is connected (for $d_1$) and $n = 1$ (for $s_0$). For $d_1$, all squares of the following cube are known to commute with the exception of the back, which then must commute since $Y_{d_1 G} \to SY_1$ is injective (using that $d_1 G$ is a corolla).

$$
\begin{array}{ccc}
X_{d_1 G} & \xleftarrow{d_1} & X_G \\
\downarrow & & \downarrow \\
SX_1 & \xleftarrow{d_1} & SX_2 \\
\downarrow & & \downarrow \\
Y_{d_1 G} & \xleftarrow{d_1} & Y_G \\
\downarrow & & \downarrow \\
SY_1 & \xleftarrow{d_1} & SY_2
\end{array}
$$

The map $s_0: X_c \to X_{c_{X+}} \subset \text{mor}^c(CX)$ takes a connected object $x$ to $\text{id}_x$, so $f(s_0 x) = f(\text{id}_x) = \text{id}_{f x} = s_0(f x)$, as desired. We conclude that $X|_{L_{\text{int}}} \to Y|_{L_{\text{int}}}$ is, in fact, a map $X \to Y$. By construction, this map is sent to $f$ by $C$. □

**Theorem 5.8.** The functor $\text{Seg}(L) \to \text{SLCC}$ is an equivalence of categories.

**Proof.** This functor is fully faithful by Proposition 3.10 and Lemma 5.7, and is essentially surjective by Theorem 5.6. □

Combining the previous theorem with Proposition 1.15, we have:

**Corollary 5.9.** The category of properads is equivalent to the category of strict labelled cospan categories. □

6. 2-Categorical structures

In this section we turn our attention to the biequivalence between properads and labelled cospan categories, Corollary 6.16, which is the main result of this paper. The functor from strict labelled cospan categories into labelled cospan categories is not an equivalence of 1-categories, as it is only surjective up to equivalence, not up to isomorphism. Thus to connect properads to labelled cospan categories, we will need to view $\text{Ppd}$ as a 2-category, rather than as a 1-category. The following definition is introduced in [Hac], where it is related to the Boardman–Vogt-style tensor product of properads.

**Definition 6.1** (Natural transformation of properad maps). If $F, G: P \to Q$ are two maps of properads with the same source and target, then a **natural transformation** from $F$ to $G$ is nothing but a polynatural transformation between the underlying map of polycategories/dioperads, in the sense of [JY21, Definition 2.5.16]. This means that a natural transformation is given by a collection of unary morphisms $\gamma_c: F(c) \to
\[ G(c), \text{ indexed by the colors of } P, \text{ so that, for each } p \in P(c_1, \ldots, c_n; d_1, \ldots, d_m), \text{ we have} \]
\begin{equation}
(G_p)(\gamma_{c_1}, \ldots, \gamma_{c_n}) = (\gamma_{d_1}, \ldots, \gamma_{d_m})(F_p)
\end{equation}

in \(Q(Fc_1, \ldots, Fc_n; Gd_1, \ldots, Gd_m)\).

As in [JY21], the notation in (7) is meant to represent iterated dioperadic composition, see Figure 4.

**Remark 6.2.** In [HRY15, Section 4.4], a closed monoidal structure is given for those properads whose operations all have at least one input and at least one output, so in particular this subcategory becomes a 2-category. But this structure is not extended there to the category of all properads. Definition 6.1 is the extension of this 2-category to \(Ppd\). (See [Dun06, Definition 6.3] for a different notion.)

6.1. **Equivalences of labelled cospan categories.** Let \(LCC\) denote the 2-category of labelled cospan categories, equipped with the 2-cells of Remark 1.4.

**Proposition 6.3.** Suppose \((f, \alpha) : C \to D\) is a 1-morphism of labelled cospan categories, as depicted below.

\[
\begin{array}{c}
\xymatrix{C \ar[r]^{f} \ar[dr]_{\pi} & D \ar[dl]^{\mu} \\
& Csp \ar@{=}[u]_{\alpha} \ar@{=}[uu]_{\mu} }
\end{array}
\]

The map \((f, \alpha)\) is an equivalence in the 2-category of labelled cospan categories if and only if \(f : C \to D\) is an equivalence of categories.

**Proof.** Suppose \(f : C \to D\) is an equivalence of categories. Then \(f\) is an equivalence in the 2-category of symmetric monoidal categories, symmetric monoidal functors, and monoidal natural transformations. Choose \(g : D \to C\) along with monoidal natural isomorphisms \(\eta : \text{id}_C \cong gf\) and \(\epsilon : fg \cong \text{id}_D\).
Define a (monoidal) natural transformation $\beta: \pi g \Rightarrow \mu$ as the following composite.

$$
\begin{array}{c}
D & \xrightarrow{g} & C & \xrightarrow{f} & D & \xrightarrow{\mu} & \Csp \\
\downarrow{\alpha^{-1} \circ \gamma} & & \downarrow{\varepsilon \circ \mu} & & \downarrow{id_D} \\
\end{array}
$$

Then $(g, \beta): D \to C$ is a 1-morphism of labelled cospan categories. Since the following pasting composites are isomorphisms,

$$
\begin{array}{c}
C & \xrightarrow{gf} & C' \\
\downarrow{id_C} & \downarrow{id_{C'}} & \downarrow{id_{C \ast}} \\
\Csp & & \Csp \\
\end{array}
$$

by Remark 1.4 we have $(g, \beta) \circ (f, \alpha) \cong id_{\pi}$ and $(f, \alpha) \circ (g, \beta) \cong id_{\mu}$ in $\LCC$. □

Note that if $f$ is an isomorphism of categories, then $(f, \alpha)$ is an isomorphism of labelled cospan categories.

6.2. **Strict labelled cospan categories form a 2-category.** We now endow $\SLCC$ with the structure of a 2-category, which will serve as an intermediary between $\Ppd$ (considered as a 2-category) and $\LCC$.

**Definition 6.4.** Given two maps

$$
\begin{array}{c}
C & \xrightarrow{f} & C' \\
\downarrow{\pi} & \downarrow{\pi'} & \downarrow{\pi'} \\
C(\ast) & & C(\ast) \\
\end{array}
$$

of strict labelled cospan categories (that is, commutative triangles with $f$ and $g$ strict maps of permutative categories), we define a 2-morphism from one to the other to be a monoidal natural transformation $\gamma: f \Rightarrow g$ so that the whiskering

$$
\begin{array}{c}
C & \xrightarrow{f} & C' \\
\downarrow{\gamma \circ \psi} & \downarrow{g} & \downarrow{\pi'} \\
C(\ast) & & C(\ast) \\
\end{array}
$$

(8)

is the identity natural transformation on $\pi: C \to C(\ast)$.

If the whiskering (8) is an isomorphism, then it is automatically an identity. This follows from Remark 1.3 and the fact that the canonical inclusion $C(\ast) \to \Csp$ is injective-on-objects and faithful.

There is a functor from the 1-category of strict labelled cospan categories to the 2-category $\LCC$ which is given by composition with $C(\ast) \to \Csp$.

**Lemma 6.5.** The usual inclusion $\SLCC \to \LCC$ extends to a 2-functor.
Proof. Suppose we are given a 2-morphism $\gamma$ between 1-morphisms $f$ and $g$ in \textbf{SLCC} as in Definition 6.4. We then obtain the middle equality in the following chain of equalities of symmetric monoidal natural transformations between monoidal functors.

\[
\begin{array}{ccc}
C & \overset{\gamma}{\rightarrow} & C' \\
\downarrow & & \downarrow \\
\mathrm{Csp} & = & \mathrm{Csp}
\end{array}
\begin{array}{ccc}
C & \overset{\gamma}{\rightarrow} & C' \\
\downarrow & & \downarrow \\
\mathrm{Csp} & = & \mathrm{Csp}
\end{array}
\begin{array}{ccc}
C & \overset{\gamma}{\rightarrow} & C' \\
\downarrow & & \downarrow \\
\mathrm{Csp} & = & \mathrm{Csp}
\end{array}
\begin{array}{ccc}
C & \overset{\gamma}{\rightarrow} & C' \\
\downarrow & & \downarrow \\
\mathrm{Csp} & = & \mathrm{Csp}
\end{array}
\]

Thus $\gamma$ is a 2-morphism between the 1-morphisms of labelled cospan categories $(f, \text{id})$ and $(g, \text{id})$ as in Remark 1.4.

We will return to the relationship between \textbf{SLCC} and \textbf{LCC} in Section 6.3, but first we will give more detail to facilitate the comparison with natural transformations of properads. Given a 2-morphism $\gamma$ of \textbf{SLCC} as in Definition 6.4, the underlying data consists of connected morphisms

\[\gamma_x: f(x) \to g(x)\]

for each connected object $x$ of $C$. We know these maps must all be connected, as $\pi'(\gamma_x) = (\text{id}_x) = \text{id}_{\pi(x)} = \text{id}_1$ is a connected map in $C(*)$. These maps are also the full extent of the data: if $c$ is an arbitrary object of $C$, it can be uniquely written as $c = x_1 \otimes \cdots \otimes x_n$, where the $x_i$ are connected objects. Since $\gamma$ is a monoidal natural transformation and $f$ and $g$ are strict monoidal functors, the diagram

\[
\begin{array}{ccc}
f(x_1) \otimes \cdots \otimes f(x_n) & \xrightarrow{\gamma_{x_1} \otimes \cdots \otimes \gamma_{x_n}} & g(x_1) \otimes \cdots \otimes g(x_n) \\
\downarrow & = & \downarrow \\
f(x_1 \otimes \cdots \otimes x_n) & \xrightarrow{\gamma_c} & g(x_1 \otimes \cdots \otimes x_n)
\end{array}
\]

commutes, that is, $\gamma_c = \gamma_{x_1} \otimes \cdots \otimes \gamma_{x_n}$. We now observe that one only needs to check compatibility of $\gamma$ with connected morphisms, rather than arbitrary morphisms.

**Proposition 6.6.** Suppose $f, g: C \to C'$ are two 1-morphisms of strict labelled cospan categories, and for each connected object $x$ of $C$ we have a connected morphism $\gamma_x: f(x) \to g(x)$. If the diagram

\[
\begin{array}{ccc}
f(x_1) \otimes \cdots \otimes f(x_n) & \xrightarrow{f(p)} & f(y_1 \otimes \cdots \otimes y_m) \\
\downarrow{\gamma_{x_1} \otimes \cdots \otimes \gamma_{x_n}} & = & \downarrow{\gamma_{y_1} \otimes \cdots \otimes \gamma_{y_m}} \\
g(x_1) \otimes \cdots \otimes g(x_n) & \xrightarrow{g(p)} & g(y_1 \otimes \cdots \otimes y_m)
\end{array}
\]

commutes for every connected morphism $p: x_1 \otimes \cdots \otimes x_n \to y_1 \otimes \cdots \otimes y_m$ of $C$, then the collection $\{ \gamma_{y_1} \otimes \cdots \otimes \gamma_{y_m} \}$ constitutes a 2-morphism $f \Rightarrow g$.

**Proof.** Any endomorphism $p$ of the tensor unit $1 \in C$ can be written as a composition of connected morphisms by Definition 1.2(2). Hence our assumption implies that $g(p) \gamma_1 = \gamma_1 f(p)$ for all such endomorphisms; since $\gamma_1$ is the identity on the tensor unit of $C'$, this implies that $f(p) = g(p)$. Now by Definition 1.2(3), any morphism $p: c \to d$ can be written uniquely as $p' \otimes p'': c \otimes 1 \to d \otimes 1$ where $p'$ is a reduced
morphism and \( p'' \) is an endomorphism of \( 1 \), so it remains to show that \( \gamma_d f(p') = g(p') \gamma_c \) for reduced morphisms \( p' \).

Let \( \sigma \) be a bijection on \( n = \{1, \ldots, n\} \) and \( x_1, \ldots, x_n \) be connected objects in \( C \). Naturality of symmetry implies that the diagram
\[
\begin{array}{ccc}
\gamma_{x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}} & \xrightarrow{\gamma_{x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}}} & \gamma_{g(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)})} \\
\downarrow \hat{\sigma} & & \downarrow \hat{\sigma} \\
\gamma_{(x_1 \otimes \cdots \otimes x_n)} & \xrightarrow{\gamma_{(x_1 \otimes \cdots \otimes x_n)}} & \gamma_{g(x_1 \otimes \cdots \otimes x_n)}
\end{array}
\]
commutes in the permutative category \( C' \). Since \( f \) and \( g \) are maps of permutative categories, the preceding diagram is equal to the following,
\[
\begin{array}{ccc}
f(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}) & \xrightarrow{\gamma_{x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}}} & g(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}) \\
\downarrow f(\hat{\sigma}) & & \downarrow g(\hat{\sigma}) \\
f(x_1 \otimes \cdots \otimes x_n) & \xrightarrow{\gamma_{x_1 \otimes \cdots \otimes x_n}} & g(x_1 \otimes \cdots \otimes x_n)
\end{array}
\]
where \( \hat{\sigma} : x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \to x_1 \otimes \cdots \otimes x_n \) is the permutation map in \( C \). Thus the proposed natural transformation is compatible with permutation maps.

Suppose \( p : x_1 \otimes \cdots \otimes x_n \to y_1 \otimes \cdots \otimes y_m \) is an arbitrary reduced morphism of \( C \) lying over \( n \to k \leftarrow m \) in \( C(*) \). By choosing appropriate bijections \( \sigma \) of \( n \) and \( \rho \) of \( m \), we can arrange things so that \( \hat{\rho}^{-1} \circ p \circ \hat{\sigma} \) is sent by \( \pi \) to a \( k \)-fold tensor product of connected morphisms in \( C(*) \) (see Lemmas 5.1 and 5.5). Once we’ve done this, by Definition 1.2(4) we can find unique connected morphisms \( p_1, \ldots, p_k \) of \( C \) so that the diagram
\[
\begin{array}{ccc}
x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} & \xrightarrow{p_1 \otimes \cdots \otimes p_k} & y_{\rho(1)} \otimes \cdots \otimes y_{\rho(m)} \\
\downarrow \hat{\sigma} & & \downarrow \hat{\sigma} \\
x_1 \otimes \cdots \otimes x_n & \xrightarrow{p} & y_1 \otimes \cdots \otimes y_m
\end{array}
\]
commutes. By assumption each of the connected morphisms \( p_i \) is compatible with \( \gamma \), hence so is \( p_1 \otimes \cdots \otimes p_k \). Combining this with the previous paragraph, we see that
\[
p = \hat{\rho} \circ (p_1 \otimes \cdots \otimes p_k) \circ \hat{\sigma}^{-1} = \hat{\rho} \circ (p_1 \otimes \cdots \otimes p_k) \circ \hat{\sigma}^{-1}
\]
is also compatible with \( \gamma \). \( \square \)

We have a composite of equivalences of 1-categories
\[
\mathbb{P}pd \xrightarrow{\simeq} \text{Seg}(L) \xrightarrow{\simeq} \text{SLCC}
\]
by Proposition 1.15 and Theorem 5.8. This extends to a strict 2-equivalence between 2-categories.

**Theorem 6.7.** The 1-functor \( \mathbb{P}pd \to \text{SLCC} \) extends to a 2-functor. This 2-functor induces isomorphisms on hom-categories, hence is a strict 2-equivalence.

The last conclusion uses the previously established fact that the functor is surjective on objects up to isomorphism.

**Proof.** Suppose \( F, G : P \to P' \) are two maps of properads, and \( f, g : C \to C' \) are the associated maps of strict labelled cospans. The colors of \( P \) are precisely the connected objects of \( C \), and the operations of \( P \) are precisely the connected
morphisms of \( C \). Under this correspondence, the diagram from Proposition 6.6 is exactly the condition for a family of unary operations in \( P' \) to constitute a natural transformation \( F \Rightarrow G \) of properad maps (Definition 6.1). It follows that

\[
Ppd(P, P')(F, G) \to \text{SLCC}(C, C')(f, g)
\]

is a bijection, that is, \( Ppd \to \text{SLCC} \) is locally fully faithful. But we already know that \( Ppd(P, P') \to \text{SLCC}(C, C') \) is bijective on objects, hence is an isomorphism of categories. \( \square \)

6.3. **Comparison with labelled cospan categories.** In this section we show that the inclusion of strict labelled cospan categories into all labelled cospan categories is a biequivalence of 2-categories.

**Notation 6.8.** If \( \pi: C \to C(*) \) is a strict labelled cospan category, we write

\[
\tilde{\pi}: C \to C(*) \to \text{Csp}
\]

for the composition of \( \pi \) with the canonical inclusion \( C(*) \to \text{Csp} \) from Examples 2.17 and 2.34. Given a 1-morphism of strict labelled cospan categories, that is a commutative triangle below left

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow \pi & & \downarrow \pi' \\
C(*) & \to & \text{Csp}
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow \tilde{\pi} & & \downarrow \tilde{\pi}' \\
\text{Csp} & \xrightarrow{id_{\text{Csp}}} & \text{Csp}
\end{array}
\]

where \( f \) is a strict permutative functor, we have the 1-morphism \( \tilde{f} := (f, id_{\text{Csp}}) \) of labelled cospan categories above right. Finally, if \( \gamma: f \Rightarrow g \) is a 2-morphism as in Definition 6.4, we write \( \tilde{\gamma}: (f, id_{\text{Csp}}) \Rightarrow (g, id_{\text{Csp}}) \) for the same natural transformation, but now thought of as a 2-morphism in \( \text{LCC} \).

We already observed in Lemma 6.5 that \( \text{SLCC} \to \text{LCC} \) is a 2-functor under these assignments.

**Lemma 6.9.** The 2-functor \( \text{SLCC} \to \text{LCC} \) is locally fully faithful.

**Proof.** Suppose \( f, g: C \to C' \) are two 1-morphisms of labelled cospan categories. We wish to show that

\[
\text{SLCC}(C, C')(f, g) \to \text{LCC}(C, C')((\tilde{f}, \tilde{g}))
\]

is a bijection. It is automatically injective since the elements on both sides are just certain natural transformations \( f \Rightarrow g \). Suppose we have a 2-morphism \( \tilde{f} \Rightarrow \tilde{g} \) as in Remark 1.4, that is a monoidal natural transformation \( \gamma: f \Rightarrow g \) so that the composite natural transformation

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow \gamma & \searrow \psi & \downarrow \pi' \\
\text{Csp} & \xrightarrow{id_{\text{Csp}}} & \text{Csp}
\end{array}
\]

is a bijection. Under this correspondence, the diagram from Proposition 6.6 is exactly the condition for a family of unary operations in \( P' \) to constitute a natural transformation \( F \Rightarrow G \) of properad maps (Definition 6.1). It follows that

\[
Ppd(P, P')(F, G) \to \text{SLCC}(C, C')(f, g)
\]

is a bijection, that is, \( Ppd \to \text{SLCC} \) is locally fully faithful. But we already know that \( Ppd(P, P') \to \text{SLCC}(C, C') \) is bijective on objects, hence is an isomorphism of categories. \( \square \)

6.3. **Comparison with labelled cospan categories.** In this section we show that the inclusion of strict labelled cospan categories into all labelled cospan categories is a biequivalence of 2-categories.

**Notation 6.8.** If \( \pi: C \to C(*) \) is a strict labelled cospan category, we write

\[
\tilde{\pi}: C \to C(*) \to \text{Csp}
\]

for the composition of \( \pi \) with the canonical inclusion \( C(*) \to \text{Csp} \) from Examples 2.17 and 2.34. Given a 1-morphism of strict labelled cospan categories, that is a commutative triangle below left

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow \pi & & \downarrow \pi' \\
C(*) & \to & \text{Csp}
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow \tilde{\pi} & & \downarrow \tilde{\pi}' \\
\text{Csp} & \xrightarrow{id_{\text{Csp}}} & \text{Csp}
\end{array}
\]

where \( f \) is a strict permutative functor, we have the 1-morphism \( \tilde{f} := (f, id_{\text{Csp}}) \) of labelled cospan categories above right. Finally, if \( \gamma: f \Rightarrow g \) is a 2-morphism as in Definition 6.4, we write \( \tilde{\gamma}: (f, id_{\text{Csp}}) \Rightarrow (g, id_{\text{Csp}}) \) for the same natural transformation, but now thought of as a 2-morphism in \( \text{LCC} \).

We already observed in Lemma 6.5 that \( \text{SLCC} \to \text{LCC} \) is a 2-functor under these assignments.

**Lemma 6.9.** The 2-functor \( \text{SLCC} \to \text{LCC} \) is locally fully faithful.

**Proof.** Suppose \( f, g: C \to C' \) are two 1-morphisms of labelled cospan categories. We wish to show that

\[
\text{SLCC}(C, C')(f, g) \to \text{LCC}(C, C')((\tilde{f}, \tilde{g}))
\]

is a bijection. It is automatically injective since the elements on both sides are just certain natural transformations \( f \Rightarrow g \). Suppose we have a 2-morphism \( \tilde{f} \Rightarrow \tilde{g} \) as in Remark 1.4, that is a monoidal natural transformation \( \gamma: f \Rightarrow g \) so that the composite natural transformation

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow \gamma & \searrow \psi & \downarrow \pi' \\
\text{Csp} & \xrightarrow{id_{\text{Csp}}} & \text{Csp}
\end{array}
\]

is a bijection, that is, \( Ppd \to \text{SLCC} \) is locally fully faithful. But we already know that \( Ppd(P, P') \to \text{SLCC}(C, C') \) is bijective on objects, hence is an isomorphism of categories. \( \square \)

6.3. **Comparison with labelled cospan categories.** In this section we show that the inclusion of strict labelled cospan categories into all labelled cospan categories is a biequivalence of 2-categories.

**Notation 6.8.** If \( \pi: C \to C(*) \) is a strict labelled cospan category, we write

\[
\tilde{\pi}: C \to C(*) \to \text{Csp}
\]

for the composition of \( \pi \) with the canonical inclusion \( C(*) \to \text{Csp} \) from Examples 2.17 and 2.34. Given a 1-morphism of strict labelled cospan categories, that is a commutative triangle below left

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow \pi & & \downarrow \pi' \\
C(*) & \to & \text{Csp}
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow \tilde{\pi} & & \downarrow \tilde{\pi}' \\
\text{Csp} & \xrightarrow{id_{\text{Csp}}} & \text{Csp}
\end{array}
\]

where \( f \) is a strict permutative functor, we have the 1-morphism \( \tilde{f} := (f, id_{\text{Csp}}) \) of labelled cospan categories above right. Finally, if \( \gamma: f \Rightarrow g \) is a 2-morphism as in Definition 6.4, we write \( \tilde{\gamma}: (f, id_{\text{Csp}}) \Rightarrow (g, id_{\text{Csp}}) \) for the same natural transformation, but now thought of as a 2-morphism in \( \text{LCC} \).

We already observed in Lemma 6.5 that \( \text{SLCC} \to \text{LCC} \) is a 2-functor under these assignments.

**Lemma 6.9.** The 2-functor \( \text{SLCC} \to \text{LCC} \) is locally fully faithful.

**Proof.** Suppose \( f, g: C \to C' \) are two 1-morphisms of labelled cospan categories. We wish to show that

\[
\text{SLCC}(C, C')(f, g) \to \text{LCC}(C, C')((\tilde{f}, \tilde{g}))
\]

is a bijection. It is automatically injective since the elements on both sides are just certain natural transformations \( f \Rightarrow g \). Suppose we have a 2-morphism \( \tilde{f} \Rightarrow \tilde{g} \) as in Remark 1.4, that is a monoidal natural transformation \( \gamma: f \Rightarrow g \) so that the composite natural transformation

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow \gamma & \searrow \psi & \downarrow \pi' \\
\text{Csp} & \xrightarrow{id_{\text{Csp}}} & \text{Csp}
\end{array}
\]

is a bijection, that is, \( Ppd \to \text{SLCC} \) is locally fully faithful. But we already know that \( Ppd(P, P') \to \text{SLCC}(C, C') \) is bijective on objects, hence is an isomorphism of categories. \( \square \)
is id$_S$. This composite is just the whiskering of $\tilde{\pi}'$ with $\gamma$. To show that $\gamma$ is a 2-morphism of $\text{SLCC}$, we need to show that its whiskering with $\pi'$ is the identity on $\pi$

$$
\begin{array}{cccc}
C & \xrightarrow{\gamma \circ g} & C' & \xrightarrow{\pi'} & C(\ast) & \longrightarrow & \text{Csp} \\
\end{array}
$$

which follows since $C(\ast) \to \text{Csp}$ is faithful. $\square$

**Lemma 6.10.** The 2-functor $\text{SLCC} \to \text{LCC}$ is locally an equivalence of categories.

**Proof.** Let $\pi: C \to C(\ast)$ and $\pi': C' \to C(\ast)$ be strict labelled cospan categories. From the previous lemma we know that $\text{SLCC}(C, C') \to \text{LCC}(C, C')$ is fully faithful, so it remains to prove that it is essentially surjective. Suppose $(f, \alpha)$ as depicted below left is 1-morphism in $\text{LCC}$.

$$
\begin{array}{cccc}
C & \xrightarrow{\pi} & \text{Csp} & \xrightarrow{\gamma} & C' \xrightarrow{\pi'} & C(\ast) \\
\end{array}
$$

Here $f$ is a symmetric monoidal functor (which does not need to be a strict map of permutative categories) and $\alpha$ is a monoidal natural isomorphism. Since $C(\ast) \to \text{Csp}$ is fully faithful, there is a (unique) monoidal natural isomorphism $\beta$ as displayed above right, whose whiskering with $C(\ast) \to \text{Csp}$ is $\alpha$. Generically write $\Phi : f(c_1) \otimes \cdots \otimes f(c_n) \to f(c_1 \otimes \cdots \otimes c_n)$ for the structural isomorphisms of $f$ (including the case $n = 0$ for the tensor unit).

We define a new functor of permutative categories $g: C \to C'$. On a general object $x_1 \otimes \cdots \otimes x_n$ of $C$ (with each $x_i$ connected), $g$ is given by

$$
g(x_1 \otimes \cdots \otimes x_n) := f(x_1) \otimes \cdots \otimes f(x_n).
$$

If $p: x_1 \otimes \cdots \otimes x_n \to y_1 \otimes \cdots \otimes y_m$ is any morphism of $C$, define $g(p)$ to sit in the following commutative square.

$$
\begin{array}{cccc}
f(x_1) \otimes \cdots \otimes f(x_n) & \xrightarrow{g(p)} & f(y_1) \otimes \cdots \otimes f(y_m) \\
\Phi \downarrow \cong \quad \cong \Phi \downarrow & & \cong \Phi \downarrow \\
f(x_1 \otimes \cdots \otimes x_n) & \xrightarrow{f(p)} & f(y_1 \otimes \cdots \otimes y_m) \\
\end{array}
$$

This $g$ is automatically a functor, and the commutativity of the diagram

$$
\begin{array}{cccc}
\left( \bigotimes_{i=1}^{k} f(x_i) \right) \otimes \left( \bigotimes_{i=k+1}^{n} f(x_i) \right) & \longrightarrow & \left( \bigotimes_{i=1}^{j} f(y_i) \right) \otimes \left( \bigotimes_{i=j+1}^{m} f(y_i) \right) \\
\Phi \downarrow \quad \Phi \downarrow & \quad \Phi \downarrow & \quad \Phi \downarrow \\
\left( \bigotimes_{i=1}^{k} x_i \right) \otimes \left( \bigotimes_{i=k+1}^{n} x_i \right) & \longrightarrow & \left( \bigotimes_{i=1}^{j} y_i \right) \otimes \left( \bigotimes_{i=j+1}^{m} y_i \right) \\
\Phi \downarrow & \quad \Phi \downarrow & \quad \Phi \downarrow \\
f(x_1 \otimes \cdots \otimes x_n) & \xrightarrow{f(p \otimes p')} & f(y_1 \otimes \cdots \otimes y_m)
\end{array}
$$
implies that the unique dashed map must be both \( g(p \otimes p') \) and \( g(p) \otimes g(p') \), hence \( g \) is a strict monoidal functor. A similar diagram shows how to infer from \( f \) being symmetric monoidal functor that the same is true for \( g \).

We will return to checking that \( g \) is a 1-morphism in \( \text{SLCC} \) in a moment. First, we define \( \gamma : g \Rightarrow f \) to be the monoidal natural isomorphism given by \( \Phi \). That is, if \( c = x_1 \otimes \cdots \otimes x_n \), then \( \gamma_c \) is the map

\[
g(c) = f(x_1) \otimes \cdots \otimes f(x_n) \to f(x_1 \otimes \cdots \otimes x_n) = f(c).
\]

We now calculate the following composite.

\[
(9) \quad C \xrightarrow{\gamma} C' \xrightarrow{\pi'} \xrightarrow{\pi} C(*)
\]

Since \( \pi \) and \( \pi' \) are strict monoidal functors and \( \beta_x \) is the identity on \( 1 \) for a connected object \( x_i \), commutativity of the following square

\[
\pi'(f(x_1) \otimes \cdots \otimes f(x_n)) \quad \xrightarrow{\beta_{x_1} \otimes \cdots \otimes \beta_{x_n}} \quad \pi(x_1) \otimes \cdots \otimes \pi(x_n)
\]

implies that the unique dashed map must be both \( g(p \otimes p') \) and \( g(p) \otimes g(p') \), hence \( g \) is a strict monoidal functor. A similar diagram shows how to infer from \( f \) being symmetric monoidal functor that the same is true for \( g \).

We will return to checking that \( g \) is a 1-morphism in \( \text{SLCC} \) in a moment. First, we define \( \gamma : g \Rightarrow f \) to be the monoidal natural isomorphism given by \( \Phi \). That is, if \( c = x_1 \otimes \cdots \otimes x_n \), then \( \gamma_c \) is the map

\[
g(c) = f(x_1) \otimes \cdots \otimes f(x_n) \to f(x_1 \otimes \cdots \otimes x_n) = f(c).
\]

We now calculate the following composite.

\[
(9) \quad C \xrightarrow{\gamma} C' \xrightarrow{\pi'} C(*)
\]

Since \( \pi \) and \( \pi' \) are strict monoidal functors and \( \beta_x \) is the identity on \( 1 \) for a connected object \( x_i \), commutativity of the following square

\[
\pi'(f(x_1) \otimes \cdots \otimes f(x_n)) \quad \xrightarrow{\pi'(\Phi)} \quad \pi(x_1 \otimes \cdots \otimes x_n)
\]

tells us that \( \beta_{x_1} \otimes \cdots \otimes \beta_{x_n} \) is the inverse of \( \pi'(\Phi_{x_1, \ldots, x_n}) \). Thus \( \beta_c \circ \pi'(\gamma_c) = \text{id}_{\pi_c} \) for all objects \( c \in C \), so it follows that the composite of (9) is the identity 2-morphism on the 1-morphism \( \pi \). We then have \( \pi' \circ g = \pi \) and so \( g \) is a 1-morphism in \( \text{SLCC} \), and \( \gamma \) is an isomorphism in \( \text{LCC}(C, C') \) between \( (g, \text{id}_x) \) and \( (f, \alpha) \).

It remains to prove that \( \text{SLCC} \to \text{LCC} \) is surjective up to equivalence. We begin with a special case.

**Lemma 6.11.** Suppose that \( C \) is a permutative category whose set of objects is a free monoid on the set \( S \) under the monoidal product, and that \( \pi : C \to \text{Csp} \) is a symmetric monoidal functor. If \( \pi \) is a labelled cospan category such that the set of connected objects is precisely \( S \subset \text{ob} C \), then there is a strict labelled cospan category \( \pi' : C \to C(*) \) such that the labelled cospan category \( \pi' \) is isomorphic to \( \pi \).

**Proof.** In this proof we write \( i : C(*) \to \text{Csp} \) for the canonical inclusion, and, whenever \( k + j = n \), write

\[
\Psi_{k,j} : i_k \otimes i_j \to i(k + j) = in
\]

for the natural monoidal structure isomorphism of \( i \). We similarly write \( \Phi_{c, d} : \pi c \otimes \pi d \to \pi(c \otimes d) \) for the structure isomorphism of the symmetric monoidal functor \( \pi \). We now aim to simultaneously define a more strict version of \( \pi \) called \( \kappa : C \to \text{Csp} \), along with a natural isomorphism \( \beta : \pi \cong \kappa \). On objects, \( \kappa \) takes a word \( c = [x_1 x_2 \ldots x_n] \) of length \( n \) (where each \( x_i \in S \)) to \( in \). We now inductively, based on word length, define isomorphisms \( \beta_c : \pi c \to \kappa c \). For length zero and length one words, we declare that \( \beta_{[\ ]} : \pi [\ ] \to \kappa [\ ] = \emptyset \) is the identity, and \( \beta_{[x]} : \pi [x] \to \kappa [x] = i1 \) is the unique isomorphism. Suppose \( c \) has length \( k > 0 \) and \( d \) has length \( j > 0 \), and
that \( \beta_c \) and \( \beta_d \) have been defined. We then define \( \beta_{c \otimes d} \) as the unique map fitting into the following square

\[
\begin{array}{ccc}
\pi c \otimes \pi d & \xrightarrow{\beta_c \otimes \beta_d} & i k \otimes i j \\
\Phi_{c,d} & \cong & \Psi_{c,d} \\
\pi(c \otimes d) & \xrightarrow{\beta_{c \otimes d}} & i(k + j).
\end{array}
\]

One must check that this is well-defined, that is, for any positive-length words \( c, d, e \in \text{ob } C \) that \( \beta_{(c \otimes d) \otimes e} = \beta_{c \otimes (d \otimes e)} \). This follows by using the hexagon constraints for \( \Phi \) and \( \Psi \) and the fact that \( C \) is strict monoidal.

We define \( \kappa \) on morphisms \( p: c \to d \) by declaring

\[ \kappa(p) := \beta_d \circ \pi(p) \circ \beta_c^{-1}. \]

So defined, \( \kappa \) is automatically a functor and \( \beta \) is a natural transformation. Since \( i \) is fully faithful, there is a unique functor \( \pi': C \to C(*) \) so that \( i \pi' = \kappa \). Using the defining equations, one can show that \( \pi' \) is a strict monoidal functor, which is also symmetric monoidal. From the diagram above one concludes that \( \beta \) is a monoidal natural transformation \( \pi \Rightarrow i \pi' \).

The existence of the monoidal natural isomorphism \( \beta \) allows one to check that \( \kappa \) is a labelled cospan category. Further, \( \pi' \) is a strict labelled cospan category and \((\text{id}_C, \beta)\) is an isomorphism \( \tilde{\pi}' = \kappa \cong \pi \) of labelled cospan categories. \( \square \)

**Lemma 6.12.** The 2-functor \( \text{SLCC} \to \text{LCC} \) is surjective up to equivalence.

**Proof.** Suppose \( \pi: C \to \text{Csp} \) is a labelled cospan category. A variation on the proof of [May74, Proposition 4.2] lets us define a permutative category \( D \) and a symmetric monoidal equivalence \( f: D \to C \) so that \( \text{ob } D \) is the free monoid on the set \( \text{ob } C \) of connected objects of the labelled cospan category \( C \). The functor \( f \) takes an object \([x_1 x_2 \cdots x_{n-1} x_n]\) (where each \( x_i \in \text{ob } C \)) to \( x_1 \otimes (x_2 \otimes (\cdots \otimes (x_{n-1} \otimes x_n) \cdots)) \). One can check that \( \pi f: D \to \text{Csp} \) is a labelled cospan category. By Proposition 6.3, \( \pi \) and \( \pi f \) are equivalent objects in \( \text{LCC} \), and by Lemma 6.11, \( \pi f \) is isomorphic to a strict labelled cospan category. Hence \( \pi \) is equivalent to a strict labelled cospan category. \( \square \)

We now have enough machinery to establish the following uniqueness result for (strict) labelled cospan categories with a given domain.

**Proposition 6.13** (Barkan–Steinebrunner). If \( D \) is a permutative category, there is at most one strict labelled cospan category \( D \to C(*) \). If \( C \) is a symmetric monoidal category, then any two labelled cospan categories \( C \to \text{Csp} \) are equivalent in \( \text{LCC} \).

**Proof.** Suppose \( \pi, \mu: C \to \text{Csp} \) are two labelled cospan categories with the same source category; we will reduce the existence of an equivalence \( \pi \cong \mu \) to the first statement. Note that \( \pi(c) \) has cardinality zero if and only if \( \mu(c) \) does by Definition 1.2(1); then if a \( \pi \)-connected object \( c \) decomposes as \( c \cong c_1 \otimes \cdots \otimes c_n \) for \( \mu \)-connected objects \( c_i \), we must have \( n = 1 \). It follows that the permutative category \( D \) and the functor \( f \) from the proof of Lemma 6.12 are the same for both \( \pi \) and \( \mu \). Applying Lemma 6.11 we have \( \pi \cong \pi f \cong \tilde{\pi}' \) and \( \mu \cong \mu f \cong \tilde{\mu}' \) for strict labelled cospan categories \( \pi', \mu': D \to C(*) \). These last two maps will turn out to be equal, establishing the second claim.
Given two strict labelled cospan categories $\pi', \mu': D \to C(*)$ with the same domain permutative category $D$, we now show that $\pi' = \mu'$. Since free monoids have unique generating sets and the set of objects of $D$ is the free monoid on the set of connected objects, the connected objects of $\pi'$ and $\mu'$ are the same. Hence, on objects, the maps $\pi'$ and $\mu'$ coincide; further, since $\pi'$ and $\mu'$ are strict maps of permutative categories, they agree on the maps $\hat{\sigma}: x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \to x_1 \otimes \cdots \otimes x_n$ for each permutation $\sigma$ of $n$. A free abelian monoid has a unique generating set, so the connected endomorphisms of $1$ coincide for $\pi'$ and $\mu'$ by Definition 1.2(2).

Since $\pi', \mu': \text{hom}(1, 1) \to \text{hom}(0, 0)$ send the generators to the unique generator $0 \to 1 \leftarrow 0$, these maps coincide on endomorphisms of the tensor unit. Suppose that $p: c \to d$ is a morphism of $D$ which is connected with respect to $\pi'$. We can write $p = \hat{\sigma} \circ (p_1 \otimes \cdots \otimes p_k) \circ \hat{\rho}$ where each $p_i$ is connected with respect to $\mu'$. Then since no $\pi'(p_i)$ is the identity on $\bar{0}$, we must have $k = 1$, so $p$ is also connected with respect to $\mu'$. Since $\pi'$ and $\mu'$ agree on objects and have the same connected morphisms, they agree on connected morphisms. But then $\pi'(p) = \mu'(p)$ for arbitrary morphisms $p$, since we can write any such $p$ as $\hat{\sigma} \circ (p_1 \otimes \cdots \otimes p_k) \circ \hat{\rho}$ with the $p_i$ connected, and $\pi'$ and $\mu'$ are strict maps of permutative categories. Thus $\pi' = \mu'$.

Remark 6.14. Suppose $f: C \to C'$ is a symmetric monoidal equivalence and $\pi: C \to \text{Csp}$, $\pi': C' \to \text{Csp}$ are labelled cospan categories. Then $f$ automatically determines a 1-morphism in $\text{LCC}$. Indeed, in this situation it is automatic that $\pi'f$ is a labelled cospan category, so unraveling the proof of Proposition 6.13 we obtain a natural isomorphism $\pi'f \cong \pi$ as the following pasting composite.

Here, the functor $D \to C(*)$ is the strict labelled cospan category constructed in the first paragraph of the proof.

**Theorem 6.15.** The 2-functor $\text{SLCC} \to \text{LCC}$ is a biequivalence.

**Proof.** This follows from Lemma 6.10 and Lemma 6.12. □

We conclude by combining this theorem with the biequivalence of Theorem 6.7.

**Corollary 6.16.** The composite 2-functor $\text{Ppd} \to \text{LCC}$ is a biequivalence. □

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