THE QUATERNION GROUP HAS GHOST NUMBER THREE

FATMA ALTUNBULAK AKSU AND DAVID J. GREEN

Abstract. We prove that the group algebra of the quaternion group $Q_8$ over any field of characteristic two has ghost number three.

1. Introduction

The study of ghost maps in stable categories originated with Freyd’s generating hypothesis in homotopy theory [6], which is still an open question. In this paper we are concerned with ghosts in modular representation theory. Let $G$ be a group and $K$ a field of characteristic $p$. A map $f: M \to N$ in the stable category $\text{stmod}(KG)$ of finitely generated $KG$-modules is called a ghost if it vanishes under Tate cohomology, that is if $f_*: \check{H}^*(G, M) \to \check{H}^*(G, N)$ is zero. The ghost maps then form an ideal in $\text{stmod}(KG)$, and Chebolu, Christensen and Mináč [3] define the ghost number of $KG$ to be the nilpotency degree of this ideal.

Determining the exact value of the ghost number is hard in all but the simplest cases. In [4], Christensen and Wang studied ghost numbers for $p$-group algebras. They gave conjectural upper and lower bounds for the ghost number of an arbitrary $p$-group, and also showed that the ghost number (over a field of characteristic two) of the quaternion group $Q_8$ is either three or four. In our earlier paper [1], we established most cases of their conjectural bounds. In this paper, we shall prove the following theorem.

**Theorem 1.1.** Let $K$ be any field of characteristic two. Then the group algebra $KQ_8$ has ghost number three.

We claim therefore that every threefold ghost map $M \xrightarrow{f} N$ is stably trivial. To show this, we take any embedding $M \hookrightarrow I$ of $M$ in a finitely generated $KQ_8$-module and show that $f$ factors through $I$.

In Section 2 we recall Dade’s presentation of the group algebra $KQ_8$ and derive some properties of ghost maps, including the crucial Lemma 2.4. In Section 3 we recall a theorem of Kronecker which classifies the linear relations on a vector
space $V$. This leads us to the construction of the lift in Section 4. We have $I = KQ_8 \otimes_K V$ for some $K$-vector space $V$. As we may assume $M$ to be projective-free, we have $M \subseteq J \otimes_K V$ for $J$ the Jacobson radical $J = J(KQ_8)$. Since a threefold ghost kills $\text{soc}^3(M)$, it follows that $f$ factors through $M/\text{soc}^3(M)$, which is a subspace of $(J/J^2) \otimes_K V \cong V^2$. That is, $M/\text{soc}^3(M)$ is a linear relation on $V$; and using Lemma 2.4 we are able to construct a lift for each indecomposable summand in its Kronecker decomposition, thus proving the theorem.

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2. Ghost maps and Dade’s generators

We only need the following property of ghost maps.

Lemma 2.1 ([3], Proposition 2.1). Let $G$ be a $p$-group, $K$ a field of characteristic $p$, and $M \xrightarrow{f} N$ a ghost map between projective-free $KG$-modules. Then $\text{Im}(f) \subseteq \text{rad}(N)$ and $\text{soc}(M) \subseteq \ker(f)$. □

The next result is presumably well-known.

Lemma 2.2. Let $G$ be a finite group, $K/k$ a finite field extension, and $M \xrightarrow{f} N$ a map in $\text{stmod}(kG)$. If $K \otimes_k M \xrightarrow{\text{Id}_K \otimes f} K \otimes_k N$ is trivial in $\text{stmod}(KG)$, then $f$ is trivial in $\text{stmod}(kG)$. Hence $\text{ghost number}(kG) \leq \text{ghost number}(KG)$.

Proof. As a map of $k$-vector spaces, inclusion $k \xrightarrow{i} K$ is a split monomorphism; let $K \xrightarrow{\pi} k$ be a splitting. Suppose that $\text{Id}_K \otimes f$ factors through a finitely generated $KG$-projective module $P$. Then $f = (\pi \otimes \text{Id}_N) \circ (\text{Id}_K \otimes f) \circ (i \otimes \text{Id}_M)$ also factors through $P$, which is also a finitely generated $kG$-projective module. The last part follows, since extending scalars preserves ghost maps. □

Consider now the quaternion group $Q_8 = \langle i, j \rangle$. Let $K$ be a field of characteristic 2 which contains $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$. In [32, (1.2)], Dade defines $x, y \in J(KQ_8)$ by

$$x = \omega i + \omega j + ij \quad y = \bar{\omega} i + \omega j + ij.$$ 

He then shows that $KQ_8$ is the $K$-algebra generated by $x, y$ with relations

$$x^2 = yxy \quad y^2 = xyx \quad xy^2 = y^2x = x^2y = yx^2 = 0.$$ 

Hence $1, x, y, xy, yx, xyx, xyy, yxy$ is a $K$-basis of $KQ_8$.

Notation. From now on, we write $R = KQ_8$ and $J = J(R) = \text{rad}(R) = \langle x, y \rangle \trianglelefteq R$.

Lemma 2.3. Suppose that $[t + J^2(R)] \in \mathbb{P}(J/J^2)$ is neither $[x + J^2]$ nor $[y + J^2]$. Then for all $R$-modules $M$, the map $\text{rad}(M) \rightarrow \text{rad}^2(M)$, $m \mapsto tm$ is surjective.
Proof. It is enough to prove the case $M = R$; and by Nakayama it suffices to prove that the map $J/J^2 \rightarrow J^2/J^3$, $r \mapsto tr + J^3$ is surjective. As $J/J^2$ and $J^2/J^3$ are both two-dimensional, $r \mapsto tr$ is surjective if and only if it is injective.

If $t \in ax + \beta y + J^2(R)$ and $r \in \lambda x + \mu y + J^2(R)$ and then $tr \in \alpha \mu xy + \beta \lambda yx + J^3(R)$. So if $tr \in J^3$ then $\alpha \mu = 0 = \beta \lambda$. But the assumption on $t$ means that $\alpha, \beta$ are both non-zero: so $r \in 0 + J^2$. \hfill \Box

Lemma 2.4. Suppose that $M \xrightarrow{f} N$ is a threefold ghost for $KQ_8$, with $M, N$ projective-free. Embedding $M$ in an injective module $R \otimes_K V$ for some $K$-vector space $V$, we have $M \subseteq J \otimes_K V$. Suppose further that $m \in M$ satisfies $m \in t \otimes v + J^2 \otimes_K V$ with $v \in V$ and $t \in \{x, y\}$. Then there is an $n \in N$ such that

$$f(m) = \begin{cases} xyxn & t = x \\ yxyn & t = y \end{cases}.$$

Proof. We treat the case $t = x$; the other case is analogous. Hence $m = x \otimes v + xyu + yxw$ for some $u, w \in R \otimes_K V$, and so $yxm = xyxyw \in \text{soc}(M)$. Let

$$M = N_0 \xrightarrow{f_0} N_1 \xrightarrow{f_2} N_2 \xrightarrow{f_3} N_3 = N$$

be a realisation of $f$ as a threefold ghost, with $N_1$ and $N_2$ projective-free. Recall from Lemma 2.3 that $\text{soc}(N_{i-1}) \subseteq \text{ker}(f_i)$ and $\text{Im}(f_i) \subseteq \text{rad}(N_i)$.

Since $\text{soc}(M) \subseteq \text{ker}(f_1)$ it follows that $yx f_1(m) = 0$. As $\text{Im}(f_1) \subseteq \text{rad}(N_1)$ there are $\alpha, \beta \in N_1$ with $f_1(m) = xa + y \beta$. Since $yx f_1(m) = 0$, we deduce that $xy \beta = 0$ and hence $xy \beta \in \text{soc}(N_1) \subseteq \text{ker}(f_2)$.

Therefore $yx f_2(\beta) = 0$. But $\text{Im}(f_2) \subseteq \text{rad}(N_2)$, and so $f_2(\beta) = x \gamma + y \delta$ with $\gamma, \delta \in N_2$. From $yx f_2(\beta) = 0$ it follows that $xy \gamma = 0$, hence $yx \gamma \in \text{soc}(N_2) \subseteq \text{ker}(f_3)$ and $yx f_3(\gamma) = 0$. It follows that

$$f(m) = xf_3f_2(\alpha) + yxf_3(\gamma) + yxf_3(\delta) = xf_3f_2(\alpha),$$

since $f_3(\delta) \in \text{rad}(N)$ and therefore $yx f_3(\delta) \in \text{rad}^4(N) = 0$. So $f(m) = xny'$ for $n' = f_3f_2(\alpha) \in \text{rad}^2(N)$. But then $n' = xny_1' + yx n_2'$ for some $n_1', n_2' \in N$, and so $f(m) = xyn_1'$. \hfill \Box

3. Kronecker’s Theorem

Theorem 3.1 (Kronecker). Let $K$ be a field, $V$ a finite-dimensional $K$-vector space, and $L \subseteq V^2$ a subspace. Suppose further that the pair $(V, L)$ is indecomposable, in the following sense: $V \neq 0$, and there is no proper direct sum decomposition $V = V_1 \oplus V_2$ such that $L = (L \cap V_1^2) \oplus (L \cap V_2^2)$. Then there is a basis $e_1, \ldots, e_n$ of $V$ such that one of the following cases holds:

1. $L$ has basis $(e_1, 0), (e_2, e_1), (e_3, e_2), \ldots, (e_n, e_{n-1}), (0, e_n)$.
2. $L$ either has basis $(e_1, 0), (e_2, e_1), (e_3, e_2), \ldots, (e_n, e_{n-1})$ or it has basis $(0, e_1), (e_1, e_2), (e_2, e_3), \ldots, (e_{n-1}, e_n)$.
3. $L$ has basis $(e_2, e_1), (e_3, e_2), \ldots, (e_n, e_{n-1})$. 

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(4) \( L = \{(v, F(v)) \mid v \in V\} \) for an automorphism \( F \) of \( V \) which has indecomposable rational canonical form with respect to the basis \( e_1, \ldots, e_n \). A rational canonical form is indecomposable if it consists of only one block, whose characteristic polynomial is moreover a power of an irreducible element of \( K[X] \).

**Proof.** In the language of [2, p. 112], the assumptions say that \( L \) is an indecomposable linear relation on \( V \), which is the same thing as an indecomposable representation of the Kronecker quiver with \( \ker(a) \cap \ker(b) \neq 0 \). So the result can be read off from Kronecker’s Theorem (Theorem 4.3.2 of [2]): note that Case (i) in [2] corresponds to our cases (2) and (4). □

**Corollary 3.2.** For every subspace \( L \subseteq V^2 \) there is a direct sum decomposition \( V = \bigoplus_{i=1}^r V_i \) such that

1. \( L = \bigoplus_{i=1}^r L_i \) for \( L_i = L \cap V_i^2 \).
2. For each \( 1 \leq i \leq r \) the pair \( (V_i, L_i) \) is indecomposable in the sense of Theorem 3.1.

We write \( (V, L) = \bigoplus_{i=1}^r (V_i, L_i) \). □

4. **Constructing the lift**

Recall that \( x + J^2, y + J^2 \) is a basis of \( J/J^2 \). Let \( V \) be a finite dimensional \( K \)-vector space. Then any submodule \( M \subseteq J \otimes_K V \) defines a subspace of \( V^2 \):

\[
L_{x,y}(M) := \{(u, v) \in V^2 \mid x \otimes u + y \otimes v \in M + J^2 \otimes_K V\}.
\]

The proof of the following result is then immediate.

**Lemma 4.1.** Let \( M \subseteq J \otimes_K V \). Then

1. \( \text{soc}^3(M) = M \cap (J^2 \otimes_K V) \).
2. Set \( L = L_{x,y}(M) \), and let \( (V, L) = \bigoplus_{i=1}^r (V_i, L_i) \) be the direct sum decomposition of Corollary 3.2. If each \( L_i \) has basis \( (u_{i1}, w_{i1}), \ldots, (u_{id_i}, w_{id_i}) \), then for any choice of elements

\[
m_{ij} \in M \cap (x \otimes u_{ij} + y \otimes w_{ij} + J^2 \otimes_K V),
\]

we have \( M = \text{soc}^3(M) + \sum_{i=1}^N M_i \), where \( M_i = \sum_{j=1}^{d_i} Rm_{ij} \). □

**Proposition 4.2.** For \( M \subseteq J \otimes_K V \) set \( L = L_{x,y}(M) \). Let \( (V, L) = \bigoplus_{i=1}^r (V_i, L_i) \) be a decomposition into indecomposables. Suppose additionally that for each indecomposable pair \( (V_i, L_i) \) which satisfies Case (i) of Theorem 3.1, the roots of the characteristic polynomial of the automorphism \( F \) all lie in \( K \).

Suppose further that \( N \) is projective-free. Then every threefold ghost \( M \xrightarrow{f} N \) extends to a map \( R \otimes_K V \xrightarrow{f'} \text{rad}^2(N) \).
Proof. Suppose first that the indecomposable \((V_i, L_i)\) satisfies Case (1) of Theorem 3.1. Then \(V_i\) has a basis \(e_1, \ldots, e_n\) such that \(L_i\) has basis \((0, e_1), (e_1, e_2), (e_2, e_3), \ldots, (e_{n-1}, e_n), (e_n, 0)\). By construction of \(L\), there are \(m_0, \ldots, m_n \in M\) such that \(m_j \in x \otimes e_j + y \otimes e_{j+1} + J^2 \otimes_K V\), where \(e_0 = e_{n+1} = 0\). Since \(\text{Im}(f) \subseteq \text{rad}^2(N)\) there are \(a_j, b_j \in N\) for \(0 \leq j \leq n\) such that
\[
f(m_j) = xy a_j + yx b_j;
\]
and by Lemma 2.4 we may take \(a_0 = b_n = 0\). We then define \(\bar{f}\) on \(R \otimes_K V_i\) by
\[
\bar{f}(1 \otimes e_j) = xy b_{j-1} + yx a_j.
\]
The two subcases of Case (2) are analogous to each other, so we only consider the case where \(L_i\) has basis \((0, e_1), (e_1, e_2), (e_2, e_3), \ldots, (e_{n-1}, e_n)\). This corresponds to the case \(f(m_n) = 0\) of Case (1) above, where we may take \(a_n = 0\).

Case (3) is even simpler: this time we have \(f(m_0) = f(m_n) = 0\) and therefore \(b_0 = a_n = 0\).

Case (4): By assumption, the matrix of \(F\) with respect to the basis \(e_1, \ldots, e_n\) of \(V_i\) is a rational canonical form which has only one block, and the minimal polynomial of this block is \((X - \lambda)^n\) for some \(\lambda \in K^*\). It follows that there is a basis \(e'_1, \ldots, e'_n\) of \(V_i\) with respect to which the matrix of \(F\) is the \((n \times n)\) Jordan block for the eigenvalue \(\lambda\). Consequently, \(L_i\) has basis
\[
(e_1', \lambda e_1'), \quad (e_j', e_{j-1}' + \lambda e_j')\quad \text{for } 2 \leq j \leq n.
\]
We may therefore pick elements \(m_1, \ldots, m_n \in M\) such that
\[
m_1 \in (x + \lambda y) \otimes e_1' + J^2 \otimes_K V
\]
\[
m_j \in y \otimes e_{j-1}' + (x + \lambda y) \otimes e_j' + J^2 \otimes_K V\quad \text{for } 2 \leq j \leq n.
\]
So since \(f(m_j) \in \text{rad}^2(N)\) for all \(j\), and since \([x + \lambda y + J^2] \text{ is neither } [x + J^2]\) nor \([y + J^2]\), Lemma 2.3 tells us that we can inductively pick \(\bar{f}(1 \otimes e_1'), \ldots, \bar{f}(1 \otimes e_n') \in \text{rad}^2(N)\) such that
\[
\bar{f}((x + \lambda y) \otimes e_1') = f(m_1)
\]
\[
\bar{f}((x + \lambda y) \otimes e_j') = f(m_j) + \bar{f}(y \otimes e_{j-1}')\quad \text{for } 2 \leq j \leq n.
\]
Treating each summand \((V_i, L_i)\) in this way we obtain a map \(\bar{f}: R \otimes_K V \to \text{rad}^2(N)\), which therefore satisfies \(\bar{f}(J^2 \otimes_K V) = 0\). It follows that all the equations above such as \(\bar{f}(x \otimes e_j + y \otimes e_{j+1}) = f(m_j)\) can be simplified to \(\bar{f}(m_j) = f(m_j)\). As \(f\) and \(\bar{f}\) are also both zero on \(\text{soc}^3(M) \subseteq J^2 \otimes_K V\), it follows by Lemma 4.1 that \(\bar{f}|_M = f\).

Proof of Theorem 1.1. By [3], the ghost number is at least three. So we have to show that every threefold ghost \(M \xrightarrow{f} N\) is stably trivial. Stripping projective summands if necessary, we may assume that \(M, N\) are projective free. Taking
an injective hull, we see that \( M \) embeds in \( R \otimes_K V \) for some finite-dimensional \( K \)-vector space \( V \). Since \( M \) is projective free, we actually have \( M \subseteq J \otimes_K V \).

By Lemma 2.2, we may replace \( K \) by a finite extension field: so we may assume that \( \mathbb{F}_4 \subseteq K \). Set \( L = L_{x,y}(M) \). Corollary 3.2 says that \((V, L)\) is a direct sum of indecomposables. Replacing \( K \) by a finite extension field again if necessary, we may assume in Case (1) of Theorem 3.1 that the characteristic polynomial of the automorphism \( F \) always splits over \( K \). By Proposition 4.2, it follows that \( f \) extends to a map \( \bar{f}: R \otimes_K V \rightarrow \text{rad}^2(N) \), meaning that \( f \) is stably trivial. \( \square \)

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E-mail address: altunbulak@cankaya.edu.tr

DEPT OF MATHEMATICS AND COMPUTER SCIENCE, ÇANKAYA UNIVERSITY, ANKARA, TURKEY

E-mail address: david.green@uni-jena.de

INSTITUT FÜR MATHEMATIK, FRIEDRICH-SCHILLER-UNIVERSITÄT JENA, 07737 JENA, GERMANY