Generalized Complex Spherical Harmonics, Frame Functions, and Gleason Theorem

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Abstract. Consider a finite dimensional complex Hilbert space $\mathcal{H}$, with $\dim(\mathcal{H}) \geq 3$, define $S(\mathcal{H}) := \{ x \in \mathcal{H} \mid ||x|| = 1 \}$, and let $\nu_{\mathcal{H}}$ be the unique regular Borel positive measure invariant under the action of the unitary operators in $\mathcal{H}$, with $\nu_{\mathcal{H}}(S(\mathcal{H})) = 1$. We prove that if a complex frame function $f : S(\mathcal{H}) \to \mathbb{C}$ satisfies $f \in L^2(S(\mathcal{H}), \nu_{\mathcal{H}})$, then it verifies Gleason’s statement: There is a unique linear operator $A : \mathcal{H} \to \mathcal{H}$ such that $f(u) = \langle u|Au \rangle$ for every $u \in S(\mathcal{H})$. $A$ is Hermitean when $f$ is real. No boundedness requirement is thus assumed on $f$ \textit{a priori}.

1 Introduction

In the absence of superselection rules, the states of a quantum system described in the Hilbert space $\mathcal{H}$ are defined as generalized probability measures $\mu : \mathfrak{P}(\mathcal{H}) \to [0,1]$ on the lattice $\mathfrak{P}(\mathcal{H})$ of orthogonal projectors in $\mathcal{H}$. By definition $\mu$ is required to verify (i) $\mu(I) = 1$ and (ii) $\mu(\sum_{k \in K} P_k) = \sum_{k \in K} \mu(P_k)$, where $\{P_k\}_{k \in K} \subset \mathfrak{P}(\mathcal{H})$, with $K$ finite or countable, is any set satisfying $P_i P_j = 0$ for $i \neq j$ and the sum in the right-hand side in (ii) is computed respect to the strong operator topology if $K$ is infinite. Normalized, positive trace-class operators, namely density or statistical operators, very familiar to physicists, define such measures. However, the complete characterization of those measures was established by Gleason [Gle57], with a milestone theorem whose proof is unexpectedly difficult.

Theorem 1 (Gleason’s theorem) Let $\mathcal{H}$ be a (real or complex) separable Hilbert space with $3 \leq \dim(\mathcal{H}) \leq +\infty$. For every generalized probability measures $\mu : \mathfrak{P}(\mathcal{H}) \to [0,1]$, there exist a unique positive, self-adjoint trace class operator $T_\mu$, with unit trace, such that:

$$\mu(P) = \text{tr}(T_\mu P) \quad \forall P \in \mathfrak{P}(\mathcal{H}).$$

The key-tool exploited in Gleason’s proof is the notion of frame function that will be the object of this paper.
Definition 2 Let $\mathcal{H}$ be a complex Hilbert space and $\mathbb{S}(\mathcal{H}) := \{\psi \in \mathcal{H} \mid ||\psi|| = 1\}$. $f : \mathbb{S}(\mathcal{H}) \to \mathbb{C}$ is a frame function on $\mathcal{H}$ if $W_f \in \mathbb{C}$ exists, called weight of $f$, with:

$$\sum_{x \in N} f(x) = W_f \quad \text{for every Hilbertian basis } N \text{ of } \mathcal{H}.$$  \hspace{1cm} (1)

(If $\mathcal{H}$ is non-separable, the series is the integral with respect to the measure counting the points of $N$.)

With the hypotheses of Gleason’s theorem, the restriction $f_\mu$ of $\mu$ to the set of the projectors on one-dimensional subspaces is a real and bounded frame function. It is known that on a real Hilbert space $\mathcal{H}$ with $dim(\mathcal{H}) = 3$, a frame function which is bounded (even from below only or only from above only) is continuous and can be uniquely represented as a quadratic form [Gle57, Dvu92]. That result is very difficult to be established and is the kernel of the original proof of the Gleason theorem. The last non-trivial step in Gleason’s proof is passing from 3 real dimensions to any (generally complex) dimension, this is done exploiting Riesz theorem, establishing that there is a unique positive, self-adjoint trace-class operator $T_\mu$ with $tr(T_\mu) = 1$ such that $f_\mu(x) = \langle x | T_\mu x \rangle$ for all $x \in \mathbb{S}(\mathcal{H})$. The final step is the easiest one: if $P \in \mathfrak{P}(\mathcal{H})$, there is a Hilbert basis $N$ such that in the strong operator topology $P = \sum_{z \in N_P} f(z) = tr(P T_\mu)$. 

Frame functions are therefore remarkable tools to manipulate generalized measures. However, they are interesting on their own right [Dvu92]. An important difference, distinguishing the finite-dimensional case from the infinite-dimensional one, is that a frame function on an infinite dimensional Hilbert space has to be automatically bounded [Dvu92]. Whereas in the finite-dimensional case ($dim(\mathcal{H}) \geq 3$), as proved by Gudder and Sherstnev, there exist infinitely many unbounded frame functions [Dvu92]. The bounded ones are the only representable as quadratic forms.

In the rest of the paper we prove a proposition concerning sufficient conditions to assure that a frame function, on a complex finite-dimensional Hilbert space $\mathcal{H}$, with $dim(\mathcal{H}) \geq 3$, is representable as a quadratic form without assuming the boundedness requirement a priori. Instead we treat the topic from another point of view. The sphere $\mathbb{S}(\mathcal{H})$, up to a multiplicative constant, admits a unique regular Borel measure invariant under the action of all unitary operators in $\mathcal{H}$. We prove that, for $dim(\mathcal{H}) \geq 3$, a complex frame function $f$ is representable as a quadratic form whenever it is Borel-measurable and belongs to $L^2(\mathbb{S}(\mathcal{H}), \nu_\mathcal{H})$. In particular it holds when $f \in L^p(\mathbb{S}(\mathcal{H}), \nu_\mathcal{H})$ for some $p \in [2, +\infty]$. The proof is direct and relies upon the properties of the spaces of generalized complex spherical harmonics [Rud86] and on some results due to Watanabe [Wat00] on zonal harmonics, beyond standard facts on Hausdorff compact topological group representations (the classic Peter-Weyl theorem).
2 Generalized Complex Spherical Harmonics

Let us introduce a \( n \)-dimensional generalization of spherical harmonics defined on:

\[
\mathbb{S}^{2n-1} := \{ x \in \mathbb{C}^n \mid ||x|| = 1 \}.
\] (2)

\( \mathbb{S}^{2n-1} \subset \mathbb{R}^{2n} \) is, in fact, a \( 2n-1 \)-dimensional real smooth manifold.

\( \nu_n \) denotes the \( U(n) \)-left-invariant regular Borel measure on \( \mathbb{S}^{2n-1} \), normalized to \( \nu_n(\mathbb{S}^{2n-1}) = 1 \), obtained from the two-sided Haar measure on \( U(n) \) on the homogeneous space given by the quotient \( U(n)/U(n-1) \equiv \mathbb{S}^{2n-1} \). That measure exists and is unique as follows from general results by Mackey (e.g., see Chapter 4 of [BR00], noticing that both \( U(n) \) and \( U(n-1) \) are compact and thus unimodular).

**Lemma 3** \( \nu_n(A) > 0 \) if \( A \neq \emptyset \) is an open subset of \( \mathbb{S}^{2n-1} \).

**Proof.** \( \{gA\}_{g \in U(n)} \) is an open covering of \( \mathbb{S}^{2n-1} \). Compactness implies that \( \mathbb{S}^{2n-1} = \bigcup_{k=1}^{N} g_k A \) for some finite \( N \). If \( \nu_n(A) = 0 \), sub-additivity and \( U(n) \)-left-invariance would imply \( \nu_n(\mathbb{S}^{2n-1}) = 0 \) that is false. \( \square \)

As \( \nu_n \) is \( U(n) \)-left-invariant,

\[
U(n) \ni g \rightarrow D_n(g) \quad \text{with} \quad D_n(g)f := f \circ g^{-1} \quad \text{for} \quad f \in L^2(\mathbb{S}^{2n-1}, d\nu_n)
\] (3)

defines a faithful unitary representation of \( U(n) \) on \( L^2(\mathbb{S}^{2n-1}, d\nu_n) \).

**Lemma 4** For every \( n = 1, 2, \ldots \) the unitary representation (3) is strongly continuous.

**Proof.** It is enough proving the continuity at \( g = I \). If \( f : \mathbb{S}^{2n-1} \rightarrow \mathbb{C} \) is continuous, \( U(n) \times \mathbb{S}^{2n-1} \ni (g, u) \mapsto f(g^{-1} u) \) is jointly continuous and thus bounded by \( K < +\infty \) since the domain is compact. Exploiting Lebesgue dominated convergence theorem as \( |f \circ g^{-1}(u) - f(u)|^2 \leq K \) and the constant function \( K \) being integrable since the measure \( \nu_n \) is finite:

\[
||D_n(g)f - f||_2^2 = \int_{\mathbb{S}^{2n-1}} |f \circ g^{-1} - f|^2 d\nu_n \rightarrow 0 \quad \text{as} \quad g \rightarrow I,
\]

If \( f \) is not continuous, due to Luzin’s theorem, there is a sequence of continuous functions \( f_n \) converging to \( f \) in the norm of \( L^2(\mathbb{S}^{2n-1}, d\nu_n) \). Therefore

\[
||f \circ g^{-1} - f||_2 \leq ||f \circ g^{-1} - f_n \circ g^{-1}||_2 + ||f_n \circ g^{-1} - f_n||_2 + ||f_n - f||_2.
\]

If \( \epsilon > 0 \), there exists \( k \) with \( ||f \circ g^{-1} - f_k \circ g^{-1}||_2 = ||f - f_k||_2 \leq \epsilon/3 \) where we have also used the \( U(n) \)-invariance of \( \nu_n \). Since \( f_k \) is continuous we can apply the previous result getting \( ||f_k \circ g^{-1} - f_k||_2 \leq \epsilon/3 \) if \( g \) is sufficiently close to \( I \). \( \square \)
We are in a position to define the notion of spherical harmonics we shall use in the rest of the paper. If, \( p, q = 0, 1, 2, \ldots \), \( P^{p,q} \) denotes the set of polynomials \( h: S^{2n-1} \to \mathbb{C} \) such that \( h(\alpha z_1, \ldots, \alpha z_n) = \alpha^p \overline{\alpha}^q h(z_1, \ldots, z_n) \) for all \( \alpha \in \mathbb{C} \). The standard Laplacian \( \Delta_{2n} \) on \( \mathbb{R}^{2n} \) can be applied to the elements of \( P^{p,q} \) in terms of decomplexified \( \mathbb{C}^n \). Now, we have the following known result (see Theorems 12.2.3, 12.2.7 in [Rud86] and theorem 1.3 in [JW77]):

**Theorem 5** If \( \mathcal{H}_n^{p,q} := \text{Ker} \Delta_{2n} |_{P^{p,q}} \), the following facts hold.

(a) The orthogonal decomposition is valid, each \( \mathcal{H}_n^{p,q} \) being finite-dimensional and closed:

\[
L^2(S^{2n-1}, d\nu_n) = \bigoplus_{p,q=0}^{+\infty} \mathcal{H}_n^{p,q}.
\]

(b) Every \( \mathcal{H}_n^{p,q} \) is invariant and irreducible under the representation (3) of \( U(n) \), so that the said representation correspondingly decomposes as

\[
D_n(g) = \bigoplus_{p,q=0}^{+\infty} D_n^{(p,q)}(g) \quad \text{with} \quad D_n^{(p,q)}(g) := D_n(g)|_{\mathcal{H}_n^{p,q}}.
\]

(c) If \( (p, q) \neq (r, s) \) the irreducible representations \( D_n^{p,q} \) and \( D_n^{r,s} \) are unitarily inequivalent: no unitary operator \( U : \mathcal{H}_n^{p,q} \to \mathcal{H}_n^{r,s} \) exists such that \( UD_n^{(p,q)}(g) = D_n^{(r,s)}(g)U \) for every \( g \in U(n) \).

**Definition 6** For \( j \equiv (p, q) \), with \( p, q = 0, 1, 2, \ldots \), the **generalized complex spherical harmonics** of order \( j \) are the elements of \( \mathcal{H}_n^{p,q} \).

A useful technical lemma is the following.

**Lemma 7** For \( n \geq 3 \), \( \mathcal{H}_n^{(1,1)} \) is made of the restrictions to \( S^{2n-1} \) of the polynomials \( h^{(1,1)}(z, \overline{z}) = \overline{z} A z \), \( A \) being any traceless \( n \times n \) matrix and \( z \in \mathbb{C}^n \).

**Proof.** \( h^{(1,1)} \) is of first-degree in each variables so \( h^{(1,1)}(z, \overline{z}) = \overline{z} A z \) for some \( n \times n \) matrix \( A \). \( \Delta_{2n} h^{(1,1)} = 0 \) is equivalent to \( trA = 0 \) as one verifies by direct inspection.

For \( n \geq 3 \), there is a special class of spherical harmonics in \( \mathcal{H}_j^n \) that are parametrized by elements \( t \in S^{2n-1} \) [Wat00].

**Definition 8** For \( n \geq 3 \), the **zonal spherical harmonics** are elements of \( \mathcal{H}_j^n \) defined, for every \( t \in \mathbb{C}^n \), as

\[
F^j_{n,t}(u) := R^n_j(\overline{u} \cdot t) \quad \forall u \in S^{2n-1},
\]

where the polynomials \( R^n_j(z) \) have the generating function

\[
(1 - \xi z - \eta \overline{z} + \xi \eta)^{1-n} = \sum_{p,q=0}^{+\infty} R^n_{p,q}(z) \xi^p \eta^q
\]
with $|z| \leq 1$, $|\eta| < 1$, $|\xi| < 1$.

These zonal spherical harmonics are a generalization of the eigenfunctions of orbital angular momentum with $L_z$-eigenvalue $m = 0$. From (5) we get two identities useful later:

\[ p!q!R_{p,q}^n(1) = (-1)^{p+q}(n-1)n(n+1)\cdots(n+p-2)(n-1)n(n+1)\cdots(n+q-2), \]

\[ p!q!R_{p,q}^n(0) = (-1)^{p}\delta_{pq}p!(n-1)n(n+1)\cdots(n+p-2). \quad (7) \]

### 3 Generalized complex Harmonics and Frame Functions

To prove our main statement in the next section we need the following preliminary technical result that relies on the technology presented in Chapter 7 of [BR00].

**Proposition 9** If \( f \in L^2(S^{2n-1}, d\nu_n) \), each projection \( f_j \) on \( H_j^n \) verifies, \( \mu \) being the Haar measure on \( U(n) \) normalized to \( \mu(U(n)) = 1 \):

\[ f_j(u) = \dim(H_j^n) \int_{U(n)} \text{tr}(\overline{D_j(g)}) f(g^{-1}u) d\mu(g) \quad \text{a.e. in } u \text{ with respect to } \nu_n, \quad (8) \]

where the right-hand side is a continuous function of \( u \in S^{2n-1} \).

If \( f \in L^2(S^{2n-1}, d\nu_n) \) is a frame function, then \( f_j \) (possibly re-defined on a zero-measure set in order to be continuous) is a frame function as well with \( W_{f_j} = 0 \) when \( j \neq (0,0) \).

**Proof.** First of all notice that, if \( f \in L^2(S^{2n-1}, d\nu_n) \), the right-hand side of (5) is well defined and continuous as we go to prove. \( U(n) \ni g \mapsto \text{tr}(\overline{D_j(g)}) \) is continuous – and thus bounded since \( U(n) \) is compact – in view of lemma 4 and \( \dim(H_j^n) \) is finite for theorem 5. Furthermore, for almost all \( u \in S^{2n-1} \) the map \( U(n) \ni g \mapsto f(g^{-1}u) \) is \( L^2(U(n), d\mu) \) – and thus \( L^1(U(n), d\mu) \) because the measure is finite – as follows by Fubini-Tonelli theorem and the invariance of \( \nu_n \) under \( U(n) \), it being

\[ \int_{U(n)} d\mu(g) \int_{S^{2n-1}} |f(g^{-1}u)|^2 d\nu_n(u) = \int_{U(n)} d\mu(g) \int_{S^{2n-1}} |f(u)|^2 d\nu_n(u) = \mu(U(n)) \|f\|^2 < +\infty. \]

Consequently, in view of the fact that \( \mu \) is invariant, and \( U(n) \) transitively acts on \( S^{2n-1} \), the map \( U(n) \ni g \mapsto f(g^{-1}u) \) is \( L^2(U(n), d\mu) \) and thus \( L^1(U(n), d\mu) \) for all \( u \in S^{2n-1} \). Continuity in \( u \) of the right-hand side of (5) can be proved as follows. Let \( u_0 = [I] \in S^{2n-1} \equiv U(n)/U(n-1) \). Since \( U(n) \) and \( U(n-1) \) are Lie groups, for any fixed \( u_1 \in S^{2n-1} \) there is an open neighbourhood \( W_{u_1} \) of \( u_1 \) and a smooth map \( W_{u_1} \ni u \mapsto g_u \in U(n) \) such that \( [g_u] = u \) (Theorem 3.58 in [War83]). As a consequence \( g_u u_0 = [g_u I] = [g_u] = u \). Therefore, using the invariance of the Haar measure and for \( u = g_u u_0 \in W_{u_1} \):

\[ \int_{U(n)} \text{tr}(\overline{D_j(g)}) f(g^{-1}u) d\mu(g) = \int_{U(n)} \text{tr}(\overline{D_j(g_u g)}) f(g^{-1}u_0) d\mu(g). \]
Since $W_{u_1} \times U(n) \ni (u, g) \mapsto \text{tr}(D^j(gu))$ is continuous due to lemma, the measure is finite and $g \mapsto f(g^{-1}u_0)$ is integrable, Lebesgue dominated convergence theorem implies that, as said above, $W_{u_1} \ni u \mapsto \int_{U(n)} \text{tr}(D^j(gu))f(g^{-1}u_0)d\mu(g)$ is continuous in $u_1$ and thus everywhere on $S^{2n-1}$ since $u_1$ is arbitrary.

Let us pass to prove (8) for $f$ containing a finite number of components. So $F$ is finite, $f \in L^2(S^{2n-1}, d\nu_n)$ and:

$$f(u) = \sum_{j \in F} f_j(u) = \sum_{j \in F} \sum_{m=1}^{\dim(H_j^n)} f^j_m Z^j_m(u), \quad f^j_m \in \mathbb{C}$$

where $\{Z^j_m\}_{m=1, \ldots, \dim(H_j^n)}$ is an orthonormal basis of $H_j^n$, with $Z^{0,0}_n = 1$, made of continuous functions (it exists in view of the fact that $P_{p,q}$ is a space of polynomials and exploiting Gramm-Schmidt’s procedure). Then

$$\overline{D^j_{m_0m_0'}}f(g^{-1}u) = \sum_{j \in F} \sum_{m,m'} D^j_{m_0m_0'}(g)D^j_{m_m'}(g)f^j_m Z^j_m(u).$$

In view of (c) of theorem and Peter-Weyl theorem, taking the integral over $g$ with respect to the Haar measure on $U(n)$ one has:

$$\int \overline{D^j_{m_0m_0'}}f(g^{-1}u)d\mu(g) = \dim(H_j^n) f^j_{m_0} Z^j_{m_0}(u),$$

that implies (8) when taking the trace, that is summing over $m_0 = m_0'$. To finish with the first part, let us generalize the obtained formula to the case of $F$ infinite. In the following $P_j : L^2(S^{2n-1}, \nu_n) \to L^2(S^{2n-1}, \nu_n)$ is the orthogonal projector onto $H_j^n$. The convergence in the norm $|| ||_2$ implies that in the norm $|| ||_1$, since $\nu_n(S^{2n-1}) < +\infty$. So if $h_m \to f$ in the norm $|| ||_2$, as $P_j$ is bounded:

$$\lim_{m \to +\infty} (1) P_j h_m = \lim_{m \to +\infty} (2) P_j h_m = P_j \left( \lim_{m \to +\infty} (2) h_m \right) = P_j f.$$

We specialize to the case $h_m = \sum_{(p,q)=(0,0)}^{p+q=m} f_{(p,q)}$ so that $h_m \to f$ as $m \to +\infty$ in the norm $|| ||_2$. As every $h_m$ has a finite number of harmonic components the identity above yields:

$$\dim(H_j^n) \lim_{m \to +\infty} (1) \int_{U(n)} \text{tr}(D^j(g))h_m(g^{-1}u)d\mu(g) = P_j f =: f_j.$$
\[ \leq K \int_{S^{2n-1}} dv(u) \int_{U(n)} d\mu(g) \left| h_m(g^{-1}u) - f(g^{-1}u) \right| \]
\[ = K \int_{U(n)} d\mu(g) \int_{S^{2n-1}} dv(u) \left| h_m(g^{-1}u) - f(g^{-1}u) \right| \]
\[ = K \int_{U(n)} d\mu(g) \int_{S^{2n-1}} dv(u) \left| h_m(u) - f(u) \right| = K\mu(U(n)) \| h_m - f \|_1 \to 0. \]

We have found that, as desired, (8) holds for \( f \), because
\[ \left\| f_j - \text{dim}(\mathcal{H}_j^n) \int_{U(n)} \text{tr}(D_j^m(g)) f(g^{-1}u) d\mu(g) \right\|_1 = 0. \]

To prove the last statement, assume \( j \neq (0,0) \) otherwise the thesis is trivial (since \( f_{(0,0)} \) is a constant function). We notice that, when \( f_j \) is taken to be continuous (and it can be done in a unique way in view of lemma 3 referring the the Hilbert basis of continuous functions \( Z^j_m \) as before), the identity (8) must be true for every \( u \in S^{2n} \). Therefore, if \( e_1, e_2, \ldots, e_n \) is any Hilbert basis of \( \mathbb{C}^n \)
\[ \frac{1}{\text{dim}(\mathcal{H}_j^n)} \sum_k f_j(e_k) = \int_{U(n)} \text{tr}(D_j^m(g)) \sum_k f(g^{-1}e_k) d\mu(g) = \int_{U(n)} \text{tr}(D_j^m(g)) W_f d\mu(g) = 0 \]
because \( W_f \) is a constant and thus it is proportional to \( 1 = D^{(0,0)} \) which, in turn, is orthogonal to \( D_j^{m'} \) for \( j \neq (0,0) \) in view of Peter-Weyl theorem and (c) of theorem 3.

4 The main result

If \( \mathcal{H} \) is a finite-dimensional complex Hilbert space \( \mathcal{H} \), with \( \text{dim}(\mathcal{H}) = n \geq 3 \), there is only a regular Borel measure, \( \nu_\mathcal{H} \), on \( \mathcal{S}(\mathcal{H}) \) which is left-invariant under the natural action of every unitary operator \( U : \mathcal{H} \to \mathcal{H} \) and \( \nu_\mathcal{H}(\mathcal{S}(\mathcal{H})) = 1 \). It is the \( U(n) \)-invariant measure \( \nu_n \) induced by any identification of \( \mathcal{H} \) with a corresponding \( \mathbb{C}^n \) obtained by fixing an orthonormal basis in \( \mathcal{H} \). The uniqueness of \( \nu_\mathcal{H} \) is due to the fact that different orthonormal bases are connected by means of transformations in \( U(n) \).

**Theorem 10** If \( f : \mathcal{S}(\mathcal{H}) \to \mathbb{C} \) is a frame function on a finite-dimensional complex Hilbert space \( \mathcal{H} \), with \( \text{dim}(\mathcal{H}) \geq 3 \) and \( f \in L^2(\mathcal{S}(\mathcal{H}), d\nu_\mathcal{H}) \), then there is a unique linear operator \( A : \mathcal{H} \to \mathcal{H} \) such that:
\[ f(z) = \langle z | Az \rangle \quad \forall z \in \mathcal{S}(\mathcal{H}), \]  
(9)
where \( \langle \quad \| \rangle \) is the inner product in \( \mathcal{H} \). A turns out to be Hermitean if \( f \) is real.

**Remark.** Since \( \nu_\mathcal{H} \) is finite, \( f \in L^2(\mathcal{S}(\mathcal{H}), d\nu_\mathcal{H}) \) holds in particular when \( f \in L^p(\mathcal{S}(\mathcal{H}), d\nu_\mathcal{H}) \) for some \( p \in [2, +\infty] \), as a classic result based on Jensen’s inequality.
Proof. We start from the uniqueness issue. Let \( B \) be another operator satisfying the thesis, so that \( \langle z | (A - B) z \rangle = 0 \; \forall z \in \mathcal{H} \). Choosing \( z = x + y \) and then \( z = x + iy \) one finds \( \langle x | (A - B) y \rangle = 0 \) for every \( x, y \in \mathcal{H} \), that is \( A = B \). We pass to the existence of \( A \) identifying \( \mathcal{H} \) to \( \mathbb{C}^n \) by means of an orthonormal basis \( \{ e_k \}_{k=1, \ldots, n} \subset \mathcal{H} \). As \( f \in L^2(\mathbb{S}^{2n-1}, dv_n) \), \( f \) can be decomposed as: \( f = \sum_j f_j \) with \( f_j \in \mathcal{H}_j^n \). Lemma \([9]\) implies that, if \( g \in U(n) \):

\[
\sum_{k=1}^{n} (D^j(g)f_j)(e_k) = \sum_{k=1}^{n} f_j(g^{-1}e_k) = 0 \quad \text{if} \quad j \neq (0,0) \tag{10}
\]

Assuming \( f_j \neq 0 \), since the representation \( D^j \) is irreducible, the subspace of \( \mathcal{H}_j^n \) spanned by all the vectors \( D^j(g)f_j \in \mathcal{H}_j^n \) is dense in \( \mathcal{H}_j^n \) when \( g \) ranges in \( U(n) \). As \( \mathcal{H}_j^n \) is finite-dimensional, the dense subspace is \( \mathcal{H}_j^n \) itself. So it must be \( \sum_{k=1}^{n} Z(e_k) = 0 \) for every \( Z \in \mathcal{H}_j^n \). In particular it holds for the zonal spherical harmonic \( F_{n,e}^{j} \) individuated by \( e_1 \):

\[
\sum_{k=1}^{n} F_{n,e}^{j}(e_k) = 0.
\]

By definition of zonal spherical harmonics the above expression can be written in these terms: \( R^j_{p,q}(1)+(n-1)R^j_{p,q}(0)=0 \), and using relations \([7]\):

\[
(-1)^{p+q}(n-1)n(n+1)\cdots(n+p-2)(n-1)n(n+1)\cdots(n+q-2) = (-1)^{p}\delta_{pq}!(n-1)^2n(n+1)\cdots(n+p-2).
\]

\([11]\) implies \( p = q \). Indeed, if \( p \neq q \) the right hand side vanishes, while the left does not. Now, for \( n \geq 3 \) and \( j \neq (0,0) \) we can write:

\[
(n-1)^2n^2(n+1)^2\cdots(n+p-2)^2 = (-1)^{p}\delta_{pq}!(n-1)^2n(n+1)\cdots(n+p-2).
\]

The identity \([12]\) is verified if and only if \( p = 1 \). In view of lemma \([7]\) we know that the functions \( f_{(1,1)} \in \mathcal{H}_{(1,1)}^{n} \) have form \( f(x) = \langle x | A_0 x \rangle \) with \( trA_0 = 0 \). We conclude that our frame function \( f \) can only have the form:

\[
f(x) = c + f_{(1,1)}(x) = \langle x | cI x \rangle + \langle x | A_0 x \rangle = \langle x | A x \rangle \quad x \in \mathbb{S}^{2n-1}.
\]

If \( f \) is real valued \( \langle x | Ax \rangle = \langle x | A^* x \rangle = \langle x | A^* x \rangle \) and thus \( \langle x | (A - A^*) x \rangle = 0 \). Exploiting the same argument as that used in the proof of the uniqueness, we conclude that \( A = A^* \).

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