LIE $\infty$-ALGEBRAS FROM LIE - RINEHART PAIRS

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Abstract. We generalize the Schouten calculus of multivector fields to commutative Lie Rinehart pairs and define a non negatively graded Lie $\infty$-algebra on their exterior power.

1. Introduction

Lie Rinehart pairs generalize the algebraic structure of vector fields and smooth functions to commutative algebras and Lie algebras, which are some kind of modules with respect to each other. In particular any Lie algebra together with its underlying field defines a Lie Rinehart pair.

Given such a pair, we look at its exterior algebra, that is the exterior power of the Lie partner seen as a module with respect to the commutative algebra. In case of vector fields and smooth functions, this is precisely the algebra of multivector fields.

The exterior power of an ordinary Lie algebra has many structures. Scientists with a more algebraic background eventually look on it as a particular codifferential graded coalgebra, where the codifferential encodes the Lie algebra structure [6], while scientists coming more from differential geometry likely see it as another graded Lie algebra with respect to the Schouten-Nijenhuis bracket [8].

We will show that the codifferential approach is not necessarily well defined with respect to the additional module structure, but that the Schouten-Nijenhuis bracket is natural.

Then we provide a non negatively graded Lie $\infty$-algebra on the exterior power, which comes with a natural injection of the original Lie Rinehart pair. In contrast to ordinary Lie theory this injection is not a single map, but a whole sequence of maps. Those functions are usually called weak Lie $\infty$-morphisms and we explain them in more detail in appendix A.

On the level of objects, this will merely be a shift in perspective, but we get a considerably richer theory on the level of morphisms, since we gain access to functions, which are much more flexible then ordinary map of (graded) Lie algebras.

2. The Schouten-Nijenhuis Algebra of a Lie Rinehart Pair

We start our work with a short introduction to Lie Rinehart pairs. We look at their exterior powers and show that in contrast to ordinary Lie algebras, in general there is no well defined (co)differential in this setting anymore. Then we introduce the Schouten-Nijenhuis bracket, well known from differential geometry.

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2.1. Lie Rinehart pairs. In what follows \( \mathfrak{g} \) will always be a real Lie algebra, that is a \( \mathbb{R} \)-vector space together with an antisymmetric, bilinear map,
\[
\left[ \cdot, \cdot \right] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}
\] (1)
called Lie bracket, such that for any three vector \( x_1, x_2, x_3 \in \mathfrak{g} \) the Jacobi identity 
\[
\left[ x_1, \left[ x_2, x_3 \right] \right] + \left[ x_2, \left[ x_3, x_1 \right] \right] + \left[ x_3, \left[ x_1, x_2 \right] \right] = 0
\]
is satisfied.

In addition \( A \) will always be a real associative and commutative algebra with unit, that is a \( \mathbb{R} \)-vector space together with an associative and commutative, bilinear map
\[
\cdot : A \times A \to A
\] (2)
called multiplication and a unit \( 1_A \in A \). According to a better readable text, we frequently suppress the symbol of the multiplication in \( A \) and just write \( ab \) instead of \( a \cdot b \).

Moreover \( \text{Der}(A) \) will be the Lie algebra of derivations of \( A \), that is the vector space of linear endomorphisms of \( A \), with \( D(ab) = D(a)b + aD(b) \) and Lie bracket 
\[
\left[ D, D' \right](a) := D(D'(a)) - D'(D(a))
\]
for any \( a, b \in A \) and \( D, D' \in \text{Der}(A) \).

Before we get to Lie Rinehart pairs, it is handy to define Lie algebra modules first:

**Definition 2.1 (Lie algebra module).** Let \( \mathfrak{g} \) be a real Lie algebra, \( A \) an \( \mathbb{R} \)-algebra and \( D : \mathfrak{g} \to \text{Der}(A) \) a Lie algebra morphism. Then \( A \) is called a Lie algebra module (or just \( \mathfrak{g} \)-module) and \( D \) is called the \( \mathfrak{g} \)-scalar multiplication.

Now a Lie Rinehart pair is nothing but a Lie algebra and an associative algebra, each of them being a module with respect to the other, such that a particular compatibility equation of their multiplications is satisfied:

**Definition 2.2 (Lie Rinehart Pair).** Let \( A \) be an associative and commutative algebra with unit, \( \mathfrak{g} \) a Lie algebra and \( \cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \) as well as \( D : \mathfrak{g} \to \text{Der}(A) \); \( \cdot \) and \( D \), \( x \mapsto D_x \) maps, such that \( A \) is a \( \mathfrak{g} \)-module with \( \mathfrak{g} \)-scalar multiplication \( D \), the vector space \( \mathfrak{g} \) is an \( A \)-module with \( A \)-scalar multiplication \( \cdot \) and the Leibniz rule
\[
[x, a \cdot_A y] = D_x(a) \cdot_A y + a \cdot_A [x, y]
\] (3)
is satisfied for any \( x, y \in \mathfrak{g} \) and \( a \in A \). Then \( (A, \mathfrak{g}) \) is called a Lie Rinehart pair.

This can be defined more general over arbitrary ground rings with unit and also with respect to non commutative algebras \( A \). We stick to the commutative situation, since we need that property to define exterior powers later on. A more general introduction can be found in [3] and in the references therein.

The two most extreme examples coming from commutative algebras on one side and Lie algebras on the other:

**Example 1.** For any commutative and associative algebra with unit \( A \), a Lie pair is given by \( (A, \text{Der}(A)) \), together with the standard action of \( \text{Der}(A) \) on \( A \) and the standard \( A \)-module structure of \( \text{Der}(A) \).

**Example 2.** Any real Lie algebra \( \mathfrak{g} \) is a \( \mathbb{R} \)-module with respect to its ordinary scalar multiplication and together with the trivial action of \( \mathfrak{g} \) on \( \mathbb{R} \), given by
\[
D : \mathfrak{g} \times \mathbb{R} \to \mathbb{R} ; (x, \lambda) \mapsto D_x(\lambda) := 0,
\]
the pair \( (\mathbb{R}, \mathfrak{g}) \) becomes a Lie Rinehart pair.
As mentioned before the archetypical example is provided by smooth functions and vector fields on a differentiable manifold:

**Example 3.** Let $M$ be a differentiable manifold, $C^\infty(M)$ the algebra of smooth, real valued functions and $\mathfrak{X}(M)$ the Lie algebra of vector fields on $M$. $\mathfrak{X}(M)$ is a $C^\infty(M)$-module and vector fields acts as derivations on smooth functions, that is the map $D : \mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M) ; (X, f) \mapsto D_X(f) := X(f)$ satisfies the equation $D_X fg = D_X f g + f D_X g$. Moreover the Leibniz rule $[X, fY] = D_X(f)Y + f[X, Y]$ holds and it follows that $(C^\infty(M), \mathfrak{X}(M))$ is a Lie Rinehart pair.

The Lie structure can be extended into a graded Lie algebra on the direct sum of the partners, concentrated in degrees zero and one. This appears in [9]:

**Definition 2.3 (Associated Lie algebra).** Let $(A, g)$ be a Lie Rinehart pair. Its associated (graded) Lie algebra is the direct sum $A \oplus g$ seen as a graded vector space, with $A$ concentrated in degree zero, $g$ concentrated in degree one and Lie bracket defined by

$$[\cdot, \cdot] : A \oplus g \times A \oplus g \to A \oplus g$$

$$(a, x), (b, y) \mapsto (D_x(a) + D_y(b), [x, y]).$$

In particular this means, that we can see any Lie Rinehart pair as a graded Lie algebra and that it makes sense to talk about (graded) Lie algebra morphisms in their context. The following proposition justifies the definition:

**Proposition 2.4.** $(A \oplus g, [\cdot, \cdot])$ is a graded Lie algebra.

Proof. $A \oplus g$ is a graded vector space by definition. To see graded symmetry of the bracket, we only need to consider mixed expressions, where we compute $[(a, 0), (0, x)] = (D_x(a), 0) = [(0, x), (a, 0)] = (-1)^{|a||x|}[(x, 0), (a, 0)]$ on scalars $a \in A$ and vectors $x \in g$.

To see the graded symmetric Jacobi identity, observe that it has to vanish, whenever at least two arguments are scalars, since the left side of the identity is an expression, homogeneous of degree $-2$. If all arguments are vectors, it becomes the usual Jacobi identity of $g$ and the remaining cases are seen from $D_y D_x - D_x D_y + D_{[x,y]} = 0$. 

Morphisms of Lie Rinehart pairs are pairs of appropriate algebra maps, which interact properly with respect to the additional module structures [9]:

**Definition 2.5 (Lie Rinehart Morphism).** Let $(A, g)$ and $(B, h)$ be two Lie Rinehart pairs. A morphism of Lie Rinehart pairs is a pair of maps $(f, g)$, such that $f : A \to B$ is a morphism of associative and commutative, real algebras with unit, $g : g \to h$ is a morphism of Lie algebras and the equations

$$g(a \cdot_A x) = f(a) \cdot_B g(x) \quad \text{and} \quad f(D_x(a)) = D_{g(x)}(f(a))$$

are satisfied for any $a \in A$ and $x \in g$.

This is the correct definition of a morphism in the setting of Lie Rinehart pairs, since all structure is respected properly:
Corollary 2.6. Let \((f, g) : (A, g) \to (B, h)\) be a morphism of Lie Rinehart pairs. The image \((f(A), g(g))\) is a Lie Rinehart pair and \((f, g) : A \oplus g \to B \oplus h\) is a morphism of graded Lie algebras.

Proof. The first structure equation of (5) implies that the vector space \(g(g)\) is a \(f(A)\)-module and the second that \(f(A)\) is a \(g(g)\)-module. To see the Leibniz equation, compute
\[
[g(x), f(a) \cdot_B g(y)] = [g(x), g(a \cdot_A y) = g([x, a \cdot_A y]) = g(D_x(a) \cdot_A y + a \cdot_A [x, y]) = f(D_x(a)) \cdot_B g(y) + f(a) \cdot_B g([x, y]) = D_{g(x)}(f(a)) \cdot_B g(y) + f(a) \cdot_B [g(x), g(y)].
\]
The second part is a consequence of (5).

2.2. The Exterior Algebra. For any \(n \in \mathbb{N}\), let \(\otimes^n_A g\) be the \(n\)-fold tensor product of the \(A\)-module \(g\) with \(\otimes^0_A g := A\). Since \(A\) is commutative, \(\otimes^n_A g\) is an \(A\)-module and we write \(a \cdot_A x\) for the \(A\)-scalar multiplication of any \(a \in A\) and \(x \in \Lambda g_A\). Note that in general any tensor can be expressed in terms of vectors:

**Proposition 2.7.** Let \(A\) be an associative and commutative algebra with unit, \(M\) an \(A\)-module and \(\otimes^n_A M\) the appropriate \(n\)-fold tensor product. Then any \(x \in \otimes^n_A M\) is an \(A\)-linear combination of simple tensors, i.e. there is a finite index set \(I\), tensors \(x_{i,1} \otimes_A \cdots \otimes_A x_{i,n} \in \otimes^n_A M\) and scalars \(a_i \in A\), such that
\[
x = \sum_{i \in I} a_i \cdot_A x_{i,1} \otimes_A \cdots \otimes_A x_{i,n}.
\]

Proof. See for example ([2]).

We call such a sum an \textbf{\(A\)-linear combination}. However in general these \(A\)-linear combinations are not unique.

Back on Lie Rinehart pairs \((A, g)\), the \textbf{tensor algebra} of the \(A\)-module \(g\) is the direct sum of all \(n\)-fold \(A\)-tensor products
\[
T_A g := \bigoplus_{n=0}^\infty \otimes^n_A g,
\]

\(\text{together with an associative but not commutative multiplication given by concatenation of tensors } \otimes_A : T_A g \times T_A g \to T_A g \) \((x, y) \mapsto x \otimes_A y\). This product has a unit \(1_A \in \otimes_0^A g \simeq A\).

As usual we get the exterior power as the quotient of the tensor power and the submodule generated by all simples tensors with 'repeated vector products':

**Definition 2.8** (Exterior Algebra). For any Lie Rinehart pair \((A, g)\) and \(n \in \mathbb{N}_0\), let \(\Lambda^0_A g := \otimes_0^A g / J^n\) be the quotient module of the \(n\)-th tensor product and the submodule \(J^n\), spanned by all \(x_1 \otimes \cdots \otimes x_n\) with \(x_i = x_j\) for some \(i = j\). Then the direct sum
\[
\Lambda g_A := \bigoplus_{n=0}^\infty \Lambda^n_A g
\]
\(\text{together with the quotient } \wedge : \Lambda g_A \times \Lambda g_A \mapsto \Lambda g_A \) \((x, y) \mapsto x \wedge y\) of the \(A\)-tensor multiplication, is called the \textbf{exterior algebra} of \((A, g)\) and the product is called the \textbf{exterior product}.

We write \(x_1 \wedge \cdots \wedge x_n \in \Lambda^n_A g\) for the coset of any tensor \(x_1 \otimes \cdots \otimes x_n \in \otimes^n_A g\) and in particular \(\Lambda^0_A g \simeq A\) and \(\Lambda^1_A g \simeq g\), since \(J^0 = \{0\}\) and \(J^1 = \{0\}\). If at least one factor in an exterior product is of tensor degree zero, we sometimes write \(a \cdot_A x\) instead of \(a \wedge x\), to stress that the exterior product is just \(A\)-scalar multiplication in that case.
Example 4. If \((C^\infty(M), \mathfrak{X}(M))\) is the Lie Rinehart pair of smooth functions and vector field, the exterior algebra \(\bigwedge \mathfrak{X}(M)_{C^\infty(M)}\) is the algebra of \textbf{multivector fields}.

The exterior algebra is an \(A\)-module and any exterior tensor can be written as a sum (not just a linear combination) of simple exterior tensors:

**Proposition 2.9.** Let \((A, \mathfrak{g})\) be a Lie Rinehart pair and \(\bigwedge \mathfrak{g}_A\) its exterior algebra. Then \(\bigwedge \mathfrak{g}_A\) is an \(A\)-module and any tensor \(x \in \bigwedge \mathfrak{g}_A\) is an \(A\)-linear combination of simple exterior products. In particular there is a finite index set \(I\) and simple exterior tensors \(x_{i,1} \wedge \cdots \wedge x_{i,n_i} \in \bigwedge \mathfrak{g}_A\), such that

\[
x = \sum_{i \in I} a_i \cdot x_{i,1} \wedge \cdots \wedge x_{i,n_i} .
\]  
(8)

Moreover any tensor \(x \in \bigwedge \mathfrak{g}_A\) is a finite sum (not just a linear combination) of simple tensors, that is there is a finite index set \(I\) and simple exterior tensors \(x_{i,1} \wedge \cdots \wedge x_{i,n_i} \in \bigwedge \mathfrak{g}_A\), such that

\[
x = \sum_{i \in I} x_{i,1} \wedge \cdots \wedge x_{i,n_i} .
\]  
(9)

**Proof.** Since \(T_A \mathfrak{g}\) is an \(A\)-module, so is \(\bigwedge \mathfrak{g}_A\). For the first equation, observe that by prop (2.7) the \(A\)-tensor algebra \(T_A \mathfrak{g}\) is spanned by simple tensors \(x_1 \otimes_A \cdots \otimes_A x_n\). It follows that the quotient \(\bigwedge \mathfrak{g}_A\) is spanned by the appropriate cosets \(x_1 \wedge \cdots \wedge x_n\).

To see the second equation apply \(\cdot x_1 \wedge \cdots \wedge x_n = (a \cdot x_1) \wedge \cdots \wedge x_n\) to the first one. \(\square\)

Morphisms of Lie Rinehart pairs prolong naturally to morphisms of exterior algebras defined as the direct sum of the scalar part and the exterior tensor power of the Lie algebra part. The compatibility conditions (5) then guarantee that this map is well defined as a morphism of exterior algebras over different scalars.

**Definition 2.10.** \textbf{(Associated Morphism)} Let \((f, g) : (A, \mathfrak{g}) \to (B, \mathfrak{h})\) be a morphism of Lie Rinehart pairs. The map

\[
\bigwedge g_f : \bigwedge \mathfrak{g}_A \to \bigwedge \mathfrak{h}_B
\]  
(10)

defined on scalars \(a \in \mathbb{N}^0 \mathfrak{g}_A\) by \(f(a)\) and on simple tensors \(x_1 \wedge_A \cdots \wedge_A x_n \in \bigwedge^n \mathfrak{g}_A\) by \(g(x_1) \wedge_B \cdots \wedge_B g(x_n)\) and then extended to all of \(\bigwedge \mathfrak{g}_A\) by \(A\)-additivity, is called the \textbf{associated morphism} of \((f, g)\).

The following proposition shows that associated morphisms are well defined as morphisms of exterior algebras over different scalars and that the construction is natural:

**Proposition 2.11.** Let \((f, g) : (A, \mathfrak{g}) \to (B, \mathfrak{h})\) be a morphism of Lie Rinehart pairs. Its associated morphism \(\bigwedge g_f\) is a well defined morphism of exterior algebras over modules of different rings and in particular the equations

\[
\bigwedge g_f((a \cdot_A x) \wedge_A y) = \bigwedge g_f(x \wedge_A (a \cdot_A y)) \quad \text{and} \quad \bigwedge g_f(x \wedge_A y) = \bigwedge g_f(x) \wedge_B \bigwedge g_f(y)
\]

are satisfied for all \(a \in A\) and \(x, y \in \bigwedge \mathfrak{g}_A\). If \((h, i) : (B, \mathfrak{h}) \to (C, \mathfrak{i})\) is another morphism of Lie Rinehart pairs then

\[
\bigwedge h \circ \bigwedge g_f = \bigwedge (i \circ g)_{hof} .
\]

**Proof.** Follows from (5), since the exterior product \(\bigwedge g\) is natural. \(\square\)
Exterior algebras are equipped with a \( \mathbb{Z} \)-grading coming from the tensor degree, which we need in the definition of a \textit{graded symmetric} Lie \( \infty \)-algebra later on. In addition we need a 'reduced' grading to understand the symmetry of the traditional Schouten-Nijenhuis bracket:

**Definition 2.12. (Gradings)** Let \((A, g)\) be a Lie Rinehart pair and \( \Lambda g_A \) its exterior algebra. An element \( x \in \Lambda^n g_A \) is called \textbf{homogeneous} and the integer \( n \) is called the \textbf{tensor degree} of \( x \), written as

\[
| x | := n .
\] (11)

In addition the \textbf{antisymmetric} degree of a homogeneous element \( x \in \Lambda^n g_A \) is the tensor degree, but reduced by one and written as

\[
\text{deg}(x) := n - 1 .
\] (12)

Calling the reduced grading 'antisymmetric', will become clear in the next section, since the traditional Schouten-Nijenhuis bracket behaves antisymmetric, with respect to this grading. The reader familiar with \( \mathbb{Z} \)-graded abelian groups will further notice, that the term is the correct one, related to the transition between symmetric and antisymmetric [1].

With respect to the tensor grading the exterior algebra is concentrated in non negative degrees and tensors of degree zero are precisely the scalars \( a \in A \).

Note that the exterior product is mostly seen as some kind of antisymmetric operation, but since

\[
x \wedge y = (-1)^{|x||y|} y \wedge x
\] (13)

holds on homogeneous tensors, it is in fact a \textbf{graded symmetric} product with respect to the tensor grading, while it has no symmetry at all with respect to the antisymmetric grading. Moreover it is a graded bilinear map, homogeneous of degree zero with respect to the tensor grading and homogeneous of degree one with respect to the antisymmetric grading.

### 2.3. The nonexistence of a natural codifferential.

Every graded Lie algebra comes with a (co)differential on its reduced symmetric tensor coalgebra \([6]\) and considering an ordinary Lie algebra as \( \mathbb{Z} \)-graded but concentrated in degree one only, we get a (co)differential on its exterior power.

Taking this into account, one would guess that such a (co)differential is defined for any Lie Rinehart pair, but as we will see the technique does not apply here anymore. In fact on the archetypical example of smooth functions and vector fields, we can not naturally define a (co)differential on the exterior algebra other than the zero operator.

This is not a contradiction to the previous mentioned coalgebraic approach, since we have to deal with the additional \( A \)-module structure, which happens to be trivial on the Lie Rinehart pair \((\mathbb{R}, g)\) of a Lie algebra.

The following theorem should be seen as a counterexample, giving a Lie Rinehart pair, without a non trivial (co)differential on its exterior algebra:
Theorem 2.13. Let $M$ be a Hausdorff, separable, finite dimensional and differentiable manifold, $(C^\infty(M), \mathfrak{X}(M))$ the Lie Rinehart pair of smooth functions and vector fields on $M$ and $\Lambda \mathfrak{X}(M)_{C^\infty(M)}$ its exterior algebra of multivector fields. Then the only naturally defined (co)differential

$$d : \Lambda \mathfrak{X}(M)_{C^\infty(M)} \to \Lambda \mathfrak{X}(M)_{C^\infty(M)},$$

homogeneous of degree $-1$ with respect to the tensor grading, is the zero operator.

Proof. We look at a slightly more general situation and proof that for any $n \in \mathbb{N}$ there is no naturally defined map $d : \Lambda^n \mathfrak{X}(M)_{C^\infty(M)} \to \Lambda^{n-1} \mathfrak{X}(M)_{C^\infty(M)}$ other than the zero map at all. Since the codifferential is homogeneous of degree $-1$, this will include the situation of the theorem.

As we require this map to be a natural operator, we can use the technique of [4]. According to theorem 14.18 in [4], those operators of order $r$ are in one to one correspondence with maps $f : T_{m+1}^r(\Lambda^n \mathbb{R}^m) \to \Lambda^{n-1} \mathbb{R}^m$, equivariant with respect to the associated action of the jet group $G_m^r$ on domain and codomain for any $m \in \mathbb{N}$. From [4] prop 14.20 we know further

$$T_{m}^r(\Lambda^n \mathbb{R}^m) \simeq (\Lambda^n \mathbb{R}^m) \oplus (\Lambda^n \mathbb{R}^m \otimes \mathbb{R}^m) \oplus \cdots \oplus (\Lambda^n \mathbb{R}^m \otimes S^r \mathbb{R}^m)$$

and write $x_1^{j_1} \cdots x_n^{j_n}, x_1^{j_1} \cdots x_n^{j_n}, \ldots, x_1^{j_1} \cdots x_n^{j_n}$ for the local coordinates of $T_{m}^r(\Lambda^n \mathbb{R}^m)$, that are the prolongations of the canonical coordinates $x_i^j$ in $\mathbb{R}^m$. In particular any such expression is antisymmetric in all upper and symmetric in all lower indices.

The required actions $l^{r+1}$ of $G_m^r$ on $T_{m}^r(\Lambda^n \mathbb{R}^m)$ are pretty involved, but fortunately we only need the actions of the general linear group $Gl(m)$, seen as a subset of the $(r + 1)$-th order jet group $G_m^r$. According to [4] proposition 14.20 this restricted action is purely tensorial and given by $\tilde{x}_1^{j_1} \cdots x_n^{j_n} = x_1^{j_1} \cdots x_n^{j_n}, \tilde{x}_1^{j_1} \cdots x_n^{j_n}, \ldots, \tilde{x}_1^{j_1} \cdots x_n^{j_n}$, $\tilde{x}_1^{j_1} \cdots x_n^{j_n}$, $\tilde{x}_1^{j_1} \cdots x_n^{j_n}$, $\tilde{x}_1^{j_1} \cdots x_n^{j_n}$, and the restricted action on $\Lambda^{n-1} \mathbb{R}^m$ is tensorial, too.

Now choose some real $\lambda > 0$ and consider the general linear transformations $a^i_j \in Gl(m)$, defined by $a^i_j = \lambda$ and $a^i_j = 0$ for $i \neq j$. The action on $\Lambda^{n-1} \mathbb{R}^m$ and $T_{m}^r(\Lambda^n \mathbb{R}^m)$ is particularly easy to compute and given by $l^{r+1}(a^i_j; x_1^{j_1} \cdots x_n^{j_n}) = \lambda x_1^{j_1} \cdots x_n^{j_n}$ and

$$l^{r+1}(a^i_j; x_1^{j_1} \cdots x_n^{j_n}, x_1^{j_1} \cdots x_n^{j_n}, \ldots, x_1^{j_1} \cdots x_n^{j_n}) = (\lambda x_1^{j_1} \cdots x_n^{j_n}, \lambda^{n+1} x_1^{j_1} \cdots x_n^{j_n}, \ldots, \lambda^n x_1^{j_1} \cdots x_n^{j_n}).$$

It follows that any map $f : T_{m}^r(\Lambda^n \mathbb{R}^m) \to \Lambda^{n-1} \mathbb{R}^m$, equivariant with respect to the action $l^{r+1}$ has to satisfy the homogeneity condition

$$\lambda^{n-1} f(x_1^{j_1} \cdots x_n^{j_n}, x_1^{j_1} \cdots x_n^{j_n}, \ldots, x_1^{j_1} \cdots x_n^{j_n}) = f(\lambda x_1^{j_1} \cdots x_n^{j_n}, \lambda^{n+1} x_1^{j_1} \cdots x_n^{j_n}, \ldots, \lambda^n x_1^{j_1} \cdots x_n^{j_n}),$$

for all real $\lambda > 0$, but by the homogeneous function theorem [4] (24.1) such a map (other than the zero morphism) only exists if the equation

$$nd_1 + (n + 1)d_2 + \cdots + (n + r)d_r = n - 1$$

has solutions $d_1, \ldots, d_r \in \mathbb{N}$, which it hasn’t. It follows that the zero map is the only equivariant function and consequently the only natural operator is the zero operator. □
In [9] Rinehart defined a (co)differential for any Lie Rinehart pair, but on the tensor product of the exterior algebra and the universal enveloping algebra of the Lie algebra instead. In case of smooth functions and vector fields this gives a structure dual to the usual De Rham complex of differential forms.

Remark. On the Lie Rinehart pair \((\mathbb{R}, g)\) of a Lie algebra with its trivial \(g\)-module structure on \(\mathbb{R}\), a (co)differential \(d : \Lambda g \to \Lambda g\) is defined by \(d(\lambda) = 0\) on scalars \(\lambda \in \mathbb{R}\) as well as \(d(x) = 0\) on vectors \(x \in g\) and by

\[
d(x_1 \wedge \cdots \wedge x_n) := \sum_{s \in Sh(2n-2)} e(s)[x_{s(1)}, x_{s(2)}] \wedge x_{s(3)} \wedge \cdots \wedge x_{s(n)}
\]
on simple tensors \(x_1 \wedge \cdots \wedge x_n \in \Lambda g\) and is then extend to all of \(\Lambda g\) by linearity. Except for the degree zero part (which is trivial) this is the 'coalgebraic' (co)differential as it appears for example in [6].

If we try to define a similar map on the exterior power of an arbitrary Lie Rinehart pair, the operator is not necessarily well defined with respect to the additional module structure and we could face situations like

\[
d((a \cdot x_1) \wedge x_2) \neq d((x_1 \wedge (a \cdot x_2))
\]

2.4. The Schouten-Nijenhuis bracket. The Lie bracket on vector fields can be extended to a graded Lie bracket on multivector fields, usually called Schouten-Nijenhuis bracket [7],[8]. We show that this can be generalized verbatim to arbitrary Lie Rinehart pairs.

Definition 2.14 (Schouten-Nijenhuis bracket). Let \((A, g)\) be a Lie Rinehart pair and \(\Lambda g_A\) its exterior algebra. The map

\[
\cdot \cdot : \Lambda g_A \times \Lambda g_A \to \Lambda g_A,
\]
defined by \([a, b] = 0\) as well as \([x, a] = [a, x] = D_x(a)\) on scalars \(a, b \in A\) and vectors \(x \in g\) and by

\[
[x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_m] = \\
\sum_{i,j}(-1)^{i+j}[x_i, y_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \wedge y_1 \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_m
\]
on simple tensors \(x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_m \in \Lambda g_A\) and then extend to all of \(\Lambda g_A\) by \(A\)-additivity, is called the (antisymmetric) Schouten-Nijenhuis bracket of \((A, g)\).

This is the traditional definition as it appears for example in [7]. In case of simple tensors an equivalent but more symmetric expression for (15) is given by

\[
\sum_{s \in Sh(1,n-1)} \sum_{t \in Sh(1,m-1)} e(s)e(t)[x_{s(1)}, y_{t(1)}] \wedge x_{s(2)} \wedge \cdots \wedge x_{s(n)} \wedge y_{t(2)} \wedge \cdots \wedge y_{t(m)}.
\]

Care has to be taken, to get the symmetry of the Schouten-Nijenhuis bracket right. In fact we have to consider the antisymmetric grading \((12)\) of \(\Lambda g_A\) to understand it properly. With this grading the common commutation equation

\[
[x, y] = -(-1)^{(|x| - 1)(|y| - 1)}[y, x]
\]
suddenly becomes more conceptual and just says that the Schouten-Nijenhuis bracket is graded antisymmetric with respect to the antisymmetric grading. Later we have to deal with a graded symmetric incarnation of the bracket and that’s why we call this one the antisymmetric bracket.

Proofing its properties has been done in the situation of multivector fields at many places before [7], [8] and we only recapitulate the basic facts for completeness:
Theorem 2.15. Let \((A, g)\) be a Lie Rinehart pair with exterior algebra \(\Lambda g\). The Schouten-Nijenhuis bracket \([\cdot, \cdot]\) is a \(\mathbb{R}\)-bilinear, graded antisymmetric operator, homogeneous of degree zero with respect to the antisymmetric grading and in particular the equation
\[
\{x, y\} = (-1)^{\deg(x)\deg(y)}\{y, x\}, \quad [x, y \wedge z] = [x, y] \wedge z + (-1)^{\deg(x)(\deg(y)-1)}y \wedge [x, z]
\]
as well as the graded Jacobi equation in its antisymmetric incarnation
\[
(-1)^{\deg(x)\deg(z)}[x, [y, z]] + (-1)^{\deg(x)\deg(y)}[y, [z, x]] + (-1)^{\deg(y)\deg(z)}[z, [x, y]] = 0
\]
are satisfied for any homogeneous tensors \(x, y, z \in \Lambda g\).

Proof. On homogeneous tensors \(x\) and \(y\) we get \(|[x, y]| = |x| + |y| - 1\) and the bracket is homogeneous of tensor degree \(-1\). This in turns gives \(\deg([x, y]) = |x| + |y| - 2 = \deg(x) + \deg(y)\) and consequently the bracket is homogeneous of degree zero with respect to the antisymmetric grading.

All other properties are computed verbatim as for the Schouten-Nijenhuis bracket of multivector fields.

Definition 2.16. Let \((A, g)\) be a Lie Rinehart pair and \([\cdot, \cdot]\) the antisymmetric Schouten-Nijenhuis bracket on \(\Lambda g\). Then the operator
\[
\{\cdot, \cdot\} : \Lambda g \times \Lambda g \to \Lambda g
\]
defined by \(\{x, y\} = e(x)[y, x]\) on homogeneous tensors \(x, y \in \Lambda g\) and extended to all of \(\Lambda g\) by \(A\)-additivity is called the symmetric Schouten-Nijenhuis bracket.

All properties of the antisymmetric bracket transform properly under the decalgné morphism and both brackets coincides with the original Lie bracket on vectors:

Corollary 2.17. The symmetric Schouten-Nijenhuis bracket \([\cdot, \cdot]\) is a \(\mathbb{R}\)-bilinear, graded symmetric operator, homogeneous of degree \(-1\) with respect to the tensor grading. In particular the symmetry equation as well as the Jacobi equation
\[
\{x, y\} = e(x, y)\{y, x\}, \quad \sum_{s \in S_h(2,1)} e(s; x_1, x_2, x_3)\{x_{s(1)}, x_{s(2)}, x_{s(3)}\} = 0
\]
is satisfied for all homogeneous tensors \(x, y, z \in \Lambda g\) and on vectors \(x, y \in \Lambda^1 g\), the bracket equals the original Lie bracket of \(g\) that is \(\{x, y\} = [x, y]\).

Proof. This follows from the properties of the decalgné morphism or can else be verified by simple computations.

According to a better readable text we will always use the Koszuls sign conventions \((24)\), when it comes to expressions, which are graded symmetric with respect to the tensor grading.

Any morphism of Lie Rinehart pairs prolongs to a morphism of exterior algebras and the following corollary shows that this is in fact a morphism of graded Lie algebras with respect to the Schouten-Nijenhuis bracket:
Proposition 2.18. Let \((f, g) : (A, g) \to (B, h)\) be a morphism of Lie Rinehart pairs. The associated morphism \(\wedge g f : \wedge g A \to \wedge h B\) of exterior algebras is a morphism of graded Lie algebras, with respect to the Schouten-Nijenhuis bracket.

Proof. \(\wedge g f\) is homogeneous of degree zero with respect to the tensor grading. The rest follows, since \(\wedge g\) is natural and \(g\) a Lie algebra morphism. \(\square\)

Remark. Since \(\wedge 0 g \simeq A\) and \(\wedge 1 g A \simeq g\), there is a natural injection \(A \oplus g \hookrightarrow \wedge g A\) of graded vector spaces and since the Schouten-Nijenhuis bracket coincides with the bracket of \(A \oplus g\) on scalars and vectors, this is in fact a morphism of graded Lie algebras.

3. The Lie \(\infty\)-Algebra of a Lie Rinehart Pair

We expand the Schouten-Nijenhuis algebra into a Lie \(\infty\)-algebra with non trivial higher brackets. Since the zero morphism is the only general (co)differential in this setting, the ‘unary’ bracket has to vanish.

The structure we obtain is particularly simple and merely a change in perspective. Its real advantage lies in the fact, that we gain access to a lot more morphisms. Morphisms which are just not there, when we restrict to the Schouten-Nijenhuis picture.

Finally we show that there is a weak injection of any Lie Rinehart pair into its associated Lie \(\infty\)-algebra.

Definition 3.1 (Higher Lie Brackets). Let \((A, g)\) be a Lie Rinehart pair with exterior algebra \(\wedge g A\) and \([\cdot, \cdot]\) the antisymmetric Schouten-Nijenhuis bracket. Then the Lie \(n\)-bracket

\[
\{\cdot, \cdot, \cdot \}^n : \wedge g A \times \cdots \times \wedge g A \to \wedge g A
\]

is defined for any integer \(n \geq 2\) and homogeneous tensors \(x_1, \ldots, x_n \in \wedge g A\) by

\[
\{x_1, \ldots, x_n\}_n := \sum_{s \in \text{Sh}(2n-2)} e(s; x_1, \ldots, x_n)e(x_{s(1)}) x_{s(2)} \wedge \cdots \wedge x_{s(3)} \wedge [x_{s(2)}, x_{s(1)}]
\]

and then extended to all of \(\wedge g A\) by \(A\)-additivity.

In particular the Lie 2-bracket is just the symmetric Schouten-Nijenhuis bracket. If we refere to the Lie \(n\)-bracket for any \(n \in \mathbb{N}\), we consider the zero operator as the 'Lie 1-bracket', that is we write \(\{ \cdot \}_1\) for the zero operator in this context.

The following proposition provides the technical details to show that the sequence of Lie \(n\)-brackets defines a non negatively graded Lie \(\infty\)-algebra on the exterior power of any Lie Rinehart pair.

Proposition 3.2. The Lie \(n\)-bracket \(\{\cdot, \cdot, \cdot \}_n\) is a graded symmetric, \(n\)-linear operator, homogeneous of tensor degree \(-1\) for any \(n \in \mathbb{N}\) and for \(p, q \in \mathbb{N}\) with \(p + q = n + 1\) and \(p > 1\) as well as \(q > 1\) the equation

\[
\sum_{s \in \text{Sh}(pq-1)} e(s; x_1, \ldots, x_n)\{x_{s(1)}, \ldots, x_{s(q)}\}_{q} x_{s(q+1)}, \ldots, x_{s(n)}\}_p = 0
\]

is satisfied.

Proof. The exterior product is homogeneous of degree zero and the antisymmetric Schouten-Nijenhuis bracket is homogeneous of degree \(-1\), with respect to the tensor grading. It follows that any \(n\)-ary bracket is homogeneous of degree \(-1\).
To proof graded symmetry observe that the exterior product is graded symmetric with respect to the tensor grading and that the expression \( e(x_1) [x_2, x_1] \) is precisely the graded symmetric Schouten-Nijenhuis bracket. The symmetry of the Lie \( n \)-bracket then follows since it is a graded symmetric composition of both operators.

Now to see that any of the 'Jacobi-like' shuffle sums

\[
\sum_{s \in Sh(q, p-1)} e(s; x_1, \ldots, x_n) \{ \{ x_s(1), \ldots, x_s(q) \} q, x_s(q+1), \ldots, x_s(n) \}_p
\]

vanishes, observe that for \( n = 3 \) and \( p = q = 2 \) this is nothing but the ordinary (graded symmetric) Jacobi expression (2.17). To see it for \( n \geq 4 \), let \( x_1, \ldots, x_n \in \wedge g_A \) be homogeneous and \( p, q \in \mathbb{N} \) with \( p + q = n + 1 \) and \( p, q \geq 2 \). First use the graded symmetry of the brackets, to rewrite the shuffle sum into a sum over arbitrary permutations. This gives

\[
\frac{1}{q! (p-1)!} \sum_{s \in S_n} e(s; x_1, \ldots, x_n) \{ \{ x_s(1), \ldots, x_s(q) \} q, x_s(q+1), \ldots, x_s(n) \}_p.
\]

Then apply the definition of \( \{ \ldots, \} \). Writing \( Sh_{\{s(1), \ldots, s(q)\}}(i, j) \) for the set of all \((i, j)\)-shuffles but explicit as permutations of the set \( \{s(1), \ldots, s(q)\} \) the previous expression becomes

\[
\frac{1}{q! (p-1)!} \sum_{s \in S_n} \sum_{t \in Sh_{\{s(1), \ldots, s(q)\}}(2, q-2)} e(s; x_1, \ldots, x_n) e(t; x_s(1), \ldots, x_s(q)) \cdot e(x_{ts(1)}) \{ x_{ts(q)} \wedge \cdots \wedge x_{ts(3)} \wedge [x_{ts(2)}, x_{ts(1)}], x_{s(q+1)}, \ldots, x_{s(n)} \}_p.
\]

Now observe, that for any \( s \in S_n \) and shuffle \( t \in Sh_{\{s(1), \ldots, s(q)\}}(2, q-2) \), the permutation \( (ts(1), \ldots, ts(q), s(q+1), \ldots, s(n)) \) is again an element of \( S_n \) and since there are precisely \( \frac{q! (p-1)!}{(q-2)!} \) many shuffles in \( Sh_{\{s(1), \ldots, s(q)\}}(2, q-2) \) we can just 'absorb' the second sum over shuffles in the previous expression into the sum over general permutation. After reindexing we get:

\[
\frac{1}{q! (p-1)! (q-2)!} \sum_{s \in S_n} e(s; x_1, \ldots, x_n) e(x_{s(1)}) \cdot \{ x_{s(q)} \wedge \cdots \wedge x_{s(3)} \wedge [x_{s(2)}, x_{s(1)}], x_{s(q+1)}, \ldots, x_{s(n)} \}_p.
\]

Then apply the definition of the bracket a second time. Care has to be taken regarding the first element. In fact there are two possible positions for it: At the most right position, i.e as the second argument inside of the Schouten-Nijenhuis bracket or at the most right position outside of the bracket. Taking this into account split the expression into two parts according to the position of the first element. If \( p = 2 \) the second shuffle sum is omitted:

\[
\frac{1}{2(p-1)! (q-2)!} \sum_{s \in S_n} \sum_{t \in Sh_{\{s(q+1), \ldots, s(n)\}}(1, p-2)} e(s; x_1, \ldots, x_n) \cdot e(t; x_s(q+1), \ldots, x_s(n)) e(x_s(1)) e(x_s(q) \wedge \cdots \wedge x_s(3) \wedge [x_s(2), x_s(1)]) \cdot x_{ts(n)} \wedge \cdots \wedge x_{ts(q+2)} \wedge [x_{ts(q+1)}, x_s(q) \wedge \cdots \wedge x_s(3) \wedge [x_s(2), x_s(1)]]
\]

\[
+ \frac{1}{2(p-1)! (q-2)!} \sum_{s \in S_n} \sum_{t \in Sh_{\{s(q+1), \ldots, s(n)\}}(2, p-3)} e(s; x_1, \ldots, x_n) \cdot e(t; x_s(q+1), \ldots, x_s(n)) e(x_s(q) \wedge \cdots \wedge x_s(3) \wedge [x_s(2), x_s(1)], x_{ts(q+1)} e(x_s(1)) e(x_{ts(q+1)}) \cdot x_{ts(n)} \wedge \cdots \wedge x_{ts(q+3)} \wedge x_{s(q)} \wedge \cdots \wedge x_{s(3)} \wedge [x_{s(2)}, x_{s(1)}] \wedge [x_{ts(q+2)}, x_{ts(q+1)}]
\]
Again we ‘absorb’ the additional shuffle sums in both cases into the appropriate sum over general permutations. After reindexing and simplification this becomes

\[-\frac{1}{2(q-2)!} \sum_{s \in S_n} e(s; x_1, \ldots, x_n) e(x_{s(2)}) \cdots e(x_{s(q)}) x_{s(n)} \wedge \cdots \wedge x_{s(q+2)} \wedge [x_{s(q+1)}, x_{s(q)} \wedge \cdots \wedge x_{s(3)} \wedge [x_{s(2)}, x_{s(1)}]]\]

(19)

+ \frac{1}{4(q-2)!} \sum_{s \in S_n} e(s; x_1, \ldots, x_n) e(x_{s(2)}) e(x_{s(3)}) x_{s(n)} \wedge \cdots \wedge x_{s(5)} \wedge [x_{s(4)}, x_{s(3)}] \wedge [x_{s(2)}, x_{s(1)}]

and again the second sum is omitted for \( p = 2 \). Now let’s look on both sums separately. From

\[e(x_2) e(x_3) [x_4, x_3] \wedge [x_2, x_1] = e(x_2) e(x_3) e([x_2, x_1], [x_4, x_3]) [x_2, x_1] \wedge [x_4, x_3]\]

\[= -e(x_1, x_3) e(x_1, x_4) e(x_2, x_3) e(x_2, x_4) e(x_1) e(x_4) [x_2, x_1] \wedge [x_4, x_3]\]

we see that the expression

\[\sum_{s \in S_n} e(s; x_1, \ldots, x_n) e(x_{s(2)}) e(x_{s(3)}) \cdot x_{s(n)} \wedge \cdots \wedge x_{s(5)} \wedge [x_{s(4)}, x_{s(3)}] \wedge [x_{s(2)}, x_{s(1)}]\]

(20)

vanishes, since for any permutation \( s \in S_n \) there is precisely one permutation \( t \in S_n \) with \( t = (s(3), s(4), s(1), s(2), s(5), \ldots, s(n)) \) and then the term for \( t \) cancel against the term for \( s \).

Now only the first sum in (19) remains, but for \( q = 2 \) it vanishes too due to the graded Jacobi equation of the symmetric Schouten-Nijenhuis bracket. This proofs the equation for \( q = 2 \).

For \( q \geq 3 \) use the Poisson identity \([x, y \wedge z] = [x, y] \wedge z + e(x, y)e(y)y \wedge [x, z]\) of the antisymmetric Schouten-Nijenhuis bracket and omit the constant factor in (19). After simplification we get

\[\sum_{s \in S_n} e(s; x_1, \ldots, x_n) e(x_{s(2)}) \cdots e(x_{s(q)}) \cdot x_{s(n)} \wedge \cdots \wedge x_{s(q+2)} \wedge [x_{s(q+1)}, x_{s(q)} \wedge \cdots \wedge x_{s(3)}] \wedge [x_{s(2)}, x_{s(1)}]\]

(21)

+ \sum_{s \in S_n} e(s; x_1, \ldots, x_n) e(x_{s(2)}) x_{s(n)} \wedge \cdots \wedge x_{s(4)} \wedge [x_{s(3)}, [x_{s(2)}, x_{s(1)}]].

The second sum vanishes due to the graded Jacobi equation of the symmetric Schouten-Nijenhuis bracket. For \( q = 3 \) the first sum vanishes too, since we arrive at situation (20). This proofs the equation for \( q = 3 \).

For \( q \geq 4 \) we apply the Poisson identity to the remaining part again and after simplification we get

\[\sum_{s \in S_n} e(s; x_1, \ldots, x_n) e(x_{s(2)}) e(x_{s(3)}) x_{s(n)} \wedge \cdots \wedge x_{s(5)} \wedge [x_{s(4)}, x_{s(3)}] \wedge [x_{s(2)}, x_{s(1)}]\]

+ \sum_{s \in S_n} e(s; x_1, \ldots, x_n) e(x_{s(2)}) \cdots e(x_{s(q-1)}) \cdot x_{s(n)} \wedge \cdots \wedge x_{s(q+1)} \wedge [x_{s(q)}, x_{s(q-1)} \wedge \cdots \wedge x_{s(3)}] \wedge [x_{s(2)}, x_{s(1)}]

Again the first sum vanishes since it is the same situation as in (20) and the second sum vanishes for \( q = 4 \) due to the same reason. For \( q > 4 \) the second sum equals the first in (21) but for \( q - 1 \) instead. Consequently we have to repeat the last computation \((q - 4)\)-times, to arrive at an expression that is equal to (20) and hence vanishes. This proofs the equation.

Taking into account, that the unary bracket \( \{\cdot\}_1 \) has to be the zero operator, we can combine the brackets into a Lie \( \infty \)-algebra on the exterior power of any Lie Rinehart pair.
Theorem 3.3 (The Lie $\infty$-algebra). Let $(A, g)$ be a Lie Rinehart pair with exterior power $\wedge g_A$. Then $(\wedge g_A, \{\cdot\}_k)_{k \in \mathbb{N}}$ is a Lie $\infty$-algebra, concentrated in non-negative degrees.

Proof. All operators are graded symmetric and homogeneous of degree $-1$ with respect to the tensor grading and the weak Jacobi identities (A.2) follow from (18), since $\{\cdot\}_1$ is the zero operator. Moreover the exterior algebra is concentrated in non-negative degrees with respect to the tensor grading. □

Remark. Note that the Lie $\infty$-structure is particularly simple in this case: The (co)differential is the zero operator and from (18) we see, that each particular shuffle sum already vanishes for fixed $p$ and $q$ in the weak Jacobi identities (A.2).

The following theorem shows, that the construction is natural with respect to morphisms of Lie Rinehart pairs. In fact any morphism of Lie Rinehart pairs gives rise to a strict morphism of Lie $\infty$-algebras:

Theorem 3.4. Let $(f, g) : (A, g) \to (B, h)$ be a morphism of Lie Rinehart pairs. The associated exterior algebra morphism $\wedge g_f : \wedge g_A \to \wedge h_B$ is a strict morphism of Lie $\infty$-algebras.

Proof. We need to show that $\wedge g_f$ commutes with the Lie $n$-bracket for all $n \in \mathbb{N}$. For $n = 1$ this is trivial and for $n = 2$ this is proposition (2.18). For $n \geq 3$ it follows, since $\wedge g$ commutes with the exterior product and the Schouten-Nijenhuis bracket. □

The natural injection $A \oplus g \hookrightarrow \wedge g_A : (a, x) \mapsto (a, x)$ can’t be a morphism of Lie $\infty$-algebras, since it has to commute with all higher brackets, but these brackets are zero on $A \oplus g$. However as the following theorem shows, a natural injection now comes as a weak morphism of Lie $\infty$-algebras:

Definition 3.5. Let $(A, g)$ be a Lie Rinehart pair with associated graded Lie algebra $A \oplus g$, exterior algebra $\wedge g_A$ and

$$i_n : A \oplus g \times \cdots \times A \oplus g \to \wedge g_A$$

$$(x_1, \ldots, x_n) \mapsto (-1)^{n-1}(n-1)! \cdot x_n \wedge \cdots \wedge x_1$$

(22)

for any $n \in \mathbb{N}$. Then the sequence $i_\infty := (i_n)_{n \in \mathbb{N}}$ is called the natural injection of the Lie Rinehart pair into its exterior Lie $\infty$-algebra.

The following theorem shown, that this sequence of multilinear maps is in fact a morphism of Lie $\infty$-algebras:

Theorem 3.6. The natural injection $i_\infty : A \oplus g \to \wedge g_A$ is a morphism of Lie $\infty$-algebras.

Proof. Any map $i_n$ is graded symmetric and homogeneous of tensor degree zero, since the same holds for the exterior product.

In the Lie $\infty$-algebra $A \oplus g$, only the binary bracket does not vanish and since $\{\cdot\}_1$ is the zero operator, the general structure equation (25) of a Lie $\infty$-algebra morphism simplifies for any $n \geq 2$ into

$$\sum_{s \in \text{Sh}(2, n-2)} e(s)i_{n-1}[\{x_{s(1)}(1), x_{s(2)}(2), x_{s(3)}, \ldots, x_{s(n)}\}] =$$

$$\sum_{p=2}^n \sum_{s \in \text{Sh}(j_1, \ldots, j_p)} e(s)\{i_{j_1}(x_{s(1)}, \ldots, x_{s(k_1)}), \ldots, i_{j_p}(x_{s(n-j_p+1)}, \ldots, x_{s(n)})\}_p .$$
Now assume \( n \geq 2 \) and \( 2 \leq p \leq n \) as well as \( j_1 + \ldots + j_p = n \) for positive integers \( j_k \). We use the graded symmetry of the Lie \( n \)-brackets and the maps \( i_s \), to rewrite the shuffle sum at the right side of the structure equation into a sum over arbitrary permutations:

\[
\sum_{s \in \text{Sh}(j_1, \ldots, j_p)} e(s) \{ i_{j_1}(x_s(1)), \ldots, x_s(j_1) \}, \ldots, i_{j_p}(x_s(n-j_p+1), \ldots, x_s(n)) \}_p = \frac{1}{j_1! \cdots j_p!} \sum_{s \in S_n} e(s) \{ i_{j_1}(x_s(1)), \ldots, x_s(j_1) \}, \ldots, i_{j_p}(x_s(n-j_p+1), \ldots, x_s(n)) \}_p
\]

To reorganize this, define \( j_0 := 0 \) and write \( X^s_k := i_{j_k}(x_{s(j_k-1)+1}, \ldots, x_{s(j_k-1)+j_k}) \) for any given vectors \( x_{1}, \ldots, x_{n} \in \mathfrak{g} \). permutation \( s \in S_n \) and \( 1 \leq k \leq p \). Then \( |X^s_k| = j_k \) and we can abbreviate the previous expression into

\[
\frac{1}{j_1! \cdots j_p!} \sum_{s \in S_n} e(s) \{ X^s_1, \ldots, X^s_p \}_p
\]

Applying the definition of the Lie \( p \)-bracket is now straight forward and leads to the expression

\[
\frac{1}{j_1! \cdots j_p!} \sum_{s \in S_n} e(s) \sum_{t \in \text{Sh}(2, p-2)} e(t, X^s_1, \ldots, X^s_p) \cdot e(X^s_t(1)X^s_{t(2)} \wedge \cdots \wedge X^s_{t(3)} \wedge [X^s_{t(2)}, X^s_{t(1)}]).
\]

Substituting the definition of each \( i_{j_k} \) back and using \( e(X^s_t(1)) = (-1)^{j_t(1)} \) this rewrites into

\[
\frac{1}{j_1! \cdots j_p!} \sum_{s \in S_n} e(s) \sum_{t \in \text{Sh}(2, p-2)} e(t, X^s_1, \ldots, X^s_p) (-1)^{j_t(1) - 1} \cdots (-1)^{j_t(p) - 1} \cdot (-1)^{j_t(1)} \cdots (-1)^{j_t(p) - 1} \cdot (-1)^{j_t(1) - 1} + \cdots + (-1)^{j_t(p) - 1} + j_t(p)
\]

\[
\cdot \left( x_{s(j_t(p)+1)}, \ldots, x_{s(j_t(p)+j_t(p)+1)} \right) \wedge \cdots \wedge x_{s(j_t(3)+1)} \wedge \cdots \wedge \left( x_{s(j_t(3)+1)}, \ldots, x_{s(j_t(3)+j_t(3)+1)} \right)
\]

and since \( e(s) \) as well as \( e(t, X^s_1, \ldots, X^s_p) \) keeps properly track of the signs we can reindex this. After simplification using \( j_1 + \cdots + j_p = n \) we get

\[
\frac{1}{j_1! \cdots j_p!} \sum_{s \in S_n} e(s) \sum_{t \in \text{Sh}(2, p-2)} (-1)^{j_t(1) + n - p} \cdot x_{s(n)} \wedge \cdots \wedge x_{s(j_t(1)+1)} \wedge x_{s(j_t(2)+1)} \wedge \cdots \wedge x_{s(n)}
\]

Since all arguments are actually vectors, we can apply the symmetric defining expression (15) of the Schouten-Nijenhuis bracket to simplify this further into

\[
(-1)^{n+p} \frac{1}{j_1! \cdots j_p!} \sum_{s \in S_n} e(s) \sum_{t \in \text{Sh}(2, p-2)} (-1)^{j_t(1) + n - p} \cdot x_{s(n)} \wedge \cdots \wedge x_{s(j_t(1)+2)} \wedge x_{s(j_t(1)+2)} \wedge \cdots \wedge x_{s(n)}
\]

Now observe, that for any \( s \in S_n \) and shuffle \( q \in \text{Sh}(1, j_t(1) - 1) \), the permutation \( (qs(1), \ldots, qs(j_t(1)), s(j_t(1)+1), \ldots, s(n)) \) is again an element of \( S_n \) and since there are precisely \( j_t(1) \) many shuffles in \( \text{Sh}(1, j_t(1) - 1) \) we can just 'absorb' the appropriate sum over shuffles in the previous expression into the sum over general permutation. The same is true for the shuffles \( r \in \text{Sh}(1, j_t(2) - 1) \). After reindexing we get:

\[
(-1)^{n+p} \frac{1}{j_1! \cdots j_p!} \sum_{s \in S_n} e(s) \sum_{t \in \text{Sh}(2, p-2)} (-1)^{j_t(1) + j_t(2) + n - p} \cdot x_{s(n)} \wedge \cdots \wedge x_{s(j_t(1)+2)} \wedge x_{s(j_t(1)+2)} \wedge \cdots \wedge x_{s(n)}
\]
Consequently the theorem follows since pair, but there is more structure on
of the higher brackets with the exterior product, to eventually come to some kind
of $\infty$-Leibniz rule to the higher brackets.

$n$ations of them $[\cdot \cdot \cdot]$, and in addition, the left side of the defining structure equation can be rewritten as
moreover useful when it comes to actual computations.

brackets’ version, since that picture fits nicely into the Schouten calculus and is

for any $n \geq 2$. (This identity was communicated by Gjergji Zaimi at mathoverflow)
To see it consider the generating function

The coefficient of $x^n$ is precisely the left side of (23) and to show that it actually equals $\frac{1}{2}$ use

the generating function to

$$\sum_{p \geq 2} \frac{(-1)^p}{p!} \binom{p}{2} p^{-2} x^2 + 2x^3 + 3x^4 + \cdots = \frac{x^2}{1-x^2}$$

and

$$\frac{x^{-1}}{2} = \sum_{p \geq 2} \frac{(-1)^p}{p!} \binom{p}{2} p^{-2}$$

to simplify the generating function to

$$\frac{\frac{x^2}{2} - \frac{x^2}{1-x^2}}{2} = \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{2} + \frac{x^5}{2} + \cdots \quad \square$$

4. Conclusion and Outlook

We defined a Lie $\infty$-structure on the exterior power $\Lambda g_A$ of any Lie Rineard pair, but there is more structure on $\Lambda g_A$. In fact one should look at the interaction of the higher brackets with the exterior product, to eventually come to some kind of $\infty$-Gerstenhaber structure. Moreover one should look for generalizations of the Leibniz rule to the higher brackets.

A. Appendix Lie $\infty$-algebras

We recall the most basic stuff about Lie $\infty$-algebras. There are many incarnations of them $[5],[6]$, but we will only look at their graded symmetric, ‘many brackets’ version, since that picture fits nicely into the Schouten calculus and is moreover useful when it comes to actual computations.

Lie $\infty$-algebras are defined on $\mathbb{Z}$-graded vector spaces and consequently we recall them first:

\[ (-1)^{n+p-1} \frac{1}{j_1 \cdots j_p} \sum_{1 \leq l < m \leq p} j_l \cdot j_m \sum_{s \in S_n} e(s) x_{s(n)} \wedge \cdots \wedge x_{s(3)} \wedge [x_{s(2)}, x_{s(1)}] \]

Using this we are able to rewrite the right side of the defining structure equation into

\[ (-1)^{n+1} \sum_{p=2}^{n} \frac{(-1)^p}{p!} \sum_{j_1 + \cdots + j_p = n} \frac{1}{j_1 \cdots j_p} \sum_{1 \leq l < m \leq p} j_l \cdot j_m \sum_{s \in S_n} e(s) \cdot x_{s(n)} \wedge \cdots \wedge x_{s(3)} \wedge [x_{s(2)}, x_{s(1)}] \]

and in addition, the left side of the defining structure equation can be rewritten as

\[ (-1)^{n-2} \frac{(n-2)!}{2(n-2)p} \sum_{s \in S_n} e(s) x_{s(n)} \wedge \cdots \wedge x_{s(3)} \wedge [x_{s(1)}, x_{s(2)}] = \]

\[ (-1)^{n+1} \frac{1}{2} \sum_{p \geq 1} \frac{(-1)^p}{p!} \sum_{j_1 + \cdots + j_p = m} \frac{1}{j_1 \cdots j_p} \sum_{1 \leq l < m \leq p} j_l \cdot j_m = \frac{1}{2} \quad (23) \]

for any $n \geq 2$. (This identity was communicated by Gjergji Zaimi at mathoverflow)
A.1. Graded Vector Spaces. In what follows \( \mathbb{K} \) will always be a field and \( \mathbb{Z} \) the Abelian group of integers with respect to addition. A \( \mathbb{Z} \)-graded \( \mathbb{K} \)-vector space \( V \) is the direct sum \( \bigoplus_{n \in \mathbb{Z}} V_n \) of \( \mathbb{K} \)-vector spaces \( V_n \). Since this is a coproduct, there are natural injections \( i_n : V_n \to V \) and a vector is called **homogeneous of degree** \( n \) if it is in the image of the injection \( i_n \). In that case we write \( \deg(v) \) or \( |v| \) for its degree.

According to a better readable text we just write graded vector space as a short-cut for \( \mathbb{Z} \)-graded \( \mathbb{K} \)-vector space.

A morphism \( f : V \to W \) of graded vector spaces, homogeneous of degree \( r \), is a sequence of linear maps \( f_n : V_n \to W_{n+r} \) for any \( n \in \mathbb{Z} \) and the integer \( r \in \mathbb{Z} \) is called the degree of \( f \), denoted by \( \deg(f) \) (or \( |f| \)).

For any \( n \in \mathbb{N} \), an \( n \)-multilinear map \( f : V_1 \times \cdots \times V_n \to W \), homogeneous of degree \( r \) is a sequence of \( n \)-multilinear maps \( f_k : (V_1)_{n_1} \times \cdots \times (V_k)_{n_k} \to W_{n_1+r} \) for all \( j_i \in \mathbb{Z} \) with \( \sum j_i = k \).

The \( \mathbb{Z} \)-graded tensor product \( V \otimes W \) of two graded vector spaces \( V \) and \( W \) is given by

\[
(V \otimes W)_n := \bigoplus_{i+j=n} (V_i \otimes W_j)
\]

and the Koszul commutativity constraint \( \tau : V \otimes W \to W \otimes V \) is on homogeneous elements \( v \otimes w \in V \otimes W \) defined by

\[
\tau(v \otimes w) := (-1)^{\deg(v)\deg(w)} w \otimes v
\]

and then extended to \( V \otimes W \) by linearity.

**Remark.** We define the symbols \( e(v) := (-1)^{\deg(v)}, e(v, w) := (-1)^{\deg(v)\deg(w)} \). The **Koszul sign** \( e(s; v_1, \ldots, v_k) \in \{-1, +1\} \) is defined for any permutation \( s \in S_k \) and any homogeneous vectors \( v_1, \ldots, v_k \in V \) by

\[
v_1 \otimes \cdots \otimes v_k = e(s; v_1, \ldots, v_k) v_{s(1)} \otimes \cdots \otimes v_{s(k)}.
\]

In an actual computation it can be determined by the following rules: When a permutation \( s \in S_k \) is a transposition \( j \leftrightarrow j+1 \) of consecutive neighbors, then \( e(s; v_1, \ldots, v_k) = (-1)^{\deg(v_j)\deg(v_{j+1})} \) and if \( t \in S_k \) is another permutation, then \( e(ts; v_1, \ldots, v_k) = e(t; v_{s(1)}, \ldots, v_{s(k)}) e(s; v_1, \ldots, v_k) \).

A graded \( k \)-linear morphism \( f : \bigwedge^k V \to W \) is called **graded symmetric** if

\[
f(v_1, \ldots, v_k) = e(s; v_1, \ldots, v_k) f(v_{s(1)}, \ldots, v_{s(k)})
\]

for all \( s \in S_k \).

A.2. Shuffle Permutation. Let \( S_k \) be the symmetric group, i.e the group of all bijective maps of the ordinal \([k]\).

**Definition A.1** (Shuffle Permutation). For any \( p, q \in \mathbb{N} \) a \((p,q)\)-shuffle is a permutation \( s \in S_{p+q} \) with \( s(1) < \cdots < s(p) \) and \( s(p+1) < \cdots < s(p+q) \). We write \( \text{Sh}(p,q) \) for the set of all \((p,q)\)-shuffles.

More generally for any \( p_1, \ldots, p_n \in \mathbb{N} \) a \((p_1,\ldots,p_n)\)-shuffle is a permutation \( s \in S_{p_1+\cdots+p_n} \) with \( s(p_{j-1}+1) < \cdots < s(p_{j-1}+p_j) \). We write \( \text{Sh}(p_1,\ldots,p_n) \) for the set of all \((p_1,\ldots,p_n)\)-shuffles.

For more on shuffles, see for example at [10].
A.3. Lie $\infty$-algebras. On the structure level Lie $\infty$-algebras generalize (differential graded) Lie-algebras to a setting where the Jacobi identity isn’t satisfied any more, but holds up to particular higher brackets. This can be defined in many different ways [5], but the one that works best for us is its ‘graded symmetric, many bracket’ version.

Definition A.2. A Lie $\infty$-algebra $(V,(D_k)_{k\in\mathbb{N}})$ is a $\mathbb{Z}$-graded $\mathbb{R}$-vector space $V$, together with a sequence $(D_k)_{k\in\mathbb{N}}$ of graded symmetric, $k$-multilinear maps $D_k : \bigotimes^k V \to V$, homogeneous of of degree $-1$, such that the weak Jacobi equations

$$\sum_{i+j=n+1} \left( \sum_{s \in \text{Sh}(j,n-j)} e(s; v_1, \ldots, v_n) D_1(D_j(v_{s_1}, \ldots, v_{s_j}), v_{s_{j+1}}, \ldots, v_{s_n}) \right) = 0$$

are satisfied for any integer $n \in \mathbb{N}$ and any vectors $v_1, \ldots, v_n \in V$.

In particular Lie $\infty$-algebras generalizes ordinary Lie algebras, if the grading is chosen right:

Example 5 (Lie Algebra). Every Lie algebra $(V,[\cdot,\cdot])$ is a Lie $\infty$-algebra if we consider $V$ as concentrated in degree one and define $D_k = 0$ for any $k \neq 2$ as well as $D_2(\cdot,\cdot) := [\cdot,\cdot]$.

Very different from common Lie theory is, that a morphism of Lie $\infty$-algebras is not necessarily just a single map. In fact such a morphism is a sequence of maps, satisfying a particular structure equation. To understand how these morphisms emerge, look for example at [6].

Definition A.3. For any two Lie $\infty$-algebras $(V,(D_k)_{k\in\mathbb{N}})$ and $(W,(l_k)_{k\in\mathbb{N}})$ a morphism of Lie $\infty$-algebras is a sequence $(f_k)_{k\in\mathbb{N}}$ of graded symmetric, $k$-multilinear maps

$$f_k : V \times \cdots \times V \to W$$

homogeneous of degree 0, such that the structure equation

$$\sum_{p+q=n+1} \left( \sum_{s \in \text{Sh}(q,p-1)} e(s; f_1(v_s(1)), \ldots, v_s(q)), v_s(q+1), \ldots, v_s(n) \right) =$$

$$\sum_{p=0}^{k_1+\cdots+k_p=n} \sum_{s \in \text{Sh}(k_1,\ldots,k_p)} e(s) f_p(f_{k_1}(v_s(1)), \ldots, v_s(k_1)), \ldots, f_{k_p}(v_s(n-k_p+1), \ldots, v_s(n))$$

(25)

is satisfied for any $n \in \mathbb{N}$ and any vectors $v_1, \ldots, v_n \in V$.

The morphism is called strict, if in addition $f_k = 0$ for all $k \geq 2$, that is, if the morphism is a single map, that commutes with all brackets.

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