Super-renormalizable Quantum Gravity

Leonardo Modesto
Perimeter Institute for Theoretical Physics, 31 Caroline St., Waterloo, ON N2L 2Y5, Canada
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In this paper we study perturbatively an extension of the Stelle higher derivative gravity involving an infinite number of derivative terms. We know that the usual quadratic action is renormalizable but suffers of the unitarity problem because of the presence of a ghost (state of negative norm) in the theory. The new theory is instead ghost-free since the introduction of (in general) two entire functions in the model with the property do not introduce new poles in the propagator. The local high derivative theory is recovered expanding the entire functions to the lowest order in the mass scale of the theory. Any truncation of the entire functions gives rise to the unitarity violation but if we keep all the infinite series we do not fall into these troubles. The theory is renormalizable at one loop and finite from two loops on. Since only a finite number of graphs are divergent then the theory is super-renormalizable. We analyze the fractal properties of the theory at high energy showing a reduction of the spacetime dimension at short scales. Black hole spherical symmetric solutions are also studied omitting the high curvature corrections in the equation of motions. The solutions are regular and the classical singularity is replaced by a “de Sitter-like core” in $r = 0$. Black holes may show a “multi-horizon” structure depending on the value of the mass.

1. INTRODUCTION TO THE THEORY

One of the biggest problems in theoretical physics is to find a theory that is able to reconcile general relativity and quantum mechanics. There are many reasons to believe that gravity has to be quantum, some of which are: the quantum nature of matter in the right-hand side of the Einstein equations, the singularities appearing in classical solutions of general relativity, etc.

The action principle for gravity we are going to introduce in this paper is the result of a synthesis of minimal requirements: (i) classical solutions must be singularity-free, (ii) Einstein-Hilbert action should be a good approximation of the theory at an energy scale much smaller than the Planck mass, (iii) the theory has to be unitary, with no other degree of freedom than the graviton in the propagator, (vi) spacetime is a single continuum of space and time and in particular the Lorentz invariance is not broken. The last requirement is supported by recent observations.

Now let us introduce the theory, step by step, starting from the perturbative non-renormalizable Einstein gravity, through high derivatives gravity theories (the Stelle theory of gravity will be our first example) onto the action which defines a complete quantum gravity theory. The impatient reader can skip to the end of the introduction for the candidate complete quantum gravity bare Lagrangian.

Perturbative quantum gravity is the quantum theory of a spin two particle on a fixed (usually for simplicity is assumed to be flat) background. Starting from the Einstein-Hilbert Lagrangian

$$\mathcal{L} = -\sqrt{-g} \kappa^{-2} R$$  \hspace{1cm} (1)

($\kappa^2 = 16\pi G_N$) we introduce a splitting of the metric in a background part plus a fluctuation

$$\sqrt{-g} g^\mu\nu = g^\mu\nu + \kappa h^\mu\nu,$$  \hspace{1cm} (2)

then we expand the action in power of the graviton fluctuation $h^\mu\nu$ around the fixed background $g^\mu\nu$. Unlike the quantum theory is divergent at two loops, producing a counter-term proportional to the Ricci tensor at the third power

$$\sqrt{-g} R_{\alpha\beta}^\gamma R_{\rho\sigma}^\delta R^{\rho\sigma}_{\alpha\beta}.$$  \hspace{1cm} (3)

In general, in $d$ dimensions the superficial degree of divergence of a Feynman diagram is $D = L d + 2V - 2I$, where $L$ is the number of loops, $V$ is the number of vertices and $I$ the number of internal lines in the graph. Using the topological relation between $V$, $I$ and $L$, $L = 1 + I - V$, we obtain $D = 2 + (d - 2)L$. In $d = 4$ the superficial degree of divergence $D = 2 + 2L$ increases with the number of loops and thus we are forced to introduce an infinite number of higher derivative counter terms and then an infinite number of coupling constants, therefore making the theory not predictive. Schematically, we can relate the loop divergences in perturbative quantum gravity to the counter terms we have to introduce to regularize the theory. In short

$$S = -\int d^4 x \sqrt{g} \left[\kappa^{-2} R + \frac{\alpha}{16} \sum_{m,n}^{m, n} \nabla^{n} R^{m}\right].$$  \hspace{1cm} (4)

with “$n$” and “$m$” integer numbers.

A first revolution in quantum gravity was introduced by Stelle [1] with the higher derivative theory

$$S = -\int d^4 x \sqrt{g} \left[\alpha R^\mu\nu R^{\mu\nu} - \beta R^2 + \kappa^{-2} R\right].$$  \hspace{1cm} (5)
This theory is renormalizable but unfortunately contains a physical ghost (state of negative norm), implying a violation of unitarity in the theory: probability, described by the scattering S-matrix, is no more conserved. The classical theory is unstable, since the dynamics can drive the system to become arbitrarily excited, and the Hamiltonian constraint is unbounded from below.

In this paper we generalize the Stelle theory to restore unitarity. This work is inspired by papers about a non-local extension of gauge theories introduced by Moffat, Cornish and their collaborators in the nineties [2]. The authors further extended the idea to gravity, having in mind the following logic [3]. They considered a modification of the Feynman rules where the coupling constants \(g_i\) for electro-weak interactions and \(G_N\) for gravity are no longer constant but function of the momentum \(p\). They checked the gauge invariance at all orders in gauge theory but only up to the second order in gravity. For particular choices of \(g_i(p)\) or \(G_N(p)\), the propagators do not show any other pole above the standard particle content of the theory, therefore the theory is unitary. On the other hand the theory is also finite if the coupling constants go sufficiently fast to zero in the ultra-violet limit. The problem with gravity is to find a covariant action that self-contains the properties mentioned before: finiteness and/or renormalizability and unitarity.

In the rest of the paper we will mainly refer to the Tomboulis paper [4] that we have discovered only recently. The theory developed in [4] is very interesting, of great generality and mainly concentrated on gauge theories but in [4] the author also concludes with an extension of the idea to gravity. Here we will use the same notations as standard definitions. We arrive to the same conclusions in a different way, but the results are exactly the same.

The action we are going to consider is a generalization of Stelle’s theory

\[
S = - \int d^4x \sqrt{-g} \{ R_{\mu\nu} \alpha(\Box_A) R^{\mu\nu} - \beta(\Box_A) R + \gamma \kappa^{-2} R \},
\]

where \(\alpha(\Box_A)\) and \(\beta(\Box_A)\) are now fixed functionals of the covariant D’Alembertian operator \(\Box_A = \Box/\Lambda^2\) and \(\Lambda\) is a mass scale in the theory.

In the remainder of this section we will summarize the steps and motivations that led us to the generalization (6) and will conclude with an extended version (6).

The covariant action (6) is a collection of terms, but the initial motivation when we started this project was the non local Barvinsky action [5]

\[
S = - \frac{1}{\kappa^2} \int d^4x \sqrt{-g} G^{\mu\nu} \frac{1}{\Box} R_{\mu\nu},
\]

where \(G_{\mu\nu} = R_{\mu\nu} - \Gamma g_{\mu\nu}/2\) is the Einstein tensor. The action (6) looks like a non-local action, but indeed reproduces the Einstein gravity at the lower order in the curvature. Barvinsky has shown that the variation of (6), with respect to the metric, gives the following equations of motion (see later in this section for details)

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + O(R^2_{\mu\nu}) = 0.
\]

The next step is to modify (6) by introducing an extra operator in the action between the Einstein tensor and the Ricci tensor in order to have a well-defined theory at the quantum level without loss of covariance.

We consider as guideline the value of the spectral dimension in the definition of a finite and/or renormalizable theory of quantum gravity which, of course, is compatible with the Einstein gravity at low energy but also manifests a natural dimensional reduction at high energy. The dimensional reduction is of primary importance in order to have a quantum theory free of divergences in the ultra violet regime. The Stelle theory is characterized by a two dimensional behavior at high energy but this is not sufficient to have a well defined theory at the quantum level because of the presence of ghosts in the propagator. A first easy generalization of (7) is the following action

\[
S = - \frac{1}{\kappa^2} \int d^4x \sqrt{-g} G^{\mu\nu} F(\Box/\Lambda^2) R_{\mu\nu},
\]

where \(F(\Box/\Lambda^2)\) is a generic function of the covariant d’Alembertian operator which satisfies the already mentioned properties that we are going to summarize below.

(I) Classical limit:

\[
\lim_{\Lambda \to +\infty} F(\Box/\Lambda^2) = 1,
\]

if the limit is satisfied, the equations of motion are the Einstein equations plus corrections in \(R^2_{\mu\nu}\) (we will explain this in more detail later in the paper).

(II) Finiteness and/or renormalizability of the quantized theory. Our guiding principle is to find a well defined quantum theory and the dimensional reduction of the spacetime at high energy. The Stelle theory, the Crane-Smolin theory [6], “asimptotiically safe quantum gravity” [7], “causal dynamical triangulation” [8], “loop quantum gravity” [9] and “string theory” already manifest this property with a high energy spectral dimension \(d_s = 2\) [11]. However such reduction is insufficient if we want a unitary theory free from negative norm states. We can anticipate that, for the model that we are going to introduce in this paper, the spectral dimension is smaller than one in the ultraviolet regime.

The general theory (7) was for the first time derived by Barvinsky [5] in the brane-world scenery and can be written in the following equivalent way

\[
S = - \frac{1}{\kappa^2} \int d^4x \sqrt{-g} G^{\mu\nu} \frac{1}{\Box} R_{\mu\nu}.
\]

In [5] the author was interested in the infra-red modifications to Einstein gravity. In the present paper, however,
we are interested to ultra-violet modifications to gravity, as stressed in [10]. The form (11) of the action is particularly useful to highlight the classical limit and in the expression of the equations of motion. In analogy to the properties satisfied by the operator $F(\Box/\Lambda^2)$, the operator $\mathcal{O}$ has to satisfy the following limit in order to reproduce the classical theory (see the discussion related to formulas (7) and (8)).

$$\lim_{\Lambda \to +\infty} \mathcal{O}(\Box/\Lambda^2) = 1.$$  

(12)

We now add some details about the Barvinsky derivation of the equation of motion. Taking the variation of (11) with respect to the metric we find [3]

$$-\frac{2}{k^2} \int d^4x \sqrt{-g} G^\mu\nu O^{-1}(\Box/\Lambda^2) \delta g_{\mu\nu} + O(R^2_{\mu\nu}).$$  

(13)

Since $\delta g_{\mu\nu} = -\frac{1}{2} \Box g_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$, integration by parts cancels the $\Box$ operator at the denominator and the contribution of the gauge parameters $\epsilon_\mu$ vanishes in view of the Bianchi identities, $\nabla^\mu G_{\mu\nu} = 0$. All the commutators of covariant derivatives with the $\Box$ operator in $O^{-1}(\Box/\Lambda)$/\Box$ give rise to curvature square operators. Also, the direct variation of the metric gives rise to curvature square terms. Then the equations of motion are very simple, if we omit the squared curvature terms:

$$O^{-1}(\Box/\Lambda) \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + O(R^2_{\mu\nu}) = 8\pi G_N T_{\mu\nu}.  

(14)

If we truncate the theory to the linear part in the Ricci curvature, the equations of motion simplify to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N O(\Box/\Lambda) T_{\mu\nu},$$

$$\nabla^\mu \left( O(\Box/\Lambda) T_{\mu\nu} \right) = 0,$$

(15)

where the second relation is a consequence of the Bianchi identities that we impose on the solution. In particular, such relation implies that the conserved quantity has to be $S_{\mu\nu} = O(\Box/\Lambda^2) T_{\mu\nu}$ in the truncation of the theory. These equations of motion are very interesting because black hole solutions, at least for a particular choice of the operator $O$, are not singular anymore, as recently showed in [11].

At the classical level, equations (14) can be derived from the action (9) or (11) but at the quantum level also the Einstein-Hilbert action and the high derivative terms introduced by Stelle are generated. Thus, the complete Lagrangian we are going to study is

$$\mathcal{L} = -\sqrt{-g} \left[ \frac{\beta}{k^2} R - \beta_2 \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \right] + \beta_0 R^2$$

$$+ \left( R_{\mu\nu} h_2 (\Box/\Lambda) R^{\mu\nu} - \frac{1}{3} R h_2 (\Box/\Lambda) R \right)$$

$$- R h_0 (\Box/\Lambda) R - \frac{1}{2k^2} F^\mu_\mu (\Box/\Lambda) F^\nu_\nu + \bar{\partial}^\mu M_{\mu\nu} C^\nu.$$  

(16)

The operator $\Box/\Lambda^2$ encapsulates the D’Alembertian of the flat fixed background, whereas $F^\mu_\mu$ is the gauge fixing function with the weight functional $\omega$. In general we introduce two different functions $h_2$ and $h_0$. Those functions have not to be polynomial but entire functions without poles or essential singularities. While nonlocal kernels can lead to unitary problems, the functions $h_2$ and $h_0$ do introduce an effective non-locality. However, since $h_2$ and $h_0$ are transcendental entire functions, their behavior is quite similar to polynomial functions and unitary problems do not occur.

Let us assume for a moment that $h_i(x) = p_n(x)$, where $p_n(x)$ is a polynomial of degree $n$. In this case, as it will be evident in the next section, the propagator takes the following form

$$\frac{1}{k^2(1 + p_n(k^2))} = \frac{c_0}{k^2} + \sum_i \frac{c_i}{k^2 - M_i^2},$$  

(17)

where we used the factorization theorem for polynomial and the partial fraction decomposition [3]. When multiplying by $k^2$ the left and right side of (17) and considering the ultraviolet limit, we find that at least one of the coefficients $c_i$ is negative, therefore the theory contains a ghost in the spectrum. The conclusion is that $h_2$ and $h_0$ can not be polynomial.

We can also add to the action the Kretschmann scalar $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ but, for spacetimes topologically equivalent to the flat space, we can use the Gauss-Bonnet topological invariant

$$\int d^4x \sqrt{-g} [R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2]$$

(18)

to rephrase the Kretschmann invariant in terms of $R^2$ and $R_{\mu\nu} R^{\mu\nu}$ already present in the action.

II. BRST INVARIANCE AND GRAVITON PROPAGATOR

We start by considering the quadratic expansion of the Lagrangian (16) in the graviton field fluctuation without specifying the explicit form of the functionals $h_2$ and $h_0$ (if not necessary). Following the Stelle paper, we expand around the Minkowski background $\eta^\mu_\nu$ in power of the graviton field $h^{\mu\nu}$ defined in the following way

$$\sqrt{-g} h^{\mu\nu} = \eta^{\mu\nu} + k h^{\mu\nu}.$$  

(19)

The form of the propagator depends not only on the gauge choice but also on the definition of the gravitational fluctuation [3]. In the quantum theory the gauge choice is the familiar harmonic gauge $\partial_\mu h^{\mu\nu} = 0$ and the Green’s functions are defined by the generating functional
\[ Z(T_{\mu\nu}) = N \int \prod_{\mu < \nu} dh^{\mu\nu} |dC^\alpha| |d\bar{C}_\tau| |de^\tau| \delta(F^\tau - e^\tau) \mu^4 \{ S_g - \frac{1}{2} \int d^4 x \epsilon^\tau \omega(-\Box^\tau) e^\tau + \int d^4 x \bar{C}_\tau F_{\mu\nu} D^{\mu\nu\alpha} C^\alpha + \kappa \int d^4 x T_{\mu\nu} h^{\mu\nu} \} \] \quad (20)

where \( S_g \) is the gravitational action defined in (10) subtracted of the gauge and ghost terms and \( F^\tau = F^{\mu\nu} h^{\mu\nu} \) with \( F_{\mu\nu} = \delta_{\mu\nu} \partial_{\tau} \). \( D_{\alpha} \) is the operator which generates the gauge transformations in the graviton fluctuation \( h^{\mu\nu} \). Given the infinitesimal coordinates transformation \( x^{\mu} = x^{\mu} + \kappa x^{\mu} \), the graviton field transforms as follows:

\[ \delta h^{\mu\nu} = D_{\alpha} \eta^{\mu\nu} \xi^\alpha + (\xi_{\alpha} \partial_{\alpha} \xi_{\mu} - \xi_{\mu} \partial_{\alpha} \xi_{\alpha}) \delta h^{\mu\nu} \]

(21)

We have also introduced a weighting gauge functional (to be precise the second term in (20)), which depends on a weight function \( \omega(-\Box^\lambda) \) with the property to fall off at least like the entire functions \( h_2(k^2/L^2), h_0(k^2/L^2) \) for large momenta [1].

When the gauge symmetry is broken by the addition of the gauge-fixing term, a residual transformation survives for the effective action which involves the gravitational, gauge-fixing and ghost actions terms. This is the BRST symmetry defined by the following transformation, which is appropriate for the gauge-fixing term,

\[ \delta_{\text{BRST}} h^{\mu\nu} = \kappa D_{\alpha} C^{\alpha} \delta \lambda, \]
\[ \delta_{\text{BRST}} C^{\alpha} = -\kappa^2 \delta \partial_{\alpha} C^{\alpha} \theta^\alpha, \]
\[ \delta_{\text{BRST}} \bar{C}_\tau = -\kappa \xi^{-1} \omega(-\Box^\lambda) F^\tau \xi \lambda, \]

(22)

where \( \delta \lambda \) is a constant infinitesimal anticommuting parameter. The first transformation in (22) is nothing but a gauge transformation generated by \( \kappa C^\alpha \delta \lambda \), so the functional \( S_g \) is BRST invariant since it is a function of \( h^{\mu\nu} \) alone. Other two BRST invariant quantities are

\[ \delta_{\text{BRST}}^2 C^{\alpha} \sim \delta_{\text{BRST}} (\partial \beta C^{\alpha} B^\beta) = 0, \]
\[ \delta_{\text{BRST}}^2 h^{\mu\nu} \sim \delta_{\text{BRST}} (D_{\alpha} C^{\alpha}) = 0. \] \quad (23)

The above transformation follows from the anticommuting nature of \( C^\alpha \), \( \delta \lambda \) and the following commutation relation of two gauge transformations generated by \( \xi^\mu \) and \( \eta^\mu \):

\[ \delta D_{\alpha} \delta_{\text{BRST}} (\xi^\alpha \eta^\beta - \eta^\alpha \xi^\beta) = \kappa D_{\alpha} (\partial \beta C^{\alpha} \eta^\alpha - \partial \alpha \eta^\alpha \xi^\alpha). \] \quad (24)

Given the second of (23), only the antighost \( \bar{C}_\tau \) transforms under the BRST transformation to cancel the variation of the gauge-fixing term. The entire effective action is BRST invariant

\[ \delta_{\text{BRST}} (S_g - \frac{1}{2} \int d^4 x \omega(-\Box^\lambda) F^\tau + \int d^4 x \bar{C}_\tau F_{\mu\nu} D^{\mu\nu\alpha} C^\alpha) = 0. \]

(25)

Let us list the mass dimension of the fields in the gauge-fixed Lagrangian: \( [h^{\mu\nu}] = \text{mass}, [C^\alpha] = \text{mass}, [\bar{C}_\tau] = \text{mass}, [\kappa] = \text{mass}^{-1} \).

Now we Taylor-expand the gravitational part of the action (11) to the second order in the gravitational perturbation \( h^{\mu\nu}(x) \) to obtain the graviton propagator. In the momentum space, the action which is purely quadratic in the gravitational field, reads

\[ S^{(2)} = \int d^4 k \ h^{\mu\nu}(k) \left( -[\beta - \beta_2 k^2 + \kappa^2 k^2 h_2(k^2/L^2)] k^2 P_{\mu\nu\rho\sigma}^{(2)}(k) + \xi^{-1} \omega(k^2/L^2) P_{\mu\nu\rho\sigma}^{(1)}(k) \right) \]
\[ + \{3 k^2 [\beta - 2 - 3\beta_3 k^2 + 3\kappa k^2 h_0(k^2/L^2)] + 2\xi^{-1} \omega(k^2/L^2) \} P_{\mu\nu\rho\sigma}^{(0-\omega)}(k) \]
\[ + k^2 [\beta - 2 - 3\beta_3 k^2 + 3\kappa^2 k^2 h_0(k^2/L^2)] \{ P_{\mu\nu\rho\sigma}^{(0-\omega)}(k) + \sqrt{3} P_{\mu\nu\rho\sigma}^{(0-\omega)}(k) \} \}
\[ h^{\rho\sigma}(k), \]

(26)

where we used the gauge \( F^\tau = \partial_{\mu} h^{\mu\tau} \) and we have introduced the projectors [12]

\[ P_{\mu\nu\rho\sigma}^{(2)}(k) = \frac{1}{2} \left( \theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho} \right) - \frac{1}{3} \theta_{\mu\nu} \theta_{\rho\sigma}, \]
\[ P_{\mu\nu\rho\sigma}^{(1)}(k) = \frac{1}{2} \left( \theta_{\mu\nu} \omega_{\rho\sigma} + \theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} \right), \]

\[ P_{\mu\nu\rho\sigma}^{(0-\omega)}(k) = \frac{1}{3} \theta_{\mu\nu} \theta_{\rho\sigma}, \]
\[ P_{\mu\nu\rho\sigma}^{(0-\omega)}(k) = \frac{1}{3} \theta_{\mu\rho} \theta_{\nu\sigma}, \]

(27)

where we defined the transverse and longitudinal pro-
jectors for vector quantities

\[ \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{k^2}, \quad \omega_{\mu\nu} = \frac{k_{\mu} k_{\nu}}{k^2}. \]  

(28)

Using the orthogonality properties of (27) we can now invert the kinetic matrix in (26) and obtain the graviton propagator. In the following expression the graviton propagator is expressed in the momentum space according to the quadratic action (26).

\[ D_{\mu\nu\rho\sigma}(k) = \frac{-i}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} \left( 2\xi_{\mu\nu\rho\sigma}^{(3)}(k) \frac{2}{\beta - \beta_2k^2} h_2(k^2/\Lambda^2) - \frac{4}{\beta - 6\beta_0k^2} h_0(k^2/\Lambda^2) \right) - \frac{2\xi\xi_{\mu\nu\rho\sigma}^{(1)}(k)}{\omega(k^2/\Lambda^2)} - \frac{\xi^2\xi_{\mu\nu\rho\sigma}^{(2)}(k) - \sqrt{3}P_{\mu\nu\rho\sigma}(k) + P_{\mu\nu\rho\sigma}^{0}(0)}{\omega(k^2/\Lambda^2)} \right). \]  

(29)

Let us consider the graviton propagator in the gauge \( \xi = 0 \). In this particular gauge, only the first two terms in (29) survive. We will show in the next section that only the physical massless spin-2 pole occurs in the propagator when the theory is renormalized at a certain scale \( \mu_0 \). The renormalization group invariance preserves unitarity in the dressed physical propagator at any energy scale and no other physical pole emerges at any other scale.

### III. RENORMALIZABILITY

In this section we want to find an upper bound to the divergences in quantum gravity; before doing this, we have to construct the entire functions \( h_2 \) and \( h_0 \). Looking at the first two gauge invariant terms in (29), we introduce the following notation

\[ \begin{align*}
\bar{h}_2(z) &= \beta - \beta_2\kappa^2\Lambda^2 z + \kappa^2\Lambda^2 z (z^2), \\
\bar{h}_0(z) &= \beta - 6\beta_0\kappa^2\Lambda^2 z + 6\kappa^2\Lambda^2 z (z^2),
\end{align*} \]  

(30)

where \( z \) will be identified with \(-\Omega\).

Considering (4), we require the following general properties for the transcendental entire functions \( \bar{h}_i (i = 2, 0)\):

(i) \( \bar{h}_i(z) \) is real and positive on the real axis, it has no zeroes on the whole complex plane \(|z| < \infty\).

This requirement implies that there are no gauge-invariant poles other than the transverse massless physical graviton pole.

(ii) \(|\bar{h}_i(z)|\) has the same asymptotic behavior along the real axis at \( \pm\infty\).

(iii) There exists \( \Theta > 0 \) such that

\[ \lim_{|z| \to \infty} |\bar{h}_i(z)| \to |z|^\gamma, \quad \gamma \geq 2 \]

for the argument of \( z \) in the cones

\[ C = \{ z | -\Theta < \arg z < +\Theta, \pi - \Theta < \arg z < \pi + \Theta \}, \]

for \( 0 < \Theta < \pi/2 \).

This condition is necessary in order to achieve the super-renormalizability of the theory. The necessary asymptotic behavior is imposed not only on the real axis, (ii) but also in conic regions surrounding the real axis. In an Euclidean spacetime, the condition (ii) is not strictly necessary if (iii) applies.

Given the above properties, let us study the ultraviolet behavior of the quantum theory. From the property (iii) in the high energy regime, the propagator in momentum space goes as \( 1/k^{2\gamma+4} \) (see (29)), but also the \( n \)-graviton interaction has the same scaling, since it can be written in the following schematic way,

\[ \mathcal{L}^{(n)} \sim h^n \square h \bar{h}_{i} (-\square) \square h \]
\[ \sim h^n \square h (\square h + h^{m} \partial h \partial)^{\gamma} \square h, \]

(31)

where \( h \) is the graviton field and \( \bar{h}_i \) is the entire function defined by the properties (i)-(iii). From (31), the superficial degree of divergence (in four spacetime dimensions) is

\[ D = 4L - (2\gamma + 4)I + (2\gamma + 4)V = 4 - 2\gamma(L - 1). \]

(32)

In (32) we used again the topological relation between vertices \( V \), internal lines \( I \) and number of loops \( L \): \( I = V + L - 1 \). Thus, if \( \gamma > 3 \), only 1-loop divergences exist and the theory is super-renormalizable. In this theory the quantities \( \beta, \beta_2, \beta_0 \) and eventually the cosmological constant are renormalized,

\[ \mathcal{L}_{\text{Ren}} = \mathcal{L} - \sqrt{-g} \left\{ \beta (Z - 1) + \lambda (Z - 1) \right\} \]
\[ - \beta_2 (Z - 1) R_{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 + \beta_0 (Z_0 - 1) R^2 \],

(33)

where all the coupling must be understood as renormalized at an energy scale \( \mu \). On the other hand, the functions \( \bar{h}_i \) are not renormalized. In order to better understand this point we can write the generic entire functions as series, \( \bar{h}_i(z) = \sum_{r=0}^{+\infty} a_r z^r \). For \( r > 1 \) there are no counterterms that renormalize \( a_r \), because of the superficial degree of divergence (32). Only the coefficient \( a_0 \) is
renormalized but this is just a normalization convention. The non-trivial dependence of the entire functions $h_i$ on their argument is preserved at quantum level.

Imposing the conditions (i)-(iii) we have the freedom to choose the following form for the functions $h_i$,

$$h_2(z) = \frac{h_2(z) - \alpha + \alpha_2 z}{\kappa^2 \Lambda^2 z^2},$$
$$h_0(z) = \frac{h_0(z) - \alpha + \alpha_0 z}{6 \kappa^2 \Lambda^2 z},$$

(34)

for three general parameters $\alpha$, $\alpha_2$ and $\alpha_0$. In general in order to be compatible with the conditions (i)-(iii), we have to take $h_i(z) = \alpha \exp H_i(z)$, where $H_i(z)$ is an entire function that exhibits logarithmic asymptotic behavior in the conical region $C$. Since $H(z)$ is an entire function, $h_i(z)$ has no zeros in all complex plane for $|z| < +\infty$. Furthermore, the non-locality in the action is actually a kind of non locality, because $h_i(z)$ is an exponential function and a Taylor expansion of $h_i(z)$ erases the denominator $\Box \Lambda$ at any energy scale. For this reason, the entire functions $h_i(z)$ can be called quasi polynomial.

The entire function $H(z)$ which satisfies the property (iii), can be defined as

$$H(z) = \int_0^{p_{\gamma+1}(z)} \frac{1 - \zeta(\omega)}{\omega} d\omega,$$

(35)

where the following requirements have to be satisfied:

a. $p_{\gamma+1}(z)$ is a real polynomial of degree $\gamma + 1$ with $p_{\gamma+1}(0) = 0$,

b. $\zeta(z)$ is an entire and real function on the real axis with $\zeta(0) = 1$,

c. $|\zeta(z)| \rightarrow 0$ for $|z| \rightarrow \infty$ in the conical region $C$ defined in (iii).

Let us assume now that the theory is renormalized at some scale $\mu_0$. If we want that the bare propagator to possess no other gauge-invariant pole than the transverse physical graviton pole, we have to set

$$\alpha = \beta(\mu_0),$$
$$\frac{\alpha_2 \kappa^2}{\Lambda} = \beta_2(\mu_0),$$
$$\frac{\alpha_0}{6 \kappa^2 \Lambda} = \beta_0(\mu_0).$$

(36)

As pointed out in [4], the relations (36) can be used to fix the introduced scale $\Lambda$ in terms of the Planck scale $\kappa^{-2}$. If we fix

$$\beta(\mu_0) = \alpha,$$
$$\frac{\beta_2(\mu_0)}{\alpha_2} = \frac{6 \beta_0(\mu_0)}{\alpha_0},$$

(37)

the two mass scales are linked by the following relation

$$\Lambda^2 = \frac{\alpha_2 \kappa}{\kappa^2 \beta_2(\mu_0)}.$$

(38)

If the energy scale $\mu_0$ is taken as the renormalization point we find $h_2 = h_0 = \beta(\mu_0) \exp H(z) := \bar{h}(z)$ and only the physical massless spin-2 graviton pole occurs in the bare propagator. In the gauge $\xi = 0$ the propagator in (29) simplifies to

$$D_{\mu\nu\rho\sigma}(k) = -\frac{i}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} \left( 2 P_{\mu\nu\rho\sigma}(k) - 4 P_{\mu\nu\rho\sigma}(0) \right)$$
$$= -\frac{i}{(2\pi)^4} \frac{e^{-H(k^2/\Lambda^2)}}{\alpha(k^2 + i\epsilon)} \left( 2 P_{\mu\nu\rho\sigma}(k) - 4 P_{\mu\nu\rho\sigma}(0) \right).$$

(39)

If we choose another renormalization scale $\mu$, then the bare propagator acquires poles; however, these poles cancel in the dressed physical propagator because the shift in the bare part is cancelled with a corresponding shift in the self energy. This follows easily from the renormalization group invariance. The same procedure is not applicable to the case $h_2(z) = h_0(z) = 0$ [1], because the theory fails to be renormalizable when the unitarity requirement $\beta_2 = \beta_0 = 0$ is imposed and an infinite tower of counterterms has to be added to the action.

An explicit example of $\bar{h}(z) = \exp H(z)$ that satisfies the properties (i)-(iii) can be easily constructed. There are of course many ways to choose $\zeta(z)$, but we focus here on the obvious exponential choice $\zeta(z) = \exp(-z^2)$, which satisfies property c. in a conical region $C$ with $\Theta = \pi/4$. The entire function $H(z)$ is the result of the integral defined in (55)

$$H(z) = \frac{1}{2} \left[ \gamma_E + \Gamma(0, p_{\gamma+1}(z)) \right] + \log[p_{\gamma+1}(z)],$$
$$\text{Re}(p_{\gamma+1}(z)) > 0,$$

(40)

where $\gamma_E = 0.577216$ is the Euler’s constant and $\Gamma(a, z) = \int_z^{+\infty} t^{a-1}e^{-t}dt$ is the incomplete gamma function. If we choose $p_{\gamma+1}(z) = z^{\gamma+1}$, $H(z)$ simplifies to:

$$H(z) = \frac{1}{2} \left[ \gamma_E + \Gamma(0, z^{2\gamma+2}) \right] + \log(z^{\gamma+1}),$$
$$\text{Re}(z^{2\gamma+2}) > 0.$$ (41)

Another equivalent expression for the entire function $H(z)$ is given by the following series

$$H(z) = \sum_{n=1}^{+\infty} (-1)^n - \frac{p_{\gamma+1}(z)^{2n}}{2n n!},$$
$$\text{Re}(p_{\gamma+1}(z)) > 0.$$ (42)

For $p_{\gamma+1}(z) = z^{\gamma+1}$ the $\Theta$ angle, which defines the cone $C$, is $\Theta = \pi/(4\gamma + 4)$. According to the above expression (42) we find the following behavior near $z = 0$ for the particular choice $p_{\gamma+1}(z) = z^{\gamma+1}$,

$$H(z) = \frac{z^{2\gamma+2}}{2} - \frac{z^{4\gamma+4}}{8} + \frac{z^{6\gamma+6}}{36} + O(z^{6\gamma+7}),$$

(43)

where the Taylor expansion is the exact one for $\text{Arg}(z) < \pi/8$ but we already have a stronger constraint on the
In the first plot $|h(z)|^2$ for $z$ real and $\alpha = \alpha_2 = 1$ (solid line). In the first plot $z \in [-10, 10]$ and in the second one $z \in [-2, 2]$. The dashed line represents the asymptotic limit for large real positive and negative values of $z$. The asymptotic behavior is $|h(z)|^2 \approx 1.8 z^6$.

$\text{cone } C, \text{ Arg}(z) < \Theta = \pi/(4\gamma + 4)$. In particular $\lim_{z \to 0} H(z) = 0^1$.

### IV. SPECTRAL DIMENSION

In this section, we calculate the spectral dimension of the spacetime at short distances, showing that the renormalizability, together with the unitarity of the theory, implies a spectral dimension smaller than one. Let us summarize the definition of spectral dimension in quantum gravity. The definition of spectral dimension is borrowed from the theory of diffusion processes on fractals $^{11}$ and easily adapted to the quantum gravity context. Let us study the Brownian motion of a test particle moving on a $d$-dimensional Riemannian manifold $\mathcal{M}$ with a fixed smooth metric $g_{\mu\nu}(x)$. The probability density for the particle to diffuse from $x'$ to $x$ during the fictitious time $T$ is the heat-kernel $K_g(x, x'; T)$, which satisfies the heat equation

$$\partial_T K_g(x, x'; T) = \Delta_g^{\text{eff}} K_g(x, x'; T)$$

where $\Delta_g^{\text{eff}}$ denotes the effective covariant Laplacian. It is the usual covariant Laplacian at low energy but it can undergo strong modification in the ultra-violet regime. In particular, we will be interested in the effective Laplacian at high energy and in relation to the flat background. The heat-kernel is a matrix element of the operator $\exp(T \Delta_g)$, acting on the real Hilbert space $L^2(\mathcal{M}, \sqrt{g} \, dt^4)$, between position eigenstates

$$K_g(x, x'; T) = \langle x' | \exp(T \Delta_g^{\text{eff}}) | x \rangle.$$  

Its trace per unit volume,

$$P_g(T) \equiv V^{-1} \int d^4x \sqrt{g(x)} \, K_g(x, x; T)$$

$$\equiv V^{-1} \text{Tr} \exp(T \Delta_g^{\text{eff}})$$

has the interpretation of an average return probability. Here $V \equiv \int d^4x \sqrt{g}$ denotes the total volume. It is well known that $P_g(T)$ possesses an asymptotic expansion for $T \to 0$ of the form $P_g(T) = (4\pi T)^{-d/2} \sum_{n=0}^{\infty} A_n T^n$. The coefficients $A_n$ have a geometric meaning, i.e. $A_0$ is the volume of the manifold and, if $d = 2$, $A_1$ is then proportional to the Euler characteristic. From the knowledge of the function $P_g(T)$ one can recover the dimensionality of the manifold as the limit for small $T$ of

$$d_s \equiv -\frac{2}{\text{dln} P_g(T)} \frac{\text{dln} T}{\text{dln} T}. \quad (49)$$

If we consider arbitrary fictitious times $T$, this quantity might depend on the scale we are probing. Formula (49) is the definition of fractal dimension we will use.

From the bare graviton propagator $^{10}$ we can easily obtain the heat-kernel and then the spectral dimension of the quantum spacetime. In short, in the momentum space the graviton propagator, omitting the tensorial structure that does not affect the spectral dimension, reads

$$D(k) \propto \frac{1}{k^2 \hbar(k^2/\Lambda^2)}.$$  

We also know that the propagator (in the coordinate space) and the heat-kernel are related by $^{13}$

$$G(x, x') = \int_0^{+\infty} dT K_g(x, x'; T)$$

$$\propto \int d^4k \, e^{ik(x-x')} \int_0^{+\infty} dT K_g(k; T), \quad (51)$$

where $G(x, x') \propto \int d^4k \exp[ik(x-x')] \, D(k)$ is the Fourier transform of $^{14}$. Given the propagator $^{15}$, it is easy to invert $^{16}$ with the heat-kernel in the momentum space,

$$K_g(k; T) \propto \exp[-k^2 \bar{h}(k^2/\Lambda^2) \, T], \quad (52)$$

$$^1$$ We can do a simple choice of the entire function $H(z)$, which gives rise to a condition stronger than (iii). If we take $H(z) = z^2$, then

$$\bar{h}_2(z) = h_0(z) = \alpha e^z.$$  

where again $z = -\Box/\Lambda^2$. Another possible choice we wish to analyze is $\bar{h}_2 = h_0 = \alpha e^z$, because of its connection with the regular Nicolini-Spallucci black holes $^{10}$. For the functions given in $^{13}$, the upper bound in $^{22}$ can be derived as a truncation of the exponentials and the result does not change: we have divergences at one loop, but the theory is finite for $L > 1$. The exponentials in $^{23}$ improve the convergence properties of the theory and the propagator is:

$$D_{\mu\nu\rho\sigma}(k) = \frac{-i}{(2\pi)^4} \frac{2 e^{-k^4/\Lambda^4}}{\alpha(k^2 + i\epsilon)} \left( P_{\mu\nu\rho\sigma}^{(2)}(k) - 2 P_{\mu\nu\rho\sigma}^{(10)}(k) \right).$$

For $\bar{h}_1(z) = \alpha \exp(z)$ the exponential in the above propagator is replaced with $\exp(-k^2/\Lambda^2)$.
which is the solution of the heat-kernel equation \( \Delta_g \) with the effective operator

\[
\Delta_{\text{eff}}^g = \tilde{h}(-\Delta_g / \Lambda^2) \Delta_g ,
\]

which goes like \((-\Delta_g)^{\gamma+1} \Delta_g\) at high energy. The necessary trace to calculate the average return probability is obtained from the Fourier transform of \( \tilde{h} \),

\[
K_g(x,x';T) \propto \int d^4k e^{-k^2 \tilde{h}(k^2/\Lambda^2) T} e^{ik(x-x')} .
\]

Now we are ready to calculate the average return probability defined in \( \text{(48)} \)

\[
P_g(T) \propto \int d^4k e^{-k^2 \tilde{h}(k^2/\Lambda^2) T} .
\]

From the requirement (iii) we know that, at high energy, \( \tilde{h}(k) \sim k^2 \) and then \( \tilde{h}(k) \sim k^{2\gamma+2} \); therefore, we can calculate the integral \( \text{(55)} \) and then the spectral dimension defined in \( \text{(49)} \) for small \( T \) will be

\[
P_g(T) \propto \frac{1}{T^{\frac{\gamma}{\gamma+2}}} \implies \quad d_s = \frac{4}{\gamma+2} .
\]

The parameter \( \gamma \geq 3 \) implies that the spectral dimension is \( d_s < 1 \), manifesting a fractal nature of the spacetime at high energy.

We can calculate the spectral dimension at all energy scales as a function of the fictitious time \( T \) using the explicit form of the entire function \( H(k^2/\Lambda^2) \) given in \( \text{(12)} \). Integrating numerically \( \text{(55)} \), we can plot directly the spectral dimension achieving the graphical result in Fig. 2.

V. BLACK HOLES

In this section, we want to solve the equation of motion coming from the renormalized theory in the case of a spherically symmetric spacetime. Let us start from the classical Lagrangian that we rewrite rearranging the parameters in a different way,

\[
\mathcal{L} = -\sqrt{-g} \left\{ \frac{\beta}{\kappa^2} R - \left( \frac{\alpha_2}{\kappa^2 \Lambda^2} \right) (R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2) \right. \\
+ \left. \left( \frac{\beta_0}{6 \kappa^2 \Lambda^2} \right) R^2 - \alpha G^\mu_\nu \tilde{h}(-\Box_{\Lambda}) - \frac{1}{\kappa^2} R_{\mu\nu} \right\} ,
\]

where \( \tilde{h}(z) := \exp H(z)^3 \). From [\( \text{3} \)], the equations of motion for the above theory up to square curvature terms are

\[
G_{\mu\nu} + O(R_{\mu\nu}^2) + O(\nabla_\mu \nabla_\nu R_{\rho\sigma}) = 8\pi G_N \tilde{h}^{-1} T_{\mu\nu} ,
\]

where we omitted the argument of \( \tilde{h}(\Box_{\Lambda}) \). Since we are going to solve the Einstein equations neglecting curvature square terms, then we have to impose the conservation \( \nabla_\mu (\tilde{h}^{-1} T_{\mu\nu}) = 0 \) in order for the theory to be compatible with the Bianchi identities. For the exact equations of motion the Bianchi identities are of course satisfied.

\[ \text{Footnote 2: For the operators introduced in the previous section, \( \exp(-\Box/\Lambda^2)^n \) \( (n = 1, 2) \), the propagator scales as} \]

\[
D(k) \propto e^{-k^2/\Lambda^2} ,
\]

and the spectral dimension goes to zero at high energy. In particular, for \( n = 1 \) the heat-kernel can be calculated analytically,

\[
K(x,x';T) \propto \frac{(x-x')^2}{\left[ 4T(x+1/\Lambda^2) \right]^2} ,
\]

as it is easy to verify by going back to the propagator \( \text{(47)} \). Now, employing Eq. \( \text{(49)} \), we find that the spectral dimension is

\[
d_s = \frac{4T}{T + 1/\Lambda^2} ,
\]

which clearly goes to zero for \( T \to 0 \) and approaches \( d_s = 4 \) for \( T \to +\infty \).
because of the diffeomorphism invariance. The condition \( \nabla_\mu (\tilde{h}^{-1} T_{\mu \nu}) = 0 \) compensates the truncation in the modified Einstein equations \( [64] \).

Our main purpose is to solve the field equations by assuming a static source, which means that the four-velocity field \( u^\mu \) has only a non-vanishing time-like component \( u^0 = (u^0, 0) \). We consider the component \( T^0_0 \) of the energy-momentum tensor for a static source of mass \( m \) in polar coordinates to be \( T^0_0 = -m \delta(r)/4\pi r^2 \) \([10],[13]\). The metric of our spacetime is assumed to be given by the usual static, spherically symmetric Schwarzschild form

\[
ds^2 = -F(r) dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2,
\]

where

\[
F(r) = 1 - \frac{2m(r)}{r}.
\]  

(62)

The effective energy density and pressures are defined by

\[
\tilde{h}^{-1} T_{\mu \nu} = \frac{G\tilde{\rho}_{\nu}}{8\pi G_N} = \text{Diag}(\rho, P_T, P_r, P_\perp).
\]

(63)

For later convenience, we temporarily adopt free-falling Cartesian-like coordinates \([17]\), we calculate the effective energy density assuming \( p_{\perp+1}(z) = z^2 \) in \( H(z) \),

\[
\rho^{\text{eff}}(\tilde{x}) := -\tilde{h}^{-1}(-\square(x)_{A}) T^0_0 = m \tilde{h}^{-1}(-\square(x)_{A}) \delta(\tilde{x})
\]

\[
= m \int \frac{d^3 k}{(2\pi)^3} e^{-i k \cdot \tilde{x}}
\]

\[
= \frac{2m}{(2\pi)^2} \int_0^{+\infty} e^{-H(r^2/r^2 \Lambda^2)} p \sin(p) dp,
\]

(64)

where \( r = |\tilde{x}| \) is the radial coordinate. Here we introduced the Fourier transform for the Dirac delta function and we also introduced a new dimensionless variable in the momentum space, \( p = k r \), where \( k \) is the physical momentum. The energy density distribution defined in \([64]\) respects spherical symmetry. We evaluated numerically the integral in \([64]\) and the resulting energy density is plotted in Fig.\( 3 \). In the low energy limit we can expand \( H(z) \) for \( z = -\square/\Lambda^2 \ll 1 \) and we can integrate analytically \([64]\)

\[
\rho'(r) = \frac{2m}{(2\pi)^2} \frac{1}{r^2} \int_0^{+\infty} e^{-p^2/(2r^2 \Lambda^2)} p \sin(p) dp.
\]

(65)

The result is really involved and the plot is given in Fig.\( 3 \) however, the Taylor expansion near \( r \approx 0 \) gives a constant leading order

\[
\rho'(r) \approx \frac{m\Lambda^3}{3227/16\Gamma(11/16)\Gamma(7/8)\Gamma(5/4)} + O(r^2).
\]

(66)

![FIG. 3: Plot of the energy density for \( m = 10 \) in Planck units assuming \( \Lambda = m_P \). The solid line is a plot of \([64]\) without any approximation, the dashed line refers to the energy density profile \([65]\) in the limit \( -\square/\Lambda^2 \ll 1 \).](image)

The covariant conservation and the additional condition, \( g_{00} = -g_{01} \), completely specify the form of \( \tilde{h}^{-1} T_{\mu \nu} \) and the Einstein’s equations reads

\[
\frac{dm(r)}{dr} = 4\pi \rho^r r^2,
\]

\[
\frac{1}{F} \frac{dF}{dr} = 2 \frac{(m(r) + 4\pi P_r^r r^2)}{r(r - 2m(r))},
\]

\[
\frac{dP_r^r}{dr} = -\frac{1}{2F} \frac{dF}{dr} (\rho^s + P_r^s) + \frac{2}{r} (P_\perp - P_r^s),\]

(67)

Because of the complicated energy density profile, it is not easy to integrate the first Einstein equation in \([67]\)

\[
\frac{m(r)}{r} = 4\pi \int_0^r dr' r'^2 \rho'(r').
\]

(68)

However, the energy density goes to zero at infinity, reproducing the asymptotic Schwarzschild spacetime with \( m(r) \approx m \) (constant). On the other hand, it is easy to calculate the energy density profile close to \( r \approx 0 \) since \( H(z) \to \log z^4 \) for \( z \to +\infty \) (or \( r \to 0 \) in \([64]\)). In this regime \( m(r) \propto m \Lambda^3 r^8 \) and for a more general monomial \( p_{\gamma+1}(z) = z^{\gamma+1} \), \( m(r) \propto m(\Lambda r)^{2\gamma+2} \). The function \( F(r) \) in the metric, close to \( r \approx 0 \), is

\[
F(r) \approx 1 - cm \Lambda^{2\gamma+2} r^{2\gamma+1},
\]

where \( c \) is a dimensionless constant.
We show now that the metric has at least two horizons, an event horizon and a Cauchy horizon. The metric interpolates two asymptotic flat regions, one at infinity and the other in \( r = 0 \), so that we can write the \( g_{rr}^{-1} = F \) component in the following way

\[
F(r) = 1 - \frac{2mf(r)}{r},
\]

where \( f(r) \to 1 \) for \( r \to \infty \), \( f(r) \propto r^{2\gamma + 2} \) for \( r \to 0 \) and \( f(r) \) does not depend on the mass \( m \). The function \( F(r) \) goes to \( -1 \) in both limits (for \( r \to +\infty \) and \( r \to 0 \)) and, since \( m \) is a multiplicative constant, we can always choose the mass \( m \) for a fixed value of the radial coordinate \( r \), such that \( F(r) \) becomes negative everywhere. From this it follows that the function \( F(r) \) must change sign at least twice. The second equation in (67) is solved by \( \rho^c = -\rho^n \) and the third one defines the transversal pressure once known the energy density \( \rho^c \). Given the lapse function \( F(r) \) in (60), we can calculate the Ricci scalar and the Kretschmann invariant

\[
R = c m \Lambda^{2\gamma + 2} (2\gamma + 2)(2\gamma + 3)r^{2\gamma - 1},
\]

\[
R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 4c^2m^2\Lambda^{4\gamma + 4} (4\gamma^4 + 4\gamma^3 + 5\gamma^2 + 4\gamma + 2)r^{4\gamma}.
\]

By evaluating the above curvature tensors at the origin one finds that they are finite for \( \gamma > 1/2 \) and in particular for the minimal super-renormalizable theory with \( \gamma \geq 3 \).

The entire function \( h(z) \) is able to tame the curvature singularity of the Schwarzschild solution at least at the truncation of the theory here analyzed. We think that the higher order corrections to the Einstein equation will not change the remarkable feature of the solutions found in this section.

Besides the analysis exposed above, we can integrate numerically the modified Einstein equation of motions (61) for the two energy densities defined respectively in (64) and (65). Using the integral form of the mass function (68), we achieve the metric component \( F(r) \) defined in (69). The numerical results are plotted in Fig.4 and Fig.5 for different values of the ADM mass \( m \). The metric function \( F(r) \) can intersect zero times, twice or more than twice the horizontal axis relative to the value of the ADM mass \( m \). This opens the possibility to have “multi-horizon black holes” as an exact solution of the equation of motions (15).

**VI. STRUCTURE OF THE INTERACTIONS**

We have already shown that the theory is well-defined and power-counting super-renormalizable. However, the calculations are not easy beyond the second order in the graviton expansion. In this section, we give a sketch of how to proceed in the graviton expansion. The reason for the plots in Fig.3 is to give an operative definition of \( \exp H(z) \). One possible approximation in the interactions is the following replacement in the graviton expansion for the minimal renormalizable theory with \( \gamma = 3 \),

\[
e^{H(z)} \approx \begin{cases} 
  z^4 & \text{for } z \gtrless 1.3, \\
  1 + \frac{8}{z} - \frac{24}{z^2} + \frac{32}{z^3} & \text{for } z \lesssim 1.3.
\end{cases}
\]

At tree level, or loop amplitudes at high energy, we can just replace \( \exp H(z) \) with \( z^4 \), at low energy with the second expansion defined in (72) for \( z \lesssim 1.3 \) and proceed in the calculation, gluing together the results in the two different regimes.

Let us recall again the classical Lagrangian,

\[
\mathcal{L} = -\sqrt{-g}\left\{ \frac{\beta}{\kappa^2}R - \left( \beta_2 - \frac{\alpha_2}{\kappa^2\Lambda^2} \right) (R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2) + \left( \beta_0 - \frac{\alpha_0}{6\kappa^2\Lambda^2} \right) R^2 - \alpha G^{\mu\nu} e^{H(-\Box)} - \frac{1}{\kappa^2} R_{\mu\nu} \right\},
\]

FIG. 4: The two plots show the function \( F(r) \), assuming the infrared energy density profile \( \rho^c \). The two plots differ in the value of the ADM mass, which is \( m = 1 \) in the first plot and \( m = 10 \) in the second plot (in Planck units, assuming the fundamental scale \( \Lambda \) to be the Planck mass). A crucial property of those black holes is the possibility to have “multi-horizon black holes” depending on the mass value. For \( m = 10 \), for example, we have six horizons according to the second plot.
where \( \exp H(z) \) is defined in \( \text{(72)} \). This Lagrangian interpolates between the Einstein-Hilbert Lagrangian at low energy and a high energy theory living in a spacetime of spectral dimension \( d_s = 4/(\gamma + 2) \).

A first approximation (but also an operative way to proceed) is to replace the interaction Lagrangian in the UV with the following truncation

\[
\mathcal{L}_{\text{UV}}^{\text{int}} \approx \frac{\alpha}{\kappa^2 \Lambda^2} \sqrt{-g} G^{\mu\nu} \left( \frac{-\Box}{\Lambda^2} \right)^\gamma R_{\mu\nu} \quad \text{for} \quad k \gtrsim \Lambda, \tag{74}
\]

(\( k \) is the energy scale) and the infra-red Lagrangian with

\[
\mathcal{L}_{\text{IR}}^{\text{int}} = -\sqrt{-g} \left( \frac{\beta}{\kappa^2} R - \left( \beta_2 - \frac{\alpha_2}{\kappa^2 \Lambda^2} \right) (R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2) + \left( \beta_0 - \frac{\alpha_0}{6 \kappa^2 \Lambda^2} \right) R^2 - \frac{\alpha}{2 \kappa^2 \Lambda^16} G^{\mu\nu} \Box^{2\gamma+1} R_{\mu\nu} \right) \tag{75}
\]

for \( k \lesssim \Lambda \). In \( \text{(74)} \) and \( \text{(75)} \) we used the same expansion of \( \text{(72)} \) but for a general value of \( \gamma \). On the other hand, the propagator is the same in both regimes and is given in \( \text{(29)} \). The three graviton interactions can be obtained performing an \( h^{\mu\nu} \) power expansion in \( \text{(74)} \) and \( \text{(75)} \). At tree level, \( n \)-points functions will be obtain by an interpolation of the amplitude calculated in the two different regimes \( k \lesssim \Lambda \) and \( k \gtrsim \Lambda \), using respectively the \( \mathcal{L}_{\text{IR}}^{\text{int}} \) and \( \mathcal{L}_{\text{UV}}^{\text{int}} \). In loop amplitudes, we should integrate the interactions terms coming from \( \mathcal{L}_{\text{IR}}^{\text{int}} \) up to \( k \lesssim \Lambda \) and the interactions coming from \( \mathcal{L}_{\text{UV}}^{\text{int}} \) in the range \( \Lambda \lesssim k < +\infty \) in the same amplitude.

VII. CONCLUSIONS

In this paper we studied a new perturbative quantum gravity theory with a "gentle non-local character." We have shown that it is possible to build a perfectly well-defined quantum gravity theory without extra poles in the graviton propagator.

The properties required for the theory introduced and studied in this paper were the following:

(i) the theory should reproduce general relativity in the infra-red limit;

(ii) black hole solutions of the classical theory have to be singularity free;

(iii) the theory should be perturbatively renormalizable or super-renormalizable or finite;

(iv) the spectral dimension should decrease at short distances;

(v) the theory has to be unitary with no other degrees of freedom than the graviton.

All the above properties are satisfied by our quite restrictive class of actions, which differ uniquely for the choice of an entire function \( H(z) \). This class of theories can not be renormalizable or finite but such theories turn out to be super-renormalizable, since only one
loop Feynman diagrams diverge, implying a renormalization of just three coupling constants. The propagator has only one pole in the graviton mass shell. The minimal super-renormalizable theory we built has spectral dimension $d_s = 4/5$ in the ultra-violet regime and is four dimensional in the infrared limit.

We have also considered a truncation of the classical theory, showing that spherically symmetric black hole solutions are singularity-free. As for the solutions in [19], we have black holes only if the mass is bigger than the Planck mass (if we assume the fundamental scale in the theory to be the Planck mass). The new black hole solutions are more properly “multi-horizon black holes,” showing a very reach spacetime structure depending on the value of the ADM mass.

Future work will focus on the following subjects:

- the cosmological singularity problem at the classical or semiclassical level. Some preliminary work has been already done in [20];
- the high curvature corrections to the black hole solutions presented in this paper. Future work will also take the following directions:
- we will consider more in detail the connection between non-locality and the fractality of the spacetime (see in particular the recent work by Calcagni in [12]);
- we will reconsider super-renormalizable gauge theories, with particular attention to the grand-unification of the fundamental interactions with or without gravity. It this work, we will try to put together ideas coming from [4] and the more recent paper [21].

Possible simplifications of the theory are:

- a scalar field theory with the same non-local structure;
- a simplification of the metric to the conformal form $g_{\mu\nu} = \Omega(x) \eta_{\mu\nu}$ and therefore a quantization of the conformal factor $\Omega(x)$.

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[1] K.S. Stelle, Phys. Rev. D **16**, 953-969 (1977).
[2] J.W. Moffat, Phys. Rev. D41, 1177-1184 (1990); B. J. Hand, J.W. Moffat, Phys. Rev. D43, 1896-1900 (1991); D. Evans, J.W. Moffat, G. Kleppe, R.P. Woodard, Phys. Rev. D43, 499-519 (1991); N.J. Cornish, Mod.Phys.Lett.A7, 1895-1904 (1992); N.J. Cornish, Int. J. Mod. Phys. A7, 6121-6158 (1992).
[3] N.J. Cornish, Mod. Phys. Lett. **A7**, 631-640 (1992); J.W. Moffat, Eur. Phys. J. Plus **126**, 43 (2011) [arXiv:1008.2182 [hep-th]].
[4] E. T. Tomboulis, [hep-th/9702146v1].
[5] A. O. Barvinsky [arXiv:1107.1463 [hep-th]]; A. O. Barvinsky, Phys. Rev. D **71** (2005) 084007 [hep-th/0501093v2]; A. O. Barvinsky, Phys. Lett. B **572** (2003) 109-116 [hep-th/0304229v3]; H. W. Hamber, (UC, Irvine) and R. M. Williams, Phys. Rev. D **72** (2003) 044026 (2005) [hep-th/0507017]; N. Arkani-Hamed, S. Dimopoulos, G. Dvali, G. Gabadadze, [hep-th/0209227v1]; G. Calcagni and G. Nardelli, Phys. Rev. D **82**, 123518 (2010) [arXiv:1004.5144v1 [hep-th]]; G. Calcagni, G. Nardelli, JHEP **1002**, 093 (2010) [arXiv:0910.2160 [hep-th]].
[6] L. Crane and L. Smolin, Nucl.Phys. B **267**, 714-757 (1986).
[7] O. Lauscher and M. Reuter, JHEP **0510**, 050 (2005); D. Benedetti, Pedro F. Machado, F. Saueressig, Mod. Phys. Lett. A24, 2233-2241 (2009) [arXiv:0901.2084 [hep-th]].
[8] I. L. Buchbinder, Sergei D. Odintsov, I. L. Shapiro, “Effective action in quantum gravity”, IOP Publishing Ltd 1992.
[9] L. Modesto, Class. Quant. Grav. **26**, 242002 (2009) [arXiv:0812.2214 [gr-qc]].
[10] D. ben-Avraham and S. Havlin, Diffusion and reactions in fractals and disordered systems, Cambridge University Press, Cambridge U.K. (2000); J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Lett. B607, 205 (2005) [hep-th/0411152].
[11] J. Ambjorn, J. Jurkiewicz and R. Loll, Phys. Rev. Lett. **95**, 171301 (2005).
[12] S. Carlip, [arXiv:0909.3329 [gr-qc]]; B. Benedetti, Phys. Rev. Lett. **102**, 111303 (2009) [arXiv:0811.1396 [hep-th]]; L. Modesto, Piero Nicolini, Phys. Rev. D81, 104040 (2010) [arXiv:0912.0220 [hep-th]]; L. Modesto, Class. Quant. Grav. **26**, 242002 (2009); L. Modesto, [arXiv:0905.1655 [gr-qc]]; F. Caravelli and L. Modesto, [arXiv:0905.2170 [gr-qc]]; E. Magliaro, C. Perini and L. Modesto, [arXiv:0911.0437 [gr-qc]]; G. Calcagni, Phys. Rev.Lett.104:251301 (2010) [arXiv:0912.3142 [hep-th]]; G. Calcagni, JHEP 1003, 120 (2010) [arXiv:1001.0577 [hep-th]]; G. Calcagni, Phys. Lett. B 697, 251-253 (2011) [arXiv:1012.1244 [hep-th]]; G. Calcagni, [arXiv:1106.0295 [hep-th]]; G. Calcagni, [arXiv:1106.5787v1 [hep-th]]; L. Anchordoqui, De Chang Dai, M. Fairbairn, G. Landsberg, D. Stojkovic, [arXiv:1003.5914 [hep-ph]]; J. R. Mureika, D. Stojkovic, Phys. Rev. Lett. 106, 101101 (2011) [arXiv:1102.3434 [gr-qc]].
[13] P. Van Nieuwenhuizen, Nuclear Physics B 60 478-492 (1973).
[14] L. Modesto, J. W. Moffat, P. Nicolini, Phys. Lett. B 695, 397-400 (2011) [arXiv:1010.0680 [gr-qc]].
[15] D. V. Vassilevich, Phys. Rept. 388, 279-360 (2003) [hep-th/0306138].

[16] A. DeBenedictis, Classical and Quantum Gravity Research, 371-426, (2008), Nova Sci. Pub. ISBN 978-1-60456-366-5 [arXiv:0711.2279v2 [gr-qc]].

[17] P. Gaete, J. A. Helayel-Neto, E. Spallucci, Phys. Lett. B 693, 155-158 (2010) [arXiv:1005.0234 [hep-ph]].

[18] N. Barnaby, N. Kamran, JHEP 0802, 008 (2008) [arXiv:0709.3963v3 [hep-th]]; G. Calcagni, M. Montobbio, G. Nardelli, Phys. Rev. D76, 126001 (2007) [arXiv:0705.3033 [hep-th]]; G. Calcagni, M. Montobbio, G. Nardelli, Phys. Lett. B662, 285-289 (2008) [arXiv:0712.2237 [hep-th]].

[19] P. Nicolini, A. Smailagic and E. Spallucci, arXiv:hep-th/0507226; P. Nicolini, J. Phys. A 38, L631 (2005) [arXiv:hep-th/0507266]; P. Nicolini, A. Smailagic and E. Spallucci, Phys. Lett. B 632, 547 (2006) [arXiv:gr-qc/0510112]; T. G. Rizzo, dimensions,” JHEP 0609, 021 (2006) [arXiv:hep-ph/0606051]; E. Spallucci, A. Smailagic and P. Nicolini, Phys. Rev. D 73, 084004 (2006) [arXiv:hep-th/0604094]; S. Ansoldi, P. Nicolini, A. Smailagic and E. Spallucci, Phys. Lett. B 645, 261 (2007) [arXiv:gr-qc/0612035]; E. Spallucci, A. Smailagic and P. Nicolini, Phys. Lett. B 670, 449 (2009) [arXiv:0801.3510 [hep-th]]; P. Nicolini, Int. J. Mod. Phys. A 24, 1229 (2009) [arXiv:0807.1939 [hep-th]]; R. Casadio and P. Nicolini, JHEP 0811, 072 (2008) [arXiv:0809.2471 [hep-th]]; I. Arraut, D. Batic and M. Nowakowski, Class. Quant. Grav. 26, 245006 (2009) [arXiv:0902.3481 [gr-qc]]; P. Nicolini and E. Spallucci, Class. Quant. Grav. 27, 015010 (2010) [arXiv:0902.4654 [gr-qc]]. A. Smailagic and E. Spallucci, Phys. Lett. B 688, 82 (2010) [arXiv:1003.3918 [hep-th]]; D. M. Gingrich, higher dimensions at the JHEP 1005, 022 (2010) [arXiv:1003.1798 [hep-th]]; I. Arraut, D. Batic and M. Nowakowski, J. Math. Phys. 51, 022503 (2010) [arXiv:1001.2226 [gr-qc]]; R. Banerjee, S. Gangopadhyay and S. K. Modak, Phys. Lett. B 686, 181 (2010) [arXiv:0911.2123 [hep-th]]; P. Nicolini, Phys. Rev. D 82, 044030 (2010) [arXiv:1005.2996 [gr-qc]]; L. Modesto, A. Randono, [arXiv:1003.1998 [hep-th]]; F. Caravelli, L. Modesto, [arXiv:1001.4364 [gr-qc]]; L. Modesto and P. Nicolini, Phys. Rev. D 82, 104035 (2010) [arXiv:1005.5605 [gr-qc]]; R. B. Mann and P. Nicolini, arXiv:1002.5096 [gr-qc].

[20] T. Biswas, T. Koivisto, A. Mazumdar, JCAP 1011, 008 (2010) [arXiv:1005.0990v2 [hep-th]].

[21] D.V. Shirkov, Phys. Part. Nucl. Lett. 7, 379-383 (2010) [arXiv:1004.1510 [hep-th]].