REFINEMENTS AND REVERSES OF FÉJER’S INEQUALITIES
FOR CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Let \( f \) be a convex function on the convex set \( C \) in a linear space and \( x, y \in C \), with \( x \neq y \). If \( p : [0, 1] \to [0, \infty) \) is Lebesgue integrable and symmetric, namely \( p(1-t) = p(t) \) for all \( t \in [0, 1] \), then

\[
0 \leq \frac{1}{2} \left[ \nabla_+ f_{x+y} (y-x) - \nabla_- f_{x+y} (y-x) \right] \int_0^1 \left| t - \frac{1}{2} \right| p(t) \, dt
\]

and

\[
0 \leq \frac{1}{2} \left[ \nabla_+ f_{y} (y-x) - \nabla_- f_{y} (y-x) \right] \int_0^1 \left( \frac{1}{2} - t - \frac{1}{2} \right) p(t) \, dt,
\]

where \( \nabla_{\pm} f (\cdot) \) are the Gâteaux lateral derivatives.

Some applications for norms and semi-inner products are also provided.

1. Introduction

Let \( X \) be a real linear space, \( x, y \in X \), \( x \neq y \) and let \([x, y] := \{(1 - \lambda) x + \lambda y, \ \lambda \in [0, 1]\}\) be the segment generated by \( x \) and \( y \). We consider the function \( f : [x, y] \to \mathbb{R} \) and the attached function \( \varphi_{(x,y)} : [0,1] \to \mathbb{R}, \varphi_{(x,y)}(t) := f \left[ (1-t) x + ty \right], \ t \in [0,1]. \)

It is well known that \( f \) is convex on \([x, y]\) iff \( \varphi_{(x,y)} \) is convex on \([0,1]\), and the following lateral derivatives exist and satisfy

(i) \( \varphi'_{-}(x,y)(s) = \nabla_{-} f_{(1-s)x+sy} (y-x), \ s \in [0,1), \)
(ii) \( \varphi'_{+}(x,y)(0) = \nabla_{+} f_{x} (y-x), \)
(iii) \( \varphi'_{-}(x,y)(1) = \nabla_{-} f_{y} (y-x), \)

where \( \nabla_{\pm} f_{x} (y) \) are the Gâteaux lateral derivatives, we recall that

\[
\nabla_{+} f_{x} (y) \ := \ \lim_{h \to 0^+} \frac{f(x + hy) - f(x)}{h},
\]

\[
\nabla_{-} f_{x} (y) \ := \ \lim_{k \to 0^-} \frac{f(x + ky) - f(x)}{k}, \ x, y \in X.
\]
The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[x, y] \subset X$:

\[(HH) \quad f \left( \frac{x + y}{2} \right) \leq \int_0^1 f[(1 - t)x + ty] \, dt \leq \frac{f(x) + f(y)}{2},\]

which easily follows by the classical Hermite-Hadamard inequality for the convex function $\varphi(x, y) : [0, 1] \rightarrow \mathbb{R}$

\[\varphi(x, y) \left( \frac{1}{2} \right) \leq \int_0^1 \varphi(x, y)(t) \, dt \leq \frac{\varphi(x, y)(0) + \varphi(x, y)(1)}{2}.\]

For other related results see the monograph on line [6]. For some recent results in linear spaces see [1], [2] and [8]-[11].

We have the following result [4] related to the first Hermite-Hadamard inequality in (HH):

**Theorem 1.** Let $X$ be a linear space, $x, y \in X, x \neq y$ and $f : [x, y] \subset X \rightarrow \mathbb{R}$ be a convex function on the segment $[x, y]$. Then for any $s \in (0, 1)$ one has the inequality

\[(1.1) \quad \frac{1}{2} \left[ (1-s)^2 \nabla_{+} f[(1-s)x + sy] (y-x) - s^2 \nabla_{-} f[(1-s)x + sy] (y-x) \right] \leq \int_0^1 f[(1 - t)x + ty] \, dt - f[(1-s)x + sy] \]

\[\leq \frac{1}{2} \left[ (1-s)^2 \nabla_{y} f(y-x) - s^2 \nabla_{+} f(x) (y-x) \right].\]

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $s = 0$ or $s = 1$.

If $f : [x, y] \rightarrow \mathbb{R}$ is as in Theorem 1 and Gâteaux differentiable in $c := (1 - \lambda) x + \lambda y, \lambda \in (0, 1)$ along the direction $y - x$, then we have the inequality:

\[(1.2) \quad \left( \frac{1}{2} - \lambda \right) \nabla f_{c}(y-x) \leq \int_0^1 f[(1 - t)x + ty] \, dt - f(c).\]

If $f$ is as in Theorem 1, then

\[(1.3) \quad 0 \leq \frac{1}{8} \left[ \nabla_{+} f_{x+y} (y-x) - \nabla_{-} f_{x+y} (y-x) \right] \]

\[\leq \int_0^1 f[(1 - t)x + ty] \, dt - f \left( \frac{x + y}{2} \right) \]

\[\leq \frac{1}{8} [\nabla_{-} f_{y} (y-x) - \nabla_{+} f_{x} (y-x)].\]

The constant $\frac{1}{8}$ is sharp in both inequalities.

Also we have the following result [5] related to the second Hermite-Hadamard inequality in (HH):

**Theorem 2.** Let $X$ be a linear space, $x, y \in X, x \neq y$ and $f : [x, y] \subset X \rightarrow \mathbb{R}$ be a convex function on the segment $[x, y]$. Then for any $s \in (0, 1)$ one has the
inequality

\[
(1.4) \quad \frac{1}{2} \left[ (1-s)^2 \nabla_+ f_{(1-s)x+sy} (y-x) - s^2 \nabla_- f_{(1-s)x+sy} (y-x) \right] \\
\leq (1-s) f(x) + s f(y) - \int_0^1 f[(1-t) x + ty] dt \\
\leq \frac{1}{2} \left[ (1-s)^2 \nabla_- f_y (y-x) - s^2 \nabla_+ f_x (y-x) \right].
\]

The constant \( \frac{1}{2} \) is sharp in both inequalities. The second inequality also holds for \( s = 0 \) or \( s = 1 \).

If \( f : [x, y] \to \mathbb{R} \) is as in Theorem 2 and Gâteaux differentiable in \( c := (1 - \lambda) x + \lambda y \), \( \lambda \in (0, 1) \) along the direction \( y - x \), then we have the inequality:

\[
(1.5) \quad \left( \frac{1}{2} - \lambda \right) \nabla f_c (y-x) \leq (1-\lambda) f(x) + \lambda f(y) - \int_0^1 f[(1-t) x + ty] dt.
\]

If \( f \) is as in Theorem 2, then

\[
(1.6) \quad 0 \leq \frac{1}{8} \left[ \nabla_+ f_{\frac{x+y}{2}} (y-x) - \nabla_- f_{\frac{x+y}{2}} (y-x) \right] \\
\leq \frac{f(x) + f(y)}{2} - \int_0^1 f[(1-t) x + ty] dt \\
\leq \frac{1}{8} \left[ \nabla_- f_y (y-x) - \nabla_+ f_x (y-x) \right].
\]

The constant \( \frac{1}{8} \) is sharp in both inequalities.

By the convexity of \( f \) we have for all \( t \in [0,1] \) that

\[
f \left( \frac{x+y}{2} \right) \leq \frac{f[(1-t) x + ty] + f[(1-t) y + tx]}{2} \leq \frac{f(x) + f(y)}{2}.
\]

If we multiply this inequality by \( p : [0,1] \to [0,\infty) \), a Lebesgue integrable function on \([0,1]\), and integrate on \([0,1]\) over \( t \in [0,1] \), then we get

\[
(1.7) \quad f \left( \frac{x+y}{2} \right) \int_0^1 p(t) dt \\
\leq \int_0^1 f[(1-t) x + ty] p(t) dt + \int_0^1 f[(1-t) y + tx] p(t) dt \\
\leq \frac{f(x) + f(y)}{2} \int_0^1 p(t) dt.
\]

By changing the variable \( s = 1-t \), then we get

\[
\int_0^1 f[(1-t) y + tx] p(t) dt = \int_0^1 f[sy + (1-s) x] p(1-s) dt
\]

and by (1.7) we obtain

\[
(1.8) \quad f \left( \frac{x+y}{2} \right) \int_0^1 p(t) dt \leq \int_0^1 f[(1-t) x + ty] \tilde{p}(t) dt \\
\leq \frac{f(x) + f(y)}{2} \int_0^1 p(t) dt,
\]

where \( \tilde{p}(t) := \frac{1}{2} [p(t) + p(1-t)], t \in [0,1] \).
If \( p \) is symmetric on \([0, 1]\), namely \( p(t) = p(1 - t) \) for \( t \in [0, 1] \), then (1.8) becomes the Féjer’s inequality

\[
(1.9) \quad f\left(\frac{x + y}{2}\right) \int_0^1 p(t) \, dt \leq \int_0^1 f((1 - t)x + ty) p(t) \, dt \leq \frac{f(x) + f(y)}{2} \int_0^1 p(t) \, dt.
\]

Motivated by the above results, we establish in this paper some refinements and reverses of Féjer’s inequalities (1.9). Some applications for norms and semi-inner products are also provided.

## 2. Refinements and Reverse Féjer Inequalities

We have:

**Theorem 3.** Let \( f \) be an convex function on \( C \) and \( x, y \in C \) with \( x \neq y \). If \( p : [0, 1] \to [0, \infty) \) is Lebesgue integrable and symmetric, namely \( p(1 - t) = p(t) \) for all \( t \in [0, 1] \), then

\[
(2.1) \quad 0 \leq \frac{1}{2} \left[ \nabla_f \frac{x + y}{2} (y - x) - \nabla_f \frac{x + y}{2} (y - x) \right] \int_0^1 \left| t - \frac{1}{2} \right| p(t) \, dt
\]

\[
\leq \int_0^1 f((1 - t)x + ty) p(t) \, dt - f\left(\frac{x + y}{2}\right) \int_0^1 p(t) \, dt
\]

\[
\leq \frac{1}{2} \left[ \nabla_f y (y - x) - \nabla_f x (y - x) \right] \left( \int_0^1 \left| t - \frac{1}{2} \right| p(t) \, dt \right).
\]

**Proof.** Let \( x, y \in C \), with \( x \neq y \). Since \( \varphi_{(x,y)} \) is differentiable everywhere on \([0,1]\) except a countable number of points, by using the integration by parts formula for Lebesgue integral, we have

\[
\int_{1/2}^1 \left( \int_t^1 p(s) \, ds \right) \varphi'_{(x,y)}(t) \, dt
\]

\[
= \left( \int_t^1 p(s) \, ds \right) \varphi_{(x,y)}(t) \left|_{1/2}^1 \right. + \int_{1/2}^1 p(t) \varphi_{(x,y)}(t) \, dt
\]

\[
= - \left( \int_{1/2}^1 p(s) \, ds \right) \varphi_{(x,y)}(1/2) + \int_{1/2}^1 p(t) \varphi_{(x,y)}(t) \, dt
\]

\[
= - \left( \int_{1/2}^1 p(s) \, ds \right) f\left(\frac{x + y}{2}\right) + \int_{1/2}^1 p(t) \varphi_{(x,y)}(t) \, dt
\]
and
\[
\int_0^{1/2} \left( \int_0^t p(s) \, ds \right) \varphi_{(x,y)}(t) \, dt
= \left( \int_0^t p(s) \, ds \right) \varphi_{(x,y)}(t) \bigg|_0^{1/2} - \int_0^{1/2} p(t) \varphi_{(x,y)}(t) \, dt
= \left( \int_0^{1/2} p(s) \, ds \right) \varphi_{(x,y)}(1/2) - \int_0^{1/2} p(t) \varphi_{(x,y)}(t) \, dt
= \left( \int_0^{1/2} p(s) \, ds \right) f \left( \frac{x + y}{2} \right) - \int_0^{1/2} p(t) \varphi_{(x,y)}(t) \, dt.
\]

By subtracting the second identity from the first, we get
\[
\int_{1/2}^1 \left( \int_0^t p(s) \, ds \right) \varphi_{(x,y)}(t) \, dt - \int_0^{1/2} \left( \int_0^t p(s) \, ds \right) \varphi_{(x,y)}(t) \, dt
= \int_{1/2}^1 p(t) \varphi_{(x,y)}(t) \, dt + \int_0^{1/2} p(t) \varphi_{(x,y)}(t) \, dt
- \left( \int_{1/2}^1 p(s) \, ds \right) f \left( \frac{x + y}{2} \right) - \left( \int_0^{1/2} p(s) \, ds \right) f \left( \frac{x + y}{2} \right).
\]

By the symmetry of \( p \) we get
\[
\int_{1/2}^1 p(s) \, ds = \int_0^{1/2} p(s) \, ds = \frac{1}{2} \int_0^1 p(s) \, ds
\]
and then we get the following identity of interest in itself
\[
(2.2) \quad \int_0^1 p(t) \varphi_{(x,y)}(t) \, dt - \int_0^1 p(s) \, ds \frac{x + y}{2}
= \int_{1/2}^1 \left( \int_0^t p(s) \, ds \right) \varphi_{(x,y)}(t) \, dt - \int_0^{1/2} \left( \int_0^t p(s) \, ds \right) \varphi_{(x,y)}(t) \, dt.
\]

By the convexity of \( \varphi_{(x,y)} \) on \([0, 1]\) we have
\[
\nabla_+ f_x \left( y - x \right) = \varphi_{+(x,y)}(0) \leq \varphi_{(x,y)}(t) \leq \varphi_{-(x,y)} \left( \frac{1}{2} \right) = \nabla_+ f_{\frac{1}{2}} \left( y - x \right),
\]
for almost every \( t \in [0, 1/2] \) and
\[
\nabla_+ f_{\frac{1}{2}} \left( y - x \right) = \varphi_{+(x,y)} \left( \frac{1}{2} \right) \leq \varphi_{(x,y)}(t) \leq \varphi_{-(x,y)}(1) = \nabla_+ f_1 \left( y - x \right),
\]
for almost every \( t \in [1/2, 1] \).

This implies that
\[
\left( \int_0^t p(s) \, ds \right) \nabla_+ f_x \left( y - x \right) \leq \left( \int_0^t p(s) \, ds \right) \varphi_{(x,y)}(t)
\leq \left( \int_0^t p(s) \, ds \right) \nabla_+ f_{\frac{1}{2}} \left( y - x \right),
\]
for almost every $t \in [0, 1/2]$ and
\[
\left( \int_t^1 p(s) \, ds \right) \nabla_x f_{x+y} (y-x) \leq \left( \int_t^1 p(s) \, ds \right) \varphi'(x,y)(t)
\leq \left( \int_t^1 p(s) \, ds \right) \nabla_y f_y (y-x),
\]
for almost every $t \in [1/2, 1]$.

By integrating these two inequalities on the corresponding intervals, we obtain
\[
\int_{1/2}^1 \left( \int_t^1 p(s) \, ds \right) dt \nabla_x f_{x+y} (y-x) \leq \int_{1/2}^1 \left( \int_t^1 p(s) \, ds \right) \varphi'(x,y)(t) \, dt
\leq \int_{1/2}^1 \left( \int_t^1 p(s) \, ds \right) dt \nabla_y f_y (y-x)
\]
and
\[
- \int_0^{1/2} \left( \int_0^t p(s) \, ds \right) dt \nabla_y f_{x+y} (y-x) \leq - \int_0^{1/2} \left( \int_0^t p(s) \, ds \right) \varphi'(x,y)(t) \, dt
\leq - \int_0^{1/2} \left( \int_0^t p(s) \, ds \right) dt \nabla_x f_x (y-x).
\]

If we add these inequalities, then we get

(2.3)
\[
\int_{1/2}^1 \left( \int_t^1 p(s) \, ds \right) dt \nabla_x f_{x+y} (y-x) - \int_0^{1/2} \left( \int_0^t p(s) \, ds \right) dt \nabla_y f_{x+y} (y-x)
\leq \int_{1/2}^1 \left( \int_t^1 p(s) \, ds \right) \varphi'(x,y)(t) \, dt - \int_0^{1/2} \left( \int_0^t p(s) \, ds \right) \varphi'(x,y)(t) \, dt
\leq \int_{1/2}^1 \left( \int_t^1 p(s) \, ds \right) dt \nabla_y f_y (y-x) - \int_0^{1/2} \left( \int_0^t p(s) \, ds \right) dt \nabla_x f_x (y-x)
\]
for all $x, y \in C$, with $x \neq y$.

Further, integrating by parts in the Lebesgue integral, we have
\[
\int_{1/2}^1 \left( \int_t^1 p(s) \, ds \right) dt = \left( \int_t^1 p(s) \, ds \right) \bigg|_{t=1/2}^1 + \int_{1/2}^1 tp(t) \, dt
= \int_{1/2}^1 tp(t) \, dt - \frac{1}{2} \int_{1/2}^1 p(s) \, ds
= \int_{1/2}^1 \left( t - \frac{1}{2} \right) p(t) \, dt
\]
and
\[
\int_0^{1/2} \left( \int_0^t p(s) \, ds \right) dt = \left( \int_0^t p(s) \, ds \right) \bigg|_{t=0}^{1/2} - \int_0^{1/2} p(t) \, t \, dt
= \frac{1}{2} \int_0^{1/2} p(s) \, ds - \int_0^{1/2} p(t) \, t \, dt
= \int_0^{1/2} \left( \frac{1}{2} - t \right) p(t) \, dt.
\]
We have

\[ \int_0^1 \left| t - \frac{1}{2} \right| p(t) \, dt = \int_{1/2}^1 \left( t - \frac{1}{2} \right) p(t) \, dt + \int_0^{1/2} \left( \frac{1}{2} - t \right) p(t) \, dt. \]

Since \( p \) is symmetric on \([0, 1]\), hence by changing the variable \( s = 1 - t \), we have

\[ \int_{1/2}^1 \left( \frac{1}{2} - t \right) p(t) \, dt = \int_{1/2}^1 \left( s - \frac{1}{2} \right) p(s) \, ds = \int_{1/2}^1 \left( t - \frac{1}{2} \right) p(t) \, dt, \]

which shows that

\[ \int_{1/2}^1 \left( t - \frac{1}{2} \right) p(t) \, dt = \int_0^{1/2} \left( \frac{1}{2} - t \right) p(t) \, dt = \frac{1}{2} \int_0^1 \left| t - \frac{1}{2} \right| p(t) \, dt. \]

By utilising (2.3) we then obtain (2.1). \( \square \)

**Remark 1.** If we put \( p \equiv 1 \) in (2.1), then we recapture the earlier result (1.3). If we take \( p(t) = \left| t - \frac{1}{2} \right|, \, t \in [0, 1] \) then we get

\[ 0 \leq \frac{1}{24} \left[ \nabla f_{x+x} \left( y - x \right) - \nabla f_{x+x} \left( y - x \right) \right] \]

\[ \leq \int_0^1 f ((1 - t) x + ty) \left| t - \frac{1}{2} \right| dt - \frac{1}{4} f \left( \frac{x + y}{2} \right) \]

\[ \leq \frac{1}{24} \left[ \nabla f_y \left( y - x \right) - \nabla f_x \left( y - x \right) \right]. \]

We also have:

**Theorem 4.** Let \( f \) be an convex function on \( C \) and \( x, y \in C \), with \( x \neq y \). If \( p : [0, 1] \to [0, \infty) \) is Lebesgue integrable and symmetric, namely \( p(1 - t) = p(t) \) for all \( t \in [0, 1] \), then

\[ 0 \leq \frac{1}{2} \left[ \nabla f_{x+x} \left( y - x \right) - \nabla f_{x+x} \left( y - x \right) \right] \int_0^1 \left( \frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t) \, dt \]

\[ \leq \frac{f(x) + f(y)}{2} \int_0^1 \left( 1 - t \right) p(t) \, dt - \int_0^1 f ((1 - t) x + ty) p(t) \, dt \]

\[ \leq \frac{1}{2} \left[ \nabla f_y \left( y - x \right) - \nabla f_x \left( y - x \right) \right] \int_0^1 \left( \frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t) \, dt. \]
Proof. Using the integration by parts for Lebesgue’s integral, we have

\[
\begin{aligned}
&\int_0^1 \left( \int_0^t p(s) \, ds - \frac{1}{2} \int_0^1 p(s) \, ds \right) \varphi'_{(x,y)}(t) \, dt \\
= &\left( \int_0^1 p(s) \, ds - \frac{1}{2} \int_0^1 p(s) \, ds \right) \left( \varphi_{(x,y)}(t) \right)_0^1 - \int_0^1 p(t) \varphi_{(x,y)}(t) \, dt \\
= &\left( \int_0^1 p(s) \, ds - \frac{1}{2} \int_0^1 p(s) \, ds \right) \varphi_{(x,y)}(1) + \left( \int_0^1 p(s) \, ds \right) \varphi_{(x,y)}(0) \\
&- \int_0^1 p(t) \varphi_{(x,y)}(t) \, dt \\
= &\left( \int_0^1 p(t) \, dt \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) \varphi_{(x,y)}(t) \, dt.
\end{aligned}
\]

We also have

\[
\begin{aligned}
&\int_0^1 \left( \int_0^t p(s) \, ds - \frac{1}{2} \int_0^1 p(s) \, ds \right) \varphi'_{(x,y)}(t) \, dt \\
= &\int_0^1 \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) \varphi'_{(x,y)}(t) \, dt \\
= &\int_0^{1/2} \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) \varphi'_{(x,y)}(t) \, dt \\
&+ \int_0^{1/2} \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) \varphi'_{(x,y)}(t) \, dt \\
= &\int_0^{1/2} \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) \varphi'_{(x,y)}(t) \, dt \\
&- \int_0^{1/2} \left( \int_0^{1/2} p(s) \, ds - \int_0^t p(s) \, ds \right) \varphi'_{(x,y)}(t) \, dt.
\end{aligned}
\]

Observe that

\[
\int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \geq 0 \text{ for } t \in [1/2, 1]
\]

and

\[
\int_0^{1/2} p(s) \, ds - \int_0^t p(s) \, ds \geq 0 \text{ for } t \in [0, 1/2].
\]

By the convexity of \( \varphi_{(x,y)} \) on the interval \([0, 1]\), we deduce

\[
\begin{aligned}
&\int_{1/2}^1 \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) dt \nabla_x f_{x+y} (y - x) \\
\leq &\int_{1/2}^1 \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) \varphi'_{(x,y)}(t) \, dt \\
\leq &\int_{1/2}^1 \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) dt \nabla_y - f_y (y - x)
\end{aligned}
\]
and

\[-\int_0^{1/2} \left( \int_0^{1/2} p(s) \, ds - \int_0^t p(s) \, ds \right) dt \nabla - f_{x+y} (y - x) \]

\[\leq -\int_0^{1/2} \left( \int_0^{1/2} p(s) \, ds - \int_0^t p(s) \, ds \right) \varphi'_{(x,y)} (t) \, dt \]

\[\leq -\int_0^{1/2} \left( \int_0^{1/2} p(s) \, ds - \int_0^t p(s) \, ds \right) dt \nabla + f_x (y - x).\]

If we add these inequalities, then we get

\[(2.6) \quad \int_{1/2}^1 \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) dt \nabla + f_{x+y} (y - x) \]

\[-\int_0^{1/2} \left( \int_0^{1/2} p(s) \, ds - \int_0^t p(s) \, ds \right) dt \nabla - f_{x+y} (y - x) \]

\[\leq \int_{1/2}^1 \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) \varphi'_{(x,y)} (t) \, dt \]

\[-\int_{1/2}^1 \left( \int_0^{1/2} p(s) \, ds - \int_0^t p(s) \, ds \right) \varphi'_{(x,y)} (t) \, dt \]

\[\leq \int_{1/2}^1 \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) dt \nabla - f_y (y - x) \]

\[-\int_0^{1/2} \left( \int_0^{1/2} p(s) \, ds - \int_0^t p(s) \, ds \right) dt \nabla + f_x (y - x).\]

Since

\[\int_{1/2}^1 \left( \int_0^t p(s) \, ds - \int_0^{1/2} p(s) \, ds \right) dt \]

\[= \int_{1/2}^1 \left( \int_0^t p(s) \, ds \right) dt - \frac{1}{2} \int_0^{1/2} p(s) \, ds \]

\[= \left( \int_0^t p(s) \, ds \right) t \right]_{1/2}^1 - \int_{1/2}^1 tp(t) \, dt - \frac{1}{2} \int_0^{1/2} p(s) \, ds \]

\[\quad \geq \int_0^1 p(s) \, ds - \frac{1}{2} \int_0^{1/2} p(s) \, ds - \int_{1/2}^1 tp(t) \, dt - \frac{1}{2} \int_0^{1/2} p(s) \, ds \]

\[\quad = \int_0^1 p(s) \, ds - \int_{1/2}^{1/2} p(s) \, ds - \int_{1/2}^1 tp(t) \, dt \]

\[\quad = \int_{1/2}^1 p(s) \, ds - \int_0^{1/2} p(s) \, ds - \int_{1/2}^1 tp(t) \, dt \]

\[\quad = \int_{1/2}^1 p(s) \, ds - \int_{1/2}^1 tp(t) \, dt = \int_{1/2}^1 (1 - t) p(t) \, dt \]
and
\[
\int_0^{1/2} \left( \int_0^{1/2} p(s) \, ds - \int_0^t p(s) \, ds \right) \, dt \\
= \frac{1}{2} \int_0^{1/2} p(s) \, ds - \frac{1}{2} \int_0^{1/2} \left( \int_0^t p(s) \, ds \right) \, dt \\
= \frac{1}{2} \int_0^{1/2} p(s) \, ds - \left( \left( \int_0^t p(s) \, ds \right) \bigg|_0^t - \int_0^{1/2} tp(t) \, dt \right) \\
= \frac{1}{2} \int_0^{1/2} p(s) \, ds - \frac{1}{2} \int_0^{1/2} p(s) \, ds + \int_0^{1/2} tp(t) \, dt = \int_0^{1/2} tp(t) \, dt.
\]

If we change the variable \( s = 1 - t \), then
\[
(2.7) \quad \int_0^{1/2} tp(t) \, dt = - \int_1^{1/2} (1 - s) p(1 - s) \, ds = \int_{1/2}^{1} (1 - s) p(1 - s) \, ds \\
= \int_{1/2}^{1} (1 - s) p(s) \, ds.
\]

Therefore
\[
\frac{1}{2} \int_0^{1} \left( \frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t) \, dt \\
= \frac{1}{2} \int_0^{1/2} \left( \frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t) \, dt + \frac{1}{2} \int_{1/2}^{1} \left( \frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t) \, dt \\
= \frac{1}{2} \int_0^{1/2} \left( \frac{1}{2} - \frac{1}{2} + t \right) p(t) \, dt + \frac{1}{2} \int_{1/2}^{1/2} \left( \frac{1}{2} - t + \frac{1}{2} \right) p(t) \, dt \\
= \frac{1}{2} \int_0^{1/2} tp(t) \, dt + \frac{1}{2} \int_{1/2}^{1/2} (1 - t) p(t) \, dt = \int_0^{1/2} tp(t) \, dt \text{ (by 2.7)}
\]

and by (2.6) we get the desired result (2.5).

\[\square\]

**Remark 2.** If we put \( p \equiv 1 \) in (2.5), then we recapture the earlier result (1.6). If we take \( p(t) = \left| t - \frac{1}{2} \right| \), \( t \in [0, 1] \) in (2.5), then we get
\[
(2.8) \quad 0 \leq \frac{1}{48} \left[ \nabla_+ f_{x+y} (y-x) - \nabla_- f_{x+y} (y-x) \right] \\
\leq \frac{f(x) + f(y)}{8} - \int_0^1 f \left( (1-t) x + ty \right) \left| t - \frac{1}{2} \right| \, dt \\
\leq \frac{1}{48} \left| \nabla_- f_{y} (y-x) - \nabla_+ f_{x} (y-x) \right|.
\]

3. **Examples for Norms**

Now, assume that \((X, \| \cdot \|)\) is a normed linear space. The function \( f_0 (s) = \frac{1}{2} \| x \|^2 \), \( x \in X \) is convex and thus the following limits exist
\[
(iv) \quad \langle x, y \rangle_x := \nabla_+ f_{0,y} (x) = \lim_{t \to 0^+} \frac{\| x+t y \|^2 - \| y \|^2 }{2t}; \\
(v) \quad \langle x, y \rangle_y := \nabla_- f_{0,y} (x) = \lim_{s \to 0^-} \frac{\| x+s y \|^2 - \| y \|^2 }{2s};
\]
for any \( x, y \in X \). They are called the lower and upper semi-inner products associated to the norm \( \| \cdot \| \).

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2] or [7]), assuming that \( p, q \in \{ s, i \} \) and \( p \neq q \):

(a) \( \langle x, x \rangle_p = \| x \|^2 \) for all \( x \in X \);

(\text{aaa}) \( \langle \alpha x, \beta y \rangle_p = \alpha \beta \langle x, y \rangle_p \) if \( \alpha, \beta \geq 0 \) and \( x, y \in X \);

(\text{aa}) \( \langle \alpha x + y, \beta x + y \rangle_p = \alpha \beta \langle x, x \rangle_p + \langle y, y \rangle_p + \langle y, x \rangle_p \) if \( x, y \in X \) and \( \alpha \in \mathbb{R} \);

(\text{v}) \( \langle -x, y \rangle_p = - \langle x, y \rangle_p \) for all \( x, y \in X \);

(va) \( \langle x + y, z \rangle_p \leq \| x \| \| z \| + \langle y, z \rangle_p \) for all \( x, y, z \in X \);

(vaa) The mapping \( \langle \cdot, \cdot \rangle_p \) is continuous and subadditive (superadditive) in the first variable for \( p = s \) (or \( p = i \));

(vaaa) The normed linear space \( (X, \| \cdot \|) \) is smooth at the point \( x_0 \in X \setminus \{ 0 \} \) if and only if \( \langle y, x_0 \rangle_s = \langle y, x_0 \rangle_i \) for all \( y \in X \); in general \( \langle y, x \rangle_i \leq \langle y, x \rangle_s \) for all \( x \), \( y \in X \);

(ax) If the norm \( \| \cdot \| \) is induced by an inner product \( \langle \cdot, \cdot \rangle \), then \( \langle y, x \rangle_i = \langle y, x \rangle = \langle y, x \rangle_s \) for all \( x, y \in X \).

The function \( f_r(x) = \| x \|^r \) \((x \in X \) and \( 1 \leq r < \infty \)) is also convex. Therefore, the following limits, which are related to the superior (inferior) semi-inner products,

\[
\nabla_{\pm} f_{r, y}(x) := \lim_{t \to \pm 0} \frac{\| y + tx \|^r - \| y \|^r}{t} = r \| y \|^{r-1} \lim_{t \to 0} \frac{\| y + tx \| - \| y \|}{t} = r \| y \|^{r-2} \langle x, y \rangle_{s(i)}
\]

exist for all \( x, y \in X \) whenever \( r \geq 2 \); otherwise, they exist for any \( x \in X \) and nonzero \( y \in X \). In particular, if \( r = 1 \), then the following limits

\[
\nabla_{\pm} f_{1, y}(x) := \lim_{t \to 0 \pm} \frac{\| y + tx \| - \| y \|}{t} = \frac{\langle x, y \rangle_{s(i)}}{\| y \|}
\]

exist for \( x, y \in X \) and \( y \neq 0 \).

If we write the inequalities (2.1) for the function \( f_r(x) = \| x \|^r \) \((x \in X \) and \( 1 \leq r < \infty \)) \(, \) then we get

\[
0 \leq \frac{1}{2} r \left\{ \frac{x + y}{2} \right\}^{r-2} \left[ \left( y - \frac{x + y}{2} \right) s - \left( y - \frac{x + y}{2} \right) i \right] \\
\times \int_0^1 \left( t - \frac{1}{2} \right) p(t) \, dt \\
\leq \int_0^1 \| (1 - t) x + ty \|^r p(t) \, dt - \left\{ \frac{x + y}{2} \right\}^r \int_0^1 p(t) \, dt \\
\leq \frac{1}{2} r \left[ \| y \|^{r-2} \langle y - x \rangle_s - \| x \|^{r-2} \langle y - x \rangle_i \right] \\
\times \int_0^1 \left( t - \frac{1}{2} \right) p(t) \, dt,
\]

for any \( p : [0, 1] \to [0, \infty) \) \( \) a Lebesgue integrable and symmetric function.
If \( r \geq 2 \), then the inequality (3.1) holds for any \( x, y \in X \). If \( r \in [1, 2) \), then the inequality (3.1) holds for any \( x, y \in X \) with \( x, y, x + y \neq 0 \).

If we take \( r = 2 \), then we get the simpler inequality

\[
0 \leq \left( y - x, \frac{x + y}{2} \right)_s - \left( y - x, \frac{x + y}{2} \right)_i \int_0^1 \left| t - \frac{1}{2} \right| p(t) \, dt
\]

\[
\leq \int_0^1 \| (1 - t) x + ty \|^{r-2} p(t) \, dt - \left\| \frac{x + y}{2} \right\|^{r-2} \int_0^1 p(t) \, dt
\]

\[
\leq \left( \langle y - x, y \rangle_s - \langle y - x, x \rangle_i \right) \left( q \int_0^1 t - \frac{1}{2} \right) p(t) \, dt,
\]

for any \( x, y \in X \).

If we write the inequalities (2.5) for the function \( f_r(x) = \| x \|^r \) (\( x \in X \) and \( 1 \leq r < \infty \)), then for any \( p : [0, 1] \rightarrow [0, \infty) \) a Lebesgue integrable and symmetric function we get

\[
0 \leq \frac{1}{2} r \left\| \frac{x + y}{2} \right\|^{r-2} \left( \langle y - x, \frac{x + y}{2} \rangle - \langle y - x, \frac{x + y}{2} \rangle_i \right)
\]

\[
\times \int_0^1 \left( \frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t) \, dt
\]

\[
\leq \frac{1}{2} r \left( \| y \|^{r-2} \langle y - x, y \rangle_s - \| x \|^{r-2} \langle y - x, x \rangle_i \right)
\]

\[
\times \int_0^1 \left( \frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t) \, dt.
\]

If \( r \geq 2 \), then the inequality (3.3) holds for any \( x, y \in X \). If \( r \in [1, 2) \), then the inequality (3.3) holds for any \( x, y \in X \) with \( x, y, x + y \neq 0 \).

If \( (H; \langle \cdot, \cdot \rangle) \) is a real inner product space and \( p : [0, 1] \rightarrow [0, \infty) \) a Lebesgue integrable and symmetric function on \([0, 1]\), then for (3.1) we have

\[
0 \leq \int_0^1 \| (1 - t) x + ty \|^{r} p(t) \, dt - \left\| \frac{x + y}{2} \right\|^{r} \int_0^1 p(t) \, dt
\]

\[
\leq \frac{1}{2} r \left( \langle y - x, y \rangle_s^{r-2} - \langle x \rangle^{r-2} x \right) \int_0^1 \left| t - \frac{1}{2} \right| p(t) \, dt,
\]

while from (3.3) we get

\[
0 \leq \frac{\| x \|^{r} + \| y \|^{r}}{2} \int_0^1 p(t) \, dt - \int_0^1 \| (1 - t) x + ty \|^{r} p(t) \, dt
\]

\[
\leq \frac{1}{2} r \left( \langle y - x, y \rangle^{r-2} - \langle x \rangle^{r-2} x \right) \int_0^1 \left( \frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t) \, dt.
\]

In particular, for \( r = 2 \), we derive the simpler inequalities

\[
0 \leq \int_0^1 \| (1 - t) x + ty \|^{2} p(t) \, dt - \left\| \frac{x + y}{2} \right\|^{2} \int_0^1 p(t) \, dt
\]

\[
\leq \| y - x \|^{2} \int_0^1 \left| t - \frac{1}{2} \right| p(t) \, dt,
\]
while from (3.3) we get
\[
0 \leq \frac{\|x\|^2 + \|y\|^2}{2} \int_0^1 p(t) \, dt - \int_0^1 \| (1-t) x + ty \|^2 \, p(t) \, dt \\
\leq \|y - x\|^2 \int_0^1 \left( \frac{1}{2} - \left| t - \frac{1}{2} \right| \right) \, p(t) \, dt
\]
for all $x, y \in H$.

4. Examples for Functions of Several Variables

Now, let $\Omega \subset \mathbb{R}^n$ be an open convex set in $\mathbb{R}^n$. If $F : \Omega \to \mathbb{R}$ is a differentiable convex function on $\Omega$, then, obviously, for any $\bar{c} \in \Omega$ we have
\[
\nabla F_{\bar{c}}(\bar{y}) = \sum_{i=1}^n \frac{\partial F(\bar{c})}{\partial x_i} \cdot y_i, \, \bar{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n,
\]
where $\frac{\partial F}{\partial x_i}$ are the partial derivatives of $F$ with respect to the variable $x_i$ ($i = 1, \ldots, n$).

Using the inequalities (2.1), we get for all $\bar{a}, \bar{b} \in \Omega$ and $p : [0,1] \to [0, \infty)$ a Lebesgue integrable and symmetric function on $[0,1]$ that
\[
0 \leq \int_0^1 F((1-t)\bar{a} + t\bar{b}) \, p(t) \, dt - F(\frac{\bar{a} + \bar{b}}{2}) \int_0^1 p(t) \, dt \\
\leq \frac{1}{2} \left( \int_0^1 \left| t - \frac{1}{2} \right| \, p(t) \, dt \right) \sum_{i=1}^n \left( \frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i} \right) (b_i - a_i)
\]
and by (2.5) we obtain
\[
0 \leq \frac{F(\bar{a}) + F(\bar{b})}{2} \int_0^1 p(t) \, dt - \int_0^1 F((1-t)\bar{a} + t\bar{b}) \, p(t) \, dt \\
\leq \frac{1}{2} \int_0^1 \left( \frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t) \, dt \sum_{i=1}^n \left( \frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i} \right) (b_i - a_i).
\]

For $p \equiv 1$ we recapture the results obtained in [4] and [5] while for $p(t) = \left| t - \frac{1}{2} \right|$, $t \in [0,1]$ we get
\[
0 \leq \int_0^1 F((1-t)\bar{a} + t\bar{b}) \left| t - \frac{1}{2} \right| \, dt - \frac{1}{4} F\left( \frac{\bar{a} + \bar{b}}{2} \right) \\
\leq \frac{1}{24} \sum_{i=1}^n \left( \frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i} \right) (b_i - a_i)
\]
and
\[
0 \leq \frac{F(\bar{a}) + F(\bar{b})}{8} - \int_0^1 F((1-t)\bar{a} + t\bar{b}) \left| t - \frac{1}{2} \right| \, dt \\
\leq \frac{1}{48} \sum_{i=1}^n \left( \frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i} \right) (b_i - a_i)
\]
for all $\bar{a}, \bar{b} \in \Omega$. 
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