ESSENTIAL $\mathcal{F}$ SETS AND MIXING PROPERTIES

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Abstract. There is a long history of studying Ramsey theory using the algebraic structure of Stone-Čech compactification of discrete semigroup. It has been shown that various Ramsey theoretic structures are contained in different algebraic large sets. In this article we will study elementary characterization of essential $\mathcal{F}$ sets. It is known that for an $IP^*$ sets $A$ in $(\mathbb{N}, +)$ and sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$. Then there exists sum subsystem $\langle y_n \rangle_{n=1}^{\infty}$ of $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle y_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A$. In present work, We shall prove some analogous result for essential $\mathcal{F}^*$ set for some particular type of sequences. It is well known that weak mixing( central* mixing or $D^*$ mixing ) implies all order weak mixing. In this article we will prove essential $\mathcal{F}^*$ mixing implies all order mixing.

1. INTRODUCTION

The Stone-Čech compactification of the set of natural number $\mathbb{N}$ denoted by $\beta\mathbb{N}$ can be imposed with the two operations $'+'$ and '$\cdot'$ which is an extension of those operations on $\mathbb{N}$. The members of $\beta\mathbb{N}$ are the ultrafilters which are the subsets of the power sets of $\mathbb{N}$. It can be shown that $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$ are two semigroups and they contains smallest two-sided ideals denoted by $K(\beta\mathbb{N})$. The ultrafilters $p \in (K(\beta\mathbb{N}), +)$ is called additively minimal ultrafilter and $p \in (K(\beta\mathbb{N}), \cdot)$ is multiplicatively minimal.

It can be shown that there is an one to one correspondence between closed subsets of $\beta\mathbb{N}$ and certain collections of subsets of $\mathbb{N}$. This certain family of subsets is called families denoted by $\mathcal{F}$ and that closed subset will be denoted by $\beta\mathcal{F}$. Any idempotent ultrafilter $p \in \beta\mathcal{F}$, if exists, is called essential idempotent and any $A \in p$ is called essential $\mathcal{F}$ set. It can be shown that if $\mathcal{F}$ is a shift invariant family, $\beta\mathcal{F}$ is an ideal of

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$\beta \mathbb{N}$. An essential $\mathcal{F}^*$ set is the set which belongs to every idempotent in $\beta \mathcal{F}$.

There is an elementary characterization of the central sets, $C$ sets, $D$ sets etc. which shows that those sets contains a chain of sets with some $IP$ type property. In this article we will characterize the essential $\mathcal{F}$-sets in terms of $\mathcal{F}$ sets, i.e. the sets belongs to the $\mathcal{F}$-family.

The famous Hindman’s theorem says for any $A \in p$, where $p$ is an idempotent ultrafilter contains $FS\langle x_n \rangle_{n=1}^\infty$, the all possible finite sum of some sequence $\langle x_n \rangle_{n=1}^\infty$. But if we partition $\mathbb{N}$ into finitely many sets, any partition may not contain simultaneously finite sum and finite product of a sequence. it was proved that any $IP^*$, central$^*$ or $C^*$ sets will contain these type of configuration of some sequences. In this article we will generalize this statement for essential $\mathcal{F}^*$ sets.

For a measure preserving system $(X, B, \mu, T)$ the study of mixing along essential $\mathcal{F}^*$ sets can be found in [6]. Different type of mixing gives comes from different large sets associated with the algebraic structure of $\beta \mathbb{N}$, such as the recurrence of strong mixing comes from co finite sets, weak mixing comes from $D^*$ sets, mild mixing comes from $IP^*$ sets etc. It was proved due to H.Furstenberg in [8] that weak mixing implies all order weak mixing. H.Furstenberg in [8] proved mild mixing implies all order mild mixing. In this article we will show that any essential $\mathcal{F}^*$ mixing implies all order essential $\mathcal{F}^*$ mixing.

2. PRELIMINARIES

Let $S$ be a discrete semigroup. In [9], the author studied a detailed analysis of the closed ideals in $(\beta S, \cdot)$. In this article we will consider $(S, \cdot) = (\mathbb{N}, +)$.

The upward hereditary families $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ which possesses the Ramsey property and the closed subsets of $\beta \mathbb{N}$ are in one to one correspondence in nature. A collection $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is upward hereditary if whenever $A \in \mathcal{F}$ and $A \subseteq B \subseteq \mathbb{N}$ then it follows that $B \in \mathcal{F}$. A nonempty and upward hereditary collection $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ will be called a family. If $\mathcal{F}$ is a family, the dual family $\mathcal{F}^*$ is given by,

$$\mathcal{F}^* = \{ E \subseteq \mathbb{N} : \forall A \in \mathcal{F}, E \cap A \neq \emptyset \}.$$ 

A family $\mathcal{F}$ possesses the Ramsey property if whenever $A \in \mathcal{F}$ and $A = A_1 \cup A_2$ there is some $i \in \{1, 2\}$ such that $A_i \in \mathcal{F}$.

There are many families $\mathcal{F}$ with Ramsay property.

- The infinite sets,
- The piecewise syndetic sets,
- The sets of positive upper density,
- The set containing arbitrary large arithmetic progression,
The set with property that $\sum_{n \in A} \frac{1}{n} = \infty$,

- The $J$ sets,
- The $IP$ sets.

It will be easy to check that the family $\mathcal{F}$ has the Ramsey property iff the family $\mathcal{F}^*$ is a filter. For a family $\mathcal{F}$ with the Ramsey property, let $\beta(\mathcal{F}) = \{ p \in \beta\mathbb{N} : p \subseteq \mathcal{F} \}$. We state and prove the following well-known theorem:

**Theorem 2.1.** For every family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ with the Ramsey property, $\beta(\mathcal{F}) \subseteq \beta\mathbb{N}$ is closed and if $\mathcal{F}$ is translation invariant then $\beta(\mathcal{F})$ is left ideal.

**Proof.** Let $q \in \beta\mathbb{N} \setminus \beta(\mathcal{F})$. Then there is $E \subseteq \mathbb{N}$ with $E \notin \mathcal{F}$ and $E \in q$. Now $\overline{E}$ is a neighborhood of $q$ with the property that $\overline{E} \subseteq \beta\mathbb{N} \setminus \beta(\mathcal{F})$, This implies that $\beta\mathbb{N} \setminus \beta(\mathcal{F})$ is open, hence $\beta(\mathcal{F})$ is closed in $\beta\mathbb{N}$. In order to prove that $\beta(\mathcal{F})$ is a closed ideal of $\beta\mathbb{N}$, it suffices to prove that $n + \beta(\mathcal{F}) \subseteq \beta(\mathcal{F})$ for every $n \in \mathbb{N}$. Let $p \in \beta(\mathcal{F})$ and $A \in n + p$ implies that $-n + A \in p \subseteq \mathcal{F}$. Since $\mathcal{F}$ is translation invariant, we have $A \in \mathcal{F}$. This finishes the proof.

□

Now for translation invariant $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$, $\beta(\mathcal{F})$ having compact sub-semigroup in $\beta\mathbb{N}$, $\beta(\mathcal{F})$ contains at least one idempotent. From this comment, we get the following definition:

**Definition 2.2.** Let $\mathcal{F}$ be a translation invariant Ramsey family and $p$ be an idempotent in $\beta(\mathcal{F})$, then each member of $p$ is called essential $\mathcal{F}$ set. And $A \subseteq \mathbb{N}$ is called essential $\mathcal{F}^*$ set if $A$ intersects with all essential $\mathcal{F}$ sets.

Let us abbreviate the family of piecewise syndetic sets as $\mathcal{PS}$, the family of positive density sets as $\Delta$ and the family of $J$ sets as $\mathcal{J}$.

From the above definition together with the abbreviations, we get quasi central set is an essential $\mathcal{PS}$ set, $D$ set is an essential $\Delta$ set and $C$ set is an essential $\mathcal{J}$ set.

Let us revisit the algebraic operation on $\beta\mathbb{N}$ in short for our purpose. Identifying the principal ultrafilters with the points of $\mathbb{N}$ and thus pretending that $\mathbb{N} \subseteq \beta\mathbb{N}$. Given $A \subseteq \mathbb{N}$ let us set,

$$\overline{A} = \{ p \in \beta\mathbb{N} : A \subset p \}.$$  

Then the set $\{ \overline{A} : A \subseteq \mathbb{N} \}$ is a basis for a topology on $\beta\mathbb{N}$. The operation $+$ on $\mathbb{N}$ can be extended to the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ so that $(\beta\mathbb{N}, +)$ is a compact right topological semigroup (meaning that for any $p \in \beta\mathbb{N}$, the function $\rho_p : \beta\mathbb{N} \to \beta\mathbb{N}$ defined by $\rho_p(q) = q + p$ is continuous) with $\mathbb{N}$ contained in its topological center (meaning that
for any $x \in \mathbb{N}$, the function $\lambda_x : \beta \mathbb{N} \to \beta \mathbb{N}$ defined by $\lambda_x(q) = x + q$ is continuous). Given $p, q \in \beta \mathbb{N}$ and $A \subseteq \mathbb{N}$, $A \in p + q$ if and only if 
\[ \{x \in \mathbb{N} : -x + A \in q\} \in p, \text{ where } -x + A = \{y \in \mathbb{N} : x + y \in A\}. \]

A nonempty subset $I$ of a semigroup $(T, \cdot)$ is called a left ideal of $T$ if $T \cdot I \subseteq I$, a right ideal if $I \cdot T \subseteq I$, and a two sided ideal (or simply an ideal) if it is both a left and right ideal. A minimal left ideal is the left ideal that does not contain any proper left ideal. Similarly, we can define minimal right ideal and smallest ideal.

Any compact Hausdorff right topological semigroup $(T, \cdot)$ has a smallest two sided ideal $K(T) = \bigcup\{L : L$ is a minimal left ideal of $T\}$
\[ = \bigcup\{R : R$ is a minimal right ideal of $T\}. \]

Given a minimal left ideal $L$ and a minimal right ideal $R$, $L \cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is called a minimal idempotent. If $p$ and $q$ are idempotents in $T$, we write $p \leq q$ if and only if $p \cdot q = q \cdot p = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal.

3. ELEMENTARY CHARACTERIZATION OF ESSENTIAL $\mathcal{F}$ SET

In [6] established dynamical characterization of essential $\mathcal{F}$ sets and elementary characterization of essential $\mathcal{F}$ sets are still unknown. Although elementary characterization of quasi central sets and $C$ sets are known from [10] and [11] respectively. Since quasi central sets and $C$ sets are coming from by the setting of essential $\mathcal{F}$ set and this fact confines the fact that essential $\mathcal{F}$ sets might have elementary characterization. In this section we will prove the supposition that elementary characterization of essential $\mathcal{F}$-sets could be found exactly the same way what the Hindman did in [11] for $C$ set.

Let $\omega$ be the first infinite ordinal and each ordinal indicates the set of all it’s predecessor. In particular, $0 = \emptyset$, for each $n \in \mathbb{N}$, $n = \{0, 1, ..., n - 1\}.$

**Definition 3.1.** (a) If $f$ is a function and $\text{dom}(f) = n \in \omega$, then for all $x$, $f^\sim x = f \cup \{(n, x)\}$.

(b) Let $T$ be a set functions whose domains are members of $\omega$. For each $f \in T$, $\mathcal{B}_f(T) = \{x : f^\sim x \in T\}.$

**Lemma 3.2.** Let $p \in \beta \mathbb{N}$. Then $p$ is an idempotent if and only if for each $A \in p$ there is a non-empty set $T$ of functions such that

1. For all $f \in T$, $\text{dom}(f) \in \omega$ and $\text{range}(f) \subseteq A$.
2. For all $f \in T$, $\mathcal{B}_f(T) \in p$. 

(3) For all \( f \in T \) and any \( x \in B_f(T), B_{f-x}(T) \subseteq x^{-1}B_f(T) \).

**Theorem 3.3.** Let \( A \subseteq \mathbb{N} \) then, all the statements are equivalent.

(a) \( A \) is an essential \( \mathcal{F} \) set.

(b) There is a non-empty set \( T \) of function such that:

(i) For all \( f \in T, \text{domain}(f) \in \omega \) and \( \text{rang}(f) \subseteq A \).

(ii) For all \( f \in T \) and all \( x \in B_f(T), B_{f-x} \subseteq -x + B_f(T) \)

(iii) For all \( F \in \mathcal{P}_f(T), \bigcap_{f \in F} B_f(T) \) is a \( \mathcal{F} \) set.

There is an FP tree \( T \) in \( A \) such that for each \( F \in \mathcal{P}_f(T), \bigcap_{f \in F} B_f \)

is a \( \mathcal{F} \) set.

(c) There is a downward directed family \( \langle C_F \rangle_{F \in I} \) of subsets of \( A \) such that

(i) for each \( F \in I \) and each \( x \in C_F \) there exists \( G \in I \) with \( C_G \subseteq x^{-1}C_F \) and

(ii) for each \( F \in \mathcal{P}_f(I), \bigcap_{F \in F} C_F \) is a \( \mathcal{F} \)-set.

(d) There is a decreasing sequence \( \langle C_n \rangle_{n=1}^\infty \) of subsets of \( A \) such that

(i) for each \( n \in \mathbb{N} \) and each \( x \in C_n \), there exists \( m \in \mathbb{N} \) with \( C_m \subseteq x^{-1}C_n \) and

(ii) for each \( n \in \mathbb{N} \), \( C_n \) is a \( \mathcal{F} \) set.

**Proof.** (a) \( \Rightarrow \) (b). As \( A \) be an essential \( \mathcal{F} \) set, then there exists an

idempotent \( p \in \beta(\mathcal{F}) \) such that \( A \in p \). Pick a set \( T \) of functions as

guaranteed by Lemma 2.3 conclusions (i) and (ii) hold directly. Given \( F \in \mathcal{P}_f(T), B_f \in p \) for all \( f \in F \), hence \( \bigcap_{f \in F} B_f \in p \) so \( \bigcap_{f \in F} B_f \) is a \( \mathcal{F} \)-set.

(b) \( \Rightarrow \) (c). Let \( T \) be guaranteed by Let \( I = \mathcal{P}_f(T) \) and for each

\( F \in I \), let \( C_F = \bigcap_{f \in F} B_f \). Then directly each \( C_F \) is a \( \mathcal{F} \)-set. Given \( F \in \mathcal{P}_f(I) \), if \( G = \bigcup F \), then \( \bigcap_{F \in F} C_F = C_G \) and is therefore a \( \mathcal{F} \)-set. To verify (i), let \( F \in I \) and let \( x \in C_F \). Let \( G = \{ f \mapsto x : f \in F \} \). For each \( f \in F \), \( B_f \subseteq -x + B_f \) and so \( C_G \subseteq -x + C_F \).

(c) \( \Rightarrow \) (a). Let \( \langle C_F \rangle \) is guaranteed by (c). Let \( M = \bigcap_{F \in F} \overline{C_F} \). By \[14\] Theorem 4.20], \( M \) is a subsemigroup of \( \beta\mathbb{N} \). By \[14\] Theorem 3.11] there is some \( p \in \beta\mathbb{N} \) such that \( \{ C_F : F \in I \} \subseteq p \subseteq \mathcal{F} \). Therefore \( M \cap \beta(\mathcal{F}) \neq \emptyset \); and so \( M \cap \beta(\mathcal{F}) \) is a compact subsemigroup of \( \beta\mathbb{N} \). Thus there is an idempotent \( p \in M \cap \beta(\mathcal{F}) \) and so , \( A \) is an essential \( \mathcal{F} \)-set.

It is trivial that (d) implies (c). Assume now that \( S \) is countable. We shall show that (b) implies (d). So let \( T \) be as guaranteed by (b). Then \( T \) is countable so enumerate \( T \) as \( \{ f_n : n \in \mathbb{N} \} \). For \( n \in \mathbb{N} \), let \( C_n = \bigcap_{k=1}^n B_{f_k} \). Then each \( C_n \) is a \( \mathcal{F} \) set. Let \( n \in \mathbb{N} \) and let \( x \in C_n \). Pick \( m \in \mathbb{N} \) such that

\( \{ f_k \mapsto x : k \in \{1, 2, \ldots, n\} \} \subseteq \{ f_1, f_2, \ldots, f_m \} \).
Then $C_m \subseteq -x + C_n$. \hfill \Box

4. COMBINED ADDITIVE AND MULTIPLICATIVE STRUCTURE

In this section and in the next section, we consider $\mathcal{F}$ is dilation invariant with translation invariant.

Given a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$, we say that $\langle y_n \rangle_{n=1}^\infty$ is a sum subsystem of $\langle x_n \rangle_{n=1}^\infty$ provided there exists a sequence $\langle H_n \rangle_{n=1}^\infty$ of non-empty finite subset such that $\max H_n < \min H_{n+1}$ and $y_n = \sum_{t \in H_n} x_t$ for each $n \in \mathbb{N}$. In [8] Hindman and Bergelson characterized $IP^*$ sets by introducing the following theorem.

**Theorem 4.1.** Let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ and $A$ be $IP^*$ set in $(\mathbb{N}, +)$. Then there exists a subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that $FS(\langle y_n \rangle_{n=1}^\infty) \cup FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A$.

In [8], D. De showed that central* sets also possess some $IP^*$ set-like properties for some specified sequences called minimal sequence:

**Definition 4.2.** A sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ is minimal sequence if

$$\cap_{m=1}^\infty cl(FS(\langle x_n \rangle_{n=m}^\infty)) \cap K(\beta N) \neq \emptyset.$$ 

It is known that $\langle 2^n \rangle_{n=1}^\infty$ is a minimal sequence while the sequence $\langle 2^{2^n} \rangle_{n=1}^\infty$ is not a minimal sequence. In [1] it is proved that in $(\mathbb{N}, +)$ minimal sequences are nothing but those for which $FS(\langle x_n \rangle_{n=1}^\infty)$ is picewise syndetic i.e. $cl(FS(\langle x_n \rangle_{n=1}^\infty)) \cap K(\beta N) \neq \emptyset$. And in [8] D. De proved the following substantial multiplicative result of central* sets:

**Theorem 4.3.** Let $\langle x_n \rangle_{n=1}^\infty$ be a minimal sequence in $\mathbb{N}$ and $A$ be central* set in $(\mathbb{N}, +)$. Then there exists a subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that $FS(\langle y_n \rangle_{n=1}^\infty) \cup FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A$.

In [7] D. De established an analog version of the above theorem in case of $C^*$ sets for some specific type of sequences called almost minimal sequence.

**Definition 4.4.** A sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ is almost minimal sequence if

$$\cap_{m=1}^\infty cl(FS(\langle x_n \rangle_{n=m}^\infty)) \cap J(N) \neq \emptyset.$$ 

In [7], with help of N. Hindman, D. De introduced an example of almost minimal sequence which is not minimal sequence and characterized the almost minimal sequences by the following theorem:

**Theorem 4.5.** In $(\mathbb{N}, +)$ the following conditions are equivalent:

1. $\langle x_n \rangle_{n=1}^\infty$ is almost minimal sequence.
2. $FS(\langle x_n \rangle_{n=1}^\infty)$ is a $J$ set.
3. There is an idempotent in $\cap_{m=1}^\infty cl(FS(\langle x_n \rangle_{n=m}^\infty)) \cap J(N)$.
Now it is write place to state the main theorem of [7]:

**Theorem 4.6.** Let \( \langle x_n \rangle_{n=1}^{\infty} \) be a minimal sequence in \( \mathbb{N} \) and \( A \) be \( C^* \) set in \( (\mathbb{N}, +) \). Then there exists a subsystem \( \langle y_n \rangle_{n=1}^{\infty} \) of \( \langle x_n \rangle_{n=1}^{\infty} \) such that

\[
FS(\langle y_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A.
\]

As we know that \( C \) sets are essential \( J \) sets, the above theorem motives us to think some analog result for essential \( F \) sets. First let us define almost \( F \) minimal sequence.

**Definition 4.7.** A sequence \( \langle x_n \rangle_{n=1}^{\infty} \) in \( \mathbb{N} \) is almost \( F \) minimal sequence if

\[
\cap_{m=1}^{\infty} cl(FS(\langle x_n \rangle_{n=m}^{\infty})) \cap \beta(F) \neq \emptyset
\]

We can characterize almost \( F \) minimal sequences as like as almost minimal sequence given below and can be proved in the same way as the author did in [7] for almost minimal sequences:

**Theorem 4.8.** In \( (\mathbb{N}, +) \) the following conditions are equivalent:

(a) \( \langle x_n \rangle_{n=1}^{\infty} \) is almost \( F \) minimal sequence.

(b) \( FS(\langle x_n \rangle_{n=1}^{\infty}) \in q \), for some \( q \in \beta(F) \).

(c) There is an idempotent in \( \cap_{m=1}^{\infty} cl(FS(\langle x_n \rangle_{n=m}^{\infty})) \cap \beta(F) \).

**Proof.** (a) \( \Rightarrow \) (b) follows from definition.

(b) \( \Rightarrow \) (c): Since \( FS(\langle x_n \rangle_{n=1}^{\infty}) \in q \in \beta(F) \) we get

\[
\cap_{m=1}^{\infty} cl(FS(\langle x_n \rangle_{n=m}^{\infty})) \cap \beta(F) \neq \emptyset.
\]

By [14] Lemma 5.11 Choose \( \cap_{m=1}^{\infty} cl(FS(\langle x_n \rangle_{n=m}^{\infty})) \)

It will easy to see that \( \cap_{m=1}^{\infty} cl(FS(\langle x_n \rangle_{n=1}^{\infty})) \) is a closed subsemigroup of \( \beta\mathbb{N} \) and as well as \( \beta(F) \) is also closed subsemigroup of \( \beta\mathbb{N} \). Hence

\[
\cap_{m=1}^{\infty} cl(FS(\langle x_n \rangle_{n=m}^{\infty})) \cap \beta(F) \text{ is a compact subsemigroup of } (\beta\mathbb{N}, +).
\]

So it will be sufficient to check that \( \cap_{m=1}^{\infty} cl(FS(\langle x_n \rangle_{n=m}^{\infty})) \cap \beta(F) \neq \emptyset \).

Now choose arbitrarily \( m \in \mathbb{N} \) and then

\[
FS(\langle x_n \rangle_{n=1}^{\infty}) = FS(\langle x_n \rangle_{n=m}^{\infty}) \cup FS(\langle x_n \rangle_{n=1}^{m-1}) \cup \{ t + FS(\langle x_n \rangle_{n=m}^{\infty}) : t \in FS(\langle x_n \rangle_{n=1}^{m-1}) \}
\]

and so we have one of the followings:

1. \( FS(\langle x_n \rangle_{n=m}^{\infty}) \in p \)
2. \( FS(\langle x_n \rangle_{n=m-1}^{\infty}) \in p \)
3. \( t + FS(\langle x_n \rangle_{n=m}^{\infty}) \in p \) for some \( t \in FS(\langle x_n \rangle_{n=1}^{m-1}) \)

Now (1) is not possible as in that case \( p \) will be a member of principle ultrafilter. (2) holds then we have done. Now if we assume (3) holds then for some \( t \in FS(\langle x_n \rangle_{n=1}^{m-1}) \), we have \( t + FS(\langle x_n \rangle_{n=m}^{\infty}) \in p \). Choose \( q \in cl(FS(\langle x_n \rangle_{n=m}^{\infty})) \)

so that \( t + q = p \). Now for every \( F \in q, t \in \{ n \in \mathbb{N} : -n + (t + F) \in q \} \) so that \( t + F \in p \). Since \( F \)-sets are translation invariant, \( F \) is a \( F \)-sets. We have \( q \in \beta(F) \cap cl(FS(\langle x_n \rangle_{n=m}^{\infty})) \).

(c) \( \Rightarrow \) (a) follows from definition of \( F \) minimal sequence and condition (3).
To prove the main theorem of this section following to lemmas are essential.

Lemma 4.9. If $A$ be an essential $\mathcal{F}$ set in $(\mathbb{N}, +)$ then $nA$ is also an essential $\mathcal{F}$ set in $(\mathbb{N}, +)$ for any $n \in \mathbb{N}$.

Proof. If $A$ be an essential $\mathcal{F}$ set, then by elementary characterization of essential $\mathcal{F}$ set we get a sequence of $\mathcal{F}$ sets $\langle C_k \rangle_{k=1}^{\infty}$ with $A \supseteq C_1 \supseteq C_2 \supseteq \cdots$ such that for each $k \in \mathbb{N}$ and each $t \in C_k$, there exists $p \in \mathbb{N}$ with $C_p \subseteq -t + C_k$. Now consider the sequence $\langle nC_k \rangle_{k=1}^{\infty}$ of $\mathcal{F}$ sets which satisfies $nA \supseteq nC_1 \supseteq nC_2 \supseteq \cdots$ and for each $k \in \mathbb{N}$ and each $t \in nC_k$, there exists $p \in \mathbb{N}$ with $nC_p \subseteq -t + nC_k$. This proves that $nA$ is an essential $\mathcal{F}$ set in $(\mathbb{N}, +)$ for any $n \in \mathbb{N}$.

We get another lemma given below.

Lemma 4.10. If $A$ be an essential $\mathcal{F}^*$ set in $(\mathbb{N}, +)$ then $n^{-1}A$ is also a essential $\mathcal{F}^*$ set in $(\mathbb{N}, +)$ for any $n \in \mathbb{N}$.

Proof. It is sufficient to show that for any essential $\mathcal{F}$ set $B$, $B \cap n^{-1}A \neq \emptyset$. Since $B$ is essential $\mathcal{F}$ set , $nB$ is essential $\mathcal{F}$ set and $A \cap nB \neq \emptyset$. Choose $m \in A \cap nB$ and $k \in B$ such that $m = nk$. Therefore $m = nk$. Therefore $k = m/n \in n^{-1}A$ so $B \cap n^{-1}A \neq \emptyset$.

We now show that all $\mathcal{F}^*$ set have a substantial multiplicative property.

Theorem 4.11. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a $\mathcal{F}$ minimal sequence and $A$ be a in $(\mathbb{N}, +)$. Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^{\infty}$ of $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle y_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A$.

Proof. since $FS(\langle x_n \rangle_{n=1}^{\infty})$ is almost $\mathcal{F}$ minimal sequence in $\mathbb{N}$. We can find some essential idempotent $p \in \beta(\mathcal{F})$ for when $FS(\langle x_n \rangle_{n=1}^{\infty}) \in p$. Since $A$ be a $\mathcal{F}^*$ set for every $n \in \mathbb{N}$, $n^{-1}A \in p$. Let $A^* = \{n \in A : -n + A \in p\}$, then $A^* \in p$. We can choose $y_i \in A^* \cap FS(\langle x_n \rangle_{n=1}^{\infty})$. Inductively, let $m \in \mathbb{N}$ and $\langle y_i \rangle_{i=1}^{m}$, $\langle H_i \rangle_{i=1}^{m}$ in $\mathcal{P}_f(\mathbb{N})$ be chosen with the following property:

1. $i \in \{1, 2, \ldots, m-1\}$ \text{max}H_i < \text{min}H_{i+1}
2. If $y_i = \sum_{t \in H_i} x_t$ then $\sum_{t \in H_i} x_t \in A^*$ and $FS(\langle y_i \rangle_{i=1}^{m}) \subseteq A$.

We observe that $\{ \sum_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}), \text{min}H > \text{max}H_m \}$. It follows that we can choose $H_{m+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\text{min}H_{m+1} > \text{max}H_m$.\[\square\]
max\(H_m\), \(\sum_{t \in H_{m+1}} x_t \in A^*\), \(\sum_{t \in H_{m+1}} x_t \in -n + A^*\) for every \(n \in FS(\langle y_i \rangle_{i=1}^m)\) and \(\sum_{t \in H_{m+1}} x_t \in n^{-1}A^*\) for every \(n \in FS(\langle y_i \rangle_{i=1}^m)\). Putting \(y_{m+1} = \sum_{t \in H_{m+1}} x_t\). Show the induction can be continued and proves the theorem.

\[\square\]

5. MIXING IMPLIES ALL ORDER MIXING

By a measure preserving dynamical system (MDS) we mean set \((X, B, \mu, T)\), where \((X, B, \mu)\) is a Lebesgue space and \(T\) is invertable and measure preserving space from compact matric space \(X\) to \(X\). We say a measurable function \(f \in L^2(X)\) is rigid for \(T\) if for some sequence \(\langle n_k \rangle_{k=1}^\infty\) such that \(T^{n_k}f \to f\) in \(L^2(X)\). Before starting the discussion about this section, let us define strong mixing, mild mixing and weak mixing.

(1) A measure preserving dynamical system \((X, B, \mu, T)\) is weak mixing if for all \(A, B \in B\) with \(\mu(A) \mu(B) > 0\)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} | \mu(A \cap T^{-n}B) - \mu(A) \mu(B) | = 0
\]

(2) A measure preserving dynamical system \((X, B, \mu, T)\) is strong mixing if for all \(A, B \in B\) with \(\mu(A) \mu(B) > 0\)

\[
\lim_{n \to \infty} \mu(A \cap T^{-n}B) = \mu(A) \mu(B)
\]

(3) A measure preserving dynamical system \((X, B, \mu, T)\) is mild mixing if there are no non-constant rigid function \(f \in L^2(X)\).

It is interesting and motivational result that mild mixing and weak mixing can be characterized in terms of \(IP^*\) sets, central \(\star\) sets and \(D^*\) sets. We know the following theorem from [10] proved by H. Frustenberg, connects weak mixing with \(D^*\) set.

**Theorem 5.1.** The measure preserving dynamical system \((X, B, \mu, T)\) is weak mixing iff for any \(A, B \in B\) with \(\mu(A) \mu(B) > 0\) and any \(\varepsilon > 0\), the set

\[
\{ n \in \mathbb{N} : | \mu(A \cap T^{-n}B) - \mu(A) \mu(B) | < \varepsilon \}
\]

is a \(D^*\) set.

Another same potentially significant equivalent condition was proved by V. Bergelson in [3] connects weak mixing with central \(\star\) set.

**Theorem 5.2.** The measure preserving dynamical system \((X, B, \mu, T)\) is weak mixing iff for any \(A, B \in B\) with \(\mu(A) \mu(B) > 0\) and any \(\varepsilon > 0\), the set

\[
\{ n \in \mathbb{N} : | \mu(A \cap T^{-n}B) - \mu(A) \mu(B) | < \varepsilon \}
\]

is a central \(\star\) set.
Mild mixing is connected with $IP^*$ set was proved by H. Frustenberg in [10] as follow:

**Theorem 5.3.** The measure preserving dynamical system $(X, B, \mu, T)$ is Weak mixing iff for any $A, B \in B$ with $\mu(A) \mu(B) > 0$ and any $\varepsilon > 0$, the set
$$\{ n \in \mathbb{N} : | \mu(A \cap T^{-n}B) - \mu(A) \mu(B) | < \varepsilon \}$$
is an $IP^*$ set.

Since $D$ sets are essential · set, and we connect a mixing with $D$ set, naturally a question arises whether we can associate a specific mixing with a specific essential $F$-set. In [6] C. Chistoferson first introduced essential $F^*$ mixing and proved some essential results of $F^*$-mixing.

**Theorem 5.4.** The measure preserving dynamical system $(X, B, \mu, T)$ is essential $F^*$ mixing if for any $A, B \in B$ (or $f, g \in L^2(X)$) with $\mu(A) \mu(B) > 0$ and any $\varepsilon > 0$, the set
$$\{ n \in \mathbb{N} : | \mu(A \cap T^{-n}B) - \mu(A) \mu(B) | < \varepsilon \}$$
or
$$\{ n \in \mathbb{N} : | \int_X f(x) g(T^n x) d\mu(x) - \int_X f(x) d\mu(x) \int_X g(x) d\mu(x) | < \varepsilon \}$$
is an essential $F^*$ set.

Now, We know from [10], weak mixing implies all order weak mixing.

**Theorem 5.5.** The measure preserving dynamical system $(X, B, \mu, T)$ is Weak mixing iff for any $A_0, A_1, \ldots, A_k \in B$ with $\mu(A_0) \mu(A_1) \cdots \mu(A_k) > 0$ and $n_1, \ldots, n_k \in \mathbb{N}$ with $n_1 < \ldots < n_k$ any $\varepsilon > 0$, the set
$$\{ n \in \mathbb{N} : | \mu(A_0 \cap T^{-n_1}A_1 \cap \ldots \cap T^{-n_k}A_k) - \mu(A_0) \mu(A_1) \cdots \mu(A_k) | < \varepsilon \}$$
is a $D^*$ set.

Following is an analog version of mild mixing known from [10]:

**Theorem 5.6.** The measure preserving dynamical system $(X, B, \mu, T)$ is Mild mixing iff for any $A_0, A_1, \ldots, A_k \in B$ with $\mu(A_0) \mu(A_1) \cdots \mu(A_k) > 0$ and $n_1, \ldots, n_k \in \mathbb{N}$ with $n_1 < \ldots < n_k$ any $\varepsilon > 0$, the set
$$\{ n \in \mathbb{N} : | \mu(A_0 \cap T^{-n_1}A_1 \cap \ldots \cap T^{-n_k}A_k) - \mu(A_0) \mu(A_1) \cdots \mu(A_k) | < \varepsilon \}$$
is an $IP^*$ set.

From the Theorem 5.5, a question arises, whether essential $F^*$-mixing set implies all order essential $F^*$-mixing. The main result of this section is an affirmative answer of this question. The following lemma was proved in [15] by C. Schnell in a sophisticated manner and posted in the blog of J. Moreira, is main ingredient of our main result.
Lemma 5.7. Let $p$ be an idempotent ultrafilter and let $\{x_n\}_n$ be a bounded sequence in $H$ (Hilbert Space) such that for each $d \in \mathbb{N}$ we have $p \lim_n \langle x_{n+d}, x_n \rangle = 0$. Then also $p \lim_n x_n = 0$ weakly.

Proof. For each $N \in \mathbb{N}$ we have that,

$$p \lim_n x_n = p \lim_{n_1} p \lim_{n_2} \cdots p \lim_{n_N} \frac{1}{N} \sum_{k=1}^{N} x_{n_k + \cdots + n_N}$$

Taking norms and using the Cauchy-Schwartz inequality

$$\left\| p \lim_{n \to \infty} x_n \right\|^2 \leq p \lim_{n_1} p \lim_{n_2} \cdots p \lim_{n_N} \left\| \sum_{k=1}^{N} x_{n_k + \cdots + n_N} \right\|^2$$

$$= \frac{1}{N^2} \sum_{k,l=1}^{N} p \lim_{n_1} p \lim_{n_2} \cdots p \lim_{n_N} \left\langle x_{n_k + \cdots + n_N}, x_{n_l + \cdots + n_N} \right\rangle$$

$$= p \lim_{n_1} p \lim_{n_2} \cdots p \lim_{n_N} \frac{1}{N^2} \left( \sum_{k=1}^{N} x_{n_k + \cdots + n_N} \left\langle \sum_{l=1}^{N} x_{n_l + \cdots + n_N} \right\rangle \right)$$

$$= p \lim_{n_1} p \lim_{n_2} \cdots p \lim_{n_N} \frac{1}{N^2} \left( \sum_{k=1}^{N} x_{n_k + \cdots + n_N} \left\langle \sum_{l=1}^{N} x_{n_l + \cdots + n_N} \right\rangle \right)$$

$$= \frac{1}{N^2} \sum_{k=1}^{N} p \lim_{n_k} \left\| x_{n_k} \right\|^2 + \frac{2}{N^2} \sum_{k<l} p \lim_{n_k} p \lim_{n_l} \left\langle x_{n_k + n_l}, x_{n_l} \right\rangle$$

$$= \frac{1}{N} p \lim_n \left\| x_n \right\|^2$$

Since $\{N\}$ was chosen arbitrarily we conclude that $\left\| p \lim_n x_n \right\|^2 = 0$. \hfill \Box

Also the following simple lemma is same important to prove our main theorem of this section and follows from definition of essential $\mathcal{F}^*$ mixing and lemma 4.10.

Lemma 5.8. The measure preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is essential $\mathcal{F}^*$ mixing iff $(X, \mathcal{B}, \mu, T^n)$ is also essential $\mathcal{F}^*$-mixing for all $n \in \mathbb{N}$.

Now, we are in the right position of proving the main theorem of this section. The technique of this proof is traditional.

Theorem 5.9. The measure preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}^*$ mixing iff for any $A_0, A_1, \ldots, A_k \in \mathcal{B}$ with $\mu(A_0) \mu(A_1) \cdots \mu(A_k) >$
and \( n_1, \ldots, n_k \in \mathbb{N} \) with \( n_1 < \ldots < n_k \) any \( \varepsilon > 0 \), the set

\[
\{ n \in \mathbb{N} : | \mu \left( A_0 \cap T^{-n_1} A_1 \cap \ldots \cap T^{-n_k} A_k \right) - \mu (A_0) \mu (A_1) \cdots \mu (A_k) | < \varepsilon \}
\]

is an \( \mathcal{F}^* \) set.

Proof. In this theorem we prove for \( k = 2 \).

It remains to show \( p - \lim_n \mu (A_0 \cap T^{-n_1} A_1 \cap T^{-n_2} A_2) = \mu (A_0) \mu (A_1) \mu (A_2) \)

Let \( a_n (x) = 1_{A_1} (T^{n_1} x) 1_{A_2} (T^{n_2} x) - \mu (A_1) \mu (A_2) \).

We will show \( p - \lim_n a_n \) with respect to the weak topology.

Since \( T \) is strongly mixing we have

\[
p - \lim \left( p - \lim_n \int 1_{A_1} (T^{(n+m)n_1} x) 1_{A_2} (T^{(n+m)n_2} x) d\mu \right) = \mu (A_1)^2 \mu (A_2)^2
\]

\[
= p - \lim \left( p - \lim_n \int 1_{A_1} (T^{mn_1} x) 1_{A_2} (T^{mn_2} x) d\mu \right) - \mu (A_1)^2 \mu (A_2)^2
\]

\[
= p - \lim \left( \int 1_{A_1} (T^{mn_1} x) 1_{A_2} (T^{mn_2} x) d\mu \right) - \mu (A_1)^2 \mu (A_2)^2
\]

\[
= 0
\]

By Lemma the above lemma, we have \( p - \lim_n a_n = 0 \) in the weak topology. This proves the theorem.

\[\square\]

From the discussion of section 2, we know that quasi central sets and \( C \) sets are essential \( \mathcal{PS} \) sets and essential \( C \) sets respectively. So it is reasonable and practical to define quasi central* mixing and \( C^* \) mixing.

Following is the definition of quasi central* mixing:

**Theorem 5.10.** The measure preserving dynamical system \( (X, B, \mu, T) \) is quasi central* mixing if for any \( A, B \in B \) with \( \mu (A) \mu (B) > 0 \) and any \( \varepsilon > 0 \), the set \( \{ n \in \mathbb{N} : | \mu (A \cap T^{-n} B) - \mu (A) \mu (B) | < \varepsilon \} \) is a quasi central* set.

Another definition:

**Theorem 5.11.** The measure preserving dynamical system \( (X, B, \mu, T) \) is \( C^* \) mixing if for any \( A, B \in B \) with \( \mu (A) \mu (B) > 0 \) and any \( \varepsilon > 0 \), the set

\[
\{ n \in \mathbb{N} : | \mu (A \cap T^{-n} B) - \mu (A) \mu (B) | < \varepsilon \}
\]

is a \( C^* \) set.

We know that, central set \( \Rightarrow \) quasi central set \( \Rightarrow \) \( D^* \) set, so \( D^* \) set \( \Rightarrow \) quasi central* set \( \Rightarrow \) central* set. From theorem 5.1 and theorem 5.2 we
get quasi central* mixing is nothing but weak mixing. Further we know that $IP^*$ set $\Rightarrow C^*$ set $\Rightarrow$ central* set. From theorem 5.3 and theorem 5.1, we get mild mixing $\Rightarrow C^*$ mixing $\Rightarrow$ weak mixing. We don’t know whether $C^*$ mixing is strictly intermediate between mild mixing and weak mixing.

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