New 1:1:1 periodic solutions in 3-dimensional galactic-type Hamiltonian systems

Jaume Llibre · Claudio Vidal

Abstract Applying the averaging theory, we prove the existence of new families of periodic orbits for 3-dimensional type-galactic Hamiltonian systems.

Keywords Periodic orbits · Galactic-type Hamiltonian · Averaging theory

Mathematics Subject Classification Primary 34C25 · 70F15

1 Introduction and statements of main results

In this work, we prove analytically the existence of families of periodic solutions for galactic Hamiltonian systems with 3 degrees of freedom. For a good introduction to the galactic-type Hamiltonians here studied, the reader can look at the paper of Caranicolas [4] and at the references therein for a detailed deduction and implications about the importance of these Hamiltonians.

Let

\[ H = H(q_1, q_2, q_3, p_1, p_2, p_3) = \frac{1}{2} (q_1^2 + q_2^2 + q_3^2) + \frac{1}{2} (q_1^2 + q_2^2 + q_3^2) + H_1 \]

be a Hamiltonian in resonance 1 : 1 : 1, where \( H_1 \) will have two different expressions. First, we consider

\[ H_1 = H_1(q_1, q_2, q_3, p_1, p_2, p_3) = -\left( q_1^2 q_2^2 + q_1^2 q_3^2 + q_2^2 q_3^2 \right), \]

(2)

with \( q_1, q_2, q_3, p_1, p_2, p_3 \in \mathbb{R} \). As Caranicolas said, the Hamiltonian (1)–(2) have been used by several authors (see for instance, Van der Aa and Sanders [12]; Martinet and Magnenat [8]; Magnenat [7]; Martinet et al [9]; Hayli et al. [5]) for studying the local dynamical properties of galaxies. Other models for studying the motion in non-axially symmetric galaxies and disk galaxies with non-spherical nuclei have been considered in [18] and [19], respectively.

The Hamiltonian systems associated to (1)–(2) are

\[ \dot{q}_1 = p_1, \]
\[ \dot{q}_2 = p_2, \]
\[ \dot{q}_3 = p_3, \]
\[ \dot{p}_1 = -q_1 + 2q_1 \left( q_2^2 + q_3^2 \right), \]
\[ \dot{p}_2 = -q_2 + 2q_2 \left( q_1^2 + q_3^2 \right), \]
\[ \dot{p}_3 = -q_3 + 2q_3 \left( q_1^2 + q_2^2 \right). \]

(3)
As usual, the dot denotes derivative with respect to the independent variable \( t \in \mathbb{R} \), the time.

In order to apply the averaging theory to system (3), we introduce a small parameter \( \varepsilon \) doing the rescaling \((q_1, q_2, q_3, p_1, p_2, p_3) = (\sqrt{\varepsilon} X, \sqrt{\varepsilon} Y, \sqrt{\varepsilon} Z, \sqrt{\varepsilon} p_X, \sqrt{\varepsilon} p_Y, \sqrt{\varepsilon} p_Z)\). Since this change of coordinates is \( \varepsilon^{-1} \)-symplectic (see for more details [10]), the Hamiltonian function (1)–(2) in these new variables writes

\[
\mathcal{H} = \frac{1}{2}(p_X^2 + p_Y^2 + p_Z^2) + \frac{1}{2}(X^2 + Y^2 + Z^2) - \varepsilon \left( X^2 Y^2 + X^2 Z^2 + Y^2 Z^2 \right),
\]

(4)

and system (3) becomes

\[
\dot{X} = p_X, \\
\dot{Y} = p_Y, \\
\dot{Z} = p_Z, \\
\dot{p}_X = -X + 2\varepsilon X(Y^2 + Z^2), \\
\dot{p}_Y = -Y + 2\varepsilon Y(X^2 + Z^2), \\
\dot{p}_Z = -Z + 2\varepsilon Z(X^2 + Y^2).
\]

(5)

The periodic orbits are the most simple non-trivial solutions of a differential system. Their study is of particular interest because the motion in their neighborhood can be determined by their kind of stability. In general, it is very difficult to study analytically the existence of periodic orbits and their kind of stability for a given Hamiltonian system. In this work, we use the averaging theory of first order for computing periodic orbits and their kind of stability, see Appendix 1 for a summary of this theory. The averaging theory allows to find analytically periodic orbits of this galactic model (5) in any positive Hamiltonian level. Roughly speaking, this method reduces the problem of finding periodic solutions of some differential system to the one of finding zeros of some convenient finite-dimensional function.

Our main result about the periodic orbits of the Hamiltonian system (5) is summarized as follows.

**Theorem 1** For \( \varepsilon > 0 \) sufficiently small at every positive Hamiltonian level \( H = h \), the galactic-type Hamiltonian system (5) has 16 periodic solutions \((X(t, \varepsilon), Y(t, \varepsilon), Z(t, \varepsilon), p_X(t, \varepsilon), p_Y(t, \varepsilon), p_Z(t, \varepsilon))\) given by

\[
X(t, \varepsilon) = r^* \cos t + O(\varepsilon), \\
Y(t, \varepsilon) = \rho^* \cos(t + \alpha^*) + O(\varepsilon), \\
Z(t, \varepsilon) = R^* \cos(t + \beta^*) + O(\varepsilon),
\]

where \( r^*, \rho^*, R^*, \alpha^*, \beta^* \) are

(i) \( r^* = 0, \rho^* = \sqrt{h}, \alpha^* = \pi/3, 4\pi/3, \beta^* = \pi/3 \),
(ii) \( r^* = \sqrt{h}, \rho^* = 0, \alpha^* = \pi/3, \beta^* = \pi, \)
(iii) \( r^* = \sqrt{h}, \rho^* = \sqrt{h}, R^* = 0, \alpha^* = 0, \pi, \beta^* = \pi/3 \),
(iv) \( r^* = \rho^* = R^* = \sqrt{2h/3} \) and \( (\alpha^*, \beta^*) \) are given by \((\pi/3, -2\pi/3), (2\pi/3, -2\pi/3), (-2\pi/3, -\pi/3), (\pi/3, -\pi/3), (-\pi/3, \pi/3), (2\pi/3, \pi/3), (\pi/3, 2\pi/3), (\pi, 0), (0, \pi), (\pi, \pi)\).

Moreover, on each Hamiltonian level \( H = h \), the 16 periodic solutions are unstable.

Theorem 1 is proved in Sect. 2 using the averaging theory of first order. Note that the change of variables is only a scale transformation; for \( \varepsilon > 0 \), the original system (3) and the transformed systems (5) have essentially the same phase portrait. Note that system (5) for \( \varepsilon > 0 \) sufficiently small is close to an integrable one.

Caranicas in [4] using a semi-numerical method found 4 families of periodic orbits for system (5). These 4 families of periodic orbits are straight line periodic orbits living on the invariant sets \( x = \pm y = \pm z \). These 4 families correspond to the 4 families \((\alpha^*, \beta^*) \in \{(0, 0), (\pi, 0), (0, \pi), (\pi, \pi)\}\) of Theorem 1, but we additionally have obtained 12 new families of periodic solutions parametrized by \( h \).

In [4], the Hamiltonian was also studied (1) with

\[
H_1 = H_1(q_1, q_2, q_3, p_1, p_2, p_3) = -(q_1 + q_2)q_3^2.
\]

(7)

The Hamiltonian system associated to (1)–(7) is

\[
\dot{q}_1 = p_1, \\
\dot{q}_2 = p_2, \\
\dot{q}_3 = p_3, \\
\dot{p}_1 = -q_1 + q_3^2, \\
\dot{p}_2 = -q_2 + q_3^2, \\
\dot{p}_3 = -q_3 + 2(q_1 + q_2)q_3.
\]

(8)

In order to apply the averaging method of second order to system (8), we introduce a small parameter \( \varepsilon \) by
the change of the variables \((q_1, q_2, q_3, p_1, p_2, p_3) = (\varepsilon X, \varepsilon Y, \varepsilon Z, \varepsilon p_X, \varepsilon p_Y, \varepsilon p_Z)\). Since this change of coordinates is \(\varepsilon^{-2}\)-symplectic, the Hamiltonian (1)–(7) in these new variables takes the form

\[
\mathcal{H} = \frac{1}{2}(p_X^2 + p_Y^2 + p_Z^2) + \frac{1}{2}(X^2 + Y^2 + Z^2) - \varepsilon(X + Y)Z^2,
\]

and system (8) becomes

\[
\begin{align*}
\dot{X} &= p_X, \\
\dot{Y} &= p_Y, \\
\dot{Z} &= p_Z, \\
\dot{p}_X &= -X + \varepsilon Z^2, \\
\dot{p}_Y &= -Y + \varepsilon Z^2, \\
\dot{p}_Z &= -Z + 2\varepsilon(X + Y)Z.
\end{align*}
\]

Our main result about the periodic orbits of the Hamiltonian system (9) is summarized as follows.

**Theorem 2** For \(\varepsilon > 0\) sufficiently small at every positive Hamiltonian level \(H = h\), the galactic-type Hamiltonian system (9) has 8 periodic solutions \((X(t, \varepsilon), Y(t, \varepsilon), Z(t, \varepsilon), p_X(t, \varepsilon), p_Y(t, \varepsilon), p_Z(t, \varepsilon))\) given by (6) where \(r^*, \rho^*, R^*, \alpha^*, \beta^*\) are given by

\[
\begin{align*}
(i) & \quad r^* = \sqrt{h/3}, \quad \rho^* = \sqrt{h/3}, \quad R^* = 2\sqrt{h/3}, \quad \alpha^* = 0, \quad \beta^* = 0, \quad \pi; \\
(ii) & \quad r^* = \sqrt{(39 + \sqrt{17})h/94}, \quad \rho^* = \sqrt{(57 - 13\sqrt{17})h/94}, \quad R^* = 2(23 + 3\sqrt{17})h/47, \quad \alpha^* = 0, \quad \beta^* = 0, \quad \pi; \\
(iii) & \quad r^* = \sqrt{(39 - \sqrt{17})h/94}, \quad \rho^* = \sqrt{(57 + 13\sqrt{17})h/94}, \quad R^* = 2(23 - 3\sqrt{17})h/47, \quad \alpha^* = 0, \quad \beta^* = 0, \quad \pi; \\
(iv) & \quad r^* = \left(\frac{5\sqrt{113} - 729 - \sqrt{9153}}{\sqrt{308\sqrt{2}}}, \frac{\sqrt{113} - 729 + 113\sqrt{89}}{242}\right), \quad \rho^* = \left(\frac{730 - 70\sqrt{87}}{730 - 70\sqrt{87}}\right) \alpha^* = \pi, \quad \beta^* = \pm\pi/2.
\end{align*}
\]

Moreover, on each Hamiltonian level \(H = h\), the two periodic solutions given by conditions (i) are linearly stable, while the 6 ones given by conditions (ii)–(iv) are unstable.

The proof of Theorem 2 is given in Sect. 3.

Caranicolas in [4] using his semi–numerical method found 2 families of rectilinear periodic solutions on the invariant sets \(x = y = \pm z/2\). These two families correspond to the two families (i) of Theorem 2. Additionally, we obtain 6 new families of periodic solutions parametrized by the Hamiltonian level \(h\).

As we shall see one of the main problems for applying the averaging theory for studying, the periodic orbits of a given differential system are to find the changes of variables which allow to write the original differential system in the normal form for applying the averaging theory. For more details in this direction, see the book [11].

In [15–17], dynamics aspects have been studied in Hamiltonian systems in 2D and 3D; in particular, the existence of periodic orbits escapes, and chaos was considered.

## 2 Proof of Theorem 1

For proving Theorem 1, we shall apply Theorem 3 to the Hamiltonian system (5). Generically, the periodic orbits of a Hamiltonian system with more than one degree of freedom are on cylinders filled of periodic orbits parametrized by the value \(h\) of Hamiltonian level. Therefore, we cannot apply directly Theorem 3 to a Hamiltonian system, because the Jacobian of the function \(f\) defined in the Appendix 1 is always zero. Then, we must apply Theorem 3 to every Hamiltonian level where the periodic orbits generically are isolated. For more details on the families of periodic solutions of Hamiltonian systems, see [1].

First, we change the Hamiltonian (4) and the equations of motion (5) to convenient generalized polar coordinates in such a way that for \(\varepsilon = 0\), we have a pair of harmonic oscillators. Thus, we consider the change of variables,

\[
\begin{align*}
X &= r \cos \theta, \quad Y = \rho \cos(\theta + \alpha), \quad Z = R \cos(\theta + \beta), \\
p_X &= r \sin \theta, \quad p_Y = \rho \sin(\theta + \alpha), \\
p_Z &= R \sin(\theta + \beta),
\end{align*}
\]

with \((r, \theta, \rho, \alpha, R, \beta) \in \mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1\). Note that this is a change of variables when \(r, \rho, \) and \(R\) are positive, and that this change of coordinates is not canonical. So, we lose the Hamiltonian structure of the differential equations. Moreover, doing this change of variables appear in the system the angular variables \(\theta, \alpha\)
and \( \beta \). Later on, the variable \( \theta \) will be used for obtaining the periodicity necessary for applying the averaging theory.

The fixed value of the Hamiltonian level \( H = h \) in polar coordinates is

\[
h = \frac{1}{2}(r^2 + \rho^2 + R^2) + \varepsilon \left[ -r^2 \cos^2(\theta) \left( R^2 \cos^2(\beta + \theta) + \rho^2 \cos^2(\alpha + \theta) \right) - R^2 \rho^2 \cos^2(\alpha + \theta) \left( r^2 \cos^2(\beta + \theta) + \rho^2 \cos^2(\alpha + \theta) \right) \right].
\]  

(11)

and the equations of motion are given by

\[
\begin{align*}
\dot{r} &= 2R \varepsilon \rho \sin(\alpha + \theta) \cos(\alpha + \theta) (r^2 \cos^2(\theta) + \rho^2 \cos^2(\alpha + \theta)) \\
\dot{\rho} &= 2R \varepsilon \rho \sin(\alpha + \theta) \cos(\alpha + \theta) (r^2 \cos^2(\theta) + \rho^2 \cos^2(\alpha + \theta)) \\
\dot{R} &= 2R \varepsilon R \sin(\beta + \theta) \cos(\beta + \theta) (r^2 \cos^2(\theta)) \\
\dot{\theta} &= 1 - \varepsilon \rho \cos^2(\theta) R^2 \cos^2(\beta + \theta) + \rho^2 \cos^2(\alpha + \theta) \\
\dot{\rho} &= 2R \varepsilon \rho \sin(\alpha + \theta) \cos(\alpha + \theta) (r^2 \cos^2(\theta) + \rho^2 \cos^2(\alpha + \theta)) \\
\dot{\beta} &= 4R \varepsilon \rho \cos^2(\alpha + \theta) \cos^2(\beta + \theta) - \rho^2 \cos^2(\alpha + \theta) (r^2 - \rho^2) \cos^2(\theta) + \rho^2 \cos^2(\alpha + \theta) (r^2 - \rho^2) \cos^2(\theta)
\end{align*}
\]

(13)

Note that the derivatives on the left-hand side of these equations are with respect to the time \( t \), and that the system is not periodic in \( t \). For \( \varepsilon > 0 \) sufficiently small in a neighborhood of \( r = \rho = R = 0 \), we have that \( \dot{\theta} > 0 \).

In such a neighborhood, we take \( \theta \) as the new independent variable, and we denote by a prime the derivative with respect to \( \theta \). The angular variables \( \alpha, \beta \) cannot be used as the independent variable because the new differential system would not have the normal form (23) for applying the averaging theory given in Theorem 3.

By fixing the value of the first integral \( H \) at \( h \) such that \( 2h - r^2 - R^2 > 0 \), solving Eq. (11) with respect to \( \rho \), and expanding \( \rho \) in Taylor series of \( \varepsilon \), we obtain

\[
\rho = \sqrt{2h - r^2 - R^2} + \varepsilon \frac{1}{\sqrt{2h - r^2 - R^2}} \left[ 2r^2 R^2 \cos^2(\theta) \cos^2(\beta + \theta) - (2h + r^2 + R^2) \cos^2(\alpha + \theta) \left( r^2 \cos^2(\theta) + R^2 \cos^2(\beta + \theta) \right) \right] + O(\varepsilon^2).
\]

(12)

We write the differential system \((r', R', \alpha', \beta')\), substitute it in the expression of \( \rho \) given in (12), and expanding in Taylor series in \( \varepsilon \), we obtain the differential system

\[
\begin{align*}
r' &= -2R \varepsilon \sin(\theta) \left( (2h - r^2 - R^2) \cos^2(\alpha + \theta) + R^2 \cos^2(\beta + \theta) \right) + O(\varepsilon^2) \\
&= F_{11} + O(\varepsilon^2), \\
R' &= -2R \varepsilon \sin(2(\beta + \theta)) \left( (2h - r^2 - R^2) \cos^2(\alpha + \theta) + R^2 \cos^2(\beta + \theta) \right) + O(\varepsilon^2) \\
&= F_{12} + O(\varepsilon^2), \\
\alpha' &= -2R \varepsilon \sin^2(\theta) \left( (2h + 2r^2 + R^2) \cos^2(\alpha + \theta) + R^2 \sin(\alpha + 2\theta) \cos^2(\beta + \theta) \right) + O(\varepsilon^2) \\
&= F_{13} + O(\varepsilon^2), \\
\beta' &= 2R \varepsilon \cos^2(\beta + \theta) \left( (2h + 2r^2 + R^2) \cos^2(\alpha + \theta) + (R^2 - r^2) \cos^2(\theta) \right) + O(\varepsilon^2) \\
&= F_{14} + O(\varepsilon^2).
\end{align*}
\]

Clearly, system (13) satisfies the assumptions of Theorem 3, and it has the normal form (23) of the averaging theory with \( F_1 = (F_{11}, F_{12}, F_{13}, F_{14}) \).

The function \( F_1 \) is analytical. Furthermore, it is \( 2\pi \)-periodic in the variable \( \theta \), the independent variable of system (13). In order to apply the averaging theory of first order, we must calculate the averaged functions of \( f_j = f_j(r, R, \alpha, \beta) \) for \( j = 1, 2, 3, 4 \), i.e.,

\[
\begin{align*}
f_1 &= \frac{1}{2\pi} \int_0^{2\pi} F_{11} d\theta \\
&= -\frac{1}{4} \varepsilon \left[ \sin(2\alpha) \left( (2h + 2r^2 + R^2) - R^2 \sin(2\beta) \right) \right], \\
f_2 &= \frac{1}{2\pi} \int_0^{2\pi} F_{12} d\theta
\end{align*}
\]
We compute the real solutions \((r^*, \alpha^*, \beta^*)\) of the system \(f_j(r, R, \alpha, \beta) = 0\) for \(j = 1, 2, 3, 4\). It is important to remember that at order 0 in \(\epsilon\), we have \(\rho = \sqrt{2h - r^2 - R^2}\), so we must have to present that \(r, R\), and \(\rho\) cannot be simultaneously zero. Solving the first equation of (14), we obtain the following possibilities. The first case, \(r = 0\); the second case, \(\sin(2\alpha) \neq 0\) and \(r = \sqrt{\csc(2\alpha) \left[(2h - R^2)\sin(2\alpha) + R^2\sin(2\beta)\right]}\); the third case, \(\sin(2\alpha) = 0\) and \(R = 0\); and finally, the fourth case \(\sin(2\alpha) = 0\) and \(\sin(2\beta) = 0\).

**Case I** \(r = 0\). From equation \(f_2\) of (14), we have \(R \left(R^2 - 2h\right) \sin(2\alpha - \beta) = 0\).

**Subcase I.1** \(R = 0\). From equation \(f_3\) of (14), we get

\[
h[\cos(2\alpha) + 2]/2, \text{ so } r = R = 0 \text{ is not a solution.}
\]

**Subcase I.2** \(R = \sqrt{2h}\). From equation of \(f_4\) in (14), we obtain \(h[\cos(2\alpha) + 2]/2\), then \(r = 0\), and \(R = \sqrt{2h}\) is not solution.

**Subcase I.3** \(\alpha = \beta + k\pi/2\) with \(k \in \mathbb{Z}\). Then, equations \(f_2\) and \(f_4\) of (14) become

\[
\begin{align*}
\left(2h - R^2\right) \left((-1)^k \cos(2\beta) + 2\right) - (-1)^k R^2 \\
+ R^2 \cos(2\beta) &= 0, \\
-(-1)^k \left(2h - R^2\right) \sin^2(\beta) + R^2 \cos(2\beta) \\
+ 2R^2 &= 0.
\end{align*}
\]

Solving this system, we verify that \(k\) must be even, i.e., \(k = 2l\) and the solutions are \(R^* = \sqrt{h}, \beta^* = \pm 2\pi/3, \pm\pi/3, \alpha^* = l\pi + \beta\). Note that in this case, \(\rho^* = \sqrt{h}\), and the corresponding Jacobian satisfies

\[
|Dr, R, \alpha, \beta(f_1, f_2, f_3, f_4)|_{(r^*, R^*, \alpha^*, \beta^*)} = \frac{9}{16} h^4 > 0.
\]

These families of periodic solutions can be reduced only to two periodic solutions, because they are the same in first-order approximation, so we have proved item (i).

**Case II** \(r = \sqrt{\csc(2\alpha) \left[(2h - R^2)\sin(2\alpha) + R^2\sin(2\beta)\right]}\) and \(\sin(2\alpha) \neq 0\). The function \(f_3\) takes the form

\[
\frac{1}{4} R \left[-2h + R^2 \sec(\alpha)(-\cos(\alpha - 2\beta)) + R^2\right]
\]

\(\sin(2\beta) = 0\),

where we get \(R = 0\), or \(-2h + R^2 \sec(\alpha - 2\beta) + R^2 = 0\), or \(\sin(2\beta) = 0\).

**Subcase II.1** If we consider the case \(R = 0\), then equation \(f_2\) can be written as \(-\frac{1}{2}h \cos(2\alpha) + 2 = 0\); therefore, there is no solutions.

**Subcase II.2** The equation \(-2h + R^2 \sec(\alpha - 2\beta) + R^2 = 0\) has the solution \(R = \sqrt{2h}/(1 - \sec(\alpha) \cos(\alpha - 2\beta))\). Substituting these values of \(r\) and \(R\) in equations \(f_2\) and \(f_4\), we obtain the system

\[
\begin{align*}
-\frac{3}{4} R \cot(\alpha) \csc(\alpha - \beta) \sin(\alpha - \beta) &= 0, \\
-\frac{3}{4} R \csc(\alpha) \cot(\alpha - \beta) \csc(\alpha - \beta) &= 0.
\end{align*}
\]

Under the restriction \(\sin(\alpha) \sin(\beta) (\alpha - \beta) \neq 0\), we obtain the solutions \(r^* = \rho^* = R^* = \sqrt{2h}/3\), with \((\alpha^*, \beta^*)\) given by \((-\frac{\pi}{4}, -\frac{2\pi}{3}), (-\frac{\pi}{4}, \frac{2\pi}{3}), (-\frac{\pi}{4}, -\frac{\pi}{3}), (-\frac{\pi}{4}, \frac{\pi}{3}), (\frac{2\pi}{3}, \frac{2\pi}{3}), (\frac{2\pi}{3}, \frac{2\pi}{3})\). It is verified that the Jacobian at these solutions satisfies

\[
|Dr, R, \alpha, \beta(f_1, f_2, f_3, f_4)|_{(r^*, R^*, \alpha^*, \beta^*)} = \frac{h^4}{4}.
\]

Therefore, Theorem 3 guarantees the existence of 8 periodic solutions, and then we have proved part of (iv).

**Subcase II.3** Here, we have \(\beta = k\pi/2\) with \(k \in \mathbb{Z}\). Then, \(r = \sqrt{2h - R^2}\) and substituting this value in equation \(f_4\), we arrive to the equation \((-h - R^2)(-1)^k + 2)/2 = 0\). So \(R = \sqrt{h}\). Next, we substitute the value of \(\beta\) and \(R\) in equation \(f_2\), and we obtain that \(k\) must be even, i.e., \(k = 2l\) and then \(\alpha = \pm 2\pi/3, \pm\pi/3, \beta = l\pi\). Therefore, \(r = \sqrt{h}\). Moreover, the Jacobian at these solutions satisfies

\[
|Dr, R, \alpha, \beta(f_1, f_2, f_3, f_4)|_{(r^*, R^*, \alpha^*, \beta^*)} = \frac{9}{8} h^4.
\]
These families of periodic solutions can be reduced only to two periodic solutions, because they are the same in first-order approximation, so we have proved item (ii).

Case III $\sin(2\alpha) = 0$ and $R = 0$, or equivalently, $\alpha = k\pi/2$ with $k \in \mathbb{Z}$. Then, equation $f_2$ writes $(2h - 2r^2)[\cos(2\alpha) + 2]/4 = 0$. So then $r = \sqrt{h}$. Equation for $f_3$ implies $-h(-2(1-k)^2 + 2)/4 = 0$, where we get that $k$ must be even, i.e., $k = 2l$ with $l \in \mathbb{Z}$, and $\beta = \pm 2\pi/3, \pm \pi/3$ and $\alpha = l\pi$. We verify that the Jacobian at these solutions satisfies

$$|D_{\alpha, \beta}(f_1, f_2, f_3, f_4)|_{(\alpha^*, R^*, \alpha^*, \beta^*)} = \frac{9}{16} h^4.$$  

By the same arguments as in the previous cases, we conclude the proof of (iii).

Case IV $\sin(2\alpha) = 0$ and $\sin(2\beta) = 0$, or equivalently, $\alpha = k\pi/2$ and $\beta = m\pi/2$ with $k, m \in \mathbb{Z}$. Therefore, $f_3 = 0$, and $f_2, f_3$ take the form

$$f_2 = \frac{1}{4} \left( (-1)^k + 2 \right) \left( 2h - 2r^2 - R^2 \right) - \left( (-1)^k - 1 \right) (-1)^m R^2,$$

$$f_4 = \frac{1}{4} \left( (-1)^k \left( (-1)^m - 1 \right) \left( -2h + r^2 + R^2 \right) + (-1)^m - 2 \right) \left( r - R \right) \left( r + R \right).$$

Solving this system, we get that $k = 2l$, $m = 2n$, so $\alpha = l\pi$, $\beta = n\pi$ with $l, n \in \mathbb{Z}$, and $r^* = R^* = \sqrt{2h/3}$. In any of these points, we have that the Jacobian at this critical point satisfies

$$|D_{\alpha, \beta}(f_1, f_2, f_3, f_4)|_{(\alpha^*, R^*, \alpha^*, \beta^*)} = -h^4.$$  

These periodic solutions can be reduced to only four periodic solutions, so we conclude the proof of (iv).

The proof of the second part of the theorem is as follows. For the family (1), we have the characteristic polynomial of the linearization of the averaged system (18) at the points $(r^* = 0, R^* = \sqrt{h}, \rho^* = \sqrt{h}, \alpha^*0, \pi, \beta^* = \pi/3)$ is $\lambda^4 - \frac{1}{2}(h - 3)\sqrt{h} \lambda^3 - \frac{1}{16} h^{3/2} \left( 2\sqrt{3h} + 15\sqrt{h} - 6\sqrt{3} \right) \lambda^2 + \frac{3}{16} h^{5/2} (h - 3) \sqrt{3\sqrt{h} - 3} \lambda + \frac{9}{16}$. We verify that one of the roots is $\lambda_1 = \sqrt{3}/4h > 0$. Thus, the two periodic orbits of (i) are unstable.

The characteristic polynomial for any solution of (ii) is $\lambda^4 + \sqrt{3} \lambda^3 + \frac{9}{8} h^2 \lambda^2 + \frac{27}{16} h^{1/2} + \frac{3\sqrt{3}}{8} \lambda^2 h^2 + \frac{9}{8} h^4$. Using the criterium described in [14], we obtain that for each $h > 0$, this polynomial has complex roots with positive real part; thus, the periodic orbits are unstable.

The characteristic polynomial for any solution in (iii) is $p_3(\lambda) = \lambda^4 + \sqrt{3} \lambda^3 + \frac{9}{8} h^2 \lambda^2 + \frac{27}{16} h^{1/2} + \frac{3\sqrt{3}}{8} \lambda^2 h^2 + \frac{9}{8} h^4$. Since $p_3(0) < 0$ and $p_3(\lambda) \to +\infty$ as $\lambda \to +\infty$, it follows that $p_3(\lambda)$ as at least one positive root; thus, the periodic orbits of (iii) are unstable.

The characteristic polynomial for the first 8 solutions of (iv) is $p_{4,1}(\lambda) = \lambda^4 + \sqrt{3} \lambda^3 + \frac{9}{8} h^2 \lambda^2 + \frac{27}{16} h^{1/2} + \frac{3\sqrt{3}}{8} \lambda^2 h^2 + \frac{9}{8} h^4$. Since $p_{4,1}(0) < 0$ and $p_{4,1}(\lambda) \to +\infty$ as $\lambda \to +\infty$, it follows that $p_{4,1}(\lambda)$ has at least one positive root; thus, these periodic orbits are unstable. For the rest of the solutions in (iv), the characteristic polynomial is $p_{4,2}(\lambda) = \lambda^4 - \sqrt{3} \lambda^3 + \frac{3\sqrt{3}}{8} h^3 \lambda + \frac{9}{16} h^4$.

Since $p_{4,2}(\lambda) = (\lambda - \sqrt{3}/2h)^2(\lambda + [\sqrt{3}/3i]h/4)(\lambda + [\sqrt{3}/3i]h/4)$, the corresponding periodic orbits are unstable. In short, we have proved Theorem 1.

3 Proof of Theorem 2

We continue using the polar coordinates given in (10), and we observe that in the Hamiltonian level $H = \frac{1}{2}(r^2 + \rho^2 + R^2) - \varepsilon \cos^2(\beta + \theta)$

$$[r \cos \theta + \rho \cos(\alpha + \theta)],$$

and the equations of motion write

$$\dot{r} = -\varepsilon R^2 \sin \theta \cos^2(\beta + \theta),$$

$$\dot{\rho} = -\varepsilon \sin(\alpha + \theta) \cos^2(\beta + \theta),$$

$$\dot{R} = \varepsilon R \sin(2(\beta + \theta))(r \cos \theta + \rho \cos(\alpha + \theta)), $$

$$\dot{\theta} = -1 + \frac{1}{r} \left[ R^2 \cos \theta \cos^2(\beta + \theta) \right],$$

$$\dot{\alpha} = -1 \left[ R^2 \cos \theta \cos^2(\beta + \theta) \right] \left( r \rho \cos(\alpha + \theta) - \rho \cos \theta \right),$$

$$\dot{\beta} = \frac{1}{r} \left[ \cos^2(\beta + \theta) \left( 2r^2 - R^2 \right) \cos \theta \right]$$
and expanding it in Taylor series of $\theta$ system, we take the variable $\rho$ as the new independent variable, and we use a prime to denote the derivative with respect to $\theta$. The angular variables $\alpha$ and $\beta$ cannot be used as independent variable because the new differential system would not be in the normal form for applying the averaging theory described in Theorem 3.

Of course, the new system has now only five equations because we do not need the $\theta$ equation. Writing it in Taylor series of $\epsilon$, we get

\[
\begin{align*}
\rho' &= -\epsilon R^2 \sin \theta \cos^2 (\beta + \theta) \\
&\quad - \epsilon^2 \frac{1}{r} R^4 \sin \theta \cos \theta \cos^4 (\beta + \theta) + O(\epsilon^3), \\
\rho &= -\epsilon R^2 \sin (\alpha + \theta) \cos^2 (\beta + \theta) \\
&\quad - \epsilon^2 \frac{1}{r} R^4 \cos \theta \sin (\alpha + \theta) \cos^4 (\beta + \theta) + O(\epsilon^3), \\
R' &= -\epsilon R \sin (2(\beta + \theta))(r \cos \theta + \rho \cos (\alpha + \theta)) \\
&\quad - \epsilon^2 \frac{1}{r} R^3 \cos \theta \sin (\beta + \theta) \cos^3 (\beta + \theta) (r \cos \theta \\
&\quad + \rho \cos (\alpha + \theta)) + O(\epsilon^3), \\
\alpha' &= \epsilon \frac{1}{r \rho} R^2 \cos^2 (\beta + \theta) (s \cos \theta - r \cos (\alpha + \theta)) \\
&\quad + \epsilon^2 \frac{1}{r \rho} R^4 \cos \theta \cos^4 (\beta + \theta) \\
&\quad (\rho \cos \theta - r \cos (\alpha + \theta)) + O(\epsilon^3), \\
\beta' &= \epsilon \frac{1}{r} \cos^2 (\beta + \theta) \left( R^2 - 2r^2 \cos \theta - 2r \rho \cos (\alpha + \theta) \right) \\
&\quad + \epsilon^2 \frac{1}{r^2} R^2 \cos \theta \cos^4 (\beta + \theta) \left( R^2 - 2r^2 \cos \theta - 2r \rho \cos (\alpha + \theta) \right) + O(\epsilon^3). \\
\end{align*}
\]

Therefore, system (16) is $2\pi$–periodic in the variable $\theta$. In order to apply Theorem 3, we fix the value of the first integral at $h > 0$, and by solving equation (15) for $\rho$ and expanding it in Taylor series of $\epsilon$, we obtain

\[
\rho = \sqrt{2h - r^2 - R^2} \\
+ \epsilon R \cos^2 (\beta + \theta) \left( \frac{r \cos \theta}{\sqrt{2h - r^2 - R^2}} \cos (\alpha + \theta) \right) \\
- \epsilon^2 R^2 \cos^3 (\beta + \theta) \left( (-2h + r^2 + R^2) \cos^2 (\alpha + \theta) + r^2 \cos^2 \theta \right) \\
+ O(\epsilon^3). \\
\]

Using this value of $\rho$ in equations (16), we obtain the following 4-dimensional differential system

\[
\begin{align*}
r' &= -\epsilon R^2 \sin \theta \cos^2 (\beta + \theta) \\
&- \epsilon^2 \frac{1}{r} R^4 \sin \theta \cos \theta \cos^4 (\beta + \theta) + O(\epsilon^3), \\
R &= \epsilon F_{11} + \epsilon^2 F_{21} + O(\epsilon^3), \\
R' &= -\epsilon R \sin (2(\beta + \theta)) \left( \sqrt{2h - r^2 - R^2} \cos (\alpha + \theta) + r \cos \theta \right) \\
&- \epsilon^2 \frac{1}{r^2} R^3 \sin (\beta + \theta) \left( \sqrt{2h - r^2 - R^2} \cos (\alpha + \theta) + r \cos \theta \right) + O(\epsilon^3), \\
\alpha' &= \epsilon R^2 \cos^2 (\beta + \theta) \left( \frac{\cos \theta}{r} - \frac{\cos (\alpha + \theta)}{\sqrt{2h - r^2 - R^2}} \right) \\
&+ \epsilon^2 \frac{R^4 \cos^4 (\beta + \theta)}{r^2 (2h - r^2 - R^2)^{3/2}} \\
&\left( \sqrt{2h - r^2 - R^2} \cos (\alpha + \theta) \cos \theta \right. \\
&\left. + r \cos \theta \left( -2h + 2r^2 + R^2 \right) \cos (\alpha + \theta) \right) \\
&+ \epsilon \frac{R^4 \cos^4 (\beta + \theta)}{r^2 (2h - r^2 - R^2)^{3/2}} \cos \theta \left( \cos^2 \theta \right) \\
&\left( \sqrt{2h - r^2 - R^2} \cos (\alpha + \theta) + r \cos \theta \right) + O(\epsilon^3), \\
\beta' &= \epsilon \frac{1}{r} \cos^2 (\beta + \theta) \left( R^2 - 2r^2 \cos \theta \right) \\
&\quad - 2r \sqrt{2h - r^2 - R^2} \cos (\alpha + \theta) + O(\epsilon^3). \\
\end{align*}
\]

Using the notation $x = (r, R, \alpha, \beta) \in \mathbb{D} = (0, \sqrt{2h}) \times (0, \sqrt{2h}) \times \mathbb{R} \times \mathbb{R}$ and $t = \theta$, system (18) has the normal form of the averaging of Theorem 3.

In system (18), $r$ and $\rho$ cannot be zero, so for $r > 0$ and $\rho > 0$, the functions $F_1 = (F_{11}, F_{12}, F_{13}, F_{14})$ and $F_2 = (F_{21}, F_{22}, F_{23}, F_{24})$ are analytical and $2\pi$–periodic in $\theta$, being $\theta$ the independent variable of system (18). After some computations and since the com-
ponents of $F_1$ are trigonometric polynomials of degree 3 in the variables $\cos \theta$ and $\sin \theta$, we observe that the averaging theory of first order does not apply because the average function of $F_1$ is identically zero, i.e.,

$$f_1(r, R, \alpha, \beta) = \int_0^{2\pi} (F_{11}, F_{12}, F_{13}, F_{14}) \, d\theta = (0, 0, 0, 0).$$

Now, we proceed to calculate the function $f_2$ of the Appendix 1 by applying the second-order averaging theory. Then

$$f_2(r, R, \alpha, \beta) = \int_0^{2\pi} \left[D_{rR\alpha\beta}F_1(\theta, r, R, \alpha, \beta), y_1(\theta, r, R, \alpha, \beta) + F_2(\theta, r, R, \alpha, \beta)\right] \, d\theta,$$

where

$$y_1(\theta, r, R, \alpha, \theta) = \int_0^\theta F_1(s, r, R, \alpha, \beta) \, ds.$$

In particular, the four components of the vector function $y_1$ are

$$y_{11} = \int_0^\theta F_{11}(t, r, R, \alpha, \beta) \, dt = -\frac{1}{3} R^2 \sin^2 (t/2) \left( \cos(2(\theta + \beta)) + 2 \cos(\theta + 2\beta) + 3 \right),$$

$$y_{12} = \int_0^\theta F_{12}(t, r, R, \alpha, \beta) \, dt = \frac{1}{12r} R^2 \left( 3 \sin(\theta + 2\beta) + \sin(3\theta + 2\beta) + 6 \sin(\theta) - 4 \sin(2\beta) \right) - \frac{1}{12\sqrt{2h - r^2 - R^2}} R^2 \cdot \left( 3 \sin(\theta - \alpha + 2\beta) + \sin(3\theta + \alpha + 2\beta) + 6 \sin(\theta + \alpha) + 3 \sin(\alpha - 2\beta) - \sin(\alpha + 2\beta) - 6 \sin(\alpha) \right),$$

$$y_{13} = \int_0^\theta F_{13}(t, r, R, \alpha, \beta) \, dt = -\frac{1}{6} r R \left( -3 \cos(\theta + 2\beta) - \cos(3\theta + 2\beta) + 4 \cos(2\beta) \right) - \frac{1}{6} R \sqrt{2h - r^2 - R^2} \left( -3 \cos(\theta - \alpha + 2\beta) - \cos(3\theta + \alpha + 2\beta) + 3 \cos(\alpha - 2\beta) + \cos(\alpha + 2\beta) \right),$$

$$y_{14} = \int_0^\theta F_{14}(t, r, R, \alpha, \beta) \, dt = -\frac{1}{6} r \left( 3 \sin(\theta + 2\beta) + \sin(3\theta + 2\beta) + 6 \sin(\theta - 4 \sin(2\beta)) + \frac{1}{12r} R^2 \left( 3 \sin(\theta + 2\beta) + \sin(3\theta + 2\beta) + 6 \sin(\theta) - 4 \sin(2\beta) \right) - \frac{1}{6} R \sqrt{2h - r^2 - R^2} \left( 3 \sin(\theta - \alpha + 2\beta) + \sin(3\theta + \alpha + 2\beta) + 3 \sin(\alpha - 2\beta) + \sin(\alpha + 2\beta) - 6 \sin(\alpha) \right) \right).$$

Using Theorem 3, we obtain that the function $f_2 = (f_{21}, f_{22}, f_{23}, f_{24})$ is

$$f_{21} = \frac{1}{24r} R^2 \left( -6r \sqrt{2h - r^2 - R^2} \sin(\alpha - 2\beta) + 4r \sin \alpha \sqrt{2h - r^2 - R^2} + 6r^2 \sin(2\beta) + 3R^2 \sin(2\beta) \right),$$

$$f_{22} = \frac{R^2}{48r^2(-2h + r^2 + R^2)} \left( 6r \sqrt{2h - r^2 - R^2} \cos(\alpha - 2\beta) + r \sqrt{2h - r^2 - R^2} \left( -16h + 16r^2 + 17R^2 \right) \cos \alpha + 24h \cos(2(\alpha - \beta)) - 24h \cos(2\beta) - 12h R^2 \cos(2\beta) - 8h R^2 - 12r^4 \cos(2(\alpha - \beta)) + 12r^4 \cos(2\beta) - 12r^2 R^2 \cos(2(\alpha - \beta)) + 18r^2 R^2 \cos(2\beta) - r^2 R^2 + 6R^4 \cos(2\beta) + 4R^4 \right),$$

$$f_{23} = \frac{1}{8r \sqrt{2h - r^2 - R^2}} \left( \sin(\alpha - 2\beta) + 2r \sqrt{2h - r^2 - R^2} - 2h + r^2 + R^2 \right) \cos \alpha + r \sin(2\beta) \sqrt{2h - r^2 - R^2},$$

$$f_{24} = \left( f_{21}, f_{22}, f_{23}, f_{24} \right).$$
\[ f_{24} = \frac{1}{24r^2} R^2 \left( -5r\sqrt{2h-r^2-R^2} \cos\alpha - 15r^2 + 3R^2 \cos(2\beta) + 2R^2 \right). \]

The next step is to find the zeros \((r^*, R^*, \alpha^*, \beta^*)\) of \(f_2(r, R, \alpha, \beta)\), and then we must check that the Jacobian determinant

\[ |D_{r,R,\alpha,\beta} f_2(r^*, R^*, \alpha^*, \beta^*)| \neq 0. \]

Since \(R^2\) is a common factor in (21), we must eliminate the case \(R = 0\) (because the Jacobian is zero if \(R = 0\), and also we must eliminate the case \(r = 0\) (because \(r\) appears in some denominators). Consequently, \(\rho = \sqrt{2h}\). In order to solve system (21) and due to its complexity, we use the Groebner basis. In the expressions of the numerators of the functions \(f_{2j}\) for j = 1, 2, 3, 4, we use \(\rho\) instead of \(\sqrt{2h-r^2-R^2}\). Then, the system whose zeros we must study becomes polynomial in the variables \((r, \rho, R, \cos\alpha, \sin\alpha, \cos(\beta), \sin(\beta))\), we denote these polynomials by \(h_j\) for \(j = 1, 2, 3, 4\). Thus, we compute the Groebner basis of the functions \(h_1, h_2, h_3, h_4, h_5 = \cos^2\alpha + \sin^2\alpha - 1, h_6 = \cos^2\beta + \sin^2\beta - 1\) and \(h_7 = \rho^2 - (2h - r^2 - R^2)\) in the mentioned seven variables. Three of the functions of the Groebner basis are

\[
(2h - \rho^2) \left( 3\rho^2 - h \right) \left( h + 9\rho^2 \right) \left( 1250h^2 - 729h \rho^2 \right) \nonumber
\]

\[
- 121\rho^4 \left( 2h^2 - 57h \rho^2 + 47\rho^4 \right) \left( 1866240000h^{10} + 3723356160h^9 \rho^2 - 28399449600h^8 \rho^4 - 1496101633920h^7 \rho^6 + 622267844616h^5 \rho^8 - 1034509687400h^5 \rho^{10} + 9413129629117h^7 \rho^{12} - 6151478550944h^3 \rho^{14} + 3479008638842h^2 \rho^{16} - 1267934032134h \rho^{18} + 134591551875h \rho^{20} \right),
\]

\[
(2h - \rho^2) \left( 3\rho^2 - h \right) \left( h + 9\rho^2 \right) \left( 1250h^2 - 729h \rho^2 - 121\rho^4 \right) \left( 1866240000h^{10} + 3723356160h^9 \rho^2 - 28399449600h^8 \rho^4 - 1496101633920h^7 \rho^6 + 622267844616h^5 \rho^8 - 1034509687400h^5 \rho^{10} + 9413129629117h^7 \rho^{12} - 6151478550944h^3 \rho^{14} + 3479008638842h^2 \rho^{16} - 1267934032134h \rho^{18} + 134591551875h \rho^{20} \right) \sin\beta,
\]

and the third function is given in the Appendix 1 due to its extension. Note, that is, a polynomial in the mentioned variables of degree 2 in the variable \(\sin\beta\), and of degree 30 in the variable \(\rho\).

Since \(\rho \in (0, \sqrt{2h})\), the zeros of \(\rho\) in that interval of the polynomial (21) are \(\rho_1^* = \sqrt{\frac{h}{3}}, \)

\[
\rho_2^* = \sqrt{\left(\frac{57 - 13\sqrt{17}}{94}\right)h}, \rho_3^* = \sqrt{\left(\frac{57 + 13\sqrt{17}}{94}\right)h}, \rho_4^* = \sqrt{\left(-\frac{729 + 13\sqrt{89}}{242}\right)h},
\]

and it is not difficult to check that the polynomial of degree 20 in the variable \(\rho\) which appears as factor in (21) has no real roots in the interval \((0, \sqrt{2h})\). From (21) and (22), we obtain that sin \(\beta = 0\), i.e., \(\beta = 0, \pi\), for the solutions \(\rho_j^*\) for \(j = 1, 2, 3, 4\). Now, we consider the third polynomial of the Groebner basis given in Appendix 1, and we substitute it in the value of \(\rho_j^*\) obtaining that sin \(\beta = \pm 1\), i.e., \(\beta = \pm \pi/2\).

For each value of \(\rho_3^*\), we compute \(R = \sqrt{2h - r^2 - \rho^2}\), and substituting it in the equation of \(f_{11}\), we obtain for \(j = 1, 2, 3, 4\) that sin \(\alpha\) = 0, i.e., \(\alpha = 0, \pi\). Continuing the analysis as in the proof of Theorem 1, we get the solutions (i)–(vii) given in the statement of Theorem 2.

The characteristic polynomial of (i) and (ii) is

\[
\frac{6400h^4}{243} + \frac{320h^2}{27} \lambda^2 + \lambda^4,
\]

whose roots are \(\pm i4\sqrt{5h}/3\) and \(\pm i4\sqrt{7h}/3\). So, the periodic solutions of (i) and (ii) are linearly stable. For (iii)–(vi), the characteristic polynomial is

\[
- \frac{25 \left(1216497781 + 295040429\sqrt{17}\right) h^4}{119604096} - \frac{5 \left(2620033 + 635097\sqrt{17}\right) h^2}{318096} - \lambda^2 + \lambda^4.
\]

This polynomial has two real solutions (one positive and one negative) and two pure imaginary pure; therefore, the periodic solutions of (iii)–(vi) are unstable.

The characteristic polynomial for (vii) and (viii) is

\[
\frac{204565823331\sqrt{89} - 1935553558577}{10204191360} h^4 + \frac{16564885419 - 1745833457\sqrt{89}}{263538000} h^2 - \lambda^2 + \lambda^4.
\]

Again, this polynomial has two real solutions (one positive and one negative) and two pure imaginary pure; therefore, the periodic solutions of (vii) and (viii) are unstable. This completes the proof of Theorem 2.
4 Conclusions

The objective of this work was to prove the existence of periodic orbits and its type of stability, in the galactic Hamiltonian of three degrees of freedom with $1 : 1 : 1$ resonance

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2) + H_1,$$

where $H_1$ is either $-(q_1^2 q_2^2 + q_1^2 q_3^2 + q_2^2 q_3^2)$, or $-(q_1 + q_2)q_3^2$. The families of periodic solutions of these two 3-dimensional galactic-type Hamiltonian systems started to be studied in [4].

We have used an important tool from the area of dynamical systems, the averaging theory for studying the existence of periodic orbits and their stability, and we have applied it for studying the families of periodic solutions of the Hamiltonian systems defined by the two previous Hamiltonians. Our main results are summarized in Theorems 1 and 2.

In Theorem 1, we have recuperated the 4 families of straight line periodic orbits found by Caranicolas in [4], but we also have obtained 12 new families of periodic solutions parametrized by the value of the Hamiltonian, and we also have proved that these 16 families of periodic orbits are unstable in each Hamiltonian level.

In Theorem 2, we again reobtained the 2 families of straight line periodic orbits found by Caranicolas, but we have obtained 6 new families of periodic solutions parametrized by the value of the Hamiltonian. Furthermore, we have proved that the 2 families of straight line periodic orbits are linearly stable, and the other 6 families of periodic orbits are unstable in each Hamiltonian level.

Acknowledgments The first author is partially supported by a MINECO/FEDER Grant MTM2008–03437, an AGAUR Grant number 2009SGR-0410, an ICREA Academia, FP7–PEOPLE–2012–IRSES–316338 and 318999, and FEDER-UNAB10-4E-378. The second author is partially supported by Fondecyt project number 2012–IRSES–316338 and 318999, and FEDER-UNAB10-4E-378. An ICREA Academia, FP7–PEOPLE–MTM2008–03437, and AGAUR Grant number 2009SGR-0410.

Appendix 1

Averaging theory of first and second order

In this section, we recall the results of the averaging theory that we have used for proving the results of this paper. For a general introduction to the averaging theory, see Chapter 11 of the book [13], and mainly the book [11]. The averaging theory up to second order stated in what follows, with the weak assumptions used, was proved in [3].

Theorem 3 Consider the differential system

$$\dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, $T$-periodic is the first variable, and $D$ is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold:

(i) $F_1(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, $F_1$, $F_2$, and $D, F_1$ are locally Lipschitz with respect to $x$, and $R$ is differentiable with respect to $\varepsilon$. We define $f_1, f_2 : D \to \mathbb{R}^n$ as

$$f_1(z) = \int_0^T F_1(s, z) \, ds,$n$$

$$f_2(z) = \int_0^T \left[ Dz F_1(s, z) \int_0^s F_1(t, z) \, dt + F_2(s, z) \right] \, ds.$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus {0}$, there exists $a \in V$ such that $f_1(a) + \varepsilon f_2(a) = 0$ and $d_B(f_1 + \varepsilon f_2, V, a) \neq 0$ (the Brouwer degree of $f_1 + \varepsilon f_2$ at $a$).

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a $T$-periodic solution $x(t, \varepsilon)$ of the system such that $x(0, \varepsilon) \to a$ when $\varepsilon \to 0$.

As usual, we have denoted by $d_B(f_1 + \varepsilon f_2)$, the Brouwer degree of the function $f_1 + \varepsilon f_2 : V \to \mathbb{R}^n$ at its zero $a$; for more details on the Brouwer degree, see [2]. A sufficient condition for showing that the Brouwer degree of a function $f$ at its zero $a$ is non-zero is that the Jacobian of the function $f$ at $a$ (when it is defined) is non-zero, see for more details [6].

If the function $f_1$ is not identically zero, then the zeros of $f_1 + \varepsilon f_2$ are essentially the zeros of $f_1$ for $\varepsilon$ sufficiently small. In this case, Theorem 3 provides the so-called averaging theory of first order.

If the function $f_1$ is identically zero and $f_2$ is not identically zero, then the zeros of $f_1 + \varepsilon f_2$ are the zeros of $f_2$. In this case, Theorem 3 provides the so-called averaging theory of second order.
In the case of the averaging theory of first order, we consider in $D$ the averaged differential equation

$$\dot{y} = \varepsilon f_1(y), \quad y(0) = x_0 \tag{24}$$

where

$$f_1(y) = \frac{1}{T} \int_0^T F_1(t, y) \, dt.$$

Then, Theorem 3 gives information about the stability or instability of the periodic solution $x(t, \varepsilon)$. In fact, it is given by the stability or instability of the equilibrium point $p$ of the averaged system (24). In fact, the singular point $p$ has the stability behavior of the Poincaré map associated to the periodic solution $x(t, \varepsilon)$ in the case of the averaging theory of first order, i.e., $f_1 \equiv 0$ and $f_2$ non-identically zero, we have that the stability and instability of the limit cycle $\varphi(\varepsilon)$ coincide with the type of stability or instability of the equilibrium point $p$ of the averaged system.

$$\dot{y} = \varepsilon^2 f_2(y), \quad y(0) = x_0 \tag{25}$$

i.e., it is the same that the singular point $p$ associated to the Poincaré map of the periodic solution $\varphi(t, \varepsilon)$, see for instance the Chapter 11 of [13].

**Appendix 2**

**Third factor in the basis of Groebner**

$$(\rho^2 - 2h)(1324064332683572715589511912023780901661294$$

$0577871677776597464834377207450246851818760580$$

$466041449665141699927330219066355649814760971176$$

$2346923641507965802835976717568000!\beta^3 \sin^2 \beta$$

$-445555371861769399126043019594164429503175893218$$

$44918086932760888236827200918596097240401590466387$$

$462626262058931567152521412968632989190583154858797$$

$76911281117568000!\beta^6 \rho^2$$

$+ 82281049273172290501256789$$

$0767490566390004230751453215943733$$

$449004331497969007299161431362840750844433535488383$$

$113091155436953552385107507681005752640000!\rho^2$$

$+ 146854966680133498121873072166014078719115370872304204991$$

$547561141769983411102853164130289144832522007890950$$

$668646966532744433866799728253539949059574505554184894$$

$56289312000!\beta^6 \rho^2$$

$- 83038956452186299875760982111\)
References

1. Abraham, R., Marsden, J.E., Ratiu, T.: Manifolds, Tensor Analysis, and Applications. Applied Mathematical Sciences, 2nd edn. Springer-Verlag, New York (1988)
2. Browder, F.: Fixed point theory and nonlinear problems. Bull. Amer. Math. Soc. 9, 1–39 (1983)
3. Buică, A., Llibre, J.: Averaging methods for finding periodic orbits via Brouwer degree. Bull. Sci. Math. 128, 7–22 (2004)
4. Caranicolas, N.: 1 : 1 : 1 resonant periodic orbits in 3-dimensional galactic-type Hamiltonians. Astronom. Astrophys. 282, 34–36 (1994)
5. Hayli, A., Desolneux, N., Galleta, G.: Orbites périodiques dans un potentiel à trois dimensions. Astronom. Astrophys. 122, 137–142 (1983)
6. Lloyd, N.G.: Degree Theory. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (1978)
7. Magenat, P.: Numerical study of periodic orbit properties in a dynamical system with three degrees of freedom. Celestial Mech. 28, 319–343 (1982)
8. Martinet, L., Magenat, P.: Invariant surfaces and orbital behaviour in dynamical systems with 3 degrees of freedom. Astronom. Astrophys 96, 68–77 (1981)
9. Martinet, L., Magenat, P., Verhulst, F.: On the number of isolating integrals in resonant systems with 3 degrees of freedom. Celestial Mech. 25, 93–99 (1981)
10. Meyer, K.R., Hall, G.R., Offin, D.: Introduction to Hamiltonian Dynamical Systems and the N-Body Problem. Applied Mathematical Sciences, 2nd edn. Springer, New York (2009)
11. Sanders, J., Verhulst, F., Murdock, J.: Averaging Methods in Nonlinear Dynamical Systems. Applied Mathematical Sciences, 2nd edn. Springer, Berlin (2007)
12. Van der Aa, E., Sanders, J.: Lecture Notes in Mathematics. Springer, Berlin (1979)
13. Verhulst, F.: Nonlinear Differential Equations and Dynamical Systems. Springer, Berlin (1991)
14. Yang, L.: Recent advances on determining the number of real roots of parametric polynomials. J. Symb. Comput. 28, 225–242 (1999)
15. Zotos, E.: Application of new dynamical spectra of orbits in Hamiltonian systems. Nonlinear Dyn. 69, 2041–2063 (2012)
16. Zotos, E.: The fast norm vector indicator (FNVI) method: a new dynamical parameter for detecting order and chaos in Hamiltonian systems. Nonlinear Dyn. 70, 951–978 (2012)
17. Zotos, E.: Revealing the evolution, the stability, and the escapes of families of resonant periodic orbits in Hamiltonian systems. Nonlinear Dyn. 73, 931–962 (2013)
18. Zotos, E., Caranicolas, N.: Order and chaos in a new 3D dynamical model describing motion in non-axially symmetric galaxies. Nonlinear Dyn. 74, 1203–1221 (2013)
19. Zotos, E., Caranicolas, N.: Determining the nature of orbits in disk galaxies with non-spherical nuclei, Nonlinear Dyn. (2013) DOI10.1007/s11071-013-1129-8