Salpeter amplitudes
in a Wilson-loop context

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**Abstract**

The bound state problem for a gauge invariant quark-antiquark system is considered in the instantaneous rest frame. Focus here is on the long range non-perturbative interaction. A two-time Green’s function is constructed for Salpeter amplitudes. The corresponding Schrödinger equation is found to be in the Salpeter form with a Wilson-loop term as the instantaneous kernel.
1 Introduction

The observed mass of a hadron results from the underlying dynamics of its constituent quarks and gluons. Understanding the process is made difficult however by the complexities of low-energy QCD which is widely believed to provide a mechanism for confinement. This is one of the most interesting and difficult studies in quantum field theory. The main problem stems from the essentially non-perturbative nature of confinement. This is so except in certain cases. For example, in the limit of heavy quark mass the theory is dynamically reduced to that of a single particle, and at short distances asymptotic freedom suggests that the binding is dominated by one gluon exchange which in the static limit yields a coulombic interaction. Thus heavy quarks forming a bound state remain largely insensitive to the details of confinement, and their static properties are well described by non-relativistic constituent-quark potential models [1]. Unfortunately, neither condition is met for the general case of deeply bound states, and a more fundamental approach is required.

The starting point for a relativistic description of the meson spectrum is the covariant Bethe-Salpeter(BS) equation. It is the most orthodox framework for addressing the two-body problem in quantum field theory and has played a central role in the discussion for over three decades. On the other hand, the appearance of the relative time variable lacks a clear interpretation in a Hamiltonian setting, and the usual approach is to make a three-dimensional reduction of the kernel while retaining relativistic kinematics. In addition, one or both single-particle propagators are often placed on an effective mass shell. Symmetric treatment of this last constraint leads to the Salpeter equation [2] whose basic statement on the analytic form of its single-particle components is given by $\psi_{\pm} = \Lambda^\pm \psi_{\pm}$, also recognized as the no-pair condition [3]. Apart from these issues are questions about the Lorentz structure of the confining kernel. Though the success of phenomenological models suggest and lattice simulations confirm static linear confinement, going beyond this limit requires knowledge of the kernel’s dominant Lorentz components. But even this knowledge
would shed no light on specifics of its dynamical origin.
What is needed and is essential for a discussion of this problem in non-perturbative
dynamics is a manifestly gauge invariant formulation. This is provided by the path
integral method. Defining the bound quark-antiquark amplitude in this way as
a color singlet has led to some of the most fruitful work in the non-perturbative
regime beginning with the pioneering observations of Wilson [4]. The simple picture
offered in his area law for static quarks is compelling: Contributions to the meson
propagator fall off exponentially in the area swept out by lines of chromoelectric flux
joining the constituents world lines; hence widely separated paths are suppressed.
The dominant effect is stated as an average over the transport of color along the
contour of the minimum area, \( \log(\langle W(C) \rangle) \approx \sigma S_{\text{min}} \). The main difficulties here are
technical and directly related to the fact that analytic expressions for the full single
fermion propagator are not known. For example, completely static propagators lead
to the well-known linear potential [5], and leading corrections yield \( O(m^{-2}) \) spin-
dependent [6], and more recently as given in the works of Brambilla and Prosperi [7],
spin-independent contributions. On another front, a full spinless propagator leads
to relativistic flux-tube dynamics [8]. Here the following question is asked: Can the Lorentz and dynamical uncertainties
present in the Salpeter approach be resolved and the technical difficulties of the path
integral method overcome in a complimentary formulation which retains relativistic
kinematics and spin structure? The answer seems to be in the affirmative with a
resulting Hamiltonian similar to that of a relativistic flux-tube model [9].

2 Two-body propagator

The beginning parts of this section follow closely the treatment and notations found
in ref. [7] where the starting point is also the gauge invariant four-point Green’s
The $\psi$'s here are fermion fields in the Heisenberg representation, $U$ straight-line path ordered exponentials, and $S$ single particle propagators in the external gauge field $A$. $C$ is the charge conjugation matrix while $c$ denotes charge conjugation. The average is taken over gauge fields with virtual quark loops ignored. Positive and negative frequency components of the full single particle propagator

\begin{align}
\mathbb{i}S(x, y|A) &= \theta(x_0 - y_0)S^+(x, y|A) - \theta(y_0 - x_0)S^-(x, y|A) \\
\text{satisfy the homogeneous Dirac equation} \quad \left(\mathbb{i} \mathbb{\not}D_x - m\right) S^\pm(x, y|A) &= 0
\end{align}
so that each term in (3) independently has the property of a propagator. By standard methods [10] the solutions to (4) satisfying (7) are expressed as the phase-space path integrals

$$S^\pm(x,y|A) = \int_y^x D[z,p] T_A T_s \Lambda_\pm(p - gA)$$

$$\times \exp \left\{ i \int_{y^0}^{x^0} dt [p \cdot \dot{z} \mp E_0(p - gA) - gA_0] \right\} \gamma^0 .$$

In this expression $T_A$ time orders gauge-field operators for $x_0 > y_0$ and antitime orders for $x_0 < y_0$, while $T_s$ does the same for Dirac matrices. Terms relevant to the minimal-area relation enter upon a semiclassical reduction. It is convenient to first isolate Wilson-loop factors by translating the integration variable $p \to p + gA$.

The momentum integration is then performed in the Gaussian approximation around stationary paths $\dot{z}_\pm = \pm p/E_0(p)$ yielding

$$S_{cl}^\pm(x,y|A) = \int_y^x D[z] T_A T_s \Lambda_\pm^\prime \exp \left\{ i \int_{y^0}^{x^0} dt [\mp m(1 - \dot{z}^2)^{1/2}] - ig \int_y^x A_\mu dr^\mu \right\} \gamma^0 \gamma_0,$$

The primed projector stands for evaluation at $p_q \equiv \pm m\dot{z}_\pm (1 - \dot{z}^2)^{-1/2}$ and an unimportant change in the integration measure has been suppressed [7]. Consider now the action of the four-gradient

$$i \partial_\mu S_{cl}^\pm = (p_\mu)_{cl} S_{cl}^\pm.$$  

Equation (4) with constraint (3) follow from the quantization of (10) by the obvious transformations, $S_{cl}^\pm \to S^\pm$, $p_{cl} \to -i \nabla$ and $\Lambda_\pm p_0 S^\pm \to 0$. Equation (3) along with the relation $C^{-1} S^\mp(y,x|A)C = [-S^\pm(x,y| - A^\tau)]^\tau$, where $\tau$ is the transpose operator, combine in the two-time Green’s function to give

$$G(x_0; \mathbf{x}_1, \mathbf{x}_2|y_0; \mathbf{y}_1, \mathbf{y}_2) = \theta(x_0 - y_0) \hat{G}_+ + \theta(y_0 - x_0) \hat{G}_-$$

with definition

$$\hat{G}_\pm(x_0; \mathbf{x}_1, \mathbf{x}_2|y_0; \mathbf{y}_1, \mathbf{y}_2) =$$

$$\int_{y_1}^{x_1} \int_{y_2}^{x_2} D[z_1] D[z_2] T_A \Lambda_\pm^{(1)} \Lambda_\pm^{(2)} \exp \left\{ i \int_{y_0}^{x_0} dt \sum_{j=1}^2 [\mp m_j(1 - \dot{z}_j^2)^{1/2}] + \log \langle W(C_\pm) \rangle \right\} \gamma_1^{0} \gamma_2^{0}$$

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for the closed contours $C_{\pm}$
	ime increasing to the right. Here the compatibility of the instantaneous Salpeter picture with that of the Wilson loop is apparent: In addition to the intermediate propagation of two particles, both allow for the propagation of two antiparticles (double Z-graphs in time-ordered perturbation theory) evidently represented here by $\hat{G}_-$. Single Z-graph components do not enter. As mentioned earlier, the straight-line approximation to the minimal-area law,

$$\iota \log \langle W(C_{\pm}) \rangle = \sigma S_{\text{min}} \equiv \int_{t^*}^t dt L_g = \pm \int_{x_0}^{x_2} dt L_g$$

is taken. In the center-of-momentum frame

$$L_g = \sigma |r| \int_0^1 ds \gamma^{-1}(v_t)$$

where $\gamma(v) = (1 - v^2)^{-1/2}$ and $v_t = s\dot{x}_1 + \cdots$ $v_j$ as the i-th component of v perpendicular to $r = x_1 - x_2$. With $L_q \equiv \sum_{j=1}^2 m_j (1 - \dot{z}_j^2)^{1/2}$ the exponential argument in (12) can be written

$$\iota A_{\pm} = \mp i \int_{y_0}^{x_0} dt(L_q + L_g) = \mp i \int_{y_0}^{x_0} dt L.$$ (14)

From the stationary condition, action of the four-gradient yields

$$\sum_j -i \partial_j (\iota A_{\pm}) = \sum_j -i \frac{d}{dx_0} \partial_{\dot{x}_j} (\iota A_{\pm})$$ (15)

$$= \sum_j \pm m_j \dot{x}_j \gamma(\dot{x}_j) + \sigma |r| \int_0^1 ds \gamma(v_t) v_t$$ (16)

$$\equiv \sum_j p_{qj} + p_g$$ (17)
and

\begin{align}
\imath \partial_{x_0} (\imath A_\pm) &= \pm (\sum_j \dot{x}_{j\pm} \cdot \frac{\partial L}{\partial \dot{x}_{j\pm}} - L) \\
&= \pm (\sum_j m_j \gamma(\dot{x}_j) + \sigma |\mathbf{r}| \int_0^1 ds \gamma(v_s)) \\
&\equiv \pm (\sum_j m_j \gamma(\dot{x}_j) + V). 
\end{align}

Then

\begin{align}
\sum_j -\imath \partial_j \hat{G}_\pm &\rightarrow (\sum_j p_{qj} + p_q) \hat{G}_\pm \\
\imath \partial_{x_0} \hat{G}_\pm &\rightarrow (H'_{01} + H'_{02} \pm V_\pm) \hat{G}_\pm 
\end{align}

with \( V_\pm = \Lambda_\pm^{(1)} \Lambda_\pm^{(2)} V \Lambda_\pm^{(1)} \Lambda_\pm^{(2)}. \)

### 3 Eigenvalue equation

Given \( \chi^{(+)}(t; \mathbf{r}_1, \mathbf{r}_2) \) and \( \chi^{(-)}(t'; \mathbf{r}_1, \mathbf{r}_2) \) as positive and negative frequency components of the single-time bound state \( q\bar{q} \) amplitude with \( t < t' \), the freely propagating amplitude at intermediate times is found with help from the Green’s function

\begin{equation}
\chi(x_0; \mathbf{x}_1, \mathbf{x}_2) = \imath \int d^3 r_1 d^3 r_2 \left[ G(x_0; \mathbf{x}_1, \mathbf{x}_2 | t; \mathbf{r}_1, \mathbf{r}_2) \chi^{(+)}(t; \mathbf{r}_1, \mathbf{r}_2) \right. \\
- \left. G(x_0; \mathbf{x}_1, \mathbf{x}_2 | t'; \mathbf{r}_1, \mathbf{r}_2) \chi^{(-)}(t'; \mathbf{r}_1, \mathbf{r}_2) \right] 
\end{equation}

and so satisfies the time-independent Schrödinger equation

\begin{equation}
\imath \partial_{x_0} \chi = H \chi 
\end{equation}

where the Hamiltonian is given by

\begin{equation}
H = H'_{01} + H'_{02} + V_+ - V_- .
\end{equation}
Here $H'_0$ represents the quark’s kinetic energy and $V$ the gluon field contribution from the Wilson-loop (which apparently enters as the instantaneous BS kernel). Projection operators on $V$ prevent mixing with the non-normalizable continuum states of the same energy.

This result should be compared with others from the Wilson-loop formalism. The classical limit to the spinless Hamiltonian of ref[8] is recovered from (25) by the obvious reduction, $H_0 \rightarrow +E_0$, which incorporates the no-backtracking constraint; that is, the second loop of fig(1) does not contribute ($V_- \rightarrow 0$) in this approximation. $O(1/m^2)$ terms of $(H)_{12}$ in the standard Dirac representation give leading relativistic corrections to the static $\sigma r$ interaction; the reduction is equivalent to a Foldy-Wouthuysen transformation. In the center of momentum frame the corrections are

$$V_{\text{corr}} \simeq -\sigma(1/m_1^2 + 1/m_2^2 - 1/(m_1 m_2))L^2/(6r) \quad \text{(26)}$$

$$-\sigma(L \cdot s_1/m_1^2 + L \cdot s_2/m_2^2)/(6r).$$

The spin-independent term above agrees with the semirelativistic result of ref[7] omitting a angular-momentum independent contribution resulting from their particular operator ordering prescription, $V_{cl}(\mathbf{r}, \mathbf{p}) \rightarrow \{V_{cl}\}_{\text{ord}} \equiv V_{qm}$. An alternative prescription better suited to quantization of (25) would be in terms of classical coordinate and velocity observables, $(\mathbf{r}, \mathbf{v})$. In this case the $\mathbf{v}_\perp$ operators are found as truncated matrices in a suitable basis from the symmetrized orbital angular momentum equation, $\mathbf{L}(\mathbf{r}, \mathbf{v}_\perp) = \mathbf{x}_i \times \mathbf{p}_i$, for a given state. The energy eigenvalue equation thus becomes a matrix equation in the radial coordinate only and is solved variationally. For details of the method see e.g. ref[11].

Spin corrections responsible for fine and hyperfine structure of the spectrum are of course an important and delicate concern. They must be handled carefully. The numerical coefficient of the spin-orbit term in (26), -1/6, is at variance with the -1/2 factor found in most of the literature. Most often the -1/2 follows from an assumption of dominant scalar confinement[12] which in relativized models based on
a BS reduction is thought to be spectroscopically favored over other Lorentz forms. On the other hand, this same disagreement appears with the $O(1/m^2)$ Wilson-loop result of ref[7] where no such assumption is made. From a purely mathematical point of view this results from the different ways in which the holonomic straight-line condition is applied to the minimal-area relation; a factor of -1/6 (-1/2) results when the constraint is imposed before (after) the variation implied here in equation(21) and in appendix B of ref[7]. Only the former option, followed here, is in line with proper procedures for the mechanics of constrained systems. It should be pointed out that Gromes has deduced a relation[13] in support of the -1/2 factor (and scalar confinement) from arguments of Lorentz covariance following a reasoning in Representation Theory on the group structure of Poincaré transformations. The relevance of this formalism in the present rest-frame Hamiltonian context is however not entirely clear and the question will not be entered into here.

4 Summary

Salpeter amplitudes have been introduced into the path integral formulation of QCD beginning from the gauge-invariant four-point function. A Hamiltonian description for mesons as bound quark-antiquark states has been derived in a systematic and straightforward manner. Interestingly, a time derivative of the Wilson-loop operator appears as the instantaneous BS kernel. Both relativistic kinematics and spinor structure of the amplitude have been preserved. The focus here has been on long range non-perturbative effects of the gluon dynamics, though a realistic calculation of the spectrum must take medium and short range contributions into account as well. These enter asymptotically as a coulombic interaction which is easily added to the present result. An apparent discrepancy between the $O(1/m^2)$ spin-orbit coefficient and the relation of Gromes has been noted and is under study.
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