A novel strong coupling expansion of the QCD Hamiltonian

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Introducing an infinite spatial lattice with box length $a$, a systematic expansion of the physical QCD Hamiltonian in $\lambda = g^{-2/3}$ can be obtained. The free part is the sum of the Hamiltonians of the quantum mechanics of spatially constant fields for each box, and the interaction terms proportional to $\lambda^n$ contain discretised spatial derivatives connecting different boxes. As an example, the energy of the vacuum and the lowest scalar glueball is calculated up to order $\lambda^2$ for the case of $SU(2)$ Yang-Mills theory.

I. INTRODUCTION

The quantum Hamiltonian of $SU(2)$ Yang-Mills theory, to which we limit ourselves here for simplicity, can be obtained \cite{1} by exploiting the time-dependence of the gauge transformations to put $A_{a0}(x) = 0$, $(a = 1, 2, 3)$, and quantizing the spatial fields in the Schrödinger representation, $\Pi_{ai}(x) = -E_{ai}(x) = -i\delta/\delta A_{ai}(x)$. The physical states $\Psi$ have to satisfy the Schrödinger equation and the three non-Abelian Gauss law constraints

$$H\Psi = E\Psi, \quad H = \int d^3x \sum_{a,i} \left[ \left( \frac{\delta}{\delta A_{ai}(x)} \right)^2 + B_{ai}^2(A(x)) \right]$$

$$G_a(x)\Psi = 0, \quad G_a(x) = -i(\delta_{ac}\partial_i + g\epsilon_{abc}A_{bi}(x)) \frac{\delta}{\delta A_{ci}(x)}$$

with the chromo-magnetic fields $B_{ai}(A) = \epsilon_{ijk}(\partial_j A_{ak} + \frac{1}{2}g\epsilon_{abc}A_{bj}A_{ck})$ and the generators $G_a(x)$ of the residual time-independent gauge transformations, satisfying $[G_a(x), H] = 0$, and $[G_a(x), G_b(y)] = ig\delta^3(x-y)\epsilon_{abc}G_c(x)$, The matrix elements have Cartesian measure

$$\langle \Phi_1|O|\Phi_2 \rangle = \int \prod_x \prod_i dA_{ik}(x) \Phi_1^*O\Phi_2.$$  \hspace{1cm} (3)

In order to implement the Gauss laws (2) into (1), to obtain the physical Hamiltonian, it is very useful to Abelianise them by a suitable point transformation of the gauge fields.

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II. THE PHYSICAL HAMILTONIAN OF SU(2) YANG-MILLS THEORY AND ITS STRONG COUPLING EXPANSION

Point transformation to the new set of adapted coordinates \([2]\), the 3 \(q_j\) \((j = 1, 2, 3)\) and the 6 elements \(S_{ik} = S_{ki}\) \((i, k = 1, 2, 3)\) of the positive definite symmetric 3 \(\times\) 3 matrix \(S\),

\[
A_{ai}(q, S) = O_{ak}(q) S_{ki} - \frac{1}{2g} \epsilon_{abc} \left( O(q) \partial_t O^T(q) \right)_{bc},
\]

where \(O(q)\) is an orthogonal 3 \(\times\) 3 matrix parametrised by the \(q_i\), leads to an Abelianisation of the Gauss law constraints

\[
G_a \Phi = 0 \quad \Leftrightarrow \quad \frac{\delta}{\delta q_i} \Phi = 0 \quad \text{(Abelianisation)}
\]

Equ. \([4]\) corresponds to the symmetric gauge \(\chi_i(A) = \epsilon_{ijk} A_{jk} = 0\). It has been proven in \([3]\) that, at least for strong coupling, the symmetric gauge exists, i.e. any time-independent gauge field can be carried over uniquely into the symmetric gauge, and in \([2]\) that both indices of the tensor field \(S\) are spatial indices, leaving \(S\) a colorless local field.

According to the general scheme of \([1]\), the correctly ordered physical quantum Hamiltonian in the symmetric gauge in terms of the colorless physical variables \(S_{ik}(x)\) and the corresponding canonically conjugate momenta \(P_{ik}(x) \equiv -i\delta/\delta S_{ik}(x)\) reads \([4]\)

\[
H(S, P) = \frac{1}{2} J^{-1} \int d^3x \ P_{ai} \ J P_{ai} + \frac{1}{2} \int d^3x (B_{ai}(S))^2 \\
- J^{-1} \int d^3x \int d^3y \ \left\{ (D_i(S)_{ma} P_{m}) (x) J (x a | D^{-2}(S) y b) (D_j(S)_{m} P_{nj}) (y) \right\}
\]

with the covariant derivative \(D_i(S)_{kl} \equiv \delta_{kl} \partial_i - g \epsilon_{klm} S_{mi}\), the Faddeev-Popov (FP) operator

\[
*D_{kl}(S) \equiv \epsilon_{kml} D_i(S)_{ml} = \epsilon_{kml} \partial_i - g \gamma_{kl}(S) , \quad \gamma_{kl}(S) \equiv S_{kl} - \delta_{kl} \text{tr} S
\]

and the Jacobian \(J \equiv \det |^D|\). The matrix element of a physical operator \(O\) is given by

\[
\langle \Psi'| O | \Psi \rangle \propto \int \prod_x \left[ dS(x) J \Psi'^x[S] O \Psi[S] \right].
\]

The inverse of the FP operator can be expanded in the number of spatial derivatives

\[
\langle x \ k | D^{-1} (S) | y \ l \rangle = -\frac{1}{g^2} \gamma_{kl}^{-1}(x) \delta(x - y) + \frac{1}{g^2} \gamma_{ka}^{-1}(x) \epsilon_{abc} \partial_c \left[ \gamma_{bl}^{-1}(x) \delta(x - y) \right] \\
- \frac{1}{g^2} \gamma_{ka}^{-1}(x) \epsilon_{abc} \partial_c \left[ \gamma_{bl}^{-1}(x) \delta(x - y) \right] + \ldots
\]

In order to perform a consistent expansion, also the non-locality in the Jacobian \(J\) has to be taken into account \([4]\). The Jacobian \(J\) factorizes \(J = J_0 \tilde{J}\) with the local

\[
J_0 \equiv \det |\gamma| = \prod_x \prod_{i < j} (\phi_i(x) + \phi_j(x)) , \quad (\phi_i = \text{eigenvalues of } S)
\]
and the non-local $\tilde{J}$, which can be included into the wave functional $\tilde{\Psi}(S) := \tilde{J}^{-1/2}\Psi(S)$ leading to the corresponding transformed Hamiltonian

$$\tilde{H}(S,P) := \tilde{J}^{1/2}H(S,P)\tilde{J}^{-1/2} = H(S,P)\bigg|_{J\to J_0} + V_{\text{measure}}(S).$$  \hspace{1cm} (10)

It is Hermitean with respect to the local measure $J_0$ on the cost of extra terms $V_{\text{measure}}$, and can be expanded in the number of spatial derivatives using (8).

Next, an ultraviolet cutoff $a$ is put by introducing an infinite spatial lattice of granulas $G(n,a)$, here cubes of length $a$, situated at sites $x = a n$ ($n \in Z^3$), and considering the averaged variables

$$S(n) := \frac{1}{a^3} \int_{G(n,a)} dx \ S(x)$$  \hspace{1cm} (11)

and discretised spatial derivatives relating the $S(n)$ of different granulas (see [4] for details).

After an appropriate rescaling of the dynamical fields a novel strong coupling expansion of the Hamiltonian in $\lambda = g^{-2/3}$ can be obtained \[4\]

$$\tilde{H} = \frac{g^{2/3}}{a} \left[ H_0 + \lambda \sum_{\alpha} V^{(\partial)}_{\alpha} + \lambda^2 \left( \sum_\beta V^{(\Delta)}_{\beta} + \sum_\gamma V_{\gamma}^{(\partial\partial\Delta)} \right) + O(\lambda^3) \right].$$  \hspace{1cm} (12)

as an alternative to existing strong coupling expansions \[5 \] and \[6\] based on Wilsonian lattice QCD. The "free part" in (12) is just the sum of Hamiltonians $H_0 = \sum_n H_0^{QM}(n)$ of Yang-Mills quantum mechanics of spatially constant fields \[7]-[10] at each site, and the $V^{(\partial)}_{\alpha}$ and $V^{(\Delta)}_{\beta}$ are interaction parts, relating different sites. The local measure $J_0 = \prod_n J_0^{QM}(n)$ is correspondingly the product of the quantum mechanical measures at each site. In terms of the principal-axes variables of the positive definite symmetric $3 \times 3$ matrix field $S$

$$S = R^T(\alpha, \beta, \gamma) \text{ diag } (\phi_1, \phi_2, \phi_3) \ R(\alpha, \beta, \gamma),$$  \hspace{1cm} (13)

with the $SO(3)$ matrix $R$ parametrized by the three Euler angles $\chi = (\alpha, \beta, \gamma)$, we find

$$J_0^{QM} \to \sin \beta \prod_{i<j} \left( \phi_i^2 - \phi_j^2 \right) \to 0 < \phi_1 < \phi_2 < \phi_3 \text{ (principle orbits)}. \hspace{1cm} (14)$$

and (with the intrinsic spin angular momenta $\xi_i$ )

$$H_0^{QM} = \frac{1}{2} \sum_{ij,k} \left[ \pi_i^2 - \frac{2i}{\phi_j - \phi_k} (\phi_j \pi_j - \phi_k \pi_k) + \xi_i^2 \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} + \phi_j^2 \phi_k^2 \right].$$  \hspace{1cm} (15)

Its low energy spectrum and eigenstates at any site $n$

$$H_0^{QM}(n)|\Phi_{i,M}^{(S)\pm}(n) = \epsilon^{(S)\pm}_i(n)|\Phi_{i,M}^{(S)\pm}(n),$$  \hspace{1cm} (16)

characterised by the quantum numbers of spin $S, M$ and parity $P$, are known with high accuracy \[10\]. Hence the eigenstates of $H_0$ in (12) are free glueball excitations of the lattice. The interactions $V$ \[12\] can be included using perturbation theory in $\lambda$. 

III. CALCULATION OF THE GLUEBALL SPECTRUM UP TO ORDER $\lambda^2$

Using 1st and 2nd order perturbation theory in $\lambda$ give the results [4]

$$E_{\text{vac}}^+ = \mathcal{N} \frac{2^{2/3}}{a} \left[ 4.1167 + 29.894\lambda^2 + \mathcal{O}(\lambda^3) \right],$$  \hspace{1cm} (17)$$

for the energy of the interacting glueball vacuum and

$$E_{1}^{(0)+}(k) - E_{\text{vac}}^+ = \left[ 2.270 + 13.511\lambda^2 + \mathcal{O}(\lambda^3) \right] \frac{g^{2/3}}{a} \left[ \frac{g^{2/3}}{a} k^2 + \mathcal{O}(\lambda^3) \right] + 0.488 \frac{a}{g^{2/3}} k^2 + \mathcal{O}((a^2 k^2)^2),$$  \hspace{1cm} (18)$$

for the energy spectrum of the interacting spin-0 glueball, up to $\lambda^2$ for the (+) b.c. and similar results for the (−) b.c. The first, zeroth order numbers, correspond to the result of Yang-Mills quantum mechanics. Note that Lorentz invariance asks for energy momentum relation $E = \sqrt{M^2 + k^2} \simeq M + (2M)^{-1} k^2$. The result, shown here for the scalar glueball, which limits itself to the terms in the Hamiltonian containing the Laplace-operator $\Delta$ as a first step, violates this condition by about a factor of two. Including all spin-orbit coupling terms in the Hamiltonian dropped in this first approach and considering all possible $J = L + S = 0$ states is expected restore Lorentz invariance.

To study the coupling constant renormalisation in the IR, consider the physical glueball mass

$$M = \frac{g_0^{2/3}}{a} \left[ \mu + cg_0^{-4/3} \right].$$  \hspace{1cm} (19)$$

Independence of the box size $a$ is given for the two cases, $g_0 = 0$ or $g_0^{4/3} = -c/\mu$. The first solution corresponds to the perturbative fixed point, and the second, if it exists ($c < 0$), to an infrared fixed point. My result for the lowest spin-0 glueball $c_1^{(0)}(\mu_1^{(0)}) = 5.95$ suggests, that no infrared fixed points exist, in accordance with the corresponding result of Wilsonian lattice QCD [11]. Solving the above equation (19) for positive ($c > 0$) one obtains

$$g_0^{2/3}(Ma) = \frac{Ma}{2\mu} + \sqrt{\left( \frac{Ma}{2\mu} \right)^2 - \frac{c}{\mu}}, \hspace{1cm} a > a_c := 2\sqrt{c\mu}/M$$  \hspace{1cm} (20)$$

with the physical glueball mass $M$. For a typical [12] $M \sim 1.6$ GeV we find $a_c \sim 1.4$ fm. Comparing the behaviour of the bare coupling constant (20), obtained for boxes of large size $a$, with those obtained for small boxes in [7, 8], should lead to information about the intermediate region, including the possibility of the existence of phase transitions.

The generalisation of the approach to include quarks is possible [13] and under current investigation.
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