Re-scale boosting for regression and classification

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Abstract—Boosting is a learning scheme that combines weak prediction rules to produce a strong composite estimator, with the underlying intuition that one can obtain accurate prediction rules by combining “rough” ones. Although boosting is proved to be consistent and overfitting-resistant, its numerical convergence rate is relatively slow. The aim of this paper is to develop a new boosting strategy, called the re-scale boosting (RBoosting), to accelerate the numerical convergence rate and, consequently, improve the learning performance of boosting. Our studies show that RBoosting possesses the almost optimal numerical convergence rate in the sense that, up to a logarithmic factor, it can reach the minimax nonlinear approximation rate. We then use RBoosting to tackle both the classification and regression problems, and deduce a tight generalization error estimate. The theoretical and experimental results show that RBoosting outperforms boosting in terms of generalization.

Index Terms—Boosting, re-scale boosting, numerical convergence rate, generalization error

I. INTRODUCTION

Contemporary scientific investigations frequently encounter a common issue of exploring the relationship between a response and a number of covariates. Statistically, this issue can be usually modeled to minimize either an empirical loss function or a penalized empirical loss. Boosting is recognized as a state-of-the-art scheme to attack this issue and has triggered enormous research activities in the past twenty years [11], [15], [18], [26].

Boosting is an iterative procedure that combines weak prediction rules to produce a strong composite learner, with the underlying intuition that one can obtain accurate prediction rules by combining “rough” ones. The gradient descent view [18] of boosting shows that it can be regarded as a step-wise fitting scheme of additive models. This statistical viewpoint connects various boosting algorithms to optimization problems with corresponding loss functions. For example, $L_2$ boosting [7] can be interpreted as a stepwise learning scheme to the $L_2$ risk minimization problem. Also, AdaBoost [16] corresponds to an approximate optimization of the exponential risk.

Although the success of the initial boosting algorithm (Algorithm 1 below) on many data sets and its “resistance to overfitting” were comprehensively demonstrated [7], [16], the problem is that its numerical convergence rate is usually a bit slow [24]. In fact, Livshits [24] proved that for some sparse target functions, the numerical convergence rate of boosting lies in $(C_0 k^{-0.1898}, C'_0 k^{-0.182})$, which is much slower than the minimax nonlinear approximation rate $O(k^{-1/2})$. Here and hereafter, $k$ denotes the number of iterations, and $C_0, C'_0$ are absolute constants. Various modified versions of boosting have been proposed to accelerate its numerical convergence rate and then to improve its generalization capability. Typical examples include the regularized boosting via shrinkage (RSBoosting) [12] that multiplies a small regularization factor to the step-size deduced from the linear search, regularized boosting via truncation (RTBoosting) [34] which truncates the linear search in a small interval and $\varepsilon$-boosting [20] that specifies the step-size as a fixed small positive number $\varepsilon$ rather than using the linear search.

The purpose of the present paper is to propose a new modification of boosting to accelerate the numerical convergence rate of boosting to the near optimal rate $O(k^{-1/2} \log k)$ . The new variant of boosting, called the re-scale boosting (RBoosting), bears the philosophy behind the faith “no pain, no gain”, that is, to derive the new estimator, we always take a shrinkage operator to re-scale the old one. This idea is similar to the “greedy algorithm with free relaxation” [30] or “sequential greedy algorithm” [33] in sparse approximation and is essentially different from Zhao and Yu’s Blasso [35], since the shrinkage operator is imposed to the composite estimator rather than the new selected weak learner. With the help of the shrinkage operator, we can derive different types of RBoosting such as the re-scale AdaBoost, re-scale Logitboost, and re-scale $L_2$ boosting for regression and classification.

We present both theoretical analysis and experimental verification to classify the performance of RBoosting with convex loss functions. The main contributions can be concluded as four aspects. At first, we deduce the
The rest of paper can be organized as follows. In Section 2, we introduce RBoosting and compare it with other related algorithms. In Section 3, we study the theoretical behaviors of RBoosting, where its numerical convergence, consistency and generalization error bound are derived. In Section 4, we employ a series of simulations to verify our assertions. In the last section, we draw a simple conclusion and present some further discussions.

II. RE-SCALE BOOSTING

In classification or regression problems with a covariate or predictor variable \( X \) on \( \mathcal{X} \subseteq \mathbb{R}^d \) and a real response variable \( Y \), we observe \( m \) i.i.d. samples \( \mathbf{Z}^m = \{(X_1, Y_1), \ldots, (X_m, Y_m)\} \) from an unknown distribution \( \mathcal{D} \). Consider a loss function \( \phi(f, y) \) and define \( Q(f) \) (true risk) and \( Q_m(f) \) (empirical risk) as

\[
Q(f) = \mathbb{E}_D \phi(f(X), Y),
\]

and

\[
Q_m(f) = \mathbb{E}_\mathbf{Z} \phi(f(X), Y) = \frac{1}{m} \sum_{i=1}^{m} \phi(f(X_i), Y_i),
\]

where \( \mathbb{E}_D \) is the expectation over the unknown true joint distribution \( \mathcal{D} \) of \( (X, Y) \) and \( \mathbb{E}_\mathbf{Z} \) is the empirical expectation based on the sample \( \mathbf{Z}^m \).

Let \( S = \{g_1, \ldots, g_m\} \) be the set of weak learners (classifiers or regressors) and define

\[
\text{Span}(S) = \left\{ \sum_{j=1}^{n} a_j g_j : g_j \in S, a_j \in \mathbb{R}, n \in \mathbb{N} \right\}.
\]

We assume that \( \phi \), therefore \( Q_m \), is Fréchet differentiable and denote by \( Q_m'(f, h) = (\nabla Q_m(f), h) \) the value of linear functional \( \nabla Q_m(f) \) at \( h \), where \( \nabla Q_m(f) \) satisfies, for all \( f, g \in \text{Span}(S) \),

\[
\lim_{t \to 0} \frac{1}{t} (Q_m(f + th) - Q_m(f)) = (\nabla Q_m(f), h).
\]

Then the gradient descent view of boosting \cite{18} can be interpreted as the following Algorithm \cite{11}.

**Algorithm 1. Boosting**

1. **Step 1 (Initialization):** Given data \{\((X_i, Y_i) : i = 1, \ldots, m\)\}, weak learner set (or dictionary) \( S \), iteration number \( k^* \), and \( f_0 \in \text{Span}(S) \).
2. **Step 2 (Projection of gradient):** Find \( g_k^* \in S \) such that
   \[
   -Q_m'(f_{k-1}, g_k^*) = \sup_{g \in S} -Q_m'(f_{k-1}, g).
   \]
3. **Step 3 (Linear search):** Find \( \beta_k^* \in \mathbb{R} \) such that
   \[
   Q_m(f_k + \beta_k^* g_k^*) = \inf_{\beta_k \in \mathbb{R}} Q_m(f_k + \beta_k g_k^*).
   \]
4. **Step 4 (Iteration):** Increase \( k \) by one and repeat Steps 2 and 3 if \( k < k^* \).

Although this original boosting algorithm was proved to be consistent \cite{3} and overfitting resistant \cite{17}, a series of studies \cite{9, 24, 29} showed that its numerical convergence rate is far slower than that of the best nonlinear approximant. The main reason is that the linear search in Algorithm \cite{11} makes \( f_k \) be not always the greediest one. In particular, as shown in Fig.1, if \( f_{k-1} \) walks along the direction of \( g_k \) to \( \theta_0 g_k \), then there usually exists a weak learner \( g \) such that the angle \( \alpha = \beta \). That is, after \( \theta_0 g_k \), continuing to walk along \( g_k \) is no more the greediest one. However, the linear search makes \( f_k-1 \) go along the direction of \( g_k \) to \( \theta_1 g_k \).

![Fig. 1. The drawback of boosting](image)

Under this circumstance, an advisable method is to control the step-size in the linear search step of Algorithm \cite{11}. Thus, various variants of boosting, comprising the RTboosting, RSboosting and \( \varepsilon \)-boosting, have been
developed based on different strategies to control the step-size. It is obvious that the main difficulty of these schemes roots in how to select an appropriate step-size. If the step size is too large, then these algorithms may face the same problem as that of Algorithm. On the other hand, if the step size is too small, then the numerical convergence rate is also fairly slow. Different from the aforementioned strategies that focus on controlling the step-size of $g_k$, we drive a novel direction to improve the numerical convergence rate and consequently, the generalization capability of boosting. The core idea is that if the approximation (or learning) effect of the $k$-th iteration is not good, then we regard $f_k$ to be too aggressive and therefore shrink it within a certain extent. That is, if a new iteration is employed, then we impose a re-scale operator on the estimator $f_k$. This is the reason why we call our new strategy as the re-scale boosting (RBoosting). The following Algorithm 2 depicts the main idea of RBoosting.

**Algorithm 2 Re-scale boosting**

Step 1 (Initialization): Given data $\{(X_i,Y_i) : i = 1,\ldots,m\}$, weak learner set $S$, a set of shrinkage degree $\{\alpha_k\}_{k=1}^\infty$, iteration number $k^*$, and $f_0 \in \text{Span}(S)$.

Step 2 (Projection of gradient): Find $g^* \in S$ such that

$$-Q'_m(f_{k-1},g^*) = \sup_{g \in S} -Q'_m(f_{k-1},g).$$

Step 3 (Linear search): Find $\beta^*_k \in \mathbb{R}$ such that

$$Q_m((1-\alpha_k)f_k+\beta^*_kg^*_k) = \inf_{\beta \in \mathbb{R}} Q_m((1-\alpha_k)f_k+\beta g^*_k).$$

Update $f_{k+1} = (1-\alpha_k)f_k + \beta^*_k g^*_k$. 

Step 4 (Iteration): Increase $k$ by one and repeat Step 2 and Step 3 if $k < k^*$.

It should be also pointed out that the present paper is not the first one to apply relaxed greedy-type algorithms in the realm of boosting. In particular, for the $L_2$ loss, XGAR has already been utilized to design a boosting-type algorithm for regression in. Since in both GAFR and XGAR, one needs to tune two parameters simultaneously in an optimization problem, GAFR and XGAR are time-consuming when faced with a general convex loss function. This problem is successfully avoided in RBoosting.

### III. Theoretical behaviors of RBoosting

In this section, we study the theoretical behaviors of RBoosting. We hope to address three basic issues regarding RBoosting, including its numerical convergence rate, consistency and generalization error estimate.

To state the main results, some assumptions concerning the loss function $\phi$ and dictionary $S$ should be presented. The first one is a boundedness assumption of $S$.

**Assumption 1**: For arbitrary $g \in S$ and $x \in X$, there exists a constant $C_1$ such that

$$\sum_{i=1}^n g_i^2(x) \leq C_1.$$

Assumption 1 is certainly a bit stricter than the assumption $\sup_{g \in S,x \in X} |g_i(x)| \leq 1$ in. Introducing such a condition is only for the purpose of deriving a fast numerical convergence rate of RBoosting with general convex loss functions. In fact, for a concrete loss function such as the $L_p$ loss with $1 \leq p \leq \infty$, Assumption 1 can be relaxed to $\sup_{g \in S,x \in X} |g_i(x)| \leq 1$. Assumption 1 essentially depicts the localization properties of the weak learners. Indeed, it states that, for arbitrary fixed $x \in X$, expert for a small number of weak learners, all the $|g_i(x)|$’s are very small. Thus, it holds for almost all the widely used weak learners such as the trees, stumps, neural networks and splines. Moreover, for arbitrary dictionary $S' = \{g_1,\ldots,g_n\}$, we can rebuild it as $S = \{g_1,\ldots,g_n\}$ with $g_i = g_i/(\sqrt{\sum_{i=1}^n (g_i)^2(x)})$. It should be noted that Assumption 1 is the only condition concerning the dictionary throughout the paper, which is different from that additionally imposed either VC-dimension or Rademacher complexity constraints to the weak learner set $S$.

We then give some restrictions to the loss function, which have already adopted in. These restrictions are given by

**Assumption 2**: (i) If $|f(x)| \leq R_1$, $|y| \leq R_2$, then there exists a continuous function $H_\phi$ such that

$$|\phi(f,y)| \leq H_\phi(R_1,R_2).$$ (1)
\begin{enumerate}
\item [(ii)] Let $D = \{f : Q_m(f) \leq Q_m(0)\}$ and $f^* = \min_{f \in D} Q_m(f)$. Assume that $\forall c_1, c_2$ satisfying $Q_m(f^*) \leq c_1 < c_2 \leq Q_m(0)$, there holds
\begin{align*}
0 & \leq \inf \{Q''_m(f, g) : c_1 < Q(f) < c_2, g \in S\} \\
& \leq \sup \{Q''_m(f, g) : Q_m(f) < c_2, h \in S\} < \infty.
\end{align*}

It should be pointed out that (i) concerns the boundedness of $\phi$ and therefore is mild. In fact, if $R_1$ and $R_2$ are bounded, then (i) implies that $\phi(f, y)$ is also bounded. It is obvious that (i) holds for almost all commonly used loss functions. Once $\phi$ is given, $H_\phi(R_1, R_2)$ can be determined directly. For example, if $\phi$ is the $L_2$ loss for regression, then $H_\phi(R_1, R_2) \leq (R_1 + R_2)^2$; if $\phi$ is the exponential loss for classification, then $H_\phi(R_1, R_2) \leq \exp\{R_1\}$; if $\phi$ is the logistic loss for classification, then $H_\phi(R_1, R_2) \leq \log(1 + \exp\{R_1\})$.

As $Q_m(f) = \sum_{i=1}^m \phi(f(X_i), Y_i)$, conditions (2) and (3) actually describe the strict convexity and smoothness of $\phi$ as well as $Q_m$. Condition (2) guarantees the strict convexity of $Q_m$ in a certain direction. Under this condition, the maximization (and minimization) in projection of gradient step (and linear search step) of Algorithms 1 and 2 are well defined. Condition (3) determines the smoothness property of $Q_m(f)$. For arbitrary $f(x) \in [-\lambda, \lambda]$, define the first and second moduli of smoothness of $Q_m(f)$ as
\begin{align*}
\rho_1(Q_m, u) &= \sup_{f, ||h||=1} |Q_m(f + uh) - Q_m(f)|, \\
\rho_2(Q_m, u) &= \sup_{f, ||h||=1} |Q_m(f + uh) - Q_m(f - uh) - 2Q_m(f)|, \quad (4)
\end{align*}

where $||\cdot||$ denotes the uniform norm. It is easy to deduce that if (3) holds, then there exist constants $C_2$ and $C_3$ depending only on $\lambda$ and $c_2$ such that
\begin{align*}
\rho_1(Q_m, u) &\leq C_2 ||u||, \quad \text{and} \quad \rho_2(Q_m, u) \leq C_3 ||u||^2. \quad (4)
\end{align*}

It is easy to verify that all the widely used loss functions such as the $L_2$ loss, exponential loss and logistic loss satisfy Assumption 2.

By the help of the above stations, we are in a position to present the first theorem, which focuses on the numerical convergence rate of RBoosting.

\textbf{Theorem 1}: Let $f_k$ be the estimator defined by Algorithm 2. If Assumptions 1 and 2 hold and $\alpha_k = \frac{3}{k+3}$, then for any $h \in \text{Span}(S)$, there holds
\begin{align*}
Q_m(f_k) - Q_m(h) \leq C(||h||_1^2 + \log k)k^{-1}, \quad (5)
\end{align*}

where $C$ is a constant depending only on $c_1, c_2, C_1$, and
\begin{align*}
||h||_1 &= \inf_{(a_j)_{j=1}^n \in \mathbb{R}^n} \sum_{j=1}^n |a_j|, \quad \text{for} \quad h = \sum_{j=1}^n a_jg_j.
\end{align*}

If $\phi(f, y) = (f(x) - y)^2$ and $S$ is an orthogonal basis, then there exists an $h^* \in \text{Span}(S)$ with bounded $||h^*||_1$ such that (5) holds.

\begin{align*}
|Q_m(f_k) - Q_m(h^*)| \geq Ck^{-1},
\end{align*}

where $C$ is an absolute constant. Therefore, the numerical convergence rate deduced in (5) is almost optimal in the sense that for at least some loss functions (such as the $L_2$ loss) and certain dictionaries (such as the orthogonal basis), up to a logarithmic factor, the deduced rate is optimal. Compared with the relaxed greedy algorithm for convex optimization \cite{10}, \cite{20} that achieves the optimal numerical convergence rate, the rate derived in (5) seems a bit slower. However, in \cite{10}, \cite{20}, the set $D = \{f : Q_m(f) \leq Q_m(0)\}$ is assumed to be bounded. This is a quite strict assumption and, to the best of our knowledge, it is difficult to verify whether the widely used $L_2$ loss, exponential loss and logistic loss satisfy this condition. In Theorem 1 we omit this condition in the cost of adding an additional logarithmic factor to the numerical convergence rate and some other easy-checked assumptions to the loss function and dictionary.

Finally, we give an explanation why we select the shrinkage degree $\alpha_k$ as $\alpha_k = \frac{3}{k+3}$. From the definition of $f_k$, it follows that the numerical convergence rate may depend on the shrinkage degree. In particular, Bagirov et al. \cite{1}, Barron et al. \cite{2} and Temlyakov \cite{28} used different $\alpha_k$ to derive the optimal numerical convergence rates of relaxed-type greedy algorithms. After checking our proof, we find that our result remains correct for arbitrary $\alpha_k = \frac{C_4}{c_kk^{\frac{1}{4}} + C_5} < 1$ with $C_4, C_5, C_6$ some finite positive integers. The only difference is that the constant $C$ in (5) may be different for different $\alpha_k$. We select $\alpha_k = \frac{3}{k+3}$ only for the sake of brevity.

Now we turn to derive both the consistency and learning rate of RBoosting. The consistency of the boosting-type algorithms describes whether the risk of boosting can approximate the Bayes risk within arbitrary accuracy when $m$ is large enough, while the learning rate depicts its convergence rate. Several authors have shown that Algorithm 1 with some specific loss functions is consistent. Three most important results can be found in \cite{3}, \cite{4}, \cite{22}. Jiang \cite{22} proved a process consistency property for Algorithm 1 under certain assumptions. Process consistency means that there exists a sequence $\{t_m\}$ such that if boosting with sample size $m$ is stopped after $t_m$ iterations, its risk approaches the Bayes
risk. However, Jiang imposed strong conditions on the underlying distribution: the distribution of $X$ has to be absolutely continuous with respect to the Lebesgue measure. Furthermore, the result derived in [22] didn’t give any hint on when the algorithm should be stopped since the proof was not constructive. [3], [4] improved the result of [22] and demonstrated that a simple stopping rule is sufficient for consistency: the number of iterations is a fixed function of $m$. However, it can also be found in [3], [4] that the deduced learning rate was fairly slow. [3, Th.6] showed that the risk of boosting converges to the Bayes risk within a logarithmic speed.

Without loss of generality, we assume $|Y_i| \leq M$ almost surely with $M > 0$. The following Theorem 2 plays a crucial role in deducing both the consistency and fast learning rate of RBoosting.

**Theorem 2:** Let $f_k$ be the estimator obtained in Algorithm 2. If $\alpha_k = \frac{3}{k+3}$ and Assumptions 1 and 2 hold, then for arbitrary $h \in \text{Span}(S)$, there holds

$$E\{Q(f_k) - Q(h)\} \leq C(||h||_1^2 + \log k)k^{-1} + C'(H_\phi(\log k, M) + H_\phi(||h||_1, M)) \frac{k(\log m + \log k)}{m},$$

where $C$ and $C'$ are constants depending only on $c_1, c_2$ and $C_1$.

Before giving the consistency of RBoosting, we should give some explanations and remarks to Theorem 2. Firstly, we present the values of $H_\phi(\log k, M)$ and $H_\phi(||h||_1, M)$. Taking $H_\phi(\log k, M)$ for example, if $\phi$ is the $L_2$ loss for regression, then $H_\phi(\log k, M) = (\log k + M)^2$, if $\phi$ is the logistic loss for classification, then $H_\phi(\log k, M) = \log(k + 1)$ and if $\phi$ is the exponential loss for classification, then $H_\phi(\log k, M) = k$. Secondly, we provide a simple method to improve the bound in Theorem 2. Let $\pi_M f(x) := \min\{M, |f(x)|\}\text{sgn}(f(x))$ be the truncation operator at level $M$. As $Y \in [-M, M]$ almost surely, there holds [36]

$$E\{Q(\pi_M f_k) - Q(h)\} \leq E\{Q(f_k) - Q(h)\}.$$  

Noting that there is not any computation to do such a truncation, this truncation technique has been widely used to rebuild the estimator and improve the learning rate of boosting [1-4]. However, this approach has a drawback: the usage of the truncation operator entails that the estimator $\pi_M f_k$ is (in general) not an element of $\text{Span}(S)$. That is, one aims to build an estimator in a class and actually obtains an estimator out of it. This is the reason why we do not introduce the truncation operator in Theorem 2. Indeed, if we use the truncation operator, then the same method as that in the proof of Theorem 2 leads to the following Corollary 1.

**Corollary 1:** Let $f_k$ be the estimator obtained in Algorithm 2. If $\alpha_k = \frac{3}{k+3}$ and Assumptions 1 and 2 hold, then for arbitrary $h \in \text{Span}(S)$, there holds

$$E\{Q(\pi_M f_k) - Q(h)\} \leq C(||h||_1^2 + \log k)k^{-1} + C'(H_\phi(\log k, M) + H_\phi(||h||_1, M)) \frac{k(\log m + \log k)}{m},$$

where $C$ and $C'$ are constants depending only on $c_1, c_2$ and $C_1$.

By the help of Theorem 2, we can derive the consistency of RBoosting.

**Corollary 2:** Let $f_k$ be the estimator obtained in Algorithm 2. If $\alpha_k = \frac{3}{k+3}$, Assumptions 1 and 2 hold and

$$k \to \infty, \frac{H_\phi(\log k, M)k\log m}{m} \to 0, \text{when } m \to \infty,$$

then

$$E\{Q(f_k)\} \to \inf_{f \in \text{Span}(S)} Q(f), \text{ when } m \to \infty.$$  

Corollary 2 shows that if the number of iterations satisfies (6), then RBoosting is consistent. We should point out that if the loss function is specified, then, we can deduce a concrete relation between $k$ and $m$ to yield the consistency. For example, if $\phi$ is the logistic function, then the condition (6) becomes $k \sim m^\gamma$ with $0 < \gamma < 1$. This condition is somewhat looser than the previous studies concerning the consistency of boosting [3], [4], [22] or its modified version [2], [34].

When used to both classification and regression, there usually is an overfitting resistance phenomenon of boosting as well as its modified versions [7], [34]. Our result shown in Corollary 2 looks to contradict it at the first glance, as $k$ must be smaller than $m$. We illustrate that this is not the case. It can be found in [7], [34] that expect for Assumption 1 there is another condition such as the covering number, VC-dimension, or Rademacher complexity imposed to the dictionary. We highlight that if the dictionary of RBoosting is endowed with a similar complexity imposed to the dictionary. We highlight that whether RBoosting is overfitting resistant depends on the dictionary. At last, we give a learning rate analysis of RBoosting, which is also a consequence of Theorem 2.

**Corollary 3:** Let $f_k$ be the estimator obtained in Algorithm 2. Suppose that $\alpha_k = \frac{3}{k+3}$ and Assumptions 1 and 2 hold. For arbitrary $h \in \text{Span}(S)$, if $k$ satisfies

$$k \sim \sqrt{\frac{m}{H_\phi(\log k, M) + H_\phi(||h||_1, M)}},$$

then for arbitrary $h \in \text{Span}(S)$, there holds

$$E\{Q(\pi_M f_k) - Q(h)\} \leq C(||h||_1^2 + \log k)k^{-1} + C'(H_\phi(\log k, M) + H_\phi(||h||_1, M)) \frac{k(\log m + \log k)}{m},$$

where $C$ and $C'$ are constants depending only on $c_1, c_2$ and $C_1$.
we can derive

\[ E\{Q(f_k) - Q(h)\} \leq C'(\sqrt{H_\phi(\log m, M)} + H_\phi(\|h\|_1, M)) \]

\[ + \|h\|^2_1 m^{-1/2} \log m, \]

where \( C \) and \( C' \) are constants depending only on \( c_1, c_2 \) and \( C_1 \) and \( M \).

The learning rate \( (7) \) together with the stopping criteria \( (7) \) depends heavily on \( \phi \). If \( \phi \) is the logistic loss for classification, then \( H_\phi(\log m, M) = \log(m + 1) \) and \( H_\phi(\|h\|_1, M) = \log(\|h\|_1 + 1) \), we thus derive from \( (8) \) that,

\[ E\{Q(f_k) - Q(h)\} \leq C'(\log(m+1) + \|h\|^2_1) m^{-1/2} \log m. \]

We encourage the readers to compare our result with [34] Th.3.2. Without the Rademacher assumptions, RBoosting theoretically performs at least the same as that of RTBoosting. If \( \phi \) is the \( L_2 \) loss for regression, we can deduce that

\[ E\{Q(f_k) - Q(h)\} \leq C'(\log(m + \|h\|^2_1)) m^{-1/2} \log m, \]

which is almost the same as the result in [11]. If \( \phi \) is the exponential loss for classification, by setting \( k \sim m^{1/3} \), we can derive

\[ E\{Q(f_k) - Q(h)\} \leq C'(\log m + e^{\|h\|_1}) m^{-1/3} \log m, \]

which is much faster than AdaBoost [3]. It should be noted that if the truncation operator is imposed to the RBoosting estimator, then the learning rate of the re-scale AdaBoost can also be improved to

\[ E\{Q(\pi_M f_k) - Q(h)\} \leq C'(\log m + e^{\|h\|_1}) m^{-1/2} \log m. \]

IV. Numerical Results

In this section, we conduct a series of toy simulations and real data experiments to demonstrate the promising outperformance of the proposed RBoosting over the original boosting algorithm. For comparison, three other popular boosting-type algorithms, i.e., \( \epsilon \)-boosting [20], RBoosting [18], and RTBoosting [34], are also considered. In the following experiments, we utilize the \( L_2 \) loss function for regression (namely, L2Boost) and logistic loss function for classification (namely, LogitBoost).

Furthermore, we use the CART [6] (with the number of splits \( J = 4 \)) to build up the weak learners for regression tasks in the toy simulations and decision stumps (with the number of splits \( J = 1 \)) to build up the weak learners for regression tasks in real data experiments and all classification tasks.

A. Toy simulations

We first consider numerical simulations for regression problems. The data are drawn from the following model:

\[ Y = m(X) + \sigma \cdot \varepsilon, \]

where \( X \) is uniformly distributed on \([-2, 2]^d\) with \( d \in \{1, 10\} \), \( \varepsilon \), independent of \( X \), is the standard gaussian noise and the noise level \( \sigma \) varies among in \( \{0, 0.3, 0.6, 1\} \). Two typical regression functions [11] are considered in the simulations. One is a univariate piecewise function defined by

\[ m_1(x) = \begin{cases} 10 \sqrt{-x} \sin(8\pi x), & x < 0, \\ 0, & \text{else} \end{cases} \]

and the other is a multivariate continuously differentiable sine function defined as

\[ m_2(x) = \sum_{j=1}^{10} (-1)^{j-1} x_j \sin(x_j^2). \]

For these regression functions and all values of \( \sigma \), we generate a training set of size 500, and then collect an independent validation data set of size 500 to select the parameters of each boosting algorithms: the number of iterations \( k \), the regularization parameter \( \nu \) of RSboosting, the truncation value of RTboosting, the shrinkage degree of RBoosting and \( \varepsilon \) of \( \varepsilon \)-boosting. In all the numerical examples, we chose \( \nu \) and \( \varepsilon \) from a 20 points set whose elements are uniformly localized in \([0.01, 1]\). We select the truncated value of RTboosting the same as that in [34]. To tune the shrinkage degree, \( \alpha_k = 2/(k + u) \), we employ 20 values of \( u \) which are drawn logarithmic equally spaced between 1 to 10^6. To compare the performances of all the mentioned methods, a test set of 1000 noiseless observations is used to evaluate the performance in terms of the root mean squared error (RMSE).

Table I documents the mean RMSE over 50 independent runs. The standard errors are also reported (numbers in parentheses). Several observations can be easily drawn from Table I. Firstly, concerning the generalization capability, all the variants essentially outperform the original boosting algorithm. This is not a surprising result since all the variants introduce an additional parameter. Secondly, RBoosting performs as the almost optimal variant since its RMSEs are the smallest or almost smallest for all the simulations. This means that, if we only focus on the generalization capability, then RBoosting is a preferable choice.

In the second toy simulation, we consider the “orange data” model which was used in [37] for binary classification. We generate 100 data points for each class to build
up the training set. Both classes have two independent standard normal inputs \( x_1, x_2 \), but the inputs for the second class conditioned on \( 4.5 \leq x_1^2 + x_2^2 \leq 8 \). Similarly, to make the classification more difficult, independent feature noise \( q \) were added to the inputs. One can find more details about this data set in [37].

Table II reports the classification accuracy of five boosting-type algorithms over 50 independent runs. Numbers in parentheses are the standard errors. In this simulation, for \( q \) varies among \( \{0, 2, 4, 6\} \), we generate a validation set of size 200 for tuning the parameters, and then 4000 observations to evaluate the performances in terms of classification error. For this classification task, RBoosting outperforms the original boosting in terms of the generalization error. It can also be found that as far as the classification problem is concerned, RBoosting is at least comparable to other variants. Here we do not compare the performance with the performance of SVMs reported in [37], because the main purpose of our simulation is to highlight the outperformance of the proposed RBoosting over the original boosting.

All the above toy simulations from regression to classification verify the theoretical assertions in the last section and illustrate the merits of RBoosting.

### B. Real Data Examples

In this subsection, we pursue the performance of RBoosting on eight real data sets (the first five data sets for regression and the others for classification). The first data set is the Diabetes data set [13]. This data set contains 442 diabetes patients that are measured on ten independent variables, i.e., age, sex, body mass index etc. and one response variable, i.e., a measure of disease progression. The second one is the Boston Housing data set created from a housing values survey in suburbs of Boston by Harrison and Rubinfeld [21]. This data set contains 506 instances which include thirteen attributions, i.e., per capita crime rate by town, proportion of non-retail business acres per town, average number of rooms per dwelling etc. and one response variable, i.e., median value of owner-occupied homes. The third one is the Concrete Compressive Strength (CCS) data set created from [32]. The data set contains 4177 instances which were measured on eight independent variables, i.e., age and ingredients etc. and one dependent variable, i.e., quantitative concrete compressive strength. The fourth one is the Prostate cancer data set derived from a study of prostate cancer by Blake et al. [5]. The data set consists of the medical records of 97 patients who were about to receive a radical prostatectomy. The predictors are eight clinical measures, i.e., cancer volume, prostate weight, age etc. and one response variable, i.e., the logarithm of prostate-specific antigen. The fifth one is the Abalone data set, which comes from an original study in [25] for predicting the age of abalone from physical measurements. The data set contains 4177 instances which were measured on eight independent variables, i.e., length, sex, height etc. and one response variable, i.e., the number of rings. For classification task, three benchmark data sets are considered, namely Spam, Ionosphere and WDBC, which can be obtained from UCI Machine Learning Repository. Spam data contains 4601 instances, and 57 attributes. These data are used to measure whether an instance is considered to be spam. WDBC (Wisconsin Diagnostic Breast Cancer) data contains 569 instances, and 30 features. These data are used to identify whether an instance is diagnosed to be malignant or benign. Ionosphere data contains 351 instances, and 34 attributes. These data are used to measure whether an instance was “good” or “bad”. For each real data, we randomly (according to the uniform distribution) select 50\% data for training, 25\% data to build the validation set for tuning the parameters and the remainder 25\% data as the test set for evaluating the performances of different boosting-type algorithms. We repeat such randomization 20 times and report the average errors and standard errors (numbers from physical measurements).
in parentheses) in Table III. The parameter selection strategies of all boosting-type algorithms are the same as those in the toy simulations. It can be easily observed that, all the variants outperform the original boosting algorithm to a large extent. Furthermore, RBoosting at least performs as the second best algorithm among all the variants. Thus, the results of real data coincide with our theoretical assertions. That is, all the experimental results show that the new idea “re-scale” of RBoosting is numerically efficient and comparable to the idea “regularization” of other variants of boosting. This paves a new road to improve the performance of boosting.

V. PROOF OF THEOREM 1

To prove Theorem 1 we need the following three lemmas. The first one is a small generalization of [28, Lemma 2.3]. For the sake of completeness, we give a simple proof.

**Lemma 1:** Let \( j_0 > 2 \) be a natural number. Suppose that three positive numbers \( c_1 < c_2 \leq j_0 \), \( C_0 \) be given. Assume that a sequence \( \{a_n\}_{n=1}^{\infty} \) has the following two properties:

(i) For all \( 1 \leq n \leq j_0 \),

\[
    a_n \leq C_0 n^{-c_1},
\]

and, for all \( n \geq j_0 \),

\[
    a_n \leq a_{n-1} + C_0 (n - 1)^{-c_1}.
\]

(ii) If for some \( v \geq j_0 \) we have

\[
    a_v \geq C_0 v^{-c_1},
\]

then

\[
    a_{v+1} \leq a_v (1 - c_2/v).
\]

Then, for all \( n = 1, 2, \ldots \), we have

\[
    a_n \leq 2^{1 + \frac{2 + c_1}{2 - c_2}} C_0 n^{-c_1}.
\]

**Proof:** For \( 1 \leq v \leq j_0 \), the inequality

\[
    a_v \geq C_0 v^{-c_1}
\]

implies that the set

\[
    V = \{ v : a_v \geq C_0 v^{-c_1} \}
\]

does not contain \( v = 1, 2, \ldots, j_0 \). We now prove that for any segment \( [n, n + k] \subset V \), there holds

\[
    k \leq (2^{\frac{c_1}{2} - c_1} - 1)n.
\]

Indeed, let \( n \geq j_0 + 1 \) be such that \( n - 1 \notin V \), which means

\[
    a_{n+j} \geq C_0 (n + j)^{-c_1}, \quad j = 0, 1, \ldots, k.
\]

Then by the conditions (i) and (ii), we get

\[
    a_{n+k} \leq a_n \Pi_{v=n}^{n+k-1} (1 - c_2/v) \leq (a_{n-1} + C_0 (n - 1)^{-c_1}) \Pi_{v=n}^{n+k-1} (1 - c_2/v).
\]

Thus, we have

\[
    (n + k)^{-c_1} \leq \frac{a_{n+k}}{C_0} \leq 2(n - 1)^{-c_1} \Pi_{v=n}^{n+k-1} (1 - c_2/v),
\]

where \( c_2 \leq j_0 \leq v \). Taking logarithms and using the inequalities

\[
    \ln(1 - t) \leq -t, \quad t \in [0, 1);
\]

TABLE II

| \( q \) | Boosting | RSboosting | RBoosting | RBoosting | \( \epsilon \)-boosting |
|-------|----------|------------|-----------|-----------|-----------------|
| 0     | 11.19(1.32)% | 10.36(1.16)% | 10.50(1.19)% | 10.44(1.12)% | **10.29(1.17)%** |
| 2     | 11.27(1.29)% | **10.48(1.24)%** | 10.71(1.19)% | 10.59(1.25)% | 10.60(1.28)% |
| 4     | 11.79(1.54)% | **10.79(1.21)%** | 11.07(1.41)% | 10.90(1.24)% | 10.94(1.26)% |
| 6     | 12.02(1.62)% | 10.93(1.21)% | 11.20(1.23)% | **10.91(1.28)%** | 11.02(1.32)% |

TABLE III

| dataset | Boosting | RSboosting | RBoosting | RBoosting | \( \epsilon \)-boosting |
|---------|----------|------------|-----------|-----------|-----------------|
| Diabetes | 59.0371(4.1959) | **55.3109(3.6591)** | 56.1343(3.2543) | 55.6552(4.5351) | 57.7947(3.3970) |
| Housing  | 4.4126(0.5311)  | 4.2742(0.7026)  | 4.3685(0.3929)  | 4.1752(0.3406)  | **4.1244(0.3322)** |
| CCS     | 5.4345(0.5473)  | **5.2049(0.1678)** | 5.5826(0.1901)  | 5.3711(0.1807)  | 5.9621(0.1960)  |
| Prostate | 0.3131(0.0598)  | 0.1544(0.0682)  | 0.2450(0.0631)  | **0.1193(0.0360)** | 0.1939(0.0545)  |
| Abalone  | 2.2180(0.0710)  | 2.1934(0.0504)  | 2.3633(0.0762)  | **2.1922(0.0574)** | 2.2998(0.0474)  |
| Spam    | 6.06(0.60)%     | 5.13(0.52)%     | 5.24(0.48)%     | **5.06(0.55)%**  | 5.02(0.51)%    |
| Ionosphere | 8.27(2.88)%   | 5.89(1.92)%     | 6.09(2.24)%     | **5.23(2.31)%**  | 5.92(2.64)%    |
| WDBC    | 5.31(2.11)%     | 2.45(1.39)%     | 2.69(1.58)%     | **2.09(1.55)%**  | 2.52(1.33)%    |
$$\sum_{v=n}^{m-1} v^{-1} \geq \int_{n}^{m} t^{-1} dt = \ln(m/n),$$

we can derive that

$$-c_1 \ln n + k \ln n - 1 \leq \ln 2 + \sum_{v=n}^{n+k-1} \ln(1 - c_2 v/n)$$

$$\leq \ln 2 - \sum_{v=n}^{n+k-1} c_2/v \leq \ln 2 - c_2 \ln n + k/n.$$  

Hence,

$$(c_2 - c_1) \ln(n + k) \leq \ln 2 + (c_2 - c_1) \ln n + c_1 \ln n - 1,$$

which implies

$$n + k \leq 2^{(c_1 + 1)/(c_2 - c_1)} n$$

and

$$k \leq \left(2^{(c_1 + 1)/(c_2 - c_1)} - 1\right) n.$$  

Let us take any $m \in \mathbb{N}$. If $m \notin V$, we have the desired inequality. Assume $m \in V$ and let $[n, n + k]$ be the maximal segment in $V$ containing $m$, then we obtain

$$a_m \leq a_n \leq a_{n-1} + C_0(n-1)^{-c_1} \leq 2C_0(n-1)^{-c_1}$$

$$\leq 2C_0 m^{-c_1} \left(\frac{n-1}{m}\right)^{-c_1}.$$  

Since $k \leq \left(2^{c_1+1} - 1\right) n$, we then have

$$\frac{m}{n-1} \leq \frac{n + k}{n} \leq 2^{c_1+1}.$$  

This means that

$$a_m \leq 2C_0 m^{-c_1} 2^{c_1+1},$$

which finishes the proof of Lemma 1. \hfill \blacksquare

The convexity of $Q_m$ implies that for any $f, g$,

$$Q_m(g) \geq Q_m(f) + Q'_m(f, g - f),$$

or, in other words,

$$Q_m(f) - Q_m(g) \leq Q'_m(f, g - f) = -Q'_m(g, f - f).$$

Based on this, we can obtain the following lemma, which was proved in [30] Lemma 1.1.

**Lemma 2:** Let $Q_m$ be a Fréchet differential convex function. Then the following inequality holds for $f \in D$

$$0 \leq Q_m(f + ug) - Q_m(f) - uQ'_m(f, g) \leq 2\rho(A, u\|g\|).$$

To aid the proof, we also need the following lemma, which can be found in [27]Lemma 2.2.

**Lemma 3:** For any bounded linear $F$ and any dictionary $S$, we have

$$\sup_{g \in S} F(g) = \sup_{f \in M_1(S)} F(f),$$

where $M_1(S) = \{\text{span}(S) : \|f\|_1 \leq 1\}$.

**Proof of Theorem:** We divide the proof into two steps. The first step is to deduce an upper bound of $f_k$ in the uniform metric. Since $f_{k+1} = (1 - \alpha_{k+1})f_k + \beta_{k+1}g_{k+1}^*$, we have

$$f_k = f_{k+1} + \frac{\alpha_{k+1} f_{k+1} - \beta_{k+1}^* g_{k+1}^*}{1 - \alpha_{k+1}}.$$  

Noting $Q_m(f)$ is twice differential, if we use the Taylor expansion around $f_{k+1}$, then

$$Q_m(f_k) = Q_m(f_{k+1}) + \frac{Q''_m(f_{k+1}, f_{k+1})}{2}$$

$$\leq Q_m(f_{k+1}) + \frac{\alpha_{k+1} Q'_m(f_{k+1}, f_{k+1})}{2(1 - \alpha_{k+1})}$$

$$\leq Q_m(f_{k+1}) + \frac{Q'_m(f_{k+1}, f_{k+1})}{2(1 - \alpha_{k+1})} \geq Q_m(f_{k+1}, g_{k+1}).$$  

where

$$\hat{f} = (1 - \theta) Q_m(f_{k+1}, g_{k+1}) + \theta f_{k+1}$$

for some $\theta \in (0, 1)$. For the convexity of $Q_m$, we have

$$\frac{(\beta_{k+1}^*)^2}{2(1 - \alpha_{k+1})^2} Q_m(f_{k+1}, g_{k+1}) \geq 0.$$  

Furthermore, if we use the fact that $f_{k+1}$ is the minimum on the path from $(1 - \alpha_{k+1})f_k$ along $g_{k+1}^*$, then it is easy to see that

$$Q'_m(f_{k+1}, g_{k+1}) = 0.$$  

According to the convexity of $Q_m$ again, we obtain

$$Q'_m(f_{k+1}, f_{k+1}) \geq Q_m(f_{k+1}) - Q_m(0).$$

Noting that $\frac{\alpha_{k+1}}{1 - \alpha_{k+1}} = \frac{4}{k}$, we obtain

$$Q_m(f_k) \geq Q_m(f_{k+1}) + \frac{4}{k} (Q_m(f_{k+1}) - Q_m(0))$$

$$+ \frac{(\beta_{k+1}^*)^2}{2} Q_m(f_{k+1}, g_{k+1}).$$

If we write $B = \inf \{Q'_m(f, g) : c_1 < Q_m(f) < c_2, g \in S\}$, then we have

$$(\beta_{j+1}^*)^2 \leq \frac{2}{B} \left( Q_m(f_j) - Q_m(f_{j+1}) + \frac{4}{3} Q_m(0) \right).$$
Therefore, 
\[ \sum_{j=0}^{k} (\beta_j^*)^2 \leq \frac{2 \log k}{B}. \]

Then it follows from the definition of \( f_k \) that
\[
\begin{align*}
    f_k &= (1 - \alpha_k)(1 - \alpha_{k-1}) \cdots (1 - \alpha_2) \beta_1^* g_1^* \\
    &+ (1 - \alpha_k)(1 - \alpha_{k-1}) \cdots (1 - \alpha_3) \beta_2^* g_2^* \\
    &+ \ldots + (1 - \alpha_k) \beta_{k-1}^* g_{k-1}^* + \beta_k^* g_k^*.
\end{align*}
\]

Therefore, it follows from the Assumption 1 that
\[ |f_k(x)| \leq \sqrt{\sum_{j=0}^{k} (\beta_j^*)^2 \sum_{j=0}^{n} |g_j^*(x)|^2} \leq \sqrt{2C_1 \log k / B}. \] (12)

Now we turn to the second step, which derives the numerical convergence rate of RBoosting. For arbitrary \( \beta_k \in \mathbb{R} \) and \( g_k \in S \), it follows from Lemma 2 that
\[
Q_m((1 - \alpha_{k+1}) f_k + \beta_{k+1} g_{k+1}) = Q_m(f_k - \alpha_{k+1} f_k + \beta_{k+1} g_{k+1}) \leq Q_m(f_k) - \beta_{k+1} (-Q'_m(f_k; g_{k+1})) - \alpha_{k+1} Q'_m(f_k, f_k) + 2\rho(Q_m, \beta_{k+1} g_{k+1} - \alpha_{k+1} f_k)).
\]

From Step 2 in Algorithm 2, \( g_{k+1} \) satisfies
\[ -Q'_m(f_k, g_{k+1}) = \sup_{g \in S} -Q'_m(f_k, g) \]

Set \( \beta_k = \|h\|_1 \alpha_k \). It follows from Lemma 3 that
\[ \sup_{g \in S} -Q'_m(f_k, g) = \sup_{\phi \in \mathcal{M}(S)} -Q'_m(f_k, \phi) \geq -\|h\|_1 \|h\|_1 Q'_m(f_k, h). \]

Under this circumstance, we get
\[
\begin{align*}
    Q_m((1 - \alpha_{k+1}) f_k + \beta_{k+1} g_{k+1}) &\leq Q_m(f_k - \alpha_{k+1} (-Q'_m(f_k; h - f_k)) + 2\rho(Q_m, \beta_{k+1} g_{k+1} - \alpha_{k+1} f_k)).
\end{align*}
\]

Based on Lemma 2 we obtain
\[ -Q'_m(f_k, h - f_k) \geq Q_m(f_k) - Q_m(h). \]

Thus,
\[
\begin{align*}
    Q_m(f_{k+1}) &= Q_m((1 - \alpha_{k+1}) f_k + \beta_{k+1} g_{k+1}) \\
    &\leq Q_m(f_k) - \alpha_{k+1} Q_m(f_k - Q - m(h)) + 2\rho(Q_m, \|h\|_1 \alpha_{k+1} g_{k+1} - \alpha_{k+1} f_k)).
\end{align*}
\]

Furthermore, according to (12), we obtain
\[
\begin{align*}
    \|h\|_1 \alpha_{k+1} g_{k+1} &\leq \|h\|_1 \alpha_{k+1} f_k \\
    &\leq \|h\|_1 \alpha_{k+1} + \alpha_{k+1} \|f_k\|_1 \\
    &\leq \|h\|_1 \alpha_{k+1} + \alpha_{k+1} \|f_k\|_1 \\
    &\leq (\|h\|_1 + \sqrt{2C_1 \ln k}) \alpha_{k+1}.
\end{align*}
\]

Therefore,
\[
\begin{align*}
    &Q_m(f_{k+1}) - Q_m(h) \leq Q_m(f_k) - Q_m(h) \\
    &- \alpha_{k+1} (Q_m(f_k) - Q_m(h)) + 2\rho(Q_m, \|h\|_1 + \sqrt{2C_1 \log k / B}) \alpha_{k+1} \end{align*}
\]

Now, we use the above inequality and Lemma 1 to prove Theorem 1. Let \( a_k = Q_m(f_{k+1}) - Q_m(h) \). Let \( c_3 \in (1, 2] \) and \( C_0 \) be selected later. We then prove the conditions (i) and (ii) of Lemma 1 hold for an appropriately selected \( C_0 \). Set
\[ C_0 = 2 + \frac{A(0)}{2} + \frac{72C_2}{25} \|h\|_1^2 + \frac{288C_1 C_2 \log k}{25B^2}. \]

Then, it follows from (13) and \( \rho(Q_m, u) \leq C_2 u^2 \) that
\[ a_1 \leq \frac{A(0)}{4} + \frac{9C_2}{8} \|h\|_1^2 \leq C_0, \quad a_2 \leq C_0 2^{-1}, \]

and for \( v \geq 2 \), there holds
\[ a_v \leq a_{v-1} - C_0 (v - 1)^{-1}. \]

Thus, the condition (i) of Lemma 1 holds with \( j_0 = 2 \), and \( a_v \geq C_0 v^{-1} \), then by (13), we get for \( v \geq 6 \),
\[
\begin{align*}
av_{v+1} &\leq a_v (1 - 3v + 3 + \frac{1}{v} + 2v + 2) \\
    &\leq a_v \left( 1 - \frac{3}{2v} \right).
\end{align*}
\]

Thus the condition (ii) of Lemma 1 holds with \( c_2 = \frac{A(0)}{2} \). Applying Lemma 1 we obtain
\[ Q_m(f_k) - Q_m(h) \leq C(\|h\|_1^2 + \log k) k^{1-1}, \]

where \( C \) is a constant depending only on \( B, C_1 \) and \( C_2 \). This finishes the proof of Theorem 1.

VI. PROOF OF THEOREM 2

To aid the proof of Theorem 2, we need the following two technical lemmas, both of them can be found in [36].

Let \( R > 0 \), we denote \( B_R \) as the closed ball of \( V_k = \text{Span}\{g_1^*, \ldots, g_k^*\} \) with radius \( R \) centered at origin:
\[ B_R = \{ f \in V_k : \|f\| \leq R \}. \]

Lemma 4: For \( R > 0 \) and \( \eta > 0 \), we have
\[ \log N(B_R, \eta) \leq C_3 k \log \left( \frac{4R}{\eta} \right), \]

where \( N(B_R, \eta) \) denotes the covering number of \( B_R \) with radius \( \eta \) under the uniform norm.

The following ratio probability inequality is a standard result in learning theory (see [36]).
Lemma 5: Let \( G \) be a set of functions on \( Z \) such that, for some \( c \geq 0 \), \( |g - E(g)| \leq B \) almost everywhere and \( E(g^2) \leq cE(g) \) for each \( g \in G \). Then, for every \( \varepsilon > 0 \), there holds
\[
P \left\{ \sup_{f \in \mathcal{G}} \frac{E(g) - \frac{1}{m} \sum_{i=1}^{m} g(z_i)}{\sqrt{E(g) + \varepsilon}} \geq \sqrt{\varepsilon} \right\}
\leq \mathcal{N}(\mathcal{G}, \varepsilon) \exp \left\{ -\frac{m\varepsilon}{2c + 2B^2\varepsilon} \right\}.
\]

Proof of Theorem 2: At first, we use the concentration inequality in Lemma 5 to bound
\[
Q(f_k) - Q(h) - (Q_m(f_k) - Q_m(h)).
\]
We need apply Lemma 5 to the set of functions \( \mathcal{F}_R \), where
\[
\mathcal{F}_R := \{ \psi(Z) = \phi(f(X), Y) - \phi(h(X), Y) : f \in B_R \}.
\]
Using the obvious inequalities \( \|f\|_{\infty} \leq R, \|y\| \leq M \) and \( \|h\|_{\infty} \leq \|h\|_1, \) from Assumption 1 it follows the inequalities
\[
|\psi(Z)| \leq \|h\|_1, M
\]
and
\[
E\psi^2 \leq (H_\phi(R, M) + H_\phi(\|h\|_1, M))E\psi.
\]
For \( \psi_1, \psi_2 \in \mathcal{F}_R \), it follows from Assumption 2 that there exists a constant \( C_4 \) such that
\[
|\psi_1(Z) - \psi_2(Z)| = |\phi(f_1, Y) - \phi(f_2, Y)|
\leq C_4|f_1(X) - f_2(X)|.
\]
We then get
\[
\mathcal{N}(\mathcal{F}_R, \varepsilon) \leq \mathcal{N}(B_R, \varepsilon/C_4).
\]
According to Lemma 4 there holds
\[
\log \mathcal{N}(\mathcal{F}_R, \varepsilon) \leq C_3k \log \left( \frac{4C_4R}{\varepsilon} \right).
\]
Employing Lemma 5 with \( B = c = H_\phi(R, M) + H_\phi(\|h\|_1, M) \)
and
\[
E\psi = Q(f) - Q(h), \quad \frac{1}{m} \sum_{i=1}^{m} \psi(Z_i) = Q_m(f) - Q_m(h),
\]
we have, for every \( \varepsilon > 0 \),
\[
\sup_{f \in B_R} \frac{Q(f) - Q(h) - (Q_m(f) - Q_m(h))}{\sqrt{Q(f) - Q(h) + \varepsilon}} \leq \sqrt{\varepsilon}
\]
with confidence at least
\[
1 - \exp \left\{ C_3k \log \left( \frac{4C_4R}{\varepsilon} \right) \right\} \exp \left\{ -\frac{3m\varepsilon}{8C(h, R, M)} \right\},
\]
where \( C(h, R, M) = (H_\phi(R, M) + H_\phi(\|h\|_1, M)) \).
It follows from (12) that \( f_k \in B_R \) with \( R = C_5 \log k \), then with confidence at least
\[
1 - \exp \left\{ C_3k \log \left( \frac{C_6 \log k}{\varepsilon} \right) \right\} \exp \left\{ -\frac{3m\varepsilon}{8C_1} \right\},
\]
there holds
\[
Q(f_k) - Q(h) - (Q_m(f_k) - Q_m(h)) \leq \sqrt{\varepsilon} \sqrt{Q(f_k) - Q(h) + \varepsilon}
\leq \frac{1}{2} (Q(f_k) - Q(h)) + \varepsilon,
\]
where \( C_1 = C(h, C_5 \log k, M) \). Therefore, with the same confidence, there holds
\[
Q(f_k) - Q(h) \leq 2(Q_m(f_k) - Q_m(h)) + 2\varepsilon.
\]
Since Assumptions 1 and 2 hold, it follows from Theorem 1 that for any function \( h \in \text{Span}(S) \), there holds
\[
Q_m(f_k) - Q_m(h) \leq C(\|h\|_2^2 + \log k)k^{-1},
\]
where \( C \) is a constant depending only on \( c_1, c_2 \) and \( C_1 \). Combining the last two inequalities yields that
\[
\mathcal{T} \leq \varepsilon
\]
holds with at least
\[
1 - \exp \left\{ C_3k \log \left( \frac{C_6 \log k}{\varepsilon} \right) \right\} \exp \left\{ -\frac{3m\varepsilon}{8C_1} \right\},
\]
where
\[
\mathcal{T} = \frac{Q(f_k) - Q(h) - C(\|h\|_2^2 + \log k)k^{-1}}{2}.
\]
For arbitrary \( \mu > 0 \), there holds
\[
E_{\rho^n}(\mathcal{T}) = \int_0^\infty \mathbb{P}\{\mathcal{T} > \varepsilon\} d\varepsilon
\leq \mu + \int_\mu^\infty \exp \left\{ C_3k \log \left( \frac{C_6 \log k}{\varepsilon} \right) - \frac{3m\varepsilon}{8C_1} \right\} d\varepsilon
\leq \mu + \exp \left\{ -\frac{3m\mu}{8C_1} \right\} \int_\mu^\infty \left( \frac{C_6 \log k}{\varepsilon} \right)^{C_3k} d\varepsilon
\leq \mu + \exp \left\{ -\frac{3m\mu}{8C_1} \right\} \left( \frac{C_6 \log k}{\mu} \right)^{C_3k} \mu.
\]
By setting \( \mu = \frac{C_1C_3k \log m + \log k}{3m} \), direct computation yields
\[
E(\mathcal{T}) \leq \frac{2C_1C_3k(\log m + \log k)}{3m}.
\]
That is,
\[
E\{Q(f_k) - Q(h)\}
\leq C(\|h\|_2^2 + \log k)k^{-1} + \frac{4C_1C_3k(\log m + \log k)}{3m},
\]
which finishes the proof Theorem 2. \( \blacksquare \)
VII. CONCLUSION AND FURTHER DISCUSSIONS

In this paper, we proposed a new idea to conquer the slow numerical convergence rate problem of boosting and then develop a new variant of boosting, named as the re-scale boosting (RBoosting). Different from other variants such as the $\varepsilon$-boosting, RTboosting, RS-boosting that control the step-size in the linear search step, RBoosting focuses on alternating the direction of linear search via implementing a re-scale operator on the composite estimator obtained by the previous iteration step. Both theoretical and experimental studies illustrated that RBoosting outperformed the original boosting and performed at least comparable to other variants of boosting. Theoretically, we proved that the numerical convergence rate of RBoosting was almost optimal in the sense that it cannot be essentially improved. Using this property, we then deduced a fairly tight generalization error bound of RBoosting, which was a new “record” for boosting-type algorithms. Experimentally, we showed that for a number of numerical experiments, RBoosting outperformed boosting, and performed at least the second best of all variants of boosting. All these results implied that RBoosting was an reasonable improvement of Boosting and the idea “re-scale” provided a new direction to improve the performance of boosting.

To stimulate more opinions from others on RBoosting, we present the following two remarks at the end of this paper.

Remark 1: Throughout the paper, up to the theoretical optimality, we can not provide any essential advantages of RBoosting in applications, which makes it difficult to persuade the readers to use RBoosting rather than other variants of boosting. We highlight that there may be two merits of RBoosting in applications. The first one is that, due to the good theoretical behavior, if the parameters of RBoosting are appropriately selected, then RBoosting may outperform other variants. This conclusion was partly verified in our experimental studies in the sense that for all the numerical examples, RBoosting performed at least the second best. The other merit is that, compared with other variants, RBoosting cheers a totally different direction to improve the performance of boosting. Therefore, it paves a new way to understand and improve boosting. Furthermore, we guess that if the idea of the “re-scale” in RBoosting and “regularization” in other variants of boosting are synthesized to develop a new boosting-type algorithm, such as the re-scale $\varepsilon$-boosting, re-scale RTboosting, then the performance may be further improved. We will keep working on this issue and report our progress in a future publication.

Remark 2: According to the “no free lunch” philosophy, all the variants improve the learning performance of boosting at the cost of introducing additional parameters, such as the truncated parameter in RTboosting, regularization parameter in RSboosting, $\varepsilon$ in $\varepsilon$-Boosting, and shrinkage degree in RBoosting. To facilitate the use of these variants, one should also present strategies to select such parameters. In particular, Elith et al. [14] showed that 0.1 is a feasible choice of $\varepsilon$ in $\varepsilon$-Boosting; Buhlmann and Hothorn [8] recommended the selection of 0.1 for the regularization parameter in RSboosting; Zhang and Yu [34] proved that $O(k^{-2/3})$ is a good value of the truncated parameter in RTboosting. One may naturally ask: how to select the shrinkage degree $\alpha_k$ in RBoosting? This is a good question and we find a bit headache to answer it. Admittedly, in this paper, we do not give any essential suggestion to practically attack this question. In fact, $\alpha_k$ plays an important role in RBoosting. If $\alpha_k$ is too small, then RBoosting performs similar as the original boosting, which can be regarded as a special RBoosting with $\alpha_k = 0$. If $\alpha_k$ is too large, an extreme case is $\alpha_k$ close to 1, then the numerical convergence rate of RBoosting is also slow. Although we theoretically present some values of the $\alpha_k$, the best one in applications, we think, should be selected via some model selection strategies. We leave this important issue into a future study [31], where the concrete role of the shrinkage degree will be revealed.

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