THE COHOMOLOGICAL HALL ALGEBRA OF A SURFACE AND FACTORIZATION COHOMOLOGY

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Abstract. For a smooth quasi-projective surface $S$ over $\mathbb{C}$ we consider the Borel-Moore homology of the stack of coherent sheaves on $S$ with compact support and make this space into an associative algebra by a version of the Hall multiplication. This multiplication involves data (virtual pullbacks) governing the derived moduli stack, i.e., the perfect obstruction theory naturally existing on the non-derived stack. By restricting to sheaves with support of given dimension, we obtain several types of Hecke operators. In particular, we study $R(S)$, the Hecke algebra of 0-dimensional sheaves. For the case $S = \mathbb{A}^2$, we show that $R(S)$ is an enveloping algebra and identify it, as a vector space, with the symmetric algebra of an explicit graded vector space. For a general $S$, we find the graded dimension of $R(S)$, using the techniques of factorization cohomology.

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0. Introduction

0.1. Motivation. A large part of the classical theory of automorphic forms for $GL_n$ over functional fields can be interpreted in terms of Hall algebras of abelian categories [32], [33]. Relevant here is $Coh(C)$, the category of coherent sheaves on a smooth projective curve $C/F_q$. Taking the Hall algebra of $Bun(C)$, the subcategory of vector bundles, produces (unramified) automorphic forms, while $Coh_0(C)$, the category of torsion sheaves, gives rise to the Hecke algebra.

The classical Hall algebra of a category such as $Coh(C)$ consists of functions on $(F_q)$-points of the moduli stack of objects and so admits various modifications, cf.[14, Ch. 8]. Most important is the cohomological Hall algebra (COHA) where we take the cohomology of the stack instead of the space of functions on the set of its points [37]. This allows us to work over more general fields such as $\mathbb{C}$.

Study of Hall algebras (classical or cohomological) of the categories $Coh(S)$ for varieties $S$ of dimension $d > 1$ can be therefore considered as a higher-dimensional analog of the theory of automorphic forms. In this paper we consider the case of surfaces ($d = 2$) over $\mathbb{C}$ and study their COHA. In this case we have a whole new range of motivations coming from gauge theory, where cohomology of the moduli spaces of instantons is an object of longstanding interest [49], [1], [8].

0.2. Description of the results. The familiar 2-fold subdivision into automorphic forms vs. Hecke operators now becomes 3-fold: we have categories $Coh_m(S)$, $m = 0, 1, 2$, of purely $m$-dimensional sheaves, see §4.1. Here, $Coh_2(S)$ consists of vector bundles, while $Coh_0(S)$ is the category of punctual sheaves. An important feature is that the COHA of $Coh_{m-1}(S)$ acts on that of $Coh_m(S)$ by Hecke operators.

We denote by $R(S)$ the COHA of the category $Coh_0(S)$. It is the most immediate analog of the unramified Hecke algebra of the classical theory and we relate it to objects studied before.

In the flat case $S = \mathbb{A}^2$, the algebra $R(\mathbb{A}^2)$ is identified with the direct sum, over $n \geq 0$, of the $GL_n$-equivariant Borel-Moore homology of the commuting varieties of $gl_n$.

Our first main result, Theorem 6.1.4, shows that $R(\mathbb{A}^2)$ is an enveloping algebra and is identified, as a graded vector space, with the symmetric algebra of an explicit graded vector space $\Theta$. It is convenient to write $\Theta = H^\text{BM}_*(\mathbb{A}^2) \otimes \Theta'$, where the first factor is 1-dimensional, in homological degree 4.

For a general surface $S$, the algebra $R(S)$ is non-commutative. Our second main result, Theorem 7.1.6, provides a version of Poincaré–Birkhoff–Witt theorem for $R(S)$. It exhibits a system of generators as well as determines the graded dimension of $R(S)$. More precisely, it establishes an isomorphism of graded vector spaces

$$(0.2.1) \quad \sigma : \text{Sym}(H^\text{BM}_*(S) \otimes \Theta') \cong R(S).$$

Like the classical PBW isomorphism for enveloping algebras, $\sigma$ is given by the symmetrized product map on the space of generators.

0.3. Role of factorization algebras. Our proof of Theorem 6.1.4 is based on the techniques of factorization homology [9], [19], [20], [43]. More precisely, we consider the cochain lift $\mathcal{R}(S)$ of $R(S)$. This can be seen as a homotopy associative algebra whose cohomology is $R(S)$. For any open set $U \subset S$ we have a similarly defined algebra $\mathcal{R}(U)$. Further, one can consider $U$ to be any open set in the complex analytic topology. In this case $Coh_0(U)$ can be considered as an analytic stack and so its Borel-Moore homology and our entire construction of the COHA make sense.

We prove in Theorem 7.5.4 that the assignment $U \mapsto \mathcal{R}(U)$ is a factorization coalgebra in the category of $E_1$- (i.e., homotopy associative dg-) algebras. This is a reflection of a more fundamental fact: $U \mapsto Coh_0(U)$ is a factorization algebra in the category of analytic stacks, see Proposition 7.4.3. These considerations allow us to lift $\sigma$ to a morphism of factorization coalgebras in the category of dg-vector spaces and deduce the global isomorphism from the local one, i.e., from the case when $S$ is an open ball which is equivalent to that of $S = \mathbb{A}^2$. 
In fact, the identification (0.2.1) is suggestive of *non-abelian Poincaré duality* (NAPD), compare [43, thm. 5.5.6.6], although it does not seem to be a formal consequence of it. NAPD can be extended to include, for instance, the classical Atiyah-Bott theorem on the cohomology of \( \text{Bun}_G(\Sigma) \), the moduli stack of holomorphic \( G \)-bundles on a compact Riemann surface \( \Sigma \), see [20]. In that latter setting, we have that 

\[
H^\bullet(BG) = \text{Sym}(V) \quad \text{(with } V \text{ being the space of characteristic classes for } G\text{-bundles)},
\]

\[
H^\bullet(\text{Bun}_G(\Sigma)) \cong \text{Sym}(H_\bullet(\Sigma) \otimes V).
\]

### 0.4. Derived nature of the COHA.

As a vector space, our COHA is the Borel-Moore homology of the Artin stack \( \text{Coh}(S) \) (the moduli stack of objects of \( \text{Coh}(S) \)), i.e., it is the cohomology of the dualizing complex:

\[
H^\bullet_{\text{BM}}(\text{Coh}(S)) = H^{-\bullet}(\text{Coh}(S), \omega_{\text{Coh}(S)}).
\]

Since \( S \) is a surface, \( \text{Coh}(S) \) is singular due to obstructions encoded by \( \text{Ext}^2 \), so the dualizing complex is highly non-trivial. However, \( \text{Coh}(S) \) is in fact a truncation of a finer object, the *derived moduli stack* \( R\text{Coh}(S) \), smooth in the derived sense, see [63], [61]. While the vector space underlying our COHA depends on \( \text{Coh}(S) \) alone, the multiplication makes appeal to the derived structure: we use the refined pullbacks corresponding to the perfect obstruction theories on \( \text{Coh}(S) \) and on the related stack of short exact sequences. So our construction has appearance of applying some cohomology theory to the derived stack \( R\text{Coh}(S) \) itself and using its natural functorialities for morphisms of derived stacks. More recently, this approach has been implemented by M. Porta and F. Sala [55] at the K-theoretical level.

### 0.5. Relation to other work.

The COHA of a surface that we consider here is a non linear analog of the COHA associated to the preprojective algebra of the Jordan quiver considered in [58], see, e.g., [59] for the case of arbitrary quivers. M. Kontsevich and Y. Soibelman introduced in [37] cohomological Hall algebras for 3-dimensional Calabi-Yau categories, by taking cohomology of the moduli stack of objects with coefficients in the natural perverse sheaf of “vanishing cycles” with respect to the Chern-Simons functional. Although the details of the approach have been worked out only for quiver-type situations, see, e.g., [10] for a comparison with [58], it seems applicable, in principle, to the category of compactly supported coherent sheaves on any 3-dimensional Calabi-Yau manifold \( M \). In particular, our COHA for a surface \( S \) should be related to the Kontsevich-Soibelman COHA for \( M \) the total space of the anticanonical bundle on \( S \).

Instead of Borel-Moore homology of the stack \( \text{Coh}(S) \), one can take its Chow groups or its algebraic K-theory, in particular, study K-theoretic analogs of the Hecke operators. This approach was developed by A. Negut [48] who studied the K-theoretic effect of explicit Hecke correspondences on the moduli spaces and, very recently, by Y. Zhao [65] who defined independently the K-theoretic Hall algebra of 0-dimensional sheaves by a method similar to ours. On the other hand, algebraic K-theory, being a more rigid object than homology, does not easily localize on the complex analytic topology and so determining the size of the resulting objects is more difficult.

In the particular case where \( S \) is the cotangent bundle to a smooth curve, other versions of the COHA (of 0-dimensional sheaves and of purely 1-dimensional sheaves) of \( S \) appeared recently in [47], [57].

After this paper first appeared on arXiv, there have been some important new developments. Thus, M. Porta and F. Sala [55] have defined a categorical and a K-theoretical version of the COHA for surfaces, using the derived enhancement of the stack of coherent sheaves. Further, A. Khan [36] introduced a motivic framework for Borel-Moore cohomology for Artin stacks which could potentially simplify the treatment of some questions considered in this paper.

### 0.6. Structure of the paper.

In §1 we discuss the basic generalities on groupoids and stacks, including higher stacks understood as homotopy sheaves of simplicial sets. We pay special attention to Dold-Kan and Maurer-Cartan (Deligne) stacks associated to 3-term complexes and dg-Lie algebras. These
constructions are used in §2 to describe stacks of extensions (needed for defining the Hall multiplication) and filtrations (needed to prove associativity).

In §3 we define and study the Borel-Moore homology of Artin stacks. This concept, which is a topological analog of A. Kresch’s concept of Chow groups for Artin stacks [38], can be defined easily once we have a good formalism of constructible derived categories and their functorialities \( f^{-1}, Rf_*, Rf_c, f! \). While in the “classical” approach (sheaves first, complexes later) this may present complications, cf. [50], [41] for a discussion, the modern point of view of homotopy descent cf. [21], allows a straightforward definition of the enhanced derived category of a stack as the \( \infty \)-categorical limit of the corresponding categories for schemes. The desired functorialities are also inherited from the case of schemes. We study virtual pullback in this context.

The COHA is defined in §4, first as a vector space, then as an associative algebra.

In §5 we consider subalgebras in the COHA corresponding to sheaves with various condition on the dimension of support. These subalgebras play the role of Hecke algebras, since they act on other subspaces in COHA (corresponding to sheaves whose dimension of support is bigger) by natural “Hecke operators” (operators formally dual to those of the Hall multiplication).

In §6 we study the flat Hecke algebra \( R(\mathbb{A}^2) \) by relating it to the earlier work on commuting varieties in \( \mathfrak{g}_n \). Here we prove Theorem 6.1.4.

Finally, in §7 we globalize the consideration of §6 by describing the global Hecke algebra \( R(S) \) as the factorization (co)homology of an appropriate factorization (co)algebra. This leads to the proof of Theorem 7.1.6.

The paper has two Appendices. Appendix A, logically preceding the entire paper, provides a reminder on \( \infty \)-categories and dg-categories. Appendix B spells out the homotopy unique nature of Euler (top Chern) classes and orientation classes at the cochain level. It logically depends on §1-3 (i.e., assumes the formalism of stacks presented in these sections) but precedes §7 for which it provides necessary material.

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1. Generalities on stacks

1.1. Groupoids and simplicial sets. A groupoid is a category \( G \) in which all morphisms are invertible. We write \( G = \{ G_1 \triangleright\triangleleft G_0 \} \) where \( G_0 = \text{Ob}(G) \) is the class of objects and \( G_1 = \text{Mor}(G) \) is the class of morphisms. For an essentially small groupoid \( G \) let \( \pi_0(G) \) be the set of isomorphisms classes of objects of \( G \). For any object \( x \in G_0 \) let \( \pi_1(G, x) = \text{Aut}_G(x) \) be the automorphism group of \( x \). All groupoids in the sequel will be assumed essentially small.

Small groupoids form a 2-category \( \mathcal{Gpd} \). For each groupoids \( G_1, G_2 \) we have a groupoid whose objects are functors \( G_1 \to G_2 \) and morphisms are natural transformations of functors. We will refer to 1-morphisms of \( \mathcal{Gpd} \) as simply morphisms of groupoids. Considered with this notion of morphisms, groupoids form a category which we denote \( \text{Gpd} \). Let \( \text{Eq} \subset \text{Mor}(\text{Gpd}) \) be the class of equivalences of groupoids.
Proposition 1.1.1. Let $f : G 	o G'$ be a morphism of groupoids. Suppose that $f$ induces a bijection of sets $\pi_0(G) \to \pi_0(G')$ and, for any $x \in \text{Ob}(G)$, an isomorphism of groups $\pi(G, x) \to \pi_1(G', f(x))$. Then $f$ is an equivalence of groupoids.

**Proof.** The conditions just mean that $f$ is essentially surjective and fully faithful hence an equivalence. \qed

For a category $C$ let $\Delta^cC$ be the category of simplicial objects in $C$. In particular, we will use the category $\Delta^cSet$ of simplicial sets and $\Delta^cAb$ of simplicial abelian groups. For a simplicial set $X$ let $|X|$ be its geometric realization. A morphism $f : X \to X'$ of simplicial sets is called a weak equivalence, if it induces a homotopy equivalence $|X| \to |X'|$. In this case we write $X \sim X'$. Let $W$ be the class of weak equivalences.

We also associate to any simplicial set $X$ its fundamental groupoid $\Pi X$. Objects of $\Pi X$ are vertices of $X$, i.e., elements $x \in X_0$, and, for $x, y \in X_0$, the set $\text{Hom}_{\Pi X}(x, y)$ consists of homotopy classes of paths in $|X|$ joining $x$ and $y$. Let $\pi_0(X)$ be the set of connected components of $|X|$, and, for each $i \geq 1$ and $x \in X_0$ let $\pi_i(X, x)$ be the topological homotopy groups of $|X|$ at $x$.

Dually, the nerve $NG$ of a groupoid $G$ is a simplicial set with the set of $m$-simplices being

$$N_mG = G_1 \times_{G_0} \cdots \times_{G_0} G_1 \quad (m \text{ times}).$$

The topological homotopy groups of $NG$ match those defined above algebraically for $G$:

$$\pi_0(NG) = \pi_0(G), \quad \pi_1(NG, x) = \pi_1(G, x), \quad \pi_i(NG, x) = 0, \quad i \geq 2.$$ A simplicial set is of groupoid type, if it is weak equivalent to the nerve of some groupoid. We denote by $\Delta^cSet^{\leq 1} \subset \Delta^cSet$ the full subcategory of simplicial sets of groupoid type.

**Proposition 1.1.3.**

(a) A simplicial set $X$ is of groupoid type if and only if $\pi_i(X, x) = 0$ for each $i \geq 2$, $x \in X_0$. Then, we have $X \simeq \text{NILX}$.\qed

(b) The functors $\Pi$, $N$ yield quasi-inverse equivalences of homotopy categories $\Delta^cSet^{\leq 1}[W^{-1}] \simeq \text{Gpd}[\text{Eq}^{-1}]$.

Let $\mathcal{A}$ be an abelian category. We denote by $\text{C}(\mathcal{A})$ the category of cochain complexes $K = (K^n, d^n : K^{n-1} \to K^n)_{n \in \mathbb{Z}}$ over $\mathcal{A}$ bounded below, with morphisms being morphisms of complexes. For $n \in \mathbb{Z}$ we denote by $\text{C}^{\leq n}(\mathcal{A})$ the category of complexes concentrated in degrees $\leq n$. For $K \in \text{C}(\mathcal{A})$ we denote by

$$K^{\leq n} = \{ \cdots \to K^{n-1} \xrightarrow{d^n} K^n \xrightarrow{0} \cdots \} \in \text{C}^{\leq n}(\mathcal{A}),$$

$$\tau_{\leq n}K = \{ \cdots \to K^{n-1} \xrightarrow{d^n} \text{Ker}(d^{n+1}) \xrightarrow{0} \cdots \} \in \text{C}^{\leq n}(\mathcal{A})$$

its stupid and cohomological truncation in degrees $\leq n$. Note that $\tau_{\leq n}$ sends quasi-isomorphisms of complexes to quasi-isomorphisms.

**Examples 1.1.4 (Dold-Kan groupoids).** Let $\text{Ab}$ denote the category of abelian groups.

(a) Given a 3-term complex over $\text{Ab}$

$$K = \{ K^{-1} \xrightarrow{d^0} K^0 \xrightarrow{d^1} K^1 \},$$

we have the action groupoid

$$\pi K = \text{Ker}(d^1)/K^{-1} := \{ K^{-1} \times \text{Ker}(d^1) \Rightarrow \text{Ker}(d^1) \}$$

whose set of objects is $\text{Ker}(d^1)$ and whose morphisms $s \to t$ are given by $\{ h \in K^{-1}; s + d^0(h) = t \}$. Then we have

$$\pi_0(\pi K) = H^0(K), \quad \pi_1(\pi K, s) = H^{-1}(K), \quad \forall s \in \text{Ob } \pi K.$$
(b) The Dold-Kan correspondence $\text{DK} : \text{dg}^{-\leq 0} \text{Ab} \to \Delta^\circ \text{Ab}$ associates to a $\mathbb{Z}_{\leq 0}$-graded complex $K$ the simplicial abelian group $\text{DK}(K)$ such that

- $\text{DK}(K)_0 = K^0$,
- the set of edges joining $s, t \in K^0$ is $\{ h \in K^{-1} ; s + d^0(h) = t \}$,
- 2-simplices with given 1-faces are in bijection with certain elements of $K^{-2}$, and so on, see, e.g., [64, §8.4.1].

For each $i \geq 0$, we have an isomorphism $\pi_i(\text{DK}(K)) \simeq H^{-i}(K)$ which is independent of the base point. In fact, the correspondence preserves the respective standard model structures. In particular, for a 3-term complex $K$ as in (a), we have

$$\varkappa K = \Pi \text{DK}(\tau_{\leq 0} K).$$

Examples 1.1.6 (Maurer-Cartan groupoids). We will use a non-abelian generalization of Examples 1.1.4, due to Deligne, see [22], [23] and references therein, Hinich [28] and Getzler [18].

(a) Consider a (possibly infinite dimensional) dg-Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ situated in degrees $[0, 2]$: $$\mathfrak{g} = \{ \mathfrak{g}^0 \xrightarrow{d^0} \mathfrak{g}^1 \xrightarrow{d^1} \mathfrak{g}^2 \}. $$

Thus $\mathfrak{g}^0$ is an ordinary complex Lie algebra. We assume that it is nilpotent, so we have the nilpotent group $G^0 = \exp(\mathfrak{g}^0)$. By definition, $G^0$ consists of formal symbols $e^y, y \in \mathfrak{g}^0$ (so $G^0$ is identified with $\mathfrak{g}^0$ as a set), with the multiplication given by the Campbell-Hausdorff formula. The set of Maurer-Cartan elements of $\mathfrak{g}$ is $$\text{mc}(\mathfrak{g}) = \left\{ x \in \mathfrak{g}^1 ; d^1 x + \frac{1}{2} [x, x] = 0 \right\}.$$ The group $G^0$ acts on $\text{mc}(\mathfrak{g})$ by the formula

$$e^y \cdot x = e^{\text{ad}(y)}(x) + \frac{1 - e^{\text{ad}(y)}}{\text{ad}(y)}(d^1(y)),$$

see [23, p. 45]. We define the Maurer-Cartan groupoid $^1$ (or Deligne groupoid) of $\mathfrak{g}$ to be the action groupoid

$$\text{MC}(\mathfrak{g}) = \text{mc}(\mathfrak{g})/G^0 := \{ G^0 \times \text{mc}(\mathfrak{g}) \rightrightarrows \text{mc}(\mathfrak{g}) \}.$$ Note that if the dg-Lie algebra $\mathfrak{g}$ is abelian, i.e., if it reduces to a 3-term cochain complex, then $G^0 = \mathfrak{g}^0$ and it acts on $\text{mc}(\mathfrak{g}) = \text{Ker}(d^1)$ by translation, so we have $\text{MC}(\mathfrak{g}) = \varkappa(\mathfrak{g}[1])$ where $\varkappa$ is as in Example 1.1.4 (a).

(b) More generally, let $\mathfrak{g}$ be any nilpotent dg-Lie algebra over $\mathbb{C}$. The Maurer-Cartan simplicial set $\text{mc}_\bullet(\mathfrak{g})$ is defined by $$\text{mc}_\bullet(\mathfrak{g}) = \text{mc}(\mathfrak{g} \otimes_{\mathbb{C}} \Omega^\bullet_{\text{pol}}(\Delta^n)),$$

where $\Omega^\bullet_{\text{pol}}(\Delta^n)$ is the commutative dg-algebra of polynomial differential forms on the standard $n$-simplex, see [28], [18]. Further, in [18] it is proved that if $\mathfrak{g}$ is concentrated in degrees $[0, 2]$ then $N_\bullet(\text{MC}(\mathfrak{g}))$, is weak equivalent to $\text{mc}_\bullet(\mathfrak{g})$.

Proposition 1.1.8. A quasi-isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ of nilpotent dg-Lie algebras induces a weak equivalences of simplicial sets $\text{mc}_\bullet(\mathfrak{g}_1) \to \text{mc}_\bullet(\mathfrak{g}_2)$. In particular:

(a) If $\mathfrak{g}_1$, $\mathfrak{g}_2$ are concentrated in degrees $[0, 2]$, then $\phi$ induces an equivalence of groupoids $\text{MC}(\mathfrak{g}_1) \to \text{MC}(\mathfrak{g}_2)$.

(b) A quasi-isomorphism $K_1 \to K_2$ of cochain complexes as in Example 1.1.4(a) induces an equivalence of groupoids $\varkappa K_1 \to \varkappa K_2$.

$^1$In this paper we use the terms “Maurer-Cartan groupoid” and “Maurer-Cartan stack” in order to avoid clashes with the algebro-geometric notion of Deligne-Mumford stacks.
Let now \( p : g \to h \) be a surjective morphism of dg-Lie algebras, both situated in degrees \([0,2]\). Let \( n \subset g \) be the kernel of \( p \) and assume that there is an embedding \( i : h \to g \) with \( p \circ i = 1 \) such that \( g = h \ltimes n \) is the semi-direct product.

We have a functor of groupoids \( p_* : MC(g) \to MC(h) \). Recall that for a functor \( \phi : C \to D \) and an object \( x \in Ob(D) \), the fiber category \( \phi/x \) consists of pairs \((y,h)\) with \( y \in Ob(C) \) and \( h : \phi(y) \to x \) a morphism in \( D \), with the obvious notion of morphisms of such pairs. If \( C, D \) are groupoids, so is \( \phi/x \). We apply this when \( C = MC(g) \), \( D = MC(h) \) and \( \phi = p_* \). We get the fiber category \( p_*/x \). On the other hand, the object \( x \in Ob(MC(h)) \) being an element of \( mc(h) \), it gives a new differential \( d_x = d - \text{ad}(x) \) on \( n \), where we abbreviate \( x = i(x) \). Let \( n_x \) be the dg-Lie algebra with underlying Lie algebra \( n \) and differential \( d_x \).

**Proposition 1.1.9.** The fiber category \( p_*/x \) is equivalent to the groupoid \( MC(n_x) \).

### 1.2. Stacks and homotopy sheaves

Let \( \mathcal{S} \) be a Grothendieck site. We recall that a stack (of essentially small groupoids) on \( \mathcal{S} \) is a presheaf of groupoids \( B : T \to p_1(T), T \in Ob(\mathcal{S}) \), satisfying the 2-categorical descent condition extending that for sheaves of sets, see [...] for background. Stacks on \( \mathcal{S} \) form a 2-category \( St_\mathcal{S} \). We will refer to 1-morphisms of \( St_\mathcal{S} \) as morphisms of stacks and will denote by \( St_\mathcal{S} \) the category of stacks on \( \mathcal{S} \) with these morphisms. Let \( Eq \subset Mor(St_\mathcal{S}) \) be the class of equivalences of stacks.

**Remark 1.2.1.** For most purposes, the above 1-categorical point of view on stacks will be sufficient. However, in various constructions below such as gluing, the full 2-categorical structure on \( St_\mathcal{S} \) becomes important. In particular, as with objects of any 2-category, to define a stack “uniquely” (e.g., naively, in a way “independent” on some choices) means, more formally, to define it uniquely up to an equivalence which is defined uniquely up to a unique isomorphism.

A stack of groupoids \( B \) gives rise to a sheaf of sets \( \pi_0(B) \) on \( \mathcal{S} \), obtained by sheafifying the presheaf \( T \mapsto \pi_0(B(T)) \). Similarly, for any \( T \in Ob(\mathcal{S}) \) and any object \( x \in B(T) \) we have a sheaf of groups \( \pi_1(B, x) \) on \( T \), i.e., on the site \( \mathcal{S}/T \), obtained by sheafifying the presheaf \( T' \mapsto \pi_1(B(T'), x|_{T'}) \), where \( x|_{T'} \) is the pullback by the morphism \( T' \to T \).

**Proposition 1.2.2.** Let \( f : B \to B' \) be a morphism in \( St_\mathcal{S} \) which induces an isomorphism of sheaves of sets \( \pi_0(B) \to \pi_0(B') \) and an isomorphism of sheaves of groups \( \pi_1(B, x) \to \pi_1(B', f(x)) \) for any \( T \in Ob(\mathcal{S}), x \in Ob(B(T)) \). Then \( f \) is an equivalence of stacks.

**Proof.** Follows from Proposition 1.1.1 by sheafification.

Let \( \Delta^\circ Set_\mathcal{S} \) be the category of presheaves of simplicial sets on \( \mathcal{S} \). Recall [63] that such a presheaf \( X \) is called a homotopy sheaf or an \( \infty \)-stack, if it satisfies descent in the homotopy sense. We denote by \( St_\mathcal{S}^\infty \) the category of homotopy sheaves of simplicial sets on \( \mathcal{S} \) and by \( W \subset Mor(St_\mathcal{S}^\infty) \) the class of weak equivalences (defined stalk-wise). A homotopy sheaf \( X \) gives rise to a sheaf of sets \( \pi_0(X) \) on \( \mathcal{S} \) and, for any \( T \in Ob(\mathcal{S}) \) and any vertex \( x \in X(T) \), a sheaf of groups \( \pi_1(X, x) \) on \( \mathcal{S}/T \). We have:

**Proposition 1.2.3.** Let \( f : X \to X' \) be a morphism in \( St_\mathcal{S}^\infty \). Suppose \( f \) induces an isomorphism of sheaves of sets \( \pi_0(X) \to \pi_0(X') \) and, for each \( T \in Ob(\mathcal{S}) \) and \( x \in X(T) \), an isomorphism of sheaves of groups \( \pi_1(X, x) \to \pi_1(X', f(x)) \). Then \( f \) is a weak equivalence.

**Proof.** If \( \mathcal{S} \) is a point, this is the standard: a map of simplicial sets is a weak equivalence iff it induces isomorphism on homotopy groups. The case of general \( \mathcal{S} \) is obtained from this by sheafification.
Any homotopy sheaf \( X \) gives a stack of groupoids \( \Pi X \) on \( \mathcal{S} \), defined by taking \( T \mapsto \Pi X(T) \). Any stack of groupoids \( B \) on \( \mathcal{S} \) gives rise to a homotopy sheaf \( N(B) \) taking \( T \) to the nerve of the groupoid \( B(T) \). A homotopy sheaf \( X \) is called of groupoid type, if it is weak equivalent to \( N(B) \) for some stack \( B \). We denote by \( St_{\mathcal{S}}^{\leq 1} \subset St_{\mathcal{S}}^{\leq 1} \) the full category of homotopy sheaves of groupoid type.

**Proposition 1.2.4.**

(a) A homotopy sheaf \( X \) is of groupoid type if and only if \( \pi_i(X, x) = 0 \) for each \( T \in \text{Ob}(\mathcal{S}) \), \( x \in X(T)_0 \) and \( i \geq 2 \).

(b) The functors \( \Pi, N \) induce mutually quasi-inverse equivalences of homotopy categories \( St_{\mathcal{S}}^{\leq 1}[W^{-1}] \cong St_{\mathcal{S}}[\text{Eq}^{-1}] \).  

\[ \Box \]

### 1.3. Artin and f-Artin stacks.

In this paper all schemes, algebras, etc., will be considered over the base field \( \mathbb{C} \) of complex numbers. Let \( \mathcal{Aff} \) be the category of affine schemes over \( \mathbb{C} \) equipped with the étale topology. We refer to [40], [52] for general background on Artin stacks, i.e., stacks of groupoids on \( \mathcal{Aff} \) with a smooth atlas and a representable, quasi-compact, quasi-separated diagonal.

**Examples 1.3.1.**

(a) Let \( G = \{ G_1 \xrightarrow{s} G_0 \} \) be a groupoid the category of schemes of finite type such that the source and target maps \( s, t \) are smooth morphisms. It gives rise to an Artin stack which we denote by \( \|G\| \). By definition, \( \|G\| \) is the stack associated with the prestack

\[ T \mapsto \{ \text{Hom}(T, G_1) \Rightarrow \text{Hom}(T, G_0) \}. \]

(b) In particular, let \( G \) be an affine algebraic group acting on a scheme \( Z \) of finite type. Then we have the action groupoid \( \{ G \times Z \rightrightarrows Z \} \) in the category of schemes of finite type. The corresponding Artin stack is denoted \( Z//G \) and is called the quotient stack of \( Z \) by \( G \). Explicitly, for \( T \in \mathcal{Aff} \) the groupoid \( (Z//G)(T) \) is identified with the category of pairs \( (P, u) \), where \( P \) is a \( G \)-torsor over \( T \) (locally trivial in étale topology) and \( u : P \to Z \) is a \( G \)-equivariant map.

**Definition 1.3.2.** An Artin stack \( B \) is called:

(a) Of finite type, if it is equivalent to the stack of the form \( \|G\|\) for a groupoid \( G \) as in Example 1.3.1(a).

(b) An f-Artin stack, if it is locally of finite type.

(c) A quotient (resp. locally quotient) stack is it is equivalent (resp. locally equivalent) to the stack of the form \( Z//G \) where \( Z, G \) are in Example 1.3.1(b).

All the stacks we will use will be f-Artin. Let the 2-category \( \mathcal{St} \) and the category \( St \) be the full 2-subcategory in \( \mathcal{St}_{\mathcal{Aff}} \) and the full subcategory in \( St_{\mathcal{Aff}} \) formed by f-Artin stacks.

Let \( \mathcal{Aff} \subset \mathcal{Aff} \) be the category of affine schemes of finite type with its étale topology. We note that f-Artin stacks are determined by their restrictions to \( \mathcal{Aff} \), and so we can and will consider them as stacks of groupoids on \( \mathcal{Aff} \).

Given an f-Artin stack \( B \), let \( \mathcal{St}_B \) be the 2-category of f-Artin stacks over \( B \), i.e., of f-Artin stacks \( X \) together with a morphism of stacks \( X \to B \). Objects of \( \mathcal{St}_B \) can, equivalently, be seen as stacks of groupoids over the Grothendieck site \( \mathcal{Aff}_B \) formed by affine schemes \( T \) of finite type together with a morphism of stacks \( f : T \to B \). Thus, an f-Artin stack \( X \) over \( B \) can be seen as associating to each \( T \in \mathcal{Aff}_B \) a groupoid \( X(T) \).
2. Stack of extensions and filtrations

2.1. Cone stacks. We refer to [50, 52] for general background on quasi-coherent sheaves on Artin stacks. For an f-Artin stack \( B \) we denote by \( QCoh(B) \), resp. \( Coh(B) \) the category of quasi-coherent, resp. coherent sheaves of \( O_B \)-modules. By a vector bundle we mean a locally free sheaf of finite rank.

Let \( B \) be an f-Artin stack and \( R = \bigoplus_{i \in \mathbb{N}} R^i \) be a graded quasi-coherent sheaf of \( O_B \)-algebras such that \( R^0 = O_B \), \( R^1 \) is coherent and \( R \) is generated by \( R^1 \) locally over \( B \). The relative affine \( B \)-scheme \( C = \text{Spec } R \) is called a cone over \( B \), see, e.g., [5, §1].

If \( \mathcal{E} \) is a coherent sheaf over \( B \), we get the associated cone \( C(\mathcal{E}) = \text{Spec}(\text{Sym}_{O_B}(\mathcal{E})) \) which is an affine group scheme over \( B \). Its value (the set of points) on \( (T \xrightarrow{f} B) \in \mathcal{A}ff_B \) is \( \text{Hom}_{O_T}(f^* \mathcal{E}, \mathcal{O}_T) \). We call such a cone an abelian cone.

For instance, the total space of a vector bundle \( \mathcal{E} \) over \( X \) is defined as

\[
\text{Tot}(\mathcal{E}) = C(\mathcal{E}^\vee) = \text{Spec} \text{Sym}_{O_B}(\mathcal{E}^\vee)
\]

where \( \mathcal{E}^\vee \) is the dual sheaf of \( O_B \)-modules. For any affine \( B \)-scheme \( f : T \rightarrow B \) we have

\[
\text{Tot}(\mathcal{E})(T) = H^0(T, f^* \mathcal{E}).
\]

Thus, a section \( s \in H^0(B, \mathcal{E}) \) is the same as a morphism \( B \rightarrow \text{Tot}(\mathcal{E}) \) of schemes over \( B \).

Any cone \( C = \text{Spec } R \) is canonically a closed subcone of the abelian cone \( \text{Spec}(\text{Sym}_{O_B}(R^1)) \), called the abelian hull of \( C \).

Example 2.1.2. Let \( d : \mathcal{E} \rightarrow \mathcal{F} \) be a morphism of vector bundles on \( B \). We denote by \( \text{Ker}(d) \subset \mathcal{E} \) the sheaf-theoretic kernel of \( d \). On the other hand, let \( \pi : \text{Tot}(\mathcal{E}) \rightarrow B \) be the projection. The morphism \( d \) determines a section \( s \) of the vector bundle \( \pi^* \mathcal{F} \) on \( \text{Tot}(\mathcal{E}) \), and we define the abelian cone \( \text{Ker}(d) \subset \text{Tot}(\mathcal{E}) \) as the zero locus of this section. We note that \( H^0(B, \text{Ker}(d)) \subset H^0(B, \mathcal{E}) \) consists precisely of those sections \( s \) which, considered as morphisms \( B \rightarrow \text{Tot}(\mathcal{E}) \), factor through the substack \( \text{Ker}(d) \).

A morphism of abelian cones over \( B \) is, by definition a morphism of group schemes over \( B \). Given a morphism of abelian cones \( E \rightarrow F \), we have an action of the affine group scheme \( E \) over \( B \) on \( F \). Hence, we can form the quotient Artin stack \( F//E \). Stacks of this form are called abelian cone stacks.

2.2. Total spaces of perfect complexes. Let \( B \) be an f-Artin stack. We denote \( C_{qcoh}(B) \) the category formed by complexes of \( O_B \)-modules with quasi-coherent cohomology. Let \( \text{qis} \) be the class of quasi-isomorphisms in \( C_{qcoh}(B) \) and \( D_{qcoh}(B) = C_{qcoh}(B)[\text{qis}^{-1}] \) be the corresponding derived category. For any integers \( p \leq q \) let \( C_{p,q}^{qcoh}(B) \subset C_{qcoh}(B) \) be the full subcategory formed by complexes situated in degrees from \( p \) to \( q \).

Definition 2.2.1. Let \( C \in C_{qcoh}(B) \) and \( p \leq q \) be integers.

(a) \( C \) is strictly \([p, q]\)-perfect, if \( C \) is quasi-isomorphic to a complex of vector bundles

\[
\{ C^p \xrightarrow{d^p} C^{p+1} \xrightarrow{d^{p+2}} \cdots \xrightarrow{d^q} C^q \}
\]

situated in degrees from \( p \) to \( q \). This complex is called a presentation of \( C \).

(b) \( C \) is \([p, q]\)-perfect, if, locally on \( B \), it is strictly \([p, q]\)-perfect and, moreover, the set of open substacks \( U \subset B \) such that \( C|_U \) is strictly \([p, q]\)-perfect, is filtering with respect to the partial order by inclusion.

For a \([p, q]\)-perfect complex \( C \) and an open \( U \subset B \) as above we will refer to a quasi-isomorphism \( C|_U \rightarrow C_U \), with \( C_U \) strictly \([p, q]\)-perfect, as a presentation of \( C \) over \( U \).

A \([p, q]\)-perfect complex \( C \) has a virtual rank \( \text{vrk}(C) \) which is a \( \mathbb{Z} \)-valued locally constant function on \( B \), i.e., a function constant on each connected component of \( B \). It is defined in terms of a presentation of \( C \) as \( \text{vrk}(C) = \sum_{i=p}^q (-1)^i \text{rk}(C^i) \).
We will be interested in making sense of total spaces of perfect complexes using (2.1.1) as a motivation, cf. [61, §3.3].

**Definition 2.2.2.**
(a) Let $\mathcal{C} \in \mathcal{C}_{qcoh}^{0}(B)$. We define the simplicial presheaf $\text{Tot}^{\infty}(\mathcal{C})$ on $\mathcal{A}ff_B$ by

$$\text{Tot}^{\infty}(\mathcal{C})(T) = \text{DK}(H^0(T, f^*\mathcal{C})), \quad (T \xrightarrow{f} B) \in \mathcal{A}ff_B.$$ 

(b) Let $\mathcal{C} \in \mathcal{C}_{qcoh}^{[-1,0]}(B)$. We define the pre-stack of groupoids $\text{Tot}(\mathcal{C})$ on $\mathcal{A}ff_B$ by

$$\text{Tot}(\mathcal{C})(T) = \varpi(H^0(T, f^*\mathcal{C})), \quad (T \xrightarrow{f} B) \in \mathcal{A}ff_B.$$ 

We call $\text{Tot}(\mathcal{C})$ the total space of $\mathcal{C}$.

**Proposition 2.2.3.**
(a) Let $\mathcal{C} \in \mathcal{C}_{qcoh}^{0}(B)$. The simplicial presheaf $\text{Tot}^{\infty}(\mathcal{C})$ is a homotopy sheaf. For any $x \in \text{Tot}^{\infty}(\mathcal{C})(T)_0$ we have (independently on the choice of base points)

$$\varpi(\text{Tot}^{\infty}(\mathcal{C})) = H^{-i}(\mathcal{C}), \quad i \geq 0.$$ 

A morphism $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ in $\mathcal{C}_{qcoh}^{0}(B)$ induces a morphism of homotopy sheaves $\phi_\ast : \text{Tot}^{\infty}(\mathcal{C}_1) \to \text{Tot}^{\infty}(\mathcal{C}_2)$ which is an equivalence, if $\phi$ is a quasi-isomorphism.

(b) Let $\mathcal{C} \in \mathcal{C}_{qcoh}^{[-1,0]}(B)$. The pre-stack $\text{Tot}(\mathcal{C})$ on $\mathcal{A}ff_B$ is a stack. The homotopy sheaf $\text{Tot}^{\infty}(\mathcal{C})$ is of groupoid type and $\Pi \text{Tot}^{\infty}(\mathcal{C}) = \text{Tot}(\mathcal{C})$. In particular, the total space is functorial and takes quasi-isomorphisms $\phi$ to isomorphisms $\phi_\ast$.

**Proof.** Part (a) follows from the fact that $\mathcal{C}$ is a sheaf and from the properties of the Dold-Kan correspondence. Part (b) follows by Proposition 1.2.4. \hfill \square

Recall that a stack morphism $f$ is called an l.c.i., i.e., a locally complete intersection morphism, if it factorizes as $f = p \circ i$ where $p$ is a smooth map and $i$ is a regular immersion.

**Proposition 2.2.4.**
(a) Let $\mathcal{C} \in \mathcal{C}_{qcoh}^{[-1,0]}(B)$ be strictly $[-1,0]$-perfect. Then we have a canonical equivalence of stacks of groupoids $u : \text{Tot}(\mathcal{C}) \to \text{Tot}(\mathcal{C}^0)//\text{Tot}(\mathcal{C}^{-1})$ on $\mathcal{A}ff_B$.

(b) Let $\mathcal{C} \in \mathcal{C}_{qcoh}^{[-1,0]}(B)$ be $[-1,0]$-perfect. Then $\text{Tot}(\mathcal{C})$ is an Artin stack over $B$.

(c) For any morphism $\phi$ of $[-1,0]$-perfect complexes, the induced morphism $\phi_\ast$ of stacks is an l.c.i.

**Proof.** Part (a) is similar to the proof of [26, lem 0.1]. That is, look at any $(T \xrightarrow{f} B) \in \mathcal{A}ff_B$. By definition, the groupoid $\text{Tot}(\mathcal{C})(T)$ is the category whose objects are elements of $x \in H^0(T, f^*\mathcal{C}^0)$ and a morphism $x \to x'$ is an element of $H^0(T, f^*\mathcal{C}^{-1})$ mapping by $d^0$ to $x'-x$. At the same time, the groupoid $(\text{Tot}(\mathcal{C}^1)//\text{Tot}(\mathcal{C}^0))(T)$ is the category of pairs consisting of an $f^*\mathcal{C}^{-1}$-torsor $P$ over $T$ and an $f^*\mathcal{C}^{-1}$-equivariant morphism $P \to \mathcal{C}^0$ of sheaves over $T$. We see that the former category is the full subcategory of the second consisting of data with the torsor $P$ being the standard trivial one, $P = f^*\mathcal{C}^{-1}$. This defines a fully faithful functor $u_T$, and such functors for all $T$ give the sought-for morphism of stacks $u$. Now, since $T$ is affine, $H^1(T, f^*\mathcal{C}^{-1}) = 0$ and so any torsor $P$ above is trivial. This means that the functor $u$ is (locally) essentially surjective hence an equivalence of stacks. This proves (a). Parts (b) and (c) follow from (a). \hfill \square

**Example 2.2.5.** Now, let $\mathcal{C}$ be a strictly $[-1,1]$-perfect complex

$$(2.2.6) \quad \mathcal{C} = \{ \mathcal{C}^{-1} \xrightarrow{d^0} \mathcal{C}^0 \xrightarrow{d^1} \mathcal{C}^1 \}.$$
The stupid truncation $\mathcal{C}^\leq 0 = \{C^{-1} \to C^0\}$ is strictly $[-1,0]$-perfect. We denote by
\[ \pi : \text{Tot}(C^0) \to B, \quad \overline{\pi} : \text{Tot}(\mathcal{C}^\leq 0) = \text{Tot}(C^0)/\text{Tot}(C^{-1}) \to B \]
the projections. We recall from Example 2.1.2(c) the abelian cone $\text{Ker}(d^1) \subset \text{Tot}(C^0)$ given as the zero locus of the section $s$ of $\pi^*C^1$ induced by $d^1$.

**Proposition 2.2.7.**
(a) If $\mathcal{C}$ is strictly $[-1,1]$-perfect, then we have a canonical equivalence of stacks $\text{Ker}(d^1)/C^{-1} \to \text{Tot}(\tau_{\leq 0}\mathcal{C})$, i.e., the section $s$ descends to a section $\overline{s}$ of $\overline{\pi}^*C^1$, and $\text{Tot}(\tau_{\leq 0}\mathcal{C})$ is the zero locus of $\overline{s}$.

(b) If $\mathcal{C}$ is $[-1,1]$-perfect, then $\text{Tot}(\tau_{\leq 0}\mathcal{C})$ is an Artin stack over $B$.

*Proof.* Part (a) is completely analogous to the proof of Proposition 2.2.4(a), with $\mathcal{C}^0$ replaced by $\text{Ker}(d^1)$. Part (b) follows from (a). \[ \square \]

We call $\text{Tot}(\tau_{\leq 0}\mathcal{C})$ the *truncated total space* of $\mathcal{C}$.

**Proposition 2.2.8.** Let $\mathcal{C}$ be a $[-1,1]$-perfect complex and $(T \xrightarrow{f} B) \in \text{Aff}_B$.

(a) For all $s \in \text{Ob}(\text{Tot}(\tau_{\leq 0}\mathcal{C})(T))$ we have
\[ \overline{\pi_0}(\text{Tot}(\tau_{\leq 0}\mathcal{C}))(s) = H^0(\mathcal{C}), \quad \overline{\pi_1}(\text{Tot}(\tau_{\leq 0}\mathcal{C}), s) = H^{-1}(f^*\mathcal{C}). \]

(b) The truncated total space of $[-1,1]$-perfect complexes is functorial and takes quasi-isomorphisms $\phi$ to isomorphisms $\phi_\mathcal{C}$.

*Proof.* Part (a) is a consequence of Proposition 2.2.7. Part (b) follows from (c). More precisely, a morphism (resp. quasi-isomorphism) $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ of $[-1,1]$-perfect complexes yields a morphism (resp. quasi-isomorphism) $\tau_{\leq 0}\mathcal{C}_1 \to \tau_{\leq 0}\mathcal{C}_2$ and the statement follows from Proposition 2.2.3(b). \[ \square \]

### 2.3. Stacks of extensions.
We now consider the following general situation. Let $B$ be an $f$-Artin stack and $p : Y \to B$ be a scheme of finite type over $B$. Let $E, F$ be coherent sheaves over $Y$ which are flat over $B$. We can form the object $\mathcal{C} \in D^b_{\text{qcoh}}(B)$ given by
\[ \mathcal{C} = Rp^* R\text{Hom}_{O_Y}(F,E)[1]. \]

Let SES be the stack over $B$ classifying short exact sequences $0 \to E \to G \to F \to 0$ of coherent sheaves over $Y$. That is, for any $B$-scheme $T \in \text{Aff}_B$ the objects of the groupoid SES(T) are short exact sequences
(2.3.1)
\[ 0 \to E|_T \to G \to F|_T \to 0 \]
of coherent sheaves of $O_Y \times_B T$-modules, and the morphisms are the isomorphisms of such sequences identical on the boundary terms. We then have
\[ \pi_0(\text{SES}(T)) = \text{Ext}^1_{O_Y \times_B T}(F|_T,\mathcal{E}|_T), \quad \pi_1(\text{SES}(T), G) = \text{Ext}^0_{O_Y \times_B T}(F|_T,\mathcal{E}|_T), \]
for any object $\mathcal{G}$ of SES(T). This implies identifications of sheaves of sets on $\text{Aff}_B$ and of sheaves of groups on $\text{Aff}_T$:
\[ \pi_0(\text{SES}) = H^0(\mathcal{C}), \quad \pi_1(\text{SES} \times_B T, \mathcal{G}) = H^{-1}(\mathcal{C}|_T). \]

These identifications, together with those of Proposition 2.5.2 (b), suggest the following.

**Proposition 2.3.4.** Assume that the complex $\mathcal{C}$ is $[-1,1]$-perfect. Then, we have an equivalence $\text{Tot}(\tau_{\leq 0}\mathcal{C}) = \text{SES}$ of cone stacks over $B$.

*Proof.* As pointed out, the $\overline{\pi}_0$ and $\overline{\pi}_1$ of the two stacks $\text{Tot}(\tau_{\leq 0}\mathcal{C})$ and SES are isomorphic. So it remains to construct a morphism of stacks inducing these identifications. For this, we first make some general discussion.
We recall [7], [35], [62] that for any Artin stack \( Z \) the category \( D^b_{qcoh}(Z) \) has a \( dg \)-thickening, i.e., there is a pre-triangulated \( dg \)-category \( C_{qcoh}(Z) \) with the same objects and spaces of morphisms being upgraded to complexes \( \text{RHom}_{C_{qcoh}(Z)}(\mathcal{K}, \mathcal{L}) \) of \( \mathbb{C} \)-vector spaces such that

\[
\text{Hom}_{\mathcal{O}_Z}(\mathcal{K}, \mathcal{L}) = H^0 \text{RHom}_{C_{qcoh}(Z)}(\mathcal{K}, \mathcal{L}).
\]

The complex \( \text{RHom} \) above can be explicitly found as

\[
\text{RHom}_{C_{qcoh}(Z)}(\mathcal{K}, \mathcal{L}) = \text{Hom}^\bullet_{\mathcal{O}_Z}(I(\mathcal{K}), I(\mathcal{L})),
\]

where \( I(\mathcal{K}) \) is a fixed injective resolution of \( \mathcal{K} \) for each \( \mathcal{K} \).

We now specialize to the case \( \mathcal{K} = \mathcal{F} \big|_T, \mathcal{L} = \mathcal{E} \big|_T[1] \), where \( T \in \mathcal{A}ff_B \) is an affine \( B \)-scheme. The complex of \( \mathbb{C} \)-vector spaces

\[
\tau_{\leq 0} \text{RHom}_{C_{qcoh}(Z)}(\mathcal{F}|_T, \mathcal{E}|_T[1])
\]

has cohomology only in degrees 0 and \(-1\), given by the Ext groups in (2.3.2). We consider the simplicial set

\[
X(T) = DK(\tau_{\leq 0} \text{RHom}_{C_{qcoh}(Z)}(\mathcal{F}|_T, \mathcal{E}|_T[1])),
\]

which is of groupoid type by Proposition 1.1.3(a). Its vertices are morphisms of complexes \( I(\mathcal{F}|_T) \to I(\mathcal{E}|_T[1]) \). The cone of such a morphism is a complex of sheaves which has only one cohomology sheaf, in degree \(-1\), and this sheaf \( \mathcal{G} \) fits into a short exact sequence as in (2.3.1). In this way we get a morphism of groupoids

\[
h(T) : \Pi X(T) \to \text{SES}(T).
\]

At the same time, by (1.1.5), the groupoid \( \Pi X(T) \) is equivalent to the groupoid \( \Gamma H^0(T, \mathcal{C}|_T) \) in Example 1.1.4(a), hence to \( \text{Tot}(\tau_{\leq 0} \mathcal{C})(T) \) by Proposition 2.2.8(a). Combining these constructions for all \( T \in \mathcal{A}ff_B \), we get a homotopy sheaf \( X \) of simplicial sets on \( \mathcal{A}ff_B \) of groupoid type, together with an equivalence and a morphism of stacks

\[
\text{Tot}(\tau_{\leq 0} \mathcal{C}) \simeq \Pi X \xrightarrow{h} \text{SES}.
\]

The morphism \( h \) induces the required identification on \( \underline{\pi}_0 \) and \( \underline{\pi}_1 \), so it is an equivalence of stacks. Proposition 2.3.4 is proved.

\[ \square \]

### 2.4. Maurer-Cartan stacks.

We now describe a non-abelian generalization of the construction of §2.2. Let \( B \) be an f-Artin stack and \( (\mathcal{G}, d, [\ - , \ - ] \) be an \( \mathcal{O}_B \)-\( dg \)-Lie algebra with quasi-coherent cohomology. In other words, \( \mathcal{G} \) is a Lie algebra object in the symmetric monoidal category \( (C_{qcoh}(B), \otimes_B) \). We will assume that \( \mathcal{G} \) is nilpotent. We define the \textit{Maurer-Cartan} \( \infty \)-\textit{stack} of \( \mathcal{G} \) to be the simplicial presheaf \( \text{mc}_\bullet(\mathcal{G}) \) on \( \mathcal{A}ff_B \) defined by

\[
\text{mc}_\bullet(\mathcal{G})(T) = \text{mc}_\bullet(H^0(T, f^*\mathcal{G})).
\]

Here \( (T \xrightarrow{f} B) \) is an object of \( \mathcal{A}ff_B \), and we apply the functor \( \text{mc}_\bullet \) to the \( dg \)-Lie algebra \( H^0(T, f^*\mathcal{G}) \) over \( \mathbb{C} \).

**Proposition 2.4.1.**

(a) The simplicial presheaf \( \text{mc}_\bullet(\mathcal{G}) \) is a homotopy sheaf.

(b) A morphism (resp. quasi-isomorphism) \( \phi : \mathcal{G}_1 \to \mathcal{G}_2 \) of nilpotent \( \mathcal{O}_B \)-\( dg \)-Lie algebras induces a morphism (resp. weak equivalence) of homotopy sheaves \( \phi_* : \text{mc}_\bullet(\mathcal{G}_1) \to \text{mc}_\bullet(\mathcal{G}_2) \).

**Proof.** Part (b) follows from Proposition 1.1.8 by sheafification. \[ \square \]
Assume that the dg-Lie algebra $\mathcal{G}$ is situated in degrees $[0, 2]$, i.e.,

$$
\mathcal{G} = \{ \mathcal{G}^0 \xrightarrow{d_0} \mathcal{G}^1 \xrightarrow{d_1} \mathcal{G}^2 \}.
$$

Then we define the stack $MC(\mathcal{G})$ of groupoids on $\mathscr{A}ff_B$ by

$$
MC(\mathcal{G})(T) = MC(H^0(T, \mathcal{G}|_T)).
$$

We call $MC(\mathcal{G})$ the Maurer-Cartan stack associated to a 3-term $\mathcal{O}_B$-dg-Lie algebra $\mathcal{G}$.

**Proposition 2.4.3.** If $\mathcal{G}$ is situated in degrees $[0, 2]$, then the simplicial sheaf $mc_\bullet(\mathcal{G})$ is of groupoid type and $\Pi mc_\bullet(\mathcal{G}) = MC(\mathcal{G})$. \hfill $\square$

Let $\mathcal{G}$ be any $\mathcal{O}_B$-dg-Lie algebra with quasi-coherent cohomology. As for complexes, we call $\mathcal{G}$ strictly $[0, 2]$-perfect, if it is quasi-isomorphic, as an $\mathcal{O}_B$-dg-Lie algebra, to a 3-term dg-Lie algebra (2.4.2) with each $\mathcal{G}^i$ being a vector bundle on $B$. We say that $\mathcal{G}$ is $[0, 2]$-perfect, if, locally on $B$, it is strictly $[0, 2]$-perfect and, moreover, the set of open substacks $U \subset B$ such that $\mathcal{G}|_U$ is strictly $[0, 2]$-perfect, is filtering with respect to the partial order by inclusion.

We now assume that $\mathcal{G}$ be a strictly $[0, 2]$-perfect dg-Lie algebra as in (2.4.2). Then, we have the closed substack $mc(\mathcal{G}) \subset \text{Tot}(\mathcal{G}^1)$ “given by the equation $d^1 x + \frac{1}{2}[x, x] = 0$”, with two equivalent definitions:

1. For any affine $B$-scheme $T \xrightarrow{f} B$ we have a dg-Lie algebra $H^0(T, \mathcal{G}^i|_T)$, and we define $mc(\mathcal{G})(T) = mc(H^0(T, \mathcal{G}^i|_T))$.

2. The stack $mc(\mathcal{G})$ is the zero locus of the section $s^G$ of $\pi^*\mathcal{G}^2$ given by the curvature

$$
\mathcal{G}^1 \rightarrow \mathcal{G}^2, \quad x \mapsto d^1 x + \frac{1}{2}[x, x].
$$

Since the Lie algebra $\mathcal{G}^0$ is nilpotent, we have a sheaf of groups $G^0 = \exp(\mathcal{G}^0)$ on $B$ by Malcev theory, which acts on the stack $mc(\mathcal{G})$ as in (1.1.7), and we can consider the quotient stack $mc(\mathcal{G})/G^0$. Consider also the quotient stack

$$
\text{Tot}(\mathcal{G}^\leq 1) = \text{Tot}(\mathcal{G}^1)/G^0
$$

and denote its projection to $B$ by $\pi$.

**Proposition 2.4.5.**

(a) Let $\mathcal{G}$ be a strictly $[0, 2]$-perfect dg-Lie algebra as in (2.4.2).

1. We have an equivalence of stacks $u : MC(\mathcal{G}) \rightarrow mc(\mathcal{G})/G^0$, so $MC(\mathcal{G})$ is an Artin stack.

2. The section $s^G$ of the bundle $\pi^*\mathcal{G}^2$ on $\text{Tot}(\mathcal{G}^1)$ descends to a section $\pi_G$ of the bundle $\pi^1 \mathcal{G}^2$ on $\text{Tot}(\mathcal{G}^\leq 1)$, and the substack $MC(\mathcal{G}) \subset \text{Tot}(\mathcal{G}^\leq 1)$ is the zero locus of $\pi_G$.

(b) If $\mathcal{G}$ is a $[0, 2]$-perfect $\mathcal{O}_B$-dg-Lie algebra, then the simplicial sheaf $mc_\bullet(\mathcal{G})$ is of groupoid type. The stack of groupoids $MC(\mathcal{G}) := \Pi mc_\bullet(\mathcal{G})$ is an Artin stack over $B$.

**Proof.** Part (a1) is proved similarly to Proposition 2.2.4(a), using the fact that, $G^0$ being a unipotent sheaf of groups, any $f^*G^0$-torsor over any $T \in \mathscr{A}ff_B$ is trivial. Part (a2) follows from (a) and from the equivalence of the two definitions (mc1) and (mc2) of the stack $mc(\mathcal{G})$. Part (b) follows because being of groupoid type and being an Artin stack over $B$ are properties local on $B$. \hfill $\square$

**Example 2.4.6.** If the dg-Lie algebra $\mathcal{G}$ is abelian, i.e., it reduces to a $[0, 2]$-perfect complex on $B$, then $MC(\mathcal{G}) = \text{Tot}(\tau_{\leq 0}(\mathcal{G}[1]))$.

Let us now globalize the considerations of Proposition 1.1.9 as follows. Let $p : \mathcal{G} = \mathcal{H} \times \mathcal{N} \rightarrow \mathcal{H}$ be a split extension of strictly $[0, 2]$-perfect dg-Lie algebras on $B$. The $B$-scheme $\pi_\mathcal{H} : mc(\mathcal{H}) \rightarrow B$ carries a strictly $[0, 2]$-perfect dg-Lie algebra $\tilde{\mathcal{N}}$ which is equal to $\pi^*_{\mathcal{H}}\mathcal{N}$ as a sheaf graded of $\mathcal{O}_{mc(\mathcal{H})}$-Lie algebras, with the differential $d_x$ at a point $x \in mc(\mathcal{H})$ defined as above. The action of the sheaf of groups $H^0$ on $mc(\mathcal{H})$ extends to a compatible action on $\tilde{\mathcal{N}}$, so that $\tilde{\mathcal{N}}$ descends to a strictly $[0, 2]$-perfect dg-Lie
algebras on the stack $\text{MC}(\mathcal{H})$. We denote this descended dg-Lie algebra by the same symbol $\tilde{N}$. Note that $\text{MC}(\tilde{N})$ is a stack over $\text{MC}(\mathcal{H})$, hence over $B$. Now, we have the following global analogue of Proposition 1.1.9.

**Proposition 2.4.7.** The stacks $\text{MC}(\mathcal{G})$ and $\text{MC}(\tilde{N})$ over $B$ are isomorphic.

**Proof.** For each affine $B$-scheme $T \in \text{aff}_B$, we have a split exact sequence of dg-Lie algebras

$$0 \rightarrow H^0(T, \mathcal{N}|_T) \rightarrow H^0(T, \mathcal{G}|_T) \rightarrow H^0(T, \mathcal{H}|_T) \rightarrow 0$$

which gives rise to a functor $p_* : \text{MC}(H^0(T, \mathcal{G}|_T)) \rightarrow \text{MC}(H^0(T, \mathcal{H}|_T))$ with the fiber category over an object $x$ equivalent to $\text{MC}(H^0(T, \mathcal{H}|_T)_x)$. This yields the following isomorphism of groupoids over $\text{MC}(H^0(T, \mathcal{H}|_T))$

$$\text{MC}(H^0(T, \mathcal{G}|_T)) = \text{MC}(H^0(T, \tilde{N}|_T)).$$

$\square$

2.5. **Stacks of filtrations.** Let $B$ be an $f$-Artin stack and $p : Y \rightarrow B$ be a scheme over $B$, locally of finite type. Let $\mathcal{E}_{01}, \mathcal{E}_{12}, \mathcal{E}_{23}$ be coherent sheaves over $Y$ which are flat over $B$. We define $\text{FILT}$ to be the stack over $B$ classifying filtered coherent sheaves $\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}$ over $Y$, together with identifications $\mathcal{E}_{0j}/\mathcal{E}_{0i} \cong \mathcal{E}_{ij}$ for $ij = 12, 23$. We have a sheaf of associative dg-algebras over $B$ defined by

$$(2.5.1) \quad \mathcal{G} = \bigoplus_{ij < kl} R_{p*}R\text{Hom}(\mathcal{E}_{kl}, \mathcal{E}_{ij}), \quad 01 < 12 < 23.$$ 

We’ll consider $\mathcal{G}$ as a sheaf of dg-Lie algebras using the supercommutator. Then, we have the following generalization of Proposition 2.3.4.

**Proposition 2.5.2.** Assume that $\mathcal{G}$ is a strictly $[0, 2]$-perfect dg-Lie algebra on $B$. Then, we have an equivalence $\text{MC}(\mathcal{G}) = \text{FILT}$ of stacks over $B$.

**Proof.** Let $\text{SES}_{012}$ be the stack over $B$ classifying short exact sequences

$$(2.5.3) \quad \mathcal{E}_{012} = \{0 \rightarrow \mathcal{E}_{01} \rightarrow \mathcal{E}_{02} \rightarrow \mathcal{E}_{12} \rightarrow 0\}$$

of coherent sheaves over $Y$. Then $\text{FILT}$ is the stack over $\text{SES}_{012}$ classifying short exact sequences

$$(2.5.4) \quad \mathcal{E}_{0123} = \{0 \rightarrow \mathcal{E}_{02} \rightarrow \mathcal{E}_{03} \rightarrow \mathcal{E}_{23} \rightarrow 0\},$$

and $\mathcal{G} = \mathcal{H} \times \mathcal{N}$ where

$$\mathcal{N} = R_{p*}\text{Hom}(\mathcal{E}_{23}, \mathcal{E}_{01} \oplus \mathcal{E}_{12}), \quad \mathcal{H} = R_{p*}\text{Hom}(\mathcal{E}_{12}, \mathcal{E}_{01}).$$

Since the dg-Lie algebra $\mathcal{H}$ is abelian, by Example 2.4.6 and Proposition 2.3.4 the stacks $\text{MC}(\mathcal{H}), \text{SES}_{012}$ are equivalent, and $\mathcal{N}$ gives an abelian strictly $[0, 2]$-perfect dg-Lie algebra $\tilde{N}$ over $\text{SES}_{012}$. Further, by Proposition 2.4.7, we have $\text{MC}(\mathcal{G}) = \text{MC}(\tilde{N})$ as stacks over $\text{SES}_{012}$. So we are reduced to prove that $\text{MC}(\tilde{N})$ is the stack over $\text{SES}_{012}$ classifying short exact sequences (2.5.4).

Let $T \xrightarrow{f} B$ be an affine $B$-scheme. Suppose the object $\mathcal{E}_{012}$ of $\text{SES}_{012}(T)$ is the cone of a morphism $u_{012}$ in $R\text{Hom}_{Y \times B}(f^*\mathcal{E}_{12}, f^*\mathcal{E}_{01})$. Thus, given injective resolutions of $f^*\mathcal{E}_{ij}$ for each $i, j$, the complex $\mathcal{E}_{02}$ is quasi-isomorphic to the complex $C(u_{012}) = I_{12} \oplus I_{01}$ where the differential is the sum of the differentials of $I_{12}, I_{01}$ and the composition with $u_{012}$, viewed as a morphism of complexes of sheaves $I_{12} \rightarrow I_{01}[1]$.

Next, we have $\tilde{N} = \pi^*_p \mathcal{N}$ as a graded sheaf, and the differential $d_{012}$ at the point $\mathcal{E}_{012}$ is given by

$$d_{012}(u) = d(u) - \text{ad}(u_{012})(u), \quad \forall u \in \text{Hom}_{Y \times B}(f^*\mathcal{E}_{23}, f^*\mathcal{E}_{01} \oplus f^*\mathcal{E}_{12}),$$

where $\text{ad}(u_{012})(u)$ denotes the differentiated morphism $u_{012}$.
see Proposition 1.1.9 and the discussion before it. In our case \( \text{ad}(u_{012})(u) \) reduces to the composition \( u_{012}u \). Thus, the condition for \( u \) to satisfy the equation \( u_{012}(u) = 0 \) is equivalent to saying that it lifts to a morphism of complexes \( f^*\mathcal{E}_{23} \to C(u_{012}) \), i.e., to a dotted arrow \( u_{0123} \) in the diagram.

\[
\begin{array}{ccc}
\mathcal{E}_{02} & \xrightarrow{+1} & \mathcal{E}_{01} \\
\xrightarrow{f^*} & & \xleftarrow{u_{012}} \\
\mathcal{E}_{12} & \xrightarrow{+1} & \mathcal{E}_{23}
\end{array}
\]

The cone of such an arrow defines \( \mathcal{E}_{03} \) with a short exact sequence (2.5.4). We have thus constructed a morphism \( MC(\mathcal{N}) \to \text{FILT} \) of stacks over \( \text{SES}_{012} \), and it is easy to check that this morphism is an equivalence. \( \square \)

3. Borel-Moore homology of stacks and virtual pullbacks

3.1. BM homology and operations for schemes. We fix a field \( k \) of characteristic 0 which will serve as the field of coefficients for (co)homology. The cases \( k = \mathbb{Q} \) or \( k = \mathbb{Q}_l \) will be the most important. For basics on simplicial categories, \( \infty \)-categories and dg-categories, see §A and the references there. By \( \text{dgVect} = \text{dgVect}_k \) we denote the dg-category of cochain complexes over \( k \). We recall the standard formalism of constructible derived categories of complexes of \( k \)-vector spaces and their functorialities [34], together with its \( \infty \)-categorical enhancement.

Let \( \text{Sch} \) denote the category of schemes of finite type over \( \mathbb{C} \). For a scheme \( T \in \text{Sch} \) we denote by \( \mathcal{C}(T) \) the category of constructible complexes of sheaves of \( k \)-vector spaces on \( T(\mathbb{C}) \). Let \( D(T) = \mathcal{C}(T)[\text{Qis}^{-1}] \) be the constructible derived category, i.e., the localization of \( \mathcal{C}(T) \) by the class of quasi-isomorphisms. We denote by \( D(T)_{\text{dg}} \) and \( D(T)_{\omega} \) the dg- and \( \infty \)-categorical enhancements of \( D(T) \) defined as in §A.2. If \( k = \mathbb{Q}_l \), we can use the étale \( l \)-adic version of the constructible derived category, see [50], [51]. It admits similar enhancements.

These categories carry the Verdier duality functor which we denote by \( \mathbb{D} \). For a morphism \( f : S \to T \) in \( \text{Sch} \) we have the usual functorialities

\[
\text{D}(S) \xrightarrow{Rf_{*}, f_{!}} \text{D}(T)
\]

with their standard adjunctions, see [34] for the case of classical topology or [50], [51] for the case of étale topology. They extend to the above enhancements and we will be using these extensions.

We denote by \( \omega_T = p^! k, p : T \to \text{pt} \), the dualizing complex of \( T \). The Borel-Moore homology of \( T \) and its complex of Borel-Moore chains are defined by

\[
H_{BM}^{\bullet}(T) = H_{-\bullet}(T, \omega_T), \quad C_{BM}^{\bullet}(T) = R\Gamma(T, \omega_T).
\]

The Poincaré-Verdier duality implies that

\[
H_{BM}^{\bullet}(T) = H_{-\bullet}^*(T)^{\ast}.
\]

A morphism \( f : S \to T \) in \( \text{Sch} \) is called strongly orientable of relative dimension \( m \in \mathbb{Z} \), if there is an isomorphism \( k_S \to f^! k_T[m] \) in \( \text{D}(S) \). A choice of such an isomorphism is called a strong orientation of \( f \). For not necessarily connected \( S \) we can speak of relative dimension being a locally constant function on \( S \), with the obvious modifications of the above.

Recall that \( H_{BM}^\bullet \) is covariantly functorial with respect to proper morphisms. By (3.1.1), an oriented morphism \( f : S \to T \) of relative dimension \( m \) gives rise to a pullback map \( f^* : H_{BM}^{\bullet}(T) \to H_{BM}^{\bullet+m}(S) \), and such maps are compatible with compositions of oriented morphisms.
Examples 3.1.3.

(a) A smooth morphism \( f \) of dimension \( d \) is strongly oriented of relative dimension \( 2d \).
(b) An l.c.i. (locally complete intersection) morphism is a morphism \( f : S \to T \) represented as a composition \( f = p \circ i \) where \( p \) is smooth and \( i \) is a regular embedding. Thus an l.c.i. morphism \( f \) has a well defined dimension \( d \), which is a locally constant \( \mathbb{Z} \)-valued function on \( S \). If the embedding \( i \) is strongly oriented, then \( f \) is also strongly oriented of relative dimension \( 2d \), hence gives rise to a pullback morphism \( f^* \). Note that the map \( f^* \) still make sense for any l.c.i. morphism, see, e.g., [51, §2.17].

Example 3.1.4. Let \( \mathcal{E} \) be a rank \( r \) vector bundle on \( T \). We recall that the \( r \)th Chern class \( c_r(\mathcal{E}) \in H^{2r}(T, \mathbb{k}) \) is the obstruction to the existence of a continuous section of \( \mathcal{E} \) which does not vanish anywhere. Let \( s \) be any section of \( \mathcal{E} \). We denote the zero locus of \( s \) with its embedding into \( T \) by \( T_s \overset{i_s}{\to} T \). In this situation we have the refined \( r \)th Chern class

\[
c_r(\mathcal{E}, s) \in H^{2r}_{T_s}(T, \mathbb{k}) = H^{2r}(T_s, i_s^! \mathbb{k}_T)
\]

whose image in \( H^{2r}(T, \mathbb{k}) \) is \( c_r(\mathcal{E}) \), yielding a virtual pullback map \( s^! : H^*_{BM}(T) \to H^*_{BM-2r}(T_s) \). More precisely, following [17, §7.3], we introduce the bivariant cohomology of any morphism \( f : S \to T \) to be

\[
H^*(S \overset{f}{\to} T) = H^*(S, f^! \mathbb{k}_T).
\]

Recall that

(a) We have \( H^*(S \overset{\text{Id}}{\to} S) = H^*(S, \mathbb{k}) \) while \( H^*(S \to \text{pt}) = H^*_{BM}(S) \).
(b) For a composable pair of maps \( S \overset{f}{\to} T \overset{g}{\to} U \) we have the product

\[
H^*(S \overset{f}{\to} T) \otimes H^*(T \overset{g}{\to} U) \to H^*(S \overset{gf}{\to} U).
\]

So, taking \( U = \text{pt} \), each \( h \in H^d(S \overset{f}{\to} T) \) gives rise to a map \( u_h : H^*_{BM}(T) \to H^*_{BM}(S) \).

We deduce that \( c_r(\mathcal{E}, s) \in H^{2r}(T_s \overset{i_s}{\to} T) \) defines a map \( H^*_{BM}(T) \to H^*_{BM-2r}(T_s) \).

The construction of \( c_r(\mathcal{E}, s) \) is as follows. We consider the embedding \( T \overset{0}{\to} \text{Tot}(\mathcal{E}) \) as the zero section. It is strongly oriented of relative dimension \( 2r \), see [17, prop. 4.1.3, 7.3.2], hence we get a class \( \eta \in H^{2r}_T(\text{Tot}(\mathcal{E})) \). Now \( T_s \) is the intersection of \( T \) with \( \Gamma_s \), the graph of \( s \) inside \( \text{Tot}(\mathcal{E}) \), and \( c_r(\mathcal{E}, s) \) is the image of \( \eta \) under the restriction map

\[
H^{2r}_T(\text{Tot}(\mathcal{E}), \mathbb{k}) \to H^{2r}_{T \cap \Gamma_s}(\Gamma_s, \mathbb{k}) = H^{2r}_{T_s}(T_s, \mathbb{k}).
\]

See also [51, §2.17] for a different approach.

Proposition 3.1.5. Let \( \mathcal{E} \) be a vector bundle on \( T \) of rank \( r \) and let \( p : \text{Tot}(\mathcal{E}) \to B \) be the projection. The pullback \( p^* : H^*_{BM}(T) \to H^*_{BM}(\text{Tot}(\mathcal{E})) \) is an isomorphism. \qed

Remark 3.1.6. For \( T \in \text{Sch} \) let \( A_m(T) \) be the Chow group of \( m \)-dimensional cycles in \( T \). We have the canonical class map \( \text{cl} : A_m(T) \to H^*_{BM}(T) \). All the above constructions (proper pushforwards, l.c.i. pullbacks, virtual pullbacks) have natural analogs for the Chow groups, see [16], which are compatible, via \( \text{cl} \), with the sheaf-theoretical constructions described above.
3.2. BM homology and operations for stacks. The formalism of constructible derived categories and their functorialities extends to $f$-Artin stacks. For the case $k = \mathbb{Q}_l$ and étale topology this is done in [50, 51]. Another approach using $\infty$-categorical limits, which we outline below, is applicable for the complex analytic topology, any $k$, as well as for the case of analytic stacks in §7.3. It is an adaptation of the approach used in [21], §3.1.1 for ind-coherent sheaves, to the constructible case. All stacks in this sections will be $f$-Artin.

Let $B$ be a stack. By $\text{Sch}_B$ we denote the category formed by schemes $T$ of finite type over $\mathbb{C}$ together with a morphism of stacks $T \to B$. We define

$$D(B)_\infty = \lim_{(T \to B)} D(T)_\infty, \quad D(B)_{dg} = \lim_{(T \to B)} D(T)_{dg},$$

the $\infty$-categorical projective limit, resp. dg-categorical (homotopy) projective limit over the category $\text{Sch}_B$, with respect to the pullback functors. Note that $D(B)_\infty$, resp. $D(B)_{dg}$ also carries the Verdier duality $\mathbb{D}$ induced by such dualities on the $D(T)_\infty$, resp. $D(T)_{dg}$ above.

We compare this with the following. Let $Z$ be a scheme of finite type over $\mathbb{C}$ with an action of an affine algebraic group $G$. Then we have action groupoid $\{G \times Z \rightrightarrows Z\}$ in the category of schemes, so its nerve $N_\bullet\{G \times Z \rightrightarrows Z\}$ is a simplicial scheme defined as in (1.1.2). The Bernstein-Lunts equivariant derived constructible $\infty$-category of $Z$ is

$$D(Z, G)_\infty = \lim_{[n] \in \Delta^c} D(N_n\{G \times Z \rightrightarrows Z\})_\infty.$$ 

It is an $\infty$-categorical version of the definition from [6]. Just as in [6], given a $G(\mathbb{C})$-equivariant constructible complex $F^\bullet$ on $Z(\mathbb{C})$, then

$$\text{Ext}_D^\bullet(D(Z, G)_\infty(k_Z, F^\bullet)) = H^\bullet_G(Z(\mathbb{C}), F^\bullet)$$

is the topological equivariant (hyper)cohomology.

**Proposition 3.2.2.** The $\infty$-category $D(Z, G)_\infty$ is identified with $D(Z/\pi(G))_\infty$.

**Proof.** Each $N_n\{G \times Z \rightrightarrows Z\}$ is an affine scheme over $Z$, therefore over $Z/\pi(G)$. In fact

$$N_n\{G \times Z \rightrightarrows Z\} = Z \times_{Z/\pi(G)} \cdots \times_{Z/\pi(G)} Z \quad (n \text{ times}).$$

So $N_\bullet\{G \times Z \rightrightarrows Z\}$ is the nerve of the (smooth) morphism $Z \to Z/\pi(G)$, which we can see as a 1-element covering of $Z/\pi(G)$ in the smooth topology. Our statement therefore means that $D(-)_\infty$ satisfies ($\infty$-categorical) descent with respect to this covering. A more general statement if true: $D(-)_\infty$ as a functor from stacks to $\infty$-categories satisfies descent (for any covering) in the smooth topology. This statement is a formal consequence of the corresponding, obvious, statement for schemes: $D(-)_\infty$ as a functor from Sch to $\infty$-categories satisfies descent (for any covering) in the smooth topology. \qed

Given a morphism of stacks $f : B \to C$, the composition with $f$ defines a functor $f_\circ : \text{Sch}_B \to \text{Sch}_C$, hence a functor which we denote

$$f^{-1} : D(C)_\infty = \lim_{(U \to C)} D(U) \longrightarrow \lim_{(T \to B)} D(T) = D(B)_\infty.$$

The right adjoint functor to $f^{-1}$ is denoted by $Rf_* : D(C)_\infty \to D(B)_\infty$.

We further define the functors

$$f^! = \mathbb{D} \circ f^{-1} \circ \mathbb{D} : D(C)_\infty \longrightarrow D(B)_\infty, \quad Rf_! = \mathbb{D} \circ Rf_* \circ \mathbb{D} : D(B)_\infty \longrightarrow D(C)_\infty.$$

In particular, we have the dualizing complex $\omega_B = \mathbb{D}(k_B) = p^!k$, where $p : B \to pt$, cf. [41]. Note that, for each affine algebraic group $G$ over $\mathbb{C}$, then $\omega_BG \simeq kBG[-2\text{dim}(G)]$, while for each smooth complex variety $S$ we have $\omega_S \simeq kS[2\text{dim}(S)]$. 

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We define the Borel-Moore homology, resp. cohomology with compact support of an (f-Artin) stack $B$ as

\[(3.2.3) \quad H^\text{BM}_*(B) = H^{-*}(B, \omega_B), \quad H^*_c(B, \mathbb{k}_B) = H^*(Rp_!\mathbb{k}_B).\]

The Poincaré-Verdier duality extends from schemes of finite type to f-Artin stacks and implies that $H^\text{BM}_*(B) = H^*_c(B, \mathbb{k}_B)^\ast$. By gluing the corresponding properties of schemes, we get that $H^\text{BM}_*$ is covariantly functorial for proper morphisms and has pullbacks with respect to l.c.i. morphisms.

**Remark 3.2.4.** The BM homology for stacks is the topological analog of the Chow groups for stacks as defined by Kresch [38].

We also note the following, cf. [38, thm. 2.1.12].

**Proposition 3.2.5.** Let $\mathcal{C}^\bullet = \{C^{-1} \to C^0\}$ be a two-term strictly perfect complex on $B$ of virtual rank $r$, with the total space $\text{Tot}(\mathcal{C}^\bullet) = C^0/C^{-1} \to B$. Then $\pi$ is a smooth morphism, hence it is strongly oriented of relative dimension $2r$, and $\pi^\ast : H^\text{BM}_*(B) \to H^\text{BM}_*(\text{Tot}(\mathcal{C}))$ is an isomorphism if $B$ admits a stratification by global quotients ([38, def. 3.5.3]), in particular if $B$ is locally quotient.

\[\square\]

### 3.3. Virtual pullback for a perfect complex.

Let $B$ be a stack and $\mathcal{E}$ be a vector bundle of rank $r$ over $B$. Let $s \in H^0(B, \mathcal{E})$ be a section of $\mathcal{E}$ and

\[i : B_s = \{s = 0\} \hookrightarrow B\]

be the inclusion of the zero locus of $s$, which is a closed substack. The section $s$ gives a regular embedding in the total space of $\mathcal{E}$, which we denote also $s : B \to \text{Tot}(\mathcal{E})$. The construction of Example 3.1.4 extends (by gluing) from schemes to stacks and gives the refined pullback morphism, or refined Gysin morphism

\[(3.3.1) \quad s^! : H^\text{BM}_*(B) \to H^\text{BM}_{*-2r}(B_s),\]

making the following diagram commute

\[
\begin{array}{ccc}
H^\text{BM}_*(B) & \xrightarrow{s^!} & H^\text{BM}_{*-2r}(B_s) \\
\downarrow & & \downarrow \\
H^\text{BM}_*(\text{Tot}(\mathcal{E})) & \xrightarrow{\pi^\ast} & H^\text{BM}_{*-2r}(B)
\end{array}
\]

**Remark 3.3.2.** The map $s^!$ is the BM-homology analog of the refined pullback on Chow groups for Artin stacks which is a particular case of Construction 3.6 of [45], or of [38, §3.1] which uses deformation to the normal cone.

Now, let $\mathcal{C}$ be a strictly $[-1, 1]$-perfect complex on $B$ and

\[\pi : \text{Tot}(\mathcal{C}^\leq 0) \to B, \quad q : \text{Tot}(\tau_{\leq 0}\mathcal{C}) \to B\]

be the obvious projections. The differential $d^1$ of $\mathcal{C}$ gives a section $s_{\mathcal{C}}$ of the vector bundle $\pi^\ast \mathcal{C}^1$ on $\text{Tot}(\mathcal{C}^\leq 0)$ whose zero locus is the cone stack $\text{Tot}(\tau_{\leq 0}\mathcal{C})$, yielding the diagram

\[
\begin{array}{c}
\pi^\ast \mathcal{C}^1 \\
\downarrow^{s_{\mathcal{C}}} \\
\text{Tot}(\mathcal{C}^\leq 0) \\
\downarrow^{i} \\
B \xrightarrow{\pi} \text{Tot}(\mathcal{C}^\leq 0) \xleftarrow{i} \text{Tot}(\tau_{\leq 0}\mathcal{C})
\end{array}
\]

such that $q = \pi \circ i$. By Proposition 3.2.5, see also [38, thm. 2.1.12], the pullback along $\pi$ defines a morphism

\[\pi^\ast : H^\text{BM}_*(B) \xrightarrow{\sim} H^\text{BM}_{*-2\text{vrk}(\mathcal{C}^\leq 0)}(\text{Tot}(\mathcal{C}^\leq 0)),\]
which is an isomorphism if $B$ admits a stratification by global quotients. Further, we have the refined pullback map on Borel-Moore homology

$$s^1_C : H_{\bullet + 2\text{rk}(C \leq 0)}(\text{Tot}(C \leq 0)) \to H_{\bullet + 2\text{rk}(C)}(\text{Tot}(\tau_{\leq 0} C)).$$

We define the virtual pullback associated with $C$ to be the composite map

$$q^1_C = s^1_C \circ \pi^* : H_{\bullet}^\text{BM}(B) \to H_{\bullet + 2\text{rk}(C)}(\text{Tot}(\tau_{\leq 0} C)).$$

By Proposition 2.2.8, the stack $\text{Tot}(\tau_{\leq 0} C)$ depends only on the isomorphism class of the complex $C$ in $D^b_{\text{coh}}(B)$ and not on the choice of the presentation (2.2.6).

**Proposition 3.3.3.** Let $C$ be a strictly $[-1, 1]$-perfect complex on $B$. The virtual pullback $q^1_C$ depends only on the isomorphism class of the strictly $[-1, 1]$-perfect complex $C$ in $D^b_{\text{coh}}(B)$.

**Proof.** Fix two presentations $C_1, C_2$ of the complex $C$ as in (2.2.6), with

$$C_k = \{ C_k^{-1} \xrightarrow{d_k} C_k^0 \xrightarrow{d_k^1} C_k^1 \}, \quad k = 1, 2$$

and fix a quasi-isomorphism $\phi : C_1 \to C_2$. By functoriality of the total space and the truncated total space, we have the commutative diagram

$$\begin{array}{ccc}
\text{Tot}(\tau_{\leq 0} C_1) & \xrightarrow{\phi_\circ} & \text{Tot}(\tau_{\leq 0} C_2) \\
\downarrow i_1 & & \downarrow i_2 \\
\text{Tot}(C^0_1) & \xrightarrow{\phi_\circ} & \text{Tot}(C^0_2) \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
B & & B,
\end{array}$$

We claim that the following triangle commutes

$$\begin{array}{ccc}
H_{\bullet}^\text{BM}(B) & \xrightarrow{q^1_{C_1}} & H_{\bullet + 2\text{rk}(C_1)}(\text{Tot}(\tau_{\leq 0} C_1)) \\
\downarrow & & \downarrow \phi_\circ \\
H_{\bullet + 2\text{rk}(C_2)}(\text{Tot}(\tau_{\leq 0} C_2)) & \xrightarrow{q^1_{C_2}} & \text{Tot}(\tau_{\leq 0} C_1) \circ \pi^*_1.
\end{array}$$

To prove this, we must prove that we have

$$s^1_{C_1} \circ \pi^*_1 = \phi_\circ \circ s^1_{C_2} \circ \pi^*_2.$$

By Proposition 2.2.4, the map $\phi_\circ : \text{Tot}(C^0_1) \to \text{Tot}(C^0_2)$ is an l.c.i. Hence there is a Gysin map $(\phi_\circ)_*$ and we have expressions through the local Chern classes associated to the sections $s_{C_1}$ of $\pi^*_i C^1_i$, $i=1,2$:

$$s^1_{C_1} \circ \pi^*_1 = c_{\text{rk}(C_1^1)}(\pi^*_1 C^1_1) \circ \phi_\circ \circ \pi^*_2,$$

$$\phi_\circ \circ s^1_{C_2} \circ \pi^*_2 = \phi_\circ \circ c_{\text{rk}(C_2^1)}(\pi^*_2 C^1_2) \circ \pi^*_2.$$
a vector bundle homomorphism such that \( h \circ s_1 = s_2 \circ f \), which yields a commutative square

\[
\begin{array}{ccc}
(B_2)_{s_2} & \xrightarrow{i_2} & B_2 & \xrightarrow{s_2} & \text{Tot}(\mathcal{E}_2) \\
\downarrow g & & \downarrow f & & \downarrow h \\
(B_1)_{s_1} & \xrightarrow{i_1} & B_1 & \xrightarrow{s_1} & \text{Tot}(\mathcal{E}_1)
\end{array}
\]

where \( g \) is an isomorphism. Then, we have a commutative square

\[
\begin{array}{ccc}
H^\bullet_{\text{BM}}(B_1) & \xrightarrow{c_{r_1}(\mathcal{E}_1,s_1)} & H^\bullet_{\text{BM}}(B_1) \\
\downarrow f^* & & \downarrow g^* \\
H^\bullet_{-2r_1+2r_2}(B_2) & \xrightarrow{c_{r_2}(\mathcal{E}_2,s_2)} & H^\bullet_{-2r_1}(B_2) \\
\end{array}
\]

Finally, let now \( B \) be an Artin stack and let \( C \) be any \([-1,1]\)-perfect complex on \( B \). Let \( \mathcal{U} \) be a filtering open cover of \( B \) consisting of open substacks \( U \) such that \( C|_U \) is strictly \([-1,1]\)-perfect. We have

\[
(3.3.5) \quad H^\bullet_{\text{BM}}(B) = \varprojlim_{U \in \mathcal{U}} H^\bullet_{\text{BM}}(U), \quad H^\bullet_{\text{BM}}(\text{Tot}(\tau_{\leq 0}C)) = \varprojlim_{U \in \mathcal{U}} H^\bullet_{\text{BM}}(\text{Tot}(\tau_{\leq 0}C|_U)).
\]

**Definition 3.3.6.** A coherent perfect system on a \([-1,1]\)-perfect complex \( C \) on \( B \) is a collection of quasi-isomorphisms \( \phi_U : C|_U \rightarrow C_U \) and \( \phi_V \circ \phi_U : C_U|_V \rightarrow C_V \) for each \( U, V \in \mathcal{U} \) with \( V \subset U \) such that \( C_U \) is a strictly \([-1,1]\)-perfect complex on \( U \) with a presentation as in (2.2.6), and \( \phi_V = \phi_V \circ \phi_U|_V \).

Given a coherent perfect system on \( C \), we define the virtual pullback

\[
q^!_C : H^\bullet_{\text{BM}}(B) \rightarrow H^\bullet_{\text{BM}}(\text{Tot}(\tau_{\leq 0}C))
\]

as the map

\[
(3.3.7) \quad q^!_C = \varprojlim_{U \in \mathcal{U}} ((\phi_U)^* \circ q^!_{C|_U}).
\]

**Remark 3.3.8.** If \( C \) is a strictly \([-1,1]\)-perfect complex on the stack \( B \), then its total space has a \( dg \)-stack structure given by

\[
(3.3.9) \quad \text{Tot}(C) = \left( \text{Tot}(C^{\leq 0}), \left( \text{Sym}(\pi^*(C^1)^\vee [1]), \partial_s \right) \right),
\]

that is, the stack \( \text{Tot}(C^{\leq 0}) \) equipped with the sheaf of commutative \( dg \)-algebras which is the Koszul complex of the section \( s \) above. This \( dg \)-stack gives rise to a derived stack in the sense of [61]. The derived stack \( \text{Tot}(C) \) depends, up to a natural equivalence, only on the isomorphism class of the complex \( C \) in \( D^b_{\text{coh}}(B) \). We expect a direct conceptual interpretation of the virtual pullback \( q^!_C \) in terms of the derived stack \( \text{Tot}(C) \). However, this would require a well behaved Borel-Moore homology theory for derived stacks and we do not know how to do it.
3.4. Virtual pullback for Maurer-Cartan stacks. Let $B$ be an Artin stack of finite type and $\mathcal{G}$ be a strictly $[0,2]$-perfect dg-Lie algebra over $B$ as in (2.4.2). We define now a virtual pullback

$$q'_G : H^*_{BM}(B) \to H^*_{BM}(\mathcal{G})$$

using the diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\pi} & \text{Tot}((\mathcal{G}^{\leq 1})) \\
\downarrow q & & \downarrow i \\
\pi_G & = & \text{MC}(\mathcal{G}).
\end{array}
$$

In order to define the map $q'_G = s_G^1 \circ \pi^*$ as in §3.3, we must check that the pullback morphism

$$\pi^* : H^*_{BM}(B) \to H^*_{BM}(\text{Tot}(\mathcal{G}^{\leq 1}))$$

and the refined pullback

$$s_G^1 : H^*_{BM}(\text{Tot}(\mathcal{G}^{\leq 1})) \to H^*_{BM}(\text{MC}(\mathcal{G}))$$

are well-defined. The refined pullback is defined as in the previous sections, using the fact that MC$(\mathcal{G})$ is the zero locus of the section $s$ of the bundle $\pi^*\mathcal{G}^2$ on $\text{Tot}(\mathcal{G}^{\leq 1})$ associated with the curvature (2.4.4). The pullback map $\pi^*$ is well-defined, because $\pi$ is a vector bundle stack, hence is smooth although non-representable.

Next, we study the behavior of the virtual pullback under extensions of dg-Lie algebras. Let $\mathcal{G} = H \times N$ and $\tilde{\mathcal{N}} = \pi^*_HN$ be as in §2.4. Note that Proposition 2.4.7 allows to write the commutative diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\pi_H} & \text{Tot}(\mathcal{H}^{\leq 1}) \\
\downarrow \pi_G & & \downarrow \pi_N \\
\text{Tot}(\mathcal{G}^{\leq 1}) & \xrightarrow{i_G} & \text{MC}(\mathcal{G}).
\end{array}
$$

The virtual pullback maps $q'_G$, $q'_N$, and $q'_H$ are defined as above.

**Proposition 3.4.1.** We have the equality $q'_G = q'_N \circ q'_H$.

**Proof.** Let $s_G$, $s_N$, $s_H$ be the sections of the bundles $\pi^*_G\mathcal{G}^2$, $\pi^*\mathcal{N}^2$, $\pi^*_H\mathcal{H}^2$ associated with the curvature maps of $\mathcal{G}$, $\mathcal{N}$, $\mathcal{H}$ respectively. We must prove that

$$s_G^1 \circ \pi^*_G = s_N^1 \circ \pi^* \circ s_H^1 \circ \pi^*_H.$$

First, observe that we have the diagram whose square is a fiber square

$$
\begin{array}{ccc}
B & \xrightarrow{\pi_H} & \text{Tot}(\mathcal{H}^{\leq 1}) \\
\downarrow \pi_G & & \downarrow \pi_N \\
\text{Tot}(\mathcal{G}^{\leq 1}) & \xrightarrow{i_G} & \text{MC}(\mathcal{G}),
\end{array}
$$

and the maps $p_b$, $j_b$ are given by the functoriality of the total space of a $[-1,0]$-complex. Note further that we have vector bundle homomorphisms

$$\pi^*_G\mathcal{G}^2 \to (p_b)^*\pi^*_H\mathcal{H}^2, \quad \pi^*\mathcal{N}^2 \to (j_b)^*\pi^*_G\mathcal{G}^2.$$
These vector bundle homomorphisms being compatible with the sections $s_G$, $s_N$ and $s_H$, the claim follows from the functoriality of the refined pullback with respect to pullback by smooth maps.

4. THE COHA OF A SURFACE

4.1. The COHA as a vector space. Let $S$ be a smooth connected quasi-projective surface over $\mathbb{C}$. Let $\text{Coh}(S)$ be the stack of coherent sheaves on $S$ with proper support. It is not smooth because the deformation theory can be obstructed due to $\text{Ext}^2$.

**Proposition 4.1.1.** \(\text{Coh}(S)\) is a locally quotient f-Artin stack.

**Proof.** This is standard, see [40, thm. 4.6.2.1]. Here are the details for future use in Prop. 4.3.2. Let \(\overline{S}\) be a smooth projective variety containing $S$ as an open set. Then $\text{Coh}(S)$ is an open substack in $\text{Coh}(\overline{S})$.

So it is enough to assume that $S$ is projective which we will. Let $\mathcal{O}(1)$ be the ample line bundle on $S$ induced by a projective embedding. The stack $\text{Coh}(S)$ splits into disjoint union

$$\text{Coh}(S) = \bigsqcup_{h \in \mathbb{K}[[t]]} \text{Coh}^{(h)}(S),$$

where $\text{Coh}^{(h)}(S)$ consists of sheaves $\mathcal{F}$ with Hilbert polynomial $h$, i.e., of $\mathcal{F}$ such that

$$\dim H^0(S, \mathcal{F}(n)) = h(n), \quad n \geq 0.$$

For any $N \in \mathbb{N}$, let $\text{Coh}^{(h,N)}(S) \subset \text{Coh}^{(h)}(S)$ be the open substack formed by $\mathcal{F}$ such that for each $n \geq N$ two conditions hold:

(a) $H^i(S, \mathcal{F}(n)) = 0$, $i > 0$,

(b) the canonical map $H^0(S, \mathcal{F}(n)) \otimes \mathcal{O}(-n) \to \mathcal{F}$ is surjective.

Now, for any coherent sheaf $\mathcal{E}$ on a scheme $B$, let $\text{Quot}_B$ be the scheme such that, for any $B$-scheme $T \to B$, the set of $T$-points $\text{Quot}_B(T)$ is the set of surjective sheaf homomorphisms $\mathcal{E}|_T \to \mathcal{F}$ where $\mathcal{F}$ is flat over $T$, modulo the equivalence relation

$$(q : \mathcal{E}|_T \to \mathcal{F}) \sim (q' : \mathcal{E}|_T \to \mathcal{F}') \iff \text{Ker}(q) = \text{Ker}(q').$$

Let $\text{Quot}^{(h,N)}(S)$ be the open subscheme of $\text{Quot}_{\mathcal{O}(-N)^\oplus h(N)}$ formed by equivalence classes of surjections $\phi : \mathcal{O}(-N)^\oplus h(N) \to \mathcal{F}$ with $\mathcal{F} \in \text{Coh}^{(h,N)}(S)$ such that $\phi(N)$ induces an isomorphism $H^0(S, \mathcal{O})^\oplus h(N) \to H^0(S, \mathcal{F}(N))$. Then, the stack $\text{Coh}^{(h,N)}(S)$ is isomorphic to the quotient stack of $\text{Quot}^{(h,N)}(S)$ by the obvious action of the group $GL_{h(N)}$. It is a stack of finite type and, as $N \to \infty$, the substacks $\text{Coh}^{(h,N)}(S)$ form an open exhaustion of $\text{Coh}^{(h)}(S)$.

4.2. The induction diagram. Let SES be the Artin stack classifying short exact sequences

$$(4.2.1) \quad 0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{F} \to 0$$

of coherent sheaves with proper support over $S$. Morphisms in SES are isomorphisms of such sequences. We then have the *induction diagram*

$$(4.2.2) \quad \text{Coh}(S) \times \text{Coh}(S) \xrightarrow{q} \text{SES} \xrightarrow{p} \text{Coh}(S),$$

where the map $p$ projects a sequence (4.2.1) to $\mathcal{G}$, while $q$ projects it to $(\mathcal{E}, \mathcal{F})$.

**Proposition 4.2.3.** The morphism $p$ is schematic (representable) and proper.

**Proof.** For any coherent sheaf $\mathcal{G}$ on $S$ with proper support, the Grothendieck Quot scheme $\text{Quot}_\mathcal{G}$ is proper. 

\qed
4.3. The derived induction diagram. We have the projections

\[ \text{Coh}(S) \times \text{Coh}(S) \xrightarrow{\text{pr}_1} \text{Coh}(S) \times \text{Coh}(S) \xrightarrow{\text{pr}_2} \text{Coh}(S) \times S. \]

Consider the tautological coherent sheaf \( \mathcal{U} \) over \( \text{Coh}(S) \times S \) and the complex of coherent sheaves over \( \text{Coh}(S) \times \text{Coh}(S) \) given by

\[
\mathcal{C} = R(p_{12})_* R\text{Hom}(p_{23}^* \mathcal{U}, p_{13}^* \mathcal{U})[1].
\]

Its fiber at a point \((\mathcal{E}, \mathcal{F})\) is the complex of vector spaces \( R\text{Hom}_S(\mathcal{F}, \mathcal{E})[1] \). Given a substack \( X \subset \text{Coh}(S) \), let \( \mathcal{U}_X = \mathcal{U}|_{X \times S} \) and \( \mathcal{C}_X = \mathcal{C}|_{X \times X} \) be the restrictions of \( \mathcal{U} \) and \( \mathcal{C} \).

**Proposition 4.3.2.**

(a) The complex \( \mathcal{C} \) is \([-1, 1]\)-perfect and admits a perfect coherent system.

(b) The complex \( \mathcal{C}_X \) is strictly \([-1, 1]\)-perfect if \( X = \text{Coh}_0(S) \).

**Proof.** As in the proof of Proposition 4.1.1, the statements reduce to the case when \( S \) is projective which we assume. We also keep the notation from that proof. Fix two polynomials \( h, h' \in \mathbb{k}[t] \) and let \( \mathcal{E} \in \text{Coh}_r(S) \), \( \mathcal{F} \in \text{Coh}_{r'}(S) \) be two fixed coherent sheaves on \( S \) with Hilbert polynomials \( h, h' \). Since \( S \) is smooth of dimension 2, we can fix a locally free resolution \( \mathcal{P}^* = \{ \mathcal{P}^{-2} \to \mathcal{P}^{-1} \to \mathcal{P}^0 \} \) of \( \mathcal{F} \). If we know that the \( \mathcal{P}^i \) are "sufficiently negative" with respect to \( \mathcal{E} \), i.e., for each \( i \in [-2, 0] \) and \( j > 0 \) the space \( \text{Ext}^2_{\mathcal{S}}(\mathcal{P}^i, \mathcal{E}) = H^j(S, (\mathcal{P}^i)^\vee \otimes \mathcal{E}) \) vanishes, then the complex of vector spaces \( R\text{Hom}_S(\mathcal{F}, \mathcal{E})[1] \) is represented by the complex

\[
\text{Hom}_S(\mathcal{P}^0, \mathcal{E}) \to \text{Hom}_S(\mathcal{P}^{-1}, \mathcal{E}) \to \text{Hom}_S(\mathcal{P}^{-2}, \mathcal{E})
\]

situated in degrees \([-1, 1]\). In order to achieve this, we define, in a standard way, the complex \( \mathcal{C}_X \) of vector bundles whose ranks are \( h, h' \) and \( \mathcal{C}_o \) of vector bundles whose ranks are \( h, h' \) and \( N_0, N_1 \).

We have the projections \( \mathcal{U}_X = \mathcal{U}|_{X \times S} \) and \( \mathcal{C}_X = \mathcal{C}|_{X \times X} \) be the restrictions of \( \mathcal{U} \) and \( \mathcal{C} \).

To see (b), we notice that for 0-dimensional \( \mathcal{E} \) and \( \mathcal{F} \) with given \( h \) and \( h' \), i.e., with given dimensions of \( H^0(S, \mathcal{E}) \) and \( H^0(S, \mathcal{F}) \), one can choose \( N_0, N_1 \) in a universal way. \( \square \)

Let now \( X \subset \text{Coh}(S) \) be a substack whose points are closed under extensions in \( \text{Coh}(S) \). Let \( \text{SES}_X \subset \text{SES} \) be the substack which classifies all short exact sequences of coherent sheaves over \( S \) which belong to \( X \). We abbreviate \( \mathcal{U} = \mathcal{U}_X \), \( \mathcal{C} = \mathcal{C}_X \) and \( \text{SES} = \text{SES}_X \). Assume further that the complex \( \mathcal{C} \) over \( X \times X \) is strictly \([-1, 1]\)-perfect. Fix a presentation of \( \mathcal{C} \) as in Example 2.2.5.

**Proposition 4.3.4.** The stack \( \text{Tot}(r_{\leq 0} \mathcal{C}) \) is isomorphic to \( \text{SES} \).

**Proof.** Apply Proposition 2.3.4 with \( Y = X \times X \times S \) and \( \mathcal{F} = p_{23}^* \mathcal{U}, \mathcal{E} = p_{13}^* \mathcal{U} \). \( \square \)
Thus, for all $X$ as above we have the following diagram of $f$-Artin stacks

\[(4.3.5)\]

\[
\begin{array}{c}
X \times X \xrightarrow{\pi} \text{Tot}(\mathcal{C}^{\leq 0}) \xrightarrow{i} \text{SES} \xrightarrow{p} X
\end{array}
\]

with $q = \pi \circ i$, which can be viewed as a refinement of the induction diagram (4.2.2). We call this diagram the derived induction diagram.

4.4. **The COHA as an algebra.** We apply the analysis of §3.3 to all diagrams (5.2.1) as $X$ runs over the set of open substacks of finite type of $\text{Coh}(S)$ such that the complex $\mathcal{C}$ in (4.3.1) is strictly $[-1, 1]$-perfect over $X \times X$. Note that the stack $\text{Coh}(S)$ is covered by all such $X$'s by the proof of Proposition 4.3.2.

Since the map $p$ is representable and proper, the pushforward $p_*$ in Borel-Moore homology is well-defined. Hence, we have the maps

\[
H_{\bullet}^\text{BM}(X \times X) \xrightarrow{q_*^!} H_{\bullet + 2v(rk(C))}^\text{BM}(\text{SES}) \xrightarrow{p_*} H_{\bullet + 2v(rk(C))}^\text{BM}(X),
\]

which, by (3.3.5), give rise to the maps

\[
H_{\bullet}^\text{BM}(\text{Coh}(S) \times \text{Coh}(S)) \xrightarrow{q_*^!} H_{\bullet + 2v(rk(C))}^\text{BM}(\text{SES}) \xrightarrow{p_*} H_{\bullet + 2v(rk(C))}^\text{BM}(\text{Coh}(S)).
\]

Composing the maps $q_*^!$, $p_*$ and the exterior product

\[
m : H_{\bullet}^\text{BM}(X) \otimes H_{\bullet}^\text{BM}(X) \to H_{\bullet}^\text{BM}(X \times X),
\]

we get the map

\[(4.4.1)\]

\[
m : H_{\bullet}^\text{BM}(X) \otimes H_{\bullet}^\text{BM}(X) \to H_{\bullet + 2v(rk(C))}^\text{BM}(X),
\]

and, by (3.3.5), the map

\[
m : H_{\bullet}^\text{BM}(\text{Coh}(S)) \otimes H_{\bullet}^\text{BM}(\text{Coh}(S)) \to H_{\bullet + 2v(rk(C))}^\text{BM}(\text{Coh}(S)).
\]

The first main result of this paper is the following theorem. It is proved in the next section.

**Theorem 4.4.2.** The map $m$ equips $H_{\bullet}^\text{BM}(X)$ and $H_{\bullet}^\text{BM}(\text{Coh}(S))$ with an associative $k$-algebra structure.

\[\square\]

4.5. **Proof of associativity.** We must prove the associativity of the map $m$. It is enough to do it for $H_{\bullet}^\text{BM}(X)$. To do that, we consider the Artin stack FILT classifying flags of coherent sheaves $\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}$ over $S$ such that the sheaves $\mathcal{E}_{01}, \mathcal{E}_{12}, \mathcal{E}_{23}$ defined by $\mathcal{E}_{ij} = \mathcal{E}_{ij}/\mathcal{E}_{0i}$ belong to the substack $X \subset \text{Coh}(S)$. For any $i < j$ we introduce a copy $X_{ij}$ of the stack $X$ parametrizing sheaves $\mathcal{E}_{ij}$. For any $i < j < k$ we introduce a copy $\text{SES}_{ijk}$ of the stack SES parametrizing short exact sequences

\[
0 \to \mathcal{E}_{ij} \to \mathcal{E}_{ik} \to \mathcal{E}_{jk} \to 0.
\]

Then, we have the fiber diagrams of stacks

\[(4.5.1)\]
and

\[
\begin{array}{ccc}
\text{FILT} & \xrightarrow{v} & \text{SES}_{013} \\
\downarrow{w} & & \uparrow{p} \\
X_{01} \times \text{SES}_{123} & \xrightarrow{1 \times p} & X_{01} \times X_{13} \\
\downarrow{1 \times q} & & \\
X_{01} \times X_{12} \times X_{23}
\end{array}
\]  

(4.5.2)

given by

\[
x(\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}) = (\mathcal{E}_{02} \subset \mathcal{E}_{03}), \quad y(\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}) = (\mathcal{E}_{01} \subset \mathcal{E}_{02}, \mathcal{E}_{23}),
\]

\[
v(\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}) = (\mathcal{E}_{01} \subset \mathcal{E}_{03}), \quad w(\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}) = (\mathcal{E}_{01}, \mathcal{E}_{12} \subset \mathcal{E}_{13}).
\]

We must prove that we have

\[
p_* \circ q^!_C \circ (p_* \times 1) \circ (q^!_C \times 1) = p_* \circ q^!_C \circ (1 \times p_*) \circ (1 \times q^!_C).
\]

Note that the morphisms \( x, z \) are both proper and representable and that we have the following equalities of stack homomorphisms

\[
(q \times 1) \circ y = (1 \times q) \circ w, \quad p \circ v = p \circ x.
\]

We claim that there are virtual pullback homomorphisms \( y^!_C \) and \( w^!_C \) such that

\[
x_* \circ y^!_C = q^!_C \circ (p_* \times 1),
\]

(4.5.3)

\[
v_* \circ w^!_C = q^!_C \circ (1 \times p_*),
\]

\[
y^!_C \circ (q^!_C \times 1) = w^!_C \times (1 \times q^!_C).
\]

The complex \( C_{023} = (p \times 1)^* \mathcal{C} \) on \( \text{SES}_{012} \times X_{23} \) and the complex \( C_{013} = (1 \times p)^* \mathcal{C} \) on \( X_{01} \times \text{SES}_{123} \) are both strictly \([-1,1]\)-perfect. Since the squares in the diagrams (4.5.1), (4.5.2) are Cartesian, by Proposition 2.3.4 we have stack isomorphisms

\[
\text{Tot}(\tau_{\leq 0} C_{023}) = \text{SES}_{012} \times X_{02} \text{SES}_{023} = \text{FILT},
\]

\[
\text{Tot}(\tau_{\leq 0} C_{013}) = \text{SES}_{123} \times X_{13} \text{SES}_{13} = \text{FILT}.
\]

Therefore, we have virtual pullback maps

\[
y^!_C = y^!_{C_{023}} : H^BM_\bullet(\text{SES}_{012} \times X_{23}) \to H^BM_{\bullet+2\text{vrk}(\mathcal{C})}(\text{FILT}),
\]

\[
w^!_C = w^!_{C_{013}} : H^BM_\bullet(X_{01} \times \text{SES}_{123}) \to H^BM_{\bullet+2\text{vrk}(\mathcal{C})}(\text{FILT})
\]

associated with the complexes \( C_{023} \) and \( C_{013} \). Then, the first two equations in (4.5.3) follow from the following base change property of virtual pullbacks.

**Lemma 4.5.4.** Let \( B, B' \) be Artin stacks of finite type, \( C \) be a strictly \([-1,1]\)-perfect complex on \( B \), and \( f : B' \to B \) be a representable and proper morphism of stacks. Then, the complex \( C' := f^* C \) on \( B' \) is strictly \([-1,1]\)-perfect and gives rise to the following Cartesian square

\[
\begin{array}{ccc}
\text{Tot}(\tau_{\leq 0} C') & \xrightarrow{g} & \text{Tot}(\tau_{\leq 0} C) \\
\downarrow{q'} & & \downarrow{q} \\
B' & \xrightarrow{f} & B.
\end{array}
\]

Further, we have the following equality of maps

\[
g_* \circ q^!_{C'} = q^!_C \circ f_* : H^BM_\bullet(B') \to H^BM_{\bullet+2\text{vrk}(\mathcal{C})}(\text{Tot}(\tau_{\leq 0} C)).
\]

\[\square\]
Now, we concentrate on the third equation in (4.5.3). To do this, we first apply Proposition 2.5.2 to the stack homomorphism
\[ p : Y = X_{01} \times X_{12} \times X_{23} \times S \to B = X_{01} \times X_{12} \times X_{23} \]
and to the coherent sheaves \( \mathcal{E}_{ij} = p^*_i \mathcal{U} \) with \( ij = 01, 12, 23 \) given by the pullback of the tautological sheaf \( \mathcal{U} \) by the obvious projections \( Y \to X \times S \). The sheaf \( \mathcal{G} \) of associative dg-algebras in (2.5.1) is a strictly \([0,2]\)-perfect dg-Lie algebra on \( B \). So, Proposition 2.5.2 yields an equivalence of stacks over \( B \)
\[ \text{MC}(\mathcal{G}) \cong \text{FILT}. \]

More precisely, we realize \( \mathcal{G} \) as a semi-direct product in two ways \( \mathcal{G} = \mathcal{H} \ltimes \mathcal{N} = \mathcal{H}' \ltimes \mathcal{N}' \) where
\[ \mathcal{N}' = Rp_* \text{Hom}(\mathcal{E}_{23}, \mathcal{E}_{01} \oplus \mathcal{E}_{12}), \quad \mathcal{H} = Rp_* \text{Hom}(\mathcal{E}_{12}, \mathcal{E}_{01}), \]
\[ \mathcal{N} = Rp_* \text{Hom}(\mathcal{E}_{12} \oplus \mathcal{E}_{23}, \mathcal{E}_{01}), \quad \mathcal{H}' = Rp_* \text{Hom}(\mathcal{E}_{23}, \mathcal{E}_{12}). \]

Then, the proof of Proposition 2.5.2 yields the following isomorphism of stacks
\[ \text{MC}(\mathcal{H}) = \text{SES}_{012} \times X_{23}, \]
\[ \text{MC}(\mathcal{H}') = X_{01} \times \text{SES}_{123}, \]
\[ \text{MC}(\mathcal{G}) = \text{MC}(\mathcal{N}) = \text{SES}_{012} \times X_{02} \text{SES}_{023} = \text{FILT}, \]
\[ \text{MC}(\mathcal{G}) = \text{MC}(\mathcal{N}') = \text{SES}_{123} \times X_{13} \text{SES}_{013} = \text{FILT}. \]

In particular, we can identify the diagram
\[ \pi^* C_{023}^{1} \to \text{Tot}(C_{023}^{\leq 0}) \]
\[ \text{SES}_{012} \times X_{23} \overset{\pi}{\longrightarrow} \text{Tot}(C_{023}^{\leq 0}) \overset{i}{\longrightarrow} \text{FILT} \]

with the diagram
\[ \pi^* \mathcal{N}^2 \to \text{Tot}(\mathcal{N}^{1})//\mathcal{N}^0 \]
\[ \text{MC}(\mathcal{H}) \overset{\pi \mathcal{N}}{\longrightarrow} \text{Tot}(\mathcal{N}^{1})//\mathcal{N}^0 \overset{i \mathcal{N}}{\longrightarrow} \text{MC}(\mathcal{G}). \]

We deduce that \( y_C^1 = q_{\mathcal{N}}^1 \). Similarly, we get
\[ q_C^1 \times 1 = q_H^1, \quad w_C^1 = q_{\mathcal{N}'}^1, \quad 1 \times q_C^1 = q_{\mathcal{H}'}^1. \]

So the third equation in (4.5.3) follows from Proposition 3.4.1. This finishes the proof of Theorem 4.4.2.

4.6. **Chow groups and K-theory versions of COHA.** Given an f-Artin stack \( B \), we denote by \( A_*(B) \) its rational Kresch-Chow groups, as in [38]. By \( K(B) \) we denote the Grothendieck group of the category of coherent sheaves on \( B \). The construction in §3.3 makes sense as well for \( A_* \) and K-theory, yielding virtual pullback morphisms
\[ q_C^1 : A_*(\text{Coh}(S) \times \text{Coh}(S)) \to A_{*+\text{vrk}(C)}(\text{SES}), \]
\[ q_C^1 : K(\text{Coh}(S) \times \text{Coh}(S)) \to K(\text{SES}), \]
associated with the complex $\mathcal{C}$ in (4.3.1). Composing them with the pushforward $p_* : A_\bullet(\text{SES}) \to A_\bullet(\text{Coh}(S))$ and $p_* : K(\text{SES}) \to K(\text{Coh}(S))$ by the map $p$ in (4.2.2), we get an associative ring structure on $A_\bullet(\text{Coh}(S))$ and on $K(\text{Coh}(S))$.

A definition of the K-theoretic COHA of finite length coherent sheaves over $S$ was independently proposed along these lines in the recent paper of Zhao [65].
5. Hecke operators

5.1. Hecke patterns and Hecke diagrams. We continue to assume that \( S \) is a smooth quasi-projective surface over \( \mathbb{C} \). Recall that \( \text{Coh}(S) \) is the stack of coherent sheaves on \( S \) with proper support.

**Definition 5.1.1.** A Hecke pattern for \( S \) is a pair \( (X,Y) \) of subsheaves in \( \text{Coh}(S) \) with the following properties:

(H1) \( X \) is open and \( Y \) is closed.
(H2) For any short exact sequence

\[
0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{F} \to 0
\]

with \( \mathcal{G} \in X \) and \( \mathcal{F} \in Y \), we have \( \mathcal{E} \in X \).
(H3) \( Y \) is closed under extensions, i.e., if in the sequence (5.1.2) we have \( \mathcal{E}, \mathcal{F} \in Y \), then \( \mathcal{G} \in Y \).

To a Hecke pattern \( (X,Y) \) we associate a version of the induction diagram (4.2.2) which we call the Hecke diagram

\[
X \times Y \xrightarrow{q} \text{SES}_{XXY} \xrightarrow{p} X.
\]

Here \( \text{SES}_{XXY} \) is the moduli stack of short exact sequences (5.1.2) with \( \mathcal{E}, \mathcal{G} \in X \) and \( \mathcal{F} \in Y \), and the projections \( q : \text{SES}_{XXY} \to X \times Y \), \( p : \text{SES}_{XXY} \to Y \) associate to a sequence (5.1.2) the pair of sheaves \( (\mathcal{E}, \mathcal{F}) \) and to the sheaf \( \mathcal{G} \) respectively. We note the following analog of Propositions 4.2.3 and 4.3.4.

**Proposition 5.1.4.**

(a) The morphism \( p \) is schematic and proper.
(b) The morphism \( q \) identifies \( \text{SES}_{XXY} \) with an open substack in \( \text{Tot}(\tau_{\leq 0}\mathcal{C}_{XY}) \), where \( \mathcal{C}_{XY} \) is the \([0,2]\)-perfect complex on \( X \times Y \) defined as in (4.3.1).

**Proof.** The fiber of \( p \) over \( \mathcal{G} \) consists of subsheaves \( \mathcal{E} \subset \mathcal{G} \) such that \( \mathcal{E} \in X \) and \( \mathcal{G}/\mathcal{E} \in Y \). Because of the property (H2) we can say that it consists of \( \mathcal{E} \subset \mathcal{G} \) such that \( \mathcal{G}/\mathcal{E} \in Y \). Since \( Y \) is closed in \( \text{Coh}(S) \), our fiber is a closed part of the Quot scheme of \( \mathcal{G} \) hence proper. Parts (a) is proved. To prove (b), note that, similarly to Proposition 4.3.4, the full \( \text{Tot}(\tau_{\leq 0}\mathcal{C}_{XY}) \) is the stack \( \text{SES}_{XY} \) formed by short exact sequences (5.1.2) with \( \mathcal{E} \in X \), \( \mathcal{F} \in Y \) but \( \mathcal{G} \) being an arbitrary coherent sheaf. Now, \( \text{SES}_{XY} \) in the intersection of \( \text{SES}_{XY} \) with the preimage of \( X \subset \text{Coh}(S) \) under the projection to the middle term. Since \( X \) is open in \( \text{Coh}(S) \), we see that \( \text{SES}_{XY} \) is open in \( \text{Tot}(\tau_{\leq 0}\mathcal{C}_{XY}) \). \( \square \)

5.2. The derived Hecke action. Let \( (X,Y) \) be a Hecke pattern for \( S \). Denote \( \mathcal{H}_X = H_{BM}^\bullet(X) \) and \( \mathcal{H}_Y = H_{BM}^\bullet(Y) \). From the property (H3) we see, as in Theorem 4.4.2, that the derived induction diagram (5.2.1) for \( Y \) makes \( \mathcal{H}_Y \) into an associative algebra. Further, similarly to (5.2.1), we have the derived diagram of f-Artin stacks which we call the derived Hecke diagram:

\[
X \times Y \xrightarrow{\pi} \text{Tot}(\tau_{\leq 0}\mathcal{C}_{XY}) \xrightarrow{i} \text{SES}_{XXY} \xrightarrow{p} X
\]

Here \( i \) identifies \( \text{SES}_{XY} \) with an open subset of the zero locus of a section of the vector bundle \( \pi^*\mathcal{C}_{XY} \) and so gives rise to the virtual pullback \( i' \). So as in §4.4, we define the map

\[
\nu : \mathcal{H}_X \otimes \mathcal{H}_Y = H_{BM}^\bullet(X) \otimes H_{BM}^\bullet(Y) \to H_{BM}^{\bullet + 2\text{vrk}\mathcal{C}_{XY}}(X) = \mathcal{H}_X.
\]

**Theorem 5.2.2.** The map \( \nu \) makes \( \mathcal{H}_X \) into a right module over the algebra \( \mathcal{H}_Y \).

**Proof.** Completely similar to that of Theorem 4.4.2. It is based on considering \( \text{FILT}_{XY} \), the stack of flags of coherent sheaves \( \mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03} \) with \( \mathcal{E}_{01} \subset X \) and \( \mathcal{E}_{02}/\mathcal{E}_{01}, \mathcal{E}_{03}/\mathcal{E}_{02} \subset Y \). \( \square \)
5.3. Examples of Hecke patterns. The general phenomenon is that sheaves with support of lower dimension act, by Hecke operators, on sheaves with support of higher dimension. We consider several refinements of the condition on dimension of support.

Definition 5.3.1. Let $0 \leq m \leq 2$.

(a) A coherent sheaf $\mathcal{F}$ on $S$ with proper support is called $m$-dimensional, if $\dim \text{Supp}(\mathcal{F}) \leq m$. We denote by $\text{Coh}_{\leq m} = \text{Coh}_{\leq m}(S) \subset \text{Coh}$ the substack formed by $m$-dimensional sheaves.

(b) We say that $\mathcal{F}$ is purely $m$-dimensional, if any non-zero $\mathcal{O}_S$-submodule $\mathcal{F}' \subset \mathcal{F}$ is $m$-dimensional.

(c) We further say that $\mathcal{F}$ is homologically $m$-dimensional, if it is $m$-dimensional and for any $\mathbb{C}$-point $x \in S$ we have $\text{Ext}^j_{\mathcal{O}_S}(\mathcal{O}_x, \mathcal{F}) = 0$ for $0 \leq j < m$. We denote by $\text{Coh}_m = \text{Coh}_m(S) \subset \text{Coh}$ the substack formed by $m$-dimensional sheaves.

Proposition 5.3.2.

(a) For $m = 0$, the conditions “0-dimensional”, “purely 0-dimensional” and “homologically 0-dimensional” sheaves are the same.

(b) For $m = 1$, the conditions “purely 1-dimensional” and “homologically 1-dimensional” are the same.

(c) For $m = 2$, the condition “purely 2-dimensional” is the same as “torsion-free” while “homologically 2-dimensional” is the same as “vector bundle”.

Proof. Parts (a) and (b) are obvious, as is the first statement in (c). Let us show the second statement. Notice that condition of being homologically 2-dimensional, i.e., $\text{Ext}^j(\mathcal{O}_x, \mathcal{F}) = 0$ for $j < 2$ and all $x$, is nothing but the maximal Cohen-Macaulay condition. Since $S$ is assumed to be smooth, any maximal Cohen-Macaulay sheaf is locally free. □

We denote by $\text{Coh}_m(S)$ the moduli stack of homologically 2-dimensional sheaves with proper support, and by $\text{Coh}_{\text{tf}}(S)$ denote the moduli stack of torsion-free (i.e., purely 2-dimensional) sheaves.

Proposition 5.3.3. The following pairs of substacks are Hecke patterns: $(\text{Coh}_1(S), \text{Coh}_0(S), (\text{Coh}_2(S), \text{Coh}_1(S)))$, $(\text{Coh}_{\text{tf}}(S), \text{Coh}_0(S))$ and $(\text{Coh}_{\text{tf}}(S), \text{Coh}_1(S))$.

To prove the proposition, we note that $\text{Coh}_1(S)$ and $\text{Coh}_0(S)$ are both open and closed in $\text{Coh}(S)$. Further, $\text{Coh}_2(S)$, the stack of vector bundles, is open, as is $\text{Coh}_{\text{tf}}(S)$. Further, all these stacks are closed under extensions. So it remains to prove the following.

Lemma 5.3.4. Suppose we have a short exact sequence as in (5.1.2).

(a) If $\mathcal{G} \subset \text{Coh}_m(S)$ and $\mathcal{F} \in \text{Coh}_{m-1}(S)$, then $\mathcal{E} \in \text{Coh}_m(S)$.

(b) If $\mathcal{G} \subset \text{Coh}_{\text{tf}}(S)$, then $\mathcal{E} \in \text{Coh}_{\text{tf}}(S)$.

Proof. (a) Since $\mathcal{E} \subset \mathcal{G}$, it is clear that $\dim \text{Supp}(\mathcal{E}) \leq m$. The vanishing of $\text{Ext}^j(\mathcal{O}_x, \mathcal{E}')$ for $j < m$ follows at once from the long exact sequence of $\text{Ext}^*(\mathcal{O}_x, -)$ induced by the short exact sequence above. Part (b) is obvious: any subsheaf of a torsion free sheaf is torsion free. □

This ends the proof of Proposition 5.3.3.

Remark 5.3.5. The non-trivial part of the proposition says that homologically (or, what is the same, purely) 1-dimensional sheaves govern Hecke modifications of vector bundles on a surface.

5.4. Stable sheaves and Hilbert schemes. Let $S$ be a smooth connected projective surface and $m = 0, 1$. We can apply the construction in §4.4 to the substack of $m$-dimensional sheaves $X = \text{Coh}_{\leq m}(S)$ of $\text{Coh}(S)$. We have derived the induction diagram (5.2.1), hence the formula (4.4.1) yields an associative multiplication on $H_{BM}^*(\text{Coh}_{\leq m}(S))$.

Now, let $P(\mathcal{E}) : m \mapsto \chi(\mathcal{E}(m))$ be the Hilbert polynomial of a coherent sheaf $\mathcal{E}$ on $S$, and $p(\mathcal{E}) = P(\mathcal{E})/(\text{leading coefficient})$ be the reduced Hilbert polynomial. The sheaf $\mathcal{E}$ is stable if it is pure and $p(\mathcal{F}) < p(\mathcal{E})$ for any proper subsheaf $\mathcal{F} \subset \mathcal{E}$. Let $M_S(r, d, n)$ be the moduli space of rank $r$ semi-stable...
sheaves with first Chern number $d$ and second Chern number $n$. See [30] for a general background on these moduli spaces.

**Theorem 5.4.1.**

(a) The direct image by the closed embeddings $\text{Coh}_0(S) \subset \text{Coh}_{\leq 1}(S) \subset \text{Coh}(S)$ gives algebra homomorphisms $H^\bullet_{BM}(\text{Coh}_0(S)) \to H^\bullet_{BM}(\text{Coh}_{\leq 1}(S)) \to H^\bullet_{BM}(\text{Coh}(S))$.

(b) The algebra $H^\bullet_{BM}(\text{Coh}_{\leq 1}(S))^{\text{op}}$ acts on $\bigoplus_{d,n} H^\bullet_{BM}(\text{M}_S(1,d,n))$.

(c) The algebra $H^\bullet_{BM}(\text{Coh}_0(S))^{\text{op}}$ acts on $\bigoplus_{d,n} H^\bullet_{BM}(\text{M}_S(1,d,n))$ for each $d$.

**Proof.** Part (a) follows from base change. Parts (b), (c) are proved as in §5.2. Let us give more details on (b), part (c) is proved in a similar way.

First, let us consider the following more general setting: let $X = \text{Coh}(S)$ and $Y \subset \text{Coh}(S)$ the substack consisting of torsion free sheaves. Note that the substack $Y \subset X$ is both open and stable by subobjects. We claim that the algebra $H^\bullet_{BM}(X)^{\text{op}}$ acts on $H^\bullet_{BM}(Y)$. To prove this, we consider the restrictions of $\text{Tot}(C^{\leq 0})$ and SES to the stack $Y \times X$ given by

$$\text{Tot}(C^{\leq 0})|_{Y \times X} = \pi^{-1}(Y \times X), \quad \text{SES}|_{Y \times X} = q^{-1}(Y \times X).$$

Then, the derived induction diagram (5.2.1) gives rise to the following commutative diagram

$$\begin{array}{ccc}
\text{Coh}(S) \times \text{Coh}(S) & \overset{\pi}{\longrightarrow} & \text{Tot}(C^{\leq 0}) \overset{i}{\longleftarrow} \text{SES} \\
Y \times X & \overset{\pi}{\longleftarrow} & \text{Tot}(C^{\leq 0})|_{Y \times X} \overset{i}{\longleftarrow} \text{SES} \\
& \overset{p}{\longrightarrow} & \text{Coh}(S) \\
\end{array}$$

where $\text{SES} = p^{-1}(Y)$ and $j$ is the obvious open immersion of stacks $j: \text{SES} \subset \text{SES}|_{Y \times X}$. Let $\tilde{s}_C$ be the restriction of the section $s_C$ of $\pi^*C^1$ to $Y \times X$. We define a map

$$\tilde{m} : H^\bullet_{BM}(Y) \otimes H^\bullet_{BM}(X) \longrightarrow H^\bullet_{BM}(Y_{+2\text{vrk}(C)}(Y))$$

as the composition of the exterior product and the composed map $\tilde{p} \circ \tilde{j}^* \circ \tilde{s}_C^1 \circ \tilde{\pi}^*$. We claim that the map $\tilde{m}$ above defines an action of the algebra $H^\bullet_{BM}(X)^{\text{op}}$ on $H^\bullet_{BM}(Y)$. Then, the diagrams (4.5.1), (4.5.2) yield the following fiber diagrams of stacks

$$\begin{array}{ccc}
\text{FILT} & \overset{x}{\longrightarrow} & \text{SES} \overset{p}{\longrightarrow} Y \\
\downarrow{y} & \downarrow{q} & \downarrow{p} \\
\text{SES} \times X & \overset{p \times 1}{\longrightarrow} & Y \times X \\
\downarrow{q \times 1} & & \downarrow{p} \\
Y \times X \times X
\end{array}$$

and

$$\begin{array}{ccc}
\text{FILT} & \overset{w}{\longrightarrow} & \text{SES} \overset{p}{\longrightarrow} Y \\
\downarrow{v} & \downarrow{q} & \downarrow{p} \\
Y \times \text{SES} & \overset{1 \times p}{\longrightarrow} & Y \times X \\
\downarrow{1 \times q} & & \downarrow{p} \\
Y \times X \times X
\end{array}$$

(5.4.4)
where $\overline{\text{filt}} \subset \text{filt}$ is the open substack classifying flags of coherent sheaves $\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}$ over $S$ such that $\mathcal{E}_{01}, \mathcal{E}_{02}, \mathcal{E}_{03} \in Y$. Then, the claim is proved as in §4.5, replacing the diagrams (4.5.1), (4.5.2) by (5.4.3), (5.4.4).

Now, a rank 1 coherent sheaf is stable if and only if it is torsion free. Thus, setting $X = \text{Coh}_{\leq 1}(S)$ and $Y \subset \text{Coh}(S)$ to be the substack consisting of rank 1 torsion free sheaves, the argument above proves the part (b).

□

Remark 5.4.5.

(a) The moduli space $M_S(1, O_S, n)$ of rank one sheaves with trivial determinant and second Chern number $n$ is canonically isomorphic to the Hilbert scheme $\text{Hilb}^n(S)$. If $S$ is a $K3$ surface, then $\text{Hilb}^n(S)$ is further isomorphic to $M_S(1, 0, n)$.

(b) The rings $A_\bullet(\text{Coh}_{\leq 1}(S))^{\text{op}}$ and $K(\text{Coh}_{\leq 1}(S))^{\text{op}}$ act on

$$\bigoplus_{d,n} A_\bullet(M_S(1, d, n)), \quad \bigoplus_{d,n} K(M_S(1, d, n))$$

respectively, as in Theorem 5.4.1. The proofs are analogous to the proof in Borel-Moore homology.

6. The flat COHA

6.1. $R(\mathbb{A}^2)$ and commuting varieties. In this section we assume $S = \mathbb{A}^2$ and denote

$$R(\mathbb{A}^2) = H_{BM}^\bullet(\text{Coh}_0(\mathbb{A}^2))$$

the COHA of 0-dimensional coherent sheaves on $\mathbb{A}^2$. We note that

$$\text{Coh}_0(\mathbb{A}^2) = \bigsqcup_{n \geq 0} \text{Coh}^{(n)}_0(\mathbb{A}^2),$$

where $\text{Coh}^{(n)}_0(\mathbb{A}^2)$ is the stack of 0-dimensional sheaves $\mathcal{F}$ such that the length of $\mathcal{F}$, i.e., $\dim H^0(\mathcal{F})$, is equal to $n$. We further recall that

$$\text{Coh}^{(n)}_0(\mathbb{A}^2) \simeq C_n//GL_n,$$

where $C_n$ is the $n \times n$ commuting variety

$$C_n = \{(A, B) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) ; [A, B] = 0\},$$

acted upon by $GL_n$ (simultaneous conjugation). Indeed, a 0-dimensional coherent sheaf $\mathcal{F}$ on $\mathbb{A}^2$ of length $n$ is the same as a $\mathbb{C}[x, y]$-module $H^0(\mathcal{F})$ which has dimension $n$ over $\mathbb{C}$, i.e., can be represented by the space $\mathbb{C}^n$ with two commuting operators $A, B$, the actions of $x$ and $y$. We recall.

Proposition 6.1.1. $C_n$ is an irreducible variety of dimension $n^2 + n$. Therefore $\text{Coh}^{(n)}_0(\mathbb{A}^2)$ is an irreducible stack of dimension $n$.

□

Accordingly, we have a direct sum decomposition

$$R(\mathbb{A}^2) = \bigoplus_{n \geq 0} R^n(\mathbb{A}^2), \quad R^n(\mathbb{A}^2) = H_{BM}^\bullet(\text{Coh}^{(n)}_0(\mathbb{A}^2)) = H_{BM}^\bullet(C_n//GL_n),$$

where on the right we have the equivariant Borel-Moore homology of the topological space $C_n$. The algebra $R(\mathbb{A}^2)$ has a $\mathbb{Z}^2$ grading (compatible with multiplication), consisting of (in this order):

(a) the length degree, by the decomposition into the $H_i^{(n)}_{\{x\}}$,

(b) the homological degree, where we put $H_i^{BM}$ in degree $i$. 

Define the $\mathbb{Z}^2$-graded vector space
\begin{equation}
\Theta = q^{-1} t \cdot k[q, t], \quad \text{deg}(q) = (0, -2), \quad \text{deg}(t) = (1, 0).
\end{equation}

The following is well known, see, e.g., [59, §5.3] and the references there, and goes back to the Feit-Fine formula for the number of points in the commuting varieties over finite fields [15, (2)] and the purity of the Borel-Moore homology of the commuting stack $C_n//GL_n$ proved in [11].

**Proposition 6.1.3.** As a $\mathbb{Z}^2$-graded vector space, $R(\mathbb{A}^2) \simeq \text{Sym}(\Theta)$. \hfill \Box

The goal of this section is to prove the following.

**Theorem 6.1.4.**
\begin{enumerate}[(a)]  
\item $\Theta$ has a natural structure of a graded Lie algebra.  
\item $R(\mathbb{A}^2)$ is isomorphic to $U(\Theta)$ as a graded algebra.  
\item The symmetrized product map yields a graded vector space isomorphism $\text{Sym}(\Theta) \simeq R(\mathbb{A}^2)$.  
\end{enumerate}

Before to do this, let us observe the following.

**Proposition 6.1.5.** The algebra $R(\mathbb{A}^2)$ is the same as the COHA considered in [58, §4.4] in the case of the Jordan quiver.

**Proof.** To prove this, we abbreviate $X_n = C_n//GL_n$, $S = \mathbb{A}^2$, and note that the tautological sheaf $U$ over $X_n \times S$ is identified with the $GL_n$-equivariant sheaf over $C_n \times S$ given by $U = \mathbb{C}^n \otimes O_{C_n}$, with the $O_{C_n}$-linear action of $O_S = \mathbb{C}[x, y]$ such that $x, y$ act as $A \otimes 1, B \otimes 1$ respectively on the fiber $U|(A, B)$. Let $p$ be the Lie algebra consisting of $(n, m)$-uper triangular matrices in $\mathfrak{gl}_{n+m}$, and let $u, I$ be its nilpotent radical and its standard Levi subalgebras. Let $P, U$ and $L$ be the corresponding linear groups. Write $X_{n,m} = X_n \times X_m$ and $C_{n,m} = C_n \times C_m$. We identify $C_{n,m}$ with the commuting variety of the Lie algebra $I$ and $X_{n,m}$ with the moduli stack $C_{n,m}//L$. We have $u = \text{Hom}_C(C^n, C^m)$, and the perfect [-1,1]-complex $C$ over $X_{n,m}$ in (4.3.1) is identified with the $L$-equivariant Koszul complex of vector bundles over $C_{n,m}$ given by
\[
\begin{array}{ccc}
  u \otimes O_{C_{n,m}} & \xrightarrow{d^0} & u^2 \otimes O_{C_{n,m}} \\
  & \xrightarrow{d^1} & u \otimes O_{C_{n,m}}
\end{array}
\]
where the differentials over the $\mathbb{C}$-point $(A, B)$ in $C_{n,m}$ are given respectively by
\[
d^0(u) = ([A, u], [B, u]), \quad d^1(v, w) = [A, u] - [B, v] = [A \oplus v, B \oplus w],
\]
and the direct sum holds for the canonical isomorphism $I \times u \to p$. The total space $\text{Tot}(C)$ of this complex, defined in (3.3.9), is a smooth derived stack over $X_{n,m}$ such that:
\begin{enumerate}[(a)]  
\item The underlying Artin stack is the vector bundle stack $C^0//[C^{-1}$ over $X_{n,m}$ such that
\[
C^{-1} = (C_{n,m} \times u)\!/L, \quad C^0 = (C_{n,m} \times u^2)\!/L.
\]
It is isomorphic to the following quotient relatively to the diagonal $P$-action
\[
\text{Tot}(C^0) = (C_{n,m} \times u^2)\!/P.
\]
\item The structural sheaf of derived algebras is the free $P$-equivariant graded-commutative $O_{C_{n,m} \times u^2}$-algebra generated by the elements of $u^r$ in degree -1. The differential is given by the transpose of the Lie bracket $u \times u \to u$.
\end{enumerate}

Therefore, the derived induction diagram (5.2.1) is
\begin{equation}
\begin{array}{ccc}
  C_{n,m}//L & \xrightarrow{\pi} & (C_{n,m} \times u^2)\!/P \\
  \xrightarrow{\iota} & \tilde{C}_{n,m}//P & \to \tilde{C}_{n,m}//GL_{n+m},
\end{array}
\end{equation}
where $\tilde{C}_{n,m}$ is the commuting variety of the Lie algebra $p$. We can now compare the product
\[
m : H^*_BM(X_n) \otimes H^*_BM(X_m) \to H^*_BM(X_{n+m})
\]
in (4.4.1) with the multiplication in [58, §4.4]. We have the fiber diagram of stacks
\[
(C_{n,m} \times \mathfrak{u})//P \xrightarrow{f} (C_{n,m} \times \mathfrak{u}^3)//P \xrightarrow{s} (C_{n,m} \times \mathfrak{u}^2)//P
\]
where 1 is the identity, 0 is the zero section, \( f \) is the projection to the third component of \( \mathfrak{u}^3 \) (which is a local complete intersection morphism) and \( s = 1 \times d_1 \). Hence, the composed map \( g = f \circ s \) is the Lie bracket \( (A, B; v, w) \mapsto [A \oplus v, B \oplus w] \) and the composition rule of refined pullback morphisms implies that
\[
g^\prime(x) = s^\prime f^\prime(x) = s^\prime \pi^\prime(x)
\]
in \( H^\bullet_{BM}(\widetilde{C}_{n,m}//P) \) for any class \( x \in H^\bullet_{BM}(X_n \times X_m) \). We deduce that the multiplication map \( m \) is the same as the multiplication considered in [58, §4.4]. \( \square \)

6.2. \( R(\mathbb{A}^2) \) as a Hopf algebra. As a first step in the proof of Theorem 6.1.4, we introduce on \( R(\mathbb{A}^2) \) a compatible comultiplication.

Let \( U \subset \mathbb{C}^2 \) be any open set in the complex analytic topology. We denote by \( Coh_0(U) \) the category of 0-dimensional coherent analytic sheaves on \( U \). The corresponding moduli stack \( Coh_0(U) \) can be understood as a complex analytic stack in the sense of [54], i.e., as a stack of groupoids on the site of Stein complex analytic spaces. It can also be understood in a more elementary way, as follows.

Let \( C_n(U) \subset C_n \) be the open subset (in the complex analytic topology) formed by pairs \((A, B)\) of commuting matrices for which the joint spectrum (the support of the corresponding coherent sheaf on \( \mathbb{C}^2 \)) is contained in \( U \). It is, therefore, a complex analytic space. Then we can define

\[
Coh_0^{(n)}(U) = C_n(U)//GL_n(\mathbb{C}),
\]
as the quotient analytic stack, and put

\[
Coh_0(U) = \bigsqcup_{n \geq 0} Coh_0^{(n)}(U).
\]

Using this understanding, we define directly

\[
R(U) = H^\bullet_{BM}(Coh_0(U)) = \bigoplus_{n \geq 0} H^\bullet_{BM}(C_n(U)//GL_n(\mathbb{C})) = \bigoplus_{n \geq 0} R^n(U).
\]

The same considerations as in §4 make \( R(U) \) into a graded associative algebra.

If \( U' \subset U \subset \mathbb{C}^2 \) are two open sets, then \( C_n(U') \hookrightarrow C_n(U) \) is an open embedding, and we have maps of \( \mathbb{Z} \)-graded, resp. \( \mathbb{Z}^2 \)-graded vector spaces

\[
\rho_{U,U'}^n : H^\bullet_{BM}(C_n(U)//GL_n(\mathbb{C})) \longrightarrow H^\bullet_{BM}(C_n(U')//GL_n(\mathbb{C})),
\]

\[
\rho_{U,U'} = \bigoplus_{n \geq 0} \rho_{U,U'}^n : R(U) \longrightarrow R(U').
\]

Proposition 6.2.1.

(a) \( \rho_{U,U'} \) is an algebra homomorphism.
(b) If the embedding \( U' \hookrightarrow U \) is a homotopy equivalence, then \( \rho_{U,U'} \) is an isomorphism.
(c) If \( U \) is a disjoint union of open subsets \( U_1, \cdots, U_m \), then

\[
R(U) \simeq R(U_1) \otimes \cdots \otimes R(U_m).
\]
Proof. Part (a) is clear from definitions. To show (b), we note that \( C_n(U) \) and \( C_n(U') \) are naturally stratified (by singularities), and, under our assumption, the embedding \( C_n(U') \hookrightarrow C_n(U) \) is a homotopy equivalence relative to the stratifications, i.e., it induces homotopy equivalences on all the strata. By \( \partial \) (spectral sequence argument) this implies that the map
\[
H^*_{BM, GL_n(C)}(C_n(U)) = H^*_{GL_n(C)}(C_n(U), \omega_{C_n(U)}) \rightarrow H^*_{GL_n(C)}(C_n(U'), \omega_{C_n(U')}) = H^*_{BM, GL_n(C)}(C_n(U'))
\]
is an isomorphism.

We abbreviate \( GL_{n_1, \ldots, n_m} = GL_{n_1} \times \cdots \times GL_{n_m} \). Then, part (c) follows from the \( GL_n(C) \)-equivariant identifications
\[
C_n(U) = \bigcup_{n_1 + \cdots + n_m = n} \left( GL_n(C) \times GL_{n_1, \ldots, n_m}(C) C_{n_1}(U_1) \times \cdots \times C_{n_m}(U_m) \right),
\]
which reflect the fact that a length \( n \) sheaf \( F \) on \( U \) consists of sheaves \( F_i \) on \( U_i \) of lengths \( n_i \) summing up to \( n \).

\( \square \)

Corollary 6.2.2. If an open \( U \subset \mathbb{C}^2 \) is homeomorphic to a 4-ball, then \( \rho_{\mathbb{C}^2, U} : R(\mathbb{C}^2) \rightarrow R(U) \) is an isomorphism.

\( \square \)

Let us now choose, once and for all, two disjoint round balls \( U_1, U_2 \subset \mathbb{C}^2 \). Define a morphism of \( \mathbb{Z}^2 \)-graded vector spaces \( \Delta : R(\mathbb{C}^2) \rightarrow R(\mathbb{C}^2) \otimes R(\mathbb{C}^2) \) as the composition
\[
R(\mathbb{C}^2) \xrightarrow{\rho_{\mathbb{C}^2, U_1, U_2}} R(U_1 \sqcup U_2) \cong R(U_1) \otimes R(U_2) \xrightarrow{\rho_{\mathbb{C}^2, U_1, U_2}^{-1} \otimes \rho_{\mathbb{C}^2, U_2}^{-1}} R(\mathbb{C}^2) \otimes R(\mathbb{C}^2).
\]

Proposition 6.2.3.

(a) \( \Delta \) does not depend on the choice of the balls \( U_1, U_2 \) provided they are disjoint.

(b) \( \Delta \) makes \( R(\mathbb{C}^2) \) into a cocommutative Hopf algebra.

Proof. Any two admissible choices of \( U_1, U_2 \) are connected by a path of admissible choices, and \( \Delta \) does not change along this path. This proves (a). To prove (b), note that all the maps in the above chain are compatible with the Hall multiplication, so \( \Delta \) is a homomorphism of algebras. Its cocommutativity follows from (a) by interchanging \( U_1 \) and \( U_2 \), i.e., by connecting \( (U_1, U_2) \) and \( (U_2, U_1) \) by a path of admissible choices. Coassociativity is proved similarly by considering triples of disjoint balls. This proves that \( R(\mathbb{C}^2) \) into a cocommutative bialgebra.

It remains to prove that \( R(\mathbb{C}^2) \) has an antipode. This is a standard argument using co-nilpotency, see, e.g., [42, §1.2]. That is, define
\[
\Delta : R(\mathbb{C}^2) \rightarrow R(\mathbb{C}^2) \otimes R(\mathbb{C}^2), \quad \Delta(x) = x - (x \otimes 1 + 1 \otimes x),
\]
and let \( \Delta^n : R(\mathbb{C}^2) \rightarrow R(\mathbb{C}^2)^{\otimes n} \) be the \( n \)-fold iteration of \( \Delta \). Then \( R(\mathbb{C}^2) \) is co-nilpotent, that is, for any \( x \in R(\mathbb{C}^2) \) there is \( n \) such that \( \Delta^n(x) = 0 \) for \( m \geq n \). Therefore the antipode \( \alpha : R(\mathbb{C}^2) \rightarrow R(\mathbb{C}^2) \) is given by the following geometric series, terminating upon evaluation on any \( x \in R(\mathbb{C}^2) \):
\[
\alpha = \sum_{n=1}^{\infty} (-1)^n m_n \circ \Delta^n,
\]
where \( m_n : R(\mathbb{C}^2)^{\otimes n} \rightarrow R(\mathbb{C}^2) \) is the \( n \)-fold multiplication.

\( \square \)

Let \( R(\mathbb{C}^2)_{\text{prim}} = \{ a \in R(\mathbb{C}^2) ; \Delta(a) = a \otimes 1 + 1 \otimes a \} \) be the Lie algebra of primitive elements with the bracket \( [a, b] = ab - ba \).

Corollary 6.2.4.

(a) \( R(\mathbb{C}^2) \) is isomorphic, as a Hopf algebra, to the universal enveloping algebra of \( R(\mathbb{C}^2)_{\text{prim}} \).

(b) \( R(\mathbb{C}^2)_{\text{prim}} \cong \Theta \) as a \( \mathbb{Z}^2 \)-graded vector space.
Proof. Part (a) follows from the Milnor-Moore theorem. Part (b) follows from the Poincaré-Birkhoff-Witt theorem and from Proposition 6.1.3.

6.3. Explicit primitive elements in \( R(\mathbb{A}^2) \). For any open \( U \subset \mathbb{C}^2 \) let \( \text{Coh}_{1,\text{pt}}^{(n)}(U) \subset \text{Coh}_0^{(n)}(U) \) be the closed analytic substack formed by 1-point coherent sheaves, i.e., sheaves whose support consists of precisely one point. In other words,

\[
\text{Coh}_{1,\text{pt}}^{(n)}(\mathbb{C}^2) = \text{Coh}_{n,1,\text{pt}}(U)/\text{GL}_n(\mathbb{C}),
\]

where \( \text{Coh}_{n,1,\text{pt}}(U) \subset \text{Coh}_n(U) \) is the closed analytic subspace formed by pairs \((A, B)\) of commuting matrices whose joint spectrum reduces to one point in \( \mathbb{C}^2 \) (but can be degenerate). Still more explicitly,

\[
\text{Coh}_{n,1,\text{pt}}(U) = U \times NC_n,
\]

where \( NC_n \) is the \( n \) by \( n \) nilpotent commuting variety

\[
NC_n = \{(A, B) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}); [A, B] = A^n = B^n = 0\}.
\]

In particular, we have the closed subvariety

\[
C_{n,1,\text{pt}} = C_{n,1,\text{pt}}(\mathbb{C}^2) = \mathbb{C}^2 \times NC_n \subset C_n,
\]

invariant under \( \text{GL}_n(\mathbb{C}) \). We recall.

**Proposition 6.3.2** ([3]). \( NC_n \) is an irreducible algebraic variety of dimension \( n^2 - 1 \).

The proposition implies that \( C_{n,1,\text{pt}} \) is an irreducible variety of dimension \( n^2 + 1 \). So \( \text{Coh}_{1,\text{pt}}^{(n)}(\mathbb{C}^2) \), its image in \( \text{Coh}_0^{(n)}(\mathbb{C}^2) \), is an irreducible stack of dimension 1, and it has the equivariant fundamental class

\[
\theta_n = [C_{n,1,\text{pt}}] \in H_2^{BM}(C_n/\text{GL}_n).
\]

Further, let \( \mathcal{E}_n \) be the trivial vector bundle of rank \( n \) on the \( \text{GL}_n \)-variety \( C_n \), equipped with the vectorial representation of \( \text{GL}_n \). We call \( \mathcal{E}_n \) the tautological sheaf. Being an equivariant vector bundle, it has the equivariant Chern characters

\[
ch_i(\mathcal{E}_n) \in H^{2i}(C_n/\text{GL}_n), \quad i \geq 0,
\]

and, for \( i \geq 0, n \geq 1 \), we define

\[
\theta_{n,i} = ch_i(\mathcal{E}_n) \cap \theta_n \in H_i^{BM}(C_n/\text{GL}_n) = R^{n,2-2i}(\mathbb{C}^2).
\]

Comparing the \( \mathbb{Z}^2 \)-grading of \( \Theta \), we see that the map

\[
\alpha : \Theta \rightarrow R(\mathbb{C}^2), \quad t^n q^i \mapsto \theta_{n,i},
\]

is a morphism of \( \mathbb{Z}^2 \)-graded vector spaces.

**Proposition 6.3.5.**

(a) \( \alpha \) is injective, i.e., each \( \theta_{n,i} \) is non-zero.

(b) \( \theta_{n,i} \) is primitive.

**Proof.** The claim (a) follows from [11, thm. C] and the explicit computations in [11, §5] in the case of the Jordan quiver. More precisely, let \( Q_g \) be the quiver with one vertex and \( g \) loops. For each integer \( n \geq 0 \), let \( \mathcal{M}(Q_g)_n \) be the coarse moduli space of semisimple \( n \)-dimensional representations of \( \mathbb{C}Q_g \), i.e., the categorical quotient of \( (\mathfrak{gl}_n)^g \) by the adjoint action of \( \text{GL}_n \). We’ll abbreviate \( \mathcal{M}(Q_g) = \bigsqcup_{n \geq 0} \mathcal{M}(Q_g)_n \).

The direct sum of representations yields a finite morphism \( \mathcal{M}(Q_g) \times \mathcal{M}(Q_g) \to \mathcal{M}(Q_g) \), hence a symmetric monoidal structure on the category \( \text{Perv}(\mathcal{M}(Q_g)) \) of perverse sheaves on \( \mathcal{M}(Q_g) \), which allows to consider
the \( n \)-th symmetric power \( \text{Sym}^n(\mathcal{E}) \) for any object \( \mathcal{E} \) in \( \text{Perv}(\mathcal{M}(Q_g)) \). Let \( \text{Sym}(\mathcal{E}) = \bigoplus_{n \geq 0} \text{Sym}^n(\mathcal{E}) \). Set \( g = 3 \) and fix an embedding \( Q_2 \subset Q_3 \). By [11], we have

\[
\bigoplus_{n \geq 0} H^*_c(C_n//GL_n) = H^*_c(\mathcal{M}(Q_3), \text{Sym}(\mathcal{BPS} \otimes H^*_c(B\mathbb{C}^\times)))
\]

(6.3.6)

\[
\bigoplus_{n \geq 0} H^*_c(C_{n,1\text{pt}}//GL_n) = H^*_c(\mathcal{M}(Q_3)_{1\text{pt}}, \text{Sym}(\mathcal{BPS} \otimes H^*_c(B\mathbb{C}^\times)))
\]

where \( \mathcal{BPS} = \bigoplus_{n \geq 0} \mathcal{BPS}_n \) and \( \mathcal{BPS}_n \) is, up to some shift, the constant sheaf supported on the small diagonal \( \mathbb{C}^3 \subset \mathcal{M}(Q_3)_n \). For each \( n \), the closed subset \( \mathcal{M}(Q_3)_{n,1\text{pt}} \subset \mathcal{M}(Q_3)_n \) is the coarse moduli space of semisimple representations of \( \mathbb{C}Q_3 \) for which the underlying \( \mathbb{C}Q_2 \)-module has a punctual support in \( \mathbb{C}^2 \). We have

\[
\mathbb{C}^3 \subset \mathcal{M}(Q_3)_{n,1\text{pt}} \subset \mathcal{M}(Q_3)_n.
\]

Taking the direct summand in (6.3.6)

\[
\mathcal{BPS}_n \otimes H^*_c(B\mathbb{C}^\times) \subset \text{Sym}(\mathcal{BPS} \otimes H^*_c(B\mathbb{C}^\times)),
\]

we get the commutative diagram

\[
\begin{array}{ccc}
H^*_c(\mathcal{M}(Q_3)_n, \mathcal{BPS}_n \otimes H^*_c(B\mathbb{C}^\times))^{*} & \xrightarrow{f} & H^*_{\text{BM}}(C_n//GL_n) \\
\downarrow & & \downarrow h \\
H^*_c(\mathcal{M}(Q_3)_{n,1\text{pt}}, \mathcal{BPS}_n \otimes H^*_c(B\mathbb{C}^\times))^{*} & \xrightarrow{g} & H^*_{\text{BM}}(C_{n,1\text{pt}}//GL_n).
\end{array}
\]

The map \( f \) is invertible, and \( h \) is the pushforward by the closed embedding \( C_{n,1\text{pt}} \subset C_n \). We deduce that the class \( \text{ch}_i(\mathcal{E}_n) \cap [C_{n,1\text{pt}}] \) is non-zero in \( H^*_{\text{BM}}(C_{n,1\text{pt}}//GL_n) \) and that its image by \( h \) is non-zero. This image is equal to the class \( \theta_{n,i} \).

To prove (b), given an open \( U \subset \mathbb{C}^2 \), we define, in the same way as before, elements

\[
\theta_{n,i}(U) \in R^{n,2-2i}(U) = H^*_{2-2i}(C_n(U)\cap GL_n(\mathbb{C})).
\]

For \( U' \subset U \) we have

\[
\rho_{U',U}(\theta_{i,n}(U)) = \theta_{n,i}(U').
\]

For \( U = U_1 \cup U_2 \) being a disjoint union of two opens, a length \( n \) \( 0 \)-dimensional sheaf \( \mathcal{F} \) on \( U \) consists of two sheaves \( \mathcal{F}_i \) on \( U_i \) of lengths \( n_i \), \( i = 1, 2 \) such that \( n_1 + n_2 = n \). This can be expressed by saying that

\[
C_n(U_1 \cup U_2) = \bigsqcup_{n_1+n_2=n} (\text{GL}_n(\mathbb{C}) \times_{\text{GL}_{n_1,n_2}(\mathbb{C})} (C_{n_1}(U_1) \times C_{n_2}(U_2))),
\]

from which we deduce the following identification

\[
R^n(U) = \bigoplus_{n_1+n_2=n} R^{n_1}(U_1) \otimes R^{n_2}(U_2),
\]

(6.3.8)

Let \( \mathcal{E}_{n,U} \) be the tautological sheaf of \( C_n(U) \) and similarly for \( U_1, U_2 \). With respect to the identification (6.3.7), we have

\[
\mathcal{E}_{n,U} = \bigsqcup_{n_1+n_2=n} (\mathcal{E}_{n_1,U_1} \boxtimes \mathcal{O} \boxplus \mathcal{O} \boxtimes \mathcal{E}_{n_2,U_2}).
\]

Thus, the additivity of the Chern character gives

\[
\text{ch}_i(\mathcal{E}_{n,U}) = \sum_{n_1+n_2=n} (\text{ch}_i(\mathcal{E}_{n_1,U_1}) \otimes 1 + 1 \otimes \text{ch}_i(\mathcal{E}_{n_2,U_2})), \quad \forall i \geq 0.
\]

(6.3.9)

Since, under the identification (6.3.8), we have

\[
\theta_n(U) = \theta_n(U_1) \otimes 1 + 1 \otimes \theta_n(U_2)
\]

\[
\theta_n(U) = \theta_n(U_1) \otimes 1 + 1 \otimes \theta_n(U_2)
\]
we deduce that we have also
\[ \theta_{n,i}(U) = \theta_{n,i}(U_1) \otimes 1 + 1 \otimes \theta_{n,i}(U_2), \quad \forall i \geq 0. \]

Our statement follows from this and from the definition of $\Delta$ via $\rho$. \hfill \Box

**Corollary 6.3.10.** The space $R(\mathbb{C}^2)_{\text{prim}}$ of primitive elements of $R(\mathbb{C}^2)$ coincides with the image $\alpha(\Theta)$. It is closed under the commutator $[a, b] = ab - ba$. \hfill \Box

Theorem 6.1.4 is proved. The symmetrized product map $\sigma : \text{Sym}(\Theta) \to R(\mathbb{A}^2)$ is considered in details in (7.1.5) below.

### 7. The COHA of a Surface $S$ and Factorization Homology

#### 7.1. Statement of results.
Let $S$ be an arbitrary smooth quasi-projective surface and $R(S) = H^\bullet_{BM}(\text{Coh}_0(S))$ be the corresponding cohomological Hall algebra. It is $\mathbb{Z}^2$-graded by (length, homological degree). We introduce a global analog of the space $\Theta$ generating the flat COHA $R(\mathbb{A}^2)$ from §6.3. Let

\[ S \overset{p_n}{\leftarrow} \text{Coh}_1^{(n)}(S) \overset{i_n}{\rightarrow} \text{Coh}_0^{(n)}(S) \]

be the stack of 1-pointed, length $n$ sheaves on $S$ with its canonical closed embedding $i_n$ into $\text{Coh}_0^{(n)}(S)$ and projection $p_n$ to $S$ (so $p_n(\mathcal{F})$ is the unique support point of $\mathcal{F}$). Proposition 6.3.2 implies that $p_n$ is a morphism with all fibers irreducible of relative dimension $(-1)$ and therefore $\text{Coh}_1^{(n)}(S)$ is irreducible of dimension $+1$. Moreover, we have a natural fundamental class in $H^2_{BM}(\text{Coh}_1^{(n)}(S))$ constructed as follows.

We consider the open subscheme $\text{FCoh}_0^{(n)}(S) := \text{Quot}^{(n,0)}(S)$ of the quot-scheme formed by equivalence classes of surjections $\phi : \mathcal{O}^n \to \mathcal{F}$ with $\mathcal{F} \in \text{Coh}_0^{(n)}(S)$ such that $\phi$ induces an isomorphism $\mathbb{C}^n \to H^0(S, \mathcal{F})$. Let $\text{FCoh}_1^{(n)}(S) \subset \text{FCoh}_0^{(n)}(S)$ be the closed subscheme formed by equivalence classes of $\phi$ such that $\mathcal{F}$ is a 1-pointed sheaf. Then, the stack $\text{Coh}_0^{(n)}(S)$ is isomorphic to the quotient stack $\text{FCoh}_0^{(n)}(S)/\text{GL}_n$ and $\text{Coh}_1^{(n)}(S)$ is isomorphic to the quotient stack $\text{FCoh}_1^{(n)}(S)/\text{GL}_n$. Further, we have the projection $\text{FCoh}_1^{(n)}(S) \to S$ with fibers being identified with the variety $N_C^n$ of pairs of nilpotent commuting matrices, see §6.3. Since this variety is irreducible of dimension $n^2 - 1$, the scheme $\text{FCoh}_1^{(n)}(S)$ is an irreducible variety of dimension $n^2 + 1$ and has the fundamental class in $H^2_{BM}(\text{FCoh}_1^{(n)}(S))$. So the quotient stack by $\text{GL}_n$ has the fundamental class in $H^2_{BM}(\text{Coh}_1^{(n)}(S))$.

Therefore we have the pullback map $p_n^!$ given by the composition

\[ H^*_{BM}(S) = H^{4-*}(S) \overset{p_n^!}{\rightarrow} H^{4-*}(\text{Coh}_1^{(n)}(S)) \rightarrow H^*_{BM}(\text{Coh}_1^{(n)}(S)), \]

where the last arrow is the cap-product with the fundamental class of $\text{Coh}_1^{(n)}(S)$.

Define the subspace
\[ \Theta_n(S) = i_n \ast p_n^! H^*_{BM}(S) \subset H^*_{BM}(\text{Coh}_0^{(n)}(S)) = R^n(S). \]

Let $\mathcal{E}_n$ denote also the tautological sheaf on $\text{Coh}_0^{(n)}(S)$ and further put, for $i \geq 0$,

\[ \Theta_{n,i}(S) = \Theta_n(S) \cap ch_i(\mathcal{E}_n) \subset R^n(S). \]

**Proposition 7.1.3.** The canonical map $H^*_{BM}(S) \to \Theta_{n,i}(S)$ is an isomorphism.
Proof. As before, we use the subscheme $\text{FCoh}_0^{(n)}(S)$ whose quotient stack by $GL_n$ is $\text{Coh}_0^{(n)}(S)$. Let $T \subset GL_n$ be a maximal torus. Then, the fixed points locus $\text{FCoh}_0^{(n)}(S)^T$ is isomorphic to $\text{FCoh}_0^{(1)}(S)^n = S^n$. Thus, we have a commutative diagram

$$
\begin{array}{ccc}
H^\bullet_{BM}(S) & \overset{p_n^*}{\longrightarrow} & H^\bullet_{BM, GL_n}(\text{FCoh}_1^{(n)}(S))_{loc} \\
\downarrow a & & \downarrow b \\
H^\bullet_{BM}(S) \otimes H^\bullet_{GL_n, loc} & \overset{\Delta}{\longrightarrow} & (H^\bullet_{BM}(S^n) \otimes H^\bullet_T(S^n))_{loc},
\end{array}
$$

where $H^\bullet_G = H^\bullet(BG)$ and the subscript $\text{loc}$ means the tensor product by the fraction field $H^\bullet_{GL_n, loc}$ of $H^\bullet_{GL_n}$ over $H^\bullet_{GL_n}$. The maps $b, c$ are the pushforward by the closed embeddings $S \subset \text{FCoh}_1^{(n)}(S)$ and $S^n \subset \text{FCoh}_0^{(n)}(S)$, which are invertible by the localization theorem in equivariant cohomology. The map $\Delta$ is the diagonal embedding. It is injective. The map $a$ is equal to $\text{Id} \otimes 1$, up to the cap-product by an invertible element in $H^\bullet(S) \otimes H^\bullet_{GL_n, loc}$. It is injective. We deduce that the map

$$i_n* p_n^* : H^\bullet_{BM}(S) \rightarrow H^\bullet_{BM, GL_n}(\text{FCoh}_1^{(n)}(S))$$

is injective as well. \qed

We define

$$\Theta(S) = \bigoplus_{n,i} \Theta_{n,i}(S) \subset R(S).$$

Thus, for $S = \mathbb{A}^2$ we have that $\Theta(\mathbb{A}^2)$ is identified with the graded space $\Theta$ from (6.1.2), embedded into $R$ by the map $\alpha$ as in (6.3.4). We recall that $H^\bullet_{BM}(\mathbb{A}^2)$ is 1-dimensional, concentrated in homological degree 4. Thus shifting the grading by putting

$$\Theta' = \Theta[0,-4] = qt \cdot k[t],$$

we have by Proposition 7.1.3, an identification of $\mathbb{Z}_2$-graded vector spaces

$$\Theta(S) \simeq H^\bullet_{BM}(S) \otimes \Theta' \simeq H^\bullet_{BM}(S//\mathbb{C}^\times) \otimes k[t].$$

We now consider the symmetrized product map $\sigma = \sigma : \text{Sym}(\Theta(S)) \rightarrow R(S)$ defined as

$$\sigma = \sum_{n \geq 0} \sigma_n, \quad \sigma_n : \text{Sym}^n(\Theta(S)) \rightarrow R(S), \quad \sigma_n(v_1 \cdots v_n) = \frac{1}{n!} \sum_{s \in S_n} v_{s(1)} \cdot \cdots \cdot v_{s(n)}.$$

Here $\cdot$ is the product in the symmetric algebra and $\ast$ is the Hall multiplication. The second main result of this paper is a version of the Poincaré-Birkhoff-Witt theorem for $R(S)$ which allows us to compute its graded dimension. It is proved in the next sections.

**Theorem 7.1.6.** $\sigma : \text{Sym}(\Theta(S)) \rightarrow R(S)$ is an isomorphism of $\mathbb{Z}_2$-graded vector spaces. \qed

### 7.2. Reminder on factorization algebras.

We follow the approach of [9] and [19]. Let $\mathcal{C}, \otimes, \mathbf{1}$ be a symmetric monoidal model category. In particular, it has a class $W$ of weak equivalences. We will consider three examples:

(a) $\mathcal{C} = \text{Top}$ is the category of topological spaces (homotopy equivalent to a CW-complex), $\otimes$ is cartesian product, and weak equivalence have the usual topological meaning.

(b) $\mathcal{C}$ is the category of Artin stacks, $\otimes$ is the Cartesian product of stacks and weak equivalences are equivalences of stacks.

(c) $\mathcal{C} = \text{dgVect}$ is the category of cochain complexes, $\otimes$ is the usual tensor product and weak equivalences are quasi-isomorphisms.

Let $M$ be a $C^\infty$ manifold of dimension $n$. 

Definition 7.2.1. A prefactorization algebra on $M$ valued in $\mathcal{C}$ is a rule $\mathcal{A}$ which associates
(a) to any open set $U \subset M$ an object $\mathcal{A}(U) \in \mathcal{C}$, so that $\mathcal{A}(\emptyset) = 1$.
(b) to any system $U_1, \ldots, U_p$ of disjoint open sets contained in an open set $U_0$, a morphism $\mu_U^{U_0} : \mathcal{A}(U_1) \otimes \cdots \otimes \mathcal{A}(U_p) \to \mathcal{A}(U_0)$, such that
(c) the morphisms $\mu_U^{U_0}$ satisfy associativity.

A morphism of prefactorization algebra $\sigma : \mathcal{A} \to \mathcal{A}'$ is a datum of morphisms $\sigma_U : \mathcal{A}(U) \to \mathcal{A}'(U)$ compatible with the structures. It is a weak equivalence if each $\sigma_U$ is a weak equivalence.

A prefactorization algebra is, in particular, a precosheaf via the maps $\mu_U^{U_0}$, i.e., it is a covariant functor from the category of open subsets in $M$ to $\mathcal{C}$.

Definition 7.2.2. An open covering of $M$ is called a Weiss covering if any finite subset of $M$ is contained in an open set of the covering.

Example 7.2.3.
(a) Let $D \subset \mathbb{R}^n$ be the standard unit disk $\|x\| < 1$. A disk in $M$ is an open subset which is homeomorphic to $D$. The open covering $\mathcal{D}(M)$ of $M$ generated by the disks of $M$ is a Weiss covering. By definition, an open covering of $\mathcal{D}(M)$ consists of a finite disjoint union of disks.

(b) A prefactorization algebra is called locally constant, if for any inclusion of disks $U_0 \subset U_1$ the map $\mu_U^{U_1}$ is a weak equivalence.

Definition 7.2.4.
(a) A prefactorization algebra $\mathcal{A}$ is called a (homotopy) factorization algebra if:

(1) For any Weiss covering $\mathcal{U} = \{U_i\}_{i \in I}$ of any open set $U \subset M$ the natural morphism
$$ \text{holim} \mathcal{N}_*(\mathcal{U}, \mathcal{A}) \to \mathcal{A}(U), $$

with $U_{ij} = U_i \cap U_j$, etc., is a weak equivalence (co-descent).

(b) The factorization homology of $M$ with coefficients in a factorization algebra $\mathcal{A}$ is the object of global cosections of $\mathcal{A}$ which we denote
$$ \int_M \mathcal{A} = \mathcal{A}(M) \in \mathcal{C}. $$

Remark 7.2.5.
(a) A multiplicative prefactorization algebra $\mathcal{A}$ is a factorization algebra if and only if for the particular Weiss covering $\mathcal{D}(U)$ of any open subset $U \subset M$, the object $\mathcal{A}(U)$ is the homotopy colimit of the diagram
$$ \coprod_{U_1, U_2 \in \mathcal{D}(U)} \mathcal{A}(U_1 \cap U_2) \to \coprod_{U_1 \in \mathcal{D}(U)} \mathcal{A}(U_1). $$

In particular, we have
$$ \int_M \mathcal{A} = \text{holim}_{U \in \mathcal{D}(M)} \mathcal{A}(U). $$

See [9, §A.4.3] for details.

(b) Any locally constant prefactorization algebra has a unique extension as a locally constant factorization algebra taking the same value on any disk, but possibly different values on other open sets, see [19, rem. 24].
Sometimes it is convenient to use the dual language. By a \((\text{pre})\text{factorization coalgebra}\) \(\mathcal{B}\) in \(\mathcal{C}\) we will mean a \((\text{pre})\text{factorization algebra}\) in \(\mathcal{C}^{\text{op}}\). Thus, we have maps

\[ \nu_{U_0, \ldots, U_p} : \mathcal{B}(U_0) \rightarrow \mathcal{B}(U_1) \otimes \cdots \otimes \mathcal{B}(U_p) \]

yielding a presheaf on \(M\). For a factorization coalgebra \(\mathcal{B}\) we have the \textit{factorization cohomology} which we denote as

\[ \int_{\mathcal{B}} \mathcal{B} = \mathcal{B}(M) = \holim_{U \in \mathcal{D}(M)} \mathcal{B}(U). \]

Let us record the following two statements for later use.

**Proposition 7.2.6.** If \(\mathcal{F}\) is a locally constant sheaf of \(k\)-\(dg\)-vector spaces, then \(\text{Sym}(\mathcal{F}) : U \mapsto \text{Sym}_k(\mathcal{F}(U))\) is a locally constant factorization coalgebra.

Note that \(\text{Sym}(\mathcal{F})\) as we define it, is not the same as the symmetric algebra of \(\mathcal{F}\) in the symmetric monoidal category of sheaves of \((dg-)\)vector spaces, in fact it is not a sheaf in the usual sense.

**Proof.** This is an analog of [9, thm. 5.2.1] which deals with sheaves corresponding to \(C^\infty\) sections of vector bundles, and their symmetric products in the sense of bornological vector spaces. In our case the proof is similar but easier due to the absence of analytic difficulties. That is, call a covering \(\mathcal{U}\) an \(n\)-Weiss covering, if each subset \(I \subset M\) of cardinality \(\leq n\) is contained in one of the opens of \(\mathcal{U}\). Then it suffices to show that \(\text{Sym}^n(\mathcal{F}) : U \mapsto \text{Sym}_k^n(\mathcal{F}(U))\) satisfies descent for \(n\)-Weiss coverings. This follows, as in the proof of [9, thm. 5.2.1], from the fact that \(\mathcal{F}_{\text{Weiss}}^{\infty}\) is a sheaf of \(M^n\).

**Proposition 7.2.7.** Let \(\sigma : \mathcal{B} \rightarrow \mathcal{B}'\) be a morphism of factorization coalgebras. Suppose that for any disk \(U \subset M\) the morphism \(\sigma_U : \mathcal{B}(U) \rightarrow \mathcal{B}'(U)\) is a weak equivalence. Then \(\sigma\) is a weak equivalence of factorization coalgebras, in particular, \(\sigma\) induces a weak equivalence \(\sigma_M : \int_M \mathcal{B} \rightarrow \int_M \mathcal{B}'\).

**Proof.** For any open \(U\) we realize \(\sigma_U\) by descent from the Weiss cover \(\mathcal{D}(U)\).

### 7.3. Analytic stacks

For the analytic version of the theory of algebraic stacks we follow [54] (where, in fact, the case of higher and derived stacks is also considered).

An \textit{analytic stack} is a stack of groupoids on the category of (possibly singular) Stein analytic spaces over \(\mathbb{C}\), equipped with the Grothendieck topology consisting of open covers in the usual sense. Analytic stacks form a 2-category \(\text{Stan}\) as well as a model category \(\text{Stan}\) where weak equivalences are equivalences of stacks.

For every scheme \(Y\) of finite type over \(\mathbb{C}\) we have the analytic space \(Y^{\text{an}}\), the analytification of \(Y\). Further, or any Artin stack \(X\) over \(\mathbb{C}\) we have the analytic stack \(X^{\text{an}}\), the analytification \(X^{\text{an}}\) defined as the Kan extension of \(X\) from the category of affine schemes of finite type to the category of Stein analytic spaces, see [54] §4 or [29] §3. Note that we have a canonical map

\[ (7.3.1) \quad \eta_X : R\Gamma(X, \omega_X) \hookrightarrow R\Gamma(X^{\text{an}}, \omega_X^{\text{an}}). \]

If \(X = Y\) is a scheme of finite type considered as an Artin stack, then \(X^{\text{an}} = Y^{\text{an}}\) is the corresponding analytic space considered as an analytic stack.

We will need only analytic stacks of special form, namely the \textit{quotient analytic stacks} \(Z//K\), where \(Z\) is an analytic space and \(K\) is a complex Lie group. For such stacks various concepts such as Borel-Moore homology, etc., can be defined directly in terms of equivariant homology of the topological spaces of \(\mathbb{C}\)-points, that is,

\[ (7.3.2) \quad H^\bullet(Z//K, \omega_{Z//K}) = H^\bullet_{BM,K}(Z(\mathbb{C}), \mathbb{C}) \]

is the equivariant Borel-Moore homology in the topological sense.
If $Y$ is a scheme of finite type over $\mathbb{C}$ and $G$ is an algebraic group over $\mathbb{C}$ acting on $Y$, then $G^{an}$ is a complex Lie group and we have the quotient analytic stack $Y^{an}/G^{an}$. Note that in this case we have
\[(7.3.3) \quad (Y/G)^{an} \simeq Y^{an}/G^{an},\]
and the map $\eta_{Y//G}$ is a quasi-isomorphism, so that
\[(7.3.4) \quad H^\bullet(R\Gamma((Y/G)^{an},\omega_{(Y/G)^{an}})) \simeq H^\bullet(R\Gamma(Y//G,\omega_{Y//G})) \simeq H^\bullet_{BM,G}(\mathbb{C})(Y(\mathbb{C}), \mathbb{C})\]
is the equivariant Borel-Moore homology in the topological sense, as above.

### 7.4. The stack $\text{Coh}_0$ and factorization algebras.

Let $\Sigma$ be a complex analytic surface. We view it as a $C^\infty$ manifold of dimension 4 and consider open subsets $U \subset \Sigma$ in the complex analytic topology. For any such nonempty $U$ we have the category $\text{Coh}_0(U)$ of 0-dimensional coherent sheaves on $U$ (with finite support). We set $\text{Coh}_0(\emptyset) = \{ \bullet \}$. We also have the analytic moduli stack $\text{Coh}_0(U) = \bigsqcup_{n \geq 0} \text{Coh}^{(n)}_0(U)$ parametrizing objects of $\text{Coh}_0(U)$, with its components given by the length, as in the algebraic case. Each component is explicitly realized as a quotient analytic stack
\[\text{Coh}^{(n)}_0(U) = F\text{Coh}^{(n)}_0(U)/GL_n(\mathbb{C}),\]
where $F\text{Coh}^{(n)}_0(U)$ is the analytic space parametrizing pairs $(\mathcal{F}, \phi)$, where $\mathcal{F}$ is a 0-dimensional coherent sheaf on $U$ and $\phi : \mathbb{C}^n \to H^0(U, \mathcal{F})$ is an isomorphism. To see that $F\text{Coh}^{(n)}_0(U)$ is well defined as an analytic space, we note that the datum of $\phi$ is equivalent to the datum of the corresponding surjection $\phi : \mathcal{O}^{\oplus n}_U \to \mathcal{F}$. Thus $F\text{Coh}^{(n)}_0(U)$ is a locally closed analytic subspace in $\text{Quot}^{(n)}(\mathcal{O}^{\oplus n}_U)$, the analytic analog of the Grothendieck Quot scheme parametrizing all length $n$ quotients of $\mathcal{O}^{\oplus n}_U$. This makes clear the following fact.

**Proposition 7.4.1.** Let $S$ be a smooth quasi-projective algebraic surface over $\mathbb{C}$. Then we have an equivalence of analytic stacks
\[\text{Coh}_0(S^{an}) \simeq \text{Coh}_0(S)^{an}.\]
In particular, $\text{Coh}_0(\mathbb{C}^2)$ is identified with the analytification of the Artin stack $\text{Coh}_0(\mathbb{A}^2)$.

If $U_1, \ldots, U_n$ are disjoint open sets contained in the open subset $U_0 \subset \Sigma$, then we have an open embedding of analytic stacks
\[(7.4.2) \quad \alpha^U_1, \ldots, U_n : \text{Coh}_0(U_1) \times \cdots \times \text{Coh}_0(U_n) \to \text{Coh}_0(U_0).\]

**Proposition 7.4.3.** $\text{Coh}_0$ is a factorization algebra on $\Sigma$ with values in the category $\text{Stan}$.

**Proof.** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a Weiss open cover of $U$. Let us understand more explicitly the analytic stack $\text{holim}_i \mathcal{N}_i(\mathcal{U}, \text{Coh}_0)$, a homotopy colimit in the model category $\text{Stan}$, or, equivalently, the 2-categorical colimit of $\mathcal{N}_i(\mathcal{U}, \text{Coh}_0)$ in the 2-category $\text{Stan}$. It is parametrized by pairs $(i \in I, \mathcal{F} \in \text{Coh}_0(U_i))$, the leftmost term in the diagram $\mathcal{N}(\mathcal{U}, \text{Coh}_0)$, subject to coherent systems of identifications given by the rest of the diagram. These identifications say that two pairs $(i \in I, \mathcal{F} \in \text{Coh}_0(U_i))$ and $(j \in J, \mathcal{F} \in \text{Coh}_0(U_j))$ are identified, whenever in the second pair $\mathcal{F}$ is the same sheaf but living on $U_j$. This happens whenever $\mathcal{F}$ lives in fact on $U_{ij} = U_i \cap U_j$. Further terms in the diagram $\mathcal{N}_i(\mathcal{U}, \text{Coh}_0)$ impose coherence conditions on such identifications. This means that this homotopy colimit parametrizes 0-dimensional coherent sheaves which live on some $U_i$. But $\mathcal{U}$ is a Weiss cover and every $\mathcal{F} \in \text{Coh}_0(U)$, has finite support which, therefore, must lie in some $U_i$. Thus, our homotopy colimit is identified with $\text{Coh}_0(U)$.

\[\square\]
7.5. **Chain-level COHA as a factorization coalgebra.** For each open set \( U \subset \Sigma \) as above we consider the complex of Borel-Moore chains of \( \text{Coh}_0(U) \)

\[
R(U) = C^\text{BM}_*(\text{Coh}_0(U)) := R\Gamma(\text{Coh}_0(U), \omega_{\text{Coh}_0(U)}).
\]

**Proposition 7.5.1.** The assignment \( R : U \mapsto R(U) \) is a locally constant factorization coalgebra on \( S \) in the category \( C(\text{Vect}_k) \) (complexes of \( k \)-vector spaces).

**Proof.** The fact that \( R \) it is a factorization algebra follows from Proposition 7.4.3. The fact that \( R \) is locally constant is proved in the same way as Proposition 6.2.1(b). \( \square \)

Next, we upgrade this statement to take into account the Hall multiplication. The relevant concept here is that of a *homotopy associative* \((E_1)\)-algebra which we now recall. We will use the language of operads, see, e.g., [9] for a brief background and additional references.

**Definition 7.5.2.** Let \((C, \otimes, 1)\) be a symmetric monoidal category.

(a) An operad \( \mathcal{P} \) in \( C \) is a system of:

1. **Objects** \( \mathcal{P}(r) \in C \) with actions of \( S_r \), given for \( r \geq 0 \).
2. **Unit** morphism \( 1 \rightarrow \mathcal{P}(1) \).
3. **Operadic compositions** for any \( k, r_1, \ldots , r_k \)

\[
\mathcal{P}(k) \otimes \mathcal{P}(r_1) \otimes \cdots \otimes \mathcal{P}(r_k) \rightarrow \mathcal{P}(r_1 + \cdots + r_k).
\]

These data must satisfy the axioms of equivariance, associativity and unit.

(b) An algebra over an operad \( \mathcal{P} \) is an object \( A \in C \) together with \( S_r \)-equivariant morphisms \( \mathcal{P}(r) \otimes A^\otimes r \rightarrow A \), \( r \geq 0 \) satisfying the axioms of unit and associativity.

We will use the case when \( C = \Delta^\circ \text{Set} \), \( C = \text{Top} \) or \( C = C(\text{Vect}_k) \). We will refer to these cases as *simplicial*, *topological* and *dg-operads*. Any topological operad \( \mathcal{P} \) gives a simplicial operad \( \text{Sing}(\mathcal{P}) \) by passing to the singular simplicial sets of the \( \mathcal{P}(r) \)'s. It further gives a dg-operad \( C_*(\mathcal{P}) \) formed by the singular chain complexes of the \( \mathcal{P}(r) \) (considered, as usual, as cochain complexes with reverse indexation).

A *weak equivalence* of simplicial operads is a morphism \( \mathcal{P} \rightarrow \mathcal{Q} \) of such operads such that for each \( r \) the morphism of simplicial sets \( \mathcal{P}(r) \rightarrow \mathcal{Q}(r) \) is a weak equivalence, i.e., it induces a homotopy equivalence on the realizations.

Recall (A.1.2) that the category \( C(\text{Vect}_k) \) is enriched in the category \( \Delta^\circ \text{Set} \) of simplicial sets. Thus, for any simplicial operad \( \mathcal{P} \) we can speak about \( \mathcal{P} \)-algebras in \( \text{dgVect} \). Such an algebra is a cochain complex \( A \) together with morphisms of simplicial sets

\[
\mathcal{P}(r) \rightarrow \text{Map}(A^\otimes r, A)
\]

compatible with the \( S_r \)-actions and operadic compositions. It sends the image of \( 1 = pt \) to the identity map. Dually, a \( \mathcal{P} \)-coalgebra in \( \text{dgVect} \) is a complex \( B \) with morphisms of simplicial sets

\[
\mathcal{P}(r) \rightarrow \text{Map}(B, B^\otimes r)
\]

satisfying similar compatibilities. If \( \mathcal{P} \) is a topological operad, its (co)algebras in \( C(\text{Vect}_k) \) are understood as (co)algebras over the simplicial operad \( \text{Sing}(\mathcal{P}) \).

Let \( m \geq 1 \). Let \( D_m \) be the topological *operad of little \( m \)-disks*. The space \( D_m(r) \) parametrizes families \((B_1, \ldots , B_r)\) of round \( m \)-dimensional open balls disjointly embedded into the standard unit ball \( B = \{|x| < 1\} \) of \( \mathbb{R}^m \), see, e.g., [9] for more details including the definition of the operadic compositions.

**Definition 7.5.3.** By a \( E_m \)-operad we mean a topological operad weakly equivalent to \( D_m \). An \( E_m \)-(co)algebra in \( \text{dgVect} \) is a (co)algebra over an \( E_m \)-operad.

We can now formulate our upgrade of the chain level COHA.

**Theorem 7.5.4.** \( R \) is a locally constant factorization coalgebra on \( \Sigma \) in the category of \( E_1 \)-algebras.
An $E_1$-algebra can be seen as a weakly (homotopy) associative dg-algebra, see discussion below.

7.6. **Proof of Theorem 7.5.4.** We first note that $D_1(r)$ is the union of $r!$ contractible components which are permuted by $S_r$. This means that algebras over $D_1$ (and so over any $E_1$-operad) can be described using the concept of a non-symmetric (non-$\Sigma$) operad [46, Def. 9]. A non-symmetric operad in a monoidal (not necessarily symmetric) category $\mathcal{C}$ is a datum $\mathcal{Q}$ of objects $\mathcal{Q}(r)$, $r \geq 0$ (no symmetric group action is required) together with a unit morphism $1 \to \mathcal{Q}(1)$ and the compositions as in (O3) satisfying the axioms of associativity and unit. Similarly, an algebra over a non-symmetric operad $\mathcal{Q}$ is an object $A \in \mathcal{C}$ together with morphisms $\mathcal{Q}(r) \otimes A^\otimes r \to A$ satisfying the axioms of unit and associativity.

Let $ND_1(r) \subset D_1(r)$ be the connected component formed by families $(B_1, \cdots, B_r)$ of disjoint 1-disks (i.e., open intervals) in $B = (-1, 1)$ such that the centers of the $B_i$ are positioned in the increasing order. Then $ND_1 = (ND_1(r))$ is a non-symmetric operad in Top with each $ND_1(r)$ contractible. Let us call an $NE_1$-operad any non-symmetric operad $Q$ in Top with each $Q(r)$ contractible. Given an $NE_1$-operad $\mathcal{Q}$, we can “symmetrize” it, forming an $E_1$-operad $SQ$ with $SQ(r) = \mathcal{Q}(r) \times S_r$ and the $S_r$ acting via the second factor. This establishes an equivalence between the categories of $NE_1$-operads and $E_1$-operads, with the categories of algebras over the corresponding operads being identified as well.

Let us now consider dg-versions of the topological operads above and use slightly different notation for these versions. Let us call an $\text{ne}_1$-operad a non-symmetric dg-operad $\mathcal{K}$ such that each cochain complex $Q(r)$ is situated in degrees $\leq 0$ and quasi-isomorphic to $k$. Because of Dold-Kan equivalence between $C^{\leq 0}(\text{Vect}_k)$ and $\Delta^\circ \text{Vect}_k$, see Example 1.1.4(b), equipping a complex with a structure of an algebra over a $NE_1$-operad is the same as equipping it with the structure of an algebra over an $\text{ne}_1$-operad.

An example of a $\text{ne}_1$-operad is given by the non-symmetric associative operad $\text{Ass}$ with $\text{Ass}(r) = k$ for all $r$ and all the compositions being the identities. Dg-algebras over $\text{Ass}$ are the same as associative dg-algebras.

So for the proof of Theorem 7.5.4 we exhibit an $\text{ne}_1$-operad $\mathcal{K}$ and equip each $\mathcal{R}(U)$ with the structure of a $\mathcal{K}$-algebra in a way compatible with factorization coalgebra structure. The argument is an upgrade of the proof of Theorem 4.4.2 (associativity of COHA) so parts of the treatment will be brief.

For $r \geq 1$ let $\text{FILT}^{(r)} = \text{FILT}^{(r)}(U)$ be the stack $\text{FILT}^{(r)}$ parametrizing flags of objects of $\text{Coh}_0(U)$

$$E_1 \subset E_2 \subset \cdots \subset E_r.$$ 

For $r = 0$ we put $\text{FILT}^{(0)} = \text{pt}$. The stack $\text{FILT}^{(r)}$ comes with the projections

$$\begin{array}{ccc}
\text{FILT}^{(r)} & \xrightarrow{\rho_r} & \text{Coh}_0(U) \\
q_r \downarrow & & \downarrow \\
\text{Coh}_0(U)^r
\end{array}$$

$$\rho_r(E_1 \subset E_2 \subset \cdots \subset E_r) = E_r,$$

$$q_r(E_1 \subset E_2 \subset \cdots \subset E_r) = (E_1, E_2/E_1, \cdots, E_r/E_{r-1}).$$

For $r = 0$ we have $\text{Coh}(U)^0 = \text{pt}$ and we define $q_0 : \text{pt} \to \text{pt}$ to be the identity map and $\rho_0 : \text{pt} \to \text{Coh}(U)$ to sent pt to the zero sheaf.

Let $\mathcal{E}_i$, $i = 1, \cdots, r$, be the ith tautological sheaf on $\text{Coh}_0(U)^r \times U$ and $p_r : \text{Coh}_0(U) \times U \to \text{Coh}_0(U)$ be the projection.

Similarly to §2.5, we form the sheaf of associative dg-algebras (and, passing to the super-commutator, of dg-Lie algebras) on $\text{Coh}_0(U)^r$

$$\mathcal{G}_r = \mathcal{G}_r(U) = \bigoplus_{1 \leq i < j \leq r} R p_{r*} \mathcal{RHom}(\mathcal{E}_j, \mathcal{E}_i)$$
and find that $\text{FILT}^{(r)} = \text{MC}(\mathcal{G}_r)$ so that $q_r$ is identified with the projection of the Maurer-Cartan stack. Therefore we have the diagram

$$
\text{Coh}_0(U)^r \xleftarrow{\pi_r} \text{Tot}(\mathcal{G}_r^{\leq 1}) \xleftarrow{i_r} \text{FILT}^{(r)} \xrightarrow{\rho_r} \text{Coh}_0(U),
$$

in which the map $i_r$ realizes $\text{FILT}^{(r)}$ as the zero locus of the section $s_r$ of $\pi_r^* \mathcal{G}_r^2$ given by the curvature map. This gives a virtual pullback $i^r_r$ on Borel-Moore homology. We get, so far at the level of BM-homology, the map

$$m_r = \rho_{r \times} \circ i_r \circ \pi_r^* : R(U)^{\otimes r} \to R(U), \quad R(U) = H_*^{\text{BM}}(\text{Coh}_0(U)).$$

As in §4.5, we see that $m_r$ is the $r$-fold product in the (associative) COHA $R(U)$.

Next, we notice that the family $(\mathcal{G}_r)_{r \geq 0}$ of dg-Lie algebras carries a kind of operadic structure. For $r_1, \ldots, r_n \geq 0$ consider the summation map

$$\sigma_{r_1, \ldots, r_n} : \text{Coh}_0(U)^{r_1 + \cdots + r_n} \to \text{Coh}(U)^n, \quad (\mathcal{F}_1, \ldots, \mathcal{F}_{r_1 + \cdots + r_n}) \mapsto \left( \bigoplus_{i=1}^{r_1} \mathcal{F}_i, \bigoplus_{i=r_1+1}^{r_1+r_2} \mathcal{F}_i, \ldots, \bigoplus_{i=r_1+\cdots+r_{n-1}+1}^{r_1+\cdots+r_n} \mathcal{F}_i \right).$$

**Proposition 7.6.3.** We have a semidirect product decomposition, more precisely, an isomorphism

$$\lambda_{r_1, \ldots, r_n} : (\sigma_{r_1, \ldots, r_n}^* \mathcal{G}_n) \times (\mathcal{G}_{r_1} \boxplus \cdots \boxplus \mathcal{G}_{r_n}) \to (\mathcal{G}_{r_1} \boxplus \cdots \boxplus \mathcal{G}_{r_n}).$$

of sheaves of dg-Lie algebras on $\text{Coh}(U)^{r_1 + \cdots + r_n}$.

The proposition means that we have a split exact sequence

$$0 \to \sigma_{r_1, \ldots, r_n}^* \mathcal{G}_n \xrightarrow{a} \mathcal{G}_{r_1+\cdots+r_n} \xrightarrow{b} \mathcal{G}_{r_1} \boxplus \cdots \boxplus \mathcal{G}_{r_n} \to 0$$

in which $\sigma_{r_1, \ldots, r_n}^* \mathcal{G}_n$ is a dg-Lie ideal with quotient $\mathcal{G}_{r_1} \boxplus \cdots \boxplus \mathcal{G}_{r_n}$.

**Proof of Proposition 7.6.3:** By construction, $\mathcal{G}_{r_1, \ldots, r_n}$ consists of upper-triangular square matrices of size $r_1 + \cdots + r_n$. We can decompose such a matrix into blocks of sizes $r_i \times r_j$, $1 \leq i \leq j \leq n$. Of these, the diagonal blocks (of sizes $r_i \times r_i$) are upper-triangular, since the total matrix must be upper-triangular. These correspond to the $\mathcal{G}_{r_i}$. So the block-diagonal part is $\mathcal{G}_{r_1} \boxplus \cdots \boxplus \mathcal{G}_{r_n}$. Similarly, the over-diagonal blocks are seen as representing $\sigma_{r_1, \ldots, r_n}^* \mathcal{G}_n$.

**Proposition 7.6.5.** The isomorphisms $\lambda_{r_1, \ldots, r_n}$ satisfy operadic associativity. That is, suppose that each $r_i$ is decomposed as $r_i = r_{i,1} + \cdots + r_{i,m_i}$. Then the isomorphisms

$$\lambda_{r_{i,1}, \ldots, r_{i,m_i}} : (\sigma_{r_{i,1}, \ldots, r_{i,m_i}}^* \mathcal{G}_{m_i}) \times (\mathcal{G}_{r_{i,1}} \boxplus \cdots \boxplus \mathcal{G}_{r_{i,m_i}}) \to \mathcal{G}_{r_{i,1}+\cdots+r_{i,m_i}} = \mathcal{G}_{r_i}, \quad i = 1, \ldots, n$$

together with $\lambda_{r_1, \ldots, r_n}$, compose to $\lambda_{r_{1,1}, \ldots, r_{1,m_1}, r_{2,1}, \ldots, r_{2,m_2}, \ldots, r_{n,1}, \ldots, r_{n,m_n}}$.

**Proof.** Straightforward verification in terms of matrices whose blocks are decomposed into further blocks. \[\square\]

Next, we study the compatibility of the curvature sections $s_r$ on the $\pi_r^* \mathcal{G}_r^2$ with the semidirect product decompositions $\lambda_{r_1, \ldots, r_n}$. Let $r = r_1 + \cdots + r_n$. On $\text{Tot}(\mathcal{G}_r^{\leq 1})$ the sequence (7.6.4) gives a short exact sequence of vector bundles

$$0 \to \pi_r^* \sigma_{r_1, \ldots, r_n}^* \mathcal{G}_n^2 \xrightarrow{\alpha} \pi_r^* \mathcal{G}_r^2 \xrightarrow{\beta} \pi_r^* (\mathcal{G}_{r_1}^2 \boxplus \cdots \boxplus \mathcal{G}_{r_n}^2) \to 0$$

pulled back from $\text{Coh}_0(U)^r$. We apply to this situation the analysis of §B.4, taking $X = \text{Tot}(\mathcal{G}_r^{\leq 1})$ and viewing (7.6.6) as an instance of the sequence (B.4.1). The curvature section $s = s_r$ of the middle bundle gives rise to the section $s'' = s''_r = \beta(s)$ of $\pi_r^* (\mathcal{G}_{r_1}^2 \boxplus \cdots \boxplus \mathcal{G}_{r_n}^2)$ with zero locus $X_{s''} = \text{Tot}(\mathcal{G}_r^{\leq 1})_{s''}$ and the
section $s' = s'_r$ of $\pi^*_r \sigma^*_r \cdots \sigma^*_1 G^2_0$ over $X_{s'}$. To describe them we consider, for each $i = 1, \ldots, n$, the stack $\text{FILT}^{(r)}_i$ parametrizing flags $E_{i,1} \subset E_{i,2} \subset \cdots \subset E_{i,r_i} = E_i$ of objects of $\text{Coh}_0(U)$. Let

$$\phi : \prod_{i=1}^n \text{FILT}^{(r_i)} \longrightarrow \text{Coh}_0(U)^r = \prod_{i=1}^n \text{Coh}_0(U)$$

be the projection which sends a tuple of flags as above to the tuple $(E_1, \ldots, E_n)$ of their maximal elements. We also denote by

$$\pi_{r_1, \ldots, r_n} : \text{Tot}(\phi^* G^{\leq 1}_n) \longrightarrow \prod_{i=1}^n \text{FILT}^{(r_i)}$$

the projection.

**Proposition 7.6.7.** (a) $s'_r$ equal to (the pullback of) the tuple $(s_{r_1}, \ldots, s_{r_n})$ considered as a section of the external direct sum.

(b) $X_{s'}$ is identified with $\text{Tot}(\phi^* G^{\leq 1}_n)$.

(c) Under the identification of (b), the restriction of $\pi^*_r \sigma^*_r \cdots \sigma^*_1 G^2_0$ to $X_{s'}$ is identified with $\pi^*_{r_1, \ldots, r_n} \phi^* G^2_n$.

(d) Under the identification of (c), the section $s'_r$ is identified with the pullback of $s_n$.

**Proof.** (a) As in the proof of Proposition 7.6.3, let us view sections of $G_r$ as upper-triangular $r \times r$ matrices subdivided into blocks of sizes $r_i \times r_j$. The projection $b$ (whose pullback is $\beta$) associates to such a matrix $x$ its block-diagonal part which we denote $x_\Delta$. Thus $\beta(s_r)$ associates to $x$ the block-diagonal part of the curvature, i.e., $(dx + (1/2)[x, x])_\Delta$. Since the block-diagonal subspace of a dg-Lie subalgebra, this equals $d(x_\Delta) + (1/2)[x_\Delta, x_\Delta]$ which corresponds to the pullback of $(s_{r_1}, \ldots, s_{r_n})$.

(b) Let us represent a point of $\text{Coh}_0(U)^r$, $r = r_1 + \cdots + r_n$ as a sequence of sheaves

$$\mathcal{F}_{1,1}, \ldots, \mathcal{F}_{1,r_1}, \mathcal{F}_{2,1}, \ldots, \mathcal{F}_{2,r_2}, \cdots, \mathcal{F}_{n,1}, \ldots, \mathcal{F}_{n,r_n}.$$ In terms of matrices $x \in G^1_r$, vanishing of the block-diagonal part of the curvature of $x$ means that the Ext-data for the $\mathcal{F}_{i,j}$ provided by $x$, integrate to $n$ filtrations

$$E_{i,1} \subset E_{i,2} \subset \cdots \subset E_{i,r_i} = E_i, \quad E_{i,j}/E_{i,j-1} \cong \mathcal{F}_{i,j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, r_i,$$

i.e., we have a point of $\prod_{i=1}^n \text{FILT}^{(r_i)}$. Further, the summation map $\sigma_{r_1, \ldots, r_n}$ on the sequence of the $\mathcal{F}_{i,j}$ corresponds to the projection $\phi$. The over-diagonal blocks of $x$ assemble into a section of $\phi^* G^{\leq 1}_n$, whence the statement.

Part (c) is clear from the above. To see (d), notice that the pullback of $s_n$ represents the over-diagonal blocks of the curvature of $x$.

We now apply the formalism of homotopy canonical Euler classes from Appendix B. Let $K_r(U) = K_{F^r} G^2_r$ be the parameter complex for the homotopy canonical orientation class of the bundle $\pi^*_r G^2_r$ on $\text{Tot}(G^{\leq 1}_r)$, see §B.3. Here $G_r = G_r(U)$, as above. By construction, each $K_r(U)$ maps quasi-isomorphically to $k$. The semidirect product decomposition of Proposition 7.6.3 and the pairings (B.4.8) of the $K$-complexes give morphisms of complexes

$$K_n(U) \otimes K_{r_1}(U) \otimes \cdots \otimes K_{r_n}(U) \longrightarrow K_{r_1, \ldots, r_n}(U).$$

The operadic associativity of the isomorphisms $\lambda_{r_1, \ldots, r_n}$ and the associativity of the pairings (B.4.8) imply that $K(U) = (K_r(U))_{r \geq 0}$ is a (non-symmetric) dg-operad. Since each $K_r(U)$ is quasi-isomorphic to $k$, we see that $K(U)$ is an $\text{nc}$-operad. Further, the correspondence $U \mapsto K(U)$ forms a presheaf (in fact, a sheaf up to homotopy, by the above) of dg-operads on the analytic surface $\Sigma$. Let $K = K(\Sigma)$ be the operad of global sections.
Finally, let us upgrade the r-fold multiplication map (7.6.2) to the cochain level by analyzing the ambiguity. This maps involves the virtual pullback \( i_r \) which is defined in terms of the refined Chern class \( c_d(\pi^*_r \mathcal{G}_r^2) \), \( d = \text{rk}(\mathcal{G}_r^2) \). Using the homotopy canonical cochain lifting \( \tilde{c}_d(\pi^*_r \mathcal{G}_r^2) \), see (B.3.4), we define a cochain level multiplication

\[
\tilde{\mu}_{r,U} : \mathcal{K}_r(U) \otimes \mathcal{R}(U)^{\otimes r} = \mathcal{K}_r(U) \otimes C^\text{BM}(\text{Coh}_0(U))^{\otimes r} \rightarrow C^\text{BM}(\text{Coh}_0(U)) = \mathcal{R}(U).
\]

The multiplicativity of the \( \tilde{c}_d \) in short exact sequences (B.4.10) implies that the \( \mu_{r,U} \) make \( \mathcal{R}(U) \) into an algebra over the \( n_1 \)-operad \( \mathcal{K}(U) \) and therefore over \( \mathcal{K}(\Sigma) \). Further, these \( \mathcal{K} \)-algebra structures are clearly compatible with the factorization coalgebra structure on the presheaf \( \mathcal{R} = (\mathcal{R}(U)) \). This finishes the proof of Theorem 7.5.4. \( \square \)

### 7.7. Proof of Theorem 7.1.6

As before, let \( \Sigma = S_{\text{an}} \). For any open subset \( U \subset \Sigma \) (in the complex analytic topology) we have the \( \mathbb{Z}^2 \)-graded space \( \Theta(U) \) and the symmetrized product map \( \sigma_U : \text{Sym}(\Theta(U)) \rightarrow \mathcal{R}(U) \). Because of Proposition 7.4.1 and the identification (7.3.4), the map \( \sigma \) of Theorem 7.1.6 is identified with the global map \( \sigma_{\Sigma} \), corresponding to \( U = \Sigma \). Now, if \( U \) is a disk, then \( \sigma_U \) is an isomorphism by Theorem 6.1.4. We will deduce the global statement (for \( U = \Sigma \)) from these local ones.

For this, we upgrade the correspondence \( U \rightarrow \Theta(U) \) to a complex of sheaves \( \mathcal{V} \) on \( \Sigma \) so that \( \Theta(U) = \mathbb{H}^{-\bullet}(U, \mathcal{V}) \) is the hypercohomology of \( U \) with coefficients in \( \mathcal{V} \). That is, we define

\[
\mathcal{V} = \omega_{\Sigma} \otimes_k \Theta',
\]

the tensor product of the dualizing complex \( \omega_{\Sigma} \) and the graded vector space \( \Theta' = \Theta[0, -4] \), see (7.1.4). Recall that \( \Theta \) and therefore \( \Theta' \) is spanned by the basis vectors \( t^n q^{i-1} \), and such a vector is identified with \( \theta_{n,i} = ch_i(\mathcal{E}_n) \cap \theta_n \in R^{n,2-2i}(\mathbb{C}^2) \), see (6.3.3). Here, as we recall, \( \mathcal{E}_n \) is the tautological rank \( n \) bundle on \( \text{Coh}_0^{(n)}(\mathbb{C}^2) \) whose fiber at a point represented by a coherent sheaf \( \mathcal{F} \) is \( H^0(\mathcal{F}) \) and \( \theta_n \) is the fundamental class of \( \text{Coh}_1^{(n)}(\mathbb{C}^2) \).

As before, we denote by the same symbol \( \mathcal{E}_n \) the analogous tautological bundle on \( \text{Coh}_0^{(n)}(\Sigma) \) and, if necessary, its restriction to \( \text{Coh}_1^{(n)}(\Sigma) \).

Extending the construction of (7.1.2), we choose a cocycle representing the fundamental class of \( \text{Coh}_1^{(n)}(\Sigma) \) in \( H^2_{BM}(\Sigma) \) and define the morphism

\[
p^\dagger_n : p_n^* \omega_{\Sigma} \longrightarrow p_n^* \omega_{\Sigma}[2] = \omega_{\text{Coh}_1^{(n)}(\Sigma)}[2]
\]

as the cup-product with this cocycle. Here \( p_n : \text{Coh}_1^{(n)}(S) \rightarrow S \) is the projection defined in (7.1.1). Further, for each \( i \) we fix a cocycle representative \( \widetilde{c}_h_i(\mathcal{E}_n) \) of \( ch_i(\mathcal{E}_n) \in H^{2i}(\text{Coh}_1^{(n)}(\Sigma), k) \).

The sheaf \( \mathcal{V} \) and the factorization coalgebra \( \mathcal{R} \) are both presheaves with values in the category of cochain complexes. We define a morphism of presheaves \( \tilde{\alpha} : \mathcal{V} \rightarrow \mathcal{R} \) as the composition of the two morphisms: first, the morphism

\[
R\Gamma(U, \omega_{\Sigma}) \otimes t^n q^{i-1} \rightarrow R\Gamma(\text{Coh}_1^{(n)}(U), \omega_{\text{Coh}_1^{(n)}(U)}), \quad \gamma \otimes t^n q^{i-1} \mapsto \widetilde{c}_h_i(\mathcal{E}_n) \cap p_n^*(\gamma),
\]

(here \( p_n^*(\gamma) \) is an element of \( R\Gamma(\text{Coh}_1^{(n)}(U)), p_n^*\omega(U) \)) and, second, the direct image morphism

\[
\varepsilon_* : R\Gamma(\text{Coh}_1^{(n)}(U), \omega_{\text{Coh}_1^{(n)}(U)}) \rightarrow R\Gamma(\text{Coh}_0^{(n)}(U), \omega_{\text{Coh}_0^{(n)}(U)}) = \mathcal{R}(U)^{(n)},
\]

where \( \varepsilon : \text{Coh}_1^{(n)}(U) \rightarrow \text{Coh}_0^{(n)}(U) \) is the (closed) embedding.

Since \( \mathcal{V} \) is a sheaf with values in the category of cochain complexes, its symmetric algebra \( \text{Sym}(\mathcal{V}) \) is a factorization coalgebra with values in this category, by Proposition 7.2.6. Since \( \mathcal{R} \) is a factorization
algebra in the category of $E_1$-algebras, we can define the symmetrized product $\tilde{\sigma} : \text{Sym}(\mathcal{V}) \to \mathcal{R}$ by setting
\[
\tilde{\sigma} = \sum_{n \geq 0} \tilde{\sigma}_n,
\]
where
\[
\tilde{\sigma}_n : \text{Sym}^n(\mathcal{V}) \to \mathcal{R}, \quad \tilde{\sigma}_n(v_1 \cdots v_n) = \frac{1}{n!} \sum_{s \in S_n} \mu_n(\tilde{\alpha}(v_{s(1)}) \otimes \cdots \otimes \tilde{\alpha}(v_{s(n)})),
\]
lifting the map $\sigma$ from (7.1.5). In other words, $\tilde{\sigma}_n$ is the symmetrization of the map
\[
\mu_n \circ (\tilde{\alpha} \otimes \cdots \otimes \tilde{\alpha}) : \mathcal{V}^\otimes n \to \mathcal{R}.
\]
The map $\tilde{\sigma}$ is a morphism of factorization coalgebras in the category of $\mathbb{Z}^2$-graded cochain complexes. Note that we do not claim (and it is not true) that $\tilde{\sigma}$ is a morphism of factorization coalgebras in the category of $E_1$-algebras.

By the above, $\tilde{\sigma}_U$ is a weak equivalence (of $\mathbb{Z}^2$-graded cochain complexes) for any $U$ which is, topologically, a disk. Therefore $\tilde{\sigma}$ is a weak equivalence (of factorization coalgebras in the category of $\mathbb{Z}^2$-graded cochain complexes) by Proposition 7.2.7. Taking $U = \Sigma$ we obtain Theorem 7.1.6.

7.8. $E_4$-structure on the flat COHA. By [19], [43], locally constant factorization (co)algebras on $\mathbb{R}^m$ with values in $C(\text{Vect}_k)$ can be identified with $E_m$-(co)algebras in $C(\text{Vect}_k)$, the identification associating to a (co)algebra $B$ the object $B(B)$ where $B \subset \mathbb{R}^m$ is the standard unit $m$-ball. Note that $B(B)$ is weak equivalent to $B(\mathbb{R}^d)$.

Let us specialize this to the case when $B = \mathcal{R}$ and $m = 4$, since $\mathbb{C}^2 \simeq \mathbb{R}^4$. In this case we form the cochain complex $\mathcal{R}(B) \simeq \mathcal{R}(\mathbb{C}^2)$ whose cohomology is the flat Hecke algebra $R(B) \simeq R(\mathbb{C}^2)$ studied in §6. The general results above, applied to the category $\mathcal{C}$ of $E_1$-algebras, imply:

**Corollary 7.8.1.** $\mathcal{R}(\mathbb{C}^2)$ is $E_1$-algebra in the category of $E_4$-coalgebras.

**Remarks 7.8.2.**

(a) The $E_4$-coalgebra structure on $\mathcal{R}(\mathbb{C}^2)$ is a cochain level refinement of the comultiplication $\Delta$ on $R(\mathbb{C}^2)$, see §6.2. While $\Delta$ is cocommutative, because it is independent on the choice of two distinct disks $U_1, U_2 \subset \mathbb{C}^2$, at the cochain level we do not seem to have cocommutativity since the space of choices of such pairs of disks is not contractible (it is precisely the space of binary operations in the operad $D_4$).

(b) By forming the Koszul dual to the $E_1$-algebra structure on $\mathcal{R}(\mathbb{C}^2)$, we obtain an $E_1$-coalgebra in the category of $E_4$-coalgebras, i.e., an $E_5$-coalgebra. Alternatively, forming the Koszul dual to the $E_4$-algebra structure, we obtain an $E_5$-algebra. This suggest that some 5-dimensional field theory may be relevant to this picture.

**Appendix A. Basics on $\infty$-categories and dg-categories**

A.1. $\infty$-categories. Let $k$ be a field of characteristic 0. By $\text{Vect} = \text{Vect}_k$ we denote the category of $k$-vector spaces and by $C(\text{Vect}) = C(\text{Vect}_k)$ the category of complexes of $k$-vector spaces bounded below, with morphisms being morphisms of complexes. By $\Delta^\circ \text{Set}$ we denote the category of simplicial sets. For a simplicial set $Y$ we denote by $|Y|$ the geometric realization of $Y$. We say that $Y$ is contractible, if $|Y|$ is a contractible topological space. For a topological space $T$ we denote by $\text{Sing}(T)$ the singular simplicial set of $T$.

An $\infty$-category $\mathcal{C}$ is a simplicial set $(\mathcal{C}_n)_{n \geq 0}$ satisfying the partial Kan condition, with elements of $\mathcal{C}_0$ called objects and elements of $\mathcal{C}_1$ called morphisms. Every $\infty$-category $\mathcal{C}$ gives rise to an ordinary category $\text{h}\mathcal{C}$ known as the homotopy category of $\mathcal{C}$. It has the same objects as $\mathcal{C}$ and its morphisms are certain equivalence classes of morphisms in $\mathcal{C}$. Further, $\mathcal{C}$ contains the maximal Kan simplicial subset $\mathcal{C}^\text{Kan}$ with $\mathcal{C}_0^\text{Kan} = \mathcal{C}_0$, having the meaning of the subgroupoid of (weakly) invertible morphisms. We refer to [44] for more details.
A simplicial category is a category \( C \) enriched in \( \Delta^\circ \text{Set} \), so that for any two objects \( \mathcal{F}, \mathcal{G} \in C \) we are given a simplicial set \( \text{Map}_C(\mathcal{F}, \mathcal{G}) \) with standard properties. A simplicial category \( C \) gives an \( \infty \)-category \( \mathit{\hat{C}} \) with the same objects, as explained in [44].

A dg-category is a category \( C \) enriched in \( \mathbb{C}(\text{Vect}) \), so that for any two objects \( \mathcal{F}, \mathcal{G} \in C \) we are given a cochain complex \( \text{Hom}_C^*(\mathcal{F}, \mathcal{G}) \) with standard properties. Any dg-category \( C \) gives rise to a \( \mathbb{k} \)-linear category \( H^0C \) with the same objects as \( C \) and

\[(A.1.1) \quad \text{Hom}_{H^0C}(\mathcal{F}, \mathcal{G}) = H^0 \text{Hom}_C^*(\mathcal{F}, \mathcal{G}).\]

Further, \( C \) can be made into a simplicial category (with the same objects) by

\[(A.1.2) \quad \text{Map}(\mathcal{F}, \mathcal{G}) = \text{DK}(\tau_{\leq 0} \text{Hom}_C^*(\mathcal{F}, \mathcal{G}))\]

where \( \text{DK} \) is the Dold-Kan simplicial set associated to a \( \mathbb{Z}_{\leq 0} \)-graded complex, see [64, §8.4.1] and a discussion in Example 1.1.4. So it gives rise to an \( \infty \)-category denoted \( \mathbb{N}_C(\mathcal{C}) \), see [43].

A.2. Enhanced derived categories. Let \( A \) be a \( \mathbb{k} \)-linear abelian category. We denote by \( C(A) \) the category of complexes over \( A \) with morphisms being morphisms of complexes. By \( C(A)_{\text{dg}} \) we denote the dg-category with the same objects as \( C(A) \). For any two objects \( \mathcal{F}, \mathcal{G} \) of \( C(A)_{\text{dg}} \), the complex \( \text{Hom}_{C(A)_{\text{dg}}}(\mathcal{F}, \mathcal{G}) \) is the graded \( \mathbb{k} \)-vector space \( \text{Hom}_A(\mathcal{F}, \mathcal{G}) \) with the differential given by the commutation with \( d_\mathcal{F} \) and \( d_\mathcal{G} \). Thus \( C(A) = H^0C(A)_{\text{dg}} \). By \( D(A) = C(A)[\text{Qis}^{-1}] \) we denote the derived category of \( A \), i.e., the localization of \( C(A) \) by the class \( \text{Qis} \) of quasi-isomorphisms. There are three closely related enhancements of \( D(A) \) with the same objects:

(a) The derived dg-category \( D(A)_{\text{dg}} \) with the property that \( D(A) = H^0 D(A)_{\text{dg}} \). If \( A \) has canonical injective resolutions \( A \mapsto I(A) \), then we define, see [7],

\[
\text{Hom}_{D(A)_{\text{dg}}}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{C(A)_{\text{dg}}}^*(I(\mathcal{F}), I(\mathcal{G})).
\]

The complex in the RHS is also denoted \( \text{RHom}^*(\mathcal{F}, \mathcal{G}) \).

(b) The simplicial derived category \( D(A)_{\Delta} \) with the property that \( \text{Hom}_{D(A)_{\Delta}}(\mathcal{F}, \mathcal{G}) = \pi_0 \text{Map}_{D(A)_{\Delta}}(\mathcal{F}, \mathcal{G}) \).

There are two homotopy equivalent ways of constructing \( \text{Map}_{D(A)_{\Delta}}(\mathcal{F}, \mathcal{G}) \):

(b1) Given the data in (a), we can define, as in (A.1.2),

\[
\text{Map}_{D(A)_{\Delta}}(\mathcal{F}, \mathcal{G}) = \text{DK}(\tau_{\leq 0} \text{RHom}^*(\mathcal{F}, \mathcal{G})).
\]

(b2) The Dwyer-Kan simplicial localization procedure [12],[13] produces simplicial sets \( \text{Map}(\mathcal{F}, \mathcal{G}) \), starting from the category \( C(A) \) and the class of morphisms \( \text{Qis} \). We can take \( \text{Map}_{D(A)_{\Delta}}(\mathcal{F}, \mathcal{G}) \) to be the simplicial sets \( \text{Map}(\mathcal{F}, \mathcal{G}) \). Further, we can use them to get an intrinsic definition of the \( \text{RHom}^*(\mathcal{F}, \mathcal{G}) \) in (a) by taking the normalized chain complexes and stabilizing with respect to the shift. This allows one to define \( D(A)_{\text{dg}} \) even without the use of canonical injective resolutions.

(c) The derived \( \infty \)-category \( D(A)_{\infty} \) with the property that \( hD(A)_{\infty} = D(A) \). As in (b2), it can be defined intrinsically, as the \( \infty \)-categorical localization of \( C(A) \) by \( \text{Qis} \), see [43].

Appendix B. Homotopy canonical Euler classes

The concept of coherent homotopy uniqueness of objects, morphisms, cohomology classes, etc., is implicit in the formalism of \( \infty \)-categories, as well as in homotopical algebra in general. In this appendix we spell out some instances of this concept in the dg-context.
B.1. Cocycles defined up to a contractible choice. Let \( V \) be a cochain complex over \( k \), and \( a \in H^d(V) \). Viewing \( a \) as a morphism \( a : k \to V[d] \) in \( \text{D}^b(\text{Vect}_k) \), we can represent \( a \) (non-uniquely) by a diagram of morphisms of complexes \( k \xrightarrow{q} K \xrightarrow{\alpha} V[d] \), where \( q \) is a quasi-isomorphism. Such a diagram is just a right fraction representing the morphism \( a \) in \( \text{D}^b(\text{Vect}_k) = C(\text{Vect}_k)[\text{Qis}^{-1}] \) as \( a = \alpha q^{-1} \). We will refer to any such diagram as a \( d \)-cocycle defined up to a contractible choice and say that it represents \( a \) up to a contractible choice.

Examples B.1.1. (a) Suppose that \( a \neq 0 \) and \( H^j(V) = 0 \) for \( j < d \). Let \( Z^d(V) \subset V^d \) be the space of \( d \)-cocycles and \( \gamma : Z^d(V) \to H^d(V) \) be the projection. Let
\[
V_{a}^{\leq d} = \{ \cdots \to V^{d-2} \to V^{d-1} \to \gamma^{-1}(ka) \} \subset V^{\leq d}.
\]
Then the projection to \( ka \cong k \) gives a quasi-isomorphism \( V_{a}^{\leq d}[d] \xrightarrow{\sim} k \) and the diagram
\[
k \leftarrow V_{a}^{\leq d}[d] \to V[d]
\]
represents \( a \) up to a contractible choice.

(b) In particular, let \( C \) be a dg-category and \( x, y \in \text{Ob}(C) \) be such that \( \text{Hom}^\bullet_c(x, y) \) has \( H^j = 0 \) for \( j < 0 \). Then any nonzero morphism \( f : x \to y \) in \( \text{H}^0C \) is represented, up to a contractible choice, by the diagram
\[
k \leftarrow \text{Hom}^\bullet_{c}^{\leq 0}(x, y)f \leftarrow \text{Hom}^\bullet_c(x, y).
\]

B.2. Homotopy canonical orientation classes. Let \( X \xrightarrow{i} Y \) be a regular embedding of stacks of codimension \( d \). We have then the canonical orientation isomorphism \( \eta_{X/Y} : \underline{k}_X \xrightarrow{\sim} i^! k_Y[2d] \) in the derived category \( \text{D}(X) \). If \( X \xrightarrow{i} Y \xrightarrow{j} Z \) are two composable regular embeddings, with \( i \) of codimension \( d \) and \( j \) of codimension \( e \), then \( ji \) is a regular embedding of codimension \( d + e \) and \( \eta_{X/Z} : \underline{k}_X \to (ji)! \underline{k}_Z[2(d + e)] \) is equal to the composition
\[
\underline{k}_X \xrightarrow{\eta_{X/Y}} i^! k_Y[2d] \xrightarrow{i^! \eta_{Y/Z}[2d]} i^! j^! k_Z[2d + 2e].
\]

Passing to the dg-enhancements, we notice that \( \eta_{X/Y} \) connects two objects which are quasi-isomorphic to single sheaves in degree 0 and so negative Ext’s between these objects vanish. We are therefore in the situation of Example B.1.1(b) and so the diagram
\[
k \xleftarrow{\sim} K_{X/Y} := \text{Hom}^\leq_{\text{D}(X)_{dg}}(\underline{k}_X, i^! k_Y[2d])_{\eta_{X/Y}} \xrightarrow{\sim} \text{Hom}^\bullet_{\text{D}(X)_{dg}}(\underline{k}_X, i^! k_Y[2d])
\]
represents \( \eta_{X/Y} \) up to a contractible choice. We can write it as a canonical closed morphism in \( \text{D}(X)_{dg} \) of degree 0
\[
\tilde{\eta}_{X/Y} : K_{X/Y} \otimes \underline{k}_X \xrightarrow{\sim} i^! k_Y[2d].
\]

If \( X \xrightarrow{i} Y \xrightarrow{j} Z \) are two composable regular embeddings as before, then the composition of Hom-complexes in the dg-category \( \text{D}(X)_{dg} \) induces a composition
\[
m_{X,Y,Z} : K_{Y/Z} \otimes K_{X/Y} \to K_{X/Z}
\]
and such compositions are associative for any composable triple of regular embeddings. The composition \( m_{X,Y,Z} \) fits into the commutative square
\[
\begin{array}{ccc}
K_{Y/Z} \otimes K_{X/Y} \otimes k_X & \xrightarrow{K_{Y/Z} \otimes \tilde{\eta}_{X/Y}} & K_{Y/Z} \otimes i^! k_Y[2d] \\
\downarrow m_{X,Y,Z} \otimes k_X & & \downarrow i^! \tilde{\eta}_{Y/Z} \\
K_{X/Z} \otimes k_X & \xrightarrow{\tilde{\eta}_{X/Z}} & i^! j^! k_Z[2d + 2e]
\end{array}
\]
which underlies the identification of \( \eta_{X/Z} \) with the composition B.2.1.
B.3. Homotopy canonical Euler classes. Let $\mathcal{E}$ be a rank $d$ vector bundle over a stack $X$. Let $s \in H^0(X, \mathcal{E})$ be a section. We consider it as a morphism $s : X \to \text{Tot}(\mathcal{E})$. Let $i_s : X_s \to X$ be the embedding of the zero locus of $s$. We have then a Cartesian square of closed embeddings

\[
\begin{array}{ccc}
X_s & \xrightarrow{i_s} & X \\
\downarrow & & \downarrow 0 \\
X & \xrightarrow{s} & \text{Tot}(\mathcal{E}).
\end{array}
\]

The zero section embedding $0 : X \to \text{Tot}(\mathcal{E})$ is regular of codimension $d$, so we have the orientation isomorphism in $\text{D}(X)$

\[
\eta_{\mathcal{E}} := \eta_{X/\text{Tot}(\mathcal{E})} : k_X \xrightarrow{i_s^{-1}} k_{\text{Tot}(\mathcal{E})}[2d].
\]

Applying $i_s$ to $\eta_{\mathcal{E}}$, we get a morphism in $\text{D}(X_s)$

\[
k_{X_s} = i_s^{-1} k_X \xrightarrow{i_s^{-1} \eta_{\mathcal{E}}} k_{X/\text{Tot}(\mathcal{E})}[2d] \xrightarrow{\text{B.C.}} i_s^{-1} k_{\text{Tot}(\mathcal{E})}[2d] = i_s k_X[2d],
\]

where “B.C.” means the base change morphism for the square (B.3.1), see [34] Prop. III.1.9(iii). The morphism (B.3.3) can be viewed as an element $c_d(\mathcal{E}, s) \in H^{2d}_X(X, k)$ which is known as the refined Euler (top Chern) class of $(\mathcal{E}, s)$. Its image in $H^{2d}(X, k)$ is the usual Euler (top Chern) class $c_d(\mathcal{E})$.

Passing to dg-enhancements, we denote $K_{\mathcal{E}} := K^{\bullet}_{X/\text{Tot}(\mathcal{E})}$. We can think of objects of the dg-categories $\text{D}(Y)_{\text{dg}}$ associated to various stacks $Y$ as (systems of, see (3.2.1)) complexes consisting of flabby sheaves. Now, for a flabby sheaf the $!$-inverse image under a closed embedding is given by the sheaf of sections with support. With this understanding, the base change morphism in a Cartesian square of closed embeddings of topological spaces is a canonical morphism of sheaves. Therefore our conventions imply that the base change morphism in (B.3.3) is defined canonically (no choice needed). So lifting $\eta_{\mathcal{E}}$ to $\tilde{\eta}_{\mathcal{E}} := \tilde{\eta}_{X/\text{Tot}(\mathcal{E})}$ as defined in §B.2, we upgrade the composite morphism (B.3.3) to a closed degree 0 morphism in $\text{D}(X)_{\text{dg}}$

\[
\tilde{c}_d(\mathcal{E}, s) : K_{\mathcal{E}} \otimes k_{X_s} \xrightarrow{i_s} k_{\text{Tot}(\mathcal{E})}[2d],
\]

representing $c_d(\mathcal{E}, s)$ up to a contractible choice.

B.4. Multiplicativity of homotopy canonical Euler classes. Let

\[
\begin{array}{ccc}
0 & \xrightarrow{a} & \mathcal{E}' \\
& \xrightarrow{b} & b \\
& \xrightarrow{c} & \mathcal{E}'' & \xrightarrow{0} \\
\end{array}
\]

be a short exact sequence of vector bundles on a stack $X$, of ranks $d', d, d''$ respectively, so $d = d' + d''$. We explain how to upgrade the multiplicativity relation $c_d(\mathcal{E}) = c_{d'}(\mathcal{E}') c_{d''}(\mathcal{E}'')$ in $H^*(X, k)$ to the level of homotopy canonical refined classes.

Let $s \in H^0(X, \mathcal{E})$ be a section. Then $s'' := b(s)$ is a section of $\mathcal{E}''$. Its zero locus $i_{s''} : X_{s''} \to X$ can be described, informally, as the locus of points $x \to X$ such that $s(x) \in \mathcal{E}'$. In particular, the bundle $i_{s''}^{*} \mathcal{E}'$ on $X_{s''}$ carries a section $s'$ given by the restriction of $s$. The zero locus $(X_{s'})_{s'}$ of this latter section is nothing but $X_s$, so we have a commutative triangle of closed embeddings

\[
\begin{array}{ccc}
X_s = (X_{s'})_{s'} & \xrightarrow{i_s} & X_{s''} \\
\downarrow i_{s''} & & \downarrow i_{s''} \\
X & & X
\end{array}
\]
The multiplicativity of refined Euler classes at the cohomology level can be expressed as the commutativity of the triangle in $D(X_a)$

\[(B.4.3) \quad k_{X_s} \xrightarrow{c_d(\mathcal{E}, s)} i_s^! k_{X_s}[2d'] \xrightarrow{i_s^! c_d(\mathcal{E}', s') [2d']} i_s^! k_{X} [2d' + 2d'']. \]

To prove this commutativity and to lift it to the homotopy canonical level, we denote by \[(B.4.4) \quad \text{Tot}(\mathcal{E'}) \xrightarrow{a} \text{Tot}(\mathcal{E}) \xrightarrow{b} \text{Tot}(\mathcal{E''}) \]
the diagram of the total spaces induced by (B.4.1). We note that \[(B.4.5) \quad \text{Tot}(\mathcal{E'}) \xrightarrow{a} \text{Tot}(\mathcal{E}) \xrightarrow{b} \text{Tot}(\mathcal{E''}) \]
is a Cartesian square. Therefore the same base change argument as used in (B.3.3) gives a morphism of complexes

\[\text{Hom}_{D(X_a)_{dg}} (k_X, 0^{i!} k_{\text{Tot}(\mathcal{E''})} [2d'']) \longrightarrow \text{Hom}_{D(\text{Tot}(\mathcal{E'}))_{dg}} (k_{\text{Tot}(\mathcal{E'})}, a^! k_{\text{Tot}(\mathcal{E})} [2d'']).\]

This morphism induces a morphism \[(B.4.6) \quad K_{\mathcal{E''}} = K_{X/\text{Tot}(\mathcal{E''})} \longrightarrow K_{\text{Tot}(\mathcal{E'})/\text{Tot}(\mathcal{E})}.\]

Also, \[(B.4.7) \quad X \xrightarrow{0_{\mathcal{E}'}} \text{Tot}(\mathcal{E'}) \xrightarrow{a} \text{Tot}(\mathcal{E}) \]
is a composable pair of regular embeddings with composition $0_{\mathcal{E}}$. Therefore composing the pairing (B.2.4) of this composable pair with the morphism (B.4.6), we get a pairing \[(B.4.8) \quad m_{\mathcal{E'}, \mathcal{E''}} : K_{\mathcal{E''}} \otimes K_{\mathcal{E'}} \longrightarrow K_{\mathcal{E}}.\]

These pairings are associative for any admissible (locally split) filtration $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}$ of vector bundles.

Further, (B.4.2) and (B.4.7) combine into a diagram \[(B.4.9) \quad X \xrightarrow{0_{\mathcal{E}'}} \text{Tot}(\mathcal{E'}) \xrightarrow{a} \text{Tot}(\mathcal{E}) \]
consisting of two Cartesian squares, whose concatenation (i.e., the outer perimeter diagram with horizontal edges composed) is the Cartesian square (B.3.1). We now notice that:

- The right square recovers $\tilde{c}_d''(\mathcal{E}'', s'')$ by pullback, as in (B.3.3), from $\tilde{\eta}_{\text{Tot}(\mathcal{E'})/\text{Tot}(\mathcal{E})}$. This follows from the square (B.4.5) which shows that $\tilde{\eta}_{\text{Tot}(\mathcal{E'})/\text{Tot}(\mathcal{E})}$ is the image of $\tilde{\eta}_{\mathcal{E}}$ under (B.4.6).
The left square recovers $\tilde{c}_d(i_{\mu}^*\mathcal{E}', s')$ by pullback from $\tilde{\eta}_{\mathcal{E}'} = \tilde{\eta}_{X/Tot(\mathcal{E}')}$. This is because we can subdivide the square into two Cartesian squares

\[ \begin{array}{ccc}
X & \xrightarrow{0_{\mathcal{E}'}} & Tot(\mathcal{E}') \\
i_{\mu} & & \downarrow s' \\
X_s & \xrightarrow{0_{i_{\mu}^*\mathcal{E}'}} & Tot(i_{\mu}^*\mathcal{E}') \\
i_{\mu}' & & \downarrow s'' \\
X_{s''} & \xrightarrow{i_{\mu}'} & X_{s''} 
\end{array} \]

which show that $\tilde{\eta}_{i_{\mu}^*\mathcal{E}'}$ is the pullback of $\tilde{\eta}_{\mathcal{E}'}$.

The composite (outer) square (B.3.1) recovers $\tilde{c}_d(\mathcal{E}, s)$ by pullback from $\tilde{\eta}_{\mathcal{E}}$ by definition.

So applying (B.2.5), we obtain a commutative square

\[ \begin{array}{c}
K_{\mathcal{E}'}^s \otimes K_{\mathcal{E}'} \otimes \mathbb{K}_{X_s}^s \xrightarrow{\eta_{\mathcal{E}'}^s \otimes c_d(i_{\mu}^*\mathcal{E}', s')} i_{s'}^! \mathbb{K}_{X_{s''}}^s [2d'] \\
m_{\mathcal{E}', \mathcal{E}, s''} \otimes \mathbb{K}_{X_s}^s \\
K_{\mathcal{E}} \otimes \mathbb{K}_{X_s}^s \xrightarrow{c_d(\mathcal{E}, s)} i_{s'}^! i_{s''}^! \mathbb{K}_{X}^s [2d' + 2d'']. 
\end{array} \]

which is a lift of (B.4.3) to the homotopy canonical level.

**References**

[1] Alday, L., Gaiotto, D., Tachikawa, Y., Liouville correlation functions from four-dimensional gauge theories, *Lett. Math. Phys.* 91 (2010), 167-197.

[2] Artin, M., Grothendieck, A., Verdier, J.L., Théorie des topos et cohomologie étale des schémas (SGA 4), Lectures Notes in Mathematics, vol. 305, Springer-Verlag, Berlin-New York, 1973.

[3] Baranovsky, V., The variety of pairs of commuting nilpotent matrices is irreducible, *Transform. Groups* 6 (2001), 3-8.

[4] Behrend, K., Cohomology of stacks, in: Intersection Theory and Moduli, ICTP Lect. Notes XIX (2004), 249-294.

[5] Behrend, K., Fantechi, B., The intrinsic normal cone, *Invent. Math.* 128 (1997), 45-88.

[6] Bernstein, J., Lunts, V. Equivariant Sheaves and Functors.

[7] Bondal, A. I., Kapranov, M. M., Enhanced triangulated categories, *Mat. Sb.* 181 (1990), 669-683.

[8] Bondal, A. I., Kapranov, M. M., Higher Segal Spaces, *Lecture Notes in Math.* 1753, Springer-Verlag, 1994.

[9] Costello, K., Gwilliam, O. Factorization Algebras in Quantum Field Theory, vol. I, Cambridge Univ. Press, 2017.

[10] Davison, B., Comparison of the Shiffmann-Vasserot product with the Kontsevich-Soibelman product, *Appendix to [56].*

[11] Davison, B., The integrality conjecture and the cohomology of preprojective stacks, arXiv:1602.0211v3.

[12] Dwyer, W. G., Kan, D. M., Simplicial localizations of categories, *J. Pure Appl. Algebra* 17 (1980) 267-284.

[13] Dwyer, W. G., Kan, D. M., Function complexes in homotopical algebra, *Topology* 19 (1980) 427-440.

[14] Etingof, T., Kapranov, M., Higher Segal Spaces, Lecture Notes in Math. 2244, Springer-Verlag, Berlin, 1999.

[15] Feit, W., Fine, N. J., Pairs of commuting matrices over a finite filed, *Duke Math. J.* 27 (1960) 91-94.

[16] Fulton, W., Intersection theory. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1998.

[17] Fulton, W., MacPherson, R. W., Categorical framework for the study of singular spaces, *Mem. Amer. Math. Soc.* 31 (1981).

[18] Getzler, E., Lie theory for nilpotent $L$-$\mathcal{E}$-algebras, *Ann. of Math.* 170 (2009) 271-301.

[19] Ginot, G., Notes on factorization algebras, factorization homology and applications, arXiv:1307.5213.

[20] Gaitsgory, D., Lurie, J., Weil Conjecture for Function Fields, Princeton Univ. Press, 2019.

[21] Gaiotto, D., Rozenblyum, N., A Study in Derived Algebraic Geometry I and II, *Amer. Math. Soc. Publ.* 2017.

[22] Goldman, W. M., Millson, J.J., The deformation theory of representations of fundamental groups of compact Kähler manifolds, *Bull. AMS.* 18 (1988) 153-158.

[23] Goldman, W.M., Millson, J.J., The deformation theory of representations of fundamental groups of compact Kähler manifolds, *Inst. Hautes Etudes Sci. Publ. Math.* 67 (1988), 43-96.
[24] Goresky, M., Kottwitz, R., MacPherson, R., Equivariant cohomology, Koszul duality and the localization theorem, Invent. Math. 131 (1998) 25-83.
[25] Graber, T., Pandharipande, R., Localization of virtual classes, Invent. Math. 135 (1999), 487-518.
[26] Heinloth, J., Coherent sheaves with parabolic structures and construction of Hecke eigensheaves for some ramified local systems, Ann. Inst. Fourier 54 (2004), 2235-2325.
[27] Hinich, V., Homological algebra of homotopy algebras, Comm. Algebra 25 (1997), 3291-3323.
[28] Hinich, V., Descent of Deligne groupoids, Internat. Math. Res. Notices 1997, 223-239.
[29] Holstein, J., Porta, M., Analytification of mapping stacks, arXiv:1812.09300.
[30] Huybrechts, D., Lehn, M., The geometry of moduli spaces of sheaves, Second edition, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010.
[31] Jardine, J.F., Simplicial Presheaves, J. Pure Appl. Algebra 47 (1987), 35-87.
[32] Kapranov, M., Eisenstein series and quantum affine algebras, Algebraic geometry, 7. J. Math. Sci. (New York) 84 (1997), 1311-1360.
[33] Kapranov, M., Schifflmann, O., Vasserot, E., The Hall algebra of a curve, Selecta Math. (N.S.) 23 (2017), 117-177.
[34] Kontsevich, M., Soibelman, Y., Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants, Comm. Numb. Theory and Physics 5 (2011), Number 2, 231-352 (arXiv:1006.2706).
[35] Kresch, A., Cycle groups for Artin stacks, Invent. math. 138 (1999), 495-536.
[36] Laumon, G., Moret-Bailly, L., Champs algébriques, Springer-Verlag, 2000.
[37] Laszlo, Y., Olsson, M., The six operations for sheaves on Artin stacks. I, Finite coefficients, Publ. Math. Inst. Hautes Études Sci. 107 (2008), 109-168.
[38] Lurie, J., Higher Algebra, Harvard University 2017.
[39] Lieblich, M., Remarks on the stack of coherent algebras, Int. Math. Res. Not. 2006 (2006), 1-12.
[40] Manolache, C., Virtual pull-backs, J. Algebraic Geom. 21 (2012), 201-245.
[41] Minets, A., Cohomological Hall algebras for Higgs torsion sheaves, moduli of triples and sheaves on surfaces, arXiv:1801.09437.
[42] Negut, A., Shufflee algebras associated with surfaces, arXiv:1703.02027.
[43] Nekrasov, N. A., Seiberg-Witten prepotential from instanton counting. Adv. Theor. Math. Phys. 7 (2003) 831-864.
[44] Olsson, M., Sheaves on Artin stacks, J. reine angew. Math. 603 (2007), 55-112.
[45] Olsson, M., Borel-Moore homology, Riemann-Roch transformations, and local terms, Adv. in Math 273 (2015), 56-123.
[46] Porta, M., Derived complex analytic geometry I: GAGA theorems, arXiv:math/0405330.
[47] Porta, M., Sala, F., Categorification of two-dimensional cohomological Hall algebras, arXiv:1903.07253.
[48] Ren, J., Soibelman, Y., Cohomological Hall algebras, semicanonical bases and Donaldson-Thomas invariants for 2-dimensional Calaby-Yau categories (with an Appendix by Ben Davison). In: Auroux D., Katzarkov L., et al. Eds. “Algebra, Geometry, and Physics in the 21st Century”. Progress in Mathematics, vol 324. Birkhäuser, Boston 2015.
[49] Sala, F., Schifflmann, O., Cohomological Hall algebra of Higgs sheaves on a curve, arXiv:1801.03482.
[50] Schifflmann, O., Vasserot, E., Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on $\mathbb{A}^2$, Publ. Math. Inst. Hautes Études Sci. 118 (2013), 213-342 (arXiv:1202.2756).
[51] Schifflmann, O., Vasserot, E., On cohomological Hall algebras of quivers : generators, J. reine angew. Math. (2018).
[52] Tabuada, G., Théorie homotopique des dg-categories, arXiv:0710.4303.
[53] Toën, B., Derive Algebraic Geometry, EMS Surv. Math. Sci. 1 (2014), 153-240.
[54] Toën, B., The homotopy theory of dg-categories and derived Morita theory, Invent. Math. 167 (2007) 615-667.
[55] Toën, B., Vezzosi, G., Homotopical algebraic geometry, II, Geometric stacks and applications. Mem. Amer. Math. Soc. 193 (2008).
[56] Weibel, C., An Introduction to Homological Algebra, Cambridge Univ. Press, 1995.
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