Flatness-based deformation control of an Euler–Bernoulli beam with in-domain actuation

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Abstract: This study addresses the problem of deformation control of an Euler–Bernoulli beam with in-domain actuation. The proposed control scheme consists in first relating the system model described by an inhomogeneous partial differential equation to a system under a standard boundary control form. Then, a combination of closed-loop feedback control and flatness-based motion planning is used for stabilising the closed-loop system around reference trajectories. The validity of the proposed method is assessed through well-posedness and stability analysis of the considered systems. The performance of the developed control scheme is demonstrated through numerical simulations of a representative micro-beam.

Nomenclature

- $\mathbb{R}$ field of all real numbers
- $\mathbb{R}^n$ n-dimensional Euclidean space
- $\Omega$ non-empty open subset of $\mathbb{R}^n$
- $\partial\Omega$ boundary of $\Omega$
- $H, X, Y$ Hilbert spaces
- $L(X, Z)$ set of continuous linear maps from the normed linear space $X$ to the normed linear space $Z$
- $L^p(\Omega)$ $p$-integrable functions
- $C^m(\Omega)$ space of $m$-times continuously differentiable functions from $\Omega$ into $\mathbb{R}$
- $H^m(\Omega)$ short for Sobolev space $H^{m,2}$, i.e. space of $m$-times weakly differentiable functions with its derivatives and itself in $L^2(\Omega)$
- $S(t)$ $C_0$-semi-group
- $G(x, \xi)$ Green’s function
- $\mathcal{D}(A)$ domain of the operator $A$
- $w(x, t)$ transversal displacement of the beam
- $\dot{w}(x, t)$ desired trajectory of the beam
- $w^d(x, t)$ steady-state of $w^d(x, t)$
- $w_x$ derivative of $w(x, t)$ with respect to $x$
- $w_t$ derivative of $w(x, t)$ with respect to $t$
- $\delta(x - \xi)$ Dirac delta function, or impulse function concentrated at the point $\xi$
- $\varphi(x - \xi)$ Blob function centered at the point $\xi$

1 Introduction

The present work addresses the control of an Euler–Bernoulli beam with in-domain actuation described by an inhomogeneous partial differential equation (PDE). A motivating example of such a problem arises from the deformation control of micro-beams, as shown in Fig. 1. This device is a simplified case of deformable micro-mirrors that are extensively used in adaptive optics [1, 2]. Due to technological restrictions in the design, the fabrication and the operation of micro-devices, design methods that may lead to control structures with a large number of sensors are not applicable to microsystems with currently available technologies.

One of the standard methods to deal with the control of inhomogeneous PDEs is to discretise the PDE model in space to obtain a system of lumped ordinary differential equations (ODEs) [3, 4], also known as early truncations. Then, a variety of techniques developed for the control of finite-dimensional systems can be applied. However, in addition to the possible instability due to the phenomenon of spillover [5] introduced by early truncations, the increase of modelling accuracy may lead to high-dimensional and complex feedback control structures, requiring a considerable number of actuators and sensors for the implementation. Therefore, it is of great interest to directly deal with the control of PDE models, which may result in control schemes with simple structures.

There exists an extensive literature on the control of different settings of flexible beams described by PDEs in which the majority of the reported work deal with the stabilisation problem by means of boundary control (see, e.g. [3, 6–15]). Placing actuators in the domain of the system will lead to inhomogeneous PDEs [3, 16, 17]. The stabilisation problem of Euler–Bernoulli beams with in-domain actuation is considered in, e.g. [16, 18–21]. The deformation control of beams has also gained in recent years an increasing attention and importance (see, e.g. [22–24]). Nevertheless, deformation control with pointwise in-domain actuation still remains an open topic, particularly in the cases where early truncations should be avoided.

In the present work, we employ the method of flatness-based control, which has been applied to a variety of infinite-dimensional systems (see, e.g. [23, 25–32]). Nevertheless, applying this tool to systems controlled by multiple in-domain actuators leads essentially to a multiple-input multiple-output (MIMO) problem, which is a challenging topic. It is noticed that a scheme proposed recently in [23, 24] tackles this issue by utilising the Weierstrass-factorised representation of the spectrum of the input-output dynamics. Nevertheless, an early truncation is still required in order to obtain a finite-dimensional input–output map, which is essential to apply the technique of flat systems.

The control scheme developed in this paper is based on an approach aimed at avoiding early truncations in control design procedure. Specifically, the design consists in first relating the original inhomogeneous model with pointwise actuation to a

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Fig. 1 Schematic of a deformable microbeam
system in a standard boundary control form to which the technique of flatness-based motion planning may be applied for feedforward control. By using the technique of lifting, this boundary-controlled system can be transformed into a regularised inhomogeneous system driven by sufficiently smooth functions. It is shown that the steady-state solution of the regularised system can approximate that of the original system. A standard closed-loop feedback control is used to stabilise the original inhomogeneous system. Moreover, the MIMO issue is solved by splitting the reference trajectory into a set of sub-trajectories based on an essential property of the Green's function. Well-posedness and stability analysis of the considered systems is also carried out, which is essential for the validation of the developed control scheme.

The main contributions of the present work can be summarised in three-pronged:

i. By performing the design directly with the PDE model, the approach proposed in this paper does not involve any early truncation in the control design procedure. As a result, the obtained control has a simple structure that requires only one measurement on the boundary for stabilisation.

ii. We extend the tool of flat systems for tracking control of a PDE system controlled by multiple in-domain actuators. To the best of the authors' knowledge, design schemes without requiring early truncations for tracking control of this type of PDE systems have not yet been reported in the open literature.

iii. The application of the Green's function enables the one-to-one property of the Green’s function. Well-posedness and stability of the considered problem in a weak sense. To this end, we start by defining the convergence on $C^0_0(\Omega)$.

Definition 1: Let $\Omega$ be a domain in $\mathbb{R}$. A sequence $\{\varphi_n\}$ of functions belonging to $C^0_0(\Omega)$ converge to $\varphi \in C^0_0(\Omega)$, if

i. there exists $K \subset \Omega$ such that $\text{supp}(\varphi_n - \varphi) \subset K$ for every $n$,

ii. $\lim_{n \to \infty} \partial^0\varphi_n(x) = \partial^0\varphi(x)$ uniformly on $K$ for all $n \geq 0$.

The linear space $C^0_0(\Omega)$ having the above property of convergence is called fundamental space, denoted by $\mathcal{D}(\Omega)$. The space of all linear continuous functionals on $\mathcal{D}(\Omega)$, denoted by $\mathcal{D}'(\Omega)$, is called the space of (Schwartz) distributions on $\Omega$, which is the dual of $\mathcal{D}(\Omega)$ (see, e.g. Chapter 1 of [36] for more properties of $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$).

Denote the set $\Phi$ by

$$\Phi = \{\varphi \in H^2((0,1)) : \varphi(0) = \varphi(1) = 0\}.$$  

(2)

We give the definition of weak solution of (1).

Definition 2: Let $T > 0$ and $\alpha_i \in L^2(0,T)$ for $i = 1, 2, \ldots, N + 1$. Let $h_0 \in \Phi$, $h_1 \in L^2(0,1)$. A weak solution to the problem (1) is a function $w \in C([0,T]; \Phi) \cap C^1([0,T]; L^2(0,1))$, satisfying

$$w(x,0) = h_0(x), \quad x \in (0,1),$$

satisfying, for every $\psi \in C^0([0,T];D(\Omega))$, one has for almost every $\tau \in [0,T]$

$$\int_0^1 w_\tau(x,\tau)\psi(x,\tau) \, dx - \int_0^1 h_\tau(x)\psi(x,0) \, dx$$

$$-\int_0^1 \int_0^1 w_\tau(x,\tau)\psi(x,\tau) \, dx \, dt + \int_0^1 \int_0^1 w_\tau(x,\tau)\psi_x(x,\tau) \, dx \, dt$$

$$= \sum_{i=1}^{N+1} \int_0^1 \int_0^1 \alpha_i(\tau)\psi(x,\tau) \, dx \, dt.$$

(3)

3 In-domain actuation design via boundary control

3.1 Relating in-domain actuation to boundary control

In the scheme developed in the present work, we use $N$ feedforward control signals to deform the beam, while dedicating the feedback stabilising control to one actuator. For notational simplicity, we assign the feedback control to the $(N + 1)$th
A weak solution of (4) can be defined in a similar way as that of (1) described in Definition 2. Note that it is shown later that the initial condition given in (1c) will be captured by the regulation error dynamics. Therefore, feedforward control design can be carried out based on System (4) with zero-initial conditions.

Due to the fact that the model given in (4) is driven by unbounded inputs, we will apply a sequence of integrable functions \( \varphi(x - x_j) \), also called \textit{blobs} [37], to approximate \( \delta(x - x_j) \) in the sense of distributions. Consider the steady-state model corresponding to (4) given by

\[
\overline{w}_{xxx}(x) = \sum_{j=1}^{N} \overline{\alpha}(x - x_j), \quad x \in (0, 1), \tag{5a}
\]

\[
\overline{w}(0) = \overline{w}(1) = \overline{w}'(0) = \overline{w}'(1) = 0, \tag{5b}
\]

where \( \overline{\alpha} \) is the static control at the position \( x_j \). We will then show in Theorem 1 that the solution to (5), \( \overline{w}(x) = \lim_{t \to \infty} w^d(x, t) \), can be approximated by a sufficiently smooth function. Towards this end, we consider first the following boundary controlled PDE:

\[
u_{x} (x,t) + u_{xxx} (x,t) = 0, \quad x \in (0, 1), \tag{6a}
\]

\[
u(0,t) = u(1,t) = u_x (0,t) = 0, \quad t > 0 \tag{6b}
\]

\[
u_{xx} (1) = g(t), \quad t > 0 \tag{6c}
\]

\[
u (0) = u (x, 0) = 0, \quad x \in (0, 1) \tag{6d}
\]

where \( g(t) = \sum_{j=1}^{N} g_j(t) \). Throughout this paper, without special statements, we assume that \( g_j(t) \in C([0, +\infty)) \) and \( g_j(0) = g_{j,0} = 0 \) for \( j = 1, 2, ..., N \). Notice that the motivation behind considering the system of the form given in (6) is that it allows employing the techniques of boundary control for feedforward control design while avoiding early truncations of dynamic model and/or controller structure.

A weak solution to (6) can be defined in a similar way as that of (1) described in Definition 2.

\textbf{Definition 3:} Let \( T > 0 \). A weak solution to the problem (6) is a function \( \psi \in C([0, T]; \Phi) \cap C([0, T]; L^2(0, 1)) \) satisfying

\[
u (0) = u (x, 0) = 0, \quad x \in (0, 1) \tag{6d}
\]

such that, for every \( v \in C([0, T]; \Phi) \), one has for almost every \( t \in [0, T] \)

\[
u_{x} (x,t) + u_{xxx} (x,t) = \sum_{j=1}^{N} \varphi_j (t) H_{xx} (x,j), \quad x \in (0, 1), \tag{6a}
\]

\[
u(0,t) = u(1,t) = u_x (0,t) = 0, \quad t > 0 \tag{6b}
\]

\[
u_{xx} (1) = g(t), \quad t > 0 \tag{6c}
\]

\[
u (0) = u (x, 0) = 0, \quad x \in (0, 1) \tag{6d}
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such that, for every \( v \in C([0, T]; \Phi) \), one has for almost every \( t \in [0, T] \)

\[
u_{x} (x,t) + u_{xxx} (x,t) = \sum_{j=1}^{N} \varphi_j (t) H_{xx} (x,j), \quad x \in (0, 1), \tag{6a}
\]

\[
u(0,t) = u(1,t) = u_x (0,t) = 0, \quad t > 0 \tag{6b}
\]

\[
u_{xx} (1) = g(t), \quad t > 0 \tag{6c}
\]

\[
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\[
u_{x} (x,t) + u_{xxx} (x,t) = \sum_{j=1}^{N} \varphi_j (t) H_{xx} (x,j), \quad x \in (0, 1), \tag{6a}
\]

\[
u(0,t) = u(1,t) = u_x (0,t) = 0, \quad t > 0 \tag{6b}
\]

\[
u_{xx} (1) = g(t), \quad t > 0 \tag{6c}
\]

\[
u (0) = u (x, 0) = 0, \quad x \in (0, 1) \tag{6d}
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\[
u (0) = u (x, 0) = 0, \quad x \in (0, 1) \tag{6d}
\]

such that, for every \( v \in C([0, T]; \Phi) \), one has for almost every \( t \in [0, T] \)

\[
u_{x} (x,t) + u_{xxx} (x,t) = \sum_{j=1}^{N} \varphi_j (t) H_{xx} (x,j), \quad x \in (0, 1), \tag{6a}
\]

\[
u(0,t) = u(1,t) = u_x (0,t) = 0, \quad t > 0 \tag{6b}
\]

\[
u_{xx} (1) = g(t), \quad t > 0 \tag{6c}
\]

\[
u (0) = u (x, 0) = 0, \quad x \in (0, 1) \tag{6d}
\]

where \( g(t) = \sum_{j=1}^{N} g_j(t) \). Throughout this paper, without special statements, we assume that \( g_j(t) \in C([0, +\infty)) \) and \( g_j(0) = g_{j,0} = 0 \) for \( j = 1, 2, ..., N \). Notice that the motivation behind considering the system of the form given in (6) is that it allows employing the techniques of boundary control for feedforward control design while avoiding early truncations of dynamic model and/or controller structure.

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\[
u (0) = u (x, 0) = 0, \quad x \in (0, 1) \tag{6d}
\]

such that, for every \( v \in C([0, T]; \Phi) \), one has for almost every \( t \in [0, T] \)

\[
u_{x} (x,t) + u_{xxx} (x,t) = \sum_{j=1}^{N} \varphi_j (t) H_{xx} (x,j), \quad x \in (0, 1), \tag{6a}
\]

\[
u(0,t) = u(1,t) = u_x (0,t) = 0, \quad t > 0 \tag{6b}
\]

\[
u_{xx} (1) = g(t), \quad t > 0 \tag{6c}
\]

\[
u (0) = u (x, 0) = 0, \quad x \in (0, 1) \tag{6d}
\]
Proof: In the steady state, we have
\[
\overline{w}_m^{\alpha}(x) = -\sqrt{f} H_m^{\alpha}(x) = \overline{\varphi}_m(x - x_f), \tag{13a}
\]
\[
\overline{w}_f^{\alpha}(0) = \overline{w}_m^{\alpha}(1) = \overline{w}_f^{\alpha}(0) = \overline{w}_f^{\alpha}(1) = 0, \tag{13b}
\]
and
\[
\overline{w}_m^{\alpha}(x) = \overline{\varphi}(x - x_f), \tag{14a}
\]
\[
\overline{w}_m^d(0) = \overline{w}_m^d(1) = \overline{w}_m^d(0) = \overline{w}_m^d(1) = 0. \tag{14b}
\]
Taking \(v(x) \in D(0,1)\), with \(v(0) = v_f(1) = v_{xx}(0) = 0\), as a test function and integrating by parts, we obtain (see equation below) Since \(\varphi_m(x - x_f) \rightarrow \delta(x - x_f)\) in the sense of distributions as \(m \rightarrow +\infty\) and \(v_{xx} \in D(0,1)\), it follows that \(\overline{w}_m^{\alpha} \rightarrow \overline{w}_f^{\alpha}\) in the sense of distributions as \(m \rightarrow +\infty\) for \(j = 1, 2, \ldots, N\). Furthermore, if \(\overline{w}_m^{\alpha} \in L^2(0,1)\) and \(\overline{w}_f^{\alpha} \in L^2(0,1)\), we have \(\overline{w}_m^{\alpha} \rightarrow \overline{w}_f^{\alpha}\) a.e. in \((0,1)\) (see Lemma 3.31 of [36], page 74). Then by the continuity of \(\overline{w}_m^{\alpha}\) and \(\overline{w}_f^{\alpha}\), we conclude that \(\overline{w}_m^{\alpha} \rightarrow \overline{w}_f^{\alpha}\) pointwise in \((0,1)\). Therefore \(\overline{w}_m^{\alpha} \rightarrow \overline{w}_f^{\alpha}\) in \(C([0,1])\) as \(m \rightarrow +\infty\) for \(j = 1, 2, \ldots, N\). □

Remark 2: Note that describing the control acting on discrete points in the domain by Dirac delta functions is a mathematical abstraction, which allows for the beam equation to possess some fundamental properties, in particular the exact controllability (see, e.g. [38]), and facilitates control design. However, real physical devices cannot generate unbounded control concentrated on a point. From this point of view, Theorem 1 provides an assessment of the gap between the performance achievable by utilising real physical devices and that predicted by the corresponding idealised mathematical model.

3.2 Well-posedness of Cauchy problems

Well-posedness analysis is essential to the approach developed in this work. In this subsection, we establish the existence and the uniqueness of weak solutions of (1), (4), (6) and (9).

Theorem 2: The following statements hold true:

i. Assume \(a_j \in L^2(0, T)\) for \(j = 1, 2, \ldots, N + 1\). Let \(h_0 \in \Phi, h_1 \in L^2(0,1)\) and \(T > 0\). Then System (1) and System (4) has a unique weak solution \(w \in C([0,T];\Phi) \cap C'([0,T]; L^2(0,1))\) and \(w^d \in C([0,T];\Phi) \cap C'([0,T]; L^2(0,1))\), respectively.

ii. Let \(T > 0\) and \(g \in C([0,T])\). Then System (6) has a unique weak solution \(u \in C([0,T];\Phi) \cap C'([0,T]; L^2(0,1))\).

Furthermore, if \(g \in C([0,T])\) and \(H_{mm}(x)\), \(j = 1, 2, \ldots, N\), is defined as in (8b), then System (6) has a unique solution \(w_m \in C([0,T];\Phi) \cap C'([0,T]; L^2(0,1))\).

Proof: The proof of (i) can be proceeded step-by-step as in Proposition 3.1 of [16]. We prove the first result of (ii) and the second part can be proceeded in the same way. Consider the following system:
\[
v_{xx}(x,t) + v_{xxxx}(x,t) = \left(\frac{1}{2} x - \frac{1}{6} x^3\right) g(t), \quad x \in (0,1), t > 0, \tag{15a}
\]
\[
v(0,t) = v_f(1,t) = v_{xx}(0,t) = v_{xxx}(1,t) = 0, \tag{15b}
\]
\[
\int_0^1 (\overline{w}_m^{\alpha}(x) - \overline{w}_f^{\alpha}(x)) v_{xx}(x) \, dx = \overline{\alpha} \int_0^1 (\varphi_m(x - x_f) - \delta(x - x_f)) v(x) \, dx.
\]

Let \(X = \Phi \times L^2(0,1)\) and \(H^{\alpha}(0,1) = \{u \in H^1(0,1); u(0) = u_f(1) = u_{xx}(0) = u_{xxx}(1) = 0\}\).

Define the inner product on \(X\) by \((u, v)_X = \int_0^1 u(x) v(x) + u'(x) v'(x) \, dx\). Define the subspace \(D(A) \subset X\) by \((A, u, v) = \{u, v\}; (u, v) \in H^{\alpha}(0,1) \times \Phi\), with the corresponding operator \(A^{*} D(A) \rightarrow X\) defined as
\[
A^{*}(u,v) = \left(\begin{array}{c}
u \\
-v_{xx}
\end{array}\right).
\]

One may directly check that \(D(A)\) is dense in \(X\), \(A^{*}\) is closed, and \((A, z, z) = 0\). Thus, by Stone’s theorem, \(A^{*}\) generates a semi-group of isometries on \(X\). Note that for fixed \(x \in (0,1)\) and \(g \in C([0,T])\),
\[
f(t) = \left(1 - \frac{1}{2} x - \frac{1}{6} x^3\right) g(t) \in C([0,T]; X).
\]

Based on a classical result on perturbations of linear evolution equations (see, e.g. Theorem A.7, Appendix A, page 375, [39] and Theorem 1.5, Chapter 6, page 187, [40]), there exists a unique \(z = (z, z) \in C([0,T]; X) \cap C([0,T]; D(A^{*}))\), such that
\[
\frac{dz}{dt} = A^{*} z + f(t), \quad z(0,0) = (0,0),
\]
which implies that (15) has a unique solution \(v = z \in C([0,T]; H^{\alpha}(0,1)) \cap C([0,T]; \Phi)\) in the usual sense. Particularly, \(v \in C([0,T]; \Phi) \cap C([0,T]; L^2(0,1))\) is a weak solution. A direct computation shows that \(u = z - (1/2) x - (1/6) x^3\) is a solution of (6) in the usual sense and, in particular, it is a weak solution. □

4 Feedback control and stability of the inhomogeneous system

The validity of the proposed scheme requires a suitable closed-loop control to guarantee the stability of the original inhomogeneous system. As the \((N + 1)\)th actuator is dedicated to stabilising control, (1a) can be written as
\[
w_{tt}(x,t) + w_{xxxx}(x,t) - \alpha_{N+1}(t) \delta(x - x_{N+1}) = \sum_{j=1}^{N+1} \alpha(t) \delta(x - x_j), \quad x \in (0,1), t > 0. \tag{16}
\]

Suppose further that the feedback control is taken as [16, 20]
\[
\alpha_{N+1}(t) = -k w_{N+1}(x_{N+1}, t) \tag{17}
\]
where \(k\) is a positive-valued constant. Then in closed loop, (16) becomes
\[
w_{tt}(x,t) + w_{xxxx}(x,t) + k w_{N+1}(x_{N+1}, t) \delta(x - x_{N+1}) = \sum_{j=1}^{N+1} \alpha(t) \delta(x - x_j), \quad x \in (0,1), t > 0. \tag{18}
\]

Let

\[
\mathcal{D}(A) = \{(w,v); (w,v) \in (H^1(0,1) \cup \mathcal{H}(0,x_1) \cup \mathcal{H}(x_1,x_2) \cup \cdots)
\]

\[
\int_0^1 (\overline{w}_m^{\alpha}(x) - \overline{w}_f^{\alpha}(x)) v_{xx}(x) \, dx = \overline{\alpha} \int_0^1 (\varphi_m(x - x_f) - \delta(x - x_f)) v(x) \, dx.
\]
\[ H'(x_N, x_{N+1}) \times H'(x_{N+1}, 1)) \times H'(0, 1), w(0) = w_x(1) = w_{xx}(0) = w_{xx}(1) = 0, v(0) = v_x(1) = 0 \]
and \( X \) are defined as in the proof of Theorem 2. Let \( T > 0 \) and \( \alpha \in L^2(0, T) \) for \( j = 1, ..., N \). Assume \((h, h) \in X\). To address the stability of System (18) with the boundary conditions (1b) and initial conditions (1c), we consider the corresponding linear control system under the following abstract form:

\[ \dot{z} = Az + B\alpha, \quad t > 0, \]

(19a)

\[ z(0) = z^0 = (h_h, h_h)^T, \]

(19b)

where \( z = (w, v, \alpha)^T, A = (\alpha_1, ..., \alpha_N)^T, A: D(A) \rightarrow X \) is defined as

\[ A(w, v) = \left( -w_{xxx} - k\alpha(\delta(x - x_{N+1})) \right)^T \]

(20)

with \( w = w_x \) and \( B: \mathbb{R}^N \rightarrow D(A^*) \), where \( A^* \) is the adjoint of \( A \), which is defined as:

\[ (Ba)y = \left( \sum_{j=1}^N \alpha_j \delta(x - x_j) \right)^T y, \quad \forall y \in D(A^*). \]

(21)

Note that \( B^* : D(A) \rightarrow \mathbb{R}^N \) is defined by

\[ B^*y = (y_1(x_1), ..., y_j(x_j))^T, \quad \forall y = (y_1, y_j) \in D(A). \]

(22)

Let \( U = \mathbb{R}^N \). Then the solution of the Cauchy problem (19) can be defined as follows (see [39], Definition 2.36, p. 53):

\[ (x(t), y)^N = (z^0, S(t)x^0) + \int_0^t (\alpha(t), B^* S(t) - \delta(t)y^N) dt, \quad t \in [0, T] \]

(23)

where \( \alpha \) is a rational number with coprime factorisation, there exist positive constants \( C_\alpha, C_\delta \) and \( \lambda \) such that for any \( 0 \leq \tau \leq T < \infty \), there holds

\[ \| x(\tau) \|_X \leq C/e^{\lambda \tau} \| z^0 \|_X + C_\delta \| \alpha \|_{L^\infty(0, T)} . \]

(24)

**Proof:** The proof is a slight modification of Theorem 2.37 in [39]. First, the admissible property of \( B \) can be obtained as (3.38) in [16]. A direct computation gives

\[ \left( A(u, v), \left( u \right)_N \right)_X = -k \| x_N \|_X^2 + k \left( u \right)_N \in D(A) . \]

Therefore, \( A \) is a dissipative operator. Moreover, it can be directly verified that \( A \) is onto and hence, according to Theorems 4.3 and 4.6 of [35] (p. 14–15), it generates a \( C_e \)-semi-group of linear contractions \( S(t) \) acting on \( X \). Furthermore, \( S(t) \) is exponentially stable if \( x_{N+1} \in (0, 1) \) is a rational number with coprime factorisation, in particular for \( x_{N+1} = 1 \) (see, e.g. [16, 20]), i.e., there exist two positive constants \( C_\delta \) and \( \lambda \), such that

\[ \| x(\tau) \|_X \leq C/e^{\lambda \tau}, \quad \forall \tau \geq 0 . \]

Then for a weak solution defined in (23), we have that for all \( 0 \leq \tau \leq T < \infty \) (see equation below) Applying the interpolation formula

\[ \| u - \| _{L^2} \leq C \left( \| u \| _{L^2} + \| u_x \| _{L^2} \right) , \]

where \( c \) is a positive constant, and the Sobolev embedding theorem yields

\[ \| y \| _{L^\alpha} \leq C \| y \| _{L^\alpha} + \| y \| _{L^\alpha} \]

which implies that (24) holds with \( C_\delta = C/c/\lambda \). Now consider the regulation error defined as \( e(x, t) = w(x, t) - w^0(x) \). Denoting \( \Delta t(x, t) = \alpha(\delta(x, t) - \bar{\alpha}(x)) \), \( j = 1, ..., N \), then from (18), (1b), (1c) and the steady-state model of (4), the regulation error dynamics satisfy

\[ e(x, t) + e_x x + \delta(x, t) \delta(x - x_{N+1}) = \sum_{\alpha j} \Delta t(x, t) \delta(x - x_j), \quad x \in (0, 1), \quad t > 0, \]

(25a)

\[ e(0, t) = e_x(1, t) = e_x(0, t) = e_x(1, t) = 0, \quad t > 0, \]

(25b)

\[ e(x, 0) = e_x(x, 0) = h_h(x) - \bar{\alpha}(h), \quad e_x(0) = e_x(h, x) = h_h(h), \quad x \in (0, 1) . \]

(25c)

Obviously, the regulation error dynamics are in an identical form as (18) with the same type of boundary conditions. We can then consider the solution of System (25) defined in the same form given by (23) with \( \xi = (e_x, e_x) \) and \( \xi^N = (e_x, e_x) \).

**Corollary 1:** Assume that all the conditions in Theorem 3 are fulfilled and \( \xi^N \in X \). Then there exist positive constants \( C_\xi, C_\delta \) and \( \lambda \), independent of \( t \), such that for any \( t \geq 0 \), there holds

\[ \| x(t) \|_X \leq C/e^{\lambda t} \| z^0 \|_X + C_\delta \| \delta \|_{L^\infty(0, + \infty)} . \]

(26)
Moreover, if \( \lim_{\tau \to \infty} \Delta \omega_\tau = 0 \) for all \( j = 1, \ldots, N \), then \( \lim_{\tau \to \infty} e(x, \tau) = 0, \forall \tau \in (0, 1) \).

The feedforward control satisfying the conditions for closed-loop stability and regulation error convergence can be obtained through motion planning, as presented in the following section.

5 Motion planning and feedback control

According to the principle of superposition for linear systems, we consider in feedforward control design the dynamics of System (6) corresponding to the input \( g_j(t) \) described in the following form:

\[
\begin{align*}
    u_{jcl}(x, t) + u_{jfeed}(x, t) &= 0, \quad (27a) \\
    u_j(0, t) &= u_j(t) = u_{jcl}(0, t) = 0, \quad (27b) \\
    u_{jfeed}(1, t) &= g_j(t), \quad (27c) \\
    u_j(x, 0) &= u_j(x, 0) = 0. \quad (27d)
\end{align*}
\]

The required control signal \( g_j(t) \) should be designed so that the output of System (27) fulfills the prescribed function \( \sum_{k=0}^{\infty} \left( \frac{1}{4k+1} \right) (4(n-k) + 2)! \) if \( t \leq 0 \) and to generating the full-state trajectory of the subsystem \( T \). By applying a standard Laplace transform-based procedure (see, e.g. [25, 32, 41]), we can obtain the full-state trajectory with zero initial values expressed in terms of the so-called flat output, \( y_j(t) \), and its time-derivatives

\[
\begin{align*}
    u_j(x, t) &= \left( \frac{1}{2} - \frac{1}{6} x^2 \right) y_j(t) \\
    &+ \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)! (4(n-k) + 2)!} \left( -1 \right)^n y_j^{(2n)}(t),
\end{align*}
\]

Now let \( y_j(t) = \tilde{\psi} \phi_j(t) \), where \( \phi_j(t) \) is a smooth function evolving from 0 to 1. The full-state trajectory can then be written as

\[
\begin{align*}
    u_j(x, t) &= \tilde{\psi} \left( \frac{1}{2} - \frac{1}{6} x^2 \right) \phi_j(t) \\
    &+ \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(4k+1) x^{4k+1}}{(4(n-k) + 2)!} \left( -1 \right)^n \phi_j^{(2n)}(t).
\end{align*}
\]

The corresponding input can be computed from (27c), which yields

\[
g_j(t) = -\tilde{\psi} \phi_j(t) + \sum_{n=1}^{\infty} \frac{1}{(4k+1)! (4(n-k) + 2)!} \left( -1 \right)^n \phi_j^{(2n)}(t).
\]

For set-point control, we need an appropriate class of trajectories enabling a rest-to-rest evolution of the system. A convenient choice of \( \phi_j(t) \) is the following smooth function:

\[
\phi_j(t) = \begin{cases} 
0 & \text{if } t \leq 0 \\
\int_0^t \exp \left( -1 / (\tau(1-\tau)) \right) d\tau & \text{if } t \in (0, T) \\
1 & \text{if } t \geq T
\end{cases}
\]

which is known as Gevrey function of order \( \sigma = 1 + 1/\varepsilon, \varepsilon > 0 \) (see, e.g. [32]).

For the convergence of (29) and (30), we have

\[
\text{Proposition 1: If } \phi_j(t) \text{ in the basic output } y_j(t) = \tilde{\psi} \phi_j(t) \text{ is chosen as a Gevrey function of order } 1 < \sigma < 2, \text{ then the infinite series (29) and (30) are convergent.}
\]

The detailed convergence analysis can follow a standard procedure (see, e.g. [25, 26, 32, 41]) and hence, it is omitted here.

Remark 1: In general, the Gevrey bounds are unknown, but it can be estimated following the way presented in [26]. Furthermore, a symmetric function in the transient phase can be considered to improve convergence analysis [26].

Now let \( \psi_m(x, t) = \Phi_m(x, t) \), \( m > 0 \). By the definition of \( \Phi(x, t) \), we obtain

\[
\psi_m(x, t) = \tilde{\psi} \sum_{n=1}^{\infty} \Phi_m(x) \Phi_n(x) \psi_m(x - x_j)
\]

To compute \( I_{j,m}(x) \), we note that the blob defined in (12) can be expressed as

\[
\Phi_m(x) = \frac{1}{(2k-1)!} (x - x_j)^{2k-2}.
\]

Therefore

\[
\begin{align*}
    P_j(x) &= \frac{1}{2} x - \frac{1}{6} x^3, \\
    I_{j,m}(x) &= \int_{x_j}^{x} \int_{x_j}^{y} \int_{x_j}^{y} \psi_m(s - x_j) ds \, dt \, dy, \\
    \Phi_m(x) &= \left( \frac{n}{\sum_{k=0}^{\infty} (4k+1)! (4(n-k) + 2)!} \right) \left( \frac{n}{\sum_{k=0}^{\infty} (4k+1)! (4(n-k) + 2)!} \right) \left( -1 \right)^n, \\
    \Psi_m(x) &= \left( \frac{n}{\sum_{k=0}^{\infty} (4k+1)! (4(n-k) + 2)!} \right) \left( \frac{n}{\sum_{k=0}^{\infty} (4k+1)! (4(n-k) + 2)!} \right) \left( -1 \right)^n.
\end{align*}
\]

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The map of (36) is invertible for all \( d \neq 0 \), can be expressed as

\[
G(x, \xi) = \begin{bmatrix}
\frac{x^3}{6} + x^2 \xi \left(1 - \frac{\xi}{2}\right) & 0 & \leq \xi < \xi_j; \\
\frac{\xi_j^3}{6} + \xi_j x \left(1 - \frac{x}{2}\right) & \xi_j \leq x \leq 1.
\end{bmatrix}
\]

Due to the principle of superposition for linear systems, the solution to (5), \( \tilde{w}^n(x) \), can be expressed as

\[
\tilde{w}^n(x) = \sum_{j=1}^{N} \int_{a}^{b} G(x, \xi)\beta(x - \xi)\,d\xi,
\]

Now taking \( N \) points on \( \tilde{w}^n(x) \) and letting \( \tilde{w}^n(x) = \tilde{w}^n(x_j) \), \( j = 1, \ldots, N \), yield

\[
\begin{bmatrix}
G(x_{1}, \xi_{1}) & \ldots & G(x_{N}, \xi_{1}) & \bar{\beta}_{1} \\
\vdots & \ddots & \vdots & \vdots \\
G(x_{1}, \xi_{N}) & \ldots & G(x_{N}, \xi_{N}) & \bar{\beta}_{N}
\end{bmatrix}
\begin{bmatrix}
\tilde{w}^{n}(x_1) \\
\vdots \\
\tilde{w}^{n}(x_N)
\end{bmatrix}
= \begin{bmatrix}
\bar{\beta}_{1} \\
\vdots \\
\bar{\beta}_{N}
\end{bmatrix}
\tag{36}
\]

which represents a steady-state input to output map.

Claim 2: The map of (36) is invertible for all \( x_{j}, \xi_{j} \in (0, 1) \), \( j = 1, \ldots, N \), and \( x_{j} \neq x_{j'}, \xi_{j} \neq \xi_{j'} \), if \( j \neq j' \). The proof of this claim is given in Appendix 2.

As \( \bar{\beta}_{j} = -\bar{\beta}_{j} \) and \( \lim_{t \to \infty} \beta(t) = \bar{\beta}_{j} = -\bar{\beta}_{j} \) for all \( j = 1, \ldots, N \), we obtain from Claim 2 that

\[
\begin{bmatrix}
\bar{\beta}_{1} \\
\vdots \\
\bar{\beta}_{N}
\end{bmatrix} = \begin{bmatrix}
G(x_{1}, \xi_{1}) & \ldots & G(x_{N}, \xi_{1}) \\
\vdots & \ddots & \vdots \\
G(x_{1}, \xi_{N}) & \ldots & G(x_{N}, \xi_{N})
\end{bmatrix}^{-1}
\begin{bmatrix}
\tilde{w}^{n}(x_1) \\
\vdots \\
\tilde{w}^{n}(x_N)
\end{bmatrix}
\tag{37}
\]

6 Simulation study

In the simulation study, we consider the deformation control in which the desired shape is given by

\[
\tilde{w}^{n}(x) = -10^{-4}(e^{-100(x-0.8)} + 2e^{-100(x-0.6)}) + 3e^{-100(x-0.5)} \quad x \in (0, 1),
\]

as shown in Fig. 2a. Note that the maximum amplitude of the shape given by (38) is \( 3.8 \times 10^{-4} \), which represents a typical micro-structure for which the beam length is of few centimetres and the displacement is of micrometre order.

In order to obtain an exponential closed-loop convergence, the actuator for feedback stabilisation is located at the position \( x_{N+1} = 1 \). To evaluate the effect of the number of actuators to interpolation accuracy, measured by \( \| \tilde{w}^{n}(x) - \tilde{w}^{n}(x) \|_{L_{2}(a,b)} \) and control effort, we considered 3 setups with, respectively, 8, 12 and 16 actuators evenly distributed in the domain. It can be seen from Fig. 2a that the setup with 8 actuators exhibits an important interpolation error, especially around \( x = 0.7 \), and the one with 16 actuators requires a high control effort in spite of a high interpolation accuracy. The setup with 12 actuators provides an appropriate compromise between the interpolation accuracy and the required control effort, which is used in control algorithm validation. Note that this study may provide a guideline for the trade-off between the overall system performance and the complexity in implementation and operation at the early design stages.

A MATLAB Toolbox for dynamic Euler–Bernoulli beams simulation provided in Chapter 14 of [42] is used in numerical implementation. With this Toolbox, the simulation accuracy can be adjusted by choosing the number of modes used in implementation. In the simulation, we implement System (1) with initial conditions \( h_{i}(x, 0) = -3 \times 10^{-10}e^{-100(x-0.9)} \) and \( h_{i}(x, 0) = 0 \), which represent initial disturbances acting on the beam.

The controller tuning is started by choosing a suitable closed-loop control gain \( k \). As the gain-margin of the stabilising loop is \( k \in (0, \infty) \), it is determined by experiments such that the closed-loop system has a sufficient decay-rate. Finally, a value of \( k = 5 \) is selected for numerical simulation. Note that other techniques, such as optimal control [3, 20], Lyapunov functions [9, 13], and backstepping [7], can eventually be applied in feedback control design and tuning.

To derive the feed-forward control signals, we start by determining the amplitude of basic outputs, \( \tilde{w}^{n}_{j}, j = 1, \ldots, 12 \), from (37). The functions used to generate the basic outputs, \( \phi_{j}(t) \), are chosen as Gevrey functions of the same order. To meet the
convergence condition given in Proposition 1, the parameter \( \alpha \) in (31) is set to \( \varepsilon = 1.111 \). The feed-forward control \( \sigma_c(t) = -g_1(t) \) is then computed by (30). The corresponding feedforward control signals with \( T = 5 \), \( \alpha_1, \ldots, \alpha_{12} \), that steer the beam to the desired shape are illustrated in Fig. 3.

The evolution of beam shapes and the regulation errors are depicted in Fig. 4. As illustrated in this figure, the initial disturbances have been quickly damped out, the beam is deformed to the desired shape, and the regulation error vanishes identically along the entire beam. The simulation results confirm the expected performance of the developed control scheme.

7 Conclusion

This paper presented a solution to the problem of in-domain control of a deformable beam described by an inhomogeneous PDE. A relationship between the original model and a system expressed in a standard boundary control form has been established. Flatness-based motion planning and feedforward control are then employed to explore the degrees-of-freedom offered by the system, while a closed-loop control is used to stabilise the system around the reference trajectories. The validity of the developed approach has been assessed through well-posedness and stability analysis. System performance is evaluated by numerical simulations, which confirm the applicability of the proposed approach. Note that in order to improve the resolution of manipulation, more actuators may be expected. Nevertheless, for the control of one-dimensional (1D) beams, the proposed scheme contains only one closed loop, allowing a drastic simplification of control system implementation and operation. This is an important feature for practical applications, such as the control of large-scale deformable micro-mirrors in adaptive optics systems.

One of the further perspectives of the approach developed in this work is to extend the control scheme to a plate equation, or in other words to a case of more than one row of actuators. A straight forward strategy towards this goal is to decouple the 2D plate equation to the Cartesian products of two 1D systems of Euler–Bernoulli beams as suggested in [19]. Then, for each set of equations the proposed control scheme may be applicable.

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9 References

...
\[
\frac{(-1)^{k+1}m^{k-1}(x-x_j)^{2k+1}}{(2k-1)!} \leq m^{2k-1}(2k+1)!
\]

Since \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \), it follows that for any \( x > 0 \)
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} = a \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0.
\]

Thus \( \sum_{k=0}^{\infty} \frac{m^{2k-1}}{(2k-1)!} \) is convergent. We conclude that the first series in (35) is uniformly convergent. The convergence of the other terms in (35) can be proved in the same way.

10.2 Appendix 2: Proof of Claim 2

We denote by \( [G(x, \xi_N)]_{N \times N} \) the matrix in the left-hand side of (36) formed by Green's functions. We argue by contradiction. If, otherwise, \( [G(x, \xi_N)]_{N \times N} \) is not invertible, then it is of rank less than \( N \). Without loss of generality, assume that there exist \( N - 1 \) constants, \( k_2, k_3, \ldots, k_{N-1} \), such that

\[
G(x, \xi_N) = \sum_{i=1}^{N-1} k_i G(x, \xi_i),
\]

\[
G(x, \xi_N) = \sum_{i=1}^{N-1} k_i G(x, \xi_i),
\]

\[
G(x, \xi_N) = \sum_{i=1}^{N-1} k_i G(x, \xi_i).
\]

The above equations show that

\[
G(x, \xi_N) = \sum_{i=1}^{N-1} k_i G(x, \xi_i)
\]

has \( N \) different positive solutions \( x_1, x_2, \ldots, x_N \), \( x_i \in (0, 1) \), \( i = 1, \ldots, N \).

We consider two cases:

1. If \( N > 3 \), since \( \xi_i, i = 1, \ldots, N \), are distinguished, \( G(x, \xi_i), i = 1, \ldots, N \), are all different from each other. Hence

\[
G(x, \xi_N) \equiv \sum_{i=1}^{N-1} k_i G(x, \xi_i) + k_N G(x, \xi_N)
\]

\[
\sum_{i=1}^{N-1} k_i G(x, \xi_N) + k_N G(x, \xi_N)
\]

Note that \( G(x, \xi_j), j = 1, \ldots, N, \) are of order at most 3, then (39) has at most three different solutions in \( \mathbb{R} \), which is a contradiction.

2. If \( N \leq 3 \), it is easy to check that (39) has a solution \( x = 0 \), and a pair of solutions \( x = x^0 \) and \( x = -x^0 \) near the origin 0. By the assumption, (39) has \( N \) different positive solutions, then it must be \( N = 1 \), which leads to a contradiction with the non-invertible property of \( G(x, \xi_j) \neq 0 \).

10 Appendix

10.1 Appendix 1: Proof of Claim 1

Consider the first series in (35). Fixing \( m > 0 \), for \( x, x_j \in (0, 1) \), \( j = 1, 2, \ldots, N \), we have