A Rokhlin Lemma for Noninvertible Totally-Ordered Measure-Preserving Dynamical Systems

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Abstract

Let \((X, \mathcal{F}, \mu, T)\) be a not necessarily invertible non-atomic measure-preserving dynamical system where the \(\sigma\)-algebra \(\mathcal{F}\) is generated by the intervals according to some total order. The main result is that the classical Rokhlin lemma may be adapted to such a situation assuming a slight extension of aperiodicity. This result is compared to previous noninvertible versions of the Rokhlin lemma.

1 Introduction

Rokhlin’s lemma traces its history to the work of Vladimir Abramovitch Rokhlin, who utilized it in studies of generators in ergodic theory [8]. In its original context, it is a statement that in a measure-preserving dynamical system \((X, \mathcal{F}, \mu, T)\) with \(T\) surjective, aperiodic, and invertible, for any \(n \in \mathbb{N}\) and \(\varepsilon > 0\), it is possible to find a measurable subset \(E\) such that each of the sets \(T^j(E)\) for \(j = 0, \ldots, n-1\) are disjoint and these together cover all but a subset of measure at most \(\varepsilon\) of the space \(X\); we say that such an \(E\) induces a \((n, \varepsilon)\)-Rokhlin tower. For nearly 40 years, the Rokhlin lemma took this form in most treatments (see [5, pp. 71-72], [3, p. 158]) but it entered the mathematical folklore that minor adjustments to the proof would permit it to be applied to a non-invertible context.

A published proof of a Rokhlin lemma for non-invertible endomorphisms does not seem to have manifested until 2000, when Stefan-Maria Heinemann and Oliver Schmitt published a proof in the context when \(X\) is a separable metric space with \(T\) measure preserving, aperiodic and surjective (but not necessarily invertible or even forward-measurable) that there exists a set \(E\) inducing an \((n, \varepsilon)\)-Rokhlin tower, in the sense that \(\mu(E \cap T^{-j}E) = 0\) for all \(j = 1, 2, \ldots, n-1\) and the sets \(T^{-j}E\) cover all but a measure \(\varepsilon\) subset of \(X\).

We wish to highlight the flexibility of the proof strategy utilized by Heinemann and Schmitt in [6] by adapting the Rokhlin lemma to a similar but distinct setting, that of systems generated by the intervals according to some total ordering with non-invertible dynamics, in the hopes that this will inspire further
investigation into conditions and contexts where other analogous versions of Rokhlin’s lemma hold.

In section 3 we compare the variants of Rokhlin’s lemma produced by Heine mann and Schmitt and which we prove in this paper, and also compare our result to a similar but distinct version of Rokhlin’s lemma obtained by Avila and Candela [2] which applies to \( \mathbb{N}^d \)-actions on (nonatomic) standard probability spaces.

For those so inspired we also suggest Avila and Bochi [1] which contains a distinct approach from [6] in proving an interesting variation of the Rokhlin lemma for compact manifolds with \( C^1 \)-endomorphism dynamics against a normalized Lebesgue measure (which is not necessarily preserved by the dynamics).

## 2 Rokhlin lemma

**Definition 1.** Suppose \( X \) is totally ordered by \(<\). We define an interval in \( X \) to be any set \( J \subseteq X \) such that if \( a, c \in J \), and \( a < b < c \) for some \( b \in X \), then \( b \in J \) also.

We utilize standard interval notation; in particular, we use the symbols \( -\infty \) and \( \infty \) to denote the absence of a lower bound or upper bound. We also introduce the notation \( \langle a, b \rangle \) to denote the closed interval between \( a \) and \( b \) regardless of their relative order; that is, \( \langle a, b \rangle = [\min(a, b), \max(a, b)] \). When \( a = b \), we let it be a singleton set \( \langle a, a \rangle = \{a\} \).

**Definition 2.** Let \( \Sigma = (X, \mathcal{F}(<), \mu, T) \) be a probability measure-preserving dynamical system such that \( \mathcal{F}(<) \) (or simply \( \mathcal{F} \)) is the \( \sigma \)-algebra generated by the intervals according to some total order \(<\) on \( X \), \( T \) is a measurable endomorphism, and \( \mu \) is non-atomic. We call \( \Sigma \) a shuffleable system.

Shuffleable systems emanate from the fact that they are the most general class of measure-preserving dynamical system in which questions of shuffling in the sense of Persi Diaconis’s geometric model of the Gilbert-Shannon-Reds model of the riffle shuffle (see [4]) may be studied. Shuffleable systems include classical examples such as the doubling map and the one-sided full shift \( \Sigma_2^+ \), but the class of shuffleable systems is general enough to also include interval maps with non-atomic preserved measures as well as systems on \( \mathcal{I} = [0, 1] \setminus \mathbb{Q} \), as well as full-measure subsets of standard probability spaces, which is essential as we will see in Section 3.

We note as a warning that, unlike with Lebesgue-Stieljes measures on the real line and standard probability spaces, shuffleable spaces do not necessarily play well with the topology of open intervals.

**Example 3.** The measure space \( (X = [0, 1] \times (0, 1), \mathcal{F}(<\text{lex}), m_1) \), where \( x <\text{lex} y \) if \( x_1 < y_1 \) or if both \( x_1 = y_1 \) and \( x_2 < y_2 \), and where \( m_1 \) is the probability measure induced on \( X \) by the premeasure on intervals given by \( m_1(x, y] = y_1 - x_1 \), is a shuffleable space. In particular, if \( \pi : X \to [0, 1] \) is the projection \( \pi(x) = x_1 \), and \( m \) the Lebesgue measure, then for any interval
$J \subseteq [0,1]$, $m_1(\pi^{-1}(J)) = m(J)$. However, if $F \subseteq [0,1]$ is a non-measurable subset of $[0,1]$, then $\pi^{-1}(F)$ is a non-measurable subset of $X$. Also, every set $\{x_1\} \times (0,1) \subseteq X$ is open, and so $\pi^{-1}(F) = \bigcup_{x_1 \in F} \{x_1\} \times (0,1)$ is open in the $<_{\text{lex}}$-order topology, yet non-measurable in $X$. \hfill \Box$

For a shuffleable system, we define $F_\mu(x) := \mu(-\infty, x]$, the distribution function of $\mu$.

**Proposition 4.** If $(X, \mathcal{F}, \mathcal{F}(\langle \rangle), \mu)$ is a non-atomic probability space with $\mathcal{F}$ the $\sigma$-algebra generated by the intervals according to the total order $\langle$, then for any open interval $J = (a, b) \subseteq [0,1]$, there exists some point $x_J \in X$ such that

$$F_\mu(x_J) = \mu(-\infty, x_J] \in J.$$ 

**Proof.** By way of contradiction, suppose no value in $(a, b) \subseteq [0,1]$ is the measure of any left-ray $(-\infty, x]$; that is, $F_\mu(X) \cap (a, b) = \emptyset$. Then the sets

$$L = \bigcup \{(-\infty, y] : y \in Y, F_\mu(y) = \mu(-\infty, y] \leq a\}$$

and

$$U = \bigcap \{(-\infty, y] : y \in Y, F_\mu(y) = \mu(-\infty, y] \geq b\}$$

are monotone unions and intersections of intervals, and thus are intervals themselves, and $\mu(L) \leq a < b \leq \mu(U)$, so that $I = U \setminus L$ is also an interval with positive measure $\mu(I) = \mu(U) - \mu(L) \geq b - a > 0$.

We note, however, that $U$ contains at most one element $y'$ such that $\mu(-\infty, y') \geq b$; if there were two, $y_1, y_2 \in U$, then $y_1 < y_2$ (without loss of generality) and then $U \subseteq (-\infty, y_1]$ cannot contain $y_2$, a contradiction. Thus, for every $x \in U \setminus \{y'\}$, $F_\mu(x) = \mu(-\infty, x] < b$. However, by assumption, no element $x$ satisfies $a < F_\mu(x) < b$, so this implies further that $F_\mu(x) \leq a$, so that $x \in L$. Consequently, $I = U \setminus L$ is a set containing at most one element, yet possessing positive measure. If $I = \{y'\}$ then $I$ is atomic; if $I = \emptyset$, then $\mu(\emptyset) > 0$ is absurd. We arrive at a contradiction: there must exist $x \in X$ such that $F_\mu(x) \in (a, b)$. \hfill \Box

Following the general construction of Stefan Heinemann and Oliver Schmitt [6], we require some analogue to the metric structure utilized; we resolve this by utilizing a pseudometric, the measure of the interval between two points. However, we must earn permission to utilize a pseudometric in place of a metric by strengthening the aperiodicity condition.

**Definition 5.** We say $T$ is $n$-nigh aperiodic if $\mu \{x : \mu \langle x, T^n x \rangle = 0\} = 0$, and that $T$ is nigh aperiodic if it is $n$-nigh aperiodic for every $n \geq 1$.

We define nigh aperiodicity in this way to highlight intervals of the form $\langle x, T^n x \rangle$ which will be key to our proof of Rokhlin’s lemma, but the measure
therein has a much more familiar form in terms of the distribution function $F_{\mu}$.
In particular,

$$
\mu(a, b) = \begin{cases} 
\mu[a, b] = F_{\mu}(b) - F_{\mu}(a) & \text{ if } a < b \\
\mu[b, a] = F_{\mu}(a) - F_{\mu}(b) & \text{ if } b < a \\
\mu\{a\} = \mu\{b\} = 0 & \text{ if } a = b
\end{cases}
= |F_{\mu}(b) - F_{\mu}(a)|.
$$

In particular this shows that $\{x : \mu(x, T^nx) = 0\}$ is a measurable set, since it is the preimage of 0 for the map $|F_{\mu}(T^nx) - F_{\mu}(x)|$, and thus nigh aperiodicity is well-defined. This also permits us to view the nigh aperiodicity condition as a relation between the distribution function $F_{\mu}$ and its orbit in $L^1(X)$ under the operator $U_T$ defined by $U_Tf(x) = f(T(x))$: $T$ is nigh aperiodic iff for every $n$,

$$
\mu\{x \in X : F_{\mu}(x) = U^n_TF_{\mu}(x)\} = 0.
$$

**Proposition 6.** Suppose $(X, \mathcal{F}(\prec), \mu, T)$ is a nigh aperiodic shuffleable system.

For every $n$ there exists $E \in \mathcal{F}$ of positive measure such that $\mu(E \cap T^{-n}E) = 0$.

Further, for any set $U$ of positive $\mu$-measure such that $\mu(U \triangle T^{-n}U) = 0$, and $\nu = \mu|_U$, there exists a set $E$ of positive $\nu$-measure such that $\nu(E \cap T^{-n}E) = 0$, so that $E' = U \cap E$ is a positive $\mu$-measure set satisfying $\mu(E' \cap T^{-n}E') = 0$.

**Proof.** By Proposition 4, we know that every open subset in $[0, 1]$ contains some value $r$ which is the value of $F_{\mu}(x_r) = \mu(-\infty, x_r)$ for some $x_r \in X$, so that we can find a countable dense subset $S$ of $[0, 1]$ of values in the range of $F_{\mu}$. For each $r \in S$, we select one such $x_r$, and collect them together in the countable set $Q \subseteq X$.

We now define a collection of intervals with endpoints in $Q$ which have measure no greater than some fixed $\varepsilon > 0$. Let us define

$$
\mathcal{R} = \{(-\infty, b], [a, b], [a, \infty) : a, b \in Q\}
$$

and from this we refine to

$$
\mathcal{R}_{\varepsilon} = \{I \in \mathcal{R} : \mu(I) < \varepsilon\}.
$$

$\mathcal{R}$ is in bijection with the disjoint union $Q \sqcup Q^2 \sqcup Q$, and thus it is countable, and $\mathcal{R}_\varepsilon$ inherits countability also.

For every $\varepsilon > 0$, we know $\mathcal{R}_\varepsilon$ covers $X$; for any $x \in X$ with $0 < F_{\mu}(x) < 1$, we know there are $a, b \in Q$ such that

$$
F_{\mu}(x) - \varepsilon \frac{2}{2} < F_{\mu}(a) < F_{\mu}(x) < F_{\mu}(b) < F_{\mu}(x) + \varepsilon \frac{2}{2}.
$$

Thus, in particular,

$$
\mu[a, b] = F_{\mu}(b) - F_{\mu}(a) < \varepsilon.
$$

By monotonicity, since $F_{\mu}(a) < F_{\mu}(x) < F_{\mu}(b)$, $x \in [a, b]$. On the other hand, if $F_{\mu}(x) = \mu(-\infty, x] = 0$, then there exists $b$ such that $0 < F_{\mu}(b) = \mu(-\infty, b] < \varepsilon$,
and so again \( x \in (\infty, b) \), and if \( F_\mu(x) = \mu(-\infty, x) = 1 \), then there exists \( a \) such that \( 1 > F_\mu(a) = \mu(-\infty, a) > 1 - \epsilon \), so that \( x \in [a, \infty) \) and \( \mu[a, \infty) < \epsilon \). In any case, there exists some interval in \( \mathcal{R}_\epsilon \) containing \( x \), and thus the collection of “small” intervals \( \mathcal{R}_\epsilon \) covers all of \( X \).

Suppose there is some \( n \) such that for every \( A \in \mathcal{F} \) with positive measure, \( \mu(A \cap T^{-n} A) > 0 \). Consider \( A \setminus T^{-n} A \); then \( \mu([A \setminus T^{-n} A] \cap T^{-n} [A \setminus T^{-n} A]) = \mu(\emptyset) = 0 \), which implies that \( \mu(A \setminus T^{-n} A) = \mu(T^{-n} A \setminus A) = 0 \).

Let us fix \( \epsilon > 0 \) and apply these observations to the collection \( \mathcal{R}_\epsilon \). Letting \( I \in \mathcal{R}_\epsilon \), we know that \( \mu(T^{-n} I \setminus I) = 0 \). Thus, for \( \mu \)-a.e. \( x \in T^{-n} I \), we also have \( x \in I \). Since \( I \) and \( \langle x, T^n x \rangle \) are both closed intervals, we have \( \langle x, T^n x \rangle \subseteq I \), so that \( \mu(x, T^n x) \leq \mu(I) < \epsilon \) for \( \mu \)-a.e. \( x \in T^{-n} I \). That is to say,

\[
\mu \{ x \in X : T^n x \in I, \mu(x, T^n x) \geq \epsilon \} = 0.
\]

Since \( \mathcal{R}_\epsilon \) covers \( X \), so also does \( \{ T^{-n} I : I \in \mathcal{R}_\epsilon \} \), and we can write

\[
\mu \{ x \in X : \mu(x, T^n x) \geq \epsilon \} = \mu \left( \bigcup_{I \in \mathcal{R}_\epsilon} \{ x \in X : T^n x \in I, \mu(x, T^n x) \geq \epsilon \} \right) \leq \sum_{I \in \mathcal{R}_\epsilon} \mu \{ x \in X : T^n x \in I, \mu(x, T^n x) \geq \epsilon \} = 0.
\]

Given that \( \epsilon \) was arbitrary, by letting this go to 0 we find that in general it must be that

\[
\mu \{ x \in X : \mu(x, T^n x) > 0 \} = 0.
\]

However, this implies that \( \mu(x, T^n x) = 0 \) for \( \mu \)-a.e. \( x \), contradicting our assumption that the system was n-high aperiodic. Therefore, our supposition that \( \mu(A \cap T^{-n} A) \neq 0 \) for every positive measure \( A \) must be false, and consequently there must exist some set \( F \in \mathcal{F} \) such that \( \mu(E \cap T^{-n} E) = 0 \), proving the first claim.

We now begin the second claim. Since \( \mu(U \cup T^{-1} U) = 0 \) it follows that \( \nu = \mu|_U \) is a \( T \)-preserved measure. Suppose that for every \( A \in \mathcal{F} \), \( \nu(A \cap T^{-n} A) > 0 \). By the same reasoning, \( \nu(T^{-n} A \setminus A) = 0 \), and thus letting \( I \in \mathcal{R}_\epsilon \), for \( \nu \)-a.e. \( x \in I, T^n x \in I \), and so \( \mu(x, T^n x) < \epsilon \), so that

\[
\nu \{ x \in X : T^n x \in I, \mu(x, T^n x) \geq \epsilon \} = 0.
\]

The same reasoning as before allows us to conclude that \( \nu \{ x \in X : \mu(x, T^n x) \geq \epsilon \} = 0 \) for every \( \epsilon > 0 \), and thus \( \nu \{ x \in X : \mu(x, T^n x) > 0 \} = 0 \). Then the comple-
ment of this set must have $\nu$-measure 1, so that we have
\[
\nu\{x \in X : \mu(x, T^n x) = 0\} = 1
\]
\[
\frac{\mu(U \cap \{x \in X : \mu(x, T^n x) = 0\})}{\mu(U)} = 1
\]
\[
\mu(U \cap \{x \in X : \mu(x, T^n x) = 0\}) = \mu(U) > 0
\]
\[
\mu\{x \in X : \mu(x, T^n x) = 0\} > 0.
\]
This however contradicts our claim that $(X, F, \mu, T)$ was nigh aperiodic. Thus, there must exist some set $E$ with positive $\nu$-measure such that
\[
\nu(E \cap T^{-n} E) = \frac{\mu(U \cap (E \cap T^{-n} E))}{\mu(U)} = \frac{\mu([U \cap E] \cap T^{-n}[U \cap E])}{\mu(U)} = 0
\]
and therefore it follows that for $E' = U \cap E$, $\mu(E' \cap T^{-n} E') = 0$. The set $E'$ must also have positive $\mu$-measure since
\[
0 < \nu(E) = \frac{\mu(U \cap E)}{\mu(U)} = \frac{\mu(E')}{\mu(U)}
\]
so that $E'$ satisfies the second claim. □

**Definition 7.** If $E \in \mathcal{F}$ is a set of positive measure such that $\mu(T^{-m} E \cap E) = 0$ for all $m = 1, \cdots, n - 1$, then we say that $E$ induces an $n$-chain. □

**Proposition 8.** Let $(X, \mathcal{F}(<), \mu, T)$ be a nigh aperiodic shuffleable system. For any $n \in \mathbb{N}_+$, and any positive measure set $G \in \mathcal{F}$, there exists $F \subseteq G$ which induces an $n$-chain.

**Proof.** The proof will proceed by induction. The case $n = 1$ is trivial: let $F = G$. Suppose then that $G$ induces an $n - 1$-chain, but does not contain any subset inducing an $n$-chain. By our observation from the previous proposition, if every positive measure subset $B \subseteq G$ satisfies $\mu(B \cap T^{-(n-1)} B) > 0$, then $\mu(G \setminus T^{-(n-1)} G) = 0$, so that $G$ and $T^{-(n-1)} G$ are identical up to a set of measure 0. For any $j \in \{0, \cdots, n - 2\}$, $\mu(T^{-j} G \triangle T^{-j-k(n-1)} G) = 0$. Thus,
\[
\mu\left(\bigcup_{j=0}^{\infty} T^{-j} G \setminus \bigcup_{j=0}^{n-2} T^{-j} G\right) = 0.
\]
Then, we restrict $\mu$ to the set $G^* = \bigcup_{j=0}^{\infty} T^{-j} G$. By the Poincaré recurrence theorem, almost every point in $G$ recurs, and thus is in some set $T^{-j} G$, so that
\[
\mu(G^* \triangle T^{-1} G^*) = \mu\left(G \setminus \bigcup_{j=1}^{\infty} T^{-j} G\right) = 0
\]
The set $G^*$ is measure theoretically equivalent to $\bigcup_{j=0}^{n-2} T^{-j} G$, and applying the second claim of Proposition 6 to the new system $(X, \mathcal{F}, \mu|_{G^*}, T)$, we find a
positive $\mu|_{G^*}$-measure subset $F' \subseteq G^*$ such that $\mu|_{G^*}(F' \cap T^{-(n-1)}F') = 0$. In particular, there exists a $j \in \{0, 1, \cdots, n-2\}$ such that $\mu|_{G^*}(F' \cap T^{-j}G) > 0$; we translate this positive measure part of $F'$ back into $T^{-(n-1)}G$ and thus $G$ itself by taking $(n-j) + 1$ preimages, and obtain the set

$$F \overset{\text{def}}{=} G \cap T^{-(n-j)-1}(F' \cap T^{-j}G) = T^{-(n-j)-2}(F') \cap G \cap T^{-(n-1)}G.$$  

We know $\mu|_{G^*}(F) = \mu|_{G^*}(T^{-j}(F)) = \mu|_{G^*}(F' \cap T^{-j}G) > 0$ and $\mu|_{G^*}(F \cap T^{-(n-1)}F) = 0$. Since $G$ generates an $n-1$-chain, $G, T^{-1}G, \ldots, T^{-(n-2)}G$ are all disjoint equal measure components of $G^*$, and so for any $A \in F$, 

$$\mu|_{G^*}(A) = \frac{\mu(A \cap G^*)}{\mu(G^*)} = \frac{\mu(A \cap G^*)}{(n-1)\mu(G)},$$

which implies that $\mu(F) > 0$ and $\mu(F \cap T^{-(n-1)}F) = 0$, since the corresponding claims in $\mu|_{G^*}$ hold also. However, this implies that $F$ does in fact induce an $n$-chain, contrary to our previous assumption. Therefore, if $G$ is a set of positive measure inducing an $n-1$-chain, then $G$ contains a subset $F \subseteq G$ of positive measure inducing an $n$-chain, and consequently $(X, \mathcal{F}(<), \mu, T)$ contains $n$-chains of all possible lengths $n$.

**Theorem 9** (Rokhlin lemma for shufleable systems). Let $(X, \mathcal{F}(<), \mu, T)$ be a nigh aperiodic shufleable system. Then for any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists a set $E$ inducing an $n$-chain such that

$$\mu \left( \bigcup_{j=0}^{n-1} T^{-j}E \right) > 1 - \varepsilon.$$

We remind the reader that we build on the basic structure of the proof of the Rokhlin Lemma offered by Heinemann and Schmitt [6]; their proof utilized the structure of a separable metric space with aperiodic dynamics to obtain the claim that $n$-chains exist for every $n$, as we have with Proposition 8, but beyond this the context is no longer necessary.

**Proof.** Let $m \in \mathbb{N}$ be such that $m \geq n$ and $1/m < \varepsilon/(n - 1)$. We know by Proposition 8 that there exist $m$-chains in $\mathcal{F}$. We may equip the collection of sets in $\mathcal{F}$ which induce $m$-chains, $\mathcal{C}_m$, with the partial order

$$F \prec \mu F' \iff (F \subseteq F') \land (\mu(F) < \mu(F')).$$

Suppose that $C$ is a chain in $\mathcal{C}_m$ in the sense of Zorn’s lemma; we wish to show that $\mathcal{C}_m$ contains an upper bound for the Zorn chain $C$. If $C$ contains an element $F^*$ with maximal measure for $C$, then we know that $\mu(F) < \mu(F^*)$ for all $F \in C$, so that $F^*$ is a unique upper bound for $C$. Suppose instead that $\sup_{F \in C} \mu(F) = M$, but $\mu(F) < M$ for every $F \in C$. For every $k \in \mathbb{N}$, we select some $F_k$ such that $\mu(F_k) > M - 1/k$, and let $B = \bigcup_{k \in \mathbb{N}} F_k$. This is a countable union and therefore it resides in $\mathcal{F}$, and furthermore, for any $F \in C$, there
Suppose that it were not; then the complement \( Y \) of \( m \) and which satisfy the recursive formula

\[
F_m \cap T^{-j}F \subseteq \bigcup_{j=0}^{\infty} T^{-j}F \subseteq T^{-k} \left( G \cap \bigcup_{j=0}^{\infty} T^{-j}F \right) = \emptyset.
\]

It follows that \( \mu([F \cup G] \cap T^{-k}[F \cup G]) = 0 \), and thus \( F \cup G \) induces an \( m \)-chain. Since \( \mu(G) > 0 \), we have \( \mu(F) < \mu(F \cup G) \), and thus \( F < \mu F \cup G \), in contradiction of the assumption that \( F \) was maximal according to the partial order \( \prec \mu \). Thus, it follows that \( \mu \left( \bigcup_{j=0}^{\infty} T^{-j}F \right) = 1 \). It follows that almost every point in \( X \) recurs to \( F \) infinitely often.

We next define the sets \( F_k = T^{-k}F \setminus \bigcup_{j=0}^{k-1} T^{-j}F \), so that \( F_k \) is the set of all \( x \in X \) such that \( T^{k}x \in F \) with \( k \) the least natural number achieving this, and which satisfy the recursive formula

\[
F_{k+1} = T^{-k}F \setminus \bigcup_{j=0}^{k} T^{-j}F = \left( T^{-k-1}F \setminus T^{-1} \left( \bigcup_{j=0}^{k-1} T^{-j}F \right) \right) \setminus F
\]

Utilizing their original definition, it is clear that the sets \( F_k \) are pairwise disjoint, and that \( \bigcup_{j=0}^{N} F_j = \bigcup_{j=0}^{N} T^{-j}F \), so that \( \mu \left( \bigcup_{j=0}^{\infty} F_j \right) = 1 \).

From this, we can tell that since \( F_{k+1} \subseteq T^{-1}F_k \), more generally we have \( F_{k+j} \subseteq T^{-j}F_k \); and since \( T^{-1}F_k \subseteq F_{k+1} \cup F \), so also

\[
T^{-j}F_k \subseteq T^{-j+1}(F_{k+1} \cup F) \subseteq T^{-j+2}(F_{k+2} \cup T^{-1}F \cup F) \subseteq \cdots \subseteq F_{k+j} \cup \bigcup_{l=0}^{j-1} T^{-l}F.
\]
It follows that
\[ F^{k+j} \subseteq T^{-j}F^k \subseteq F^{k+j} \cup \bigcup_{l=0}^{j-1} T^{-l}F = F^{k+j} \cup \bigcup_{l=0}^{j-1} F^l. \]  

(1)

We now assert that the set
\[ E = \bigcup_{k=1}^{\infty} F^{kn-1} \]
induces an \( n \)-chain. Suppose that for some \( j \in \{1, \cdots, n-1\} \), we have
\[ x \in E \cap T^{-j}E. \]

Then in particular, it must be that for some \( p, q \in \mathbb{N}, x \in F^{pn-1} \cap T^{-j}(F^{qn-1}). \)

We know \( F^{pn-1} \) is disjoint from \( F^{qn-1+j} \), since \( pn-1 \neq qn-1+j \) (which would imply \( j = n(p-q) \), but \( j \in \{1, 2, \cdots, n-1\} \) which is disjoint from \( n\mathbb{Z} \)) and if \( k_1 \neq k_2 \) then \( F^{k_1} \cap F^{k_2} = \emptyset \) by construction. Since these two sets are disjoint, (1) tells us that
\[ x \in F^{pn-1} \cap T^{-j}(F^{qn-1}) \subseteq F^{pn-1} \cap \left( F^{j+q-1} \cup \bigcup_{l=0}^{j-1} F^l \right) = F^{pn-1} \cap \bigcup_{l=0}^{j-1} F^l. \]

By the disjointness of the different \( F^k \), and that \( j \leq n-1 \), we know that \( pn-1 \in \{0, \cdots, j-1\} \subseteq \{0, \cdots, n-2\} \). But this would imply \( pn \in \{1, \cdots, n-1\} \), which is disjoint from \( n\mathbb{Z} \), a contradiction. Thus, such an \( x \) cannot exist, and so \( E \) must induce an \( n \)-chain.

Finally, we estimate the measure of the \( n \)-chain induced by \( E \), by using (1) together with the fact that \( F \) induces an \( m \)-chain for \( m \geq n \), and the fact that \( \mu(F) \leq 1/m < \varepsilon/(n-1) \), and find that it satisfies the theorem:
\[
\begin{align*}
\mu\left( \bigcup_{j=0}^{n-1} T^{-j}E \right) &= \mu\left( \bigcup_{j=0}^{n-1} \bigcup_{k=1}^{\infty} T^{-j}F^{kn-1} \right) \geq \mu\left( \bigcup_{j=0}^{n-1} \bigcup_{k=1}^{\infty} F^{kn-1+j} \right) \\
&= \mu\left( \bigcup_{i=n-1}^{\infty} F^i \right) = 1 - \mu\left( \bigcup_{i=0}^{n-2} F^i \right) = 1 - \sum_{i=0}^{n-2} \mu(T^{-i}F) \\
&= 1 - (n-1)\mu(F) \geq 1 - (n-1)\frac{\varepsilon}{n-1} = 1 - \varepsilon.
\end{align*}
\]

Thus, for arbitrary \( n \in \mathbb{N} \) and \( \varepsilon > 0 \), there exists a set \( E \) inducing an \( n \)-chain which covers all but a subset of measure \( \varepsilon \) of the space \( X \).

3 Connections

The contexts of the separable metric space of the Rokhlin lemma of Heinemann and Schmitt [6] and of our version for shuffleable space are not so dissimilar as
they may originally seem; a system on a separable metric space equipped also with completeness, a non-atomic measure, and without isolated points must be conjugate to a shuffleable system. The connection develops through the following proposition, originally that of K. Kuratowski [7, p. 440] but refined into a shape more useful for our purposes by H.L. Royden [9].

**Proposition 10.** [9, Prop. 15.8] Let $X$ be a complete separable metric space without isolated points, and let $I$ be the set of irrational points in the unit interval. Then there is a one-to-one continuous map $\varphi$ of $I$ onto $X$ such that $\varphi(O)$ is an $F_\sigma$ set in $X$ for each open subset $O$ of $I$.

**Proposition 11.** Let $\Sigma = (X, \mathcal{B}(X), \mu, T)$ be a measure-preserving dynamical system with $X$ a complete separable metric space without isolated points, $\mathcal{B}(X)$ the Borel $\sigma$-algebra on $X$, $\mu$ a non-atomic probability measure, and $T$ surjective. Then $\Sigma$ is measure-theoretically conjugate to a shuffleable system on the measurable space $(I, \mathcal{B}(I))$.

**Proof.** We have assumed all of the conditions of Proposition 10, so there exists a bijective map $\varphi : I \to X$ satisfying all of its conclusions. In particular, since $\varphi$ is continuous, it is also measurable according to the Borel $\sigma$-algebras $\mathcal{B}(X)$ and $\mathcal{B}(I)$, and since the image of any open set $O$ is a $F_\sigma$ set, we also know that it is forward measurable (if $E \in \mathcal{B}(I)$, then $\varphi(E) \in \mathcal{B}(X)$). Thus, both $\varphi$ and $\varphi^{-1}$ are measurable.

We then define a measure on $I$ by $\mu_0(E) = \mu(\varphi(E))$ and dynamics on $I$ by $T_0(x) = \varphi^{-1}(T(\varphi(x)))$. We know $T_0$ preserves the measure $\mu_0$ since $T$ originally did:

$$\mu_0(T_0^{-1}E) = \mu(\varphi\varphi^{-1}T^{-1}\varphi(E)) = \mu(T^{-1}\varphi(E)) = \mu(\varphi(E)) = \mu_0(E)$$

Thus, it follows that $(I, \mathcal{B}(I), \mu_0, T_0)$ forms a measure-preserving dynamical system. It admits the natural total order $<$, which generates its topology and also $\mathcal{B}(I)$, and it is conjugate through $\varphi$ to $(X, \mathcal{B}(X), \mu, T)$ which is non-atomic so that $\mu_0$ must also be non-atomic also. Further, it must also be surjective since $T$ is surjective: for any $y \in I$, there must be $x \in X$ such that $T(x) = \varphi(y)$, and since $\varphi$ is bijective, there exists $z \in I$ such that $\varphi(z) = x$, so that $\varphi^{-1}T\varphi(z) = T_0(z) = y$. Thus, $(I, \mathcal{B}(I), \mu_0, T_0)$ is in fact a shuffleable system which is conjugate to $\Sigma$. \hfill \Box

As is, it is not clear that this system is necessarily nigh aperiodic. Since Proposition 10 is agnostic to measure theoretic concerns, and its structure does not immediately lend itself to any obvious criteria which can guarantee this.

In 2016, Avila and Candela proved a version of the Rokhlin lemma for free measure-preserving actions on atomless standard probability space. Their version of the Rokhlin lemma connects to the shuffleable Rokhlin lemma in the cyclic subcase. To see this connection, a few definitions are necessary for context.

**Definition 12.** Two probability spaces $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ are **isomorphic** if there exists a bimeasurable bijection $f : X \to Y$ such that both $f$ and $f^{-1}$
are measure-preserving maps. If instead there exist subsets $E_X$ and $E_Y$ with $\mu(E_X) = \nu(E_Y) = 0$ such that $X \setminus E_X$ is isomorphic to $Y \setminus E_Y$, then we say $(X, F, \mu)$ and $(Y, G, \nu)$ are isomorphic mod 0.

Definition 13. [10, Definition 2.1] $(X, L, \mu)$ is a standard probability space$^2$ if $X$ is isomorphic mod 0 to the interval $[0, 1]$ with the standard Lebesgue measure $m$.

Theorem 14. [2, Theorem 1.2] Let $\varepsilon > 0$ and let $n \in \mathbb{N}_d$ be a $d$-tuple of positive integers. Then for every measure-preserving action $f$ of the monoid of $d$-tuples of nonnegative integers $\mathbb{N}_0^d$ on a standard probability space such that for any $k, l \in \mathbb{N}_0^d$, $k \neq l$,

$$\mu(\{x \in X : f_k(x) = f_l(x)\}) = 0$$

there exists an $n$-tower for $f$ of measure at least $1 - \varepsilon$.

When we assume $d = 1$, this reduces to a measure-preserving transform of an atomless standard probability space, the more traditional form of the Rokhlin lemma, and the condition (2) reduces to aperiodicity.

The context of a standard probability space is very similar to that of a shuffleable system, but not identical; the standard probability space must be on a complete $\sigma$-algebra, while shuffleable systems are equipped with a Borel $\sigma$-algebra instead, which is not complete. We therefore may extend our definition of shuffleable spaces to $\sigma$-algebras which are complete.

Definition 15. We say $\Sigma = (X, L(\prec), \mu, T)$ is a completed shuffleable system if it is a probability measure-preserving dynamical system where $L$ is the completion of $F(\prec)$ according to $\mu$, $T$ is a measurable endomorphism, and $\mu$ is non-atomic.

Proposition 16. If $(X, L(X), \mu, T)$ is an aperiodic measure-preserving dynamical system on a standard probability space, then there exists a full-measure subset $Y \subseteq X$ on which $(Y, L(Y), \mu|_Y, T)$ is a nigh aperiodic completed shuffleable system.

Proof. Since $(X, L(X), \mu)$ is a standard probability space, there exists an isomorphism mod 0 from $X$ to $[0, 1]$, $f : X \setminus E_X \rightarrow [0, 1] \setminus E_I$. We can assume without loss of generality that $T(X \setminus E_X) = X \setminus E_X$; if necessary, we substitute $E'_X = \bigcup_{j=0}^{\infty} T^{-j} E_X$ for $E_X$, and let $E'_I = [0, 1] \setminus f(X \setminus E'_X)$.

The isomorphism mod 0 between $X$ and $[0, 1]$ then induces a total order $\prec_Y$ on $Y = X \setminus E_X$ via $f$, by

$$a \prec_Y b \iff f(a) < f(b).$$

By definition, $f$ is monotone according to the orders $\prec_Y$ and $\prec$. Consequently, the intervals in $Y$ according to $\prec_Y$ are measurable since they are the preimages

$^2$It is sometimes permitted that $X$ also include an atomic component, but this is explicitly excluded by Avila and Candela’s theorem.
of intervals in $[0, 1] \setminus E_I$ through $f$, a bimeasurable map, and thus $\prec_Y$ generates $\mathcal{F}(\prec)$. In particular, $f(\langle x, T^n(x) \rangle) = \langle f(x), f(T^n(x)) \rangle$, so $\mu(f(\langle x, T^n(x) \rangle)) = m(f(x), f(T^n(x))) = |f(T^n(x)) - f(x)|$. Since $f$ is bijective, $f(T^n(x)) = f(x)$ iff $T^n(x) = x$, and since $T$ is aperiodic, this only occurs on a set of zero measure. Therefore, in $Y$ according to the total order $\prec_Y$,

$$\mu \{ x \in Y : \mu(\langle x, T^n(x) \rangle) = 0 \} = 0$$

and thus it follows that $T$ is nigh aperiodic on $Y$ also.

Our proof of Rokhlin’s lemma for shuffleable systems applies equally well to completed shuffleable systems; applying this version of Rokhlin’s lemma to $Y$ yields an alternative proof of Rokhlin’s lemma for aperiodic dynamics on standard probability spaces.

References

[1] A. Avila and J. Bochi, A generic $C^1$ map has no absolutely continuous invariant probability measure, Nonlinearity, 19 (2006), arXiv:math/0605729.

[2] A. Avila and P. Candela, Towers for commuting endomorphisms, and combinatorial applications, Ann. Inst. Fourier (Grenoble), 66 (2016), 1529–1544, arXiv:1507.07010.

[3] J. Brown, Ergodic theory and topological dynamics, Academic Press, 1976.

[4] D. Bayer and P. Diaconis, Trailing the Dovetail Shuffle to its Lair, Ann. Appl. Probab., 2 (1992), 294–313.

[5] P. Halmos, Lectures on Ergodic Theory, Amer. Math. Soc., 1956.

[6] S-M. Heinemann and O. Schmitt, Rokhlin’s Lemma for Non-Invertible Maps, Dynam. Systems Appl., 10 (2000), 201–214.

[7] K. Kuratowski, Topology, Vol. I. Academic Press, 1966.

[8] V. Rokhlin, Generators in ergodic theory, Vestnik Leningrad Gos. Univ., 1 (1963), 26-32.

[9] H. Royden, Real Analysis, Third edition. Macmillan New York, 1988.

[10] I. Sinai, Topics in Ergodic Theory (PMS-44), Princeton University Press, Princeton, 2017.