Decoherence and Irreversibility: the role of the reservoir effective Hilbert space size

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We show that decoherence is determined by the effective Hilbert space size, and demonstrate that a few degrees of freedom system can simulate a “N” degrees of freedom environment if they have the same effective Hilbert space. The effective Hilbert space size of the environment also determines its long time efficiency as generator of irreversible coherence loss.

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I. INTRODUCTION

There are many situations where the direct application of the Schroedinger equation leads to unsatisfactory results, a well known example is dissipation. Dissipation can be phenomenologically modeled by two main approaches, the complex Hamiltonian [1, 2] and open system [3, 4, 5]. It is also possible to quantize a non-Hamiltonian system [6], but this approach is similar to the open system one. The main idea is based on the fact that once we are dealing with a physical phenomenon there are many particles involved. Due to the difficulty of dealing with a large number of particles, it is common to treat the system of interest as an open system, i.e., coupled with a reservoir, which we may call the environment. It is a common belief that the environment should be modeled by a system with infinity degrees of freedom.

While dissipation is a classical phenomena, in a real quantum system we have also another effect, decoherence, which is an independent phenomena [7]. The main idea of decoherence can be summarized as Omnes’s definition: “Decoherence is a dynamic effect through which the states of the environment associated with different collective states become rapidly orthogonal. It comes from a loss of local phase correlation between the corresponding wave functions of the environment, which is due to the interaction between the collective system and the environment, also responsible for dissipation. It depends essentially upon the fact that the environment has a very large number of degrees of freedom.” See also [9, 10]. This loss of coherence has been experimentally observed [11].

For many practical situations the environment has been chosen as a collection of harmonic oscillators [3, 8, 9, 10, 11, 12] and its phenomenological success has been also proved in some cases [13, 14, 15]. One of the problems of quantum open systems is the difficulty of modeling the environment by first principles and therefore its basis is phenomenological. There are also many technical difficulties to obtain the so called master equations and there is not a unique receipt [16, 17, 18]. From the theoretical point of view there is a strong debate on the necessity of using an environment in order to recover classical dynamics from Quantum Mechanics. This is a subject of the classical limit problem [20, 21, 22, 23]. It was demonstrated that diffusion can produce only quantum coherence attenuation, being not able to eliminate it [12], thus we are forced to include a finite experimental resolution to reach the classical limit [25].

Decoherence of a simple system can be quantified in terms of entropy, or linear entropy [12, 24]. If the linear entropy of a single system is zero we can say that it has not lost quantum coherence, while the system energy has no direct relation with it. Usually the term decoherence is applied to an irreversible loss of coherence of a quantum system, but can we always measure the reversibility? Is it possible, for any practical or philosophic reason, to use a few degrees of freedom system as an environment? The main purpose of this work is to investigate these questions. As a model, we use the quartic oscillator coupled with a phase reservoir. The use of this model has two practical reasons, it has analytical solutions and its classical limit is characterized by a unique time [26, 27], while in general we have many
time scales \[28\]. Also this model has experimental interest: it is used to study Bose-Einstein condensate \[29\] and Kerr like medium \[30\].

II. PHASE RESERVOIR MODEL

Let us consider a general phase reservoir Hamiltonian

\[ H = H_1 + H_2 + H_{12}, \]

where \( H_1 \) refers to the system of interest, \( H_2 \) to the reservoir and \( H_{12} \) to their interaction. The phase reservoir is characterized by the commutation relations

\[ [H, H_1] = [H, H_2] = [H, H_{12}] = 0. \]

Taking \( |\phi_n\rangle \) and \( |\psi_k\rangle \) as eigenvectors of \( H_1 \) and \( H_2 \), respectively, relations \[2\] permit us to write

\[ H |\phi_n\rangle |\psi_k\rangle = E_{n,k} |\phi_n\rangle |\psi_k\rangle. \]

If \( H_2 \) concerns \( M \) independent subsystems, we may assume

\[ H_2 = \sum_{l=1}^{M} H_{2,l}, \quad H_{12} = \sum_{l=1}^{M} H_{12,l}, \]

with \( [H_{2,l}, H_{2,l}^\dagger] = [H_{12,l}, H_{12,l}^\dagger] = 0 \). The eigenvectors and eigenvalues will be given by \( H_2^\dagger |\psi_{l,r}\rangle = E_{l,r}^\dagger |\psi_{l,r}\rangle \) and \( H_{12,l} |\phi_n\rangle |\psi_{l,r}\rangle = E_{n,l,r} |\phi_n\rangle |\psi_{l,r}\rangle \). In order to quantify the entanglement between the system of interest and the reservoir and their interaction, the linear entropy is calculated as \( H \)

\[ H = \sum_{l=1}^{M} |\Delta_{l}\rangle \langle \Delta_{l}|. \]

Taking \( |\phi_n\rangle \) and \( |\psi_{l,r}\rangle \) as eigenvectors of \( H_{12} \) and \( H_1 \), respectively, relations \[2\] permit us to write

\[ H_{12} |\phi_n\rangle |\psi_{l,r}\rangle = E_{n,l,r} |\phi_n\rangle |\psi_{l,r}\rangle. \]

If \( H_2 \) concerns \( M \) independent subsystems, we may assume

\[ H_2 = \sum_{l=1}^{M} H_{2,l}, \quad H_{12} = \sum_{l=1}^{M} H_{12,l}, \]

where \( \Delta_{l} \) refers to the system of interest, \( \Delta_{l} \) to the reservoir and \( \Delta_{12} \) to their interaction. The phase reservoir is characterized by the commutation relations

\[ [H, H_1] = [H, H_2] = [H, H_{12}] = 0. \]

Taking \( |\phi_n\rangle \) and \( |\psi_{l,r}\rangle \) as eigenvectors of \( H_1 \) and \( H_2 \), respectively, relations \[2\] permit us to write

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\[ \rho(0) = \sum_{v,w} B_{v,w} |\phi_v\rangle \langle \phi_w| \prod_{l=1}^{M} \left\{ \sum_{r,s} A_{r,s}^l |\psi_{l,r}\rangle \langle \psi_{s,l}| \right\}, \]

we find

\[ \delta(t) = 1 - \prod_{l=1}^{N} \left\{ \sum_{r,s} A_{r,s}^l \sum_{v,w} A_{r,s}^l B_{v,w} B_{v,w} \right\} \times \exp \left\{ -\frac{it}{\hbar} \left( E_{v,r}^l - E_{w,r}^l - E_{v,s}^l + E_{w,s}^l \right) \right\}. \]

Notice that \( \delta(t) \) depends only on the interaction part of the whole Hamiltonian.

In order to analyze a specific case, we now assume that \( H_{12} = \sum_{l} \lambda_l (\hbar N_1)^x (\hbar N_2^y)^y \), where \( N_1 |\phi_v\rangle = v |\phi_v\rangle \) and \( N_2^y |\psi_{l,r}\rangle = r_l |\psi_{l,r}\rangle \) ( \( v \) and \( r_l \) are integer numbers), leading to

\[ \delta(t) = 1 - \prod_{l=1}^{M} \left\{ \sum_{r,s} A_{r,s}^l \sum_{v,w} A_{r,s}^l B_{v,w} B_{v,w} \right\} \times \exp \left\{ -\hbar^{-1} \left( \sum_{l} \lambda_l [(v^2 - w^2)(r_l^y - s_l^y)] \right) \right\}. \]

The linear entropy behavior is usually characterized by decoherence time \[24\] and revival time. Performing the expansion \( \delta(t) \approx \delta(0) + \delta_1 t + \delta_2 t^2 + O(t^3) \), it is easy to see that \( \delta_1 = 0 \). Thus, decoherence time is defined as \( t_D = 1/\sqrt{\delta_2} \) and, for the system of interest in initially pure state \( \delta(0) = 0 \), the explicit expression for this time reads

\[ t_D = \frac{1}{\hbar x + y - 1} 2 \sum_{l=1}^{M} \left( \lambda_l^2 \lambda_l^2 \right)^{1/2} \]

where \( (\Delta_1^2) \) is \( (N_1)^x \) operator variance and \( (\Delta_2^2) \) is \( (N_2^y)^y \) operator variance for \( \rho(0) \). Decoherence time \[28\] depends solely on the coupling Hamiltonian and the initial state of system plus environment.
Supposing that the coupling Hamiltonian frequency has a least multiple frequency $\Lambda$, it recovers its purity at the revival time

$$t_R = 2 \frac{s \pi}{\hbar x + y - 1} \Lambda.$$  

(9)

In this case, we may define another important characteristic time, the revival life time $\tau_R$, as the time interval when the recovered purity can be observed. By expanding $\delta(t)$ around $t_R$, we obtain $\tau_R = 2t_D$, which may be written as

$$\tau_R = \frac{\sqrt{2}}{\hbar x + y - 1} \Lambda \Delta_1 \Delta_2,$$

(10)

where $\Delta_2 = \sqrt{\sum_{i=1}^{M} (\Delta^2_i/k_i)}$ and $k_i = \Lambda/\lambda_i$. The expressions for $t_D$ and $\tau_R$ indicate that decoherence process can occur even for environments with few degrees of freedom: the crucial factor is the product $\Delta_1 \Delta_2$ compared to $1/ (\hbar x + y - 1)$. From equations (8) and (10) we can say that, in what concerns the main behavior of linear entropy of the system of interest, the reservoir composed of $M$ systems is phenomenologically equivalent to one degree of freedom with an effective coupling constant $\Lambda$ and state variance $(\Delta_2)^2$.

We get some insight by investigating the non-linear oscillator Hamiltonian:

$$H = \hbar \omega a^\dagger a + \hbar^2 g (a^\dagger a)^2 + \sum_{i=1}^{M} \hbar \Omega b^\dagger_i b_i + \sum_{j=1}^{M} \hbar^2 \lambda_a a^\dagger b_j^\dagger b_j,$$

(11)

where $a^\dagger (a)$ and $b^\dagger_i (b_i)$ are creation (annihilation) bosonic operators. This Hamiltonian fulfills the conditions discussed above with $x = y = 1$, and its eigenstates are the same as the ones of the harmonic oscillator. Considering that the states of all environmental bosons are initially the same, and defining the recurrence time $t_r$ as $\rho_1(t_r) = \rho_1(0)$, we get: if $\lambda = 0$, then $t_r = n \pi/\hbar g$; else, for $g/\lambda \in \mathbb{Q}$ we have $t_r = \frac{2n\pi}{\hbar \lambda m}$, where $\frac{n}{m} = 2^k s$ and $s, n, m \in \mathbb{N}$.

From now on we assume the system of interest in initial state

$$\rho(0) = \frac{1}{2} \{ |0\rangle \langle 0| + |1\rangle \langle 1| + |0\rangle \langle 1| + |1\rangle \langle 0| \} \otimes \prod_{k=0}^{M} r_k(0),$$

(12)

where $|0\rangle$ and $|1\rangle$ are Fock states and $r_k(0)$ concerns the state of the $k$-th environmental oscillator. Considering the reservoir in thermal equilibrium, i.e.,

$$r_k(0) = \left[ 1 - \exp \left( - \frac{\hbar \Omega}{k_B T} \right) \right] \sum_{n_k=0}^{\infty} \exp \left( - \frac{\hbar \Omega}{k_B T} n_k \right) |n_k\rangle \langle n_k|,$$

(13)

where the $|n_k\rangle$ are Fock states, $k_B$ is Boltzmann’s constant and $T$ the absolute temperature, we obtain

$$\delta(t) = \frac{1}{2} \left\{ 1 - \frac{\left[ 1 - \exp \left( - \frac{\hbar \Omega}{k_B T} \right) \right]^2}{1 + \exp \left( - \frac{2\hbar \Omega}{k_B T} \right) - 2 \exp \left( - \frac{\hbar \Omega}{k_B T} \cos(\lambda t) \right)} \right\}^M,$$

(14)

which explicitly shows that even for a large $M$ the system recovers purity at $t = t_R$. In Fig. 11, we show the linear entropy for this thermal initial condition [33]. The insert shows the linear entropy around $t_R$: as we can observe, the quantum behavior is attenuated as the total number of oscillators is increased, as expected, and also at higher temperatures. It is clear from this figure that a unique oscillator at high temperature and $M$ oscillators with low energy are phenomenologically equivalent as an environment (since $\Delta_2$ increases with the temperature). In order to stress the fact that (for fixed $\Delta_2$) the main features of the evolution of linear entropy is independent of the specific reservoir initial state, we plotted in Fig. 2 the evolution of $\delta(t)$ for the reservoir in Pegg-Barnett phase state [31] $\langle \varphi_m \rangle (r)$, defined as

$$|\varphi_m\rangle (r) = \frac{1}{\sqrt{r+1}} \sum_{n=0}^{r} e^{im \varphi_m} |n\rangle,$$

(15)
FIG. 1: It shows the linear entropy for thermal initial environmental state. Thick dotted line: $M = 201$, $\Delta_2 = 3.16$, $\lambda t_D \approx 0.032$. Dash-dotted line: $M = 1$, $\Delta_2 = 44.83$, $\lambda t_D \approx 0.032$. Full line: $M = 1$, $\Delta_2 = 6.61$, $\lambda t_D \approx 0.214$. Thin dotted line: $M = 15$, $\Delta_2 = 1.71$, $\lambda t_D \approx 0.214$. The insert shows linear entropy around $t_R$. The horizontal axis corresponds to $\lambda t$.

where $\varphi_m = \frac{2\pi m}{r+1}$, $m = 0, 1, ..., r$.

The Pegg-Barnett phase state has a uniform distribution over the lowest number states; also they form a truncated $r + 1$ dimensional Hilbert space, with $\Delta_2 = \frac{r (r + 2)}{12}$. That decoherence properties of the evolutions of Fig. (2) depend on the quantity $\Delta_1 \Delta_2$. For other initial states, we see, by the characteristics times calculated, that decoherence properties depend also on this quantity. Based on that, we define the effective Hilbert space size as the truncated Hilbert space size of the equivalent phase state that generates the same purity loss of the reservoir state considered:

$$H_s = \sqrt{1 + 12 (\Delta_2)^2}.$$  \hspace{1cm} (16)

If $y \neq 1$, we can use the same receipt: for each specific $N_y$ variance, there is an equivalent $r$ which defines $H_s$.

Now let us say a word about quantum uncertainty and reversibility. The above result shows that the revival mean time $\tau_R$ goes to zero in the limit $\Delta_2 \rightarrow \infty$. The important point that shall be stressed is that this limit can be obtained even for an environment with $M = 1$, i.e., a one system reservoir in infinity temperature. In fact we have not infinity temperature in laboratories. However, measuring this revival for high temperatures (large $(\Delta_2)^2$) demands high time precision. The time uncertainty is related to the system energy uncertainty by Heisenberg uncertainty principle as $\Delta t \Delta E \geq \hbar/2$. Considering our initial state (12), and as we are interested in revival dynamics, we get $\Delta t \geq \frac{1}{\lambda h}$. For $\Delta t \gg \tau_R$, or $\frac{\sqrt{\tau}}{\Delta_1 \Delta_2} \ll 1$, we would have a vanishing probability of observing the revival. The Heisenberg uncertainty relation gives a fundamental limitation in experimental resolution; in real experiments, time uncertainty is always greater [32]. This suggests a classical limit based on coarse-grained measurement [22, 23].

Another way for revival vanishing is to assume that $x$ or $y$ are not integers. In figure (3), we show the linear entropy evolution for initial phase state and a new coupling term: $H_{12} = \sum_{j=1}^{M} \hbar^2 \lambda a_i \sqrt{b_j b_j}$. The revivals are absent and partial purity recover is attenuated as we increase $\Delta_2$. Again we see that decoherence depends almost solely on $\lambda$ and $\Delta_2$.
FIG. 2: Same as figure 11 for phase state reservoir with $m = 0$.

FIG. 3: It shows the linear entropy for phase state reservoir with $m = 0$. Thick dotted line: $r = 10$, $M = 20$, $\lambda t_D \approx 0.49$. Dash-dotted line: $r = 289$, $M = 1$, $\lambda t_D \approx 0.49$. Full line: $r = 2$, $M = 2$, $\lambda t_D \approx 2.37$. Thin dotted line: $r = 8$, $M = 1$, $\lambda t_D \approx 2.37$. In all graphics we used $y = 1/2$. The horizontal axis corresponds to $\lambda t$.

III. CONCLUSION

We showed that decoherence is determined by the effective Hilbert space size characterized by the reservoir state variance and the coupling Hamiltonian. Thus the specific reservoir state is not relevant for linear entropy dynamics. Also we showed that a simple system can be an effective reservoir. Uncertain relations are invoked to demonstrate that recoherence cannot be measured in a poor resolution regime.
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