Thermodynamic Limit of the Nekrasov-type Formula for E-string Theory

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Introduction

Nekrasov Formula
(Instanton Partition Function) (N.A. Nekrasov ’04)

Seiberg-Witten description
(N. Seiberg-E. Witten ’94)

Low Energy Theories of 4D N=2 SYM

Nekrasov formula $\hbar \rightarrow 0$ → SW description
(N.A. Nekrasov ’04, N.A. Nekrasov-A. Okounkov ’03)
Introduction

Nekrasov-type Expression
(BPS partition function) (K. Sakai '12)

Seiberg-Witten description
A Low Energy Theory of 4D N=2 SYM
for E-string theory on $\mathbb{R}^4 \times T^2$
(O.J. Ganor '97)

$\hbar \to 0$

Nekrasov-type expression $\rightarrow$ SW description?
OUTLINE

1. E-string Theory and SW description
2. The Nekrasov-type Expression for E-string Theory
3. Reproducing SW description
4. Summary
E-string Theory

6D (1,0) theory with a tensor multiplet

Torus Compactification

4D N=2 U(1) Gauge Theory

SW-curve and Prepotential for E-string theory

\[ y^2 = 4x^3 - \frac{E_4(\tau)}{12} u^4 x - \frac{E_6(\tau)}{216} u^6 + 4u^5 \]

\[ \frac{\partial F_0}{\partial \varphi} = 8\pi^3 i (\varphi_D - \tau \varphi) \]

\( \tau \): complex structure of the torus

\( u \): Coulomb modulus

\( E_4, E_6 \): Eisenstein Series

(\( \varphi \): Higgs vev

\( \varphi_D \): dual Higgs vev

(O.J.Ganor-A.Hanany ’96)

(N.Seiberg-E.Witten ’96)

(O.J.Ganor ’97)

(O.J.Ganor-D.Morrison-N.Seiberg ’96)
The Nekrasov-type Expression for E-string Theory

(K.Sakai ’12)

\[ Z = \sum_{R} (-e^{2\pi i \varphi}) |R| \prod_{k=1}^{3} \prod_{(i,j) \in R_k} \frac{\vartheta_1 \left( \frac{1}{2\pi} (a_k + (j - i)\tilde{h}), \tau \right)^6}{\prod_{l=1}^{3} \vartheta_1 \left( \frac{1}{2\pi} (a_k - a_l + h_{k,l}(i,j)\tilde{h}), \tau \right)^2} \]

\( R = (R_1, R_2, R_3) \): 3-sets of partition

\( h_{k,l}(i,j) \): relative hook-length

for E-string theory

\( \varphi \): Higgs vev

\( a_k \): loci on the torus fixed as \( a_1 = \pi, \ a_2 = -\pi - \pi \tau, \ a_3 = \pi \tau \)

\( \tau \): complex structure of the torus (= IR gauge coupling)

E-string theory has a distinct physical interpretation for the parameters from usual gauge theories
Reproducing SW description

(N.A.Nekrasov ’04, N.A.Nekrasov-A.Okounkov ’03)

\[ F_0 = (2\hbar^2 \ln Z)|_{\hbar=0} \quad Z = \sum_{\mathcal{R}} e^{2\pi i \varphi}|_{\mathcal{R}} Z_{\mathcal{R}} \]

\[ Z_{\mathcal{R}} = \exp \left[ -\frac{1}{4} \int dzdw f''(z)f''(w)\gamma(z-w;\hbar) + 3 \int dzf''(z)\gamma(z;\hbar) + \sum_{k,l=1}^{3} \gamma(a_k - a_l;\hbar) - 6 \sum_{k=1}^{3} \gamma(a_k;\hbar) \right] \]

This can be viewed as the partition function of a matrix model

\[ \gamma(z;\hbar) : \text{a certain function} \]

\[ f(z) : \text{a profile function} \]

\[ f''(z) : \text{a density function} \]

determine the most dominant profile in the limit \( \hbar \to 0 \)
(a saddle point approximation)
Solving the saddle point equation (e.o.m. in a matrix model)

**Resolvent** $\omega(z) := \Omega'(z)$

**Antiderivative** $\Omega := \int_C f''(w) \ln \vartheta_1 \left( \frac{z - w}{2\pi} \right) dw - 6 \ln \vartheta_1 \left( \frac{z}{2\pi} \right)$

$2\pi i f''(z) = \omega(z - i\epsilon) - \omega(z + i\epsilon), \; z \in C \quad C : \text{support of } f''$

$\int_C dw f''(w) \gamma_0(z - w) - 6\gamma_0(z) - \pi i \varphi z^2 = 0, \; z \in C$

$\Leftrightarrow \frac{1}{2} (\Omega(z - i\epsilon) + \Omega(z + i\epsilon)) - 2\pi i \varphi = 0$

$\Omega(z) = 2 \ln \left( \sqrt{-\frac{1}{4} u_\varphi'(z)^2} + \sqrt{-\frac{1}{4} u_\varphi'(z)^2 - 1} \right) dz + 2\pi i \varphi$
An integral over $\alpha$-cycle

\[
\frac{1}{4\pi^2 i} \oint_\alpha \Omega(z) dz = 0 \mod \mathbb{Z}
\]

\[
\frac{\partial \varphi}{\partial u} = \frac{i}{4\pi^2 u} \oint_\alpha \frac{\varphi'(z) dz}{\sqrt{\varphi'(z)^2 + 4u^{-1}}}
\]

SW-curve with genus four

\[
\varphi(z) = u^{-2}x
\]

a variable change

\[
\frac{\partial \varphi}{\partial u} = \frac{i}{4\pi^2} \int_{\tilde{\alpha}} \frac{dx}{y}
\]

\[
y^2 = 4x^3 - \frac{E_4}{12} u^4 x - \frac{E_6}{216} u^6 + 4u^5
\]

SW-curve with genus one
An integral over $\beta$-cycle

\[
\frac{1}{4\pi^2i\tau} \oint_{\beta} \Omega(z) dz = \begin{cases} 
\frac{i}{8\pi^3\tau} \left( \frac{\partial F_0}{\partial \varphi} \right) \mod \mathbb{Z} \\
\frac{1}{2\pi^2i\tau} \oint_{\beta} \ln \left( \sqrt{-\frac{1}{4}u\varphi'(z)^2} + \sqrt{-\frac{1}{4}u\varphi'(z)^2 - 1} \right) dz + \varphi
\end{cases}
\equiv \varphi_D
\]

\[
\frac{\partial F_0}{\partial \varphi} = 8\pi^3i(\varphi_D - \tau \varphi)
\]

the SW-description has reproduced from the Nekrasov-type Expression
Summary

We proved that

\[ Z = \sum_{\mathcal{R}} (-e^{2\pi i \varphi}) |\mathcal{R}| \prod_{k=1}^{3} \prod_{(i,j) \in R_k} \frac{\vartheta_1 \left( \frac{1}{2\pi} (a_k + (j - i) \hbar), \tau \right)^6}{\prod_{l=1}^{3} \vartheta_1 \left( \frac{1}{2\pi} (a_k - a_l + h_{k,l}(i, j) \hbar), \tau \right)^2} \]

is the Nekrasov-type formula for E-string theory.

Similarly, SW description for E-string theory with E6 or E7 symmetry can be correctly reproduced.