Probability Distinguishes Different Types of Conditional Statements

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1 INTRODUCTION

Conditional statements, including subjunctive and counterfactual conditionals, are the source of many enduring challenges in formal reasoning. The language of probability can distinguish among several different kinds of conditionals, thereby strengthening our methods of analysis. Here we shall use probability to define four principal types of conditional statements: SUBJUNCTIVE, MATERIAL, EXISTENTIAL, and FEASIBILITY. Each probabilistic conditional is quantified by a fractional parameter between zero and one that says whether it is purely affirmative, purely negative, or intermediate in its sense. We shall consider also TRUTH-FUNCTIONAL conditionals constructed as statements of material implication from the propositional calculus; these constitute a fifth principal type. Finally there is a sixth type called BOOLEAN-FEASIBILITY conditionals, which use Boole’s mathematical logic to analyze the sets of possible truth values of formulas of the propositional calculus. Each TRUTH-FUNCTIONAL or BOOLEAN-FEASIBILITY conditional can be affirmative or negative, with this sense indicated by a binary true/false parameter. There are other kinds of conditional statements besides the six types addressed here. In particular, some conditionals ought to be regarded as recurrence relations that generate discrete dynamical systems [12].

Besides its principal type and the value of its (fractional or binary) sense-parameter, each conditional statement is further characterized by its content and by its role in analysis. Two important aspects of content are factuality and exception handling. We shall consider the factuality of a conditional relative to some proposition whose truth value is known. If the known proposition is included in the conditional statement (usually as part of the antecedent clause), then the conditional is declared ‘factual’ relative to that proposition; if the negation of the proposition is included then the conditional is ‘antifactual’ (strongly counterfactual) relative to it; and if the proposition is omitted then the conditional is ‘afactual’ (weakly counterfactual) relative to it. Like their indicative counterparts, subjunctive conditionals may be factual or counterfactual; and furthermore any given conditional statement may be correct or incorrect. These three distinct properties—mood, factuality, and correctness—may be correlated with one another. We shall consider two mechanisms to address potential exceptions that may confound conditional relationships: first, allowing the revision of old conditional statements as new information becomes available; and second, expressing conditionals in a cautious, defeasible manner in the first place. Regarding their use in analysis, there are two basic roles for conditional statements: a conditional may be asserted as a constraint itself, or provided as a query whose truth or falsity is to be evaluated subject to some other set of constraints.

Recognizing the diverse types of conditional statements helps to clarify several subtle semantic distinctions. Using the tools of probability and algebra, each semantically distinct conditional statement is represented as a syntactically distinct mathematical expression. These expressions include symbolic probability expressions, polynomials with real-number coefficients, sets of real numbers defined by polynomial constraints, and systems of equations and inequalities built from such formulas. Various algorithmic methods can be used to compute interesting results from conditional statements represented in mathematical form. These computational methods include linear and nonlinear optimization, arithmetic with symbolic polynomials, and manipulation of relational-database tables. Among other benefits, these methods of analysis offer paraconsistent procedures for logical deduction that produce such familiar results as modus ponens, transitivity, disjunction introduction, and disjunctive syllogism—while avoiding any explosion of consequences from inconsistent premises.

The proposed method of analysis is applied to several example problems from Goodman and Adams [7, 2, 1].
To begin, let us review some computational methods that will be useful for performing analysis.

2.1 A Basic Parametric Probability Model

For the purpose of representing the various types of conditional statements, we develop a basic probability model with two true/false variables $A$ and $B$ symbolizing logical propositions. We use the real-valued parameter $x$ to specify the probability that $A$ is true; $y$ for the conditional probability that $B$ is true given that $A$ is true; and $z$ for the conditional probability that $B$ is true given that $A$ is false. Truth and falsity are abbreviated $T$ and $F$. In order to enforce the laws of probability, each real-valued parameter $x$, $y$, and $z$ is constrained to lie between zero and one. This gives a parametric probability network that is specified as the following graph, two input probability tables, and some associated parameter constraints:

\[
\begin{align*}
A & \quad \text{Pr}_0(A) \\
T & \quad x \\
F & \quad 1 - x \\
\end{align*}
\]

\[
\begin{align*}
\text{Pr}_0(B|A) & \\
A & \quad B = T \quad B = F \\
T & \quad y \quad 1 - y \\
F & \quad z \quad 1 - z \\
\end{align*}
\]

There are two sets of variables in this parametric probability network: the primary variables $A$ and $B$, which represent logical propositions; and the parameters $x$, $y$, and $z$, which are used to specify probabilities associated with these propositions. For this model, each input probability is a polynomial in the ring $\mathbb{R}[x,y,z]$. The subscript 0 in $\text{Pr}_0(A)$ and $\text{Pr}_0(B|A)$ indicates that these expressions refer to input probabilities used to specify the probability model. In contrast, output probabilities computed from these inputs are written without a subscript. For example the output table $\text{Pr}(B|A)$ shown in Equation 6 was computed from both input tables $\text{Pr}_0(A)$ and $\text{Pr}_0(B|A)$ from Equation 1.
of such sums (hence a fractional polynomial). For example, the unconditioned probability that $A$ is true is computed as the sum of elements 1 and 2 of the full-joint probability table $\Pr(A,B)$ from Equation 2

$$\Pr(A = T) \Rightarrow (xy) + (x-xy) \Rightarrow x$$  (3)

The unconditioned probability that $B$ is true is the sum of elements 1 and 3 of the table $\Pr(A,B)$:

$$\Pr(B = T) \Rightarrow (xy) + (z-xz) \Rightarrow z + xy - xz$$  (4)

The conditional probability that $B$ is true given that $A$ is true is computed by first recalling the definition $\Pr(B|A) = \Pr(A,B)/\Pr(A)$ from the general laws of probability, and then evaluating the numerator and denominator separately:

$$\Pr(B = T|A = T) \Rightarrow \frac{\Pr(A = T, B = T)}{\Pr(A = T)} \Rightarrow \frac{xy}{(xy) + (x-xy)} \Rightarrow \frac{xy}{x}$$  (5)

You may be tempted to simplify this computed quotient $xy/x$ to the elementary expression $y$, but such a premature step would discard valuable information. Because the constraints in Equation 1 allow zero as a feasible value for $x$, the quotient $xy/x$ could have the value $0/0$ (which is not the same mathematical object as the expression $y$). Avoiding premature simplification allows a conditional probability to be recognized as indefinite when its condition is impossible (in exactly the same sense that the numerical quotient $0/0$ is indefinite). For example, when $\Pr(A = T) = 0$, we compute explicitly that $\Pr(B = T|A = T)$ does not have any particular real-number value.

Here is the table of computed values for the probability query $\Pr(B|A)$, incorporating all four true/false combinations of the variables $A$ and $B$:

| $\Pr(B|A)$ | $A$ | $B = T$ | $B = F$ |
|------------|-----|---------|---------|
|            | $T$ | $xy/x$  | $(x-xy)/x$ |
|            | $F$ | $(z-xz)/(1-x)$ | $(1-x-z+xz)/(1-x)$ |

Note that the computed output probabilities in the table $\Pr(B|A)$ of Equation 5 are different symbolic expressions from the input probabilities in the table $\Pr(B|A)$. The output probabilities contain factors of $x/y$ or $(1-x)/(1-x)$, which make it possible to catch the exception of division by zero that would be caused by an impossible condition.

2.3 Boolean Polynomial Translation

Boole provided a method to translate logical formulas of the propositional calculus into polynomials with real coefficients [3]. Among other benefits, this formulation makes it possible to perform logical deduction by solving equations. Boolean polynomial translation uses the following rules, with some notation here modified from Boole’s original text. The rules are summarized in Table 1. Elementary truth maps to the real number one, and elementary falsity maps to the real number zero. Each propositional variable $X_i$ translates into a real-valued algebraic variable $x_i$. Because two-valued logic mandates that each $X_i \in \{T,F\}$, each translated $x_i$ is subject to the constraint $x_i \in \{0,1\}$. This constraint $x_i \in \{0,1\}$ may be specified as the polynomial equation $x_i^2 = x_i$ (which Boole called the ‘fundamental law of thought’ in [3]). Propositional functions map to arithmetical functions according to the rules shown in Table 1 which were derived according to Boole’s polynomial interpolation method (which he called ‘function development’ in [3]). The rules can be applied by top-down or bottom-up parsing. While simplifying polynomials translated from logical formulas, any squared variable $x_i^2$ (or any higher power $x_i^n$ and so on) can be replaced with the unadorned $x_i$ (because $x_i^2 = x_i$ for $x_i \in \{0,1\}$).

For example, consider the propositional-calculus formula $X \land (X \rightarrow Y)$. To prepare for translation, we declare polynomial variables $x$ and $y$ which are subject to constraints $x \in \{0,1\}$ and $y \in \{0,1\}$. The logical formula $X$ maps to the polynomial $x$, and $Y$ maps to the polynomial $y$. The material implication $x \rightarrow y$ maps to the polynomial $1 - x + xy$. The conjunction $x \land (1 - x + xy)$ maps to the polynomial expression $x(1 - x + xy)$, which expands to the polynomial $x - x^2 + x^2y$. Taking advantage of the identity $x^2 = x$, this simplifies to $x - x + xy$, which then simplifies to $xy$. This polynomial result is a member of the ring $\mathbb{R}[x,y]$. 

3
Logic Function | Propositional Form | Polynomial Form
---|---|---
Truth | T | 1
Falsity | F | 0
Atomic formula | $X_i$ | $x_i$
Negation | $\neg p$ | $1 - p$
Conjunction | $p \land q$ | $pq$
Exclusive disjunction | $p \oplus q$ | $p + q - 2pq$
Inclusive disjunction | $p \lor q$ | $p + q - pq$
Material implication | $p \rightarrow q$ | $1 - p + pq$
Biconditional | $p \leftrightarrow q$ | $1 - p - q + 2pq$

Table 1: Rules for Boolean polynomial translation of propositional-calculus formulas. Here $p$ and $q$ are polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ whose indeterminates match the propositional variables $X_1, \ldots, X_n$. Because each $X_i \in \{T, F\}$, each $x_i \in \{0, 1\}$.

Let us use $\mathcal{B}_1(\varphi)$ to denote the Boolean polynomial translation of a propositional-calculus formula $\varphi$. For this example we write:

$$\mathcal{B}_1(X \land (X \rightarrow Y)) \Rightarrow xy$$

understanding the real-valued variables $x$ and $y$ to be limited to values in $\{0, 1\}$. Note that the polynomial $xy$ is also the Boolean translation of the propositional-calculus formula $X \land Y$, which has the same truth table as the formula $X \land (X \rightarrow Y)$. The polynomials generated by Boolean translation provide a useful normal form for propositional-calculus formulas which offers advantages over the traditional conjunctive and disjunctive normal forms.

Boole’s polynomial translation scheme offers a way to represent unknown functions of the propositional calculus in a parametric fashion. For example with variables $x$ and $y$ and coefficients $c_1, c_2, c_3, c_4$, each limited to real values in the set $\{0, 1\}$, the following polynomial expression represents an unknown propositional-calculus function of two variables $X$ and $Y$:

$$c_1xy + c_2x(1 - y) + c_3(1 - x)y + c_4(1 - x)(1 - y)$$

To illustrate one instantiation, with $(c_1, c_2, c_3, c_4) = (1, 0, 1, 1)$ the above expression simplifies to $1 - x + xy$ indicating the statement of material implication $X \rightarrow Y$. Boole described this technique in [3].

2.4 Embedding Formulas from the Propositional Calculus

We can extend parametric probability networks to incorporate logical formulas from the propositional calculus, by using conditional probability tables that mimic logical truth tables. For example, let us amend the probability model in Equation 1 to include the formulas $A \rightarrow B$ and $A \land B$ from the propositional calculus. The truth tables of these statements of material implication give the following conditional probability tables:

| $A$ | $B$ | $\langle A \rightarrow B \rangle$ | $\langle A \land B \rangle$ |
|---|---|---|---|
| T | T | 1 | 0 |
| T | F | 0 | 1 |
| F | T | 0 | 1 |
| F | F | 0 | 1 |

The subscript 1 indicates that elementary logical truth is mapped to the number one; there is an alternative translation scheme with truth mapped to the number zero. Another option is to use polynomial coefficients in the finite field $\mathbb{F}_2 = \{0, 1\}$ instead of the real numbers $\mathbb{R}$. Numbers in $\mathbb{F}_2$ require integer arithmetic modulo two; there are some computational advantages to this choice.
Using these tables together with the original probability model in Equation 1, symbolic probability inference yields the probabilities that the embedded propositional-calculus formulas are true:

| \(A \rightarrow B\) | \(\Pr (A \rightarrow B)\) | \(A \wedge B\) | \(\Pr (A \wedge B)\) |
|---------------------|---------------------|---------------------|---------------------|
| T                    | \(1 - x + xy\)      | T                    | \(xy\)              |
| F                    | \(x - xy\)          | F                    | \(1 - xy\)          |

(10)

For convenience we can embed elementary logical truth and falsity within a parametric probability model, using the following conditional probability tables:

| \(T\) | \(\Pr_0 (\langle T \rangle)\) |
|-------|-------------------------------|
| T     | 1                             |
| F     | 0                             |

| \(F\) | \(\Pr_0 (\langle F \rangle)\) |
|-------|-------------------------------|
| T     | 0                             |
| F     | 1                             |

(11)

These input probability tables say that truth is certainly true and falsity is certainly false. Note that conditioning on the truth of embedded truth (or the falsity of embedded falsity) leaves the polynomial formula computed for any probability expression unchanged: this operation simply adds 1 to the numerator and denominator of the computed polynomial quotient. For example, given any main variable \(A\) the conditional probability \(\Pr (A = T \mid \langle T \rangle = T)\) evaluates to the same polynomial expression as the unconditioned probability \(\Pr (A = T)\). Adding to consideration any variable \(B\), the conditional probability \(\Pr (A = T \mid B = T, \langle T \rangle = T)\) yields the same polynomial quotient as the simpler conditional probability \(\Pr (A = T \mid B = T)\).

2.4.1 Boole’s Polynomial Coincidence

Using a parametric probability model with a certain independence structure, the polynomial formula for the probability that an embedded propositional-calculus formula is true must coincide with the Boolean polynomial translation of that formula. Boole noted this property in [3] without stating precisely the probabilistic independence condition involved. The required probability-network structure reflects the assertion that each propositional variable is probabilistically independent of the others. Consider propositional variables \(X_1, \ldots, X_m\). The probabilistic independence assertion can be specified by a network graph in which the nodes for these variables are not directly connected with one another. Hence the full-joint probability table \(\Pr (X_1, \ldots, X_m)\) is factored into individual input tables \(\Pr_0 (X_1)\) through \(\Pr_0 (X_m)\). Let each input \(\Pr_0 (X_i = T)\) be defined as the similarly-named parameter \(x_i\) (and correspondingly each \(\Pr_0 (X_i = F)\) defined as \(1 - x_i\), with \(0 \leq x_i \leq 1\). Now, for any propositional-calculus formula \(\phi\) whose propositional variables are \(X_1, \ldots, X_m\), the polynomial formula computed as the probability that the embedded formula \(\phi\) is true (using the embedding method of this section and the symbolic probability inference method of Section 2.2) must coincide with the Boolean polynomial translation of \(\phi\) (computed by the method of Section 2.2).

\[
\Pr (\langle \phi \rangle = T) = \mathcal{B}_1 (\phi)
\]

(12)

The assertion that each propositional variable \(X_i\) is probabilistically independent from the others is not ‘uninformative’. Instead this independence assertion conveys nontrivial prior information about these variables, even when the relevant parameters \(x_i\) are not constrained beyond the bounds 0 and 1 required by the laws of probability. Indeed factoring the full-joint probability table \(\Pr (X_1, \ldots, X_m)\) in any way provides prior information. The truly uninformative option is to specify the full-joint probability directly as a single input table \(\Pr_0 (X_1, \ldots, X_m)\) (using \(2^m\) parameters) and provide only those constraints required by the laws of probability (each parameter between zero and one, and the sum of parameters equal to one).

2.5 Bounded Global Polynomial Optimization

Polynomials, such as those generated by symbolic probability inference and those translated from the propositional calculus by Boole’s method, can be used to build optimization problems and solution sets which will be useful for performing logical deduction. Let us formulate a general polynomial optimization problem with bounded variables. Consider a list \(x = (x_1, \ldots, x_m)\) of real-valued variables with each \(x_i\) bounded by finite lower and upper limits \(a_i\) and
\( \beta \). Consider also an objective function \( f \) and several constraint functions \( g_1, \ldots, g_m \), each of which is a polynomial in \( \mathbb{R}[x] \). These components provide the following template for polynomial optimization problems:

Maximize : \( f(x) \)
subject to : \( g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \)
and : \( \alpha_i \leq x_1 \leq \beta_1, \ldots, \alpha_n \leq x_n \leq \beta_n \) \( (13) \)

Certain variations on this template are allowed. Any given optimization problem may request either the minimum or maximum feasible value of the objective, and each constraint may use either equality = or weak inequality \( \geq \) or \( \leq \). Strict inequalities are approximated by the introduction of a small numerical constant \( \varepsilon \) such as \( 0.001 \) or \( 1 \times 10^{-6} \), using for example \( g(x) > \varepsilon \) to represent \( g(x) > 0 \). Any variable \( x_i \) may be restricted to take only integer values within its range (such as 0 and 1 for a variable \( x_i \) bounded by \( 0 \leq x_i \leq 1 \)).

Optimization problems like Equation \( (13) \) can be challenging to solve, because with nonlinear polynomials there can exist local solutions which are not globally optimal. The author has developed a suitable algorithm for bounded global polynomial optimization, based on earlier reformulation and linearization methods and dependent on a separate mixed integer-linear programming solver \[10\]. The associated software implementation was used to compute the problems addressed here, the author’s solver takes trivial amounts of time to compute solutions of ample numerical precision. Problems with inconsistent constraints are reported directly as being infeasible.

Moving on, let us next consider a polynomial set-comprehension expression which is related to the optimization problem in Equation \( (13) \). Now, instead of seeking the minimum or maximum feasible value of the objective \( f \), we seek the set \( \Phi \) of all feasible values of the objective function subject to the constraints:

\[
\Phi \leftarrow \{ f(x) : g(x) \geq 0, \ldots, g(x) \geq 0; \alpha_i \leq x_1 \leq \beta_1, \ldots, \alpha_n \leq x_n \leq \beta_n \} \quad (14)
\]

As before each constraint may use either relation \( =, \geq, \) or \( \leq \); also any variable \( x_i \) can be restricted to a finite set of integer values. By its construction, the solution set defined in Equation \( (14) \) is a subset of the real numbers: \( \Phi \subseteq \mathbb{R} \).

A solution set like Equation \( (14) \) can be described using a pair of optimization problems like Equation \( (13) \) one problem to compute the minimum feasible value of the objective \( f \), and another to compute the maximum feasible value. In the case that the specified constraints are inconsistent, both optimization problems will be infeasible. Otherwise let us designate the computed minimum and maximum feasible values of \( f \) as \( \alpha^* \) and \( \beta^* \) respectively. The solution set \( \Phi \) must be a subset of the real interval bounded by these computed solutions, and it must contain at least those two points: hence \( \Phi \subseteq [\alpha^*, \beta^*] \) and \( \alpha^* \in \Phi \) and \( \beta^* \in \Phi \). In the special case \( \alpha^* = \beta^* \) that the computed minimum and maximum values coincide, then the solution set \( \Phi \) is the singleton \( \{\alpha^*\} \) containing that common value.

To illustrate, let us return to the basic probability model from Equation \( (1) \). We choose as our objective \( f(x, y, z) \) the probability \( \Pr(B = T) \), which is the polynomial \( z + xy - xz \) given in Equation \( (4) \). We choose as constraints the parameter bounds given in Equation \( (1) \) joined with two additional constraints: \( x = 1 \) and \( xy = x \) (whose provenance will be discussed later). Thus our goal is to find the solution set \( \Phi \) given by:

\[
\Phi \leftarrow \{ z + xy - xz : x = 1; xy = x; x, y, z \in [0, 1] \} \quad (15)
\]

In order to characterize this solution set \( \Phi \) we solve two optimization problems patterned after Equation \( (13) \)

Minimize : \( z + xy - xz \)
subject to : \( x = 1 \)
and : \( 0 \leq x \leq 1 \)
\( (16) \)

Maximize : \( z + xy - xz \)
subject to : \( x = 1 \)
and : \( 0 \leq x \leq 1 \)
\( (16) \)

The author’s polynomial optimization solver determines that both problems are feasible. The computed minimum \( \alpha^* \) is 1.000 and the computed maximum \( \beta^* \) is also 1.000. It follows from these results that the solution set \( \Phi \) specified by Equation \( (15) \) is a subset of the real interval \([1.000, 1.000]\) containing at least the point 1.000; in other words \( \Phi \) evaluates
Taking advantage of the computational methods presented in Section 2, we can develop formal definitions of several different types of conditional statements. These types and their mathematical definitions are summarized in Table 2.

We define four principal types of conditionals in terms of constraints on probabilities: these types are called ‘subjunctive’, ‘material’, ‘existential’, and ‘feasibility’. Each of these probabilistic conditionals includes a real-valued parameter $k$ between zero and one that quantifies whether it is purely affirmative ($k = 1$), purely negative ($k = 0$), or intermediate in its sense; the default is $k = 1$ for an affirmative conditional. Also we shall consider two types of conditionals based on the propositional calculus. ‘Truth-functional’ conditionals, the fifth principal type, use material implication. ‘Boolean-feasibility’ conditionals, the sixth principal type, use Boole’s algebraic method to analyze polynomial expressions translated from propositional-calculus formulas. Each truth-functional or Boolean conditional includes a true/false parameter $K$, rather than a real-valued parameter $k$, to indicate the affirmative ($K = T$) versus the negative ($K = F$) sense; by default $K = T$. The five types of conditionals that are not called ‘subjunctive’ are classified (

| Type            | Symbolic | Probability or Logical Equations | Affirmative ($k = 1$), in Algebra |
|-----------------|----------|---------------------------------|----------------------------------|
| Subjunctive     | $A \rightsquigarrow_k B$ | $\Pr_0(B = T | A = T) = k$ | $y = 1$ |
| Material        | $A \rightarrow_k B$ | $\Pr(A = T, B = T) = k \cdot \Pr(A = T)$ | $xy = x$ |
| Existential     | $A \nrightarrow_k B$ | $\Pr(A = T, B = T) = k \cdot \Pr(A = T)$ | $xy = x$ and $x > 0$ |
| Feasibility     | $\Gamma \vdash_k B$ | $\{ \Pr(B = T : \Gamma) \} = \{k\}$ | $\{ z + xy - xz : \Gamma, \Gamma_0 \} = \{1\}$ |
| Quotient-feasibility | $\Gamma : A \vdash_k B$ | $\{ \Pr(B = T | A = T) : \Gamma \} = \{k\}$ | $\{ xy/x : \Gamma, \Gamma_0 \} = \{1\}$ |
| Truth-functional | $A \rightarrow_k B$ | $A \rightarrow (K \leftrightarrow B) = T$ | $1 - a + ab = 1$ |
| Boolean-feasibility | $A \rightarrow_k B$ | $\{ B : A = T \} = \{K\}$ | $\{ b : a = 1, \Gamma_0 \} = \{1\}$ |

Table 2: Several types of conditional statements relating true/false antecedent $A$ and consequent $B$, specializing the statement ‘if $A$ then $B$’. The probabilistic conditionals are quantified by a fractional parameter $0 < k < 1$, with $k = 1$ indicating affirmative and $k = 0$ negative statements. The symbol $\Gamma$ denotes a set of polynomial equality and inequality constraints supplied by the user. The displayed algebraic translations of affirmative conditionals were derived using symbolic probability inference from the probability model from Equation 1 or Boolean polynomial translation from the propositional calculus. The set $\Gamma_0$ consists of the constraints $x, y, z \in [0, 1]$ required by the laws of probability for the model in Equation 1; the set $\Gamma_0'$ consists of the constraints $a, b \in \{0, 1\}$ required by two-valued logic.

to the singleton $\{1\}$. For the set-comprehension expression in Equation 15, this result is also evident from manual calculations. For example, after substituting 1 for $x$ (according to the first constraint $x = 1$) the second constraint $xy = x$ becomes $y = 1$. Then when 1 is substituted for $x$ and also for $y$ the objective $z + xy - xz$ simplifies to the constant real number 1.

3 TYPES OF CONDITIONAL STATEMENTS

In the following sections, the six principal types of conditionals are defined mathematically. The four probabilistic conditionals use probability expressions as intermediate representations, and the two others use the propositional calculus as an intermediate form. Ultimately each type of conditional statement is translated into a system of equations and inequalities involving polynomials with real coefficients. These algebraic formulas can be used for analysis. Due to their different algebraic definitions, the various types of conditionals exhibit distinctive patterns of behavior (in terms of what they say about conditional probabilities, how affirmative and negative conditionals interact with one another, what information each conditional statement provides about its antecedent, and what happens with false antecedents). These properties are discussed in Section 4.

Note that conditional statements need not have any temporal or causal significance. It is not necessary that the consequent event should come after the antecedent event in time, nor is it necessary that the antecedent should be a cause of the consequent.
3.1 Subjunctive Conditionals

The **subjunctive** conditional \( A \leadsto_k B \) with antecedent term \( A \), consequent term \( B \), and fractional parameter \( k \) is defined by the equation that the input probability that \( B \) is true given that \( A \) is true must equal \( k \):

\[
Pr_0(B = T | A = T) = k
\]  

(17)

The **subjunctive** conditional defined by Equation(17) says ‘If \( A \) then \( B \)’ in the following more specific sense: ‘If there were any \( A \), then \( k \) would also be \( B \).’ With \( k = 1 \) this becomes the affirmative statement \( A \leadsto B \) that if there were any \( A \), then all would also be \( B \). With \( k = 0 \) this becomes the negative statement \( A \leadsto_0 B \) that if there were any \( A \), then none would also be \( B \). Using the basic probability model from Equation(1) the definition in Equation(17) becomes the following polynomial equation:

\[
y = k
\]  

(18)

Hence \( y = 1 \) for the affirmative conditional and \( y = 0 \) for its negative counterpart.

Even when there is no input table \( Pr_0(B | A) \) in a particular probability model relating the antecedent \( A \) and consequent \( B \), it may still be possible to express a subjunctive conditional using that model. The essential requirement is that the polynomial quotient computed as the value of the conditional probability \( Pr(B = T | A = T) \) must contain its denominator as a factor in its numerator. In the probability network graph the antecedent must be an ancestor of the consequent. The appropriate polynomial expression to constrain is then constructed by eliminating this common term from the numerator and denominator (without assuming it is nonzero). That is, the polynomial quotient computed for \( Pr(B = T | A = T) \) must have the form \( p_q/q \) for some polynomials \( p \) and \( q \), where \( q \) is also the computed value of \( Pr(A = T) \). The subjunctive conditional is formed by constraining the factor \( p \) alone to equal the sense-parameter \( k \).

Multiple antecedent terms can be accommodated in a **subjunctive** conditional by including additional variables in the condition part of the input probability from Equation(17). Thus the **subjunctive** conditional \( \{A_1,\ldots,A_m \} \leadsto_k B \) is defined by the equation:

\[
Pr_0(B = T | A_1 = T,\ldots,A_m = T) = k
\]  

(19)

For all types of conditionals it is acceptable to invert the truth or falsity of any antecedent terms \( A \) or \( A_j \) and/or of the consequent term \( B \). Thus for example to express the **subjunctive** interpretations of the conditionals ‘If \( A \) then not-\( B \)’ and ‘If \( A_1 \) and not-\( A_2 \) then \( B \)’ we would use:

\[
A \leadsto \overline{B} \equiv Pr_0(B = F | A = T) = 1
\]  

(20)

\[
\{A_1,\overline{A_2}\} \leadsto B \equiv Pr_0(B = T | A_1 = T,A_2 = F) = 1
\]  

(21)

Note that there are two ways to oppose the affirmative conditional \( A \leadsto B \): first by inverting the truth of the consequent term \( B \), as in \( A \leadsto \overline{B} \), second by using the fractional parameter value \( k = 0 \) instead of \( k = 1 \), as in \( A \leadsto_0 B \). Both approaches lead to equivalent polynomial constraints. It is a different thing to negate an entire conditional: the negation \( \overline{(A \leadsto_k B)} \) means the constraint \( Pr_0(B = T | A = T) \neq k \).

3.2 Material Conditionals

The **material** conditional \( A \leftrightarrow_k B \) with antecedent term \( A \), consequent term \( B \), and fractional parameter \( k \) is defined by the equation that the probability that \( A \) and \( B \) are both true must equal \( k \) times the probability that \( A \) is true:

\[
Pr(A = T, B = T) = k \cdot Pr(A = T)
\]  

(22)

The **material** conditional defined by Equation(22) says ‘If \( A \) then \( B \)’ in the following more specific sense: ‘Either there are no \( A \), or \( k \) of the \( A \) are also \( B \).’ With \( k = 1 \) this is the affirmative statement \( A \leftrightarrow B \) that either all \( A \) are also \( B \), or there are no \( A \). With \( k = 0 \) this is the negative statement \( A \leftrightarrow_0 B \) that either no \( A \) are also \( B \), or there are no \( A \). The **material** conditional with an intermediate value of its parameter \( k \) asserts that either there are no \( A \), or the specified
fraction $k$ of $A$ are also $B$. Using the basic probability model from Equation \[1\] the definition in Equation \[22\] becomes the polynomial equation:

$$xy = kx$$

(23)

Hence $xy = x$ for the affirmative MATERIAL conditional and $xy = 0$ for its negative counterpart.

Multiple antecedent terms can be accommodated in a MATERIAL conditional by including additional variables in the probability expressions from Equation \[22\]. Thus the MATERIAL conditional $\{A_1, \ldots, A_m\} \rightarrow_k B$ is defined by the equation:

$$\Pr(A_1 = T, \ldots, A_m = T, B = T) = k \cdot \Pr(A_1 = T, \ldots, A_m = T)$$

(24)

3.3 Existential Conditionals

The EXISTENTIAL conditional $A \rightarrow_k B$ adds to its MATERIAL counterpart the requirement that the probability that the antecedent $A$ is true cannot be zero. Therefore the EXISTENTIAL conditional with antecedent term $A$, consequent term $B$, and fractional parameter $k$ is defined by the following equation and inequality:

$$\Pr(A = T, B = T) = k \cdot \Pr(A = T)$$

$$\Pr(A = T) > 0$$

(25)

The EXISTENTIAL conditional defined by Equation \[25\] says ‘If $A$ then $B$’ in the following more specific sense: ‘There are some $A$, of which $k$ are also $B$.’ With $k = 1$ this is the affirmative statement $A \rightarrow B$ that all $A$ are $B$, and there are some $A$. With $k = 0$ this is the negative statement $A \rightarrow_0 B$ that no $A$ are $B$, and there are some $A$. Intermediate values of $k$ assert that $B$ is true for the given proportion of cases in which $A$ is also true. Using the basic probability model from Equation \[1\] the definition in Equation \[25\] becomes the polynomial equation and inequality:

$$xy = kx$$

$$x > 0$$

(26)

Equation \[25\] can be modified to accommodate multiple antecedent terms in an EXISTENTIAL conditional. The EXISTENTIAL conditional $\{A_1, \ldots, A_n\} \rightarrow_k B$ is defined by:

$$\Pr(A_1 = T, \ldots, A_m = T, B = T) = k \cdot \Pr(A_1 = T, \ldots, A_m = T)$$

$$\Pr(A_1 = T, \ldots, A_m = T) > 0$$

(27)

3.4 Feasibility Conditionals

The FEASIBILITY conditional $\Gamma \vdash_k B$ with antecedent $\Gamma$, consequent $B$, and parameter $k$ is defined by the equation that $k$ is the only feasible value of the probability that $B$ is true, subject to the polynomial equality and inequality constraints in the set $\Gamma$:

$$\Gamma \vdash_k B \quad \Leftrightarrow \quad \{ \Pr(B = T) : \Gamma \} = \{k\}$$

(28)

The FEASIBILITY conditional defined by Equation \[28\] says ‘If $\Gamma$ then $B$’ in the following more specific sense: ‘Subject to the constraints in $\Gamma$, the only feasible value of the probability that $B$ is true is $k$’. The symbolic probability inference methods from Section \[22\] and the polynomial optimization methods from Section \[25\] can be used to evaluate the solution set on the left-hand side of Equation \[28\] and thereby determine whether or not the equation is satisfied. The general constraints $\Gamma_0$ required by the laws of probability must be included during analysis (restricting each probability to lie between zero and one, and constraining appropriate sums of probability to equal one).

Using the probability model from Equation \[1\] and symbolic probability inference, the definition in Equation \[28\] for the FEASIBILITY conditional $\Gamma \vdash_k B$ becomes the following equation about sets of real numbers:

$$\{ z + xy - xz : \Gamma; x, y, z \in [0, 1] \} = \{k\}$$

(29)
Here the general constraints \( \Gamma_0 = \{ x \in [0,1], y \in [0,1], z \in [0,1] \} \) required by the laws of probability for the model in Equation[1] have been included.

The antecedent set \( \Gamma \) may include arbitrary polynomial equality and inequality constraints, perhaps derived from other probability expressions and probabilistic conditions. For example we might use the constraint-set \( \{ \Pr(A = T) = 1, \Pr_0(B = T | A = T) = 1 \} \) to provide the premises that \( A \) is certainly true and that the affirmitive SUBJUNCTIVE conditional \( A \rightarrow B \) from Equation[17] is true. After symbolic probability inference this example set \( \Gamma \) consists of the polynomial constraints \( \{ x = 1, y = 1 \} \).

Note that unlike the parts of the other types of conditionals, the antecedent \( \Gamma \) and consequent \( B \) of a FEASIBILITY conditional have different data types: \( \Gamma \) is a set of polynomial constraints using the real-valued parameters of the probability model in use, whereas \( B \) is a true/false variable that is one of the primary variables in the probability model. However we can use \( A \vdash^+ B \) to abbreviate the FEASIBILITY conditional \( \{ \Pr(A = T) = 1 \} \vdash^+_k B \) whose antecedent set \( \Gamma \) consists of the solitary constraint that \( A \) must certainly be true.

3.4.1 Quotient-Feasibility Conditionals

As a variation on the theme we can formulate a QUOTIENT-FEASIBILITY conditional \( \Gamma : A \vdash^+_k B \) which says that, subject to the constraints in the set \( \Gamma \), the only feasible value of the computed conditional probability that \( B \) is true given that \( A \) is true is \( k \). Thus the definition:

\[
\Gamma : A \vdash^+_k B \equiv \{ \Pr(B = T | A = T) : \Gamma \} = \{ k \}
\]  

(30)

It is necessary to use fractional polynomial optimization to evaluate directly the solution set in this definition, since its objective is a quotient of polynomials. A suitable optimization method is discussed in Appendix[A]. Notably by this fractional polynomial optimization method, it is considered infeasible for the denominator of the objective function to equal zero. Therefore if the denominator is otherwise constrained to equal zero, the entire problem is considered infeasible by the solver. Otherwise, the computed solution set describes the possible values of the objective function when the denominator is not zero and the given constraints \( \Gamma \) are satisfied. This arrangement means that the conditional \( \Gamma : A \vdash^+_k B \) could be satisfied without excluding zero as a possible value for the denominator \( \Pr(A = T) \) subject to the constraints \( \Gamma \); it is just that other values besides zero would need to be possible too.

Alternatively, the QUOTIENT-FEASIBILITY conditional can be evaluated using two different solution sets. In order for the right-hand side of Equation[30] to be satisfied, all three of the following must be true:

\[
\{ \Pr(A = T) : \Gamma \} \neq \emptyset \\
\{ \Pr(A = T) : \Gamma \} \neq \{0\} \\
\{ \Pr(A = T, B = T) - k \cdot \Pr(A = T) : \Gamma \} = \{0\}
\]

(31)

In other words: the constraints \( \Gamma \) must be feasible; it must be feasible that the denominator probability \( \Pr(A = T) \) is not zero, subject to \( \Gamma \); and it must be true that the ratio \( \Pr(A = T, B = T) / \Pr(A = T) \) equals \( k \) whenever it is defined (that is, when \( \Pr(A = T) \) is not zero), also subject to \( \Gamma \). This reformulation follows from the laws of probability. Using the probability model in Equation[1] the QUOTIENT-FEASIBILITY conditional \( \Gamma : A \vdash^+_k B \) is given by the following three equations about sets of real numbers:

\[
\{ x : \Gamma; x,y,z \in [0,1] \} \neq \emptyset, \quad \{ x : \Gamma; x,y,z \in [0,1] \} \neq \{0\}, \quad \{ xy - kx : \Gamma; x,y,z \in [0,1] \} = \{0\}
\]

(32)

These equations are similar to the EXISTENTIAL conditional \( A \vdash^+_k B \) from Equation[25].

The QUOTIENT-FEASIBILITY conditional \( \Gamma : (T) \vdash^+_k B \) whose antecedent is the embedded propositional-calculus formula for elementary truth is equivalent to the simple FEASIBILITY conditional \( \Gamma \vdash^+_k B \) defined above.
3.5 Truth-Functional Conditionals

The affirmative \textsc{truth-functional} conditional \( A \rightarrow B \) with antecedent \( A \) and consequent \( B \) is defined as the logical equation that the corresponding statement of material implication from the propositional calculus is true\(^\dagger\)

\[
A \rightarrow B \equiv A \rightarrow B = \top
\]  
(33)

Here \( A \) and \( B \) denote formulas from the propositional calculus; these may be atomic formulas (propositional variables) or compound formulas. The affirmative \textsc{truth-functional} conditional has the same meaning as the affirmative \((k = 1)\) probabilistic \textsc{material} conditional: ‘Either \( A \) is false or \( B \) is true (or both).’

When \( A \) and \( B \) are atomic formulas, Boole’s polynomial translation method from Section 2.3 yields the following polynomial version of Equation (33) representing the conditional \( A \rightarrow B \):

\[
1 - a + ab = 1
\]  
(34)

The real-valued variables \( a \) and \( b \) are subject to the constraints \( a \in \{0, 1\} \) and \( b \in \{0, 1\} \).

A \textsc{truth-functional} conditional is negated by inverting the sense of its consequent term \( B \). Hence the negative \textsc{truth-functional} conditional \( A \rightarrow \overline{B} \) is defined by the equation that the propositional-calculus statement that \( A \) materially implies the negation of \( B \) is true:

\[
A \rightarrow \overline{B} \equiv A \rightarrow \neg B = \top
\]  
(35)

When \( A \) and \( B \) are atomic formulas, Boolean translation provides the following polynomial equation for the negative \textsc{truth-functional} conditional \( A \rightarrow \overline{B} \):

\[
1 - ab = 1
\]  
(36)

We can express the affirmative and negative \textsc{truth-functional} conditionals in an integrated way with the introduction of a logical parameter \( K \) which can be either true or false, using \( K = \top \) to indicate an affirmative conditional and \( K = \bot \) to indicate a negative conditional:

\[
A \rightarrow_K B \equiv A \rightarrow (K \leftrightarrow B) = \top
\]  
(37)

Note that the biconditional \( \top \leftrightarrow B \) has the same truth table as \( B \) itself, and the biconditional \( \bot \leftrightarrow B \) has the same truth table as the negation \( \neg B \). Boolean translation maps the true/false value \( K \) to a real number \( k \) which is either 0 or 1. When \( A \) and \( B \) are atomic formulas, the Boolean polynomial translation of the definiens in Equation (37) is as follows:

\[
1 - ak - ab + 2kab = 1
\]  
(38)

Now \( k = 1 \) designates an affirmative conditional and \( k = 0 \)designates a negative conditional. Note that whereas the parameter \( k \) for the probabilistic conditionals above takes continuous values in the real interval \([0, 1]\), the parameter \( k \) for \textsc{truth-functional} conditionals is limited to the integers \([0, 1]\).

When the antecedent \( A \) and consequent \( B \) of a \textsc{truth-functional} conditional are not atomic formulas but instead general propositional-calculus functions of some other propositional variables \( X_1, \ldots, X_n \), it is necessary to compute the appropriate Boolean polynomial translation using the rules in Table 1. In this case the polynomial translation of the \textsc{truth-functional} conditional defined by Equation (37) is given by:

\[
A \rightarrow_K B \equiv B_1(A \rightarrow (K \leftrightarrow B)) = 1
\]  
(39)

using the value of \( K \) supplied by the user (with \( K = \top \) for an affirmative conditional and \( K = \bot \) for a negative one). The translated propositional-calculus formula on the left-hand side of the above equation will be a polynomial function of the real-valued variables \( x_1, \ldots, x_n \) corresponding to the propositional variables used by the formulas \( A \) and \( B \).

Additional antecedent terms can be accommodated through logical conjunction. Considering antecedent formulas \( A_1, \ldots, A_m \), the \textsc{truth-functional} conditional with consequent \( B \) and sense-parameter \( K \) is defined by:

\[
\{A_1, \ldots, A_m\} \rightarrow_K B \equiv B_1((A_1 \land \cdots \land A_m) \rightarrow (K \leftrightarrow B)) = 1
\]  
(40)

\(^\dagger\)The \textsc{truth-functional} conditional is a special case of the probabilistic \textsc{material} conditional. Note \( \Pr (A \rightarrow B = \top) = 1 \) gives same constraint as \( \Pr (A = \top, B = \top) = \Pr (A = \top) \), and \( \Pr (A \rightarrow \neg B = \top) = 1 \) same as \( \Pr (A = \top, B = \bot) = 0 \). Convenient to calculate with \( \mathbb{F}_2 \) after Boolean translation of \textsc{truth-functional} version.
3.6 Boolean-Feasibility Conditionals

The boolean-feasibility conditional \( A \vdash_K B \) with antecedent \( A \), consequent \( B \), and parameter \( K \) is defined by the constraint that \( K \) is the only feasible truth value for \( B \), subject to the constraint that \( A \) is true:

\[
A \vdash_K B \equiv \{ B : A = T \} = \{ K \}
\]  

(41)

Here \( A \) and \( B \) are formulas of the propositional calculus, and \( K \) is either elementary truth value \( T \) or \( F \). The boolean-feasibility conditional defined by Equation (41) says ‘If \( A \) then \( B \)’ in the following more specific sense: ‘Assuming that \( A \) is true, then \( B \) must be \( K \).’ The affirmative boolean-feasibility conditional is indicated by \( K = T \) and its negative counterpart by \( K = F \). The definiens of Equation (41) relates two sets of elementary truth values, each of which is a (non-strict) subset of \{T,F\}. Note that Equation (41) imposes both ‘positive’ and ‘negative’ criteria on the possible truth values of the consequent \( B \) subject to the constraint that the antecedent \( A \) is true. Positively speaking, there must be at least one feasible case in which \( A = T \) and \( B = K \). Negatively speaking, there must be no feasible case in which \( A = T \) and \( B = \neg K \). If there are no feasible cases in which \( A = T \) at all then the set \{ \( B : A = T \) \} is empty and the conditional \( A \vdash_K B \) fails. Here a ‘case’ means a valuation of the propositional variables involved. We can have a qualitative system in which we mark some valuations as possible and others as impossible; or we can assign quantitative probabilities to the possible valuations. Implicit in this framing is the idea that there may be a separate declaration of which valuations of the propositional variables are feasible and which are not. There are several possible ways to supply such prior information, for example by adding additional terms to the antecedent part of the conditional or by introducing a probability distribution over the set of valuations of the propositional variables.

Boolean polynomial translation of the definiens of Equation (41) gives the following equation:

\[
\{ \mathfrak{B}_1(B) : \mathfrak{B}_1(A) = 1; \Gamma_0 \} = \{ k \}
\]  

(42)

where \( k = 1 \) indicates an affirmative conditional and \( k = 1 \) indicates a negative conditional. Reflecting two-valued logic, the set \( \Gamma_0 \) includes a constraint that limits the value of each variable to either zero or one:

\[ \Gamma_0 \leftarrow \{ x_1 \in \{ 0,1 \}, \ldots, x_n \in \{ 0,1 \} \} \]  

(43)

Following Boole each constraint \( x_i \in \{ 0,1 \} \) could be specified as \( x_i^2 = x_i \). Equation (42) relates two sets of real numbers, each a (non-strict) subset of the set \{0,1\}. Note that there are four possible values for the solution set included in Equation (42): the set \{0\}; the set \{1\}; the set \{0,1\}; and the empty set \( \emptyset \). This set of solution sets can be exploited to develop a system of modal logic [12].

Multiple antecedent terms can be accommodated in boolean-feasibility conditionals through conjunction or equivalently through additional constraints. Thus we define:

\[
\{ A_1, \ldots, A_m \} \vdash_K B \equiv \{ B : (A_1 \land \cdots \land A_m) = T \} = \{ K \}
\]  

(44)

\[
\{ B : A_1 = T, \ldots, A_m = T \} = \{ K \}
\]  

(45)

The Boolean polynomial translation method from Section 2.3 could be applied to the conjunction \( (A_1 \land \cdots \land A_m) \) following the first version, or applied to the separate antecedent formulas \( A_1 \) through \( A_m \) following the second version.

3.7 Relationships Among the Various Types of Conditionals

Let us consider two sorts of relationships among the several types of conditionals just defined. First, the conditionals based on the propositional calculus turn out to be closely related to probabilistic conditionals: truth-functional conditionals are special cases of material conditionals, and boolean-feasibility conditionals are special cases of quotient-feasibility conditionals. Second, there is a hierarchy among the simple probabilistic conditionals: existential conditionals subsume subjunctive ones, which in turn subsume material ones. Feasibility conditionals stand outside this hierarchy; they can be used as a metalevel statements to reason about conditionals of other types.
3.7.1 Truth-Functional and Material Conditionals

The truth-functional conditional $A \rightarrow_k B$ defined by Equation 33 is a special case of the probabilistic material conditional $A \rightarrow_k B$ from Equation 22 in the following sense. Recall that the affirmative ($K = T$) truth-functional conditional is defined as the equation that the propositional-calculus statement of material implication $A \rightarrow B$ is true; similarly the negative ($K = F$) truth-functional conditional is defined as the equation that $A \rightarrow \neg B$ is true. These propositional-calculus formulas can be embedded within a probability network by the technique from Section 2.4. It happens that the equation stating that the embedded formula $A \rightarrow B$ is certainly true is the same as the equation that specifies the affirmative ($k = 1$) material conditional (according to Definition 22):

$$
\Pr(\langle A \rightarrow B \rangle = 1) = 1 \equiv \Pr(A = T, B = T) = \Pr(A = T)
$$

Likewise the equation stating that the embedded formula $A \rightarrow \neg B$ is true is the same as the equation specifying the negative ($k = 0$) material conditional:

$$
\Pr(\langle A \rightarrow \neg B \rangle = 1) = 1 \equiv \Pr(A = T, B = T) = 0
$$

In other words, the affirmative truth-functional conditional $A \rightarrow B$ specifies the same equation as the affirmative material conditional $A \rightarrow B$; and the negative truth-functional conditional $A \rightarrow^* B$ (equivalently $A \rightarrow \overline{B}$) specifies the same equation as the negative material conditional $A \rightarrow 0 B$ (equivalently $A \rightarrow \overline{B}$).

The general relationships from Equations 46 and 47 hold using any parametric probability model that includes the main variables $A$ and $B$ and additional main variables for the embedded propositional-calculus formulas $A \rightarrow B$ and $A \rightarrow \neg B$. For example using the probability model in Equation 1 both affirmative statements give the constraint $xy = x$ (symbolic probability inference yields $\Pr(A \rightarrow B) = T \Rightarrow 1 - x + xy$). Likewise using this same probability model both negative statements give the constraints $xy = 0$ (symbolic probability inference yields $\Pr(A \rightarrow \neg B) = T \Rightarrow 1 - xy$).

Note that intermediate values of the sense-parameter $k$ allow probabilistic material conditionals to express constraints that have no direct propositional-calculus counterparts. For example by Equation 22 the material conditional $A \leftrightarrow 0.5 B$ specifies the equation $\Pr(A = T, B = T) = 0.5 \cdot \Pr(A = T)$, which does not have a simple relationship to the equation $\Pr(A \rightarrow B) = T = 0.5$, nor to the equation $\Pr(A \rightarrow \neg B) = T = 0.5$. Using the probability model in Equation 1 the respective polynomial equations are:

$$
xy = 0.5x, \quad 0.5 + xy = x, \quad 0.5 = xy
$$

These three equations express the respective ideas: first, ‘Either $A$ is false or half the time $B$ is true’ (also, ‘Either $A$ is false or half the time $B$ is false’); second, ‘Half the time, it is the case that either $A$ is false or $B$ is true’; and third, ‘Half the time, it is the case that either $A$ is false or $B$ is false’. These are some halfway points in various truth-functional interpretations of the conditionals ‘If $A$ then $B$’ and ‘If $A$ then not-$B$’.

3.7.2 Boolean-Feasibility and Feasibility Conditionals

The boolean-feasibility conditional $A \vdash_k B$ defined by Equation 41 is a special case of the probabilistic quotient-feasibility conditional from Equation 33 using an empty set $\Gamma$ of additional constraints. In this case the affirmative quotient-feasibility conditional $\emptyset : A \vdash^* B$ and its affirmative boolean-feasibility counterpart $A \vdash B$ both make the similar demand that, assuming that $A$ is certainly true, truth must be the only feasible value for $B$:

$$
\{ \Pr(B = T | A = T) \} = 1
$$

Likewise the negative quotient-feasibility conditional $\emptyset : A \vdash^* \overline{B}$ (equivalently $\emptyset : A \vdash \overline{B}$) and its boolean-feasibility counterpart $A \vdash \overline{B}$ (equivalently $A \vdash \overline{B}$) both make the similar demand that, assuming that $A$ is certainly true, falsity must be the only feasible value for $B$:

$$
\{ \Pr(B = T | A = T) \} = 0
$$
Recall from Section 3.4.1 and Appendix A that an optimization problem with the fractional polynomial objective \( Pr(B = T | A = T) \), meaning the quotient \( Pr(A = T, B = T) / Pr(A = T) \), is considered infeasible if zero is the only feasible value of the objective’s denominator \( Pr(A = T) \). Hence both types of conditionals \( A \rightarrow_k B \) and \( \emptyset : A \rightarrow^*_k B \) are false if it is certain \( a \text{ priori} \) that the antecedent \( A \) cannot be true, regardless of the value of the respective parameter \( K \) or \( k \). However if it is feasible but not necessary that \( Pr(A = T) = 0 \) (meaning \( 0 \in \{ Pr(A = T) \} \) and \( \{ Pr(A = T) \} \neq \{ 0 \} \)), then the indefinite value \( 0/0 \) is a feasible value of the computed conditional probability \( Pr(B = T | A = T) \) even if a conditional of either type is satisfied. If desired, an explicit constraint \( Pr(A = T) > 0 \) could be added in order to forbid this possibility of an indefinite quotient.

Propositional-calculus conditionals (TRUTH-FUNCTIONAL and BOOLEAN-FEASIBILITY) allow some results to be computed by techniques which are different from the algebraic and numerical calculations required by their probabilistic counterparts (MATERIAL and BOOLEAN-FEASIBILITY). For small problems the propositional-calculus formulations support direct and exhaustive enumeration of various sets of formulas and valuations. For larger problems, Boolean translation of propositional-calculus formulas (now into polynomials with coefficients in the finite field \( F_2 \) instead of the real numbers \( \mathbb{R} \)) facilitates automated search through finite sets of equivalence classes of logical formulas.

3.7.3 Hierarchy Among Probabilistic Conditionals

Next let us discuss the subsumption relationships among conditionals of three of the probabilistic types (SUBJUNCTIVE, MATERIAL, and EXISTENTIAL), using their affirmative variants. Consider two binary distinctions: first \( Pr(A = T) = 0 \) versus \( Pr(A = T) > 0 \); and second \( Pr_0(B = T | A = T) < 1 \) versus \( Pr_0(B = T | A = T) = 1 \). Using the probability model in Equation 1 let us denote the first distinction as \( x = 0 \) versus \( x > 0 \), and the second distinction as \( y < 1 \) versus \( y = 1 \). Integrating these two binary distinctions gives four possible configurations:

\[
\begin{array}{c|c}
\text{MATERIAL} & x = 0, y = 1 \\
\hline
\text{SUBJUNCTIVE} & x > 0, y = 1 \\
\hline
\text{EXISTENTIAL} & x > 0, y = 1 \\
\end{array}
\]

The MATERIAL conditional defined by Equation 24 is satisfied in three of these four configurations (either \( x = 0 \), or \( y = 1 \)); it is the least restrictive statement. The SUBJUNCTIVE conditional defined by Equation 17 is satisfied in two of these four configurations (\( y = 1 \)); it is intermediate. The EXISTENTIAL conditional defined by Equation 25 is the most restrictive statement, being satisfied by only one of the four configurations (\( x > 0 \) and \( y = 1 \)).

Therefore we can identify the affirmative forms of the three basic probabilistic conditionals by the configurations of the values \( x \) (representing \( Pr(A = T) \)) and \( y \) (representing \( Pr_0(B = T | A = T) \)) consistent with each:

\[
\begin{array}{c|c|c}
\text{MATERIAL} & A \leftrightarrow B & x = 0, y = 1 \\
\hline
\text{SUBJUNCTIVE} & A \rightarrow B & x > 0, y = 1 \\
\hline
\text{EXISTENTIAL} & A \Rightarrow B & x > 0, y = 1 \\
\end{array}
\]

You can see regarding these affirmative conditionals that if the EXISTENTIAL conditional holds then the SUBJUNCTIVE conditional must also, and in turn if the SUBJUNCTIVE conditional holds then the MATERIAL conditional must also.

4 DISTINCTIVE FEATURES OF CONDITIONALS

Let us compare a few features of the various types of conditionals just defined. We shall examine four features:

- **Conditional probability**: what a probabilistic conditional ‘If \( A \) then \( B \)’ says about the conditional probability that \( B \) is true given that \( A \) is true.

- **Consistency of opposites**: what happens when opposing conditionals ‘If \( A \) then \( B \)’ and ‘If \( A \) then not-\( B \)’ are asserted together.

- **Existential import**: what information the conditionals ‘If \( A \) then \( B \)’ and ‘If \( A \) then not-\( B \)’ (considered separately or together) convey about the truth of their antecedent \( A \).
Table 3 Features of various types of probabilistic conditionals with antecedent A and consequent B: the possible values of the computed conditional probability shown; the result of asserting opposite (affirmative and negative) conditionals simultaneously; the type of existential import; and the result of an antecedent that is certainly false.

| Type      | Cond. Prob. | Opposites | Existential Import | False Antecedent |
|-----------|-------------|-----------|--------------------|------------------|
| Subjunctive | k or 0/0   | inconsistent | none               | $A \leadsto_k B$ unaffected |
| Material  | k or 0/0   | consistent  | indirect           | $A \leftarrow_k B$ true |
| Existential | k          | inconsistent | direct             | $A \leftrightharpoonup_k B$ false |
| Feasibility | k or 0/0   | inconsistent | direct             | $A \vdash_k B$ false |

○ **False antecedents**: what a false antecedent $A$ means for the conditionals ‘If $A$ then $B’’ and ‘If $A$ then not-$B’’.

Different types of conditionals exhibit different patterns of behavior. No particular set of features is intrinsically right or wrong. Instead it behooves the user to choose the type of conditional statement that provides the desired features for any given instance of analysis. Table 3 summarizes the features which are discussed.

### 4.1 Implications for Conditional Probability

Each probabilistic conditional with antecedent $A$, consequent $B$, and parameter $k$ says in a slightly different way that the conditional probability that $B$ is true given that $A$ is true equals $k$. Recall from Equation 5 in Section 2.2 that the conditional probability $Pr(B = T|A = T)$ is computed from the probability model in Equation 1 as the following quotient:

$$Pr(B = T|A = T) \Rightarrow \frac{xy}{x}$$

(55)

Here $x$ is the input $Pr_0(A = T)$ and $y$ is the input $Pr_0(B = T|A = T)$, both from Equation 1. There are three different ways to fix the value of this quotient $xy/x$ to equal some specified constant value $k$: the SUBJUNCTIVE conditional $A \leadsto_k B$ says $y = k$; the MATERIAL conditional $A \leftarrow_k B$ says $xy = kx$; and the EXISTENTIAL conditional $A \leftrightharpoonup_k B$ says both $xy = kx$ and $x > 0$. The SUBJUNCTIVE and MATERIAL constraints allow two possible results for the computed value of $Pr(B = T|A = T)$: either this value is $k$, or it is undefined (because of division by zero):

$$Pr(B = T|A = T) \Rightarrow \begin{cases} k, & \text{if } Pr(A = T) > 0 \\ 0/0, & \text{if } Pr(A = T) = 0 \end{cases}$$

(56)

On the other hand the EXISTENTIAL constraint requires that the computed conditional probability must have the definite value $k$ (since having the denominator equal zero is specifically forbidden):

$$Pr(B = T|A = T) \Rightarrow k$$

(57)

In contrast to their MATERIAL and EXISTENTIAL counterparts, probabilistic FEASIBILITY conditionals express a different idea of ‘conditional’ probability—now using the algebraic operation of constraint instead of the arithmetical operation of division as the mechanism to implement conditioning. Thus instead of considering the possible values of the conditional-probability expression $Pr(B = T|A = T)$, for the FEASIBILITY conditional we consider the possible values of the unconditioned-probability expression $Pr(B = T)$ subject to the constraint that the unconditioned-probability expression $Pr(A = T)$ must equal 1. Following Equation 28 the FEASIBILITY version of the conditional ‘If $A$ then $B’’ is defined using the template $\Gamma \vdash_k B$ with the antecedent constraint-set $\Gamma \equiv \{ Pr(A = T) = 1 \}$ containing the assertion that $A$ must certainly be true:

$$\{ Pr(A = T) = 1 \} \vdash_k B = \{ Pr(B = T) : Pr(A = T) = 1 \} = \{ k \}$$

(58)

This particular FEASIBILITY conditional can be written in abbreviated form as $A \vdash_k B$. There are two possible computed values for the solution set that is constrained by $A \vdash_k B$. If it is feasible that $Pr(A = T) = 1$ (satisfying also
the general constraints of probability for the model in use), then the solution set will have the computed value \( \{ k \} \); otherwise the computed solution set will be empty. Thus in parallel with Equation \( 56 \) above we have two possible consequences of the asserted \textsc{feasibility} conditional \( A \vdash_k B \):

\[
\{ \Pr (B = T) : \Pr (A = T) = 1 \} \Rightarrow \begin{cases} \{ k \}, & \text{if } \Pr (A = T) = 1 \text{ is feasible} \\ \emptyset, & \text{if } \Pr (A = T) = 1 \text{ is infeasible} \\ (\text{but } \Pr (A = T) > 0 \text{ is feasible}) \end{cases}
\]

(59)

After symbolic probability inference using the probability model in Equation \([1]\) the \textsc{feasibility} conditional \( A \vdash_k B \) says:

\[
\{ z + xy - xz : x = 1; x, y, z \in [0, 1] \} = \{ k \}
\]

(60)

Note that after substituting \( 1 \) for \( x \) the objective \( z + xy - xz \) simplifies to the polynomial \( y \). Thus using the probability model in Equation \([1]\) the conditional \( A \vdash_k B \) requires that \( y = k \) and that it is feasible that \( x = 1 \).

4.2 Consistency of Opposites

Whether it is consistent or inconsistent to assert opposing conditionals like ‘If \( A \) then \( B \)’ and ‘If \( A \) then not-\( B \)’ depends upon the type of the conditional statements.\(^2\) For four of the six principal types of conditionals, it is inconsistent in a simple algebraic sense to assert opposing conditionals; for the other two types it is consistent. For \textsc{subjunction} conditionals as defined by Equation \([17]\) the affirmative statement \( A \leadsto B \) (meaning \( A \leadsto B \)) and the negative statement \( A \leadsto_0 B \) together specify the following pair of equations:

\[
\Pr_0 (B = T | A = T) = 1 \quad (61)
\]

\[
\Pr_0 (B = T | A = T) = 0 \quad (62)
\]

meaning that the same input value must simultaneously equal the real number \( 1 \) and the real number \( 0 \); this is obviously impossible. For example using the probability model from Equation \([1]\) the relevant equations are \( y = 1 \) and \( y = 0 \), which cannot be satisfied simultaneously by any real value of \( y \).

For \textsc{existential} conditionals as defined by Equation \([25]\) the affirmative statement \( A \lrarrows B \) and the negative statement \( A \lrarrows_0 B \) together specify the following system of equations and inequalities:

\[
\Pr (A = T, B = T) = \Pr (A = T) \quad (63)
\]

\[
\Pr (A = T, B = T) = 0 \quad (64)
\]

\[
\Pr (A = T) > 0 \quad (65)
\]

This system is algebraically inconsistent. The first two equations require \( \Pr (A = T) \) to equal \( 0 \) yet the final inequality requires \( \Pr (A = T) \) to be strictly greater than \( 0 \). For example using the probability model from Equation \([1]\) the relevant constraints are \( xy = x, xy = 0 \), and \( x > 0 \), which cannot be satisfied by any real values of \( x \) and \( y \).

For \textsc{feasibility} conditionals the affirmative statement \( A \vdash^* B \) and the negative statement \( A \vdash^*_0 B \) as defined by Equation \([28]\) together specify the following pair of equations:

\[
\{ \Pr (B = T) : \Pr (A = T) = 1 \} = \{ 1 \} \quad (66)
\]

\[
\{ \Pr (B = T) : \Pr (A = T) = 1 \} = \{ 0 \} \quad (67)
\]

meaning that the same set of real numbers must simultaneously equal \( \{ 1 \} \) and \( \{ 0 \} \), which is algebraically impossible. For example using the probability model from Equation \([1]\) the relevant equations are:

\[
\{ z + xy - xz : x = 1; x, y, z \in [0, 1] \} = \{ 1 \} \quad (68)
\]

\[
\{ z + xy - xz : x = 1; x, y, z \in [0, 1] \} = \{ 0 \} \quad (69)
\]

\(^2\)Note that with every type of conditional, it is different to negate the entire conditional statement versus negating just its consequent. That is, the statement ‘Not (If \( A \) then \( B \))’ is mathematically distinct from the statement ‘If \( A \) then not-\( B \)’ in every interpretation.
There are no real values \((x, y, z)\) that satisfy both equations; that would require \(y = 1\) and \(y = 0\) simultaneously.

Likewise for BOOLEAN-FEASIBILITY conditionals the affirmative statement \(A \vdash B\) and the negative statement \(A \vdash_0 B\) (meaning \(A \vdash \neg B\)) together specify the following pair of equations:

\[
\begin{align*}
\{ B : A = T \} &= \{ T \} \\
\{ B : A = 0 \} &= \{ F \}
\end{align*}
\]

This system is inconsistent because the same set of truth values cannot simultaneously equal \(\{ T \}\) and \(\{ F \}\). Using the Boolean polynomial translation method from Section 2.3 these equations about logical formulas become equations about polynomial formulas:

\[
\begin{align*}
\{ b : a = 1; a^2 = a, b^2 = b \} &= \{ 1 \} \\
\{ b : a = 1; a^2 = a, b^2 = b \} &= \{ 0 \}
\end{align*}
\]

There are no real values \((a, b)\) that satisfy both equations; that would require \(b = 1\) and \(b = 0\) simultaneously.

In contrast, opposing conditionals of the remaining two types are not inconsistent with one another. For MATERIAL conditionals as defined by Equation 22 the affirmative statement \(A \leftrightarrow B\) and the negative statement \(A \leftrightarrow_0 B\) together specify the following pair of equations using probability expressions (which map to polynomials using the symbolic probability inference method of Section 2.2):

\[
\begin{align*}
\Pr(A = T, B = T) &= \Pr(A = T) \\
\Pr(A = T, B = T) &= 0
\end{align*}
\]

These equations are satisfied simultaneously when \(\Pr(A = T) = 0\). For example using the probability model from Equation 1 the relevant equations are \(xy = x\) and \(xy = 0\), which are both satisfied when \(x = 0\). For TRUTH-FUNCTIONAL conditionals as defined by Equation 33 the affirmative statement \(A \rightarrow B\) and the negative statement \(A \rightarrow_0 B\) together specify the following pair of equations using propositional-calculus formulas (which map to polynomials using the Boolean translation method of Section 2.3):

\[
\begin{align*}
(A \rightarrow B) &= T \\
(A \rightarrow \neg B) &= T
\end{align*}
\]

Both formulas \(A \rightarrow B\) and \(A \rightarrow \neg B\) are true when \(A\) is false. For example after Boolean translation these two equations about logical formulas become a system of equations about polynomials with real coefficients:

\[
\begin{align*}
1 - a + ab &= 1 \\
1 - a + a(1 - b) &= 1 \\
a^2 &= a \\
b^2 &= b
\end{align*}
\]

You can see by inspection that this system is satisfied for \(a = 0\) and any value of \(b\). The number 0 of course represents logical falsity in Boole’s translation scheme.

4.3 Existential Import

Something interesting happens when opposing conditionals of the MATERIAL or TRUTH-FUNCTIONAL types are combined: they tell us that the antecedent must be false. In the MATERIAL case the opposing conditionals \(A \leftrightarrow B\) (meaning \(A \leftrightarrow_1 B\)) and \(A \leftrightarrow_0 B\) are satisfied simultaneously if and only if \(\Pr(A = T) = 0\) (meaning that the antecedent \(A\) is certainly false). In fact this conclusion would follow from any material conditionals \(A \leftrightarrow_{k_1} B\) and \(A \leftrightarrow_{k_2} B\) with different sense-parameters \(k_1 \neq k_2\). In the TRUTH-FUNCTIONAL case the opposing conditionals \(A \rightarrow B\) and \(A \rightarrow_0 B\) (meaning \(A \rightarrow \neg B\)) are satisfied simultaneously if and only if \(A = F\) (meaning that the antecedent \(A\) is identically false). Let us call this phenomenon ‘indirect existential import’: whereas no individual conditional statement limits the possible values of the antecedent, pairs of opposite conditionals indeed do so.
In contrast, EXISTENTIAL, FEASIBILITY, and BOOLEAN-FEASIBILITY conditionals possess ‘direct existential import’. If an individual conditional of one of these types is true, then it must be at least possible for the antecedent to be true (though it need not be certain that the antecedent is true). An EXISTENTIAL conditional \( A \vdash_k B \) requires directly \( \Pr(A = T) > 0 \), stating that the probability that the antecedent \( A \) is true must be strictly greater than zero. A FEASIBILITY conditional \( \Gamma \vdash_k B \) requires it to be possible for the antecedent constraints \( \Gamma \) to be satisfied. Using \( \{ \Pr(A = T) = 1 \} \) as the antecedent constraint-set \( \Gamma \), the resulting FEASIBILITY conditional \( A \vdash_k B \) requires that it is feasible for the the antecedent \( A \) to be certainly true (which is different from the requirement \( \Pr(A = T) > 0 \)). Likewise the BOOLEAN-FEASIBILITY conditional \( A \vdash B \) requires that it is possible for its antecedent \( A \) to be true.

Finally, SUBJUNCTIVE conditionals have no intrinsic existential import of either the direct or the indirect kind. In order to express a SUBJUNCTIVE conditional the probability model must have the antecedent \( A \) as an ancestor of the consequent \( B \) in the probability network graph, as in the model in Equation 11. With this setup the input \( \Pr_0(B = T | A = T) \) does not concern the probability \( \Pr(A = T) \) at all; when \( \Pr(A = T) \) is calculated by symbolic probability inference (as described in Section 2.2) the input \( \Pr_0(B = T | A = T) \) cancels out algebraically. A particular probability model could introduce direct existential import to a SUBJUNCTIVE conditional by using the same parameters to specify the input \( \Pr_0(B = T | A = T) \) and for example the input \( \Pr_0(A = T) \), or by including constraints relating these input values. In the absence of such features, SUBJUNCTIVE conditionals remain free of existential import.

4.4 Consequences of False Antecedents

Conditionals of the various types behave differently when their antecedents are false. Usually, the truth or falsity of its antecedent \( A \) has no effect on the truth or falsity of a SUBJUNCTIVE conditional \( A \rightarrow B \). As stated in the last section the input \( \Pr_0(B = T | A = T) \) constrained by \( A \rightarrow B \) is unrelated to the computed probability \( \Pr(A = T) \), unless the probability model includes specific constraints (or common parameters) that link these two values.

With \( \Pr(A = T) = 0 \), a MATERIAL conditional \( A \rightarrow_k B \) using any value of \( k \) must be true. In this case Equation 22 defining the conditional \( A \rightarrow_k B \) simplifies to the trivial requirement \( 0 = 0 \). Likewise with \( A = F \) both TRUTH-FUNCTIONAL conditionals \( A \rightarrow B \) simplifies to \( B \) must be true. Using the propositional calculus, both equations \( (F \rightarrow B) = T \) and \( (F \rightarrow \neg B) = T \) described by Equation 23 are correct, because each of the included formulas of material implication simplifies to the elementary truth-value \( T \).

Conditionals of the remaining types are always false when their antecedents are false. With \( \Pr(A = T) = 0 \) the constraint \( \Pr(A = T) > 0 \) required by Equation 25 for the EXISTENTIAL conditional \( A \vdash_k B \) is violated. Also with \( \Pr(A = T) = 0 \) the constraint \( \Pr(A = T) = 1 \) included in the solution set from Equation 28 for the FEASIBILITY conditional \( A \vdash_k B \) (meaning \( \{ \Pr(A = T) = 1 \} \vdash_k B \)) is violated, leading to an empty solution set which therefore cannot equal any set \( \{ k \} \). Similarly with \( A = F \) the constraint \( A = T \) in the solution set from Equation 41 for the BOOLEAN-FEASIBILITY conditional \( A \vdash_k B \) is violated, leading to an empty solution set that cannot equal \( \{ T \} \) nor \( \{ F \} \).

5 DETAILS: FACTUALITY, BASIC MEASURES, AND REVISION

Let us now address a few details of the mathematical framework for analyzing conditional statements which was just introduced: factuality and counterfactuality; correctness and exception handling; and diverse basic measures underlying probabilities. There are several related properties to appreciate about conditional statements. We have already considered mood, using the distinction between subjunctive and several kinds of indicative conditionals. These other properties are conceptually separate. For example, a conditional statement can be factual or counterfactual, independently of whether it is subjunctive or indicative. Moreover, a conditional could be correct or incorrect, whatever combination of mood and factuality it has. What is believed to be correct or incorrect may change over time as the knowledge of the analyst changes. Finally, different conditional statements, even those describing the same set of events or propositions, may in fact concern measures of different basic properties: observed frequency, subjective belief, causal propensity, theoretical symmetry, and so on. All of these features can be mixed and matched in order to express precisely the semantic message that the analyst wishes to express.
Table 4  Selected properties of common United States coins, using data from [http://www.usmint.gov](http://www.usmint.gov). Here 'Cu' abbreviates the element copper and 'M' stands for million. Each penny has one image of President Lincoln on it; coins of the other denominations have none. Reeds are the small ridges along the edges of some coins.

| Name  | Cu-plating? | Edge   | Value | Mass   | 2013 Production | Lincolns | Fraction Cu |
|-------|-------------|--------|-------|--------|-----------------|----------|-------------|
| Penny | yes         | smooth | 1¢    | 2.500 g | 7070.00 M coins | 1 image  | 2.50 %      |
| Nickel| no          | smooth | 5¢    | 5.000 g | 1223.04 M coins | 0 images | 75.00 %     |
| Dime  | no          | reeded | 10¢   | 2.268 g | 2112.00 M coins | 0 images | 91.77 %     |
| Quarter| no       | reeded | 25¢   | 5.670 g | 1455.20 M coins | 0 images | 91.77 %     |

5.1 Factuality and Counterfactuality

Let us say that the 'factuality' of a conditional statement pertains to specific identified facts, and describes whether each fact or its negation is included explicitly in the conditional statement. We recognize three states of factuality for a conditional statement relative to a given fact:

- **factuality** if the fact appears in the conditional;
- **antifactuality** if the negation of the given fact appears in the conditional; and
- **afactuality** if neither the fact nor its negation appears in the conditional. Afactuality and antifactuality are our two kinds of counterfactuality; the former is weaker and the latter is stronger. A single conditional could be both factual and antifactual with respect to some particular fact, if both the fact and its negation were to appear in that conditional (though legal, that would be an odd construction to make). Although factuality generally concerns antecedent part of the conditional in question, we shall also include the consequent part in our deliberations.

For example if the truth of some logical proposition $C$ were accepted as a fact, then all of these conditionals would be factual with respect to this fact $C = T$:

\[
\{A, C\} \rightarrow B \quad C \rightarrow B \quad C \rightarrow B \quad \{A_1, A_2, C\} \vdash B \quad \{A, C\} \vdash^* \overline{B} \quad A \rightarrow C
\]

However if the known fact were instead $A = T$, then relative to this fact: the first conditional would be antifactual; the second, third, and fourth conditionals would be afactual; and the fifth and sixth conditionals would be factual.

As mentioned above, **factuality** and **mood** are two different properties. In principle any combination of {factual, afactual, antifactual} and {subjunctive, indicative} can occur within a single conditional statement. However certain other features of conditionals may correlate with these two properties. For example, an antifactual indicative conditional (whose antecedent contains the negation of a known fact) would have to be correct if its type were MATERIAL or TRUTH-FUNCTIONAL, but incorrect if it had any other type. Anyway, it is important not to confuse subjunctiveness with counterfactuality; they are two different properties.

5.2 Diverse Basic Measures

Probabilities can be understood as ratios of basic measures (in the measure-theoretic sense). Basic measures may refer to many different properties of objects or events. Therefore, probabilistic conditionals may refer to different properties of the propositions that they concern. Even TRUTH-FUNCTIONAL and BOOLEAN-FEASIBILITY conditionals can be understood to invoke probabilities and hence diverse basic measures. It can be very important to understand which basic measure is described by a particular conditional statement.

For a simple illustration, consider certain properties of several U.S. coins which are listed in Table 4. Let us first ask, what is the probability of a dime? Using face monetary value as the basic measure, we compute the following measures for the relevant events:

\[
\mu(\{\text{dime}\}) \implies 10\, \epsilon \\
\mu(\{\text{penny, nickel, dime, quarter}\}) \implies 41\, \epsilon
\]

The probability of the event \{dime\} relative to the universal event \{penny, nickel, dime, quarter\} (generally denoted $\Omega$) is given by the ratio of these measures: $10\epsilon/41\epsilon$, which is approximately 0.244. Different basic measures give different numerical probabilities. For example using mass as the basic measure, $\Pr(\{\text{dime}\})$ evaluates to $2.268\, \text{g}/15.438\, \text{g}$.
which is approximately 0.147. Or, using the cardinality of the set that constitutes each event as its measure, the probability \( \Pr(\{\text{dime}\}) \), which in fact designates the conditional probability \( \Pr(\{\text{dime}\} \mid \{\text{penny, nickel, dime, quarter}\}) \), evaluates to 1/4.

Note a few important features of probabilities: probabilities are inherently conditional (the universal set \( \Omega \) containing all elementary events under consideration gives the default denominator); basic measures have units of measure (cents, grams, and so on), which cancel out during division; measures are assigned to sets of elementary events; measures are by definition additive with the union of disjoint sets; the empty set must have measure zero; other sets of elementary events may also have measure zero. Note also that not every numerical property behaves like a measure. For example the fraction of copper is not additive with set union: although a dime contains 91.77% copper and a quarter also contains 91.77% copper, it is not the case that the set containing both a dime and a quarter itself contains 183.54% copper. Moving on, in the special case that \( \mu(\Omega) = 0 \) all probabilities for the system in question have the indefinite value 0/0. Uncountable or countably infinite sets of elementary events require special attention as well.

Next let us ask, what is the conditional probability of copper plating (denoted \( B \)), given smooth edge (denoted \( A \))? Using monetary value as the basic measure gives \( \Pr(B = T \mid A = T) = 1/6 \). Using Lincoln images as the basic measure gives \( \Pr(B = T \mid A = T) = 1 \). Returning our attention to conditionals, let us observe that the affirmative \textsc{material} conditional \( A \leftrightarrow B \) (defined by Equation \ref{eq:22} with \( k = 1 \)) is correct using Lincoln images as the basic measure, but incorrect using monetary value as the basic measure. The choice of basic measure can affect the truth or falsity of a conditional statement.

Note that for some very specific basic measures we do not know quite what the appropriate units of measure are. For example, what is the unit of subjective belief? Or of causal propensity? One potential basic measure for historical events is an indicator-type measure of what actually happened: we might assign one (anonymous) unit of measure to the event that actually happened, and zero units to every event that did not happen. Here we see one benefit of the construction of probability: it is possible to reason about ratios of measures whose units we do not know, since those units cancel out during the division used to compute the ratios. We must take care to recognize the potential for division by zero, and to handle that exception appropriately.

We come to an important point. Different types of conditionals tend to use different basic measures. \textsc{subjunctive} conditionals tend to use basic measures such as subjective belief, symmetry (symmetrical cases, as in probability 1/2 for a coin to land on heads because it has two symmetrical sides), or causal propensity. In contrast, the various indicative conditionals tend to use basic measures such as occurrence (e.g. the indicator just discussed), observed cases (whose ratio gives frequency), or physical properties such as mass.

5.3 Correctness and Revision for Exceptions

Any given conditional statement may be correct or incorrect (true or false). In some cases that correctness or incorrectness derives purely from the syntax of the statement. For example, the \textsc{boolean-feasibility} conditional \( \text{A} \models \overline{\text{A}} \), which says ‘Assuming that \text{A} is true then \text{A} must be false’ according to Equation \ref{eq:41} is intrinsically incorrect without regard to which logical proposition or real-world event the variable \text{A} represents. In other cases the correctness or incorrectness of a conditional statement depends upon the referents of its variables. For example, the \textsc{boolean-feasibility} conditional \( \text{A} \models \text{B} \) might or might not be true, depending on which propositions or events the variables \text{A} and \text{B} represent.

Critically, an analyst’s impression about the correctness of any given conditional statement may change over time as that analyst’s knowledge changes. It is appropriate that our models should change as our knowledge does. It is important to allow some mechanisms for revision or exception handling. Thus we might supplement the statistician Box’s observation that “All models are wrong, but some are useful” with a paraphrase of Shakespeare: Some models are born wrong, and some have wrongness thrust upon them.

Considering the diverse basic measures above, it is natural that some of these measures should change over time. The physical universe changes. And the subjective beliefs of an analyst might change too. Here it is helpful to think in terms of absolute units of belief rather than ratios of them. A common motif in problems is learning about an exception to a rule. This may seem tricky in terms of probabilities, but it is more straightforward in terms of basic measures.

\footnote{You may be tempted to insert a self-reference here. That would be a fine way to formulate a recurrence relation that defines a discrete dynamical system, amenable to algebraic analysis as discussed in [12].}
Also note that it may not be possible or practical to verify whether any given conditional statement is true or false, because the underlying basic measure cannot practically be measured. We have as yet no instruments to test subjective belief. It is a matter of scientific experimentation and statistical analysis to establish measures of causal propensity. Even symmetry may take some nontrivial investigation to understand.

5.3.1 Revision of Old Conditionals

Our models may change as our knowledge does. Consider two options which may be useful. First, have the ability to retract an old conditional statement upon learning about an exception (and then assert a new one). This requires non-monotonicity. Second, formulate (some or all) conditionals as defeasible statements in the first place (using nullifier terms in the antecedents), so that any statement can be rendered ineffectual without being retracted (‘defeated without being deleted’). Now we need only the ability to add fixed values for propositional variables at run time (and to answer queries conditioned on the values of the nullifier terms, so that they can be ignored when it is desired to consider the default case).

5.3.2 Defeasible Conditionals

One option is to include anonymous ‘nullifier’ terms to represent otherwise unexpected exceptions. For example instead of \( A \vdash B \) we could use \( (A \land \neg Z) \vdash B \) to say ‘If \( A \) then \( B \), unless \( Z \)’. Another option is to use probabilities other than zero and one, for example always 0.001 and 0.999 instead (or a fancier solution such as \( \delta \) instead of 0, and \( 1 - \delta \) instead of 1, with \( 0 \leq \delta \leq 0.001 \)).

6 ALGEBRAIC DEDUCTION

A conditional of any principal type can be interpreted as a statement of logical deduction. Among the various types, FEASIBILITY and BOOLEAN-FEASIBILITY conditionals are especially well-suited for this purpose: they reproduce intuitive patterns of inference such as modus ponens; they avoid explosions of consequences from inconsistent premises; and they provide frameworks within which conditionals of other types may be included. In this section we shall investigate how certain methods of algebra can be used to compute logical deductions involving probabilistic, truth-functional, and Boolean conditionals. As we shall see, the solutions to various systems of polynomial equalities and inequalities provide interesting results.

6.1 Probabilistic Deduction

Probabilistic FEASIBILITY conditionals can be interpreted as statements of logical deduction. Using this interpretation we can read the following affirmative conditional from Equation 28 (with \( k = 1 \)):

\[
\Gamma \vdash^* B \equiv \{ \Pr (B = T) : \Gamma \} = \{1\}
\] (85)

as the deductive statement that ‘\( B \) is a consequence of \( \Gamma \)’. In this interpretation we consider the members of the antecedent set \( \Gamma \) to be premises, and the consequent \( B \) to be the conclusion. Recalling the earlier description this deductive statement also says that ‘Subject to the constraints in \( \Gamma \), the probability that \( B \) is true must be exactly 1’. As with all conditionals, any particular deductive statement \( \Gamma \vdash^* B \) itself may be true or false. The methods of Section 2.2 and 2.5 allow us to compute which is the case. Let us say that this formulation provides one of many possible notions of consequentiality involving logical propositions.

For example, to demonstrate modus ponens using the probability model from Equation 1, let us specify as the antecedent set \( \Gamma \) the constraint \( \Pr (A = T) \) that \( A \) is certainly true, and some version of the affirmative conditional ‘If \( A \) then \( B \)’. Using the MATERIAL version of this affirmative conditional gives the constraint \( \Pr (A = T, B = T) = \Pr (A = T) \) from Equation 22. We identify \( \Pr (B = T) \) as the objective function. The resulting solution set is given by:

\[
\Phi \Leftarrow \{ \Pr (B = T) : \Pr (A = T) = 1, \Pr (A = T, B = T) = \Pr (A = T) \}
\] (86)

After symbolic probability inference, this set-comprehension expression becomes precisely one presented in Equation 15. As demonstrated in Section 2.5 the computed minimum solution \( \alpha^* = 1.000 \) and the maximum solution
\[ \beta^* = 1.000. \] By these calculations the solution set \( \Phi \Rightarrow \{1\} \). In other words, we have computed that the following equation about sets of real numbers is satisfied:

\[ \{ \Pr (B = T) : \Pr (A = T) = 1, \Pr (A = T, B = T) = \Pr (A = T) \} = \{1\} \]  

(87)

It follows that the equivalent FEASIBILITY conditional from Equation\[85\]is true:

\[ \{ A, A \rightarrow B \} \models B \]

(88)

Therefore it is correct to say that, assuming the premises that \( A \) is certainly true and that the affirmative MATERIAL conditional ‘If \( A \) then \( B \)’ holds, it is a consequence that \( B \) must certainly be true. For this example, it happens that substituting either the SUBJUNCTIVE or the EXISTENTIAL interpretation of the inner conditional ‘If \( A \) then \( B \)’ would also give a true outer FEASIBILITY conditional; both of the following statements are correct:

\[ \{ A, A \rightarrow B \} \models B \]

\[ \{ A, A \leftrightarrow B \} \models B \]

(89)

### 6.1.1 Probabilistic Conditionals as Consequences

We can use probability equations similar to the FEASIBILITY conditional in Equation\[23\]in order to determine whether or not certain proposed SUBJUNCTIVE, MATERIAL, or EXISTENTIAL conditionals are correct subject to given sets of constraints. Let us focus on affirmative conditionals (with fractional parameter \( k = 1 \)). Consider the following solution sets which make use of a given set \( \Gamma \) of input constraints, with certain probability expressions involving the primary variables \( A \) and \( B \) as objective functions and additional constraints:

\[ \Phi \leftarrow \{ \Pr_0 (B = T | A = T) : \Gamma \} \]  

(90)

\[ \Psi \leftarrow \{ \Pr (A = T) - \Pr (A = T, B = T) : \Gamma \} \]  

(91)

\[ \Upsilon \leftarrow \{ \Pr (A = T) : \Pr (A = T) = \Pr (A = T, B = T), \Gamma \} \]  

(92)

If (and only if) the input constraints \( \Gamma \) are inconsistent (when considered together with the default probability constraints \( \Gamma_0 \)), then all three solution sets \( \Phi, \Psi, \) and \( \Upsilon \) will evaluate to the empty set. Otherwise the values of these solution sets reveal which probabilistic conditional statements are necessarily correct and which are possibly correct.

The affirmative SUBJUNCTIVE conditional \( A \rightarrow B \) defined by Equation\[41\]requires \( \Pr_0 (B = T | A = T) = 1 \), hence \( \Phi = \{1\} \) in terms of Equation\[90\]Thus it is necessary that \( A \rightarrow B \) is correct if the computed value of \( \Phi \) is precisely the set \( \{1\} \). If \( \Phi \) merely contains \( 1 \) then it is possible that \( A \rightarrow B \) is correct. If \( \Phi \) does not contain \( 1 \) then it is impossible that \( A \rightarrow B \) is correct. Likewise, the affirmative MATERIAL conditional \( A \rightarrow B \) defined by Equation\[22\]requires \( \Pr (A = T) = \Pr (A = T, B = T) \), hence \( \Psi = \{0\} \) in terms of Equation\[91\]It follows that \( A \rightarrow B \) is necessarily correct if \( \Psi = \{0\} \); possibly correct if \( 0 \notin \Psi \); and impossibly correct if \( 0 \notin \Psi \). The EXISTENTIAL conditional \( A \leftrightarrow B \) from Equation\[22\]adds to its MATERIAL counterpart the requirement \( \Pr (A = T) > 0 \). In terms of Equation\[92\]this means the solution set \( \Upsilon \) must contain only nonzero values; hence \( \Upsilon \neq \emptyset \) and \( 0 \notin \Upsilon \). It follows that \( A \leftrightarrow B \) is necessarily correct if \( \Upsilon \neq \emptyset \) and \( 0 \notin \Upsilon \); possibly correct if \( \Upsilon \neq \emptyset \) and \( \Upsilon \notin \{0\} \); and impossibly correct if \( \Upsilon = \emptyset \) or \( \Upsilon = \{0\} \).

### 6.2 Boolean Deduction

BOOLEAN FEASIBILITY conditionals can be interpreted as statements of logical deduction. This is in effect what Boole did throughout his *Laws of Thought* by drawing logical conclusions from the solutions to polynomial equations [3]. Considering an antecedent \( A \) and a consequent \( B \) which are formulas of the propositional calculus (using some propositional variables \( X_1, \ldots, X_n \), let us read the affirmative conditional \( A \vdash B \) from Equation\[41\]as the deductive statement ‘\( B \) is a consequence of \( A \)’:

\[ A \vdash B \equiv \{ B : A = T \} = \{T\} \]

(93)

Recalling the earlier description, this statement \( A \vdash B \) also says that ‘Assuming that \( A \) is true, then \( B \) must be true’. After Boolean polynomial translation, this definition becomes:

\[ A \vdash B \equiv \{ \mathcal{B}_1 (B) : \mathcal{B}_1 (A) = 1; x_1 \in \{0, 1\}, \ldots, x_n \in \{0, 1\} \} = \{1\} \]

(94)
The set $\Gamma_0$ of constraints expanded into the above definition includes a constraint $x_i \in \{0, 1\}$ for each real-valued variable $x_i$ translated from a propositional variable $X_i$. Note that the affirmative BOOLEAN-FEASIBILITY conditional $A \vdash B$ in Equation (22) is not the same as the affirmative TRUTH-FUNCTIONAL conditional from Equation (33). In particular, when the antecedent formula $A$ is unsatisfiable (identically false) then the relation $A \rightarrow B$ is true but the relation $A \vdash B$ is not.

We could also modal statements of consequiallity, such as ‘$B$ is a potential consequence of $A$’ meaning $T \in \{ B : A = \top \}$, thus satisfied by either solution set $\{ T \}$ or $\{ T, F \}$ but not by $\{ F \}$ or $\emptyset$.

Boolean deduction as defined by Equation (93) reproduces the familiar pattern of modus ponens. Considering propositional variables $X$ and $Y$, let us take as the antecedent $A$ the assertion that the formulas $X$ and $X \rightarrow Y$ are both true: hence the conjunction $X \land (X \rightarrow Y)$. Does it follow that $Y$ must also be true? Using the formula $Y$ as the consequent $B$, Equation (93) says that the Boolean-deductive statement $X \land (X \rightarrow Y) \vdash Y$ signifies the following equation about sets of truth values:

$$\{ X : X \land (X \rightarrow Y) = \top \} = \{ T \}$$

(95)

After Boolean translation according to Section 2.3, this becomes an equation about sets of real numbers:

$$\{ x : xy = 1, x \in \{0, 1\}, y \in \{0, 1\} \} = \{ 1 \}$$

(96)

Section 2.3 showed the Boolean polynomial translation $\mathcal{B}_Y (X \land (X \rightarrow Y)) \Rightarrow xy$. Algebraic analysis confirms that the value of the set-comprehension expression above is indeed the set $\{1\}$. The corresponding optimization problems:

Minimize : $y$
subject to : $xy = 1$
and : $x \in \{0, 1\}$

Maximize : $y$
subject to : $xy = 1$
and : $x \in \{0, 1\}$

(97)

have computed solutions $\alpha^* = 1.000$ and $\beta^* = 1.000$. Thus it is demonstrated that the statement of Boolean deduction $X \land (X \rightarrow Y) \vdash Y$, interpreted according to Equation (93) is correct. This statement says that, assuming the premise that the formula $X \land (X \rightarrow Y)$ is true, then it is a consequence that the formula $Y$ must be true also.

Logical deduction based on BOOLEAN-FEASIBILITY conditionals exhibits several desirable properties. This methodology provides a paraconsistent system of logic (for formulas of the propositional calculus) that avoids explosion from inconsistent premises, yet retains the intuitive properties of disjunction introduction, disjunctive syllogism, and transitivity.

6.2.1 Boolean Consequences and Sets of Valuations

There is an important difference between the TRUTH-FUNCTIONAL conditionals from Section 3.5 and the BOOLEAN-FEASIBILITY conditionals from Section 3.6. Whereas each TRUTH-FUNCTIONAL conditional operates on individual valuations of the propositional variables in use, each BOOLEAN-FEASIBILITY conditional operates on set of such valuations. Understanding this feature we can develop algebraic problems to determine when proposed BOOLEAN-FEASIBILITY conditionals are true subject to given constraints.

Considering a system with some list $(X_1, \ldots, X_n)$ of propositional variables, let us say that a valuation or valuation vector consists of an assignment of a truth value $K_i \in \{ T, F \}$ to each propositional variable $X_i$; thus a vector $(K_1, \ldots, K_n)$ of truth values. With $n$ propositional variables there are $2^n$ possible valuation vectors. For example with 2 propositional variables $(A, B)$ there are 4 possible valuations $(T, T)$, $(T, F)$, $(F, T)$, and $(F, F)$. The affirmative TRUTH-FUNCTIONAL conditional $A \rightarrow B$ from Equation (33) operates on one valuation at a time. For example with $(A, B) = (T, T)$ the conditional $A \rightarrow B$ is true and with $(A, B) = (T, F)$ the conditional $A \rightarrow B$ is false. A logical truth table summarizes such results. In this case:

| $A$ | $B$ | $A \rightarrow B$ |
|-----|-----|-------------------|
| $T$ | $T$ | $T$               |
| $T$ | $F$ | $F$               |
| $F$ | $T$ | $T$               |
| $F$ | $F$ | $T$               |

(98)
In contrast, the affirmative BOOLEAN-FEASIBILITY conditional from Equations 41 and 93 properly operates on one set of valuations at a time. For example, given the valuation \((A, B) = (T, T)\), is the affirmative BOOLEAN-FEASIBILITY conditional \(A \vdash B\) true or false? Well, that depends on which other valuations are possible. If the valuation \((A, B) = (T, F)\) were also possible then the solution set \(\{B : A = T\}\) from Equation 41 would contain \(F\) as well as \(T\), hence the solution set \(\{T, F\}\). This solution set \(\{T, F\}\) does not equal the set \(\{T\}\) required by the affirmative conditional; thus \(A \vdash B\) would be false. On the other hand, if the valuation \((A, B) = (T, F)\) were impossible, then the solution set \(\{B : A = T\}\) from Equation 41 would contain only the value \(T\), hence the solution set \(\{T\}\) which would make the affirmative conditional \(A \vdash B\) true.

We can use a modified truth table to describe the sets of variable valuations that make a given BOOLEAN-FEASIBILITY conditional true. To this end let us evaluate the set comprehension expression for the conditional \(A \vdash B\) from Equation 41 four separate times, using as input the singleton set \(\{(K_1, K_2)\}\) tabulated by each row of the table:

\[
\begin{array}{ccc}
A & B & \{B : A = T\} \\
T & T & \{T\} \\
T & F & \{F\} \\
F & F & \emptyset \\
F & T & \emptyset \\
\end{array}
\]

The last two sets are empty because the constraint \(A = T\) is violated. These values give instructions for constructing the sets of valuations of \((A, B)\) that satisfy the conditional \(A \vdash B\): each set must include the point \((A, B) = (T, T)\); it must not include the point \((T, F)\); and it may or may not include the points \((F, T)\) and \((F, F)\). These instructions are like the four-part set comprehension expressions that Boole used in [3]. However in this context, in general the mandatory part of the set of valuations is specified as one or more members of an identified subset. Anyway for this example, 4 of the 16 possible sets of valuations match the instructions:

\[
\begin{array}{cccc}
\{(T, T)\}, & \{(T, T), (F, T)\}, & \{(T, T), (F, F)\}, & \{(T, T), (F, T), (F, F)\} \\
\end{array}
\]

For each of these four sets of valuation vectors \((A, B)\) it is true that the set of values of \(B\) subject to the constraint \(A = T\) equals the set \(\{T\}\). Thus each set of valuations satisfies the relation \(\{B : A = T\} = \{1\}\) that defines the affirmative BOOLEAN-FEASIBILITY conditional \(A \vdash B\) according to Equation 41. Note that only one of these four sets of valuations consists of the valuations that satisfy the material-implication formula \(A \rightarrow B\).

6.2.2 Probabilistic Expression of Boolean Feasibility

There is a different way to specify the set of sets of valuations that satisfy a given BOOLEAN-FEASIBILITY conditional. Let us adopt a new probability model in which the joint probability table \(Pr_0(A, B)\) is input directly:

\[
\begin{array}{ccc}
A & B & Pr_0(A, B) \\
T & T & x_1 \\
T & F & x_2 \\
F & T & x_3 \\
F & F & x_4 \\
\end{array}
\]

The constraints \(0 \leq x_i \leq 1\) and \(x_1 + x_2 + x_3 + x_4 = 1\) are added to enforce the laws of probability. We embed the propositional-calculus formulas \(A \rightarrow B\) and \(A \land B\) by the method of Section 2.4:

\[
\begin{array}{ccc}
Pr_0(\langle A \land \lnot B \rangle | A, B) & Pr_0(\langle A \land B \rangle | A, B) \\
A & B & \langle A \land \lnot B \rangle = T & \langle A \land \lnot B \rangle = F \\
T & T & 0 & 1 \\
T & F & 1 & 0 \\
F & T & 0 & 1 \\
F & F & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
Pr_0(\langle A \land B \rangle | A, B) \\
A & B & \langle A \land B \rangle = T & \langle A \land B \rangle = F \\
T & T & 1 & 0 \\
T & F & 0 & 1 \\
F & T & 0 & 1 \\
F & F & 0 & 1 \\
\end{array}
\]

24
Parametric probability analysis gives the following results for $\Pr(\langle A \land \neg B \rangle)$ and $\Pr(\langle A \land B \rangle)$:

| $\langle A \land B \rangle$ | $\Pr(\langle A \land \neg B \rangle)$ | $\langle A \land B \rangle$ | $\Pr(\langle A \land B \rangle)$ |
|-----------------------------|---------------------------------|-----------------------------|---------------------------------|
| $\top$                      | $x_2$                           | $\top$                      | $x_1$                           |
| $\bot$                      | $x_1 + x_3 + x_4$               | $\bot$                      | $x_2 + x_3 + x_4$               |

The output table $\Pr(A, B)$ is identical to the input table $\Pr_0(A, B)$. It happens in this model that all polynomials are linear functions of the parameters $x_1, \ldots, x_4$; hence conditional probability queries yield fractional linear functions.

For the affirmative BOOLEAN-FEASIBILITY conditional $A \vdash B$ we can define the set of satisfactory sets of valuations of $\langle A, B \rangle$ by two probability constraints involving the embedded logical formulas:

$$
\Pr(\langle A \land \neg B \rangle = \top) = 0
$$

$$
\Pr(\langle A \land B \rangle = \top) > 0
$$

Let us call these constraints the ‘negative’ and ‘positive’ criteria, respectively. Regarding the probability model defined by the input table $\Pr_0(A, B)$ above, you can see that these two equations are necessary and sufficient conditions for the probabilistic FEASIBILITY conditional $A \vdash^* B$ to hold. Note that the embedded event $\langle A \land \neg B \rangle = \top$ is equivalent to the embedded events $\langle A \rightarrow B \rangle = \bot$ and $\langle A \rightarrow \neg B \rangle = \top$. The material nonimplication relation $A \rightarrow B$ means the negation $\neg(\langle A \rightarrow B \rangle)$.

We consider a set of valuations to be satisfactory if each of its members (corresponding to an element of the table $\Pr(A, B)$) can be assigned a nonzero probability such that these constraints are satisfied. After symbolic probability inference these become the algebraic constraints $x_2 = 0$ and $x_1 > 0$. These relations are useful because they can be used as constraints or as objectives in optimization problems, in order to assert or test whether the conditional $A \vdash B$ holds. For example to query whether the conditional $A \vdash B$ is true given some set $\Gamma$ of logical constraints, we can evaluate the solution sets:

$$
\Phi \leftarrow \{ \Pr(\langle A \land \neg B \rangle = \top) : \Gamma \}
$$

$$
\Psi \leftarrow \{ \Pr(\langle A \land B \rangle = \top) : \Pr(\langle A \land \neg B \rangle = \top) = 0, \Gamma \}
$$

If $\Phi = \{0\}$ and $\Psi \neq \emptyset$ and $0 \notin \Psi$ then given $\Gamma$ it is necessary that $A \vdash B$ is true; if $0 \in \Phi$ and $\Psi \neq \emptyset$ and $\Psi \neq \{0\}$ then given $\Gamma$ it is possible that $A \vdash B$ is true; and if $0 \notin \Phi$ or $\Psi = \{0\}$ or $\Psi = \emptyset$ then given $\Gamma$ it is impossible that $A \vdash B$ is true. When the full-joint probability distribution concerning the propositional variables has been specified using a single input table as $\Pr_0(A, B)$ above, the desired solutions can be computed by linear optimization (there are no nonlinear polynomials involved, as is the general case for parametric probability models).

Note that there are several ways for a set of constraints $\Gamma$ to fail to confirm a conditional $A \vdash B$. It could be that the constraints $\Gamma$ are inconsistent, in which case we do not consider any conditional statement to hold. Next there is the ‘sin of commission’ that it is possible (probability greater than zero) for the consequent $B$ to be false while the antecedent $A$ is true. Finally there is the ‘sin of omission’ that it is impossible (probability zero) for the consequent $B$ to be true while the antecedent $A$ is true. We can consider these mechanisms of failure in a modal way by the above analysis (thus computing whether it is necessary, possible, or impossible for $A \vdash B$ to hold subject to the constraints $\Gamma$).

### 6.2.3 Disjunction Introduction and Disjunctive Syllogism

Logical deduction based on BOOLEAN-FEASIBILITY conditionals provides disjunction introduction and disjunctive syllogism. To check the former we evaluate the conditional $A \vdash A \lor B$; for the latter we evaluate $\{A \lor B, \neg A\} \vdash B$. We clarify that restrictions on the joint prior probabilities of logical propositions may invalidate these or other deductions. For example if $A = B$ then disjunctive syllogism fails. To be explicit about prior probabilities we could use probabilistic FEASIBILITY conditionals instead of BOOLEAN-FEASIBILITY conditionals, with the aid of embedded propositional-calculus functions to express the desired premises.
6.2.4 Transitivity of Boolean Deduction

To illustrate transitivity, let us investigate the relationship between the boolean-feasibility conditionals $A \vdash B$ and $B \vdash C$ asserted as premises, and the conditional $A \vdash C$ queried as an objective. We desire to evaluate the solution set:

$$
\Phi_0 \iff \{ (A \vdash C) : (A \vdash B) = T, (B \vdash C) = T \} 
$$

(108)

If the computed value of $\Phi_0$ is $\{T\}$ then we will declare $A \vdash C$ a necessary consequence of the premises $A \vdash B$ and $B \vdash C$, according to Equation (93). It turns out that we can formulate a series of linear optimization problems whose solutions will reveal the computed value of the truth-value set $\Phi_0$.

Consider a new probability model with the following input table $Pr_0(A, B, C)$, including parameter constraints $\Gamma_0 \iff \{ 0 \leq y_i \leq 1, \sum_i y_i = 1 \}$ to enforce the laws of probability:

|   | $A$ | $B$ | $C$ | $Pr_0(A, B, C)$ |
|---|-----|-----|-----|-----------------|
| T | T   | T   | y₁  |                 |
|   | T   | F   | y₂  |                 |
|   | F   | T   | y₃  |                 |
|   | F   | F   | y₄  |                 |
| F | T   | T   | y₅  |                 |
|   | T   | F   | y₆  |                 |
| F | T   | F   | y₇  |                 |
| F | F   | F   | y₈  |                 |

Symbolic probability inference with embedded propositional-calculus formulas yields:

| $\langle A \land B \rangle$ | $Pr(\langle A \land B \rangle)$ | $\langle A \rightarrow B \rangle$ | $Pr(\langle A \rightarrow B \rangle)$ |
|-----------------------------|---------------------------------|-----------------------------|---------------------------------|
| T                           | $y_1 + y_2$                     | T                           | $y_1 + y_2 + y_5 + y_6 + y_7 + y_8$ |
| F                           | $y_3 + y_4 + y_5 + y_6 + y_7 + y_8$ | F                           | $y_3 + y_4$                     |
| $\langle B \land C \rangle$ | $Pr(\langle B \land C \rangle)$ | $\langle B \rightarrow C \rangle$ | $Pr(\langle B \rightarrow C \rangle)$ |
| T                           | $y_1 + y_5$                     | T                           | $y_1 + y_3 + y_4 + y_5 + y_7 + y_8$ |
| F                           | $y_2 + y_3 + y_4 + y_5 + y_7 + y_8$ | F                           | $y_2 + y_6$                     |
| $\langle A \land C \rangle$ | $Pr(\langle A \land C \rangle)$ | $\langle A \rightarrow C \rangle$ | $Pr(\langle A \rightarrow C \rangle)$ |
| T                           | $y_1 + y_3$                     | T                           | $y_1 + y_3 + y_5 + y_6 + y_7 + y_8$ |
| F                           | $y_2 + y_4$                     | F                           | $y_2 + y_4$                     |

Following Equations (105) and (104) Let us use the following set $\Gamma$ of probability constraints to express the premises that $A \vdash B$ and $B \vdash C$ are both true:

$$
\Gamma \iff \{ Pr(\langle A \land B \rangle = T) > 0, Pr(\langle A \rightarrow B \rangle = F) = 0, Pr(\langle B \land C \rangle = T) > 0, Pr(\langle B \rightarrow C \rangle = F) = 0 \} 
$$

(113)

Parametric probability analysis yields the corresponding algebraic expressions:

$$
\Gamma \implies \{ y_1 + y_2 > 0, y_3 + y_4 = 0, y_1 + y_5 > 0, y_2 + y_6 = 0 \}
$$

(114)

First we query the feasible values of $Pr(\langle A \rightarrow C \rangle = F)$ subject to the constraints $\Gamma$ and $\Gamma_0$. The paired linear optimiza-
This confirms the transitivity of \textit{BOOLEAN}-\textit{FEASIBILITY} problems:

\[
\begin{align*}
\text{Minimize:} & \quad y_2 + y_4 \\
\text{subject to:} & \quad y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 = 1 \\
& \quad y_1 + y_2 \geq \varepsilon \\
& \quad y_3 + y_4 = 0 \\
& \quad y_1 + y_5 \geq \varepsilon \\
& \quad y_2 + y_6 = 0 \\
\text{and:} & \quad 0 \leq y_1 \leq 1 \\
& \quad 0 \leq y_2 \leq 1 \\
& \quad 0 \leq y_3 \leq 1 \\
& \quad 0 \leq y_4 \leq 1 \\
& \quad 0 \leq y_5 \leq 1 \\
& \quad 0 \leq y_6 \leq 1 \\
& \quad 0 \leq y_7 \leq 1 \\
& \quad 0 \leq y_8 \leq 1 \\
& \quad \varepsilon = 0.001
\end{align*}
\]

have computed solutions $\alpha^* = 0.000$ and $\beta^* = 0.000$. Thus it was computed that the solution set:

\[
\{ \Pr((A \rightarrow C) = F) : \Gamma \} \Rightarrow \{0\}
\]  

(116)

Using similar technique, linear optimization reveals that the solution set:

\[
\{ \Pr((A \land C) = T) : \Pr((A \rightarrow C) = F) = 0, \Gamma \} \subseteq [0.001, 1.000]
\]

(117)

(with both end points 0.001 and 1.000 feasible), indicating that the queried objective must be strictly greater than zero given the provided constraints (the precise numerical results reflect our choice of the small constant $\varepsilon$). In other words, these optimization results reveal that, subject to the constraints that both conditionals $A \vdash B$ and $B \vdash C$ are true, it is necessary that both of the following constraints are satisfied:

\[
\begin{align*}
\Pr((A \rightarrow C) = F) &= 0 \\
\Pr((A \land C) = T) &> 0
\end{align*}
\]

(118) (119)

These two constraints are precisely the criteria under which the BOOLEAN-FEASIBILITY conditional $A \vdash C$ is true. Hence truth is the only feasible value for the conditional $A \vdash C$, subject to the premises $(A \vdash B) = T$ and $(B \vdash C) = T$; in other words $\Phi_0 \Rightarrow \{T\}$. Thus we have computed using parametric probability and linear optimization that the following metalevel BOOLEAN-FEASIBILITY conditional is correct:

\[
\{ (A \vdash B), (B \vdash C) \} \vdash (A \vdash C)
\]

(120)

This confirms the transitivity of BOOLEAN-FEASIBILITY conditionals interpreted as statements of logical deduction.

Marginal probabilities:

\[
\begin{array}{ccc}
A & \Pr(A) \\
T & y_1 + y_2 + y_3 + y_4 \\
F & y_5 + y_6 + y_7 + y_8
\end{array}
\quad
\begin{array}{ccc}
B & \Pr(B) \\
T & y_1 + y_2 + y_5 + y_6 \\
F & y_3 + y_4 + y_5 + y_8
\end{array}
\quad
\begin{array}{ccc}
C & \Pr(C) \\
T & y_1 + y_3 + y_5 + y_7 \\
F & y_2 + y_4 + y_6 + y_8
\end{array}
\]

(121)

Computed joint probabilities:

\[
\begin{array}{ccc}
A & B & \Pr(A, B) \\
T & T & y_1 + y_2 \\
T & F & y_3 + y_4 \\
F & T & y_5 + y_6 \\
F & F & y_7 + y_8
\end{array}
\quad
\begin{array}{ccc}
B & C & \Pr(B, C) \\
T & T & y_1 + y_5 \\
T & F & y_2 + y_6 \\
F & T & y_3 + y_7 \\
F & F & y_4 + y_8
\end{array}
\quad
\begin{array}{ccc}
A & C & \Pr(A, C) \\
T & T & y_1 + y_3 \\
T & F & y_2 + y_4 \\
F & T & y_3 + y_7 \\
F & F & y_6 + y_8
\end{array}
\]

(122)
Here are a few problems from the literature, analyzed by the proposed methodology [7, 1, 2]. These clever and well-crafted problems raise myriad issues in modeling and analysis, all of which we can address using the methods introduced above.

7.1 Hot Buttered Conditionals

We begin with Goodman’s first two problems from Fact, Fiction, and Forecast [7]. From his page 4:

What, then, is the problem about counterfactual conditionals? Let us confine ourselves to those in which antecedent and consequent are inalterably false—as, for example, when I say of a piece of butter that was eaten yesterday, and that had never been heated,

If that piece of butter had been heated to 150° F., it would have melted.

Considered as truth-functional compounds, all counterfactuals are of course true, since their antecedents are false. Hence

If that piece of butter had been heated to 150° F., it would not have melted

would also hold. Obviously something different is intended, and the problem is to define the circumstances under which a given counterfactual holds while the opposing counterfactual with the contradictory consequent fails to hold.

Probability makes it easy to express ‘something different’. Let us use the probability model from Equation 11 with A denoting the event that the considered piece of butter was heated, and B denoting the event that the piece of butter melted. We can distill four statements from Goodman’s problem description:

| ID | PROBABILITY | POLYNOMIAL | DESCRIPTION |
|----|-------------|------------|-------------|
| 1  | Pr(A = T) = 0 | x = 0      | It was not heated |
| 2  | Pr(B = T) = 0 | z + xy - xz = 0 | It did not melt |
| 3  | Pr₀(B = T | A = T) = 1 | y = 1      | It would melt if heated |
| 4  | Pr₀(B = T | A = T) = 0 | y = 0      | It would not melt if heated |

Here the premises 1 and 2 assert the facts that the butter was not heated and that it did not melt. The conditionals 3 and 4 are counterfactual (more specifically, antifactual) relative to the fact 1 that A is known to be false. More importantly, 3 and 4 are also subjunctive: they concern the input probability Pr₀(B|A) which does not involve the probability Pr(A). The constraints in Equation 123 behave exactly as Goodman intended. It is consistent to assert either subjunctive/counterfactual statement 3 or 4 along with the facts 1 and 2, however it is inconsistent to assert both of the opposite subjunctives together.

We can also consider Goodman’s statement:

Since that butter did not melt, it wasn’t heated to 150° F.

Let us assume Goodman intends that the affirmative subjunctive stating that the butter would have melted had it been heated is in effect here. In order to confirm the quoted statement of deduction, we can query the feasible values of Pr(A = F) subject to the constraints Pr(B = T) = 0 and Pr₀(B = T | A = T) = 1. Using symbolic probability inference to find polynomial formulas for these probability expressions reveals the solution set:

\[ \{ 1 - x : z + xy - xz = 0; y = 1; x, y, z \in [0, 1] \} \Rightarrow \{ 1 \} \] (124)
Here are the optimization problems generated to evaluate this solution set:

\[
\begin{align*}
\text{Minimize} : & \quad 1 - x \\
\text{subject to} : & \quad z + xy = xz \\
& \quad y = 1 \\
& \quad 0 \leq x \leq 1 \\
& \quad 0 \leq y \leq 1 \\
& \quad 0 \leq z \leq 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{Maximize} : & \quad 1 - x \\
\text{subject to} : & \quad z + xy = xz \\
& \quad y = 1 \\
& \quad 0 \leq x \leq 1 \\
& \quad 0 \leq y \leq 1 \\
& \quad 0 \leq z \leq 1 \\
\end{align*}
\]

Solving the optimization problems gives computed minimum \(a^* = 1.000\) and maximum \(b^* = 1.000\), hence the solution set \(\{1\}\) containing just the real number one. In other words, it is a probabilistic consequence of the premises that the butter did not melt \((\Pr (B = T) = 0)\) and that the butter would have melted had it been heated \((\Pr_B (B = T | A = T) = 1)\) that the butter certainly was not heated \((\Pr (A = F) \in \{1\})\). It might be clearer to rephrase Goodman’s statement of this deductive result as:

Since that butter did not melt, and it would have melted had it been heated—therefore it must not have been heated to 150° F.

Separately, the facts that \(A\) and \(B\) are both false allow us to deduce the subjunctive conditional \(\Pr_B (B = T | A = F) = 0\) (that is, \(z = 0\)). In other words, the stated facts require that the piece of butter certainly would not have melted, had it not been heated.

7.2 Conspiracy Theories

We continue with Adams’s problems from [2] involving indicative and subjunctive conditionals. For these examples, probabilistic SUBJUNCTIVE conditionals provide the desired meanings. There are several alternative ways to render the indicative statements: as MATERIAL, EXISTENTIAL, or FEASIBILITY conditionals. This author’s intuition is that the FEASIBILITY interpretation is best; from this perspective the indicative conditionals are regarded as statements of deduction (which happen to be incorrect, for the examples that Adams provided).

7.2.1 Murder Most Subjunctive

For this example we are asked to consider a murder victim \(V\) and two suspects \(A\) and \(B\), with the evidence strongly favoring \(A\) as the culprit. We are given the following subjunctive conditional \((\text{Su1})\) and its indicative counterpart \((\text{I1})\):

\((\text{Su1})\) If \(A\) hadn’t murdered \(V\), \(B\) would not have either.

\((\text{I1})\) If \(A\) didn’t murder \(V\), then \(B\) didn’t either.

Adams suggests that \((\text{Su1})\) is ‘justified’ whereas \((\text{I1})\) is ‘unjustified’. Let us calculate. We identify the following propositions as the main variables in our probability model:

\[
\begin{align*}
A : & \text{ A murdered V,} \\
B : & \text{ B murdered V,} \\
V : & \text{ V was murdered}
\end{align*}
\]

Adams’s description suggests that \(V\) requires either \(A\) or \(B\) (or both)—a murder needs a murderer. Hence we set the input \(\Pr_B (V | A, B)\) as though for the embedded propositional-calculus formula \((A \lor B)\). The probability network is as follows:

\[
\begin{array}{c|cc}
A & B & V \\
\hline
T & T & 1 \\
T & F & 0 \\
F & T & 0 \\
F & F & 1 \\
\end{array}
\]

with each parameter \(x, y,\) and \(z\) constrained to the real interval \([0, 1]\).
Consider the following, which includes two ways to render the indicative I1 (MATERIAL or EXISTENTIAL):

| ID | PROBABILITY | POLYNOMIAL | DESCRIPTION |
|----|-------------|------------|-------------|
| Su1 | Pr(B = T | A = F) = 0 | z = 0 | If A hadn’t, B wouldn’t have |
| I1-M | Pr(V = T, A = F, B = T) = 0 | z - xz = 0 | Either A did or B didn’t |
| I1-E | Pr(V = T, A = F, B = T) = 0 | z - xz = 0 | If A didn’t then B didn’t |
|     | Pr(V = T, A = F) > 0 | z - xz > 0 | (nonzero chance that A didn’t) |

Both indicative constraints assert Pr(B = T | V = T, A = F) = 0, or equivalently Pr(B = F | V = T, A = F) = 1 (it is the same to say ‘certainly-not true’ as ‘certainly false’); the MATERIAL interpretation I1-M also allows the indefinite value 0/0 for these conditional probabilities. Anyway under the MATERIAL probabilistic interpretation I1-M, we would accept proposition I1 as a true statement and conclude that A must have murdered V, if in fact V was murdered. Under the EXISTENTIAL probabilistic interpretation I1-E, we would reject proposition I1 as inconsistent: it asserts that there is zero probability that B murdered V and A didn’t, and simultaneously that this same probability is strictly greater than zero. Note what the indicative I1 says when it is interpreted as a MATERIAL conditional:

(I1) Either A murdered V, or B didn’t murder V

We can also interpret I1 as a statement of deduction, let us say I1-F, claiming that ‘Assuming that V was murdered and A didn’t do it—therefore B didn’t either.’ This statement of deduction asserts that the following solution set is:

\[
\{ \Pr(B = F) \mid Pr(V = T, A = F) = 1 \} = \{1\} \tag{128}
\]

We shall see presently that this assertion is unsatisfiable. After substituting the results of symbolic probability inference the requested solution set is given by:

\[
\{ 1 - z - xy + xz : z - xz = 1; x, y, z \in [0, 1] \} \Rightarrow \{0\} \tag{129}
\]

In this case the only feasible value of the objective is 0. It follows that Equation \[128\] could be satisfied only if 0 = 1. Therefore the correct deductive statement is the opposite of I1-F, namely ‘Assuming V was murdered and A didn’t do it—therefore B must have’ or more succinctly ‘If A didn’t murder V, then B did’. We would get the same result having started from a slightly different set comprehension expression that splits the constraints on A and V into two equations:

\[
\{ \Pr(B = F) \mid Pr(V = T) = 1; Pr(A = F) = 1 \} \tag{130}
\]

All of the indicative interpretations I1-M, I1-E, and I1-F are factual with respect to the fact that V was murdered (‘factual’ meaning that the conditions explicitly include the known event V = T). On the other hand the subjunctive interpretation of Su1 is a factual (that is, weakly counterfactual) about the fact that V was murdered (‘afactual’ meaning that the known event V = T does not appear in the condition part of the input table Pr0(B | A)).

The following output probability tables were used to generate the results just presented:

| # | V | A | Pr(V, A) |
|---|---|---|---------|
| 1 | T | T | xzxy    |
| 2 | T | F | x - xy  |
| 3 | T | F | z - xz  |
| 4 | F | T | 0       |
| 5 | F | T | 0       |
| 6 | F | F | 0       |
| 7 | F | F | 0       |

| # | V | A | B | Pr(V, A, B) |
|---|---|---|---|-------------|
| 1 | T | T | T | xzxy        |
| 2 | T | T | F | x - xy     |
| 3 | T | F | T | z - xz     |
| 4 | F | T | F | 0          |

Pr(V) A Pr(A) B Pr(B)

\[
\begin{align*}
T & : x + z - xz & T & : x & T & : z + xy - xz \\
F & : 1 - x - z + xz & F & : 1 - x & F & : 1 - z - xy + xz \\
\end{align*} \tag{132}
\]

30
Also note:

| # | A | V | B | \( \text{Pr}(V, B | A) \) |
|---|---|---|---|------------------|
| 1 | T | T | T | \( xy / x \) |
| 2 | T | T | F | \( (x - xy) / (x) \) |
| 3 | T | F | T | \( 0 / (x) \) |
| 4 | F | F | F | \( 0 / (1 - x) \) |
| 5 | F | T | T | \( (z - xz) / (1 - x) \) |
| 6 | F | T | F | \( 0 / (1 - x) \) |
| 7 | F | F | T | \( 0 / (1 - x) \) |
| 8 | F | F | F | \( (1 - x - z + xz) / (1 - x) \) |

Consider also the matter of explaining inconsistency and revising the probability model to address it. For example, if it is asserted that Su1 holds but that V certainly has been murdered and that A certainly is innocent, what can we do to avoid the implied contradiction? One course might be to adjust probabilities, for example to interpret Su1 as the constraint \( z \approx 0 \) instead of as a strict equality (allowing for example \( z = \delta_1 \) for some small constant \( \delta_1 \)). Another revision might be to change \( \text{Pr}_0 (V = T | A = F, B = F) \) from 0 to some small constant \( \delta_2 \) to represent the idea that someone other than A or B might have murdered V.

### 7.2.2 Kennedy and Oswald

We next consider the problems in [2] about the late President John F. Kennedy and his assassin Lee Harvey Oswald. It is generally accepted as historical fact that Kennedy was shot and killed by Oswald in Dallas in 1963. Adams provided two pairs of conditionals. The first pair:

(Su2) If Oswald hadn’t shot Kennedy in Dallas, then no one else would have.

(I2) If Oswald didn’t shoot Kennedy in Dallas, then no one else did.

And the second pair:

(Su3) If Oswald hadn’t shot Kennedy in Dallas, Kennedy would be alive today.

(I3) If Oswald didn’t shoot Kennedy in Dallas, then Kennedy is alive today.

In order to analyze these conditionals, we revise the probability model from Equation 126 to change the referents of the variables \( A, B, \) and \( V \):

\( A \): Oswald shot Kennedy,

\( B \): Someone else shot Kennedy,

\( V \): Kennedy was shot

We add a new variable \( L \):

\( L \): Kennedy is alive today

The updated probability-network graph and the additional input table for \( L \) are as follows:

\[
\begin{array}{ccc}
A & B & \text{Pr}_0(L | V) \\
V & L = T & L = F \\
T & 0 & 1 \\
F & w & 1 - w \\
\end{array}
\] (134)

The input table \( \text{Pr}_0(L | V) \) says that Kennedy certainly would not be alive today had he been shot; however had he not been shot, there is some probability \( w \) that he would now be alive. As with the other parameters \( x, y, \) and \( z \), we have the constraint \( 0 \leq w \leq 1 \).

In terms of probability, Su2 and I2 are just like Su1 and I1 above. The subjunctive conditional Su2 is the probability constraint \( \text{Pr}_0 (B = T | A = F) = 0 \), equivalently \( \text{Pr}_0 (B = F | A = F) = 1 \), which yields the polynomial constraint \( z = 0 \). As with I1, the indicative statement I2 may be interpreted in a variety of ways: as a MATERIAL,
EXISTENTIAL, or FEASIBILITY conditional. The EXISTENTIAL interpretation I2-E offers the infeasible constraints
\( \Pr(B = T \mid V = T, A = F) = 0 \) and \( \Pr(V = T, A = F) > 0 \), which together assert the incompatible premises that there is
nonzero probability that Kennedy was shot by someone other than Oswald, yet that in that case no one else could have
done it. The MATERIAL interpretation I2-M also forbids the event that someone else shot Kennedy if Kennedy was not
shot by Oswald, but allows that there could be zero probability that Kennedy was shot by someone other than Oswald;
in this zero-probability case the conditional probabilities \( \Pr(B = T \mid V = T, A = F) \) and \( \Pr(B = F \mid V = T, A = F) \) have
the indeterminate value 0/0. The FEASIBILITY interpretation I2-F provides the incorrect deductive statement that it
is a probabilistic consequence of the premise that Kennedy was shot, and not by Oswald, that someone else must not
have shot Kennedy either. The correct deduction is that indeed someone else must have shot Kennedy in that case.

Moving along we now consider statements Su3 and I3. Following Table 2 the subjunctive conditional Su3 should
be rendered as the constraint \( \Pr_0(L = T \mid A = F) = 1 \) which says that the input probability that Kennedy is alive given
that Oswald had not shot him must be one. However in our probability model there is no such input table \( \Pr_0(L \mid A) \).
We can find an alternative means to express this particular subjective conditional Su3 using members of the input
tables \( \Pr_0(L \mid V) \) and \( \Pr_0(B \mid A) \) from Equations 126 and 134. Following the logic of the example, Su3 requires both
\( \Pr_0(B = F \mid A = F) = 1 \) and \( \Pr_0(L = T \mid V = F) = 1 \): the first saying that no one else would have shot Kennedy had
Oswald not done it, and the second saying that Kennedy would still be alive had he not been shot. These probability
constraints yield the polynomial constraints: \( z = 0 \) and \( w = 1 \). Because the variables \( w \) and \( z \) are constrained to lie
between zero and one, it is equivalent to provide the following single constraint to represent the subjunctive conditional
Su3:

\[
\Pr(L = T \mid A = F) \Rightarrow \frac{w - xw - zw + xzw}{1 - x}
\]

This quotient factors into the following expression, whose numerator and denominator are products of input probabil-
ities:

\[
\frac{(1 - x)(1 - z)(1)(w)}{1 - x}
\]

Here the denominator is \( \Pr_0(A = F) \) and the numerator is the product:

\[
\Pr_0(A = F) \cdot \Pr_0(B = F \mid A = F) \cdot \Pr_0(V = F \mid A = F, B = F) \cdot \Pr_0(L = T \mid V = F)
\]

Eliminating the input probability \( \Pr_0(A = F) \) from the numerator and denominator (without assuming it must be
nonzero!) yields the desired input-probability expression to constrain for the subjunctive conditional Su3.

There are several ways to interpret the indicative conditional I3.

| ID   | PROBABILITY | POLYNOMIAL | DESCRIPTION                          |
|------|-------------|------------|--------------------------------------|
| Su3  | \( \Pr_0(L = T \mid V = F) \cdot \Pr_0(B = F \mid A = F) = 1 \) | \( w(1 - z) = 1 \) | If O. hadn’t shot, K. would be alive |
| I3-M | \( \Pr(V = T, A = F, L = T) = \Pr(V = T, A = F) \) | \( 0 = z - xz \) | Either O. shot or K. is alive        |
| I3-E | \( \Pr(V = T, A = F, L = T) = \Pr(V = T, A = F) \) | \( 0 = z - xz \) | If O. didn’t shoot then K. is alive |

\[
(139)
\]
Consider the joint probability:

| # | V | A | L | \( \Pr(V, A, L) \) |
|---|---|---|---|------------------|
| 1 | T | T | T | 0                |
| 2 | T | T | F | x                |
| 3 | T | F | T | 0                |
| 4 | T | F | F | \( z - xz \)      |
| 5 | F | T | T | 0                |
| 6 | F | T | F | 0                |
| 7 | F | F | T | \( w - xw - zw + xzw \) |
| 8 | F | F | F | \( 1 - x - z - w + xz + xw + zw - xzw \) |

Interpreted as a statement of deduction, the indicative I3-F asserts that one is the only feasible value of the probability that Kennedy is still alive, given that he was shot and not by Oswald:

\[
\{ \Pr(L = T) : \Pr(V = T, A = F) = 1 \} = \{ 1 \} \quad (141)
\]

But this equation is incorrect; the actual solution set is \( \{ 0 \} \). Here is the set-comprehension expression after symbolic probability inference:

\[
\{ w - xw - zw + xzw : z - xz = 1 ; w, x, y, z \in [0, 1] \} \Rightarrow \{ 0 \} \quad (142)
\]

You can see by inspection that the constraints require \( x = 0 \) and \( z = 1 \); with these substitutions the objective simplifies to the constant 0. The same result would come from specifying the premise as two probability constraints instead of one:

\[
\{ \Pr(L = T) : \Pr(V = T) = 1, \Pr(A = F) = 1 \} \Rightarrow \{ 0 \} \quad (143)
\]

Note that omitting the fact \( V = T \) that Kennedy was shot yields a different result for the indicative conditional I3-F (which thereby becomes a factual—weakly counterfactual—with respect to Kennedy’s shooting). In this case we evaluate the solution set:

\[
\{ \Pr(L = T) : \Pr(A = F) = 1 \} \quad (144)
\]

The algebraic expression is the following:

\[
\{ w - xw - zw + xzw : 1 - x = 1 ; w, x, y, z \in [0, 1] \} \Rightarrow [0, 1] \quad (145)
\]

In this case the constraints require \( x = 0 \), with which substitution the objective \( \Pr(L = T) \) simplifies to the product \( w(1-z) \). Subject to the constraint that Oswald did not shoot him, but allowing that he may not have been shot at all, the probability that Kennedy is alive today depends on the probability \( z \) that someone else would have shot him (had Oswald not done so) and the probability \( w \) that he would still be alive had he not been shot. Absent other constraints on the values of these parameters, the value of the objective \( w(1-z) \) could have any value between zero and one. In this case I3 still would not be a correct statement of deduction, because the computed solution set \([0, 1]\) is not exactly \( \{ 1 \} \). However it would be a correct statement of probabilistic deduction that:

Ignoring the fact that Kennedy was shot, if Oswald didn’t shoot Kennedy in Dallas, then Kennedy may or may not be alive today.

Here ‘may or may not be’ reflects the idea that it is feasible for the queried probability to be anywhere between zero and one (including these end points), subject to the given constraints.

For clarity we might adopt the convention to preface factual conditionals with a phrase like, ‘Accepting the fact that …’ when stating them in natural language. Thus we might clarify which indicative I3 is intended. For example, there is the factual MATERIAL conditional interpretation of I3:

Accepting the fact that Kennedy was shot, either Oswald shot Kennedy in Dallas or Kennedy is alive today (or both).
And there is a corresponding afactual MATERIAL conditional:

Ignoring the fact that Kennedy was shot, either Oswald shot Kennedy in Dallas or Kennedy is alive today (or both).

Both of these happen to be correct given the information provided.

7.2.3 Soft, What Conditional Breaks?

We now consider the role of observed evidence. We are given the following conditionals:

(Su4) $X$ is soft at time $t = df$ if $X$ should be (were, had been, depending on the relation of $t$ to the present) subject to moderate deforming pressure at time $t$, then it would be (would have been) significantly deformed.

(I4) If $X$ is (was) subject to moderate deforming pressure at time $t$, then it will be (is, was) significantly deformed.

We are asked to “... suppose that we have observed that at time $t$, $X$ was not deformed, but that we don’t know whether it was subject to deforming stress at that time.” Given this observation, we are asked to evaluate the truth of the conditionals Su4 and I4. In order to analyze this problem let us return to the basic probability model from Equation 1. Let us assume that we are concerned with just one moment in time.

We update the referents of the main variables $A$ and $B$ to the following propositions:

$A : X$ was subject to moderate deforming pressure, $B : X$ was significantly deformed

The observation that $X$ was not deformed constitutes the evidence that $B$ is certainly false, in other words $\Pr (B = T) = 0$ or equivalently $\Pr (B = F) = 1$. By symbolic probability inference this evidence becomes the constraint $z + xy = xz$.

What can we say about the conditionals Su4 and I4 given this evidence? Following Equation 17 the affirmative subjunctive conditional Su4 corresponds to the equation $\Pr(B = T | A = T) = 1$. Therefore, in order to evaluate the conditional Su4, let us analyze the set $\Phi_1$ of feasible values for the input probability $\Pr(B = T | A = T)$ subject to the given evidence $\Pr(B = T) = 0$ along with the general constraints $\Gamma_0$ of the probability model:

$$\Phi_1 \leftarrow \{ \Pr(B = T | A = T) : \Pr(B = T) = 0, \Gamma_0 \} \quad (146)$$

Using symbolic probability inference with the probability model from Equation 1, we generate the following pair of polynomial optimization problems to characterize this solution set $\Phi_1$:

Minimize : $y$ \hspace{1cm} Maximize : $y$
subject to : $z + xy = xz$ \hspace{1cm} subject to : $z + xy = xz$
and : $0 \leq x \leq 1$ \hspace{1cm} and : $0 \leq x \leq 1$
$0 \leq y \leq 1$ \hspace{1cm} $0 \leq y \leq 1$
$0 \leq z \leq 1$ \hspace{1cm} $0 \leq z \leq 1$

(147)

The computed minimum and maximum solutions are $\alpha^* = 0.00$ and $\beta^* = 1.00$, from which it follows that the solution set $\Phi_1 \subseteq [0, 1]$ with $\Phi_1$ including at least the points 0 and 1. In other words, the evidence that $B$ is certainly false does not constrain the value of the objective $\Pr(B = T | A = T)$ to any particular value. Therefore the provided evidence does not tell us anything about whether the subjunctive Su4 is true or false, leaving it possible that Su4 “might well be justified” by the evidence (to borrow Adams’s phrasing).

In order to check whether the affirmative MATERIAL or conditional interpretation of I4 might hold given the provided evidence $\Pr(B = T) = 0$, we can evaluate the following solution set $\Phi_2$:

$$\Phi_2 \leftarrow \{ \Pr(A = T) - \Pr(A = T, B = T) : \Pr(B = T) = 0, \Gamma_0 \} \quad (148)$$

Analysis according to Section 2.5 gives minimum and maximum solutions $\alpha^* = 0.00$ and $\beta^* = 1.00$, indicating $\Phi_2 \subseteq [0, 1]$ with 0 $\in \Phi_2$ and 1 $\in \Phi_2$. Therefore, subject to the given evidence, it is possible but not necessary that the constraint $\Pr(A = T, B = T) = \Pr(A = T)$ from Equation 12 defining the affirmative MATERIAL conditional is satisfied. Hence the MATERIAL interpretation of the indicative conditional I4 is consistent with the supplied evidence (though not required by the evidence). Note what the MATERIAL interpretation of the indicative I4 says:
Either \( X \) was not subject to moderate deforming pressure at time \( t \) or it was significantly deformed.

In order to investigate the existential interpretation of \( I_4 \) subject to the provided evidence \( \Pr (B = T) = 0 \), we can evaluate the following solution set \( \Phi_3 \):

\[
\Phi_3 \Leftarrow \{ \Pr (A = T) : \Pr (A = T, B = T) = \Pr (A = T), \Pr (B = T) = 0, \Gamma_0 \} \tag{149}
\]

Analysis according to Section 2.5 gives minimum and maximum solutions \( \alpha^* = 0.000 \) and \( \beta^* = 0.000 \), indicating \( \Phi_3 \Rightarrow \{0\} \). In other words, it is not feasible for \( \Pr (A = T) \) to be greater than zero. Therefore, subject to the provided evidence, it is not possible to satisfy Equation 25 with \( k = 1 \) indicating an affirmative existential conditional, because the constraint \( \Pr (A = T) > 0 \) is violated. Hence the existential interpretation of the indicative conditional \( I_4 \) is not compatible with the supplied evidence.

By the foregoing calculations, probabilistic analysis has demonstrated that the subjunctive conditional \( Su_4 \) is compatible with the provided evidence, as is the material interpretation of the indicative \( I_4 \). However the existential interpretation of the indicative conditional \( I_4 \) is not compatible with the stated evidence. Neither can the feasibility interpretation of the indicative conditional \( I_4 \) hold, unless the stated evidence \( \Pr (B = T) = 0 \) is ignored.

7.2.4 A Brown Bird in the Bush

Finally we consider the last example from [2]. We are given “a natural law that all ravens are black” and the observation of “a brown bird in the distance,” along with two propositions:

(Su5) If that were a raven, it would be black

(I5) If that is a raven, it is black

We recycle the probability model from Equation 1 with the referents of the variables modified to the following:

\( A : \) It is a raven, \( B : \) It is black

This example is similar to the last one except that the truth of the subjunctive conditional is now given as evidence. Using Equation 17 with \( k = 1 \) to indicate an affirmative statement, we render the subjunctive conditional \( Su_5 \) as the probability equation \( \Pr_0 (B = T | A = T) = 1 \), which yields the polynomial equation \( y = 1 \). We interpret the natural law about ravens being black as the assertion that the subjunctive \( Su_5 \) is true (hence as the constraint \( y = 1 \)). We also have as evidence the assertion \( \Pr (B = T) = 0 \) that the observed bird was certainly not black; this gives the polynomial constraint \( z + xy = xz \).

In order to check whether the affirmative material interpretation of \( I_5 \) might hold given the provided evidence, we can evaluate the following solution set \( \Phi_5 \):

\[
\Phi_5 \Leftarrow \{ \Pr (A = T) - \Pr (A = T, B = T) : \Pr_0 (B = T | A = T) = 1, \Pr (B = T) = 0, \Gamma_0 \} \tag{150}
\]

Analysis according to Section 2.5 gives minimum and maximum solutions \( \alpha^* = 0.000 \) and \( \beta^* = 1.000 \). Hence \( \Phi_5 \subseteq [0, 1] \) with \( 0 \in \Phi_5 \) and \( 1 \in \Phi_5 \) and it is possible but not necessary that the material interpretation of the indicative conditional \( I_5 \) holds, subject to the provided evidence.

For the affirmative existential interpretation of \( I_5 \), we evaluate the following solution set \( \Phi_6 \):

\[
\{ \Pr (A = T) : \Pr (A = T) = \Pr (A = T, B = T), \Pr_0 (B = T | A = T) = 1, \Pr (B = T) = 0, \Gamma_0 \} \tag{151}
\]

Analysis according to Section 2.5 gives minimum and maximum solutions \( \alpha^* = 0.000 \) and \( \beta^* = 0.000 \), hence \( \Phi_6 \Rightarrow \{0\} \) and the existential interpretation of \( I_5 \) is inconsistent with the given evidence (because the constraint \( \Pr (A = T) > 0 \) cannot be satisfied).
Finally we address Adams’s earlier problems from [1]. Rather than trying to figure out when the propositional calculus may be ‘safely’ used, we can dispense with it altogether and model these problems directly with conditional probabilities. These problems point out two kinds of fallacies: first, confusion about division by zero; second, mishandling specializations of general rules. We address these by calling out explicitly when a quotient has the indefinite value 0/0; and by understanding that general conditional statements may have exceptions when they are specialized to account for new antecedent terms. Note that such revision of conditional statements can be done within the propositional calculus too. However, proper treatment of impossible antecedents would be cumbersome in the propositional calculus.

All but one of the problems from [1] can be solved using the BOOLEAN-FEASIBILITY conditionals along with the probability and linear optimization methods described in Section 6.2.2. The exception is F4 which illustrates a different point: sometimes an initial conditional statement ought to be revised in order to account for new information (or alternatively it should have been stated in a defeasible manner in the first place, to allow for future exceptions). Stubborn adherence to prematurely-made statements is a special kind of fallacy.

F1. John will arrive on the 10 o’clock plane. Therefore, if John does not arrive on the 10 o’clock plane, he will arrive on the 11 o’clock plane.

Variables:

\[ A : \text{John arrives on the 10:00 plane} \]
\[ B : \text{John arrives on the 11:00 plane} \]

Using the TRUTH-FUNCTIONAL interpretation, F1 asserts that the following conditional is true:

\[ A \rightarrow (\neg A \rightarrow B) \] (152)

Using the BOOLEAN-FEASIBILITY interpretation we consider the corresponding conditional:

\[ A \vdash (\neg A \vdash B) \] (153)

Unlike the familiar TRUTH-FUNCTIONAL conditional, the BOOLEAN-FEASIBILITY conditional defined by Equation 41 is false when its antecedent is false. So, using the BOOLEAN-FEASIBILITY interpretation, the inner conditional \( \neg A \rightarrow B \) must be false subject to the constraint \( A = T \) imposed by the outer conditional. It follows that the outer conditional is false too (true antecedent, false consequent).

We can arrive at the same conclusion regarding the BOOLEAN-FEASIBILITY interpretation of F1 using the probability and linear-programming approach from Section 6.2.2. This has the advantage of offering some explanation as to why the conditional stated as F1 is incorrect. Following Equations 104 and 105 we regard the inner conditional \( \neg A \rightarrow B \) as equivalent to the following pair of constraints concerning a parametric probability model including the primary variables \( A \) and \( B \):

\[ \Pr (\langle \neg A \land \neg B \rangle = T) = 0 \] (154)
\[ \Pr (\langle \neg A \land B \rangle = T) > 0 \] (155)

The antecedent \( A \) of the outer conditional signifies the following singleton set of constraints \( \Gamma \):

\[ \Gamma \leftarrow \{ \Pr (A = T) = 1 \} \] (156)

In order to determine whether the outer conditional \( A \vdash (\neg A \vdash B) \) holds we evaluate the following solution sets \( \Phi \) and \( \Psi \):

\[ \Phi \leftarrow \{ \Pr (\langle \neg A \land \neg B \rangle = T) : \Gamma \} \] (157)
\[ \Psi \leftarrow \{ \Pr (\langle \neg A \land B \rangle = T) : \Pr (\langle \neg A \land \neg B \rangle = T) = 0, \Gamma \} \] (158)
Using the probability model from Section 6.2.2 supplemented by additional embedded propositional-calculus functions, and including the general probability constraints \( \Gamma_0 = \{ 0 \leq x_i \leq 1, \sum x_i = 1 \} \), symbolic probability inference yields:

\[
\Phi \Rightarrow \{ x_4 : x_1 + x_2 = 1; 0 \leq x_i \leq 1, \sum x_i = 1 \} \quad (159)
\]

\[
\Psi \Rightarrow \{ x_3 : x_4 = 0, x_1 + x_2 = 1; 0 \leq x_i \leq 1, \sum x_i = 1 \} \quad (160)
\]

Linear optimization computes \( \Phi \Rightarrow \{ 0 \} \) and \( \Psi \Rightarrow \{ 0 \} \). It follows that the BOOLEAN-FEASIBILITY conditional \( A \vdash (\neg A \vdash B) \) cannot hold. These solution sets provide some explanation for this impossibility. Subject to the constraint \( \Gamma \) representing the antecedent \( A \) of the outer conditional, the result \( \Phi = \{ 0 \} \) shows that the negative requirement of the inner conditional is always satisfied: it is infeasible for the consequent \( B \) and the antecedent \( \neg A \) to be true simultaneously. However the result \( \Psi = \{ 0 \} \) indicates that the positive requirement of the inner conditional cannot be satisfied: there must be zero probability that the consequent \( B \) is true and the antecedent \( \neg A \) is true. Therefore the inner conditional \( \neg A \vdash B \) is necessarily false, subject to the constraint that the antecedent \( A \) of the outer conditional is true. In other words the set-comprehension expression \( \{ (\neg A \vdash B) : A = T \} \) evaluates to \( \{ F \} \). Because this solution set is not \( \{ T \} \), Equation 41 tells us that the outer conditional \( A \vdash (\neg A \vdash B) \) is false.

For the solution set \( \Phi \), linear optimization gives solutions \( \alpha^* = 0.000 \) and \( \beta^* = 0.000 \) to the optimization problems:

Minimize : \( x_4 \)
subject to : \( x_1 + x_2 + x_3 + x_4 = 1 \)
\( x_1 + x_2 = 1 \)
and : \( 0 \leq x_1 \leq 1 \)
\( 0 \leq x_2 \leq 1 \)
\( 0 \leq x_3 \leq 1 \)
\( 0 \leq x_4 \leq 1 \)

Maximize : \( x_4 \)
subject to : \( x_1 + x_2 + x_3 + x_4 = 1 \)
\( x_1 + x_2 = 1 \)
and : \( 0 \leq x_1 \leq 1 \)
\( 0 \leq x_2 \leq 1 \)
\( 0 \leq x_3 \leq 1 \)
\( 0 \leq x_4 \leq 1 \)

(161)

For the solution set \( \Psi \), linear optimization gives solutions \( \alpha^* = 0.000 \) and \( \beta^* = 0.000 \) to the optimization problems:

Minimize : \( x_3 \)
subject to : \( x_1 + x_2 + x_3 + x_4 = 1 \)
\( x_4 = 0 \)
\( x_1 + x_2 = 1 \)
and : \( 0 \leq x_1 \leq 1 \)
\( 0 \leq x_2 \leq 1 \)
\( 0 \leq x_3 \leq 1 \)
\( 0 \leq x_4 \leq 1 \)

Maximize : \( x_3 \)
subject to : \( x_1 + x_2 + x_3 + x_4 = 1 \)
\( x_4 = 0 \)
\( x_1 + x_2 = 1 \)
and : \( 0 \leq x_1 \leq 1 \)
\( 0 \leq x_2 \leq 1 \)
\( 0 \leq x_3 \leq 1 \)
\( 0 \leq x_4 \leq 1 \)

(162)

Note these inputs and outputs:

| \( A \) | \( B \) | \( Pr_0(A,B) \) | \( \neg A \land \neg B \) | \( Pr(\neg A \land \neg B) \) | \( \neg A \land B \) | \( Pr(\neg A \land B) \) |
|---|---|---|---|---|---|---|
| T | T | \( x_1 \) | T | \( x_4 \) | T | \( x_3 \) |
| T | F | \( x_2 \) | T | \( x_4 \) | T | \( x_3 \) |
| F | T | \( x_3 \) | F | \( x_1 + x_2 + x_3 \) | F | \( x_1 + x_2 + x_4 \) |
| F | F | \( x_4 \) |

(163)

Alternatively, in order to evaluate the inner conditional \( \neg A \vdash B \) subject to the constraint \( A = T \) from the outer conditional, we could evaluate the following solution set:

\( \Upsilon \Leftarrow \{ \Pr(B = T | A = F) : \Pr(A = T) = 1 \} \)

(164)
These are the resulting fractional linear optimization problems:

**Minimize:** \( \frac{x_3}{x_3 + x_4} \) \hspace{1cm} **Maximize:** \( \frac{x_3}{x_3 + x_4} \)

subject to: \( x_1 + x_2 + x_3 + x_4 = 1 \) \hspace{1cm} subject to: \( x_1 + x_2 + x_3 + x_4 = 1 \)

\( x_1 + x_2 = 1 \) \hspace{1cm} \( x_1 + x_2 = 1 \)

and: \( 0 \leq x_1 \leq 1 \) \hspace{1cm} and: \( 0 \leq x_1 \leq 1 \)

\( 0 \leq x_2 \leq 1 \) \hspace{1cm} \( 0 \leq x_2 \leq 1 \)

\( 0 \leq x_3 \leq 1 \) \hspace{1cm} \( 0 \leq x_3 \leq 1 \)

\( 0 \leq x_4 \leq 1 \) \hspace{1cm} \( 0 \leq x_4 \leq 1 \)

The author’s solver [10] returns the status **infeasible** for both problems, because the denominator of their common objective function has been constrained to equal zero. Hence the solution set \( \Upsilon \Rightarrow / 0 \) meaning that, subject to the constraint \( A = T \) from the outer conditional, the inner conditional \( \neg A \vdash B \) cannot be true. It follows that the outer conditional is incorrect also.

F2. **John will arrive on the 10 o’clock plane. Therefore, if John misses his plane in New York, he will arrive on the 10 o’clock plane.**

Updated variables:

- **A**: John arrives on the 10:00 plane
- **B**: John misses his plane in New York

The **truth-functional** interpretation of F2 is the correct assertion that the following propositional-calculus formula is true:

\[ A \to (B \to A) \]  

(166)

Let us evaluate the corresponding **boolean-feasibility** conditional:

\[ A \vdash (B \vdash A) \]  

(167)

Following Equations 104 and 105 we interpret the inner conditional \( B \vdash A \) as this pair of constraints:

\[ \Pr((B \land \neg A) = T) = 0 \]  

\[ \Pr((B \land A) = T) > 0 \]  

(168)  

(169)

In order to evaluate the satisfiability of these constraints given the antecedent \( A \) of the outer conditional, we consider these two solution sets:

\[ \Phi \Leftarrow \{ \Pr((B \land \neg A) = T) : \Pr(A = T) = 1 \} \]  

\[ \Psi \Leftarrow \{ \Pr((B \land A) = T) : \Pr((B \land \neg A) = T) = 0, \Pr(A = T) = 1 \} \]  

(170)  

(171)

Symbolic probability inference with the probability model from Section 6.2.2 yields the algebraic expressions:

\[ \Phi \Rightarrow \{ x_3 : x_1 + x_2 = 1, 0 \leq x_3 \leq 1, \sum_i x_i = 1 \} \]  

\[ \Psi \Rightarrow \{ x_1 : x_3 = 0, x_1 + x_2 = 1, 0 \leq x_1 \leq 1, \sum_i x_i = 1 \} \]  

(172)  

(173)

Note these inputs and outputs:

| A | B | \( \Pr_0(A, B) \) |
|---|---|---|
| T | T | \( x_1 \) |
| T | F | \( x_2 \) |
| F | T | \( x_3 \) |
| F | F | \( x_4 \) |

| (B \land \neg A) | \Pr((B \land \neg A)) | (B \land A) | \Pr((B \land A)) |
|---|---|---|---|
| T | \( x_3 \) | T | \( x_1 \) |
| F | \( x_1 + x_2 + x_4 \) | F | \( x_2 + x_3 + x_4 \) |
For the solution set $\Phi$, linear optimization gives solutions $\alpha^* = 0.000$ and $\beta^* = 0.000$ to the optimization problems:

$$
\begin{align*}
\text{Minimize:} & \quad x_1 & \text{Maximize:} & \quad x_1 \\
\text{subject to:} & \quad x_1 + x_2 + x_3 + x_4 = 1 & \text{subject to:} & \quad x_1 + x_2 + x_3 + x_4 = 1 \\
& \quad x_1 + x_2 = 1 & & \quad x_1 + x_2 = 1 \\
& \quad 0 \leq x_1 \leq 1 & \quad 0 \leq x_1 \leq 1 \\
& \quad 0 \leq x_2 \leq 1 & \quad 0 \leq x_2 \leq 1 \\
& \quad 0 \leq x_3 \leq 1 & \quad 0 \leq x_3 \leq 1 \\
& \quad 0 \leq x_4 \leq 1 & \quad 0 \leq x_4 \leq 1 \\
\end{align*}
$$

For the solution set $\Psi$, linear optimization gives solutions $\alpha^* = 0.000$ and $\beta^* = 1.000$ to the optimization problems:

$$
\begin{align*}
\text{Minimize:} & \quad x_1 & \text{Maximize:} & \quad x_1 \\
\text{subject to:} & \quad x_1 + x_2 + x_3 + x_4 = 1 & \text{subject to:} & \quad x_1 + x_2 + x_3 + x_4 = 1 \\
& \quad x_3 = 0 & & \quad x_3 = 0 \\
& \quad x_1 + x_2 = 1 & \quad x_1 + x_2 = 1 \\
& \quad 0 \leq x_1 \leq 1 & \quad 0 \leq x_1 \leq 1 \\
& \quad 0 \leq x_2 \leq 1 & \quad 0 \leq x_2 \leq 1 \\
& \quad 0 \leq x_3 \leq 1 & \quad 0 \leq x_3 \leq 1 \\
& \quad 0 \leq x_4 \leq 1 & \quad 0 \leq x_4 \leq 1 \\
\end{align*}
$$

These optimization results reveal $\Phi \Rightarrow \{0\}$ and $\Psi \subseteq \{0, 1\}$ with $0 \in \Psi$ and $1 \in \Psi$. According to the criteria of Section 6.2.2, these results indicate that, assuming the outer antecedent $A$ is true, the inner conditional $B \vdash A$ is possibly true, but not necessarily true. Because $\Psi$ does not exclude 0 there is ambiguity about whether the positive requirement $Pr ((B \land A)) = T$ is met.

You can see by inspection of the objective function $x_1$ of the linear programs used to compute $\Psi$ that the inner conditional $B \vdash A$ must be false (subject to the outer antecedent $A = T$) precisely when $x_1 = 0$, that is when the input $Pr_0 (A = T, B = T) = 0$. Conversely if $Pr_0 (A = T, B = T) > 0$ (note the strict inequality) then the solution set $\Psi$ must exclude zero and hence the inner conditional is necessarily true.

In terms of the example 3, what would the constraint $Pr_0 (A = T, B = T) = 0$ mean? Well, this assertion states that it is impossible a priori for John to arrive on the 10:00 plane and also to have missed his plane in New York. This assertion would be quite correct if the plane from New York is the same one that is scheduled to arrive at 10 o’clock.

We can analyze 3 in a slightly different way using a modified truth table to compute which sets of valuations satisfy the BOOLEAN-FEASIBILITY conditionals involved. Consider the following view of the inner conditional $B \vdash A$:

$$
\begin{array}{c|c|c}
A & B & \{A : B = T\} \\
\hline
T & T & \{1\} \\
T & F & \emptyset \\
F & T & \{F\} \\
F & F & \emptyset \\
\end{array}
$$

This indicates that 4 of the 16 possible sets of $(A, B)$ valuations satisfy the requirements for $B \vdash A$:

$$
\{ (T, T) \}, \quad \{ (T, T), (T, F) \}, \quad \{ (T, T), (F, F) \}, \quad \{ (T, T), (T, F), (F, F) \}
$$

All 4 sets include the valuation $(A, B) = (T, T)$. On the other hand it is possible to satisfy the premise $A = T$ without including this valuation $(T, T)$. These are the 3 valuations satisfying the premise $A = T$, equivalently $T \vdash A$ (meaning $\{A : T = T\} = \{T\}$):

$$
\{ (T, T) \}, \quad \{ (T, F) \}, \quad \{ (T, T), (T, F) \}
$$
Note that the probability constraint $\Pr(A = T) > 0$ gives the algebraic constraint $x_1 + x_2 > 0$ which is satisfied by either $x_1 > 0$ or $x_2 > 0$ or both: these strict inequalities indicate which valuations must be included in the sets of satisfactory valuations.

The valuation set \{(T, F)\} satisfies the outer antecedent $A$ but not the outer consequent $B \vdash A$. Therefore $B \vdash A$ is not a consequence of $A$. In order to achieve consequentiality we would have to declare that the valuation $(A, B) = (T, T)$ must be possible: that it is feasible for John to miss his plane in New York and still arrive on the 10:00 plane (presumably not the same aircraft!).

For yet another approach we can evaluate the inner conditional $B \vdash A$ subject to the constraint $A = T$ from the outer conditional using the following solution set:

$$\Upsilon \iff \{ \Pr(A = T \mid B = T) : \Pr(A = T) = 1 \} \quad (180)$$

These are the resulting fractional linear optimization problems:

\[
\begin{align*}
\text{Minimize:} & \quad \frac{(x_1)}{(x_1 + x_3)} & \quad \text{Maximize:} & \quad \frac{(x_1)}{(x_1 + x_3)} \\
\text{subject to:} & \quad x_1 + x_2 + x_3 + x_4 = 1 & \quad \text{subject to:} & \quad x_1 + x_2 + x_3 + x_4 = 1 \\
& \quad x_1 + x_2 = 1 & \quad x_1 + x_2 = 1 \\
& \quad 0 \leq x_1 \leq 1 & \quad 0 \leq x_1 \leq 1 \\
& \quad 0 \leq x_2 \leq 1 & \quad 0 \leq x_2 \leq 1 \\
& \quad 0 \leq x_3 \leq 1 & \quad 0 \leq x_3 \leq 1 \\
& \quad 0 \leq x_4 \leq 1 & \quad 0 \leq x_4 \leq 1 \\
\end{align*}
\]

(181)

The computed solutions $\alpha^* = 1.000$ and $\beta^* = 1.000$ indicate $\Upsilon \Rightarrow \{1\}$. However in this case it is feasible for the denominator $x_1 + x_3$ of the objective function to equal zero. A separate optimization problem would confirm this. Thus we amend the results to say that either $\Upsilon = \{1\}$ or $\Upsilon = \emptyset$.

F3.  
*If Brown wins the election, Smith will retire to private life. Therefore, if Smith dies before the election and Brown wins it, Smith will retire to private life.*

Variables:

- $A$ : Brown wins the election
- $B$ : Smith retires to private life
- $C$ : Smith dies before the election

The TRUTH-FUNCTIONAL interpretation of $F3$ gives antecedent $A \rightarrow B$ and consequent $(A \land C) \rightarrow B$, for an integrated conditional statement:

$$ (A \rightarrow B) \rightarrow ((A \land C) \rightarrow B) \quad (182) $$

As a statement of material implication, this is tautologically true. However the content of the example suggests that propositions $B$ and $C$ cannot be true simultaneously: Smith could not possibly retire to private life if he had already died before the election. In this circumstance it is unintuitive to regard $B$ as a consequence of $A$ and $C$.

Using the BOOLEAN-FEASIBILITY interpretation provides a different result. Consider the conditional:

$$ (A \vdash B) \vdash ((A \land C) \vdash B) \quad (183) $$

Following Equations 104 and 105, we interpret the inner antecedent conditional $A \vdash B$ as the pair of probability constraints, designated as the set $\Gamma$:

$$\Pr(\langle A \land \neg B \rangle = T) = 0$$

$$\Pr(\langle A \land B \rangle = T) > 0 \quad (184)$$

Likewise we interpret the inner consequent conditional $(A \land C) \vdash B$ as the pair of constraints:

$$\Pr(\langle (A \land C) \land \neg B \rangle = T) = 0$$

$$\Pr(\langle (A \land C) \land B \rangle = T) > 0 \quad (186)$$
In order to compute whether the second pair of constraints follows from the first, we evaluate these two solution sets:

\[
\Phi \iff \{ \text{Pr}\left(\langle (A \land C) \land \lnot B \rangle = T \right) : \Gamma \}\}
\]

\[
\Psi \iff \{ \text{Pr}\left(\langle (A \land C) \land B \rangle = T \right) : \text{Pr}\left(\langle (A \land C) \land \lnot B \rangle = T \right) = 0, \Gamma \}\}
\]

(188) \hspace{1cm} (189)

Using the constraint set \(\Gamma \iff \{ \text{Pr}\left(\langle A \land \lnot B \rangle = T \right) = 0, \text{Pr}\left(\langle A \land B \rangle = T \right) > 0 \}\). The probability model from Section 6.2.4 adds the general constraints \(\Gamma_0 \iff \{ 0 \leq y_i \leq 1, \sum_i y_i = 1 \}\).

For the solution set \(\Phi\) linear optimization yields minimum and maximum feasible values \(\alpha^* = 0.000\) and \(\beta^* = 0.000\) indicating \(\Phi \Rightarrow \{0\}\). For the solution set \(\Psi\) linear optimization yields minimum and maximum feasible values \(\alpha^* = 0.000\) and \(\beta^* = 1.000\) indicating \(\Psi \subseteq [0, 1]\). These results \(\Phi = \{0\}\) and \(\Psi \subseteq [0, 1]\) indicate that \(\langle A \land C \rangle \vdash B\) is a possible but not necessary consequence of the premise \(A \vdash B\).

Inspection of the linear programming problems used to calculate \(\Psi\) helps to characterize when this deduction is correct and when it is not. These are the problems:

**Minimize:** 
\[y_1\] 
subject to:
\[
y_3 = 0
\]
\[
y_3 + y_4 = 0
\]
\[
y_1 + y_2 \geq \epsilon
\]
and:
\[0 \leq y_1 \leq 1\]
\[0 \leq y_2 \leq 1\]
\[0 \leq y_3 \leq 1\]
\[0 \leq y_4 \leq 1\]
\[0 \leq y_5 \leq 1\]
\[0 \leq y_6 \leq 1\]
\[0 \leq y_7 \leq 1\]
\[0 \leq y_8 \leq 1\]
\[\epsilon = 0.001\]

**Maximize:** 
\[y_1\] 
subject to:
\[
y_3 = 0
\]
\[
y_3 + y_4 = 0
\]
\[
y_1 + y_2 \geq \epsilon
\]
and:
\[0 \leq y_1 \leq 1\]
\[0 \leq y_2 \leq 1\]
\[0 \leq y_3 \leq 1\]
\[0 \leq y_4 \leq 1\]
\[0 \leq y_5 \leq 1\]
\[0 \leq y_6 \leq 1\]
\[0 \leq y_7 \leq 1\]
\[0 \leq y_8 \leq 1\]
\[\epsilon = 0.001\]

(190)

You can see that the constraint \(y_1 = 0\) would force the solutions to both optimization problems to zero, hence producing \(\Psi = \{0\}\) and rendering the outer conditional false. On the other hand the constraint \(y_1 > 0\) (implemented as \(y_1 \geq \sigma\) with a small numerical constant \(\sigma\) for the optimization solver) would force the minimum solution \(\alpha^*\) to be strictly greater than zero, hence producing \(0 \notin \Psi\) (yet still \(\Psi \neq \emptyset\)) and thereby rendering the outer conditional true.

In terms of the example F3 these results say the following. It is correct to deduce the conclusion ‘If Smith dies before the election and Brown wins it, Smith will retire to private life’ from the premise ‘If Brown wins the election, Smith will retire to private life’—unless it is known a priori that it would be impossible for all three events to occur together (Brown’s victory, Smith’s retirement, and Smith’s death), in which case the deduction is incorrect. If the prior probability of these three events is not known (meaning that it could be anywhere between zero and one, as limited by the laws of probability), then it is possible but not necessary that the stated conclusion is a consequence of the stated premise. For the content of this example it would be appropriate to impose the constraint \(\text{Pr}\left(\langle B = T, C = T \rangle = 0\right)\) to specify that it would be impossible for Smith to retire after having died. This would yield the algebraic constraint \(y_1 + y_3 = 0\) which would force \(y_1 = 0\) (since the laws of probability constrain \(0 \leq y_1 \leq 1\) and \(0 \leq y_3 \leq 1\)). Equivalently the valuations \((A, B, C) = (T, T, T)\) and \((F, T, T)\) could be declared to be impossible.

**F4. If Brown wins the election, Smith will retire to private life. If Smith dies before the election, Brown will win it. Therefore, if Smith dies before the election, then he will retire to private life.**

We continue to use the variables from F3

\[
A : \text{Brown wins the election}
\]
\( B : \) Smith retires to private life  
\( C : \) Smith dies before the election  

The **TRUTH-FUNCTIONAL** interpretation asserts the following formula of the propositional calculus:

\[
((A \rightarrow B) \land (C \rightarrow A)) \rightarrow (C \rightarrow B)
\]  

(191)

And indeed this formula is tautologically true. For this problem the corresponding **BOOLEAN-FEASIBILITY** conditional is also correct:

\[
((A \vdash B) \land (C \vdash A)) \vdash (C \vdash B)
\]  

(192)

This compound conditional statement is simply a permutation of the transitivity example from Section 6.2.4. For this problem the issue is not the way the conditionals involved are interpreted and analyzed; the issue is the content of those conditionals. In this case it is essential to revise the formal model as it is being developed, in order to account for information that is revealed during the course of the problem description.

Let us examine the process of formulating conditional statements to model this problem. We invoke the idea of an agent called ‘the Analyst’ which develops the formal model. First, the Analyst splits the problem description into three English-language sentences, each containing a conditional statement:

\( E_1 : \) If Brown wins the election, Smith will retire to private life.

\( E_2 : \) If Smith dies before the election, Brown will win it.

\( E_3 : \) If Smith dies before the election, then he will retire to private life.

The Analyst then sets out to model each English sentence as a formal conditional statement. Since the type of conditionals is not important here, the familiar **TRUTH-FUNCTIONAL** interpretation is used. Upon reading the first sentence \( E_1 \) the Analyst defines the variable \( A \) to mean that Brown wins, and \( B \) to mean that Smith retires. The Analyst then introduces the following conditional \( S_1 \) (with antecedent \( A \) and consequent \( B \)) to represent \( E_1 \):

\[ S_1 : \quad A \rightarrow B \]  

Upon reading \( E_3 \) the Agent adds the variable \( C \) to mean that Smith dies. The Agent then introduces the following conditionals \( S_2 \) and \( S_3 \) to represent the English sentences \( E_2 \) and \( E_3 \):

\[ S_2 : \quad C \rightarrow A \]  

\[ S_3 : \quad C \rightarrow B \]

Assembling these three conditionals \( S_1, S_2, \) and \( S_3 \) according to the connective ‘therefore’ included in the original problem statement yields the compound conditional shown in Equation (191) which is tautologically true (though perhaps unintuitive).

Upon reflection, the Agent now realizes that there is a problem with the initial conditional \( S_1 \). It was revealed in the second sentence \( E_2 \) that Smith could die before the election. The Agent has prior knowledge that a dead person could not possibly retire to private life. Therefore the Agent revises the initial statement \( S_1 \), which was \( A \rightarrow B \), to account for the exceptional circumstance of Smith’s death:

\[ S_1' : \quad (A \land \neg C) \rightarrow B \]

This revised conditional \( S_1' \) reflects an updated English sentence \( E_1' \), which might be stated as:

\( E_1' : \) If Brown wins the election, and Smith does not die before the election, then Smith will retire to private life.

\( E_1' : \) If Brown wins the election, then Smith will retire to private life—unless Smith dies before the election.
Assembling this revised conditional $\mathcal{S}'_1$ with $\mathcal{S}_2$ and $\mathcal{S}_3$ results in the following compound conditional:

$$(((A \land \neg C) \to B) \land (C \to A)) \to (C \to B)$$

Unlike the formula in Equation 191 this revised statement of material implication is not tautologically true. In fact, if $A$ and $C$ are true and $B$ is false (Brown won, Smith died and did not retire) then the antecedent $((A \land \neg C) \to B) \land (C \to A)$ is true but the consequent $C \to B$ is false.

The Agent, now worried about other exceptions, may wish to revise $\mathcal{S}_2$ and the understanding of $\mathcal{E}_2$ also to state that Brown could not win the election even after Smith had died, if some exceptional event $Z$ happens that renders Brown unable to win:

$\mathcal{S}'_2 : (C \land \neg Z) \to A$

$\mathcal{E}'_2 :$ If Smith dies before the election, then Brown will win it—unless something exceptional happens.

For the content of this example it would make sense to add some new premises that say that Smith cannot retire after he has died (regardless of whether or not Brown won), and that the hypothetical exception $Z$ is indeed incompatible with Brown’s victory:

$\mathcal{S}_4 : \neg (B \land C)$

$\mathcal{S}_5 : \neg (Z \land A)$

So there are at least two ways to handle potential exceptions to conditional statements: first, to revise preexisting conditional statements in order to account for specific exceptions, once those exception have been discovered (as in revising $A \to B$ into $(A \land \neg C) \to B$ to account for the exception $C$); second, to express conditional statements in a defeasible manner in the first place, in order to account for unknown exceptions (as in starting with $(C \land \neg Z) \to A$ instead of $C \to A$, anticipating that some exception $Z$ might later be discovered). After including unknown exceptions such as $Z$ it may be appropriate to analyze each problem twice, once with the constraint that $Z$ is true and once with the constraint that $Z$ is false, in order to see how the potential exception would affect the results of inference.

F5. If Brown wins, Smith will retire. If Brown wins, Smith will not retire. Therefore, Brown will not win.

Interpreted using TRUTH-FUNCTIONAL conditionals we have the tautologically-true formula:

$$((A \to B) \land (A \to \neg B)) \to \neg A$$

BOOLEAN-FEASIBILITY conditionals behave differently. We consider the corresponding formula:

$$((A \vdash B) \land (A \vdash \neg B)) \vdash \neg A$$

Following Equation 195 each inner conditional is defined by an equation about sets of truth values:

$$A \vdash B \equiv \{ B : A = T \} = \{ T \}$$

$$A \vdash \neg B \equiv \{ \neg B : A = T \} = \{ T \}$$

Applying the negation operator both sides of the second equation gives the revised form $\{ B : A = T \} = \{ F \}$ for the conditional $A \vdash \neg B$. The same solution set cannot equal $\{ T \}$ and $\{ F \}$ simultaneously, hence the antecedent of the outer conditional in Equation 195 is necessarily false. A BOOLEAN-FEASIBILITY conditional with a false antecedent is false itself; hence the deduction described by F5 is incorrect using the BOOLEAN-FEASIBILITY interpretation of its conditional statements.

F6. Either Dr. A or Dr. B will attend the patient. Dr. B will not attend the patient. Therefore, if Dr. A does not attend the patient, Dr. B will.

Here are the variables:
Using TRUTH-FUNCTIONAL conditionals gives the tautologically-true formula:

\[(A \vee B) \land \neg B \rightarrow (\neg A \rightarrow B)\]  (198)

Using the BOOLEAN-FEASIBILITY interpretation we consider the isomorphic formula:

\[((A \vee B) \land \neg B) \vdash (\neg A \vdash B)\]  (199)

This new conditional statement is necessarily false because a BOOLEAN-FEASIBILITY conditional with a false premise is not considered true; in this case the truth of the outer antecedent \((A \vee B) \land \neg B\) requires the falsity of the inner antecedent \(\neg A\). One way to confirm this result is with probability and linear programming. Following Equations 104 and 105 the inner conditional \(\neg A \vdash B\) requires that these two constraints are both satisfied:

\[
\Pr((\neg A \land \neg B) = T) = 0
\]  (200)

\[
\Pr((\neg A \land B) = T) > 0
\]  (201)

Thus in order to test the conditional in Equation 199 we evaluate these two solution sets:

\[
\Phi \iff \{ \Pr((\neg A \land \neg B) = T) : \Gamma \}
\]  (202)

\[
\Psi \iff \{ \Pr((\neg A \land B) = T) : \Pr((\neg A \land \neg B) = T) = 0, \Gamma \}
\]  (203)

with \(\Gamma \iff \{ \Pr((A \vee B) \land \neg B) = T) = 1 \) representing the antecedent of the outer conditional.

Using the probability model from Section 6.2.2 linear programming gives solutions \(\alpha^* = 0.000\) and \(\beta^* = 0.000\) for \(\Phi\) and \(\alpha^* = 0.000\) and \(\beta^* = 0.000\) for \(\Psi\), indicating both \(\Phi \Rightarrow \{0\}\) and \(\Psi \Rightarrow \{0\}\). This pattern means that, when the outer antecedent \((A \vee B) \land \neg B\) from Equation 199 is true, the inner conditional \(\neg A \vdash B\) must be false (because it is not feasible that the inner antecedent and consequent are simultaneously true, as the inner conditional requires). Therefore the outer conditional in Equation 199 must always be false.

F7. It is not the case that if John passes history, he will graduate. Therefore John will pass history.

The variables are designated as follows:

\(A\): John passes history

\(B\): John graduates

Using TRUTH-FUNCTIONAL conditionals gives this tautologically true formula of the propositional calculus:

\[\neg(A \rightarrow B) \rightarrow A\]  (204)

We consider the isomorphic statement using BOOLEAN-FEASIBILITY conditionals:

\[\neg(A \vdash B) \vdash A\]  (205)

The insight here is that it is possible to invalidate the inner conditional \(A \vdash B\) without requiring the truth of proposition \(A\). Recall from Equation 41 that the conditional \(A \vdash B\) is defined by the constraint that truth is the only feasible value for \(B\), subject to the constraint that \(A\) is true:

\[A \vdash B \iff \{ B : A = T \} = \{T\}\]  (206)

There are three different ways to achieve the negation \(\{ B : A = T \} \neq \{T\}\) of this equation about sets of truth values: the solution set could equal \(\{T, F\}\), or \(\{F\}\), or \(\emptyset\). In neither of these three cases is it required that truth is the one and only feasible value for \(A\).
We can evaluate the possible truth values of the antecedent and consequent of the outer conditional in Equation 205 considering various sets of valuations of the variables A and B to be possible. Let us begin with the assumption that both valuations \( (A,B) = (F,T) \) and \( (F,F) \) are possible. If no other valuations are possible this gives the set \( V_1 = \{ (F,T), (F,F) \} \) of possible valuation vectors. Considering these two possible valuations: the set \( \{ B : A = T \} \) evaluates to the empty set, and A has possible values \( \{ F \} \). Because \( \{ B : A = T \} = \emptyset \) the BOOLEAN-FEASIBILITY conditional \( \vdash B \) defined by Equation 41 is false in the circumstance that the set \( V_1 \) gives the possible valuations of \( (A,B) \). In terms of the example F7 if it is known a priori that John will not pass history, then the BOOLEAN-FEASIBILITY interpretation of ‘If John passes history, he will graduate’ must be false whereas the proposition ‘John will pass history’ must be false; hence the overall conditional statement F7 is incorrect.

Similarly, assuming the set \( V_2 = \{ (T,F), (F,T), (F,F) \} \) of possible valuations for \( (A,B) \) yields the results \( \{ B : A = T \} \Rightarrow \{ F \} \) and \( \{ A \} \Rightarrow \{ T,F \} \). In this case \( A \vdash B \) must be false, hence the outer antecedent \( \neg(A \vdash B) \) in Equation 205 must be true. Yet the outer consequent \( A \) may be either true or false. In other words the solution set \( \{ A : \neg(A \vdash B) = \{ T \} \} = \{ T,F \} \), which by Definition 41 means that the BOOLEAN-FEASIBILITY conditional in Equation 205 is not correct.

Finally, assuming the set \( V_3 = \{ (T,T), (T,F), (F,T), (F,F) \} \) of possible valuations for \( (A,B) \) yields the results \( \{ B : A = T \} \Rightarrow \{ T,F \} \) and \( \{ A \} \Rightarrow \{ T,F \} \). Again \( A \vdash B \) is necessarily false while \( A \) could be either true or false; the conditional in Equation 205 fails.

In terms of the content of this example F7 these results from the valuation sets \( V_2 \) and \( V_3 \) say that, if it is possible that John could not pass history, and furthermore possible that John could pass history and not graduate, then it is not correct to deduce from the premise ‘It is not the case that if John passes history, he will graduate’ the conclusion that ‘John will pass history’. Whereas the premise is true given either set of possibilities \( V_2 \) or \( V_3 \), in both of these cases the conclusion could be either true or false.

F8. If you throw both switch S and switch T, the motor will start. Therefore, either if you throw switch S the motor will start, or if you throw switch T the motor will start.

These are the variables:

\[ A : \text{You throw switch S} \]
\[ B : \text{You throw switch T} \]
\[ C : \text{The motor starts} \]

Using the TRUTH-FUNCTIONAL interpretation of conditionals gives this statement of material implication, which is tautologically true:

\[ ((A \land B) \rightarrow C) \rightarrow (A \rightarrow C) \lor (B \rightarrow C) \] (207)

We consider the isomorphic BOOLEAN-FEASIBILITY formula:

\[ ((A \land B) \vdash C) \lor (A \vdash C) \lor (B \vdash C) \] (208)

Each conditional statement included in Equation 208 constrains the set of possible valuations of the variables \( (A,B,C) \). Here we will use a table-based approach to computing and describing the relevant sets of valuations. Table 5 shows a worksheet that describes the sets of valuations that describe each inner conditional that is included in Equation 208. We begin with the conditional \( (A \land B) \vdash C \) which is defined by Equation 41 as the following equation about sets of truth values:

\[ \{ C : (A \land B) = T \} = \{ T \} \] (209)

Considering binary variables \( A, B, \) and \( C \) there are \( 2^3 = 256 \) possible sets of \( (A,B,C) \) valuation vectors; let us use \( V^* \) to designate this set of valuations. The relevant column in Table 5 includes the values of the solution set \( \{ C : (A \land B) = T \} \) assuming that only the valuation indicated by each row is possible. The tabulated results
describe the features of the set $V_1 \subseteq V^*$ of valuations that satisfy Equation 209; each valuation set must contain $(A, B, C) = (T, T, T)$; it must omit $(A, B, C) = (T, T, F)$; and it may or may not contain any of the other valuations. There are $2^6$ or 64 such sets of valuations in the set $V_1$, for example the sets $(T, T, T)$ and $(F, F, F)$.

Regarding the conditional $A \vdash C$, Equation 41 gives the defining equation

$$\{ C : A = T \} = \{ T \} \quad (210)$$

Table 5 shows that each member of the set $V_2$ of valuations satisfying Equation 210 must contain either $(A, B, C) = (T, T, T)$ or $(T, F, T)$ or both; it must omit $(T, T, F)$ and $(T, F, F)$; and it may or may not contain any of the other valuations. There are 48 such sets of valuations.

Finally for the conditional $B \vdash C$, Equation 41 gives the defining equation

$$\{ C : B = T \} = \{ T \} \quad (211)$$

Table 5 shows that each member of the set $V_3$ of valuation-sets satisfying Equation 211 must contain either $(A, B, C) = (T, T, T)$ or $(F, T, T)$ or both; it must omit $(T, T, F)$ and $(F, T, F)$; and it may or may not contain any of the other valuations. There are 48 such sets of valuations.

Integrating these results for the valuation sets $V_1$, $V_2$, and $V_3$, it is clear how to construct a set of valuations that satisfies the conditional $(A \land B) \vdash C$ from the antecedent of Equation 208 but neither of the conditionals $A \vdash C$ nor $B \vdash C$ from the consequent. Such a valuation-set must include $(A, B, C) = (T, T, T)$, $(T, F, F)$, and $(F, T, F)$; it must omit $(T, T, F)$; and it may or may not contain the other valuations. There are 16 such sets of valuations, including for example:

$$\langle A, B, C \rangle \in \{ (T, T, T), (T, F, F), (F, T, F) \} \quad (212)$$

In terms of the content of this example [8], there are many conceivable circumstances in which the antecedent of the outer conditional in Equation 208 holds but the consequent does not; in other words using the BOOLEAN-CONDITIONAL interpretation, the stated deduction is incorrect. In each of the counterexamples it is possible to throw switch S alone without starting the motor (the valuation $(A, B, C) = (T, F, F)$ is possible) and it is also possible to throw switch T alone without starting the motor ($(A, B, C) = (F, T, F)$ is possible); yet if both switches are thrown the motor must start ($(A, B, C) = (T, T)$ is possible and $(T, T, F)$ is impossible).

F9. If John will graduate only if he passes history, then he won’t graduate. Therefore, if John passes history he won’t graduate.

We return to the variables from F7

$A :$ John passes history
Let us interpret ‘B only if A’ to mean ‘It is not the case that B is true and A is false’. Using the truth-functional interpretation of conditionals, this problem F9 asserts the following formula of the propositional calculus:

\[
\neg (B \land \neg A) \rightarrow \neg B \rightarrow (A \rightarrow \neg B)
\]

Indeed the above formula is tautologically true. We consider the isomorphic formula using boolean-feasibility conditionals:

\[
\neg (B \land \neg A) \vdash \neg B \vdash (A \vdash \neg B)
\]

The worksheet in Table 6 shows that the first inner conditional \(\neg (B \land \neg A) \vdash \neg B\) requires that either valuation \((A, B) = (T, F)\) or \((F, F)\) or both must be possible, and that \((A, B) = (T, F)\) must be impossible, in order to satisfy the equation \(\{ \neg B : \neg (B \land \neg A) = T \}\) that defines this particular boolean-feasibility conditional according to Equation 211. There are 6 such sets of valuations. Also, Table 6 shows that the second inner conditional \(A \vdash \neg B\) requires that the valuation \((A, B) = (T, T)\) must be possible and that \((A, B) = (T, F)\) must be impossible. There are 4 such sets of valuations. There are altogether 16 possible sets of valuations for the binary variables \((A, B)\).

There two sets of possible valuations for \(A\) and \(B\) that satisfy the antecedent inner conditional in Equation 214 but not the consequent. The requisite condition is that \((A, B) = (F, F)\) is possible, while both \((A, B) = (T, T)\) and \((T, F)\) are impossible. The remaining valuation \((A, B) = (F, T)\) may be possible or impossible. Here are the two matching sets of valuations:

\[
V_1 \iff \{ (F, F) \}
\]

\[
V_2 \iff \{ (F, T), (F, F) \}
\]

In terms of the content of F9 these results say that it is incorrect to deduce the conclusion ‘If John passes history he won’t graduate’ from the premise ‘If John will graduate only if he passes history, then he won’t graduate’. The reason is that in the case that it is impossible for John to pass history (both valuations \((A, B) = (T, T)\) and \((T, F)\) are impossible), and also possible for him not to graduate (the valuation \((A, B) = (F, F)\) is possible), then the stated premise would be satisfied yet the stated conclusion would be necessarily false (because the inner antecedent \(A\) itself would be necessarily false).

Analysis would have produced the same ultimate result, had we interpreted ‘B only if A’ to mean ‘B if and only if A’, as in the propositional-calculus formula \(A \leftrightarrow B\) (with the biconditional operation).
8 DISCUSSION

8.1 Respecting Diversity Among Conditional Statements

There are many different types of conditional statements. The distinction between subjunctive and indicative conditionals is important. But then among the indicatives there still several distinct types. Besides the principal types there are many potential specializations of conditional statements, for example to account for factual evidence that has been provided. Anyway with appropriate use of probability we can express each semantically distinct type and subtype of conditional with a syntactically distinct algebraic expression.

There are yet other kinds of conditional statements. Recurrence relations are an important omission from this document as they are out of scope. Indeed there are other mathematical means of deduction too, which generate different (data) types of solutions (arithmetical, algebraic, probabilistic, dynamical).

8.2 Curating the Contents of Conditionals

Independently of the type of each conditional statement, it is important to get its content right. That is, to ensure that the formal model expresses the information that the modeler actually intends. It is a subtle but terrible fallacy to use certain pieces of information during the informal phase of analysis, but then discard that very information during the formal phase of analysis. If you think in English that it would be impossible for someone to arrive on an airplane that he had failed to board in the first place, then you should say that in formal language. If you discover some exceptional event that you had not previously considered, then you should revise your earlier formal model to account for that event. Or, if you insist on never retracting any formal statement, then you should be make those statements in a cautious and limited way that allows for future exceptions.

8.3 Polylogicism

Appreciating the diversity of conditional statements and using the methods of algebra, probability, and optimization to compute deductions provides a new perspective on mathematical logic. Let us call this ‘polylogicism’: an updated view of the close relationship between logic and algebra, and the essentially mathematical nature of logic.

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We consider also polynomial optimization problems with fractional objectives. Let us update the problem template from Equation 13 by modifying the objective function to be a quotient of polynomials:

\[
\begin{align*}
\text{Maximize :} & \quad f(x)/h(x) \\
\text{subject to :} & \quad g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \\
& \quad \alpha_i \leq x_1 \leq \beta_1, \ldots, \alpha_n \leq x_n \leq \beta_n
\end{align*}
\]  

Here the functions \( f, h \), and each \( g_j \) are polynomials in \( \mathbb{R}[x] \). As before each variable \( x_i \) is bounded by finite limits \( \alpha_i \) and \( \beta_i \). The author’s solver accommodates polynomial optimization problems with fractional objectives by applying the Charnes-Cooper transformation \( [5] \) to the reformulated and linearized problems.

The possibility of division by zero requires special attention. Using the Charnes-Cooper transformation, it is considered infeasible for the denominator of the objective function to have the exact value zero. However it can be detected when the value of a fractional objective is unbounded as its denominator approaches zero. These behaviors are perhaps best explained by a few examples, in which the numerical results returned by the author’s solver have been annotated with the status computed for each optimization problem:

\[
\begin{align*}
\text{Minimize :} & \quad y/x \\
\text{subject to :} & \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\
& \quad \Rightarrow [0,0] \text{ optimal} 
\end{align*}
\]  

\[
\begin{align*}
\text{Maximize :} & \quad y/x \\
\text{subject to :} & \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\
& \quad \Rightarrow \text{ unbounded}
\end{align*}
\]  

\[
\begin{align*}
\text{Minimize :} & \quad y/x \\
\text{subject to :} & \quad x = 0, \quad 0 \leq y \leq 1 \\
& \quad \Rightarrow \text{ infeasible}
\end{align*}
\]  

\[
\begin{align*}
\text{Minimize :} & \quad xy/x \\
\text{subject to :} & \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\
& \quad \Rightarrow [0,0] \text{ optimal}
\end{align*}
\]  

\[
\begin{align*}
\text{Maximize :} & \quad xy/x \\
\text{subject to :} & \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\
& \quad \Rightarrow [1,1] \text{ optimal}
\end{align*}
\]  

\[
\begin{align*}
\text{Minimize :} & \quad xy/x \\
\text{subject to :} & \quad x = 0, \quad 0 \leq y \leq 1 \\
& \quad \Rightarrow \text{ infeasible}
\end{align*}
\]

This behavior regarding division by zero requires two practical considerations. First, if a problem with a fractional objective has been found to be infeasible, then it may be necessary to solve a second problem with the same constraints and an arbitrary non-fractional objective in order to distinguish whether the original infeasibility occurred because the denominator of the fractional objective must be zero, or because the other constraints in the problem are infeasible. Second, if it is important to establish whether some particular indeterminate form such as 0/0 or 1/0 is a feasible value for a fractional objective \( f(x)/h(x) \), then a separate problem must be solved in order to determine this (for example, using the numerator \( f(x) \) alone as a non-fractional objective and adding the constraint \( h(x) = 0 \)).
