D-HOMOTHETIC DEFORMATION OF \((\kappa, \mu)\) MANIFOLD

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Abstract. In this paper, we study the invariance of certain curvature conditions in \((\kappa, \mu)\)-contact metric manifold under \(D\)-homothetic deformation. Finally we give an example to verify the results.

Keywords: \((\kappa, \mu)\)-manifold; \(D\)-homothetic deformation; extended Ricci-recurrent; \(\eta\)-parallel.

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1. INTRODUCTION

The class of \((\kappa, \mu)\)-contact metric manifolds encases both Sasakian and non-Sasakian structures. This class of manifolds are invariant under \(D\)-homothetic transformation. It is noted that the class of spaces acquired through \(D\)-homothetic deformation [13] is a contact metric manifold whose curvature satisfies \(R(X, Y)\xi = 0\). In [13], [14], the authors used \(D\)-homothetic deformation on Sasakian and \(K\)-contact structures to get results on the first Betti number, second Betti number and harmonic forms. A plane section in the tangent space \(T_p(M)\) is called a \(\phi\)-section if there exist a unit vector \(X\) in \(T_p(M)\) orthogonal to \(\xi\) such that \(\{X, \phi X\}\) is an orthonormal basis of the plane section. Then the sectional curvature \(K(X, \phi X) = g(R(X, \phi X)X, \phi X)\) is called a \(\phi\)-sectional curvature. A contact metric manifold \(M(\phi, \xi, \eta, g)\) is said to be of constant \(\phi\)-sectional curvature if at any point \(p \in M\), the sectional curvature \(K(X, \phi X)\) is independent of
choice of non-zero \( X \in \mathcal{D}_p \), where \( \mathcal{D} \) denotes the contact distribution of the contact metric manifold defined by \( \eta = 0 \).

The Riemannian curvature tensor \( R \) of Sasakian manifold of constant \( \phi \)-sectional curvature is determined by Ogiue [9]. The geometry of contact Riemannian manifolds of constant \( \phi \)-sectional curvature is obtained by Tanno [15]. If the \( \phi \)-sectional curvature \( H \) is constant on \( K \)-contact Riemannian manifold \( M(\phi, \xi, \eta, g) \) then \( H \) can be deformed by a \( D \)-homothetic deformation of the structure tensors [16]. An extensive research about \( D \)-homothetic deformation on contact geometry is carried out in recent years. The \( D \)-homothetic deformation is related to the following tensor structures. In other words, it means that the changing of the tensor form

\[
\eta' = a\eta, \quad \xi' = \left(\frac{1}{a}\right)\xi, \quad \phi' = \phi, \quad g' = ag + (a - 1)\eta \otimes \eta, \quad (1.1)
\]

where \( a \) is a positive constant. In particular, some authors (Carriazo et al [3]), (De et al [4]) studied \( D \)-homothetic deformations of certain structures. An almost contact metric manifolds is said to be \( \eta \)-Einstein if its Ricci tensor \( S \) is of the form

\[
S = \alpha g + \beta \eta \otimes \eta, \quad (1.2)
\]

where \( \alpha \) and \( \beta \) are smooth functions on the manifold.

The notion of local symmetry of a Riemannian manifold has been studied by many authors in several ways to different structures. As a weaker version of local symmetry Takahashi [12] introduced the notion of a local \( \phi \)-symmetry on a Sasakian manifold. Generalizing the notion of a local \( \phi \)-symmetry of Takahashi [12]. De et al. [6] introduced the idea of \( \phi \)-recurrent Sasakian manifolds. The notion of a generalized recurrent manifold has been introduced by Dubey [7] and studied by others. Again, the notion of a generalized Ricci recurrent manifold has been introduced and studied by De et. al. [5]. The properties of the extended generalized \( \phi \)-recurrent \( \beta \)-Kenmotsu, Sasakian and \((LCS)_{2n+1}\)-manifolds have been studied in [11], [10] and [18] respectively. Motivated by the above studies, in this paper we characterize the \((\kappa, \mu)\)-contact metric manifolds under \( D \)-homothetic deformation. We study the invariance properties of extended generalized \( \phi \)-recurrent, locally \( \phi \)-Ricci symmetric \((\kappa, \mu)\) manifolds under \( D \)-homothetic deformation. Also \( \eta \)-parallel Ricci tensor is considered in \((\kappa, \mu)\)-contact metric manifolds. Finally, we give an example of such manifold.
2. Preliminaries

Let $M$ be $(2n+1)$-dimensional almost contact metric manifold. Then it carries two fields $\phi$ and $\xi$ and a 1-form $\eta$. The field $\phi$ represents the endomorphism of the tangent spaces, the field $\xi$ is called characteristic vector field and $\eta$ is a 1-form satisfying

\begin{align}
\phi^2 &= -I + \eta \otimes \xi, \quad g(X, \xi) = \eta(X), \\
\eta(\xi) &= 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\
g(\phi X, Y) &= -g(X, \phi Y),\quad g(X, \phi Y) = d\eta(X, Y),
\end{align}

for any vector fields $X, Y \in \chi(M)$. In a contact metric manifold, we characterize a $(1,1)$ tensor field $h$ by $h = \frac{1}{2} \mathcal{L}_\xi \phi$, where $\mathcal{L}$ denotes the Lie differentiation. At this point $h$ is symmetric and satisfies $h\phi = -\phi h$. Also we have $Tr = Tr h = 0$ and $h\xi = 0$. The $(\kappa, \mu)$-nullity distribution of a Riemannian manifold $(M, g)$ is a distribution

\begin{align}
N(\kappa, \mu) : p \mapsto N_p(\kappa, \mu) = \{Z \in \chi_p(M) : R(X, Y)Z = \kappa[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY]\}
\end{align}

for any $X, Y, Z \in \chi_p(M)$ and $\kappa$ and $\mu$ being constants, where $R$ denotes the Riemannian curvature tensor and $\chi_p(M)$ denotes the tangent vector space of $M$ at any point $p \in M$. If the characteristic vector field of a contact metric manifold belongs to the $(\kappa, \mu)$ nullity distribution, then the relation

\begin{align}
R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)
\end{align}

holds. A contact metric manifold with $\xi \in N(\kappa, \mu)$ is called a $(\kappa, \mu)$-contact metric manifold [1]. In a $(\kappa, \mu)$-contact metric manifold $M$ the following relations hold [1], [2]:

\begin{align}
h^2 &= (\kappa - 1)\phi^2, \\
\nabla X \xi &= -\phi X - \phi hX,
\end{align}
(2.9) \((\nabla_X \phi) Y = g(X + hX, Y) \xi - \eta(Y)(X + hX)\),

(2.10) \((\nabla_X \eta) Y = g(X + hX, \phi Y)\),

(2.11) \(R(\xi, X) Y = \kappa(g(X, Y) \xi - \eta(Y)X) + \mu(g(hX, Y) \xi - \eta(Y)hX)\),

(2.12) \(S(X, Y) = [2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y) + [2(1-n) + n(2\kappa + \mu)]\eta(X)\eta(Y), n \geq 1\)

(2.13) \(S(X, \xi) = 2n\kappa\eta(X)\),

(2.14) \(r = 2n[2n - 2 + \kappa - n\mu]\),

(2.15) \((\nabla_X h)(Y) = [(1 - \kappa)g(X, \phi Y) + g(X, h\phi Y)]\xi + \eta(Y)X(1 - \kappa)\phi X + \phi hX - \mu \eta(X)\phi hY\),

where \(S\) and \(r\) are the Ricci tensor and scalar curvature respectively and \(Q\) is the Ricci operator, i.e., \(g(QX, Y) = S(X, Y)\).

3. THE D-HOMOTHEtic DEFORMATION IN \((\kappa, \mu)\) CONTACT METRIC MANIFOLD

Let \((M, \phi, \xi, \eta, g)\) be \((2n+1)\) dimensional \((\kappa, \mu)\)-contact metric manifold and \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) be obtained from \((M, \phi, \xi, \eta, g)\) by homothetic deformation (1.1). Throught the paper the quantity with bar denote quantities in \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) and the quantity without bar are for \((M, \phi, \xi, \eta, g)\). The relation between \(\bar{R}\) and \(R\) of \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) as follows: [8].

\[
\bar{R}(X, Y) Z = R(X, Y) Z + (1-a)[g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y + 2\eta(X)\eta(Z)hY
- 2\eta(Y)\eta(Z)hX + 2g(\phi Y, X)\phi Y + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi]
+ \frac{(1-a)}{a}[2\eta(Y)g(hX, Z)\xi - 2\eta(X)g(hY, Z)\xi + (1 - \kappa)\{\eta(Y)g(X, Z)\xi
- \eta(X)g(Y, Z)\xi\} + g(\phi hX, Z)\phi Y - g(\phi hY, Z)\phi hX] + (a^2 - 1)[\eta(Y)\eta(Z)X
- \eta(X)\eta(Z)Y],
\]
for any vector fields $X, Y, Z$ on $M$.

Using (3.1), we derive

$$
\bar{S}(Y, Z) = aS(Y, Z) + (a - 1)[(a^2 - 2a - \kappa + 1)g(Y, Z) + (2na^2 + 2na + 2a - a^2 + \kappa - 1)\eta(Y)\eta(Z) + a(2 + \mu)g(hY, Z)].
$$

(3.2)

**Theorem 3.1.** Under a D-homothetic deformation the expression $Q\phi - \phi Q$ of a $(2n+1)$-dimensional $(\kappa, \mu)$-contact metric manifold is invariant, provided $\mu = -2$.

**Proof:** From (3.1) we have

$$
\bar{Q}X = QX + \frac{a-1}{a}[(a^2 - 2a - \kappa + 1)X + (2na^2 + 2na + 2a - a^2 + \kappa - 1)\eta(X)\xi + a(2 + \mu)hX].
$$

Operating $\bar{\phi} = \phi$ on both sides of above equation from the left, we have,

$$
\bar{\phi}\bar{Q}X = \phi QX + \frac{a-1}{a}[(a^2 - 2a - \kappa + 1)\phi X + a(2 + \mu)\phi hX].
$$

Again, putting $\bar{\phi}X = \phi X$ in (3.2) we have

$$
\bar{Q}\phi X = Q\phi X + \frac{a-1}{a}[(a^2 - 2a - \kappa + 1)\phi X + a(2 + \mu)\phi hX].
$$

From (3.3) and (3.5) we get

$$
(\bar{\phi}\bar{Q} - \bar{Q}\phi)X = (\phi Q - Q\phi)X + 2a(a - 1)(2 + \mu)\phi hX.
$$

(3.6)

Hence the proof.

**Lemma 3.1.** In a $(2n+1)$-dimensional $\eta$-Einstein $(\kappa, \mu)$ manifold $M(\phi, \xi, \eta, g)$, the Ricci tensor is expressed as

$$
S(X, Y) = (r - \kappa)g(X, Y) - (\frac{r}{2n} - 2n\kappa - \kappa)\eta(X)\eta(Y).
$$

(3.7)

**Proof:** On contracting (1.2) we have

$$
r = (2n+1)\alpha + \beta,
$$

(3.8)

where $r$ is the scalar curvature of the manifold. Again putting $X = \xi$ in (2.13) we obtain,

$$
\alpha + \beta = 2n\kappa.
$$

(3.9)
Solving (3.8) and (3.9) we obtain values for \( \alpha = \frac{r}{2n} - \kappa \) and \( \beta = -\frac{r}{2n} + (2n + 1)\kappa \). Putting the values of \( \alpha \) and \( \beta \) in (1.2), we get (3.7).

**Theorem 3.2.** Under D-homothetic deformation, a \((2n + 1)\)-dimensional \(\eta\)-Einstein \((\kappa, \mu)\)-contact metric manifold transforms to \(\eta\)-Einstein \((\kappa, \mu)\)-contact metric manifold provided \(\mu = -2\).

**Proof:** Let \(M(\phi, \xi, \eta, g)\) be a \((2n + 1)\)-dimensional \(\eta\)-Einstein \((\kappa, \mu)\)-contact metric manifold which becomes \(M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) under a D-homothetic deformation. Then from (3.1) it follows by virtue of (3.7) that

\[
\bar{S}(X, Y) = \bar{A}\bar{g}(X, Y) = \bar{B}\bar{\eta}(X)\bar{\eta}(Y) + (\frac{2 + \mu}{a})\bar{g}(hX, Y),
\]

where \(\bar{A}\) and \(\bar{B}\) are smooth functions given by

\[
\bar{A} = \frac{1}{a}(\frac{r}{2n} - \kappa + (\frac{a - 1}{a})(a^2 - 2a - \kappa - 1))
\]

and

\[
\bar{B} = -\left(\frac{a - 1}{a}\right)(\frac{r}{2n} - \kappa + (\frac{a - 1}{a})(a^2 - 2a - \kappa + 1)) - \frac{1}{a^2}\left(\frac{r}{2n} - 2n\kappa - \kappa - (\frac{a - 1}{a})\right)[2na^2 + 2na + 2a - a^2 + \kappa - 1]).
\]

The Proof follows by (3.10).

**Theorem 3.3.** Under D-homothetic deformation, the \(\phi\)-sectional curvature of a \((2n + 1)\)-dimensional \((\kappa, \mu)\)-contact metric manifold is invariant, provided \(\kappa = (1 - 3a)\).

**Proof:** Here we consider the \(\phi\)-sectional curvature on a \((2n + 1)\)-dimensional \((\kappa, \mu)\)-contact metric manifold. From (3.1) it can be easily seen that

\[
\bar{K}(X, \phi X) - K(X, \phi X) = -(1 - a)(3a + \kappa - 1).
\]

Hence we have the proof of the theorem.
4. EXTENDED GENERALIZED $\phi$-RECURRENT, LOCALLY $\phi$-RICCI SYMMETRY AND $\eta$-PARALLEL $(\kappa, \mu)$-MANIFOLD

Firstly, we study the properties of the extended generalized $\phi$-reccurrent $(\kappa, \mu)$-manifolds under $D$-homothetic deformation.

**Definition 4.1.** A $(\kappa, \mu)$-manifold $M(\phi, \xi, \eta, g)$, is said to be an extended generalized $\phi$-reccurrent manifold under $D$-homothetic deformation if its curvature tensor $\bar{R}$ satisfies

$$\phi^2((\nabla_W \bar{R})(X,Y)Z = A(W)\phi^2(\bar{R}(X,Y)Z) + B(W)\phi^2(G(X,Y)Z),$$

for $X, Y, Z, W \in \chi(M)$, where $A$ and $B$ are non-vanishing 1-forms such that $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$ and $G$ is a tensor field of type $(1,3)$ defined as

$$G(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$$  

The 1-forms $A$ and $B$ are called the associated 1-forms of the manifold.

**Definition 4.2.** A $(\kappa, \mu)$ manifold $M(\phi, \xi, \eta, g)$ is said to be generalized Ricci-recurrent manifold under $D$-homothetic deformation if its non-vanishing Ricci tensor $\bar{S}$ satisfies the relation

$$\nabla_W \bar{S}(Y,Z) = A(W)\bar{S}(Y,Z) + B(W)g(Y,Z),$$

for all vector fields $W, X, Y \in \chi(M)$.

**Theorem 4.1.** If an extended generalized $\phi$-recurrent $(\kappa, \mu)$-manifold $M$ under $D$-homothetic deformation is a generalized Ricci-recurrent manifold, then the 1-forms $A$ and $B$ are related as

$$2n(1 - a^2 - \kappa a)A(W) + (4n^2 - 2n - 1)B(W) = 0.$$  

**Proof:** Let us suppose that the manifold $M(\phi, \xi, \eta, g)$, is an extended generalized $\phi$-recurrent $(\kappa, \mu)$-manifold under $D$-homothetic deformation. Then from (2.1), (2.2), (2.3) and (4.1), we have

$$-(\nabla_W \bar{R})(X,Y)Z + \eta((\nabla_W \bar{R})(X,Y)Z)\xi = A(W)[-\bar{R}(X,Y)Z + \eta(\bar{R}(X,Y)Z)\xi]$$

$$+ B(W)[-G(X,Y)Z + \eta(G(X,Y)Z)\xi],$$

$$\eta = \frac{1}{2}(1 - \phi^2 - \kappa).$$

$$\eta = \frac{1}{2}(1 - \phi^2 - \kappa).$$
from which it follows that
\[-g((\nabla_W \tilde{R})(X,Y)Z,U) + \eta((\nabla_W \tilde{R})(X,Y)Z)\eta(U) = A(W)[-g(\tilde{R}(X,Y)Z,U)
\]
\[+ \eta(\tilde{R}(X,Y)Z)\eta(U)] + B(W)[-g(G(X,Y)Z,U)
\]
\[+ \eta(G(X,Y)Z)\eta(U)].
\]

Let \{e_i, i = 1, 2, ..., 2n + 1\} be an orthonormal basis of the tangent space at any point of the manifold. Replacing \(X = U = e_i\) in (4.4) and taking summation over \(i, 1 \leq i \leq 2n + 1\), we have
\[(\nabla_W \tilde{S})(Y,Z) - g((\nabla_W \tilde{R})(\xi,Y)Z,\xi) = A(W)[\tilde{S}(Y,Z) - \eta(\tilde{R}(\xi,Y)Z)]
\]
\[+ B(W)[(2n - 1)g(Y,Z) + \eta(Y)\eta(Z)].
\]

In consequence of (2.1), (2.2), (3.1) we have
\[
(4.6) \quad \eta(\tilde{R}(\xi,Y)Z) = (\frac{a^2 - 1 + \kappa}{a})[g(Y,Z) - \eta(Y)\eta(Z)] + (\mu - \frac{2(1-a)}{a})g(hY,Z).
\]

The covariant derivative of the above equation along the vector field \(W\) gives
\[
g((\nabla_W \tilde{R})(\xi,Y)Z,\xi) = (\frac{a^2 - 1 + \kappa}{a} - \mu(1 - \kappa) + \frac{2(1 - \kappa)(1-a)}{a})g(\phi W,Y)\eta(Z)
\]
\[+ [\frac{a^2 - 1 + \kappa}{a} - \mu + \frac{2(1-a)}{a}]g(\phi hW,Y)\eta(Z) - \mu(1 - \kappa)
\]
\[\mu(\mu - \frac{2(1-a)}{a})g(\phi hY,Z)\eta(W).
\]

In view of (4.6), (4.7), (4.5) becomes
\[
(4.8) \quad (\nabla_W \tilde{S})(Y,Z) - (\frac{a^2 - 1 + \kappa}{a} - \mu(1 - \kappa) + \frac{2(1 - \kappa)(1-a)}{a})g(\phi W,Y)\eta(Z)
\]
\[- (\frac{a^2 - 1 + \kappa}{a} - \mu + \frac{2(1-a)}{a})g(\phi hW,Y)\eta(Z) + \mu(1 - \kappa)g(\phi W,Z)\eta(Y) + \mu g(\phi hW,Z)\eta(Y)
\]
\[+ \mu g(\phi hW,Z)\eta(Y) + \mu(\mu - \frac{2(1-a)}{a})g(\phi hY,Z)\eta(W)
\]
\[= A(W)[\tilde{S}(Y,Z) - (\frac{a^2 - 1 + \kappa}{a})[g(Y,Z) - \eta(Y)\eta(Z)] + (\mu - \frac{2(1-a)}{a})g(hY,Z)]
\]
\[+ B(W)[(2n - 1)g(Y,Z) + \eta(Y)\eta(Z)].
\]
From (4.8) and the definition (4.2), it follows that an extended generalized $\phi$-recurrent $(\kappa, \mu)$-manifold under $D$-homothetic deformation is a generalized Ricci-recurrent manifold if and only if

\[
\frac{a^2 - 1 + \kappa}{a} - \mu(1 - \kappa) + \frac{2(1 - \kappa)(1 - a)}{a} g(\phi W, Y) \eta(Z) + \frac{a^2 - 1 + \kappa}{a} - \mu + \frac{2(1 - a)}{a} g(\phi h W, Y) \eta(Z) - \mu(1 - \kappa) g(\phi W, Z) \eta(Y) - \mu g(\phi h W, Z) \eta(Y) - \mu g(\phi W, Z) \eta(Y) - \mu \eta(W) \eta(Y) = 0.
\]

Let $\{e_i : i = 1, 2, \ldots, 2n + 1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $Y = Z = e_i$ in (4.9) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we have

\[
2n(1 - a^2 - \kappa a)A(W) + (4n^2 - 2n - 1)B(W) = 0.
\]

Next, we deal with the study of locally $\phi$-Ricci symmetric $(\kappa, \mu)$-manifolds under $D$-homothetic deformation.

**Theorem 4.2.** The property of locally $\phi$-Ricci symmetry on an $(\kappa, \mu)$-manifold is invariant under the $D$-homothetic deformation provided $\mu = -2$.

**Proof:** Differentiating (3.2) covariantly with respect to $W$ we have

\[
(\nabla_W \bar{Q}) X = (\nabla_W Q) X + \left(\frac{a - 1}{a}\right)(2na^2 + 2na + 2a - a^2 + \kappa - 1)((\nabla_W \eta)(X)\xi
\]

\[
+ \eta(X)(-\phi W - \phi h W)) + (2 + \mu)(\nabla_W h) X.
\]

Simplifying by using (2.10) and (2.15) and operating $\phi^2$ on both sides and suppose that $X$ is orthogonal to $\xi$, we find that

\[
\hat{\phi}^2(\nabla_W \bar{Q})(X) = \phi^2(\nabla_W Q)(X) + (2 + \mu)\mu \eta(W) \phi h X.
\]

Hence the proof.

Now, we deal with the study of $\eta$-parallel $(\kappa, \mu)$-manifolds under $D$-homothetic deformation.
Theorem 4.3. Under D-homothetic deformation, an \( \eta \)-Parallel Ricci tensor in a \((\kappa, \mu)\)-manifold remains \( \eta \)-parallel, provided \( \mu = -2 \).

Proof: Differentiating (3.1) covariantly with respect to \( W \) and then using (2.10) and (2.15) we have

\[
(\nabla_W \tilde{S})(X, Y) = (\nabla_W S)(X, Y) + (\frac{a-1}{a})(2na^2 + 2na + 2a - a^2 + \kappa - 1)(\eta(Y)(\nabla_W \eta)(X)
+ \eta(X)(\nabla_W \eta)(Y)) + (a-1)(2+\mu)[(1-\kappa)g(W, \phi X)\eta(Y) - g(W, \phi h X)\eta(Y)
- (1-\kappa)\eta(X)g(\phi W, Y) - \eta(X)g(\phi h W, Y) - \mu \eta(W)g(\phi h X, Y)].
\]

Replacing the vector fields \( X \) by \( \phi X \) and \( Y \) by \( \phi Y \) in (4.13) and then by using (2.1) and (2.2) we obtain

\[
(\nabla_W \tilde{S})(X, Y) = (\nabla_W S)(X, Y) - (a-1)(2+\mu)\mu \eta(W)g(X, \phi Y).
\]

Hence the Proof.

5. Example

We consider 3-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3\} \), where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). Let \(\{E_1, E_2, E_3\}\) be linearly independent global frame on \( M \) given by \( E_1 = \frac{\partial}{\partial x} \), \( E_2 = \frac{\partial}{\partial y} \) and \( E_3 = 2y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \). \([E_1, E_2] = 0, [E_2, E_3] = 2E_1, [E_1, E_3] = 2E_2\). Let \( g \) be a metric defined by \( g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0, g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1 \).

Let \( \eta \) be the 1-form defined by \( \eta(V) = g(V, E_1) \) for any \( V \in \mathfrak{X}(M) \). Let \( \phi \) be the \((1, 1)\)-tensor field defined by \( \phi E_1 = 0, \phi E_2 = E_3, \phi E_3 = -E_2 \) and \( h E_1 = 0, h E_2 = E_2 \) and \( h E_3 = -E_3 \).

Using the linearity of \( \phi \) and \( g \), we have \( \eta(E_1) = 1, \phi^2 V = -V + \eta(V)\xi \) and \( g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W) \), for any \( V, W \in \mathfrak{X}(M) \).

The Riemannian connection \( \nabla \) of the metric tensor \( g \) is given by

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).
\]

Using Koszul’s formula, we get the following,

\[
\nabla_{E_1} E_3 = 2E_2, \nabla_{E_1} E_2 = -2E_3, \nabla_{E_1} E_1 = 0, \nabla_{E_2} E_3 = 0, \nabla_{E_2} E_2 = 0, \nabla_{E_2} E_1 = -2E_3,
\]

\[
\nabla_{E_3} E_3 = 0, \nabla_{E_3} E_2 = 0, \nabla_{E_3} E_1 = 0.
\]

\[
(5.1)
\]
From (5.1) it can be easily seen that \((\phi, \xi, \eta, g)\) is a \((\kappa, \mu)\) manifold. Next we find the curvature tensor as follows:

\[
R(E_1, E_2)E_3 = 0, R(E_2, E_3)E_3 = -4E_2, R(E_1, E_3)E_3 = 0, \\
R(E_1, E_2)E_2 = 0, R(E_2, E_3)E_2 = 4E_3, R(E_1, E_3)E_2 = 0, \\
R(E_1, E_2)E_1 = -4E_2, R(E_2, E_3)E_1 = 0, R(E_1, E_3)E_1 = 4E_3.
\]

(5.2)

In view of the expression of the curvature tensor we find the Ricci tensor as follows:

\[
S(E_1, E_1) = g(R(E_1, E_2)E_2, E_1) + g(R(E_1, E_3)E_3, E_1) = 0.
\]

(5.3)

Similarly we find \(S(E_2, E_2) = -4 = S(E_3, E_3)\). Hence \(r = -8\).

It is well known that in a 3-dimensional manifold, the curvature tensor \(R\) satisfies the relation

\[
R(X, Y)Z = S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y].
\]

(5.4)

From (2.12) we have

\[
S(X, Y) = -\mu g(X, Y) + \mu g(hX, Y) + (2\kappa + \mu)\eta(X)\eta(Y).
\]

(5.5)

From (5.5) we can find that

\[
R(X, Y)Z = 2\mu [g(X, Z)Y - g(Y, Z)X] + \mu [g(hX, Z)X - g(hX, Z)Y + g(Y, Z)hX - g(X, Z)hY] \\
+ (2\kappa + \mu)[\eta(Y)X - \eta(X)Y]\eta(Z) + (2\kappa + \mu)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi \\
- \frac{r}{2}[g(Y, Z)X - g(X, Z)Y].
\]

which is equivalent to

\[
^tR(X, Y, Z, W) = \mu [g(X, Z)g(Y, W) - g(Y, Z)g(X, W)] + \mu [g(hY, Z)g(X, W) \\
- g(hX, Z)g(Y, W) + g(Y, Z)g(hX, W) - g(X, Z)g(hY, W)] \\
+ (2\kappa + \mu)[\eta(Y)g(X, W) - \eta(X)g(Y, W)]\eta(Z) \\
+ (2\kappa + \mu)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\eta(W) \\
- \frac{r}{2}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].
\]

(5.7)

In view of above relation we get

\[
K(E_1, \phi E_1) = 0,
\]
\[ K(E_2, \phi E_2) = g(R(E_2, \phi E_2)E_2, \phi E_2) = g(R(E_1, E_3)E_2, E_3) = 2\mu + \frac{r}{2}. \]

Similarly we have \( K(E_3, \phi E_3) = 2\mu + \frac{r}{2}. \) Again from (3.1) it can be easily shown that
\[ \bar{K}(E_2, \phi E_2) - K(E_2, \phi E_2) = -(1 - a)(3a - 1). \]
Similarly we have \( \bar{K}(E_3, \phi E_3) - K(E_3, \phi E_3) = -(1 - a)(3a - 1) \)
Therefore \((\kappa, \mu)\)-manifold satisfies the relation (3.13) and hence Theorem (3.3) is verified.

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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