DIFFERENTIAL GRADED ALGEBRA OVER QUOTIENTS OF SKEW POLYNOMIAL RINGS BY NORMAL ELEMENTS

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Abstract. Differential graded algebra techniques have played a crucial role in the development of homological algebra, especially in the study of homological properties of commutative rings carried out by Serre, Tate, Gulliksen, Avramov, and others. In this article, we extend the construction of the Koszul complex and acyclic closure to a more general setting. As an application of our constructions, we shine some light on the structure of the Ext algebra of quotients of skew polynomial rings by ideals generated by normal elements. As a consequence, we give a presentation of the Ext algebra when the elements generating the ideal form a regular sequence, generalizing a theorem of Bergh and Oppermann. It follows that in this case the Ext algebra is noetherian, providing a partial answer to a question of Kirkman, Kuzmanovich and Zhang.

Introduction

Differential graded (DG) algebra techniques have played a crucial role in the development of homological algebra, especially in the study of homological properties of commutative rings carried out by Serre [17], Tate [19], Gulliksen [10], Avramov [1], and others. Central to much of this work is the notion of Koszul complex, or more generally the process of adjoining variables to remove unwanted homology classes. The Shafarevich complex, introduced by Golod-Shafarevich [8] and further studied by Golod [7], generalizes this notion to an arbitrary associative algebra over a field filtered by a semigroup. Unfortunately, the DG algebras obtained using the Shafarevich construction can be far from minimal, and therefore do not convey the amount of information one is accustomed to in the commutative case.

The first main result of this paper shows that under some hypotheses on the cycle(s) in question (see Definition 2.2 and Proposition 2.9), one may adjoin a set of exterior or divided powers variables (rather than free) to kill cycles. In particular, this extends the notion of Koszul complex to a broad class of rings of interest in noncommutative algebraic geometry, in such a way that the DG algebras one obtains are minimal. When restricted to the case of a sequence of skew commuting variables in a skew polynomial ring, our construction differs from (but is inspired by) previous ones (cf. [2]) in that it carries a natural DG algebra structure.

In the commutative case, a useful feature of adjoining variables to kill homology classes is that one may always repeat this process. That is, if one has adjoined variables in degree $n+1$ in order to kill homology classes in degree $n$, the cycles of degree $n+1$ representing the homology classes of the resulting complex can then be killed by variables adjoined in degree $n+2$, and so on. Unfortunately, the technical hypotheses required by our construction may not be satisfied in the

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extension, and hence we may not be able to continue this procedure in general. However, this difficulty is no longer present if one works in the context of color commutative rings. As such, the remainder of the paper is devoted to applications of our construction in this setting; we recall the definition.

Let $k$ be a field, $G$ an abelian group, and let $\chi : G \times G \to k^*$ be a skew bicharacter. A $k$-algebra $R$ is called $\chi$-color commutative (or simply color commutative if $\chi$ is understood) provided $A$ admits a $G$-grading $R = \bigoplus_{\sigma \in G} R_\sigma$ such that if $x \in R_\sigma$ and $y \in R_\tau$, then $xy = \chi(\sigma, \tau)yx$ (see Definition 3.3). While these perhaps seem exotic at first glance, such $k$-algebras are essentially just quotients of skew polynomial rings by an ideal generated by normal elements (see Proposition 3.6). The benefit of incorporating the skew bicharacter $\chi$ into the study of such algebras is that they help to contain the proliferation of constants that appear when commuting elements past one another.

Our goal is to develop the differential graded framework over a color commutative finitely generated connected graded $k$-algebra $R$ in a way which parallels the theory in the case of a commutative noetherian local ring. Our treatment is inspired heavily by the lecture notes of Avramov [1] as well as the text of Gulliksen-Levin [9].

Section 4 is devoted to developing the required machinery of color DG algebras needed in later sections. In Section 5 we recall the definition of color derivations (see [6]) and use them to prove that an acyclic closure (see Definition 5.6) of $k$ over $R$ is minimal, generalizing a fundamental result of Gulliksen [10]. In particular, when $R = Q/(f_1, \ldots, f_c)$ for $Q$ a skew polynomial ring and $(f_1, \ldots, f_c)$ a regular sequence of normal elements (we call such an algebra a skew complete intersection), an acyclic closure may be obtained by adjoining variables only in homological degree one and two, in a manner analogous to that of the Tate resolution in the commutative case. As a consequence we obtain a rational expression for the Poincaré series of $k$ over $R$, see Corollary 10.8.

In Section 6 we introduce the category of color commutative DG algebras with divided powers, and prove that an acyclic closure of $k$ over a noetherian connected graded color $k$-algebra $R$ is unique up to isomorphism, generalizing a result of Gulliksen-Levin [9]. Given that the acyclic closure $R\langle Y \rangle$ of $k$ over $R$ is unique, it is natural to consider the $R$-module of color derivations from $R\langle Y \rangle$ to itself as an invariant of $R$. Section 7 is devoted to proving that it is a color DG Lie $R$-algebra, which is a natural generalization of a DG Lie algebra that incorporates the skew bicharacter into the skew symmetry and Jacobi identities [15] (see Definition 7.1 for details). Its homology is therefore a graded color Lie algebra which we call the homotopy color Lie algebra of $R$, denoted by $\pi(R)$. This object carries several gradings, and the dimension of the components record the number of variables adjoined to the acyclic closure in each respective degree, just as in the commutative case.

In Section 8 we generalize [18, Theorem 3] and show that the Ext algebra $\text{Ext}_R(k, k)$ (where $k$ is viewed as a right $R$-module, see 8.1) is the universal enveloping algebra of $\pi(R)$, proving a version of the classical Poincaré-Birkhoff-Witt theorem for $U(\pi(R))$ along the way. As a corollary, we obtain that $\text{Ext}_R(k, k)$ is a graded color Hopf algebra, which is a generalization of a graded Hopf algebra that incorporates the skew bicharacter $\chi$ into the bialgebra structure.

In Section 9 we compute the Lie bracket of elements in cohomological degree one in the homotopy color Lie algebra, generalizing a result of Sjödin [18, Theorem 4].
When $R$ is a skew complete intersection the homotopy color Lie algebra is concentrated in cohomological degree one and two and thus the Lie bracket computations provide a presentation of $\text{Ext}_R(k,k)$ as a $k$-algebra. We provide this presentation in Section 10; our presentation extends the one given by Bergh-Opperman in [3] which was obtained using other methods. This presentation also shows that the Ext algebra of a skew complete intersection is noetherian, giving a partial answer to a question of Kirkman-Kuzmanovich-Zhang in [13].

1. Background

Let $k$ be a field. Until further notice, all $k$-algebras in this paper are assumed to be unital and associative, and unadorned tensor products are assumed to be defined over $k$.

**Definition 1.1.** A $k$-algebra $R$ is graded if $R = \bigoplus_{n \in \mathbb{Z}} R_n$, where each $R_n$ is a $k$-module and $R_m R_n \subseteq R_{m+n}$. A graded algebra is connected if $R_0 = k$ and $R_i = 0$ for $i < 0$. A graded $k$-algebra $R$ is bigraded if each component $R_n$ has a further vector space decomposition $R_n = \bigoplus_{m \in \mathbb{Z}} R_{m,n}$ such that $R_m R_{k,l} \subseteq R_{m+k,n+l}$. In particular, this implies that $R = \bigoplus_{m,n \in \mathbb{Z}} R_{m,n}$.

In this paper, all objects will be bigraded. We will call the first grading the homological grading, and the second the internal grading. For example, a connected graded algebra $R$ may be considered a bigraded algebra by concentrating it in homological degree zero. The homological grading is the one relevant to many of the constructions that follow in this paper, but we will also need the internal grading for arguments that involve the minimality of certain constructions. We will often suppress the internal grading from our notation, however, see Remark 4.2.

**Definition 1.2.** Let $R$ be as in Definition 1.1. We say an element $x \in R$ is homogeneous of (internal) degree $n$, and write $\deg x = n$, if $x \in R_n$. We say an element $x$ is bihomogeneous of (internal) degree $n$ and homological degree $m$, and write $\deg x = n$ and $|x| = m$, if $x \in R_{m,n}$.

**Definition 1.3.** Let $R$ be a graded $k$-algebra. A differential graded (DG) $R$-algebra $A$ is a bigraded unital associative $k$-algebra with $R \subseteq A_0$ equipped with a graded $R$-linear differential $\partial$ of homological degree $-1$ such that $\partial^2 = 0$, and such that the Leibniz rule holds:

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b).$$

We denote the underlying $R$-algebra of $A$ as $A^\natural$. Note that every graded $k$-algebra may be considered a DG algebra with trivial differential and homological grading. One may also consider DG modules over $A$, cf. [1].

The Leibniz rule shows that the cycles $Z(A)$ form a graded $R$-subalgebra of $A$, and that the boundaries $B(A)$ are a two-sided ideal in $Z(A)$. Therefore $H(A) = Z(A)/B(A)$ is a graded $H_0(A)$-algebra.

2. The Process of Killing a Graded Normal Cycle

In a manner similar to the commutative case, we would like to adjoin an exterior or divided powers variable to a DG algebra in order to kill a cycle in homology. It turns out that the process of killing a graded normal cycle (whose definition follows) is quite similar to the commutative case.
Definition 2.1. Let $A$ be a bigraded $k$-algebra. A bihomogeneous element $z \in A$ is said to be \textit{graded normal} if there exists a bigraded automorphism $\sigma$ of $A$ such that for all bihomogeneous elements $y \in A$, one has $zy = (-1)^{|y||z|} \sigma(y)z$ and $\sigma(z) = z$. In such a case, we call such an automorphism $\sigma$ a \textit{normalizing automorphism} of $z$.

Note that ‘graded normal’ agrees with the usual definition of normal, except that a normalizing automorphism associated to a graded normal element may differ from the usual one by a sign when the normal element is of odd homological degree.

Definition 2.2. Let $A$ be a DG algebra. A cycle $z$ is called \textit{killable} if it is a graded normal cycle whose normalizing automorphism is a chain map. If the homological degree of $z$ is odd, we assume in addition that $z^2 = 0$.

In order to adjoin a variable, we first must define the underlying algebra of the extension. To do this, we use twisted tensor products.

Definition 2.3. Let $A$ and $B$ be $k$-algebras with multiplication maps $\mu_A$ and $\mu_B$, respectively. A $k$-linear homomorphism $\tau : B \otimes A \to A \otimes B$ is a \textit{twisting map} provided $\tau(b \otimes 1_A) = 1_A \otimes b$ and $\tau(1_B \otimes a) = a \otimes 1_B$ for all $a \in A, b \in B$. A multiplication on $A \otimes B$ is then given by $\mu_\tau := (\mu_A \otimes \mu_B) \circ (\text{id}_A \otimes \tau \otimes \text{id}_B)$. By [4, Proposition 2.3], $\mu_\tau$ is associative if and only if
\[
\tau \circ (\mu_B \otimes \mu_A) = \mu_\tau \circ (\tau \otimes \tau) \circ (\text{id}_B \otimes \tau \otimes \text{id}_A)
\]
as maps $B \otimes B \otimes A \otimes A \to A \otimes A$. The pair $(A \otimes B, \mu_\tau)$ is a twisted tensor product of $A$ and $B$, denoted by $A \otimes^\tau B$.

We first treat the case of killing an even degree cycle.

Construction 2.4. Let $A$ be a DG $k$-algebra, and let $z$ be a killable cycle with $\sigma$ a normalizing automorphism of $z$.

We construct a DG $k$-algebra $A(y)$ by setting $A(y)^z = A \otimes^\tau k(y)$, where $k(y)$ is the exterior algebra on a variable $y$ in homological degree $|z| + 1$, $\tau$ is given by
\[
\tau((b + cy) \otimes a) = a \otimes b + (-1)^{|a|} c \sigma(a) \otimes y, \quad a \in A, b, c \in k,
\]
and $a$ is bihomogeneous. We remark that in $A(y)$, one has $ya = (-1)^{|a|} \sigma(a)y$ for all bihomogeneous $a$. Note that this makes $A(y)$ a free left (and right) $A$-module. The differential of $A(y)$ is given by
\[
\partial(a_1 + a_2 y) = \partial(a_1) + \partial(a_2)y + (-1)^{|a_2|} a_2 z.
\]

Proposition 2.5. \textit{Construction 2.4 gives $A(y)$ the structure of a DG $k$-algebra.}

\textit{Proof.} First, we check that $\partial^2 = 0$. Indeed, for $a = a_1 + a_2 y$ bihomogeneous of homological degree $|a|$, one has
\[
\partial(\partial(a_1 + a_2 y)) = \partial(\partial(a_1) + \partial(a_2)y + (-1)^{|a_2|} a_2 z)
\]
\[
= (-1)^{|a_2|}\partial(a_2)z + (-1)^{|a_2| - 1}\partial(a_2)z = 0.
\]

Next, we verify the Leibniz rule. Let $a = (a_1 + a_2 y)$ and $b = (b_1 + b_2 y)$ be bihomogeneous elements of $A(y)$. Note that since $y$ is of odd degree, $|a_2|$ and $|a|$ have opposite parity, so that $(-1)^{|a_2|} = (-1)^{|a| - 1}$. Hence
\[
\partial(a_1 + a_2 y)(b_1 + b_2 y) = (\partial(a_1) + \partial(a_2)y + (-1)^{|a_2| - 1} a_2 z)(b_1 + b_2 y)
\]
\[
= \partial(a_1)b_1 + \partial(a_1)b_2 y + (-1)^{|b|}\partial(a_2)\sigma(b_1)y + (-1)^{|a| - 1} a_2 z b_1 + (-1)^{|a| - 1} a_2 z b_2 y.
\]
Similarly, one has
\[(a_1 + a_2 y)\partial(b_1 + b_2 y) = (a_1 + a_2 y)(\partial(b_1) + \partial(b_2)y + (-1)^{|b|-1}b_2 z)\]
\[= a_1\partial(b_1) + a_1\partial(b_2)y + (-1)^{|b|-1}a_1 b_2 z + \]
\[(-1)^{|b|-1}a_2\partial(\partial(b_1))y + a_2\partial(b_2)zy.\]

Finally, we consider \(\partial\) of the product \(ab\):
\[
\partial((a_1 + a_2 y)(b_1 + b_2 y)) = \partial(a_1 b_1 + (a_1 b_2 + (-1)^{|b|}a_2 b_1) y) + \]
\[\partial((a_1 b_2 + (-1)^{|b|}a_1 b_2) y) + (-1)^{|a|+|b|-1}(a_1 b_2 + (-1)^{|b|}a_2 b_1) y + \]
\[(-1)^{|b|}\partial(a_2 \sigma(b_1))y + (-1)^{|a|+|b|-1}a_2 \partial(\sigma(b_1))y + \]
\[(-1)^{|a|+|b|-1}a_2 b_2 z + (-1)^{|a|-1}a_2 \sigma(b_1))z.\]

That \(\partial\) satisfies the Leibniz rule now follows from the fact that \(\sigma\) is a chain map and that \(z\) is graded normal. \(\square\)

One also has a similar construction when \(z\) is a graded normal cycle of odd degree.

**Construction 2.6.** Consider the same setup as in Construction 2.4, except that \(z\) is killable of odd degree.

In this case, let \(A(y)^\tau = A \otimes^\tau k\langle y \rangle\), where \(k\langle y \rangle\) is the divided powers algebra on a variable \(y\) in degree \(|z| + 1\), and \(\tau\) is given by
\[
\tau(\sum_i c_i y^{(i)} \otimes a) = \sum_i c_i \sigma^i(a) \otimes y^{(i)}.\]

Recall that the divided powers algebra \(k\langle y \rangle\) has \(k\)-basis the set of all “divided powers” \(y^{(i)}\) with multiplication given by \(y^{(i)}y^{(j)} = i^i j^{j-1} y^{(i+j)}\), for \(i, j\) nonnegative integers. As is customary, we let \(y^{(0)} = 1\). We remark that in \(A(y)\), one has \(y^{(i)}a = \sigma^i(a)y^{(i)}\) for all bihomogeneous \(a\). Note that again, \(A(y)\) is a free left (and right) \(A\)-module. The differential of \(A(y)\) is given by
\[
\partial(a_0 + \sum_{i \geq 1} a_i y^{(i)}) = \partial(a_0) + \sum_{i \geq 1} (\partial(a_i)y^{(i)} + (-1)^{|a_i|}a_izy^{(i-1)}).\]

A computation similar to the proof of Proposition 2.5 shows that the following proposition holds.

**Proposition 2.7.** Construction 2.6 gives \(A(y)\) the structure of a DG \(k\)-algebra.

If we wish to include the differential of \(y\) as part of the notation for \(A(y)\), we write \(A(y \mid \partial(y) = z)\).

**Definition 2.8.** We call an extension \(A \subseteq B\) of DG algebras obtained by successive application of either Construction 2.4 or 2.6 is called a semi-free extension of \(A\). We denote such an extension by \(A(Y)\) where \(Y\) is the set of variables we have adjoined.

For such an extension and a total order \(<\) on the exterior and divided powers variables \(Y\), a monomial in the variables \(Y\) is said to be in normal form with respect to \(<\) if the variables appearing in it are written in increasing order with respect to \(<\). If the ordering on \(Y\) is understood, we say that such a monomial in \(Y\) is in normal form.
Now suppose one has two killable cycles $z_1$ and $z_2$ that we wish to remove in homology. In order to iterate the above procedure, we must ensure that $z_2$ remains graded normal in the semi-free extension used to kill the cycle $z_1$. This is achieved in the next proposition under a skew commuting hypothesis.

**Proposition 2.9.** Let $A$ be a DG algebra and let $z_1, z_2$ be killable cycles with normalizing automorphisms $\sigma_1, \sigma_2$ respectively. If $\sigma_1 \sigma_2 = \sigma_2 \sigma_1$ and $z_1, z_2$ skew commute, then $z_2$ is a killable cycle in $A(y \mid \partial(y) = z_1)$.

**Proof.** We set the notation $z_1 z_2 = (-1)^{|z_1||z_2|} q z_2 z_1$ and prove the theorem in the case $|z_1|$ even, the odd case is similar. We define the following map

$$\hat{\sigma}_2 : A(y) \to A(y)$$

$$a + by \mapsto \sigma_2(a) + q^{-1} \sigma_2(b)y,$$

and prove that $\hat{\sigma}_2$ is a normalizing chain automorphism for $z_2$ in $A(y)$. Let $a + by$ be a bihomogeneous element of $A(y)$, we check that $z_2$ is graded normal of $z_2$ in $A(y)$:

$$z_2(a + by) = z_2a + z_2by$$

$$= (-1)^{|z_2||a|} \sigma_2(a) z_2 + (-1)^{|z_2|(|a| - 1)} \sigma_2(b) z_2 y$$

$$= (-1)^{|z_2||a|} \sigma_2(a) z_2 + (-1)^{|z_2||a|} q^{-1} \sigma_2(b) y z_2$$

$$= (-1)^{|z_2||a|} \hat{\sigma}_2(a + by) z_2.$$

It is a straightforward verification that $\hat{\sigma}_2$ is an automorphism of the DG algebra $A(Y)$.

$\square$

As an application of the process of killing cycles, we develop the Koszul complex of a skew commuting sequence of normal elements with commuting normalizing automorphisms in an arbitrary connected graded algebra.

**Construction 2.10.** Let $R$ be a connected graded algebra, considered as a DG algebra $A^0$ concentrated in homological degree zero with trivial differential. Let $f_1, \ldots, f_c$ be a skew commuting sequence of normal elements of $R$, and let $\sigma_i$ denote a normalizing automorphism of $f_i$. Finally, assume that $\sigma_i \sigma_j = \sigma_j \sigma_i$ for all $i, j$.

For $i = 1, \ldots, c$, set $A^i = A^{i-1}(y_i \mid \partial(y_i) = f_i)$. By Proposition 2.9, $f_{i+1}, \ldots, f_c$ are normal in $A^i$ for each $i$, which allows for Construction 2.4 to continue after adjoining the $t$th variable. We define the skew Koszul complex of $f_1, \ldots, f_c$ over $R$ to be the DG algebra $A^c$, and denote it by $K^R(f)$.

It is clear that $K^R(f)$ is a free left and right $R$-module with basis given by $1 \in R = K^R(f)_0$ together with monomials of the form $y_{i_1} \cdots y_{i_r}$ for $1 \leq i_1 < i_2 < \cdots < i_r \leq c$. The differential on this basis is:

$$\partial(y_{i_1} \cdots y_{i_r}) = \sum_{j=1}^r (-1)^{j-1} f_{i_j} \prod_{s=1}^{j-1} r_{i_s, i_{s+1}} y_{i_1} \cdots \hat{y}_{i_s} \cdots y_{i_r},$$

where $f_{i_s} f_{i_j} = r_{i_s, i_j} f_{i_s} f_{i_j}$. Note that the differential is both left and right $R$-linear by the Leibniz rule and that this differential differs from the one given in [2].

Next, we recall the definition of skew polynomial ring.
Definition 2.11. Let $q = (q_{i,j})$ be a $n \times n$ matrix with entries in $\mathbb{k}$ such that $q_{i,i} = 1$ for all $i = 1, \ldots, n$ and $q_{i,j} = q_{j,i}^{-1}$ for all $i, j = 1, \ldots, n$. Then the skew polynomial ring associated to the matrix $q$ is

$$\mathbb{k}_q[x_1, \ldots, x_n] = \frac{T(\mathbb{k}x_1 \oplus \cdots \oplus \mathbb{k}x_n)}{(x_ix_j - q_{i,j}x_jx_i \text{ for all } i, j = 1, \ldots, n)},$$

where $T(\mathbb{k}x_1 \oplus \cdots \oplus \mathbb{k}x_n)$ is the tensor algebra of $\mathbb{k}x_1 \oplus \cdots \oplus \mathbb{k}x_n$. It is clear that each $x_i$ is normal in $\mathbb{k}_q[x_1, \ldots, x_n]$, and that the normalizing automorphisms of the variables commute with one another.

More generally (see Section 3), we will show that any pair of normal elements in a skew polynomial ring skew commute and their normalizing automorphisms commute with one another.

Definition 2.12. Let $Q = \mathbb{k}_q[x_1, \ldots, x_n]$ and $R = Q/I$ for some homogeneous ideal $I$ of $Q$, and denote by $\bar{x}_1, \ldots, \bar{x}_n$ the images of the variables of $Q$ in $R$. We denote the Koszul complex of $\bar{x}_1, \ldots, \bar{x}_n$ over $R$ by $K^R$ and call it the Koszul complex of $R$.

Next we examine the effect that adjoining a variable has on $H(A)$. First, we must contend with a notion of regularity, suitably modified for the DG setting.

Definition 2.13. Let $A$ be a graded algebra, and let $z \in A$ be a bihomogeneous normal element. We denote by $(0 :_A z)$ the left annihilator of $z$ in $A$. If $|z|$ is even, we say $z$ is regular provided $(0 :_A z) = 0$. If $|z|$ is odd and $z^2 = 0$, then we say that $z$ is regular if $(0 :_A z) = zA$.

Remark 2.14. It is worth noticing that in the previous definition, the left annihilator of $z$ is zero (even case) or $zA$ (odd case) if and only if the right annihilator of $z$ is zero (even case) or $zA$ (odd case).

Theorem 2.15. Let $A$ be a DG algebra, and let $z$ be a killable cycle of degree $d \geq 0$ such that $w = \text{cls}(z)$ is regular. Then there is a canonical isomorphism $\overline{H(A)} \cong H(A\langle y \rangle)$.

Proof. We follow the treatment given in [11 §6.1]. Let $\sigma$ be a normalizing chain automorphism associated to $z$. We consider the case when $d$ is even and odd separately. If $d$ is even, define $\vartheta : A\langle y \rangle \to A\langle y \rangle$ by $\vartheta(a + by) = (-1)^{|b|}\sigma^{-1}(b)$. A straightforward verification shows that $\vartheta$ is a chain map of degree $-d - 1$. This gives a short exact sequence of DG algebras

$$0 \to A \xrightarrow{\partial} A\langle y \rangle \xrightarrow{\vartheta} A \to 0.$$ 

Computing the connecting map $\overline{\vartheta}$, one sees that for $b \in H(A)$, $\overline{\vartheta}(b) = \sigma(b)w = wb$, so that the connecting map is left multiplication by $w$. Therefore the homology long exact sequence is

$$\cdots \to H_{n-d}(A) \xrightarrow{w} H_n(A) \xrightarrow{H_n(\sigma)} H_n(A\langle y \rangle) \xrightarrow{H_n(\vartheta)} H_{n-d-1}(A) \to \cdots$$

which implies the theorem.

When $d$ is odd, we define $\vartheta(\sum a_i y^{(i)}) = \sum a_i y^{(i-1)}$, which is again a chain map of degree $-d - 1$ and which in turn gives a short exact sequence of DG algebras

$$0 \to A \xrightarrow{i} A\langle y \rangle \xrightarrow{\vartheta} A\langle y \rangle \to 0.$$
While multiplication by $w$ does not appear in the long exact sequence associated to the long exact sequence above, we can determine the effect of killing the homology class $w$ using the spectral sequence associated to the filtration $F^pA\langle y \rangle = \sum_{i \leq p} A^i(y)$, c.f. [20, Section 5.4].

By definition, $\partial_p = F_pA\langle y \rangle_{p+1} / F_{p-1}A\langle y \rangle_{p+1} \cong A_{q-dp}(y)^{[p]}$, and the differential $\partial_p$ sends $ay(p)$ to $\partial(a)y(p)$, so that we have $E_{p,q} = H_{q-dp}(A)$. The differential $d_{p,q}$ in turn is induced by the differential on $A\langle y \rangle$ and sends $ay(p)$ to $(-1)^{|a|}azy(p-1)$. Therefore $E_{p,q}$ is the homology of the complex
\[ H_{q-d(p+1)}(A) \xrightarrow{w} H_{q-dp}(A) \xrightarrow{w} H_{q-d(p-1)}(A). \]

Therefore for all $q$ one has
\[ 2E_{0,q} = \frac{H_q(A)}{wH_{q-d}(A)} \quad \text{and} \quad 2E_{p,q} = \frac{(0 : H(A) \langle w \rangle_{q-pd})}{wH_{q-(p+1)d}(A)} \quad \text{when} \quad p \geq 1. \]

If $w$ is regular, then we have $2E_{p,q} = 0$ for $p \geq 1$, giving us the desired conclusion, since the spectral sequence converges to $H(A\langle y \rangle)$.

A straightforward consequence of Theorem 2.15 is

**Corollary 2.16.** If $f_1, \ldots, f_c$ is a regular sequence of normal elements in the skew polynomial ring $Q$, then $K^Q(f_1, \ldots, f_c)$ is a $Q$-resolution of $Q/(f_1, \ldots, f_c)$.

### 3. Color Commutative Rings

As mentioned in the previous section, in order to iterate the procedure of adjoining variables, one requires the cycles to skew commute, and that their normalizing automorphisms commute. This leads one to consider algebras for which this hypothesis is always satisfied. In this section, we introduce such a class of algebras which in the end turns out to be familiar.

For the rest of the paper, $G$ will denote an abelian group, written multiplicatively, and we will use $e_G$ to denote the identity of $G$.

**Definition 3.1.** A function $\chi: G \times G \to k^*$ is called a *skew bicharacter* provided for all $\alpha, \beta, \sigma \in G$, one has
\[
\begin{align*}
\chi(\sigma, \alpha)\chi(\sigma, \beta) &= \chi(\sigma, \alpha\beta) \\
\chi(\alpha, \beta) &= \chi(\beta, \alpha)^{-1} \\
\chi(\alpha, \alpha) &= 1
\end{align*}
\]

Note that the last condition is not typically part of the definition, but we require it for the proof of Proposition 3.3. In particular, note that for all $\alpha, \beta \in G$, one has $\chi(\alpha, \beta) = \chi(\alpha, \beta^{-1})^{-1}$ and $\chi(e_G, \beta) = 1$. Our interest in skew bicharacters comes from the fact that an algebra that is generated by skew commuting elements defines a skew bicharacter on a subgroup of the automorphism group of the algebra, as seen in the next example.

**Example 3.2.** Let $Q = k[x_1, \ldots, x_n]$ be a skew polynomial ring. Then a basis of $Q$ consists of the set of all (ordered) monomials in these variables, and are hence normal as well. Let $G$ be the subgroup of $\text{Aut}(Q)$ generated by the normalizing automorphisms associated to the variables. Then it follows that $G$ is an abelian group, and that $Q$ admits a $G$-grading by associating to a monomial of $Q$ its corresponding normalizing automorphism.
We use the skew commutativity of the variables to define a skew bicharacter on $G$. To do this, first suppose that $\sigma$ and $\tau$ are normalizing automorphisms of monomials $m_\sigma$ and $m_\tau$ respectively. Then $m_\sigma m_\tau = q_{\sigma,\tau} m_\tau m_\sigma$ for some $q_{\sigma,\tau} \in \mathbb{k}^*$. We define $\chi(\sigma, \tau) = q_{\sigma,\tau}$.

For general $\sigma, \tau \in G$, we have that $\sigma = \sigma_1 \sigma_2^{-1}$ for some $\sigma_1, \sigma_2$ such that each $\sigma_i$ is the normalizing automorphism of a monomial. Likewise, $\tau = \tau_1 \tau_2^{-1}$ for some $\tau_1, \tau_2$ where $\tau_j$ is the normalizing automorphism of a monomial. We then define

$$\chi(\sigma, \tau) = \chi(\sigma_1, \tau_1) \chi(\sigma_2, \tau_1)^{-1} \chi(\sigma_1, \tau_2)^{-1} \chi(\sigma_2, \tau_2).$$

A routine verification shows that $\chi$ is well-defined and that it is a skew bicharacter on $G$.

To capture the commutation behavior present in the previous example relative to the skew bicharacter $\chi$, we introduce the concept of $\chi$-color commutativity.

**Definition 3.3.** Let $A$ be a $G$-graded $\mathbb{k}$-algebra with decomposition $A = \bigoplus_{\sigma \in G} A_\sigma$, and let $\chi$ be a skew bicharacter defined on $G$. We say that $A$ is $\chi$-color commutative (or simply color commutative if $\chi$ is understood) if for every $x \in A_\sigma$ and $y \in A_\tau$, one has $xy = \chi(\sigma, \tau)yx$. An element $x \in A_\sigma$ is said to be $G$-homogeneous. We call the $G$-degree of a $G$-homogeneous element $x$ the color of $x$, and we denote this by $\mathcal{G}(x)$. If $x$ and $y$ are $G$-homogeneous we abuse notation and use $\chi(x,y)$ to denote $\chi(\mathcal{G}(x), \mathcal{G}(y))$.

We regard $Q$ as a color commutative ring corresponding to the $G$-grading and associated skew bicharacter defined as in Example 3.2.

**Notation 3.4.** Let $n \geq 1$ be an integer. For each $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, we denote the product $x_1^{i_1} \cdots x_n^{i_n}$ by $x^I$, which is a $\mathbb{k}$-basis of $Q$. Given an element $f \in Q$, we let $\text{supp}(f)$ denote the set of multiindices $I$ such that the coefficient of $x^I$ is nonzero in the unique expression of $f$ as a linear combination of monomials.

The following lemma regarding normal elements of $Q$ generalizes [12, Lemma 3.5], and characterizes the components of the $G$-grading of $Q$ in terms of its normal elements.

**Lemma 3.5.** Let $Q$ be a skew polynomial ring and let $\chi$ be its associated skew bicharacter. Then:

1. A homogeneous element $f \in Q$ is normal with normalizing automorphism $\sigma$ if and only if $f$ is $G$-homogeneous of color $\sigma$.
2. If $f, g \in Q$ are normal and homogeneous, then there exists $p \in \mathbb{k}$ such that $fg = pgf$.
3. If $f \in Q$ is normal then $\chi(x^I, x^J) = 1$ for all $I, J \in \text{supp}(f)$.

**Proof.** If a homogeneous element $f$ of $Q$ is $G$-homogeneous, then $f$ is normal, since it skew commutes with the variables. For the converse, suppose that $f = \sum_{I \in \mathbb{N}^n} c_I x^I$. It suffices to show that $\chi(x^I, x^J) = \chi(x^I, x^J)$ for all $I, J \in \text{supp}(f)$, since then the $G$-degree of each monomial in the support of $f$ will be the same.

This claim is easily verified if $|\text{supp}(f)| = 1$, therefore we assume $|\text{supp}(f)| \geq 2$. If there is an $x^K$ such that $x^K$ divides $x^I$ for all $I \in \text{supp}(f)$ then we can write $f$ as $x^K g$ with $g$ normal. Indeed, if $h \in Q$ then $fh = \tau(h)f$ where $\tau$ is the normalizing automorphism of $f$. Thus $x^K gh = \tau(h)x^K g$, but since $x^K$ is normal we can write $x^K gh = x^K \tau'(h) g$, hence $x^K (gh - \tau'(h) g) = 0$. Since $x^K$ is regular we deduce...
$gh = \tau'(h)g$, i.e. $g$ is normal. If the claim is true for $g$ then it is true for $f$, therefore we can assume that there is no $x^k$ dividing $x^I$ for all $I \in \text{supp}(f)$.

In what follows, we use $\overline{\alpha}$ to denote the image of an element in $Q/(x_j)$; notice that $\overline{f}$ is nonzero for any choice of $x_j$. Since $f$ is normal we have

$$fx_j = (\sum_k \beta_k x_k)f,$$

which modulo $x_j$ gives

$$0 = (\sum_{k \neq j} \beta_k x_k)\overline{f}.$$

Hence $\beta_k = 0$ for all $k \neq j$ and

$$fx_j = \beta_j x_j f.$$ 

It follows that $\chi(I^I, x_j) = \chi(I^{I'}, x_j) = \beta_j$ for all $I, I' \in \text{supp}(f)$.

The second claim follows since by (1), $f$ and $g$ are $G$-homogeneous, therefore $fg = \chi(f, g)g f$. The final claim follows since $\chi(\sigma, \sigma) = 1$ for all $\sigma \in G$. □

The following proposition shows that color commutative algebras are not exotic. Our motivation for their introduction is to contain the proliferation of constants that arise in computations with skew commutative polynomial rings.

**Proposition 3.6.** Let $A$ be a finitely generated connected graded $k$-algebra. Then $A$ is color commutative if and only if it is a quotient of a skew polynomial ring by an ideal generated by homogeneous normal elements.

**Proof.** Suppose $A$ is color commutative and generated by $\{x_1, \ldots, x_n\}$. Since $A$ is $G$-graded, we can decompose each $x_i$ according to the $G$-grading and hence may assume this generating set is $G$-homogeneous. Since $A$ is color commutative, we have that $x_i x_j = \chi(x_i, x_j) x_j x_i$ so that $A$ is a quotient of a skew polynomial ring $Q$. Since the projection from $Q$ to $A$ is $G$-homogeneous, its kernel is $G$-homogeneous and hence is generated by normal elements. The converse follows from Example 3.2 and Lemma 3.5(1). □

4. Color DG Algebras

In this section, we introduce the main tool we use to study homological properties of color commutative algebras. It is a natural extension of the theory of DG algebras over a commutative ring.

**Definition 4.1.** Let $R$ be a color commutative connected graded $k$-algebra. Let $A$ be a $R$-algebra as in Definition 1.3. We say that $A$ is a $\chi$-color $DG$ $R$-algebra provided $A$ is $G$-graded with a grading compatible with the bigrading of $A$, and the differential on $A$ is also $G$-homogeneous of color $e_G$. We similarly define the notion of left (or right) $\chi$-color $DG$ $A$-module. If the bicharacter $\chi$ is understood, we simply call the above notions color $DG$ algebras/modules.

We say that an element of $A$ is trihomogeneous if it is bihomogeneous and $G$-homogeneous.

We also assume that a color DG $R$-algebra $A$ is graded color commutative. That is, for all trihomogeneous $x, y \in A$, we assume that $xy = (-1)^{|x||y|}\chi(x,y)yx$, and that $x^2 = 0$ when $x$ is trihomogeneous of odd homological degree. As in the commutative case, the first condition implies the second when the characteristic of $k$ is not 2.
It follows that if $A$ is a color DG algebra, then $Z(A)$ and $B(A)$ are $G$-graded, and hence $H(A)$ carries the structure of a $G$-graded algebra. Similarly, if $U$ is a color DG $A$-module, one also has that $H(U)$ is a color left $H(A)$ module. An important (but simple) observation is that $Z(A)$ (and hence $H(A)$) is color commutative so that $G$-homogeneous elements of $Z(A)$ skew commute with one another. This is essentially the reason for working in the color commutative setting, since now the hypotheses on the cycles appearing in Proposition $2.9$ are automatically satisfied.

Note that while the skew bicharacter $\chi$ does not enter the definition of a $\chi$-color DG algebra, it does play a role in the notion of an $A$-linear map; see Definition $4.6$ below. When $x$ and $y$ are $G$-homogeneous of $G$-degree $\sigma$ and $\tau$ respectively, we continue to abuse notation and write $\chi(x,y)$ for $\chi(\sigma,\tau)$.

Remark 4.2. The notion of color DG algebra brings a third grading into the mix: an internal, a homological, and a $G$-grading. For elements of a color DG module we will use the same terminology that has been introduced in Definition $1.2$, $3.3$ and $4.1$. Given a trihomogeneous element $u$ in a color DG module $U$, we denote the internal, homological and group degree of $u$ by $\text{deg}(u)$, $|u|$ and $\mathcal{G}(u)$, respectively.

We denote the component of $U$ in homological degree $m$, group degree $\sigma$, and internal degree $n$ by $U_{m,\sigma,n}$. These indices are listed in order of relevance for our computations, and so we also adopt the convention that when fewer indices are used, they are left off of the end. That is, we use $U_{m,\sigma}$ to denote the component of $U$ in homological degree $m$ and group degree $\sigma$, and $U_m$ to denote the component of $U$ in homological degree $m$.

Let $U$ and $V$ be color DG modules over the color DG algebra $A$. A homomorphism of color DG modules $\varphi : U \to V$ is said to be homogeneous of (internal) degree $n \in \mathbb{Z}$ if $\text{deg} \varphi(u) = \text{deg} u + n$ for all homogeneous elements $u \in U$, in which case we write $\text{deg} \varphi = n$. It is said to be bihomogeneous of homological degree $m \in \mathbb{Z}$ if it is homogeneous and $|\varphi(u)| = |u| + m$ for all bihomogeneous elements $u \in U$, in which case we write $|\varphi| = m$. It is said to be trihomogeneous of color $\sigma$ if it is bihomogeneous and $\mathcal{G}(\varphi(u)) = \sigma \mathcal{G}(u)$ for all trihomogeneous elements $u \in U$, in which case we write $\mathcal{G}(\varphi) = \sigma$. The map $\varphi$ is said to be $G$-homogeneous of color $\sigma$ if $\mathcal{G}(\varphi(u)) = \sigma \mathcal{G}(u)$ for all $G$-homogeneous elements $u \in U$, in which case we write $\mathcal{G}(\varphi) = \sigma$.

Before continuing with some basic remarks on color DG modules, we record some facts regarding our constructions in Section $2$.

Proposition 4.3. Suppose that $A$ is a color DG algebra, and $z \in Z(A)$ is a trihomogeneous cycle. Then $A\langle y \mid \partial(y) = z \rangle$ is a color DG algebra. In particular, if $R$ is a color commutative connected graded $k$-algebra, and $f_1, \ldots, f_c$ is a sequence of normal elements of $R$, then the Koszul complex $K^R(f)$ given in Construction $2.14$ is a color DG algebra.

Proof. By assigning the internal and $G$-degree of $y$ to that of $z$, the differential of $A\langle y \rangle$ is homogeneous and $G$-homogeneous. Showing that $A\langle y \rangle$ is color commutative is a straightforward verification using Lemma $3.5$ and the fact that the normalizing automorphism of $y$ is the same as that of $z$.

It remains to prove that trihomogeneous elements of $A\langle y \rangle$ of odd homological degree square to zero. We prove the case that $y$ has even degree, the odd degree
case is similar. Suppose that \( z = \sum a_i y^{(i)} \) is trihomogeneous of odd degree. Then

\[
    z^2 = \sum_{i,j} (a_i y^{(i)})(a_j y^{(j)}) = \sum_{i<j} ((a_i y^{(i)})(a_j y^{(j)}) + (a_j y^{(j)})(a_i y^{(i)})) + \sum_i (a_i y^{(i)})^2.
\]

Each term in the first sum in the last line of the display is zero since for all \( i, a_i y^{(i)} \) has the same \( G \)-degree as \( z, \chi(z, z) = 1 \), and \( z \) has odd homological degree. Each term in the second sum is zero since \( a_i^2 = 0 \) for each \( i \).

**Construction 4.4.** Let \( R \) be a color commutative connected graded \( k \)-algebra. Let \( F_1 \) be the Koszul complex as constructed in Construction 2.10. For \( n \geq 1 \), set

\[
    F_{n+1} = F_n(y_1, \ldots, y_n) \mid \partial(y_i) = z_i, \ i = 1, \ldots, m
\]

where \( H_n(F_n) = \langle \cls(z_1), \ldots, \cls(z_m) \rangle \). This iterative process is possible since the set of representative cycles chosen as generators must skew commute because they are \( G \)-homogeneous.

Then the complex \( F = \lim_{\to} F_n \) is a free resolution of \( k \) over \( R \). In Section 5 we will prove that if the generators of \( H_n(F_n) \) are chosen minimally for all \( n \) then the resolution \( F \) will be minimal.

Next, we develop some basic properties of color DG modules over a color DG algebra. This involves the next construction, which helps control the introduction of some constants in the development that follows.

**Definition 4.5.** Let \( A \) be a color DG algebra. Let \( U \) be a left \( A \)-module. Then \( U \) may be given the structure of an \( A \)-bimodule by setting \( ua = (-1)^{|u||a|} \chi(a, \tau)au \) for all \( u \) of \( G \)-degree \( \sigma \) and \( a \) of \( G \)-degree \( \tau \). Note that if \( A \to B \) is a homomorphism of color DG algebras, and \( B \) is viewed as a left \( A \)-module, then this bimodule structure is compatible with the usual one.

**Definition 4.6.** Let \( U \) and \( V \) be left color DG modules over a color DG algebra \( A \). A homomorphism of complexes \( \beta : U \to V \) is said to be (left) \( A \)-linear of color \( \sigma \) if it is trihomogeneous of group degree \( \sigma \), and if for all trihomogeneous elements \( a \in A \) and \( u \in U \), one has \( \beta(au) = (-1)^{|a||\beta|} \chi(a, \beta) \beta(au) \). The subspace of \( \Hom_A(U, V) \) spanned by all \( A \)-linear trihomogeneous homomorphisms of complexes is denoted \( \Hom_A(U, V) \). By definition, this set is \( G \)-graded, with \( \sigma \) component given by the set of all \( A \)-linear homomorphisms of color \( \sigma \) from \( U \) to \( V \). By definition, this set also has a natural internal and homological grading.

For all trihomogeneous \( a, \beta, u \) as above, the action

\[
    (a\beta)(u) = a(\beta(u)) = (-1)^{|a||\beta|} \chi(a, \beta) \beta(au)
\]

and differential \( \partial(\beta) = \partial V \beta - (-1)^{|\beta|} \beta \partial U \) gives \( \Hom_A(U, V) \) the structure of a color DG module over \( A \). As usual, there is a correspondence between cycles of \( \Hom_A(U, V) \) and \( A \)-linear chain maps from \( U \) to \( V \), as well as a correspondence between homotopy classes of \( A \)-linear chain maps from \( U \) to \( V \) and homology classes in \( \Hom_A(U, V) \).

**Remark 4.7.** Let \( A \) be a color DG algebra with \( A_0 = R \) a color commutative \( k \)-algebra. Note that while an \( A \)-linear map \( \psi \in \Hom_A(U, V) \) is not left \( R \)-linear, it is right \( R \)-linear using the right action introduced in Definition 4.5. Indeed, for all
trihomogeneous elements \( u \in U \) and homogeneous and \( G \)-homogeneous elements \( r \in R \), one has

\[
\psi(ur) = \psi(\chi(u, r)ru) = \chi(u, r)\psi(ru) = \chi(u, r)\psi(\psi(u)) = \psi(u)r.
\]

It follows that we may view elements of \( \text{Hom}_A(U, V) \) (which are defined using a left color linearity condition) as right linear homomorphisms of \( R \)-modules.

**Definition 4.8.** Let \( U \) and \( V \) be color DG modules over \( A \). The tensor product \( U \otimes_A V \) is the quotient of the (trigraded) tensor product \( U \otimes_k V \) by the vector space spanned by all elements of the form \( ua \otimes_k v - u \otimes_k av \). The left action of \( A \) on \( U \) provides the \( A \)-action on \( U \otimes_A V \), and the differential

\[
\partial(u \otimes v) = \partial(u) \otimes v + (-1)^{|u|} u \otimes \partial v
\]

gives \( U \otimes_A V \) the structure of a color DG module over \( A \).

**Proposition 4.9.** Let \( A \) and \( B \) be color DG algebras, and \( A \rightarrow B \) a morphism of DG algebras. Let \( U \) be a color DG \( A \)-module, and \( V, W \) color DG \( B \)-modules. Then the map

\[
\Phi : \text{Hom}_A(U, \text{Hom}_B(V, W)) \rightarrow \text{Hom}_B(U \otimes_A V, W)
\]

\[
\varphi \mapsto (u \otimes v \mapsto \varphi(u)(v)),
\]

is an isomorphism of color DG \( A \)-modules.

**Proof.** The map \( \Phi(\varphi) \) is clearly \( k \)-linear and trihomogeneous. To see that it preserves the relations of \( U \otimes_A V \), note that if \( \varphi \) is trihomogeneous, one has

\[
\Phi(\varphi)(ua \otimes v) = \varphi(ua)(v) = (-1)^{|a||u|} \chi(u, a)\varphi(au)(v) = (-1)^{|a||u|+|a|} \chi(u, a)\chi(\varphi, a)\varphi(a)(u)(v) = \varphi(u)(av) = \Phi(\varphi)(u \otimes av).
\]

The remaining claims are easily checked. \( \square \)

To finish the section, we record some properties of an important class of color DG modules - those whose underlying module is is free.

**Definition 4.10.** A bounded below color DG module \( F \) over \( A \) is semi-free if its underlying \( A^\natural \)-module \( F^\natural \) has an \( A^\natural \)-basis \( \{ e_\lambda \}_{\lambda \in A} \) which is trihomogeneous.

The following proposition appears as [1, Proposition 1.3.1] in the case of DG algebras. The same proof that is given there is applicable here as well.

**Proposition 4.11.** Suppose \( F \) is a semi-free color DG module over a color DG algebra \( A \). Then each diagram of morphisms of color DG modules over \( A \) represented
by solid arrows

\[
\begin{array}{c}
\gamma \\
\beta \\
\alpha \\
\end{array}
\]

with a surjective quasi-isomorphism \(\beta\) can be completed to a commutative diagram by a morphism \(\gamma\) that is uniquely defined up to \(A\)-linear homotopy.

Standard arguments (c.f. [11 Proposition 1.3.2]) also provide the following proposition.

**Proposition 4.12.** If \(F\) is a semi-free color DG module, then each quasi-isomorphism \(\beta : U \to V\) of color DG modules over \(A\) induces quasi-isomorphisms

\[
\text{Hom}_A(F, \beta) : \text{Hom}_A(F, U) \to \text{Hom}_A(F, V); \quad F \otimes_A \beta : F \otimes_A U \to F \otimes_A V.
\]

5. **The acyclic closure and its properties**

In this section, we explore properties of semi-free extensions of the form \(A\langle Y \rangle\) where \(A\) is a color DG algebra, and \(Y\) is a set of divided powers variables whose differential is compatible with all gradings; see Constructions 2.4 and 2.6.

**Definition 5.1.** Let \(A\) be a color DG algebra, and let \(U\) be a color module over \(A\langle Y \rangle\), where \(Y\) is a set of exterior and divided power variables of positive degree. A trihomogeneous \(k\)-linear map \(D : A\langle Y \rangle \to U\) such that

\[
D(a) = 0 \quad \text{for all } a \in A,
\]

\[
D(bb') = D(b)b' + (-1)^{|D||b|} \chi(D, b)bD(b') \quad \text{for all } b, b' \in A\langle Y \rangle, b, b' \text{ trihomogeneous},
\]

\[
D(y(i)) = D(y) y(i-1) \quad \text{for all } y \in Y_{\text{even}} \text{ and all } i \in \mathbb{N},
\]

is called an \(A\)-linear color derivation. Note that according to Definition 4.6 such a map is a homomorphism of left color \(A\)-modules.

Using the above properties, one sees that

\[
D(y_{\lambda_1}^{(i_1)} \cdots y_{\lambda_q}^{(i_q)}) = \sum_{j=1}^{q} (-1)^{s_j-1} c_j - 1 y_{\lambda_1}^{(i_1)} \cdots D(y_{\lambda_j}^{(i_j)}) \cdots y_{\lambda_q}^{(i_q)}
\]

where \(s_j = |D|(i_{\lambda_1}|x_{\lambda_1}| + \cdots + i_{\lambda_j}|x_{\lambda_j}|), \) and \(c_j = \chi(D, y_{\lambda_1}^{(i_1)} \cdots y_{\lambda_j}^{(i_j)})\). This implies that a derivation \(D\) is determined by its value on \(Y\). In fact, the converse is true: each trihomogeneous function \(Y \to U\) extends to a unique \(A\)-linear color derivation from \(A\langle Y \rangle\) to \(U\) using the formulas above and \(A\)-linearity.

For a fixed \(\sigma \in G\), we denote the set of all \(A\)-linear derivations of degree \(\sigma\) and homological degree \(i\) from \(A\langle Y \rangle\) to \(U\) by \(\text{Der}_A(A\langle Y \rangle, U)_i,\sigma\), which is a left \(A\)-module using the action given in Definition 4.6. Finally, we let \(\text{Der}_A(A\langle Y \rangle, U)\) denote the direct sum \(\bigoplus_{(i,\sigma) \in \mathbb{Z} \times G} \text{Der}_A(A\langle Y \rangle, U)_i,\sigma\).

One may check that if \(U\) is a color DG module over \(A\langle Y \rangle\), then \(\text{Der}_A(A\langle Y \rangle, U)\) is a color DG \(A\langle Y \rangle\)-submodule of \(\text{Hom}_A(A\langle Y \rangle, U)\). Further, if \(\beta : U \to V\) is a homomorphism of color DG modules over \(A\langle Y \rangle\), then the induced map

\[
\text{Der}_A(A\langle Y \rangle, \beta) : \text{Der}_A(A\langle Y \rangle, U) \to \text{Der}_A(A\langle Y \rangle, V)
\]
Lemma 5.5. Let 

Note that we consider 

is also a homomorphism of color DG modules over \(A(Y)\); that is, \(\text{Der}_A(A(Y),-)\) is an endofunctor of the category of color DG modules over \(A(Y)\). The next proposition shows that this functor is representable; its proof is a straightforward adaptation of [1] Proposition 6.2.3 and is omitted.

**Proposition 5.2.** Let \(A(Y)\) be a semi-free extension of the color DG algebra \(A\). Then there exists a semi-free color DG module \(\text{Diff}_A A(Y)\) over \(A(Y)\) and a degree \((0,e_G)\) chain derivation \(d : A(Y) \to \text{Diff}_A A(Y)\) such that

1. The \((A(Y))\)\(^2\)-module \((\text{Diff}_A A(Y))\)\(^2\) has a basis \(\{dy \mid y \in Y\}\), where \(dy\) and \(y\) have the same internal, homological, and group gradings.
2. \(d(y) = dy\) for all \(y \in Y\).
3. \(\partial(b(dy)) = \partial(b(dy)) + (-1)^{[b]}bd(\partial(y))\) for all \(b \in A(Y)\).
4. The map

\[
\text{Hom}_{A(Y)}(\text{Diff}_A A(Y),U) \to \text{Der}_A(A(Y),U)
\]

\[
\beta \mapsto \beta \circ d
\]

is an isomorphism (natural in \(U\)) of color DG modules over \(A(Y)\) with inverse given by

\[
\vartheta \mapsto \left( \sum_{y \in Y} a_y dy \mapsto \sum_{y \in Y} (-1)^{|\vartheta||a_y|} \chi(\partial,a_y)a_y \vartheta(y) \right).
\]

Using Propositions 4.41 and 4.12 one therefore obtains the following corollary.

**Corollary 5.3.** If \(U \to V\) is a (surjective) quasi-isomorphism of color DG modules over \(A(Y)\), then so is the induced map \(\text{Der}_A(A(Y),U) \to \text{Der}_A(A(Y),V)\).

We next wish to construct derivations corresponding to the variables \(Y\) added in the extension \(A(Y)\). We continue to follow the treatment in [1] Section 6).

**Construction 5.4.** Let \(A(Y)\) be a semi-free extension of \(A\) and fix a total order of the variables \(Y\). Let \(J\) denote the kernel of the morphism \(A \to B = H_0(A(Y))\). Note that we consider \(B\) as a color DG algebra with trivial differential. Let \(Y^{(\geq 2)}\) be the set of normal monomials that are decomposable; that is, the variables appearing in the monomial are sorted in increasing order, and their word length in the \(Y\) variables is two or more. Then \(A + JY + AY^{(\geq 2)}\) is a DG \(A\)-submodule of \(A(Y)\).

The complex of indecomposables of the extension \(A(Y)\) is defined to be the quotient complex \(A(Y)/(A + JY + AY^{(\geq 2)})\), and it is denoted \(\text{Ind}_A(A(Y))\). It is a DG module over \(A(Y)\) which is also a complex of free \(B\)-modules with basis \(Y_n\) in homological degree \(n\), where \(Y_n\) denotes the elements of \(Y\) of homological degree \(n\). We denote by \(\pi\) the projection \(A(Y) \to \text{Ind}_A(A(Y))\).

One may use the complex of indecomposables to define derivations using the following lemma.

**Lemma 5.5.** Let \(V\) be a color DG module over \(B\), let \(U\) be a color DG module over \(A\) with \(U_i = 0\) for \(i < 0\), and suppose that \(\beta : U \to V\) is a surjective quasi-isomorphism.

For each \(B\)-linear map \(\xi : \text{Ind}_A(A(Y)) \to V\) of degree \((-n,\sigma)\), there exists an \(A\)-linear chain derivation \(\vartheta : A(Y) \to U\) of the same degree such that \(\beta \vartheta = \xi \pi\), and any two such derivations are homotopic by a homotopy that is itself an \(A\)-linear derivation.
Furthermore, if \( \{y_u\} \subseteq U_0 \) is a collection of elements such that \( \beta(y_u) = \xi(y) \) for all \( y \in Y_n \) then there is a chain derivation \( \vartheta \) satisfying \( \vartheta(y) = u_y \) for all \( y \in Y_n \).

**Proof.** Set \( D = \text{Diff}_A(A(Y)) \). Since the projection \( \pi \) is an \( A \)-linear chain derivation, by Proposition 5.2 there is an induced \( A(Y) \)-linear morphism \( D \to \text{Ind}_A(A(Y)) \) such that \( \pi(dy) = \pi(y) = y \) for all \( y \in Y \). This in turn induces a morphism of \( B \)-complexes \( B \otimes_{A(Y)} D \to \text{Ind}_A(A(Y)) \). This map is bijective on a basis of each and is hence an isomorphism.

Combining this map with the adjoint isomorphism in Proposition 4.9 the universal property of \( D \), and Corollary 5.3 gives a surjective quasi-isomorphism

\[
\text{Der}_A(A(Y), U) \xrightarrow{\sim} \text{Hom}_B(\text{Ind}_A(A(Y)), V).
\]

Therefore given \( \xi \) as in our hypothesis, there exists a chain derivation \( \vartheta : A(Y) \to U \) that satisfies \( \beta \vartheta = \xi \pi \) as claimed. Any two choices must necessarily differ by a boundary of \( \text{Der}_A(A(Y), U) \), i.e. a homotopy which itself is an \( A \)-linear derivation.

For the last claim, suppose that \( \{u_y\} \subseteq U_0 \) as in the statement of the lemma, i.e. we have chosen a lifting \( u_y \) of each \( \xi(y) \) along \( \beta \). Since \( U_i = 0 \) for \( i < 0 \), we are free to choose \( \vartheta_i \) to be any map that satisfies \( (\beta_0 \vartheta_i)(y) = (\xi_0 \pi_i)(y) \) for all \( y \in Y_n \), with setting \( \vartheta_0(y) = u_y \) one such choice.

\[\Box\]

We finish this section by showing that if the cycles in the construction of \( A(Y) \) are chosen minimally (in a sense made more precise in the following definition), then the underlying complex is minimal.

**Definition 5.6.** Let \( A \) be a nonnegatively graded color DG \( k \)-algebra such that \( R = A_0 \) is a quotient of a skew polynomial ring by a sequence of homogeneous normal elements, and suppose each right \( R \)-module \( H_n(A) \) is finitely generated. Let \( A \to B \) be a surjective map of color DG algebras with \( B \) concentrated in degree zero (i.e. \( B \) is a color commutative \( k \)-algebra) with \( J = \ker(A \to B) \) as before, such that \( J_0 \) is generated by a sequence of normal elements.

Construction 4.4 can be applied to produce a resolution \( A(Y) \) of \( B \) over \( A \). If \( A(Y) \) satisfies the following two conditions:

1. \( \partial_0(Y_1) \) minimally generates \( J_0 \) modulo \( \partial_1(A_1) \), and
2. \( \{\text{cls}(\partial(y)) \mid y \in Y_{n+1}\} \) minimally generates \( H_n(A(Y_{\leq n})) \) for \( n \geq 1 \),

then \( A(Y) \) is called an acyclic closure of \( B \) over \( A \).

When naming the variables we adjoin in an acyclic closure, we do so using the natural numbers such that \( |y_i| \leq |y_j| \) when \( i < j \). It follows that we may totally order the normal monomials of \( Y \) using the graded lexicographic order induced by this total order of \( Y \).

We will denote by \( y^{(I)} \) the normal monomial \( y_1^{(i_1)} \cdots y_q^{(i_q)} \) when \( I = (i_1, \ldots, i_q) \) is a finite indexing sequence. Any finite indexing sequence may be padded at the end with zeroes in order to make comparisons using graded lexicographic order. The following lemma appears in [1] Lemma 6.3.3. Its proof is the same as the one given in loc. cit, except one uses Lemma 5.5 above, which adapted the commutative setting to the color commutative one.

**Lemma 5.7.** Let \( A(Y) \) be an acyclic closure of \( k \) over \( A \). Then there exist \( A \)-linear chain derivations \( \vartheta_i : A(Y) \to A(Y) \) for \( i \geq 1 \) such that:
logical degree there is associated a sequence of elements
divided powers
if to every trihomogeneous element $R$
then $\chi$: $A$

However, since $\partial$
Definition 6.1.
appears in [9] for the case of commutative rings.

commutative DG algebras with divided powers. We follow the developme
t that $\partial$
where the second equality follows from Lemma 5.7(2). Therefore
Proof.
Let $b$ be a trihomogeneous element of $A(Y)$, and write $\partial(b) = \sum H a_H y^{(H)}$
where each $y^{(H)}$ is a normal monomial. Proving that $\partial(A(Y))$
is equivalent to showing that if $|y^{(H)}| = |b| - 1$ and $a_H \neq 0$, then deg($a_H$) > 0.
Suppose that there exists an index $I$ such that $|y^{(I)}| = |b| - 1$, $a_I \neq 0$, deg($a_I$) = 0,
and that $I$ is chosen to be least in the graded lexicographic order with respect to
this property.
Applying the chain derivation $\partial^I$ from Lemma 5.7 to $\partial(b)$ we obtain
$$\pm \partial_1(\partial^I(b)) = \partial^I(\partial(b)) = a_I + \sum_{H > I} a_H \partial^I(y^{(H)}) \equiv a_I \mod R_+ A(Y)$$
where the second equality follows from Lemma 5.7(2). Therefore $\partial_1(\partial^I(b)) \notin R_+$.
However, since $A_0 = R$ and $A(Y)$ is an acyclic closure of $k$, we must have that
$\partial_1(A(Y)) = R_+$, giving us our contradiction.

6. The category of color commutative DG algebras with divided powers

In this section we prove the uniqueness of acyclic closures in the category of color
commutative DG algebras with divided powers. We follow the development that
appears in [9] for the case of commutative rings.

Definition 6.1. Let $A$ be a color DG $R$-algebra with respect to a skew bicharacter
$\chi: G \times G \to k^*$, and with differential $\partial$. We say that $A$ is a color DG algebra with
divided powers if to every trihomogeneous element $x \in A$ of even positive homological degree
there is associated a sequence of elements $x^{(k)} \in A$ ($k = 0, 1, 2, \ldots$)
satisfying:

(1) $x^{(0)} = 1, x^{(1)} = x, |x^{(k)}| = k|x|, \partial(x^{(k)}) = (\partial(x))^k$,
(2) $x^{(h)} x^{(k)} = \binom{h+k}{h} x^{(h+k)}$, 

(3) Each $\partial_i$ is unique up to an $A$-linear homotopy which is a color derivation.
(3) if \(|x|, \mathcal{G}(x) = (|y|, \mathcal{G}(y))\) then
\[(x + y)^{(k)} = \sum_{i+j=k} x^{(i)} y^{(j)},\]

(4) for \(k \geq 2\) and \(y\) trihomogeneous
\[(xy)^{(k)} = \begin{cases} 0 & \text{if } |x| \text{ and } |y| \text{ are odd} \\ \chi(y, x) \binom{k}{\frac{k}{2}} x^k y^{(k)} & \text{if } |x| \text{ is even and } |y| \text{ is even and positive}, \end{cases}\]
\[\text{where } \binom{k}{\frac{k}{2}} = 0 \text{ if } k < 2,\]

(5) for \(k \geq 1\)
\[\partial(x^{(k)}) = (\partial(x)) x^{(k-1)}\]

The scalars \(\binom{h}{k}\) and \([h]\) are computed in \(\mathbb{Z}\) and then reduced modulo the characteristic of \(k\).

It remains to notice that \([h]\) is an integer, this follows since \([h]\) = 1 and for \(k \geq 1\) from the recursive relation
\[\binom{h}{k} = \left(\begin{array}{c} h \\ k - 1 \end{array}\right) \binom{(k-1)h}{h-1}\]

**Definition 6.2.** Let \(A\) and \(B\) be color DG \(R\)-algebras with divided powers, with the same skew bicharacter \(\chi\). A map \(f : A \rightarrow B\) is a morphism of color DG algebras with divided powers if it is a trihomogeneous morphism of color DG \(R\)-algebras, such that
\[f(x^{(k)}) = f(x)^{(k)}, \text{ for } k \geq 0 \text{ and } x \in A, |x| \text{ even and positive}.\]

**Definition 6.3.** The category \(\mathrm{DGA}_\chi^Y(R)\) is the category with objects color DG algebras with divided powers with skew bicharacter \(\chi\), and morphisms as defined in Definition 6.2.

**Lemma 6.4.** Let
\[
\begin{array}{ccc}
A(Y) & \xrightarrow{f} & C \\
g & & h \\
A & \xrightarrow{f} & B
\end{array}
\]
be a diagram in \(\mathrm{DGA}_\chi^Y(R)\) with \(C\) acyclic, with \(g\) and \(h\) inclusions. Then there exists a morphism \(\tilde{f} : A(Y) \rightarrow C\) in \(\mathrm{DGA}_\chi^Y(R)\) making the diagram commutative.

**Proof.** By induction it suffices to prove the case \(Y = \{y\}\), therefore \(A(Y) = A(y | \partial A(y)(y) = z)\). The acyclicity of \(C\) implies that there exists \(c \in C\) such that \(\partial C(c) = f(z)\). Since \(\partial C, \partial A\) and \(f\) are \(G\)-homogeneous maps of degree \(e_G\), it follows that \(\mathcal{G}(c) = \mathcal{G}(y)\). If \(|y|\) is even then \(A(y)\) is a free \(A\)-module with basis \(\{y^{(k)}\}_{k \geq 0}\). Define the map \(\tilde{f} : A(y) \rightarrow C\) as: \(\tilde{f}(ay^{(k)}) = f(a)c^{(k)}\) for \(k \geq 0, a \in A\) and then extend by left color \(R\)-linearity. It is straightforward to check that \(\tilde{f}\) is a map of algebras which preserves divided powers, and that \(\tilde{f}\) fits into the diagram above. \(\square\)
Lemma 6.5. Let $R$ be a noetherian color commutative connected $\k$-algebra. Let

$$
\begin{array}{c}
A(Y) \xrightarrow{f} A'(Y') \\
\downarrow g \hspace{1cm} \downarrow h \\
A \xrightarrow{f} A'
\end{array}
$$

be a commutative diagram in $\text{DGA}_r^\Gamma(R)$ of algebras with $g$ and $h$ free extensions with $Y$ and $Y'$ concentrated in positive degree with finitely many variables in each degree. Assume further that $f$ an isomorphism, and $H_0(A'(Y')) \neq 0$. Then $f$ is surjective if and only if $H_0(\tilde{f})$ is an isomorphism and the induced map $\tilde{f} : \text{Ind}_A(A(Y)) \to \text{Ind}_A(A'(Y'))$ is an isomorphism.

Proof. Set $B = H_0(A(Y))$ and $B' = H_0(A'(Y'))$. Clearly if $\tilde{f}$ is an isomorphism then $H_0(\tilde{f})$ and $\tilde{f}$ are isomorphisms as well. For the converse, as noted in Construction 5.3 one has that $\text{Ind}_A(A(Y))$ is a complex of free $B$-modules, and since $B \cong B'$, we have that $\text{Ind}_A(A'(Y'))$ is as well. Since $\tilde{f}$ is an isomorphism, it follows that for each $q$, $Y$ and $Y'$ contain the same number of variables in homological degree $q$. Since $f$ is an isomorphism, this implies that $A(Y)_q$ and $A'(Y')_q$ are isomorphic as well. Since $R$ is noetherian and connected graded, in order to show that $\tilde{f}_q$ is an isomorphism, it suffices to show that $\tilde{f}_q$ is surjective.

We proceed by induction on $q$, with the case $q = 0$ being clear. Assume that $\tilde{f}_p$ is surjective for all $p < q$. Since $\tilde{f}$ is surjective, we have that

$$A'(Y') \subseteq \tilde{f}(A(Y)_q) + A'_q + (J'Y')_q + (A'Y'_{\geq 2})_q,$$

where $J' = \ker A' \to B'$ as in the definition of $\text{Ind}_A(A'(Y'))$. Since $f$ is surjective, we have $A'_q \subseteq \tilde{f}(A(Y)_q)$. Since $f$ preserves divided powers, the induction hypothesis and surjectivity of $f$ implies $(J'_{\geq 1}Y')_q$ and $(A'Y'_{\geq 2})_q$ are contained in $\tilde{f}(A(Y)_q)$. Since $H_0(A'(Y'))$ is nonzero, we know that $J'_0$ is contained in $R_0A'_0$, hence $(J'_0Y') \subseteq R_0A'_0Y' \subseteq R_+A'(Y')$. Putting these facts together, we have that

$$A'(Y')_q \subseteq \tilde{f}(A(Y)_q) + R_+A'(Y')_q,$$

which gives surjectivity of $\tilde{f}_q$ by Nakayama’s Lemma.

We are finally ready to prove uniqueness of acyclic closures.

Theorem 6.6. Let $R(Y)$ and $R(Y')$ be acyclic closures of $\k$ over $R$. Then there exists an isomorphism $\tilde{f} : R(Y) \to R(Y')$ in the category $\text{DGA}_r^\Gamma(R)$ such that $f|_R = \text{id}_R$.

Proof. The existence of $\tilde{f}$ follows from extending $\text{id}_R : R \to R$ using Lemma 6.3. The map $f$ induces homomorphisms in $\text{DGA}_r^\Gamma(R)$ for all $i \geq 0$

$$\tilde{f}_i : R(Y_{\leq i}) \to R(Y'_{\leq i}).$$

It suffices to prove by induction that $\tilde{f}_i$ is an isomorphism since then it will follow from the exactness of direct limits that $\tilde{f}$ is also an isomorphism.
The map $\tilde{f}_0 = \text{id}_R$ is clearly an isomorphism. Let $i \geq 1$ and consider the following commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \rightarrow & \text{Ind}_R(R\langle Y_{\leq i-1}\rangle) & \rightarrow & \text{Ind}_R(R\langle Y_{\leq i}\rangle) & \rightarrow & 0 \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \rightarrow & \text{Ind}_R(R\langle Y'_{\leq i-1}\rangle) & \rightarrow & \text{Ind}_R(R\langle Y'_{\leq i}\rangle) & \rightarrow & 0 \\
\end{array}
$$

where the vertical maps are induced by $\tilde{f}$.

Notice that by construction, the rows of the following diagram are isomorphisms of $k$-vector spaces induced by the differential of the acyclic closure

$$
\begin{array}{cccccc}
\text{Ind}_R(R\langle Y_{\leq i-1}\rangle) & \rightarrow & H_{i-1}(R\langle Y_{\leq i-1}\rangle) \otimes_R k & \rightarrow & \\
\downarrow \gamma & & \downarrow \eta & & \\
\text{Ind}_R(R\langle Y'_{\leq i-1}\rangle) & \rightarrow & H_{i-1}(R\langle Y'_{\leq i-1}\rangle) \otimes_R k & \\
\end{array}
$$

It follows that if $\tilde{f}_{i-1}$ is an isomorphism then so is $\eta$ and hence $\gamma$. The map $\alpha$ is also an isomorphism, hence we deduce that $\beta$ is one as well. By Lemma 6.5 it follows that $\tilde{f}_i$ is an isomorphism, completing the induction argument. □

7. Homotopy Color Lie Algebra

We start this section by giving the definition of graded color Lie algebra over an associative ring $R$, regardless of the characteristic of $R$. Since the Lie algebras of interest to us arise in cohomology, we adopt cohomological conventions for our gradings. We continue to use the same notational conventions appearing in Remark 4.2.

In our applications the Lie algebras will come equipped with an internal grading which, for the sake of readability, will be dropped in the definition of graded color Lie algebra.

Let $G$ be an abelian group, with identity $e_G$, and let $\chi$ be a skew bicharacter on $G$ as defined in Section 5. Let $L = \bigoplus_{(i, \sigma) \in \mathbb{Z} \times G} L^{i, \sigma}$ be a $(\mathbb{Z} \times G)$-graded left $R$-module. If $R$ comes equipped with an internal grading then $L$ is assumed to also have an internal grading, in which case all the maps in this section are assumed to be compatible with respect to this grading and all the elements that follow are assumed to be homogeneous. As before, if $x \in L^{i, \sigma}$ and $y \in L^{j, \tau}$ then we abuse notation and write $\chi(x, y)$ for $\chi(\sigma, \tau)$. If $x \in L^{i, \sigma}$ then we denote by $|x|$ its cohomological degree, i.e. $|x| = i$.

**Definition 7.1.** The $(\mathbb{Z} \times G)$-graded $R$-module $L$ is said to be a *graded color Lie algebra* if it is endowed with a $R$-bilinear operation

$$[-, -] : L \times L \to L,$$

and square maps

$$(-)^{[2]} : L^{2i+1, \sigma} \to L^{4i+2, \sigma^2},$$

such that for all $x \in L^{i, \sigma}$ and $y \in L^{j, \tau}$, one has

1. $[L^{i, \sigma}, L^{j, \tau}] \subseteq L^{i+j, \sigma \tau}$,
(2) The bracket is color anti-commutative:
\[ [x, y] = -(-1)^{|x||y|} \chi(x, y)[y, x], \]

(3) The color Jacobi identity holds:
\[
(-1)^{|x||z|} \chi(z, x)[[x, y], z] + (-1)^{|y||x|} \chi(x, y)[[y, z], x] + (-1)^{|y||z|} \chi(y, z)[[z, x], y] = 0,
\]

(4) if \(|x|\) is even then \([x, x] = 0\),

(5) if \(|x|\) is odd then \([x, x], x) = 0,

(6) if \(x, y \in L^i\sigma\) with \(i\) odd then \((x + y)^2 = [x, y] + x^2 + y^2\),

(7) if \(|x|\) is odd and \(a \in R\) then \((ax)^2 = a^2x^2\),

(8) if \(|x|\) is odd then \([x^2, y] = [x, [x, y]]\) for all \(y \in L\).

**Remark 7.2.** We point out that if the characteristic of \(R\) is not 2 nor 3 it is possible to give the previous definition using a skew bicharacter defined on the group \(\mathbb{Z} \times G\).

**Remark 7.3.** Property (3) in the previous definition can also be expressed as
\[ [x, [y, z]] = [[x, y], z] + (-1)^{|x||y|} \chi(x, y)[y, [x, z]]. \]
This is equivalent to saying the map \(\text{ad}_x(y) = [x, y]\) is a color derivation of \(L\).

**Remark 7.4.** Let \(B\) be an associative \(\mathbb{Z} \times G\)-graded \(R\)-algebra. Then \(\text{Lie}(B)\) is the graded color Lie algebra with underlying \(R\)-module \(B\), bracket given by \([x, y] = xy - (-1)^{|x||y|} \chi(x, y)yx\), with \(x\) and \(y\) trihomogeneous and \(z^2 = z^2\) for \(z\) trihomogeneous with \(|z|\) odd. One can check that \(\text{Lie}(B)\) is indeed a graded color Lie algebra.

**Definition 7.5.** Let \(L, L'\) be graded color Lie algebras over \(R\). A trihomogeneous map of left \(R\)-modules \(f : L \to L'\) is a morphism of graded color Lie algebras if \(f([x, y]) = [f(x), f(y)]\) for all \(x, y \in L\) and \(f(z^2) = f(z)^2\) for all trihomogeneous \(z \in L\) with \(|z|\) odd.

**Definition 7.6.** Let \(L\) be a graded color Lie algebra. The universal enveloping algebra of \(L\) is the following quotient of the tensor algebra \(T(L)\):
\[
U(L) = \frac{T(L)}{x \otimes y - (-1)^{|x||y|} \chi(x, y)y \otimes x - [x, y], |z| \text{ odd}}.
\]

**Remark 7.7.** The universal enveloping algebra of \(L\) satisfies the following universal property. Given a \(\mathbb{Z} \times G\)-graded associative \(R\)-algebra \(B\) and morphism of graded color Lie algebras \(g : L \to \text{Lie}(B)\), there is a unique homomorphism of \(\mathbb{Z} \times G\)-graded associative algebras \(g' : U(L) \to B\), such that \(g = g' f\), where \(f\) is the canonical inclusion \(f : L \to U(L)\). We call \(g'\) the universal extension of \(g\).

**Remark 7.8.** Assume that \(L^n = 0\) for \(n \leq 0\), and that \(L\) is a free \(R\)-module. Fix a trihomogeneous basis of \(L\), denoted by \(\Theta = \{\theta_i\}_{i \geq 1}\) ordered in such a way that \(|\theta_i| \leq |\theta_j|\) for \(i < j\). Let \(I = (i_1, i_2, \ldots)\) be a sequence of nonnegative integers such that \(i_j \in \{0, 1\}\) if \(|\theta_j|\) is odd and \(i_j = 0\) for \(j \gg 0\). Fix an indexing sequence \(I\) and a \(q\) such that \(i_j = 0\) for \(j > q\), a normal monomial on \(\Theta\) is an element of \(U(L)\) of the form \(\theta^I = \theta_1^{i_1} \cdots \theta_q^{i_q}\) (we are dropping the tensor product sign). It is a straightforward check that the set of normal monomials on \(\Theta\) span \(U(L)\).

**Definition 7.9.** A color DG Lie algebra over a ring \(R\) is a graded color Lie algebra \(L\) over \(R\) with a degree \((-1, e_G)\) \(R\)-linear map \(\partial : L \to L\), such that \(\partial^2 = 0\) and
\[
(\partial([x, y]) = [\partial(x), y] + (-1)^{|x||\partial(y)|}, \text{ and } \partial(z^2) = [\partial(x), z] \text{ for } |z| \text{ odd}.
\]
Our interest in color DG Lie algebras arises from the following lemma. For the remainder of the paper $R$ will denote the quotient of a skew polynomial ring by an ideal generated by a sequence of homogeneous normal elements.

**Lemma 7.10.** Let $R \to R(Y)$ be a semi-free extension. The inclusion

$$\text{Der}_R(R(Y), R(Y)) \subseteq \text{Lie}(\text{Hom}_R(R(Y), R(Y)))$$

is one of color DG Lie algebras.

**Proof.** Let $x, y$ be trihomogeneous elements of $R(Y)$ and $\theta$ be a derivation of odd degree. Then

$$\theta^2(xy) = \theta(\theta(x)y + (-1)^{|\theta|x}\chi(\theta, x)x\theta(y))$$

$$= \theta^2(x)y + (-1)^{|\theta|(|\theta|+|x|)}\chi(\theta, \theta(x))\theta(x)\theta(y) + (-1)^{|\theta||x|}\chi(\theta, x)x\theta^2(y))$$

$$= \theta^2(x)y - (-1)^{|x|}\chi(\theta, \theta)\chi(\theta, x)x\theta(y) + (-1)^{|x||\theta|}\chi(\theta, x)\theta(x)\theta(y) + \chi(\theta, x)\chi(\theta, x)x\theta^2(y)$$

$$= \theta^2(x)y + \chi(\theta^2, x)x\theta^2(y).$$

If $|y|$ is an even variable then

$$\theta^2(y^{(i)}) = \theta(\theta(y)y^{(i-1)}) = \theta^2(y)y^{(i-1)} - (\theta(y))\theta(y)^{(i-2)} = \theta^2(y)y^{(i-1)}.$$

Now we prove that the bracket of two color derivations is a color derivation

$$[\theta, \xi](xy) = (\theta \xi - (-1)^{|\theta||\xi|}\chi(\theta, \xi)\xi\theta)(xy)$$

$$= \theta(\xi(x)y + (-1)^{|\xi||x|}\chi(\xi, x)x\xi(y)) - \chi(\theta, \theta(\xi)(x)\xi(y) + (-1)^{|\theta||x|}\chi(\theta, x)x\theta(y))$$

$$= \partial\xi(x) + (-1)^{|\xi||x|}\chi(\theta, \xi(x))\xi\theta(y) + \chi(\theta, \theta(x)\xi(y) + (-1)^{|\theta||x|}\chi(\theta, x)x\theta(y))$$

$$= \alpha(\xi(x)y + (-1)^{|\xi||\theta|}\chi(\theta, \xi(x)x\xi(y))$$

$$= \theta(\xi(x)y + (-1)^{|\xi||\theta|}\chi(\theta, \xi(x)x\xi(y) + \chi(\theta, \theta(x)\xi(y) + (-1)^{|\xi||\theta|}\chi(\theta, x)x\theta(y))$$

$$= \theta\xi(x)y - (-1)^{|\xi||\theta|}\chi(\theta, \xi(x)x\theta(y)$$

$$= \theta(\xi(x)y + (-1)^{|\xi||\theta|}\chi(\theta, \xi(x)x\xi(y))$$

This shows that the derivations form a graded color Lie algebra, we now prove that they form a DG color Lie algebra. We denote by $\bar{\partial}^{\text{Der}}_R$ the differential of the acyclic closure and by $\partial^{\text{Der}}_R$ the differential of the complex of derivations. We notice that $\partial^{\text{Der}}_R$ is a color derivation and that if $\theta$ is a color derivation then $\partial^{\text{Der}}_R(\theta) = [\partial^{\text{Der}}_R, \theta]$. Now the conditions in (7.3) follow from the color Jacobi identity (using Remark 7.3) and property (8) in the definition of graded color Lie algebras. \qed
Since the variables adjoined in the acyclic closure of $k$ over $R$ are trigraded, and since the acyclic closure is unique up to isomorphism, we obtain a family of invariants of a color commutative algebra $R$.

**Definition 7.11.** Let $R(Y)$ be an acyclic closure of $k$ over $R$. The following invariants of $R$ are called the *deviations* of $R$:

$$
\varepsilon_{i,\sigma,j}(R) = \{|y \in Y \mid |y| = i, \mathcal{G}(y) = \sigma, \deg y = j\}, \quad i, j \in \mathbb{N}, \sigma \in G.
$$

**Definition 7.12.** The *homotopy color Lie algebra* of $R$ is

$$\pi(R) = \text{H}(\text{Der}_R(R(Y), R(Y))),$$

where $R(Y)$ is an acyclic closure of $k$ over $R$. Note that since the acyclic closure is trigraded, $\pi(R)$ possesses an additional internal grading which is not part of our definition of graded color Lie algebra above. Also, it is an invariant of $R$ by Theorem 6.6. We denote its graded components as

$$\pi^{i,\sigma,j}(R),$$

where $i$ denotes the cohomological degree, $\sigma$ denotes the group degree and $j$ the internal degree.

**Theorem 7.13.** Let $R(Y)$ be an acyclic closure of $k$ over $R$, where $Y = \{y_i\}_{i \geq 1}$ and $|y_i| \leq |y_j|$ for $i < j$.

1. $\text{rank}_k \pi^{i,\sigma,j}(R) = \varepsilon_{i,\sigma,j}(R)$, for $i, j \in \mathbb{N}$ and $\sigma \in G$.
2. $\pi(R)$ has a $k$-basis

$$\Theta = \{\vartheta_i = \text{cls}(\vartheta_i) \mid \vartheta_i \in \text{Der}_R(R(Y), R(Y)), \vartheta_i(y_j) = \delta_{ij} \text{ for } j \leq i\}_{i \geq 1}.$$

**Proof.** For the first claim, notice that $(\text{Diff}_R R(Y)) \otimes_R k \cong kY$ as complexes with trivial differential by Proposition 5.2. There is a chain of graded isomorphisms

$$\pi(R) = \text{H}(\text{Der}_R(R(Y), R(Y)))
\cong \text{H}(\text{Der}_R(R(Y), k))
\cong \text{H}(\text{Hom}_{R(Y)}(\text{Diff}_R R(Y), k))
\cong \text{H}(\text{Hom}_k((\text{Diff}_R R(Y)) \otimes_R k, k))
\cong \text{H}(\text{Hom}_k(kY, k))
= \text{Hom}_k(kY, k)$$

where the first isomorphism follows by Corollary 5.3, the second by Proposition 5.2, the third by Proposition 4.9, and the fourth by the observation at the start of the proof. Now we are done by definition of deviation.

For the second claim, by Lemma 5.7(1) there are derivations $\vartheta_i$ such that $\vartheta_i(y_j) = \delta_{ij}$ for $j \leq i$. If $\vartheta_{i_1}, \ldots, \vartheta_{i_m}$ all have the same (tri-)degree and $\sum_k \beta_k \vartheta_{i_k} = 0$ for some $\beta_k \in k$, then by evaluating at $y_j$, we deduce that $\beta_j = 0$ for all $j = 1, \ldots, m$. This proves that $\Theta$ is a linearly independent set. By part (1) we know that in each (tri-)degree $\Theta$ has the same dimension of $\pi(R)$ and therefore forms a $k$-basis.

**8. Ext Algebra**

In this section, we study the graded $k$-algebra $\text{Ext}_R(k, k)$, which is the homology of the DG algebra $\text{Hom}_R(R(Y), R(Y))$. In what follows, we denote the augmentation map $R(Y) \to k$ by $\epsilon$. 
Remark 8.1. Since \( \text{Hom}_R(R(Y), R(Y)) \) denotes the set of left color \( R \)-linear maps from \( R(Y) \) to itself, Remark 8.7 shows that the Ext algebra mentioned above is \( \text{Ext}_R(k_R, k_R) \), the Ext algebra of \( k \) over \( R \) where \( k \) is considered a right \( R \)-module.

However, there are isomorphisms (cf. [14, Pg. 5])
\[
\text{Ext}_R(Rk, Rk) \cong \text{Ext}_{R^e}(R^eR, R^eR) \cong \text{Ext}_R(k_R, k_R)^{op}
\]
as graded \( k \)-algebras so that one may convert the descriptions of right Ext algebras given below into a description of left Ext algebras by taking the opposite ring. This is especially important when comparing our results with those that exist in the literature.

**Theorem 8.2.** The inclusion \( \text{Der}_R(R(Y), R(Y)) \rightarrow \text{Hom}_R(R(Y), R(Y)) \) induces an injective morphism of graded color Lie algebras
\[
\iota : \pi(R) \rightarrow \text{Lie}(\text{Ext}_R(k, k)).
\]

**Proof.** We consider the following diagram
\[
\begin{array}{ccc}
\text{Der}_R(R(Y), R(Y)) & \rightarrow & \text{Hom}_R(R(Y), R(Y)) \\
\text{Der}_R(R(Y), k) & \cong & \text{Hom}_R(R(Y), k) \\
\end{array}
\]
where the top and bottom maps are just inclusions and the left and right maps are the quasi-isomorphisms given by Corollary 5.3 and Proposition 4.12 respectively. A straightforward computation shows that this diagram is commutative. By taking (co-)homology in the diagram it follows that the bottom map is (isomorphic to) \( \iota \), which is therefore injective. \( \square \)

We prove a version of the Poincaré-Birkhoff-Witt Theorem for the color Lie algebra \( \pi(R) \). In the next theorem we will use the same notation used in Theorem 7.13.

**Theorem 8.3.** The normal monomials on \( \Theta \) form a \( k \)-basis of \( U\pi(R) \). Moreover the universal extension of the map
\[
\iota : \pi(R) \rightarrow \text{Lie}(\text{Ext}_R(k, k)),
\]
is an isomorphism of associative algebras
\[
\iota' : U\pi(R) \rightarrow \text{Ext}_R(k, k).
\]

**Proof.** We identify \( \iota \) with the inclusion
\[
\iota : \text{Der}_R(R(Y), k) \rightarrow \text{Hom}_R(R(Y), k).
\]
By Theorem 7.13 a \( k \)-basis of \( \text{Der}_R(R(Y), k) \) is given by
\[
\Theta = \{ e\vartheta_i \mid \vartheta_i \in \text{Der}_R(R(Y), R(Y)), \vartheta_i(y_j) = \delta_{ij} \text{ for } j \leq i \}_{i \geq 1}.
\]
Since \( \text{Hom}_R(R(Y), k) \) is the graded \( k \)-dual of \( R(Y) \), the “dual elements” to the normal monomials of the acyclic closure form a \( k \)-basis.

We will use \( H \) and \( I \) to denote indexing sequences of normal monomials (in both \( U\pi(R) \) and \( R(Y) \)). Denoting \( e\vartheta_i \) by \( \theta_i \), let \( \theta^I \in U\pi(R) \) be a normal monomial. Note that since \( \iota' \) is the universal extension of the inclusion \( \iota \), \( \iota' \) sends a normal monomial \( \theta^I \) to itself. By Lemma 6.7, \( \theta^I(y^H) = 0 \) if \( H < I \) and \( \theta^I(y^H) = 1 \) if
$H = I$. Therefore the coordinate vectors of the normal monomials on the elements of $\Theta$ with respect to the dual basis of the normal monomials of the acyclic closure are linearly independent. We had previously noted that normal monomials span, hence they are a $k$-basis of $U \pi(R)$.

To prove that $\iota'$ is an isomorphism we first notice that it is injective since the images of the normal monomials on $\Theta$ are $k$-linearly independent. By Theorem 7.13(1) and since the normal monomials on $\Theta$ are a $k$-basis of $U \pi(R)$, we deduce that in each degree the algebras $U \pi(R)$ and $\text{Ext}_R(k, k)$ have the same $k$-dimension, so that $\iota'$ is an isomorphism. □

We recall the definition of graded color Hopf algebra. In our applications the Hopf algebras will come equipped with an internal grading which, for the sake of readability, will be dropped in the definition of color Hopf algebra.

**Definition 8.4.** Let $G$ be an abelian group with a bicharacter $\chi : G \times G \to k^*$. Let $H = \bigoplus_{i,\sigma} H^{(i,\sigma)}$ be a $(\mathbb{Z} \times G)$-graded connected $k$-algebra with product $m$ and unit $u$. We denote $m(a \otimes b)$ by $ab$. If $H$ comes equipped with an internal grading, then all the maps that follow in this definition are assumed to be compatible with respect to this grading and all the elements that follow are assumed to be homogeneous. If $a \in H^{(i,\sigma)}$, then we denote by $|a|$ its $\mathbb{Z}$-degree, i.e. $|a| = i$. If $H$ is a $(\mathbb{Z} \times G)$-graded coalgebra with coproduct $\Delta$ and counit $\epsilon$, then we say that $H$ is a graded color coalgebra if

$$\epsilon(b) = \chi(a, b)\epsilon(b), \text{ and } \epsilon(a) = \chi(a, b)\epsilon(a), \quad \text{for all } a \in H^{(i,\sigma)}, b \in H^{(j,\tau)}.$$  

We let $H \otimes^X H$ be the $(\mathbb{Z} \times G)$-graded algebra which is $H \otimes H$ as a vector space, with product given by the following, for $a \in H^{(i,\sigma)}$, $b \in H^{(j,\tau)}$, $c \in H^{(k,\rho)}$, $d \in H^{(l,\gamma)}$:

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}\chi(b, c)(ac) \otimes (bd).$$

The algebra $H$ is a graded color bialgebra if $\Delta : H \to H \otimes^X H$ and $\epsilon$ are maps of $(\mathbb{Z} \times G)$-graded algebras.

A graded color Hopf algebra is a graded color bialgebra with an antipode map $S$, i.e. with a map $S : H \to H$ such that

$$m(S \otimes \text{id}_H)\Delta = u\epsilon = m(\text{id}_H \otimes S)\Delta.$$  

**Remark 8.5.** A color Hopf algebra is just a special case of the more general notion of braided Hopf algebra; see [11].

**Remark 8.6.** Let $\pi(R)$ be the graded color Lie algebra of Definition 7.12. Then $U \pi(R)$ is a graded color Hopf algebra with the following structure:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad x \in \pi(R),$$  

$$\epsilon(x) = 0, \quad x \in \pi(R),$$  

$$S(x) = -x, \quad x \in \pi(R).$$

Where $\Delta$ and $\epsilon$ are extended to all of $U \pi(R)$ multiplicatively and $S$ is extended to all of $U \pi(R)$ color anti-multiplicatively, i.e.

$$S(ab) = (-1)^{|a||b|}\chi(a, b)S(b)S(a), \quad a, b \in U \pi(R) \text{ trihomogeneous.}$$

A remarkable consequence of Theorem 8.3 and the previous remark is

**Corollary 8.7.** The algebra $\text{Ext}_R(k, k)$ is a graded color Hopf algebra.
9. Lie operations on \( \pi_1(R) \)

In this section, we carry out the computations necessary to compute the bracket on \( \pi_1(R) \) in homological degree one and obtain results analogous to those of Sjödin [18]. The theorem statement will come at the end of the section, after all the necessary notation has been introduced.

Recall that \( R = Q/I \) with \( I = (f_1, \ldots, f_c) \) a homogeneous ideal generated by normal elements. We also assume that \( I \subseteq R_{>2} \), and therefore for each \( j = 1, \ldots, c \), there exist homogeneous and \( G \)-homogeneous elements \( a_{h,i,j} \in Q \) such that

\[
    f_j = \sum_{1 \leq h \leq i \leq n} a_{h,i,j} x_h x_i.
\]

If \( f \in Q \) then we denote the image in \( R \) by \( \tilde{f} \). Let \( K^R(\mathfrak{x}) \) be the Koszul complex on \( \bar{x}_1, \ldots, \bar{x}_n \) as in Definition 2.12, i.e.,

\[
    K^R(\mathfrak{x}) = R(y_1, \ldots, y_n) \mid \partial(y_i) = \bar{x}_i
\]

Let \( T_2 \) be the complex that one obtains from \( K^R(\mathfrak{x}) \) by killing a minimal generating set of \( \text{H}_1(K^R(\mathfrak{x})) \), i.e.,

\[
    T_2 = K^R(\mathfrak{x}) \left\{ y_{n+1}, \ldots, y_{n+c} \mid \partial(y_{n+j}) = \sum_{1 \leq h \leq i \leq n} \tilde{a}_{h,i,j} x_h y_i \right\}.
\]

Adjoining the variables \( y_1, \ldots, y_{n+c} \) to \( R \) are the first two steps in constructing the acyclic closure of \( k \) over \( R \), which we denote by \( R(Y) \). We continue our convention of numbering the variables of an acyclic closure in a manner which respects the homological degree.

Let \( \vartheta_l \in \text{Der}_R(K^R(\mathfrak{x}), K^R(\mathfrak{x})) \) be such that \( \vartheta_l(y_j) = \delta_{l,j} \) for \( j = 1, \ldots, n \). To extend \( \vartheta_l \) to a derivation of \( T_2 \), notice that:

\[
    \vartheta_l \partial(y_{n+j}) = \vartheta_l \left( \sum_{h \leq i} \tilde{a}_{h,i,j} x_h y_i \right) = \sum_{h=1}^l \tilde{a}_{h,i,j} \partial x_h = \partial \left( \sum_{h=1}^l \tilde{a}_{h,i,j} y_h \right).
\]

Therefore by setting \( \vartheta_l(y_{n+j}) = -\sum_{h=1}^l \tilde{a}_{h,i,j} y_h \) for \( l = 1, \ldots, n \) and \( j = 1, \ldots, c \) (and extending so that the color Leibniz rule holds) we obtain an extension to a derivation of \( T_2 \) that commutes with the differential. We notice that \( \vartheta_l(y_{n+j}) = \vartheta_l(y_l) \) and therefore, if \( \vartheta_l(y_l) = \sigma_l \) and \( \vartheta_l(y_i) = \sigma_i \) we have that \( \vartheta_l(y_{n+j}) = \chi(\sigma_l, \sigma_i) = \chi(\sigma_l^{-1}, \sigma_i^{-1}) \), i.e. \( \chi(\vartheta_l, \vartheta_i) = \chi(y_l, y_i) \). We compute \( [\vartheta_l, \vartheta_i](y_{n+j}) \) for \( l < i \) and \( j = 1, \ldots, c \):

\[
    [\vartheta_l, \vartheta_i](y_{n+c}) = \left( \vartheta_l \vartheta_i + \chi(\vartheta_l, \vartheta_i) \vartheta_i \vartheta_l \right)(y_{n+c})
    = \vartheta_l \left( -\sum_{h=0}^i \tilde{a}_{h,i,j} y_h \right) + \chi(\vartheta_l, \vartheta_i) \vartheta_i \left( -\sum_{h=0}^l \tilde{a}_{h,i,j} y_h \right) = -\tilde{a}_{i,i,j}.
\]

For the square we have

\[
    \vartheta_i \vartheta_i(y_{n+c}) = \vartheta_i \left( -\sum_{h=0}^i \tilde{a}_{h,i,j} y_h \right) = -\tilde{a}_{i,i,j}.
\]

We collect the previous results in the following theorem:
Theorem 9.1. Let $Q = k_q[x_1, \ldots, x_n]$ be a skew polynomial ring, $R = Q/I$ with $I = (f_1, \ldots, f_c) \subseteq Q$ an ideal with each $f_i$ normal, homogeneous and of internal degree at least two, and let $\epsilon$ denote the augmentation from $R$ to $k$. For each $j$, write $f_j = \sum_{1 \leq h \leq l} a_{h,i,j} x_h x_i$ for normal, homogeneous $a_{h,i,j} \in Q$. Let $R(Y)$ be the acyclic closure of $k$ over $R$, and let $\theta_i = e \partial_i \in \text{Der}_R(R(Y), k)$, where $\partial_i$ is the derivation corresponding to the variable $y_i$. Then for all $1 \leq l < i \leq n$, one has equalities:

\begin{equation}
|\theta_i, \theta_l| = -\sum_{j=1}^c \epsilon(\bar{a}_{i,l,j})\theta_{n+j}, \quad \text{for } l < i \quad \text{and} \quad \theta_i^{[2]} = -\sum_{j=1}^c \epsilon(\bar{a}_{i,i,j})\theta_{n+j}.
\end{equation}

10. Skew Complete Intersections

Definition 10.1. We say that the ring $Q/I$ is a skew complete intersection if $Q$ is a skew polynomial ring and $I$ is a two-sided ideal generated by a regular sequence of homogeneous normal elements.

Remark 10.2. The definition of quantum complete intersection appearing in [3] requires the ideal $I$ to be generated by powers of the variables of $Q$. Definition [10.1] generalizes this definition.

Definition 10.3. Let $q$ be a multiplicatively antisymmetric matrix. A skew exterior algebra is an algebra of the form

\[ \bigwedge_q k^n = k_q[x_1, \ldots, x_n] / (x_1^{q_1}, \ldots, x_n^{q_n}). \]

We consider it a DG algebra with zero differential and graded cohomologically with $|x_i| = 1$ for all $i$'s.

Theorem 10.4. Let $R = Q/I$ be a skew complete intersection with $I$ generated by $\{f_1, \ldots, f_c\}$ with each $f_i$ normal, homogeneous of internal degree at least two. Let $K_R^R(\mathbf{x}) = R(y_1, \ldots, y_n | \partial(y_i) = \bar{x}_i)$. If $f_j = \sum_{i=1} a_{i,j} x_i$, then $H_1(K_R(\mathbf{x}))$ is generated by the cycles $\sum_{i=1}^n \bar{a}_{i,j} y_i$ for $j = 1, \ldots, c$. Moreover an acyclic closure of $k$ over $R$ is

\[ R(Y) = R\left\langle y_1, \ldots, y_{n+c} \left| \begin{array}{c} \partial(y_i) = \bar{x}_i \\ \partial(y_{n+j}) = \sum_{i=1}^n \bar{a}_{i,j} y_i \end{array} \right. \right\rangle. \]

Proof. We denote the sequence $f_1, \ldots, f_c$ by $f$ and the sequence $x_1, \ldots, x_n$ by $\mathbf{x}$. Let $\alpha$ be the quasi-isomorphism $\alpha : K^R(f) \to R$ and $\beta$ the quasi-isomorphism $\beta : K^R(\mathbf{x}) \to k$. By Proposition 10.12 these maps induce quasi-isomorphisms

\[ k \otimes K^Q(f) \overset{\beta \otimes K^Q(f)}{\leadsto} K^Q(\mathbf{x}) \otimes Q K^Q(f) \overset{K^Q(\mathbf{x}) \otimes \alpha}{\rightarrow} K^Q(\mathbf{x}) \otimes Q R. \]

We notice that $K^Q(\mathbf{x}) \otimes Q R \cong K^R(\mathbf{x})$, the skew Koszul complex of $\bar{x}_1, \ldots, \bar{x}_n$ over $R$, with $k \otimes Q K^Q(f)$ is a skew exterior algebra, which we denote by $\Lambda$. We fix the following notation:

\[ K^Q(f) = Q(e_1, \ldots, e_c | \partial(e_i) = f_i) \quad \text{and} \quad K^Q(\mathbf{x}) = Q(y_1, \ldots, y_n | \partial(y_i) = x_i). \]

The element $\sum_{i=1} a_{i,j} y_i \otimes 1 \otimes e_j$ is mapped by $K^Q(\mathbf{x}) \otimes \alpha$ to $\sum_{i,j} a_{i,j} y_i \otimes 1$ which corresponds to the element $\sum_{i=1}^n \bar{a}_{i,j} y_i$ of the Koszul complex of $\bar{x}_1, \ldots, \bar{x}_n$ over $R$. That same element is mapped by $\beta \otimes K^Q(f)$ to $-1 \otimes e_j$ which corresponds to the element $-e_j$ thought as one of the variables generating $\Lambda$. This shows that the
homology of the skew Koszul complex of $\bar{x}_1, \ldots, \bar{x}_n$ is generated in cohomological degree 1 by the cycles $\sum_{i=1}^n \bar{a}_{i,j}y_i$ for $j = 1, \ldots, c$, proving the first part of the theorem. These cycles are also regular because they correspond to the variables (with a negative sign) of $\Lambda$. By Theorem 10.5 once these cycles are killed we obtain a resolution of $k$, proving the last assertion of the theorem. \hfill $\Box$

As a consequence of the proof, we obtain the following:

**Corollary 10.5.** If $R = Q/I$ is a skew complete intersection then its Koszul homology algebra is isomorphic to a skew exterior algebra.

**Remark 10.6.** Let $\theta_1, \ldots, \theta_n$ be a $k$-basis of $\pi^1(R)$ and $\theta_{n+1}, \ldots, \theta_{n+c}$ be a $k$-basis for $\pi^2(R)$. Then by Theorem 10.7 and Theorem 10.4 these elements form a $k$-basis for the color Lie algebra $\pi(R)$.

With the notation from Section 10 we also have the following description of the Ext algebra of a skew complete intersection ring.

**Theorem 10.7.** If $R$ is a skew complete intersection then, as a graded color Hopf algebra

$$\text{Ext}_R(k, k) \cong \frac{\bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_{n+c})}{\left[\theta_1, \theta_l\right] + \sum_{j=1}^c \epsilon(\bar{a}_{i,j})\theta_{n+j}, \quad \text{for } l \leq n \right) \bigwedge \left[\theta_1, \theta_l\right], \quad \text{for } l \leq n \bigwedge \left[\bar{a}_{i,j}\right], \quad \text{for } l > n \right)}$$

where $[\theta_1, \theta_l] = \theta_l\theta_1 - (-1)^{l+1}\chi(\theta_1, \theta_l)\theta_l\theta_1$, with $[\theta_i] = 1$ if $i \leq n$ and 2 otherwise, and the Hopf structure on the right is obtained by identifying it with $U\pi(R)$.

**Proof.** It follows from Theorem 8.3 formula (9.1) and Remark 10.6. \hfill $\Box$

Using Remark 8.1 one sees that Theorem 10.7 generalizes [3, Theorem 5.3].

If $P^R_k(t)$ denotes the (ungraded) Poincaré series of $k$ over $R$, i.e. the Hilbert series of $\text{Ext}_R(k, k)$, then as a corollary of Theorem 10.4 and [1, Theorem 7.1.3] we deduce

**Corollary 10.8.** If $R = Q/I$ is a skew complete intersection, with $Q$ skew polynomial ring in $n$ variables $x_1, \ldots, x_n$ and $I = (f_1, \ldots, f_c) \subseteq (x_1, \ldots, x_n)^2$, then

$$P^R_k(t) = \frac{(1 + t)^n}{(1 - t^2)^c}.$$

The invariant defined below captures the growth of the minimal free resolution of $k$ over a connected graded $k$-algebra $\Omega$ and it is closely related to the Gelfand-Kirillov dimension of $\text{Ext}_\Omega(k, k)$.

**Definition 10.9.** Let $\Omega$ be a connected graded $k$-algebra. The complexity of $k$ over $\Omega$ is

$$\text{cx}_\Omega k = \inf\{d \in \mathbb{N} | \exists f(t) \in \mathbb{Z}[t], \deg f = d-1, \dim_k \text{Ext}_\Omega^n(k, k) \leq f(n) \text{ for } n \geq 1\}.$$

As a consequence of Corollary 10.8 one has

**Corollary 10.10.** If $R = Q/I$ is a skew complete intersection, with $Q$ a skew polynomial ring in $n$ variables $x_1, \ldots, x_n$ and $I = (f_1, \ldots, f_c) \subseteq (x_1, \ldots, x_n)^2$, then

$$\text{cx}_R k = c.$$
Corollary 10.11. If $R$ is a skew complete intersection then $\text{Ext}_R(k,k)$ is a noetherian algebra.

Proof. A presentation of $\text{Ext}_R(k,k)$ is given by Theorem 10.7. It is clear from that presentation that this algebra is finitely generated over the subring generated by $\theta_{n+1}, \ldots, \theta_{n+c}$, which is a skew polynomial ring and hence noetherian. The result now follows. □

Definition 10.12. [5] Let $A$ be a $k$-algebra which is finitely generated in degree 1. One says that $A$ is a $K_2$-algebra if $\text{Ext}_A(k,k)$ is generated as an algebra by $\text{Ext}_A^1(k,k)$ and $\text{Ext}_A^2(k,k)$.

The notion of a $K_2$-algebra is a generalization of the notion of Koszul algebra which has been studied in the literature. As a consequence of Theorem 10.7 we deduce the following result generalizing [5, Corollary 9.2].

Corollary 10.13. If $R$ is a skew complete intersection generated in degree one then $R$ is a $K_2$-algebra.

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