RENORMALIZED VOLUME OF MINIMALLY BOUNDED REGIONS IN ASYMPTOTICALLY HYPERBOLIC EINSTEIN SPACES

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Abstract. We define a renormalized volume for a region in an asymptotically hyperbolic Einstein manifold that is bounded by a Graham-Witten minimal surface and the conformal infinity. We prove a Gauss-Bonnet theorem for the renormalized volume, and compute its derivative under variations of the minimal hypersurface.

1. Introduction

The renormalized volume of an even-dimensional asymptotically hyperbolic Einstein (AHE) manifold \((\mathbb{X}^{n+1}, g^+_{+})\) is among its most important global invariants. Introduced in [HS98] (see also [Gra00]), it is defined by taking the order-zero term in the expansion in \(\varepsilon\) of the quantity \(\text{vol}_{g^+}(\{r > \varepsilon\})\), where \(r\) is a so-called geodesic defining function for the boundary at infinity, \(M^n\). There are many such defining functions, and the essential property of the renormalized volume \(V^+\) is that it does not depend on which one is chosen. (This is generally not true if \(X\) is odd-dimensional.)

One of the basic theorems regarding renormalized volume in dimension four is Anderson’s Gauss-Bonnet theorem ([And01], see also [CQY08]), which states that

\[
4\pi^2 \chi(X^4) = 3V^+ + \frac{1}{8} \int_X |W_{g^+}|^2_{g^+} dv_{g^+}.
\]

Here \(W_{g^+}\) is the Weyl tensor of \(g^+\); since \(|W_{g^+}|^2_{g^+}\) is a pointwise conformal invariant of weight \(-4\), the integral is guaranteed to converge notwithstanding the infinite volume of \((X, g^+)\). Anderson used (1) to compute the variation of \(V^+\) with respect to changes in \(g^+\).

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In this paper, we establish analogous results for half of an AHE manifold that has been partitioned into two by a minimal surface. Specifically, suppose \((X^4, g^+)\) is an AHE manifold with conformal infinity \((M^3, [\bar{h}])\), and suppose further that \(Y^3 \subset X\) is a minimal hypersurface that intersects \(M\) in a closed manifold \(\Sigma^2 = M \cap Y\); we further assume that \(Y\) divides \(X\) into two parts, \(X^+\) and \(X^-\), whose intersection is precisely \(Y\) (the assignment of + is arbitrary). Such a setting has been much studied in the literature on AHE manifolds, beginning with [GW99], which defined the renormalized area of \(Y\) in analogy to the renormalized volume of \(X\); it has also been and remains a setting of much interest in the physics literature, particularly in the context of the AdS/CFT correspondence.

We will be concerned, not with the renormalized area of \(Y\), but with the renormalized volume \(V^+\) of \(X^+\), which we may define as the constant term in the expansion \(\text{vol}_{g^+}(\{x \in X^+ : r(x) > \varepsilon\})\), with \(r\) a geodesic defining function. It is not immediately obvious that this quantity is independent of the choice of \(r\): the proof in the global case depends strongly on the product decomposition \([0, \delta) \times M\) of a collar neighborhood of \(M\) in \(X\), but generically there is no such decomposition of a collar neighborhood of \(M^+ = M \cap X^+\) in \(X^+\). One could prove using rather more elaborate versions of the arguments of [Gra00] that \(V^+\) is invariant in this context, but our interest is in a Gauss-Bonnet formula, and so we approach the result by a somewhat different path, as described below.

We note that renormalized volume of regions in AH spaces divided in two by hypersurfaces was considered in [GW19] using quite different techniques. The authors showed that a volume could be defined in quite general circumstances – in particular, not assuming the Einstein or minimality conditions – but did not show that it is well-defined independent of all choices in the four-dimensional Einstein case.

Let \(N \subset \tilde{X}\) be any hypersurface, and let \(h = g^+|_{TN}\) be the induced metric on \(N\). Define an extrinsic curvature quantity \(\mathcal{C}_N\) on \(N\) by the formula

\[
\mathcal{C}_N = \hat{L}_N^{\alpha\beta} R^g_{\alpha\beta} - \hat{L}_N^{\alpha\beta} R^h_{\alpha\beta} + \frac{1}{3} H_N |\hat{L}_N|_h^2 - \frac{1}{3} \text{tr}_h \hat{L}_N^3.
\]

Here \(L_N\) is the second fundamental form of \(N\) and \(\hat{L}_N\) its tracefree part, while \(H_N = h^{\alpha\beta} L_{\alpha\beta}\) is its mean curvature. The curvature terms appearing are the Ricci tensors of the respective metrics, and \(\alpha, \beta\) are indices on \(TN\). It is easy to show (and will be shown within) that \(\mathcal{C}_N\) is a pointwise conformal invariant of weight \(-3\); indeed, in the notation of [CQ97], \(\mathcal{C}_N = -\frac{1}{2} \mathcal{L}_4 - \frac{1}{3} \mathcal{L}_5\).
Theorem 1.1. Let \( (X^4, g_+) \) be an asymptotically hyperbolic space satisfying the Einstein condition \( \text{Ric}(g_+) = -3g_+ \), with conformal infinity \((M^3, [\tilde{h}])\). Let \( Y^3 \) be a complete minimal hypersurface dividing \( X \) into two pieces \( X^+ \) and \( X^- \) such that \( X^+ \cap X^- = Y \) and such that \( Y \cap M = \Sigma^2 \neq \emptyset \). Let \( r \) be a fixed geodesic defining function for \( M \), and let \( V^+_+ \) be the constant term in the expansion

\[
\text{vol}_{g_+}\left( \{ x \in X^+ : r(x) > \varepsilon \} \right) = c_0 \varepsilon^{-3} + c_2 \varepsilon^{-1} + V^+_+ + o(1).
\]

Let \( \tilde{h} = g_+|_{TY} \). Then

\[
(2) \quad \pi^2 (4\chi(X^+) - \chi(\Sigma^2)) = 3V^+_+ + \frac{1}{8} \int_{X^+} |W_{g_+}|^2_{g_+} \text{dv}_{g_+} + \int_Y \mathcal{C}_Y \text{dv}_{\tilde{h}}.
\]

One then immediately obtains

Corollary 1.2. The renormalized volume \( V^+_+ \) is independent of the choice of geodesic defining function \( r \), and it satisfies (2).

A natural question about the newly defined renormalized volume is how it changes if \( Y \) is varied through minimal surfaces in \( X \). The second main result of the paper is as follows.

Theorem 1.3. Let \( X, M, Y, \Sigma, X^+, g_+, \tilde{h}, \) and \( V^+_+ \) be as in Theorem 1.1. Suppose that \( \mathcal{F} : (-\varepsilon, \varepsilon)_t \times Y \rightarrow X \) is a \( C^3 \) variation of \( Y \) through minimal surfaces in \( X \), so that \( \mathcal{F}(t, \Sigma) \subset M \) for all \( t \). Let

\[
\tilde{\mathcal{F}} = \mathcal{F}|_{(-\varepsilon, \varepsilon)_t \times \Sigma}.
\]

Define \( \tilde{f} \in C^\infty(\Sigma) \) by

\[
\tilde{f} = \left. \frac{d}{dt} \right|_{t=0} \tilde{\mathcal{F}}(\tilde{v}_M),
\]

where \( \tilde{v}_M \) is the inward-pointing normal vector to \( \Sigma \) in \( M^+ \) with respect to \( \tilde{h} \). Define \( f \in C^\infty(Y) \) by

\[
f = \langle \frac{d}{dt} \big|_{t=0} \mathcal{F}, \mu_Y \rangle_{g_+},
\]

where \( \mu_Y \) is the \( (X^+, g_+) \)-inward unit normal vector along \( Y \). Let \( r \) be a geodesic defining function near \( M \). Then

\[
\left. \frac{d}{dt} \right|_{t=0} V^+_+ = \frac{1}{2} \int_{\Sigma} \tilde{f} g^{(3)}(\tilde{v}_M, \tilde{v}_M) \text{dv}_{\tilde{k}} - \frac{1}{3} f.p. \int_Y f |\tilde{L}_Y|_{\tilde{h}}^2 \text{dv}_{\tilde{h}},
\]

where \( \tilde{k} = \tilde{h}|_{\Sigma}, \tilde{h} = g_+|_{TY} \), \( g^{(3)} \) is the nonlocal term in the expansion in \( r \) of \( g_+ \), and \( f.p. \int_Y f |\tilde{L}_Y|_{\tilde{h}}^2 \text{dv}_{\tilde{h}} \) denotes the zeroth-order part, in \( \varepsilon \), of \( \int_{Y \cap \{ r > \varepsilon \}} f |\tilde{L}_Y|_{\tilde{h}}^2 \text{dv}_{\tilde{h}} \).

For discussion of the nonlocal term \( g^{(3)} \), see (5).

The above theorem is stated for variations of \( Y \) through minimal surfaces, whose existence in general we do not assert. However, one can broaden the definition of \( V^+_+ \) to any dividing hypersurface by using (2). In that case, Theorem 1.3 remains valid for any variation of \( Y \).
that preserves minimality to first order; see section \[4\] where we also explain why $C^3$-regularity of such a variation is in general optimal.

In considering the existence problem for the variation of $Y$, the required boundary data would be the induced variation of $\Sigma$, so another natural question is whether the derivative $\dot{V}^+_{\nu}$ only depends on the induced normal variation $\tilde{f}$. For example, suppose there are two variations of $Y$ through minimal surfaces that induce the same variation of $\Sigma$; do the derivatives of $\dot{V}^+_{\nu}$ with respect to these variations agree? The answer is yes, at least if $|\dot{L}_Y|_{\hat{h}}^2 \leq 3$ everywhere; see Lemma \[4.1\]

These theorems may be interpreted physically within the AdS/CFT correspondence of high-energy and condensed matter physics. To do so, we assume that $(M^4, [\bar{h}])$ is a spacelike slice within a static four-dimensional conformal field theory $\Omega$; and that $(X^4, g_\pm)$ is an Einstein spacelike slice within a static asymptotically anti-de-Sitter Einstein five-dimensional spacetime $Z$ with conformal infinity $\Omega$. The surface $\Sigma$ is then known as an entangling surface between $M^+$ and $M^-$, and $Y$ is the so-called Ryu-Takayanagi surface extending $\Sigma$. According to the “volume = complexity” conjecture ([Sus16, BAC16, CFN17, ABN18, JKK120]), then, $\dot{V}^+_{\nu}$ encodes the algorithmic complexity of the quantum state of $M^\pm$. The above theorems can then be interpreted as giving formulae for this complexity and for its derivative as the entangling surface $\Sigma$ is varied continuously, so long as $Y$ also varies continuously. (As demonstrated in [BAC16], the latter will not always be the case.)

The assumption that $X$ and its five-dimensional ambient Lorentzian manifold $Z$ are both Einstein, of course, is rather restrictive. In general physical situations, one might expect that the Ricci tensor of $X$ includes some extrinsic terms. But even if so, these would have well-defined asymptotics due to the asymptotically AdS condition on $Z$, and it would be straightforward, if tedious, to carry out our calculation the same way in that context.

In section 2, we introduce our setting and notation. In section 3, we prove Theorem 1.1 and in section 4, we prove Theorem 1.3.

2. Setting and Notation

Recall that an asymptotically hyperbolic (AH) manifold is a compact manifold $X^{n+1}$ with boundary $M^n$, equipped on the interior $\tilde{X}$ with a metric $g_\pm$ such that, for any defining function $\varphi$ for $M$, the metric $\tilde{g} = \varphi^2 g_\pm$ extends to a Riemannian metric on $X = \tilde{X}$; and such that, in addition, $|d\varphi|_{\tilde{g}} = 1$ along $M$. The optimal regularity of $\tilde{g}$ is in general a delicate question, but in the context of this paper (i.e., $X$ is four dimensional) by a result of Chruściel-Delay-Lee-Skinner [CDLS05] we
may assume that there is a compactification such that $\bar{g}$ is smooth up to the boundary. The canonical example of an AH manifold is hyperbolic space itself, where $X$ is the unit ball $\mathbb{B}^{n+1}$, and the metric is $g_H = \frac{4|dx|^2}{(1-|x|^2)^2}$. Given an AH metric, the metric $\bar{h} = \bar{g}|_{TM}$ is a metric on $M$, but is not well defined since the choice of $\varphi$ is arbitrary. However, the conformal class $[\bar{h}]$ is well defined, and is called the conformal infinity.

A defining function $r$ for $M$ is called geodesic if $|dr|_{r^2g_+} = 1$ on a neighborhood of $M$. Such a function induces a diffeomorphism

$$\psi : [0, \varepsilon)_r \times M \hookrightarrow X$$

onto a neighborhood of $M$ in $X$ such that

$$\psi^*g_+ = \frac{dr^2 + \bar{h}_r}{r^2},$$

where $\bar{h}_r$ is a one-parameter family of metrics on $M$. A lemma of Graham-Lee ([GL91]) states that geodesic defining functions are in one-to-one correspondence with the representatives $\bar{h}$ of $[\bar{h}]$, according to the correspondence $\bar{h}_0 = \bar{h}$. The form $\bar{h}$ is called the geodesic normal form corresponding to $\bar{h} = \bar{h}_0$. We may assume that any geodesic compactification of $X$ is smooth ([CDLS05]).

An AH metric is called Einstein (or AHE) if it satisfies as well the condition $\text{Ric}(g) + ng = 0$. We will be concerned exclusively with four-dimensional AHE spaces, i.e. the case $n = 3$. In this case, it is known ([FG85, FG12, Gra00]) that in geodesic normal form, $\bar{h}_r$ has the expansion

$$\bar{h}_r = \bar{h} - r^2P^\bar{h} + r^3g^{(3)} + O(r^4),$$

where $\text{tr}_{\bar{h}}g^{(3)} = 0$ and where $P^\bar{h}$ is the Schouten tensor of $\bar{h}$, given by

$$P^\bar{h}_{\mu\nu} = R^\bar{h}_{\mu\nu} - \frac{1}{4}R_{\bar{h}}\bar{h}_{\mu\nu}.$$ 

Apart from the trace condition, the tensor $g^{(3)}$ is not locally determined by the geometry of $(M^3, \bar{h})$.

The renormalized volume of $(X, g_+)$ is defined as follows ([HS98, Gra00]). Choose a metric $\bar{h} \in [\bar{h}]$, and let $r$ be the corresponding geodesic defining function. Then the set $\{r > \varepsilon\}$ has volume

$$\text{vol}_{\bar{h}}(\{r > \varepsilon\}) = c_0\varepsilon^{-3} + c_2\varepsilon^{-1} + V_+ + o(1).$$

The renormalized volume is $V_+$, and it is independent of the choice of $\bar{h}$ (that is, of $r$).

In our setting of interest, there exists as well an orientable minimal surface $Y^3 \subset X$, intersecting $M$ transversely in a closed two-manifold $\Sigma^2 = Y \cap M$, and dividing $X$ into two connected pieces $X^+$ and $X^-$ such
that $Y = X^+ \cap X^-$. We write $M^+ = X^+ \cap M$ and $M^- = X^- \cap M$, so that $\Sigma = M^+ \cap M^-$. The assignment of the signs $+$ and $-$ is arbitrary, and corresponds to a choice of unit normal vector field on $Y$.

We now introduce the notations we will use. We let $(X^3, M^3, g_+)$ be an AHE space, and $Y^3 \subset X$ a minimal surface as above. We will let $[\bar{h}]$ be the conformal infinity, and corresponding to the metric $\bar{h}$ will be the geodesic defining function $r$. The compactified metric is $\bar{g} = r^2 g_+$. Furthermore, $X^+, M^+$, and $\Sigma^2$ will be as above. For $\varepsilon > 0$, we let $X_\varepsilon = \{ r > \varepsilon \}$, with $X_\varepsilon^+ = X^+ \cap X_\varepsilon$. We set $Y_\varepsilon = Y \cap X_\varepsilon$ and $M_\varepsilon = \{ r = \varepsilon \}$. Similarly we set $M_\varepsilon^+ = X^+ \cap M_\varepsilon$. Finally, $\Sigma_\varepsilon = Y \cap M_\varepsilon^+$.

Next, there are a number of metrics to name. We let $h_\varepsilon = g_+ |_{TM_\varepsilon}$, while $\bar{h}_\varepsilon = \varepsilon^2 h_\varepsilon = \bar{g} |_{TM_\varepsilon}$. We let $\bar{h} = g_+ |_{TY}$, while $\bar{h} = r^2 \bar{h} = \bar{g} |_{TY}$. We let $\bar{k} = \bar{g} |_{T X_\varepsilon}$, while $k_\varepsilon = g_+ |_{T \Sigma_\varepsilon}$ and $\bar{k}_\varepsilon = \bar{r}^2 k_\varepsilon = \varepsilon^2 k_\varepsilon$. The decorations of $\varepsilon$ will sometimes change position as needed; for example, we will write $h_{\mu \nu}^\varepsilon$, but $h_{\mu \nu}$.

Now, near $\Sigma \subset M$, we can uniquely solve the eikonal equation and find $w \in C^\infty(M)$ such that $|dw|^2_{\bar{h}} = 1$ near $\Sigma$, $w|_{\Sigma} = 0$, and $w \geq 0$ on $M^+$. The metric $\bar{h}$ then takes the form $\bar{h} = dw^2 + \bar{k}_w$, with $\bar{k}_w$ a one-parameter family of metrics on $\Sigma$. Near any point $p \in \Sigma$, we can choose coordinates $x^1, x^2$ on a neighborhood of $p$ in $\Sigma$; then by the flow of $\text{grad}_{\bar{h}} w$ on $M^+$, the system $(x^1, x^2, x^3 = w)$ extends to a coordinate system on a neighborhood of $p$ in $M$. Finally, by the flow of $\text{grad}_{\bar{g}} r$, the system $(r = x^0, x^1, x^2, x^3 = w)$ extends to a coordinate system on a neighborhood of $p$ in $X$. Now, we will regard $Y$ as given by a function

$$w = u(r, x^1, x^2),$$

where $u(0, x^1, x^2) \equiv 0$. This is the same convention as in [GW99]. In fact, we may regard a neighborhood of $\Sigma$ in this way as a product $[0, \varepsilon) \times \Sigma \times (-\varepsilon, \varepsilon) w$; when using this product identification, we will use $\zeta$ to refer to a point of $\Sigma$, so that a generic point may be written $(r, \zeta, w)$.

When using index notation locally, we will let $0 \leq i, j \leq 3$ be indices on $TX$; $1 \leq \mu, \nu \leq 3$ be indices on $TM$; and $1 \leq a, b \leq 2$ be indices on $T\Sigma$. We also let $0 \leq \alpha, \beta \leq 2$, which we will use when discussing $TY$.

Turning to extrinsic geometry, we let $\bar{\mu}_{M_\varepsilon}, \bar{\nu}_Y$ be the $X^+$-inward unit $\bar{g}$-normal to the given hypersurface; the unbarred versions will refer to the unit normal with respect to $g_+$. We let $\bar{\nu}_{M_\varepsilon}$ be the $\bar{g}$-unit normal to $\Sigma_\varepsilon$ that is directed into $M^+$, and $\bar{\nu}_Y$, similarly, the $Y_\varepsilon$-inward $\bar{g}$-unit normal to $\Sigma_\varepsilon$. We let $T_{M_\varepsilon}, T_Y$ be the second fundamental forms of the indicated hypersurfaces with respect to the inward unit normals $\bar{\mu}_{M_\varepsilon}$
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and \( \bar{\mu}_Y \), and computed with respect to \( \bar{g} \). Thus, for example,

\[
\bar{L}_Y(A, B) = -\langle \nabla^\bar{g}_A \bar{\mu}_Y, B \rangle.
\]

The tracefree parts are denoted \( \bar{T}_{M_\varepsilon} \), etc. In all of these, we will sometimes write the hypersurface in the upper position, should it be convenient to do so to place covariant indices; similarly, an unbarred \( L \) will refer to the second fundamental form with respect to \( g_+ \) instead of \( \bar{g} \).

We let \( \bar{H}_{M_\varepsilon} = h_\varepsilon^{\mu\nu}\bar{T}_{M_\varepsilon}^{\mu\nu} \) be the mean curvature of \( M_\varepsilon \) with respect to \( \bar{g} \) (or, if we omit the \( \varepsilon \), that of \( M \)); similarly for \( \bar{H}_Y \), while \( H_{M_\varepsilon} \) and \( H_Y \) are the same quantities with respect to \( g_+ \) (recall we assume \( H_Y \equiv 0 \)).

We let \( \bar{\Pi}_{Y_\varepsilon} \) be the second fundamental form of \( \Sigma_\varepsilon \) viewed as a hypersurface of \( Y_\varepsilon \) with respect to \( \bar{\tilde{h}} \), while \( \bar{\Pi}_{M_\varepsilon} \) is the same for \( \Sigma \) viewed as a hypersurface in \( M_\varepsilon \) with respect to \( \bar{\tilde{h}}_\varepsilon \). The traces of these (i.e., the mean curvatures of \( \Sigma_\varepsilon \) viewed as a hypersurface of the respective three-manifold) we denote \( \bar{\eta}_{Y_\varepsilon}, \bar{\eta}_{M_\varepsilon} \). Again, the unbarred versions are with respect to the unbarred metrics \( \bar{\tilde{h}} \) and \( \tilde{h}_\varepsilon \). We also let \( \bar{\eta}_M \) be the mean curvature of \( (\Sigma, \bar{\tilde{h}}) \subset (\bar{\tilde{h}}, \bar{\tilde{h}}) \).

We define a smooth function \( \theta_0^\varepsilon \in C^\infty(\Sigma_\varepsilon) \) to be the angle, at each point, between \( Y \) and \( M_\varepsilon \); that is, \( \cos(\theta_0^\varepsilon) = -\langle \bar{\mu}_Y, \bar{\mu}_{M_\varepsilon} \rangle \). If the \( \varepsilon \) is omitted, then it denotes the angle between \( M \) and \( Y \) at a point of \( \Sigma \). Since \( \theta_0^\varepsilon \) is manifestly a conformal invariant, we do not distinguish between barred and unbarred versions.

If \( A \) is a vector or tensor field, we write \( A = O_g(\varphi) \), for \( \varphi \) a function, whenever \( |A|_g = O(\varphi) \).

### 3. The Gauss-Bonnet Formula

We now prove Theorem 1.1. We do so by using a form of the Gauss-Bonnet formula that has good conformal invariance properties, which allows us to compute using \( \bar{g} \) instead of \( g_+ \).

Proof of Theorem 1.1. Let \( (X, M, g_+) \) be an AHE space with conformal infinity \( [\bar{h}] \), and let \( Y \) be as in the previous section. Let \( \bar{h} \in [\bar{h}] \), and let \( r \) be the corresponding geodesic defining function. Let \( \varepsilon > 0 \). Then \( X_\varepsilon^+ \) is a four-manifold with codimension-two corner \( \Sigma_\varepsilon \), and boundary hypersurfaces \( M_\varepsilon^+ \) and \( Y_\varepsilon \) (see section 2 for all notation). The Gauss-Bonnet theorem for Riemannian manifolds with corners (in this case \( X_\varepsilon^+ \)), proven first in [AW43] (and see [Che45]), can be rewritten in the
following conformally useful way ([McK20], building on [CQ97]).

\[
4\pi^2 \chi(X_\varepsilon^+) = \int_{X_\varepsilon^+} \left( \frac{1}{8} |W_{g+}|^2 + \frac{1}{2} Q_{g+} \right) dv_{g+} + \int_{Y_\varepsilon} (\mathcal{L}_Y + T_Y) dv_{h+} \\
+ \int_{M_\varepsilon^+} (\mathcal{L}_{M_\varepsilon} + T_{M_\varepsilon}) dv_{h} + \oint_{\Sigma_\varepsilon} (U_{\Sigma_\varepsilon} + G_{\Sigma_\varepsilon}) dv_{k\varepsilon}.
\]

(9)

Here, \( W_{g+} \) is the Weyl tensor of \( g_+ \), and the norm in question is its two-tensor norm \( W_{ijkl} W_{ijkl} \). Meanwhile, \( Q_{g+} \) is the \( Q \)-curvature of \( g_+ \), defined for any metric \( g \) by

\[
Q_g = -\frac{1}{6} \Delta_g R_g + \frac{1}{6} R_g^2 - \frac{1}{2} R^{ij}_g R^g_{ij}.
\]

Here, the Laplacian is a negative operator and the curvatures are respectively the scalar and Ricci curvatures of \( g \). For any metric \( g \), the quantity \( |W_g|^2 dv_g \) is a pointwise conformal invariant of weight zero.

Under a conformal transformation \( \tilde{g} = e^{2\omega} g \), the \( Q \) curvature transforms as

\[
e^{4\omega} Q_{\tilde{g}} = Q_g + P^9_{4\omega},
\]

where \( P^9_4 \) is the Paneitz operator associated to \( g \); we will not use the Paneitz operator and so omit it here.

We give the definition of \( \mathcal{L}_N \) and \( T_N \), due to [CQ97], for an arbitrary boundary hypersurface \((N^3, h)\) embedded in a four-manifold endowed with metric \( g \). The definition is

\[
\mathcal{L}_N = \tilde{L}^{\mu\nu}_N R^g_{\mu\nu} - 2\tilde{L}^{\mu\nu}_N R^h_{\mu\nu} + \frac{2}{3} H_N |\tilde{L}_N|^2_h - \text{tr}_h \tilde{L}_N^3,
\]

(10)

where \( L_N \) and \( H_N \) are the second fundamental form and the mean curvature as before, and \( \mu, \nu \) are indices on \( T N \). Similarly, the \( T \)-curvature is defined by

\[
T_N = -\frac{1}{12} \mu(R_g) - \tilde{L}^{\mu\nu}_N R^g_{\mu\nu} + \tilde{L}^{\mu\nu}_N R^h_{\mu\nu} - \frac{1}{2} H_N |\tilde{L}_N|^2_h + \frac{2}{3} \text{tr}_h \tilde{L}_N^3
\\
+ \frac{1}{6} R_h H_N - \frac{1}{27} H_N^3 - \frac{1}{3} \Delta_h H_N,
\]

(11)

where \( \mu \) is the inward-pointing unit normal to \( N \). Under the conformal change \( \tilde{g} = e^{2\omega} g \), this transforms according to the equation

\[
e^{3\omega} \tilde{T}_N = T_N + P^9_{3\omega},
\]

(12)
where $P^g_3 : C^\infty(X) \to C^\infty(N)$ is the conformally covariant boundary operator

$$P^g_3 f = \frac{1}{2} \mu \Delta_g f - \Delta_h \mu f - H_N \Delta_h f - \tilde{L}_N^{\mu} \nabla^h \nabla^h f - \frac{1}{3} H_N^2 f$$

(13)

$$+ \left( \frac{1}{6} R_g - \frac{1}{2} R_h - \frac{1}{2} \tilde{L}_N^2 + \frac{1}{3} H_N^2 \right) \mu(f).$$

Next we turn to the corner quantities. For a corner $(\Xi, k)$ that forms the intersection between two boundary hypersurfaces $N$ and $S$ making angle $\theta_0 \in C^\infty(\Xi)$, $G$ is defined by

$$G_\Xi = \frac{1}{2} \cot(\theta_0) (|II_N|^2_k + |II_S|^2_k) - \csc(\theta_0) II^{\nu\mu}_N \tilde{II}^{\nu\mu}_S,$$

where $II$, etc., are as in section 2. The $G$ curvature is a pointwise conformal invariant of weight $-2$ (when the ambient metric on the four-manifold is changed conformally). Next, $U_\Xi$ is defined by

$$U_\Xi = (\pi - \theta_0) K_\Xi \frac{1}{4} \cot(\theta_0) (\eta_0^2 + \eta_S^2) + \frac{1}{2} \csc(\theta_0) \eta_0 \eta_S - \frac{1}{3} (\nu_N H_N + \nu_S H_S).$$

Here, $K_\Xi$ is the Gaussian curvature of $\Xi$, and the other quantities are defined analogously to those in the previous section. Under a global conformal change $\tilde{g} = e^{2\omega} g$, $U$ transforms according to the equation

$$e^{2\omega} \tilde{U}_\Xi = U_\Xi + P^g_2 \omega,$$

where $P^g_2 : C^\infty(X) \to C^\infty(\Xi)$ is the conformally covariant operator

$$P^g_2 f = (\theta_0 - \pi) \Delta_k f + \nu_N \mu_N f + \nu_S \mu_S f$$

$$+ \cot(\theta_0) (\eta_0 \nu_N f + \eta_S \nu_S f) - \csc(\theta_0) (\eta_0 \nu_N f + \eta_S \nu_S f)$$

$$+ \frac{1}{3} (H_N \nu_N f + H_S \nu_S f).$$

We now analyze formula (9) in the context of our space $(X^+_\varepsilon, g_+)$. Because $|W_{g_+}|^2_{g_+} dv_{g_+}$ is a pointwise conformal invariant of weight zero, its integral converges as $\varepsilon \to 0$ to $\int_{X^+_+} |W_{g_{\varepsilon}}|^2 dv_{g}$, which in particular is finite.

In our setting, $R^{\varepsilon}_{ij} = -3g^{\varepsilon}_{ij}$ and $R_{g_+} \equiv -12$, so $\Delta_{g_+} R_{g_+} \equiv 0$ and $Q_{g_+} \equiv 6$. The integral of $\frac{1}{2} Q_{g_+}$ therefore is simply the integral of $3$, so the second integral over $X^+_+$ becomes simply $3 \text{vol}_{g_+}(\{r > \varepsilon\} \cap X^+_+)$, which is the same quantity considered in (7), except that the latter is over all of $X$ instead of $X^+$. To compute the contribution from this integral, we consider four different regions of $X$. First, let $r_0 > 0$ be small – sufficiently small, in particular, that the geodesic normal form (4) holds for $r < 2r_0$, and that the region $\mathcal{U} = \{r < 2r_0, -2r_0 < w <
\[ 2r_0 \} \text{ has the decomposition } [0, 2r_0) \times \Sigma \times (-2r_0, 2r_0), \text{ with } |u(r, \zeta)| < \frac{1}{2}r_0 \text{ on } \mathcal{U}. \] Having chosen \( r_0 \), we will leave it fixed for all time.

The first region of interest to us is then \( A = \{ p \in X^+ : r(p) \geq r_0 \} \). (This set does not depend on \( \varepsilon \), which we assume is smaller than \( r_0 \).)

Next, we want to capture the points near the boundary \( M^+ \). The obvious set to consider is \( B_\varepsilon = (\varepsilon, r_0) \times M^+ \). The problem is that this may omit points that are contained in \( X^+ \) or include points contained in \( X^- \), because \( Y \) is given not by \( w = 0 \) but by \( w = u(r, \zeta) \), where \( u \) may be positive or negative away from \( \Sigma \). To address this, we need to add the volume of the omitted points, \( C_\varepsilon \), and subtract the volume of the over-included points \( D_\varepsilon \), viz.,

\[ X^+_\varepsilon = (A \cup B_\varepsilon \cup C_\varepsilon) \setminus D_\varepsilon. \]

To proceed, we analyze the volume form \( dv_{g_+} \). First, at all points, we have \( dv_{g_+} = r^{-4}dv_{\tilde{g}} \). Near \( M \), we can write

\[ dv_{\tilde{g}} = dv_{\tilde{h}}, dr \]

using the normal-form identification \( \text{(3)} \). Now in local coordinates \((r, x^1, x^2, x^3)\) near \( M \), we may write

\[ dv_{\tilde{h}} = \sqrt{\frac{\det(\tilde{h}_r)}{\det(h)}} dv_{\tilde{h}}. \]

As shown for example in \( \text{[Gra00]} \), we have the expansion

\[ \sqrt{\frac{\det(\tilde{h}_r)}{\det(h)}} = 1 + v^{(2)}r^2 + v^{(4)}r^4 + O(r^5), \]

where \( v^{(2)}, v^{(4)} \in C^\infty(M) \) are the so-called renormalized volume coefficients. Either by direct computation using \( \text{(5)} \) or by using equation \( (4.5) \) and the equation at the top of the same page of \( \text{[Gra17]} \) (remembering that \( M \) is totally geodesic with respect to \( \tilde{g} \) and that the singular Yamabe metric for \( \tilde{g} \) is \( g_+ \)), we may show that \( v^{(2)} = -\frac{1}{8}R_{\tilde{h}} \).

Thus,

\[ dv_{g_+} = r^{-4} \left( 1 - \frac{1}{8}r^2R_{\tilde{h}} + O(r^4) \right) dv_{\tilde{h}} dr \]

\[ = \left( r^{-4} - \frac{1}{8}r^{-2}R_{\tilde{h}} + O(1) \right) dv_{\tilde{h}} dr. \]
We next derive an expression for $dv_{g}$ (and thus $dv_{g_{+}}$) near $\Sigma$. Since $\bar{h} = dw^{2} + \bar{k}_{w}$ near $\Sigma$, we have

$$dv_{\bar{h}} = \sqrt{\frac{\det(\bar{k}_{w})}{\det(k)}} dv_{\bar{k}} dw = (1 + O(w)) dv_{\bar{k}} dw.$$  

Hence, near $\Sigma$, we have

$$dv_{g_{+}} = \left( r^{-4} - \frac{1}{8} r^{-2} R_{\bar{h}} + O(1) \right) (1 + O(w)) dv_{\bar{k}} dw dr.$$  

We then have

$$\text{vol}_{g}(X_{\varepsilon}^{+}) = \text{vol}_{g_{+}}(A) + \text{vol}_{g_{+}}(B_{\varepsilon}) + \text{vol}_{g_{+}}(C_{\varepsilon}) - \text{vol}_{g_{+}}(D_{\varepsilon})$$

$$= \text{vol}_{g_{+}}(A) + \int_{M_{+}} \int_{r_{0}}^{r_{0}} \left( r^{-4} - \frac{1}{8} r^{-2} R_{\bar{h}} + O(1) \right) dr dv_{\bar{h}}$$

$$- \int_{\Sigma_{\varepsilon}} \int_{r_{0}}^{r_{0}} u(r, \zeta) \left( r^{-4} + O(r^{-2}) \right) (1 + O(w)) dw dr dv_{\bar{k}}(\zeta),$$

where the last integral represents $\text{vol}_{g_{+}}(C_{\varepsilon}) - \text{vol}_{g_{+}}(D_{\varepsilon})$. Now, by equations (2.13) and (2.14) in [GW99],

$$u(r, \zeta) = \frac{1}{4} r^{2} \bar{\eta}_{M}(\zeta) + r^{4} \log(r) v(\zeta) + O(r^{4}),$$

where $\bar{\eta}_{M}$ is the mean curvature of $\Sigma$ viewed as a hypersurface of $(M, \bar{h})$ and $v \in C^{\infty}(\Sigma)$. Thus, we find

$$3 \text{vol}_{g_{+}}(X_{\varepsilon}^{+}) = 3 \text{vol}_{g_{+}}(A) + 3 \int_{M_{+}} \int_{r_{0}}^{r_{0}} \left( r^{-4} - \frac{1}{8} r^{-2} R_{\bar{h}} + O(1) \right) dr dv_{\bar{h}}$$

$$- 3 \int_{\Sigma_{\varepsilon}} \int_{r_{0}}^{r_{0}} \left( \frac{1}{4} r^{-2} \bar{\eta}_{M} + v \log(r) + O(1) \right) dr dv_{\bar{k}}$$

$$= \varepsilon^{-3} \text{vol}_{h}(M^{+}) - \varepsilon^{-1} \left( \frac{3}{8} \int_{M_{+}} R_{\bar{h}} dv_{\bar{k}} + \frac{3}{4} \int_{\Sigma_{\varepsilon}} \bar{\eta}_{M} dv_{\bar{k}} \right)$$

$$+ 3V_{s}^{+} + o(1).$$

Here $V_{s}^{+}$ is the collection of all the order-zero terms in $\varepsilon$ in the volume expansion, and is defined to be the renormalized volume; of course, we have not shown so far that $V_{s}^{+}$ is independent of the choice of $\bar{h} \in [\bar{h}]$ (or equivalently, of $r$).

Since (as we saw above) $Q_{g_{+}} = 6$, the above right-hand side is thus the integral $\int_{X_{\varepsilon}} \frac{1}{2} Q_{g_{+}} dv_{g_{+}}$. We next turn to the boundary integrals over $Y_{\varepsilon}$ and $M_{\varepsilon}$, beginning with $Y_{\varepsilon}$. We will analyze $\mathcal{L}_{Y}$ and $T_{Y}$ with respect to the metric $g_{+}$; of course, since $\mathcal{L}_{Y}$ is a pointwise conformal
invariant, it is automatic that the integral of \( \mathcal{L}_Y \) over \( Y_\varepsilon \) will converge as \( \varepsilon \to 0 \). Now, because \( g_+ \) is Einstein and \( Y \) is minimal in \((X, g_+)\), the first and third terms in (10) vanish in this case. Thus, we get simply

\[
\mathcal{L}_Y = -2\mathcal{L}_Y^\alpha \mathcal{L}_Y^\beta \mathcal{P}_h^\alpha \mathcal{P}_h^\beta - \text{tr}_h \mathcal{L}_Y^3.
\]

Next turning to \( T_Y \), we again compute with respect to the ambient metric \( g_+ \), i.e., with respect to the non-compactified setting. Again, due to the Einstein condition of \( g_+ \) and the minimal condition on \( Y \), the first, second, fourth, sixth, seventh, and eighth terms of (11) vanish, so we get

\[
T_Y = \frac{1}{2} \mathcal{L}_Y + \frac{1}{6} \text{tr}_h \mathcal{L}_Y^3.
\]

Now, \( \mathcal{L}_Y \) and \( \text{tr}_h \mathcal{L}_Y^3 \) are both pointwise conformal invariants of weight \(-3\), so we have exhibited \( T_Y \) itself as such a pointwise conformal invariant. We define

\[
\mathcal{C}_Y = \frac{1}{2} \mathcal{L}_Y + \frac{1}{6} \text{tr}_h \mathcal{L}_Y^3.
\]

This is a pointwise conformal invariant, and the upshot of the above remarks is that

\[
(21) \quad \int_{Y_\varepsilon} (\mathcal{L}_Y + T_Y) dv_h = \int_{Y_\varepsilon} \mathcal{C}_Y dv_h = \int_{Y} \mathcal{C}_Y dv_h + O(\varepsilon).
\]

We now turn to the integral over \( M_\varepsilon^+ \) in (9). Here, we will compute \( \mathcal{T}_{M_\varepsilon} \) and \( \mathcal{L}_{M_\varepsilon} \), the extrinsic curvature quantities with respect to the compactified metrics \( \bar{g} \) and \( \bar{h}_\varepsilon \); then we will compute the transformation to \( g_+, h_\varepsilon \) using equation (12), which in particular implies that

\[
\int_{M_\varepsilon^+} (\mathcal{L}_M + T_M) dv_{g_+} = \int_{M_\varepsilon^+} (\mathcal{T}_M + \mathcal{L}_M + P^g_3(-\log r)) dv_{\bar{g}}.
\]

Our goal is thus to compute the right-hand side of this equation. We begin by computing some basic quantities. Recalling that \( \bar{g} = dr^2 + \bar{h}_r \) and \( M_\varepsilon = \{ r = \varepsilon \} \), we find that

\[
\mathcal{T}_{M_\varepsilon} = -\frac{1}{2} \partial_r \bar{h}_r |_{r=\varepsilon} = \varepsilon P^h + O(\varepsilon^2),
\]

where \( P^h \) is the Schouten tensor of \( \bar{h} \), and we have used (14). Thus,

\[
\mathcal{T}_{M_\varepsilon} = \varepsilon (P^h_\mu)^\mu + O(\varepsilon^3) = \frac{1}{4} \varepsilon R_h + O(\varepsilon^3).
\]
The reason the error is $O(\varepsilon^3)$ is that the $r^3$ term in the expansion of $\tilde{h}_r$ is trace-free. We also have

$$\mathcal{T}_{M_\varepsilon} = \varepsilon \tilde{P}^h + O(\varepsilon^2).$$

We next wish to compute $R_{g}$ on $M_\varepsilon$. To do this, we use the fact that $R_{g_+} \equiv -12$ and that $g_+ = r^{-2} \tilde{g}$. Thus, we will use the conformal transformation formula for scalar curvature. Let $\omega = -\log(r)$. It will be useful to record that

$$\Delta g \omega = r^{-2} + \frac{1}{4} R_{\tilde{h}} + O(r^2),$$

which follows easily from (5). Thus, from the conformal change formula, we find

$$-12 = r^2 (R_\tilde{g} - 6 \Delta g \omega - 6 |d\omega|^2)$$

$$= r^2 \left( R_\tilde{g} - 6r^{-2} - \frac{3}{2} R_{\tilde{h}} - 6r^{-2} + O(r^2) \right),$$

whence

$$R_\tilde{g} = \frac{3}{2} R_{\tilde{h}} + O(r^2).$$

We next compute the tracefree tangential Ricci tensor $\tilde{R}_{\mu \nu}^\theta$. We will use again the same technique of conformal transformation and the fact that $\text{Ric}(g_+) = -3g_+$. We first find using (11) that

$$\nabla_\mu \nabla_\nu \tilde{g} \omega = \tilde{P}_{\mu \nu}^h + O(r).$$

It then follows from the equation

$$R_{\mu \nu}^\theta = R_{\mu \nu}^\theta - 2 \nabla_\mu \nabla_\nu \omega + 2 \omega_\mu \omega_\nu - (\Delta g \omega - 2 |d\omega|^2) \tilde{g}_{\mu \nu}$$

that

$$\tilde{R}_{\mu \nu}^\theta = 2 \tilde{P}_{\mu \nu}^h + O(r).$$

We are ready to analyze the curvature integrands on $M_\varepsilon$. First, we easily find using (10) and the above that

$$\mathcal{L}_{M_\varepsilon} = O(\varepsilon),$$

where the first-order contribution is from the first two terms of (10), and the last two terms provide contributions of order $O(\varepsilon^3)$. Next, we compute $\mathcal{T}_{M_\varepsilon}$, recalling that $\tilde{\mu}_{M_\varepsilon} = \frac{\partial}{\partial r}$. Then it again follows from the above computations that

$$\mathcal{T}_{M_\varepsilon} = O(\varepsilon).$$

The lowest-order contributions come once again from the first three terms of (11), as well as the sixth.
We next turn to computing $P_3^\theta(\omega) = -P_3^\theta(\log(r))$ for $P_3^\theta$ associated to $M_\varepsilon$. First, observe that $\omega|_{M_\varepsilon} \equiv -\log(\varepsilon)$, and $\bar{\mu}_{M_\varepsilon}(\omega) \equiv \frac{1}{\varepsilon}$. Thus, all tangential derivatives of both quantities vanish, which means the second through fifth terms of (13) vanish. Thus, only the first and last remain.

It follows from (23) that

$$\frac{1}{2} \bar{\mu}_{M_\varepsilon} \Delta_{\bar{g}} \omega = -\varepsilon^{-3} + O(\varepsilon).$$

Next, using again the facts that $R_\bar{g} = \frac{3}{2} R_h + O(r^2)$ and our above calculations, we find that the last term of (13) simplifies to

$$\left(\frac{1}{6} R_\bar{g} - \frac{1}{2} R_h - \frac{1}{2} \bar{L}_{M_\varepsilon} h_{\bar{M}_\varepsilon}^2 + \frac{1}{3} \bar{H}_{M_\varepsilon}^2 \right) \bar{\mu}(-\log(r)) = \frac{1}{4} \varepsilon^{-1} + O(\varepsilon).$$

Now, we wish to perform the integral over $M^+_\varepsilon$, not $M_\varepsilon$. Just as for the interior integral, the simplest approach will be first to compute the integral over $\{\varepsilon\} \times M^+$, and then subtract or add whatever was missed near the corner due to turning of $Y$ away from $\Sigma$. First, we observe that from our above computations, it is clear that

$$\int_{M^+_\varepsilon} (\mathcal{T}_{M_\varepsilon} + \mathcal{L}_{M_\varepsilon} + P_3^\theta(-\log(r))) dv_{h_\varepsilon} = \int_{M^+_\varepsilon} P_3^\theta(-\log(r)) dv_{\bar{h}_\varepsilon} + O(\varepsilon).$$

We may focus therefore only on contributions from $P_3^\theta(-\log(r))$. We write

$$\int_{M^+_\varepsilon} P_3^\theta(\omega) dv_{h_\varepsilon} = \int_{\{\varepsilon\} \times M^+} P_3^\theta(\omega) dv_{h_\varepsilon} - \oint_{\Sigma} \int_{u(\varepsilon, \zeta)} P_3^\theta(\omega)(1 + O(w)) dw dv_{\bar{h}}(\zeta).$$

(Compare (18).) We compute the first term first. Recall that $dv_{h_\varepsilon} = (1 - \frac{1}{8} \varepsilon^2 R_h + O(\varepsilon^4)) dv_h$. Thus,

$$\int_{\{\varepsilon\} \times M^+} P_3^\theta(\omega) = \int_{M^+} \left( -\varepsilon^{-3} + \frac{1}{8} \varepsilon^{-1} R_h + O(\varepsilon) \right) \left( 1 - \frac{1}{8} \varepsilon^2 R_h + O(\varepsilon^4) \right) dv_h$$

$$= -\varepsilon^{-3} \text{vol}_h(M^+) + \frac{3}{8} \varepsilon^{-1} \int_{M^+} R_h dv_h + O(\varepsilon).$$
As for the corner integral, we find using (19)

\begin{align*}
\oint_{\Sigma} \int_{0}^{u(\varepsilon, \zeta)} P_3^\theta(\omega)(1 + O(w))dw dv_k(\zeta) &= \oint_{\Sigma} (-\varepsilon^{-3} + O(\varepsilon^{-1})) \cdot \left(\frac{1}{4} \varepsilon^2 \eta_M + O(\varepsilon^4 \log(\varepsilon))\right) dv_k \\
&= -\frac{1}{4} \varepsilon^{-1} \oint_{\Sigma} \eta_M dv_k + O(\varepsilon \log \varepsilon).
\end{align*}

Thus, we have found that

(24)

\begin{align*}
\int_{M^+} (T_M + L_M) dv_{\theta_+} &= -\varepsilon^{-3} \text{vol}_h(M^+) \\
&+ \varepsilon^{-1} \left(\frac{3}{8} \int_{M^+} R_h dv_{\theta} + \frac{1}{4} \oint_{\Sigma} \eta_M dv_k \right) + o(1).
\end{align*}

We are finally ready to evaluate the corner terms $U_{\Sigma}^\varepsilon$ and $G_{\Sigma}^\varepsilon$ in (22). Just as for $M^\varepsilon$, our strategy will be to evaluate first with respect to $\bar{g}$, and then use the conformal transformation formula (16) and the pointwise conformal invariance of $G$. Thus, we will find

\begin{align*}
\oint_{\Sigma_e} (G_k + U_k) dv_k &= \oint_{\Sigma_e} (\bar{G}_{\Sigma_e} + \bar{U}_{\Sigma_e} + P_2^\theta (\log r)) dv_{\theta_e}.
\end{align*}

To begin, we wish to estimate $\theta_0^\varepsilon$, which enters the formulas for $U, G$, and $P_2$. To do this, we find normal vectors $\bar{\mu}_M^\varepsilon$ and $\bar{\mu}_Y$. The first is easy: $\bar{\mu}_M^\varepsilon = \frac{\partial}{\partial r}$. For the second, we observe that, for $\varepsilon$ small, we can write $Y$ as the zero level set of $F = w - u(r, \zeta)$ (where, again, $\zeta \in \Sigma$).

Now,

\begin{align*}
\text{grad}_{\bar{g}} F &= (1 + O(r^2)) \frac{\partial}{\partial w} - \frac{\partial u}{\partial r} \frac{\partial}{\partial r} - \bar{k}^{ab} \frac{\partial u}{\partial x^a} \frac{\partial}{\partial x^b} + O^i(r^3 \log(r)) \frac{\partial}{\partial x^i} \\
&= (1 + O(r^2)) \frac{\partial}{\partial w} - \frac{1}{2} \bar{r} \bar{\eta}_M \frac{\partial}{\partial r} - \frac{1}{4} r^2 \bar{k}^{ab} \bar{\eta}_M \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} + O_{\bar{g}}(r^3 \log(r)).
\end{align*}

Since $\left| \frac{\partial}{\partial w} \right|_{\bar{g}} = 1 + O(r^2)$, we have

\begin{align*}
\left| \text{grad}_{\bar{g}} F \right|_{\bar{g}} &= 1 + O(r^2).
\end{align*}

Consequently,

(25)

\begin{align*}
\bar{\mu}_Y = \frac{\text{grad}_{\bar{g}} F}{\left| \text{grad}_{\bar{g}} F \right|_{\bar{g}}} &= (1 + O(r^2)) \frac{\partial}{\partial w} - \left(\frac{1}{2} r \bar{\eta}_M + O(r^3 \log(r))\right) \frac{\partial}{\partial r} + O_{\bar{g}}(r^2).
\end{align*}

Thus,

\begin{align*}
\cos(\theta_0^\varepsilon) &= -\langle \bar{\mu}_M^\varepsilon, \bar{\mu}_Y \rangle = \frac{1}{2} \varepsilon \bar{\eta}_M + O(\varepsilon^3 \log(\varepsilon)).
\end{align*}
Next we wish to estimate the second fundamental form $\overline{\Pi}_{Y_{\varepsilon}}$ of $\Sigma_{\varepsilon}$ viewed as a submanifold of $Y_{\varepsilon}$. To do this, we first want to know the inward-pointing unit normal vector $\overline{\nu}_{Y_{\varepsilon}}$ to $\Sigma_{\varepsilon}$ in $Y_{\varepsilon}$. By inspection, we can see that

$$V = \frac{\partial}{\partial r} - \frac{\partial F}{\partial r} \frac{\text{grad}_g F}{|dF|^2_g}$$

is normal to $\Sigma_{\varepsilon}$ and tangent to $Y_{\varepsilon}$, so

$$\overline{\nu}_{Y_{\varepsilon}} = V \left| V \right| \overline{g} = (1 + O(\varepsilon^2)) \frac{\partial}{\partial r} + \frac{1}{2} \varepsilon \eta M \frac{\partial}{\partial w} + O(\varepsilon^3 \log \varepsilon).$$

Now, a local frame for $T\Sigma_{\varepsilon}$ is given by $\{X_1, X_2\}$, where

$$X_a = \frac{\partial}{\partial x^a} - \frac{\partial F}{\partial x^a} \frac{\partial}{\partial w}.$$  

Since $\nabla^g_{\partial_{x_a}} \partial_t = O(\varepsilon) \partial_t$ (which is easy to check), we may conclude that

$$\langle \nabla^g_{X_a} \overline{\nu}_{Y_{\varepsilon}} , X_b \rangle = O(\varepsilon).$$

Thus, by Weingarten’s equation,

$$|\overline{\Pi}_{Y_{\varepsilon}}|_{\overline{g}} = O(\varepsilon).$$

It now follows that $\overline{G}_{\Sigma_{\varepsilon}} = O(\varepsilon)$: the first term in (14) because $\cot(\theta_{\varepsilon}^0) = O(\varepsilon)$, and the second because of the estimate on $\overline{\Pi}_{Y_{\varepsilon}}$.

We next turn to $\overline{U}_{\Sigma_{\varepsilon}}$. The second and third terms in (13) are $O(\varepsilon)$ for the same reason. Turning to the fourth term, $\overline{\nu}_M \overline{\Pi}_{M_{\varepsilon}} = O(\varepsilon)$ by (22).

To compute $\overline{\nu}_{Y_{\varepsilon}} \overline{H}_{Y_{\varepsilon}}$, we first compute $\overline{H}_{Y_{\varepsilon}}$ using the conformal change formula. Recall that $H_{Y_{\varepsilon}} \equiv 0$. Then again taking $\omega = -\log r$, we find from the conformal transformation formula $H_{Y_{\varepsilon}} = e^{-\omega}(\overline{H}_{Y_{\varepsilon}} - 3\overline{\mu}_{Y_{\varepsilon}}(\omega))$ that

$$0 = r(\overline{H}_{Y_{\varepsilon}} - 3/2 \overline{\mu}_M + O(r^2 \log(r))),$$

whence

$$\overline{H}_{Y_{\varepsilon}} = \frac{3}{2} \overline{\mu}_M + O(r^2 \log(r)).$$

Thus, $\overline{\nu}_{Y_{\varepsilon}} \overline{H}_{Y_{\varepsilon}} = O(\varepsilon \log(\varepsilon))$; so since $\theta_{\varepsilon}^0 = \frac{\pi}{2} + O(\varepsilon)$, we have

$$\overline{U}_{\Sigma_{\varepsilon}} = \frac{\pi}{2} K_{\overline{\kappa}} + O(\varepsilon \log \varepsilon).$$

Consequently,

$$\int_{\Sigma_{\varepsilon}} (G_{\Sigma_{\varepsilon}} + U_{\Sigma_{\varepsilon}}) d\nu_{\Sigma_{\varepsilon}} = \pi^2 \chi(\Sigma) + O(\varepsilon \log \varepsilon).$$

We still need to compute the integral of $P_2(\omega)$. First, still letting $\omega = -\log r$, observe that $\omega|_{M_{\varepsilon}} \equiv -\log \varepsilon$ and that $\overline{\mu}_M \omega \equiv -\frac{1}{\varepsilon}$. Thus,
the first and second terms of (17) in $P_2^\beta(\omega)$ vanish identically, as do the terms $\eta_M \bar{\nu}_M \omega$, $\bar{\eta}_Y \bar{\nu}_M \omega$, and $\bar{H}_M \bar{\nu}_M \omega$.

Now, the third term takes the form

$$\bar{\nu}_Y \bar{\nu}_M \omega = \frac{1}{2} \bar{\eta}_M \partial_\omega \bar{\eta}_M + O(\varepsilon \log \varepsilon)$$

$$= O(\varepsilon \log \varepsilon).$$

Next, $\bar{\nu}_Y \omega = -\frac{1}{\varepsilon} + O(\varepsilon)$, so $\cot(\theta_0^\beta) \bar{\eta}_Y \bar{\nu}_Y \omega = O(\varepsilon)$. On the other hand, $-\csc(\theta_0^\beta) \bar{\eta}_M \bar{\nu}_Y \omega = \varepsilon^{-1} \bar{\eta}_M + O(\varepsilon)$, since $\bar{\eta}_M = \bar{\eta}_M + O(\varepsilon^2)$ and $\csc(\theta_0^\beta) = 1 + O(\varepsilon^2)$.

Finally,

$$\frac{1}{3} \int_Y \bar{\nu}_Y \omega = -\frac{1}{2} \varepsilon^{-1} \bar{\eta}_M + O(\varepsilon \log \varepsilon).$$

Adding together all these terms, we therefore find that $P_2^\beta \omega = \frac{1}{2} \varepsilon^{-1} \bar{\eta}_M + O(\varepsilon \log \varepsilon)$. Thus,

(27)

$$\int_{\Sigma_\varepsilon} (\bar{G}_\varepsilon + \bar{U}_\varepsilon + P_2^\beta (-\log r)) \, dv_{\bar{\varepsilon}} = \frac{1}{2} \varepsilon^{-1} \int_{\Sigma_\varepsilon} \bar{\eta}_M \, dv_{\bar{\varepsilon}} + \pi^2 \chi(\Sigma) + O(\varepsilon \log \varepsilon).$$

Combining (9), (20), (21), (24), and (27), we find

$$\pi^2 (4 \chi(X^+_{\varepsilon}) - \chi(\Sigma)) = 3V^+_\omega + \frac{1}{8} \int_{X^+_\varepsilon} |W_{g+}|^2_{g+} \, dv_{g+} + \int_{Y^+_{\varepsilon}} C_Y \, dv_{\bar{\varepsilon}} + O(\varepsilon \log \varepsilon).$$

Letting $\varepsilon \to 0$ yields the result. \qed

4. Variation of Renormalized Volume

In this section we give a proof of Theorem 1.3. Since this will require extensive calculations we begin by establishing some new notational conventions.

In addition to using the coordinate system $(r, x^1, x^2, w)$, it will be convenient to use the system $(x^0, x^1, x^2, x^3) = (r, x^1, x^2, w - u)$, where $u$ is as in (8). We will still use $0 \leq i, j \leq 3$ to refer to coordinate fields on $X$, but will use $0 \leq \bar{\alpha}, \bar{\beta} \leq 2$ to refer to the coordinate fields tangent to $Y$. It will also be useful on the interior $\tilde{X}$ to let $x^8$ be the $g_+$-distance to $\tilde{Y}$, so that $\frac{\partial}{\partial x^8} = \mu_Y$ is the $g_+$-unit inward normal vector to $\tilde{Y}$. The system $(r, x^1, x^2, x^8)$ is clearly another coordinate system near $\tilde{Y}$, and the corresponding coordinate vector fields tangent to $Y$ are the same.
As in the introduction, suppose $\mathcal{F} : (-\varepsilon, \varepsilon) \times Y \to X$ is a $C^3$ variation of $Y$ through minimal surfaces in $X$ such that $\mathcal{F}(t, \Sigma) \subset M$ for all $t$. For each $t \in (-\varepsilon, \varepsilon)$, $\mathcal{F}_t(Y) = Y_t$ splits $X$ into two disjoint sets, $X^+_t$, $X^-_t$ and we can make our choice of $X^+_t$ consistent by fixing a point $p \in X^+_0$ and requiring that $p \in X^+_t$ for $t$ in a possibly smaller time interval $t \in (-\delta, \delta)$. Let $V^+_t(\cdot) = V^+_t(X^+_t)$. We will also use the notation $V^+_t(\mathcal{F}_t(Y))$. Our goal is to use the formula (2) to compute a formula for the first variation, $\dot{V}^+_t$.

Before proceeding we recall that strictly speaking, the formula for $V^+_t$ given by (2) only holds for minimal $Y$. However, as we remarked in the introduction, one can use this formula to define $V^+_t$ for any dividing hypersurface, in particular for $Y_t = \mathcal{F}_t(Y)$, where $\mathcal{F}_t$ is a general variation of $Y$.

We begin by showing that we can make two simplifying assumptions about the variation $\mathcal{F}$. First, we show that it suffices to consider normal variations. We then show that we can weaken the assumption that $Y_t = \mathcal{F}_t(Y)$ is minimal for each $t$, and only assume that minimality is preserved infinitesimally.

To see why it suffices to consider normal variations, let $Z$ be the variation field of $\mathcal{F}$:

$$
\frac{d}{dt} \mathcal{F}_t \bigg|_{t=0} = Z.
$$

We may uniquely write $Z = Z^\perp + Z^\top$, where the two fields are respectively orthogonal to $Y$ (with respect to any compactification $\tilde{g}$) and tangential to $Y$. Clearly, it does not matter which compactification is chosen when defining $Z^\perp$, since orthogonality is a conformally invariant notion.

We claim that $\dot{V}^+_t$ only depends on $Z^\perp$. By Theorem 9.34 of [Lee13] and the fact that $\mathcal{F}_t(\Sigma) \subset M$ for all $t$, there is a smooth flow $\mathcal{F}^\top : (-\varepsilon, \varepsilon) \times Y \to X$ generated by $Z^\top$, such that $\mathcal{F}^\top_t(Y) = Y$. Therefore,

$$
\frac{d}{dt} V^+_t(\mathcal{F}^\top_t(Y)) \bigg|_{t=0} = 0.
$$

Also, if $\mathcal{F}^\perp : (-\varepsilon, \varepsilon) \times Y \to X$ is any normal variation of $Y$ such that $\frac{d}{dt} \bigg|_{t=0} \mathcal{F}^\perp_t = Z^\perp$, then by linearity of the derivative

$$
\frac{d}{dt} V^+_t(\mathcal{F}_t(Y)) \bigg|_{t=0} = \frac{d}{dt} V^+_t(\mathcal{F}^\top_t(Y)) \bigg|_{t=0} + \frac{d}{dt} V^+_t(\mathcal{F}^\perp_t(Y)) \bigg|_{t=0} = \frac{d}{dt} V^+_t(\mathcal{F}^\perp_t(Y)) \bigg|_{t=0}.
$$

Therefore, the derivative only depends on $Z^\perp$, as claimed.
The same argument shows that if \( H_{Y^t} \) is the mean curvature of \( \mathcal{F}_t(Y) \) and \( H_{Y^t}^\perp \) is the mean curvature of \( \mathcal{F}_t^\perp(Y) \), then

\[
0 = \left. \frac{d}{dt} H_{Y^t} \right|_{t=0} = \left. \frac{d}{dt} H_{Y^t}^\perp \right|_{t=0}.
\]

The upshot is that it suffices to consider normal variations \( \mathcal{F} \) of \( Y \) with \( Y^t = \mathcal{F}_t(Y) \) minimal to first order; i.e., such that

\[
\left. \frac{d}{dt} H_{Y^t} \right|_{t=0} = 0.
\]

Let \( \mathcal{F} : (-\varepsilon, \varepsilon) \times Y \to X \), be a \( C^3 \) normal variation satisfying (29). As in the statement of Theorem 1.3 we let \( f = \langle \mu_Y, \left. \frac{d}{dt} \right|_{t=0} \mathcal{F} \rangle_{g^+} \), where \( \mu_Y \) is the \((X^+, g_+)-inward unit normal vector along Y. Since \( \mathcal{F} \) is normal, we can write

\[
\left. \frac{d}{dt} \right|_{t=0} \mathcal{F}_t = f \mu_Y.
\]

Also, let \( \tilde{\mathcal{F}} = \mathcal{F}|_{(-\varepsilon, \varepsilon) \times \Sigma} \). Then \( \tilde{\mathcal{F}} \) determines \( \tilde{f} \in C^\infty(\Sigma) \) given by

\[
\tilde{f} = \left. \frac{d}{dt} \right|_{t=0} \langle \tilde{\mathcal{F}}, \nu_M \rangle,
\]

where \( \nu_M \) is the inward-pointing normal vector to \( \Sigma \) in \( M^+ \) with respect to \( \tilde{h} \).

From now on, to simplify notation we will let primes denote \( \left. \frac{d}{dt} \right|_{t=0} \).

By the formulas (76), (83), and (84) in the appendix, the variations of the induced metric, second fundamental form, and mean curvature of \( Y \) are given by

\[
\tilde{h}^\alpha_{\tilde{\beta}} = -2f L^\alpha_{\tilde{\beta}},
\]

\[
L^\alpha_{\tilde{\beta}} = \nabla^\alpha \tilde{h}^\tilde{\beta} - \tilde{h}^{\alpha\gamma} L^\gamma_{\tilde{\alpha}\tilde{\beta}} f + R^g_{\tilde{\alpha}^\gamma \tilde{\beta}^\delta} \tilde{h}^{\alpha\gamma} f,
\]

\[
H' = \Delta_{\tilde{h}} f + (|L_Y|^2_{\tilde{h}} - 3) f.
\]

By (28), \( H' = 0 \), so the last formula above implies that \( f \) must satisfy

\[
\Delta_{\tilde{h}} f = (3 - |L_Y|^2_{\tilde{h}}) f.
\]

**Lemma 4.1.** \( f \in C^\infty(\tilde{Y}) \) has an asymptotic expansion of the form

\[
f = r^{-1} \tilde{f} + o(1),
\]

where \( \tilde{f} \in C^\infty(\Sigma) \) is given by (31).

Conversely, if \( |L_Y|^2_{\tilde{h}} \leq 3 \) on \( \tilde{Y} \), then given \( \tilde{f} \in C^\infty(\Sigma) \), there is a unique solution \( f \) to (33) satisfying the expansion (34).
Proof. We first observe that near $M$,
\begin{equation}
|L_Y|^2_{\bar{h}} = O(r^2).
\end{equation}
This follows from (64) below, but it can also be seen by using the fact that $L_Y$ is trace-free (since $Y$ is minimal), and the the trace-free second fundamental form is a conformal invariant (of weight 1). Using (35), it is easy to see that the indicial roots of the operator
\begin{equation}
P = \Delta_{\bar{h}} - (3 - |L_Y|^2_{\bar{h}})
\end{equation}
are $-1$ and $3$. It follows that $f$ has an expansion of the form
\begin{equation}
f = r^{-1}f_{-1} + O(1),
\end{equation}
for some $f_{-1} \in C^\infty(\Sigma)$. However, using the expansion of the metric $\bar{h}$ near $M$ in (5), we have $\bar{h}^{00} = 1 + O(r^2)$, and using this it is easy to see that
\begin{equation}
f - r^{-1}f_{-1} = o(1).
\end{equation}
as in (34). Since $\mu_Y = r\bar{\mu}_Y$, (30) implies
\begin{equation}
\left. \frac{d}{dt} \right|_{t=0} F_t = f\mu_Y
= \left[ r^{-1}f_{-1} + o(1) \right] r\bar{\mu}_Y
= f_{-1}\bar{\mu}_Y + o(r),
\end{equation}
and it follows from (31) and the definition of $\tilde{F}$ that $f_{-1} = \tilde{f}$.

Conversely, given $\tilde{f}$, if we let
\begin{equation}
f_{-1} = r^{-1}\tilde{f}
\end{equation}
then $Pf_{-1} = O(1)$. It then follows from standard arguments (see [Lee06]) that there is a unique solution of $Pf = 0$ with $f = r^{-1}f_{-1} + O(1)$. Again using the expansion of the metric it is readily checked that $f = r^{-1}\tilde{f} + o(1)$.

\begin{remark}
Although $f \in C^\infty(\tilde{Y})$, since the indicial roots of the equation satisfied by $f$ are $-1$ and $3$, the expansion of $f$ must in general be expected to have a term $r^3 \log r$, so $rf \in C^{3,\alpha}(\tilde{Y})$, and optimal regularity of $\mathcal{F}$ is $C^3$.
\end{remark}

Proof of Theorem 1.3. The statement of Theorem 1.3 consists of two claims: the formula for the derivative of $V_+^+$, and the assertion that $\tilde{f}$ determines $f$. Since the latter follows from the uniqueness claim in Lemma 4.1, to complete the proof of the theorem we just need to carry out the calculation of $\dot{V}_+^+$.\qed
By Theorem 1.1,

\[ 3V^+_t(X_t) = \pi^2(4\chi(X^+_t) - \chi(\partial Y^t)) - \frac{1}{8} \int_{X^+_t} |W_{g^+_t}|^2_{g^+_t} dv_{g^+_t} - \int_{Y^t} C_{Y^t} dv_{h^t}. \]

We let \( \tilde{h}_t = g^+_t|_{T_t}. \) For \( \varepsilon > 0 \) small, recall that \( X_\varepsilon = \{ x \in X : r(x) > \varepsilon \}. \)

We let \( Y^t_\varepsilon = Y^t \cap X_\varepsilon, \) and define

\[ 3V_\varepsilon(t) = \pi^2(4\chi(X^+_t \cap X_\varepsilon) - \chi(\partial Y^t_\varepsilon)) - \frac{1}{8} \int_{X^+_t \cap X_\varepsilon} |W_{g^+_t}|^2_{g^+_t} dv_{g^+_t} - \int_{Y^t_\varepsilon} C_{Y^t} dv_{\tilde{h}^t}. \]

Then

\[ 3 \frac{d}{dt} V_\varepsilon(t) \bigg|_{t=0} = -\frac{1}{8} \frac{d}{dt} \int_{X^+_t \cap X_\varepsilon} |W_{g^+_t}|^2_{g^+_t} dv_{g^+_t} \bigg|_{t=0} - \frac{d}{dt} \int_{Y^t_\varepsilon} C_{Y^t} dv_{\tilde{h}^t} \bigg|_{t=0}. \]

For the first integral,

\[ (36) \quad -\frac{1}{8} \frac{d}{dt} \int_{X^+_t \cap X_\varepsilon} |W_{g^+_t}|^2_{g^+_t} dv_{g^+_t} = \frac{1}{8} \int_{Y_\varepsilon} |W_{g^+_t}|^2_{g^+_t} dv_{\tilde{h}^t}. \]

To analyze the second integral, we let \( dv_{\tilde{h}^t} = \psi dv_{h^t}, \) where \( \psi = \theta(r - \varepsilon), \) with \( \theta \) the Heaviside function. Then

\[ \frac{d}{dt} \bigg|_{t=0} \int_{Y^t_\varepsilon} C_{Y^t} dv_{\tilde{h}^t} = \frac{d}{dt} \bigg|_{t=0} \int_{Y^t} C_{Y^t} dv_{\tilde{h}^t} \]

\[ = \lim_{\tau \to 0} \frac{1}{\tau} \left[ \int_Y (C_{Y^t} \circ F_\tau)(\psi \circ F_\tau)(F_\tau^* dv_{h^t} - dv_{\tilde{h}^t}) \right. \]

\[ + \int_Y (C_{Y^t} \circ F_\tau - C_Y)(\psi \circ F_\tau) dv_{\tilde{h}} \]

\[ + \int_Y C_Y(\psi \circ F_\tau - \psi) dv_{\tilde{h}} \]

\[ = \int_{Y_\varepsilon} C_Y \left( \frac{d}{dt} \bigg|_{t=0} dv_{\tilde{h}} \right) \]

\[ + \int_{Y_\varepsilon} \frac{d}{dt} C_Y^t \bigg|_{t=0} dv_{\tilde{h}} + \lim_{\tau \to 0} \frac{1}{\tau} \int_Y C_Y(\psi \circ F_\tau - \psi) dv_{\tilde{h}}. \]

Now by the Implicit Function Theorem, the equation \( r(F(t, r, \zeta)) = \varepsilon \)

can be written \( r = \xi(t, \zeta) \) for some smooth \( \xi : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R}. \)

Writing \( dv_{\tilde{h}} = \eta r^{-3} dr dv_{\xi}, \) for some smooth correction factor \( \eta \) that is one on \( \Sigma_\varepsilon, \) we may use the fundamental theorem of calculus to write the last
term as
\[
\lim_{\tau \to 0} \frac{1}{\tau} \int_Y C_Y(\psi \circ F_\tau - \psi) \, dv_h = - \lim_{\tau \to 0} \frac{1}{\tau} \int \xi \int_{\varepsilon} C_Y(r, \zeta) \eta(r, \zeta) r^{-3} \, dr \, dv_{k_\varepsilon}(\zeta)
\]
\[
= - \int_{\Sigma_{\varepsilon}} \frac{d}{dt} \bigg|_{t=0} \xi \int_{\varepsilon} C_Y(r, \zeta) \eta(r, \zeta) r^{-3} \, dr \, dv_{k_\varepsilon}(\zeta)
\]
\[
= - \int_{\Sigma_{\varepsilon}} C_Y(\varepsilon, \zeta) \varepsilon^{-3} \frac{\partial \xi}{\partial t} \bigg|_{t=0} \, dv_{k_\varepsilon}(\zeta)
\]
\[
= \int_{\Sigma_{\varepsilon}} C_Y \varepsilon^{-1} \, dr \, (f_{\mu_Y}) \, dv_{k_\varepsilon}
\]
\[
= \int_{\Sigma_{\varepsilon}} C_Y \langle r \partial_r, f_{\mu_Y} \rangle \, g_{k_\varepsilon} \, dv_{k_\varepsilon}.
\]
Therefore
\[
\frac{d}{dt} \int_{Y_{\varepsilon}} C_Y \, dv_{\tilde{h}} \bigg|_{t=0} = \int_{Y_{\varepsilon}} \left( \frac{d}{dt} C_Y \bigg|_{t=0} \right) \, dv_{\tilde{h}} + \int_{Y_{\varepsilon}} C_Y \left( \frac{d}{dt} \tilde{h} \right) \bigg|_{t=0} + \int_{\Sigma_{\varepsilon}} C_Y \langle r \partial_r, f_{\mu_Y} \rangle \, g_{k_\varepsilon} \, dv_{k_\varepsilon}.
\]
We dispose of the last term with

Claim 1.
\[
\lim_{\varepsilon \to 0} \int_{\Sigma_{\varepsilon}} C_Y \langle r \partial_r, f_{\mu_Y} \rangle \, g_{k_\varepsilon} \, dv_{k_\varepsilon} = 0.
\]

Proof. We know that \( \mu_Y = r \bar{\mu}_Y \) and that \( C^g_{Y^\varepsilon} = r^3 C_Y^g \). We also know from (25) that
\[
\langle r \partial_r, \bar{\mu}_Y \rangle^g = O(\varepsilon^2).
\]
So we get
\[
C^g_{Y^\varepsilon} \langle r \partial_r, \mu_Y \rangle^g = r^3 C_Y^g \langle r \partial_r, \mu_Y \rangle^g = O(\varepsilon^4).
\]
Therefore, taking into account the asymptotics of \( f \), we get
\[
\int_{\Sigma_{\varepsilon}} C_Y \langle r \partial_r, f_{\mu_Y} \rangle \, g_{k_\varepsilon} \, dv_{k_\varepsilon} = O(\varepsilon).
\]
\]

By (29) and the formula for the variation of the volume form (85) in
the appendix we have
\[
\frac{d}{dt} dv_{\tilde{h}} \bigg|_{t=0} = H_Y \, dv_{\tilde{h}} = 0,
\]
since $Y$ is minimal. The minimality of $Y$ to first order also implies $H_{Y^t} = O(t^2)$. Since $g_{\cdot \cdot}$ is Einstein, the formula for $C_{Y^t}$ thus simplifies to

\[
C_{Y^t} = -(L_{Y^t})^{\alpha \beta} R_{\alpha \beta}^{\dot{h}_t} - \frac{1}{3} \text{tr}_{\dot{h}_t} (L_{Y^t})^3 + O(t^2),
\]

where $L_{Y^t}$ is the second fundamental form of $Y^t$ with respect to $\mu_Y$ and $R^{\dot{h}_t}$ is the Ricci tensor of $\dot{h}_t$. Combining (37), (38), (39) and (40) we obtain

\[
\frac{d}{dt} \int_{Y^t} C_{Y^t} dv_{\dot{h}_t} \bigg|_{t=0} = -\int_{Y^t} \frac{d}{dt} \left( (L_{Y^t})^{\alpha \beta} R_{\alpha \beta}^{\dot{h}_t} \right) \bigg|_{t=0} dv_{\dot{h}} - \frac{1}{3} \int_{Y^t} \frac{d}{dt} \text{tr}_{\dot{h}_t} (L_{Y^t})^3 \bigg|_{t=0} dv_{\dot{h}} + O(\varepsilon).
\]

We intend to apply integration by parts on the integrand of this expression to write quantities in terms of boundary integrals on $\Sigma$. We first write the integrands in terms of geometric quantities on $Y$.

Define

\[
A = (L_{Y^t})^{\alpha \beta} R_{\alpha \beta}^{\dot{h}_t}
\]

\[
B = \text{tr}_{\dot{h}_t} (L_{Y^t})^3.
\]

Differentiating $A$ gives

\[
A' = \tilde{h}^{\tilde{\alpha} \tilde{\gamma}} \tilde{h}^{\tilde{\beta} \tilde{\delta}} \tilde{R}^{\tilde{h}}_{\tilde{\gamma} \tilde{\delta}} \nabla^{\tilde{h}}_{\tilde{\alpha}} \nabla^{\tilde{h}}_{\tilde{\beta}} f + 3 f (L^2) \tilde{R}^{\tilde{h}}_{\tilde{\alpha} \tilde{\beta}} + f \tilde{h}^{\tilde{\alpha} \tilde{\gamma}} \tilde{h}^{\tilde{\beta} \tilde{\delta}} \tilde{R}^{\tilde{h}}_{\tilde{\gamma} \tilde{\delta}} R_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta}}
\]

\[
+ \tilde{h}^{\tilde{\alpha} \tilde{\gamma}} \tilde{h}^{\tilde{\beta} \tilde{\delta}} L_{\tilde{\gamma} \tilde{\delta}} (R_{\tilde{\alpha} \tilde{\beta}}^3)' \nabla^{\tilde{h}}_{\tilde{\alpha}} \nabla^{\tilde{h}}_{\tilde{\beta}} (\text{tr}_{\tilde{h}} \tilde{h}').
\]

A standard formula for the variation of the Ricci tensor (see e.g. Top06) gives us

\[
(R_{\tilde{\alpha} \tilde{\beta}}^3)' = -\frac{1}{2} \left[ \Delta_{\tilde{h}} \tilde{h}'_{\tilde{\alpha} \tilde{\beta}} - \nabla_{\tilde{\alpha}} (\tilde{h} \tilde{h}') - \nabla_{\tilde{\beta}} (\tilde{h} \tilde{h}') + \nabla_{\tilde{\alpha}} \nabla_{\tilde{\beta}} (\text{tr}_{\tilde{h}} \tilde{h}') \right]
\]

\[
- \tilde{h}^{\tilde{\alpha} \tilde{\gamma}} \tilde{h}^{\tilde{\beta} \tilde{\delta}} \tilde{R}^{\tilde{h}}_{\tilde{\gamma} \tilde{\delta} \tilde{\gamma} \tilde{\delta} \tilde{\eta} \tilde{\zeta}} + \frac{1}{2} \tilde{h}^{\tilde{\eta} \tilde{\zeta}} \tilde{R}^{\tilde{h}}_{\tilde{\alpha} \tilde{\eta} \tilde{h}'} + \frac{1}{2} \tilde{h}^{\tilde{\eta} \tilde{\zeta}} \tilde{R}^{\tilde{h}}_{\tilde{\beta} \tilde{n} \tilde{h}'}.
\]

Here $\tilde{\delta}$ is the divergence with respect to $\tilde{h}$. Now, by (32), $\Delta_{\tilde{h}} \tilde{h}_{\tilde{\alpha} \tilde{\beta}} = -2 f L_{\tilde{\alpha} \tilde{\beta}}$. By the same equation,

\[
\text{tr}_{\tilde{h}} \tilde{h}' = 0.
\]

Taking the divergence of both sides above gives us

\[
\tilde{\delta}_{\tilde{\beta}} \tilde{h}' = \tilde{h}^{\tilde{\alpha} \tilde{\gamma}} \nabla_{\tilde{\gamma}} \tilde{h}'_{\tilde{\alpha} \tilde{\beta}}
\]

\[
= -\nabla_{\tilde{h}}^2 (2 f L_{\tilde{\alpha} \tilde{\beta}})
\]

\[
= -2 f \nabla_{\tilde{h}}^2 L_{\tilde{\alpha} \tilde{\beta}} - 2 L_{\tilde{\alpha} \tilde{\beta}} \nabla_{\tilde{h}}^2 f.
\]
Now by Codazzi, we have
\[ R^g_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} = \nabla^\tilde{\alpha}_{\tilde{\beta}} L_{\tilde{\gamma}\tilde{\delta}} - \nabla^\tilde{\delta}_{\tilde{\alpha}} L_{\tilde{\beta}\tilde{\gamma}} \]
along \( Y \). Contracting \( \tilde{\alpha} \) and \( \tilde{\gamma} \) and using the Einstein condition on \( g_+ \) along with the fact that \( Y \) is minimal gives
\[ 0 = R^g_{\tilde{\beta}\tilde{d}} = -\tilde{h}^{\tilde{\alpha}\tilde{\gamma}} \nabla^\tilde{\alpha}_{\tilde{\beta}} L_{\tilde{\gamma}\tilde{\delta}} + \tilde{h}^{\tilde{\alpha}\tilde{\gamma}} \nabla^\tilde{\beta}_{\tilde{\alpha}} L_{\tilde{\gamma}\tilde{\delta}} = -\nabla^\tilde{\gamma}_{\tilde{h}} L_{\tilde{\beta}\tilde{\delta}} + \nabla^\tilde{\delta}_{\tilde{h}} L_{\tilde{\beta}\tilde{\gamma}} - \nabla^\tilde{\alpha}_{\tilde{h}} L_{\tilde{\beta}\tilde{\gamma}} + \nabla^\tilde{\gamma}_{\tilde{h}} L_{\tilde{\beta}\tilde{\delta}}. \]
Hence
\[ (44) \]
\[ \nabla^\tilde{\alpha}_{\tilde{h}} L_{\tilde{\beta}\tilde{\gamma}} = 0 \]
and
\[ \delta_{\tilde{\beta}} \tilde{h'} = -2 \nabla^\tilde{\gamma}_{\tilde{h}} f. \]

Turning to the fifth term of \((42)\), we consider the Riemann tensor on \( Y \). As the dimension of \( Y \) is three, it follows that the Weyl tensor of \( \tilde{h} \) vanishes, giving us
\[ R^\tilde{h}_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = \tilde{h}_{\tilde{a}\tilde{b}} \tilde{R}^\tilde{h}_{\tilde{c}\tilde{d}} - \tilde{h}_{\tilde{a}\tilde{d}} \tilde{R}^\tilde{h}_{\tilde{b}\tilde{c}} - \tilde{h}_{\tilde{b}\tilde{c}} \tilde{R}^\tilde{h}_{\tilde{a}\tilde{d}} + \tilde{h}_{\tilde{c}\tilde{d}} \tilde{R}^\tilde{h}_{\tilde{a}\tilde{b}} - \frac{1}{2} \tilde{R}^\tilde{h}_{\tilde{a}\tilde{b}} \tilde{h}_{\tilde{c}\tilde{d}} \]
\[ + \frac{1}{2} \tilde{R}^\tilde{h}_{\tilde{c}\tilde{d}} \tilde{h}_{\tilde{a}\tilde{b}}. \]
Thus,
\[-\tilde{h}^{\tilde{a}\tilde{b}} \tilde{h}^{\tilde{c}\tilde{d}} \tilde{R}^\tilde{h}_{\tilde{a}\tilde{b}} \tilde{h}_{\tilde{c}\tilde{d}} = -R^\tilde{h}_{\tilde{h}} \tilde{h}_{\tilde{h}} \tilde{h}_{\tilde{h}} + \tilde{h}^{\tilde{a}\tilde{b}} \tilde{h}^{\tilde{c}\tilde{d}} \tilde{R}^\tilde{h}_{\tilde{a}\tilde{b}} \tilde{h}_{\tilde{c}\tilde{d}} + \frac{1}{2} \tilde{R}^\tilde{h}_{\tilde{a}\tilde{b}} \tilde{h}_{\tilde{c}\tilde{d}} = \]
\[-R^\tilde{h}_{\tilde{h}} \tilde{h}_{\tilde{h}} \tilde{h}_{\tilde{h}} - \frac{1}{2} \tilde{R}^\tilde{h}_{\tilde{a}\tilde{b}} \tilde{h}_{\tilde{c}\tilde{d}} \]
So we can write the last three terms of \((42)\) as
\[-R^\tilde{h}_{\tilde{h}} \tilde{h}_{\tilde{h}} \tilde{h}_{\tilde{h}} - \frac{1}{2} \tilde{R}^\tilde{h}_{\tilde{a}\tilde{b}} \tilde{h}_{\tilde{c}\tilde{d}} = \]
\[-R^\tilde{h}_{\tilde{h}} \tilde{h}_{\tilde{h}} \tilde{h}_{\tilde{h}} \tilde{h}_{\tilde{h}} - \frac{1}{2} \tilde{R}^\tilde{h}_{\tilde{a}\tilde{b}} \tilde{h}_{\tilde{c}\tilde{d}} = \]
Therefore we have found
\[(R^h_{\tilde{a}\tilde{b}})' = \Delta^\tilde{h}(f L_{\tilde{a}\tilde{b}}) - \nabla^\tilde{h}_{\tilde{a}} (L_{\tilde{b}\tilde{c}} \nabla^\tilde{c} f) - \nabla^\tilde{b}_{\tilde{c}} (L_{\tilde{a}\tilde{c}} \nabla^\tilde{a} f) + 2 f R^\tilde{h}_{\tilde{h}} \nabla^\tilde{a} \nabla^\tilde{b} f - 3 f L_{\tilde{a}} \nabla^\tilde{c} L^\tilde{c} f - 3 f L_{\tilde{b}} \nabla^\tilde{c} L^\tilde{c} f + f R^\tilde{h}_{\tilde{a}\tilde{b}}. \]
This then lets us write down an expression for \( \langle L, (\text{Ric}^\tilde{h})' \rangle_{\tilde{h}} : \)
\[ L^{\tilde{a}\tilde{b}} (R^h_{\tilde{a}\tilde{b}})' = L^{\tilde{a}\tilde{b}} \Delta^\tilde{h}(f L_{\tilde{a}\tilde{b}}) - 2 L^{\tilde{a}\tilde{b}} \nabla^\tilde{c}_{\tilde{a}} (L_{\tilde{b}\tilde{c}} \nabla^\tilde{c} f) - 6 f L^{\tilde{a}\tilde{b}} R^h_{\tilde{a}\tilde{b}} + f R^h |L|^2; \]
hence
\[ A' = R^h_{\tilde{a}\tilde{b}} \nabla^\tilde{a}_{\tilde{a}} \nabla^\tilde{b}_{\tilde{b}} f - 3 f (L^2)^{\tilde{a}\tilde{b}} R^h_{\tilde{a}\tilde{b}} + f R^h_{\tilde{a}\tilde{b}} R_{\tilde{a}\tilde{b}} + \]
\[ + L^{\tilde{a}\tilde{b}} \Delta^\tilde{h}(f L_{\tilde{a}\tilde{b}}) - 2 L^{\tilde{a}\tilde{b}} \nabla^\tilde{c}_{\tilde{a}} (L_{\tilde{b}\tilde{c}} \nabla^\tilde{c} f) + f R^h |L|^2. \]
Rather more straightforwardly, we may now write $B'$ as

$$
B' = (\text{tr} L^2)' = 3(\tilde{h}^{\alpha\gamma})\tilde{h}^{\beta\delta}L_{\alpha\beta}L_{\gamma\delta}L_{\eta\zeta} + 3\tilde{h}^{\alpha\gamma\delta}\tilde{h}^{\beta\delta}L'_{\alpha\beta}L_{\gamma\delta}L_{\eta\zeta}
$$

$$
= 6 \left| L^2 \right|_{h}^2 + 3\tilde{h}^{\alpha\gamma\delta}\tilde{h}^{\beta\delta}\left[ \nabla_{\alpha}^{\tilde{h}}\nabla_{\beta}^{\tilde{h}}f - L^2_{\alpha\beta}f \right]
$$

$$
+ (R^{g+}_{\alpha\beta\gamma\delta} L^{\alpha}_{\alpha} L^{\beta}_{\beta}) f L_{\gamma\delta}L_{\eta\zeta}
$$

$$
= 3 \left| L^2 \right|_{h}^2 + 3(\nabla_{\alpha}^{\tilde{h}}\nabla_{\beta}^{\tilde{h}}f)(L^2)_{\alpha\beta} + 3 f R^{g+}_{\alpha\beta\gamma\delta} (L^2)^{\alpha\beta}.
$$

It will also be useful to record that, by Gauss’s equation and the Einstein condition,

$$
R^{\tilde{h}}_{\eta\zeta} = -3\tilde{h}_{\eta\zeta} - R^{g+}_{\eta\mu\zeta\nu} - (L^2)_{\eta\zeta}.
$$

It then also follows that

$$
R_{\tilde{h}} = -6 - |L|^2.
$$

We now focus on rewriting four terms in $A'$ and $B'$ to make them amenable to integration by parts. We thus make the following definitions:

$$
D_1 = \int_{Y_{\varepsilon}} \tilde{h}^{\alpha\beta} \nabla_{\alpha}^{\tilde{h}} \nabla_{\beta}^{\tilde{h}} f dv_{\tilde{h}}
$$

$$
D_2 = \int_{Y_{\varepsilon}} L^{\alpha\beta} \Delta_{\tilde{h}} (f L_{\alpha\beta}) dv_{\tilde{h}}
$$

$$
D_3 = -\int_{Y_{\varepsilon}} 2L^{\alpha\beta} \nabla_{\alpha}^{\tilde{h}} (L_{\beta\gamma} \nabla_{\gamma}^{\tilde{h}} f) dv_{\tilde{h}}
$$

$$
D_4 = \int_{Y_{\varepsilon}} 3(L^2)^{\alpha\beta} \nabla_{\alpha}^{\tilde{h}} \nabla_{\beta}^{\tilde{h}} f dv_{\tilde{h}}.
$$

We will write each of the above terms as an integral over $Y_{\varepsilon}$ plus an integral over $\Sigma_{\varepsilon}$. Recall that $\nu_{Y_{\varepsilon}}$ is the inward pointing $\tilde{h}$ unit-normal
vector field to Σε in Ye. We find

\[ D_1 = \int_{Y_ε} \tilde{h}^{\alpha\gamma} \tilde{h}^{\beta\delta} R^{h}_{\gamma\delta} \nabla^{h}_\alpha \nabla^{h}_\beta f \, dv_h \]

\[ = - \int_{Y_ε} \tilde{h}^{\alpha\gamma} \tilde{h}^{\beta\delta} \nabla^{h}_\alpha \tilde{R}^{\alpha}_{\gamma\delta} \nabla^{h}_\beta f \, dv_h - \oint_{\Sigma_ε} R^{h}_{\alpha\beta} \nu_Y \nabla^{h}_\beta f \, dv_{kε} \]

\[ = - \int_{Y_ε} \frac{1}{2} (\nabla_h^\epsilon \tilde{R}_h) \nabla^{h}_\alpha f \, dv_h - \oint_{\Sigma_ε} R^{h}_{\alpha\beta} \nu_Y \nabla^{h}_\beta f \, dv_{kε} \]

\[ = \int_{Y_ε} \frac{1}{2} R_h h^2 \Delta h_\epsilon f \, dv_h + \oint_{\Sigma_ε} \left[ \frac{1}{2} R_h \nu_{\gamma_ε} (f) - \tilde{R}_h (\nu_Y, \nabla^{h_\gamma} f) \right] dv_{kε} \]

\[ = \int_{Y_ε} \left( \frac{6 + |L|^2}{2} \right) (|L|^2 - 3) f \, dv_h - \oint_{\Sigma_ε} \left[ \tilde{R}_h \nu_{\gamma_ε} (f) \right] dv_{kε} \]

\[ = \int_{Y_ε} \left( \frac{|L|^4}{2} + \frac{3|L|^2}{2} - 9 \right) f \, dv_h - \oint_{\Sigma_ε} \left( \tilde{R}_h - \frac{1}{2} R_h \tilde{h}_\epsilon \right) \left( \nabla^{h_\gamma} f, \nu_{\gamma_ε} \right) \, dv_{kε}. \]

Next,

\[ D_2 = \int_{Y_ε} L^{\tilde{\alpha}\tilde{\beta}} \Delta_h (f L_{\tilde{\alpha}\tilde{\beta}}) \, dv_h \]

\[ = \int_{Y_ε} \left[ |L|^2 \Delta_h f + f L^{\tilde{\alpha}\tilde{\beta}} \Delta_h L_{\tilde{\alpha}\tilde{\beta}} + 2 L^{\tilde{\alpha}\tilde{\beta}} \nabla^{\tilde{h}}_\tilde{\alpha} f \nabla^{\tilde{h}}_\tilde{\beta} L_{\tilde{\alpha}\tilde{\beta}} \right] \, dv_h \]

\[ = \int_{Y_ε} \left[ |L|^2 \Delta_h f + L^{\tilde{\alpha}\tilde{\beta}} \Delta_h (L_{\tilde{\alpha}\tilde{\beta}}) f + \langle \nabla^{\tilde{h}}_\tilde{\alpha}, \nabla^{\tilde{h}}_\tilde{\beta} |L|^2 \rangle \right] \, dv_h \]

\[ = \int_{Y_ε} f L^{\tilde{\alpha}\tilde{\beta}} \Delta_h L_{\tilde{\alpha}\tilde{\beta}} \, dv_h - \oint_{\Sigma_ε} |L|^2 \nu_{\gamma_ε} (f) \, dv_{kε}. \]

We now recall Simon’s identity, which in this setting reads

\[ \Delta_h L_{\tilde{\alpha}\tilde{\beta}} = 2 R^{h}_{\tilde{\alpha} \tilde{\gamma}} L^{\tilde{\gamma}}_{\tilde{\beta}} + L^{\tilde{\alpha}} \tilde{h}^{\tilde{\beta}} R^{h}_{\tilde{\beta} \tilde{\gamma}} - \langle L, \tilde{R}_h \rangle \tilde{h}_{\tilde{\alpha} \tilde{\beta}} - L^{\tilde{\gamma}}_{\tilde{\alpha}} R^{h}_{\tilde{\gamma} \tilde{\beta} \tilde{\gamma}} - \frac{1}{2} R^{h} L_{\tilde{\alpha}\tilde{\beta}}. \]

We provide a derivation for the reader’s convenience, as there are many versions in different conventions in the literature. By Codazzi we know

\[ R^{h}_{\tilde{\gamma} \tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\alpha}} = \nabla^{h}_{\tilde{\alpha}} L_{\tilde{\gamma} \tilde{\beta}} - \nabla^{h}_{\tilde{\gamma}} L_{\tilde{\alpha} \tilde{\beta}}, \]

so we may write

\[ \tilde{h}^{\tilde{\alpha} \tilde{\beta}} = \nabla^{h}_{\tilde{\alpha}} L_{\tilde{\gamma} \tilde{\beta}} - \nabla^{h}_{\tilde{\gamma}} L_{\tilde{\alpha} \tilde{\beta}} = \nabla^{h}_{\tilde{h}} L_{\tilde{\gamma} \tilde{\beta}} - \nabla^{h}_{\tilde{h}} L_{\tilde{\alpha} \tilde{\beta}} = \Delta_h L_{\tilde{\alpha}\tilde{\beta}}. \]
Now, letting $\Gamma^k_{ij}$ be the Christoffel symbols of $g_+$, we observe the following:

\[ \nabla^h_\delta R^g_{\gamma\alpha\beta\bar{n}} = \nabla^g_\delta R^g_{\gamma\alpha\beta\bar{n}} - \Gamma^\gamma_\delta R^g_{\eta\alpha\beta\bar{n}} - \Gamma^\eta_\delta R^g_{\gamma\alpha\beta\bar{n}} - \Gamma^\gamma_\delta R^g_{\bar{n}\alpha\beta\bar{n}}. \]

Therefore we get

\[ \tilde{h}^{\delta\gamma} \nabla^h_\delta R^g_{\gamma\alpha\beta\bar{n}} = \tilde{h}^{\delta\gamma} \nabla^g_\delta R^g_{\gamma\alpha\beta\bar{n}} + L^\gamma_\delta R^g_{\gamma\alpha\beta\bar{n}}. \]

The second Bianchi identity gives

\[ \nabla^g_\delta R^g_{\gamma\alpha\beta\bar{n}} + \nabla^g_\beta R^g_{\gamma\alpha\bar{n}\delta} + \nabla^g_{\bar{n}} R^g_{\gamma\alpha\beta\bar{n}} = 0. \]

Contracting $\tilde{\delta}$ and $\tilde{\gamma}$ yields

\[ \tilde{h}^{\delta\gamma} \nabla^g_\delta R^g_{\gamma\alpha\beta\bar{n}} + \nabla^g_\beta R^g_{\gamma\alpha\bar{n}\delta} + \nabla^g_{\bar{n}} R^g_{\gamma\alpha\beta\bar{n}} = 0. \]

By the Einstein condition, the last two terms of the above equation vanish, so

\[ \tilde{h}^{\delta\gamma} \nabla^g_\delta R^g_{\gamma\alpha\beta\bar{n}} = 0. \]

Thus we may write

\[ \tilde{h}^{\delta\gamma} \nabla^h_\delta R^g_{\gamma\alpha\beta\bar{n}} = L^\gamma_\delta R^g_{\gamma\alpha\beta\bar{n}}, \]

which gives us

\[ \Delta_h L_{\alpha\beta} = \tilde{h}^{\gamma\delta} \nabla^h_\gamma \nabla^h_\alpha L_{\gamma\beta} - \tilde{h}^{\delta\gamma} \nabla^h_\delta R^g_{\gamma\alpha\beta\bar{n}} = \tilde{h}^{\delta\gamma} \nabla^h_\delta \nabla^h_\alpha L_{\gamma\beta} - L^\gamma_\delta R^g_{\gamma\alpha\beta\bar{n}}. \]

Now we want to commute the covariant derivatives in the first term on the right-hand side of this equation. By the Ricci identity,

\[ \nabla^\delta_\alpha \nabla^\delta_\gamma L_{\gamma\beta} - \nabla^\delta_\gamma \nabla^\delta_\alpha L_{\gamma\beta} = R^\delta_{\alpha\gamma} L_{\gamma\beta} + R^\delta_{\alpha\gamma} L_{\gamma\beta}. \]

Contracting $\tilde{\delta}$ and $\tilde{\gamma}$ and using (44) gives

\[ \tilde{h}^{\delta\gamma} \nabla^\delta_\alpha \nabla^\delta_\gamma L_{\gamma\beta} = \tilde{h}^{\delta\gamma} R^\delta_{\alpha\gamma} L_{\gamma\beta} + R^\delta_{\alpha\gamma} L_{\gamma\beta}. \]
Now the second term on the right-hand side of (51) is
\[
\bar{h}^{\bar{\gamma}\bar{\delta}} \bar{h}^{\bar{\gamma}\bar{\delta}} R_{\bar{\delta}\bar{\alpha}\bar{\beta}} R_{\bar{\delta}\bar{\alpha}\bar{\beta}} \bar{h}_{\bar{\gamma}\bar{\eta}} = \bar{h}^{\bar{\gamma}\bar{\delta}} \bar{h}^{\bar{\gamma}\bar{\delta}} \bar{h}_{\bar{\gamma}\bar{\beta}} R_{\bar{\delta}\bar{\alpha}} - \bar{h}^{\bar{\gamma}\bar{\delta}} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} + \bar{h}^{\bar{\gamma}\bar{\delta}} R_{\bar{\delta}\bar{\alpha}} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} - \bar{h}^{\bar{\gamma}\bar{\delta}} R_{\bar{\delta}\bar{\alpha}} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} - \frac{1}{2} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} + \frac{1}{2} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} \bar{h}_{\bar{\gamma}\bar{\beta}}
\]
\[
= \bar{h}^{\bar{\gamma}\bar{\delta}} R_{\bar{\delta}\bar{\alpha}} L_{\bar{\gamma}\bar{\eta}} - \frac{1}{2} L_{\bar{\alpha}\bar{\beta}} \bar{h}_{\bar{\gamma}\bar{\beta}} - \frac{1}{2} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} \bar{h}_{\bar{\gamma}\bar{\beta}}
\]
\[
= \bar{h}^{\bar{\gamma}\bar{\delta}} R_{\bar{\delta}\bar{\alpha}} L_{\bar{\gamma}\bar{\eta}} - \langle L, \text{Ric}_h \rangle \bar{h}_{\bar{\alpha}\bar{\beta}} + \frac{1}{2} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} \bar{h}_{\bar{\gamma}\bar{\beta}} - \frac{1}{2} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} \bar{h}_{\bar{\gamma}\bar{\beta}}.
\]
This yields (47).

We express each term in Simon’s identity in terms of $L$ and $\text{Riem}_{g_+}$ using (45), which allows us to write the first term on the right-hand side of (47) as
\[
2L_{\bar{\beta}} \bar{h}^{\bar{\beta}} R_{\bar{\alpha}\bar{\gamma}} = 2L_{\bar{\beta}} \bar{h}^{\bar{\beta}} [-3\bar{h}_{\bar{\alpha}\bar{\gamma}} - R_{\bar{\alpha}\bar{\beta}} - (L^2)_{\bar{\alpha}\bar{\gamma}}]
\]
\[
= -6L_{\bar{\alpha}\bar{\beta}} - 2R_{\bar{\alpha}\bar{\beta}} L_{\bar{\alpha}} - 2(L^2)_{\bar{\alpha}\bar{\beta}}.
\]
Then (45) again allows us to write
\[
-\langle L, \text{Ric}_h \rangle \bar{h}_{\bar{\alpha}\bar{\beta}} = L \bar{h}^{\bar{\gamma}} R_{\bar{\alpha}\bar{\beta}} \bar{h}_{\bar{\gamma}} + \text{tr}_h (L^2),
\]
which gives us for the third term of (47)
\[
-\langle L, \text{Ric}_h \rangle \bar{h}_{\bar{\alpha}\bar{\beta}} = L \bar{h}^{\bar{\gamma}} R_{\bar{\alpha}\bar{\beta}} \bar{h}_{\bar{\gamma}} + \text{tr}_h (L^2) \bar{h}_{\bar{\alpha}\bar{\beta}}.
\]
Using (46) gives us that the last term on the right-hand side of (47) may be written as
\[
\frac{1}{2} R_{\bar{\delta}\bar{\alpha}} L_{\bar{\alpha}\bar{\beta}} = \frac{1}{2} L_{\bar{\alpha}\bar{\beta}} [-6 - |L|^2]
\]
\[
= 3L_{\bar{\alpha}\bar{\beta}} + \frac{1}{2} |L|^2 L_{\bar{\alpha}\bar{\beta}}.
\]
Putting the above formulas into (47) gives
\[
\Delta_h L_{\bar{\alpha}\bar{\beta}} = \bar{h}^{\bar{\gamma}\bar{\delta}} \nabla_{\bar{\delta}} \nabla_{\bar{\alpha}} \nabla_{\bar{\beta}} L_{\bar{\gamma}\bar{\beta}} - L_{\bar{\alpha}} \bar{h}^{\bar{\gamma}} R_{\bar{\gamma}\bar{\beta}}
\]
\[
= \bar{h}^{\bar{\gamma}\bar{\delta}} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} + \frac{1}{2} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} \bar{h}_{\bar{\gamma}\bar{\beta}} - \frac{1}{2} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} \bar{h}_{\bar{\gamma}\bar{\beta}}
\]
\[
= 2R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} + \frac{1}{2} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} \bar{h}_{\bar{\gamma}\bar{\beta}} - \frac{1}{2} R_{\bar{\delta}\bar{\alpha}} \bar{h}_{\bar{\gamma}\bar{\beta}} \bar{h}_{\bar{\gamma}\bar{\beta}}
\]
\[
+ L \bar{h}^{\bar{\gamma}} R_{\bar{\alpha}\bar{\beta}} \bar{h}_{\bar{\gamma}} + \text{tr}(L^2) \bar{h}_{\bar{\alpha}\bar{\beta}} + \frac{1}{2} |L|^2 L_{\bar{\alpha}\bar{\beta}}.
\]
Therefore,

\[ L^{\dot{\alpha} \dot{\beta}} \Delta_h L_{\dot{\alpha} \dot{\beta}} = -4(L^2)^{\dot{\alpha} \dot{\beta}} R^{\dot{\gamma}+}_{\ddot{\alpha} \ddot{\beta} \ddot{\gamma} \ddot{\delta}} - 6|L|^2 - 3|L^2|^2 + \frac{1}{2} |L|^4. \]

This allows us to write

\[
\int_{Y^c} f L^{\dot{\alpha} \dot{\beta}} \Delta_h L_{\dot{\alpha} \dot{\beta}} dv_h - \oint_{\Sigma^c} |L|^2 \nu_Y (f) dv_k = \\
\int_{Y^c} \left[ -4f(L^2)^{\dot{\alpha} \dot{\beta}} R^{\dot{\gamma}+}_{\ddot{\alpha} \ddot{\beta} \ddot{\gamma} \ddot{\delta}} - 6f|L|^2 - 3f|L^2|^2 + \frac{1}{2} f|L|^4 \right] dv_h \\
- \oint_{\Sigma^c} |L|^2 \nu_Y (f) dv_k.
\]

Therefore,

\[
D_2 = \int_{Y^c} \left[ -4f(L^2)^{\dot{\alpha} \dot{\beta}} R^{\dot{\gamma}+}_{\ddot{\alpha} \ddot{\beta} \ddot{\gamma} \ddot{\delta}} - 6f|L|^2 - 3f|L^2|^2 + \frac{1}{2} f|L|^4 \right] dv_h \\
- \oint_{\Sigma^c} |L|^2 \nu_Y (f) dv_k.
\]

Meanwhile, more straightforwardly,

\[
D_3 = - \int_{Y^c} 2L^{\dot{\alpha} \dot{\beta}} \nabla^h_{\dot{\alpha}} (L^{\dot{\gamma}+}_{\ddot{\alpha} \ddot{\beta} \ddot{\gamma} \ddot{\delta}} \nabla^h \bar{f}) dv_h \\
= 2 \int_{Y^c} \left[ L^{\dot{\alpha} \dot{\beta}} \nabla^h_{\dot{\alpha}} L^{\dot{\gamma}+}_{\ddot{\alpha} \ddot{\beta} \ddot{\gamma} \ddot{\delta}} \nabla^h \bar{f} + (L^2)^{\dot{\alpha} \dot{\gamma}} \nabla^h_{\dot{\alpha}} \nabla^h \bar{f} \right] dv_h \\
- \oint_{\Sigma^c} 2L^2 (\nabla^h \bar{f}, \nu_Y) dv_k.
\]

Now using integration by parts and applying (44) we see

\[
D_4 = \int_{Y^c} 3(L^2)^{\dot{\alpha} \dot{\beta}} \nabla^h_{\dot{\alpha}} \nabla^h \bar{f} dv_h \\
= - \int_{Y^c} 3L^{\dot{\alpha} \dot{\gamma}} \nabla^h_{\dot{\alpha}} L^{\dot{\gamma}+}_{\ddot{\gamma} \ddot{\delta}} \nabla^h \bar{f} dv_h \\
- \oint_{\Sigma^c} 3L^{\dot{\alpha} \dot{\gamma}} L^{\dot{\gamma}+}_{\ddot{\gamma} \ddot{\delta}} \nu_Y \nabla^h \bar{f} dv_k.
\]
Next we consider the following:

\[(53)\]

\[-3L\bar{\alpha}^\gamma \nabla^h_{\hat{\alpha}^\gamma} L_{\hat{\beta}^\gamma} \nabla^\gamma_h f = -3L\bar{\alpha}^\gamma (\nabla^h_{\hat{\beta}^\gamma} L_{\hat{\beta}^\gamma} - \nabla^h_{\hat{\beta}^\gamma} L_{\hat{\beta}^\gamma}) \nabla^\gamma_h f - 3L\bar{\alpha}^\gamma \nabla^h_{\hat{\beta}^\gamma} L_{\hat{\beta}^\gamma} \nabla^\gamma_h f = -3L\bar{\alpha}^\gamma R^g_{\hat{\beta}^\gamma} \nabla^\gamma_h f - \frac{3}{2} \nabla^\gamma_h |L|^2 \nabla^\gamma_h f.\]

Then (50) allows us to write

\[(54)\]

\[3L\bar{\alpha}^\gamma \nabla^\beta_h R^{g\gamma}_{\hat{\beta}^\gamma} f = 3fL_{\hat{\alpha}^\gamma} L_{\hat{\beta}^\gamma} R^{g\gamma}_{\hat{\beta}^\gamma}.\]

We also observe that

\[(55)\]

\[3f \nabla^\beta_h L_{\hat{\alpha}^\gamma} R^{g\gamma}_{\hat{\beta}^\gamma} = \frac{3}{2} f(\nabla^\beta_h L_{\hat{\alpha}^\gamma} - \nabla^\beta_h L_{\hat{\beta}^\gamma}) R_{g\gamma} \hat{\beta}_{\hat{\alpha}^\gamma} = \frac{3}{2} f R^g_{\hat{\alpha}^\gamma} R_{g\gamma} \hat{\beta}_{\hat{\alpha}^\gamma} = -\frac{3}{2} f W^g_{\hat{\alpha}^\gamma} W_{g\gamma} \hat{\beta}_{\hat{\alpha}^\gamma}.\]

So if we apply integration by parts to \(D_4\), use (53), apply integration by parts again (to both terms on the RHS of (53)) and then use (54) and (55) we get

\[D_4 = \int_{Y_\varepsilon} \left[ \frac{3}{2} f(L^2) \nabla^\gamma_h f \hat{\beta} - \frac{3}{2} f W^g_{\hat{\alpha}^\gamma} W_{g\gamma} \hat{\beta}_{\hat{\alpha}^\gamma} + \frac{3}{2} |L|^2 \Delta_h f \right] dv_h - \int_{Y_\varepsilon} \left[ \frac{3}{2} f L_{\hat{\alpha}^\gamma} L_{\hat{\beta}^\gamma} R^{g\gamma}_{\hat{\beta}^\gamma} \hat{\beta}_{\hat{\alpha}^\gamma} \right] dv_h + \frac{3}{2} L^2 (\nabla^\gamma_h f, \nu_{Y_\varepsilon}) \right] dv_{k\varepsilon}.\]

Now we want to compute \(\int_{Y_\varepsilon} C dv_h = -\int_{Y_\varepsilon} A' dv_h - \frac{1}{3} \int_{Y_\varepsilon} B' dv_h\). Using our expressions for \(D_1, D_2, D_3\) and \(D_4\) and gathering together all of the terms that appear as integrals over \(Y_\varepsilon\) we get:

\[(56)\]

\[I_Y = -\int_{Y_\varepsilon} \left[ \left( \frac{3}{2} |L|^2 + \frac{1}{2} L^2 f - 4f(L^2) \nabla^\gamma_h f \hat{\beta} - 6f |L|^2 \right) L^2 + \frac{3}{2} f(L^2) \nabla^\gamma_h f \hat{\beta} - f R^g_{\hat{\alpha}^\gamma} R_{g\gamma} \hat{\beta}_{\hat{\alpha}^\gamma} - f R^g_{\hat{\alpha}^\gamma} R_{g\gamma} \hat{\beta}_{\hat{\alpha}^\gamma} + f R^g_{\hat{\alpha}^\gamma} L^2 \right] dv_h.\]
Next, we apply (46), (45), and (56) becomes
\[ R^\theta_{\alpha n \beta n} = W^g_{\alpha n \beta n} - \tilde{h}_{\alpha \beta}, \]
which implies
\[ R^\theta_{\alpha \beta} = -L^2_{\alpha \beta} - 2\tilde{h}_{\alpha \beta} - W^g_{\alpha n \beta n}; \]
and (56) becomes
\[
I_Y = -\int_{Y_\epsilon} \left[ \frac{3|L|^2}{2} f + \frac{|L|^4}{2} f - 9f - f(L^2)\tilde{h}_{\alpha \beta} W^g_{\alpha n \beta n} f - W^{\beta \gamma}_{+} \tilde{h}_{\alpha \beta} W^{g+}_{\alpha n \beta n} - 4f|L|^2 - 6f|L|^2 + 3f|L|^2 f + 6|L|^2 f - 3f|L|^2 + \frac{1}{2} f|L|^4 - 6f|L|^2 f + 2|L|^2 f + (L^2)\tilde{h}_{\alpha \beta} W^{g+}_{\alpha n \beta n} f - |L|^2 f - \frac{1}{2} W^{g+}_{\alpha \beta n} W^{\tilde{h}_{\alpha \beta} n} f - W^{g+}_{\alpha n \beta n} W^{\tilde{h}_{\alpha \beta} n} f \right] dv_{\tilde{h}}.
\]

We may simplify this helpfully:

\[ \frac{1}{2} W^{g+}_{\alpha \beta n} W^{\tilde{h}_{\alpha \beta} n} f - W^{g+}_{\alpha n \beta n} W^{\tilde{h}_{\alpha \beta} n} f = \frac{1}{8} |W^{g+}_{\alpha n}|^2. \]

**Proof.** Observe that
\[
|W^{g+}_{\alpha n}|^2 = W^{g+}_{\alpha n} W^{ijkl}_{g+} W^{g+}_{ijkl} g_{ijkl} = \begin{cases} 4W^{g+}_{\alpha n} W^{\tilde{h}_{\alpha \beta} n} + 4W^{g+}_{\alpha n} W^{\tilde{h}_{\alpha \beta} n} W^{\tilde{n}_{\alpha \beta} n} + 4W^{g+}_{\alpha n} W^{\tilde{h}_{\alpha \beta} n} W^{\tilde{h}_{\alpha \beta} n} + 4W^{g+}_{\alpha n} W^{\tilde{h}_{\alpha \beta} n} W^{\tilde{h}_{\alpha \beta} n} W^{\tilde{h}_{\alpha \beta} n} \\
+ 4W^{g+}_{\alpha n} W^{\tilde{h}_{\alpha \beta} n} W^{\tilde{h}_{\alpha \beta} n} W^{\tilde{h}_{\alpha \beta} n} W^{\tilde{h}_{\alpha \beta} n} \end{cases}.
\]

Note that
\[ W^{g+}_{\alpha \beta n} = -W^{g+}_{\tilde{n} \tilde{n} \delta}. \]

Therefore
\[
4W^{g+}_{\tilde{n} \tilde{n} \delta} W^{\tilde{n}_{\alpha \beta} n} W^{\tilde{n}_{\alpha \beta} n} + 4W^{g+}_{\alpha n} W^{\tilde{h}_{\alpha \beta} n} W^{\tilde{h}_{\alpha \beta} n} = 4W^{g+}_{\tilde{n} \tilde{n} \delta} W^{\tilde{n}_{\alpha \beta} n} g_{\alpha} + 4W^{g+}_{\tilde{n} \tilde{n} \delta} W^{\tilde{n}_{\alpha \beta} n} g_{\alpha} = 8W^{g+}_{\tilde{n} \tilde{n} \delta} W^{\tilde{n}_{\alpha \beta} n} g_{\alpha}.
\]
which gives
\[ |W_{g^+}|^2 = 4W^g_{\tilde{\alpha}\tilde{\beta}\gamma} W^g_{\tilde{\alpha}\tilde{\beta}\gamma} + 8W^g_{\tilde{\alpha}\tilde{\beta}\gamma} W^g_{\tilde{\alpha}\tilde{\beta}\gamma}. \]

It follows from this and (53) that (56) is equal to
\[ I_Y = \int_{Y_\epsilon} \left[ |L|^2 f + \frac{1}{8} |W_{g^+}|^2 f \right] dv_h + \oint_{\Sigma_\epsilon} \nu_Y(f) dv_{k_\epsilon}. \]

Gathering the boundary terms from \( D_1, D_2, D_3 \) and \( D_4 \) and the normal derivative term on the above line we get
\[ \oint_{\Sigma_\epsilon} \left[ \left( \frac{1}{2} R^\epsilon_{\tilde{h}} - \text{Ric}_{\tilde{h}} \right) (\nabla^\tilde{h} f, \nu_Y) - |L|^2 \nu_Y(f) + 2L^2 (\nabla^\tilde{h} f, \nu_Y) \right. \\
+ fL^\tilde{\alpha} R^g_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta} \nu_{Y \epsilon}} + \frac{1}{2} |L|^2 \nu_{Y \epsilon}(f) - L^2 (\nabla f, \nu_{Y \epsilon}) + \nu_{Y \epsilon}(f) \left. \right] dv_{k_\epsilon}. \]

Now we use (57) to re-write \(-L^2 (\nabla^\tilde{h} f, \nu_{Y \epsilon})\), and we use \( R^g_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta} \nu_{Y \epsilon}} = W^g_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta} \nu_{Y \epsilon}} \)
to re-write \( fL^\tilde{\alpha} R^g_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta} \nu_{Y \epsilon}} \), giving us
\[ \oint_{\Sigma_\epsilon} \left[ - |L|^2 \nu_{Y \epsilon}(f) + 2L^2 (\nabla f, \nu_{Y \epsilon}) - fL^\tilde{\alpha} W^g_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta} \nu_{Y \epsilon}} + W^g_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta} \nu_{Y \epsilon}} f \nu_{Y \epsilon} \right] dv_{k_\epsilon}. \]

Combining this with (36), (37) and (59) gives us
\[ 3 \frac{d}{dt} V(t)|_{t=0} = \int_{Y_\epsilon} - |L|^2 f dv_h + \oint_{\Sigma_\epsilon} \left[ - W^g_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta} \nu_{Y \epsilon}} \nabla^\tilde{h} f + |L|^2 \nu_{Y \epsilon}(f) \right. \\
- 2L^\tilde{\alpha} W^g_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta} \nu_{Y \epsilon}} f \nu_{Y \epsilon} \left. \right] dv_{k_\epsilon} + O(\epsilon). \]

Next we will examine the asymptotics of the term \(-W^g_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta} \nu_{Y \epsilon}} f \nu_{Y \epsilon}\). Now, it follows from (5), (6), and the second-last equation on the bottom of page 52 of [FG12] that
\[ W^g_{0\mu0\nu} = O(r). \]

Moreover, from the first equation on p. 53 of the same book, we may conclude that
\[ W^g_{0\mu0\nu} = -\frac{3}{2} r^{-1} g^{(3)}_{\mu\nu}, \]
with \( g^{(3)} \) as in (5). By the conformal change formula for the Weyl tensor, therefore, we find
\[ W^g_{\mu00} = -\frac{3}{2} r^{-1} g^{(3)}_{\mu0} + O(1). \]
Now by (34),

\[-W_{g+}(\nu_{Yt}, \mu_Y, \nabla^h f, \mu_Y) = -r^3 W_{g+}(\bar{\nu}, \bar{\mu}_Y, \nabla^Y f, \bar{\mu}_Y) + O(r^4)\]

\[= -r^3 W_{g+} \bar{\alpha}_{\bar{\beta} \bar{\gamma}} \bar{\nu}_{\bar{T}} \bar{f} \bar{\beta} + O(r^4)\]

\[= -r^3 W_{g+} \bar{\alpha}_{\bar{\beta} \bar{\gamma}} \bar{g}_{\bar{Y}} \bar{\beta} \bar{\gamma} \partial \bar{\gamma} f + O(r^4)\]

\[= r^3 W_{0000} \bar{g} 0 \bar{f} + O(r^4),\]

where \(\bar{n}\) corresponds to \(\bar{\mu}_Y\). Taking (25), (26), (62), and (34), we see that the first corner term of (61) may be written

\[(63) \oint_{\Sigma} (\nu_{Yt}, \mu_Y, \nabla^h f, \mu_Y) dv_k = \oint_{\Sigma} g^{(3)}(\bar{\nu}_M, \bar{\mu}_M)^{\bar{f}} dv_k + O(\varepsilon).\]

We now simplify the remaining terms of (61):

**Claim 3.**

\[\int_{Y} -|L|^2 f dv_k = \oint_{\Sigma} \left[ W_{g+} \bar{\alpha}_{\bar{\beta} \bar{\gamma}} \bar{\nu}_{\bar{T}} \nabla^\beta f - |L|^2 \nu_{Yt}(f) + 2L\bar{\alpha}_\bar{\gamma} L_{\bar{\beta} \bar{\gamma}} \nu_{Yt} \nabla^\beta f - L\bar{\gamma} W_{g+} \bar{\alpha}_{\bar{\beta} \bar{\gamma}} f \nu_{Yt} \right] dv_k = f.p. \int_{Y} -|L|^2 f dv_k + O(\varepsilon \log \varepsilon). \]

**Proof.** Observe that

\[(64) L_{\bar{\alpha} \bar{\beta}} = \frac{T_{\bar{\alpha} \bar{\beta}}}{r^2} + \frac{\bar{\mu}_Y(r)}{r^2} \bar{g}_{\bar{\alpha} \bar{\beta}}.\]

Now

\[(65) \bar{\mu}_Y = (1 + O(r^2)) \partial_w - \left( \frac{\bar{\mu}_M r}{2} + O(r^3 \log r) \right) \partial_r + O(a(r^2)) \partial_a.\]

Therefore

\[(66) \bar{\mu}_Y(r) = -\frac{1}{2} \left[ \bar{\mu}_M r + O(r^3 \log r) \right].\]

Now using the fact that \(\bar{g}_{\alpha \beta} = \bar{g}_{\alpha \beta} + O(r^2)\) we may write

\[(67) T_{\bar{\alpha} \bar{\beta}} = -\frac{1}{2} \bar{\mu}_Y \bar{g}_{\alpha \beta} + O(r^2) = -\frac{1}{2} \partial_w \bar{g}_{\alpha \beta} + O(r^2).\]

Therefore we may write

\[(68) L_{\bar{\alpha} \bar{\beta}} = -\frac{\partial_w \bar{g}_{\alpha \beta}}{2r} - \frac{\bar{\mu}_M \bar{g}_{\alpha \beta}}{2r} + O(\log r).\]
Hence
\begin{equation}
|L|^2_{\tilde{h}} \nu_{\gamma}(f) dv_k = \varepsilon^4 \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} \left[ \frac{\partial_w \bar{g}_{\alpha\beta}}{2\varepsilon} + \frac{\bar{\eta}_M \bar{g}_{\alpha\beta}}{2\varepsilon} + O(\varepsilon \log \varepsilon) \right] \cdot \left[ \frac{\partial_v \bar{g}_{\gamma\delta}}{2\varepsilon} + \frac{\bar{\eta}_M \bar{g}_{\gamma\delta}}{2\varepsilon} + O(\varepsilon \log \varepsilon) \right] [1 + O(\varepsilon)] dv_k,
\end{equation}
so we may write
\begin{equation}
\int_{\Sigma^{\varepsilon}} |L|^2_{\tilde{h}} \nu_{\gamma}(f) dv_k = \int_{\Sigma^{\varepsilon}} \bar{T}_M |\tilde{f}|^2 dv_k + O(\varepsilon \log \varepsilon).
\end{equation}
Now,
\begin{equation}
|L|_{\tilde{h}}^2 |L|_{\tilde{h}} f dv_k = r^4 \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} \left[ \frac{\partial_w \bar{g}_{\alpha\beta}}{2r} + \frac{\bar{\eta}_M \bar{g}_{\alpha\beta}}{2r} + O(r \log r) \right] \cdot \left[ \frac{\partial_v \bar{g}_{\gamma\delta}}{2r} + \frac{\bar{\eta}_M \bar{g}_{\gamma\delta}}{2r} + O(r \log r) \right] \left[ \tilde{f} + O(r) \right] dv_k,
\end{equation}
so
\begin{equation}
\int_{\Sigma^{\varepsilon}} |L|_{\tilde{h}}^2 f dv_k = C + \int_{\varepsilon}^{r_0} \int_{\Sigma} |L|_{\tilde{h}}^2 f dv_k dr
\end{equation}
\begin{equation}
= C + \int_{\varepsilon}^{r_0} \int_{\Sigma} r^{-2} |\tilde{T}_M|_{\tilde{h}}^2 \tilde{f} + O(\log r) dv_k dr
\end{equation}
\begin{equation}
= C' - \varepsilon^{-1} \int_{\Sigma} |\tilde{T}_M|_{\tilde{h}}^2 \tilde{f} dv_k + O(\varepsilon \log \varepsilon)
\end{equation}
for some constants $C$ and $C'$ and $r_0 > 0$ chosen small enough. Observe that
\[ C' = f.p. \int_{\Sigma} |L|_{\tilde{h}}^2 f dv_k. \]
By (68) we can write
\[ L_{\tilde{a} \tilde{b}} = O(1). \]
Now we can write
\begin{align*}
L_{\tilde{a} \tilde{b}} L_{\tilde{b} \tilde{c}} \nu_{\tilde{c}} \nabla \tilde{b} f &= L_{\tilde{a} \tilde{b}} L_{\tilde{b} \tilde{c}} \nu_{\tilde{c}} \tilde{f} + L_{\tilde{a} \tilde{c}} L_{\tilde{b} \tilde{c}} \nu_{\tilde{c}} \tilde{f} + L_{\tilde{a} \tilde{c}} L_{\tilde{b} \tilde{c}} \nu_{\tilde{c}} \tilde{f} + L_{\tilde{a} \tilde{c}} L_{\tilde{b} \tilde{c}} \nu_{\tilde{c}} \tilde{f} = O(r^3)
\end{align*}
It follows that
\[ L_{\tilde{a} \tilde{b}} L_{\tilde{b} \tilde{c}} \nu_{\tilde{c}} \nabla \tilde{b} f dv_k = O(\varepsilon) dv_k, \]
so
\[
\int_{\Sigma_r} L_{\alpha\gamma} \tilde{L}_{\beta} \nu_{\alpha} \nabla \tilde{\beta} f dv_k = O(\varepsilon).
\]

Now we turn our attention to the term \( L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma}\tilde{\alpha}} \nu_{\tilde{\alpha}} \). First observe that
\[
W_{\tilde{\gamma}\tilde{\alpha}} = \rho W_{\tilde{\gamma}\tilde{\alpha}}
\]
and
\[
W_{\tilde{\gamma}\tilde{\alpha}} = \frac{W_{\tilde{\gamma}}}{\tilde{\gamma}}.
\]
where \( \hat{n} \) corresponds to \( \tilde{\mu} \). Now
\[
W_{\tilde{\gamma}\tilde{\alpha} \tilde{\beta} \tilde{\delta}} = R_{\tilde{\gamma}}^\theta - R_{\tilde{\gamma}}^\theta - \nabla \tilde{\gamma} L_{\tilde{\alpha} \tilde{\beta} \tilde{\delta}}
\]
\[
= \partial_\tilde{\gamma} L_{\tilde{\alpha} \tilde{\beta} \tilde{\delta}} - \Gamma_\tilde{\alpha}^{\beta}_{\tilde{\gamma}} \partial_{\tilde{\beta} \tilde{\delta}} - \Gamma_\tilde{\alpha}^{\beta}_{\tilde{\gamma}} \partial_{\tilde{\beta} \tilde{\delta}} - \Gamma_\tilde{\alpha}^{\beta}_{\tilde{\gamma}} \partial_{\tilde{\beta} \tilde{\delta}} - \Gamma_\tilde{\alpha}^{\beta}_{\tilde{\gamma} \tilde{\delta} \tilde{\beta}} L_{\tilde{\beta} \tilde{\delta}}
\]
\[
= \partial_\tilde{\gamma} L_{\tilde{\alpha} \tilde{\beta} \tilde{\delta}} - \Gamma_\tilde{\alpha}^{\beta}_{\tilde{\gamma} \tilde{\delta}} \partial_{\tilde{\beta} \tilde{\delta}} - \Gamma_\tilde{\alpha}^{\beta}_{\tilde{\gamma} \tilde{\delta}} \partial_{\tilde{\beta} \tilde{\delta}} - \Gamma_\tilde{\alpha}^{\beta}_{\tilde{\gamma} \tilde{\delta} \tilde{\beta}} L_{\tilde{\beta} \tilde{\delta}}
\]
\[
= O(r).
\]
This gives us
\[
L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma} \tilde{\alpha} \tilde{\beta} \tilde{\delta}} f \nu_{\tilde{\alpha}} = O(r^2)
\]
and
\[
L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma} \tilde{\alpha} \tilde{\beta} \tilde{\delta}} f \nu_{\tilde{\alpha}} = L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma} \tilde{\alpha} \tilde{\beta} \tilde{\delta}} f \nu_{\tilde{\alpha}} + L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma} \tilde{\alpha} \tilde{\beta} \tilde{\delta}} f \nu_{\tilde{\alpha}}
\]
\[
= L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma} \tilde{\alpha} \tilde{\beta} \tilde{\delta}} f \nu_{\tilde{\alpha}} + O(r^2)
\]
\[
= O(r^2).
\]
Therefore we may write
\[
L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma} \tilde{\alpha} \tilde{\beta} \tilde{\delta}} f \nu_{\tilde{\alpha}} dv_k = O(\varepsilon) dv_k.
\]
We then get that
\[
\int_{\Sigma_r} L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma} \tilde{\alpha} \tilde{\beta} \tilde{\delta}} f \nu_{\tilde{\alpha}} dv_k = O(\varepsilon).
\]
This proves the claim.

Combining Claim 3 with (61) and (63) and letting \( \varepsilon \to 0 \) yields the theorem.
5. Appendix

In this appendix we give a brief summary of the formulas needed in the proof of Theorem 1.3 based on notes provided by Nicholas Edelen. Although they are all standard, due to differences in notation and convention we have decided to present a summary of the calculations.

Let \((X, g)\) be a Riemannian manifold of dimension \(n + 1\), and \(\nabla\) denote the Riemannian connection. Let \(Y\) be a smooth manifold of dimension \(n\), and consider a one-parameter family of smooth immersions \(F_t : (-\epsilon, \epsilon) \times Y \to X\). Let \(h = (F_t)^* g\) be the induced metric on \(Y\), and \(\nabla_Y\) the corresponding connection.

Let \(V\) denote the variation field of \(F_t\):

\[
V = \left. \frac{d}{dt} F_t \right|_{t=0}.
\]

Eventually we will assume that \(F_t\) is a normal variation; i.e., if \(\nu\) is a choice of unit to \(Y\) then there is a function \(f \in C^\infty(Y)\) such that \(V = f \nu\).

Let \(\{x^1, \ldots, x^n\}\) be local coordinates near a point \(0 \in Y\). They induce coordinates on \(F_t(Y)\) defined via \((t, x^1, \ldots, x^n) \mapsto F_t(x^1, \ldots, x^n)\), and we have the corresponding coordinate vector fields \(\{\partial_1, \ldots, \partial_n\}\), along with \(\partial_t = V\). Let

\[
h_{\alpha\beta}(t, x) = g_{F_t(Y)}(\partial_\alpha, \partial_\beta).
\]

Then

\[
h'_{\alpha\beta} = \left. \frac{\partial}{\partial t} h_{\alpha\beta} \right|_{t=0}
\]

\[
= g(\nabla_{\partial_t} \partial_\alpha, \partial_\beta) + g(\partial_\alpha, \nabla_{\partial_t} \partial_\beta)
\]

\[
= g(\nabla_{\partial_\alpha} V, \partial_\beta) + g(\partial_\alpha, \nabla_{\partial_\beta} V).
\]

If \(V = f \nu\), then this becomes

\[
h'_{\alpha\beta} = f g(\nabla_{\partial_\alpha} \nu, \partial_\beta) + g(\partial_\alpha, \nabla_{\partial_\beta} \nu).
\]

Given a choice of normal \(\nu\) our definition of the second fundamental form of \(Y\) is

\[
L(\partial_\alpha, \partial_\beta) = g(\nu, \nabla_{\partial_\alpha} \partial_\beta) = -g(\nabla_{\partial_\alpha} \nu, \partial_\beta).
\]

Therefore, by (74) we conclude

\[
h'_{\alpha\beta} = -2f L_{\alpha\beta}.
\]

By the standard formula for the inverse, this implies

\[
(h^{\alpha\beta})' = 2f L^{\alpha \gamma} L^{\beta \gamma}.
\]
By our definition of second fundamental form,

\[ L'_{\alpha\beta} = \frac{\partial}{\partial t} L_{\alpha\beta} \bigg|_{t=0} = g(\nabla_{\partial\alpha} \nu, \nabla_{\partial\alpha} \partial_\beta) + g(\nu, \nabla_{\partial\alpha} \nabla_{\partial\alpha} \partial_\beta). \]  

(78)

The first term on the right is easily seen to vanish, since \(0 = \partial_t g(\nu, \nu) = 2 g(\nabla_{\partial\alpha} \nu, \nu)\) implies that

\[ g(\nabla_{\partial\alpha} \nu, \nabla_{\partial\alpha} \partial_\beta) = -L_{\alpha\beta} g(\nabla_{\partial\alpha} \nu, \nu) = 0. \]  

(79)

For the second term, we commute derivatives to get

\[ g(\nu, \nabla_{\partial_\alpha} \nabla_{\partial_\alpha} \partial_\beta) = g(\nu, \nabla_{\partial_\alpha} \nabla_{\partial_\alpha} \nu) + R(V, \partial_\alpha, \partial_\beta, \nu) \]

(80)

where \(R\) is the curvature tensor of \(g\). If \(V = f \nu\) then by (79) and (80), (78) simplifies to

\[ L'_{\alpha\beta} = g(\nu, \nabla_{\partial_\alpha} \nabla_{\partial_\alpha} \nu) + f R(\nu, \partial_\alpha, \partial_\beta, \nu), \]

(81)

where in the last line we used the fact that \(\partial_\alpha g(\nu, \nu) = 0\). Using this fact again we also find

\[ g(\nu, \nabla_{\partial_\alpha} \nabla_{\partial_\alpha} \nu) = -g(\nabla_{\partial_\alpha} \nu, \nabla_{\partial_\alpha} \nu). \]  

(82)

It follows from the definition of the second fundamental form that

\[ \nabla_{\partial_\alpha} \nu = -L_\gamma^\alpha \partial_\gamma, \]

hence

\[ -g(\nabla_{\partial_\alpha} \nu, \nabla_{\partial_\beta} \nu) = -L_\alpha^\gamma L_{\beta\gamma}. \]

Substituting this into (82) and combining with (81), we arrive at

\[ L'_{\alpha\beta} = \nabla_\alpha^\gamma \nabla_\beta^\nu f - f L_\gamma^\alpha L_{\beta\gamma} + f R(\nu, \partial_\alpha, \partial_\beta, \nu). \]  

(83)

For the variation of the mean curvature \(H = h^\alpha{}_{\alpha\beta} L_{\alpha\beta}\) we use (77) and (83) to obtain

\[ H' = \Delta_Y f + \left(|L|^2 + \text{Ric}(\nu, \nu)\right) f. \]  

(84)

Finally, using the standard formula for the derivative of the volume form, we have

\[ (dv_h)' = -f H dv_h. \]  

(85)
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