LINEAR TRIANGLE DYNAMICS: THE PEDAL MAP AND BEYOND.

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Abstract. We present a moduli space for similar triangles, then classify triangle maps \( f \) that arise from linear maps on this space, with the well-studied pedal map as a special case. Each linear triangle map admits a Markov partition, showing that \( f \) is mixing, hence ergodic.

1. Introduction

Because our main result exploits the symmetries a triangle can possess, we collect a few definitions. For us, a triangle \( T = z_1 z_2 z_3 \) is an ordered triple \((z_1, z_2, z_3)\) of distinct points in the complex plane \( \mathbb{C} \), and \( T \) is flat if \( z_1, z_2, z_3 \) are collinear. Let \( T \) be the collection of triangles. Our convention is to draw edges cyclically from \( z_i \) to \( z_{i+1} \), (indices modulo 3). Two triangles are similar if the sets of edge lengths are proportional. The normalized principal angle between complex numbers \( z_1, z_2 \) is given by

\[
\theta_{z_1, z_2} = \frac{1}{\pi} \arccos \left( \frac{\text{Re}(z_1 \overline{z_2})}{|z_1||z_2|} \right), \quad 0 \leq \theta_{z_i, z_{i+1}} \leq 1
\]

and the (normalized) interior angle of a triangle at vertex \( z_i \) is \( \alpha_i := \theta_{z_{i+1} z_i} - \theta_{z_{i+1} z_{i+2}} \). The shape of \( T \) is the ordered triple of interior angles \((\alpha_1, \alpha_2, \alpha_3)\) with \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \). By the law of sines, two triangles \( T_1, T_2 \) are similar if the shape of \( T_1 \) is equal to the shape of \( T_2 \), up to permutation of vertices. If we let \( S \) be the group generated by affine transformations \( z \mapsto az + b, a, b \in \mathbb{C}, a \neq 0 \) and the group \( \Sigma_3 \) of permutations on 3 letters, then the diagonal action of \( S \) on \( \mathbb{C}^3 \) partitions \( T \) into collections of similar triangles. Call \([T] \in T/S\) the similarity class of \( T \). By means of stereographic projection, we append to \( T \) degenerate triangles with a vertex at infinity. If we declare the interior angle at infinity to be 0 then the interior angles of degenerate triangles sum to 1 and the \( S \)-action formulation of similarity extends to these triangles as well. We say that a triangle map \( f : T/S \to T/S \). In other words, a function \( g : T \to T \) is a triangle map if for every \( s \in S \) there exists \( u \in S \) such that \( T_1 = sT_2 \) implies \( f(T_1) = uf(T_2) \). We introduce three sets to describe similarity classes in \( T/S \). Define \( A \) be the plane

\[
A = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \mid \alpha_1 + \alpha_2 + \alpha_3 = 1\},
\]

\( A_p \subset A \) be the set of interior angles

\[
A_p = \{(\alpha_1, \alpha_2, \alpha_3) \in [0,1]^3 \mid \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0\},
\]

and \( D \subset A_p \) be the set of all ordered interior angles

\[
D = \{(\alpha_1, \alpha_2, \alpha_3) \in [0,1]^3 \mid \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq 0\}.
\]

Notice that the interior angle function establishes a bijection \( \phi : D \to T/S \); each shape \((\alpha_1, \alpha_2, \alpha_3) \in D\) describes a unique similarity class \([T] \in T/S \) by assigning \( \alpha_1 \) to a largest interior angle, then \( \alpha_2 \) to the next largest, then \( \alpha_3 \). Thus, for each triangle map \( f \) there exists a unique map \( \overline{f} : D \to D \) such that the following diagram commutes:

\[
\begin{array}{ccc}
D & \xrightarrow{T} & D \\
\downarrow_{\psi} & & \downarrow_{\psi} \\
T/S & \xrightarrow{f} & T/S
\end{array}
\]

and we refer to a triangle map by either \( f \) or \( \overline{f} \). Our interest is in tracking the shape of \( f^n([T]) \) for various \( f, n, \) and \([T] \). The paper is organized as follows. Section 2 reviews the construction of pedal triangles, whose triangle map \( P : D \to D \) we call the pedal map. In section 3 we show how to identify an element of \( A \) to a
shape in $D$ by describing a similarity class $\{T\}$ with nonprinciple angles. These identifications arise as the action of the wallpaper group $G = p6m$ on $A$, with $D$ homeomorphic $A/G$. Borrowing terminology from toral automorphisms, call $f : D \rightarrow D$ a linear triangle map if $f$ is the quotient map of an invertible linear map $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that leaves $A$ invariant and preserves the identifications induced by $G$. Coordinatizing, we call $M$ an angle transition matrix (ATM) of $f$. Section 4 contains our main theorem, which classifies all possible ATM’s.

**Theorem 1.** [Classification of angle transition matrices] Suppose $M : A \rightarrow A$ is an ATM. Then there exists $g \in G$ such that $gM$ is either:

1. **(Type I)** A circulant and symmetric matrix

$$
\begin{bmatrix}
c_0 & c_1 & c_1 \\
c_1 & c_0 & c_1 \\
c_1 & c_1 & c_0
\end{bmatrix}
$$

with $c_0 \in \mathbb{Z}, c_1 \in \mathbb{Z}, c_0 + c_1 + c_1 = 1$

2. **(Type II)**

$$
T_w^{-1} \begin{bmatrix}
c_0/3 & c_1/3 & c_1/3 \\
c_1/3 & c_0/3 & c_1/3 \\
c_1/3 & c_1/3 & c_0/3
\end{bmatrix}
$$

with $c_0 \in \mathbb{Z}, c_1 \in \mathbb{Z}, c_0 + c_1 + c_1 = 1, c_0 \text{ congruent to } 1 \text{ mod } 3$, and $T_w$ is the matrix that acts on $A$ by translation in the direction of $w = (1/3, 1/3, -2/3)$.

3. **(Type III)**

$$
\begin{bmatrix}
0 & k & -k \\
-k & 0 & k \\
k+1 & -k+1 & 1
\end{bmatrix}
$$

with $k \in \mathbb{Z}$

The three types are displayed in Figure 1. Many articles, some appearing in the *Monthly*, have studied the measurable dynamics of the pedal map $P$, showing $P$ is ergodic [Un90], mixing [Un90], and semiconjugate to a Bernoulli shift on 4 symbols [Un90, La90]. Using the classification (Theorem 1) and well-known results about Markov partitions all such triangle maps are semiconjugate to a one-sided Bernoulli shift, hence mixing and ergodic (2). Since the (extended) pedal map is Type I linear (1), 2 includes the dynamics of $P$ as a special case.

2. **The Pedal Map**

Our motivating example begins with a reference triangle $T_0 = x_1 x_2 x_3$. For each vertex $x_i \in \mathbb{C}$ of $T_0$, consider the line $\overrightarrow{x_{i+1} x_{i+2}}$ passing through the other two points (indices modulo 3). Let $y_1$ be the unique point of intersection between $\overrightarrow{x_{i+1} x_{i+2}}$ and its perpendicular through $x_i$. Adjoining edges between the feet of the three perpendiculars forms the first pedal triangle $T_1 = \Delta y_1 y_2 y_3$ from $T_0$. Iterating $n$ times creates the $n$-th pedal triangle $T_n$ from $T_0$. Hobson correctly wrote down a formula for the (normalized) interior angles of $T_1$ in terms of $T_0$, namely,

$$
\alpha_1 = 1 - 2\alpha_0, \beta_1 = 1 - 2\beta_0, \gamma_1 = 1 - 2\gamma_0
$$
Figure 2. The figure on the left shows a reference triangle with first pedal triangle. The figure on the right is the space $A_p$ of interior angles, where the central triangle represents acute triangles and three outer triangles for obtuseness in $\alpha, \beta, \gamma$ respectively.

when $T_0$ is acute, and

$$\alpha_1 = 2\alpha_0 - 1, \beta_1 = 2\beta_0, \gamma_1 = 2\gamma_0$$

if $\alpha_0$ is obtuse, with similar formulas when $\beta_0, \gamma_0$ is obtuse. Notice that the formula degenerates for right triangles, when the feet of the perpendiculars coincide. Hobson then went on to write down a formula for the interior angles $(\alpha_n, \beta_n, \gamma_n)$ of the $n$th pedal triangle $T_n$. Kingston and Synge [KSSS] recognized that Hobson’s formula for $T_n$ was flawed, and in correcting it they proved that the sequence $\{(\alpha_n, \beta_n, \gamma_n)\}$ of interior angles is eventually periodic if angles of $T_0$ are rational and not dyadic. They introduced a simple way of parametrizing the shape of triangles by the set $A_p$, pictured in Figure footnote 1. The 3 perpendiculars of $T_0$ meet at a point, called the orthocenter $O$ of $T$. Observe that Hobson’s formulas define a piecewise four-to-one mapping $P : A_p \to A_p$, because the triangles $Oz_2z_3, z_1Oz_3, z_1z_2O$ also have first pedal triangle equal to $T$. Following [KSSS], we call any one of these four preimages an ancestor of $T$. Each nonflat $T_0$ has exactly one acute ancestor which we call the antipedal triangle $T_{-1}$ of $T_0$. If we include similarity by permuting vertices then Hobson’s formulas define a map $P : D \to D$ which we also call the pedal map.

Our intuition stems from the following two observations. First, the substitution $1 = \alpha_n + \beta_n + \gamma_n$ into Hobson’s formula yields, when $T_n$ is acute,

$$\begin{align*}
\alpha_{n+1} &= -\alpha_n + \beta_n + \gamma_n \\
\beta_{n+1} &= +\alpha_n - \beta_n + \gamma_n = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{bmatrix}.
\end{align*}$$

so that the angles of $T_{n+1}$ are linear combinations of $T_n$. This was done implicitly in [Ma10] Section 3]. If we let $M$ be the matrix

$$M = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

then $M$ agrees with $P$ on the part of $A_p$ that describes acute triangles. While $M$ is invertible, $P$ is not. We leave it to the reader to verify that the angles $\alpha_{-1}, \beta_{-1}, \gamma_{-1}$ of $T_{-1}$ are given by

$$\begin{bmatrix} \alpha_{-1} \\ \beta_{-1} \\ \gamma_{-1} \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{bmatrix}.$$

Our second observation is that

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Periodicity in iterated pedal triangles was observed for specific $T_0$ in [Tu33] and [Ta45].
and we see the inverse nature between pedaling and antipedaling in the angles of $T$ when $T$ is acute. Since $M^{-1}$ is a regular Markov chain with steady state vector $(1/3, 1/3, 1/3)$ and two eigenvalues of $-1/2$, the antipedal map is a contraction map on $A_p$, and $M^{-1}A_p$ excludes obtuse triangles. Indeed, for each $T_0$, $T_{-n}$ tends to equilateral as $n \to \infty$. Inverting, $M$ fixes $(1/3, 1/3, 1/3)$ and expands $A_p$ by a factor of $-2$, sending obtuse triples in $A_p$ to points in $A$ but outside $A_p$. Since $A_p$ describes all triangle shapes (up to permutation), we seek to identify triples in $A \setminus A_p$ to points in $A_p$ that describe the same similarity class. For $T_n$ obtuse in $\alpha_n$, again substitute $1 = \alpha_n + \beta_n + \gamma_n$ into Hobson’s formula to get

$$
\begin{bmatrix}
\alpha_{n+1} \\
\beta_{n+1} \\
\gamma_{n+1}
\end{bmatrix} =
\begin{bmatrix}
1 & -1 & -1 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
\alpha_n \\
\beta_n \\
\gamma_n
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
\alpha_n \\
\beta_n \\
\gamma_n
\end{bmatrix}.
$$

In section 2, we shall see that

$$R_n :=
\begin{bmatrix}
-1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
$$

arises naturally as an action on $A$, identifying points that describe the same similarity class in $T/S$.

When studying the measurable dynamics of $P$, previous authors ignored those $T_0$ which were eventually flat since that subset of $A_p$ is a set of (normalized Lebesgue) measure zero. Manning (Ma10) augmented $A_p$ to include nonprincipal (at least one of $\alpha, \beta, \gamma < 0$) and flat $T_0$ by identifying triples via orientation changes. They then showed that $P$ could be continuously extended to flat triangles and found the limit point of the circumcenters of $T_n$ as $n \to \infty$ (Ma10, Theorem 3.2). In view of observation 6, we recast and complete the identifications carried out in Ma10, in the next section.

### 3. Moduli space for triangles

Loosely speaking, a moduli space is a collection of numerical ingredients that encode the salient features of geometric objects. We have already seen one moduli space for $T/S$: the set $D$. Let $\epsilon_1, \epsilon_2, \epsilon_3$ be the standard coordinate vectors of $\mathbb{R}^3$. Whether we treat $v \in \mathbb{R}^3$ as a row or column vector will be clear from context. Define the equivalence relation $\sim$ on $A$ by writing $v = (\alpha_1, \alpha_2, \alpha_3), v' = (\alpha_1', \alpha_2', \alpha_3')$. Then we identify $v \sim v'$ if and only if, $\alpha_i \equiv -\alpha_{i-1}'$ mod 1 or $\alpha_i \equiv -\alpha_{i-1}'$ mod 1, $\sigma \in \Sigma_3$

and say that $v$ is a re-expression of $v'$ if $v \sim v'$. The reader can verify that $\sim$ is an equivalence relation that identifies points in $A$ to shapes in $D$. We provide a geometric explanation of $\sim$. Let $T = z_1z_2z_3$ be a triangle with shape $(\alpha_1, \alpha_2, \alpha_3) \in A_p$. Permuting the vertices of $T$ by any $\sigma \in \Sigma_3$ will not change the shape of $T$. The induced $\Sigma_3$-action on $\mathbb{R}^3$, generated by

$$P_{12} =
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
P_{13} =
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix},
P_{23} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
$$

leaves $A$ invariant, reflecting across the planes $x - y = 0, x - z = 0, y - z = 0$. These planes intersect $A$ through the medians of $A_p$. Thus $\{\sigma D \mid \sigma \in \Sigma_3\}$ tiles $A_p$ by 6 copies of $D$ and $v \sim w$ in $A_p$ if and only if $v = \sigma w$ for some $\sigma \in \Sigma_3$.

Identifying points between $A_p$ and $A \setminus A_p$ requires two separate identifications. Let $(\alpha_1, \alpha_2, \alpha_3) \in A$. If we treat $\alpha_i \geq 0$ as rotation about the vertex $z_i$ (either clockwise of counterclockwise, depending on $T$) that takes the line $\frac{z_i + z_1}{2}$ to $\frac{z_i + z_2}{2}$ (indices modulo 3), then $\alpha_i \pm k, k \in \mathbb{Z}$, describes another rotation with the same orientation as $\alpha_1$ that takes $\frac{z_i + z_1}{2}$ to $\frac{z_i + z_2}{2}$. Notice that the difference of two vectors in $A$ lie in the orthogonal hyperplane $(1/3, 1/3, 1/3)$ of $T$. So two vectors $v, v'$ in $A$ describe the same similarity class if $v - v' \in \mathbb{Z}\{e_1 - e_2, e_2 - e_3\}$: we recover interior angles by $\alpha_i \equiv \alpha_i'$ mod 1. So $v \sim v'$. Notice further, if $(\alpha_1, \alpha_2, \alpha_3) \in A$ then rotation about $z_i$ by either $-\alpha_i$ (opposite orientation as $\alpha_i$) or $1 - \alpha_i$ (same orientation as, but supplementary to, $\alpha_i$) takes $\frac{z_i + z_2}{2}$ to $\frac{z_i + z_1}{2}$. Ensuring that the angles sum to 1, we make the identification

$$(\alpha_1, \alpha_2, \alpha_3) \sim (-\alpha_1, 1 - \alpha_2, 1 - \alpha_3).$$
In this case, we recover the interior angles of $T$ by $\alpha_i \equiv -\alpha'_i \mod 1$. Substituting $\alpha_1 + \alpha_2 + \alpha_3 = 1$ into (9),

\begin{equation}
(\alpha_1, \alpha_2, \alpha_3) \sim (-\alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2).
\end{equation}

or

\[
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix}
\sim
\begin{bmatrix}
-1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix}
= R_{\alpha_1}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix}.
\]

We have found the matrix $R_{\alpha_1}$ in (9). Making similar observations for $\alpha_2, \alpha_3$, we identify points in $A$ via the actions of

\begin{align}
R_{\alpha_1} &= \begin{bmatrix}
-1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix} = P_{23} \begin{bmatrix}
-1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix} \\
R_{\alpha_2} &= \begin{bmatrix}
0 & 1 & 1 \\
0 & -1 & 0 \\
1 & 1 & 0
\end{bmatrix} = P_{13} \begin{bmatrix}
1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 1 & 1
\end{bmatrix} \\
R_{\alpha_3} &= \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} = P_{12} \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\end{align}

Geometrically, $R_{\alpha_1}$ is $P_{23}$ composed with a reflection in $\mathbb{R}^3$ across the plane that passes through $x = 0$ and leaves $A$ invariant. Notice that $R_{\alpha_1}^2 = R_{\alpha_2}^2 = R_{\alpha_3}^2 = id$, and $R_{\alpha_1}R_{\alpha_2}R_{\alpha_3}$ translates $A$ by $e_1 - e_2, e_2 - e_3$ respectively. If we let

\begin{equation}
R = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\end{equation}

then $R$ acts on $A$ by reflection across the line $z = 0$ in $A$. The reader can quickly verify that $R_{\alpha_1} = P_{23}P_{13}RP_{13}, R_{\alpha_2} = P_{23}RP_{13}P_{12}, R_{\alpha_3} = P_{12}R$. If we let $G$ be the group generated by the involutions $P_{13}, P_{23}$, and $R$, then each $g \in G$ leaves $A$ invariant. Each generator of $G$ reflects $D$ along one of its boundary edges and the $G$-action on $A$ is generated by reflections across lines that are intersections of $A$ with the planes

\begin{equation}
x, y, z \in \mathbb{Z}, x - y, y - z, z - x \in \mathbb{Z}
\end{equation}

We have shown that $v \sim w$ if and only if $w = gv$ for some $g \in G$. In other words, the $G$-orbit $Gv$ is the set of all re-expressions of $v \in A$. Under these identifications, the set of orbits $A/G$ is a quotient of $A$ by the rank 2 lattice $\Lambda := e_1 + \mathbb{Z}\{e_1 - e_2, e_2 - e_3\}$, along with the 12 identifications from $v \sim R_{\alpha_1}v, v \sim \sigma v$ ($\sigma \in \Sigma_3$) as seen in Figure eq. (11)(a). Every $G$-orbit $O$ has exactly one representative $p \in D$, so we call $D$ the fundamental domain of $A/G$. Observe $GD := \{gp \mid g \in G, p \in D\} = A$ and $D$ is the (closed) 30 - 60 - 90 triangle with vertices at $v_1 = b = (1/3, 1/3, 1/3)$ and $v_2 = (1, 0, 0), v_3 = (1/2, 1/2, 0)$. Thus the $G$-action on $A$ is that of the wallpaper group $p6m$. For any $v \in A$, the point group $H_v \subset G$ is the subgroup of $G$ that stabilizes $v$. As $GD = A$ and $gH_vg^{-1} = H_{gv}$, $H_v$ is conjugate to exactly one $H_p, p \in D$. Since $G$ is generated by reflections along the 3 boundary lines of $D$, $H_p$ is isomorphic.
to a dihedral group. Moreover, $|H_p| = 1, 2, 4, 6$ or 12 depending on whether $p$ lies, respectively, in the interior $D$, on an edge of $D$ (but not a vertex), or $p = v_3, b,$ or $v_2$. The next key lemma, the local-point group property articulates the $G$-action on $A$. Denote by $B_\epsilon(v)$ the ball of radius $\epsilon$ centered at $v$.

**Theorem 2.** [Local point group property] Let $G = p6m$ act on $A$ by reflection across the lines in $[14]$. There exists a constant $\epsilon_0 > 0$, depending only on $G$, such that, if $v_1, \ldots, v_k \in A$ are contained in some orbit and some ball $B_{\epsilon_0}(v)$ then $v_1, \ldots, v_k$ are elements in the orbit of some point group $H_v$ of $G$.

**Proof.** Since each $g \in G$ is an isometry and $GD = A$, we need only consider balls with center in the fundamental domain $D$. Consider the vector $v_1 = (1/3,1/3,1/3)$, and let $c_1 = \sqrt{1/6}$ so that $B_{c_1}(v_1)$ is tangent to the line $\overline{v_2 v_3}$. Since $B_{c_1}(v_1)$ is contained in the region $H_{v_1}D$ and $H_{v_1}$ acts transitively on $H_{v_1}D$, any collection of points in $B_{c_1}(v_1)$ that are $G$-equivalent must be $H_{v_1}$-equivalent. Thus, $G$-equivalence implies $H_{v_1}$ equivalence for any collection of points in a ball of radius $B_\epsilon(w)$, so long as $0 < \epsilon < \sqrt{1/6}$ and $w \in B_{c_1 - \epsilon}(v_1)$. We can make analogous conclusions for $v_2, e_2 = \sqrt{1/2}$ and $v_3, c_3 = \sqrt{1/8}$. It remains to show $B_{c_1 - \epsilon_0}(v_1), B_{c_2 - \epsilon_0}(v_2), B_{c_3 - \epsilon_0}(v_3)$ cover $D$ for some $\epsilon_0 > 0$. Consider the barycenter $b = (11/18, 5/18, 2/18)$ of $D$, and let $d_1, d_2, d_3$ be the distances from $b$ to $v_1, v_2, v_3$ respectively. Explicit calculation shows $d_i < c_i, i = 1, 2, 3$. By convexity, $B_d(v_1)$ contains the convex hull of $v_1, (v_1 + v_{i+1})/2, (v_1 + v_{i+2})/2$ and $b, i = 1, 2, 3$, indices mod 3. These 3 convex regions cover $D$, seen by expressing $w \in D$ as a convex combination of $v_1, v_2, v_3$. Thus $\epsilon_0 = \min\{c_1 - d_1, c_2 - d_2, c_3 - d_3\} = c_1 - d_1 = \frac{3\sqrt{3}}{3\sqrt{6}}$. The theorem follows. \qed

**Remark.** Theorem 2 shows that $G$ acts properly discontinuously on $A$, giving $D$ is has the structure of the orbifold $A/G$ with covering space $A$ [14, Thm 13.2.1]. Each open set $U \subset D$ in the relative topology lifts to an open set $G\ U \subset A$. The identifications unfold $D$ along the lines of reflection in $[14]$.

### 4. Linear maps on $A/G$

A function $M : A \to A$ preserves re-expression if $v \sim w$ implies $Mv \sim Mw$. Any function that preserves re-expression induces a map on $A/G$, hence a triangle map on $T/S$. We say that a triangle map $f$ is linear if $f : D \to D$ is the quotient map of an invertible linear map $M : \mathbb{R}^3 \to \mathbb{R}^3$ that leaves $A$ invariant and the restriction of $M$ to $A$ preserves re-expression. We shall henceforth refer to $M$ as an angle transition matrix, writing ATM for short. Since $M$ leaves $A$ invariant, each column of an ATM sums to 1. Recall that $A_p$ is the equilateral triangle in $A$ with vertices $e_1, e_2, e_3$. As $M$ is invertible, linear, and leaves $A$ invariant, $MA_p$ is a nonflat triangle in $A$ and the columns of $M$ are the vertices of $MA_p$. The geometry of the moduli space $A/G$ (Theorem 2) imposes strong conditions on the type of triangle $MA_p$, capable of yielding a classification. Our first theorem requires two observations. Define $\Lambda = e_1 + \mathbb{Z}\{e_2 - e_1, e_3 - e_2\}$ to be the rank two sublattice of $A$ consisting of integer entries. Notice that each $v \in A$ is met by 6 lines of $[14]$. Since $\Lambda = Ge_1$ and $H_{e_1}$ is the only point group from $D$ with order 12, $|H_{e_1}| = 12$ if and only if $v \in \Lambda$. Take any point group $H_v$ of $G$ and consider an $H_v$-orbit, $O$. The average $a := \frac{1}{|O|} \sum_{w \in O} w$ is a fixed point of $H_v$. If $|H_v| > 2$, then $v$ is the only fixed point of $H_v$, so $a = v$.

**Lemma 1.** For any ATM $M$, $M\Lambda \subset \Lambda$.

**Proof.** Let $\lambda \in \Lambda$ be arbitrary. Since the point group $H_\lambda$ is $G$-conjugate to $H_{e_1}$, $|H_\lambda| = 12$. Let $\epsilon_0$ be the constant of Theorem 2. As $M$ is linear, $M$ is uniformly continuous. Thus, there exists $\delta > 0$ such that, for all $w_1, w_2 \in A$, $|w_1 - w_2| < \delta$ implies $|Mw_1 - Mw_2| < \epsilon_0$. Select any orbit $O$ of $H_\lambda$ that has 12 elements $p_1, \ldots, p_{12}$ contained in $B_{\min\{\epsilon_0, \delta\}}(\lambda)$ which average to $\lambda$; such an $O$ exists. As $M$ is invertible, $Mp_1, \ldots, M_{p_{12}}$ are 12 distinct points contained in $B_{\epsilon_0}(M\lambda)$. Since $M$ preserves re-expression, $GMp \subset GMp$ so $MP_1, \ldots, MP_{12}$ belong to some $G$-orbit. By Theorem 2, $MO = \{MP_1, \ldots, MP_{12}\}$ belongs to an orbit of some point group $H_v$. By the orbit-stabilizer theorem, the cardinality of an $H_v$-orbit is at most $|H_v|$. Since $|H_v| \leq 12$ and $|MO| = 12$, $|H_v| = 12$. The preceding paragraph shows $v \in \Lambda$, and that the average

$$a = \frac{1}{12} \sum_{i=1}^{12} M_p = M \left(\frac{1}{12} \sum_{i=1}^{12} p_i\right) = M\lambda$$

must be equal to $v$. Thus $M\lambda \subset \Lambda$, proving the proposition. \qed
Specilaizing Lemma 1 to $e_1, e_2, e_3 \in \Lambda$ shows that the entries of $M$ must be integers. Now, let $R \subset A$ be the set of all $w \in A$ with nontrivial point group, meaning $w \in R$ if and only if $w$ is fixed point of at least one reflection in $G$. Thus, $R$ is the union of lines described in \([14]\).

**Lemma 2.** For any ATM $M$, $MR \subset R$.

*Proof.* The point group of any $w \in A \setminus R$ is trivial. Since $M$ is uniformly continuous and preserves re-expression, we must have $MR \subset R$ by Theorem 2. \qed

The local point group property (Theorem 2) imposed a strong condition on $M$ by looking at the corners of $A_p$; the next proposition examines the edges of $A_p$. Recall that $v_1 = b = (1/3, 1/3, 1/3)$ is the barycenter of $A_p$.

**Proposition 1.** The columns of $M$ are vertices of an equilateral triangle in $A$. Moreover, a subgroup of $H_{Mb} \subset G$ permutes the columns of $M$.

*Proof.* By Lemma 1 the vertices of $MA_p$ lie in $\Lambda$. Since the boundary of $A_p$ is a subset of $R$, Lemma 2 states that each edge of $MA_p$ must lie on one of the 12 lines of reflection through a vertex of $MA_p$. Taking pairs of edges, the interior angles of $MA_p$ are positive integer multiples of $\pi/6$. Up to permutation, the angles of $MA_p$ are either $(\pi/6, \pi/3, \pi/2), (\pi/3, \pi/3, \pi/3)$, or $(\pi/6, \pi/6, 2\pi/3)$. Since a point $v \in A$ on a median of $A_p$ is fixed by the reflection in $G$ that transposes the equal components, $v \in R$. As $MR \subset R$, $Mv \in R$. By linearity, Lemma 2 shows $MA_p$ cannot have an angle of $\pi/6$. So the vertices of $MA_p$ must form an equilateral triangle. Again by linearity, the medians of $MA_p$ are concurrent at $Mb$. Since $MA_p$ is equilateral, reflections about its medians permute the vertices of $MA_p$, fixing $Mb$. By Lemma 2 each median of $MA_p$ lies on a line in $R$, so the medial reflections are elements of $G$. So $H_{Mb}$ contains the subgroup of medial reflections which permute the vertices of $MA_p$. \qed

**Definition.** A $3 \times 3$ matrix $M$ is *circulant* if

$$M = \begin{bmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{bmatrix}$$

and $M$ is *symmetric* if $M^T = M$.

If $M$ is circulant and symmetric, then $M$ looks like

$$M = \begin{bmatrix} c_0 & c_1 & c_1 \\ c_1 & c_0 & c_1 \\ c_1 & c_1 & c_0 \end{bmatrix}.$$ 

To complete the classification, notice that each element $g \in G$ is the rather uninteresting ATM that re-expresses the angles of a triangle. Since each $g \in G$ corresponds to the identity on $T/S$, we say $M, gM$ are *equivalent* ATM’s. As $|H_b| = 6$, uniform continuity of $M$ and Theorem 2 imply $|H_{Mb}| = 6$ or 12. We are now ready to prove the main Theorem.

**Proof of Theorem 4.** We draw the boundary of $MA_p$, citing appropriate theorems along the way. By Lemma 1, $Me_1 \in \Lambda$. As $MR \subset R$ (Lemma 2), $Me_1 - Me_2$ is a integer multiple of one of the 12 directions

$$e_1 - e_2, e_2 - e_3, e_3 - e_1, -2e_1 + e_2 + e_3, e_1 - 2e_2 + e_3, e_1 + e_2 - 2e_3$$

that span lines of reflection through $Me_1$. By $1 MA_p$ is equilateral whose medians lie in $R$. Computing the angles between any of the 12 vectors above, replacing $M$ by $hM$ if necessary ($h \in H_{Mb}$), the boundary directions take one of the two following forms:

**Case 1.** $Me_1 - Me_2 = k(e_1 - e_2), Me_2 - Me_3 = k(e_2 - e_3), Me_3 - Me_1 = k(e_3 - e_1)$. We separate into subcases based on whether $|H_{Mb}| = 6$ or 12.

**Case 1.1.** $|H_{Mb}| = 6$. 


Recall that $b$ is the only point in the fundamental domain $D$ with point group of order 6. Thus $|H_{Mb}| = 6$ implies $Mb \in Gb$, so there exists $g \in G$ such that $gM$ is matrix with $b$ as a fixed point. By Lemma [1], $gMe_1 \in \Lambda$, and by Lemma [2] the medial line from $gMb = b$ to $gMe_1$ must lie on one of the lines $x - y = 0, x - z = 0, y - z = 0$ in $R$ through $b$. Thus, $gMe_1$ has integer components (Lemma [1]) with two components equal. By [1] $H_{Mb}$ permutes the columns $gMe_1, gMe_2, gMe_3$, and $H_{Mb} = H_b$ is the group of permutation matrices in $\mathbb{R}^3$ [8]. Thus there exists $Q \in H_b$ such that $QgM$ is circulant and symmetric, i.e., a Type I matrix [11].

**Case 1.2.** $|H_{Mb}| = 12$.

Thus $Mb \in \Lambda$. Since the translation subgroup $Z\{e_1 - e_2, e_2 - e_3\}$ acts transitively on $\Lambda$, there exists a translation $t \in G$ sending $Mb$ to $e_3$. Consider the subset of $R' \subset R$ of reflection lines given by

$$ x - y \in \mathbb{Z}, y - z \in \mathbb{Z}, x - z \in \mathbb{Z}. $$

Recall $w = (1/3, 1/3, -2/3)$ and the translation $T_w$ sending $e_3$ to $b$. Notice that $T_w$ leaves $A$ invariant but $T_w \notin G$. Nevertheless, straightforward calculations verify that the medians of $MA_p$, by the assumption of column differences, are subsets of $R'$. More straightforward calculations show $tR' = R', T_wR' = R'$, and that $T_w\Lambda$ is the subset of elements whose point group is order 6, consisting of triples of third integers $(a/3, b/3, c/3)$ in $A$ with $a \equiv b \equiv c \equiv 1 \mod 3$. Thus, the vertices $T_w tMe_i$ are elements of $T_w A$ that lie on the lines $y - z = 0, x - z = 0, y - z = 0$ in $R'$ through $b$. As $(T_w tM)A_p$ is equilateral,

$$ T_w tM = \begin{bmatrix} c_0/3 & c_1/3 & c_1/3 \\ c_1/3 & c_0/3 & c_1/3 \\ c_1/3 & c_1/3 & c_0/3 \end{bmatrix} $$

yielding [2].

**Case 2.** $Me_1 - Me_2 = k(1, 1, -2), Me_2 - Me_3 = k(-2, 1, 1), Me_3 - Me_1 = k(1, -2, 1)$.

Observe,

$$ Mb = \frac{1}{3} (Me_1 + Me_2 + Me_3) = \frac{1}{3} (Me_1 + Me_1 - k(1, 1, -2) + Me_1 + k(1, -2, 1)). $$

Thus $Mb \in \Lambda$, so $|H_{Mb}| = 12$. Let $b^\perp$ be the orthogonal hyperplane to $b$ in $\mathbb{R}^3$. Direct calculation shows $Me_i - Mb = k(e_{i+2} - e_{i+1})$ (indices modulo 3), thus $Me_i - Mb \in b^\perp$. Similar to subcase [122] there exists a translation $g \in G$ of $A$ with $gMb = e_3$. Since $g$ is given by matrix multiplication that acts as the identity on $(1/3, 1/3, 1/3)^t$, we have $gMe_i = gMb + gb(e_{i+2} - e_{i+1}) = e_3 + k(e_{i+2} - e_{i+1})$, or

$$ gM = \begin{bmatrix} 0 & k & -k \\ -k & 0 & k \\ k + 1 & -k + 1 & 1 \end{bmatrix} $$

yielding [3].

**Corollary 1.** The pedal mapping $P : D \to D$ may be defined where $P(v)$ is flat so that $P$ is linear.

**Proof.** The observations in section [2] verify that $P : D \to D$ agrees with the quotient of the Type 1 ATM

$$ M = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} $$

for triangles $T$ whose first pedal triangle is not flat. This ATM shows precisely how to define the $P$ when $P([T])$ is flat so that the extended map is linear. \hfill \square

The dynamics of linear triangle maps now follow from well known results (cf [KH97] pp 80 156).

**Corollary 2.** Let $f : T/S \to T/S$, with ATM $M$. the image $MD$ of the fundamental domain $D$ is a 30 – 60 – 90 triangle in $A$ whose edges lie in $R$. Consequently, there exist $|\det(M)|$ elements of $GD$ which tile $MD$. The preimages of these $|\det(M)|$ triangles form a Markov partition of $D$, making $M$ semiconjugate to a one-sided Bernoulli shift on $|\det(M)|$ symbols. Thus $M$ is mixing, and hence, ergodic.

See [A93] for a detailed discussion of a Markov partition for $P$. 


5. Type I triangle maps

Theorem \[ \text{1} \] found triangle maps abstractly as angle transition matrices on \( A \). We offer one method for constructing triangles with prescribed ATM \( M \). We focus on Type I ATM’s, following the intuition for the pedal map developed in Section \[ \text{2} \]. In this section, we find it easier to work in \( A_p \) instead of \( D \). Let \( M \) be a nonidentity Type I ATM, i.e., \( M \) is circulant and symmetric, with integer entries and columns that sum to 1. Observe that \( M \) has one eigenvalue of 1 with eigenvector \( b = (1/3, 1/3, 1/3) \), and repeated eigenvalue \( c_0 - c_1 = 1 - 3c_1 \in Z \) with eigenvectors \( e_1 - e_2, e_2 - e_3 \). As \( c_1 \neq 0, M \) is expanding. Thus \( M^{-1} \)
is a contraction map on \( A \), hence \( A_p \), where \( M^{-1} A_p \) is an equilateral triangle with barycenter \( b \). Notice that \( M^{-1} \) does not preserve re-expression:, if \( v \) is a nearest neighbor to \( b \) with \( v \sim b \) then \( b = M b \sim M v \) because \( M \) is a contraction. We seek invertible matrices \( M_1 \cdots M_k \) such that \( M^{-1} = M_1^{-1} \cdots M_k^{-1} \) and each \( M_i \) is easily recognized by some triangle construction. Inverting, \( M = M_k \cdots M_1 \). If we allow for intermediate products \( M_i \) to not preserve re-expression, then any triangle map with Type I matrix can be constructed as a composition inverse processes. The intermediate constructions will be similar to the construction of Hofstadter triangles \[ \text{K92} \]. Let \( T_0 = ABC \) be a triangle with shape \((\alpha_0, \beta_0, \gamma_0) \in A_p \) and let \( 0 < r < 1 \). To construct \( T_{-1} \), rotate the line \( \overline{AB} \) about \( B \) and towards \( C \) and rotate the line \( \overline{AC} \) about \( C \) and towards \( B \) until the lines intersect at a point \( A’ \) in the interior of \( T_0 \), so that \( T_{-1} = A’BC \) has shape \((\alpha_{-1}, \beta_{-1}, \gamma_{-1}) = ((1-r)\alpha + (1-r)\beta + \gamma, r, r) \), or

\[
H_{A,r} = \begin{bmatrix}
\alpha_{-1} \\
\beta_{-1} \\
\gamma_{-1}
\end{bmatrix} = \begin{bmatrix}
1 & 1 - r & 1 - r \\
0 & r & 0 \\
0 & 0 & r
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\
\beta_0 \\
\gamma_0
\end{bmatrix}.
\]

(15)

Similarly, define \( H_{B,r} \) and \( H_{C,r} \). We call and \( H_{A,r}, H_{B,r}, H_{C,r} \) Hofstadter matrices. If we include the antipedal map \( P^{-1} \), we claim that for any Type I ATM \( M \), there exists \( r_1, r_2, r_3 \) such that

\[
M^{-1} = H_{A,r_1} H_{B,r_2} H_{C,r_3} P^{-1} \quad \text{or} \quad M^{-1} = H_{A,r_1} H_{B,r_2} H_{C,r_3}
\]

(15)

To see this, notice that \( M^{-1} \) remains circulant and symmetric. Thus \( M^{-1} A_p \) is an equilateral triangle with barycenter \( b \) and edges parallel to \( A_p \). Since the repeated eigenvalue of \( P \) is \(-2 < 0\), either \( MA_p \) or \( MP A_p \) is homothetic to \( A_p \) with positive scaling factor \( r \) that fixes \( b \). Notice, \( H_{A,r} A_p \) is an equilateral triangle with edges parallel to \( e_1, (1-r)e_1 + re_2, (1-r)e_1 + re_3 \), so \( H_{A,r} \) shrinks \( A_p \), fixing \( e_1 \) and the lines \( e_1 e_2 \) and \( e_1 e_3 \) with \( H_{A,r} e_2 e_3 \) parallel to \( e_1 e_2 \) and \( e_1 e_3 \). Similarly, \( H_{B,r} \) and \( H_{C,r} \) are defined.

Notice that for any \( r_2, H_{A,r_1} H_{B,r_2} A_p \) will be an equilateral triangle with \( H_{A,r_1} H_{B,r_2} e_2 e_3 \) lying on \( M^{-1} e_2 e_3 \). If \( M^{-1} P^{-1} e_2 e_3 \) shares a vertex and two edges of \( M^{-1} A_p \), then \( M^{-1} P^{-1} A_p \) preserves re-expression, since none leave \( \Lambda \) invariant. However, we may extend \( H_{A,r_1}^{-1} \) by linearity to a linear map defined on \( A \). Notice that none of \( H_{A,r_1}^{-1}, H_{B,r_2}^{-1}, H_{A,r_3}^{-1} \) preserve re-expression, since none leave \( \Lambda \) invariant. However, \( M = PH_{C,r_3}^{-1} H_{B,r_2}^{-1} H_{A,r_1}^{-1} \) shows \( PH_{C,r_3}^{-1} H_{B,r_2}^{-1} H_{A,r_1}^{-1} \) preserves re-expression. As an example, consider the Type I matrix

\[
N = \begin{bmatrix}
-3 & 2 & 2 \\
2 & -3 & 2 \\
2 & 2 & -3
\end{bmatrix}.
\]

with repeated eigenvalue \(-5\). \( N \) induces a \( 25 = \mid \det(N) \mid \) to 1 map on \( A/G \), with inverse

\[
N^{-1} = \begin{bmatrix}
1/5 & 2/5 & 2/5 \\
2/5 & 1/5 & 2/5 \\
2/5 & 2/5 & 1/5
\end{bmatrix}.
\]
The proof that outlines the decomposition (15) also provides an algorithm, and

\[
N^{-1} = H_{A,4/5}H_{B,3/4}H_{C,2/3}P^{-1}
\]

\[
= \begin{bmatrix}
1 & 1/5 & 1/5 \\
0 & 4/5 & 0 \\
0 & 0 & 4/5
\end{bmatrix}
\begin{bmatrix}
3/4 & 0 & 0 \\
1/4 & 1 & 1/4 \\
0 & 0 & 3/4
\end{bmatrix}
\begin{bmatrix}
2/3 & 0 & 0 \\
0 & 2/3 & 0 \\
1/3 & 1/3 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 1/2 & 1/2 \\
1/2 & 0 & 1/2 \\
1/2 & 1/2 & 0
\end{bmatrix}.
\]

The effect of this composition on $A_p$ is shown in Figure 5.

6. Future Research

The construction in section section 5 is not canonical. Indeed, the Hofstadter matrices $H_{A,r}, H_{B,r}, H_{C,r}$ are conjugate by permutation matrices, and

\[
N^{-1} = H_{C,4/5}H_{B,3/4}H_{A,2/3}P^{-1}
\]

is another way of decomposing $N^{-1}$. We wonder whether there is a canonical construction whose inverse is $N^{-1}$, similar to the pedal mapping. We guess that its discoverer would be rewarded with a wonderful picture. We also wonder about connections between triangle maps and constructable numbers.

Obviously, one could relax the assumption of linearity, to find, e.g., continuous maps $f : A \to A$ that preserve re-expression. One might investigate linear $n$-gon maps (though convexity might be an issue). [DHZ03] examined convergent polygon constructions and our inversion approach could unveil new constructions.

In [Al93], the author posed a question about the dynamics of pedal tetrahedron and we include linear tetrahedron constructions to the list of open questions.

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