Improved electronic measurement of the Boltzmann constant by Johnson noise thermometry

Jifeng Qu, Samuel P Benz, Alessio Pollarolo, Horst Rogalla, Weston L Tew, Rod White and Kunli Zhou

1 National Institute of Metrology (NIM), Beijing 100029, People’s Republic of China
2 National Institute of Standards and Technology (NIST), 325 Broadway, Boulder, CO 80305–3328, USA
3 ECEE Department, University of Colorado at Boulder, Boulder, CO 80303, USA
4 National Institute of Standards and Technology (NIST), 100 Bureau Drive, Gaithersburg, MD 20899, USA
5 Measurement Standards Laboratory of New Zealand, Lower Hutt, New Zealand

E-mail: qujf@nim.ac.cn

Received 10 February 2015, revised 28 April 2015
Accepted for publication 30 April 2015
Published 19 August 2015

Abstract
The unit of thermodynamic temperature, the kelvin, will be redefined in 2018 by fixing the value of the Boltzmann constant, $k$. The present CODATA recommended value of $k$ is determined predominantly by acoustic gas-thermometry results. To provide a value of $k$ based on different physical principles, purely electronic measurements of $k$ were performed by using a Johnson noise thermometer to compare the thermal noise power of a 200 $\Omega$ sensing resistor immersed in a triple-point-of-water cell to the noise power of a quantum-accurate pseudo-random noise waveform of nominally equal noise power. Measurements integrated over a bandwidth of 575 kHz and a total integration time of about 33 d gave a measured value of $k = 1.3806513(53) \times 10^{-23}$ J K$^{-1}$, for which the relative standard uncertainty is $3.9 \times 10^{-6}$ and the relative offset from the CODATA 2010 value is $+1.8 \times 10^{-6}$.

Keywords: Boltzmann constant, Johnson noise, quantum voltage, noise thermometry

1. Introduction
Johnson noise is the electronic noise caused by thermally induced fluctuations in voltage and current that occurs in all electrical conductors at finite temperature. Although Einstein predicted the noise in his 1905 explanation of Brownian motion [1], it was more than two decades before it was measured by Johnson [2, 3] and explained in detail by Nyquist [4]. Both Johnson noise and Brownian motion are examples of a fundamental thermodynamic relation called the fluctuation-dissipation theorem [5], which relates microscopic fluctuations to linear dissipative mechanisms.

For Johnson noise, the fluctuation-dissipation theorem describes the power spectral density of the noise voltage across the resistor [4, 5]

$$ S_R = 4hR\left[\frac{1}{2} + \frac{1}{\exp(\hbar f/kT)} - 1\right] $$

where $\hbar$ is Planck’s constant, $f$ is frequency, $R$ is the resistance, $k$ is Boltzmann’s constant, and $T$ is the temperature of the resistor. Usually, Johnson noise is characterized by its mean-square voltage, conventionally called the noise power. For temperatures near 300 K and frequencies below 1 GHz, the noise power is approximated with a relative error of less than $1 \times 10^{-9}$ by Nyquist’s law,

$$ V_f^2 = 4kTR\Delta f. $$

where $\Delta f$ is the bandwidth over which the noise is measured. Johnson noise is described as a ‘white noise’ since the power spectral density $S_R = 4kTR$ is independent of frequency.
Johnson noise thermometers (JNT) are purely electronic, they offer an appealing alternative to the various forms of primary gas thermometry: constant volume gas thermometry [6], acoustic gas thermometry [7–11], and dielectric constant gas thermometry [12, 13], all of which are limited by the non-ideal properties of real gases. Almost certainly, the numerical value assigned to Boltzmann’s constant with the introduction of the ‘New SI’ (new international system) [14] will be primarily determined by the values obtained by acoustic gas thermometry [15]. However, there remains the possibility of unknown systematic effects that might bias the gas thermometry results, and therefore an alternative determination using a different physical technique and different principles provides valuable assurance that any unrecognised systematic effects must be small. The consultative committee on thermometry (CCT) of the international committee for weights and measures (CIPM) has proposed that the kelvin redefinition should proceed when the next CODATA adjustment assigns a value of \( k \) with a relative uncertainty below \( 1 \times 10^{-6} \), supported by at least one determination from a second technique reporting a relative uncertainty below \( 3 \times 10^{-6} \) [16].

Historically, noise thermometry has not been a practical option for Boltzmann constant determinations because of three main challenges. Firstly, the noise voltages are extremely small, typically less than 2 \( \mu \)V rms (root-mean-square). To practically measure such a signal, it has to be amplified by factors of \( 10^5 \) or more while maintaining the relative accuracy. The second and most significant breakthrough was the development of ac-Josephson voltage synthesizers [19, 20]. The most successful JNT technique for the medium- and high-temperature ranges is the switched-input correlator pioneered by Brixy for application in nuclear reactors [17, 36] and is now used routinely for most metrological noise thermometry. The switched correlator combines the amplifier-noise immunity of cross-correlators, first exploited by Fink [37], and the gain-instability immunity of the Dicke radiometer [38]. Digital noise thermometers also eliminate the analog multiplier traditionally employed in correlators by digitizing the signals from the two correlator channels and performing the multiplication and averaging functions in software.

Figure 1 shows a simplified schematic diagram of the QVNS-JNT. At the input there are two noise sources. The first is a resistor maintained in a TPW cell, producing Johnson noise with a power spectral density

\[ S_R = 4kT_W R_K, \]  

where \( T_W \) is the temperature of the TPW, which currently defines the kelvin, and the sensing resistance is expressed as the ratio \( R_K \) in units of the von Klitzing resistance \( R_K \equiv h/le^2 \) [39], where \( e \) is the charge of the electron, and \( h \) is Planck’s constant. The second noise source is the quantum-accurate voltage-noise source (QVNS) [26, 27], producing a pseudo-random noise for which the calculable power spectral density is...
\[ S_{Q\text{-calc}} = D^2 N_j^2 M / K \bar{f}, \]  

where \( K \equiv 2e/h \) is the Josephson constant, \( f_s \) is a clock frequency, \( M \) is the bit length of the digital code for the noise waveform, \( D \) is an adjustable parameter of the software that generates the digital code that sets the amplitude of the synthesized QVNS waveform, and \( N_j \) is the number of junctions in the Josephson array used in the QVNS. The JNT is normally operated with the power spectral densities closely matched, \( S_{Q\text{-calc}} \approx S_R \). For calculation purposes we adopt the 1990 conventional electrical units \( V_{90} \) and \( \Omega_{90} \) such that \( K = 483597.9 \) GHz/V, and \( R_{k} = 25812.807 \) \( \Omega_{90} \) [40].

The two correlator channels alternately amplify, filter, and sample the respective noise signals from the thermal and QVNS sources, which are then digitized by the ADCs, Fourier transformed, and cross-correlated in software. The output of the correlator is proportional to the noise powers of the respective thermal and QVNS signals. Since the bandwidth of the system is defined digitally and is the same in the two configurations, the ratio of the cross-correlated discrete-Fourier transforms for each source yields the ratio of the power spectral densities \( S_Q / S_Q \). The Boltzmann constant is then determined by

\[ k = \frac{D^2 N_j^2 M}{4T_{90}X_{90}K \bar{f}^2} \langle S_R \rangle / \langle S_Q \rangle. \]  

where \( \langle S_R \rangle / \langle S_Q \rangle \) is the ratio of the averaged measured spectral densities. The factor of \( 4i / R_k \bar{f}^2 \) in (5) is equivalent to the Planck constant expressed in conventional 1990 units. Contributions to the uncertainty in the determination of \( k \) are therefore due to the uncertainties in the QVNS clock frequency, the noise-power-ratio measurements, the realization of the triple point of water, and the resistance measurement. The remaining parameters in (5) are known exactly and contribute no uncertainty. By far the most significant sources of uncertainty are associated with the measurement of the ratio of the power spectral densities (details are given in section 4).

2.2. The thermal noise source

The sensing resistor consists of two Ni–Cr-alloy foils on an alumina substrate, each with a nominal resistance value of 100 \( \Omega \). Two pairs of gold-coated leads on the hermetically-sealed package are connected to the two ends of the series-connected resistors and a fifth lead is connected between the two resistors to form the ground connection (figure 1). The resistor is mounted on a copper header soldered at the end of a 6 mm diameter and 0.1 mm thick-wall stainless-steel probe (‘R-probe’). The R-probe is about 50 cm long, filled with argon gas and hermetically sealed. Two pairs of very thin coaxial cables, also of 50 cm length, transmit the differential noise signals from the resistor to the connectors at the top of the probe. The fifth lead is connected directly to the probe head and then to the aluminium box that shields the switch assembly (figure 1). The NIM probe is immersed in a 14 mm diameter TPW cell maintained in an ice bath contained within a stainless-steel Dewar.

Both the NIM and NIST R-probes use the same resistor package, but the NIST probe uses a pair of shielded twisted-pair copper leads and has the fifth wire connected to the shields for each twisted pair. The NIST resistor sense probe uses demountable O-ring seals and a continuous supply of dry nitrogen gas to keep the probe at a positive internal pressure. The probe is inserted into a 13 mm diameter TPW cell whose ice mantle is maintained with a thermo-electric cooler [27].

2.3. The QVNS

The QVNS is a quantized delta–sigma (\( \Delta–\Sigma \)) digital-to-analog converter (DAC) using oversampling techniques to produce a programmed sequence of high-speed pulses, typically with frequency of a few gigahertz. With appropriate algorithms and biasing [41], it produces a pseudo-random noise waveform with the desired harmonic content over a band from dc to several megahertz, well beyond the operating bandwidth of the JNT. The primary advantage of the QVNS is that each pulse, \( V(t) \), from the DAC has a quantized integral

\[ \int V(t) dt = nh/2e, \]  

where \( n \) is an integer (\( n = 1 \) under normal operating conditions) [42]. This enables the synthesis of waveforms calculable exactly from the known sequence of pulses, the clock frequency of the pulse generator, and fundamental physical constants, according to (4).

The synthesis technique underlying the QVNS was originally developed for ac-Josephson voltage standards [19, 20].
Conventionally, a high-speed pulse train, obtained through a software $\Delta - \Sigma$ modulator, is applied to the Josephson junctions to synthesize the desired voltage waveform, which contains the low-frequency component dominated by the expected waveform. The low-frequency component in the driven signal increases the system complexity and induces unwanted voltages through the on-chip inductances. Though the inductive error is negligible at frequency of a few kilohertz, it becomes significant at the higher frequencies desired for JNT measurements having larger bandwidth. Instead of the conventional synthesis method, we use a novel zero-composition method to synthesize the pseudo noise waveform [43]. The zero-compensation pulse pattern is obtained through two-level $\Delta - \Sigma$ modulation and then reconstructed by padding each positive pulse with two negative pulses of half amplitude for cancellation. This approach significantly reduces the low-frequency components of the resulting bipolar, ternary-level pulse train and thus the inductive voltage error [44].

In the QVNS chip that was fabricated at NIST in Boulder there are ten superconductor-normal metal-superconductor (SNS) Josephson junctions ($N_J = 10$) in each of two arrays [41]. The critical current of the Josephson junctions is around 6 mA and the characteristic frequency is about 5 GHz. The chip is mounted on a flexible package in a 1 m long magnetically-shielded probe (QVNS probe). The probe is cooled to 4 K in a 100 L liquid-helium Dewar. The spectrum of the QVNS waveform is composed of a series of odd harmonic tones at multiples of the pattern repetition frequency, $f_1 = f_o/M = 90$ Hz, up to 4 MHz. All tones have the same amplitude but random phase (see figure 2). When the QVNS is used to measure $k$, the rms voltage amplitude of each of the tones is set to be 23.3034565 nV, so that the synthesized waveform’s average power spectral density matches the thermal noise power spectral density to about 0.01%.

In addition to providing the link to Planck’s constant, the QVNS has important advantages that have enabled significant improvements in the bandwidth and accuracy of the noise thermometer. Unlike a resistor noise source, the QVNS output voltage is inherently independent of its output impedance. This makes it possible to match both the noise powers and the output impedances of the thermal and QVNS sources, overcoming the matching conflict inherent in conventional Johnson noise thermometers that use two thermal noise sources [29]. To achieve the impedance match, four resistors terminate the QVNS transmission line such that one resistor is placed on-chip in each of the four QVNS output leads. Since these resistors are at 4 K, they produce only a small amount of uncorrelated noise. There are also four small-valued resistors placed in each lead of the sensing-resistor transmission line (figure 1). These resistors ensure the thermal and QVNS sources have both the same uncorrelated noise powers to minimize effects of any non-linearity. To ensure a good match of the frequency responses of the thermal and QVNS probes across a wide bandwidth, small parallel-connected capacitors and series-connected inductors with carefully trimmed values are inserted at the input of the switch circuit. Close matching of the frequency responses of the two probes and their transmission lines minimises the ‘spectral mismatch’ error and ensures that the JNT can be operated over a greater bandwidth and with a reduced measurement time.

### 2.4. The correlator

The respective noise powers of the resistor and the QVNS are alternately measured by the correlator, depending on the relay configuration of the switching network, as indicated in figure 1. Each channel of the correlator is composed of a low-noise preamplifier and gain blocks with a total gain of about 70 dB, filters that define the overall measurement bandwidth and prevent aliasing, and a fast ADC.

The switching circuit consists of four latching relays mounted on a FR4 printed circuit board (PCB). The relays are controlled by a field-programmable gate array (FPGA) to alternate between the noise sources. The power supply of the FPGA is turned off during the measurement and controlled via optical fiber so as to eliminate the FPGA as a source of electromagnetic interference close to the input signal leads. On the PCB, the signal traces are symmetrically placed in the top layer and the total length from input lead to output lead is about 15 mm. The copper ground plane is in the bottom layer, and the area opposite to the signal traces is removed. This design minimizes the capacitance to ground of the signal leads and hence also minimizes the effect of dielectric losses [27].

The preamplifier, based on a common-source-common-base field-effect transistor (FET)-bipolar cascode input stage operated without feedback [45], has been designed for lower noise, higher bandwidth and higher common-mode rejection [30]. These features are necessary to meet the demanding JNT requirements of low input noise voltage, very low input noise current, low noise-current-noise-voltage correlation, high input resistance, low input capacitance, and high common-mode rejection ratio. The filters are passive LC-ladder filters implementing an 11th-order Butterworth response with a
cut-off frequency of 1.8 MHz. Two filters are placed in each amplifier chain to ensure that the contribution of aliased signals to the measured noise power is negligible.

Custom-made 16-bit-ADC boards sample the signals in each channel for 1 s periods with a sampling frequencies of 4 MHz (NIM) and 2.083 MHz (NIST). The ADCs are clocked and triggered externally via optical fibres, with the phases of the clocks carefully adjusted so that the acquisitions of the two correlator channels are synchronized. Although the ADC is only 16 bits, dithered by both correlated and uncorrelated noise and that quantization effects and differential non-linearity have no practical effect on the measurement [46]. Large-scale and integral non-linearity effects are still a factor and these are discussed in section 4.1.3.

Once each signal has been sampled for 1 s, fast Fourier transforms (FFT) of the signals are computed yielding complex spectra with 1 Hz frequency-resolved FFT bins and a 2 MHz Nyquist frequency. To ensure that the QVNS tones are located in a single FFT bin, the ADC clocks are locked to the same frequency reference as the QVNS clock. In addition to the FFTs, the computer carries out a complex frequency-domain cross correlation of the FFT spectra for the two channels, which reduces the uncorrelated amplifier noise voltages. The autocorrelation for each channel is also calculated for diagnostic purposes. All of these spectra are then accumulated for 100 s into averaged spectra to provide a compact form of data storage for post processing and for final computation and diagnostic checks. These computations are carried out in real time, and every 100 s, the correlator is switched between the resistor probe and the QVNS. The combination of 100 s of resistor data and 100 s of QVNS data is defined as one ‘chop’ and for the purposes of indexing the integration time 1 chop = 200 s.

2.5. Shielding and grounding

Very good shielding and grounding are necessary because electromagnetic interference (EMI) adversely affects the measurements by inducing preamplifier overloads and by creating systematic errors due to coupling into both correlator channels [33]. The switching circuit, the amplifier chains and the lithium-ion batteries powering the analog parts of the correlator are mounted in separate aluminium boxes. The whole of the measurement electronics is then placed in a high-permeability nickel-alloy box. The signal connections between the electronics box and the QVNS and the resistor probe are heavily shielded and use separately shielded twisted pairs to minimize any electromagnetic interference (EMI) and crosstalk between the channels. To eliminate ground loops, only the ground of the pulse generator that drives the Josephson junctions through micro-wave cables is directly connected to the earth point via a low-resistance connection.

The whole of the JNT system at NIM is housed in a large underground screened room that has a filtered mains supply and fully isolated communications systems. The room is also deliberately located away from major magnetic machinery such as transformers and electric motors, to reduce the effects of low-frequency magnetic EMI. With all of these precautions, no EMI signals are observed in the integrated spectra for both thermal and QVNS noise measurements.

3. Measurement results

Currently, the two JNT measurement systems in operation at NIST and at NIM utilize different methods for long-term measurements. The different methods are a result of practical requirements for maintaining the TPW cells and differences in the battery technologies as utilized at the two laboratories. The first subsection below is included to highlight the differences in the engineering and operation of the NIST and NIM JNT systems, to describe some of the diagnostic measurements that have been carried out on the NIST system (which is largely replicated at NIM), and to record some of the recent engineering and operational improvements shared between the two sites.

3.1. NIST results

The most recent NIST determination of k is reported in [27]. At NIST, the measurement time is limited only by the He-boil-off in the storage Dewar used to cool the QVNS circuit. The batteries of the JNT electronics are automatically switched and recharged and the triple-point cell is continuously cooled by a Peltier cooler, allowing for measurements of more than 200 h. The entire measurement system and probes are interconnected through one continuous shield but are located in a typical un-shielded laboratory.

Figure 3 shows an example for a 135 h continuous measurement of 2439 chops for a filter bandwidth of 800 kHz. In the top-most plot, the statistical uncertainty in the measured ratio versus chop number still follows the normal $1/\sqrt{N}$ dependence as expected for uncorrelated random noise. In past experiments [24], failure to follow this trend has been an indicator of distortion-limited statistics or the presence of spectral aberrations. Much of the ongoing work at NIST is dedicated to the identification and removal of such distortions and aberrations and to the optimization of the pulse code and signal processing in order to further increase the bandwidth and decrease the influence of foreign signals. Results of the extended measurements (a limited set is shown in figure 3) are analysed in this respect for occurrences at certain frequencies and signal levels and for a correlation with certain frequency dependent system aspects.

These diagnostic results were then used to further optimize the system at NIST. Information about the correlator setup, the matching of the impedance of the signal lines and their frequency response and the microwave circuitry was shared with NIM and allowed the high-precision measurements described in this article. Among other results, recent work at NIST suggests that some spectral aberrations can be attributed to ferrite-cored inductors in the input circuits in front of the preamplifier, which may be used to suppress the common-mode Colpitt’s oscillation that can occur from interaction between the differential FET input stages during the sense resistor measurement. The ferrites can introduce losses and non-linearities that may arise at lower frequencies. Although
the inductors are connected in common mode, there is sufficient differential-mode leakage inductance for the cores to introduce frequency dependence. The results of the distortion analysis and further optimization of the measurement system will be published elsewhere. The remainder of this paper describes JNT measurements performed at NIM and the resulting Boltzmann constant value and its uncertainty are entirely from these data.

3.2. NIM results

At NIM, individual measurements were performed, each having an integration period of about 15 to 20 h, which is determined by the capacity of the batteries for the digitizers and the maintenance period of the triple point of water cell in the ice bath. For the results reported here, 45 such measurements were completed to reduce the statistical uncertainty, such that the combined total integration period of about 777 h, or 33 d, were accumulated with 13,982 chops of data. The resistance value of the sense resistor was checked before and after every measurement with a dc resistance bridge.

The accumulated auto-correlation and cross-correlation spectra for one measurement with 380 chops are shown in figure 4. Both the QVNS and thermal noise spectra are very clean such that they show no evidence of EMI signals in the 2 MHz measurement bandwidth. Note also that the amplitude of the spectra decreases visibly above 1.6 MHz due to the frequency response of the low-pass filter (LPF). The QVNS spectrum shows that the aliased part of the spectrum is quickly attenuated such that it is already 60 dB lower than the in-band signal at 1.6 MHz.

Once the measurements are complete, all of the cross-correlation and auto-correlation spectra, 13,982 chops of data for both of the thermal and QVNS sources, are averaged respectively, and the real part of each of the thermal and QVNS cross-spectra are averaged and reduced in resolution by summing the 1800 FFT bins, including those with the 10 QVNS tones, to form a spectral ‘block’. This rebinning is necessary because most of the bins in the QVNS spectrum are largely empty due to the absence of the tones, and a direct ratio of the QVNS and thermal spectra would not have similar ratios in each bin. The ratio of the rebinned thermal and QVNS spectra is then computed. Forming the ratio spectrum is not only essential for computing $S_b/S_Q$ in (5), it also removes the amplifier and analog filter frequency responses from the calculations, and ensures equal weighting of all spectral elements in the calculation. The equal weighting has the effect of maximizing the correlation bandwidth of the thermometer [32] and yields a small reduction in the statistical uncertainty. Note, particularly, that all of the spectral averages are computed before the noise-power ratio spectrum is calculated. This calculation sequence is necessary to avoid a bias caused by the (non-linear) division calculations operating on the stochastic data [47, 48].

The noise-power ratio spectra calculated with the rebinned thermal noise and QVNS spectra are shown in figure 5.
deviations from the fits determined the relative statistical uncertainty. Nevertheless, there remained small unexplained spectral aberrations manifest as variations in the values of the fitted parameter values when the fits were performed over different bandwidths. These effects of spectral aberrations dominated the final measurement uncertainty [27]. For the data and analysis presented here, a higher-order correction model is employed to fit the data, and described in detail in section 4.1.2.

4. Sources of error and uncertainty

Because all of the variables in equation (5) for the Boltzmann constant appear as multiplying or dividing factors, the propagation-of-uncertainty expression for the uncertainty in $k$ is most simply expressed in terms of relative uncertainties:

$$
\frac{u^2(k)}{k^2} = \frac{u^2(S_R)}{S_R^2} + \frac{u^2(S_Q)}{S_Q^2} + \frac{u^2(f_c)}{f_c^2} + \frac{u^2(T_{TPW})}{T_{TPW}^2} + \frac{u^2(X_K)}{X_K^2},
$$

(7)

where each relative uncertainty component can be expressed as a dimensionless fraction. Note that there are no uncertainty terms in $K_j$, $R_K$, $D$, $N_j$ or $M$ because they are either defined conventional values or exact integers or numbers.

The contributions occur in four main groups: 1) those associated with the measurement of the ratio of the power spectral densities, 2) those associated with the QVNS waveform generation, 3) those associated with the realization of the water-triple-point temperature, and 4) those influencing the measurement of the sensor resistance in terms of $R_K$. All of these uncertainty contributions are discussed in detail in the following sections and in all cases the stated values are standard uncertainties.

4.1. The power spectral ratio

The uncertainty associated with the measurement of the ratio of the power spectral densities originates from four effects, the statistical uncertainty, the spectral mismatch and aberrations, the non-linearity, and the electromagnetic interference, as described below.

4.1.1. Statistical uncertainty. Because the thermal noise is random, it results in a purely statistical uncertainty that makes the largest contribution to the total uncertainty. The two measurement phases: one for the measurement of the thermal noise power and one for the QVNS power measurement, have slightly different contributions [32]:

$$
\frac{u^2(S_R)}{S_R^2} = \frac{1}{2\pi R \Delta f}
\left[ 1 + \frac{S_{n1}}{S_R} \left( 1 + \frac{S_{n2}}{S_R} \right) + 1 \right],
$$

(8)

and

$$
\frac{u^2(S_Q)}{S_Q^2} = \frac{1}{2\pi Q \Delta f}
\left[ 1 + \frac{S_{n1}}{S_Q} \left( 1 + \frac{S_{n2}}{S_Q} \right) - 1 \right].
$$

(9)
where $r_0$ and $r_k$ are the integration times for the thermal and QVNS measurement phases respectively, $\Delta f_c$ is the correlation bandwidth, and $S_{n1}$ and $S_{n2}$ are the power spectral densities of the uncorrelated equivalent input noise voltages in the two channels of the correlator. It is assumed that the various power spectral densities are independent of frequency. Because the noise-power ratios are measured in the frequency domain, the correlation bandwidth is simply defined by the FFT bins selected for inclusion in the measurements. Note that $r_0 \Delta f_c = r_0 \Delta f_c$ is the number of selected FFT bins, and the uncertainty decreases as $1/\sqrt{n \Delta f_c}$, as expected for random noise.

The uncertainty contributions of the two measurement phases are different, as evident from the differing signs of the last term in each of (8) and (9). The sign is positive for the thermal measurement and negative for the QVNS measurement. The negative sign for the QVNS measurement arises because the QVNS signal is not truly random but has a constant noise power when integrated over multiples of the code recycle time, and hence, the uncertainty is lower than for the thermal noise-power measurement. The uncertainty in the QVNS measurement is zero when the uncorrelated noise, largely due to the preamplifier equivalent input noise voltage, is zero.

Because the two measurement phases make different contributions to the uncertainty, it is possible to minimise the total uncertainty by spending a greater fraction of the measurement time integrating the thermal noise signal [32]. However, the advantage is modest and for the measurements reported here, the integration times were the same for both phases. Since the total measurement time, $\tau$, is the sum of the integration times for the thermal and QVNS measurements, and $S_{n0} \approx S_{n\ell}$, the statistical uncertainty is approximately

$$\frac{\mu^2(S_{n0})}{S_{n\ell}} + \frac{\mu^2(S_{n\ell})}{S_{n0}} \approx \frac{2}{r \Delta f_c} \left( \frac{1}{S_{n1}} + \frac{S_{n1}}{S_{n\ell}} \right) \left( \frac{1}{S_{n2}} + \frac{S_{n2}}{S_{n\ell}} \right)$$

(10)

However, this result applies only when every selected QVNS-tone block in the ratio spectrum contributes equally to the estimate of $S_{n0}/S_{n\ell}$. The use of the least-squares fit to correct for any frequency dependence in the power spectra (see discussion below) changes the weighting of the different blocks, and the statistical uncertainty is larger than indicated by equation (10).

### 4.12. Spectral mismatch and aberrations.

Because the same correlator is used to measure the noise power for the thermal and QVNS signals, the ratio spectrum is independent of the gain and frequency response of the correlator. Frequent ($\approx 100\text{ s}$) switching between the two sources also eliminates the effects of drifts in gain and frequency response. However, because the response of the transmission lines connecting the sensor and QVNS to the correlator are independent, small differences in the frequency responses of these components do have an impact on the measured ratio spectrum.

There are two general categories of mechanisms that result in departures from a perfectly flat noise power ratio spectrum. The first category is the ordinary spectral mismatch that occurs when time constants associated with the R-probe input network (i.e. probe plus cabling and switches) are not perfectly matched with those associated with the QVNS probe input network. As discussed in section 3.3 above, these mismatches result in a differential filtering response in the ratio spectrum which is manifest as even-order frequency correction terms. The second category is the spectral aberrations that occur due to more complex effects that are not readily parameterized as simple filtering. We include dielectric loss effects, block-summation errors [29], aliasing effects, and other more complex frequency-dependent errors in this category. For this work, the spectral mismatch contributions are the largest source of frequency dependence.

Spectral aberrations associated with finite dielectric loss originate from various shunt capacitances with loss tangents $\tan \delta$. These losses can occur in the components used to assemble either the QVNS probe, the R-probe, or the common portions of the input circuit, leading to contributions in the transfer functions for the R and QVNS networks that are linear in $1/\tan \delta$. The differences in such terms between the R and QVNS lines can then be manifest in the ratio spectrum [49]. In the previous measurement [27], the relative uncertainty due to this effect was estimated to be about $2 \times 10^{-6}$. With this in mind, every effort has been made to minimise the lengths of PCB tracks in the input circuits and to minimise the use of fiberglass PCB, which is a major contributor of this effect. The net effect can be simulated, given reasonable estimates for the various impedances, but this modelling has not been completed. It is expected that the relative error in the present measurement caused by dielectric loss is less than $1 \times 10^{-6}$.

Spectral-mismatch contributions to frequency dependence in the ratio spectrum arise from differences between the source impedances (thermal and QVNS), and differences in the associated load impedances presented to each source by their respective transmission lines connecting them to the preamplifiers, and differences in how those impedances couple to the input impedances of the preamplifiers. Ideally, the very small attenuations that occur should be the same for both sensors so that the ratio of two spectra is flat. When the transmission lines have been carefully matched, any remaining spectral mismatches occur only at high frequencies, and ideally the ratio spectra have the appearance of the response of a high-order Butterworth filter, which is maximally flat throughout its passband. Figure 5, which shows the measured ratio spectrum, exhibits these features. For frequencies up to about 800kHz, the ratio spectrum is flat, and then above 800kHz the ratio begins to fall rapidly.

Another more subtle source of spectral mismatch occurs due to correlated noise terms that originate in the preamplifiers and are coupled between the two ($j = 1, 2$) correlator channels through the finite source impedance of the R-probe [45]. These are small accidental correlations which occur from preamplifier noise currents $i_{n\ell}$ and noise voltages $v_{nj}$ that couple and produce frequency dependent errors proportional to $i_{n1}i_{n2}$ and $i_{n1}v_{n2}$. These correlated noise error terms are primarily quadratic in frequency, but higher even-order powers are likely present at lower levels. These errors do not occur in the QVNS probe input since the ideal source
impedance of the QVNS decouples the preamplifier noise in the two channels. Hence, the ratio spectrum includes correlated noise contributions from the R-probe input only. The frequency dependence from these noise-error terms is practically indistinguishable from that of the other filtering-related error terms.

The ratio spectrum, including corrections from spectral mismatches and correlated noise terms, can be modelled by

\[
R(f) = \frac{S_R}{S_Q} (1 + a_2 f^2 + a_4 f^4 + a_6 f^6 + ...),
\]

where \( S_R/S_Q \) is the desired low-frequency limiting value of the power-spectral ratio required for the measurement of \( k \) (equation (5)), and the coefficients \( a_2, a_4, a_6 \) represent the effects of the various impedance mismatches, and are ideally very small. Only even order terms are included in the series expansion because it is assumed that the transmission lines are short enough to be modelled by lumped inductances and capacitances, which would give rise only to even order terms in the real parts of the power spectra.

There are two approaches for managing the spectral mismatches. The first is to limit the bandwidth so that the mismatch effects are small, and this was the approach adopted with most conventional noise thermometers prior to the NIST measurement. With this approach, \( S_R/S_Q \) is measured simply as the average of the ratio spectrum between zero frequency and the chosen correlation bandwidth, \( \Delta f_c \),

\[
\frac{S_R}{S_Q} \mid_{\text{meas}} = \frac{S_R}{S_Q} \left(1 + \frac{a_2}{3} \Delta f_c^2 + \frac{a_4}{5} \Delta f_c^4 + \frac{a_6}{7} \Delta f_c^6 + ...\right),
\]

so that the measured value of \( S_R/S_Q \) is biased by the frequency dependent terms. Typically, when the measured value of \( S_R/S_Q \) is plotted against frequency, the bias will increase rapidly and emerge from the noise, very much as shown in figure 5.

As is well known, if the spectral ratio alone (a constant) is fitted to the data, the uncertainty in the fitted value of \( S_R/S_Q \) should fall in proportion to the square root of the number of data points

\[
u(S_R/S_Q) = \frac{\sigma}{\sqrt{N}},
\]

where \( \sigma \) is the variance in the spectral ratio for each QVNS-tone block, and \( N \) is the number of QVNS-tone blocks included in the fit. Operation of the noise thermometer therefore requires a compromise between the statistical uncertainty, which decreases as the square root of the bandwidth (equation (10)), and the uncertainty due to spectral mismatches, which increases according to a power law, according to (12).

The second approach is to apply a least-squares fit of (11) to the measured ratio spectrum, so that values are determined for all of the parameters in the model, including the desired low-frequency power spectral ratio. The cost of this approach is that the uncertainty in the fitted value of \( S_R/S_Q \) increases with the number of fitted parameters. Suppose that the least-squares fit estimates the low-frequency power spectral ratio \( S_R/S_Q \) by fitting the function

\[
R(f_i) = \frac{S_R}{S_Q} (1 + a f_i^2)
\]

to the ratio spectrum, where \( f_i \) are the frequencies at the centre of each QVNS tone block, and hence the low-frequency power spectral ratio is estimated as

\[
\frac{S_R}{S_Q} = \frac{1}{N} \sum_i \left( R(f_i) - \sum_i f_i^2 \sum_i \frac{R(f_i)}{N} \right)^2.
\]

Since, \( i \) ranges from 1 to \( N \) and \( f_i = f_0(2i - 1) \) where \( f_0 \) is the QVNS-tone pattern repetition frequency, equation (15) identifies the sensitivity coefficients for \( S_R/S_Q \) with respect to each \( R(f_i) \) as

\[
\frac{d(S_R/S_Q)}{dR(f_i)} = \frac{1}{N} \sum_i \left( f_i^4 - f_i^2 \sum_i f_i^2 \right).
\]

The total uncertainty in \( S_R/S_Q \) is proportional to the quadrature sum of the sensitivity coefficients (assuming that the uncertainty for the ratio in each block is independent of frequency) the uncertainty in the fitted value of \( S_R/S_Q \) is

\[
\sigma^2 = \frac{1}{N^2} \sum_i \left( \frac{d(S_R/S_Q)}{dR(f_i)} \right)^2 \approx \frac{\sigma^2}{N^2}.
\]

so for large \( N \), the variance in \( S_R/S_Q \) determined by a second-order least-squares fit is larger than \( 1/N \) by the factor 9/4. If the fit is extended to the fourth- and sixth-order terms, the variances increase by the factors 225/64 and 1225/256 respectively. As the order of the fit increases, the uncertainty in the calculated value of \( S_R/S_Q \) increases. The least-squares fit therefore adds an additional complication in the compromise between bandwidth, uncertainty, and the effects of the spectral mismatches.

The NIST analysis in [27] was the first to combine the two approaches by using a second-order least squares fit combined with a bandwidth restriction. With the second order least-squares fit applied to the ratio spectrum, the bias in the measured spectral ratio is

\[
\frac{S_R}{S_Q} \mid_{\text{meas}} = \frac{S_R}{S_Q} \left(1 - \frac{3a_4}{35} \Delta f_c^4 - \frac{2a_6}{21} \Delta f_c^6 - \frac{a_8}{11} \Delta f_c^8 + ...\right)
\]

Note that there is no longer any bias due to the second-order terms of equation (11). Least-squares fits using higher-order models produce similar equations where the terms in the expressions for the bins omit the terms included in the fit.

There remains the problem of determining the best compromise between bandwidth and complexity of the fit. Figure 6 below shows the fitted value for \( S_R/S_Q \) (the thick lines with marked points) as a function of correlation bandwidth and the order of the least-squares fit for the data reported here. The fits included in the analysis are zeroth-, second-, fourth-, sixth- and eighth-order. The data are all plotted as deviations from the point representing the value of \( S_R/S_Q \) for the fourth-order fit and a bandwidth of 575 kHz. The envelopes formed
by matching pairs of dotted lines in figure 6 show the relative statistical uncertainty measured for the corresponding fits.

Ideally, each of the $S_p/S_Q$ curves shown in the figure should exhibit a characteristic shape. At low frequencies, the points should follow a type of random walk that lies, more or less, within the corresponding uncertainty envelope and does not sit close to the part of the curve that diverges rapidly. Amongst all of the curves in figure 6 exhibit this shape to some degree. Also shown in figure 6 is a rectangular box enclosing a consistent set of data points from all but the zeroth-order fit. As expected, the low-order fits converge on the box at lowest frequencies before diverging, while the higher-order fits converge at higher frequencies. Amongst these points, the one that has the lowest statistical uncertainty and does not sit close to the part of the curve that diverges rapidly, is the point for fourth-order fit and 575 kHz bandwidth. The uncertainty in the fitted value for $S_p/S_Q$ for this point is $3.2 \times 10^{-6}$.

There are two aspects of the uncertainty envelopes in figure 6 worthy of further discussion. Firstly, none of the measured envelopes follow the expected $\Delta f_c^{-1/2}$ dependence. At high frequencies especially, the residuals in the fit include the effects of the spectral mismatches as well as the effects of the random noise. The effect of the spectral mismatches is most apparent in the uncertainty envelopes for the zeroth- and second-order fits where they diverge rapidly above 800 kHz. Secondly, the uncertainty envelopes in figure 6 describe the uncertainty in the fitted value of $S_p/S_Q$, but they do not describe the extent of the random walk of the fitted values because they do not account for the correlations between adjacent data points. For example, the data used for the 500 kHz fits includes 100% of the data used for all the fits for smaller bandwidths. When the effects of the correlations are included in the zeroth-order fit, the envelopes are modified by the factor $1 - \Delta f_c/\Delta f_{c, \text{ref}}^{1/2}$ where $\Delta f_{c, \text{ref}}$ is the bandwidth for the point to which the plots are referenced.

The necessity of introducing a frequency-dependence correction model presents a new difficulty in the analysis since there is some degree of ambiguity in the choice of both the order $n_{\text{max}}$ of the model and the bandwidth, or fit limit $f_{\text{max}}$, over which it can be applied. With this in mind, we have consulted with a NIST statistician to analyze the data and estimate additional random and systematic uncertainties due to imperfect knowledge of the model. This analysis also considers even polynomial correction models (up to 14th order) including correction models shown in figure 6. The analysis estimates a selection probability for each candidate model based on the observed data using a five-fold cross-validation method [50]. The approach is essentially a hybrid of standard cross-validation techniques and data resampling without replacement (permutation). Using this method, a fourth order model yields the maximum estimated selection probability, 0.936, for $f_{\text{max}} = 575$ kHz which supports our observations above regarding the characteristics of the fitted value of $S_p/S_Q$ plot shown in figure 6.

The weighted average of the estimates of $S_p/S_Q$ with different models is calculated by

$$
\frac{S_p}{S_Q} = \sum_d p(d) \frac{S_p}{S_Q}(d),
$$

where $p(d)$ and $S_p/S_Q(d)$ is the selection probability and the fitted value of $S_p/S_Q$ with the $d$-th order correction model. For the fourth-order fit, the uncertainty of $S_p/S_Q$ due to ambiguity of the fitting model is then determined by

$$
u_{4\text{th, amb}}^2 = \sum_d p(d) \nu_{4\text{th, sta}}^2 + \sum_d p(d) \left( \frac{S_p}{S_Q}(d) - \frac{S_p}{S_Q} \right)^2 = \nu_{4\text{th, sta}}^2,
$$

where $\nu_{4\text{th, sta}}$ is the statistical uncertainty of the $S_p/S_Q$ from the $4\text{th}$-order fit. With a fitting bandwidth of 575 kHz, the estimated additional uncertainty due to model ambiguity is $1.8 \times 10^{-6}$. The additional statistical analysis, which includes examination of the effects of fitting out to bandwidths as high as $1.8$ MHz with a range of models up to 14th order even polynomials, will be published elsewhere [51].

4.1.3. Non-linearity. In order to amplify the approximately $2 \mu$V rms signals to a level appropriate for the ADCs, the gain in each channel of the correlator must be on the order of $10^5$. Inevitably, the non-linearity in the various amplifier stages, filters, and ADCs accumulates to introduce a significant error in the noise-power measurement.

The effects of non-linearity are managed by operating the correlator so that the effects on both the thermal and QVNS signals are, ideally, identical, and therefore the ratios of the noise powers used to compute $S_p/S_Q$ are unaffected by the non-linearity. There are three main operating requirements to ensure the effects of non-linearity are minimised [29]. Firstly, the average power spectral densities of the QVNS and thermal sources must be the same to ensure that the total noise powers are the same in the two measurement phases. The match is achieved by adjusting the amplitude of the QVNS signal.
Secondly, the QVNS must produce a signal that closely approximates the Gaussian distribution of the thermal noise voltage. This ensures that all of the various distortion products (effectively the moments of the distribution) are the same for both the thermal and QVNS signals. For a pseudo-random noise signal composed of a frequency comb with \( m \) sinusoids of constant amplitude and random phase, the moments of the frequency comb converge to the Gaussian moments as \( 1/m \). Thus, for a noise thermometer with, say, 100 \( \mu \text{V}/\sqrt{\text{V}} \) distortion, more than 22,200 sinusoids used in the QVNS waveform are sufficient to ensure that the differences in distortion products are below \( 0.025 \times 10^{-6} \) of the measured signal. Thirdly, because some of the distortion products involve the uncorrelated preamplifier noise voltages, the uncorrelated noise powers in each channel must also be the same for the two measurements.

While carefully matched operating conditions ensure that the effects of non-linearity are small, there remains the problem of knowing how well the matching conditions are met, and of estimating the uncertainty due to the remaining mismatches. A study of the various causes and effects of non-linearity [52] concluded that the causes were so numerous, and the effects so variable, that a physical model of the non-linearity was not practical. Instead, any model of the non-linearity must be empirical and based on measurements made with deliberate mismatches of the noise power. One of the advantages of a noise thermometer operating a QVNS is that it is possible to change the QVNS voltage without changing any of the other operating conditions of the JNT, and this makes it possible to measure the non-linearity.

Suppose the (voltage) transfer function for each channel of the correlator, when referred to the correlator input, is represented as the Taylor series [29]

\[
V_i = \sum_{r=0}^{\infty} a_{ir}(V + V_{n,i})^r, \quad a_{i1} = 1,
\]

(21)

where \( V \) is the voltage to be measured, the index \( s \) indicates the channel number (\( s = 1, 2 \)), the index \( t \) indicates the polynomial order of the distortion coefficient, and \( V_{n,i} \) is the equivalent input noise voltage for channel \( s \). The coefficients \( a_{it} \) therefore represent the offset voltages, and \( a_{i1} \) represent the linear gains and are assumed to be equal to 1 for the purpose of analysis. All of the \( a_{it} \) except \( a_{i1} \) represent unwanted non-linear terms, and are assumed to be small.

When the two Taylor series are multiplied and averages calculated, the most significant terms in measurements of noise power arise from the third order non-linearities:

\[
\overline{V_{\text{meas}}^2} \approx \overline{V^2} + 3(a_{11} + a_{22})\overline{(V^2)^2} + 3(a_{12}V_{n,1} + a_{23}V_{n,2}^2)\overline{V^2}.
\]

(22)

The ratio of the QVNS and thermal noise-power measurements is therefore

\[
\frac{V_{\text{Q,meas}}^2}{V_{\text{Th,meas}}^2} \approx \frac{V_{\text{Q}}^2}{V_{\text{Th}}^2} [1 + 3(a_{11} + a_{22})(\overline{V_{\text{Q}}^2} - \overline{V_{\text{Th}}^2}) + 3a_{12}(\overline{V_{n,1}^2} - \overline{V_{n,2}^2})] + 3a_{23}(\overline{V_{n,2}^2} - \overline{V_{n,2,\text{Th}}^2})],
\]

(23)

which clearly shows the requirement for matching the various noise powers. The equation also shows that the non-linearity in the noise power ratio should be proportional to the noise-power mismatch.

To measure the effects of the non-linearity, the Boltzmann constant measurement was repeated with several deliberately mismatched values of \( S_p/S_Q \). The additional measurements were run only for a single day, so the uncertainties are much larger than that for the primary determination. Figure 7 above shows the variations in the measured value of \( k \) versus the degree of mismatch. The line

\[
k_{\text{meas}} = k[1 + \epsilon(S_p/S_Q - 1)],
\]

(24)

which assumes that the amplifier noises are matched, was fitted to the data and the non-linearity coefficient was determined to be \( \epsilon = (39 \pm 20) \times 10^{-6} \). Since the measured ratio of \( S_p/S_Q \) used for the determination deviates from unity by less than 0.0002, the correction is negligible. A total uncertainty of \( 0.1 \times 10^{-6} \) is assigned for non-linearity effects due to both the correlated noise-power mismatch and the uncorrelated noise-power mismatch, with no correction performed.

4.1.4. Electromagnetic interference (EMI). EMI is an insidious source of error. It is typically intermittent, caused by nearby electrical machinery not associated with the JNT, and for magnetically-coupled EMI, very difficult to shield [52, 53]. While statistical tests on the averaged power spectra can readily detect stationary single-frequency EMI, in general spectral tests are not sufficiently powerful to detect all types of EMI [54].

Evidence of the absence of EMI effects in the QVNS measurements was obtained by operating the QVNS so that it generates zero volts. Any non-zero noise power then indicates an error due to EMI. A similar test was carried out with the thermal noise source but this requires a ‘dummy’ thermal sensor that forms a four-terminal zero of resistance (so that it generates zero correlated noise), and has the same geometric layout as the real thermal sensor, but generates no correlated...
noise [52]. After a full day of integration, these tests yielded residual correlated-noise powers of relative magnitude less than $0.2 \times 10^{-6}$ in both the QVNS and resistor probes. We estimate a relative standard uncertainty of $0.4 \times 10^{-6}$ contributed by possible EMI effect.

4.2. QVNS Waveform

The quantized nature of the voltages produced by the QVNS and its very wide bandwidth ensure that the uncertainties arising from the QVNS are small. The most significant contribution comes from the quantization noise due to the generation of a continuous baseband signal from an integer number of high-frequency QVNS pulses. The software generating the code for the QVNS exploits first-order noise shaping to push the quantization error to the high-frequency end of the spectrum (see figure 2). The resulting error integrated over the bandwidth used in the JNT contributes about $1 \times 10^{-7}$ to the relative uncertainty. Another negligible source of uncertainty includes the uncertainty in the reference frequency.

4.3. Triple point of water

4.3.1. Triple point realisation. The TPW cell used in present measurements is made of borosilicate glass and contains distilled and de-gassed water. The diameter of the thermometer well is 14 mm and the height of the ice mantle above the thermal noise sensor is about 23 cm. By definition, the water-triple point temperature is 273.16 K exactly. However, there are errors and uncertainties associated with the practical realization of the triple point, all of which are summarised in detail in [55]. Because the cell was manufactured without an isotopic analysis, and chemical analysis would require the destruction of the cell, the temperature realized in the cell was determined by comparing it to the NIM standard TPW cell [56, 57] with a long-stem standard platinum resistance thermometer (SPRT). The temperature realized in the cell was found to be 1.46 mK higher than that in the NIM standard cell. A small platinum thermometer sensor was used to investigate the immersion characteristics of the probe. The measurement result is shown in figure 8. The immersion error is expected to decay exponentially with the increasing immersion depth [59]. The intercept of the fitted line in figure 8 indicates an immersion error of 0.05 mK when the probe head is at the bottom of the thermal well. A relative standard uncertainty of $0.18 \times 10^{-6}$ is assigned to account for immersion errors.

4.4. Resistance measurement

Equation (3) expresses Nyquist’s law in terms of the resistance of the sensor, but strictly, it relates the noise power spectral density to the real part of the sensor impedance, $\text{Re}(Z(f))$:

$$S_n = 4kT \text{Re}(Z(f)).$$

(25)

Thus, in principle, the resistance or impedance of the sensor should be measured over the full operating bandwidth of the thermometer. While dc-resistance measurements can be made with relative uncertainties of 0.01 $\mu\Omega/\Omega$ or less [60], sensible measurements of resistance at the required accuracies and at ac, especially at frequencies above a few tens of kHz, are far more difficult [61]. For the JNT, the solution is to use a resistor with the widest practical bandwidth. This is achieved with very small thin film resistors manufactured with a meander pattern designed to balance the already small inductance and capacitance in such a way to maximise the bandwidth. The thin film construction also ensures that skin effects are minimised. This structure ensures that the ac frequency dependence of the noise spectrum is negligible and dominated instead by the inductance and capacitance of the connecting leads (transmission line), for which the frequency dependence is accommodated by the least-squares fit, as described in previous sections.

The dc resistance was measured with a dc resistance bridge and a standard resistor. The statistical relative uncertainty was
typically \(5 \times 10^{-8}\) for the average value of 200 individual measurements. The standard resistor is a hermetic encapsulated type with a calibration history traceable to the NIM quantum Hall resistance standard which maintained the value of SI ohm with an uncertainty of \(2 \times 10^{-8}\). The drift rate of the standard resistor has been determined to be \(3 \times 10^{-8}\) yr\(^{-1}\).

The relative uncertainty in the standard resistance is estimated to be \(1 \times 10^{-7}\) including its calibration and instability. The uncertainties from other effects, such as relaxation effect, thermoelectric, frequency dependence, had been analyzed in detail, and in this paper we assigned the same relative uncertainty to account for the effects [27].

### 4.5. Final result

With the analysis as described above, the fitted result of the noise-power ratio with a fourth-order correction model over 575 kHz bandwidth, associated with the carefully calibrated value of the resistance \(R\) and the temperature \(T\) that are traceable to the quantum Hall resistance and the definition of the kelvin, and the calculated value of the noise power \(S_Q_{\text{calc}}\), the value of the Boltzmann constant is determined to be \(k = 1.3806513 \times 10^{-23} \text{ J K}^{-1}\), with a combined relative uncertainty of \(3.9 \times 10^{-6}\). Table 1 lists the value of parameters that are involved in the calculation.

Table 1. Value of the parameters that are involved in the calculation of the Boltzmann constant.

| Parameter | Value Comment |
|-----------|---------------|
| \(S_Q/S_Q\) | Least-square fit with 4th-order model and 575 kHz bandwidth |
| \(S_Q_{\text{calc}}\) | Calculated with \(K_J = 483597.9\) GHz V\(^{-1}\) |
| \(T\) | Compared with the NIM standard TPW cell |
| \(R\) | Calculated with \(R_K = 25812.807\) Ω |

Table 2. Summary uncertainty budget for a determination of Boltzmann constant by QVNS noise thermometry. All uncertainties are expressed as relative uncertainties in parts per million.

| Component | Term | Relative uncertainty |
|-----------|------|----------------------|
| Ratio of the power spectral densities, \(S_R/S_Q\) | Statistical | 3.2 |
| | Model Ambiguity | 1.8 |
| | Aberrations / Dielectric losses | 1.0 |
| | Non-linearity | 0.1 |
| | EMI | 0.4 |
| | Total \(u_r(S_R/S_Q)\) | 3.8 |
| QVNS waveform \(S_Q\) | Frequency reference | \(< 0.001\) |
| | Quantization effects | 0.1 |
| | Total \(S_Q\) | 0.11 |
| TPW temperature \(T\) | Reference standard TPW cell | 0.29 (0.08 mK) |
| | Temperature measurement | 0.04 (0.01 mK) |
| | Hydrostatic pressure correction | 0.08 (0.02 mK) |
| | Immersion effects | 0.18 (0.05 mK) |
| | Total \(u_r(T_W)\) | 0.35 |
| Resistance \(R\) | Ratio measurement | 0.05 |
| | Transfer Standard | 0.1 |
| | AC–dc difference | 0.1 |
| | Relaxation effect | 0.5 |
| | Thermoelectric effect | 0.1 |
| | Total \(u_r(R)\) | 0.53 |
| TOTAL (\(k_B\)) | 3.9 |

5. Conclusion

This paper reports the results of a measurement of Boltzmann constant, by Johnson noise thermometry, yielding a value of \(k = 1.3806513(53) \times 10^{-23} \text{ J K}^{-1}\) with a relative uncertainty of \(3.9 \times 10^{-6}\). This result is \(1.8 \times 10^{-6}\) higher than the current (2010) CODATA value of Boltzmann constant, and within the uncertainty of the measurement. Although the uncertainty in this result is not as low as recent results obtained using acoustic gas thermometry [8, 9], this measurement is purely electronic, based on different physical principles, uses different types of measuring equipment, and therefore provides very strong assurance that there are no major systematic errors affecting any of the recent \(k\) determinations.
The measurement has been made possible by NISTs development of the quantum-accurate voltage noise source (QVNS) based on the ac Josephson voltage synthesiser. The QVNS enables (i) matching of the source impedances for the thermal and QVNS sources, which makes it possible to operate at wider bandwidths and reduced measurement times, (ii) a match of noise power enabling a reduction in the effects of non-linearities, (iii) a programmable noise source enabling linearity measurements, and (iv) a direct link to the fundamental constants underpinning the electrical units of measurement.

Several improvements have been made in the process of carrying out this measurement. Firstly, very close attention has been given to the matching of the frequency responses of the correlator to the thermal and QVNS signals, with the match achieved by including trimming resistors, capacitors and inductors to match the impedances of the transmission lines. Additionally, care was taken to minimise the dielectric loss in stray capacitances associated with the engineering of the thermal probe and the switch circuit.

Secondly, the entire JNT system at NIM is housed in an underground screened room remote from electromagnetic interference sources. For the first time, noise spectra have been obtained that are free of evidence of EMI. Subsidiary experiments using the QVNS at zero volts and a dummy thermal sensor confirm that any residual EMI effects are negligible.

Thirdly, greater attention has been given to the spectral analysis to achieve a better compromise between the complexity of the spectral model and the bandwidth. We have extended the frequency correction model to include a fourth-order term to better account for the effects of spectral mismatches. A statistical analysis was performed to estimate the uncertainty associated with the possible biases and ambiguity in these frequency dependence correction models. We have quantified an additional component of uncertainty which accounts for imperfect knowledge of the spectral mismatch model. This approach was facilitated by expanding the experimental passband via a new ADC board, operating with a Nyquist frequency of 2 MHz, which acquired more than 42 TB of data over a total integration time of about 33 d.

Finally, the non-linearity of a noise thermometer has been measured directly for the first time. The non-linearity is remarkably low, and combined with a very close match of the correlated and uncorrelated noise powers, ensures the uncertainty due to non-linearity is negligible. The non-linearity measurements are made possible by the QVNS, which can provide quantum accurate noise voltages of (practically) any amplitude without modifying any other JNT operating condition.

We plan to now to make a measurement of $k$ with a relative uncertainty below $3 \times 10^{-8}$ to meet the CCT second requirement (different physical technique) for the implementation of the new kelvin definition. This will be achieved by giving further attention to the spectral mismatches, and increasing the integration time.

Acknowledgments

The research team at NIM are deeply indebted to NIST for their support. The project was made possible with critical hardware, the QVNS chip fabricated by NIST, software, and the opportunity for team members to enjoy guest researcher positions at NIST, Boulder. The authors wish to acknowledge Kevin Coakley of the NIST Statistical Engineering Division for providing a model uncertainty analysis, and Jianping Sun, Xiaoke Yan, and Zhengkun Li of NIM for helping with the TPW temperature and resistance calibration. The work at NIM is supported by NSFC (61372041 and 61010304) and the public welfare scientific research project (201010008).

References

[1] Einstein A 1905 On the motion—required by the molecular kinetic theory of heat—of small particles suspended in a stationary liquid Annalen der Physik 17 549–60
[2] Johnson J B 1927 Thermal agitation of electricity in conductors Nature 119 50–1
[3] Johnson J B 1928 Thermal agitation of electricity in conductors Phys. Rev. 32 97–109
[4] Nyquist H 1928 Thermal agitation of electric charge in conductors Phys. Rev. 32 110–3
[5] Callen H B and Welton TA 1951 Irreversibility and generalized noise Phys. Rev. 83 33–40
[6] Edsinger R E and Schooley J F 1989 Differences between thermodynamic temperature and t (IPTS-68) in the Range 230°C to 660°C Metrologia 26 95–106
[7] Moldover M R, Trusler J P M, Edwards T J, Mehl J B and Davis R S 1988 J. Res. Natl Bur. Stand. 93 85–144
[8] Pitre L, Sparasci F, Truong D, Guillou A, Riesegari L and Himbert M 2011 Determination of the Boltzmann constant using a quasi-spherical acoustic resonator Int. J. Thermophys. 32 1825–86
[9] de Podesta M, Underwood R, Sutton G, Morantz P, Harris P, Mark D F, Stuart F M, Varga H and Machin G 2013 Metrologia 50 354–76
[10] Gavioso R M, Benedetto G, Albo P A G, Ripa D M, Merlone A, Guainvarc’h C, Moro F and Cuccaro R 2010 Metrologia 47 387–409
[11] Lin H, Feng X J, Gillis K A, Moldover M R, Zhang J T, Sun J P and Duan Y Y 2013 Metrologia 50 417–32
[12] Gaiser C, Zandi T, Fellmuth B, Fischer J, Jusko O and Sabuga W 2013 Improved determination of the Boltzmann constant by dielectric-constant gas thermometer Metrologia 50 L7–11
[13] Gaiser and Fellmuth B 2012 Low-temperature determination of the Boltzmann constant by dielectric-constant gas thermometer Metrologia 49 L4–7
[14] Mills I M, Mohr P J, Quinn T J, Taylor B N and Williams E R 2006 Redefinition of the kilogram, ampere, kelvin and mole: a proposed approach to implementing CIPM recommendation 1 (CI-2005) Metrologia 43 227–46
[15] Fischer J et al 2008 Preparative steps towards the new definition of the kelvin in terms of the Boltzmann constant Int. J. Thermophys. 28 1753–65
[16] Fischer J et al 2014 The CCT report to the CIPM in 2014 BIPM, Sevres, France
[17] Brixy H, Hecker R, Oehmen J, Rittinghaus K F, Setiawan W and Zimmermann E 1992 Noise thermometer for industrial and metrological applications at KFA Jülich Temperature, Its Measurement and Control in Science and Industry ed J F Schooley vol 6 (New York: Am. Inst. Phys.) pp 993–6
[18] White D R et al 1996 The status of Johnson noise thermometer Metrologia 33 325–35
[19] Benz S P and Hamilton C A 1996 A pulse-driven programmable Josephson voltage standard Appl. Phys. Lett. 68 3171–3
Benz S P, Hamilton C A, Burroughs C J, Harvey T E, Christian L A and Przybysz J X 1998 Pulse-driven Josephson digital/analog converter IEEE Trans. Appl. Supercond. 8 42–7

[21] Benz S P, Martinis J M, Nam S W, Tew W L and White D R 2002 A new approach to Johnson noise thermometry using a Josephson quantized voltage source for calibration Proc. TEMPMEKO 2001 The 8th Int. Symp. on Temperature and Thermal Measurements in Industry and Science ed B Fellmuth et al (Berlin: VDE Verlag) pp 37–44

[22] Nam S W, Benz S P, Dresselhaus P D, Tew W L, White D R and Martinis J M 2003 Johnson noise thermometry measurements using a quantum voltage noise source for calibration IEEE Trans. Instrum. Meas. 52 550–3

[23] Benz S P, Dresselhaus P D and Martinis J M 2003 An ac Josephson source for Johnson noise thermometry IEEE Trans. Instrum. Meas. 52 545–4

[24] Nam S W, Benz S P, Dresselhaus P D, Burroughs C J, Tew W L, White D R and Martinis J M 2005 Progress on Johnson noise thermometry using a quantum voltage noise source for calibration IEEE Trans. Instrum. Meas. 54 653–7

[25] Nam S W, Benz S P, Martinis J M, Dresselhaus P D, Tew W L and White D R 2003 A ratiometric method for Johnson noise thermometry using a quantized voltage noise source Temperature: Its Measurement and Control In Science and Industry 7 ed D C Ripple (Melville, New York: American Institute of Physics) pp 37–42

[26] Benz S P, White D R, Qu J, Rogalla H and Tew W L 2009 Electronic measurement of the Boltzmann constant with a quantum-voltage-calibrated Johnson-noise thermometer C. R. Phys. 10 849–58

[27] Benz S P, Pollarolo A, Qu J F, Rogalla H, Urano C, Tew W L, Dresselhaus P D and White D R 2011 An electronic measurement of the Boltzmann constant Metrologia 48 142–53

[28] Mohr P J, Taylor B N and Newell D B 2012 CODATA recommended values of the fundamental physical constants Rev. Mod. Phys. 84 1527–605

[29] White D R and Benz S P 2008 Constraints on a synthetic-noise source for Johnson noise thermometry Metrologia 45 93–101

[30] Qu J F, Benz S P, Rogalla H and White D R 2009 Reduced nonlinearities and improved temperature measurements for the NIST Johnson noise thermometer Metrologia 46 512–24

[31] Qu J F et al 2011 Reduced nonlinearity effect on the electronic measurement of the Boltzmann constant IEEE Trans. Instrum. Meas. 60 2427–33

[32] White D R, Benz S P, Labenski J R, Nam S W, Qu J F, Rogalla H and Tew W L 2008 Measurement time and statistics for a noise thermometer with a synthetic-noise reference Metrologia 45 395–405

[33] Pollarolo A, Jeong T, Benz S P and Rogalla H 2013 Johnson noise thermometer measurement of the Boltzmann constant with a 200Ω sense resistor IEEE Trans. Instrum. Meas. 62 1512–7

[34] Qu J F, Fu Y F, Zhang J Q, Rogalla H, Pollarolo A and Benz S P 2013 Flat frequency response in the electronic measurement of the Boltzmann constant measurement IEEE Trans. Instrum. Meas. 62 1518–23

[35] Qu J F, Zhang J T, Fu Y F, Rogalla H, Pollarolo A and Benz S P 2013 Development of a quantum-voltage-calibrated noise thermometer at NIM AIP Temp.: Meas. Control Sci. Ind. 8 29–33

[36] Brixey H 1971 Temperature measurement in nuclear reactors by noise thermometer Nucl. Instrum. Methods 97 75–80

[37] Fink H J 1959 A new absolute noise thermometer at low temperatures Can. J. Phys. 37 1397–1406

[38] Dicke R H 1946 The measurement of thermal radiation at microwave frequencies Rev. Sci. Instrum. 17 268–75

[39] Klitzing K V, Dorda G and Pepper M 1980 New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance Phys. Rev. Lett. 45 494–7

[40] Taylor B N and Witt T J 1989 New international electrical reference standards based on the Josephson and quantum Hall effect Metrologia 26 47–62

[41] Benz S P, Dresselhaus P D and Burroughs C J 2011 Multitone waveform synthesis with a quantum voltage noise source IEEE Trans. Appl. Supercond. 21 681–6

[42] Josephson B D 1962 Possible new effects in superconducting tunneling Phys. Lett. 1 251–3

[43] Benz S P and Walmann S B 2014 Pulse-bias electronics and techniques for a Josephson arbitrary waveform synthesizer IEEE Trans. Appl. Supercond. 24 1400107

[44] Zhou K L, Qu J F, Dong X Y and Benz S P in preparation

[45] White D R and Zimmermann F 2000 Preamplifier limitations on the accuracy of Johnson noise thermometers Metrologia 37 11–23

[46] White D R and Pickup C P 1987 Systematic errors in digital cross correlators due to quantization and differential nonlinearity IEEE Trans. Instrum. Meas. 36 47–53

[47] Pollarolo A, Tew, W, Rogalla H, Underwood J M and Benz S P 2014 Systematic error resolved in NIST Johnson noise thermometer Conf. on Precision Electromagnetic Measurements 26–27

[48] Stuart A, Ord K and Arnold S 1997 Kendall’s Advanced Theory of Statistics 6th edn (London: Wiley) vol 1 pp 351

[49] Labenski J R, Tew W L, Nam S W, Benz S P, Dresselhaus P D and Burroughs C J 2007 IEEE Trans. Instrum. Meas. 56 481–5

[50] Hjort N L and Claeskens G 2008 Model Selection and Model Averaging (Cambridge: Cambridge University Press)

[51] Coakley K 2015 Uncertainty of electronic measurement of the Boltzmann constant due to imperfect knowledge of spectral aberration models in preparation

[52] White D R and Mason R S 2004 An EMI test for Johnson noise thermometer Proc. TEMPMEKO 2004 (Cavtat-ubrovnik, Croatia) ed D Zvijdic et al (Zagreb: Faculty of Mechanical Engineering and Naval Architecture) pp 485–90

[53] Ott H W 1988 Noise Reduction Techniques in Electronic Systems 2nd edn (New York: Wiley)

[54] Willink R and White D R 1998 Detection of corruption in Gaussian processes with application to noise thermometry Metrologia 35 787–98

[55] Consultative Committee for Thermometry 2006 Mise en pratique for the definition of the Kelvin (Sèvres, France: Bureau International des Poids et Mesures)

[56] Yan X, Qu P, Wang Y, Wu H, Feng Y and Zhang Z 2004 NIM development of the triple point of water cells of high quality Proc. of TEMPMEKO 2004, 9th Int. Symp. on Temperature and Thermal Measurements in Industry and Science (FSBA/PM, Zagreb, Croatia, 2004) ed D Zvijdić et al pp 283–8

[57] Stock M et al 2006 Final report on CCT-K7 key comparison of water triple point cell Metrologia 43 03001 (http://kcdt.bipm.org/appendixC/T/CN_T_CN.pdf)

[58] White D R and Jongeneelen A 2010 The immersion characteristics of industrial PRT’s Int. J. Thermophys. 31 1685–95

[59] Sasaki H, Nishinaka H and Shida K 1987 A modified wheatstone bridge for high-precision automated resistance measurement Japan. J. Appl. Phys. 26 L1947–9 Pt. 2, Letters

[60] Kibble B P and Raynor G H 1984 Coaxial AC Bridges (Bristol: Hilger)