Polynomial to non-Darcian flow study

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Abstract. The theoretical modeling of fluid flow in low-permeability porous media is a common problem within engineering such as passages of fluid flow. Some factors lead the flow behaviors to deviate from the classical Darcy’s law. Therefore, it is of great scientific significance to carry out the research of non-Darcian flow and specific analysis the fluid flow in porous media that do not satisfy the linear constitution law of flow. Compared with existing research results, this paper mainly shows that polynomials with powers of no more than 4 degrees have smaller mean squared errors (MSE) and higher correlation coefficients (R\textsuperscript{2}) in experimental data.

1. Introduction
Since Henry Darcy’s remarkable modeling of linear relation between water flux and the hydraulic gradient, e.g., Darcian flow, in 1856, many researchers found the Darcy’s law is not good enough for description of water and gas flow in low-permeability media like clay and shale. So extensive efforts have been devoted to approaches of nonlinear relation between water flux and hydraulic gradient called non-Darcian flow in past decades. Hansbo\textsuperscript{[1]}\textsuperscript{[2]} proposed a power relationship between water flux and hydraulic gradient for non-Darcian flow in clay media. Aiming at low-permeability media like clay, Miller and Low\textsuperscript{[3]} supposed a threshold gradient for water flow in clays to distinguish linear and nonlinear flow. They found that water flow rate is linearly related to hydraulic gradient at gradients above the threshold gradient, but no flow occurs below threshold gradient. By analyzing data sets for water flow in clay soils, Swartzendruber\textsuperscript{[4]} proposed an exponential function to validate Darcy’s law, resulting in a nonlinear relation of water flux versus gradient. Recently, Deng et al.\textsuperscript{[5]} suggested a new equation of nonlinear flow in saturated clays that can describe characteristics of flow curve of the nonlinear flow from low to high hydraulic gradients. In order to capture the non-Darcian flow behavior, Liu and Birkholzer\textsuperscript{[6]} developed a new relationship between water flux and hydraulic gradient by generalizing the currently existing relationships. The new relationship is shown to be consistent with experimental observations for both saturated and unsaturated conditions. Wang et al.\textsuperscript{[7]} developed an experimental study to investigate the non-Darcian behavior of water flow in soil-rock mixtures (SRM) with various rock block percentages. Their work presented the data set of water flux as a power function of hydraulic gradient. Zhou H W and Yang S\textsuperscript{[8]} proposed the fractional porous
medium equations and explained that the equation was consistent with the data in Wang et al. [7].

To validate Darcy’s law and develop non-Darcian models seem to be an endless challenge. Non-Darcian flow in different porous media as a nonlinear phenomenon requires many new mathematical approaches. In this paper, the experimental data in [7] were used to fit polynomials because Hansbo [1][2] proposed that non-darcy seepage flow can be represented by polynomials. In this process, it was found that when the power of the polynomials did not exceed the fourth degree, mean squared errors (MSE) was smaller and correlation coefficients (R^2) was higher than that of Swartzendruber equation [4] and fractional porous medium equations [8].

2. Parameter determination for non-Darcian flow

Hansbo [1][2] suggested a relationship between water flux and hydraulic gradient to represent the non-Darcian flow behavior in low-permeability clay material,

\[ q = \begin{cases} \text{ki}^n, & i \leq i_1 \\ \text{ki}^{n+1}(i-I), & i \geq i_1 \end{cases} \]

where \( i_1 = \frac{nI}{n-1}, I \) is threshold hydraulic gradient, n (–) is an exponent generally greater than 1.

Swartzendruber [3] proposed an exponential relation between water flux and hydraulic gradient to modify Darcy’s law,

\[ \begin{align*}
\frac{dq}{di} &= k(1 - e^{\frac{-I}{i}}) \\
q(0) &= 0
\end{align*} \]

where I is the threshold gradient and actually refers to the intersection of the linear part in plot of the hydraulic gradient and the water flux. H.W.Zhou indicated that the fractional derivative flow model

\[ \frac{d^\gamma q}{di^\gamma} = k(1 - e^{\frac{-I}{i}}), 0 \leq \gamma \leq 1 \]

where I is the threshold gradient. And (3) is better agreement with the experimental data [7] than the Swartzendruber equation (2) with lower mean squared errors in [8]. Under the same conditions, it is concluded that a series of polynomial by polynomial data fitting for marketers rock block percentages 20%-70% and get correlation coefficients (R^2) and mean squared errors (MSE) in table 1.

First, take \( n = 2 \) and combine with the data [7], the polynomials with different rock block percentages 20%-70% were

\[ q_{12} = (0.3328i^2 + 28.34i - 655.3) \times 10^{-8}, \]
\[ q_{22} = (0.2964i^2 + 15.79i - 610.4) \times 10^{-8}, \]
\[ q_{32} = (0.1369i^2 + 11.7i - 299.6) \times 10^{-8}, \]
\[ q_{42} = (0.9209i^2 - 62.85i - 1944) \times 10^{-8}, \]
\[ q_{52} = (0.4609i^2 + 2.087i - 215.8) \times 10^{-8}, \]
\[ q_{62} = (1.431i^2 + 0.552i - 1702) \times 10^{-6}. \]

Second, take \( n = 3 \) and combine with the data [7], the polynomials with different rock block percentages 20%-70% were

\[ q_{13} = (0.007877i^3 - 1.519i^2 + 159.6i - 3353) \times 10^{-8}, \]
\[ q_{23} = (-0.00373i^3 + 1.261i^2 - 58.53i - 991.2) \times 10^{-8}, \]
\[ q_{33} = (0.001482i^3 - 20.95i^2 + 36.27i - 807.3) \times 10^{-8}, \]
\[ q_{43} = (0.0271i^3 - 4.603i^2 + 281.4i - 4456) \times 10^{-8}, \]
\[ q_{53} = (0.01523i^3 - 2.325i^2 + 159.4i - 2463) \times 10^{-8}, \]
\[ q_{63} = (0.05258i^3 - 6.009i^2 + 277.9i - 2968) \times 10^{-6}. \]

Last, take \( n = 4 \) and combine with the data\[7]\[4], the polynomials with different rock block percentages 20\%-70\% were
\[ q_{44} = (0.0002099i^4 - 0.06486i^3 + 7.225i^2 - 264.8i - 3504) \times 10^{-8}, \]
\[ q_{24} = (0.00009855i^4 - 0.006501i^3 + 1.57i^2 - 0.7238i - 1199) \times 10^{-8}, \]
\[ q_{34} = (0.001018i^4 - 0.0321i^3 + 3.447i^2 - 126.5i + 1649) \times 10^{-8}, \]
\[ q_{44} = (0.0002585i^4 - 0.04368i^3 + 2.232i^2 + 9.273i - 731.6) \times 10^{-8}, \]
\[ q_{54} = (0.0009895i^4 - 0.2246i^3 + 18.38i^2 - 588.2i + 6980) \times 10^{-8}, \]
\[ q_{64} = (0.007487i^4 - 883.4i^3 + 3.588i^2 - 1.056i - 59.35) \times 10^{-6}. \]

### Table 1 Correlation coefficients (R²) and mean squared errors (MSE) with the polynomials

| SRM specimens | n=2   | n=3   | n=4   |
|---------------|-------|-------|-------|
|               | R²    | MSE   | R²    | MSE   | R²    | MSE   |
| SRM20-1       | 0.9916| 0.9973e-11 | 0.9914| 0.1258e-10 | 0.9980| 0.2377e-11 |
| SRM30-1       | 0.9925| 0.4057e-11 | 0.9964| 0.1956e-11 | 0.9957| 0.2312e-11 |
| SRM40-1       | 0.9801| 0.1964e-11 | 0.9805| 0.1916e-11 | 0.9826| 0.1715e-11 |
| SRM50-1       | 0.9467| 0.1694e-10 | 0.9928| 0.229e-11   | 0.9939| 0.1929e-11 |
| SRM60-1       | 0.9751| 0.4212e-11 | 0.9825| 0.2958e-11 | 0.9896| 0.1733e-11 |
| SRM70-1       | 0.9743| 0.3184e-07 | 0.9864| 0.1676e-07 | 0.9867| 0.1142e-07 |

### Table 2 Correlation coefficients (R²) and mean squared errors (MSE) with the existed methods

| SRM specimens | Swartzendruber equation | Fractional derivative flow model |
|---------------|-------------------------|---------------------------------|
|               | R²    | MSE   | R²    | MSE   |
| SRM20-1       | 0.9817| 0.2242| 0.9817| 0.2242|
| SRM30-1       | 0.9898| 0.0531| 0.9898| 0.0531|
| SRM40-1       | 0.9869| 0.1358| 0.9908| 0.0095|
| SRM50-1       | 0.9326| 0.2322| 0.9326| 0.2322|
| SRM60-1       | 0.9876| 0.0363| 0.9819| 0.0307|
| SRM70-1       | 0.9710| 0.0291| 0.9807| 0.0193|

As can be seen from table 1, with the increase of the rock block percentage, mean squared errors (MSE) was smaller and correlation coefficients (R²) was higher. By comparing table 1 and table 2, the polynomial is more effective than Swartzendruber equation and fractional derivative flow model as far as mean squared errors (MSE) is concerned. It is the more effective with the higher the power of the polynomial as far as correlation coefficients (R²) is concerned. But with the increase of the rock block...
percentage, the polynomial and Swartzendruber equation and fractional derivative flow model have a downward trend in mean squared errors (MSE) and correlation coefficients ($R^2$).

3. Conclusion
The object of this paper is to discuss how to describe non-Darcian flow behavior between water flux and hydraulic gradient. Through the above research, it is found that polynomials can be used to solve non-Darcian flow problems, which is easy to understand. However, there is no uniform fixed value to determine the coefficient according to the specific situation of seepage.

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References
[1] Hansbo S. (1960) Consolidation of clay, with special reference to influence of vertical sand drains. Swedish Geotechnical Institute Proc. Stockholm. volume 18.
[2] Hansbo S. (2001) Consolidation equation valid for both Darcian and non-Darcian flow. Geotechnique. 51(1): 51-4.
[3] Miller R, Low P. (1963) Threshold gradient for water flow in clay systems. Soil Sci Soc Am Proc. 27(6): 605-9.
[4] Swartzendruber D. (1962) Modification of Darcy’s law for the flow of water in soils. Soil Science. 93(1): 22-9.
[5] Deng Ye, Xie Hp, Huang Rq, Liu Cq. (2007) Law of nonlinear flow in saturated clays and radial consolidation. Applied Mathematics and Mechanics. 28(11): 1427-36.
[6] Liu H.H, Lai B., Chen J.. (2016) Unconventional spontaneous imbibition into shale matrix: Theory and a methodology to determine relevant parameters. Transport in Porous Media, 111(1): 41-57.
[7] Wang Y, Li X, Zheng B, Zhang YX, Li GF, Wu YF. (2016) Experimental study on the non-Darcian flow characteristics of soil-rock mixture. Environmental Earth Sciences. 75(9): 756.
[8] Zhou, H.W., Yang, S., Wang, R., Zhong, J.C. (2018) Non-Darcian flow or fractional derivative? ArXiv e-prints 2018a; <arXiv:1806.00977>