Abstract. The Jordan totient $J_k(n)$ can be defined by $J_k(n) = n^k \prod_{p|n}(1 - p^{-k})$. In this paper, we study the average behavior of fractions $P/Q$ of two products $P$ and $Q$ of Jordan totients, which we call Jordan totient quotients. To this end, we prepare some general and ready-to-use methods to deal with a wider class of totient functions first by an elementary method, and then by using an advanced method due to Balakrishnan and Pétermann. As an application, we determine the average behavior of the Jordan totient quotient, the $k$th normalized derivative of the $n$th cyclotomic polynomial $\Phi_n(z)$ at $z = 1$, the second normalized derivative of the $n$th cyclotomic polynomial $\Phi_n(z)$ at $z = -1$, and the average order of the Schwarzian derivative of $\Phi_n(z)$ at $z = 1$.

1. Introduction

Jordan totient quotients. Let $k \geq 1$ be an integer. The $k$th Jordan totient function $J_k(n)$ is the number of $k$-tuples chosen from a complete residue system modulo $n$ such that the greatest common divisor of each set is coprime to $n$. It is not difficult to show that

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right),$$

where $p$ here, and indeed in the whole paper, denotes a prime number. The Jordan function first showed up in the work of Camille Jordan in 1870 in formulas for the order of finite matrix groups (such as $GL(m, \mathbb{Z}/n\mathbb{Z})$). For an introduction to Jordan totients see Section 2.

Definition. Let $r \geq 1$ be an integer and $e = (e_1, \ldots, e_r)$ be a vector with integer entries. Put $w = \sum_i i e_i$. An arithmetic function $J_e$ of the form

$$J_e(n) = \prod_{i=1}^r J_{e_i}^{e_i}(n) = n^w \prod_{p|n} \prod_{i=1}^r \left(1 - \frac{1}{p^{e_i}}\right)^{e_i},$$

is said to be a Jordan totient quotient of weight $w$. If $w = 0$, then we say that it is a balanced Jordan totient quotient. If the weight is different from 0 we call it unbalanced.

Note that if $J_e$ is balanced, then $J_e(n)$ depends only on the square-free kernel of $n$. A famous (unbalanced) Jordan totient quotient is the Dedekind $\Psi$-function defined by

$$\Psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right) = \frac{J_2(n)}{J_1(n)},$$

which showed up in the work of Dedekind on modular forms.

In this paper we study the average behavior of Jordan totient quotients. In the remainder of the introduction we describe our main results, including an application to the study of the average of the normalized derivative of cyclotomic polynomials.

2010 Mathematics Subject Classification. 11N37, 11Y60.
Our first result gives an asymptotic formula for the summatory function of any balanced Jordan totient quotient $J_e(n)$, which implies that $J_e(n)$ is constant on average.

**Theorem 1.** Let $r \in \mathbb{N}$, $e = (e_1, \ldots, e_r) \in \mathbb{Z}^r$ be a vector of integers, and $J_e$ be a Jordan totient quotient of weight $w = \sum_i ie_i = 0$. Then asymptotically

$$\sum_{n \leq x} J_e(n) = \mathcal{S}_e x + \sum_{r=1}^{|e_1|} C_{e,r} (\log x)^r + O_e((\log x)^{|e_1|/3}(\log \log x)^{4|e_1|/3}),$$

where the constant

$$\mathcal{S}_e = \prod_p \left(1 + \frac{J_e(p)p^{-w} - 1}{p}\right),$$

is positive and the $C_{e,r}$ are some constants.

Note that the convergence of $\mathcal{S}_e$ is ensured since

$$1 + \frac{J_e(p)p^{-w} - 1}{p} = 1 + O(p^{-2}).$$

As $J_e(p)p^{-w} > 0 > 1 - p$, we have $\mathcal{S}_e > 0$. This constant can be expanded as a product of partial zeta values, see Moree and Niklasch [7, 8]. As partial zeta values can be easily evaluated up to high precision (say, with thousand decimals), this then allows one to do the same for $\mathcal{S}_e$.

In case $e = (0)$ is the zero vector, then $J_e(n) = 1$ for every $n \geq 1$, $\mathcal{S}_e = 1$ and Theorem 1 merely states that $\sum_{n \leq x} 1 = x + O(1)$.

The proof of Theorem 1 uses the method of Balakrishnan and Pétermann [2]. We consider not only the balanced Jordan totient quotients, but also a more general class of totient functions (see Section 3 for the definitions), a similar to the one earlier studied by Kaczorowski [5] in the context of the inverse theorems for the Selberg class. An analog of Theorem 1 for $w \neq 0$ can be easily established on invoking Lemma 6 and partial summation. As this is a long and rather inelegant result, we leave it to the interested reader to write it down.

Before applying the Balakrishnan-Pétermann method as in Section 4, we develop a simpler argument (see Section 3), which actually applies to a wider class of totients. This method allows us to get the main term of Theorem 1 however only with a weaker error term.

**Theorem 2.** Let $r \in \mathbb{N}$, $e = (e_1, \ldots, e_r) \in \mathbb{Z}^r$ be a vector of integers, and $J_e$ be a Jordan totient quotient of weight $w = \sum_i ie_i = 0$. Then asymptotically

$$\sum_{n \leq x} J_e(n) = \mathcal{S}_e x + O_e((\log x)^{|e_1|}),$$

where the constant $\mathcal{S}_e$ is positive and given by (3).

By elementary means, we also obtain the mean value estimate of Theorem 2 for non-zero weight (see Proposition 1).

It is an open problem to obtain a result at least as strong as Theorem 1 by more elementary methods than used by Balakrishnan and Pétermann.

---

1Work in progress by the fourth author [13] suggests that the exponent $4|e_1|/3$ of $\log \log x$ in the error term can be decreased to $|e_1|/3$. 
Applications. In Section 5 of the present paper, we consider normalized higher derivatives of cyclotomic polynomials at 1. Our main result shows that they are constant on average. We use the standard notation $\Phi_n$ and $B_n$ for the $n^{th}$ cyclotomic polynomial and $n^{th}$ Bernoulli number, respectively (cf. Section 5.1).

**Theorem 3.** Let $k \geq 1$. There exist a computable constant $\mathcal{S}_k(\Phi)$ and constants $C_0, \ldots, C_r$ such that asymptotically

$$
\sum_{1 < n \leq x} \frac{\Phi_n^{(k)}(1)}{\varphi(n)^k \Phi_n(1)} = \mathcal{S}_k(\Phi)x + \sum_{r=1}^{k} C_r (\log x)^r + O_k((\log x)^{2k/3}(\log \log x)^{4k/3}),
$$

where the constant $\mathcal{S}_k(\Phi)$ is defined by

$$
\mathcal{S}_k(\Phi) = (-1)^k k! \prod_{i=1}^{k} \frac{1}{\lambda_i!} \left(\frac{B_i}{i!} \right)^{\lambda_i} \mathcal{G}_{e(\lambda)}
$$

with the summation $\sum_{(s)}$ taken over all non-negative $\lambda_1, \ldots, \lambda_k \geq 0$ such that $\lambda_1 + 2\lambda_2 + \ldots + k\lambda_k = k$, and the indices $e(\lambda) = e(\lambda_1, \ldots, \lambda_k)$ are defined by

$$
e(\lambda) = (e_i(\lambda))_{i=1}^{\infty}, \quad e_i(\lambda) = \begin{cases} 
\lambda_i - k, & i = 1, \\
\lambda_i, & 2 \leq i \leq k, \\
0, & i > k.
\end{cases}
$$

Note that the vectors $e(\lambda)$ appearing as summands in (4) are all balanced. We can not predict the sign of $\mathcal{S}_k(\Phi)$. Theoretically we could even have $\mathcal{S}_k(\Phi) = 0$.

Although some part of the sum $\sum_{r=1}^{k} C_r (\log x)^r$ can be swamped by the error term, it turns out to be easier to work with this full series rather than an appropriately truncated one.

In case $k = 1$, we have by (17)

$$
\sum_{1 < n \leq x} \frac{1}{\varphi(n)} \frac{\Phi_n'(1)}{\Phi_n(1)} = \sum_{1 < n \leq x} \frac{1}{2} = \frac{x}{2} + O(1),
$$

improving Theorem 3. However, as our method of proof naturally includes the case $k = 1$, we have not excluded it from our formulation of Theorem 3.

Theorem 3 is a simple consequence of Lemma 8 and Theorem 1. We expect that an analogous result can be obtained with 1 replaced by any root of unity, and that this would involve averages of generalized Jordan totients (introduced in Bzdęga et al. [3]) of the form

$$
J_k(\chi; n) = \sum_{d|n} \mu(n/d)\chi(d)d^k,
$$

with $\chi$ a Dirichlet character of modulus $m$ and $m$ is the order of the root of unity. We will see such a result for $-1$ in case $k = 2$ in the proof of Theorem 4 which is due to Herrera-Poyatos and the first author [1].

Finally, in Theorem 5, we determine the average of the Schwarzian derivative of $\Phi_n(z)$ evaluated at $z = 1$.

2. THE TOTIENT FUNCTIONS

Let $k \geq 1$ be an integer and $J_k(n)$ be the $k^{th}$ Jordan totient function. This is one of many generalizations of Euler’s totient function (the case $k = 1$), see Sivaramakrishnan [11]. It is
easy to see, cf. [12, p. 91], that
\[ n^k = \sum_{d|n} J_k(d), \]
which, by Möbius inversion, yields
\[ J_k(n) = \sum_{d|n} \mu(d) \left( \frac{n}{d} \right)^k. \]  
Thus \( J_k \) is a Dirichlet convolution of two multiplicative functions and hence is itself multiplicative. By the Euler product formula, it then follows from (7) that (1) holds true.

Given a Jordan totient quotient function of weight \( w = \sum_i i e_i \) as in (2), we normalize it by dividing by \( n^w \), resulting in
\[ \frac{J_e(n)}{n^w} = \prod_{p|n} \prod_{i=1}^r \left( 1 - \frac{1}{p^i} \right)^{e_r} = \prod_{p|n} \left( 1 - \frac{e_1}{p} + O \left( \frac{1}{p^2} \right) \right). \]
Although our focus is the study of this particular function, our methods easily allow a more general class of totients to be dealt with.

**Definition (General totient).** Let \( \theta_n \) be a complex valued multiplicative function supported on square-free numbers. Define the \( \theta \)-totient \( \phi_\theta(n) \) by
\[ \phi_\theta(n) = \prod_{p|n} (1 + \theta_p) = \sum_{d|n} \theta_d. \]

It is easy to see that any arithmetic function \( f \) that only depends on the square-free kernel of \( n \) for every \( n \geq 1 \), is of the form \( \phi_\theta \) for some \( \theta \).

We next describe the conditions we impose on \( \theta \) throughout the paper.

**Condition \( \Theta_1 \).** There exist non-negative constants \( \sigma, \kappa, A \) with \( 0 \leq \sigma < 1 \) such that for any \( x \geq 2 \) we have
\[ \sum_{p \leq x} \frac{|\theta_p|}{p^\sigma} \leq \kappa \log \log x + A. \]

**Condition \( \Theta_2 \).** There exist \( 0 < \lambda < 1/2 \) and \( \alpha \in \mathbb{R} \) with \( |\alpha| \geq 1 \) such that for all primes \( p \) we have
\[ \theta_p = \alpha/p + r_p, \quad r_p = O(p^{-1-\lambda}). \]

**Condition \( \Theta_3 \).** With respect to \( p \) the function \( p \theta_p \) is ultimately monotonic.

Note that if Condition \( \Theta_2 \) is satisfied, then so is Condition \( \Theta_1 \) with \( \sigma = 0 \) and \( \kappa = |\alpha| \).

We point out that in order to prove Theorem 2, only Condition \( \Theta_1 \) is needed, whereas to prove Theorem 1 we shall impose the stronger Condition \( \Theta_2 \). Notice that if \( \theta \) is defined by \( J_e(n)/n^w = \phi_\theta(n) \), then Condition \( \Theta_2 \) is satisfied with \( \alpha = -e_1 \) and \( \lambda = 1 \), cf. (8).

3. **Mean values of general totients via an elementary method**

In this section, we give a simple method to obtain asymptotic formulas for the mean value of multiplicative functions of a certain type. The ideas and techniques are not new, but our aim is to provide a quick way to translate the definition of multiplicative functions to the asymptotic formula of its mean value. As we have seen, our \( \theta \)-totient is modeled on the normalized Jordan totient quotient \( J_e \). Thus we need to introduce a weight factor \( n^\beta \).

\(^2\text{Condition } \Theta_3 \text{ can be removed at the expense of more technicalities, see [13].}\)
Lemma 1. Let \( \beta \) be an arbitrary real number. For \( x \geq 1 \) we have
\[
\sum_{n \leq x} n^\beta = M_\beta(x) + C_0(\beta) + O(x^\beta),
\]
where \( C_0(\beta) \) is a constant depending only on \( \beta \), \( M_{-1}(x) = \log x \) and
\[
M_\beta(x) = \frac{x^{\beta+1}}{\beta + 1}, \text{ if } \beta \neq -1.
\]

Proof. Follows from parts (a), (b), and (d) of [1, Theorem 3.2]. \( \square \)

Lemma 2. Let \( \phi \) be a \( \theta \)-totient and \( \beta \) be an arbitrary real number. Assume that \( \theta \) satisfies Condition \( \ominus \). We then have
\[
\sum_{n \leq x} n^\beta \phi(n) = S_\theta M_\beta(x) + C(\theta, \beta) + O(\sigma, \kappa, A, \beta) \left( x^{\beta+1} (\log x)^\kappa \right),
\]
where \( S_\theta \) is given by the absolutely convergent product
\[
S_\theta = \prod_p \left( 1 + \frac{\theta_p}{p} \right),
\]
and \( C(\theta, \beta) \) is a constant depending only on \( \theta \) and \( \beta \).

Proof. By the definition of \( \theta \)-quotient, we have
\[
\sum_{n \leq x} n^\beta \phi(n) = \sum_{n \leq x} \sum_{d \mid n} \theta_d = \sum_{d \leq x} d^\beta \sum_{m \leq x/d} m^\beta.
\]
Thus, by Lemma 1 we have
\[
\sum_{n \leq x} n^\beta \phi(n) = \frac{x^{\beta+1}}{\beta + 1} \sum_{d \leq x} \frac{\theta_d}{d} + C_0(\beta) \sum_{d \leq x} d^\beta \theta_d + O(\frac{1}{x} \sum_{d \leq x} |\theta_d|)
\]
for the case \( \beta \neq -1 \), and
\[
\sum_{n \leq x} n^\beta \phi(n) = \sum_{d \leq x} \frac{\theta_d}{d} \log \frac{x}{d} + C_0(\beta) \sum_{d \leq x} \frac{\theta_d}{d} + O(\frac{1}{x} \sum_{d \leq x} |\theta_d|)
\]
for the case \( \beta = -1 \). Using Condition \( \ominus \) we find that
\[
\sum_{d \leq x} \left| \frac{\theta_d}{d} \right| \leq \prod_{p \leq x} \left( 1 + \frac{|\theta_p|}{p^\sigma} \right) \leq \exp \left( \sum_{p \leq x} \frac{|\theta_p|}{p^\sigma} \right) \ll_{\sigma, \kappa, A} (\log x)^\kappa.
\]
This implies that for \( \beta \geq -\sigma \)
\[
\sum_{d \leq x} d^\beta |\theta_d| \leq x^{\sigma+\beta} \sum_{d \leq x} \frac{|\theta_d|}{d^\sigma} \ll_{\sigma, \kappa, A} x^{\sigma+\beta} (\log x)^\kappa,
\]
and that for \( \beta < -\sigma \)
\[
\sum_{d > x} d^\beta |\theta_d| \ll_{\sigma, \beta} \sum_{d > x} \frac{|\theta_d|}{d^\sigma} \int_x^\infty u^{\sigma+\beta-1} du \ll_{\sigma, \beta} \int_x^\infty \left( \sum_{d \leq u} \frac{|\theta_d|}{d^\sigma} \right) u^{\sigma+\beta-1} du \\
\ll_{\sigma, \kappa, A, \beta} x^{\sigma+\beta} (\log x)^\kappa.
\]
Hence, in particular,
\[
\sum_{d \leq x} |\theta_d| \ll \sigma (\log x)^\kappa,
\]
and
\[ \sum_{d \leq x} \frac{\theta_d}{d} = \mathcal{S}_\theta + r(x), \quad r(x) \ll_{\sigma, \kappa} x^{\sigma - 1} (\log x)^\kappa. \]

By combining the above, we obtain the assertion for the case $\beta \neq -1$.

For the case $\beta = -1$, we have to evaluate the main term. We have
\[
\sum_{d \leq x} \frac{\theta_d}{d} \log \frac{x}{d} = \sum_{d \leq x} \frac{\theta_d}{d} \int_{d}^{x} \frac{du}{u} = \int_{1}^{x} \left( \sum_{d \leq u} \frac{\theta_d}{d} \right) \frac{du}{u}
\]
\[= \mathcal{S}_\theta \log x + \int_{1}^{x} \frac{r(u)}{u} \, du = \mathcal{S}_\theta \log x + \int_{1}^{\infty} \frac{r(u)}{u} \, du - \int_{x}^{\infty} \frac{r(u)}{u} \, du. \]

The last integral can be estimated as
\[\int_{x}^{\infty} \frac{r(u)}{u} \, du \ll \int_{x}^{\infty} u^{\sigma - 2} (\log u)^\kappa \, du \ll x^{\sigma - 1} (\log x)^\kappa\]

since $0 \leq \sigma < 1$ as in Condition $\Theta_1$. This completes the proof for the case $\beta = -1$. \qed

As a special case we obtain the following result involving the Jordan totient quotient.

**Proposition 1.** Let $e = (e_1, \ldots, e_r) \in \mathbb{Z}^r$ be a vector of integers and $J_e(n)$ be the associated Jordan totient quotient of weight $w = \sum_i i e_i$. For any real number $\beta$ we have
\[\sum_{n \leq x} J_e(n)n^\beta = \mathcal{S}_e M_{\beta + w}(x) + C(e, \beta) + O_{e, \beta}(x^{\beta + w} (\log x)^{\lfloor e \rfloor}), \]

where $\mathcal{S}_e$ is given by $\mathcal{S}$ and $C(e, \beta)$ is a constant depending only on $e$ and $\beta$.

**Proof.** We can regard $J_e(n)n^{-w}$ as a general totient $\phi_{\theta}(n)$ with components $\theta_p = -e_1/p + O(p^{-2})$.

This gives
\[\sum_{p \leq x} |\theta_p| = \sum_{p \leq x} \frac{|e_1|}{p} + O(1) = |e_1| \log \log x + O(1), \]
i.e. the current $\theta$ satisfies Condition $\Theta_1$ with $\sigma = 0$, $\kappa = \lfloor e_1 \rfloor$. Note that
\[\theta_p = J_e(p)p^{-w} - 1, \]

and so the comparison of $\theta$ and $\mathcal{S}$ yields $\mathcal{S}_\theta = \mathcal{S}_e$. Under the above setting, we can rewrite the left-hand side of the assertion as
\[\sum_{n \leq x} J_e(n)n^\beta = \sum_{n \leq x} n^{\beta + w} \left( \frac{J_e(n)}{n^w} \right) = \sum_{n \leq x} n^{\beta + w} \phi_{\theta}(n)\]

and the proposition follows by Lemma $\Theta_2$. \qed

**Corollary 1.** For $k \geq 1$ we have
\[\sum_{n \leq x} n^{k - 1} \phi(n)^k = \mathcal{S}_{(-k)} \log x + C_k + O_k \left( \frac{(\log x)^k}{x} \right),\]

where $\mathcal{S}_{(-k)}$ is given by $\mathcal{S}$ and $C_k$ is a constant depending on $k$.

**Proof.** Apply Proposition $\Theta_1$ with $J_e(n) = \phi(n)^{-k}$, $w = -k$ and $\beta = k - 1$. \qed
4. MEAN VALUES OF GENERAL TOTIENTS BY BALAKRISHNAN-PÉTERMANN

In this section we use the method of Balakrishnan and Pétermann [2] in order to prove Theorem 1. This method yields an asymptotic formula for the mean value of \( \theta \)-totients, provided some condition stronger than Condition \( \Theta_1 \) is satisfied. It consists of Propositions 2 and 3 below.

**Proposition 2** (Balakrishnan and Pétermann [2, Theorem 1]). Let

\[
f(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}
\]

be a Dirichlet series that converges absolutely for \( \sigma > 1 - \lambda \), with \( \lambda \) a positive real number. Define two arithmetic functions \( a_n \) and \( v_n \) by

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \zeta(s)\zeta(s + 1)^\alpha f(s + 1), \quad \sum_{n=1}^{\infty} \frac{v_n}{n^s} = \zeta(s)^\alpha f(s),
\]

where \( \sigma > 1, \alpha \) is an arbitrary real number and the branch of \( \zeta(s + 1)^\alpha \) is taken by the one for which \( \text{arg} \, \zeta(s + 1) \) equals zero on the positive real line. Then we have

\[
\sum_{n \leq x} a_n = \zeta(2)^\alpha f(2)x + \sum_{r=0}^{[\alpha]} A_r (\log x)^{\alpha-r} + R(x) + o(1)
\]

as \( x \to \infty \), where the coefficients \( A_r \) are computable from the Laurent expansion of \( \zeta(s)^\alpha f(s) \) at \( s = 1 \), the remainder term \( R(x) \) is given by

\[
R(x) = \sum_{n \leq y} \frac{v_n}{n^s} \psi \left( \frac{x}{n} \right),
\]

with \( y = x \exp(-{(\log x)^{1/6}}) \), and \( \psi(x) = \{x\} - 1/2 \). The implicit constant in the error term might depend on all the input data.

**Remark.** In [2], there are several places to use the zero-free region for the Riemann zeta function. Throughout this paper, we use a specific zero-free region

\[
\text{Re } s \geq 1 - \frac{1}{(\log t)^{4/5}} \quad \text{and} \quad \text{Im } s \geq t_0,
\]

where \( t_0 \) is some large constant. See (6.15.1) of [14]. This zero-free region enables us to take \( b = 1/6 \) in [2]. See Subsection 1.4, Lemma 3, and Lemma 5 of [2].

**Lemma 3** (Balakrishnan and Pétermann [2, Lemma 3]). In the notation of Proposition 2 we have

\[
\sum_{n \leq x} \frac{v_n}{n} = \sum_{0 \leq r \leq \log x^{1/6}} V_r (\log x)^{\alpha-r} + O(\exp(-{(\log x)^{1/6}})),
\]

with \( |V_r| \leq (cr)^r \) for every \( r \geq 1 \) and \( c \geq 1 \) a constant possibly depending on \( v \).

Now we prove Theorem 1. As already mentioned, we need to assume that \( \theta \) satisfies a stronger condition than Condition \( \Theta_1 \). In this section, we use Conditions \( \Theta_2 \) and \( \Theta_3 \) and hence all implicit constants in this section will depend on the constants \( \alpha, \lambda \) and the implicit constant appearing in Condition \( \Theta_2 \).
Lemma 4. Let $\phi_\theta$ be a $\theta$-totient with $\theta$ satisfying Condition Θ2. Consider the formal Dirichlet series
\[
f(s + 1) = \sum_{n=1}^{\infty} \frac{b_n}{n^{s+1}} = \zeta(s)^{-1} \zeta(s + 1)^{-\alpha} \sum_{n=1}^{\infty} \frac{\phi_\theta(n)}{n^s},
\]
where $\alpha$ is the same one as in Condition Θ2. Then $f(s)$ converges absolutely for $\text{Re } s > 1 - \lambda$.

Proof. By the definition of $\phi_\theta$ we have
\[
\sum_{n=1}^{\infty} \frac{\theta_n}{n^s} = \zeta(s + 1)^{\alpha} f(s + 1).
\]
If we consider the Dirichlet series given by
\[
\zeta(s)^{-\alpha} = \sum_{n=1}^{\infty} \frac{\tau_{-\alpha}(n)}{n^s},
\]
then, using (10) for the coefficients of $f(s)$, we obtain
\[
b_n = \sum_{dm=n} \tau_{-\alpha}(d) \theta_m m.
\]
Using the Euler product expansion and the generalized binomial formula, we see that
\[
\zeta(s)^{-\alpha} = \prod_p \left(1 - \frac{1}{p^s}\right)^{\alpha} = \prod_p \left(1 + \sum_{\nu=1}^{\infty} (-1)^\nu \binom{\alpha}{\nu} \frac{1}{p^{\nu s}}\right) = \prod_p \left(1 + H_\alpha(p^{-s})\right),
\]
where
\[
\binom{\alpha}{\nu} = \frac{\nu!}{\nu! \prod_{\ell=0}^{\nu-1} (\alpha - \ell)},
\]
is a generalized binomial coefficient. Since
\[
|H_\alpha(p^{-s})| = \frac{|\alpha|}{p^\sigma} + O_\alpha \left(\frac{1}{p^{2\sigma}}\right),
\]
the Euler product (12) is absolutely convergent for $\sigma = \text{Re } s > 1$ and
\[
\tau_{-\alpha}(p^\nu) = (-1)^\nu \binom{\alpha}{\nu}.
\]
Note that
\[
|\tau_{-\alpha}(p^\nu)| \leq \left|\binom{\alpha}{\nu}\right| \leq \frac{1}{\nu!} \prod_{\ell=1}^{\nu} (|\alpha| + \ell - 1) \leq \prod_{\ell=1}^{\nu} \left(1 + \frac{|\alpha|}{\ell}\right) \leq \exp \left(\sum_{\ell=1}^{\nu} \frac{|\alpha|}{\ell}\right) \ll \nu^{|\alpha|} \ll_{\epsilon} p^{\nu \epsilon}
\]
for every $\epsilon > 0$. Substituting $n = p$ and $n = p^\nu$ into (11) and using Condition Θ2, we find that
\[
b_p = \tau_{-\alpha}(p) + p\theta_p = -\alpha + \alpha + pr_p = O(p^{-\lambda}),
\]
respectively, $b_p \ll |\tau_{-\alpha}(p^\nu)| \ll_{\epsilon} p^{\nu \epsilon}$ for $\nu \geq 2$ and every $\epsilon > 0$. As
\[
\sum_{n=1}^{\infty} \frac{|b_n|}{n^\sigma} = \prod_p \left(1 + \frac{|b_p|}{p^\sigma} + \sum_{\nu=2}^{\infty} \frac{|b_{p^\nu}|}{p^{\nu \sigma}}\right)
\]
is bounded when both $\sigma + \lambda > 1$ and $2\sigma > 1$, the result follows since $\lambda < 1/2$. \qed
Lemma 5. Let $\phi_\theta$ be a $\theta$-totient with $\theta$ satisfying Condition (2). Then we have

$$\sum_{n \leq x} \phi_\theta(n) = G_\theta x + \sum_{r=0}^{[\alpha]} C_r(\theta)(\log x)^{\alpha-r} + R(x) + o(1),$$

where $G_\theta$ is given by (3), $R(x)$ by

$$R(x) = \sum_{n \leq y} \theta_n \psi \left( \frac{x}{n} \right),$$

and $y = x \exp(-\log x)^{1/6}$.

Proof. With the choice $a_n = \phi_\theta(n)$, we are in the scope of Proposition 2 by Lemma 4, and on applying it and noting that $v_n = n\theta_n$, the proof is completed. $\square$

We next estimate the error term in Proposition 2. For this purpose, we need Theorem 1 of Pétermann [10], which we state below.

Proposition 3 (Pétermann [10, Theorem 1]). Let $v_n$ be a real-valued multiplicative function. Assume that there exist real numbers $\alpha_1, \beta \geq 0$, and a sequence of real numbers $\{V_r\}_{r=0}^\infty$, such that for every integer $B > 0$ and real number $x \geq 4$, we have

(h1) $\sum_{n \leq x} |v_n| = x \sum_{r=0}^{B+\lfloor \alpha_1 \rfloor} V_r(\log x)^{\alpha_1-r} + O_B(x(\log x)^{-B}),$

(h2) $\sum_{n \leq x} |v_n|^2 \ll x(\log x)^\beta,$

(h3) $v_p$ is ultimately monotonic with respect to $p,$

$v_{p^s}$ is bounded as $p^s$ runs over the prime powers.

Then, for $x \geq 4$, we have

$$\sum_{n \leq y} \frac{v_n}{n} \psi \left( \frac{x}{n} \right) \ll (\log x)^{2(\alpha_1+1)/3}(\log \log x)^{4(\alpha_1+1)/3},$$

where $y = x \exp(-\log x)^{1/6}$ and the implicit constant depends on the constants in Conditions (h1), (h2) and (h3).

We now apply Proposition 3 to our setting. For this purpose, we need Lemma 3 (which can, in principle, also be proven via the Selberg–Delange method).

Lemma 6. Let $\phi_\theta$ be a $\theta$-totient. Assume that $\theta$ satisfies Conditions (2) and (3). Then

$$\sum_{n \leq x} \phi_\theta(n) = G_\theta x + \sum_{r=0}^{[\alpha]} C_r(\theta)(\log x)^{\alpha-r} + O((\log x)^{2[\alpha]/3}(\log \log x)^{4[\alpha]/3}),$$

where $G_\theta$ is given by (3). Furthermore, for $\beta$ real,

$$\sum_{n \leq x} n^\beta \phi_\theta(n) = G_\theta M_\beta x + C(\theta, \beta) + \sum_{r=0}^{[\alpha]} C_r(\theta, \beta) x^\beta (\log x)^{\alpha-r} + E(x; \beta),$$

Note that [2, Theorem 2] contains an error. See the errata of [2] and [10].
where $M_{\beta}(x)$ is defined in Lemma $[7]$ and 
\[ E(x; \beta) \ll x^\beta (\log x)^{2|\alpha|/3} (\log \log x)^{4|\alpha|/3}. \]

**Proof.** By Lemma $[5]$ it is sufficient to show that $R(x) = O((\log x)^{2|\alpha|/3} (\log \log x)^{4|\alpha|/3})$, which we do via Proposition $[3]$. Hence, we need to check that Conditions $[\text{l1}, \text{l2}]$, and $[\text{l3}]$ are all satisfied. Since $\theta_n$ satisfies Condition $[\Theta2]$, $|\theta_n|$ also satisfies Condition $[\Theta2]$ but with $|\alpha|$ instead of $\alpha$. Thus, we can apply Lemma $[3]$ with $|\theta_n|$ instead of $\theta_n$. Then, as $v_n = n\theta_n$, we can replace $v_n$ in Proposition $[2]$ by $|v_n|$.

We start with Condition $[\text{h1}]$. We apply Lemma $[3]$ and obtain 
\[ \sum_{n \leq x} \frac{|v_n|}{n} = \sum_{0 \leq r \leq (\log x)^{1/6}} V_r (\log x)^{|\alpha|-r} + O((\log x)^{1/6}), \]
where the $V_r$ are some constants satisfying $|V_r| \leq (cr)^r$ with some $c \geq 1$. Let $B > 0$ be an integer that is kept fixed. Then it is easy to see that for $x$ larger than some constant depending on $B$ and $\alpha$,
\[ \sum_{B+|\alpha|-r \leq (\log x)^{1/6}} V_r (\log x)^{|\alpha|+B-r} \]
are bounded above as $\ll 2^{-r}$ and so the sum is bounded by some constant, and we infer that
\[ \sum_{B+|\alpha|-r \leq (\log x)^{1/6}} V_r (\log x)^{|\alpha|-r} \ll_B (\log x)^{-B}. \]

This enables us to truncate the sum over $r$ to obtain
\[ S(x) := \sum_{n \leq x} \frac{|v_n|}{n} = \sum_{0 \leq r \leq B+|\alpha|} V_r (\log x)^{|\alpha|-r} + R_S(x), \quad R_S(x) \ll_B (\log x)^{-B}. \]

By partial summation,
\[ \sum_{n \leq x} |v_n| = \int_2^x u \, dS(u) + O(1) = \sum_{0 \leq r \leq B+|\alpha|} (|\alpha|-r)V_r \int_2^x (\log u)^{|\alpha|-r-1} \, du + \int_2^x u \, dR_S(u) + O(1). \]

The main terms can be evaluated using integration by parts as
\[ \int_2^x (\log u)^{|\alpha|-r-1} \, du = \sum_{0 \leq m \leq B+|\alpha|-r-1} C_m x (\log x)^{|\alpha|-r-1-m} + O_B(x (\log x)^{-B}), \]
with some constants $C_m$ depends on $\alpha$ and $r$. The error term can be estimated as
\[ \int_2^x u \, dR_S(u) \ll_B x (\log x)^{-B} + \int_2^x (\log u)^{-B} \, du \ll_B x (\log x)^{-B}. \]

By combining the above estimates, we arrive at
\[ \sum_{n \leq x} |v_n| = x \sum_{r=0}^{B+|\alpha|-1} \tilde{V}_r (\log x)^{|\alpha|-1-r} + O(x (\log x)^{-B}), \]
where the $\tilde{V}_r$ are constants. By Condition $[\Theta2]$ we have $|\alpha| \geq 1$. Hence, Condition $[\text{h1}]$ of Proposition $[3]$ is satisfied with $\alpha_1 = |\alpha| - 1 \geq 0$. 


As to Condition \((h_2)\), we start with the string of estimates
\[
\sum_{n \leq x} |v_n|^2 = \sum_{n \leq x} n^2 |\theta_n|^2 \leq x \sum_{n \leq x} n |\theta_n|^2 \leq x \prod_{p \leq x} (1 + p |\theta_p|^2) \leq x \exp \left( \sum_{p \leq x} p |\theta_p|^2 \right).
\]
Now Condition \((\Theta_2)\) implies that
\[
\sum_{p \leq x} p |\theta_p|^2 \ll \sum_{p \leq x} \frac{1}{p} \ll \log \log x.
\]
By combining \((13)\) and \((14)\), we see that Condition \((h_2)\) is satisfied as well.

The remaining Condition \((h_3)\) follows immediately from our setting and Condition \((\Theta_3)\).

Thus Conditions \((h_1)\), \((h_2)\) and \((h_3)\) are satisfied and we get the claimed upper bound for \(R(x)\), which on insertion in Lemma 5 yields the first assertion of the lemma. The second claim now follows by partial summation. \(\square\)

**Proof of Theorem 1.** Consider the \(\theta\)-quotient \(\phi_{\theta}(n)\) defined by \(\phi_{\theta}(n) = J_\varepsilon(n) n^{-w}\). Note that \(\theta\) satisfies Condition \((\Theta_2)\) with \(\alpha = -e_1\) and \(\lambda = 1\) and, moreover, satisfies Condition \((\Theta_3)\). Thus, in case \(e_1 \neq 0\), Theorem 1 follows immediately from Lemma 6. The case \(e_1 = 0\) is just a corollary of Theorem 2. \(\square\)

5. Applications

**Definition.** Let \(f(X) \in \mathbb{Z}[X]\) be a polynomial and let \(\deg f\) denote its degree with respect to \(X\). For any complex number \(z\) such that \(f(z) \neq 0\), we define
\[
F^{(k)}(z) = \frac{1}{(\deg f)^k} \frac{f^{(k)}(z)}{f(z)}
\]
as the normalized \(k^{th}\) derivative of \(f\) at \(z\).

In case \(f(X) \in \mathbb{Z}_{\geq 0}[X]\), \(z \geq 1\) is real, and \(f(z) \neq 0\), it is easy to show that \(F^{(k)}(z) \leq 1\). This observation leads to the following problem.

**Problem.** Let \(z\) be given. Let \(\mathcal{F}\) be an infinite family of polynomials \(f\) with \(f(z) \neq 0\). Study the average behavior and value distribution of \(F^{(k)}(z)\) in the family \(\mathcal{F}\).

Here we consider the family \(\mathcal{F} = \{\Phi_n : n \geq 2\}\), where \(\Phi_n\) denotes the \(n^{th}\) cyclotomic polynomial. It can be defined by
\[
\Phi_n(X) = \prod_{1 \leq j \leq n \atop (j,n)=1} (X - \zeta_n^j) = \sum_{k=0}^{\varphi(n)} a_n(k) X^k,
\]
with \(\zeta_n\) any primitive \(n^{th}\) root of unity. Note that \(\Phi_n(1) \neq 0\) for \(n > 1\) and that \(\Phi_n(-1) \neq 0\) for \(n > 2\). Theorem 3 shows that
\[
\frac{1}{\varphi(n)^k} \Phi_n^{(k)}(1),
\]
the \(k^{th}\) normalized derivative of \(\Phi_n\) at 1, is constant on averaging over \(n\).
5.1. The $k^{th}$ derivative of $\Phi_n$ at 1. In this section we first recall some known results on $\Phi_n^{(k)}$. For a survey (and some new results) see Herrera-Poyatos and Moree [4].

The Bernoulli numbers $B_n$ can be recursively defined by

$$B_n = -\sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1},$$

with $B_0 = 1$. The coefficients $c(k, j)$ of the polynomial

$$X(X - 1) \cdots (X - k + 1) = \sum_{j=0}^{k} c(k, j) X^j$$

are called the signed Stirling numbers of the first kind.

**Lemma 7** (Lehmer [6, Theorems 2 and 3]). For $n > 1$ and $k \geq 1$, we have

$$\frac{\Phi_n^{(k)}(1)}{\Phi_n(1)} = k! \sum_{(\ast)} \prod_{i=1}^{k} \frac{(-s_i(n))^{\lambda_i}}{\lambda_i! \lambda_i}$$

where the summation $\sum_{(\ast)}$ is as in Theorem 2 and

$$s_i(n) := -\frac{1}{(i-1)!} \sum_{h=1}^{i} (-1)^h \frac{B_{2h}}{2h} c(i, 2h) J_{2h}(n).$$

**Remark.** Theorem 2 of [6] gives the formula

$$s_i(n) = \frac{(-1)^i}{2} \varphi(n) - \frac{1}{(i-1)!} \sum_{h=1}^{[i/2]} \frac{B_{2h}}{2h} c(i, 2h) J_{2h}(n).$$

The expression above is slightly different from (16), but we can simplify this as in Lemma 7 since $B_{2h} = 0$ for odd $h > 1$, $B_1 = -1/2$, and $c(i, 1) = (-1)^{i-1}(i-1)!$.

In particular, using Lemma 7 with $k = 1, 2$ for $n > 1$ we obtain

$$(17) \quad \frac{\Phi_n'(1)}{\Phi_n(1)} = \frac{\varphi(n)}{2},$$

and

$$\frac{\Phi_n''(1)}{\Phi_n(1)} = \frac{\varphi(n)}{4} \left( \varphi(n) + \frac{\Psi(n)}{3} - 2 \right).$$

**Lemma 8.** For $n > 1$ and $k \geq 1$, we have

$$\frac{1}{\varphi(n)^k} \frac{\Phi_n^{(k)}(1)}{\Phi_n(1)} = k! \sum_{(\ast)} \prod_{i=1}^{k} \frac{(-1)^i \lambda_i}{\lambda_i!} \left( \frac{B_i}{i!} \right)^{\lambda_i} \frac{J_i(n)^{\lambda_i}}{\varphi(n)^k} + O_k \left( \frac{n^{k-1}}{\varphi(n)^k} \right),$$

where the summation $\sum_{(\ast)}$ is as in Theorem 3.

**Proof.** Since $J_h(n) \leq n^h$ and $c(i, i) = 1$, it follows from (16) that

$$-s_i(n) = (-1)^i \frac{B_i}{i!} J_i(n) + O_k(n^{i-1}).$$
JORDAN TOTIENT QUOTIENTS

Hence, by raising this to the $\lambda_i$-th power,

$$(-s_i(n))^{\lambda_i} = (-1)^{i\lambda_i} \left( \frac{B_i}{i!} J_i(n) \right)^{\lambda_i} + O_k(n^{i\lambda_i - 1}).$$

By substituting this estimate into (15), the proof of the lemma is concluded by taking the product over $1 \leq i \leq k$ and noting that the error term is $O_k(n^{\sum_{i=1}^k i\lambda_i - 1}) = O_k(n^{k-1})$ for each choice of $\lambda_1, \ldots, \lambda_k$ contributing to the sum $\sum_{(\star)}$.

Proof of Theorem 3. By (6), we may assume $k \geq 2$. By Lemma 8 and Corollary 1,

$$\sum_{1 < n \leq x} \frac{1}{\varphi(n)^k \Phi_n(1)} = k! \sum_{(\star)} \prod_{i=1}^k \left( \frac{(-1)^{i\lambda_i}}{i!} \cdot \frac{B_i}{i!} \right)^{\lambda_i} \sum_{n \leq x} J_{e(\lambda)}(n) + O_k((\log x)^k),$$

where we used the summation $\sum_{(\star)}$ and the indices $e(\lambda)$ defined in Theorem 3. Note that every index $e(\lambda)$ appearing on the right-hand side has weight

$$w = \sum_{i=1}^\infty i e_i(\lambda) = \sum_{i=1}^k i \lambda_i - k = 0.$$

Trivially $|e_1(\lambda)| \leq k$ and hence, by applying Theorem 3 and using $k \geq 2$,

$$\sum_{1 < n \leq x} \frac{1}{\varphi(n)^k \Phi_n(1)} = \mathcal{S}_k(\Phi)x + \sum_{r=1}^k C_r((\log x)^r + O_k((\log x)^{2k/3}(\log \log x)^{4k/3}),$$

where

$$\mathcal{S}_k(\Phi) := (-1)^k k! \sum_{(\star)} \prod_{i=1}^k \frac{1}{i!} \left( \frac{B_i}{i!} \right)^{\lambda_i} \mathcal{S}_{e(\lambda)}.$$

5.2. The second derivative of $\Phi_n$ at $-1$. We prove an analogous result for the normalized second derivative of $\Phi_n$ at $-1$.

Theorem 4. We have

$$\sum_{2 < n \leq x} \frac{1}{\varphi(n)^2 \Phi_n(-1)} = \frac{x}{48} (5\mathcal{S}_{(-2,1)} + 12) + c_2 \log^2 x + O((\log x)^{4/3}(\log \log x)^{8/3}),$$

where $c_2$ is a constant and $\mathcal{S}_{(-2,1)}$ computed via (3) equals

$$\mathcal{S}_{(-2,1)} = \prod_p \left( 1 + \frac{2}{p(p-1)} \right).$$

Proof. By [4, Corollary 22] it follows that for $n \geq 3$ we have

$$\frac{\Phi''_n(-1)}{\Phi_n(-1)} = \frac{\varphi(n)}{4} \left( \varphi(n) + a_n \Psi(n) - 2 \right),$$

where

$$a_n = \begin{cases} 1 & \text{if } n \text { is odd,} \\ 1/9 & \text{if } 2 \parallel n, \\ 1/3 & \text{otherwise.} \end{cases}$$
Using the above and Lemma 1 it now follows that

\[ \sum_{2 < n \leq x} \frac{1}{\varphi(n)^2} \Phi_n(-1) = \frac{x}{4} + \frac{1}{3} \sum_{n \leq x} \frac{\Psi(n)}{\varphi(n)} + O(\log x). \]

Note that

\[ \sum_{n \leq x} \left( a_n - \frac{1}{3} \right) \frac{\Psi(n)}{\varphi(n)} = \frac{2}{3} \sum_{n \leq x} \frac{\Psi(n)}{\varphi(n)} - \frac{2}{9} \sum_{n \leq x} \frac{\Psi(n)}{\varphi(n)} = \frac{2}{3} \sum_{n \leq x} \frac{\Psi(n)}{\varphi(n)} - \frac{2}{3} \sum_{n \leq x/2} \frac{\Psi(n)}{\varphi(n)}, \]

and so

\[ \sum_{n \leq x} a_n \frac{\Psi(n)}{\varphi(n)} = \frac{1}{3} \sum_{n \leq x} \frac{\Psi(n)}{\varphi(n)} + \frac{2}{3} \sum_{n \leq x/2} \frac{\Psi(n)}{\varphi(n)} - \frac{2}{3} \sum_{n \leq x/2} \frac{\Psi(n)}{\varphi(n)}. \]

By Theorem 2 for the first sum, we have

\[ \sum_{n \leq x} \frac{\Psi(n)}{\varphi(n)} = \mathcal{G}(-2, 1)x + c_1 \log^2 x + O((\log x)^{4/3}(\log \log x)^{8/3}). \]

On noting that

\[ 1_{2n^2} \frac{\Psi(n)}{\varphi(n)} = \prod_{p \mid n} (1 + \theta_p), \quad \theta_p = \frac{2}{p - 1} (p \neq 2), \quad \theta_2 = -1, \]

we get on applying Lemma 3

\[ \sum_{n \leq x} \frac{\Psi(n)}{\varphi(n)} = \frac{1}{4} \mathcal{G}(-2, 1)x + c_2^2 \log^2 x + O((\log x)^{4/3}(\log \log x)^{8/3}). \]

Combining the results above we get

\[ \sum_{n \leq x} a_n \frac{\Psi(n)}{\varphi(n)} = \frac{5}{12} \mathcal{G}(-2, 1)x + 4c_2 \log^2 x + O((\log x)^{4/3}(\log \log x)^{8/3}), \]

which, together with (18), concludes the proof. \( \square \)

5.3. Schwarzian derivative of \( \Phi_n \) at 1. Given a holomorphic function \( f \) of one complex variable \( z \), we define its Schwarzian derivative, cf. [9], as

\[ S(f(z)) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2. \]

**Theorem 5.** We have

\[ \sum_{n \leq x} \frac{S(\Phi_n(1))}{\varphi(n)^2} = -\frac{1}{24}(\mathcal{G}(-4, 2) + 3)x + c_4 \log^4 x + c_3 \log^3 x + O((\log x)^{8/3}(\log \log x)^{16/3}), \]

where \( c_3, c_4 \) are constants and \( \mathcal{G}(-4, 2) \) computed via [3] equals

\[ \mathcal{G}(-4, 2) = \prod_p \left( 1 + \frac{4}{(p - 1)^2} \right). \]
Proof. By Lemma 7, we have for $n \geq 2$

$$S(\Phi_n(1)) = -\frac{\varphi(n)^2}{8} - \frac{\Psi(n)^2}{24} + \frac{1}{2},$$

and thus

$$\sum_{n \leq x} S(\Phi_n(1)) \varphi(n)^2 = -\frac{1}{8} \sum_{n \leq x} 1 - \frac{1}{24} \sum_{n \leq x} \Psi(n)^2 + \frac{1}{2} \sum_{n \leq x} \varphi(n)^2.$$

The last sum is bounded by a constant by Proposition 1 with $J_e(n) = 1/\varphi(n)^2$ and $\beta = 0$. The result now follows on applying Theorem 1 with $e = (−4, 2)$. □

Remark. On applying the elementary Theorem 2, we obtain Theorems 4 and 5 with error terms $O(\log^2 x)$ and $O(\log^4 x)$, respectively.

Acknowledgement

A large portion of this paper was written during the stay of the second and the fourth author at the Max Planck Institute for Mathematics (MPIM) in September 2018. They would like to thank Pieter Moree for inviting them and they gratefully acknowledge the support, hospitality as well as the excellent environment for collaboration at the MPIM. The second author is supported by the Austrian Science Fund (FWF): Project F5505-N26 and Project F5507-N26, which are part of the special Research Program “Quasi Monte Carlo Methods: Theory and Application”. The fourth author is supported by Grant-in-Aid for JSPS Research Fellow (Grant Number: JP16J00906).

References

[1] T.M. Apostol, Introduction to analytic number theory, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.

[2] U. Balakrishnan and Y.-F.S. Pétermann, The Dirichlet series of $\zeta(s)\zeta(s + 1)^\alpha f(s + 1)$: On an error term associated with its coefficients, Acta Arith. 75 (1) (1996), 39–69, Errata: ibid. 87 (3) (1999), 287–289.

[3] B. Bzdég, A. Herrera-Poyatos and P. Moree, Cyclotomic polynomials at roots of unity, arXiv:1611.06783, (extended version of ibid. Acta Arith. 184 (3) (2018), 215–230).

[4] A. Herrera-Poyatos and P. Moree, Coefficients and higher order derivatives of cyclotomic polynomials: old and new, arXiv:1805.05207, submitted for publication.

[5] J. Kaczorowski, On a generalization of the Euler totient function, Monatsh. Math. 170 (1) (2013), 27–48.

[6] D. H. Lehmer, Some properties of cyclotomic polynomials, J. Math. Anal. Appl. 15 (1) (1966), 105–117.

[7] P. Moree, Approximation of singular series and automata, With an appendix by Gerhard Niklasch, Manuscripta Math. 101 (3) (2000), 385–399.

[8] P. Moree and G. Niklasch, Webpage on high precision numerical evaluation of Euler products, http://guests.mpim-bonn.mpg.de/moree/Moree.en.html

[9] V. Ovsienko and S. Tabachnikov, What is ... the Schwarzian derivative?, Notices Amer. Math. Soc. 56 (1) (2009), 34–36.

[10] Y.-F. S. Pétermann, On an estimate of Walfisz and Saltykov for an error term related to the Euler function, J. Théor. Nombres Bordeaux 10 (1) (1998), 203–236.

[11] R. Sivaramakrishnan, The many facets of Euler’s totient. I. A general perspective, Nieuw Arch. Wisk. 4 (4) (1986), 175–190; II. Generalizations and analogues, Nieuw Arch. Wisk. 8 (4) (1990), 169–187.

[12] R. Sivaramakrishnan, Classical theory of arithmetic functions, Monographs and Textbooks in Pure and Applied Mathematics 26, Marcel Dekker, Inc., New York, 1989.

[13] Y. Suzuki, On error term estimates à la Walfisz for mean values of arithmetic functions, in preparation.

[14] E. C. Titchmarsh, The theory of the Riemann-zeta function, 2nd ed., Claredon Press, Oxford, 1986.
(P. Moree, A. Sedunova) Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany.
E-mail address, P. Moree: moree@mpim-bonn.mpg.de
E-mail address, A. Sedunova: alisa.sedunova@phystech.edu

(S. Saad Eddin) Institute of Financial Mathematics and Applied Number Theory, JKU Linz, Altenbergerstrasse 69, 4040 Linz, Austria.
E-mail address: sumaia.saad_eddin@jku.at

(Y. Suzuki) Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, Japan.
E-mail address: suzuyu1729@gmail.com