HOW DO HYPERBOLIC HOMOCLINIC CLASSES COLLIDE AT HETERODIMENSIONAL CYCLES?

LORENZO J. DÍAZ
Departamento de Matemática, PUC-Rio
Marquês de S. Vicente 225
22453-900 Rio de Janeiro RJ, Brazil

JORGE ROCHA
Departamento de Matemática Pura, Universidade do Porto
Rua Campo Alegre 687
4169-007 Porto, Portugal

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ABSTRACT. We present a model illustrating heterodimensional cycles (i.e., cycles associated to saddles having different indices) as a mechanism leading to the collision of hyperbolic homoclinic classes (of points of different indices) and thereafter to the persistence of (heterodimensional) cycles. The collisions are associated to secondary (saddle-node) bifurcations appearing in the unfolding of the initial cycle.

1. Introduction.

1.1. Context. A diffeomorphism $f$ has a heterodimensional cycle if there are hyperbolic sets $\Lambda$ and $\Sigma$ with different indices (dimension of the unstable bundle) such that the stable manifold of $\Lambda$ meets the unstable one of $\Sigma$ and vice-versa (i.e., $W^s(\Lambda) \cap W^u(\Sigma) \neq \emptyset$ and $W^u(\Lambda) \cap W^s(\Sigma) \neq \emptyset$). The simplest case of heterodimensional cycle occurs when $\Lambda$ and $\Sigma$ are both periodic saddles. We analyze how the phenomena of collision of non-trivial hyperbolic sets, intermingled homoclinic classes, and persistence of heterodimensional cycles arise at simple heterodimensional cycles.

We consider one-parameter families of diffeomorphisms $(f_t)_{t \in [-1,1]}$ starting in a hyperbolic region (Morse-Smale systems) and crossing the boundary of hyperbolicity at a first bifurcation (say $t = 0$) corresponding to a heterodimensional cycle associated to a pair of saddles, say $P$ and $Q$. We study the dynamics in a neighbourhood of the cycle set (i.e., the $f$-invariant set defined as closure of the orbits of the saddles $P$ and $Q$ in the cycle and the intersections $W^s(P) \cap W^u(Q)$ and $W^u(P) \cap W^s(Q)$ between their invariant manifolds). We fix a small neighbourhood $W$ of the cycle set and, for small $t > 0$, study the part of the non-wandering set...
\( \Omega(f_t) \) of \( f_t \) in \( W \), the so-called resulting non-wandering set for the parameter \( t \), denoted by \( \Omega(f_t)' \).

The goal (see [36, Chapter 7]) is to describe the dynamics of the resulting non-wandering set \( \Omega(f_t)' \) having some persistence or prevalence (in terms of the parameter \( t \)) after the bifurcation (i.e., for small \( t > 0 \)). Such a dynamic depends essentially on the choice of the parameter and on the restriction of the bifurcating diffeomorphism to the cycle set.

We adopt the following strategy for studying bifurcations. The type of bifurcation of the diffeomorphism \( f \) (saddle-node, flip, Hopf, homoclinic tangency, heterodimensional cycle) defines a (local) bifurcation submanifold \( \Sigma \) (i.e., any \( g \in \Sigma \) has a bifurcation as the one of \( f \)). We assume that \( \Sigma \) is in the boundary of the hyperbolic systems and has codimension one. More precisely, we consider arcs \( (f_t)_{t \in [-1,1]} \) intersecting transversely \( \Sigma \) for (say) \( t = 0 \) and such that, for every \( t < 0 \), the diffeomorphism \( f_t \) is hyperbolic. Then we say that \( (f_t)_{t \in [-1,1]} \) crosses the boundary of hyperbolicity at \( t = 0 \) and unfolds the bifurcation for positive \( t \). [32] conjectured that the generic way of crossing such a boundary is either by the loss of hyperbolicity of some periodic orbit or by the loss of transversality of the intersection between the invariant manifolds of hyperbolic periodic points (creation of cycles). This conjecture holds for surface diffeomorphisms in the boundary of the Morse-Smale systems and it remains open in other cases.

The first sort of bifurcation above (loss of hyperbolicity) is local (the dynamics on a finite orbit) while the second one (creation of cycles) is semi-global involving the orbit of a non-periodic point (a non-transverse intersection between invariant manifolds). Interesting dynamical features arise when there is some interplay between these local and semi-global bifurcations. For instance, bifurcations homoclinic bifurcation (tangencies) generate infinitely many saddle-node, period doubling and Hopf bifurcations, see [11] and [36, Chapter 3]. For the converse, loss of hyperbolicity bifurcations generate cycles, some semi-global hypotheses on the dynamics are necessary. Such an interaction occurs, for instance, in the so-called saddle-node cycles introduced in [33]: there is a saddle-node whose unstable manifold intersects its stable manifold (or vice-versa). Especially interesting saddle-node cycles are the saddle-node horseshoes introduced in [42], where the saddle-node belongs to a (topological) horseshoe (the archetypal saddle-node horseshoe is depicted in Figure 1).

In some cases, saddle-node horseshoes lead to a string of bifurcations and phenomena as Hénon-like attractors and persistence of tangencies, see [10, 21, 11, 8, 10, 23, 15], about saddle-node cycles and saddle-node horseshoes of surface diffeomorphisms, and [11, 17], for partially hyperbolic saddle-node cycles in higher dimensions. For a survey on this subject see [20]. We study the occurrence of saddle-node horseshoes at heterodimensional cycles. In fact, saddle-node cycles are a simple type of the collision bifurcations we will discuss in the next paragraphs.

1.2. **Collision bifurcations and heterodimensional cycles.** We study partially hyperbolic heterodimensional cycles: the cycle set has a partially hyperbolic splitting \( E^s \oplus E^c \oplus E^u \), where \( E^s \) and \( E^u \) are uniformly hyperbolic (non-trivial) and \( E^c \) is one-dimensional and is not hyperbolic. Thus the dimension of the ambient is at least three (heterodimensional cycles only occur in dimensions strictly greater than two). We describe how the unfolding of these cycles generate twin saddle-node horseshoes with a common saddle-node, corresponding to the collision of horseshoes.
of different indices. These collisions can be thought as the simultaneous occurrence of two saddle-node cycles associated to the same saddle-node, see Figure 2.

This paper illustrates the interplay between local (saddle-node) and semi-global (heterodimensional cycles) bifurcations: the unfolding of a heterodimensional cycle accomplishes a sequence \((t_n, t_n \to 0^+)\), of \textit{colliding twin saddle-node horseshoes}. Each saddle-node bifurcation generates \textit{persistence of cycles} and \textit{intermingled homoclinic classes} (see the definitions below). In rough terms, the each cycle generates two horseshoes having different indices. For each \(t_n\), the two horseshoes become saddle-node horseshoes sharing a saddle-node (following [22], this is a \textit{collision bifurcation of hyperbolic sets}). Finally, these saddle-nodes generate persistent of heterodimensional cycles.

Let us state our main theorem informally. First, recall that the \textit{homoclinic class} of a (hyperbolic) saddle \(P\) of a diffeomorphism \(f\), denoted by \(H(P, f)\), is the closure of the transverse intersections of the stable and unstable manifolds of the orbit of \(P\). Important properties of a homoclinic classes \(H(P, f)\) are: (i) \(f\)-invariance, (ii)\textit{transitivity} (i.e., existence of a dense orbit), and (iii) the subset of periodic points with the same index as \(P\) is dense in \(H(P, f)\).

Concerning (iii), note that a homoclinic class may have saddles of different indices, thus being non-hyperbolic. This phenomenon occurs at heterodimensional cycles, see [12, 13, 18]. This paper gives examples of such a situation. Finally, a homoclinic class is \textit{trivial} if it consists of a single orbit, and two homoclinic classes are \textit{intermingled} if they have non-empty intersection.

\textbf{Theorem 1.} \textit{There are arcs} \((f_t)_{t \in [-1,1]}\) \textit{unfolding at} \(t = 0\) \((t \geq 0)\) a \textit{partially hyperbolic heterodimensional cycle associated to saddles} \(P\) \textit{and} \(Q\) \textit{having sequences of scaled parameter intervals}

\[C_n = [a_n, t_n] = \lambda^n [1 + \mu^-, 1 + \mu^+],\]

\[H_n = (t_n, b_n] = \lambda^n (1 + \mu^*, 1 + \mu^+),\]

\textit{such that:}
(A) Hyperbolicity: for every \( n \) and for all \( t \in H_n \), the resulting non-wandering set of \( f_t \) is hyperbolic and equal to the disjoint union of the (non-trivial) homoclinic classes of \( P \) and \( Q \).

(B) Secondary saddle-node collision bifurcations: for every \( n \), \( f_{t_n} \) has a saddle-node \( S_n \) such that the intersection of the homoclinic classes of \( P \) and \( Q \) is exactly the orbit of \( S_n \).

(C) Persistence of cycles at collision bifurcations: for every \( n \), \( C_n = (a_n, t_n) \) is an interval of persistence of heterodimensional cycles: for every parameter \( t \in C_n \), the diffeomorphism \( f_t \) has a heterodimensional cycle associated to hyperbolic sets \( \Upsilon_t \) of index one contained in the homoclinic class of \( P \) and \( \Xi_t \) of index two contained in the homoclinic class of \( Q \).

We observe that, for every saddle-node parameter \( t_n \), every compact invariant set contained in either \( H(P, f_{t_n}) \) or in \( H(Q, f_{t_n}) \) and disjoint from the saddle-node is hyperbolic.

Theorem 1 shows how homoclinic classes may change from the metric point of view. One aims to understand the map \((P, f) \rightarrow H(P, f)\) that associates to a saddle \( P \) (depending continuously on \( f \)) its homoclinic class. The map \( H(\cdot, \cdot) \) is lower semi-continuous. This follows from the continuous dependence on the dynamics of the invariant manifolds of saddles and on their transverse intersections (on compact parts). A metric bifurcation of a homoclinic class \( H(P, f) \) is discontinuity of \( H(\cdot, \cdot) \). A well-known metric bifurcation is the creation of cycles. The theorem describes bifurcations of homoclinic classes via collision. Let us shortly discuss this notion.

Consider an arc of \((f_t)_{t \in [a, b]}\) such that, for all \( t \in [a, b] \), there are saddles \( P_t \) and \( Q_t \) (depending continuously on \( t \)). The homoclinic classes of \( P_t \) and \( Q_t \) collide at \( t = c \) if \( H(P_t, f_t) \cap H(Q_t, f_t) = \emptyset \) for every \( t \in [a, c] \), and \( H(P_c, f_c) \cap H(Q_c, f_c) \neq \emptyset \). In the collisions in the theorem the following holds: (i) the colliding homoclinic classes are both non-trivial and hyperbolic; (ii) at the collision parameter, there are twin saddle-node horseshoes intersecting along the orbit of a saddle-node; and (iii) after the collision, the homoclinic classes simultaneously explode. See Figure 3.

![Figure 3. Collision of homoclinic classes](image)

In our definition of collision the saddles \( P_c \) and \( Q_c \) are defined, thus their homoclinic classes are defined at and after the collision. Thus it is stronger than the following definition: the homoclinic classes of \( P_t \) and \( Q_t \) have a weak collision at \( t = c \) if \( H(P_t, f_t) \cap H(Q_t, f_t) = \emptyset \) for all \( t < c \) and the distance between \( H(P_t, f_t) \) and \( H(Q_t, f_t) \) goes to zero as \( t \to c \). A priori, the saddles \( P_c \) and \( Q_c \) to be defined. The simplest case of weak collision is the collision of two isolated saddles via a saddle-node bifurcation. A more interesting case happens when the saddle-node is
non-isolated. In dimension two, the main examples are the Derived from Anosov bifurcations, where a repeller and a non-trivial hyperbolic attractor collide, and the saddle-node horseshoes above.

In the weak collisions above, at least a homoclinic class is trivial (a sink or a repeller) while in our construction both colliding classes are hyperbolic and non-trivial. Collisions of two non-trivial and non-hyperbolic homoclinic classes were studied in \[22\]. In \[22\] the collision leads to intermingled homoclinic classes: after the collision, the two classes explode and coincide.

1.3. Dynamics at partially hyperbolic heterodimensional cycles. We now focus on the dynamics generated by partially hyperbolic heterodimensional cycles (with one-dimensional central direction) of codimension one (the bifurcating submanifold has codimension one). From now on, we assume these two conditions, which imply that the cycle is related to saddles \(P\) and \(Q\) of indices \(s\) and \(s+1\). In such a case, the unfolding does not generate homoclinic tangencies and all the cycles in the unfolding are heterodimensional ones. In fact, the unfolding accomplishes infinitely many cycles. We next discuss the transitions from hyperbolicity to intermingled homoclinic as the parameter evolves. We review some results and see how they are used in this paper.

In this setting, for all small \(t > 0\) the homoclinic classes of \(P\) and \(Q\) explode and are both non-trivial (the dependence of the saddles on \(t\) is omitted). Thus, by the Smale’s homoclinic theorem, there are horseshoes \(\Lambda_P\) and \(\Lambda_Q\) such that \(P \in \Lambda_P \subset H(P, f_t)\) and \(Q \in \Lambda_Q \subset H(Q, f_t)\). These horseshoes have different indices and, in some cases, the homoclinic classes of \(P\) and \(Q\) are equal to \(\Lambda_P\) and \(\Lambda_Q\), thus the homoclinic classes are disjoint. But there are some cases where the homoclinic classes of \(P\) and \(Q\) coincide and properly contain the horseshoes \(\Lambda_P\) and \(\Lambda_Q\). A goal of this paper is to study how these hyperbolic homoclinic classes can bifurcate and describe the dynamics following these secondary bifurcations. In this direction, three crucial steps are:

: \(\text{(i)}\) to determine whether the homoclinic classes of \(P\) and \(Q\) are hyperbolic;
: \(\text{(ii)}\) to describe the resulting non-wandering set of \(f_t\) after the bifurcation;
: \(\text{(iii)}\) to study secondary bifurcations generated by the unfolding of the first cycle.

We now recall some results related to \(\text{(i)}-\text{(iii)}\) in the simplest case of bifurcations from Morse-Smale systems. We first fix some notations. Consider \((f_t)_{t \in [-1,1]}\) unfolding a cycle, say at \(t = 0\) and for positive \(t\). Given a dynamical property \(\mathcal{P}\), consider the sets of parameters

\[
\mathcal{P} = \{ s \in [0,1]: f_s \text{ satisfies } \mathcal{P} \} \quad \text{and} \quad \mathcal{P}(t) = \mathcal{P} \cap [0,t].
\]

Following \[36\] Chapter 7, we say that the property \(\mathcal{P}\) is robust (after the bifurcation) if there is some \(t > 0\) such that \(\mathcal{P}(t) = (0,t]\). Similarly, property \(\mathcal{P}\) is prevalent if \(\liminf_{t \to 0^+} |\mathcal{P}(t)|/t > 0\) and totally prevalent \(\liminf_{t \to 0^+} |\mathcal{P}(t)|/t = 1\) (\(|\cdot|\) stands for the Lebesgue measure).

We are interested in three dynamical features: hyperbolic dynamics of the resulting non-wandering set (property \(\mathcal{H}\)), intermingled homoclinic classes (prop. \(\mathcal{I}\)), and existence of heterodimensional cycles associated to hyperbolic sets (prop. \(\mathcal{C}\)).
As above, let
\[
\begin{align*}
\mathbb{H} &= \{ t \in [0,t_0] : \Omega(f_t)' \text{ is hyperbolic (} f_t \text{ verifies } \mathcal{H} \} , \\
\mathbb{I} &= \{ t \in [0,t_0] : H(P, f_t) \cap H(Q, f_t) \neq \emptyset (f_t \text{ verifies } I) \} , \quad \text{and} \\
\mathbb{C} &= \{ t \in [0,t_0] : f_t \text{ has a heterodimensional cycle (} f_t \text{ satisfies } C) \} .
\end{align*}
\]
Note that \( \mathbb{H} \cap \mathbb{I} = \emptyset \) and \( \mathbb{H} \cap \mathbb{C} = \emptyset. \)

Concerning question (i) above, [12, 13] construct open sets of arcs \( (f_t)_{t \in [-1,1]} \) such that there is \( t_0 > 0 \) with \( \mathbb{I}(t_0) = (0,t_0) \). Moreover, [13] states that the phenomenon \( \mathcal{I} \) of intermingled homoclinic classes is always prevalent at the bifurcation.

On the other hand and in the opposite direction, [16] gives open sets of arcs \( (f_t)_{t \in [-1,1]} \) which are prevalently hyperbolic at the bifurcation. Since \( \mathbb{I} \cap \mathbb{H} = \emptyset \), [18] above implies that \( \lim \sup_{t \to 0^-} (|H(t)|/t) < 1 \). Thus hyperbolicity is not totally prevalent at the bifurcation. But [16] assures that one can construct \( (f_t)_{t \in [-1,1]} \) with frequency of hyperbolicity close to one: fixed any \( \varepsilon > 0 \), there are arcs with \( \lim_{t \to 0^+} (|H(t)|/t) > (1-\varepsilon). \)

We now discuss questions (ii) and (iii) above. In [16] above, for every \( t > 0 \), the parameter interval \( [0,t] \) contains infinitely many intervals corresponding alternately to hyperbolic and non-hyperbolic diffeomorphisms. Thus a natural problem is to understand the transition from hyperbolic to non-hyperbolic dynamics (secondary bifurcations). For the hyperbolic parameters \( t \) in [16] the non-wandering set of \( f_t \) is the (disjoint) union of the homoclinic classes of \( P \) and \( Q \). Thus, in this case, there are two natural possibilities for secondary bifurcations: new heterodimensional cycles and loss of hyperbolicity of some saddle. These two possibilities correspond to the semi-global and local bifurcations discussed in Section [13]. We now focus on local bifurcations.

Typically, the loss of hyperbolicity occurs via saddle-node bifurcations (in this context flip and Hopf bifurcations are forbidden). The saddle-node may correspond to either an emerging saddle-node or to the collision of a pair of old saddles in the homoclinic classes of \( P \) and \( Q \). In the emerging case, the saddle-node generates a pair of saddles \( A \) and \( B \) of different indices. In some cases, these new saddles may be independent of \( P \) and \( Q \) (i.e., the saddles do not belong to the homoclinic classes of \( P \) and \( Q \)). Moreover, throughout the unfolding, cycles related to \( A \) and \( B \) can appear. Then this new cycle generates homoclinic points of \( A \) and \( B \). Therefore non-trivial homoclinic classes independent of the ones of \( P \) and \( Q \) arise. Hence the hyperbolicity of the homoclinic classes of the saddles \( P \) and \( Q \) does not imply the global hyperbolicity of the non-wandering set (the homoclinic classes of \( A \) and \( B \) may fail to be hyperbolic). This sort of bifurcations and dynamics are described in [19].

In this paper, we study the non-emerging case: the saddle-node corresponds to a collision of saddles in the hyperbolic homoclinic classes of \( P \) and \( Q \) (thus to a collision of homoclinic classes). We describe how the phenomena of intermingled homoclinic classes and of persistence of heterodimensional cycles arise from these bifurcations.

1.4. Ingredients of the proof. We next describe some ideas behind the results above we use in this paper. The hypotheses (partial hyperbolicity and bifurcation from Morse-Smale) imply that the intersection \( W^s(P, f_0) \cap W^u(Q, f_0) \) is the orbit of a periodic curve \( \gamma \), named connection, tangent to the central direction \( E^c \) and whose extremes are the saddles \( P \) and \( Q \). The heuristic principle in [12, 13, 16, 19] is that the restriction of the diffeomorphism \( f_0 \) to the connection \( \gamma \) (shortly, the central
(dynamics) determines the dynamics after the bifurcation. The different kinds of dynamics after the bifurcation we described before correspond to different choices of parameters and of dynamics of $f_0$ along the connection. In rough terms: (a) small distortion along the connection $\gamma$ generates robustly intermingled homoclinic classes (of $P$ and $Q$); (b) to get hyperbolic parameters one needs appropriate big distortion along $\gamma$.

It is interesting to compare the previous results, suggesting that (for heterodimensional cycles) the central dynamics determines the dynamics following the bifurcation, to the papers [28, 34, 35, 26, 38, 27] showing that, in the setting of homoclinic tangencies, the dynamics after the bifurcation is determined by the fractal dimensions of the bifurcating hyperbolic set. For heterodimensional cycles, the role of fractal dimensions is unknown. Here we describe the creation of thick Cantor sets at heterodimensional cycles and derive consequences from this fact.

More precisely, the restriction of $f_0$ to the connection $\gamma$ induces a one-parameter family of maps $(L_\mu)_{\mu \in [-\delta, \delta]}$, we call limit central one-dimensional dynamics, as follows. There are sequences $t^+_n$ of parameters, $t^-_n \to 0$, and of reparametrizations $\varrho_n : [-\delta, \delta] \to [t^-_n, t^+_n]$, such that the dynamics of $f_{\varrho_n(\mu)}$ in the resulting non-wandering set is the product of $L_\mu$ (the central part) by a hyperbolic saddle dynamics. The central part $L_\mu$ is associated to a system of iterated functions generated by:

: (i) the linearizations of $f_0$ in the central directions of $Q$ and $P$;
: (ii) (assume that $\text{index}(Q) = \text{index}(P) + 1$) a transition map given by the iterates from a (scaled) fundamental domain of $Q$ in $\gamma$ to a (scaled) fundamental domain of $P$ in $\gamma$;
: (iii) a translation corresponding to the rescaling of the parameter $t$ unfolding the cycle.

The family $(L_\mu)_{\mu \in [-\delta, \delta]}$ consists of non-critical maps having infinitely many discontinuities. See Section 3.2. For an informal discussion about the limit family, see [2] Chapter 6.1.2.

The role of the family $(L_\mu)_{\mu \in [-\delta, \delta]}$ is similar to the one of the quadratic family for homoclinic tangencies (see [36, Chapter 3.4]) or the limit ghost dynamics of saddle-node cycles (see [20] Section 11.3). Dynamical properties after the bifurcation (hyperbolicity, of saddle-node bifurcations, and existence of cycles) are derived from similar properties for $(L_\mu)_{\mu \in [-\delta, \delta]}$.

The persistent (heterodimensional) cycles are associated to the creation of thick Cantor sets (see Definition 6.1 and the Gap Lemma 7.1). Thus the mechanism is somewhat similar to the one of persistence of homoclinic tangencies. However, the dynamics generating these thick sets are different. In the homoclinic setting, the thick Cantor sets are related to the critical dynamics and obtained using the quadratic family, see [36, Chapter 6]. Here the limit dynamics is non-critical, but we prove that the family $(L_\mu)_{\mu \in [-\delta, \delta]}$ also generates thick Cantor sets. We see that every $L_\mu$ has two hyperbolic sets $\Upsilon_\mu$ and $\Xi_\mu$ with non-empty intersection. Such an intersection is assured by a condition on the product of their thickness (see the Gap Lemma 7.1). The sets $\Upsilon_\mu$ and $\Xi_\mu$ are the projections along the stable and unstable directions of two hyperbolic sets $\Upsilon_{\varrho_n(\mu)}$ and $\Xi_{\varrho_n(\mu)}$ of $f_{\varrho_n(\mu)}$ having different indices. Thus intersections of $\Upsilon_\mu$ and $\Xi_\mu$ correspond to (heterodimensional) cycles associated to $\Upsilon_{\varrho_n(\mu)}$ and $\Xi_{\varrho_n(\mu)}$. 
Some comments about our constructions arise naturally. In rough terms, our construction is a skew-product, where we fix a one-dimensional dynamics and construct a cycle whose central limit dynamics is the fixed one. We do not know the flexibility of our construction. For instance, whether our result holds for arcs of diffeomorphisms in a neighbourhood of the arcs in this paper. The skew-product structure allows us to estimate the thickness of Cantor sets created in the unfolding of the cycle, concluding the persistence of cycles thereafter. But, in general, estimates of the thickness in dimensions strictly bigger than two are tough and subtle, depending on the regularity of the invariant foliations, see [37]. Probably, the ideas sketched in [27] for homoclinic tangencies in higher dimensions could be an ingredient for extending our results to a wider setting.

Finally, related to our results, in the context of vector fields, [3] gives a vector field with two homoclinic classes whose intersection is the closure of the unstable manifold of a hyperbolic singularity. See also [24], where it is proved that a singular hyperbolic set (see [25] for the definition) containing a unique singularity is either transitive or the union of two homoclinic classes.

This paper is organized as follows. In Section 2 we describe a model arc unfolding a heterodimensional cycle. In Section 3 we introduce a one-parameter system of iterated functions describing the central dynamics after the bifurcation (limit dynamics). After introducing the basic terminology, we sketch the proof of the theorem in Section 3.3. In Section 4 we choose the central dynamics by considering saddle-node arcs. In Section 5 we construct two hyperbolic sets (of different indices) and describe their collision via a saddle-node bifurcation. In Section 6 we construct two one-dimensional hyperbolic Cantor sets (contracting and expanding), associated to the system of iterated functions, the expanding one with large thickness. Finally, we see that these Cantor sets correspond to two hyperbolic sets of diffeomorphisms of the arc (see Sections 6.1 and 6.3) and prove, in Section 7 that these hyperbolic sets have heterodimensional cycles.

2. The model one-parameter family. For notational simplicity, we will carry our constructions in dimension 3, the extension to higher dimensions is straightforward. We consider a one-parameter family of (local) diffeomorphisms \((f_t)_{t \in [-\tau, \tau]}\) defined on the cube \(R = [-1,1] \times [-1,5] \times [-1,1]\) of \(\mathbb{R}^3\). Using local coordinates, this construction can be carried to any manifold of dimension three.

The definition of the arc \((f_t)_{t \in [-\tau, \tau]}\) is done in three steps: (a) semi-local product partially hyperbolic dynamics, (b) existence and unfolding of the heterodimensional cycle, and (c) semi-global hypotheses (existence of a filtration). The filtration enables us to control the dynamics after the bifurcation. The product partial hyperbolicity allow us to reduce the study of the dynamics of the resulting non-wandering set to a one-dimensional family of iterated functions.

The cycle is related to hyperbolic fixed saddles \(Q = (0,0,0)\) and \(P = (0,4,0)\) of indices 2 and 1, respectively, and the semi-local dynamics satisfies the following properties.

(a) Partially hyperbolic product dynamics. In the cube \(R = [-1,1] \times [-1,5] \times [-1,1]\), for every \(t \in [-\tau, \tau]\), it holds \(f_t = f\), where

\[
f(x, y, z) = (\lambda_s x, F(y), \lambda_u z), \quad 0 < \lambda_s < 1 < \lambda_u,
\]

where \(F\) is a strictly increasing function to be fixed in Section 4 such that:
Thus \( W \) of dimension one) heterodimensional cycle related to \( P \). Thus, as \( f \) the composition of \( \lambda \)

The domination condition implies that \( \lambda < \beta < \lambda_n \) and that the restriction of \( f \) to \( R \) has a partially hyperbolic splitting \( E^s \oplus E^c \oplus E^u \) with three non-trivial bundles given by the coordinate directions. Observe also that, by definition of the diffeomorphism \( f \),

\[
\begin{align*}
\{0\} \times [-1, 4] \times [-1, 1] & \subset W^u(0) \quad \text{and} \quad [-1, 1] \times \{0\} \subset W^s(0), \\
\{(0, 4)\} \times [-1, 1] & \subset W^u(0) \quad \text{and} \quad [-1, 1] \times \{0, 5\} \times \{0\} \subset W^s(0).
\end{align*}
\]

Thus \( W^s(P) \) and \( W^u(Q) \) meet transversely along the connection \( \gamma = \{0\} \times (0, 4) \times \{0\} \).

(b) Existence and unfolding of the heterodimensional cycle. Consider the points \( A = (0, 4, -1/2) \in W^u(P) \) and \( B = (1/2, 0, 0) \in W^s(Q) \).

We assume that there is \( k_0 \in \mathbb{N} \) with \( f^{k_0}(A) = B \) and \( f^{j}(A) \notin R \) for all \( 0 < j < k_0 \). So \( A, B \in W^u(P) \cap W^s(Q) \neq \emptyset \) and there is a heterodimensional cycle associated to \( P \) and \( Q \).

Take a small compact neighbourhood \( U_A \) of \( A \) such that \( U_A, \ldots, f^{k_0}(U_A) \) are pairwise disjoint, \( U_A, f^{k_0}(U_A) \subset R \), and \( f^{j}(U_A) \cap R = \emptyset \), for all \( 0 < j < k_0 \). We assume that the dynamics of \( f^{k_0} \) from \( U_A \) to \( f^{k_0}(U_A) \) is of the form:

: affine return: \( f^{k_0}(x, y, z) = (x + \frac{1}{2}, y - 4, z + \frac{1}{2}) \), for every \( (x, y, z) \in U_A \).

This implies that, for some \( \delta > 0 \), \( \{(1/2, 0)\} \times (-\delta, \delta) \subset W^u(P) \). Thus \( W^s(P) \) and \( W^s(Q) \) meet quasi-transversely throughout the orbit of the heteroclinic point \( A \),

\[
T_A W^u(P) + T_A W^s(Q) = T_A W^u(P) \oplus T_A W^s(Q) = \mathbb{R}^3.
\]

Thus, as \( W^s(P) \) and \( W^u(Q) \) have non-empty transverse intersection, \( f \) has a (codimension one) heterodimensional cycle related to \( P \) and \( Q \).

We next define the arc arc \((f_t)_{t \in [-\tau, \tau]} \) unfolding the cycle above at \( t = 0 \). We let \( f_t(x, y, z) = f(x, y, z) \) for all \( (x, y, z) \in R \) and define the restriction of \( f^{k_0}_t \) to \( U_A \) the composition of \( f^{k_0} \) and a \( t \)-translation parallel to the \( \mathbb{R} \) direction:

\[
f^{k_0}_t(x, y, z) = f^{k_0}(x, y, z) + (0, t, 0) = (x + \frac{1}{2}, y - 4 + t, z + \frac{1}{2}). \tag{1}
\]

Note that \([-1, 1] \times (0, 5) \times \{0\} \subset W^s(P, f_t) \) and, by construction, \( \{(1/2, t)\} \times (-\delta, \delta) \subset W^u(P, f_t) \). Thus, for \( t > 0 \), \( (1/2, t, 0) = f^{k_0}_t(0, 4, -\delta) = f^{k_0}_t(A) \) is a

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{model_cycle.png}
\caption{The model cycle and its unfolding}
\end{figure}
transverse homoclinic point of $P$. Similarly, $B$ is a transverse homoclinic point of $Q$. Thus, for $t > 0$, $H(P, f_1)$ and $H(Q, f_1)$ are non-trivial.

We now assume some properties of the bifurcating diffeomorphism $f$ that allow us to control the global dynamics of $f$. First, a pair $(M_1, M_2)$ is a filtration of $f$ if $M_1$ and $M_2$ are compact manifolds with boundary of the same dimension as $M$ such that $f(M_i) \subset \text{int}(M_i)$, $i = 1, 2$.

(c) Semi-global dynamics: filtrations and relative Morse-Smale dynamics. The neighbourhood of the cycle

$$W = R \cup (\cup_{j \in \{1, \ldots, k_0-1\}} f^j(U_A))$$

is a filtrating set: there is a filtration $(M_1, M_2)$ of $f$ such that $W = M_2 \setminus \text{int}(M_1)$.

This implies that if $x \in W$ and $f(x) \notin W$ then $f^n(x) \notin W$, for every $n \in \mathbb{N}$. Similarly, if $x \in W$ and $f^{-1}(x) \notin W$ then $f^{-n}(x) \notin W$, for all $n \in \mathbb{N}$. It follows that, for every small $|t|$, the set $W$ is also a filtrating neighbourhood of $f_t$. Hence, for all small $|t|$,\n
$$\Omega(f_t) = \Omega(f_1) \cap W = \Omega(f_t|_W) \subset \bigcap_{n \in \mathbb{Z}} f_t^n(W), \quad H(P, f_1) \cup H(Q, f_1) \subset \bigcap_{n \in \mathbb{Z}} f_t^n(W).$$

The semi-local dynamics of $f_t$ implies that, for $t < 0$, the restriction of $f_t$ to $W$ is Morse-Smale (in particular, the homoclinic classes of $P$ and $Q$ are both trivial). Thus at $t = 0$ there is a $\Omega$-explosion: for positive $t$, the classes of $H(P, f_t)$ and $H(Q, f_t)$ are both non-trivial.

We finish this section with a remark about invariant foliations. For $X = (x_0, y_0, z_0) \in R$, define its strong stable, strong unstable and central leaves, denoted by $F^s(X)$, $F^u(X)$, and $F^c(X)$, by

$$F^s(X) = \{(x, y, z) \in R : y = y_0, z = z_0\}, \quad F^u(X) = \{(x, y, z) \in R : x = x_0, y = y_0\}, \quad F^c(X) = \{(x, y, z) \in R : x = x_0, z = z_0\}.$$\n
This defines three foliations (strong stable, strong unstable, and central) on $R$. For each $t$, we extend them, via $f_t$, to the whole $W$. These foliations (denoted by $F^s_t$, $F^u_t$, $F^c_t$, $\sigma = s, u, c$) are $f_t$-invariant in the following sense: if $X \in R$ and $f_t(X) \in R$ then $f_t(F^s(X)) \subset F^s_t(f_t(X))$, $f_t(F^u(X)) \supset F^u_t(f_t(X))$ and $f_t(F^c(X)) \cap F^c_t(f_t(X))$ is a neighbourhood of $f_t(X)$ in $F^c_t(f_t(X))$; if $X \in R$, $f_t(X) \notin R$ and $f_t^{k_0}(X) \in R$ (i.e., $X \in U_A$) then $f_t(F^c_t(X)) \subset F^c_t(f_t(X))$, $\sigma = s, u, c$.

Let $\Lambda_t = \cap_{n \in \mathbb{Z}} f_t^n(W)$. Note that if $X \in R$ then $Df_t(X)$ uniformly contracts in the $X$-direction and uniformly expands in the $Z$-direction, and if $X \in U_A$ then $Df_t^{k_0}(X)$ is the identity. These remarks and the partial hyperbolicity of $f_t$ in $R$ imply that $\Lambda_t$ is partially hyperbolic: the $X$ and $Z$-directions are hyperbolic and dominate the (central) $Y$-direction. Thus the dynamics of $f_t$ is mainly determined by its central dynamics. We study the central dynamics in the next section.

3. Normalized central dynamics, selection of parameters, and outline of the construction. In this section, we introduce the main ideas in the proof of Theorem 1 and sketch its proof. We first present the main terminology in our constructions: itineraries (Section 3.1) and normalized central dynamics (Section 3.2). Finally, we outline the proof of Theorem 1 in Section 3.3.
3.1. **Itineraries.** Take fundamental domains \( D^+ = [\beta^{-1}, 1] \) and \( D^- = [3, 4 - \lambda] \) of \( F \) and assume that \( F^n(D^+) = D^- \) for some \( n \in \mathbb{N} \). Consider the reference cubes
\[
\Delta^\pm = [-1, 1] \times D^\pm \times [-1, 1].
\]

Fix small \( t > 0 \) and \( X \in \Lambda_t \cap \Delta^+ \). There are two possibilities for positive iterates of \( X \): either there is \( n > 1 \) with \( F^n(X) \in \Delta^+ \) or not. In the first case, the smallest \( n > 1 \) with \( F^n(X) \in \Delta^+ \) is the *forward return time of \( X \)*, denoted by \( r^+(X) \). In the second case, \( X \in W^s(P) \cup W^s(Q) \). Similarly, for negative iterates of \( X \), either there is \( n > 1 \) with \( F^{-n}(X) \in \Delta^+ \) or not. In the first case, the *backward return time of \( X \)*, \( r^-(X) \), is defined as above. In the second case, \( X \in W^u(P) \cup W^u(Q) \). These properties follow using the filtration and the geometry of the cycle. See [16, Lemma 4.1]. Since, by the definition of \( f_t \), the invariant manifolds of \( P \) and \( Q \) meet transversely or quasi-transversely, according to the case, the previous results may be read as follows:

**Lemma 3.1.** Given small \( t > 0 \) and any point \( X \in \Lambda_t \cap \Delta^+ \) there are the following possibilities:

- \( X \) has infinitely many forward or backward returns to \( \Delta^+ \),
- \( X \) has finitely many forward and finitely many backward returns to \( \Delta^+ \), then there are three possibilities:
  1. \( X \) is a transverse homoclinic point of \( P \) or \( Q \) (thus \( X \in H(P, f_t) \) or \( X \in H(Q, f_t) \)),
  2. \( X \in W^u(P) \cap W^s(Q) \) (heteroclinic cycle point) and \( W^u(P, f_t) \) and \( W^s(Q, f_t) \) meet quasi-transversely at \( X \), and
  3. \( X \in W^s(P) \cap W^u(Q) \) (heteroclinic connection point) and \( W^s(P, f_t) \) and \( W^u(Q, f_t) \) meet transversely at \( X \).

Given \( X = (x, y, z) \), write \( x_i = (x_i, y_i, z_i) = f_t^i(X) \). Assume that \( X \) has a first forward return \( r^+(X) = \bar{q} + k_0 + \bar{p} + N \) to \( \Delta^+ \), \( \bar{q} = \bar{q}(X) \) and \( \bar{p} = \bar{p}(X) \), where \( k_0 \) is as in the definition of the cycle, \( F^N(D^+) = D^- \), and

\[
\begin{align*}
X_N \in \Delta^-,
X_N, \ldots, X_{\bar{q} - 1 + N}, X_{\bar{q} + N} \in R \quad \text{and} \quad X_{\bar{p} + N} \in U_A, \\
X_{k_0 + \bar{p} + N} \in f_t^{k_0}(U_A) \subset R, \\
X_{i + k_0 + \bar{p} + N} \in (R \setminus \Delta^+) \quad \text{for all} \quad i = 0, 1, \ldots, \bar{q} - 1, \text{and} \\
X_{\bar{q} + k_0 + \bar{p} + N} \in \Delta^+.
\end{align*}
\]

**Figure 5.** Itineraries
3.2. Choice of parameters and normalized central dynamics. For special parameters, we can estimate \( \bar{p} \) and \( \bar{q} \), controlling the returns in the central direction.

**Lemma 3.2.** Consider small \( \delta > 0 \) and let \( t \in B_n = [t_n (1 - \delta), t_n (1 + \delta)] \), where \( t_n = \lambda^n \).

- If \( X \in \Delta^+ \) has a first forward return \( r^+(X) = \bar{q} + k_0 + \bar{p} + N \) to \( \Delta^+ \) (for \( f_t \)) then \( \bar{p}, \bar{q} \geq (n - 1) \).
- Moreover, if \( t = t_n(1 + \mu) \), with \( \mu > 0 \), then \( \bar{p}, \bar{q} \geq n \).

**Proof:** Fix \( t = t_n(1 + \mu) \), \( \mu \in (-\delta, \delta) \), and \( X = (x, y, z) \). By definition of \( f_t \) and since \( F \) is affine in \([3, 5]\),

\[
y_{\bar{p} + n} = \lambda^\bar{p} (y_N - 4) + 4.
\]

Since \( X \) returns to \( \Delta^+ \), the unfolding condition \( \text{(1)} \) implies that

\[
0 < y_{k_0 + \bar{p} + n} = \lambda^\bar{p} (y_N - 4) + 4 + (t - 4) = \lambda^\bar{p} (y_N - 4) + t.
\]

Finally, as \( y_N \in D^- = [3, 4 - \lambda] \), \(-1 \leq (y_N - 4) \leq -\lambda \), it follows

\[
-t = -t_n (1 + \mu) = -\lambda^n (1 + \mu) < \lambda^{\bar{p}} (y_N - 4) \leq -\lambda^{\bar{p} + 1} \lambda = -\lambda^{\bar{p} + 1}, \quad \lambda^n (1 + \mu) > \lambda^{\bar{p} + 1}.
\]

So, if \( \mu \) is small, \( \bar{p} \geq (n - 1) \). Moreover, if \( \mu > 0 \) one has \( \bar{p} \geq n \).

Finally, the estimate for \( \bar{q} \) follows similarly noting that \( y_{k_0 + \bar{p} + n} < t \).

Fix \( t = t_n(1 + \mu) \in \lambda^n (1 - \delta, 1 + \delta) \). Using Lemma 3.2 we write

\[
r^+(X) = q(X) + n + k_0 + p(X) + n + N, \quad q(X), p(X) \geq -1,
\]

and we say that the first return \( r^+(X) \) of \( X \) is of type \( (q(X), p(X)) \). Since \( \beta^n = t_n^{-1} \) and \( \lambda^n = t_n \), the y-coordinate of \( X_{r^+(X)} \) is

\[
y_{r^+(X)} = \beta^{q(X) + n} [t_n (1 + \mu) - 4 + (\lambda^{n+p(X)} (F^N (y) - 4) + 4)] = \beta^{q(X) + n} [t_n (1 + \mu) + (\lambda^{n + p(X)} (F^N (y) - 4))] = \beta^{p(X)} [1 + \mu + \lambda^{p(X)} (F^N (y) - 4)].
\]

Equation 2 leads to the following one-parameter family of maps:

\[
\Phi^{q,p}_\mu : D^{q,p}_\mu \to D^+, \quad \Phi^{q,p}_\mu (y) = \beta^{p} [1 + \mu + \lambda^{p} (F^N (y) - 4)], \quad \mu \in [-\delta, \delta], \quad p, q \geq -1,
\]

where \( D^{q,p}_\mu \) is the maximal subset of \( D^+ \) consisting of points \( y \) with \( \Phi^{q,p}_\mu (y) \in D^+ \) (see Figure 3).

The family \( (\Phi^{q,p}_\mu)_{\mu \in (-\delta, \delta)} \) is the normalized central dynamics of \( (f_t)_{t \in [-\tau, \tau]} \). Remark 3.3 justifies this name and implies that \( \Phi^{q,p}_\mu \) gives the y-coordinate of the \((q,p)\)-returns of \( f_{t_n(1 + \mu)} \).

**Remark 3.3. (Central dynamics).** Let \( t = \lambda^n (1 + \mu) \in B_n \).

- By the monotonicity of \( F \), \( D^{q,p}_\mu \) is either empty or a closed subinterval of \( D^+ \).
- Suppose that \( X = (x_0, y_0, z_0) \in \Delta^+ \) has a \((q,p)\)-return. Then \( y_0 \in D^{q,p}_\mu \) and the y-coordinate of \( f_{t_n + n + k_0 + p + n}^+ (X) \) is \( \Phi^{q,p}_\mu (y_0) \). Conversely, given any \( y_0 \in D^{q,p}_\mu \), there is at least one point of the form \( X = (x, y, z) \in \Delta^+ \) with a \((q,p)\)-return.
- **(Consecutive returns)** Assume that \( X_0 = (x_0, y_0, z_0) \in \Delta^+ \) has a \((q_0,p_0)\)-return \( X^1 \) and \( X^1 \) has a \((q_1,p_1)\)-return \( X^2 \).
By Remark 3.3, the maps $\Phi_{\mu}^{p,q}$ give the $Y$-coordinate (central dynamics) of returns of $f_t$ for appropriate $t$. Now $Df_t$ is uniformly contracting in the $X$-direction and uniformly expanding in the $Z$-direction. These facts together with the product structure imply that some dynamical properties of subsets of $\Lambda_t$, such as hyperbolicity, existence of periodic points, and bifurcations, have a translation to dynamical properties of the maps $\Phi_{\mu}^{p,q}$, for appropriate $\mu$, $p$, and $q$. For example, to a fixed point of $\Phi_{\mu}^{p,q}$ corresponds a periodic point of $f_t$, where $p$ and $q$ determines the central coordinate of the iterates of the point as well as its index. The dynamics of the maps $\Phi_{\mu}^{p,q}$ depend only on $F^N$, $\lambda$, and $\mu$. In Section 4, we choose $F$ and $\lambda$ accordingly.

3.3. Outline of the constructions. Sketch of the proof. In this section, we present the main steps of our constructions and sketch the proof of Theorem 1.

**Selection of the central dynamics.** To define the map $F$, corresponding to the central dynamics, we first consider the auxiliary saddle-node arc $g_\alpha$:

$$g_\alpha(y) = y - \varepsilon(y - 1/2)^2 + \alpha(1 - y),$$

with a saddle-node at the point $1/2$ for the parameter $\alpha = 0$ (here $\varepsilon > 0$ is small). The saddle-node is unfolded for negative $\alpha$: for $\alpha > 0$, the map $g_\alpha$ has two hyperbolic fixed points close to $1/2$, collapsing to $1/2$ for $\alpha = 0$, and disappearing thereafter. We select an interval $L$ containing the point $1/2$ in the reference fundamental domain $D^+ = [\beta^{-1}, 1]$. To each parameter $\alpha$ (close to 0), we consider a map $F_\alpha$ defining a possible central dynamics of $f = f_\alpha$ in the cube $R$ as follows,

$$f(x, y, z) = (\lambda_s x, F_\alpha(y), \lambda_u z), \quad \text{where } F_\alpha^N : L \to D^- = [3, 4 - \lambda],$$

$$F_\alpha^N(y) = g_\alpha(y) + 3. \quad (4)$$

This is a key property of $F_\alpha$, see condition (F3) in Section 4.1. We will choose $\alpha^* > 0$ close to 0 and let $F = F_{\alpha^*}$. Then, by equation (2), for parameters $t_n = \lambda^n$ the map $g_{\alpha^*}$ gives the $y$-coordinate of the points having a return of type $(0, 0)$.

In very rough and purposely vague terms, the choice of $\alpha^*$ is done for guaranteeing the creation of one dimensional (central) Cantor sets with large thickness after unfolding the saddle-node bifurcations that appear throughout the unfolding of the initial heterodimensional cycle (for details, see the next paragraphs). The creation of these thick Cantor sets is a key step for obtaining the persistence of heterodimensional cycles in Theorem 1.
Dynamics of the returns. The selection of $\alpha$ in the definition of the central dynamics is done as follows. We fix small $\mu^r < 0$ and consider the auxiliary saddle-node family

$$\varphi_\alpha(y) = g_\alpha(y) + \mu^r.$$ 

This family has a saddle-node bifurcation for some $\alpha^* = \alpha(\mu^r) > 0$. Fixed $\alpha^* > 0$, the central dynamics is given by $F = F_{\alpha^*}$. Thus the arc $(f_t)_{t \in [-\tau, \tau]}$ is now completely defined.

FIGURE 7. Saddle-node arcs

We consider a new saddle-node arc $(\mathcal{L}_\mu)_{\mu \in (\alpha^*, \alpha]}$, $\mathcal{L}_\mu = g_{\alpha^*} + \mu$, with a saddle-node for $\mu^l < 0$. By equations (2) and (11) and Remark 5.3 for $t = t_n (1 + \mu) \in B_n$ and $X \in [-1, 1] \times L \times [-1, 1]$ with a return of type $(0, 0)$, the map $\mathcal{L}_\mu$ defines the $y$-coordinate of such returns,

$$y_{+}(X) = \beta^{n+1} \left[ t_n (1 + \mu) - 4 + (\lambda^{n+1} (F^{n}(y) - 4) + 4) \right] = 1 + \mu + (F^{n}(y) - 4) = \mu + F^{n}(y) - 3 = g_{\alpha^*}(y) + \mu = \mathcal{L}_\mu(y).$$

Secondary (rescaled) saddle-node bifurcations. Take the saddle-node arc $(\mathcal{L}_\mu)_{\mu \in (\alpha^*, \alpha]}$ and let $s_{\mu^*} \in L$ (close to $1/2$) be the corresponding saddle-node. The construction of $(f_t)_{t \in [-\tau, \tau]}$ and the comments above imply that, for any $t_n^* = t_n (1 + \mu^*)$, the diffeomorphism $f_{t_n^*}$ has a saddle-node at some point $S_{t_n^*}$ in $\Delta^+$ whose $y$-coordinate is $s_{\mu^*}$. For every $t_n^*$, the saddle-node is essentially the same, the only difference being that the periods $\pi_n$ of $S_{t_n^*}$ increase as $n$, but the local central dynamics of the $f_{t_n^*}^{n}$ at the saddle-nodes $S_{t_n^*}$ are always the same (given by $h_{\mu^*}$).

Hyperbolicity before the unfolding of the saddle-node. By Remark 5.3 for every $t = t_n (1 + \mu)$ with small $\mu$, the system $(\Phi^{\mu^*}_\mu)$ gives the central returns to the reference cube $\Delta^+$. We prove that there is $\mu^+, \mu^+ > \mu^*$, such that the system $(\Phi^{\mu^*}_{\mu^+})$ is hyperbolic for all $\mu \in (\mu^*, \mu^+]$. For the precise meaning of this assertion see Section 5.4. The relevant fact is that the hyperbolicity of $(\Phi^{\mu^*}_{\mu^+})$ implies the hyperbolicity of $f_t$ for $t = t_n (1 + \mu)$. Let us explain this point.

For $\mu > \mu^*$, the map $\mathcal{L}_\mu = \Phi^{\mu^*}_{\mu}$ has two fixed points $s_{\mu}^-$ (repelling) and $s_{\mu}^+$ (contracting) collapsing to the saddle-node $s_{\mu^*}$ at $\mu^*$. As above, for $t = t_n (1 + \mu)$, $\mu > \mu^*$, the points $s_{\mu}^-$ and $s_{\mu}^+$ correspond to saddles $S_{t}^-$ and $S_{t}^+$ of $f_t$, of indices 2 and 1, whose $y$-coordinate are $s_{\mu}^-$ and $s_{\mu}^+$.

Using Remark 5.3, the product structure of the dynamics, and noting that every $f_t$ uniformly contracts in the $X$-direction and uniformly expands in the $Z$-direction, we translate the hyperbolicity of $(\Phi^{\mu^*}_{\mu^+})$ to $f_t$ for every $t = t_n (1 + \mu)$ with $\mu \in (\mu^*, \mu^+]$ (any $n$). Finally, using the family $(\Phi^{\mu^*}_{\mu^+})$, we prove that for such parameters one has that

- $H(P, f_t) = H(S_{t}^+, f_t)$ and $H(Q, f_t) = H(S_{t}^-, f_t)$;
- the homoclinic classes $H(P, f_t)$ and $H(Q, f_t)$ are both hyperbolic (thus disjoint); and
- the resulting non-wandering set of $f_t$ is the union of $H(P, f_t)$ and $H(Q, f_t)$.
Finally, we see that these hyperbolic homoclinic classes collide for $t^*_n$.

**Persistence of cycles at the collisions.** To get the intervals of persistence of cycles (associated to the saddle-nodes $t^*_n$) we choose carefully the parameter $\alpha^*$ (of the arc $(g_n)$) in the definition of $F$. In fact, in the previous steps about hyperbolicity the choice of $\alpha^*$ is essentially irrelevant.

For $\mu < \mu^*$ close to $\mu^*$ and appropriate $(q, p)$, the system $(\bar{\Phi}^{q,p})$ induces a family of (uniformly) expanding maps from a subset of $D^+$ of the form $[\beta^{-1} + \varepsilon(\mu), 1]$ to the whole $D^+$, for some small $\varepsilon(\mu) > 0$. The choice of $\alpha^*$ is done in such a way that $\varepsilon(\mu) \approx |\alpha^*|$, where $|\alpha^*|$ is close to zero.

Recall that, roughly, the thickness of a Cantor set is the quotient $\text{length(bridge)}/\text{length(gap)}$, where in each step of the construction of the Cantor set a gap is a removed open interval and its two bridges are the adjacent resulting connected components, see Definition 6.1.

We consider (essentially) the maximal invariant set $\Sigma_\mu$ of $\bar{\Phi}^{q,p}$ in $[\beta^{-1} + \varepsilon(\mu), 1]$. Then, roughly, the gaps of the Cantor set $\Sigma_\mu$ are the pre-images of $[\beta^{-1}, \beta^{-1} + \varepsilon(\mu)]$ and the corresponding bridges are pre-images of $[\beta^{-1} + \varepsilon(\mu), 1]$. Figure 8 gives a naive representation of some of these gaps. Using bounded distortion properties, we see that these quotients are of order of

$$\frac{1 - \beta^{-1} - \varepsilon(\mu)}{\varepsilon(\mu)} \approx \frac{1}{\varepsilon(\mu)} \approx \frac{1}{|\alpha^*|},$$

see Proposition 6.9. Thus, taking $\alpha^*$ close to zero, one has that $\Sigma_\mu$ has large thickness.

![Figure 8. Naive gaps](image)

Remark 3.3 now implies that $\Sigma_\mu$ is contained in the the projection in the central direction of a hyperbolic set $\Xi_t$, $t = t_n (1 + \mu)$, of index two.

We also consider a hyperbolic set $\Upsilon_t$ of index one in the homoclinic class of $P$ whose projection in the central direction is a Cantor set with thickness uniformly bounded from 0, see Section 6.4.

The Gap Lemma (Lemma 7.1) claims that two Cantor sets with non-disjoint convex hulls are and whose product of their thickness is larger than one have non-empty intersection. The previous constructions (for small $|\alpha^*|$) imply that product of the thickness of the projections of $\Xi_t$ and $\Upsilon_t$ is bigger than one. One checks that for small $|\alpha^*|$ the convex hulls of these sets are non-disjoint. Thus they verify the Gap Lemma. Hence the projections of $\Xi_t$ and $\Upsilon_t$ are non-disjoint. The product structure now gives the persistence of cycles (associated to $\Xi_t$ and $\Upsilon_t$) in Theorem 1.

4. **Definition of the map $F$. Saddle-node arcs.** In this section, we define the map $F$ giving the central dynamics of the diffeomorphisms $f_t$. 
4.1. A saddle-node family. For small positive \( \varepsilon \) (further restrictions on the size of \( \varepsilon \) will appear throughout the text) consider the saddle-node arc \((g_\alpha)\), small \(|\alpha|\),

\[
g_\alpha(y) = y - \varepsilon(y - 1/2)^2 + \alpha(1 - y).
\]

The arc \(g_\alpha\) unfolds, at \( \alpha = 0 \), a saddle-node at \( 1/2 \) as follows. For \( \alpha > 0 \), the map \(g_\alpha\) has two hyperbolic points. These points collapse to the saddle-node at \( \alpha = 0 \) and disappear for negative \( \alpha \).

For \( \alpha \) and \( \varepsilon \) small, let \( d_\alpha \) be the critical point of \( g_\alpha \),

\[
d_\alpha = 1/2 + (1 - \alpha)/(2 \varepsilon) > 1.
\]

Note that the maps \( g_\alpha \) are increasing in \((-\infty, 1]\), \( g_0(1/2) = 1/2 > \beta^{-1} \), and \( g_0(\beta^{-1}) < \beta^{-1} \) (small \( \varepsilon \)). Thus, for every \( \alpha \) close to \( 0 \), there is a unique \( c_\alpha \in (\beta^{-1}, 1/2) \) with \( g_\alpha(c_\alpha) = \beta^{-1} \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{saddle_node_arc.png}
\caption{The saddle-node arc \( g_\alpha \)}
\end{figure}

Write \( D^+ = [\beta^{-1}, c_\alpha] \cup [c_\alpha, 1] \). We take

\[
\lambda = \beta^{-1} = \varepsilon/4
\]

and, for small \( \alpha \), choose a \( C^2 \)-map \( F_\alpha : [-1, 5] \to \mathbb{R} \) verifying conditions (F1)–(F5) below:

- (F1) \( F_\alpha \) is strictly increasing and \( F_\alpha^N(D^+) = D^- = [3, 4 - \lambda] \),
- (F2) \( F_\alpha(y) = \beta y \) in \([-1, \beta^{-1}]\), \( F_\alpha(y) = \lambda (y - 4) + 4 \) in \([3, 5] \),
- (F3) the restriction of \( F_\alpha^N \) to the interval \([c_\alpha, 1]\) is equal to \( g_\alpha + 3 \),
- (F4) \( (F_\alpha^N)'(y) > 1 \) for all \( y \in [\beta^{-1}, c_\alpha] \), and
- (F5) \( \lambda (F_\alpha^N)'(\beta^{-1}) = \beta (F_\alpha^N)'(1) \).

We will take the map \( F \) giving the central dynamics, see Section 2, equal to some \( F_\alpha \), for a convenient \( \alpha \), to be chosen in Proposition 6.3.

For the existence of a map \( F_\alpha \) satisfying the properties above, note first that (F1) and (F3) are compatible. On one hand, by definition, \( g_\alpha(1) = 1 - \varepsilon/4 = 1 - \lambda \), so \( F_\alpha(1) = 4 - \lambda = g_\alpha(1) + 3 \). On the other hand, note that \( g_\alpha(c_\alpha) + 3 = \beta^{-1} + 3 > 3 \). So there is space for extending \( F_\alpha^N \) to the subinterval \([\beta^{-1}, c_\alpha]\) of \( D^+ \) in such a way \( F_\alpha^N([\beta^{-1}, c_\alpha]) = [3, 3 + \beta^{-1}] \).

By continuity, it suffices to check (F4) for \( \alpha = 0 \). Since \( F_0^N([\beta^{-1}, c_0]) = [3, 3 + \beta^{-1}] \), it is enough to see that \( c_0 - \beta^{-1} < \beta^{-1} \), that is, \( c_0 < 2 \beta^{-1} \). Since \( g_0 \) is strictly increasing in \([0, 1]\), \( c_0 > 0 \), and \( 2 \beta^{-1} < 1 \), this condition is equivalent to

\[
g_0(c_0) = \beta^{-1} < g_0(2 \beta^{-1}).
\]

Since \( \beta^{-1} = \varepsilon/4 \),

\[
g_0(2 \beta^{-1}) = g_0(\varepsilon/2) = \varepsilon/2 - \varepsilon(\varepsilon/2 - 1/2)^2 = \varepsilon(1/2 - (1/4)(\varepsilon - 1)^2).
\]
Thus, \( g_0(2\beta^{-1}) > \beta^{-1} \) is equivalent to
\[
\varepsilon (1/2 - (1/4) (\varepsilon - 1)^2) > \varepsilon / 4,
\]
or
\[
1/4 > (1/4) (\varepsilon - 1)^2,
\]
or
\[
1 > (\varepsilon - 1)^2,
\]
which is automatically verified for small positive \( \varepsilon \).

Finally, to check condition (F5), observe that by (F3), \( (F_0^N)'(1) = g_0'(1) \), so (F5) is equivalent to \( (F_0^N)'(\beta^{-1}) = 16 (1 - \varepsilon - \alpha) / \varepsilon^2 \), which is clearly compatible with the previous conditions.

4.2. Estimates on the derivative of the saddle-node arc. For \( \alpha \) close to 0, consider the fundamental domains of \( g_\alpha \)
\[
J_\alpha = [c_\alpha, g_\alpha^{-1}(c_\alpha)] \quad \text{and} \quad I_\alpha = [g_\alpha(1), 1] = [1 - \lambda, 1],
\]
and the constants
\[
d_m(\alpha) = \min\{g_\alpha'(y), y \in [c_\alpha, 1]\} \quad \text{and} \quad d_M(\alpha) = \max\{g_\alpha'(y), y \in [c_\alpha, 1]\}.
\]
As the critical point \( d_\alpha \) of \( g_\alpha \) is strictly bigger than 1 (for small \( |\alpha| \)), we have \( d_M(\alpha) = d_m(\alpha) > 0 \).

Note that for \( \alpha < 0 \), given any \( w \in (g_\alpha^{-1}(c_\alpha), 1] \) there is a first \( n_\alpha(w) \in \mathbb{N} \) with \( g_n(\omega)(w) \in J_\alpha \).

**Proposition 4.1.** Fix \( \varepsilon \in (0, \frac{1}{10}) \). There is \( \alpha_\varepsilon > 0 \) such that
\[
(g_\alpha^n(w))'(w) > 1/3, \quad \text{for all } \alpha \in [-\alpha_\varepsilon, 0] \text{ and every } w \in (g_\alpha^{-1}(c_\alpha), 1].
\]

This proposition will be used to estimate the thickness of the dynamically defined Cantor sets to be constructed in Section 6.

The main step in the proof of Proposition 4.1 is the next estimate of the derivative of \( g_\alpha^n(w) \):

**Lemma 4.2.** There is \( \alpha_\varepsilon > 0 \) such that
\[
(g_\alpha^n(w))'(w) \geq \frac{d_m(\alpha)}{(d_M(\alpha))^2 |J_\alpha|} > \frac{1}{3}, \quad \text{for any } \alpha \in [-\alpha_\varepsilon, 0)
\]
\[
\text{and any } w \in I_\alpha = [g_\alpha(1), 1].
\]

**Proof of the lemma:** To prove the second inequality it is enough to check it for \( \alpha = 0 \). The general case follows from the continuous dependence of \( d_m(\alpha) \), \( d_M(\alpha) \), \( |J_\alpha| \), and \( |I_\alpha| \) on the parameter \( \alpha \). Since \( g_0 \) is decreasing, \( I_0 \) and \( J_0 \) are contained in \( (-\infty, d_0) \), and \( c_0 > \beta^{-1} \),
\[
d_m(0) = g_0'(1) = (1 - \varepsilon),
\]
\[
d_M(0) = g_0'(c_0) < g_0'(\beta^{-1}) = g_0'(\varepsilon/4) = 1 - 2 \varepsilon (\varepsilon/4 - 1/2)
\]
\[
= (1 + \varepsilon - \varepsilon^2/2) < (1 + \varepsilon).
\]
By definition, \( |I_0| = \varepsilon / 4 \). It remains to estimate \( |J_0| \).

**Claim 4.3.** \( |J_0| \geq \varepsilon / 8 \).

**Proof:** Write \( J_0 = [c_0, g_0^{-1}(c_0)] = g^{-1}([\beta^{-1}, c_0]) \) and observe that
\[
|J_0| = \frac{|c_0 - \beta^{-1}|}{g_0(\xi)}, \quad \text{for some } \xi \in [\beta^{-1}, c_0].
\]
Since \( g_0'(\xi) > 1 \), one has
\[
|J_0| \geq c_0 - \beta^{-1} = c_0 - g_0(c_0) = \varepsilon (c_0 - 1/2)^2.
\]
We will see that \( c_0 \in [0, 1/8] \). In this case one immediately gets that \(|J_0| > \varepsilon (3/8)^2 > \varepsilon /8\), proving the claim. To see that \( c_0 \in (0, 1/10) \),

\[ g_0(1/8) = 1/8 - \varepsilon (3/8)^2 > \varepsilon /4 = \beta^{-1} = g_0(c_0). \]

Thus, by the monotonicity of \( g_0 \), \( c_0 \in [0, 1/8] \).

Putting all the previous inequalities together and noting that \( \varepsilon < 1/10 \), we get:

\[ \frac{|J_0|}{|I_0|} \frac{d_m(0)}{(d_M(0))^2} \geq \frac{(\varepsilon /8)}{(\varepsilon /4)} \frac{(1 - \varepsilon)}{(1 + \varepsilon)^2} > \frac{1}{3}. \]

This completes the proof of the second inequality in the lemma.

We now prove the first inequality in the lemma. Given \( \alpha < s \) we will see that \( g_0' \) is 1 (if \( \alpha \) is small),

\[ |J_0| \geq |g_0(J_0)| \geq |g^{-1}(J_0)| \geq \frac{|J_0|}{d_m(\alpha)}. \]

Therefore, taking \( z_\alpha \in I_\alpha \) such that \( (g_\alpha^{N_\alpha})'(z_\alpha) = |g_\alpha^{N_\alpha}(I_\alpha)|/|I_\alpha| \), one has

\[ (g_\alpha^{N_\alpha})'(z_\alpha) \geq \frac{|J_\alpha|}{|I_\alpha| d_M(\alpha)}. \]  

Define \( b_\alpha \in I_\alpha \) by \( g_\alpha^{N_\alpha}(b_\alpha) = g_\alpha^{-1}(c_\alpha) \). For a given \( w \in I_\alpha \) there are three possibilities:

- \( w \in [g_\alpha(1), z_\alpha] \). Note that \( n_\alpha(w) = N_\alpha \) or \( n_\alpha(w) = N_\alpha + 1 \). Observe that if \( n_\alpha(w) = N_\alpha + 1 \) then \( g_\alpha'(g_\alpha^{N_\alpha}(w)) > 1 \). Thus using the monotonicity of \( g_\alpha' \), one gets

\[ (g_\alpha^{N_\alpha}(w))'(w) \geq (g_\alpha^{N_\alpha})'(w) \geq (g_\alpha^{N_\alpha})'(z_\alpha) \geq \frac{|J_\alpha|}{|I_\alpha| d_M(\alpha)} \cdot \]

- \( w \in [b_\alpha, 1] \). In this case, \( n_\alpha(w) = N_\alpha + 1 \). Since \( g_\alpha(w) < z_\alpha \), from the monotonicity of \( g_\alpha' \), \( (g_\alpha^{N_\alpha})'(g_\alpha(w)) \geq (g_\alpha^{N_\alpha})'(z_\alpha) \). Thus from equation (5) and \( g_\alpha'(w) \geq d_m(\alpha) \),

\[ (g_\alpha^{N_\alpha+1})'(w) = (g_\alpha^{N_\alpha})'(g_\alpha(w)) g_\alpha'(w) \geq (g_\alpha^{N_\alpha})'(z_\alpha) g_\alpha'(w) \geq \frac{|J_\alpha|}{|I_\alpha| d_M(\alpha)} d_m(\alpha). \]

- \( w \in [z_\alpha, b_\alpha] \) if \( z_\alpha < b_\alpha \). In this case \( n_\alpha(w) = N_\alpha \). Arguing as in the previous cases,

\[ (g_\alpha^{N_\alpha})'(w) \geq (g_\alpha^{N_\alpha})'(1) = \frac{(g_\alpha^{N_\alpha+1})'(1)}{(g_\alpha')'(g_\alpha^{N_\alpha})'(1)} \geq \frac{1}{d_M(\alpha)} \frac{|J_\alpha|}{|I_\alpha| d_M(\alpha)} d_m(\alpha). \]

The proof of the first inequality in the lemma now follows noting that \( d_m(\alpha) < 1 < d_M(\alpha) \).

**Proof of the proposition:** Let \( s_\alpha \) be the (unique) point in \( (g_\alpha^{-1}(c_\alpha), 1) \) with \( g_\alpha'(s_\alpha) = 1 \). Take any \( w \in (g_\alpha^{-1}(c_\alpha), g_\alpha(1)) \). First, if \( w \leq s_\alpha \) the result is obvious: all the derivatives throughout the orbit of \( w \) are bigger or equal than 1. If \( w \geq s_\alpha \) take \( k \geq 0 \) with \( g_\alpha^{-k}(w) = \bar{w} \in I_\alpha \). The result follows from
Lemma 4.2 noting that \((g^n_{\alpha}(w))'(w) = (g^n_{\alpha}(w)+k)'(w) (g^{-k}(w))'(w), \ n_{\alpha}(w) + k = n_{\alpha}(w)\), and \((g^{-k}(w))'(w) \geq 1\). This completes the proof of Proposition 4.1. \(\square\)

5. Hyperbolic dynamics. In this section, we prove the parts of Theorem 1 about hyperbolicity and collisions of homoclinic classes. First, in Section 5.1, we select the map \(F\) and state a hyperbolicity result for the family \((\Phi_{p,q}^{\alpha})\), for parameters \(\mu \in (\mu^*, \mu^+)\), where \(\mu^*\) is the saddle-node parameter of \((L_\mu)\) in Section 3.3 (see equation (5)). In Section 5.2, using Remark 3.3 and the product structure, we translate the hyperbolicity of \((\Phi_{p,q}^{\alpha})\) to \(f_t\), for \(t \in t_n (1 + \mu^*, 1 + \mu^+)\). Finally, in Section 5.3, we state the collision of homoclinic classes, related to the saddle-node bifurcations \(\mu^*\) of \(L_\mu = \Phi_{0,0}^{\alpha}\). Since we will follow closely [16, Section 4] and Sections 3, 7, and 8 we will omit some details of the proofs, just giving the corresponding precise references.

5.1. Hyperbolicity of the normalized family. Take small \(\mu^* < 0\) and consider the saddle-node family \((\varphi_{\alpha})_{\alpha = g_{\alpha} + \mu^*}\), where \((\varphi_{\alpha})_{\alpha}\) is the saddle-node family in Section 4.1. The arc \((\varphi_{\alpha})_{\alpha}\) has a saddle-node bifurcation (close to 1/2) for some \(\alpha^* = \alpha(\mu^*) > 0\) close to zero. Fixed such an \(\alpha^*\), define now the auxiliary saddle-node arc

\[L_\mu = h_{\mu}^{\alpha^*} = g_{\alpha^*} + \mu. \tag{7}\]

By construction, the arc \((L_\mu)\) has a saddle-node close to 1/2 for the parameter \(\mu^*\). For small \(\mu > \mu^*\), the map \(L_\mu\) has (exactly) a pair of (hyperbolic) fixed points in \(D^+\), \(s^+_\mu\) and \(s^-_\mu\), \(s^-_\mu < s^+_\mu\), where \(s^-_\mu\) is contracting and \(s^+_\mu\) is expanding. These points collapse to the saddle-node \(s_{\mu^*}\). For small \(\mu \geq \mu^*\), consider the partition of \(D^+\) given by the intervals

\[L_u(\mu) = [\beta^{-1}, s^-_\mu], \quad L_c(\mu) = (s^-_\mu, s^+_\mu), \quad \text{and} \quad L_s(\mu) = [s^+_\mu, 1].\]

Note that \(L_c(\mu^*) = \emptyset\).

As in Section 4.1, define \(c_{\mu^*}\) by \(L_\mu(\mu^*) = \beta^{-1}\). Consider the maps \(\Phi_{p,q}^{\alpha}\), defined on \(D^+\), induced by \(F = F_{\mu^*}\) associated to \(L_\mu\), i.e., the restriction of \(F^N\) to \([c_{\mu^*}, 1]\). In Section 4.1, Equation 3 implies that, for all \(y \in [c_{\mu^*}, 1]\) and \(\mu \) close to \(\mu^*\), one has

\[\Phi_{p,q}^{\alpha}(1 + \mu + \lambda^0(F^N - 4)) = (1 + \mu + (g_{\alpha^*} + 3) - 4) = g_{\alpha^*} + \mu = L_\mu.\]

Thus, for \(\mu > \mu^*\), \(s^+_\mu\) are (hyperbolic) fixed points of \(\Phi_{p,q}^{\alpha}\) in \([c_{\mu^*}, 1]\).

**Definition 5.1.** A point \(y\) is a \(\omega\)-limit point of \(\{\Phi_{p,q}^{\alpha}\}\) if there are a point \(x_0\) and sequences of non-negative numbers \((p_n)\) and \((q_n)\) such that \(x_n \rightarrow y\), where \(x_{n+1} = \Phi_{p_n,q_n}^{\alpha}(x_n)\). The (forward) limit set of \(\{\Phi_{p,q}^{\alpha}\}\), \(L^+(\{\Phi_{p,q}^{\alpha}\})\), is the closure of the set of \(\omega\)-limit points of \(\{\Phi_{p,q}^{\alpha}\}\).

Lemmas 5.2, 5.3, and 5.4 mean that there is \(\mu^+\), \(0 > \mu^+ > \mu^*\), such that, for every \(\mu \in [\mu^*, \mu^+]\), the forward limit set \(L^+(\{\Phi_{p,q}^{\alpha}\})\) is contained in \(L_s(\mu)\). Moreover, if \(\mu \in (\mu^*, \mu^+)\) this set is hyperbolic (the part contained in \(L_s(\mu)\) is contracting and the part in \(L_u(\mu)\) is expanding). By Remark 3.3, these properties can be translated to diffeomorphisms \(f_t\) for parameters \(t \in [t_n (1 + \mu^*), t_n (1 + \mu^+)]\).

**Lemma 5.2.** There is \(\mu^+\), \(0 > \mu^+ > \mu^*\), such that, for every \(\mu \in H = [\mu^*, \mu^+]\), it holds:

- \(\Phi_{p,q}^{\alpha}(L_s(\mu)) \subset L_s(\mu)\) for all \(p \geq 0\). If \(\mu \neq \mu^*\), these maps are uniformly contracting.
All the returns of points of $L_s(\mu)$ to $D^+$ are of the form $(0, p)$ with $p \geq 0$, i.e., there are no returns of type $(q, p)$ with $q > 0$.

Proof: To prove the first item, recall that, by definition of $F$, $F^N(D^+) = D^-$, so $F^N(1) = (4 - \lambda)$. The definition of $\Phi^0_{\mu, p}$, see (3), now implies that

$$\Phi^0_{\mu, p}(s_+^0, 1) = [s_+^0, 1 - \lambda + \mu] \subset [s_+^0, 1]$$

and that $\Phi^0_{\mu, p+1}([s_+^0, 1])$ is at the right of $\Phi^0_{\mu, p}([s_+^0, 1])$, for all $p \geq 0$. This implies that, for all $y \in D^+$ and every $n \geq 1$, $s_+^n \leq \Phi^0_{\mu, p}(y)$. Thus to prove the inclusion in the first item it remains to check that $\Phi^0_{\mu, p}(y) \leq 1$ for all $p \geq 1$ and $y \in D^+$.

As $\beta^{-1} = \lambda = \varepsilon/4$, small $\varepsilon$, it holds

$$1 > (1 - \lambda) + \mu > \beta^{-1}, \quad \text{for every small } |\mu|.$$    \hspace{1cm} (8)

Note that, for all $y \in D^+$, $F^N(y) \in [3, 4 - \lambda]$, thus $(F^N(y) - 4) \in (-1, -\lambda)$. Hence, by equation (3),

$$\Phi^0_{\mu, p}(y) \in (1 + \mu - \lambda^p, 1 + \mu - \lambda^{p+1}) \subset (\beta^{-1}, 1) \subset D^+.$$    \hspace{1cm} (9)

Thus, $\Phi^0_{\mu, p}([s_+^0, 1]) \subset [s_+^0, 1]$ for all $p \geq 0$, proving the inclusions. The hyperbolicity follows from the definition of the maps $\Phi^0_{\mu, p}$ and that $\Phi^0_{\mu, p}(y) \leq 1$ if $y \in L_s(\mu)$ and $q \geq 1$. Thus $\Phi^0_{\mu, p}(y) \notin D^+$, implying the claim.

The proof of the next lemma follows arguing as in the previous one, so it will be omitted.

Lemma 5.3. For every $\mu \in H = [\mu^*, \mu^+]$, $\mu \neq \mu^*$, it holds:

- $\Phi^0_{\mu, p}(L_c(\mu)) = L_c(\mu)$ and the restriction of $\Phi^0_{\mu, p}$ to $L_c(\mu)$ is strictly increasing.
- $\Phi^0_{\mu, p}(L_c(\mu)) \subset L_s(\mu)$ for all $p \geq 1$.
- Any return of points of $L_c(\mu)$ to $D^+$ is of the form $(0, p)$ with $p \geq 0$. Moreover, the returns of $L_c(\mu)$ to $L_c(\mu)$ are of the form $(0, 0)$.

Lemma 5.4. For every $\mu \in H = [\mu^*, \mu^+]$ it holds:

- $\Phi^0_{\mu, p}(L_u(\mu)) \subset L_s(\mu)$ for every $p \geq 1$.
- Any return of points of $L_u(\mu)$ to $D^+$ is of the form $(0, p)$ or $(q, 0)$, with $p, q \geq 0$.
- If $\mu \neq \mu^*$, the restriction of $\Phi^0_{\mu, p}$ to $L_u(\mu)$ is uniformly expanding for all $q \geq 0$.

Proof: For the first item just note that, if $p \geq 1$, then $\Phi^0_{\mu, p}(L_u(\mu))$ is at the right of $\Phi^0_{\mu, p}(L_s(\mu)) \subset L_s(\mu)$. The second assertion follows from the first one and the proof of Lemma 5.2. The expansion of $\Phi^0_{\mu, p}$ in $D^\mu_{\mu, p} \cap L_u(\mu)$ is the interval of definition of $\Phi^0_{\mu, p}$ follows noting that $D^\mu_{\mu, p} \cap L_u(\mu) = [c_\mu, s_\mu]$ and using the expansivity of $h_\mu$ in $[c_\mu, s_\mu]$, where (as above) $c_\mu$ is defined by $L_u(c_\mu) = \beta^{-1}$. The expansivity of $\Phi^0_{\mu, p}$ in $D^\mu_{\mu, p} \subset (\beta^{-1}, c_\mu)$ follows from its definition and condition (F4).
5.2. Prevalent hyperbolicity for the arc \((f_t)_{t \in [-\tau, \tau]}\). In this section, using Remark \[3.3\] we translate the hyperbolicity of the system \((\Phi^t_p), \mu \in (\mu^*, \mu^+)\), to the diffeomorphisms \(f_t\) for \(t \in (t_n (1 + \mu^*), t_n (1 + \mu^+))\).

To each point \(X_0 = (x_0, y_0, z_0) \in \Delta^+ \cap \Lambda_t\) (recall that \(\Lambda_t\) is the maximal invariant set of \(f_t\) in the neighbourhood \(W\) of the cycle) we associate a sequence \(i(X_0) = (i_k(X_0))_{k \in E(X_0)}\) of symbols \(s, c\) and \(u\), as follows. We let \(i_0(X_0) = j\) if \(y_0 \in L_j(\mu)\).

Let \(X_1 = (x_1, y_1, z_1)\) and \(X_{-1} = (x_{-1}, y_{-1}, z_{-1})\) be the first forward and backward returns of \(X_0\) to \(\Delta^+\) by \(f_t\) (if they are defined); write \(i_{\pm 1}(X_0) = j\) if \(y_{\pm 1} \in L_j(\mu)\). If possible, we inductively define first forward and backward returns \(X_{k(n+1)}\) of \(X_0\) and let \(i_{n+1}(X_0) = i_1(X_n) = i_n(X_1)\) and \(i_{-1}(X_{n+1}) = i_{-1}(X_{-n}) = i_{-1}(X_{-1})\). The set \(E(X_0) \subset Z\) is the maximal interval (in \(Z\)) where these indices are defined. The sequence \(i(X_0) = (i_k(X_0))_{k \in E(X_0)}\) is the itinerary of \(X_0\). We also write \(r_i(X) = r_i = (q, p)\) if the \(i\)-th return of \(X\) is of type \((q, p)\).

Using the cycle configuration, it is easy to see that every \(X \in \Lambda_t\), \(X \neq P, Q\), has some (backward or forward) iterate in \(\Delta^+\). So we can associate to each \(X \in (\Lambda_t \setminus \{P, Q\})\) a sequence \(i(Y)\), where \(Y\) is some iterate of \(X\) in \(\Delta^+\). These sequences are well defined and coincide (up to a shift) for points whose returns do not have central coordinates equal to \(\beta^{-1}\) or 1. In fact, for the parameters that we select, we will see that the central coordinates \(\beta^{-1}\) and 1 correspond to wandering points. Thus, from now on, we assume that \(X \in \Delta^+\), otherwise we replace \(X\) by some iterate of it.

Define now the sets:
\[\Omega(t) = \{X \in \Omega(f_t) \cap W: E(X) \neq \emptyset \text{ and } i_k(X) = s \text{ for all } k \in E(X)\} \cup \{P\},\]
\[\Omega_u(t) = \{X \in \Omega(f_t) \cap W: E(X) \neq \emptyset \text{ and } i_k(X) = u \text{ for all } k \in E(X)\} \cup \{Q\},\]
\[\Omega_c(t) = \{X \in \Omega(f_t) \cap W: E(X) \neq \emptyset \text{ and } i_k(X) = c \text{ for all } k \in E(X)\}.\]

For each \(n\), write
\[H_n = \{t = t_n (1 + \mu), \mu \in H\}, \quad H = [\mu^*, \mu^+], \quad \mu^* \text{ and } \mu^+ \text{ as in Lemma } 5.2\]

Fix \(t = t_n (1 + \mu) \in H_n\) and take \(X = (x, y, z) \in \Delta^+\) with a first return \(X_0 = (x_1, y_1, z_1)\) to \(\Delta^+\). Assume that this return is of type \((q, p)\). Then, by Remark \[3.3\] \(y_1 = \Phi^p_p(y)\). This fact and Lemmas \[5.2\], \[5.3\] and \[5.4\] imply that the itinerary of any non-wandering point \(X\) of \(\Lambda_t\) is constant (equal to \(s, c\) or \(u\)). These lemmas also imply the following:
- If \(i_k(X) = s\) for all \(k \in E(X)\) then all the returns are of type \((0, p), p \geq 0\).
  Thus, if \(\mu \neq \mu^*\), the returns are uniformly contracting in the \(Y\)-direction.
- If \(i_k(X) = u\) for all \(k \in E(X)\) then all the returns are of type \((q, 0), q \geq 0\).
  Thus, if \(\mu \neq \mu^*\), the returns are uniformly expanding in the \(Y\)-direction.
- If \(i_k(X) = c\) for all \(k \in E(X)\) then all the returns are of type \((0, 0)\).

Actually, there is no non-wandering points \(X\) with \(i_k(X) = c\), for all \(k \in E(X)\). This follows from the first item in Lemma \[5.3\] the restriction of \(\Phi^p_p\) to \(L_c(\mu)\) is monotone without fixed points, see [10] Section 8 and Lemma 8.3]. Similarly, any point having a return with central coordinate equal to \(\beta^{-1}\) or 1 is wandering. These facts are the ingredients of the proof of the next lemma (see [10] Section 4]). Note that the hyperbolicity result in Theorem [10] follows from Lemma \[5.3\].

\textbf{Lemma 5.5.} ([10] Steps B-G) For every \(t \in H_n\),
- \(\Omega(f_t) \cap W = \Omega_s(t) \cup \Omega_u(t)\). In particular, \(\Omega_c(t) = \emptyset\).
- \(\Omega_u(t) = H(P, f_t)\) and \(\Omega_u(t) = H(Q, f_t)\).
- If \(t \neq \lambda^* (1 + \mu^*)\), the homoclinic classes of \(P\) and \(Q\) are hyperbolic.
There are no cycles associated to the homoclinic classes of $P$ and $Q$.

### 5.3. Sequence of saddle-node bifurcations.

Let $t = \lambda^n(1 + \mu)$, $\mu \in (\mu^+, \mu^-]$, and consider the corresponding hyperbolic fixed points of $\Phi_{\mu}^{0,0}$, $s^+_\mu$ (attracting) and $s^-_\mu$ (repelling). Having in mind Remark 5.3 and borrowing Lemma 8.1 of [19], we obtain that associated to these points there are hyperbolic periodic points $S^\pm_t$ of $f_t$ in $\Delta^+$ whose central coordinates are $s^\pm_\mu$. Moreover, the returns of the points $S^+_t$ and $S^-_t$ are of type (0,0), thus the central dynamics of these returns is given by $\Phi_{\mu}^{0,0}$. The points $S^\pm_t$ are hyperbolic ($S^+_t$ of index 1 and $S^-_t$ of index 2) with period $n + k_0 + n + N$. Finally, by Lemma 5.6, $S^+_t \in H(P, f_t)$ and $S^-_t \in H(Q, f_t)$.

Analogously, for $t^*_n = t_n(1 + \mu^*)$, there is a saddle-node $S^+_n = S^-_n = S_{t^*_n}$, corresponding to the collision of $S^+_t$ and $S^-_t$, whose central coordinate is $s_{\mu^*}$. The next lemma implies (B) in Theorem 4.

**Lemma 5.6.** The intersection $H(P, f_{t^*_n}) \cap H(Q, f_{t^*_n})$ is the orbit of the saddle-node $S^+_n$. Moreover, every compact $f_{t^*_n}$-invariant subset of $H(P, f_{t^*_n}) \cup H(Q, f_{t^*_n})$ disjoint from $S^+_n$ is hyperbolic.

This lemma follows as Lemma 5.4 using Remark 5.3 and Lemmas 5.2, 5.3, and 5.4. First, note that $L_c(\mu^*) = \emptyset$. For $p \geq 0$, the map $\Phi_{\mu^*}^{0,p}$ is uniformly contracting in any compact subset of $L_a(\mu^*)$ that does not contain $s_{\mu^*}$. For $q \geq 0$, the map $\Phi_{\mu^*}^{q,0}$ is uniformly expanding in any compact subset of $L_u(\mu^*)$ disjoint from $s_{\mu^*}$. By the product structure, these properties can be translated to $f_{t^*_n}$. Finally, the fact that $H(P, f_{t^*_n}) \cap H(Q, f_{t^*_n})$ is the orbit of $S^+_n$ follows from $s_{\mu^*} = L_s(\mu^*) \cap L_u(\mu^*)$, Lemmas 5.2, 5.3 and 5.4 and that $s_{\mu^*}$ is a saddle-node of $\Phi_{\mu^*}^{0,0}$.

6. Dynamically defined Cantor sets. In this section, we construct dynamically defined Cantor sets for the family $(\Phi_{\mu}^{p,q})$ having arbitrarily big thickness. In this step, the selection of the parameter $\alpha^*$ in the definition of $F$ (so in the definition of $(\Phi_{\mu}^{p,q})$) is crucial. Note that in the previous sections we just need to consider small $\alpha$. Therefore, for the sake of clearness, we write explicitly the two parameters $\mu$ and $\alpha$ involved in our constructions. In this way, we consider a two parameter family of maps $\Phi_{\mu,\alpha}^{p,q}$ defined as $\Phi_{\mu}^{p,q}$ just taking the central map $F$ equal to $F_{\alpha}$. Similarly, we also have a two-parameter family of diffeomorphisms $f_{t,\alpha}$.

In what follows, we use dynamically defined Cantor sets and their thickness. For further information on this topic we refer to [30], Chapter 4]. We begin by recalling some definitions.

**Definition 6.1.** A presentation of a Cantor set $\mathcal{Y} \subset \mathbb{R}$ is an enumeration $\mathcal{G} = \{G_n\}$ of the bounded gaps of $\mathcal{Y}$ (i.e., the bounded components of $\mathbb{R} \setminus \mathcal{Y}$). The thickness of the presentation $\mathcal{G}$, $\tau(\mathcal{Y}, \mathcal{G})$, is defined as follows. Let $I$ be the convex hull of $\mathcal{Y}$. For each $n$ and each point $g$ in the boundary of a gap $G_n$, let $B_g$ be the connected component of $I \setminus (G_1 \cup \cdots \cup G_n)$ containing $g$. The interval $B_g$ is the left or right bridge of $G_n$, according to its relative position. Then

$$
\tau(\mathcal{Y}, \mathcal{G}) = \inf_{n \in \mathbb{N}, g \in \partial G_n} \tau(\mathcal{Y}, \mathcal{G}, g), \quad \tau(\mathcal{Y}, \mathcal{G}, g) = |B_g|/|G_n|.
$$

Finally, the thickness of $\mathcal{Y}$ is

$$
\tau(\mathcal{Y}) = \sup \tau(\mathcal{Y}, \mathcal{G}),
$$

where the supremum is taken over all the presentations of $\mathcal{Y}$. Thus, for any presentation $\mathcal{G}$ of $\mathcal{Y}$, $\tau(\mathcal{Y}, \mathcal{G})$ is a lower bound of $\tau(\mathcal{Y})$. 

We define the central localization of an $f_{t,\alpha}$-invariant set $\Lambda$ in the neighbourhood $W$ by

$$\Lambda^c = \{y_0 \in D^+: \text{there is } X \in \Lambda \text{ of the form } (x, y_0, z)\}. \quad (10)$$

In this section, for appropriate $\alpha$, we get a sequence $(C_n)_n$ of $t$-intervals such that for every $t \in C_n$ there are two (transitive) hyperbolic sets $\Upsilon_{t,\alpha}$ and $\Xi_{t,\alpha}$ of $f_{t,\alpha}$ such that

- $\Upsilon_{t,\alpha}$ has index one and is contained in the homoclinic class of $P$;
- $\Xi_{t,\alpha}$ has index two and is contained in the homoclinic class of $Q$; and
- the product of the thickness of the central localizations of $\Upsilon_{t,\alpha}$ and $\Xi_{t,\alpha}$ is bigger than one.

These facts (and Lemma 7.1) are the key for getting a $t$-parameter interval with persistence of cycles, (item (C) in Theorem 1).

### 6.1. Construction of the hyperbolic set $\Upsilon_{t,\alpha}$ of index one.

**Lemma 6.2.** There are parameters $\mu_0$ and $\alpha_0 > 0$ and strictly positive constants $d^c$ and $\tau^c$ such that, for every $t \in B_n = [t_n(1 - \mu_0), t_n(1 + \mu_0)]$, where $\alpha \in [-\alpha_0, \alpha_0]$, there exists a (transitive) hyperbolic set $\Upsilon_{t,\alpha}$ of $f_{t,\alpha}$ of index one, contained in the homoclinic class of $P$, such that the diameter and the thickness of $(\Upsilon_{t,\alpha})^c$ are lower bounded by $d^c$ and $\tau^c$, respectively.

**Proof:** The proof of this lemma has two parts. We first construct a dynamically defined Cantor set for the system $\Phi_{\mu,\alpha}$, for $(\mu, \alpha)$ close to $(0,0)$. Next, using Remark 3.3, we see that such a set is the central localization of a hyperbolic set of $f_{t_n(1 + \mu),\alpha}$ contained in the homoclinic class of $P$.

We claim that there is $p_1$ such that, for all $p \geq p_1$, (i) $\Phi_{\mu,\alpha}^{0,p}$ is contracting, (ii) $D_{\mu,\alpha}^{0,p} = D^+$, and (iii) $\Phi_{\mu,\alpha}^{0,p}(D^+)$ is in the interior of $D^+$. To get the contraction in (i), take big $p_1$ in order that $\lambda^{p_1}$ compensates any expansion introduced by $F_\alpha$, any small $|\alpha|$, see the definition of $\Phi_{\mu,\alpha}^{0,p}$ in (3). Note that equation (7) holds for small $|\alpha|$, thus this equation implies that $\Phi_{\mu,\alpha}^{0,p}(D^+) \subset \text{int}(D^+)$, for $p \geq 1$, which (by continuity) implies (ii) and (iii), for small $|\mu|$ and $|\alpha|$ and $p \geq 1$.

Define $K_{\mu,\alpha}^{p_1,p_1+1}$ as the maximal invariant set in $D^+$ of $\Phi_{\mu,\alpha}^{0,p_1}$ and $\Phi_{\mu,\alpha}^{0,p_1+1}$, that is, $K_{\mu,\alpha}^{p_1,p_1+1} = \bigcup_{j \geq 0} D_{\mu,\alpha}^+(j)$, where $D_{\mu,\alpha}^+(0) = D^+$ and $D_{\mu,\alpha}^+(i+1) = \bigcup_{p=p_1}^{p_1+1} \Phi_{\mu,\alpha}^{0,p}(D^+(i))$. By construction, $K_{\mu,\alpha}^{p_1,p_1+1}$ is a hyperbolic dynamically defined Cantor set conjugate to a shift of two symbols. In particular, this set is transitive and has a dense subset of periodic points. Let $d^c(0,0) > 0$ and $\tau^c(0,0) > 0$ be the diameter and the thickness of $K_{\mu,\alpha}^{0,0+1}$. As the thickness and the diameters of $K_{\mu,\alpha}^{p_1,p_1+1}$ depend continuously on $\Phi_{\mu,\alpha}^{0,p_1}$ and $\Phi_{\mu,\alpha}^{0,p_1+1}$, thus on $\mu$ and $\alpha$, there are small $\mu_0$ and $\alpha_0$ such that, for all $(\mu, \alpha) \in [-\mu_0, \mu_0] \times [-\alpha_0, \alpha_0]$, it holds

$$\text{diam}(K_{\mu,\alpha}^{p_1,p_1+1}) > d^c(0,0)/2 = \frac{\nu}{2}, \quad \tau(K_{\mu,\alpha}^{p_1,p_1+1}) > \tau^c(0,0)/2 = \tau^c.$$

To prove the lemma, it suffices to find a hyperbolic set $\Upsilon_{t,\alpha}$ (contained in $H(P, f_{t,\alpha})$) whose central localization is $K_{\mu,\alpha}^{p_1,p_1+1}$. Let us explain this in details.

Fix an interval $B_n = [t_n(1 - \mu_0), t_n(1 + \mu_0)]$. By Remark 3.3, for each $t(\mu) = t_n(1 + \mu) \in B_n$, the central coordinate of a return of type $(0, p)$ by $f_{t(\mu),\alpha}$ is given by $\Phi_{\mu,\alpha}^{0,p}$. Define $\Upsilon_{t(\mu),\alpha}$ as the $f_{t(\mu),\alpha}$-invariant set of points with infinitely many
consecutive returns to $\Delta^{+} = [-1,1] \times D^{+} \times [-1,1]$ of types $(0,p)$ or $(0,p+1)$, i.e., $E(X) = Z$ and $r_{t}(X) = (0,p)$ or $r_{t}(z) = (0,p+1)$. By construction, the central localization of $\mathcal{Y}_{t(\mu),\alpha}$ is $K^{\mu,p+1}$. Since the directions parallel to the axis $X$ and $Z$ are contracting and expanding and, by the previous step, the $Y$-direction is contracting for the restriction of $f_{t(\mu),\alpha}$ to $\mathcal{Y}_{t(\mu),\alpha}$, this set is hyperbolic of index one. The claims for the diameter and the thickness now are obvious.

It remains to prove that $\mathcal{Y}_{t(\mu),\alpha}$ is a transitive subset of the homoclinic class of $P$. As a consequence of the contraction in the $X$-direction and the expansion in the $Z$-direction and the $f_{t,\alpha}$-invariance of the planes $XZ$ (product structure), the set $\mathcal{Y}_{t(\mu),\alpha}$ intersects every plane parallel to the $XZ$-plane in at most one point. Thus there is a one-to-one correspondence between $\mathcal{Y}_{t(\mu),\alpha}$ and $(\mathcal{Y}_{t(\mu),\alpha})^{c}$. Hence the dynamics (via $f_{t(\mu),\alpha}$) of the returns of the points of $\mathcal{Y}_{t(\mu),\alpha}$ to $\Delta^{+}$ and the dynamics of $(\mathcal{Y}_{t(\mu),\alpha})^{c}$ is $K^{\mu,p+1}$. By the system $\Phi^{0,\mu}_{p}$ and $\Phi^{0,\mu+1}_{p}$ are conjugate. Hence, the periodic points are dense in $\mathcal{Y}_{t(\mu),\alpha}$ and this set is transitive.

Finally, to see that any periodic point $A \in \mathcal{Y}_{t(\mu),\alpha}$ is homoclinically related to $P$ note that, by construction, the orbit of the unstable manifold of $A$ meets the rectangle $[-1,1] \times D^{+} \times \{0\} \subset W^{s}(P,f_{t(\mu),\alpha})$. Thus $W^{u}(A,f_{t(\mu),\alpha})$ transversely meets $W^{s}(P,f_{t(\mu),\alpha}) \cap \Delta^{+} \neq \emptyset$. To see that $W^{u}(P,f_{t(\mu),\alpha})$ and $W^{s}(A,f_{t(\mu),\alpha})$ have some transverse intersection note that, by construction and considering negative iterations by $f_{t(\mu),\alpha}$, the unstable manifold of $A$ contains a disk of the form $[-1,1] \times D^{+} \times \{z_{0}\}$ and that $W^{u}(P,f_{t(\mu),\alpha})$ intersects any disk of this form (any $z_{0} \in [-1,1]$). This completes the proof of the lemma. □

6.2. Construction of the thick hyperbolic set $\Xi_{t,\alpha}$ of index two. As in Section 6.1 we first construct a dynamically defined Cantor set $\Sigma_{\mu,\alpha}$ associated to the family $\Phi^{0,\mu}_{\mu,\alpha}$, $q \geq 0$, with large thickness (Proposition 6.1). Next we see that this set corresponds to the central localization of a transitive hyperbolic set of index two (the announced set $\Xi_{t,\alpha}$).

As $\Phi^{0,\mu}_{\mu,\alpha} = F^{\mu}_{\alpha} + 3 + \mu$, see 6.1, by condition (F1), it holds $\Phi^{0,\mu}_{\mu,\alpha}(\beta^{-1}) = \mu$. So, for small $\mu < 0$, there are $c_{\mu,\alpha}$ (the continuation of $c_{\alpha}$ in Section 4) and $\ell_{\mu,\alpha} \in D^{+}$, $\beta^{-1} < \ell_{\mu,\alpha} < c_{\mu,\alpha}$, with

$$\Phi^{0,\mu}_{\mu,\alpha}(\ell_{\mu,\alpha}) = 0 \quad \text{and} \quad \Phi^{0,\mu}_{\mu,\alpha}(c_{\mu,\alpha}) = \beta^{-1}.$$  

Lemma 6.3. (Bounded distortion lemma) There is a constant $L$, independent of small $|\alpha|$ and $|\mu|$, such that

$$L^{-1} \frac{|I|}{|J|} \leq \frac{|(\Phi^{0,\mu}_{\mu,\alpha})^{k}(I)|}{|(\Phi^{0,\mu}_{\mu,\alpha})^{k}(J)|} \leq L \frac{|I|}{|J|},$$

for every pair of intervals $I$ and $J$ in the same fundamental domain of $\Phi^{0,\mu}_{\mu,\alpha}$ in $[c_{\mu,\alpha},1]$ with $(\Phi^{0,\mu}_{\mu,\alpha})^{j}(I \cup J) \subset [c_{\mu,\alpha},1]$ for all $j = 1, \ldots, k - 1$.

The proof of this lemma is quite standard and uses that the map $\Phi^{0,\mu}_{\mu,\alpha}$ is $C^2$ and monotone in $[c_{\mu,\alpha},1]$, which follows from condition (F3) and equation 4. For details see 6.3 pages 58-59.

Define, for $\mu < 0$, the interval $G_{0}(\mu, \alpha) = (\beta^{-1}, \ell_{\mu,\alpha})$. By definition, $\Phi^{0,\mu}_{\mu,\alpha}(G_{0}(\mu, \alpha)) = (\mu, 0)$. Thus $|\mu| = |(\Phi^{0,\mu}_{\mu,\alpha})^{\gamma}(\xi)| |G_{0}(\mu, \alpha)|$, for some $\xi \in [\beta^{-1}, \ell_{\mu,\alpha}] \subset [\beta^{-1}, 1]$, and there is a constant $K$ (independent of small $\alpha$ and $\mu$) such that

$$|\mu| < |G_{0}(\mu, \alpha)| = (\ell_{\mu,\alpha} - \beta^{-1}) < K |\mu|.$$  

(11)
We take small $\mu < 0$ with

$$K |\mu| < \min \{d^e, \tau^e\}, \quad \text{where } d^e \text{ and } \tau^e \text{ are as in Lemma 62}$$

(12)

6.2.1. The induced map. Fix small $\mu < 0$ satisfying (12) and let $\alpha(\mu)$ be the parameter corresponding to the saddle-node bifurcation of the family $(h_\alpha = g_\alpha + \mu)_\alpha$. Take $\alpha < \alpha(\mu)$ and note that $\Phi_{\mu,\alpha}^{0,0} = g_\alpha + \mu$ has no fixed points in $[c_{\mu,\alpha}, 1]$. Define

$I_0(\mu, \alpha) = [\ell_{\mu,\alpha}, c_{\mu,\alpha}]$ and $H_0(\mu, \alpha) = [\beta^{-1}, c_{\mu,\alpha}] = \{\beta^{-1}\} \cup G_0(\mu, \alpha) \cup I_0(\mu, \alpha)$.

Since $H_0(\mu, \alpha)$ is a fundamental domain of $\Phi_{\mu,\alpha}^{0,0}$ and the graph of $\Phi_{\mu,\alpha}^{0,0}$ is under the diagonal, there is a first $k_0 \in \mathbb{N}$ such that $(\Phi_{\mu,\alpha}^{0,0})^{k_0}(1) \in H_0(\mu, \alpha)$. For each $k \in \{1, \ldots, k_0\}$, let

$$L_k(\mu, \alpha) = (\Phi_{\mu,\alpha}^{0,0})^{-k}(L_0(\mu, \alpha)) \cap [\beta^{-1}, 1], \quad L = G, H, I.$$  

(13)

Note that $I_{k_0}(\mu, \alpha) = \emptyset$ if $1 \in G_{k_0}(\mu, \alpha)$ and $\Phi_{\mu,\alpha}^{0,0}(\ell_{\mu,\alpha}, c_{\mu,\alpha}) = (0, \beta^{-1})$. Take the following (infinite) partition $\{J_i(\mu, \alpha)\}_{i \in \mathbb{N}}$ of $[\ell_{\mu,\alpha}, c_{\mu,\alpha}] \subset I_0(\mu, \alpha)$ by intervals accumulating to $\ell_{\mu,\alpha}$ whose interiors are pairwise disjoint:

$$(\ell_{\mu,\alpha}, c_{\mu,\alpha}] = \bigcup_{i \geq 1} J_i(\mu, \alpha), \quad \text{where } J_i(\mu, \alpha) = (\Phi_{\mu,\alpha}^{0,0})^{-1}(\beta^{-i}([\beta^{-1}, 1]).$$

(14)

Using this partition, we define the following induced map (depicted in Figure 10)

$$\Psi_{\mu,\alpha} : (\ell_{\mu,\alpha}, c_{\mu,\alpha}] \to [\beta^{-1}, 1], \quad x \in J_i(\mu, \alpha) \mapsto \Psi_{\mu,\alpha}(x) = \beta^i(\Phi_{\mu,\alpha}^{0,0}(x)) = \Phi_{\mu,\alpha}^{0,0}(x).$$

(15)

Note that the map $\Psi_{\mu,\alpha}$ is bi-valuated in boundary of $J_i(\mu, \alpha)$ and maps the interior of each $J_i(\mu, \alpha)$ onto $(\beta^{-1}, 1)$.

Remark 6.4. Let $m^+$ and $m^- > 0$ be upper and lower bounds of the derivative of $\Phi_{\mu,\alpha}^{0,0}$ in $[\beta^{-1}, 1]$ independent of small $\mu$ and $\alpha$. Then $\beta m^- \leq \Psi'_{\mu,\alpha}(x) \leq \beta m^+$ for all $x \in J_i(\mu, \alpha)$.

**Figure 10.** The induced map $\Psi_{\mu,\alpha}$
Define the map $\Gamma_{\mu, \alpha}$ (which is bi-valuated at infinitely many points) as follows (see Figure 11):

$$\Gamma_{\mu, \alpha}: I(\mu, \alpha) = \bigcup_{i=0}^{k_{0}} I_i(\mu, \alpha) \to [\beta^{-1}, 1], \quad x \in I_i(\mu, \alpha) \to \Gamma_{\mu, \alpha}(x) = \Psi_{\mu, \alpha} \circ (\Phi_{\mu, \alpha}^{-1})^i(x).$$

(16)

\begin{center}
\begin{tikzpicture}
\draw[help lines] (0,0) grid (4,4);
\draw[thick] (0,0) -- (0,1) -- (1,1) -- (1,0) -- (4,0) -- (4,1) -- (3,1) -- (3,0) -- (0,0);
\end{tikzpicture}
\end{center}

\textbf{Figure 11.} The map $\Gamma_{\mu, \alpha}$

\textbf{Lemma 6.5.} The map $\Gamma_{\mu, \alpha}$ verifies the following:

- its lateral derivatives are defined at every point in the interior of $I(\mu, \alpha)$;
- it is uniformly expanding (i.e., there is $\rho > 1$ such that every lateral derivative is strictly bigger than $\rho$); and
- $\Gamma_{\mu, \alpha}'(x) \to +\infty$ as $x \to (\Phi_{\mu, \alpha}^{-1})^{-1}(\ell(\mu, \alpha)), i = 0, \ldots, k_0$.

\textbf{Proof:} The first and the third items follow immediately from the definition of $\Gamma_{\mu, \alpha}$. For the second assertion, we begin by estimating the derivative of $\Psi_{\mu, \alpha}$ in $I_0(\mu, \alpha)$. Take any $x \in I_0(\mu, \alpha), x \in J_k(\mu, \alpha)$ for some $k \geq 1$, by definition,

$$\Psi_{\mu, \alpha}(x) = \beta^k (\Phi_{\mu, \alpha}^0(x)) = \beta^k (F_{\mu, \alpha}^N(x) + 3 + \mu) = \beta^k (g_{\alpha}(x) + \mu).$$

Since $J_k(\mu, \alpha) \subset [\beta^{-1}, c_{\mu, \alpha}]$, condition (F4) implies that $\Psi_{\mu, \alpha}'(y) > \beta$.

Suppose now that $x \in I_j(\mu, \alpha), j \geq 1$, then $x, (\Phi_{\mu, \alpha}^0)(x), \ldots, (\Phi_{\mu, \alpha}^0)^{-1}(x)$ belong to $[c_{\mu, \alpha}, 1]$, thus $\Phi_{\mu, \alpha}^0$ coincides with $g_{\alpha} + \mu$ along the first $j$ iterates of $x$. Therefore,

$$\Gamma_{\mu, \alpha}(x) = \Psi_{\mu, \alpha} \circ (g_{\alpha} + \mu)^j(x).$$

We claim that $((g_{\alpha} + \mu)^{-1})(g_{\alpha} + \mu)(x)) > 1/4$, which implies the result. As $((g_{\alpha} + \mu)^j(x) \in I_0(\mu, \alpha), \Psi_{\mu, \alpha}'(g_{\alpha} + \mu)^j(x)) > \beta$, it is enough to note that $\beta/9 > 1$ and $g_{\alpha}'(x) > 1/2$ for all $x$.

To prove the claim, note that, fixed small $\mu < 0, (g_{\alpha} + \mu)$ is a saddle-node arc bifurcating at $\alpha(\mu) = \varepsilon (-1 + \sqrt{1 - 4\mu})$ close to 0. Repeating the proof of Proposition 6.1 and noting that, with the notation of the proposition, $j - 1 = n_{\alpha}((g_{\alpha} + \mu)(x))$, one gets, for small $\alpha < \alpha(\mu), ((g_{\alpha} + \mu)^{-1})((g_{\alpha} + \mu)(x)) > 1/4$, ending the proof of the claim and of the lemma. \hfill \Box

We are now ready to construct a dynamically defined Cantor set $\Sigma_{\mu, \alpha}$, associated to $\Gamma_{\mu, \alpha}$, with large thickness and small gaps. We see that, for $t = \lambda \alpha (1 + \mu)$, the set $\Sigma_{\mu, \alpha}$ is the central localization is a hyperbolic set $\Omega_{\mu, \alpha}$ of index 2. The key step in this construction is Proposition 6.6. This proposition together with Lemma 6.2
and (Gap) Lemma 7.1 are the ingredients of the persistence of cycles in Theorem 1, see Section 7.

Archetypally, the set $\Sigma_{\mu,\alpha}$ consists of the points in $D^+$ whose orbits by $\Gamma_{\mu,\alpha}$ do not meet $G_0(\mu,\alpha)$. In general, this set fails to be dynamically defined (and thus we do not know how to calculate its thickness). To bypass this difficulty, we consider a dynamically defined subset of it. This is done by constructing a Markov partition for $\Gamma_{\mu,\alpha}$ and defining $\Sigma_{\mu,\alpha}$ as the maximal invariant set associated to such a partition.

Recall that, fixed small $\mu^* < 0$, $\alpha(\mu^*)$ is the parameter corresponding to the saddle-node bifurcation of $(g_\alpha + \mu^*)\alpha$. So the arc $(L_\mu = g_\alpha(\mu^*) + \mu)\mu$ unfolds a saddle-node at $\mu^*$.

**Proposition 6.6.** There are $\tau_0$, $\tau_1 > 0$, and $0 < \eta \leq 1$, independent of small $\mu^* < 0$, such that, for all $\mu < \mu^*$, the map $\Gamma_{\mu,\alpha(\mu^*)}$ has a dynamically defined Cantor set $\Sigma_{\mu,\alpha(\mu^*)}$ such that:

: (a) $\tau(\Sigma_{\mu,\alpha(\mu^*)}) \geq \tau_0/|\mu|^\eta$, where $\tau$ denotes the thickness of the Cantor set,

: (b) every (bounded) gap of the Cantor set $\Sigma_{\mu,\alpha(\mu^*)}$ has length less than $\tau_1 |\mu|^\eta$,

: (c) the diameter of $\Sigma_{\mu,\alpha(\mu^*)}$ is bigger than $(1 - \beta^{-1})/2$.

**Notational remark:** From now on, the parameter $\alpha$ is fixed and equal to $\alpha(\mu^*)$, the saddle-node parameter of the arc $(g_\alpha + \mu^*)\alpha$, for some small negative $\mu^*$. Thus for notational simplicity, the dependence on $\alpha = \alpha(\mu^*)$ of the maps and the associated points and sets will be omitted.

For $\mu < \mu^* < 0$ and close to $\mu^*$, to construct the set $\Sigma_{\mu}$ we consider a Markov partition. There are two cases, according to the position of $G_{k_0}(\mu)$.

**6.2.2. Construction of the Markov partition.** Case 1: $1 \in G_{k_0}(\mu)$ (i.e., $I_{k_0}(\mu) = \emptyset$). Let $q_\mu \in I_{k_0-1}(\mu)$ be the closest point to 1 with $\Gamma_{\mu}(q_\mu) = q_\mu$. The existence of $q_\mu$ is guaranteed by $\Gamma_{\mu}(I_{k_0-1}(\mu)) = [\beta^{-1}, 1]$. By construction, there is a sequence $x_n \in (\ell_\mu, 1)$ of fixed points of $\Gamma_{\mu}$ converging to $\ell_\mu$. In particular, there is a fixed point $p_\mu$ of $\Gamma_{\mu}$ arbitrarily close to $\ell_\mu$ satisfying

$$\Gamma_{\mu}(p_\mu) = p_\mu, \quad p_\mu - \ell_\mu < |\mu|. \quad (17)$$

![Figure 12. Choice of $p_\mu$](image)

**Lemma 6.7.** There are constants $K_1$, $K_2$, and $K_3$, independent of small $\mu^* < 0$, such that, for all $\mu < \mu^*$ close to $\mu^*$,

1. $|G_{k_0}(\mu)| < K_1 |G_0(\mu)| < K_2 |\mu|.$
2. \((1-q_\mu) \leq \frac{K_1 \rho}{\rho - 1} |G_0(\mu)| \leq \frac{K_2 \rho}{\rho - 1} |\mu| \leq K_3 |\mu|\), where \(\rho > 1\) is as in Lemma 6.5.

Proof: The inequalities in item (1) follow from Proposition 4.1, the first one, and equation (1), the second one. To prove item (2), let \(G_{k_0}(\mu) = (1 - v_\mu, 1)\). By construction, considering the branch of \(\Gamma_{\mu}\) in \(I_{k_0 - 1}(\mu)\), one has \(\Gamma_{\mu}([q_\mu, 1 - v_\mu]) = [q_\mu, 1]\). Since \(\Gamma_{\mu}\) has no discontinuities in \([q_\mu, 1 - v_\mu]\), by Lemma 6.5 there is \(x \in (q_\mu, 1 - v_\mu)\) such that

\[
(1 - q_\mu) = \Gamma_{\mu}'(x)(1 - v_\mu - q_\mu) \geq \rho(1 - v_\mu - q_\mu).
\]

Thus, since \(v_\mu = |G_{k_0}(\mu)|\), using the first part of the lemma, one has that

\[
(1 - q_\mu) \leq \frac{\rho}{\rho - 1} v_\mu \leq \frac{\rho}{\rho - 1} K_1 |G_0(\mu)| \leq \frac{\rho}{\rho - 1} K_2 |\mu| \leq K_3 |\mu|,
\]

ending the proof of the inequalities in the second item of the lemma. \(\square\)

Let \(A_0(\mu) = (\beta^{-1}, p_\mu)\) and \(A_{k_0}(\mu) = (q_\mu, 1)\). Equations (11) and (17) and Lemma 6.8 imply:

**Lemma 6.9.** There is a constant \(K_4 > 0\), independent of small \(\mu^*\), such that

\[
\max\{|A_0(\mu)|, |A_{k_0}(\mu)|\} \leq K_4 |\mu|, \quad \text{for all } \mu < \mu^* \text{ close to } \mu^*.
\]

We now define the intervals corresponding to the initial gaps and bridges of the Cantor set \(\Sigma_\mu:\)

- For \(j \in \{1, \ldots, (k_0 - 1)\}\), \(A_j(\mu) = (a_j, b_j)\) is an interval containing \(G_j(\mu)\) such that \(\Gamma_{\mu}(a_j) = q_\mu\) and \(\Gamma_{\mu}(b_j) = p_\mu\). More precisely, \(a_j\) is the biggest point in \(I_{j - 1}(\mu)\) with \(\Gamma_{\mu}(a_j) = q_\mu\) and \(b_j \in I_j(\mu)\) is chosen close to the right extreme of \(G_j(\mu)\). A priori, there are infinitely many possibilities for the choice of \(b_j\), we will explicit this choice below. The gaps \(A_0(\mu)\) and \(A_1(\mu)\) are depicted in Figure 13.
- For \(j = 0, \ldots, k_0 - 1\), \(B_j(\mu) = [b_j, a_{j + 1}]\), where \(a_{k_0} = q_\mu\) and \(b_0 = p_\mu\).

**Remark 6.9.** By construction, \((\Phi^{0,0}_{\mu})^*(B_{k_0}(\mu)) = B_0(\mu)\).

In \([p_\mu, c_\mu] \subset I_0(\mu)\), the map \(\Psi_{\mu} = \Gamma_{\mu}\) has finitely many discontinuities, say \(r\). Note that the number \(r\) increases as \(p_\mu\) approaches to \(\ell_\mu\), so \(r\) can be taken arbitrarily large after an appropriate choice of \(p_\mu\). We choose the points \(b_j\) above such that the restriction of \(\Gamma_{\mu}\) to each \(B_j(\mu)\) also has \(r\) discontinuity points \(d_{j,1} < d_{j,2} < \cdots < d_{j,r}\), where \(\Gamma_{\mu}\) is bi-valuated and takes the values \(\beta^{-1}\) and 1. Thus the restriction of \(\Gamma_{\mu}\) to each \(B_j(\mu)\) has \((r + 1)\) (injective) branches, where

- \(\Gamma_{\mu}([b_j, d_{j,1}]) = [p_\mu, 1]\),
- \(\Gamma_{\mu}([d_{j,i}, d_{j,i+1}^\mu]) = [\beta^{-1}, 1]\), for \(i = 1, \ldots, r - 1\), and
- \(\Gamma_{\mu}([d_{j,r}, a_{j + 1}]) = [\beta^{-1}, q_\mu]\).

To get a Markov partition of \(\Gamma_{\mu}\), we refine the bridges \(B_j(\mu)\) by adding new gaps (corresponding to the discontinuities \(d_{j,i}\)), see Figure 13.

- For each \(j \in \{0, \ldots, k_0 - 1\}\) and \(i \in \{1, \ldots, r\}\), \(A_{j,i}(\mu)\) is the smallest interval \((a_{j,i}^\mu, b_{j,i}^\mu)\) containing the discontinuity \(d_{j,i}\) such that \(\Gamma_{\mu}(a_{j,i}^\mu) = q_\mu\) and \(\Gamma_{\mu}(b_{j,i}^\mu) = p_\mu\).
- The intervals \(B_j^0(\mu), B_j^1(\mu), \ldots, B_j^r(\mu)\) are the connected components of \(B_j(\mu)\) in \(\cup_{i=1}^r A_{j,i}(\mu)\), ordered in such a way \(B_j^0(\mu)\) is at the left of \(B_j^{i+1}(\mu)\). By construction, \(\Gamma_{\mu}(B_j^0(\mu)) = [p_\mu, q_\mu]\). Note that \(B_j^1(\mu) = [b_{j,1}^\mu, a_{j+1}^\mu]\) and that \(b_0^0 = p_\mu\).

The previous constructions and Lemma 6.5 imply the following:
Proposition 6.10. Let $B_\mu = \{ B_i^j(\mu) \}_{j=0, \ldots, (k_0-1)}$, $B_\mu = \cup_{i,j} B_i^j(\mu)$, and $\Sigma_\mu$ be the maximal invariant set of $\Gamma_\mu$ in $B_\mu$. The set $\Sigma_\mu$ is a (expanding) dynamically defined Cantor set associated to the Markov partition $B_\mu$.

Throughout this section, in our estimates of the thickness, we use repeatedly the next result:

Lemma 6.11. (Bounded distortion property, Theorem 1, page 58) Let $\Sigma$ be a dynamically defined Cantor set of a $C^{1+\varepsilon}$-expanding map $\Gamma$. Given any $\delta > 0$, there is $\kappa(\delta) > 0$ such that: for any pair of points $x$ and $y$ and $n \geq 1$ such that $|\Gamma^n(x) - \Gamma^n(y)| \leq \delta$ and the interval of extremes $\Gamma^i(x)$ and $\Gamma^i(y)$ is contained in the domain of $\Gamma$, for all $i = 0, \ldots, (n-1)$, it holds

$$|\log |(\Gamma^n)'(x)| - \log |(\Gamma^n)'(y)|| \leq \kappa(\delta).$$

Moreover, the constant $\kappa(\delta)$ can be taken such that $\kappa(\delta) \to 0$ as $\delta \to 0$.

We apply this lemma to the dynamically defined Cantor set $\Sigma_\mu$ of the expanding map $\Gamma_\mu$ (restricted to the Markov partition $B_\mu$). In our context, it is enough to fix $\delta = (1 - \beta^{-1})$, which is an upper bound for the length of any interval of the partition, and let $\kappa_\mu = \kappa_\mu(\delta)$, the constant given by the lemma. Using the definition of $\Gamma_\mu$, see and , and that it is $C^2$, one gets

$$\kappa_\mu \leq \max \frac{|\Psi'_\mu(x)|}{(\min |\Psi'_\mu(y)|)^2} (1 - \beta^{-1}) \frac{1}{\rho - 1} = \kappa,$$

where $\rho > 1$ is as in Lemma Thus we can take $\kappa$ instead of $\kappa_\mu$, so the distortion constant can be taken independent of small $\mu$. This inequality (whose proof we omit here) is obtained adapting straightforwardly the proof of Lemma 6.11 to the $C^2$-case (see page 59).

To estimate the thickness of $\Sigma_\mu$, we choose the following presentation $A_\mu$ of it:

The presentation $A_\mu$ of the set $\Sigma_\mu$: The generation $(0,1)$ of gaps is $\{ A_1(\mu), A_2(\mu), \ldots, A_{k_0}(\mu) \}$. The generation $(0,2)$ of gaps is defined by

$$\{ A_3(\mu), A_4(\mu), \ldots, A_\mu(\mu), A_0(\mu), A_1(\mu), A_2(\mu), \ldots, A_{k_0}(\mu), A_{k_0}(\mu), \ldots, A_{k_0}(\mu) \}.$$

Inductively, for each $i = 1$ or $2$, we define the $(k+1,i)$-generation of gaps as the pre-images by $\Gamma_\mu$ of the gaps of the $(k,i)$-generation, enumerated according to the ordering in $\mathbb{R}$. Moreover, any gap of generation $(k+1,2)$ is posterior to any gap of generation $(k+1,1)$.

The heuristic principle in our construction is that to estimate the thickness of the Cantor set $\Sigma_\mu$ it is enough to calculate the quotients bridge/gap for the first
generation of gaps. These gaps are the generating ones: gaps of higher generations are pre-images of them. This principle uses that the maps defining set $\Sigma_\mu$ have small distortion. This follows from Remark 6.12 below.

Figure 14. New generation of gaps

Take a gap $G$ of $(n, i)$-generation and a bridge $B$ of $G$. Then the number $n$ and any pair of points $x \in G$ and $y \in B$ verify the hypotheses of Lemma 6.11 Moreover, by construction, $\Gamma_\mu(G)$ is a gap of $(0, i)$-generation and $\Gamma_\mu(B)$ is a bridge of it. Using Lemma 6.11 one gets the following (which is a standard argument):

$$e^{-\kappa} \frac{|B|}{|G|} \leq \frac{|\Gamma_\mu^n(B)|}{|\Gamma_\mu^n(G)|} \leq e^\kappa \frac{|B|}{|G|}$$

(18)

Remark 6.12. Take $\tau > 0$ such that $|B|/|G| > \tau$ for any gap $G$ of generation $(0, 1)$ or $(0, 2)$ and any bridge $B$ of $G$. Equation (18) implies that the thickness of $\Sigma_\mu$ is bigger than $\tau e^{-\kappa}$.

By Remark 6.12 to get a lower bound for $\tau(\Sigma_\mu)$ (i.e. to prove item (a) of Proposition 6.6 in Case 1) it suffices to estimate the ratios between bridges and gaps of generations $(0, 1)$ and $(0, 2)$. This is done in the next lemma, which immediately implies item (a) of Proposition 6.4 in Case 1.

Lemma 6.13. There is $K_0$, independent of small $\mu^* < 0$, such that, for every gap $G$ of generation $(0, 1)$ or $(0, 2)$ and any bridge $B$ of $G$, it holds

$$|B|/|G| > K_0 \frac{|\mu|}{|\mu^*|}, \text{ for every } \mu < \mu^* \text{ close to } \mu^*.$$

Proof: We first estimate the quotients $|B_0(\mu)|/|A_1(\mu)|$ and $|B_1(\mu)|/|A_1(\mu)|$. Note that $B_0(\mu)$ is the left bridge of $A_1(\mu)$ and $B_1(\mu)$ is contained in the right bridge of $A_1(\mu)$. To estimate the length of $A_1(\mu) = (a_1, b_1)$, write $G_1(\mu) = (c_\mu, g^*_1) \subset A_1(\mu)$ (see equation (13)). Then

$$|A_1(\mu)| = b_1 - a_1 = (b_1 - g^*_1) + |G_1(\mu)| + (c_\mu - a_1).$$

Note that $c_\mu$ and $a_1$ are in the same branch of $\Gamma_\mu$, $\Gamma_\mu(c_\mu) = 1$, and $\Gamma_\mu(a_1) = q_\mu$. Thus Lemmas 6.5 and 6.7 imply

$$(c_\mu - a_1) \leq \frac{1 - q_\mu}{\rho} \leq K_3 \frac{|\mu|}{\rho}.$$  (19)

Also, since by construction, $\Gamma_\mu(G_1(\mu)) = G_0(\mu)$, from Lemmas 6.5 and 6.7

$$|G_1(\mu)| \leq \frac{|G_0(\mu)|}{\rho} \leq \frac{K_2 |\mu|}{K_1 \rho} = \frac{K_5 |\mu|}{\rho}.$$
By construction, (recall again equation (14)) \( \Phi^{0,0}_\mu(g^*_i) = \ell_\mu \) and \( \Phi^{0,0}_\mu(b_1) = p_\mu \). Proposition 6.4 and equation (17) imply that

\[
|A_1(\mu)| \leq \frac{K_3 + K_5 + 3\rho}{\rho} |\mu| = K_6 |\mu|.
\]

(20)

Putting these inequalities together, one gets

\[
|A_1(\mu)| \leq \frac{K_3 + K_5 + 3\rho}{\rho} |\mu| = K_6 |\mu|.
\]

To estimate \( B_0(\mu) = [p_\mu, a_1] \), write

\[
|B_0(a_\mu)| = (c_\mu - \ell_\mu) - (p_\mu - \ell_\mu) - (c_\mu - a_1).
\]

Note that the size of \( |\ell_\mu, c_\mu| \) is uniformly bounded from zero (independent of small \( \mu^* \)). By (14) and (19), \( (p_\mu - \ell_\mu) + (c_\mu - a_1) \leq (1 + (K_3/\rho)) |\mu| \). Thus, for small \( \mu \), there is \( K_7 > 0 \) such that

\[
|B_0(a_\mu)| > K_7 > 0.
\]

Finally, by Remark 6.3 and Lemma 6.3, the previous inequality implies that

\[
|B_1(\mu)| > K_8 > 0.
\]

All the estimates above lead to

\[
\min \left\{ \frac{|B_1(\mu)|}{|A_1(\mu)|}, \frac{|B_0(\mu)|}{|A_1(\mu)|} \right\} \geq \frac{K_9}{|\mu|}.
\]

(21)

Similarly, using Lemma 6.3, Remark 6.3 and equation (21), one gets

\[
\min \left\{ \frac{|B_i(\mu)|}{|A_i(\mu)|}, \frac{|B_{i-1}(\mu)|}{|A_i(\mu)|} \right\} \geq \frac{L K_9}{|\mu|} = \frac{K_{10}}{|\mu|}.
\]

(22)

Observing that, for each \( i \), \( B_{i-1}(\mu) \) is the left bridge of \( A_i(\mu) \) and \( B_i(\mu) \) is contained in the right bridge of \( A_i(\mu) \), this ends the estimates for the generation \((0, 1)\).

We now estimate the ratio between bridges and gaps of generation \((0, 2)\). In this step, each interval \( B_j(\mu) \) splits into \((r + 1)\) new bridges \( B^*_j(\mu), i = 0, \ldots, r \), obtained from \( B_j(\mu) \) by removing the intervals \( A^*_j(\mu) \) around the discontinuities \( d^*_j \), \( i = 1, \ldots, r \), where \( r \) is the number of \( \mu \)-discontinuities of \( \Psi_\mu \) in \([p_\mu, a_1]\). As before, bearing in mind Lemma 6.3 we first estimate these ratios for the bridges \( B_0^*(\mu) \).

Observe that, by definition, \( B_0^*(\mu) = [b_0^*, a_0^{i+1}] \subset J_{r, i+1}(\mu) \) (recall equation (14)) and

\[
\Gamma_\mu((d^*_0, b_0^*)) = (\beta^{-1}, p_\mu) \quad \text{and} \quad \Gamma_\mu((a_0^*, d^*_0)) = (q_\mu, 1).
\]

Recall that \( |(\beta^{-1}, p_\mu)| < (K + 1) |\mu| \) (see (11) and (14)) and \( |(q_\mu, 1)| < K_3 |\mu| \) (Lemma 6.7). Note that \((a_0^*, d^*_0)\) \( \cup \) \( B_0^*(\mu) \subset J_{r+1-i}(\mu) \) and that \((d^*_0, b_0^*) \subset J_{r+1-i-1}(\mu) \).

Remark 6.3 now implies that

\[
|A_0^*(\mu)| = (b_0^* - a_0^*) = (b_0^* - d_0^*) + (d_0^* - a_0^*) \leq \frac{(K + 1 + K_3) \mu}{\beta^{-i+1} - m} = \frac{K_{11}|\mu|}{\beta^{-i+1} - m}.
\]

(23)

Analogously, as \( \Psi_\mu(B_0^*(\mu)) = [p_\mu, q_\mu] \), and the size of \([p_\mu, q_\mu]\) is uniformly lower bounded, one has (independently of small \( \mu \))

\[
|B_0^*(\mu)| \geq \frac{K_{12}}{\beta^{r-i+1} - m}. \tag{24}
\]

Now, from inequalities (24) and (25), it follows that,

\[
\min \left\{ \frac{|B_0^{-1}(\mu)|}{|A_0^*(\mu)|}, \frac{|B_0(\mu)|}{|A_0^*(\mu)|} \right\} \geq \frac{K_{13}}{|\mu|}.
\]

(25)
We consider the dynamically defined Cantor set $\Sigma^\mu$ and repeat the construction of Case 1 replacing $G$ as in Case 1. In this way, the Cantor set $\Sigma^\mu$ but in this case one gets exactly the same as in Case 1. The estimate for (1) leads to the relative big gap $G_{k_0}(\mu)$ obtaining the new gap $G'_{k_0}(\mu) = G_{k_0}(\mu) \cup I_{k_0}(\mu)$ and repeat the construction of Case 1 replacing $G_{k_0}(\mu)$ by $G'_{k_0}(\mu)$.

The definitions of $p_\mu$ and $q_\mu$ and the estimate for $p_\mu - \xi_\mu$ (equation (17)) are exactly the same as in Case 1. The estimate for $(1 - q_\mu)$ follows as in Lemma 6.7 but in this case one gets 

$$(1 - q_\mu) \leq (K_3 + \xi) |\mu|.$$ 

We consider the dynamically defined Cantor set $\Sigma_\mu$ associated to the map $\Gamma_\mu$ in $[p_\mu, q_\mu]$ (the Markov partition and the gaps of generations (0, 1) and (0, 2) are defined as in Case 1). In this way, the Cantor set $\Sigma_\mu$ verifies Proposition 6.6 (taking $\eta$ equal to $\frac{1}{3}$).

**Case 2.1.** $|I_{k_0}(\mu)| \leq \xi |\mu|$. In this case the construction is similar to the one of Case 1. We modify the construction as follows. We add the relatively small interval $I_{k_0}(\mu)$ to the relatively big gap $G_{k_0}(\mu)$ obtaining the new gap

$$G'_{k_0}(\mu) = G_{k_0}(\mu) \cup I_{k_0}(\mu)$$

and repeat the construction of Case 1 replacing $G_{k_0}(\mu)$ by $G'_{k_0}(\mu)$.

The definitions of $p_\mu$ and $q_\mu$ and the estimate for $p_\mu - \xi_\mu$ (equation (17)) are exactly the same as in Case 1. The estimate for $(1 - q_\mu)$ follows as in Lemma 6.7 but in this case one gets

$$(1 - q_\mu) \leq (K_3 + \xi) |\mu|.$$ 

We consider the dynamically defined Cantor set $\Sigma_\mu$ associated to the map $\Gamma_\mu$ in $[p_\mu, q_\mu]$ (the Markov partition and the gaps of generations (0, 1) and (0, 2) are defined as in Case 1). In this way, the Cantor set $\Sigma_\mu$ verifies Proposition 6.6 (taking $\eta$ equal to $\frac{1}{3}$).

**Case 2.2.** $|I_{k_0}(\mu)| > \xi |\mu|$. We split $I_{k_0}(\mu)$ into two intervals $I_{k_0}(\mu)$ and $I'_{k_0}(\mu)$ with disjoint interiors, where $I_{k_0}(\mu)$ is the domain of all (infinitely many) surjective branches of the restriction of $\Gamma_\mu$ to $I_{k_0}(\mu)$ and $I'_{k_0}(\mu)$ is its complement (corresponding to the only non surjective branch of $\Gamma_\mu$). See Figure 15. Write $\Gamma_\mu(I'_{k_0}(\mu)) = [\beta^{-1}, \eta]$, where $\eta < 1$. By Lemma 6.6

$$|I'_{k_0}(\mu)| \leq \frac{(\eta - \beta^{-1})}{\rho}.$$ 

There are now two sub-cases, according to the ratios between the sizes of $I'_{k_0}(\mu)$ and $I'_{k_0}(\mu)$: either $|I'_{k_0}(\mu)|/|I_{k_0}(\mu)| \leq \xi |\mu|$ (Case 2.2.1) or $|I'_{k_0}(\mu)|/|I_{k_0}(\mu)| > \xi |\mu|$ (Case 2.2.2).
Case 2.2.1.  \(|I_{k_0}^* (\mu)|/|\ell_{k_0}^* (\mu)| \leq \xi |\mu|\).  This condition means the interval \(I_{k_0}^* (\mu)| is small in comparison with \(\ell_{k_0}^* (\mu)| and the arguments are similar to the ones in Case 1, considering \(I_{k_0}^* (\mu)| an extremal gap which plays a role similar to the gap \(G_{k_0}^* (\mu)| in Case 1.  Let \(q_{k_0}'\) be the fixed point of \(\Gamma_{\mu}| in \(I_{k_0}^* (\mu)| closest to 1.  Consider the dynamically defined Cantor set \(\Sigma_{\mu}| in \([p_{\mu}, q_{k_0}']\)| associated to \(\Gamma_{\mu}| and the Markov partition obtained by modifying the one in Case 1 as follows:

- We consider \(k_0| gaps \(A_i (\mu)| (i = 1, \ldots, k_0)| of generation (0, 1) (instead of \((k_0 - 1)| as in Case 1), where each gap \(A_i (\mu)| contains \(G_i (\mu)| and is defined as in Case 1 replacing \(q_{\mu}| by \(q_{k_0}'\).

- The gaps of generation (0, 2) are obtained as the ones in Case 1, but now adding the discontinuities of \(\Gamma_{\mu}| in \(I_{k_0}^* (\mu)|.

By Remark 6.12, it is enough to estimate the ratios bridge/gap for the generations (0, 1) and (0, 2).  We first consider the generation (0, 1).  Since \(I_{k_0}^* (\mu)| is (at most) of order of \(|\xi| |\mu| \simeq |\mu|^{2/5},| arguing as in (2) in Lemma 6.7, it follows that \(|1 - q_{k_0}'| is of the same order as \(|I_{k_0}^* (\mu)|.\)  Hence \(|1 - q_{k_0}'| is at most of order of \(|\mu|^{2/5}.| This implies that the ratios bridge/gap of generation (0, 1), excluded \(|I_{k_0}^* (\mu)|/|A_{k_0}^* (\mu)|, are of order of (at least) \(1/|\mu|^{2/5}.| As in Case 1, to estimate \(|I_{k_0}^* (\mu)|/|A_{k_0}^* (\mu)| it suffices to estimate \(|I_{k_0}^* (\mu)|/|G_{k_0}^* (\mu)| (these quotients are of the same order).  Note that

\[
\xi |\mu| \leq |I_{k_0} (\mu)| = |I_{k_0}^* (\mu)| + |I_{k_0}^* (\mu)| = |I_{k_0}^* (\mu)| \left(\frac{|I_{k_0}^* (\mu)|}{|I_{k_0}^* (\mu)|} + 1\right) \leq |I_{k_0}^* (\mu)| (\xi |\mu| + 1).
\]

Thus,

\[
|I_{k_0}^* (\mu)| \geq \frac{\xi |\mu|}{1 + \xi |\mu|}.
\]

Arguing as in Lemma 6.7, one gets \(|G_{k_0}^* (\mu)| < K_2 |\mu|.\)  Finally, recalling the definition of \(\xi| in Lemma 6.7,\)

\[
\frac{|I_{k_0}^* (\mu)|}{|G_{k_0}^* (\mu)|} \geq \frac{\xi}{K_2 (1 + \xi |\mu|)} = K_2 \frac{1}{|\mu|^{2/5}} \geq K_2 \frac{1}{|\mu|^{2/5}},
\]

ending the estimates for the generation (0, 1).

The estimates for the ratios bridge/gap of generation (0, 2) follow as in Case 1, but now these ratios are of order of (at least) \(1/|\mu|^{2/5}.\)  The previous calculations and the choice of \(\xi| in (27) imply that, for small \(\mu^*| and \(\mu < \mu^*| close to \(\mu^*|, the ratios bridge/gap of generations (0, 1) and (0, 2) are at least of order of \(1/|\mu|^{2/5}.\)  Thus, by Remark 6.12, the set \(\Sigma_{\mu}| verifies Proposition 6.6| taking \(n = 2/5.\)  Finally, the lower bound for the diameter of \(\Sigma_{\mu}| is obtained as in Case 1 replacing \(q_{\mu}| by \(q_{k_0}'.\)

This completes the proof of the proposition in Case 2.2.1.

Case 2.2.2.  \(|I_{k_0}^* (\mu)|/|\ell_{k_0}^* (\mu)| > \xi |\mu|.\)

Claim 6.14.  Suppose that \(|I_{k_0}^* (\mu)|/|\ell_{k_0}^* (\mu)| > \xi |\mu|.\)  Then \(\Gamma_{\mu}(I_{k_0}^* (\mu)| contains (at least) two surjective branches of \(\Gamma_{\mu}.\)

Proof:  Note first that \(|J_i (\mu)| \to 0| as \(i \to \infty.\)  Thus we can choose the fixed point \(p_{\mu}| of \(\Gamma_{\mu}| arbitrarily close to \(\ell_{\mu}| in such a way that there is \(s_{\mu} > p_{\mu}| such that \([p_{\mu}, s_{\mu}]| contains two consecutive branches of \(\Gamma_{\mu}| and \(|s_{\mu} - \ell_{\mu}| < |\mu|.\)  Since \(\Gamma_{\mu}(I_{k_0}^* (\mu)| = [\beta^{-1}, \eta]| and \(G_{0}(\mu)| = (\beta^{-1}, \ell_{\mu}),| to get \((p_{\mu}, s_{\mu}) \subset \Gamma_{\mu}(I_{k_0}^* (\mu)| it is enough to see that \((\beta^{-1}, s_{\mu}) \subset \Gamma_{\mu}(I_{k_0}^* (\mu)|.\)  By equation (11) and the choice of \(s_{\mu},\)

\[
(s_{\mu} - \beta^{-1}) = (s_{\mu} - \ell_{\mu}) + (\ell_{\mu} - \beta^{-1}) \leq |G_{0}(\mu)| + |\mu| \leq K |\mu| + |\mu|.
\]
Thus it is enough to see that
\[ |\Gamma_\mu(I_{k_0}(\mu))| > (K + 1) |\mu|. \]

We argue by contradiction, if the last inequality does not hold, then, by Lemma 6.5
\[ |I_{k_0}(\mu)| \leq \frac{(1 + K) |\mu|}{\rho}. \]

Thus, by hypothesis,
\[ \xi |\mu| < |I_{k_0}(\mu)| = |I_{k_0}(\mu)| \leq |I_{k_0}(\mu)| \left( 1 + \frac{|I_{k_0}(\mu)|}{|I_{k_0}(\mu)|} \right) \leq \frac{(K + 1) |\mu|}{\rho} \left( 1 + \frac{1}{\xi |\mu|} \right). \]

Hence,
\[ \xi \leq \frac{(K + 1)}{\rho} \left( 1 + \frac{1}{\xi |\mu|} \right). \]

This contradicts the definition of \( \xi \) in (27) for small \( \mu \) close to \( \mu^* \). This proves the claim. \( \square \)

By the claim, there is \( h \geq 3 \) such that \( I_{k_0}(\mu) \) is of the form
\[ I_{k_0}(\mu) = \bigcup_{j=1}^{h} (G_{k_0} + j(\mu) \cup I_{k_0} + j(\mu)), \]
where
- the interval \( I_{k_0+h}(\mu) \) may be empty,
- \( \Gamma_\mu(G_{k_0} + j(\mu)) = G_{j-1}(\mu) \) and \( \Gamma_\mu(I_{k_0} + j(\mu)) = I_{j-1}(\mu) \), if \( j \neq h \),
- if \( j = h \) and \( I_{k_0+h}(\mu) \neq \emptyset \), then \( \Gamma_\mu(G_{k_0} + h(\mu)) = G_{h-1}(\mu) \) and \( \Gamma_\mu(I_{k_0} + h(\mu)) \subset I_{k_0}(\mu) \),
- if \( j = h \) and \( I_{k_0+h}(\mu) = \emptyset \), then \( \Gamma_\mu(G_{k_0} + h(\mu)) \subset G_{h-1}(\mu) \).

\[ \text{Figure 15. New gaps and the map } \Gamma_\mu^{(1)} \]

Consider now the map (see Figure 15):
\[ \Gamma_\mu^{(1)} : [\beta^{-1}, 1] \setminus (\cup_{j=0}^{k_0+h} G_j(\mu)) \to [\beta^{-1}, 1], \]
\[ \begin{cases} \Gamma_\mu^{(1)}(y) = \Gamma_\mu(y), & \text{if } y \notin I_{k_0}(\mu), \\ \Gamma_\mu^{(1)}(y) = \Gamma_\mu \circ \Gamma_\mu(y), & \text{if } y \in I_{k_0}(\mu). \end{cases} \]

**Remark 6.15.** By construction and Lemma 6.5, the map \( \Gamma_\mu^{(1)} \) is \( \rho \)-expanding. Moreover, its restriction to \( I_{k_0+1}(\mu), \ldots, I_{k_0+h}(\mu) \) is \( \rho^2 \)-expanding.
As in the case of the map $\Gamma_{\mu}$, we consider two cases according to the position of $1$.

- **Case (a):** the interval $I_{k_0 + h}(\mu)$ is empty, or, equivalently, $1 \in G_{k_0 + h}(\mu)$.
- **Case (b):** $I_{k_0 + h}(\mu) \neq \emptyset$.

**Case (a).** In this case, we argue as in Case 1 (Section 6.2.2) with the following modifications.

- Take the fixed points $p_\mu$ as above and $q_\mu^1$ (the fixed point of $\Gamma_{\mu}^{(1)}$ closest to 1). The gaps of generation $(0,1)$ are the intervals $A_0(\mu), \ldots, A_{k_0 + h}(\mu)$, where each $A_j(\mu)$ contains the corresponding $G_j(\mu)$ and the extremes of $A_j(\mu)$ are mapped by $\Gamma_{\mu}^{(1)}$ into $p_\mu$ and $q_\mu^1$.
- To define the gaps of generation $(0,2)$, consider the discontinuities $d_1^\mu, d_2^\mu \in I_1(\mu)$, of $\Gamma_{\mu}^{(1)}$ and, for each $d_1^\mu$, the interval $A_1^\mu(\mu)$ containing $d_1^\mu$ and mapped onto $[\beta^{-1},p_\mu) \cup (q_\mu^1, 1]$ by $\Gamma_{\mu}^{(1)}$.

As in Case 1, the key step is to estimate the ratios bridges/gaps of generations $(0,1)$ and $(0,2)$. As before, it is enough to estimate the ratios $|I_j(\mu)|/|G_j(\mu)|$ and $|I_{j-1}(\mu)|/|G_j(\mu)|$ instead of the ratios of the bridges and gaps (for notational simplicity, we let $I_{k_0}(\mu) = I_{k_0}(\mu)$). First, as in Case 1, one has $|1 - q_\mu^1| < (K_{15} |\mu|)/\rho^2$, where $K_{15}$ is independent of small $\mu$. We have the following:

- For $j = 1, \ldots, k_0 - 1$, the estimates for the quotients
  \[ \frac{|I_j(\mu)|}{|G_j(\mu)|} \quad \text{and} \quad \frac{|I_{j-1}(\mu)|}{|G_j(\mu)|} \]
  are the similar to the ones made in Case 1. This also holds for $|I_{k_0-1}(\mu)|/|G_{k_0}(\mu)|$.
- Since the old interval $I_{k_0}(\mu)$ was subdivided, we need to estimate the ratios
  \[ \frac{|I_{k_0}(\mu)|}{|G_{k_0}(\mu)|} \quad \text{and} \quad \frac{|I_{k_0}(\mu)|}{|G_{k_0+1}(\mu)|} \]
  considering the new $I_{k_0}(\mu) = I_{k_0}^\ell(\mu)$. This is done using Claim 6.16 below.
- The ratios $|I_{k_0+j}(\mu)|/|G_{k_0+j}(\mu)|$ and $|I_{k_0+j}(\mu)|/|G_{k_0+1+j}(\mu)|, j \geq 1$, are calculated exactly as the ratios bridge/gap of generation $(1,1)$ in Case 1.

**Claim 6.16.** $|I_{k_0}^\ell(\mu)| < K_{16} |I_{k_0}^\ell(\mu)|, K_{16}$ is independent of $\mu$ close to $\mu^*$.

The claim follows immediately from equation (18), taking $B = I_{k_0}^\ell(\mu)$ and $A = I_{k_0}^\ell(\mu)$, and noting that $I_{k_0}^\ell(\mu) \cup I_{k_0}^\ell(\mu)$ is contained in one interval (branch) of definition of $\Gamma_{\mu}$, see Lemma 6.11. We omit the details of this proof.

First, by Lemma 6.7 $|G_{k_0}(\mu)| < K_2 |\mu|$. Next, by the claim and hypothesis (Case 2.2), one has

\[ \frac{\xi}{K_2} = \frac{\xi |\mu|}{K_2 |\mu|} \leq \frac{|I_{k_0}(\mu)|}{|G_{k_0}(\mu)|} = \frac{|I_{k_0}^\ell(\mu)| + |I_{k_0}^r(\mu)|}{|G_{k_0}(\mu)|} \leq \frac{(K_{16} + 1) |I_{k_0}^\ell(\mu)|}{|G_{k_0}(\mu)|}. \]

Thus

\[ \frac{|I_{k_0}^\ell(\mu)|}{|G_{k_0}(\mu)|} \geq \frac{\xi}{K_2 (1 + K_{16})}. \]

We now estimate $|I_{k_0}^\ell(\mu)|/|G_{k_0+1}(\mu)|$. From the definition of $G_{k_0+1}(\mu)$, inequality (11), and Lemma 6.3

\[ |G_{k_0+1}(\mu)| \leq \frac{|G_0(\mu)|}{\rho} \leq \frac{K |\mu|}{\rho}. \]
By hypothesis (Case 2.2) and Claim 6.16
\[ |I(\mu)| \leq |I_{k_\ast}(\mu)| = |I_{k_\ast}^0(\mu)| + |I_{k_\ast}^\ell(\mu)| \leq (1 + K_{16}) |I_{k_\ast}^\ell(\mu)|. \]

Putting these inequalities together we have,
\[
\frac{|I_{k_\ast}^\ell(\mu)|}{|G_{k_\ast+1}(\mu)|} > \frac{\xi \rho}{(1 + K_{16}) K}.
\]

The estimates for the ratios bridge/gap of generation (0, 2) follow as in Case 1, using Lemma 6.11 and noting that the sizes of \([\beta^{-1}, p_\mu]\) and \([q_\mu, 1]\) are of order of (at most) \(|\mu|\). Arguing as in the previous case, this implies that all the ratios \(|\mu|/|G|\) of generations (0, 1) and (0, 2) are of order of (at least) \(1/|\mu|^{2/5}\). This ends the estimate of the thickness of \(\Sigma_{\mu}\) in Case (a).

Case (b). In this case the situation similar to Case 2, now considering the map \(\Gamma_\mu^{(1)}\) and 1 \(\in I_{k_\ast+1}(\mu) = I_{k_1}(\mu)\). We proceed exactly as before, if we are in Cases (similar to) 1, 2.1, 2.2, 2.2.1 or 2.2.2(a) then the construction goes as above ends and the proof of Proposition 6.16 is finished.

If we are in Case 2.2.2(b), we define a new map \(\Gamma_{\mu}^{(2)}\), obtained from \(\Gamma_{\mu}^{(1)}\) exactly as \(\Gamma_{\mu}^{(2)}\) from \(\Gamma_{\mu}\). There is a new interval \(I_{k_\ast}(\mu)\) containing the point 1. Since the definition of \(I_{k_\ast}(\mu)\) involves two backwards iterations by \(\Gamma_{\mu}\), by Lemma 6.16 \(|I_{k_\ast}(\mu)| \leq 1/\rho^2\).

Lemma 6.11 and equation (18) guarantee that all the constants in Case 2.2.2(b) for \(\Gamma_{\mu}^{(1)}\) are the same as the ones obtained in the first inductive step. We now proceed inductively: in each step \(n\) either the construction is finished (i.e., we fall in new Cases 1, 2.1, 2.2, 2.2.1 or 2.2.2(a)) or we define a new map \(\Gamma_{\mu}^{(n)}\) and an interval \(I_{k_\ast}(\mu)\) of size less than \(1/\rho^n\) containing the point 1. Therefore, there is a first \(n\) such that \(|I_{k_\ast}(\mu)| < 1/\rho^n < \xi |\mu|\), so we are in a situation similar to Case 2.1 and the recurrent procedure ends.

The proof of item (a) (lower bound for the thickness) of Proposition 6.16 is now complete. By construction, all the gaps of \(\Sigma_{\mu}\) (in all sub-cases) are at most of order of \(|\mu|^{2/5}\), implying (b). The claim on the diameter follows noting that the convex hull of \(\Sigma_{\mu}\) always contains \([p_\mu, q_\mu]\).

6.3. The thick hyperbolic unstable set \(\Xi_t\). Consider a parameter \(t \in B_\alpha, t = \lambda^\mu (1 + \mu)\), where \(\mu < \mu^*\) and close to \(\mu^*\). Recall that \(\alpha\) is fixed and equal to \(\alpha(\mu^*)\) and the transition map \(F\) associated to the arc \(f_t\) is \(F_{\alpha(\mu^*)}\). We consider the (transitive) hyperbolic set \(\Xi_t\) of index two defined as the (minimal) \(f_t\)-invariant set such that:

: (a) The central localization of \(\Xi_t\) (see (10) for the definition) is \((\Xi_t)^c = \Sigma_{\mu}\).

: (b) Every \(X = (x, y, z) \in \Xi_t \cap \Delta^+\) has consecutive returns (by \(f_t\)) to \(\Delta^+\) such that the central coordinate of the composition of these consecutive returns \(\Gamma_{\mu}^{(n)}\) (where \(n\) is the last step in the inductive process of the construction of \(\Sigma_{\mu}\)),

\[ f_t^n(X) = (x', y', z') \quad \text{and} \quad y' = \Gamma_{\mu}^{(n)}(y), \]

here \(r\) is the total number of iterations for obtaining such a final return.

Let us explain a bit more condition (b) and why the set \(\Xi_t\) is hyperbolic (of index two). Let us assume, for simplicity, that \(\Sigma_{\mu} = (\Xi_t)^c\) is obtained in the first step of the process. This means that the set is associated to \(\Gamma_{\mu}\) and we are either in Case 1 or in Case 2.2.1. Take \(X = (x, y, z) \in \Xi_t \cap \Delta^+\) and note that \(y \in \cup_{j=0}^{k_\ast} I_j(\mu)\), see
First, if \( y \in I_0(\mu) \) then \( y \in J_i(\mu) \) for some \( i \), thus the point \( X \) has a return of type \((i,0)\), recall the definitions of the intervals \( J_i(\mu) \) in Section 6.3 (just before equation (2)). Using Remark 3.3, one has

\[
f_t^{i+n+k_0+0+n+N}(x, y, z) = (x', y', z'), \quad y' = \Psi_\mu(y) = \Phi_\mu^{0}(y) = \Gamma_\mu(y).
\]

By Lemma 6.3, \( Df_t^{i+n+k_0+0+n+N}(X) \) expands in the central direction (at least) by a factor \( \rho > 1 \).

Second, if \( y \in I_k(\mu) \) then \((\Phi_\mu^0)^k(y)\) belongs to some subinterval \( J_i(\mu) \) of \( I_0(\mu) \).

Therefore, using Remark 3.3, the point \( X = (x, y, z) \in \Delta^+ \) has \( k \) consecutive returns to \( \Delta^+ \) of type \((0,0)\) (each one corresponding to \((0+n+k_0+0+n+N)\) iterations of \( f_t \)) followed by a return of type \((i,0)\) (corresponding to \((i+n+k_0+0+n+N)\) iterations).

Thus

\[
f_t^{i+n+k_0+n+N}(x, y, z) = (x', y', z'), \quad y' = \Gamma_\mu(y).
\]

By Lemma 6.3 the central direction is (at least) \( \rho \)-expanding, \( \rho > 1 \). As the number of iterations involved in those returns is upper bounded (the bound depending on the localization of \( p_\mu \) and the number \( k_0 \) of initial bridges of the Markov partition), one gets the hyperbolicity of the set \( \Xi_t \).

If the set \( \Sigma_\mu \) is obtained in the step \( n \) of the process (associated to \( \Gamma_\mu^{(n)} \)), one needs to consider other types of returns (defined similarly). The hyperbolicity follows from the fact that \( \Gamma_\mu^{(n)} \) is \( \rho \)-expanding and that the returns involve a uniformly upper bounded number of iterates of \( f_t \).

Finally, it remains to check that the hyperbolic set \( \Xi_t \), \( t = \lambda^n(1+\mu) \), is contained in the homoclinic class \( H(Q, f_t) \) of \( Q \). Consider the periodic point \( P_t \in \Xi_t \) corresponding to the fixed point \( p_\mu \) of the map \( \Gamma_\mu \). We claim that \( P_t \) and \( Q_t \) are homoclinically related. First, by the cycle configuration, one has immediately that \( W^s(P_t, f_t) \) transversely meets \( W^u(Q, f_t) \). The fact that \( W^s(Q, f_t) \) transversally meets \( W^u(P_t, f_t) \) is a consequence of the following remark.

By construction, the unstable manifold of \( p_\mu \) contains the segment \([\beta^{-1}, 1]\). Hence, by the product structure, the unstable manifold of \( P_t \) contains a disk of the form \( \{x_0\} \times [\beta^{-1}, 1] \times [-1, 1] \). Finally, the cycle configuration implies that, for every \( t > 0 \), the stable manifold of \( Q \) intersects transversely any two disk \( \Delta \) of the form \( \Delta = \{a_0\} \times [\beta^{-1}, 1] \times [-1, 1], a_0 \in [-1, 1] \). This last assertion follows just noting that the projection of \( \Delta \) in the central \( \Psi \) direction contains a fundamental domain of the connection \( \gamma = \{0\} \times [0, 4] \times \{0\} \).

7. Linked sets. Persistence of heterodimensional cycles. We prove that, for \( \mu^- < \mu^* \) close to \( \mu^* \), for every \( t \in C_n = [\lambda^n(1+\mu^-), \lambda^n(1+\mu^*)] \), the hyperbolic sets \( \Upsilon_t = \Upsilon_{t, a(\mu^*)} \) in Lemma 6.2 (of index one and contained in the homoclinic class of \( P \)) and \( \Xi_t \) in Section 6.3 (of index two) are homoclinically related. This will end the proof of the theorem.

**Lemma 7.1.** (Gap Lemma, [29] and [30] page 63) Let \( K_1 \) and \( K_2 \) be Cantor sets of \( \mathbb{R} \) with thickness \( \tau_1 \) and \( \tau_2 \), respectively. If \( \tau_1 \cdot \tau_2 > 1 \) then one of the following three alternatives occur: (i) \( K_1 \) is contained in a gap of \( K_2 \); (ii) \( K_2 \) is contained in a gap of \( K_1 \); (iii) \( K_1 \cap K_2 \neq \emptyset \).

Write \( t = \lambda^n(1+\mu) \) and recall that \( \Upsilon_\mu = (\Upsilon_t)^c \) and \( \Sigma_\mu = (\Sigma_t)^c \). By Lemma 6.2 and Proposition 5.3 the product of the thickness of \( \Upsilon_\mu \) and \( \Sigma_\mu \) is bigger than one. Moreover, by construction, the convex hull of \( \Upsilon_\mu \) and \( \Sigma_\mu \) are non-disjoint. Finally,
Lemma 6.2 and Proposition 6.6 imply that each Cantor set is not contained in a gap of the other one (recall the conditions on the size of the gaps of $\Sigma_\mu$ and on the diameters of these two sets). Thus, by Lemma 7.1 $\Sigma_\mu \cap \Upsilon_\mu \neq \emptyset$.

By construction, it is immediate that $W^s(\Upsilon_i) \cap W^w(\Xi_i) \neq \emptyset$. To see that $W^u(\Upsilon_i) \cap W^s(\Xi_i) \neq \emptyset$ take any $y_0 \in (\Upsilon_i \cap \Sigma_\mu$. By Remark 6.6 there are points $X_1 = (x_1, y_0, z_1) \in \Upsilon_1$ and $X_2 = (x_2, y_0, z_2) \in \Xi_1$. By construction,

$$\{(x_1, y_0)\} \times [-1, 1] \subset W^u(X_1) \quad \text{and} \quad [-1, 1] \times \{(y_0, z_2)\} \subset W^s(X_2).$$

So $(x_1, y_0, z_2) \in W^u(\Upsilon_i) \cap W^s(\Xi_i)$ and there is a (heterodimensional) cycle related to $\Upsilon_i$ and $\Xi_i$.

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E-mail address: lodiaz@mat.puc-rio.br
E-mail address: jrocha@fc.up.pt