Isotopies of planar compacta need not extend to isotopies of planar continua

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Abstract

We construct, as a step toward a theory of braids over planar compacta, an isotopy of a planar compactum which cannot be extended to an isotopy of any planar continuum.

1 Introduction

In this paper we address an obstruction to developing a satisfactory theory of braids over planar compacta by establishing a technical sense in which all braids over planar continua are not wild. We then construct an example that is apparently the first of its kind: An isotopy of a planar compactum shown not to be the restriction of an isotopy of any planar continuum.

Given a finite set $F \subset \mathbb{R}^2$ with $|F| = n$, Artin’s pure braid group on $n$-strands $G_n$ is $\pi_1(I(F, \mathbb{R}^2), \text{id}_F)$, where $I(F, \mathbb{R}^2)$ is the space of embeddings of $F$ into $\mathbb{R}^2$. Artin’s braid group on $n$ strands is $\pi_1(Y, F)$ where $Y$ is the space of planar sets of size $n$ in the Hausdorff metric. The simplest cases where $|F| = \infty$ are still being explored. In [1] the author constructs a group of wild braids $B_\infty$ which completes $B_\infty$, Artin’s braid group on infinitely many strands. The full inverse limit of Artin’s pure braid groups is a group $G \subset B_\infty$ consisting of the pure braids on infinitely many strands. However, a typical braid representative $g \in [g] \in G$ is not the restriction of an ambient isotopy of the plane.

In hopes of eliminating this particular brand of wildness, suppose instead $F \subset \mathbb{R}^2$ is the disjoint union of finitely many nonseparating planar continua $X_1 \cup \ldots X_n$, and a braid representative is an isotopy $h_t : F \rightarrow \mathbb{R}^2$ such that $h_0 = \text{id}_F$ and $h_1(F) = F$. Though indirectly settled in case $F$ is PL (see [2]), a thorough understanding of this braid group hinges on the following question:

Problem 1 Can every isotopy starting at $\text{id}_W$ of a planar continuum $W$ be extended to an isotopy of the plane?
We do not settle this challenging problem but prove in section 3 Theorem 4, a weaker technical result which says roughly, that loops in the complements of finite subsets of the continuum $W$ cannot be forcibly stretched by large amounts via isotopy of $W$. We then construct in Corollary 2 a wild isotopy over a convergent sequence which is unextendable to an isotopy of any planar continuum.

The proof of Theorem 4 relies in part on Theorem 3, a result in section 2 which establishes mild conditions under which homotopy classes in $X\setminus A$ are naturally transported via homotopy of $A$. In particular Theorem 4 eliminates reliance on canonical parameterizations of $X\setminus A$.

The final section 4 includes Lemmas useful for the proof of Theorem 6.

2 Theorem 4: a path lifting theorem.

2.1 Motivation

Given an isotopy $h_t : A \to \mathbb{R}^2$ and $[f] \in \pi_1(R^n\setminus A)$, we would like $[f]$ to ‘come along for the ride’ via a natural choice $[f_t] \in \pi_1(R^n\setminus A_t)$. The choice is clear in the special case $A$ is finite since we can canonically extend $h_t$ to an isotopy of the plane. However we would like another scheme for selecting $[f_t]$ since it’s generally impossible to canonically parameterize $X\setminus A_t$ in an arbitrary space $X$. Fortunately we can generally recover $[f_t]$ under weak hypothesis as seen in Theorem 4.

2.2 $\mathcal{B}$ and $\mathcal{E}$

Suppose $X$ is a metric space and $\mathcal{B}$ is a collection of compact subsets of $X$ such that $\forall A \in \mathcal{B}$ there exists a compact $F \subset X$ and a homotopy $H : X\setminus A \times [0,1] \to X\setminus A$ such that $\forall x \in X\setminus A$ $H(x,0) = x$ and $H(x,1) \in F$.

Decide $A_n \to A \in B$ if $A_n \to A$ in the Hausdorff metric and there exists $N$ and homeomorphisms $h_n : X\setminus A \to X\setminus A_n$ for $n \geq N$ such that $h_n \to id$ in the compact open topology. For $\{A,B\} \subset \mathcal{B}$ let $\mathcal{H}(A,B)$ denote the Hausdorff distance.

Fixing a compact metric space $Z$ let $\mathcal{P}(A,f)$ denote the path component of $f$ in $C(Z,X\setminus A)$. Let $\mathcal{E}$ denote the space of ordered pairs $(A,\mathcal{P}(A,f))$ such that $A \in \mathcal{B}$ and $f \in C(Z,X\setminus A)$. Decide that the sequence $(A_n,\mathcal{P}(A_n,f_n)) \to (A,\mathcal{P}(A,f))$ iff $A_n \to A$ in $\mathcal{B}$ and if there exists $g \in \mathcal{P}(A,f)$ and $N$ such that $g \in \mathcal{P}(A_n,f_n)$ $\forall n \geq N$. Define $\Pi_1 : \mathcal{E} \to B$ as $\Pi_1(A,\mathcal{P}(A,f)) = A$. Let $\Pi_2(A,\mathcal{P}(A,f)) = \mathcal{P}(A,f)$. Topologize $\mathcal{B}$ and $\mathcal{E}$ by sequential convergence declaring a set $\mathcal{F}$ to be closed if and only the limit of each convergent sequence in $\mathcal{F}$ is an element of $\mathcal{F}$.

Lemma 2 Suppose $\alpha : [0,1] \to \mathcal{B}$ is continuous $T \in [0,1]$, $\varepsilon > 0$, and $F \subset X\setminus \alpha(T)$ is compact. Then there exists $\gamma > 0$ so that if $|t-T| < \gamma$ then $\mathcal{H}(\alpha(t),\alpha(T)) < \varepsilon$ and there exists a homeomorphism $h_t : X\setminus \alpha(T) \to X\setminus \alpha(t)$ such that $d(h_t(x),x) < \varepsilon \forall x \in F$. 2
Lemma 3 Suppose $\beta : [0, 1] \to \mathcal{E}$ is continuous and $g \in \Pi_2(\beta(T))$. Then there exists $\gamma > 0$ such that if $|T - t| < \delta$ then $g \in \Pi_2(\beta(t))$.

Proof. Let $T \in [0, 1]$. Suppose in order to obtain a contradiction that there does not exist $\gamma > 0$ so that $g \in \Pi_2(\beta(t))$ if $|T - t| < \gamma$. Choose $t_n \to T$ with $g \notin \Pi_2(\beta(t_n)) \forall n$. Since $\beta(t_n) \to \beta(T)$ choose $M$ and $f_T \in \Pi_2(\beta(T))$ so that if $n \geq M$ then $f_T \in \Pi_2(\beta(t_n))$. Let $\beta(t) = (A_t, \mathcal{P}(A_t, f_t))$. Let $H : \mathbb{Z} \times [0, 1] \to \mathbb{X} \setminus A_T$ be any homotopy between $g$ and $f_T$. Since $\text{im}(H)$ is compact choose $\varepsilon > 0$ so that each point of $\text{im}(H)$ is at least $\varepsilon$ from each point of $A_T$. Choose by Lemma 3 $N \geq M$ so that if $n \geq N$ then $\mathcal{H}(A_T, A_{t_n}) < \varepsilon$. Thus $H$ is a homotopy between $f_T$ and $g$ in $\mathbb{X} \setminus A_{t_n}$, Hence $g \in \Pi_2(\beta(t_n))$ for $n \geq N$ and we have a contradiction. ■

Theorem 4 Suppose $\alpha : [0, 1] \to \mathcal{B}$ is continuous, $e \in \mathcal{E}$ and $\Pi_1(e) = \alpha(0)$. Then there exists a unique continuous function $\beta : [0, 1] \to \mathcal{E}$ such that $\alpha = \Pi_1(\beta)$.

Proof. Let $I = \{t \in [0, 1] \text{ such that there exists a continuous function } \beta : [0, t] \to \mathcal{E} \text{ such that } \alpha_{[0,t]} = \Pi_1(\beta) \text{ and } \beta(0) = e\}$. Note $I \neq \emptyset$ since $0 \in I$ and $I$ is connected since $I$ is the union of intervals each of which contains 0. To prove $I$ is open in $[0, 1]$, suppose $T \in I$ and $T \in [0, 1]$. Suppose $\beta : [0, T] \to \mathcal{E}$ satisfies $\alpha_{[0,T]} = \Pi_1(\beta)$. Let $f \in \Pi_2(\beta(T))$. Since $\alpha(T)$ and $\text{im}(f)$ are disjoint compacta choose $\varepsilon > 0$ so that each point of $\alpha(T)$ is at least a distance $\varepsilon$ from each point of $\text{im}(f)$. Choose by Lemma 3 $\delta > 0$ so that if $|T - t| < \delta$ then $\mathcal{H}(\alpha(t), \alpha(T)) < \varepsilon/2$. Hence $\text{im}(f) \subset \mathbb{X} \setminus \alpha(T)$ and $\beta$ can be continuously extended to $[0, T + \delta]$ by defining $\beta(t) = (\alpha(t), \mathcal{P}(\alpha(t), f))$ for $t \in [T, T + \delta]$.

To see that $I$ is closed suppose $[0, T] \subset I$. Choose a compact $F \subset \mathbb{X} \setminus \alpha(T)$, a homotopy $H : \mathbb{X} \setminus \alpha(T) \times [0, 1] \to \mathbb{X} \setminus \alpha(T)$ such that $\forall x \in \mathbb{X} \setminus A \mathcal{H}(x, 0) = x$ and $H(x, 1) \in F$. Since $\alpha(T)$ and $F$ are disjoint compacta choose $\varepsilon > 0$ so that each point of $\alpha(T)$ is at least a distance $\varepsilon$ from each point of $F$. Choose by Lemma 3 $S < T$ and a homeomorphism $h_S : \mathbb{X} \setminus \alpha(T) \to \mathbb{X} \setminus \alpha(S)$ so that if $S \leq t \leq T$ then $\mathcal{H}(\alpha(t), \alpha(S)) < \varepsilon/2$ and $d(h_S(x), x) < \varepsilon/2 \forall x \in F$. Let $\beta : [0, S] \to \mathcal{E}$ satisfy $\Pi_1(\beta) = \alpha_{[0,S]}$. Let $f \in \Pi_2(\beta(S))$. Define for $s \in [0, 1]$ $g^s \in C(Z, \mathbb{X} \setminus \alpha(S))$ via $g^s(z) = h_S H(h_S^{-1} f(z), s)$. Note $g^0 = f$ and hence $g^1 \in \mathcal{P}(\alpha(S), f) = \Pi_2(\beta(S))$. Note $\text{im}(g^1) \subset h_S(F)$. Hence $\text{im}(g^1) \cap \alpha(t) = \emptyset$ for $t \in [S, T]$ Defining $\gamma(t) = (\alpha(t), \mathcal{P}(\alpha(t), g^1))$ for $t \in [S, T]$ creates a continuous lift $\gamma \cup \beta_{[0,S]}$ of $\alpha_{[0,T]}$ and proves $T \in I$. Hence $I = [0, 1]$ and the existence of $\beta$ is established.

For uniqueness suppose $\gamma$ and $\beta$ are two lifts of $\alpha$ such that $\beta(0) = \gamma(0) = e$. Let $J = \{j \in [0, 1] | \gamma(j) = \beta(j)\}$. Note $J$ is nonempty and closed. It suffices to prove $J$ is open. Suppose $T \in J$. Since $\gamma(T) = \beta(T)$ let $g \in \Pi_2(\beta(T)) \cap \Pi_2(\gamma(T))$. Choosing $\gamma$ as in Lemma 3 it follows that $\beta(t) = \gamma(t) = (\alpha(t), \mathcal{P}(\alpha(t), g))$ for $|T - t| < \gamma$. Hence $J$ is open and $\gamma = \beta$. ■
3 Main results: Theorem 6 and Corollary 7

Let $B$ denote the space of planar compacta, topologized as in section 2, each of which is the disjoint union of finitely many nonseparating planar continua. (i.e. $X \in B$ iff $\exists n$ and nonseparating planar continua $X_1, \ldots, X_n$ such that $X = \cup_{i=1}^{n} X_i$ and $X_i \cap X_j = \emptyset$ whenever $i \neq j$).

Fixing $Z = S^1$, define $E$ as in section 3. For $\delta > 0$ let $B(p, \delta) \subset \mathbb{R}^2$ and $\text{int}(B(p, \delta)) \subset \mathbb{R}^2$ denote respectively the closed and open disks centered at $p$ of radius $\delta$. Let $\partial B(p, \delta)$ denote $B(p, \delta) \setminus \text{int}B(p, \delta)$, the circle of radius $\delta$ centered at $p$.

Given $A \subset \mathbb{R}^2$, an isotopy $h_t : A \leftrightarrow \mathbb{R}^2$ is a homotopy such that $h_0 = \text{id}_A$ and $h_t$ is an embedding. We do not require that $h_t(A) = A$. Given an isotopy of a planar continuum $W$, Theorem 6 establishes a technical sense in which small loops in the complements of finite subsets of $W$ cannot be forcibly stretched by large amounts.

Care is required since in general a small isotopy $h_t : A \leftrightarrow \mathbb{R}^2$ of a finite set $A \subset \mathbb{R}^2$ can dramatically stretch complementary loops when $h_t$ is extended to the plane. However if the loop $S \subset \mathbb{R}^2 \setminus A$ encloses only one point of the finite set $A$, then an isotopy of $A$ cannot not stretch $S$ very much. The latter fact is exploited in our proof of Lemma 5 very roughly sketched as follows:

Assuming $W$ is nonseparating and the isotopy $h_t : W \leftrightarrow \mathbb{R}^2$ fixes $p \in W$ throughout consider $X = B(p, \varepsilon_1) \cup (B(p, \varepsilon_5) \cap W)$ with $Y_1 \subset X$ the component containing $p$. Take a simple closed curve $S \subset \mathbb{R}^2 \setminus X$ such that $S$ is inside $X$ and the remaining components $Y_2, \ldots, Y_6$ of $X$ lie outside $S$. Let $p = y_1$ and for $i \geq 2$ pick $y_i \in Y_i$ such that $|y_i - p| = \varepsilon_5$. Let $A = \{y_1, y_2, \ldots, y_6\}$. Let $H_t : \mathbb{R}^2 \setminus X_1 \to \mathbb{R}^2 \setminus A$ be an $\varepsilon$ homeomorphism mapping a neighborhood of $Y_1^t$ to a neighborhood of $y_1^t$. Now suppose $F \subset W$ is finite and $f : S^1 \to \mathbb{R}^2 \setminus F$ satisfies $\text{im}(f) \subset B(p, \varepsilon_1)$. Observe $S_t = H_t^{-1}(\partial B(p, \varepsilon_3))$ has small diameter and encloses $\text{im}(f_t)$. Hence $\text{im}(f_t)$ has small diameter.

Lemma 5 Suppose $W \subset \mathbb{R}^2$ is a continuum, $h_t : W \leftrightarrow \mathbb{R}^2$ is an isotopy, $p \in W$, $h_t(p) = p \forall t \in [0, 1]$ and $\varepsilon > 0$. Suppose $\forall n \geq 1$ $F_n \subset W$ is finite, $f_n : S^1 \to \mathbb{R}^2 \setminus F_n$ is an embedding, $\text{im}(f_n) \to \{p\}$ in the Hausdorff metric, $\alpha_n : [0, 1] \to B$ is defined as $\alpha_n(t) = h_t(F_n)$ and $\beta_n : [0, 1] \to E$ is the (unique) lift of $\alpha_n$ such that $\Pi \beta_n(0) = \mathcal{P}(F_n, f_n)$. Then there exists $N$ such that if $n \geq N$ and $T \in [0, 1]$ then there exists $f^n_T \in \Pi \beta_n(T)$ such that $\text{diam}(\text{im}(f^n_T)) < \varepsilon$.

Proof. Note first that $\beta_n$ is well defined by Theorem 4. By uniform continuity of the isotopy $h$ over $[0, 1] \times W$, and by injectivity of $h_t$, choose $0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{10} = \varepsilon$ such that $\forall \{s, t\} \subset [0, 1], \forall \{x, y\} \subset W$ if $|h_t(x) - h_t(y)| \leq \varepsilon_n$ then $|h_t(x) - h_t(y)| < \varepsilon_{n+1}$ and such that $\varepsilon_n < \varepsilon_{n+1}/2$. Note if $W \subset B(p, \varepsilon_3)$ the theorem is proved since $h_t(W) \subset \text{int}(B(p, \varepsilon_6)) \subset B(p, \varepsilon)$ and we can canonically deform all loops into $B(p, \varepsilon_6)$ via radial contraction outside $\mathbb{R}^2 \setminus \text{int}(B(p, \varepsilon_6))$. So we assume $W \setminus B(p, \varepsilon_6) \neq \emptyset$. Choose $N$ so that if $n \geq N$ then $\text{im}(f_n) \subset \text{int}(B(p, \varepsilon_1))$. Suppose $n \geq N$. Let $X = B(p, \varepsilon_1) \cup F_n \cup (B(p, \varepsilon_5) \cap W)$. Let $P = X \cup V$ where $V$ is the union of the bounded complementary domains of $X$. For clarity note
1. Each component $P_i$ of $P$ is a nonseparating planar continuum of diameter no greater than $2\varepsilon_5$.

2. (By Lemma 3) each component of $P_i$ of $P$ contains a point $y_i$ such that $|y_i - p| \geq \varepsilon_5$.

Let $Y \subset P$ consist of those components $Y_1,...,Y_K$ of $P$ such that $Y_i \cap F_n \neq \emptyset$ indexed so that $p \in Y_1$. Since $2\varepsilon_5 < \varepsilon_6$ each component of $h_t(Y)$ has diameter no more than $\varepsilon_7$. Hence we may choose $\forall t \in [0,1]$ a collection of disjoint closed topological disks $D^t_1,...,D^t_K$ such that $\text{diam}(D^t_i) < \varepsilon_8$ and $\text{int}(D^t_i)$ contains exactly one component of $h_t(Y)$, namely $h_t(Y_i)$. Choose $y_i \in Y_i$ such that $|p - y_i| \geq \varepsilon_5$ for $i \geq 2$. Let $y_1 = p$. Let $A = \{y_1, y_2,...,y_K\}$. Choose $\forall t \in [0,1]$ a surjective ‘decomposition’ map $H_t : R^2 \rightarrow R^2$ such that, preserving orientation, $H_t$ maps $R^2 \setminus h_t(Y)$ homeomorphically onto $R^2 \setminus h_t(A)$, $H_t$ maps $h_t(Y)$ onto $\{h_t(y_i)\}$ and $H_t$ fixes $R^2 \setminus (\cup_{i=1}^K \text{int}(D^t_i))$ pointwise. Let $r : S^1 \rightarrow \partial(B(p, \varepsilon_3))$ be any homeomorphism. Let $g^*_{n} = H_t^{-1}(r)$. Note $g^*_{n}$ is well defined since $h_t(A) \cap \partial B(p, \varepsilon_3) = \emptyset$. Note $g^*_{n} \in \mathcal{P}(A, r)$ since $A \cap \partial B(p, \varepsilon_3) = \{p\}$. Hence $Y_1$ is contained within the bounded complementary domain of $\text{im}(g^*_{n})$. Moreover since $B(p, \varepsilon_3) \subset Y_1$ it follows that $\text{im}(f_n)$ is contained within the bounded complementary domain of $\text{im}(g^*_{n})$. By Lemma 4 the map $\beta_n^2 : [0,1] \rightarrow \mathcal{E}$ defined as $\beta_n^2(t) = (h_t(Y), \mathcal{P}(h_t(Y), g^*_{n}))$ is continuous. Hence there exists $f_n^T \in \Pi_2\beta_n(T)$ such that $\text{im}(f_n^T)$ is contained inside the bounded complementary domain of $\text{im}(g^*_{n})$. In particular $\text{diam}(f_n^T) \leq \text{diam}(g^*_{n}) < 2\varepsilon_8 < \varepsilon_9 < \varepsilon$.

**Theorem 6** Suppose $h_t : W \hookrightarrow R^2$ is an isotopy of the continuum $W \subset R^2$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $F \subset W$ is finite, $f : S^1 \rightarrow R^2 \setminus F$ is an embedding $\alpha : [0,1] \rightarrow B$ is defined as $\alpha(t) = h_t(F)$ and $\beta : [0,1] \rightarrow \mathcal{E}$ is the (unique) lift of $\alpha$ such that $\beta(0) = (F, \mathcal{P}(F,f))$ then $\forall T \in [0,1]$ there then exists $f^T \in \Pi_2\beta(T)$ such that $\text{diam}(\text{im}(f^T)) < \varepsilon$.

**Proof.** We reduce the problem to Lemma 4 as follows. If Theorem 4 were false we could construct an isotopy $h_t : W \hookrightarrow R^2$, $\varepsilon > 0$, finite sets $F_n \subset W$, embeddings $f_n : S^1 \rightarrow R^2 \setminus F_n$ such that $\text{diam}(\text{im}(f_n)) \rightarrow 0$ and so that each element of $\Pi_2(\beta_n(t_n))$ has diameter greater than $\varepsilon$, where $\beta_n$ is the unique lift guaranteed by Theorem 4 of $\alpha_n : [0,1] \rightarrow B$ defined as $\alpha_n(t) = h_t(F_n)$. In such a counterexample each bounded complementary domain of $\text{im}(f_n)$ must include at least two points of $F_n \subset W$ to ensure each element of $\Pi_2(\beta_n(t_n))$ has diameter at least $\varepsilon$. Since $W$ is compact, passing to a subsequence if necessary, we may further assume $\text{im}(f_n) \rightarrow p$ in the Hausdorff metric. Finally if $h_t$ is a counterexample to the theorem then the isotopy $g_t : W \hookrightarrow R^2$ defined as $g_t(w) = h_t(w) + p - h_t(p)$ is also a counterexample. Hence we may further assume that $h_t(p) = p \forall t \in [0,1]$. Now apply Lemma 4 to conclude that no such counterexample exists.

**Corollary 7** There exists an isotopy $h_t$ of a compactum $W \subset R^2$ such that $h_t$ cannot be extended to an isotopy of any planar continuum.
Proof. Consider $W = \{0\} \cup \{1, 1/2, 1/3\} \subset \mathbb{R}^2$ and the following isotopy $h_t : A \hookrightarrow \mathbb{R}^2$. For $n \geq 2$ let the point $1/n$ perform a counterclockwise orbit around the point $\frac{1}{n}$ for $t \in [1/n, 1/n - 1]$ while all other points remain fixed. Taking $F_n = \{1/n, 1/n - 1, \ldots, 1/2, 1\}$, Let $f_n : S^1 \hookrightarrow \mathbb{R}^2 \setminus F$ be an embedding onto the boundary of a small convex set containing $\left(\frac{1}{n + 1}, \frac{1}{n} \right)$. Define $\alpha : [0, 1] \to B$ as $\alpha(t) = h_t(F_n)$. Now observe that each point of $\Pi_2 \beta(1)$ has diameter at least $1 - \frac{1}{n - 1}$ where $\beta : [0, 1] \to E$ is the unique lift of $\alpha$ such that $\beta(0) = (F_n, \mathcal{P}(F_n, f))$. Hence, for any continuum $W$ such that $A \subset W$, we cannot extend $h_t$ to an isotopy of $W$, since fixing $\varepsilon < 1$ there does not exist $N$ guaranteed by Theorem 3.

4 Lemmas

Lemma 8 Suppose $X$ is a metric continuum, $p \in X$ and $\overline{B(p, 1)} \neq X$ then each component of $B(p, 1) \cap X$ contains a point whose distance from $p$ is exactly 1. ($B(p, 1)$ denotes the closed metric ball of radius 1 centered at $p$.)

Proof. To obtain a contradiction let $Y$ be a component of $B(p, 1) \cap X$ violating the conclusion. Since $X$ is compact, the components of $B(p, 1) \cap X$ are exactly the quasicomponents. Let $U$ and $V$ be nonempty open sets separating $B(p, 1) \cap X$ such that $Y \subset U$. Now map the connected space $X$ onto the disconnected space $U \cup \{\infty\}$ via $f(x) = x$ if $x \in U$ and $f(x) = \infty$ otherwise, and we have a contradiction.

Lemma 9 Suppose $Y = \bigcup_{i=1}^{N} Y_i$ is the disjoint union of finitely many nonseparating planar continua $Y_1, \ldots, Y_N$ and $A \subset Y$ contains exactly one point from each continuum $Y_i$. Then $F_Y$ is nonempty and path connected where, in the compact open topology, $F_Y$ is the space of maps $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $h$ is an orientation preserving homeomorphism between $\mathbb{R}^2 \setminus Y$ and $\mathbb{R}^2 \setminus A$, $h$ maps $Y$ onto $A$, and there exists a sequence of disjoint closed topological disks $D_1, D_2, \ldots, D_N$ (which may vary with $h$) such that $Y_i \subset \text{int}(D_i)$ and $h$ fixes $A \cup (\mathbb{R}^2 \setminus (\bigcup_{i=1}^{N} \text{int}(D_i)))$ pointwise.

Proof. Fixing $f \in F_Y$ the map $H : F_Y \rightarrow F_A$ defined as $H(h) = hf^{-1}$ is a homeomorphism. Thus it suffices to show $F_A$ is path connected. Take any $g \in F$ and disjoint closed disks $D_i$ outside of which $g$ is pointwise fixed and isotop to $id$ via the ‘Alexander isotopy’ on each $D_i$.

Lemma 10 Suppose $\alpha : [0, 1] \rightarrow B$ is continuous and $B$ is the collection of planar compacta $Y$ such that $Y$ is the disjoint union of finitely many nonseparating planar continua $Y_1 \cup \ldots \cup Y_n$. Suppose $\gamma : [0, 1] \rightarrow B$ is continuous and $\gamma(t)$ contains exactly one point from each component of $\alpha(t)$. Suppose $\forall t \in [0, 1]$ $H_t : \mathbb{R}^2 \setminus \alpha(t) \rightarrow \mathbb{R}^2 \setminus \gamma(t)$ is a homeomorphism such that $H_t$ fixes pointwise an open set $U_t$ such that $\mathbb{R}^2 \setminus U_t$ is a collection of disjoint closed topological disks $D^1_t, \ldots, D^n_t$ such that $\text{int}(D^i_t)$ contains exactly one component of $\alpha(t)$. Suppose $Z = S^1$ and $r : S^1 \rightarrow \mathbb{R}^2$ is continuous and $\forall t \in [0, 1]$ $\text{im}(r) \subset \mathbb{R}^2 \setminus \gamma(t)$. Then the map $\beta : [0, 1] \rightarrow E$ defined as $\beta(t) = (\alpha(t), \mathcal{P}(\alpha(t), H_t^{-1}r))$ is continuous.
Proof. Suppose \( T \in [0,1] \). By Lemma 6, \( \beta(T) \) does not depend on our choice of \( D^T_1, \ldots, D^T_n \) as long as \( \text{int}(D^T_i) \) contains exactly one component of \( \alpha(T) \). Thus for \( t \) sufficiently close to \( T \), \( \beta(t) = (\alpha(t), \mathcal{P}(\alpha(t), H^{-1}_T r)) \) and in particular \( H^{-1}_T r \in \Pi_2 \beta(t) \). Thus \( \beta \) is continuous.

References

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