Some new hybrid power mean formulae of trigonometric sums

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Abstract
We apply the analytic method and the properties of the classical Gauss sums to study the computational problem of a certain hybrid power mean of the trigonometric sums and to prove several new mean value formulae for them. At the same time, we also obtain a new recurrence formula involving the Gauss sums and two-term exponential sums.

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1 Introduction
For any integer $m$ and odd prime $p \geq 3$, the cubic Gauss sums $A(m,p) = A(m)$ are defined as follows:

$$A(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right),$$

where, as usual, $e(y) = e^{2\pi i y}$.

We found that several scholars studied the hybrid mean value problems of various trigonometric sums and obtained many interesting results. For example, Chen and Hu [1] studied the computational problem of the hybrid power mean

$$S_k(p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right)^k \cdot \left| \sum_{c=1}^{p-1} e\left(\frac{mc + \bar{c}}{p}\right) \right|^2,$$

where $\bar{c}$ denotes the multiplicative inverse of $c$ mod $p$, that is, $c \cdot \bar{c} \equiv 1 \mod p$.

For $p \equiv 1 \mod 3$, they proved an interesting third-order linear recurrence formula for $S_k(p)$.
Li and Hu [2] studied the computational problem of the hybrid power mean
\[
\sum_{b=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left( \frac{ba^3}{p} \right) \right|^2 \cdot \left| \sum_{c=1}^{p-1} e\left( \frac{bc + c}{p} \right) \right|^2
\]
and proved an exact computational formula for (1).

Zhang and Zhang [3] proved the identity
\[
\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left( \frac{ma^3 + na}{p} \right) \right|^4 = \begin{cases} 
2p^3 - p^2 & \text{if } 3 \nmid p - 1, \\
2p^3 - 7p^2 & \text{if } 3 \mid p - 1.
\end{cases}
\]

Other related contents can also be found in [4–12], which will not be repeated here.

In this paper, inspired by [1] and [2], we consider the following mean value:
\[
H_k(c, p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{ma^3}{p} \right) \right) \left( \sum_{b=0}^{p-1} e\left( \frac{mb + b}{p} \right) \right)^{3k}.
\]

We do not know whether there exists a precise computational formula for (2), where \( c \) is any integer with \((c, p) = 1\), and \( p \equiv 1 \mod 3 \).

Actually, there also exists a third-order linear recurrence formula of \( H_k(c, p) \) for all integers \( k \geq 1 \) and \( c \). But for some integers \( c \), the initial value of \( H_k(c, p) \) is very simple, whereas for other \( c \), the initial value of \( H_k(c, p) \) is more complex. So a satisfactory recursive formula for \( H_k(c, p) \) is not available.

The main purpose of this paper is using an analytic method and the properties of classical Gauss sums to give an effective calculation method for \( H_k(c, p) \) with some special integers \( c \). We will prove the following two theorems.

**Theorem 1** Let \( p \) be a prime with \( p \equiv 1 \mod 3 \). If 3 is not a cubic residue \( \mod p \), then we have
\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{3ma^3}{p} \right) \right)^3 \cdot \left( \sum_{b=0}^{p-1} e\left( \frac{mb^3 + b}{p} \right) \right)^3 = 3p^2 + dp^2,
\]
\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{ma^3}{p} \right) \right) \cdot \left( \sum_{b=0}^{p-1} e\left( \frac{mb^3 + b}{p} \right) \right)^3 = p^3(3p - 5d),
\]
and
\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{3ma^3}{p} \right) \right)^3 \cdot \left( \sum_{b=0}^{p-1} e\left( \frac{mb^3 + b}{p} \right) \right)^3 = p^3(5dp + 9p - d^2).
\]

**Theorem 2** Let \( p \) be an odd prime with \( p \equiv 1 \mod 3 \). If 3 is a cubic residue \( \mod p \), then for any integer \( k \geq 3 \), we have the third-order linear recurrence formula
\[
H_k(1, p) = 3pH_{k-2}(1, p) + dpH_{k-3}(1, p),
\]
where the first three terms are \( H_0(1, p) = 2p^2 - pd \), \( H_1(1, p) = p^2(d - 6) \), and \( H_2(1, p) = p^2(6p - 5d) \).
Some notes: First, in Theorem 1, if \((3, p - 1) = 1\), then the question we are discussing is trivial, because in this case, we have
\[
\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) = \sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) = 0.
\]
Second, in the first and third formulas of Theorem 1, we take \(c = 3\) (and \(c = 1\) in the second formula). These are all for getting the exact value of the mean value. Otherwise, the results will not be pretty.

2 Several lemmas
To complete the proofs of our theorems, several lemmas are essential. Hereafter, we will use related properties of the classical Gauss sums and the third-order character \(\mod p\), all of which can be found in books concerning elementary number theory or analytic number theory, such as [13] and [14]. First we have the following:

**Lemma 1** Let \(p\) be a prime with \(p \equiv 1 \mod 3\). Then for any third-order character \(\psi \mod p\), we have the identity
\[
\sum_{m=1}^{p-1} \psi(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right)^3 = \overline{\psi}(3)p\tau^2(\overline{\psi}) - 3p\tau(\psi).
\]

**Proof** First, applying the trigonometric identity
\[
\sum_{m=1}^q e\left(\frac{nm}{q}\right) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{if } q \nmid n \end{cases}
\]
and noting that \(\psi^3 = \chi_0\), the principal character \(\mod p\), we have
\[
\sum_{m=1}^{p-1} \psi(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right)^3
= \sum_{m=1}^{p-1} \psi(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right)^2
\]
\[
+ \sum_{m=1}^{p-1} \psi(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right) \left(\sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right)
\]
\[
= 2 \sum_{m=1}^{p-1} \psi(m) \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) + \sum_{m=1}^{p-1} \psi(m) \left(\sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right)^2
\]
\[
+ \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=1}^{p-1} \overline{\psi}(a^3 + b^3 + c^3) e\left(\frac{a + b + c}{p}\right)
\]
\[
= -2\tau(\psi) + \tau(\psi) \sum_{a=1}^{p-1} \overline{\psi}(a^3 + 1) \sum_{b=1}^{p-1} e\left(\frac{b(a + 1)}{p}\right)
\]
\[ + \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \overline{\psi}(a^3 + b^3 + 1) \sum_{c=1}^{p-1} e\left( \frac{c(a + b + 1)}{p} \right) \]

\[ = -2\tau(\psi) - \tau(\psi) \sum_{a=1}^{p-1} \bar{\psi}(a^3 + 1) + p\tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \overline{\psi}(a^3 + b^3 + 1) \]

\[ - \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \overline{\psi}(a^3 + b^3 + 1) \]

\[ = -2\tau(\psi) - \tau(\psi) \sum_{a=1}^{p-1} \bar{\psi}(a^3 + 1) + p\tau(\psi) \sum_{a=0}^{p-1} \overline{\psi}(a^3 - (a + 1)^3 + 1) \]

\[ - \tau(\psi) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \overline{\psi}(a^3 + b^3 + 1). \quad (4) \]

Noting that \( \psi^2 = \overline{\psi} \) and \( \tau(\psi)\tau(\overline{\psi}) = p \), from the properties of Gauss sums we have

\[ \sum_{a=1}^{p-1} \overline{\psi}(a^3 + 1) = \sum_{a=1}^{p-1} \overline{\psi}(a + 1)(1 + \psi(a) + \bar{\psi}(a)) \]

\[ = \sum_{a=1}^{p-1} \overline{\psi}(a + 1) + \sum_{a=1}^{p-1} \overline{\psi}(1 + a) + \sum_{a=1}^{p-1} \overline{\psi}(a^2 + a) \]

\[ = -2 + \frac{1}{\tau(\psi)} \sum_{b=1}^{p-1} \psi(b) \sum_{a=1}^{p-1} \overline{\psi}(a) e\left( \frac{b(a + 1)}{p} \right) \]

\[ = -2 + \frac{\tau^2(\overline{\psi})}{\tau(\psi)} = -2 + \frac{\tau^2(\overline{\psi})}{p}, \quad (5) \]

\[ \sum_{a=0}^{p-1} \overline{\psi}(a^3 - (a + 1)^3 + 1) = \sum_{a=0}^{p-1} \overline{\psi}(-3a(a + 1)) \]

\[ = \overline{\psi}(3) \sum_{a=1}^{p-1} \overline{\psi}(a(a + 1)) = \frac{\overline{\psi}(3)\tau^3(\overline{\psi})}{p}. \quad (6) \]

Since \( \psi \) is a third-order character mod \( p \), for any integer \( c \) with \( (c,p) = 1 \), from the properties of the classical Gauss sums we have

\[ \sum_{a=0}^{p-1} e\left( \frac{ca^3}{p} \right) = 1 + \sum_{a=1}^{p-1} (1 + \psi(a) + \overline{\psi}(a)) e\left( \frac{ca}{p} \right) = \overline{\psi}(c)\tau(\psi) + \psi(c)\tau(\overline{\psi}). \quad (7) \]

Applying (7), we have

\[ \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \overline{\psi}(a^3 + b^3 + 1) \]

\[ = \frac{1}{\tau(\psi)} \sum_{c=1}^{p-1} \psi(c) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} e\left( \frac{ca^3 + cb^3 + c}{p} \right) \]

\[ = \frac{1}{\tau(\psi)} \sum_{c=1}^{p-1} \psi(c) e\left( \frac{c}{p} \right) \left( \sum_{a=0}^{p-1} e\left( \frac{ca^3}{p} \right) \right)^2. \]
\[
\tau = \frac{1}{\tau(\psi)} \sum_{c=1}^{p-1} \psi(c) e\left(\frac{c}{p}\right) (\psi(c) \langle c \rangle^2 (\psi) + 2p + \overline{\psi}(c) \langle c \rangle^2 (\overline{\psi})) = \tau(\psi) \tau(\overline{\psi}) + 2p - \frac{\tau^3(\psi)}{p} = 3p - \frac{\tau^3(\psi)}{p}. \tag{8}
\]

Combining (4), (5), (6), and (8), we have the identity
\[
\sum_{m=1}^{p-1} \psi(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^3 = \overline{\psi}(3)p \tau^2(\psi) - 3p \tau(\psi).
\]

This proves Lemma 1. \(\square\)

**Lemma 2** Let \(p\) be a prime with \(p \equiv 1 \text{ mod } 3\), and let \(\psi\) be any third-order character \(\text{mod } p\). Then we have
\[
\tau^3(\psi) + \tau^3(\overline{\psi}) = dp,
\]
where \(\tau(\psi)\) denotes the classical Gauss sums, and \(d\) is uniquely determined by \(4p = d^2 + 27b^2\) and \(d \equiv 1 \text{ mod } 3\).

**Proof** See [4] or [9]. \(\square\)

**Lemma 3** Let \(p\) be a prime with \(p \equiv 1 \text{ mod } 3\). Then we have the identity
\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^3 = 2p^2 - pd.
\]

**Proof** Since the congruence equation \(x^3 + 1 \equiv 0 \text{ mod } p\) has three solutions in a reduced residue system \(\text{mod } p\), from (3) we have
\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^3
= \sum_{m=0}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^3
= \sum_{m=0}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2 + \sum_{m=0}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right) \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p}\right)
= p + 2 \sum_{m=0}^{p-1} \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) + \sum_{m=0}^{p-1} \left( \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right)^2
+ \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=1}^{p-1} \sum_{m=0}^{p-1} e\left(\frac{mc^3(a^3 + b^3 + 1) + c(a + b + 1)}{p}\right)
= p + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=0}^{p-1} e\left(\frac{mb^3(a^3 + 1) + b(a + 1)}{p}\right).
\]
+ p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=1}^{p-1} \frac{c(a + b + 1)}{p} \\
= p + p(p - 1) - 2p + p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 - p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1. \quad (9)

It is clear that the conditions \(a^3 + b^3 + 1 \equiv 0 \mod p\) and \(a + b + 1 \equiv 0 \mod p\) (\(0 \leq a, b \leq p - 1\)) imply \(a + (a + 1) \equiv 0 \mod p\) and \(a + b + 1 \equiv 0 \mod p\), or \((a, b) = (0, p - 1)\) and \((a, b) = (p - 1, 0)\).

So we have

\[ p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 = 2p^2. \quad (10) \]

From (3), (7), Lemma 2, and the properties of Gauss sums we have

\[ p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 = \sum_{m=0}^{p-1} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} e \left( \frac{ma^3 + b^3 + 1}{p} \right) \]

\[ = p^3 + \sum_{m=1}^{p-1} e \left( \frac{m}{p} \right) \sum_{a=0}^{p-1} e \left( \frac{ma^3}{p} \right)^2 \]

\[ = p^2 + \sum_{m=1}^{p-1} e \left( \frac{m}{p} \right) \left( \overline{\psi}(m) \tau(\psi) + \psi(m) \tau(\overline{\psi}) \right)^2 \]

\[ = p^2 + \sum_{m=1}^{p-1} e \left( \frac{m}{p} \right) \left( \psi(m) \tau^2(\psi) + 2p + \overline{\psi}(m) \tau^2(\overline{\psi}) \right) \]

\[ = p^2 + \tau^3(\psi) + 2p + \tau^3(\overline{\psi}) = p^2 - 2p + dp. \quad (11) \]

Combining (9), (10), and (11), we have the identity

\[ \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3}{p} + a \right) \right)^3 = 2p^2 - pd. \]

This proves Lemma 3.

\[ \square \]

3 Proofs of the theorems

We achieve our main results in this part. First, we prove Theorem 1. For any integer \(m\) with \((m, p) = 1\), from (7) and Lemma 2 we have

\[ A^3(3m) = \left( \sum_{a=0}^{p-1} e \left( \frac{3ma^3}{p} \right) \right)^3 = \left( \overline{\psi}(3m) \tau(\psi) + \psi(3m) \tau(\overline{\psi}) \right)^3 \]

\[ = \tau^3(\psi) + \tau^3(\overline{\psi}) + 3p \left( \overline{\psi}(3m) \tau(\psi) + \psi(3m) \tau(\overline{\psi}) \right) = dp + 3pA(3m). \quad (12) \]
Applying (7) and Lemmas 1 and 2, we have

\[
\sum_{m=1}^{p-1} A(3m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^3
= \sum_{m=1}^{p-1} \left( \psi(3m) \tau(\psi) + \psi(3m) \tau(\overline{\psi}) \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^3
= \psi(3) \tau(\psi) \left( \psi(3) \tau(\psi) \right) + \psi(3) \tau(\overline{\psi}) \left( \psi(3) \tau(\overline{\psi}) \right) - 3p \tau(\psi)
= p \left( r^3(\psi) + r^3(\overline{\psi}) \right) - 3p^2 \left( \psi(3) + \overline{\psi}(3) \right) = dp^2 + 3p^2 - 3p^2 \left( 1 + \psi(3) + \overline{\psi}(3) \right)
= p^2(d + 3),
\] (13)

where we have used the identity \( 1 + \psi(3) + \overline{\psi}(3) = 0 \).

Applying Lemmas 1, 2, and 3 and (7), we have

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^2 \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^3
= \sum_{m=1}^{p-1} \left( \psi(m) \tau(\psi) + \psi(m) \tau(\overline{\psi}) \right)^2 \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^3
= 2p \left( 2p^2 - dp \right) + r^2(\psi) \left( \psi(3) \tau^2(\overline{\psi}) \right) - 3p \tau(\psi)
+ r^2(\overline{\psi}) \left( \psi(3) \tau^2(\psi) - 3p \tau(\psi) \right)
= 2p^2(2p - d) + \left( \psi(3) + \overline{\psi}(3) \right)p^3 - 3p \left( r^3(\psi) + r^3(\overline{\psi}) \right)
= p^2(3p - 5d).
\] (14)

Applying Lemmas 1, 2, and 3 and (12), we have

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{3ma^3}{p} \right) \right)^3 \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^3
= \sum_{m=1}^{p-1} \left( \psi(3m) \tau(\psi) + \psi(3m) \tau(\overline{\psi}) \right)^3 \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^3
= dp \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^3 + 3p \sum_{m=1}^{p-1} A(3m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^3
= dp \left( 2p^2 - pd \right) + 3p \left( 3p^2 + dp^2 \right) = p^2 \left( 5dp + 9p - d^2 \right).
\] (15)

Now Theorem 1 follows from (13), (14), and (15).

If \( p \equiv 1 \mod 3 \) and 3 is a cubic residue mod \( p \), then \( \psi(3) = \overline{\psi}(3) = 1 \). From Lemma 3 we have

\[
H_0(1, p) = 2p^2 - pd.
\] (16)
From (7) and Lemmas 1 and 2 we have

\[ H_1(1, p) = \tau(\psi)(pr^2(\psi) - 3p\tau(\psi)) + \tau(\psi)(pr^2(\psi) - 3p\tau(\psi)) \]
\[ = p(\tau^3(\psi) + \tau^3(\psi)) - 6p^2 = dp^2 - 6p^2. \quad (17) \]

From (7) and Lemmas 1, 2, and 3 we also have

\[ H_2(1, p) = 2pH_0(1, p) + \tau^2(\psi)(pr^2(\psi) - 3p\tau(\psi)) + \tau^2(\psi)(pr^2(\psi) - 3p\tau(\psi)) \]
\[ = 2p^2(2p - d) + 2p^3 - 3p(\tau^3(\psi) + \tau^3(\psi)) = p^2(6p - 5d). \quad (18) \]

If \( k \geq 3 \), then applying (12), we have

\[ H_k(1, p) = \sum_{m=1}^{p-1} A^k(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^3 \]
\[ = \sum_{m=1}^{p-1} A^{k-3}(m)(dp + 3pA(m)) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right)^3 \]
\[ = 3pH_{k-2}(1, p) + dpH_{k-3}(1, p). \quad (19) \]

Now Theorem 2 follows from (16), (17), (18), and (19).

This completes the proofs of all our results.

4 Conclusion

The main work of this paper includes two theorems. In Theorem 1, we obtained some exact values of (2) when \( k = 1, 2, \) and 3. In Theorem 2, we showed that \( H_k(1, p) \) satisfies an interesting third-order linear recurrence formula. These works not only profoundly reveal the regularity of a certain hybrid power mean of the trigonometric sums, but also provide some new ideas and methods for further study of such problems.

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Authors’ contributions

Both authors have equally contributed to this work. Both authors read and approved the final manuscript.

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