Bäcklund-type superposition and free particle $n$-susy partners

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Abstract

The higher order susy partners of Schrödinger Hamiltonians can be explicitly constructed by iterating a nonlinear difference algorithm coinciding with the Bäcklund superposition principle used in soliton theory. As an example, it is applied in the construction of new higher order susy partners of the free particle potential, which can be used as a handy tool in soliton theory.

Recent studies confirm that the higher order supersymmetric (susy) partners of Schrödinger Hamiltonians are most easily constructed by a simple algebraic tool named intertwining technique [1]. One of the keys of this method is an algebraic nonlinear expression which links solutions of different Riccati equations (see, e.g. [2–4]). In a previous paper [3], we have studied the application of this method to the free particle potential. The ‘building blocks’ of some of the resulting potentials are the well known soliton solutions of the Korteweg-de Vries (KdV) equation: $\kappa^2\text{sech}^2[\kappa(x-a)]$ and $\kappa^2\text{csch}^2[\kappa(x-a)]$. In this work we shall sketch the main steps of the approach in order to present some of the potentials derived in [3].

First, consider the intertwining relationship $H_1A_1 = A_1H_0$, where the intertwiner $A_1$ is the first order differential operator $A_1 = \frac{d}{dx} + \beta_1(x, \epsilon)$. All the available information concerning the Hamiltonians $H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + V_0(x)$ and $H_1 = -\frac{1}{2} \frac{d^2}{dx^2} + V_1(x, \epsilon)$ is encoded in the beta function, which satisfies the Riccati equation

$$-\beta_1'(x, \epsilon) + \beta_1^2(x, \epsilon) = 2[V_0(x) - \epsilon].$$

(1)

The arbitrary integration constant $\epsilon$ plays the role of a factorization energy. It is very simple to check that the potentials are related by the first order susy relationship

$$V_1(x, \epsilon) = V_0(x) + \beta_1'(x, \epsilon).$$

(2)

Equations (1) and (2) are necessary and sufficient conditions for the Hamiltonians to be factorized as $H_0 - \epsilon = (1/2) A_1^\dagger A_1$, and $H_1 - \epsilon = (1/2) A_1 A_1^\dagger$. Suppose now that $V_0(x)$ is a known solvable potential with eigenvalues $E_n$ and eigenfunctions $\psi_n$, $n = 0, 1, 2, \ldots$. Let us assume that we have found a general solution of (1) for a given factorization energy $\epsilon_1 \neq E_n, \forall n$. Then, the potential $V_1(x, \epsilon_1)$ is also given [1–3]. The iteration of this procedure
starts by considering now $V_1(x, \varepsilon_1)$ as the known solvable potential and looking for a new one $V_2(x, \varepsilon_1, \varepsilon)$ satisfying the second order susy relationship

$$V_2(x, \varepsilon_1, \varepsilon) = V_1(x, \varepsilon_1) + \beta'_2(x, \varepsilon_1, \varepsilon).$$

(3)

Therefore, the new beta function must fulfill the Riccati equation

$$- \beta'_2(x, \varepsilon_1, \varepsilon) + \beta''_2(x, \varepsilon_1, \varepsilon) = 2[V_1(x, \varepsilon_1) - \varepsilon],$$

(4)

where $\varepsilon$ is again an arbitrary factorization energy. The corresponding solution is given by

$$\beta_2(x, \varepsilon_1, \varepsilon) = -\beta_1(x, \varepsilon_1) - \frac{2(\varepsilon_1 - \varepsilon)}{\beta_1(x, \varepsilon_1) - \beta_1(x, \varepsilon)}.$$  

(5)

The finite difference expression (5) is a nonlinear superposition of two general solutions of (4), one for each factorization energy $\varepsilon_1$ and $\varepsilon$, transforming equation (4) into (3) by the change of $V_0(x)$ by $V_1(x, \varepsilon_1)$, and $\beta_1(x, \varepsilon)$ by $\beta_2(x, \varepsilon_1, \varepsilon)$. This transformation can be used to link the higher order susy partners of $V_0(x)$ with the first order superpotentials $\beta_1(x, \varepsilon)$, just by solving (4) for different values of the factorization energy $\varepsilon$. For instance, providing $n$ different general solutions of (4), one for each $\varepsilon_k$, $k = 1, 2, ..., n$, we are able to iterate $n - 1$ times the algorithm (5) acquiring a new beta function in each step, given by

$$\beta_k(x, \varepsilon_k) = -\beta_{k-1}(x, \varepsilon_{k-1}) - \frac{2(\varepsilon_{k-1} - \varepsilon_k)}{\beta_{k-1}(x, \varepsilon_{k-1}) - \beta_{k-1}(x, \varepsilon_k)}, \quad k = 2, 3, ... n.$$  

(6)

We have adopted here an abbreviated notation making explicit only the dependence of $\beta_k$ on the factorization constant introduced in the very last step, keeping implicit the dependence on the previous factorization constants (henceforth, the same criterion will be used for any other symbol depending on $k$ factorization energies). Therefore, given any initial potential $V_0$, the corresponding $n$-susy partner potential $V_n$ can be written as

$$V_n(x, \varepsilon_n) = V_0(x) + \sum_{k=1}^{n} \beta'_k(x, \varepsilon_k),$$

(7)

provided that the master equations for $\beta_k$ and $V_k$ are given by

$$- \beta'_k(x, \varepsilon_k) + \beta''_k(x, \varepsilon_k) = 2[V_{k-1}(x, \varepsilon_{k-1}) - \varepsilon_k], \quad k = 1, 2, ..., n,$$  

(8)

$$V_k(x, \varepsilon_k) = V_{k-1}(x, \varepsilon_{k-1}) + \beta'_k(x, \varepsilon_k), \quad k = 1, 2, ..., n.$$  

(9)

Now, let us stress that every general solution of the Riccati equation (4), for a given $\varepsilon$, depends on an additional implicit integration parameter $\alpha$, hence, the process accumulates as many of these integration parameters as many general solutions of (4) have been used.

Observe the coincidence of our nonlinear algorithm (5) and the Wahlquist and Estabrook superposition principle expression (see equation (16) of [5]), derived from the Bäcklund transformation (BT) of the KdV equation $w_t = 6w^2_x - w_{xxx}$; subscripts $t$ and $x$ denote partial derivatives. The method has been typically used to generate new, multisoliton solutions $w_{12}, ..., w_{(n)}$ of the KdV equation from a given one-soliton solution $w \equiv w_1$ of the same
equation. It is thus quite interesting that the validity of the same algorithm in the intertwining problem (supersymmetry) is much easier to demonstrate without worrying at all about the nonlinear equations! Moreover, its physical applicability in susy seems much wider. Thus, e.g., the singular solutions of KdV (singular water waves) would be of marginal physical interest. The singular potentials in the Schrödinger equation are not! Therefore, the possibility of reducing the \( n \)-th intertwining iteration to the multiple applications of the Bäcklund superposition principle means that \( n \)-susy could be a universal method generating the “multisoliton deformations” of any initial potential.

We shall now focus on the vacuum case, presenting some simplifications which the method offers in deriving the \( n \)-susy partners for the potential \( V_0(x) = 0 \). In this case, the Riccati equation (1) has the general solution

\[
\beta_1(x, \epsilon) = -\sqrt{2\epsilon} \cot[\sqrt{2\epsilon}(x - \alpha)],
\]

where \( \alpha \) is an integration constant (in general complex). It is well known that the superpotential (10) gives four different first order susy partners of \( V_0(x) = 0 \) by taking different values of \( \epsilon \) and \( \alpha \). This information is summarized in Table I.

| Case | \( \epsilon \) | \( \sqrt{2\epsilon} \) | \( \alpha \) | \( \beta_1(x, \epsilon) \) |
|------|----------------|----------------|----------|-------------------|
| S    | \( \epsilon < 0 \) | \( i\sqrt{2|\epsilon|} = i\kappa \) | \( a \) | \( -\kappa \coth[\kappa(x - a)] \) |
| R    | \( \epsilon < 0 \) | \( i\sqrt{2|\epsilon|} = i\kappa \) | \( -b - \frac{i\pi}{2\kappa} \) | \( -\kappa \tanh[\kappa(x + b)] \) |
| P    | \( \epsilon > 0 \) | \( \sqrt{2\epsilon} = k \) | \( a \) | \( -k \cot[k(x - a)] \) |
| N    | 0              | 0              | \( a \) | \( -1 \) |

As an example, notice that the regular case (R) leads to the well known modified Pöschl-Teller type susy partner \( V_1^R(x, \epsilon) = -\kappa^2 \text{sech}^2[\kappa(x + b)] \), while the null case (N) leads to the potential barrier \( V_1^N(x, 0) = (x - a)^{-2} \). Now, in order to give an example of second order susy partner potentials \( V_2(x, \epsilon) \), let us consider the superpotentials R and S as given in Table I. By introducing them in (2) and (3) we get

\[
V_2(x, \epsilon_2) = -\left(\kappa_1^2 - \kappa_2^2\right) \frac{\kappa_1^2 \text{csch}^2[\kappa_1(x + b)] + \kappa_2^2 \text{sech}^2[\kappa_2(x - a)]}{(-\kappa_1 \coth[\kappa_1(x + b)] + \kappa_2 \tanh[\kappa_2(x - a)])^2}.
\]

(11)

The potential (11) has two finite wells which can be modulated by changing the values of \( \kappa_1 \) and \( \kappa_2 \) under the condition \( \kappa_2 < \kappa_1 \). A Taylor expansion of (11) shows a singularity at \( x = a \) when \( \kappa_2 > \kappa_1 \). The case \( \kappa_2 = \kappa_1 \) gives a potential \( V_2(x, \epsilon_1) = 0 \).
Let us remark that, for the periodic superpotentials $\beta_1$ in Table I, equation (7) leads to a natural classification of two kinds of potentials depending on the parity of $n$. For $n$ even, the periodic superpotential $\beta_1$ does not appear as a separate term in (7), affecting only one of denominators. The resulting susy partners have only a finite quantity of singularities. This fact has been used by Stalhofen [8] by constructing potentials with bound states embedded in the continuum. On the other hand, for $n$ odd, the function $\beta_1$ is a separate term in the sum (7) and its global effect is not canceled by any similar term. The corresponding susy partners become singular periodic potentials.

In conclusion, the nonlinear difference algorithm (6) allows the construction of higher order susy partners of any initial potential $V_0(x)$, provided that a certain number of solutions of (1) have been given. This finite difference algorithm generalizes the superposition principle reported in [5] extending its applications to the susy construction of new solvable potentials. In particular, the higher order susy partners $V_n(x, \epsilon_n)$ of the free particle potential represent a wide set of transparent wells in the terms discussed in [7–9], as well as multisoliton solutions of the KdV equation as given in [5].

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References

[1] D.J. Fernández C, Int. J. Mod. Phys. A 12, 171 (1997); D.J. Fernández C., M.L. Glasser and L.M. Nieto, Phys. Lett. A 240, 15 (1998)

[2] D.J. Fernández C., V. Hussin and B. Mielnik, Phys. Lett. A 244, 1 (1998); J.O. Rosas-Ortiz, J. Phys. A 31, L507 (1998); J. Phys. A 31, 10163 (1998); D.J. Fernández C. and V. Hussin, J. Phys. A 32, 3603 (1999)

[3] B. Mielnik, L.M. Nieto and O. Rosas-Ortiz, The finite difference algorithm for higher order supersymmetry, Universidad de Valladolid Preprint, Spain (1998)

[4] V. E. Adler, Physica D 73, 335 (1994)

[5] H. D. Wahlquist and F. B. Estabrook, Phys. Rev. Lett. 31, 1386 (1973)

[6] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. Miura, Phys. Rev. Lett. 19, 1095 (1967)

[7] V.B. Matveev and M.A. Salle, Darboux Transformations and Solitons, Springer-Verlag, Berlin (1991)

[8] A. Stahlhofen, Phys. Rev. A 51, 934 (1995)

[9] B.N. Zakhariev and V. M. Chabanov, Inverse Problems 13, R47 (1997)