A theory of the dark matter

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Abstract

In an earlier paper I proposed a highly symmetric semi-classical initial condition to describe the universe in the period leading up to the electroweak transition and completely determine all cosmology after that. Nothing beyond the Standard Model is assumed. Inflation is not needed. The initial symmetry allows no adjustable parameters. It is a complete theory of the Standard Model cosmological epoch, predictive and falsifiable. Here, the time evolution of the initial condition is calculated in the classical approximation. The fields with nontrivial classical values are the SU(2)-weak gauge field (the cosmological gauge field or CGF) and the Higgs field. The CGF produces the electroweak transition then evolves as a non-relativistic perfect fluid ($w_{\text{CGF}} \approx 0$). At the present time, i.e. when $H = H_0$, the CGF energy density satisfies $\Omega_\Lambda + \Omega_{\text{CGF}} = 1$. The CGF is the dark matter. The dark matter is a classical phenomenon of the Standard Model. The classical universe contains only the dark matter, no ordinary matter. At next to leading order the fluctuations of the Standard Model fields will provide a calculable, relatively small amount of ordinary matter such that $\Omega_\Lambda + \Omega_{\text{CGF}} + \Omega_{\text{ordinary}} = 1$.

Contents

1 Introduction 2
  1.1 Assumptions ........................................... 3
  1.2 The CGF cosmology .................................... 4
  1.3 Physical parameters ................................... 6
  1.4 Organization of the paper ......................... 6

2 Spin(4)-symmetric initial condition 7
  2.1 Spin(4)-symmetric physics on the 3-sphere ....... 7
  2.2 Initial CGF energy ................................... 9
  2.3 Coordinates for local physics .................... 9
  2.4 Start of the electroweak transition at $a = a_{\text{EW}}$ 10
  2.5 CGF density and pressure for $a \leq a_{\text{EW}}$ .... 10
  2.6 Realizing the electroweak transition ............ 11
  2.7 Adiabatic condition for $a \leq a_{\text{EW}}$ .......... 12
  2.8 Temperature from the CGF ....................... 12
1 Introduction

I proposed in [1] that the initial condition of the universe leading up to the electroweak transition was a specific highly symmetric semi-classical state of the Standard Model and General Relativity. The initial condition is a classical SU(2)-weak cosmological gauge field (the CGF). The CGF cosmology has some attractive features:

- Nothing beyond the Standard Model and General Relativity is assumed.
- There are no adjustable parameters.

The initial symmetry and sufficiently large initial energy completely determine the initial state. The initial condition is a specific state whose time evolution describes all of the Standard Model cosmological epoch, i.e. from the electroweak transition onward — the period when the Standard Model and General Relativity govern physics.

- The state of the universe is semi-classical. Its time evolution can be calculated by a systematic expansion around the classical CGF.
- It is a complete and calculable theory of cosmology, predictive and falsifiable.
- It is an alternative to inflation. The initial symmetry implies spatial isotropy and
homogeneity. We will see here that a sufficiently large initial energy in the CGF results in the observed flatness of the present universe.

Here we calculate the time evolution of the initial condition in the leading order, classical approximation, i.e. ignoring the fluctuations of the fields.

- After the CGF produces the electroweak transition, it drives the expanding universe as a non-relativistic fluid \( w_{\text{CGF}} = 0 \) to become the dark matter in the present universe. The dark matter is a classical phenomenon in the Standard Model.

- The universe contains only dark matter in the leading order, classical approximation. The ordinary matter appears at the next order as a relatively small perturbation of the dark matter universe by the fluctuations of the Standard Model fields.

- We derive the dark matter equation of state in analytic form.

In [2] the Tolman-Oppenheimer-Volkoff stellar structure equations for dark matter stars made of the CGF are solved numerically using the CGF equation of state derived here.

- The possible CGF dark matter stars — the solutions of the TOV equations — have mass \( M \) ranging from 0 up to \( 9.14 \times 10^{-6} \, M_\odot \) with radius \( R \) taking certain specific values between 5.23 cm and 13.6 cm. Presumably the CGF has collapsed gravitationally into an ensemble of such dark matter stars. For \( M > 5.09 \times 10^{-6} \, M_\odot \) multiple values of \( R \) are possible, suggesting the possibility of transitions that would release gravitational energy on the order of \( 10^{41} \, \text{J} \) in times on the order of \( 10^{-10} \, \text{s} \).

In [3] it is shown that

- The initial condition is thermodynamically stable under fluctuations of the SU(2)-weak gauge field, a first step towards showing that the initial condition is physically natural.

The classical CGF is the skeleton of cosmology — a universe containing only dark matter. The fluctuations of the Standard Model fields will flesh out the skeleton with ordinary matter. The skeleton has the right form. It remains to calculate the higher order corrections to check whether the CGF cosmology matches the observed universe.

### 1.1 Assumptions

The only assumptions are:

1. The universe is governed by the Standard Model and General Relativity (with cosmological constant). Nothing beyond the Standard Model is assumed. Nothing beyond the known laws of physics is assumed.

2. The universe is a 3-sphere.

3. The state of the universe is invariant under a Spin(4) symmetry group that acts on the 3-sphere as SO(4) and on the Standard Model fields such that the SU(2)-weak doublets transform as spinors.

4. The initial energy in the Standard Model fields is \( > 10^{107} \) in natural units.
1.2 The CGF cosmology

These assumptions result in a unique cosmology. There are no adjustable parameters. The cosmology is semi-classical. The leading order approximation — the classical approximation — is solvable in terms of elliptic functions and elliptic integrals. The universe at leading order contains only the CGF, no ordinary matter. The ordinary matter appears at next to leading order as a perturbation due to the fluctuations of the Standard Model fields around the classical solution.

1. The Spin(4) symmetry and high initial energy imply that the state of the universe is semi-classical — a solution of the classical equations of motion plus small fluctuations. The fluctuations are ignored in the leading order, classical approximation.

2. The only nontrivial fields in the Spin(4)-symmetric classical solution are
   - the homogeneous, isotropic space-time metric characterized by the radius $R(\hat{t})$ of the 3-sphere universe as a function of conformal time $\hat{t}$,
   - the Higgs field $\phi$ which is fixed by the Spin(4) symmetry at the local maximum $\phi = 0$ of the Higgs potential,
   - a Spin(4)-symmetric SU(2)-weak gauge field (the CGF).

3. The Spin(4)-symmetric SU(2)-weak gauge field is described by a single degree of freedom $\hat{b}(\hat{t})$. The equation of motion for $\hat{b}(\hat{t})$ is solved analytically by an elliptic function. $\hat{b}(\hat{t})$ oscillates anharmonically with dimensionless energy $\hat{E}_{\text{CGF}}$ in the natural energy unit $\hbar/R(\hat{t})$ (with $c = 1$). It will turn out that $\hat{E}_{\text{CGF}} > 10^{107}$ is needed for the electroweak transition to take place and for the present curvature of the universe to be consistent with the observed flatness.

4. There is a natural parametrization of the scale of the universe, $a(\hat{t}) = \epsilon R(\hat{t})$, where $\epsilon = (8\hat{E}_{\text{CGF}})^{-1/4}$. The local physics in terms of $a(\hat{t})$ is independent of the particular value of the energy $\hat{E}_{\text{CGF}} > 10^{107}$ so the CGF cosmology has no adjustable parameters.

5. The universe expands, driven by the gauge field energy and the energy in the Higgs field at $\phi = 0$. The CGF behaves effectively as a perfect fluid, a mixture of radiation with a small amount of vacuum energy, $w_{\text{CGF}} \approx 1/3$.

6. The CGF oscillation is much faster than the expansion so an adiabatic approximation is justified, averaging over the CGF oscillation. The rapidly oscillating CGF contributes an effective $\dot{\phi}^2\phi$ term to the Higgs potential which stabilizes $\phi$ at 0. The effective $\dot{\phi}^2\phi$ term is proportional to $1/a^2$ so the stabilizing effect weakens as the universe expands.

7. The stabilizing effect becomes insufficient when the scale reaches $a_{\text{EW}} = 0.585 \hbar/m_{\text{Higgs}}$. The electroweak transition starts. The Higgs field begins to move away from 0 towards its vacuum expectation value.

8. The elliptic function $\hat{b}(\hat{t})$ is periodic in both imaginary time and real time. The imaginary time period defines an inverse temperature. The fluctuations of the fields are initially in the specific thermal state defined by this periodicity in imaginary time.

The initial condition leading up to $a_{\text{EW}}$ is a specific semi-classical state of the Standard Model and General Relativity. All of cosmology after $a_{\text{EW}}$ is calculable from first principles starting from this initial condition.
Figure 1: The CGF is a radiation fluid at high density, before $a_{ EW }$. At low density, after about $10^{ 2 } a_{ EW }$, it is a non-relativistic fluid.

9. The adiabatic approximation continues to be valid after $a_{ EW }$. The adiabatic equations of motion for the CGF after $a_{ EW }$ are solved analytically by elliptic functions and elliptic integrals.

10. After $a_{ EW }$, over the next 1 or 2 ten-folds of expansion, $\phi^\dagger \phi$ tracks the minimum of the evolving effective Higgs potential to the vacuum expectation value $v^2/2$. The CGF evolves adiabatically to become harmonic oscillation at the bottom of the massive gauge field potential.

11. From about $10^{ 2 } a_{ EW }$ onward the expansion of the universe is driven by the CGF energy density $\rho_{ CGF } = 0.890 m_{ \text{Higgs} } / a^3$. From about $10^{ 2 } a_{ EW }$ onward the CGF is a non-relativistic perfect fluid — a perfect fluid with equation of state parameter $w_{ CGF } = 0$. Figure 1 shows the evolution of $w_{ CGF }$ and $\phi^\dagger \phi$ with the scale of the universe.

12. The rate of expansion $H$ decreases with time, reaching the Hubble constant $H_0$ at $a_0 = 1.40 \times 10^{-8}$ s. The condition $H = H_0$ identifies the present.

13. The present curvature is

$$ \frac{ 1 }{ H_0^2 R_0^2 } = \frac{ 1 }{ H_0 a_0^2 (8 \dot{ E }_{ CGF } )^{ 1/2 } } $$

(1.1)

The observed flatness of the universe puts a lower bound on the initial energy.

$$ \frac{ 1 }{ H_0^2 R_0^2 } < 0.001 \iff \dot{ E }_{ CGF } > 10^{ 107 } $$

(1.2)

14. In the leading, classical approximation the only matter in the universe is the CGF. The present curvature is negligible so the present normalized energy density is

$$ \Omega = \Omega_\Lambda + \Omega_{ CGF } = 1 $$

(1.3)

There is only dark matter, the CGF.
15. The fluctuations will provide the ordinary matter. The total energy density in the semi-classical state at \( H = H_0 \) will be \( \Omega_\Lambda + \Omega_{\text{CGF}} + \Omega_{\text{ordinary}} = 1 \). The ratio \( \Omega_{\text{ordinary}}/\Omega_{\text{CGF}} \) will have to agree with the observed ratio \( \Omega_{\text{ordinary}}/\Omega_{\text{CDM}} \).

There are no adjustable parameters in the theory. The energy \( \hat{E}_{\text{CGF}} \), the one apparently adjustable parameter, is irrelevant to the local physics. Everything in the theory is calculable. Every calculation of an observable quantity is an opportunity to confirm or refute the theory. If the theory works, it will provide a first principles cosmology of the Standard Model epoch. And it will be a sharp target for theories of cosmology before the electroweak transition. A theory of the early universe will be successful if and only if it produces the Standard Model plus General Relativity in a spherical universe with Spin(4) symmetry and high initial energy.

1.3 Physical parameters

Table 1 shows the physical parameters (in \( c = 1 \) units) that are used in the calculations. The values are taken from [4]. \( \kappa = 8\pi G \) is the gravitational constant. \( m_{\text{Higgs}} \) is the mass of the Higgs boson. \( H_0 \) is the Hubble constant. \( g \) is the SU(2) gauge coupling constant. \( \lambda \) is the Higgs coupling constant. \( \Omega_{\text{CDM}} = \rho_{\text{CDM}}/\rho_c \) is the present dark matter density normalized to the critical density \( \rho_c = 3H_0^2/\kappa \). \( \Omega_{\text{ordinary}} \) is the normalized density of ordinary matter. \( \Omega_\Lambda \) is the normalized dark energy density (assumed due to the cosmological constant). The normalized curvature is \( -\Omega_{\text{curvature}} \). The normalization is such that \( \Omega_\Lambda + \Omega_{\text{CDM}} + \Omega_{\text{ordinary}} + \Omega_{\text{curvature}} = 1 \).

| \( t_{\text{grav}} = (\hbar\kappa)^{\frac{1}{2}} = 2.70 \times 10^{-43} \text{s} \) | \( g^2 = 0.426 \) | \( \lambda^2 = 0.258 \) |
| \( t_{\text{Higgs}} = \frac{\hbar}{m_{\text{Higgs}}} = 5.26 \times 10^{-27} \text{s} \) | \( \Omega_{\text{CDM}} = 0.266 \) | \( \Omega_{\text{ordinary}} = 0.049 \) |
| \( t_{\text{Hubble}} = \frac{\Omega_\Lambda}{H_0} = 4.58 \times 10^{17} \text{s} \) | \( \Omega_\Lambda = 0.685 \) | \( |\Omega_{\text{curvature}}| < 0.001 \) |

Table 1: Physical parameters (in \( c = 1 \) units)

The calculations in the paper are in the leading order, classical approximation, subject to higher order corrections. The results are shown to three decimal places to indicate that experimental accuracy will be the only limitation once the higher order corrections are calculated.

1.4 Organization of the paper

The paper is organized as follows. Section 2 summarizes [1], describing the Spin(4)-symmetric initial condition, covering points 1 through 8 above. Section 3 cites some mathematical literature. Section 4 calculates the adiabatic time evolution of the CGF after the start of the electroweak transition at \( a_{\text{EW}} \). Section 5 identifies the present time by the condition \( H = H_0 \), expresses the present curvature in terms of the CGF initial
energy, and shows that the present energy density in the CGF gives the observed flatness, \( \Omega_{\text{CGF}} + \Omega_\Lambda - 1 < 0.001 \). The CGF equation of state calculated in Sections 2 and 4 is summarized. Section 6 collects comments and questions. Appendix A lists some identities for the elliptic functions and elliptic integrals.

The numerical calculations are done in SageMath [5] using the mpmath arbitrary-precision floating-point arithmetic library [6]. The Sagemath notebooks along with print-outs are provided in the Supplemental Material [7].

2 Spin(4)-symmetric initial condition

This section is a summary of [1] with some small changes of notation. Details of the calculations are shown in the Supplemental Material of [1].

2.1 Spin(4)-symmetric physics on the 3-sphere

Let space be a 3-sphere. Write \( S^3 \) for the unit 3-sphere in \( \mathbb{R}^4 \) with metric \( \hat{g}_{ij}(\hat{x}) \).

\[
\hat{x} \in \mathbb{R}^4 \quad (\hat{x}^\mu) = (\hat{x}^i, \hat{x}^4) \quad \delta_{\mu\nu}\hat{x}^\mu\hat{x}^\nu = 1 \quad \hat{g}_{ij}(\hat{x})d\hat{x}^id\hat{x}^j = \delta_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu \quad (2.1)
\]

Identify \( S^3 \) with SU(2) by

\[
\hat{x} \in S^3 \leftrightarrow g_{\hat{x}} = \hat{x}^4 + \hat{x}^k\sigma_k \in \text{SU}(2) \quad (2.2)
\]

\( \sigma_k \) being the Pauli matrices. Spin(4) is \( \text{SU}(2)_L \times \text{SU}(2)_R \). It acts on \( S^3 \) as SO(4).

\[
U = (g_L, g_R) \quad g_{\hat{x}} = g_L g_\hat{x} g_R^{-1} \quad (2.3)
\]

The general SO(4)-symmetric space-time metric is

\[
ds^2 = R(\hat{t})^2 (-dt^2 + \hat{g}_{ij}(\hat{x})d\hat{x}^id\hat{x}^j) \quad (2.4)
\]

\( \hat{t} \) is conformal cosmological time. \( R(\hat{t}) \) is the radius of the spatial 3-sphere at conformal time \( \hat{t} \). Co-moving time \( t \) is given by \( dt = R(\hat{t})d\hat{t} \).

Let Spin(4) act on the SU(3) and U(1) gauge bundles of the Standard Model as product bundles, so the only Spin(4)-symmetric gauge fields are the trivial gauge fields, \( A_{\mu}^{\text{SU}(3)} = A_{\mu}^{U(1)} = 0 \). Let the SU(2) gauge bundle be identified with the spinor bundle over \( S^3 \). An SU(2) doublet field such as the Higgs field \( \phi \) transforms under Spin(4) by

\[
U = (g_L, g_R) \quad \phi \mapsto U\phi \quad U\phi(\hat{t}, \hat{x}) = g_L \phi(\hat{t}, U^{-1}\hat{x}) \quad (2.5)
\]

The only Spin(4)-symmetric Higgs field is \( \phi = 0 \). There is a one parameter family of Spin(4)-symmetric SU(2) gauge fields on \( S^3 \) with covariant derivative

\[
D_{i}^{\text{SU}(2)} = \hat{\nabla}_i + \hat{\gamma}_i(\hat{x}) \quad (2.6)
\]

where the \( \hat{\gamma}_i(\hat{x}) \) are the Dirac matrices at \( \hat{x} \) and \( \hat{\nabla} \) is the metric covariant derivative,

\[
\hat{\gamma}_i(\hat{x}) = \frac{1}{2} g_{\hat{x}}\hat{\partial}_i(g_{\hat{x}}^{-1}) \quad \hat{\gamma}_i(\hat{x}) = \frac{1}{2} g_{\hat{x}}\hat{\partial}_i \quad (\hat{\gamma}_i(\hat{x}) + \hat{\gamma}_j(\hat{x})) = -\frac{1}{2} g_{\hat{x}}(\hat{x}) \quad (2.7)
\]

\[
\hat{\nabla}_i = \hat{\partial}_i + \hat{\gamma}_i \quad \hat{\partial}_i = \frac{\partial}{\partial x_i} \quad \hat{\nabla}_i\hat{\gamma}_j = 0
\]
In space-time the general Spin(4)-symmetric $SU(2)$ gauge field in unitary gauge is

$$D_0^{SU(2)} = 0 \quad D_i^{SU(2)} = \hat{\nabla}_i + \hat{b}(\hat{t})\hat{\gamma}_i(\hat{x})$$  \hspace{1cm} (2.8)

The cosmological gauge field (the CGF) is $B_i^{\text{CGF}} = \hat{b}(\hat{t})\hat{\gamma}_i(\hat{x})$ or, for short, $\hat{b}(\hat{t})$.

The general Spin(4)-symmetric classical state is characterized by the pair of functions $R(\hat{t})$ and $\hat{b}(\hat{t})$ subject to the classical equations of motion. The Yang-Mills action (as written in [4]) is

$$\frac{1}{\hbar} S_{\text{gauge}} = \int \frac{1}{2g^2} \text{tr}(-F_{\mu\nu}F^{\mu\nu}) \sqrt{-g} \, d^4x$$  \hspace{1cm} (2.9)

The action of the Spin(4)-symmetric gauge field is

$$\frac{1}{\hbar} S_{\text{gauge}} = \text{Vol}(S^3) \frac{3}{g^2} \int \left[ -\frac{1}{2} \left( \frac{d\hat{b}}{d\hat{t}} \right)^2 + \frac{1}{2} (\hat{b}^2 - 1)^2 \right] d\hat{t} \quad \text{Vol}(S^3) = 2\pi^2$$  \hspace{1cm} (2.10)

The action is independent of $R$ because the Yang-Mills theory is conformally invariant. $\hat{b}$ is an anharmonic oscillator with equation of motion

$$\frac{d^2\hat{b}}{d\hat{t}^2} + 2\hat{b}(\hat{b}^2 - 1) = 0$$  \hspace{1cm} (2.11)

The dimensionless energy

$$\hat{E}^{\text{CGF}} = \frac{1}{2} \left( \frac{d\hat{b}}{d\hat{t}} \right)^2 + \frac{1}{2} (\hat{b}^2 - 1)^2$$  \hspace{1cm} (2.12)

is conserved. In co-moving time the action is

$$S_{\text{gauge}} = \frac{\hbar}{R} \text{Vol}(S^3) \frac{3}{g^2} \int \left[ -\frac{1}{2} \left( \frac{d\hat{b}}{d\hat{t}} \right)^2 + \frac{1}{2} (\hat{b}^2 - 1)^2 \right] d\hat{t}$$  \hspace{1cm} (2.13)

The physical energy is dual to co-moving time so

$$E^{\text{phys}}_{\text{CGF}} = \frac{\hbar}{R} \text{Vol}(S^3) \frac{3}{g^2} \hat{E}^{\text{CGF}}$$  \hspace{1cm} (2.14)

The general classical solution (for $\hat{E}^{\text{CGF}} > 1/2$) is the elliptic function

$$\hat{b}(\hat{t}) = \frac{b(z)}{\epsilon} \quad b(z) = k \text{cn}(z, k)$$  \hspace{1cm} (2.15)

where $k = K(k)$ is the complete elliptic integral of the first kind.
2.2 Initial CGF energy

\( \hat{E}_{\text{CGF}} \) is the only adjustable parameter in the classical solution. We start out assuming \( \hat{E}_{\text{CGF}} \gg 1 \) so that the CGF is semi-classical. Later we will find that \( \hat{E}_{\text{CGF}} \) must be very large if the cosmology is to be physically realistic. We first find \( \hat{E}_{\text{CGF}} > 10^{67} \) is needed in order that the CGF produces the electroweak transition in an expanding universe. Then we derive a formula for the present curvature of the universe,

\[
- \Omega_{\text{curvature}} = \frac{1}{H_0^2 R_0^2} = 1.07 \times 10^{51} \epsilon^2 \tag{2.16}
\]

The observed flatness of the present universe, \(|\Omega_{\text{curvature}}| < 0.001\), implies

\[
\epsilon < 10^{-27} \quad \hat{E}_{\text{CGF}} > 10^{107} \tag{2.17}
\]

The bound (2.17) on \( \hat{E}_{\text{CGF}} \) replaces a handwaving argument in [1] that gave a similar bound. When \( \hat{E}_{\text{CGF}} \) is very large, \( \epsilon \) is very small, so to very high accuracy

\[
k^2 = \frac{1}{2} \quad K = K(1/\sqrt{2}) = \frac{\Gamma(1/4)^2}{4\pi^{1/2}} = 1.854075\ldots \tag{2.18}
\]

2.3 Coordinates for local physics

Scale the spatial coordinates \( x = \hat{x}/\epsilon \) to match the scaling of time \( z = \hat{t}/\epsilon \). The space-time metric in the scaled coordinate system is

\[
ds^2 = a(z)^2 \left( -dz^2 + g_{ij}(x)dx^idx^j \right) \quad a(z) = \epsilon R(\hat{t}) \tag{2.19}
\]

The geometry of the scaled 3-sphere is

\[
x \in \frac{1}{\epsilon} S^3 \quad x = \frac{\hat{x}}{\epsilon} \quad \delta_{\mu\nu} x^\mu x^\nu = \frac{1}{\epsilon^2} \quad g_{ij}(x)dx^idx^j = \delta_{\mu\nu}dx^\mu dx^\nu
\]

\[
g_{ij}(x)dx^idx^j = \frac{1}{\epsilon^2} \hat{g}_{ij}(\hat{x})d\hat{x}^i d\hat{x}^j \quad g_{ij}(x) = \hat{g}_{ij}(\hat{x}) \tag{2.20}
\]

\[
\gamma_i(x)dx^i = \frac{1}{\epsilon} \hat{\gamma}_i(\hat{x})d\hat{x}^i \quad \gamma_i(x) = \hat{\gamma}_i(\hat{x}) \quad \gamma_i \gamma_j + \gamma_j \gamma_i = -\frac{1}{2}g_{ij}
\]

The CGF covariant derivative is

\[
D_i = \nabla_i + b(z)\gamma_i(x) \quad \nabla_i = \partial_i + \epsilon\gamma_i \quad b(z) = k \text{cn}(z, k) \tag{2.21}
\]

The Spin(4)-symmetric gauge field action (2.10) becomes

\[
\frac{1}{\hbar} S_{\text{gauge}} = \frac{\text{Vol}(S^3)}{\epsilon^3} \frac{3}{g^2} \int \left[ -\frac{1}{2} \left( \frac{db}{dz} \right)^2 + \frac{1}{2} \left( b^2 - \epsilon^2 \right)^2 \right] dz \tag{2.22}
\]

The local physics in the scaled coordinate system is independent of \( \epsilon \) when \( \epsilon \) is very small. Only global properties of the universe can depend on \( \epsilon \).

\[\footnote{Note that the complex conformal time variable \( z \) is not the redshift.}\]
2.4 Start of the electroweak transition at \( a = a_{\text{EW}} \)

The action of the Higgs field (as written in [4]) is

\[
\frac{1}{\hbar} S_{\text{Higgs}} = \int \left[ a^{-2} D_{\mu} \phi^\dagger D^{\mu} \phi + \frac{1}{2} \lambda^2 \left( \phi^\dagger \phi - \frac{v^2}{2} \right)^2 \right] a^4 \sqrt{-g} d^4 x
\]  
(2.23)

where \( x^\phi = z \), \( D_0 = \partial_0 \), and \( D_i \) is the SU(2)-weak gauge covariant derivative. The Higgs mass is \( m_{\text{Higgs}} = \hbar \lambda v \). In the presence of the CGF the covariant derivative (2.21) gives

\[
D_{\mu} \phi^\dagger D^{\mu} \phi = \partial_{\mu} \phi^\dagger \partial^{\mu} \phi + \frac{1}{2} b(z) ( \partial^i \phi^\dagger \gamma_i \phi - \phi^\dagger \gamma_i \partial^i \phi ) + \frac{3}{4} b(z)^2 \phi^\dagger \phi
\]  
(2.24)

with \( \partial_i \) replacing \( \nabla_i = \partial_i + \epsilon \gamma_i \) because \( \epsilon \) is very small.

We make two assumptions to be verified later. We assume that \( a \) and \( \phi \) change very slowly compared to the CGF oscillation. And we assume that the energy in the CGF and in the Higgs field at \( \phi = 0 \) drives an expanding universe.

Given that \( a \) and \( \phi \) are changing slowly, \( b(z) \) and \( b(z)^2 \) can be replaced in the action (2.23) by their expectation values over a period of oscillation.

\[
\langle b \rangle = 0 \quad \langle b^2 \rangle = \frac{1}{4K} \int_0^{4K} k^2 \text{cn}^2(z,k) \, dz = \frac{\pi}{4K^2}
\]  
(2.25)

Equation (A.11) is the relevant identity. The effective action for \( \phi \) is

\[
\frac{1}{\hbar} S_{\text{Higgs}}^{\text{eff}} = \int \left[ a^{-2} \partial_{\mu} \phi^\dagger \partial^{\mu} \phi + V_{\text{eff}}^{\text{Higgs}}(\phi) \right] a^4 \sqrt{-g} d^4 x
\]  
(2.26)

\[
V_{\text{Higgs}}^{\text{eff}}(\phi) = \frac{\lambda^2 v^4}{8} + \left( \frac{3}{4} \frac{\langle b^2 \rangle}{a^2} - \frac{\lambda^2 v^2}{2} \right) \phi^\dagger \phi + \frac{\lambda^2}{2} (\phi^\dagger \phi)^2
\]

When \( a(z) \) is small the coefficient of \( \phi^\dagger \phi \) is positive so \( \phi = 0 \) is stable. When the coefficient of \( \phi^\dagger \phi \) becomes negative, \( \phi = 0 \) becomes unstable. The electroweak transition starts when the quadratic term vanishes, at \( a = a_{\text{EW}} \) given by

\[
a_{\text{EW}} = \left( \frac{3}{2} \frac{\langle b^2 \rangle}{\lambda^2 v^2} \right)^{1/2} = \frac{(6\pi)^{1/2}}{4K} t_{\text{Higgs}} = 0.5854 t_{\text{Higgs}} = 3.08 \times 10^{-27} \text{ s}
\]  
(2.27)

2.5 CGF density and pressure for \( a \leq a_{\text{EW}} \)

The energy-momentum tensors (derived in [1]) are

\[
\frac{1}{\hbar} T^{\text{gauge}}_{\mu\nu} = g^2 \text{tr} \left( -2 F_{\mu\sigma} F^{\nu\sigma} + \frac{1}{2} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right)
\]

\[
\frac{1}{\hbar} T^\phi_{\mu\nu} = 2 D_{\mu} \phi^\dagger D_{\nu} \phi - g_{\mu\nu} D_{\sigma} \phi^\dagger D^{\sigma} \phi - g_{\mu\nu} \frac{1}{2} \lambda^2 \left( \phi^\dagger \phi - \frac{v^2}{2} \right)^2
\]  
(2.28)
For the Spin(4)-symmetric state in the classical, adiabatic approximation the total energy-momentum tensor is expressed by the CGF energy density and pressure

\[
T_{\mu\nu}^{\text{CGF}} = T_{\mu\nu}^{\text{gauge}} + T_{\mu\nu}^{\phi}
\]

\[
T_{00}^{\text{CGF}} = \rho_{\text{CGF}}(-g_{00})
\]

\[
T_{ij}^{\text{CGF}} = p_{\text{CGF}} g_{ij}
\]

\[
T_{i0}^{\text{CGF}} = 0
\]

\[
\frac{1}{\hbar} \rho_{\text{CGF}} = \frac{3}{g^2} \frac{\hat{E}_{\text{CGF}}}{R^4} + \frac{\lambda^2 v^4}{8} = \frac{3}{8g^2 a^4} + \frac{1}{8\lambda^2 t_{\text{Higgs}}^4}
\]

\[
\frac{1}{\hbar} p_{\text{CGF}} = \frac{1}{g^2} \frac{\hat{E}_{\text{CGF}}}{R^4} - \frac{\lambda^2 v^4}{8} = \frac{1}{8g^2 a^4} - \frac{1}{8\lambda^2 t_{\text{Higgs}}^4}
\]

The first term in each expression is due the gauge field which is a perfect fluid with \( w_{\text{gauge}} = 1/3 \). The second term is due to the Higgs field vacuum energy, a perfect fluid with \( w_{\phi} = -1 \). The equation of state for \( a \leq a_{\text{EW}} \) is

\[
p_{\text{CGF}} = \frac{1}{3} \rho_{\text{CGF}} - \frac{1}{6\lambda^2} \frac{\hbar}{t_{\text{Higgs}}^4}
\]

\[
(2.29)
\]

### 2.6 Realizing the electroweak transition

We have to show that the expanding universe driven by the CGF and the Higgs field \( \phi = 0 \) actually passes through \( a = a_{\text{EW}} \). The Friedmann equation is

\[
H^2 + \frac{1}{R^2} = \frac{1}{3} \kappa \rho_{\text{CGF}} \quad H = \frac{1}{R} \frac{dR}{dt} = \frac{1}{R} \frac{dR}{dt}
\]

\[
(2.31)
\]

In terms of \( z = \dot{t}/\epsilon \), \( a = \epsilon R \), and \( t_{\text{grav}}^2 = \hbar \kappa \) and using equation (2.29) for the density,

\[
H^2 + \frac{\epsilon^2}{a^2} = \frac{1}{8g^2} \frac{t_{\text{grav}}^2}{a^4} + \frac{1}{24\lambda^2} \frac{t_{\text{Higgs}}^2}{t_{\text{Higgs}}^4}
\]

\[
H = \frac{1}{a^2} \frac{da}{dz}
\]

\[
(2.32)
\]

or, equivalently,

\[
a^2 H^2 = \frac{t_{\text{grav}}^2}{t_{\text{Higgs}}^2} \left( \frac{1}{8g^2} \frac{t_{\text{Higgs}}^2}{a^2} + \frac{1}{24\lambda^2} \frac{a^2}{t_{\text{Higgs}}^2} \right) - \epsilon^2
\]

\[
(2.33)
\]

There is a solution as long as the right hand side is positive. Equation (2.33) at \( a = a_{\text{EW}} \) is

\[
a_{\text{EW}}^2 H_{\text{EW}}^2 = \frac{t_{\text{grav}}^2}{t_{\text{Higgs}}^2} \left( \frac{1}{8g^2} \frac{t_{\text{Higgs}}^2}{a_{\text{EW}}^2} + \frac{1}{24\lambda^2} \frac{a_{\text{EW}}^2}{t_{\text{Higgs}}^2} \right) - \epsilon^2 = 2.40 \times 10^{-33} - \epsilon^2
\]

\[
(2.34)
\]

so the expansion passes through \( a_{\text{EW}} \) if \( \epsilon^2 < 2.40 \times 10^{-33} \). The expansion passes through all values of \( a \) if the minimum of \( a^2 H^2 \) in (2.33) is positive,

\[
\epsilon^2 < \frac{t_{\text{grav}}^2}{t_{\text{Higgs}}^2} \left( \frac{1}{8g^2} \frac{g}{\sqrt{3}\lambda} + \frac{1}{24\lambda^2} \frac{\sqrt{3}\lambda}{g} \right) = \frac{t_{\text{grav}}^2}{t_{\text{Higgs}}^2} \frac{1}{4\sqrt{3}g\lambda} = 1.15 \times 10^{-33}
\]

\[
(2.35)
\]

So we assume \( \epsilon^2 < 10^{-33} \) which is \( \hat{E}_{\text{CGF}} > 10^{67} \) in order that the expansion passes through \( a_{\text{EW}} \) and in order that the CGF oscillates for some considerable period before \( a_{\text{EW}} \).
2.7 Adiabatic condition for $a \leq a_{\text{EW}}$

In order to justify averaging over the CGF oscillation, we need to show that the expansion time $1/H$ is much longer than the oscillation period $4Ka$ for $a \leq a_{\text{EW}}$, i.e.

$$4KaH \ll 1 \quad (2.36)$$

For $a$ on the order of $a_{\text{EW}}$, equation (2.34) gives the ratio of time scales $4KaH$ to be on the order of $10^{-16}$ so the adiabatic condition (2.36) is well satisfied. For $a$ significantly less than $a_{\text{EW}}$, the term in (2.33) proportional to $1/a^2$ dominates, so the adiabatic condition is satisfied as long as

$$(4K)^2a^2H^2 \sim (4K)^2 \frac{t_{\text{grav}}^2}{t_{\text{Higgs}}^2} \left( \frac{1}{8g^2} \frac{t_{\text{Higgs}}^2}{a^2} \right) \ll 1 \quad (2.37)$$

which is $a \gg t_{\text{grav}}$ so there is plenty of time for expansion up to $a_{\text{EW}}$ with the adiabatic condition well satisfied.

2.8 Temperature from the CGF

The cosmological gauge field $b(z) = k \operatorname{cn}(z,k)$ is an elliptic function of the time $z$, periodic in real time with period $4K$ and also periodic in imaginary time with period $4K' i$. The imaginary period in co-moving time defines an inverse temperature $T_{\text{CGF}}$.

$$\frac{\hbar}{k_B T_{\text{CGF}}} = 4K' a \quad (2.38)$$

This is to be the origin of the cosmological temperature. The CGF acts as thermal bath for the fluctuations of the Standard Model fields. The correlation functions of the fluctuations are periodic in imaginary time with the period $4K' i$. This defines a specific semi-classical quantum state of the Standard Model plus General Relativity whose time evolution describes the universe from $a_{\text{EW}}$ onwards.

In [3] the initial thermal state of the SU(2) gauge field fluctuations is constructed and shown to be thermodynamical stable, a first step towards showing that the proposed Spin(4)-symmetric initial condition is thermodynamically stable and thus physically natural.

3 Mathematical literature

The SO(4)-invariant classical solution of pure SU(2) gauge theory on $S^3$ (without the Higgs field) was first discovered as a mathematical object by Alfaro, Fubini, and Furlan [8], Cervero, Jacobs, and Nohl [9], and Lüscher [10]. Ivanova, Lechtenfeld, and Popov [11, 12] examined properties of the SO(4)-invariant gauge theory solution on de Sitter and anti-de Sitter space. Lechtenfeld described the back reaction in General Relativity [13]. The symmetry group was SO(4) in all of these works because Spin(4) symmetry can be posited only when there are SU(2) doublet fields such as the Higgs field. None of these mathematical works made or suggested any connection between the mathematical solutions and physical cosmology or the electroweak transition or thermal physics. I thank O. Lechtenfeld for bringing these mathematical works to my attention after the release of [1].

12
4 Adiabatic time evolution of the CGF

4.1 Oscillating CGF at $a \geq a_{\text{EW}}$

As the scale $a(z)$ approaches $a_{\text{EW}}$ the fluctuations of the Higgs field around $\phi = 0$ grow large. After $a_{\text{EW}}$ the Higgs field $\phi$ fluctuates in and around the minima of the effective potential (2.26) which is

$$\phi \phi = (\phi \phi) = \frac{v^2}{2} - \frac{3}{4\lambda^2} \frac{\langle b^2 \rangle}{a^2}$$

(4.1)

As $a(z)$ increases slowly, the potential well deepens and the $\phi$ fluctuations concentrate at $\phi \phi = (\phi \phi)$. The $\phi$ fluctuations are Spin(4)-symmetric so the effective action remains Spin(4)-symmetric. The CGF continues to evolve as a Spin(4)-symmetric SU(2) gauge field $b(z)$ but now with effective action

$$\frac{1}{\hbar} S_{\text{gauge}}^{\text{eff}} = \frac{\text{Vol}(S^3)}{e^3} \frac{3}{g^2} \int \left[ -\frac{1}{2} \left( \frac{db}{dz} \right)^2 + \frac{1}{4} g^2 a^2 (\phi \phi) b^2 + \frac{1}{2} b^4 \right] dz$$

(4.2)

comprising the gauge action (2.22) plus the $b$-dependent term of the $\phi$ action (2.23). After $a_{\text{EW}}$ the CGF $b(z)$ oscillates in a quartic potential whose quadratic term is changing very slowly compared to the oscillation. The dimensionless conserved energy is

$$E_{\text{CGF}} = \frac{\mu^2}{2} E_{\phi} = \frac{1}{2} \left( \frac{db}{dz} \right)^2 + \frac{1}{2} \mu^2 b^2 + \frac{1}{2} b^4 \quad \mu^2 = \frac{1}{2} g^2 a^2 (\phi \phi)$$

(4.3)

Again by equation (A.8) the oscillating solution for fixed $\mu$ and $E_{\text{CGF}}$ is

$$b(z) = k \frac{\text{cn}(u, k)}{\alpha} \quad dz = \alpha du$$

(4.4)

with parameters $k^2$, $\alpha$ depending on $\mu$, $E_{\text{CGF}}$ by

$$\mu^2 = \frac{1 - 2 k^2}{2 \alpha^2} \quad E_{\text{CGF}} = \frac{K(k) - E}{2 \alpha^2}$$

(4.5)

$\mu^2 \geq 0$ so $k^2 \leq 1/2$. Equations (4.1) and (4.3) determine $\mu$ in terms of $\langle b^2 \rangle$ which is the average over a period of oscillation,

$$\langle b^2 \rangle = \frac{1}{4K} \int_0^{4K} b^2 du = \frac{1}{4K} \int_0^{4K} \frac{1}{\alpha^2} k^2 \text{cn}^2(u, k) du = \frac{1}{\alpha^2} \left( k^2 - 1 + \frac{E}{K} \right)$$

(4.6)

evaluated using equation (A.11). $E = E(k)$ is the complete elliptic integral of the second kind. We are now re-using the variables $k$ and $K = K(k)$. From now on we will write $k_{\text{EW}}, K_{\text{EW}},$ and $E_{\text{EW}}$ for the values at $a \leq a_{\text{EW}}$, the constants

$$k^2_{\text{EW}} = \frac{1}{2} \quad K_{\text{EW}} = K(k_{\text{EW}}) \quad E_{\text{EW}} = E(k_{\text{EW}}) = \frac{K_{\text{EW}}}{2} + \frac{\pi}{4K_{\text{EW}}}$$

(4.7)

After $a_{\text{EW}}$ the parameters $\mu$, $E_{\text{CGF}}$ and $k$, $\alpha$ evolve slowly away from their values at $a_{\text{EW}}$

$$\mu_{\text{EW}} = 0 \quad E_{\text{CGF,EW}} = \frac{1}{8} \quad k^2_{\text{EW}} = \frac{1}{2} \quad \alpha_{\text{EW}} = 1$$

(4.8)
The equations derived so far,

\[
(\phi^\dagger \phi)_0 = \frac{v^2}{2} - \frac{3}{4}\frac{\langle b^2 \rangle}{a^2} \quad \mu^2 = \frac{1}{2} g^2 a^2 (\phi^\dagger \phi)_0
\]

\[
\mu^2 = \frac{1-2k^2}{\alpha^2} \quad E_{\text{CGF}} = \frac{k^2(1-k^2)}{2\alpha^4} \quad \langle b^2 \rangle = \frac{1}{\alpha^2} \left( k^2 - 1 + \frac{E}{K} \right)
\]

are one equation short of determining the time evolution completely.

### 4.2 Adiabatic invariant

The adiabatic invariant

\[
I = \frac{1}{2\pi} \oint p \, dq
\]

is a constant of motion in the adiabatic time evolution of an oscillating classical degree of freedom \( q \) with conjugate momentum \( p \). The integral is over one oscillation. Here, from the action (4.2),

\[
q = b \quad \frac{1}{\hbar} p = \frac{\text{Vol}(S^3)}{e^3} \frac{3}{g^2} \frac{db}{dz}
\]

so, using (A.11),

\[
\frac{1}{\hbar} I = \frac{1}{2\pi} \frac{\text{Vol}(S^3)}{e^3} \frac{3}{g^2} \int_0^{4K} \frac{db}{dz} \frac{db}{du} = \frac{\text{Vol}(S^3)}{2\pi e^3} \frac{3}{g^2} \alpha^3 \int_0^{4K} k^2 \text{cn}'(u,k)^2 \, du
\]

\[
= \frac{\text{Vol}(S^3)}{2\pi e^3} \frac{3}{g^2} \frac{1}{\alpha^3} 3 \left[ (1-k^2)K + (2k^2-1)E \right]
\]

Dropping the constant factors, we use

\[
\tilde{I} = \frac{1}{\alpha^3} \left[ (1-k^2)K + (2k^2-1)E \right]
\]

as the adiabatic constant of motion. At \( a = a_{\text{ew}}, k_{\text{ew}}^2 = 1/2 \) and \( \alpha = 1 \) so

\[
\tilde{I} = \frac{1}{2} K_{\text{ew}}
\]

so the adiabatic equation is

\[
\frac{1}{\alpha^3} \left[ (1-k^2)K + (2k^2-1)E \right] = \frac{1}{2} K_{\text{ew}}
\]

Now there are enough equations to parametrize the time evolution by \( k^2 \).

### 4.3 \( a \) as a function of the elliptic parameter \( k^2 \)

The adiabatic equation (4.15) gives \( \alpha \) as a function of \( k^2 \). Then (4.9) gives \( \mu^2, E_{\text{CGF}}, \) and \( \langle b^2 \rangle \) as functions of \( k^2 \), and then \( a \) as a function of \( k^2 \),

\[
a = t_{\text{mess}} \hat{a} \quad \hat{a}^2 = \frac{3}{2} \langle b^2 \rangle + \frac{4\lambda^2}{g^2} \mu^2
\]

Numerical calculations graphed in Figure 2 show that \( a \) increases monotonically from \( a_{\text{ew}} \) to \( \infty \) as \( k^2 \) decreases from 1/2 to 0. So the parametrization by \( k^2 \) implicitly gives the time evolution as a function of the scale \( a \). The \( k^2 \to 0 \) regime is reached after about two ten-folds of expansion from \( a_{\text{ew}} \).
Figure 2: The left graph shows that $a$ increases monotonically as $k^2$ decreases from $1/2$ to 0. The right graph shows that the $k^2 \to 0$ regime is $a \geq 10^2 a_{EW}$.

4.4 CGF density and pressure after $a_{EW}$

For $a \geq a_{EW}$, the energy-momentum tensors (2.28) for the Spin(4)-symmetric state in the classical and adiabatic approximation give

$$\rho_{CGF} = \frac{\hbar}{a^4} \left( \frac{3E_{CGF}}{g^2} + \frac{9\langle b^2 \rangle^2}{32\lambda^2} \right)$$

$$p_{CGF} = \frac{\hbar}{a^4} \left( \frac{E_{CGF} - \mu^2 \langle b^2 \rangle}{g^2} - \frac{9\langle b^2 \rangle^2}{32\lambda^2} \right)$$ (4.17)

The dimensionless density and pressure are

$$\hat{\rho}_{CGF} = \frac{t^4_{Higgs} \rho_{CGF}}{\hbar} = \frac{1}{\hat{a}^4} \left( \frac{3E_{CGF}}{g^2} + \frac{9\langle b^2 \rangle^2}{32\lambda^2} \right)$$

$$\hat{p}_{CGF} = \frac{t^4_{Higgs} p_{CGF}}{\hbar} = \frac{1}{\hat{a}^4} \left( \frac{E_{CGF} - \mu^2 \langle b^2 \rangle}{g^2} - \frac{9\langle b^2 \rangle^2}{32\lambda^2} \right)$$ (4.18)

These two equations parametrize the density and pressure by $k^2$. The equation of state relating $p$ to $\rho$ is given implicitly by the two equations.

4.5 Parametrize the time evolution by $k^2$

Table 2 summarizes the parametrization by $k^2$. The leading terms in the $k^2 \to 0$ regime are calculated in [7] using the Taylor series expansions (A.4) of $K$ and $E$. 

15
\[ k_{EW}^2 = \frac{1}{2} \quad k^2 \to 0 \]

\[
\begin{align*}
\alpha^3 &= \frac{2(1 - k^2)K + 2(2k^2 - 1)E}{K_{EW}} \\
\alpha^2\langle b^2 \rangle &= k^2 - 1 + \frac{E}{K} \\
\alpha^2 \mu^2 &= 1 - 2k^2 \\
\alpha^4 E_{\text{CGF}} &= \frac{k^2(1 - k^2)}{2} \\
\alpha^2 a^2 &= \frac{3\alpha^2 \langle b^2 \rangle}{2} + \frac{4\lambda^2 \alpha^2 \mu^2}{g^2} \\
\hat{\rho}_{\text{CGF}} &= \frac{1}{a^4} \left( \frac{3E_{\text{CGF}}}{g^2} + \frac{9\langle b^2 \rangle^2}{32\lambda^2} \right) \\
\hat{\rho}_{\text{CGF}} &= \frac{1}{a^4} \left( \frac{E_{\text{CGF}} - \mu^2 \langle b^2 \rangle}{g^2} - \frac{9\langle b^2 \rangle^2}{32\lambda^2} \right) \\
\frac{2(\phi^4 \phi)_0}{v^2} &= 1 - \frac{3\langle b^2 \rangle}{2a^2} \\
\end{align*}
\]

Table 2: The parametrization by \( k^2 \).

\[
\begin{array}{|c|c|c|}
\hline
\alpha^3 &= \frac{2(1 - k^2)K + 2(2k^2 - 1)E}{K_{EW}} & 1 \to \frac{3\pi k^2}{2K_{EW}} \\
\alpha^2 \langle b^2 \rangle &= k^2 - 1 + \frac{E}{K} & \frac{\pi}{4K_{EW}^2} \to \frac{k^2}{2} \\
\alpha^2 \mu^2 &= 1 - 2k^2 & 0 \to 1 \\
\alpha^4 E_{\text{CGF}} &= \frac{k^2(1 - k^2)}{2} & \frac{1}{8} \to \frac{k^2}{2} \\
\alpha^2 a^2 &= \frac{3\alpha^2 \langle b^2 \rangle}{2} + \frac{4\lambda^2 \alpha^2 \mu^2}{g^2} & \frac{3\pi}{8K_{EW}^2} \to \frac{4\lambda^2}{g^2} \\
\hat{\rho}_{\text{CGF}} &= \frac{1}{a^4} \left( \frac{3E_{\text{CGF}}}{g^2} + \frac{9\langle b^2 \rangle^2}{32\lambda^2} \right) & 8K_{EW}^4 + \frac{1}{8\lambda^2} \to \frac{3g^2 k^2}{32\lambda^4} \\
\hat{\rho}_{\text{CGF}} &= \frac{1}{a^4} \left( \frac{E_{\text{CGF}} - \mu^2 \langle b^2 \rangle}{g^2} - \frac{9\langle b^2 \rangle^2}{32\lambda^2} \right) & 8K_{EW}^4 + \frac{1}{8\lambda^2} \to \frac{9g^2(8\lambda^2 - g^2)k^4}{2048\lambda^6} \\
\frac{2(\phi^4 \phi)_0}{v^2} &= 1 - \frac{3\langle b^2 \rangle}{2a^2} & 0 \to 1 - \frac{3g^2 k^2}{16\lambda^2} \\
\hline
\end{array}
\]
5 CGF as the dark matter

5.1 $\Omega_{\text{CGF}} + \Omega_\Lambda = 1$ in the present

The Friedmann equation is

$$H^2 + \frac{1}{R^2} = \frac{1}{3}\kappa(\rho_m + \rho_\Lambda) \quad H = \frac{1}{R} \frac{dR}{dt} \quad (5.1)$$

$\rho_m$ is the matter density and $\rho_\Lambda$ is the dark energy density (assumed due to the cosmological constant). In scaled coordinates,

$$R = \frac{a}{\epsilon} \quad dt = adz \quad (5.2)$$

We are ignoring fluctuations so the entire matter density is

$$\rho_m = \rho_{\text{CGF}} \quad (5.3)$$

The Friedmann equation normalized by $H_0^2$ is

$$\frac{H^2}{H_0^2} = \Omega_{\text{CGF}} + \Omega_\Lambda + \Omega_{\text{curvature}} \quad (5.4)$$

$\rho_c$ is the critical density. The observed present curvature is close to zero, $|\Omega_{\text{curvature}}| < 0.001$. The observed dark energy density is $\Omega_\Lambda = 0.685$. We assume that the dark energy is due to the cosmological constant, i.e. that $\Omega_\Lambda$ is constant in time.

Using the $k \to 0$ asymptotic formulas,

$$\Omega_{\text{CGF}} = \frac{\kappa}{3H_0^2}\rho_{\text{CGF}} = \frac{t_{\text{Hubble}}^2}{3t_{\text{Higgs}}^4}\rho_{\text{CGF}} = \frac{t_{\text{Hubble}}^2}{3t_{\text{Higgs}}^4}\frac{g^2 k^2}{32\lambda^4} = \frac{K_{\text{EW}}}{6\pi g\lambda} \frac{t_{\text{Hubble}}^2}{t_{\text{Higgs}} a^3} \quad (5.5)$$

The present time is identified by the condition $H = H_0$. By the Friedmann equation (5.4), $H = H_0$ is equivalent to $\Omega_{\text{CGF}} = 0.315$ assuming that $|\Omega_{\text{curvature}}| < 0.001$. So the present values of $k^2$ and $a$ are

$$k_0^2 = 0.315 \frac{t_{\text{Higgs}}^4}{t_{\text{Hubble}}^2} \frac{32\lambda^4}{g^2} = 7.89 \times 10^{-56}$$

$$a_0 = \left( \frac{0.315 \frac{6\pi g\lambda}{K_{\text{EW}}}}{t_{\text{Higgs}}^2 t_{\text{Hubble}}^2} \right)^{-\frac{1}{4}} = 1.40 \times 10^{-8} \text{ s} = 2.66 \times 10^{18} t_{\text{Higgs}} = 4.54 \times 10^{18} a_{\text{EW}}$$

$k_0^2$ is very small so using the asymptotic formulas is justified. The present curvature is

$$-\Omega_{\text{curvature}} = \frac{t_{\text{Hubble}}^2}{a_0^2} \frac{\epsilon^2}{\kappa} = 1.07 \times 10^{51} \epsilon^2 \quad (5.7)$$
so the present flatness condition, $|\Omega_{\text{curvature}}| < 0.001$, is equivalent to $\epsilon < 10^{-27}$ which is $\hat{E}_{\text{CGF}} > 10^{107}$.

$$\hat{E}_{\text{CGF}} > 10^{107} \iff |\Omega_{\text{curvature}}| < 0.001 \quad (5.8)$$

Then

$$\Omega_{\text{CGF}} + \Omega_\Lambda = 1 \quad (5.9)$$

The CGF is the dark matter.

### 5.2 $w_{\text{CGF}} \approx 0$ after $10^2 a_{\text{EW}}$

The equation of state parameter is

$$w_{\text{CGF}} = \frac{p_{\text{CGF}}}{\rho_{\text{CGF}}} \quad (5.10)$$

The formulas of Table 2 give, in the $k \to 0$ regime,

$$w_{\text{CGF}} = \left(\frac{3}{8} - \frac{3g^2}{64\lambda^2}\right) k^2 \quad (5.11)$$

so the CGF evolves as a non-relativistic fluid ($w = 0$) after the first one or two ten-folds of expansion from $a_{\text{EW}}$. Figure 3 shows $w_{\text{CGF}}$ approaching 0 as the electroweak transition approaches completion, i.e. as $(\phi^\dagger \phi)_0$ approaches the vacuum expectation value $v^2/2$.

![Figure 3](image-url)

In the $k \to 0$ regime the formulas of Table 2 give

$$\rho_{\text{CGF}} = \frac{K_{\text{EW}}}{2\pi g \lambda \frac{\hbar}{t_{\text{Higgs}}}} \frac{1}{a^3} = \frac{0.890 m_{\text{Higgs}}}{a^3} \quad (5.12)$$

showing the $1/a^3$ behavior of a $w = 0$ fluid, which follows from the conservation of energy-momentum

$$\rho + \frac{3}{a} (\rho + p) = 0 \quad \frac{d \ln \rho}{d \ln a} = -3(1 + w) \quad (5.13)$$
Let $x^\mu_{\text{phys}}$ be the dimensionful physical coordinates

$$x^0_{\text{phys}} = t \quad x^i_{\text{phys}} = a x^i \quad ds^2 = dt^2 - \delta_{ij} dx^i_{\text{phys}} dx^j_{\text{phys}}$$  \hspace{1cm} (5.14)

The CGF is

$$B_{\text{CGF}} = B^{\text{phys}}_{\text{CGF},i} dx^i_{\text{phys}}$$  \hspace{1cm} (5.15)

The dimensionful gauge field $B^{\text{phys}}_{\text{CGF},i}$ is given by

$$B^{\text{phys}}_{\text{CGF},i} dx^i_{\text{phys}} = b(z) \gamma_i dx^i$$  \hspace{1cm} (5.16)

which is

$$B^{\text{phys}}_{\text{CGF},i} = \frac{b(z)}{a} \gamma_i = \frac{k \cn(u, k)}{\alpha a} \gamma_i \quad du = \frac{1}{\alpha} dz = \frac{1}{\alpha a} dt$$  \hspace{1cm} (5.17)

In the $k \to 0$ regime ($a > 10^2 a_{\text{EW}}$),

$$\alpha a = \frac{2 \lambda_{\text{Higgs}}}{g} = \frac{2 \lambda h}{g m_{\text{Higgs}}} = \frac{h}{m_W} = \frac{1}{\omega_W} \quad \cn(u, k) = \cos(u) = \cos(\omega_W t)$$  \hspace{1cm} (5.18)

so the dimensionful CGF is

$$B^{\text{phys}}_{\text{CGF},i} = k \omega_W \cos(\omega_W t) \gamma_i \quad \omega_W = \frac{g}{2 \lambda_{\text{Higgs}}} = \frac{m_W}{h} = 1.22 \times 10^{26} \text{s}^{-1}$$  \hspace{1cm} (5.19)

The present CGF oscillates harmonically at the bottom of the massive gauge field potential. The oscillation period is

$$4K\alpha a = \frac{2\pi}{\omega_W} = 5.15 \times 10^{-26} \text{s}$$  \hspace{1cm} (5.20)

The present value of $k$ is $k_0 = 2.81 \times 10^{-28}$ so, presently,

$$B^{\text{phys}}_{\text{CGF},i} = 2.26 \times 10^{-26} \text{GeV} \frac{\cos(\omega_W t) \gamma_i}{h}$$  \hspace{1cm} (5.21)

Again, this is a leading order calculation, without fluctuations. Higher order effects presumably cause the fluctuating CGF to collapse gravitationally into an ensemble of self-gravitating bodies such as the dark matter stars described in [2].

### 5.4 CGF equation of state

The density scale of the CGF is

$$\rho_b = \frac{h}{t^2_{\text{Higgs}}} = 5.68 \times 10^{28} \frac{\text{kg}}{\text{m}^3}$$  \hspace{1cm} (5.22)

The dimensionless CGF density and pressure are

$$\hat{\rho}_{\text{CGF}} = \frac{\rho_{\text{CGF}}}{\rho_b} \quad \hat{p}_{\text{CGF}} = \frac{p_{\text{CGF}}}{\rho_b}$$  \hspace{1cm} (5.23)

The dimensionless density at $a_{\text{EW}}$ from Table 2 is

$$\hat{\rho}_{\text{EW}} = \frac{8 K^4_{\text{EW}}}{3 \pi^2 g^2} + \frac{1}{8 \lambda^2} = 7.97$$  \hspace{1cm} (5.24)
Equation (2.30) gives the equation of state for $\hat{\rho} \geq \hat{\rho}_{EW}$

$$\hat{\rho} \geq \hat{\rho}_{EW} \quad \hat{\rho} = \frac{1}{3} (\hat{\rho} - c_b \hat{\rho}_{EW}) \quad c_b = \frac{1}{2\lambda^2 \hat{\rho}_{EW}} = 0.243 \quad (5.25)$$

For $\hat{\rho} \leq \hat{\rho}_{EW}$ the equation of state is given implicitly by the analytic functions of Table 2.

$$\hat{\rho} \leq \hat{\rho}_{EW} \quad \hat{\rho}, \hat{p} = \hat{\rho}_{CGF}(k^2), \hat{p}_{CGF}(k^2) \quad 0 \leq k^2 \leq \frac{1}{2} \quad (5.26)$$

The equation of state is well-defined because $\hat{\rho}_{CGF}(k^2)$ is monotonic in $k^2$ as shown in Figure 4. The limit $k^2 \to 0$ is $\hat{\rho} \to 0$. From Table 2, the equation of state in the limit is

$$\rho \to 0 \quad p = \frac{c_a}{2} \rho^2 \quad c_a = \frac{\lambda^2(8\lambda^2 - g^2)}{g^2} = 0.992 \quad (5.27)$$

### 5.5 Adiabatic condition for $a \geq a_{EW}$

Finally, we need to verify that the adiabatic condition is satisfied for $a \geq a_{EW}$, that the adiabatic approximation of the time evolution is justified. The ratio of the oscillation period $4K\alpha a$ to the expansion time $1/H$ is $4K\alpha aH$. The Friedmann equation (5.2) can be written

$$a^2 H^2 + \epsilon^2 = \frac{t_{Higgs}^2}{t_{Higgs}^2} \frac{1}{3} a^2 \hat{\rho}_{CGF} + \frac{t_{Higgs}^2}{t_{Hubble}^2} \Omega_\Lambda a^2 \quad (5.28)$$

$$\dot{a} = \frac{a}{t_{Higgs}} \quad \hat{\rho}_{CGF} = \frac{1}{h t_{Higgs}} \hat{\rho}_{CGF}$$

The quantity $4K\alpha \sqrt{a^2 H^2 + \epsilon^2}$ is an upper bound on the ratio of time scales $4K\alpha aH$. It is plotted in Figure 5 for the first two ten-folds of expansion after $a_{EW}$. 

Figure 4: $\hat{\rho}_{CGF}$ increases monotonically with $k^2$. 
At \( a_{\text{EW}} \) the bound is \( 3.64 \times 10^{-16} \). The \( \epsilon^2 \) term is negligible, so this is the actual ratio of time scales at \( a_{\text{EW}} \). The Friedmann equation (5.28) in the \( k \to 0 \) regime gives

\[
(4K\alpha a H)^2 + (4K\alpha \epsilon)^2 = \frac{t_{\text{grav}}^2}{r_{\text{Higgs}}^2} \frac{\pi^2}{2\lambda^2} k^2 + \frac{t_{\text{Higgs}}^2}{t_{\text{Hubble}}^2} \frac{16\pi^2 \lambda^2}{g^2} \Omega \lambda
\]

\[
= 5.04 \times 10^{-32} k^2 + 8.65 \times 10^{-87}
\]

so the ratio of time scales \( 4K\alpha a H \) decreases monotonically from \( 10^{-16} \) as \( a \) increases from \( a_{\text{EW}} \). The present ratio of time scales is

\[
4K_0\alpha_0 a_0 H_0 = 2\pi \frac{2\lambda}{g} \frac{t_{\text{Higgs}}}{t_{\text{Hubble}}} = 1.12 \times 10^{-43}
\]

The adiabatic condition is well satisfied from \( a_{\text{EW}} \) onward (and, as we saw earlier, for some ten-folds before \( a_{\text{EW}} \)). The adiabatic approximation of the CGF time evolution is justified.

### 6 Questions and comments

6.1 Testing the CGF cosmology
6.2 Spin(4) and fluctuations
6.3 Thermalization before \( a_{\text{EW}} \)
6.4 CGF temperature after \( a_{\text{EW}} \)
6.5 CP violation
6.6 Semi-classical approximation
6.7 Neutrinos
6.1 Testing the CGF cosmology

Two approaches to checking the theory seem obvious.

1. The initial thermal state of the fluctuations should be constructed and its time evolution calculated to see if the right amount of ordinary matter results. As a start, methods are developed in [3] to construct the initial thermal state of the fluctuations.

2. If the CGF is indeed the dark matter, can it be detected? What form does it take in the present? What are its interactions with ordinary matter? It seems reasonable to suppose that fluctuations in the CGF have collapsed gravitationally to stable self-gravitating structures. As a first step, the Tolman-Oppenheimer-Volkoff stellar structure equations for stars made of the CGF are solved numerically in [2]. It should be possible to model a purely dark matter universe populated with galaxies of dark matter stars. The actual universe would be a perturbation of the dark matter universe.

If the theory survives testing then first principles cosmology will become possible. All of the Standard Model cosmological epoch will be calculable from first principles. Eventually, discrepancies between the theory and cosmological observation can become clues to more fundamental physical principles.

6.2 Spin(4) and fluctuations

The Spin(4) symmetry is a global symmetry of the universe in the CGF cosmology. Presumably we live in a fluctuation where the global Spin(4) is not apparent. We see the Poincaré symmetry of the Standard Model with the Higgs field $\phi$ transforming as a scalar, not a spinor.

Before the electroweak transition, as the universe expands towards $a_{\text{ew}}$, the fluctuations of $\phi$ around 0 grow large. After $a_{\text{ew}}$ the $\phi$ fluctuations concentrate again, now at the bottom of the effective Higgs potential, the set $\phi^\dagger \phi = (\phi^\dagger \phi)_0$ which evolves slowly towards the vacuum expectation value $v^2/2$.

Spin(4) continues to be a global symmetry of the universe. The fluctuations of $\phi(x)$ at the bottom of the effective Higgs potential are Spin(4)-symmetric. The crucial quantity is the correlation length of the $\phi$ fluctuations after the electroweak transition. This is the largest distance $\xi$ such that

$$\text{dist}(x_1, x_2) < \xi \implies \langle \phi^\dagger(x_1) \phi(x_2) \rangle = \frac{v^2}{2}$$

(6.1)

The Standard Model as we observe it operates within regions of size $\xi$.

6.3 Thermalization before $a_{\text{ew}}$

The imaginary time periodicity of the CGF defines a specific thermal state of the fluctuations of the Standard Model fields. That state is thermodynamically stable against fluctuations of the SU(2) gauge field [3]. It remains to be checked that it is stable against all fluctuations. A generic state presumably thermalizes to this state. Is it possible to determine how long the thermalization takes? how long before $a_{\text{ew}}$ the CGF must have been oscillating to ensure that the fluctuations are in the thermal state at $a_{\text{ew}}$?
My first motivation for formulating the CGF cosmology was to explain the origin of cosmological temperature as the imaginary time periodicity of the CGF \([1]\). I made a handwaving estimate of the redshift \(a_0/a_{EW}\) by assuming that the CGF temperature \(T_{CGF}\) at \(a_{EW}\) redshifted to the CMB temperature \(T_{CMB}\) in the present. We can be more definite now that we know the time evolution of the CGF after \(a_{EW}\).

The imaginary period in the time variable \(u\) is \(\Delta u = 4K'i\), so the imaginary period in \(z\) is \(\Delta z = \alpha \Delta u = 4K'\alpha i\), so the imaginary period in co-moving time \(\Delta t = a \Delta z = 4K'\alpha a i\). So the inverse CGF temperature after \(a_{EW}\) is

\[
\frac{\hbar}{k_B T_{CGF}} = 4K'\alpha a
\]  

The present CGF temperature is

\[
T_{CGF,0} = \frac{\hbar}{k_B 4K'_0 \alpha_0 a_0} = 3.60 \times 10^{12} \text{ K} \quad k_B T_{CGF,0} = 310 \text{ MeV}
\]  

so the ordinary matter must have decoupled from the CGF some time in the past.

Figure 6 shows the CGF temperature after \(a_{EW}\).

In the \(k \to 0\) regime, the asymptotic formulas and equation (A.4) give

\[
T_{CGF} = \frac{\hbar}{k_B 4 \ln(4/k)} \frac{g}{2 \lambda_{Higgs}} = 2.33 \times 10^{14} \text{ K} \quad k_B T_{CGF} = \frac{20.1 \text{ GeV}}{\ln(4/k)}
\]  

Redshifting the asymptotic CGF temperature \(T_{CGF}\) to the present gives

\[
\frac{a}{a_0} T_{CGF} = \frac{\hbar}{k_B 4 \ln(4/k)} \frac{g}{2 \lambda_{Higgs}} \frac{a}{a_0} = 8.77 \times 10^{-5} \text{ K} \quad \frac{\hat{a}}{\ln(4/k)}
\]  

The redshifted CGF temperature will equal the CMB temperature,

\[
\frac{a}{a_0} T_{CGF} = T_{CMB} = 2.7255 \text{ K}
\]  

when \(\hat{a} \approx 10^6\) which is well within the asymptotic \(k \to 0\) regime. Numerical solution of

\[
\frac{a}{a_0} T_{CGF} = \frac{\hbar}{k_B 4K'\alpha a_0} = 1.36 \times 10^{-4} \text{ K} \quad \frac{K'\alpha}{K'} = 2.7255 \text{ K}
\]
finds [7]

\[ k^2 = 5.12 \times 10^{-18} \quad \frac{a}{a_{\text{EW}}} = 1.13 \times 10^6 \quad k_B T_{\text{CGF}} = 0.944 \text{ GeV} \quad (6.8) \]

which is indeed within the asymptotic regime. The CGF has to decouple from the ordinary matter at \( a = 10^6 a_{\text{EW}} \) when \( k_B T_{\text{CGF}} = 1 \text{ GeV} \).

This is a very simplistic calculation. It ignores fluctuations and supposes that all of the ordinary matter decouples at the same time and that the temperature simply redshifts from that time on.

### 6.5 CP violation

If the CGF cosmology is to succeed as a complete theory of the Standard Model epoch, it will have to explain the baryon-antibaryon asymmetry. Everything after \( a = a_{\text{EW}} \) is completely determined by the Spin(4)-symmetric initial condition leading up to the electroweak transition. What comes before the Standard Model epoch is immaterial.

My thought in [1] was that the discrete CP symmetry takes \( \hat{b} \rightarrow -\hat{b} \) (because the Dirac matrices \( \gamma_i \) change sign when the orientation of space is reversed). During the transition \( \hat{b} \) settles in one of the minima \( \hat{b} = \pm 1 \) of the double well potential (2.13), breaking CP as it settles. But the two minima are indistinguishable in local coordinates, i.e. \( \hat{b} = \pm \epsilon \). So any CP violating effects have to be global.

The Higgs field fluctuations play a role in this. The symmetry group Spin(4) is SU(2) × SU(2). The Higgs field \( \phi(x) \) decomposes into a sum of irreducible representations

\[ (1/2, 0) \otimes \sum_j (j, j) = \sum_j (j + 1/2, j) \oplus (j, j + 1/2) \quad (6.9) \]

CP exchanges \( (j + 1/2, j) \leftrightarrow (j, j + 1/2) \). The orientation of the fluctuation in the vector space \( (j + 1/2, j) \oplus (j, j + 1/2) \) can break \( CP \). Again, any effects will be global since \( (j + 1/2, j) \) and \( (j, j + 1/2) \) become indistinguishable when \( j \) is large.

The expectation value of the baryon number density will be zero. The question again is whether the baryon number density is correlated over large enough regions and whether the magnitude of the two point function in such regions is big enough.

### 6.6 Semi-classical approximation

A quantitative measure of the validity of the semi-classical approximation is the adiabatic invariant \( I \) of equation (4.10) An oscillating quantum system is semi-classical when \( \frac{1}{\hbar} I \gg 1 \). For the CGF, equations (4.12) and (4.14) and the bound (5.8) on \( \hat{E}_{\text{CGF}} \) give

\[ \frac{1}{\hbar} I = \frac{\text{Vol}(S^3)}{2\pi\epsilon^3} \frac{2K_{\text{EW}}}{g^2} > 10^{82} \quad (6.10) \]

So the CGF is semi-classical.

### 6.7 Neutrinos

It is not entirely accurate that the theory assumes nothing beyond the Standard Model. This is true in the classical approximation. When fluctuations are taken into account, a mechanism will have to be added to the Standard Model to produce neutrino masses and mixing.
A Elliptic integrals and elliptic functions

The following formulas are from Gradsteyn and Ryzhik (G&R) [14] and the Digital Library of Mathematical Functions (DLMF) [15]:

G&R 8.11, DLMF 19.2(ii)

The complete elliptic integrals of the first and second kinds are

\[
K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \alpha}} \, d\alpha \quad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \alpha} \, d\alpha
\]

(A.1)

\[k^2 + k'^2 = 1 \quad K = K(k) \quad E = E(k) \quad K' = K(k') \quad E' = E(k')\]

\(k\) is called the elliptic modulus, \(k'\) the complementary modulus. \(k^2\) is called the elliptic parameter (usually written \(m\)). We can assume \(0 < k, k' < 1\).

G&R 8.122, DLMF 19.7.1

\[EK' + E'K - KK' = \frac{\pi}{2}\]  

(A.2)

G&R 8.129

\[k^2 = \frac{1}{2} \quad K = K' = \frac{\Gamma \left( \frac{1}{4} \right)^2}{4 \sqrt{\pi}} \quad E = E' = \frac{K}{2} + \frac{\pi}{4K}\]  

(A.3)

G&R 8.113-4, DLMF 19.5.1-2,19.12.1

\[k \to 0 \quad K(k) \to \frac{\pi}{2} \left( 1 + \frac{k^2}{4} + \frac{9k^4}{64} \right) + O(k^6)\]

\[E(k) \to \frac{\pi}{2} \left( 1 - \frac{k^2}{4} - \frac{3k^4}{64} \right) + O(k^6)\]  

(A.4)

\[K' \to \ln \left( \frac{4}{k} \right) + O(k^2 \ln k)\]

G&R 8.123, DLMF 19.4(i)

\[k \frac{dK}{dk} = \frac{E}{k'^2} - K \quad k \frac{dE}{dk} = E - K\]  

(A.5)

which imply

\[k \frac{d E}{dk K} = -\frac{1}{k'^2} \left( \frac{E}{K} \right)^2 + \frac{2E}{K} - 1\]  

(A.6)

G&R 8.159, DLMF 22.13.2

The Jacobi elliptic function \(cn(z) = cn(z, k)\) is the analytic function (with poles) satisfying

\[cn'(z)^2 = (1 - cn^2)(k'^2 + k^2 cn^2) \quad cn(0) = 1\]  

(A.7)
which is to say that \( f(z) = k \cn(z, k) \) solves

\[
f'^2 = (k^2 - f^2)(k'^2 + f^2)
\]  

(A.8)

G&R 8.151, 8.146, DLMF 22.4(i), 22.5(ii)

\( \cn(z, k) \) is doubly periodic in \( z \).

\[
\cn(z) = \cn(z + 4K) = \cn(z + 4K' i) = \cn(z + 2K + 2K' i)
\]  

(A.9)

On the real axis, \( \cn(z, k) \) oscillates between \( \pm 1 \) with period \( 4K \). In the limit \( k \to 0 \),

\[
k \to 0 \quad \cn(z, k) \to \cos(z)
\]  

(A.10)

G&R 5.134, 5.131

\[
\int_0^{4K} k^2 \cn(z)^2 \, dz = 4 \left[(k^2 - 1)K + E\right]
\]  

(A.11)

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