SINGULARITY STRUCTURE ANALYSIS, INTEGRABILITY, 
SOLITONS and DROMIONS in (2+1)-DIMENSIONAL 
ZAKHAROV EQUATIONS

Ratbay MYRZAKULOV

Centre for Nonlinear Problems, PO Box 30, 480035, Alma-Ata-35, Kazakhstan

Abstract

In this paper, a singularity structure analysis of the (2+1)-dimensional 
Zakharov equation is carried out and it is shown that it admits the Painleve 
property. The bilinear form of this equation is derived from the Painleve 
analysis in a straightforward manner. Using this bilinear form, we have 
constructed the simply one soliton solution by the Hirota method. We have 
then presented the localized solution (dromion). The (2+1)-dimensional 
Fokas equation is shown to be nothing but the particular case of the Za-
kharov equation. Finally, we have presented the associated integrable spin 
equations.
1 Introduction

In recent years there has been considerable interest in (2+1)-dimensional soliton equations [1-2]. The study of these equations has thrown up new ideas in soliton theory, because, they, though, are much richer than their 1+1 dimensional counterparts. Particularly, the introduction of exponentially localized structures (dromions) has triggered renewed interest in these integrable equations [3-12]. An specially important subject related to the study of nonlinear differential equations (NLDE) is that concerning the singularity structure of them [13-14]. The singularity structure analysis appears as a systematic procedure for constructing Bäcklund, Darboux and Miura transformations, Lax representations, different types solutions etc of given NLDE [14]. At the same time, the Painleve test allows us identify integrable equations (see, e.g., [3] and refs therein). Notable amongst (2+1)-dimensional soliton equations are the Davey-Stewartson (DS) equation, the Zakharov equations (ZE), the Nizhnik-Novikov-Veselov equation, the Ishimori equation, the Kadomtsev-Petviashvili equation and so on.

In this paper, we consider the following ZE

\[ iq_t + M_1 q + vq = 0 \]  
\[ ip_t - M_1 p - vp = 0 \]  
\[ M_2 v = -2M_1(pq) \]

where

\[ M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} + 4\alpha(b-a) \frac{\partial^2}{\partial x \partial y} + 4(a^2 - 2ab - b) \frac{\partial^2}{\partial x^2}, \]
This equation was may be introduced in [5] and is integrable. Equation (1) contains several important particular cases. So, for example, we have the following cases:

(i) $a = b = -\frac{1}{2}$, it yields the DS equation

$$i q_t + q_{xx} + \alpha^2 q_{yy} + v q = 0 \quad (2a)$$

$$\alpha^2 v_{yy} - v_{xx} = -2(\alpha^2 (pq)_{yy} + (pq)_{xx}). \quad (2b)$$

(ii) When $a = b = -1$, we obtain the following equation

$$i q_t + q_{YY} + v q = 0 \quad (3a)$$

$$i p_t - p_{YY} - vp = 0 \quad (3b)$$

$$v_x + v_Y + 2(pq)_Y = 0 \quad (3c)$$

where $X = x/2, \ Y = y/\alpha$.

(iii) If $a = b = -1, X = t$, then equation (1) reduces to the following (1+1)-dimensional Yajima-Oikawa equation [19]

$$i q_t + q_{YY} + v q = 0 \quad (4a)$$

$$i p_t - p_{YY} - vp = 0 \quad (4b)$$

$$v_t + v_Y + 2(pq)_Y = 0 \quad (4c)$$

and so on [16].

Even though the ZE (1) is known to be completely integrable, its Painleve property has not yet been established. Also the interesting question arises naturally, whether there exist dromion solutions in equation (1) as well. In this paper, following Lakshmanan and coworkers (see, e.g. [3] and references therein), we address ourselves to these problems and carry out the singularity structure analysis and confirm its Painleve nature. We also deduce its bilinear form from the Painleve analysis. Next we construct soliton and dromion solutions using the Hirota method. We also show that the Fokas equation (FE) is the particular case of the ZE as $a = -\frac{1}{2}, \alpha^2 = 1$.

The present work falls into seven parts. In section II, we present the equivalent forms of the ZE (1) apart from studying its properties. In section III, we carry out the singularity structure analysis of equation (1) and confirm its Painleve nature. We then obtain the Hirota bilinear form directly from the Painleve analysis in section IV. In section V, we generate the simplest one soliton solution (1-SS), the dromion and 1-rational solutions. A connection between the ZE and the FE we will discuss in section VI. The associated integrable spin systems we present in section VII. Section VIII contains a short discussion of the results.
2 Lax representation and equivalent forms

Equation (1) has the following Lax representation [5]

\[ \alpha \Psi_y = 2B_1 \Psi_x + B_0 \Psi \quad (5a) \]
\[ \Psi_t = 4iC_2 \Psi_{xx} + 2C_1 \Psi_x + C_0 \Psi \quad (5b) \]

with
\[ B_1 = \begin{pmatrix} a + 1 & 0 \\ 0 & a \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \]
\[ C_2 = \begin{pmatrix} b + 1 & 0 \\ 0 & b \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & iq \\ ip & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \]
\[ c_{12} = i[2(2b - a + 1)q_x + i\alpha q_y] \quad c_{21} = i[2(a - 2b)p_x - i\alpha p_y] \]

and \( v = i(c_{22} - c_{11}) \). Here \( c_{jj} \) are the solution of the following equations

\[ 2(a + 1)c_{11x} - \alpha c_{11y} = i[2(2b - a + 1)(pq)_x + \alpha (pq)_y] \quad (6a) \]
\[ 2ac_{22x} - \alpha c_{22y} = i[2(a - 2b)(pq)_x - \alpha (pq)_y]. \quad (6d) \]

For our future algebra the ZE in the form (1) is rather complicated. Because it has sense to look for a other more convenient and elegant forms of equation (1). The first form we obtain from the compatibility condition of equations (5).

We have
\[ iq_t + M_1q + i(c_{22} - c_{11})q = 0 \quad (6a) \]
\[ ip_t - M_1p - i(c_{11} - c_{22})p = 0 \quad (6b) \]
\[ 2(a + 1)c_{11x} - \alpha c_{11y} = i[2(2b - a + 1)(pq)_x + \alpha (pq)_y] \quad (6c) \]
\[ 2ac_{22x} - \alpha c_{22y} = i[2(a - 2b)(pq)_x - \alpha (pq)_y]. \quad (6d) \]

Now, if we introduce the following transformations
\[ V' = c_{22} - i(2b + 1)pq, \quad U' = c_{11} - i(2b + 1)pq \quad (7) \]

then equation (6) takes the form
\[ iq_t + M_1q + i(V' - U')q = 0 \quad (8a) \]
\[ ip_t - M_1p - i(V' - U')p = 0 \quad (8b) \]
\[ 2aV'_x - \alpha V'_y = -2ib[2(a + 1)(pq)_x - \alpha (pq)_y] \quad (8c) \]
\[ 2(a + 1)U'_x - \alpha U'_y = -2i(b + 1)[2a(pq)_x - \alpha (pq)_y]. \quad (8d) \]

Let us now rewrite this equation in the following form
\[ iq_t + (1 + b)q \xi \xi - bq_{\eta \eta} + [2bV - 2(b + 1)U]q = 0 \quad (9a) \]
\[ ip_t - (1 + b)p \xi \xi + bq_{\eta \eta} - [2bV - 2(b + 1)U]p = 0 \quad (9b) \]
\[ V_\xi = (pq)_\eta \quad (9c) \]
\[ U_\eta = (pq)_\xi \quad (9d) \]
where $U, V, \xi$ and $\eta$ are defined by

\[
\begin{align*}
U' &= -2i(b + 1)U, \quad V' = -2ibV, \\
\xi &= \frac{x}{2} + \frac{a + 1}{\alpha}y, \quad \eta = -\frac{x}{2} - \frac{a}{\alpha}y.
\end{align*}
\] (10)

Having this form of the ZE, we are in a convenient position to explore the singularity structure of it. Note that in terms of $\xi, \eta$, equation (1) takes the form

\[
\begin{align*}
iq_t &+ (1 + b)q\xi - bq\eta + vq = 0 \quad (11a) \\
ipt &- (1 + b)p\xi + bq\eta - vq = 0 \quad (11b) \\
v_{\xi\eta} &= -2[(1 + b)(pq)\xi - b(pq)\eta]. \quad (11c)
\end{align*}
\]

In particular, from this equation as $b = 0$, we get the other ZE [5]

\[
\begin{align*}
iq_t &+ q\xi + vq = 0 \quad (12a) \\
ipt &- p\xi - vq = 0 \quad (12b) \\
v_{\eta} &= -2(pq)\xi. \quad (12c)
\end{align*}
\]

### 3 Singularity structure analysis

In order to carry out a singularity structure analysis, following [3], we effect a local Laurent expansion in the neighbourhood of a noncharacteristic singular manifold $\phi(\xi, \eta, t) = 0$, $\phi_{\xi}, \phi_{\eta}, \phi_t \neq 0$. We assume the leading orders of the solutions of equation (9) to take the form

\[
q = q_0\phi^m, \quad p = p_0\phi^n, \quad V = V_0\phi^\gamma, \quad U = U_0\phi^\delta
\] (13)

where $q_0$, $p_0$, $V_0$ and $U_0$ are analytic functions of $(\xi, \eta, t)$. In (13) $m, n, \gamma$ and $\delta$ are integers (if they exist) to be evaluated. Substituting expressions (13) into equation (9) and balancing the most dominant terms, we get

\[
m = n = -1, \quad \gamma = \delta = -2
\] (14)

and the following equations

\[
p_0q_0 = \phi_{\xi}\phi_{\eta}, \quad V_0 = \phi_{\eta}^2, \quad U_0 = \phi_{\xi}^2.
\] (15)

To evaluate the resonances, we consider the Laurent series of the solutions

\[
q = \sum_{j=0} q_j\phi^{j-1}, \quad p = \sum_{j=0} p_j\phi^{j-1}, \quad V = \sum_{j=0} V_j\phi^{j-2}, \quad U = \sum_{j=0} U_j\phi^{j-2}.
\] (16)

Then we substitute these expansions into equation (9) and equate the coefficients ($\phi^{j-3}, \phi^{j-3}, \phi^{j-3}, \phi^{j-3}$) to zero to give

\[
\begin{bmatrix}
j(j-3)((b+1)\phi_{\xi}^2 - b\phi_{\eta}^2) & 0 & 2bq_0 & -2(b+1)q_0 \\
(j-2)p_0\phi_{\eta} & (j-2)q_0\phi_{\eta} & b\phi_{\eta} & -2(b+1)p_0 \\
(j-2)p_0\phi_{\xi} & (j-2)q_0\phi_{\xi} & 0 & -2(b+1)\phi_{\xi}
\end{bmatrix}
\begin{bmatrix}
q_j \\
p_j \\
V_j \\
U_j
\end{bmatrix} = 0
\] (17)
From the condition for the existence of nontrivial solutions to equation (17), we get the resonance values as

\[ j = -1, 0, 2, 2, 4. \]  

(18)

Obviously, the resonance at \( j = -1 \) represents the arbitrariness of the singularity manifold \( \phi(\xi, \eta, t) = 0 \). At the same time, the resonance at \( j = 0 \) is associated with the arbitrariness of the functions \( q_0, p_0, V_0 \) or \( U_0 \) (cf equation (15)). To prove the existence of arbitrary functions at the other resonance values \( j = 2, 2, 3, 4 \), we use the Laurent expansion (16) into equation (9).

Now, gathering the coefficients of \((\phi^{-2}, \phi^{-2}, \phi^{-2}, \phi^{-2})\), we obtain

\[ 2[bV_0 - (b + 1)U_0]q_1 + 2bq_0V_1 - 2(b + 1)q_0U_1 = A_1 \]  

(19a)

\[ 2[bV_0 - (b + 1)U_0]p_1 + 2bp_0V_1 - 2(b + 1)p_0U_1 = B_1 \]  

(19b)

\[ p_0\phi_\eta q_1 + q_0\phi_\eta p_1 - \phi_\xi V_1 = C_1 \]  

(19c)

\[ p_0\phi_\xi q_1 + q_0\phi_\xi p_1 - \phi_\eta U_1 = D_1 \]  

(19d)

where

\[ A_1 = iq_0\phi_t + (b + 1)[2q_0\phi_\xi + q_0\phi_\xi_\xi] - b[2q_0\phi_\eta + q_0\phi_\eta_\eta] \]  

(20a)

\[ B_1 = -ip_0\phi_t + (b + 1)[2p_0\phi_\xi + p_0\phi_\xi_\xi] - b[2p_0\phi_\eta + p_0\phi_\eta_\eta] \]  

(20b)

\[ C_1 = \phi_\xi \phi_\eta - \phi_\eta \phi_\xi_\eta \]  

(20c)

\[ D_1 = \phi_\eta \phi_\xi_\xi - \phi_\xi \phi_\eta_\eta. \]  

(20d)

The solution of equation (19) has the form

\[ q_1 = \frac{iq_0\phi_t + (b + 1)[2q_0\phi_\xi + q_0\phi_\xi_\xi] - b[2q_0\phi_\eta + q_0\phi_\eta_\eta]}{2[b\phi_\eta^2 - (b + 1)\phi_\xi^2]} \]  

(21a)

\[ p_1 = \frac{-ip_0\phi_t + (b + 1)[2p_0\phi_\xi + p_0\phi_\xi_\xi] - b[2p_0\phi_\eta + p_0\phi_\eta_\eta]}{2[b\phi_\eta^2 - (b + 1)\phi_\xi^2]} \]  

(21b)

\[ V_1 = -\phi_\eta \]  

(21c)

\[ U_1 = -\phi_\xi_\xi. \]  

(21d)

Similarly, collecting the coefficients of \((\phi^{-1}, \phi^{-1}, \phi^{-1}, \phi^{-1})\), we obtain

\[ 2[bV_0 - (b + 1)U_0]q_2 + 2bq_0V_2 - 2(b + 1)q_0U_2 = A_2 \]  

(22a)

\[ 2[bV_0 - (b + 1)U_0]p_2 + 2bp_0V_2 - 2(b + 1)p_0U_2 = B_2 \]  

(22b)

\[ V_1\xi = (p_0q_1 + p_1q_0)_\eta \]  

(22c)

\[ U_1\eta = (p_0q_1 + p_1q_0)_\xi \]  

(22d)

where

\[ A_2 = -iq_0 + bq_0\eta - (b + 1)q_0\xi - [2bV_1 - 2(b + 1)U_1]q_1 \]  

(23a)

\[ B_2 = ip_0 + bp_0\eta - (b + 1)p_0\xi - [2bV_1 - 2(b + 1)U_1]p_1. \]  

(23b)
In this case, from (16) we have

\[ 2bV_3 - (b + 1)U_3q_0 = A_3 \] (24a)
\[ 2bV_3 - (b + 1)U_3p_0 = B_3 \] (24b)
\[ \phi_\eta(p_3q_0 + p_0q_3) - V_3\phi_\xi = C_3 \] (24c)
\[ \phi_\xi(p_3q_0 + p_0q_3) - U_3\phi_\eta = D_3 \] (24d)

with

\[ A_3 = -iq_{1t} - iq_2\phi_t - (b + 1)[q_1\xi + 2q_2\phi_\xi + q_2\phi_\xi_\xi] + b[q_1\eta + 2q_2\phi_\eta + q_2\phi_\eta_\eta] \]
\[ -2b[V_1q_2 + V_2q_1] + 2(b + 1)[U_1q_2 + U_2q_1] \] (25a)
\[ B_3 = ip_{1t} + ip_2\phi_t - (b + 1)[p_1\xi + 2p_2\phi_\xi + p_2\phi_\xi_\xi] + b[p_1\eta + 2p_2\phi_\eta + p_2\phi_\eta_\eta] \]
\[ -2b[V_1p_2 + V_2p_1] + 2(b + 1)[U_1p_2 + U_2p_1] \] (25b)
\[ C_3 = V_2\xi - [(p_0q_2)_\eta + (p_1q_1)_\eta + (p_2q_0)_\eta + (p_2q_1 + p_1q_2)\phi_\eta] \] (25c)
\[ D_3 = U_2\eta - [(p_0q_2)_\xi + (p_1q_1)_\xi + (p_2q_0)_\xi + (p_2q_1 + p_1q_2)\phi_\xi]. \] (25d)

This system can be reduced to the three equations in four unknown functions, hence follows that one of the functions \( q_3, p_3, V_3 \) and \( U_3 \) is arbitrary. Proceeding further to the coefficients of \( (\phi^1, \phi^2, \phi^3, \phi^4) \), we have checked that one of the functions \( q_4, p_4, V_4 \) and \( U_4 \) is arbitrary. Thus the general solution \((q, p, V, U)(\xi, \eta, t)\) of equation (9) admits the required number of arbitrary functions without the introduction of any movable critical manifold, thereby passing the Painleve property. Thus the ZE (9) is expected to be integrable.

4 Bilinearization and Bäcklund transformation

Using the results of the previous section, we can investigate the other integrability properties of equation (9). Particularly, we can construct Bäcklund, Darboux and Miura transformations, Lax representations, bilinear form, different types solutions of the ZE. For example, to obtain the Bäcklund transformation of equation (9), we truncate the Laurent series at the constant level term, that is

\[ q_{j-1} = p_{j-1} = V_j = U_j = 0, \quad j \geq 3 \] (26)

In this case, from (16) we have

\[ q = q_0\phi^{-1} + q_1, \quad p = p_0\phi^{-1} + p_1 \] (27a)
\[ V = V_0\phi^{-2} + V_1\phi^{-1} + V_2, \quad U = U_0\phi^{-2} + U_1\phi^{-1} + U_2 \] (27b)

where \((q, q_1), (p, p_1), (V, V_2)\) and \((U, U_2)\) satisfy equation (9) with \((q_0, p_0, V_0, U_0)\) and \((V_1, U_1)\) satisfying equations (15). If we take the vacuum solution \( q_1 = p_1 = V_2 = U_2 = 0 \), then from the above Bäcklund transformation (27) we have

\[ q = q_0\phi^{-1} \] (28a)
\[ p = p_0 \phi^{-1} \]  
\[ V = V_0 \phi^{-2} + V_1 \phi^{-1} = -\partial_{\eta\eta} \log \phi \]  
\[ U = U_0 \phi^{-2} + U_1 \phi^{-1} = -\partial_{\xi\xi} \log \phi. \]  

Hence and from (9), in the case, when \( \phi \) is real, follows

\[ [i D_t + (b + 1) D^2_\xi - b D^2_\eta] g_0 \circ \phi = 0 \]  
\[ [i D_t - (b + 1) D^2_\xi + b D^2_\eta] p_0 \circ \phi = 0 \]  
\[ D_\xi D_\eta \phi \circ \phi = -2p_0q_0 \]

which is the desired Hirota bilinear form for equations (9). Note that the bilinear form of the ZE for its form (1) is given by

\[ [i D_t - 4(a^2 - 2ab - b) D^2_x - 4\alpha(b - a) D_x D_y - \alpha^2 D^2_y] \left(G \circ \phi\right) = 0 \]  
\[ [i D_t - 4(a^2 - 2ab - b) D^2_x - 4\alpha(b - a) D_x D_y - \alpha^2 D^2_y] \left(P \circ \phi\right) = 0 \]  
\[ 4a(a + 1) D^2_x - 2\alpha(2a + 1) D_x D_y + \alpha^2 D^2_y \left(\phi \circ \phi\right) = -2PG \]

where

\[ q = \frac{G}{\phi}, \quad p = \frac{P}{\phi} \]

with

\[ v = 2M_2 \log \phi. \]

Hereafter (29) \( \equiv (29a, b, c). \)

5 Simplest solutions

Equations (29) allow us to obtain the interesting classes of solutions for the ZE (9) [25]. Below we find some simplest solutions of equation (9), when \( p = Eq^*, E = \pm 1, \alpha^2 = 1. \) In this case, the Hirota bilinear equations (29) take the form (\( q_0 \equiv g \))

\[ [i D_t + (b + 1) D^2_\xi - b D^2_\eta] g_0 \circ \phi = 0 \]  
\[ D_\xi D_\eta \phi \circ \phi = -2Egg^* \]

5.1 The 1-soliton solution

The construction of the soliton solutions is standard. One expands the functions \( g \) and \( \phi \) as a series of \( \epsilon \)

\[ g = \epsilon g_1 + \epsilon^3 g_3 + \epsilon^5 g_5 + \cdots \]  
\[ \phi = 1 + \epsilon^2 \phi_2 + \epsilon^4 \phi_4 + \epsilon^6 \phi_6 + \cdots \]

Substituting these expansions into (30) and equating the coefficients of \( \epsilon \), in the 1-SS case, one obtains the following system of equations:

\[ \epsilon^1: \quad [i D_t + (b + 1) D^2_\xi - b D^2_\eta] g_1 \circ 1 = 0 \]
\[ \epsilon^3 : \ [iD_t + (b + 1)D^2_{\xi} - bD^2_{\eta}]g_1 \circ \phi_2 = 0 \]  
\[ \epsilon^2 : \ \ D_{\xi}D_{\eta}(1 \circ \phi_2 + \phi_2 \circ 1) = -2Eg^* \]  
\[ \epsilon^4 : \ \ D_{\xi}D_{\eta}(\phi_2 \circ \phi_2) = 0. \]  

Using these equations we can construct the 1-SS of equation (5). In order to construct exact 1-SS of equation (9), we take the ansatz
\[ g_1 = \exp \chi_1 \]  

where
\[ \chi_1 = p_1 \xi + s_1 \eta + c_1 t + e_1, \quad p_1 = p_{1R} + ip_{1I}, \quad s_1 = s_{1R} + is_{1I}. \]  

Sustituting (33) into (32a), we obtain
\[ c_1 = i[(b + 1)p^2_{1I} - bs^2_{1I}]. \]  

The expression for \( \phi_2 \), we get from (32c)
\[ \phi_2 = \exp(\chi_1 + \chi^*_1 + 2\psi) \]  

with
\[ \exp(2\psi) = -E/4p_{1R}s_{1R}. \]  

Equations (32b,d) are identically satisfied. Finally, from (28), (33) and (36), we get the 1-SS of equation (9) in the form
\[ q(\xi, \eta, t) = \frac{1}{2} \exp(-\psi)sech(\chi_{1R} + \psi)\exp(i\chi_{1I}) \]  
\[ V(\xi, \eta, t) = -s^2_{1R}sech^2(\chi_{1R} + \psi) \]  
\[ U(\xi, \eta, t) = -p^2_{1R}sech^2(\chi_{1R} + \psi) \]  

and for the hybrid potential
\[ v(\xi, \eta, t) = 2[(b + 1)p^2_{1R} - bs^2_{1R}]sech^2(\chi_{1R} + \psi) \]  

where \( \chi_{1R} = Re\chi_1 = p_{1R}\xi + s_{1R}\eta - [2(b + 1)p_{1R}p_{1I} - 2bs_{1R}s_{1I}]t. \) This algebra can be used to construct \( N \)-line soliton solutions as well. As shown in [3], the above 1-SS reveals the fact that
\[ q \to 0, \quad U \to 0, \quad V = -s^2_{1R}sech^2(\chi'_{1R} + \psi') \to v_1(\eta, t) \quad as \quad p_{1R} \to 0 \]  

where \( \chi'_{1R} = s_{1R}[\eta + 2bs_{1I}t] + e_{1R} \) and \( \psi' \) is a new phase constant. Similarly, we have
\[ q \to 0, \quad V \to 0, \quad U = -p^2_{1R}sech^2(\chi''_{1R} + \psi'') \to u_1(\eta, t) \quad as \quad p_{1R} \to 0 \]  

where \( \chi''_{1R} = p_{1R}[\xi - 2(b + 1)p_{1I}t] + e_{1R} \) and \( \psi'' \) is another phase constant. Thus, as in [3], the solution is composed of two ghost solitons \( v_1(\eta, t) \) and \( u_1(\xi, t) \) driving the potentials \( V \) and \( U \) respectively in the absence of the physical field \( q \). Note that these results are the same as in [3].
5.2 The (1,1)-dromion solutions

Let us now construct a dromion solution of the ZE. For example, to get a simple (1, 1) dromion solution, following Radha and Lakshmana (see, e.g. [3]), we take the ansatz

\begin{align}
q_{11D} & = \rho \exp(\chi_1 + \chi_2) \\
\phi_{11D} & = 1 + j \exp(\chi_1 + \chi_1^*) + k \exp(\chi_2 + \chi_2^*) + l \exp(\chi_1 + \chi_1^* + \chi_2 + \chi_2^*)
\end{align}

(41a)
(41b)

where \( j, k, l \) are real positive constants and

\[
\chi_1 = p_1 \xi + i(b + 1)p^2 t + \chi_0^1, \quad \chi_2 = s_2 \eta - ibs^2_2 t + \chi_0^2.
\]

(42)

Here \( p_1 = p_{1R} + i p_{1I}, s_2 = s_{2R} + is_{2I} \) are complex constants. Substituting (41) into (30), we get the following conditions

\[
| \rho |^2 = 4p_{1R}s_{1R}(jk - l)/E, \quad (l - jk) \exp(-2\psi) > 0.
\]

(43)

At last, from (41) and (28), we obtain the (1, 1) dromion solution in the form

\[
q_{11D} = \frac{g_{11D}}{\phi_{11D}}, \quad V_{11D} = -\partial_{\eta\eta} \log \phi_{11D}, \quad U_{11D} = -\partial_{\xi\xi} \log \phi_{11D}
\]

(44a)

or

\[
q_{11D} = \frac{\rho \exp(\chi_1 + \chi_2)}{1 + j \exp(\chi_1 + \chi_1^*) + k \exp(\chi_2 + \chi_2^*) + l \exp(\chi_1 + \chi_1^* + \chi_2 + \chi_2^*)}
\]

(44b)

\[
V_{11D} = \frac{-4s^2_{2R} \exp(2\chi_2R)[(k + l \exp(2\chi_2R)][1 + j \exp(2\chi_1R)]}{[1 + j \exp(2\chi_1R) + k \exp(2\chi_2R) + l \exp(2(\chi_1R + \chi_2R))]^2}
\]

(44c)

\[
U_{11D} = \frac{-4p^2_{1R} \exp(2\chi_1R)[j + l \exp(2\chi_1R)][1 + k \exp(2\chi_2R)]}{[1 + j \exp(2\chi_1R) + k \exp(2\chi_2R) + l \exp(2(\chi_1R + \chi_2R))]^2}.
\]

(44d)

From the last two equations, we can get the expression for the hybrid potential \( v \)

\[
v_{11D} = 2bV_{11D} - 2(b + 1)U_{11D}.
\]

(44e)

5.3 The 1-rational solution

In this subsection, we want present the simple 1-rational solution of equation (9). Let \( g_1 = b_0 = \text{const.} \). Then, from (32) we get

\[
\phi_2 = -E|b_0|^2\xi\eta + b_1\eta, \quad b_1 = \text{const.}
\]

(45)

So, the 1-rational solution has the form

\[
q = \frac{b_0}{1 - E|b_0|^2\xi\eta + b_1\eta}
\]

(46a)

\[
V = \left[ \frac{b_1 - E|b_0|^2\xi}{1 - E|b_0|^2\xi\eta + b_1\eta} \right]^2
\]

(46b)

\[
U = \left[ \frac{|b_0|^4\eta^2}{1 - E|b_0|^2\xi\eta + b_1\eta} \right]^2.
\]

(46c)
Note that in this case, we have the following boundary conditions

\[(q, U, V)_{|\xi = \pm \infty} = (0, 0, \frac{1}{\eta^2} = v_2(\eta)) \quad (47a)\]

and

\[(q, U, V)_{|\eta = \pm \infty} = (0, \frac{1}{\xi^2} = u_2(\xi), 0). \quad (47b)\]

### 6 A connection between the ZE and the FE

Now let us consider the FE [4]

\[i \psi_t = \psi_{\xi \xi} + (\gamma + \beta)\psi_{\eta \eta} - 2\lambda \psi[(\gamma + \beta)(\int_{-\infty}^{\xi} (pq)_{\eta} d\xi')]\]

\[+ v_1(\eta, t) - (\gamma - \beta)(\int_{-\infty}^{\eta}(pq)_{\xi} d\eta') + v_2(\xi, t)] = 0 \quad (48a)\]

\[i \psi_t + (\gamma - \beta)\psi_{\xi \xi} - (\gamma + \beta)\psi_{\eta \eta} + 2\lambda \psi[(\gamma + \beta)(\int_{-\infty}^{\xi} (pq)_{\eta} d\xi')\]

\[+ v_1(\eta, t) - (\gamma - \beta)(\int_{-\infty}^{\eta}(pq)_{\xi} d\eta') + v_2(\xi, t)] = 0 \quad (48b)\]

with \(p = \bar{q}\) and in contrast with the equation (9), in this case \(\xi, \eta\) are the characteristic coordinates defined by

\[\xi = x + y, \quad \eta = x - y. \quad (49)\]

This equation also contains several interesting particular cases. Let us recall these cases.

(i) \(\gamma = \beta = \frac{1}{2}, v_1 = v_2 = 0\), yields equation

\[i \psi_t + q_{\eta \eta} - 2\lambda q \int_{-\infty}^{\xi} (pq)_{\eta} d\xi' = 0, \quad \lambda = \pm 1. \quad (50)\]

As noted by Fokas, equation (50) is perhaps the simplest complex scalar equation in 2+1 dimensions, which can be solved by the IST method. It is also worth pointing out that when \(x = \eta\) this equation reduces to the (1+1)-dimensional integrable NLSE.

(ii) \(\gamma = 0, \beta = 1\), yields the celebrated DSI equation

\[i \psi_t + q_{\eta \eta} - 2\lambda q \int_{-\infty}^{\xi} (pq)_{\eta} d\xi' + v_1(\eta, t)(\int_{-\infty}^{\eta}(pq)_{\xi} d\eta' + v_2(\xi, t)) = 0. \quad (51)\]

This equation has the Painlevé property and admits exponentially localized solutions including dromions for nonvanishing boundaries.

(iii) \(\gamma = 1, \beta = 0\) yields the DSIII equation

\[i \psi_t + q_{\eta \eta} - 2\lambda q \int_{-\infty}^{\xi} (pq)_{\eta} d\xi' + v_1(\eta, t) - (\int_{-\infty}^{\eta}(pq)_{\xi} d\eta' + v_2(\xi, t)) = 0. \quad (52)\]
Equation (52) also supports certain localized solutions.

Now we return to the ZE (9) and make the simplest scaling transformation: from \((t, \xi, \eta, q, p, v)\) to \((Ft, C\xi, D\eta, Aq, Bp, HV, EU)\). Then, for example, equation (9) takes the form

\[
iq - (\gamma - \beta)q_{\xi\xi} + (\gamma + \beta)q_{\eta\eta} - 2\lambda[(\gamma + \beta)V - (\gamma - \beta)U]q = 0 \tag{53a}
\]

\[
-ip + (\gamma - \beta)p_{\xi\xi} - (\gamma + \beta)p_{\eta\eta} + 2\lambda[(\gamma + \beta)V - (\gamma - \beta)U]p = 0 \tag{53b}
\]

\[
\begin{align*}
V_{\xi} &= (pq)_{\eta} \\
U_{\eta} &= (pq)_{\xi}
\end{align*}
\tag{53c,d}
\]

where

\[
\lambda = \frac{ABCD}{F}, \quad F = \frac{\beta - \gamma}{1 + b}C^2, \quad D^2 = \frac{b(\gamma - \beta)}{(1 + b)(\gamma + \beta)}C^2,
\]

\[
\gamma = -\frac{1}{2}F[(b + 1)D^2 + bC^2]C^{-2}D^{-2}, \quad \beta = \frac{1}{2}F[(b + 1)D^2 - bC^2]C^{-2}D^{-2}.
\]

From (53c,d), we get

\[
V = \int_{-\infty}^{\xi} (pq)_{\eta} d\xi' + v_1(\eta, t) \tag{54a}
\]

\[
U = \int_{-\infty}^{\eta} (pq)_{\xi} d\eta' + v_2(\xi, t). \tag{54b}
\]

Substituting (54) into (53a,b), we obtain the FE (48). Thus, we have proved that the ZE and the FE are equivalent to each other, as \(\alpha^2 = 1, a = -\frac{1}{2}\).

In particular, this is why the ZE contains and at the same time the FE not contains the DSII equation. Recently it was proved by Radha and Lakshmanan [3] that the FE (48) satisfies the Painleve property and hence it is expected to be integrable. From these results follow that the ZE also satisfies the Painleve property and is integrable.

### 7 Associated integrable spin systems

In this section, we wish present, in a briefly form, the spin equivalent counterpart of the ZE and its reductions. It is well known that the ZE (1) is gauge equivalent to the Myrzakulov IX (M-IX) equation [16]

\[
iS_t + \frac{1}{2}[S, M_1S] + A_2S_x + A_1S_y = 0 \tag{55a}
\]

\[
M_2u = \frac{\alpha^2}{2i} tr(S[S_x, S_y]) \tag{55b}
\]

where \(\alpha, b, a = \) consts,

\[
S = \begin{pmatrix} S_4 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^\pm = S_1 \pm iS_2, \quad S^2 = EI, \quad E = \pm 1, \quad r^2 = \pm 1,
\]

\[
A_1 = i\{\alpha(2b + 1)a_y - 2(2ab + a + b)a_x\},
\]
\[ A_2 = i\{4a^{-1}(2a^2b + a^2 + 2ab + b)u_x - 2(2ab + a + b)u_y \}. \]

This equation is integrable and also admits several integrable reductions. There are some of them:

(i) The Myrzakulov VIII (M-VIII) equation. First, let us consider the reduction of the M-IX equation (55) as \( a = b = -1 \). We have [16]

\[
\begin{align*}
    iS_t + \frac{1}{2}[S, S_{YY}] + iwS_Y &= 0 \quad (56a) \\
    w_x + w_y + \frac{1}{4t} tr(S[S_X, S_Y]) &= 0 \quad (56b)
\end{align*}
\]

where \( X = x/2, \quad Y = y/\alpha, \quad w = -\alpha^{-1}u_Y. \)

(ii) The Ishimori equation. Now let \( a = b = -\frac{1}{2} \). Then equation (4) reduces to the known Ishimori equation [15]

\[
\begin{align*}
    iS_t + \frac{1}{2}[S, (S_{xx} + \alpha^2 S_{yy})] + iu_yS_x + iu_xS_y &= 0 \quad (57a) \\
    \alpha^2 u_{yy} - u_{xx} &= \frac{\alpha^2}{2t} tr(S[S_x, S_y]) \quad (57b)
\end{align*}
\]

(iii) The Myrzakulov XXXIV (M-XXXIV) equation. This equation has the form

\[
\begin{align*}
    iS_t + \frac{1}{2}[S, (S_{xx} + \alpha^2 S_{yy})] + iu_yS_x + iu_xS_y &= 0 \quad (58a) \\
    w_t + w_y + \frac{1}{4} \{tr(S^2_x)\}_y &= 0 \quad (58b)
\end{align*}
\]

The M-XXXIV equation (19) was proposed in [16] to describe nonlinear dynamics of compressible magnets. It is integrable and has the different soliton solutions [24].

(iv) The Myrzakulov XIX (M-XVIII) equation. Now we consider the reduction: \( a = -\frac{1}{2} \). Then the equation (55) reduces to the M-XVIII equation [16]

\[
\begin{align*}
    iS_t + \frac{1}{2}[S, S_{xx} + 2\alpha(2b + 1)S_{xy} + \alpha^2 S_{yy}] + A'_2S_x + A'_1S_y &= 0 \quad (59a) \\
    \alpha^2 u_{yy} - u_{xx} &= \frac{\alpha^2}{2t} tr(S[S_x, S_y]) \quad (59b)
\end{align*}
\]

where \( A'_j = A_j \) as \( a = -\frac{1}{2} \).

(v) The Myrzakulov XIX (M-XIX) equation. Let us consider the case: \( a = b \). Then we obtain the M-XIX equation [16]

\[
\begin{align*}
    iS_t + \frac{1}{2}[S, \alpha^2 S_{yy} - 4a(a + 1)S_{xx}] + A''_2S_x + A''_1S_y &= 0 \quad (60a) \\
    M_2u &= \frac{\alpha^2}{2t} tr(S[S_x, S_y]) \quad (60b)
\end{align*}
\]

where \( A''_j = A_j \) as \( a = b \).
(vi) The Myrzakulov XX (M-XX) equation. This equation has the form [16]

\[ iS_t + \frac{1}{2}[S, (b + 1)S_{\xi \xi} - bS_{\eta \eta}] + ibw_{\eta}S_{\eta} + i(b + 1)w_{\xi}S_{\xi} = 0 \]  

and so on [16]. The gauge equivalent counterparts of equations (56), (57), (58) and (61) are the equations (3), (2), (4) and (11), respectively. Note that from (61) as \( b = 0 \), we get the M-VIII equation in the following form [16]

\[ iS_t + \frac{1}{2}[S, S_{\xi \xi}] + iw_{\xi}S_{\xi} = 0 \]  

the gauge equivalent of which is the equation (12). If we put \( \eta = t \), then equations (61) and (12) take the forms

\[ iS_t + \frac{1}{2}[S, S_{\xi \xi}] + iw_{\xi}S_{\xi} = 0 \]  

\[ w_{\xi \eta} = \frac{1}{4i}tr(S[S_{\xi}, S_{\eta}]) \]  

\[ w_{\xi \eta} = \frac{1}{4i}tr(S[S_{\xi}, S_{\eta}]) \]  

\[ w_{\xi \eta} = \frac{1}{4i}tr(S[S_{\xi}, S_{\eta}]) \]  

Equation (63) is the equivalent form of the M-XXXIV equations. At the same time, its gauge equivalent (64) is the Ma equation [20], which is also the equivalent form of the YOE (4).

Note that these spin systems admit the different types solutions (see, e.g. [21-25]).

8 Conclusion

We have investigated the integrability aspects of the (2+1)-dimensional ZE by the singularity structure analysis and shown that it admits the Painlevé property. We have also derived its bilinear form directly from the Painlevé analysis. We have then generated the simplest 1-SS using the Hirota method. We have constructed and the (1,1) dromion solution. Finally, in last section we have presented the associated integrable spin systems, which are gauge equivalent counterparts of the ZE and its reductions. Here, we would like note that between these spin systems and the NLSE-type equations can take place the so-called Lakshmanan equivalence. This problem we will consider, in detail, in other places (see, e.g., [26-29]).
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