LINEAR TRANSFORMATIONS OF VERTEX OPERATOR PRESENTATIONS OF HALL-LITTLEWOOD POLYNOMIALS

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ABSTRACT. We study the effect of linear transformations on quantum fields with applications to vertex operator presentations of symmetric functions. Properties of linearly transformed quantum fields and corresponding transformations of Hall-Littlewood polynomials are described, including preservation of commutation relations, stability, explicit combinatorial formulas and generating functions. We prove that specializations of linearly transformed Hall-Littlewood polynomials describe all polynomial tau functions of the KP and the BKP hierarchy. Examples of linear transformations are related to multiparameter symmetric functions, Grothendieck polynomials, deformations by cyclotomic polynomials, and some other variations of Schur symmetric functions that exist in the literature.

1. INTRODUCTION

The language of quantum fields is a widely used tool in representation theory of infinite-dimensional algebraic structures. Actions of algebras of fermions, Heisenberg algebra, Virasoro algebra, \( gl_\infty \), affine Lie algebras, boson-fermion correspondence, vertex algebras and their representations are some examples that use this language. Presentation of families of symmetric functions as results of application of quantum fields to a vacuum vector allowed researchers to prove many important results with applications in integrable systems and representation theory. A classical example is the vertex operator presentation of Schur functions by the action of charged free fermions \([24, 58]\), which is the base of the construction of the boson-fermion correspondence. From the existing numerous vertex operator presentations of other families of symmetric functions we mention through the paper the ones that are most related to our construction.

In this note we study the effect of a linear transformation of quantum fields of vertex operators on the properties of the resulting symmetric functions. Surprisingly, this simple modification covers a broad class of Schur-like symmetric functions that appear in the literature. At the same time, the simplicity of this transformation allows one to get many important properties of these transformed families of symmetric functions “almost for free” from the properties of the original classical family.

The initial motivation for this project was the study of tau-functions of the KP and the BKP hierarchy \([7, 8, 9, 10, 23, 54]\) by generalizing the methods of \([52]\), where the author proved that the multiparameter Schur \( Q \)-functions are tau-functions of the BKP hierarchy. We aimed to provide a description of all polynomial tau-functions of the KP and the BKP hierarchies unifying the ideas of \([52]\) and \([31]\). This is done in Section 6. Along the way it was convenient to consider a more general set up applying linear transformations to vertex operators of Hall-Littlewood polynomials that were first constructed in \([25]\). This allowed us not only to formulate and prove in a uniform way statements for the KP and the BKP tau-functions, but to prove a number of properties of linearly transformed Hall-Littlewood polynomials, such as combinatorial formulas, generating functions, stability property, preservation of commutation relations of quantum fields under linear transformations. Moreover,
these properties carry on to particular specializations that provide deformations of Hall-Littlewood polynomials, Schur and Schur $Q$-functions that appear in the literature.

The paper is organized as follows. In Section 2 we review the necessary facts on symmetric functions, quantum fields, formal distributions. In Section 3 we introduce linear transformations of quantum fields and describe their basic properties. In Section 4 we review Hall-Littlewood polynomials and apply linear transformations to their vertex operator presentations to define a new family of symmetric functions that depend on a parameter $t$. The properties of these new symmetric functions are discussed. In Section 5 we formulate the properties of specializations at $t = 0$ and $t = -1$. In Section 6 we describe all polynomial tau-functions of the KP and the BKP hierarchies as results of linear transformations of vertex operator presentations of Schur functions and Schur $Q$-functions. In Section 7 we discuss particular examples of linear transformations matching with the existing literature, that include multiparameter symmetric functions, Grothendieck polynomials, deformations by cyclotomic polynomials.

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2. Symmetric functions and quantum fields

2.1. Symmetric functions. We review properties of symmetric functions following [37, 55]. The setup is similar to [26, 31, 48, 52]. Consider the algebra of formal power series $\mathbb{C}[[x]] = \mathbb{C}[[x_1, x_2, \ldots]]$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l > 0)$ be a partition of length $l$. The monomial symmetric function is a formal series

$$m_\lambda = \sum_{(i_1, \ldots, i_l) \in \mathbb{N}^l} x_{i_1}^{\lambda_1} \cdots x_{i_l}^{\lambda_l}.$$ 

Let $\Lambda$ be the subalgebra of $\mathbb{C}[[x]]$ spanned as a vector space by all monomial symmetric functions. It is called the algebra of symmetric functions. Note that elements of $\Lambda$ are invariant with respect to any permutation of a finite number of indeterminates $x_1, x_2, \ldots$. The following families of symmetric functions play important role in our study.

For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l > 0)$, Schur symmetric function $s_\lambda$ is defined as

$$s_\lambda(x_1, x_2, \ldots) = \sum_T x^T,$$

where the sum is over all semistandard tableaux of shape $\lambda$.

Complete symmetric functions $h_k = s_{(k)}$ are given by the formula

$$h_k(x_1, x_2 \ldots) = \sum_{1 \leq i_1 \leq \cdots \leq i_k < \infty} x_{i_1} \cdots x_{i_k}, \quad k \in \mathbb{N},$$

while elementary symmetric functions $e_k = s_{(1^k)}$ by

$$e_k(x_1, x_2 \ldots) = \sum_{1 \leq i_1 < \cdots < i_k < \infty} x_{i_1} \cdots x_{i_k}, \quad k \in \mathbb{N}.$$ 

Power sums $p_k$ are symmetric functions defined by

$$p_k(x_1, x_2, \ldots) = \sum_{i \in \mathbb{N}} x_i^k, \quad k \in \mathbb{N}.$$ 

It is convenient to set $h_{-k}(x_1, x_2 \ldots) = e_{-k}(x_1, x_2 \ldots) = p_{-k}(x_1, x_2 \ldots) = 0$ for $k \in \mathbb{N}$ and $h_0 = e_0 = p_0 = 1$.

Algebra $\Lambda$ is a polynomial algebra in any of these three families of generators:

$$\Lambda = \mathbb{C}[h_1, h_2, \ldots] = \mathbb{C}[e_1, e_2, \ldots] = \mathbb{C}[p_1, p_2, \ldots].$$
Schur symmetric functions \( \{ s_\lambda \} \) labeled by all partitions form a linear basis of \( \Lambda \). They can be also expressed through complete symmetric functions by the Jacobi - Trudi identity

\[
s_\lambda = \det[h_{\lambda_i - \lambda_j}]_{1 \leq i,j \leq l}
\]  

(2.2)

We will use (2.2) as the extension of the definition of \( s_\lambda \) for any integer vector \( \lambda \in \mathbb{Z}^l \).

There is a natural scalar product on \( \Lambda \) where the set of Schur symmetric functions \( \{ s_\lambda \} \) labeled by partitions \( \lambda \) form an orthonormal basis, \( \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \). Then for any linear operator acting on the vector space \( \Lambda \) one can define the corresponding adjoint operator. In particular, any symmetric function \( f \in \Lambda \) defines an operator of multiplication \( f : g \mapsto fg \) for any \( g \in \Lambda \). The corresponding adjoint operator \( f^\perp \) is defined by the standard rule \( \langle f^\perp g_1, g_2 \rangle = \langle g_1, fg_2 \rangle \) for all \( g_1, g_2 \in \Lambda \).

It is known \([37], I.5\) Example 3, that \( p^\perp \lambda = n \frac{\partial}{\partial p_n} \). Since any element \( f \in \Lambda \) can be expressed as a polynomial function of power sums

\[
f = F(p_1, p_2, p_3, \ldots),
\]

the corresponding adjoint operator \( f^\perp \) is a polynomial differential operator with constant coefficients

\[
f^\perp = F(\partial/\partial p_1, 2\partial/\partial p_2, 3\partial/\partial p_3, \ldots).
\]

In particular, \( e_k \) and \( h_k \) are homogeneous polynomials of degree \( k \) in \( (p_1, p_2, p_3, \ldots) \), so the adjoint operators \( e_k^\perp \) and \( h_k^\perp \) are homogeneous polynomials of degree \( k \) in \( (\partial/\partial p_1, 2\partial/\partial p_2, \ldots) \), which implies the following statement.

**Lemma 2.1.** For any symmetric function \( f \in \Lambda \) there exists a positive integer \( N = N(f) \), such that

\[
e^\perp_l(f) = 0 \quad \text{and} \quad h^\perp_l(f) = 0 \quad \text{for all} \quad l \geq N.
\]

2.2. **Formal distributions and quantum fields.** For more details see \([27, 32]\). Let \( W \) be a vector space. A \( W \)-valued formal distribution is a bilaternal series in the indeterminate \( u \) with coefficients in \( W \):

\[
a(u) = \sum_{n \in \mathbb{Z}} a_n u^n, \quad a_n \in W.
\]

We denote as \( W[[u, u^{-1}]] \) the vector space of all \( W \)-valued formal distributions. We also use the notation \( W[u] \) for the space of polynomials, \( W[[u]] \) for the space of power series, \( W[u, u^{-1}] \) for the space of Laurent polynomials, and \( W((u)) \) for the space of formal Laurent series.

A special case of a formal distribution is a quantum field, which is an End \( W \)-valued formal distribution \( \Gamma(u) = \sum_{k \in \mathbb{Z}} \Gamma_k u^{-k} \), such that for any \( f \in W \), \( \Gamma_k(f) = 0 \) for \( k >> 0 \).

A formal distribution in two and more indeterminates is defined similarly. The formal delta-function \( \delta(u, v) \) is the \( \mathbb{C} \)-valued formal distribution in variables \( u \) and \( v \)

\[
\delta(u, v) = \sum_{i,j \in \mathbb{Z}} u^i v^j = i_{u,v} \left( \frac{1}{u-v} \right) - i_{v,u} \left( \frac{1}{u-v} \right),
\]

(2.3)

where \( i_{u,v} \) (resp. \( i_{v,u} \)) denotes the expansion of a rational function of \( u, v \) in the domain \(|u| > |v|\) (resp. \(|u| > |v|\))

\[
i_{u,v} \left( \frac{1}{u-v} \right) = \sum_{k=0}^{\infty} \frac{u^k}{u^{k+1}}.
\]

(2.4)
2.3. Generating series of polynomial differential operators acting on \( \Lambda \). Denote by \( \mathcal{D} \) the algebra of differential operators acting on \( \Lambda = \mathbb{C}[p_1, p_2, \ldots] \), which consists of finite sums
\[
\sum_{i_1, \ldots, i_m} F_{i_1, \ldots, i_m}(p_1, p_2, \ldots) \partial_{p_1}^{i_1} \cdots \partial_{p_m}^{i_m},
\]
where coefficients \( F_{i_1, \ldots, i_m}(p_1, p_2, \ldots) \) are polynomials in \( (p_1, p_2, \ldots) \). Then operators of multiplication \( p_n, h_n, e_n \), their adjoints \( p_n^\perp, h_n^\perp, e_n^\perp \) along with their products are elements of \( \mathcal{D} \).

Consider the generating series of complete and elementary symmetric functions
\[
H(u) = \sum_{k \in \mathbb{Z}_{\geq 0}} h_k u^k = \prod_{i \in \mathbb{N}} \frac{1}{1 - x_i u}, \quad E(u) = \sum_{k \in \mathbb{Z}_{\geq 0}} e_k u^k = \prod_{i \in \mathbb{N}} (1 + x_i u), \quad (2.5)
\]
which are elements of \( \Lambda[[u]] \). We will use the same notation for the corresponding multiplication operators \( H(u), E(u) \in \mathcal{D}[[u]] \). Similarly, we define \( E^\perp(u), H^\perp(u) \in \mathcal{D}[[u^{-1}]] \) as
\[
E^\perp(u) = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{e_k}{u} \partial p_k u^k, \quad H^\perp(u) = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{h_k}{u} \partial p_k u^k. \quad (2.6)
\]

The following properties of these generating series with coefficients in \( \mathcal{D} \) are well known (e.g. [37], I.5).

**Proposition 2.1.** We have in \( \mathcal{D}[[u]] \) (resp. in \( \mathcal{D}[[u^{-1}]] \))
\[
H(u) E(-u) = 1, \quad H^\perp(u) E^\perp(-u) = 1,
\]
\[
H(u) = \exp \left( \sum_{n \in \mathbb{N}} \frac{p_n}{n} u^n \right), \quad E(u) = \exp \left( - \sum_{n \in \mathbb{N}} \frac{(-1)^n p_n}{n} u^n \right), \quad (2.7)
\]

**Lemma 2.2.** ([37], I.5 Example 29). We have the following commutation relations in \( \mathcal{D}[[u^{-1}, v]] \):
\[
\left(1 - \frac{v}{u}\right) E^\perp(u) E(v) = E(v) E^\perp(u), \quad \left(1 - \frac{v}{u}\right) H^\perp(u) H(v) = H(v) H^\perp(u),
\]
\[
E^\perp(u) E(v) = \left(1 + \frac{v}{u}\right) E(v) E^\perp(u), \quad H^\perp(u) H(v) = \left(1 + \frac{v}{u}\right) H(v) H^\perp(u).
\]

2.4. Schur symmetric \( Q \)-functions. The elements \( \{q_k(x_1, x_2, \ldots)\}_{k \in \mathbb{Z}} \) are the coefficients of the expansion of \( Q(u) \in \Lambda[[u]] \), where
\[
Q(u) = \sum_{k \in \mathbb{Z}} q_k u^k = E(u) H(u). \quad (2.8)
\]

Note that \( q_k = \sum_{i=0}^{k} e_i h_{k-i} \) for \( k > 0 \), \( q_0 = 1 \), and \( q_k = 0 \) for \( q < 0 \). For \( a, b \in \mathbb{Z}_{\geq 0} \) let
\[
q_{a,b} = q_a q_b + 2 \sum_{i \in \mathbb{Z}} (-1)^i q_{a+i} q_{b-i}. \quad (2.9)
\]
Then, [37], III.8,
\[ q_{a,b} = -q_{b,a}, \quad q_{a,a} = 0. \]  

**Proposition 2.2. ([37], III.8)** We have in \( \mathcal{D}[[u]] \) (resp. in \( \mathcal{D}[[u^{-1}]] \))

\[ Q(u) = S_{\text{odd}}(u)^2, \quad \text{where} \quad S_{\text{odd}}(u) = \exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} \frac{p_n}{n} u^n \right), \]

\[ S_{\text{odd}}^+(u) = \exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} \frac{\partial}{\partial p_n} \frac{1}{u^n} \right). \]

Here \( \mathbb{N}_{\text{odd}} = \{1, 3, 5, \ldots\} \).

Recall that the Pfaffian of a skew-symmetric matrix \( M = [M_{ij}] \) of size \( 2l \times 2l \) is defined as

\[ \text{Pf} [M] = \sum_{\sigma \in S_{2l}} \text{sgn}(\sigma) M_{\sigma(1)\sigma(2)} \cdots M_{\sigma(2l-1)\sigma(2l)}, \]

where \( S_{2l} \) is the subset of the permutation group \( S_{2l} \) that consists of \( \sigma \in S_{2l} \) such that \( \sigma(2k-1) < \sigma(2k) \) for \( 1 \leq k < l \) and \( \sigma(2k-1) < \sigma(2k+1) \) for \( 1 \leq k \leq l - 1 \).

If \( \lambda = (\lambda_1, \ldots, \lambda_{2m}) \) is a strict partition, i.e. \( \lambda_1 > \cdots > \lambda_{2m} \geq 0 \), then the matrix \( M_\lambda = (q_{\lambda_i, \lambda_j}) \) is skew-symmetric by (2.10), and the Schur symmetric \( Q \)-function \( q_\lambda \) is defined as

\[ q_\lambda(x_1, x_2, \ldots) = \text{Pf} M_\lambda. \]  

**2.5. Charged free fermions.** Let \( z \) and \( u \) be formal indeterminants. Consider the boson Fock space \( \mathcal{B} = \mathbb{C}[z, z^{-1}] \otimes \Lambda \), where \( \Lambda \) is the ring of symmetric functions. A number of important algebraic structures act on the space \( \mathcal{B} \). We review the action of charged free fermions and refer to [23, 27, 32] for more details.

Let \( R(u) \) and its inverse be formal distributions in variable \( u \) of operators acting on the elements of the form \( z^m f \), where \( f \in \Lambda \), \( m \in \mathbb{Z} \), by the rule

\[ R(u)(z^m f) = z^{m+1} u^{m+1} f, \quad R^{-1}(u)(z^m f) = z^{m-1} u^{-m} f. \]

Define formal distributions \( \psi^\pm(u) \) of operators acting on the space \( \mathcal{B} \) through the action of \( R^\pm(u) \) and the \( \mathcal{D} \)-valued generating series (2.5), (2.6):

\[ \psi^+(u) = u^{-1} R(u) H(u) E^+(u), \]

\[ \psi^-(u) = R^{-1}(u) E(-u) H^+(u), \]

or, in other words, for any \( m \in \mathbb{Z} \) and any \( f \in \Lambda \),

\[ \psi^+(u)(z^m f) = z^{m+1} u^{m} H(u) E^+(u)(f), \]

\[ \psi^-(u)(z^m f) = z^{-m-1} u^{-m} E(-u) H^+(u)(f). \]

Let the operators \( \{\psi^+_i\}_{i \in \mathbb{Z} + 1/2} \) be the coefficients of the expansions

\[ \psi^\pm(u) = \sum_{i \in \mathbb{Z} + 1/2} \psi^+_i u^{-i-1/2}. \]

These operators are called the charged free fermions. Formal distributions \( \psi^\pm(u) \) of operators acting on the space \( \mathcal{B} \) are quantum fields that satisfy relations

\[ \psi^+(u) \psi^+(v) + \psi^+(v) \psi^+(u) = 0, \]

\[ \psi^+(u) \psi^-(v) + \psi^-(v) \psi^+(u) = \delta(u, v), \]
or, equivalently,
\[ \psi_k^+ \psi_l^+ + \psi_l^- \psi_k^- = 0, \quad \psi_k^+ \psi_l^- = \delta_{k,-l}, \quad k, l \in \mathbb{Z} + 1/2. \] (2.12)

2.6. **Heisenberg algebra.** The Heisenberg algebra is the complex Lie algebra with a basis \( \{ \alpha_k \}_{k \in \mathbb{Z} \cup \{ 1 \}} \) and commutation relations

\[ [1, \alpha_n] = 0, \quad [\alpha_m, \alpha_n] = m \delta_{m,-n}, \quad m, n \in \mathbb{Z}. \] (2.13)

This is equivalent to

\[ [\alpha(z), \alpha(w)] = \partial_w \delta(z, w) \cdot 1, \]

where \( \alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1} \). The Heisenberg algebra acts on the space \( B \) by differentiation and multiplication operators

\[ \alpha_k = \partial/\partial p_k, \quad \alpha_{-k} = kp_k, (k = 1, 2, \ldots), \quad \alpha_0 = a_0 \cdot 1, \quad 1 = 1. \]

2.7. **Neutral fermions.** Consider the boson Fock space generated by odd power sums:

\[ \mathcal{B}_{\text{odd}} = \mathbb{C}[p_1, p_3, p_5, \ldots]. \]

Recall [37] III.8 (8.3) that \( q_k \in \mathcal{B}_{\text{odd}} \), and that \( \mathcal{B}_{\text{odd}} = \mathbb{C}[q_1, q_3, \ldots] \). From (2.7) it is clear that \( \mathcal{B}_{\text{odd}} \) is invariant with respect to action of \( e_k^+ \) and \( h_k^\pm \), and one can prove that restrictions to \( \mathcal{B}_{\text{odd}} \) of the operators \( E^\pm(u), H^\pm(u), S^+_{\text{odd}}(u) \) coincide.

Define a quantum field \( \phi(u) \) of operators acting on \( \mathcal{B}_{\text{odd}} \):

\[ \phi(u) = E(u)H(u)E^\dagger(u) = Q(u)S_{\text{odd}}^+(u). \] (2.14)

Let \( \{ \phi_i \}_{i \in \mathbb{Z}} \) be coefficients of the expansion \( \phi(u) = \sum_{j \in \mathbb{Z}} \phi_j u^{-j} \).

One has relations

\[ \phi(u) \phi(v) + \phi(v) \phi(u) = 2v \delta(v, -u), \]

where \( \delta(u, v) \) is the formal delta function. Hence (2.14) is the action of the Clifford algebra of neutral fermions on the space \( \mathcal{B}_{\text{odd}} \):

\[ \phi_m \phi_n + \phi_n \phi_m = 2(-1)^m \delta_{m+n,0} \quad \text{for} \quad m, n \in \mathbb{Z}. \] (2.15)

3. **Linear transformations of quantum fields**

3.1. **Linear transformations of quantum fields.** Let \( \Gamma(u) = \sum_{i \in \mathbb{Z}} \Gamma_i u^{-i} \) be a quantum field of operators \( \{ \Gamma_i \}_{i \in \mathbb{Z}} \) acting on a vector space \( W \). Fix \( f \in W \) and set

\[ \Gamma(u_1) \ldots \Gamma(u_l) (f) = F(u_1, \ldots, u_l). \] (3.1)

Note that (3.1) is a well-defined formal distribution with coefficients in \( W \):

\[ F(u_1, \ldots, u_l) = \sum_{\lambda \in \mathbb{Z}^l} F_\lambda u_1^{\lambda_1} \ldots u_l^{\lambda_l}, \quad F_\lambda = \Gamma_{-\lambda_1} \ldots \Gamma_{-\lambda_l} (f) \in W. \] (3.2)

Let \( A = (A_{ij})_{i,j \in \mathbb{Z}} \) be an infinite complex-valued matrix. Set formally

\[ \hat{\Gamma}_i = \sum_{i \in \mathbb{Z}} A_{i,j} \Gamma_j. \]

For any integer vector \( \lambda \in \mathbb{Z}^l \) consider a formal infinite sum

\[ \hat{F}_\lambda = \sum_{\alpha \in \mathbb{Z}^l} A_{-\lambda_1,-\alpha_1} \ldots A_{-\lambda_l,-\alpha_l} F_\alpha, \]

where \( \{ F_\alpha \} \subset W \) are coefficients of the expansion (3.2).

We say that \( B = (B_{ij})_{i,j \in \mathbb{Z}} \) is the (left) inverse of the infinite matrix \( A \) and write \( B = A^{-1} \) if \( \sum_{k \in \mathbb{Z}} B_{ik} A_{kj} = \delta_{i,j} \).
Theorem 3.1. Assume that for any fixed $i \in \mathbb{Z}$, $A_{i,j} = 0$ for $j << 0$:

\begin{align*}
\downarrow i \\
| j \rightarrow & \ldots -2 -1 0 1 2 3 \ldots \\
\vdots & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
\vdots & \ldots 0 \ast \ast \ast \ast \ast \ast \ldots \\
-3 & \ldots 0 0 0 \ast \ast \ast \ast \ast \ldots \\
-2 & \ldots 0 0 \ast \ast \ast \ast \ast \ast \ldots \\
-1 & \ldots 0 0 \ast \ast \ast \ast \ast \ast \ast \ldots \\
0 & \ldots 0 0 0 \ast \ast \ast \ast \ast \ast \ast \ldots \\
1 & \ldots 0 0 0 0 \ast \ast \ast \ast \ast \ast \ast \ldots \\
2 & \ldots 0 0 0 0 \ast \ast \ast \ast \ast \ast \ast \ldots \\
3 & \ldots 0 0 0 \ast \ast \ast \ast \ast \ast \ast \ast \ldots \\
\vdots & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 
\end{align*}

For each $i \in \mathbb{Z}$, let $A_{i,M(i)}$ be the first non-zero term in the $i$-th row of the matrix $A$, reading from left to right:

$$M(i) = \max \{ k \in \mathbb{Z} | A_{i,r} = 0 \text{ for all } r < k \}. \quad (3.3)$$

Then

(a) $\tilde{\Gamma}_i$ is a well-defined linear operator acting on the space $W$ for any $i \in \mathbb{Z}$.

(b) $\tilde{F}_\lambda$ is a well-defined finite linear combination of $F_\alpha$'s, i.e. $\tilde{F}_\lambda \in W$, and

$$\tilde{F}_\lambda = \tilde{\Gamma}_{-\lambda_1} \cdots \tilde{\Gamma}_{-\lambda_l}(f).$$

(c) If $A$ is invertible, then

$$F_\lambda = \sum_{\alpha \in \mathbb{Z}^l} (A^{-1})_{-\lambda_1,-\alpha_1} \cdots (A^{-1})_{-\lambda_l,-\alpha_l} \tilde{F}_\alpha.$$ 

(d) If $A$ is invertible, then formal distribution $F(u_1, \ldots, u_l)$ can be re-expanded:

$$F(u_1, \ldots, u_l) = \sum_{\lambda \in \mathbb{Z}^l} \tilde{F}_\lambda g_{\lambda_1}(u_1) \cdots g_{\lambda_l}(u_l),$$

where $g_k(u) = \sum_{s \in \mathbb{Z}} (A^{-1})_{-s,-ku}$ are formal complex-valued distributions.

(e) If $M(i)$ is a strictly increasing function of $i$, then $\tilde{\Gamma}(u) = \sum_{i \in \mathbb{Z}} \tilde{\Gamma}_i u^{-i}$ is a quantum field. In that case $\tilde{F}_{(i,\lambda)} = 0$ for any $\lambda \in \mathbb{Z}^l$ and $i << 0$. 

\begin{align*}
\vdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
\vdots \ast \ast \ast \ast \ast \ast \ast \ast \ldots \\
\vdots \ast \ast \ast \ast \ast \ast \ast \ast \ast \ldots \\
\vdots 0 \ast \ast \ast \ast \ast \ast \ast \ast \ldots \\
\vdots 0 0 0 \ast \ast \ast \ast \ast \ast \ast \ldots \\
\vdots 0 0 0 0 0 \ast \ast \ast \ast \ast \ldots \\
\vdots 0 0 0 0 0 0 \ast \ast \ast \ast \ast \ldots \\
\vdots 0 0 0 0 0 0 0 \ast \ast \ast \ast \ldots \\
\vdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 
\end{align*}
Proof. (a) Since $\Gamma(u)$ is a field, for any $f \in W$ there exists an integer $N(f)$ such that $\Gamma_j(f) = 0$ for all $j > N(f)$. For any fixed $i \in \mathbb{Z}$ $A_{i,j} = 0$ for $j < M(i)$. Then

$$\tilde{\Gamma}_i(f) = \sum_{j \in \mathbb{Z}} A_{i,j} \Gamma_j(f) = \sum_{M(i) \leq j \leq N(f)} A_{i,j} \Gamma_j(f)$$

(3.4)

is a well-defined finite sum of elements in $W$.

(b) This follows from (a).

(c) One has

$$\sum_{\alpha \in \mathbb{Z}^l} (A^{-1})_{-\lambda_1, -\alpha_1} \cdots (A^{-1})_{-\lambda_l, -\alpha_l} \tilde{F}_\alpha$$

$$= \sum_{\alpha, \beta \in \mathbb{Z}^l} (A^{-1})_{-\lambda_1, -\alpha_1} \cdots (A^{-1})_{-\lambda_l, -\alpha_l} A_{-\alpha_1, -\beta_1} \cdots A_{-\alpha_l, -\beta_l} F_{\beta} = \sum_{\beta \in \mathbb{Z}^l} \delta_{\lambda, \beta} F_{\beta} = F_\lambda.$$

(d) From (c),

$$F(u_1, \ldots, u_t) = \sum_{\lambda \in \mathbb{Z}^l} \sum_{\alpha \in \mathbb{Z}^l} (A^{-1})_{-\lambda_1, -\alpha_1} \cdots (A^{-1})_{-\lambda_l, -\alpha_l} \tilde{F}_\alpha u_1^{\lambda_1} \cdots u_t^{\lambda_l}$$

$$= \sum_{\alpha \in \mathbb{Z}^l} \tilde{F}_\alpha g_{\alpha_1}(u_1) \cdots g_{\alpha_l}(u_t).$$

(e) If $M(i)$ is a strictly increasing function of $i$, then for $i > 0$, $M(i) > N(f)$ and all the terms in (3.4) vanish. Then $\tilde{\Gamma}_i(f) = 0$ for $i > 0$, so $\tilde{\Gamma}(u)$ is a quantum field. For the second statement observe that $\tilde{\Gamma}_{(i, \lambda)} = \tilde{\Gamma}_{-1}(\tilde{F}_\lambda)$.

\[\square\]

3.2. Re-expansion of formal delta-function. For a complex-valued infinite matrix $A = (A_{ij})_{i,j \in \mathbb{Z}}$ consider the collection of formal complex-valued distributions

$$f_k(x) = \sum_{s \in \mathbb{Z}} A_{-k, -s} x^s, \quad k \in \mathbb{Z}.$$

If $A$ is invertible, we also introduce formal complex-valued distributions

$$g_k(x) = \sum_{s \in \mathbb{Z}} A^{-1}_{-s, -k} x^s, \quad k \in \mathbb{Z}.$$

Lemma 3.1. (a) Assume that $A$ is invertible. Then we have the equality of formal complex-valued distributions

$$\sum_{k \in \mathbb{Z}} g_k(x^{-1}) f_k(y) = x \delta(x, y).$$

where $\delta(x, y)$ is the formal delta function (2.3).

(b) If $A$ and $A^{-1}$ are block matrices of the form

$$A = \begin{bmatrix}
A^- & \vdots & 0 \\
\vdots & 0 & 1 & 0 & \cdots \\
0 & \vdots & A^+
\end{bmatrix}, \quad A^{-1} = \begin{bmatrix}
(A^-)^{-1} & \vdots & 0 \\
\vdots & 0 & 1 & 0 & \cdots \\
0 & \vdots & (A^+)^{-1}
\end{bmatrix},$$

(3.5)
with $A^- = (A_{i,j})_{i,j<0}$, $A^+ = (A_{i,j})_{i,j\geq 0}$, and $A_{0,0} = 1$, then
\[
\sum_{k \geq 0} g_k(x^{-1}) f_k(y) = \sum_{k \geq 0} \frac{y^k}{x^k} = i_{x,y} \left( \frac{x}{x-y} \right),
\]
\[
\sum_{k < 0} g_k(x^{-1}) f_k(y) = \sum_{k \geq 0} \frac{y^k}{x^k} = i_{y,x} \left( \frac{x}{y-x} \right),
\]
where $i_{x,y} \left( \frac{1}{x-y} \right)$ is the expansion of a rational function (2.4).

Proof. We check the first statement:
\[
\sum_{k \in \mathbb{Z}} g_k(x^{-1}) f_k(y) = \sum_{k,r,p \in \mathbb{Z}} (A^{-1})_{r-k} A_{-k,-p} x^{-r} y^p = \sum_{r,p \in \mathbb{Z}} \delta_{r,p} x^{-r} y^p = x \delta(x, y).
\]
Other identities follow from similar computations. □

Examples of identities of type Lemma 3.1 can be found in Section 7.

Remark 3.1. If $A$ is a two-block matrix as in Lemma 3.1 (b), then $f_k(x)$ is a power sum in $x$ for $k > 0$, and in $1/x$ for $k < 0$. If, in addition for any fixed $i \in \mathbb{Z}$, $A_{i,j} = 0$ for $j << 0$, then for $k > 0$ $f_k(x)$ is a polynomial.

We define matrix $A^\vee = (A^\vee)_{ij \in \mathbb{Z}}$ by
\[
(A^\vee)_{ij} = A_{-j,-i}.
\]
Next statement is obvious.

Lemma 3.2. Assume that $A$ is invertible and $A^{-1} = A^\vee$. Then $g_k(x) = f_{-k}(x^{-1})$.

3.3. Commutation relations of transformed vertex operators. In some cases we can formulate conditions when the linearly transformed quantum fields keep the commutation relations of the original quantum fields. In this section we state these conditions for commonly used algebraic structures of charged free fermions, neutral fermions and the Heisenberg algebra.

Proposition 3.1. Let $\{\tilde{\psi}_{-1/2}^\pm\}_{r \in \mathbb{Z}}$ be charged free fermions satisfying relations (2.12). Let $A = (A_{ij})_{i,j \in \mathbb{Z}}$ and $B = (B_{ij})_{i,j \in \mathbb{Z}}$ be two matrices, such that for any fixed $i$, $A_{i,j} = 0$ and $B_{i,j} = 0$ for $j << 0$. Set
\[
\tilde{\psi}_{k-1/2}^+ = \sum_{i \in \mathbb{Z}} A_{k,i} \tilde{\psi}_{i-1/2}^+, \quad \tilde{\psi}_{k-1/2}^- = \sum_{i \in \mathbb{Z}} B_{k,i} \tilde{\psi}_{i-1/2}^-.
\]
Then $\{\tilde{\psi}_{r-1/2}^\pm\}_{r \in \mathbb{Z}}$ satisfy the anti-commutation relations of charged free fermions if and only if $(A^{-1})_{i,j} = B_{1-j,-i}$.

Proof. Relations $[\tilde{\psi}_{k-1/2}^\pm, \tilde{\psi}_{m-1/2}^\pm]_+ = 0$, for $k, m \in \mathbb{Z}$, are immediate. One has
\[
[\tilde{\psi}_{k-1/2}^+, \tilde{\psi}_{m-1/2}^-]_+ = \sum_{i,j \in \mathbb{Z}} A_{k,i} B_{m,j} [\tilde{\psi}_{i-1/2}^+, \tilde{\psi}_{j-1/2}^-]_+ = \sum_{i,j \in \mathbb{Z}} A_{k,i} B_{m,j} \delta_{i+j,1} = \sum_{i \in \mathbb{Z}} A_{k,i} B_{m,1-i}.
\]
Then $[\tilde{\psi}_{k-1/2}^+, \tilde{\psi}_{m-1/2}^-]_+ = \delta_{k+m,1}$ if and only if $\sum_{i \in \mathbb{Z}} A_{k,i} B_{1-m,1-i} = \delta_{k,m}$ □

Proposition 3.2. Let $\{\varphi_k\}$ be neutral fermions, satisfying relations (2.15). Let $A = (A_{ij})_{i,j \in \mathbb{Z}}$ with the property that for any fixed $i$, $A_{i,j} = 0$ for $j << 0$. Set
\[
\hat{\varphi}_k = \sum_{i \in \mathbb{Z}} A_{k,i} \varphi_i.
\]
Proposition 3.3. Let \( \{ \tilde{\varphi}_i \}_{i \in \mathbb{Z}} \) satisfy the anticommutation relations of neutral fermions if and only if \((A^{-1})_{i,j} = (-1)^{i-j}A_{-j,-i}\).

Proof. 

\[
[\tilde{\varphi}_k, \tilde{\varphi}_m]_\pm = \sum_{i,j \in \mathbb{Z}} A_{k,i} A_{m,j} [\varphi_i, \varphi_j]_\pm = 2 \sum_{i,j \in \mathbb{Z}} A_{k,i} A_{m,j} (-1)^j \delta_{i,j,0} = 2 \sum_{i \in \mathbb{Z}} A_{k,i} (-1)^i A_{m,-i}.
\]

Then \( 2(1)^m \delta_{m+k,0} = [\tilde{\varphi}_k, \tilde{\varphi}_m]_\mp \) is equivalent to \( \sum_{i \in \mathbb{Z}} A_{k,i} (-1)^{m-i} A_{m,-i} = \delta_{k,m}. \)

Proposition 3.3. Let \( \{ \alpha_k \} \) be generators of Heisenberg algebra, satisfying relations (2.13). Let \( A = (A_{ij})_{i,j \in \mathbb{Z}} \) with the property that for any fixed \( i, A_{i,j} = 0 \) for \( j < 0 \). Set

\[
\tilde{\alpha}_k = \sum_{i \in \mathbb{Z}} A_{k,i} \alpha_i.
\]

Then \( \{ \tilde{\alpha}_i \}_{i \in \mathbb{Z}} \) satisfy the relations of type (2.13) if and only if \( APA^T = P \), where \( P = (P_{ij})_{i,j \in \mathbb{Z}} \) is a matrix with entries \( P_{ij} = i \delta_{i,-j} \) and \( A^T_{ij} = A_{j,i} \).

Proof. The statement follows from this calculation:

\[
k \delta_{k,-m} = [\tilde{\alpha}_k, \tilde{\alpha}_m] = \sum_{i,j \in \mathbb{Z}} A_{k,i} A_{m,j} [\alpha_i, \alpha_j] = \sum_{i,j \in \mathbb{Z}} A_{k,i} i \delta_{i,-j} A_{m,j} = \sum_{i,j \in \mathbb{Z}} A_{k,i} i \delta_{i,-j} (A^T)_{j,m}.
\]

4. Linear transformations of vertex operator presentation of Hall-Littlewood polynomials

In this section we consider vertex operator presentation of Hall-Littlewood polynomials, constructed first in [25]. We apply linear transformations of Section 3 to vertex operators of Hall-Littlewood polynomials to obtain new symmetric polynomials depending on parameter \( t \) and deduce their properties. In the subsequent sections we match specializations with different families that are studied by other authors.

4.1. Hall-Littlewood polynomials. First, we review necessary facts about Hall-Littlewood polynomials in a setup similar to [37, 48]. Let \( E(u) \), \( H(u) \), \( E^\pm(u) \), \( H^\pm(u) \) be quantum fields of operators acting on the space of symmetric functions \( \Lambda \) defined in Section 2.3. Define quantum fields \( \Gamma^\pm(u) = \sum_{k \in \mathbb{Z}} \Gamma_k u^{-k} \) of operators acting on \( \Lambda[[t]] \)

\[
\Gamma^+(u) = E(-tu)H(u)E^\dagger(-u), \quad (4.1) \\
\Gamma^-(u) = H(tu)E(-u)H^\dagger(u). \quad (4.2)
\]

Consider a formal distribution with coefficients in \( \Lambda[[t]] \)

\[
\mathcal{F}(u_1, \ldots, u_l; t) = \prod_{1 \leq i < j \leq l} i_{u_i, tu_j} \left( \frac{u_i - u_j}{u_i - tu_j} \right) \prod_{i=1}^l E(-tu_i)H(u_i), \quad (4.3)
\]

with the series expansion of rational functions \( \frac{u_i - u_j}{u_i - tu_j} \) in the regions \( |tu_j| < |u_i| \) for \( 1 \leq i < j \leq l \):

\[
i_{u_i, tu_j} \left( \frac{u_i - u_j}{u_i - tu_j} \right) = 1 + \sum_{s \geq 1} \left( t^s - t^{s-1} \right) \left( \frac{u_j}{u_i} \right)^s.
\]
Theorem 4.1. (a) Quantum fields $\Gamma^\pm(u)$ satisfy generalized fermion relations
\[
(u - vt) \Gamma^\pm(u) \Gamma^\pm(v) + (v - ut) \Gamma^\pm(v) \Gamma^\pm(u) = 0,
\]
\[
(v - ut) \Gamma^+(u) \Gamma^-(v) + (u - vt) \Gamma^-(v) \Gamma^+(u) = \delta(u, v)(1 - t)^2.
\]
(b) \[
\Gamma^+(u_1) \ldots \Gamma^+(u_t)(1) = \mathcal{F}(u_1, \ldots, u_t; t).
\](4.4)
(c) Coefficients of the formal distribution (4.4)
\[
\mathcal{F}(u_1, \ldots, u_t; t) = \sum_{\lambda \in \mathbb{Z}^t} F_{\lambda} u_1^{\lambda_1} \ldots u_t^{\lambda_t},
\](4.5)
have the form
\[
F_{\lambda} = \Gamma_{+\lambda_1}^{\vdash} \Gamma_{+\lambda_2}^{\vdash} \ldots \Gamma_{+\lambda_t}^{\vdash}(1) \in \Lambda[t].
\]
(d) For any $\lambda \in \mathbb{Z}^t$, $F_{(m, \lambda)} = 0$ for $m < 0$.
(e) \{\(F_{\lambda}\}_{\lambda \in \mathbb{Z}^t}\} defines a family of symmetric polynomials $F_{\lambda} = F_{\lambda}(x_1, x_2, \ldots, x_n; t)$, $n \in \mathbb{N}$, with the stability property:
\[
F_{\lambda}(x_1, \ldots, x_n; t) = F_{\lambda}(x_1, \ldots, x_n, 0; t).
\]
(f) Let $\lambda \in \mathbb{N}^t$ be an integer vector of length $t$ with positive coordinates. Let $n \geq l$ and set $\lambda_{l+1} = \cdots = \lambda_n = 0$. Then the corresponding coefficient $F_{\lambda}$ in the expansion (4.5) can be identified with the symmetric polynomial in variables $(x_1, x_2, \ldots, x_n)$ with coefficients in $\mathbb{C}[t]$
\[
F_{\lambda} = F_{\lambda}(x_1, \ldots, x_n; t) = \frac{(1 - t)^n}{\prod_{i=1}^{n-1}(1 - t^i)} \sum_{\sigma \in S_n} \sigma \left( x_1^{\lambda_1} \ldots x_n^{\lambda_n} \prod_{1 \leq i < j} (x_i - t x_j) \right).
\](4.6)
When $\lambda$ is a partition, (4.6) is called Hall-Littlewood polynomial.

(g) The set \{\(F_{\lambda}\}\} of Hall-Littlewood polynomials labeled by partitions form a linear basis of $\Lambda[t]$.
(h) When $\lambda$ is a partition, the specializations of Hall-Littlewood polynomials provide important families of symmetric functions: $F_{\lambda}(x_1, \ldots, x_n; 0) = s_{\lambda}(x_1, x_2, \ldots)$ is Schur symmetric function (2.1), and $F_{\lambda}(x_1, \ldots, x_n; -1) = q_{\lambda}(x_1, x_2, \ldots)$ is Schur $Q$-function (2.11).

Proof. (a) For the original proof see [25]. One can also deduce relations from the definition (4.1), (4.2) and Lemma 2.2, see [48] for this approach.
(b) Follows from (4.1), (4.2) and Lemma 2.2 applied to $\Gamma^+(u_1) \ldots \Gamma^+(u_t)(1)$.
(c) Immediately follows from (4.4).
(d) Follows from the fact that $\Gamma^+(u)$ is a quantum field.
(e) Is proved directly in [37]. For a shorter proof observe that vertex operators (4.1), (4.2) and their coefficients do not depend on $(x_1, x_2, \ldots, x_n)$, hence the resulting symmetric polynomial $F_{\lambda}$ also does not depend on them.
(f) For a partition $\lambda$ this statement is proved in [25, 37]. The proof in [37], III.2 carries without any changes to show that the statement is true for any vector $\lambda \in \mathbb{N}^t$.
(g) and (h) are discussed in [37], III.2.

\[\square\]

Remark 4.1. Note that if $\lambda \in \mathbb{Z}^t$ contains zero or negative entries, the coefficient $F_{\lambda}$ of (4.5) is still an element of $\Lambda[t]$, but it is not described by the formula (4.6), which in this case would involve negative powers of $x_i$'s. For example, coefficients $F(k) = 0$ for $k < 0$, and $F(-1,3) = (t^3 - t^2 + t - 1)F_2 + (t^2 - t)F^2_{(1)}$.

\[\text{Polynomials (4.6) correspond to polynomials } Q_{\lambda}(x, t) \text{ in notations of [37] III.2 (2.11).}\]
4.2. Linear transformation of vertex operators of Hall-Littlewood polynomials. Let $A = (A_{i,j})_{i,j \in \mathbb{Z}}$ be a complex-valued matrix with the property that for any $i \in \mathbb{Z}$, $A_{i,j} = 0$ for $j << 0$. Following Section 3, we define $\Gamma^+_i = \sum_{j \in \mathbb{Z}} A_{i,j} \Gamma^+_j$, where operators $\Gamma^+_j$ are coefficients of quantum fields (4.1) that realize Hall-Littlewood polynomials.

**Theorem 4.2.** Let $\lambda \in \mathbb{Z}^l$ and let

$$F_\lambda = \prod_{i=1}^{l-\lambda_1} \cdots \prod_{i=1}^{l-\lambda_l} (1).$$

(a) $\tilde{F}_\lambda$ is a finite element of $\Lambda[t]$ and

$$\tilde{F}_\lambda = \sum_{\alpha \in \mathbb{Z}^l} A_{-\lambda_1,-\alpha_1} \cdots A_{-\lambda_l,-\alpha_l} t^\alpha,$$

where $\{F_\alpha \in \Lambda[t]\}$ are coefficients in the expansion (4.5).

(b) The family of symmetric polynomials $F_\lambda (\lambda \in \mathbb{Z}^l)$ satisfy the stability property:

$$F_\lambda(x_1, \ldots, x_n; t) = F_\lambda(x_1, \ldots, x_n; 1).$$

(c) If $A$ is invertible, then for any $\alpha \in \mathbb{Z}^l$

$$F_\alpha = \sum_{\lambda \in \mathbb{Z}^l} (A^{-1})_{-\lambda_1,-\alpha_1} \cdots (A^{-1})_{-\lambda_l,-\alpha_l} \tilde{F}_\lambda.$$ (4.8)

(d) If $A$ is invertible, then we can write a re-expansion of the formal distribution (4.3) with coefficients $\tilde{F}_\lambda$:

$$F(u_1, \ldots, u_l; t) = \sum_{\lambda \in \mathbb{Z}^l} \tilde{F}_\lambda g_{\lambda_1}(u_1) \cdots g_{\lambda_l}(u_l),$$

where $g_k(u) = \sum_{s \in \mathbb{Z}} (A^{-1})_{-s,-k} u^s$ are complex-valued formal distributions.

(e) Assume that $A_{i,j} = 0$ for all $i < 0, j \geq 0$, and that $A_{0,j} = \delta_{0,j}$:

| $\downarrow$ i | $\rightarrow$ j | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | ... |
|----------------|----------------|-----|-----|-----|---|---|---|---|-----|
| ...            | ...            | ... | ... | ... | ...| ...| ...| ...| ... |
| $-3$           | ...            | *   | *   | *   | 0 | 0 | 0 | 0 | ... |
| $-2$           | ...            | *   | *   | *   | 0 | 0 | 0 | 0 | ... |
| $-1$           | ...            | *   | *   | *   | 0 | 0 | 0 | 0 | ... |
| 0              | ...            | 0   | 0   | 0   | 1 | 0 | 0 | 0 | ... |
| 1              | ...            | *   | *   | *   | * | * | * | * | ... |
| 2              | ...            | *   | *   | *   | * | * | * | * | ... |
| 3              | ...            | *   | *   | *   | * | * | * | * | ... |
| 4              | ...            | *   | *   | *   | * | * | * | * | ... |
| ...            | ...            | ... | ... | ... | ...| ...| ...| ...| ... |

Let $\lambda \in \mathbb{N}^l$, let $n \geq l$. Set $\lambda_{l+1} = \cdots = \lambda_n = 0$. Then the element $\tilde{F}_\lambda \in \Lambda[t]$ can be identified with a symmetric polynomial in variables $(x_1, \ldots, x_n)$ with coefficients in $\mathbb{C}[t]$ given by

$$\tilde{F}_\lambda(x_1, \ldots, x_n; t) = \frac{(1 - t)^n}{\prod_{i=1}^{n-l}(1 - t^i)} \sum_{\sigma \in S_n} \sigma \left( f_{\lambda_1}(x_1) \cdots f_{\lambda_n}(x_n) \prod_{i=1}^{n-l} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right),$$

where $f_k(x) = \sum_{j=1}^{-M(k)} A_{-k,-j} x^j$ ($k \in \mathbb{N}$) are complex-valued polynomials with zero constant coefficient ($f_k(0) = 0$), $f_0(x) = 1$, and $M(k)$ is defined as in (3.3).
Proof. Statements (a), (c) and (d) immediately follow from Theorem 3.1.

(b) Due to stability property of symmetric polynomials $F_\alpha$ and expansion (4.8) that involves coefficients that do not depend on $(x_1, \ldots, x_n)$, polynomials $\tilde{F}_\lambda$, also satisfy stability property.

(c) Let $\lambda \in \mathbb{N}^l$. With the imposed restriction on matrix $A$, all the terms $F_\alpha$ in (4.8) have the form (4.6), and

$$\tilde{F}_\lambda(x_1, x_2, \ldots, x_n; t) = \sum_{\alpha \in \mathbb{N}^l} A_{-\lambda_1, -\alpha_1} \cdots A_{-\lambda_l, -\alpha_l} F_\alpha(x_1, x_2, \ldots, x_n; t)$$

$$= \sum_{\alpha \in \mathbb{N}^l} A_{-\lambda_1, -\alpha_1} \cdots A_{-\lambda_l, -\alpha_l} \frac{(1-t)^n}{\prod_{i=1}^{n-l(\alpha)} (1-t)} \sum_{\sigma \in S_n} \sigma \left( x_1^{\alpha_1} \cdots x_i^{\alpha_i} \cdots x_n^{0} \prod_{1 \leq i < j} x_i - t x_j \right).$$

Note that due to the restriction on the matrix $A$, for all non-trivial terms in the sum $l(\alpha) = l$, and that we can interpret $x_k^0 = \sum_{\alpha_k \in \mathbb{Z}} A_{0, -\alpha_k} x_k^{\alpha_k}$. Then we can write

$$\tilde{F}_\lambda(x_1, x_2, \ldots, x_n; t) = \frac{(1-t)^n}{\prod_{i=1}^{n-l(\alpha)} (1-t)} \sum_{\sigma \in S_n} \left( \prod_{i=1}^{n-l(\alpha)} A_{-\lambda_i, -\alpha_i} \prod_{1 \leq i < j} x_i - t x_j \right).$$

The general condition that for any $A_{i,j} = 0$ for any $i < j$, and the specific form of the matrix $A$ imply that $\sum_{\alpha \in \mathbb{Z}^n} A_{-\lambda_i, -\alpha_i} x_i^{\alpha_i}$ are polynomials with zero constant term for $\lambda_i > 0$, and just constant polynomial 1 for $\lambda_i = 0$. This proves (4.9). \(\square\)

Remark 4.2. Representation theory of infinite-dimensional algebraic structures is based on applications of symmetric functions that do not depend on a number of variables, rather than on symmetric polynomials. This can be seen in the formulation of the boson-fermion correspondence, actions of $GL_{\infty}$, $S_{\infty}$, centers of universal enveloping algebras, etc. Hence, for such applications the stability property of generalizations of classical families of symmetric functions is essential, and we pay special attention to it through the text.

Corollary 4.1. Let $\{f_k(x)\}_{k \in \mathbb{Z} \geq 0}$ be a sequence of complex-valued polynomials with the property that $f_0(x) = 1$ and $f_k(0) = 0$ for all $k \in \mathbb{N}$. Then the family of symmetric polynomials in variables $(x_1, \ldots, x_n)$ labeled by partitions $\lambda$

$$\tilde{F}_\lambda(x_1, \ldots, x_n; t) = \frac{(1-t)^n}{\prod_{i=1}^{n-l(\alpha)} (1-t)} \sum_{\sigma \in S_n} \left( f_{\lambda_1}(x_1) \cdots f_{\lambda_n}(x_n) \prod_{1 \leq i < j} x_i - t x_j \right),$$

satisfies stability property

$$\tilde{F}_\lambda(x_1, \ldots, x_n; t) = \tilde{F}_\lambda(x_1, \ldots, x_n, 0; t).$$

Proof. Due to Theorem 4.2, (e) such polynomials can be interpreted as a result of a linear transformation of vertex operator presentation of Hall-Littlewood polynomials with a matrix $A$ defined by the coefficients of the given sequence of polynomials. Hence by Theorem 4.2, (b), they form a family of stable symmetric polynomials. \(\square\)

Remark 4.3. If $A$ is not of the form as in Theorem 4.2, (e), formula (4.9) cannot be applied to compute the values of $\tilde{F}_\lambda(x_1, \ldots, x_n)$, as illustrated by the next example.
Example 4.1. Let $a \neq 0$ and let
\[
A = Id + aE_{-2,0} = \begin{pmatrix}
\ldots & 1 & 0 & 0 & 0 & 0 & \\
\ldots & 0 & 1 & 0 & a & 0 & 0 \\
\ldots & 0 & 0 & 1 & 0 & 0 & 0 \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Then
\[
f_2(x) = x^2 + a, \quad \text{and} \quad f_k(x) = x^k \quad \text{for} \quad k \neq 2.
\]
By definition, $\tilde{F}(\alpha) = F(\alpha) + aF_0 = F(\alpha) + a$. However, $\tilde{F}(x_1, \ldots, x_n)$ cannot be computed by (4.9) since, using [37] III.1 (1.4),
\[
\tilde{F}(\alpha) = \frac{(1-t)^n}{\prod_{i=1}^{n-1}(1-t^i)} \sum_{\sigma \in S_n} \sigma \left( (x_1^2 + a) \prod_{i=1}^{n} \frac{x_i - t x_j}{x_i - x_j} \right) = F(\alpha) + a(1 - t^n).
\]
This example illustrates that if the condition $f_k(0) = 0$ is omitted in the Corollary 4.1, the resulting polynomial may depend on the number of variables $n$.

5. Specializations of linearly transformed Hall-Littlewood Polynomials

5.1. Specialization $t = 0$. Linearly transformed Hall-Littlewood polynomials specialized at $t = 0$ correspond to Schur-like symmetric functions. In addition to the properties implied by Theorem 4.2, several nice properties are specific to this specialization. Observe that $i_{u_i, t_{u_j}}(u_i - u_j)/(u_i - t u_j)|_{t=0} = 1 - \frac{u_i}{u_j}$. Using the Vandermonde determinant and the Jacobi-Trudi identity (2.2) the specialization of (4.3) at $t = 0$ is reduced to the formal distribution
\[
S(u_1, \ldots, u_l) = \Gamma^+(u_1)|_{t=0} \cdots \Gamma^+(u_l)|_{t=0}(1) = \prod_{1 \leq i < j \leq l} \left( \frac{u_i - u_j}{u_i} \right)^j H(u_i)
\]
\[
= \sum_{\alpha \in \mathbb{Z}^l} \det[h_{\alpha_i, -i+j}] u^\alpha = \sum_{\alpha \in \mathbb{Z}^l} s_\alpha u^\alpha.
\]
When $\alpha$ is a partition, by the Jacobi-Trudi identity the coefficient $s_\alpha$ coincides with a Schur symmetric function, so $S(u_1, \ldots, u_l)$ can be viewed as the generating function for Schur symmetric functions. It is also known that
\[
S(u_1, \ldots, u_l) = \det[u_i^{-j+i} H(u_i)].
\]
Let $A = (A_{i,j})_{i,j \in \mathbb{Z}}$ be a complex-valued matrix with the property that for any $i \in \mathbb{Z}$, $A_{i,j} = 0$ for $j < 0$. Consider the specialization of (4.7) at $t = 0$:
\[
\tilde{s}_\lambda = \tilde{\Gamma}^+_{-\lambda_i}|_{t=0} \cdots \tilde{\Gamma}^+_{-\lambda_l}|_{t=0}(1).
\]

Corollary 5.1. (a) For any integer vector $\lambda \in \mathbb{Z}^l$
\[
\tilde{s}_\lambda = \sum_{\alpha \in \mathbb{Z}^l} A_{-\lambda_1, -\alpha_1} \cdots A_{-\lambda_l, -\alpha_l} s_\alpha,
\]
and, if $A$ is invertible,
\[
s_\alpha = \sum_{\lambda \in \mathbb{Z}^l} (A^{-1})_{-\alpha_1, -\lambda_1} \cdots (A^{-1})_{-\alpha_l, -\lambda_l} \tilde{s}_\lambda.
\]
(b) (Analogue of the Jacobi-Trudi formula) For any integer vector \( \lambda \in \mathbb{Z}^l \)

\[
\tilde{s}_\lambda = \det \left[ \tilde{h}_{\lambda_i; i-j} \right]_{i,j=1\ldots l},
\]

where \( \tilde{h}_{k;m} = \sum_{r \in \mathbb{Z}} A_{-k,-r} h_{r-m} \). If \( A \) is invertible, then \( h_{k;m} \) are coefficients of expansion

\[
\sum_{k \in \mathbb{Z}} \tilde{h}_{k;m} g_k(u) = u^m H(u),
\]

where \( g_k(u) = \sum_{s \in \mathbb{Z}} (A^{-1})_{-s,-k} u^s \) and \( H(u) \) is defined by (2.5).

(c) Symmetric polynomials \( \{ \tilde{s}_\lambda \}_{\lambda \in \mathbb{Z}^l} \) satisfy stability property

\[
\tilde{s}_\lambda(x_1, \ldots, x_n, 0) = \tilde{s}_\lambda(x_1, \ldots, x_n).
\]

(d) Let \( A \) be invertible. Then we have the equality of \( \Lambda \)-valued formal distributions

\[
S(u_1, \ldots, u_l) = \sum_{\lambda \in \mathbb{Z}^l} \tilde{s}_\lambda g_{\lambda_1}(u_1) \cdots g_{\lambda_l}(u_l).
\]

(e) Let \( A \) be of the form as in Theorem 4.2, (e). Let \( \lambda \in \mathbb{N}^l \) and let \( n \geq l \). Set \( \lambda_{l+1} = \cdots = \lambda_n = 0 \). Then \( \tilde{s}_\lambda \in \Lambda \) can be identified with a symmetric polynomial in variables \((x_1, \ldots, x_n)\)

\[
\tilde{s}_\lambda(x_1, \ldots, x_n) = \frac{\det[f_{\lambda_i}(x_i)x_i^{n-j}]}{\det[x_i^{n-j}]},
\]

where \( f_k(x) = \sum_{j=1}^{M(k)} A_{-k,-j} x^j \) \((k \in \mathbb{N})\) are complex-valued polynomials with zero constant coefficient, \( f_0(x) = 1 \), and \( M(k) \) is defined as in (3.3).

Proof. (a), (c), (d) (e) follow from Theorem 4.2 by specialization at \( t = 0 \). Let us prove (b). From (a) and Jacobi-Trudi formula (2.2),

\[
\tilde{s}_{(\lambda_1, \ldots, \lambda_l)} = \sum_{\alpha \in \mathbb{Z}^l} A_{-\lambda_1,-\alpha_1} \cdots A_{-\lambda_l,-\alpha_l} \det[h_{\alpha_1-i+j}]

= \sum_{\sigma \in S_l} \sum_{\alpha \in \mathbb{Z}^l} (-1)^{n} A_{-\lambda_1,-\alpha_1} \cdots A_{-\lambda_l,-\alpha_l} h_{\alpha_1-1+\sigma(1)} \cdots h_{\alpha_l-1+\sigma(l)}

= \det \left[ \sum_{r \in \mathbb{Z}} A_{-\lambda_i,-r} h_{r-t+j} \right]_{1 \leq i, j \leq l},
\]

which proves (5.5). Direct calculation proves the second part of the statement. \( \square \)

5.2. Specialization \( t = -1 \). Linearly transformed Hall-Littlewood polynomials at \( t = -1 \) correspond to generalizations of Schur \( Q \)-functions. Formula (4.3) at \( t = -1 \) reduces to the generating function of Schur \( Q \)-functions

\[
Q(u_1, \ldots, u_l) = F(u_1, \ldots, u_l; t)_{t=-1} = \Gamma^+(u_1)_{t=-1} \cdots \Gamma^+(u_l)_{t=-1}(1)

= \prod_{1 \leq i < j \leq l} i_{u_i, u_j} \left( \frac{u_i - u_j}{u_i + u_j} \right) \prod_{i=1}^l Q(u).
\]

Here \( Q(u) = \sum_{k \in \mathbb{Z}} q_k u^k \) is is a formal distribution with coefficients in \( \mathcal{B}_{odd} \) defined by (2.8), and the series expansion of rational functions \( \frac{u_i - u_j}{u_i + u_j} \) in the regions \( |u_j| < |u_i| \) for \( 1 \leq i < j \leq l \) is

\[
i_{u_i, u_j} \left( \frac{u_i - u_j}{u_i + u_j} \right) = 1 + 2 \sum_{s \geq 1} (-1)^s \left( \frac{u_j}{u_i} \right)^s \in \mathbb{C}[|u_j/u_i|].
\]
Expand
\[ Q(u_1, \ldots, u_l) = \sum_{\alpha \in \mathbb{Z}^l} Q_{\alpha} u^\alpha. \tag{5.7} \]

When \( \alpha \) is a strict partition, coefficient \( Q_{\alpha} \) coincides with Schur \( Q \)-function (2.11), see [37] III.8. In particular, consider the coefficients of the formal distribution
\[ Q(u, v) = \left( 1 + \sum_{s \geq 1} (-1)^s \frac{u^s}{s!} \right) Q(u)Q(v) = \sum_{r,s \in \mathbb{Z}} Q_{r,s} u^r v^s. \]

For \( r > s \geq 0 \) the coefficient \( Q_{r,s} = q_{r,s} \) defined by (2.9).

Since
\[ Q(u_1, \ldots, u_{2l-1}) = -Q(u_1, \ldots, u_{2l-1}, 0), \]
it is sufficient to consider the case of even number of variables \((u_1, \ldots, u_{2l})\).

Let \( M = (M_{i,j})_{i,j=1,\ldots,2l} \) be a skew-symmetric matrix with entries
\[ M_{i,j} = \begin{cases} Q(u_i, u_j), & i < j, \\ 0, & i = j, \\ -Q(u_j, u_i), & i > j. \end{cases} \]

It is known that
\[ Q(u_1, \ldots, u_{2l}) = \text{Pf}[M_{i,j}]. \]

Remark 5.1. Note that for \( i > j \) both \( M_{i,j} = -M_{j,i} \) are elements of \( \mathbb{C}[[u_j/u_i]] \).

For any \( \alpha \in \mathbb{Z}^{2l} \) the coefficient \( Q_{\alpha} \) of the expansion (5.7) is
\[ Q_{\alpha} = \text{Pf}[q_{\alpha_i, \alpha_j}]_{i,j=1,\ldots,2l}, \tag{5.8} \]
where \( q_{ab} \) is defined by (2.9).

Let \( A \) be a complex-valued matrix with the property that for any \( i \in \mathbb{Z}, A_{i,j} = 0 \) for \( j << 0 \). Consider specialization of (4.7) at \( t = -1 \):
\[ \tilde{Q}_\lambda = \tilde{\Gamma}_+^{\lambda_1} |_{t=-1} \cdots \tilde{\Gamma}_+^{\lambda_l} |_{t=-1}(1). \tag{5.9} \]

Corollary 5.2. (a) For any \( \lambda \in \mathbb{Z}^l \),
\[ \tilde{Q}_\lambda = \sum_{\alpha \in \mathbb{Z}^l} A_{-\lambda_1,-\alpha_1} \cdots A_{-\lambda_l,-\alpha_l} Q_{\alpha}. \tag{5.10} \]

If \( A \) is invertible, then
\[ Q_{\alpha} = \sum_{\lambda \in \mathbb{Z}^l} (A^{-1})_{-\alpha_1,-\lambda_1} \cdots (A^{-1})_{-\alpha_l,-\lambda_l} \tilde{Q}_\lambda. \]

(b) (Analogue of Pfaffian formula) For any integer vector \( \lambda \in \mathbb{Z}_{2l} \)
\[ \tilde{Q}(\lambda_{1}, \ldots, \lambda_{2l}) = \text{Pf} \left[ \sum_{k,r \in \mathbb{Z}} A_{-\lambda_1,-k} A_{-\lambda_2,-r} q_{k,r} \right]_{i,j=1,\ldots,2l}. \tag{5.11} \]

(c) Symmetric polynomials \( \{\tilde{Q}_\lambda\}_{\lambda \in \mathbb{Z}^l} \) satisfy the stability property:
\[ \tilde{Q}_\lambda(x_1, \ldots, x_n, 0) = \tilde{Q}_\lambda(x_1, \ldots, x_n). \]
(d) Let $A$ be invertible. Consider $g_k(u) = \sum_{s \in \mathbb{Z}} (A^{-1})_{-s,-k} u^{-s}$. Then
\[
Q(u_1, \ldots, u_l) = \sum_{\lambda \in \mathbb{Z}^l} \tilde{Q}_\lambda g_{\lambda_1}(u_1) \cdots g_{\lambda_l}(u_l).
\] (5.12)

(e) Let $A$ be as in Theorem 4.2, (e). Let $\lambda \in \mathbb{N}^l$. For $n \geq l$ and set $\lambda_{l+1} = \cdots = \lambda_n = 0$. Then the element $\tilde{Q}_\lambda \in \Lambda$ can be identified with a symmetric polynomial in variables $(x_1, \ldots, x_n)$
\[
\tilde{Q}_\lambda(x_1, \ldots, x_n) = 2^l \sum_{\sigma \in S_n} \sigma \left( f_{\lambda_1}(x_1) \cdots f_{\lambda_n}(x_n) \prod_{i=1}^n \prod_{1<j<n} \frac{x_i + x_j}{x_i - x_j} \right),
\]
where $f_k(x) = \sum_{i=1}^{M(k)} A_{-k,-i} x^i$ $(k \in \mathbb{N})$, are complex-valued polynomials with zero constant coefficient, $f_0(x) = 1$, and $M(k)$ is defined as in (3.3).

Proof. (a), (c), (d), (e) follow from Theorem 4.2 by evaluation at $t = -1$.

Let $M_{i,j}^\lambda = \sum_{k,r} A_{-\lambda_i,-k} A_{-\lambda_j,-r} g_{k,r}$. Note that $M_{i,j}^\lambda = -M_{j,i}^\lambda$ and that $M_{i,i}^\lambda = 0$. From (a) and (5.8),
\[
\tilde{Q}_{\lambda_1, \ldots, \lambda_{2l}} = \sum_{\alpha \in \mathbb{Z}^{2l}} A_{-\lambda_1,-\alpha_1} \cdots A_{-\lambda_{2l},-\alpha_{2l}} Q_{\alpha_1, \ldots, \alpha_{2l}} = \sum_{\alpha \in \mathbb{Z}^{2l}} A_{-\lambda_1,-\alpha_1} \cdots A_{-\lambda_{2l},-\alpha_{2l}} Pf[q_{\alpha_i, \alpha_j}]
\]
\[
= \sum_{\sigma \in S_{2l}} sgn(\sigma) \sum_{\alpha \in \mathbb{Z}^{2l}} A_{-\lambda_{\sigma(1)},-\alpha_{\sigma(1)}} A_{-\lambda_{\sigma(2)},-\alpha_{\sigma(2)}} q_{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}} \cdots A_{-\lambda_{\sigma(2l-1)},-\alpha_{\sigma(2l-1)}} A_{-\lambda_{\sigma(2l)},-\alpha_{\sigma(2l)}} q_{\alpha_{\sigma(2l-1)}, \alpha_{\sigma(2l)}}
\]
\[
= Pf[M_{i,j}^\lambda]_{i,j=1, \ldots, 2l}.
\]
We used that $A_{-\lambda_1,-\alpha_1} \cdots A_{-\lambda_{2l},-\alpha_{2l}} = A_{-\lambda_{\sigma(1)},-\alpha_{\sigma(1)}} \cdots A_{-\lambda_{\sigma(2l)},-\alpha_{\sigma(2l)}}$ for any $\sigma \in S_{2l}$. This computation proves (b).

6. POLYNOMIAL TAU-FUNCTIONS OF THE KP AND THE BKP HIERARCHY

In [54] M. Sato introduced the KP hierarchy of evolution equations. The ideas were further developed for this and other examples of soliton type hierarchies by the Kyoto school [7, 9, 10, 23], and later by many other authors. In [54] solutions of the KP hierarchy were expressed through tau-functions. Polynomial tau-functions form an infinite Grassmann manifold. Schur polynomials, which are Schur symmetric functions expressed as polynomials of power sums, are examples of tau-functions of the KP hierarchy [54]. Similarly in [56, 57] it is proved that Schur $Q$-functions expressed through polynomials of power sums are examples of polynomial tau-functions of the BKP, DKP and MDKP hierarchies.

Recently all polynomial tau-functions of several soliton hierarchies were described. In [28, 29, 30] it is demonstrated that any polynomial tau-function of the KP-hierarchy is obtained from a Schur polynomial by certain shifts of arguments in the Jacobi-Trudi formula, and that any polynomial tau-function of the BKP or the DKP hierarchy is described by a shift of arguments in the Pfaffian formula for Schur $Q$-polynomials. Another approach in [31] proves that any polynomial tau-function of the KP, the BKP and the $s$-component KP hierarchy can be interpreted as a zero-mode of an appropriate combinatorial generating function.

In this note we show that specializations considered in Sections 5.1, 5.2 provide one more description of all polynomial tau-functions of the KP and the BKP hierarchies. This observation will easily follow from the results of [31] with the advantage that the new formulation immediately implies that a number of Schur-like symmetric functions that can be found in the literature provide polynomial
tau-functions of the KP hierarchy (when these symmetric functions are expressed as polynomials in power sums). For example, we recover as a particular case our earlier result [52] that multiparameter Schur $Q$-functions are tau-functions of the BKP hierarchy. More examples can be found in the end of this note.

6.1. **Polynomial tau-functions of the bilinear KP identity.** Let $\left\{ \psi_k^\pm \right\}$ be the charged free fermions those action on the boson Fock space $B$ is defined in Section 2.5. Let $\Omega = \sum_{k \in \mathbb{Z}+1/2} \psi_k^+ \otimes \psi_{-k}$. The bilinear KP identity [9, 10, 32] is the equation of the form

$$\Omega (\tau \otimes \tau) = 0 \quad (6.1)$$

on a function $\tau = z^m \sigma(p_1, p_2, \ldots)$ from the formal completion of the space $B^{(m)} = z^m \Lambda$, where the ring of symmetric functions $\Lambda$ is identified with the ring $\mathbb{C}[p_1, p_2, \ldots]$ of polynomials in power sums (see Section 2.1). Non-zero solutions of (6.1) are called tau-functions of the KP hierarchy. Accordingly, we will say that a non-zero solution of (6.1) is a polynomial tau-function, if it is a polynomial function in the variables $(p_1, p_2, \ldots)$ times $z^m$ (hence, an element of $B^{(m)}$ rather than a completion of the space $B^{(m)}$).

In [31] polynomial tau-functions of the KP hierarchy are described as coefficients of formal distributions of the form $A_1(u_1) \cdots A_l(u_l) S(u_1, \ldots, u_l)$. Here we state that polynomial tau-functions of the KP hierarchy are linear transformations of vertex operators presentation of Schur symmetric functions, and in particular cases, are coefficients of re-expansions of the formal distribution $S(u_1, \ldots, u_l)$ in another basis of formal distributions.

Let $A$ be a complex-valued matrix with the property that for any $i \in \mathbb{Z}$, $A_{i,j} = 0$ for $j << 0$. For any integer vector $\lambda \in \mathbb{Z}^l$ let $s_{\lambda}$ be defined by (5.3). Recall that $s_{\lambda}$ can be also computed by (5.4) or by (5.5).

**Theorem 6.1.**

a) $s_{\lambda}$ is a polynomial tau-function of the KP hierarchy. Any polynomial tau-function of the KP hierarchy is of the form $s_{\lambda}$ for an appropriate choice of matrix $A$.

b) If $A$ is invertible and $g_k(u) = \sum_{r \in \mathbb{Z}} (A^{-1})_{r-k} u^r$, then any coefficient $s_{\lambda}$ of the re-expansion of the generating function (5.1) of Schur symmetric functions

$$S(u_1, \ldots, u_l) = \sum_{\lambda \in \mathbb{Z}^l} (-1)^{l+1} s_{\lambda} g_{\lambda_1}(u_1) \cdots g_{\lambda_l}(u_l)$$

is a polynomial tau-function of the KP bilinear hierarchy.

**Proof.**

a) In [31] Theorem 3.1 provides the following description of polynomial tau-functions of the KP hierarchy. Consider a collection of complex-valued Laurent series $A_1(u), \ldots, A_l(u)$. Let $\alpha \in \mathbb{Z}^l$, and let $T_\alpha$ be the coefficient$^2$ of $u^\alpha$ in the expansion

$$\sum_{\alpha \in \mathbb{Z}^l} T_\alpha u_1^{\alpha_1} \cdots u_l^{\alpha_l} = \prod_{i=1}^l A_i(u_i) S(u_1, \ldots, u_l). \quad (6.2)$$

Then all coefficients $T_\alpha$ are polynomial tau-functions of the KP hierarchy, and for any polynomial tau-function of the KP hierarchy there exists a collection of Laurent polynomials $A_1(u), \ldots, A_l(u) \in \mathbb{C}[u, u^{-1}]$ such that $\tau = T(0, \ldots, 0)$ in the Laurent series expansion of (6.2).

Let $s_{\lambda}$ be defined by (5.3). By (5.4), $s_{\lambda} = \sum_{\alpha \in \mathbb{Z}^l} A_{-\lambda_1, -\alpha_1} \cdots A_{-\lambda_l, -\alpha_l} s_\alpha$, and the sum is finite. Then $s_{\lambda}$ is the coefficient of $u_1^{\alpha_1} \cdots u_l^{\alpha_l}$ in (6.2) with the choice $A_i(u) = A_{-\lambda_i, k} u^k$, $i = 1, \ldots, l$ (note $A_i(u)$ are Laurent series by the property $A_{ij} = 0$ for $j << 0$). Then by the results of [31], $s_{\lambda}$ is a polynomial tau-function.

---

$^2T_{(\alpha_1, \ldots, \alpha_l)}$ here corresponds to $T_{(\alpha_1-(+1, \ldots, 0))}$, and (6.2) to $T(u_1, \ldots, u_l)/u_1^{\alpha_1} \cdots u_l^{\alpha_l}$ in notations of [31] (3.12).
The other way, let \( \tau \) be a polynomial KP tau-function. Then for an appropriate choice of Laurent polynomials \( A_i(u) = \sum_{k \in \mathbb{Z}} a_{i,k} u^k, \ i = 1, \ldots, l \), it is a zero-mode of (6.2) and \( \tau = \sum_{a \in \mathbb{Z}} a_1, -a_1 \ldots a_l, -a_l s_a \). Then by (5.4) \( \tau = \hat{s}_l \) for the transformation matrix

\[
A_{-i,j} = \begin{cases} 
    a_{i,j}, & i = 1, \ldots, l, j \in \mathbb{Z}, \\
    \delta_{i,j}, & \text{otherwise.} 
\end{cases}
\]  

(6.3)

Since \( A_i(u) \) are Laurent polynomials, the matrix \( A \) satisfies the condition \( A_{ij} = 0 \) for \( j << 0 \).

b) The coefficients of the re-expansion (5.6) are exactly of the form (5.3), hence they are polynomial tau-functions of the KP hierarchy.

\[ \square \]

**Remark 6.1.** We use invertibility of matrix \( A \) to define basis terms \( g_k(u) \) in re-expansion of the generating function \( S(u_1, \ldots, u_l) \). This restriction on \( A \) does not allow us to interpret any polynomial tau-function of the KP hierarchy as a coefficient of such re-expansion. Yet, we will see in the end of the note that many interesting symmetric functions correspond to invertible linear transformations, and for this reason are polynomial tau-functions that can be interpreted as such coefficients.

**Remark 6.2.** The authors of [14] study properties of transformations acting on the quantum fields in a fermionic Fock space within the view of applications to the description of the tau-functions of the KP and BKP hierarchy. The transformations in [14] depend on an invertible upper-triangular matrix. The authors relate with their construction Schur-like symmetric polynomials, where in the top alternating monomials \( x^k \) are substituted by an arbitrary sequence of monic polynomials [14] (3.5). In some cases these Schur-like symmetric polynomials may depend on the number of variables \( n \), in which case they do not extend to symmetric functions.

### 6.2. Polynomial tau-functions of the bilinear BKP identity.

Let \( \{ \varphi_k \} \) be neutral fermions those action on the boson Fock space \( \mathcal{B}_{odd} \) is defined by (2.14). Let \( \Omega = \sum_{n \in \mathbb{Z}} \varphi_n \otimes (-1)^n \varphi_{-n} \). The **bilinear BKP identity** [9, 10] is the equation of the form

\[ \Omega(\tau \otimes \tau) = \tau \otimes \tau \]  

(6.4)
on elements \( \tau = \tau(p_1, p_2, p_3 \ldots) \) from the completion of \( \mathcal{B}_{odd} \). Non-zero solutions of (6.4) are called the **tau-functions of the BKP hierarchy**. We will say that a solution of (6.4) is a polynomial tau-function if it is a polynomial function in the variables \( (p_1, p_2, p_3 \ldots) \) (hence, it is an element of \( \mathcal{B}_{odd} \) rather than its completion).

All polynomial tau-functions of the BKP hierarchy are described in [29, 31]. Similarly, to \( t = 0 \) case, we state here that these polynomial tau-functions are results of linear transformations of vertex operators of Schur \( Q \)-functions and, in invertible cases, are coefficients of re-expansions of \( Q(u_1, \ldots, u_l) \). The proof is again based on the results of [31].

Let \( A \) be a complex-valued matrix with the property that for any \( i \in \mathbb{Z}, A_{i,j} = 0 \) for \( j << 0 \). For any \( \lambda \in \mathbb{Z}^l \) let \( \hat{Q}_\lambda \) be defined by (5.9). Recall that \( \hat{Q}_\lambda \) can be also computed by (5.10) or by (5.11).

**Theorem 6.2.**

a) \( \hat{Q}_\lambda \) is a polynomial tau-function of the BKP hierarchy. Any polynomial tau-function of the BKP hierarchy can be written in the form \( \hat{Q}_\lambda \) for an appropriate choice of matrix \( A \).

b) If \( A \) is invertible and \( g_k(u) = \sum_{r \in \mathbb{Z}} (A^{-1})_{-r,-k} u^r \), then any coefficient \( \hat{Q}_\lambda \) of the re-expansion of generating function (5.7) of Schur \( Q \)-functions

\[
Q(u_1, \ldots, u_l) = \sum_{\lambda \in \mathbb{Z}^l} \hat{Q}_\lambda g_{\lambda_1}(u_1) \ldots g_{\lambda_l}(u_l)
\]

is a polynomial tau-function of the BKP bilinear hierarchy.
a) The proof is based on [31] Theorem 4.1. Let $A_1(u), \ldots, A_l(u) \in \mathbb{C}[u, u^{-1}]$ be a collection of Laurent polynomials. For any $\alpha \in \mathbb{Z}^l$ let $T_\alpha$ be the coefficient in the expansion

$$
\sum_{\alpha \in \mathbb{Z}^l} T_\alpha u_1^{a_1} \cdots u_l^{a_l} = \prod_{i=1}^l A_i(u_i) \, Q(u_1, \ldots, u_l).
$$

Then all coefficients $T_\alpha$ are polynomial tau-functions of the BKP hierarchy, and for any polynomial tau-function of the BKP hierarchy there exists a collection of Laurent polynomials $A_1(u), \ldots, A_l(u) \in \mathbb{C}[u, u^{-1}]$ such that $\tau = T_{(0, \ldots, 0)}$ in the Laurent series expansion of (6.5).

By (5.10), $\tilde{Q}_\lambda$ can be interpreted as the coefficient of $u_1^0 \cdots u_l^0$ in (6.5) with the choice $A_i(u) = A_{-\lambda_i, k} u^k$, $i = 1, \ldots, l$. Hence $\tilde{Q}_\lambda$ is a polynomial tau-function of the BKP hierarchy. Any polynomial tau-function $\tau$ is a zero-mode of (6.5) for an appropriate choice of Laurent polynomials $A_i(u) = \sum_{k \in \mathbb{Z}} a_{i,k} u^k$, $i = 1, \ldots, l$. Hence by (5.10) it can be identified with $\tau = \tilde{Q}_\lambda$ with the same transformation matrix (6.3) as in the KP case.

b) is clear from (5.12).

7. Examples of Linear Transformations

In this section we relate general construction of linear transformations to examples of symmetric functions considered by other authors.

7.1. Linear transformation by a Toeplitz matrix. Let $A$ be a Toeplitz matrix with constant complex values $(a_k)_{k \in \mathbb{Z}}$ along diagonals:

$$
A_{i,j} = a_{j-i} \quad \text{for all } i, j \in \mathbb{Z}.
$$

The desired condition $A_{i,j} = 0$ for $j << 0$ imposes the restriction on the sequence $a_k = 0$ for $k << 0$. Then

$$
\tilde{\Gamma}_i = \sum_{j \in \mathbb{Z}} a_{j-i} \Gamma_j, \quad \tilde{F}_\lambda = \sum_{\alpha \in \mathbb{Z}} a_{\lambda_i-a_i} \cdots a_{\lambda_l-a_l} F_\alpha
$$

and

$$
\sum_{\alpha \in \mathbb{Z}} \tilde{F}_\alpha u_1^{a_1} \cdots u_l^{a_l} = A(u_1) \cdots A(u_l) F(u_1, \ldots, u_l),
$$

with Laurent series $A(u) = \sum_{k \in \mathbb{Z}} a_k u^k$. Since in this case matrix $A$ is not of the form Theorem 4.2 (e), we cannot use (4.9) directly to compute the corresponding transformation of Hall-Littlewood polynomial $\tilde{F}_\lambda$ as a symmetric polynomial in variables $(x_1, \ldots, x_n)$.

7.2. Change of basis $x_i \mapsto (x + \cdots + x_i)$. Let $A$ be the block matrix of the form (3.5) with upper-triangular blocks

$$
(A^-)_{ij} = \begin{cases} 1, & i \leq j < 0 \\ 0, & \text{otherwise} \end{cases}, \quad (A^+)_{ij} = \begin{cases} 1, & i = j > 0, \\ -1, & j = i + 1 > 1 \\ 0, & \text{otherwise} \end{cases}.
$$
The matrix $A^\lambda$ where we used that the particular case of this formula \((\ref{eq7.1})\) specializes to $t = 1$.

Then $A^\lambda = A^{-1}$ and by Lemma 3.2 for $k \in \mathbb{N}$

$$f_k(x) = g_{-k}(x^{-1}) = x + \cdots + x^k = \frac{x(1 - x^k)}{1 - x}, \quad f_{-k}(x) = g_k(x^{-1}) = \frac{x - 1}{x^{k+1}}, \quad f_0(x) = g_0(x) = 1.$$  

The matrix $A$ satisfies the properties of Theorem 4.2, (e). Hence for any partition $\lambda$ the corresponding transformation of Hall-Littlewood polynomials provides a symmetric polynomial in variables $(x_1, \ldots, x_n)$

$$\tilde{F}_\lambda(x_1, \ldots, x_n; t) = \frac{(1 - t)^n}{\prod_{i=1}^{n-i}(1 - t^i)} \sum_{\sigma \in S_n} \sigma \left( (1 - x_1^\lambda) \cdots (1 - x_i^\lambda) \prod_{i=1}^{n} \frac{x_i}{1 - x_i} \prod_{1 < j} \frac{x_i - tx_j}{x_i - x_j} \right), \quad (7.1)$$

with specialization at $t = 0$

$$\tilde{s}_\lambda(x_1, \ldots, x_n) = \frac{\det[x_i^{n-j+1}(1 - x_i^\lambda)(1 - x_i)^{-1}]}{\det[x_i^{n-j}]} \quad (7.2)$$

For any partition $\lambda$ the Jacobi-Trudi identity (5.5) reads in this example as

$$\tilde{s}_\lambda = \det \left[ \sum_{k=1}^{\lambda_i} h_{k-1+j} \right]_{i,j=1, \ldots, t} = \det \left[ \tilde{s}(\lambda_{i+j} - i) - \tilde{s}(j-i) \right]_{i,j=1, \ldots, t},$$

where we used that the particular case of this formula $\lambda = (m)$ gives $\tilde{s}_{(m)} = h_1 + \cdots + h_m$. Let $\{\psi_k\}_{k \in \mathbb{Z}}$ be the charged free fermions with the action on the ring of symmetric functions defined in Section 2.5. This action provides the vertex operator presentation of classical Schur functions.

Then, according to our general construction, operators

$$\tilde{\psi}_{k-1/2}^+ = \sum_i A_{k,i} \tilde{\psi}_{i-1/2}^+, \quad \tilde{\psi}_{k-1/2}^- = \sum_i A_{k-1,i-1} \tilde{\psi}_{i-1/2}^- \quad (7.3)$$

provide the vertex operator presentation of symmetric functions $\tilde{s}_\lambda$, and by Proposition 3.1 also satisfy relations (2.12) of charged free fermions.

By the results of Section 6, symmetric function (7.2), expressed as a polynomial in power sums, is a tau-function of the KP hierarchy, and the specialization of (7.1) at $t = -1$, expressed as a polynomial in odd power sums, is a tau-function of the BKP hierarchy.

We would like to illustrate also that linear transformations of quantum fields can be viewed as a source of curious identities. For example, for $|x| < |y|$ Lemma 3.1 implies

$$\sum_{k > 0} \frac{(1 - x^k)(y - 1)}{y^{k+1}(1 - x)} = \frac{1}{x - y}.$$
More generally, let’s apply Theorem 4.2 taking \( l = 1 \):

\[
\mathcal{F}(u) = \sum_{k \in \mathbb{Z}} \tilde{F}(k) g_k(u). \tag{7.4}
\]

Note that \( \tilde{F}(k) = 0 \) for \( k < 0 \), \( \tilde{F}(0) = 1 \). Using [37] III (2.9), we get for \( k \in \mathbb{N} \),

\[
\tilde{F}_k = \sum_{s=1}^{k} F_k = (1 - t) \sum_i (x_i + \cdots + x^k) \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} = (1 - t) \sum_i \frac{x_i(1 - x^k)}{1 - x_i} \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}.
\]

Recall that \( \mathcal{F}(u) = E(-tu)H(u) = \prod_i \frac{x_i - tu}{1 - x_i} \) and that \( g_k(1/u) = u^k - u^{k+1} \) for \( k > 0 \). Then (7.4) provides the identity of formal distributions

\[
\prod_i \frac{1 - x_i tu}{1 - x_i} = 1 + (1 - t) \sum_r \frac{x_r(1 - x^r)}{1 - x_r} \prod_{i \neq j} \frac{x_i - x_j}{x_i - x_j} (u^r - u^{r+1}),
\]

in analogy with the identity [37] III (2.10).

### 7.3. Multiparameter Symmetric Functions

Consider an infinite sequence of complex numbers \( a = (a_1, a_2, \ldots) \). For \( n \in \mathbb{N} \) define the multiparameter powers of variables

\[
(x|a)_n = (x - a_1)(x - a_2)\ldots(x - a_n),
\]

and set \((x|a)_0 = 1\). The following transitions are well-known ([51] Lemma 2.5, [20] (10.2), [39]) Theorem 2.1). They can be proved by direct computation.

**Lemma 7.1.** For \( n \in \mathbb{N} \)

\[
(x|a)_n = \sum_{k=0}^{n} (-1)^{n-k} e_{n-k}(a_1, \ldots, a_n) x^k,
\]

\[
\frac{1}{(x|a)_n} = \sum_{k=n}^{\infty} h_{k-n}(a_1, \ldots, a_n) x^{-k},
\]

\[
x^n = \sum_{k=0}^{n} h_{n-k}(a_1, \ldots, a_{k+1}) (x|a)_k,
\]

\[
x^{-n} = \sum_{k=n}^{\infty} (-1)^{n-k} e_{k-n}(a_1, \ldots, a_{k-1}) \frac{1}{(x|a)_k}.
\]

Let \( A \) be the upper-triangular block matrix with the non-zero entries

\[
\begin{align*}
A_{i,0} &= A_{0,i} = \delta_{i,0}, \quad i \in \mathbb{Z}, \\
A_{0,j} &= h_{j-i}(a_1, \ldots, a_i), \quad i, j \in \mathbb{N}, \\
A_{-i,j} &= (-1)^{i-j} e_{i-j}(a_1, \ldots, a_{i-1}), \quad i, j \in \mathbb{N}.
\end{align*}
\]
Using short notations $h_r[s] = h_r(a_1, \ldots, a_s)$ and $e_r[s] = e_r(a_1, \ldots, a_s)$, the matrix $A$ has the form

\[
\begin{array}{cccccc}
\cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\
-3 & \cdots & 1 & -e_1[2] & e_2[2] & 0 & 0 & 0 & 0 & \cdots \\
-2 & \cdots & 0 & 1 & -e_1[1] & 0 & 0 & 0 & 0 & \cdots \\
-1 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
1 & \cdots & 0 & 0 & 0 & 0 & 1 & h_1[1] & h_2[1] & \cdots \\
2 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & h_2[2] & \cdots \\
3 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
4 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\end{array}
\]

### Lemma 7.2
$A^{-1} = A^\top$

**Proof.** Let $n \in \mathbb{N}$. By Lemma 7.1 we can write the sequence of transitions

\[
\frac{1}{x^n} = \sum_{k=1}^{\infty} (-1)^{n-k} e_{k-n}(a_1, \ldots, a_{k-1}) \frac{1}{(x|a)_k} = \sum_{k,r=1}^{\infty} (-1)^{n-k} e_{k-n}(a_1, \ldots, a_{k-1})h_{r-k}(a_1, \ldots, a_k) \frac{1}{x^r} = \sum_{k,r=1}^{\infty} A_{k-n}A_{r,k} \frac{1}{x^r}.
\]

Comparing the coefficients of $1/x^r$ on both sides we obtain that $\delta_{n,r} = \sum_{k=1}^{\infty} A_{k-n}A_{r,k}$. □

By Lemma 3.2 for $n \in \mathbb{N}$,

\[
f_k(x) = g_{-k}(x^{-1}) = x(x|a)_{k-1}, \quad f_{-k}(x) = g_k(x^{-1}) = 1/(x|a)_k, \quad f_0(x) = g_0(x) = 1.
\]

For $|x| < |y|$ Lemma 3.1 implies the identity [50, 51]

\[
\sum_{k>0} \frac{(x|a)_{k-1}}{(y|a)_k} = \frac{1}{y-x}.
\]

The identity of type (7.4) looks in this example

\[
\prod_i \frac{u-x_i t}{u-x_i} = 1 + (1-t) \sum_{r=1}^{\infty} \sum_i x_i (x|a)_{r-1} \prod_{i \neq j} \frac{x-tx_j}{x_i-x_j}.
\]

By Theorem 4.2, (e), for any partition $\lambda$ the corresponding transformation of Hall-Littlewood polynomials provides a symmetric polynomial in variables $(x_1, \ldots, x_n)$

\[
\tilde{F}_\lambda(x_1, \ldots, x_n; t) = \frac{(1-t)^n}{\prod_{r=1}^{n-1} (1-t^r)} \sum_{\sigma \in S_n} \sigma \left( (x_1|a)_{\lambda_1-1} \cdots (x_i|a)_{\lambda_i-1} \prod_{r=1}^{n} x_i \prod_{i<j} x_i - tx_j \right), \quad (7.5)
\]

with specialization at $t = 0$

\[
\tilde{s}_\lambda(x_1, \ldots, x_n) = \frac{\det[x_i^{n-j+1}(x_1|a)_{\lambda_j-1}]}{\det[x_i^{n-j}]}. \quad (7.6)
\]

The entries of the determinant of the Jacobi-Trudi identity (5.5) in this case are given by

\[
\tilde{h}_{i:m} = \sum_{j=1}^{i} (-1)^{i-j} e_{i-j}(a_1, \ldots, a_{i-1}) h_{j-m}, \quad i \in \mathbb{N}.
\]
They can be interpreted as coefficients of re-expansion of $H(1/u)$:

$$1 + \sum_{k \in \mathbb{N}} \frac{\hat{h}_{k,m}}{(u|a)_k} = u^m H(1/u).$$

(7.7)

By the results of Section 6, symmetric function (7.6), expressed as a polynomial in power sums, is a tau-function of the KP hierarchy, and the specialization of (7.5) at $t = -1$, expressed as a polynomial in odd power sums, is a tau-function of the BKP hierarchy.

Different multiparameter analogues of symmetric functions find their applications in representation theory, algebraic geometry, combinatorics. Without aspirations to get even close to a complete overview, we mention here some of the examples that are most closely related to our construction.

Polynomials of type (7.5) appear in [44, 45, 46], where the authors generalize Hall-Littlewood polynomials by the substitution $x_i^k \rightarrow (x|a)_k$ and the change of ordinary addition to some formal group law. The authors study combinatorics and relations of these polynomials to the flag bundles in the complex cobordism theory.

One can note that the ratio of two alternants (7.6) stands between the definition of Schur symmetric functions $s_\lambda$ and the factorial Schur functions, where the later are symmetric polynomials that depend on a doubly-infinite sequence of parameters $a = (a_i)_{i \in \mathbb{Z}}$. They are defined as

$$s^F_\lambda(x_1, \ldots, x_n) = \frac{\det[(x_i|a)_{\lambda_j + n - j}]_{1 \leq i,j \leq n}}{\det[x_i^{n-j}]_{1 \leq i,j \leq n}}.$$  

(7.8)

On the contrary to $s_\lambda$ and $\tilde{s}_\lambda$, symmetric polynomials $s^F_\lambda(x_1, \ldots, x_n)$ do not enjoy the stability property. At the same time, their reach combinatorics is widely studied in the literature. For example, $s^F_\lambda(x_1, \ldots, x_n)$ are known to be a special case of the double Schubert polynomials, satisfy vanishing properties and the Jacobi-Trudi identity, and many other analogues of standard properties of Schur symmetric functions: [3, 4, 6, 12, 13, 34, 35, 38, 39, 40, 49], [37] I-3 Examples 20–21, and many others. Factorial Schur functions with the sequence of parameters $(0, 1, 2, 3, \ldots)$ are closely related to shifted Schur functions, which are a family of functions symmetric in shifted variables. Their nowadays well-developed theory started with [50]. In relation to our setup we mention that the vertex operator presentation of shifted symmetric functions was constructed in [26].

The mentioned above examples are compared in [51] with another multiparameter family of symmetric functions. The authors of [51] introduce Frobenius-Schur super-symmetric functions that correspond to a sequence of parameters $a = \frac{2i+1}{2}$, and extend the definition to a general multiparameter analogue with a general sequence $a = (a_i)_{i \in \mathbb{Z}}$. We denote here these generalizations as $s^{[ORV]}_{\mu; a}$. These symmetric functions find their applications in the asymptotic character theory of the symmetric group. By formula (3.4) in [51], this multiparameter analogue satisfies the Jacobi-Trudi identity

$$s^{[ORV]}_{\mu; a} = \det \left[ h^{[ORV]}_{\mu_i - i + j; \tau^r a} \right],$$

(7.8)

where $\mu$ is a partition, and $h^{[ORV]}_{\mu; a}$ are coefficients of the formal distribution re-expansion

$$1 + \sum_{k > 0} \frac{h^{[ORV]}_{\mu; a}}{(u - a_{1-r}) \cdots (u - a_{k-r})} = H(1/u).$$

(7.9)

One can see from (7.9) and (7.7) that $h^{[ORV]}_{\mu; a} = \tilde{h}_{k,0}$, but in general $s^{[ORV]}_{\mu; a}$ is not of the form $\tilde{s}_\mu$.

One probably can deduce a vertex operator presentation of $s^{[ORV]}_{\mu; a}$ in the spirit of (5.3) from (7.8), but it would most likely have more complicated structure involving a shift $(\tau^r a)_i = a_{i+r}$ at every step of application of a vertex operator.
Thus, (7.6) resembles different multiparameter analogues of Schur functions considered in the literature, but does not coincide with any of them.

At the same time, the specialization \( t = -1 \) of (7.5) does coincide with introduced earlier by other authors the multiparameter Schur \( \hat{Q} \)-functions for a sequence of parameters \((0, a_1, a_2, \ldots)\). These interpolation analogues of the classical Schur \( \hat{Q} \)-functions were studied combinatorially in [20, 33]. In [52] we proved that multiparameter Schur \( \hat{Q} \)-functions are tau-functions of the BKP hierarchy. The initial goal of this study, achieved in Section 6, was to generalize this result. The factorial Schur \( \hat{Q} \)-functions, which are multiparameter Schur \( \hat{Q} \)-functions corresponding to \((a_i = i - 1)\) proved to be useful in study of a number of questions of representation theory and algebraic geometry: [1, 17, 18, 19, 47, 53]

7.4. The uniform shift. Let's consider the special case of linear transformation of Section 7.3 with all parameter values \((a_i = 1)\) \( i \in \mathbb{N} \). Then for \( k \in \mathbb{N} \)

\[
f_k(x) = x(x - 1)^{k-1} = \sum_{j > 0} (-1)^{k-j} \binom{k-1}{j-1} x^j, \quad f_{-k} = \frac{1}{(x-1)^k} \sum_{j=1}^{\infty} \binom{j-1}{k-1} \frac{1}{x^j}, \quad f_0 = 0.
\]

The blocks of matrix \( A \) of Section 7.3 become Pascal’s matrices

\[
\begin{align*}
A_{i, i-j} &= (-1)^{i-j} \binom{i-1}{j-1}, \quad i, j \in \mathbb{N}, \\
A_{i,j} &= \binom{i-1}{j-1}, \quad i, j \in \mathbb{N}.
\end{align*}
\]

Then the polynomial

\[
\tilde{F}_\lambda(x_1, \ldots, x_n; t) = \frac{(1-t)^n}{\prod_{i=1}^{n} (1-t^i)} \sum_{\sigma \in S_n} \sigma \left( x_1(x_1-1)^{\lambda_1-1} \cdots x_n(x_n-1)^{\lambda_n-1} \prod_{i<j} x_i - t x_j \right)
\]

(7.11)

coincides with the inhomogeneous Hall-Littlewood polynomial that appear in [5] as a degeneration of a dual Hall-Littlewood-like rational symmetric functions. That is polynomial \( \tilde{G}_\lambda \) with \( k = 0 \) in the notations of Section 8.2 of [5]. These rational symmetric functions are rational deformations of Hall-Littlewood polynomials, defined as partition functions for path ensembles in a square grid with assigned vertex weights. Accordingly, the specialization at \( t = 0 \)

\[
\tilde{s}_\lambda(x_1, \ldots, x_n) = \frac{\det ([x_i - 1]^{\lambda_i-1} x_i^{n-j+1})}{\prod_{i<j} (x_i - x_j)}.
\]

(7.12)

corresponds to inhomogeneous Schur polynomials \( G_\lambda^{(q=0)} \) with \( k = 0 \) in Section 8.4 of [5].

Then all the general statements about linear transformations can be applied to the examples of this section:

a) Fromula (4.7) of Theorem 4.2 with \( \tilde{E}_i^+ = \sum A_{ij} \Gamma_j^+ \), matrix \( A \) given by (7.10) and \( \Gamma_i^+ \) by (4.1), provides a vertex operator presentation of (7.11), and fromula (4.8) gives the expression of (7.11) in terms of Hall-Littlewood polynomials. Theorem 4.2 (b) implies that (7.11) are stable polynomials.

b) Corollary 5.1 b) with matrix \( A \) given by (7.10) gives the Jacobi- Trudi formula of (7.12)

\[
\tilde{s}_\lambda = \det \left[ \sum_{r} (-1)^{\lambda_r-j} \binom{\lambda_{r-1}}{j-1} h_{r-i+j} \right].
\]

c) By Theorem 6.1, the family of symmetric functions (7.12), expressed as polynomials of power sums, can be interpreted as tau-functions of the KP hierarchy (6.1).
d) Since in the considered example $A^{-1} = A^\nu$, by Lemma 3.2, $g_k(x) = f_{-k}(x^{-1})$. Following Corollary 5.1, re-expand the generating function of Schur symmetric functions in the basis of monomials in formal distributions $g_k(u)$:

$$S(u_1, \ldots, u_l) = \sum_{\lambda \in \mathcal{P}_l} \tilde{s}_\lambda g_{\lambda_1}(u) \cdots g_{\lambda_l}(u).$$

Then by Corollary 5.1 d), for any partition $\lambda$ symmetric function (7.12) is the coefficient of the monomial

$$g_{\lambda_1}(u) \cdots g_{\lambda_l}(u) = \frac{u_1^{\lambda_1} \cdots u_l^{\lambda_l}}{(1 - u_1)^{\lambda_1} \cdots (1 - u_l)^{\lambda_l}}$$

of this re-expansion. As before, here we identify rational functions with the appropriate expansions: $\frac{u_1^{\lambda_1} \cdots u_l^{\lambda_l}}{(1 - u_1)^{\lambda_1} \cdots (1 - u_l)^{\lambda_l}} = \sum_{j=1}^{\infty} \frac{(j-1)!}{k!} u^j$. With the change of variables $v_i = \frac{u_i}{1-u_i}$ we can state that for any partition $\lambda$ symmetric function $\tilde{s}_\lambda$ given by (7.12) is the coefficient of the monomial $u^\lambda$ in the expansion of

$$S(v_1/1 + v_1, \ldots, v_l/1 + v_l) = \sum_{\lambda \in \mathcal{P}_l} \tilde{s}_\lambda v_1^{\lambda_1} \cdots v_l^{\lambda_l}.$$

7.5. **Grothendieck polynomials.** In [5] Section 8.4 similarity of polynomials (7.12) to Grothendieck polynomials is also pointed out. Grothendieck polynomials were introduced in [35] as polynomial representatives of Schubert classes in the Grothendieck ring. These polynomials and their variations were further studied by many authors, such as [11, 15, 16, 36, 21, 22, 41, 42, 43] etc.

In [41] formula (5.1) for a partition $\lambda$ the Grothendieck polynomial is defined as

$$G_\lambda(x_1, \ldots, x_n; \beta) = \frac{\det[x_i^{\beta_i + n-j}(1 + \beta x_i)^{j-1}]}{\prod_{i<j}(x_i - x_j)}. \quad (7.13)$$

This formula can be used to express the Grothendieck polynomials through polynomials of type (7.12) Indeed, by the change of variables $x_i = 1 + \beta y_i$ in (7.12) and introducing $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ with $\mu_n - k + 1 = \lambda_k - k$, we can write

$$\tilde{s}_\lambda(1 + y_1 \beta, \ldots, 1 + y_n \beta) e_n(1 + y_1 \beta, \ldots, 1 + y_n \beta) = \frac{\det[(\beta y_i)^{\lambda_i-1}(1 + \beta y_i)^{n-j+1}]}{\prod_{i<j}(\beta y_i - y_j)\prod_{i=1}^n(1 + y_i \beta)}$$

$$= \frac{\beta^{n(n-1)/2}}{\beta^{n(n-1)/2} \prod_{i<j}(y_i - y_j)\prod_{i=1}^n(1 + y_i \beta)} = \pm \beta^{\mu_1} G_\mu(y_1, \ldots, y_n; \beta).$$

**Remark 7.1.** Note that if $\mu = (\mu_1, \ldots, \mu_n)$ is a partition, then $\lambda \in \mathbb{N}^n$, and $|\mu| = \sum(\lambda_i - i)$.

In [21, 22] free fermions presentation of stable Grothendieck polynomials and their duals are constructed. The Jacobi-Trudi identities that are provided in these papers allow us to put these constructions in the format of this note. By Proposition 3.9 in [21], stable Grothendieck polynomials satisfy the Jacobi-Trudi identity

$$G_\lambda(x_1, \ldots, x_n; \beta) = \det \left[ \sum_{m=0}^{\infty} \binom{i-l}{m} \beta^m G_{\lambda_i - i+j+m} \right] \quad (7.14)$$

where $G_m = G_m(x_1, \ldots, x_n; \beta)$ are defined through generating function of Proposition 3.8 in [21], which we can interpret in our notations as

$$G(u) = \sum_{m=0}^{\infty} G_m u^m = i_u \left( \frac{1}{1 + \beta/u} \right) E(\beta) H(u).$$
Here \( i_{u, \beta}\left(\frac{u}{u + \beta}\right) = \sum_{n} \frac{(-\beta)^{n}}{n!} \) and \( E(\beta) = \prod_{i=1}^{\infty} (1 + \beta x_i) = \sum_{i \geq 0} e_i(x_1, x_2, \ldots) \beta^i. \)

Note that formula (7.14) does not depend on variables \((x_1, \ldots, x_n)\).

Let

\[
G(u_1, \ldots, u_l) = \prod_{i=1}^{l} E(\beta) (1 + \beta / u_i)^{i-l-1} \cdot S(u_1, \ldots, u_l),
\]

where \( S(u_1, \ldots, u_l) \) is the generating function for Schur symmetric functions given by (5.1), and we use binomial series expansion \((1 + \frac{\beta}{u})^a = i_{u, \beta}\left(1 + \frac{\beta}{u}\right) = \sum_{r=0}^{\infty} \binom{a}{r} \frac{\beta^r}{u^r} \) for any \( a \in \mathbb{C} \).

Proposition 7.1.

a) In the expansion \( G(u_1, \ldots, u_l) = \sum_{\lambda \in \mathbb{J}} G_{\lambda} u^\lambda \) the coefficients \( G_{\lambda} \) that correspond to partitions \( \lambda \) are stable Grothendieck polynomials.

b) Vertex operator presentation of stable Grothendieck polynomials can be written in the form

\[
G(u_1, \ldots, u_l) = B^+(u_1) \ldots B^+(u_l) (1),
\]

where

\[
B^+(u) = H(u)E^\perp(-u)E(\beta) = \Gamma^+(u)|_{t=0} E(\beta)
\]

in notations of Sections 2.3 and 4.1.

Proof.

a) By (5.2)

\[
G(u_1, \ldots, u_l) = \prod_{i=1}^{l} E(\beta) (1 + \beta / u_i)^{i-l-1} \cdot \det[u_i^{-j+i} H(u_i)]
\]

\[
= \prod_{i=1}^{l} (1 + \beta / u_i)^{i-l} \det[u_i^{-j+i} (1 + \beta / u_i)^{-1} E(\beta) H(u_i)] = \prod_{i=1}^{l} (1 + \beta / u_i)^{i-l} \det[u_i^{-j+i} G(u_i)]
\]

\[
= \prod_{i=1}^{l} \sum_{m_i \geq 0} \binom{i - l}{m_i} \beta^{m_i} \sum_{\sigma \in S_i} \sum_{\alpha \in \mathbb{J}} (-1)^{\sigma} G_{\alpha} u_1^{\alpha_1 + 1 - \sigma(1)} \ldots G_{\alpha_l} u_l^{\alpha_l + 1 - \sigma(l)}
\]

\[
= \sum_{\lambda \in \mathbb{J}} \sum_{\sigma \in S_i} \sum_{m_i \geq 0} (-1)^{\sigma} \binom{i - l}{m_i} \beta^{m_i} G_{\lambda - 1 + \sigma(1) + m_i} \ldots \binom{0}{m_i} \beta^{m_i} G_{\lambda - l + \sigma(l) + m_i} u_1^{\lambda_1} \ldots u_l^{\lambda_l}
\]

\[
= \sum_{\lambda \in \mathbb{J}} \det \left[ \sum_{m_i \geq 0} \binom{i - l}{m_i} \beta^{m_i} G_{\lambda_i - i+j+m_i} \right] u_1^{\lambda_1} \ldots u_l^{\lambda_l} = \sum_{\lambda \in \mathbb{J}} G_{\lambda} u^\lambda.
\]

b) Using relations of Lemma 2.2

\[
\left(1 + \frac{\beta}{u}\right) E^\perp(-u)E(\beta) = E(\beta)E^\perp(-u),
\]

move all terms \( E(\beta) \) in the product \( B^+(u_1) \ldots B^+(u_l) (1) \) to the left, and use that \( H(u_1)E^\perp(-u_1) \ldots H(u_l)E^\perp(-u_l)(1) = S(u_1, \ldots, u_l) \) by (5.1).

□

Similarly, the definition of dual stable Grothendieck polynomials \( g_{\lambda}(z_1, \ldots, z_n) \) in Section 4.1 of [21] is followed by the Jacobi-Trudi formula

\[
g_{\lambda}(x_1, \ldots, x_n; \beta) = \det \left[ \sum_{m=0}^{\infty} \binom{1 - i}{m} \beta^{m} h_{\lambda_i - i+j-m} \right],
\]
(Proposition 4.4 in [21], see also [2]). Let
\[ J(u_1, \ldots, u_t) = \prod_{i=1}^{t} (1 + \beta u_i)^{1-i} S(u_1, \ldots, u_t), \]
where \( S(u_1, \ldots, u_t) \) is the generating function for Schur symmetric functions given by (5.1) and we use the binomial expansion \((1 + \beta u)^a = \sum_{r=0}^{\infty} \binom{a}{r} \beta^r u^r \) for any \( a \in \mathbb{C} \).

**Proposition 7.2.**

a) In the expansion \( J(u_1, \ldots, u_t) = \prod_{i=1}^{t} (1 + \beta u_i)^{1-i} \) \( S(u_1, \ldots, u_t) \), the coefficients \( g_{\lambda} \) that correspond to partitions \( \lambda \) are dual stable Grothendieck polynomials.

b) \( J(u_1, \ldots, u_t) = J^+(u_1) \ldots J^+(u_t)(1) \), where
\[ J^+(u) = H(u) E^+(u) H^+(u^{-1}/\beta) = \Gamma^+(u)|_{t=0} H^+(u^{-1}/\beta) \]
in notations of Sections 2.3 and 4.1.

c) Dual stable Grothendieck polynomials \( g_{\lambda}(x_1, \ldots, x_n) \), expressed as polynomials in power sums \( p_1, p_2, \ldots \) are polynomial tau-functions of the KP hierarchy (6.1).

**Proof.** Similar calculations show that
a) \[ J(u_1, \ldots, u_t) = \prod_{i=1}^{t} (1 + \beta u_i)^{1-i} \cdot \det [u_i^{-j+i} H(u_i)] \]
\[ = \prod_{i=1}^{t} \sum_{m_i \geq 0} \binom{1-i}{m_i} h^{m_i} u_i^{m_i} \sum_{\sigma \in S_t} \sum_{a \in \mathbb{Z}} (-1)^\sigma h_{\alpha_1}^{\alpha_1+1-\sigma(1)} \ldots h_{\alpha_t}^{\alpha_t+1-\sigma(t)} \]
\[ = \sum_{\lambda \in \mathbb{Z}} \sum_{\sigma \in S_t} \sum_{m_i \geq 0} (-1)^\sigma \binom{0}{m_1} h^{m_1} h_{\lambda_1-1+\sigma(1)-m_1} \ldots \binom{1-i}{m_t} h^{m_t} h_{\lambda_t-1+\sigma(t)-m_t} \sum_{\lambda \in \mathbb{Z}} g_{\lambda} u^\lambda. \]
b) Using relations of Lemma 2.2
\[ (1 + \beta u) H^+(u^{-1}/\beta) H(u) = H(u) H^+(u^{-1}/\beta), \]
move all terms \( H^+(u^{-1}/\beta) \) in the product \( J^+(u_1) \ldots J^+(u_t)(1) \) to the right, and use that \( H^+(1) = 1 \) and that \( H(u_1) E^+(u_1) \ldots H(u_t) E^+(u_t)(1) = S(u_1, \ldots, u_t) \) by (5.1).

c) By part a), any dual stable Grothendieck polynomial \( g_{\lambda} \) is a coefficient of the series \( J(u_1, \ldots, u_t) \). Note that it is obtained from \( S(u_1, \ldots, u_t) \) by multiplication by power series \( A_i(u_i) = \sum_{r=0}^{\infty} \binom{r+i}{r} \beta^r u_i^r \). Then by [31] Theorem 3.1 symmetric function \( g_{\lambda} \) is a polynomial tau-functions of the KP hierarchy (6.1).

\[ \square \]

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