A Covering for the dKP–hyper CR Interpolating Equation and Multi-Valued Einstein–Weyl Structures

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Abstract. We apply the technique of integrable extensions to the symmetry pseudo-group of the dKP–hyper CR interpolating equation. This allows us to find a covering for this equation and to construct multi-valued Einstein–Weyl structures.

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1. Introduction

We consider the equation
\[ u_{yy} = u_{tx} + (b u_x - c u_y) u_{xx} + c u_x u_{xy}, \]  
(1)

\( b = \text{const}, \ c = \text{const}, \) found recently by M. Dunajski, [9], as a symmetry reduction of Plebański’s second heavenly equation, [23]. When \( c = 0 \) and \( b \neq 0 \), Eq. (1) coincides with the potential form of the Khokhlov–Zabolotskaya equation, [13], also known as dispersionless Kadomtsev–Petviashvili equation (dKP). For \( b = 0 \) and \( c \neq 0 \) Eq. (1) becomes the hyperCR equation studied in [24, 8]. The Einstein-Weyl structure associated to (1) is
\[ \begin{cases} h = (dy - c u_x dt)^2 - 4 (dx + (c u_y - b u_x) dt) dt, \\ \omega = -c u_{xx} dy + ((c^2 u_x + 4 b) u_{xx} - 2 c u_{xy}) dt. \end{cases} \]  
(2)

A Lax pair
\[ \begin{cases} q_t = ((c z - b) u_x + c u_y + z^2) q_x + b (z u_{xx} + u_{xy}) q_z, \\ q_y = (c u_x + z) q_x + b u_{xx} q_z \end{cases} \]  
(3)
is proposed for Eq. (1) in [9]. It contains differentiation with respect to a new independent variable \( z \). The compatibility condition \( q_{ty} = q_{yt} \) for system (3) together with the requirement \( u_z = 0 \) coincides with Eq. (1). The equation \( u_z = 0 \) does not follow from (3).

When \( b \neq 0 \) and \( c \neq 0 \), the simple scaling
\[ t = c^4 b^{-3} \tilde{t}, \quad x = c^2 b^{-1} \tilde{x}, \quad y = -c^3 b^{-2} \tilde{y} \]  
(4)
gives after dropping tildes the following equation:
\[ u_{yy} = u_{tx} + (u_x + u_y) u_{xx} - u_x u_{xy}. \]  
(5)
The aim of this paper is to find a covering, [14, 15, 16], for Eq. (5). We apply the method proposed in [22] and find a contact integrable extension for the Cartan’s structure equations of the symmetry pseudo-group of Eq. (5). Integrating the extension yields a Bäcklund transformation and a covering equation for (5). We use these results for constructing multi-valued Einstein–Weyl structures [2] likewise in [21] multi-valued solutions to dKP were found.

2. Preliminaries

2.1. Coverings of PDEs

Let \( \pi_\infty : J^\infty(\pi) \to \mathbb{R}^n \) be the infinite jet bundle of local sections of the bundle \( \pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \). The coordinates on \( J^\infty(\pi) \) are \((x^i, u_I)\), where \( I = (i_1, ..., i_k) \) are symmetric multi-indices, \( i_1, ..., i_k \in \{1, ..., n\} \), \( u_\emptyset = u \), and for any local section \( f \) of \( \pi \) there exists a section \( j_\infty(f) : \mathbb{R}^n \to J^\infty(\pi) \) such that \( u_I(j_\infty(f)) = \partial^{I-1}(f)/\partial x^{i_1}...\partial x^{i_k}, \)
The total derivatives on $J^\infty(\pi)$ are defined in the local coordinates as

$$D_i = \frac{\partial}{\partial x^i} + \sum_{#I \geq 0} u_I \frac{\partial}{\partial u_I}.$$  

We have $[D_i, D_j] = 0$ for $i, j \in \{1, \ldots, n\}$. A differential equation $F(x^i, u_K) = 0$ defines a submanifold $\mathcal{E}^\infty = \{D_I(F) = 0 \mid #I \geq 0\} \subset J^\infty(\pi)$, where $D_I = D_{i_1} \circ \ldots \circ D_{i_k}$ for $I = (i_1, \ldots, i_k)$. We denote restrictions of $D_i$ on $\mathcal{E}^\infty$ as $\bar{D}_i$.

In local coordinates, a covering over $\mathcal{E}^\infty$ is a bundle $\tilde{\mathcal{E}}^\infty = \mathcal{E}^\infty \times \mathcal{V} \to \mathcal{E}^\infty$ with fibre coordinates $v^\gamma, \gamma \in \{1, \ldots, N\}$ or $\gamma \in \mathbb{N}$, equipped with extended total derivatives

$$\tilde{D}_i = \bar{D}_i + \sum_{\gamma} T_i^\gamma(x^j, u_I, v^\beta) \frac{\partial}{\partial v^\gamma}$$

such that $[\tilde{D}_i, \tilde{D}_j] = 0$ whenever $(x^i, u_I) \in \mathcal{E}^\infty$. Action of $\tilde{D}_i$ on the fibre variables $v^\gamma$ gives

$$v^\gamma_{x^i} = T_i^\gamma(x^j, u_I, v^\beta).$$

These equations are called a Bäcklund transformation. Excluding $u$ from them yields a system of pde's for $v^\gamma$. This system is called a covering equation.

In terms of differential forms, the covering is defined by the forms, $\omega^\gamma = dv^\gamma - T_i^\gamma(x^j, u_I, v^\beta) dx^i$

such that

$$d\omega^\gamma \equiv 0 \text{ mod } \omega^\beta, \bar{\vartheta}_I \iff (x^i, u_I) \in \mathcal{E}^\infty,$$

where $\bar{\vartheta}_I$ are restrictions of contact forms $\vartheta_I = du_I - u_{I,k} dx^k$ on $\mathcal{E}^\infty$.

2.2. Cartan’s structure theory of Lie pseudo-groups

Let $M$ be a manifold of dimension $n$. A local diffeomorphism on $M$ is a diffeomorphism $\Phi : U \rightarrow \hat{U}$ of two open subsets of $M$. A pseudo-group $\mathfrak{G}$ on $M$ is a collection of local diffeomorphisms of $M$, which is closed under composition when defined, contains an identity and is closed under inverse. A Lie pseudo-group is a pseudo-group whose diffeomorphisms are local analytic solutions of an involutive system of partial differential equations.

Élie Cartan’s approach to Lie pseudo-groups is based on a possibility to characterize transformations from a pseudo-group in terms of a set of invariant differential 1-forms called Maurer–Cartan (mc) forms. In a general case mc forms $\omega^1, \ldots, \omega^m$ of a Lie pseudo-group $\mathfrak{G}$ on $M$ are defined on a direct product $\hat{M} \times G$, where $\mu : \hat{M} \times G \rightarrow M$ is a bundle, $m = \dim \hat{M}, G$ is a finite-dimensional Lie group. The forms $\omega^i$ are semi-basic w.r.t. the natural projection $\hat{M} \times G \rightarrow \hat{M}$ and define a coframe on $\hat{M}$, that is, a basis of the cotangent bundle of $\hat{M}$. They characterize the pseudo-group $\mathfrak{G}$ in the following sense: a local diffeomorphism $\Phi : U \rightarrow \hat{U}$ on $M$ belongs to $\mathfrak{G}$ whenever there exists a
Covering of the dKP–hyper CR Interpolating Equation

local diffeomorphism $\Psi : V \rightarrow \hat{V}$ on $\tilde{M} \times G$ such that $\mu \circ \Psi = \Phi \circ \mu$ and the forms $\omega^i$ are invariant w.r.t. $\Psi$, that is,

$$\Psi^* (\omega^i|_{\hat{V}}) = \omega^i|_{V}. \quad (7)$$

Expressions of exterior differentials of the forms $\omega^i$ in terms of themselves give Cartan’s structure equations of $G$:

$$d\omega^i = A^i_{\gamma j} \pi^\gamma \wedge \omega^j + B^i_{jk} \omega^j \wedge \omega^k, \quad B^i_{jk} = -B^i_{kj}. \quad (8)$$

Here and below we assume summation on repeated indices. The forms $\pi^\gamma$, $\gamma \in \{1, \ldots, \dim G\}$, are linear combinations of $\mathcal{MC}$ forms of the Lie group $G$ and the forms $\omega^j$. The coefficients $A^i_{\gamma j}$ and $B^i_{jk}$ are either constants or functions of a set of invariants $U^\kappa : M \rightarrow \mathbb{R}$, $\kappa \in \{1, \ldots, l\}$, $l < \dim M$, of the pseudo-group $\mathcal{G}$, so $\Phi^* (U^\kappa|_{U}) = U^\kappa|_{U}$ for every $\Phi \in \mathcal{G}$. In the latter case, differentials of $U^\kappa$ are invariant 1-forms, so they are linear combinations of the forms $\omega^j$,

$$dU^\kappa = C^\kappa_j \omega^j, \quad (9)$$

where the coefficients $C^\kappa_j$ depend on the invariants $U^1, \ldots, U^l$ only.

Eqs. (8) must be compatible in the following sense: we have

$$d(d\omega^i) = 0 = d\left( A^i_{\gamma j} \pi^\gamma \wedge \omega^j + B^i_{jk} \omega^j \wedge \omega^k \right), \quad (10)$$

therefore there must exist expressions

$$d\pi^\gamma = W^\gamma_{\lambda j} \chi^\lambda \wedge \omega^j + X^\gamma_{\beta j} \pi^\beta \wedge \pi^\epsilon + Y^\gamma_{\beta j} \pi^\beta \wedge \omega^j + Z^\gamma_{jk} \omega^j \wedge \omega^k, \quad (11)$$

with some additional 1-forms $\chi^\lambda$ and the coefficients $W^\gamma_{\lambda j}$ to $Z^\gamma_{jk}$ depending on the invariants $U^\kappa$ such that the right-hand side of (10) appear to be identically equal to zero after substituting for (8), (9), and (11). Also, from (9) it follows that the right-hand side of the equation

$$d(dU^\kappa) = 0 = d(C^\kappa_j \omega^j) \quad (12)$$

must be identically equal to zero after substituting for (8) and (9).

The forms $\pi^\gamma$ are not invariant w.r.t. the pseudo-group $\mathcal{G}$. Respectively, the structure equations (8) are not changing when replacing $\pi^\gamma \mapsto \pi^\gamma + z^\gamma_j \omega^j$ for certain parametric coefficients $z^\gamma_j$. The dimension $r^{(1)}$ of the linear space of these coefficients satisfies the following inequality

$$r^{(1)} \leq n \dim G - \sum_{k=1}^{n-1} (n - k) \sigma_k, \quad (13)$$

where the reduced characters $\sigma_k$ are defined by

$$\sigma_k = \max_{u_1, \ldots, u_k} \text{rank} \ A_k(u_1, \ldots, u_k)$$

with the matrices $A_k$ inductively defined by

$$A_1(u_1) = \left( A_{i j}^1 u_1^j \right), \quad A_k(u_1, \ldots, u_l) = \left( A_{i j}^k(u_1, \ldots, u_l - 1) A_{i j}^l u_l^j \right), \quad (14)$$
see \[4, \S 5\], \[23\] Def. 11.4 for the full discussion. The system of forms $\omega^k$ is involutive if \(13\) is an equality, \[4, \S 6\], \[23\] Def. 11.7.

Cartan’s fundamental theorems, \[4, \S\S 16, 22–24\], \[7\], \[27\] \S\S 16, 19, 20, 25,26\], \[26\] \S\S 14.1–14.3, state that for a Lie pseudo-group there exists a set of mc forms whose structure equations satisfy the compatibility and involutivity conditions; conversely, if Eqs. \(5\), \(9\) meet the compatibility conditions \(10\), \(12\), and the involutivity condition, then there exists a collection of 1-forms $\omega^1, \ldots, \omega^m$ and functions $U^1, \ldots, U^l$ which satisfy \(5\) and \(9\). Eqs. \(7\) then define local diffeomorphisms from a Lie pseudo-group.

**Example 1.** Consider the bundle $J^2(\pi)$ of jets of the second order of the bundle $\pi: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$, $\pi: (x^1, \ldots, x^n, u) \mapsto (x^1, \ldots, x^n, u)$. A differential 1-form $\vartheta$ on $J^2(\pi)$ is called a contact form if it is annihilated by all 2-jets of local sections $f$ of the bundle $\pi$: $j_2(f)^*\vartheta = 0$. In the local coordinates every contact 1-form is a linear combination of the forms $\vartheta_0 = du - u_i dx^i$, $\vartheta_i = du_i - u_{ij} dx^j$, $i, j \in \{1, \ldots, n\}$, $u_{ij} = u_{ji}$. A local diffeomorphism $\Delta: J^2(\pi) \rightarrow J^2(\pi)$, $\Delta: (x^i, u, u_{ij}) \mapsto (\hat{x}^i, \hat{u}, \hat{u}_{ij})$, is called a contact transformation if for every contact 1-form $\vartheta$ the form $\Delta^*\vartheta$ is also contact. We denote by Cont($J^2(\pi)$) the pseudo-group of contact transformations on $J^2(\pi)$. Consider the following 1-forms

$$
\Theta_0 = a \vartheta_0, \quad \Theta_i = g_i \Theta_0 + a B^k_i \vartheta_k, \quad \Xi^i = c^i \Theta_0 + f^{ik} \Theta_k + b^i_k dx^k,
$$

$$
\Theta_{ij} = a B^l_i B^m_j (du_{kl} - u_{klm} dx^m) + s_{ij} \Theta_0 + w^k_{ij} \Theta_k, \quad (14)
$$

defined on $J^2(\pi) \times \mathcal{H}$, where $\mathcal{H}$ is an open subset of $\mathbb{R}^{(2n+1)(n+3)(n+1)/3}$ with local coordinates $(a, b^i_k, c^i, f^{ik}, g_i, s_{ij}, u^j_k, u_{ijk})$, $i, j, k \in \{1, \ldots, n\}$, such that $a \neq 0$, $\det(b^i_k) \neq 0$, $f^{ik} = f^{ki}$, $s_{ij} = s_{ji}$, $u^j_k = w^j_k$, $u_{ijk} = u_{ikj} = u_{jik}$, while $(B^i_k)$ is the inverse matrix for the matrix $(b^i_k)$. As it is shown in \[20\], the forms \(14\) are mc forms for Cont($J^2(\pi)$), that is, a local diffeomorphism $\tilde{\Delta}: J^2(\pi) \times \mathcal{H} \rightarrow J^2(\pi) \times \mathcal{H}$ satisfies the conditions $\tilde{\Delta}^* \Theta_0 = \Theta_0$, $\tilde{\Delta}^* \Theta_i = \Theta_i$, $\tilde{\Delta}^* \Xi^i = \Xi^i$, and $\tilde{\Delta}^* \Theta_{ij} = \Theta_{ij}$ if and only if it is projectable on $J^2(\pi)$, and its projection $\Delta: J^2(\pi) \rightarrow J^2(\pi)$ is a contact transformation.

The structure equations for Cont($J^2(\pi)$) have the form

\[
\begin{align*}
\ld d\Theta_0 &= \Phi_0^0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i, \\
\ld d\Theta_i &= \Phi_i^0 \wedge \Theta_0 + \Phi_i^k \wedge \Theta_k, \\
\ld d\Xi^i &= \Phi_0^k \wedge \Xi^k - \Phi_k^k \wedge \Xi^k + \Psi^0 \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k, \\
\ld d\Theta_{ij} &= \Phi_k^i \wedge \Theta_{kj} + \Phi_k^j \wedge \Theta_{ki} - \Phi_0^i \wedge \Theta_{ij} + \Upsilon_{ij}^0 \wedge \Theta_0 + \Upsilon_{ij}^k \wedge \Theta_k + \Xi^k \wedge \Theta_{ijk},
\end{align*}
\]

where the additional forms $\Phi_0^0, \Phi_i^0, \Phi_i^k, \Psi^0, \Psi^{ik}, \Upsilon_{ij}^0, \Upsilon_{ij}^k, \Theta_{ijk}$ depend on differentials of the coordinates of $\mathcal{H}$.

**Example 2.** Suppose $\mathcal{E}$ is a second-order differential equation in one dependent and $n$ independent variables. We consider $\mathcal{E}$ as a submanifold in $J^2(\pi)$. Let Cont($\mathcal{E}$) be the group of contact symmetries for $\mathcal{E}$. It consists of all the contact transformations on $J^2(\pi)$ mapping $\mathcal{E}$ to itself. Let $\iota_0: \mathcal{E} \rightarrow J^2(\pi)$ be an embedding, and $\iota = \iota_0 \times id: \mathcal{E} \times \mathcal{H} \rightarrow J^2(\pi) \times \mathcal{H}$. The mc forms of Cont($\mathcal{E}$) can be computed from the forms $\theta_0 = \iota^*\Theta_0$, $\theta_i = \iota^*\Theta_i$, $\xi^i = \iota^*\Xi^i$, and $\theta_{ij} = \iota^*\Theta_{ij}$ algorithmically by means of Cartan’s method of equivalence, \[4, 5, 6, 7, 10, 12, 23\], see details and examples in \[11, 19, 20\].
3. Cartan's structure of the symmetry pseudo-group for the dKP-hyperCR interpolating equation

We use the method outlined in the previous section to compute mc forms and structure equations of the pseudo-group of contact symmetries for Eq. (11). We write out here the structure equations for the forms \( \theta_0, \theta_j, \xi^j \) with \( j \in \{1, 2, 3\} \) only; the full set of the structure equations is given in Appendix.

We have

\[
d\theta_0 = \eta_1 \wedge \theta_0 + \xi_1 \wedge \theta_1 + \xi_2 \wedge \theta_2 + \xi_1 \wedge \theta_1 + \xi_3 \wedge \theta_3, \\
d\theta_1 = \frac{1}{2} \eta_1 \wedge \theta_1 + \frac{1}{2} \eta_2 \wedge (\theta_0 + 8 \theta_3) + \frac{1}{16} (24 (\theta_{22} - U \xi_1 - \xi_2) - 5 \xi_3) \wedge \theta_1 + 2 \theta_2 \wedge \theta_3 \\
\quad + \frac{1}{8} (11 \theta_2 - 16 V \theta_{22} + 8 \theta_{23} + 4 (2 V - 1) \xi_2 + (8 V - 3 U) \xi_3) \wedge \theta_0 + \xi_1 \wedge \theta_11 \\
\quad + \xi_2 \wedge \theta_{12} + \xi_3 \wedge \theta_{13}, \\
d\theta_2 = \frac{1}{10} (8 (\eta_1 + \theta_{22} - U \xi_1 - \xi_2) - 3 \xi_3) \wedge \theta_2 + \xi_1 \wedge \theta_{12} + \xi_2 \wedge \theta_{22} + \xi_3 \wedge \theta_{23}, \\
d\theta_3 = \eta_2 \wedge \theta_2 + (\eta_1 + \theta_{22} - U \xi_1 - \xi_2 - \frac{1}{2} \xi_3) \wedge \theta_3 + \frac{5}{64} (8 \theta_{22} - \xi_2 + 3 \xi_3) \wedge \theta_0 + \xi_1 \wedge \theta_{13} \\
\quad + \xi_2 \wedge \theta_{23} + \xi_3 \wedge \theta_{12}, \\
d\xi_1 = - \frac{1}{10} (8 \eta_1 + 24 (\theta_{22} + \xi_2) - 5 \xi_3) \wedge \xi_1, \\
d\xi_2 = \frac{1}{5} (5 \theta_0 + 8 V \theta_2 + 8 \theta_3 - 4 U \xi_2) \wedge \xi_1 + \frac{1}{16} (8 (\eta_1 - \theta_{22}) + 3 \xi_3) \wedge \xi_2 - \eta_2 \wedge \xi_3, \\
d\xi_3 = - (2 \eta_2 + \theta_2 + U \xi_3) \wedge \xi_1 - (\theta_{22} - \xi_2) \wedge \xi_3, \\
\]

where the invariants

\[
U = u_{xx}^4 (u_{x} + u_y) u_{xxxx}^2 + (u_{tx} - u_x u_{xy} + u_{xx} (u_{xy} + 4 u_{xx}) u_{xxx} - u_{xyy}^2 - u_{xx} u_{xyy}) , \\
V = u_{xxxx} u_{xx}^{-2}
\]

satisfy

\[
dU = \frac{1}{2} U \eta_1 + \eta_2 + \eta_3 - \frac{5}{8} \theta_0 - (V + \frac{1}{8}) \theta_2 - \theta_3 - (4 V - \frac{3}{2} U \theta_{22} - \theta_{23} \\
\quad + (4 V - U + \frac{1}{2}) \xi_2 - (V - \frac{1}{16} U) \xi_3, \\
dV = \frac{1}{2} V (\eta_1 + \theta_{22} + (6 V - U) \xi_1 - \xi_2) - \frac{5}{16} V \xi_3, \\
\]

and where

\[
\theta_0 = V^2 u_{xx}^3 (du - u_t dt - u_x dx - u_y dy), \\
\theta_2 = - V (du_x - u_{tx} dt - u_{xx} dx - u_{xy} dy), \\
\theta_3 = V u_{xyy} u_{xx}^{-2} du_x - V^2 du_y + V (V u_{xy} - u_{tx} u_{xyy} u_{xx}^{-2}) dt + V (V u_{xy} - u_{xyy} u_{xx}^{-1}) dx \\
\quad + V (V (u_{tx} + (u_x + u_y) u_{xx} - u_x u_{xy}) - u_{xy} u_{xyy} u_{xx}^{-2}) dy - \frac{5}{8} \theta_0, \\
\theta_{22} = - u_{xx}^{-1} (du_x - u_{tx} dt - u_{xx} dx - u_{xy} dy), \\
\xi_1 = u_{xx} V^{-1} dt, \\
\xi_2 = u_{xx}^{-1} (u_{xxx} dx + u_{xyy} dy - (u_{xxx} (u_x + u_y) - u_{xyy} u_{xx} - u_{xx}^2 u_{xyy}^{-1}) dt), \\
\xi_3 = u_{xx} (u_x + 2 u_{xyy} u_{xx}^{-1}) dt + u_{xx} dy, \\
\eta_1 = 2 u_{xx}^{-1} du_{xxx} + (10 V - 3 U + 4 (V u_{xy} u_{xx}^{-1} - u_{xyy} u_{xx}^{-2})) \xi_1 + \frac{7}{8} \xi_3 + 3 (\theta_{22} - \xi_2), \\
\eta_2 = - u_{xx}^{-2} du_{xyy} - (V^2 (u_x + u_y + (u_{tx} - u_{xyy} (u_x - 1))) u_{xx}^{-2}).
\]
With structure equations (8), (9). Consider a system of equations \( V_\xi \) and \( U_\xi \), which are solutions to Eqs. (18), (19) together satisfy the compatibility and involutivity conditions.

4. Integrable extensions

In [3, §6], the definition of integrable extension of an exterior differential system is designed to study finite-dimensional coverings. In general case, coverings of PDEs with three or more independent variables are infinite-dimensional, [18]. To cope with infinite-dimensional coverings we use a natural generalization of the definition, [22].

Suppose \( \mathcal{G} \) is a Lie pseudo-group on a manifold \( M \) and \( \omega^1, ..., \omega^m \) are its MC forms with structure equations [8], [9]. Consider a system of equations

\[
\begin{align*}
\, & d\tau^q = D^q_{\rho\tau} \eta^\rho \wedge \tau^\rho + E^q_{\tau\rho} \tau^\rho \wedge \tau^\sigma + F^q_{\tau\beta} \tau^\beta \wedge \pi^\beta + G^q_{\tau\rho} \tau^\rho \wedge \omega^j + H^q_{\tau\beta} \pi^\beta \wedge \omega^j \\
\, & + I^q_{jk} \omega^j \wedge \omega^k; \\
\, & dV^\epsilon = J^\epsilon_j \omega^j + K^\epsilon_q \tau^q,
\end{align*}
\]

with unknown 1-forms \( \tau^q, q \in \{1, ..., Q\}, \eta^\rho, \rho \in \{1, ..., R\}, \) and unknown functions \( V^\epsilon, \epsilon \in \{1, ..., S\} \) for some \( Q, R, S \in \mathbb{N} \). The coefficients \( D^q_{\rho\tau}, ..., K^\epsilon_q \) in (18), (19) are supposed to be functions of \( U^q \) and \( V^\epsilon \).

**Definition 1.** System (18), (19) is an integrable extension of system (8), (9), if Eqs. (18), (19), (8), and (9) together satisfy the compatibility and involutivity conditions.

In this case from Cartan’s third fundamental theorem for Lie pseudo-groups it follows that there exists a set of forms \( \tau^q \) and functions \( V^\epsilon \) which are solutions to Eqs. (18) and (19). Then \( \tau^q, V^\epsilon \) together with \( \omega^j, U^q \) define a Lie pseudo-group on a manifold \( N \cong M \times \mathbb{R}^Q \).

**Definition 2.** The integrable extension is called trivial, if there is a change of variables on \( N \) such that in the new variables the coefficients \( F^q_{\tau\beta}, G^q_{\tau\rho}, H^q_{\tau\beta}, I^q_{jk}, \) and \( J^\epsilon_j \) are equal to zero, while the coefficients \( D^q_{\rho\tau}, E^q_{\tau\rho}, \) and \( K^\epsilon_q \) are independent of \( U^q \). Otherwise, the integrable extension is called non-trivial.

Let \( \theta^\gamma_j \) and \( \xi^j \) be a set of MC forms of the symmetry pseudo-group Cont(\( \mathcal{E} \)) of a PDE \( \mathcal{E} \) such that \( \xi^1 \wedge ... \wedge \xi^n \neq 0 \) on any solution manifold of \( \mathcal{E} \), while \( \theta^\gamma_j \) are contact forms. We take the following reformulation of the definition (6) of a covering.

**Definition 3.** A non-trivial integrable extension of the form

\[
d\omega^q = \Pi^q_j \wedge \omega^j + \xi^j \wedge \Omega^q_j
\]

is called a contact integrable extension (CIE) of the structure equations of Cont(\( \mathcal{E} \)) if

1. \( \Pi^q_j \) are some non-trivial differential 1-forms,
2. \( \Omega^q_j \equiv 0 \mod \theta^\gamma_j, \omega^j \) for some additional 1-forms \( \omega^q_j \),
3. \( \Omega^q_j \not\equiv 0 \mod \omega^q_j \).

Since (20) is integrable extension, Cartan’s theorem yields existence of the forms \( \omega^q \) satisfying (20). From [2, Ch. IV, Prop. 5.10] it follows that the forms \( \omega^q \) define a system of PDEs. This system is a covering for \( \mathcal{E} \).
We apply this construction to the structure equations (33), (15), and (16) of the symmetry pseudo-group of Eq. (5). We restrict our analysis to cies of the form

\[
d\omega_0 = \left( \sum_{i=0}^{3} A_i \theta_i + \sum_{j=1}^{3} B_{ij} \theta_{ij} + \sum_{s=1}^{7} C_s \eta_s + \sum_{j=1}^{3} D_j \xi^j + E \omega_1 \right) \wedge \omega_0
\]

\[+ \sum_{j=1}^{3} \left( \sum_{k=0}^{3} F_{jk} \theta_k + G_j \omega_1 \right) \wedge \xi^j, \tag{21}\]

where \(\sum^*\) means summation for all \(i, j \in \mathbb{N}\) such that \(1 \leq i \leq j \leq 3, (i, j) \neq (3, 3)\), and consider two types of these cies:

**TYPE 1** — the coefficients \(A_i\) to \(G_j\) in (21) depend on the invariants \(U, V\) of the symmetry pseudo-group of Eq. (5) only;

**TYPE 2** — the coefficients \(A_i\) to \(G_j\) depend also on one additional invariant, say \(W\). In this case, the differential of this new invariant satisfies the following equation

\[
dW = \sum_{i=0}^{3} H_i \theta_i + \sum_{j=1}^{3} I_{ij} \theta_{ij} + \sum_{s=1}^{7} J_s \eta_s + \sum_{j=1}^{3} K_j \xi^j + \sum_{q=0}^{1} L_q \omega_q, \tag{22}\]

where the coefficients \(H_i\) to \(L_q\) are functions of \(U, V, W\).

The requirements of Definitions 1 and 3 yield over-determined systems for the coefficients \(A_i\) to \(G_j\) of the cie of the first type and \(A_i\) to \(L_q\) of the cie of the second type. The results of analysis of these systems are summarized in the following theorem.

**Theorem 1.** There is no a cie of the first type for the structure equations (33), (15), and (16). Every their cie of the second type is contact-equivalent to the following one:

\[
d\omega_0 = \left( \frac{1}{2} (\eta_1 - \theta_{22}) - \omega_1 - \frac{1}{2}(W^2 + 2(V - W) - U) \xi_1 - \frac{1}{10} (8W - 11) \xi_3 \right) \wedge \omega_0
\]

\[+ \left( W^2 \omega_1 + \frac{5}{8} \theta_0 + W \theta_2 + \theta_3 \right) \wedge \xi_1 + \omega_1 \wedge \xi_2 + (W \omega_1 + \theta_2) \wedge \xi_3, \tag{23}\]

\[
dW = V \omega_1 + \frac{1}{2} W \eta_1 + \eta_2 + \frac{1}{2} W \theta_{22} + \theta_2 + \frac{1}{2} W (V(W + 4) - U) \xi_1
\]

\[+ \frac{1}{2} (V - W - 1) \xi_2 + \frac{1}{10} (W(4V - 5) + 8V) \xi_3. \tag{24}\]

Since the forms (17) are known, it is easy to find the form \(\omega_0\) explicitly:

**Theorem 2.** We have the following solution to Eq. (23) up to a contact equivalence:

\[\omega_0 = u_{xxx} u_{xx}^{-1} v_x^{-1} (dv - v_x (\ln^2 |v_x| - u_y - (\ln |v_x| + 1) u_x + 1) \ dx - v_x \ dx - v_x (\ln |v_x| - u_x) \ dy). \tag{25}\]

This form defines the following Bäcklund transformation or a Lax pair:

\[
\begin{aligned}
v_t &= v_x (\ln^2 |v_x| - u_y - (\ln |v_x| + 1) u_x + 1), \\
v_y &= v_x (\ln |v_x| - u_x).
\end{aligned} \tag{26}\]

It is easy to verify directly that Eqs. (26) are compatible whenever Eq. (5) is satisfied. From Eqs. (26) it follows that

\[
\begin{aligned}
u_x &= \ln |v_x| - v_y v_x^{-1}, \\
u_y &= (v_y (\ln |v_x| + 1) - v_t) v_x^{-1} - \ln |v_x| + 1, \tag{27}
\end{aligned}
\]
Cross-differentiating $u$ in this system yields the covering equation:

$$
v_{yy} = v_{tx} + \left( (v_y \ln|v_x| - v_t) v_x^{-1} + 1 \right) v_{xx} + \left( v_y v_x^{-1} - \ln|v_x| \right) v_{xy}. \tag{28}
$$

5. Multi-valued Einstein–Weyl Structures

We use the results of the previous section to construct a family of Einstein–Weyl structures \([2]\) depending on two arbitrary functions of one variable. We take the ansatz, \([1, \text{Ch. VIII, § 5.IV}], \ [21]\),

$$
v_t = F(v_x), \quad v_y = G(v_x). \tag{29}
$$

This system is compatible for every (smooth) functions $F$ and $G$. Substituting for (29) into (28) and denoting $v_x = s \tag{30}$ yields

$$(G'(s))^2 = F'(s) + (G(s) \ln|s| - F(s)) s^{-1} + 1 + (G(s) s^{-1} - \ln|s|) G'(s).$$

We consider this as an ODE for the unknown function $F$ and the functional parameter $G$ being an arbitrary smooth function. Then we have

$$F(s) = s \int s^{-1} \left( (G'(s))^2 + \ln|s| (G'(s) - s^{-1} G(s)) - s^{-1} G(s) G'(s) - 1 \right) ds. \tag{31}
$$

From (29) and (30) it follows that the function $s$ satisfies the compatible system of PDEs

$$s_t = F'(s) s_x, \quad s_y = G'(s) s_x. \tag{32}
$$

The general solution of this system in the implicit form reads

$$s = Q(x + t F'(s) + y G'(s)), \tag{33}
$$

where $Q$ is an arbitrary (smooth) function of one variable. For $t = 0$ and $y = 0$ we have $s = Q(x)$, so $Q$ is an initial value for Eqs. (32). In general, Eq. (33) defines $s$ as a multi-valued function of $t$, $x$, and $y$.

Then Eqs. (27), (29), and (30) give

$$
\begin{align*}
    u_x &= \ln|s| - s^{-1} G(s), \\
    u_y &= (s^{-1} G(s) - 1) \ln|s| - s^{-1} F(s) + 1,
\end{align*}
$$

where the function $F$ is defined by Eq. (31), and the function $s$ is defined by (33). Eqs. (34) together with the scaling (4) provide a family of Einstein–Weyl structures (2) depending on two arbitrary functions of one variable.

Figures 1 to 4 show graphs of $u_x$ and $u_y$ at $t = -10$ and $t = 10$ for the choice of $G(s) = -3s$, $Q(x) = x^2 + 1$, and $F(s) = (5 - \ln|s|) s$. 
Figure 1. The graph of $u_x$ at $t = -10$.

Figure 2. The graph of $u_y$ at $t = -10$. 
Figure 3. The graph of $u_x$ at $t = 10$.

Figure 4. The graph of $u_y$ at $t = 10$. 
Appendix

The structure equations of the symmetry pseudo-group for the dKP-hyperCR interpolating equation read

\[ d\theta_0 = \eta_1 \land \theta_0 + \xi_1 \land \theta_1 + \xi_2 \land \theta_2 + \xi_1 \land \theta_1 + \xi_3 \land \theta_3, \]
\[ d\theta_1 = \frac{1}{2} \eta_1 \land \theta_1 + \frac{1}{4} \eta_2 \land \theta_0 + 8 \theta_3 + \frac{1}{16} (24 (\theta_{22} - U \xi_1 - \xi_2) - 5 \xi_3) \land \theta_1 + 2 \theta_2 \land \theta_3 + \frac{1}{8} (11 \theta_2 - 16 V \theta_{22} + 8 \theta_{23} + 4 (2 V - 1) \xi_2 + (8 V - 3 U) \xi_3) \land \theta_0 + \xi_1 \land \theta_{11} + \xi_2 \land \theta_{12} + \xi_3 \land \theta_{13}, \]
\[ d\theta_2 = \frac{1}{16} (8 (\eta_1 + \theta_{22} - U \xi_1 - \xi_2) - 3 \xi_3) \land \theta_2 + \xi_1 \land \theta_{12} + \xi_2 \land \theta_{22} + \xi_3 \land \theta_{23}, \]
\[ d\theta_3 = \eta_2 \land \theta_2 + (\eta_1 + \theta_{22} - U \xi_1 - \xi_2 - \frac{1}{4} \xi_3) \land \theta_3 + \frac{24}{64} (8 \theta_{22} - \xi_2 + 3 \xi_3) \land \theta_0 + \xi_1 \land \theta_{13} + \xi_2 \land \theta_{23} + \xi_3 \land \theta_{12}, \]
\[ d\xi_1 = -\frac{1}{16} (8 \eta_1 + 24 (\theta_{22} + \xi_2) - 5 \xi_3) \land \xi_1, \]
\[ d\xi_2 = \frac{1}{8} (5 \theta_0 + 8 V \theta_2 + 8 \theta_3 - 4 U \xi_2) \land \xi_1 + \frac{1}{16} (8 (\eta_1 - \theta_{22}) + 3 \xi_3) \land \xi_2 - \eta_2 \land \xi_3, \]
\[ d\xi_3 = -(2 \eta_2 + \theta_2 + U \xi_3) \land \xi_1 - (\theta_{22} - \xi_2) \land \xi_3, \]
\[ d\theta_{11} = \frac{3}{2} \eta_2 \land \theta_1 + ((2 V - \frac{1}{2} U) \eta_2 + 2 V \eta_3 - \eta_4) \land \theta_0 + \eta_5 \land \xi_2 + \eta_6 \land \xi_3 + \eta_7 \land \xi_1 + ((U - 2 V^2 + \frac{1}{4} U) \theta_2 + (1 - 2 V) \theta_3 + \frac{7}{2} \theta_{12} - 2 V (U + 3 V) \theta_{22} + \frac{1}{4} (6 U - V) \theta_{23} + \frac{1}{8} (U (24 V - 7) + V (48 V - 1)) \eta_2 + \frac{1}{16} (44 V^2 - 15 U^2 + 28 U V) \xi_3) \land \theta_0 + (2 \theta_2 - 11 V \theta_{22} + \theta_{23} + \frac{1}{2} (22 V - 1) \xi_2 + \frac{1}{4} (4 V + U) \xi_3) \land \theta_1 + (V \theta_{12} - 3 \theta_{13}) \land \theta_2 + \frac{3}{4} V \theta_2 + \theta_{12} - 2 V \theta_{23} + (U - 2 V) \xi_2 - 2 V (U + V) \xi_3) \land \theta_3 + (2 \eta_1 + 3 \theta_{22} - 3 \xi_2 - \frac{5}{8} \xi_3) \land \theta_{11} + 2 U \xi_2 \land \theta_{12} + (4 \eta_2 + \frac{5}{2} U \xi_3) \land \theta_{13}, \]
\[ d\theta_{12} = \left(\frac{1}{4} \eta_2 - 2 V \theta_{22} - \theta_{23} + 3 V \xi_2 + \frac{1}{8} U \xi_3\right) \land \theta_2 + \eta_3 \land \xi_2 + \eta_4 \land \xi_3 + \eta_5 \land \xi_1 + \frac{5}{8} \theta_0 - \theta_{22} + U \xi_2) \land \theta_2 + (2 \eta_2 + \frac{3}{2} U \xi_3) \land \theta_{23}, \]
\[ d\theta_{13} = \frac{5}{64} (6 \eta_2 - 8 (\eta_3 - \theta_3 - U \theta_{22} - \theta_{23}) + (8 V - 7) \theta_2 - 6 U (2 \xi_2 + \xi_3) \land \theta_0 + \eta_4 \land \xi_2 + \eta_5 \land \xi_3 + \eta_6 \land \xi_1 + \frac{5}{64} (8 (\theta_{22} - \xi_2) + 3 \xi_3) \land \theta_1 + (3 \eta_2 + 2 \theta_2 + 2 U \xi_3) \land \theta_{12} - \frac{3}{8} \xi_3 \land \theta_{23} + \frac{3}{8} (2 \eta_2 - 4 V (\theta_{22} - \xi_2) + U \xi_3) \land \theta_3 + \frac{1}{16} (24 \eta_1 + 40 (\theta_{22} - \xi_2) - 9 \xi_3) \land \theta_{13}, \]

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\[d\theta_2 = \eta_3 \wedge \xi_1 + \frac{1}{16} (8 (\eta_2 - \theta_{22}) + 3 \xi_3) \wedge \xi_2 = \eta_2 \wedge \xi_3,\]
\[d\theta_{23} = \frac{1}{2} \eta_1 \wedge \theta_{23} + \eta_2 \wedge (\theta_{22} - \xi_2) + \eta_1 \wedge \xi_1 + \frac{1}{64} (8 (\theta_{22} - \xi_2) + (64 V - 1) \xi_3) \wedge \theta_2 + \frac{1}{8} (8 (\eta_3 - \theta - 3) - 5 \theta_0 + 12 U \xi_2) \wedge \xi_3 + \frac{1}{5} \theta_{22} \wedge (3 (4 \theta_{23} - \xi_2) - 8 U \xi_3) + \frac{1}{8} \theta_{23} \wedge (24 \xi_2 - 7 \xi_3),\]
\[d\eta_1 = -\frac{1}{8} (2 \eta_2 + 6 \theta_2 - 16 (V \theta_{22} + \theta_{23}) + 4 (2 V - 1) \xi_2 + (8 V - 3 U) \xi_3) \wedge \xi_1 - \frac{5}{8} (\theta_{22} - \xi_2) \wedge \xi_3,\]
\[d\eta_2 = \frac{1}{16} (8 (\eta_1 + \theta_{22} - U \xi_1 - \xi_2) + 3 \xi_3) \wedge \eta_2 + \frac{5}{10} \theta_0 \wedge \xi_1 - \frac{1}{5} \theta_2 \wedge (3 V \xi_1 - \xi_3) + \frac{1}{2} \theta_3 \wedge \xi_1 + (\theta_{12} + V \theta_{23} + V (V + U) \xi_3) \wedge \xi_1 + \frac{1}{2} (\theta_{22} + (U - 2 V) \xi_1) \wedge \xi_2 + (2 V \theta_{22} + \theta_{23} - \frac{1}{4} (8 V + 1) \xi_2) \wedge \xi_3,\]
\[d\eta_3 = \frac{1}{16} \eta_1 \wedge (8 (\eta_3 + V \theta_2 + \theta_3) + 5 \theta_0 - 2 U \xi_2) - \theta_{12} \wedge ((V + \frac{9}{8}) \xi_1 + \xi_3) - \theta_{13} \wedge \xi_1 + \frac{1}{2} \eta_2 \wedge (2 \theta_2 + 2 (U - 2 V) \xi_1 - \xi_2 + U \xi_3) + \eta_4 \wedge \xi_1 - \frac{5}{8} \theta_1 \wedge \xi_1 + \frac{1}{8} (24 \theta_{22} + 8 (3 U - 8 V) \xi_1 - 16 \xi_2 - 5 \xi_3) \wedge \eta_3 + \frac{1}{32} (19 U - 8 V) \xi_2 \wedge \xi_3 + \frac{5}{152} \theta_0 \wedge (8 \theta_{22} + 16 (U - 4 V - 1) \xi_1 - 11 \xi_3) - \frac{1}{2} (U^2 - 2 U V - 8 V^2) \xi_1 \wedge \xi_3 + \frac{1}{16} (8 \theta_2 \wedge (8 V \theta_{22} + 2 ((16 V + 9 U) - (56 V + 9) V) \xi_1 - 9 \xi_2 + 3 V \xi_3) + \frac{1}{16} \eta_3 \wedge (8 \theta_{22} + 16 (2 U - 4 V - 1) \xi_1 - 11 \xi_3) + \frac{1}{2} \theta_{23} \wedge (V (U - 12 V) \xi_1 + (\frac{1}{4} U + V) \xi_2) + \frac{1}{2} \theta_{23} \wedge ((U - 2 V) \xi_1 - \xi_2 - 2 V \xi_3) + \frac{1}{2} ((U - 2 V) (U - 1) - 24 V^2) \xi_1 \wedge \xi_2,\]
\[d\eta_4 = \eta_8 \wedge \xi_1 + \frac{3}{32} (60 \theta_0 + (96 V - 1) \theta_2 + 96 \theta_3 + 48 U \theta_{22} - 24 \theta_{23} - 96 U \xi_2 + 32 (2 V - U) \xi_3) \wedge \eta_2 + \frac{1}{10} (24 \eta_2 + 7 \theta_2 - 12 \theta_{23} + 3 \xi_2 + 4 (7 U - 8 V) \xi_3) \wedge \eta_3 + (\eta_1 + 3 (\theta_{22} - \xi_2) - \frac{7}{4} \xi_3) \wedge \eta_4 + \frac{1}{16} (17 U \theta_{22} + (24 V - 1) \theta_{23} - 6 (V + 4 U) \xi_2) \wedge \theta_2 - \frac{1}{5} (7 \theta_2 - \theta_{22} + 12 \theta_{23} + 4 (\xi_2 - (7 U - 8 V - 2) \xi_3)) \wedge \theta_0 + \frac{1}{8} (3 \theta_2 + \theta_{22} - 12 \theta_{23} - 4 \xi_2 - 8 (7 U - 8 V - 2) \xi_3) - \frac{1}{16} (72 \theta_{22} + \xi_2) + 73 \xi_3) \wedge \theta_{12} + \frac{1}{16} (U (448 V + 143) - 16 V (32 V^2 + 9)) \theta_2 \wedge \xi_3 + ((V - \frac{11}{16} U) \xi_2 - (U^2 - 13 V^2) \xi_3) \wedge \theta_{22} + \frac{1}{8} (2 (3 U + 1) \xi_2 + (U + 4 V) \xi_3) \wedge \theta_{23} + \frac{1}{4} (4 V (13 V^2 - 1) - (11 U - 8 V - 2)) \xi_2 \wedge \xi_3,\]
\[d\eta_5 = \frac{3}{2} \eta_1 \wedge \eta_5 + \frac{1}{2} \eta_2 \wedge (4 \eta_4 + \frac{1}{2} \theta_{12} - U \theta_{23} + (U - 2 V) \xi_2) + (2 \theta_2 - \xi_2 + \frac{5}{8} U \xi_3) \wedge \eta_4 + \frac{1}{16} (56 (\theta_{22} - \xi_2) - 13 \xi_3) \wedge \eta_5 + \eta_8 \wedge \xi_3 + \eta_0 \wedge \xi_1 + \frac{3}{8} \theta_1 \wedge (\theta_{22} - \xi_2) + \frac{5}{64} (16 \eta_3 - (24 V + 1) \theta_2 + 16 \theta_{12} - 8 U (\theta_{22} + \theta_{23} + (U - 4 V + 1) \xi_2)) \wedge \theta_0 + \frac{1}{16} \theta_2 \wedge (16 V \eta_3 - 2 (24 V + 1) \theta_2 - 8 (4 V + 1) \theta_{12} - 48 V (U - 2 V) \theta_{22}) - (U + V) \theta_{23} + 2 (U (2 V + 5) + V (16 V - 11)) \xi_2 + (U^2 - 4 U V - 4 V^2) \xi_3 + (2 (\eta_3 + \theta_{12}) - \theta_{23} + (4 V - 2 U + 1) \xi_2) \wedge \theta_3 + \xi_3 \wedge \theta_{22} - \xi_2) + \frac{1}{8} (16 (\eta_3 + 2 V \theta_{22} - \theta_{23}) + (4 U - 16 V + 13) \xi_2 - 4 (4 V - 3 U) \xi_3) \wedge \theta_{12} + \left(\frac{1}{2} U^2 + U V - 12 V^2\right) \xi_2 \wedge \xi_2 + \frac{1}{3} \left(3 (2 V - U) \xi_2 + (4 V (U + V) - 3 U^2) \xi_3\right) \wedge \theta_{23} - \frac{1}{2} (2 (3 U - 4 V) \eta_3 + (8 V^2 + 2 U V - U^2) \xi_3) \wedge \xi_2,\]
\[d\eta_6 = \frac{1}{64} \eta_2 \wedge (320 \eta_5 + (80 V - 85 U) \theta_0 - 60 \theta_1 + 128 V (U + V) \theta_2 + 24 U \theta_3 - 160 U \theta_{12})\]
\[ d\eta = \begin{align*}
+ & \left( 6 V \theta_3 - \frac{5}{2} \theta_{13} \right) \wedge \eta_3 + \eta_4 \wedge (V \theta_2 + 2 (\theta_3 - U \xi_2)) - 3 \eta_5 \wedge (\theta_2 + U \xi_3) \\
+ & (2 \eta_1 + 4 \theta_{22} - 4 \xi_2 - \frac{7}{8} \xi_3) \wedge \eta_6 + \eta_8 \wedge \xi_2 + \eta_9 \wedge \xi_3 + \eta_{10} \wedge \xi_1 \\
+ & \frac{1}{64} (8 (\eta_4 + (5 U - 4 V) \eta_3) + 10 \theta_1 - 2 \theta_{12} - 4 (5 U + 4 V - 2) \theta_3 + 2 (7 V - 8 U) \theta_{23} \\
- & (96 V^2 + 7 U - 10 U^2 + 8 U V - 14 V) \wedge \xi_2 + 2 (17 V^2 - 5 U^2 + 7 U V) \xi_3) \wedge \theta_0 \\
+ & \frac{5}{128} (32 (\xi_3 - \theta_3) + 8 (U \theta_{22} - \theta_{23} + U \xi_2) - 9 U \xi_3) \wedge \theta_1 \\
+ & \frac{1}{256} ((U (400 V + 310) - V (640 V + 295)) \theta_0 + 40 (8 V + 5) \xi_1) \wedge \theta_2 \\
+ & \frac{1}{32} ((14 U + 192 V^2 + 23 V) \theta_2 - 52 \theta_{12} + 288 V (2 V - U) \theta_{22} + 8 (3 V - 2 U) \theta_{23} \\
+ & 4 (3 U (16 V - 1) - 6 V (24 V + 1)) \xi_2 + 24 V (V + U) \xi_3) \wedge \theta_3 \\
- & \frac{5}{64} (8 (\theta_{22} + \xi_2) - 3 \xi_3) \wedge \theta_{11} + \theta_{12} \wedge ((U + \frac{2}{5} V) \theta_2 - 3 V \theta_{23} + 2 (U - 2 V) \xi_2 \\
+ & 2 U^2 - 2 V^2 - 2 U V) \xi_3) + \frac{1}{16} \theta_{13} \wedge (25 \theta_0 + 40 (\xi_3 + \theta_{23}) + (40 V + 23) \theta_2 \\
- & 144 V \theta_{22} + 4 (36 V - 5 U - 3) \xi_2 + 12 (2 V - U) \xi_3) + \frac{2}{8} V (U - 12 V) \theta_0 \wedge \theta_{22} \\
+ & \frac{3}{4} \theta_2 (2 U \theta_2 - U \xi_2) \wedge \theta_{23} + \frac{1}{2} V \theta_2 \wedge ((U - 6 V) \xi_2 - V (5 U + 12 V) \xi_3)
\end{align*} \]

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