THE MEANING OF MAXIMAL SYMMETRY IN PRESENCE OF TORSION

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Abstract

We show that the usual physical meaning of maximal symmetry can be made to remain unaltered even if torsion is present. All that is required is that the torsion fields satisfy some mutually consistent constraints. We also give an explicit realisation of such a scenario by determining the torsion fields, the metric and the associated Killing vectors.

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In $N$ dimensions, a metric that admits the maximum number $N(N+1)/2$ of Killing vectors is said to be maximally symmetric. A maximally symmetric space is homogeneous and isotropic about all points [1]. Such spaces are of natural interest in the general theory of relativity as they correspond to spaces of globally constant curvature which in turn is related to the concepts of homogeneity and isotropy. The requirement of isotropy and homogeneity leads to maximally symmetric metrics in the context of standard cosmologies – the most well known example being the Robertson-Walker cosmology. However, in the presence of torsion there is a drastic change in the scenario and one needs to redefine maximal symmetry itself.

In this paper we propose a way to do this such that the usual physical meaning of maximal symmetry remains the same. The only requirement is that the torsion fields satisfy some mutually consistent constraints. We also give an example of a toy model where our ideas can be realised by determining the torsion fields, the metric and the Killing vectors.

The presence of torsion implies that the affine connections $\tilde{\Gamma}^\alpha_{\mu\nu}$ are asymmetric and contain an antisymmetric part $H^\alpha_{\mu\nu}$ in addition to the usual symmetric term $\Gamma^\alpha_{\mu\nu}$ [2,3]:

$$\tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + H^\alpha_{\mu\nu}$$

The notation has been chosen keeping string theory in mind where the antisymmetric third rank tensor $H_{\alpha\mu\nu} = \partial_{(\alpha} B_{\mu\nu)}$ is identified with the background torsion and $B_{\mu\nu}$ is the second rank antisymmetric tensor field mentioned above. Note that $H^\alpha_{\mu\nu}$ are completely arbitrary to start with.

Defining covariant derivatives with respect to $\Gamma^\alpha_{\mu\nu}$ we have for a vector
field $V_{\mu}:

V_{\mu;\nu;\beta} - V_{\mu;\beta;\nu} = -\bar{R}_{\mu\nu\beta}^\lambda V_\lambda + 2H_\beta^\alpha V_{\mu;\alpha} \quad (2)

where

\begin{align*}
\bar{R}_{\mu\nu\beta}^\lambda &= R_{\mu\nu\beta}^\lambda + \bar{R}_{\mu\nu\beta}^\lambda \quad (3a) \\
R_{\mu\nu\beta}^\lambda &= \Gamma_{\mu\nu\beta}^\lambda - \Gamma_{\mu\beta,\nu}^\lambda + \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\beta}^\lambda - \Gamma_{\mu\beta}^\alpha \Gamma_{\alpha\nu}^\lambda \quad (3b) \\
\bar{R}_{\mu\nu\beta}^\lambda &= H_{\mu\nu,\beta}^\lambda - H_{\mu\beta,\nu}^\lambda + H_{\mu\nu}^\alpha H_{\alpha\beta}^\lambda - H_{\mu\beta}^\alpha H_{\alpha\nu}^\lambda \\
&\quad + \Gamma_{\mu\nu}^\alpha H_{\alpha\beta}^\lambda - H_{\mu\beta}^\alpha \Gamma_{\alpha\nu}^\lambda + H_{\mu\nu}^\alpha \Gamma_{\alpha\beta}^\lambda - \Gamma_{\mu\beta}^\alpha H_{\alpha\nu}^\lambda \quad (3c)
\end{align*}

The generalised curvature $\bar{R}_{\mu\nu\beta}^\lambda$ does not have the usual symmetry (antisymmetry) properties. Note that the last term on the right hand side (2) is obviously a tensor. Hence $\bar{R}_{\mu\nu\beta}^\lambda$ is also a tensor.

Now, the determination of all infinitesimal isometries of a metric is equivalent to determining all Killing vectors $\xi_{\mu}$ of the metric. A Killing vector is defined through the Killing condition:

\[ \xi_{\mu ; \nu} + \xi_{\nu ; \mu} = 0 \quad (4) \]

and it is easily verified that this condition is preserved also in the presence of the torsion $H_{\mu\nu}^\alpha$.

Equation (2) for a Killing vector hence takes the form:

\[ \xi_{\mu;\nu;\beta} - \xi_{\mu;\beta;\nu} = -\bar{R}_{\mu\nu\beta}^\lambda \xi_\lambda + 2H_\beta^\alpha \xi_{\mu;\alpha} \quad (5) \]

The $H_{\beta\nu}^\alpha$ are arbitrary and so we may choose them to be such that

\[ H_{\beta\nu}^\alpha \xi_{\mu;\alpha} = 0 \quad (6) \]

This is a constraint on the $H_{\beta\nu}^\alpha$ and not the Killing vectors $\xi_{\mu}$. We shall shortly see that this constraint is essential for the existence of maximal symmetry.
even in the presence of torsion. The commutator of two covariant derivatives of a Killing vector thus becomes:

\[ \xi_{\mu;\nu;\beta} - \xi_{\mu;\beta;\nu} = -\bar{R}^\lambda_{\mu\nu\beta} \xi_\lambda \]  

(7)

We now impose the cyclic sum rule on \( \bar{R}^\lambda_{\mu\nu\beta} \):

\[ \bar{R}^\lambda_{\mu\nu\beta} + \bar{R}^\lambda_{\nu\beta\mu} + \bar{R}^\lambda_{\beta\mu\nu} = 0 \]  

(8a)

As \( R^\lambda_{\mu\nu} + R^\lambda_{\nu\beta\mu} + R^\lambda_{\beta\mu\nu} = 0 \) the constraint (8a) implies

\[ \bar{R}^\lambda_{\mu\nu\beta} + \bar{R}^\lambda_{\nu\beta\mu} + \bar{R}^\lambda_{\beta\mu\nu} = 0 \]

and this reduces to

\[ H^\lambda_{\mu\nu,\beta} + H^\alpha_{\mu\nu,\beta} + H^\lambda_{\nu\beta,\mu} + H^\alpha_{\nu\beta,\mu} + H^\lambda_{\beta\mu,\nu} + H^\alpha_{\beta\mu,\nu} = 0 \]  

(8b)

Hence adding (7) and its two cyclic permutations and using (4), the relation (7) can be recast into

\[ \xi_{\mu;\nu;\beta} = -\bar{R}^\lambda_{\beta\mu\nu} \xi_\lambda \]  

(9)

Therefore given \( \xi_\lambda \) and \( \xi_{\lambda;\nu} \) at some point \( X \), we can determine the second derivatives of \( \xi_\lambda(x) \) at \( X \) from (9). Then following the usual arguments [1], any Killing vector \( \xi^n_\mu(x) \) of the metric \( g_{\mu\nu}(x) \) can be expressed as

\[ \xi^n_\mu(x) = A^\lambda_\mu(x ; X) \xi^n_\lambda(X) + C^\lambda_\mu(x ; X) \xi^n_{\lambda;\nu}(X) \]  

(10)

where \( A^\lambda_\mu \) and \( C^\lambda_\mu \) are functions that depend on the metric, torsion and \( X \) but not on the initial values \( \xi_\lambda(X) \) and \( \xi_{\lambda;\nu}(X) \), and hence are the same for all Killing vectors. Also note that the torsion fields present in \( A^\lambda_\mu(x ; X) \)
and $C^\lambda_\mu(x;X)$ obey the constraint (6). A set of Killing vectors $\xi^n_\mu(x)$ is said to be independent if they do not satisfy any relation of the form $\sum_n d_n \xi^n_\mu(x) = 0$, with constant coefficients $d_n$. It therefore follows that there can be at most $N(N+1)/2$ independent Killing vectors in $N$ dimensions, even in the presence of torsion provided the torsion fields satisfy the constraints (6) and (8b). Therefore, one can generalise the concept of maximal symmetry to the case where torsion is present, provided the torsion fields satisfy the constraints embodied in the equations (6) and (8b). This is the principal result of this paper.

Now, $\bar{\mathcal{R}}_{\lambda\mu\nu}\bar{\beta}$ is antisymmetric in the indices $(\lambda, \mu)$ and $(\nu, \beta)$. This follows from the fact that $R_{\lambda\mu\nu}\beta$ and $\bar{R}_{\lambda\mu\nu}\beta$ both have these properties. This can be verified through an elaborate but straightforward calculation. Then proceeding as in Ref.[1], we have (using $-\bar{R}_{\mu\alpha}^\alpha = \bar{R}_{\mu\nu}$ etc.)

$$(N - 1)\bar{R}_{\lambda\mu\nu}\beta = \bar{R}_{\beta\mu}g_{\lambda\nu} - \bar{R}_{\nu\mu}g_{\lambda\beta}$$

i.e.

$$(N - 1)R_{\lambda\mu\nu}\beta + (N - 1)\bar{R}_{\lambda\mu\nu}\beta$$

$$= R_{\beta\mu}g_{\lambda\nu} - R_{\nu\mu}g_{\lambda\beta} + \bar{R}_{\beta\mu}g_{\lambda\nu} - \bar{R}_{\nu\mu}g_{\lambda\beta}$$

(11)

where $N$ is the number of dimensions. $R_{\lambda\mu\nu}\beta$, $R_{\beta\mu}$ are functions of the symmetric affine connections $\Gamma$ only, whereas $\bar{R}_{\lambda\mu\nu}\beta$, $\bar{R}_{\beta\mu}$ are functions of both $\Gamma$ and $H$. Broadly, the solution space of equation (11) consists of (a) solutions with $H$ determined by $\Gamma$ or vice versa (b) solutions where $H$ and $\Gamma$ are independent of each other. All these solutions lead to maximally symmetric spaces even in the presence of torsion.

We shall now illustrate that the smaller subspace (b) of these solutions enables one to cast the definition of maximal symmetry in the presence of
torsion in an exactly analogous way to that in the absence of torsion. A particular set of such solutions of (11) can be obtained by equating corresponding terms on both sides to get:

\[(N - 1) R_{\lambda\mu\beta} = R_{\beta\mu} g_{\lambda\nu} - R_{\nu\mu} g_{\lambda\beta} \quad (12a)\]

\[(N - 1) \tilde{R}_{\lambda\mu\beta} = \tilde{R}_{\beta\mu} g_{\lambda\nu} - \tilde{R}_{\nu\mu} g_{\lambda\beta} \quad (12b)\]

The above two equations lead to

\[R_{\lambda\mu\beta} = R^\alpha_{\lambda}(g_{\lambda\nu} g_{\mu\beta} - g_{\lambda\beta} g_{\nu\mu})/N(N - 1) \quad (13a)\]

and

\[\tilde{R}_{\lambda\mu\beta} = \tilde{R}^\alpha_{\lambda}(g_{\lambda\nu} g_{\mu\beta} - g_{\lambda\beta} g_{\nu\mu})/N(N - 1) \quad (13b)\]

It is appropriate to note here that

\[\tilde{R}^\alpha_{\alpha\nu\beta} = 0 \quad (14a)\]

\[\tilde{R}^\alpha_{\mu\nu\alpha} = H^\alpha_{\mu\nu,\alpha} - H^\alpha_{\mu\gamma} H^\gamma_{\alpha\nu} + H^\alpha_{\mu\nu} \Gamma^\gamma_{\alpha\gamma} - H^\alpha_{\nu\gamma} \Gamma^\gamma_{\mu\alpha} - H^\alpha_{\mu\gamma} \Gamma^\gamma_{\alpha\nu} \quad (14b)\]

Consider the constraint (8b) with \(\beta = \lambda\). This gives:

\[H^\alpha_{\mu\nu,\alpha} + H^\alpha_{\mu\nu} \Gamma^\gamma_{\alpha\gamma} + H^\alpha_{\nu\gamma} \Gamma^\gamma_{\alpha\mu} - H^\alpha_{\mu\gamma} \Gamma^\gamma_{\alpha\nu} = 0 \quad (15)\]

It is straightforward to verify that this implies \(\tilde{R}_{\mu\nu} = \tilde{R}_{\nu\mu}\). Hence (8b) implies that \(\tilde{R}_{\mu\nu}\) is symmetric.

Under these circumstances we have

\[R_{\mu\nu} = (1/N) g_{\mu\nu} R^\alpha_{\alpha} \quad (16a)\]

\[\tilde{R}_{\mu\nu} = (1/N) g_{\mu\nu} \tilde{R}^\alpha_{\alpha} = H^\alpha_{\mu\beta} H^\beta_{\alpha\nu} \quad (16b)\]
Now using arguments similar to those given in Ref. [1] for the Bianchi identities we can conclude that in the presence of torsion fields satisfying constraints discussed before:

\[
\bar{R}_{\lambda\mu\nu\beta} = R_{\lambda\mu\nu\beta} + \tilde{R}_{\lambda\mu\nu\beta} = (K + \bar{K})(g_{\lambda\nu}g_{\mu\beta} - g_{\lambda\beta}g_{\mu\nu}) = \bar{K}(g_{\lambda\nu}g_{\mu\beta} - g_{\lambda\beta}g_{\mu\nu})
\]

(17)

where

\[
R^\alpha_\alpha = \text{constant} = KN(1 - N) \quad (18a)
\]

\[
\tilde{R}^\alpha_\alpha = H^\mu_\beta H^\beta_\mu = \text{constant} = \tilde{K}N(1 - N) \quad (18b)
\]

\[
\bar{K} = K + \tilde{K} = \text{constant} \quad (18c)
\]

(In deriving the above results from the Bianchi identities we have used the fact that for a flat metric the curvature constant \(K = 0\). Hence demanding \(\bar{K} = 0\) for a (globally) zero curvature space means that \(\tilde{K} = 0\) which in turn means that the torsion must vanish.)

We now give a simple example of a torsion field which satisfies (8b) and (18b) and is also consistent with (6). First note that any non vanishing torsion is always consistent with (6) because

\[
H^\alpha_\beta\xi_{\mu;\alpha} = 0
\]

implies

\[
H^\alpha_\beta\xi_{\alpha;\mu} = 0 \quad (6b)
\]

through the Killing condition (4). Adding (6) and (6b) gives

\[
H^\alpha_\beta(\xi_{\mu;\alpha} + \xi_{\alpha;\mu}) = 0
\]

Hence any non-zero torsion is consistent with (6) and (6b). The torsion is an antisymmetric third rank tensor obtained from a second rank antisymmetric
tensor $B_{\mu\nu}$ as follows:

$$H_\mu^\alpha = \partial^\alpha B_{\mu\nu} + \partial_\mu B_\nu^\alpha + \partial_\nu B_\mu^\alpha$$

Consider dimension $D = 3$ and a general form for the metric as

$$ds^2 = -f_0(r)dt^2 + f(r)dr^2 + f_1(r)(dx^1)^2$$

So the metric components are

$$g_{00} = -f_0(r), g_{rr} = f(r), g_{11} = f_1(r)$$

i.e. the metric components are functions of $r$ only. Further assume that all fields, including $B_{\mu\nu}$, depend only on $r$. Then the torsion is just $H_{01}^r$.

It is straightforward to verify that with our chosen metric the only non-zero components of $\Gamma^\alpha_{\mu\nu}$ are $\Gamma^r_{00}, \Gamma^r_{rr}, \Gamma^0_{0r}$ and $\Gamma^1_{1r}$.

Now (8b) with $\lambda = \beta$ gives (15) which in the case under consideration reduces to (using the antisymmetry of $H$)

$$H^r_{01,1} + H^r_{01} \Gamma^r_{rr} + H^0_{1r} \Gamma^r_{00} - H^1_{0r} \Gamma^r_{11} = 0$$

We may write

$$H^0_{1r} = g^{00} g_{rr} H^r_{01}, \quad H^1_{tr} = -g^{11} g_{rr} H^r_{01}$$

Therefore (15) becomes

$$H^r_{01,1} + H^r_{01} [\Gamma^r_{rr} + g^{00} g_{rr} \Gamma^r_{00} + g^{11} g_{rr} \Gamma^r_{11}] = 0$$

Using the values

$$\Gamma^r_{rr} = \partial_r (ln f^{1/2}), \quad \Gamma^r_{00} = (1/2)(1/f) \partial_r f_0, \quad \Gamma^r_{11} = -(1/2)(1/f) \partial_r f_1$$
yields the equation
\[ H_{01, r} + H_{01} \partial_r [\ln(f/(f_0 f_1))]^{1/2} = 0 \] (21)
whose solution is
\[ H_{01}^r = [f_0 f_1/f]^{1/2} \] (22)
(Note that the torsion can be taken proportional to the completely antisymmetric \( \epsilon \) tensor in three dimensions as follows: \( H_{r01} = g_{rr} H_{01}^r \), and so \( H_{r01} \) may be written as \( H_{r01} = [f f_0 f_1]^{1/2} \epsilon_{r01} \) and this can be further integrated to give the "magnetic field" as \( B_{01} = \epsilon_{01} \int dr [f f_0 f_1]^{1/2} \), etc.)

It is immediately verified that
\[ H_{01}^r H_{r01}^0 = -1 \ (i.e. \text{a constant}) \]
thereby satisfying the constraint (18b).

If \( \xi \) denote the the Killing vectors then the Killing equations are
\[ \xi^r \partial_r f_0 + 2 f_0 \partial_t \xi^t = 0 \] (23a)
\[ \xi^r \partial_r f + 2 f \partial_r \xi^r = 0 \] (23b)
\[ \xi^r \partial_r f_1 + 2 f_1 \partial_t \xi^t = 0 \] (23c)
\[ f_1 \partial_t \xi^1 - f_0 \partial_t \xi^t = 0 \] (23d)
\[ f \partial_t \xi^r - f_0 \partial_r \xi^t = 0 \] (23e)
\[ f_1 \partial_t \xi^1 + f \partial_1 \xi^r = 0 \] (23f)

The solutions to these equations are
\[ f_0 = \text{Exp}[-2\alpha \int dr f^{1/2}] \] (24a)
\[ f_1 = \exp[-2\gamma \int dr f^{1/2}] \quad (24b) \]
\[ \xi^r = 1/f^{1/2} \quad (24c) \]
\[ \xi^t = \alpha t + \eta(x^1) + \beta \quad (24d) \]
\[ \xi^1 = \gamma x^1 + \psi(t) + \delta \quad (24e) \]

where \( \alpha, \beta, \gamma, \delta \) are constants and \( \eta(x^1) \) and \( \psi(t) \) are functions of \( x^1 \) only and \( t \) only respectively.

We therefore have an explicit example of a scenario where our generalised definition of maximal symmetry works. Note that the metric coefficients can be any well behaved functions of \( r \) and the torsion is related to these coefficients (equation (22)). This is as it should be and the rather general nature of these solutions (i.e. not limited to any unique function of \( r \)) imply that our results are neither accidental nor ad-hoc.

Therefore, in the presence of torsion the criteria of maximal symmetry has been generalised through the equations (13b), (17) and (18). The physical meaning is still that of a globally constant curvature (which now also has a contribution coming from the torsion). The torsion fields are, however, subject to the constraints embodied in equations (6), (8b) and (18b) and these constraints are mutually consistent. Hence our usual concepts of the homogeneity and isotropy of space have a more generalised footing in the presence of torsion. The thing to note is that the existence of torsion does not necessarily jeopardise the prevalent concepts of an isotropic and homogeneous spacetime. All that is required is that the torsion fields obey certain mutually consistent constraints. These constraints, in some sense, ensure that the usual physical meaning of an isotropic and homogeneous spacetime remains
Hence, in the context of the General Theory of Relativity, the present work generalises the concept of maximal symmetry of spacetimes.

It is also worth mentioning here the relevance of the present work in the context of string theory. The low energy string effective action possesses for time dependent metric $G_{\mu\nu}(\mu, \nu = 1, 2, ..., d)$, torsion $B_{\mu\nu}$ and dilaton $\phi$ background fields a full continuous $O(d, d)$ symmetry under which "cosmological" solutions of the equations of motion are transformed into other inequivalent solutions [4]. A solution with zero torsion is necessarily connected to a solution with non-zero torsion. Therefore, in the framework of the generalised maximal symmetry discussed here it is worth studying whether this generalised maximal symmetry is preserved under the $O(d) \otimes O(d)$ transformation. The results of this investigation will be reported elsewhere.

Standard Cosmology largely supports the homogeneity and isotropy of the universe. Our results show that the presence of torsion does not necessarily jeopardise this fact. Finally, the most important consequence of this work is in the realm of cosmologies with torsion. It would be interesting to see whether the generalised maximally symmetric solutions can become suitable candidates for already existing cosmological observations.

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