Primary beam effects of radio astronomy antennas – II. Modelling the MeerKAT L-band beam

K. M. B. Asad, J. N. Girard, M. de Villiers, R. Lehmensiek, T. Ansah-Narh, K. Iheanetu, O. Smirnov, M. G. Santos, T. Ansah-Narh, K. Iheanetu, O. Smirnov, M. G. Santos, T. Ansah-Narh, K. Iheanetu, O. Smirnov, M. G. Santos, T. Ansah-Narh, K. Iheanetu, O. Smirnov, M. G. Santos, T. Ansah-Narh, K. Iheanetu, O. Smirnov, M. G. Santos

ABSTRACT

After a decade of design and construction, South Africa’s SKA-MID precursor MeerKAT has begun its science operations. To make full use of the widefield capability of the array, it is imperative that we have an accurate model of the primary beam of its antennas. We have used an available L-band full-polarization astro-holographic observation and electromagnetic simulation to create sparse representations of the beam using principal components and Zernike polynomials. The spectral behaviour of the spatial coefficients has been modelled using discrete cosine transform. We have provided the Zernike-based model over a diameter of 10 degrees in an associated software tool that can be useful for direction dependent calibration and imaging. The model is more accurate for the diagonal elements of the beam Jones matrix and at lower frequencies. As we get more accurate beam measurements and simulations in the future, especially for the cross-polarization patterns, our pipeline can be used to create more accurate sparse representations of MeerKAT beam.

Key words: Primary beam – Astro-holography – Sparse representation

1 INTRODUCTION

Observational radio astronomy is continuing its growth through the construction of new generations of radio telescopes such as LOFAR (van Haarlem et al. 2013), ASKAP (McConnell et al. 2016), MeerKAT (Jonas & MeerKAT Team 2016) and the upcoming SKA (Braun et al. 2015). The new and upcoming telescopes can offer exquisite sensitivity and resolution and the ability to image large fractions of the sky very quickly which makes them ideal for exploring new science that relies on the detection of extremely weak astrophysical signals. However, to make full use of these enhanced capabilities, observers must improve their calibration techniques and the models of instrumental effects. The first and second generations of calibration strategies (defined in Smirnov 2011a) can no longer do justice to the new interferometers.

The key science goals of these newcomers will often demand an implementation of more advanced calibration strategies where both direction independent (e.g., receiver electronics) and dependent (e.g., primary beam and ionosphere) effects are taken into account under an unified mathematical formalism such as the radio interferometer measurement equation (RIME: Hamaker et al. 1996; Smirnov 2011a), which relates the visibilities observed \( V_{pq} \) by a baseline \( pq \) (formed by the antennas \( p \) and \( q \)) to the true sky represented by the ‘brightness matrix’ \( B \). It is given by the equation (following Smirnov 2011a, eqn. 18)

\[
V_{pq} = G_B \left( \int \int E_p^* B K_{pq} E_q^* \frac{dl dm}{n} \right) G_{pq}^H
\]
where $K_{pq} = e^{-2\pi i(u_{pq}+v_{pq}m)}w_{pq}(n-1)$, $u, v, w$ are the coordinates of the baseline in a reference frame oriented toward the observing direction, $G$ and $E$ are their direction independent (DIE) and dependent systematic effects (DDE), respectively, $l, m$ are the 'direction cosines' toward the sky and $n = \sqrt{1-l^2-m^2}$. In this paper, $E$ will be used to denote only the primary beam (hereafter, only beam, which should not be confused with the synthesised point spread function of an array), neglecting other DDEs, such as the ionospheric, tropospheric and Faraday rotation effects.

The worrying assumption of traditional self-calibration, i.e. second generation calibration, is that DDEs are trivial, which implies each visibility is a measurement of the sky 'coherency function' corrupted by some multiplicative gains. In such a scenario, an interferometer array would measure the Fourier transform of one common sky. This assumption holds only if the DDEs are identical across all antennas and constant in time, which they are not. In the presence of the actual non-trivial DDEs, the observed visibilities would be a convolution of the 'sky coherency' and the DDEs (Smirnov 2011b). In our case, the beam convolves the ideal visibilities with a different kernel per antenna and per time and frequency step and, hence, we get a different $uv$-plane for every baseline, time and frequency sample. Correcting for the beam effects is not trivial and still under much experimentation (for an example, see Bhatnagar et al. 2013; Tasse et al. 2018), but a clear pre-requisite step towards this complex task would be making an accurate beam model.

A simple approximate model can be created using, e.g., Gaussian or Jinc functions that can capture the basic features of the beam. However, such models cannot account for asymmetries in the shapes of the main and side lobes and the position and level of far-out sidelobes. Capturing such spectral effects will be crucial to properly characterise weak astrophysical signals. For instance, the foreground cleaning techniques in the intensity mapping of neutral Hydrogen (HI) during the epoch of reionization (Mellema et al. 2013) and also at lower redshifts (Santos et al. 2015) rely on the spectral smoothness of the diffuse Galactic foregrounds, which can be destroyed by such frequency effects if the beam is not properly modelled and corrected for (e.g. see Asad et al. 2018, and references therein).

More realistic models of the beam, in angular space, time and frequency can be either phenomenological, based on the appearance of the far-field radiation pattern of an observing antenna, or physical, based on the actual engineering physics of the antenna (described briefly in Jflight et al. 2018, section 3.1). One of the most widely used phenomenological methods is ‘astro-holography’ (AH), where a holographic measurement of the far-field pattern of an antenna scanning an astronomical source is taken, by cross-correlating its measured voltages with that of another antenna tracking the same source (for more information see Morris et al. 1988; Cotton 1994; Harp et al. 2011; Perley 2016). Measurement of the beam directly through this method is usually called ‘beam holography’ and using the measured beam to model the figure of the reflecting surface of an antenna, through the Fourier relationship between the far-field pattern and the aperture illumination function, is usually called ‘dish holography’. The first use of AH in radio astronomy was aimed at the latter (Scott & Ryle 1977). On the other hand, the basis of most physical approaches is the electromagnetic (EM) simulation of an antenna. EM simulations can predict the beam over a large field of view relatively easily, but it is computationally very expensive to produce EM models for each antenna of an array and for every frequency channel. In addition, it is more difficult to account for the temporal and environmental behaviour of the beam in EM simulations. On the other hand, beam models created from AH observations are more accurate and easier to obtain for multiple frequency channels and time samples, but they are usually restricted to smaller fields of view and angular resolution. Therefore, information from both AH observation and EM simulation can lead us to a better representation of the beam.

This paper is the second in a series of papers dealing with the modelling and effects of the primary beams of radio astronomy antennas taking into account both the physical and phenomenological approaches. The first paper presented the modelling of the Jansky Very Large Array (JVLA) beam from AH and compared these models to those created from EM simulations and physical considerations (Iheanetu et al. 2019, hereafter ‘Paper I’). It presented two different techniques of AH beam modelling—‘data-driven’ modelling using Principal Component Analysis (PCA) and ‘basis-driven’ modelling using Zernike polynomials (ZP). In this paper, we will present different approaches of modelling available AH measurements and EM simulations of MeerKAT, an SKA-MID precursor array located in South Africa. Here, we call the data-driven approach ‘characteristic’ and the basis-driven approach ‘analytic.’ Besides the PCA and ZP approaches, we will also demonstrate the use of spherical harmonics (SH). The characteristic and analytic basis models created from the AH observations will be compared with each other and also with the EM simulations. We will also present spectral modelling of the spatial coefficients using Discrete Cosine Transform (DCT).

Section 2 gives a general introduction to MeerKAT and describes the AH observations and EM simulations used in this work. Section 3 describes the general formalism of the characteristic and analytic approaches of modelling the spatial shape of the beam. Here, we also discuss spectral modelling of the spatial coefficients and compare the different approaches. Finally, we end with the main conclusions of the paper and some remarks about our future work in Section 4. The Zernike-based model presented here is available through the openly accessible tool EIDOS.1

2 MEERKAT PRIMARY BEAM MEASUREMENT AND SIMULATION

MeerKAT is located in the Upper Karoo region of South Africa. It has 64 interlinked receptors among which, 48 are located in a core region of 1 km in diameter centred at $-30^\circ 42'47.41''$ South, $21^\circ 26'38.60''$ East. The other 16 are located outside the core giving a maximum baseline of 8 km. Fig. 1 (left) shows a satellite picture of the MeerKAT location and Fig. 1 (right) shows a recent photograph of some of its antennas. The left and right panels of Fig. 2 show

1 https://github.com/ratt-ru/eidos
MeerKAT L-band beam modelling

Figure 1. *Left:* A satellite view of the MeerKAT receptor pads before the installation of the receptors, with the Hydrogen Epoch of Reionization Array (HERA) in the neighbourhood, captured from Google Earth. *Right:* A photograph of some of the MeerKAT receptors with the flat-topped hills (Karoo Kopies) capped by dolerite sills, a reminder of the Gondwanan past, in the background (courtesy SARAO).

Figure 2. The distribution of the 64 antennas of MeerKAT, each identified by an integer ranging from 0 to 63. *Left:* The distribution outside the 1-km-core. *Right:* The distribution inside the 1-km-core. The core is loosely delimited by the hexagonal boundary visible in the satellite image of Fig. 1 (left). The West-East and South-North distances are shown relative to the arbitrary centre located at $-30^\circ 42' 47.41''$ South, $21^\circ 26' 38.00''$ East.

Each MeerKAT receptor contains three components: the antenna positioner, a steerable dish on a pedestal, a set of radio receivers and a set of associated digitisers. The antenna positioner is made of an approximately 14 m effective diameter main reflector and a sub-reflector of diameter 3.8 m (excluding extension), arranged in a classic conic section offset Gregorian configuration. The sub-reflector has an extension at the bottom end of 20° (as measured from the secondary focus) to reduce the spillover noise from the hot ground. Each receptor can accommodate up to four receivers and digitisers mounted on the receiver indexer. The receiver captures the astronomical radiation and converts it to a voltage signal that is then amplified by cryogenic low-noise receivers (Lehmensiek & Theron 2014). The amplified voltages are digitised and then sent, via underground cables, to the correlator situated in the Karoo Array Processing Building on the telescope site. We refer the readers to Jonas & MeerKAT Team (2016) and Camilo et al. (2018) for more information about MeerKAT.

Three receivers of MeerKAT are expected to cover three different bands of the radio spectrum, namely the UHF (580–1015 MHz), L (900–1670 MHz) and S (1750–3500 MHz).
band. The L-band dual-polarization receivers have been fully deployed and we will only focus on this band because substantial AH observations have been carried out at these frequencies.

2.1 MeerKAT Astro-holographic observation

AH observations are available for all 64 antennas of MeerKAT. We have taken one such observation in which the nearest bright quasar 3C 273 (Schmidt 1963) was scanned using 18 antennas and simultaneously tracked using another 36 antennas for 0.5 hours. The relevant observational parameters are given in Table 1. It is classified as an ‘astro’-holographic observation because it was targeted at an astronomical object; for an example of a holographic observation of the beam of a MeerKAT antenna using a satellite beacon instead of an astronomical source, see Jonas & MeerKAT Team (2016, figure 6).

After calibrating the observed data, the AH beam was extracted for three of the scanning antennas (M009, M012 and M015) using all the available tracking antennas. The raw noisy beam measurements are then de-noised via aperture-plane masking. Aperture-plane masking is performed relying on the Fourier relationship between the primary beam and the aperture illumination function (AIF) of an antenna. The measured beams are Fourier transformed to obtain the AIF, the Fourier modes lying outside the physical aperture plane are masked and, finally, the masked AIF is inverse-Fourier transformed to create a smooth de-noised beam. We have used the de-noised beam for all analyses in this paper.

For AH observations, the correlator is operated as in normal observing mode (while the antenna tracking strategy is necessarily different), and a set of visibilities is generated. The calibration process is broadly similar to that of a normal observation, but some important subtleties need to be taken into account (see, e. g., Paper I for an extended discussion). The data in this work was reduced using a combination of the standard MeerKAT Science Data Processing (SDP) pipeline and the MeerKAT holography tool KATHOLOG\textsuperscript{2} (M. de Villiers, in prep).

Before producing models from the AH measurements, we shift the beam centres at each frequency independently to remove the frequency-dependent offset, i. e. squint, of the beam centre from the pointing centre. The squint is different for each of the feeds of each antenna and it also varies with frequency. Therefore, for an accurate comparison between the AH and EM datasets, we remove squint

\textsuperscript{2} https://github.com/ska-sa/katholog.
from both of them. The squints were calculated by fitting 2D elliptical Gaussians on the beam measurements at each frequency. The spectral behaviour of the squints corresponding to the $E_{00}$ element of the AH measurement is shown in Fig. 4. Note that the pointing offset averaged over all frequencies is related to the mechanical pointing error of an antenna, not with its optical properties. Therefore, we have calculated the squint after taking out the average pointing error. The resulting squint varies horizontally from low to middle frequencies of the L-band, but the variation at higher frequencies is related to the mechanical pointing error of an antenna.

Finally, we average the beam measurements of the three antennas to create an antenna-averaged beam because the asymmetry of the beam is also clear from the difference between the horizontal (red solid line) and vertical (blue solid line) cuts. The first sidelobe level is more than 20 dB below the maximum and the second sidelobe more than 30 dB below. Cuts through the main-diagonal (green solid line) and anti-diagonal (magenta solid line) of the average cross-polarization pattern are shown. Cross-polarization power increases with radius and reach a maximum of $\sim -35$ dB in the central region of the cloverleafs.

We refer to this re-centred normalised three-antenna-averaged AH beam with a diameter of 10 degrees as $E^h$. It is effectively a squint-less differential beam measurement with respect to the centre. As an example, the squared amplitude of the AH beam at a particular frequency (1070 MHz) is shown in Fig. 3. The main lobe and the first three sidelobes in the diagonal elements and a cloverleaf pattern in the off-diagonal elements are clearly visible in the figure.

The sidelobe levels can be seen more clearly in Fig. 5 where a 1D cut through the centre of the Stokes I beam (average of the diagonal Jones elements) is shown. The asymmetry of the beam is also clear from the difference between the horizontal and vertical cuts. The first sidelobe level is more than 20 dB below the maximum and the second sidelobe more than 30 dB below. Cuts through the main-diagonal and the anti-diagonal of the average cross-polarization pattern are also shown. Cross-polarization power increases with radius and reach a maximum of $\sim -35$ dB in the central region of the cloverleafs. The corresponding cuts through the JVLA beam are shown in the same figure (dashed lines) and we see that the first sidelobe level of JVLA beam, at 1070 MHz, is almost an order magnitude higher than that of MeerKAT. However, their cross-polarization levels are comparable.

2.2 MeerKAT Electromagnetic simulation

The MeerKAT primary beam has been simulated using the EM simulation software GRASP\footnote{https://www.ticra.com/software/grasp}, taking into account the principles of physical optics (PO) and the physical theory of diffraction (PTD). The GRASP simulations compare well with...
EM simulations performed using the MLFMM (multilevel fast multipole method) technique of FEKO. The simulations are available for the whole L-band with a spectral resolution of 5 MHz and over the entire hemisphere. For the purpose of this paper, we restrict ourselves to within a diameter of 10 degrees. The frequency-dependent squints of the simulated beam are shown in Fig. 4 (shades of grey). Squint varies smoothly in a horizontal direction as one goes from low to high frequencies and this trend is similar to the AH measurements. We remove these squints and re-centre the EM models to the same pixel as the centre of the AH measurements and normalise them using the same convention. The re-centred and normalised EM dataset of shape \((N_ν^2 \times 2 \times 2 \times 256 \times 256)\), where \(N_ν^2\) is the number of channels in the EM simulation, is hereafter referred to as \(E^ν\).

The EM model at 1070 MHz is shown in the top row of Fig. 6 (top panel, top row) and the residuals after subtracting the model from the corresponding AH measurement are shown in the bottom row. The residuals have been multiplied by 100 for better visualisation. There is a low-level dipolar structure in the residuals and further investigation is needed to identify its cause which is not within the scope of this paper. However, it is certain that the diagonal Jones elements of \(E^ν\) and \(E^ν\) are much closer to each other than the off-diagonal elements. The good match in the diagonal elements is, on average, around four times higher than that of the diagonal elements and the error increases with frequency. Again, to what extent this discrepancy is due to actual error of the EM models cannot be known for certain because the low-level off-diagonal elements are less well-known in the AH observations and also become more noisy at higher frequencies. The NRMSE of the diagonal elements increases smoothly with frequency, but that of the off-diagonal elements increases rapidly after around 1350 MHz. The outliers in this figure are caused by Radio Frequency Interference (RFI) and not due to any intrinsic effect of the model or the measurement. Note that the NRMSE is averaged over a diameter of 10 degrees and, hence, dominated by the errors near the nulls.

\[ \text{NRMSE} = \frac{\sqrt{\sum (\text{RMSE}_ν)^2}}{\text{RMSE}_ν}, \]

as a function of frequency. The NRMSE of the off-diagonal, i.e. cross-polarization, elements is, on average, around four times higher than that of the diagonal elements and the

3 MODELLING AH OBSERVATION AND EM SIMULATION

The squint-less, normalised, thereby differential, AH measurement \(E^ν\) and EM simulation \(E^ν\) are effectively the starting point for the main body of our work. We model the spatial beamshape using characteristic and analytic basis functions and the spectral behaviour using DCT. This creates a sparse representation of the given beam datasets that can be used in direction dependent calibration and imaging effectively.

3.1 Spatial modelling

The beamshape is modelled using two types of orthonormal bases: ‘characteristic basis’ (derived from PCA) and ‘analytic bases’ (Zernike polynomials and spherical harmonics). The method and results of the two approaches are described in the following two subsections. Here, we will focus on just one frequency channel, centred at \(ν_ν = 1070\) MHz, because both \(E^ν\) and \(E^ν\) are sampled at this frequency, and the spectral behaviour of the spatial coefficients will be treated in the next section.

3.1.1 Characteristic basis

In this approach, the basis vector (eigenvector or eigenbeam) to describe a beam is created from the AH beam measurement itself, akin to the Characteristic Basis Function Pattern (CBFP) method (e.g. Maaskant & Ivashina 2012; Mutonkole et al. 2016, and references therein). Following the method described in section 4.3 of Paper I, we have used Principal Component Analysis (PCA) to create such eigenbeams with the help of Singular Value Decomposition (SVD).

As opposed to the analytic bases, it is more appropriate to apply PCA on the spatio-spectral beam-cube, containing data for all the L-band frequency channels of \(E^ν\) and \(E^ν\). We remove the RFI-affected channels from the AH measurement, stack \(E^ν\) and \(E^ν\) together and flatten the spatial dimensions to create a 4D array \(E^{be}\) of shape \(N_ν \times 2 \times 2 \times N_ν\), where \(N_ν = N_ν^0 + N_ν^c\) is the total number of RFI-free frequency channels, and \(N_ν\) is the total number of pixels in a beam image. For each complex Jones element, \(N_ν \times N_ν\) matrix \(E^{be}\) is decomposed into three complex matrices— \(U\), a \(N_ν \times N_ν\) unitary matrix, \(Σ\), a \(N_ν \times N_ν\) diagonal matrix whose diagonals contain all the singular values, and \(V^ν\), a

![](image.png)

Figure 7. Normalised root-mean-square error of the EM model with respect to the AH observation as a function of frequency averaged over a diameter of 10 degrees.

https://altairhyperworks.com/product/FEKO

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**Figure 8.** The principal component model created from the AH observation (left panels; $E^{\ell h}$) and EM simulation (right panels; $E^{\ell e}$) at 1070 MHz. The models and the residuals ($E^{\ell h} - E^{\ell h}$ and $E^{\ell e} - E^{\ell e}$) are shown in the top panels and the bottom panels show the radial profiles (averaged over concentric annuli) of the data, model and residual for the $E_{00}$ and $E_{01}$ elements. In the top panels, the power beams are shown in dB unit and the residual images are multiplied by 100 to improve visualisation.

**Figure 9.** The Zernike model created from the AH observation (left panels; $E^{\ell h}$) and EM simulation (right panels; $E^{\ell e}$) at 1070 MHz. The models and the residuals ($E^{\ell h} - E^{\ell h}$ and $E^{\ell e} - E^{\ell e}$) are shown in the top panels and the bottom panels show the radial profiles of the data, model and residual for the $E_{00}$ and $E_{01}$ elements. The units and scale are same as those of Fig. 8.
Normalised root-mean-square error (NRMSE) of the principal component model (red) and the Zernike model (blue) as a function of the number of modes used at a frequency of 1070 MHz averaged over a diameter of 10 degrees. The plus and cross signs represent the errors for the $E_{00}$ and $E_{01}$ elements, respectively.

$N_{\nu} \times N_{px}$ unitary matrix—using SVD. Here, $V^c$ is the collection of our basis vectors or principal components (PC), i.e. eigenbeams, and $C^* = U \Sigma$ contains the corresponding coefficients (for the relationship between SVD and PCA, see Jolliffe & Cadima 2016, section 2a). The models for all frequencies of L-band are reconstructed from the eigenbeams and the coefficients as $E^c(\nu) = C^*_0 V^c_0$, where $C^*_0$ contains the strongest coefficients and $V^c_0$ the corresponding PC's.

The results over the full L-band will be presented in Section 3.2; here we focus on a single channel. The top row of Fig. 8 shows the four reconstructed Jones elements of $E^{ch}(\nu_c)$ (left panels; AH) and $E^{ze}(\nu_c)$ (right panels; EM) where $\nu_c = 1070$ MHz. The corresponding residuals after subtracting the models from $E^c$ are shown below the models. Note that all the residual images are calculated from the actual amplitudes of the Jones elements, not the squared amplitudes, i.e. power. These models are created using $N^c = 15$ strongest PC's. The NRMSE ($[\langle|E^c|^2 - |E^c|^2|\rangle^{1/2}/|E^c|]$) of the model created from AH is shown in Fig. 10 as a function of number of modes used. We see that PCA can represent the beam measurements accurately with less than 15 components, but we have used 15 modes for a fair comparison with the models created using analytic bases.

The bottom panels of Fig. 8 show the radial profiles of the $E_{00}$ and $E_{01}$ Jones elements of the data (blue dots), the characteristic models (green line) and the residuals (red dots). Radial profiles are created by averaging the beam images in concentric circular annuli. In both the images of the top panels and the radial profiles of the bottom panels, we show the squared amplitude for the sake of consistency. The mean and standard deviation of the residuals are shown above the radial profile plots. If we take the square-root of these values, we see that with 15 PC’s, we can reach an rms residual level of $\sim 10^{-3}$ for the diagonal Jones elements and $\sim 5 \times 10^{-4}$ for the off-diagonal elements for both the AH and EM beams. By comparing the radial profiles of $E_{00}^{ch}$ and $E_{01}^{ch}$, we see that the residual level is lower for the off-diagonal elements of the EM model compared to the AH measurement.

Along with a faithful representation of the beam, one other advantage of using an orthogonal basis is the sparsity of the representation—the beam can be described using less information. The AH or EM datasets contained $4N_{px} \times N_{\nu}$ parameters for describing the full-Jones beam and although the characteristic basis model $E^c$ now needs $4N^c(N_{px} + N_{\nu})$ parameters to represent the full-bandwidth full-polarization beam, the frequency behaviour of the PC coefficients can be modelled using $N^c < N_{\nu}$ parameters, as described in Sec. 3.2, further reducing the information-load. However, $N_{px}$ will still be a large number, especially if a bigger field of view or better resolution is desired. This is where the ‘analytic basis’ approach comes in. It can further sparsify the model by representing the spatial dimensions via 2D analytic functions calculated over unit disks.

### 3.1.2 Analytic basis

We test two types of analytic bases to represent $E^{ch}$ and $E^{ze}$—Zernike polynomials (ZP; for an introduction, see Lakshminarayanan & Fleck 2011, and Appendix A) and spherical harmonics (SH). Unlike PCA, the analytic basis models are created for each AH and EM frequency channel independently. The basic procedure is the same for both ZP and SH. If the analytic bases are denoted by $V = [Z, S]$ for ZP and SH, respectively, and the corresponding coefficients by $C$, then the decomposition of $E^c$ gives us the coefficients

$$C = (V^T V)^+ V^T E^{ch}$$

where $^T$ stands for transpose and $^+$ for the Moore-Penrose pseudoinverse. These coefficients can be used to reconstruct a beam model as

$$E^m = \sum_{i=0}^{N^m} C_i V_i$$

where $i = 0$ denotes the strongest among the selected coefficients and $i = N^m$ the weakest, and $m = \{z, s\}$ for ZP and SH. The results of the decomposition and reconstruction of the AH and EM datasets using these bases are presented below.

**Zernike polynomial model:** ZP models $E^c$ created from the AH ($E^{ch}$) and EM ($E^{ze}$) datasets are shown in the top panels of Fig. 9, the former on the left and the latter on the right. Like the PCA model, the 15 strongest modes were used to reconstruct these models. We keep using 15 coefficients because, as shown in Fig. 10 and described in more detail below, modelling error cannot be reduced substantially by including more coefficients in the case of a single frequency channel. By comparing the residual images of Fig. 8 and 9, we immediately see that PC’s can represent the data more faithfully with the same number of modes. A comparison

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6 The python module `scipy.linalg.svd` is used.

7 In this paper, $E^c$ refers to a model created using the basis function $x = \{c, z\}$ for both AH observation and EM simulation whereas $E^{ch}$ and $E^{ze}$ refer to the AH and EM models created using the basis, individually.

8 Calculated using the python module `numpy.linalg.pinv`.
of the corresponding radial profiles reveals that the residual levels for $E^{zh}$ (left panels) and $E^{ce}$ (right panels) are, on average, an order of magnitude higher than those for $E^{zh}$ (left panels) and $E^{ce}$ (right panels).

In particular, $E_{00}$ seems to have higher residual near the edge of the beam compared to $E_{01}$ and the same goes for the $E_{10}$ elements. Moreover, $E_{00}$ has higher residual near the edge compared to $E_{01}$. Cross-polarization levels beyond the cloverleaf pattern are very low for both AH and EM and for AH, they are also dominated by noise. Therefore, we should expect the residual to be higher in this region in case of ZP because the ZP’s are less prone to model the noise than the PC’s. This is also evident from the difference between $E_{00}$ and $E_{01}$—there is no noise in EM simulation and, hence, the residual is lower, albeit still higher than that of $E_{00}$. Another difference between the $E_{00}$ and $E_{01}$ is that PC’s seems to be able to model the central region of the cloverleafs better than ZP in case of EM simulation.

NRMSE of $E^{h-c}$ (difference between AH measurement and ZP representation) is shown in Fig. 10 (blue plus markers for $E_{00}$ and blue crosses for $E_{01}$) as a function of number of modes used; $E^{h-c}$, not shown here, follows a similar trend. After using the 15 strongest Zernike modes, the NRMSE of $E_{00}$ is $\sim 0.1$ and that of $E_{01}$ around 0.3; compare them to the corresponding NRMSE for $E^{h-c}$ (red plus and cross)—around 0.02 and 0.2. Comparing the blue and red markers, we see that the NRMSE for $E_{00}^{h-c}$ and $E_{01}^{h-c}$ decreases in a similar fashion as more modes are included, but they always maintain a difference of almost an order of magnitude. The difference is lower for the off-diagonal elements.

ZP are real functions, therefore, we model both the real and imaginary parts using the real functions and store the resulting complex coefficients. The beam measurement and simulation were decomposed using the first 300 Zernike modes and the 15 strongest ones among them were selected for this particular reconstruction at 1070 MHz. As discussed before, the main reason for using analytic bases is to reduce the amount of information needed to represent the spatial structure. It is, therefore, imperative to look at the shapes of the Zernike modes needed for that representation. However, we want to find the strongest Zernike coefficients needed to model both the beam measurements and simulations for all frequencies and, hence, the shapes of the individual modes is discussed in Section 3.2.

Spherical harmonic model: Unlike the ZP, the SH are complex valued although their imaginary parts do not have any zero-frequency component and for any order (or degree) $n$, the frequency $\nu = 0$ contains the average of all the harmonics. We decompose the observed complex beam $E^h$ using the first 256 complex SH (with a maximum $n$ of 15) and select the strongest coefficients. Unlike PC’s and ZP’s, SH’s cannot reconstruct the diagonal elements accurately with 15 modes and at least 40 modes are needed for a reasonable representation. Therefore, the SH model $E^{zh}$, shown in Fig. 11 (top row), has been created using 40 and 15 coefficients for the diagonal and off-diagonal elements, respectively. Even with 40 modes, SH does not perform as well as PC and ZP—the residuals show a dipolar structure in the main lobe of the beam. The residuals for the off-diagonal elements also show more structure compared to the the ZP and PC-based models. This precludes using SH for modelling 10-degree beams across a wide bandwidth and, hence, in the next section, we will focus only on ZP and PC.

3.2 Spectral modelling

The two full-bandwidth sparse representations $E^h$, based on PC, and $E^h$, based on ZP, can be compressed even further if we model the spectral behaviour of the associated strongest coefficients in each case. PC’s are calculated from the combined dataset of the full-bandwidth AH and EM beams, but ZP decomposition is performed at each frequency channel separately. To prepare a dataset for PCA, we removed the RFI-affected channels from AH measurements, as mentioned in Section 3.1.1, and created a new dataset containing only the non-contaminated AH channels and all the EM channels.

We used three different figures of merit to detect RFI-affected channels in the AH measurement: the rms of the beam images, the position of the peak in the diagonal elements and the difference between the energies of the ZP coefficients of adjacent channels. In the RFI-affected channels, rms is comparatively high and the beam is not exactly centred at the central pixel (128,128). The dataset created after removing the unusable channels based on these two criteria was used for PCA. However, these criteria could detect only the worst channels. To identify the remaining lower-level RFI, we used ZP coefficients $C^{zh}(\nu)$. ZP decomposition was performed on all channels including the RFI-affected ones because, unlike PC, ZP coefficients of one channel do not depend on those of another. Then, we identified the channels $\nu_i$ for which $C^{zh}(\nu_i) - C^{zh}(\nu_{i+1}) > 10^{-3}$ as RFI-affected. Finally, we created a comprehensive list of RFI-affected channels and masked those channels in both $C^{zh}(\nu)$ and the PC coefficients $C^{zh}(\nu)$.

3.2.1 Principal components

The most dominant PC’s are shown in Fig. 12 as a function of frequency for all the Jones elements of $E^{zh}$ (dots) and $E^{ce}$ (solid lines). AH measurements are available from 856 to 1712 MHz with a resolution of 0.83 MHz, but we show results
Figure 12. The strongest principle components, as a function of frequency, for the AH (dots) and EM (solid lines) models. We show the strongest 5 components for the diagonal elements and the strongest 3 for the off-diagonal ones. The top four panels show the real parts of the four Jones elements and the bottom four panels the imaginary parts.

up to the effective L-band edge, 1670 MHz. EM simulations are available from 900 to 1670 MHz with a resolution of 5 MHz. The top four panels in the figure show the real parts of \( C^c(\nu) \) and the bottom ones the imaginary parts. The first 5 real PC’s of the diagonal elements of \( E^h \) and \( E^e \) match very well and their difference is only visible in the imaginary parts which are at a much lower level. The AH measurement and EM simulation diverge more in the off-diagonal elements which reiterates the trend seen in Fig. 6 (right; blue and green dots). In both these figures, \( E^h \) and \( E^e \) diverge more after around 1350 MHz because the AH measurements at higher frequencies are more dominated by noise. The cross-polarization patterns in AH measurements are still under investigation and, hence, a more accurate comparison of the off-diagonal elements of \( E^h \) and \( E^e \) over the full L-band is not possible at this stage.

The amplitude of the strongest PC (red lines in the diagonal elements of Fig. 12) models the beamwidth as a function of frequency to a large extent. This amplitude decreases smoothly with frequency because beamwidth is proportional to \( \lambda/D \), but it also exhibits a low-level frequency-dependent ripple, more clearly visible in the imaginary part (red lines in \( E^{00,\text{imag}} \) and \( E^{11,\text{imag}} \)). The ripple of MeerKAT beam is described in more detail in Section 3.2.4.
3.2.2 Zernike coefficients

To select the most dominant Zernike coefficients $C_i(\nu)$, we calculate the average of the absolute value of the real and imaginary parts separately over all frequencies for both AH measurement and EM simulation. The 15 strongest Zernike modes selected from these datasets is shown in Fig. 13 for the $E_{00}$ and $E_{01}$ elements. The red circles and blue dots show the average energies of the coefficients for $E^h$ and $E^e$, respectively. We see that the real part of the diagonal elements is symmetric, but the beam has asymmetries which are represented by coma modes and spherical modes are never used for modelling the EM. The imaginary part is also represented by the astigmatism ($m = 2$) and coma ($m = −1$) modes. The imaginary part is also clearly visible in the spherical modes. The energies of $E_{00}$ and $E_{01}$ match very well in the real part as opposed to the imaginary part.

The off-diagonal elements are mainly represented by the negative astigmatism ($m = −2$) modes that exhibit a cloverleaf pattern. Note that the asymmetries of the diagonal elements were represented by positive astigmatism where the cloverleaf pattern is arranged horizontally and vertically whereas in the negative modes they are arranged at an angle of 45°, more suitable for modelling cross-polarization patterns. Asymmetries in the cloverleaf pattern are represented by coma modes and spherical modes are never used here. There is significant difference between $E^h$ and $E^e$. One important difference is that the negative coma modes ($m = −1$) used for $E^h$ are not used for modelling the EM simulation dataset.

Spectral behaviour of the most dominant Zernike coefficients is presented in Fig. 14. Like Fig. 12, $C^h$ and $C^e$ are plotted using dots and $C^{ze}$ using solid lines. Similar to the PC’s, the strongest Zernike coefficients for modelling the real part of the diagonal elements match very well between $E^h$ and $E^e$ and the ripple of the beamwidth is also clearly visible in the imaginary parts. Off-diagonal elements of $C^{ze}$ diverge substantially from those of $C^{ze}$ at higher frequencies. Note that higher order spherical modes are needed for modelling higher frequency beams because number of sidelobes within the $10^5$ diameter increases with frequency.

Ideally, we would want to incorporate information from both EM simulation and AH observation to create a more accurate beam model, but this is not within the scope of the current paper. Additionally, the cross-polarization patterns of the two datasets are not reliable enough for such a task. Therefore, we present separate sparse representations of $E^h$ and $E^e$. The spectral behaviour of the spatial coefficients presented above is modelled using DCT as described below. We discuss spectral compression only for the $ZP \nu$ case because this is the basis we have used in our final beam models and $ZP$’s can represent the beam using less information. However, the spectral behaviour of the PC’s can be modelled using the same method when needed.

3.2.3 Discrete cosine transform

Only 15 ZP coefficients were needed to model the beam at 1070 MHz as shown in Fig. 9 and 10. However, at least 20 coefficients are necessary in case of the full L-band. Therefore, we model the spectral behaviour of the 20 most dominant spectral coefficients $C^h$, i.e. $C^{zh}$ for AH and $C^{ze}$ for EM. Before compressing the coefficients, we interpolate their energies, for both AH and EM models, to $N_\nu = 7701$ channels from 900 to 1670 MHz with a resolution of 0.1 MHz. In case of $C^{zh}$, the coefficients for the RFI-affected channels were reconstructed by interpolation from the RFI-free ones and then the coefficients of $N_\nu = 1024$ channels were used to fill $N_\nu$ channels via piecewise linear interpolation. In case of $C^{ze}$, coefficients from the original $N_\nu = 155$ channels are used to fill the $N_\nu$ channels.

Interpolated coefficients $C$ are then compressed via DCT. First, we decompose the coefficients using DCT of

\[ E^{\text{real}}_{00} \]

\[ E^{\text{imag}}_{00} \]

\[ E^{\text{real}}_{01} \]

\[ E^{\text{imag}}_{01} \]
Figure 14. The strongest Zernike coefficients, as a function of frequency, for the AH (dots) and EM (solid lines) models. The top four panels show the real parts of the four Jones elements, and the bottom four panels the imaginary parts.

Type II as

\[ C_k = \frac{2}{\sqrt{wN_i}} \sum_{n=0}^{N_i^l-1} C_n(\nu) \cos \left( \frac{\pi k (2n + 1)}{2N_i^l} \right) \]  

for \( 0 \leq k < N_i^l \), where \( w = 4 \) when \( k = 0 \) and \( w = 2 \) otherwise. We have seen that the number of resulting DCT coefficients that are more than 10 times higher than the noise level in the decomposition is usually around 30 for \( C_{zh} \) and 40 for \( C_{ze} \). Therefore, we decided to keep 40 DCT coefficients for each Jones element of the AH model and 30 for the EM model. \( C_{zh} \) needs more coefficients because it has noise.

If the resulting most dominant DCT coefficients are denoted by \( C'_n \), we can reconstruct a de-noised smooth spectral model of the coefficients via Inverse DCT (same as DCT of Type III) as

\[ \hat{C}_n(\nu) = C_n(\nu) + \sqrt{\frac{2}{n}} \sum_{k=0}^{N_i^l-1} C'_k \cos \left( \frac{\pi k (n + \frac{1}{2})}{N_i^l} \right) \]  

for \( 0 \leq n < N_i^l \). Both \( C(\nu) \) (thick coloured lines) and \( \hat{C}(\nu) \) (thin white lines) are shown in Fig. 15 for both the AH (left panels) and EM (right panels) models. The sixth to tenth most dominant coefficients for modelling \( E_{00} \) and \( \hat{E}_{00} \) are shown because the first five coefficients, already shown in Fig. 14, have even less error and are not representative of the rest of the coefficients.

From Fig. 15, we see that the smooth reconstruction of the spectral behaviour follows the original energies of the coefficients closely; the thin white lines are always inside the thick coloured lines. DCT can also reconstruct the small-
scale ripple on the coefficients to some extent, but we will focus on more accurate modelling of the ripple in our future papers.

We need only $2 \times 2 \times N^d \times N^z$ coefficients to represent the full L-band complex beam model of MeerKAT where $N^d$ is the number of DCT coefficients for modelling the spectral behaviour of each of the $N^z$ ZP coefficients. To show the accuracy of the semi-analytic beam models created from these coefficients, provided in EIDOS, we reconstruct the models for 155 channels from 900 to 1670 MHz and calculate their NRMSE with respect to the given measurement and simulation. These errors are also compared with the corresponding error of the PC-based models.

Fig. 16 shows the NRMSE of the PC and ZP-based models for the $E_{00}^{zh}$ and $E_{00}^{ch}$ elements. The left and right panels show the errors for $E^h$ and $E^e$, respectively. The error of $E_{00}^{zh}$ is almost an order of magnitude higher than that of $E_{00}^{ch}$ which reiterates the results presented in Fig. 10 for a single channel. However, we need much less information to reconstruct beam models using ZP and, hence, only Zernike coefficients are given in EIDOS. Errors of the diagonal elements increase with frequency for both $E^h$ and $E^e$ because

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**Figure 15.** Spectral modelling of the sixth to tenth strongest Zernike coefficients using DCT for AH (left panels) and EM models (right panels). Both the real (top panels) and imaginary (bottom panels) parts are shown. The thick coloured lines show the original interpolated energies of the coefficients and the thin white lines running through them represent the corresponding DCT reconstructions.

**Figure 16.** Normalised root-mean-square error of the ZP and PC-based AH and EM models as a function of frequency. Left panel shows the error of $E_{00}^{zh}$ and $E_{00}^{ch}$ with respect to the given AH measurement and the right panel the error of $E_{00}^{ze}$ $E_{00}^{ce}$ with respect to the given EM simulation.
there are more sidelobes at higher frequencies, making the modelling less optimal. Off-diagonal elements exhibit comparatively higher error, as expected. The outlier points in the left panel are caused by RFI which is present in the AH measurement, but not in our models.

By comparing the left and right panels of Fig. 16, one can see that PC’s model the AH measurement and EM simulation with similar errors (blue and green markers), but ZP’s exhibit higher error (red and magenta markers) in modelling the AH dataset. This is because the measurements have low-level noisy structure and PC’s are more prone to modelling the noise than ZP’s. Even though PC’s are more faithful to a given data, representing all the structures and irregularities in an AH observation might not be desirable.

### 3.2.4 Beamwidth and ripple

As a final test of the ZP-based models, we compare the full width at half-maximum (FWHM or ‘beamwidth’) of the beam models with that of the original datasets as a function of frequency. To calculate the beamwidth, we fit 2D elliptical Gaussian functions to the datasets and models within the region where beam amplitude is more than 0.01. The same method was used to calculate squint as described in Section 2.1 and shown in Fig. 4. The resulting beamwidth $\theta$ has a horizontal (semi-major axis) and a vertical (semi-minor axis) component and they vary from the theoretical beamwidth $\theta_0 = \lambda/D$ (where $\lambda$ denotes wavelength and $D = 14$ m, the dish diameter). Fig. 17 shows $\theta_0/\theta$, for the horizontal width only as a function of frequency for the Stokes I beams (90 element of the Mueller matrix) of the AH measurement (red dots), EM simulation (blue line), and the Zernike models created from the AH (red line) and EM (blue line) datasets. The ripple of $\theta(\nu)$ is clearly visible here because $\theta_0(\nu)$ is smooth. It is more prominent at the lower part of the band. The models follow the original datasets closely, although they diverge more at the upper part of the band. There is a clear division between the lower and upper parts of the L-band in terms of beamwidth and, hence, the band can be conveniently divided into two subbands: one before $\sim 1350$ MHz, the other after it. If we model the lower and upper subbands separately, the models will be more accurate, albeit at the expense of increasing the amount of required information. For continuum science, a continuous full-bandwidth beam is more desirable, but for emission line studies, e. g. in case of HI intensity mapping, we can use a more accurate model of the beam created from the lower subband.

The ripple of the beamwidth is caused by the interference of the electric field diffracted from the secondary reflector with the electric field of the main beam (de Villiers 2013). To show the amplitude of the MeerKAT ripple, we subtract a smooth third-order polynomial from $\theta(\nu)$ calculated from the original AH and EM datasets. The resulting ripple $\delta\theta(\nu)$ for the lower subband is shown in the inset of Fig. 17 in arcmin units. The top and bottom panels show the horizontal ($\delta\theta^H$) and vertical ($\delta\theta^V$) ripples, respectively, and the AH and EM datasets are represented by the red and blue colours, respectively. $\delta\theta^H$ is significantly larger than $\delta\theta^V$ which can be interpreted as a frequency-dependent ‘beam squish’ (Heiles et al. 2001), i.e. the beam is not squeezed symmetrically but squashed along a preferred direction. The ripples predicted by the EM simulation match reasonably well with the AH measurements. The match is relatively poor in case of $\delta\theta^V$ because it is at a lower level and, hence, more affected by the noise of the AH dataset.

### 4 CONCLUSION

This is the second in a series of papers dealing with primary beam modelling effects of radio astronomy antennas. The basic formalism was set in Paper I (Iheanetu et al. 2019) in the context of the JVLA. In this paper, we extend that formalism and use it to create MeerKAT beam models from astro-holographic (AH; $E^h$) observations and EM simulations ($E^e$). Our aim was to present a pipeline that is able to create sparse representations of the beam, given any dataset, that can be used for direction dependent calibration and imaging. This model can be called ‘eidetic’ because it reconstructs the given datasets faithfully. We have modelled the given $E^h$ (Fig. 3) and $E^e$ (Fig. 6) over a diameter of 10 degrees using characteristic and analytic basis functions, for all frequencies of L-band and all polarizations, and compared the results. Here, $E^h$ is averaged over three antennas and $E^e$ is a generic simulation for an ideal antenna.

The diagonal elements of $E^e$ matches well with $E^h$, but the off-diagonal elements, i.e. the cross-polarization patterns, vary significantly and the difference increases with frequency (Fig. 6 and 7). We have decomposed the two beam datasets using principal components (PC), Zernike polynomials (ZP) and spherical harmonics (SH). PC’s and ZP’s can represent the beam measurements and simulations very well with around 15 modes (compare Fig. 8 and 9), but 10-degree-beams cannot be modelled as well using the same number of SH modes (Fig. 11). Therefore, we focus on PC’s and ZP’s in the paper.

The coefficients corresponding to the most dominant
PC and ZP modes are modelled in frequency using DCT. The full-bandwidth full-polarization 10-degree beam model can be represented using $2 \times 2 \times 2 \times N^d \times N^b$ coefficients where $N^d$ is the number of DCT coefficients needed to model the spectral behaviour of each of the $N^b$ PC or ZP coefficients. The first PC models the beamwidth as a function of frequency to a large extent (red line in the diagonal elements of Fig. 12) and the small-scale ripple of the width can be seen clearly in the imaginary part of this component. The spectral behaviour of the most dominant PC (Fig. 12) and ZP (Fig. 14) modes representing $E^h$ (dots) and $E^e$ (solid lines) matches well in the diagonal elements, but not so well in the off-diagonals.

Beam models can be represented by analytic basis functions like ZPs’s using less information than the characteristic bases like PC’s because for the latter, we also need to store the eigenbeams. By looking at the 15 most dominant ZP modes (Fig. 13), we see that the diagonal elements of $E^h$ or $E^e$ are modelled mainly by the symmetric spherical modes and the asymmetries in the beam are modelled by the coma and astigmatism modes (first 105 ZP’s are shown in Fig. A1). On the other hand, the off-diagonal elements are modelled mainly by the negative astigmatism modes and their asymmetries are represented by the coma modes.

We have been able to represent the spectral behaviour of the coefficients using 40 DCT coefficients for $E^h$ and 30 for $E^e$ (Fig. 15). Before DCT decomposition, the coefficients for the RFI-affected channels in $E^h$ are calculated by interpolating from the RFI-free channels. Through interpolation and DCT, we have calculated coefficients for all frequencies between 900 and 1670 MHz with a resolution of 0.1 MHz.

The resulting beam models calculated from the ZP coefficients have a residual error (after subtracting the model from data) of around $10^{-5}$ and $10^{-6}$ corresponding to the power of the diagonal and off-diagonal elements, respectively (Fig. 9). For the electric field, the corresponding residual levels would be around $10^{-3}$. Comparatively, the PC-based models exhibit lower residuals (Fig. 8). The off-diagonal elements are less accurate and the errors of all the Jones elements increase with frequency, as seen in the trend of the normalised root-mean-square error (NRMSE) (Fig. 10 and 16). However, NRMSE should be used with caution because it is averaged over a diameter of 10 degrees and, hence, mostly dominated by errors in the nulls and noisy regions.

The antenna and frequency dependent squints of the beam (Fig. 4) were taken out before modelling, but they can be reintroduced by re-centering the model beams at a later stage if desired. On the other hand, the squash effect (Fig. 17) is already incorporated in the Zernike coefficients; later stage if desired. On the other hand, the squash effect can be reintroduced by re-centring the model beams at a beam (Fig. 4) were taken out before modelling, but they mostly dominated by errors in the nulls and noisy regions. However, NRMSE should be used with caution because it is averaged over a diameter of 10 degrees and, hence, mostly dominated by errors in the nulls and noisy regions.

The advantage of having a Zernike-based eidetic model of the beam is that it can be calculated quickly and instantaneously during calibration or beam-correction. The models provided in EIDOS can be used in MeerKAT-specific pipelines through the radio astronomy package STIMELA. The JVLA beam models presented in Paper I will also be included in this tool. In future papers, we will explore the spatial, spectral and polarization effects of these beams on continuum imaging and intensity mapping.

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REFERENCES
Asad K. M. B., Koopmans L. V. E., Jelić V., de Bruyn A. G., Pandey V. N., Gehlot B. K., 2018, MNRAS, 476, 3051
Bhatnagar S., Rau U., Golap K., 2013, ApJ, 770, 91
Born M., Wolf E., 1999, Principles of Optics, 7 edn
Braun R., Bourke T., Green J. A., Keane E., Wagg J., 2015, in Advancing Astrophysics with the Square Kilometre Array (AASKA14), p. 174
Camilo F., et al., 2018, ApJ, 856, 180
Cotton W. D., 1994, AIPS Memoranda, 86
Hamaker J. P., Bregman J. D., Sault R. J., 1996, A&AS, 117, 137
Harp G. R., et al., 2011, IEEE Transactions on Antennas and Propagation, 59, 2004
Heiles C., et al., 2001, PASP, 113, 1247
Ibeauetkut K., Girard J. N., Smirnov O., Asad K. M. B., de Villiers M., Thorat K., Makhathi S., Perley R. A., 2019, MNRAS, 485, 4107
Jagannathan P., Bhatnagar S., Brisken W., Taylor A. R., 2018, AJ, 155, 3
Jolliffe I. T., Cadima J., 2016, Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 374
Jonas J., MeerKAT Team 2016, in Proceedings of MeerKAT Science: On the Pathway to the SKA. 25-27 May, p. 1
Laskshminarayanan V., Fleck A., 2011, Journal of Modern Optics, 58, 545
Lohmannsiek R., Theron I. P., 2014, in 31th URSI General Assembly and Scientific Symposium (GASS), Beijing, China. pp 1–4

\[^{10}\] https://github.com/SpheMakh/Stimela
APPENDIX A: ZERNIKE POLYNOMIALS

The Zernike polynomials of order \( n \) and angular frequency \( m \) can be written in polar coordinates as (following Born & Wolf 1999, chapter IX, section 9.2.1)

\[
Z_n^m(\rho, \phi) = R_n^m(\rho)e^{im\phi}
\]  

(A1)

where \( n \) and \( m \) are non-negative integers with \( n \geq |m| \) and \( n - |m| \) is always even. The radial polynomials

\[
R_n^\pm m(\rho) = \sum_{s=0}^{\frac{n-m}{2}} \frac{(-1)^s(n-s)!}{s!(\frac{n+m}{2}-s)!(\frac{n-m}{2}-s)!} \rho^{n-2s}
\]  

(A2)

and the normalisation is such that for all permissible values of \( n \) and \( m \), \( R_n^m(1) = 1 \). The first 105 Zernike polynomials with \( n \leq 13 \) are shown in Fig. A1.

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Figure A1. The first 105 Zernike polynomials up to a maximum radial order $n_{\text{max}} = 13$ on a polar closed unit disk.