Abstract
This paper gives the most general form of the Adler–Kostant–Symes theorem, and many applications of it, both finite and infinite dimensional, the former yielding algebraic completely integrable (a.c.i.) systems, and the latter examples in random matrix theory.

Keywords: algebraic integrability, random matrix theory, Lax pairs, Virasoro constraints, integrability

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Introduction

The study of integrable systems begins with Euler’s top [14] linearizing on an elliptic curve (thought of as an algebraic group), leading eventually to Liouville’s theorem [16] concerning the connection between the existence of enough constants of the motion (integrals) in involution leading to a solution of the system by quadrature, and the more refined Arnold–Liouville theorem leading to linearizing such systems on tori and the use of action-angle coordinates. In practice, the tori were, at least in the canonical classical examples, algebraic tori and the motion was actually linear motion on the (universal covering space of the) algebraic tori. The existence of many integrals in involution was often seen to be a consequence of physical configurational symmetries, but certainly not always, the Kovalevsky top [30] the best example where this was certainly not the case, so in effect, it was often a mystery. Likewise the fact these systems linearized on algebraic tori was another mystery, starting from the case of the Euler top!

In the latter half of the twentieth century, much progress has been made, starting with the famous Fermi–Pasta–Ulam experiments and punctuated by the Lax formulation of the KdV equation [32]. Indeed the theory was broadened to new systems like the Toda lattice [41], which also was shown to have a Lax pair representation [23, 24]. Many other partial differential equations were introduced, like the KP equation [20], as well as new methods of solution involving scattering theory, the Baker Akhiezer function [31] and the $\tau$-function of Sato [38] and his school. As the subject matured it reached into new areas such as string theory, matrix models and random matrix theory (RMT) [15].

This paper will touch on a few of the above topics. First the mystery of why systems have many constants of the motion in involution is well explained by the Adler–Kostant–Symes (AKS) theorem [14], which says a Lie algebra splitting as a vector space into two Lie algebras gives rise to integrals in involution and commuting vector fields whose solutions linearize through group factorization (or splitting); although the AKS theorem does not say when systems are completely integrable. Of course Noether’s theorem [33] is the first and more basic theorem, showing configurational group symmetries yield integrals. In section 1 we give the most general form of the AKS theorem and in sections 2 and 3 we give various examples. In section 2 almost all the examples are infinite dimensional examples (like the KP equation), being either lattices or PDEs, showing the robustness of the theorem; also these examples were picked as they all have application to RMT. Then came the work on the periodic KdV equation [34] and the periodic Toda lattice [45]. Both systems were linearizable on Jacobians of hyperelliptic curves; see also [27, 28]. These systems were extended to isospectral flows on difference operators [46], also Lax pairs and Hamiltonian flows possessing many constants of the motion in involution; these flows linearize on Jacobians of more general set of curves. The Lax pair and the symplectic structure, depending on free parameter $h$, gave rise to the spectral curve, together with the whole family of invariants in involution. These and other examples pointed in the direction of Kac–Moody Lie algebras. Section 3 discusses examples coming from Kac–Moody Lie algebras, which moreover linearize on algebraic tori that are Jacobian of curves, hence are examples of algebraically integrable systems (a.c.i.) [8, 14]. General tools are discussed that are used to show when Lax-equations over Kac–Moody Lie algebras lead to systems linearizing on Jacobians of curves and hence a.c.i. systems [6, 7, 14]. Sections 4–6 deal with three examples from RMT [5, 11, 12] having to do respectively with the KdV, Toda and Pfaff Lattice, and the two-Toda system.
specialness of having coming from RMT also leads to the systems satisfying so-called Virasoro constraints, and the combination of the fact that the RMT examples can both be deformed to be integrable systems, solvable with τ-functions, which moreover satisfy Virasoro constraints, leads to PDEs for the so-called gap probabilities associated with the random matrix systems.

1. The AKS theorem

The AKS theorem first appeared in [1], then a more advanced version appeared in [40] and later an r-matrix version appeared in [39]. Some basic elements of it appeared in [29]. Interestingly enough, the first version was just an attempt to get a Lie algebra theorem that simultaneously yielded both the Toda and KdV cases, which at the time people realized were deeply related. The following version of the theorem is a synthesis of all the above, although not the most abstract version (but most useful), since we identify $g \sim g^\ast$, the Lie algebra with its dual, via an Ad-invariant non-degenerate bilinear form; it appears in [14]. The point of the theorem is that vector space decompositions of the Lie algebras, where the components are subalgebras, lead to integrable systems.

First some preliminaries are in order. Assume we are given an Ad-invariant\footnote{$\langle [X,Y],Z \rangle = \langle X,[Y,Z] \rangle$} non-degenerate bilinear form on a Lie algebra $g$, \begin{equation} \langle \ , \ \rangle : g \times g \to \mathbb{C}, \end{equation} which induces $\nabla F(X) \in g$ the gradient of $F$ at $X$ for functions $F$ on $g^* \cong g$ by \begin{equation} dF(X) = \langle \nabla F(X), dX \rangle \end{equation} and the Kostant–Kirillov Poisson structure on $g^* \cong g$ with respect to $\langle \ , \ \rangle$ by \begin{equation} \{ F,H \}(X) = \langle X,[\nabla F(X),\nabla H(X)] \rangle. \end{equation} The isomorphism $g \to g^*$ induced by $\langle \ , \ \rangle$ established a one-to-one correspondence between Ad$^\ast$-invariant functions on $g^*$ and Ad-invariant functions on $g$, and so the Hamiltonian vector fields $X_H$ on $(g^* \cong g, \{ \ , \ \})$ take the simple Lax form (which is why we identify $g^* \cong g$) \begin{equation} X_H(X) \equiv \dot{X} = [\nabla H(X),X]. \end{equation} Note that in general \begin{equation} X_H(F) = \langle \nabla F, X_H \rangle := \{ F,H \}. \end{equation} Also note that the Casimirs of $X_H$, i.e. the Hamiltonians that produce null vector-fields, and hence are constant along symplectic leaves, are precisely the Ad$^\ast$ invariant functions: $\mathcal{F}(g^* \cong g^\ast) \cong \mathcal{F}(g)$, characterized by $0 = X_H(X) = [\nabla H(X),X]$, $\forall X \in g$. Given a vector space splitting of the Lie algebra $g$ as follows, $g = g_+ \oplus g_-$ with $g_{\pm}$ Lie algebras, with the above a vector space direct sum (as opposed to a Lie algebra direct sum). In view of the non-degeneracy of $\langle \ , \ \rangle$, we furthermore have the vector space direct sum and isomorphisms:

\begin{equation} g^* \cong g^*_+ \oplus g^*_-, \quad \text{with} \quad g^\pm_\pm \cong g^*_{\mp}, \end{equation} where $g^\perp_\pm$ is the orthogonal complement (with respect to $\langle \ , \ \rangle$) of $g_\pm$, and therefore they carry the Kostant–Kirillov Poisson structure $\{ \ , \ \}_{g^\perp_\pm \cong g^*_{\mp}}$, namely,
With all the preliminaries out of the way, we can now state the AKS theorem.

Going with the decomposition \( g = g_+ \oplus g_- \) we have the projections \( P_\pm g \rightarrow g_\pm \), and now define \( R = P_+ - P_- \) and the new Lie algebra on \( g \)

\[
[X, Y]_R \overset{(*)}{=} \frac{1}{2} ([RX, Y] + [X, RY])
\]

with

\[
X_\pm = P_\pm X.
\]

Indeed, given the first definition of \([ \cdot, \cdot \]_R\), namely \((*)\), and a general linear map \( R : g \rightarrow g \), then it yields a Lie algebra (satisfies the Jacobi-identity) provided

\[
[RX, RY] - R([RX, Y] + [X, Y]) = -c[X, Y],
\]

the modified Yang–Baxter equation \((c = 0)\) in the Yang–Baxter equation. The case \( R = P_+ - P_- \) corresponds to \( c = 1 \) (which by rescaling can always be assumed if \( c \neq 0 \)), but in general if we set \( c = 1 \) and assume \( R \) satisfies \((*)\), then

\[
g_\pm := \{ X \pm RX \mid X \in g \}
\]

are always Lie subalgebras of \( g = g_+ \oplus g_- \) and then \( R = P_+ - P_- \), with \( P_\pm \) the projections onto the \( g_\pm \). The version of the AKS theorem given below shall only work for the case \( R = P_+ - P_- \). Below the subscript \((\cdot)_R\) shall indicate the structure induced by \([\cdot, \cdot]_R\).

The Kostant–Kirillov structure induced by \([\cdot, \cdot]_R\) and \((\cdot)_R\) on \( g^* \cong g \) when \( R = P_+ - P_- \) is given by (remember \( P_- \) are projections onto \( g_\pm \) along \( g_\mp \))

\[
\mathcal{X}_H(X)|_R = \hat{P}_-([\nabla H]^+, X) - \hat{P}_+([\nabla H]^-, X),
\]

since by \((1.5)\) and \((1.3)\)

\[
\langle \mathcal{X}_H(X)|_R, \nabla F \rangle = \mathcal{X}_H(F) = \{F, H\}_R
\]

\[
= \langle X, [\nabla F, \nabla H]_R \rangle = \langle X, [\nabla F]^+, [\nabla H]^+ \rangle - [\nabla F]^-, [\nabla H]^- \rangle
\]

\[
= \langle [\nabla H]^+, X], (\nabla F)^+ \rangle - ([\nabla H]^- X], \nabla F^- \rangle
\]

\[
= \langle \hat{P}_-([\nabla H]^+, X) - \hat{P}_+([\nabla H]^- X], \nabla F \rangle.
\]

With all the preliminaries out of the way, we can now state the AKS theorem.
Theorem 1.1 (AKS theorem on $g$). Suppose that $g = g_+ \oplus g_-$ is a Lie algebra splitting and that $(\cdot, \cdot)$ is an $Ad$-invariant non-degenerate bilinear form on $g$, leading to a vector space splitting
\[ g = g_+^\perp \oplus g_-^\perp. \] (1.11)

Let $F, H \in \mathcal{F}(g)^G$ and suppose that $\epsilon \in g$ satisfies
\[ [\epsilon, g_+] \in g_+^\perp, \quad [\epsilon, g_-] \in g_-^\perp. \] (1.12)

Setting $F_\epsilon(x) := F(\epsilon + X)$, then

1) $\{F, H\}_R = 0$ and $\{F_\epsilon, H\}_{g_+^\perp} = 0$; hence $[X_F, X_H] = 0$.
2) The Hamiltonian vector fields $X_H := \{\cdot, H\}_R$ and $X_{H_\epsilon} := \{\cdot, H_\epsilon\}_{g_+^\perp}$ are respectively given by the Lax equations
\[ X_H(X) = -\frac{1}{2}[X, R(\nabla H(X))] = \pm[X, (\nabla H(X))_{g_+^\perp}] \] (1.13)

and
\[ X_{H_\epsilon}(X) = -\frac{1}{2}[Y, R(\nabla H(Y))] = \pm[Y, (\nabla H(Y))_{g_+^\perp}], \] (1.14)

where $Y \in g_+^\perp \oplus \epsilon$, yielding two families of commuting vector fields.

3) For $X_0 \in g$ and for $|t|$ small, let $g_+(t)$ and $g_-(t)$ denote the smooth curves in $G_+$ resp. $G_-$ that solve the factorization problem
\[ \exp(-t\nabla H(X_0)) = g_+(t)^{-1}g_-(t), \quad g_\pm(0) = e. \] (1.15)

Then the integral curve of $X_H$ that starts at $X_0$ is given for $|t|$ small by
\[ X(t) = Ad_{g_+(t)}X_0 = Ad_{g_-(t)}X_0. \] (1.16)

Remark 1.1. Maybe the most amazing thing about the AKS theorem is the diversity of examples that it covers. The theorem should be understood to include the case where $g$ is infinite dimensional, where only the first part of (1), and (3) needs to be interpreted carefully (or formally), and we shall not dwell on such issues here.

Remark 1.2. The AKS theorem is thought of as a theorem about integrability, but classically integrable systems were solved by integration (quadrature) while the systems here are solved in (3) by the algebraic process of factoring an element in $G$, i.e. solving the splitting problem $G = G_+, G_-$, which can be done near their identity, and moreover through algebraic operations for finite dimensional groups or analytic operations for say Kac–Moody groups.

Remark 1.3. Classically complete integrability means, say on a Poisson manifold, that in addition to the Casimirs that define the symplectic leaves and lead to null Hamiltonian vector fields, there are $\frac{1}{2}$ (dimensions of the symplectic leaves) number of additional commuting integrals (generically independent). In many cases of the AKS theorem this is indeed the case, either for the generic symplectic leaves and/or degenerate symplectic leaves (like on the classical Toda system). It would be nice to have some general theorem that gives useful hypotheses to ensure complete integrability, especially as $\mathcal{F}(g)^G$ often does not provide enough integrals. An interesting and beautiful such example is [21], where the action of a parabolic subgroup supplies additional integrals.
Remark 1.4. It would be wonderful to quantize AKS in a meaningful way to actually yield in a consistent and uniform way quantized ‘integrable’ systems.

Remark 1.5. In may examples, as we shall see, you need $\epsilon \neq 0$, although in the original Toda lattice, you may take $\epsilon = 0$, even though $\epsilon \neq 0$ first appears in that case; hence it is important to include the case $\epsilon \neq 0$ in the theorem. Secondly, even though we need $R = P_+ - P_-$, the original case of the theorem, the $R$ matrix version of the theorem allows for more initial conditions $X \in g$ than the original case where $Y \in g^\perp + \epsilon$, and that flexibility is useful in many examples, as we shall see. Another source of flexibility in the theorem is that any choice of Ad-invariant inner product (i.e. not just the Killing form) will do, and that effects the flow, as it effects the form of $\nabla H$.

Remark 1.6. It is not always possible to say what examples cannot be covered by a theorem or some natural generalization of it, as for example, the Kowalevsky top was eventually shown in [18] to be an example of the AKS theorem, after some nontrivial effort by a number of authors. However, clearly the theorem in its present form does not cover any quantum integrable systems or systems related to supergroups, etc. There are, for example, many systems in random matrices that satisfy: differential equations through the use of the Riemann–Hilbert method, and it is hard to claim that these systems would not eventually be covered by some integrability theorem, whether it be AKS or some other theorem to be discovered. We do not understand integrability well enough to really answer such questions or even offer idle speculation.

2. Examples of the AKS theorem

2.1. Example 1: the KP and Gel’fand Dickey hierarchies

This subsection deals with the KP hierarchy and its invariant subsystems, the Gel’fand–Dickey hierarchies [25]. The KP equation describes water waves in two space dimensions where the water feels the bottom of its container, such as waves at the beach or tsunami waves generated by an undersea earthquake through displacement of the ocean bottom, traveling across the ocean at approximately the speed of sound while maintaining their profile to a very high degree. The KdV equation, the most famous of the Gel’fand–Dickey equations, deals with waves in one space dimension, such as you would find in a canal. In general, the Gel’fand–Dickey equations come up in various areas of mathematical physics, like string theory, random matrix theory, etc, no doubt due to their complete integrability.

Consider the Lie algebra

$$g = \left\{ \sum_{-\infty < i \leq \infty} a_i(x)D^i \mid a_i(x) \in R \right\}, \quad D = \frac{d}{dx},$$

(2.1)

the ring of pseudo-differential operators over $\mathbb{R}$ or $S^1$, with $R$ the ring of differential functions over $\mathbb{R}$ or $S^1$ and

$$g = g_+ \oplus g_-, \quad g_+ = \left\{ \sum_{0 \leq i \leq \infty} a_i(x)D^i \mid a_i(x) \in R \right\} = g^+ \cong g^-$$

$$g_- = \left\{ \sum_{-\infty < i \leq -1} a_i(x)D^i \mid a_i(x) \in R \right\} = g_-^+ \cong g^+_-.$$

(2.2)

\(^i \ll \infty\) means all $i$ are less than some finite $N$ for any element of $g_+$, but $N$ varies with the element.
The latter equalities are a consequence of (the Adler trace [1])

\[ \langle a, b \rangle = \text{tr}(ab), \quad \text{tr}(\Sigma a_i(x)D^f) = D^{-1}a_{-1}(x) := \int a_{-1}(x)dx, \]

(2.3)

and note that \( \text{tr}[a, b] = 0 \), so \( \langle \cdot, \cdot \rangle \) is Ad-invariant.

Also set \( \epsilon = D \), which satisfies (1.12). The functions

\[ H^{(\ell)}(a) = \frac{\text{tr} a^{\ell+1}}{\ell + 1} \in \mathcal{F}(g)^G, \]

(2.4)

since

\[ dH^{(\ell)}(a) = \langle a^\ell, da \rangle, \quad \nabla H^{(\ell)}(a) = a^\ell \implies [\nabla H^{(\ell)}(a), a] = 0, \]

and so the Hamiltonian vector fields of theorem 1.1

\[ X_{H^{(\ell)}} := \{\cdot, H^{(\ell)}\}_{\epsilon} \]  

on \( Y = D + \sum_{i=1}^{\infty} a_iD^{-i} \in g^\perp + \epsilon, \)

given by (1.14), take the form

\[ \frac{\partial Y}{\partial t_\ell} := X_{H^{(\ell)}}(Y) = \pm[(Y^\ell)_\pm, Y], \]

(2.5)

which is nothing but the \( \ell \)th flow of the KP hierarchy [38].

Hence the KP hierarchy is an ‘integrable’ Hamiltonian system, with Hamiltonian structure given by the Kostant–Kirillov symplectic structure on \( g_- = g^\perp \cong g^*_+, \) namely, (1.7). Indeed, setting \( X = \sum_{i=1}^{\infty} a_iD^{-i} \in g_- \), the Hamiltonian vector field generated by \( H(X) \) is given by:

\[ X_{H}(X) = P_-[\nabla_+ H, X], \quad \nabla_+ H(\bullet) = \sum_{i=1}^{\infty} D^{-1}(\frac{p_i}{D_{a_i}} \bullet), \]

(2.6)

The latter using

\[ dH(X) := \sum_{i=1}^{\infty} \frac{DH}{D_{a_i}}da_i =: (dX, \nabla_+ H), \quad \nabla_+ H \in g^+_+, \quad dX = \sum_{i=1}^{\infty} da_iD^{-i}, \]

while the \( \ell \)th flow of the hierarchy is generated by the Hamiltonian

\[ H^{(\ell)}_D(X) = (\text{tr}(D + X)^{\ell+1})/({\ell + 1}). \]

(2.7)

Of course (1.16) of the AKS theorem is only a formal solution to the hierarchy. As far as we know, the above result is new.

Note that \( Y^\prime \in g^+_+ \) is invariant under the KP hierarchy, and then the flows are called the \( n \)-Gel’fand–Dickey hierarchy. We can think of these as flows in their own right, living in \( g^+_+ = g^+_+ \cong g^*_-, \) so in the AKS theorem we just interchange the role of \( g^+_+ \). Now the induced Hamiltonian structure (1.7) on \( g^+_+ \) is given by

\[ X_{H}(X) = P_+[\nabla_- H, X], \quad X = \sum_{0 \leq i < \infty} a_iD^i, \quad \nabla_- H = \sum_{i=1}^{\infty} D^{-i-1} \left( \frac{DH}{D_{a_i}} \bullet \right). \]

(2.8)

\[ \text{Note that } D^{-a}(D^a(g^*_-)) = \sum_{i=0}^{\infty} (-i)(D^a)D^{-a-i}(g^*_-). \]
Note

\[ A_n = \left\{ \sum_{i=0}^{n} a_i D^i, \text{ with } a_n = 1, a_{n-1} = 0 \right\} \quad (2.9) \]

is an invariant manifold of the (co-adjoint and hence the) Hamiltonian action on \( g_\perp^+ \simeq g_\perp^- \) given by \( \mathcal{X}_H \), i.e. a union of symplectic leaves, sometimes called a Poisson subspace. So we may consider \( \mathcal{X}_{H|_{B_p}} \) (the Gel’fand–Dickey symplectic structure as described in [1]) and setting \( \epsilon = 0 \) in (1.12) and

\[ H_0^{(\ell)}(X) = -\frac{n}{n+\ell} \text{tr} X^{\ell+1}, X \in A_n, \quad (2.10) \]
yields by (1.14), with \( \ell \mapsto \ell \pm \) in AKS the \( \ell \)th flow of the Gel’fand–Dickey hierarchy [25]

\[ \frac{\partial X}{\partial t^\ell} := \mathcal{X}_{H_0^{(\ell)}}(X) = \pm \left[ X, \left( X^{\ell+1} \right)_{\pm} \right]. \quad (2.11) \]

Once again (1.16) of the AKS theorem provides only a formal solution.

2.2. Example 2: the Toda lattice on \( g^\ell(n+1) \)

The famous Toda lattice was invented by Toda [41] as a simple model for a one-dimensional crystal in solid state physics with nearest neighbor interactions given by the exponential function. Both the periodic and nonperiodic lattice, as well as various Lie algebra versions have proven to be important in mathematical physics and [17] enumerative geometry dealing with string theory. The system is also important in orthogonal polynomial theory [10] and random matrix theory [9].

Here we set \( g = g^\ell(n+1) = g_+ \oplus g_- \), where \( g_+ \) (respectively \( g_- \) = Lie algebra of lower triangular matrices (resp. Lie algebra of strictly upper triangular matrices), while \( \langle A, B \rangle = \text{tr} AB \); hence \( g_+^\epsilon \) (resp. \( g_-^\epsilon \)) \( \cong g_+^\epsilon \) (resp. \( g_-^\epsilon \)) = Lie algebra of strictly lower triangular matrices (resp. upper triangular matrices), and \( \epsilon = \text{matrix with all 1’s one below the main diagonal and all other entries are 0. Here } n \text{ may be infinite.}

Since the induced Hamiltonian structure on \( g_\perp^\pm \cong g_+^\epsilon \) is given by (1.7)

\[ \mathcal{X}_H(X) = \hat{P}_- [\nabla_+ H, X], \]

with \( X \in g_\perp^\pm \), \( \hat{P}_- \) projection onto \( g_\perp^\pm (g = g_\perp^\pm \oplus g_\perp^\pm) \) and

\[ dH = \langle dX, \nabla_+ H \rangle, \nabla_+ H \in g_+, \]

then

\[ B_p = \{ X \in g_\perp^\pm \text{ with at most } p \text{ bands} \} \]
is an invariant manifold of the Hamiltonian action given by \( \mathcal{X}_H \), so we may consider \( \mathcal{X}_H|_{B_p} \).

If we set

\[ H_0^{(\ell)}(X) = -\frac{\text{tr}(\epsilon + X)^{\ell+1}}{\ell + 1}, \quad Y = \epsilon + X, \]

the equations (1.14) of the AKS theorem yield

\[ \frac{\partial}{\partial t^\ell} Y := \mathcal{X}_{H_0^{(\ell)}}(Y) = \pm [Y, (Y^\ell)_{\pm}], \quad X \in B_2 \quad (2.12) \]
(in the style of [29]) the Toda hierarchy, with \( \ell = 1 \) the classical Toda equation:

\[
\dot{Y} = [Y, Y_+] = -[Y, Y_-],
\]

where

\[
Y := \begin{pmatrix}
  b_1 & a_1 & 0 \\
  1 & b_2 & a_2 \\
  & \ddots & \ddots \\
  & & 1 & b_n & a_n \\
  0 & & & 1 & b_{n+1}
\end{pmatrix}
\]

and where

\[
Y_- = \begin{pmatrix}
  0 & a_1 & 0 \\
  0 & 0 & a_2 \\
  & \ddots & \ddots \\
  & & 0 & 0 & a_n \\
  0 & & & 0 & 0
\end{pmatrix}, \quad Y_+ = \begin{pmatrix}
  b_1 & 0 & 0 \\
  1 & b_2 & 0 \\
  & \ddots & \ddots \\
  & & 1 & b_n & 0 \\
  0 & & & 1 & b_{n+1}
\end{pmatrix}.
\]

Formula (1.16) of AKS yields the explicitly solution of the hierarchy. It is worth noting that the Toda equations can be gotten from another decomposition in the style of [1], namely,

\[
g = g_\ell (n + 1) = g_+ \oplus g_-.
\]

where

\[
g_+ (\text{resp. } g_-) = \text{lower triangular matrices with diagonal }\text{resp. skew-symmetric matrices},
\]

with \((\cdot, \cdot)\) the same, hence

\[
g_+^* (\text{resp. } g_-^*) \cong g_+^\perp (\text{resp. } g_-^\perp) = \text{lower triangular matrices with no diagonal }\text{resp. symmetric matrices},
\]

and \(\epsilon = 0\). The induced Hamiltonian structure (1.7) on \(g_+^\perp\) is given by

\[
H_\ell(X) = \tilde{P}_- [\nabla_+ H, X], \quad dH =: (dX, \nabla_+ H), \quad \nabla_+ H \in g_+.
\]

Observe

\[
A_p = \{X \in g_+^\perp \text{ with at most } p \text{ bonds above and below the diagonal} \}
\]

is an invariant manifold of the Hamiltonian action, so we may consider \(X_\ell|_{A_p}\). If we set

\[
H_\ell^{(\ell)}(X) = -\frac{1}{2} \text{ tr } X^{\ell+1}, \quad X \in g_+^\perp,
\]

the equations (1.14) of the AKS theorem

\[
\frac{\partial X}{\partial t_\ell} := X_\ell^{(\ell)}(X) = \pm \frac{1}{2} [X, (X^\ell)_\pm], \quad X \in A_2
\]
yield the Toda hierarchy with $\ell = 1$ the classical Toda equations, but in slightly different coordinates than with the previous splitting. This Lax equation first appeared in [23, 24]. Once again (1.16) of AKS yields the explicit solution of the hierarchy.

2.3. Example 3: the two-Toda lattice

The two-Toda lattice, a deep generalization of Ueno and Tagasaki [44] of the one-Toda lattice, occurs in various areas of mathematical physics, such as random matrix theory [22, 11], Gromov–Witten theory and Hurwitz number [35], as well as matrix models in string theory, to name a few examples. We finally need the $R$-matrix version of the AKS theorem.

Consider the splitting of the algebra $g$ of pairs $(P_1, P_2)$ of infinite $(\mathbb{Z} \times \mathbb{Z})$ or semi-infinite $(N \times \mathbb{N})$ matrices such that $(P_1)_{ij} = 0$ for $j - i \gg 0$ and $(P_2)_{ij} = 0$ for $i - j \gg 0$, used in [11]; with:

\[
\begin{align*}
g &= g_+ + g_-, \\
g_+ &= \{(P_1, P_2) \mid P_{ij} = 0 \text{ if } |i - j| \gg 0\} = \{(P_1, P_2) \in g \mid P_1 = P_2\}, \\
g_- &= \{(P_1, P_2) \mid (P_1)_{ij} = 0 \text{ if } j \geq i, \ (P_2)_{ij} = 0 \text{ if } i > j\},
\end{align*}
\]

with $(P_1, P_2) = (P_1, P_2)_+ + (P_1, P_2)_-$ given by

\[
(P_1, P_2)_+ = (P_{1u} + P_{2u}, P_{1u} + P_{2u}),
\]
\[
(P_1, P_2)_- = (P_{1\ell} - P_{2\ell}, P_{2u} - P_{1u});
\]

$P_+$ and $P_-$ denote the upper (including diagonal) and strictly lower triangular parts of the matrix $P$, respectively.

Take for $(\ , \ )$ on $g$, $(\ , \ )_1 + (\ , \ )_2$, with $(A, B)_i = \operatorname{tr} AB$ on the $i$th components of $g$; i.e. $(\ , \ )_1$ just decouples (as does the Lie bracket) so it is Ad-invariant. Then let $L = (L_1, L_2)$ be the running variables on $g \cong g^*$ and consider the (formal) Hamiltonians

\[
H_n^{(i)}(L) = \frac{\operatorname{tr} L_n^{i+1}}{n + 1}, \quad i = 1, 2, n = 1, 2, \ldots
\]

Then under the Hamiltonian vector-fields $X_{H_n^{(i)}(L)}$ of (1.13) we find

\[
\frac{\partial L}{\partial t_n} =: \{X_{H_n^{(1)}(L)}\}_{[0, L_n^{1+1}]},
\]
\[
\frac{\partial L}{\partial s_n} =: \{X_{H_n^{(2)}(L)}\}_{[0, L_n^{2+1}]},
\]

which are deformations of a pair of infinite matrices, restricted as follows:

\[
L = (L_1, L_2) = \left( \sum_{-\infty < i \leq 1} a_i^{(1)} \Lambda^i, \sum_{-1 \leq i < \infty} a_i^{(2)} \Lambda^i \right) \in g,
\]

with $\Lambda$ the shift operator and where $a_i^{(1)}$ and $a_i^{(2)}$ are diagonal matrices depending on $i = (i_1, i_2, \ldots)$ and $s = (s_1, s_2, \ldots)$, such that

\[
a_i^{(1)} = I \text{ and } (a_i^{(2)})_{nn} \neq 0 \text{ for all } n;
\]

that is to say matrices of the above form are an invariant manifold of the flows, and the flows restricted to this manifold are called the two-Toda flows [44]. While $\Lambda = (\delta_{j=i+1})$, in the

\footnote{So $(P_i)_{ij} = 0$ for $j > i$ eventually, i.e. when $j - i$ is sufficiently large, etc.}
semi-infinite case we need to set \( \Lambda^\dagger - i = (\Lambda^\dagger)^\dagger \) for \( i \geq 1 \), and once again, the flows restrict to (2.23) in the semi-infinite case, yielding the semi-infinite two-Toda flows. Of course the Hamiltonians may not converge, in which case they are ‘formal Hamiltonians’, but the flows make perfectly good sense in any case, and they all commute.

2.4. Example 4: the Pfaff lattice

The Pfaff lattice comes up as the natural integrable system that is the deformation class for the GOE and GSE examples in random matrix theory [13], just as the Toda system is the natural integrable system with which to deform the GUE case of random matrix theory. Similarly it is the natural deformation class of skew-orthogonal polynomials, just as the Toda system is the natural deformation class of orthogonal-polynomials. There is an associated tau-function theory for this hierarchy and it fits in naturally to the other Sato hierarchies [20].

First consider the Lie algebra \( D = g_{\ell \infty} \) of semi-infinite matrices, viewed as being composed of \( 2 \times 2 \) blocks. It admits the natural decomposition into subalgebras as follows [2]:

\[
D = D_\pm \oplus D_0 \oplus D_\pm = D_0^\pm \oplus D_0^\mp \oplus D_+^\mp D_+, \tag{2.24}
\]

where \( D_0 \) has \( 2 \times 2 \) blocks along the diagonal with zeroes everywhere else, and where \( D_+ \) (resp. \( D_- \)) is the subalgebra of upper-triangular (resp. lower-triangular) matrices with \( 2 \times 2 \) zero matrices along \( D_0 \) and zero below (resp. above). As we point out in (2.24), \( D_0 \) can further be decomposed into two Lie subalgebras as follows:

\[
D_0^- = \{ \text{all } 2 \times 2 \text{ blocks } \in D_0 \text{ are proportional to Id} \},
\]

\[
D_0^+ = \{ \text{all } 2 \times 2 \text{ blocks } \in D_0 \text{ have trace 0} \}. \tag{2.25}
\]

Consider the following: the semi-infinite skew-symmetric matrix \( J \), zero everywhere, except for the following \( 2 \times 2 \) blocks, along the ‘diagonal’,

\[
J = \begin{pmatrix}
0 & 1 & & \\
-1 & 0 & & \\
& & \ddots & \\
0 & 1 & & -1 & 0
\end{pmatrix} \in D_0^+, \text{ with } J^2 = -I; \tag{2.26}
\]

and the associated Lie algebra order 2 involution

\[
\mathcal{J} : D \to D : a \mapsto \mathcal{J}(a) := Ja^\dagger J. \tag{2.27}
\]

The splitting of \( D \) into two Lie subalgebras\(^7\) (with corresponding projections \( P_{\pm} \))

\[
g = g_+ + g_- \text{ with } g_+ = D_- + D_0^- \text{ and } g_- = \{ a + \mathcal{J}(a), \ a \in D \} = \text{sp}(\infty), \tag{2.28}
\]

with corresponding Lie groups\(^8\) \( G_+ \) and \( G_- = \text{Sp}(\infty) \), plays a crucial role here. Notice that \( g_- = \text{sp}(\infty) \) and \( G_- = \text{Sp}(\infty) \) stand for the infinite rank affine symplectic algebra group. Let

\(^7\)Note that \( g_- \) is the fixed point set of \( \mathcal{J} \).

\(^8\)\( G_+ \) is the group of invertible elements in \( g_+ \), i.e. lower-triangular matrices, with nonzero \( 2 \times 2 \) blocks proportional to \( \text{Id} \) along the diagonal.
formally speaking) $\langle A, B \rangle = \text{tr} AB$ be the Ad-invariant inner product. The applying (1.13) of the AKS theorem yields the flows [2]:

$$\frac{\partial L}{\partial t_i} = [P_+ \nabla H_i, L] = [-P_- \nabla H_i, L], \quad H_i = -\frac{\text{tr} L_i^{i+1}}{i+1}$$

(2.29)

on matrices $L = Q \wedge Q^{-1}$, with $Q \in G_+$ and $\Lambda$ the customary shift operator. We call these equations the Pfaff Lattice. Note that $L$ of the above form are preserved by (2.29), as follows from proposition 2.1.

Remembering the decomposition (2.24), write $a \in D, a = a_- + a_0 + a_+$, while for any element $a \in g = g_+ + g_-$, write $a = P_+ a + P_- a$, and then we have that

$$a = a_- + a_0 + a_+ = P_+ a + P_- a$$

and so we can write the Pfaff lattice (2.29) more explicitly:

$$\frac{\partial L}{\partial t_i} = \left[ -\left( (L)_- \mathcal{J} (L')_+ \right) - \frac{1}{2} \left( (L')_0 - \mathcal{J} (L')_0 \right), L \right]$$

$$= \left[ \left( (L')_+ + \mathcal{J} (L')_+ \right) + \frac{1}{2} \left( (L')_0 + \mathcal{J} (L')_0 \right), L \right].$$

(2.30)

We have the followings proposition [2]:

**Proposition 2.1.** For the matrices

$L := Q \wedge Q^{-1}$ and $m := Q^{-1} J Q^{-1} T$, with $Q \in G_+$,

the following three statements are equivalent:

(i) $\frac{\partial Q}{\partial t_i} = -P_+ L_i$,
(ii) $L_i + \frac{\partial Q}{\partial t_i} Q^{-1} \in g_-$,
(iii) $\frac{\partial m}{\partial t_i} = \Lambda^i m + m \Lambda^N$

which yields the following:

**Theorem 2.2.** Consider the skew-symmetric solution

$m_\infty(t) = e^{\sum t^i \Lambda^i} m_\infty(0) e^{\sum t^i \Lambda^i},$

to the commuting equations

$$\frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty + m_\infty \Lambda^N,$$

(2.31)

with skew-symmetric initial condition $m(0)$ and its skew-Borel decomposition

$$m_\infty = Q^{-1} J Q^{-1} T,$$ with $Q \in G_+$.

(2.32)

Then the matrix $Q$ evolves according to the equations

$$\frac{\partial Q}{\partial t_i} Q^{-1} = -P_+ (Q \wedge^i Q^{-1})$$

(2.33)

9 The Hamiltonians $H_i$ are viewed as formal sums; the convergence of this formal sum would require some sufficiently fast decay of the entries of $L$. Since $\nabla H_i = L_i$, one does not need to be concerned about this point.
and the matrix \( L := Q \wedge Q^{-1} \) provides a solution to the Lax pair

\[
\frac{\partial L}{\partial t_i} = [-P_i^L, L] = [P_i^L, L].
\] (2.34)

Conversely, if \( Q \in G_\infty \) satisfies (2.33), then \( m_\infty \), defined by (2.32), satisfies (2.31). \( \square \)

### 3. A.C.I. examples of the AKS theorem

This section deals with integrable systems [14] that linearize on algebraic tori and, in particular, are solved by Lax-equations involving a formal parameter, i.e. Lax-equations on Kac–Moody Lie algebras. This gives rise to an invariant algebraic curve (constant as time evolves) such that the systems are linearized on the Jacobians of these curves. Historically, the periodic Toda and KdV were studied in [27, 28, 34] foreshadowing more general techniques, such as in [44]. The periodic matrices defining the Jacobians ultimately yield the fundamental physical quasi-periods of these integrable systems. Effective techniques for solving these equations are given.

Set

\[
g := \left\{ \sum_{-\infty < i < \infty} a_i h^i \mid a_i + \text{gl}(n, \mathbb{R} \text{ or } \mathbb{C}) \right\} = g_+ + g_-, \]

with

\[
g_- := \left\{ \sum_{0 \leq i < \infty} a_i h^i \in g \right\} = g_{-} \cong g_{+}^*, \]

\[
g_+ := \left\{ \sum_{-\infty < i < -1} a_i h^i \in g \right\} = g_+ \cong g_-.\]

Here \( h \) is a formal parameter, with the Ad-invariant form on \( g \)

\[
\langle a, b \rangle := \text{coef}_{h^0} (\text{tr}(abh)) = \sum_{i+j+1=0} (a_i, b_j),
\]

with \( (\ ,\ ) \) being the Killing form on \( \text{gl}(n) \), and

\[
\left[ \sum_i a_i h^i, \sum_j b_j h^j \right] = \sum_{i,j} [a_i, b_j] h^{i+j}.
\]

Thus the Hamiltonian structure on \( g_{-} \cong g_{+}^* \) given by (1.7) is

\[
\mathcal{H}(X) = P_- [\nabla_+ H(X), X],
\]

with the manifold

\[
C_m(\alpha, \gamma) := \alpha h^m + \gamma h^{m-1} + A_{m-1}
\]

invariant under the Hamiltonian action, so \( \mathcal{H} \) restricts to \( C_m(\alpha, \gamma) \), with (Casimirs) \( \alpha \) and \( \gamma \) two fixed diagonal matrices and

\[
A_{m-1} := \left\{ \sum_{j=0}^{m-1} a_j h^j \mid \text{diag}(a_{m-1}) = 0 \right\},
\]

with \( \alpha = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) \) with \( \prod_{i<j} (\alpha_i - \alpha_j) \neq 0 \). Taking Hamiltonians of the form
\[ H(a) = \langle f(ah^{-1}), h^2 \rangle, \quad a \in C_m(\alpha, \gamma), \] (3.1) with \( f \) ‘nice’ and applying (1.14) of the AKS theorem we find [6]
\[ \dot{a} = \mathcal{X}_\alpha(a) = [a, f'(ah^{-1})h^{k-1}]_-, \quad a \in C_m(\alpha, \gamma), \]
(3.2)
and upon setting \( j = m, k = m + 1 \), this yields
\[ \dot{a} = [a, b + \beta h], \quad \beta := f'(\alpha), \quad b := \text{ad}_\alpha^{-1}a_{m-1} + f''(\alpha)\gamma, \]
\[ b_{ij} = (1 - \delta_i)(\beta_i - \beta_j)(\alpha_i - \alpha_j)^{-1}(a_{m-1})_{ij} + \delta_i\gamma d''(\alpha_i), \]
(3.3)
\[ H(a) = \langle f(ah^{-m}), h^{m+1} \rangle. \]
(3.4)
If in the above we set \( m = 1, \gamma = 0, \beta = \alpha^{1/2} \), we find the Euler–Arnold spinning top for the Lie algebra \( \text{gl}(n) \), while if we set \( A_0 = -A_0^\dagger \), we arrive at the Euler top for \( u(n) \). Moreover let us set for \( x, y \in \mathbb{H}^n \) or \( \mathbb{C}^n \)
\[ \Gamma_{xy} = x \otimes y - y \otimes x, \quad \Gamma_{xx} = x \otimes x, \quad \Gamma_{yy} = y \otimes y, \quad \Delta_{xy} = x \otimes y + y \otimes x, \]
and let us call the following differential equations,
\[ \dot{\Gamma}_{xy} = [\Gamma_{xy}, \Gamma], \quad \Gamma_{ij} = (\Gamma_{xy})_0(J_i + J_j)^{-1}, \]
(3.6) for \( J_i \) > 0, the ‘special’ Euler equations. In what follows \( J_i = \sqrt{\alpha_i} \), and we arrive at the following theorem of [6]:

**Theorem 3.1.** The Lax equation \( \dot{a} = [a, \Gamma + \beta h] \), with \( a = a(h), \beta = \text{diag}(\beta_1, \ldots, \beta_n) \), \( \Gamma = \text{ad}_\beta \text{ad}_\alpha^{-1} \Gamma_{xy} \) of (3.3) corresponds to (a) Euler equations, (b) the geodesic flow on the ellipsoid and the Neumann problem, and (c) the central force problem on the ellipsoid respectively, with

(a) \[ a = \alpha h + \Gamma_{xy} \]
(b) \[ a = \alpha h^2 + h\Gamma_{xy} - \Gamma_{xx} \]
(c) \[ a = \alpha h^2 + h\Gamma_{xy} + \Delta_{xy} - \alpha \]
(3.7)
and with the respective Hamiltonians (3.4) of the form

(a) \[ H = \left( \frac{2}{3} (ah^{-1})^{3/2}, h^2 \right), \quad f(x) = \frac{2}{3} x^{3/2} \]
(b) \[ H = \langle \ln(ah^{-1}), h^2 \rangle, \quad f(x) = \ln(x) \text{ (geodesic)} \]
(c) \[ H = \left( \frac{1}{2} (ah^{-1})^2, h^2 \right), \quad f(x) = \frac{1}{2} x^2 \text{ (Neumann)} \]
(3.8)

In order to study the equations (3.2) we could apply (1.16) of the AKS theorem, thinking of \( h \) as a complex variable and use the Birkhoff factorization theorem in the style of [36, 37], but instead we should take advantage of the additional algebraic geometrical structure in (3.2). This leads [7, 14] us to the following propositions, theorems and corollaries, which enable us to solve flows of the form (3.2) as linear flows on algebraic tori which are Jacobians of curves.

**Proposition 3.2.** Given a Lax pair defining the flow
\[ \dot{X}(h) = [X(h), Y(h)], \quad X(h), Y(h) \in \text{gl}(N)[h, h^{-1}], \]
(3.9)
the functions $q_{k\ell}$ that are defined by the coefficients of the characteristic polynomial of $X(h)$,

$$\det(z \mathbf{Id}_N - X(h)) = z^N + \sum_{\ell \leq \ell' \leq \ell} q_{k\ell} h^\ell z^\ell$$

are constants of motion of the flow. The plane algebraic curve, associated to each $X(h)$,

$$\Gamma_X := \{(h, z) \in \mathbb{C} \times \mathbb{C} \mid \det(z \mathbf{Id}_N - X(h)) = 0\},$$

is preserved by the flow. Similarly, for each $X(h)$ the ‘isospectral’ variety of matrices $A_c \subset M$

defined by

$$A_c := \{X'(h) \mid X(h) \text{ and } X'(h) \text{ have the same characteristic polynomial}
\text{ with all } q_{k\ell} = c_{k\ell}\}$$

is preserved by the flow. For $X \in A_c$ such that $\Gamma_X =: \Gamma_c$ is smooth, let us denote its smooth
compactification by $\overline{\Gamma}_c$ and let

$$\{p_1, \ldots, p_s\} := \overline{\Gamma}_c \setminus \Gamma_c$$

denote the points at infinity. At each of these points $h$ has a zero or a pole, i.e. (possibly after
relabeling) we have that

$$\text{ord}_{p_i}(h) = \begin{cases} -\mu_i & 1 \leq i \leq s' \\ \mu_i & s' + 1 \leq i \leq s \end{cases}$$

where $\mu_i > 0$ for $i = 1, \ldots, s$.

Assume generically in $c$, that $\Gamma_c$ is nonsingular, and for a generic point $(h, z)$ on $X(h) \in A_c$,
the eigenspace $\xi = \xi(z, X(h))$ of $X(h)$ with eigenvalue $z$ is one dimensional. By Cramer’s rule,
$\xi = (\xi_1 \cdots \xi_k)$, normalized such that $\xi_1 = 1$, is a meromorphic function on $\overline{\Gamma}_c$.

For a generic $X(h) \in A_c$, with corresponding normalized eigenvector $\xi$, let $D_X$ be the minimal
effective divisor on $\Gamma_c$ such that

$$(\xi_\ell)_{\Gamma_c} \geq -D_X \quad \text{for all } \ell = 1, \ldots, N;$$

by continuity, $d := \deg(D_X)$ is independent of $X = X(h) \in A_c$ and thus, $D_X$ defines an effective
divisor of degree $d$ in $\Gamma_c$ for any $X = X(h) \in A_c$. The point is to study the motion of the
divisor $D_X$ in $\Gamma_c$, when $X(h)$ is moving in $A_c$. Roughly speaking, $D_X$ is the divisor of poles of
the normalized eigenvector $\xi_\ell(z; X(h))$ on $\Gamma_c$, not at infinity. Note for non-generic $X(h)$ the divisor
$D_X$ may contain one or several of the points $p_i$ at infinity.

Choose a divisor $D_0 \in \text{Div}^d(\overline{\Gamma}_c)$ and a basis $(\omega_1, \ldots, \omega_k)$ of holomorphic differentials on $\overline{\Gamma}_c$ and let $\vec{\omega} := (\omega_1, \ldots, \omega_k)^\top$. Define the linearizing map

$$J_c : A_c \to \text{Jac}(\overline{\Gamma}_c)$$

$$X \mapsto \int_{D_0}^{D_X} \vec{\omega}.$$ 

For example, one may choose a base point $q$ on $\overline{\Gamma}_c$ and take $D_0 := dq$. Then the map is given by

$$J_c(X) = \sum_{i=1}^d \int_{q}^{q_i} \vec{\omega} \in \text{Jac}(\overline{\Gamma}_c),$$
where $\mathcal{D}_X = q_1 + \cdots + q_d$.

It is easy to check from (3.9) that $\xi(t) = \xi(z, X(h, t))$ satisfies and defines a function $\lambda$ as follows:

\[ \dot{\xi} + Y\xi = \lambda\xi, \quad (3.10) \]

$Y = Y(h, X(h, t))$, with $\lambda$ a scalar function of $(h, z, t)$. This leads to the following theorems.

**Theorem 3.3.** Along the integral curves $X(t)$ of the Lax equation $\dot{X} = [X, Y]$, the derivative of the linearizing map is given by

\[ \frac{d}{dt} \int_{\mathcal{D}_X(0)} \overline{\omega} = s \sum_{i=1}^{s} \text{Res}_{p_i} \lambda(h, z, t)\overline{\omega}. \]

**Theorem 3.4 (Linearization criterion).** The map $J_c$ linearizes the spectral flow $\dot{X} = [X, Y]$ on $A_c$, that is to say

\[ \int_{\mathcal{D}_X(0)} \overline{\omega} = t \sum_{i=1}^{s} \text{Res}_{p_i} \lambda(h, z, X(h, 0))\overline{\omega}, \]

if and only if there exists for each $X \in A_c$ a meromorphic function $\phi_X$ on $\Gamma_c$ with $(\phi_X)_{\Gamma_c} > -n \sum_{i=1}^{d} \mu p_i + n' \sum_{i=d+1}^{s} \mu p_i$ such that for all $p_i$,

\[ (\text{Laurent tail of } \frac{d\lambda(h, z, X)}{dt} \text{ at } p_i) = (\text{Laurent tail of } \phi_X \text{ at } p_i), \]

where $(\Delta_{k,f}(z, X(h, t)))$ being the $(k, f)$ cofactor of $\lambda \text{Id}_N - X(h, t))$

\[ \frac{d\lambda(h, z, X)}{dt} = \frac{d}{dt} \left( \sum_{f=1}^{N} Y_{1f}(h, z, X(h, t)) \frac{\Delta_{1f}(z, X(h, t))}{\Delta_{11}(z, X(h, t))} \right), \]

$d/dt$ being computed using the Lax equation $\dot{X} = [X, Y]$.

**Corollary 3.5.** Suppose that $h$ has no zero at infinity and that there exists a polynomial $p(x, y, z)$ whose coefficients are arbitrary constants of the motion, and that there exists an algebraic function $\Psi$, whose coefficients are arbitrary constants of the motion, such that

\[ Y(h) = \Psi(p(X, h, h^{-1})) + (C_0 + C_1 h^{-1} + C_2 h^{-2} + \cdots) \]

where $C_0$ is a lower triangular matrix, and where the matrices $C_1, C_2, \ldots$ are arbitrary. If $\xi/h$ has no pole at the points $p_i$, then the linearization criterion is satisfied.

Theorem 3.4 (due to Griffiths [26]) can be used to show that the flows in (3.2) and hence (3.3) and theorem 3.1 linearize on Jac$(\Gamma_c)$, while corollary 3.5 is more special but actually applies to many other cases found in [31, 32, 40, 46], like the general periodic Toda flows. This brings us to the following definitions, where $\{,\}$ is a Poisson bracket on the manifold $M$ and the $F_i$ are integrals in involution.

**Definition 3.6.** Let $(M, \{,\}, F)$ be a complex integrable system, where $M$ is a non-singular affine variety and where $F = (F_1, \ldots, F_s)$. We say that $(M, \{,\}, F)$ is an *algebraic completely integrable system* or an a.c.i. system if for generic $c \in \mathbb{C}^s$ the fiber $F_c$ defined by $F = c$ is an affine part of an Abelian variety and if the Hamiltonian vector fields $\chi_{F_i}$ are translation
invariant, when restricted to these fibers. In the particular case in which $M$ is an affine space $\mathbb{C}^n$ we will call $(\mathbb{C}^n, \{\cdot, \cdot\}, F)$ a polynomial a.c.i. system. When the generic Abelian variety of the a.c.i. system is irreducible we speak of an irreducible a.c.i. system.

**Definition 3.7.** Let $(M, \{\cdot, \cdot\}, F)$ be a complex integrable system, where $M$ is a (non-singular) affine variety, and where $F = (F_1, \ldots, F_k)$. We say that $(M, \{\cdot, \cdot\}, F)$ is a generalized a.c.i. system if for generic $c \in \mathbb{C}^n$ the integrable vector fields $\mathcal{X}_{F_1}, \ldots, \mathcal{X}_{F_k}$ define the local action of an algebraic group on $F_c$.

Theorems 3.4 and corollary 3.5 are general tools to show systems like (3.2) and (3.3) are a.c.i. systems in the appropriate coordinates (those of $M$), which in fact are meromorphic functions on the Abelian variety $\text{Jac}(\Gamma_c)$. For instance, in the case of theorem 3.1, the appropriate coordinates are $\chi_i^1, y_i^2$ and $x_{ij}, 1 \leq i \leq n$. The Toda hierarchy (2.12) is a generalized a.c.i. system, with the appropriate coordinates being the $a_i$ and $b_i$.

### 4. Random matrices, limiting distributions and KdV

The point of this section is to get partial differential equations for the spectral gap probabilities for the Airy and Bessel processes, which are “universal” limiting processes in the GUE ensemble respectively at the hard and soft edges. In particular, for the one-interval gap case we recover respectively the Painlevé II and V equations going with the Tracy–Widom distribution and the Bessel distribution. Since these distributions are universal, they appear in many other contexts in statistical mechanics. The integrable deformation class of these universal distributions is seen to be the KdV equation and hence its vertex operator and Virasoro symmetries as well as Sato’s KP theory play a crucial role in the derivation of the partial differential equations.

Define on the ensemble $\mathcal{H}_N = \{N \times N \text{ Hermitian matrices}\}$ the probability

$$ P(M \in dM) = ce^{-TrV(M)}dM, $$

where $c$ is a normalization constant. Then for $z_1, \ldots, z_N \in \mathbb{R}$ we have

$$ P(\text{one eigenvalue in each } [z_i, z_i + dz_i], i = 1, \ldots, N) = c \text{ vol}(U(N))e^{-\sum_i V(z_i)} \Delta^2(z)dz_1 \cdots dz_N $$

and for $0 \leq k \leq N$,

$$ P(\text{one eigenvalue in each } [z_i, z_i + dz_i], i = 1, \ldots, k) = c' \left( \int_{\mathbb{R}^{n-k}} \cdots \int e^{-\sum_{i} V(z_i)} \Delta^2(z)dz_{k+1} \cdots dz_N \right) dz_1 \cdots dz_k $$

$$ = c'' \det(K_N(z_i, z_j))_{1 \leq i, j \leq k} dz_1 \cdots dz_k, $$

and if $J \subset \mathbb{R}$, then

$$ P(\text{exactly } k \text{ eigenvalues in } J) = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \lambda^k} \det(I - \lambda K_N^J) |_{\lambda = 1}, $$

where

$$ K_N^J(z, z') = K_N(z, z')I_J(z'). $$

$I_J$ the indicator function of $J$, and $K_N$ is the Schwartz kernel of the orthogonal projector $\mathbb{C}[z] \to \mathbb{C} + \mathbb{C}z + \cdots + \mathbb{C}z^{N-1}$ with respect to the measure $e^{-TrV(z)}dz$, namely
\[ K_N(z, z') = \sum_{k=0}^{N-1} \varphi_k(z) \varphi_k(z') \]

in terms of orthonormal functions \( \varphi_k(z) = e^{-V(z)/2}p_k(z) \) with respect to \( dz \) or orthogonal polynomials \( p_k(z) = (1/\sqrt{\theta_k})z^k + \cdots \) with respect to \( e^{-V(z)}dz \).

When \( V(z) \) is quadratic, and more generally convex, we have for large \( N \) [47]

\[
P(\text{an eigenvalue } \in [z, z + dz]) = K_N(z, z)dz \\
\sim \begin{cases} 
\frac{1}{\pi} (2N - z^2)^{1/2}dz & \text{if } |z| < (2N)^{1/2} \\
0 & \text{if } |z| > (2N)^{1/2} 
\end{cases}
\]

is given by the circular distribution (Wigner’s semi-circle law). We have that for \( z \sim 0 \) the average spacing between the eigenvalues near the origin is \( \sim (K_N(0, 0))^{-1} = \pi/\sqrt{2N} \) and near the edge \( (z \sim \sqrt{2N}) \) is \( 1/(2^{1/2}N^{1/6}) \), leading to

\[
\lim_{N \to \infty} \frac{1}{K_N(0, 0)} K_N \left( \frac{z}{K_N(0, 0)}, \frac{z'}{K_N(0, 0)} \right) \\
= K(z, z') = \frac{1}{\pi} \frac{\sin \pi (z - z')}{z - z'} \quad \text{(bulk scaling limit)},
\]

\[
\lim_{N \to \infty} \frac{1}{K_N(0, 0)} K_N \left( \sqrt{2N} + \frac{z}{21/2N^{1/6}}, \sqrt{2N} + \frac{z'}{21/2N^{1/6}} \right) \\
= \int_0^\infty A(x + z)A(x + z')dx \quad \text{(edge scaling limit)},
\]

in terms of the classical Airy function. In a similar context one also finds the Bessel kernel; for background on such matters, consult [15].

This section deals with computing PDEs for the gap probabilities given by Fredholm determinants involving the limiting ‘universal’ Airy and Bessel kernel, which appear now in a variety of circumstances. Miraculously these gap probabilities are given by ‘continuous soliton formulas’ of the KP equation and in particular the KdV equation, i.e. the 2-Gel’fand–Dickey equations. This plus Virasoro constraints built into the very special KdV solutions associated with these kernels yield the PDEs. We shall be dealing with the KP hierarchy, briefly explained in section 2: remember it is a hierarchy of isospectral deformations of a pseudodifferential operator \( L = D + \sum_{j \geq 1} a_j(x, t)D^{-j} \), with \( D := d/dx \),

\[
\frac{\partial L}{\partial t_n} = [[L^n]_+, L] \quad \text{for } t \in C^\infty.
\]

We also consider the \( p \)-Gel’fand–Dickey hierarchy, i.e. the reduction to \( L \)’s such that \( L^p \) is a differential operator for some fixed \( p \geq 2 \). Sato tells us that the solution \( L \) to the KP equations can ultimately be expressed in terms of a \( \tau \)-function. The wave and adjoint wave functions, expressed in terms of the \( \tau \)-function [20]

\[
\Psi(x, t, z) = e^{z\xi + \sum_{j=1}^\infty \omega_j \tau(t - [z^{-1}])}/\tau(t), \quad \Psi^*(x, t, z) = e^{-z\xi - \sum_{j=1}^\infty \omega_j \tau(t + [z^{-1}])}/\tau(t)
\]

satisfy

\[
z\Psi = L\Psi \quad \frac{\partial \Psi}{\partial t_n} = (L^n)_+ \Psi \quad \frac{\partial \Psi^*}{\partial t_n} = -(L^n)_+ \Psi.
\]

(4.1)
As in the general theory of integrable systems, vertex operators play a prominent role: they are Darboux transforms involving all times. In particular, for the KP equation, the vertex operator

\[
X(t, y, z) := \frac{1}{z - y} e^{\sum_{\nu} (\zeta - y^{\nu})}\kappa_{\nu}^{\sum_{\nu} (\nu^{\nu} - y^{\nu})^{\nu}} \frac{\partial}{\partial \nu}
\]  

(4.3)

has the striking feature that \(X(t, y, z)\tau\) and \(\tau + X(t, y, z)\tau\) are both \(\tau\)-functions. Given distinct roots of unity \(\omega, \omega' \in \mathbb{C}_p := \{\omega \mid \omega^p = 1\}\), the vertex operator \(X(t, \omega, \omega')\) maps the space of \(p\)-Gel’fand–Dickey \(\tau\)-functions into itself.

We also note that the 2-Gel’fand–Dickey KP equation satisfied by \(\tau\) is

\[
\left(\frac{\partial}{\partial t_1}\right)^4 - 4 \omega^2 \frac{\partial^2}{\partial t_1 \partial t_3} \log \tau + 6 \left(\frac{\partial^2}{\partial t_1^2} \log \tau\right)^2 = 0.
\]  

(4.4)

We have the two basic theorems [5]:

**Theorem 4.1.** Define the \((x, t)\)-dependent kernel \(k_{x,t}(y, z)\) and \(k^E_{x,t}(y, z) := k_{x,t}(y, z)1_E(z)\) with \(x \in \mathbb{R}, t \in \mathbb{C}^\infty, y, z \in \mathbb{C},\) and \(E \subset \mathbb{R}^+\) a Borel subset:

\[
k_{x,t}(y, z) := \int dx \sum_{\omega \in \mathbb{C}_p} a_\omega \Psi^*(x, t, \omega y) \sum_{\omega' \in \mathbb{C}_p} b_{\omega'} \Psi(x, t, \omega' z),
\]  

(4.5)

where \(\Psi(x, t, z)\) and \(\Psi^*(x, t, z)\) are the wave and adjoint wave function for the \(p\)-Gel’fand–Dickey hierarchy and where the coefficients \(a_\omega, b_{\omega'} \in \mathbb{C}\) are subjected to \(\sum_{\omega \in \mathbb{C}_p} a_\omega b_{\omega'} = 0\).

Then the following holds:

(i) The kernel \(k(y, z)\) is a Fredholm determinant and its Fredholm determinant is all three expressible in terms of the vertex operators

\[
Y(t, y, z) := \sum_{\omega, \omega' \in \mathbb{C}_p} a_\omega b_{\omega'} X(t, \omega y, \omega' z)
\]  

(4.6)

function:

\[
k_{x,t}(y, z) = \frac{1}{\tau} Y(t, y, z)\tau
\]

\[\det(k_{x,t}(y_i, z_j))_{1 \leq i, j \leq n} = \frac{1}{\tau} Y(t, y_1, z_1) \cdots Y(t, y_n, z_n)\tau
\]

\[\det(I - \mu E_{x,t}) = \frac{1}{\tau} e^{-\mu \int dx \cdot Y(t, z)\tau} \quad \text{('continuous' soliton formula)}.
\]  

(4.7)

(ii) Let the kernel \(k_{x,t}(y, z)\) in (4.5) be such that the underlying \(\tau\)-function of \(\Psi\) and \(\Psi^*\) satisfies a Virasoro constraint\(^\text{10}\):

\[
W_{bp}^{(2)} \tau = c_{bp}\tau \text{ for a fixed } k \geq -1.
\]

\(^{10}\text{Define } W_{ab}^{(n)} := \delta_{ab}.\)

\[
J_n^{(1)} := W_n^{(1)} = \begin{cases} \frac{\partial}{\partial n} & \text{if } n > 0 \\ (-n)^{-n} & \text{if } n < 0 \\ 0 & \text{if } n = 0 \end{cases}, \quad J_n^{(2)} := W_n^{(2)} + (n + 1)W_n^{(1)} = \sum_{i+j=n} J_i^{(1)} f_j^{(1)}.
\]  

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Then for the disjoint union $E = \bigcup_{i=1}^{r} [a_{2i-1}, a_{2i}] \subset \mathbb{R}^+$, the Fredholm determinant $\det(I - \mu_k^E)$ satisfies the following Virasoro constraint for that same $k \geq -1$:

$$
\left(-\sum_{i=1}^{2r} a_{2i}^{p+1} \frac{\partial}{\partial a_i} + \frac{1}{2} (W_{2p}^{(2)} - c_{2p}) \right) (\tau \det(I - \mu_k^E)) = 0;
$$

(4.8)

\( \text{note the boundary a-part and the time t-part decouple.} \)

In the next theorem we apply equation (4.8) to compute the partial differential equations for the distribution of the spectrum for matrix ensembles whose probability is given by a kernel. To state the problem, consider a first-order differential operator $A$ in $z$ of the form

$$
A = A_z = \frac{1}{2} z^{m+1} \left( \frac{\partial}{\partial z} + V'(z) \right) + \sum_{i \geq 1} c_{-2i} z^{-2i},
$$

(4.9)

with

$$
V(z) = \frac{\alpha}{2} z + \frac{\beta}{6} z^j \neq 0, \quad m = \deg V' = 0 \text{ or } 2,
$$

(4.10)

and the differential part of its ‘Fourier’ transform

$$
\hat{A} = \hat{A}_z = \left( \frac{1}{2} (x + V'(D)) D^{-m+1} + \sum_{i \geq 1} c_{-2i} D^{-2i} \right) \quad \text{with } D = \frac{\partial}{\partial x}.
$$

(4.11)

Given a disjoint union $E = \bigcup_{i=1}^{r} [a_{2i-1}, a_{2i}] \subset \mathbb{R}^+$, define differential operators $A_n$, which we declare to be of homogeneous ‘weight’ $n$, as follows:

$$
A_n := \sum_{i=1}^{2r} A_i^{n+1} \frac{\partial}{\partial A_i}, \quad n = 1, 3, 5, \ldots
$$

We can now state the second main theorem.

**Theorem 4.2.** Let $\Psi(x, z), \ x \in \mathbb{R}, \ z \in \mathbb{C}$ be a solution of the linear partial differential equation

$$
A_z \Psi(x, z) = \hat{A}_z \Psi(x, z),
$$

(4.12)

with holomorphic (in $z^{-1}$) initial condition at $x = 0$, subjected to the following differential equation for some $a, b, c \in \mathbb{C}$,

$$
(a A_z^2 + b A_z + c) \Psi(0, z) = z^2 \Psi(0, z), \ \text{with} \ \Psi(0, z) = 1 + O(z^{-1}).
$$

(4.13)

Then

- $\Psi(x, z)$ is a solution of a second order problem for some potential $q(x)$

$$
(D^2 + q(x)) \Psi(x, z) = z^2 \Psi(x, z).
$$

(4.14)

- Given the kernel

$$
K^E_k(y, z) := I_k^E(z) \int y \Phi(x, \sqrt{y} \Phi(x, \sqrt{z}) \frac{dx}{2^{y/4} z^{1/4}},
$$

(4.15)
with
\[\Phi(x, u) := \sum_{\omega = \pm 1} b_\omega e^{i\omega V(x)} \Psi(x, \omega u),\]
the Fredholm determinant \( f(A_1, \ldots, A_{2r}) := \det(I - \lambda K^E_x) \) satisfies a hierarchy of bilinear partial differential equations\(^{11}\) in the \( A_i \) for odd \( n \geq 3 \):
\[
f \cdot A_n A_1 f - A_n f \cdot A_1 f - \sum_{i+j=n+1} p_i(\tilde{A}) f \cdot p_j(-\tilde{A}) f = 0,
\]
\[\text{where } x \text{ appears in the coefficients of lower weight terms only.}\]

As an application of theorem 4.2 we find [5]

**Theorem 4.3.** Given the Airy kernel
\[K^E_y(z) = I_\varepsilon(z) \frac{1}{2\pi} \int_x^y A(x + y) A(x + z) dx,\]
the Fredholm determinant \( f(A_1, \ldots, A_{2r}) := \det(I - \lambda K^E_x) \) satisfies the hierarchy of bilinear partial differential equations in the \( A_i \) for odd \( n \geq 3 \):
\[
f \cdot A_n A_1 f - A_n f \cdot A_1 f - \sum_{i+j=n+1} p_i(\tilde{A}) f \cdot p_j(-\tilde{A}) f = 0,
\]
with
\[A_n = \sum_{i=1}^{2r} A_i^{n+1} \frac{\partial}{\partial A_i}, \quad n = 1, 3, 5, \ldots.\]

The variables appearing in the Schur polynomials \( p_i \) are non-commutative and are written according to a definite order. Finally, the first equation in the hierarchy (4.16) takes on the following form:
\[
\left(A_1^3 - 4 \left(A_3 - \frac{1}{2}\right)\right) R + 6(A_1 R)^2 = 0
\]
(4.19)
for
\[R := A_1 \log f = \sum_{i=1}^{2r} \frac{\partial \log \det(I - \lambda K^E_x)}{\partial A_i}.
\]

When \( E = (-\infty, A) \), the function \( R = A_1 \log f = \frac{\partial}{\partial A_1} \log \det(I - K^E) \) satisfies
\[R'' - 4AR' + 2R + 6R^2 = 0 \quad \text{(Painlevé II).}\]

\(^{11}\) The \( p_i \) are the elementary Schur polynomials \( e^{\sum_{r=1}^\infty c^r} = \sum_{r=0}^\infty p_r(t) z^r \), and \( p_i(\pm \tilde{A}) := p_i(\pm A_1, 0, \pm 1, A_3, 0, \ldots) \).
The Painlevé II equation for the logarithmic derivative had been obtained previously by Tracy and Widom [42]; equation (4.19), which leads to Painlevé, is new.

**Proof.** Setting in (4.10) \( V(z) = \frac{2}{3}z^3 \), from (4.9) and (4.11) find

\[
A_z := \frac{1}{2z} \left( \frac{\partial}{\partial z} + 2z^2 \right) - \frac{1}{4}z^{-2} \quad \text{and} \quad \hat{A}_x = \frac{\partial}{\partial x}.
\]  

(4.20)

Then in terms of the Airy function

\[
\Psi(x, z) = \frac{1}{\sqrt{\pi}} e^{-\frac{2}{3}z^3} \sqrt{z} F(x + z^2)
\]

and is a solution of

\[
A_z \Psi(x, z) = \hat{A}_x \Psi(x, z),
\]

with \( \Psi(0, z) \) satisfying

\[
A_z^2 \Psi(0, z) = z^2 \Psi(0, z) \quad \text{and} \quad \Psi(0, z) = 1 + O(z^{-1}).
\]

Setting in (4.5) \( b_+ = 1 \) and \( b_- = 0 \), we find for \( \Phi(x, u) \) and \( \mathcal{K}_x^E(y, z) \) in (4.15),

\[
\Phi(x, u) = \sqrt{u} \sqrt{\pi} A(x + u^2)
\]

Then from theorem 4.2, \( f(A_1, \ldots, A_{2r}) := \text{det}(I - \lambda \mathcal{K}_x^E) \) satisfies the hierarchy of equations (4.16), with lower weight terms; the \( A_n \) are defined by (4.18). However, upon rewriting the variables of \( p_n \) in an appropriate way, all lower weight terms can be removed, as follows from a combinatorial argument.

Finally from (4.8) with \( p = 2, \ k p \rightarrow k, \ a_{i^p} \rightarrow A_i^k, \ E \rightarrow E^2, \ t_3 \rightarrow t_3 + \frac{2}{3}, \ \tau^E := \tau \text{det}(I - \lambda \mathcal{K}_x^E) \) satisfies

\[
A_1 \log \tau^E = \frac{1}{2} \left( \sum_{i \geq 3} i t_i \frac{\partial}{\partial t_{i-2}} + 2 \frac{\partial}{\partial t_1} \right) \log \tau^E + \frac{t_1^2}{4}
\]

\[
A_3 \log \tau^E = \frac{1}{2} \left( \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} + 2 \frac{\partial}{\partial t_3} \right) \log \tau^E + \frac{1}{16}
\]

from which the partial derivatives \( \frac{\partial}{\partial t_0} \log \tau^E, \frac{\partial}{\partial t_0} \log \tau^f \) and \( \frac{\partial^2}{\partial t_0 \partial t_0} \log \tau^f \) at \( t = 0 \) can be extracted. Putting these partials in the KP-equation (4.4) leads to the equation (4.19). In the particular case of a semi-infinite interval \( (-\infty, A) \), one finds the Painlevé II equation.

□

We also have another application [5] of theorem 4.2:
Theorem 4.4. Given the (Bessel) kernel
\[ KE(y, z) = -\frac{1}{2} I_e(z) \int_0^1 s J_\nu(s \sqrt{y}) J_\nu(s \sqrt{z}) ds, \]  
(4.21)
the Fredholm determinant \( f(A_1, \ldots, A_{2r}) := \det(I - \lambda K^E) \) satisfies the hierarchy (4.16) of bilinear partial differential equations in the \( A_i \) for odd \( n \geq 3 \), with
\[ A_n := \sum_{i=1}^{2r} A_i \frac{\partial}{\partial A_i}, \quad n = 1, 3, 5, \ldots \]  
(4.22)
The first equation in the hierarchy (4.16) for \( F := \log \det(I - \lambda K^E) \) takes on the following form:
\[ \left( A_1 - 2A_2^2 + (1 - \nu^2)A_3 + A_3 \left( A_1 - \frac{1}{2} \right) \right) F - 4(A_1 F)(A_2^2 F) + 6(A_2^2 F)^2 = 0; \]  
(4.23)
when \( E = (0, A) \), we have for \( R := -A_1 F = -A_2 \frac{\partial}{\partial A_1} \log \det(I - \lambda K^E) \), the equation [43]
\[ A^2 R''' + AR'' + (A - \nu^2)R' - R \frac{3}{2} + 4RR' - 6AR^2 = 0 \]  
(Painlevé V).

Proof. Pick in (4.10) \( V(z) = -z \); then from (4.9) and (4.11)
\[ A_z = \frac{1}{2} \left( \frac{\partial}{\partial z} - 1 \right) \]
and \( \hat{A}_z = \frac{1}{2} (x - 1) \frac{\partial}{\partial x} \).
We look for a function \( \Psi(x, z) \) satisfying
\[ A_z \widehat{\psi}(x, z) = \hat{A}_z \psi(x, z) \]
(4.24)
with initial condition \( \psi(0, z) \) satisfying
\[ \left( 4A_z^2 - 2A_z - \frac{1}{4} \right) \psi(0, z) = z^2 \psi(0, z), \quad \psi(0, z) = 1 + O \left( \frac{1}{z} \right). \]  
(4.25)
The solution to the differential equation (4.25) is given by
\[ \psi(0, z) = B(z) = \varepsilon \sqrt{z} H_{\nu}(iz) = \frac{e^{\nu + 1/2}}{\Gamma(-\nu + 1/2)} \int_1^{\infty} u^{-\nu + 1/2} e^{-uc} du \]  
\[ = 1 + O \left( \frac{1}{z} \right) \]
with \( \varepsilon = i \sqrt{\pi/2} e^{i\pi/4} \), \( -\frac{1}{2} < \nu < \frac{1}{2} \). Then
\[ \Psi(x, z) = e^{x} B((1 - x)z) \]
satisfies (4.24); from theorem 4.2, \( \psi(x, z) \) satisfies a second order spectral problem, which can explicitly be computed:
\[
\left( \frac{d^2}{dx^2} - \frac{(\nu^2 - \frac{1}{4})}{(x-1)^2} \right) \Psi(x, z) = z^2 \Psi(x, z).
\]

Picking in (4.5) \( b_+ = e^{-i\pi \nu /2 \sqrt{\pi}} \) and \( b_- = i \bar{b}_+ \), yields for (4.15)

\[
\Phi(x, z) = \frac{e^{-i\pi \nu/2}}{2\sqrt{\pi}} e^{-iz} \Psi(x, z) + \frac{i e^{i\pi \nu/2}}{2\sqrt{\pi}} e^{iz} \Psi(x, -z)
\]

\[
= \sqrt{\frac{(x-1)z}{2}} J_\nu((1-x)iz),
\]

and

\[
K_E^E = I_E(z) \int_1^x \Phi(x, \sqrt{y}) \Phi(x, \sqrt{z}) \frac{dy}{2\sqrt{2z^{1/4}}} dx
\]

\[
= -\frac{1}{2} I_E(z) \int_0^{1-x} d^{-1}J_\nu(s\sqrt{y})J_\nu(s\sqrt{z}) ds.
\]

The special value \( x = i + 1 \) leads to the standard Bessel kernel:

\[
K_{i+1}^E = -\frac{1}{2} I_E(z) \int_0^1 d^{-1}J_\nu(s\sqrt{y})J_\nu(s\sqrt{z}) ds
\]

\[
= I_E(z) \left( J_\nu(\sqrt{z})\sqrt{z} J'_\nu(\sqrt{z}) - J_\nu(\sqrt{y})\sqrt{y} J'_\nu(\sqrt{y}) \right) \frac{2}{2(z-y)}.
\]

From theorem 4.2, the Fredholm determinant

\[
f(A_1, \ldots, A_{2r}) := \det(I - \lambda K_{i+1}^E)
\]

satisfies equation (4.16), with

\[
A_n = \sum_{i=1}^{2r} A_{n+i} \frac{\partial}{\partial A_i}, \quad n = 1, 3, 5, \ldots
\]

Picking the special value \( x = i + 1 \) leads to the shift \( t_i \mapsto t_i + i + 1 \) and (4.8) with \( p = 2 \), \( kp \mapsto k \), \( a_i^{p} \mapsto A_i^{p} \), etc leads to

\[
A_1 \log \tau^E = \frac{1}{2} \left( \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} + \sqrt{-1} \frac{\partial}{\partial t_1} \right) \log \tau^E + \frac{1}{4} \left( \frac{1}{4} - \nu^2 \right)
\]

\[
A_3 \log \tau^E = \frac{1}{2} \left( \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_{i+2}} + \frac{1}{2} \frac{\partial^2}{\partial t_1^2} + \sqrt{-1} \frac{\partial}{\partial t_3} \right) \log \tau^E
\]

\[
+ \frac{1}{4} \left( \frac{\partial}{\partial t_1} \log \tau^E \right)^2;
\]

expressing the partial derivatives appearing in (4.4) at \( t = 0 \) in terms of the operators \( A_1 = \sum_{i=1}^{2r} A_i \frac{\partial}{\partial A_i} \) and \( A_3 = \sum_{i=1}^{2r} A_i^2 \frac{\partial}{\partial A_i} \) leads to the partial differential equation (4.23), which for \( E = (0, A) \) leads to the Painlevé V equation, ending the proof of theorem 4.4.  \qed
5. The distribution of the spectrum in Hermitian, symmetric and symplectic random ensembles and Toda and Pfaff lattices

In this section we derive partial differential equations and partial differential-recursion relations for the spectral gap probabilities for the Gaussian and Laguerre ensemble for the Hermitian, symmetric and symplectic cases of random matrix ensemble [12]. The crucial tool is that the deformation classes for these ensembles are the (integrable) Toda and Pfaff lattices, and we rely heavily on the tau-function theory of these lattices as well as the Virasoro symmetries coming from the gauge transformations inherent in random matrix integrals.

Consider the weights of the form \( \rho(z) dz := e^{-V(z)} dz \) on an interval \( F = [A, B] \subseteq \mathbb{R} \), with rational logarithmic derivative and subjected to the following boundary conditions:

\[
- \frac{\rho'(z)}{\rho(z)} = V' = \sum_{j=0}^{\infty} b_j z^j, \quad \lim_{z \to A, B} \frac{f(z)}{\rho(z)} z^k = 0 \text{ for all } k \geq 0,
\]

(5.1)
together with a disjoint union of intervals,

\[
E = \bigcup_{i=1}^r [c_{2i-1}, c_{2i}] \subseteq F \subseteq \mathbb{R}.
\]

(5.2)

The data (5.1) and (5.2) define an algebra of differential operators

\[
B_k = \sum_{j=1}^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i}.
\]

(5.3)

Let \( \mathcal{H}_n, \mathcal{S}_n \) and \( \mathcal{T}_n \) denote the Hermitian \( (M = M^\top) \), symmetric \( (M = M^\top) \) and ‘symplectic’ \( (M = M^\top, M = JMJ^{-1}) \) ensembles, respectively. Traditionally, the latter is called the ‘symplectic ensemble’, although the matrices involved are not symplectic! These conditions guarantee the reality of the spectrum of \( M \). Then, \( \mathcal{H}_n(E), \mathcal{S}_n(E) \) and \( \mathcal{T}_n(E) \) denote the subsets of \( \mathcal{H}_n, \mathcal{S}_n \) and \( \mathcal{T}_n \) with spectrum in the subset \( E \subseteq F \subseteq \mathbb{R} \). The aim of this section is to find PDEs for the probabilities

\[
P_n(E) := P_n(\text{all spectral points of } M \in E) = \int_{\mathcal{H}_n(E), \mathcal{S}_n(E) \text{ or } \mathcal{T}_n(E)} e^{-\text{tr} V(M)} dM
\]

\[
= \int_{\mathcal{H}_n(F), \mathcal{S}_n(F) \text{ or } \mathcal{T}_n(F)} e^{-\text{tr} V(M)} dM
\]

\[
= \int_{F_1} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-V(z_k)} dz_k, \quad \beta = 2, 1, 4 \text{ respectively}
\]

(5.4)

for the Gaussian, Laguerre and Jacobi weights.

The method used in [12] to obtain these PDEs involves inserting time-parameters into the integrals, appearing in (5.4) and to notice that the integrals obtained satisfy

- Virasoro constraints: linear PDEs in \( t \) and the boundary points of \( E \), and
- integrable hierarchies: satisfied by matrix integrals:

| ensemble       | \( \beta \) | lattice |
|----------------|------------|---------|
| Hermitian      | \( \beta = 2 \) | Toda    |
| symmetric      | \( \beta = 1 \) | Pfaff   |
| symplectic     | \( \beta = 4 \) | Pfaff   |
As a consequence of duality between $\beta$-Virasoro generators under the map $\beta \mapsto 4/\beta$, the PDEs obtained have a remarkable property: the coefficients $Q$ and $Q_i$ in the PDEs are functions of the variables $n, \beta, a, b$, and have the invariance property under the map

$$n \rightarrow -2n, \ a \rightarrow -\frac{a}{2}, \ b \rightarrow -\frac{b}{2};$$

to be precise,

$$Q_i\left(-2n, \beta, -\frac{a}{2}, -\frac{b}{2}\right)\bigg|_{\beta=1} = Q_i(n, \beta, a, b)\bigg|_{\beta=4}.$$ (5.5)

**Important remark.** For $\beta = 2$, the probabilities satisfy PDEs in the boundary points of $E$, whereas in the case $\beta = 1, 4$, the equations are inductive. Namely, for $\beta = 1$ (resp. $\beta = 4$), the probabilities $P_{n+2}$ (resp. $P_{n+1}$) are given in terms of $P_{n-2}$ (resp. $P_{n-1}$) and a differential operator acting on $P_n$.

### 5.1. Virasoro constraints

**Theorem 5.1 (Adler–van Moerbeke [11]).** The multiple integrals

$$I_n(t, c; \beta) := \int_{E^n} |\Delta_n(z)|^\beta \prod_{k=1}^n \left( e^{\sum_{i=1}^{\infty} t_i^i \rho(z_k)} \right) \text{ for } n > 0$$ (5.6)

and

$$I_n\left(t, c; \frac{4}{\beta}\right) := \int_{E^n} |\Delta_n(z)|^{4/\beta} \prod_{k=1}^n \left( e^{\sum_{i=1}^{\infty} t_i^i \rho(z_k)} \right) \text{ for } n > 0,$$ (5.7)

with $I_0 = 1$, satisfying respectively the following Virasoro constraints$^{12}$ for all $k \geq -1$:

$$\left(-B_k + \sum_{i \geq 0} \left(a_i, \beta \delta_{k+i,n}^{(2)}(t, n) - b_i, \beta \delta_{k+i+1,n}^{(1)}(t, n)\right) \right) I_n(t, c; \beta) = 0,$$

$$\left(-B_k + \sum_{i \geq 0} \left(a_i, \beta \delta_{k+i,n}^{(2)} \left(-\frac{\beta t}{2}, -\frac{2n}{\beta}\right) \right) + \frac{\beta b_i}{2}, \beta \delta_{k+i+1,n}^{(1)} \left(-\frac{\beta t}{2}, -\frac{2n}{\beta}\right) \right) I_n\left(t, c; \frac{4}{\beta}\right) = 0,$$ (5.8)

in terms of the coefficients $a_i, b_i$ of the rational function $(-\log \rho)'$ and the end points $c_i$ of the subset $E$, as in (5.1)–(5.3). For all $n \in \mathbb{Z}$, the $\beta \delta_{k,n}^{(2)}(t, n)$ and $\beta \delta_{k,n}^{(1)}(t, n)$ form a Virasoro and a Heisenberg algebra respectively, with central charge

$$c = 1 - 6 \left(\frac{\beta}{2}\right)^{1/2} - \left(\frac{\beta}{2}\right)^{-1/2}\right)^2.$$ (5.9)

$^{12}$ When $E$ equals the whole range $F$, then the $B_k$’s are absent in the formula (5.8).
Remark 5.1. The $\beta j_k^{(2)}$'s are defined as follows:

$$
\beta j_{k,n}^{(2)} = \beta \sum_{i+j=k} \beta j_{i,n}^{(1)} j_{j,n}^{(1)} + \left(1 - \frac{\beta}{2}\right) \left( k + 1 \right) j_{k,n}^{(0)} - k j_{k,n}^{(0)}.
$$

Componentwise, we have

$$
\beta j_{k,n}^{(1)}(t,n) = \beta j_{k}^{(1)}(t) + n j_{k}^{(0)}(t) \text{ and } \beta j_{k,n}^{(0)} = n \delta_{k,n}
$$

and hence

$$
\beta j_{k,n}^{(2)}(t,n) = \left(\frac{\beta}{2}\right) \beta j_{k}^{(2)}(t) + \left(n \beta + (k + 1) \left( 1 - \frac{\beta}{2} \right) \right) \beta j_{k}^{(1)}(t) + n \left( n - 1 \right) \beta j_{k}^{(0)}(t).
$$

Setting

$$
d I_n(x) := |\Delta_n(x)|^\beta \prod_{k=1}^n \left( e^{\sum_{i=1}^{\infty} \beta z_i^k \rho(x_k) dx_k} \right).
$$

Theorem 5.1 is based on the following variational formula:

$$
\frac{d}{dt} d I_n(\xi_i \mapsto \xi_i + \epsilon f(\xi_i) \xi_i^{k+1}) \bigg|_{\epsilon=0} = \sum_{\ell=0}^\infty \left( a_{\ell} \beta j_{k}^{(2)}(t) - b_{\ell} \beta j_{k}^{(1)}(t) \right) d I_n.
$$

5.2. Matrix integrals and associated integrable systems

5.2.1. Hermitian matrix integrals and the Toda lattice. Given a weight $\rho(z) = e^{-\lambda(z)}$ defined as in (5.1), the inner-product

$$
\langle f, g \rangle_i = \int_E f(z) g(z) \rho_i(z) dz, \quad \text{with } \rho_i := e^{\sum_{i=1}^\infty \beta z_i^k \rho(z)}
$$

leads to a moment matrix

$$
m_n(t) = (\mu_{ij})_{0 \leq i,j < n} = (\langle z_i^k, z_j^l \rangle)_{0 \leq i,j < n},
$$

which is a Hänkel matrix, thus symmetric. Hänkel is tantamount to $\Lambda m_\infty = m_\infty \Lambda^\top$. The semi-infinite moment matrix $m_\infty$ evolves in $t$ according to the equations

$$
\frac{\partial \mu_{ij}}{\partial t} = \mu_{i+k,j}, \text{ and thus } \frac{\partial m_\infty}{\partial t} = \Lambda^4 m_\infty \left( \text{commuting vector fields} \right).
$$

Another important ingredient is the factorization of $m_\infty$ into a lower- times an upper-triangular matrix

$$
m_\infty(t) = S(t)^{-1} S(t)^\top.
$$

13 $\beta j_{k}^{(1)} = \frac{\mu_{k}}{\mu_{k+1}} + \frac{1}{\mu_{k}} (-\kappa) r_{-k}$

$$
\beta j_{k}^{(2)} = \sum_{i+j=k} \beta \frac{\partial}{\partial \mu_{i+j}} + \frac{1}{\mu_{k}} \sum_{i+j=k} \mu_{i} \frac{\partial}{\partial \mu_{i+j}} + \frac{1}{\mu_{k}} \sum_{i+j=k} \mu_{j} \frac{\partial}{\partial \mu_{i+j}}.
$$

14 Hänkel means $\mu_{ij}$ depends on $i+j$ only.

15 This factorization is possible for those $t$'s for which $\tau_n(t) := det m_n(t) = 0$ for all $n > 0$. 27
where $S(t)$ is lower-triangular with nonzero diagonal elements. The following theorem can be found in [9].

**Theorem 5.2.** The vector $\tau(t) = (\tau_n(t))_{n \geq 0}$, with

$$
\tau_n(t) := \det m_n(t) = \frac{1}{n!} \int_{\mathbb{C}^n} \Delta_n^2(z) \prod_{k=1}^{n} \rho_k(z_k) d\zeta_k
$$

satisfies

(i) Virasoro constraints (5.8) for $\beta = 2$,

$$
\left( -2 \sum_{i=1}^{2} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_1} + \sum_{i \geq 0} \left( a_{i,k+1}^{(2)} - b_{i,k+1}^{(1)} \right) \right) \tau = 0.
$$

(ii) The KP-hierarchy\(^{16}\)

$$
\left( p_{k+4}(\partial) - \frac{1}{2} \frac{\partial^2}{\partial t_k \partial t_{k+3}} \right) \tau_n \circ \tau_n = 0,
$$
of which the first equation reads

$$
\left( \frac{\partial}{\partial t_1} \right)^4 + 3 \left( \frac{\partial}{\partial t_2} \right)^2 - 4 \left( \frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \tau_n + 6 \left( \frac{\partial^2}{\partial t_1^2} \log \tau_n \right)^2 = 0,
$$

$k = 0, 1, 2, \ldots$.

(iii) The standard Toda lattice; i.e. the tridiagonal matrix

$$
L(t) := S(t) \Lambda S(t)^{-1} = \begin{pmatrix}
\frac{\partial}{\partial n_1} \log \frac{n_1}{n_2} & \left( \frac{n_2}{n_1} \right)^{1/2} & 0 \\
\left( \frac{n_2}{n_1} \right)^{1/2} & \frac{\partial}{\partial n_2} \log \frac{n_2}{n_3} & \left( \frac{n_3}{n_2} \right)^{1/2} \\
0 & \left( \frac{n_3}{n_2} \right)^{1/2} & \frac{\partial}{\partial n_3} \log \frac{n_3}{n_4} & \ddots
\end{pmatrix}
$$

satisfies the commuting equations\(^{17}\) (2.19)

$$
\frac{\partial L}{\partial t_k} = \left[ \frac{1}{2} (L^k)_{\cdots} , L \right].
$$

(iv) Orthogonal polynomials: The $n^{th}$ degree polynomials $p_n(t,z)$ in $z$, depending on $t \in \mathbb{C}^\infty$, orthonormal with respect to the $t$-dependent inner product (5.11)

$$
\langle p_k(t,z), p_l(t;z) \rangle = \delta_{kl},
$$

are eigenvectors of $L$, i.e. $(L(t) p(t;z))_n = \lambda p_n(t;z), n \geq 0$, and enjoy the following representations:

\(^{16}\) For the customary Hirota symbol $p(\partial_1) f \circ g := p \left( \frac{\partial}{\partial t} \right) f(t + y) g(t - y) \big|_{y=0}$, the $p_i$’s are the elementary Schur polynomials $\varphi_{\sum_{i=1}^{\infty} i^r} := \sum_{i=0}^{\infty} \varphi_{(i_1,i_2,\ldots)} z^i$ and $p_i(\partial_1) := p_i \left( \frac{\partial}{\partial n_1}, \frac{\partial}{\partial n_1}, \ldots \right)$.

\(^{17}\) ( )\(^{\ast}\) means take the skew-symmetric part of ( ) in the decomposition ‘skew-symmetric‘ + ‘lower-triangular‘.
The functions \( q_n(t; z) := z \int_{\mathbb{R}} \frac{p_n(zu)}{z-u} \rho(t) \, du \) are ‘dual eigenvectors’ of \( L \), i.e. \( \{L(t)q_n(t; z)\}_n = zq_n(t; z), \ n \geq 1 \), and have the following \( \tau \)-function representation (see the remark after (5.20)):

\[
q_n(t; z) := z \int_{\mathbb{R}} \frac{p_n(t; u)}{z-u} \rho(t) \, du = (S^{-1}(t)\chi(z^{-1}))_n = z^{-n}h_n^{-1/2} \frac{\tau_{n+1}(t + [z^{-1}])}{\tau_n(t)}, \tag{5.18}
\]

(v) Bilinear relations: for all \( n, m \geq 0 \), and \( a, b \in \mathbb{C}^\infty \), such that \( a - b = t - t' \),

\[
\int_{z=\infty} \frac{\tau_n(t - [z^{-1}]) \tau_{m+1}(t' + [z^{-1}]) e^{\sum_{\chi} a_\chi z^{n-\chi}}}{2\pi i} \, dz = \int_{z=0} \frac{\tau_{n+1}(t + [z]) \tau_m(t' - [z]) e^{\sum_{\chi} k \chi z^{n-m-1}}}{2\pi i} \, dz. \tag{5.19}
\]

In the case \( \beta = 2 \), the Virasoro expressions take on a particularly elegant form, namely, for \( n \geq 0 \),

\[
\mathcal{J}^{(2)}_{k,n} (t) = \sum_{i+j=k} : \mathcal{J}^{(1)}_{i,n} (t) \mathcal{J}^{(1)}_{j,n} (t) : = J_k^{(2)} (t) + 2n J_k^{(1)} (t) + n^2 \delta_{0k}, \]

\[
\mathcal{J}^{(1)}_{k,n} (t) = J_k^{(1)} (t) + n \delta_{0k},
\]

with\(^{18}\)

\[
J_k^{(1)} = \frac{\partial}{\partial t_k} + \frac{1}{2} (-k) t_{-k},
\]

\[
J_k^{(2)} = \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \sum_{i+j=k} \text{ii} \frac{\partial}{\partial j} + \sum_{i+j=k} \text{ii} \frac{\partial}{\partial j}.
\tag{5.20}
\]

**Remark.** The vectors \( p \) and \( q \) are eigenvectors of \( L \). Indeed, remembering \( \chi(z) = (1, z, z^2, \ldots)^T \), we have

\[
\Lambda \chi(z) = z\chi(z) \text{ and } \Lambda^T \chi(z^{-1}) = z\chi(z^{-1}) - ze_0, \text{ with } e_0 = (1, 0, 0, \ldots)^T.
\]

\(^{18}\)The expression \( J_k^{(1)} = 0 \) for \( k = 0 \).
Therefore, \( p(z) = S\chi(z) \) and \( q(z) = S^{T-1}\chi(z^{-1}) \) are eigenvectors, in the sense
\[
Lp = S\Delta S^{-1}S\chi(z) = zS\chi(z) = zp,
\]
and
\[
L^T q = S^{T-1}\Lambda^T S^T S^{T-1}\chi(z^{-1})
= zS^{T-1}\chi(z^{-1}) - zS^{T-1}e_0 = zq - zS^{T-1}e_0.
\]
Then, using \( L = L^T \), one is lead to
\[
((L - zI)p)_n = 0 \text{ for } n \geq 0 \text{ and } ((L - zI)q)_n = 0 \text{ for } n \geq 1.
\]

5.2.2. Symmetric/symplectic matrix integrals and the Pfaff lattice. Consider an inner-product with a skew-symmetric weight \( \rho(y, z) \),
\[
\langle f, g \rangle_t = \int f(y)g(z)e^{\sum_{i=1}^n \epsilon_i(y')^2} \rho(y, z)dydz, \quad \text{with } \rho(z, y) = -\rho(y, z).
\] (5.21)
Then, since
\[
\langle f, g \rangle_t = -\langle g, f \rangle_t,
\]
the (semi-infinite) moment matrix, depending on \( t = (t_1, t_2, \ldots) \),
\[
m_n(t) = (\mu_{ij}(t))_{0 \leq i, j \leq n-1} = (\langle y^i, z^j \rangle_t)_{0 \leq i, j \leq n-1}
\]
is skew-symmetric and the semi-infinite matrix \( m_\infty \) evolves in \( t \) according to the commuting vector fields
\[
\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+kj} + \mu_{ij+k}, \quad \text{i.e. } \frac{\partial m_\infty}{\partial t_k} = \Lambda^{t}m_\infty + m_\infty \Lambda^{Tt}.
\] (5.22)
It is well known that the determinant of an odd skew-symmetric matrix equals 0, whereas the determinant of an even skew-symmetric matrix is the square of a polynomial in the entries, the Pfaffian, with a sign specified below. So
\[
\det(m_{2n-1}(t)) = 0
\]
\[
(\det m_{2n}(t))^{1/2} = \text{pf}(m_{2n}(t)) = \frac{1}{n!}(dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{2n-1})^{-1}
\]
\[
\left( \sum_{0 \leq i < j \leq 2n-1} \mu_{ij}(t)dx_i \wedge dx_j \right)^n.
\]
Define now the Pfaffian \( \tau \)-functions:
\[
\tau_{2n}(t) := \text{pf } m_{2n}(t).
\] (5.23)
Considering as a special skew-symmetric weight (5.21)
\[
\rho(y, z) := 2D^\alpha \delta(y - z)\tilde{\rho}(y)\tilde{\rho}(z), \quad \text{with } \alpha = \mp 1, \quad \tilde{\rho}(y) = e^{-\tilde{V}(y)},
\] (5.24)
the inner-product (5.21) becomes\(^19\)
\[^{19} \varepsilon(y) = \text{sign}(y) \text{ and } \{ f, g \} := f'g - fg'. \text{ Also notice that } \varepsilon' = 2\delta(x). \]
\[ \langle f, g \rangle_t = \int f(y) g(z) e^{\sum (\epsilon + \epsilon^\prime)} 2D^{\alpha} \delta(y - z) \tilde{\rho}(y) \tilde{\rho}(z) dy dz \]
\[ = \left\{ \begin{array}{ll}
\int_{\mathbb{R}^2} f(y) g(z) e^{\sum (\epsilon + \epsilon^\prime)} \tilde{\epsilon}(y - z) \tilde{\rho}(y) \tilde{\rho}(z) dy dz & \text{for } \alpha = -1 \\
\int_{\mathbb{R}} \langle f, g \rangle(y) e^{\sum 2\alpha y} \tilde{\rho}(y)^2 dy & \text{for } \alpha = +1,
\end{array} \right. \]

and
\[
\text{pf} \left( \langle y', z' \rangle \right)_{0 \leq i, j \leq 2n-1} = \left\{ \begin{array}{ll}
\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\Delta_{2n}(z)| \prod_{k=1}^{2n} e^{\sum \epsilon_k} \tilde{\rho}(z_k) dz_k & \text{for } \alpha = -1, \\
\frac{1}{n!} \int_{\mathbb{R}^n} |\Delta_n(z)| \prod_{k=1}^{n} e^{\sum 2\alpha y} \tilde{\rho^2}(z_k) dz_k & \text{for } \alpha = +1.
\end{array} \right. \] \tag{5.25}

Setting
\[
\begin{align*}
\tilde{\rho}(z) &= \rho(z) I_E(z) & \text{for } \alpha = -1 \\
\tilde{\rho}(z) &= \rho^{1/2}(z) I_E(z), \ t \mapsto t/2 & \text{for } \alpha = +1
\end{align*}
\]
in the identities (5.25), we are led to the identities between integrals and Pfaffians, which are spelled out in [13] theorem 5.3:

**Theorem 5.3.** The integrals \( I_n(t, c) \),

\[
I_n = \int |\Delta_n(z)|^{1/2} \prod_{k=1}^{n} e^{\sum \epsilon_k} \rho(z_k) dz_k
\]

\[
= \left\{ \begin{array}{ll}
n! \text{pf} \left( \int_{\mathbb{R}^n} y' e^{\sum \epsilon_k} (y - z) e^{\sum \epsilon_k} \rho(y) \rho(z) dy dz \right)_{0 \leq i, j \leq 2n-1} & \text{n even for } \beta = 1 \\
n! \tau_n(t, c) & \text{n arbitrary for } \beta = 4
\end{array} \right.
\]

and the \( \tau_n(t, c) \)'s above satisfy the following equations:

(i) The Virasoro constraints\(^{20}\) (5.8) for \( \beta = 1, 4 \),

\[
\left( -2 \sum_{i=1}^{2n} c_i f(c) \frac{\partial}{\partial c_i} + \sum_{j \geq 0} \left( a_i^{(2)} b_j^{(2)}_{k+l, n} - b_i^{(1)} b_j^{(1)}_{k+l+1, n} \right) \right) I_n = 0. \] \tag{5.26}

\(^{20}\) Here the \( a_i \)'s and \( b_i \)'s are defined in the usual way, in terms of \( \rho(z) \); namely

\[ \frac{\partial}{\partial c_i} = \frac{\sum_j b_j^{(1)}_{k+l, n}}{\sum_j b_j^{(2)}_{k+l, n}}. \]
(ii) The Pfaff-KP hierarchy: (see footnote 11)

\[
\left( p_{k+1}(\partial) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tau_n \circ \tau_n = p_k(\partial) \tau_{n+2} \circ \tau_{n-2} \tag{5.27}
\]

of which the first equation reads

\[
\left( \frac{\partial}{\partial t_1} \right)^4 + 3 \left( \frac{\partial}{\partial t_2} \right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \tau_n + 6 \left( \frac{\partial^2}{\partial t_1^2} \log \tau_n \right)^2 = 12 \frac{\tau_{n-2} \tau_{n+2}}{\tau_n^2}, \quad n \text{ even.}
\]

(iii) The Pfaff lattice: the time-dependent matrix

\[
L(t) = Q(t) \Lambda Q(t)^{-1}
\]

satisfies the Hamiltonian commuting equations, as in (2.29)

\[
\frac{\partial L}{\partial t_i} = [-P_+(L^i), L], \quad \text{(Pfaff lattice).}
\]

(iv) Skew-orthogonal polynomials: the vector of time-dependent polynomials \( q(t; z) := (q_n(t; z))_{n \geq 0} \) satisfies the eigenvalue problem

\[
L(t)q(t; z) = zq(t; z) \tag{5.29}
\]

and enjoy the following representations:

\[
q_{2n}(t; z) = z^{n} h^{2n}_{2n} \frac{\tau_{2n}(t - [z^{-1}])}{\tau_{2n}(t)}, \quad h_{2n} = \frac{\tau_{2n+2}(t)}{\tau_{2n}(t)}
\]

\[
q_{2n+1}(t; z) = z^{n} h^{2n+1/2}_{2n} \frac{1}{\tau_{2n}(t)} \left( z + \frac{\partial}{\partial t_1} \right) \tau_{2n}(t - [z^{-1}]).
\]

They are skew-orthogonal polynomials in \( z \); i.e.

\[
(q_i(z; t), q_j(t; z)) = J_{ij}.
\]

(v) The bilinear identities: for all \( n, m \geq 0 \), the \( \tau_{2n} \)'s satisfy the following bilinear identity:

\[
\oint_{\gamma = 0} \tau_{2n}(t - [z^{-1}]) \tau_{2m+2}(t' + [z^{-1}]) e^{\sum_{i=1}^{n} (t - t'_i)'} \ z^{2n-2m-2} \frac{dz'}{2\pi i} + \oint_{\gamma = 0} \tau_{2n+2}(t + [z]) \tau_{2m}(t' - [z]) e^{\sum_{i=1}^{n} (t - t'_i)'} \ z^{2n-2m} \frac{dz}{2\pi i} = 0. \tag{5.30}
\]

5.3. Expressing \( t \)-partials in terms of \( \partial \)-partials

Given first-order linear operators \( D_1, D_2, D_3 \) in \( c = (c_1, \ldots, c_2) \in \mathbb{R}^2 \) and a function \( F(t, c) \), with \( t \in \mathbb{C}^\infty \), satisfying the following partial differential equations in \( t \) and \( c \):

\[
D_1 F = \frac{\partial F}{\partial t_k} + \sum_{-1 \leq j < k} \gamma_j V_j(F) + \gamma_k + \delta_k t_1, \quad k = 1, 2, 3, \ldots. \tag{5.31}
\]
with $V_j(F)$ nonlinear differential operators in $t_i$ of which the first few are given here:

$$V_j(F) = \sum_{i+j \geq 1} t_i \frac{\partial F}{\partial t_{i+j}} + \frac{\beta}{2} \delta_{2j} \left( \frac{\partial^2 F}{\partial t_1^2} + \left( \frac{\partial F}{\partial t_1} \right)^2 \right), \quad -1 \leq j \leq 2. \quad (5.32)$$

In (5.31) and (5.32), $\beta > 0$, $\gamma_{ij}, \gamma_k, \delta_k$ are arbitrary parameters; also $\delta_{2j} = 0$ for $j \neq 2$ and $\delta_{2} = 1$ for $j = 2$. The claim is that the equations (5.31) enable one to express all partial derivatives

$$\left. \frac{\partial^{i_1 + \cdots + i_n} F(t, c)}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}} \right|_{L}, \quad \text{along } L := \{ \text{all } t_i = 0, \ c = (c_1, \ldots, c_{2r}) \ \text{arbitrary} \},$$

uniquely in terms of polynomials in $D_{j_1} \cdots D_{j_r} F(0, c)$. Indeed, the method consists of expressing $\frac{\partial F}{\partial t_k} \bigg|_{t=0}$ in terms of $D_{k}F \big|_{t=0}$, using (5.31). Second derivatives are obtained by acting on $D_{k}F$ with $D_{\ell}$, by noting that $D_{\ell}$ commutes with all $t$-derivatives, by using the equation for $D_{\ell}F$, and by setting in the end $t = 0$:

$$D_{\ell}D_{k}F = D_{\ell} \frac{\partial F}{\partial t_k} + \sum_{-1 \leq j < k} \gamma_{kj} D_{\ell} (V_j(F)) \quad (5.34)$$

When the nonlinear term is present, it is taken care of as follows:

$$D_{\ell} \left( \frac{\partial F}{\partial t_1} \right)^2 = 2 \frac{\partial F}{\partial t_1} \frac{\partial F}{\partial t_1} = 2 \frac{\partial F}{\partial t_1} D_{\ell} \frac{\partial F}{\partial t_1}$$

$$= 2 \frac{\partial F}{\partial t_1} \frac{\partial^2 F}{\partial t_1 \partial t_1}$$

$$= 2 \left( \frac{\partial F}{\partial t_1} + \sum_{-1 \leq j < \ell} \gamma_{kj} V_j(F) + \gamma_{\ell} + \delta_{\ell} t_1 \right); \quad (5.35)$$

higher derivatives are obtained in the same way.

5.4. Using the KP-like equations

Let

$$\delta_{1,4}^\beta := 2 \left( \left( \frac{\beta}{2} \right)^{1/2} - \left( \frac{\beta}{2} \right)^{-1/2} \right)^2 = \begin{cases} 0 & \text{for } \beta = 2, \\ 1 & \text{for } \beta = 1, 4. \end{cases}$$
From theorems 5.2 and 5.3, the integrals $I_\beta(t,c)$, depending on $\beta = 2, 1, 4$, on $t = (t_1, t_2, \ldots)$ and on the boundary points $c = (c_1, \ldots, c_{2\beta})$ of $E$, relate to $\tau$-functions, as follows:

$$I_n(t, c) = \int_{E_n} |\Delta_n(z)|^\beta \prod_{k=1}^n (e^{\sum_{\gamma} \tau_\gamma(z_k) d_{z_k}})$$

$$= \begin{cases} n! \tau_n(t, c), & n \text{ arbitrary, } \beta = 2 \\ n! \tau_n(t, c), & n \text{ even, } \beta = 1 \\ n! \tau_{2n}(t/2, c) & n \text{ arbitrary, } \beta = 4. \end{cases}$$

(5.36)

$I_\beta(t)$ refers to the integral (5.36) over the full range. It also follows that $\tau_n(t, c)$ satisfies the KP-like equation

$$12 \frac{\tau_{n-2}(t, c) \tau_{n+2}(t, c)}{\tau_n(t, c)^2} \delta^3 V = (\text{KP})_t \log \tau_n(t, c), \quad \begin{cases} n \text{ arbitrary for } \beta = 2 \\ n \text{ even for } \beta = 1, 4 \end{cases}$$

(5.37)

where

$$(\text{KP}), F := \left( \frac{\partial}{\partial t_1} \right)^4 + 3 \left( \frac{\partial}{\partial t_2} \right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) F + 6 \left( \frac{\partial^2}{\partial t_1^2} F \right)^2.$$

$\beta = 2, 1$. Evaluating the left-hand side of (5.37) (for $\beta = 1$) yields, taking into account $P_n := P_n(E) = I_0(0, c)/I_0(0)$,

$$12 \frac{\tau_{n-2}(t, c) \tau_{n+2}(t, c)}{\tau_n(t, c)^2} \bigg|_{t=0} = 12 \frac{(n!)^2}{(n-2)!(n+2)!} \frac{I_{n-2}(t, c)I_{n+2}(t, c)}{I_n(t, c)^2} \bigg|_{t=0}$$

$$= 12 \frac{n(n-1)}{(n+1)(n+2)} \frac{P_{n-2}(0)P_{n+2}(0)}{P_n(0)^2} \frac{P_{n-2}P_{n+2}}{P_n^2}$$

$$= 12 b_n^{(1)} \frac{P_{n-2}(E)P_{n+2}(E)}{P_n^2(E)}$$

with $b_n^{(1)}$ a constant. The case $\beta = 4$ is gotten by duality. Concerning the right-hand side of (5.37), it follows from theorem 5.1 that $F_n(t; c) = \log I_n(t; c)$, as in (5.36), satisfies Virasoro constraints. As explained in (5.31)–(5.35), we express

$$\frac{\partial^2 F}{\partial t_1^4} \bigg|_{t=0}, \; \frac{\partial^2 F}{\partial t_2^2} \bigg|_{t=0}, \; \frac{\partial^2 F}{\partial t_1 \partial t_3} \bigg|_{t=0}, \; \frac{\partial^2 F}{\partial t_1^2} \bigg|_{t=0}, \; F = \log I_n(t, c)$$

in terms of $\mathcal{D}_3$ then $\mathcal{B}_k$, which are linear combinations of the $\mathcal{D}_3$, which when substituted in the right-hand side of (5.37), i.e. in the KP-expressions, leads to the following theorems.

5.4.1. Hermitian, symmetric and symplectic Gaussian ensembles. Given the disjoint union $E \subset \mathbb{R}$ and the weight $e^{-bc^2}$, the differential operators $\mathcal{B}_k$ take on the form

$$\mathcal{B}_k = \sum_{l=1}^{2r} c_l^{k+1} \frac{\partial}{\partial c_l}.$$ 

Also, define the invariant polynomials

$$Q(-2n, \beta, 2\beta - \frac{1}{2}, \frac{1}{2}) \bigg|_{\beta=4} = Q(n, \beta, a, b) \bigg|_{\beta=4}.$$
\[ Q = 12b^2n \left( n + 1 - \frac{2}{\beta} \right), \quad Q_2 = 4 \left( 1 + \delta_{1,4}^\beta \right) b \left( 2n + \delta_{1,4}^\beta \left( 1 - \frac{2}{\beta} \right) \right) \]

and

\[ Q_1 = \left( 2 - \delta_{1,4}^\beta \right) b^2 \beta. \]

**Theorem 5.4 (Adler–van Moerbeke [12]).** The following probabilities for \((\beta = 2, 1, 4)\)

\[
P_n(E) = \frac{\int_{E_n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-kz_k} \, dz_k}{\int_{\mathbb{R}^n} |\Delta_n(z)|^\beta \prod_{k=1}^n e^{-kz_k} \, dz_k},
\]

satisfy the PDEs \((F := F_n = \log P_n)\):

\[
\delta_{1,4}^\beta \frac{Q_{n-1}^\beta P_{n+1}}{P_n^\beta} - 1 \text{ with index } \begin{cases} 0 & \text{when } \beta = 2 \\ n \text{ is even and } \beta = 1 \\ 1 & \text{when } n \text{ is arbitrary and } \beta = 4 \\ \end{cases}
\]

\[
= (B_{n+1}^\beta + (Q_2 + 6B_2^2)B_{2}^\beta + 4Q_1 (3B_2^2 - 4B_{-1}^2B_1 + 6B_0)) F.
\]

5.4.2. Hermitian, symmetric and symplectic Laguerre ensembles. Given the disjoint union \(E \subset \mathbb{R}^+\) and the weight \(za e^{-bz}\), the \(B_k\) take on the form

\[
B_k = \sum_{i=1}^{2^k} \gamma_i^{k+2} \frac{\partial}{\partial \gamma_i}.
\]

Also define the polynomials, again respecting the duality (see footnote 20)

\[
Q = \begin{cases} \frac{3}{2}n(n-1)(n+2a)(n+2a+1) & \text{for } \beta = 1 \\ \frac{3}{2}n(2n+1)(2n+a)(2n+a-1) & \text{for } \beta = 4, \end{cases}
\]

\[
Q_2 = \left( 3\beta n^2 - \frac{a^2}{\beta} + 6an + 4 \left( 1 - \frac{3}{2} \right) a + 3 \right) \delta_{1,4}^\beta + (1 - a^2)(1 - \delta_{1,4}^\beta),
\]

\[
Q_1 = \left( \frac{3}{2}n^2 + 2an + \left( 1 - \frac{a}{\beta} \right) a \right), \quad Q_0 = b(2 - \delta_{1,4}^\beta) \left( n + \frac{a}{\beta} \right),
\]

\[
Q_{-1} = \frac{b^2}{2} \left( 2 - \delta_{1,4}^\beta \right).
\]

**Theorem 5.5 (Adler–van Moerbeke [12]).** The following probabilities

\[
P_n(E) = \frac{\int_{E_n} |\Delta_n(z)|^\beta \prod_{k=1}^n za e^{-kz_k} \, dz_k}{\int_{\mathbb{R}^n} |\Delta_n(z)|^\beta \prod_{k=1}^n za e^{-kz_k} \, dz_k}
\]

satisfy the PDE\(^{22}\) \((F := F_n = \log P_n)\)

\(^{22}\) With the index convention for \(\delta_{1,4}^\beta\) as in theorem 5.4.
\[
\delta_{1,4}^\beta Q \left( \frac{P_{n-1}P_{n+1}}{P_n^2} - 1 \right) \\
= \left( B_n^2 - 2(\delta_{1,4}^\beta + 1)B_{n-1}^2 \\
+ (Q_2 + 6B_n^2)F - 4(\delta_{1,4}^\beta + 1)B_{n-1}F)B_{n-1}^2 - 3\delta_{1,4}^\beta (Q_1 - B_{n-1}F)B_{n-1} \\
+ Q_1(3B_n^2 - 4B_nB_{n-1} - 2B_{n-1}) + Q_0(2B_nB_{n-1} - B_{n-1}) \right) F.
\]

5.4.3. ODEs, when \( E \) has one boundary point. Assume the set \( E \) consists of one boundary point \( c = x \), besides the boundary of the full range. In that case the PDEs above lead to ODEs in \( x \):

(1) **Gaussian** \((n \times n)\) matrix ensemble (for the function \( \beta = 2, 1, 4 \)):

\[
f_n(x) = \frac{d}{dx} \log P_n(\max \lambda_i \leq x)
\]

satisfies

\[
\delta_{1,4}^\beta Q \left( \frac{P_{n-1}P_{n+1}}{P_n^2} - 1 \right) \\
= f_n'' + f_n^2 + \left( 4\frac{b^2x^2}{\beta} (\delta_{1,4}^\beta - 2) + Q_2 \right) f_n' - 4\frac{b^2x}{\beta} (\delta_{1,4}^\beta - 2) f_n.
\]

(2) **Laguerre ensemble** (for \( \beta = 2, 1, 4 \)): all eigenvalues \( \lambda_i \) satisfy \( \lambda_i \geq 0 \), and

\[
f_n(x) = x \frac{d}{dx} \log P_n(\max \lambda_i \leq x)
\]

satisfies (with \( f := f_n(x) \))

\[
\delta_{1,4}^\beta Q \left( \frac{P_{n-1}P_{n+1}}{P_n^2} - 1 \right) \\
= x^3 f''' - (2\delta_{1,4}^\beta - 1)x^2 f'' + 6x f' \\
- x \left( 4(\delta_{1,4}^\beta + 1)f - \frac{b^2x^2}{\beta} (\delta_{1,4}^\beta - 2) - 2Q_0x - Q_2 + 2\delta_{1,4}^\beta + 1 \right) f'.
\]

For \( \beta = 2 \), \( f_n(x) \) satisfies the third-order equation (of the so-called Chazy-type) with quadratic nonlinearity in \( f_n' \). Then \( f_n \) also satisfies an equation, which is the second-order in \( f \) and quadratic in \( f'' \), which after some rescaling can be put in a canonical form. Namely,

\[
\text{Gauss} \quad g_n(z) = b^{-1/2}f_n(zb^{-1/2}) + \frac{2}{3}nz,
\]

\[
\text{Laguerre} \quad g_n(z) = f_n(z) = \frac{b}{4}(2n + a)z + \frac{a^2}{4}.
\]
satisfies the respective canonical equations of Cosgrove and Cosgrove–Scoufis,

\[
g'' = -4g'^3 + 4(zg' - g)^2 + A_1g' + A_2 \quad \text{(Painlevé IV)}
\]

\[
(zg'' - g)'(4g'^2 + A_1(zg' - g) + A_2) + A_3g' + A_4 \quad \text{(Painlevé V)}.
\]

The ‘Jacobi’ case can be done in a similar fashion [12].

6. The spectrum of coupled random matrices and the two-Toda lattice

The purpose of this section is to derive a partial differential equation for the spectral gap probabilities for coupled-GUE-Hermitian random matrices [11]. The two-Toda lattice is the integrable deformation class of such coupled matrices and the tau-function theory, vertex theory and Virasoro symmetries play a crucial role in the derivation of the partial differential equations. There is a huge literature on coupled random matrices, see for example [19], as they play a huge role in matrix models, and string theory, as well as random matrix theory.

6.1. Matrix integrals and two-Toda structure

Consider the general weight \( \rho(y, z) \) dy dz := \( \rho_{ts}(y, z) \) dy dz := \( e^{V_{ts}(y, z)} \) dy dz on \( \mathbb{R}^2 \), with \( \rho_0 = e^{V_0} \), where

\[
V_{ts}(y, z) := V_0(y, z) + \sum_{i=1}^{\infty} t_i y^i - \sum_{i=1}^{\infty} s_i z^i = \sum_{i,j \geq 1} c_{ij} y^i z^j + \sum_{i=1}^{\infty} t_i y^i - \sum_{i=1}^{\infty} s_i z^i,
\]

with \( \rho_0 \) and the inner product with regard to a subset \( E \subset \mathbb{R}^2 \)

\[
\langle f, g \rangle_E = \int_E dy dz \rho_{ts}(y, z) f(y) g(z).
\]

Given the moment matrix (over \( E \)),

\[
m_{t, s}(t, s, c) = \langle (y', z') \rangle_E = \langle (y', z') \rangle_E \mid_{y' \leq y, z' \leq z},
\]

according to [10], the Borel decomposition of the semi-infinite matrix\(^23\)

\[
m_{\infty} = S_1^{-1} S_2 \quad \text{with} \quad S_1 \in \mathcal{D}_{-\infty, 0}, \quad S_2 \in \mathcal{D}_{0, \infty},
\]

\[
m_{\infty}(t, s, c) = e^{\sum_{i=1}^{\infty} s_i^t \Lambda^t} m_{\infty}(0, 0, c) e^{-\sum_{i=1}^{\infty} s_i^t \Lambda^t},
\]

with \( S_1 \) having \( 1 \)'s on the diagonal and \( S_2 \) having \( h_i \)'s on the diagonal, leads to two strings \( (p^{(1)}(y), p^{(2)}(z)) \) of monic polynomials in one variable (dependent on \( E \)), constructed, in terms of the character \( \tilde{\chi}(z) = (z^n)_{n \in \mathbb{Z}, n \geq 0} \), as follows:

\[
p^{(1)}(y) = S_1 \tilde{\chi}(y), \quad p^{(2)}(z) = h(S_2^{-1})^T \tilde{\chi}(z).
\]

We call these two sequences \textit{bi-orthogonal polynomials}; in fact, according to [10], the Borel decomposition of \( m_{\infty} = S_1^{-1} S_2 \) above is equivalent to the ‘orthogonality’ relations of the polynomials

\[
\langle p^{(1)}_n, p^{(2)}_m \rangle_E = \delta_{n, m} h_n.
\]

\(^23\) \( \mathcal{D}_{k, \ell} \) denotes the set of band matrices with zeros outside the strip \( (k, \ell) \), \( \Lambda = (\delta_{m+1})_{ij} \).
The matrices
\[ L_1 := S_1 \Lambda S_1^{-1} \quad \text{and} \quad L_2 := S_2 \Lambda^\top S_2^{-1} \]
interact with the vector of string orthogonal polynomials as follows:
\[ L_1 p^{(1)}(y) = y p^{(1)}(y), \quad \hbar L_2 h^{-1} p^{(2)}(z) = z p^{(2)}(z). \] (6.6)

Also define wave vectors \( \Psi_1 \) and \( \Psi_2 \) as follows:
\[ \Psi_1(z) := e^{\sum_n \psi_n^p t^{(1)}(z)} \quad \text{and} \quad \Psi_2(z) := e^{\sum_n \psi_n^v h^{-1} p^{(2)}(z)} \]
\[ = e^{\sum_n \psi_n^v S_1 \chi(x)} = e^{\sum_n \psi_n^v (S_2^{-1})^\top \chi(z^{-1})}. \] (6.7)

As a function of \((t, s)\), the couple \( L := (L_1, L_2) \) satisfies the two-Toda lattice equations (2.22), and \( \Psi_1 \) and \( \Psi_2 \) satisfy [11] (remember that \( L, \Psi_1 \) and \( \Psi_2 \) all depend on \( E \))
\[ \begin{cases} \frac{\partial}{\partial t_1} \Psi = (L_1, 0)_+ \Psi = ((L_1^a)_a(L_1^a)_{a_1}) \Psi \\ \frac{\partial}{\partial s_1} \Psi = (0, L_2)_+ \Psi = ((L_2^a)_a(L_2^a)_{a_1}) \Psi \end{cases} \]
\[ \begin{cases} \frac{\partial}{\partial t_2} \Psi^* = -((L_1^a, 0)_+)^\top \Psi^* \\ \frac{\partial}{\partial s_2} \Psi^* = -((0, L_2^a)_+)^\top \Psi^* \end{cases} \] (6.8)

where in the above we introduced also \( \Psi_1^*, \Psi_2, \Psi = (\Psi_1, \Psi_2), \Psi^* = (\Psi_1^*, \Psi_2^*) \). Moreover \((\tau_0^E := 1)\)
\[ n! \det m_n(t, s, c) = \int_{(u, v) \in \mathbb{R}^{2n}} \Delta_n(u) \Delta_n(v) \prod_{k=1}^{n} e^{V_{n}(u_k, v_k)} du_k dv_k \]
\[ = n! \det (E_n(t) m_\infty(0, 0, c) E_n(-s)^\top) \]
\[ = \prod_{n-1}^{0} h_n(t, s, c) \]
\[ = \tau^E_n(t, s, c), \] (6.9)

where \( E_n(t) := (\text{the first } n \text{ rows of } e^{\sum_n i \tau_n A^s}) \) is a matrix of Schur polynomials \( p_n(t) \). Also \( \tau_n(t, s, c) \) is a \( \tau \)-function with regard to \( t \) and \( s \) and
\[ (L_1^a)_a = \frac{\partial}{\partial t_k} \log \frac{\tau_{n+1}}{\tau_n}, \quad (L_2^a)_a = -\frac{\partial}{\partial s_k} \log \frac{\tau_{n+1}}{\tau_n}, \quad n \geq 0. \]

In particular,
\[ L_1 = \cdots + \left( \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} \right)_{n \geq 0} + \Lambda \quad \text{and} \quad L_2 = \left( \frac{\tau_{n-1} \tau_{n+1}}{\tau_n} \right)_{n \geq 0} \Lambda^\top - \left( \frac{\partial}{\partial s_1} \log \frac{\tau_{n+1}}{\tau_n} \right)_{n \geq 0} + \cdots, \]

with the wave vectors parametrized by the \( \tau \)-functions as follows:
\[ \Psi_1(z) = \left( \frac{\tau_n(t - [z]^{-1}, s)}{\tau_n(t, s)} e^{\sum_{n \geq 0} \zeta_n z^{-n}} \right)_{n \geq 0}, \]
\[ \Psi_2(z) = \left( \frac{\tau_n(t, s + [z])}{\tau_{n+1}(t, s)} e^{\sum_{n \geq 0} \zeta_n z^{-n}} \right)_{n \geq 0}, \] (6.10)
etc for $\Psi_1^*, \Psi_2$. Introducing the wave matrices

$$W_i = S(t, s)e^{\xi(A)}$$

$$\xi_1(z) = \sum_{i=1}^{\infty} i z^i, \quad \xi_2(z) = \sum_{i=1}^{\infty} s_k z^{-k},$$  \hspace{1cm} (6.11)

$$\Psi_i(t, s; z) = W_i \tau(z) = e^{\xi(z)} S_i \tau(z), \quad \Psi_i^*(t, s; z) = (W_i^*)^{-1} \tau(z) = e^{-\xi(z)} (S_i^*)^{-1} \tau(z),$$  \hspace{1cm} (6.12)

from the relations (6.8) the pair of matrices $W = (W_1, W_2)$ satisfies the bilinear relation (in the $\pm$ splitting of (2.20))

$$(W(t, s) W(t', s')^{-1})_- = 0$$

or equivalently,

$$W_i(t, s) W_i(t', s')^{-1} = W_i(t, s) W_i(t', s')^{-1},$$  \hspace{1cm} (6.13)

from which one proves proposition 6.1; for details see [3, 4].

**Proposition 6.1 (Bi-infinite and semi-infinite).** The wave and adjoint wave functions satisfy, for all $m, n \in \mathbb{Z}$ (bi-infinite) and $m, n \geq 0$ (semi-infinite) and $t, s, t', s' \in \mathbb{C}^\infty$,

$$\oint_{z=\infty} \Psi_{in}(t, s; z) \Psi_{in}^*(t', s'; z') \frac{dz}{2\pi i z} = \oint_{z=0} \Psi_{2in}(t, s; z) \Psi_{2in}^*(t', s'; z') \frac{dz}{2\pi i z}. \hspace{1cm} (6.14)$$

**Proposition 6.2.** Two-Toda $\tau$-functions satisfy the following bilinear identities:

$$\oint_{z=\infty} \tau_n(t - [z^{-1}], s) \tau_{m+1}(t' + [z^{-1}], s') e^{\sum_{i=1}^{\infty} (t_i - t'_i) z^{-i-1}} dz = \oint_{z=0} \tau_{n+1}(t, s - [z]) \tau_{m}(t', s' + [z]) e^{\sum_{i=1}^{\infty} (s_i - s'_i) z^{-i-1}} dz, \hspace{1cm} (6.15)$$

or, expressed in terms of the Hirota symbol$^{24}$,

$$\sum_{j=0}^{\infty} p_{m-n+j} (-2a) p_j (\partial_t) e^{\sum_{i=1}^{\infty} (a_i \frac{d}{dt} + b_i \frac{d}{ds})} \tau_{m+1} \circ \tau_n$$

$$= \sum_{j=0}^{\infty} p_{-m-n+j} (-2b) p_j (\partial_s) e^{\sum_{i=1}^{\infty} (a_i \frac{d}{dt} + b_i \frac{d}{ds})} \tau_m \circ \tau_{n+1}, \hspace{1cm} (6.16)$$

both for the bi-infinite ($m, n \in \mathbb{Z}$) and the semi-infinite case ($m, n \in \mathbb{Z}, m, n \geq 0$).

**Proof.** Equation (6.15) follows at once from proposition 6.1 and the $\tau$-function representations (6.10), whereas (6.16) follows from the shifts $t \mapsto t - a$, $t' \mapsto t' + a$, $s \mapsto s - b$, $s' \mapsto s' + b$, combined with the definition of the Hirota symbol.

$^{24}$For the customary Hirota symbol $p(\partial_t) f \circ g := p \left( \frac{d}{dt} \right) f(t + y) g(t - y) \big|_{y=0}$, with $\partial_t = (\partial_t, \frac{1}{2} \partial_t, \frac{1}{3} \partial_t, \ldots)$,

$\partial_s = (\partial_s, \frac{1}{2} \partial_s, \frac{1}{3} \partial_s, \ldots)$.
This has as a direct consequence the following: Two-Toda $\tau$-functions $\tau(t, s)$ satisfy the KP-hierarchy in $t$ and $s$ separately, of which the first equation reads

$$\left( \frac{\partial}{\partial t_i} \right)^4 \log \tau + 6 \left( \frac{\partial}{\partial t_i} \right)^2 \log \tau + 3 \left( \frac{\partial}{\partial t_i} \right)^2 \log \tau - 4 \frac{\partial^2}{\partial t_i \partial t_j} \log \tau = 0.$$  

(6.17)

But they also satisfy the following identities [11]:

**Theorem 6.3.** Two-Toda $\tau$-functions satisfy

$$\left\{ \frac{\partial^2 \log \tau}{\partial t_i \partial t_j}, \frac{\partial^2 \log \tau}{\partial s_i \partial s_j} \right\}_{t_i} + \left\{ \frac{\partial^2 \log \tau}{\partial s_i \partial s_j}, \frac{\partial^2 \log \tau}{\partial t_i \partial t_j} \right\}_{s_j} = 0$$

and

$$-\frac{\partial}{\partial s_1} \log \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\partial^2}{\partial s_1^2} \log \tau_n, \quad \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\partial^2}{\partial t_1 \partial t_1} \log \tau_n.$$  

(6.18)

With the two-Toda lattice, we associate four different vertex operators $X_i(\lambda, \mu)$ for $1 \leq i, j \leq 2$; they map infinite vectors of $\tau$-functions into $\tau$-vectors. The vertex operators $X_{11}$ and $X_{22}$ are basic vertex operators for Toda, and KP as well, whereas $X_{12}$ and $X_{21}$ are vertex operators, native to two-Toda. In particular, we construct for the semi-infinite two-Toda

$$X_{12}(\mu, \lambda) = \Lambda^{-1}X(\lambda)X(-s, \mu)\overline{X}(\mu),$$  

(6.19)

with $\Lambda$ the customary shift-operator $(\Lambda \nu)_n = \nu_{n+1}$, and with

$$X(t, \lambda) := e^{\sum_{i=0}^{\infty} \lambda^i} e^{-\sum_{i=0}^{\infty} \lambda^{-i} \frac{d^x_i}{dy}}.$$  

Given a two-Toda lattice $\tau$-vector $\tau = (\tau_0, \tau_1, \ldots)$, we have that $\tau + X_{12}(y, z)\tau$ is another $\tau$-vector. But more is true. We show that the kernels $K_{12, n}(y, z)$, defined by the ratios $(X_{12}\tau)_n/\tau_n$, have *eigenfunction expansions* in terms of the eigenfunctions $\Psi$, reminiscent of the Christoffel–Darboux formula for orthogonal polynomials; to be precise [11].

**Theorem 6.4.** We have for $\tau_n = \tau_n^R$

$$K_{12, n}(y, z) := \frac{1}{\tau_n} X_{12}(y, z)\tau_n = \sum_{0 \leq j < n} \Psi_j^z(z^{-1})\overline{\Psi}_j(y).$$  

(6.20)

together with a Fredholm determinant-like formula

$$\det \left( K_{12, n}(y_{\alpha}, z_{\beta}) \right)_{1 \leq \alpha, \beta \leq k} = \frac{1}{\tau_n} \left( \sum_{n=1}^{k} X_{12}(y_n, z_n)\tau \right)_n.$$  

(6.21)

**Corollary 6.5.** The vector of Fredholm determinants equals

$$\det(I - \lambda K^E) = \frac{1}{\tau} e^{-\lambda \int_{x} dxdy \rho_{s}(x,y) X_{12}(x,y) \tau}$$

for the kernel $K^E = K_{12, n}(y, z)I_k(z), \tau_n = \tau_n^R$, with the measure $\rho_{s}(x,y)dxdy$. 

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Proof. Putting the corresponding determinant obtained in theorem 6.4 in the Fredholm formula below, we find for a subset of the form $E = E_1 \times E_2 \subset \mathbb{R}^2$,

$$
(\det(I - \lambda K^k))_{n \in \mathbb{Z}} = 1 + \sum_1^{\infty} \frac{(-\lambda)^k}{k!} \prod_{t} \int_{E_t} \det(K_{12n}(x_t, y_t))_{1 \leq i, j \leq k} \Pi_1^{\lambda} \rho_{t,n}(x_t, y_t) d x_t d y_t
$$

$$
= \sum_0^{\infty} \frac{(-\lambda)^k}{k!} \prod_{t} \int_{E_t} \frac{1}{\tau} \left( \int_{E} X_{12}(x, y) \right)^{\tau} \Pi_1^{\lambda} \rho_{t,n}(x_t, y_t) d x_t d y_t
$$

$$
= \frac{1}{\tau} \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\lambda \int_{E} X_{12}(x, y) \rho_{t,n}(x, y) d x d y \right)^{k} \tau
$$

$$
= \frac{1}{\tau} e^{-\lambda \int_{E} \rho_{t,n}(x, y) X_{12}(x, y) d x d y} \tau.
$$

□

6.2. PDEs for the gap probabilities of coupled random matrices

Given the space of Hermitian matrices $H_N$, and given

- spectrum $M_1 = \{x_1, \ldots, x_N\}$ and
- spectrum $M_2 = \{y_1, \ldots, y_N\}$, with $M_1, M_2 \in H_N$,

we define, for a set $E \subset \mathbb{R}^2$,

$$H_{N,E}^2 = \{(M_1, M_2) \in H_N^2 \text{ with all } (x_k, y_k) \in E\}.$$  

Consider the product Haar measure $dM_1 dM_2$ on the product space $H_N^2$, with each $dM_i$, decomposed into its radial part and its angular part. Also define the probability measure

$$
\frac{dM_1 dM_2 e^{\text{Tr} V_{n}(M_1, M_2)}}{\int_{H_N^2} dM_1 dM_2 e^{\text{Tr} V_{n}(M_1, M_2)}} \quad (6.22)
$$

and the Virasoro operators (see footnote 10)

$$
\mathcal{J}^{(2)}_{k} = \left( J^{(2)}_{k,n} \right)_{n \in \mathbb{Z}} = \frac{1}{2} \left( J^{(2)}_{k} + (2n + k + 1) J^{(1)}_{k} + n(n + 1) J^{(0)}_{k} \right)_{n \in \mathbb{Z}},
$$

$$
\mathcal{J}^{(2)}_{n} = \left( J^{(2)}_{n,k} \right)_{n \in \mathbb{Z}} = \frac{1}{2} \left( J^{(2)}_{n,k} + (2n + k + 1) J^{(1)}_{n,k} + n(n + 1) J^{(0)}_{n,k} \right)_{n \in \mathbb{Z}}.
$$

Given the disjoint union

$$E = E_1 \times E_2 := \bigcup_{i=1}^{r} [a_{2i-1}, a_{2i}] \times \bigcup_{i=1}^{s} [b_{2i-1}, b_{2i}] \subset \mathbb{R}^2, \quad (6.23)$$

$^{25} d_{k}^{(0)} = d_{k}^{(0)} |_{n \rightarrow -i}, \quad i = 1, 2.$
define the following integral:

\[ \mathcal{U}_E := \int_E \mathcal{X}_{12}(x, y) \rho_0(x, y) \, dx \, dy, \]  

(6.24)
of the vertex operator \( \mathcal{X}_{12} \), defined in (6.19).

This brings us to the following theorems:

**Theorem 6.6.** Given a set \( E \), as in (6.23), the probability

\[ P(\text{all } M_1 \text{-eigenvalues } \in E_1 \text{ and all } M_2 \text{-eigenvalues } \in E_2) \]

\[ = \frac{\int_{M_1} dM_1 dM_2 e^{Tr(V_1(M_1, M_2))}}{\int_{M_1} dM_1 dM_2 e^{Tr(V_1(M_1, M_2))}} = \frac{\tau_n^E}{\tau_n} \]

(6.25)
is a ratio of two \( \tau \)-functions \( \tau_n^E \) and \( \tau_n \) such that

\[ \tau_n^E = ((\mathcal{U}_E)^n) \tau_n. \]

Moreover, \( \tau_n \) and \( \tau_n^E \) satisfy the partial differential equations, labeled for \( k \geq -1 \), and when all \( c_{ij} = 0 \), but \( c_{11} = \infty \), we find

\[ \left( -\sum_{i=1}^r a_i^{k+1} \frac{\partial}{\partial a_i} + \mathfrak{J}_k^{(2)} \right) \tau_n^E + c p_{k+n} (\tilde{\partial}_1) p_n (\tilde{\partial}_1) \tau_n^E \circ \tau_{n-1}^E = 0, \]

\[ \left( -\sum_{i=1}^r b_i^{k+1} \frac{\partial}{\partial b_i} + \mathfrak{J}_k^{(2)} \right) \tau_n^E + c p_{n} (\tilde{\partial}_1) p_{k+n} (\tilde{\partial}_1) \tau_n^E \circ \tau_{n-1}^E = 0. \]

(6.26)

**Remark.** Whenever some \( a_i = \infty \), we must interpret \( c_{i}^{k+1} \frac{\partial}{\partial c_i} \) or \( b_i^{k+1} \frac{\partial}{\partial b_i} \equiv 0 \); in particular, \( \tau_n \) satisfies the same equations, but without the boundary terms.

The above formula (6.26) depends on many results; namely, setting

\[ \mathcal{V}_k : = -b^{k+1} \frac{\partial}{\partial b} - a^{k+1} \frac{\partial}{\partial a} + \sum_{i,j \geq 1} i c_{ij} \frac{\partial}{\partial c_i^{k+1} + k} \].

(6.27)

**Theorem 6.7.** For all \( k \geq -1 \) and \( n \geq 1 \),

\[ [\mathcal{V}_k, (\mathcal{U}_E)^n] = 0, \]

with the vector \( \mathfrak{J}_k^{(2)} \) forming a Virasoro algebra of central charge \( c = -2 \):

\[ [\mathfrak{J}_k^{(2)}, \mathfrak{J}_\ell^{(2)}] = (k - \ell) \mathfrak{J}_{k+\ell}^{(2)} + (-2) \left( \frac{k^3 - k}{12} \right) \delta_{k,-\ell} \]

and the remarkable identity

\[ \frac{\partial \tau_n^E}{\partial c_{\alpha \beta}} = \tau_n^E \sum_{i=0}^{n-1} (L_+^i L_2^j) \rho_0 (\tilde{\partial}_1) p_{\alpha+n-1} (\tilde{\partial}_1) p_{\beta+n-1} (\tilde{\partial}_1) \tau_n^E \circ \tau_{n-1}^E. \]

(6.28)
To prove the last formula of (6.25) first observe that
\[
\int_{\Omega} dU e^{\text{Tr}U^T U} = \frac{(2\pi)^{n(n-1)}/2 \det(e^{x_i y_j})_{1 \leq i, j \leq n}}{n! e^{\pi(x_i y_j) / 2} \Delta(x) \Delta(y)}
\]
which implies that for \( E = E_1 \times E_2 \subset \mathbb{R}^2 \), the following holds:
\[
\int_{\mathbb{R}^2} e^{\text{Tr}(M_1 M_2)} e^{\text{Tr} \sum_{i=0}^{\infty} (s_i M_1^i + t_i M_2)} dM_1 dM_2 = \int_{E_1} \prod_{k=1}^{N} \left( dx_k dy_k e^{\sum_{i=1}^{\infty} (s_i x_k^i - t_i y_k^i) + c_k x_k^i} \right) \Delta_N(x) \Delta_N(y). \tag{6.29}
\]
We now can prove [11]:

**Proposition 6.8.** For \( E = E_1 \times E_2 \subset \mathbb{R}^2 \), we have
\[
\tau^E_n = (\langle U \rangle^n)^n_n. \tag{6.30}
\]

**Proof.** In what follows, we use the monic bi-orthogonal polynomials (6.4) \( p_1^{(1)}, p_1^{(2)} \), defined by \( \rho_{s,c}(x, y) \) on \( \mathbb{R}^2 \); therefore the \( h_l(t, s, c) \) are the \( \mathbb{R}^2 \) inner products. We first compute, using (6.9) for \( E = \mathbb{R}^2 \), and remembering the notation (6.20), (6.24) and formulae (6.9) and (6.7),
\[
\tau^E_n = \prod_{k=1}^{n-1} \left( \int_{E_1} \prod_{k=1}^{n} (dx_k dy_k \rho_{s,c}(x_k, y_k)) \Delta_n(x) \Delta_n(y) \right)
\]
\[
= \prod_{k=1}^{n-1} \left( \int_{E_1} \prod_{k=1}^{n} (dx_k dy_k \rho_{s,c}(x_k, y_k)) \det(p^{(1)}_{i-1}(x_k) \rho^{(2)}_{i-1}(y_k), 1 \leq i \leq n) \right)
\]
\[
= \prod_{k=1}^{n} \left( \int_{E_1} \prod_{k=1}^{n} (dx_k dy_k \rho_0(x_k, y_k)) \det(\sum_{i=1}^{n} p^{(1)}_{i-1}(x_k) h_{i-1}^{-1} p^{(2)}_{i-1}(y_k) e^{-\sum_{i=1}^{n} h_{i-1}^{-1} \rho^{(2)}_{i-1}(y_k)}) \right)
\]
\[
= \prod_{k=1}^{n} \left( \int_{E_1} \prod_{k=1}^{n} (dx_k dy_k \rho_0(x_k, y_k)) \det(\sum_{i=1}^{n} \Psi_{1i}(x_k) \Psi_{2i}^{*}(y_k^{-1})) \right)
\]
\[
= \prod_{k=1}^{n} \left( \int_{E_1} \prod_{k=1}^{n} (\rho_0 dx_k dy_k) \det(K_{x_k, y_k}, 1 \leq k, \ell \leq n) \right)
\]
\[
= \prod_{k=1}^{n} \left( \frac{1}{\tau} \int_{E_1} \prod_{k=1}^{n} \left( \sum_{i=1}^{n} \Psi_{1i}(x_k) \Psi_{2i}^{*}(y_k) \right) \right) \tag{6.31}
\]
establishing (6.30). \( \Box \)

To give the next result, we define differential operators \( A_k, B_k \) of weight \( k \), in terms of the coupling constant \( c \), and the boundary of the set
\[
E = E_1 \times E_2 := \bigcup_{i=1}^{r} [a_{2i-1}, a_{2i}] \times \bigcup_{i=1}^{s} [b_{2i-1}, b_{2i}] \subset \mathbb{R}^2; \tag{6.31}
\]
A_1 = \frac{1}{c^2 - 1} \left( \sum_{j=1}^r \frac{\partial}{\partial a_j} + c \sum_{j=1}^s \frac{\partial}{\partial b_j} \right), \quad B_1 = \frac{1}{1-c^2} \left( \sum_{j=1}^r \frac{\partial}{\partial a_j} + \sum_{j=1}^s \frac{\partial}{\partial b_j} \right);

A_2 = \sum_{j=1}^r \frac{\partial}{\partial a_j} - c \frac{\partial}{\partial c}, \quad B_2 = \sum_{j=1}^s b_j \frac{\partial}{\partial b_j} - c \frac{\partial}{\partial c};

they form a Lie algebra parametrized by \( c \):

\[ [A_1, B_1] = 0, \quad [A_1, A_2] = \frac{1 + c^2}{1 - c^2} A_1, \quad [A_2, B_1] = \frac{2c}{1 - c^2} A_1 \]

\[ [A_2, B_2] = 0, \quad [A_1, B_2] = - \frac{2c}{1 - c^2} B_1, \quad [B_1, B_2] = \frac{1 + c^2}{1 - c^2} B_1. \]

The following theorem deals with the joint distribution (6.22), with

\[ V_{1,t}(M_1, M_2) = -\frac{1}{2} (M_1^2 + M_2^2) + cM_1 M_2, \]

\[ P_n(E) := P(\text{all}(M_1\text{-eigenvalues}) \in E_1, \text{all}(M_2\text{-eigenvalues}) \in E_2), \]

and leads to a formula in [11], which is the ‘mirror image’ of theorem 6.3.

**Theorem 6.9 (Gaussian probability).** The statistics (6.33) satisfy the \( n \)-independent nonlinear third-order partial differential equation\(^{26}\) \( F_n := \frac{1}{n} \log P_n(E) \):

\[ \left\{ B_2 A_1 F_n, \ B_1 A_1 F_n + \frac{c}{c^2 - 1} \right\}_{A_1} - \left\{ A_2 B_1 F_n, \ A_1 B_1 F_n + \frac{c}{c^2 - 1} \right\}_{B_1} = 0. \]

**Remark 6.1.** Since the equation above for the joint statistics is independent of the size \( n \), the same joint statistics for infinite coupled ensembles should presumably be given by the same partial differential equation.

**Remark 6.2.** For \( E = E_1 \times E_2 := (-\infty, a] \times (-\infty, b] \) equation (6.34) takes on the following form: Upon introducing the new variables \( x := -a + cb, \ y := -ac + b \), the differential operators \( A_1 \) and \( B_1 \) take on the simple form \( A_1 = \partial/\partial x, \ B_1 = \partial/\partial y \) and (6.34) becomes

\[ \frac{\partial}{\partial x} \left( \left( c^2 - 1 \right) \frac{\partial^2 E}{\partial x^2} + 2cx + \left( 1 + c^2 \right) y \right) = \frac{\partial}{\partial y} \left( \left( c^2 - 1 \right) \frac{\partial^2 E}{\partial x \partial y} + 2cy + \left( 1 + c^2 \right) x \right). \]

We sketch the proof of (6.34).

**Proof.** From (6.25), it clearly follows that

\[ P_n(E) = \left| \frac{\delta E(t, s, c \eta)}{\delta n} \right|_{L}, \]

\(^{26}\) Using the following relation for non-commutative operators \( X \) and \( Y \),

\[ XY \log f = \frac{1}{f} (XYf - XfY) \text{ and } f, g = f'g - fg'. \]
where $\tau_n^E$ is an integral over $E^n \subset \mathbb{R}^{2n}$, i.e. $(x,y) \in E_1^r \times E_2^r = E^n$,

$$
\tau_n^E(t, s, c_j) = \int_{E^n} dx dy \sum_{k=1}^n e^{-\frac{1}{2}(x_i^2 + y_i^2 - 2x_1x_i + \sum_{i=0}^{n-1}(x_{i+1} - x_i)^2) + \sum_{i,j \in \{1,2\} \land i \neq j} c_i c_j y_i y_j}.
$$

(6.35)

and where $L$ denotes the locus

$$
L = \{ t_i = s_i = 0, \; c_{11} = c \text{ and all other } c_j = 0 \}.
$$

We need to write down the Virasoro constraints (6.26):

$$
\left. \frac{\partial}{\partial t_i} \log \tau_n \right|_L = A_1 \log \tau_n \bigg|_L, \quad \left. \frac{\partial}{\partial s_j} \log \tau_n \right|_L = B_1 \log \tau_n \bigg|_L
$$

$$
\left. \frac{\partial^2}{\partial t_i \partial s_j} \log \tau_n \right|_L = B_2 A_1 \log \tau_n + \frac{nc}{c^2 - 1},
$$

(6.36)

Setting (6.36)–(6.38) into the formula of proposition 6.3 one is led to an expression for $B_1 \log \tau_n^{E_1}$ and a dual expression for $A_1 \log \tau_n^{E_2}$:

$$
- A_1 \log \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{A_2 B_1 \log \tau_n}{A_1 B_2 \log \tau_n + \frac{nc}{c^2 - 1}}
$$

$$
- B_1 \log \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{B_2 A_1 \log \tau_n}{B_2 A_1 \log \tau_n + \frac{nc}{c^2 - 1}}.
$$

(6.39)

Upon taking $A_1$ of the second expression, subtracting from it $B_1$ of the first one and using $\{A_1, B_1\} = 0$, one finds the following identity:

$$
A_1 \frac{B_2 A_1 \log \tau_n}{B_1 A_1 \log \tau_n + \frac{nc}{c^2 - 1}} - B_1 \frac{A_2 B_1 \log \tau_n}{A_1 B_2 \log \tau_n + \frac{nc}{c^2 - 1}} = 0.
$$

This difference amounts to the equality of two Wronskians ($G_n := \frac{1}{n} \log \tau_n$):

$$
\left\{ B_2 A_1 G_n, B_1 A_1 G_n + \frac{c}{c^2 - 1} \right\}_{A_1} = \left\{ A_2 B_1 G_n, A_1 B_1 G_n + \frac{c}{c^2 - 1} \right\}_{B_1}.
$$

(6.40)

Because of the fact that

$$
\log P_n(E) = \log(\tau_n(E)/\tau_n(\mathbb{R}^2)) = \log \tau_n(E) - \log \tau_n(\mathbb{R}^2),
$$

together with the fact that $A_1 \tau_n(\mathbb{R}^2) = B_1 \tau_n(\mathbb{R}^2) = 0$, we have that $F_n(E) := \frac{1}{n} \log P_n(E)$ satisfies (6.40) as well, thus leading to (6.34).
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