Colliding axisymmetric \( pp \)-waves

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(8 May 1997)

An exact solution is found describing the collision of a class of axisymmetric \( pp \)-waves. They are impulsive in character and their coordinate singularities become point curvature singularities at the boundaries of the interaction region. The solution is conformally flat. Concrete examples are given, involving an ultrarelativistic black hole against a burst of pure radiation or two colliding beam-like waves.

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I. INTRODUCTION

The problem of colliding plane waves in general relativity has been thoroughly investigated by now [1], [2]. Even more interesting and realistic is the collision of the more general class of \( pp \)-waves of finite extent and energy. One particular example, the collision of ultrarelativistic black holes, has been studied by approximate methods [3], [4], [5]. The main reason for the lack of exact solutions is that \( pp \)-waves are written easily in Brinkmann coordinates, but the analog of the Rosen transformation has not been known. Recently, the diagonalization of axisymmetric \( pp \)-waves was achieved [6]. They are described by the line element in cylindrical coordinates

\[
ds^2 = 2dudv - e^{-U} (e^{V} dr^2 + e^{-V} d\varphi^2)
\]

where \( u = \frac{1}{\sqrt{2}}(t - z) \), \( v = \frac{1}{\sqrt{2}}(t + z) \) and \( U \), \( V \) depend on \( u \), \( r \) for a left-coming wave and on \( v \), \( r \) for a right-coming wave.

The standard description of a head-on collision of two waves divides the \( u \), \( v \) space into four regions [1]. Regions II \((u > 0, v < 0)\) and III \((u < 0, v > 0)\) are occupied by the approaching waves with line element (1). Region I \((u < 0, v < 0)\) represents the flat spacetime between the waves. Region IV \((u > 0, v > 0)\) describes their collision and interaction. We suppose that in the interaction region the line element preserves its axial symmetry and is described by functions \( g_{uv} = e^{-M} \), \( U \) and \( V \) which depend on \( u \), \( v \), \( r \). In the present paper we shall find all solutions with \( M = 0 \), in a manner similar to the classification of diagonal plane waves with \( M = 0 \) [6].

In Sec.II the general solution with \( M = 0 \) is found in the interaction region. It is extended to a global solution in Sec.III and its parameters are linked to the characteristics of the approaching waves. The structure of the solution is elucidated further in Sec.IV by studying its invariants. In Sec.V two examples are given. Sec.VI contains some conclusions.

II. SOLUTION WITH \( M = 0 \) IN THE INTERACTION REGION

The vacuum Einstein equations in the interaction region simplify when (1) is rewritten as

\[
ds^2 = 2dudv - Q^2 dr^2 - P^2 d\varphi^2
\]

Then they read:

\[
PQ_{uu} + Q P_{uu} = 0 \tag{3}
\]

\[
PQ_{vv} + Q P_{vv} = 0 \tag{4}
\]

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When \( P_r = Q_r = 0 \) eqs(7,8) become trivial and the others reduce to the equations for plane waves [7].

Eqs(7,8) are easily integrated and give \( Q = e^{h(r)} P_r \) with an arbitrary \( h(r) \). However, when \( u = v = 0 \) we are in region I with Minkowskian background and \( P(0,0,0) = r, Q(0,0,0) = 1 \). This condition fixes \( h(r) \) to zero and the result is

\[
Q = P_r \tag{10}
\]

coinciding with the condition for a single \( pp \)-wave [3]. In fact, the final conclusion drawn below does not depend on \( h(r) \). Now (3,4) become

\[
(P_{uu} P_{vv})_r = 0 \tag{11}
\]

\[
(P_{vv} P_{vu})_r = 0 \tag{12}
\]

Eqs(5,6) show that the \( u \) and \( v \) dependence separate:

\[
P = f(u,r) + g(v,r) \tag{13}
\]

The remaining eq(9) may be written in two equivalent forms:

\[
(e^{-U})_{uv} = 0 \tag{14}
\]

\[
(P_u P_v)_r = 0 \tag{15}
\]

The first form is well-known from the study of colliding plane waves.

We want to prove that the solution of (11-15) is

\[
P = b_0(r) - b_1(r) u - b_2(r) v \tag{16}
\]

for some functions \( b_i(r) \). If \( P_{uu} = P_{vv} = 0 \) eq(16) immediately follows. Suppose that \( P_{vv} \neq 0 \). Then (11,12) and (13-15) give

\[
\left( \frac{P_{uu}}{P_{vv}} \right)_r = 0 \tag{17}
\]

\[
(P_{uu} P_{vv})_r = 0 \tag{18}
\]

which are equivalent to

\[
P_{uu} P_{vv} = 0 \tag{19}
\]

\[
P_{uu} P_{uv} = 0 \tag{20}
\]

There are two possibilities: \( P_{uur} = P_{aur} = 0 \), \( P_{uu} \neq 0 \) or \( P_{uv} = 0 \). The first possibility combined with (13) means that \( P_r = 0 \). Hence (3-9) reduce to the plane wave case, as was already mentioned, which is discussed in [3]. The second possibility means that \( f = c_1(r) u + c_2(r) v \). Putting this result into (13-15) we get \( c_1(r) g_v = g_1(v) \). Eqs(11,12) become

\[
g_1 v \left( u + \frac{c_2 + g}{c_1} \right)_r = 0 \tag{21}
\]

If \( g_1 v = 0 \) it easily follows that \( P \) takes the form (16). If \( g_1 v \neq 0 \), \( g = c_1 g_2(v) - c_2 \) and (17,18) give \( (c_1^2 g_2)_r = 0 \). Again, \( P \) is of the form (16).

Suppose, at last, that \( P_{vv} = 0 \) but \( P_{uu} \neq 0 \). Since the equations are symmetric with respect to \( u, v \) the same argument leads to the same conclusion. Inserting (16) into (15) we obtain the constraint

\[
b_1(r) b_2(r) = a \tag{22}
\]

where \( a \neq 0 \) is some constant.
III. GLOBAL SOLUTION WITH M=0

It is obtained by taking into account the Minkowski boundary condition and extending the solution from region IV to regions II and III with the help of the Penrose ansatz:

\[ P = r - b_1 (r) u \theta (u) - b_2 (r) v \theta (v) \]  \hspace{1cm} (23)

\[ Q = 1 - b_{1r} u \theta (u) - b_{2r} v \theta (v) \]  \hspace{1cm} (24)

Going to region II or region III we see that the approaching waves are impulsive \[ b_i (r) = H_i (r)_r \]  \hspace{1cm} (25)

\[ ds^2 = 2 du_i dw_i + 2 H_i (P) \delta (u_i) du_i^2 - dP^2 - P^2 d\varphi^2 \]  \hspace{1cm} (26)

where \[ u_1 = u, u_2 = v \]. They are induced by some impulse with energy-density

\[ \rho_i (r, u_i) = \frac{1}{2r} (r H_i (r)_r)_r \delta (u_i) \]  \hspace{1cm} (27)

due to a beam of pure radiation \[ r \], light \[ v \] or a point-particle moving with the speed of light \[ q \]. Then (22) becomes

\[ H_2 (r)_r = \frac{a}{H_1 (r)_r} \]  \hspace{1cm} (28)

It is clear that (28) prevents the study of two equal colliding waves, e.g. two ultrarelativistic black holes. This is a consequence of the simplifying assumption \[ M = 0 \]. Positive energy-density induces positive and increasing \[ H_i \], hence \[ b_i > 0, b_{ir} > 0, a > 0 \]. Applying the constraint (22) to (23,24) and changing notation to \[ b_1 \equiv b \] yields

\[ P = r - b u \theta (u) - \frac{a}{b} v \theta (v) \]  \hspace{1cm} (29)

\[ Q = 1 - b_{1r} u \theta (u) + \frac{ab_{ir}}{b^2} v \theta (v) \]  \hspace{1cm} (30)

The change of the relative sign in \[ Q \] is reminiscent of the similar change in the Babala solution \[ b \] which is one of the three diagonal vacuum plane waves with \[ M = 0 \]. Thus (29,30) may be considered as a one-function analog of the Babala solution, although they do not reduce to it when \[ b \] is constant. Eqs(29,30) also give

\[ e^{-U} = r - (rb_r + b - bb_r) u \theta (u) + \frac{a}{b^2} \left( rb_r - b - \frac{ab_{ir} v}{b} \right) v \theta (v) \]  \hspace{1cm} (31)

The presence of null matter at the boundaries is signalled in the coordinates (1) by the discontinuities in \[ U_u \] which break the O’Brien-Synge boundary conditions \[ b \]. The terms linear in \[ u \] and \[ v \] disappear from (31) when \[ rb_r \pm b = 0 \] and these conditions are satisfied simultaneously only by a trivial \[ b \].

IV. STRUCTURE OF THE INVARIANTS

More information about the solution may be learned from its invariants. The only non-trivial Ricci scalars are

\[ \Phi_{22} = \frac{1}{2} R_{uu} = -\frac{1}{2} e^U (P_{uu} P)_r = re^U \rho_1 (r, u) \]  \hspace{1cm} (32)

\[ \Phi_{00} = \frac{1}{2} R_{vv} = -\frac{1}{2} e^U (P_{vv} P)_r = re^U \rho_2 (r, v) \]  \hspace{1cm} (33)

from which one can deduce the energy-momentum tensor:
\[ T_{\mu\nu} = 2re^U (\rho_1 l_{\mu} l_{\nu} + \rho_2 n_{\mu} n_{\nu}) \] (34)

where \( l_{\mu}, n_{\mu} \) are the first two vectors of the usual NP tetrad for (2) [1]. There are two planes of null dust with variable energy densities. In regions II, III they coincide with (27) but along the boundaries of IV they become dependent on the other null coordinate because \( re^U \rho_1 \neq \rho_1 \). The factor \( e^U \) in (32-34) is well-known in plane wave solutions with thin shells of null-matter [1], [2]. Eq(29) shows that in regions II and III the single pp-waves have coordinate singularities \( P = 0 \). Eqs(32,33) tell that they turn into curvature singularities on the boundaries of the interaction region at points \( v = 0, u = \frac{r}{b} \) and \( u = 0, v = \frac{r}{a} \).

The only non-trivial Weyl scalars are

\[ \Psi_4 = \frac{P_{uu}}{P} + \Phi_{22} = -\frac{b^2 \delta (u)}{rb - au\theta (v)} + \Phi_{22} \] (35)

\[ \Psi_0 = \frac{P_{vv}}{P} + \Phi_{00} = -\frac{a\delta (v)}{b(r - bu\theta (u))} + \Phi_{00} \] (36)

From (27,32,33,35,36) it is clear that the interaction region is conformally flat. The Weyl scalars confirm that the approaching waves are impulsive. They, like the Ricci scalars, become singular when \( P \) vanishes. These point curvature singularities are generic and cannot be avoided by a careful choice of \( b \). In regions II, III eqs(35,36) coincide with the expressions derived in [3].

V. SOME EXAMPLES

The energy-densities may be given as functions of \( b \) by (22,25,27):

\[ \rho_1 = \frac{(rb)}{2r} \delta (u) \] (37)

\[ \rho_2 = \frac{a}{2r} \left( \frac{r}{b} \right) \delta (v) \] (38)

The second density is positive when \( \frac{r}{b} \) is an increasing function. One possible solution includes an ultrarelativistic black hole approaching from region II and is given by

\[ b = \frac{4\mu}{r} \] (39)

\[ H_1 = 4\mu \ln r \] (40)

\[ H_2 = \frac{ar^2}{8\mu} \] (41)

\[ \rho_1 = \frac{\mu}{2} \delta (r) \delta (u) \] (42)

\[ \rho_2 = \frac{a}{4\mu} \delta (v) \] (43)

\[ P = r - \frac{4\mu}{r} a\theta (u) - \frac{ar}{4\mu} v\theta (v) \] (44)

\[ Q = 1 + \frac{4\mu}{r^2} a\theta (u) - \frac{a}{4\mu} v\theta (v) \] (45)
\[ \Phi_{00} = \frac{ar^4 \delta(v)}{4\mu(r^4 - 16\mu^2 u^2 \theta(u))} \]  

(46)

\[ \Phi_{22} = \frac{8\mu^3 \delta(r) \delta(u)}{(4\mu - av\theta(v))^2} \]  

(47)

\[ \Psi_0 = -\frac{ar^2 u\theta(u) \delta(v)}{r^4 - 16\mu^2 u^2 \theta(u)} \]  

(48)

\[ \Psi_4 = \frac{8\mu^2 \delta(u)}{4\mu - av\theta(v)} \left( \frac{\mu\delta(r)}{4\mu - av\theta(v)} - \frac{2}{r^2} \right) \]  

(49)

where \( \mu \) is the momentum of the null point-particle. The wave arriving from region III is induced by a pure radiation burst of constant density across the wavefront. In fact, this is a plane wave [6]. Speaking loosely, this example describes the collision of an almost pure exterior solution with the simplest interior solution.

There is another solution with positive \( \rho_1 \) which are finite on the axis \( r = 0 \) and decrease when \( r \to \infty \). It is given by

\[ b = \frac{1}{r} \ln (c + r^2) \]  

(50)

\[ \rho_1 = \frac{1}{c + r^2} \delta(u) \]  

(51)

\[ \rho_2 = \frac{a}{\ln (c + r^2)} \left[ 1 - \frac{r^2}{(c + r^2) \ln (c + r^2)} \right] \delta(v) \]  

(52)

with \( c > e \). One can say that the waves are beam-like i.e. they have finite transverse extent, but their energy diverges. It seems impossible to arrange for finite energy of both waves when \( M = 0 \). We omit the lengthy expressions for the metric and its invariants because the solution possesses the general features established above.

**VI. CONCLUSION**

The assumption \( M = 0 \) in the case of colliding axisymmetric \( pp \)-waves is almost as restrictive as in the case of colliding plane waves, although the freedom in the solution extends to an arbitrary function \( b \) instead of arbitrary constants. The solution in the interaction region is still conformally flat and linear in \( u \) and \( v \). This indicates the presence of null matter along the boundaries, but anyway for \( pp \)-waves the distinction between pure gravitational and matter field components is not so clean-cut in view of relations like (27). The mechanism by which the coordinate singularities turn into curvature singularities is the same as for plane waves and has its roots in (14). The constraint (22) does not allow to study the simplest possible case of two equivalent approaching waves. Obviously, more exact solutions are necessary and non-trivial interactions between \( pp \)-waves of finite energy should be possible when the condition \( M = 0 \) is relaxed.

**Acknowledgements**

This work was supported by the Bulgarian National Fund for Scientific Research under contract F-632.

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