CONVOLUTION OPERATORS WITH SINGULAR MEASURES OF FRACTIONAL TYPE ON THE HEISENBERG GROUP

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Abstract. We consider the Heisenberg group $H^n = \mathbb{C}^n \times \mathbb{R}$. Let $\mu_{\gamma}$ be the fractional Borel measure on $H^n$ defined by

$$\mu_{\gamma}(E) = \int_{\mathbb{C}^n} \chi_E(z, w) \prod_{j=1}^{n} \eta_j(|w_j|^2 |w_j|^{-\frac{\gamma}{2}} dw,$$

where $0 < \gamma < 2n$, $\varphi(w) = \sum_{j=1}^{n} a_j |w_j|^2$, $w = (w_1, ..., w_n) \in \mathbb{C}^n$, $a_j \in \mathbb{R}$, and $\eta_j \in C_c^\infty(\mathbb{R})$. In this paper we study the set of pairs $(p, q)$ such that the right convolution operator with $\mu_{\gamma}$ is bounded from $L^p(H^n)$ into $L^q(H^n)$.

1. Introduction

Let $H^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with group law $(z, t) \cdot (w, s) = (z + w, t + s + (z, w))$ where $(z, w) = \frac{1}{2} Im(\sum_{j=1}^{n} z_j \cdot \overline{w_j})$. For $x = (x_1, ..., x_{2n}) \in \mathbb{R}_{2n}$, we write $x = (x', x'')$ with $x' \in \mathbb{R}^n$, $x'' \in \mathbb{R}^n$. So, $\mathbb{R}_{2n}$ can be identified with $\mathbb{C}^n$ via the map $\Psi(x', x'') = x' + ix''$. In this setting the form $(z, w)$ agrees with the standard symplectic form on $\mathbb{R}_{2n}$. Thus $H^n$ can be viewed as $\mathbb{R}_{2n} \times \mathbb{R}$ endowed with the group law

$$(x, t) \cdot (y, s) = (x + y, t + s + \frac{1}{2} B(x, y))$$

where the symplectic form $B$ is given by $B(x, y) = \sum_{j=1}^{n} (y_{n+j} x_j - y_j x_{n+j})$, with $x = (x_1, ..., x_{2n})$ and $y = (y_1, ..., y_{2n})$, with neutral element $(0, 0)$, and with inverse $(x, t)^{-1} = (-x, -t)$.

Let $\varphi : \mathbb{R}_{2n} \to \mathbb{R}$ be a measurable function, and let $\mu_{\gamma}$ be the fractional Borel measure on $H^n = \mathbb{R}_{2n} \times \mathbb{R}$ supported on the graph of $\varphi$, given by

$$(1) \quad \langle \mu_{\gamma}, f \rangle = \int_{\mathbb{R}_{2n}} f(x, \varphi(x)) \prod_{j=1}^{n} \eta_j(|w_j|^2 |w_j|^{-\frac{\gamma}{2}} dw$$

with $0 < \gamma < 2n$, and where the $\eta_j$'s are functions in $C_c^\infty(\mathbb{R})$ such that $0 \leq \eta_j \leq 1$, $\eta_j(t) \equiv 1$ if $t \in [-1, 1]$ and supp($\eta_j$) $\subset (-2, 2)$.

Let $T_{\mu_{\gamma}}$ be the right convolution operator by $\mu_{\gamma}$, defined by

$$(2) \quad T_{\mu_{\gamma}} f(x, t) = (f \ast \mu_{\gamma})(x, t) = \int_{\mathbb{R}_{2n}} f((x, t) \cdot (w, \varphi(w))^{-1}) \prod_{j=1}^{n} \eta_j(|w_j|^2 |w_j|^{-\frac{\gamma}{2}} dw.$$

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We are interested in studying the type set

\[ E_{\mu, \gamma} = \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in [0,1] \times [0,1] : \|T_\mu\|_{L^p - L^q} < \infty \right\} \]

where the \( L^p \) spaces are taken with respect to the Lebesgue measure on \( \mathbb{R}^{2n+1} \).

We say that the measure \( \mu_\gamma \) defined in (1) is \( L^p \)-improving if \( E_{\mu, \gamma} \) does not reduce to the diagonal \( \frac{1}{p} = \frac{1}{q} \).

This problem is well known if in [2] we consider \( \gamma = 0 \) and replace the Heisenberg group convolution with the ordinary convolution in \( \mathbb{R}^{2n+1} \). If the graph of \( \varphi \) has non-zero Gaussian curvature at each point, a theorem of Littman (see [4]) implies that \( E_{\mu, \gamma} \) is the closed triangle with vertices \((0,0), (1,1), \) and \( \left( \frac{2n+1}{2n+2}, \frac{1}{2n+2} \right) \) (see [5]). A very interesting survey of results concerning the type set for convolution operators with singular measures can be found in [6].

Returning to our setting \( \mathbb{H}^n \), in [8] and [9] S. Secco obtains \( L^p \)-improving properties of measures supported on curves in \( \mathbb{H}^1 \), under certain assumptions. In [7] F. Ricci and E. Stein showed that the type set of the measure given by (1), for the case \( \varphi(w) = 0 \), \( \gamma = 0 \) and \( n = 1 \), is the triangle with vertices \((0,0), (1,1), \) and \( \left( \frac{1}{2}, \frac{1}{2} \right) \).

In [2], the authors adapt the work of Ricci and Stein for the case of manifolds quadratic hypersurfaces in \( \mathbb{R}^{2n+1} \), there we also give some examples of surfaces with degenerate curvature at the origin.

We observe that if \( \left( \frac{1}{p}, \frac{1}{q} \right) \in E_{\mu, \gamma} \) then

\[ p \leq q, \quad \frac{1}{q} \geq \frac{2n+1}{p} - 2n, \quad \frac{1}{q} \geq \frac{1}{(2n+1)p}. \]

Indeed, the first inequality follows from Lemma 3 in [2], replacing the sets \( A_\delta \) and \( F_{\delta, \gamma} \) in the proof of Lemma 4 in [2] by the sets

\[ A_\delta' = \left\{ (x, t) \in \mathbb{R}^{2n} \times \mathbb{R} : x \in \tilde{D} \cap |t - \varphi(x)| \leq \frac{1}{4} \right\} \]

and

\[ F_{\delta, \gamma} = \left\{ y \in \tilde{D} : \|x - y\|_{\mathbb{R}^{2n}} \leq \frac{\delta}{4n(1 + \|
abla \varphi\|_{L^\infty})} \right\} \]

where \( \tilde{D} \) is a closed disk in \( \mathbb{R}^{2n} \) contained in the unit disk centered in the origin such that the origin not belongs to \( \tilde{D} \), we observe that the argument utilized in the proof of Lemma 4 in [2] works in this setting so we get the others two equalities.

Since \( 0 < \gamma < 2n \) it is clear that \( \|T_\mu f\|_p \leq c\|f\|_p \) for all Borel function \( f \in L^p(\mathbb{H}^n) \) and all \( 1 \leq p \leq \infty \), so \( \left( \frac{1}{p}, \frac{1}{q} \right) \in E_{\mu, \gamma} \).

In Lemma 4, section 2 below, we obtain the following necessary condition for the pair \( \left( \frac{1}{p}, \frac{1}{q} \right) \) to be in \( E_{\mu, \gamma} \):

\[ \frac{1}{q} \geq \frac{1}{p} \geq \frac{2n - \gamma}{2n + 2}. \]

Let \( D \) be the point of intersection, in the \( \left( \frac{1}{p}, \frac{1}{q} \right) \) plane, between the lines \( \frac{1}{q} = \frac{2n+1}{p} - 2n, \quad \frac{1}{q} = \frac{1}{p} = \frac{2n - \gamma}{2n + 2} \) and let \( D' \) be its symmetric with respect to the non principal diagonal. So

\[ D = \left( \frac{4n^2 + 2n + \gamma}{2n(2n + 2)}, \quad \frac{2n + (2n + 1)\gamma}{2n(2n + 2)} \right) = \left( \frac{1}{pd}, \frac{1}{qd} \right) y D' = \left( 1 - \frac{1}{qd}, 1 - \frac{1}{pd} \right). \]

Thus \( E_{\mu, \gamma} \) is contained in the closed trapezoid with vertices \((0,0), (1,1), D \) and \( D' \).

Finally, let \( C_\gamma \) be the point of intersection of the lines \( \frac{1}{q} = 1 - \frac{1}{p} \) and \( \frac{1}{q} = \frac{1}{p} = \frac{2n - \gamma}{2n + 2} \), thus \( C_\gamma = \left( \frac{4n + 2 - 2n - \gamma}{2n + 2}, \frac{2n - \gamma}{2n + 2} \right) \).
In section 3 we prove the following results:

**Theorem 1.** If \( \mu_\gamma \) is the fractional Borel measure defined in (4), supported on the graph of the function \( \varphi(w) = \sum_{j=1}^{n} a_j |w_j|^2 \), with \( n \in \mathbb{N}, a_j \in \mathbb{R} \) and \( w_j \in \mathbb{R}^2 \), then the interior of the type set \( E_{\mu_\gamma} \) coincide with the interior of the trapezoid with vertices \((0,0), (1,1), D \) and \( D' \). Moreover the semi-open segments \([1,1); (1, q_D^{-1})\) and \([(0,0); (1 - q_D^{-1}, 1 - p_D^{-1})\) are contained in \( E_{\mu_\gamma} \).

**Theorem 2.** Let \( \mu_\gamma \) be a fractional Borel measure as in Theorem 1. Then \( C_\gamma \in E_{\mu_\gamma} \).

Let \( \tilde{\mu_\gamma} \) be the Borel measure given by

\[
(\tilde{\mu_\gamma}, f) = \int_{\mathbb{R}^{2n}} f(w, |w|^{2m}) \eta(|w|) dw,
\]

where \( m \in \mathbb{N}, \gamma = \frac{2m - 1}{(n+1)m} \), and \( \eta \) is a function in \( C_c^\infty(\mathbb{R}) \) such that \( 0 \leq \eta \leq 1 \), \( \eta(t) \equiv 1 \) if \( t \in [-1, 1] \) and \( \text{supp}(\eta) \subset (-2, 2) \).

In a similar way we characterize the type set of the Borel measure \( \tilde{\mu_\gamma} \), supported on the graph of the function \( \varphi(w) = |w|^{2m} \). In fact we prove

**Theorem 3.** Let \( \tilde{\mu_\gamma} \) be the Borel measure defined in (4) with \( \gamma = \frac{2m - 1}{(n+1)m} \), where \( n \in \mathbb{N} \) and \( m \in \mathbb{N}_{\geq 2} \). Then the type set \( E_{\tilde{\mu_\gamma}} \) is the closed triangle with vertices

\[
A = (0,0), \quad B = (1,1), \quad C = \left( \frac{2n + 1}{2n + 2}, \frac{1}{2n + 2} \right).
\]

This result improves to the one obtained in Theorem 2 in [2].

Throughout this work, \( c \) will denote a positive constant not necessarily the same at each occurrence.

2. Auxiliary results

**Lemma 4.** Let \( \mu_\gamma \) be the fractional Borel measure defined by (2), where \( \varphi(w) = \sum_{j=1}^{n} a_j |w_j|^2 \) and \( 0 < \gamma < 2n \). If \( \left( \frac{1}{\mu_\gamma}, \frac{1}{\eta} \right) \in E_{\mu_\gamma} \), then \( \frac{1}{\mu_\gamma} \geq \frac{1}{\mu_\gamma} - \frac{2n - 2}{2n + 2} \).

**Proof.** For \( 0 < \delta \leq 1 \), we define \( Q_\delta = D_{\delta} \times \left[-(4M + n)\delta^2, (4M + n)\delta^2 \right] \), where \( D_{\delta} = \{ x \in \mathbb{R}^{2n}; \|x\| \leq \delta \} \) and \( M = \max \{|\varphi(y)|; y \in D_1\} \). We put

\[
A_\delta = \left\{ (x,t) \in D_{\frac{\delta}{2}} \times \mathbb{R}; |t - \varphi(x)| \leq 2M\delta^2 \right\}.
\]

Let \( f_\delta = \chi_{Q_\delta} \). We will prove first that \( |(f_\delta * \mu_\gamma)(x,t)| \geq c\delta^{2n-\gamma} \) for all \((x,t) \in A_\delta \), where \( c \) is a constant independent of \( \delta \).

If \( (x,t) \in A_\delta \), we have that

\[
(x,t) \cdot (y, \varphi(y))^{-1} \in Q_\delta \text{ for all } y \in D_{\frac{\delta}{2}};
\]

indeed, \((x,t) \cdot (y, \varphi(y))^{-1} = (x - y, t - \varphi(y) - \frac{1}{\delta}B(x,y))\), from the homogeneity of \( \varphi \) and since \( \frac{1}{\delta}B(x,y) \leq n \|x\|_{\mathbb{R}^{2n}} \|x - y\|_{\mathbb{R}^{2n}} \), (3) follows . So

\[
|(f_\delta * \mu)(x,t)| = \int_{\mathbb{R}^{2n}} f_\delta \left((x,t) \cdot (y, \varphi(y))^{-1}\right) \prod_{j=1}^{n} \eta_j(|y_j|^2 |y_j|^{-\gamma} dy
\geq \int_{D_{\frac{\delta}{2}}} |y|^{-\gamma} \prod_{j=1}^{n} \eta_j(|y_j|^2) dy = \int_{D_{\frac{\delta}{2}}} |y|^{-\gamma} dy = c\delta^{2n-\gamma}
\]
for all \((x, t) \in A_\delta\) and all \(0 < \delta < 1/2\). Thus
\[
\|f_\delta \ast \mu_\gamma\|_q \geq \left( \int_{A_\delta} |f \ast \mu_\gamma|^q \right)^{\frac{1}{q}} \geq c\delta^{2n-\gamma} |A_\delta|^{\frac{1}{q}} = c\delta^{2n-\gamma + \frac{1}{q}(2n+2)}.
\]

On the other hand, \(\frac{1}{p}, \frac{1}{q} \in E_\mu\) implies
\[
\|f_\delta \ast \mu_\gamma\|_q \leq c \|f_\delta\|_p = c\delta^{\frac{1}{p}(2n+2)},
\]
therefore \(\delta^{2n-\gamma + \frac{1}{q}(2n+2)} \leq c\delta^{\frac{1}{p}(2n+2)}\) for all \(0 < \delta < 1\) small enough, then
\[
\frac{1}{q} \geq \frac{1}{p} - \frac{2n - \gamma}{2n + 2}.
\]

The following two lemmas deal on certain identities that involve to the Laguerre polynomials. We recall the definition of these polynomials: the Laguerre polynomials \(L_n^\alpha(x)\) are defined by the formula
\[
L_n^\alpha(x) = e^x \frac{d^n}{dx^n} \left( e^{-x} x^n \right), \quad n = 0, 1, 2, ...
\]
for arbitrary real number \(\alpha > -1\).

**Lemma 5.** If \(\text{Re}(\beta) > -1\), then
\[
\int_0^\infty \sigma^\beta L_k^{-1}(\sigma) e^{-\sigma(\frac{x}{r} + i\epsilon)} d\sigma = \frac{1}{k!} \left[ \frac{d^k}{dr^k} \left( \frac{\Gamma(\beta + 1)}{(1-r)^n} \right) \right]_{r=0},
\]
for \(n \in \mathbb{N}\) and \(k \in \mathbb{N} \cup \{0\}\).

**Proof.** Let \(0 < \epsilon < 1\) be fixed. From the generating function identity (4.17.3) in [3] p. 77, we have
\[
\sum_{j \geq 0} \sigma^\beta L_j^{-1}(\sigma) e^{-\sigma(\frac{x}{r} + i\epsilon)} r^j = \frac{1}{(1-r)^n} \sigma^\beta e^{-\sigma(\frac{x}{r} + i\epsilon)} , \quad |r| < 1.
\]
Since \(\left| L_j^{-1}(\sigma) e^{-\sigma} \right| \leq \frac{(j+n-1)!}{j!(n-1)!} \) for all \(\sigma > 0\) (see proposition 4.2 in [13]), the series in \(\mathbb{R}\) is uniformly convergent on the interval \([\epsilon, \frac{1}{\epsilon}]\). Integrating on this interval we obtain
\[
\sum_{j \geq 0} \left( \int_\epsilon^{\frac{1}{\epsilon}} \sigma^\beta L_j^{-1}(\sigma) e^{-\sigma(\frac{x}{r} + i\epsilon)} d\sigma \right) r^j = \frac{1}{(1-r)^n} \int_\epsilon^{\frac{1}{\epsilon}} \sigma^\beta e^{-\sigma(\frac{x}{r} + i\epsilon)} d\sigma,
\]
so
\[
\int_\epsilon^{\frac{1}{\epsilon}} \sigma^\beta L_k^{-1}(\sigma) e^{-\sigma(\frac{x}{r} + i\epsilon)} d\sigma = \frac{1}{k!} \left[ \frac{d^k}{dr^k} \left( \frac{1}{(1-r)^n} \right) \right]_{r=0} \int_\epsilon^{\frac{1}{\epsilon}} \sigma^\beta e^{-\sigma(\frac{x}{r} + i\epsilon)} d\sigma,
\]
now let us computation \(\left[ \frac{d^k}{dr^k} \left( \int_\epsilon^{\frac{1}{\epsilon}} \sigma^\beta e^{-\sigma(\frac{x}{r} + i\epsilon)} d\sigma \right) \right]_{r=0}\). We start to compute first the derivatives of the function \(u \to \int_\epsilon^{\frac{1}{\epsilon}} \sigma^\beta e^{-\sigma u} d\sigma\), where \(\text{Re}(u) > 0\). We define
Since $\lim_{\epsilon \to 0} I_1(u, \epsilon) = 0$ for all $\epsilon > 0$, we have that

$$\lim_{\epsilon \to 0} \frac{d^k}{du^k} I_1(u, \epsilon) = 0$$

for $j = 1, 2$ and all $k \geq 0$. It is easy to check that

$$I_2(u, \epsilon) = e^{\beta+1} \int_{[1, u]} z^\beta e^{-cz} dz.$$

Since $Re(\beta) > -1$ we have that $\lim_{\epsilon \to 0} I_2(u_0, \epsilon) = 0$. From the analyticity of the function $z \to z^\beta e^{-cz}$ on the region $\{z : Re(z) > 0\}$ it follows that

$$\left[ \frac{d^k}{du^k} I_2(u, \epsilon) \right]_{u=u_0} = e^{\beta+1} \left[ \frac{d^{k-1}}{du^{k-1}} u^\beta e^{-cu} \right]_{u=u_0},$$

and

$$\lim_{\epsilon \to 0} \left[ \frac{d^k}{du^k} I_2(u, \epsilon) \right]_{u=u_0} = 0$$

for all $k \geq 0$.

Analogously and taking account the rapid decay of the function $z \to e^{-cz}$ on the region $\{z : Re(z) > 0\}$ we obtain that $\lim_{\epsilon \to 0} \left[ \frac{d^k}{du^k} I_1(u, \epsilon) \right]_{u=u_0} = 0$ for all $k \geq 0$, so (9) follows. To derive in (8), from the Leibniz’ formula and (9) it follows that

$$\int \sigma^\beta e^{-\epsilon u} d\sigma = u^{-(\beta+1)} \int z^\beta e^{-cz} dz$$

to apply the Cauchy’s Theorem we have

$$\int_{\epsilon}^{R} \sigma^\beta e^{-\sigma u} d\sigma = u^{-(\beta+1)} \int_{\epsilon}^{R} z^\beta e^{-cz} dz$$

where

$$I_1(u, \epsilon) = \int_{[1, u]} z^\beta e^{-cz} dz$$

and

$$I_2(u, \epsilon) = \int_{[1, u]} z^\beta e^{-cz} dz$$

are line integrals on $C$. Now we will prove that for each $u_0 \in C$ with $Re(u_0) > 0$ the following identity holds

$$\lim_{\epsilon \to 0} \left[ \frac{d^k}{du^k} I_1(u, \epsilon) \right]_{u=u_0} = 0$$

for $j = 1, 2$ and all $k \geq 0$. It is easy to check that

$$I_2(u, \epsilon) = \epsilon^{\beta+1} \int_{[1, u]} z^\beta e^{-cz} dz.$$

Finally, from (10), to apply the chain rule to the function $r \to \int_{\epsilon}^{R} \sigma^\beta e^{-\epsilon u(r)} d\sigma$ where

$$u(r) = \frac{1}{2} + \frac{r}{1-r} + i\xi$$

and the Leibniz’ formula give, to do $\epsilon \to 0$ in (10), that

$$\int_{0}^{\infty} \sigma^\beta L_k^{-1}(\sigma) e^{-\sigma(\frac{1}{2} + i\xi)} d\sigma = \lim_{\epsilon \to 0} \int_{\epsilon}^{R} \sigma^\beta L_k^{-1}(\sigma) e^{-\sigma(\frac{1}{2} + i\xi)} d\sigma$$

$$= \frac{1}{k!} \left[ \frac{d^k}{dr^k} \left( \frac{\Gamma(\beta+1)}{(1-r)^n(\frac{1}{2} + \frac{r}{1-r} + i\xi)^{\beta+1}} \right) \right]_{r=0}.$$
Lemma 6. If \( \text{Re}(\beta) > -1 \) and \( w(\xi) = -\frac{\xi}{e^\beta + \xi} \) (\( \xi \in \mathbb{R} \)), then
\[
\int_0^\infty \sigma^\beta L_k^{n-1}(\sigma) e^{-\sigma (\frac{1}{2} + i\xi)} d\sigma = \frac{\Gamma(\beta + 1)}{\left(\frac{1}{2} + i\xi\right)^{\beta+1}} \sum_{j+l=k} \frac{\Gamma(n-1-\beta+j)\Gamma(\beta+1+l)\,w(\xi)^j}{\Gamma(n-1-\beta)\Gamma(\beta+1)\,j!l!},
\]
for \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \cup \{0\} \).

Proof. We will start find the power series centered at \( r = 0 \) of the following function
\[
Q(r) = \frac{1}{(1-r)^n\left(\frac{1}{2} + \frac{\xi}{1-r} + i\xi\right)^{\beta+1}}, \quad |r| < 1.
\]
We observe that
\[
Q(r) = \frac{1}{(1-r)^n\beta^{-1}\left(\frac{1}{2} + i\xi + r\left(\frac{1}{2} - i\xi\right)\right)^{\beta+1}},
\]
doing \( w = -\frac{\xi}{e^\beta + \xi} \), we obtain
\[
Q(r) = \frac{1}{\left(\frac{1}{2} + i\xi\right)^{\beta+1}(1-r)^n\beta^{-1}(1-rw)^{\beta+1}}.
\]
A simple computation gives
\[
(1-r)^{-n+\beta+1} = 1 + \sum_{j \geq 1} (n-1-\beta)(n-1-\beta+1)...(n-1-\beta+j-1)\frac{r^j}{j!}
\]
\[(1)
= \sum_{j \geq 0} \frac{\Gamma(n-1-\beta+j)\,r^j}{\Gamma(n-1-\beta)\,j!}.
\]
Analogously we have
\[
(1-rw)^{-\beta-1} = \sum_{j \geq 0} \frac{\Gamma(\beta+1+j)\,(rw)_j^j}{\Gamma(\beta+1)\,j!}.
\]
Thus
\[
Q(r) = \frac{1}{\left(\frac{1}{2} + i\xi\right)^{\beta+1}} \left(\sum_{j+l \geq 0} \frac{\Gamma(n-1-\beta+j)\Gamma(\beta+1+l)\,r^j\,w^l}{\Gamma(n-1-\beta)\Gamma(\beta+1)\,j!l!}\right).
\]
Finally, from Lemma 5 it follows
\[
\int_0^\infty \sigma^\beta L_k^{n-1}(\sigma) e^{-\sigma (\frac{1}{2} + i\xi)} d\sigma = \frac{\Gamma(\beta + 1)}{\left(\frac{1}{2} + i\xi\right)^{\beta+1}} \sum_{j+l=k} \frac{\Gamma(n-1-\beta+j)\Gamma(\beta+1+l)\,w^l}{\Gamma(n-1-\beta)\Gamma(\beta+1)\,j!l!}.
\]

Lemma 7. If \( \text{Re}(\beta) > -1 \), then
\[
\sum_{j+l=k} \frac{\Gamma(n-1-\text{Re}(\beta)+j)\Gamma(\text{Re}(\beta)+1+l)\,1}{\Gamma(n-1-\text{Re}(\beta))\Gamma(\text{Re}(\beta)+1)\,j!l!} = \frac{(n+k-1)!}{(n-1)!k!},
\]
for \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \cup \{0\} \).
Proof. From (11) it obtains
\[
\sum_{j+l=k} \frac{\Gamma(n-1-\text{Re}(\beta)+j)\Gamma(\text{Re}(\beta)+1+l)}{\Gamma(n-1-\text{Re}(\beta))\Gamma(\text{Re}(\beta)+1)j!l!} = \frac{1}{k!} \left[ \frac{d^k}{dr^k} (1-r)^{-n+\text{Re}(\beta)+1}(1-r)^{-\text{Re}(\beta)-1} \right]_{r=0} = \frac{1}{k!} \left[ \frac{d^k}{dr^k} (1-r)^{-n} \right]_{r=0}.
\]
Since
\[
(1-r)^{-n} = \sum_{j\geq0} \frac{(n+j)!}{\Gamma(n)} \frac{r^j}{j!},
\]
we have
\[
\sum_{j+l=k} \frac{\Gamma(n-1-\text{Re}(\beta)+j)\Gamma(\text{Re}(\beta)+1+l)}{\Gamma(n-1-\text{Re}(\beta))\Gamma(\text{Re}(\beta)+1)j!l!} = \frac{(n+k)!}{(n-1)!k!}.
\]

3. The main results

To prove Theorem 1 we will decompose the operator \(T_{p\mu} \) of the following way: we consider a family \(\{T_{p\mu_k}\}_{k\in\mathbb{N}}\) of operators such that \(T_{p\mu} = \sum_{k\in\mathbb{N}} T_{p\mu_k} \|T_{p\mu}\|_{L^1} \sim 2^{-k(2n-\gamma)}\) and \(\|T_{p\mu}\|_{p,q} \sim 2^{k\gamma} \|T_{p\mu_0}\|_{p,q}\) where \(T_{p0}\) is the operator defined by (2), taking there \(\gamma = 0\) and \(\varphi(w) = \sum_{j=1}^n |a_j| |w|^2\). Then Theorem 1 will follow from Theorem 1 in [2], the Riesz-Thorin convexity Theorem and Lemma 4.

Proof of Theorem 1. For each \(k\in\mathbb{N}\) we define
\[
A_k = \{y = (y_1, \ldots, y_n) \in (\mathbb{R}^2)^n : 2^{-k} < |y_j| \leq 2^{-k+1}, j = 1, 2, \ldots, n\}
\]
Let \(\mu_k\) be the fractional Borel measure given by
\[
\mu_k(E) = \int_{A_k} \chi_E(y, \varphi(y)) \prod_{j=1}^n \eta_j \left(|y_j|^2 \right) |y_j|^{-\theta} \, dy
\]
and let \(T_{p\mu_k}\) be its corresponding convolution operator, i.e: \(T_{p\mu_k} f = f \ast \mu_k\). Now, it is clear that \(\mu_\gamma = \sum_k \mu_k\) and \(\|T_{p\mu}\|_{p,q} \leq \sum_k \|T_{p\mu_k}\|_{p,q}\). For \(f \geq 0\) we have that
\[
\int f(y, s) d\mu_k(y, s) \leq 2^{k\gamma} \int_{\mathbb{R}^2} f(y, \varphi(y)) \prod_{j=1}^n \eta_j \left(|y_j|^2 \right) \, dy.
\]
Thus \(\|T_{p\mu}\|_{p,q} \leq c2^{k\gamma} \|T_{p\mu_0}\|_{p,q}\) from Theorem 1 in [2] it follows that
\[
\|T_{p\mu}\|_{2n+2, 2n+2} \leq c2^{k\gamma}.
\]
It is easy to check that \(\|T_{p\mu}\|_{L^1} \leq |\mu_k(\mathbb{R}^{2n+1})| \sim \int_{A_k} |y|^{-\gamma} \, dy = c2^{-k(2n-\gamma)}\).

For \(0 < \theta < 1\), we define
\[
\left(\frac{1}{p^\theta} \frac{1}{q^\theta}\right) = \left(\frac{2n+1}{2n+2} \frac{1}{2n+2}\right) (1-\theta) + (1, 1)\theta,
\]
by the Riesz convexity Theorem we have
\[
\|T_{p\mu}\|_{p^\theta, q^\theta} \leq c2^{k\gamma(1-\theta) - k(2n-\gamma)\theta}
\]
choosing \(\theta\) such that \(k\gamma(1-\theta) - k(2n-\gamma)\theta = 0\) result \(\sup_{k\in\mathbb{N}} \|T_{p\mu_k}\|_{p^\theta, q^\theta} \leq c \leq \infty\). A simple computation gives \(\theta = \frac{2n-\gamma}{2n}\), then \(\left(\frac{1}{p^\theta}, \frac{1}{q^\theta}\right) = \left(\frac{1}{pD}, \frac{1}{qD}\right)\), so \(\|T_{p\mu}\|_{pD, qD} \leq c\),
where $c$ no depend on $k$. Interpolating once again, but now between the points $\left(\frac{1}{p_0}, \frac{1}{q_0}\right)$ and $(1, 1)$ we obtain, for each $0 < \tau < 1$ fixed
\[
\|T_{\mu_k}\|_{p_\tau, q_\tau} \leq c 2^{-k(2n-\gamma)\tau},
\]
since $\|T_{\mu_k}\|_{p, q} \leq \sum_k \|T_{\mu_k}\|_{p, q}$ and $0 < \gamma < 2n$, it follows that
\[
\|T_{\mu}\|_{p_\tau, q_\tau} \leq c \sum_{k \in \mathbb{N}} 2^{-k(2n-\gamma)\tau} < \infty,
\]
by duality we also have
\[
\|T_{\mu}\|_{\frac{p}{2}, \frac{1}{1-\frac{\gamma}{2}}} \leq c_\tau < \infty.
\]
Finally, the theorem follows from the Riesz convexity Theorem, the restrictions that appear in \cite{3} and Lemma 4. \hfill \Box

To prove Theorem 2, we will consider an auxiliary operator $T_N$, with $N \in \mathbb{N}$ fixed, which will be embedded in an analytic family $T_{N,z}$ of operators on the strip $-\frac{2n-\gamma}{2+\gamma} \leq \Re(z) \leq 1$ such that
\[
\begin{align*}
\|T_{N,z}(f)\|_{L^\infty(\mathbb{H}^n)} &\leq A_z \|f\|_{L^1(\mathbb{H}^n)} &\Re(z) = 1 \\
\|T_{N,z}(f)\|_{L^2(\mathbb{H}^n)} &\leq A_z \|f\|_{L^2(\mathbb{H}^n)} &\Re(z) = -\frac{2n-\gamma}{2+\gamma}
\end{align*}
\]
where $A_z$ will depend admissibly on the variable $z$ and it will not depend on $N$. We denote $T_N = T_{N,0}$. By Stein’s theorem on complex interpolation, it will follow that the operator $T_N$ will be bounded from $L^{p_\alpha}(\mathbb{H}^n)$ into $L^{p_\beta}(\mathbb{H}^n)$, where $\left(\frac{1}{p_0}, \frac{1}{p'_0}\right) = C_\gamma$, uniformly in $N$. If we see that $T_N f(x,t) \rightarrow cT_{\mu}\ f(x,t)$ a.e. $(x,t)$ as $N \rightarrow \infty$, then Theorem 2 will follow from Fatou’s Lemma.

To prove the second inequality in (12) we will see that such a family will admit the expression
\[
T_{N,z}(f)(x,t) = (f * K_{N,z})(x,t),
\]
where $K_{N,z} \in L^1(\mathbb{H}^n)$, moreover it is a poliradial function (i.e. the values of $K_{N,z}$ depend on $|w_1|, \ldots, |w_n|$ and $\tau$). Now our operator $T_{N,z}$ can be realized as a multiplication of operators via the group Fourier transform, i.e.
\[
\widehat{T_{N,z}(f)}(\lambda) = \hat{f}(\lambda) \widehat{K_{N,z}}(\lambda)
\]
where, for each $\lambda \neq 0$, $\widehat{K_{N,z}}(\lambda)$ is an operator on the Hilbert space $L^2(\mathbb{R}^n)$ given by
\[
\widehat{K_{N,z}}(\lambda)g(\xi) = \int_{\mathbb{H}^n} K_{N,z}(\xi, t) \pi_\lambda(\xi, t) g(\xi) d\xi dt.
\]
It then follows from Plancherel’s theorem for the group Fourier transform that
\[
\|T_{N,z}f\|_{L^2(\mathbb{H}^n)} \leq A_z \|f\|_{L^2(\mathbb{H}^n)}
\]
if and only if
\[
\left\|\widehat{K_{N,z}}(\lambda)\right\|_{op} \leq A_z
\]
uniformly over $N$ and $\lambda \neq 0$. Since $K_{N,z}$ is a poliradial integrable function, then by a well known result of Geller (see Lemma 1.3, p. 213 in \cite{1}), the operators $\widehat{K_{N,z}}(\lambda) : L^2(\mathbb{H}^n) \rightarrow L^2(\mathbb{H}^n)$ are, for each $\lambda \neq 0$, diagonal with respect to a Hermite basis for $L^2(\mathbb{R}^n)$. This is
\[
\widehat{K_{N,z}}(\lambda) = C_\gamma (\delta_{\gamma, \alpha} \nu_{N,z}(\alpha, \lambda))_{\gamma, \alpha \in \mathbb{N}_0^2}
\]
where $C_n = (2\pi)^n$, $\alpha = (\alpha_1, ..., \alpha_n)$, $\delta_{\gamma, \alpha} = 1$ if $\gamma = \alpha$ and $\delta_{\gamma, \alpha} = 0$ if $\gamma \neq \alpha$, and the diagonal entries $\nu_{N,z}(\alpha_1, ..., \alpha_n, \lambda)$ can be expressed explicitly in terms of the Laguerre transform. We have in fact
\[
\nu_{N,z}(\alpha_1, ..., \alpha_n, \lambda) = \int_0^\infty ... \int_0^\infty K_{N,z}^\lambda(r_1, ..., r_n) \prod_{j=1}^n (r_j^j |\lambda| r_j^e)e^{-\frac{1}{2} |\lambda| r_j^f} \, dr_1 ... dr_n
\]
where $L_k^0(s)$ are the Laguerre polynomials, i.e. $L_k^0(s) = \sum_{i=0}^k \binom{k-i}{i} (-1)^i$ and $K_{N,z}^\lambda(s) = \int \mathcal{K}_{N,z}(s,t)e^{i\lambda t} dt$. Now (17) implies that (18) holds with $\lambda, \nu$ independent of $N, \alpha_1$ and $\alpha_j$, and then we obtain the boundedness of $L^2(\mathbb{R}^n)$ that is stated in (12).

We consider the family $\{I_z\}_{z \in \mathbb{C}}$ of distributions on $\mathbb{R}$ that arises by analytic continuation of the family $\{\gamma\}$ of functions, initially given when $Re(z) > 0$ and $s \in \mathbb{R} \setminus \{0\}$ by
\[
I_z(s) = \frac{2^{-s} \pi}{\Gamma(s)} |s|^{s-1}.
\]
In particular, we have $\tilde{I}_z = I_{1-z}$, also $I_0 = c\delta$ where $\tilde{\gamma}$ denotes the Fourier transform on $\mathbb{R}$ and $\delta$ is the Dirac distribution at the origin on $\mathbb{R}$.

Let $H \in S(\mathbb{R})$ such that $supp(H) \subseteq (-1,1)$ and $\int \tilde{H}(t) dt = 1$. Now we put $\phi_N(t) = H(\frac{t}{N})$ thus $\phi_N(\xi) = N\tilde{H}(N\xi)$ and $\phi_N \to \delta$ in the sense of the distribution, as $N \to \infty$.

For $z \in \mathbb{C}$ and $N \in \mathbb{N}$, we also define $J_{N,z}$ as the distribution on $\mathbb{R}^n$ given by the tensor products
\[
J_{N,z} = \delta \otimes ... \otimes \delta \otimes \left( I_z \ast_\mathbb{R} \phi_N \right)
\]
where $\ast_\mathbb{R}$ denotes the usual convolution on $\mathbb{R}$ and $I_z$ is the fractional integration kernel given by (15). We observe that
\[
J_{N,0} = \delta \otimes ... \otimes \delta \otimes c\phi_N \to \delta \otimes ... \otimes \delta \otimes c\delta
\]
in the sense of the distribution as $N \to \infty$.

Proof of Theorem 2. Let $\{T_{N,z}\}$ be the family operators on the strip $-\frac{2n-\gamma}{2+\gamma} \leq Re(z) \leq 1$, given by
\[
T_{N,z} f = f \ast_\mathbb{R} \mu_{\gamma,z} \ast J_{N,z},
\]
where $J_{N,z}$ is given by (16) and $\mu_{\gamma,z}$ by
\[
\mu_{\gamma,z}(E) = \int_{\mathbb{R}^n} \chi_E(w, \varphi(w)) \prod_{j=1}^n \eta_j (|w_j|^2)^{\frac{z-1}{2}} dw.
\]
Now (17) implies that $T_{N,0} f(x,t) \to c\mu_{\gamma,z} f(x,t)$ a.e. $(x,t)$ as $N \to \infty$.

For $Re(z) = 1$ we have
\[
\mu_{\gamma,z} \ast J_{N,z}(x,t) = \left( I_z \ast_\mathbb{R} \phi_N \right) (t - \varphi(x)) \prod_{j=1}^n \eta_j (|x_j|^2)^\frac{z}{2} |x_j|^{|Im(z)|} \hat{\phi}_N(x),
\]
so \( \| \mu_{\gamma,z} \ast J_{N,z} \|_\infty \leq c \| \Gamma \left( \frac{z}{2} \right) \|_1^{-1} \). Then, for \( \Re(z) = 1 \), we obtain

\[
\| T_{N,z} f \|_\infty \leq \| f \ast \mu_{\gamma,z} \ast J_{N,z} \|_\infty \leq \| f \|_1 \| \mu_{\gamma,z} \ast J_{N,z} \|_\infty \leq c \| \Gamma \left( \frac{z}{2} \right) \|_1^{-1} \| f \|_1
\]

where \( c \) is a positive constant independent of \( N \) and \( z \).

We put \( K_{N,z} = \mu_{\gamma,z} \ast J_{N,z} \), for \( \Re(z) = -\frac{2n-\gamma}{2+\gamma} \) we have that \( K_{N,z} \in L^1(\mathbb{H}^n) \). Indeed

\[
K_{N,z}(x,t) = \left( I_z \ast \phi_N \right) \left( t - \varphi(x) \right) \prod_{j=1}^n \eta_j (|x_j|^2) |x_j|^{(z-1)\frac{1}{2}}
\]

since \( 0 < \gamma < 2n \) it follows that \( 2 + \Re((z-1)\frac{2}{n}) = 2 - \frac{2n+2}{2+\gamma} \) and hence \( \prod_{j=1}^n \eta_j (|x_j|^2) |x_j|^{(z-1)\frac{1}{2}} \in L^1(\mathbb{R}^{2n}) \), in the proof of Lemma 5 in \( 2 \) it shows that \( \left( I_z \ast \phi_N \right) \in L^1(\mathbb{R}) \). These two facts imply that \( K_{N,z} \in L^1(\mathbb{H}^n) \). In addition \( K_{z,N} \) is a polyanalogue function. Thus the operator \( \tilde{K}_{z,N}(\lambda) \) is diagonal with respect to a Hermite base for \( L^2(\mathbb{R}^n) \), and its diagonal entries \( \nu_{z,N}(\alpha, \lambda) \), with \( \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n \), are given by

\[
\nu_{z,N}(\alpha, \lambda) = \int_0^\infty \cdots \int_0^\infty K_{N,z}(r_1, ..., r_n) \prod_{j=1}^n \left( r_j L_0^0(|\lambda|r_j^2/2) e^{-\frac{1}{2} |\lambda|r_j^2} \right) dr_1 \cdots dr_n
\]

Thus, it is enough to study the integral

\[
\int_0^\infty \eta_1(r^2) L_0^0(|\lambda|r^2/2) e^{-\frac{1}{2} |\lambda| r^2} e^{i\lambda_1 r^2} r^{1+(z-1)\frac{1}{2}} dr,
\]

where \( \alpha_1 \in \mathbb{R} \) and \( \eta_1 \in C^\infty_c(\mathbb{R}) \). We make the change of variable \( \sigma = |\lambda|r^2/2 \) in such an integral to obtain

\[
\int_0^\infty \eta_1(r^2) L_0^0(|\lambda|r^2/2) e^{-\frac{1}{2} |\lambda| r^2} e^{i\lambda_1 r^2} r^{1+(z-1)\frac{1}{2}} dr
\]

\[
= 2^{\frac{n+1}{2+\gamma} \frac{1}{2(n+1)}} \int_0^\infty \eta_1 \left( \frac{2\sigma}{|\lambda|} \right) L_0^0(\sigma) e^{-\frac{\sigma}{2} e^{i2 \text{sgn}(\lambda) \alpha_1 \sigma \frac{|\lambda|}{|\lambda|} - \frac{i}{2(n+1)}}} d\sigma
\]

\[
= 2^{\frac{n+1}{2+\gamma} \frac{1}{2(n+1)}} \left( F_{\alpha_1,\beta} G_\lambda \right)(-2 \text{sgn}(\lambda) \alpha_1)
\]

\[
= 2^{\frac{n+1}{2+\gamma} \frac{1}{2(n+1)}} \left( F_{\alpha_1,\beta} * G_\lambda \right)(-2 \text{sgn}(\lambda) \alpha_1)
\]

where

\[
F_{\alpha_1,\beta}(\sigma) := \chi_{(0,\infty)}(\sigma) L_0^0(\sigma) e^{-\frac{\sigma}{2} e^{i\beta}}
\]

with \( \beta = \frac{(z-1)\frac{1}{2}}{2n} \), and

\[
G_\lambda(\sigma) := \eta_1 \left( \frac{2\sigma}{|\lambda|} \right).
\]

Now

\[
\left( F_{\alpha_1,\beta} * G_\lambda \right)(-2 \text{sgn}(\lambda) \alpha_1) \leq \left\| F_{\alpha_1,\beta} \ast G_\lambda \right\|_1 \leq \left\| F_{\alpha_1,\beta} \right\|_\infty \left\| G_\lambda \right\|_1 = \left\| F_{\alpha_1,\beta} \right\|_\infty \left\| \eta_1 \right\|_1.
\]
So it is enough to estimate $\|F_{\alpha, \beta}\|_\infty$. Since

$$F_{\alpha, \beta}(\xi) = \int_0^\infty \sigma^\beta L_{\alpha, \beta}^0 (\sigma) e^{-\sigma(\frac{1}{2} + i\xi)} d\sigma,$$

from Lemma 6, with $n = 1$, $k = \alpha_1$ and $\beta = \frac{(\alpha_1-1)\gamma}{2n}$ we obtain

$$F_{\alpha, \beta}(\xi) = \frac{\Gamma(\beta+1)}{(\frac{1}{2} + i\xi)^{\beta+1}} \sum_{j=0}^\infty \frac{\Gamma(-\beta+j)\Gamma(\beta+1+l)}{\Gamma(-\beta)\Gamma(\beta+1)} \frac{w^j}{j!}.$$

to take modulo in this expression and since $|w| = 1$ it follows that

$$|F_{\alpha, \beta}(\xi)| \leq \frac{\Gamma(-\beta)\Gamma(\beta+1)}{(\frac{1}{2} + i\xi)^{\beta+1}} \sum_{j=0}^\infty \frac{\Gamma(-\beta+j)\Gamma(\beta+1+l)}{\Gamma(-\beta)\Gamma(\beta+1)} \frac{1}{j!}.$$

From Lemma 7, with $n = 1$ and $k = \alpha_1$ we have

$$\sum_{j=0}^\infty \frac{\Gamma(-\beta+j)\Gamma(\beta+1+l)}{\Gamma(-\beta)\Gamma(\beta+1)} \frac{1}{j!} = 1.$$

So

$$|F_{\alpha, \beta}(\xi)| \leq \frac{\Gamma(-\beta)\Gamma(\beta+1)}{(\frac{1}{2} + i\xi)^{\beta+1}} \sum_{j=0}^\infty \frac{\Gamma(-\beta+j)\Gamma(\beta+1+l)}{\Gamma(-\beta)\Gamma(\beta+1)} \frac{1}{j!}.$$

Finally, for $Re(z) = -\frac{2n-\gamma}{2+\gamma}$, we obtain

$$|v_{\nu, N}(k, \lambda)| \leq c_{n, \gamma}|\nu_{1-z}(-\lambda)\phi_N(\lambda)||\lambda|^{-1+n(1+\frac{\gamma}{2})} \prod_{j=1}^\infty \|F_{\alpha, \beta}\|_\infty \|\tilde{\nu}\|_1$$

\[ \leq c_{n, \gamma} e^{\frac{1n(z)}{4}} \|H\|_\infty \left[ \Gamma\left(\frac{n+1}{2(2+\gamma)n}\right) \right]^n \left[ \Gamma\left(\frac{2n-\gamma}{2(2+\gamma)n}\right) \right]^n \prod_{j=1}^n \|\tilde{\nu}\|_1. \]

By (13) it follows, for $Re(z) = -\frac{2n-\gamma}{2+\gamma}$, that

$$\|T_{N, z} f\|_{L^2(\mathbb{H}^n)} \leq c_{n, \gamma} e^{\frac{1n(z)}{4}} \left[ \Gamma\left(\frac{1-z}{2+\gamma}\right) \right]^n \|f\|_{L^2(\mathbb{H}^n)}.$$
Proceeding as in the proof of Theorem 2 it follows, for \( \text{Re}(z) = 1 \), that \( ||U_{N,z}||_{1,\infty} \leq c ||\Gamma(z/2)||^{-1} \). Also it is clear that, for \( \text{Re}(z) = -n \), the kernel \( \tilde{\mu}_{(1-z)\gamma} \cdot J_{N,z} \in L^1(\mathbb{R}^n) \) and it is a radial function. Now, our operator \( (\tilde{\mu}_{(1-z)\gamma} \cdot J_{N,z})(\lambda) \) is diagonal, with diagonal entries \( \nu_{N,z}(k,\lambda) \) given by

\[
\nu_{N,z}(k,\lambda) = \frac{k!}{(k+n-1)!} \left( \int_0^\infty (\tilde{\mu}_{(-1-z)\gamma} \cdot J_{N,z})(s,-\lambda) L_k^{-n-1}(|\lambda|s^2/2) e^{-|\lambda|s^2/4} s^{2n-1} ds \right)
\]

which follows, for \(\gamma\) is a.e.\((x,\lambda)\) as \(N \to \infty\). Now, we estimate \( \eta \) as

\[
\eta(s^2) L_k^{-n-1}(|\lambda|s^2/2) e^{-|\lambda|s^2/4} e^{i\lambda s^2 m} s^{2n-1+(1-z)\gamma} ds.
\]

Now we study this integral, the change of variable \(\sigma = |\lambda| s^2 / 2\) gives

\[
\eta(s^2) L_k^{-n-1}(|\lambda|s^2/2) e^{-|\lambda|s^2/4} e^{i\lambda s^2 m} s^{2n-1+(1-z)\gamma} ds = \int_0^\infty \eta(\frac{2\sigma}{|\lambda|}) L_k^{-n-1}(\sigma) e^{-\frac{\sigma}{|\lambda|}} e^{i(2\sigma)|\lambda|^{1-m} \text{sgn}(\lambda) |\lambda|^{-n} \left( 1 + \frac{1-z\gamma}{2} \right)} d\sigma,
\]

where

\[
F_{k,\beta}(\sigma) := \chi_{(0,\infty)}(\sigma)L_k^{-n-1}(\sigma) e^{-\frac{\sigma}{|\lambda|}} e^{i(2\sigma)|\lambda|^{1-m} \text{sgn}(\lambda) |\lambda|^{-n} \left( 1 + \frac{1-z\gamma}{2} \right)},
\]

with \(\beta = n - 1 + \frac{1-z\gamma}{2}\),

\[
G_{\lambda}(\sigma) := \eta(2\sigma/|\lambda|)
\]

and

\[
R_{\lambda}(\sigma) = \chi_{(0,|\lambda|)}(\sigma)e^{i(2m \text{sgn}(\lambda)|\lambda|^{1-m}}}.
\]

Now

\[
||F_{k,\beta} \ast (G_{\lambda} \ast R_{\lambda})||_{\infty} \leq ||F_{k,\beta}||_1 ||G_{\lambda}||_1 ||R_{\lambda}||_{\infty}
\]

so it is enough to estimate the right side of this inequality. From Lemma 6 and Lemma 7 we obtain

\[
|F_{k,\beta}(\xi)| \leq \frac{\Gamma(n-1-\text{Re}(\beta)) \Gamma(\text{Re}(\beta)+1)}{(\frac{1}{2} + \xi)^{n+1}|\Gamma(n-1-\beta)| (n-1)!}.
\]

Since \(\text{Re}(z) = -n, \gamma = \frac{2(m-1)}{(n+1)m}\) and \(\beta = n - 1 + \frac{1-z\gamma}{2}\) we have \(\text{Re}(\beta) = n - \frac{1}{m}\), thus it follows that

\[
||F_{k,\beta}||_1 \leq \frac{c e^{i(2m \text{sgn}(\lambda)|\lambda|^{1-m})}}{G_{\lambda}(\frac{1}{2} - \gamma)} (n+1-k)! \int_0^\infty \frac{1}{(\frac{1}{2} + \xi)^{n+1-k}} d\xi,
\]

the last integral is finite for all \(n \geq 1\) and all \(m \geq 2\). It is clear that \(||G_{\lambda}||_1 = ||\tilde{\eta}||_1\).

Now, we estimate \(||R_{\lambda}||_{\infty}\). Taking account of Proposition 2, p. 332, in \(\text{[14]}\) we note that

\[
|R_{\lambda}(\xi)| \leq \int_0^{|\lambda|} e^{i(2m \text{sgn}(\lambda)|\lambda|^{1-m}) - \xi \sigma} d\sigma \leq \frac{C_m}{|\lambda|^{1-m}}
\]

and...
where the constant $C_n$ does not depend on $\lambda$. Then, for $Re(z) = -n$, from (21), (22) and (23) we obtain
\[
|\nu_{N,z}(k,\lambda)| \leq e^{\frac{k!}{(k+n-1)!}} |I_{1-z}(-\lambda)\phi_N(\lambda)||\lambda|^{-(n+\frac{1}{2})}\|G_{k,\lambda} \ast (\overline{G_{k,\lambda}})\|_{L^2}
\]
\[
\leq c_{n,m} \|H\|_{L^2} \|\tilde{\eta}\|_1 \int \frac{1}{\Gamma\left(\frac{1}{2}\right)} \left|\Gamma\left(\frac{2-1-\gamma}{2}\right)\right| \int_0^{\infty} \frac{1}{\left(\frac{1}{4} + \xi^2\right)^\frac{1}{2}(n+1-\gamma)} d\xi.
\]
By (11) it follows, for $Re(z) = -n$, that
\[
\|U_{N,z}f\|_{L^2(\mathbb{H}^n)} \leq c_{n,m} \int e^{\frac{im(z)+\gamma}{2}} \left|\Gamma\left(\frac{1}{2}\right)\right| \left|\Gamma\left(\frac{1}{2} - \gamma\right)\right| \left|f\right|_{L^2(\mathbb{H}^n)}.
\]
It is clear that the family $\{U_{N,z}\}$ satisfies, on the strip $-n \leq Re(z) \leq 1$, the hypothesis of the complex interpolation theorem. Thus $U_{N,0}$ is bounded from $L^{\frac{2n+2}{n+2}}(\mathbb{H}^n)$ into $L^{2n+2}(\mathbb{H}^n)$ uniformly in $N$, and letting $N$ tend to infinity we conclude that the operator $U_{R_n}$ is bounded from $L^{\frac{2n+2}{n+2}}(\mathbb{H}^n)$ into $L^{2n+2}(\mathbb{H}^n)$ for $n \in \mathbb{N}$. Finally, the theorem follows from the restrictions that appear in (3). \hfill $\Box$

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