A possible test for quadratic gravity in $d \geq 4$ dimensions

Janusz Garecki

Institute of Physics, University of Szczecin, Wielkopolska 15; 70-451 Szczecin, POLAND

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Abstract

In the letter we consider the Einsteinian strengths and dynamical degrees of freedom for quadratic gravity. We show that the purely metric quadratic gravity theories are much more stronger in Einsteinian sense than the competitive quadratic gravity theories which admit torsion.

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The notion of “strength of the field equations” was introduced in past by Einstein [1] in order to analyze systems of partial differential equations for physical fields. Later this notion was examined and effectively used in field theory by several authors [2–6]. In particular Schutz [3] pointed out that the Einsteinian strength of the field equations is closely connected with the number of the dynamical degrees of freedom which these equations admit in *Cauchy problem*.

In this letter we analyze the Einsteinian strength and related numbers for a typical purely metric 4th-order gravity [7–14] which follows from the Lagrangian

\[ L_g = \chi R + c_0 R^2 + c_1 |Ric|^2 + c_2 |Riem|^2, \]

where \( \chi, c_0, c_1, c_2 \) are some dimensionals constants, and for a typical “Poincaré gauge quadratic field theory of gravity” (PGT) with torsion [15–20].

The gravitational Lagrangian \( L_g \) for PGT can only be quadratic in curvature like (1) (but admitting torsion) or can contains terms quadratic in curvature (like (1)) plus terms quadratic in irreducible components of torsion.

It is commonly known that one can always get a symmetric energy–momentum tensor for matter \( T^{ik} = T^{ki} \) starting from the canonical pair

\[ cT^{ik}, cS^{ikl} = (-)cS^{kil} : cT^{ik} - cT^{ki} = \nabla_{lc} S^{ikl}. \]

\( cT^{ik} \neq cT^{ki} \) means here a canonical energy–momentum tensor for matter and \( cS^{ikl} = (-)cS^{kil} \) its canonical spintensor (see e.g. [22]). It can be easily done by use of the Belinfante symmetrization procedure [21,22]. The symmetric energy–momentum tensor \( T^{ik} \) gives at least as well description of the energy–momentum and angular momentum of matter as the canonical pair \( (cT^{ik}, cS^{ikl}) \) gives; but it is simpler and has more symmetry and better conservative properties.

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1Concerning the most general Lagrangian \( L_g \) for PGT see e.g. [16].
To geometrize the symmetric energy–momentum tensor $T^{ik} = T^{ki}$ the metric $g_{ik}$ (and Levi–Civita connection) is sufficient, i.e., the (pseudo)-Riemannian geometry is sufficient. A geometrization of such a kind leads us to general relativity (GR) and to its quadratic (higher–order) purely metric generalizations. The field equations are here obtained by use Hilbert (or metric) variational principle and have the following general form

$$\delta \sqrt{|g|} L_g \frac{\delta g^{ab}}{\delta g^{ab}} = \delta \sqrt{|g|} L_{\text{mat}} \frac{\delta g^{ab}}{\delta g^{ab}} (= 1/2 \sqrt{|g|} T_{ab}).$$

(3)

The ten field equations (3) are, in general, of the 4th–order.

In order to geometrize the canonical pair $(\varepsilon T^{ik}, \varepsilon S^{ikl})$ the Palatini variational principle and a something more general metric geometry, namely Riemann–Cartan geometry with torsion are needed.

In the Palatini variational principle we take $g_{ik}, \Gamma^i_{kl}$ as independent variables (or, equivalently, an orthonormal tetrad $h^{(a)}(x)$ and “Lorentz connection” $\Gamma_{(i)(k)}^l$ [14],[20]). This variational principle leads us to Einstein–Cartan–Sciama–Kibble (ECSK) theory of gravity and to its generalizations —- Poincaré gauge quadratic field theories of gravity (PGT). The forty field equations are here of the 2nd–order and they have the following general form

$$\frac{\delta \sqrt{|g|} L_g}{\delta g^{ab}} = \frac{\delta \sqrt{|g|} L_{\text{mat}}}{\delta g^{ab}} (= \sqrt{|g|} T_{(ab)}),$$

$$\frac{\delta \sqrt{\text{vert} g} L_g}{\delta \Gamma^{ikl}} = \frac{\delta \sqrt{|g|} L_{\text{mat}}}{\delta \Gamma^{ikl}} (= \sqrt{|g|} S_{ik}^l)$$

(4)

plus additional metricity constraints

$$\nabla_i g_{kl} = 0.$$  

(5)

The antisymmetric part $\varepsilon T_{(ab)}$ of the canonical energy–momentum is determined by covariant divergence of the canonical spin tensor $\varepsilon S^{ikl} = (-) \varepsilon S^{ikl}$ and by vectorial part of torsion (see e.g. [6],[20]). The field equations (4)-(5) are of the 2nd-order with respect to $g_{ik}$ and $\Gamma^i_{kl}$

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If we take metric $g_{ik}$ and connection $\Gamma^i_{kl}$ as independent variables.
or, equivalently, they are of the 3rd-order with respect to the really independent variables: metric and contorsion (see e.g. [6,16]).

It is remarkable that the both initial theories in these two geometrization schemes — *GR* and *ECSK* theory of gravity — *have the same Einsteinian strengths* (12 in four dimensions) and admit *the same numbers dynamical degrees of freedom* (4 in four dimensions) in Cauchy problem. But, as we will see, the pure metric scheme to geometrize symmetric energy–momentum tensor of matter leads us to the quadratic gravity theories (in general of 4th–order) which have much more smaller strenghts (48 in four dimensions) and numbers dynamical degrees of freedom (16 in four dimensions) than the competitive *PGT* (120 and 40 in four dimensions respectively).

Thus, following Einstein [1] the purely metric geometrization scheme gives us *much more stronger*, i.e., *better* from the physical point of view field equations than the competitive *PGT*. This fact can be used as a *possible test* for quadratic gravity. Namely, following Einstein one should choose the purely metric quadratic gravity theories as *the better ones* from the all set of the quadratic gravity theories.

In general, one can easily see that the Palatini variational principle leads us to the field equations (of the 2nd-order but much more greater in number) which *are not equivalent* for the same Lagrangian $L_g$ to the ten purely metric field equations obtained by use of the Hilbert variational principle (exception is the general relativity Lagrangian $L_g = \chi R$).

The Einsteinian strengths $S_E(d)$ and numbers of the dynamical degrees of freedom $N_{DF}(d)$ for the field equations obtained by Palatini variational principle are much more greater then the corresponding quantities for the purely metric gravity theories obtained by use Hilbert variational principle. This means that the Palatini variational principle gives

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3If strenght is *smaller* then the corresponding field equations are *stronger*, i.e., then the field equations *more precisely determine physical field*.

4For extended discussion of this problem see [14].
much more weaker, i.e., worse from the physical point of view gravitational field equations than the Hilbert variational principle. Only the so-called “constrained Palatini variational principle” [14] with Lagrange multipliers gives gravitational field equations which are fully equivalent to that obtained by use of the Hilbert variational principle.

II. STRENGTHS AND DYNAMICAL DEGREES OF FREEDOM FOR QUADRATIC GRAVITY THEORIES IN $D \geq 4$ DIMENSIONS

Using the definitions and formulas given in [1–6] one can very easy find the following number $Z_n(d)$ of the free coefficients of order $n$ in Taylor’s expansion of an analytic solution for the pure metric quadratic theory of gravity with the Lagrangian (1) in the general case

$$Z_n(d) = \frac{d(d + 1)}{2} \binom{d}{n} - d \binom{d}{n+1} - \frac{d(d + 1)}{2} \binom{d}{n-4} + d \binom{d}{n-5}$$

$$\times \binom{d}{n} - \frac{2d(d - 1)(d - 2)}{n} \approx 2d(d - 2) \binom{d - 1}{n},$$

(6)

where

$$\binom{d}{n} := \frac{(n + d - 1)!}{n!(d - 1)!}. \quad (7)$$

The symbol $\approx$ means equality in the highest powers of $n$. $n \rightarrow \infty$.

In the formula (6) the first term on the right gives the total number of the nth–order coefficients and the other terms, before the sign $\approx$, give numbers of independent conditions imposed on these nth–order coefficients: $d \binom{d}{n+1}$ conditions follows from gauge freedom and $\left\{ \binom{d+1}{2} \binom{d}{n-4} - d \binom{d}{n-5} \right\}$ conditions follow from vacuum field equations and from differential identities which are satisfied by them (see e.g. [1–6]).

One can easily read from these expressions that the Einsteinian strength $S_E(d)$ for such a theory is equal

$^5S_E(d)$ is defined as the coefficient of $1/n$ in the ratio $Z_n(d)/[d^n]$.
\[ S_E(d) = 2d(d - 1)(d - 2) \] (8)

and that the number dynamical degrees of freedom \( N_{DF}(d) \) equals

\[ N_{DF}(d) = 2d(d - 2). \] (9)

The formula (6) can be expanded in the other form proposed by Schutz [3]

\[
Z_n(d) = 2d(d - 2) \left[ \frac{d - 1}{n} \right] + (9d - 3d(d + 1)) \left[ \frac{d - 2}{n} \right] \\
+ (2d(d + 1) - 11d) \left[ \frac{d - 3}{n} \right] + \left(4d - \frac{d(d + 1)}{2}\right) \left[ \frac{d - 4}{n} \right] \\
- 2d \left[ \frac{d - 5}{n} \right] - d\left( \sum_{k=d-6}^{d-1} \frac{k}{n} \right). \] (10)

The physical meaning of the coefficients in the above expansion, except of the first coefficient \( 2d(d - 2) =: N_{DF}(d) \), is unclear.\(^7\)

In four dimension \( (d = 4) \) we have from (6) or from (10)

\[
Z_n(4) = 16 \left[ \frac{3}{n} \right] - 24 \left[ \frac{2}{n} \right] - 4 \left[ \frac{1}{n} \right] \\
\asymp \left[ \frac{4}{n} \right] \frac{48}{n} \asymp 16 \left[ \frac{3}{n} \right], \] (11)

i.e., we have here the strength equal 48 and 16 degrees of freedom.\(^8\)

We must emphasize that there exist an interesting example of the quadratic and pure metric theory of gravity called the \textit{Einstein–Gauss–Bonnet} theory [22] (EGB) which has specific gravitational Lagrangian \( L_g \) of the form

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\(^6\) \( N_{DF}(d) \) is the limit for large \( n \) of \( Z_n(d)/\left[ \frac{(d - 1)}{n} \right] \). \( N_{DF}(d) \) is the number of free functions of \( (d - 1) \) variables in the theory (see e.g. [3]).

\(^7\) There were given trials to understand the physical meaning of the rest coefficients of the \( Z_n(d) \) (see e.g. [4,5]).

\(^8\) In special cases these numbers can be smaller [11,12,22].
\[ L_g = L_E + L_{GB} = L_E \\
+ \alpha (R_{iklm} R^{iklm} - 4 R_{ik} R^{ik} + R^2), \quad (12) \]

where \( \alpha \) is a new coupling constant.

The Lagrangian \( L_{GB} \) is called Gauss–Bonnet or Lovelock Lagrangian.

The field equations of this theory are of the 2nd–order for \( d \geq 4 \) (iff \( d = 4 \), then these field equations are simply Einstein equations) and they have the same strength \( S_E(d) \) and number dynamical degrees of freedom \( N_{DF}(d) \) as Einstein equations have, i.e., they have

\[ S_E(d) = d(d-1)(d-3), \quad N_{DF}(d) = d(d-3). \quad (13) \]

Iff \( d = 4 \) we have for this theory

\[ S_E(4) = 12, \quad N_{DF}(4) = 4. \quad (14) \]

We see that the Lagrangian (12) leads us to the strongest field equations. Moreover the quadratic theory of gravity with Lagrangian (12) admits no ghosts or tachyons in its linear approximation.

Following Einstein [1] the EGB quadratic theory of gravity is the best one theory from the all set of the purely metric gravity theories which have quadratic gravitational Lagrangian of the general form (1). It is because the EGB theory of gravity has the strongest field equations.

On the other hand for a standard PGT\(^9\) we have [6,26]

\[ Z_n(d) = \frac{d(d+1)}{2} \left[ \begin{array}{c} d \\ n \end{array} \right] + \frac{d(d-1)}{2} d \left[ \begin{array}{c} d \\ n-1 \end{array} \right] - d \left[ \begin{array}{c} d \\ n+1 \end{array} \right] \]

\[ - \left\{ d^2 \left[ \begin{array}{c} d \\ n-2 \end{array} \right] + \frac{d(d-1)}{2} d \left[ \begin{array}{c} d \\ n-3 \end{array} \right] - \frac{d(d-1)}{2} \left[ \begin{array}{c} d \\ n-4 \end{array} \right] \right\} \]

\[ - d \left[ \begin{array}{c} d \\ n-3 \end{array} \right] \times \left[ \begin{array}{c} d(d+1)(d-1)(d-2) \\ n \end{array} \right] \approx d(d+1)(d-2) \left[ \begin{array}{c} d-1 \\ n \end{array} \right]. \quad (15) \]

\(^9\)With or without terms quadratic in torsion in its Lagrangian
This gives for $d = 4$

$$Z_n(4) \simeq \left[ \frac{4}{n} \right] \frac{120}{n} \simeq 40 \left[ \frac{3}{n} \right],$$

(16)

i.e., we have here $S_E(d) = 120$, $N_{DF}(d) = 40$.

In the formula (15), likely as it was in the formula (6), the first two terms on the right give the total number of the $n$th–order coefficients and the other terms before the sign $\simeq$ give numbers of independent conditions imposed on these $n$th–order coefficients:

$$d \left[ \begin{array}{c} d \\ n + 1 \\ d \\ n - 3 \end{array} \right]$$

follows from gauge freedom and

$$\left\{ d^2 \left[ \begin{array}{c} d \\ n - 2 \end{array} \right] + d \left[ \begin{array}{c} d \\ n - 3 \end{array} \right] - \left( \begin{array}{c} d \\ n - 4 \end{array} \right) - \left( \begin{array}{c} d \\ n - 3 \end{array} \right) \right\}$$

conditions follow from the vacuum field equations and from differential identities which are satisfied by them (see e.g. [6]).

Comparing (11) and (16) we see that a typical 3rd–order PGT has in four dimensions almost three times greater strength and number dynamical degrees of freedom than a typical 4th–order purely metric quadratic gravity theory.

Note also that the formal limes

$$\lim_{d \to \infty} \frac{Z_n^{PGT}}{Z_n^{FOTH}} = \infty,$$

(17)

i.e., it is infinite. This means that if $d$ is growing than the field equations of the purely metric quadratic gravity becomes more and more stronger in comparison with the field equations of a PGT.

III. CONCLUSION

In the letter we have considered Einsteinian strengths $S_E(d)$ and number dynamical degrees of freedom $N_{DF}(d)$ for a typical 4th–order purely metric theory of gravity with general gravitational Lagrangian of the form (1). Such theory of gravity gives us a typical example of a purely metric, quadratic theory of gravity. We have compared these numbers $S_E(d)$ and $N_{DF}(d)$ with the analogous numbers for a typical PGT with torsion. As we have seen, the numbers $S_E(d)$ and $N_{DF}(d)$ for a typical PGT are much more greater than for a
pure quadratic metric gravity. This means that the purely metric quadratic gravity theories obtained by use Hilbert variational principle have much more stronger field equations than the competitive quadratic theories of gravity with torsion obtained by use Palatini variational principle. Following Einstein [1], if one have no other criterion, one choose as the better this theory of gravity, which has stronger field equations. So, following Einstein, one should treat the purely metric quadratic theories of gravity as the better quadratic theories of gravity than the competitive quadratic theories of gravity with torsion.

Among these purely metric quadratic theories of gravity in \( d \geq 4 \) the 2nd–order EGB theory has the strongest field equations, i.e., this is the best one theory from the all set of the purely metric quadratic theories of gravity.

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