Bondi mass in classical field theory

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Abstract

We analyze three classical field theories based on the wave equation: scalar field, electrodynamics and linearized gravity. We derive certain generating formula on a hyperboloid and on a null surface for them. The linearized Einstein equations are analyzed around null infinity. It is shown how the dynamics can be reduced to gauge invariant quantities in a quasi-local way. The quasi-local gauge-invariant “density” of the hamiltonian is derived on the hyperboloid and on the future null infinity $S^+$. The result gives a new interpretation of the Bondi mass loss formula [2], [3], [10]. We show also how to define angular momentum.

Starting from affine approach for Einstein equations we obtain variational formulae for Bondi-Sachs type metrics related with energy and angular momentum generators. The original van der Burg asymptotic hierarchy is revisited and the relations between linearized and asymptotic nonlinear situations are established. We discuss also supertranslations, Newman-Penrose charges and Janis solutions.

1 Introduction

In the papers [2], [3], [4] from the series “Gravitational waves in general relativity” Bondi, van der Burg, Metzner and Sachs have analyzed asymptotic behaviour of the gravitational field at null infinity. The energy in this regime so called Bondi mass was defined and the main property – loss of the energy was proved. See also discussion on p. 127 in the last paper [10] in this series. The energy at null infinity was also proposed by Trautman [1] and it will be called a Trautman-Bondi energy.

We interprete their result from symplectic point of view and it is shown that the concept of Trautman-Bondi energy arises not only in gravity but can be also defined for other fields. In this case the TB energy can be treated formally as a “hamiltonian” and the loss of energy formula has a natural interpretation given by eq. (19). We apply similar technique to define angular momentum.

We introduce here the language of generating functions which simplifies enormously our calculations. This point of view on dynamics is due to W. M. Tulczyjew (see [28]).

We start from an example of a scalar field for which we define TB energy as a “hamiltonian” on a hyperboloid. The motivation for concerning hyperboloids in gravitation one can find in [12], [15] and [16].

In section 3 we give an example from electrodynamics.

Next we prove analogous formulae for the linearized gravity. The result is formulated in a nice gauge-independent way. We show how the formula (19) can be related with original Bondi-Sachs result – mass loss eqn (35) of [2] (cf. also eqn (4.16) in [3], eqn (13) in [4] and eqn (3.8) in [10]). Our result is an important gauge-independent generalization of this original mass loss equation. It shows the straightforward relation between the Weyl tensor on scri and the flux of the radiation energy through it. We show how to define angular momentum from this point of view.

In section 8 we give covariant on a sphere formulation of the asymptotic equations from [4]. We discuss several features of the theory like supertranslations, charges etc. and also the relations between linear and nonlinear theory.
2 Scalar field

Consider a scalar field theory derived from a Lagrangian $L = L(\varphi, \varphi_\mu)$, where $\varphi_\mu := \partial_\mu \varphi$. The entire information about field dynamics may be encoded in equation

$$\delta L(\varphi, \varphi_\mu) = \partial_\mu (p^\mu \delta \varphi) = (\partial_\mu p^\mu) \delta \varphi + p^\mu \delta \varphi_\mu .$$

(1)

The above generating formula is equivalent to the system of equations:

$$\partial_\mu p^\mu = \frac{\partial L}{\partial \varphi} ; \quad p^\mu = \frac{\partial L}{\partial \varphi_\mu} .$$

(2)

Hamiltonian description of the theory is based on a chronological analysis, i.e. on a (3+1)-foliation of space-time. Treating separately time derivative and the space derivatives, we rewrite (1) as

$$\delta L = (\pi \delta \varphi)^\cdot + \partial_\nu (p^\nu \delta \varphi) ,$$

(3)

where we denoted $\pi := p^0$. Integrating over a 3-dimensional space–volume $V$ we obtain:

$$\delta \int_V L = \int_V (\dot{\pi} \delta \varphi + \pi \dot{\delta \varphi}) + \int_{\partial V} p^\perp \delta \varphi = \int_V (\dot{\pi} \delta \varphi - \dot{\varphi} \delta \pi + \delta (\pi \varphi)) + \int_{\partial V} p^\perp \delta \varphi .$$

(4)

where by $p^\perp$ we denote normal part of the momentum $p^k$. Hence, Legendre transformation between $\pi$ and $\dot{\varphi}$ gives us:

$$- \delta \mathcal{H}(\varphi, \pi) = \int_V (\dot{\pi} \delta \varphi - \dot{\varphi} \delta \pi) + \int_{\partial V} p^\perp \delta \varphi ,$$

(5)

where

$$\mathcal{H} = \int_V \pi \dot{\varphi} - L .$$

(6)

Equation (5) is equivalent to Hamilton equations:

$$\dot{\pi} = - \frac{\delta \mathcal{H}}{\delta \varphi} ; \quad \dot{\varphi} = \frac{\delta \mathcal{H}}{\delta \pi}$$

(7)

provided no boundary terms remain when integration by parts is performed. To kill these boundary terms we restrict ourselves to an infinitely dimensional functional space of initial data $(\varphi, \pi)$, which are defined on $V$ and fulfill the Dirichlet boundary conditions $\varphi|_{\partial V} \equiv f$ on its boundary. Imposing these conditions, we kill the boundary integral in (5), because $\delta \varphi \equiv 0$ within the space of fields fulfilling boundary conditions. This way formula (5) becomes an infinitely dimensional Hamiltonian formula. Without any boundary conditions, the field dynamics in $V$ can not be formulated in terms of any Hamiltonian system, because the evolution of initial data in $V$ may be influenced by the field outside of $V$.

Physically, a choice of boundary conditions corresponds to an insulation of a physical system composed of a portion of the field contained in $V$. The choice of Dirichlet conditions is not unique. Performing e.g. Legendre transformation between $\varphi$ and $p^\perp$ in the boundary term of (5), we obtain:

$$\int_{\partial V} p^\perp \delta \varphi = \delta \int_{\partial V} p^\perp \varphi - \int_{\partial V} \varphi \delta p^\perp .$$

(8)

Hence, we have

$$- \delta \mathcal{\overline{H}} = \int_V (\dot{\pi} \delta \varphi - \dot{\varphi} \delta \pi) - \int_{\partial V} \varphi \delta p^\perp .$$

(9)

The new Hamiltonian

$$\mathcal{\overline{H}} = \mathcal{H} + \int_{\partial V} p^\perp \varphi$$

(10)
generates formally the same partial differential equations governing the dynamics, but the evolution takes place in a different phase space. Indeed, to derive Hamiltonian equations from (9) we have now to kill \( \delta p^\perp \) at the boundary. For this purpose we have to impose Neumann boundary condition \( p^\perp|_{\partial V} = \tilde{f} \).

The space of fields fulfilling this condition becomes now our infinite dimensional phase space, different from the previous one.

The difference between the above two dynamical systems is similar to the difference between the evolution of a thermodynamical system in two different regimes: in an adiabatic insulation and in a thermal bath (see [22]). As another example we may consider the dynamics of an elastic body: Dirichlet conditions mean controlling exactly the position of its surface, whereas Neumann conditions mean controlling only the forces applied to the surface. We see that the same field dynamics may lead to different Hamiltonian systems according to the way we control the boundary behaviour of the field. Without imposing boundary conditions field dynamics can not be formulated in terms of a Hamiltonian system.

2.1 Coordinates in Minkowski space

We shall consider flat Minkowski metric of the following form in spherical coordinates:

\[
\eta_{\mu\nu}dy^\mu dy^\nu = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)
\]

Minkowski space \( M \) has a natural structure of spherical foliation around null infinity, more precisely, the neighbourhood of \( S^+ \) looks like \( S^2 \times M_2 \).

We shall use several coordinates on \( M_2 \): \( s, t, r, \rho, \omega, v, u \). They are defined as follows:

\[
\begin{align*}
  r &= \sinh \omega = \rho^{-1} \\
  t &= s + \cosh \omega = s + \rho^{-1}\sqrt{1 + \rho^2} \\
  u &= t - r = s + \frac{\rho}{1 + \sqrt{1 + \rho^2}} \\
  v &= t + r = s + \rho^{-1}(\sqrt{1 + \rho^2} + 1).
\end{align*}
\]

2.2 Scalar field on a hyperboloid

We shall consider a scalar field \( \varphi \) in a flat Minkowski space \( M \) with the metric:

\[
\eta_{\mu\nu}dx^\mu dx^\nu = \rho^{-2} \left( -\rho^2 ds^2 + \frac{2dsd\rho}{\sqrt{1 + \rho^2}} + \frac{d\rho^2}{1 + \rho^2} + d\theta^2 + \sin^2\theta d\phi^2 \right)
\]

(11)

Let us fix a coordinate chart \( (x^\mu) \) on \( M \) such that \( x^1 = \theta, x^2 = \phi \) (spherical angles), \( x^3 = \rho \) and \( x^0 = s \) and let us denote by \( \tilde{\gamma}_{AB} \) a metric on a unit sphere \( (\tilde{\gamma}_{AB} dx^A dx^B := d\theta^2 + \sin^2\theta d\phi^2) \).

We shall consider an initial value problem on a hyperboloid \( \Sigma \):

\[
\Sigma_s := \{ x \in M \mid x^0 = s = const. \}
\]

for our scalar field \( \varphi \) with density of the lagrangian (corresponding to the wave equation)

\[
L := -\frac{1}{2} \sqrt{-\det \eta_{\mu\nu}} \eta^{\mu\nu} \varphi_\mu \varphi_\nu = -\frac{1}{2} \rho^{-2} \sin \theta \left[ \rho^2 (\varphi_3)^2 - \frac{(\varphi_0)^2}{1 + \rho^2} + \frac{2\varphi_3 \varphi_0}{\sqrt{1 + \rho^2}} + \varphi_{AB} \right]
\]

We use the following convention for indices: greek indices \( \mu, \nu, \ldots \) run from 0 to 3; \( k, l, \ldots \) are coordinates on a hyperboloid \( \Sigma_s \) and run from 1 to 3; \( A, B, \ldots \) are coordinates on \( S(s, \rho) \) and run from 1 to 2, where \( S(s, \rho) := \{ x \in \Sigma_s \mid x^3 = \rho = const. \} \).
The formula (1) can be written as follows:

$$
\delta \int_V L = \int_V (p^0 \delta \varphi)_0 + \int_{\partial V} p^3 \delta \varphi
$$

and

$$
p^0 = \frac{\partial L}{\partial \varphi_0} = \rho^{-2} \sin^2 \theta \left( \frac{\varphi_0}{1 + \rho^2} - \frac{\varphi_3}{\sqrt{1 + \rho^2}} \right)
$$

$$
p^3 = -\frac{\partial L}{\partial \varphi_3} = \rho^{-2} \sin^2 \theta \left( \frac{1}{\sqrt{1 + \rho^2}} \varphi_0 + \rho^2 \varphi_3 \right)
$$

Let us observe that the integral $$\int_V L$$ is in general not convergent on $$\Sigma$$ if we assume that $$\varphi = O(\rho)$$ and $$\varphi_3 = O(1)$$. We can “renormalize” $$L$$ adding a full divergence:

$$
\bar{L} := -\frac{1}{2} \sin^2 \theta \left[ \rho^2 (\psi_0^2) - \frac{1}{1 + \rho^2} (\psi_0^2)^2 + \frac{2}{\sqrt{1 + \rho^2}} \psi_3 \psi_0 + \gamma^{AB} \psi_A \psi_B \right]
$$

$$
= L + \frac{1}{2} \partial_0 \left( \sin^2 \theta \rho^2 \psi_0^2 \right) - \frac{1}{2} \partial_3 \left( \sin^2 \theta \rho^2 \psi_3^2 \right)
$$

(12)

where we have introduced a new field variable $$\psi := \rho^{-1} \varphi$$ which is natural close to null infinity. The generating formula takes the following form:

$$
\delta \int_V \bar{L} = \int_V (\pi^0 \delta \psi)_0 + \int_{\partial V} \pi^3 \delta \psi
$$

and

$$
\pi^0 = \frac{\partial \bar{L}}{\partial \psi_0} = \sin^2 \theta \left( \frac{\psi_0}{1 + \rho^2} - \frac{\psi_3}{\sqrt{1 + \rho^2}} \right)
$$

$$
\pi^3 = \frac{\partial \bar{L}}{\partial \psi_3} = -\sin^2 \theta \left( \frac{1}{\sqrt{1 + \rho^2}} \psi_0 + \rho^2 \psi_3 \right)
$$

$$
\pi^A = \frac{\partial \bar{L}}{\partial \psi_A} = -\sin^2 \theta \gamma^{AB} \psi_B
$$

$$
\frac{\partial \bar{L}}{\partial \psi} = 0
$$

(12)

It is easy to check that all terms are finite at null infinity provided $$\psi = O(1)$$ and $$\psi_3 = O(1)$$.

From the above equations (in hamiltonian form) one can easily obtain wave equation:

$$
\Box \psi = 0
$$

(13)

where the wave operator $$\Box$$ is defined with respect to the metric

$$
\pi_{\mu\nu} dx^\mu dx^\nu := -\rho^2 ds^2 + \frac{2 ds d\rho}{\sqrt{1 + \rho^2}} + \frac{d\rho^2}{1 + \rho^2} + d\theta^2 + \sin^2 \theta d\phi^2 = \rho^2 \eta_{\mu\nu} dx^\mu dx^\nu
$$

(14)

which is conformally related to the original flat metric $$\eta_{\mu\nu}$$.
Remark Let us observe that
\[
\mathcal{L} = -\frac{1}{2}\sqrt{-\det \eta_{\mu\nu}} \psi_{\mu} \psi_{\nu} = \mathcal{L} + \frac{1}{2} \left( \sqrt{-\det \eta_{\mu\nu}} (\ln \rho), \psi_{\mu} \psi_{\nu}^2 \right)_{,\mu}
\]
so we are not surprised that (13) holds. It can be easily checked that equation (13) is equivalent to the original wave equation:
\[
\Box \varphi = 0 \tag{15}
\]
by the usual conformal transformation for the conformally invariant operator $\Box + \frac{1}{6} R$ because the scalar curvature $R$ of the metric $\eta_{\mu\nu}$ vanishes.

If we want to have a closed Hamiltonian system we have to assume that $\dot{\psi}|_{\partial V} = 0$ and then the energy will be conserved in time. But we would like to describe the situation with any data on $S^+$. In this case we can define Bondi energy which is no longer conserved, formally we can treat it as a Hamiltonian and formula (6) is useful as a definition of the Bondi energy together with its changes in time. In our case boundary condition $f$ depends on time (see discussion after formula (7)) and an interesting case for us is to compare the data with different boundary conditions. Although the energy defined on a hyperboloid is not a Hamiltonian in a usual sense it plays an important role for the description of the radiation at null infinity. The method is useful to the construction of the other generators of the Poincare group and will be applied for the angular momentum.

Let us define “Hamiltonian density”:
\[
H := \frac{1}{2} \sin \theta \left[ (\rho \psi, 3)^2 + \frac{1}{1 + \rho^2} (\psi_0)^2 + \gamma^{AB} \psi_A \psi_B \right] = \pi^0 \psi_0 - \mathcal{L}
\]
and formally the following variational relation holds:
\[
-\delta \int_V H = \int_V \left( \pi \delta \dot{\psi} - \dot{\pi} \delta \psi \right) + \int_{\partial V} \pi^3 \delta \psi, \tag{16}
\]
here $\pi := \pi^0$.

Remark Relation between (16) coming from $\mathcal{L}$ and (6) with respect to $L$ can be described by following observations
\[
\dot{\pi} \delta \psi - \dot{\psi} \delta \pi = \rho^0 \delta \varphi - \varphi \delta \rho^0
\]
\[
\pi^3 \delta \psi - \rho^3 \delta \varphi = \frac{1}{2} \delta \sin \theta \rho \psi^2
\]
So the formulae give the same Hamiltonian because $\rho \psi^2$ vanishes on $S^+$.

We should express our Hamiltonian as a functional of $(\pi, \psi)$:
\[
H := \frac{1}{2} \sin \theta \left[ (\rho \psi, 3)^2 + \left( \frac{\pi \sqrt{1 + \rho^2}}{\sin \theta} + \psi_3 \right)^2 + \gamma^{AB} \psi_A \psi_B \right]
\]
and the equations (7) are the following:
\[
\dot{\psi} = \frac{\pi}{\sin \theta} (1 + \rho^2) + \psi_3 \sqrt{1 + \rho^2} \tag{17}
\]
\[
\dot{\pi} = (\pi \sqrt{1 + \rho^2}), 3 + \left( (1 + \rho^2) \sin \theta \psi_3 \right), 3 + (\sin \theta \gamma \psi_B), A \tag{18}
\]
and they correspond to the wave equation (13).
Although formally the formula (16) looks similar to the usual hamiltonian system in our case there is no possibility to kill boundary term. Our “hamiltonian” is not conserved in time:

$$- \partial_0 \left( \int_{\Sigma} H \right) = \int_{\partial \Sigma} \pi^3 \dot{\psi} = \int_{S(s,0)} \sin \theta(\dot{\psi})^2$$  \hspace{1cm} (19)

($\partial \Sigma=S(s,0)$ is odd oriented).

Nevertheless this formal calculation is very useful. For example we can easily define angular momentum along the $z$-axis as a generator for the vector field $\frac{\partial}{\partial \phi}$:

$$\int_{\Sigma} (\pi_\phi \delta \psi - \psi_\phi \delta \pi) = -\delta \int_{\Sigma} \pi \psi_\phi = -\delta J_z$$

Using equations of motion we can check that the angular momentum is not conserved in time:

$$- \partial_0 J_z = - \partial_0 \left( \int_{\Sigma} \pi \psi_\phi \right) = \int_{\partial \Sigma} \pi^3 \psi_\phi \to \int_{S(s,0)} \sin \theta \dot{\psi} \psi_\phi \hspace{1cm} (20)$$

We will show in the sequel that the formulae (19), (20) can be written for linearized gravity and have also interpretation as the TB mass loss formula and angular momentum loss equation.

Equations (19) and (20) are examples of the general formula which has the following form for a vector field $X = X^\mu \partial_\mu$:

$$- \partial_0 \left( \int_{\Sigma} X^0 H + \pi X^k \psi_k \right) = \int_{\partial \Sigma} \pi X^3 \dot{\psi} - \pi^3 X^k \psi_k - \pi^3 X^0 \dot{\psi}$$ \hspace{1cm} (21)

**Remark** The formula (21) defines usual hamiltonian system if the field $X$ vanishes on the boundary. For example this situation take place for the so-called CMC (constant mean curvature) foliation which “ends” on the same sphere at $S$.

We can take a boost generator along $z$-axis restricted to the hyperboloid $\Sigma_0$ (with coordinate $s=0$)

$$X = -\rho \sqrt{1 + \rho^2 \cos \theta} \partial_3 - \sqrt{1 + \rho^2 \sin \theta} \partial_\theta$$

and the formula (21) takes the form:

$$\partial_0 \left( \int_{\Sigma} \pi X^k \psi_k \right) = \int_{\partial \Sigma} \pi X^3 \dot{\psi} - \pi^3 X^k \psi_k = \int_{S(s,0)} \sin^2 \theta \dot{\psi} \psi_\theta$$ \hspace{1cm} (22)

Similarly the generator for linear momentum in $z$ direction:

$$X = -\frac{\cos \theta}{\sqrt{1 + \rho^2}} \partial_\theta - \rho^2 \cos \theta \partial_3 - \rho \sin \theta \partial_\theta$$

gives the loss formula:

$$\partial_0 P_z = \int_{\partial \Sigma} \pi X^3 \dot{\psi} - \pi^3 X^k \psi_k - \pi^3 X^0 \dot{\psi} = \int_{S(s,0)} \sin \theta \cos \theta(\dot{\psi})^2$$ \hspace{1cm} (23)

where $P_z := \int_{\Sigma} X^0 H + \pi X^k \psi_k$.

The equations (19), (20), (22) and (23) express non-conservation law of the “hamiltonians” defined at null infinity.

Let us formulate the following theorem:

**Theorem.** If the TB mass is conserved than angular momentum is conserved too.

This means that it is impossible to radiate away angular momentum without loss of mass. The proof is a simple consequence of (19) and (20). If the TB mass is conserved than (from (19)) $\dot{\psi}$ have to vanish on $S$ and from (20) we get that angular momentum is conserved.

We shall see in the sequel that this theorem also holds for Bondi-Sachs type metrics describing asymptotically flat solutions at null infinity for the full (nonlinear) Einstein equations.
2.3 Scalar field on a null cone

We shall consider an initial value problem on a null surface $N$:

$$N := \left\{ x \in M \mid v = s + \rho^{-1}(1 + \sqrt{1 + \rho^2}) = \text{const} \right\}$$

(24)

where we have introduced a null coordinate $v := s + \rho^{-1}(1 + \sqrt{1 + \rho^2})$ which plays a role of time in our analysis. Formally $S^+$ corresponds to the surface $\rho = 0$. Let us rewrite the Minkowski metric (11) using new coordinates $v, u$ instead of $s, \rho$:

$$\eta_{\mu\nu} dx^\mu dx^\nu = \rho^{-2} \left( -\rho^2 dv^2 - \rho^2 du^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

The relation between coordinates $(v, u)$ and $(x^0, x^3)$ used in the previous subsection is the following:

$$v = x^0 + \rho^{-1}(1 + \sqrt{1 + \rho^2}), \quad u = -\rho^{-1}, \quad \rho = x^3$$

$$\partial_0 = \partial_v, \quad \partial_3 = 2\rho^{-2} \partial_u - \rho^{-2} \left( 1 + \frac{1}{\sqrt{1 + \rho^2}} \right) \partial_v$$

$$dx^0 = dv + \frac{1}{2} \left( 1 + \frac{1}{\sqrt{1 + \rho^2}} \right) du, \quad dx^3 = \frac{1}{2} \rho^2 du$$

The density of the lagrangian takes the following form:

$$L := -\frac{1}{2} \sqrt{-\det \Pi_{\mu\nu} \Pi^{\mu\nu}} \psi^\mu \psi^\nu = \sin \theta \left[ \psi_\pi \psi_v - \frac{\psi_v^2}{\pi} - \frac{1}{4} \rho^2 \gamma^{AB} \psi_A \psi_B \right]$$

The formula (1) can be written as follows:

$$\delta \int_N L = \int_N (\pi^v \delta \psi)_v + \int_{\partial N} \pi^\pi \delta \psi$$

and

$$\pi^v = \frac{\partial L}{\partial \psi_v} = \sin \theta \psi_\pi$$

$$\pi^\pi = \frac{\partial L}{\partial \psi_\pi} = \sin \theta (\psi_v - 2\psi_\pi) = -\sin \theta \left( \rho^2 \psi_3 + \frac{\psi_0}{\sqrt{1 + \rho^2}} \right)$$

In this way we obtain density of the hamiltonian on a cone $N$

$$H = \pi^v \psi_v - L = \sin \theta \left[ (\psi_\pi)^2 + \rho^2 \gamma^{AB} \psi_A \psi_B \right]$$

$$\lim_{\rho \to 0^+} H = \sin \theta (\psi_\pi)^2 = \sin \theta \hat{\psi}^2$$

(25)

$$\int_N \pi \delta \psi = \int_N \sin \theta \psi_\pi \delta \psi \xrightarrow{\rho \to 0^+} \int_{S^+} \sin \theta \hat{\psi} \delta \psi d\pi d\theta d\phi$$

(26)

We will show in the sequel that the above formulae exist in electrodynamics and linearized gravity. Similarly, the equation

$$\int_N \pi \psi_\phi = \int_N \sin \theta \psi_\pi \psi_\phi \xrightarrow{\rho \to 0^+} \int_{S^+} \sin \theta \hat{\psi}_\phi d\pi d\theta d\phi$$

(27)

describes the flux of angular momentum through $S^+$. 

7
2.4 ADM mass

We have tried to treat separately hyperboloid and scri and we have learned that there is no possibility to get a nice hamiltonian system. Let \( N \) denote in this subsection a “piece” of \( S^+ \) between \( \Sigma \) and \( i^0 \). If we take the surface \( \Sigma \cup N \) together:

\[
- \delta \left( \int_{\Sigma} H + \int_{N} H \right) = \int_{\Sigma} \left( \pi \delta \psi - \dot{\psi} \delta \pi \right) + \int_{N} \left( \pi \delta \psi - \dot{\psi} \delta \pi \right) + \int_{\partial \Sigma} \pi^3 \delta \psi + \int_{\partial N} \pi \delta \psi \tag{28}
\]

we will obtain hamiltonian system provided we can kill boundary term. This can be achieved assuming for example that

\[
\lim_{u \to -\infty} \dot{\psi} = 0
\]

which simply means that \( \dot{\psi} \) is vanishing at spatial infinity. This usually happens for initial data on Cauchy surface \( t = \text{const} \) with compact support or vanishing sufficiently fast at spatial infinity. The following relations confirm our theorem:

\[
\pi^3 \big|_{\partial \Sigma} = - \sin \theta \dot{\psi} = \pi \pi
\]

\( \partial \Sigma = S(s, 0), \partial N = S(s, 0) \cup S(-\infty, 0) \)

\[
- \delta m_{\text{ADM}} = \int_{\Sigma \cup N} \left( \pi \delta \psi - \dot{\psi} \delta \pi \right) + \int_{\partial (\Sigma \cup N)} \sin \theta \dot{\psi} \delta \psi \tag{29}
\]

where \( m_{\text{ADM}} := \int_{\Sigma \cup N} H \).

2.4.1 one-parameter family of hamiltonian systems and their limit

\( \Sigma_{\tau, \epsilon} := \{ s = \tau, \rho \geq \epsilon \} \)

\( N_{\tau, \epsilon} := \left\{ v = \tau + \frac{1 + \sqrt{1 + \epsilon^2}}{\epsilon}, \frac{\epsilon}{1 + \sqrt{1 + \epsilon^2}} \leq \rho \leq \epsilon \right\} \)

\( I_{\tau, \epsilon} := \left\{ t = \tau, 0 < \rho \leq \frac{\epsilon}{1 + \sqrt{1 + \epsilon^2}} \right\} \)

\( \lim_{\epsilon \to 0^+} \Sigma_{\tau, \epsilon} = \Sigma_{\tau} \)

\( \lim_{\epsilon \to 0^+} N_{\tau, \epsilon} = N_{\tau} \subset S^+ \)

\( \lim_{\epsilon \to 0^+} I_{\tau, \epsilon} = i^0 \)

\( N_{\tau, \epsilon} := \left\{ v = \tau + \frac{1 + \sqrt{1 + \epsilon^2}}{\epsilon}, -2\sqrt{1 + \epsilon^2} + \tau \leq u \leq \tau \right\} \)

\( I_{\tau, \epsilon} := \left\{ t = \tau, r \geq \frac{1 + \sqrt{1 + \epsilon^2}}{\epsilon} \right\} \)

\( \Sigma_{\tau, \epsilon} \cup N_{\tau, \epsilon} \cup I_{\tau, \epsilon} \) is an explicit example of a one-parameter family of surfaces (with respect to \( \tau \)) and the hamiltonian related to this family is an ADM mass. On the other hand the hamiltonian system \([28]\) is a limit of these systems with respect to the second parameter \( \epsilon (\epsilon \to 0^+) \). This way we have certain “finite” procedure for the hamiltonian system \([28]\) at infinity.
2.5 Energy–momentum tensor

Let us consider energy–momentum tensor for the scalar field \( \phi \):

\[
T^\mu_\nu = \frac{1}{\sqrt{-\eta}} (\eta^\mu \phi_\nu - \delta^\mu_\nu L)
\]

where \( \eta := \det \eta_{\mu\nu} \) and by \( \delta^\mu_\nu \) we have denoted Kronecker delta. In our case (for scalar field) symmetric and canonical energy momentum tensors are equal.

For a Killing vector field \( X^\mu \) we can integrate the equation

\[
\partial_\mu (\sqrt{-\eta}T^\mu_\nu X^\nu) = 0
\]

and obtain as follows:

\[
\partial_0 \int_\Sigma \sqrt{-\eta}T^0_\nu \cdot X^\nu = - \int_\partial \Sigma \sqrt{-\eta}T^3_\nu \cdot X^\nu
\]

It can be easily verified that for energy and angular momentum we have respectively

\[
\int_\Sigma \sqrt{-\eta}T^0_0 = \int_\Sigma H
\]

\[
\int_\Sigma \sqrt{-\eta}T^0_\phi = \int_\Sigma \pi_\psi \phi
\]

and the boundary terms can be expressed in terms of energy–momentum tensor

\[
- \int_{\partial \Sigma} T^3_0 \rho^{-4} \sin \theta \theta \varphi \, d\theta \varphi = -\frac{1}{2} \int_{\partial \Sigma} T^v_v \rho^{-2} \sin \theta \theta \varphi \, d\theta \varphi \quad \left( = \int_{\partial \Sigma \subset S^+} \dot{\psi}^2 \sin \theta \theta \varphi \, d\theta \varphi \right)
\]

\[
\int_N T^v_v \rho^{-4} \, d\rho \sin \theta \theta \varphi \, d\theta \varphi = \int_N d\pi \left( \frac{1}{2} T^v_v \rho^{-2} \sin \theta \theta \varphi \, d\theta \varphi \right) \quad \text{and} \quad \frac{1}{2} \rho^{-2} T^v_v \big|_{S^+} = \dot{\psi}^2
\]

Similarly for angular momentum:

\[
- \int_{\partial \Sigma} T^3_\phi \rho^{-4} \sin \theta \theta \varphi \, d\theta \varphi = -\frac{1}{2} \int_{\partial \Sigma} T^v_v \rho^{-2} \sin \theta \theta \varphi \, d\theta \varphi \quad \left( = \int_{\partial \Sigma \subset S^+} \dot{\psi}_\psi \phi \sin \theta \theta \varphi \, d\theta \varphi \right)
\]

\[
\int_N T^v_v \rho^{-4} \, d\rho \sin \theta \theta \varphi \, d\theta \varphi = \int_N d\pi \left( \frac{1}{2} T^v_v \rho^{-2} \sin \theta \theta \varphi \, d\theta \varphi \right) \quad \text{and} \quad \frac{1}{2} \rho^{-2} T^v_v \big|_{S^+} = \dot{\psi}_\psi \phi
\]

Some of the equalities above hold for any energy-momentum tensor not only for the scalar field. Compare with (19–20) and (25–27) for scalar field.

This calculation shows that \textbf{quasi-local} density of the energy on \( S^+ \) has two different interpretations. It is a boundary term which describes non-conservation of the “hamiltonian” on a hyperboloid \( \Sigma \) or a density of a “hamiltonian” on \( S^+ \) as a limit of \( N \). More precisely, it is a density with respect to the parameter \( \pi \) but integrated over sphere. This is an example of an object which is local on \( M_2 \) but non-local on \( S^2 \). We call such objects \textit{quasi-local}. It will be shown in the sequel that this concept of \textit{quasi-locality} is useful in electrodynamics and gravitation.
3 Electrodynamics

This section should convince the reader that the TB mass and angular momentum at null infinity can be described in classical electrodynamics in a similar way as the scalar field in previous section.

Field equations for linear electrodynamics may be written as follows:
\[
\delta L = \partial_\mu (F^{\nu \mu} \delta A_\nu) = \partial_\mu (F^{\nu \mu}) \delta A_\nu + F^{\nu \mu} \delta A_{\nu \mu},
\]
where \( A_{\nu \mu} := \partial_\mu A_\nu \) and \( L \) is the Lagrangian density of the theory. The above formula (see [25]) is a convenient way to write the Euler-Lagrange equations
\[
\partial_\mu F^{\nu \mu} = \frac{\delta L}{\delta A_\nu},
\]
(31)
together with the relation between the electromagnetic field \( f_{\nu \mu} = A_{\nu \mu} - A_{\mu \nu} \) and the electromagnetic induction density \( F^{\nu \mu} \) describing the momenta canonically conjugate to the potential:
\[
F^{\nu \mu} = \frac{\delta L}{\delta A_{\nu \mu}}.
\]
(32)

For the linear Maxwell theory the Lagrangian density is given by the standard formula:
\[
L = -\frac{1}{4} \sqrt{-\eta} f_{\nu \mu} f^{\nu \mu},
\]
(33)
and relation (32) reduces in this case to \( F^{\nu \mu} := \sqrt{-\eta} \eta^{\alpha \beta} f_{\alpha \beta} \).

Integrating (30) over \( V \) we obtain:
\[
\delta \int_V L = \int_V \partial_\mu (F^{k0} \delta A_k) + \int_{\partial V} F^{03} \delta A_3 =
\int_V \partial_\mu (F^{B0} \delta A_B + F^{30} \delta A_3) + \int_{\partial V} (F^{B3} \delta A_B + F^{03} \delta A_0).
\]
(34)

We assume that the charge \( e \) defined by the surface integral
\[
e := \int_{S(s, \rho)} F^{03}
\]
vanishes. The situation with \( e \neq 0 \) can be described similarly as in [24] but we are interested in “wave” degrees of freedom and we are going to show how the volume part of (34) can be reduced to the gauge-invariant quantities.

Let \( a := \rho^{-2} \Delta \) where \( \Delta \) denotes the 2-dimensional Laplace-Beltrami operator on a sphere \( S(s, \rho) \) and one can easily check that the operator \( a \) does not depend on \( \rho \) and is equal to the Laplace-Beltrami operator on the unit sphere \( S(1) \). Operator \( a \) is invertible on the space of monopole-free functions (functions with a vanishing mean value on each \( S(s, \rho) \)).

Let us denote by \( \varepsilon^{AB} \) the Levi–Civita antisymmetric tensor on a sphere \( S(s, \rho) \). We can rewrite (34) in the following way provided that the electric charge \( e \) vanishes.
\[
\delta \int_V L = \int_V \partial_\mu \left[ F^{B0},B a^{-1} \rho^{-2} A_B || B + F^{30} A_3 + F^{0B},C \varepsilon_B^C a^{-1} \rho^{-2} (\varepsilon^{AB} A_A || B) \right] +
+ \int_{\partial V} \left[ F^{3B},B a^{-1} \rho^{-2} A_B || B + F^{03} A_0 + F^{3B},C \varepsilon_B^C a^{-1} \rho^{-2} (\varepsilon^{AB} A_A || B) \right]
\]
(36)

Here, by “\( || \)” we denote the 2-dimensional covariant derivative on each sphere \( S(s, \rho) \). Using identities \( \partial_B F^{B0} + \partial_3 F^{30} = 0 \) and \( \partial_B F^{B3} + \partial_3 F^{03} = 0 \) implied by the Maxwell equations and again integrating by parts we finally obtain:
\[
\delta \int_V L = \int_V \partial_\mu \left[ F^{30} a^{-1} \delta (a A_3 - (\rho^{-2} A_B || B), s) + (F^{0B},C \varepsilon_B^C a^{-1} \rho^{-2} (\varepsilon^{AB} A_A || B) \right] +
+ \int_{\partial V} \left[ F^{03} a^{-1} \delta (a A_0 - \rho^{-2} A_B || B, 0) + (F^{3B},C \varepsilon_B^C a^{-1} \rho^{-2} (\varepsilon^{AB} A_A || B) \right].
\]
(37)
The quantities $\mathbf{a} \mathbf{A}_0 - \rho^{-2} \mathbf{A}^B \mathbf{B}_{0,0}$ and $(\mathbf{a} \mathbf{A}_3 - (\rho^{-2} \mathbf{A}^B \mathbf{B})_3)$ are gauge invariant and it may be easily checked that

$$\sin \theta \left( \mathbf{a} \mathbf{A}_0 - \rho^{-2} \mathbf{A}^B \mathbf{B}_{0,0} \right) = \rho^2 \mathcal{F}^A_{0,0} (= \pi^3)$$

and

$$\sin \theta \left[ \mathbf{a} \mathbf{A}_3 - (\rho^{-2} \mathbf{A}^B \mathbf{B})_3 \right] = \rho^2 \mathcal{F}^A_{3,0} (= \pi)$$

Let us introduce the following gauge invariants:

$$\psi := \mathcal{F}^{30} / \sin \theta$$

$$\pi := - \rho^2 \mathcal{F}^A_{3,0}$$

$$\star \psi := \rho^{-2} \varepsilon^{AB} \mathbf{A}^B \mathbf{B}_{A,0} = - \mathcal{F}^{30} / \sin \theta$$

$$\star \pi := \mathcal{F}^{0B} \varepsilon_{BC} = \rho^2 \mathcal{F}^A_{3,0}$$

where

$$\star \mathcal{F}^{\mu \nu} = \frac{1}{2} \varepsilon^{\mu \nu \lambda \sigma} \mathcal{F}_{\lambda \sigma}$$

Now we will show how the vacuum Maxwell equations

$$\partial_\mu \mathcal{F}^{\mu \nu} = 0, \quad \partial_\mu \star \mathcal{F}^{\mu \nu} = 0$$

allow to introduce equations for gauge-invariants and the result is analogous to (13) and (18) describing scalar field

$$\dot{\psi} = \frac{\pi}{\sin \theta} (1 + \rho^2) + \psi_3 \sqrt{1 + \rho^2}$$

$$\dot{\pi} = \left( \pi \sqrt{1 + \rho^2} \right)_3 + \left[ (1 + \rho^2) \sin \theta \psi_3 \right]_3 + \sin \theta \mathbf{a} \psi$$

$$\star \dot{\psi} = \frac{\pi^3}{\sin \theta} (1 + \rho^2) + \psi_3 \sqrt{1 + \rho^2}$$

$$\star \dot{\pi} = \left( \pi \sqrt{1 + \rho^2} \right)_3 + \left[ (1 + \rho^2) \sin \theta \star \psi_3 \right]_3 + \sin \theta \mathbf{a} \star \psi$$

The proof of (38) is based on the following observations:

$$\sin \theta \psi_{0,0} = \mathcal{F}^{30,0} = - \mathcal{F}^{3A} \mathbf{A}^A_0 = \frac{\pi}{\sin \theta} (1 + \rho^2) + \psi_3 \sqrt{1 + \rho^2}$$

$$- \pi = \rho^2 \mathcal{F}^A_{3,0} \mathbf{B}_A = \frac{\mathcal{F}^{3A} \mathbf{A}^A_0}{1 + \rho^2} + \sqrt{1 + \rho^2}$$

and

$$\sin \theta \psi_3 = \mathcal{F}^{30,3} = \mathcal{F}^{3A} \mathbf{B}_3 \mathbf{A}^A$$

Similarly for (39) we have the following relations:

$$- \left( \star \mathcal{F}^{0A} \mathbf{B}^A_{0,0} \right)_0 - \left( \star \mathcal{F}^{3A} \mathbf{B}^A \mathbf{B}_3 \mathbf{3,0} \right) + (\star \mathcal{F}^{AB} \mathbf{B}^C_{BC} \mathbf{C})_0 = 0$$

$$\star \mathcal{F}^{0A} = - \rho^2 \varepsilon^{AB} \mathcal{F}_{AB}$$

$$\star \mathcal{F}^{3A} = \rho^2 \varepsilon^{AB} \mathcal{F}_{0B}$$
\[ *\mathcal{F}^{AB} = \rho^2 \varepsilon^{AB} \mathcal{F}_{03} \]
\[ \mathcal{F}_{03} = -\rho^{-4} \mathcal{F}^{03} \]

which allow to get an equation
\[ \left( \frac{\mathcal{F}^{3A}}{1 + \rho^2} + \frac{\mathcal{F}^{0A}}{\sqrt{1 + \rho^2}} \right)_0 + \left( \rho^2 \mathcal{F}^{0A} \right)_{V} - \frac{\mathcal{F}^{3A}}{\sqrt{1 + \rho^2}} + a \mathcal{F}^{30} = 0 \]

which is equivalent to [37].
For \((\psi, \pi)\) the proof is the same provided we apply the Hodge dual \(*\) for the variables and equations:
\[ (\pi, \psi) \xrightarrow{*} (\pi, *\psi) \xrightarrow{*} (\pi, \psi) \]

Now we will show how our variables appear in formula (37). Let us perform the Legendre transformation in the volume \(V\)
\[ -\mathcal{F}^{03} \delta \left[ A_3 - a^{-1}(\rho^{-2} A^B)_{|B,3} \right] = -\delta \left[ \mathcal{F}^{03} \left( A_3 - a^{-1}(\rho^{-2} A^B)_{|B,3} \right) \right] + \left[ A_3 - a^{-1}(\rho^{-2} A^B)_{|B,3} \right] \delta \mathcal{F}^{03} \]

and on the boundary \(\partial V\)
\[ \mathcal{F}^{03} \delta \left( A_0 - a^{-1} \rho^{-2} A^B_{|B,0} \right) = \delta \left[ \mathcal{F}^{03} \left( A_0 - a^{-1} \rho^{-2} A^B_{|B,0} \right) \right] - \left( A_0 - a^{-1} \rho^{-2} A^B_{|B,0} \right) \delta \mathcal{F}^{03} . \]

This way the formula [37] may be written as follows:
\[ \delta \int_V \left[ L - \partial_0 (\psi a^{-1} \pi) - \partial_3 (\psi a^{-1} \rho^2 \mathcal{F}^{A}_{0|A}) \right] = -\int_V \partial_0 \left[ \pi a^{-1} \delta \psi + *\pi a^{-1} \delta *\psi \right] + \int_{\partial V} \left[ \rho^2 \mathcal{F}^{A}_{0|A} a^{-1} \delta \psi - \mathcal{F}^{3A}_{|B} \varepsilon_{AB} \delta *\psi \right] . \]

Finally we have obtained the following variational principle:
\[ \delta \int_V L = -\int_V \partial_0 (\pi a^{-1} \delta \psi + *\pi a^{-1} \delta *\psi) + \int_{\partial V} \pi^3 a^{-1} \delta \psi + *\pi^3 a^{-1} \delta *\psi , \]

where the lagrangian \(L\) is defined by
\[ L = L - \partial_0 (\psi a^{-1} \pi) - \partial_3 (\psi a^{-1} \rho^2 \mathcal{F}^{A}_{0|A}) . \]

and boundary momenta:
\[ \pi^3 := -\rho^2 \mathcal{F}^{0A}_{|A} = \rho^2 \mathcal{F}^{0A}_{|A} - \frac{\mathcal{F}^{3A}_{|A}}{\sqrt{1 + \rho^2}} = \sin \theta \left( \frac{\psi}{\sqrt{1 + \rho^2}} + \rho^2 \psi^3 \right) \]
\[ \ast \pi^3 := \mathcal{F}^{3A}_{|B} \varepsilon_{AB} = \rho^2 \ast \mathcal{F}^{0A}_{|A} = \sin \theta \left( \frac{\ast \psi}{\sqrt{1 + \rho^2}} + \rho^2 \ast \psi^3 \right) \]

From lagrangian relation [43] we immediately obtain the hamiltonian one performing the Legendre transformation:
\[ -\delta \int_V H = -\int_V \pi a^{-1} \delta \psi - \psi a^{-1} \delta \pi + *\pi a^{-1} \delta *\pi - \ast \pi a^{-1} \delta *\pi + \int_{\partial V} \pi^3 a^{-1} \delta \psi + *\pi^3 a^{-1} \delta *\psi \]

where
\[ H := -\pi a^{-1} \psi - *\pi a^{-1} \ast \psi - L , \]
is the density of the hamiltonian of the electromagnetic field on a hyperboloid.

The value of \( I \) is equal to the amount of electromagnetic energy contained in a volume \( V \) and defined by the energy–momentum tensor.

\[
T^\mu_{\nu} = f^{\mu\lambda} f_{\lambda\nu} + \frac{1}{4} \delta^\mu_\nu f^{\kappa\lambda} f_{\kappa\lambda}
\]

We are not surprised that the quantity \( H \) is related to \( T^0_0 \)

\[
\int_{S(s,\rho)} H d\theta d\phi = \int_{S(s,\rho)} \sqrt{-\eta} T^0_0 d\theta d\phi
\]

and to prove it we can use the following identity

\[
\rho^{-4} \sin \theta \left[ \pi a^{-1} \phi \hat{\pi} a^{-1} \psi - \partial_3 \left( \psi a^{-1} \rho^2 F^A_{0||A} \right) \right] = \]

\[
= F^{03} F_{03} - F^{0A} || A a^{-1} \rho^2 F^A_{0||A} - F^{0A} || B \varepsilon_{AB} a^{-1} F_{0A} || B \varepsilon^{AB}
\]

The non-conservation law for the energy we can denote as follows:

\[
- \partial_0 \int_{\Sigma} H = \int_{\partial \Sigma} \sin \theta \left( \psi a^{-1} \psi + \phi a^{-1} \phi \right) = - \int_{S(s,0)} \left( \psi a^{-1} \psi + \phi a^{-1} \phi \right) \sin \theta \theta d\phi d\phi
\]

(47)

For angular momentum defined by

\[
J_3 := - \int_{\Sigma} \pi a^{-1} \psi,\phi + \pi a^{-1} \phi,\phi
\]

we have a similar formula

\[
- \partial_0 J_3 = \partial_0 \int_{\Sigma} \pi a^{-1} \psi,\phi + \pi a^{-1} \phi,\phi = - \int_{S(s,0)} \sin \theta \left( \psi a^{-1} \psi,\phi + \phi a^{-1} \phi,\phi \right) d\theta d\phi
\]

(48)

but the relation with symmetric energy–momentum tensor is not so obvious.

\[
\hat{J}_3 := \int_{\Sigma} \sqrt{-\eta} T^0_3 = \int_{\Sigma} F^{03} f_{3\phi} + * F^{03} * f_{3\phi}
\]

Using the relations

\[
\pi = -\rho^{-2} \sin \theta f_3^A || A \quad \psi,3 = -\rho^{-2} \varepsilon^{AB} f_{3A} || B
\]

\[
\pi \hat{\pi} = \rho^{-2} \sin \theta f_3^A || A \quad \phi,3 = -\rho^{-2} \varepsilon^{AB} * f_{3A} || B
\]

we can express \( \hat{J}_3 \) in terms of \((\pi, \psi, \pi, \phi)\) as follows:

\[
\hat{J}_3 = \int_{\Sigma} \psi \left( a^{-1} \pi,\phi + \sin \theta \partial_\phi a^{-1} \phi,3 \right) + \int_{\Sigma} \phi \left( a^{-1} \pi,\phi - \sin \theta \partial_\phi a^{-1} \phi,3 \right) =
\]

\[
- \int_{\Sigma} \pi a^{-1} \psi,\phi + \pi a^{-1} \phi,\phi + \int_{\partial \Sigma} \sin \theta \psi \partial_\phi a^{-1} \psi
\]

where \( \partial A := \varepsilon^A_B \partial_B \) and we have used the identity

\[
\int_{S^2} \sin \theta \psi,3 \partial_\phi a^{-1} \phi = - \int_{S^2} \sin \theta \phi \partial_\phi a^{-1} \psi,3
\]

The boundary term \( \int_{\partial \Sigma} \sin \theta \psi \partial_\phi a^{-1} \psi \) usually has to vanish if we want to interprete the integral \( \int_{\Sigma} \sqrt{-\eta} T^0_3 \) as an angular momentum generator but in our case \( \psi, \phi \) do not vanish on \( S^+ \) and we obtain in general two different definitions of angular momenta \( J_3 \) and \( \hat{J}_3 \).
Let us observe that
\[ A_\phi = (\rho^{-2} A^B B)_\phi - \hat{\theta}_\phi \psi \]
and
\[ \int V (F^{\lambda 0} A_\phi)_\lambda = \int_{\partial V} \sin \theta \psi a^{-1} \rho^{-2} (A^B B)_\phi - \int_{\partial V} \sin \theta \psi \hat{\theta}_\phi a^{-1} * \psi \]
so the angular momentum \( J_z \) is related rather to the canonical energy-momentum tensor with gauge \( A^B B = 0 \) than to the symmetric one. More precisely, the canonical energy-momentum density \( T^\mu \nu \) is related with symmetric one as follows:
\[ T^\mu \nu = F^{\lambda \mu} A_{\lambda \nu} - \frac{1}{4} \delta^\mu \nu L = \sqrt{-\eta} T^\mu \nu + (F^{\lambda \mu} A_{\lambda \nu})_\lambda \]
For angular momentum we obtain:
\[ \int \Sigma T^0 \phi = - \int \Sigma \pi a^{-1} \psi, \phi + \pi a^{-1} \delta \psi, \phi - \int_{\partial \Sigma} \sin \theta \psi, \phi a^{-1} \rho^{-2} A^B B = J_z + \]
\[ + \int_{S(s,0)} \psi, \phi a^{-1} \rho^{-2} A^B B \sin \theta \theta \phi \]
Let us observe that if \( \psi, \phi \) and \( \delta \psi, \phi \) are vanishing on \( S^+ \) then \( J_z \) is well defined in terms of the canonical energy-momentum tensor density \( T^0 \phi \) and conserved.

### 3.1 Electrodynamics on a null surface

We will show now how the formula \( (24) \) can be obtained in classical electrodynamics. Let us consider volume \( V \subset N \) where \( N \) has been already defined by \( (24) \).

\[ \int V F^{\mu \nu} \delta A_\nu d\rho \theta d\phi = \int_{\partial V} \int F^{\mu \nu} a^{-1} \delta \rho^{-2} A^B B d\theta d\phi + \]
\[ + \int V F^{\mu \nu} \delta [A_\mu - a^{-1} (\rho^{-2} A^B B)_\mu] d\rho \theta d\phi - \int V F^{\mu A} ||B \varepsilon_{AB} a^{-1} \delta \rho^{-2} A_{AB} B d\rho \theta d\phi \]
and similarly as on \( \Sigma \)
\[ a A_\rho - (\rho^{-2} A^B B)_\rho = \frac{1}{\sin \theta} F^{\mu A} ||A = \psi, \rho = 2 \rho^{-2} \psi, \pi \]
where \( \pi = -\frac{2}{\rho} \) and \( \partial \rho = 2 \rho^{-2} \partial \pi \). For dual degree of freedom the similar relation holds
\[ F^{\mu A} ||B \varepsilon_{AB} = a F^{\mu A} ||A = 2 \sin \theta \rho^{-2} \psi, \pi \]
We obtain gauge–independent part \( + \) boundary term \( + \) full variation
\[ \int V F^{\mu \nu} \delta A_\nu d\rho \theta d\phi = \int_{\partial V} \int F^{\mu \nu} a^{-1} \delta \rho^{-2} A^B B d\theta d\phi - \delta \int V \sin \theta \psi a^{-1} \psi, \rho d\rho d\phi + \]
\[ + \int V \sin \theta (\psi, \rho a^{-1} \delta \psi + \delta \psi, \rho a^{-1} \delta * \psi) d\rho d\phi \]
(49)

This equality means that modulo boundary term and full variation we can reduce our form on \( S^+ \) and the final form is similar to \( (20) \) and posseses quasi-local character. \( u = v + \pi \) so on the surface \( v = \text{const.} \) we can use coordinate \( u \) as well as \( \pi \) and this observation refers to the objects on \( N \) but not on \( M \).
Now we will show how the flux of energy through \( S^+ \) is related with energy-momentum tensor, similarly to section 2.5.

\[
T^v_v = \frac{1}{2} \rho^{-4} (f^v)^2 + \frac{1}{2} \rho^{-4} (\ast f^v)^2 + \frac{1}{2} \eta_{AB} f^v f^B
\]

\[
\int_V T^v_v \rho^{-4} \sin \theta d\rho d\theta d\phi = - \int_V \sin \theta d\rho d\theta d\phi (\psi_u a^{-1} \psi_u + \ast \psi_u a^{-1} \ast \psi_u) + \\
+ \frac{1}{4} \int_V \rho^2 \sin \theta d\rho d\theta d\phi (\psi^2 + \ast \psi^2)
\]

the last term vanishes on scri

\[
\rho^2 (\psi^2 + \ast \psi^2) \xrightarrow{\rho \to 0^+} 0
\]

so

\[
\int_{S^+} \sqrt{-\eta} T^v_v = - \int_{S^+} du \left[ \sin \theta d\rho d\phi (\psi_u a^{-1} \psi_u + \ast \psi_u a^{-1} \ast \psi_u) \right]
\]

(50)

The integral on a sphere in quadratic brackets represents \textit{quasi-local} density of the flux of the energy through \( S^+ \). The main difference comparing with scalar field is that here there is no possibility to work with local density because of the operator \( a^{-1} \) and only quasi-local object assigned to a sphere can be well defined.

\section{Linearized gravity on a hyperboloid}

We start from ADM formulation of the initial value problem for Einstein equations [3]. In subsection 1 we introduce the hyperboloidal slicing and in subsection 2 we consider an initial value problem for the linearized Einstein equations on it. In subsection 3 we discuss “charges” on the hyperboloid and in the next two sections we introduce invariants which describe reduced dynamics. In subsection 6 we derive “hamiltonian” in terms of gauge invariant quantities.

\subsection{Hyperboloidal conventions}

The flat Minkowski metric of the following form in spherical coordinates:

\[
\eta_{\mu\nu} dy^\mu dy^\nu = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

(51)

with \( r = \sinh \omega, \ t = s + \cosh \omega \) already defined in section 2, can be expressed in coordinates \( s, \omega \) well adopted to “hyperboloidal” slicing of Minkowski spacetime \( M \):

\[
\eta_{\mu\nu} dy^\mu dy^\nu = -ds^2 - 2 \sinh \omega d\omega ds + d\omega^2 + \sinh^2 \omega (d\theta^2 + \sin^2 \theta d\phi^2)
\]

(52)

In this section we use different coordinate \( \omega \) instead of \( \rho \) used previously but at the end we return to \( \rho \) to compare the results for the scalar field and linearized gravity. Let us fix a coordinate chart \( (y^\mu) \) on \( M \) such that \( y^1 = \theta, \ y^2 = \phi \) (spherical angles), \( y^3 = \omega \) and \( y^0 = s \). So we have

\[
\Sigma_s := \{ y \in M : y^0 = s \} = \bigcup_{\omega \in [0, \infty]} S_s(\omega) \quad \text{where} \quad S_s(\omega) := \{ y \in \Sigma_s : y^3 = \omega \}
\]

(53)

and \( \Sigma_s \) is a three–dimensional hyperboloid, \( S_s(\omega) = S(\omega, \frac{1}{\sinh \omega}) \) and \( \partial \Sigma_s = S_s(\infty) = S(s, 0) \).

We use the similar convention for indices (as for coordinates \( (x^\mu) \)), namely: greek indices \( \mu, \nu, \ldots \) run from 0 to 3; \( k, l, \ldots \) are coordinates on \( \Sigma \) and run from 1 to 3; \( A, B, \ldots \) are coordinates on \( S(r) \) and run from 1 to 2.
Hyperboloid $\Sigma$ has a very simple geometry. The induced Riemannian metric $\eta_{kl}$ on $\Sigma$ in our coordinates takes the following form:

$$\eta_{kl}dy^k dy^l = d\omega^2 + \sinh^2 \omega (d\theta^2 + \sin^2 \theta d\phi^2)$$ (54)

The hypersurface $\Sigma$ is a constant curvature space and the three-dimensional curvature tensor of $\Sigma$ can be expressed by the metric:

$$^{3}R_{ijkl} = \eta_{jk} \eta_{il} - \eta_{ik} \eta_{jl}$$ (55)

### 4.2 ADM formulation for linearized gravity on a hyperboloid

Let $(g_{kl}, P^{kl})$ be the Cauchy data for Einstein equations on a 3-dimensional hyperboloid $\Sigma$. This means that $g_{kl}$ is a Riemannian metric on $\Sigma$ and $P^{kl}$ is a symmetric tensor density which we identify with the A.D.M. momentum [5], i.e.

$$P^{kl} = \sqrt{\text{det} g_{mn}} (k^{kl})$$

where $K_{kl}$ is the second fundamental form (external curvature) of the embedding of $\Sigma$ into a spacetime $M$ which is now curved.

The 12 functions $(g_{kl}, P^{kl})$ must fulfill 4 Gauss–Codazzi constraints:

$$P^l_{|l} = 8\pi \sqrt{\text{det} g_{mn}} T_{i\mu} n^\mu$$ (56)

$$\text{det} g_{mn} \mathcal{R} - P^{kl} P_{kl} + \frac{1}{2} (P^{kl} g_{kl})^2 = 16\pi (\text{det} g_{mn}) T_{\mu\nu} n^\mu n^\nu$$ (57)

where $T_{\mu\nu}$ is an energy momentum tensor of the matter, by $\mathcal{R}$ we denote the (three–dimensional) scalar curvature of $g_{kl}$, $n^\mu$ is a future timelike four–vector normal to the hypersurface $\Sigma$ and the calculations have been made with respect to the three–metric $g_{kl}$ ("|" denotes covariant derivative, indices are raised and lowered etc.).

Einstein equations and the definition of the metric connection imply the first order (in time) differential equations for $g_{kl}$ and $P^{kl}$ (see [3] or [6] p. 525) and contain the lapse function $N$ and the shift vector $N^k$ as parameters:

$$\dot{g}_{kl} = \frac{2N}{\sqrt{g}} \left( P_{kl} - \frac{1}{2} g_{kl} P \right) + N_{k|l} + N_{l|k}$$ (58)

where $g := \text{det} g_{mn}$ and $P := P^{kl} g_{kl}$

$$\dot{P}^{kl} = -N \sqrt{g} \left( R^{kl} - \frac{1}{2} g^{kl} R \right) - \frac{2N}{\sqrt{g}} \left( P^{km} P^m_{\ |l} - \frac{1}{2} P P^{kl} \right) + \left( P^{kl} N^m \right)_{|m} +$$

$$+ \frac{N}{2\sqrt{g}} g^{kl} \left( P^{kl} P_{kl} - \frac{1}{2} P^2 \right) - N^k_{\ |m} P^{ml} - N^l_{\ |m} P^{mk} + \sqrt{g} \left( N^{[kl} - g^{kl} N^m \right)_{|m} +$$

$$+ 8\pi N \sqrt{g} T_{mn} g^{km} g^{ln}$$ (59)

We want to consider an initial value problem for the linearized Einstein equations on the hyperboloidal slicing introduced in the previous section. For this purpose let us check first that on this slicing the ADM momentum $P^{kl}$ for background flat Minkowski spacetime on each hyperboloid $\Sigma_s$ is no longer trivial:

$$P^{kl} = -2\sqrt{g} g^{kl}$$ (60)

and

$$g_{kl} dy^k dy^l = \eta_{kl} dy^k dy^l = d\omega^2 + \sinh^2 \omega (d\theta^2 + \sin^2 \theta d\phi^2)$$ (61)
where \( g^{kl} \) is the three-dimensional inverse of \( g_{kl} \).

Let us define the linearized variations \( (h_{kl}, \varpi^{kl}) \) of the full nonlinear Cauchy data \( (g_{kl}, P^{kl}) \) around background data \( (\hat{g}_{kl}, \hat{P}^{kl}) \):

\[
h_{kl} := g_{kl} - \eta_{kl}, \quad \varpi^{kl} := P^{kl} + 2\Lambda\eta^{kl}
\]

where \( \Lambda := \sqrt{\det(\eta_{kl})} (= \sin^2 \omega \sin \theta) \).

We should now rewrite equations (50–53) in a linearized form in terms of \( (h_{kl}, \varpi^{kl}) \). Let us denote by \( \varpi := \eta_{kl}\varpi^{kl} \) and by \( h := \eta^{kl}h_{kl} \). The vector constraint (57) can be linearized as follows:

\[
P_{|l} \approx \varpi_{|l} - 2\Lambda h^{k}_{|k} + \Lambda h_{|l}
\]

Let us stress that the symbol “\(|\)" has a different meaning on the left-hand side and on the right-hand side of the above formula. It denotes the covariant derivative with respect to the full nonlinear metric \( g_{kl} \) when applied to the \( P^{kl} \) but on the right hand side it means covariant derivative with respect to the background metric \( \eta_{kl} \). The scalar constraint (58) after linearization takes the form

\[
\sqrt{g}R - \frac{1}{\sqrt{g}} \left( P^{kl}P_{kl} - \frac{1}{2}(P^{kg}g_{kl})^2 \right) \approx \Lambda \left( h^{kl}_{|l} - h^{l|k} \right) - 2\varpi
\]

Linearized constraints for vacuum \( (T_{\mu\nu} = 0) \) have the following form:

\[
\varpi_{|k} - 2\Lambda h_{|k} + \Lambda h_{|l} = 0 (= 8\pi\Lambda T_{l\mu}n^\mu)
\]

\[
\Lambda \left( h^{kl}_{|l} - h^{l|k} \right) - 2\varpi = 0 (= 16\pi\Lambda T_{\mu\nu}n^\mu n^\nu)
\]

The linearization of (58) leads to the equation:

\[
\dot{h}_{kl} = \frac{2N}{\Lambda} \left( \varpi_{kl} - \frac{1}{2}\eta_{kl}\varpi \right) + h_{0k|l} + h_{0l|k} + 2N\eta_{kl}(n + \frac{1}{2}h) - 2Nh_{kl} +

- N^m(h_{mk|l} + h_{ml|k} - h_{kl|m})
\]

where \( N := \frac{1}{\sqrt{-\eta^{00}}} = \cosh \omega, \ N_3 = \eta_{03} = - \sinh \omega, \ N_A = \eta_{0A} = 0 \) are lapse and shift for the background and \( n := \frac{n^0}{\sqrt{-\eta^{00}}} \) is the linearized lapse.

\[
\ddot{\varpi}^{kl} = - N\Lambda h^{kl} + N\varpi^{kl} + N^m\varpi^{kl}_{|m} + 2\Lambda \left( h_{0k|l} + h_{0l|k} - \eta^{kl}h_{0m|m} \right) +

+ \Lambda \left( N\eta^{kl} - \eta^{kl}(N^m|m) \right) - \frac{\Lambda}{2}N \left( h^{mk|l}_{m} + h^{ml|k}_{m} - h^{kl|m} - h^{kl} \right) +

- \frac{\Lambda}{2}N_{m} \left[ h^{kl|m} + 3h^{ml|k} + 3h^{mk|l} - \eta^{kl}(h^{lm} + 2h^{mn}|n) \right]
\]

It is well known (see for example [8]) that linearized Einstein equations are invariant with respect to the “gauge” transformation:

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi_{\mu\nu} + \xi_{\nu\mu}
\]

where \( \xi_{\mu} \) is a covector field, pseudoriemannian metric \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) and “\(|\)" denotes four–dimensional covariant derivative with respect to the flat Minkowski metric \( \eta_{\mu\nu} \). There is no \((3+1)\) splitting of the gauge for hyperboloidal slicing similar to the situation described in [8]. The \((3+1)\) decomposition of the gauge acts on Cauchy data in the following way:

\[
\Lambda^{-1}\varpi^{kl} \rightarrow \Lambda^{-1}\varpi^{kl} + N\xi^{0|kl} - N\eta^{kl}\xi_{0|m} - 2N\xi_{0}^{0}h^{kl} - N^{k}\xi^{0|l} - N^{l}\xi^{0|k} +

+ 2\eta^{kl}N_{m}\xi_{0|m} + 2\xi^{k|l} - 2\eta^{kl}\xi_{m}^{m}
\]

\[
h_{kl} \rightarrow h_{kl} + \xi_{|k} + \xi_{|k} + 2N\eta_{kl}\xi^{0}
\]
It can be easily checked that scalar constraint (66) and vector constraint (65) are invariant with respect to the gauge transformations (70) and (71). The Cauchy data \( (h_{kl}, \varpi^{kl}) \) and \( (\tilde{h}_{kl}, \tilde{\varpi}^{kl}) \) on \( \Sigma \) are equivalent to each other if they can be related by the gauge transformation \( \xi_\mu \). The evolution of canonical variables \( \varpi^{kl} \) and \( h_{kl} \) given by equations (67), (68) is not unique unless the lapse function \( n \) and the shift vector \( h^{0}_{k} \) are specified.

Let us define the "new momentum" \( p^{kl} \):

\[
p^{kl} := \varpi^{kl} - \Lambda (2h^{kl} - \eta^{kl}h) \quad (p := \varpi + \Lambda h)
\]

This object can be also introduced in full nonlinear theory as \( P^{kl} + 2\sqrt{gg^{kl}} \) and after linearization we obtain \( p^{kl} \):

\[
P^{kl} + 2\sqrt{gg^{kl}} \approx p^{kl}
\]

Let us first observe that the new momentum is trivial for flat Minkowski data. Secondly the symplectic structure is preserved:

\[
dp^{kl} \land dg_{kl} - d(P^{kl} + 2\sqrt{gg^{kl}}) \land dg_{kl} = -4d^2\sqrt{g} = 0
\]

Moreover, the gauge transformation for \( p^{kl} \) is simpler than for \( \varpi^{kl} \):

\[
\Lambda^{-1} p^{kl} \rightarrow \Lambda^{-1} p^{kl} + N\xi^{0|kl} - N\eta^{kl}\xi^{0|m}m - N k\xi^{0|l} - N l\xi^{0|k} + 2\eta^{kl}N_m\xi^{0|m}
\]

and the vector constraint has a familiar form:

\[
p^{l}_{|l} = 0 (= 8\pi\Lambda T_{\mu}n^{\mu})
\]

We can also rewrite the dynamical equation (68) in terms of the new momentum:

\[
p^{kl} = N^m p^{kl} + n^k(\eta^{kl}p - 3p^{kl}) + \Lambda \left[(Nn)^{kl} - \eta^{kl}(Nn)^{|m}m + 2Nn\eta^{kl}\right] + N\Lambda (\eta^{kl}h - 3h^{kl}) - \frac{\Lambda}{2} N \left[ h^{mk|l} + h^{ml|k} - h^{kl|m} - h^{lk|m} \right] + \frac{\Lambda}{2} N_m \left[ h^{mk|l} + h^{ml|k} - h^{kl|m} + \eta^{kl}(h^{im} - 2h^{mn|n}) \right]
\]

We will show in the sequel that it is possible to define reduced dynamics in terms of invariants which is no longer sensitive on gauge conditions. The construction is analogous to the analysis given in [7].

### 4.3 “Charges” on a hyperboloid

The equation (73) allows to introduce “charges” related to the symmetries of the hyperboloid. There are six generators of the Lorentz group which are simultaneously Killing vectors on the hyperboloid \( \Sigma \). Let us denote this Killing field by \( X^k \). It fulfills the following equation:

\[
X_{k|l} + X_{l|k} = 0
\]

Let \( V \subset \Sigma \) be a compact region in \( \Sigma \). For example \( V := \bigcup_{r \in [r_0, r_1]} S_s(r) \) and \( \partial V = S_s(r_0) \cup S_s(r_1) \). From (73) and (74) we get:

\[
(8\pi \int_V \Lambda T_{\mu}n^{\mu}) = \int_V p^{kl}_{|l}X_k = \int_V (p^{kl}X_k)_{|l} = \int_{\partial V} p^{3k}X_k
\]

The equation (74) expresses the “Gauss” law for the charge “measured” by the flux integral.

In particular for angular momentum when \( X = \partial/\partial \phi \) we can show the relation of this charge with dipole part of invariant \( y \) which we introduce in the sequel (subsection 4.4).

\[
16\pi s^2 := 16\pi j^{xy} = -2 \int_{\partial V} p^3_\phi = -2 \int_{\partial V} p^3_A(r^2 \varepsilon^{AB} \cos \theta)_{|B} =
\]
The time translation defines a mass charge as follows:

\[(16\pi \int_V AT_\mu n^\mu) = 0 = \int_V N \left[ \Lambda \left( h^{kl}_{\mid l} - h^{lk}_{\mid k} \right) - 2\pi \right] + 2N_k p^k_{\mid l} = \]

\[= \int_V \left[ 2N_k p^k_{\mid l} + N\Lambda \left( h^{kl}_{\mid k} - h^{lk}_{\mid l} \right) + \Lambda \left( N_k h^{kl} - N^l h^k \right) \right]_{\mid l} = \]

\[= \int_{\partial V} 2N_k p^{k3} + \Lambda \left( Nh^{3k}_{\mid k} - Nh^{3|3} + N_k h^{k3} - N^3 h \right) \]

and it can be related to monopole part of an invariant \( x \) (subsection 4.4).

\[16\pi p^0 = \int_{\partial V} 2N_k p^{k3} + \Lambda \left( Nh^{3k}_{\mid k} - Nh^{3|3} + N_k h^{k3} - N^3 h \right) = \]

\[= \int_{\partial V} \frac{\Lambda}{\sinh \omega} \left( 2\cosh^2 \omega h^{33} - \cosh \omega \sinh \omega H_{1,3} - H - \frac{2\sinh^2 \omega}{\Lambda} p^{33} \right) = \]

\[= \int_{\partial V} \frac{\Lambda}{r} x \]

Remark: Traceless part of \( h_{kl} \) and \( p^{kl} \) has nice properties, gauge splits into 0-component (transversal to \( \Sigma \)) which acts on \( p^{kl} \) and space components (tangent to \( \Sigma \)) which act on \( h_{kl} - \frac{1}{3} \eta_{kl} h \). But \( h \) and \( \omega \) remain nontrivial unless we impose gauge conditions. The most popular gauge condition, which allows to obtain scalar constraint as a full divergence (see below (81)), is to assume that \( \omega = 0 \). Assuming this gauge we can define different “mass” charge as a surface integral coming from the scalar constraint (66) (but we obtain totally nonlocal object). More precisely, one can analyze the scalar constraint (66) in the same way as (76).

\[2 \int_V \omega = \int_V \Lambda \left( h^{kl}_{\mid l} - h^{lk}_{\mid k} \right) = \int_{\partial V} \Lambda \left( h^{3l}_{\mid l} - h^{3|3} \right) \]

and there is no “Gauss” law for the “mass” defined by the surface integral of the right-hand side of (80) unless we impose gauge condition \( \omega = 0 \). This means that such definition of the mass charge measured by the flux integral at null infinity is not gauge invariant like ADM mass at spatial infinity.

### 4.4 2+1 decomposition and reduction

Now we introduce reduced gauge invariant data on \( \Sigma \) for the gravitational field similar to the invariants introduced in [7]. For this purpose we use spherical foliation of \( \Sigma \) (see equations (53) and (54)).

In this section we present mainly results without detailed proofs as in the section about electrodynamics.

Let \( \kappa := \coth \omega \). The gauge (71) splits in the following way:

\[ h_{33} \rightarrow h_{33} + 2\xi_{3,3} + 2N\xi^0 \]  
\[ h_{3A} \rightarrow h_{3A} + \xi_{3,A} + \xi_{A,3} - 2\kappa\xi_A \]  
\[ h_{AB} \rightarrow h_{AB} + \xi_{A||B} + \xi_{B||A} + 2\kappa\eta_{AB}\xi_3 + 2N\eta_{AB}\xi^0 \]
where by ‘||’ we denote covariant derivative with respect to the two–metric $\eta_{AB}$ on $S(r)$. Similarly the gauge (72) can be splitted as follows:

$$
\Lambda^{-1} p^{33} \rightarrow \Lambda^{-1} p^{33} + N\xi^{[33]} - N\xi^{[m]} = \Lambda^{-1} p^{33} - N\xi^{[A]} - 2N\kappa\xi^{0,3} \quad (84)
$$

$$
\Lambda^{-1} p^{3A} \rightarrow \Lambda^{-1} p^{3A} + N\xi^{[03]} - N^3\xi^{[A]} = \Lambda^{-1} p^{3A} + N\xi^{0,3} - \frac{1}{\sinh\omega}\xi^{0,A} \quad (85)
$$

$$
\Lambda^{-1} p^{AB} \rightarrow \Lambda^{-1} p^{AB} + N\xi^{[AB]} - N\eta^{AB}_m m + 2\eta^{AB} N_m \xi^{[0]} = \Lambda^{-1} p^{AB} + +N\xi^{[AB]} - N\eta^{AB} (\xi^{0,3} + \xi^{[C]} + (\kappa - 2N^3)\xi^{0,3}) \quad (86)
$$

It is also quite easy to rewrite (2+1) decomposition of (67):

$$
\hat{h}_{33} = \frac{2N}{\Lambda} \left( p_{33} - \frac{1}{2}p \right) + 2h_{033} + 2N(n + h_{33}) - N^3h_{333} = \quad (87)
$$

$$
\hat{h}_{3A} = \frac{2N}{\Lambda} p_{3A} + h_{03|A} + h_{0A|3} + 2Nh_{3A} - N^3h_{333,A} = \quad (88)
$$

$$
\hat{h}_{AB} = \frac{2N}{\Lambda} \left( p_{AB} - \frac{1}{2}\eta_{AB}p \right) + h_{0A|B} + h_{0B|A} + 2N\eta_{AB}n + 2Nh_{AB} + -N^3(h_{3A|B} + h_{3B|A} - h_{AB|3}) = \frac{2N}{\Lambda} \left( p_{AB} - \frac{1}{2}\eta_{AB}p \right) + h_{0A|B} + +h_{0|B} + 2\kappa\eta_{AB}h_{03} + 2N\eta_{AB}(n + h_{33}) + 2Nh_{AB} + -N^3(h_{3A|B} + h_{3B|A} - h_{AB|3}) \quad (89)
$$

The vector constraint (72) can be splitted in the similar way:

$$
p_k^k|k = p_k^k, 3 + p_3^3|A - \kappa p^{AB}\eta_{AB} = 0 \quad (90)
$$

$$
p_A^k|k = p_A^k, 3 + p_A^3|B = p_{3A,3} + S_A^B|B + \frac{1}{2}S_{||A} = 0 \quad (91)
$$

where $S := p^{AB}\eta_{AB}$ and $S^{AB} := p^{AB} - \frac{1}{2}\eta^{AB} S$. Similarly let us denote $H := \hat{h}_{AB}\eta^{AB}$ and $\chi_{AB} := \hat{h}_{AB} - \frac{1}{2}\eta_{AB} H$. The invariants are defined as follows:

$$
X = 2\cosh^2\omega h_{33} + 2\cosh\omega\sinh\omega h_{3C}||C + \sinh^2\omega\chi^{AB}||AB - \cosh\omega\sinh\omega H, 3 + -\frac{1}{2}(a + 2H) - \frac{2\sinh^2\omega}{\Lambda}p^{33} \quad (92)
$$

$$
Y = 2\sinh^2\omega S^{3A}||AB + 2\cosh\omega\sinh\omega p^{3A}||A + ap^{33} \quad (93)
$$

$$
Y = 2\Lambda^{-1}\sinh^2\omega p^{3A}||B \varepsilon_{AB} \quad (94)
$$

$$
Y = \Lambda(a + 2h_{3A}|B \varepsilon_{AB} - \sinh^2\omega(\Lambda\chi^{C}_A||CB\varepsilon^{AB}), 3 \quad (95)
$$

(2+1) decomposition of the scalar constraint (66) can be written in the form:

$$
\Lambda \left( h^{3}|3 - h^{0|k} \right) + 2\omega = \left[ \Lambda(H, 3 - 2h^{3A}||A - 2\kappa h^{33} + \kappa H) \right], 3 - 2\Lambda(h^{33} + H) + +2(p^{33} + S) - \Lambda(\chi^{AB}||AB + 2\kappa h^{3A}||A) + \Lambda \left( h^{33}||A + \frac{1}{2}H||C \right) = 0 \quad (96)
$$

20
The dynamical equations \[74\) take the following (2 + 1) form:

\[
\Lambda^{-1} \ddot{p}^{33} = \Lambda^{-1} N_3 (p^{33}, 3 - \kappa S) - (N n)^{\Lambda A} - 2 \kappa N n, _{3} + \\
\frac{N}{2} \left[ h^{33}|^{|A} + H_{33} + 2 \kappa H_{33} - 2 \kappa (2 h^{3A}|_{A} + h^{33}, 3) - (2 h^{3A}|_{A}, 3) \right] + \\
\frac{1}{2} N_3 \left[ H, _{3} - 2 h^{3A}|_{A} \right]
\]

\[97\]

\[
\Lambda^{-1} \mathbf{p}^{3A} = \Lambda^{-1} N^3 (p^{3A}, 3 + 2 \kappa p^{3A}) + [(N n), 3 - \kappa N n]|^{|A} + \frac{N}{2} \left[ h^{3A}|_{A} - \kappa h^{33}|_{A} + \\
\frac{2}{\sin h^2 \omega} h^{3A} - h^{AB}|_{B, 3} - 2 \kappa h^{AB}|_{B} - h^{3B}|_{A} + h^{3A}|_{B} \right] + \frac{1}{2} N_3 h^{33}|_{A}
\]

\[98\]

\[
\Lambda^{-1} \mathbf{p}_{AB} = \Lambda^{-1} N^3 p_{AB, 3} + \Lambda^{-1} N \left[ \eta_{AB} (p^{3} + S) + p_{AB} \right] + \\
\frac{N (n_{||AB} - \eta_{AB} n^{(C)} C)}{N_3 + N \eta_{AB} (n, _{33} + \kappa n, 3)} + 2 \eta_{AB} N^3 n, _{3} + \\
\frac{N^3}{2} \left[ h^{3A}|_{B} + h^{3B} + h^{3C} + \eta_{AB} (h^{3A}|_{A} - h^{3A}|_{B, 3} - \chi A^{C} \eta_{CB}) + \\
\frac{1}{N} \left[ (\chi^{B} C, 3 \eta_{CA}), 3 + \chi_{AB} |^{C} C - \chi^{C} A^{||BC} - \chi^{C} B^{||AC} + h^{33}|_{AB} + \frac{1}{2} \eta_{AB} H^{||C} C + \\
\frac{1}{2} \eta_{AB} \left( h^{3A}|_{B} + h^{3B} + h^{3C} \right) \right] \right] + \\
\frac{1}{\sin h^2 \omega} \frac{1}{2} \left( h^{3A}|_{B} + h^{3B} + h^{3C} \right) + \frac{1}{2} \frac{1}{\sin h^2 \omega} \chi_{AB} + \\
\frac{1}{\sin h^2 \omega} \frac{1}{2} \left( h^{3A}|_{B} + h^{3B} + h^{3C} \right) \right] \right] \right] + \\
\frac{1}{2} \frac{1}{\sin h^2 \omega} \chi_{AB} \]

\[99\]

where we can check the reduced field equations for our invariants:

\[
\dot{x} = \frac{N}{\Lambda} x + (N^3 x), _{3}
\]

\[100\]

\[
\dot{\mathbf{x}} = N^3 \mathbf{x}, _{3} + \Lambda N \Delta_{\Sigma} x - \Lambda N^3 (x, _{3} + 2 \kappa x)
\]

\[101\]

\[
\dot{y} = \frac{N}{\Lambda} y + \frac{N^3}{\Lambda} (\Lambda y), _{3}
\]

\[102\]

\[
\dot{\mathbf{y}} = \Lambda (N^3 \Lambda^{-1} \mathbf{y}), _{3} + \Lambda N \Delta_{\Sigma} \mathbf{y} - \Lambda N^3 (\mathbf{y}, _{3} + 2 \kappa \mathbf{y})
\]

\[103\]

where \(\Delta_{\Sigma}\) is a Laplacian on a hyperboloid \(\Sigma\).

It can be easily verified that the invariants \(x\) and \(y\) fulfill usual d’Alambert equation (as a consequence of the above dynamical equations):

\[
\Box x = 0
\]

\[
\Box y = 0
\]

Let us notice that \(x\) and \(y\) are scalars on each sphere \(S_{\varepsilon}(r)\) with respect to the coordinates \(y^{A}\).

For the scalar \(f\) on a sphere we can define “monopole” part \(\text{mon}(f)\) and a “dipole” part \(\text{dip}(f)\) as a corresponding component with respect to spherical harmonics on \(S^2\). Similarly “dipole” part of a vector \(v^{A}\) corresponds to the dipole harmonics for the scalars \(v^{A}|_{A}\) and \(\varepsilon^{AB} v_{A}|_{B}\). Let us denote by \(f\) monopole, dipole–free part of \(f\). According to this decomposition we have:

\[
x = \text{mon}(x) + \text{dip}(x) + \overline{x}
\]
\[ y = \text{mon}(y) + \text{dip}(y) + y \]

Then mono-dipole part of the each scalar can be solved explicitly from the equations \((100-103)\) and the solution has the form:

\[ x - x = \frac{4m}{\sinh \omega} + \frac{12k}{\sinh^2 \omega} \]

\[ y - y = \frac{12s}{\sinh^2 \omega} \]

Let \( p := k \) then \( \dot{m} = \dot{p} = \dot{s} = 0 \)

and

\[ k = p(s + \cosh \omega) + k_0 \]

Moreover \( am = 0, (a + 2)p = (a + 2)k_0 = (a + 2)s = 0 \) which simply means that \( m \) is a monopole and \( k_0, p, s \) are dipoles and they are constant on \( M_2 \). They correspond to the charges introduced in \([8]\). Let us rewrite the solution in coordinates \( u, r \) which will be more useful in the sequel:

\[ x = \bar{x} + \frac{4m + 12p}{r^2} + \frac{12(k_0 + pu)}{r^2} \]  

\[ y = \bar{y} + \frac{12s}{r^2} \]  

(104) \hspace{1cm} (105)

Let us remind the relation between spatial constant three-vectors in cartesian coordinates and dipole harmonics

\[ k_0 = \frac{j^{0i}z_i}{r}, \quad p = \frac{p^i z_i}{r}, \quad s = \frac{s^i z_i}{r} \]

where \( z_i \) are cartesian coordinates and \( j^{0i}, p^i, s^i \) are corresponding three-vectors representing our charges (see \([8]\)).

### 4.5 Reduction of symplectic form on a hyperboloid

We want to show the relation between the symplectic structure and the invariants introduced in the previous subsection. Let \( (p^{kl}, h_{kl}) \) and \( (s^{kl}, q_{kl}) \) denote two pairs of a Cauchy data on a hyperboloid.

The \((2 + 1)\)-splitting of the tensor \( q_{kl} \) gives the following components on a sphere: \( \tilde{q} := \eta^{AB} q_{AB} \), \( q_{33} \) — scalars on \( S^2 \), \( q_{3A} \) — vector and \( \tilde{q}_{AB} := q_{AB} - \frac{1}{2} \eta_{AB} \tilde{q} \) — symmetric traceless tensor on \( S^2 \). Similarly we can decompose tensor density \( p^{kl} \). The quadratic form \( \int_V p^{kl} q_{kl} \) can be decomposed into monopole part, dipole part and the remainder in a natural way.

The “monodipole” part we write separately:

\[
\text{mon}(\int_V p^{kl} q_{kl}) = \int_V \frac{1}{2 \cosh^2 \omega} p^{33} \text{mon}(\xi) + \int_V \frac{\tanh^2 \omega}{\Lambda} p^{33} \text{mon}(s^{33}) + \\
\frac{1}{2} \int_{\partial V} \text{tan} \omega p^{33} \text{mon}(\tilde{q}) \]

\[
\text{dip}(\int_V p^{kl} q_{kl}) = \int_V \frac{1}{2 \cosh^2 \omega} p^{33} \text{dip}(\xi) - 2 \int_V \text{dip}(\text{sinh}^2 \omega p^{3A} [B \varepsilon_{AB}) a^{-1} (q_{3A}) [B \varepsilon^{AB}]) + 
\]

22
we can partially reduce our form:

\[ + \int_V \frac{\tanh^2 \omega}{\Lambda} q^{33} \text{dip}(s^{33}) + \frac{1}{2} \int_{\partial V} \tanh \omega p^{33} \text{dip}(\tilde{q}) + \]

\[ + \int_V \frac{p^{33}}{2 \cosh^2 \omega} + \tanh \omega a^{-1} p^{3A} ||_A \left( \frac{1}{2} a^2 \tilde{q} - 2 \sinh \omega \cosh \omega q^{3A} ||_A \right) \]

(107)

where invariant $\xi$ is defined as follows:

\[ \xi := 2 \cosh^2 \omega q^{33} + 2 \cosh \omega \sinh \omega q^{3A} ||_A + \sinh^2 \omega \tilde{q}^{AB} ||_{AB} - \cosh \omega \sinh \omega \tilde{q},_{3} + \]

\[ - \frac{1}{2} (a + 2) \tilde{q} - 2 \sinh^2 \omega \Lambda^{-1} s^{33} \]

From vector constraints:

\[ \sinh \omega p^{33},_3 + \sinh \omega p^{3A} ||_A - \cosh \omega \tilde{p} = 0 \]  (108)

\[ (\sinh^2 \omega p^{3A} ||_A),_3 + (\sinh^2 \omega \tilde{p}^{AB} ||_{AB}) + \frac{1}{2} a \tilde{p} = 0 \]  (109)

\[ (\sinh^2 \omega p^{3A} ||_{AB}),_3 + (\sinh^2 \omega \varepsilon^{AC} \tilde{p}_A B ||_{BC}) = 0 \]  (110)

we can partially reduce our form:

\[ \int_V p^{k l} q_{k l} = \int_V p^{33} q_{33} + 2 p^{3A} q_{3A} + \frac{1}{2} 2 \tilde{q} + \tilde{p}^{AB} \tilde{q}_{AB} = \]

\[ = \int_V p^{33} q_{33} - 2(\sinh \omega p^{3A} ||_A) a^{-1}(\sinh \omega q^{3A} ||_A) - 2(\sinh \omega p^{3A} ||_{AB}) a^{-1}(\sinh \omega q_{3A} ||_{AB}) + \]

\[ + \int_V \frac{1}{2} \tilde{p} q + 2(\sinh^2 \omega \varepsilon^{AC} \tilde{p}_A B ||_{BC}) a^{-1} (a + 2)^{-1}(\sinh^2 \omega \varepsilon^{AC} \tilde{q}_A B ||_{BC}) + \]

\[ + 2 \int_V (\sinh^2 \omega \tilde{p}^{AB} ||_{AB}) a^{-1} (a + 2)^{-1}(\sinh^2 \omega \tilde{q}^{AB} ||_{AB}) = \]

\[ = \int_V p^{33} q_{33} - 2(\sinh \omega p^{3A} ||_A) a^{-1}(\sinh \omega q^{3A} ||_A) - 2(\sinh \omega p^{3A} ||_{AB}) a^{-1}(\sinh \omega q_{3A} ||_{AB}) + \]

\[ + \int_V \frac{1}{2} (\tanh \omega p^{33},_3 + \tanh \omega p^{3A} ||_A) q - 2 \int_V (\sinh^2 \omega p^{3A} ||_{AB}) a^{-1}(a + 2)^{-1}(\sinh^2 \omega \varepsilon^{AC} \tilde{q}_A B ||_{BC}) + \]

\[ - 2 \int_V \left[ (\sinh^2 \omega p^{3A} ||_A),_3 + \frac{1}{2} a(\tanh \omega p^{33},_3 + \tanh \omega p^{3A} ||_A) \right] a^{-1}(a + 2)^{-1}(\sinh^2 \omega \tilde{q}^{AB} ||_{AB}) = \]

\[ = \int_{\partial V} \tanh \omega p^{33} \left[ \frac{1}{2} q - (a + 2)^{-1}(\sinh^2 \omega \tilde{q}^{AB} ||_{AB}) \right] + \]

\[ - 2 \int_{\partial V} (\sinh^2 \omega p^{3A} ||_A) a^{-1} (a + 2)^{-1}(\sinh^2 \omega \tilde{q}^{AB} ||_{AB}) + \]

\[ - 2 \int_{\partial V} (\sinh^2 \omega p^{3A} ||_{AB}) a^{-1} (a + 2)^{-1}(\sinh^2 \omega \varepsilon^{AC} \tilde{q}_A B ||_{BC}) + \]

\[ - 2 \int_V (\sinh^2 \omega p^{3A} ||_{AB}) a^{-1} \left[ q ||_{3A} ||_{AB} - (a + 2)^{-1}(\sinh^2 \omega \varepsilon^{AC} \tilde{q}_A B ||_{BC}),_3 \right] + \]
The volume term in the framebox is mono–dipole–free and corresponds to the invariants \( y, Y \). The last two terms we can proceed further but let us first write a scalar constraint in two equivalent forms:

\[
\begin{aligned}
&\left(\frac{\sinh^3 \omega}{\cosh \omega} \hat{q}^{AB}_{||AB}, 3 + (a + 2)^{-1} \left(\frac{\sinh^3 \omega}{\cosh \omega} \hat{q}^{AB}_{||AB}, 3 - \frac{1}{2}(\tanh \omega \hat{q}), 3\right) + \right. \\
&+ \left. \int_{V} \tanh \omega p^{3A}_{||A} \left[ \frac{1}{2} \hat{q}^2 + 2 \sinh \omega \cosh \omega a^{-1}(a + 2)^{-1}(\sinh^2 \omega \hat{q}^{AB}_{||AB}, 3 + \right. \\
&\left. - (a + 2)^{-1}(\sinh^2 \omega \hat{q}^{AB}_{||AB}) - 2a^{-1}(\sinh \omega \cosh \omega q^{3A}_{||A}) \right] \right)
\end{aligned}
\]

In “radiation” part we get the following result:

\[
\begin{aligned}
&\int_{V} p^{33} \left[ q_{33} + (a + 2)^{-1} \left(\frac{\sinh^3 \omega}{\cosh \omega} \hat{q}^{AB}_{||AB}, 3 - \frac{1}{2}(\tanh \omega \hat{q}), 3\right) + \right. \\
&\left. + \int_{V} \tanh \omega p^{3A}_{||A} \left[ \frac{1}{2} \hat{q}^2 + 2 \sinh \omega \cosh \omega a^{-1}(a + 2)^{-1}(\sinh^2 \omega \hat{q}^{AB}_{||AB}, 3 + \right. \\
&\left. - (a + 2)^{-1}(\sinh^2 \omega \hat{q}^{AB}_{||AB}) - 2a^{-1}(\sinh \omega \cosh \omega q^{3A}_{||A}) \right] \right)
\end{aligned}
\]
Hyperboloid around $S^+$

Let us return to the coordinate $\rho := \frac{1}{\sinh \omega}$. The metric on $M$ takes the starting form (11). It is convenient to introduce new canonical field variables similar to the variables for the scalar field and electrodynamics:

$$
\Psi_x := \rho^{-1} \xi_x, \quad \Psi_y := \rho^{-1} \xi_y
$$

$$
\Pi_x := \frac{\xi_x}{\sqrt{1 + \rho^2}}, \quad \Pi_y := \frac{\xi_y}{\sqrt{1 + \rho^2}}
$$

Equations of motion are the same for both degrees of freedom:

$$
\frac{1}{\sqrt{1 + \rho^2}} \Psi_\Upsilon - \Psi_{\Upsilon, \rho} = \frac{\Pi_\Upsilon \sqrt{1 + \rho^2}}{\sin \theta} \quad \Upsilon = x, y
$$
\[ \dot{\Pi}_\psi - (\Pi_\psi \sqrt{1 + \rho^2})_\rho = \sin \theta \left[ a\Psi_\psi + ((1 + \rho^2)\Psi_{\psi,\rho})_\rho \right] \]

and they are similar to (17), (18) for the scalar field and (38), (39) for electrodynamics.

The reduction of symplectic form from the previous section allows to formulate the hamiltonian relation in terms of new canonical variables:

\[
\sum_{Y=x,y} \int_V \dot{\Psi}_Y a^{-1}(a + 2)^{-1}\delta \Pi_Y - \dot{\Pi}_Y a^{-1}(a + 2)^{-1}\delta \Psi_Y = \delta \mathcal{H} \]

\[
- \sum_{Y=x,y} \int_{\partial V} \left( \Pi_\psi \sqrt{1 + \rho^2} + \sin \theta (1 + \rho^2) \Psi_{\psi,\rho} \right) a^{-1}(a + 2)^{-1}\delta \Psi_Y + \frac{1}{2} \sum_{Y=x,y} \int_V \rho^2 \sin \theta \Psi_{\psi,\rho} a^{-1}(a + 2)^{-1}\Psi_{\psi,\rho} - \sin \theta \Psi_\psi (a + 2)^{-1}\Psi_\psi
\]

(113)

where

\[ \mathcal{H} := \frac{1}{2} \sum_{Y=x,y} \int_V \left( \frac{\Pi_\psi}{\sin \theta} + \Psi_{\psi,\rho} \right) a^{-1}(a + 2)^{-1} \left( \frac{\Pi_\psi}{\sin \theta} + \Psi_{\psi,\rho} \right) + + \frac{1}{2} \sum_{Y=x,y} \int_V \rho^2 \sin \theta \Psi_{\psi,\rho} a^{-1}(a + 2)^{-1}\Psi_{\psi,\rho} - \sin \theta \Psi_\psi (a + 2)^{-1}\Psi_\psi \]

(114)

Similarly for angular momentum we propose the following expression:

\[ J_z = \sum_{Y=x,y} \int_\Sigma \Pi_\psi a^{-1}(a + 2)^{-1}\Psi_{\psi,\phi} \]

(115)

The non-conservation laws for the energy and angular momentum

\[-\partial_0 \mathcal{H} = \sum_{Y=x,y} \int_{S(0)} \sin \theta \dot{\Psi}_Y a^{-1}(a + 2)^{-1}\Psi_Y \]

\[-\partial_0 J_z = \sum_{Y=x,y} \int_{S(0)} \sin \theta \dot{\Psi}_Y a^{-1}(a + 2)^{-1}\Psi_{\psi,\phi} \]

are similar to (19), (20) and (47), (48). It should be also possible to formulate linear momentum \( P_z \) in a similar way as (23)

\[-\partial_0 P_z = \sum_{Y=x,y} \int_{S(0)} \sin \theta \cos \theta \dot{\Psi}_Y a^{-1}(a + 2)^{-1}\Psi_Y \]

but this will be analyzed in a separate paper. It is obvious that all these formulae are \textit{quasi-local}.

4.7 Appendix – explicit formulae on a hyperboloid

\[ \Lambda = \sinh^2 \omega \sin \theta \]

\[ N_{|m} = -N_m \quad N_{|kl} = -N_{k|l} = N_{\eta_{kl}} \quad N^{|k}_{|k} = -N^{|k}_{|k} = 3N \]

\[ N = \cosh \omega \quad N^3 = N_3 = -\sinh \omega \quad N^A = 0 \quad \eta_{AB,3} = 2\kappa \eta_{AB} \]

\[ \eta^{AB,3} = -2\kappa \eta^{AB} \quad \varepsilon_{AB,3} = 2\kappa \varepsilon_{AB} \quad \varepsilon^{AB,3} = -2\kappa \varepsilon^{AB} \quad \kappa = \coth \omega \]

\[ \Lambda,3 = 2\kappa \Lambda \quad \kappa N_{|3} = -\kappa N_3 = N \quad \kappa_{,3} = -\frac{1}{\sinh^2 \omega} \]

\[ \Gamma_{AB,3} = -\kappa \eta_{AB} \quad \Gamma^A_{B3} = \kappa \delta^A_B \quad \Gamma^A_{BC,3} = 0 \]
\[ 2R_{ABCD} = \frac{1}{\sinh^2 \omega} (\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}) \]
\[ 2R_{AB} = \frac{1}{\sinh^2 \omega} \eta_{AB} \]
\[ \xi^{33} = \xi^{33}, \quad \xi^{3A} = \xi^{3A} - \kappa \xi^A \quad \xi^{|AB} = \xi^{||AB} + \kappa \eta^{AB} \xi^3 \]
\[ \xi_{3|3} = \xi_{3,3}, \quad \xi_{3|A} = \xi_{3|A} - \kappa \xi_A \quad \xi_{A|3} = \xi_{A,3} - \kappa \xi_A \]
\[ \xi_{A|B} = \xi_{A||B} + \kappa \eta^{AB} \xi^3 \]
\[ h_{33|3} = h_{33,3}, \quad h_{33|A} = h_{33,A} - 2\kappa h_{3A} \]
\[ h^{AB}_{|3} = h^{AB,3} + 2\kappa h^{AB} \quad h_{AB|3} = h_{AB,3} - 2\kappa h_{AB} \quad h^{A|B}_{|3} = h^{A,B,3} \]
\[ h^{3A}_{|3} = h^{3A,3} + \kappa h^{3A} \quad h_{3A|3} = h_{3A,3} - \kappa h_{3A} \]
\[ h_{3A|B} = h_{3A||B} - \kappa h_{3B} + \kappa \eta_{AB} h_{33} \]
\[ h_{AB|C} = h_{AB||C} + \kappa \eta_{AC} h_{3B} + \kappa \eta_{BC} h_{3A} \]

5 Linearized gravity in null coordinates

We are going to follow the idea from subsection 2.2 and apply it to linearized gravity.

5.1 Minkowski metric in null coordinates

Let us introduce null coordinates: \( u := t - r, \ v := r + t \) together with the index \( a \) corresponding to the coordinates \( (u,v) \). The spherical foliation is the same as previously and the coordinates on a sphere \((x^A), (A = 1,2), (x^1 = \theta, x^2 = \phi)\) are the same.

For convenience we need also some more denotations: \( \rho := r^{-1} = \frac{2}{\sqrt{-\rho^2}}, \rho_a = \rho^2 \epsilon_a \) where \( \epsilon_u := \frac{1}{2}, \epsilon_v := -\frac{1}{2}, \eta^{ab} \epsilon_a \epsilon_b = 1 \). We will also need \( e^a := \eta^{ab} \epsilon_b \) and we can check that \( e^u = 1, e^v = -1, \eta_{ab} e^a e^b = 1 \).

The explicit formulae for the components of Minkowski metric can be denoted as follows:
\[ \eta_{AB} = \rho^{-2} \eta_{AB}, \quad \eta_{ab} = -\frac{1}{2} |E_{ab}|, \quad \eta_{A} = 0 \]
where \( E_{uu} = 0 = E_{vv} \) and \( E_{uv} = 1 = -E_{vu} \). Similarly the inverse metric has the following components
\[ \eta^{AB} = \rho^2 \eta^{AB}, \quad \eta^{ab} = -2|E^{ab}|, \quad \eta^{A} = 0 \]
where \( E^{uu} = 0 = E^{vv} \) and \( E^{uv} = 1 = -E^{vu} \). We shall also need derivatives
\[ \eta^{AB} , a = 2 \rho \epsilon_a \eta^{AB}, \quad \eta_{AB} , a = -2 \rho \epsilon_a \eta_{AB} \]
and finally the nonvanishing Christoffel symbols except \( \Gamma^{A}_{BC} \) are the following:
\[ \Gamma^{a}_{AB} = \rho \epsilon_a \eta_{AB}, \quad \Gamma^{A}_{aB} = -\rho \epsilon_a \delta^{A}_{B} \]
5.2 Riemann tensor in null coordinates

We need to derive linearized Riemann tensor in null coordinates:

\[ 2R_{abcd} = h_{ad,bc} - h_{bd,ac} + h_{bc,ad} - h_{ac,bd} \]

\[ 2R_{abdc} = h_{aD,be} - h_{bD,ae} + h_{be,AD} - h_{AE,bD} + \]

\[ + \rho e_b (h_{aD,c} + h_{eD,a} - h_{ac,D}) - \rho e_a (h_{bD,c} + h_{eD,b} - h_{bc,D}) \]

\[ 2R_{ABCD} = h_{dA||C,b} + h_{BC||A,d} - h_{bd||AC} - h_{AC,bd} + \]

\[ + \rho e_b (h_{dA||C} - h_{dC||A} - h_{AC,d}) - \rho e_d (h_{BC||A} - h_{bA||C} - h_{AC,b}) + \]

\[ + \rho \eta_{ACE}^a (h_{bD,a} - h_{ad,b} - h_{ab,d} - 2\rho^2 \epsilon_{d}^{ab} \epsilon_{AC}^{d} h_{AC}) \]

\[ 2R_{ABCD} = h_{dA||BC} + h_{BC||A,d} - h_{BD||AC} - h_{AC||BD} + 2\rho \eta_{AB}^a (h_{D||A} - h_{AC||B}) + \]

\[ + \rho \eta_{BCA}^a (h_{a,b} - h_{dA,a} + h_{ad,A} + 2\rho \eta_{AB}^a (h_{aB,d} - h_{dB,a} + h_{ad,B} + 2\rho \eta_{AB}^a (h_{aB,A} - h_{AD||BC} - h_{BC||AD}) + 2\rho^2 \epsilon_{d}^{ab} \epsilon_{AC}^{d} h_{AC}) \]

5.3 Ricci tensor in null coordinates

The Ricci tensor takes the following form:

\[ 2R_{ab} = h^{c}_{b,ac} + h^{c}_{a,c,b} - h^{c}_{ab,c} - h^{c}_{c,ab} + h_{A,B} A || A + h_{bA,a} A || A - h_{ab} A || A - H_{ab} + \]

\[ + \rho e_a H_{b} + \rho e_b H_{a} + 2\rho \epsilon_{a} (h_{b,c} - h_{ac,b} - h_{bc,a}) \]

\[ 2R_{ab} = h_{b,B,a} - h_{AB|B} A + h_{c,B} c + h_{c,C,B} - h_{c,aB} + h_{a} A || B - h_{AB || A} + \chi_{B} A || A,a - 1/2 H_{B,a} + \]

\[ + \rho e_a (2h_{b,B} - h_{b,B}) - 2\rho h_{b,B,a} - 2\rho \epsilon_{a} \epsilon_{b} h_{b,B} \]

\[ 2R_{AB} = (h_{A||B} + h_{a||B})_{,a} - h_{a,||AB} - \chi_{AB} a_{a} - 2\rho \epsilon_{a} \chi_{AB,a} + \chi_{A} C || B + \chi_{B} C || C + \]

\[ - \chi_{AB} C || C + \eta_{AB} \left[ -1/2 (H||C + H_{c} a) + 2\rho \epsilon_{a} (H_{a} - h_{a} A || A) + \rho^2 (2\epsilon_{a} \epsilon_{b} h_{ab} - H) \right] \]
5.4 Gauge in null coordinates

The gauge transformation $\xi_\mu$

$$h_{\mu\nu} \to h_{\mu\nu} + \xi_{(\mu;\nu)}$$

splits in the following way:

$$h_{ab} \to h_{ab} + \xi_{a,b} + \xi_{b,a}$$

$$h_{aA} \to h_{aA} + \xi_{a,A} + \xi_{A,a} + 2\rho\epsilon^a_a\xi_a$$

$$h_{AB} \to h_{AB} + \xi_{A||B} + \xi_{B||A} - 2\rho\eta_{AB}\epsilon^a_a\xi_a$$

and it would be also useful the following formulae

$$\chi_{AB} \to \chi_{AB} + \xi_{A||B} + \xi_{B||A} - \eta_{AB}\xi^C_{||C}$$

$$\frac{1}{2}H \to \frac{1}{2}[H + \xi^A_{||A} - 2\rho\epsilon^a_a\xi_a]$$

$$h_{a}^A \to h_{a}^A + \xi_{a}^A + \xi_{A,a}$$

which are straightforward consequences of the previous one.

5.5 Invariants

Let us introduce the following gauge invariant quantities:

$$y_a := (a + 2)h_{aA||B}e^{AB} - (\rho^{-2}\chi^C_{||C}e^{AB})_a$$

$$y := 2\rho^{-2}(h_{bB||A}e^{AB})_aE^{ab}$$

$$x := \rho^{-2}\chi_{AB||BA} - \frac{1}{2}aH + \rho^{-1}\epsilon^a_aH_{,a} - H + 2\epsilon^a_bh_{ab} - 2\rho^{-1}\epsilon^a_a\chi^a_{A||A}$$

$$x_{ab} := a(a + 2)h_{ab} - (a + 2)\left[(\rho^{-2}\chi^A_{||A})_b + (\rho^{-2}\chi_{bA||A})_a\right] + \left[(\rho^{-2}\chi^A_{||AB})_b + \rho^{-2}\chi^{A||AB}\right]_a$$

They fulfill the following equations:

$$(\rho^{-2}y^a)_a = 0$$

$$2E^{ab}(\rho^{-2}y)_b + \rho^{-2}y^a = 0$$

$$[\rho^{-4}(y_{a,b} - y_{b,a})]_b + \rho^{-2}(a + 2)y_a = 0$$

$$(\rho^{-1}y)^a_a + \rho^{-1}ay = 0$$

$$(\rho^{-1}x)^a_a = -\rho^{-1}ax$$

$$\rho^{-2}x^a_{ab} = a(a + 2)x$$

$$\eta^{ab}x_{ab} = 0$$

$$x_{ab} = 2(\rho^{-2}x)_a - \eta_{ab}(\rho^{-2}x)^c_c$$

if we assume vacuum equations $R_{\mu\nu} = 0$. 

29
5.6 Reduction of symplectic form on $S^+$

Now we will show how linearized symplectic form on $N$, which was introduced by Kijowski in the full nonlinear theory, can be reduced to the invariants in “wave” part similarly to the hyperboloid case. We shall calculate this form in a convenient gauge but final result will be gauge-invariant. This way we shall prove that modulo boundary terms depending on the gauge the gauge-invariant part of the symplectic form can be obtained in the demanded shape in “wave” part.

5.6.1 gauge conditions

We would like to work with the following gauge conditions:

$$\chi_{AB} = 0, \quad h^A_{||A} = 0$$

It is easy to verify that they are compatible for “wave” part:

$$\rho^{-2} \chi_{AB} \rightarrow \rho^{-2} \chi_{AB} + (a + 2)\xi^A_{||A}$$

$$\rho^{-2} \chi_{A} C_{||CB} \rightarrow \rho^{-2} \chi_{A} C_{||CB} + (a + 2)\xi_{A||B}$$

$$h^A_{||A} \rightarrow h^A_{||A} + \rho^{-2}a\xi_a + (\xi^A_{||A}).a$$

More precisely, mono-dipole free part of $\xi_a$ and $\xi^A_A$ is uniquely defined under these gauge conditions.

5.6.2 partial reduction to extract gauge invariant part

The linearized $\pi^{\mu\nu}$ has the following form

$$\pi^{\mu\nu} = -\Lambda h^{\mu\nu} + \frac{1}{2} \eta^{\mu\nu}(h^A_{||A} + H)A$$

and it can be simplified in our gauge. Let us observe that invariants (116) and (119) have simple form in our gauge in terms of $h^\mu_{\nu}$. From (119) and (120) we obtain $h^3_{\nu} = -4h^4_{\nu} = 0$. Moreover $\pi^{AB} = \frac{1}{2} \Lambda \eta^{AB} h^a_{||a} - \Lambda \chi^{AB}$ vanishes. Similarly $\pi^A_{||A} = 0$ because $\pi^{AB} = -A h^{AB}$. Finally from the above considerations we obtain:

$$\int_{S(s,\rho)} \pi^{\mu\nu} \delta A^a_{||\mu\nu} = \int_{S(s,\rho)} \pi^{cd} \delta A^a_{||cd} - 2\pi^{cAB}_{||B} \Lambda^a_{||B} \rho^{-2}a^{-1} \delta A^a_{||cB}$$

One can show the following relation:

$$\int_V \pi^{cd} \delta A^v_{||cd} \sim \int_V \Lambda \rho^2 (\rho^{-1}x^a)_{,a} a^{-2}(a + 2)^{-2} \delta (\rho^{-1}x^a)$$

where $\sim$ denotes equality modulo boundary terms and full variation. Similarly one can prove

$$\int_V \pi^{cAB}_{||B} \rho^{-2}a^{-1} \delta A^v_{||B} \rho^{-2}a^{-1} \delta A^v_{||B} \sim \int_V \Lambda \rho^2 y^b a^{-1}(a + 2)^{-2} \delta (\rho^{-4}y^b)_{,b} - (\rho^{-2}y^b)v$$

5.6.3 full reduction to $x,y$

We would like to obtain similar formula to (24). The “curl” part (122) reduces easily to the demanded form

$$\int_V \Lambda \rho^2 y^b a^{-1}(a + 2)^{-2} \delta (\rho^{-4}y^b)_{,b} - (\rho^{-2}y^b)v \sim \int_V 2\Lambda \rho^2 (\rho^{-1}y^b)_{,b} a^{-1}(a + 2)^{-1} \delta (\rho^{-1}y^b)$$
On the other hand the second part \[121\] can be rewritten in the following way
\[
\int V \Lambda \rho^2 (\rho^{-1} x_{ab}), u_{ab} - 2 (a + 2)^{-2} \delta \left( \rho^{-1} x_{ab} \right) + \int V 16 \Lambda \rho^2 \left[ \rho^{-1} (\rho^{-1} x), u_{ab} + \frac{1}{2} a (\rho^{-1} x), v_{ab} + 1 \right] = 2 \Lambda \rho^2 (\rho^{-1} x), u_{ab} - 2 (a + 2)^{-1} \delta \left( \rho^{-1} x \right)
\]

Let us observe that the last term vanishes on \(S^+\), more precisely \(\rho^{-1} x, v = O(\rho^2)\). Presented calculations should convince the reader that the following formula
\[
\int N \pi_{\mu\nu} \delta A_{\mu\nu} \sim \int N 2 \Lambda \rho^2 \left[ (\Psi_x), u_{ab} - 1 (a + 2)^{-2} \delta (\Psi_x) + (\Psi_y), u_{ab} - 1 (a + 2)^{-1} \delta (\Psi_y) \right]
\]
holds and this is quasi-local form which is similar to \[26\] and \[49\].

## 6 Generating formula for Einstein equations

There are different variational principles which may be used to derive Einstein equations. They may be classified as belonging to three basic approaches:

1. The purely metric approach, where the variation is performed with respect to the metric tensor. As a Lagrangian one can use the second order Hilbert Lagrangian or the first order (gauge dependent) Lagrangian, quadratic with respect to the Christoffel symbols.

2. The metric–affine approach, based on the Palatini variational principle, where the variation is performed independently with respect to the metric tensor and to the connection.

3. The purely affine approach where the variation is performed with respect to the connection. The metric tensor arises as a momentum canonically conjugate to the connection – see \[23\].

Each of these variational principles leads to the same Hamiltonian description of the theory. In this paper we use the Hilbert variational principle. At the end of this section we will show how the different variational principles converge to the same generating formula. Hence, the canonical structure, derived from this formula, does not depend upon the variational principle we begin with.

The variation of the Hilbert Lagrangian
\[
L = \frac{1}{16\pi} \sqrt{|g|} \ R
\]
may be calculated as follows:
\[
\delta L = \delta \left( \frac{1}{16\pi} \sqrt{|g|} \ g^{\mu\nu} R_{\mu\nu} \right) = - \frac{1}{16\pi} G^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{16\pi} \sqrt{|g|} \ g^{\mu\nu} \delta R_{\mu\nu}
\]
where
\[
G^{\mu\nu} := \sqrt{|g|} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right).
\]

We are going to prove that the last term in \[124\] is a boundary term (a complete divergence). For this purpose we denote:
\[
\pi^{\mu\nu} := \frac{1}{16\pi} \sqrt{|g|} \ g^{\mu\nu},
\]
and
\[
A_{\mu\nu} := \Gamma_{\mu\nu} - \delta_{(\rho} \Gamma_{\nu)\rho}.
\]
(do not try to attribute any sophisticated geometric interpretation to \( A^\lambda_{\mu\nu} \); it is merely a frequently appearing combination of the connection coefficients, which we introduce in order to simplify the derivation of the final formula). We have:

\[
\partial_\lambda A^\lambda_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_{(\mu} \Gamma^\lambda_{\nu)\lambda} = R_{\mu\nu} - \Gamma^\lambda_{\sigma\nu} \Gamma^\sigma_{\mu\lambda} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\lambda}
\]

(128)

Hence, we obtain an identity

\[
\partial_\lambda \left( \pi^{\mu\nu} \delta A^\lambda_{\mu\nu} \right) = \pi^{\mu\nu} \delta R_{\mu\nu} + \pi^{\mu\nu} \delta \left( A^\lambda_{\mu\sigma} A^\sigma_{\nu\lambda} - \frac{1}{3} A^\lambda_{\mu\lambda} A^\lambda_{\nu\sigma} \right) + \left( \partial_\lambda \pi^{\mu\nu} \right) \delta A^\lambda_{\mu\nu}
\]

(129)

Due to metricity of \( \Gamma \) we have \( \nabla_\lambda \pi^{\mu\nu} = 0 \). This way we obtain

\[
\pi^{\mu\nu} \delta R_{\mu\nu} = \partial_\lambda \left( \pi^{\mu\nu} \delta A^\lambda_{\mu\nu} \right) = \partial_\kappa \left( \pi^\mu_{\lambda^{\mu\kappa}} \delta \Gamma^\lambda_{\mu\nu} \right) ,
\]

(130)

where we denote

\[
\pi^\mu_{\lambda^{\mu\kappa}} := \pi^{\mu\nu} \delta \delta^\kappa_{\lambda} - \pi^\lambda_{\kappa^{\nu} \delta^\mu}_{\lambda} .
\]

(131)

Inserting (130) into (124) we have:

\[
\delta L = - \frac{1}{16\pi} G^{\mu\nu} \delta g_{\mu\nu} + \partial_\lambda \left( \pi^{\mu\nu} \delta A^\lambda_{\mu\nu} \right) .
\]

(132)

We conclude that Euler-Lagrange equations \( G^{\mu\nu} = 0 \) are equivalent to the following generating formula, analogous to (1) in field theory:

\[
\delta L = \partial_\lambda \left( \pi^{\mu\nu} \delta A^\lambda_{\mu\nu} \right)
\]

(133)

or, equivalently,

\[
\delta L = \partial_\kappa \left( \pi^\mu_{\lambda^{\mu\kappa}} \delta \Gamma^\lambda_{\mu\nu} \right) .
\]

(134)

This formula is a starting point for our derivation of canonical gravity. Let us observe, that it is valid not only in the present, purely metric, context but also in any variational formulation of General Relativity. For this purpose let us rewrite (132) without using \textit{a priori} the metricity condition \( \nabla_\lambda \pi^{\mu\nu} = 0 \). This way we obtain the following, universal formula:

\[
\delta L = - \frac{1}{16\pi} G^{\mu\nu} \delta g_{\mu\nu} - \left( \nabla_\kappa \pi^\mu_{\lambda^{\mu\kappa}} \right) \delta \Gamma^\lambda_{\mu\nu} + \partial_\lambda \left( \pi^\mu_{\lambda^{\mu\kappa}} \delta \Gamma^\lambda_{\mu\nu} \right)
\]

(135)

It may be proved that in this form, the formula remains valid also in the metric-affine approach and in the purely-affine one. In metric-affine formulation, the vanishing of \( \nabla_\lambda \pi^{\mu\nu} \) is not automatic: it is a part of field equations. We see that, again, the entire field dynamics is equivalent to (134). Finally, in the purely affine formulation of General Relativity the Einstein equations are satisfied “from the very beginning” whereas the metricity condition for the connection becomes the dynamical equation. We conclude that also in this case the entire information about the field dynamics is contained in generating formula (134).

This formula, compared with (1) suggests that the role of field potentials in General Relativity should be rather played by the connection \( \Gamma \), whereas the metric \( g \) should rather remain on the side of canonical momenta. This observation was the origin of the purely affine formulation of the theory. Also in the multisymplectic formulation (i. e. formulation in terms of Poincaré-Cartan form – see [24]) the connection appears on the side of field configurations. We stress, however, that the results presented in this paper do not depend upon the choice of a variational formulation.
7 Metrics of Bondi–Sachs type

In this section we shall consider the initial value problem for the curved space-time $M$ with a metric of the form:

$$g_{\mu\nu}dx^\mu dx^\nu = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}(dx^A - U^Adu)(dx^B - U^Bdu) \quad (136)$$

on the null cone $C = \{ x \in M \mid x^0 = u = \text{const} \}$ (see [10], [2], [4]) and boundary at null infinity $\partial C = S(u, 0)$. We have the following non-vanishing components of the inverse metric $g^{\mu\nu}$:

$$
\begin{align*}
g^{33} &= \frac{V}{r}e^{-2\beta} \\
g^{03} &= -e^{-2\beta} \\
g^{3A} &= -e^{-2\beta}U^A \\
g^{AB} &= \frac{1}{r^2}\gamma^{AB}
\end{align*}
$$

where $\gamma^{AB}$ is the inverse metric to $\gamma_{AB}$.

Let us define the “covector” $U_B$ as follows:

$$U_B := g_{BA}U^A = r^2\gamma_{BA}U^A$$

We have in our coordinate system the following non-vanishing components of the metric $g_{\mu\nu}$:

$$
\begin{align*}
g_{00} &= -\frac{V}{r}e^{2\beta} + U_AU^A \\
g_{03} &= -e^{2\beta} \\
g_{3A} &= -U^A \\
g_{AB} &= r^2\gamma_{AB}
\end{align*}
$$

We also assume that

$$\sqrt{\det \gamma_{AB}} = \sin \theta$$

The metric (136) implies the following expressions for $16\pi\gamma^{\mu\nu}$ and $A^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu}\Gamma^\nu_{\nu)\sigma}$ defined by (126) and (127):

$$
\begin{align*}
\sqrt{-g} &= e^{2\beta}r^2\sin \theta \\
16\pi\gamma_{03} &= -r^2\sin \theta \\
16\pi\gamma_{AB} &= e^{2\beta}\sin \theta\gamma^{AB} \\
16\pi\gamma_{33} &= rV\sin \theta \\
16\pi\gamma_{3A} &= -r^2\sin \theta U^A \\
16\pi\gamma_{03} &= -r^2\sin \theta \\
16\pi\gamma_{AB} &= e^{2\beta}\sin \theta\gamma^{AB} \\
A^0_{33} &= A^0_{0A} = 0
\end{align*}
$$
\[ A^0_{03} = -\beta_{33} - \frac{1}{r} \]
\[ A^0_{AB} = \frac{1}{2} e^{-2\beta} (r^2 \gamma_{AB})_3 \]
\[ A^3_{33} = -\frac{2}{r} \]
\[ A^3_{3A} = \frac{1}{2} e^{-2\beta} U^B_{3} g_{BA} - \frac{1}{2} (\ln \sin \theta)_A \]
\[ A^3_{03} = \frac{V}{r} \beta_{33} + \left( \frac{V}{2r} \right)_3 - U^B \beta_{3B} - \frac{1}{2} e^{-2\beta} U_A U^A_{3} \]
\[ A^3_{AB} = \frac{1}{2} e^{-2\beta} \left( g_{AB} - \frac{V}{r} g_{AB,3} + U_{A||B} + U_{B||A} \right) \]

The following expression was proposed by Bondi to call the mass:
\[ m_{TB} := \frac{1}{8\pi} \int_{\partial C} r - V \] (137)

Choose a (3+1)-foliation of space-time and integrate (133) over a 3-dimensional null-volume \( V \subset C = \{ x^0 = \text{const} \} \):
\[ \delta \int_V L = \int_V \left( \pi^{\mu
u} \delta A^0_{\mu\nu} \right) + \int_{\partial V} \pi^{\mu
u} \delta A^3_{\mu\nu} . \] (138)

Similarly as in the case of electrodynamics, we use here adapted coordinates; this means that coordinate \( x^3 \) is constant on the boundary \( \partial V \). Adapted coordinates simplify considerably derivation of the final formula. We stress, however, that all our results have an independent, geometric meaning. To rewrite them in a coordinate-independent form it is sufficient to replace “dots” by Lie derivatives \( \mathcal{L}_X \), where \( X \) is the vector field generating our one-parameter group of transformations which we are describing. In adapted coordinates \( X := \frac{\partial}{\partial x^0} \). Moreover, the upper index “3” has to be replaced everywhere by the sign \( \perp \), denoting the transversal component with respect to the world tube. This way our results have a coordinate-independent meaning as relations between well defined geometric objects and not just their specific components.

Because the translation between these two notations is so simple, we have decided to use much simpler language, based on adapted coordinates. The volume part of the formula (138) can be simplified (or reduced) as follows:
\[
16\pi \pi^{\mu\nu} \delta A^0_{\mu\nu} = 16\pi \delta A^0_{k3} + 32\pi \delta A^0_{0k} + 16\pi \delta A^0_{00} \\
= 32\pi \delta A^0_{03} + 16\pi \delta A^0_{AB} \\
= -\frac{1}{2} \sin \theta (r \gamma_{AB})_3 \delta (r \gamma^{AB}) + \\
+ \delta \left[ 2r^4 \sin \theta \left( \frac{\beta}{r^2} \right)_3 \right] . \] (139)

The last term in the above formula is a full variation of the quantity which logarithmically diverges when we try to integrate it, \( \beta = O(r^{-2}) \) and \( 2r^4 \sin \theta \left( \frac{\beta}{r^2} \right)_3 = O(r^{-1}) \). Removing of this term (– the renormalization of the symplectic form) corresponds to the renormalization of the lagrangian for scalar field (12).
On the other hand the boundary part in (138) can be rewritten as follows:

\[ 16\pi\pi^{\mu\nu} \delta A^3_{\mu\nu} = 16\pi\pi^{33} \delta A^3_{33} + 32\pi\pi^{03} \delta A^3_{03} + 32\pi\pi^{3A} \delta A^3_{3A} + 16\pi\pi^{AB} \delta A^3_{AB} = \]

\[ = 2\sin\theta \left( 2V - r^2 U_{||B} \right) \delta\beta + \sin\theta \gamma^{AB} \delta U_{A||B} - \frac{1}{2} \sin\theta \left( \dot{g}_{AB} - \frac{V}{r} \dot{g}_{AB,3} \right) \delta\gamma^{AB} + \]

\[ + r^2 \sin\theta e^{-2\beta} U_{,3} g_{BA} \delta U^A - \delta \left[ 2r^2 \sin\theta \left( \frac{V}{r^2} + \frac{V}{2r} \right) ,3 + \frac{V}{r} \beta_{,3} - \dot{\beta} - U^B_{,3} \right] \]  

(140)

where we have denoted by "\(\partial\)" a covariant derivative with respect to the two-metric \(g_{AB}\) on \(\partial V\). Inserting these results into (138) we obtain:

\[ 16\pi\delta \int_V L = -\int_V \frac{1}{2} \sin\theta \left[ (r\gamma_{AB})_3 \delta (r\gamma^{AB})_3 \right] + \int_{\partial V} r^2 \sin\theta e^{-2\beta} U_{,3} g_{BA} \delta U^A + \]

\[ + \int_{\partial V} 2\sin\theta \left( 2V - r^2 U_{||B} \right) \delta\beta + \frac{1}{2} \sin\theta \left( rV \gamma_{AB,3} - r^2 \dot{\gamma}_{AB} - 2U_{A||B} \right) \delta\gamma^{AB} + \]

\[ + \delta \int_{\partial V} 4r^2 \sin\theta \beta_{,3} - \delta \int_{\partial V} r^2 \sin\theta \left[ 2\frac{V}{r^2} + \left( \frac{V}{r} \right) ,3 + \frac{V}{r} \beta_{,3} - 2U^B_{,3} \right] \]  

(141)

because 2-dimensional divergencies "\(\partial_A f^A\)" vanish when integrated over the boundary \(\partial V\).

It is convenient to introduce the following asymptotic variables \((\Pi_{AB}, \psi^{AB})\) related to asymptotic degrees of freedom:

\[ \psi^{AB} := r\gamma^{AB} \]

\[ \Pi_{AB} := -\frac{1}{2} \sin\theta (r\gamma_{AB})_3 \]

From (139) we get the following relation:

\[ 16\pi\pi^{\mu\nu} A^0_{\mu\nu} = -\frac{1}{2} \sin\theta (r\gamma_{AB})_3 \left( r\dot{\gamma}^{AB} + 2r^4 \sin\theta \left( \frac{\dot{\beta}}{r^2} \right)_3 \right) \]  

(142)

On the other hand from (38) we know that:

\[ 16\pi \int_V \pi^{\mu\nu} L_X A^0_{\mu\nu} = \int_{\partial V} \sqrt{-g} \left( \nabla^2 X^0 - \nabla^0 X^3 \right) = \]

\[ = \int_{\partial V} r^2 \sin\theta \left[ \frac{V}{r} ,3 + 2\frac{V}{r} \beta_{,3} - 2U^B_{,3} - 2\dot{\beta} - e^{-2\beta} U_A U^A_{,3} \right] \right) = 2r^2 \sin\theta A^3_{03} \]  

(143)

where the last equality can be checked directly for the metric (136) and \(X^0 = \delta^0_0\). From (142) and (143) we obtain final formula:

\[ 16\pi\delta \int_V L = \int_V \frac{1}{2} \sin\theta \left[ (r\dot{\gamma}^{AB})_3 \delta (r\gamma_{AB})_3 - (r\dot{\gamma}_{AB})_3 \delta (r\gamma^{AB})_3 \right] - \delta \int_{\partial V} 2V \sin\theta + \]

\[ + \frac{1}{2} \int_{\partial V} \sin\theta \left( rV \gamma_{AB,3} - r^2 \dot{\gamma}_{AB} - 2U_{A||B} + r^2 e^{-2\beta} U^C_3 \gamma_{CA} U_B \right) \delta\gamma^{AB} + \]

\[ + \int_{\partial V} 2r^2 \sin\theta \left( \frac{2V}{r^2} - U^B_{||B} + U_A U^A_{,3} \right) \delta\beta - r^2 \sin\theta e^{-2\beta} U_A \delta U^A_{,3} \]  

(144)
Remark It seems to me that more natural “control mode” in the above formula corresponds to the control of the term \((r^2 U^A)_{,3}\) than \(U^A_{,3}\) and it can be achieved by the following manipulation:

\[
-r^2 \sin \theta e^{-2\beta} U_A \delta U^A_{,3} = - \sin \theta e^{-2\beta} U_A \delta (r^2 U^A)_{,3} + \delta (r \sin \theta e^{-2\beta} U_A U^A) + \\
+2r \sin \theta e^{-2\beta} U_A U^A \delta \beta + \frac{1}{r} \sin \theta e^{-2\beta} U_A U^A \delta \gamma_{AB}
\]

If we pass to the limit the formula (144) takes the following form:

\[
-16\pi \delta m_{\text{TB}} = - \delta \int_{\partial C} 4M \sin \theta = \int_C \Pi_{AB} \delta \psi^{AB} - \psi^{AB} \delta \Pi_{AB} - \frac{1}{2} \int_{\partial C} \sin \theta r^2 \gamma_{AB} \delta \gamma_{AB} \tag{145}
\]

where \(V = r - 2M + O(r^{-1})\) and the asymptotic conditions are given in [4] and will be summarized in the next section. We can denote non-conservation law for the \(TB\) mass:

\[
-16\pi \partial_0 m_{\text{TB}} = - \frac{1}{2} \int_{\partial C} \sin \theta r^2 \gamma_{AB} \gamma^{AB} \left( = \frac{1}{2} \int_{\partial C} \sin \theta \chi_{AB,u} \chi^{AB}_{,u} \right) \tag{146}
\]

where the last form in the brackets becomes clear when we learn about asymptotics presented in the next section.

Similarly for angular momentum we get the answer from the superpotential proposed by Komar [20]:

\[
16\pi \int_V \pi^{\mu\nu} L_X A^0_{\mu\nu} = \int_{\partial V} \sqrt{-g} \left( \nabla^3 X^0 - \nabla^0 X^3 \right)
\]

where now \(X = \partial/\partial \phi\).

The right-hand side can be expressed in terms of the Bondi-Sachs type metric:

\[
\int_{\partial V} \sqrt{-g} \left( \nabla^3 X^0 - \nabla^0 X^3 \right) = \int_{\partial V} r^4 \sin \theta e^{-2\beta} \gamma_{\phi A} U^A_{,3} \rightarrow 16\pi J_z
\]

the limit is taken on \(S^+\) and according to the asymptotics presented in the next section we obtain

\[
16\pi J_z = - \int_{\partial C} (6 N_{\phi} + \frac{1}{2} \chi_{AB,u} \chi^{AB}_{,u} \parallel C) \sin \theta d\theta d\phi
\]

But on the other hand

\[
16\pi \int_V \pi^{\mu\nu} L_X A^0_{\mu\nu} = \int_V \pi^{\mu\nu} A^0_{\mu\nu,\phi} = \int_V \Pi_{AB} \psi^{AB},\phi
\]

and

\[
16\pi \partial_0 \int_C \pi^{\mu\nu} A^0_{\mu\nu,\phi} = \int_C \Pi_{AB} \psi^{AB},\phi - \Pi_{AB,\phi} \psi^{AB} = \frac{1}{2} \int_{\partial C} \sin \theta r^2 \gamma_{AB} \gamma^{AB},\phi
\]

We will show in the next section that non-conservation law for angular momentum agrees in terms of the asymptotics:

\[
16\pi J_z = - \int_{\partial C} \frac{1}{2} \chi_{AB,u} \chi^{AB}_{,\phi} \sin \theta d\theta d\phi \tag{147}
\]
7.1 Symplectic structure on scri

Let us observe that we can use previous results (139) and (140) to reduce the form

\[ \pi^{\mu\nu} \delta A^v_{\mu\nu} = \pi^{\mu\nu} \delta(A^0_{\mu\nu} + 2A^3_{\mu\nu}) \]

Let us also remind coordinate system which can be used to describe the situation in a similar way as in section 2.3 for scalar field and 3.1 for electrodynamics. \((u, r) \rightarrow (v, \overline{r}),\overline{r} = -2r, v = u + 2r, \partial_u = \partial_v,\partial_r = -2\partial_{\overline{r}} + \partial_v, du \wedge dv = \frac{1}{2} d\overline{r} \wedge dv\) and finally \(\pi^{\mu\nu} \delta A^v_{\mu\nu} dvd\phi = \frac{1}{2} \pi^{\mu\nu} \delta A^v_{\mu\nu} d\overline{r}d\phi.d\phi\).

If we put

\[
16\pi \pi^{\mu\nu} \delta A^0_{\mu\nu} = -\frac{1}{2} \sin \theta (r\gamma_{AB})_3 \delta (r\gamma_{AB}) + \delta \left[ 2r^4 \sin \theta \left( \frac{\beta}{r^2} \right)_3 \right]
\]

and

\[
16\pi \pi^{\mu\nu} \delta A^3_{\mu\nu} = 2 \sin \theta \left( 2V - r^2U^B \right) \delta \beta + \sin \theta \gamma_{AB} \delta U_{A||B} - \frac{1}{2} \sin \theta \left( \gamma_{AB} - \frac{V}{r} g_{AB,3} \right) \delta \gamma_{AB} + \frac{1}{2} \sin \theta (149)
\]

assuming asymptotic behaviour on \(S^+\) we obtain the following formula on the future null infinity:

\[
16\pi \pi^{\mu\nu} \delta A^v_{\mu\nu}|_{S^+} = -\sin \theta r\gamma_{AB} \delta \gamma_{AB} + 4\delta (\sin \theta M)
\]

Let \(N = \{ u_i, u_f \} \times S^2 \subset S^+\) is a “finite piece” of \(S^+\). The relation with the TB mass is based on the following observations. First of all from (150) we obtain

\[
16\pi \int_N \frac{1}{2} \pi^{\mu\nu} A^v_{\mu\nu,\theta} d\theta d\phi = -\frac{1}{2} \int_N r^2 \sin \theta \gamma_{AB} \gamma_{AB} d\theta d\phi + 2 \int_{\partial N} M \sin \theta d\theta d\phi
\]

and secondly

\[
16\pi \int_N \pi^{\mu\nu} \mathcal{L}_X A^v_{\mu\nu} = \int_{\partial N} \sqrt{-g} (\nabla^\pi X^v - \nabla^v X^\pi) = -\int_{\partial N} 2M \sin \theta d\theta d\phi
\]

where \(X = \partial_0\), so finally

\[
-4 \int_{\partial N} M \sin \theta d\theta d\phi = -\frac{1}{2} \int_N r^2 \sin \theta \gamma_{AB} \gamma_{AB} d\theta d\phi
\]

The left-hand side of the above formula represents the change of Bondi mass from initial state \(u_i\) to final state \(u_f\) \((\partial N = \{ u_f \} \times S^2 \cup \{ u_i \} \times S^2)\) but the right-hand side is a flux of the energy through \(N\) which is a piece of \(S^+\) between initial and final state.

Similarly for angular momentum we have

\[
16\pi \int_N \pi^{\mu\nu} \mathcal{L}_X A^v_{\mu\nu} = 16\pi \int_N \frac{1}{2} \pi^{\mu\nu} A^v_{\mu\nu,\phi} d\phi d\phi = -\frac{1}{2} \int_N r^2 \sin \theta \gamma_{AB} \gamma_{AB} d\theta d\phi
\]

where now \(X = \partial_\phi\).
8 Multipole structure of Bondi–van der Burg–Metzner–Sachs equations

Let \( v = u + 2r \) than the metric (136) takes the following form:

\[
g_{\mu \nu} dx^\mu dx^\nu = \left( -\frac{V}{r} + r^2 \gamma_{AB} U^A U^B + e^{2\beta} \right) du^2 - e^{2\beta} dv^2 - 2r^2 \gamma_{AB} U^B du dx^A + r^2 \gamma_{AB} dx^A dx^B
\]

We shall rewrite formulae from van der Burg paper \[4\] in a “spherically covariant” way. More precisely we denote:

0. \( M \) are scalars
1. pairs of functions \( U, W \) and \( N, P \) can be combined in two vectors \( U^A \) and \( N^A \) respectively:

\[
\begin{align*}
U^\theta &= U^\theta = U \\
U^\phi &= \sin^2 \theta U^\phi = W \sin \theta \\
N^\theta &= N^\theta = N \\
N^\phi &= \sin^2 \theta N^\phi = P \sin \theta
\end{align*}
\]

2. pairs of functions \( c, d, C, H \) and \( D, K \) correspond to the symmetric traceless tensors \( \chi_{AB}, C_{AB} \) and \( D_{AB} \):

\[
\begin{align*}
\chi^\theta_\theta &= -\chi^\phi_\phi = 2c \\
\chi^\phi_\phi &= \sin^2 \theta \chi^\theta_\phi = 2d \sin \theta
\end{align*}
\]

Similarly \( C^\theta_\theta = C, \ D^\theta_\theta = D \) etc. The reason of this notation arises in a natural way if we change the parameterization of the 2-dimensional metric \( \gamma_{AB} \). Let us remind that van der Burg in \[4\] (p. 112) proposed the following parameterization:

\[
\gamma_{AB} dx^A dx^B = e^{2\gamma} \cosh(2\delta) d\theta^2 + 2 \sinh(2\delta) \sin \theta d\theta d\phi + e^{-2\gamma} \cosh(2\delta) \sin^2 \theta d\phi^2
\]

which differs from original Sachs formulation by linear transformation of functions \( \gamma \) and \( \delta \) (see \[3\] p. 107). Next the used functions \( \gamma \) and \( \delta \) are expanded as follows

\[
\begin{align*}
\gamma &= c/r + \left( C - \frac{1}{6} d^3 - \frac{3}{2} e^2 d^2 \right) r^{-3} + Dr^{-4} + O(r^{-5}) \\
\delta &= d/r + \left( H - \frac{1}{6} f^3 + \frac{1}{2} e^2 d \right) r^{-3} + Kr^{-4} + O(r^{-5})
\end{align*}
\]

Let us notice that there is no \( r^{-2} \) term which was analyzed in \[10\] and vanishing of this term is called “outgoing radiation condition”.

We propose to change this parameterization in such a way that for original Bondi axially symmetric metric both formulations are the same and the main advantage of our change is that the expansion terms take a nice geometric form (mainly the term of order \( r^{-3} \) takes a nice form).

Let us fix the frame \( d\theta, \sin \theta d\phi \) which is orthonormal with respect to the background metric \( \gamma_{AB} \).

The symmetric matrix (close to unity)

\[
\begin{pmatrix}
e^{2\gamma} \cosh(2\delta) & \sinh(2\delta) \\
\sinh(2\delta) & e^{-2\gamma} \cosh(2\delta)
\end{pmatrix}
\]

with determinant equal 1 can be also parameterized in a natural way by exponential mapping

\[
\exp(a\sigma_x + b\sigma_z)
\]
where $\sigma_x$ and $\sigma_z$ are Pauli matrices:

\[
\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

The solution of the matrix equation

\[
\begin{pmatrix}
e^{2\gamma} \cosh(2\delta) & \sinh(2\delta) \\
\sinh(2\delta) & e^{-2\gamma} \cosh(2\delta)
\end{pmatrix} = \exp(a\sigma_x + b\sigma_z)
\]

leads to the relation between $a, b$ and $\gamma, \delta$ in the following form:

\[
a = \sinh(2\gamma) \cosh(2\delta) \frac{\arccosh(\cosh(2\delta) \cosh(2\gamma))}{\sqrt{\sinh^2(2\delta) + \cosh^2(2\delta) \sinh^2(2\gamma)}}
\]

\[
b = \sinh(2\delta) \frac{\arccosh(\cosh(2\delta) \cosh(2\gamma))}{\sqrt{\sinh^2(2\delta) + \cosh^2(2\delta) \sinh^2(2\gamma)}}
\]

but the asymptotic relation for small $\gamma, \delta$ is simpler:

\[
a = 2\gamma + \frac{8}{3} \gamma \delta^2 + O(\gamma, \delta)^5
\]

\[
b = 2\delta + \frac{4}{3} \gamma^2 \delta + O(\gamma, \delta)^5
\]

and it gives only correction in $r^{-3}$ in our expansion. More precisely

\[
\frac{1}{2} a = c/r + (C - \frac{1}{6}(c^2 + d^2)c)r^{-3} + Dr^{-4} + O(r^{-5})
\]

\[
\frac{1}{2} b = d/r + (H - \frac{1}{6}(d^2 + c^2)d)r^{-3} + Kr^{-4} + O(r^{-5})
\]

but now we can write the expansion in a matrix form:

\[
\log \gamma_{AB} = \chi_{AB}/r + \left(2C_{AB} - \frac{1}{48} \chi_{CD} \chi_{CD} \right)_{AB} r^{-3} + 2D_{AB} r^{-4} + O(r^{-5})
\]

and each term of the expansion is a traceless symmetric tensor on a sphere. The indices are raised with respect to the inverse $\chi^{AB}$ of the background metric (which is a standard unit sphere). It is diagonal in our coordinates $\gamma_{\theta\theta} = 1$ and $\gamma_{\phi\phi} = \sin^2 \theta$. Metric connection of the $\gamma^{\phi}_{AB}$ has the following non-vanishing components

\[
\Gamma^\theta_{\phi\phi} = - \sin \theta \cos \theta \quad \Gamma^\phi_{\theta\theta} = \cot \theta
\]

We are ready to show the asymptotic expansions for the rest of the quantities which appear in Bondi–Sachs type metric \([134]\). They were introduced in \([3]\) (p.114) but now we can rewrite them in a covariant way on $S^2$:

\[
U^A = \frac{1}{2r^2} \chi_{AB} ||B + \frac{2N_A}{r^3} + \frac{1}{r^3} \left[ \frac{1}{2} \chi_{B} \chi_{BC} ||C + \frac{1}{16} \left( \chi_{CD} \chi_{CD} \right) ||A \right]
\]

\[
U_A := r^2 \gamma_{AB} U^B = - \frac{1}{2} \chi_{A} ||B + \frac{2N_A}{r^3} + \frac{1}{16r} \left( \chi_{CD} \chi_{CD} \right) ||A
\]

\[
1 - \frac{V}{r} = 2M/r + \frac{N_A ||A}{r} - \frac{1}{r^2} \left[ \frac{1}{4} \chi_{BC} ||B \chi_{A} ||C + \frac{1}{16} \chi_{CD} \chi_{CD} \right]
\]
\[ \beta = -\frac{1}{32} \cdot \frac{1}{r^2} \chi_{AB} X \]

Basic equations, (eq. 13–15 in [4]):

\[ M_{\nu} = -\frac{1}{8} \chi_{AB,\nu} \chi_{AB} + \frac{1}{4} \chi_{AB} |AB,\nu \]

\[ 3N^A_{\nu} = -M^A_{\nu} - \frac{1}{4} \epsilon^B D\epsilon^C \left( \chi_{C,DE} \right)_{\nu} - \frac{3}{4} \chi_{B} D \chi_{C,\nu} - \frac{1}{4} \chi_{C,\nu} \chi_{C||D} \]

(eq. 8–9 i 11–12 in [4]):

\[ -4\dot{C}_{AB} - \frac{1}{8} \chi_{CD} \chi_{AB,\nu} + \frac{1}{4} \chi_{AB} \chi_{CD,\nu} = N_A||_B + N_B||_A - \gamma_{AB} N^C || C^+ \]

\[ -M \chi_{AB} - \frac{1}{4} \chi_{AC} \chi_{B,\nu} \chi_{E,F || FG} \]

\[ -4\dot{D}_{AB} = (a + 4)C_{AB} - \gamma_{AC} N_B ||_C - \gamma_{AC} N_A || C^+ \gamma_{AB} (\chi_{CD} N_D) || C \]

Let us observe that mono-dipole but also quadrupole part of the right-hand side of the equation (159) vanishes. More precisely, from the relation

\[ C^{AB} || C_{AB} = C^{AB} ABC^C + 2C^{AB} || AB \]

we obtain that

\[ [(a + 4)C^{AB}] || AB = (a + 6) (C^{AB} || AB) \]

and similarly for \( \dot{C}_{AB} := \epsilon_{AB} C^D ||_B \). Let us rewrite equation (159) in the following way:

\[ -4\dot{D}_{AB} = (a + 4)C_{AB} - S^C_{AB} || B \]

where

\[ S_{ABC} := \chi_{AC} N_B + \chi_{BC} N_A - \gamma_{AB} \chi_{CD} N_D \]

It is easy to check that \( S_{ABC} \) is traceless symmetric tensor (in each pair of indices) and the same holds for \( S_{ABC} := \epsilon_{AD} S^{DB} || BC \). One can prove that \( S^{ABC} || C_{AB} \) and \( S^{ABC} || C_{AB} \) are orthogonal to the first three eigenvalue spherical harmonics (with \( l = 0, 1, 2 \)). This way we get 10-dimensional space of quadrupole Newman–Penrose charges in \( D_{AB} \) which are conserved (114, 114). More precisely, quadrupole (and also mono-dipole) part of \( \partial_\nu D^{AB} || AB \) and \( \partial_\nu D^{AB} || AB \) have to vanish. However for the polyhomogeneous asymptotics it may be not conserved (see [10]).

The linearized metric of Bondi-Sachs type

\[ H \cong \frac{1}{2} \cdot \frac{1}{r^2} \cdot \chi_{AB} X \]

\[ h_{uu} \cong 1 - \frac{V}{r} + 2\beta + \gamma_{AB} U^A U^B \cong \frac{2M}{r} + \frac{N^A}{r^2} \]

\[ h_{uv} \cong -\beta \cong 0 \]

\[ h_{uA} \cong -r^2 \gamma_{AB} U^B - r \epsilon_{AB} U^B \cong -\frac{1}{2} \epsilon_{AB} U^B \]

\[ \chi_{AB} \cong r \chi_{AB} \quad h_{AB} \cong r \chi_{AB} + \frac{1}{2} r^2 \gamma_{AB} H \cong r \chi_{AB} \]
and linearized asymptotics of invariants

\[ x \simeq \frac{4M}{r} + \frac{6N_A}{r^2} \left[ \frac{1}{2r} \left( \varphi \chi_{AB} \right)_{,u} + \frac{3}{8r^2} \varphi \chi_{AB} \chi_{CD} - \frac{1}{8r^2} a \left( \varphi \chi_{CD} \right) \right] \]

\[ y \simeq -\frac{1}{r} \varphi \chi_{AB} \chi_{CD} + \frac{6}{r^2} N_A ||B \varphi \chi_{AB} \]

\[ \Psi_x = r x \simeq 4M, \quad \Phi_x = r x, u \simeq A \chi_{AB} \]

\[ \Psi_y = r y \simeq -\frac{1}{r} \varphi \chi_{AB} \chi_{CD} , \quad \Phi_y = r y, u \simeq -\left( \varphi \chi_{AB} \chi_{CD} \right) \]

give an indication how to relate linearized theory with van der Burg asymptotics. This observation will be used in the sequel.

### 8.1 Supertranslations

Let us consider \( S^+ \) as a cartesian product \( S^2 \times R^1 \) or rather trivial affine bundle over \( S^2 \) with typical fiber \( R^1 \) then the supertranslation corresponds to the null section of this affine bundle. On the other hand the boost transformation leads to the nontrivial scaling factor in a fiber and a conformal transformation on a base manifold \( S^2 \) (see [3] p. 111).

Prolongation of the supertranslation from scri “to the center” in Bondi coordinates (metric (136)) leads to the following asymptotic relations (see also [3] p. 119):

\[ \pi^A = x^A + \frac{1}{r} \alpha^{|A|} - \frac{1}{2r^2} \left( \varphi \chi^{|A|} \alpha_0^{|B|} - 2\alpha^{|AB|} \alpha_0^{|B|} + \Gamma^A_{BC} \alpha^{|B|} \alpha^{|C|} \right) + ... \]

\[ \pi = u - \alpha - \frac{1}{2r} \alpha^{|A|} \alpha_0^{|A|} + \frac{1}{4r^2} \left[ \varphi \chi^{|A|} \alpha^{|B|} - \alpha^{|AB|} \alpha_0^{|B|} \alpha^{|A|} \right] + ... \]

\[ \tau = r - \frac{1}{2} \alpha^{|A|} \alpha_0^{|A|} + \frac{1}{2r^2} \left[ \varphi \chi^{|A|} \alpha^{|B|} + \frac{1}{2} \varphi \chi^{|A|} \alpha^{|AB|} + \frac{1}{2} \varphi \chi^{|A|} \alpha^{|AB|} \right] + \frac{1}{4} (\alpha \alpha)^{|A|} + ... \]

Now we can check the transformation law for \( \varphi \chi_{AB} \) and \( M \):

\[ \overline{\varphi} \chi_{AB} = \varphi \chi_{AB} - 2a_{|AB|} + 2a_{|AB|} \alpha_0 \]

\[ \overline{\partial_A} = \partial_A + \alpha_{,A} \partial_0 \]

and finally we obtain that certain combination

\[ 4M - \varphi \chi_{||AB}^{|A|} = 4M - \varphi \chi_{||AB} + a (a + 2) \alpha \]

has a simple transformation law with respect to the supertranslations. Moreover, mono-dipole part of \( M \) is invariant with respect to the supertranslations. It corresponds to the mass and linear momentum at null infinity. This way we have proved the following:
Theorem 1. The energy-momentum 4-vector at null infinity is invariant with respect to the supertranslations.

On the other hand angular momentum is not invariant with respect to the supertranslations but it transforms probably as follows:

\[ 16\pi J_z = 16\pi J_z \] 

Remark The supertranslation gauge freedom also exists in the linearized theory. The linearized part of the supertranslation corresponds to the gauge condition which preserves five components of the linearized metric: \( h_{ur}, H, h_{rr}, h_{rA} \). More precisely, it is a solution of the gauge conditions:

\[ \xi^u, r = 0, \quad \xi^{u||A} = (\xi^A)||r, \quad (\xi_A||B e^{AB}), r = 0 \]

\[ \xi^A||A + \frac{2}{r}\xi^r = 0, \quad \xi^r, r + \xi^u, u = 0 \]

We use here Minkowski background metric in the form:

\[ \eta_{\mu\nu} dx^\mu dx^\nu = -du^2 - 2du \, dr + r^2 \gamma_{AB} \, dx^A dx^B \]

and the solution of the gauge equations is the following:

\[ \xi^A = \frac{1}{r} \gamma^{AB} \alpha||B, \quad \xi^u = -\alpha, \quad \xi^r = -\frac{1}{2} a \alpha \]

where \( \alpha \) is any real mapping \( \alpha : S^2 \rightarrow \mathbb{R} \) and mono-dipole part of \( \alpha \) corresponds to the usual translations in Minkowski space. The gauge transformation for traceless symmetric tensor \( \chi_{AB} \)

\[ \chi_{AB} \rightarrow \chi_{AB} - 2r \alpha||AB + r \gamma_{AB} a \alpha \]

is similar to the nonlinear case.

How to remove supertranslation gauge freedom?

Assume at time \( u_0 \) that \( 4M - \chi^{AB}||AB = 0 \) then stationary solution becomes simple stationary solution.

This procedure allows to treat \( \chi^{AB} \) as invariant asymptotic degrees of freedom.

Remark The Kerr-Newman metric in Bondi-Sachs coordinates can be asymptotically represented in such a way that \( M = 0 = \chi^{AB} \).

8.2 Hierarchy of asymptotic solution on scri for scalar wave equation

Let us rewrite wave equation in null coordinates \((u, v)\)

\[ \rho^{-1}(\rho^{-1}\varphi)^a^a_a + a \varphi = 0 \] (161)

and suppose we are looking for a solution of the wave equation (161) as a series

\[ \varphi = \varphi_1 \rho + \varphi_2 \rho^2 + \varphi_3 \rho^3 + \ldots \] (162)

where each \( \varphi_n \) is a function on scri, \( \partial_v \varphi_n = 0 \).

If we put the series (162) into the wave equation (161) we obtain the following recursion:

\[ \partial_u \varphi_{n+1} = -\frac{1}{2n}[a + (n - 1)n] \varphi_n \] (163)

Compare with equations 2, 3, 4 in \([2]\).

Remark The kernel of the operator \([a + l(l + 1)]\) corresponds to the \( l \)-th spherical harmonics. The right-hand side of (163) vanishes on the \( n - 1 \) spherical harmonics subspace. This means that the corresponding multipole in \( \varphi_{n+1} \) does not depend on \( u \). In particular for \( n = 3 \) we have quadrupole charge in the fourth order. The nonlinear counterpart of this object is called Newman-Penrose charge. We discuss some features related with NP charges in section 8.5.
8.3 Linear theory, asymptotic hierarchy, “charges”

Let us first check that linearized theory can be obtained if we reject nonlinear terms in asymptotic hierarchy (157–159):

\[ 4 \dot{M} = \dot{\chi}^{AB}_{||AB,u} \]

\[ 3 \dot{N}^A_{||A} = -aM \]

\[ 4 \dot{C}^{AB}_{||AB} = (a + 2)N^A_{||A} - 4 \dot{C}^{AB}_{||AB,u} = (a + 2)N^A_{||B} \]

\[ 4 \dot{D}^{AB}_{||AB} = (a + 6)C^{AB}_{||AB} - 4 \dot{D}^{AB}_{||AB,u} = (a + 6)\dot{C}^{AB}_{||AB} \]

\[ x = 4M \rho + 6N^A_{||A} \rho^2 + 6\dot{C}^{AB}_{||AB} \rho^3 + 4D^{AB}_{||AB} \rho^4 + O(\rho^5) \]

\[ y = \dot{\chi}^{AB}_{||AB} \rho + 6N^A_{||B} \rho^2 + 6\dot{C}^{AB}_{||AB} \rho^3 + 4\dot{D}^{AB}_{||AB} \rho^4 + O(\rho^5) \]

Full agreement with (163) up to the 4-th order.

\[ M = m + 3p + \dot{M} \]

\[ 4 \dot{M} = \dot{\chi}^{AB}_{||AB,u} \]

\[ 3 \dot{N}^A_{||A} = -aM \]

\[ 4 \dot{C}^{AB}_{||AB} = (a + 2)N^A_{||A} - 4 \dot{C}^{AB}_{||AB,u} = (a + 2)N^A_{||B} \]

\[ 4 \dot{D}^{AB}_{||AB} = (a + 6)C^{AB}_{||AB} - 4 \dot{D}^{AB}_{||AB,u} = (a + 6)\dot{C}^{AB}_{||AB} \]

\[ x = 4M \rho + 6N^A_{||A} \rho^2 + 6\dot{C}^{AB}_{||AB} \rho^3 + 4D^{AB}_{||AB} \rho^4 + O(\rho^5) \]

\[ y = \dot{\chi}^{AB}_{||AB} \rho + 6N^A_{||B} \rho^2 + 6\dot{C}^{AB}_{||AB} \rho^3 + 4\dot{D}^{AB}_{||AB} \rho^4 + O(\rho^5) \]

Let us notice that mono-dipole part of invariants:

\[ x = 4m \rho + 12 j^{10} x_{1} \cdot \rho^3 + 12 p^j x_{1} \cdot \rho^3 \cdot \left( u + \frac{2}{\rho} \right) \]

\[ y = 12 s^j x_{1} \cdot \rho^3 \]

or

\[ x = 4(m + 3p) \cdot \rho + 12(k_0 + p \cdot u) \cdot \rho^2 \]

\[ y = 12 s \cdot \rho^2 \]

is the same as (104) and (105).

8.3.1 nonradiating solutions

Suppose \( \int_{S_2} \dot{M} = 0 \) then from basic equation (156) we know

\[ 0 = \int_{S_2} \dot{M} = -\frac{1}{8} \int_{S_2} \dot{\chi}^{AB}_{||AB,u} \]

and we get \( \dot{\chi}^{AB}_{||AB,u} = 0 \) and finally also \( \dot{M} = 0 \). Moreover equation (157) gives the following relation:

\[ 3 \dot{N}^A_{||B} = \frac{1}{4} \dot{\chi}^{CD}_{||CD} \]

so the dipole part \( \text{dip}(\dot{N}^A_{||B} \dot{\chi}^{AB}_{||CD}) \) vanishes and this means that the angular momentum is conserved.

This way we have proved the theorem formulated at the end of subsection 2.2, namely:

Theorem 2. If the TB mass is conserved then angular momentum is conserved too.

The general solution of this type (namely \( \dot{M} = 0 = \dot{\chi}^{AB}_{||AB,u} \)) will be called nonradiating solution and it has the following form:

\[ M = m + 3p + \dot{M} \]
From (156) we get non-conservation law for the TB mass:

\[ N^A = -p^{||A}u - k^{||A} - \xi^{AB} s^{||B} + \tilde{N}^A + \frac{u}{3} \left( \frac{1}{4} \chi^{CD} \chi ||CDB - \tilde{M}^{||A} \right) \]

\[ 4\dot{C}_{AB} = -N_{A||B} - N_{B||A} + \tilde{\gamma}_{AB} N^C ||C + (m + 3p + \tilde{M}) \tilde{\gamma}_{AB} + \frac{1}{4} \chi^{CD} \chi ||CD \]

\[ 4\dot{D}_{AB} = -(a + 4) C_{AB} + S_A C^C_{B||C} \]

\[ S_{ABC} = \tilde{\chi}_{AC} N_B + \tilde{\chi}_{BC} N_A - \tilde{\gamma}_{AB} \tilde{\chi}_{CD} N^D \]

### 8.4 How to relate linearized theory with van der Burg equations, stationary solutions

The “first order” asymptotics of the Bondi-Sachs type metric on \( S^+ \) is described by three functions \( M, \chi_{AB} \). We shall try now to relate these data with boundary value on \( S^+ \) of our invariants \( \Psi_T \) in the linearized theory in such a way that non-conservation laws for the mass and angular momentum are similar in both cases. Suppose we know \( M, \chi_{AB} \) at the moment \( u = u_0 \) and \( \tilde{\chi}_{AB,u} \) on \( S^+ \) (we need only in the neighbourhood of \( u_0 \)). We propose to perform supertranslation which is related with data at \( u_0 \) in such a way that

\[ (4M - \chi^{AB} ||AB)(u_0) = 0 \]  \( (164) \)

Let us call the condition (164) supertranslation gauge at the moment \( u_0 \). This way we have removed quasi-locally supertranslation ambiguity at the moment \( u_0 \). We stress that the relation \( 4M - \chi^{AB} ||AB = 0 \) holds only at \( u_0 \) because \( QF(\chi^{AB} ||AB,u) := 4M_u - \chi^{AB} ||AB,u \) is not vanishing in general however for the nonradiating solutions it may be fullfilled globally. This is the main difference between linearized theory where the condition \( 4M - \chi^{AB} ||AB = 0 \) can be fullfilled globally and nonlinear data where we can only demand this condition to be fullfilled at one moment \( u_0 \). Nevertheless these procedure allows us to relate nonlinear data at \( u_0 \) with linearized theory namely

\[ 4M = \tilde{\chi}^{AB} ||AB \rightarrow \Psi_x, \quad \chi^{AB} ||AB,u \rightarrow \tilde{\Psi}_x \]

\[ \tilde{\chi}^{AB} ||AB \rightarrow \Psi_y, \quad \tilde{\chi}^{AB} ||AB,u \rightarrow \tilde{\Psi}_y \]

Now it is easy to verify analogy. The calculations from the previous sections devoted to the linearized gravity should convince the reader that in linear theory we can believe in the following equations:

\[ -16\pi \partial_h m_{TB} = \int_{S(u,0)} \sin \theta d\theta d\phi [\tilde{\Psi}_x a^{-1}(a + 2)^{-1} \tilde{\Psi}_x + \tilde{\Psi}_y a^{-1}(a + 2)^{-1} \tilde{\Psi}_y] \]

\[ -16\pi \partial_h J_z = \int_{S(u,0)} \sin \theta d\theta d\phi \left[ \tilde{\Psi}_x a^{-1}(a + 2)^{-1} \tilde{\Psi}_{x,\phi} + \tilde{\Psi}_y a^{-1}(a + 2)^{-1} \tilde{\Psi}_{y,\phi} \right] \]

On the other hand the TB energy can be defined in terms of the asymptotics on \( S^+ \)

\[ 16\pi m_{TB} = \int_{S^2} 4M \sin \theta d\theta d\phi \]

From (156) we get non-conservation law for the TB mass:

\[ -16\pi \partial_h m_{TB} = - \int_{S^2} 4M \sin \theta d\theta d\phi = \frac{1}{2} \int_{S^2} \tilde{\chi}^{AB} ||AB,u \sin \theta d\theta d\phi = \]

44
\[
\begin{align*}
&= \int_{S^2} \sin \theta d\theta d\phi \left[ \sigma^{AB}_{\|AB},u a^{-1}(a + 2)^{-1} \sigma^{AB}_{\|AB},u + \langle \hat{\chi}^{AB}_{\|AB},u a^{-1}(a + 2)^{-1} \rangle_{AB},u \right] \\
\text{Similarity between (165) and (167) is obvious provided that supertranslation ambiguity is removed.}
\end{align*}
\]  

Similarly angular momentum (around z-axis) can be defined as follows:

\[
8\pi J_z = 3 \int_{S^2} \tilde{\mathcal{N}}_{A||B} \sigma^{AB} \cos \theta \sin \theta d\theta d\phi
\]

where

\[
\tilde{\mathcal{N}}_A := N_A + \frac{1}{12} \sigma_{AB} \chi^{BC} ||C
\]

We have promised at the end of the previous section to show the relation (147) for angular momentum. The following sequence of equalities holds:

\[
16\pi \hat{J}_z = - \int_{S^2} 6\tilde{\mathcal{N}}_\phi + \frac{1}{2} \partial_u (\tilde{\mathcal{N}}_{AB} \chi^{BC} ||C) = \int_{S^2} \sigma_{AB} \chi^{BC} ||C ||u + \frac{1}{2} \sigma^{AB} \chi^{BC} ||u \tilde{\mathcal{N}}_{AB} ||B - \frac{1}{2} \sigma_{AB} \chi^{BC} ||C =
\]

where in the middle we have used equation (157). The last equality (in the above sequence) is a nontrivial identity and can be generally denoted:

\[
\int_{S^2} X^A \chi_{AB} \chi^{BC} ||C + \frac{1}{2} \chi^{BC} \chi_{AB} ||C X^A - \frac{1}{2} \chi^A \chi_{AB} ||C \chi^{BC} ||C =
\]

\[
= - \frac{1}{2} \int_{S^2} \chi^{AB} (X^C \chi_{AB} ||C + X^C ||B \chi_{CA} + X^C ||A \chi_{CB})
\]

where now \( \chi_{AB} \) and \( \chi_{AB} \) are any symmetric traceless tensors on a unit sphere, \( X^A \partial_A := \partial \phi \) and

\[
\chi_{AB,\phi} = X^C \chi_{AB} ||C + X^C ||B \chi_{CA} + X^C ||A \chi_{CB}
\]

Another form of (168) can be transformed as follows:

\[
\frac{1}{2} \int_{S^2} \chi^{BC} (X^A \chi_{AB} ||C + X_C \chi_B ||A - X^A \chi_{BC} ||A) = 0
\]

is equivalent to

\[
\int_{S^2} \chi^{BC} X^A (\chi_{BC} ||A - \chi_{BA} ||C) = \int_{S^2} \chi^{BC} X_C \chi_B ||A
\]

and last equality holds for integrands

\[
\chi^{BC} X^A (\chi_{BC} ||A - \chi_{BA} ||C) = \chi^{BC} X^A \varepsilon_{AC} \chi_B ||D = X^A \chi^{BC} \varepsilon_{AC} \chi_B \chi_F ||D = X^A \chi_{AF} \chi^{FD} ||D
\]

This way we have proved (168) and finally (147) which can be rewritten as follows

\[
-16\pi J_z = \frac{1}{2} \int_{S^2} \chi_{AB,\phi} =
\]

\[
= \int_{S^2} \sin \theta d\theta d\phi \left[ \sigma^{AB}_{\|AB},u a^{-1}(a + 2)^{-1} \sigma^{AB}_{\|AB},u + \langle \hat{\chi}^{AB}_{\|AB},u a^{-1}(a + 2)^{-1} \rangle_{AB},u \right]
\]

The similarity with (164) is obvious provided that supertranslation ambiguity is removed.
Let us introduce the following objects in the full nonlinear asymptotics:

\[ 4M = 4m + 12p + Q + B_x \]
\[ \chi^{AB} ||_{AB} = B_x + Q_0 \quad \tilde{\chi}^{AB} ||_{AB} = B_y \]

where \( m \) – monopole (mass), \( p \) – dipole (linear momentum), \( Q, Q_0, B_x, B_y \) – mono-dipole free. \( Q_0 \) represents supertranslation ambiguity:

\[ \tilde{Q}_0 = Q_0 - a(a + 2)\alpha, \quad \dot{Q}_0 = 0 \]

and equation (156) is equivalent to

\[ 4\dot{m} + 12\dot{p} + \dot{Q} = -\frac{1}{2} \chi^{AB} \tilde{\chi}^{AB} = QF(\dot{B}_x, \dot{B}_y) \]

where \( QF \) is a quadratic quasi-local functional. Supertranslation gauge \( \Psi_T \) allows to relate \( \Psi_T \) at \( S^+ \) with \( \Psi_T \). The corresponding names \( m \) – mass, \( p \) – linear momentum, \( Q \) – “supertranslation charge” are obvious. This decomposition is chosen in a convenient manner for the situation of the so-called “sandwich-wave”.

Suppose \( B_x \) and \( B_y \) have compact support on \( S^+ \). Let us also suppose that below \( u_i \) and upper \( u_f \) our gravitating system is stationary. These two assumptions define “sandwich-wave”.

When nonradiating solution becomes stationary?

From \( \dot{N}^A = 0 \) we obtain \( \dot{p} = \frac{M}{\dot{\chi}^{AB} ||_{AB} = 0} \), \( m, k_0, s \) are not restricted but also \( \dot{\tilde{\chi}}^{AB} ||_{AB} \) does not vanish. From \( \dot{D}_{AB} = 0 \) we get \( C_{AB} = (a + 4)^{-1}S_{AC}B ||_{BC} \). Similarly \( \dot{C}_{AB} = 0 \) gives \( m \dot{\chi}_{AB} = \tilde{N}_A ||_{B} + \tilde{N}_B ||_{A} - \tilde{\gamma}_{AB} \tilde{N}^C ||_{C} \) or \( \tilde{N}_A ||_{B} \tilde{\gamma}_{AB} = 0 \) and \( \dot{\chi}^{AB} ||_{AB} = (a + 2)\tilde{N}^A ||_{A} \). Let us call simple stationary solution the situation when \( \dot{\chi}_{AB} = 0 \) and \( M = m = \text{const.} \) described by van der Burg in static case (sec.5 in [4]).

Remark: The equations related with Newman-Penrose charge in static situation presented in [4] at the end of page 119 can be denoted in our notation as follows:

\[ (a + 10)D_{AB} = 15(MC_{AB} - N_A\tilde{N}_B + \frac{1}{2} \tilde{\gamma}_{AB} N^C\tilde{N}_C) \]

We have defined three categories of special solutions:

simple stationary solutions \( \subset \) stationary solutions \( \subset \) nonradiating solutions

and let us observe that the supertranslation gauge leads to the conclusion that every stationary solution in supertranslation gauge (164) is simple.

\( \text{(nonradiating sol.in supertr. gauge)} \cup \text{(stationary sol.)} = \text{simple stationary sol.} \)

On the other hand in the case of the “sandwich wave” the supertranslation gauge at \( u_i \) and at \( u_f \) is not the same in general. The difference depends on

\[ \int_{u_i}^{u_f} \dot{Q}du = \int_{u_i}^{u_f} QFdu \]

so in general the initial and final states cannot be simple in the same Bondi coordinates.
8.5 Special solutions of asymptotic hierarchy, Newman–Penrose charges

Equations (156–159) represent nonlinear analogue (up to the fourth order) of the hierarchy (163) for usual wave equation.

We could define as a generalized NP charge any solution which starts in the \( n+1 \)-th order from “multipole constant”. More precisely, if \( \varphi \in \ker[a + (n - 1)n] \) then from (163) \( \varphi_{n+1,u} = 0 \). Let us observe that if this charge vanishes we can derive “finite” Janis solution [13], which is obtained by “cutting the series” and derive hierarchy “upward”. More precisely,

\[
\varphi = \varphi_1 \rho + \varphi_2 \rho^2 + \ldots + \varphi_n \rho^n
\]

\[\varphi_n \in \ker[a + n(n - 1)] \implies \varphi_{n+1} = 0, \quad \varphi_n = C(u)Y_{n-1}(\theta, \phi)\]

\[
\varphi_{k-1} = \frac{2k - 2}{n(n - 1) - (k - 1)(k - 2)} \dot{\varphi}_k \quad k \leq n
\]

and \( Y_l \) is a spherical harmonics \( (a + l(l + 1))Y_l = 0 \). In particular when \( \dot{\varphi}_1 = 0 \) then \( C(u) \) is a polynomial of degree \( n - 1 \).

On the other hand if NP charge is not vanishing the solution \( \varphi \) can not be stationary. Moreover, monopole and dipole examples show that these solutions are singular (but on \( S^- \)). The monopole example is the following

\[
\varphi = 2\varphi_2 \rho \varepsilon^{-1} = \varphi_2 \frac{2\rho^2}{2 + \rho u} \quad a\varphi_2 = 0 \quad \partial_u \varphi_2 = 0
\]

similarly the dipole one

\[
\varphi = \frac{4}{3} \varphi_3 \rho \varepsilon = -\varphi_3 \frac{4}{3} \frac{\rho^3}{(2 + \rho u)^2} (a + 2) \varphi_3 = 0 \quad \partial_u \varphi_3 = 0
\]

and generaly

\[
\varphi_{n+1} \in \ker[a + n(n - 1)] \implies \dot{\varphi}_{n+1} = 0, \quad \varphi_{n+1} = CY_{n-1}(\theta, \phi)
\]

but now \( \dot{C} = 0 \) and \( \dot{\varphi}_{n+2} = -\frac{n}{n+1} \varphi_{n+1} \neq 0 \).

For gravity we have:

1. \( D_{AB} = 0 \implies \) Janis solution
2. \( D_{AB} - \) pure quadrupole \( \implies \) NP charge solution.

In Janis paper there are only linearized solutions. We shall try now to construct asymptotic quadrupole solution of nonlinear hierarchy (156–159). Let us assume that \( M \) and \( \chi^{CD}_{||CD} \) are given quadrupoles \( (0 = (a + 6)M = (a + 6)\chi^{CD}_{||CD}) \).

\[
M_{,u} = \delta_{AB,u} = 0
\]

\[
4M = \bar{x}(\theta, \phi) \quad \delta^D_{C, ||DE} \delta^{CE} = \bar{y}(\theta, \phi)
\]

\[
D_{AB} = 0
\]

\[
S_{ABC} = \delta_{AC} \quad N^B + \delta_{BC} \quad N^A - \gamma_{AB} \chi_{CD} N^D
\]

\[
C_{AB} = (a + 4)^{-1}(S_{AC} B_{||C}) - \frac{1}{24} u^2 (n^A ||B + n^B ||A - \gamma_{AB} n^C ||C)
\]

\[
N^A = -p^{(A} u^{||A} - \varepsilon^{AB} s^{||B} + \bar{N}^A + \frac{u}{3} n^A
\]

\[
M = m + 3p + M
\]
\[ n^A := \frac{1}{4} \varepsilon^{AB} \chi^{CD} ||CDB - \hat{M}||^A \]

\[ \hat{N}_{A||B} + \hat{N}_{B||A} - \hat{\gamma}_{AB} \hat{N}^C ||C||_{CD} - 4(a + 4)^{-1}(S_A^{C, B||C, a}) \]

This a special example of general nonradiating asymptotic solution defined by the condition that the TB mass is conserved.

**List of symbols**

- \( V \) – three-dimensional volume or function in Bondi-Sachs type metric
- \( L \) – lagrangian density
- \( \Sigma \) – hyperboloid
- \( \varphi \) – scalar field
- \( \psi \) – rescaled scalar field
- \( \delta \) – “variational” derivative
- \( \partial_{\mu} \) – partial derivative
- \( T^{\mu}_{\nu} \) – symmetric energy-momentum tensor
- \( \eta_{\mu\nu} \) – flat Minkowski metric
- \( \eta \) – det \( \eta_{\mu\nu} \)
- \( T_{\mu\nu} \) – canonical energy-momentum density
- \( \delta_{\mu\nu} \) – Kronecker’s delta
- \( p^\mu \) – canonical field momenta
- \( \pi \) – canonical momenta
- \( \mathcal{H} \) – hamiltonian, energy generator
- \( H \) – density of a hamiltonian or two-dim. trace of \( h_{AB} \)
- \( x^\mu, y^\nu \) – coordinates on \( M \)
  - \( t \) – time coordinate on \( M_2 \)
  - \( r \) – radial coordinate on \( M_2 \)
  - \( \omega \) – related radial coordinate on \( M_2, r = \sinh \omega \)
  - \( \rho \) – “inverse” radial coordinate on \( M_2, r = \rho^{-1} \)
  - \( s \) – “hyperboloidal time” coordinate on \( M_2, s = t - \sqrt{1 + r^2} \)
- \( u, v \) – null coordinates on \( M_2, u = t - r, v = t + r \)
- \( \overline{\nu} \) – coordinate on \( M_2, \overline{\nu} = -2r \)
- \( \theta, \phi \) – spherical coordinates on \( S^2 \)
- \( d \) – exterior derivative
- \( \mu, \nu, \ldots \) – four-dimensional indices running 0, \ldots, 3
- \( k, l, \ldots \) – three-dimensional indices running 1, \ldots, 3
- \( A, B, \ldots \) – two-dimensional indices on a sphere
- \( a, b, \ldots \) – two-dimensional “null” indices on \( M_2 \)
- \( \Box \) – d’Alambertian, wave operator
- \( \Box \) – conformally related wave operator
- \( \overline{\eta}_{\mu\nu} \) – conformally related metric
- \( R \) – scalar curvature
$X$ – vector field
$s^0$ – spatial infinity
$S$ – null infinity
$S^+$ – future null infinity
$S^-$ – past null infinity
$N$ – null surface “parallel” to $S^+$ or a piece of $S^+
$m_{\text{ADM}}$ – ADM mass
$S^2$ – sphere parameterized by $\theta, \phi$
$S(s, \rho)$ – sphere in $M$ corresponding to coordinates $s, \rho$
$S(s, \omega)$ – sphere in $M$ corresponding to coordinates $s, \omega$
$S(s, 0)$ – sphere on $S^+$
$S(1)$ – unit sphere
$\hat{\gamma}_{AB}$ – metric on a unit sphere
$\mathbf{a}$ – two-dimensional laplacian on a unit sphere
$\hat{\varepsilon}^{AB}$ – skew-symmetric tensor on a unit sphere, $\sin \theta \varepsilon^{\theta \phi} = 1$
$\varepsilon^{AB}$ – two-dimensional skew-symmetric tensor, $r^2 \sin \theta \varepsilon^{\theta \phi} = 1$
$||$ – two-dimensional covariant derivative on a sphere
$\hat{\partial}_A$ – dual of $\partial_A$, $\hat{\partial}_A = \varepsilon^{B \hat{A}} \partial_B$
$F_{\mu\nu}$ – electromagnetic induction density
$f_{\mu\nu}$ – electromagnetic field
$A_{\mu}$ – electromagnetic potential
$\psi, \pi$ – gauge-invariant positions for electromagnetism
$\pi, \pi^*$ – gauge-invariant momenta for electromagnetism
$\hat{J}_k$ – angular momentum
$g_{kl}$ – three-dimensional riemannian metric
$P_{kl}$ – ADM momentum
$K_{kl}$ – extrinsic curvature
$R_{\nu\lambda\sigma}$ – curvature tensor
$R_{\mu\nu}$ – Ricci tensor
$\Gamma^{\lambda}_{\mu\nu}$ – Christoffel symbol
$n^\mu$ – normal unit future directed vector
$\mathcal{R}$ – three-dimensional scalar curvature
$h_{kl}$ – linearized metric
$\pi^{kl}$ – linearized momentum
$p^{kl}$ – “new” linearized momentum
$g$ – det $g_{kl}$
$\Lambda$ – volume element, $\Lambda = r^2 \sin \theta$
$\xi_\mu$ – gauge in linearized gravity
$\kappa$ – $\kappa = \coth \omega$
$x, X, y, Y$ – invariants
$\chi_{AB}$ – traceless part of $h_{AB}$
$S_{AB}$ – traceless part of $p_{AB}$
\( S \) – trace of \( p_{AB} \)
\( H \) – trace of \( h_{AB} \)
\( \triangle \) – laplacian on a hyperboloid
\( J_z \) – angular momentum generator
\( P_z \) – linear momentum generator
\( \Psi_x, \Psi_y \) – “asymptotic position” on a hyperboloid
\( \Pi_x, \Pi_y \) – “asymptotic momenta” on a hyperboloid
\( \Upsilon \) – abstract index, \( \Upsilon = x, y \)
\( E_{ab} \) – \( \frac{1}{2} E_{ab} dx^a \wedge dx^b = du \wedge dv \)
\( y_a \) – invariant in null coordinates
\( x_{ab} \) – invariant in null coordinates
\( \beta, V, U^A, \gamma_{AB} \) – parameters describing Bondi-Sachs type metric
\( C \) – null cone or van der Burg asymptotics
\( m_{TB} \) – the Trautman-Bondi mass
\( \Psi^{AB} \) – nonlinear asymptotic position on a null cone
\( \Pi_{AB} \) – nonlinear asymptotic momenta on a null cone
\( \mathcal{L}_X \) – Lie derivative with respect to vector field \( X \)
\( s^s, s^i, s \) – spin charge
\( \mathbf{m} \) – mass charge
\( p^s, p^i, p \) – linear momentum charge
\( j^0, k_0 \) – static momentum charge (center of mass)
\( M \) – Minkowski space or asymptotics of function \( V \) in van der Burg notation
\( \gamma, \delta \) – van der Burg parameterization of \( \gamma_{AB} \)
\( U, W \) – van der Burg parameterization of \( U^A \)
\( N, P \) – van der Burg parameterization of \( N^A \)
\( N^A \) – asymptotics of \( U^A \)
\( c, C, D \) – van der Burg notation for the asymptotics of \( \gamma \)
\( d, H, K \) – van der Burg notation for the asymptotics of \( \delta \)
\( \hat{\chi}_{AB}, C_{AB}, D_{AB} \) – asymptotics of \( \gamma_{AB} \)
\( \sigma_x, \sigma_z \) – Pauli matrices
\( S_{ABC} \) – traceless symmetric tensor appearing in eq. \( \text{[59]} \)
\( \hat{S}_{ABC} \) – “dual” of \( S_{ABC} \), \( \hat{S}_{ABC} = \hat{\varepsilon}_A \cdot D S_{DBC} \)
\( \hat{C}_{AB} \) – “dual” of \( C_{AB} \), \( \hat{C}_{AB} = \hat{\varepsilon}_A \cdot D C_{DB} \)
\( \hat{\chi}_{AB} \) – “dual” of \( \chi_{AB} \), \( \hat{\chi}_{AB} = \hat{\varepsilon}_A \cdot D \hat{\chi}_{DB} \)
\( \hat{D}_{AB} \) – “dual” of \( D_{AB} \), \( \hat{D}_{AB} = \hat{\varepsilon}_A \cdot C D_{CB} \)
\( \text{mon}(F) \) – monopole part of \( F \)
\( \text{dip}(F) \) – dipole part of \( F \)
\( \overline{F} \) – supertranslation of \( F \)
\( Y_l \) – spherical harmonics with eigenvalue \(-l(l+1)\) of the laplacian \( \Delta \)
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