Isoparametric functions and harmonic unit vector fields in K-Contact Geometry

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Abstract

We provide some examples of harmonic unit vector fields as normalized gradients of isoparametric functions coming from a K-contact geometry setting.

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Introduction

in [4], the authors showed that given an isoparametric function \( f \) on an Einstein manifold, the normalized gradient vector field \( \frac{\nabla f}{\| \nabla f \|} \) is a harmonic unit vector field. In this paper, without the Einstein assumption, we present explicit isoparametric functions on double K-contact structures and also show that their normalized gradient vector fields are harmonic unit vector fields. We may ask whether or not there are examples of double K-contact structures on non-Einstein manifolds. The answer to this question is negative if K-contact is replaced by Sasakian. A compact, double Sasakian manifold is of constant curvature 1 (see [5], [8]).
1 Transnormal functions

A smooth function $f$ on $M$ is said to be transnormal if there exists a real, smooth function

$$b : \mathbb{R} \to \mathbb{R}^+$$

such that

$$\|\nabla f\|^2 = b(f),$$

(1)

where $\nabla f$ is the gradient vector field of $f$.

The function $f$ is said to be isoparametric if there is another continuous function

$$a : \mathbb{R} \to \mathbb{R}$$

such that the Laplacian of $f$, $\Delta f$ satisfies:

$$\Delta f = a(f).$$

(2)

Proposition 1 Let $f$ be a transnormal function on a Riemannian manifold $(M, g)$. Then $N = \frac{\nabla f}{\|\nabla f\|}$ is a geodesic unit vector field defined on the complement of the critical set of $f$.

Proof Let $H$ be any vector field such that $g(\nabla f, H) = df(H) = 0$. Then, using transnormality,

$$g(\nabla \nabla f, H) = \nabla f g(\nabla f, H) - g(\nabla f, \nabla \nabla f H)$$

$$= -g(\nabla f, \nabla H \nabla f) - g(\nabla f, [\nabla f, H])$$

$$= -\frac{1}{2} Hg(\nabla f, \nabla f) - g(\nabla f, [\nabla f, H])$$
\[
- \frac{1}{2} b'(f) g(\nabla f, H) - g(\nabla f, [\nabla f, H])
= -g(\nabla f, [\nabla f, H])
\]
\[
g(\nabla_{\nabla f} \nabla f, H) = -g(\nabla f, [\nabla f, H]) \tag{3}
\]

But also, the Lie derivative of \( df \) satisfies:
\[
L_{\nabla f} df(H) = \nabla f df(H) - df([\nabla f, H]) = -df([\nabla f, H]) = -g(\nabla f, [\nabla f, H])
\]
and
\[
L_{\nabla f} df(H) = (di_{\nabla f} df)(H) = H\|\nabla f\|^2 = b'(f) df(H) = 0.
\]

We deduce from (3) that
\[
\nabla_{\nabla f} \nabla f = K \nabla f
\]
for some function \( K \) on \( M \). Hence
\[
\nabla N \cdot N = \frac{1}{\|\nabla f\|} \nabla_{\nabla f} \nabla f \frac{\nabla f}{\|\nabla f\|}
= \left[ \frac{1}{\|\nabla f\|} \nabla f \left( \frac{1}{\|\nabla f\|} \right) + \frac{K}{\|\nabla f\|^2} \right] \nabla f
\]

Since \( \nabla N \cdot N \) is orthogonal to \( N \), and therefore to \( \nabla f \), it follows that
\( \nabla N \cdot N = 0. \) \( \square \)

**Lemma 1** If \( f \) is a transnormal function on a Riemannian manifold \((M, g)\), with \( \|\nabla f\|^2 = b(f) \), then the mean curvature \( h \) of every regular level surface satisfies:
\[
h = \frac{\Delta f}{\|\nabla f\|} + \frac{b'(f)}{2\sqrt{b}}.
\]
Proof  Let $E_i, i = 1, 2, ..., m-1, N = \nabla f/\|\nabla f\|$ be an adapted orthonormal frame field, where $E_i \perp \nabla f$.

\[
    h = -\sum_{i=1}^{m-1} g(\nabla E_i, N, E_i) = -\sum_{i=1}^{m-1} g(\nabla E_i, N, E_i) - g(\nabla N, N) = -\frac{1}{\|\nabla f\|} \left( \sum_{i=1}^{m-1} g(\nabla E_i, \nabla f, E_i) + g(\nabla N, \nabla f, N) \right) + \frac{N(\|\nabla f\|)}{\|\nabla f\|^2} g(\nabla f, N) = \frac{1}{\|\nabla f\|} \Delta f + \frac{b'(f)}{2\sqrt{b}}.
\]

\[\square\]

2 Harmonic unit vector fields

Let $(M, g)$ be an $m$-dimensional Riemannian manifold. A unit vector field $Z$ on $M$ can be regarded as an immersion $Z: M \to T_1 M$ of $M$ into its unit tangent bundle, which is itself a Riemannian manifold with its Sasaki metric $g_S$. In this setting, the induced metric on $M$ is given by

\[
    Z^* g_S(X, Y) = g(X, Y) + g(\nabla_X Z, \nabla_Y Z).
\]

Denote by $A_Z$ and $L_Z$ the operators

\[
    A_Z X = -\nabla_X Z
\]

and

\[
    L_Z X = X + A'_Z(A_Z X).
\]
The energy $E(Z)$ is given by

$$E(Z) = \frac{1}{2} \int_M tr L_Z dV_g = \frac{m}{2} Vol(M) + \frac{1}{2} \int_M \|\nabla Z\|^2 dV_g$$

where $dV_g$ is the Riemannian volume form on $M$. A critical point for the functional $E$ is called a harmonic unit vector field.

The critical point condition for $E$ have been derived in [9]. More precisely, $Z$ is a harmonic unit vector field on $(M,g)$ if and only if the one form $\nu_Z$,

$$\nu_Z(X) = tr (u \mapsto (\nabla_u A^t_Z) X)$$

vanishes on $Z^\perp$. Equivalently, the critical point condition for harmonic unit vector fields is again

$$\sum_{u_i} g((\nabla_{u_i} A^t_Z) X, u_i) = 0, \quad \forall X \in Z^\perp$$

(4)

where the $u_i$s form an orthonormal basis.

Let $N$ be a geodesic vector field with integrable orthogonal complement $N^\perp$. The endomorphism $A_N = -\nabla N$ is then symmetric. Indeed, for any $X, Y \perp N$,

$$g(A_N X, Y) = g(-\nabla_X N, Y)$$

$$= -X g(N, Y) + g(N, \nabla_X Y)$$

$$= g(N, \nabla_Y X)$$

$$= Y g(N, X) - g(\nabla_Y N, X)$$

$$= g(X, A_N Y)$$

Since $A_N N = 0$ and $g(A_N X, N) = 0$, it follows that $A_N$ is symmetric. Let $\lambda_i, \ i = 1, 2, 3, \ldots m$ be the eigenvalues of $A_N$ on $N^\perp$. Let also $E_1, \ldots, E_m$ be
an orthonormal frame of $N^\perp$ consisting of eigenvectors. One has

$$A_N N = A_N^t N = 0, \ A_N E_i = A_N^t E_i, \ i = 1, 2, ..., m$$

and $N$ is a harmonic unit vector field if and only if, for $j = 1, ..., n$

$$0 = \nu_N(E_j) = \sum_{i=1}^{m} g((\nabla_{E_i} A_N^t)E_j, E_i) + g((\nabla_N A^t)E_j, N) \quad (5)$$

If $\tau$ is a field of symmetric endomorphisms, then so is $\nabla_E \tau$ for any $E$. We continue the above calculation

$$0 = g((\nabla_{E_i} A_N)E_i, E_j) + g((\nabla_N A_N)N, E_j)$$

$$= \sum_{i=1}^{m} g(\nabla_{E_i}(A_N E_i), E_j) - g(A_N(\nabla_{E_i} E_i), E_j)$$

$$= \sum_{i=1}^{m} g(\nabla_{E_i}(\lambda_i E_i), E_j) - \lambda_j \sum_{i=1}^{m} g(\nabla_{E_i} E_i, E_j)$$

$$= E_j(\lambda_j) + \sum_{i=1}^{m} (\lambda_i - \lambda_j) g(\nabla_{E_i} E_i, E_j)$$

$$= E_j(\lambda_j) + \sum_{i=1}^{m} (\lambda_i - \lambda_j) g(\nabla_{E_i} E_i, E_j)$$

$$0 = E_j(\lambda_j) + \sum_{i=1}^{m} (\lambda_i - \lambda_j) g(\nabla_{E_i} E_i, E_j) \quad (6)$$

On the other hand, Codazzi equations imply that

$$g(R(E_i, E_j)E_i, N) = -g(R(E_i, E_j)N, E_i)$$

$$= g(\nabla_{E_i}(A_N E_j) - \nabla_{E_j}(A_N E_i) - A_N[E_i, E_j], E_i)$$

$$= E_i g(A_N E_j, E_i) - g(A_N E_j, \nabla_{E_i} E_i) -$$

$$E_j g(A_N E_i, E_i) + g(A_N E_i, \nabla_{E_j} E_i)$$

$$- g([E_i, E_j], A_N E_i)$$
\[ \begin{align*}
E_i(\lambda_j g(E_j, E_i) - \lambda_j g(E_j, \nabla E_i) - E_j(\lambda_i) + \\
\lambda_i g(E_i, \nabla E_j E_i) - \lambda_i g([E_i, E_j], E_i)
\end{align*} \]

\[ \begin{align*}
E_i(\lambda_j g(E_j, E_i) + \lambda_j E_i g(E_j, E_i) - \\
\lambda_j g(E_j, \nabla E_i E_i) - E_j(\lambda_i) - \lambda_i g([E_i, E_j], E_i)
\end{align*} \]

\[ \begin{align*}
E_i(\lambda_j \delta_{ij} + \lambda_j g(\nabla E_i E_j, E_i) - E_j(\lambda_i) - \\
\lambda_i g(\nabla E_i E_j, E_i)
\end{align*} \]

Summing over \( i \) and using \( g(R(N, E_j)N, N) = 0 \), we get:

\[ \begin{align*}
-\rho(E_j, N) &= \sum_{i=1}^{m} (\lambda_j - \lambda_i) g(\nabla E_i E_j, E_i) + E_j(\lambda_j) - E_j(\sum_{i=1}^{m} \lambda_i)
\end{align*} \]

\[ \begin{align*}
= \sum_{i=1}^{m} (\lambda_i - \lambda_j) g(\nabla E_i E_j, E_j) + E_j(\lambda_j) - E_j(\sum_{i=1}^{m} \lambda_i)
\end{align*} \]

Now, combining with the previous identity (6), one sees that

\[ \begin{align*}
-\rho(E_j, N) &= 0 - E_j(\sum \lambda_i)
\end{align*} \]

\[ \begin{align*}
\rho(E_j, N) &= E_j(\sum \lambda_i)
\end{align*} \]

(7)

Therefore, we see that \( N \) is harmonic if and only if

\[ \begin{align*}
X(h) = \rho(X, N)
\end{align*} \]

for all \( X \perp N \), where \( h = \sum_{i=1}^{m} \lambda_i \) is the mean curvature of \( N^\perp \) and \( \rho \) is the Ricci tensor.

For a geodesic vector field \( N \), with integrable orthogonal complement, Identity (4) reduces to

\[ \begin{align*}
X(h) = \rho(X, N), \quad \forall X \perp N
\end{align*} \]

(8)
where $h$ is the mean curvature of $N^\perp$ and $\rho$ is the Ricci tensor.

3 Double K-Contact structures

A contact metric structure on an odd-dimensional $(2n+1)$ manifold $M$ is determined by the data of a 1-form $\alpha$ with Reeb field $Z$ together with a Riemannian metric $g$, called the adapted contact metric, and a partial complex structure $J$ such that the following identities hold:

i) $\alpha \wedge (d\alpha)^n$ is a volume form on $M$.

ii) $J^2A = -A + \alpha(A)Z$

iii) $d\alpha(A,B) = 2g(A,JB)$ for any tangent vectors $A$ and $B$.

If $Z$ is an infinitesimal isometry for $g$, then the structure is called K-contact. If in addition, the identity

$$(\nabla_A J)B = g(A,B)Z - \alpha(B)A$$

is satisfied for any two tangent vectors $A$ and $B$, then the structure is called Sasakian.

**Definition 1** A double K-contact structure on a manifold $M$ is a pair of K-contact forms $\alpha$ and $\beta$ with same contact metric $g$ and commuting Reeb vector fields $Z$ and $X$.

An example: On $S^3 \hookrightarrow \mathbb{R}^4$ with coordinates $x_1, y_1, x_2, y_2$, $x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$. The standard K-contact form is $\alpha = y_1 dx_1 - x_1 dy_1 + y_2 dx_2 - x_2 dy_2$ with Reeb field $Z = y_1 \partial x_1 - x_1 \partial y_1 + y_2 \partial x_2 - x_2 \partial y_2$. Another K-contact form
with same adapted metric is $\beta = -y_1 dx_1 + x_1 dy_1 + y_2 dx_2 - x_2 dy_2$ with Reeb field $X = -y_1 \partial x_1 + x_1 \partial y_1 + y_2 \partial x_2 - x_2 \partial y_2$.

$[X, Z] = 0$ and the angle function $g(X, Z) = -y_1^2 - x_1^2 + y_2^2 + x_2^2$ is isoparametric. Its gradient vector field is

$$2JX = 2((-x_1 \partial x_1 - y_1 \partial y_1 + x_2 \partial x_2 + y_2 \partial y_2) +$$

$$2(x_1^2 + y_1^2 x_2^2 - y_2^2)(x_1 \partial x_1 + y_1 \partial y_1 + x_2 \partial x_2 + y_2 \partial y_2).$$

$J$ is the standard partial complex structure on $S^3$.

$N = \frac{2JX}{\|2JX\|}$ is a harmonic unit vector field as it will follow from results in the following sections. Similar examples as the above one can be repeated on any odd dimensional unit sphere.

**Proposition 2** In the case of a double K-contact structure $(M, \alpha, Z, \beta, X)$, the angle function $f = g(X, Z)$ is always transnormal.

**Proof** Let’s denote by $J$ and $\phi$ the respective complex structures on the contact sub-bundles. Then, the gradient of $f$ is given by

$$\nabla f = 2JX = 2\phi Z.$$

Its norm square is therefore

$$\|\nabla f\|^2 = 2\|JX\|^2 = 4(1 - g(X, Z)^2) = 4(1 - f^2) = b(f)$$

with $b(t) = 4(1 - t^2)$.

**Lemma 2** The Laplacian of the transnormal function $f = g(Z, X)$ satisfies:

$$\Delta f = (4n + 4)f + 2 \sum_{i=1}^{2n-2} g(J\phi E_i, E_i),$$

9
where \( E_i \) are orthonormal and each is perpendicular to \( Z, X, \) and \( N \).

**Proof** Let \( E_i \perp Z, X, JX \) for \( i=1, \ldots, 2n-2 \), \( E_{2n-1} = Z, E_{2n} = \frac{X-\alpha(X)}{\sqrt{1-\alpha^2(X)}}, N = \frac{JX}{\sqrt{1-\alpha^2(X)}} \) be an orthonormal frame field.

\[
\Delta f = -2\sum_{i=1}^{2n} g(\nabla_{E_i} JX, E_i) + g(\nabla_N JX, N)
\]

\[
= -2\sum_{i=1}^{2n} g(R(Z, E)X, E) - g(J\phi E_i, E_i) + g(R(Z, N)X, N) - g(J\phi N, N)
\]

\[
= 2\text{Ricci}(Z, X) + 2\sum_{i=1}^{2n} g(J\phi E_i, E_i) + 2g(J\phi N, N)
\]

\[
= 2\text{Ricci}(Z, X) + 2\sum_{i=1}^{2n-2} g(J\phi E_i, E_i) + 2\alpha(X)g(N, N) + 2g(J\phi N, N)
\]

\[
= 2\text{Ricci}(Z, X) + 2\sum_{i=1}^{2n-2} g(J\phi E_i, E_i) + 2\alpha(X)g(N, N) + 2\alpha(X)g(JN, JN)
\]

\[
= (2(2n) + 4)g(X, Z) + 2\sum_{i=1}^{2n-2} g(J\phi E_i, E_i)
\]

\[
= (4n + 4)g(X, Z) + 2\sum_{i=1}^{2n-2} g(J\phi E_i, E_i)
\]

\[
\Delta f = (4n + 4)g(X, Z) + 2\sum_{i=1}^{2n-2} g(J\phi E_i, E_i). \quad (9)
\]

\[
\square
\]

4 Double K-contact structures in dimensions 3 and 5

A transnormal function doesn’t have to be isoparametric except in some particular cases. One of these cases is the angle function of double K-contact structures in lower dimensions.
**Theorem 1** Let \((M, \alpha, Z, \beta, X, g)\) be a double K-contact structure on a closed 3-dimensional or 5-dimensional manifold \(M\). Then the transnormal angle function \(f = g(X, Z)\) is isoparametric.

**Proof** In dimension 3, Identity (9) reduces to

\[
\Delta f = 8g(X, Z) = 8f.
\]

In dimension 5, Identity (9) reduces to

\[
\Delta f = 12g(X, Z) + 2 \sum_{i=1}^{2} g(J\phi E_i, E_i) = 12f \pm 4
\]

since in this case one has \(J = \pm \phi\) on the orthogonal complement of \(\{Z, X, JX\}\).

5 Harmonic Unit vector fields in K-contact geometry

On any Riemannian manifold \((M, g)\), with Ricci tensor \(\rho\), one defines the Ricci endomorphism \(Q\) by

\[
\rho(A, B) = g(QA, B).
\]

If \((M, g, \alpha, Z, J)\) is a K-contact structure on \(M\), then one has

\[
QZ = 2nZ.
\]  \hspace{1cm}  (10)

If the K-contact structure is Sasakian one has also the following identity:

\[
QJ = JQ,
\]  \hspace{1cm}  (11)
that is, the Ricci endomorphism commutes with the transverse complex structure. (See [2] for these and more identities on K-contact structures.)

**Lemma 3** On a double K-contact manifold \((M, \alpha, Z, \beta, X)\), suppose that one of the contact forms, say \(\alpha\), is Sasakian. Then one has \(\phi J = J\phi\) on the subbundle orthogonal to \(\{Z, X, JX = \phi Z\}\). Moreover, \(\phi J\) is symmetric and its only eigenvalues are \(\pm 1\).

**Proof** Let’s denote by \(\mathcal{H}\) the tangent sub-bundle orthogonal complement of \(\{Z, X, JX\}\). It is easily seen that \(\mathcal{H}\) is \(\phi\) and \(J\) invariant. The gradient \(\nabla f\) of \(f = g(X, Z)\) is given by \(\nabla f = 2JX\). The Hessian, \(\text{Hess}_f\) is given by

\[
\text{Hess}_f(A, B) = (\nabla df)(A, B) = g(\nabla_A \nabla f, B) = 2g(\nabla_A (JX), B).
\]

Ultimately, using Sasakian identities like \((\nabla_U J)V = g(U, V)Z - \alpha(V)U\), one obtains:

\[
\text{Hess}_f(A, B) = 2g(A, X)g(Z, B) - 2\alpha(X)g(A, B) - 2g(J\phi A, B).
\]  

(12)

Since \(\text{Hess}_f(., .)\) is a symmetric bilinear form, one deduce from Identity (12) that for any two sections \(A\) and \(B\) of \(\mathcal{H}\),

\[
g(J\phi A, B) = g(J\phi B, A)
\]

that is \(J\phi\) is a symmetric endomorphism of \(\mathcal{H}\). Moreover,

\[
g(J\phi A, B) = g(J\phi B, A) = g(B, \phi JA),
\]

which means that \(\phi\) and \(J\) commute on \(\mathcal{H}\).
As a consequence,

$$ (\phi J)^2 = \phi J\phi J = \phi J^2 = Id $$

on $\mathcal{H}$ and hence the only eigenvalues are $\pm 1$.

\begin{proof}

**Theorem 2** Let $(M, \alpha, Z, \beta, X)$ be a double K-contact structure on a $2n+1$ -
dimensional closed manifold $M$ with one of the contact forms, say $\alpha$, Sasakian.

Then the function $f = g(Z, X)$ is isoparametric and the vector field $N = \frac{\nabla f}{\|\nabla f\|}$
is a harmonic unit vector field.

**Proof**

Isoparametricity holds thanks to the fact that on the distribution orthogonal to $\{Z, X JX\}$, the endomorphisms $J$ and $\phi$ commute by Lemma 3. Using

an orthonormal basis of eigenvectors of $J\phi$, Identity (9) reduces to

$$ \Delta f = (4n + 4)g(X, Z) + 2 \sum_{i=1}^{2n-2} g(\pm E_i, E_i) = (4n + 4)f + 2 \sum_{i=1}^{2n-2} (\pm 1). $$

Harmonicity follows from the critical point condition for a geodesic unit vector field with integrable orthogonal complement (see Identity (8)).

$$ E(h) = \rho(E, N) $$

for any $E \perp N$. From the isoparametric condition, $h$ is a function of $f$, hence

$E(h) = 0$. But also, using identities (10) and (11),

$$ \rho(E, N) = \rho(E, \frac{JX}{\|JX\|}) = \frac{1}{\|JX\|} g(E, QJX) = \frac{1}{\|JX\|} g(E, JQX) $$

$$ = \frac{2n}{\|JX\|} g(E, JX) = 0. $$

\end{proof}
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