Infinite families of 2-designs and 3-designs from linear codes

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Abstract

The interplay between coding theory and \( t \)-designs started many years ago. While every \( t \)-design yields a linear code over every finite field, the largest \( t \) for which an infinite family of \( t \)-designs is derived directly from a linear or nonlinear code is \( t = 3 \). Sporadic 4-designs and 5-designs were derived from some linear codes of certain parameters. The major objective of this paper is to construct many infinite families of 2-designs and 3-designs from linear codes. The parameters of some known \( t \)-designs are also derived. In addition, many conjectured infinite families of 2-designs are also presented.

Keywords: Difference family, cyclic code, linear code, \( t \)-design.

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1. Introduction

Let \( \mathcal{P} \) be a set of \( v \geq 1 \) elements, and let \( \mathcal{B} \) be a set of \( k \)-subsets of \( \mathcal{P} \), where \( k \) is a positive integer with \( 1 \leq k \leq v \). Let \( t \) be a positive integer with \( t \leq k \). The pair \( \mathcal{D} = (\mathcal{P}, \mathcal{B}) \) is called a \( t \)-(\( v, k, \lambda \)) design, or simply \( t \)-design, if every \( t \)-subset of \( \mathcal{P} \) is contained in exactly \( \lambda \) elements of \( \mathcal{B} \). The elements of \( \mathcal{P} \) are called points, and those of \( \mathcal{B} \) are referred to as blocks. We usually use \( b \) to denote the number of blocks in \( \mathcal{B} \). A \( t \)-design is called simple if \( \mathcal{B} \) does not contain repeated blocks. In this paper, we consider only simple \( t \)-designs. A \( t \)-design is called symmetric if \( v = b \).

It is clear that \( t \)-designs with \( k = t \) or \( k = v \) always exist. Such \( t \)-designs are trivial. In this paper, we consider only \( t \)-designs with \( v > k > t \). A \( t \)-(\( v, k, \lambda \)) design is referred to as a Steiner system if \( t \geq 2 \) and \( \lambda = 1 \), and is denoted by \( S(t, k, v) \).

A necessary condition for the existence of a \( t \)-(\( v, k, \lambda \)) design is that

\[
\binom{k-i}{t-i} \text{ divides } \lambda \binom{v-i}{t-i}
\]

(1)

for all integer \( i \) with \( 0 \leq i \leq t \).

There has been an interplay between codes and \( t \)-designs for decades. The incidence matrix of any \( t \)-design spans a linear code over any finite field \( GF(q) \). A lot of progress in this direction has been made and documented in the literature (see, for examples, [1], [6], [18, 19]). On the
other hand, both linear and nonlinear codes may hold \( t \)-designs. Some linear and nonlinear codes were employed to construct 2-designs and 3-designs \([1, 18, 19]\). Binary and ternary Golay codes of certain parameters hold 4-designs and 5-designs \([11, \text{p. } 303}\). However, the largest \( t \) for which an infinite family of \( t \)-designs is derived directly from codes is \( t = 3 \). It looks that not much progress on the construction of \( t \)-designs from codes has been made so far, while many other constructions of \( t \)-designs are documented in the literature \((3, 5, 14, 17)\).

The main objective of this paper is to construct infinite families of 2-designs and 3-designs from linear codes. In addition, we determine the parameters of some known \( t \)-designs, and present many conjectured infinite families of 2-designs that are based on projective ternary cyclic codes.

2. The classical construction of \( t \)-designs from codes and highly nonlinear functions

Let \( C \) be a \([v, \nu, d]\) linear code over \( \text{GF}(q) \). Let \( A_i := A_i(C) \), which denotes the number of codewords with Hamming weight \( i \) in \( C \), where \( 0 \leq i \leq v \). The sequence \((A_0, A_1, \cdots, A_v)\) is called the weight distribution of \( C \), and \( \sum_{i=0}^v A_i z^i \) is referred to as the weight enumerator of \( C \). For each \( k \) with \( A_k \neq 0 \), let \( \mathcal{B}_k \) denote the set of supports of all codewords of Hamming weight \( k \) in \( C \), where the coordinates of a codeword are indexed by \((0, 1, 2, \cdots, v - 1)\). Let \( \mathcal{P} = \{0, 1, 2, \cdots, v - 1\} \). The pair \((\mathcal{P}, \mathcal{B}_k)\) may be a \( t-(v, k, \lambda) \) design for some positive integer \( \lambda \). The following theorems, developed by Assmus and Mattson, show that the pair \((\mathcal{P}, \mathcal{B}_k)\) defined by a linear code is a \( t \)-design under certain conditions.

Theorem 1. [Assmus-Mattson Theorem \([3, \text{p. } 303}\)] Let \( C \) be a binary \([v, \nu, d]\) code. Suppose \( C^\perp \) has minimum weight \( d^\perp \). Suppose that \( A_i = A_i(C) \) and \( A_i^\perp = A_i(C^\perp) \), for \( 0 \leq i \leq v \), are the weight distributions of \( C \) and \( C^\perp \), respectively. Fix a positive integer \( t \) with \( t < d \), and let \( s \) be the number of \( i \) with \( A_i^\perp \neq 0 \) for \( 0 < i \leq v-t \). Suppose that \( s \leq d-t \). Then

- the codewords of weight \( i \) in \( C \) hold a \( t \)-design provided that \( A_i \neq 0 \) and \( d \leq i \leq v \), and
- the codewords of weight \( i \) in \( C^\perp \) hold a \( t \)-design provided that \( A_i^\perp \neq 0 \) and \( d^\perp \leq i \leq v \).

The Assmus-Mattson Theorem for nonbinary codes is given as follows [Assmus-Mattson Theorem \([2, \text{p. } 303}\)]

Theorem 2. Let \( C \) be a \([v, \nu, d]\) code over \( \text{GF}(q) \). Suppose \( C^\perp \) has minimum weight \( d^\perp \). Let \( w \) be the largest integer with \( w \leq v \) satisfying

\[
\frac{w + q - 2}{q - 1} < d.
\]

(\( w = v \) when \( q = 2 \).) Define \( w^\perp \) analogously using \( d^\perp \).

Suppose that \( A_i = A_i(C) \) and \( A_i^\perp = A_i(C^\perp) \), for \( 0 \leq i \leq v \), are the weight distributions of \( C \) and \( C^\perp \), respectively. Fix a positive integer \( t \) with \( t < d \), and let \( s \) be the number of \( i \) with \( A_i^\perp \neq 0 \) for \( 0 < i \leq v-t \). Suppose that \( s \leq d-t \). Then

- the codewords of weight \( i \) in \( C \) hold a \( t \)-design provided that \( A_i \neq 0 \) and \( d \leq i \leq w \), and
- the codewords of weight \( i \) in \( C^\perp \) hold a \( t \)-design provided that \( A_i^\perp \neq 0 \) and \( d^\perp \leq i \leq \min\{v-t, w^\perp\} \).
The Assmus-Mattson Theorems documented above are very powerful tools in constructing \( t \)-designs from linear codes. We will employ them heavily in this paper. It should be noted that the conditions in Theorems 1 and 2 are sufficient, but not necessary for obtaining \( t \)-designs.

To construct \( t \)-designs via Theorems 1 and 2, we will need the following lemma in subsequent sections, which is a variant of the MacWilliam Identity [20, p. 41].

**Theorem 3.** Let \( C \) be a \([v, \kappa, d]\) code over \( GF(q) \) with weight enumerator \( A(z) = \sum_{i=0}^{v} A_i z^i \) and let \( A^\perp(z) \) be the weight enumerator of \( C^\perp \). Then

\[
A^\perp(z) = q^{-\kappa} \left( 1 + (q-1)z \right)^{r} A \left( \frac{1-z}{1+(q-1)z} \right).
\]

A function \( f \) from \( GF(q^m) \) to itself is called planar or perfect nonlinear (PN) if

\[
\max_{0 \neq a \in GF(q^m)} \max_{b \in GF(q^m)} |\{ x \in GF(q^m) : f(x+a) - f(x) = b \}| = 1,
\]

and almost perfect nonlinear (APN) if

\[
\max_{0 \neq a \in GF(q^m)} \max_{b \in GF(q^m)} |\{ x \in GF(q^m) : f(x+a) - f(x) = b \}| = 2.
\]

Later in this paper, we will employ such functions in the constructions of linear codes and thus our constructions of \( t \)-designs.

3. Infinite families of 3-designs from the binary RM codes

It was known that Reed-Muller codes give families of 3-(\( 2^m, k, \lambda \)) designs ([16, Chapter 15], [19]). However, the parameters of \( k \) and \( \lambda \) may not be specifically given in the literature. The purpose of this section is to determine the parameters of some 3-designs derived from binary Reed-Muller codes.

We use RM\( (r, m) \) to denote the binary Reed-Muller code of length \( 2^m \) and order \( r \). Note that RM\( (m-r, m) = RM(r-1, m) \), where \( 2 \leq r < m \). The definition and information about binary Reed-Muller codes can be found in [20, Section 4.5] and [16, Chapters 13 and 14].

**Lemma 4.** The weight distribution of RM\( (m-2, m) \) (except \( A_i = 0 \)) is given by

\[
A_{4k} = \frac{1}{2^{m+1}} \left[ 2 \left( \frac{2^m}{4k} \right) + (2^{m+1}-2) \left( \frac{2^{m-1}}{2k} \right) \right]
\]

for \( 0 \leq k \leq 2^{m-2} \), and by

\[
A_{4k+2} = \frac{1}{2^{m+1}} \left[ 2 \left( \frac{2^m}{4k+2} \right) - (2^{m+1}-2) \left( \frac{2^{m-1}}{2k+1} \right) \right]
\]

for \( 0 \leq k \leq 2^{m-2} - 1 \).

**Proof.** It is well known that the weight enumerator of RM\( (1, m) \) is

\[
1 + (2^{m+1}-2)z^{2^{m-1}} + z^{2^m}.
\]
By Theorem [3] the weight enumerator of \( \text{RM}(m - 2, m) \), which is the dual of \( \text{RM}(1, m) \), is given by

\[
B(z) = \frac{1}{2^{m+1}} (1 + z)^{2m} \left[ 1 + (2^{m+1} - 2) \left( \frac{1 - z}{1 + z} \right)^{2m-1} + \left( \frac{1 - z}{1 + z} \right)^{2m} \right]
\]

\[
= \frac{1}{2^{m+1}} \left[ (1 + z)^{2m} + (2^{m+1} - 2)(1 - z)^{2m-1} + (1 - z)^{2m} \right]
\]

\[
= \frac{1}{2^{m+1}} \left[ \sum_{i=0}^{2m-1} \binom{2m}{2i} z^{2i} + (2^{m+1} - 2) \sum_{i=0}^{2m-1} \binom{2m-1}{i} (-1) z^{2i} \right]
\]

\[
= \frac{1}{2^{m+1}} \sum_{k=0}^{2m-2} \left[ 2 \left( \frac{2^{m}}{4k} \right) + (2^{m+1} - 2) \binom{2m-1}{2k} \right] z^{4k} + \frac{1}{2^{m+1}} \sum_{k=0}^{2m-1} \left[ 2 \left( \frac{2^{m}}{4k+2} \right) - (2^{m+1} - 2) \binom{2m-1}{2k+1} \right] z^{4k+2}.
\]

The desired conclusion then follows. \( \square \)

The following theorem gives parameters of all the 3-designs in both \( \text{RM}(m - 2, m) \) and \( \text{RM}(1, m) \).

**Theorem 5.** Let \( m \geq 3 \). Then \( \text{RM}(m - 2, m) \) has dimension \( 2^m - m - 1 \) and minimum distance 4. For even positive integer \( \kappa \) with \( 4 \leq \kappa \leq 2^m - 4 \), the supports of the codewords with weight \( \kappa \) in \( \text{RM}(m - 2, m) \) hold a 3-(2^m, \kappa, \lambda) design, where

\[
\lambda = \begin{cases} 
\frac{1}{2^{m+1}} \binom{\kappa}{2} \left( \frac{2^{m+1} - 2}{2^t} \right) & \text{if } \kappa = 4k, \\
\frac{1}{2^{m+1}} \binom{\kappa}{2} \left( \frac{2^{m+1} - 2}{2^t} \right) \left( \frac{2^{m+1} - 2}{2^t} \right) & \text{if } \kappa = 4k + 2.
\end{cases}
\]

The supports of all codewords of weight \( 2^{m-1} \) in \( \text{RM}(1, m) \) hold a 3-(2^m, 2^{m-1}, 2^{m-2} - 1) design.

**Proof.** Note that the weight distribution of \( \text{RM}(1, m) \) is given by

\[
A_0 = 1, A_{2^m} = 1, A_{2^{m-1}} = 2^{m+1} - 2, \text{ and } A_i = 0 \text{ for all other } i.
\]

It is known that the minimum distance \( d \) of \( \text{RM}(m - 2, m) \) is equal to 4. Put \( t = 3 \). The number of \( i \) with \( A_i \neq 0 \) and \( 1 \leq i \leq 2^m - 3 \) is \( s = 1 \). Hence, \( s = d - t \). Notice that two binary vectors have the same support if and only if they are equal. The desired conclusions then follow from Theorem [1] and Lemma [4]. \( \square \)

As a corollary of Theorem [5] we have the following [16, p. 63], which is well known.

**Corollary 6.** The minimum weight codewords in \( \text{RM}(m - 2, m) \) form a 3-(2^m, 4, 1) design, i.e., a Steiner system.

The following theorem is also well known, and tells us that Reed-Muller codes give much more 3-designs [19].
Theorem 7. Let $m \geq 4$ and $2 \leq r < m$. Then $RM(m-r,m)$ has dimension $2^m - \sum_{i=0}^{r-1} \binom{m}{i}$ and minimum distance $2^r$. For every nonzero weight $\kappa$ in $RM(m-r,m)$, the codewords of weight $\kappa$ in $RM(m-r,m)$ hold a $3-(2^m, \kappa, \lambda)$ design.

Proof. Since $2 \leq r < m$, by Theorem 24 in [10, p. 400], the automorphism group of $RM(m-r,m)$ is triply transitive. The desired conclusion then follows from Theorem 8.4.7 in [11, p. 308]. □

Determining the weight distribution of $RM(m-r,m)$ may be hard for $3 \leq r \leq m-3$ in general. Therefore, it may be difficult to find out the parameters $(\kappa, \lambda)$ of all the 3-designs. The following problem is open in general.

Open Problem 1. Determine the weight distribution of $RM(m-r,m)$ for $3 \leq r \leq m-3$.

Some progress on the open problem above was made by Kasami and Tokura [12] and Kasami, Tokura and Azumi [13]. Detailed information on this problem can be found in [16, Chapter 15].

4. Designs from cyclic Hamming codes

Let $\alpha$ be a generator of $GF(q^m)^\ast$. Set $\beta = \alpha^{q-1}$. Let $g(x)$ be the minimal polynomial of $\beta$ over $GF(q)$. Let $C_{q,m}$ denote the cyclic code of length $v = (q^m-1)/(q-1)$ over $GF(q)$ with generator polynomial $g(x)$. Then $C_{q,m}$ has parameters $[(q^m-1)/(q-1), (q^m-1)/(q-1) - m, d]$, where $d \in \{2, 3\}$. When $\gcd(q-1, m) = 1$, $C_{q,m}$ has minimum weight 3 and is equivalent to the Hamming code.

Lemma 8. The weight distribution of $C_{q,m}$ is given by

$$A_k = \frac{1}{q^m} \sum_{0 \leq i \leq q^m-1} \sum_{0 \leq j \leq q^m-1} \left[ \frac{q^m-1}{q^m-1-i} \right] \binom{q^m-1}{j} (q-1)^k + (-1)^j (q^m-1) \binom{q^m-1}{j}$$

for $0 \leq k \leq (q^m-1)/(q-1)$.

Proof. $C_{q,m}$ is the simplex code, as $\gcd(q-1, (q^m-1)/(q-1)) = 1$. Its weight enumerator is

$$1 + (q^m-1)z^{q^m-1}.$$ 

By Theorem 3 the weight enumerator of $C_{q,m}$ is given by

$$A(z) = \frac{1}{q^m} \left[ 1 + (q-1)z \right] \left[ 1 + (q^m-1) \left( \frac{1}{1+(q-1)z} \right)^{q^m-1} \right]$$

$$= \frac{1}{q^m} \left[ (1 + (q-1)z)^r + (q^m-1)(1 - z)^{q^m-1} \left( 1 + (q-1)z \right)^{q^m-1} \right]$$

$$= \frac{1}{q^m} \left[ (1 + (q-1)z)^{q^m-1} + (q^m-1)(1 - z)^{q^m-1} \right].$$

The desired conclusion then follows. □
A code of minimum distance $d = 2e + 1$ is perfect, if the spheres of radius $e$ around the codewords cover the whole space. The following theorem introduces a relation between perfect codes and $t$-designs and is due to Assmus and Mattson [2].

**Theorem 9.** A linear $q$-ary code of length $v$ and minimum distance $d = 2e + 1$ is perfect if and only if the supports of the codewords of minimum weight form a simple $(e + 1) \cdot (v, 2e + 1, (q - 1)^s)$ design. In particular, the minimum weight codewords in a linear or nonlinear perfect code, which contains the zero vector, form a Steiner system $S(e + 1, 2e + 1, v)$.

It is known that the Hamming code over $GF(q)$ is perfect, and the codewords of weight 3 hold a 2-design by Theorem 5. The 2-designs documented in the following theorem may be viewed as an extension of this result.

**Theorem 10.** Let $m \geq 3$ and $q = 2$ or $m \geq 2$ and $q > 2$, and let $\gcd(q - 1, m) = 1$. Let $\mathcal{P} = \{0, 1, 2, \ldots, (q^m - q)/(q - 1)\}$, and let $\mathcal{B}$ be the set of the supports of the codewords of Hamming weight $k$ with $A_k \neq 0$ in $C_{(q,m)}$, where $3 \leq k \leq w$ and $w$ is the largest such that $w - \lceil (w + q - 2)/(q - 1) \rceil < 3$. Then $(\mathcal{P}, \mathcal{B})$ is a $2-(q^m - 1)/(q - 1), k, \lambda)$ design. In particular, the supports of codewords of weight 3 in $C_{(q,m)}$ form a $2-(q^m - 1)/(q - 1), 3, q - 1)$ design.

The supports of all codewords of weight $q^m - 1$ in $C_{(q,m)} \perp$ form a $2-((q^m - 1)/(q - 1), q^{m - 1}, \lambda)$ design, where

$$\lambda = (q - 1)q^{m - 2}.$$  

*Proof.* $C_{(q,m)} \perp$ is the simplex code, as $\gcd(q - 1, (q^m - 1)/(q - 1)) = 1$. Its weight enumerator is

$$1 + (q^m - 1)z^{q^{m - 1}}.$$  

A proof of this weight enumerator is straightforward and can be found in [8, Theorem 15].

Recall now Theorem 2 and the definition of $w$ for $C_{(q,m)}$ and $w^\perp$ for $C_{(q,m)} \perp$. Since $C_{(q,m)}$ has minimum weight 3. Given that the weight enumerator of $C_{(q,m)} \perp$ is $1 + (q^m - 1)z^{q^{m - 1}}$, we deduce that $w^\perp = q^{m - 1}$. Put $t = 2$. It then follows that $s = 1 = d - t$. The desired conclusion on the 2-design property then follows from Theorem 2 and Lemma 8.

We now prove that the supports of codewords of weight 3 in $C_{(q,m)}$ form a $2-((q^m - 1)/(q - 1), 3, q - 1)$ design. We have already proved that these supports form a $2-((q^m - 1)/(q - 1), 3, \lambda)$ design. To determine the value $\lambda$ for this design, we need to compute the total number $b$ of blocks in this design. To this end, we first compute the total number of codewords of weight 3 in $C_{(q,m)}$. It follows from Lemma 8 that

$$A_3 = \frac{(q^m - 1)(q^m - q)}{6}.$$  

Since 3 is the minimum nonzero weight in $C_{(q,m)}$, it is easy to see that two codewords of weight 3 in $C_{(q,m)}$ have the same support if and only one is a scalar multiple of another. Thus, the total number $b$ of blocks is given by

$$b := \frac{A_3}{q - 1} = \frac{(q^m - 1)(q^m - q)}{6(q - 1)}.$$  

It then follows that

$$\lambda = \frac{b \binom{q}{3}}{\binom{q - 1}{2}} = q - 1.$$
Let \( \alpha \) be a generator of \( \text{GF}(q^m)^* \), and set \( \beta = \alpha^{q - 1} \). Then \( \beta \) is a \( v \)-th primitive root of unity, where \( v = (q^m - 1)/(q - 1) \). It is known that

\[
C_{(q,m)}^\perp = \{ \mathbf{c}_u : u \in \text{GF}(q^m) \},
\]

where \( \mathbf{c}_u = (\text{Tr}(u), \text{Tr}(u\beta), \cdots, \text{Tr}(u\beta^{q-1})) \) and \( \text{Tr}(x) \) is the trace function from \( \text{GF}(q^m) \) to \( \text{GF}(q) \).

It is then easily seen that \( \mathbf{c}_u \) and \( \mathbf{c}_v \) have the same support if and only if \( u = av \) for some \( a \in \text{GF}(q)^* \). We then deduce that the total number \( b^\perp \) of blocks in the design is given by

\[
b^\perp = \frac{q^m - 1}{q - 1}.
\]

Consequently,

\[
\lambda^\perp = \frac{q^m - 1}{q - 1} \frac{q^m - 1}{2} = (q - 1)q^{m-2}.
\]

Thus, the supports of all codewords of weight \( q^m - 1 \) in \( C_{(q,m)}^\perp \) form a 2-design with parameters

\[
((q^m - 1)/(q - 1), q^{m-1}, (q-1)q^{m-2}).
\]

\[\square\]

Theorem [10] tells us that for some \( k \geq 3 \) with \( A_k \neq 0 \), the supports of the codewords with weight \( k \) in \( C_{(q,m)} \) form \( 2-((q^m - 1)/(q - 1), k, \lambda) \) design. However, it looks complicated to determine the parameter \( \lambda \) corresponding to this \( k \geq 4 \). We draw the reader’s attention to the following open problem.

**Open Problem 2.** Let \( q \geq 3 \) and \( m \geq 2 \). For \( k \geq 4 \) with \( A_k \neq 0 \), determine the value \( \lambda \) in the 2-((q^m - 1)/(q - 1), k, \lambda) design, formed by the supports of the codewords with weight \( k \) in \( C_{(q,m)} \).

Notice that two binary codewords have the same support if and only if they are equal. When \( q = 2 \), Theorem [10] becomes the following.

**Corollary 11.** Let \( m \geq 3 \). Let \( \mathcal{P} = \{0, 1, 2, \cdots, 2^m - 2\} \), and let \( \mathcal{B} \) be the set of the supports of the codewords with Hamming weight \( k \) in \( C_{(2,m)} \), where \( 3 \leq k \leq 2^m - 3 \). Then \((\mathcal{P}, \mathcal{B})\) is a 2-(2^m - 1, k, \lambda) design, where

\[
\lambda = \frac{(k - 1)kA_k}{(2^m - 1)(2^m - 2)}
\]

and \( A_k \) is given in Lemma [5].

The supports of all codewords of weight \( 2^m - 1 \) in \( C_{(2,m)}^\perp \) form a 2-(2^m - 1, 2^m - 2, 2^m - 2) design.

Corollary [11] says that each binary Hamming code \( C_{(2,m)} \) and its dual code give a total number \( 2^m - 4 \) of 2-designs with varying block sizes.

The following are examples of the 2-designs held in the binary Hamming code.

**Example 1.** Let \( m \geq 4 \). Let \( \mathcal{P} = \{0, 1, 2, \cdots, 2^m - 2\} \), and let \( \mathcal{B} \) be the set of the supports of the codewords with Hamming weight \( 3 \) in \( C_{(2,m)} \). Then \((\mathcal{P}, \mathcal{B})\) is a 2-(2^m - 1, 3, 1) design.
Proof. By Lemma 8, we have

\[ A_3 = \frac{(2^{m-1} - 1)(2^m - 1)}{3}. \]

The desired value for \( \lambda \) then follows from Corollary 11.

Example 2. Let \( m \geq 4 \). Let \( \mathcal{P} = \{0, 1, 2, \ldots, 2^m - 2\} \), and let \( \mathcal{B} \) be the set of the supports of the codewords with Hamming weight 4 in \( C_{2,m} \). Then \( (\mathcal{P}, \mathcal{B}) \) is a \( 2-(2^m - 1, 4, 2^{m-1} - 2) \) design.

Proof. By Lemma 8, we have

\[ A_4 = \frac{(2^{m-1} - 1)(2^{m-1} - 2)(2^m - 1)}{6}. \]

The desired value for \( \lambda \) then follows from Corollary 11.

Example 3. Let \( m \geq 4 \). Let \( \mathcal{P} = \{0, 1, 2, \ldots, 2^m - 2\} \), and let \( \mathcal{B} \) be the set of the supports of the codewords with Hamming weight 5 in \( C_{2,m} \). Then \( (\mathcal{P}, \mathcal{B}) \) is a \( 2-(2^m - 1, 5, \lambda) \) design, where

\[ \lambda = \frac{2(2^{m-1} - 2)(2^{m-1} - 4)}{3}. \]

Proof. By Lemma 8, we have

\[ A_5 = \frac{(2^{m-1} - 1)(2^{m-1} - 2)(2^{m-1} - 4)(2^m - 1)}{15}. \]

The desired value for \( \lambda \) then follows from Corollary 11.

Example 4. Let \( m \geq 4 \). Let \( \mathcal{P} = \{0, 1, 2, \ldots, 2^m - 2\} \), and let \( \mathcal{B} \) be the set of the supports of the codewords with Hamming weight 6 in \( C_{2,m} \). Then \( (\mathcal{P}, \mathcal{B}) \) is a \( 2-(2^m - 1, 6, \lambda) \) design, where

\[ \lambda = \frac{(2^{m-1} - 2)(2^{m-1} - 3)(2^{m-1} - 4)}{3}. \]

Proof. By Lemma 8, we have

\[ A_6 = \frac{(2^{m-1} - 1)(2^{m-1} - 2)(2^{m-1} - 3)(2^{m-1} - 4)(2^m - 1)}{45}. \]

The desired value for \( \lambda \) then follows from Corollary 11.

Example 5. Let \( m \geq 4 \). Let \( \mathcal{P} = \{0, 1, 2, \ldots, 2^m - 2\} \), and let \( \mathcal{B} \) be the set of the supports of the codewords with Hamming weight 7 in \( C_{2,m} \). Then \( (\mathcal{P}, \mathcal{B}) \) is a \( 2-(2^m - 1, 7, \lambda) \) design, where

\[ \lambda = \frac{(2^{m-1} - 2)(2^{m-1} - 3)(4 \times 2^{2(m-1)} - 30 \times 2^{m-1} + 71)}{30}. \]

Proof. By Lemma 8, we have

\[ A_7 = \frac{(2^{m-1} - 1)(2^{m-1} - 2)(2^{m-1} - 3)(2^{m} - 1)(4 \times 2^{2(m-1)} - 30 \times 2^{m-1} + 71)}{630}. \]

The desired value for \( \lambda \) then follows from Corollary 11.
5. Designs from a class of binary codes with two zeros and their duals

In this section, we construct many infinite families of 2-designs and 3-designs with several classes of binary cyclic codes whose duals have two zeros. These binary codes are defined by almost perfect nonlinear (APN) functions over GF(2^m).

Table 1: Weight distribution for odd m.

| Weight w | No. of codewords A_w |
|----------|----------------------|
| 0        | (2^m - 1)(2^m - 1)   |
| 2^m - 1 | (2^m - 1)(2^m + 1)   |
| 2^m - 1 + 2^m | (2^m - 1)(2^m - 1) |

Lemma 12. Let m ≥ 5 be odd. Let C_m be a binary linear code of length 2^m - 1 such that its dual code C_m^⊥ has the weight distribution of Table 1. Then the weight distribution of C_m is given by

\[2^{2m}A_k = \sum_{0 \leq i \leq 2^{m-1} - 2^{(m-1)/2} \atop 0 \leq j \leq 2^{m-1} - 2^{(m-1)/2} - 1} (-1)^ia \left(\begin{array}{c}2^{m-1} - 2^{(m-1)/2} \\ i\end{array}\right) \left(\begin{array}{c}2^{m-1} + 2^{(m-1)/2} - 1 \\ j\end{array}\right) + \sum_{0 \leq i \leq 2^{m-1} \atop 0 \leq j \leq 2^{m-1} - 1} (-1)^ib \left(\begin{array}{c}2^{m-1} \\ i\end{array}\right) \left(\begin{array}{c}2^{m-1} - 1 \\ j\end{array}\right) + \sum_{0 \leq i \leq 2^{m-1} + 2^{(m-1)/2} \atop 0 \leq j \leq 2^{m-1} - 2^{(m-1)/2} - 1} (-1)^ic \left(\begin{array}{c}2^{m-1} + 2^{(m-1)/2} \\ i\end{array}\right) \left(\begin{array}{c}2^{m-1} - 2^{(m-1)/2} - 1 \\ j\end{array}\right) \]

for 0 ≤ k ≤ 2^m - 1, where

\[a = (2^m - 1)(2^{(m-1)/2} + 1)2^{(m-3)/2}, \]
\[b = (2^m - 1)2^{(m-1)/2} + 1, \]
\[c = (2^m - 1)(2^{(m-1)/2} - 1)2^{(m-3)/2}. \]

In addition, C_m has parameters [2^m - 1, 2^m - 1 - 2m, 5].

Proof. By assumption, the weight enumerator of C_m^⊥ is given by

\[A^⊥(z) = 1 + az^{2^{m-1} - 2^{(m-1)/2}} + bz^{2^m - 1} + cz^{2^{m-1} + 2^{(m-1)/2}}. \]
It then follows from Theorem 3 that the weight enumerator of $C_m$ is given by

$$A(z) = \frac{1}{2^m} (1 + z)^{2m-1} \left[ 1 + a \left( \frac{1 - z}{1 + z} \right)^{2^m - 2^m(1/2)} \right] + \frac{1}{2^m} (1 + z)^{2m-1} \left[ b \left( \frac{1 - z}{1 + z} \right)^{2^m - 1} + c \left( \frac{1 - z}{1 + z} \right)^{2^m - 1 + 2^m(1/2)} \right]$$

$$= \frac{1}{2^m} \left[ (1 + z)^{2m-1} + a (1 - z)^{2m-1} - 2^m(1/2) (1 + z)^{2m-1} + 2^m(1/2) - 1 
+b(1 - z)^{2m-1} (1 + z)^{2m-1} + c (1 - z)^{2m-1} + 2^m(1/2) (1 + z)^{2m-1} - 2^m(1/2) - 1 \right].$$

Obviously, we have

$$(1 + z)^{2m-1} = \sum_{k=0}^{2m-1} \left( \begin{array}{c} 2m - 1 \\ k \end{array} \right) z^k.$$

It is easily seen that

$$(1 - z)^{2m-1} - 2^m(1/2) (1 + z)^{2m-1} + 2^m(1/2) - 1$$

and

$$(1 - z)^{2m-1} + 2^m(1/2) (1 + z)^{2m-1} - 2^m(1/2) - 1$$

Similarly, we have

$$(1 - z)^{2m-1} - (1 + z)^{2m-1} = \sum_{k=0}^{2m-1} \left[ \sum_{0 \leq i \leq 2^m-1} (-1)^i \left( \begin{array}{c} 2^m - 1 \\ i \end{array} \right) \left( \begin{array}{c} 2^m - 1 - 2^m(1/2) - 1 \right) \right] z^k.$$
The weight distribution in Table 1 tells us that the dimension of \( C_m^\perp \) is \( 2m \). Therefore, the dimension of \( C_m \) is equal to \( 2^{m-1} - 2m \). Finally, we prove that the minimum distance \( d \) of \( C_m \) equals 5.

After tedious computations with the formula of \( A_k \) given in Lemma 12 one can verify that \( A_1 = A_2 = A_3 = A_4 = 0 \) and
\[
A_5 = \frac{4 \times 2^{3m-5} - 22 \times 2^{2m-4} + 26 \times 2^{m-3} - 2}{15}.
\]

When \( m \geq 5 \), we have
\[
4 \times 2^{3m-5} = 4 \times 2^{m-1} 2^{2m-4} \geq 64 \times 2^{2m-4} > 22 \times 2^{2m-4}
\]
and
\[
26 \times 2^{m-3} - 2 > 0.
\]
Consequently, \( A_5 > 0 \) for all odd \( m \). This proves that \( d = 5 \).

\[ \square \]

**Theorem 13.** Let \( m \geq 5 \) be odd. Let \( C_m \) be a binary linear code of length \( 2^m - 1 \) such that its dual code \( C_m^\perp \) has the weight distribution of Table 1. Let \( \mathcal{P} = \{0, 1, 2, \cdots, 2^m - 2\} \), and let \( \mathcal{B} \) be the set of the supports of the codewords of \( C_m \) with weight \( k \), where \( A_k \neq 0 \). Then \( (\mathcal{P}, \mathcal{B}) \) is a \( 2-(2^m-1, k, \lambda) \) design, where
\[
\lambda = \frac{k(k-1)A_k}{(2^m-1)(2^m-2)},
\]
where \( A_k \) is given in Lemma 12.

Let \( \mathcal{P} = \{0, 1, 2, \cdots, 2^m - 2\} \), and let \( \mathcal{B}^\perp \) be the set of the supports of the codewords of \( C_m^\perp \) with weight \( k \) and \( A_k^\perp \neq 0 \). Then \( (\mathcal{P}, \mathcal{B}^\perp) \) is a \( 2-(2^m-1, k, \lambda) \) design, where
\[
\lambda = \frac{k(k-1)A_k^\perp}{(2^m-1)(2^m-2)},
\]
where \( A_k^\perp \) is given in Lemma 12.

**Proof.** The weight distribution of \( C_m \) is given in Lemma 12 and that of \( C_m^\perp \) is given in Table 1. By Lemma 12 the minimum distance \( d \) of \( C_m \) is equal to 5. Put \( t = 2 \). The number of \( i \) with \( A_i \neq 0 \) and \( 1 \leq i \leq 2^m - 1 - s \) is \( s = 3 \). Hence, \( s = d - t \). The desired conclusions then follow from Theorem 1 and the fact that two binary vectors have the same support if and only if they are equal. \[ \square \]

**Example 6.** Let \( m \geq 5 \) be odd. Then \( C_m^\perp \) gives three \( 2 \)-designs with the following parameters:

- \((n, k, \lambda) = \left(2^m - 1, 2^{m-1} - 2(m-1)/2, 2^{m-3}(2^{m-1} - 2(m-1)/2 - 1)\right)\).
- \((n, k, \lambda) = \left(2^m - 1, 2^{m-1} + 2(m-1)/2, 2^{m-3}(2^{m-1} + 2(m-1)/2 - 1)\right)\).
- \((n, k, \lambda) = \left(2^m - 1, 2^{m-1}, (2^{m-1} - 1)(2^{m-1} + 1)\right)\).
Example 7. Let \( m \geq 5 \) be odd. Then the supports of all codewords of weight 5 in \( C_m \) give a \( 2-(2^m - 1, 5, (2^{m-1} - 4)/3) \) design.

Proof. By Lemma\[12\]
\[
A_5 = \frac{(2^{m-1} - 1)(2^{m-1} - 4)(2^m - 1)}{30}
\]
The desired value for \( \lambda \) then follows from Theorem\[13\].

Example 8. Let \( m \geq 5 \) be odd. Then the supports of all codewords of weight 6 in \( C_m \) give a \( 2-(2^m - 1, 6, \lambda) \) design, where

\[
\lambda = \frac{(2^{m-2} - 2)(2^{m-1} - 3)}{3}
\]

Proof. By Lemma\[12\]
\[
A_6 = \frac{(2^{m-1} - 1)(2^{m-1} - 4)(2^{m-1} - 3)(2^m - 1)}{90}
\]
The desired value for \( \lambda \) then follows from Theorem\[13\].

Example 9. Let \( m \geq 5 \) be odd. Then the supports of all codewords of weight 7 in \( C_m \) give a \( 2-(2^m - 1, 7, \lambda) \) design, where

\[
\lambda = \frac{2 \times 2^3(m-1) - 25 \times 2^2(m-1) + 123 \times 2^{m-1} - 190}{30}
\]

Proof. By Lemma\[12\]
\[
A_7 = \frac{(2^{m-1} - 1)(2^m - 1)(2 \times 2^3(m-1) - 25 \times 2^2(m-1) + 123 \times 2^{m-1} - 190)}{630}
\]
The desired value for \( \lambda \) then follows from Theorem\[13\].

Example 10. Let \( m \geq 5 \) be odd. Then the supports of all codewords of weight 8 in \( C_m \) give a \( 2-(2^m - 1, 8, \lambda) \) design, where

\[
\lambda = \frac{(2^{m-2} - 2)(2 \times 2^3(m-1) - 25 \times 2^2(m-1) + 123 \times 2^{m-1} - 190)}{45}
\]

Proof. By Lemma\[12\]
\[
A_8 = \frac{(2^{m-1} - 1)(2^{m-1} - 4)(2^m - 1)(2 \times 2^3(m-1) - 25 \times 2^2(m-1) + 123 \times 2^{m-1} - 190)}{8 \times 315}
\]
The desired value for \( \lambda \) then follows from Theorem\[13\].
Lemma 14. Let \( m \geq 5 \) be odd. Let \( C_m \) be a linear code of length \( 2^m - 1 \) such that its dual code \( C_m^\perp \) has the weight distribution of Table 7. Denote by \( \overline{C}_m \) the extended code of \( C_m \) and let \( \overline{C}_m^\perp \) denote the dual of \( \overline{C}_m \). Then the weight distribution of \( \overline{C}_m^\perp \) is given by

\[
2^{2m+1} A_k = (1 + (-1)^k) \binom{2^m}{k} + \frac{1 + (-1)^k}{2} (-1)^{\lfloor k/2 \rfloor} \binom{2^{m-1} - 2^{(m-1)/2}}{i} \binom{2^{m-1} + 2^{(m-1)/2}}{j} \nu +
\]

\[
u \sum_{0 \leq i < 2^{m-1 - 2^{(m-1)/2}}} \sum_{0 \leq j < 2^{m-1 + 2^{(m-1)/2}}} (-1)^{i+j} \binom{2^{m-1} - 2^{(m-1)/2}}{i} \binom{2^{m-1} + 2^{(m-1)/2}}{j}
\]

for \( 0 \leq k \leq 2^m \), where

\[
u = 2^{2m-1} - 2^{m-1} \quad \text{and} \quad \nu = 2^{2m} + 2^{m} - 2.
\]

In addition, \( \overline{C}_m \) has parameters \([2^m, 2^m - 1 - 2m, 6]\).

The code \( \overline{C}_m \) has weight enumerator

\[
A(z) = 1 + u z^{2^m - 2^{(m-1)/2}} + v z^{2^{m-1} + 2^{(m-1)/2}} + z^{2^m}, \quad (3)
\]

and parameters \([2^m, 2m + 1, 2^{m-1} - 2^{(m-1)/2}]\).

Proof. It was proved in Lemma 12 that \( C_m \) has parameters \([2^m - 1, 2^m - 1 - 2m, 5]\). By definition, the extended code \( \overline{C}_m \) has parameters \([2^m, 2^m - 1 - 2m, 6]\). By Table 7 all weights of \( C_m^\perp \) are even.

Note that \( C_m \) has length \( 2^m - 1 \) and dimension \( 2m \), while \( \overline{C}_m \) has length \( 2^m \) and dimension \( 2m + 1 \). By definition, \( \overline{C}_m \) has only even weights. Therefore, the all-one vector must be a codeword in \( \overline{C}_m^\perp \). It can be shown that the weights in \( \overline{C}_m^\perp \) are the following:

\[
0, w_1, w_2, w_3, 2^{m-1} - 2^{(m-1)/2}, 2^{m-1}, 2^{m-1} + 2^{(m-1)/2}, 2^m,
\]

where \( w_1, w_2 \) and \( w_3 \) are the three nonzero weights in \( C_m^\perp \). Consequently, \( \overline{C}_m^\perp \) has the following four weights

\[
2^{m-1} - 2^{(m-1)/2}, 2^{m-1}, 2^{m-1} + 2^{(m-1)/2}, 2^m.
\]

Recall that \( \overline{C}_m \) has minimum distance 6. Employing the first few Pless Moments, one can prove that the weight enumerator of \( \overline{C}_m \) is the one given in (3).

By Theorem 3 the weight enumerator of \( \overline{C}_m \) is given by

\[
2^{2m+1} A(z) = (1 + z) 2^m [1 + u \left( \frac{1 - z}{1 + z} \right)^{2^{m-1} - 2^{(m-1)/2}} + v \left( \frac{1 - z}{1 + z} \right)^{2^{m-1}}] +
\]

\[
(1 + z)^2 2^m \left[ u \left( \frac{1 - z}{1 + z} \right)^{2^{m-1} + 2^{(m-1)/2}} + \left( \frac{1 - z}{1 + z} \right)^{2^m} \right]
\]

\[
= (1 + z)^2 2^m + (1 - z)^2 2^m + v(1 - z) 2^{2m-1} +
\]

\[
u(1 - z)^2 2^{m-1} - 2^{(m-1)/2} (1 + z)^2 2^{m-1} + 2^{(m-1)/2} +
\]

\[
u(1 - z)^2 2^{m-1} + 2^{(m-1)/2} (1 + z)^2 2^{m-1} - 2^{(m-1)/2}. \quad (4)
\]
We now treat the terms in (4) one by one. We first have
\[
(1 + z)^{2^m} + (1 - z)^{2^m} = \sum_{k=0}^{2^m} (1 + (-1)^k) \binom{2^m}{k}.
\] (5)

One can easily see that
\[
(1 - z)^{2^m-1} = \sum_{i=0}^{2^m-1} (-1)^i \binom{2^m-1}{i} z^i = \sum_{k=0}^{2^m} \frac{1 + (-1)^k}{2} (-1)^k \binom{2^m-1}{k/2} z^k.
\] (6)

Notice that
\[
(1 - z)^{2^m-2(m-1)/2} = \sum_{i=0}^{2^m-2(m-1)/2} \binom{2^m-2(m-1)/2}{i} (-1)^i z^i
\]
and
\[
(1 + z)^{2^m-2(m-1)/2} = \sum_{i=0}^{2^m-2(m-1)/2} \binom{2^m-2(m-1)/2}{i} z^i.
\]
We have then
\[
(1 - z)^{2^m-2(m-1)/2} (1 + z)^{2^m-2(m-1)/2} = \sum_{k=0}^{2^m} \sum_{0 \leq i \leq 2^m-2(m-1)/2 \atop 0 \leq j \leq 2^m-2(m-1)/2 \atop i + j = k} (-1)^i \binom{2^m-2(m-1)/2}{i} \binom{2^m-2(m-1)/2}{j} z^k.
\] (7)

Similarly, we have
\[
(1 - z)^{2^m-2(m-1)/2} (1 + z)^{2^m-2(m-1)/2} = \sum_{k=0}^{2^m} \sum_{0 \leq i \leq 2^m-2(m-1)/2 \atop 0 \leq j \leq 2^m-2(m-1)/2 \atop i + j = k} (-1)^i \binom{2^m-2(m-1)/2}{i} \binom{2^m-2(m-1)/2}{j} z^k.
\] (8)

Plugging (5), (6), (7), and (8) into (4) proves the desired conclusion. □

**Theorem 15.** Let \( m \geq 5 \) be odd. Let \( C_m \) be a linear code of length \( 2^m - 1 \) such that its dual code \( C_m^⊥ \) has the weight distribution of Table 7. Denote by \( \overline{C}_m \) the extended code of \( C_m \) and let \( \overline{C}_m^⊥ \) denote the dual of \( \overline{C}_m \). Let \( \mathcal{P} = \{0, 1, 2, \ldots, 2^m - 1\} \), and let \( \overline{\mathcal{B}} \) be the set of the supports of the codewords of \( \overline{C}_m \) with weight \( k \), where \( \overline{\mathcal{B}}_k \neq 0 \). Then \( (\mathcal{P}, \overline{\mathcal{B}}) \) is a 3-(2^m, k, \lambda) design, where
\[
\lambda = \frac{\overline{\mathcal{B}}(k)}{\binom{2^m}{k}/14}.
\]
where $\overline{A}_k$ is given in Lemma 14.

Let $P = \{0, 1, 2, \cdots, 2^m - 1\}$, and let $\overline{B}$ be the set of the supports of the codewords of $\overline{C}_m$ with weight $k$ and $\overline{A}_k \neq 0$. Then $(P, \overline{B})$ is a $3$-$\left(2^m, k, \lambda\right)$ design, where

$$\lambda = \frac{A_k \left(\frac{k}{3}\right)}{\binom{2^m}{3}},$$

where $\overline{A}_k$ is given in Lemma 14.

**Proof.** The weight distributions of $C_m$ and $\overline{C}_m$ are described in Lemma 14. Notice that the minimum distance $d$ of $C_m$ is equal to 6. Put $t = 3$. The number of $i$ with $\overline{A}_i \neq 0$ and $1 \leq i \leq 2^m - t$ is $s = 3$. Hence, $s = d - t$. The desired conclusions then follow from Theorem 1 and the fact that two binary vectors have the same support if and only if they are identical.

**Example 11.** Let $m \geq 5$ be odd. Then $\overline{C}_m$ gives three $3$-designs with the following parameters:

- $(v, k, \lambda) = \left(2^m, 2^m - 1 - 2^m(1/2)(m-3/2)(m-1/2), \lambda\right)$.
- $(v, k, \lambda) = \left(2^m, 2^m - 1 + 2^m(1/2)(m-3/2)(m-1/2), \lambda\right)$.
- $(v, k, \lambda) = \left(2^m, 2^m - 1 + 1(2^m - 1), \lambda\right)$.

**Example 12.** Let $m \geq 5$ be odd. Then the supports of all codewords of weight 6 in $\overline{C}_m$ give a $3$-$\left(2^m, 6, \lambda\right)$ design, where

$$\lambda = \frac{2^m - 1}{3}.$$

**Proof.** By Lemma 14

$$\overline{A}_6 = \frac{2^m - 1(2^m - 1)(2^m - 4)(2^m - 1)}{90}$$

The desired value for $\lambda$ then follows from Theorem 15.

**Example 13.** Let $m \geq 5$ be odd. Then the supports of all codewords of weight 8 in $\overline{C}_m$ give a $3$-$\left(2^m, 8, \lambda\right)$ design, where

$$\lambda = \frac{2 \times 2^3(m-1) - 25 \times 2^2(m-1) + 123 \times 2^m - 190}{30}.$$
Example 14. Let \( m \geq 5 \) be odd. Then the supports of all codewords of weight 10 in \( C_m \) give a 3\((2^m, 10, \lambda)\) design, where
\[
\lambda = \frac{(2^{m-1} - 4)(2 \times 2^4(m-1) - 34 \times 2^3(m-1) + 235 \times 2^2(m-1) - 931 \times 2^{m-1} + 1358)}{315}.
\]

Proof. By Lemma 14,
\[
A_{10} = \frac{2h(2^h - 1)(2^h - 4)(2^{h+1} - 1)(2 \times 2^4h - 34 \times 2^3h + 235 \times 2^2h - 931 \times 2^h + 1358)}{4 \times 14175},
\]
where \( h = m - 1 \). The desired value for \( \lambda \) then follows from Theorem 15.

To demonstrate the existence of the 2-designs and 3-designs presented in Theorems 13 and 15 respectively, we describe a list of binary codes that have the weight distribution of Table 1 when \( m \) is odd and \( s \) takes on the following values [8]:

1. \( s = 2^h + 1 \), where \( \gcd(h, m) = 1 \) and \( h \) is a positive integer.
2. \( s = 2^h - 2^h + 1 \), where \( h \) is a positive integer.
3. \( s = 2^{(m-1)/2} + 3 \).
4. \( s = 2^{(m-1)/2} + 2^{(m-1)/4} - 1 \), where \( m \equiv 1 \pmod{4} \).
5. \( s = 2^{(m-1)/2} + 2^{(3m-1)/4} - 1 \), where \( m \equiv 3 \pmod{4} \).

In all these cases, \( C_m \) has parameters \([2^m - 1, 2^m - 1 - 2m, 5]\) and is optimal. It is also known that the binary narrow-sense primitive BCH code with designed distance \( 2^m - 1 - 2^{(m-1)/2} \) has also the weight distribution of Table 1 when \( m \) is odd and \( s \) takes on the following values [8]:

It is known that \( C_m \) has parameters \([2^m - 1, 2^m - 1 - 2m, 5]\) if and only if \( x^s \) is an APN monomial over GF\((2^m)\). However, even if \( x^s \) is APN, the dual code \( C_m^\perp \) may have many weights, and thus the code \( C_m \) and its dual \( C_m^\perp \) may not give 2-designs. One of such examples is the inverse APN monomial.

6. Infinite families of 2-designs from a type of ternary linear codes

In this section, we will construct infinite families of 2-designs with a type of primitive ternary cyclic codes.

| Weight \( w \) | No. of codewords \( A_w \) |
|---------------|--------------------------|
| 0             | 1                        |
| \( 2 \times 3^{m-1} - 3^{(m-1)/2} \) | \( (3^m - 1)(3^{m-1} + 3^{(m-1)/2}) \) |
| \( 2 \times 3^{m-1} \) | \( (3^m - 1)(3^{m-1} + 1) \) |
| \( 2 \times 3^{m-1} + 3^{(m-1)/2} \) | \( (3^m - 1)(3^{m-1} - 3^{(m-1)/2}) \) |
Lemma 16. Let $m \geq 3$ be odd. Assume that $C_m$ is a ternary linear code of length $3^m - 1$ such that its dual code $C_m^\perp$ has the weight distribution of Table 2. Denote by $\overline{C}_m$ the extended code of $C_m$ and let $\overline{C}_m^\perp$ denote the dual of $\overline{C}_m$. Then we have the following conclusions.

1. The code $C_m$ has parameters $[3^m - 1, 3^m - 1 - 2m, 4]$.
2. The code $C_m^\perp$ has parameters $[3^m - 1, 2m, 2 \times 3^{m-1} - 3(m-1)/2]$.
3. The code $\overline{C}_m$ has parameters $[3^m, 2m + 1, 2 \times 3^{m-1} - 3(m-1)/2]$, and its weight distribution is given in Table 3.
4. The code $\overline{C}_m^\perp$ has parameters $[3^m, 3^{m-1} - 2m, 5]$, and its weight distribution is given by

$$3^{2m+1} A_k = \binom{3^m}{k} \left( 2^k + (-1)^k 2 \right) \binom{3^m}{k} +$$

$$v \sum_{0 \leq r \leq 2 \times 3^{m-1}} (-1)^r \binom{2 \times 3^{m-1} - 3^{m-1}/2}{i} 2^i \binom{3^{m-1} - 3^{m-1}/2}{j} +$$

$$u \sum_{0 \leq r \leq 2 \times 3^{m-1} - 3^{m-1}/2} (-1)^r \binom{2 \times 3^{m-1} - 3^{m-1}/2}{i} 2^i \binom{3^{m-1} - 3^{m-1}/2}{j} +$$

$$u \sum_{0 \leq r \leq 2 \times 3^{m-1} - 3^{m-1}/2} (-1)^r \binom{2 \times 3^{m-1} + 3^{m-1}/2}{i} 2^i \binom{3^{m-1} - 3^{m-1}/2}{j}$$

for $0 \leq k \leq 3^m$, where

$$u = 3^{2m} - 3^m \quad \text{and} \quad v = (3^m + 3)(3^m - 1).$$

Proof. The proof is similar to that of Lemma 14 and is omitted here.

Theorem 17. Let $m \geq 3$ be odd. Let $C_m$ be a linear code of length $3^m - 1$ such that its dual code $C_m^\perp$ has the weight distribution of Table 2. Denote by $\overline{C}_m$ the extended code of $C_m$ and let $\overline{C}_m^\perp$ denote the dual of $\overline{C}_m$. Let $\mathcal{P} = \{0, 1, 2, \ldots, 3^m - 1\}$, and let $\mathcal{B}$ be the set of the supports of the codewords of $\overline{C}_m$ with weight $k$, where $5 \leq k \leq 10$ and $\mathcal{P}_k \neq 0$. Then $(\mathcal{P}, \mathcal{B})$ is a $2$-$(3^m, k, \lambda)$ design for some $\lambda$.\]
Let \( \mathcal{P} = \{0, 1, 2, \ldots, 3^m - 1\} \), and let \( \overline{\mathcal{B}} \) be the set of the supports of the codewords of \( \overline{\mathcal{C}}_m \) with weight \( k \) and \( \overline{A}_k \neq 0 \). Then \( (\mathcal{P}, \overline{\mathcal{B}}) \) is a \( 2-(3^m, k, \lambda) \) design for some \( \lambda \).

Proof. The weight distributions of \( \overline{\mathcal{C}}_m \) and \( \overline{\mathcal{C}}_m \) are described in Lemma 16. Notice that the minimum distance \( d \) of \( \overline{\mathcal{C}}_m \) is equal to 5. Put \( t = 2 \). The number of \( i \) with \( \overline{A}_i \neq 0 \) and \( 1 \leq i \leq 3^m - t \) is \( s = 3 \). Hence, \( s = d - t \). The desired conclusions then follow from Theorem 2.

Corollary 18. Let \( m \geq 3 \) be odd. Let \( \mathcal{C}_m \) be a ternary linear code of length \( 3^m - 1 \) such that its dual code \( \mathcal{C}_m^\perp \) has the weight distribution of Table 2. Denote by \( \overline{\mathcal{C}}_m \) the extended code of \( \mathcal{C}_m \) and let \( \overline{\mathcal{C}}_m \) denote the dual of \( \overline{\mathcal{C}}_m \).

Let \( \mathcal{P} = \{0, 1, 2, \ldots, 3^m - 1\} \), and let \( \overline{\mathcal{B}} \) be the set of the supports of the codewords of \( \overline{\mathcal{C}}_m \) with weight \( 2 \times 3^{m-1} - 3^{(m-1)/2} \). Then \( (\mathcal{P}, \overline{\mathcal{B}}) \) is a \( 2-(3^m, 2 \times 3^{m-1} - 3^{(m-1)/2}, \lambda) \), where

\[
\lambda = \frac{(2 \times 3^{m-1} - 3^{(m-1)/2})(2 \times 3^{m-1} - 3^{(m-1)/2} - 1)}{2}.
\]

Proof. It follows from Theorem 17 that \( (\mathcal{P}, \overline{\mathcal{B}}) \) is a 2-design. We now determine the value of \( \lambda \). Note that \( \overline{\mathcal{C}}_m \) has minimum weight \( 2 \times 3^{m-1} - 3^{(m-1)/2} \). Any two codewords of minimum weight \( 2 \times 3^{m-1} - 3^{(m-1)/2} \) have the same support if and only if one is a scalar multiple of the other. Consequently,

\[
|\overline{\mathcal{B}}| = \frac{3^{2m} - 3^m}{2}.
\]

It then follows that

\[
\lambda = \frac{2^{3m-3^m}}{2^2} = \frac{(2 \times 3^{m-1} - 3^{(m-1)/2})(2 \times 3^{m-1} - 3^{(m-1)/2} - 1)}{2}.
\]

Corollary 19. Let \( m \geq 3 \) be odd. Let \( \mathcal{C}_m \) be a ternary linear code of length \( 3^m - 1 \) such that its dual code \( \mathcal{C}_m^\perp \) has the weight distribution of Table 2. Denote by \( \overline{\mathcal{C}}_m \) the extended code of \( \mathcal{C}_m \) and let \( \overline{\mathcal{C}}_m \) denote the dual of \( \overline{\mathcal{C}}_m \). Let \( \mathcal{P} = \{0, 1, 2, \ldots, 3^m - 1\} \), and let \( \overline{\mathcal{B}} \) be the set of the supports of the codewords of \( \overline{\mathcal{C}}_m \) with weight 5. Then \( (\mathcal{P}, \overline{\mathcal{B}}) \) is a \( 2-(3^m, 5, \lambda) \) design, where

\[
\lambda = \frac{5(3^{m-1} - 1)}{2}.
\]

Proof. It follows from Theorem 17 that \( (\mathcal{P}, \overline{\mathcal{B}}) \) is a 2-design. We now determine the value of \( \lambda \). Using the weight distribution formula in Lemma 16 we obtain that

\[
\overline{A}_5 = \frac{3^{3m-1} - 4 \times 3^{2m-1} + 3^m}{4}.
\]

Recall that \( \overline{\mathcal{C}}_m \) has minimum weight 5. Any two codewords of minimum weight 5 have the same support if and only if one is a scalar multiple of the other. Consequently,

\[
|\overline{\mathcal{B}}| = \frac{\overline{A}_5}{18}.
\]
Table 4: The weight distribution for odd \( m \geq 3 \)

| Weight               | Frequency                      |
|----------------------|--------------------------------|
| \( 3^{m-1} - 3^{(m-1)/2} \) | \( \frac{(3^{m+1}+3^{(m-1)/2})(3^m-1)}{2} \) |
| \( 3^{m-1} \)       | \( (3^m - 3^{m-1} + 1)(3^m - 1) \) |
| \( 3^{m-1} + 3^{(m-1)/2} \) | \( \frac{(3^{m+1}-3^{(m-1)/2})(3^m-1)2}{2} \) |

It then follows that

\[
\lambda = \frac{\tilde{A}_5}{2} \binom{5}{2} = \frac{5(3^{m-1} - 1)}{2}.
\]

Theorem gives more 2-designs. However, determining the corresponding value \( \lambda \) may be hard, as the number of blocks in the design may be difficult to derive from \( \tilde{A}_k \) or \( \tilde{A}_k^* \).

**Open Problem 3.** Determine the value of \( \lambda \) of the 2-(3\( m \), \( k \), \( \lambda \)) design for \( 6 \leq k \leq 10 \), which are described in Theorem 17.

**Open Problem 4.** Determine the values of \( \lambda \) of the 2-(3\( m \), 3\( m-1 \), \( \lambda \)) design and the 2-(3\( m \), 2 \( \times \) 3\( m-1 \) - 3\( (m-1)/2 \), \( \lambda \)) design, which are described in Theorem 17.

To demonstrate the existence of the 2-designs presented in Theorem 17, we present a list of ternary cyclic codes that have the weight distribution of Table 2 below.

Put \( n = 3^{m} - 1 \). Let \( \alpha \) be a generator of GF\((3^m)\)\(^*\). Let \( g_s(x) = M_{3^{m}-1}(x) \prod_{a \in \alpha} \overline{M}_{3^{h}/2}(x) \), where \( M_{h}(x) \) is the minimal polynomial of \( \alpha^h \) over GF\((3)\). Let \( C_m \) denote the cyclic code of length \( n = 3^{m} - 1 \) over GF\((3)\) with generator polynomial \( g_s(x) \). It is known that \( C_m^* \) has dimension 2\( m \) and the weight distribution of Table 2 when \( m \) is odd and \( s \) takes on the following values [4, 21]:

1. \( s = 3^h + 1 \), \( h \geq 0 \) is an integer.
2. \( s = (3^h + 1)/2 \), where \( h \) is a positive integer and gcd\((m, h) = 1\).

In these two cases, \( x^s \) is a planar function on GF\((3^m)\). Hence, these ternary codes are extremal in the sense that they are defined by planar functions whose differentiability is extremal.

More classes of ternary codes such that their duals have the weight distribution of Table 2 are documented in [8]. They give also 2-designs via Theorem 17. There are also ternary cyclic codes with three weights but different weight distributions in [8]. They may also hold 2-designs.

7. Conjectured infinite families of 2-designs from projective cyclic codes

Throughout this section, let \( m \geq 3 \) be an odd integer, and let \( v = (3^m - 1)/2 \). The objective of this section is to present a number of conjectured infinite families of 2-designs derived from linear projective ternary cyclic codes.
Lemma 20. Let $C_m$ be a linear code of length $v$ over GF(3) such that its dual $C_m^\perp$ has the weight distribution in Table 4. Then the weight distribution of $C_m$ is given by

$$3^{2m}A_k = \sum_{0 \leq j \leq \frac{3^m-1}{2}} (-1)^{j/2}a\left(\frac{3^m-1}{2} - \frac{3(m-1)/2}{j}\right)\left(\frac{3^m-1+2^j}{2}\right)$$

$$+ \left(\frac{3^m-1}{k}\right)2^k + \sum_{0 \leq j \leq \frac{3^m-1}{2}} (-1)^{j/2}b\left(\frac{3^m-1}{2}\right)\left(\frac{3^m-1}{j}\right)$$

$$+ \sum_{0 \leq j \leq \frac{3^m-1}{2}} (-1)^{j/2}c\left(\frac{3^m-1}{2} + \frac{3(m-1)/2}{j}\right)\left(\frac{3^m-1+2^j}{2}\right)$$

for $0 \leq k \leq \frac{3^m-1}{2}$, where

$$a = \frac{(3^m-1+3(m-1)/2)(3^m-1)}{2}$$

$$b = \frac{(3^m-3(m-1)+1)(3^m-1)}{2}$$

$$c = \frac{(3^m-3(m-1)+3(m-1)/2)(3^m-1)}{2}$$

In addition, $C_m$ has parameters $[(3^m-1)/2, (3^m-1)/2-2m, 4]$.

**Proof.** Note that the weight enumerator of $C_m^\perp$ is

$$1 + az^{3^m-3(m-1)/2} + bz^{3^m-1} + cz^{3^m-1+3(m-1)/2}.$$ 

The proof of this theorem is similar to that of Lemma 12 and is omitted. \hfill \Box

Below we present two examples of ternary linear codes $C_m$ such that their duals $C_m^\perp$ have the weight distribution of Table 4.

**Example 15.** Let $m \geq 3$ be odd. Let $\alpha$ be a generator of GF($3^m$)$^\ast$. Put $\beta = \alpha^2$. Let $M_i(x)$ denote the minimal polynomial of $\beta^i$ over GF(3). Define

$$\delta = 3^m-1 - \frac{3^{m+1}/2-1}{2}$$

and

$$h(x) = (x-1)\text{lcm}(M_1(x), M_2(x), \cdots, M_{\delta-1}(x)),$$

where lcm denotes the least common multiple of the polynomials. Let $C_m$ denote the cyclic code of length $v = (3^m-1)/2$ over GF(3) with generator polynomial $g(x) := (x^\delta - 1)/h(x)$. Then $C_m$ has parameters $[(3^m-1)/2, (3^m-1)-2m, 4]$ and $C_m^\perp$ has the weight distribution of Table 4.
Conjecture 22. Let \( m \geq 3 \) be odd. Let \( \alpha \) be a generator of \( \text{GF}(3^m)^* \). Let \( \beta = \alpha^2 \). Let \( g(x) = M_{m-1}(x)M_{m-2}(x) \), where \( M_i(x) \) is the minimal polynomial of \( \beta^i \) over \( \text{GF}(3) \). Let \( C_m \) denote the cyclic code of length \( v = (3^m - 1)/2 \) over \( \text{GF}(3) \) with generator polynomial \( g(x) \). Then \( C_m \) has parameters \([3^m - 1)/2, (3^m - 1) - 2m, 4]\) and \( C_m^\perp \) has the weight distribution of Table 4.

Proof. A proof of the desired conclusions was given in [15]. \( \square \)

Example 16. Let \( m \geq 3 \) be odd. Let \( \alpha \) be a generator of \( \text{GF}(3^m)^* \). Let \( \beta = \alpha^2 \). Let \( g(x) = M_{m-1}(x)M_{m-2}(x) \), where \( M_i(x) \) is the minimal polynomial of \( \beta^i \) over \( \text{GF}(3) \). Let \( C_m \) denote the cyclic code of length \( v = (3^m - 1)/2 \) over \( \text{GF}(3) \) with generator polynomial \( g(x) \). Then \( C_m \) has parameters \([3^m - 1)/2, (3^m - 1) - 2m, 4]\) and \( C_m^\perp \) has the weight distribution of Table 4.

Proof. The desired conclusions can be proved similarly as Theorem 19 in [15]. \( \square \)

Conjecture 21. Let \( \mathcal{P} = \{0, 1, 2, \ldots, v-1\} \), and let \( \mathcal{B} \) be the set of the supports of the codewords of \( C_m \) with Hamming weight \( k \), where \( A_k \neq 0 \). Then \( (\mathcal{P}, \mathcal{B}) \) is a \( 2-(v,k,\lambda) \) design for all odd \( m \geq 3 \).

Conjecture 22. Let \( \mathcal{P} = \{0, 1, 2, \ldots, v-1\} \), and let \( \mathcal{B} \) be the set of the supports of the codewords of \( C_m \) with Hamming weight \( 4 \). Then \( (\mathcal{P}, \mathcal{B}) \) is a Steiner system \( S(2,4,(3^m - 1)/2) \) for all odd \( m \geq 3 \).

There are a survey on Steiner systems \( S(2,4,v) \) [17] and a book chapter on Steiner systems [5]. It is known that a Steiner system \( S(2,4,v) \) exists if and only if \( v \equiv 1 \) or \( 4 \pmod{12} \) [10].

If Conjecture 21 is true, so is Conjecture 22. In this case, a coding theory construction of a Steiner system \( S(2,4,(3^m - 1)/2) \) for all odd \( m \geq 3 \) is obtained.

Conjecture 23. Let \( \mathcal{P} = \{0, 1, 2, \ldots, v-1\} \), and let \( \mathcal{B} \) be the set of the supports of the codewords of \( C_m \) with Hamming weight \( k \), where \( A_k \neq 0 \). Then \( (\mathcal{P}, \mathcal{B}) \) is a \( 2-(v,k,\lambda) \) design for all odd \( m \geq 3 \).

Even if some or all of the three conjectures are not true for ternary codes with the weight distribution of Table 4, these conjectures might still be valid for the two classes of ternary cyclic codes described in Examples 15 and 16. Note that Theorem 14 does not apply to the three conjectures above. We need to develop different methods for settling these conjectures.

8. Summary and concluding remarks

In the last section of this paper, we mention some applications of \( t \)-designs and summarize the main contributions of this paper.

8.1. Some applications of \( t \)-designs

Let \( \mathcal{P} \) be an Abelian group of order \( v \) under a binary operation denoted by \( + \). Let \( \mathcal{B} = \{B_1, B_2, \ldots, B_k\} \), where all \( B_i \) are \( k \)-subsets of \( \mathcal{P} \) and \( k \) is a positive integer. We define \( \Delta(B_i) \) to be the multiset \( \{x - y : x \in B_i, y \in B_i\} \). If every nonzero element of \( \mathcal{P} \) appears exactly \( \delta \) times in the multiset \( \bigcup_{i=1}^k \Delta(B_i) \), we call \( \mathcal{B} \) a \( (v,k,\delta) \) difference family in \((\mathcal{P},+)\).

The following theorems are straightforward and should be well known.

Theorem 24. Let \( \mathcal{P} \) be an Abelian group of order \( v \) under a binary operation denoted by \( + \). Let \( \mathcal{B} = \{B_1, B_2, \ldots, B_k\} \), where all \( B_i \) are \( k \)-subsets of \( \mathcal{P} \) and \( k \) is a positive integer. Then \( (\mathcal{P}, \mathcal{B}) \) is a \( 2-(v,k,\lambda) \) design if and only if \( \mathcal{B} \) is a \( (v,k,\lambda v) \) difference family in \((\mathcal{P},+)\).

Theorem 25. Let \( (\mathcal{P}, \mathcal{B}) \) be a \( t-(v,k,\lambda) \) design, where \( \mathcal{P} \) is an Abelian group. If \( t \geq 2 \), then \( \mathcal{B} \) is a \( (v,k,\delta) \) difference family in \( \mathcal{P} \), where

\[
\delta = \frac{v\lambda \binom{v-2}{k-2}}{\binom{k-2}{t-2}}
\]
Difference families have applications in the design and analysis of optical orthogonal codes, frequency hopping sequences, and other engineering areas. By Theorems 24 and 25, \( t \)-designs with \( t \geq 2 \) have also applications in these areas. In addition, 2-designs give naturally linear codes \([1, 6]\). These show the importance of 2-designs in applications.

8.2. Summary

It is well known that binary Reed-Muller codes hold 3-designs. Hence, the only contribution of Section 3 is the determination of the specific parameters of the 3-designs held in \( \text{RM}(m - 2, m) \) and its dual code, which are documented in Theorem 5.

It has also been known for a long time that the codewords of weight 3 in the Hamming code hold a 2-(\( (q^m - 1)/(q - 1), 3, q - 1 \)) design. The contribution of Section 4 is Theorem 10, which may be viewed as an extension of the known 2-(\( (q^m - 1)/(q - 1), 3, q - 1 \)) design held in the Hamming code, and also the parameters of the infinite families of 2-designs derived from the binary Hamming codes, which are documented in Examples 2, 3, 4, and 5.

A major contribution of this paper is presented in Section 5, where Theorems 13 and 15 document many infinite families of 2-design and 3-designs. The parameters of these 2-designs and 3-designs are given specifically. These designs are derived from binary cyclic codes that are defined by special almost perfect nonlinear functions.

Another major contribution of this paper is documented in Section 6, where Theorem 17 and its two corollaries describe several infinite families of 2-designs. These 2-designs are related to planar functions.

It is noticed that the total number of 3-designs presented in this paper (see Theorems 5 and 15) are exponential. All of them are derived from linear codes. After comparing the list of infinite families of 3-designs in [14] with the 3-designs presented in this paper, one may conclude that many, if not most, of the known infinite families of 3-designs are from coding theory.

Section 7 presents many conjectured infinite families of 2-designs. The reader is cordially invited to attack these conjectures and solve other open problems presented in this paper.

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