EXISTENCE AND UNIQUENESS OF SOLUTIONS IN FRACTIONAL ORLICZ-SOBOLEV SPACES FOR A NONLOCAL SINGULAR ELLIPTIC PROBLEM

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ABSTRACT

In this article, we investigate the existence and uniqueness of a positive solution for a class of singular nonlinear elliptic problem with boundary condition. Our result holds in fractional Orlicz-Sobolev spaces.

1 Introduction

We are interested here with the existence and uniqueness of solution to a non-homogeneous problem in fractional Orlicz-Sobolev spaces

\[
(P_a) \begin{cases} 
(-\Delta)_a^s u + a(|u|)u = \frac{a(x)}{u^\gamma(x)} & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases} 
\]  

(1.1)

where \( \Omega \) is an open bounded subset in \( \mathbb{R}^N \) with Lipschitz boundary \( \partial \Omega \), \( 0 < s < 1 \), \( \gamma : \bar{\Omega} \to (0, 1) \) is a continuous function, \( g \) is continuous functions and \( (-\Delta)_a^s \) is the fractional \( a \)-Laplacian operator defined as

\[
(-\Delta)_a^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} a \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|x - y|^s} \frac{dy}{|x - y|^{N+s}},
\]

where \( a : \mathbb{R}^+ \to \mathbb{R}^+ \) which will be specified later. Recently, a great attention has been focused on the study of nonlocal fractional operators of elliptic type (see, as an example \([10, 12]\)). This sort of operators arises in various types applications; for instance, continuum mechanics, phase-change phenomena, population dynamics, minimal surfaces and anomalous diffusion, since they’re the standard outcome of stochastically stabilization of Levy processes, see \([6, 16, 27]\) and the references cited therein. Moreover, such equations and therefore the associated fractional operators allow us to develop a generalization of quantum mechanics and also to explain the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media, for more details see \([6, 16]\) and the references therein. Indeed, the nonlocal fractional operators have been extensively studied by many authors in many various cases: bounded domains and unbounded domains, different behavior of the nonlinearity, and so on. Before giving our main results, allow us to briefly recall the literature concerning related singular problems. The following research problem has been investigated in several previous works:

\[
\begin{cases} 
(-\Delta)^s u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, 
\end{cases} 
\]  

(1.2)

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where $f$ is a Carathéodory function. The questions of existence, multiplicity and regularity of solutions to problem (1.2) have been extensively studied in [1, 4, 25, 28, 30, 31] and references therein. Concerning the existence and multiplicity of solutions to doubly nonlocal problems, a lot of works have been done. For a detailed state of art, one can refer [18, 20, 21, 32, 37] and references therein. On the other hand, Barrios et al. [13] started the work on nonlocal problems in this class of fractional Orlicz-Sobolev spaces. Within the third section, applying several important properties of fractional a-Laplacian operator, we obtain the existence and uniqueness of weak solutions of problem (1.3) have been extensively studied in [1, 4, 25, 28, 30, 31] and references therein. Concerning the existence and multiplicity of solutions to doubly nonlocal problems, a lot of works have been done. For a detailed state of art, one can refer [18, 20, 21, 32, 37] and references therein. On the other hand, Barrios et al. [13] started the work on nonlocal problems in this class of fractional Orlicz-Sobolev spaces.

The rest of the paper is organized as follows. In the second section we make a brief introduction for each of the following spaces: Orlicz-Sobolev spaces and fractional Orlicz-Sobolev. Within the third section, applying several important properties of fractional a-Laplacian operator, we obtain the existence and uniqueness of weak solutions of problem (1.3) have been extensively studied in [1, 4, 25, 28, 30, 31] and references therein. Concerning the existence and multiplicity of solutions to doubly nonlocal problems, a lot of works have been done. For a detailed state of art, one can refer [18, 20, 21, 32, 37] and references therein. On the other hand, Barrios et al. [13] started the work on nonlocal problems in this class of fractional Orlicz-Sobolev spaces.

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2 Some preliminaries results

2.1 Orlicz-Sobolev Spaces.

First, we briefly recall the definitions and some elementary properties of the Orlicz-Sobolev spaces. We refer the reader to [2, 3, 34], for additional reference and for some of the proofs of the results in this subsection. Let $\Omega$ be an open subset of $\mathbb{R}^N$, $N \geq 1$. We assume that $a : \mathbb{R} \rightarrow \mathbb{R}$ in $(P_a)$ is such that : $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$\varphi(t) = \begin{cases} a(|t|)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0 \end{cases}$$

is increasing homeomorphism from $\mathbb{R}$ onto itself. Let

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau.$$  

Then $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an N-function, that is, $\Phi$ is continuous, convex, with $\Phi(t) > 0$ for $t > 0$, $\frac{\Phi(t)}{t} \rightarrow 0$ as $t \rightarrow 0$ and $\frac{\Phi(t)}{t} \rightarrow \infty$ as $t \rightarrow \infty$. Equivalently, $\Phi$ admits the representation;

$$\Phi(t) = \int_0^t \varphi(s)ds,$$

where $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is non-decreasing, right continuous, with $\varphi(0) = 0$, $\varphi(t) > 0$ $\forall t > 0$ and $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$. The conjugate N-function of $\Phi$ is defined by

$$\overline{\Phi}(t) = \int_0^t \overline{\varphi}(s)ds,$$

where $\overline{\varphi} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is given by $\overline{\varphi}(t) = \sup\{s : \varphi(s) \leq t\}$. Evidently we have

$$st \leq \Phi(t) + \overline{\Phi}(s),$$  

(2.1)
which is known Young’s inequality. Equality holds in (2.1) if and only if either \( t = \varphi(s) \) or \( s = \varphi(t) \).

We say that the N-function \( \Phi \) satisfies the global \( \Delta_2 \)-condition if, for some \( k > 0 \)
\[
\Phi(2t) \leq k\Phi(t), \quad \forall t \geq 0.
\]

When this inequality holds only for \( t \geq t_0 > 0 \), \( \Phi \) is said to satisfy the \( \Delta_2 \)-condition near infinity. We call the pair \((\Phi, \Omega)\) is \( \Delta \)-regular if either:

(a) \( \Phi \) satisfies a global \( \Delta_2 \)-condition, or
(b) \( \Phi \) satisfies a \( \Delta_2 \)-condition near infinity and \( \Omega \) has finite volume.

Throughout this paper, we assume that \( \Phi \):
\[
1 < \varphi^- := \inf_{s \geq 0} \frac{s\varphi(s)}{\Phi(s)} \leq \varphi^+ := \sup_{s \geq 0} \frac{s\varphi(s)}{\Phi(s)} < +\infty,
\]
which assures that \( \Phi \) satisfies the global \( \Delta_2 \)-condition. We assume that:
\[
\text{the function } t \mapsto \Phi(\sqrt{t}) \text{ is convex for all } t \geq 0.
\]

The above relation assures that \( L_\Phi(\Omega) \) is an uniformly convex space (see [29]). Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). The Orlicz class \( K_\Phi(\Omega) \) (resp. the Orlicz space \( L_\Phi(\Omega) \)) is defined as the set of (equivalence classes of) real-valued measurable functions \( u \) on \( \Omega \) such that
\[
\int_\Omega \Phi(|u(x)|)dx < \infty \quad \text{(resp. } \int_\Omega \Phi(\lambda|u(x)|)dx < \infty \text{ for some } \lambda > 0 \text{)}.
\]

\( L_\Phi(\Omega) \) is a Banach space under the Luxemburg norm
\[
\|u\|_\Phi = \inf \left\{ \lambda > 0 : \int_\Omega \Phi \left( \frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}
\]
and \( K_\Phi(\Omega) \) is a convex subset of \( L_\Phi(\Omega) \). The closure in \( L_\Phi(\Omega) \) of the set of bounded measurable functions on \( \Omega \) with compact support in \( \Omega \) is denoted by \( E_\Phi(\Omega) \). The equality \( E_\Phi(\Omega) = L_\Phi(\Omega) \) holds if and only if \((\Phi, \Omega)\) is \( \Delta \)-regular.

Using Young’s inequality, we have the following Hölder type inequality,
\[
\left| \int_\Omega u v dx \right| \leq 2\|u\|_\Phi \|v\|_{\overline{\Phi}}, \quad \forall u \in L_\Phi(\Omega) \text{ and } v \in L_{\overline{\Phi}}(\Omega).
\]

**Proposition 2.1.** (see: [29]) Assume condition (2.2) is satisfied. Then the following relations holds true
\[
\|u\|_{\Phi}^\varphi^- \leq \int_\Omega \Phi(|u|)dx \leq \|u\|_{\Phi}^\varphi^+, \quad \forall u \in L_\Phi(\Omega) \text{ with } \|u\|_\Phi > 1.
\]

**Lemma 2.2.** (c.f. [15]) Assume that \( \Phi \in \Delta_2 \). Then we have,
\[
\Phi(\varphi(t)) \leq c\Phi(t) \text{ for all } t \geq 0 \text{ where } c > 0.
\]

To simplify the notation, we put
\[
h(x, y) := \frac{u(x) - u(y)}{|x - y|^s}, \quad d\mu = \frac{dxdy}{|x - y|^N}, \quad \forall (x, y) \in \Omega \times \Omega, \quad \int_{\Omega^2} := \int_\Omega \int_\Omega.
\]

### 2.2 Fractional Orlicz-Sobolev space.

This subsection is devoted to the definition of the fractional Orlicz-Sobolev spaces, and we recall some result of continuous and compact embedding of fractional Orlicz-Sobolev spaces. We refer the reader to [29] [33] for further reference and for some of the proofs of these results.

\[
W^s L_\Phi(\Omega) = \left\{ u \in L_\Phi(\Omega) : \int_{\Omega^2} \Phi \left( \frac{\lambda|u(x) - u(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^N} < \infty \right\}
\]
for some \( \lambda > 0 \). This space is equipped with the norm,
\[
\|u\|_{s, \Phi} = \|u\|_\Phi + [u]_{s, \Phi},
\]
where \([\cdot]_{s, \Phi}\) is the Gagliardo semi-norm, defined by
\[
[u]_{s, \Phi} = \inf \left\{ \lambda > 0 : \int_{\Omega^2} \Phi \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dxdy}{|x - y|^N} \leq 1 \right\}.
\]

By [15], \( W^s L_\Phi(\Omega) \) is Banach space, also separable (resp. reflexive) space if and only if \( \Phi \in \Delta_2 \) (resp. \( \Phi \in \Delta_2 \) and \( \Phi(\sqrt{t}) \) is convex), then the space \( W^s L_\Phi(\Omega) \) is uniformly convex.
Proposition 2.3. (cf. [11]) Assume condition (2.2) is satisfied. Then the following relations holds true,
\[ [u]_{s, \Phi}^* \leq \phi(u) \leq [u]_{s, \Phi}^+, \forall u \in W^s L_\Phi(\Omega) \text{ with } [u]_{s, \Phi} > 1 \]
\[ [u]_{s, \Phi}^- \leq \phi(u) \leq [u]_{s, \Phi}^-, \forall u \in W^s L_\Phi(\Omega) \text{ with } [u]_{s, \Phi} < 1 \]
where \( \phi(u) = \int_{\Omega^2} \Phi\left(\frac{|u(x) - u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N} \).

Proposition 2.4. (cf. [11]) The following relations holds true
\[ \Psi(u) \geq \|u\|_{s, \Phi}^+ \forall u \in W^s L_\Phi(\Omega), \|u\|_{s, \Phi} > 1 \]
\[ \Psi(u) \geq \|u\|_{s, \Phi}^- \forall u \in W^s L_\Phi(\Omega), \|u\|_{s, \Phi} < 1. \]
Where \( \Psi(u) := \int_{\Omega^2} \Phi\left(\frac{|u(x) - u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N} + \int_{\Omega} \Phi(|u(x)|)dx \).

Theorem 2.5. (cf. [11]) Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \). Then,
\[ C^0_0(\Omega) \subset W^s L_\Phi(\Omega). \]

Remark 2.6. A trivial consequence of Theorem 2.5, \( C^\infty_0(\Omega) \subset W^s L_\Phi(\Omega) \) and \( W^s L_\Phi(\Omega) \) is non-empty.

Theorem 2.7. (cf. [11]) Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), \( N \geq 1 \) with \( C^{0,1} \) regularity and bounded boundary, let \( 0 < s' < s < 1 \). Assume condition (2.2) is satisfied. We define
\[ \varphi_{s'}^* = \begin{cases} \frac{N\varphi^-}{N-s'}, & \text{if } N > s' \varphi^- \\ \frac{N\varphi^-}{N-s'}, & \text{if } N \leq s' \varphi^- \end{cases} \]

- If \( s' \varphi^- < N \), then \( W^s L_\Phi(\Omega) \hookrightarrow L^{s'}(\Omega) \), for all \( q \in [1, \varphi_{s'}^*] \) and the embedding \( W^s L_\Phi(\Omega) \hookrightarrow L^q(\Omega) \), is compact for all \( q \in [1, \varphi_{s'}^*] \).
- If \( s' \varphi^- = N \), then \( W^s L_\Phi(\Omega) \hookrightarrow L^q(\Omega) \), for all \( q \in [1, \infty] \) and the embedding \( W^s L_\Phi(\Omega) \hookrightarrow L^q(\Omega) \), is compact for all \( q \in [1, \infty] \).
- If \( s' \varphi^- > N \), then the embedding \( W^s L_\Phi(\Omega) \hookrightarrow L^\infty(\Omega) \), is compact.

Lemma 2.8. (cf. [11]) Assume condition (2.2) is satisfied. Then the following relations holds true
\[ \Phi(\sigma t) \geq \sigma \varphi^- \Phi(t), \forall t > 0, \sigma > 1 \]
\[ \Phi(\tau t) \geq \tau \varphi^+ \Phi(t), \forall t > 0, \tau \in (0, 1). \]

Remark 2.9. First, we can apply Theorem 2.7 to obtain that, \( W^s L_\Phi(\Omega) \) is continuously and compactly embedded in the classical Lebesgue space \( L^{s'}(\Omega) \) for all \( 1 \leq r^+ < \varphi_{s'}^* \). That fact combined with continuous embedding of \( L^{r^+}(\Omega) \) in \( L^{r^-}(\Omega) \), with \( r \in C(\Omega) \) and \( 1 < r^- \leq r(x) \leq r^+ < +\infty \), with \( r^+ = \sup_{x \in \Omega} r(x) \) and \( r^- = \inf_{x \in \Omega} r(x) \), as a result, \( W^s L_\Phi(\Omega) \) is compactly embedded in \( L^{r^+}(\Omega) \).

Lemma 2.10. (cf. [11]) The functional \( I_1 : W^s L_\Phi(\Omega) \rightarrow \mathbb{R} \) is of class \( C^1 \) and
\[ \langle I_1'(u), v \rangle = \int_{\Omega^2} a(h_{x,y}(u)) h_{x,y}(u) h_{x,y}(v) d\mu + \int_{\Omega} a(|u|) uv dx \]
for all \( u, v \in W^s L_\Phi(\Omega) \).

The following Proposition generalizes the definition of convexity, and therefore, we give a proof for the convenience.

Proposition 2.11. Let \( X \) be a vector space and let \( I : X \rightarrow \mathbb{R} \). Then \( I \) is convex if and only if
\[ I((1-\lambda)u + \lambda v) < (1-\lambda)I(u) + \lambda I(v), \quad 0 < \lambda < 1 \]
whenever \( I(u) < \theta \) and \( I(v) < \beta \), for all \( u, v \in X \) and \( \theta, \beta \in \mathbb{R} \).

Proof. Assume that functional \( I : X \rightarrow \mathbb{R} \) is convex. Moreover, since \( I \) is a real-valued functional, there are real numbers \( \theta, \beta \in \mathbb{R} \) such that \( I(u) < \theta \) and \( I(v) < \beta \). Then
\[ I((1-\lambda)u + \lambda v) < (1-\lambda)I(u) + \lambda I(v) < (1-\lambda)\theta + \lambda \beta, \quad 0 < \lambda < 1 \]
On the other hand, assume that (2.5) holds. Since $I( u ) < \theta$ and $I( v ) < \beta$, we can write, for all $\varepsilon > 0$
\[ I( u ) < I( u ) + \varepsilon := \theta \]
\[ I( v ) < I( v ) + \varepsilon := \beta. \]
Therefore,
\[ I((1 - \lambda) u + \lambda v) < (1 - \lambda)I( u ) + \lambda I( v ) + \varepsilon, \quad 0 < \lambda < 1 \]
If we consider that (2.6) holds for any $\varepsilon > 0$, we conclude
\[ I((1 - \lambda) u + \lambda v) \leq (1 - \lambda)I( u ) + \lambda I( v ). \]
\[ \square \]

3 The main results and their proof

Now, we give the definition of weak solutions for problem (P_a)

**Definition 3.1.** A function $u$ is called a weak solution to problem $(P_a)$ if $u \in W^s L_p(\Omega)$ such that $u > 0$ in $\Omega$ and
\[ \int_{\Omega} a ( |h_{x,y}(u)| ) h_{x,y}(u) h_{x,y}(v) d\mu + \int_{\Omega} a(|u|)uv dx = \int_{\Omega} \frac{g(x)u^{1-\gamma}v}{1 - \gamma(x)} dx, \]
for all $v \in W^s L_p(\Omega)$.

The main result of present paper are the following.

**Theorem 3.2.** Suppose that the following assumption hold: (G0) : $g(x) \in C^1(\overline{\Omega})$ is a nontrivial nonnegative function. Then problem $(P_a)$ has a positive ground state solution in $W^s L_p(\Omega)$ with a negative energy level.

Let us consider the functional $J : W^s L_p(\Omega) \rightarrow \mathbb{R}$ associated to problem $(P_a)$ as follows
\[ J(u) = \int_{\Omega} \Phi \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) d\mu + \int_{\Omega} \Phi(|u(x)|)dx - \int_{\Omega} \frac{g(x)|u|^{1-\gamma(x)}}{1 - \gamma(x)} dx. \]

Now, we give an auxiliary result

**Lemma 3.3.** Assume the hypotheses of Theorem are fulfilled. Then the functional $J$ is coercive.

**Proof.** For each $u \in W^s L_p(\Omega)$ with $\|u\| > 1$, relations (2.4) and the condition (G0) imply
\[ J(u) = \int_{\Omega} \Phi \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) d\mu + \int_{\Omega} \Phi(|u(x)|)dx - \int_{\Omega} \frac{g(x)|u|^{1-\gamma(x)}}{1 - \gamma(x)} dx \]
\[ \geq \|u\|^\gamma_{s,\Phi} - \frac{|g|}{1 - \gamma^+} \|u\|^{1-\gamma(x)} \frac{1}{L^{1-\gamma(x)}(\Omega)} \]
\[ \geq \|u\|^\gamma_{s,\Phi} - c\|u\|^{1-\gamma(x)}, \]
since $\varphi^- > 1 - \gamma^+$ the above inequality implies that $J(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$, that is, $J$ is coercive. \[ \square \]

**Lemma 3.4.** Assume the hypotheses of Theorem 3.2 are fulfilled. Then there exists $\phi \in W^s L_p(\Omega)$ such that $\phi > 0$ and $J(\phi) < 0$ for $t > 0$ small enough.

**Proof.** Let $\Omega_0 \subset \subset \Omega$, for $x_0 \in \Omega_0$, $0 < R < 1$ satisfy $B_{2R}(x_0) \subset \Omega_0$, where $B_{2R}(x_0)$ is the ball of radius $2R$ with center at the point $x_0$ in $\mathbb{R}^N$. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ satisfies $0 \leq \phi \leq 1$ and $\phi \equiv 1$ in $B_{2R}(x_0)$. Theorem 2.3 implies that $\|\phi\| < \infty$. Then for $0 < t < 1$, we have
\[ J(t\phi) = \int_{\Omega_0} \Phi \left( \frac{|t\phi(x) - t\phi(y)|}{|x - y|^s} \right) d\mu + \int_{\Omega} \Phi(|t\phi|)dx - \int_{\Omega} \frac{g(x)|t\phi|^{1-\gamma(x)}}{1 - \gamma(x)} dx \]
\[ \leq |t\phi|^{\gamma_{s,\Phi}} - \frac{t^{1-\gamma^+}}{1 - \gamma^+} \int_{\Omega_0} g(x)|\phi|^{1-\gamma(x)} dx \]
\[ \leq t^{\varphi^-} |\phi|^{\gamma_{s,\Phi}} - \frac{t^{1-\gamma^+}}{1 - \gamma^+} \int_{\Omega} g(x)|\phi|^{1-\gamma(x)} dx, \]
since $\varphi^- > 1 - \gamma^+$ and $\int_{\Omega_0} g(x)|\phi|^{1-\gamma(x)} dx > 0$ we have $J( t_0 \phi ) < 0$ for $t_0 \in (0, t)$ sufficiently small. \[ \square \]
Lemma 3.5. The functional $J$ attains the global minimizer in $W^s L^p(\Omega)$, that is, there exists a function $u_* \in W^s L^p(\Omega)$ such that

$$m = J(u_*) = \inf_{u \in W^s L^p(\Omega)} J(u) < 0.$$ 

Proof. By (G0), Hölder inequality, Proposition 2.4 and the continuous embedding $W^s L^p(\Omega) \hookrightarrow L^{r(x)}(\Omega)$, it follows

$$|J(u)| \leq I_1(u) + \frac{|g|_\infty}{1 - \gamma^+} \int_\Omega |u|^{1 - \gamma(x)} \, dx$$

$$\leq 2\|u\|_{s, \Phi}^+ + \frac{|g|_\infty}{1 - \gamma^+} \|u\|_{\gamma(x)}^{1 - \gamma(x)} \|1\|_{L^{r(x)}(\Omega)} \|u\|_{s, \Phi}^{1 - \gamma^-} < +\infty,$$

which shows that $J$ is well-defined on $W^s L^p(\Omega)$. Considering the fact that the functional $I_1(u)$ is of class $C^1 (W^s L^p(\Omega), \mathbb{R})$ (see [15]). Further, by the well-known inequality

$$|a^p - b^p| \leq |a - b|^p,$$

for any real numbers $a, b \geq 0$ and $0 < p < 1$, we obtain

$$|J(u) - J(v)| \leq |I_1(u) - I_1(v)| + \frac{|g|_\infty}{1 - \gamma^+} \int_\Omega |u|^{1 - \gamma(x)} - |v|^{1 - \gamma(x)} \, dx$$

$$\leq |I_1(u) - I_1(v)| + \frac{|g|_\infty}{1 - \gamma^+} \int_\Omega |u - v|^{1 - \gamma(x)} \, dx$$

$$\leq |I_1(u) - I_1(v)| + \frac{|g|_\infty}{1 - \gamma^+} \|u - v\|_{L^{r(x)}(\Omega)}^{1 - \gamma(x)} \|1\|_{L^{r(x)}(\Omega)}^{1 - \gamma(x)}$$

$$\leq |I_1(u) - I_1(v)| + \frac{c|g|_\infty}{1 - \gamma^+} \|u - v\|_{s, \Phi},$$

for any $u, v \in W^s L^p(\Omega)$. Therefore $J$ is continuous on $W^s L^p(\Omega)$. Now, we shall show that $J$ is convex on $W^s L^p(\Omega)$. To this end, considering the assumptions for $g$ and $\gamma$, we have

$$J(u) = I_1(u) - \int_\Omega g(x)|u|^{1 - \gamma(x)} \, dx$$

$$\leq \max \left\{ \|u\|_{s, \Phi}^-, \|u\|_{s, \Phi}^+ \right\} := \theta$$

and

$$J(v) = I_1(v) - \int_\Omega g(x)|v|^{1 - \gamma(x)} \, dx$$

$$\leq \max \left\{ \|v\|_{s, \Phi}^-, \|v\|_{s, \Phi}^+ \right\} := \beta,$$

for all $u, v \in W^s L^p(\Omega)$. On the other hand, since $\Phi$ is convex, so

$$I_1(u) = \int_\Omega \Phi(x, |\nabla u|) \, dx + \int_\Omega \Phi(u) \, dx,$$

(see [22] is also convex. Therefore, for $0 < \lambda < 1$, we have

$$(I_1((1-\lambda)u + \lambda v)) \leq (1-\lambda)I_1(u) + \lambda I_1(u).$$

Therefore, considering the all pieces of information obtained above along, it follows

$$J((1-\lambda)u + \lambda v) = I_1((1-\lambda)u + \lambda v) - \int_\Omega g(x)((1-\lambda)u + \lambda v)^{1 - \gamma(x)} \, dx$$

$$\leq (1-\lambda) \max \left\{ \|u\|_{s, \Phi}^-, \|u\|_{s, \Phi}^+ \right\} + \lambda \max \left\{ \|v\|_{s, \Phi}^-, \|v\|_{s, \Phi}^+ \right\}$$

$$\leq (1-\lambda)\theta + \lambda\beta.$$
Hence, by Proposition 2.11, \( J \) is convex on \( W^s L_\Phi(\Omega) \). As the functional \( J \) is continuous, coercive and convex, it has a global minimum belonging to \( W^s L_\Phi(\Omega) \), which in turn becomes a solution to problem \((P_t)\). Let us denote 

\[
m = \inf_{u \in W^s L_\Phi(\Omega)} J(u),
\]

which is well-defined due to \((3.2)\). Now, applying the Lemma 3.4, we obtain that \( J(t \phi) < 0 \). If we set \( t \phi = u \) with \( \|u\|_\Phi < 1 \), we obtain that 

\[
m = \inf_{u \in W^s L_\Phi(\Omega)} J(u) < 0.
\]

On the other hand, if we take into account the definition of \( m \), there exists a minimizing sequence \((u_n)\) of \( W^s L_\Phi(\Omega) \) such that 

\[
m = \lim_{n \to \infty} J(u_n) < 0.
\]

Moreover, since \( J(u_n) = J(|u_n|) \) we may assume that \( u_n \geq 0 \). Due to the coercivity of \( J \), \( u_n \) must be bounded in \( W^s L_\Phi(\Omega) \) other wise we would have that \( J(u_n) \to +\infty \) as \( \|u_n\|_\Phi \to \infty \) which contradicts \((3.3)\). Since \( W^s L_\Phi(\Omega) \) is reflexive there exists a subsequence, not relabelled, and \( u_\ast \in W^s L_\Phi(\Omega) \) such that 

\[u_n \rightharpoonup u_\ast \text{ in } W^s L_\Phi(\Omega),\]

since \( J \) is continuous and convex on \( W^s L_\Phi(\Omega) \), it is weakly lower semi-continuous on \( W^s L_\Phi(\Omega) \). Therefore, 

\[
m \leq J(u_\ast) = \int_{\Omega^2} \Phi \left( \frac{|u_\ast(x) - u_\ast(y)|}{|x-y|^s} \right) d\mu + \int_{\Omega} \Phi(|u_\ast|) dx - \int_{\Omega} g(x) \frac{|u_\ast|^{1-\gamma(x)}}{1-\gamma(x)} dx = \liminf_{n \to \infty} J(u_n) = m,
\]

which means 

\[
m = J(u_\ast) = \inf_{W^s L_\Phi(\Omega)} J(u) < 0.
\]

\[\square\]

of Theorem 3.2 Since \( m = J(u_\ast) < 0 = J(0) \), it must be \( u_\ast \geq 0, u_\ast \neq 0 \). For \( \phi \in W^s L_\Phi(\Omega), \phi \geq 0 \) and \( t > 0 \), we obtain 

\[
0 \leq \liminf_{t \to 0} \frac{J(u_\ast + t \phi) - J(u_\ast)}{t} \leq \int_{\Omega^2} a \left( |h_{x,y}(u_\ast)| \right) h_{x,y}(u_\ast)h_{x,y}(\phi) d\mu + \int_{\Omega} a(|u_\ast|) u_\ast \phi dx - \limsup_{t \to 0} \int_{\Omega} g(x) \frac{(u_\ast + t \phi)^{1-\gamma(x)} - u_\ast^{1-\gamma(x)}}{1-\gamma(x)} dx.
\]

Or 

\[
\limsup_{t \to 0} \int_{\Omega} g(x) \frac{(u_\ast + t \phi)^{1-\gamma(x)} - u_\ast^{1-\gamma(x)}}{1-\gamma(x)} dx \leq \int_{\Omega^2} a \left( |h_{x,y}(u_\ast)| \right) h_{x,y}(u_\ast)h_{x,y}(\phi) d\mu + \int_{\Omega} a(|u_\ast|) u_\ast \phi dx.
\]

By the mean value theorem, there exists \( \theta \in (0, 1) \) such that 

\[
\int_{\Omega} g(x) \frac{(u_\ast + t \phi)^{1-\gamma(x)} - u_\ast^{1-\gamma(x)}}{1-\gamma(x)} dx = \int_{\Omega} g(x) (u_\ast + t \theta \phi)^{-\gamma(x)} \phi dx.
\]

Thus, we deduce that 

\[
(u_\ast + t \theta \phi)^{-\gamma(x)} \phi \geq 0, \quad \forall x \in \Omega,
\]

and 

\[
(u_\ast + t \theta \phi)^{-\gamma(x)} \phi \to u_\ast^{-\gamma(x)} \phi, \text{ as } t \to 0, \text{ a.e. } x \in \Omega.
\]
using Fatou’s Lemma in (3.8), we find that,

$$
\limsup_{t \to 0} \int_{\Omega} g(x) \frac{(u_* + t\phi)^{1-\gamma(x)} - u_*^{1-\gamma(x)}}{1-\gamma(x)} \, dx \\
\geq \liminf_{t \to 0} \int_{\Omega} g(x) \frac{(u_* + t\phi)^{1-\gamma(x)} - u_*^{1-\gamma(x)}}{1-\gamma(x)} \, dx \\
= \liminf_{t \to 0} \int_{\Omega} g(x) (u_* + t\phi)^{-\gamma(x)} \phi \, dx \geq \int_{\Omega} g(x) u_*^{-\gamma(x)} \phi \, dx \geq 0.
$$

(3.9)

Taking into account relations (3.7) and (3.9), we deduce that

$$
\int_{\Omega^2} a\left(|h_{x,y}(u_*)|\right) h_{x,y}(u_*)h_{x,y}(\phi) d\mu + \int_{\Omega} a(|u_*|)u_*\phi \, dx \\
- \int_{\Omega} g(x) u_*^{-\gamma(x)} \phi \, dx \geq 0, \quad \forall \phi \in W^sL_\phi(\Omega), \ \phi \geq 0.
$$

(3.10)

As above, we have that function $u_* \in W^sL_\phi(\Omega)$ satisfies

$$
(-\Delta)_s^* u_* + a(|u_*|)u_* \geq 0 \quad \text{in} \ \Omega,
$$

(3.11)

in the weak sense. Since $u_* \geq 0$ and $u_* \neq 0$, by the strong maximum principle for weak solutions (see [7]), we must have $u_*(x) > 0, \ \forall x \in \Omega$. Next, we show that $u_* \in W^sL_\phi(\Omega)$ satisfies (3.1). The proof below has been adapted from one given in [26]. For given $\delta > 0$, define $\Lambda : [-\delta, \delta] \to (-\infty, \infty)$ by $\Lambda(t) = J(u_* + tu_*)$. Then $\Lambda$ achieves its minimum at $t = 0$. Thus,

$$
\frac{d}{dt} \Lambda(t) \bigg|_{t=0} = \frac{d}{dt} J(u_* + tu_*) \bigg|_{t=0} = 0. \quad \text{Or}
$$

$$
\int_{\Omega^2} a\left(|h_{x,y}(u_*)|\right) h_{x,y}(u_*)h_{x,y}(\phi) d\mu + \int_{\Omega} a(|u_*|)u_*dx - \int_{\Omega} g(x) u_*^{-\gamma(x)} \, dx = 0.
$$

(3.12)

Indeed, let $\phi \in W^sL_\phi(\Omega)$, and define $\Theta \in W^sL_\phi(\Omega)$ such that $\Theta := (u_* + \varepsilon \phi)^+ = \max\{0, u_* + \varepsilon \phi\}, \varepsilon > 0$. Evidently, $\Theta \geq 0$. If we replace $\Theta$ both in (3.10) and (3.12), we have

$$
0 \leq \int_{\{u_* + \varepsilon \phi \geq 0\}} \int_{\{u_* + \varepsilon \phi \geq 0\}} a\left(|h_{x,y}(u_*)|\right) h_{x,y}(u_*)h_{x,y}(u_* + \varepsilon \phi) \, d\mu \\
+ \int_{\{u_* + \varepsilon \phi \geq 0\}} a(|u_*|) u_* (u_* + \varepsilon \phi) \, dx - \int_{\{u_* + \varepsilon \phi \geq 0\}} g(x) u_*^{-\gamma(x)} (u_* + \varepsilon \phi) \, dx \\
= \left( \int_{\Omega^2} - \int_{\{u_* + \varepsilon \phi < 0\}} \int_{\{u_* + \varepsilon \phi < 0\}} a\left(|h_{x,y}(u_*)|\right) h_{x,y}(u_*)h_{x,y}(u_* + \varepsilon \phi) \, d\mu \\
+ \left( \int_{\Omega} - \int_{\{u_* + \varepsilon \phi < 0\}} \right) a(|u_*|) u_* (u_* + \varepsilon \phi) \, dx \\
- \left( \int_{\Omega} - \int_{\{u_* + \varepsilon \phi < 0\}} \right) g(x) u_*^{-\gamma(x)} (u_* + \varepsilon \phi) \, dx
$$
we have

\[ = \int_{\Omega^2} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(u_*) d\mu + \int_{\Omega} a(|u_*|) u_*^2 dx - \int_{\Omega} g(x) u_*^{1-\gamma(x)} dx \]

\[ + \varepsilon \int_{\Omega^2} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(u_*) \cdot h_{x,y}(\varepsilon\phi) d\mu - \int_{\{u_* + \varepsilon\phi < 0\}} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(u_*) \cdot h_{x,y}(\varepsilon\phi) d\mu \]

\[ - \int_{\{u_* + \varepsilon\phi < 0\}} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(u_*) - \int_{\{u_* + \varepsilon\phi < 0\}} g(x) u_*^{1-\gamma(x)} (u_* + \varepsilon\phi) dx \]

\[ = \varepsilon \left( \int_{\Omega^2} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(\varepsilon\phi) d\mu \right) + \varepsilon \left( \int_{\Omega} a(|u_*|) u_* \phi dx - \int_{\Omega} g(x) u_*^{1-\gamma(x)} \phi dx \right) \]

\[ - \varepsilon \int_{\{u_* + \varepsilon\phi < 0\}} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(\varepsilon\phi) d\mu - \int_{\{u_* + \varepsilon\phi < 0\}} a(|u_*|) u_* \phi dx. \]

where

\[ \ell_0(\varepsilon) = \varepsilon \left( \int_{\Omega^2} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(\varepsilon\phi) d\mu \right) \]

\[ + \varepsilon \left( \int_{\Omega} a(|u_*|) u_* \phi dx - \int_{\Omega} g(x) u_*^{1-\gamma(x)} \phi dx \right) \]

\[ \ell_1(\varepsilon) = \varepsilon \int_{\{u_* + \varepsilon\phi < 0\}} \int_{\{u_* + \varepsilon\phi < 0\}} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(\varepsilon\phi) d\mu \]

\[ - \varepsilon \int_{\{u_* + \varepsilon\phi < 0\}} a(|u_*|) u_* \phi dx. \]

Considering that \( u_* > 0 \) and Lebesgue measure of the domain of integration \( \{u_* + \varepsilon\phi < 0\} \) tends to zero as \( \varepsilon \to 0 \), we have

\[ \int_{\{u_* + \varepsilon\phi < 0\}} \int_{\{u_* + \varepsilon\phi < 0\}} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(\varepsilon\phi) d\mu \to 0, \text{ as } \varepsilon \to 0. \]

and

\[ \int_{\{u_* + \varepsilon\phi < 0\}} a(|u_*|) u_* \phi dx \to 0, \text{ as } \varepsilon \to 0. \]

On the other hand, considering that \( a(\cdot) \in (0, \infty) \), we can drop the term

\[ - \int_{\{u_* + \varepsilon\phi < 0\}} \int_{\{u_* + \varepsilon\phi < 0\}} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(u_*) d\mu - \int_{\{u_* + \varepsilon\phi < 0\}} a(|u_*|) u_*^2 dx, \]

in \( \ell_0(\varepsilon) \) since it is negative. Thus, dividing \( \ell_1(\varepsilon) \) by \( \varepsilon \) and letting \( \varepsilon \to 0 \), we find

\[ \int_{\Omega^2} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(\varepsilon\phi) d\mu \]

\[ + \int_{\{u_* \phi < 0\}} a(|u_*|) u_* \phi dx - \int_{\Omega} g(x) u_*^{1-\gamma(x)} \phi dx \geq 0. \]  \hspace{1cm} (3.13)

Considering that \( \phi \in W^s L_\phi(\Omega) \) is arbitrary, \eqref{3.13} holds for \( -\phi \) as well. As a conclusion, we obtain

\[ \int_{\Omega^2} a(|h_{x,y}(u_*)|) h_{x,y}(u_*) \cdot h_{x,y}(\phi) d\mu \]

\[ + \int_{\{u_* |u_*| < 0\}} a(|u_*|) u_* \phi dx - \int_{\Omega} g(x) u_*^{1-\gamma(x)} \phi dx = 0 \]  \hspace{1cm} (3.14)
that is to say, $u_* \in W^s L_\varphi(\Omega)$ is a weak solution to problem $(P_\alpha)$. Additionally, since $J$ is coercive and bounded below on $W^s L_\varphi(\Omega)$, $u_*$ is a positive ground state solution to problem $(P_\alpha)$, i.e., a solution with minimum action among all nontrivial solutions. Additionally, since $J(u_*) < 0$ this solution has a negative energy level.

**Theorem 3.6.** Assume the hypotheses of Theorem 3.2 hold. Additionally, assume the following conditions hold:

(a1) There exists a real number $\alpha > 0$ such that $a(x,t) \geq \alpha > 0$ holds for any $t \in \mathbb{R}$. Then $u_* \in W^s L_\varphi(\Omega)$ is the unique solution to problem $(P_\alpha)$.

**Proof.** Let us assume $v_*$ is another solution to problem $(P_\alpha)$. Then, from (3.1), we have

\[\int_{\Omega} a((h_{x,y}(u_*))) h_{x,y}(u_* - v_*) \, d\mu + \int_{\Omega} a(|u_*|) u_* (u_* - v_*) \, dx
\]
\[\quad - \int_{\Omega} g(x) u_*^{-\gamma(x)} (u_* - v_*) \, dx - \int_{\Omega} a((h_{x,y}(v_*))) h_{x,y}(v_* - u_*) \, d\mu
\]
\[\quad - \int_{\Omega} a(|v_*|) v_* (u_* - v_*) \, dx + \int_{\Omega} g(x) v_*^{-\gamma(x)} (u_* - v_*) \, dx = 0.\]

Or

\[\int_{\Omega} a((h_{x,y}(u_*))) h_{x,y}(u_* - v_*) - a((h_{x,y}(v_*))) h_{x,y}(v_* - u_*) \, d\mu
\]
\[\quad + \int_{\Omega} (a(|u_*|) u_* - a(|v_*|) v_*) (u_* - v_*) \, dx \quad (3.15)
\]
\[= \int_{\Omega} g(x) \left( u_*^{-\gamma(x)} - v_*^{-\gamma(x)} \right) (u_* - v_*) \, dx. \quad (3.16)
\]

For $\alpha \in (0,1)$ and $x, y \geq 0$, we have the elementary inequality

\[(x^{-\alpha} - y^{-\alpha}) (x - y) \leq 0. \quad (3.17)
\]

On the other hand, by Lemma 2.4 given in [29], we have the following inequality: for any $k, l > 0$, there exists a positive constant $C(\delta)$, $\delta = \min \{1, a_0, k, l\}$, such that

\[(ka(|\xi|) \xi - la(|\eta|) \eta) \cdot (\xi - \eta) \geq C(\delta) \Phi(|\xi - \eta|) \quad \forall \xi, \eta \in \mathbb{R}^N, \quad (3.18)
\]

holds, provided that (a1) hold. Therefore, if we apply (3.18) and (3.17) to the lines (3.15) and (3.16) respectively, we deduce that

\[0 \leq C(\delta) \int_{\Omega} \Phi(h_{x,y}(u_* - v_*)) \, d\mu + \int_{\Omega} \Phi(|u_* - v_*|) \, dx \leq 0.
\]

Or

\[\Psi(u_* - v_*) = 0.
\]

As consequence of Proposition 2.3, we deduce that

\[0 \leq \min \left\{ \|u_* - v_*\|^{-}, \|u_* - v_*\|^{+} \right\} \leq \Psi(u_* - v_*),
\]

it follows that

\[\|u_* - v_*\| = 0.
\]

Therefore, we obtain $u_* = v_*$ in $W^s L_\varphi(\Omega)$, that is, $u_*$ is the unique solution to problem $(P_\alpha)$. □

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