Renormalization of an effective model Hamiltonian by a counter term

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An ill-defined integral equation for modeling the mass-spectrum of mesons is regulated with an additional but unphysical parameter. This parameter dependence is removed by renormalization. Illustrative graphical examples are given.

We focus on the integral equation
\[ M^2 - 4m^2 - 4\vec{k}^2 \phi(\vec{k}) = \int d^3\vec{k}' U(\vec{k}', \vec{k}) \phi(\vec{k}), \]
with the attractive kernel
\[ U(\vec{k}', \vec{k}) = -\frac{4}{3\pi^2} \frac{\alpha}{m} \left[ \frac{2m^2}{(\vec{k}' - \vec{k})^2} + 1 \right]. \]

It has two parameters \( \alpha, m \).

From a physical point of view the equation is a QCD-inspired effective one-body bound-state equation for modeling mesons with different constituent quark flavors \cite{1}. \( M^2 \) are the invariant mass squares of the physical mesons, while \( m = m_1 = m_2 \) is the effective mass of the quark and anti-quark. It takes this explicit form due to an over-simplification by the \( \uparrow\downarrow \)-model \cite{2}. If one Fourier transforms the kernel \( U \) to configuration space, the interaction potential consists of a long-ranged Coulomb-interaction and a short-ranged delta-interaction. It is this latter part, which generates all the well known trouble. In order to get reasonable solutions one has to regulate the high momentum transfers \( Q^2 = (\vec{k}' - \vec{k})^2 \). Therefore we substitute the number 1 by a regulating function, 1 \( \to R(\Lambda, Q) \), for which the soft cut-off
\[ R(\Lambda, Q^2) = \frac{\Lambda^2}{\Lambda^2 + Q^2} = \frac{\Lambda^2}{\Lambda^2 + (\vec{k}' - \vec{k})^2} \]
is chosen. In configuration space the short-ranged delta is now smeared out to a Yukawa interaction.

Since the regulator \( \Lambda \) is an additional but unphysical parameter, one has to renormalize the equation in order to restore the original problem in the limit \( \Lambda \to \infty \). For getting a greater transparent way we want to interpret the physical parameters \( \alpha \) and \( m \) as renormalization constants.

That we are dealing here with a bound-state equation on the light-cone, can not be seen explicitly. The above equation results from a variable transform in the longitudinal momentum fraction \( x \). For equal masses the relation is given by
\[ x(k_z) = \frac{1}{2} \left( 1 + \frac{k_z}{\sqrt{m^2 + \vec{k}_\perp^2 + k_z^2}} \right). \]
The relationship between the light-cone wavefunction \( \psi \) and the function \( \phi \) is given by
\[ \psi(x, \vec{k}_\perp) = \frac{\phi(\vec{k})}{\sqrt{x(1-x)}} \left[ 1 + \frac{\vec{k}^2}{m^2} \right]^+. \]
The function \( \phi \) has no physical meaning in the sense of a probability amplitude and is refered to as the reduced wavefunction.

After regularization one faces an integral equation with three parameters \( \alpha, m \) and \( \Lambda \). For simplification the functions \( \phi \) are restricted to the calculation of s-waves: \( \phi(\vec{k}) = \phi(|\vec{k}|) \) and by reasons explained below, we fix \( m = 406 \text{ MeV} \).

The spectrum of the bound-state mass squares \( M^2_i(\alpha, \Lambda) \) are then calculated numerically. For
the ground state $M_0^2(\alpha, \Lambda)$ this is displayed in Figure 1. A similar graph could be given for the first excited state $M_1^2(\alpha, \Lambda)$.

### 1. Example for local renormalization

The new parameter $\Lambda$ appears due to regularization. According to renormalization theory the spectrum may not depend on this formal parameter, thus we must require

$$\delta_\Lambda M^2(\Lambda) = 0. \quad (1)$$

To achieve this, we extend the model interaction by adding to $R$ a counter term $C(\Lambda, Q)$. We choose this function according to three criteria. First, the new function $\tilde{R} = R + C$ must again be a regulator. Second, we require that a zero is added for a particular value of $\Lambda$, say for $\Lambda = \mu$. Third, we require the first $\Lambda$-derivative of $\tilde{R}$ to vanish at $\Lambda = \mu$. The conditions are met by

$$C(\Lambda, Q) = -Q^2 \frac{(\Lambda^2 - \mu^2)}{((\Lambda^2 + Q^2)^2)}.$$ 

The kernel of the integral equation becomes then

$$U = -\frac{4 \alpha}{3 \pi^2 \Lambda} \cdot \frac{2m^2}{Q^2} \left( R + C \right).$$

The lowest eigenvalue of the corresponding integral equation is displayed in Figure 2 as function of $\alpha$ and $\Lambda$.

Based on the Hellmann-Feynman theorem, and

$$\frac{d\tilde{R}}{d\Lambda^2} = 2Q^2 \frac{(\Lambda^2 - \mu^2)}{(\Lambda^2 + Q^2)^2},$$

one expects that the derivative of the eigenvalues change sign at $\Lambda = \mu$. The numerical results in Figure 2 illustrate this very convincingly. In fact, for the numerical value $\mu = 1330$ MeV ($\mu/\Delta = 3.8$), the eigenvalues satisfies Eq.(1). The Hamiltonian is thus partially renormalized in the vicinity of $\Lambda \sim \mu$ for all $\alpha$.

### 2. Exact renormalization by counter terms

Above, we have constructed a local renormalization counter term in the region of $\Lambda/\Delta = 3.8$. Now our aim is to renormalize globally, i.e. for all possible $\Lambda$. This can be achieved by requiring that the $\Lambda^2$-derivatives of all orders have to vanish in the point $\Lambda = \mu$. Besides that, we will take up an easier and more straightforward way to derive a global counter term.
The regularization function $\tilde{R}$ is defined by:

$$\tilde{R}(\Lambda, Q) = R(\Lambda, Q) + C(\Lambda, Q),$$

with $R(\Lambda, Q) = \frac{\Lambda^2}{\Lambda^2 + Q^2}$.

Goal is to construct a counter term $C$ such that

$$C(\Lambda = \mu, Q) = 0,$$

and $\frac{d\tilde{R}}{d\Lambda^2} \bigg|_{\forall \Lambda} = 0$.

The requirements are satisfied by the differential equation

$$\frac{dC}{d\Lambda^2} = -\frac{dR}{d\Lambda^2} = -\frac{Q^2}{(\Lambda^2 + Q^2)^2}.$$

The boundary conditions are included by its integral form

$$C(\Lambda, Q) = -\int_{\mu}^{\Lambda} d\lambda^2 \frac{dR(\lambda^2, Q)}{d\lambda^2}$$

$$= \frac{\mu^2}{\mu^2 + Q^2} - \frac{\Lambda^2}{\Lambda^2 + Q^2}.$$

The regularization function $\tilde{R}$ becomes

$$\tilde{R}(\Lambda, Q) = \frac{\mu^2}{\mu^2 + Q^2},$$

which is to be used in the integral equation of the $\uparrow\downarrow$-model, i.e.

$$[M^2 - 4m^2 - 4\vec{k}^2] \phi(\vec{k}) = \int d^3 \vec{k}' \ U(\vec{k}', \vec{k}) \phi(\vec{k}'),$$

with $U(\Lambda, Q) = -\frac{4}{3\pi^2} \frac{4\alpha}{m} \left( \frac{2m^2}{Q^2} + \frac{\mu^2}{\mu^2 + Q^2} \right)$.

The equation is now manifestly independent of $\Lambda$ and the limit $\Lambda \to \infty$ can be taken trivially. In line with the theory of renormalization, the three parameters $\alpha$, $\mu$ and $m$ have to be determined by experiment, i.e. in principal three experimental values are needed to fix them.

### 3. Determining the parameters $\alpha$ and $\mu$

We fix the two unknown parameters $\alpha$ and $\mu$ by the experimental values of the ground and excited state mass of the pion. The pion has the mass $M_\pi = 140$ MeV. The precise empirical value of the excited pion mass is not known very well. We choose here $M_{\pi^*} = M_\rho = 768$ MeV for no good reason other than convenience. This large value is the reason for our comparatively large quark mass $m = 406$ MeV, which is fixed here once and for all.

Each of the two equations, $M_0^2(\alpha, \mu) = M_\pi^2$ and $M_1^2(\alpha, \mu) = M_{\pi^*}^2$, determine a function $\alpha(\mu)$, as illustrated in Figure 3. Their intersection point determines the solution, that is $\alpha_0 = 0.761$ and $\mu_0 = 1.15$ GeV, or $\mu_0/\Delta \sim 3.28$, as displayed in the figure.

Important to note is that the two contours in Figure 3 are intersecting only once. This crossing of the contours is unique, even for $\mu \to \infty$.

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**REFERENCES**

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