(2+1) dimensional stars

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Abstract

We investigate, in the framework of (2+1) dimensional gravity, stationary, rotationally symmetric gravitational sources of the perfect fluid type, embedded in a space of arbitrary cosmological constant. We show that the matching conditions between the interior and exterior geometries imply restrictions on the physical parameters of the solutions. In particular, imposing finite sources and absence of closed timelike curves privileges negative values of the cosmological constant, yielding exterior vacuum geometries of rotating black hole type. In the special case of static sources, we prove the complete integrability of the field equations and show that the sources’ masses are bounded from above and, for vanishing cosmological constant, generally equal to one. We also discuss and illustrate the stationary configurations by explicitly solving the field equations for constant mass–energy densities. If the pressure vanishes, we recover as interior geometries G"odel like metrics defined on causally well behaved domains, but with unphysical values of the mass to angular momentum ratio. The introduction of pressure in the sources cures the latter problem and leads to physically more relevant models.

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1 Introduction

They are several, good or not so good, reasons to pay attention to gravity in lower dimensions. In particular, as quoted by R. Jackiw \[1\], systems in a hot phase are phenomenologically described in a manifold of topology $\Sigma \times S^1$ instead of $\Sigma \times \mathbb{R}$; in the limit of infinite temperature the circle $S^1$ shrinks into a point and we are left with an Euclidean theory on $\Sigma$. Here we shall however not consider Euclidean, but only Lorentzian $(2+1)$ solutions. Yet, the latter may be relevant \[1\] for the description of large $1$–dimensional structures that seem to be observed in the Universe, such as strings and vortices, whose interactions are governed by $(2+1)$–gravity as far as we may ignore their extension in the third spacelike dimension.

On the other hand, pure gravity in $(2+1)$ dimensions is not as trivial as it seems at first sight. It has globally defined degrees of freedom whose physics is far from being totally understood (see for example ref. \[2\]) and may thus possess some relevant features of $(3+1)$–gravity. It seems therefore worthwhile to improve our understanding of the classical physics it defines before tackling the $(3+1)$ problem, all the more because $(2+1)$–gravity is undoubtedly simpler than $(3+1)$ \[3, 4, 5, 6\]. In absence of matter (vacuum solutions), the $(2+1)$ geometry is locally de Sitterian, anti–de Sitterian or Minkowskian. Globally however, things are more complicated. The full space–time appears, in general, as the quotient of a covering space and a discrete isometry group. For instance, as shown in \[7\], well–chosen identifications in anti–de Sitter (AdS) space yield the exterior geometry of rotating black holes in $(2+1)$ dimensions.

The present work was originally motivated by the finding \[8\] of a one parameter family of $(2+1)$–dimensional Gödel like metrics containing the AdS geometry. This suggested the possibility of truncating the former and matching it to the latter, so as to obtain non vacuum solutions of $(2+1)$–gravity without causal pathologies. A preliminary investigation has shown that such solutions indeed exist, and has led to a more systematic search for interior solutions connected to a space describing the exterior of black holes. These solutions are the $(2+1)$ analogues of stars (and will be so called hereafter), as they correspond to finite extended objects whose geometry matches an appropriate exterior geometry.

In section 2, we establish the field equations and junction conditions, assuming an abelian two parameter symmetry group of the metric, a perfect fluid as gravitational source and (anti) de Sitter exterior geometries.
In section 3, we first consider static configurations and show that, due to the simplicity of the geometry and the physics in (2+1) dimensions, they can be solved by quadratures, locally for positive values of the cosmological constant, and globally otherwise, extending previous works [3, 4, 9, 10]. In section 4, we consider rotating sources. For constant mass–energy density, we were able to obtain analytic solutions (contrary to what happens in (3+1) dimensions, where, to our knowledge, no analytical solutions for compact objects have been obtained). One class of interior solutions contains the aforementioned one parameter family of Gödel like geometries, the other being expressible in terms of elliptic functions. All the obtained solutions are causally well behaved; some are nevertheless physically unacceptable, as their angular momentum is too large compared to their mass. Indeed, such solutions would lead to naked causal singularities in case of collapse, unless centrifugal forces prohibit them to evolve into black holes. This point requires studying the singularity theorems in (2+1) dimensions, which is beyond the goal of this paper. Section 5 presents a few concluding words.

2 Stellar structure equations

We assume the gravitational field equation to be

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = \pi T_{\mu\nu} , \]

where we have conventionally fixed the gravitational coupling constant equal to \( \pi \) and introduced a cosmological constant \( \Lambda \). The metrics we shall consider will be supposed stationary and rotation invariant, i.e. admitting a 2-dimensional abelian isometry group. This means that in adapted coordinates, the metric components will depend only on a single variable. The interior geometry of the star is driven by the matter energy momentum tensor, assumed to be that of a perfect fluid:

\[ T_{\mu\nu} = (\sigma + p) \, u_\mu u_\nu + p \, g_{\mu\nu} , \]

where \( \sigma \) is the mass–energy density of the fluid constituting the star, \( p \) the pressure, and \( \mathbf{u} \) the 3-velocity of the fluid, satisfying \( \mathbf{u} \cdot \mathbf{u} = -1 \). Moreover, we shall also suppose that \( \sigma \) and \( p \) are positive definite and related by an equation of state:

\[ \sigma = \sigma(p) \quad , \quad \sigma \geq 0 \quad , \quad p \geq 0 \quad . \]
The interior metrics we shall consider hereafter take the general form:

\[ ds^2_{in} = -T[r]^2 dt^2 + B[r]^2 dr^2 + (Y[r]^2 - Z[r]^2)d\phi^2 - 2T[r]Z[r]dtd\phi \]

where the functions \( T[r], B[r], \) and \( Y[r] \) are assumed to be non-negative. The function \( B[r] \) is a gauge function that can be fixed at our convenience, at least locally. If we assume this metric to possess a symmetry axis (located on \( r = \bar{r} \)) without conical singularities, we have to impose the conditions:

\[
\lim_{r \to \bar{r}} \frac{Y[r]^2}{B[r]^2 (r - \bar{r})^2} = 1 \quad , \quad \lim_{r \to \bar{r}} \frac{Z[r]}{B[r]^2 (r - \bar{r})^2} = \omega
\]

where the constant \( \omega \) is the angular velocity of the star near its center of rotation.

The most general expression of the metric for the vacuum exterior geometry has been established in [7] (for \( \Lambda \neq 0 \)). It reads:

\[ ds^2_{ex} = -[(N^\perp)^2 - \rho^2 (N^\theta)^2]d\tau^2 + \frac{1}{(N^\perp)^2} d\rho^2 + \rho^2 d\theta^2 + 2\rho^2 N^\theta d\tau d\theta \]

with

\[ N^\theta = -\frac{J}{2\rho^2} + L_\infty \]

\[ (N^\perp)^2 = -M - \Lambda \rho^2 + \frac{J^2}{4\rho^2} \]

where \( M \) is the mass, \( J \) the angular momentum, and \( L_\infty \) an integration constant (denoted by \( N^\phi(\infty) \) in ref. [7]). This metric describes, for \( \Lambda < 0 \), \( (2+1) \)-dimensional black holes. Note that, as stressed in [7], \( \rho = 0 \) does not correspond to a line in the geometry (3), but to a 2-dimensional surface, a cylinder whose circular sections are lightlike, on which the Killing vector \( \partial_\theta \) becomes null but not zero. This explains why this metric does not present a conical singularity, despite the fact that the conditions (5) are not satisfied at \( \rho = 0 \).

The matching conditions on the surface, separating the regions where the internal and external geometries are defined, impose the equality of the intrinsic geometry and of the extrinsic curvature with respect to both geometries [12]. For all the solutions we shall discuss, the equation of these
surfaces will be of constant radial coordinate \( r = r_\ast \) in terms of coordinates \( \{t, r, \phi\} \) covering the interior region. Such surfaces are orbits of the 2 parameter isometry group, and their equations are thus also simply \( \rho = \rho_\ast \) in terms of the exterior coordinates. On the junction surface, the \( \{t, r\} \) and \( \{\tau, \rho\} \) coordinates are related by linear transformations, whereas the angular variables \( \phi \) and \( \theta \) may be identified, as they are both assumed to vary between 0 and \( 2\pi \). Hence, the coordinate transformation relations on the connection surface read:

\[
\phi = \theta, \quad \tau = T_0 \, t \quad \text{and} \quad \rho = R_0 (r + r_0) \quad , \tag{9}
\]

with \( T_0, R_0 \) and \( r_0 \) constants to be determined. The sign of \( T_0 \) will be supposed positive, the time \( t \) and \( \tau \) flowing in the same direction. The sign of \( R_0 \) will indicate what part (\( \rho > \rho_\ast \) or \( \rho < \rho_\ast \)) of the so-called exterior geometry will be glued at the junction surface. If \( R_0 < 0 \), we have to take the part \( \rho < \rho_\ast \) and obtain a closed space, presenting generically a singularity at \( \rho = 0 \), hidden or naked. In the following we shall limit ourselves to the cases \( R_0 > 0 \), except when stated otherwise. Note furthermore that we have used the space–time isometry group to fix at zero the arbitrary additive constants in the two first equalities; \( r_0 \) becomes meaningful once the remaining gauge freedom in the internal metric is removed by completely specifying the internal \( r \) coordinate.

Owing to the transformations (9), the equality of the induced metric on the junction surface implies the continuity of the interior and exterior metric components expressed in the same coordinates:

\[
g^{\text{in}}_{\mu \nu}[r_\ast] = g^{\text{ex}}_{\mu \nu}[\rho_\ast] \quad , \quad \mu, \nu \in \{t, r, \phi\} \quad . \tag{10}
\]

The equality of the extrinsic curvature with respect to the two space–time geometries reduces, in the particular case considered here (junction surface at \( r = r_\ast \)), to require the continuity of some of the metric component derivatives:

\[
\partial_r g^{\text{in}}_{\mu \nu}[r] \bigg|_{r = r_\ast} = \partial_r g^{\text{ex}}_{\mu \nu}[\rho[r]] \bigg|_{r = r_\ast} \quad , \quad \mu, \nu \in \{t, \phi\} \quad . \tag{11}
\]

Conditions (10, 11) imply seven equations that must be satisfied on the surface of the star \( r = r_\ast \):

\[
T_\ast^2 = \left[ - M - \Lambda \rho_\ast^2 + JL_\infty - L_\infty^2 \rho_\ast^2 \right] T_0^2 \quad , \tag{12}
\]
\[ \frac{T_* T'_*}{R_0} = - (\Lambda + L_\infty^2) \rho_* T_0^2 , \]  
\[ B_*^2 = \frac{R_0^2}{- M - \Lambda \rho_*^2 + \frac{J^2}{4 \rho_*^2}} , \]  
\[ \frac{1}{R_0} (Y_* Y'_* - Z_* Z'_*) = \rho_* , \]  
\[ \frac{1}{R_0} (T_* Z_* + T'_* Z'_*) = -2 \rho_* L_\infty T_0 . \]

These seven equations fix the six unknowns \{L_\infty, \rho_*, T_0, R_0, J, M\}:

\[ L_\infty = - \frac{(T Z)'}{2 B T Y} \bigg|_{r=r_*} , \]  
\[ \rho_* = (Y^2 - Z^2)^{1/2} \bigg|_{r=r_*} , \]  
\[ T_0 = \frac{2 B T Y}{(Y^2 - Z^2)'} \bigg|_{r=r_*} , \]  
\[ R_0 = \frac{(Y^2 - Z^2)'}{2 (Y^2 - Z^2)^{1/2}} \bigg|_{r=r_*} , \]  
\[ J = \left( \frac{Y^2 - Z^2}{T Z} \right)' \frac{T Z^2}{B Y} \bigg|_{r=r_*} , \]  
\[ M = - \frac{R_0^2}{B^2} - \Lambda \rho_*^2 + \frac{J^2}{4 \rho_*^2} , \]

where we have used the positivity assumptions of \( R_0 \) and \( T_0 \). We obtain moreover the additional consistency relation:

\[ \frac{T' Y'}{B^2 T Y} + \frac{1}{4} \left[ \frac{T}{B Y} \left( \frac{Z}{T} \right)' \right]^2 \bigg|_{r=r_*} + \Lambda = 0 . \]

In order to solve the Einstein equations (1), we found useful to introduce the triad:

\[ \theta^0 = T[r] \, dt + Z[r] \, d\phi \quad , \quad \theta^1 = B[r] \, dr \quad , \quad \theta^2 = Y[r] \, d\phi \quad , \]

where we have used the positivity assumptions of \( R_0 \) and \( T_0 \). We obtain moreover the additional consistency relation:

\[ \frac{T' Y'}{B^2 T Y} + \frac{1}{4} \left[ \frac{T}{B Y} \left( \frac{Z}{T} \right)' \right]^2 \bigg|_{r=r_*} + \Lambda = 0 . \]
such that:
\[ ds^2_m = -(\theta^0)^2 + (\theta^1)^2 + (\theta^2)^2 \]

(27)

With respect to the basis \([24]\), the Einstein tensor has 4 non-trivially vanishing components:

\[
G^0_0 = \frac{1}{BY} \left( \frac{Y'}{B} \right)' - \frac{3}{4} \left[ \frac{T}{BY} \left( \frac{Z}{T} \right)' \right]^2, \tag{28}
\]

\[
G^1_1 = \frac{T'Y'}{B^2TY} + \frac{1}{4} \left[ \frac{T}{BY} \left( \frac{Z}{T} \right)' \right]^2, \tag{29}
\]

\[
G^2_2 = \frac{1}{BT} \left( \frac{T'}{B} \right)' + \frac{1}{4} \left[ \frac{T}{BY} \left( \frac{Z}{T} \right)' \right]^2, \tag{30}
\]

\[
G^2_0 = \frac{1}{2BT^2} \left[ \frac{T^3}{BY} \left( \frac{Z}{T} \right)' \right]' \tag{31}
\]

Our symmetry assumptions imply that the fluid’s velocity \( u \) has to be a linear combination of the two Killing vector fields. Accordingly, the corresponding one-form \( u \) may be written:

\[
u = \cosh[v] \theta^0 + \sinh[v] \theta^2 \]

(32)

where \( v \) may depend on \( r \). Using the radial \( G^1_1 \)-Einstein equation,

\[
G^1_1 = \pi p - \Lambda \]

(33)

we see that the consistency equation \([25]\) is satisfied if and only if the pressure vanishes at the surface of the star. This condition is a quite natural requirement from a physical point of view. Indeed, a discontinuity of the pressure is incompatible with the hydrostatic equilibrium equation of the fluid (the Bianchi identity):

\[
p' = -\frac{p + \sigma}{TY} [\cosh^2[v]TY' - \sinh^2[v]TY' - \cosh[v] \sinh[v]T^2 \left( \frac{Z}{T} \right)'] \]

(34)

In what follows, we shall build explicit stellar models, under the simplifying ansatz that the fluid’s velocity one-form is aligned with \( \theta^0 \), i.e. \( v = 0 \), but not necessarily integrable. We shall consider both the static (\( u \) integrable) and stationary (\( u \) non integrable) cases. Under this assumption, eq.(34) simplifies into:

\[
p' = -\frac{T'}{T}(\sigma + p) \]

(35)
This equation and the inequalities (3) imply that, if the star possesses a surface, \( T' \) must be positive on it. As a consequence, under the assumption of the positivity of \( R_0 \), we obtain the bound:

\[
L_\infty^2 \leq -\Lambda,
\]

and thus \( \Lambda \leq 0 \). This shows that finite perfect fluid structures of the type described here need an AdS or Minkowskian exterior space, unless \( R_0 < 0 \), in which case the surrounding space is closed.

To close the general considerations of this section, let us remind that in the AdS case, the metric (8) presents black hole type horizons located at:

\[
\rho_{H\pm}^2 = \frac{M}{-2\Lambda} \left( 1 \pm \sqrt{1 + \frac{J^2 \Lambda}{M^2}} \right),
\]

if \( M > 0 \) and \(-\Lambda J^2/M^2 < 1\). It also presents a surface of infinite redshift, an ergosphere, at:

\[
\rho_{\text{erg}}^2 = \frac{M - JL_\infty}{-\Lambda - L_\infty^2}.
\]

Equation (12) shows that the radius of the junction surface is always in the static region, i.e. in a region of finite redshift. The star models considered here can thus never present an ergosphere (nor a blackhole horizon), if \( R_0 \) is positive. The origin of this constraint lies in our choice of the interior metric (4), for which \( g_{tt} \) is explicitly negative.\(^5\)

### 3 Static stars

The most general (2+1)-dimensional, static, rotation invariant metric can be written as:

\[
ds^2_{\text{in}} = -T^2[r] \, dt^2 + B^2[r] \, dr^2 + Y^2[r] \, d\phi^2,
\]

with the range of the variable \( \phi \) fixed to \([0, 2\pi]\). The Einstein equations (1) become, taking this ansatz and eqs (28–31) into account, and with as matter

\(^5\)Lorentzian rotating solution with \( g_{tt} > 0 \) could be obtained by exchanging the definitions of \( \theta^0 \) with \( \theta^2 \) in eq. (26), but they never present centers of rotation.
source a perfect fluid at rest [eqs (2 and 32) with \( v = 0 \)]:

\[
G^0_0 \equiv \frac{1}{BY} \left( \frac{Y'}{B} \right)' = -\pi \sigma - \Lambda \equiv -S \quad , \quad (40)
\]

\[
G^1_1 \equiv \frac{T'Y'}{B^2TY} = \pi p - \Lambda \equiv P \quad , \quad (41)
\]

\[
G^2_2 \equiv \frac{1}{BT} \left( \frac{T'}{B} \right)' = \pi p - \Lambda \equiv P \quad , \quad (42)
\]

with as initial conditions:

\[
T[0] = 1 \quad , \quad B[0] = 1 \quad , \quad Y[0] = 0 \quad \text{and} \quad Y'[0] = 1 \quad , \quad (43)
\]

in order to avoid conical singularities (eq. 5) and to maintain the central pressure \( p[0] \) finite. By equating the lefthand sides of eqs (41, 42) and using the initial conditions (43), we obtain:

\[
T' = P[0] BY \quad . \quad (44)
\]

If the function \( Y[r] \) is constant, \( S \) and \( P \) vanish and the space is flat. So, without loss of generality, we shall hereafter assume that \( Y[r] \) is not constant. Mimicking the Oppenheimer–Volkoff [13] integration of the stellar structure equations in (3+1) dimensions, we may define a radial coordinate \( Y[r] = r \).

We immediately obtain from eqs (40 and 43):

\[
B^{-2}[r] = 1 - 2\pi \int^r_0 \sigma(x)x dx - \Lambda r^2 \quad . \quad (45)
\]

The Bianchi identity (35) becomes:

\[
\frac{dP}{dr} = -rP(P + S)B^2 \quad . \quad (46)
\]

This equation shows that, unless \( \Lambda = 0 \), we cannot have a static, pressureless star. Indeed, a cloud of dust cannot remain static in an expanding universe, driven here by a cosmological constant. Using eq. (40), written in the form

\[
\frac{dB}{B} = r S B^2 dr \quad , \quad (47)
\]

we obtain from eq. (46):

\[
\left( \frac{1}{P + S} - \frac{1}{P} \right) dP = \frac{dB}{B} \quad . \quad (48)
\]
In order to go ahead, we now use the equation of state (3) describing the model. It allows us to integrate eq. (48), which gives:

\[
\frac{B[r]}{P[r]} = \frac{P[0]}{W[0]},
\]

where \(W[r] = w[p[r]]\) and \(w[p]\) is the index of the fluid (the thermodynamical temperature in case of radiation) defined by:

\[
w[p] = w[p_0] \exp \int_{p_0}^p \frac{dq}{\sigma(q) + q}.
\]

The \(G_1^-\)–Einstein equation (41) can be integrated in the same way:

\[
\frac{dT}{T} = \mathcal{P} B^2 r dr = - \frac{d\mathcal{P}}{\mathcal{P} + \mathcal{S}} = - \frac{dW}{W}.
\]

The solution of this equation yields, using (43), the well known Tolman thermal equilibrium condition [14]:

\[
T[r]W[r] = W[0],
\]

Equations (49, 50 and 52) give a parametric representation of the solution as a function of a thermodynamical variable, let say the pressure. A geometrical parametrisation, in terms of the radial coordinate \(r\), is obtained from eqs (49 and 51), which combine into:

\[
\frac{\mathcal{P}}{(\mathcal{P} + \mathcal{S})} \frac{d\mathcal{P}}{W^2} = \frac{\mathcal{P}}{W^3} \frac{dW}{W[0]} = - \left( \frac{\mathcal{P}[0]}{W[0]} \right)^2 \left( \frac{r^2}{2} \right) \frac{d}{(r/2)}.
\]

and furnish, after integration, the implicit dependence of \(p\) as a function of \(r\).

Let us emphasize that we have here only discussed the local integration of the interior field equations. We have still to consider their domain of validity and the matching of the internal solutions with appropriate (AdS, flat or de Sitter) external spaces.

### 3.1 AdS-like stars

We first consider the AdS case \(\Lambda < 0\), already discussed in detail by Cruz and Zanelli (CZ) [10]. As \(p \geq 0\), we have \(\mathcal{P}[r] > 0\), and thus \(Y'[r] \neq 0\).
0. Accordingly, the choice of the $r$ variable is valid inside the whole star. Moreover, eq. (53) shows that the radius of the star will be finite or infinite according to the behavior of the index $w$ near $p = 0$. If $\lim_{p \to 0} w[p] = 0$, the star will have an infinite radius. For instance, if the star is of pure radiation ($\sigma = 2$, $w \propto p^{1/3}$), we obtain, following the previous scheme:

$$T[r] = \frac{p[0]^{1/3}}{p[r]^{1/3}}, \quad (54)$$

$$B[r] = \frac{p[r]^{1/3}}{p[0]^{1/3}} \frac{\pi p[0] - \Lambda}{\pi p[r] - \Lambda}, \quad (55)$$

and after elementary integration:

$$\left(\frac{\Lambda + 2\pi p[0]}{p[0]^{2/3}}\right) - \left(\frac{\Lambda + 2\pi p[r]}{p[r]^{2/3}}\right) = \left(\frac{\pi p[0] - \Lambda}{p[0]^{1/3}}\right)^2 r^2, \quad (56)$$

which reduces to a cubic equation for $p[r]^{1/3}$. Otherwise, when $w[0] \neq 0$, we shall denote by $r_\star$ the finite value of the radial coordinate of the connection surface and by the same subscript ($\star$) the values that the various functions ($P, T, B, Y, T', Y', \ldots$) take on it. As $B$ must be positive on $r_\star$, we obtain the CZ upper bound for the mass [10]:

$$m_\star \equiv 2\pi \int_0^{r_\star} \sigma(x)dx \leq 1 - \Lambda r_\star^2. \quad (57)$$

For a static star, the angular momentum $J$ and the integration constant $L_\infty$ obviously vanish and the exterior geometry around the star is given by the metric:

$$ds_{ex}^2 = -(-M - \Lambda \rho^2) d\tau^2 + \frac{d\rho^2}{-M - \Lambda \rho^2} + \rho^2 d\phi^2. \quad (58)$$

The matching conditions (eqs [49, 54] and eqs [45 and 57]) imply the mass relation:

$$M = m_\star - 1, \quad (59)$$

already given in ref. [10]. This equation illustrates the fact that the exterior mass parameter $M$ is always larger than $-1$ (under the assumption $\sigma > 0$), with the extreme value $M = -1$ corresponding to the usual, singularity–free, AdS space [7].
However, we would like to stress that we do not find a mass gap between $M = -1$ and $M \geq 0$, contrary to what happens for black holes. Note also that eqs (57) and (59) directly imply the absence of blackhole horizons for $M > 0$, the horizon $\rho_{H+}$ (eq. 37) corresponding to a surface located inside the star:

$$\rho_{H+}^2 = \frac{M}{-\Lambda} < \rho_\star^2 = r_\star^2 \quad . \quad (60)$$

We see furthermore that an observer at rest on the surface of such a star (an observer whose world line is $\phi = Cst$ and $r = r_\star$) follows an accelerated trajectory, with an invariant acceleration:

$$a = \sqrt{a_{\alpha}a^{\alpha}} = \left| \frac{T_r'}{B_\star T_\star} \right| = \frac{-\Lambda r_\star}{\sqrt{-M - \Lambda r_\star^2}} \quad , \quad (61)$$

directed in the direction of increasing $r$. It is interesting to notice that, while the gravitational interaction does not propagate in (2+1) dimensions, the acceleration of an observer at the surface of the star depends both on $\Lambda$ and $M$. More precisely, as the radial coordinate $r_\star$ has a geometrical meaning, this equation illustrates the fact that the acceleration depends on the presence or not of a star. Indeed, though gravity does not propagate, it manifests itself by holonomy effects [1], affecting here the definition of $r_\star$ as lengths of circles around the star.

### 3.2 de Sitter-like stars

As shown at the end of section 2, finite objects embedded in a de Sitter environment require $R_0$ to be negative. This implies that the full space is closed, the so-called exterior region being described by a metric like (58) but with $\Lambda > 0$ and $\rho \in [0, \rho_\star]$. In other words, there is no concept of interior or exterior regions for such spaces; all constant $t$–time sections are compact.

To discuss these models, we skip to the gauge where $B = 1$ and perform the change of radial coordinate : $\xi = \int B[r]dr$. Indeed, the function $Y[\xi]$ is here not everywhere increasing inside the star and the previous parametrisation, $Y[\xi] = r$, remains only locally valid. In the $B = 1$ gauge, the Einstein equations (40–42) become:

$$\frac{Y''}{Y} = -(\pi \sigma + \Lambda) = -S \quad , \quad (62)$$
\[ T \ddot{Y} = \pi p - \Lambda = +P, \quad (63) \]
\[ \frac{T''}{T} = \pi p - \Lambda = +P, \quad (64) \]

with the same initial conditions (43). The first equation (62) shows that \( Y[\xi] \) is a convex function, bounded by \( \Lambda^{(-1/2)} \sin[\Lambda^{1/2} \xi] \) on its physical (i.e. \( Y \geq 0 \)) interval of definition, which is included in the interval \([0, \pi \Lambda^{1/2}]\). This justifies a posteriori our gauge choice. Indeed, using eqs (44 and 63), we see that \( Y[\xi] \) reaches its maximum when \( P = 0 \); its derivative vanishes and starts to become negative. But as \( \Lambda > 0 \), the pressure is still positive and we have thus not yet reached the star’s surface.

Once the solution \( Y[\xi] \) of eq. (62) is obtained, we may integrate eq. (44) which gives:
\[ T[\xi] = 1 + P[0] \int_0^\xi Y(x) \, dx. \quad (65) \]
This equation, introduced into the junction condition (13), yields:
\[ T' T_* = P[0] Y_* T_* = -\Lambda R_0 T_0^2 \rho_* \quad , \quad (66) \]
implying \( P[0] > 0 \). The function \( T[\xi] \) is thus increasing. Combining eqs (53 and 14), we now obtain:
\[ p' = -P[0](\sigma + p) \frac{Y}{T} \quad . \quad (67) \]
The pressure hence decreases with \( \xi \) and it depends on the specific form of the equation of state whether or not a singularity occurs.

Solving the junction conditions (12–16), taking care of the sign of the radial derivative, which is opposite to the one used in the previous case, we obtain:
\[ T_0 = -\frac{T_*}{Y_*^2} \quad , \quad R_0 = -Y'_* \quad , \quad \rho_* = Y_* \quad , \quad M = -\Lambda Y_*^2 - Y'^2 < 0 \quad . \quad (68) \]
This closed space possesses two centers of rotation. One is located at \( \xi = 0 \), the other at \( \rho = 0 \). By assumption (eq. 4), there is no conical singularity on the first axis, but in general the second suffers from an angular defect of:
\[ \delta = (1 - \sqrt{-M})2\pi \quad . \quad (69) \]
To be complete, note that if the initial conditions are such that $\mathcal{P}[0] < 0$, the function $T[\xi]$ is decreasing and $p'$ is positive, so that $p$ cannot vanish. We have thus generically a singularity there where $T[\xi]$ vanishes (see eq. [52]), except if $Y[\xi]$ re–vanishes first, and moreover if at this second zero $Y'[\xi]$ is equal to $-1$, in which case we obtain a closed space with 2 antipodal centers of rotation and free of conical singularities.

So we conclude that, in general, there do not exist static stars whose matter satisfies the phenomenological conditions (2, 3), in a singularity–free (2+1)–dimensional space–time with positive cosmological constant.

By way of illustration, suppose the energy density $\sigma$ constant, so that $S$ is a positive constant. We easily obtain the expression of the interior metric components and the radial dependence of the pressure:

\[
Y[r] = \frac{\sin(\sqrt{S} r)}{\sqrt{S}} ,
\]

\[
T[r] = 1 + \frac{\mathcal{P}[0](1 - \cos[\sqrt{S} r])}{S} ,
\]

\[
p[r] = \frac{\mathcal{P}[0] + \sigma}{T[r]} - \sigma .
\]

The space admitting such a geometry appears as the product of a 2–sphere of radius $S^{(-1/2)}$ with a line of time. The surface of vanishing pressure is given by the solution of

\[
\cos[\sqrt{S} r_{\star}] = 1 - \frac{S \mathcal{P}[0]}{\sigma \mathcal{P}[0]} ,
\]

which, according to the above discussion, requires $\mathcal{P}[0] > 0$ to exist. Using eq. [63], we obtain for this model

\[
M = -1 + \frac{\pi \sigma}{\pi \sigma + \Lambda} \sin^2[\sqrt{S} r_{\star}] ,
\]

which shows that the conical singularity is unavoidable.

### 3.3 Minkowskian–like stars

If $\Lambda = 0$, we may either use the gauge $B = 1$ or choose $Y[\xi] = r$ to discuss the global properties of the internal solutions. The function $Y[\xi]$ is indeed everywhere increasing inside the star. Let us first assume that the pressure is
not identically zero; the maximum of \( Y[ξ] \) then coincides with the vanishing of the pressure, occurring just on the surface of the star. Using eqs (50,53), it is easy to see that the radial coordinate \( r^⋆ \) of the surface of the star is infinite when the index of the fluid vanishes with the pressure as \( p^α \) with \( α > 1/2 \). This criterion generalizes the results obtained for polytropic fluids in ref. [4].

As the curves \( (ϕ = Cst, \ r = r^⋆) \) are accelerated trajectories, it is impossible to match these interior geometries across a surface \( ρ = Cst \) to a flat space of metric:

\[
ds_{ex}^2 = -dt^2 + dρ^2 + dy^2 \quad ,
\]

expressed in minkowskian coordinates. The exterior metrics that continue the interior geometries are given by:

\[
ds_{ex}^2 = -ρ^2dτ^2 + dρ^2 + dy^2 \quad ,
\]
corresponding to flat geometries written in Rindler coordinates, in a \((2+1)\)-dimensional Minkowski space whose spacelike direction \( y \) has been rendered periodic by identifying \( y \) with \( y + 2πY^⋆ \). This periodicity condition results from the matching conditions for the \( g_{ϕϕ} \) component of the metrics, while the continuity of their derivatives reduces to \( p^⋆ = 0 \). The junction conditions due to the \( g_{tt} \) metric components are

\[
T^⋆ = T_0ρ^⋆ \quad , \quad T'^⋆ = T_0 \quad .
\]

An unexpected result that emerges from the hydrostatic equilibrium equation (46) is that, assuming a (reasonable) equation of state, a star with pressure and finite radius has always its mass parameter \( m^⋆ \) equal to 1 if \( Λ = 0 \). Indeed, if \( B^{-2}[r] \) does not vanish with \( p \) at the star’s surface, Cauchy theorem applied to eq. (46) implies that \( p = 0 \) everywhere in the star. As a consequence, \( B^⋆^{-2} = 0 \) and, using eqs (15 and 51), we get \( m^⋆ = 1 \) and, by virtue of eq.(53), \( M = 0 \). The metric (58), with \( Λ = 0 \), becomes thus singular, which confirms the necessity of using the metric (75). This universal value of the mass can also be directly obtained from the Einstein equation (62) as follows:

\[
m^⋆ = π \int_0^{r^2} σdr^2 = π \int_0^{Y^2} σdY^2 = -2 \int_0^{Y^⋆} Y''dY = - \int_1^0 dY'^2 = 1 \quad .
\]

This result has already been noticed in the special case of a uniform density star in [3] and for polytropic fluids in [4], and demonstrated in a different
way in [2]. But, contrary to what is claimed there, we see from eq. (77) that the surface of the star follows an accelerated trajectory with respect to the exterior flat space. Physically, this is due to the fact that in absence of gravitational interaction, a fluid shearing an internal pressure has to expand. Moreover, an observer at the surface of such a star feels an acceleration directed towards the center of the star, of magnitude \[ a = \frac{\xi^{-1} - T_z}{T^\prime} \]. If this observer\(^6\) jumps from the surface of the star, he will follow an inertial geodesic trajectory in the surrounding flat space. Nevertheless, he will fall again on the accelerated surface of the star because he will be recaptured by it. This is actually also what happens for observers in (3+1)–dimensional Schwarzschild metric except that in (2+1) dimensions the escape velocity is the velocity of light.

If the star is pressureless (a special case also studied in ref. [4]), the function \( T[\xi] \) is constant and \( \sigma \) becomes an arbitrary function of \( \xi \). Indeed, as there is no gravitational interaction between the particles of the the dust constituting the star, they may be distributed with an arbitrary radial dependence. If the convex function \( Y[\xi] \) is such that at the surface of the star \( Y\prime \) is still non negative, the exterior solution is given by the flat space geometry:

\[
 ds^2_{\text{ex}} = -(1 - m_{\star}) \, dt^2 + \frac{dr^2}{1 - m_{\star}} + \rho^2 \, d\phi^2 \quad \text{with} \quad \rho \geq Y_{\star} . \tag{79}
\]

This geometry presents an angular deficit \( \delta = (1 - \sqrt{1 - m_{\star}})2 \pi \), i.e. \( m_{\star} = 1 \) remains the maximal mass allowed for (a circularly symmetric) object in this kind of universe, if we want to preserve its locally Minkowskian character. Note that if \( Y\prime \) is negative, the interior space matches the cylindrical portion \( \rho < Y_{\star} \) of a flat space, presenting a conical singularity on the axis \( \rho = 0 \).

\section{Rotating star}

When the star is uniformly rotating, its metric stops being static but remains stationary. In the B=1 gauge, it can be written:

\[
 ds^2_{\text{m}} = -T[r]^2 \, dt^2 + dr^2 + (Y[r]^2 - Z[r]^2) \, d\phi^2 - 2T[r]Z[r] \, dt \, d\phi \quad , \tag{80}
\]

with the condition \( Z[r]T[r] \neq 0 \). Under the assumption \( \Theta \) that the fluid 3-velocity is not tilted with respect to the frame \( \Theta \), Einstein’s equations

\(^6\)For more information about the life of inhabitants in a (2+1)–dimensional world, see ref. [15]
impose that $G^2_0$ (31) be equal to zero. This implies that $Y[r]$ is of the form:

$$Y[r] = \frac{1}{c} T[r]^3 \left( \frac{Z[r]}{T[r]} \right)' ,$$

where $c$ is a constant. When a center of rotation exists, eq. (5) yields:

$$\omega = \frac{c}{2} ,$$

leading to interpret the constant $c$ as the angular velocity at the center of the star. The remaining three non-trivially satisfied Einstein equations are:

$$G^0_0 \equiv -\frac{3c^2}{4T^4} + \frac{T'}{T} \left[ \frac{(T^2(Z/T)')'}{T^2(Z/T)'} \right] + \frac{TZ'' - T''Z}{TZ' - T'Z} = -\pi \sigma - \Lambda ,$$  \hspace{1cm} (83)

$$G^1_1 \equiv \frac{c^2}{4T^4} + \frac{T'}{T} \left[ \frac{(T^3(Z/T)')'}{T^3(Z/T)'} \right] = \pi p - \Lambda ,$$  \hspace{1cm} (84)

$$G^2_2 \equiv \frac{c^2}{4T^4} + \frac{T''}{T} = \pi p - \Lambda .$$  \hspace{1cm} (85)

Our choice of the velocity field $u$ imposes the equality of $G^1_1$ and $G^2_2$, from which we deduce that:

$$\frac{T''}{T} = \frac{T'}{T} \left[ \frac{(T^3(Z/T)')'}{T^3(Z/T)'} \right] .$$  \hspace{1cm} (87)

This is satisfied by two types of solutions,

$$T[r] = Cst \quad \text{or} \quad (88)$$

$$Z[r] = \alpha T[r] + \beta T^{-1}[r] ,$$  \hspace{1cm} (89)

where $\alpha$ and $\beta$ are constants. They are hereafter referred to as dust-like and pressurized (star) models, respectively. These interior solutions can match various “exterior” spaces, but, like for the static case, only those of AdS type avoid singularities. We shall thus limit the subsequent analysis to $R_0$ positive and $\Lambda$ negative. Rotating perfect fluid solutions with $\Lambda = 0$, which exemplify the occurrence of pathologies, are given in ref. [16].
4.1 Dust–like models

These solutions are characterized by eq. (88) and hence, without loss of generality, by \( T[r] = 1 \). The pressure \( p \) is thus constant, and we put it equal to zero in order that the star admits a boundary. Let us emphasize that this model implies \( \Lambda = -c^2/4 < 0 \); a negative cosmological constant stabilizes the system by counterbalancing the centrifugal force. The remaining function \( Z[r] \) has to satisfy eq. (83) which reduces to:

\[
Z'''' + (\pi \sigma + 4\Lambda)Z' = 0.
\] (90)

This means that \( Z[r] \) either defines the pressureless matter (dust) repartition, or, conversely, is itself defined by the matter repartition. If we assume the existence of a center of rotation at \( r = 0 \), the regularity conditions (3) imply the initial conditions for eq. (90):

\[
Z[0] = 0, \quad Z'[0] = 0, \quad Z''[0] = 2\omega = c = 2\sqrt{-\Lambda}.
\] (91)

As the pressure vanishes everywhere, the consistency condition (23) is identically satisfied by virtue of the Einstein equations. The radius \( r_* \) of the surface of the star remains a free parameter. Applying the general matching conditions (eqs 19–24) to this specific case, we find all parameters of the exterior metric as functions of \( Z[r] \) and its first two derivative evaluated on \( r_* \), except the integration constant which is constant:

\[
L_\infty = -\sqrt{-\Lambda},
\] (92)

and reaches its upper bound allowed by eq. (36). This equality implies, by virtue of eq. (12), the additional condition:

\[
-\Lambda \frac{J^2}{M^2} > 1,
\] (93)

when \( M > 0 \). This relation severely limits the physical relevance of all dust–like star solutions. Indeed, in case of collapse, a naked singularity occurs, whatever the sign of \( M \).

Nevertheless, we find it worthwhile to illustrate these models in the special case \( \sigma = Cst \), if only because we recover Gödel like geometries analogous to those considered in [8, 17, 18, 19]. The function \( Z \) is then obtained from the Einstein equation (90):

\[
Z[r] = \begin{cases} 
  g \exp[r/a] + h \exp[-r/a] + k & \text{if } a^{-2} = -\pi \sigma - 4\Lambda > 0, \\
  g \sin[r/a] + h \cos[r/a] + k & \text{if } a^{-2} = +\pi \sigma + 4\Lambda > 0.
\end{cases}
\] (94)
Changing, if necessary, $r$ into $r + Cst$, the metric (80) simplifies into four forms:

$$ds^2_{in} = -dt^2 + dr^2 + \gamma^2 \left\{ \begin{array}{l}
\sinh^2 \left[ \frac{r}{a} \right] - a^2 c^2 (\cosh \left[ \frac{r}{a} \right] + \kappa)^2 \\
\exp \left[ \frac{2r}{a} \right] - a^2 c^2 (\exp \left[ \frac{r}{a} \right] + \kappa)^2 \\
\cosh^2 \left[ \frac{r}{a} \right] - a^2 c^2 (\sinh \left[ \frac{r}{a} \right] + \kappa)^2 \\
\sin^2 \left[ \frac{r}{a} \right] - a^2 c^2 (\cos \left[ \frac{r}{a} \right] + \kappa)^2 \\
\end{array} \right\} d\phi^2 
$$

The first three metrics correspond to low density stars, i.e. $\pi \sigma < -4\Lambda$, with the product $gh$ being $>0$, $=0$ or $<0$, respectively. The last metric corresponds to a high density star: $\pi \sigma > -4\Lambda$. Locally, the three metrics labeled (a,b,c) are equivalent, as they have the same Einstein tensor (while the coordinate transformations\footnote{If we accept to consider complex coordinate transformations, then we may go from the metric (B5 a) to (B5 c) by the following change: $r \mapsto r + i \frac{\pi}{a}, \phi \mapsto i \phi$ and $\kappa \mapsto i \kappa$. Such transformation may be relevant in a semi-classical theory, where it could describe tunneling transitions.} that link them is not always obvious, see for instance [8, 18]). But as the interior solutions are given by restricting the domain of the $r$ variable to $r < r_*$, they correspond to different non-diffeomorphic subsets of a larger space on which $r$ is unrestricted. As a consequence, the metrics (95 a,b,c) have to be considered as distinct solutions.

It is easy to convince oneself that only the geometries (95 a) and (95 d) can have a non-singular center of rotation, if we fix:

$$\gamma = a \quad , \quad \kappa = -1 \quad ,$$

in which case, using eq. (5), the angular velocity $\omega = c/2 = \pm \sqrt{\Lambda}$. Setting

$$\mu = a c = \left\{ \begin{array}{l}
(1 - \frac{\pi \sigma}{4|\Lambda|})^{-1/2} \geq 1 \quad \text{if} \quad \sigma \leq 4|\Lambda|/\pi \\
(\frac{\pi \sigma}{4|\Lambda|} - 1)^{-1/2} \geq 0 \quad \text{if} \quad \sigma \geq 4|\Lambda|/\pi \\
\end{array} \right\} ,$$

we end up with two 1–parameter families of metrics. The first family, corresponding to $\sigma \leq 4|\Lambda|/\pi$, is given by:

$$ds^2_{in} = -dt^2 + dr^2 + 4 a^2 \left( \sinh^2 \left[ \frac{r}{2a} \right] + (1 - \mu^2) \sinh^4 \left[ \frac{r}{2a} \right] \right) d\phi^2$$
This family has been studied in \[8\], essentially from a geometrical point of view; it also appears as the non trivial part of solutions of Einstein–Maxwell/Einstein–Maxwell–scalar field equations [17, 18] and more recently in the framework of a low energy string effective action [19]. The elements of this family can be viewed [8] as squashed AdS geometries: for \(\mu = 1\), the metric describes a regular AdS space in unusual coordinates, whereas for \(\mu^2 = 2\), it corresponds to the non trivial 3–dimensional part of the Gödel metric [20]. All these geometries, except the AdS one, are \(SO(2, 1) \times SO(2)\) invariant. Moreover, closed timelike curves pass through each of their points. In particular, the circles centred on the origin whose radii exceed the threshold value:

\[ r_c = 2a \arcsinh[(\mu^2 - 1)^{-1/2}] = a \log \left[ \frac{\mu + 1}{\mu - 1} \right], \quad (99) \]

are typical examples of such curves. To avoid these causality pathologies, we have to fix the radius \(r_*\) of the surface of the star to be less than \(r_c\). Using eqs (23 and 24), we find that the mass and angular momentum parameters of the connected external AdS space are expressed in terms of the parameter \(a\) and \(\mu\) as:

\[ J = -4a \mu (\mu^2 - 1) \sinh^4 \left( \frac{r}{2a} \right), \quad (100) \]

\[ M = -1 + 4 (\mu^2 - 1) \sinh^2 \left( \frac{r}{2a} \right) - 2 (\mu^2 - 1)(\mu^2 - 2) \sinh^4 \left( \frac{r}{2a} \right). \quad (101) \]

The second 1–parameter family of metrics, which corresponds to \(\sigma \geq 4|\Lambda|/\pi\), reads:

\[ ds_{in}^2 = -dt^2 + dr^2 + 4a^2 \left( \sin^2 \left( \frac{r}{2a} \right) - (1 + \mu^2) \sin^4 \left( \frac{r}{2a} \right) \right) d\phi^2 + 2a \left( 2 \mu \sin^2 \left( \frac{r}{2a} \right) \right) dt d\phi. \quad (102) \]

Due to the higher mass–energy density, the maximally analytic extensions of these spaces are topologically \(S^2 \times \mathbb{R}\) instead of \(\mathbb{R}^3\). These geometries also contain closed timelike curves. Again, to avoid causality inconsistencies, the radius of the surface of the star has to be chosen less than a critical value:

\[ r_c = 2a \arcsin \left[ (\mu^2 + 1)^{-1/2} \right]. \quad (103) \]
The mass and angular momentum parameters of the connected external AdS space are:

\[ J = 4a \mu \left( \mu^2 + 1 \right) \sin^4 \left( \frac{r}{2a} \right), \]  
\[ M = -1 + 4 \left( \mu^2 + 1 \right) \sin^2 \left( \frac{r}{2a} \right) - 2 \left( \mu^2 + 1 \right)(\mu^2 + 2) \sin^4 \left( \frac{r}{2a} \right). \]

By way of illustration, plots of \(|J|\) and \(M\) as functions of \(r^\star\), for the 2 families of metrics (98 and 102) and for various values of \(\mu\) and thus of the mass–energy density \(\sigma\), are depicted in figs [1-2]. All curves are stopped at the limiting value:

\[ r_{\text{lim}} = \begin{cases} 
\frac{\mu}{2} \arccosh \left[ \frac{\mu^2}{\mu^2 - 1} \right] & \text{if } \sigma \leq 4|\Lambda|/\pi, \\
\mu \arcsin \left[ \frac{1}{\sqrt{2(\mu^2 + 1)}} \right] & \text{if } \sigma \geq 4|\Lambda|/\pi,
\end{cases} \]

beyond which \(R_0\) becomes negative.

### 4.2 Pressurized models

The other types of solutions to eq. (87) are given by eq. (89) and are characterized by \(p \neq 0\). If the regularity conditions (5) at the origin are imposed, we obtain furthermore that:

\[ \alpha = -\beta, \quad T'[0] = 0, \quad c/2 = \alpha T''[0] = \omega. \]

As \(T[0]\) may set equal to 1, the metric reads:

\[ ds^2 = -T^2 dt^2 + dr^2 - 2\alpha(T^2 - 1)d\phi dt + \alpha^2 \left[ \frac{T'^2}{\omega^2} - \left( T - \frac{1}{T} \right)^2 \right] d\phi^2. \]

In this case, the matching condition (19) yields:

\[ L_\infty = -\omega, \]

with

\[ \omega^2 < -\Lambda, \]

according to eq. (36).

Let us again illustrate this class of solutions by considering the special case of constant mass–energy density. By integrating twice the Einstein equation
using the expression of $Z[r]$ and eq. (ef{85}) to fix an integration constant, we obtain:

$$T'^2 = \omega^2 \left( \frac{1}{T^2} - 1 \right) + 2 \pi (p[0] + \sigma)(T - 1) - (\pi \sigma + \Lambda)(T^2 - 1) \quad , \quad \text{(111)}$$

showing that the explicit solution could be expressed in terms of elliptic functions. More interesting is the relation resulting from the elimination of $T''[r]$ in eq. (85):

$$T = \frac{p[0] + \sigma}{p[r] + \sigma} \quad , \quad \text{(112)}$$

which allows to express the matching parameters in terms of the mass–energy density $\sigma$, the central pressure $p[0]$ and the central angular velocity $\omega$, $\alpha$ being obtained from eq. (107):

$$\alpha = \frac{\omega}{(-\Lambda + \pi p[0] - \omega^2)} \quad . \quad \text{(113)}$$

We get:

$$J = -\frac{2 \pi^2 p[0]^2 \omega}{\sigma(\omega^2 + \Lambda - \pi p[0])^2} \quad , \quad \text{(114)}$$

$$M = \frac{\pi p[0]^2 (-\Lambda + \omega^2) - \sigma(\omega^2 + \Lambda)^2}{\sigma(\omega^2 + \Lambda - \pi p[0])^2} \quad . \quad \text{(115)}$$

We leave to the reader the check that the limit of $M$ for vanishing $\omega$ can also be obtained by applying the method exposed in section 3 to the special case of constant mass–energy density.

The causality requirement $Y^2_r - Z^2_r > 0$ restricts the range of the variable $\omega$ to:

$$\omega^2 < -\Lambda + \frac{\pi \sigma p[0]}{p[0] + 2\sigma} \quad , \quad \text{(116)}$$

which is automatically satisfied owing to the inequality (110). From the latter, it is easy to verify that $M$ is always larger than $-1$ for positive $\sigma$ and $p[0]$. However, it becomes positive only if the central pressure is high enough:

$$p[0]^2 > \frac{\sigma (\Lambda + \omega^2)^2}{\pi (-\Lambda + \omega^2)} \quad . \quad \text{(117)}$$
Moreover, for \( M > 0 \), the requirement \( \sqrt{-\Lambda}|J|/M \leq 1 \) imposes:
\[
p[0]^2 > \frac{\sigma(\Lambda + \omega^2)^2}{\pi(\sqrt{-\Lambda} - |\omega|)^2},
\]
which implies condition (117). This establishes the existence of physically acceptable solutions, for sufficiently large values of the pressure. Finally, the condition ensuring the absence of tachyonic matter, \( \sigma > p \) [21], imposes a lower bound for the mass–energy density:
\[
\sigma_{\text{min}} = \frac{(\Lambda + \omega^2)^2}{\pi(\sqrt{-\Lambda} - |\omega|)^2},
\]
to avoid any causality pathology.

The above discussion is illustrated in fig. [3] where we have plot, in a \((\sigma, p^2[0])\) plane, the boundary curves \( \sigma = p[0], M = 0 \) and \(-\Lambda (J/M)^2 = 1\) for \( \omega = 0.1 \sqrt{-\Lambda} \). These curves delimit the physically allowed values of the mass–energy density and central pressure.

### 5 Conclusion

This incursion in (2+1)–dimensional gravity confirms the qualitative difference between the roles played by positive and negative cosmological constants. We have indeed shown that relevant elementary star models, obeying the physical criteria that space–time be free of naked singularities and regions of causality violations, require \( \Lambda < 0 \). We have also seen that, while the hydrostatic equilibrium equation (see eq. [35]) is similar to the one we obtain in (3+1) dimensions, the gravitational potential \( T'/T \) it involves manifests itself only through the cosmological expansion or contraction of the space and holonomy effects, in accord with our understanding of (2+1)–gravity. The interpretation of the kinematics of the static star embedded in flat space is particularly illustrative from this point of view. It inflates, its surface follows a curve a constant acceleration and its mass is a universal constant. The fact that such star models with “time” independent mass–energy density do exist, results from a subtle cancellation between volume expansion and Lorentz contraction. Indeed, the equal “time” planes are not the usual Minkowskian \( t = Cst \) sections, but the boosted Rindler time sections.

When the surrounding space is AdS, the physics is richer. The cosmological constant acts as an attractive gravitational force, increasing with the
radial distance, equilibrated by the mechanical effects of the pressure and the centrifugal forces. But pressure plays another, paradoxical role. As seen in section 4, it also acts as a "gravitational source", which counterbalances the centrifugal effect. It was indeed shown that taking pressure into account allows to obtain physically acceptable solutions, i.e. singularity free solutions without closed timelike curves, which, in case of subsequent collapse, lead to black holes without naked singularities. The main reason is that, owing to the pressure, the angular velocity of the star may be less than its upper bound (see eq. (10)). The deep significance of the pressure’s dual behavior is far from being completely clear (for us). In (3+1) dimensions, we know that the pressure contributes, together with the mass–energy density, to the gravitational attraction. In contrast, the Newtonian limit of (2+1)–dimensional Einstein gravity is a theory in which only the pressure is the source of the potential $\mathcal{F}$, thereby enlightening its special role. However, the relative signs between the cosmological constant, the squared angular velocity, the pressure and the mass–energy density render difficult to draw from eqs (114, 115) a complete physical intuition, similar to the one that has been built from (3+1)–dimensional experiences. These differences must be kept in mind before extrapolating to (3+1) dimensions physical conclusions from the (toy) model that (2+1)–gravity theory is.

Note that we have focused here on circular star models admitting a center. However, other models that do not satisfy the regularity conditions (3) are surely interesting to consider. They may indeed present geometrical peculiarities relevant to illustrate various aspects of wormholes, ergospheres, etc. They will be considered elsewhere.

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Figure captions

- Figure 1. Plots of the asymptotic angular momentum $|J|$ (in units $|\Lambda| = 1$), for constant mass–energy density dust–like star models, as a function of the radial surface coordinate $r_\star$, for various values of $\mu$:
  a) $\mu = 1.1, 1.5, 2, 100$ and $\sigma \leq 4|\Lambda|/\pi$;
  b) $\mu = .25, 1, 2, 10$ and $\sigma \geq 4|\Lambda|/\pi$.
  All curves are stopped at the critical radius $r_{\text{lim}}$ given in eq. (106).

- Figure 2. Plots of the asymptotic mass $M$ (in units $|\Lambda| = 1$), for constant mass–energy density dust–like star models, as a function of the radial surface coordinate $r_\star$, for various values of $\mu$:
  a) $\mu = 1.1, 1.5, 2, 100$ and $\sigma \leq 4|\Lambda|/\pi$;
  b) $\mu = .25, 1, 2, 10$ and $\sigma \geq 4|\Lambda|/\pi$.
  All curves are stopped at the critical radius $r_{\text{lim}}$ given in eq. (106).

- Figure 3. Plot, in the $(\sigma, p^2[0])$–plane, of the limiting curves $\sigma = p[0]$, $M = 0$ (eq. 117) and $-\Lambda J^2/M^2 = 1$ (eq. 118) (in units $|\Lambda| = 1$), for rotating, constant density, pressurized star models with central angular velocity $|\omega| = 0.1 \sqrt{-\Lambda}$. The gray area corresponds to the domain of physically allowed values of $\sigma$ and $p[0]$. 

26
Figure 1

(a) \( p = 0, \sigma \leq 4 |\Lambda| / \pi \)

(b) \( p = 0, \sigma \geq 4 |\Lambda| / \pi \)
Figure 2

(a) $p = 0, \sigma \leq 4 |\Lambda| / \pi$

(b) $p = 0, \sigma \geq 4 |\Lambda| / \pi$
Figure 3