Abstract: This note presents an analytic technique for proving the linear independence of certain small subsets of real numbers over the rational numbers. The applications of this test produce simple linear independence proofs for the subsets of triples \( \{1, e, \pi \} \), \( \{1, e, \pi^{-1} \} \), and \( \{1, \pi^r, \pi^s \} \), where \( 1 \leq r < s \) are fixed integers.

1 Introduction

The algebra and number theory literature has many elementary techniques used to verify the linear independence of small finite subsets of algebraic numbers \( \{\alpha_1, \alpha_2, \ldots, \alpha_d \} \subset \mathbb{Q} \) over the rational numbers \( \mathbb{Q} \). A few examples of these algebraic subsets are

1. \( \{1, \sqrt{2}, \sqrt{3}, \sqrt{2} \} \subset \mathbb{Q} \),
2. \( \{1, \sqrt{2}, \sqrt{3}, \sqrt{5} \} \subset \mathbb{Q} \),
3. \( \{1, \alpha, \alpha^2, \ldots, \alpha^d \} \subset \mathbb{R} \), where \( f(\alpha) = 0 \),
4. \( \{1, \omega, \omega^2, \ldots, \omega^{p(n)} \} \subset \mathbb{R} \), where \( \omega^n = 1 \).

But, these techniques are intrinsically algebraic, and do not seem to be applicable to small subsets of nonalgebraic numbers \( \{\alpha_1, \alpha_2, \ldots, \alpha_d \} \subset \mathbb{R} \). This note presents an analytic technique for proving the linear independence of certain small subsets of nonalgebraic numbers \( \{\alpha_1, \alpha_2, \alpha_3 \} \subset \mathbb{R} \) over the rational numbers \( \mathbb{Q} \). The applications of this test produce simple proofs for the followings subsets of triples.

Theorem 1.1. The real numbers \( 1, e \) and \( \pi \) are rationally independent.

Theorem 1.2. The real numbers \( 1, e \) and \( \pi^{-1} \) are rationally independent.

Theorem 1.3. For any pair of integers \( 1 \leq r < s \), the real numbers \( 1, \pi^r \) and \( \pi^s \) are rationally independent.

The proofs are presented in Section \( \text{Section 6} \), Section \( \text{Section 8} \), and Section \( \text{Section 10} \) respectively.

Since both \( e \) and \( \pi \) are irrational numbers, it is immediate that at least one, the trace \( Tr(\alpha) = e + \pi \) or the norm \( N(\alpha) = e\pi \) of the polynomial \( f(x) = (x + e)(x + \pi) \), is irrational. Simple applications of the above linear independence results demonstrate that both of these numbers are irrational, see Corollary \( \text{Corollary 7.1} \) and Corollary \( \text{Corollary 9.1} \) respectively. Similar application demonstrates that the real number

\[
\pi + \pi^2
\]

is irrational, see Corollary \( \text{Corollary 11.1} \).
2 The Irrational Limit Test

The *irrational limit test* converts some apparently intractable decision problems in the real domain \( \mathbb{R} \) to simpler decision problems in the finite field domain \( \mathbb{F}_2 = \{0, 1\} \).

**Definition 2.1.** Let \( \alpha \in \mathbb{R} \) be a real number. The *irrational limit test* is a map \( I : \mathbb{R} \rightarrow \mathbb{F}_2 = \{0, 1\} \) defined by

\[
I(\alpha) = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} e^{i\alpha n}.
\]  

(2)

The normalization is intrinsic to the number \( \pi \). But, it can be modified as needed. The *irrational limit test* is a point map or equivalently a class map, and it is not invertible. But, inversion is not required in the applications to decision problems.

**Lemma 2.1.** For any real number \( \alpha \in \mathbb{R} \), the irrational limit test satisfies the followings.

\[
I(2\pi m \alpha) = \begin{cases} 
1 & \text{if and only if } \alpha \in \mathbb{Q}, \text{ for some } m \in \mathbb{Z}, \\
0 & \text{if and only if } \alpha \notin \mathbb{Q}, \text{ for any } m \in \mathbb{Z}.
\end{cases}
\]

(3)

**Proof.** Given any rational number \( \alpha \in \mathbb{Q} \), there is an integer \( m \in \mathbb{Z} \) such that \( \alpha m \in \mathbb{Z} \), and the limit is

\[
I(2\pi m \alpha) = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} e^{i2\pi\alpha m n} = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} 1 = 1.
\]

(4)

The above proves that for any rational number \( \alpha \in \mathbb{Q} \), and any integer \( m \), the sequence

\[
\{\alpha mn : n \in \mathbb{Z}\}
\]

(5)

is not uniformly distributed. While for any irrational number \( \alpha \notin \mathbb{Q} \), and any integer \( m \neq 0 \), the sine function \( \sin(\alpha \pi m) \neq 0 \). Hence, applying Lemma 5.1, the evaluation of the limit is

\[
I(2\pi \alpha m) = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} e^{i2\pi\alpha mn} \\
\leq \lim_{x \to \infty} \frac{1}{2x} \frac{1}{|\sin(\alpha \pi m)|} = 0.
\]

(6)

The above proves that for any irrational number \( \alpha \in \mathbb{Q} \), and any integer \( m \neq 0 \), the sequence

\[
\{\alpha mn : n \in \mathbb{Z}\}
\]

(7)

is uniformly distributed. This proof is equivalent to the Weil criterion, see [9, Theorem 2.1].

As it is evident, the class function \( I \) maps the class of rational numbers \( \mathbb{Q} \) to 1 and the class of irrational numbers \( \mathbb{I} = \mathbb{R} - \mathbb{Q} \) to 0. The *irrational limit test* induces an equivalence relation on the set of real numbers \( \mathbb{R} \):

- A pair of real numbers \( a \) and \( b \) are equivalent \( a \sim b \) if and only if \( I(2\pi a) = I(2\pi b) \). This occurs if either both \( a \) and \( b \) are rational numbers or both \( a \) and \( b \) are irrational numbers.

- A pair of real numbers \( a \) and \( b \) are not equivalent \( a \not\sim b \) if and only if \( I(2\pi a) \neq I(2\pi b) \). This occurs if either \( a \) or \( b \) is a rational numbers but not both.

Some standard irrationality tests, criteria, and proofs are given in [1, Chapter 7], [3], [7], [11], [16], [17], et alii.
3 Approximation By Lattice Points

A handful of elementary integer relations and integers points approximations are considered in this Section.

Lemma 3.1. The numbers $e$ and $\pi$ are not integer multiple. Specifically, $ek \neq m\pi$ for any integers $k, m \geq 1$.

Proof. Numerically $\sin(e) = 0.410781 \ldots \neq 0$. Computing it using the infinite product yields

$$0 \neq \sin(e) = \frac{1}{e} \prod_{n \geq 1} \left(1 - \frac{e^2}{\pi^2 n^2}\right).$$  \hfill (8)

Ergo, for each integer $n \geq 1$, the local factor

$$1 - \frac{e^2}{\pi^2 n^2} = 1 - \left(\frac{e}{\pi n}\right)^2 \neq 0 \quad \hfill (9)$$

cannot vanish. This proves that $e/\pi n \neq 1$. Equivalently, these numbers are not integer multiple: $ek \neq m\pi$ for any integers $k, m \geq 1$. \hfill ■

The discrete lines $L_0(r) = \{(2r+1)\pi/2 : r \in \mathbb{Z}\}$ and $L_1(r) = \{(2r+1)\pi : r \in \mathbb{Z}\}$
never intercept the discrete lattice $L(k, m) = \{ke + m : k, m \in \mathbb{Z}\}$, \hfill (10)
but comes arbitrarily close. A proof, based on the simplest form of the Kronecker approximation theorem, see Theorem [12.1] is given below.

Lemma 3.2. If $k$ and $m$ are nonzero integers, and let $r \in \mathbb{Z}$, then

1. $ke + m \neq (2r+1)\pi/2$.
2. $ke + m \neq r\pi$.

Proof. (i) It is sufficient to verify the inequality (3.2)-i on the first quadrant, which is specified by $k \geq 1$ and $m \geq 1$. The verification in any quadrant is almost the same. Let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the irrational number $e$. The Diophantine approximation inequalities

$$\frac{1}{2q_{n+1}} \leq |q_ne - p_n| \leq \frac{1}{q_n} \quad \hfill (12)$$

and

$$|q_ne - p_n| \leq |ke - m| \quad \hfill (13)$$

for $k \leq q_n$, see Lemma [12.4] and Lemma [12.6] lead to the lattice points approximation

$$|ke + m - (2r+1)\pi/2| \geq ||ke + m| - (2r+1)\pi/2| \geq \left|q_ne - p_n| - (2r+1)\pi/2\right|, \quad \hfill (14)$$

where $r \in \mathbb{Z}$, and $|2r + 1| \geq 1$. Rearranging it, and applying the reverse triangle inequality $|X - Y| \geq ||X| - |Y||$, yield

$$\left|\frac{e - p_n}{q_n} - \frac{(2r + 1)\pi}{2q_n}\right| \geq \left|\frac{1}{2q_{n+1}} - \frac{(2r + 1)\pi}{2q_n}\right| \geq \frac{1}{2q_{n+1}} > 0. \quad \hfill (15)$$

Therefore, relation $ke + m = (2r+1)\pi/2$ is false for any nontrivial integer point $(k, m) = (k \neq 0, m \neq 0)$ and $r \in \mathbb{Z}$. (ii) The proof for this case is similar. \hfill ■
Observe that the continued fraction \( e = [a_0, a_1, a_2, \ldots] \) has arbitrary long arithmetic progressions, very visibly, see Theorem 12.1 the relation \( ke + m = (2r + 1)\pi/2 \) would implies that continued fraction \( \pi = [b_0, b_1, b_2, \ldots] \) has arbitrary long arithmetic progressions. But, this is unknown. These elementary results seem to be implied by a more advanced technique given in 14 about certain equivalence of irrational numbers.

**Lemma 3.3.** Let \( 1 \leq u < v \) be a pair of integers, and let \( k, m, \) and \( r \) be any nonzero integers, then

1. \( k\pi^u + m\pi^v \neq r\pi \).
2. \( ke^u + me^v \neq r\pi \).

**Proof.** (i) It is sufficient to verify the inequality (3.3)-i on the first quadrant, which is specified by \( k \geq 1 \) and \( m \geq 1 \). The verification in any quadrant is almost the same. Let \( \{p_n/q_n: n \geq 1\} \) be the sequence of convergents of the irrational number \( \pi^{v-u} \). The Diophantine approximation inequalities

\[
\frac{1}{2q_{n+1}} \leq |q_n\pi^{v-u} - p_n| \leq \frac{1}{q_n} \tag{16}
\]

and

\[
|q_n\pi^{v-u} - p_n| \leq |k\pi^{v-u} - m| \tag{17}
\]

for \( k \leq q_n \), see Lemma 12.4 and Lemma 12.5 lead to the lattice points approximation

\[
|k\pi^u + m\pi^v - r\pi| = |\pi^u(k\pi^{v-u} + m) - r\pi| \tag{18}
\]

\[
\geq |\pi^u(k\pi^{v-u} - m) - r\pi| \geq |\pi^u(q_n\pi^{v-u} - p_n) - r\pi|,
\]

where \( r \in \mathbb{Z} \) and \( \pi^u \geq 1 \). Rearranging it, and applying the reverse triangle inequality \( |X - Y| \geq ||X| - |Y|| \), yield

\[
|\pi^u p_n/q_n - r\pi| \geq \left| \frac{\pi^u}{2q_{n+1}} - \frac{r\pi}{q_n} \right| \geq \frac{1}{2q_{n+1}} > 0.
\]

Therefore, relation \( k\pi^u + m\pi^v = r\pi \) is false for any nontrivial integer point \( (k, m) = (k \neq 0, m \neq 0) \) and \( r \in \mathbb{Z} \). (ii) The proof for this case is similar. \[\blacksquare\]

### 4 Nonvanishing Sine Function Values

The nonvanishing of the sine function at certain real numbers are required in the proofs of certain results. These are verified using either the irrationality of the real number \( \pi \) or via the infinite product \( \sin(x) = x^{-1}\prod_{n \geq 1}(1 - (x\pi^{-1}n^{-1})^2) \) for any real number \( x \in \mathbb{R} \) or the reflection formula \( \Gamma(1 - z)\Gamma(z) = \pi/\sin\pi z \) of the gamma function \( \Gamma(z) \) for \( z \in \mathbb{C} \).

**Lemma 4.1.** For any pair of integers \( r \neq 1 \), and \( m \neq 0 \), the sine function satisfies the followings.

1. \( \sin(m) \neq 0 \).
2. \( \sin(\pi m) \neq 0 \).

**Proof.** (i) The verification, using the reflection formula of the gamma function, yields

\[
\sin(m) = \sin(\pi \cdot m/\pi) = \frac{\Gamma(1-m/\pi)}{\Gamma(m/\pi)} \neq 0
\]

\[
\left(1 - \frac{m}{\pi}\right)\Gamma\left(\frac{m}{\pi}\right) \neq 0
\]
for any integer \( m \geq 1 \) since the gamma function \( \Gamma(z) \) has its poles at the negative integers \( z = n \leq 0 \), and

\[
1 - \frac{m}{\pi} \quad \text{and} \quad \frac{m}{\pi}
\]

are irrational numbers, not negative integers. (ii) Similar to the previous case.

**Lemma 4.2.** If \( k \) and \( m \) are nonzero integers, then

\[
\sin(ke + m) \neq 0.
\]

**Proof.** The task to prove that the set of nontrivial integer solutions \((k, m) \neq (0,0)\) of the equation

\[
\sin(ke + m) = 0
\]

is empty splits into three different cases.

**Case 1.** \( k = 0 \), and \( m \neq 0 \). The relation

\[
\sin(ke + m) = \sin(m) \neq 0
\]

is true, see Lemma 4.1.

**Case 2.** \( k \neq 0 \), and \( m = 0 \). By Lemma 3.1, \( e \neq a\pi \) for any integer \( a \geq 1 \). Thus, the multiple \( ke \neq ak\pi = n\pi \). This implies that the equation

\[
\sin(ke + m) = \sin(ke) = \sin(n\pi) = 0
\]

is impossible.

**Case 3.** \( k \neq 0 \), and \( m \neq 0 \). By Lemma 3.2, \( ke + m \neq r\pi \) for any integers \( k, m \), and \( r \in \mathbb{Z} \). This implies that the equation

\[
\sin(ke + m) = \sin(r\pi)
\]

is impossible.

**Lemma 4.3.** If \( k \) and \( m \) are nonzero integers, then

\[
\sin(ke\pi + m) \neq 0.
\]

**Proof.** The task to prove that the set of nontrivial integer solutions \((k, m) \neq (0,0)\) of the equation

\[
\sin(ke\pi + m) = 0
\]

is empty splits into three different cases.

**Case 1.** \( k = 0 \), and \( m \neq 0 \). The relation

\[
\sin(ke\pi + m) = \sin(m) \neq 0
\]

is true, see Lemma 4.1.

**Case 2.** \( k \neq 0 \), and \( m = 0 \). Since \( e \) is irrational, the relation \( ke\pi = n\pi \), where \( n \neq 0 \), is impossible. This implies that the equation

\[
\sin(ke\pi + m) = \sin(ke\pi) = \sin(n\pi) = 0
\]

is impossible.
is impossible.

**Case 3.** \( k \neq 0, \text{ and } m \neq 0. \) By Lemma 3.2 \( k\pi + m \neq (2r + 1)\pi \) for any integers \( k, m, \) and \( r \in \mathbb{Z}. \) This implies that the equation

\[
\sin(ke\pi + m) = \sin ((2r + 1)\pi)
\]

is impossible. \( \blacksquare \)

**Lemma 4.4.** Let \( 1 \leq r < s \) be a pair of integers, and let \( k \) and \( m \) be nonzero integers, then

\[
\sin(k\pi r + 1 + m\pi s + 1) \neq 0
\]

\( (32) \)

**Proof.** The task to prove that the set of nontrivial integer solutions \( (k, m) \neq (0, 0) \) of the equation

\[
\sin(k\pi r + 1 + m\pi s + 1) = 0
\]

is empty splits into three different cases.

**Case 1.** \( k = 0, \text{ and } m \neq 0. \) Since \( s \geq 2 \) is an integer, the relation

\[
\sin(k\pi r + 1 + m\pi s + 1) = \sin(m\pi s + 1) = 0,
\]

where \( \pi^s \geq \pi^3, \) is false, see Lemma 4.1.

**Case 2.** \( k \neq 0, \text{ and } m = 0. \) Since \( r \geq 1 \) is an integer, the relation

\[
\sin(k\pi r + 1 + m\pi s + 1) = \sin(k\pi r + 1) = 0
\]

where \( \pi^r \geq \pi^2, \) is false, see Lemma 4.1.

**Case 3.** \( k \neq 0, \text{ and } m \neq 0. \) By Lemma 3.2 \( k\pi r + 1 + m\pi s + 1 \neq a\pi \) for any integers \( k \neq 0, m \neq 0, \) and \( a \in \mathbb{Z}. \) This implies that the equation

\[
\sin(k\pi r + 1 + m\pi s + 1) = \sin(a\pi)
\]

is impossible. \( \blacksquare \)

## 5 Finite Sine Sums

**Lemma 5.1.** For any real number \( t \neq k\pi, k \in \mathbb{Z}, \) and a large integer \( x \geq 1, \) the finite sum

1. \( \sum_{-x \leq n \leq x} e^{2\pi tn} = \frac{\sin((2x + 1)t)}{\sin(t)}. \)

2. \( \left| \sum_{-x \leq n \leq x} e^{2\pi tn} \right| \leq \frac{1}{|\sin(t)|}. \)

**Proof.** (i) Expand the complex exponential sum into two subsums:

\[
\sum_{-x \leq n \leq x} e^{2\pi tn} = e^{-i2t} \sum_{0 \leq n \leq x-1} e^{-i2tn} + \sum_{0 \leq n \leq x} e^{i2tn}.
\]

Lastly, use the geometric series to determine the closed form. \( \blacksquare \)
6 Linear Independence Of 1, $e$, and $\pi$

Proof. (Theorem 1.1) On the contrary, the numbers 1, $e$ and $\pi$ are linearly dependent over the rational numbers, and the equation

$$1 \cdot A + e \cdot B + \pi \cdot C = 0,$$  \hspace{1cm} (38)

where $A, B, C \in \mathbb{Z}^\times$ are integers, has a nontrivial rational solution $(A, B, C) \neq (0, 0, 0)$. Rewrite it in the equivalent form

$$-2\pi C = 2(\pi B + A).$$  \hspace{1cm} (39)

Take the irrational limit test, see Lemma 2.1, in both sides to obtain

$$I(-2\pi C) = I(2(\pi B + A)).$$  \hspace{1cm} (40)

The left side and the right side are evaluated separately.

Left Side. The verification is based on the identity $e^{-i2\pi C} = 1$, where $C$ is an integer. The evaluation of the limit is

$$I(-2\pi C) = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} e^{-i2\pi Cn} = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} 1 = 1.$$  \hspace{1cm} (41)

Right Side. The verification is based on the nonvanishing of the sine function $\sin(\pi B + A) \neq 0$, see Lemma 4.2. An application of Lemma 5.1 yields

$$I(2(\pi B + A)) = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} e^{i(2\pi B + A)n} \leq \lim_{x \to \infty} \frac{1}{2x} \frac{1}{|\sin(\pi B + A)|} = 0.$$  \hspace{1cm} (42)

Clearly, these distinct evaluations

$$1 = I(-2\pi C) \neq I(2(\pi B + A)) = 0$$  \hspace{1cm} (43)

contradict equation (40). This implies that equation (38) does not have a nontrivial rational solution $(A, B, C) \in \mathbb{Z}^\times \times \mathbb{Z}^\times \times \mathbb{Z}^\times$. Hence, the numbers 1, $e$ and $\pi$ are linearly independent over the rational numbers $\mathbb{Q}^\times$.  

7 The Real Number $e + \pi$

The continued fraction of the number understudy is

$$e + \pi = [5; 1, 6, 7, 3, 21, 2, 1, 2, 1, 1, 2, 3, 3, 2, 5, 2, 1, 1, 1, 3, 1, 8, \ldots].$$  \hspace{1cm} (44)

The previous result immediately implies that this continued fraction is infinite.

Corollary 7.1. The real number $e + \pi$ = 5.859874\ldots is irrational number.

Proof. By Theorem 1.1 the equation

$$1 \cdot A + e \cdot B + \pi \cdot C = 0,$$  \hspace{1cm} (45)

has no nontrivial integer solutions $(A, B, C) \neq (0, 0, 0)$. 

Conjecture 7.1. The real number $e + \pi$ is transcendental.

Conjecture 7.2. The irrationality measure of the real number $e + \pi$ is $\mu(e + \pi) = 2$.

A few values were computed to illustrate the prediction in this conjecture, see Table 1. The fourth column displays the numerical approximation $\mu_0(e + \pi)$ of the actual value $\mu(e + \pi)$.  

\[ \text{Table 1} \]

\begin{tabular}{|c|c|}
\hline
$\mu_0(e + \pi)$ & $\mu(e + \pi)$ \\
\hline
2 & 2 \\
\hline
\end{tabular}

\[ \text{Table 1} \]
Table 1: Numerical Data For Irrationality Measure $|p_n/q_n - e - \pi| \geq q_n^{\mu_0(e + \pi)}$.

| $n$ | $p_n$ | $q_n$ | $\mu_0(e + \pi)$ |
|-----|-------|-------|-----------------|
| 1   | 5     | 1     |                 |
| 2   | 6     | 1     |                 |
| 3   | 41    | 7     | 3.033470        |
| 4   | 93    | 50?   | 3.153443        |
| 5   | 920   | 157   | 2.608509        |
| 6   | 19613 | 3347  | 2.124717        |
| 7   | 40146 | 6851  | 2.382347        |
| 8   | 59759 | 10198 | 2.073126        |
| 9   | 379087| 64692 | 2.067776        |
| 10  | 538751| 91939 | 2.066541        |

8 Linear Independence Of 1, $e$, And $\pi^{-1}$

Proof. (Theorem 1.2) On the contrary, the numbers 1, $e$ and $\pi^{-1}$ are linearly dependent over the rational numbers, and the equation

$$1 \cdot A + e \cdot B + \pi^{-1} \cdot C = 0,$$

where $A, B, C \in \mathbb{Z}$ are integers, has a nontrivial rational solution $(A, B, C) \neq (0, 0, 0)$. Rewrite it in the equivalent form

$$-2\pi A = 2(e\pi B + C).$$

Take the irrational limit test, see Lemma 2.1, in both sides to obtain

$$I(-2\pi A) = I(2(e\pi B + C)).$$

The left side and the right side are evaluated separately.

**Left Side.** The verification is based on the identity $e^{-i2\pi A} = 1$, where $A$ is an integer. The evaluation of the limit is

$$I(-2\pi A) = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} e^{-i2\pi An} = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} 1 = 1.$$

**Right Side.** The verification is based on the nonvanishing $\sin(e\pi B + C) \neq 0$ of the sine function, see Lemma 4.3. An application of Lemma 5.1 yields

$$I(2(e\pi B + C)) = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} e^{i(2(e\pi B + C))n} \leq \lim_{x \to \infty} \frac{1}{2x} \frac{1}{|\sin(e\pi B + C)|} = 0.$$

Clearly, these distinct evaluations

$$1 = I(-2\pi A) \neq I(2(e\pi B + C)) = 0$$

contradict equation (48). This implies that equation (46) does not have a nontrivial rational solution $(A, B, C) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Hence, the numbers 1, $e$ and $\pi^{-1}$ are linearly independent over the rational numbers $\mathbb{Q}$. ■
9 The Real Number $e\pi$

The continued fraction of the number under study is

$$e\pi = [8; 1, 1, 5, 1, 3, 1, 4, 12, 3, 2, 1, 5, 2, 12, 1, 1, 10, 2, 2, 1, 3, 2, 2, 2, 29, 1, \ldots].$$  \(52\)

The previous result immediately implies that this continued fraction is infinite.

**Corollary 9.1.** The real number $e\pi = 8.539734 \ldots$ is irrational.

**Proof.** By Theorem 1.2, the equation

\[1 \cdot A + e \cdot B + \pi^{-1} \cdot C = 0,\] \(53\)

has no nontrivial integer solutions $(A, B, C) \neq (0, 0, 0)$.

**Conjecture 9.1.** The real number $e\pi$ is transcendental.

**Conjecture 9.2.** The irrationality measure of the real number $e\pi$ is $\mu(e\pi) = 2$.

A few values were computed to illustrate the prediction in this conjecture, see Table 2. The fourth column displays the numerical approximation $\mu_0(e\pi)$ of the actual value $\mu(e\pi)$.

| $n$ | $p_n$ | $q_n$ | $|p_n/q_n - e\pi| \geq q_n^{\mu(e\pi)}$ |
|-----|-------|-------|----------------------------------------|
| 1   | 8     | 1     |                                        |
| 2   | 9     | 1     |                                        |
| 3   | 17    | 2     | 4.653474                               |
| 4   | 94    | 11    | 3.153443                               |
| 5   | 111   | 13    | 2.599126                               |
| 6   | 427   | 50    | 2.104500                               |
| 7   | 538   | 63    | 2.382347                               |
| 8   | 2579  | 302   | 2.442400                               |
| 9   | 31486 | 3687  | 2.150201                               |
| 10  | 97037 | 11363 | 2.123550                               |

10 Linear Independence Of $1$, $\pi^r$, And $\pi^s$

**Proof.** (Theorem 1.3) On the contrary, the numbers 1, $\pi^r$ and $\pi^s$ are linearly dependent over the rational numbers, and the equation

\[1 \cdot A + \pi^r \cdot B + \pi^s \cdot C = 0,\] \(54\)

where $A, B, C \in \mathbb{Z}^\times$ are integers, has a nontrivial rational solution $(A, B, C) \neq (0, 0, 0)$. Rewrite it in the equivalent form

\[-2\pi A = 2 \left( \pi^r+1B + \pi^s+1C \right).\] \(55\)

Take the irrational limit test, see Lemma 2.1, in both sides to obtain

\[I(-2\pi A) = I \left( 2(\pi^r+1B + \pi^s+1C) \right).\] \(56\)

The left side and the right side are evaluated separately.
The verification is based on the identity $e^{-i2\pi A} = 1$, where $A$ is an integer. The evaluation of the limit is

$$I(-2\pi A) = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} e^{-i2\pi An} = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} 1 = 1. \quad (57)$$

The verification is based on the nonvanishing $\sin \left( \pi r + 1 \pi B + \pi s + 1 \pi C \right) \neq 0$ of the sine function, see Lemma 4.4. An application of Lemma 5.1 yields

$$I \left( 2(\pi^{r+1} + \pi^{s+1} B + \pi^{t+1} C) \right) = \lim_{x \to \infty} \frac{1}{2x} \sum_{-x \leq n \leq x} e^{i(2(\pi^{r+1} B + \pi^{s+1} C))n} \leq \lim_{x \to \infty} \frac{1}{2x} \frac{1}{|\sin (\pi r + 1 \pi B + \pi s + 1 \pi C)|} = 0. \quad (58)$$

Clearly, these distinct evaluations

$$1 = I(-2\pi A) \neq I \left( 2(\pi^{r+1} B + \pi^{s+1} C) \right) = 0 \quad (59)$$

contrast equation (56). This implies that equation (53) does not have a nontrivial rational solution $(A, B, C) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Hence, the numbers $1, \pi^r$ and $\pi^s$ are linearly independent over the rational numbers $\mathbb{Q}$. \[\Box\]

### 11 The Real Number $\pi + \pi^2$

The continued fraction of the number under study is

$$\pi + \pi^2 = [13, 89, 3, 4, 3, 1, 2, 3, 1, 9, 2, 1, 1, 3, 1, 12, 1, 1, 4, 2748, 6, 91, 18, 19, 2, 12, 1, \ldots]. \quad (60)$$

The previous result immediately implies that this continued fraction is infinite.

**Corollary 11.1.** The real number $\pi + \pi^2 = 13.011197\ldots$ is irrational.

**Proof.** Set $r = 1$ and $s = 2$. By Theorem 4.3, the equation

$$1 \cdot A + \pi \cdot B + \pi^2 \cdot C = 0, \quad (61)$$

has no nontrivial integer solutions $(A, B, C) \neq (0, 0, 0).$ \[\Box\]

**Conjecture 11.1.** The real number $\pi + \pi^2$ is transcendental.

**Conjecture 11.2.** The irrationality measure of the real number $\pi + \pi^2$ is $\mu(\pi + \pi^2) = 2$.

A few values were computed to illustrate the prediction in this conjecture, see Table 3. The fourth column displays the numerical approximation $\mu_0(\pi + \pi^2)$ of the actual value $\mu(\pi + \pi^2)$.

### 12 Basic Diophantine Approximations Results

All the materials covered in this section are standard results in the literature, see [7], [10], [12], [15], [16], [17], et alii.
Table 3: Numerical Data For Irrationality Measure \(|p_n/q_n - e\pi| \geq q_n^\mu(\pi + \pi^2)\).

| n | p_n | q_n | \mu(\pi + \pi^2) |
|---|-----|-----|------------------|
| 1 | 13  | 1   | 2.262270         |
| 2 | 1158| 89  | 2.273061         |
| 3 | 3487| 268 | 2.193714         |
| 4 | 15106| 1161| 2.068191         |
| 5 | 48805| 3751| 2.152449         |
| 6 | 63911| 4912| 2.211219         |
| 7 | 176627| 13575| 2.152449         |
| 8 | 593792| 45637| 2.031677         |
| 9 | 770419| 59212| 2.211219         |
| 10| 7527563| 578545| 2.073701        |

12.1 Rationals And Irrationals Numbers Criteria

A real number \(\alpha \in \mathbb{R}\) is called rational if \(\alpha = a/b\), where \(a, b \in \mathbb{Z}\) are integers. Otherwise, the number is irrational. The irrational numbers are further classified as algebraic if \(\alpha\) is the root of an irreducible polynomial \(f(x) \in \mathbb{Z}[x]\) of degree \(\text{deg}(f) > 1\), otherwise it is transcendental.

**Lemma 12.1.** If a real number \(\alpha \in \mathbb{R}\) is a rational number, then there exists a constant \(c = c(\alpha)\) such that

\[
\frac{c}{q} \leq |\alpha - p/q| \tag{62}
\]

holds for any rational fraction \(p/q \neq \alpha\). Specifically, \(c \geq 1/b\) if \(\alpha = a/b\).

This is a statement about the lack of effective or good approximations for any arbitrary rational number \(\alpha \in \mathbb{Q}\) by other rational numbers. On the other hand, irrational numbers \(\alpha \in \mathbb{R} - \mathbb{Q}\) have effective approximations by rational numbers. If the complementary inequality \(|\alpha - p/q| < c/q\) holds for infinitely many rational approximations \(p/q\), then it already shows that the real number \(\alpha \in \mathbb{R}\) is irrational, so it is sufficient to prove the irrationality of real numbers.

**Lemma 12.2** (Dirichlet). Suppose \(\alpha \in \mathbb{R}\) is an irrational number. Then there exists an infinite sequence of rational numbers \(p_n/q_n\) satisfying

\[
0 < |\alpha - p_n/q_n| < \frac{1}{q_n^2} \tag{63}
\]

for all integers \(n \in \mathbb{N}\).

**Lemma 12.3.** Let \(\alpha = [a_0, a_1, a_2, \ldots]\) be the continued fraction of a real number, and let \(\{p_n/q_n : n \geq 1\}\) be the sequence of convergents. Then

\[
0 < |\alpha - p_n/q_n| < \frac{1}{a_n+1q_n^2} \tag{64}
\]

for all integers \(n \in \mathbb{N}\).

This is standard in the literature, the proof appears in [7, Theorem 17], [16, Corollary 3.7], [8, Theorem 9], and similar references.

**Lemma 12.4.** Let \(\alpha = [a_0, a_1, a_2, \ldots]\) be the continued fraction of a real number, and let \(\{p_n/q_n : n \geq 1\}\) be the sequence of convergents. Then
1. \[ \frac{1}{2q_{n+1}q_n} \leq \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}, \]

2. \[ \frac{1}{2a_{n+1}q_n^2} \leq \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}, \]

for all integers \( n \in \mathbb{N} \).

The recursive relation \( q_{n+1} = a_{n+1}q_n + q_{n-1} \) links the two inequalities. Confer [13] Theorem 3.8, [8] Theorems 9 and 13, et alia. The proof of the best rational approximation stated below, appears in [13] Theorem 2.1, and [16] Theorem 3.8.

**Lemma 12.5.** Let \( \alpha \in \mathbb{R} \) be an irrational real number, and let \( \{p_n/q_n : n \geq 1\} \) be the sequence of convergents. Then, for any rational number \( p/q \in \mathbb{Q} \times \mathbb{R} \),

1. \[ |\alpha q_n - p_n| \leq |\alpha q - p|, \]
2. \[ \frac{1}{\alpha - \frac{p_n}{q_n}} \leq \frac{1}{\alpha - \frac{p}{q}} \]

for all sufficiently large \( n \in \mathbb{N} \) such that \( q \leq q_n \).

**Theorem 12.1.** (Kronecker approximation theorem) Let \( \alpha, \beta \in \mathbb{R} \times \mathbb{R} \) be real numbers, and \( \alpha \) irrational. Given a small number \( \varepsilon > 0 \), there exists infinitely many pairs of integers \( p, q \in \mathbb{N} \) such that

\[ |\alpha q - p - \beta| < \varepsilon. \] (65)

The \( n \)th dimensional version and related problems are studied in [2], [6], and similar references.

### 12.2 Irrationalities Measures

The concept of measures of irrationality of real numbers is discussed in [17] p. 556, [3] Chapter 11, et alii. This concept can be approached from several points of views.

**Definition 12.1.** The irrationality measure \( \mu(\alpha) \) of a real number \( \alpha \in \mathbb{R} \) is the infimum of the subset of real numbers \( \mu(\alpha) \geq 1 \) for which the Diophantine inequality

\[ \frac{1}{|\alpha - \frac{p}{q}|} \ll \frac{1}{q^{\mu(\alpha)}} \] (66)

has finitely many rational solutions \( p \) and \( q \). Equivalently, for any arbitrary small number \( \varepsilon > 0 \)

\[ \frac{1}{|\alpha - \frac{p}{q}|} \gg \frac{1}{q^{\mu(\alpha)+\varepsilon}} \] (67)

for all large \( q \geq 1 \).

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