Particle production and AGK relations in the Color Glass Condensate picture

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Abstract

In this talk, we discuss some general properties of particle production in a field theory coupled to strong time dependent sources, and techniques to compute the spectrum of the produced particles in such theories. We also discuss the application of these results to the description of hadron or heavy ion collisions in the Color Glass Condensate framework.

1 Introduction

At high energy, all the internal timescales of a hadron are time dilated. Therefore, more and more soft fluctuations – carrying a smaller and smaller fraction $x$ of the hadron momentum – become long-lived and become relevant in interactions with another hadron. On the contrary, on the timescales relevant for such an interaction process, the large $x$ partons can be seen as completely frozen degrees of freedom, whose only role is to act as sources that radiate more small $x$ gluons. Moreover, the small $x$ modes will eventually have an occupation number larger than unity, and undergo recombinations – a process known as saturation \cite{1}.

In the Color Glass Condensate (CGC) framework \cite{2,3}, one thus divides the degrees of freedom of a hadron in static color sources – described by a density denoted $\rho$ – that represent the large $x$ partons, and dynamical gauge fields that represent the small $x$ partons. The CGC can thus be seen as an effective theory of gauge fields coupled to external sources. The details of this separation of degrees of freedom can change with the separation scale, but this should not

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affect physical quantities. This leads to a renormalization group equation – known as the JIMWLK equation [3] – that governs the evolution with $x$ of the distribution $W[\rho]$ of hard color sources.

In these proceedings, we consider the collision at high energy of two hadrons (or heavy ions) described in the CGC framework. We assume that the distributions of hard color sources that describe the two projectiles are known, and we address the question of calculating physical observables in given configurations of the two sources. Moreover, we will consider only the regime where the two projectiles are saturated, in which the two sources $\rho_{1,2}$ are strong – both of order $g^{-2}$.

As stated before, one must consider an effective theory described by the following Lagrangian,

$$
\mathcal{L} \equiv -\frac{1}{2} \text{tr} F^{\mu\nu} F_{\mu\nu} + A_\mu (J_1^\mu + J_2^\mu) ,
$$

with currents given at lowest order in the sources by

$$
J_1^\mu = g\delta^{\mu+}\delta(x^-)\rho_1(x_\perp) , \quad J_2^\mu = g\delta^{\mu-}\delta(x^+)\rho_2(x_\perp) .
$$

These currents must be covariantly conserved – $[D_\mu, J_1^\mu] = 0$ – which leads to a feedback of the gauge field on the currents. Therefore, in general, the eqs. (2) get modified by corrections of higher order in $\rho_{1,2}$. An important observation is that the strength of the sources lead to non-perturbative effects, in the sense that an infinite set of diagrams must be summed in order to calculate a quantity at a fixed order in $g$. However, the large strength of the sources has also a valuable consequence: as we shall see later, the leading order is dominated by tree diagrams only, and it can be studied by classical methods.

## 2 AGK cancellations

Following [4], let us first consider the theory of a real scalar field coupled to strong sources. Most of the structural properties we want to discuss can indeed already be studied in this simpler framework. Some general results – that are to a large extent independent of the details of the theory under consideration – can be obtained by considering the generating function for the probabilities $P_n$ of producing a given number $n$ of particles:

$$
\mathcal{F}(z) \equiv \sum_{n=0}^{\infty} P_n z^n .
$$

In [4], we have proven that, if we denote $b_r/g^2$ the sum of all the cut connected vacuum-vacuum diagrams\(^1\) with exactly $r$ cut lines, the logarithm of $\mathcal{F}(z)$ reads

$$
\ln \mathcal{F}(z) = \frac{1}{g^2} \sum_{r=1}^{\infty} b_r (z^r - 1) .
$$

\(^1\)The explicit factor $1/g^2$ represents the natural order of these cut diagrams when the sources are strong.
From eqs. (3) and (4), one gets the following expression\(^2\) for the probability \(P_n\):

\[
P_n = e^{-\sum_r b_r/g^2} \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{r_1+\cdots+r_p=n} \prod_{r=1}^{p} \frac{b_{r_1} \cdots b_{r_p}}{g^{2p}}.
\]

(5)

In this formula, the index \(p\) is the number of connected cut subdiagrams from which the \(n\) particles are produced: \(r_1\) particles are produced by the first of these \(p\) subdiagrams, \(r_2\) by the second one, etc... If we further expand the prefactor \(\exp(-\sum_r b_r/g^2)\), we obtain:

\[
P_n = \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \left( \sum_r \frac{b_r}{g^2} \right)^q \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{r_1+\cdots+r_p=n} \prod_{r=1}^{p} \frac{b_{r_1} \cdots b_{r_p}}{g^{2p}}.
\]

(6)

In this expression for \(P_n\), the index \(q\) is the number of connected subdiagrams that are not cut (this can be seen from the fact that it comes from the absorptive correction, whose sole role is to preserve unitarity).

The term of fixed \(p\) and \(q\) in this formula can therefore be interpreted as the probability of producing \(n\) particles from \(p+q\) connected subdiagrams, \(p\) of which are cut and \(q\) of which are not cut. By summing this probability over \(n\), one “integrates out” some degrees of freedom in order to keep only the information about the probability of having \(p\) cut subdiagrams and \(q\) that are not cut, regardless of the number of produced particles. This probability reads:

\[
\mathcal{R}_{p,q} = \frac{(-1)^q}{p!q!} \left( \sum_r \frac{b_r}{g^2} \right)^{p+q}.
\]

(7)

Summing this expression over \(q\) from 0 to \(\infty\), one finally obtains the probability of having \(p\) cut subdiagrams:

\[
\mathcal{R}_p = e^{-\sum_r b_r/g^2} \frac{1}{p!} \left( \sum_r \frac{b_r}{g^2} \right)^p.
\]

(8)

One therefore sees that the number of cut subdiagrams has a Poissonian distribution, with an average of \(\langle n \rangle_{\text{cut}} = \sum_r b_r/g^2\). Eqs. (7) and (8) are the essence of the Abramovsky-Gribov-Kancheli cancellations\(^1\). As one can see from the above derivation, they are simply a consequence of the factorization of a generic diagram in terms of its connected subdiagrams. Therefore, we expect them to be much more general than the context of reggeons field theories in which they have first been discussed. Another point should also be obvious at this point: in order to obtain the eqs. (7) and (8), one has “integrated out” the number \(n\) of produced particles. By doing this, the infinite sequence \(b_1, b_2, b_3, \cdots\) has reduced to the single combination \(\sum_r b_r\). This means that a lot of the dynamical information about the theory under consideration has been lost in this process, and that there are certain questions that cannot be answered anymore by the sole knowledge of the \(\mathcal{R}_{p,q}\)’s.

\(^2\)A model for the coefficients \(b_r\) has recently been proposed in\(^3\).
3 Generating function

Let us now consider the generating function \( F(z) \) per se, and discuss what it would take to calculate it. The following results on this question have been established in [4]:

(i) Diagrammatically, \( F(z) \) is the sum of all the cut vacuum-vacuum diagrams, where each cut propagator is weighted by a factor \( z \).

(ii) At leading order, the derivative of \( \ln F(z) \), \( F'(z)/F(z) \) can be calculated from a pair of solutions \( \Phi_{\pm}(z|x) \) of the classical equation of motion. For a scalar field theory with a cubic coupling and a source \( j \), the classical EOM reads

\[
(\Box_x + m^2)\Phi_{\pm}(z|x) + \frac{g^2}{2}\Phi_{\pm}^2(z|x) = j(x) .
\]

The expression of \( F'(z)/F(z) \) is simpler if written in terms of the Fourier modes \( f_{\pm}(\pm)(z|x_0,p) \) and \( f_{\pm}(\pm)(z|x_0,p) \) of these classical fields,

\[
\Phi_{\pm}(z|x) = \int\frac{d^3p}{(2\pi)^{3/2}2p} \left\{ f_{\pm}(\pm)(z|x_0,p) e^{-ip\cdot x} + f_{\pm}(\pm)(z|x_0,p) e^{+ip\cdot x} \right\},
\]

and reads

\[
\frac{F'(z)}{F(z)} \bigg|_{LO} = \int\frac{d^3p}{(2\pi)^{3/2}2p} f_{\pm}(\pm)(z|+\infty,p) f_{\pm}(\pm)(z|+\infty,p).
\]

(iii) The two solutions \( \Phi_{\pm} \) of the classical EOM must obey the following boundary conditions:

\[
\begin{align*}
&f_{\pm}^{(+)}(z|0,-\infty,p) = 0, \quad \ f_{\pm}^{(-)}(z|0,-\infty,p) = 0, \\
&f_{\pm}^{(+)}(z|0,+\infty,p) = z f_{\pm}^{(+)}(z|0,+\infty,p), \\
&f_{\pm}^{(-)}(z|0,+\infty,p) = z f_{\pm}^{(-)}(z|0,+\infty,p).
\end{align*}
\]

(iv) From unitarity, \( F(1) = \sum_n P_n = 1 \). This property serves as the initial condition for going from \( F'(z)/F(z) \) to \( F(z) \), by writing:

\[
F(z) = \exp \left\{ \int_1^z d\tau \frac{F'(\tau)}{F(\tau)} \right\} .
\]

Unfortunately, finding the pair of solutions of the classical equation of motion that obey the boundary conditions of eqs. [12] is a difficult numerical problem, that has not yet been studied in this context. However, as we shall see in the next section, one can obtain from here a formula for the average multiplicity which is much easier to evaluate numerically.
4 Average multiplicity at leading order

4.1 General method

The moments of the distribution of multiplicities, in particular the average multiplicity \( \langle n \rangle \equiv \sum_n n P_n \), enjoy a special status because extra simplifications occur in their calculation. Let us consider the multiplicity since it is the simplest one. One can get the multiplicity from the generating function as

\[
\langle n \rangle = \mathcal{F}'(1) = \frac{\mathcal{F}'(1)}{\mathcal{F}(1)}.
\]

(The second equality is of course simply due to \( \mathcal{F}(1) = 1 \).) Therefore, calculating \( \langle n \rangle \) is a special case – with \( z = 1 \) – of the calculation of \( \mathcal{F}'(z)/\mathcal{F}(z) \). At \( z = 1 \), the third and fourth of the boundary conditions in eqs. (12) simply reduce to

\[
\begin{align*}
\Phi^+(1|x^0 = +\infty, p) &= \Phi^+(1|x^0 = +\infty, p), \\
\Phi^-(1|x^0 = +\infty, p) &= \Phi^-(1|x^0 = +\infty, p),
\end{align*}
\]

which means that the fields \( \Phi^+ \) and \( \Phi^- \) are equal at large positive times, as well as their first time derivative. Since the classical equation of motion is deterministic, this implies that these two classical fields are equal at all times. The first two boundary conditions in eqs. (12) then imply that \( \Phi^\pm \) are vanishing at large negative times, as well as their first time derivative.

Therefore, at leading order, the multiplicity \( \langle n \rangle \) is given by eqs. (10) and (11) with \( \Phi^+ = \Phi^- \) the retarded solution of the classical EOM with a null initial condition at \( x_0 = -\infty \). One should emphasize the following: it was obvious from the beginning that the multiplicity at leading order would be somehow related to solutions of the classical equation of motion – since it involves only tree diagrams at this order – but it is a non-trivial result that this is with retarded boundary conditions. The retarded nature of the boundary condition is crucial in practice, in order to solve this problem numerically.

4.2 Gluon multiplicity at leading order

Going back to QCD, it is straightforward to generalize the previous results to the case of gluon production. The inclusive gluon spectrum is given at leading order by

\[
E_p \frac{d\langle n_{\text{gluons}} \rangle_{\text{LO}}}{d^3 \vec{p}} = \frac{1}{16\pi^3} \sum_\lambda \int_{x,y} e^{i\vec{p} \cdot (x-y)} \Box_x \Box_y \varepsilon_\lambda \cdot A_R(x) \varepsilon_\lambda \cdot A_R(y).
\]

In this formula, \( A_R^\mu(x) \) is the retarded solution of the classical Yang-Mills equations – in the presence of the sources \( \rho_{1,2} \) – and with \( A_R^\mu = 0 \) and \( \partial^0 A_R^\mu = 0 \) at \( x_0 = -\infty \). This problem was solved numerically in [7]. The diagrams\(^3\) that are

\[^3\text{The propagators that appear in this diagrammatic representation are not Feynman propagators, but retarded propagators.}\]
4.3 Quark production at leading order

Similarly, the production of quarks at leading order has been considered in [8]. Note that we use “leading order” somewhat abusively here, since quark production is strictly speaking a Next-to-Leading Order effect – indeed, in the saturated regime, the number of produced gluons scales like $g^{-2}$ while the number of produced quarks scales like $g^0$. For quark production at leading order, one must use the following formula,

$$E_p \frac{d\langle n_{\text{quarks}} \rangle_{\text{LO}}}{d^3 \vec{p}} = \frac{1}{16\sqrt{\pi}} \int_{x,y} \int_q e^{ip \cdot (x-y)} (i\partial_x - m)(i\partial_y + m) \overline{\psi_q}(x)\psi_q(y),$$

(17)

where $\psi_q(x)$ is the retarded solution of the Dirac equation – with the classical field obtained in the calculation of gluon production in the background – with a free negative energy spinor, $v(q)e^{iq \cdot x}$, as the initial condition. The diagrams involved in eq. (17) are sketched in the left of figure 2. Also represented in the right part of this figure is the resulting quark spectrum, for various quark masses.
Figure 2: Left: space-time representation of the diagrams involved in the calculation of the quark multiplicity at leading order. Right: the resulting quark spectra for various quark masses.

5 Further developments

5.1 Gluon multiplicity at NLO

In fact, the production of quarks is one of the pieces – the simplest one – that contribute to particle production at NLO, i.e. at order $g^0$. In [4], we have detailed the principles of a full NLO calculation of the particle yield, in the case of scalar fields. Of course, things will be more complicated in QCD with gluons, but one crucial property of the result will survive: the calculation of particle production at NLO can be done from retarded solutions of the classical EOM and retarded solutions of the EOM of a small fluctuation on top of the classical field. The crucial point again is the fact that these objects are needed with retarded boundary conditions, which means that it is a problem which is tractable numerically in a straightforward way.

Two types of topologies, sketched in figure 3, contribute to the gluon multiplicity at NLO [9]. One of them (left diagram) is very similar to that already encountered in quark production – it corresponds to the production of pairs of gluons, and involves retarded solutions for the equation of motion of small gluonic fluctuations on top of the classical field. The diagram on the right can be seen as a 1-loop virtual correction to the classical field – the field in the complex conjugate amplitude remaining the tree-level one. It was shown in [4] that the latter contribution can also be expressed in terms of retarded solutions of the small fluctuations equation. Therefore, the result according to which the inclusive multiplicity can be calculated from retarded solutions of some equations of motion remain true at NLO. To this diagram with a virtual gluon loop, one must add two similar diagrams, respectively with a quark loop and a ghost loop (only in gauges that have ghosts for the latter).

One additional issue arises when one considers these one-loop corrections
to the gluon yield: some of the contributions have a divergence of the form
\( \alpha_s \int dx/x \) which is reminiscent of the divergences already resummed by the
JIMWLK evolution of the distribution of sources \( W[\rho_1] \) and \( W[\rho_2] \) for the two
projectiles. For the CGC framework to be self-consistent, one must prove that
these divergences that appear in the gluon yield for fixed \( \rho_1 \) and \( \rho_2 \) can be
absorbed in the evolution of the \( W[\rho_1, \rho_2] \) \[9\].

5.2 Survival probabilities

One can also consider exclusive processes in this framework. For instance, in-\nstead of the plain – fully inclusive – probability \( P_n \) of producing \( n \) particles, one
may define a probability \( P_n(\Omega) \) of producing \( n \) particles in a certain region \( \Omega \)
of the phase-space and none outside of \( \Omega \), and construct a generating function
for these exclusive probabilities, \( F_\Omega(z) \equiv \sum_{n=0}^{\infty} P_n(\Omega) z^n \). One can show \[10\]
that this generating function can in principle be calculated at leading order by
methods that are similar to those described in section 3 modulo two differences.

(i) The boundary conditions for the two classical fields in terms of which one
can express \( F'_{\Omega}(z)/F_{\Omega}(z) \) read:
\[
\begin{align*}
  f_+^{(+)}(z|x^0 = -\infty, p) &= 0, & f_-^{(-)}(z|x^0 = -\infty, p) &= 0, \\
  f_-^{(+)}(z|x^0 = +\infty, p) &= z \Omega(p) f_+^{(+)}(z|x^0 = +\infty, p), \\
  f_+^{(-)}(z|x^0 = +\infty, p) &= z \Omega(p) f_-^{(-)}(z|x^0 = +\infty, p),
\end{align*}
\]
where \( \Omega(p) \) is a function which is 1 in the region \( \Omega \) and zero outside.

(ii) The “integration constant” \( F_{\Omega}(1) \) is no longer unity. In fact, this quantity
is the total probability of not producing particles outside of \( \Omega \). It will
appear as a prefactor in all the probabilities $P_n(\Omega)$, and it can therefore be interpreted as a survival probability for the empty region outside $\Omega$.

6 Conclusions

Multiparticle production in field theories coupled to external time-dependent sources – e.g. in the Color Glass Condensate framework for hadronic collisions at high energy – exhibits some non-trivial features when these sources are as strong as the inverse coupling: even in the weak coupling regime, one must resum at each order an infinite set of diagrams. Quite generically, one recovers in this type of model the combinatoric relations among the probabilities of cut subdiagrams that lead to the AGK cancellations.

At leading order, both the generating function of the probability distribution and the average multiplicity can be calculated from solutions of the classical equation of motion in the presence of the external sources. However, while these solutions must be found with complicated boundary conditions in the case of the generating function, the problem can be reduced to finding retarded solutions in the case of the multiplicity, leading to straightforward algorithms for calculating the multiplicity numerically. This has been done at leading order in the CGC framework for the production of gluons and quarks in nucleus-nucleus collisions.

The present study can be extended in several directions. One of these extensions is the production of gluons at NLO (NLO particle production has in fact been studied in [4], but the techniques developed there must be extended to gauge theories). Another extension is the study of exclusive reactions, where one enforces some constraints on the phase-space of the produced particles.

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