We introduce the concepts of Bayesian lens, characterizing the bidirectional structure of exact Bayesian inference, and statistical game, formalizing the optimization objectives of approximate inference problems. We prove that Bayesian inversions compose according to the compositional lens pattern, and exemplify statistical games with a number of classic statistical concepts, from maximum likelihood estimation to generalized variational Bayesian methods. This paper is the first in a series laying the foundations for a compositional account of the theory of active inference, and we therefore pay particular attention to statistical games with a free-energy objective.

1. Introduction

Those systems that we might classify as ‘cybernetic’, ‘adaptive’, or ‘alive’ all display a fundamental property: they resist perturbations that would push them away from their goals or render their existence unsustainable. In order to do so, such systems are somehow able to sense their current state of affairs (through perception) and respond appropriately (through action). In the series of papers of which this is the first part, we seek to supply new compositional foundations for a theory of active inference adequate to describe such systems, with a particular focus on the framework that has come to be known in the compositional neuroscience and artificial life communities as the free energy principle [1], whose structures we seek to make precise.

A central feature of active inference is the use of the statistical procedure called Bayesian inference, which supplies a recipe by which a system might invert a statistical model (say, of how causes generate observations) in order to form beliefs about the causes of observed data. It is easy to see how such a process of inferring causes could be understood as a process of perception, but the central dogma of active inference is that both perception and action can be rendered as problems of Bayesian inference, with action being ‘dual’ to perception: instead of changing its internal state (its beliefs about causes) to match its observations better, a system might act to change the external state (the state of the world) so that the observations that it expects or desires obtain. In the free energy framework, both perception and action emerge through the optimization of a single quantity, the free energy.
Such processes of optimization, and perception and action more generally, are inherently dynamical processes. In this first paper of the series, we put the dynamics temporarily aside, and lay the statistical foundations, characterizing the compositional structure of the generative models instantiated by cybernetic systems and the algebra of their inversion. This algebra is formalized by our concept of Bayesian lens, which we introduce to characterize the inherently bidirectional structure of Bayesian inversion, drawing on the 'lens' pattern that structures bidirectional systems from economic games [2], to databases [3], and machine-learners [4].

This lens structure serves more than an organizing purpose: we prove that the inversion of a composite or 'hierarchical' statistical model is equivalently given (up to almost-equality) by the lens composition rule. This means that cybernetic systems embodying complex composite models can simply invert each component factor of their models and then combine these inversions, in order to obtain an inversion of the whole. In turn, this explains the observation that hierarchical systems in the brain (such as much of the visual cortex) can be explained as a composite of 'local' circuits each performing a form of approximate Bayesian inference call predictive coding [5].

Having established the structures required to state and prove that “Bayesian updates compose optically”\(^1\), we formalize the "algebra of statistical inference problems" as categories of statistical games. These 'games' consist of a lens paired with a contextual fitness function, which define the quantities that we often think of cybernetic systems as optimizing, and where the 'context' formalizes the system’s interaction with its environment. In this development, we draw much inspiration from compositional game theory [2, 7]. We exemplify these statistical games with a range of examples from maximum likelihood estimation to generalized variational Bayesian methods.

This paper is the first of a series of papers. The next instalment introduces the structures necessary to supply statistical games with "dynamical semantics", and thus breathe some life into those systems that perform approximate inference. A subsequent paper will then explain how such systems can perform action, and thereby affect the worlds that they inhabit.

**Overview of this paper** We begin in §2.1 by introducing the basics of compositional probability theory. In §2.2, we introduce the structures necessary to describe the lens pattern, and recall that in ‘nice’ situations, the resulting categories of lenses are monoidal. In §2.3, we introduce the Para construction, which has been proposed [8] as foundational for categorical cybernetics, and often plays an important role for us, too. Then, in §3, we define Bayesian lenses and prove the theorem that Bayesian inversions compose according to the lens pattern; we also give a detailed description of the low-dimensional structure of parameterized Bayesian lenses (§3.4). In §4, we define contexts for Bayesian lenses, fitness functions, and the resulting monoidal categories of statistical games. Finally, in §5, we give a number of examples.

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\(^1\)Although we do not use the heavier machinery of ‘optics’ here; for that, see our preprint [6] and Remark 2.41 below.
2. Mathematical background for statistical games

2.1. Compositional probability, concretely and abstractly

In order to define Bayesian lenses and statistical games and prove some basic results about them, we will work at a high level of abstraction; then, to exemplify them with applications, we will need to work more concretely. In both instances, our basic categorical setting will be a copy-delete or Markov category, whose morphisms we will call stochastic channels or Markov kernels and which behave like functions with uncertain outputs; for us, these model the processes by which observational data are (believed to be) generated by processes in the world, and composite channels model composite (sequences of) processes.

We will typically be interested in applications where the sample spaces are continuous and where the probability measures may have infinite support. A stochastic channel \( c : X \to Y \) will be something like a function taking values in probability measures over a space, but in applications one will often fix a reference measure and then work with a density function \( p_c : X \times Y \to [0, 1] \) representing the channel \( c \) with respect to that measure. We begin this section by introducing the category \( \text{sfKrn} \) of s-finite kernels, where these concepts have precise meanings, before generalizing graphically to the abstract setting of copy-delete categories.

2.1.1. S-finite kernels

We sketch the basic structure of the category \( \text{sfKrn} \) of s-finite kernels between measurable spaces, and refer the reader to Cho and Jacobs [9] and Staton [10] for elaboration of the details.

**Definition 2.1.** Suppose \((X, \Sigma_X)\) and \((Y, \Sigma_Y)\) are measurable spaces, with \(X, Y\) sets and \(\Sigma_X, \Sigma_Y\) the corresponding \(\sigma\)-algebras. A kernel \(k\) from \(X\) to \(Y\) is a function \(k : X \times \Sigma_Y \to [0, \infty]\) satisfying the conditions:

- for all \(x \in X\), \(k(x, -) : \Sigma_Y \to [0, \infty]\) is a measure; and
- for all \(B \in \Sigma_Y\), \(k(-, B) : X \to [0, \infty]\) is measurable.

A kernel \(k : X \times \Sigma_Y \to [0, \infty]\) is finite if there exists some \(r \in [0, \infty]\) such that, for all \(x \in X\), \(k(x, Y) \leq r\). And \(k\) is s-finite if it is the sum of at most countably many finite kernels \(k_n\), \(k = \sum_{n \in \mathbb{N}} k_n\).

**Proposition 2.2.** Measurable spaces and s-finite kernels \((X, \Sigma_X) \to (Y, \Sigma_Y)\) between them form a category, denoted \(\text{sfKrn}\); note that often we will just write \(X\) for a measurable space, leaving the \(\sigma\)-algebra \(\Sigma_X\) implicit. Identity morphisms \(\text{id}_X : X \to X\) in \(\text{sfKrn}\) are Dirac kernels \(\delta_X : X \times \Sigma_X \to [0, \infty] := x \times A \mapsto 1\) iff \(x \in A\) and 0 otherwise. Composition is given by a Chapman-Kolmogorov equation: suppose \(c : X \to Y\) and \(d : Y \to Z\). Then

\[
d \circ c : X \times \Sigma_Z \to [0, \infty] := x \times C \mapsto \int_{y \in Y} d(C|y) c(dy|x)
\]

where we have used ‘conditional probability’ notation \(d(C|y) := d(y, C)\).

**Remark 2.3.** In the scientific literature, one often encounters the term conditional probability distribution, which can almost always be interpreted as indicating a probability kernel: we can think of a probability kernel \(c : X \to Y\) as a function that emits a probability distribution \(c(x) : \Sigma_Y \to [0, \infty]\) over \(Y\) for each \(x : X\). We can then see \(c(x)\) as a probability distribution conditional on the choice or observation of \(x : X\), hence the notation \(c(-|x)\). One reads this notation "\(x\) given \(-\)."

**Proposition 2.4.** There is a monoidal structure \((\otimes, 1)\) on \(\text{sfKrn}\), the unit of which is the singleton set 1 with its trivial sigma-algebra. On objects, \(X \otimes Y\) is the Cartesian product \((X \times Y, \Sigma_{X \times Y})\) of measurable spaces,
with the product sigma-algebra. On morphisms, \( f \otimes g : X \otimes Y \to A \otimes B \) is given by
\[
f \otimes g : (X \times Y) \times \Sigma_{A \times B} := (x \times y) \times E \mapsto \int_{a:A} \int_{b:B} \delta_{A \otimes B}(E|x, y) \ f(da|x) \ g(db|y)
\]
where, as above, \( \delta_{A \otimes B}(E|a, b) = 1 \) iff \((a, b) \in E\) and 0 otherwise. Note that \( (f \otimes g)(E|x, y) = (g \otimes f)(E|y, x) \) for all s-finite kernels (and all \( E, x \), and \( y \)), by the Fubini-Tonelli theorem for s-finite measures [9, 10], and so \( \otimes \) is symmetric on sfKrn.

**Remark 2.5** (States and effects). We will call kernels with domain 1 states. A kernel \( 1 \to X \) is equivalently a function \( \Sigma_X \to [0, \infty] \), which is simply a (possibly improper) measure on the space \((X, \Sigma_X)\). Occasionally we will say distribution to mean 'state'. We call a state on a product space, such as \( \omega : 1 \to X \otimes Y \), a joint state or joint distribution.

Dually, kernels with codomain 1 will be called effects. Note that although 1 is the unit of the preceding monoidal structure, this unit is not terminal in sfKrn: s-finite kernels \( X \to 1 \) are equivalently measurable functions \( X \to [0, \infty] \), and there are of course many nontrivial examples; of which density functions will form an important class.

**Definition 2.6** (Probability kernel, probability measure, probability space). If a kernel \( k : X \times \Sigma_Y \to [0, \infty] \) satisfies the additional conditions that it takes values in the unit interval \([0, 1]\) and, for all \( x : X \), \( k(Y) = 1 \), then we call \( k \) a probability kernel. If \( k \) is a state (i.e., \( X = 1 \)), then we call it a probability measure. A probability space is a pair \((\Omega, \pi)\) of a measurable space \( \Omega \) with a probability measure \( \pi : 1 \to \Omega \).

**Remark 2.7** (Giry monad). Probability measures \( 1 \to X \) on each \( X \) form the points of a space which we will denote \( G X \). This space can be equipped with a canonical \( \sigma \)-algebra, making \( G \) into a functor \( \text{Meas} \to \text{Meas} \) which acts on each measurable function \( f : X \to Y \) by returning its pushforward \( f_* : G X \to G Y \), defined by \( f_* : \Sigma_Y \to [0, 1] : B \mapsto \nu(f^{-1}(B)) \). \( G \) can in turn be equipped with a monad structure \( (\mu, \eta) \). Every measurable function \( X \to G Y \) corresponds to a probability kernel \( X \to \Sigma_Y \), making the Kleisli category \( K \ell(G) \) a subcategory of sfKrn. Composition of probability kernels \( X \overset{f}{\to} Y \overset{g}{\to} Z \) corresponds accordingly to Kleisli composition \( X \overset{g \circ f}{\to} Z \). \( G \) is a probability Preserving kernel, making the Kleisli category \( K \ell(G) \) under \( 1 \) is the category of probability spaces and measure-preserving kernels between them, called \( \text{ProbStoch} \) by Fritz [11].

**Remark 2.8** (Convex spaces and expectations). A convex space is an algebra of the Giry monad; that is, a space \( X \) equipped with a measurable function \( G X \to X \) called the algebra evaluation or expected value. Each measurable function \( f : X \to X \) induces an expected value \( \mathbb{E}_f : G X \to X \) defined as
\[
\mathbb{E}_f(\pi) := \int_{x : X} f(x) \, \pi(dx).
\]
We will typically instead write
\[
\mathbb{E}_{x \sim \pi} [f] := \mathbb{E}_f(\pi)
\]
where the notation \( x \sim \pi \) should be read as “\( x \) distributed according to \( \pi \)”. More generally, we have an operator \( \mathbb{E} : \text{Meas}(\Omega, X) \times G \Omega \to X \) defined similarly by
\[
\mathbb{E}_{\omega \sim \pi} [p] := \int_{\omega : \Omega} p(\omega) \, \pi(d\omega)
\]
where \( p : \Omega \to X \) and \( \pi : G \Omega \). Note that this subsumes the case where \( p \) is an \( X \)-valued random variable defined on a probability space \((\Omega, \pi)\). Commonly, we will have \( X = \mathbb{R} \) or \( X = [0, \infty] \).
Remark 2.9 (Effects and validities). A special case of the preceding expectation operator occurs when \( X = [0, \infty) \). In this case, maps \( \Omega \to [0, \infty) \) are of course effects \( \Omega \to 1 \) in \text{sfKrnn}, and for each state \( \pi : 1 \to \Omega \) and effect \( p : \Omega \to 1 \), the expectation operator simply computes the composite \( p \circ \pi \). That is, we have in this case \( E_{\omega \sim \pi}[p] = p \circ \pi \), and we might then call \( p \) a predicate on \( \Omega \) and the expectation \( E_{\omega \sim \pi}[p] \) the validity of \( p \) in the state \( \pi \). We refer the reader to Cho et al. [12, §5] for more on this perspective, where the validity is written \( \pi \models p \).

Observation 2.10. Each space \( X \) in \text{sfKrnn} is equipped with a canonical effect \( \Phi_X : X \to 1 \), the constant effect \( x \mapsto 1 \). We denote this family of effects by the `ground' symbol \( \Phi \) and call the components discarding maps with the intuition that they act by `discarding' information ("wiring to ground"). Another way to characterize channels or processes causal: they cannot affect the outcomes of processes earlier in a sequence of composites.

Not only can we discard information in \text{sfKrnn}, but we can also copy it. In other words, each object is equipped with a canonical copy-delete structure, making it into a comonoid, with which we can duplicate and discard states of the corresponding type. In the terms of Fong and Spivak [13], \text{sfKrnn} supplies comonoids.

Proposition 2.11 (\text{sfKrnn} supplies comonoids). Each object \( X \) is equipped with a canonical comonoid structure \((\check{\Phi}_X, \Phi_X)\), with \( \check{\Phi}_X : X \to X \otimes X \) and \( \Phi_X : X \to 1 \) satisfying the usual comonoid laws. Discarding is given by the family of effects \( \Phi_X : X \to [0, \infty] := x \mapsto 1 \), and copying is Dirac-like: \( \check{\Phi}_X : X \times \Sigma_X \to X \otimes X := x \times E \mapsto 1 \) iff \( (x, x) \in E \) and 0 otherwise.

Discarding part of a joint state gives us marginals (or "marginal distributions").

Definition 2.12. Given a joint distribution \( \omega : 1 \to X \otimes Y \), we call \( \omega_X := (\text{id}_X \otimes \check{\Phi}_Y) \cdot \omega : 1 \to X \) and \( \omega_Y := (\check{\Phi}_X \otimes \text{id}_Y) \cdot \omega : 1 \to Y \) the marginals of \( \omega \). We define projection (or marginalization) operators by \( \text{proj}_X := \text{id}_X \otimes \check{\Phi}_Y : X \otimes Y \to X \otimes 1 \) and \( \text{proj}_Y := \check{\Phi}_X \otimes \text{id}_Y : X \otimes Y \to 1 \otimes Y \).

In this work, we are interested in the problem of inverting stochastic channels \( X \to Y \) in order to obtain channels \( Y \to X \), and we are particularly interested in what is known as Bayesian inversion. As we will see, the Bayesian inversion of a channel \( c : X \to Y \) is determined in conjunction with a state \( \pi : 1 \to X \).

Definition 2.13. We call the pairing \( (\pi, c) \) of a state \( \pi : 1 \to X \) with a channel \( c : X \to Y \) a generative model \( X \to Y \). It induces a joint distribution \( \omega_{(\pi, c)} := (\text{id}_X \otimes c) \cdot \check{\Phi}_X \cdot \pi : 1 \to X \otimes Y \). The marginals of \( \omega_{(\pi, c)} \) are \( \pi \) and \( c \cdot \pi \).

In the informal scientific literature, the Bayesian inversion of a channel \( X \to Y \) (typically called the 'likelihood') with respect to a state \( 1 \to X \) (typically called the 'prior') is often written as the expression

\[
p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int_{x' \in X} p(y|x')p(x') \text{d}x'},
\]

but this expression is very ill-defined: what is \( p(y|x) \), and how does it relate to a channel \( c : X \to Y \)? Why are the clearly different terms \( p(x|y), p(y|x), p(x), p(y) \) all written with the same symbol \( p \)?

To answer these questions and clarify such expressions, we use density functions.

Definition 2.14 (Density functions). We will say that a kernel \( c : X \to Y \) is represented by the effect \( p_c : X \otimes Y \to 1 \) with respect to the state \( \mu : 1 \to Y \) when

\[c : X \times \Sigma_Y \to [0, \infty] := x \times B \mapsto \int_{y \in B} \mu(dy) p_c(y|x).\]

We call the corresponding function \( p_c : X \times Y \to [0, \infty] \) a density function for \( c \). Note that we also use conditional probability notation for density functions, and so \( p_c(x, y) := p_c(x, y) \).
Remark 2.15. When a channel $c$ is associated with a density function, we will adopt the convention of naming the density function $p_c$; that is, with a subscript indicating the corresponding channel. In this way, we can rewrite Equation (1) as

$$p_{c;\pi}(x|y) = \frac{p_c(y|x) p_{\pi}(x)}{p_{c;\pi}(y)} = \frac{p_c(y|x) p_{\pi}(x)}{\int_{x':X} p_c(y|x') p_{\pi}(x') \, dx'}. \quad (2)$$

Remark 2.16. We will also adopt the convention of denoting a Bayesian inversion of the channel $c$ with respect to the state $\pi$ by the symbol $c^\dagger$. We adopt this symbol because it is known that Bayesian inversion induces a ‘dagger’ functor [14] on (a quotient of) the category $1/K\ell(G)$ of probability spaces and measure-preserving functions [11, Remark 13.9]; when we ‘forget’ the measures associated with the probability spaces—which form the ‘priors’ for the inversions—then we have to explicitly incorporate them into the structure, which we indicate with the subscript $\pi$ in $c^\dagger$. We are now in a position to define Bayesian inversions for channels in $\text{sfKrn}$, although we leave the abstract definition satisfied by the following until Definition 2.30.

Proposition 2.19 (Cho and Jacobs [9, Example 8.4]). Suppose $(\pi, c)$ is a generative model $X \xrightarrow{\pi} Y$ in $\text{sfKrn}$, where $c$ is represented by the effect $p_c$ with respect to the state $\mu : 1 \rightarrow X$. Then, when it exists, the channel $c^\dagger_{\pi} : Y \rightarrow X$ defined as follows is a Bayesian inversion of $c$ with respect to $\pi$:

$$c^\dagger_{\pi} : Y \times \Sigma_X \rightarrow [0, \infty] := y \times A \mapsto \left( \int_{x:A} \pi(dx) p_c(y|x) \right) p_c^{-1}(y)$$

$$= p_c^{-1}(y) \int_{x:A} p_c(y|x) \pi(dx),$$

where $p_c^{-1} : Y \rightarrow I$ is given up to $\mu$-almost-equality by

$$p_c^{-1} : Y \rightarrow [0, \infty] := y \mapsto \left( \int_{x:X} p_c(y|x) \pi(dx) \right)^{-1}.$$

Note that from the preceding proposition we recover the informal form of Bayes’ rule (Equation (2)). Suppose $\pi$ is itself represented by a density function $p_{\pi}$ with respect to the Lebesgue measure $dx$. Then

$$c^\dagger_{\pi}(A|y) = \int_{x:A} \frac{p_c(y|x) p_{\pi}(x)}{\int_{x':X} p_c(y|x') p_{\pi}(x') \, dx'} \, dx.$$

### 2.1.2. Copy-delete categories and their graphical calculus

While most of our examples and applications will found in $\text{sfKrn}$, most of our definitions and results hold more generally, and it is in such more general terms that they are most naturally expressed. Our main language will be that of categories like $\text{sfKrn}$ in which information can be transformed, copied, and deleted.
Definition 2.20 (Cho and Jacobs [9, Def. 2.2]). A copy-delete category is a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ in which every object $X$ is supplied with a commutative comonoid structure $(\sqcap_X, \hat{\sqcap}_X)$ compatible with the monoidal structure of $(\otimes, I)$. An affine copy-delete category, or Markov category [11], is a copy-delete category in which every channel $c$ is causal in the sense that $\hat{\sqcap} \cdot c = \hat{\sqcap}$. Equivalently, a Markov category is a copy-delete category in which the monoidal unit $I$ is the terminal object.

Example 2.21. $\sf{Krn}$ is a copy-delete category, while $\mathcal{K}(\mathcal{G})$ is a Markov category.

Monoidal categories, and (co)monoids within them, admit a formal graphical calculus that substantially simplifies many calculations involving complex morphisms: proofs of many equalities reduce to visual demonstrations of isotopy, and structural morphisms such as the symmetry of the monoidal product acquire intuitive topological depictions. We make substantial use of this calculus below, and summarize its features here. For more details, see Cho and Jacobs [9, §2] or Fritz [11, §2] or the references cited therein.

Depiction 2.22 (Basic conventions). String diagrams in this paper will be read vertically, with information flowing upwards (from bottom to top); in later parts, we will have diagrams oriented left-to-right. Sequential composition is represented by connecting strings together; and parallel composition $\otimes$ by placing diagrams adjacent to one another. This way, $c : X \to Y$, $\text{id}_X : X \to X$, $d \cdot c : X \to X$, $d$, $f \otimes g : X \otimes Y \to A \otimes B$ are depicted respectively as:

We represent (the identity morphism on) the monoidal unit $I$ as an empty diagram: that is, we leave it implicit in the graphical representation.

Depiction 2.23 (States and effects). States $\sigma : I \to X$ and effects $\eta : X \to I$ will be depicted as follows:

Definition 2.24 (Causality). We say that a morphism is causal if it satisfies the following condition, where $\hat{\sqcap}$ is the canonical discarding map (supplied by the copy-delete category structure) of the appropriate type; compare Observation 2.10.

Remark 2.25. Observe that, if the monoidal unit is terminal, then every morphism is causal.

Depiction 2.26 (Monoidal symmetry). The symmetry of the monoidal structure $\text{swap}_{XY} : X \otimes Y \to Y \otimes X$ is depicted as the swapping of wires, and satisfies the equations below. The left says that swapping is an
isomorphism; the right says that it commutes with copying, making every object a *commutative* comonoid:

\[
\begin{align*}
\text{Diagram 2.27} & \quad \text{(Comonoid laws)} \\
\text{Diagram 2.28} & \quad \text{(Marginalization of joint states)} \\
\text{Diagram 2.29} & \quad \text{(Generative models)}
\end{align*}
\]

**Diagram 2.27** (Comomoid laws). The copy-delete structure \((\otimes, \top)\) is required to satisfy the comonoid laws, depicted below, of unitality (left depiction) and associativity (right depiction):

**Diagram 2.28** (Marginalization of joint states). The discarding maps induce projections \(X \otimes Y \xrightarrow{\text{id} \otimes \top} X \otimes I \xrightarrow{\top \otimes \text{id}} Y \otimes Y \xrightarrow{\text{id} \otimes \top} Y\), with which we can obtain the marginals of joint states; compare Definition 2.12. Suppose then that a joint state \(\omega : I \leftrightarrow X \otimes Y\) has marginals \(\omega_1 : I \leftrightarrow X\) and \(\omega_2 : I \leftrightarrow Y\). Then we have

**Diagram 2.29** (Generative models). A generative model \((\pi, c) : X \leftrightarrow Y\) induces a joint state \(\omega\) on \(X \otimes Y\) by

with marginals \(\pi\) and \(c \bullet \pi\) given by

**Definition 2.30** (Bayesian inversion). We say that a channel \(c : X \leftrightarrow Y\) *admits Bayesian inversion* with respect
to $\pi : I \leftrightarrow X$ if there exists a channel $c : Y \leftrightarrow X$ satisfying the following equation [9, eq. 5]:

$$
\begin{array}{c}
X \\
\downarrow c
\end{array}
= 
\begin{array}{c}
Y \\
\downarrow c_r
\end{array}

$$

We say that $c$ admits Bayesian inversion *tout court* if $c$ admits Bayesian inversion with respect to all states $\pi : I \leftrightarrow X$ such that $c \cdot \pi$ has non-empty support. We say that a category $C$ admits Bayesian inversion if all its morphisms admit Bayesian inversion *tout court*.

**Depiction 2.31** (Density functions). A channel $c : X \leftrightarrow Y$ is said to be represented by the effect $p_c : X \otimes Y \leftrightarrow I$ with respect to $\mu : I \leftrightarrow Y$ if

$$
\begin{array}{c}
Y \\
\downarrow c
\end{array}
= 
\begin{array}{c}
Y \\
\downarrow \mu
\end{array}

$$

We call $p_c$ a *density function* for $c$; compare Definition 2.14.

**Definition 2.32** (Almost-equality). Given a state $\pi : I \leftrightarrow X$, we say that two channels $c : X \leftrightarrow Y$ and $d : X \leftrightarrow Y$ are $\pi$-almost-equal, denoted $c \approx d$, if

$$
\begin{array}{c}
X \\
\downarrow c
\end{array}
\approx 
\begin{array}{c}
X \\
\downarrow d
\end{array}

$$

**Proposition 2.33** (Composition preserves almost-equality). If $c \approx d$, then $f \cdot c \approx f \cdot d$.

*Proof.* Immediate from the definition of almost-equality.
Proposition 2.34 (Bayesian inverses are almost-equal). Suppose \( \alpha : Y \leftrightarrow X \) and \( \beta : Y \leftrightarrow X \) are both Bayesian inversions of the channel \( c : X \leftrightarrow Y \) with respect to \( \pi : I \leftrightarrow X \). Then \( \alpha \sim \beta \).

Proof. Immediate from Equation (3). \qed

2.2. Lenses for dependent bidirectional processes

The Bayesian inversion of a stochastic channel \( c : X \leftrightarrow Y \) is a family of channels \( c_\pm : Y \leftrightarrow X \) in the opposite direction, indexed by states on \( X \). Pairs \((c, c_\pm)\) of a morphism \( c \) with a \( c \)-dependent opposite morphism \( c_\pm \) often fall into the compositional 'lens' pattern, and the Bayesian case is no exception. In this section, we sketch the basic theory of lenses, and refer the reader to our preprint [6] for further exposition. The central element of the structure is a (pseudo)functor picking out, for each morphism \( c \), the category in which \( c_\pm \) lives. With this piece to hand, an entire corresponding category of lenses can be defined most concisely.

Definition 2.35 (Spivak [16, Def. 3.3]). The category \( \text{GrLens}_F \) of Grothendieck lenses for a pseudofunctor \( F : \mathcal{C}^{op} \to \text{Cat} \) is the total category of the Grothendieck construction for the pointwise opposite of \( F \).

Proposition 2.36 (\( \text{GrLens}_F \) is a category). The objects \((\text{GrLens}_F)_0\) of \( \text{GrLens}_F \) are (dependent) pairs \((C, X)\) with \( C : \mathcal{C} \) and \( X : F(C) \), and its hom-sets \( \text{GrLens}_F((C, X), (C', X')) \) are dependent sums

\[
\text{GrLens}_F((C, X), (C', X')) = \sum_{f : C(C,C')} F(C)(F(f)(X'), X)
\]

so that a morphism \((C, X) \to (C', X')\) is a pair \((f, f_\dagger)\) of \( f : C(C, C') \) and \( f_\dagger : F(C)(F(f)(X'), X) \). We call such pairs Grothendieck lenses for \( F \) or \( F \)-lenses.

Proof sketch. The identity Grothendieck lens on \((C, X)\) is \( \text{id}_{(C, X)} = (\text{id}_C, \text{id}_X) \). Sequential composition is as follows. Given \((f, f_\dagger) : (C, X) \to (C', X') \) and \((g, g_\dagger) : (C', X') \to (Y, Y') \), their composite \((g, g_\dagger) \circ (f, f_\dagger)\) is defined to be the lens \((g \circ f, F(f)(g_\dagger)) : (C, X) \to (Y, Y') \). Associativity and unitality of composition follow from functoriality of \( F \). \qed

Definition 2.37. Suppose \( F(C)_0 = C_0 \), with \( F : \mathcal{C}^{op} \to \text{Cat} \) a pseudofunctor. Define \( \text{SimpGrLens}_F \) to be the full subcategory of \( \text{GrLens}_F \) whose objects are duplicate pairs \((C, C)\) of objects \( C \) in \( \mathcal{C} \). We call \( \text{SimpGrLens}_F \) the category of simple \( F \)-lenses. More generally, any lens between such duplicate pairs will be called a simple lens. Since duplicating the objects in the pairs \((X, X)\) is redundant, we will write the objects simply as \( X \).

Another name for a pseudofunctor \( F : \mathcal{C}^{op} \to \text{Cat} \) is an indexed category. When \( \mathcal{C} \) is a monoidal category with which \( F \) is appropriately compatible, then we can 'upgrade' the notions of indexed category and Grothendieck construction accordingly. In this work, the domain categories \( \mathcal{C} \) are only trivially bicategories, and the pseudofunctors \( F \) are really just functors; we will restrict ourselves to the 1-categorical case of monoidal indexed categories, too.

Definition 2.38 (Moeller and Vasilakopoulou [17, §3.2]). Suppose \((\mathcal{C}, \otimes, I)\) is a monoidal category. We say that \( F \) is a monoidal indexed category when \( F \) is a weak lax monoidal functor \((F, \mu, \mu_0) : (\mathcal{C}^{op}, \otimes^{op}, I) \to (\text{Cat}, \times, 1)\). This means that the laxator \( \mu \) is given by a natural family of functors \( \mu_{A,B} : FA \times FB \Rightarrow F(A \otimes B) \) along with, for any morphisms \( f : A \to A' \) and \( g : B \to B' \) in \( \mathcal{C} \), a natural isomorphism \( \mu_{f,g} : \mu_{A,B'} \circ (Ff \times Fg) \Rightarrow F(f \otimes g) \circ \mu_{A,B} \). The laxator and the unitor \( \mu_0 : 1 \to FI \) together satisfy standard axioms of associativity and unitality. All told, this structure makes \((F, \otimes, \mu, I, \mu_0)\) into a pseudomonoid in the monoidal 2-category of indexed categories and indexed functors.
Proposition 2.39 (Moeller and Vasilakopoulou [17, §6.1]). Suppose \((F, \mu, \mu_0) : (C^{op}, \otimes^{op}, I) \to (\text{Cat}, \times, 1)\) is a monoidal indexed category. Then the total category of the Grothendieck construction \(\int F\) obtains a monoidal structure \((\otimes_\mu, I_\mu)\). On objects, define
\[
(C, X) \otimes_\mu (D, Y) := (C \otimes D, \mu_{CD}(X, Y))
\]
where \(\mu_{CD} : FC \times FD \to F(C \otimes D)\) is the component of \(\mu\) at \((C, D)\). On morphisms \((f, f^\dagger) : (C, X) \to (C', X')\) and \((g, g^\dagger) : (D, Y) \to (D', Y')\), define
\[
(f, f^\dagger) \otimes_\mu (g, g^\dagger) := (f \otimes g, \mu_{CD}(f^\dagger, g^\dagger)).
\]

The monoidal unit \(I_\mu\) is defined to be the object \(I_\mu := (I, \mu_0(\ast))\). Writing \(\lambda : I \otimes (-) \Rightarrow (-)\) and \(\rho : C \otimes (-) \Rightarrow (-)\) for the left and right unitors of the monoidal structure on \(C\), the left and right unitors in \(\int F\) are given by \((\lambda, \text{id})\) and \((\rho, \text{id})\) respectively. Writing \(\alpha\) for the associator of the monoidal structure on \(C\), the associator in \(\int F\) is given by \((\alpha, \text{id})\).

Corollary 2.40. When \(F : C^{op} \to \text{Cat}\) is equipped with a (weak) lax monoidal structure \((\mu, \mu_0)\), its category of lenses \(\text{GrLens}_F\) becomes a monoidal category \((\text{GrLens}_F, \otimes_\mu', I_\mu)\). On objects \(\otimes_\mu'\) is defined as \(\otimes_\mu\) in Proposition 2.39, as is \(I_\mu\). On morphisms \((f, f^\dagger) : (C, X) \to (C', X')\) and \((g, g^\dagger) : (D, Y) \to (D', Y')\), define
\[
(f, f^\dagger) \otimes_\mu' (g, g^\dagger) := (f \otimes g, \mu_{CD}^{op}(f^\dagger, g^\dagger))
\]
where \(\mu_{CD}^{op} : F(C)^{op} \times F(D)^{op} \to F(C \otimes D)^{op}\) is the pointwise opposite of \(\mu_{CD}\). The associator and unitors are defined as in Proposition 2.39.

Remark 2.41. An alternative perspective on lenses is given by the family of structures known as optics [18], which generalize lenses using the kind of actegorical machinery to which we now turn. This machinery puts the categories of forwards and backwards maps on equal footing, unlike the fibrational machinery developed here (which privileges the base category), at the cost of the explicit dependence structure and somewhat heavier categorical tooling. The synthesis of Grothendieck lenses and optics, dependent optics, has recently been articulated [19–21]; while powerful, this structure demands both the machinery of fibrations and of actegories. For its relative simplicity, we therefore stick to the fibrational lens perspective in this paper.

2.3. Categories with parameters

In many applications, we will be interested in cybernetic systems, where a single system might have some freedom in the choice of forwards and backwards channel; consider the synaptic strengths or weights of a neural network, which change as the system learns about the world, affecting the predictions it makes and actions it takes. This freedom is well modelled by equipping the morphisms of a category with parameters, and gives rise to a notion of parameterized category. In general, the parameterization may have different structure to the processes at hand, and so we describe the ‘actegorical’ situation in which a category of parameters \(M\) acts on a category of processes \(C\), generating a category of parameterized processes.

Definition 2.42 (\(M\)-actegory). Suppose \(M\) is a monoidal category with tensor \(\boxtimes\) and unit object \(I\). We say that \(C\) is a left \(M\)-actegory when \(C\) is equipped with a functor \(\odot : M \to \text{Cat}(C, C)\) called the action along with natural unitor and associator isomorphisms \(\lambda_X : I \odot X \xrightarrow{\sim} X\) and \(a_{M,N,X}^{\odot} : (M \boxtimes N) \odot X \xrightarrow{\sim} M \odot (N \odot X)\) compatible with the monoidal structure of \((M, \boxtimes, I)\).

Proposition 2.43 (Capucci et al. [8]). Let \((C, \odot, \lambda, a)\) be an \((M, \boxtimes, I)\)-actegory. Then there is a bicategory of \(M\)-parameterized morphisms in \(C\), denoted \(\text{Para}(\odot)\). Its objects are those of \(C\). For each pair of objects
In this section, we define a collection of indexed categories, each denoted \( \text{Para}(\bigcirc)(X,Y) := \sum_{M : M} \mathcal{C}(M \odot X, Y) \); we denote an element \((M,f)\) of this set by \( f : X \xrightarrow{M} Y \). Given 1-cells \( f : X \xrightarrow{M} Y \) and \( g : Y \xrightarrow{N} Z \), their composite \( g \circ f : X \xrightarrow{N \odot M} Z \) is the following morphism in \( \mathcal{C} \):

\[
(N \boxtimes M) \odot X \xrightarrow{\delta_{N,M,X}^{\bigcirc}} N \odot (M \odot X) \xrightarrow{id_N \odot f} N \odot Y \xrightarrow{g} Z
\]

Given 1-cells \( f : X \xrightarrow{M} Y \) and \( f' : X \xrightarrow{M'} Y \), a 2-cell \( \alpha : f \Rightarrow f' \) is a morphism \( \alpha : M \rightarrow M' \) in \( \mathcal{M} \) such that \( f = f' \circ (\alpha \odot \text{id}_X) \) in \( \mathcal{C} \); identities and composition of 2-cells are as in \( \mathcal{C} \).

**Proposition 2.44.** When \( \mathcal{C} \) is equipped with both a symmetric monoidal structure \((\otimes, I)\) and an \((\mathcal{M}, \odot, I)\)-category structure, and there is a natural isomorphism \( \mu_{M,X,Y}^\bigcirc : M \odot (X \otimes Y) \xrightarrow{\sim} X \otimes (M \odot Y) \) called the \((\textbf{right})\) costrength\(^2\), the symmetric monoidal structure \((\otimes, I)\) lifts to \( \text{Para}(\bigcirc) \). First, from the symmetry of \( \otimes \), one obtains a (left) costrength, \( \mu_{M,X,Y}^{\bigcirc} : M \odot (X \otimes Y) \xrightarrow{\sim} (M \odot X) \otimes Y \). Second, using the two costrengths and the associator of the actegory structure, one obtains a natural isomorphism \( i_{M,N,X,Y}^{\bigcirc} : (M \boxtimes N) \odot (X \otimes Y) \xrightarrow{\sim} (M \odot X) \otimes (N \odot Y) \) called the interchanger. The tensor of objects in \( \text{Para}(\bigcirc) \) is then defined as the tensor of objects in \( \mathcal{C} \), and the tensor of morphisms (1-cells) \( f : X \xrightarrow{M} Y \) and \( g : A \xrightarrow{N} B \) is given by the composite

\[
f \otimes g : X \otimes A \xrightarrow{M \boxtimes N} Y \otimes B := (M \boxtimes N) \odot (X \otimes A) \xrightarrow{i_{M,N,X,A}^{\bigcirc}} (M \odot A) \otimes (N \odot A) \xrightarrow{\mu_{M,A,N,A}^\bigcirc} Y \otimes B.
\]

In many simple cases, the parameters will live in the same category as the morphisms being parameterized; this is formalized by the following proposition.

**Proposition 2.45.** If \((C, \otimes, I)\) is a monoidal category, then it induces a parameterization \( \text{Para}(\otimes) \) on itself. For each \( M, X, Y : C \), the morphisms \( X \xrightarrow{M} Y \) of \( \text{Para}(\otimes) \) are the morphisms \( M \otimes X \rightarrow Y \) in \( \mathcal{C} \).

**Notation 2.46.** When considering the self-parameterization induced by a monoidal category \((C, \otimes, I)\), we will often write \( \text{Para}(C) \) instead of \( \text{Para}(\otimes) \).

It will frequently be the case that we do not in fact need the whole bicategory structure. The following proposition tells us that we can also just work 1-categorically, as long as we work with equivalence classes of isomorphically-parameterized maps, in order that composition is sufficiently strictly associative.

**Proposition 2.47.** Each bicategory \( \text{Para}(\bigcirc) \) induces a 1-category \( \text{Para}(\bigcirc)_1 \) by forgetting the bicategorical structure. The hom sets \( \text{Para}(\bigcirc)_1(X,Y) \) are given by \( U\text{Para}(\bigcirc)(X,Y)/\sim \) where \( U \) is the forgetful functor \( U : \text{Cat} \rightarrow \text{Set} \) and \( f \sim g \) if and only if there is some 2-cell \( \alpha : f \Rightarrow g \) that is an isomorphism. We call \( \text{Para}(\bigcirc)_1 \) the \( 1 \)-categorical truncation of \( \text{Para}(\bigcirc) \). When \( \text{Para}(\bigcirc) \) is monoidal, so is \( \text{Para}(\bigcirc)_1 \).

### 3. The bidirectional structure of Bayesian updating

In this section, we define a collection of indexed categories, each denoted \( \text{Stat} \), whose morphisms can be seen as “generalized Bayesian inversions”. Following Proposition 2.36, these induce corresponding categories of lenses which we call \( \text{Bayesian lenses} \). We show that, for the subcategories of exact Bayesian lenses whose backward channels correspond to ‘exact’ Bayesian inversions, the Bayesian inversion of a composite of forward channels is given (up to almost-equality) by the lens composite of the corresponding backward channels. This justifies calling these lenses ‘Bayesian’, and provides the foundation for the study of approximate (non-exact) Bayesian inversion in the subsequent section.

**Remark.** An alternative account of Bayesian lenses, from an ‘optical’ perspective, is told in the preprint [6].

\(^2\)So named for its similarity to the (co)strength of certain (co)monads; and similarly, too, our costrength should satisfy certain standard coherence laws which we omit here.
3.1. State-dependent channels

A channel \( c : X \nrightarrow Y \) admitting a Bayesian inversion induces a family of inverse channels \( c^\dagger : Y \nrightarrow X \), indexed by 'prior' states \( \pi : 1 \nrightarrow X \). Making the state-dependence explicit, in typical cases where \( c \) is a probability kernel we obtain a measurable function \( c^\dagger : \mathcal{G}X \times Y \rightarrow \mathcal{G}X \). In more general situations, we obtain a morphism \( c^\dagger : \mathcal{C}(I, X) \rightarrow \mathcal{C}(Y, X) \) in the base of enrichment of the monoidal category \( (\mathcal{C}, \otimes, I) \) of \( c \). We call morphisms of this general type state-dependent channels, and structure the indexing as an indexed category.

**Definition 3.1.** Let \( (\mathcal{C}, \otimes, I) \) be a monoidal category enriched in a Cartesian category \( \mathcal{V} \). Define the \( \mathcal{C} \)-state-indexed category \( \text{Stat} : \mathcal{C}^{op} \rightarrow \mathcal{V}\text{-Cat} \) as follows.

\[
\begin{pmatrix}
\text{Stat}(X)_0 & := & \mathcal{C}_0 \\
\text{Stat}(X)(A, B) & := & \mathcal{V}(\mathcal{C}(I, X), \mathcal{C}(A, B)) \\
\text{id}_A & : & \text{Stat}(X)(A, A) := \left\{ \text{id}_A : \mathcal{C}(I, X) \rightarrow \mathcal{C}(A, A) \right\} \\
\end{pmatrix}
\]

Composition in each fibre \( \text{Stat}(X) \) is as in \( \mathcal{C} \). Explicitly, indicating morphisms \( \mathcal{C}(I, X) \rightarrow \mathcal{C}(A, B) \) in \( \text{Stat}(X) \) by \( A \xrightarrow{\omega} B \), and given \( \alpha : A \xrightarrow{\omega} B \) and \( \beta : B \xrightarrow{\omega'} C \), their composite is \( \beta \circ \alpha : A \xrightarrow{\omega \omega'} C := \rho \rightarrow \beta(\rho) \bullet \alpha(\rho) \), where here we indicate composition in \( \mathcal{C} \) by \( \bullet \) and composition in the fibres \( \text{Stat}(X) \) by \( \circ \). Given \( f : Y \nrightarrow X \) in \( \mathcal{C} \), the induced functor \( \text{Stat}(f) : \text{Stat}(X) \rightarrow \text{Stat}(Y) \) acts by pullback.

**Notation 3.2.** Just as we wrote \( X \xrightarrow{M} Y \) for an \( M \)-parameterized morphism in \( \mathcal{C}(M \otimes X, Y) \) (see Proposition 2.43), we write \( A \xrightarrow{X} B \) for an \( X \)-state-dependent morphism in \( \mathcal{V}(\mathcal{C}(I, X), \mathcal{C}(A, B)) \). Given a state \( \rho \) in \( \mathcal{C}(I, X) \) and an \( X \)-state-dependent morphism \( f : A \xrightarrow{X} B \), we write \( f_\rho \) for the resulting morphism in \( \mathcal{C}(A, B) \).

**Remark 3.3.** We can thus think of \( \text{Stat} \) as a kind of 'external' parameterization of channels in \( \mathcal{C} \), here by states in \( \mathcal{C} \). Generalizing this notion to "external parameterization by \( \mathcal{V} \) objects" gives rise to a cousin of the \textbf{Para} construction: a notion of category by proxy, denoted \textbf{Prox}, of which \( \text{Stat} \) is a (fibred) special case. Correspondingly, we think of \textbf{Para} as capturing parameterization 'internal' to \( \mathcal{C} \). Consider for instance the case where \( \mathcal{M} \) acts on \( \mathcal{C} \) by \( \otimes : \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{C} \) and suppose that, for all \( A : \mathcal{C} \), the functors \( (\_ \otimes A) : \mathcal{M} \rightarrow \mathcal{C} \) are left adjoint to \( [A, \_] : \mathcal{C} \rightarrow \mathcal{M} \). Then we have \( \textbf{Para}(\otimes)(A, B) \cong \sum_{M \in \mathcal{M}} \mathcal{M}(M, [A, B]) \), which is strongly reminiscent of the definition of \( \text{Stat} \), and comes close to a definition of \textbf{Prox}. The connections between these constructions are a matter of on-going study; a summary is available in Capucci et al. [22].

**Proposition 3.4.** \( \text{Stat} \) is lax monoidal. The components \( \mu_{XY} : \text{Stat}(X) \times \text{Stat}(Y) \rightarrow \text{Stat}(X \otimes Y) \) of the laxator are defined on objects by \( \mu_{XY}(A, A') := A \otimes A' \) and on morphisms \( f : A \xrightarrow{X} B \) and \( f' : A' \xrightarrow{X'} B' \) by \( \mu_{XY}(f, f') := f \otimes f' : A \otimes A' \xrightarrow{X \otimes X'} B \otimes B' \), where \( f \otimes f' \) is the \( \mathcal{V} \)-morphism \( \mathcal{C}(I, X \otimes Y) \rightarrow \mathcal{C}(A \otimes A', B \otimes B') \). Making the state-dependence explicit, in typical cases where \( \omega, \omega' \) are the \( X \) and \( Y \) marginals of \( \omega \), defined by \( \omega_X := \text{proj}_X \bullet \omega \) and \( \omega_Y := \text{proj}_Y \bullet \omega \) (see Definition 2.12 and Depiction 2.28). The unit \( \mu_0 : 1 \rightarrow \text{Stat}(I) \) of the lax monoidal structure is the functor mapping the unique object \( 1 \) to \( I : \text{Stat}(I) \).
Example 3.5. The category $\text{Meas}$ of general measurable spaces is not Cartesian closed, as there is no general way to make the evaluation maps $\text{Meas}(X, Y) \times X \to Y$ measurable, meaning that if we take $C = \mathcal{K}(\mathcal{G})$ above, then we are forced to take $V = \text{Set}$. In turn, this makes the inversion maps $c^! : \mathcal{K}(\mathcal{G})(1, X) \to \mathcal{K}(\mathcal{G})(Y, X)$ into mere functions. We can salvage measurability by working instead with $\mathcal{K}(\mathcal{Q})$, where $\mathcal{Q} : \text{QBS} \to \text{QBS}$ is the analogue of the Giry monad for quasi-Borel spaces [23]. The category $\text{QBS}$ is indeed Cartesian closed, and $\mathcal{K}(\mathcal{Q})$ is enriched in $\text{QBS}$, so that we can instantiate $\text{Stat}$ there, and the corresponding inversion maps are accordingly measurable. Moreover, there is a quasi-Borel analogue of the notion of $s$-finite kernel [24, §11], with which we can define a variant of the category $\text{sfKrn}$.

Remark 3.6. When $C$ is a Kleisli category $\mathcal{K}(\mathcal{T})$, it is of course possible to define a variant of $\text{Stat}$ on the other side of the product-exponential adjunction, with state-dependent morphisms $A \xrightarrow{X} B$ having the types $TX \times A \to TB$. This avoids the technical difficulties sketched in the preceding example at the cost of requiring a monad $T$. However, the exponential form makes for better exegesis, and so we will stick to that.

3.2. Bayesian lenses

We define the category of Bayesian lenses in $C$ to be the category of Grothendieck $\text{Stat}$-lenses.

Definition 3.7. The category $\text{BayesLens}_C$ of Bayesian lenses in $C$ is the category $\text{GrLens}_{\text{Stat}}$ of Grothendieck lenses for the functor $\text{Stat}$. A Bayesian lens is a morphism in $\text{BayesLens}_C$. Where the category $C$ is evident from the context, we will just write $\text{BayesLens}$.

Unpacking this definition, we find that the objects of $\text{BayesLens}_C$ are pairs $(X, A)$ of objects of $C$. Morphisms (that is, Bayesian lenses) $(X, A) \to (Y, B)$ are pairs $(c, c^!)$ of a channel $c : X \to Y$ and a “generalized Bayesian inversion” $c^! : B \xleftarrow{X} A$; that is, elements of the hom objects

$$\text{BayesLens}_C((X, A), (Y, B)) = \text{GrLens}_{\text{Stat}}((X, A), (Y, B)) \cong C(X, Y) \times \text{V}(C(I, X), C(B, A)).$$

The identity Bayesian lens on $(X, A)$ is $(\text{id}_X, \text{id}_A)$, where by abuse of notation $\text{id}_A : C(I, Y) \to C(A, A)$ is the constant map $\text{id}_A$ defined in Equation (4) that takes any state on $Y$ to the identity on $A$.

The sequential composite $(d, d^!) \circ (c, c^!)$ of $(c, c^!) : (X, A) \to (Y, B)$ and $(d, d^!) : (Y, B) \to (Z, C)$ is the Bayesian lens $((d \circ c), (c^! \circ c^*d^!)) : (X, A) \to (Z, C)$ where $(c^! \circ c^*d^!) : C \xleftarrow{X} A$ takes a state $\pi : I \to X$ to the channel $c^! \circ d^! \circ \pi : C \xleftarrow{X} A$.

Definition 3.8. Given a Bayesian lens $(c, c^!) : (X, A) \to (Y, B)$, we will call $c$ its forwards or prediction channel and $c^!$ its backwards or update channel (even though $c^!$ is really a family of channels).

Remark 3.9. Note that the definition of $\text{Stat}$ and hence the definition of $\text{BayesLens}_C$ do not require $C$ to be a copy-delete category, even though our motivating categories of stochastic channels are; all that is required for the definition is that $C$ is monoidal.

Remark 3.10. On the other hand, the structure of $C$ might be stronger than merely monoidal. For instance, when $C$ is Cartesian closed, then we can take $V = C$ and the monoidal structure to be the categorical product $(\times, 1)$. Then a Bayesian lens $(X, A) \to (Y, B)$ is equivalently given by a pair of a forwards map $X \to Y$ and a backwards map $X \times B \to A$. We call such lenses Cartesian, and they characterize the original “lens” notion; see Remark 3.16.

Proposition 3.11. $\text{BayesLens}_C$ is a monoidal category, with structure $((\otimes, (I, I)))$ inherited from $C$. On objects, define $(A, A') \otimes (B, B') := (A \otimes A', B \otimes B')$. On morphisms $(f, f^!) : (X, A) \to (Y, B)$ and
\((g, g^\dagger) : (X', A') \to (Y', B')\), define \((f, f^\dagger) \otimes (g, g^\dagger) := (f \otimes g, f^\dagger \otimes g^\dagger)\), where \(f^\dagger \otimes g^\dagger : B \otimes B' \to X \otimes X'\) acts on states \(\omega : I \to X \otimes X'\) to return the channel \(f_{\omega_X}^\dagger \otimes g_{\omega_{X'}}^\dagger\), following the definition of the laxator \(\mu\) in Proposition 3.4. The monoidal unit in \(\text{BayesLens}_C\) is the pair \((I, I)\) duplicating the unit in \(C\). When \(C\) is moreover symmetric monoidal, so is \(\text{BayesLens}_C\).

**Proof sketch.** The main result is immediate from Proposition 3.4 and Corollary 2.40. When \(\otimes\) is symmetric in \(C\), the symmetry lifts to the fibres of \(\text{Stat}\) and hence to \(\text{BayesLens}_C\).

**Remark 3.12.** Although \(\text{BayesLens}_C\) is a monoidal category, it does not inherit a copy-delete structure from \(C\), owing to the bidirectionality of its component morphisms. To see this, we can consider morphisms into the monoidal unit \((I, I)\), and find that there is generally no canonical discarding map. For instance, a morphism \((X, A) \to (I, I)\) consists in a pair of a channel \(X \to I\) (which may indeed be a discarding map) and a state-dependent channel \(I \to A\), for which there is generally no suitable choice satisfying the comonoid laws.

Note, however, that a lens of the type \((X, I) \to (I, B)\) might indeed act by discarding, since we can choose the constant state-dependent channel \(B \to I\) on the discarding map \(\hat{\Delta} : B \to I\). By contrast, the Grothendieck category \(\text{Stat}\) is a copy-delete category, as the morphisms \((X, A) \to (I, I)\) in \(\text{Stat}\) are pairs \(X \to I\) and \(A \to I\), and so for both components we can choose morphisms witnessing the comonoid structure.

### 3.3. Bayesian updates compose optically

In this section we prove the fundamental result on which the development of statistical games rests: that the inversion of a composite channel is given up to almost-equality by the lens composite of the backwards components of the associated ‘exact’ Bayesian lenses.

**Definition 3.13.** Let \((c, c^\dagger) : (X, X) \to (Y, Y)\) be a Bayesian lens. We say that \((c, c^\dagger)\) is exact if \(c\) admits Bayesian inversion and, for each \(\pi : I \to X\) such that \(c \oti \pi\) has non-empty support, \(c\) and \(c^\dagger \oti \pi\) together satisfy equation \(3\). Bayesian lenses that are not exact are said to be approximate.

**Theorem 3.14.** Let \((c, c^\dagger)\) and \((d, d^\dagger)\) be sequentially composable exact Bayesian lenses. Then the contravariant component of the composite lens \((d, d^\dagger) \circ (c, c^\dagger) = (d \oti c, c^\dagger \oti c^\dagger \circ d^\dagger)\) is, up to \(d \oti c \oti \pi\)-almost-equality, the Bayesian inversion of \(d \oti c\) with respect to any state \(\pi\) on the domain of \(c\) such that \(c \oti \pi\) has non-empty support. That is to say, Bayesian updates compose optically: \((d \oti c)^\dagger_\pi \sim d^\dagger \oti c^\dagger \oti d^\dagger_\pi\).

**Proof.** Suppose \(c^\dagger_\pi : Y \to X\) is the Bayesian inverse of \(c : X \to Y\) with respect to \(\pi : I \to X\). Suppose also that \(d^\dagger_\pi : Z \to Y\) is the Bayesian inverse of \(d : Y \to Z\) with respect to \(c \oti \pi : I \to Y\), and that \((d \oti c)^\dagger_\pi : Z \to X\) is the Bayesian inverse of \(d \oti c : X \to Z\) with respect to \(\pi : I \to X\):
The lens composite of these Bayesian inverses has the form \( c^\pi \bullet d_{c^\pi} \) : \( Z \leftrightarrow X \), so to establish the result it suffices to show that

\[
\begin{array}{c}
\bullet
\end{array}
\]

We have

\[
\begin{array}{c}
\bullet
\end{array}
\]

where the first obtains because \( d_{c^\pi} \) is the Bayesian inverse of \( d \) with respect to \( c \cdot \pi \), and the second because \( c^\pi \) is the Bayesian inverse of \( c \) with respect to \( \pi \). Hence, \( c^\pi \bullet d_{c^\pi} \) and \( (d \bullet c)^\pi \) are both Bayesian inverses of \( d \bullet c \) with respect to \( \pi \). Since Bayesian inverses are almost-equal (Prop. 2.34), we have \( c^\pi \bullet d_{c^\pi} \overset{\sim}{=} (d \bullet c)^\pi \), as required.

**Remark 3.15.** Note that, in the context of finitely-supported probability (e.g., in \( \mathcal{K}(\mathcal{D}) \), where \( \mathcal{D} \) is the finitely-supported probability distribution monad), almost-equality coincides with simple equality, and so Bayesian inverses are then just equal.

**Remark 3.16.** Lenses were originally studied in the context of database systems [3], where one thinks of the forward channel as ‘viewing’ a record in a database, and the backward channel as ‘updating’ a record by taking a record and a new piece of data and returning the updated record. In this context, lenses have often been subject to additional axioms characterizing well-behavedness; for example, that updating a record with some data is idempotent (the ‘put-put’ law). Bayesian lenses do not in general satisfy these laws, and nor even do exact Bayesian lenses. This is because Bayesian updating mixes information in the prior state (the ‘record’) with the observation (the ‘data’), rather than replacing the prior information outright. We refer the reader to our preprint [6, §6] for a more detailed discussion of this situation.

### 3.4. Parameterized Bayesian lenses

Bayesian lenses for which the component channels are equipped with parameters will play an important role in certain applications: an example which we will meet in the next section is the variational autoencoder, a neural network architecture originally developed for machine learning and which is well described by a particular class of parameterized statistical games. Since \( \text{BayesLens}_C \) is a monoidal category, it induces a self-parameterization \( \text{Para}(\text{BayesLens}_C) \) by Proposition 2.45, which is sufficient for the purposes of this paper. Therefore, in this section, we summarize the resulting structure for later reference.
**0-cells and 1-cells**  The 0-cells of the bicategory \( \text{Para}(\text{BayesLens}_C) \) are pairs \( (X, A) \) of objects in \( C \). The 1-cells \( (c,c') : (X,A) \xrightarrow{(\Theta,\Theta)} (Y,B) \) are Bayesian lenses \( (c,c') : (\Omega \otimes X, \Theta \otimes A) \rightarrow (Y,B) \). The forwards component \( c \) is a channel \( \Omega \otimes X \twoheadrightarrow Y \) in \( C \), and hence also a parameterized channel \( X \overset{\Omega}{\twoheadrightarrow} Y \) in \( \text{Para}(C) \); we often think of this channel as representing a system’s model of the process by which observations (of type \( Y \)) are generated from causes (of type \( X \)), with the parameters (of type \( \Omega \)) representing the system’s beliefs about the structure of this generative process.

Conversely, the backwards component \( c' \) is a state-dependent channel \( B \overset{\Theta}{	woheadrightarrow} \Theta \otimes A \), which means a \( V \)-morphism \( C(I, \Omega \otimes X) \rightarrow C(B, \Theta \otimes A) \). This is a generalized Bayesian inversion which takes a state (or ‘prior’ belief) jointly over parameters and causes \( \Theta \otimes X \) and returns a channel \( B \leftrightarrow \Theta \otimes A \) that we think of as taking an observation (of possibly different type \( B \)) and returning an updated joint belief about parameters and causes (of possibly different types \( \Theta \) and \( A \)). Note that this ‘update’ channel is not itself parameterized: the parameterization of the inversion is mediated through \( \Omega \), with \( \Theta \) being the type of updated parameters.

**Sequential composition**  Given parameterized Bayesian lenses \( (c,c') : (X,A) \xrightarrow{(\Omega,\Theta)} (Y,B) \) and \( (d,d') : (Y,B) \xrightarrow{(\Omega',\Theta')} (Z,C) \), their composite \( (d,d') \circ (c,c') \) is defined as the following morphism in \( \text{BayesLens}_C \):

\[
(\Omega' \otimes \Omega \otimes X, \Theta' \otimes \Theta \otimes A) \xrightarrow{(\text{id,id}) \circ (\Theta,\Theta)} (\Omega' \otimes Y, \Theta' \otimes B) \xrightarrow{(d,d')} (Z,C).
\]

**Parallel composition**  Given parameterized Bayesian lenses \( (c,c') : (X,A) \xrightarrow{(\Omega,\Theta)} (Y,B) \) and \( (d,d') : (X',A') \xrightarrow{(\Omega',\Theta')} (Y',B') \), their tensor \( (c,c') \otimes (d,d') \) is defined as the following morphism in \( \text{BayesLens}_C \):

\[
\left( \Omega \otimes \Omega' \otimes X \otimes X', \Theta \otimes \Theta' \otimes A \otimes A' \right) \xrightarrow{(c,d)} \left( \Omega \otimes X \otimes \Omega' \otimes X', \Theta \otimes A \otimes \Theta' \otimes A' \right)
\]

where we have written the pairs vertically, so that \( \left( \frac{X}{A} \right) := (X,A) \).

**Reparameterization**  Given 1-cells \( (f,f') : (X,A) \xrightarrow{(\Omega,\Theta)} (Y,B) \) and \( (g,g') : (X,A) \xrightarrow{(\Omega',\Theta')} (Y,B) \), the 2-cells \( \alpha : (f,f') \Rightarrow (g,g') \) of \( \text{Para}(\text{BayesLens}_C) \) are Bayesian lenses \( \alpha : (\Omega,\Theta) \rightarrow (\Omega',\Theta') \) such that \( (f,f') = (g,g') \circ (\alpha \otimes \text{id}_{(X,A)}) \). We can think of the 2-cells as reparameterizations, or higher-order processes that predict the parameters on the basis of yet more abstract data.

**Higher structure**  The \( \text{Para} \) construction turns a (monoidal) category into a (monoidal) bicategory, thereby “adding a dimension”. We can consider morphisms in a parameterized category (as well as morphisms in \( \text{Stat} \); see Remark 3.3) as processes by which processes are chosen, and reparameterizations witness the factorization of these choice processes. In many cybernetic and statistical situations, it is of interest to consider adding more than just one extra dimension: that is, we may be interested in the processes by which these choice processes are chosen, and, as in the Bayesian setting, we may be interested in improving the performance of these ‘meta-processes’. More concretely, in statistics, we may wish to describe meta-learning algorithms (such as “learning to learn”), or in neuroscience, we may wish to describe how neuromodulation affects synaptic plasticity (which in turn affects the generation of action potentials).

Since \( \text{Para}(\text{BayesLens}_C) \) is itself monoidal, one has a further parameterization \( \text{Para}(\text{Para}(\text{BayesLens}_C)) \), and indeed \( \text{Para} \) has a monad structure whose multiplication collapses a doubly-parameterized morphism to a singly-parameterized one [8, Prop. 3]. However, the full structure that emerges when iterating this construction \( \text{ad infinitum} \) is not yet well understood\(^3\). A similar structure appears when one considers arbitrarily

\(^3\)At present, we believe the resulting structure may have an opetopic shape, and constitute something like an ‘\( \infty \)–fibration’.
‘nested’ dynamical systems (such as cells in an organism, organisms in societies, and societies in ecosystems; or the ownership structure of a complex modern economy); see St. Clere Smithe [25, Remark 2.3]. Such cybernetic systems will be treated in a future chapter of the present series.

4. Statistical games

The Bayesian lenses of Theorem 3.14 are exact, but most physically realistic cybernetic systems do not compute exact inversions: the inversion of a channel \( c \) with respect to a prior \( \pi \) generally involves evaluating the composite \( c \bullet \pi \), which is typically computationally costly; consequently, realistic systems typically instantiate approximate Bayesian lenses. Fortunately, as a consequence of Theorem 3.14, an approximate inversion of a composite channel will be approximately equal to this lens composite; conversely, the lens composite of approximate inversions will be approximately equal to the exact inversion of the corresponding composite.

We can thus approximate the inversion of a composite channel by the lens composite of approximations to the inversions of the components. But what do we mean by “approximate”? There is often substantial freedom in the choice of approximation scheme—often manifest as some form of parameterization—and in typical situations, the ‘fitness’ of a particular scheme will be context-dependent. We think of Bayesian lenses as representing ‘open’ statistical systems, in interaction with some environment or ‘context’, and so the fitness of a particular lens then depends not only on the configuration of the system (i.e., the choice of lens), but also on the suitability of its configuration for the environment.

In this section, we introduce statistical games in order to quantify this context-dependent fitness: a statistical game will be a Bayesian lens paired with a contextual fitness function. The fitness function measures how well the lens performs in each context, and the “aim of the game” is then to choose the lens or context (depending on your perspective) that somehow optimizes the fitness. In order to define statistical games, we first therefore define our notion of context, following compositional game theory [2, 7]: a context is “everything required to make an open system closed”. In the next section, we exemplify statistical games by formalizing a number of classic and not-so-classic problems in statistics.

4.1. Contexts for Bayesian lenses

Our first step is to define the notion of simple context, with which we will be able to “close off” a lens with respect to sequential composition \( \phi \), and thereby define our categories of statistical games.

Definition 4.1. A simple context for a Bayesian lens over \( C \) is an element of the \( V \)-profunctor \( \text{BayesLens}_C \) defined by

\[
\text{BayesLens}_C \times \text{BayesLens}_C^{\text{op}} \to V \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\text{BayesLens}_C((I, I), -) \times \text{BayesLens}_C(=, (I, I)).
\]

If the lens is \((A, S) \to (B, T)\), its object of simple contexts is

\[
\text{BayesLens}_C((I, I), (A, S)) \times \text{BayesLens}_C((B, T), (I, I)).
\]

If we denote the lens by \( f \), then we can denote this object of simple contexts by

\[
\text{Ctx}(f) := \text{BayesLens}_C((A, S), (B, T)).
\]

When the monoidal unit \( I \) is terminal in \( C \), or equivalently when \( C \) is semicartesian, the object of simple contexts acquires a simplified (and intuitive) form.
Proposition 4.2. When \( I \) is terminal in \( C \), we have
\[
\text{BayesLens}_C((X, A), (Y, B)) \cong C(I, X) \times V(C(I, B), C(I, Y))
\]

Proof. A straightforward calculation which we omit: use causality (Definition 2.24).

Remark 4.3. Proposition 4.2 means that, when \( C \) is a Markov category (such as \( K\ell(\mathcal{G}) \)), a context for a Bayesian lens consists of a ‘prior’ on the domain of the forwards channel and a ‘continuation’: a \( V \)-morphism (such as a function) which takes the output of the forwards channel (possibly a ‘prediction’) and returns an observation for the update map. We think of the continuation as encoding the response of the environment given the prediction.

In order to define the sequential composition of statistical games, we will need to construct, from the context for a composite lens, contexts for each factor of the composite. The functoriality of \( \text{BayesLens}_C \) guarantees the existence of such local contexts.

Definition 4.4. Given another lens \( g : (B, T) \to (C, U) \) and a simple context for their composite \( g \circ f : (A, S) \to (C, Y) \), then we can obtain a 1-local context for \( f \) by the action of the profunctor
\[
\text{BayesLens}_C((A, S), g) : \text{BayesLens}_C((A, S), (C, U)) \to \text{BayesLens}_C((A, S), (B, T))
\]
and we can obtain a simple context for \( g \) similarly:
\[
\text{BayesLens}_C(f, (C, U)) : \text{BayesLens}_C((A, S), (C, U)) \to \text{BayesLens}_C((B, T), (C, U))
\]

Note that we can write \( g^* : \text{Ctx}(g \circ f) \to \text{Ctx}(f) \) for the former action and \( f_* : \text{Ctx}(g \circ f) \to \text{Ctx}(g) \) for the latter, since \( g^* \) acts by precomposition (pullback) and \( f_* \) acts by postcomposition (pushforwards). Note also that, given \( h \circ g \circ f \), we have \( h^* \circ f_* = f_* \circ h^* : \text{Ctx}(h \circ g \circ f) \to \text{Ctx}(g) \) by the associativity of composition.

Remark 4.5. The local contexts of the preceding definition are local with respect to sequential composition of lenses. If we view the monoidal category \( \text{BayesLens}_C \) as a one-object bicategory, then sequential composition is composition of 1-cells, which explains their formal naming as 1-local contexts.

To lift the monoidal structure of \( \text{BayesLens}_C \) to our categories of statistical games, we will similarly need ‘2-local’ contexts: from the one-object bicategory perspective, these are local contexts with respect to 2-cell composition. By analogy with the 1-local case, this means exhibiting maps of the form \( \text{Ctx}(f \otimes f') \to \text{Ctx}(f) \) and \( \text{Ctx}(f \otimes f') \to \text{Ctx}(f') \) which give the ‘left’ and ‘right’ local contexts for a parallel pair of lenses.

Without requiring extra structure from \( C \) or \( \text{BayesLens}_C \), we first need to pass from simplex contexts to complex ones: a 2-local context should allow for information to pass alongside a lens, through a process that is parallel to it. Formally, we adjoin an object to the domain and codomain of the context, and quotient by the rule that, if there is any process that ‘fills the hole’ represented by the adjoined objects, then we consider the objects equivalent: this allows us to forget about the contextually parallel processes, and keep track only of the type of information that flows. Such adjoining-and-quotienting gives us the following coend\(^3\) formula defining complex contexts.

\(^3\)An alternative is to ask for \( \text{BayesLens}_C \) to be equipped with a natural family of ‘discarding’ morphisms \((A, S) \to (I, I)\). This in turn means asking for canonical states \( I \to S \), which in general we do not have: if we are working with stochastic channels, we could obtain such canonical states by allowing the channels to emit subdistributions, and letting the canonical states be given by those which assign 0 density everywhere. But even though we can make the types check this way, the semantics are not quite what we want: rather, we seek to allow information to "pass in parallel".

\(^{\text{For more information about (co)ends and (co)end calculus, we refer the reader to Loregian [26].}}\)
Definition 4.6. We define a complex context to be an element of the profunctor

$$\text{BayesLens}_C((A, S), (B, T)) := \int^{(M, N) : \text{BayesLens}_C} \text{BayesLens}_C((M, N) \otimes (A, S), (M, N) \otimes (B, T)).$$

If \(f : (A, S) \to (B, T)\) is a lens, then we will write \(\text{Ctx}(f) := \text{BayesLens}_C((A, S), (B, T)).\) As in the case of simple contexts, we have 'projection' maps \(g^* : \text{Ctx}(g \circ f) \to \text{Ctx}(f)\) and \(f_* : \text{Ctx}(g \circ f) \to \text{Ctx}(g);\) these are defined similarly. We will call the adjoined object, here denoted \((M, N),\) the residual. There is a canonical inclusion \(\text{Ctx}(f) \hookrightarrow \text{Ctx}(f)\) given by adjoining the trivial residual \((I, I)\) to each simple context.

We immediately have the following corollary of Proposition 4.2:

Corollary 4.7. When \(I\) is terminal in \(C,\)

$$\text{BayesLens}_C((A, S), (B, T)) \cong \int^{(M, N)} C(I, M \otimes A) \times V(C(I, M \otimes B), C(I, N \otimes T)).$$

Remark 4.8. Since the coend denotes a quotient, its elements are equivalence classes. The preceding corollary says that, when \(I\) is terminal, an equivalence class of contexts is represented by a choice of residual \((M, N),\) a prior on \(M \otimes A\) in \(C,\) and a continuation \(C(I, M \otimes B) \to C(I, N \otimes T)\) in \(V.\)

Of course, when \(I\) is not terminal, then the definition says that a general complex context for a lens \((A, S) \to (B, T)\) is an equivalence class represented by: as before, a choice of residual \((M, N),\) a prior on \(M \otimes A,\) and a continuation \(C(I, M \otimes B) \to C(I, N \otimes T)\); as well as an 'effect' \(M \otimes B \to I\) in \(C,\) and what we might call a 'vector' \(C(I, I) \to C(N \otimes S, I)\) in \(V.\) The effect and vector measure the environment’s 'internal response' to the lens' outputs (as opposed to representing the environment’s feedback to the lens).

Using complex contexts, it is easy to define the 'projections' that give 2-local contexts.

Definition 4.9. Given lenses \(f : \Phi \to \Psi\) and \(f' : \Phi' \to \Psi'\) and a complex context for their tensor \(f \otimes f',\) the left 2-local context is the complex context for \(f\) given by

$$\pi_f : \text{Ctx}(f \otimes f') = \text{BayesLens}_C(\Phi \otimes \Phi', \Psi \otimes \Psi')$$

$$= \int^{\Theta : \text{BayesLens}_C} \text{BayesLens}_C(\Theta \otimes \Phi \otimes \Phi', \Theta \otimes \Psi \otimes \Psi')$$

$$\cong \int^{\Theta : \text{BayesLens}_C} \text{BayesLens}_C(\Theta \otimes \Phi' \otimes \Phi, \Theta \otimes \Psi' \otimes \Psi)$$

$$= \int^{\Theta : \text{BayesLens}_C} \text{BayesLens}_C((I, I), \Theta \otimes \Phi' \otimes \Phi) \times \text{BayesLens}_C(\Theta \otimes \Psi' \otimes \Psi, (I, I))$$

$$\cong \int^{\Theta : \text{BayesLens}_C} \text{BayesLens}_C((I, I), \Theta \otimes \Psi' \otimes \Phi) \times \text{BayesLens}_C(\Theta \otimes \Psi' \otimes \Psi, (I, I))$$

$$= \int^{\Theta' : \text{BayesLens}_C} \text{BayesLens}_C(\Theta \otimes \Psi' \otimes \Phi, \Theta \otimes \Psi' \otimes \Psi)$$

$$\cong \int^{\Theta' : \text{BayesLens}_C} \text{BayesLens}_C(\Theta' \otimes \Phi, \Theta' \otimes \Psi) = \text{Ctx}(f).$$

The equalities here are just given by expanding and contracting definitions; the isomorphism uses the symmetry of \(\otimes\) in \(C;\) the arrow marked \(f'\) is given by composing \(\text{id}_I \otimes f' \otimes \text{id}_A\) after the 'prior' part of the context; and the inclusion is given by collecting the tensor of \(\Theta\) and \(\Psi'\) together into the residual. These steps formalize the idea of filling the right-hand hole of the complex context with \(f'\) to obtain a local context for \(f.\)
The right 2-local context is the complex context for \( f' \) obtained similarly:

\[
\pi_{f'} : \text{Ctx}(f \otimes f') = \text{BayesLens}_C(\Phi \otimes \Phi', \Psi \otimes \Psi') \\
= \int \Theta : \text{BayesLens}_C(\Theta \otimes \Phi \otimes \Phi', \Theta \otimes \Psi \otimes \Psi') \\
= \int \Theta : \text{BayesLens}_C((I, I), \Theta \otimes \Phi \otimes \Phi') \times \text{BayesLens}_C(\Theta \otimes \Psi \otimes \Psi', (I, I)) \\
\rightarrow \int \Theta : \text{BayesLens}_C((I, I), \Theta \otimes \Psi \otimes \Phi') \times \text{BayesLens}_C(\Theta \otimes \Psi \otimes \Psi', (I, I)) \\
= \int \Theta : \text{BayesLens}_C(\Theta \otimes \Psi \otimes \Phi', \Theta \otimes \Psi \otimes \Psi') \\
\rightarrow \int \Theta : \text{BayesLens}_C(\Theta' \otimes \Phi', \Theta' \otimes \Psi') = \text{Ctx}(f')
\]

Note here that we do not need to use the symmetry, as both the residual and the ‘hole’ (filled with \( f \)) are on the left of the tensor. (Strictly, we do not require symmetry of \( \otimes \) for \( \pi_f \) either, only a braiding.)

**Remark 4.10.** It is possible to make the intuition of “filling the left and right holes” more immediately precise, at the cost of introducing another language, by rendering the 2-local context functions in the graphical calculus of the monoidal bicategory of \( V \)-profunctors. We demonstrate how this works, making the hole-filling explicit, in Appendix A.

Henceforth, when we say ‘context’, we will mean ‘complex context’.

### 4.2. Monoidal categories of statistical games

We are now in a position to define monoidal categories of statistical games over \( C \). In typical examples, the fitness functions will be valued in the real numbers, but this is not necessary for the categorical definition; instead, we allow the fitness functions to take values in an arbitrary monoid.

**Proposition 4.11.** Let \( C \) be a \( V \)-category admitting Bayesian inversion and let \((R, +, 0)\) be a monoid in \( V \). Then there is a category \( \mathcal{SGame}_C \) whose objects are the objects of \( \text{BayesLens}_C \) and whose morphisms \((X, A) \to (Y, B)\) are statistical games: pairs \((f, \phi)\) of a lens \( f : \text{BayesLens}_C((X, A), (Y, B)) \) and a fitness function \( \phi : \text{Ctx}(f) \to R \). When \( R \) is the monoid of reals \( \mathbb{R} \), then we just denote the category by \( \mathcal{SGame}_C \).

**Proof.** Suppose given statistical games \((f, \phi) : (X, A) \to (Y, B)\) and \((g, \psi) : (Y, B) \to (Z, C)\). We seek a composite game \((f, \phi) \circ (g, \psi) : (X, A) \to (Z, C)\). We have \( gf = g \circ f \) by lens composition. The composite fitness function \( \psi \circ \phi \) is obtained using the 1-local contexts by

\[
\psi \circ \phi := \text{Ctx}(g \circ f) \overset{(\psi, f \circ \phi)}{\longrightarrow} \text{Ctx}(g \circ f) \times \text{Ctx}(g \circ f) \overset{(g \circ \circ, f \circ \circ)}{\longrightarrow} \text{Ctx}(g) \times \text{Ctx}(g) \overset{(\phi, \psi)}{\longrightarrow} R \times R \overset{+}{\rightarrow} R
\]

The identity game \((X, A) \to (X, A)\) is given by \((\text{id}, 0)\), the pairing of the identity lens on \((X, A)\) with the unit 0 of the monoid \( R \). Associativity and unitality follow from those properties of lens composition, the coassociativity of copying \( \bigcirc \) in \( V \), and the monoid laws of \( R \).

**Definition 4.12.** We will write \( \mathcal{SimpSGame}_C \rightarrow \mathcal{SGame}_C \) for the full subcategory of \( \mathcal{SGame}_C \) defined on simple Bayesian lenses \((X, X) \to (Y, Y)\). As the case of simple lenses (Definition 2.37), we will eschew redundancy by writing the objects \((X, X)\) simply as \( X \).
Proposition 4.13. \( R\text{SGame}_C \) inherits a monoidal structure \((\otimes, (I, I))\) from \((\text{BayesLens}_C, \otimes, (I, I))\). When \((C, \otimes, I)\) is furthermore symmetric monoidal and \((R, +, 0)\) is a commutative monoid, then the monoidal structure on \(R\text{SGame}_C\) is symmetric.

Proof. The structure on objects and on the lens components of games is defined as in Proposition 3.11. On \(R\)-indexed games, we use the monoidal structure of \(R\) given games \((f, \phi) : (X, A) \rightarrow (Y, B)\) and \((f', \phi') : (X', A') \rightarrow (Y', B')\), we define the fitness function \( \phi \otimes \phi' \) of their tensor \((f, \phi) \otimes (f', \phi')\) as the composite

\[
\phi \otimes \phi' := \text{ctx}(f \otimes f') \Rightarrow \text{ctx}(f \otimes f') \times \text{ctx}(f \otimes f') \xrightarrow{(\pi_f, \pi_f)} \text{ctx}(f) \times \text{ctx}(f') \xrightarrow{(\phi, \phi')} R \times R \rightarrow R.
\]

That is, we form the left and right 2-local contexts, compute the local fitnesses, and compose them using the monoidal operation in \(R\). Unitality and associativity follow from that of \(\otimes\) in \(\text{BayesLens}_C\) and \(+\) in \(R\). \(\square\)

Corollary 4.14. Since each category \(R\text{SGame}_C\) is thus monoidal, we obtain categories \(\text{Para}(R\text{SGame}_C)\) of parameterized statistical games by Proposition 2.45, which are themselves monoidal. The structure is as described for Bayesian lenses in Section 3.4, with some minor additions to incorporate the fitness functions, the details of which we leave to the reader.

Remark 4.15 (Parameters as strategies). A parameterized statistical game of type \((X, A) \rightarrow (Y, B)\) in \(\text{Para}(R\text{SGame}_C)\) is a statistical game \(\left(\begin{array}{c} \Omega \otimes X \\ \Theta \otimes A \end{array}\right) \rightarrow \left(\begin{array}{c} Y \\ B \end{array}\right)\) in \(R\text{SGame}_C\); that is a pair of a Bayesian lens \(\left(\begin{array}{c} \Omega \otimes X \\ \Theta \otimes A \end{array}\right) \rightarrow \left(\begin{array}{c} Y \\ B \end{array}\right)\) and a fitness function \(\text{BayesLens}_C\left(\left(\begin{array}{c} \Omega \otimes X \\ \Theta \otimes A \end{array}\right), \left(\begin{array}{c} Y \\ B \end{array}\right)\right) \rightarrow R\) in \(V\). If we fix a choice of parameter \(\omega : \Theta\) and discard the updated parameters in \(\Omega\)—that is, if we reparameterize along the 2-cell induced by the lens \(\left(\begin{array}{c} \omega, \Theta \end{array}\right) : (I, I) \rightarrow (\Omega, \Theta)\)—then we obtain an unparameterized statistical game \((X, A) \rightarrow (Y, B)\). In this way, we can think of the parameters of a parameterized statistical game as the strategies by which the game is to be played: each parameter \(\omega : \Omega\) picks out a Bayesian lens, whose forwards channel we think of as a model by which the system predicts its observations and whose backwards channel describes how the system updates its beliefs. And, if we don’t just discard them, then these updated beliefs may include updated parameters (of a possibly different type \(\Theta\)). A successful strategy (a good choice of parameter) for a statistical game is then one which optimizes the fitness function in the contexts of relevance to the system: we can think of these “relevant contexts” as something like the system’s ecological niche.

Remark 4.16 (Multi-player games). In a later instalment of this series of papers, we will see how to compose the statistical games of multiple interacting agents, so that the game-playing metaphor becomes more visceral: the observations predicted by each system will then be generated by other systems, with each playing a game of optimal prediction. In this paper, however, we usually think of each statistical game as representing a single system’s model of its environment (its context), even where the games at hand are themselves sequentially or parallely composite. That is to say, our games here are fundamentally ‘two-player’, with the two players being the system and the context.

Remark 4.17. Both in the case of sequential or parallel composition of statistical games, the local fitnesses are computed independently and then summed. If a fitness function depends somehow on the residual, this might lead to ‘double-counting’ the fitness of any overlapping factors of the residual. For our purposes, this assumption of ‘independent fitness’ will suffice, and so we leave the question of gluing together correlated fitness functions for future work.

5. Examples

In this section, we describe how a number of common concepts in statistics and particularly statistical inference fit into the framework of statistical games. We begin with the simple example of maximum likelihood
estimation and progressively generalize to include ‘variational’ [27] methods such as the variational autoencoder [28] and generalized variational inference [29]. Along the way, we introduce the concepts of free energy and evidence upper bound. We do not here consider the algorithms by which statistical games may be played or optimized; that is a matter for a subsequent paper in this series. Instead, we see statistical games as providing an ‘algebra’ for the compositional construction of inference problems.

**Remark 5.1** (The role of fitness functions). Before we introduce our first example, we note that the games here are classified by their fitness functions, with the choice of lens being somewhat incidental to the classification⁶. We note furthermore that our fitness functions will tend to be of the form \( E_{k \bullet \pi}[f] \), where \((\pi, k)\) is a context for a lens, \(c\) is a channel, and \(f\) is an appropriately typed effect⁷. This form hints at the existence of a compositional treatment of fitness functions, which seem roughly to be something like “lens functionals”. We leave such a treatment, and its connection to Remark 4.17, to future work.

We first study the classic problem of maximum likelihood estimation, beginning by establishing an auxiliary results about contexts.

**Proposition 5.2**. Let \(I\) denote the monoidal unit \((I, I)\) in \(\text{BayesLens}_C\), and let \(l : I \to \Psi\) be a lens. Then

\[
\text{Ctx}(l) = \int \Theta : \text{BayesLens}_C \quad \text{BayesLens}_C(I, \Theta \otimes I) \times \text{BayesLens}_C(\Theta \otimes \Psi, I)
\]

\[
\cong \int \Theta : \text{BayesLens}_C (I, \Theta) \times \text{BayesLens}_C(\Theta \otimes \Psi, I)
\]

\[
\cong \text{BayesLens}_C(I \otimes \Psi, I)
\]

\[
\cong \text{BayesLens}_C(\Psi, I).
\]

Suppose \(\Psi = (A, S)\). Then \(\text{Ctx}(l) = C(A, I) \times V(C(I, A), C(I, S))\) by the definition of \(\text{BayesLens}_C\).

**Proof**. The first and third isomorphisms hold by unitality of \(\otimes\); the second holds by the Yoneda lemma (see Loregian [26, Prop. 2.2.1] for the argument). \[\square\]

**Example 5.3** (Maximum likelihood). When \(I\) is terminal in \(C\), a Bayesian lens of the form \((I, I) \to (X, X)\) is determined by its forwards channel, which is simply a state \(\pi : I \to X\). Following Proposition 5.2, and using that \(I\) is terminal, a context for such a lens is given simply by a continuation \(k : C(I, X) \to C(I, X)\) taking states on \(X\) to states on \(X\). A maximum likelihood game is then any statistical game \(\pi\) of the type \((I, I) \to (X, X)\) with fitness function \(\phi : \text{Ctx}(\pi) \to \mathbb{R}\) given by \(\phi(k) = E_{k(\pi)}[p_\pi]\), where \(p_\pi\) is a density function for \(\pi\). More generally, we might consider maximum \(f\)-likelihood games for monotone functions \(f : \mathbb{R} \to \mathbb{R}\), in which the fitness function is given by \(\phi(k) = E_{k(\pi)}[f \circ p_\pi]\). A typical choice here is \(f := \log\).

In order that there may be some freedom to optimize the fitness function, one typically works in the parametered category: the aim of the game is then to choose the optimal parameter for the context, as quantified by the fitness function. This gives us the notion of parameterized maximum likelihood game; but first, we define some simplifying notation.

**Notation 5.4** (Feedback). Let \(I\) be terminal in \(C\) and consider a Bayesian lens \(l = (l_1, l') : (A, S) \to (B, T)\), with a context represented by: a residual \((M, N)\); a prior \(\pi : I \to M \otimes A\); and a continuation \(k : C(I, M \otimes B) \to C(I, N \otimes T)\). Write \(\{\pi | l | k\} \) to denote \(k((\text{id}_M \otimes l_1) \bullet \pi)_T\) where \((\text{id}_M \otimes l_1) \bullet \pi\) is the map

\[
I \xrightarrow{\pi} M \otimes A \xrightarrow{\text{id}_M \otimes l_1} M \otimes B
\]

---

⁶This incidentality is lessened when we consider examples of parameterized games, but even here the parameterization only induces something of a ‘sub’-classification; the main classification remains due to the fitness functions.

⁷The resulting ‘optimization-centric’ perspective is in line with the aesthetic preference of [29], though we do not yet know what this alignment might signify; we are interested to find examples of a different flavour.
and where \((-)_{T}\) denotes the projection (marginalization) onto \(T\); here, by the channel \(N \otimes T \leftrightarrow T\).

Note that \(\{\pi \mid l \mid k\}\) therefore has the type \(I \leftrightarrow T\) in \(C\): it encodes the environment’s feedback in \(T\) to the lens, given its output in \(B\) and the context.

**Example 5.5** (Parameterized maximum likelihood). A parameterized Bayesian lens \((I, I) \xrightarrow{(\Omega, \Theta)} (X, X)\) is equivalently a Bayesian lens \((\Omega, \Theta) \rightarrow (X, X)\), and hence given by a pair of a channel (or “parameter-dependent state”) \(\Omega \leftrightarrow X\) and a parameter-update \(X \xrightarrow{\Theta} \). When \(I\) is terminal in \(C\), a context for the lens is represented by \(\pi : I \rightarrow M \otimes \Omega\) and \(k : C(I, M \otimes X) \rightarrow C(I, N \otimes X)\). We then define a *parameterized maximum f-likelihood game* to be a parameterized statistical game of the form \(l = (l_1, l') : (I, I) \xrightarrow{(\Omega, \Theta)} (X, X)\) with fitness function \(\phi : \text{Ctx} \rightarrow \mathbb{R}\) given by \(\phi(\pi, k) = \mathbb{E}_q[p_{l \circ \pi_\Omega}]\).

Here, \(p_{l \circ \pi_\Omega} : X \rightarrow [0, \infty]\) is a density function for the composite channel \(l \circ \pi_\Omega\). In applications one often fixes a single choice of parameter \(\alpha\), with the marginal state \(\pi_\Omega\) then being a Dirac delta distribution on that choice. One then writes the density function \(p_{l \circ \pi_\Omega}(-)\) as \(p_l(-|\alpha)\) or \(p_l(-|\Omega = o)\).

**Remark 5.6.** Recalling that we can think of probability density as a measure of the likelihood of an observation, we have the intuition that an “optimal strategy” (i.e., an optimal choice of lens or parameter) for a maximum likelihood game is one that maximizes the likelihood of the state obtained from the context, or in other words provides the “best explanation” of the data generated by the continuation.

Considering parameterized maximum likelihood games, which are equipped with parameter-update maps, leads one to wonder how to optimize this ’inferential’ backwards part of the game, and not just the ‘predictive’ forwards part. Such backwards optimization is approximate Bayesian inference.

**Example 5.7** (Bayesian inference). Let \(D : C(I, X) \times C(I, X) \rightarrow \mathbb{R}\) be a measure of divergence between states on \(X\). Then a (simple) \(D\)-Bayesian *inference* game is a statistical game \((c, \phi) : (X, X) \rightarrow (Y, Y)\) with fitness function \(\phi : \text{Ctx}(c) \rightarrow \mathbb{R}\) given by \(\phi(\pi, k) = \mathbb{E}_{q \sim \pi}(D(c_{\phi}(y), c_{\pi}(y)))\), where \(c = (c_1, c')\) constitutes the lens part of the game and \(c_{\pi}\) is the exact inversion of \(c_1\) with respect to \(\pi\).

Note that we say that \(D\) is a “measure of divergence between states on \(X\).” By this we mean any function of the given type with the semantical interpretation that it acts like a distance measure between states. But this is not to say that \(D\) is a metric or even pseudometric. One usually requires that \(D(\pi, \pi') = 0 \iff \pi = \pi'\), but typical choices do not also satisfy symmetry nor subadditivity. An important such typical choice is the relative entropy or Kullback-Leibler divergence, denoted \(D_{KL}\).

**Definition 5.8.** The *Kullback-Leibler divergence* \(D_{KL} : C(I, X) \times C(I, X) \rightarrow \mathbb{R}\) is defined by

\[
D_{KL}(\alpha, \beta) := \mathbb{E}_{x \sim \alpha} \left[\log p(x)\right] - \mathbb{E}_{x \sim \alpha} \left[\log q(x)\right]
\]

where \(p\) and \(q\) are density functions corresponding to the states \(\alpha\) and \(\beta\).

In many situations, computing the exact inversion \(c_{\pi}(x)\) is costly, and so is computing the divergence \(D \left(c_{\pi}(x), c_{\pi}^\dagger(x)\right)\). Consequently, approximate inversion schemes typically either approximate the divergence (as in Monte Carlo methods), or they optimize an upper bound on it (as in variational methods). In this section, we are interested in different choices of fitness function, rather than the algorithms by which the functions are exactly or approximately evaluated; hence we here consider the latter ‘variational’ choice, leaving the former for future work.

One widespread choice is to construct an upper bound on the divergence called the free energy or the evidence upper bound.
**Definition 5.9** (D-free energy). Let \((c, \pi)\) be a generative model with \(c : X \rightarrow Y\). Let \(p_c : Y \times X \rightarrow \mathbb{R}_+\) and \(p_\pi : X \rightarrow \mathbb{R}_+\) be density functions corresponding to \(c\) and \(\pi\). Let \(p_{c \pi} : Y \rightarrow \mathbb{R}_+\) be a density function for the composite \(c \cdot \pi\). Let \(c_\pi'\) be a channel \(Y \rightarrow X\) that we take to be an approximation of the Bayesian inversion of \(c\) with respect to \(\pi\) and that admits a density function \(q : X \times Y \rightarrow \mathbb{R}_+\). Finally, let \(D : \mathcal{C}(I, X) \times \mathcal{C}(I, X) \rightarrow \mathbb{R}\) be a measure of divergence between states on \(X\). Then the D-free energy of \(c_\pi'\) with respect to the generative model given an observation \(y : Y\) is the quantity

\[
\mathcal{F}_D(c_\pi', c, \pi, y) := \mathbb{E}_{x \sim c_\pi'(y)} \left[ -\log p_c(y|x) \right] + D \left( c_\pi'(y), \pi \right).
\]

We will elide the dependence on the model when it is clear from the context, writing only \(\mathcal{F}_D(y)\).

The D-free energy is an upper bound on the relative entropy \(D_{KL}\), as we now show.

**Proposition 5.10** (Evidence upper bound). The \(D_{KL}\)-free energy satisfies the following equality:

\[
\mathcal{F}_{D_{KL}}(y) = D_{KL} \left( c_\pi'(y), c_\pi^+(y) \right) - \log p_{c \pi}(y) = \mathbb{E}_{x \sim c_\pi'(y)} \left[ \log \frac{q(x|y)}{p_c(y|x) \cdot p_\pi(x)} \right]
\]

Since \(\log p_{c \pi}(y)\) is always negative, the free energy is an upper bound on \(D_{KL} \left( c_\pi'(y), c_\pi^+(y) \right)\), where \(c_\pi^+\) is the exact Bayesian inversion of the channel \(c\) with respect to the prior \(\pi\). Similarly, the free energy is an upper bound on the negative log-likelihood \(-\log p_{c \pi}(y)\). Thinking of this latter quantity as a measure of the "model evidence" gives us the alternative name evidence upper bound for the \(D_{KL}\)-free energy.

**Proof.** Let \(p_\omega : Y \times X \rightarrow \mathbb{R}_+\) be the density function \(p_\omega(y,x) := p_c(y|x) \cdot p_\pi(x)\) corresponding to the joint distribution of the generative model \((\pi, c)\). We have the following equalities:

\[
-\log p_{c \pi}(y) = \mathbb{E}_{x \sim c_\pi'(y)} \left[ -\log p_{c \pi}(y) \right]
\]

\[
\begin{align*}
= \mathbb{E}_{x \sim c_\pi'(y)} \left[ -\log \frac{p_\omega(y,x)}{p_{c_\pi}(x|y)} \right] & \quad \text{(by Bayes’ rule)} \\
= \mathbb{E}_{x \sim c_\pi'(y)} \left[ -\log \frac{p_\omega(y,x)}{q(x|y) \cdot p_{c_\pi}(x|y)} \right] \\
= -\mathbb{E}_{x \sim c_\pi'(y)} \left[ \log \frac{p_\omega(y,x)}{q(x|y)} \right] - D_{KL} \left( c_\pi'(y), c_\pi^+(y) \right)
\end{align*}
\]

\(\square\)

**Definition 5.11.** We will call \(\mathcal{F}_{D_{KL}}\) the variational free energy, or simply free energy, and denote it by \(\mathcal{F}\) where this will not cause confusion. We will take the result of Proposition 5.10 as a definition of the variational free energy, writing

\[
\mathcal{F}(y) = \mathbb{E}_{x \sim c_\pi'(y)} \left[ \log \frac{q(x|y)}{p_c(y|x) \cdot p_\pi(x)} \right]
\]

where each term is defined as in Definitions 5.9 and 5.8.

**Remark 5.12.** The name free energy is due to an analogy with the Helmholtz free energy in thermodynamics, as, when \(D = D_{KL}\), we can write it as the difference between an (expected) energy and an entropy term:

\[
\begin{align*}
\mathcal{F}(y) &= \mathbb{E}_{x \sim c_\pi'(y)} \left[ \log \frac{q(x|y)}{p_c(y|x) \cdot p_\pi(x)} \right] \\
&= \mathbb{E}_{x \sim c_\pi'(y)} \left[ -\log p_c(y|x) - \log p_\pi(x) \right] - S_X \left[ c_\pi'(y) \right] \\
&= \mathbb{E}_{x \sim c_\pi'(y)} \left[ E_{(\pi, c)}(x, y) \right] - S_X \left[ c_\pi'(y) \right] = U - TS
\end{align*}
\]
where we call \( E_{(\pi,c)} : X \times Y \to \mathbb{R}_+ \) the energy of the generative model \((\pi, c)\), and where \( S_X : \mathcal{C}(I, X) \to \mathbb{R}_+ \) is the Shannon entropy on \( X \). The last equality makes the thermodynamic analogy: \( U \) is the internal energy of the system; \( T = 1 \) is the temperature; and \( S \) is again the entropy.

Having now defined a more tractable fitness function, we can construct statistical games accordingly. Since the free energy is an upper bound on relative entropy, optimizing the former can have the side effect of optimizing the latter\(^8\). We call the resulting games autoencoder games, for reasons that will soon be clear.

**Example 5.13** (Autoencoder). Let \( D : \mathcal{C}(I, X) \times \mathcal{C}(I, X) \to \mathbb{R} \) be a measure of divergence between states on \( X \). Then a simple \( D\)-autoencoder game is a simple statistical game \((c, \phi) : (X, X) \to (Y, Y)\) with fitness function \( \phi : \text{Ctx}(c) \to \mathbb{R} \) given by \( \phi(\pi, k) = \mathbb{E}_{y \sim \{\pi | c | k\}} [\mathcal{F}_D(c_\pi', c, \pi, y)] \) where \( c = (c, c') : (X, X) \to (Y, Y) \) constitutes the lens part of the game.

One also of course has parameterized versions of the autoencoder games.

**Example 5.14** (Simply parameterized autoencoder). A simply parameterized \( D\)-autoencoder game is a simple parameterized statistical game \((c, \pi) : (X, X) \overset{(\Omega, \Theta)}{\to} (Y, Y)\) with the \( D\)-autoencoder fitness function \( \phi : \text{Ctx}(c) \to \mathbb{R} \) given by \( \phi(\pi, k) = \mathbb{E}_{y \sim \{\pi | c | k\}} [\mathcal{F}_D(c_\pi', c, \pi, y)] \). That is, a simply parameterized \( D\)-autoencoder game is just a simple \( D\)-autoencoder game with tensor product domain type.

More often in applications, one doesn’t use the same backwards channel to update both the “belief about the causes” in \( X \) and the parameters in \( \Omega \) simultaneously. Instead, the backwards channel updates only the beliefs over \( X \), and any updating of the parameters is left to another ‘higher-order’ process. The \( X\)-update channel may nonetheless still itself be parameterized in \( \Omega \): for instance, if it represents an approximate inference algorithm, then one often wants to be able to improve the approximation, and such improvement amounts to a change of parameters. The ‘higher-order’ process that performs the parameter updating is then often represented as a reparameterization: a 2-cell in the bicategory \( \text{Para}(\text{SGame}_c) \). The next example tells the first part of this story.

**Example 5.15** (Parameterized autoencoder). A (simple) parameterized \( D\)-autoencoder game is a parameterized statistical game \((c, \phi) : (X, X) \overset{(\Omega, \Theta)}{\to} (Y, Y)\) with fitness function \( \phi : \text{Ctx}(c) \to \mathbb{R} \) given by

\[
\phi(\pi, k) = \mathbb{E}_{y \sim \{\pi | c | k\}} [\mathcal{F}_D(c_\pi, c \pi, \pi_X, y)].
\]

As before, the notation \((-)_X\) indicates taking the \( X \) marginal, along the projection \( \Theta \otimes X \to X \). We also define \( c|_{\pi_X} := c \otimes (\pi_X \otimes \text{id}_X) \), indicating “\( c \) given the parameter state \( \pi_X \)”. Written out in full, the fitness function is therefore given by

\[
\phi(\pi, k) = \mathbb{E}_{y \sim \{\pi | c | k\}} [\mathcal{F}_D(\text{proj}_X \bullet c_\pi', c \bullet (\pi_X \otimes \text{id}_X), \text{proj}_X \bullet \pi, y)].
\]

Note that the pair \((c|_{\pi_X}, (c_{\pi_X})_X) : (X, X) \to (Y, Y)\), with \( c|_{\pi_X} : X \to Y \) and \((c_{\pi_X})_X : Y \to X\).

**Remark 5.16** (Meaning of ‘autoencoder’). Why do we call autoencoder games thus? The name originates in machine learning, where one thinks of the forwards channel as ‘decoding’ some latent state into a prediction

\(^8\)Strictly speaking, one can have a decrease in free energy along with an increase in relative entropy, as long as the former remains greater than the latter. Therefore, optimizing the free energy does not necessarily optimize the relative entropy. However, as elaborated in Remark 5.16, the difference between the variational free energy and the relative entropy is the log-likelihood, so optimizing the free energy corresponds to simultaneous maximum-likelihood estimation and Bayesian inference.
of some generated data, and the backwards channel as ‘encoding’ a latent state given an observation of the data; typically, the latent state space is thought to have lower dimensionality than the observed data space, justifying the use of this ‘compression’ terminology. A slightly more precise way to see this is to consider an autoencoder game where the context and forwards channel are fixed. The only free variable available for optimization in the fitness function is then the backwards channel, and the optimum is obtained when the backwards channel equals the exact inversion of the forwards channel (given the prior in the context, and for all elements of the support of the state obtained from the continuation). Conversely, allowing only the forwards channel to vary, it is easy to see that the autoencoder fitness function is then equal to the fitness function of a maximum log-likelihood game (up to a constant). Consequently, optimizing the fitness of an autoencoder game in general corresponds to performing approximate Bayesian inference and maximum likelihood estimation simultaneously. The optimal strategy (lens or parameter) can then be considered as representing an ‘optimal’ model of the process by which observations are generated, along with a recipe for inverting that model (and hence ‘encoding’ the causes of the data). The prefix auto- indicates that this model is learnt in an unsupervised manner, without requiring input about the ‘true’ causes of the observations.

Some authors (in particular, Knoblauch et al. [29]) take a variant of the $D$-autoencoder fitness function to define a generalization of Bayesian inference: in an echo of Remark 5.16, the intuition here is that Bayesian inference simply define a generalization of Bayesian inference: in an echo of Remark 5.16, the intuition here is that Bayesian inference is maximum likelihood estimation, except ‘regularized’ by the uncertainty encoded in the prior, which stops the optimum strategy being trivially given by a Dirac delta distribution. By allowing both the choice of likelihood function and divergence measure to vary, one obtains a family of generalized inference methods. Moreover, when one retains the standard choices of log-density as likelihood and relative entropy as divergence, the resulting generalized Bayesian inference games coincide with variational autoencoder games; then, when the forwards channel (or its parameter) is fixed, both types of game coincide with the Bayesian inference games of Example 5.7 above.

**Example 5.17** (Generalized Bayesian inference [29]). Let $D: C(I, X) \times C(I, X) \to \mathbb{R}$ be a measure of divergence between states on $X$, and let $l : Y \otimes X \to I$ be any effect on $Y \otimes X$. Then a simple generalized $(l, D)$-Bayesian inference game is a simple statistical game $(c, \phi) : (X, X) \to (Y, Y)$ with fitness function $\phi : \mathcal{C}(\text{tx}(c)) \to \mathbb{R}$ given by

$$
\phi(\pi, k) = \mathbb{E}_{y \sim \{ \pi \mid c \mid k \}} \left[ \mathbb{E}_{x \sim c'_\pi(y)} [l(y, x)] + D(c'_\pi(y), \pi) \right]
$$

where $(c, c') : (X, X) \to (Y, Y)$ constitutes the lens part of the game.

**Proposition 5.18.** Generalized Bayesian inversion and autoencoder games coincide when $D = D_{KL}$ and $l = -\log p_c$, where $p_c$ is a density function for the forwards channel $c$.

**Proof.** Consider the $D_{KL}$-free energy. We have

$$
\mathcal{F}_{D_{KL}}(c'_{\pi}, c, \pi, y) = \mathbb{E}_{x \sim c'_{\pi}(y)} \left[ -\log p_c(y|x) - \log p_{\pi}(x) \right] - S_X \left[ c'_{\pi}(y) \right] \quad \text{by Remark 5.12}
$$

$$
= \mathbb{E}_{x \sim c'_{\pi}(y)} \left[ -\log p_c(y|x) \right] + \mathbb{E}_{x \sim c'_{\pi}(y)} [\log q(x|y) - \log p_{\pi}(x)]
$$

$$
= \mathbb{E}_{x \sim c'_{\pi}(y)} \left[ -\log p_c(y|x) \right] + D_{KL}(c'_{\pi}(y), \pi)
$$

$$
= \mathbb{E}_{x \sim c'_{\pi}(y)} [l(y, x)] + D(c'_\pi(y), \pi)
$$

where $q$ is a density function for $c'_\pi$. \qed
Unsurprisingly, as in the autoencoder case, there are parameterized and simply parameterized variants of generalized Bayesian inference games.

Finally, we remark that, in the case where $C = \text{sfKrn}$, where $I$ is not terminal and morphisms into $I$ correspond to functions into $[0, \infty]$, composing a Bayesian lens and its context gives a lens $(I, I) \rightarrow (I, I)$: both the forwards and backwards parts of this $I$-endolens’ return positive reals (which in this context Jacobs and colleagues call validities [12, 30]), and which we can think of as “the environment’s measurements of its compatibility with the lens”. In this case, we can therefore define validity games, where the fitness function is simply given by computing the backwards validity\(^9\). Since such a fitness function measures the interaction of the lens with its environment, the corresponding statistical games may be of relevance in modelling multi-agent or otherwise interacting statistical systems—for instance, in modelling evolutionary dynamics. We leave the exploration of this for future work.

6. References

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\(^9\)Note that the backwards effect, as an $I$-state-dependent effect (or ‘vector’), already depends upon the forwards validity, so we do not need to include the forwards validity directly in the fitness computation.
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A. 2-local contexts, graphically

To clarify the idea that the 2-local contexts for the factors of a tensor product game (or morphism more generally) are obtained by “filling the hole” on the left or right of the tensor, one can work in the monoidal bicategory of $\mathcal{V}$-profunctors and use the associated graphical calculus [31] to depict the ‘hole’ in the context and its filler. In this section, we work with a general monoidal category $\mathcal{C}$, which may or may not be a category of lenses or games. Nonetheless, the basic idea is the same: a (complex) context for a morphism $X \rightarrow Y$ is given by a triple of a residual denoted $\Theta$, a state $I \rightarrow \Theta \otimes X$ and a ‘continuation’ (or ‘effect’) $\Theta \otimes Y \rightarrow I$, coupled according to the coend quotient rule.

Below, we show how to obtain the object of right local 2-contexts for a tensor product morphism $f \otimes f' : X \otimes X' \rightarrow Y \otimes Y'$, using the graphical calculus of $\mathcal{V}$-Prof. At each stage, we depict on the left the object named on the right. We start with a complex context for the tensor along with a ‘filler’ object of morphisms $X' \rightarrow Y'$, which is shown “filling the (left-hand) hole”. In the first step, we use the composition rule of $\mathcal{C}$ to connect the matching ‘ports’ on the domain $X$. We then couple the matching $Y$ port using the coend and gather $\Theta$ and $Y$ together into a single residual. Note that these steps correspond directly to factors in the definition of $\pi_{f'} : \text{Ctx}(f \otimes f') \rightarrow \text{Ctx}(f')$ (Definition 4.9).

$$
\begin{align*}
\int^{\Theta,\mathcal{C}} \mathcal{C}(I, \Theta \otimes X \otimes X') \times \mathcal{C}(X, Y) \times \mathcal{C}(\Theta \otimes Y \otimes Y', I) \\
\int^{\Theta,\mathcal{C}} \mathcal{C}(I, \Theta \otimes Y \otimes X') \times \mathcal{C}(\Theta \otimes Y \otimes Y', I) \\
\int^{\Theta',\mathcal{C}} \mathcal{C}(I, \Theta' \otimes X') \times \mathcal{C}(\Theta' \otimes Y', I)
\end{align*}
$$

We find this graphical representation to be a useful aid in comprehension, and often simplifies the symbolic ‘book-keeping’ that can complicate expressions such as those in Definition 4.9. The cost of this expressivity is the introduction of another categorical structure, and its associated cognitive load. In previous work [32], we have made more use of this representation: there, we worked with the ‘optical’ definition of Bayesian lenses described in St. Clere Smithe [6]; and we note that an earlier informal version of this graphical language was
originally used to define local contexts for tensor product games in the compositional game theory literature [7]. Since the \( \mathbf{V} \)-profunctorial setting [18] is required in order to define optics, we had already paid this extra cognitive cost. In this paper, however, we have preferred to stick with the simpler fibrational definition of Bayesian lenses.

**Remark A.1.** A different but related graphical calculus for optics is described by Boisseau [33]. However, this alternative calculus is somewhat less general than that of Román [31], and its adoption here would not eliminate the extra cognitive cost; we do nonetheless make use of it in [6].