Relations between nonlinear Riccati equations and other equations in fundamental physics

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Abstract. Many phenomena in the observable macroscopic world obey nonlinear evolution equations while the microscopic world is governed by quantum mechanics, a fundamental theory that is supposedly linear. In order to combine these two worlds in a common formalism, at least one of them must sacrifice one of its dogmas. Linearizing nonlinear dynamics would destroy the fundamental property of this theory, however, it can be shown that quantum mechanics can be reformulated in terms of nonlinear Riccati equations. In a first step, it will be shown that the information about the dynamics of quantum systems with analytical solutions can not only be obtainable from the time-dependent Schrödinger equation but equally well from a complex Riccati equation. Comparison with supersymmetric quantum mechanics shows that even additional information can be obtained from the nonlinear formulation. Furthermore, the time-independent Schrödinger equation can also be rewritten as a complex Riccati equation for any potential. Extension of the Riccati formulation to include irreversible dissipative effects is straightforward. Via (real and complex) Riccati equations, other fields of physics can also be treated within the same formalism, e.g., statistical thermodynamics, nonlinear dynamical systems like those obeying a logistic equation as well as wave equations in classical optics, Bose–Einstein condensates and cosmological models. Finally, the link to abstract “quantizations” such as the Pythagorean triples and Riccati equations connected with trigonometric and hyperbolic functions will be shown.

1. Introduction
In the 19th century electrodynamics (as formulated by James C. Maxwell) and thermodynamics (condensed in three laws derived from phenomenological experience) were a huge success. They projected an image of nature as being continuous (strongly favoured by physicists like Ernst Mach) as opposed to being built from discrete components as assumed in an atomistic approach (favoured by scientists like Ludwig Boltzmann). The ideas of Democritus (who coined the term atom for a part of our material world that is no longer divisible) and, in a more abstract way, of Plato and Pythagoras who assumed smallest units as building blocks of our world, seemed to belong purely to the realm of philosophy.

Towards the end of the 19th century and the beginning of the 20th, the situation changed. First, in order to explain the spectrum of the black-body radiation, Max Planck had to “discretize” a physical quantity called action that, for example, can be expressed as a product of position times momentum or energy times time, i.e., a product of two physical quantities that cannot be measured exactly at the same time. (Something that Heisenberg later on proved to be impossible in principle.) Planck first considered the corresponding minimal quantum of action,
denoted by “h” (for the German “Hilfsgröße”), only as a kind of temporary auxiliary quantity that could be abandoned when a more natural explanation for the spectrum could be found. It turned out this was not the case. In fact, by introducing his quantum of action, conservative Planck actually initiated a revolution in the physicists’ view of the world in a way he never expected, not to mention intended! And at the beginning of the 20th century even more peculiar effects were observed. For instance light, that (after Maxwell) was finally considered to be a continuous wave, behaved like discrete particles in certain experiments such as the photoelectric effect. Contrarily, electrons that (in the meantime) were assumed to be particles, displayed wave-like behaviour in some experiments and produced interference patterns. The dichotomy of light versus matter, or continuous versus discrete, was only resolved in the mid-twenties of the last century by Schrödinger and Heisenberg (and finally Dirac) with the development of quantum theory.

Though physically equivalent, Schrödinger’s wave mechanics turned out to be more successful and receptive to the physics community than Heisenberg’s matrix mechanics that used a less familiar mathematical description than Schrödinger’s partial differential equation. Since both formulations are closely-related to classical Hamiltonian mechanics they also have similar properties, in particular, there is no direction of time in the evolution of the system and energy is a constant of motion (at least in the cases that are usually discussed in textbooks and can be solved analytically in closed form).

However, as everyone can observe daily in the surrounding world, nature actually behaves quite differently. There is a direction of time in almost every evolutionary process (and we cannot reverse it even if we would sometimes like to). Also mechanical energy is not a conserved quantity but dissipated into heat by effects like friction. There are ways of explaining and including these phenomena into the theories mentioned earlier but, for ordinary people, they are not really convincing. Nevertheless, quantum theory (with all its technological developments) is undoubtedly the most successful theory so far, and not only in physics but also from an economic viewpoint.

The situation in physics took a different twist near the end of the 20th century with the development of nonlinear (NL) dynamics. This theory is able to describe evolutionary processes like population growth with given resources or weather patterns and other such complex systems as they occur in real life. At the same time it can also take into account phenomena like irreversibility of evolution and dissipation of energy.

So why not combine the two theories to get the best of both worlds?

From a formal, mathematical point of view, there are several aspects that must be considered. Here, only two will be mentioned that are relevant to the subsequent discussion.

1. C. N. Yang, in his lecture on the occasion of Schrödinger’s centennial celebration [1], stated that in his opinion, the major difference between classical and quantum physics is the occurrence of the imaginary unit \(i = \sqrt{-1}\) in quantum mechanics since here it enters physics in a fundamental way (not just as a tool for computational convenience) and “complex numbers became a conceptual element of the very foundation of physics”. The very meaning of the fundamental equations of wave mechanics and matrix mechanics “would be totally destroyed if one tries to get rid of \(i\) by writing them in terms of real and imaginary parts”. This, however, should not be a problem in the unification of quantum theory and NL dynamics since, also in the latter, complex numbers play an eminent role (e.g. in the complex quadratic family leading to the Mandelbrot set).

2. The contrast between reversible time-evolution in quantum theory and irreversible evolution, as possible in NL dynamics, appears more problematic. But, even if one would restrict the unification only to systems with reversible evolution the major problem seems to be that quantum theory is a linear one, i.e., it is essentially based on linear differential equations. On the other hand, NL dynamics, by definition, uses NL differential equations (or discretized versions
of such). Why then should linearity be so important for quantum theory? As mentioned above, quantum theory can explain the wave properties of material systems such as diffraction patterns that can essentially be considered as superposition of different solutions of the same equation. But this superposition principle is usually only attributed to linear differential equations!

So is there any way out of this dilemma?

In principle, NL dynamics cannot be linearized otherwise it would lose not only its linguistic meaning but also its physical properties. Therefore, the only solution seems to be a NL version of quantum theory which takes into account all the conventional properties of this theory (including a kind of superposition principle) while displaying formal compatibility with NL dynamics and which might (hopefully) provide additional information that cannot be obtained easily (or even be expected) from the linear form of quantum theory.

For this purpose, the next section will consider the time-dependent Schrödinger equation (TDSE) for cases where it possesses exact analytic solutions in the form of Gaussian wave packets (WPs). (In other words, for potentials at most quadratic in position variable, here explicitly for the one-dimensional harmonic oscillator (HO) with \( V = \frac{1}{2} \omega^2 x^2 \) with constant frequency \( \omega \), its generalization for \( \omega = \omega(t) \) and, in the limit \( \omega \to 0 \), for the free motion \( V = 0 \).) It will be shown that the information about the dynamics of these systems can equally-well be obtained from a complex, quadratically-nonlinear, inhomogeneous Riccati equation. An interesting feature of this equation is that it can be linearized, therefore being able to provide a kind of superposition principle. Conversely, due to the nonlinearity, this equation is very sensitive to the initial conditions, which is not at all obvious in the linearized form. Due to these attributes, it is predestined to bridge the gap between linear theories like quantum mechanics and NL ones. Several methods of treating the NL Riccati equation will be mentioned and formal similarities with supersymmetric (SUSY) quantum mechanics will be shown. During the course, a dynamical invariant will also be introduced with the dimension of action (the quantity that is quantized according to Planck). Further, it will be shown how factorization of the corresponding operator can lead to generalized creation/annihilation operators allowing for the construction of generalized coherent states (CSs) with TD width, i.e., TD position (and momentum) uncertainties as known from squeezed states. An application of the generalized creation/annihilation operators to obtain unusual solutions of the TDSE for the free motion will be mentioned.

The linearization of the complex Riccati equation to a complex Newton-type equation will allow for a geometric interpretation of the linearized variable as well as of the afore-mentioned invariant.

In section 3, it will be shown that a, formally, similar treatment is also possible for the time-independent (TI) SE but now without restriction to quadratic potential since \( V(x) \) corresponds to \( \omega(t) \) in the TD case and enters only as an inhomogeneity into the Riccati equation. Since the discussion in section 2 holds also for TD \( \omega(t) \), the same applies in the TISE for any \( V(x) \). With this treatment it is possible to obtain complex solutions with space-dependent phase of the wave function. Using a formalism well-known from SUSY quantum mechanics, complex (non-hermitian) potentials with real eigenvalues can be obtained in a new way. As a first example, a “complexified” Pöschl–Teller potential will be presented that is derived from the free motion SE (see [2] for further details).

Section 4 will briefly show that a generalization is possible that includes dissipation and irreversibility. Several effective models describing these systems and their interrelations will be mentioned. Using a specific NL modification of the TDSE, it will be shown how the equations of the WP dynamics change in this case, particularly the complex Riccati equation describing the TD of the WP width.

The transformation between the linear SE (TD and TI) and the NL Riccati equation allows for connections with many other fields of physics where NL Riccati equations play an important
role. This will be shown in section 5 using, as examples, systems with Bose–Einstein (BE) or Fermi–Dirac statistics in statistical thermodynamics and systems obeying the logistic equation in NL dynamics. Changing to complex Riccati equations, systems in classical optics as well as BECs will be considered and, by analogy, even cosmological problems can be treated in this way.

Complex Riccati (or related Bernoulli) equations also provide the answer to the question of finding Pythagorean triples, i.e. right-angled triangles with integer length of their sides. Furthermore, the geometry of these triangles with their description in terms of trigonometric functions allows one to find Riccati equations from their derivatives. Admitting imaginary variables as well, this can be generalized to include hyperbolic functions.

Section 6 will summarize the results and point out some areas for future developments.

2. Complex Riccati equations equivalent to the time-dependent Schrödinger equation

In the following the one-dimensional TDSE for problems with exact analytic solutions in the form of Gaussian WPs will be considered. This applies for potentials that are at most quadratic in position variable. In particular the HO \( V = \frac{m}{2} \omega^2 x^2 \) with constant frequency, \( \omega = \omega_0 \), or the parametric oscillator with TD frequency, \( \omega = \omega(t) \), will be discussed where the corresponding expressions for the free motion \( (V(x) = 0) \) are always obtained in the limit \( \omega \to 0 \).

The WP solution of the TDSE

\[
 i \hbar \frac{\partial}{\partial t} \Psi(x, t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m}{2} \omega^2 x^2 \right\} \Psi(x, t) \quad (1)
\]

(\( \hbar = \frac{h}{2\pi} \) with \( h = \) Planck’s constant) can be written as

\[
 \Psi(x, t) = N(t) \exp \left\{ i \left[ y(t) \tilde{x}^2 + \frac{<p>}{\hbar} \tilde{x} + K(t) \right] \right\} \quad (2)
\]

with the shifted coordinate \( \tilde{x} = x - <x> = x - \eta(t) \), where the mean value \( <x> = \int_{-\infty}^{\infty} \Psi^* x \Psi dx = \eta(t) \) corresponds to the classical trajectory, \( <p> = m \dot{\eta} \) represents the classical momentum and the coefficient of the quadratic term in the exponent, \( y(t) = y_R(t) + iy_I(t) \), is a complex function of time. The (possibly TD) normalization factor \( N(t) \) and the purely TD function \( K(t) \) in the exponent are not relevant to the following discussion.

The equations of motion for \( \eta(t) \) and \( y(t) \), or \( \left( \frac{2\hbar}{m} y = C \right) \), that are obtained by inserting WP (2) into Eq. (1) are relevant for our purpose and are given by

\[
 \ddot{\eta} + \omega^2 \eta = 0 , \quad (3)
\]

and

\[
 \dot{C} + C^2 + \omega^2 = 0 , \quad (4)
\]

where overdots denote derivatives with respect to time. The Newtonian equation (3) simply expresses that the maximum of the WP, located at \( x = <x> = \eta(t) \), follows the classical trajectory. The equation for the quantity \( \frac{2\hbar}{m} y(t) = C \) has the form a of a complex NL Riccati equation and describes the time-dependence of the WP width that is related to the position uncertainty via \( y_I = \frac{1}{2 <\tilde{x}^2>} \) with \( <\tilde{x}^2> = <x^2> - <x>^2 \) being the mean square deviation of position.
2.1. Direct solution of the Riccati equation

There are different ways of treating this (inhomogeneous) Riccati equation, elucidating different aspects of the equation. First, it can be solved directly by transforming it into a (homogeneous) NL (complex) Bernoulli equation if a particular solution \( \tilde{C} \) of the Riccati equation is known.

The general solution of Eq. (4) is then given by \( C = \tilde{C} + V(t) \) where \( V(t) \) fulfils the Bernoulli equation

\[
\dot{V} + 2 \tilde{C} V + V^2 = 0 .
\]  

The coefficient of the linear term depends on the particular solution \( \tilde{C} \). Equation (5) can be linearized via \( V = \frac{1}{\kappa(t)} \) to yield

\[
\dot{\kappa} - 2 \tilde{C} \kappa = 1 ,
\]

which can be solved straightforwardly. For constant \( \tilde{C} \), \( \kappa(t) \) can be expressed in terms of exponential or hyperbolic functions (for real \( \tilde{C} \)) or trigonometric functions (for imaginary \( \tilde{C} \)). In this case, \( C \) can be written as

\[
C(t) = \tilde{C} + \frac{e^{-2\tilde{C}t}}{2\tilde{C}} \left( 1 - e^{-2\tilde{C}t} \right) + \kappa_0
\]  

where the enumerator is obviously the derivative of the denominator.

For \( \tilde{C} \) being TD, \( \kappa(t) \) and hence \( V \) can be expressed in terms of \( \mathcal{I}(t) = \int^t dt' e^{-\int^{t'} dt'' 2\tilde{C}(t'')} \).

So the general solution of Eq. (4) can be written as

\[
C(t) = \tilde{C} + \frac{d}{dt} \ln [\kappa_0 + \mathcal{I}(t)]
\]

with the logarithmic derivative representing the solution of the Bernoulli equation. It defines a one-parameter family of solutions depending on the (complex) initial value \( \kappa_0 = V_0^{-1} \) as a parameter. To realize the strong qualitative influence of the initial value \( \kappa_0 \) on the solution of the Riccati equation (which is not surprising since this is a NL differential equation), a comparison with supersymmetric (SUSY) quantum mechanics [3, 4, 5] could be quite useful.

In SUSY quantum mechanics, the Hamiltonian can be represented by a \( 2 \times 2 \) diagonal matrix where the potentials \( V_i \) of the Hamiltonian operators \( H_i = - \hbar^2 2m \frac{d^2}{dx^2} + V_i \) (i=1,2) on the diagonal are determined by a (real) Riccati equation for the so-called “superpotential” \( W(x) \),

\[
V_{1/2} = \frac{1}{2} \left[ W^2 + \frac{\hbar}{\sqrt{m}} \frac{d}{dx} W \right]
\]

and differ only by the sign of the term depending on the derivative of \( W(x) \). Again, one can try to solve this Riccati equation by reducing it to a Bernoulli equation using a particular solution \( \tilde{W} \). In the case of the HO, this particular solution is given by \( \tilde{W} = \sqrt{m\omega_0} x \), leading to the two potentials

\[
\dot{V}_{1/2} = \frac{m}{2} \omega_0^2 x^2 \mp \frac{\hbar}{2 \omega_0} ,
\]

essentially the parabolic harmonic potential, only shifted by minus/plus the ground state energy \( \frac{\hbar}{2} \omega_0 \). The general solution can again be written in the form \( W(x) = W(x) + \Phi(x) \) where \( \Phi(x) \) must now fulfil the Bernoulli equation

\[
\frac{\hbar}{\sqrt{m}} \frac{d}{dx} \Phi + 2\tilde{W} \Phi + \Phi^2 = 0
\]  

[9]
(written here for the plus sign of the derivative). This can be solved in the same way by linearization to finally yield the general solution

\[ W(x) = \tilde{W}(x) + \sqrt{\frac{m}{\hbar}} \frac{d}{dx} \ln [\varepsilon + I(x)]. \]  

(12)

The integral \( I(x) \) is formally identical to the one in the TD case, only \( t \) must be replaced by \( x \) and \( \tilde{C} \) by \( \tilde{W} \). Also this solution depends on a (this time real) parameter \( \varepsilon \) (corresponding to \( \Phi^{-1}(0) \)). This generalized \( W(x) \) gives rise to a one-parameter family of isospectral potentials (e.g., for \( i = 1 \))

\[ V_1(x; \varepsilon) = \tilde{V}_1(x) - \sqrt{\frac{\hbar}{m}} \frac{d^2}{dx^2} \ln [\varepsilon + I(x)], \]  

(13)
i.e., potentials with different shapes but the same energy spectrum, namely that of the HO, only with ground state energy equal to zero (apart from \( \varepsilon = 0 \), where this state is missing).

The shape of the potentials is now, unlike the parabolic harmonic potential, no longer symmetric under the exchange \( x \to -x \); there is even a second minimum showing up for negative \( x \) whose depth increases with decreasing \( \varepsilon \) (for details, see [5]). Only for \( \varepsilon \to \infty \), the ln-term vanishes and the parabolic potential \( \tilde{V}_1(x) \) is re-gained. So, obviously the parameter \( \varepsilon \) can have drastic qualitative consequences for the solution of the Riccati equation. The same can also happen in the above-mentioned TD case when \( \kappa_0 \) is varied.

### 2.2. Treatment via Ermakov equation

There is a second way of treating the complex Riccati equation (4). Transforming it into a real NL differential equation which, in combination with the Newtonian equation (3), leads to a dynamical invariant with the dimension of an action. For this purpose a new (real) variable \( \alpha(t) \) is introduced via \( C_I = \left( \frac{2\hbar}{m} y_I \right) = \frac{1}{\alpha^2} \), i.e., \( \alpha \propto \sqrt{<\tilde{x}^2>} \) and is thus directly proportional to the WP width. Inserting this into the imaginary part of Eq. (4) allows one to determine the real part of the variable as \( C_R = \left( \frac{2\hbar}{m} y_R \right) = \frac{\dot{\alpha}}{\alpha} \), which, when inserted into the real part of (4) together with the above definition of \( C_I \), finally turns the complex Riccati equation into the real NL so-called Ermakov equation \(^1\) for \( \alpha(t) \),

\[ \ddot{\alpha} + \omega^2 \alpha = \frac{1}{\alpha^3}. \]  

(14)

It was shown by Ermakov [9] in 1880, i.e., 45 years before quantum mechanics was formulated by Schrödinger and Heisenberg, that from the pair of equations (3) and (14), coupled via \( \omega^2 \), by eliminating \( \omega^2 \) from the equations, a dynamical invariant, the Ermakov-invariant,

\[ I_L = \frac{1}{2} \left[ (\dot{\eta} \alpha - \eta \dot{\alpha})^2 + \left( \frac{\eta}{\alpha} \right)^2 \right] = \text{const}. \]  

(15)
can be obtained (this invariant was rediscovered by several authors, also in a quantum mechanical context; see, e.g., [10, 11, 12]).

This invariant has (at least) two remarkable properties: i) it is also a constant of motion for \( \omega = \omega(t) \), in the case where the corresponding Hamiltonian does not have this property; ii) apart from a missing constant factor \( m \), i.e., mass of the system, it has the dimension of an action.

\(^1\) This equation was studied already in 1874 by Adolph Steen [6]. However, Steen’s work was ignored by mathematicians and physicists for more than a century because it was published in Danish in a journal usually not containing many articles on mathematics. An English translation of the original paper [7] is available and generalizations can be found in [8].
not of an energy. The missing factor $m$ is due to the fact that Ermakov used the mathematical Eq. (3) whereas, in a physical context, Newton’s equation of motion, i.e., Eq. (3) multiplied by $m$, is relevant.

Using the definitions of $C_I$ and $C_R$, the invariant can be rewritten as

$$I_L = \frac{1}{2} \alpha^2 \left[ (\dot{\eta} - C_R \dot{\eta})^2 + (C_I \eta)^2 \right] = \frac{1}{2} \alpha^2 \left[ (\dot{\eta} - C_I \eta)^2 + (\eta - C^* \eta) \right]. \quad (16)$$

In this form it is obvious that the invariant (and consequently an operator that can be obtained from it by canonical quantization) can be factorized in a similar way as the Hamiltonian operator $H_{op}$ of the HO or an operator related to it via

$$\hat{H}_{op} = \frac{H_{op}}{\hbar \omega_0} = \left( a^+ a + \frac{1}{2} \right), \quad (17)$$

where $a^+ a$ is the so-called number operator and the creation and annihilation operators are defined by

$$a^+ = -i \sqrt{\frac{m}{2\hbar \omega_0}} \left( p_{op} \frac{m}{\hbar} + i \omega_0 x \right) = \frac{1}{\sqrt{2\hbar \omega_0}} \left( -\frac{\hbar}{\sqrt{m}} \frac{\partial}{\partial x} + \sqrt{m \omega_0} x \right), \quad (18)$$

$$a = i \sqrt{\frac{m}{2\hbar \omega_0}} \left( p_{op} \frac{m}{\hbar} - i \omega_0 x \right) = \frac{1}{\sqrt{2\hbar \omega_0}} \left( \frac{\hbar}{\sqrt{m}} \frac{\partial}{\partial x} + \sqrt{m \omega_0} x \right), \quad (19)$$

where $p_{op} = \frac{\hbar}{i} \frac{\partial}{\partial x}$ and $a$ is the adjoint operator of $a^+$.

The number that is the eigenvalue of $a^+ a$ is the number of quanta of the action $\hbar$ since $\frac{H_{op}}{\hbar \omega_0}$ has the dimension of an action! With the help of $a$, the ground state wave function can be obtained and from this, by successive application of $a^+$, the excited states can be created. Via superposition of all these states, Schrödinger obtained a stable Gaussian WP (with constant width) [13]. Generalizations of Schrödinger’s approach were achieved for the description of coherent light beams emitted by lasers in terms of what is now called coherent state (CS).

One of at least three different definitions of CSs is that these are eigenstates of the annihilation operator $a$ with (complex) eigenvalue $z$, $a^+ |z> = z |z>$. Comparing the CS $|z>$ for the HO with the minimum uncertainty WP solution in the form of Eq. (2), it shows that $\omega_0 = \frac{2\hbar}{m} y_I = \frac{1}{\alpha_0}$. So, in definitions (18) and (19), $i \omega_0$ can be replaced by $i \sqrt{2\hbar \omega_0}$. Therefore, for the more general case of WPs or CSs with TD width ($C_R \neq 0$), $i \omega_0$ must be replaced by the full complex quantity $C$ in $a$ and by $C^*$ in the adjoint operator $a^+$. If one then substitutes $\frac{1}{\sqrt{\omega_0}} = \alpha_0$ in front of the brackets with $\alpha(t)$, the generalized creation and annihilation operators take the form

$$a^+(t) = -i \sqrt{\frac{m}{2\hbar}} \alpha(t) \left( \frac{p_{op}}{m} - C^* x \right), \quad (20)$$

$$a(t) = i \sqrt{\frac{m}{2\hbar}} \alpha(t) \left( \frac{p_{op}}{m} - C x \right). \quad (21)$$

These operators can even be turned into constants of motion if an additional phase factor is taken into account. But in the case of CSs, as discussed here, this factor can be absorbed into the phase of the CS and will therefore be omitted in the following (for further details see [14]).

Employing the above definition of the CS, but now with our generalized annihilation operator, i.e., $a(t) |z> = z |z>$, the CS (in position representation, $<x|z> = \Psi_z(x,t)$) can be obtained as

$$\Psi_z(x,t) = \left( \frac{m}{\pi \hbar} \right)^{\frac{1}{4}} \lambda^{-\frac{1}{2}} \exp \left\{ i \left[ y \bar{x}^2 + \frac{<p \bar{x}^2>}{\hbar} + \frac{<p> <x>}{2\hbar} \right] \right\}, \quad (22)$$
which is in complete agreement with our WP definition (2) (here, also \( N(t) \) and \( K(t) \) are now specified where the complex quantity \( \lambda(t) \) is defined via \( C = \left( \frac{2\hbar}{m} y \right) = \frac{\lambda}{\lambda} \) and will be discussed in the next subsection).

From the mean values of position and momentum, \(<x> = \eta, <p> = m\dot{\eta}\), calculated with these CSs, the real and imaginary parts of the complex eigenvalue \( z = z_R + iz_I \) can be determined to be

\[
z_R = \sqrt{\frac{m}{2\hbar}} \left( \frac{\eta}{\alpha} \right), \quad z_I = \sqrt{\frac{m}{2\hbar}} (\dot{\eta}\alpha - \eta\dot{\alpha})
\]

which looks familiar when compared with the Ermakov invariant (15). Indeed, the absolute square of \( z \) is, up to a constant factor, identical to \( I_L \).

In the same way that the annihilation operator of the HO can be used to obtain the corresponding ground state wave function and the creation operator to obtain the excited states, the generalized annihilation and creation operators (20) and (21) can be used to obtain solutions of the TDSE where the potential of the problem enters via the complex TD variable \( C \) (or \( y \)) of the corresponding Riccati equation.

So, e.g., for the free motion \((V = 0)\) application of \( a(t) \) on the ground state wave function \( |0> \) leads (in position representation) via \( a(t)|0> \) to

\[
<x|0> = \Psi_0(x,t) = \left( \frac{m}{\pi \hbar} \right)^{\frac{1}{4}} \lambda^{-\frac{1}{2}} \exp \left\{ iyx^2 \right\}
\]

where \( y(t) \) is obtained from the Riccati equation (4) for \( \omega = 0 \). Note that this ground state (in contrast with the TI HO) has a position dependent phase via \( iy_R x^2 \).

The corresponding excited states can be obtained by applying \( a(t) \) on this ground state according to \( \Psi_n(x,t) = (a^+)^n |0> \) in position-space representation as

\[
\Psi_n(x,t) = \left[ \left( \frac{m}{\pi \hbar} \right)^{\frac{1}{4}} \frac{1}{\alpha n! 2^n} \right]^{1/2} H \left( \frac{x}{x_0} \right) \exp \left\{ -\frac{1}{2} \left( \frac{x}{x_0} \right)^2 + i\frac{\dot{\alpha}}{\alpha} x^2 \right\}
\]

with \( x_0 = \sqrt{\frac{\hbar}{m}} \alpha(t) \) being TD due to \( \alpha(t) \). Hence, the dimensionless variable \( \xi = \frac{x}{x_0} \) that usually appears in this context in the Hermite polynomials \( H(\xi) \) is now TD! Note: the \( \Psi_n(x,t) \) are solutions of the TDSE for \( V = 0 \) but they are no eigenfunctions of the operator \( H_{op} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \) (but of \( I_{L,op} \)). Similar solutions for the TD free motion SE have also been found by Guerrero et al [15] in a different way.

### 2.3. Linearization of the complex Riccati equation

Another property of the Riccati equation, particularly interesting in a quantum mechanical context, is the existence of a superposition principle for this NL differential equation [16, 17]. This is related to the fact that the Riccati equation can always be linearized. In our case, this can be achieved using the ansatz

\[
\left( \frac{2\hbar}{m} y \right) = C = \frac{\dot{\lambda}}{\lambda},
\]

with complex \( \lambda(t) \), leading to

\[
\dot{\lambda} + \omega^2(t) \lambda = 0
\]
which has the form of the Newton-type equation (3) of the corresponding problem, but now for a complex variable.

First, a kind of geometric interpretation of the motion of $\lambda$ in the complex plane shall be given. Expressed in Cartesian coordinates, $\lambda$ can be written as $\lambda = u + iz$, or in polar coordinates as $\lambda = \alpha e^{i\varphi}$. Inserting the polar form into Eq. (27) leads to

$$\left( \frac{2\hbar}{m} y \right) = \frac{\dot{\alpha}}{\alpha} + i \varphi$$

(29)

where the real part is already identical to $2\hbar m y_R$, as defined above.

The quantity $\alpha$ defined in $\frac{2\hbar}{m} y_I$ as being proportional to the position uncertainty is identical to the absolute value of $\lambda$ if it can be shown that

$$\dot{\varphi} = \frac{1}{\alpha^2}.$$  

(30)

This, however, can be proven by simply inserting real and imaginary parts of (29) into the imaginary part of the Riccati equation (4). Comparing relation (30), that can also be written in the form

$$\dot{z} u - \dot{u} z = \alpha^2 \dot{\varphi} = 1,$$

(31)

with the motion of a particle under the influence of a central force in two-dimensional physical space, it shows that this relation corresponds to the “conservation of angular momentum”, but here for the motion in the complex plane!

Relation (31) also shows that real and imaginary parts, or phase and amplitude, respectively, of the complex quantity are not independent of each other but uniquely coupled. This coupling, which, as mentioned in the Introduction, is typical for quantum systems (but not only for these) is due to the quadratic nonlinearity in the Riccati equation. We will find an analogous situation also in the TI case, discussed in the next section.

The complex variable $\lambda = u + iz$ also allows for a kind of geometric interpretation of the Ermakov invariant. For this purpose it will be used that the imaginary part of $\lambda$ is directly proportional to the classical trajectory, i.e., $z = \frac{m}{\alpha p_0} \eta(t)$ (this can be proven using the corresponding TD Green function, for details see, e.g., [18]). Therefore, the second quadratic term of the invariant (15) is proportional to $(\frac{z}{\alpha})^2 = \sin^2 \varphi$. Consequently, the first term must be $(\frac{u}{\alpha})^2 = \cos^2 \varphi$ to yield a constant value for $I_L$. So the invariant can be written as

$$I_L = \frac{1}{2} \left( \frac{\alpha_0 p_0}{m} \right)^2 \left[ \left( \frac{u}{\alpha} \right)^2 + \left( \frac{z}{\alpha} \right)^2 \right] = \text{const.}.$$  

(32)

Defining a new variable $\mathcal{Y}(\varphi) = (\frac{z}{\alpha}) = \sin \varphi$ that depends on the angle $\varphi$ instead of time $t$, the first term in the the square bracket can be expressed as

$$\mathcal{Y}'(\varphi) = \frac{d}{d\varphi} \mathcal{Y} = \cos \varphi = \left( \frac{u}{\alpha} \right).$$

(33)

In this form, the invariant (32) is formally equivalent to the Hamiltonian of a HO with angle-dependent variable $\mathcal{Y}(\varphi)$ (instead of TD variable $\eta(t)$) and unit frequency $\omega = 1$, leading to the corresponding equation of motion

$$\mathcal{Y}'' + 1^2 \mathcal{Y} = 0.$$

(34)

However, the time-dependence is implicitly contained in the time-dependence of the angle $\varphi$, i.e., $\varphi = \varphi(t)$. Therefore, expressing Eq. (34) as a differential equation with respect to time instead of with respect to angle $\varphi$ and using $ \frac{d}{dt} = \alpha^2 \frac{d}{d\varphi}$ with $\mathcal{Y}(\varphi(t)) = \mathcal{Y}(t)$ yields

$$\frac{d^2}{dt^2} \dot{\mathcal{Y}}(t) + 2 \frac{\dot{\alpha}}{\alpha} \frac{d}{dt} \dot{\mathcal{Y}}(t) + \dot{\varphi}^2 \mathcal{Y}(t) = 0.$$  

(35)
For $\dot{\alpha} = 0$, i.e. $\alpha = \text{constant}$, with $z(t) \propto \eta(t)$ and $\dot{\varphi} = \text{const.} = \omega_0$, Eq. (35) just turns into
\[ \ddot{\eta} + \omega_0^2 \eta = 0, \]
i.e., Eq. (3) for TI frequency $\omega = \omega_0$ and a WP solution with constant width.

For $\dot{\alpha} \neq 0$, i.e. $\alpha = \alpha(t)$, an additional first derivative term appears in (35) that looks like a linear velocity dependent friction force in the Langevin equation (see below) with friction coefficient $2 \dot{\alpha} \alpha$. At first sight this looks contradictory since we are not dealing so far with dissipative systems with irreversible time-evolution. A closer look shows that Eq. (35) is actually still invariant under time reversal since the coefficient of the second term also contains a time-derivative (in $\dot{\alpha}$). So, together with $\frac{d}{dt} \dot{Y}$, this term also does not change its sign under time-reversal (unlike in the Langevin equation, where the friction coefficient $\gamma$ is usually assumed to be constant). Eq. (35) takes into account that not only the angle of the system described by $\lambda(t)$, but also its radius may change in time.

3. Complex Riccati equations related to the time-independent Schrödinger equation

We have seen in the TD case that the real and imaginary parts, or phase $\varphi$ and amplitude $\alpha$, of the complex variable $\lambda(t) = \alpha e^{i\varphi}$ which fulfils the linear equation (28), obtained via Eq. (27) from the Riccati equation (4), are not independent of each other but coupled via the conservation law (30). A similar situation exists when considering the TISE, but now in the space-dependent case.

This can be shown using Madelung’s hydrodynamic formulation of quantum mechanics [19] where the wave function is written in polar form as
\[ \Psi(r, t) = \varrho^{1/2}(r, t) \exp \left( \frac{i}{\hbar} S(r, t) \right) \]
with the square root of the probability density $\varrho = \Psi^* \Psi$ as amplitude and $\frac{1}{\hbar} S$ as phase ($r$ is the position vector in three dimensions).

Inserting this form into the TDSE (1) (now in three dimensions, and replacing $\frac{\partial}{\partial x}$ by the nabla operator $\nabla$), leads to a modified Hamilton–Jacobi equation for the phase,
\[ \frac{\partial}{\partial t} S + \frac{1}{2m} (\nabla S)^2 + V - \frac{\hbar^2}{2m} \Delta \varrho^{1/2} = 0 , \]
and a continuity equation for the amplitude,
\[ \frac{\partial}{\partial t} \varrho + \frac{1}{m} \nabla (\varrho \nabla S) = 0 . \]

Already here, the coupling of phase and amplitude can be seen clearly since the Hamilton–Jacobi equation for the phase $S$ contains a term (misleadingly called “quantum potential”, $V_{\text{qu}}$) depending on $\varrho$, and the continuity equation for the density $\varrho$ contains $\nabla S$. It can be shown that also in the TI case this coupling is not arbitrary but related to a conservation law.

In 1994, G. Reinisch [20] did this in a NL formulation of TI quantum mechanics. Since in this case $\frac{\partial}{\partial t} \varrho = 0$ and $\frac{\partial}{\partial t} S = -E$ are valid, the continuity equation (38) (we now use the notation $\varrho^{1/2} = |\Psi| = a$) turns into
\[ \nabla (a^2 \nabla S) = 0 \]
and the modified Hamilton–Jacobi equation into
\[ -\frac{\hbar^2}{2m} \Delta a + (V - E)a = -\frac{1}{2m} (\nabla S)^2 a . \]
Equation (39) is definitely fulfilled for $\nabla S = 0$, turning (40) into the usual TISE for the real wave function $a = |\Psi|$ with position-independent phase $S$. (N.B.: the kinetic energy term divided by $a$ is just identical to $V_q$!)

However, Eq. (39) can also be fulfilled for $\nabla S \neq 0$ if only the conservation law

$$\nabla S = \frac{C}{a^2}$$

is fulfilled with constant (or, at least, position-independent) $C$.

This relation now shows explicitly the coupling between phase and amplitude of the wave function and is equivalent to Eq. (30) in the TD case. Inserting (41) into the rhs of Eq. (40) changes this into the Ermakov equation

$$\Delta a + \frac{2m}{\hbar^2}(E - V)a = \left(\frac{C}{\hbar}\right)^2 \frac{1}{a^3},$$

equivalent to Eq. (14) in the TD case.

Returning to the method described in [20], so far the energy $E$ occurring in Eq. (42) is still a free parameter that can take any value. However, solving this equation numerically for arbitrary values of $E$ generally leads to solutions $a$ that diverge for increasing $x$. Only if the energy $E$ is appropriately tuned to any eigenvalue $E_n$ of the TISE (see Eq. (44), below) does this divergence disappear and normalizable solutions can be found. So, the quantization condition that is usually obtained from the requirement of truncation of an infinite series in order to avoid divergence of the wave function is, in this case, obtained from the requirement of non-diverging solutions of the NL Ermakov equation (42) by variation of the parameter $E$. This has been numerically verified in the case of the one-dimensional HO and the Coulomb problem and there is the conjecture that this property is “universal” in the sense that it does not depend on the potential $V$ (see [20, 21]).

The corresponding complex Riccati equation is now given by

$$\nabla \left( \frac{\nabla \Psi}{\Psi} \right) + \left( \frac{\nabla \Psi}{\Psi} \right)^2 + \frac{2m}{\hbar^2}(E - V) = 0$$

with the complex variable $C = \left( \frac{\nabla \Psi}{\Psi} \right) = \frac{\nabla a}{a} + i \frac{1}{\hbar} \nabla S$ which corresponds to \( \frac{2\hbar m y}{m} \dot{\lambda} \dot{\lambda} = \dot{\alpha} \dot{\alpha} + i \dot{\phi} \) in the TD problem.

It is possible to show straightforwardly that Eq. (43) can be linearized to yield the usual TISE

$$-\frac{\hbar^2}{2m} \Delta \Psi + V \Psi = E \Psi,$$

but in this form the information on the coupling of phase and amplitude, expressed by Eq. (41) and originating from the quadratic NL term in Eq. (43), gets lost.

However, particularly this relation (41) now allows for the construction of complex solutions of the form

$$\Psi(r) = a(r) \exp \left[ -\frac{i}{\hbar} \int^r \frac{C}{a^2(r')} dr' \right]$$

once the solution of the Ermakov equation (42) is known. Using the techniques known from SUSY quantum mechanics it is possible to start from a real potential with real eigenvalue $E$ to construct a complex partner potential, but still with real eigenvalues, opening a new way to obtain non-hermitian Hamiltonians with real spectrum.
The procedure shall be outlined briefly for a one-dimensional example. In this case, the TISE
\[
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1 \right) \Psi = H_1 \Psi = E \Psi \tag{46}
\]
can be expressed in terms of the two linear operators
\[
A = -\frac{\hbar}{\sqrt{m}} \frac{d}{dx} + W(x) \quad \text{and} \quad B = \frac{\hbar}{\sqrt{m}} \frac{d}{dx} + W(x) \tag{47}
\]
as
\[
AB = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + W^2 - \frac{\hbar}{\sqrt{m}} \frac{d}{dx} W = 2(H_1 - E) \tag{48}
\]
where the potential \( V_1 \) is related to \( W(x) \) via
\[
V_1 = \frac{1}{2} \left( W^2 - \frac{\hbar}{\sqrt{m}} \frac{d}{dx} \right). \tag{49}
\]

The partner Hamiltonian \( H_2 \) can be obtained by reversing the order of the operators \( A \) and \( B \), i.e.,
\[
BA = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + W^2 + \frac{\hbar}{\sqrt{m}} \frac{d}{dx} W = 2(H_2 - E) \tag{50}
\]
where \( H_2 \) differs from \( H_1 \) only by an additional term in the potential, i.e.,
\[
V_2 = \frac{1}{2} \left( W^2 + \frac{\hbar}{\sqrt{m}} \frac{d}{dx} W \right) = V_1 + \frac{\hbar}{\sqrt{m}} \frac{d}{dx} W. \tag{51}
\]

In the same way as usual in SUSY quantum mechanics, the function \( W(x) \) is related to the wave function \( \Psi(x) \) of Eq. (46) via its logarithmic derivative (where \( \Psi \) here is not necessarily the ground state wave function and \( E \) not necessarily the ground state energy \( E = 0 \)),
\[
W = -\frac{\hbar}{\sqrt{m}} \frac{d}{dx} \ln \Psi. \tag{52}
\]
Particularly in our case, we chose \( \Psi(x) \) as a complex function as given in (45). Therefore, the additional term in \( V_2 \) will also be complex, turning it into a complex potential though \( V_1 \) and \( E \) are real!

As a first example the case \( V = 0 \), i.e., the free motion shall be mentioned. In this case it can be shown that the Ermakov equation (42) (for proper choice of the occurring parameters and \( \hbar = m = 1 \)) has the solution
\[
a(x) = \sqrt{\cosh(\kappa x)} \tag{53}
\]
with \( E = -\frac{\kappa}{2} \), \( \kappa > 0 \), which can be used to construct the complex wave function \( \Psi(x) \) via (45) and the function \( W(x) \) via (52) as
\[
W(x) = -\frac{\kappa}{2} \tanh(\kappa x) + i \frac{1}{2} \frac{1}{\cosh(\kappa x)}. \tag{54}
\]

Since \( V_1 = 0 \), \( V_2 \) is just the derivative of \( W(x) \) and given by
\[
V_2 = -\frac{\kappa}{2} \left( \frac{1}{2 \cosh^2(\kappa x)} - i \frac{\sinh(\kappa x)}{2 \cosh^2(\kappa x)} \right) \left[ 1 + i \frac{\sinh(\kappa x)}{\kappa} \right] V_{PT}(x), \tag{55}
\]
where \( V_{PT} \) is the well known Pöschl–Teller potential.

Whereas the real part of \( V_2 \), i.e. \( V_{PT}(x) \), is symmetric under the exchange \( x \rightarrow -x \), the imaginary part is antisymmetric under this operation. Further details will be discussed in a forthcoming publication [2].
4. Riccati equations in dissipative Schrödinger equations

Since real physical systems are always in contact with some kind of environment and this coupling usually introduces the phenomena irreversibility and dissipation (but not necessarily always both simultaneously!), the questions arise of how this can be taken into account in the formalism of classical (Hamiltonian) mechanics and, especially, in a quantum mechanical context.

In the trajectory picture, dissipation can be included by adding a phenomenological friction force that is proportional to velocity (or momentum) to the Newtonian equation of motion, thus turning it into an irreversible evolution equation, i.e.,

\[ m \ddot{x} = m \dot{v} = -m \gamma v - \frac{\partial}{\partial x} V \]

which is the Langevin equation without stochastic contribution (\( \gamma \) is the friction coefficient). This equation is used to describe Brownian motion from a trajectory point of view. An equivalent description of this phenomenon can also be given in terms of (classical) probability distributions \( \rho_{cl} \) via the Fokker–Planck equations that contain irreversible diffusion terms. Particularly in position space, this can be written in the form of the Smoluchowski equation for \( \rho_{cl}(x,t) \),

\[ \frac{\partial}{\partial t} \rho_{cl} + \frac{\partial}{\partial x} \left( \frac{F(x)}{m \gamma} \rho_{cl} \right) = 0 \]

with the conservative force \( F(x) \), Boltzmann’s constant \( k \) and temperature \( T \). Comparison with Einstein’s theory of Brownian motion [22] shows that the coefficient of the diffusion term fulfills the Einstein relation \( D = \frac{kT}{m \gamma} \).

However, Eqs. (56) and (57) do not fit consistently into the Lagrange/Hamilton formalism that is invariant under canonical transformations and provides a basis for quantization. So, these phenomenological equations do not yet provide the answer to the questions posed at the beginning of this section.

There are numerous approaches in the literature trying to answer these questions in different ways, most of them with some problematic aspects. In the following, only one approach will be discussed, that uses an effective modified TDSE that, for potentials at most quadratic in position variable, also leads to Gaussian WP solutions like the ones discussed in section 2 and thus allows for direct comparison with the corresponding conservative case. However, this approach can be uniquely linked with other similar approaches and approaches using modified Lagrangian/Hamiltonian functions (with subsequent canonical quantization) or even system-plus-reservoir approaches like the one by Caldeira and Leggett. Since a detailed discussion of these interrelations has already been given in the proceedings of the last Symmetries in Science Symposium [23], this shall not be repeated here.

For the following it will be sufficient to consider a modified TDSE that is able to take into account the effect of an environment without introducing any individual degrees of freedom of it, but changing the observable dynamics in a way, that particularly the averaged equation of motion for the position, according to Ehrenfest, contains a linear velocity dependent friction force as it occurs in the Langevin equation (56). Approaches adding a so-called “friction potential” (that usually introduces some kind of nonlinearity) to the Hamiltonian operator in a way that it provides the desired averaged equation of motion, contain a lot of ambiguity and therefore lead to sometimes rather inconsistent results.

We therefore tried (see [24]) an approach that is not based on the dissipative aspect of the trajectory picture but on the irreversible aspect of the probability picture. For this purpose, we add a time-reversal symmetry-breaking diffusion term to the continuity equation for \( \rho \), thus turning it, as previously discussed, into an irreversible Smoluchowski equation

\[ \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho \nu_-) - D \frac{\partial^2}{\partial x^2} \rho = 0 \]

(58)
but now for the quantum mechanical $\rho(x,t)$. For the continuity equation, Madelung [25] and Mrowka [26] had shown that with a bilinear ansatz for the probability current density,

$$j = \rho \left( \frac{1}{m} \frac{\partial}{\partial x} S \right) = \rho v = \left( \frac{\hbar}{2m} \right) \left( \Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \right),$$

this can be separated into the TDSE and its complex conjugate where the separation "constant" is proportional to the potential $V$. However, due to the diffusion term in (58), $\Psi$ and $\Psi^*$-dependent terms can no longer be separated in general.

For separation the condition

$$-D \frac{\partial^2 \rho}{\partial x^2} = F_1(\Psi) + F_2(\Psi^*)$$

must be fulfilled.

This can be achieved by introducing the additional separation condition

$$-D \frac{\partial^2 \rho}{\partial x^2} = \gamma (\ln \rho - < \ln \rho >) ,$$

where $\ln \rho = \ln \Psi + \ln \Psi^*$. This leads to an additional complex logarithmic term [24] in the SE for $\Psi_{NL}(x,t)$,

$$i\hbar \frac{\partial}{\partial t} \Psi_{NL}(x,t) = \{ H_L + \gamma \frac{\hbar}{i} (\ln \Psi_{NL} - < \ln \Psi_{NL} >) \} \Psi_{NL}(x,t) = \{ H_L + W_{SCH} \} \Psi_{NL}(x,t) ,$$

where $H_L$ is the usual linear Hamiltonian. The additional NL term $W_{SCH}$ can be written as real and imaginary contributions in the form

$$W_{SCH} = W_R + iW_I = \frac{\gamma \hbar}{2i} \left( \ln \frac{\Psi_{NL}}{\Psi_{NL}^*} - < \ln \frac{\Psi_{NL}}{\Psi_{NL}^*} > \right) + \frac{\gamma \hbar}{2i} \left( \ln \frac{\rho_{NL}}{\rho_{NL}^*} - < \ln \frac{\rho_{NL}}{\rho_{NL}^*} > \right)$$

where the real part only depends on the phase of the wave function and provides the friction force in the averaged equation of motion. The imaginary part does not contribute to dissipation but introduces irreversibility into the evolution of the wave function. It corresponds to the diffusion term in the Smoluchowski equation but still allows for normalizability due to the subtraction of the mean value of $\ln \rho$. Furthermore, it has properties that are usually attributed to a stochastic term occurring in the Langevin equation (details are not relevant to the following discussion and will be discussed elsewhere).

The imaginary part breaks the time-reversal symmetry on the level of the probability density, introduces a non-unitary time evolution and turns the Hamiltonian into a non-hermitian one while still guaranteeing normalizable wave functions and real energy mean values since its mean value vanishes.

By itself, the real part would provide dissipation but retain a unitary time-evolution of the wave function, whereas the imaginary part on its own would provide irreversibility via a non-unitary time-evolution but no dissipation. Consequently only the combination of real and imaginary parts provides all the desired properties of the quantum system under consideration. The reason for this is the coupling of phase and amplitude of the wave function as expressed by Eq. (41) since $W_R$ depends on the phase and $W_I$ on the amplitude.

4.1. Modification of the Gaussian WP dynamics

Inserting WP (2) into the NLSE (61) yields, for $\eta(t) = < x >_{NL}$ and hence for the motion of the WP maximum, the Newtonian equation including the friction force

$$\ddot{\eta} + \gamma \dot{\eta} + \omega^2 \eta = 0$$

(63)
and for the complex variable \( \frac{\partial}{m} y_{NL} = C_{NL} \) the modified Riccati equation

\[
\dot{C}_{NL} + \gamma C_{NL} + C_{NL}^2 + \omega^2(t) = 0 \tag{64}
\]

with an additional linear term depending on the friction coefficient \( \gamma \).

The relation between the imaginary part of this variable and the position uncertainty or Ermakov variable, respectively, remains unchanged as in the conservative case

\[
C_{NL,I} = \frac{\hbar}{2m <\hat{x}^2>_{NL}} = \frac{1}{\alpha_{NL}^2} \tag{65}
\]

but the real part differs by a contribution from the friction coefficient, i.e.,

\[
C_{NL,R} = \frac{\dot{\alpha}_{NL}}{\alpha_{NL}} - \frac{\gamma}{2} \tag{66}
\]

Inserting (65) and (66) into the Riccati equation (64) turns this into the Ermakov equation

\[
\ddot{\alpha}_{NL} + \left( \omega^2 - \frac{\gamma^2}{4} \right) \alpha_{NL} = \frac{1}{\alpha_{NL}^2} \tag{67}
\]

i.e., in comparison with the conservative case, only \( \omega^2 \) has been replaced by \( \Omega^2 = \left( \omega^2 - \frac{\gamma^2}{4} \right) \).

Again, from Eqs. (63) and (67) a dynamical invariant can be obtained that now has the form [27] [28] [29]

\[
I_{NL} = \frac{1}{2} e^{\gamma t} \left[ \left( \dot{\bar{\eta}} \alpha_{NL} - \left( \dot{\alpha}_{NL} - \frac{\gamma}{2} \alpha_{NL} \right) \bar{\eta} \right)^2 + \left( \frac{\bar{\eta}}{\alpha_{NL}} \right)^2 \right] \tag{68}
\]

Riccati equation (64) can also be linearized, now using the ansatz \( C_{NL} = \frac{\dot{\lambda}}{\lambda} = \frac{\dot{x}}{x} - \frac{\dot{y}}{y} \) to yield the Newtonian equation with linear friction term

\[
\ddot{\lambda} + \gamma \dot{\lambda} + \omega^2 \lambda = 0 \tag{69}
\]

for the complex variable \( \lambda(t) = \lambda e^{-\gamma t/2} = \alpha_{NL} e^{-\gamma t/2 + i\varphi} \). Inserting the polar form into the imaginary part of Eq. (64) leads to the unchanged conservation law

\[
\dot{\varphi} = \frac{1}{\alpha_{NL}} \tag{70}
\]

as in the conservative case.

Comparing Eqs. (63), (64) and (69) with the corresponding Eqs. (3), (4) and (28) in the conservative case, in the linear second-order differential equations for \( \eta \) and \( \lambda \), a linear term with first derivative has been added while, in the first-order Riccati equation, an additional term linear in \( C_{NL} = \left( \frac{\partial}{m} y \right) \) appears. All the additional terms depend on the coefficient \( \gamma \) of the friction force and vanish in the limit \( \gamma \to 0 \).

Similar modifications can also be obtained for the TISE (for details see [23]).

5. Riccati equations in other fields of physics

5.1. Riccati equations in statistical thermodynamics

Let us return to solution (7) of the Riccati equation (4), but now for real \( C \), and rewrite it in the form

\[
C(t) = \tilde{C} + \frac{2\tilde{C} e^{-2\tilde{C} t}}{\kappa_0 2\tilde{C} + \left( 1 - e^{-2\tilde{C} t} \right)} = \tilde{C} + \frac{2\tilde{C}}{\kappa_0 2\tilde{C} e^{2\tilde{C} t} + \left( e^{2\tilde{C} t} - 1 \right)} \tag{71}
\]
For the choice $\kappa_0 = 0$, this turns into
\[
C(t) = \tilde{C} + \frac{2\tilde{C}}{e^{2\tilde{C}t} - 1} = \tilde{C} \coth \tilde{C}t .
\]

(72)

Replacing time $t$ by $t \to \frac{1}{kT} = \beta$ (with $k =$ Boltzmann’s constant and $T =$ temperature) and the constant particular solution by $\tilde{C} = \bar{h} \frac{\omega}{2}$, one obtains
\[
\frac{\hbar}{2}\omega + \frac{\hbar}{e^{\hbar \omega / kT} - 1} = \frac{\hbar}{2} \omega \coth \left( \frac{\hbar \omega}{2kT} \right) = <E>_{th}
\]

(73)

which is the expression known from statistical thermodynamics for the average energy of a single oscillator in thermal equilibrium [30]. The first term on the lhs is just the ground state energy of the HO, the second is equal to Planck’s distribution function for the black body radiation. This type of relation between Eq. (73) and the Riccati equation has also been found by Rosu et al [31]. Equation (73) can also be expressed in terms of the partition function $Z = \sum_n e^{-\hbar \omega \beta} = \frac{1}{1 - e^{-\hbar \omega / kT}}$ as
\[
<E>_{th} = \frac{\hbar}{2}\omega + \frac{\partial}{\partial \beta} \left( \frac{1}{Z} \right) = \frac{\hbar}{2} \omega - \frac{\partial}{\partial \beta} \ln Z .
\]

(74)

The Riccati equation corresponding to solution (72) in the form (73) can be written as
\[
C' + C^2 - \tilde{C}^2 = 0
\]

(75)

with $C = C(\beta)$ depending on the variable $\beta = \frac{1}{kT}$, prime denoting derivative with respect to this variable and $\tilde{C}$, as mentioned above, being the ground state energy, $\tilde{C} = \frac{\hbar}{2} \omega$. The inverse quantity of $C$, multiplied by $-\tilde{C}^2$, i.e., $K = -\tilde{C}^2 C^{-1}$ fulfills
\[
-K' + K^2 - \tilde{C}^2 = 0 ,
\]

(76)

i.e., also a Riccati equation but now with negative sign for the derivative term and the solution
\[
K \left( \frac{1}{kT} \right) = \frac{\hbar}{2} \omega - \frac{\hbar \omega}{e^{\hbar \omega / kT} + 1} = \frac{\hbar}{2} \omega \tanh \left( \frac{\hbar \omega}{2kT} \right) .
\]

(77)

From the ground state energy, in this case, a term is subtracted that represents a Fermi–Dirac distribution whereas in the solution for $C$, a term was added to the ground state energy that represents a Bose–Einstein distribution. So both quantum statistics can be obtained from the solution of the Bernoulli equations derived from the Riccati equations for $C \left( \frac{1}{kT} \right)$ and (essentially) its inverse quantity.

5.2. The logistic equation as Riccati/Bernoulli equation

The last Riccati equation for $K \left( \frac{1}{kT} \right) = \frac{\hbar \omega}{2} \tanh \left( \frac{\hbar \omega}{2kT} \right)$ can be rewritten in terms of more neutral, general variables using $\frac{1}{kT} \to x$ and $\hbar \omega \to 1$, changing $K \left( \frac{1}{kT} \right) \to g(x)$ with $g(x) = \frac{1}{2} \tanh \left( \frac{x}{2} \right)$, fulfilling the Riccati equation
\[
g' + g^2 - \frac{1}{4} = 0
\]

(78)
with $' = \frac{d}{dx}$.

Just adding the constant $\frac{1}{2}$ to $g(x)$, one obtains the function $f(x)$ as

$$f(x) = \frac{1}{2} + g(x) = \frac{1}{2} \left(1 + \tanh \left(\frac{x}{2}\right)\right) = \frac{1}{1 + e^{-x}}$$

which describes the standard sigmoid function that fulfills the (homogeneous, Bernoulli-type) logistic equation

$$-f' + f^2 - f = 0$$

or

$$-f' = f(1 - f).$$

This equation has wide applications in NL dynamics from modelling population growth in ecology to modelling tumor growth in medicine, from autocatalytic reaction models in chemistry to fields like linguistics, economics and many others.

5.3. Complex Riccati equation in classical optics

The wave equation of classical optics can be written (see, e.g., [32]) as

$$\Delta \phi - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \phi = 0$$

with the scalar field $\phi(r, t)$, $c =$ velocity of light in vacuum, $n = \frac{c}{v} =$ diffraction index (with $v =$ velocity of light in medium). In general $n = n(r)$, i.e., the diffraction index depends on the medium and can therefore be a function of space. In this case the solutions of Eq. (82) are no longer plane waves but have a similar, but more general form. Using the ansatz

$$\phi(r, t) = \exp [A(r) + ik_0(L(r) - ct)]$$

with $L(r)$ being the optical wave length or phase, also called Eikonal, which is, as well as $A(r)$, the quantity that determines the amplitude of $\phi$, a function of the spatial variable $r$. Inserting this ansatz into the wave equation (82) leads to two coupled differential equations for $A(r)$ and $L(r)$,

$$\nabla(\nabla A) + (\nabla A)^2 - (k_0 \nabla L)^2 + (k_0 n)^2 = 0$$

$$\nabla(k_0 \nabla L) + 2(\nabla A)(k_0 \nabla L) = 0$$

where the inhomogeneity in Eq. (84) originates from the time-derivative via

$$-\frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \phi = k_0^2 n^2 \phi .$$

Since $\phi(r, t)$ is a complex function, Eqs. (84) and (85) are the real and imaginary parts of the, originally, complex differential equation (82). Similar to the situation in sections 2 and 3, the linear complex equation can be considered as a linearized form of a complex NL Riccati equation whose variable is the logarithmic derivative of the one fulfilling the linear equation.

In this sense, the complex variable in this case can be written as $C(r) = (\nabla A) + i k_0(\nabla L)$ (corresponding to $\frac{\nabla \phi}{n} + i \frac{1}{\hbar} \nabla S$ in the case of the TISE) and the complex Riccati equation (corresponding to the wave equation (82)) reads

$$\nabla C + C^2 + (k_0 \ n(r))^2 = 0 .$$

The position-dependent inhomogeneity $(k_0 n(r))^2$ corresponds to the term $\frac{2m}{\hbar^2}(E - V(r))$ in the TISE. It should be noted that the case, where a complex potential has been discussed in section 3, corresponds to the optical situation with complex diffraction index where the imaginary part describes absorption. A more detailed discussion will be given elsewhere [2].
5.4. Complex Riccati equation for Bose–Einstein condensates

In the mean field approximation, a Bose–Einstein condensate (BEC) can be described by a macroscopic WP for the condensate, $\Psi(r, t)$, which obeys the cubic NL Gross–Pitaevskii equation

$$i\hbar \frac{\partial}{\partial t} \Psi = \left\{ -\frac{\hbar^2}{2m} \Delta + V(r, t) + g|\Psi|^2 \right\} \Psi$$

(88)

where $V(r, t)$ represents the trapping potential and can be given (e.g., for a Paul trap) by $V(r, t) = \frac{m}{2}\omega^2(t)r^2$ with $r = |r|$ = absolute value of the vector $r$ and TD frequency $\omega = \omega(t)$; $g$ parametrizes the strength of the atomic interaction.

Although Eq. (88) cannot be solved analytically, the dynamics of the BEC characterized by this equation can be described in terms of so-called moments $M_n$ ($n = 1 - 4$) (for details, see e.g., [33]), where $M_1$ represents the norm, $M_2$ the width, $M_3$ the radial momentum and $M_4$ the energy of the WP. It can be shown that these moments satisfy a set of coupled first-order differential equations (where $\frac{d}{dt}M_1$ corresponds to the conservation of probability or particle number). This set can be reduced to a single equation for $M_2$ which, using a new variable $X = \sqrt{M_2}$, can be expressed in the form of an Ermakov equation,

$$\ddot{X} + \omega^2(t)X = \frac{k^2}{X^3}.$$ (89)

As we have seen already before (e.g. in section 2), the real Ermakov equation is equivalent to a complex NL Riccati equation for the variable $C = \frac{\dot{X}}{X} + i\frac{k}{X}$. Again, as shown in section 2, this can be used to define generalized creation/annihilation operators that allow one to obtain coherent states following the dynamics that corresponds to the abovementioned moments.

To include dissipative effects, one could add another NL term like the logarithmic one from Eq. (61) to the Gross–Pitaevskii equation which would correspond to adding a linear term to the Riccati equation. So, one simply has to solve this modified Riccati equation (or the corresponding Ermakov equation) to obtain all moments $M_n$ for the dissipative BEC [34].

5.5. Complex Riccati equation in cosmology

Another example shall be given where a physical system can be described by a (complex) Riccati equation, or its equivalent, a real Ermakov equation, not for a microscopic but for a really macroscopic system. For this purpose, we switch to cosmology and apply the cosmological principle, i.e., the assumption of a spatially-homogeneous and isotropic universe, to Einstein’s field equations. In this case, the Robertson–Walker metric applies [35] that contains the scale factor $a(t)$ (so to say, the radius of the universe) and the curvature $k$ that can attain the values 0 for a flat or $\pm 1$ for a closed or open universe. This finally leads to the Friedmann–Lemaître equations

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{2}{3} \varrho - \frac{k}{a^2}$$

(90)

and

$$\dot{\varrho} = -3H(\varrho + p)$$

(91)

with $H = \frac{\dot{a}}{a}$ = Hubble parameter; overdots denote derivatives with respect to cosmic proper time, $\varrho =$ energy density and $p =$ pressure (where $4\pi G = c = 1$ and the cosmological constant $\Lambda = 0$ have been set).

Assuming the matter source as a self-interacting scalar field $\Phi = \Phi(t)$, the energy density $\varrho$ and pressure $p$ can be written as

$$\varrho = \frac{1}{2} \left( \frac{d}{dt} \Phi \right)^2 + U(t),$$

(92)
\[ p = \frac{1}{2} \left( \frac{d}{dt} \Phi \right)^2 - U(t) . \]  
\[ (93) \]

Taking the time-derivative of (90) and inserting (91) using (92) and (93) leads to

\[ \frac{d}{dt} \left( \frac{\dot{a}}{a} \right) = - \left( \frac{d}{dt} \Phi \right)^2 + \frac{k}{a^2} . \]  
\[ (94) \]

Introducing a new time variable (conformal time \( \tau \)) via \( \frac{d}{dt} = \frac{a}{\dot{a}} \frac{d}{d\tau} \), Eq. (94) can be rewritten as an Ermakov equation,

\[ \frac{d^2}{d\tau^2} a + \left( \frac{d}{d\tau} \Phi \right)^2 a = \frac{k}{a^3} , \]  
\[ (95) \]

which is equivalent to a Riccati equation for the complex quantity \( C = \left( \frac{\dot{a}}{a} + i \frac{1}{\dot{a}} \right) \). The above derivation of the Ermakov equation in comparison with the afore-mentioned BEC had been given by Lidsey in [36]. Everything said about the complex Riccati or real Ermakov equation in section 2 can also be applied to this system, e.g., corresponding creation and annihilation operators and coherent states can be defined (see [14]).

5.6. Complex Riccati/Bernoulli equations, “quantized” triangles and Pythagorean triples

Looking for the origin of the idea of quantization one might first think of the Greek philosopher Democritus (ca. 460 - ca. 371 BC) and his idea of dividing our world into minute components that are not further divisible, the “atoms”.

A similar idea was formulated in a more abstract way by another equally famous Greek philosopher living around the same time. In his work *Timaios*, Plato (428/27 - 348/47 BC) gives his view of how the world is built up in terms of right-angled triangles, essentially his “quanta”. Werner Heisenberg, equally fascinated with, and puzzled by, this text, summarizes this idea [37] in the general sense as follows: “Matter is made up of right-angled triangles which, after being paired to form isosceles triangles or squares, are joined together to build the regular bodies of stereometry: cube, tetrahedron, octahedron and icosahedron. These four solids are then supposed to be the basic units of the four elements earth, fire, air and water”.

So, geometric objects like right-angled triangles should be the quanta of nature. But one could go even further and ask if particular right-angled triangles might play a special role in quantization. Yet another Greek philosopher enters the scene. We remember Pythagoras’ theorem from school, i.e., \( a^2 + b^2 = c^2 \) where \( a \) and \( b \) are the two shorter sides (catheti) and \( c \) is the longest side (hypotenuse) of a right-angled triangle. Pythagoras (around 570 - 500 BC) and his disciples were well-known for their dogma “everything is number”, with number meaning integer. So let us look for right-angled triangles where the lengths of all three sides are integers (a kind of “second quantization”) fulfilling the Pythagorean theorem. The most common example for such a Pythagorean triple is \((3, 4, 5)\) with \( 9 + 16 = 25 \). But, asked for a few more examples of the kind, even mathematically-affiliated persons have difficulties finding any – though an infinite number of triples exists! Moreover, there is even a rather simple rule of finding these triples. This rule was probably already known to the Babylonians more than 3500 years ago [38] but, at least, Diophantus of Alexandria (around 250 AD) knew of it.

What does this have to do with our complex Riccati equations? For this purpose the case of the homogeneous Bernoulli equation (5) will be considered. As shown above, with a particular solution of the Riccati equation its inhomogeneity can always be removed. The resulting additional linear term in the Bernoulli equation (at least for constant coefficient \( \tilde{C} \)) can also be removed. So we are dealing with a complex equation of the form

\[ \frac{d}{dt} C + C^2 = 0 . \]  
\[ (96) \]
Then \(-\frac{d}{dx}C\) is also a complex quantity, \(C^2\), where its real and imaginary parts as well as its absolute value again define a right-angled triangle (in the complex plane) and each side contains contributions from \(R\) and \(I\), i.e., \(R\{C^2\} = R^2 - I^2\), \(3\{C^2\} = 2RI\) and \(|C|^2 = R^2 + I^2\).

If we now assume that \(R\) and \(I\) are integers (with \(R > I\), all three sides of the right-angled triangle created by \(C^2\) in the complex plane are also integers. As examples, we choose: (a) \(R = 2, I = 1\): \(R^2 - I^2 = 3, 2RI = 4, R^2 + I^2 = 5\) with \(9 + 16 = 25\); (b) \(R = 3, I = 2\): \(R^2 - I^2 = 5, 2RI = 12, R^2 + I^2 = 13\) with \(25 + 144 = 169\).

All possible Pythagorean triples can be obtained in this way just by applying all integers \(R\) and \(I\) with \(R > I\). In a physical context this means that whenever a physical quantity obeys a complex Riccati equation and this quantity can be “quantized” in the sense that its real and imaginary parts can be expressed as multiples of some units, its evolution (in time, space or depending on other variables like inverse temperature etc.) can also be expressed in terms of the same units.

### 5.7. Scale invariance, trigonometric and hyperbolic functions and Riccati equations

In the last sub-section it was shown how particular right-angled triangles can be related to (complex) Riccati/Bernoulli equations. Yet another relation between right-angled triangles and Riccati equations can be established. For this purpose, we divide all sides of these triangles by the length of the hypotenuse so that this now has a unit length and the length of the two catheti are given (in the case of the complex numbers) by the cosine (real part) and sine (imaginary part) of the angle between the hypotenuse and the real axis. The same also applies to any right-angled triangle in a unit circle in the real plane, accordingly.

As mentioned before, in NL dynamics it is not the absolute changes that are relevant but relative ones, thus providing scale invariance of the corresponding laws. On the other hand, the variables of the Riccati equations are simply logarithmic derivatives of the quantities that fulfill the corresponding linearized second order differential equation, i.e., the enumerator of the Riccati variable is the derivative of the denominator. Looking at the catheti of our right-angled triangles, their ratio fulfills this requirement (not only for the triangles in the unit circle, but for any right-angled triangle with constant length of its sides). So the ratio of the two catheti is either \(\tan \varphi = \frac{\sin \varphi}{\cos \varphi}\) or \(\cot \varphi = \frac{\cos \varphi}{\sin \varphi}\), where \(\sin \varphi\) and \(\cos \varphi\) are (up to the sign) the derivatives of each other. Therefore, there should also be a connection to Riccati equations; and it obviously exists. For example, the derivative of \(\tan x\) with respect to \(x\) is \(\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}\). Using in addition the relation \(\cos^2 x + \sin^2 x = 1\), this can be rewritten as the Riccati equation

\[
-\frac{d}{dx} \tan x + \tan^2 x + 1 = 0 ,
\]

with unit inhomogeneity. In the same way, one obtains from \(\frac{d}{dx} \cot x = -\frac{1}{\sin^2 x}\)

\[
\frac{d}{dx} \cot x + \cot^2 x + 1 = 0 .
\]

Changing from the real argument \(x\) to an imaginary argument \(ix\) changes the trigonometric functions into hyperbolic ones. As we have seen in 5.1, these functions and corresponding Riccati equations actually occur in statistical thermodynamics where the transition from \(t\) to \(\frac{\hbar}{kT}\) actually represents a transition to an imaginary time. This concept of imaginary (or complex) time is sometimes used for computational reasons to transform the TDSE with oscillatory solutions into a diffusion-type equation and also for other purposes in, e.g., theoretical chemistry [39].

Looking at the hyperbolic functions, the situation is very similar to the trigonometric ones. Also in \(\tanh \varphi = \frac{\sinh \varphi}{\cosh \varphi}\) and \(\coth \varphi = \frac{\cosh \varphi}{\sinh \varphi}\), the enumerators and denominators are derivatives
of each other and, now with the relation \( \cosh^2 x - \sinh^2 x = 1 \), the following Riccati equations can be obtained:

\[
\frac{d}{dx} \tanh x + \tanh^2 x - 1 = 0 \tag{99}
\]
\[
\frac{d}{dx} \coth x + \coth^2 x - 1 = 0 \tag{100}
\]

The inhomogeneity now has a negative sign which would correspond in the TD case in section 2 to a change from an attractive to a repulsive quadratic potential.

6. Conclusions and perspectives

It has been shown in section 2 that the information about the dynamics of a quantum mechanical WP, i.e. the equations of motion for the time-evolution of its maximum and width, are not only obtainable from the solution of the TDSE, but equally-well from a complex NL Riccati equation. The classical particle aspect that is reflected by the Newtonian equation determining the motion of the wave packet maximum can be obtained, using that the complex Riccati equation can be linearized by the ansatz \( C = \dot{\lambda} \lambda \) to a Newtonian equation for the complex quantity \( \lambda(t) \), which can be written as \( \lambda = u + z = \alpha e^{i\phi} \). Knowing that \( z \propto \eta(t) \), the imaginary part of \( \lambda \) immediately provides (up to a constant factor) the classical trajectory \( \eta(t) \).

Reformulation of the complex Riccati equation in terms of a real variable, \( \alpha(t) \), that is proportional to the WP width (and is identical to the absolute value of \( \lambda(t) \)), the solution of the resulting NL so-called Ernakev equation for \( \alpha \) provides the information about the dynamics of this WP property. Both equations describing particle and wave aspects of the WP can be combined to yield a dynamical invariant with the dimension of “action”. Factorization of an operator corresponding to this invariant allows for the definition of generalized creation/annihilation operators leading not only to CSs with TD width, but also to unconventional solutions of the TDSE.

Formal similarities exist between this TD problem and SUSY quantum mechanics where (real) position-dependent Riccati equations play a similar role and the construction of isospectral potentials demonstrates the importance of parameters like initial conditions.

In section 3 it has been shown that the Riccati formalism established for the TDSE can also be applied to the TISE if the TD complex quantity \( C = \dot{\lambda} \lambda \) is replaced by the space-dependent complex quantity \( \nabla \Psi \Psi \) for a complex wave function \( \Psi \). In a certain way, this looks like a complex version of SUSY quantum mechanics where not only the logarithmic derivative of the ground state is considered as a variable for a Riccati equation, but also any (even complex) excited state may fulfill a now complex Riccati equation. Applying methods known from SUSY actually allows one to start from a real potential with complex wave function to obtain a system with complex potential, i.e., non-hermitian Hamiltonian, but still with real eigenvalue.

The flexibility of the Riccati formalism was illustrated in section 4 where it was shown that effects like dissipation and irreversibility can easily be included into it. In the TD case, only an additional term linear in \( C \) appears in Eq. (4). Since, in the solution via the homogeneous Bernoulli equation (5), a linear term already occurs only the coefficient of this term changes.

Finally in section 5, the formal similarity between this formulation of quantum mechanics and several other fields of physics that also can be written in the form of (real or complex) Riccati equations was demonstrated. Replacing time \( t \) with (actual imaginary) “time” \( \frac{\hbar}{\bar{t}} \) leads to well-known expressions from statistical thermodynamics as solutions of Riccati equations that are also able to distinguish between bosonic and fermionic properties.

By simply adding a constant to the solution of the Fermi–Dirac equation, the solution of an equation well-known in NL dynamics can be found, namely the logistic equation which is actually also a Riccati/Bernoulli equation. This equation, originally found in the context of population dynamics, has wide applications and not only in physics.
While the previous examples were still dealing with real Riccati equations, in the next example it was shown that also the wave equation of classical optics can be rewritten as a complex Riccati equation where the (position-dependent) diffraction index essentially corresponds to the (position-dependent) potential of the Riccati equation obtained from the TISE. The case of the complex potential would therefore correspond to a complex diffraction index with the imaginary part describing absorption.

Also the dynamics of a BEC can be described by a complex Riccati or the corresponding Ermakov equation. We then went from the microscopic scale to the really macroscopic one, to cosmology, and demonstrated that the Friedmann–Lemaître equations, which under certain assumptions describe the dynamics of our universe, can be written in terms of a real Ermakov or equivalent complex Riccati equation.

Finally, returning to the abstract mathematical roots we show how complex Riccati/Bernoulli equations are related to the Pythagorean triples and thus to the most fundamental “quantization” problem. Generalization of this problem from very particular right-angled triangles to arbitrary ones where the lengths of their sides can be expressed in terms of trigonometric functions also displayed the connection of these functions with Riccati equations. Including imaginary arguments for these functions, Riccati equations can also be formulated for hyperbolic functions. This then opens a new approach to the theory of solitons which will be discussed elsewhere.

There are many more examples like electrodynamics, quantum optics, etc., where the Riccati equation allows for a unifying formulation with the fields mentioned above as well as others. In conclusion therefore, one can say that the NL version of quantum mechanics based on a (complex) Riccati equation is able to cover all phenomena of standard quantum mechanics (including the superposition principle, due to its linearizability). In addition, the sensitivity of NL differential equations to initial conditions provides further information that gets lost in the linearized form. The complex form of the Riccati equation (via its imaginary part) also supplies a new conservation law that resembles the conservation of angular momentum, but now for the motion in the complex plane (something closely related to the quantum property spin), expressing the connections between phase and amplitude of complex quantities that are essential in quantum mechanics, but not only there.

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