RIGIDITY OF GROUPS OF CIRCLE DIFFEOMORPHISMS AND TEICHMÜLLER SPACES

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Abstract. We consider deformations of a group of circle diffeomorphisms with Hölder continuous derivative in the framework of quasiconformal Teichmüller theory and show certain rigidity under conjugation by symmetric homeomorphisms of the circle. As an application, we give a condition for such a diffeomorphism group to be conjugate to a Möbius group by a diffeomorphism of the same regularity. The strategy is to find a fixed point of the group which acts isometrically on the integrable Teichmüller space with the Weil-Petersson metric.

0. Introduction and the statement of theorems

In this paper, we prove certain rigidity of the deformation of a Fuchsian group within a group of circle diffeomorphisms. The regularity of the diffeomorphisms we consider here is such that their derivatives are Hölder continuous. This class is important not only in the quasiconformal theory of Teichmüller spaces but also for the fixed point property in geometric group theory and for certain problems on the smoothness of foliations of codimension one in closed 3-manifolds. For a constant $\alpha \in (0,1)$, we denote by $\text{Diff}_{+}^{1+\alpha}(S)$ the group of all orientation-preserving diffeomorphisms $g$ of the unit circle $S$ whose derivatives are $\alpha$-Hölder continuous. This means that there is a constant $c \geq 0$ such that

$$|g'(x) - g'(y)| \leq c|x - y|^\alpha$$

for any $x, y \in S = \mathbb{R}/\mathbb{Z}$. More precisely, $g$ is identified with its lift $\tilde{g} : \mathbb{R} \to \mathbb{R}$ and the above condition is applied to it.

We refer to rigidity in general when a weaker equivalence implies a stronger equivalence. A typical example occurs in a triangle group $\Gamma$ in the group $\text{Möb}(S) \cong \text{PSL}(2,\mathbb{R})$ of the Möbius transformations. If $f\Gamma f^{-1}$ is in $\text{Möb}(S)$ for an orientation-preserving homeomorphism $f : S \to S$, then $f$ actually belongs to $\text{Möb}(S)$. Hence the conjugation by such a homeomorphism implies the conjugation by a Möbius transformation. In this sense, $\Gamma$ is rigid.

We formulate rigidity phenomena in the framework of quasiconformal Teichmüller theory. Let $\text{QC}(\mathbb{D})$ denote the group of all quasiconformal self-homeomorphisms of the unit disk.

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disk $\mathbb{D}$, and $\text{Bel}(\mathbb{D})$ the space of Beltrami coefficients on $\mathbb{D}$. Every $w \in \text{QC}(\mathbb{D})$ extends continuously to a quasisymmetric homeomorphism of $S = \partial \mathbb{D}$. Let $\text{QS}$ be the group of all quasisymmetric self-homeomorphism of $S$. The boundary extension map $q : \text{QC}(\mathbb{D}) \to \text{QS}$ is a surjective homomorphism. The universal Teichmüller space is defined by $T = \text{Mob}(S) \setminus \text{QS}$. In Section 1, we give preliminaries for these facts on the universal Teichmüller space. We put emphasis on a fact that the Bers embedding $\beta : T \to B(\mathbb{D}^*)$ into the Banach space $B(\mathbb{D}^*)$ of hyperbolically bounded holomorphic functions on $\mathbb{D}^* = \mathbb{C} - \mathbb{D}$ realizes the action of a Möbius group on $T$ linear isometrically on $B(\mathbb{D}^*)$.

A quasiconformal homeomorphism $w \in \text{QC}(\mathbb{D})$ is called asymptotically conformal if its complex dilatation $\mu \in \text{Bel}(\mathbb{D})$ vanishes at the boundary. The subspace of $\text{Bel}(\mathbb{D})$ consisting of all Beltrami coefficients vanishing at the boundary is denoted by $\text{Bel}_0(\mathbb{D})$ and the subgroup of $\text{QC}(\mathbb{D})$ consisting of all asymptotically conformal self-homeomorphisms of $\mathbb{D}$ is denoted by $\text{AC}(\mathbb{D})$. A quasisymmetric homeomorphism $g \in \text{QS}$ is called symmetric if its quasisymmetric quotient $m_g(x, t) = g(x + t) - g(x)$ $g(x) - g(x - t)$ tends to 1 as $t \to 0$ uniformly with respect to $x \in S$. The group of all symmetric self-homeomorphisms of $S$ is denoted by $\text{Sym}$. It is known that $\text{Sym} = q(\text{AC}(\mathbb{D}))$.

In the theory of the asymptotic Teichmüller space, Gardiner and Sullivan [14] considered the little subspace $T_0 = \text{Mob}(S) \setminus \text{Sym}$ and its Bers embedding $\beta : T_0 \to B_0(\mathbb{D}^*)$ into the little subspace $B_0(\mathbb{D}^*)$ of $B(\mathbb{D}^*)$ consisting of the elements vanishing hyperbolically at the boundary. In Section 2 we summarize all these spaces involved in $T_0$. Necessary properties of asymptotically conformal homeomorphisms are also prepared. In particular, we prove a prototype of our rigidity theorem concerning the conjugation of a Möbius group by a symmetric homeomorphism (Theorem 2.2).

As Carleson [8] first clarified, the decay order of the complex dilatation $\mu \in \text{Bel}_0(\mathbb{D})$ of an asymptotically conformal homeomorphism $w \in \text{AC}(\mathbb{D})$ as $|z| \to 1$ reflects that of $m_g(x, t) - 1$ for its boundary extensions $g = q(w)$ as $t \to 0$. Moreover, the decay order of $m_g(x, t) - 1$ is related to the differentiability of elements in $\text{Sym}$, that is, the exponent $\alpha$ of the Hölder continuity of derivatives. In this way, we can bring the study of $\text{Diff}_1^{1+\alpha}(S) \subset \text{Sym}$ to theories of Teichmüller spaces, which has been done in our related work [26]. We will survey these properties in the first part of Section 3.

The main technical step for our rigidity theorem is a fact that we can choose the regularity of a conjugating circle diffeomorphism arbitrarily close to the desired one but cannot make it exactly the same when we change the base point by a symmetric homeomorphism in general. In the second part of Section 3 we consider these phenomenon mainly on the Bers embedding. Theorem 3.6 will be the basis of our later investigation.

Appreciating the importance of the class of symmetric homeomorphisms of $S$, we formulate our rigidity theorem in Section 4. This is to show that if $\Gamma \subset \text{Mob}(S)$ acting on $T = \text{Mob}(S) \setminus \text{QS}$ has a fixed point in the little subspace $T_0 = \text{Mob}(S) \setminus \text{Sym}$ besides the
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origin [id], then it is actually in the subspace \( T^0_\alpha = \text{Möb}(S) \setminus \text{Diff}^{1+\alpha}_+(S) \). For the proof, the Bers embedding of these spaces plays a significant role. In fact, it requires to consider the infinite sum of a certain orbit of \( \Gamma \) in \( B(D^*) \), which is possible on the linear space.

**Theorem 4.1.** Let \( \Gamma \) be a subgroup of \( \text{Möb}(S) \) that contains a hyperbolic element. If \( f \Gamma f^{-1} \subset \text{Diff}^{1+\alpha}_+(S) \) for \( f \in \text{Sym} \), then \( f \in \text{Diff}^{1+\alpha}_+(S) \).

As an application of our rigidity theorem, we consider the conjugation problem of a group \( G \) of circle diffeomorphisms in Section 5. This concerns a condition under which \( G \subset \text{Diff}^{1+\alpha}_+(S) \) is conjugate to \( \Gamma \subset \text{Möb}(S) \) by a conjugating element of the same regularity as the elements of \( G \). We can translate it to the fixed point problem of \( G \) acting on \( T \) as before.

We give conditions for the existence of such conjugation in terms of the integrability of complex dilatations of quasiconformal extension. For \( p \geq 2 \), the \( p \)-integrable norm of a Beltrami coefficient \( \mu \in \text{Bel}(D) \) with respect to the hyperbolic metric \( \rho_D(z)|dz| \) is defined by

\[
\|\mu\|_p = \left( \int_D |\mu(z)|^p \rho_D^2(z)dxdy \right)^{1/p}.
\]

The space of all \( p \)-integrable Beltrami coefficients on \( D \) is denoted by \( \text{Ael}^p(D) \).

The integrable Teichmüller space \( T^2 \) obtained from \( \text{Ael}^2(D) \) was introduced by Cui [9] and Takhtajan and Teo [33], in particular, the *Weil-Petersson metric* on this Teichmüller space was investigated. A recent result by Shen [32] also gives an important characterization of an element of \( T^2 \) as a mapping of \( S \) itself without mentioning quasiconformal extension. Completeness and negatively curved nature of the distance induced this metric are important for our problems.

Using the Weil-Petersson metric on \( T^2 \), we prove the following result on the conjugation problem for \( \text{Diff}^{1+\alpha}_+(S) \) with \( \alpha > 1/2 \). For the proof, an estimate of the Weil-Petersson distance \( d_{WP}^2 \) between the origin and a point given by a Beltrami coefficient \( \mu \) in terms of its integrable norm (Theorem 5.4) is essential. We will prove this in Section 7.

**Theorem 5.3.** Let \( G \) be an infinite non-abelian subgroup of \( \text{Diff}^{1+\alpha}_+(S) \) with \( \alpha \in (1/2, 1) \). Then the following conditions are equivalent:

1. There exists some \( f \in \text{Diff}^{1+\alpha}_+(S) \) such that \( f^{-1}Gf \subset \text{Möb}(S) \);
2. There exist positive constants \( \kappa_2 < \infty \) and \( \kappa_\infty < 1 \) such that
   \[
   (a) \quad \inf_{\pi(\mu)=g} \|\mu\|_2 \leq \kappa_2; \quad (b) \quad \inf_{\pi(\mu)=g} \|\mu\|_\infty \leq \kappa_\infty
   \]
   for all \( g \in G \);
3. The orbit \( G(o) \) of the base point \( o = [\text{id}] \) is bounded in \( T^2 \) with respect to \( d_{WP}^2 \).

A more restricted sufficient condition for the conjugation will be also obtained without the assumption \( \alpha > 1/2 \). To do this, we use the \( p \)-integrable Teichmüller space \( T^p \) obtained from \( \text{Ael}^p(D) \) (\( p \geq 2 \)) for \( \alpha > 1/p \) (Theorem 5.5). The proof will be postponed.
until Section 7. In a future research, we expect that this result can be generalized to the statement as in Theorem 5.3.

In the present paper, we also introduce the $p$-Weil-Petersson metric on $T^p$, which is a Finsler metric defined in a similar way to the Weil-Petersson metric on $T^2$, and prove its basic properties in Section 6. In particular, comparison with the $p$-norm (Corollary 6.6), completeness (Corollary 6.7) and continuity (Theorem 6.9) are given. Geometry of this metric will be utilized for the future problem.

Results obtained in this paper and the related work [26] have been announced in survey articles [22] and [24]. As another application of the rigidity theorem, we can embed the deformation space of a Fuchsian group $\Gamma$ within $\text{Diff}_+^r(S)$ for any $r > 1$ into the deformation space of $\Gamma$ in $\text{Sym}$. This is also sketched out in [24]; the detail will appear elsewhere. The latter space has been studied in [25] as the Teichmüller space of $\Gamma$-invariant symmetric structures on $S$.

1. The universal Teichmüller space

In this section, we review basic facts on the universal Teichmüller space. We can consult monographs by Lehto [17] and Nag [27] for quasiconformal aspects of Teichmüller spaces.

An orientation-preserving homeomorphism $w$ of a domain in the complex plane is said to be quasiconformal if partial derivatives $\partial w$ and $\bar{\partial} w$ in the distribution sense exist and if the complex dilatation $\mu_w(z) = \bar{\partial} w(z)/\partial w(z)$ satisfies $\|\mu_w\|_\infty < 1$. Let

$$\text{Bel}(\mathbb{D}) = \{\mu \in L^\infty(\mathbb{D}) \mid \|\mu\|_\infty < 1\}$$

be the space of such measurable functions on the unit disk $\mathbb{D}$, which are called Beltrami coefficients. Set the group of all quasiconformal self-homeomorphisms of $\mathbb{D}$ by $\text{QC}(\mathbb{D})$. By the measurable Riemann mapping theorem (see [1]), for every $\mu \in \text{Bel}(\mathbb{D})$, there is $w \in \text{QC}(\mathbb{D})$ satisfying $\mu_w = \mu$ uniquely up to the post-composition of elements of $\text{Möb}(\mathbb{D}) \cong \text{PSL}(2,\mathbb{R})$, the group of all Möbius transformations of $\mathbb{D}$. This gives the identification

$$\text{Möb}(\mathbb{D}) \setminus \text{QC}(\mathbb{D}) \cong \text{Bel}(\mathbb{D}).$$

Every $w \in \text{QC}(\mathbb{D})$ extends continuously to a quasisymmetric homeomorphism of $S = \partial \mathbb{D}$. Here an orientation-preserving self-homeomorphism $g : S \to S$ is called quasisymmetric if there is a constant $M \geq 1$ such that the quasisymmetric quotient $m_g(x, t)$ satisfies

$$\frac{1}{M} \leq m_g(x, t) := \frac{g(x + t) - g(x)}{g(x) - g(x - t)} \leq M$$

for every $x \in S = \mathbb{R}/\mathbb{Z}$ and for every $t > 0$. Remark that $g$ is identified with its lift $\tilde{g} : \mathbb{R} \to \mathbb{R}$. Let $\text{QS}$ be the group of all quasisymmetric self-homeomorphism of $S$. We denote the boundary extension map by

$$q : \text{QC}(\mathbb{D}) \to \text{QS},$$
which is known to be a surjective homomorphism. The universal Teichmüller space is defined by
\[ T = \text{Mob}(\mathbb{S}) \backslash \text{QS}. \]
Then the boundary extension \( q \) induces the Teichmüller projection \( \pi : \text{Bel}(\mathbb{D}) \to T \). The quotient topology of \( T \) is induced from \( \text{Bel}(\mathbb{D}) \) by \( \pi \). Actually the Teichmüller distance \( d_T \) can be defined by
\[
d_T(\tau_1, \tau_2) = \inf_{\pi(\mu_1) = \tau_1, \pi(\mu_2) = \tau_2} \log \frac{1 + \| \frac{\mu_1 - \mu_2}{1 - \mu_2 \mu_1} \|_{\infty}}{1 - \| \frac{\mu_1 - \mu_2}{1 - \mu_2 \mu_1} \|_{\infty}}
\]
for any \( \tau_1, \tau_2 \in T \), where the infimum is taken over all \( \mu_1, \mu_2 \in \text{Bel}(\mathbb{D}) \) with \( \pi(\mu_1) = \tau_1 \) and \( \pi(\mu_2) = \tau_2 \).

The group QS acts on \( T \) canonically:
\[
([f], g) \in T \times \text{QS} \mapsto g^*[f] := [f \circ g] \in T.
\]
This is regarded as the mapping class group of the universal Teichmüller space. The action is faithful and transitive. Moreover, this is isometric with respect to the Teichmüller distance on \( T \). The isotropy subgroup of QS at the origin \([id]\) is \( \text{Mob}(\mathbb{S}) \). The condition that \( g \in \text{QS} \) fixes \([f]\) in \( T \), that is, \( g^*[f] = [f] \), can be written as \([fgf^{-1}] = [id] \), and this is equivalent to the condition that \( fgf^{-1} \in \text{Mob}(\mathbb{S}) \).

For any \( \mu \in \text{Bel}(\mathbb{D}) \), we extend it to a Beltrami coefficient \( \tilde{\mu} \) on the Riemann sphere \( \hat{\mathbb{C}} \) by setting \( \tilde{\mu}(z) \equiv 0 \) for \( z \in \mathbb{D}^* = \hat{\mathbb{C}} - \overline{\mathbb{D}} \). We denote a quasiconformal homeomorphism of \( \hat{\mathbb{C}} \) with complex dilatation \( \tilde{\mu} \) by \( f_\mu \). The measurable Riemann mapping theorem guarantees the existence of such \( f_\mu \) and the uniqueness of \( f_\mu \) up to the post-composition of Möbius transformations of \( \hat{\mathbb{C}} \).

Take the Schwarzian derivative \( S_{f_\mu} : \mathbb{D}^* \to \hat{\mathbb{C}} \) of the conformal homeomorphism \( f_\mu|_{\mathbb{D}^*} \). This measures the difference of the marked complex projective structure on \( \mathbb{D}^* \) from the standard one. By the Nehari-Kraus theorem, \( S_{f_\mu} \) belongs to the complex Banach space of holomorphic functions
\[
B(\mathbb{D}^*) = \{ \varphi \in \text{Hol}(\mathbb{D}^*) \mid \| \varphi \|_{\infty} = \sup_{z \in \mathbb{D}^*} \rho_{\mathbb{D}^*}(z)|\varphi(z)| < \infty \},
\]
where \( \rho_{\mathbb{D}^*}(z) = 2/(|z|^2 - 1) \) is the hyperbolic density on \( \mathbb{D}^* \). By this correspondence \( \mu \mapsto S_{f_\mu} \), a holomorphic map
\[
\Phi : \text{Bel}(\mathbb{D}) \to B(\mathbb{D}^*)
\]
is defined, which is called the Bers projection (onto the image).

So far, we have two projections: the Teichmüller projection \( \pi : \text{Bel}(\mathbb{D}) \to T \) and the Bers projection \( \Phi : \text{Bel}(\mathbb{D}) \to B(\mathbb{D}^*) \). Then we can show that \( \Phi \circ \pi^{-1} \) is well-defined and injective, which defines the Bers embedding \( \beta : T \to B(\mathbb{D}^*) \). Actually, \( \beta \) is a homeomorphism onto the image \( \beta(T) \) and \( \beta(T) \) is a bounded domain in \( B(\mathbb{D}^*) \). This provides a complex Banach manifold structure for \( T \).
Every element $\gamma \in \text{M"ob}(S)$ acts on $B(D^*)$ linear isometrically through the Bers embedding $\beta$. This means that, for any point $[f] \in T$ with $\beta([f]) = \varphi \in \beta(T)$, the Bers embedding $\beta(\gamma^*[f])$ of the image $\gamma^*[f] = [f \circ \gamma]$ is represented by

$$(\gamma^*\varphi)(z) = \varphi(\gamma(z))\gamma'(z)^2,$$

where we regard $\gamma$ as the element of $\text{M"ob}(D^*)$ and $\gamma^*\varphi$ is the pull-back of $\varphi$ as a quadratic differential form. Clearly this action extends to $B(D^*)$ and satisfies $\|\gamma^*\varphi\|_{\infty} = \|\varphi\|_{\infty}$.

More generally, if $g \in \text{QS}$ has a fixed point $[f] \in T$, then the action of $g$ on $\beta(T)$ is conjugate to the linear isometric action of $fgf^{-1}$ on $\beta(T) \subset B(D)$ under the base point change automorphism $R_{[f]} : T \to T$ given by $[g] \mapsto [g \circ f^{-1}]$.

2. ASYMPTOTIC CONFORMALITY AND SYMMETRIC HOMEOMORPHISMS

A quasiconformal homeomorphism $w \in \text{QC}(D)$ is called asymptotically conformal if the complex dilatation vanishes at the boundary, that is, $\mu_w(z) \to 0 \ (|z| \to 1)$. The subspace of $\text{Bel}(D)$ consisting of all Beltrami coefficients vanishing at the boundary is denoted by $\text{Bel}_0(D)$ and the subgroup of $\text{QC}(D)$ consisting of all asymptotically conformal self-homeomorphisms of $D$ is denoted by $\text{AC}(D)$. On the other hand, a quasisymmetric homeomorphism $g \in \text{QS}$ is called symmetric if the quasisymmetric quotient $m_g(x,t)$ tends to 1 as $t \to 0$ uniformly with respect to $x \in S$. Note that a circle diffeomorphism is symmetric, but a symmetric homeomorphism is not necessarily absolutely continuous. The group of all symmetric self-homeomorphisms of $S$ is denoted by $\text{Sym}$. Then the restriction of the boundary extension to $\text{AC}(D)$ gives a surjective homomorphism $\pi : \text{AC}(D) \to \text{Sym}$.

Gardiner and Sullivan [14] studied the asymptotic Teichmüller space defined by

$$AT = \text{Sym} \setminus \text{QS},$$

and the little universal Teichmüller space defined by

$$T_0 = \text{M"ob}(S) \setminus \text{Sym} = \pi(\text{Bel}_0(D)).$$

They introduced $\text{Sym}$ as the characteristic topological subgroup of $\text{QS}$ consisting of all elements $g \in \text{QS}$ such that the adjoint map $\text{QS} \to \text{QS}$ given by conjugation of $g$ is continuous at the identity.

Here we summarize the characterization of symmetric homeomorphisms in $\text{Sym}$, Beltrami coefficients vanishing at the boundary in $\text{Bel}_0(D)$ and the Bers embedding of the little universal Teichmüller space $T_0 = \text{M"ob}(S) \setminus \text{Sym} = \pi(\text{Bel}_0(D))$. We set a Banach subspace of $B(D^*)$ consisting of the elements of vanishing at the boundary by

$$B_0(D^*) = \{ \varphi \in B(D^*) \mid \lim_{|z| \to 1} \rho_D^{-2}(z)|\varphi(z)| = 0 \}.$$

The following result appeared in [14], which were attributed to Fehlmann [13] and Becker and Pommerenke [3]. Condition (4) was in the last paper.
Proposition 2.1. For a quasisymmetric homeomorphism $g \in \text{QS}$, the following conditions are equivalent:

1. $g$ belongs to $\text{Sym}$;
2. there is $\mu \in \text{Bel}_0(\mathbb{D})$ such that $\pi(\mu) = [g] \in T$;
3. $\beta([g]) \in \beta(T)$ is in $B_0(\mathbb{D}^*)$;
4. $\lim_{|z| \to 1} |f''(z)/f'(z)| = 0$ for any $\mu \in \text{Bel}(\mathbb{D})$ with $\pi(\mu) = [g]$.

The proof of the rigidity theorem is carried out by using the Bers embedding of the Teichmüller space; we transfer the problems to those for the linear isometric action of Möbius transformations on $B(\mathbb{D}^*)$. We apply the Bers embedding to the little universal Teichmüller space $T_0$ and prove a certain prototype of our rigidity theorem. This is the starting point of our arguments.

Theorem 2.2. Let $\Gamma$ be a subgroup of $\text{Möb}(\mathbb{S})$ that contains hyperbolic or parabolic elements. If $f \Gamma f^{-1} \subset \text{Möb}(\mathbb{S})$ for $f \in \text{Sym}$, then $f \in \text{Möb}(\mathbb{S})$.

Proof. Set $\varphi = \beta([f])$, which belongs to the subspace $B_0(\mathbb{D}^*)$ by Proposition 2.1. The condition $f \Gamma f^{-1} \subset \text{Möb}(\mathbb{S})$ is equivalent to that $\gamma^* \varphi = \varphi$ for every $\gamma \in \Gamma \subset \text{Möb}(\mathbb{D}^*)$. Then

$$
\rho_{B^*(\mathbb{D})}^2(\varphi(z)) = \rho_{B^*(\mathbb{D})}^2(\gamma^* \varphi(z)) = \rho_{B^*(\mathbb{D})}^2(\gamma(z)) |\varphi(\gamma(z))|.
$$

Since $\varphi \in B_0(\mathbb{D}^*)$ and there is a sequence $\gamma_n \in \Gamma$ such that $|\gamma_n(z)| \to 1$ $(n \to \infty)$ for all $z \in \mathbb{D}^*$, we have $\varphi(z) \equiv 0$. This means that $[f] = [\text{id}]$, and equivalently $f \in \text{Möb}(\mathbb{S})$. \qed

In the remainder of this section, we prepare two claims on asymptotically conformal homeomorphisms, which will be used later. The corresponding results under a stronger assumption that the complex dilatation $\mu$ has an explicit decay order (that is, $\mu \in \text{Bel}_0(\mathbb{D})$, which will be defined in the next section) are given in [20, Theorem 6.4, Proposition 6.8].

Lemma 2.3. Let $f \in \text{AC}(\mathbb{D})$ be a quasiconformal homeomorphism of $\mathbb{D}$ with $f(0) = 0$ and with the complex dilatation $\mu$ in $\text{Bel}_0(\mathbb{D})$. Let $\varepsilon > 0$ be an arbitrary positive constant. Then there is a constant $A \geq 1$ depending on $\mu$ and $\varepsilon$ such that

$$
\frac{1}{A} (1 - |z|)^{1+\varepsilon} \leq 1 - |f(z)| \leq A (1 - |z|)^{1-\varepsilon}
$$

for every $z \in \mathbb{D}$.

Proof. Since $\mu \in \text{Bel}_0(\mathbb{D})$, we can find $t_0 \in (0, 1/4)$ so that $|\mu(\zeta)| \leq \varepsilon/2$ for every $\zeta \in \mathbb{D}$ with $|\zeta| > 1 - \sqrt{t_0}$. This depends on $\mu$ and $\varepsilon$. Define a Beltrami coefficient $\mu_0(\zeta)$ by setting $\mu_0(\zeta) = \mu(\zeta)$ on $|\zeta| \leq \sqrt{t_0}$ and $\mu_0(\zeta) \equiv 0$ elsewhere. Let $f_0$ be the quasiconformal homeomorphism of $\mathbb{D}$ with complex dilatation $\mu_0$ and with the normalization fixing 0 and 1. Let $f_1$ be the quasiconformal homeomorphism of $\mathbb{D}$ such that $f = f_1 \circ f_0$. For $K = (1 + \varepsilon/2)/(1 - \varepsilon/2)$, we see that $f_1$ is a $K$-quasiconformal homeomorphism of $\mathbb{D}$. Here we have

$$
\frac{1}{K} = \frac{1 - \varepsilon/2}{1 + \varepsilon/2} \geq 1 - \varepsilon.
$$
First we apply a distortion theorem to the conformal homeomorphism \( f_0(z) \) restricted to \( z \in \mathbb{D} \) with \( |z| > \sqrt{t_0} \). Actually, we may assume that \( f_0 \) is a conformal homeomorphism of an annulus \( \{ \sqrt{t_0} < |z| < 1/\sqrt{t_0} \} \) by the reflection principle. Since \( \mathbb{S} \) is compact, there is some constant \( L \geq 1 \) such that the modulus of the derivative \( |f'_0(\xi)| \) at any \( \xi \in \mathbb{S} \) is bounded by \( L \), which is actually depending on \( \mu \) and \( \varepsilon \). Now the Koebe distortion theorem (see [30, Theorem 1.3]) in the disk of radius \( \sqrt{t_0} \) and center \( \xi = z/|z| \) yields

\[
1 - |f_0(z)| \leq \frac{L(1 - |z|)}{1 - (1 - |z|)/\sqrt{t_0}^2} \leq \frac{L(1 - |z|)}{(1 - \sqrt{t_0})^2} \leq 4L(1 - |z|)
\]

for every \( z \in \mathbb{D} \) with \( 1 - |z| < t_0 \).

Next we apply the theorem of Mori (see [1, Section III.C]) to the quasiconformal self-homeomorphism \( f_1 \) of \( \mathbb{D} \). It implies that

\[
1 - |f_1(w)| \leq 16(1 - |w|)^{1/K} \leq 16(1 - |w|)^{1-\varepsilon}
\]

for every \( w \in \mathbb{D} \). Then by setting \( w = f_0(z) \) we have

\[
1 - |f(z)| \leq 16\{4L(1 - |z|)\}^{1-\varepsilon} \leq 64L(1 - |z|)^{1-\varepsilon}.
\]

If \( 1 - |z| \geq t_0 \), we simply obtain

\[
1 - |f(z)| \leq 1 \leq \frac{1}{t_0}(1 - |z|)^{1-\varepsilon}.
\]

Combined with the previous estimate, this gives the right side inequality in the statement. For the left side inequality, we apply the lower estimates in both the Koebe and the Mori theorems, or apply the above arguments to the inverse map \( f^{-1} \).

The next lemma is a variant of the Goluzin inequality. We also consider Beltrami coefficients on \( \mathbb{D}^* \) and denote their space by \( \text{Bel}(\mathbb{D}^*) \) and the subspace of all elements vanishing at the boundary by \( \text{Bel}_0(\mathbb{D}^*) \). The lemma has the statements both on \( \mathbb{D} \) and on \( \mathbb{D}^* \) and slightly different conditions are assumed for later purpose.

**Lemma 2.4.** (1) Let \( f \) be a conformal homeomorphism of \( \mathbb{D} \) into \( \mathbb{C} \) with \( f(0) = 0 \) and with \( e^{-s} \leq f'(0) \leq e^s \) for some \( s \geq 0 \) whose quasiconformal extension to \( \mathbb{D}^* \) has complex dilatation \( \mu^* \) in \( \text{Bel}_0(\mathbb{D}^*) \). Let \( \varepsilon > 0 \) be an arbitrary positive constant. Then there is a constant \( B \geq 1 \) depending only on \( \mu^*, \varepsilon \) and \( s \) such that

\[
\frac{1}{B}(1 - |z|)^{\varepsilon} \leq |f'(z)| \leq B(1 - |z|)^{-\varepsilon}
\]

for every \( z \in \mathbb{D} \).

(2) Let \( f \) be a conformal homeomorphism of \( \mathbb{D}^* \) with \( f(\infty) = \infty \) and with \( \lim_{z \to \infty} f'(z) = 1 \) whose quasiconformal extension to \( \mathbb{D} \) has complex dilatation \( \mu \) in \( \text{Bel}_0(\mathbb{D}) \). Let \( \varepsilon > 0 \) and \( R > 1 \) be arbitrary constants. Then there is a constant \( B' \geq 1 \) depending only on \( \mu \), \( \varepsilon \) and \( R \) such that

\[
\frac{1}{B'}(|z| - 1)^{\varepsilon} \leq |f'(z)| \leq B'(|z| - 1)^{-\varepsilon}
\]

for every \( z \in \mathbb{D}^* \) with \( 1 < |z| < R \).
 Proposition 2.1 implies that \( \lim_{t \to 0} \beta_{\mu^*}(t) = 0 \). Then we apply the argument in Pommerenke and Warschawski [31, p.109]. Choose \( t > 0 \) such that \( \beta_{\mu^*}(t) \leq \varepsilon \) and fix it. Consider any \( r = |z| \) with \( 1 - t \leq r < 1 \) and any \( \theta = \arg z \). The integration of the above formula along a segment between \((1 - t)e^{i\theta} \) and \( re^{i\theta} \) gives

\[
\left| \log \frac{f'(re^{i\theta})}{f'((1 - t)e^{i\theta})} \right| \leq \varepsilon \log \frac{t}{1 - r},
\]

which can be written as

\[
t^{-\varepsilon}|f'((1 - t)e^{i\theta})|(1 - |z|)^\varepsilon \leq |f'(z)| \leq t^\varepsilon|f'((1 - t)e^{i\theta})|(1 - |z|)^{-\varepsilon}
\]

for \( 1 - t \leq |z| < 1 \).

By the Koebe distortion theorem (see [30, Theorem 1.3]), we see that

\[
e^{-s}\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq e^s\frac{1 + |z|}{(1 - |z|)^3}
\]

for \( z \in \mathbb{D} \). Combining the above two inequalities, we have

\[
e^{-s}\frac{t^{1-\varepsilon}}{(2 - t)^3}(1 - |z|)^\varepsilon \leq |f'(z)| \leq e^{s}\frac{(2 - t)}{t^{3-\varepsilon}}(1 - |z|)^{-\varepsilon}
\]

for \( |z| \geq 1 - t \). Then, by taking

\[B = e^s \max\left\{ \frac{2 - t}{t^3}, \frac{(2 - t)^3}{t} \right\},\]

we obtain the required inequalities for \( |z| < 1 \).

(2) The argument proceeds parallel to that of (1). Set

\[
\beta_{\mu}(t) = \sup_{1 < |z| \leq 1 + t} \left( |z| - 1 \right) \left| \frac{f''(z)}{f'(z)} \right|.
\]

Proposition 2.1 implies that \( \lim_{t \to 0} \beta_{\mu}(t) = 0 \). Fix \( t > 0 \) such that \( \beta_{\mu}(t) \leq \varepsilon \). Take any \( r = |z| \) with \( 1 < r \leq 1 + t \) and any \( \theta = \arg z \). The integration along a segment between \( re^{i\theta} \) and \( (1 + t)e^{i\theta} \) gives

\[
t^{-\varepsilon}|f'((1 + t)e^{i\theta})|(1 - |z|)^\varepsilon \leq |f'(z)| \leq t^\varepsilon|f'((1 + t)e^{i\theta})|(1 - |z|)^{-\varepsilon}
\]

for \( 1 < |z| \leq 1 + t \).

The distortion theorem for a conformal homeomorphism of \( \mathbb{D}^* \) with the normalization above (see [5, p.41]) asserts that

\[
\frac{|z|^2 - 1}{|z|^2} \leq |f'(z)| \leq \frac{|z|^2}{|z|^2 - 1}
\]
for \( z \in \mathbb{D}^* \). Combining the above two inequalities, we have

\[
\frac{(2 + t)t^{1-\varepsilon}}{(1 + t)^2}(|z| - 1)^\varepsilon \leq |f'(z)| \leq \frac{(1 + t)^2}{(2 + t)t^{1-\varepsilon}}(|z| - 1)^{-\varepsilon}
\]

for \(|z| \leq 1 + t\). Then, by taking

\[
B' = (R - 1)^\varepsilon \frac{(1 + t)^2}{(2 + t)t},
\]

we obtain the required inequalities for \( 1 < |z| < R \). \( \square \)

3. Norm estimates of the Bers embedding of circle diffeomorphisms

We consider modification and generalization of the arguments in Theorem 2.2 to Teichmüller spaces of circle diffeomorphisms. First we give a characterization of \( \text{Diff}_{1+\alpha}^1(S) \) for \( \alpha \in (0, 1) \) analogously to Proposition 2.1. Here are spaces we have to deal with in this situation:

\[
T_0^\alpha = \text{Möb}(S) \setminus \text{Diff}_{1+\alpha}^1(S);
\]

\[
\text{Bel}_0^\alpha(\mathbb{D}) = \{ \mu \in \text{Bel}_0(\mathbb{D}) \mid \|\mu\|_{\infty, \alpha} = \text{esssup}_{z \in \mathbb{D}} \rho_0^\alpha(z)|\mu(z)| < \infty \};
\]

\[
B_0^\alpha(\mathbb{D}^*) = \{ \varphi \in B_0(\mathbb{D}^*) \mid \|\varphi\|_{\infty, \alpha} = \sup_{z \in \mathbb{D}^*} \rho_0^{-2+\alpha}(z)|\varphi(z)| < \infty \}.
\]

As usual \( \rho_0(z) = 2/(1 - |z|^2) \) is the hyperbolic density on \( \mathbb{D} \). We regard \( T_0^\alpha \) as the Teichmüller space of circle diffeomorphisms of \( \alpha \)-Hölder continuous derivatives.

**Theorem 3.1.** For a quasisymmetric homeomorphism \( g \in \text{QS} \), the following conditions are equivalent:

1. \( g \) belongs to \( \text{Diff}_{1+\alpha}^1(S) \);
2. there is \( \mu \in \text{Bel}_0^\alpha(\mathbb{D}) \) such that \( \pi(\mu) = [g] \in T \);
3. \( \beta([g]) \in \beta(T) \) is in \( B_0^\alpha(\mathbb{D}^*) \);
4. \( \sup_{z \in \mathbb{D}^*} (|z| - 1)^{1-\alpha} \frac{f''_{\mu}(z)}{f'_{\mu}(z)} \varphi(z) < \infty \) for any \( \mu \in \text{Bel}(\mathbb{D}) \) with \( \pi(\mu) = [g] \).

The statement of the theorem in this form is given in [26] Theorems 4.6 and 6.7, based on previous results on asymptotically conformal homeomorphisms by Carleson [8] and its refinement. Moreover, we can estimate the relevant quantity in each item above in terms of other quantities. In particular, the estimate of the norm of \( B_0^\alpha(\mathbb{D}^*) \) by the norm of \( \text{Bel}_0^\alpha(\mathbb{D}) \) is important in our study.

In this section, we try to generalize this estimate to the case where the base point is different from the origin. The corresponding results under a stronger assumption are obtained in [26] Section 7] and the arguments here are carried out similarly by adding necessary modifications. We prepare notation used hereafter.
Definition. For $\mu \in \text{Bel}(\mathbb{D})$, the quasiconformal self-homeomorphism of $\mathbb{D}$ whose complex dilatation coincides with $\mu$ having the normalization that fixes 0 and 1 is denoted by $f^\mu$. For $\nu \in \text{Bel}(\mathbb{D})$ the complex dilatation of the composition $f^\mu \circ f^\nu$ is denoted by $\mu \ast \nu$, and that of the inverse $(f^\nu)^{-1}$ is denoted by $\nu^{-1}$.

Remark. It is often the case that the normalization for $f^\mu$ is given by fixing three distinct points on $\mathbb{S}$, say, 1, 1, $\infty$. We adopt the above normalization in this paper for the sake of our arguments. There is no essential difference between them because of compactness of the normalized quasiconformal self-homeomorphisms of $\mathbb{D}$ with bounded dilatations.

The following inequality is a fundamental tool, which is due to Yanagishita [35, Lemma 3.1, Proposition 3.2] obtained by applying the argument of Astala and Zinsmeister [3].

Proposition 3.2. For Beltrami coefficients $\mu$ and $\nu$ in $\text{Bel}(\mathbb{D})$, let $f_\mu$ and $f_\nu$ be the quasiconformal homeomorphisms of $\hat{\mathbb{C}}$ that are conformal on $\mathbb{D}^*$ and have complex dilatations $\mu$ and $\nu$ respectively on $\mathbb{D}$. Set $\Omega = f_\nu(\mathbb{D})$ and $\Omega^* = f_\nu(\mathbb{D}^*)$. Then

$$|S_{f_\nu \circ f_\nu^{-1}|_{\Omega^*}}(\zeta)| \leq \frac{6\rho_{\Omega^*}(\zeta)}{\sqrt{\pi}} \left( \int_{\Omega} \frac{\left| \mu(f_\nu^{-1}(w)) - \nu(f_\nu^{-1}(w)) \right|^2}{(1 - |\mu(f_\nu^{-1}(w))|^2)(1 - |\nu(f_\nu^{-1}(w))|^2)} \frac{dudv}{|w - \zeta|^4} \right)^{1/2}$$

holds for $\zeta \in \Omega^*$.

The first result is as follows. We have to modify the norm $\| \cdot \|_{\infty, \alpha}$ in the Bers embedding to $\| \cdot \|_{\infty, \alpha - \epsilon}$ by an arbitrary positive $\epsilon > 0$ in the case of a different base point. Hereafter, whenever we mention the dependence of constants, we always fix $\alpha \in (0, 1)$ and omit mentioning the dependence on it.

Lemma 3.3. For arbitrary $\mu \in \text{Bel}(\mathbb{D})$, $\nu \in \text{Bel}_0(\mathbb{D})$ and $\epsilon > 0$, it holds that

$$\|\Phi(\mu) - \Phi(\nu)\|_{\infty, \alpha - \epsilon} \leq C_1 \|\mu - \nu\|_{\infty, \alpha},$$

where $C_1 > 0$ is a constant depending on $\|\mu\|_{\infty}$, $\nu$ and $\epsilon$. The right side term is assumed to be $\infty$ when $\mu - \nu \notin \text{Bel}_0(\mathbb{D})$.

Proof. Since $\nu \in \text{Bel}_0(\mathbb{D})$, Lemma 2.3 shows that there is a constant $a > 0$ depending only on $\nu$ and $\epsilon$ such that

$$\rho_{\mathbb{D}}^{-\alpha}(z) = \left\{ \frac{1 - |z|^2}{2} \right\}^\alpha \leq a \left\{ \frac{1 - |f_\nu(z)|^2}{2} \right\}^{(1-\epsilon/(4\alpha))\alpha} \leq a\rho_{\mathbb{D}}^{-\alpha}(f_\nu(z)).$$

We set $f = f_\nu \circ (f_\nu)^{-1}$, where $f_\nu$ (and $f_\mu$ below) are as in Proposition 3.2. This map $f$ is a conformal homeomorphism of $\mathbb{D}$ extending to a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ whose complex dilatation on $\mathbb{D}^*$ coincides with $(\nu^*)^{-1}$, which is the reflection of the complex dilatation of $(f^\nu)^{-1}$. By the formula of $\nu^{-1}$ in terms of $\nu$, we see that $(\nu^*)^{-1}$ belongs to $\text{Bel}_0(\mathbb{D}^*)$. We can choose $f_\nu$ so that $f_\nu(0) = f(0) = 0$ keeping the normalization $f_\nu(\infty) = \infty$ and $\lim_{z \to \infty} f_\nu'(z) = 1$. Note that $f(\mathbb{D}) = f_\nu(\mathbb{D})$.

By the normalization of $f_\nu$ appealing to the Schwarz lemma and the Koebe one-quarter theorem (cf. [30, Theorem 1.3]) on $\mathbb{D}^*$, we see that $f_\nu(\mathbb{D})$ is not strictly contained in
Let \( \eta \) be contained in the disk \( \{ |z| < 4 \} \). Hence, there is some \( x_1 \in \mathbb{S} \) such that \( 1 \leq |f_\nu(x_1)| \leq 4 \). Take \( z \) with \( |z| = 1/2 \) arbitrarily and consider the cross-ratio \( [0, x_1, \infty, z] \).

By the distortion theorem for cross-ratio due to Teichmüller (see [1, Section III.D]), the hyperbolic distance of \( \mathbb{C} - \{0, 1\} \) between \( [0, x_1, \infty, z] \) and \( [0, f_\nu(x_1), \infty, f_\nu(z)] \) is bounded by \( \log K \), where \( K = (1 + \|\nu\|_\infty)/(1 - \|\nu\|_\infty) \). This implies that there is a constant \( r > 0 \) depending only on \( \|\nu\|_\infty \) such that \( |f_\nu(z)| \geq r \) for \( |z| = 1/2 \) and hence \( f(\mathbb{D}) = f_\nu(\mathbb{D}) \) contains the disk of center at \( 0 \) and radius \( r \). By the Schwarz lemma applied to the conformal homeomorphism \( f \) of \( \mathbb{D} \), we see that there is a constant \( s = s(r) \geq 0 \) depending only on \( r \) and hence on \( \|\nu\|_\infty \) such that \( e^{-s} \leq |f'(0)| \leq 4 \).

It follows from Lemma 2.4 (1) that there is a constant \( b > 0 \) depending only on \( \nu \) and \( \varepsilon \) such that

\[
\rho_\mathbb{D}^{-\alpha/4}(f_\nu(z)) \leq b\rho_\Omega^{-\alpha/2}(f_\nu(z)).
\]

By the definition of the norm, we have

\[
|\mu(z) - \nu(z)| \leq \rho_\mathbb{D}^{-\alpha}(z)\|\mu - \nu\|_{\infty,\alpha}
\]

for \( z \in \mathbb{D} \). For \( w = f_\nu(z) \in \Omega \), these inequalities yield

\[
|\mu(f_\nu^{-1}(w)) - \nu(f_\nu^{-1}(w))| \leq ab\rho_\Omega^{-\alpha/2}(w)\|\mu - \nu\|_{\infty,\alpha}.
\]

Set \( \alpha' = \alpha - \varepsilon/2 \).

By substituting this inequality to the integral in Proposition 3.2, we will estimate

\[
\left( \int_\Omega \frac{\rho_\Omega^{-2\alpha'}(w)}{|w - \zeta|^4} \text{d}u\text{d}v \right)^{1/2}.
\]

Let \( \eta_\Omega(w) \) be the euclidean distance from \( w \in \Omega \) to \( \partial \Omega \) and \( \eta_\Omega^*(\zeta) \) the euclidean distance from \( \zeta \in \Omega^* \) to \( \partial \Omega^* \). As a consequence from the Koebe one-quarter theorem, we see that both \( \rho_\Omega(w)\eta_\Omega(w) \) and \( \rho_\Omega^*(\zeta)\eta_\Omega^*(\zeta) \) are bounded below by \( 1/2 \). We have

\[
\rho_\Omega^{-2\alpha'}(w) \leq 4\eta_\Omega^{2\alpha'}(w) \leq 4|w - \zeta|^{2\alpha'}
\]

for every \( w \in \Omega \) and for every \( \zeta \in \Omega^* \). Hence the integral can be estimated as

\[
\int_\Omega \frac{\rho_\Omega^{-2\alpha'}(w)}{|w - \zeta|^4} \text{d}u\text{d}v \leq 4 \int_\Omega \frac{\text{d}u\text{d}v}{|w - \zeta|^{4-2\alpha'}} \leq 4 \int_{|w - \zeta| \geq \eta_\Omega^*(\zeta)} \frac{\text{d}u\text{d}v}{|w - \zeta|^{4-2\alpha'}} = \frac{8\pi}{2 - 2\alpha'} \cdot \frac{1}{\eta_\Omega^*(\zeta)^{2-2\alpha'}} \leq \frac{32\pi}{\varepsilon} \cdot \rho_\Omega^*(\zeta)^{2-2\alpha'}.
\]

Plugging this estimate in the inequality of Proposition 3.2, we have

\[
\rho_\Omega^{-2}(\zeta)|S_{f_\nu f_\nu^{-1}}|_{\Omega^*}(\zeta) \leq \frac{48ab\|\mu - \nu\|_{\infty,\alpha}}{\sqrt{2\varepsilon(1 - \|\mu\|_{\infty}^2)(1 - \|\nu\|_{\infty}^2)}} \rho_\Omega^{-\alpha/2}(\zeta).
\]
For $\zeta = f_\nu(z)$ with $z \in \mathbb{D}^*$, the left side term equals to
$$\rho^{-2}(z)|S_{f_\nu|\mathbb{D}^*}(z) - S_{f_\nu|\mathbb{D}^*}(z)|.$$ For the right side term, we apply Lemma 2.4 (2) to the quasiconformal homeomorphism $f_\nu$ of $\hat{\mathbb{C}}$ which is conformal on $\mathbb{D}^*$. Then there is a constant $b' > 0$ depending only on $\nu$ and $\varepsilon$ such that
$$\rho^{-(\alpha-\varepsilon/2)}(f_\nu(z)) \leq b' \rho^{-(\alpha-\varepsilon)}(z)$$ for $1 < |z| < 2$. Therefore the above inequality turns out to be
$$\rho^{-2+(\alpha-\varepsilon)}(z)|S_{f_\nu|\mathbb{D}^*}(z) - S_{f_\nu|\mathbb{D}^*}(z)| \leq \frac{48ab\varepsilon\|\mu - \nu\|_{\infty,\alpha}}{\sqrt{2\varepsilon(1 - \|\mu\|_{\infty,\alpha})(1 - \|\nu\|_{\infty,\alpha})}} \quad (1 < |z| < 2).$$

On the other hand, we use the following estimate (see [17, Theorem II.3.2]) for large $|z|$:
$$\rho^{-2}(z)|S_{f_\nu|\mathbb{D}^*}(z) - S_{f_\nu|\mathbb{D}^*}(z)| \leq \frac{3\|\mu - \nu\|_{\infty}}{1 - \|\mu\|_{\infty} \|\nu\|_{\infty}}.$$ For $|z| \geq 2$, it is clear that $\rho^{-\varepsilon}(z) \leq 1$. Hence we have
$$\rho^{-2+(\alpha-\varepsilon)}(z)|S_{f_\nu|\mathbb{D}^*}(z) - S_{f_\nu|\mathbb{D}^*}(z)| \leq \frac{3\|\mu - \nu\|_{\infty,\alpha}}{1 - \|\mu\|_{\infty} \|\nu\|_{\infty}} \quad (|z| \geq 2).$$ Thus, including the case of $|z| \geq 2$, we can choose the required constant $C_1 > 0$ as in the statement. \qed

Under the stronger assumption that $\nu \in \text{Bel}_0^{\alpha'}(\mathbb{D})$ for some $\alpha' \in (0,1)$ in Lemma 3.3, we have the following stronger consequence than the above. The proof is along the same line except for applying the improved estimates of Lemmata 2.3 and 2.4 where the constant $\varepsilon$ is not involved. In particular, the stronger estimate for the hyperbolic density is also obtained in this argument. See [26, Theorem 6.4, Lemma 7.4].

**Proposition 3.4.** (1) For $\nu \in \text{Bel}_0^{\alpha'}(\mathbb{D})$, there is a constant $c' > 0$ such that
$$c'\rho_D(z) \leq \rho_D(f_\nu(z)) \leq c'\rho_D(z) \quad (z \in \mathbb{D}),$$ where $c'$ depends only on $\|\nu\|_{\infty,\alpha'}$ and $\alpha'$. (2) For every $\mu \in \text{Bel}(\mathbb{D})$ and every $\nu \in \text{Bel}_0^{\alpha'}(\mathbb{D})$, there is a constant $C'_1 > 0$ such that
$$\|\Phi(\mu) - \Phi(\nu)\|_{\infty,\alpha} \leq C'_1\|\mu - \nu\|_{\infty,\alpha},$$ where $C'_1$ depends only on $\|\nu\|_{\infty,\alpha'}$, $\alpha'$, $\|\mu\|_{\infty}$ and $\|\nu\|_{\infty}$.

Next, we consider the difference of the norms of Beltrami coefficients under the right translation defined below. Also in this case, an arbitrary positive constant $\varepsilon > 0$ should be involved when $\nu \in \text{Bel}_0(\mathbb{D})$, but it disappears when $\nu$ is promoted as above.

**Definition.** The right translation $r_\nu$ of $\text{Bel}(\mathbb{D})$ is defined by $r_\nu(\mu) = \mu * \nu^{-1}$, which satisfies $\pi \circ r_\nu = R_{\pi(\nu)} \circ \pi$. 
Proposition 3.5. For arbitrary $\mu_1, \mu_2 \in \text{Bel}(\mathbb{D})$, $\nu \in \text{Bel}_0(\mathbb{D})$ and $\varepsilon > 0$, it holds that
\[
\|r_\nu(\mu_1) - r_\nu(\mu_2)\|_{\infty,\alpha - \varepsilon} \leq C_2 \|\mu_1 - \mu_2\|_{\infty,\alpha},
\]
where $C_2 > 0$ is a constant depending on $\|\mu_1\|_{\infty}$, $\|\mu_2\|_{\infty}$, $\nu$ and $\varepsilon$. Moreover, if $\nu \in \text{Bel}_0'(\mathbb{D})$ for some $\alpha' \in (0,1)$ in addition, it holds that
\[
\|r_\nu(\mu_1) - r_\nu(\mu_2)\|_{\infty,\alpha} \leq C_2' \|\mu_1 - \mu_2\|_{\infty,\alpha},
\]
where $C_2' > 0$ is a constant depending on $\|\mu_1\|_{\infty}$, $\|\mu_2\|_{\infty}$, $\|\nu\|_{\infty,\alpha'}$ and $\alpha'$. The right side terms are assumed to be $\infty$ when $\mu_1 - \mu_2 \notin \text{Bel}_0'(\mathbb{D})$.

Proof. We have the following inequality for $\zeta = f^\nu(z)$:
\[
|\nu(\mu_1)(\zeta) - \nu(\mu_2)(\zeta)| = |\mu_1 * \nu^{-1}(\zeta) - \mu_2 * \nu^{-1}(\zeta)|
\]
\[
= \left| \frac{\mu_1(z) - \nu(z)}{1 - \nu(z)\mu_1(z)} - \frac{\mu_2(z) - \nu(z)}{1 - \nu(z)\mu_2(z)} \right|
\]
\[
= \frac{|\mu_1(z) - \mu_2(z)|(1 - |\nu(z)|^2)}{|1 - \nu(z)\mu_1(z)||1 - \nu(z)\mu_2(z)|} \leq \frac{|\mu_1(z) - \mu_2(z)|}{\sqrt{(1 - |\mu_1(z)|^2)(1 - |\mu_2(z)|^2)}}
\]

For $\nu \in \text{Bel}_0(\mathbb{D})$, Lemma 2.3 implies that there is some constant $c > 0$ depending on $\nu$ and $\varepsilon$ such that $\rho_\nu^{\alpha - \varepsilon}(\zeta) \leq c\rho_\nu^{\alpha}(z)$. Hence we have
\[
\rho_\nu^{\alpha - \varepsilon}(\zeta)|r_\nu(\mu_1)(\zeta) - r_\nu(\mu_2)(\zeta)| \leq C_2 \rho_\nu^{\alpha}(z)|\mu_1(z) - \mu_2(z)|
\]
for $C_2 = c/\sqrt{(1 - |\mu_1|^2)(1 - |\mu_2|^2)}$.

For $\nu \in \text{Bel}_0'(\mathbb{D})$, the better estimate as in Proposition 3.4 (1) shows that there is some constant $c' > 0$ depending on $\|\nu\|_{\infty,\alpha'}$ and $\alpha'$ such that $\rho_\nu^{\alpha}(\zeta) \leq c'\rho_\nu^{\alpha}(z)$. This yields that
\[
\rho_\nu^{\alpha}(\zeta)|r_\nu(\mu_1)(\zeta) - r_\nu(\mu_2)(\zeta)| \leq C_2' \rho_\nu^{\alpha}(z)|\mu_1(z) - \mu_2(z)|
\]
for $C_2' = c'/\sqrt{(1 - |\mu_1|^2)(1 - |\mu_2|^2)}$. $\square$

Finally, the combination of Lemma 3.3 and Proposition 3.5 simply yields an output in this section.

Theorem 3.6. For arbitrary $\mu_1 \in \text{Bel}(\mathbb{D})$, $\mu_2, \nu \in \text{Bel}_0(\mathbb{D})$ and $\varepsilon > 0$, it holds that
\[
\|\Phi(r_\nu(\mu_1)) - \Phi(r_\nu(\mu_2))\|_{\infty,\alpha - \varepsilon} \leq C\|\mu_1 - \mu_2\|_{\infty,\alpha},
\]
where $C > 0$ is a constant depending on $\|\mu_1\|_{\infty}$, $\mu_2, \nu$ and $\varepsilon$. Moreover, if $\mu_2, \nu \in \text{Bel}_0'(\mathbb{D})$ for some $\alpha' \in (0,1)$ in addition, it holds that
\[
\|\Phi(r_\nu(\mu_1)) - \Phi(r_\nu(\mu_2))\|_{\infty,\alpha} \leq C'\|\mu_1 - \mu_2\|_{\infty,\alpha},
\]
where $C' > 0$ is a constant depending on $\|\mu_1\|_{\infty}$, $\|\mu_2\|_{\infty}$, $\|\nu\|_{\infty,\alpha}$, $\|\nu\|_{\infty,\alpha'}$ and $\alpha'$. 
Thus we obtain the consequences.

Proof. For the first statement, Lemma \ref{lem:3.3} gives
$$
\|\Phi(r_\nu(\mu_1)) - \Phi(r_\nu(\mu_2))\|_{\infty, \alpha - \varepsilon} \leq C_1 \|r_\nu(\mu_1) - r_\nu(\mu_2)\|_{\infty, \alpha - \varepsilon/2},
$$
and then the first statement of Proposition \ref{prop:3.5} yields
$$
\|r_\nu(\mu_1) - r_\nu(\mu_2)\|_{\infty, \alpha - \varepsilon/2} \leq C_2 \|\mu_1 - \mu_2\|_{\infty, \alpha}.
$$
The dependence of the constants comes from both claims.

For the second statement, we first note that $r_\nu(\mu_2)$ also belongs to $\text{Bel}_0^\alpha(\mathbb{D})$ because $\text{Bel}_0^\alpha(\mathbb{D})$ is a group under the operation $\ast$. This is due to the formula of $r_\nu(\mu_2) = \mu_2 \ast \nu^{-1}$ and Proposition \ref{prop:3.4} (1). Then we apply Proposition \ref{prop:3.4} (2) to obtain
$$
\|\Phi(r_\nu(\mu_1)) - \Phi(r_\nu(\mu_2))\|_{\infty, \alpha} \leq C_1' \|r_\nu(\mu_1) - r_\nu(\mu_2)\|_{\infty, \alpha}.
$$
We also apply the second statement of Proposition \ref{prop:3.5} to obtain
$$
\|r_\nu(\mu_1) - r_\nu(\mu_2)\|_{\infty, \alpha} \leq C_2' \|\mu_1 - \mu_2\|_{\infty, \alpha}.
$$
The dependence of the constants also comes from the claims involved. \hfill \Box

4. Self-improvement arguments and rigidity of a hyperbolic cyclic group

In this section, we complete the proof of our rigidity theorem mentioned in the introduction.

Theorem 4.1. Let $\Gamma$ be a subgroup of $\text{M"ob}(\mathbb{S})$ that contains a hyperbolic element. If $f \Gamma f^{-1} \subset \text{Diff}^{1+\alpha}(\mathbb{S})$ for $f \in \text{Sym}$, then $f \in \text{Diff}^{1+\alpha}(\mathbb{S})$.

The strategy of the proof is twofold: Self-improvement arguments and the proof for a hyperbolic cyclic subgroup. The former one is based on the following claim, which is a reformulation of Theorem \ref{thm:3.6}.

Proposition 4.2. Let $g$ belong to $\text{Diff}^{1+\alpha}(\mathbb{S})$. For every $f \in \text{Sym}$ and for any $\alpha' \in (0, \alpha)$,
$$
\beta([g \circ f]) \in \beta([f]) + B_0^{\alpha'}(\mathbb{D}^\ast).
$$
Moreover, if $f \in \text{Diff}^{1+\alpha'}(\mathbb{S})$, then
$$
\beta([g \circ f]) \in \beta([f]) + B_0^{\alpha}(\mathbb{D}^\ast).
$$

Proof. By Theorem \ref{thm:3.4}, we choose $\mu \in \text{Bel}_0^\alpha(\mathbb{D})$ such that $\pi(\mu) = [g]$. For the first case, we also choose $\nu \in \text{Bel}_0^\alpha(\mathbb{D})$ such that $\pi(\nu) = [f^{-1}]$. Then $\beta([g \circ f]) = \Phi(r_\nu(\mu))$ and $\beta([f]) = \Phi(r_\nu(0))$. The first statement of Theorem \ref{thm:3.6} says that
$$
\|\beta([g \circ f]) - \beta([f])\|_{\infty, \alpha} \leq C \|\mu - 0\|_{\infty, \alpha} < \infty.
$$
For the second case, we choose $\nu \in \text{Bel}_0^{\alpha'}(\mathbb{D})$ such that $\pi(\nu) = [f^{-1}]$. The second statement of Theorem \ref{thm:3.6} says that
$$
\|\beta([g \circ f]) - \beta([f])\|_{\infty, \alpha} \leq C' \|\mu - 0\|_{\infty, \alpha} < \infty.
$$
Thus we obtain the consequences. \hfill \Box
Under the circumstances of Theorem 4.1, set \( \varphi = \beta([f]) \), which belongs to \( B_0(\mathbb{D}^*) \) by Proposition 2.1. The assumption \( f \Gamma f^{-1} \subset \text{Diff}^{1+\alpha}(S) \) is equivalent to the existence of \( g \in \text{Diff}^{1+\alpha}(S) \) with \( [f \circ \gamma] = [g \circ f] \) for every \( \gamma \in \Gamma \). Then Proposition 4.2 yields that \( \gamma^* \varphi - \varphi \in B_0'(\mathbb{D}^*) \). For the present, we want to obtain \( \varphi \in B_0'(\mathbb{D}^*) \) again by Proposition 4.2, which we call self-improvement arguments.

Our second strategy is to consider just one hyperbolic element \( \gamma \in \Gamma \) to show that \( \varphi \in B_0'(\mathbb{D}^*) \). Choose any hyperbolic element \( \gamma \in \Gamma \) and set

\[
\psi = \gamma^* \varphi - \varphi \in B_0'(\mathbb{D}^*).
\]

Then, by similar arguments for proving Theorem 2.2 we have the following representation of \( \varphi \).

**Proposition 4.3.** \( \varphi(z) = -\sum_{i=0}^{\infty} (\gamma^*)^i \psi(z) = \sum_{i=1}^{\infty} (\gamma^*)^{-i} \psi(z) \) for each \( z \in \mathbb{D}^* \).

**Proof.** For each \( i \in \mathbb{Z} \), it holds \((\gamma^*)^i \psi = (\gamma^*)^{i+1} \varphi - (\gamma^*)^i \varphi\). Summing up this from \( i = 0 \) to \( n \geq 0 \), we have

\[
\sum_{i=0}^{n} (\gamma^*)^i \psi = (\gamma^*)^{n+1} \varphi - \varphi.
\]

Here \( \lim_{n \to +\infty} (\gamma^*)^{n+1} \varphi = 0 \). Indeed, for each \( z \in \mathbb{D}^* \),

\[
\rho_{\mathbb{D}}^{-2}(z) |(\gamma^*)^{n+1} \varphi(z)| = \rho_{\mathbb{D}}^{-2}(\gamma^{n+1}(z)) |\varphi(\gamma^{n+1}(z))|,
\]

and the right side term converges to 0 as \( n \to \infty \) since \( \varphi \in B_0(\mathbb{D}^*) \). Thus \( \varphi(z) = -\sum_{i=0}^{\infty} (\gamma^*)^i \psi(z) \) follows. If we sum up the above equation from \( i = -1 \) to \(-n \leq -1 \) and take the limit as \( n \to \infty \), then we can obtain the second equation in the same reason. \( \square \)

Using this representation of \( \varphi \) in terms of \( \psi \in B_0'(\mathbb{D}^*) \), we see that \( \varphi = \beta([f]) \) also belongs to \( B_0'(\mathbb{D}^*) \) as follows.

**Lemma 4.4.** If \( \varphi(z) = -\sum_{i=0}^{\infty} (\gamma^*)^i \psi(z) = \sum_{i=1}^{\infty} (\gamma^*)^{-i} \psi(z) \) for a hyperbolic element \( \gamma \in \text{Mob}(\mathbb{D}^*) \) and if \( \psi \in B_0(\mathbb{D}^*) \), then \( \varphi \in B_0'(\mathbb{D}^*) \).

**Proof.** Take a M"{o}bius transformation \( h \) that maps \( \mathbb{D}^* \) to the upper half-space \( \mathbb{H} \), the attracting fixed point \( a_\gamma \) of \( \gamma \) to 0 and the repelling fixed point \( r_\gamma \) of \( \gamma \) to \( \infty \). We use the Banach space of holomorphic functions

\[
B_0^\alpha(\mathbb{H}) = \{ \phi \in \text{Hol}(\mathbb{H}) \mid \| \phi \|_{\infty, \alpha} = \sup_{\zeta \in \mathbb{H}} \rho_{\mathbb{H}}^{-2+\alpha}(\zeta) |\phi(\zeta)| < \infty \},
\]

where \( \rho_{\mathbb{H}}(\zeta) = 1/\text{Im} \zeta \) is the hyperbolic density on \( \mathbb{H} \). Set \( \tilde{\psi} = h_* \psi \) for \( \psi \in B_0(\mathbb{D}^*) \), where \( h_* \psi(\zeta) = \psi(h^{-1}(\zeta))(h^{-1})'(\zeta)^2 \). It satisfies

\[
\rho_{\mathbb{H}}^{-2}(\zeta) |\tilde{\psi}(\zeta)| = \rho_{\mathbb{D}}^{-2}(z) |\psi(z)|
\]
for $\zeta = h(z)$. On the other hand, there is a constant $C > 0$ depending only on $h$ such that

$$\rho_H^2(\zeta) = \rho_H^2(\zeta)|h'(z)|^\alpha \geq \frac{1}{C^\alpha \rho_H^2(\zeta)}$$

for every $z \in D^*$ except in some neighborhood of $\infty$. This shows that $\widetilde{\varphi} = h_\ast \psi \in B_0^\alpha(\mathbb{H})$ for every $\psi \in B_0^\alpha(D^*)$. Namely,

$$h_\ast : B_0^\alpha(D^*) \to B_0^\alpha(\mathbb{H})$$

is a bounded linear injection, but not surjective nor isometric; the norm $\| \cdot \|_{\infty,\alpha}$ on $B_0^\alpha(\cdot)$ is not Möbius invariant. We also consider $\tilde{\varphi} = h_\ast \varphi \in B_0^\alpha(\mathbb{H})$.

We define $\gamma = h_\ast \gamma \in \text{Möb}(\mathbb{H})$ by the conjugate $h_\ast \gamma = h\gamma h^{-1}$. Note that the attracting fixed point of $\tilde{\gamma}$ is 0 and the repelling fixed point is $\infty$; this can be represented as $\tilde{\gamma}(\zeta) = \lambda \zeta$ for the multiplier $\lambda \in (0, 1)$ of $\gamma$. Then $\gamma_\ast \widetilde{\varphi} = h_\ast (\gamma_\ast \psi)$, where $\gamma_\ast \widetilde{\psi}$ means the pull-back of $\widetilde{\psi}$ by $\tilde{\gamma}$ as a $(2, 0)$-form.

From the assumption $\varphi(z) = -\sum_{i=0}^\infty (\gamma_\ast)^i \psi(z)$, it follows that $\tilde{\varphi}(\zeta) = -\sum_{i=0}^\infty (\tilde{\gamma}_\ast)^i \tilde{\psi}(\zeta)$. Here $\|\tilde{\psi}\|_{\infty,\alpha} < \infty$ and $\tilde{\gamma}(\zeta) = \lambda \zeta$ ($0 < \lambda < 1$). Hence we have

$$\rho_H^{-2}(\zeta)|\tilde{\varphi}(\zeta)| = \rho_H^{-2}(\zeta) \sum_{i=0}^\infty (\tilde{\gamma}_\ast)^i \tilde{\psi}(\zeta)$$

$$\leq \rho_H^{-2}(\zeta) \sum_{i=0}^\infty (\tilde{\gamma}_\ast)^i \tilde{\psi}(\zeta) = \rho_H^{-2}(\zeta) \sum_{i=0}^\infty (\tilde{\gamma}_\ast)^i \tilde{\psi}(\tilde{\gamma}_\ast^i(\zeta))$$

$$\leq \|\tilde{\psi}\|_{\infty,\alpha} \sum_{i=0}^\infty (\tilde{\gamma}_\ast)^i \tilde{\psi}(\tilde{\gamma}_\ast^i(\zeta)) = \|\tilde{\psi}\|_{\infty,\alpha} \sum_{i=0}^\infty (\text{Im}(\lambda^i \zeta))^\alpha$$

$$= \|\tilde{\psi}\|_{\infty,\alpha} \sum_{i=0}^\infty (\lambda^i)^\alpha = \|\tilde{\psi}\|_{\infty,\alpha} \frac{\lambda^\alpha}{1 - \lambda^\alpha} = \rho_H^{-\alpha}(\zeta)$$

This gives $\tilde{\varphi} \in B_0^\alpha(\mathbb{H})$ (though this does not yet imply that $\varphi \in B_0^\alpha(D^*)$).

To see $\varphi \in B_0^\alpha(D^*)$, we use the other expression $\varphi(z) = \sum_{i=1}^\infty (\gamma_\ast)^{-i} \psi(z)$. We take a Möbius transformation $e \in \text{Möb}(\mathbb{H})$ with $e(\zeta) = 1/\zeta$. Then we have

$$(e \circ h)^{-1} \gamma^{-1} (e \circ h) = \tilde{\gamma}.$$ 

Set $\widetilde{\psi}_1 = (e \circ h)_\ast \psi$ and $\tilde{\varphi}_1 = (e \circ h)_\ast \varphi$. As before $\widetilde{\psi}_1$ belongs to $B_0^\alpha(\mathbb{H})$, and from $\varphi(z) = \sum_{i=1}^\infty (\gamma_\ast)^{-i} \psi(z)$, it follows that $\tilde{\varphi}_1(z) = \sum_{i=1}^\infty (\tilde{\gamma}_\ast)^i \tilde{\psi}_1(\zeta)$. Then

$$\rho_H^{-2}(\zeta)|\tilde{\varphi}_1(\zeta)| \leq \frac{\lambda^\alpha}{1 - \lambda^\alpha} \rho_H^{-\alpha}(\zeta)$$

which gives $\tilde{\varphi}_1 \in B_0^\alpha(\mathbb{H})$.

Finally, we will conclude $\varphi \in B_0^\alpha(D^*)$ from both $\tilde{\varphi} \in B_0^\alpha(\mathbb{H})$ and $\tilde{\varphi}_1 \in B_0^\alpha(\mathbb{H})$. There is a constants $c > 0$ and a neighborhood $U \subset \mathbb{C}$ of $r_\gamma$ with $a_\gamma \not\in \overline{U}$ such that $|h'(z)| \leq c$ for
every \( z \in \mathbb{D}^* - U \). Hence
\[
\rho_{\mathbb{D}}^{-2}(z)|\varphi(z)| = \rho_{\mathbb{H}}^{-2}(\zeta)|\tilde{\varphi}(\zeta)| \leq \|\tilde{\varphi}\|_{\infty, \alpha} \rho_{\mathbb{H}}^{-\alpha} (\zeta) \\
= \|\tilde{\varphi}\|_{\infty, \alpha} \rho_{\mathbb{D}}^{-\alpha} (z)|h'(z)|^\alpha \leq c^\alpha \|\varphi\|_{\infty, \alpha} \rho_{\mathbb{D}}^{-\alpha} (z)
\]
for every \( z \in \mathbb{D}^* - U \). Similarly, there is a constants \( c_1 > 0 \) and a neighborhood \( U_1 \subset \mathbb{C} \) of \( a_\gamma \) with \( U \cap U_1 = \emptyset \) such that \( |(e \circ h)'(z)| \leq c_1 \) for every \( z \in \mathbb{D}^* - U_1 \). Hence
\[
\rho_{\mathbb{D}}^{-2}(z)|\varphi(z)| = \rho_{\mathbb{H}}^{-2}(\zeta)|\tilde{\varphi}_1(\zeta)| \leq \|\tilde{\varphi}_1\|_{\infty, \alpha} \rho_{\mathbb{H}}^{-\alpha} (\zeta) \\
= \|\tilde{\varphi}_1\|_{\infty, \alpha} \rho_{\mathbb{D}}^{-\alpha} (z)|(e \circ h)'(z)|^\alpha \leq c_1^\alpha \|\varphi\|_{\infty, \alpha} \rho_{\mathbb{D}}^{-\alpha} (z)
\]
for every \( z \in \mathbb{D}^* - U_1 \). Therefore, we have
\[
\rho_{\mathbb{D}}^{-2+\alpha}(z)|\varphi(z)| \leq \max \{ c^\alpha \|\varphi\|_{\infty, \alpha}, c_1^\alpha \|\varphi_1\|_{\infty, \alpha} \} < \infty
\]
for every \( z \in \mathbb{D}^* \), which proves \( \varphi \in B_0^\alpha (\mathbb{D}^*) \).

Now we can complete the proof.

**Proof of Theorem 4.1.** By Proposition 4.3 and Lemma 4.4, we have \( \varphi \in B_0^\alpha (\mathbb{D}^*) \). This condition implies that \( f \in \text{Diff}_+^{1+\alpha}(S) \) by Theorem 3.1. Having this, we repeat the same argument as above from the beginning. At first, Proposition 4.2 yields that \( \gamma^\alpha \varphi - \varphi \in B_0^\alpha (\mathbb{D}^*) \) in this turn. Then, we see that \( \varphi \in B_0^\alpha (\mathbb{D}^*) \) by Proposition 4.3 and Lemma 4.4, which means that \( f \in \text{Diff}_+^{1+\alpha}(S) \) by Theorem 3.1. \( \square \)

5. **Conjugation of a Group of Circle Diffeomorphisms to a Möbius Group**

As an application of our rigidity theorem, we prove a certain conjugation problem of a group of circle diffeomorphisms. We use the following integral class of Beltrami coefficients.

**Definition.** A Beltrami coefficient \( \mu \in \text{Bel}(\mathbb{D}) \) is \( p \)-integrable for \( p \geq 1 \) if
\[
\|\mu\|_p = \int_{\mathbb{D}} |\mu(z)|^p \rho_0^2(z) dxdy < \infty.
\]
The space of all \( p \)-integrable Beltrami coefficients on \( \mathbb{D} \) is denoted by \( \text{Ael}^p(\mathbb{D}) \).

We find an appropriate subspace of \( T \) including \( T_0^\alpha \) where \( \text{Diff}_+^{1+\alpha}(S) \subset \text{QS} \) acts. The following Teichmüller spaces are studied by Cui [9], Guo [15], Shen [32], Takhtajan and Teo [33], Tang [34] and Yanagishita [35] among others.

**Definition.** A quasisymmetric homeomorphism \( g : S \rightarrow S \) belongs to \( \text{Sym}^p \) for \( p \geq 1 \) if \( g \) has a quasiconformal extension \( \tilde{g} : \mathbb{D} \rightarrow \mathbb{D} \) whose complex dilatation \( \mu_{\tilde{g}} \) belongs to \( \text{Ael}^p(\mathbb{D}) \). The \( p \)-integrable Teichmüller space \( T^p \) is defined by
\[
T^p = \pi(\text{Ael}^p(\mathbb{D})) = \text{Möb}(S) \setminus \text{Sym}^p \subset T.
\]
The topology on \( T^p \) is induced by \( \| \cdot \|_p + \| \cdot \|_\infty \) on \( \text{Ael}^p(\mathbb{D}) \).
We also consider the space of all $p$-integrable holomorphic functions on $\mathbb{D}^*$:

$$A^p(\mathbb{D}^*) = \{ \varphi \in \text{Hol}(\mathbb{D}^*) \mid \| \varphi \|_p^p = \int_{\mathbb{D}^*} \rho_{D^*}^{2-2p}(z)|\varphi(z)|^p dxdy < \infty \}.$$ 

Concerning the inclusion relation between $A^p(\mathbb{D}^*)$ and $B(\mathbb{D}^*)$, the following results are known. See [9, Lemma 1] and [15, Lemma 2] for example.

**Proposition 5.1.** For every $\varphi \in A^p(\mathbb{D}^*)$, it holds that $\| \varphi \|_\infty \leq c_p \| \varphi \|_p$ for every $p \geq 1$, where $c_p = (2p - 1)/(4\pi)$. In particular $A^p(\mathbb{D}^*) \subset B(\mathbb{D}^*)$.

Moreover, the inclusion of $A^p(\mathbb{D}^*)$ between $B_0^0(\mathbb{D}^*)$ and $B_0(\mathbb{D}^*)$ is also known (cf. [9], [15]).

**Proposition 5.2.** In general $A^p(\mathbb{D}^*) \subset B_0(\mathbb{D}^*)$ for $p \geq 1$. If $p\alpha > 1$ then $B_0^\alpha(\mathbb{D}^*) \subset A^p(\mathbb{D}^*)$.

**Proof.** We use a fact that Laurent polynomials at $\infty$ are dense in $A^p(\mathbb{D}^*)$. For every $\varphi \in A^p(\mathbb{D}^*)$, we choose a sequence of Laurent polynomials $\{ \varphi_n \}$ that converges to $\varphi$. Then by Proposition 5.1

$$\| \varphi - \varphi_n \|_\infty \leq c_p \| \varphi - \varphi_n \|_p \to 0 \quad (n \to \infty).$$

Since $\varphi_n \in B_0(\mathbb{D}^*)$ and $B_0(\mathbb{D}^*)$ is a closed subspace of $B(\mathbb{D}^*)$, we have $\varphi \in B_0(\mathbb{D}^*)$.

Every $\varphi \in B_0^\alpha(\mathbb{D}^*)$ satisfies $|\varphi(z)| \leq \rho_{D^*}^{2-\alpha}(z)\| \varphi \|_{\infty, \alpha}$ by definition. Then

$$\| \varphi \|_p^p = \int_{\mathbb{D}^*} \rho_{D^*}^{2-2p}(z)|\varphi(z)|^p dxdy \leq \left( \int_{\mathbb{D}^*} \rho_{D^*}^{2-p\alpha}(z) dxdy \right) \| \varphi \|_{\infty, \alpha}^p.$$ 

The last term is integrable if $2 - p\alpha < 1$, which implies that $\varphi \in A^p(\mathbb{D}^*)$ if $p\alpha > 1$. \qed

It was proved in [9] Theorem 2 and [15] Theorem 2] that the Bers embedding $\beta$ of $T^p$ is a homeomorphism onto the image and satisfies

$$\beta(T^p) = \beta(T) \cap A^p(\mathbb{D}^*)$$

for $p \geq 2$. This in particular implies that $T^p \subset T_0$ and hence $\text{Sym}^p \subset \text{Sym}$. Moreover, $\beta : T_0^\alpha \to B_0^\alpha(\mathbb{D}^*)$ is a homeomorphism onto the image and satisfies

$$\beta(T_0^\alpha) = \beta(T) \cap B_0^\alpha(\mathbb{D}^*),$$

which were shown in [26] Theorem 7.1. If $p\alpha > 1$, then $T_0^\alpha \subset T^p$ and hence $\text{Diff}^{1+\alpha}(\mathbb{S}) \subset \text{Sym}^p$.

For every point $\tau = [f] \in T$, the base point change automorphism $R_\tau : T \to T$ is defined by $[g] \mapsto [g \circ f^{-1}]$ as before. It is known that if $\tau \in T^p$ ($p \geq 2$) then $R_\tau$ preserves $T^p$ (see [9] Theorem 4], [33] Lemma 3.4, [35] Proposition 5.1]). This can be alternatively expressed as a condition $\pi(r_\nu(\mu)) \in T^p$ for any $\mu$ and $\nu$ in $\text{Ael}^p(\mathbb{D})$. The canonical coordinate of $T^p$ at each $\tau \in T^p$ as the Banach manifold is given by

$$\beta_\tau = \beta \circ R_\tau : T^p \to \beta(T) \cap A^p(\mathbb{D}^*).$$
Definition. The $p$-Weil-Petersson metric $d^p_{WP}$ on $T^p$ ($p \geq 2$) is a Finsler metric induced by the norm $\| \cdot \|_p$ on the tangent space $T\tau(T^p)$ at each $\tau \in T^p$, which is identified with $A^p(D^*)$ by the canonical coordinate $\beta_\tau$. The distance induced by this metric is also denoted by $d^p_{WP}(\cdot,\cdot)$.

Remark. The above metric can be alternatively defined by using the operator norm of $\varphi \in A^p(D^*)$ acting on $A^q(D^*)$ with $1/p + 1/q = 1$. This is done in [24, Definition 6.5]. However, the ratio of the two norms is bounded from above and below (see Kra [18, p.90] for example), and hence there is no much difference.

The topology defined by $d^p_{WP}$ coincides with the original topology on $T^p$. Continuity of the $p$-Weil-Petersson metric $d^p_{WP}$ on $T^p$ will be shown in Theorem 6.9 later. From the definition, we see that $d^p_{WP}$ is invariant under $R_\tau$ for every $\tau \in T^p$. In particular,

$$d^p_{WP}(\pi \circ r_\mu(\mu_1), \pi \circ r_\mu(\mu_2)) = d^p_{WP}(\pi(\mu_1), \pi(\mu_2))$$

for any $\mu_1, \mu_2, \nu \in Ael^p(D)$. We can also mention that $Sym^p \subset QS$ acts on $(T^p, d^p_{WP})$ isometrically.

For $p = 2$, Cui [9, Theorems 5, 6] proved that $(T^2, d^2_{WP})$ is complete and contractible. Takhtajan and Teo [33, Theorem 7.14] proved later that $(T^2, d^2_{WP})$ is negatively curved. From these properties, we see that $(T^2, d^2_{WP})$ is a CAT(0) space. Indeed, by the negatively curved condition, the exponential map at any point is surjective, which implies that there is a unique geodesic connecting any two points (see Lang [16, Chapter IX, Section 3]). Then we see that every geodesic triangle satisfies the CAT(0) property ([16, Chapter IX, Section 4]).

Now we state our result on the conjugation problem as follows.

**Theorem 5.3.** Let $G$ be an infinite non-abelian subgroup of $Diff^{1+\alpha}_1(S)$ with $\alpha \in (1/2, 1)$. Then the following conditions are equivalent:

1. There exists some $f \in Diff^{1+\alpha}_1(S)$ such that $f^{-1}Gf \subset Möb(S)$;
2. There exist positive constants $\kappa_2 < \infty$ and $\kappa_\infty < 1$ such that

$$\inf_{\pi(\mu)=g} \|\mu\|_2 \leq \kappa_2; \quad \inf_{\pi(\mu)=g} \|\mu\|_\infty \leq \kappa_\infty$$

for all $g \in G$;
3. The orbit $G(o)$ of the base point $o = [id]$ is bounded in $T^2$ with respect to $d^2_{WP}$.

Remark. We can replace the above infima of the norms of the complex dilatations with the complex dilatation $\mu_\tilde{g}$ of the barycentric extension $\tilde{g} \in QC(D)$ of $g$ introduced by Douady and Earle [10]. For condition (b), we can find this fact in [10, Proposition 7], and for condition (a), we find in Cui [9, Theorem 1]. A subgroup $G \subset QS$ satisfying condition (b) is called uniformly quasisymmetric.

For the proof of Theorem 5.3 we use the following theorem, which can be obtained for $p \geq 2$ in general. This result has its own interest because it asserts a certain kind of metrically equivalent condition between the $p$-Weil-Petersson distance $d^p_{WP}$ and the norm
\[ \| \cdot \|_p \] that defines the topology on \( T^p \). The inverse inequality will be also given later in Proposition 6.8.

**Theorem 5.4.** For every \( \mu \in \text{Ael}^p(\mathbb{D}) \), the \( p \)-Weil-Petersson distance satisfies

\[ d_{WP}^p([0], [\mu]) \leq C \| \mu \|_p, \]

where \( C > 0 \) is a constant depending only on \( \| \mu \|_{\infty} \).

The proof of this theorem will be given in Section 7 by dividing the arguments into several steps. Assuming Theorem 5.4 for the moment, we can prove Theorem 5.3.

**Proof of Theorem 5.3** : (2) \( \Rightarrow \) (3) : By Theorem 5.4, uniform boundedness conditions (a) and (b) imply that \( G \subset \text{Diff}^{1+\alpha}_+(S) \subset \text{Sym}^2 \) has a bounded orbit in \( T^2 \).

(3) \( \Rightarrow \) (1) : By a property of CAT(0) space, if the orbit of a subgroup \( G \subset \text{Sym}^2 \) in \( T^2 \) is bounded with respect to \( d_{WP}^2 \), then \( G \) has a fixed point \([f]\) in \( T^2 \). Actually, any bounded subset has a unique circumcenter by the NC-inequality for CAT(0) spaces (see Ballmann [4, Section 5]) and the unique circumcenter of the orbit of an isometric action is clearly a fixed point. This implies that there exists a symmetric homeomorphism \( f \in \text{Sym}^2 \subset \text{Sym}^2 \) such that \( f - 1 \subset \text{Möb}(S) \). Then, by applying Theorem 4.1, we see that \( f \) satisfies conditions (a) and (b).

Actually the inequality for \( \| \cdot \|_{\infty} \) is well-known. The inequality for \( \| \cdot \|_2 \) can be found in [33, Lemma 2.7] and [35, Proposition 5.1]. Here we prove this below for the sake of convenience.

Let \( \mu \in \text{Bel}_0^p(\mathbb{D}) \subset \text{Ael}^2(\mathbb{D}) \) be the complex dilatation of the barycentric extension \( \tilde{f} \) of \( f \in \text{Diff}^{1+\alpha}_+(S) \). For each \( g \in G \), we set \( \gamma = f^{-1} g f \in \text{Möb}(S) \) and regard it also as an element of \( \text{Möb}(\mathbb{D}) \). Then \( \tilde{f} \gamma \tilde{f}^{-1} \) is a quasiconformal extension of \( g \in G \). We will show that the 2-norm of the complex dilatation of this quasiconformal homeomorphism is uniformly bounded. The complex dilatation of \( \tilde{f} \gamma \) is denoted by \( \gamma^* \mu \). Then the complex dilatation of \( \tilde{f} \gamma \tilde{f}^{-1} \) is \( r_{\mu}(\gamma^* \mu) \).

We apply the formula in the proof of Proposition 3.5 for \( \nu = \mu, \mu_1 = \gamma^* \mu \) and \( \mu_2 = \mu \). Then we have

\[ |r_{\mu}(\gamma^* \mu)(\zeta)| \leq \frac{|\gamma^* \mu(z) - \mu(z)|}{\sqrt{1 - |\gamma^* \mu(z)|^2}(1 - |\mu(z)|^2)} \leq \frac{|\gamma^* \mu(z) - \mu(z)|}{1 - \|\mu\|^2_{\infty}} \]

for \( \zeta = \tilde{f}(z) \). Here we note that the Jacobian \( J_f \) of \( \tilde{f} \) satisfies

\[ \rho_0^2(\tilde{f}(z))J_f(z) \leq C \rho_0^2(z) \]
for some constant $C > 0$ depending only on $\|\mu\|_{\infty}$. See [10, Theorem 2] and [35, Proposition 2.1]. Therefore,

$$\|r_{\mu}(\gamma^* \mu)\|_2 = \left( \int_D |r_{\mu}(\gamma^* \mu)(\zeta)|^2 \rho_D^2(\zeta) d\xi d\eta \right)^{1/2} \leq \frac{C^{1/2}}{1 - \|\mu\|_{\infty}^2} \left( \int_D |\gamma^* \mu(z) - \mu(z)|^2 \rho_D^2(z) dxdy \right)^{1/2} \leq \frac{C^{1/2}}{1 - \|\mu\|_{\infty}^2}(\|\gamma^* \mu\|_2 + \|\mu\|_2).$$

Since $\|\gamma^* \mu\|_2 = \|\mu\|_2$ by $\rho_D^2(\gamma(z))J_{\gamma}(z) = \rho_D^2(z)$, we see that $\|r_{\mu}(\gamma^* \mu)\|_2$ is uniformly bounded. $\square$

Theorem 5.3 can be generalized to some extend for an arbitrary $\alpha \in (0, 1)$ as follows. The proof will be given in Section 7. Here we define

$$k_p(g) = \inf_{\pi(\mu) = g} \left( \int_D \left( \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \rho_D^2(z) dxdy \right)^{p/2} \right)^{1/p}$$

for every $g \in \text{Sym}^p$ with $p \geq 2$.

**Theorem 5.5.** If an infinite non-abelian subgroup $G \subset \text{Diff}^{1+\alpha}_+(\mathbb{S})$ for any $\alpha \in (0, 1)$ satisfies $k_p(g) \leq \varepsilon_p$ for all $g \in G$ and for a sufficiently small constant $\varepsilon_p > 0$ depending only on $p$ with $p \alpha > 1$, then there exists $f \in \text{Diff}^{1+\alpha}_+(\mathbb{S})$ such that $f^{-1}Gf \subset \text{M"ob}(\mathbb{S})$.

Finally in this section, we record another consequence from Theorem 4.1. In the conjugation problem of Navas [28], uniform integrability of the Liouville cocycle are assumed for the group $G$ of circle diffeomorphisms. Here for an orientation-preserving absolutely continuous self-homeomorphism $g$ of $\mathbb{S}$,

$$c(g)(x, y) = \frac{g'(x)g'(y)}{4 \sin^2((g(x) - g(y))/2) - 1} - \frac{1}{4 \sin^2((x - y)/2)}$$

is called the Liouville cocycle. We consider its integrable norm

$$\|c(g)\|_1 = \int_{\mathbb{S} \times \mathbb{S}} |c(g)(x, y)| dxdy.$$

**Proposition 5.6.** Let $\Gamma$ be a subgroup of $\text{M"ob}(\mathbb{S})$ that contains a hyperbolic element. If $f^{-1}f^{1+\alpha}_+(\mathbb{S})$ for an orientation-preserving absolutely continuous self-homeomorphism $f$ of $\mathbb{S}$ with $\|c(f)\|_1 < \infty$, then $f \in \text{Diff}^{1+\alpha}_+(\mathbb{S})$. Moreover, $f$ belongs to $\text{Sym}^p$ for any $p > 1$.

**Proof.** Since $\|c(f)\|_1 < \infty$, we see that $f$ belongs to $\text{Sym}$ by [21, Theorem 5.1]. Hence, Theorem 4.1 shows that $f \in \text{Diff}^{1+\alpha}_+(\mathbb{S})$. Then $f$ also belongs to $\text{Sym}^p$ for any $p > 1$ by [21, Corollary 5.4]. $\square$
6. Properties of the $p$-Weil-Petersson metric

We will show certain comparison between the $p$-Weil-Petersson metric $d_{WP}^p$ on $T^p$ and the $p$-integrable norm on $A^p(D^*)$. This verifies basic properties of the $p$-Weil-Petersson metric such as continuity of the metric $d_{WP}^p$ as the base point varies and completeness of the distance $d_{WP}^p(\cdot, \cdot)$ induced by the metric. Some of these results are used in the proof of Theorem 5.4 in the next section but also have their own interest.

The following norm estimate of the Bers projection $\Phi : \text{Ael}^p(D) \to A^p(D^*)$ at the origin is crucial, which was essentially given by Cui [9, Theorem 2] for $p = 2$ and generalized to $p \geq 2$ by Guo [15, Theorem 2].

Lemma 6.1. The Bers projection $\Phi$ satisfies

$$\|\Phi(\mu)\|_p \leq 3 \left( \int_{D} \left( \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \right)^{p/2} \rho_{D^*}^2(z) dxdy \right)^{1/p} \leq 3 \sqrt{1 - \|\mu\|_\infty^2} \|\mu\|_p$$

for every $\mu \in \text{Ael}^p(D)$.

Proof. We put $\nu = 0$ for Proposition 3.2. Then

$$|S_{\mu|D^*}(\zeta)| \leq \frac{6 \rho_{D^*}(\zeta)}{\sqrt{\pi}} \left( \int_{D} \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \frac{dxdy}{|z - \zeta|^4} \right)^{1/2}.$$

Here, applying the Hölder inequality for the integral, we have

$$\int_{D} \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \frac{dxdy}{|z - \zeta|^4} \leq \left( \int_{D} \left( \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \right)^{p/2} \frac{dxdy}{|z - \zeta|^4} \right)^{2/p} \left( \int_{D} dxdy \right)^{1 - 2/p}.$$

Moreover, it is know that

$$\int_{D} dxdy = \frac{\pi}{4} \rho_{D^*}^2(\zeta); \quad \int_{D^*} \frac{d\xi d\eta}{|z - \zeta|^4} = \frac{\pi}{4} \rho_{D}^2(z).$$

This shows that

$$\int_{D^*} \rho_{D^*}^{2-2p}(\zeta)|S_{\mu|D^*}(\zeta)|^p d\xi d\eta \leq \left( \frac{6}{\sqrt{\pi}} \right)^p \left( \int_{D} \left( \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \right)^{p/2} \frac{dxdy}{|z - \zeta|^4} \right) \left( \frac{\pi}{4} \rho_{D^*}^2(\zeta) \right)^{p/2 - 1} d\xi d\eta$$

$$= \frac{4 \cdot 3^p}{\pi} \int_{D} \left( \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \right)^{p/2} \left( \int_{D^*} \frac{d\xi d\eta}{|z - \zeta|^4} \right) dxdy$$

$$= 3^p \int_{D} \left( \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} \right)^{p/2} \rho_{D}^2(z) dxdy,$$
which implies the required estimate. \hfill \Box

**Remark.** The derivative of \( \Phi \) at 0 in the direction of \( \mu \in \mathrm{Bel}(\mathbb{D}) \) can be represented by

\[
    d_0 \Phi(\mu)(z) = -\frac{6}{\pi} \int_{\mathbb{D}} \frac{\Phi(\zeta)}{|\zeta|^2 - z} d\zeta d\eta \quad (z \in \mathbb{D}^*).
\]

See for instance [27, Section 3.4.5]. This defines a bounded linear operator \( d_0 \Phi \) from the tangent space of \( \mathrm{Ael}^p(\mathbb{D}) \) to \( A^p(\mathbb{D}^*) \). Lemma [6,1] implies that its operator norm satisfies \( \|d_0 \Phi\| \leq 3 \). The derivative \( d_0 \Phi(\mu) \) can be alternatively represented by using the Bergman kernel

\[
    K_D^*(z, \zeta) = \frac{12}{\pi (\bar{z} \zeta - 1)^4}
\]

on \( \mathbb{D}^* \); the above formula turns out to be the Bergman projection

\[
    d_0 \Phi(\mu)(z) = -\frac{1}{2} \int_{\mathbb{D}^*} (\zeta \zeta^*)^{-2} \mu(\zeta^*) K_D^*(z, \zeta) d\zeta d\eta \quad (z \in \mathbb{D}^*, \zeta^* = 1/\bar{\zeta} \in \mathbb{D}).
\]

Then by Kra [18, p.90], the estimate \( \|d_0 \Phi\| \leq 3 \) was already known.

A holomorphic local section of \( \Phi \) at the origin 0 \( \in B(\mathbb{D}^*) \) can be given explicitly by Ahlfors and Weill [2] (see also [17, Theorem II.5.1]). Note that the following form is the adaptation to the unit disk case.

**Theorem 6.2.** Let \( U^\infty(1/2) \) be the open ball of the Banach space \( B(\mathbb{D}^*) \) centered at the origin with radius 1/2. For every \( \varphi \in U^\infty(1/2) \), set

\[
    \sigma(\varphi)(z) = -2 \rho_D^2(z^*) (zz^*)^2 \varphi(z^*).
\]

Then \( \mu(z) = \sigma(\varphi)(z) \) belongs to \( \mathrm{Bel}(\mathbb{D}) \) and satisfies \( \Phi(\mu) = \varphi \). Here \( z^* = 1/\bar{z} \in \mathbb{D}^* \) is the reflection of \( z \in \mathbb{D} \) with respect to \( \mathbb{S} \). Hence \( \sigma: U^\infty(1/2) \to B(\mathbb{D}^*) \) is a holomorphic local section of \( \Phi \) around 0.

A quasiconformal self-homeomorphism of \( \mathbb{D} \) induced by the Ahlfors and Weill section is a diffeomorphism, its complex dilatation is infinitely differentiable and has a convenient property for its Jacobian. This was discovered by Takhtajan and Teo [33, Lemma 2.5], based on a fact that the partial derivative \( \partial f^\mu(z) \) at each point \( z \in \mathbb{D} \) converges to 1 as \( \mu \in \sigma(U^\infty(1/2)) \) converges to 0 within the Ahlfors and Weill section. See Bers [7, p.97].

**Lemma 6.3.** For every \( \varepsilon > 0 \), there exists \( \delta \in (0, 1/2) \) such that the quasiconformal homeomorphism \( f^\mu \) of \( \mathbb{D} \) having the complex dilatation \( \mu = \sigma(\varphi) \) or \( \sigma(\varphi)^{-1} \) for \( \varphi \in U^\infty(\delta) \) satisfies

\[
    |\rho_D^2(f^\mu(z))| \partial f^\mu(z)|^2 - \rho_D^2(z)| \leq \varepsilon \rho_D^2(z)
\]

for every \( z \in \mathbb{D} \). In particular, there exists \( \delta_0 \in (0, 1/2) \) such that \( f^\mu \) with \( \mu \) or \( \mu^{-1} \in \sigma(U^\infty(\delta_0)) \) satisfies

\[
    \rho_D^2(f^\mu(z)) J_{f^\mu}(z) \leq 2(1 - |\mu(z)|^2) \rho_D^2(z),
\]

where \( J_{f^\mu}(z) = |\partial f^\mu(z)|^2 - |\partial f^\mu(z)|^2 \) is the Jacobian of \( f^\mu(z) \).
This result in particular implies that the Jacobian with respect to the hyperbolic metric is estimated as
\[ \rho_D^2(w) dudv \leq 2\rho_D^2(z) dxdy \]
for \( w = f^\mu(z) \) with \( \mu \) or \( \mu^{-1} \in \sigma(U^\infty(\delta_0)) \). Hereafter, we choose \( \delta_0 \) not greater than 1/4 as in the above lemma and fix it.

The generalization of Lemma 6.4 can be also obtained. For \( p = 2 \), this is essentially given by [33, Lemma 2.9].

**Lemma 6.4.** Let \( \mu \in \text{Ael}^p(\mathbb{D}) \) be arbitrary and let \( \nu \) or \( \nu^{-1} \in \text{Ael}^p(\mathbb{D}) \) be in \( \sigma(U^\infty(\delta_0)) \). Then
\[
\|\Phi(\mu) - \Phi(\nu)\|_p \leq 24 \left( \int_{\Sigma} \left( \frac{|\mu(z) - \nu(z)|^2}{(1 - |\mu(z)|^2)(1 - |\nu(z)|^2)} \right)^{p/2} \rho_D^2(z) dxdy \right)^{1/p} \leq \frac{24}{\sqrt{(1 - \|\mu\|_\infty^2)(1 - \|\nu\|_\infty^2)}} \|\mu - \nu\|_p.
\]

**Proof.** Set \( \Omega = f_\nu(\mathbb{D}) \) and \( \Omega^* = f_\nu(\mathbb{D}^*) \). Applying the H"older inequality to the integral appearing Proposition 3.3, we have
\[
\int_\Omega \frac{|\mu(f_\nu^{-1}(w)) - \nu(f_\nu^{-1}(w))|^2}{(1 - |\mu(f_\nu^{-1}(w))|^2)(1 - |\nu(f_\nu^{-1}(w))|^2)} \frac{dudv}{|w - \zeta|^4} \leq \left( \int_\Omega \left( \frac{|\mu(f_\nu^{-1}(w)) - \nu(f_\nu^{-1}(w))|^2}{(1 - |\mu(f_\nu^{-1}(w))|^2)(1 - |\nu(f_\nu^{-1}(w))|^2)} \right)^{p/2} \frac{dudv}{|w - \zeta|^4} \right)^{2/p} \left( \int_\Omega \frac{dudv}{|w - \zeta|^4} \right)^{1-2/p}.
\]
Here we have the following inequalities in this case by the same arguments in the proof of Lemma 3.3
\[
\int_\Omega \frac{dudv}{|w - \zeta|^4} \leq 4\pi \rho_D^2(\zeta); \quad \int_{\Omega^*} \frac{d\xi d\eta}{|w - \zeta|^4} \leq 4\pi \rho_D^2(w).
\]
This shows that
\[
\int_{\Omega^*} \rho_D^{-2p}(\zeta) |S_{f_\nu f_\nu^{-1}|\Omega^*}(\zeta)|^p d\xi d\eta \leq \left( \frac{6}{\sqrt{\pi}} \right)^p \int_{\Omega^*} \rho_D^{-2p}(\zeta) \left[ \int_\Omega \left( \frac{|\mu(f_\nu^{-1}(w)) - \nu(f_\nu^{-1}(w))|^2}{(1 - |\mu(f_\nu^{-1}(w))|^2)(1 - |\nu(f_\nu^{-1}(w))|^2)} \right)^{p/2} \frac{dudv}{|w - \zeta|^4} \right] d\xi d\eta
\]
\[
\times \left( 4\pi \rho_D^2(\zeta) \right)^{p/2-1} d\xi d\eta
\]
\[
= \frac{12^p}{4\pi} \int_\Omega \left( \frac{|\mu(f_\nu^{-1}(w)) - \nu(f_\nu^{-1}(w))|^2}{(1 - |\mu(f_\nu^{-1}(w))|^2)(1 - |\nu(f_\nu^{-1}(w))|^2)} \right)^{p/2} \left[ \int_{\Omega^*} \frac{d\xi d\eta}{|w - \zeta|^4} \right] dudv
\]
\[
\leq 12^p \int_\Omega \left( \frac{|\mu(f_\nu^{-1}(w)) - \nu(f_\nu^{-1}(w))|^2}{(1 - |\mu(f_\nu^{-1}(w))|^2)(1 - |\nu(f_\nu^{-1}(w))|^2)} \right)^{p/2} \rho_D^2(w) dudv.
\]
derivatives, the first term in the above inequality equals to
\[
\int_{D^*} \rho_{D^*}^{2-2p}(z)|S_{f_*\nu}(z) - S_{f_\nu}(z)|^p dxdy = \frac{1}{p} \|\Phi(\mu) - \Phi(\nu)\|_p^p.
\]

On the other hand, \(f_* : \mathbb{D} \to \Omega\) can be given by the composition of the quasiconformal self-homeomorphism \(f^\nu\) of \(\mathbb{D}\) and a conformal homeomorphism \(\mathbb{D} \to \Omega\). By Lemma 5.3, the Jacobian of \(f^\nu\) satisfies \(\rho_D^2(f^\nu(z))J_{f^\nu}(z) \leq 2\rho_D^2(z)\). Hence, the change of variable \(z = f_*^{-1}(w)\) for \(w \in \Omega\) is applied under
\[
\rho_D^2(w)dudv \leq 2\rho_D^2(z)dxdy.
\]
Thus the last term in the above inequality is bounded by
\[
2 \cdot 12^p \int_{D} \left( \frac{\|\mu(z) - \nu(z)\|^2}{(1 - |\mu(z)|^2)(1 - |\nu(z)|^2)} \right)^{p/2} \rho_D^2(z) dxdy.
\]
By taking the \(p\)-th root, we obtain the desired inequality. \(\square\)

**Remark.** A similar argument to the above proof can be found in Yanagishita [35, Proposition 3.2]. In his paper, instead of taking \(\nu\) from \(\sigma(U^\infty(\delta_0))\), he assumes that \(\nu\) is obtained by the barycentric extension. Then, by the estimate of the Jacobian with respect to the hyperbolic metric as in the proof of Theorem 5.3, we can show a similar result to Lemma 6.4 also under this assumption on \(\nu\). This is used to prove the continuity of \(\Phi\). See also Cui [9] and Tang [33].

We first compare the \(p\)-Weil-Petersson distance \(d_{WP}^p\) on \(T^p\) with the \(p\)-integrable norm \(\|\cdot\|_p\) in a small open ball of \(A^p(\mathbb{D}^*)\) centered at the origin. Let \(U^p(r) \subset A^p(\mathbb{D}^*)\) and \(U^\infty(r) \subset B(\mathbb{D}^*)\) denote the open balls of radius \(r\) centered at the origin. Set \(\delta_p = \delta_0/c_p\) where \(c_p\) is the constant satisfying the condition \(\|\sigma\|_\infty \leq c_p\|\varphi\|_p\) as in Proposition 5.1 and \(\delta_0 \leq 1/4\) is the constant as in Lemma 6.3. Then \(U^p(\delta_p) \subset U^\infty(\delta_0) \subset \beta(T)\). Note that \(U^\infty(\delta_0)\) is properly contained in the domain \(U^\infty(1/2)\) of the Ahlfors-Weil section \(\sigma : U^\infty(1/2) \to \text{Bel}(\mathbb{D})\) and hence the Teichmüller distance \(d_T\) on \(T\) and the supremum norm \(\|\cdot\|_\infty\) of \(B(\mathbb{D}^*)\) are comparable there (see Lehto [17, Section III.4.2]).

For the base point change map \(R_\tau\) for \(\tau \in T^p\), which is a biholomorphic and isometric automorphism of \((T^p, d_{WP}^p)\) sending \(\tau\) to \(o = [\text{id}]\), we consider its conjugate by the Bers embedding \(\beta : T^p \to \beta(T) \cap A^p(\mathbb{D}^*)\):
\[
R_\beta^\varphi = \beta \circ R_\tau \circ \beta^{-1}.
\]
For each \(\varphi \in \beta(T) \cap A^p(\mathbb{D}^*)\), \(R_\beta^\varphi\) is a biholomorphic automorphism of \(\beta(T) \cap A^p(\mathbb{D}^*)\) sending \(\varphi\) to 0. The derivative \(d_\varphi R_\beta^\varphi : A^p(\mathbb{D}^*) \to A^p(\mathbb{D}^*)\) at any \(\psi \in \beta(T) \cap A^p(\mathbb{D}^*)\) is a bounded linear operator. The following result and its corollaries are similarly obtained by Cui [9, Theorem 4] for \(p = 2\).

**Theorem 6.5.** The operator norm of the derivative \(d_\varphi R_\beta^\varphi\) at \(\varphi\) and that of \(d_0(R_\beta^\varphi)^{-1}\) at 0 satisfy \(\|d_\varphi R_\beta^\varphi\| \leq 16\) and \(\|d_0(R_\beta^\varphi)^{-1}\| \leq 128\) for every \(\varphi \in U^\infty(\delta_0) \cap A^p(\mathbb{D}^*)\).
Proof. First, we consider \( R^*_\varphi \) and estimate of the norm of its derivative at \( \varphi \) from above. We decompose \( R^*_\varphi \) into \( R^*_\varphi = \Phi \circ r_\nu \circ \sigma \), where \( \nu = \sigma(\varphi) \in \text{Ael}^p(\mathbb{D}) \). Then

\[
d_\varphi R^*_\varphi = d_0 \Phi \circ d_\nu r_\nu \circ d_\varphi \sigma.
\]

Here the Ahlfors-Weill section \( \sigma \) is linear with \( \|d\sigma\| = 2 \) and the remark after Lemma 6.1 says that \( \|d_0 \Phi\| \leq 3 \). On the other hand, since

\[
(d_\nu r_\nu)(\lambda)(w) = \frac{\lambda(z)}{1 - |\nu(z)|^2} \frac{\partial f^\nu(z)}{\partial f^\nu(w)} \quad (w = f^\nu(z))
\]

for a tangent vector \( \lambda \) of \( \text{Ael}^p(\mathbb{D}) \) at \( \nu \), we see that

\[
\|d_\nu r_\nu\|_p \leq \frac{1}{1 - \|\nu\|^2_\infty} \left( \int_D |\lambda((f^\nu)^{-1}(w))|^p \rho_\nu^2(w)dudv \right)^{1/p}
\]

\[
= \frac{1}{1 - \|\nu\|^2_\infty} \left( \int_D |\lambda(z)|^p \rho_\nu^2(f^\nu(z))J_{f^\nu}(z)dxdy \right)^{1/p}
\]

\[
\leq \frac{2^{1/p}}{1 - (2\delta_0)^2} \left( \int_D |\lambda(z)|^p \rho_\nu^2(z)dxdy \right)^{1/p},
\]

where the last inequality comes from Lemma 6.3 and \( \|\nu\|_\infty = 2\|\varphi\|_\infty \leq 2\delta_0 \). Since \( \delta_0 \leq 1/4 \), this shows that \( \|d_\nu r_\nu\| \leq 8/3 \). Consequently, we have

\[
\|d_\varphi R^*_\varphi\| \leq \|d_0 \Phi\| \cdot \|d_\nu r_\nu\| \cdot \|d_\varphi \sigma\| \leq 16.
\]

Next, we consider \((R^*_\varphi)^{-1}\) and estimate of the norm of its derivative at 0 from above. We decompose \((R^*_\varphi)^{-1}\) into \((R^*_\varphi)^{-1} = \Phi \circ r_{\nu^{-1}} \circ \sigma\). Then

\[
d_0(R^*_\varphi)^{-1} = d_\nu \Phi \circ d_0 r_{\nu^{-1}} \circ d_0 \sigma.
\]

As before \( \|d\sigma\| = 2 \). Lemma 6.4 implies that

\[
\|d_\varphi R^*_\varphi\|_p \leq \frac{24}{1 - \|\nu\|^2_\infty} \|\lambda\|_p
\]

for a tangent vector \( \lambda \) of \( \text{Ael}(\mathbb{D}) \) at \( \nu \). Hence \( \|d_\varphi \Phi\| \leq 24/(1 - \|\nu\|^2_\infty) \leq 32 \). For the derivative \( d_0 r_{\nu^{-1}} \), we know that

\[
d_0 r_{\nu^{-1}}(\lambda_\nu)(w) = \lambda_\nu(z)(1 - |\nu^{-1}(z)|^2) \frac{\partial((f^\nu)^{-1})(z)}{\partial((f^\nu)^{-1})(z)}
\]

\[
= \lambda_\nu(f^\nu(w))(1 - |\nu(w)|^2) \frac{\partial f^\nu(w)}{\partial f^\nu(w)} \quad (w = (f^\nu)^{-1}(z))
\]
for a tangent vector $\lambda_*$ of $Ael^p(\mathbb{D})$ at $\nu^{-1}$. Then

$$
\| (d_0r_{\nu^{-1}})(\lambda_*) \|_p \leq \left( \int_{\mathbb{D}} |\lambda_*(f^\nu(w))|^p \rho^2_D(w) dudv \right)^{1/p} = \left( \int_{\mathbb{D}} |\lambda_*(z)|^p \rho^2_D(f^\nu(z)) J_{f^\nu}(z) dxdy \right)^{1/p} \leq 2^{1/p} \left( \int_{\mathbb{D}} |\lambda_*(z)|^p \rho^2_D(z) dxdy \right)^{1/p}.
$$

Hence $\| d_0r_{\nu^{-1}} \| \leq 2$. These estimates together conclude

$$
\| d_0(R_{\varphi^*})^{-1} \| \leq \| d_0 \Phi \| \cdot \| d_0r_{\nu^{-1}} \| \cdot \| d_0 \sigma \| \leq 128,
$$

which completes the proof.

\begin{corollary}
Any $\varphi_0, \varphi_1 \in U^p(\delta_p/3) \subset U^\infty(\delta_0) \cap A^p(\mathbb{D}^*)$ satisfy

$$
\frac{1}{128} \| \varphi_1 - \varphi_0 \|_p \leq d_{WP}^p(\beta^{-1}(\varphi_1), \beta^{-1}(\varphi_0)) \leq 16 \| \varphi_1 - \varphi_0 \|_p.
$$

The upper estimate is still valid for any $\varphi_0, \varphi_1 \in U^\infty(\delta_0) \cap A^p(\mathbb{D}^*)$.
\end{corollary}

\begin{proof}
For the upper estimate, we choose the segment $\gamma_0 = \{ t\varphi_1 + (1-t)\varphi_0 \}_{t \in [0,1]}$ in $U^\infty(\delta_0) \cap A^p(\mathbb{D}^*)$ connecting $\varphi_0$ and $\varphi_1$. Then the $p$-Weil-Petersson length $\ell_{WP}^p(\gamma_0)$ of $\gamma_0$ is given by

$$
\ell_{WP}^p(\gamma_0) = \int_0^1 \| (dt_{t\varphi_1+(1-t)\varphi_0} R_{t\varphi_1+(1-t)\varphi_0}^*)(\varphi_1 - \varphi_0) \|_p dt.
$$

Since Theorem 6.5 implies that

$$
\| (dt_{t\varphi_1+(1-t)\varphi_0} R_{t\varphi_1+(1-t)\varphi_0}^{*}) \|_p \leq 16 \| \varphi_1 - \varphi_0 \|_p,
$$

we see that $\ell_{WP}^p(\gamma_0) \leq 16 \| \varphi_1 - \varphi_0 \|_p$. This shows the upper estimate.

For the lower estimate, we choose a smooth path $\gamma$ in $\beta(T^p)$ connecting $\varphi_0$ and $\varphi_1$ whose length $\ell_{WP}^p(\gamma)$ is arbitrarily close to $d_{WP}^p(\beta^{-1}(\varphi_0), \beta^{-1}(\varphi_1))$. We give a length parameter $s$ for $\gamma$ with respect to the norm $\| \cdot \|_p$; its parametrization is $\{ \gamma(s) \}_{s \in [0,S]}$ for $S \geq \| \varphi_1 - \varphi_0 \|_p$, where $\gamma(0) = \varphi_0$ and $\gamma(S) = \varphi_1$. We may assume that $\gamma$ is contained in $U^p(\delta_p)$, for otherwise, we replace $\gamma$ with a sub-path $\{ \gamma(s) \}_{s \in [0,S]}$ in $U^p(\delta_p)$ that still holds $S' \geq \| \varphi_1 - \varphi_0 \|_p$. This is possible because $\| \varphi_1 - \varphi_0 \|_p < 2\delta_p/3$ and $\| \varphi_0 \|_p < \delta_p/3$. Then

$$
\ell_{WP}^p(\gamma) = \int_0^S \| (d_{\gamma(s)} R_{\gamma(s)}^*)(\dot{\gamma}(s)) \|_p ds.
$$

Here Theorem 6.5 implies that the integrand is bounded from below by 1/128. Hence $\ell_{WP}^p(\gamma) \geq S/128 \geq \| \varphi_1 - \varphi_0 \|/128$. Since $\ell_{WP}^p(\gamma)$ can be arbitrarily close to the distance $d_{WP}^p(\beta^{-1}(\varphi_0), \beta^{-1}(\varphi_1))$, we obtain the lower estimate.
\end{proof}

\begin{corollary}
The $p$-Weil-Petersson metric $d_{WP}^p$ is complete on $T^p$.
\end{corollary}
Proof. Consider any Cauchy sequence in $(T^p,d^p_{WP})$. We have only to deal with its tail whose diameter can be arbitrary small. Since the isometric automorphism group acts transitively on $T^p$, we may assume that the Cauchy sequence is contained in $\beta^{-1}(U^p(\delta_p/3))$. Then by Corollary 6.6 its Bers embedding is a convergent sequence in $U^p(\delta_p/3)$ with respect to the norm $\| \cdot \|_p$, and hence the Cauchy sequence also converges with respect to $d^p_{WP}$.

By similar arguments as above, we also obtain the following result, which is a counterpart to Theorem 5.4. We note here that the Teichmüller distance whose diameter can be arbitrary small. Since the isometric automorphism group acts transitively on $T^p$, we may assume that the Cauchy sequence is contained in $U^p(\delta_p/3)$ with respect to the norm $\| \cdot \|_p$, and hence the Cauchy sequence also converges with respect to $d^p_{WP}$.

Proposition 6.8. For every $\tau \in T^p$, there is $\mu \in \text{AEL}^p(\mathbb{D})$ with $\tau = [\mu]$ such that

$$\|\mu\|_p \leq C d^p_{WP}([0],[\mu])$$

for a constant $C > 0$ depending only on $d^p_{WP}(o,\tau)$.

Proof. We choose a finite sequence of points $\{\tau_i\}_{i=0}^n \subset T^p$ so that $\tau_0 = o$, $\tau_n = \tau$ and $d^p_{WP}(\tau_{i-1},\tau_i) \leq \delta_p/384$ for $1 \leq i \leq n$. We can choose the number $n \geq 1$ uniformly proportional to $d^p_{WP}(o,\tau)$. For $\tau_1$ with $d^p_{WP}(o,\tau_1) \leq \delta_p/384$, we see that $\|\beta(\tau_1)\|_p \leq \delta_p/3$ by Corollary 6.6. We set $\mu_1 = \sigma(\beta(\tau_1)) \in \text{AEL}^p(\mathbb{D})$, which satisfies $\pi(\mu_1) = \tau_1$. Then $\|\mu_1\|_p \leq 2\delta_p/3 < \delta_p$ and $\|\mu_1\|_\infty \leq 2\delta_0/2$.

For $\tau_2$, we consider $\tau_2' = R_{\tau_1}(\tau_2) \in T^p$. Since $d^p_{WP}(o,\tau_2') = d^p_{WP}(\tau_1,\tau_2) \leq \delta_p/384$, we see that $\|\beta(\tau_2')\|_p \leq \delta_p/3$. We set $\mu_2' = \sigma(\beta(\tau_2')) \in \text{AEL}^p(\mathbb{D})$, which satisfies $\pi(\mu_2') = \tau_2'$, $\|\mu_2'\|_p \leq \delta_p$ and $\|\mu_2'\|_\infty \leq 1/2$. Then $\mu_2 = \mu_2' \ast \mu_1$ satisfies $\pi(\mu_2) = \tau_2$ and $\|\mu_2\|_\infty \leq 4/5$. Moreover, as in the proofs of Proposition 3.5 and Theorem 5.3, we have that

$$\|\mu_2 - \mu_1\|_p = \|r_{\mu_1}^{-1}(\mu_2') - r_{\mu_1}^{-1}(0)\|_p \leq \frac{c_1}{1 - \|\mu_1\|_\infty^2} \|\mu_2'\|_p \leq \frac{c_1}{1 - \|\mu_1\|_\infty^2} \delta_p,$$

where $c_1 > 0$ is a constant depending only on $\|\mu_1\|_\infty = \|\mu_1\|_\infty^{-1}$.

For $\tau_3, \ldots, \tau_n$, we repeat the same argument. Inductively, we define $\mu_i'$ and $\mu_i$ ($i = 3, \ldots, n$) similarly, which satisfy $\|\mu_i'\|_p \leq \delta_p$ and $\|\mu_i - 1\|_\infty \leq (3^{i-1} - 1)/(3^{i-1} + 1)$. Then we also have

$$\|\mu_i - \mu_{i-1}\|_p \leq \frac{c_{i-1}}{1 - \|\mu_{i-1}\|_\infty^2} \delta_p$$

for a constant $c_{i-1} > 0$ depending only on $\|\mu_{i-1}\|_\infty$.

Therefore, by summing up these estimates, we conclude that

$$\|\mu_n\|_p \leq \|\mu_1\|_p + \|\mu_2 - \mu_1\|_p + \cdots + \|\mu_n - \mu_{n-1}\|_p \leq C' \delta_p n$$

for $\mu_n \in \text{AEL}(\mathbb{D})$ with $\pi(\mu_n) = \tau_n = \tau$, where $C' > 0$ is a constant depending only on $n$. Since $n$ is proportional to $d^p_{WP}(o,\tau)$, this yields the required inequality.

Finally, in this section, we show the continuity of the metric, by which it will be accepted as a Finsler metric.

Theorem 6.9. The $p$-Weil-Petersson metric $d^p_{WP}$ is continuous on $T^p$. 
Proof. It suffices to show that for each $\psi \in A^p(\mathbb{D}^*)$, $\|d_\varphi R^*_\varphi(\psi)\|_p$ converge to $\|\psi\|_p$ as $\varphi$ tend to 0 in $A^p(\mathbb{D}^*)$. Since

$$d_\varphi R^*_\varphi = d_0\Phi \circ d_\nu r_\nu \circ d_\varphi \sigma$$

for $\nu = d_\sigma(\varphi)$, we have

$$d_\varphi R^*_\varphi(\psi)(z) = -\frac{\pi}{6} \int_{\mathbb{D}} (d_\nu r_\nu)(\lambda)(w) \frac{\rho^2_{\mathbb{D}^*}(z)}{(w-z)^4} dudv$$

for $\lambda = d_\sigma(\psi)$ by the remark after Lemma 6.1. Hence

$$\|d_\varphi R^*_\varphi(\psi)\|_p = \int_{\mathbb{D}^*} \left| \frac{\pi}{6} \int_{\mathbb{D}} (d_\nu r_\nu)(\lambda)(w) \frac{\rho^2_{\mathbb{D}^*}(z)}{(w-z)^4} dudv \right|^p \rho^2_{\mathbb{D}^*}(z) dxdy;$$

$$\|\psi\|_p^p = \int_{\mathbb{D}^*} \left| \frac{\pi}{6} \int_{\mathbb{D}} \frac{\lambda(w)}{(w-z)^4} dudv \right|^p \rho^2_{\mathbb{D}^*}(z) dxdy.$$ 

Here, as we have seen before,

$$(d_\nu r_\nu)(\lambda)(w) = \frac{\lambda(\zeta)}{1 - |\nu(\zeta)|^2} \frac{\partial f^\nu(\zeta)}{\partial f^\nu(\zeta)}$$

for $w = f^\nu(\zeta)$, which converge to $\lambda(\zeta)$ as $\varphi \to 0$. In particular, if we restrict the inner integrals in the above formulae to a smaller disk $|w| \leq r$ for any $r \in (0, 1)$, the dominated convergence theorem can be applied to that part to see the convergence of the $p$-norm.

We consider a uniform estimate of the $p$-integral of $(d_\nu r_\nu)(\lambda)(w)$ on $r < |w| < 1$. By change of variables $\zeta = f^\nu(w)$, we have

$$\|(d_\nu r_\nu)(\lambda)\|_p^p = \int_{\mathbb{D}^*} \left| \frac{\lambda(\zeta)}{1 - |\nu(\zeta)|^2} \right|^p \rho^2_{\mathbb{D}^*}(f^\nu(\zeta)) J_{f^\nu}(\zeta) d\zeta d\eta.$$ 

Lemma 6.3 implies that for every $\varepsilon > 0$, there is $\delta \in (0, 1/2)$ such that

$$\left| \frac{1}{(1 - |\nu(\zeta)|^2)^p} \lambda(\zeta)^p \rho^2_{\mathbb{D}^*}(f^\nu(\zeta)) J_{f^\nu}(\zeta) - |\lambda(\zeta)|^p \rho^2_{\mathbb{D}^*}(\zeta) \right| \leq \varepsilon$$

for every $\zeta \in \mathbb{D}$ and for every $\varphi \in U^\infty(\delta)$. Since $f^\nu$ converge to id uniformly on $\mathbb{D}$ as $\varphi \to 0$ and since $\lambda \in \text{Ael}^p(\mathbb{D})$, we see that for every $\tilde{\varepsilon} > 0$, there is some $r \in (0, 1)$ such that

$$\left( \int_{|w| < 1} |(d_\nu r_\nu)(\lambda)(w)|^p \rho^2_{\mathbb{D}^*}(w) dudv \right)^{1/p} \leq \tilde{\varepsilon}$$

for every $\varphi \in U^\infty(\delta)$ by replacing $\delta$ with a smaller constant if necessary.

By Lemma 6.1 supporting the fact $\|d_0\Phi\| \leq 3$, we obtain that

$$\left( \int_{\mathbb{D}^*} \left| \frac{\pi}{6} \int_{|w| < 1} (d_\nu r_\nu)(\lambda)(w) \frac{\rho^2_{\mathbb{D}^*}(z)}{(w-z)^4} dudv \right|^p \rho^2_{\mathbb{D}^*}(z) dxdy \right)^{1/p} \leq 3\tilde{\varepsilon};$$

$$\left( \int_{\mathbb{D}^*} \left| \frac{\pi}{6} \int_{|w| < 1} \frac{\lambda(w)}{(w-z)^4} dudv \right|^p \rho^2_{\mathbb{D}^*}(z) dxdy \right)^{1/p} \leq 3\tilde{\varepsilon}.$$
for every $\varphi \in U^\infty(\delta)$. Then by letting $\varphi \to 0$, the dominated convergence theorem concludes that

$$\|\psi\|_p - 6\tilde{\varepsilon} \leq \liminf_{\varphi \to 0} \|d_\varphi R_\varphi^*(\psi)\|_p \leq \limsup_{\varphi \to 0} \|d_\varphi R_\varphi^*(\psi)\|_p \leq \|\psi\|_p + 6\tilde{\varepsilon}.$$ 

Since $\tilde{\varepsilon} > 0$ can be taken arbitrarily small, the proof completes. \qed

### 7. Proof of Theorems

In this section, we give the proofs of Theorems 5.4 and 5.5. First, as another consequence from Lemma 6.3, we see that the base point change $r_\nu$ of Bel($D$) is Lipschitz continuous if $\nu$ or $\nu^{-1}$ is given by the Ahlfors-Weill section of small norm. Compare with Proposition 3.5.

**Proposition 7.1.** Take $\mu_1, \mu_2$ in $Ael^p(D)$ and $\nu$ or $\nu^{-1}$ in $\sigma(U^\infty(\delta_0))$. Then

$$\|r_\nu(\mu_1) - r_\nu(\mu_2)\|_p \leq C_0 \|\mu_1 - \mu_2\|_p,$$

where $C_0 > 0$ is a constant depending only on $\|\mu_1\|_\infty$ and $\|\mu_2\|_\infty$.

**Proof.** As before, we use the following inequality for $w = f_\nu(z)$:

$$|r_\nu(\mu_1)(w) - r_\nu(\mu_2)(w)| \leq \frac{|\mu_1(z) - \mu_2(z)|}{\sqrt{(1 - \|\mu_1\|_\infty^2)(1 - \|\mu_2\|_\infty^2)}}.$$

Then Lemma 6.3 is again applied to show that

$$\int_D |r_\nu(\mu_1)(w) - r_\nu(\mu_2)(w)|^p \rho_D^2(w)dudv \leq \frac{2}{(1 - \|\mu_1\|_\infty^2)^{p/2}(1 - \|\mu_2\|_\infty^2)^{p/2}} \int_D |\mu_1(z) - \mu_2(z)|^p \rho_D^2(z)dxdy.$$

The assertion is now clear. \qed

**Remark.** More generally, the estimate in Proposition 7.1 is still true if $\nu \in$ Bel($D$) is given by the composition of elements in $\sigma(U^\infty(\delta_0))$. In this case, the constant $C_0$ also depends on $\|\nu\|_\infty$. Based on this fact and Lemma 6.4, Takhtajan and Teo [33, Theorem 2.13] proved the inclusion

$$\Phi \circ r_\nu(Ael^p(D)) \subset \Phi(\nu^{-1}) + A^p(D^*)$$

in the case of $p = 2$. This is also true for $p > 2$ in general, and moreover, the equality actually holds. As we have mentioned before, Proposition 7.1 can be obtained if we replace the assumption on $\nu$ with the condition that it is given by the barycentric extension. See Yanagishita [35, Proposition 5.1].

Under these preparations, we start the proofs of the theorems here.
Proof of Theorem 5.4. Let $\delta_0 > 0$ be the constant as in Lemma 6.3. We take the number of division $n \in \mathbb{N}$ greater than

$$\frac{3\|\mu\|_\infty}{2\delta_0(1 - \|\mu\|_\infty^2)}.$$ 

Set $t_i = i/n$ ($i = 0, 1, \ldots, n$). Then, for every $i \geq 1$,

$$\|r_{t_{i-1}\mu}(t_i\mu)\|_\infty \leq \frac{\|t_i\mu - t_{i-1}\mu\|_\infty}{1 - \|t_i\mu\|_\infty \|t_{i-1}\mu\|_\infty} \leq \frac{\|\mu\|_\infty}{n(1 - \|\mu\|_\infty^2)} < \frac{2\delta_0}{3}.$$ 

For the Bers projection $\Phi : \text{Bel}(\mathbb{D}) \to B(\mathbb{D}^*)$, we define

$$\varphi_i = \Phi(r_{t_{i-1}\mu}(t_i\mu)).$$

Note that $\|\Phi(\mu)\|_\infty \leq 3\|\mu\|_\infty/2$ ([17] Theorem II.3.2). Then $\|\varphi_i\|_\infty < \delta_0$ and hence $\varphi_i$ belongs to $U^\infty(\delta_0) \cap A^p(\mathbb{D}^*)$.

For integers $i$ and $k$ with $1 \leq i \leq n$ and $1 \leq k \leq n$, we set

$$m_p(i - 1, k) = \|r_{\sigma(\varphi_{i-1}) \circ \cdots \circ \sigma(\varphi_1)}(t_k\mu)\|_p;$$

$$m_\infty(i - 1, k) = \|r_{\sigma(\varphi_{i-1}) \circ \cdots \circ \sigma(\varphi_1)}(t_k\mu)\|_\infty.$$ 

Note that this includes the case where $i = 1$ so that $m_p(0, k) = \|t_k\mu\|_p$ and $m_\infty(0, k) = \|t_k\mu\|_\infty$. Also, when $i \leq 0$, the above definition should be understood as $m_p(i - 1, k) = 0$ and $m_\infty(i - 1, k) = 0$.

We use the following recursive representation of $\varphi_i$:

$$\varphi_i = \Phi(r_{t_{i-1}\mu}(t_i\mu)) = \Phi(r_{\sigma(\varphi_{i-1}) \circ \sigma(\varphi_{i-2}) \circ \cdots \circ \sigma(\varphi_1)}(t_i\mu)).$$

This holds true because $\Phi \circ r_\mu = \Phi \circ r_{\mu'}$ if $\pi(\mu) = \pi(\mu')$ and

$$\pi(\sigma(\varphi_{i-1}) \ast \cdots \ast \sigma(\varphi_1)) = \pi(r_{t_{i-2}\mu}(t_{i-1}\mu) \ast \cdots \ast (t_1\mu)) = \pi(t_{i-1}\mu).$$

Then Lemma 6.1 yields that

$$\|\varphi_i\|_p \leq \frac{3m_p(i - 1, i)}{\sqrt{1 - m_\infty(i - 1, i)^2}}.$$ 

and since $m(i - 1, i)_{\infty} (1 \leq i \leq n)$ are uniformly bounded by a constant less than 1 depending on $n$ and $\|\mu\|_\infty$, there is a constant $C_1 > 0$ depending only on $\|\mu\|_\infty$ such that $\|\varphi_i\|_p \leq C_1 m_p(i - 1, i)$.

We can obtain recursive inequalities for $m_p(i - 1, k)$. First we note that

$$m_p(i - 1, k) = \|r_{\sigma(\varphi_{i-1}) \circ \cdots \circ \sigma(\varphi_1)}(t_k\mu)\|_p$$

$$= \|r_{\sigma(\varphi_{i-1}) \circ \cdots \circ \sigma(\varphi_1)}(t_k\mu) - r_{\sigma(\varphi_{i-1})}(r_{\sigma(\varphi_{i-2}) \circ \cdots \circ \sigma(\varphi_1)}(t_k\mu))\|_p.$$
Then Proposition 7.1 implies that there is a constant $C_0 > 0$ depending only on $\|\mu\|_{\infty}$ such that
\[
\|r_{\sigma(\varphi_{i-1})}(r_{\sigma(\varphi_{i-2})} \circ \cdots \circ r_{\sigma(\varphi_1)}(t_k\mu)) - r_{\sigma(\varphi_{i-1})}(\sigma(\varphi_{i-1}))\|_p
\leq C_0\|r_{\sigma(\varphi_{i-2})} \circ \cdots \circ r_{\sigma(\varphi_1)}(t_k\mu) - \sigma(\varphi_{i-1})\|_p.
\]

Finally, the last $p$-norm is estimated as
\[
\|r_{\sigma(\varphi_{i-2})} \circ \cdots \circ r_{\sigma(\varphi_1)}(t_k\mu) - \sigma(\varphi_{i-1})\|_p
\leq \|r_{\sigma(\varphi_{i-2})} \circ \cdots \circ r_{\sigma(\varphi_1)}(t_k\mu)\|_p + 2\|\varphi_{i-1}\|_p
\leq m_p(i - 2, k) + 2C_1m_p(i - 2, i - 1).
\]

Hence, we have recursive inequalities
\[
m_p(i - 1, k) \leq Cm_p(i - 2, k) + Cm_p(i - 2, i - 1),
\]
where $C = \max\{C_0, 2C_0C_1\}$.

From these inequalities, we can show that
\[
m_p(i - 1, k) \leq C^{i-1}\{m_p(0, k) + \sum_{j=0}^{i-2} 2^j m_p(0, i - 1 - j)\}
\]
for $1 \leq i \leq n$ and $1 \leq k \leq n$. Indeed, this is valid for $i = k = 1$. Suppose that this is true for lower indices than $(i - 1, k)$. Then
\[
m_p(i - 1, k) \leq Cm_p(i - 2, k) + Cm_p(i - 2, i - 1)
\leq C \cdot C^{i-2}\{m_p(0, k) + \sum_{j=0}^{i-3} 2^j m_p(0, i - 2 - j)\}
\quad + C \cdot C^{i-2}\{m_p(0, i - 1) + \sum_{j=0}^{i-3} 2^j m_p(0, i - 2 - j)\},
\]
where the last term is equal to the desired one. Using
\[
m_p(0, k) = \|t_k\mu\|_p = \frac{k}{n}\|\mu\|_p,
\]
we in particular obtain
\[
m_p(i - 1, i) \leq C^{i-1}\{m_p(0, i) + \sum_{j=0}^{i-2} 2^j m_p(0, i - 1 - j)\}
\quad = C^{i-1}\{\frac{i}{n} + \sum_{j=0}^{i-2} \frac{2^j}{n} \frac{i - 1 - j}{n}\}\|\mu\|_p
\quad = C^{i-1}\frac{(2^i - 1)}{n}\|\mu\|_p.
\]

We will complete the estimate of the Weil-Petersson distance. We start with
\[
d^p_{WP}(\mu) \leq \sum_{i=1}^{n} d^p_{WP}(\delta_{i-1} \mu, [t_i \mu]) = \sum_{i=1}^{n} d^p_{WP}(\mu, [r_{t_{i-1}}(t_i \mu)]).
\]
by the invariance of $d_{WP}^0$ under the base point change. Since $\varphi_i = \Phi(r_{t_{i-1}}(t_i\mu))$ belongs to $U^\infty(\delta_0) \cap A^p(\mathbb{D}^*)$, Corollary 6.4 asserts that

$$d_{WP}^0([0], [r_{t_{i-1}}(t_i\mu)]) \leq 16\|\varphi_i\|_p.$$ 

Also $\|\varphi_i\|_p \leq C_1 m_p(i - 1, i)$ as we have seen before. Hence

$$d_{WP}^0([0], [\mu]) \leq 16C_1 \sum_{i=1}^n m_p(i - 1, i)$$

$$\leq 16C_1 \sum_{i=1}^n \frac{C^{i-1}(2^i - 1)}{n} \|\mu\|_p$$

$$\leq 16C_1 \frac{(2C)^n}{n} \|\mu\|_p.$$ 

This multiplier for $\|\mu\|_p$ is a constant depending only on $\|\mu\|_\infty$ and thus the proof completes.

\[\Box\]

**Proof of Theorem 5.5.** For every $g \in G$, set $\varphi_g = \beta([g]) \in B(\mathbb{D}^*)$. Since $g \in \text{Diff}^{1+\alpha}(S)$ and $\alpha > 1$, we have $[g] \in T^p$ and hence $\varphi_g \in A^p(\mathbb{D}^*)$. By Lemma 6.1 and the assumption $k_p(g) \leq \varepsilon_p$, we see that $\|\varphi_g\|_p \leq 3\varepsilon_p$ for every $g \in G$. Since $\|\varphi\|_\infty \leq c_p \|\varphi\|_p$ for $\varphi \in A^p(\mathbb{D}^*)$, we have $\|\varphi_g\|_\infty \leq 3c_p \varepsilon_p$. Hence, we can choose $\varepsilon_p > 0$ so small that $\varphi_g \in U^\infty(1/2)$ and then its image $\mu_g = \sigma(\varphi_g)$ of the Ahlfors-Weill section $\sigma : U^\infty(1/2) \rightarrow \text{Bel}(\mathbb{D})$ satisfies $\|\mu_g\|_p \leq 6\varepsilon_p$ and $\|\mu_g\|_\infty \leq 6c_p \varepsilon_p$ for every $g \in G$. This in particular implies that $G$ is a uniformly quasiconformal group whose elements have quasiconformal extensions to $\mathbb{D}$ with a sufficiently small dilatation bound $\kappa_\infty$ by choosing $\varepsilon_p > 0$.

Markovic [19] proved that a uniformly quasiconformal group $G \subset \text{QS}$ is conjugate into $\text{M"ob}(S)$ by a quasiconformal homeomorphism $f_0 \in \text{QS}$. This means that $\tau_0 = [f_0] \in T$ is a fixed point of $G$. Then $G$ acts on the Banach space $B(\mathbb{D}^*)$ linear isometrically through the Bers embedding $\beta_{\tau_0} = \beta \circ R_{\tau_0}$. Moreover, we can take $\tau_0$ sufficiently close to the origin $\sigma = [\text{id}]$ if $\kappa_\infty$ is sufficiently small. In particular, we may assume that $\tau_0 = [\nu]$ for $\nu \in \sigma(U^\infty(\delta_0))$.

The linear isometric action of $G$ on $B(\mathbb{D}^*)$ by $\beta_{\tau_0}$ also preserves the affine subspace $\varphi_0 + A^p(\mathbb{D}^*) \subset B(\mathbb{D}^*)$ for $\varphi_0 = \beta_{\tau_0}(o) = \Phi(\nu^{-1})$ invariant. Moreover, the orbit of $\varphi_0$ under $G$, which is $\{\beta_{\tau_0}([g]) \}_{g \in G} = \{\Phi(r_{\nu}(\mu_g))\}_{g \in G}$, is bounded with respect to the norm of $A^p(\mathbb{D}^*)$. These facts can be verified by Lemma 6.4 and Proposition 7.1 (see also the remark after this proposition) as follows. The combination of these claims yields that if $r_{\nu}(\mu_2)$, $\nu \in \sigma(U^\infty(\delta_0))$ then

$$\|\Phi(r_{\nu}(\mu_1)) - \Phi(r_{\nu}(\mu_2))\|_p \leq C\|\mu_1 - \mu_2\|_p,$$

where $C > 0$ is a constant depending only on $\|\mu_1\|_\infty$, $\|\mu_2\|_\infty$ and $\|\nu\|_\infty$. We apply this inequality for $\mu_1 = \mu_g$ ($g \in G$), $\mu_2 = 0$ and $\nu$ as above. Then

$$\|\Phi(r_{\nu}(\mu_g)) - \varphi_0\|_p \leq C\|\mu_g\|_p \leq 6\varepsilon_p C$$

for every $g \in G$.

This gives a fixed point of $G$ in $\varphi_0 + A^p(\mathbb{D}^*)$ since the Banach space $A^p(\mathbb{D}^*)$ is uniformly convex; any bounded orbit has a unique circumcenter also in this case. See [23] for a survey
of this property. Furthermore, we choose $\varepsilon_p$ so small that the closed ball in $\varphi_0 + A^p(\mathbb{D}^*)$ of center at $\varphi_0$ and radius $6\varepsilon_p C$ is contained in the open set

$$\beta_{\tau_0}(T^p) \subset \beta(T) \cap (\varphi_0 + A^p(\mathbb{D}^*)).$$

Then the fixed point of $G$, which is the circumcenter of the orbit, is also in $\beta_{\tau_0}(T^p)$. Having the new fixed point $\tau = [f] \in T^p$, we perform the same argument as in the case where $\tau \in T^2$ before. Since $f \in \text{Sym}^p \subset \text{Sym}$ conjugates $G$ into $\text{Möb}(\mathbb{S})$, Theorem 4.1 asserts that $f \in \text{Diff}^{1+\alpha}(\mathbb{S})$. This completes the proof. \hfill \Box

**Remark.** If the metric space $(T^p, d_{WP}^p)$ has a property that every isometry group with a bounded orbit has a fixed point, then the statement of Theorem 5.5 can be improved. We expect that $(T^p, d_{WP}^p)$ is uniformly convex to satisfy this property. See [23].

**References**

[1] L. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, 1966.

[2] L. Ahlfors and G. Weill, *A uniqueness theorem for Beltrami equations*, Proc. Amer. Math. Soc. 13 (1962), 975–978.

[3] K. Astala and M. Zinsmeister, *Teichmüller spaces and BMOA*, Math. Ann. 289 (1991), 613–625.

[4] W. Ballmann, *Lectures on spaces of nonpositive curvature*, DMV Seminar Band 25, Birkhäuser, 1995.

[5] J. Becker, *Conformal mappings with quasiconformal extensions*, Aspects of contemporary complex analysis, Academic Press, 1980, pp. 37–77.

[6] J. Becker and C. Pommerenke, *Über die quasikonforme Fortsetzung schlichter Funktionen*, Math. Z. 161 (1978), 69–80.

[7] L. Bers, *Fiber spaces over Teichmüller spaces*, Acta Math. 130 (1973), 89–126.

[8] L. Carleson, *On mappings, conformal at the boundary*, J. Anal. Math. 19 (1967), 1–13.

[9] G. Cui, *Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces*, Sci. China Ser. A 43 (2000), 267–279.

[10] A. Douady and C. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. 157 (1986), 23–48.

[11] C. J. Earle, F. P. Gardiner and N. Lakic, *Asymptotic Teichmüller space, Part I: The complex structure*, In the Tradition of Ahlfors and Bers, Contemporary Math. vol. 256, pp. 17–38, Amer. Math. Soc., 2000.

[12] C. J. Earle, V. Markovic and D. Saric, *Barycentric extension and the Bers embedding for asymptotic Teichmüller space*, Complex manifolds and hyperbolic geometry, Contemporary Math. vol. 311, pp. 87–105, Amer. Math. Soc., 2002.

[13] R. Fehlmann, *Über extremale quasikonforme Abbildungen*, Comment Math. Helv. 56 (1981), 558–580.

[14] F. Gardiner and D. Sullivan, *Symmetric structure on a closed curve*, Amer. J. Math. 114 (1992), 683–736.

[15] H. Guo, *Integrable Teichmüller spaces*, Sci. China Ser. A 43 (2000), 47–58.

[16] S. Lang, *Fundamentals of differential geometry*, Graduate Texts in Mathematics vol. 191, Springer, 1999.

[17] O. Lehto, *Univalent functions and Teichmüller spaces*, Graduate Texts in Mathematics vol. 109, Springer, 1986.

[18] I. Kra, *Automorphic forms and Kleinian groups*, Mathematics lecture note series, W. A. Benjamin, 1972.
[19] V. Markovic, *Quasisymmetric groups*, J. Amer. Math. Soc. **19** (2006), 673–715.

[20] K. Matsuzaki, *Symmetric groups that are not the symmetric conjugates of Fuchsian groups*, In the tradition of Ahlfors and Bers, V, Contemporary Math. **510**, pp. 239–247, Amer. Math. Soc., 2010.

[21] K. Matsuzaki, *Certain integrability of quasisymmetric automorphisms of the circle*, Comput. Methods Funct. Theory **14** (2014), 487–503.

[22] K. Matsuzaki, *The universal Teichmüller space and diffeomorphisms of the circle with Hölder continuous derivatives*, Handbook of group actions (Vol. I), Advanced Lectures in Mathematics vol. 31, pp. 333–372, Higher Education Press and International Press, 2015.

[23] K. Matsuzaki, *Uniform convexity, normal structure and the fixed point property of metric spaces*, Topology Appl. **196** (2015), part B, 684–695.

[24] K. Matsuzaki, *Circle diffeomorphisms, rigidity of symmetric conjugation and affine foliation of the universal Teichmüller space*, Geometry, Dynamics, and Foliations 2013, Advanced Studies in Pure Mathematics vol. 72, pp. 145–180, Mathematical Society of Japan, 2017.

[25] K. Matsuzaki, *The Teichmüller space of group invariant symmetric structures on the circle*, Ann. Acad. Sci. Fenn. Math. **42** (2017), 535–550.

[26] K. Matsuzaki, *Teichmüller space of circle diffeomorphisms with Hölder continuous derivative*, arXiv:1607.06300.

[27] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, John Wiley & Sons 1988.

[28] A. Navas, *On uniformly quasisymmetric groups of circle diffeomorphisms*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 437–462.

[29] A. Navas, *Groups of circle diffeomorphisms*, Chicago Lectures in Math., Univ. Chicago Press, 2011.

[30] C. Pommerenke, *Boundary behaviour of conformal maps*, Springer, 1992.

[31] C. Pommerenke and S. E. Warschawski, *On the quantitative boundary behavior of conformal maps*, Comment. Math. Helv. **57** (1982), 107–129.

[32] Y. Shen, *Weil-Petersson Teichmüller space*, preprint, arXiv:1304.3197

[33] L. Takhtajan and L. Teo, *Weil-Petersson metric on the universal Teichmüller space*, Mem. Amer. Math. Soc. **183** (2006), No. 861.

[34] S. Tang, *Some characterizations of the integrable Teichmüller space*, Sci. China Ser. A **56** (2013), 541–551.

[35] M. Yanagishita, *Introduction of a complex structure on the p-integrable Teichmüller space*, Ann. Acad. Sci. Fenn. **39** (2014), 947–971.