A braid group representation derived from handlebodies

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Abstract

We define an action of Artin’s braid group on a finite dimensional algebra.

1 Introduction

It seems that there is still interest in new representations of the braid group. One reason is the still open question if the braid group is linear. Another is the search for finite dimensional braid algebras that generalize the Birman-Wenzl algebra. In this note we describe a representation that is constructed in a new way. It uses a series of algebras associated to handlebodies.

2 The representation

In this section we construct a braid group representation that is related to the action of the braid group on a free group. So we start with the following well known definition and result:

Theorem and Definition 1. The n-string braid group $B_n$ is defined by generators $X_1, \ldots, X_{n-1}$ and relations

\begin{align*}
X_iX_jX_i &= X_jX_iX_j & |i - j| = 1 \\
X_iX_j &= X_jX_i & |i - j| > 1
\end{align*}

There is a monomorphism from the braid group into the automorphism group of the free group in n generators $\phi : B_n \to \text{Aut}(F_n)$ given by

\begin{align*}
\phi(X_i)(t_j) &:= t_j & j \neq i, i + 1 \\
\phi(X_i)(t_{i+1}) &:= t_i \\
\phi(X_i)(t_i) &:= t_it_{i+1}t_i^{-1}
\end{align*}

It is best to think of the free group $F_n$ as the fundamental group of the handlebody of genus $n$, i.e. the product of a $n$-holed disc with the unit interval. The holes times the unit interval are called the axes of the handlebody and are numbered by 1, \ldots, $n$. In
recent work [3] Temperley-Lieb algebras related to handlebodies have been studied. It is not necessary to give the complete definitions used in that work. It suffices to consider the following algebras:

**Definition 2.** Fix a natural number \( n \) and parameters \( c_{i,j} \in R, i, j \in \{1, \ldots, n\} \) in an integral domain \( R \). The algebra \( T_n \) is defined to be generated by \( f_{i,j}, i, j \in \{1, \ldots, n\} \) with the set of relations:

\[
f_{i,j} f_{k,l} = c_{j,k} f_{i,l}
\]

(6)

It follows at once that \( T_n \) is a free \( R \)-module of rank \( n^2 + 1 \).

In the handlebody setting we interpret \( f_{i,j} \) as the (1,1) tangle in the handlebody that is composed out of a string that enters axes \( j \) and one that emerges from axes \( i \). \( c_{i,j} \) is then the parameter that accounts for removing a string that connects two axes. Since Temperley-Lieb strings are not oriented one has \( c_{i,j} = c_{j,i} \). In the present paper we will find an even more restrictive condition on the parameters.

In accordance with the general philosophy of Temperley-Lieb algebras we interpret

\[
Y_i := \alpha_i f_{i,i} + \beta_i \quad i \in \{1, \ldots, n\}
\]

as the generator of the braid group of the handlebody that surrounds axes \( i \). This means that \( Y_i \) is to be considered as the image of \( t_i \) under the Kauffman bracket map for the handlebody. The \( \alpha_i, \beta_i \) are parameters in \( R \) and we require \( \alpha_i \) to be invertible. This makes \( Y_i \) invertible. It is now clear, how to define the braid group action if we postulate, that the \( c_{i,j} \) are invertible:

**Definition 3.** For each \( i \in \{1, \ldots, n-1\} \) define operators \( \phi(X_i) \) on \( T_n \) by

\[
\phi(X_i)(f_{j,j}) := f_{j,j} \quad j \neq i, i+1
\]

(7)

\[
\phi(X_i)(f_{i+1,i+1}) := f_{i,i}
\]

(8)

\[
\phi(X_i)(f_{i,i}) := \alpha_i^{-1}(Y_iY_{i+1}Y_i^{-1} - \beta_i)
\]

(9)

\[
\phi(X_i)(f_{i,j}) := \phi(X_i)(f_{i,i})\phi(X_i)(f_{j,j})c_{i,j}^{-1} \quad i \neq j
\]

(10)

\[
\phi(X_i)(1) := 1
\]

(11)

Denote by \( B(i) \) the matrix of \( \phi(X_i) \) with respect to the basis \( 1, f_{1,1}, \ldots, f_{1,n}, f_{2,1}, \ldots, f_{n,n} \).

**Lemma 4.** The operators \( \phi(X_i) \) are invertible. Inverses are given by

\[
\phi(X_i^{-1})(f_{j,j}) := f_{j,j} \quad j \neq i, i+1
\]

(12)

\[
\phi(X_i^{-1})(f_{i,i}) := f_{i+1,i+1}
\]

(13)

\[
\phi(X_i^{-1})(f_{i+1,i+1}) := \alpha_i^{-1}(Y_i^{-1}Y_{i+1}Y_i - \beta_i)
\]

(14)

\[
\phi(X_i^{-1})(f_{i,j}) := \phi(X_i^{-1})(f_{i,i})\phi(X_i^{-1})(f_{j,j})c_{i,j}^{-1} \quad i \neq j
\]

(15)

\[
\phi(X_i^{-1})(1) := 1
\]

(16)

The proof is a simple calculation. Note that the representation matrices \( B(i) \) have off-diagonal terms.

It remains to establish that the \( B(i) \) satisfy the relation of Artin’s braid group. To do this we will have to fix most of the parameters to special values. We start by checking if
the relation \( f_{i,i}^2 = c_{i,i} f_{i,i} \) is preserved under the action of \( \phi(X_i) \). In the expansion of the result the term proportional to 1 is

\[
\alpha_i^{-2}(\beta_i - \beta_{i+1})(c_{i,i}\alpha_i + \beta_i - \beta_{i+1})
\]

We choose the \( \beta_i \) to be the same:

\[
\beta_i := \beta \quad (17)
\]

Similarly, the coefficient of \( f_{i,i} \) tells us that either \( \alpha_i \) should vanish or that

\[
\alpha_i := c_{i-1,i-1}\alpha_{i-1}/c_{i,i} \quad i > 1
\]

Examining now the general relation (8) yields

\[
c_{i,j} := c \quad (19)
\]

**Proposition 5.** The map \( \phi \) is a representation of the braid group if (17), (18), (19) hold.

Again, this is a tedious calculation.

It is easy to see that the conditions on the parameters imply that the action of \( \phi \) on \( T_n \) preserves the subspace that is spanned by the \( f_{i,j} \). The matrices given below are for this subspace.

The representation depends in fact only on one parameter:

\[
\mu = c^2\alpha_1/\beta
\]

From the calculation of examples we have the conjecture: The characteristic polynomial \( \det(B(i) - \lambda) \) is given by

\[
(\lambda - 1)^{2+(n-1)^2}(\lambda + 1 + \mu)^{n-1}(\lambda + (1 + \mu)^{-1})^{n-1}
\]

The eigenvalues are thus 1, \( 1 + \mu, (1 + \mu)^{-1} \) and the representation matrices have unit determinant. For the benefit of the reader, we give the matrices for the example \( n = 3 \). Experimental calculations show that the 3-string algebra they generate is of dimension 19. Its centralizer algebra is of dimension 7. Hence, if the algebra is semi-simple (as we hope) then it has 4 two-dimensional and 3 one-dimensional representations.

\[
B(1) = 
\begin{pmatrix}
\frac{-\mu^2}{1+\mu} & \frac{\mu}{1+\mu} & 0 & -\mu & 1 & 0 & 0 & 0 & 0 \\
\mu & 0 & 0 & 1 + \mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{1+\mu} & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{-\mu}{1+\mu} & \frac{1}{1+\mu} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{1+\mu} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\mu & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 + \mu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
3 Discussion

We have found a braid group representation from a simple geometric setting. This construction raises a number of open questions that should be discussed: One should try to classify the irreducible subrepresentations.

It is sensible to expect that the same method yields much more representations when applied to other quotients of the handlebody braid group.

Another question: Is a Markov trace connected with this representation?

References

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