Cactus Graphs and Graph Complement Conjecture

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Abstract

In this paper we proof that any cactus graph satisfies graph complement conjecture by finding a orthogonal representation of its complement in $\mathbb{R}^5$.

Key words: Graph Complement Conjecture, simple connected graphs, minimum semidefinite rank, $\delta$-graph, C-$\delta$ graphs, orthogonal representation.

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1 Introduction

A graph $G$ consists of a set of vertices $V(G) = \{1, 2, \ldots, n\}$ and a set of edges $E(G)$, where an edge is defined to be an unordered pair of vertices. The order of $G$, denoted $|G|$, is the cardinality of $V(G)$. A graph is simple if it has no multiple edges or loops. The complement of a graph $G(V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all those edges of the complete graph $K_{|G|}$ that are not in $E$.

A matrix $A = [a_{ij}]$ is combinatorially symmetric when $a_{ij} = 0$ if and only if $a_{ji} = 0$. We say that $G(A)$ is the graph of a combinatorially symmetric matrix $A = [a_{ij}]$ if $V = \{1, 2, \ldots, n\}$ and $E = \{\{i, j\} : a_{ij} \neq 0\}$. The main diagonal entries of $A$ play no role in determining $G$. Define $S(G, F)$ as the set of all $n \times n$ matrices that are real symmetric if $F = \mathbb{R}$ or complex Hermitian if $F = \mathbb{C}$ whose graph is $G$. The sets $S_+(G, F)$ are the corresponding subsets of positive semidefinite (psd) matrices. The smallest possible rank of any matrix $A \in S(G, F)$ is the minimum rank of $G$, denoted $\text{mr}(G, F)$, and the smallest possible rank of any matrix $A \in S_+(G, F)$ is the minimum semidefinite rank of $G$, denoted $\text{msr}(G)$ or $\text{msr}(G)$.

In 1996, the minimum rank among real symmetric matrices with a given graph was studied by Nylen [27]. It gave rise to the area of minimum rank problems which led to the study of minimum rank among complex Hermitian matrices and positive semidefinite matrices associated with a given graph. Many results can be found for example in [1, 19, 23, 24, 27].

During the AIM workshop of 2006 in Palo Alto, CA, it was proposed question about how large can $\text{mr}(G) + \text{mr}(\overline{G})$ be [13] the that for any graph $G$ and infinite field $F$. It was conjectured that $\text{mr}(G) + \text{mr}(\overline{G}) \leq |G| + 2$ for some infinite families of graphs for which the minimum rank of both the graph and their complement were known but for an arbitrary graph $G$ it is unknown whether or not this inequality is true and therefore it is called the “Graph Complement Conjecture” denoted as GCC. When we restrict the study to $S_+(G, F)$, $F = \mathbb{C}$ or $\mathbb{R}$ this conjecture is referred to as GCC$^+_+$, which states that for any graph $G$, $\text{msr}(G) + \text{msr}(\overline{G}) \leq |G| + 2$.

In [1] it is shown that trees satisfy GCC$^+_+$ since the minimum semidefinite rank of a tree is $|G| - 1$ and that of its complement was shown to be at most 3. Hogben [22], showed that some

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other families of sparse graphs, including unicyclic graphs and 2-trees also satisfy GCC+. Sharawi [30], showed that GCC+ holds for a complete bipartite graph \( G \) with \( V(G) = R \sqcup L, |L| = n \geq 2, |R| = m \geq n \). Also, it was shown in [30] that a \( k \)-regular Harary graph, \( H_{k,n} \) which is \( k \)-connected satisfies the GCC+. Mitchell [24], showed that chordal graphs satisfy GCC+. Other results have been found in [2, 16, 22, 24, 30]. However, the general problem of graph complement conjecture is still open.

2 Graph Theory Preliminaries

In this section, we will establish some of the results for the minimum semidefinite rank (msr) of a graph \( G \) that we will be using in the subsequent sections.

A graph \( G(V, E) \) is a pair \((V(G), E(G))\), where \( V(G) \) is the set of vertices and \( E(G) \) is the set of edges together with an incidence function \( \psi(G) \) that associate with each edge of \( G \) an unordered pair of (not necessarily distinct) vertices of \( G \). The order of \( G \), denoted \(|G|\), is the number of vertices in \( G \). A graph is said to be simple if it has no loops or multiple edges. The complement of a graph \( G(V, E) \) is the graph \( \overline{G} = (V, \overline{E}) \), where \( \overline{E} \) consists of all the edges that are not in \( E \). A subgraph \( H = (V(H), E(H)) \) of \( G = (V, E) \) is a graph with \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). An induced subgraph \( H \) of \( G \), denoted \( G[V(H)] \), is a subgraph with \( V(H) \subseteq V(G) \) and \( E(H) = \{\{i, j\} \in E(G) : i, j \in V(H)\} \). Sometimes we denote the edge \{i, j\} as ij. We say that two vertices of a graph \( G \) are adjacent, denoted \( v_i \sim v_j \), if there is an edge \( \{v_i, v_j\} \) in \( G \). Otherwise we say that the two vertices \( v_i \) and \( v_j \) are non-adjacent and we denote this by \( v_i \not\sim v_j \). Let \( N(v) \) denote the set of vertices that are adjacent to the vertex \( v \) and let \( N[v] = \{v\} \cup N(v) \). The degree of a vertex \( v \) in \( G \), denoted \( d_G(v) \), is the cardinality of \( N(v) \). If \( d_G(v) = 1 \), then \( v \) is said to be a pendant vertex of \( G \). We use \( \Delta(G) \) to denote the minimum degree of the vertices in \( G \), whereas \( \Delta(G) \) will denote the maximum degree of the vertices in \( G \). Two graphs \( G(V, E) \) and \( H(V', E') \) are identical denoted \( G = H \), if \( V = V' \), \( E = E' \), and \( \psi_G = \psi_H \). Two graphs \( G(V, E) \) and \( H(V', E') \) are isomorphic, denoted by \( G \cong H \), if there exist bijections \( \theta : V \rightarrow V' \) and \( \phi : E \rightarrow E' \) such that \( \psi_G(e) = \{u, v\} \) if and only if \( \psi_H(\phi(e)) = \{\theta(u), \theta(v)\} \). A complete graph is a simple graph in which the vertices are pairwise adjacent. We will use \( nG \) to denote \( n \) copies of a graph \( G \). For example, \( 3K_1 \) denotes three isolated vertices \( K_1 \) while \( 2K_2 \) is the graph given by two disconnected copies of \( K_2 \). A path is a list of distinct vertices in which successive vertices are connected by edges. A path on \( n \) vertices is denoted by \( P_n \). A graph \( G \) is said to be connected if there is a path between any two vertices of \( G \). A cycle on \( n \) vertices, denoted \( C_n \), is a path such that the beginning vertex and the end vertex are the same. A tree is a connected graph with no cycles. A graph \( G(V, E) \) is said to be chordal if it has no induced cycles \( C_n \) with \( n \geq 4 \). A component of a graph \( G(V, E) \) is a maximal connected subgraph. A cut vertex is a vertex whose deletion increases the number of components. The union \( G \cup G_2 \) of two graphs \( G_1(V_1, E_1) \) and \( G_2(V_2, E_2) \) is the union of their vertex set and edge set, that is \( G \cup G_2(V_1 \cup V_2, E_1 \cup E_2) \). When \( V_1 \) and \( V_2 \) are disjoint their union is called disjoint union and denoted \( G_1 \sqcup G_2 \).

3 The Minimum Semidefinite Rank of a Graph

In this section, we will establish some of the results for the minimum semidefinite rank (msr) of a graph \( G \) that we will be using in the subsequent sections.

A positive definite matrix \( A \) is an Hermitian \( n \times n \) matrix such that \( x^*Ax > 0 \) for all nonzero \( x \in \mathbb{C}^n \). Equivalently, \( A \) is a \( n \times n \) Hermitian positive definite matrix if and only if all the eigenvalues of \( A \) are positive ([20], p.250).

A \( n \times n \) Hermitian matrix \( A \) such that \( x^*Ax \geq 0 \) for all \( x \in \mathbb{C}^n \) is said to be positive.
semidefinite (psd). Equivalently, $A$ is a $n \times n$ Hermitian positive semidefinite matrix if and only if $A$ has all eigenvalues nonnegative ([20], p. 182).

If $\vec{V} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subset \mathbb{R}^m$ is a set of column vectors then the matrix $A^T A$, where $A = [\begin{array}{c} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{array}]$ and $A^T$ represents the transpose matrix of $A$, is a psd matrix called the Gram matrix of $\vec{V}$. Let $G(V, E)$ be a graph associated with this Gram matrix. Then $V_G = \{v_1, \ldots, v_n\}$ correspond to the set of vectors in $\vec{V}$ and $E(G)$ correspond to the nonzero inner products among the vectors in $\vec{V}$. In this case $\vec{V}$ is called an orthogonal representation of $G(V, E)$ in $\mathbb{R}^m$. If such an orthogonal representation exists for $G$ then $\text{msr}(G) \leq m$.

The maximum positive semidefinite nullity of a graph $G$, denoted $M_{+}(G)$ is defined by $M_{+}(G) = \max \{\text{null}(A) : A$ is symmetric and positive semidefinite and $G(A) = G\}$, where $G(A)$ is the graph obtained from the matrix $A$. From the rank-nullity theorem we get $\text{msr}(G) + M_{+}(G) = |G|$.

Some of the most common results about the minimum semidefinite rank of a graph are the following:

**Result 3.1.** [19] If $T$ is a tree then $\text{msr}(T) = |T| - 1$.

**Result 3.2.** [17] The cycle $C_n$ has minimum semidefinite rank $n - 2$.

**Result 3.3.** [17] If a connected graph $G$ has a pendant vertex $v$, then $\text{msr}(G) = \text{msr}(G - v) + 1$ where $G - v$ is obtained as an induced subgraph of $G$ by deleting $v$.

**Result 3.4.** [18] If $G$ is a connected, chordal graph, then $\text{msr}(G) = \text{cc}(G)$.

**Result 3.5.** [10] If a graph $G(V, E)$ has a cut vertex, so that $G = G_1 \cdot G_2$, then $\text{msr}(G) = \text{msr}(G_1) + \text{msr}(G_2)$.

The next two definitions give us two families of graphs which are important in the study of the minimum semidefinite rank of a graph.

**Definition 3.6.** Suppose that $G = (V, E)$ with $|G| = n \geq 4$ is simple and connected such that $\overline{G} = (V, \overline{E})$ is also simple and connected. We say that $G$ is a $\delta$-graph if we can label the vertices of $G$ in such a way that

1. the induced graph of the vertices $v_1, v_2, v_3$ in $G$ is either $3K_1$ or $K_2 \sqcup K_1$, and
2. for $m \geq 4$, the vertex $v_m$ is adjacent to all the prior vertices $v_1, v_2, \ldots, v_{m-1}$ except for at most $\left\lfloor \frac{m}{2} - 1 \right\rfloor$ vertices.

**Definition 3.7.** Suppose that a graph $G(V, E)$ with $|G| = n \geq 4$ is simple and connected such that $\overline{G} = (V, \overline{E})$ is also simple and connected. We say that $G(V, E)$ is a C-$\delta$ graph if $G$ is a $\delta$-graph.

In other words, $G$ is a C-$\delta$ graph if we can label the vertices of $G$ in such a way that

1. the induced graph of the vertices $v_1, v_2, v_3$ in $G$ is either $K_3$ or $P_3$, and
2. for $m \geq 4$, the vertex $v_m$ is adjacent to at most $\left\lfloor \frac{m}{2} - 1 \right\rfloor$ of the prior vertices $v_1, v_2, \ldots, v_{m-1}$.

**Example 3.8.** The cycle $C_n$, $n \geq 6$ is a C-$\delta$ graph and its complement is a $\delta$-graph.
Note that we can label the vertices of $C_6$ clockwise $v_1, v_2, v_3, v_4, v_5, v_6$. The graph induced by $v_1, v_2, v_3$ is $P_3$. The vertex $v_4$ is adjacent to a prior vertex which is $v_3$. Also, the vertex $v_5$ is adjacent to vertex $v_4$ and the vertex $v_6$ is adjacent to two prior vertices $v_1$ and $v_5$. Hence, $C_6$ is C-$\delta$ graph. The 3-prism which is isomorphic to the complement of $C_6$, is a $\delta$-graph.

In [15] it was proved the following result which prove that any $\delta$-graph satisfies delta conjecture.

**Theorem 3.9.** Let $G(V, E)$ be a $\delta$-graph then

$$\text{msr}(G) \leq \Delta(G) + 1 = |G| - \delta(G)$$

### 4 Cactus Graphs and Graph Complement Conjecture

In this section we will prove that the complement of any cactus graph has an orthogonal representation in $\mathbb{R}^5$.

**Definition 4.1.** [12] A simple connected graph $G$ is a **cactus graph** if every edge is part of at most one cycle in $G$.

Equivalently, a simple connected graph $G$ is a cactus graph if any two simple cycles in $G$ have at most one vertex in common. From [12] we know that a cactus graph $G$ is outerplanar since $G$ cannot contain $K_4$ or $K_{2,3}$ as a minor.

**Example 4.2.** Some familiar simple connected graphs are cactus graphs. For example:

1. Trees are cactus graphs with no cycles.
2. Unicyclic graphs are cactus graphs with only one cycle.
3. Chains are cactus graphs with at least two cycles.

It is possible to label the vertices of a cactus graph $G$, $|G| > 5$, satisfying the definition of a C-$\delta$ graph. As a consequence, the graph complement of any cactus graph has an orthogonal representation in $\mathbb{R}^{\Delta(G)+1}$. However, since the number of cycles sharing a single vertex in a cactus graph could be arbitrarily large, the upper bound $\Delta(G) + 1$ for $\text{msr}(G)$ is too large to prove GCC$^+$. The next proposition gives sufficient conditions to get an orthogonal representation in $\mathbb{R}^5$ for the graph complement of a simple connected cactus graph. We will use this result to prove that any cactus graph satisfies GCC$^+$. 
Proposition 4.3. Let $G(V,E), |G| \geq 5$ be a simple connected graph that can be constructed from a path $P : v_1 v_2 v_3$ in such a way that the newly added vertex $v_m, m \geq 5$ is adjacent to at most two of the prior vertices $v_1, v_2, \ldots, v_{m-1}$. Then $\overline{G}(V,E)$ has an orthogonal representation of pairwise linearly independent vectors in $\mathbb{R}^5$.

PROOF:

If $G$ is a simple connected graph with $|G| \leq 5$ it is easy to check that $\overline{G}$ has an orthogonal representation in $\mathbb{R}^5$. So Assume that $|G| = n > 5$. Let $V = \{v_1, v_2, \ldots, v_n\}$. There is a path $P_3 : v_1 v_2 v_3$ as a subgraph of $G$ induced by $\{v_1, v_2, v_3\} \subseteq V_G$. Let $H$ be the induced graph of $G$ obtained from the vertices $v_1, v_2$ and $v_3$. Then $H$ is $K_2 \sqcup K_1$. Let $\{e_1^+, e_2^+, e_3^+, e_4^+, e_5^+\}$ be the standard orthonormal basis for $\mathbb{R}^5$.

Using a similar argument as in Theorem 3.9 we can get vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ and $\overrightarrow{v_4}$ in $\mathbb{R}^5$ for the vertices $v_1, v_2, v_3$, and $v_4$ where all the entries of these vectors are nonzero and belong to different field extensions.

Assume that for any, $Y_{m-1} = (V_{m-1}, E_{m-1})$, $V_{m-1} = \{v_1, v_2, \ldots, v_{m-1}\}, 5 \leq m \leq n$ it is possible to get an orthogonal representation of pairwise linearly independent vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_{m-1}}$ in $\mathbb{R}^5$, of the form $\overrightarrow{v_i} = k_{i,1} e_1 + k_{i,2} e_2 + k_{i,3} e_3 + k_{i,4} e_4 + k_{i,5} e_5, i = 1, \ldots, m-1$ such that all the entries of vectors can be chosen nonzero and from different field extensions. Let $Y_m$ be the induced graph of $G$ given by the vertices $v_1, v_2, \ldots, v_m$. Assume that $v_m$ has a vector $\overrightarrow{v_m} = k_{m,1} e_1 + k_{m,2} e_2 + \cdots + k_{m,5} e_5$.

Case 1. $v_m$ is adjacent in $G$ to two vertices $v_i$ and $v_j$, $i \neq j$ from $\{v_1, v_2, \ldots, v_{m-1}\}$.

Let $\rho$ be a permutation of $(1, 2, \ldots, m-1)$. Suppose $\rho(p(1)), \rho(p(2)), \ldots, \rho(p(m-1))$ are adjacent to $v_m$ in $\overline{G}$ and $\rho(p(m-2)), \rho(p(m-3))$ are not adjacent to $v_m$ in $\overline{G}$. The vectors $\overrightarrow{v_{\rho(1)}}, \overrightarrow{v_{\rho(2)}}, \ldots, \overrightarrow{v_{\rho(m-3)}}, \overrightarrow{v_{\rho(m-2)}}, \overrightarrow{v_{\rho(m-1)}}$ and $\overrightarrow{v_m}$ satisfy the non-homogeneous system $S$ given by:

\[
\begin{align*}
\langle \overrightarrow{v_{\rho(1)}}, \overrightarrow{v_m} \rangle &= g_{m,1}, g_{m,1} \neq 0 \\
\langle \overrightarrow{v_{\rho(2)}}, \overrightarrow{v_m} \rangle &= g_{m,2}, g_{m,2} \neq 0 \\
&\vdots \\
\langle \overrightarrow{v_{\rho(m-3)}}, \overrightarrow{v_m} \rangle &= g_{m,m-3}, g_{m,m-3} \neq 0 \\
\langle \overrightarrow{v_{\rho(m-2)}}, \overrightarrow{v_m} \rangle &= 0 \\
\langle \overrightarrow{v_{\rho(m-1)}}, \overrightarrow{v_m} \rangle &= 0
\end{align*}
\]

containing $m-3$ equations from the adjacency conditions in $\overline{G}$ and two equations from the orthogonal conditions in $\overline{G}$. The vectors $\overrightarrow{v_{\rho(i)}}, i = 1, 2, \ldots, m-1$ have the form $\overrightarrow{v_{\rho(i)}} = k_{\rho(i),1} e_1 + k_{\rho(i),2} e_2 + \cdots + k_{\rho(i),5} e_5$ where all $k_{\rho(i),j}, i = 1, 2, \ldots, m-1, j = 1, 2, \ldots, 5$ are not zero.

Similar argument as in the proof for $\delta$-graphs of the theorem 3.9 can be applied to get a non-homogeneous system $S$ in the variables $k_{m,1}, \ldots, k_{m,5}$ and a homogeneous system $S_H$ in the variables $k_{m,1}, \ldots, k_{m,5}, \ldots, g_{m,1}, \ldots, g_{m,m-3}$. The matrix $A$ of the homogeneous system $S_H$ is given by

$$
A = \begin{pmatrix}
    k_{\rho(1),1} & k_{\rho(1),2} & k_{\rho(1),3} & k_{\rho(1),4} & k_{\rho(1),5} & -1 & 0 & 0 & 0 & 0 & \ldots & 0 \\
    k_{\rho(2),1} & k_{\rho(2),2} & k_{\rho(2),3} & k_{\rho(2),4} & k_{\rho(2),5} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
    k_{\rho(3),1} & k_{\rho(3),2} & k_{\rho(3),3} & k_{\rho(3),4} & k_{\rho(3),5} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
    k_{\rho(4),1} & k_{\rho(4),2} & k_{\rho(4),3} & k_{\rho(4),4} & k_{\rho(4),5} & 0 & 0 & 0 & -1 & 0 & \ldots & 0 \\
    k_{\rho(5),1} & k_{\rho(5),2} & k_{\rho(5),3} & k_{\rho(5),4} & k_{\rho(5),5} & 0 & 0 & 0 & 0 & -1 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    k_{\rho(m-3),1} & k_{\rho(m-3),2} & k_{\rho(m-3),3} & k_{\rho(m-3),4} & k_{\rho(m-3),5} & 0 & 0 & 0 & 0 & 0 & \ldots & -1 \\
    k_{\rho(m-2),1} & k_{\rho(m-2),2} & k_{\rho(m-2),3} & k_{\rho(m-2),4} & k_{\rho(m-2),5} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
    k_{\rho(m-1),1} & k_{\rho(m-1),2} & k_{\rho(m-1),3} & k_{\rho(m-1),4} & k_{\rho(m-1),5} & 0 & 0 & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}
$$
Let Lemma 4.4.

**PROOF:**

To all but at most two of the prior vertices.

with an induced subgraph nonzero.

for $0 \leq m \leq 3$. As a consequence, the system $R \overrightarrow{g} = 0$ where $\overrightarrow{g} = (g_{m,1}, \ldots, g_{m,m-3})^T$, has infinitely many solutions depending on at least three free variables. Choosing all the free variables from different field extensions as in the proof of Theorem 3.9 we can get all the values $g_{m,1}, \ldots, g_{m,m-3}$ nonzero. Also, with a similar argument as with the matrix in Case 2 of the proof of 3.9 we get that the values in the diagonal of the matrix $B_2$ are nonzero. As a consequence, we get that $k_{m,1}, \ldots, k_{m,5}$ can be chosen nonzero.

Since the values $g_{m,i}, i = 3, 4, \ldots, m - 3$ can be chosen non zero and from field extensions different from all previous field extensions taken for the values of $k_{i,j}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, 5$ there exist at least one solution for the system $S$. Therefore the vector $v_m$ can be constructed satisfying all the adjacency conditions and orthogonal conditions.

As a consequence, $Y_m$ has an orthogonal representation of vectors in $\mathbb{R}^5$. Since $5 \leq m \leq n = |G|$, we get that $Y_{|G|} = \overrightarrow{G}$ has an orthogonal representation of vectors in $\mathbb{R}^5$.

**Case 2.** $v_m$ is adjacent in $G$ with only one vertex $v_i \in \{v_1, v_2, \ldots, v_{m-1}\}$. The argument is similar to the Case 1 above. Since we have only one equation from the orthogonal condition $(\overrightarrow{v}_m, \overrightarrow{v}_i) = 0$ for $\overrightarrow{G}$ we can choose the values $g_{m,1}, g_{m,2}, \ldots, g_{m,m-3}$ and the values $k_{m,i}, i = 1, 2, \ldots, 5$ nonzero.

Note that if $G$ is a graph satisfying the hypothesis of Proposition 4.3, then $\text{msr}(\overrightarrow{G}) \leq 5$.

**Lemma 4.4.** Let $G(V, E)$ be a cactus graph with $|G| \geq 5$. Then $\overrightarrow{G}$ can be constructed by starting with an induced subgraph $P_3$, by adding one vertex at a time such that the newest vertex is adjacent to all but at most two of the prior vertices.

**PROOF:**

Let $G(V, E)$ be a cactus graph, $|G| \geq 4$. Since $G$ is simple and connected, $G$ contains a path $P_3 : v_1v_2v_3$. From Proposition 4.3 it is enough to show that $G$ can be constructed from $P_3$ by adding one vertex at a time such that the newest vertex is adjacent to all previous vertices but at most two vertices. Assume that $|G| = M$. Starting with the vertices $v_1, v_2, v_3$ in $P_3$ and since $G$ is outerplanar we can get a path $P$ traveling along the graph following a clockwise orientation, that is, at any intersection we will choose the clockwise direction to continue. Then skipping the repeated vertices we can label all the remaining vertices $v_4, v_5, \ldots, v_{M-1}, v_M$ following the order in which they first appeared in the path $P$. Note that in this way all possible intersection points are counted just once because after the first time we arrived at the point we should skip it. Consider the sequence of induced subgraphs $Y_3(V_3, E_3) \subseteq Y_4(V_4, E_4) \subset \cdots \subset Y_k(V_k, E_k) \subset \cdots \subset Y_{M-1}(V_{M-1}, E_{M-1}) \subseteq Y_M(V_M, E_M) = G$ where $V_3 = \{v_1, v_2, v_3\}$ and $V_k = V_{k-1} \cup \{v_k\}, v_k \not\in Y_k$. 

Reducing this matrix to echelon form we get the matrix $B$

\[
B = \begin{pmatrix}
B_1 & B_2 \\
B_3 & R
\end{pmatrix}
\]

where $B_1$ is a square matrix of size 5, the matrix $B_2$ is a matrix of size $5 \times (m - 3)$, the zero matrix has size $(m - 6) \times 5$, and the matrix $R$ has size $(m - 6) \times (m - 3)$. As a consequence, the system $R \overrightarrow{g} = 0$ where $\overrightarrow{g} = (g_{m,1}, \ldots, g_{m,m-3})^T$, has infinitely many solutions depending on at least three free variables. Choosing all the free variables from different field extensions as in the proof of Theorem 3.9 we can get all the values $g_{m,1}, \ldots, g_{m,m-3}$ nonzero. Also, with a similar argument as with the matrix in Case 2 of the proof of 3.9 we get that the values in the diagonal of the matrix $B_2$ are nonzero. As a consequence, we get that $k_{m,1}, \ldots, k_{m,5}$ can be chosen nonzero.

Since the values $g_{m,i}, i = 3, 4, \ldots, m - 3$ can be chosen non zero and from field extensions different from all previous field extensions taken for the values of $k_{i,j}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, 5$ there exist at least one solution for the system $S$. Therefore the vector $v_m$ can be constructed satisfying all the adjacency conditions and orthogonal conditions.

As a consequence, $Y_m$ has an orthogonal representation of vectors in $\mathbb{R}^5$. Since $5 \leq m \leq n = |G|$, we get that $Y_{|G|} = \overrightarrow{G}$ has an orthogonal representation of vectors in $\mathbb{R}^5$.

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**Lemma 4.4.** Let $G(V, E)$ be a cactus graph with $|G| \geq 5$. Then $\overrightarrow{G}$ can be constructed by starting with an induced subgraph $P_3$, by adding one vertex at a time such that the newest vertex is adjacent to all but at most two of the prior vertices.

**PROOF:**

Let $G(V, E)$ be a cactus graph, $|G| \geq 4$. Since $G$ is simple and connected, $G$ contains a path $P_3 : v_1v_2v_3$. From Proposition 4.3 it is enough to show that $G$ can be constructed from $P_3$ by adding one vertex at a time such that the newest vertex is adjacent to all previous vertices but at most two vertices. Assume that $|G| = M$. Starting with the vertices $v_1, v_2, v_3$ in $P_3$ and since $G$ is outerplanar we can get a path $P$ traveling along the graph following a clockwise orientation, that is, at any intersection we will choose the clockwise direction to continue. Then skipping the repeated vertices we can label all the remaining vertices $v_4, v_5, \ldots, v_{M-1}, v_M$ following the order in which they first appeared in the path $P$. Note that in this way all possible intersection points are counted just once because after the first time we arrived at the point we should skip it. Consider the sequence of induced subgraphs $Y_3(V_3, E_3) \subseteq Y_4(V_4, E_4) \subset \cdots \subset Y_k(V_k, E_k) \subset \cdots \subset Y_{M-1}(V_{M-1}, E_{M-1}) \subseteq Y_M(V_M, E_M) = G$ where $V_3 = \{v_1, v_2, v_3\}$ and $V_k = V_{k-1} \cup \{v_k\}, v_k \not\in Y_k$. 

Reducing this matrix to echelon form we get the matrix $B$
CLAIM 4.5. The induced graph of $Y_k, k \geq 4$ in $\overline{G}$ can be constructed in such a way that $v_k$ is adjacent to all prior vertices $v_1, v_2, \ldots, v_{k-1}$ except to one or two vertices.

PROOF OF THE CLAIM:

Constructing the induced graph $Y_4$ in $G$ we realize that $v_4$ is adjacent to $v_3$ in $G$ but cannot be adjacent to more than two of the vertices $v_1, v_2, v_3$ in $G$. Assume that the induced graph $Y_{k-1}$ in $G$ was constructed as defined above. Then we can construct the induced graph $Y_k$ by adding the vertex $v_k$ to the induced graph $Y_{k-1}$. The vertex $v_k$ cannot be one of the previous vertices because we skip all the repeated vertices. Now if $v_k$ is adjacent to more than two of the prior vertices then the graph $G$ contains two cycles sharing a common edge which contradicts the definition of cactus graphs. As a consequence, $v_k$ is adjacent to all but at most one or two of the prior vertices $v_1, v_2, \ldots, v_{k-1}$ of the induced graph $Y_{k-1}$ in $\overline{G}$ and the claim holds. Finally, from the claim 4.5 since $Y_M = G$ we get that the cactus graph $G$ can be constructed as stated.

Note that Lemma 4.4 implies that a cactus graph $G, |G| \geq 5$ is a C-δ graph.

Corollary 4.6. Let $G(V, E)$ be a cactus graph then $\text{msr}(\overline{G}) \leq 5$.

PROOF:

Let $G$ be a cactus graph. If $|G| \leq 5$ it is straightforward to check that $\text{msr}(\overline{G}) \leq 5$. Then assume that $|G| \geq 5$. From the Lemma 4.4 we have that there is a orthogonal representation for $\overline{G}$ of pairwise linearly independent vectors in $\mathbb{R}^5$. As a consequence $\text{msr}(\overline{G}) \leq 5$.

Proposition 4.7. Let $G(V, E)$ be a cactus graph with exactly one cycle. Then the tree cover number $T(G) = 2$.

PROOF:

Let $G$ be a cactus graph with exactly one cycle. Then $G$ consists of the cycle $C$ and some trees joined to the cycle at some of its vertices. Consider a maximum induced tree $T_1$ of $G$. By definition of a maximum induced tree, $T_1$ contains all but a vertex $w$ of $C$. Let $T_2$ be the component of $G$ containing $w$ and all the trees joined to $G$ at $w$. Then $T_2$ is a tree since is a simple connected graph without cycles. Since all the vertices of $G$ are in $T_1 \cup T_2, \{T_1, T_2\}$ is a minimal tree cover of $G$. As a consequence $T(G) = 2$.

Proposition 4.8. Let $G(V, E)$ be a cactus graph with at least two cycles. Then the tree cover number $T(G) \geq 3$.

PROOF:

Suppose $G$ has exactly two cycles $C_1$ and $C_2$. If $v$ is a cut vertex of $C_1$ and $C_2$ then $G - v$ is a union of two trees $T_1$ and $T_2$. Hence vertices of $G$ are covered by induced trees $T_1, T_2$ and $\{v\}$. If there is a path connecting $u \in C_1$ and $v \in C_2$, then removing a vertex $w_1 \in C_1$ such that $w_1 \neq u$ and another vertex $w_2 \in C_2$ such that $w_2 \neq v$ produces three induced trees $T_1, T_2, T_3$ such that all vertices of $G$ are covered by these three trees. Hence $T(G) \geq 3$ in the case $G$ has exactly two cycles.

Suppose $G$ has three or more cycles. Then an induced subgraph consisting of exactly two cycles has tree cover number of at least three. Hence $T(G) \geq 3$.

Proposition 4.9. Let $G(V, E)$ be a cactus graph with at least two cycles. Then $G$ satisfies GCC_4.

PROOF:

Let $G(V, E)$ be a cactus graph with at least two cycles. Since $G$ is outerplanar we know from 4.4 that $\text{msr}(G) = |G| - T(G)$. From proposition 4.8 we get $T(G) \geq 3$ and from the Corollary 4.6 we get $\text{msr}(\overline{G}) \leq 5$. Therefore $|G| - T(G) + 5 \leq |G| + 2$.

From the results above we get the next result.
Theorem 4.10. Let $G(V, E)$ be a cactus graph. Then $G$ satisfies GCC$_{++}$.

PROOF:
Let $G(V, E)$ be a cactus graph. Then if $G$ has no cycles, then $G$ is a tree we know from [1] that msr($G$) $\leq$ 3 and $G$ satisfies GCC$_{++}$. If $G$ is a unicyclic graph then from [22] we get that $G$ satisfies GCC$_{++}$. Finally, if $G$ has at least two cycles then from proposition [4,9] $G$ satisfies GCC$_{++}$. □

Example 4.11. The cactus graph $G$ shown in figure [4] can be labeled in such a way that $G$ satisfies the definition of C-δ graph. From the above theorem $G$ satisfies GCC$_{++}$.

Figure 3: Cactus Graph

5 Conclusion

Proving GCC$_{++}$ for cactus graph give us other way to prove the result for several families like trees, cycles, chains of cycles, uncyclic graphs, and others which were proved by using combinatorial approach, coloring and other techniques. The way used in this paper could be used to prove GCC$_{++}$ for several others infinite families of simple connected graphs and a good approach in proving GCC$_{++}$.

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