TWO-ECHELON TRADE CREDIT WITH DEFAULT RISK IN AN EOQ MODEL FOR DETERIORATING ITEMS UNDER DYNAMIC DEMAND

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Abstract. In today’s competitive markets, offering delay payments has become a commonly adopted method. In this paper, we examine an optimal dynamic decision-making problem for a retailer selling a single deteriorating product, the demand rate of which varies simultaneously with on-hand inventory level and the length of credit period that is offered to the customers. In addition, the risk of default increases with the credit period length. In this study, not only the supplier would offer fixed credit period to the retailer, but retailer also adopt the trade credit policy to his customer in order to promote the market competition. The retailer can accumulate revenue and interest after the customer pays for the amount of purchasing cost to the retailer until the end of the trade credit period offered by the supplier. A generalized model is presented to determine the optimal trade credit and replenishment strategies that maximize the retailer’s total profit after the default risk occurs over a planning period. For the objective function sufficient conditions for the existence and uniqueness of the optimal solution are provided. Some properties of the optimal solutions are shown to find the optimal ordering policies of the considered problem. At the end of this paper, some numerical examples and the results of a sensitivity and elasticity analysis are used to illustrate the features of the proposed model; we then offer our concluding remarks.

1. Introduction. Given the increasingly competitive global market, enterprises always adopt trade credit financing policy to promote sales, increase market share, and reduce current inventory levels. Trade credit financing plays an important role as a source of funds aside from banks and other financial institutions (Yang and Brige [47]). In India, the non-state-owned enterprises often obtain limited support from banks. Therefore, the trade credit policy is adopted as a very important short term financing method. Through trade credit, vendors provide a delayed payment date for the owed amount. Typically, there is no interest charged if the balance is

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paid on or before the agreed payment date. However, if the payment is not made in full by the end of the permissible delay period, interest is charged on the balance.

Inventory policies under trade credit financing have been researched intensively. Cash on delivery is not quite practical in real markets. Since trade credit can reduce the customer’s inventory holding cost, lengthening the credit period may create reputation among potential customers and consequently gain increased market share. Therefore, trade credit is an increasingly general payment behavior in real business transactions. In recent years, a large amount of attention has been devoted to the models for inventory replenishment policies involving trade credit policy. The fundamental result in the development of the economic order quantity (EOQ) model with trade credit policy was that of Goyal [11] who studied an EOQ model under the conditions of permissible delay in payments. Chu, Chang, and Lan [3] extended the model of Goyal [11] by considering a deterministic inventory model with a constant deterioration rate. Jamal, Sarker and Wang [15] and Chang and Dye [1] allow shortage under delayed payment and deteriorating conditions. In modifying Goyal’s [11] model, Teng [35] then assumed that the selling price is not equal to the purchasing price. The important finding from Teng’s [35] model is that it makes economic sense for a well-established retailer to order small lot sizes to take advantage of the payment delay more frequently. The economic production quantity (EPQ) model is researched by Chung and Huang [5] considering the manufacturer offering the delayed payment to the retailer. Chung [4] provides the complete proof on the optimal ordering policy under cash discount and trade credit. Huang [13] first extends Goyal [11] to analyze the two levels of the trade credit: the manufacturer’s credit period available to retailers as $M$ and the retailers providing the customers a credit period $N, M > N$. Qiu, Liang, Yu and Du [29], Chung et al. [8], Huang [14], Mahata and Goswami [21], Teng and Chang [38], and Chung et al. [6] extend the model of Huang [13]. Recently, Liao, Huang, and Chung [19] attempt to determine economic order quantity for deteriorating items with two-storage facilities where trade credit is linked to order quantity. Mahata [22] discuss the EPQ model for deteriorating items with up-stream full trade credit and down-stream partial trade credit with a constant demand.

The above papers discussed the EOQ or EPQ inventory models under trade credit financing based on the assumption that the demand rate is constant over time. However, in practice, the market demand is always changing rapidly and is affected by several factors such as price, time, inventory level, and delayed payment period, etc. Some researchers realize this phenomenon and extend their studies above to build the inventory models by assuming that the demand is variable. Chung and Liao [7] discuss the inventory replenishment problems with trade credit financing by considering a price-sensitive demand. Sarkar [30] and Teng, Min, and Pan [41], Mahata and Mahata [23] build the economic quantity model with trade credit financing for time-dependent demand. Mahata [24] discuss the inventory replenishment problems with up-stream full trade credit and down-stream partial trade credit financing by considering a price-sensitive demand.

Related to this context, few papers explore the optimal replenishment policies of the retailer under trade credit financing by considering demands that are sensitive to stock and credit period. Some studies regarding trade credit consider the demand dependent on the inventory level. Teng, Krommyda, Skouri, and Lou [37] establish the optimal ordering policy for retailers by considering the stock-dependent demand and two progressive credit periods offered by suppliers. Min,
Zhou and Liu [28] develop an inventory model under conditions of permissible delay in payments, assuming that the items are replenished with the demand rate of the items dependent on the current inventory level. Soni [33] combines the price and inventory level factors to discuss the optimal replenishment policies under trade credit. Some studies focus on demand dependence on delayed payment time. The inventory model with the credit-linked demand are discussed by Su, Ouyang, Ho, and Chang [34], Jaggi, Goyal, and Goel [16], Jaggi, Kapur, Goyal and Goel [17]. Thangam and Uthayakumar [43] discuss trade credit financing for perishable items in a supply chain when demand depends on both selling price and credit period. Ho [12] propose the integrated inventory model with price and credit-linked demand with two-level trade credits. Giri and Maiti [10] discuss the supply chain model with price- and trade credit-sensitive demand with trade credit by considering the fact that a retailer shares a fraction of the profit earned during the credit period.

Indeed, when payment is delayed, the default risk that the buyer cannot pay the entire amount needs to be considered. The longer trade credit can induce more market demand and revenue, the higher the default risk is. In reality, many manufacturers using trade credit are difficult to pay off, which causes their suppliers to face the default risk. Almost all papers focusing on the inventory policy with the trade credit do not consider the effect of default risk. Recently, some researchers introduce the default risk to supply chain management. Shi and Zhang [32] discussed the optimal trade credit policy by considering the endogenous default risk of trade credit with constant demand using the boundary logistic model to describe the downstream of default risk. Lou and Wang [20] study optimal trade credit and order quantity by considering trade credit with a positive correlation of market sales, but are negatively correlated with credit risk. Teng, Lou, and Wang [42] discuss the optimal trade credit and lot size policies considering the demand and default risk sensitive to the credit period with learning curve production costs. Wu, Ouyang, Barron, and Goyal [44] explore optimal credit period and lot size by considering demand dependence on delayed payment time with default risk for deteriorating items with expiration dates. Wu and Chan [45] analyse the lot sizing policies for deteriorating items with expiration dates and partial trade credit to credit risk customer by considering demand dependence on trade credit and default risk related to credit period. Zhang, Dong, Luo, and Segerstedt [48] explore the issue of supply chain coordination by considering trade credit and its risk with the stochastic demand with quantity discount. Dye and Yang [9] discuss the sustainable trade credit and replenishment policy with credit linked demand and credit risk considering the carbon emission constraints. Mahata [25] propose an EOQ model for the retailer to obtain his/her optimal credit period and cycle time under two-level trade credit by considering demand dependent on delayed payment time with default risk for deteriorating items.

When discussing a firm’s operations under trade credit with default risk, it is necessary to consider the effect of market demand related to the inventory level and the credit period. A large pile of products in the supermarket can induce more demand and profit. Moreover, the marginal influence of the credit period on sales is associated with the unrealized potential market demand. Therefore, the market demand of the retailer is a function of the inventory level and credit period. Therefore, in-depth research is required on the inventory policy that considers demand sensitive to stock and credit period involving default risk to extend the traditional
EOQ model. In this paper, inventory models are constructed and optimal solutions for the models are discussed. Numerical examples and sensitivity analysis are conducted in association with managerial insights.

In this paper, we explore the optimal order policy and the trade credit policy for deteriorating items with stock and credit period sensitive demand involving default risk. First, inventory model is established. Second, we show how to make optimal decisions based on the model. Furthermore, numerical examples and sensitivity analysis are conducted to provide some managerial insights. However, none of the paper discusses the optimal trade credit and order policy by considering demand dependence on trade credit and inventory level involving default risk. Therefore, this paper develops an inventory model for deteriorated products under trade credit with demand dependence on trade credit and inventory level involving default risk. The remainder of this paper is organized as follows. The notation and assumptions are introduced in Section 2. The inventory models are presented in Section 3. The optimal solutions for the models are shown in Section 4. The numerical examples and sensitivity analysis are then conducted in Section 5 in association with managerial insights. Section 6 closes the paper with a conclusion.

2. Notation and assumptions. In the paper, we discuss the optimal inventory policy and the trade credit policy for deteriorated products under two-level trade credit financing with demand dependence on trade credit and inventory level involving the default risk. To build the mathematical models, the following notations and assumptions are adopted in this paper.

2.1. Notation. The following notations are used throughout this paper.

Parameters
- $A$: the ordering cost per order in dollars
- $h$: the inventory holding cost per dollar per unit per year excluding interest charges
- $v$: the unit purchasing cost in dollars
- $p$: the unit selling price in dollars with $p > v$
- $I_e$: interest earned per unit per year by the retailer
- $I_p$: interest charged per unit per year to the retailer
- $M$: the trade credit period offered to the retailer by the supplier
- $\theta$: the deterioration rate for the products, where $0 \leq \theta < 1$
- $\xi$: the elasticity of a certain parameter

Decision variables
- $N$: the credit period in years offered by the retailer to its customers
- $T$: the replenishment period in years for the retailer
- $Q$: the replenishment quantity for the retailer

Functions
- $I(t)$: the inventory level in units at time $t$
- $D(t, N)$: the annual market demand rate as a function of the inventory level, $I(t)$, and the trade credit period, $N$
- $F(N)$: the rate of default risk giving the credit period $N$
- $\Pi(N, T)$: the annual total profit in dollars of inventory system, which is a function of $N$ and $T$.

Next, the mathematical models proposed in this paper are based on the following assumptions:
1. The supplier provides the trade credit period $M$ to the retailer, and the retailer offers its customers trade credit period $N$.

2. An increase in shelf space for an item induces more consumers to buy it. This occurs because of its visibility, popularity or variety. Conversely, low stocks of certain goods (e.g., baked goods) might raise the perception that they are not fresh. Therefore, building up inventory often has a positive impact on the sales, as well as the profit. Levin et al. [18] observed that “large piles of consumer goods displayed in a supermarket will lead the customer to buy more. Yet, too much pileup in everyone’s way leaves a negative impression on buyer and employee alike”. Hence, demand rate is a function of inventory level $I(t)$. By following Chang et al. [2] and Yang et al. [46], the demand rate $D$ is considered as proportional to $I(t)$. Thus, one can have

$$D \approx D_0 + aI(t),$$

where both $D_0$ and $a > 0$.

3. Since the longer the credit period to the buyer, the higher the default risk (e.g., a 30-year mortgage has a higher default risk than a 15-year mortgage), we may assume without loss of generality that the rate of default risk giving the credit period $N$ offered by the retailer is $F(N) = 1 - e^{-bN}$, where $b$ is a positive constant. This default risk pattern is used in some studies (Lou and Wang [20], Zhang et al. [48], Mahata et al. [27]).

4. If $T \geq M$, then the retailer settles the account at time $M$ and pays for the interest charges on items in stock with rate $I_p$ over the interval $[M, T]$. If $T \leq M$, then the retailer settles the account at time $M$ and there is not interest charge in stock during the whole cycle. On the other hand, if $M > N$, the retailer can accumulate revenue and earn interest during the period from $N$ to $M$ with rate $I_e$ under the up-stream and down-stream trade credit conditions.

5. The time to deterioration of a product follows an exponential distribution with parameter $\theta$, i.e. the deterioration rate is a constant fraction of the on-hand inventory.

6. Replenishment is instantaneous.

7. In today’s time-based competition, we may assume that shortages are not allowed to occur.

8. The time horizon is infinite.
Given the above notation and assumptions, it is possible to formulate the retailer’s annual total profit as a function of the downstream trade credit $N$ and the replenishment cycle time $T$ for deteriorating items into a mathematical model.

3. **Mathematical model.** According to the notation and assumptions mentioned above, the behavior of the inventory system during the replenishment period $[0, T]$ is as follows. The retailer receives the order quantity $Q$ at $t = 0$. Hence, the inventory starts with $Q$ units at $t = 0$, and gradually reaches zero at $t = T$ due to the combined influence of the demand and deterioration.

Therefore, the variation of inventory with respect to time can be described by the following differential equation:

$$
\frac{dI(t)}{dt} + \theta I(t) = -D_0 - aI(t) - \lambda(N), \quad 0 \leq t \leq T, \quad (1)
$$

with the boundary condition $I(T) = 0$.

By solving the differential equation (1), we can obtain

$$
I(t) = \frac{\varepsilon}{\eta} \left( e^{\eta(T-t)} - 1 \right), \quad 0 \leq t \leq T, \text{ where } \eta = \theta + a, \ \varepsilon = D_0 + \lambda(N). \quad (2)
$$

Utilizing the result of (2), the ordering quantity during the replenishment period $[0, T]$, denoted by $Q$, is

$$
Q = I(0) = \frac{\varepsilon}{\eta} \left( e^{\eta T} - 1 \right). \quad (3)
$$

The annual market demand is

$$
D = \varepsilon + \frac{a\varepsilon}{\eta} \left( e^{\eta(T-t)} - 1 \right). \quad (4)
$$

The profit per cycle consists of the following elements: revenue, ordering cost and purchasing cost, holding cost, interest charged, and interest earned. The components are evaluated as follows.

1. **Ordering cost:** Annual ordering cost is $\frac{A}{T}$.

2. **Purchasing cost:** Annual purchasing cost is $\frac{vQ}{T} = \frac{v\varepsilon}{T\eta} \left( e^{\eta T} - 1 \right)$.

3. **Holding cost:** Annual stock holding cost excluding the interest charges is

$$
c_h = \frac{h}{T} \int_0^T I(t)dt = \frac{h\varepsilon}{T\eta^2} \left( e^{\eta T} - \eta T - 1 \right).
$$

4. **Sales revenue:** The sales revenue considering default risk is

$$
R = \frac{pe^{-bn}}{T} \int_0^T D(t, N)dt = \frac{pe^{-bN\varepsilon}}{T\eta^2} \left[ \theta T\eta + a(e^{\eta T} - 1) \right].
$$

5. **Interest earned and interest paid:** Based on the values of $M$ (i.e. the time at which the retailer must pay the supplier to avoid interest charge), $T$ (i.e. the replenishment cycle time), and $T + N$ (i.e. the time at which the retailer receives the payment from the last customer), we have to examine following three situations: (1) $0 < T + N \leq M$, (2) $T \leq M \leq T + N$, and $M \leq T$. Note that different approaches are available in existing literature to calculate the interest earned and interest charged. In this paper, we have employed Teng [36] approach throughout in this article.
Situation 1: $0 < T + N \leq M$ (i.e., $0 < T < M - N$).
In this case, the retailer receives the total revenue at time $T + N$ (i.e., a customer receives items at time $t$, and must pay the credit payment at time $t + N$ of sales), and is able to pay the supplier the total purchase cost at time $M$. Consequently, there is no interest charged.

On the other hand, the retailer starts selling products at time 0, but getting the money at time $N$. Consequently, the retailer accumulates revenue in an account that earns $I_e$ per dollar per year starting from $N$ through $M$. During the period $[N, T + N]$, the retailer can obtain the interest earned on the sales revenues received and on the full sales revenue during the period $[T + N, M]$. Therefore, the annual interest earned is

$$\frac{pI_e}{T} \left[ \int_0^{T+N} \int_N^{T+N} D(u - N, N) du dt + (M - T - N) \int_0^T D(u, N) du \right] - \frac{vI_p}{T} \int_M^{T+N} I(t - N) dt = \frac{pI_e}{T} \left[ \frac{\theta \eta^2}{2} + a(1 - e^{\eta T}) + Ta \eta e^{\eta T} + \eta(M - T - N) \times (T \eta^2 + a(e^{\eta T} - 1 - \eta T)) \right].$$

Situation 2: $N \leq M \leq T + N$
In this sub-case, the retailer cannot payoff the supplier by $M$ because the supplier credit period $M$ is shorter than the customer last payment time $T + N$. As a result, the retailer must finance all items sold after time $M - N$ at an interest charged $I_p$ per dollar per year. As a result, the annual interest charged is

$$\frac{vI_p}{T} \int_{M-N}^T I(t) dt = \frac{vI_p}{T} \int_{M-N}^T I(t) dt = \frac{vI_p}{T} \left[ e^{\eta (T - M + N)} - \eta (T - M + N) - 1 \right].$$

On the other hand, the retailer starts selling products at time 0, but getting the money at time $N$. Consequently, the retailer accumulates revenue in an account that earns $I_e$ per dollar per year starting from $N$ through $M$. Therefore, we have the interest earned in the following:

$$\frac{pI_e}{T} \int_N^M \int_N^{T+N} D(u - N, N) du dt = \frac{pI_e}{T \eta^3} \left[ \frac{\theta}{2} \eta^2 (M - N)^2 + a \left( e^{\eta (T - M + N)} - \eta (T - M + N) \right) \right].$$

Situation 3: $M \leq N$
Since $N \geq M$, there is no interest earned for the retailer. In addition, the retailer must finance all items ordered at time $M$ at an interest charged $I_e$ per dollar per year, and start to payoff the loan after time $N$ and pay off the loan from $[N, T + N]$. Therefore, the interest charged per cycle is

$$\frac{vI_p}{T} \left[ \int_N^{T+N} I(t - N) dt + (N - M)Q \right] = \frac{vI_p}{T \eta^2} \left[ e^{\eta T} - \eta T - 1 + \eta (N - M) (e^{\eta T} - 1) \right].$$

Combining the above results, the retailer’s annual total profit can be expressed as follows.
Retailer’s profit per cycle = net revenue after default risk - ordering cost - purchasing cost - holding cost excluding interest earned and interest charged + interest earned - interest charged.

\[ \Pi(N, T) = \begin{cases} 
\Pi_1(N, T), & \text{if } N \leq T + N \leq M \\
\Pi_2(N, T), & \text{if } N \leq M \leq T + N \\
\Pi_3(N, T), & \text{if } M \leq N \leq T + N 
\end{cases} \] (5)

where

\[
\Pi_1(N, T) = \frac{pe^{-bN}}{T \eta^2} \left[ \theta T \eta + a(e^{\eta T} - 1) \right] - \frac{A}{T} \\
- \frac{v \varepsilon}{T \eta} (e^{\eta T} - 1) - \frac{h \varepsilon}{T \eta^2} (e^{\eta T} - \eta T - 1) \\
+ \frac{pL \varepsilon}{T \eta^3} \left[ \frac{\theta}{2} T^2 \eta^2 + a \left( 1 - e^{\eta T} + T \eta e^{\eta T} + \eta (M - N - N) \right) \right] \\
\times (T \eta^2 + a(e^{\eta T} - \eta T - 1)),
\] (6)

\[
\Pi_2(N, T) = \frac{pe^{-bN}}{T \eta^2} \left[ \theta T \eta + a(e^{\eta T} - 1) \right] - \frac{A}{T} \\
- \frac{v \varepsilon}{T \eta} (e^{\eta T} - 1) - \frac{h \varepsilon}{T \eta^2} (e^{\eta T} - \eta T - 1) \\
+ \frac{pL \varepsilon}{T \eta^3} \left[ \frac{\theta}{2} (M - N) \eta^2 + a \left( e^{\eta(T-M+N)} - e^{\eta T} \right) + \eta (M - N) a e^{\eta T} \right] \\
- \frac{v \varepsilon}{T \eta^2} \left[ e^{\eta(T-M+N)} - \eta (T - M + N) - 1 \right],
\] (7)

and

\[
\Pi_3(N, T) = \frac{pe^{-bN}}{T \eta^2} \left[ \theta T \eta + a(e^{\eta T} - 1) \right] - \frac{A}{T} \\
- \frac{v \varepsilon}{T \eta} (e^{\eta T} - 1) - \frac{h \varepsilon}{T \eta^2} (e^{\eta T} - \eta T - 1) \\
- \frac{v \varepsilon}{T \eta^2} \left[ e^{\eta T} - \eta T - 1 + \eta (N - M)(e^{\eta T} - 1) \right],
\] (8)

which are functions of two variables \(N\) and \(T\). Therefore, the retailer’s objective is to determine the optimal credit period \(N^*\) and cycle time \(T^*\) such that the annual total profit \(\Pi_i(N, T)\) for \(i = 1, 2, 3\) is maximized.

4. Solution methodology. To obtain global optimal solutions \((N^*, T^*)\), we pursue the Wu et al. [44] solution method. The problem is \(\max \Pi(T, N)\). For each of the branches, we first show the existence of a unique optimal solution value of \(T^*\) for any given \(N\), and then we decide the value of \(N^*\) for any known \(T\).

4.1. Case 1. \(N \leq T + N \leq M\). For any given \(N\), to find the optimal replenishment policy \(T^*_1\), we take the first-order partial derivative of \(\Pi_1(T|N)\) with respect to \(T\). We can obtain

\[
\frac{\partial \Pi_1(T|N)}{\partial T} = \frac{1}{T} f_1(T|N),
\]
where

\[ f_1(T|N) = A + \left[ \frac{pe^{-kN\varepsilon}}{\eta^2} a - \frac{v\varepsilon}{\eta} - \frac{h\varepsilon}{\eta^2} - \frac{pIe\varepsilon}{\eta^3} a + \frac{pIe\varepsilon}{\eta^3} a + \frac{pIe\varepsilon}{\eta^2}(M - T - N)a \right] \times (T\eta^{\alpha T} - e^{\eta T} + 1) + \left[ - \frac{pIe\varepsilon}{2\eta} - \frac{pIe\varepsilon a(e^{\eta T} - 1)}{\eta^2 T} \right] T^2. \] (9)

The optimal value of \( T \), say \( T_1^* \), can be found by solving the equation \( f_1(T|N) = 0 \). It is easy to obtain the following derivative

\[ \frac{df_1(T|N)}{dT} = - \frac{pIe\varepsilon}{\eta} T - \frac{pIe\varepsilon a(e^{\eta T} - 1 + T\eta^{\alpha T})}{\eta^2} + \left[ \frac{peae^{-kN\varepsilon} - v\varepsilon - h\varepsilon}{\eta^2} - \frac{pIe\varepsilon a(M - T - N)}{\eta^2} \right] Te^{\eta T}. \] (10)

By taking the second-order partial derivative of \( \Pi_1(T|N) \) with respect to \( T \), we can obtain

\[ \frac{\partial^2 \Pi_1(T|N)}{\partial T^2} = - \frac{2}{T^3} f_1(N|T) + \frac{1}{T^2} f'_1(N|T) = - \frac{2}{T^3} [f(N|T) - Tf'_1(N|T)]. \]

If \( \hat{T}_1 \) is the root of \( \frac{\partial \Pi_1(T|N)}{\partial T} = 0 \) (this may or may not exist) and \( f'_1(\hat{T}_1|N) < 0 \), then

\[ \frac{\partial^2 \Pi_1(T|N)}{\partial T^2} = \frac{1}{T^2} f'_1(N|T) < 0. \]

\( \hat{T}_1 \) corresponds to a maximum value of \( \Pi_1(T|N) \).

If \( \hat{T}_1 \) is feasible, i.e., \( 0 < \hat{T}_1 \leq M - N \), the optimal replenishment cycle is \( T_1^* = \hat{T}_1 \). If \( \hat{T}_1 \) does not exist or is infeasible, then the optimal replenishment cycle is \( T_1^* = M - N \). The optimal value of the replenishment order quantity is \( Q = \frac{\varepsilon}{\eta}(e^{\eta T^*} - 1) \).

Based on Equation (10), when \( pa - h - \eta \varepsilon - \frac{pIe\varepsilon a}{\eta} + pIe Ma < 0 \), then \( f'_1(\hat{T}_1|N) < 0 \). Therefore, the retailer’s profit is the concave function of \( T \). Therefore, it is easy to obtain \( \lim_{T \to 0} f_1(T|N) = A > 0 \), \( \lim_{T \to \infty} f_1(T|N) = -\infty < 0 \). Based upon the above arguments, the intermediate value theorem yields that the optimal solution \( T_1^* \), not only exists but also is unique. The similar procedure as described in Case 1 can be applied to the remaining two cases.

Based on the analysis, it is easy to obtain Theorem 1.

**Theorem 1.** For any given \( N \geq 0 \), if \( pa - h - \eta \varepsilon - \frac{pIe\varepsilon a}{\eta} + pIe Ma < 0 \),

1. \( \hat{T}_1 \) does not only exist, but is also unique.
2. If \( 0 < \hat{T}_1 \leq M - N \), then \( \Pi_1(T|N) \) subject to \( T + N \leq M \) is maximized at \( T_1^* = \hat{T}_1 \).
3. If \( \hat{T}_1 \geq M - N \), then \( \Pi_1(T|N) \) subject to \( T + N \leq M \) is maximized at \( T_1^* = M - N \).
For any given \( T > 0 \), taking the first-order and second-order partial derivative of \( \Pi_1(\eta|T) \) with respect to \( \eta \), we can obtain

\[
\frac{\partial \Pi_1(\eta|T)}{\partial \eta} = (-ke^{-k\eta} + e^{-k\eta} \epsilon') \frac{p}{\eta^2} \left( \eta^2 T + a(e^{\eta T} - 1 - \eta T) \right) - \frac{\epsilon'}{T \eta} (e^{\eta T} - 1)
\]

\[
- \frac{h \epsilon'}{T \eta^2} (e^{\eta T} - 1 - \eta T) + \frac{p \lambda \epsilon'}{T} \left( \frac{\theta}{2 \eta} T^2 + a \frac{1 - e^{\eta T}}{\eta^3} + a e^{\eta T} \frac{T}{\eta^2} \right)
\]

\[
+ \frac{p \lambda \epsilon'}{T \eta^2} \left[ \eta^2 T + a \left( e^{\eta T} - 1 - \eta T \right) \right] \left[ \epsilon' (M - T - N) - \epsilon \right],
\]

\[
(11)
\]

\[
\frac{\partial^2 \Pi_1(\eta|T)}{\partial \eta^2} = (\eta^2 T + a(e^{\eta T} - 1 - \eta T)) \frac{p}{\eta^2} \left[ \eta^2 T + a(e^{\eta T} - 1 - \eta T) \right]
\]

\[
- \frac{\epsilon'}{T \eta} (e^{\eta T} - 1) - \frac{h \epsilon'}{T \eta^2} \left( e^{\eta T} - 1 \right)
\]

\[
+ \frac{p \lambda \epsilon'}{T} \left( \frac{\theta}{2 \eta} T^2 + a \frac{1 - e^{\eta T}}{\eta^3} + a e^{\eta T} \frac{T}{\eta^2} \right)
\]

\[
+ \frac{p \lambda \epsilon'}{T \eta^2} \left[ \eta^2 T + a \left( e^{\eta T} - 1 - \eta T \right) \right] \left[ \epsilon'' (M - T - N) - 2 \epsilon' \right].
\]

\[
(12)
\]

where \( \epsilon' = \lambda'(N) \) and \( \epsilon'' = \lambda''(N) \).

Finding \( \Pi_1(\eta|T) \) is a continuous function of \( N \) for \( N \in [0, \infty) \) is easy. Therefore, \( \Pi_1(\eta|T) \) has a maximum value for \( N \in [0, \infty) \). To identify whether \( N = 0 \) or positive, we define the following discrimination term.

\[
\Delta_{N_1} = \left. \frac{\partial \Pi_1(\eta|T)}{\partial \eta} \right|_{N=0}
\]

\[
= (-k e^{\eta T} + e^{\eta T} \epsilon') \frac{p}{\eta^2} \left[ \eta^2 T + a(e^{\eta T} - 1 - \eta T) \right] - \frac{\epsilon'}{T \eta} (e^{\eta T} - 1)
\]

\[
- \frac{h \epsilon'}{T \eta^2} (e^{\eta T} - 1 - \eta T) + \frac{p \lambda \epsilon'}{T} \left( \frac{\theta}{2 \eta} T^2 + a \frac{1 - e^{\eta T}}{\eta^3} + a e^{\eta T} \frac{T}{\eta^2} \right)
\]

\[
+ \frac{p \lambda \epsilon'}{T \eta^2} \left[ \eta^2 T + a \left( e^{\eta T} - 1 - \eta T \right) \right] \left[ \epsilon' (M - T - N) - \epsilon \right].
\]

\[
(13)
\]

\[
\Delta_{N_M_1} = \left. \frac{\partial \Pi_1(\eta|T)}{\partial \eta} \right|_{N=M}
\]

\[
= (-k e^{\eta T} + e^{\eta T} \epsilon') \frac{p}{\eta^2} \left[ \eta^2 T + a(e^{\eta T} - 1 - \eta T) \right] - \frac{\epsilon'}{T \eta} (e^{\eta T} - 1)
\]

\[
- \frac{h \epsilon'}{T \eta^2} (e^{\eta T} - 1 - \eta T) + \frac{p \lambda \epsilon'}{T} \left( \frac{\theta}{2 \eta} T^2 + a \frac{1 - e^{\eta T}}{\eta^3} + a e^{\eta T} \frac{T}{\eta^2} \right)
\]

\[
- \frac{p \lambda \epsilon'}{T \eta^2} \left[ \eta^2 T + a \left( e^{\eta T} - 1 - \eta T \right) \right] \left[ \epsilon' T + \epsilon \right].
\]

\[
(14)
\]

If \( \frac{\partial^2 \Pi_1(\eta|T)}{\partial \eta^2} < 0 \), \( \Pi_1(\eta|T) \) is strictly a concave function in \( N \), and hence there exists a unique maximum solution \( \hat{N}_1 \). If \( \Delta_{N_1} \leq 0 \), then \( \Pi_1(\eta|T) \) is maximized at \( N_1^* = 0 \); if \( \Delta_{N_1} > 0 \) but \( \Delta_{N_M_1} < 0 \), \( \Pi_1(\eta|T) \) is maximized with \( N_1^* = \hat{N}_1 \) where \( 0 < \hat{N}_1 < M \); if \( \Delta_{N_M_1} \geq 0 \) then \( \Pi_1(\eta|T) \) is maximized at \( N_1^* = M \). Based on Equation (12), it is easy to obtain if \( k^2 \epsilon - 2k \epsilon' + \epsilon'' \leq 0 \) and \( \epsilon'' \left( \frac{\theta}{2 \eta} T^2 + a(1 - e^{\eta T} + \frac{\theta}{2 \eta} T^2) \right) < 0 \).
with Eq. (11), we have \( \lim_{N \to \infty} \) the solution of \( \Pi \) for any given \( N \). The following theoretical results can be derived.

By taking the second-order partial derivative of \( \Pi \) with respect to \( T \), we can obtain

\[
\frac{\partial^2 \Pi_2(T|N)}{\partial T^2} = -\frac{2}{T^3} f_2(N|T) + \frac{1}{T^2} f'_2(N|T) = -\frac{2}{T^3} [f_2(N|T) - T f'_2(N|T)].
\]

From equation (15), if \( p a - h - v e - \frac{v_I e a T}{\eta} + p I e \eta a T < 0 \) and \( \frac{v_I e a}{\eta} - v I a < 0 \), then \( \frac{\partial^2 \Pi_2(T|N)}{\partial T^2} < 0 \). The retailer’s profit is a concave function of the \( T \). It is easy to
obtain \( \lim_{T \to \infty} f_2(T|N) = -\infty < 0 \). Moreover,

\[
\lim_{T \to 0} f_2(T|N) = A - \frac{pI_p}{\eta^2} \left[ \frac{\theta}{2} \eta^2 (M - N)^2 + ae^{\eta(T-M+N)} \right] - \frac{vI_p}{\eta^2} \left[ 1 - e^{\eta(-M+N)} - \eta(M - N) \right].
\]

Given that \( e^{-\eta(M-N)} > 1-\eta(M-N) \), if \( A - \frac{pI_p}{\eta^2} \left[ \frac{\theta}{2} \eta^2 (M - N)^2 + ae^{\eta(T-M+N)} \right] > 0 \), then we can obtain \( \lim_{T \to 0} f_2(T|N) > 0 \). Therefore, \( \lim_{T \to \infty} f_1(T|N) = -\infty < 0 \) and \( f_2(T = 0|N) > 0 \). Given the intermediate value theorem, \( \hat{T}_2 \) is the root of \( \partial \Pi_2(T|N) \) for \( T \in (0, \infty) \). Therefore, \( \hat{T}_2 \) not only exists, but is also unique. If \( 0 < T_2 \leq M - N \), then \( \Pi_2(T|N) \) subject to \( T + N \geq M \) is maximized at \( T^*_2 = \hat{T}_2 \).

However, if \( pa - h - v\eta - \frac{pI_p a}{\eta} + pI_c Ma < 0 \) and \( \frac{pI_p a}{\eta} - vI_p < 0 \), when

\[
A - \frac{pI_p}{\eta^2} \left[ \frac{\theta}{2} \eta^2 (M - N)^2 + ae^{\eta(T-M+N)} \right] - \frac{vI_p}{\eta^2} \left[ 1 - e^{\eta(-M+N)} - \eta(M - N) \right] \leq 0,
\]

then \( \lim_{T \to \infty} f_1(T|N) = -\infty < 0 \) and \( f_2(T = 0|N) < 0 \), then \( T^*_2 = M - N \).

Based on these analyses, Theorem 3 can be used to decide the optimal policy for Case 2.

**Theorem 3.** For any given \( N \geq 0 \),

(a) when \( A - \frac{pI_p}{\eta^2} \left[ \frac{\theta}{2} \eta^2 (M - N)^2 + ae^{\eta(T-M+N)} \right] > 0 \), if \( pa - h - v\eta - \frac{pI_p a}{\eta} + pI_c Ma < 0 \) and \( \frac{pI_p a}{\eta} - vI_p < 0 \), then

(1) \( \hat{T}_2 \) does not only exists, but is also unique.

(2) If \( 0 < \hat{T}_2 \leq M - N \), then \( \Pi_2(T|N) \) subject to \( T + N \geq M \) is maximized at \( T^*_2 = \hat{T}_2 \).

(3) If \( T_2 \geq M - N \), then \( \Pi_2(T|N) \) subject to \( T + N \geq M \) is maximized at \( T^*_2 = \hat{T}_2 \).

(b) When \( A - \frac{pI_p}{\eta^2} \left[ \frac{\theta}{2} \eta^2 (M - N)^2 + ae^{\eta(T-M+N)} \right] - \frac{vI_p}{\eta^2} \left[ 1 - e^{\eta(-M+N)} - \eta(M - N) \right] \leq 0 \), if \( pa - h - v\eta - \frac{pI_p a}{\eta} + pI_c Ma < 0 \) and \( \frac{pI_p a}{\eta} - vI_p < 0 \), the \( T^*_2 = M - N \)

For any given \( T > 0 \), taking the first-order and second-order partial derivative of \( \Pi_2(N|T) \) with respect to \( N \), we can obtain

\[
\frac{\partial \Pi_2(N|T)}{\partial N} = (-ke^{-kN} \varepsilon + e^{-kN} \varepsilon') \frac{p}{\eta^2} [\theta (\eta')^2 T + a(e^{\eta'T} - 1 - \eta T)] - \frac{v_{\varepsilon'}}{T\eta} (e^{\eta'T} - 1)
\]

\[
- \frac{h\varepsilon'}{T\eta^2} (e^{\eta'T - 1 - \eta T}) + \frac{pI_p \theta}{2T\eta} \left[ (M - N)^2 - 2\varepsilon(M - N) \right]
\]

\[
+ \frac{pI_c}{T\eta^2} \left[ (\eta\varepsilon + \varepsilon')a(e^{\eta(T-M+N)} - e^{\eta T}) + (M - N)\eta a\varepsilon e^{\eta T} \right] - \frac{v_{\varepsilon'} I_p}{T\eta^2}
\]

\[
\times \left[ e^{\eta(T-M+N)} - \eta(T - M + N) - 1 \right] - \frac{v I_p}{T\eta} \left[ e^{\eta(T-M+N)} - 1 \right].
\]
To identify whether $N_2$ is 0 or positive, we define the following discrimination term:

\[
\Delta_{N_2} = \left. \frac{\partial^2 \Pi_2(N|T)}{\partial N^2} \right|_{N=0} = (-k\varepsilon + \varepsilon') \frac{p}{\eta^2 T} \left[ \eta^2 T + a(e^{\eta T} - 1 - \eta T) \right] - \frac{\varepsilon'}{T} (e^{\eta T} - 1)
\]

\[
- \frac{h \varepsilon'}{T^2 \eta} (e^{\eta T} - 1 - \eta T) + \frac{p I_e \theta}{2 T \eta} \left[ \varepsilon' M^2 - 2\varepsilon M \right]
\]

\[
+ \frac{p I_e}{T \eta^3} \left[ (\eta \varepsilon + \varepsilon') \left( e^{\eta(T-M)} - e^{\eta T} \right) + M \eta a \varepsilon' e^{\eta T} \right]
\]

\[
- \frac{\varepsilon' I_p}{T \eta^2} \left[ e^{\eta(T-M)} - \eta(T-M) - 1 \right] - \frac{\varepsilon I_p}{T \eta} \left[ e^{\eta(T-M)} - 1 \right]
\]

(19)

\[
\Delta_{M_2} = \left. \frac{\partial^2 \Pi_2(N|T)}{\partial N^2} \right|_{N=M} = (-k\varepsilon - kM \varepsilon + e^{-kM} \varepsilon') \frac{p}{\eta^2 T} \left[ \eta^2 T + a(e^{\eta T} - 1 - \eta T) \right] - \frac{\varepsilon'}{T} (e^{\eta T} - 1)
\]

\[
- \frac{h \varepsilon'}{T^2 \eta} (e^{\eta T} - 1 - \eta T) - \frac{\varepsilon' I_p}{T^2 \eta^2} \left[ e^{\eta T} - \eta T - 1 \right] - \frac{\varepsilon I_p}{T \eta} \left[ e^{\eta T} - 1 \right]
\]

(20)

If $\frac{\partial^2 \Pi_2(N|T)}{\partial N^2} < 0$, $\Pi_2(N|T)$ is strictly a concave function in $N$, a unique maximum solution $\bar{N}_2$ exists. If $\Delta_{N_2} \leq 0$, then $\Pi_2(N,T)$ is maximized at $N^*_2 = 0$; if $\Delta_{N_2} > 0$ but $\Delta_{M_2} < 0$, then there exists a unique solution $N^*_2 = \bar{N}_2$ where $0 < \bar{N}_2 < M$, such that $\Pi_2(N,T)$ is maximized; and if $\Delta_{M_2} \geq 0$, $\Pi_2(N,T)$ is maximized at $N^*_2 = M$. Based on equation (17), any given $T$, if $k^2 \varepsilon - 2ke' + e'' \leq 0$ and $\theta \eta^2 \left[ \varepsilon'' (M-N)^2 - 2(M-N) \varepsilon' + 2\varepsilon \right] + 2a \left[ (2\eta \varepsilon' + \varepsilon') \left( e^{\eta(T-M+N)} - e^{\eta T} \right) + \eta^2 e^{\eta(T-M+N)} \varepsilon + \eta (M-N) e^{\eta T} e'' \right] \leq 0$, then $\Pi_2(N|T)$ is strictly concave function in $N$, resulting to a unique maximum solution.

If $\frac{\partial^2 \Pi_2(N|T)}{\partial N^2} \geq 0$, then $\Pi_2(N|T)$ is a convex function of $N$. Therefore, the optimal solution of $\Pi_2(N|T)$ is at one of the two boundary points (0 or $\infty$).
with Eq. (16), we have \( \lim_{N \to \infty} \frac{\partial \Pi_3(N|T)}{\partial N} = 0 \). Hence \( T = +\infty \) is not an optimal solution, which implies that the optimal solution is \( N^*_2 = 0 \). Let \( \delta \) denote that
\[
\delta = \theta \eta^2 \left[ \epsilon''(M - N)^2 - 2(M - N)\epsilon' + 2\epsilon \right] + 2a \left[ (2\eta \epsilon' + \epsilon'')(e^{\eta(T-M+N)} - e^{\eta T}) + \eta^2 e^{\eta(T-M+N)\epsilon} + \eta(M-N)e^{\eta T}\epsilon'' \right].
\]
Consequently, the following theoretical results can be derived based on the analysis above.

**Theorem 4.** For any given \( T > 0 \), if \( -k^2 \epsilon - 2k \epsilon' + \epsilon'' \leq 0 \) and \( \delta \leq 0 \); then we can have
1. \( \Pi_3(N|T) \) is strictly concave function in \( N \), and hence there exists a unique maximum solution.
2. If \( \Delta_{N_2} \leq 0 \), then \( \Pi_3(N|T) \) is maximized at \( N^*_2 = 0 \).
3. If \( \Delta_{N_2} > 0 \) but \( \Delta_{N_{M_2}} < 0 \), then there exists a unique \( 0 < \tilde{N}_2 < M \) such that \( \Pi_2(N|T) \) is maximized at \( N^*_2 = \tilde{N}_2 > 0 \).
4. If \( \Delta_{N_{M_2}} \geq 0 \), \( \Pi_3(N,T) \) is maximized at \( N^*_2 = M \).

4.3. **Case 3.** \( M \leq N \leq T + N \). To find the optimal replenishment policy \( T_3^* \), we take the first-order partial derivative of \( \Pi_3(T|N) \) with respect to \( T \). We can obtain
\[
\frac{\partial \Pi_3(T|N)}{\partial T} = \frac{1}{T^2} f_3(T|N),
\]
where
\[
f_3(T|N) = A + \frac{peae^{-kN} - h\epsilon - v\epsilon - vI_p\epsilon(1 + \eta(N-M))}{\eta^2} (\eta T e^{\eta T} - e^{\eta T} + 1).
\]
It is easy to obtain
\[
\frac{df_3(T|N)}{dT} = \left[ peae^{-kN} - h\epsilon - v\epsilon - vI_p\epsilon(1 + \eta(N-M)) \right]Te^{\eta T}.
\]
Taking the second-order partial derivative of \( \Pi_3(N,T) \) with respect to \( T \), we can obtain
\[
\frac{\partial^2 \Pi_3(T|N)}{\partial T^2} = -\frac{2}{T^2} f_3(T|N) + \frac{1}{T^2} f'_3(T|N) = -\frac{2}{T^3} [f_3(T|N) - T f'_3(T|N)].
\]
If \( \tilde{T}_3 \) is the root of \( \frac{\partial \Pi_3(T|N)}{\partial T} = 0 \) (this may or may not exist) and \( f'_3(\tilde{T}_1|N) < 0 \), then
\[
\frac{\partial^2 \Pi_3(T|N)}{\partial T^2} = \frac{1}{T^2} f'_3(T|N) < 0.
\]
Based on Equation (19), if \( peae^{-kN} - h\epsilon - v\epsilon - vI_p\epsilon(1 + \eta(N-M)) < 0 \), then \( \frac{\partial^2 \Pi_3(T|N)}{\partial T^2} < 0 \). Thus, \( \tilde{T}_3 \) does not only exists, but is also unique. Given this condition, \( \lim_{T \to \infty} f_3(T|N) = -\infty < 0 \), \( \lim_{T \to 0} f_3(T|N) = A > 0 \). Therefore, the intermediate value theorem yields that \( \tilde{T}_3 \) exists as the root of \( f_3(T|N) = 0 \) for \( T \in (0, \infty) \). Based on the analysis above, it is easy to obtain Theorem 5.

**Theorem 5.** For any given \( N \) with \( M \leq N \), if \( peae^{-kN} - h\epsilon - v\epsilon - vI_p\epsilon(1 + \eta(N-M)) < 0 \), then \( \Pi_3(T|N) \) is a strictly concave function in \( T \), which has a unique maximum solution \( \tilde{T}_3 \) for \( T \in (0, \infty) \); the optimal value of \( T_3^* \) corresponds to \( \tilde{T}_3 \).
For any given $T > 0$, taking the first-order and second-order partial derivative of $\Pi_3(N|T)$ with respect to $N$, we can obtain

$$
\frac{\partial^2 \Pi_3(N|T)}{\partial N^2} = \left( k^2 e^{-kN} \varepsilon - 2ke^{-kN} \varepsilon' + e^{-kN} \varepsilon'' \right) \frac{P}{\eta^2} \left[ \eta^2 T + a(e^{\eta T} - 1 - \eta T) \right] - \frac{u\varepsilon'}{T\eta} \left( e^{\eta T} - 1 \right) - \frac{v\varepsilon''}{T\eta^2} \left[ e^{\eta T} - \eta T - 1 + \eta(N - M)(e^{\eta T} - 1) \right] - \frac{2v\varepsilon'}{T\eta} \left( e^{\eta T} - 1 \right)
$$

To identify whether $N_3$ is equal to $M$ or $N_3 > M$, we define the following discrimination term:

$$
\Delta_{N_3} = \left. \frac{\partial^2 \Pi_3(N|T)}{\partial N^2} \right|_{N=M}
$$

$$
= \left( -k \varepsilon + \varepsilon' \right) \frac{P}{\eta^2} \left[ \eta^2 T + a(e^{\eta T} - 1 - \eta T) \right] - \frac{u\varepsilon'}{T\eta} \left( e^{\eta T} - 1 \right) - \frac{h\varepsilon'}{T\eta^2} \left( e^{\eta T - 1 - \eta T} \right) - \frac{v\varepsilon''}{T\eta^2} \left[ e^{\eta T} - \eta T - 1 \right] - \frac{2v\varepsilon'}{T\eta} \left( e^{\eta T} - 1 \right)
$$

If $k^2 \varepsilon - 2ke\varepsilon' + \varepsilon'' \leq 0$, then $\frac{\partial^2 \Pi_3(N|T)}{\partial N^2} < 0$, $\Pi_3(N|T)$ is strictly a concave function of $N$. Hence, the retailer’s profit exists given the unique maximum solution $\tilde{N}_3$.

Otherwise, the optimal solution of $\Pi_3(N|T)$ is at one of the two boundary points ($M$ or $\infty$). Substituting $\infty$ with Equation (20), we have $\lim_{N \to \infty} \frac{\partial^2 \Pi_3(N|T)}{\partial N^2} < 0$. Hence, $T = +\infty$ is not an optimal solution; the optimal solution is $N^* = M$. Consequently, the following theoretical results can be derived.

**Theorem 6.** For any given $T > 0$, if $k^2 \varepsilon - 2ke\varepsilon' + \varepsilon'' \leq 0$, then we have $\Pi_3(N|T)$ is strictly concave function in $N$, and hence there exists a unique maximum solution.

(2) If $\Delta_{N_3} \leq 0$, then $\Pi_3(N|T)$ is maximized at $N^*_3 = M$.

(3) If $\Delta_{N_3} > 0$, then there exists a unique $\tilde{N}_3 > M$ such that $\Pi_3(N|T)$ is maximized at $N^*_3 = \tilde{N}_3 > M$.

5. **Some special cases.** Firstly, if there is no deterioration (i.e., $\theta$ is approaching to zero), then the proposed model becomes for non-deteriorating items. Consequently, the retailer’s order quantity per cycle in (3) becomes

$$
Q = I(0) = \frac{\varepsilon}{a} \left( e^{\eta T} - 1 \right).
$$
As a result, we know that the retailer’s annual total profit in (6) is simplified to

$$\Pi_1(N,T) = \frac{p}{aT} - \frac{v}{aT} (e^{\theta T} - 1) - \frac{h}{aT} (e^{\theta T} - 1) - \frac{\lambda}{2aT} (e^{\theta T} - aT - 1)$$

$$+ \frac{pI_e}{aT} \left[ (1 - e^{\theta T} + a^2 Te^{\theta T} + a(M - N)) (e^{\theta T} - 1) \right].$$ (26)

Similarly, if there is no deterioration, then we get

$$\Pi_2(N,T) = \frac{pe^{-bN}}{aT} (e^{\theta T} - 1) - \frac{v}{aT} (e^{\theta T} - 1) - \frac{h}{aT} (e^{\theta T} - aT - 1)$$

$$+ \frac{pI_{Te}}{a^2T} \left[ e^{\theta(T-M+N)} - e^{\theta T} + a(M - N)e^{\theta T} \right]$$

$$- \frac{eI_{Te}}{a^2T} \left[ e^{\theta(T-M+N)} - a(T - M + N) - 1 \right],$$ (27)

and

$$\Pi_3(N,T) = \frac{pe^{-bN}}{aT} (e^{\theta T} - 1) - \frac{v}{aT} (e^{\theta T} - 1) - \frac{h}{aT} (e^{\theta T} - aT - 1)$$

$$- \frac{eI_{Te}}{a^2T} \left[ e^{\theta T} - aT - 1 + a(N - M)(e^{\theta T} - 1) \right].$$ (28)

This simplified problem with $D_0 = 0$, $a = 0$ and $\lambda(N) = ke^{cN}$ has been solved by Teng and Lou [40].

In fact, several previous models are indeed special cases of the proposed inventory model here.

1. When $\theta \to 0$, $D_0 = 0$, $a = 0$ and $\lambda(N) = ke^{cN}$, then the proposed model is reduced to that in Teng and Lou [40].
2. When $\theta \to 0$, $M = 0$, $D_0 = 0$, $a = 0$ and $\lambda(N) = ke^{cN}$, then the proposed model is the same as that in Lou and Wang [20].
3. When $\theta \to 0$, $D_0 = 0$, $a = 0$, $b = 0$ and $\lambda(N) = ke^{cN}$, then the proposed model is simplified to that in Teng and Goyal [39].
4. When $\theta \to 0$, $N = 0$, $D_0 = 0$, $a = 0$, $b = 0$, and $\lambda(N) = k$, then the proposed model is similar to that in Teng [34].
5. When $\theta \to 0$, $N = 0$, $p = v$, $D_0 = 0$, $a = 0$, $b = 0$, and $\lambda(N) = k$, then the proposed model is reduced to that in Goyal [11].

6. **Numerical analysis.** In this section, in order to show the applicability of the presented model and also the solution procedure, numerical examples are presented. In addition, these examples provide the materials for sensitivity analysis as well as extracting some managerial insights, which will be discussed in the next section.

There are two simple ways to quantify the demand function $\lambda(N)$ of credit period $N$. They include: (a) a linear form, such as $\lambda(N) = a(1 + bN)$ (e.g. see Shah and Cardenas-Barron [31], and (b) an exponential pattern, such as $\lambda(N) = ke^{cN}$ (e.g., see Teng et al. [42], Wu et al. [44], Mahata [25]. All these demand functions satisfy the following generalized assumption: the demand rate $\lambda(N)$ as a concave function in credit period $N$.

In our proposed model, we assume that $\lambda(N) = ke^{cN}$ to conduct the numerical analysis for illustrating the theoretical results and obtaining the optimal global solutions using the software LINGO.

**Example 1.** This is under the assumption that $k = 0.2$/year, $a = 0.2$/year, $\theta = 0.1$/year, $p = $20/unit, $c = $0.1$/year, b = 1/year, $A = $10/order, $D_0 = 100$ units/year, $h = $5/unit/year, $v = $10/unit, $I_c = 0.09$, $I_p = 0.14$. 

Based on the theorems, we check the conditions for obtaining the optimal solutions. First, for $T_1^*$, based on Theorem 1, we can check that $pa - h - v\eta - pI_e a/\eta + pI_e Ma = -5.02 < 0$, hence, $\Pi_1(T|N)$ is a strictly concave function in $T$ in Case 1 with $N \leq M \leq T + N$; based on Theorem 3, with $N \leq M \leq T + N$, we can check that $A - pI_e \varepsilon \left[ \frac{\theta h^2(M-N)^2}{2} + a e^{\eta(-M-N)} \right] / \eta^3 < 0$ for $T \in [0,1]$ and $N \in [0,0.5]$, $pa - h - v\eta - pI_e a/\eta + pI_e Ma = -5.02 < 0$ and $pI_e a/\eta - vI_p = -0.2 < 0$, then $\Pi_2(T|N)$ is a strictly concave function in $T$ with $T_2^* = M - N$. Based on Theorem 5, $pa - h - v\eta - vI_p(1 + \eta(N - M)) < pa - h - v\eta - vI_p(1 - \eta M) = -4.465 < 0$, then $\Pi_3(T|N)$ is a strictly concave function in $T$ in Case 3.

For $N$, based on Theorem 2, with the conditions for $T \in [0,1]$ and $N \in [0,0.5]$, we can obtain $-k^2 \varepsilon - 2k \varepsilon' + \varepsilon'' < 0$ and $\varepsilon'' \left[ \frac{9 \eta^2 T^2}{2} + a(1 - e^{\eta T} + \eta T e^{\eta T}) \right] + \eta \eta^2 T + a(e^{\eta T} - 1 - \eta T) \left[ \varepsilon''(M - T - N) - 2 \varepsilon' \right] \leq 0$. Then, we obtain $\Pi_1(N|T)$ as a strictly concave function in $N$. Also, $\Delta_{N_3} \leq 0$, then $\Pi_3(N|T)$ is maximized at $N_3^* = M = 0.5$ for $M \leq N$.

Therefore, we have the global optimal solutions to maximize $\Pi(N,T)$ in the following:

$N_1^* = 0$ years, $T_1^* = 0.1846$ years, and $\Pi_1^*(N,T) = $993.40

$N_2^* = 0$ years, $T_2^* = 0.5$ years, and $\Pi_2^*(N,T) = $932.40

$N_3^* = 0.5$ years, $T_3^* = 0.1817$ years, and $\Pi_3^*(N,T) = $709.10.

Consequently, the retailer’s optimal solution is $N^* = 0$ years, $T^* = 0.1846$ years, and $\Pi^*(N,T) = $993.40.

For this type of demand pattern, the average profit function is highly non-linear. So it is impossible to find closed type formula for $N$ and $T$. But Fig. 1 shows the concavity of the annual profit function in both $N$ and $T$. Hence, the better optimal solution is a global maximum.

**Example 2.** The parameters are the same as in Example 1, except $b = 5$/year, $c = 5$/year, $p = $30/unit, $k = 5$/year. Similar to Example 1, we have the maximum solution to $\Pi(N,T)$ in the following:

$N_1^* = 0.4441$ years, $T_1^* = 0.2040$ years, and $\Pi_1^*(N,T) = $2144.50

$N_2^* = 0$ years, $T_2^* = 0.5$ years, and $\Pi_2^*(N,T) = $2100.70

$N_3^* = 1.6345$ years, $T_3^* = 0.0125$ years, and $\Pi_3^*(N,T) = $27979.20

Consequently, the retailer’s optimal solution is $N^* = 1.6345$ years, $T^* = 0.0125$ years, and $\Pi^*(N,T) = $27979.20.

Fig. 2 reveals that $\Pi(N,T)$ is a strictly pseudo-concave function in both $N$ and $T$. Hence the better optimal solution is a global maximum.

6.1. **Sensitivity analysis and real usage of the model.** In order to discuss the real effects of this model, we visited some electronic departmental stores in West Bengal, India. These stores sell different ranges of electronic commodities like television, fridge, washing machines etc. Based on the market analysis, the following key facts are visualized.

1. Due to their attractive policies and promotions, the number of customers increases gradually with a positive rate.

2. To promote the products and stand into market competition, suppliers offer some credit period to their retailers so that they do not need to invest a lot at the threshold of the replenishment period. Paying the whole at the time of ordering may restrict the retailers to replenish a sufficiently large quantity to meet the market demand.
3. In line with the suppliers’ credit policy, the retailers also declare a “zero down payment credit scheme” to their customers that make them attracted to the qualities and features of those products.

4. Demand is simultaneously effected by inventory level and the credit period offered by retailers to their customers.

It is also to be noted that the retailers need to replenish the stock based on the market demand. Thus it is quite complicated for the retailers to decide the replenishing interval and the credit level co-relatively to earn more profit. They also need to take the time bound of the supplier to clear the purchase cost of the commodities in mind. Implications of trade credit concept:

To promote the products, retailers announce a full trade credit period to their customers namely “zero down payment credit scheme”. In this scheme the customers need not to pay to purchase the product. Only they need to show some required security documents at the time of purchase. This “buy first pay later” offer makes the promotion of a fascinating one and the customers with proper documents and eligibility may just go to the shop, choose a product and purchase it without paying at that moment. This offer attracts a large number of customers and it increases at a satisfactorily high rate as they do not need to think initially about their budgets.

Based on some previous year’s data the customer demand rate express exponential characteristics with retailer’s offering credit period to customers. The demand graph of credit period with credit period based demand factor chart is given below.

It is now evident that the demand is increasing in an approximately exponential manner with time and hence our assumption of the credit factor of demand $\lambda(N) = ke^{cN}$ is a good estimation for the real situations. Thus by properly analyzing the situations, the system parameters values may be set as follows:

**Example 3:**

- $k = 0.5$/year, $a = 0.2$/year, $\theta = 0.1$/year, $p = 30$/unit, $c = 5$/year, $b = 5$/year, $A = 10$/order, $D_0 = 100$ units/year, $h = 5$/unit/year, $v = 10$/unit, $I_c = 0.09$, $I_p = 0.14$. Using these data, we study the sensitivity analysis on the optimal solution with respect to each parameter in appropriate unit. The computational results are shown in Table 2.

The sensitivity analysis reveals that:

1. If the value of $A$ increases, then the optimal order cycle $T^*$ increases while the values of $N^*$ and the optimal profit $\Pi^*(N,T)$ decrease. When the ordering cost is higher, the retailer orders the products in the longer replenishment period to reduce the order frequency. Thus, he/she pays less ordering cost.

2. If the value of $D_0$ increases, then the values of $N^*$ and the optimal order cycle $T^*$ decrease while the value of the optimal profit $\Pi^*(N,T)$ increases. When the initial market demand is greater, the retailer can make more profit.

3. If the value of $p$, $M$, $b$, or $c$ increases, then the values of $N^*$ and the value of the optimal profit $\Pi^*(N,T)$ increase while the optimal order cycle $T^*$ decreases. When the sales price is higher, the retailer offers a shorter delayed payment time and longer order cycle for his customers to make more profit.

   When the supplier offers a longer trade credit period to the retailer, the retailer can provide more trade credit to his customers to earn more profit. When the market demand is more sensitive to the trade credit, the retailer should provide a longer delayed payment time to make more profit.

4. If the value of $v$ or $k$ increases, then the values of $N^*$ and the optimal profit $\Pi^*(N,T)$ decrease while the value of the order cycle $T^*$ increases. When the
Table 1. Demand factors over real case study

| Credit periods | Customer demand factor |
|----------------|------------------------|
| 0.1            | 1.010050167            |
| 0.2            | 1.04452134             |
| 0.3            | 1.019224534            |
| 0.4            | 1.032920774            |
| 0.5            | 1.041841096            |
| 0.6            | 1.069682147            |
| 0.7            | 1.072508181            |
| 0.8            | 1.083287068            |
| 0.9            | 1.129936284            |
| 1              | 1.112734718            |
| 1.1            | 1.16404607             |
| 1.2            | 1.123120852            |
| 1.3            | 1.173786783            |
| 1.4            | 1.085449799            |
| 1.5            | 1.166159193            |
| 1.6            | 1.173510871            |
| 1.7            | 1.177650051            |
| 1.8            | 1.203693163            |
| 1.9            | 1.209249598            |

unit purchasing price of the retailer is higher, the retailer makes less profit. When the default risk of the customers is higher, the retailer should offer a shorter delayed payment time to his customers. The retailer can take some measurements to reduce the default risk of the customer, such as by adopting the partial delayed payment policy, to make more profit.

5. If the value of $h$ or $\theta$ increases, then the values of $N^*$, the optimal order cycle $T^*$, and the value of the optimal profit $\Pi^*(N,T)$ decreases. The retailer can adopt some measurements to reduce the deteriorating rate, the purchasing cost, or the holding cost to make more profit.

6. If the value of $a$ increases, then the values of $N^*$ decrease, but the optimal order cycle $T^*$ and the value of the optimal profit $\Pi^*(N,T)$ increase. When the market demand is more sensitive to the inventory level, the retailer provides shorter delayed payment time and longer order cycle to make more profit.

To rank the impacting factors on the decisions and the profit, we adopt the following formula to carry out their elasticity:

$$\xi_i = \frac{\Delta y}{y} \cdot \frac{\Delta x}{x}.$$ 

$\xi_i$ is the elasticity of a certain parameter for term $i$ ($i = T, N, \Pi$); $x$ is the basic value of a certain parameter; $\Delta x$ is the change of a certain parameter value; $y$ and $\Delta y$ are optimal terms corresponding to $x$ and $\Delta x$. Table 3 is the elasticity analysis of the parameters in the model based on the sensitivity analysis in Table 2.

According to Table 3, the parameter $c$ of the demand dependent on the credit period is the most sensitive factor that affects the optimal profit of the retailer. Default risk $k$ is one of the most influential factors on the optimal profit. For the credit term determination, the $p$, $v$, and $k$ are the first three most sensitive factors
Table 2. Sensitivity analysis on parameters

| Parameter | $N^*$ | $T^*$ | $H^*(N,T)$ |
|-----------|-------|-------|------------|
| $A$       | 10    | 1.6346| 0.0125     | $27979.15$ |
|           | 14    | 1.6337| 0.0148     | $27685.27$ |
|           | 18    | 1.6328| 0.0168     | $27431.85$ |
|           | 22    | 1.6314| 0.0181     | $27105.88$ |
| $D_0$     | 100   | 1.6346| 0.0125     | $27979.15$ |
|           | 150   | 1.6336| 0.0124     | $28060.09$ |
|           | 200   | 1.6325| 0.0123     | $28141.27$ |
|           | 250   | 1.6317| 0.0125     | $28222.71$ |
| $p$       | 30    | 1.6345| 0.0125     | $27979.15$ |
|           | 35    | 1.6886| 0.0066     | $102463.6$ |
|           | 40    | 2.1081| 0.0038     | $314389.6$ |
|           | 45    | 2.3019| 0.0023     | $44938.8$  |
| $v$       | 10    | 1.6345| 0.0125     | $27979.15$ |
|           | 11    | 1.4765| 0.0182     | $13830.44$ |
|           | 12    | 1.3304| 0.0257     | $7283.99$  |
|           | 13    | 1.1924| 0.0353     | $4063.65$  |
| $h$       | 5     | 1.6345| 0.0125     | $27979.15$ |
|           | 7     | 1.6337| 0.0110     | $27770.88$ |
|           | 9     | 1.6330| 0.0100     | $27584.54$ |
|           | 11    | 1.6323| 0.0093     | $27414.47$ |
| $M$       | 0.5   | 1.6345| 0.0125     | $27979.15$ |
|           | 0.6   | 1.6540| 0.0119     | $30602.06$ |
|           | 0.7   | 1.6737| 0.0113     | $33493.21$ |
|           | 0.8   | 1.6934| 0.0107     | $36682.29$ |
| $k$       | 0.5   | 1.6345| 0.0125     | $27979.15$ |
|           | 0.6   | 1.3669| 0.0243     | $8658.22$  |
|           | 0.7   | 1.1563| 0.0406     | $3698.78$  |
|           | 0.8   | 0.9745| 0.0619     | $2007.86$  |
| $b$       | 5     | 1.6345| 0.0125     | $27979.15$ |
|           | 7     | 1.6358| 0.0105     | $39454.92$ |
|           | 9     | 1.6307| 0.0092     | $50971.26$ |
|           | 11    | 1.6373| 0.0083     | $62515.11$ |
| $a$       | 0.2   | 1.6345| 0.0125     | $27979.15$ |
|           | 0.3   | 1.6344| 0.0126     | $27997.37$ |
|           | 0.4   | 1.6343| 0.0127     | $28015.81$ |
|           | 0.5   | 1.6342| 0.0129     | $28034.48$ |
| $c$       | 5     | 1.6345| 0.0125     | $27979.15$ |
|           | 6     | 1.6755| 0.0048     | $151018.6$ |
|           | 7     | 1.7026| 0.0019     | $831067.7$ |
|           | 8     | 1.7221| 0.0007     | $4642183.0$|
| $\theta$ | 0.1   | 1.6345| 0.0125     | $27979.15$ |
|           | 0.2   | 1.6340| 0.0116     | $27855.04$ |
|           | 0.3   | 1.6335| 0.0109     | $27739.36$ |
|           | 0.4   | 1.6331| 0.0103     | $27630.62$ |

that enrich the research of Shi and Zhang [32], in which the default risk has less effect on the credit term with a constant demand rate. For the replenishment cycle, the parameters $p$, $v$, and $k$ are the first three most sensitive factors, which shows that considering the effect of the default risk in the inventory model of the trade credit is necessary for the decisions.

7. Conclusions. Taking care of upstream and downstream trade credits simultaneously for deteriorating items and considering demands that are dependent on
stock and credit period involving default risk has received relatively little attention from researchers. In this paper, we explore the optimal order policy and the trade credit policy for deteriorating items with stock and credit period sensitive demand under two-level trade credit financing involving default risk. First, inventory models are established. Second, we show how to make optimal decisions based on the models. Furthermore, numerical examples and sensitivity analysis are conducted to provide some managerial insights.
When the default risk of the customers is higher, a retailer should provide a shorter delayed payment time to his customers. Therefore, the retailer can take some measurements to reduce the default risk of the customer, such as partial delayed payment policy, to make more profit. When market demand is more sensitive to the trade credit, the retailer should offer a longer credit period to make more profit. When the market demand is more sensitive to the inventory level, the retailer provides a shorter delayed payment time and longer order cycle to make more profit.

When the supplier offers a longer trade credit period to the retailer, the retailer can provide more trade credit to his customers to make more profit.

Based on the elasticity analysis, for credit term determination, default risk is one of the first three most sensitive factors that enrich the research of Shi and Zhang [32], in which the default risk has less effect on the credit term with a constant demand rate. For the optimal replenishment cycle, the default risk is also one of the first three most sensitive factors, which shows that in the inventory model of the trade credit, considering the effect of the default risk is necessary in making decisions.

Future studies can be extended in several ways. For example, the expiration dates for the deteriorating items can be introduced to the inventory model with the trade credit financing. Additionally, future research can extend the full trade credit policy to partial trade credit policy. The models can also be generalized to consider shortage or partial backlogging to obtain some management insights.

Fig. 1. Optimal profit graph for Example 1
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