Equivalence principle and critical behaviour for nonequilibrium decay modes

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We generalize an orthonormality relation between decay eigenmodes of equilibrium systems to nonequilibrium markovian generators which commute with their time-reversal. Viewing such modes as tangent vectors to the manifold of statistical ensembles, we relate the result to the choice of a coordinate patch which makes the Fisher-Rao metric euclidean at the invariant state, realizing a sort of statistical equivalence principle. Finally, we classify nonequilibrium systems according to their spectrum, arguing that a degenerate Fisher matrix is a signature of the insurgence of a class of phase transitions between nonequilibrium regimes. We exhibit an order parameter with power-law critical decay and prove divergent correlations between suitable unbiased estimators.

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It is a chief objective of NonEquilibrium Statistical Mechanics (NESM) to draw lines and analogies between equilibrium and nonequilibrium systems, in search of as powerful instruments of analysis as are, for example, equilibrium ensembles. What makes a statistical ensemble equilibrum is certainly not its exponential form nor the fact that it is a constrained maximizer of the Gibbs-Shannon entropy. Jaynes claimed that “essentially all of the known results of Statistical Mechanics, equilibrium and nonequilibrium, are derivable consequences of this principle”[4]. Whether or not this program has been satisfactorily pursued, the maximum entropy inferential method might as well work for nonequilibrium steady states. Rather, nonequilibrium is discriminated by the fact that p is the terminal state of spontaneous relaxation to a static scenario where no internal currents flow. The ensemble per se only tells a part of the story. The study of relaxation modes is essential to the full characterization of nonequilibrium behaviour.

In this letter we establish a connection between thermodynamical aspects of nonequilibrium systems, encoded in their decay modes, and geometrical properties of the Fisher-Rao metric of distance between probability distributions. The former will be modelled through markovian evolution of a probability distribution. The latter will be modelled through the Fisher-Rao measure of distance between probability distributions. The former will be modelled through markovian evolution of a probability distribution.

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$\alpha\beta(\pi, t/dt) \propto \exp -\sum_t \beta_i \pi_i, \tag{1}$

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$g^{\alpha\beta} := \sum_i g^\alpha_i g^\beta_i = \delta^{\alpha\beta}, \tag{3}$

to systems whose generators commute with their time reversal. To this class there belong equilibrium systems, with real decay spectrum, and a class of p-normal, truly nonequilibrium systems, with oscillatory behaviour. We finally linger on the geometric meaning of the results, showing that (v) decay modes can be seen as local reference frames for the Fisher-Rao metric, so that p-normal systems correspond to coordinate patches which make it euclidean at one point of the manifold, thus realizing a sort of statistical equivalence principle. As is customary in differential geometry, we will use a swarm of indices which are implicitly summed over when repeated.

The system’s finite sample space consists of n + 1 samples, labelled by i,j,k. On it, a normalized probability density p^t (a state of the system) undergoes markovian evolution, \frac{d}{dt} p^t = L p^t. Conservation of probability requires the generator’s columns to add to zero,

$L_{ij} = \begin{cases} w_{ij}, & i \neq j \\ -\sum_k w_{ki}, & i = j \end{cases}, \tag{4}$
The $w$’s are real positive transition rates, with units of an inverse time. Under mild assumptions on the connectedness of the state space and on the reversibility of paths, the markovian dynamics has a unique invariant state $p$, i.e. a null eigenvector of the generator, $Lp = 0$, towards which any initial state $p_\epsilon$ tends at large times, $\lim_{t \to \infty} e^{tL} p_\epsilon = p$. The propagator $\exp tL$ is a stochastic matrix with strictly positive entries. By a standard argument in the theory of Markov chains, the Perron--Frobenius theorem can be applied to prove that the $n$ nonnull eigenvalues of $L$ have negative real part,

$$Lq^\alpha = (-1/\tau_\alpha + i\omega_\alpha) q^\alpha, \quad \alpha = 1, \ldots, n,$$

with $\tau_\alpha$ a positive characteristic decay time and $\omega_\alpha$ a frequency. For the time being, we suppose the spectrum to be nondegenerate. The decay eigenmodes $q^\alpha$, plus the invariant state $p$, form a basis of eigenvectors of $L$.

Propagating $p_\epsilon = \delta_0 p + \epsilon_\alpha q^\alpha$ up to time $t$ yields

$$\exp(tL) p_\epsilon = \delta_0 p + \sum_{\alpha} e^{-t/\tau_\alpha + i\omega_\alpha t} \epsilon_\alpha q^\alpha.$$

At late times decay modes are exponentially damped, so that by normalization $\delta_0 = 1$, and consequently the eigenmodes’ entries are seen to add to zero, $\Sigma_i q_i^\alpha = 0$.

With a slight abuse of language, we will say that $L$ is of equilibrium when its invariant state satisfies detailed balance, vanishing internal steady currents $j_{ij} = w_{ij} p_j - w_{ji} p_i$, there being no net exchange of information between samples. Following Andrieux [11], we define the diagonal matrix $\sqrt{p} = \text{diag}(\sqrt{p_1}, \ldots, \sqrt{p_n})$.

Detailed balance holds if and only if matrix

$$H := -\sqrt{p}^{-1} L \sqrt{p}.$$

is symmetrical. Transformation (7) is a similarity of matrices. By the spectral theorem, $H$ has a real spectrum which coincides with that of $-L$, $H e^\alpha = \tau_\alpha e^\alpha$, with $\omega_\alpha = 0$ and $\alpha$ ranging from $0, \ldots, n$. Eigenvector $e^\alpha_\xi = \sqrt{p_i}$ corresponds to the steady state, with $\tau_0 = \infty$. All others are related to the decay modes via $e^\alpha_\xi = q_\alpha / \sqrt{\xi}$. The spectral theorem also permits to choose orthonormal eigenvectors,

$$h^{ab} := (e^a, e^b) = \delta^{ab},$$

where $(\cdot, \cdot)$ denotes the euclidean scalar product. The l.h.s. provides the definition of the symmetric matrix $h^{ab}$. Entries along the zeroth row yield the identities

$$h_0^0 = \Sigma_i p_i = 1$$

and

$$h_0^\alpha = \Sigma_i q_i^\alpha = 0,$$

which are true by construction also of nonequilibrium generators. Imposing $h^{\alpha\beta} = \delta^{\alpha\beta}$ produces the desired result, Eq.(3). Since, if we restrict to the field of reals, the derivation follows a chain of necessary and sufficient arguments, writing

$$h^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & g^{ab} \end{pmatrix},$$

we can state that $g^{\alpha\beta}$ is diagonal in a basis of eigenmodes of a markovian generator with real non-degenerate spectrum if and only if the latter is of equilibrium.

On the complex field, the spectral theorem generalizes to normal matrices: it suffices (and is necessary) that $HH^\dagger = H^\dagger H$ to make $H$ unitarily diagonalizable, i.e. with orthonormal eigenvectors with respect to the hermitian scalar product $(\cdot, \xi) \in \Re = \sum_i v_i^* w_i$. Let us define the time-reversal (also known as p-dual) markovian generator $\bar{L} = \sqrt{p}^* L^T \sqrt{p}^{* -1}$ which also has $p$ as its invariant state [12]. Of course, there exists no generator which could possibly run the dissipative dynamics backward: $L$ is what comes closest to it, as it inverts all of the steady currents $j_{ij} = -j_{ji}$. Normality of $H$ translates in the commutation relation of p-normal generators:

$$[L, \bar{L}] = 0.$$

Since the entries of $L$ are real, its $2k \leq n$ complex eigenvectors come in complex conjugate pairs $(q^\kappa)^* = q^{\kappa *}$, $\kappa = 1, \ldots, k \leq n/2$. It is then simple to see that Eq.(10) is a sufficient and necessary condition for the real and imaginary parts of $e^\kappa$ to satisfy

$$g^{\kappa\kappa'} := (\Re e^\kappa, \Re e^{\kappa'}) = \frac{1}{2} \delta^{\kappa\kappa'} = (\Im e^\kappa, \Im e^{\kappa'}) =: g_\kappa \delta^{\kappa\kappa'},$$

to be orthogonal among themselves, $(\Re e^\kappa, \Im e^\kappa) = 0$, and to the remaining $n - 2k$ real eigenvectors $e^\ell$, which satisfy Eq.(3) on their own, $(e^\ell, e^\ell') = \delta^{\ell\ell'}$.

Consider now the late-time behavior of relative entropy with respect to the invariant state. We distinguish $[n/2]$ phases in the space of generators with nondegenerate spectra, parametrized by transition rates.

(A) There are $n$ real negative eigenvalues. Expanding Eq.(2) to second order in $e^\ell$ near the steady state, with $p^0 = \sqrt{p}(e^0 + e^\ell e^\ell)$, we obtain

$$S(p^0|p) \approx \frac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\beta\gamma} (e^\alpha, e^\gamma) =: \frac{1}{2} \| e^\ell \|^2,$$

where first order contributions vanish. To second order, relative entropy is one-half the length of vector $e^\ell$ with respect to the positive-definite metric $g^{\alpha\beta}$ defined in the l.h.s. of Eq.(3). Considering the explicit time evolution, Eq.(4), with $\omega_\alpha = 0$, we identify $e^\ell_\alpha = e^{-t/\tau_\alpha} \epsilon_\alpha$. Nonequilibrium models with real spectrum then display superposition of modes with different decay times, affecting the late time behaviour of relative entropy. This superposition disappears for equilibrium generators. In a way, $g^{\alpha\beta}$ measures the correlation between decay eigenmodes, with equilibrium modes being uncorrelated.

(Bk) There are $n - 2k$ real eigenvalues $-1/\tau_\ell$, and $k$ couples of complex conjugate eigenvalues $-1/\tau_\kappa \pm i\omega_\kappa$. In a basis of eigenmodes, the initial distribution reads $p_\epsilon = \sqrt{p} (e^0 + \epsilon_\ell e^\ell + e^\kappa e^{\kappa'})$, where $\sigma = \pm$. Notice that since $p^0$ is real one necessarily has $(\epsilon_\kappa)^* = \epsilon_{\kappa'}$. While relative entropy does not look like a real bilinear form when expressed in terms of the complex vector $\epsilon = (\epsilon_\kappa, \bar{\epsilon}_\kappa, \epsilon_\ell)$, it is indeed a positive bilinear form of the complexified vector $\bar{\epsilon} = (\Re \epsilon_\kappa, \Im \epsilon_\kappa, \epsilon_\ell)$. A direct calculation yields

$$S(p^0| p) \approx \frac{1}{2} \| \Omega^T \bar{\epsilon} \|^2,$$

where $\Omega^T$ is the matrix of orthonormalized eigenvectors, and $\epsilon_k = \sqrt{p_k}$.
where the block-diagonal matrix

$$\Omega^t = \begin{pmatrix} e^{-t/\tau} R(\omega \xi t) & 0 \\ 0 & e^{-t/\tau} \end{pmatrix}$$  \hspace{1cm} (14)$$

has \( k \) copies of the \( 2 \times 2 \) rotation matrix \( R(\varphi) \), and the norm is to be calculated using the “complexified” metric

$$\tilde{g}^{\alpha \beta} = \begin{pmatrix} g_{11}^{\kappa \kappa} & \cdots & g_{1n}^{\kappa \kappa} \\ \vdots & \ddots & \vdots \\ g_{n1}^{\kappa \kappa} & \cdots & g_{nn}^{\kappa \kappa} \end{pmatrix}.$$  \hspace{1cm} (15)$$

Eq. (13) displays mixing of decay and oscillatory times, which only disappears when the generator commutes with its reversal and \( \tilde{g}^{\alpha \beta} \) is diagonal — with \( k \) blocks \( \propto 1_{2 \times 2} \). One last case is left out from our analysis:

(C) The generator is defective, that is, degenerate eigenvalues lack a complete set of eigenvectors. This case and the phenomenology so far analyzed are better illustrated with the aid of an example. Consider the following generator, parametrized by positive rates \( \xi, \chi \)

$$L(\xi, \chi) = \begin{pmatrix} -\xi - \chi & 1 & \chi \\ \chi & -1 - \chi & 1 \\ \xi & \chi & -1 - \chi \end{pmatrix}.$$  \hspace{1cm} (16)$$

The dynamics generated by \( L(\xi, \chi) \) is that of a hopping particle with a systematic bias in the counterclockwise direction, and one perturbed clockwise rate (see Fig. 1a). Its phase space is depicted in Fig. 1b. By Kolmogorov’s criterion, transition rates satisfy detailed balance if and only if the only macroscopic affinity vanishes, in \( \xi/\chi^3 = 0 \), which traces the equilibrium line \( \ell_{eq} \). The model corresponding to \( \chi = 1 = \xi \) is known as the unbiased hopping particle, with twice degenerate eigenvalue \( \lambda = 2 \) which affords a complete basis of eigenvectors. The space of parameters is partitioned into two phases of type A (in grey in Fig. 1a) and B1 (in white), marked out by the critical lines \( \ell_1 : \chi = 3 \xi = 4 \) and \( \ell_2 : \xi = \chi \). For the first class of models, direct calculation of the eigenvectors shows that \( g^{\alpha \beta} \) is diagonal only along the equilibrium line. In phase B1 one needs to turn to the complex components of the eigenmodes to be able to expand relative entropy as a positive bilinear form. Along \( \ell^* \) are the biased hopping particle models \( L(1, \chi) \), which make the complexified matrix diagonal. Their reversal if found by inverting the bias in the clockwise direction, yielding \( L = L^T \).

With the exception of \( L(1,1) \), a generator \( L \) picked along the critical lines only has one eigenvector \( q \) relative to the degenerate eigenvalue \( -\tau^{-1} \). A generalized eigenvector \( u \) shall then be introduced, with \( Lu = -\tau^{-1} u + q \), carrying \( L \) into Jordan’s normal form. The time evolved \( \exp(tL)p_\varphi \) is seen to acquire a term \( \propto te^{-t/\tau} q \) [11]. Consider now a path \( \Gamma = \{ (\xi(s), \chi(s)) \} \) in parameter space, as depicted in Fig. 1. We first traverse the complex phase. At \( s = 1 \) we come upon an abrupt switch in the appropriate basis. Approaching the critical line \( \ell_1 \) from below, the imaginary part \( \Im q_2(s) \) becomes smaller and smaller until it vanishes; from above, the modes \( q_1(s) \) and \( q_2(s) \), respectively with higher and lower eigenvalue, tend to align. At the critical line the correlation matrix becomes degenerate and has a discontinuity

$$g^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow g^1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$  \hspace{1cm} (17)$$

An order parameter \( f_s(\infty) \) — of little physical meaning though— can also be engineered. Consider two vectors \( q^1(s) \) and \( u^1 \), respectively in the orthogonal complement of \( q_2(s) \) and \( u \). Projecting \( \exp(tL(s))p_\varphi \) along the two, and then taking the ratio, yields

$$f_{2 \gg 1}(t) \propto \left[ 1 + ce^{-t(\tau^{-1} - \tau^{-1})} \right]^{-1}, \quad f_1(t) \propto 1/(1 + c't).$$  \hspace{1cm} (18)$$

The exponential decay, reaching an arbitrary nonzero value of \( f_s(\infty) \), becomes a power-law at the critical line, with \( f_s(\infty) = 0 \). Thence we took the liberty to refer to a class of nonequilibrium phase transitions, with an acceptance that is reminiscent of that employed for simple driven lattice models [13]. This is also motivated by the equilibrium usage of the Fisher matrix, which coincides with the covariance matrix of the observable constraints \( X^\alpha \) which define the ensemble [4]. Crooks [8] commented that, as we vary the intensive parameters \( \beta, \alpha \), correlations vary smoothly except at phase transitions, where divergencies occur. On this line, further insight might come from estimation theory. Consider an unbiased estimator \( \tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_n) \), whose average \( \langle \tilde{\xi} \rangle_{p_\varphi} = \varphi \) yields precisely the vector of parameters. The Crâmer-Rao inequality [6], establishes a lower bound on the covariance matrix \( \left( \langle \tilde{\xi} - \varphi \rangle_{p_\varphi} \right)_{\tilde{\xi} = \varphi} \geq g^{-1} \), where \( A \geq B \) means that \( A - B \) is positive semidefinite. Multiplying by \( g \) and taking the trace we obtain

$$\langle \tilde{\xi}^\alpha - \varphi \rangle_{p_\varphi} \geq n, \quad s \neq 1, 2.$$  \hspace{1cm} (19)$$

A degenerate metric admits nonnull vectors of null norm. If these can be reproduced in terms of unbiased estimators, approaching the critical line degeneracy must be compensated by divergent correlations.
Indeed, a rich nonequilibrium phenomenology is marked by the peculiar representation of relative entropy near the invariant state, which can be interpreted as a metric on the manifold of statistical states $\mathcal{P}$. Let us hint at its construction. Relative entropy is not a good distance: it is not symmetrical, and the triangle inequality can be violated [4]. The way out of this puzzle is to stick to nearby distributions, as we did in Eq. [12], thus obtaining a local metric which measures the lenght of vectors $\epsilon_\alpha g^\alpha$ living on the tangent space to $\mathcal{P}$ at $p$. When moving to a different neighbourhood, one will shift the reference probability distribution to $p'$, and there define the metric in terms of $S(\cdot | p')$. If this procedure is carried on pointwise, one endows $\mathcal{P}$ with the Fisher-Rao metric. One can then assign coordinates $x^\alpha$ to neighbours of the manifold; associated to such coordinates is a basis of preferred tangent vectors $\partial x^\alpha$, which yield a matrix representative for the metric at each point of the neighbourhood. Notice that the Fisher-Rao metric is smoothly defined all over the manifold (except at boundaries and corners); it is its coordinatization that might suffer from pathologies, as is the case for our critical systems. But for $n = 1$, it can be shown that $g$ has a nonnull Riemann curvature: while one can always choose a coordinate patch which trivializes the metric at one given point, there is no such coordinate transformation which simultaneously makes $g$ diagonal all over a neighbourhood. Given the twice-contravariant transformation law for the metric, $g^{\alpha \beta} = \Lambda^\alpha_{\alpha'} \Lambda^\beta_{\beta'} g_{\alpha' \beta'}$, where $\Lambda^\alpha_{\alpha'}(x') = \partial x^\alpha / \partial x'_{\alpha'}$, is the inverse jacobian of the coordinate transformation $x \rightarrow x'(x)$, after Eqs. [10] one realizes that the components of $e_\alpha$ can be interpreted as the jacobian of an embedding patch, also called a frame, which trivializes the metric at $p$. Orthonormal frames are associated to $p$-normal systems; two such systems with the same invariant state yield different orthonormal frames, which are connected by an $O(n + 1)$ “gauge” transformation.

This mechanism lies at the heart of the Equivalence Principle of General Relativity. While gravity curves spacetime so as to prevent the definition of broad notions of “parallelism” and “simultaneity”, one can always find coordinates which make spacetime minkowskian at one point, and gravity indiscernible from a fictitious force. To a special observer, the frames’ entries provide an inertial frame of coordinate axis: in a very precise way they measure how much the orientation of these axis differs, up to Lorentz transformations, from “bent” coordinate axis. This discrepancy is the gravitational field [13].

To conclude, the present letter establishes a close connection between the Fisher-Rao metric and markovian generators: The choice of a generator induces a natural identification of a positive semi-definite matrix representation, nested in the late-time behaviour of relative entropy. A trivialized metric emerges for generators which commute with their time-reversal, including equilibrium systems; nonequilibrium phase transitions are induced by degenerate coordinatizations. This study calls for a careful treatment of the algebraic varieties of critical and trivial loci in the space of generators, and for a thermodynamical characterization of the order parameters.

Normal systems seem to enjoy special properties, and deserve more in-depth study, in particular in relation to the fluctuations and large deviations of their microscopic jump trajectories. We notice at a passing glance that the “curvature” operator $[L, L] \delta t^2$ retains a geometrical flavour, as it measures how circuitation along infinitesimal parallelograms fails to close, when we first evolve the initial state with $L$ up to time $\delta t$ and then run it back with $L$, rather then first evolving it back to $-\delta t$ with $L$ and then run it forward with $L$.

Finally, the neat framework suggests to deepen the nonequilibrium characterization of geometric objects such as Christoffel coefficients, geodesic curves, intrinsic and extrinsic curvature. To conclude with a motto, we might claim that equilibrium systems are to nonequilibrium thermodynamics what inertial frames are to gravity.

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SUPPLEMENTARY MATERIAL

On the Fisher-Rao metric

The choice of a markovian generator \(L\) identifies a point on the manifold of statistical states and a set of decay modes. We first map the probability simplex into the surface of a sphere, then interpret the pushed-forward vectors’ entries \(e_i^a\) as the Jacobian of a second coordinate transformation, in such a way that an equilibrium generator corresponds to the choice of a coordinate patch which trivializes the Fisher-Rao metric at the invariant state. Given an invariant state, there exists a whole \(O(n+1)\)-orbit of frame fields which trivialize the metric. Vice versa, two equilibrium generators with the same invariant state yield gauge-equivalent frames.

From a geometrical perspective — see for example [6] — the two matrices \(H\) and \(L\) are both representations of an operator \(H\) with eigenvectors

\[
e^a_0 = \frac{\partial}{\partial p_n} = e^a_0 \frac{\partial}{\partial x_i}.
\]

(20)

The peculiar notation employed for the basis vectors denotes that \(\sqrt{p}\) might be seen as the inverse Jacobian of the coordinate transformation \(p_j \rightarrow z_j(p) = 2\sqrt{p_j}\), which maps the probability simplex \(\{p_j \in [0,1]^{n+1} : \sum_j p_j = 1\}\) into a portion of the hypersphere with square radius \(\sum_i z_i^2 = 1\). Both are embeddings of the abstract manifold \(\mathcal{P}\) of probability distributions into \(\mathbb{R}^n_{n+1}\). Each such coordinate system (say, \(x\)) of “positions” on \(\mathcal{P}\) endows the \((n+1)\)-dimensional vector space \(V \approx \mathbb{R}^{n+1}\) of “velocities”, attached to \(p\), with a preferred basis of directions \(\partial x_i\). Vectors \(e^a\) span the \(n\)-dimensional tangent space \(T_p \mathcal{P} \subset V\), while \(e^0\) describes how a neighbourhood of \(p\) sits in the embedding space.

A metric at \(p\) is a positive semidefinite bilinear form \(h_p[v, w]\), with vectors \(v, w \in V\) attached to \(p\). A choice of coordinates endows the dual space \(V^*\) with a basis of linear forms \(dx_a\), such that \(dx_a[\partial x_i] = \delta^b_i\). Vectors’ components are then obtained by projecting \(v_a = dx_a[v]\), while the metric in coordinates reads \(h_p = h^{ab}(x)dx_a \otimes dx_b\). Again, the notation “\(dx_a\)” highlights the transformation properties of linear forms: if \(x \rightarrow x'(x)\) is a diffeomorphism, the metric’s entries transform twice contravariantly (dropping dependencies)

\[
h^{ab'} = \frac{\partial x_a}{\partial x_{a'}} \frac{\partial x_b}{\partial x_{b'}} h^{ab}.
\]

(21)

A metric is fully characterized by its action on a complete set of vectors. In our case, we define \(h_p\) via

\[
h_p[e^a, e^b] := \delta^{ij} e_i^a e_j^b = \delta^{ij} e_i^a \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}.
\]

(22)

where Eq. (8) is inbuilt in the l.h.s., while the r.h.s. helps introducing the frame forms \(e_i = \epsilon_a^i dx_a\) [6]. Given a metric, frames are determined up to \(O(n+1)\)-rotations \(e_i \rightarrow R^j_i e_j\), which leave the metric’s coefficients unaltered. When such rotations are performed point-by-point all over a neighbourhood of the manifold, in a differentiable way, one talks of a gauge transformation.

Comparison of Eq. (22) with Eq. (21) reveals that \(e_i^a\) is the inverse Jacobian of a coordinate transformation \(z \rightarrow x(z)\) which pulls back the metric’s components to the unit matrix. For example, choosing “spherical” coordinates \(x_a = \delta_i^a z_i\), one obtains the Euclidean length element \(h_p = (dz)^2\). The intrinsic metric at \(p\), \(g_p\), is defined as the restriction of \(h_p\) over the hypersurface \(\mathcal{P}\), acting on the tangent space \(T_p \mathcal{P}\). By construction \(e^0\) is orthogonal to all other \(e^a\)’s. Finally, the Fisher-Rao metric on the manifold is simply the disjoint collection of such local intrinsic metrics, \(g = \bigsqcup_{p \in \mathcal{P}} g_p\).

On relative entropy in the complex phase

Consider the probability density

\[
p_\infty = \sqrt{p_0} \left[ c^0 + e_i^a e^a_0 + \frac{1}{2} \left( e_1^\kappa + ie_2^\kappa \right) \left( e_1^\kappa - ie_2^\kappa \right) \right] + \text{c.c.},
\]

(23)

where \(e_1^\kappa = \Re e_1^\kappa\) and \(e_2^\kappa = \Im e_2^\kappa\). Here \(\kappa\) ranges among the \(n\) couples of complex conjugate eigenvectors, \(i\) spans the remaining \(n-2k\) real modes. Propagating up to time \(t\) yields, in full extent,

\[
e^{Lt} p_\infty = \sqrt{p_0} \left\{ c^0 + \sum_\iota e^{-i\Omega_\iota t} e_i^\iota + \sum_\kappa e^{-i\Omega_\kappa t} \left[ \cos(\omega_\kappa t) e_1^\kappa e_2^\kappa - \sin(\omega_\kappa t) e_1^\kappa e_2^\kappa + \sin(\omega_\kappa t) e_1^\kappa e_2^\kappa + \cos(\omega_\kappa t) e_1^\kappa e_2^\kappa \right] \right\}
\]

(24)

In short, \(e^{Lt} p_\infty = \sqrt{p_0} (c^0 + \tilde{e}^T \Omega \tilde{e})\), where \(\tilde{e} = (e_1^\kappa, e_2^\kappa, e^\iota_0)\), \(\tilde{e} = (e_1^0, e_2^0, e^\iota_0)\) and \(\Omega^\iota\) is defined in Eq. (14). Formula (13) then easily follows. Notice that when \(L\) is \(p\)-normal, from the Hermitian orthogonality relations

\[
(e_1 + ie_2, e_1 + ie_2) = 0, \quad (e_1 - ie_2, e_1 + ie_2) = \epsilon^c.
\]

(25)

there follow the real orthonormality relations

\[
(e_1, e_2) = 0, \quad (e_1, e_1) = (e_2, e_2) = \epsilon^c.
\]

(26)

Whatever the normalization chosen for the complex eigenvectors, real and imaginary parts have the same nor-
nalization, so that $\tilde{g}$ consists of blocks of type $c1_{2\times2}$. It follows that for $p$-normal systems oscillatory late-time behaviour in the relative entropy, encoded in $\Omega^T \tilde{g} \Omega^t$, will disappear, since $R(\omega_c t)^T \cdot c1_{2\times2} \cdot R(\omega_c t) = c1_{2\times2}$.

**On the three-state system**

We work out in full extent the three-state model, with $x = \xi, y = \chi$. Equilibrium holds when the Kolmogorov criterion is satisfied, that is then the ratio of products of clockwise over counterclockwise rates yields 1. Its locus defines the equilibrium line $\ell_{eq}: x - y^3 = 0$. The characteristic polynomial of $L$ is

$$\det(\lambda I - L) = \lambda^3 + (x + 2y + 3)\lambda^2 + (3y^2 + 2x + 2y + xy + 1)\lambda,$$

with roots $\lambda = 0$ and $\lambda_0 = -(x + 3y + 2 + \sqrt{\Delta})/2$ where the discriminant $\Delta = (x - y)(x + 3y - 4)$ vanishes at the critical lines $\ell_1: x + 3y - 4 = 0$ and $\ell_2: x - y = 0$. The unnormalized invariant state is

$$p = (y^2 + y + 1, y^2 + y + x, y^2 + xy + x). \quad (27)$$

We proceed analyzing the three cases of interest.

(A: $\Delta > 0$). The eigenvectors are

$$q_\pm = \begin{pmatrix} 4y - (x + y + \sqrt{\Delta})^2 \\ -4x + 2y(x + y + \sqrt{\Delta}) \\ 2x^2 + 2xy - 4y^2 + 2x\sqrt{\Delta} \end{pmatrix} \quad (28)$$

The off-diagonal element of the complexified Fisher matrix reads

$$g^{12} = \begin{pmatrix} (x - y) y + y^2 & xy + y^2 \\ 4x^2 y + y^2 & xy + y^2 + y^3 \end{pmatrix}. \quad (29)$$

It vanishes on the line $\ell^*: x = 1$, but there is another class of solutions in the first quadrant. It can be shown that those solutions correspond, as above, to systems for which $q^*$ are not the correct eigenvectors. The time-reversal operator is

$$\tilde{L}(x, y) = \begin{pmatrix} -x-y & x^2 + xy + y^2 \\ -x + y & x + y + x \end{pmatrix}.$$ (30)

Setting the commutator with $[\tilde{L}, L]$ to zero yields

$$[\tilde{L}, L] = \begin{pmatrix} (x - 1)(y^3 - x) \end{pmatrix} = 0 \quad (31)$$

with

$$M_{11} = -y(x + 2xy + 2y^2 + y^3)$$

$$M_{12} = (1 + y + y^2)(x + xy + 2y^2)$$

$$M_{13} = -(x + y^3)(1 + y + y^2)$$

$$\cdots$$

We don’t need to specify how more entries of $M$, as system $[29]$ has as its unique solutions $x = y^3$ (the equilibrium line) and $\ell^*: x = 1$, as expected.

(B: $\Delta < 0$) Eigenvectors $q_\pm$ have real and imaginary parts given by

$$Rq_\pm = \begin{pmatrix} 2x - x^2 + y^2 - 2xy \\ y^2 + xy - 2x \\ x^2 + xy - 2y^2 \end{pmatrix}, \quad Iq_\pm = \sqrt{\Delta} \begin{pmatrix} -x - y \\ y \\ x \end{pmatrix}$$

whose kernel is 1-dimensional, unless $x = 1$. Similar conclusions can be drawn on $\ell_1$.

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[Bae] J. Baez and J. P. Mumin, *Gauge Fields, Knots and Gravity* (World Scientific, Singapore, 1994), pp. 409–407.