The divergence of the barycentric Padé interpolants

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Abstract

We explain that, like the usual Padé approximants, the barycentric Padé approximants proposed recently by Brezinski and Redivo-Zaglia can diverge. More precisely, we show that for every polynomial $P(z)$ there exists a function $g(z) = \sum_{n=0}^{\infty} c_n z^n$, with $c_n$ arbitrarily small, such that the sequence of barycentric Padé approximants of $f(z) = P(z) + g(z)$ do not converge uniformly in any subset of $\mathbb{C}$ with a non-empty interior.

1 Introduction

In the recent article [1], Claude Brezinski and Michela Redivo-Zaglia proposed a barycentric version of Padé approximation and illustrated its effectiveness in practice. In the conclusion of their article they asked whether their approximants converge in theory. In this article we explain that, like the usual Padé approximants, there are entire functions for which the barycentric Padé approximants do not converge uniformly in any subset of $\mathbb{C}$ with a non-empty interior.

In the barycentric approach to Padé approximation proposed by Brezinski and Redivo–Zaglia, given $n + 1$ distinct interpolation points $x_{n,m} \in \mathbb{C} - \{0\}$, we define

$$p_n(z) := \sum_{m=0}^{n} \frac{w_{n,m} f(x_{n,m})}{z - x_{n,m}}$$

and

$$q_n(z) := \sum_{m=0}^{n} \frac{w_{n,m}}{z - x_{n,m}},$$

with weights $w_{n,m}$ chosen so that $f(z) q_n(z) = p_n(z) + O(z^n)$. The resulting barycentric approximants $p_n(z) / q_n(z)$ interpolate $f(z)$ at the points $x_{n,m}$ and match its first $n - 1$ derivatives at $z = 0$. Of course, usual Padé approximants with the same degrees of freedom would match more derivatives at $z = 0$ and the barycentric approach exchanges these derivatives by the interpolation at the points $x_{n,m}$.

Given a polynomial $P(z)$, interpolation nodes $X = \{x_{n,m}, n \in \mathbb{N}, 0 \leq m \leq n\} \subset \mathbb{C} - \{0\}$, with $x_{n,m} \neq x_{n,k}$ for $m \neq k$, and a set $\{a_k, k \in \mathbb{N}\} \subset \mathbb{C} - X$, we explain how to build functions of the form

$$f(z) = P(z) + \sum_{m=0}^{\infty} c_m z^m,$$

where $c_m$ are arbitrarily small.
with $c_m$ arbitrarily small, and indexes $\{n_k, k \in \mathbb{N}\}$, such that $f$’s barycentric Padé approximant of degree $n_k$ has a pole arbitrarily close to $\alpha_k$. This shows that the poles of $f$’s barycentric approximants can form a dense subset of $\mathbb{C}$. In this case, the sequence of approximants do not converge uniformly to $f(z)$ in any set with a non-empty interior.

In formal terms, we prove the following theorem. The $\alpha_k$ represent complex numbers to which the poles of the approximants will be arbitrarily close. We show that $\alpha_k$ can be arbitrarily chosen, as long as they do not coincide with the interpolation nodes. Once the $\alpha_k$ are chosen, the theorem guarantees the existence of approximants with poles very close to them.

**Theorem 1** Consider

1. A set $X = \{x_{n,m}, n \in \mathbb{N}, 0 \leq m \leq n\} \subset \mathbb{C} - \{0\}$ with $x_{n,m} \neq x_{n,k}$ for $m \neq k$.
2. A sequence $\{\varepsilon_m, m \in \mathbb{N}\}$ of small positive numbers.
3. A sequence $\{n_k, k \in \mathbb{N}\}$ of indexes with $n_k + 1 > 2n_k$.
4. A sequence $\{\alpha_k, k \in \mathbb{N}\} \subset \mathbb{C} - X$.

For every polynomial $P(z)$ with degree of($P$) $< n_0$ there exists a set $\{\pi_k, k \in \mathbb{N}\} \subset \mathbb{C} - X$ and coefficients $\{c_m, m \in \mathbb{N}\}$ such that the function $f(z)$ in (2) is entire and

(i) $c_m = 0$ for $m < n_0$ and $|c_m| \leq \varepsilon_m$ for $m \geq n_0$.

(ii) $|\pi_k - \alpha_k| \leq \varepsilon_{n_k}$ for $k \in \mathbb{N}$.

(iii) For all $k \in \mathbb{N}$, $\pi_k$ is a pole of the barycentric Padé approximant of $f(z)$ with nodes $x_{n_k,m}$.

In section 3 we prove Theorem 1. Our proof uses lemmas which are stated in section 4 and proved in section 5. We suggest that, at first, the reader follows the proof of the general theorem accepting the lemmas as true results. Unfortunately, the proof is very technical. In order to motivate our technical arguments, in the next section we present them in an informal way and sketch an algorithm to compute the coefficients $c_m$ and the poles $\pi_k$ mentioned in the main theorem. We hope that by reading this informal section the reader will grasp the overall structure of the proof and will no be distracted by the unavoidable technical details.

2 An informal description of the proof of Theorem 1

We can think of the proof of Theorem 1 as an algorithm to compute the coefficients $c_m$ in such way that we guarantee the existence of the poles $\pi_k$ near the points $\alpha_k$. Unfortunately, this algorithm would not work in practice because it considers a sequence of approximants for which the number of interpolation nodes grows exponentially. It also uses infinite sequences, of which most terms would underflow in finite precision arithmetic. Moreover, rounding errors would change things completely. In other words, our examples are neither robust nor practical.

The fact that our algorithm to construct the function $f(z)$ would not work in practice is an evidence of a deeper problem with this article and the theoretical analysis of the convergence of numerical algorithms in general. As we have shown in other instances [5, 6, 7, 8, 9], several algorithms which work well in practice are vulnerable to
theoretical examples like the ones presented here. Therefore, the algorithm by Brezinski and Redivo–Zaglia may be quite adequate for all the degrees of the approximants one usually considers in practice (but this would be the subject of another article.)

It is our opinion that our examples tell more about the inadequacy of the asymptotic analysis of the algorithms than they tell about the inadequacy of the algorithms themselves. In fact, articles like this one only show how far we are from an adequate theory to explain the behavior of these algorithms for large (but finite) problem instances in finite precision arithmetic.

The difference between theory and practice affects not only the convergence of numerical algorithms; it affects applied mathematics in general. In Statistics the discussion of this topic is quite old, as one can notice by reading the article [4] by C. Mallows, the commentaries after it and its references. In Computer Science the discussion is illustrated in the first 15 minutes of the provocative talk by Alan Kay at OOPSLA97 [2] and the last minutes of Donald Knuth’s talk at Google [3]. The difference of opinions of outstanding scholars like Peter Huber and Brad Efron illustrated in Mallows’s article and the opposing views of Kay and Knuth show that there are no easy answers regarding the interplay of theory and practice in applied mathematics (in a broad sense.)

That said, we now present an informal algorithm to compute the coefficients $c_m$ in Theorem 1, under the assumption that we are using exact arithmetic and can estimate constants for which our usual theories provide only existence proofs (like the radius of convergence for Newton’s method for a $C^1$ function for which we do not know how to bound the derivative or its inverse.) To keep things simple, we assume that we are concerned with the polynomial $P(z) = 0$ and take $n_k = 3^k$.

The first step is to write the entire function $f$ in theorem 1 as

$$f(z) = \sum_{k=0}^{\infty} \mu_k z^{3^k} \sum_{m=0}^{3^k} d_{k,m} z^m,$$  \hspace{1cm} (3)

where the $\mu_k$ are free parameters to be determined by our algorithm and the $d_{k,m}$ are carefully chosen constants which depend on the interpolation nodes and the $\alpha_k$, but which do not depend on the $\mu_k$. The algebraic expressions defining the constants $d_{k,m}$ are relevant for the technical details but, once the reader believes in our claims about them, they do not matter much for the overall understanding of the proof.

We build the $\mu_k$ one by one, by induction. However, we must be careful because the location of the poles of the barycentric Padé approximant with nodes

$$X_k := \left\{ x^{3^j,m}, m = 0, \ldots, 3^k \right\}$$

of the function $f$ in (3) will be also influenced by the $\mu_n$’s for $n > k$, which we do not know at the $k$-th step. The idea then is to bound the $\mu_n$ for $n > k$ so that they do not influence much the location of the poles of the previous approximants. Therefore, instead of building only the sequence $\mu_k$ we also build a family $\beta_{k,j}$ of bounds such that if, for $n > k$, $|\mu_n| \leq \beta_{k,n}$ then the location of the poles determined at step $k$ will not be significantly affected by the $\mu_n$ with $n > k$. In order to achieve this goal the $\mu_k$ and the $\beta_{k,n}$ must decay very rapidly with $k$, so rapidly in fact that they would underflow in finite precision arithmetic.

In idealized terms, the algorithm can be summarized as follows:

1. Start with $\beta_{-1,m} = 1$ for $m \in \mathbb{N}$. 
2. For \( j = 0, 1, \ldots \), do

3. Choose \( \mu_j \) with \(|\mu_j| \leq \beta_{j-1,j}\) such that the barycentric Padé approximant of

\[
f_j(z) = \sum_{k=0}^{j} \mu_k z^k \sum_{m=0}^{3j} d_{k,m} z^m
\]

with nodes \( X_j = \{x_{3j,m}, m = 0, \ldots, 3j\} \) has a pole near \( \alpha_j \) (the constants \( d_{k,m} \) are chosen so that this is possible.)

4. Use the complex version of the implicit function theorem to define \( \beta_{j,n} \leq \beta_{j,n-1} \) for \( n \in \mathbb{N} \) such that if \(|\mu_n| \leq \beta_j\) for \( n > k \) then the barycentric Padé approximant of the function \( f \) in (3) also has a pole near \( \alpha_j \).

5. goto 2.

In principle, one could try to apply the same procedure to analyze other versions of Padé approximants. However, this may not work because we may not be able to adapt step 3. We were able to build our examples because we found constants \( d_{m,k} \) with an important property, which may be specific to the barycentric Padé approximants: the \( d_{m,k} \) are such that the algebraic formulae to compute the barycentric Padé approximants imply the existence of the poles near the \( \alpha_j \)'s. In some sense, for these \( d_{m,k} \), the method causes its own demise, because the same equations that ensure its degree of approximation at the origin and the interpolation at the remaining nodes lead to the poles near the \( \alpha_j \).

Unfortunately, there are many technical details involved in turning the informal arguments above into a theorem. We tried to find a simpler way than our proof of Theorem 1 and our lemmas to achieve this goal, but we failed.

3 Proof of Theorem 1

Let us start by defining the terms we use. We are concerned with sub-sequences with indexes \( n_k \) of the sequence of barycentric Padé approximants. For \( n = n_k \), we interpolate at distinct points \( x_{n_k,0}, x_{n_k,1}, \ldots, x_{n_k,n_k} \) and define

\[
t_{k,m} := x_{n_k,m}.
\]

The Vandermonde matrix \( V_k \) corresponding to Lagrange interpolation at \( t_{k,m} \) is

\[
V_k = \begin{pmatrix}
1 & t_{k,0} & t_{k,0}^2 & \cdots & t_{k,0}^{n_k} \\
1 & t_{k,1} & t_{k,1}^2 & \cdots & t_{k,1}^{n_k} \\
1 & t_{k,2} & t_{k,2}^2 & \cdots & t_{k,2}^{n_k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & t_{k,n_k} & t_{k,n_k}^2 & \cdots & t_{k,n_k}^{n_k}
\end{pmatrix}, \quad \text{with} \quad v_{k,i,j} := t_{k,i}^j.
\]

(Our matrices have indexes \((i, j)\), with \(0 \leq i, j \leq n_k\), and we denote \((V_k)_{i,j}\) by \(v_{k,i,j}\).)

The weights for usual barycentric interpolation at \( t_{k,m} \) are

\[
\lambda_{k,m} := \frac{1}{\prod_{i \neq m} (t_{k,i} - t_{k,m})}.
\]
We also use the vectors $a_k \in \mathbb{C}^{1+n_k}$ with entries
\[ a_{k,m} := \frac{1}{\alpha_k - i_{k,m}} \quad (6) \]
and $B_k$ is the $(1+n_k) \times (1+n_k)$ diagonal matrix which has $a_{k,j}$ in its diagonal:
\[ b_{k,i,i} = a_{k,i} \quad \text{and} \quad b_{k,i,j} = 0 \quad \text{for} \quad i \neq j. \quad (7) \]
The $(1+n_k)$-dimensional vector $e_k$ has entries
\[ e_{k,i} := 0 \quad \text{for} \quad 0 \leq i \leq n_k \quad \text{and} \quad e_{k,n_k} := 1. \quad (8) \]
The coefficients $c_m$ of the function $f(z)$ in (12) are defined in terms of the vectors
\[ d_k := V_k^{-1} a_k, \quad (9) \]
and a sequence $\{ \sigma_k, k \in \mathbb{N} \}$:
\begin{align*}
    c_m & := 0 \quad \text{for} \quad m < n_0, \quad (10) \\
    c_m & := 0 \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad 2n_k < m < 2n_k+1, \quad (11) \\
    c_m & := \sigma_k d_{k,m-n_k} \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad n_k \leq m \leq 2n_k, \quad (12) 
\end{align*}
so that
\[ f(z) = f(z; \sigma) := P(z) + \sum_{k=0}^{\infty} \sigma_k z^m \sum_{m=0}^{n_k} d_{k,m} z^m. \quad (13) \]
Let us define
\[ r_k := 1 + \max_{0 \leq m \leq k} \sum_{j=0}^{\infty} \frac{|f_{m,j}|}{\max_{0 \leq m \leq j} \sum_{j=0}^{\infty} |f_{m,j}|}, \quad (14) \]
\[ \tau_k := \min_{0 \leq m \leq 2n_{k+1}} \epsilon_m \left( 1 + \sum_{j=0}^{\infty} \frac{e_m}{r_k^{n_{k+1}}} \right)^{1/n_{k+1}} \left( 1 + \|d_{k+1}\|_1 \right) \left( 1 + n_{k+1} \right)! \]
Note that $0 < \tau_k < 1$ and, if $\sigma_k$ in (12) is such that $0 < \sigma_k \leq \tau_k$, then $|c_m| < \epsilon_m$ for $n_k \leq m \leq 2n_k$. Moreover,
\[ \chi := \sum_{k=0}^{\infty} r_k^{n_k} \|d_k\|_1 \tau_k < \infty, \quad (15) \]
and the series in (13) converges for all $z$ when $0 \leq \sigma_k \leq \tau_k$ for all $k$.
Finally, by perturbing $\sigma_k$, we can assume that
\[ \sum_{m=0}^{n_k} \frac{\lambda_k,m}{\alpha_k - i_{k,m}} \neq 0 \quad \text{and} \quad \sum_{m=0}^{n_k} \frac{\lambda_k,m i_{k,m}^n}{\alpha_k - i_{k,m}} \neq 0. \quad (16) \]
We are now ready to prove Theorem 1.

**Proof of Theorem 1:** We prove the following:

**Main claim:** There exists $\{ \mu_k, k \in \mathbb{N} \}$ so that the function $f(z) = f(z; \mu)$ in (13) satisfies the requirements of Theorem 1.

In order to verify the main claim, we build $\{ \mu_k, k \in \mathbb{Z} \}$ and $\{ \rho_k, k \in \mathbb{Z} \}$ such that:

(a) $\rho_m := \mu_m := 1$ for $m < 0$.  

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(b) \(0 < p_{m+1} \leq p_m\) for all \(m \in \mathbb{Z}\).

e (c) \(0 < \mu_m \leq \mu_{m-1} \tau_{m-1}\) for \(m = 0, 1, 2, \ldots\).

(d) Let \(m \in \mathbb{N}\) and \(\{\sigma_h, h \in \mathbb{N}\}\) be such that

1. \(\sigma_h = \mu_h\) for \(0 \leq h \leq m\),
2. \(0 < \sigma_h \leq \rho_m \tau_{h-1}\) for \(h > m\),

then there exists \(\xi(\sigma) \in \mathbb{C}\) such that \(|\xi(\sigma) - \alpha_m| \leq \tau_m\) and the barycentric Padé approximant \(p_n(z)/q_n(z)\) of the function \(f(z) = f(z; \sigma)\) in \((13)\) satisfies

\[
p_n(\xi(\sigma)) \neq 0 \quad \text{and} \quad q_n(\xi(\sigma)) = 0.
\]  

The existence of \(\rho_k\) and \(\mu_k\) satisfying (a)–(d) verifies the main claim because, for each \(m \in \mathbb{N}\), we can apply item (d) to \(\sigma = \mu\) and conclude that there exists \(\pi_m = \xi(\sigma)\) as required by Theorem 1.

We have already defined \(\mu_k\) and \(\rho_k\) for \(k < 0\) and the items (a)–(d) above hold for negative \(m = k < 0\). We now assume that \(k \geq 0\) and we have defined \(\mu_m\) and \(\rho_m\) for \(m < k\) and the items (b)–(d) hold for such \(m\), and define \(\mu_k\) and \(\rho_k\) such that (b)–(d) holds for \(m = k + 1\). By the induction principle, this defines \(\mu_k\) and \(\rho_k\) for all \(k \in \mathbb{Z}\).

For \(\mu_0, \mu_1, \ldots, \mu_k\), consider the function

\[
f_k(z) := f_k(z; \mu) := P(z) + \sum_{h=0}^{k-1} \mu_h y_h + \sum_{m=0}^n d_{h,m} z^m + \mu_k \sum_{m=0}^n d_{k,m} z^m.
\]  

Let \(\{\sigma_m, m \in \mathbb{N}\}\) be such that

\[
\sigma_h = \mu_h \quad \text{for} \quad 0 \leq h \leq k \quad \text{and} \quad 0 < \sigma_h \leq \rho_k \tau_h \quad \text{for} \quad h > k.
\]  

The Lemmas 3, 4, and 5 show that the barycentric Padé approximant for the function \(f(z) = f(z; \sigma)\) in \((13)\) for \(\sigma\) in \((19)\) is defined by matrices \(Y, U\) and \(S(\sigma)\) and weights \(w(\sigma) \neq 0\) with

\[
(Y + \mu_k U + S(\sigma)) w(\sigma) = 0,
\]  

and, for \(\chi\) in \((15)\),

\[
\|S(\sigma)\|_2 < \rho_k (1 + n_k) \chi.
\]  

These lemmas show that there exist \(\mu_k \in (0, \rho_{k-1} \tau_{k-1})\) and \(v \in \mathbb{C}^{1+n_k}\) such that:

\[
\nu_m \neq 0 \quad \text{for} \quad 0 \leq m \leq n_k,
\]  

\[
\text{rank of } (Y + \mu_k U) = n_k \quad \text{and} \quad (Y + \mu_k U) v = 0,
\]  

\[
\sum_{m=0}^{n_k} \nu_m = 0 \quad \text{and} \quad \sum_{m=0}^{n_k} \nu_m f_k (t_{k,m}) \neq 0.
\]  

(To verify the second inequality in \((24)\), take \(\kappa = -\sum_{m=0}^{n_k} \nu_m f_k (t_{k,m}) / (\alpha_k - t_{k,m})\) in Lemma 6.) Since \(Y + \mu_k U\) is a \(n_k \times (1 + n_k)\) matrix with rank \(n_k\), there exists \(\zeta_{\alpha} \in (0, \tau_k)\) such that

\[
\|\delta M\|_2 \leq \zeta_{\alpha} \Rightarrow Y + \mu_k U + \delta M \quad \text{has rank} \quad n_k.
\]  

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1 The numbers \(p_m \tau_{m-1}\) are the bound \(\beta_{h,j}\) mentioned in the argument in section 2.
By continuity and (24), there exists \(\zeta_1 \in (0, \min \{\zeta_0, |v_0|, |v_1|, \ldots, |v_n|\})\) such that

\[
\max \{|\delta \nu|, |\delta \alpha|, |\delta f_m|\} \leq \zeta_1 \Rightarrow \sum_{m=0}^{n_k} \frac{v_m + \delta v_m}{\alpha_k + \theta(\delta \nu) - t_{k,m}} \neq 0. \tag{26}
\]

Lemma 3 shows that the entries in the first row of \(Y\) are all zero and that the first equation in the system \((Y + \mu_k U) v = 0\) can be written as \(\eta(\alpha_k, v) = 0\), for

\[
\eta(\alpha, v) := \sum_{m=0}^{n_k} \frac{v_m}{\alpha_k - t_{k,m}} = 0.
\]

Equation (24) shows that the function \(\eta(\alpha, v)\) has partial derivative

\[
\frac{\partial \eta}{\partial \alpha}(\alpha_k, v) = \sum_{m=0}^{n_k} \frac{v_m}{(\alpha_k - t_{k,m})^2} \neq 0.
\]

Since \(\eta(\alpha_k, v) = 0\), the (complex) implicit function theorem shows that there exists \(\zeta_2 \in (0, \zeta_1)\) such that if \(|\delta \nu| < \zeta_2\) then there exists \(\theta(\delta \nu) \in C\) with

\[
|\theta(\delta \nu)| \leq \zeta_1 \quad \text{and} \quad \eta(\alpha + \theta(\delta \nu), v + \delta \nu) = \sum_{m=0}^{n_k} \frac{v_m + \delta v_m}{\alpha_k + \theta(\delta \nu) - t_{k,m}} = 0. \tag{27}
\]

Since \(Y + \mu_k U\) has rank \(n_k\), there exists \(\zeta_3 \in (0, \zeta_2)\) such that if \(\|\delta \mathbf{M}\|_2 \leq \zeta_3\) then there exists \(\kappa(\delta \mathbf{M})\) with

\[
\kappa(\delta \mathbf{M}) \leq \zeta_2 \quad \text{and} \quad (Y + \mu_k U + \delta \mathbf{M})(v + \kappa(\delta \mathbf{M})) = 0. \tag{28}
\]

We claim that by considering \(X\) in (15) and taking

\[
\rho_k := \min \left\{\rho_{k-1}, \frac{\zeta_3}{(1 + n_k) X}\right\} \tag{29}
\]

and the \(\mu_k\) above we satisfy the requirement (d) on \(\rho_k\) and \(\mu_k\) for \(m = k\), and we end this proof validating this claim. In fact, let \(\{\sigma_h, h \in \mathbb{N}\}\) be a sequence satisfying (19).

Equations (21) and (29) show that \(\|\mathbf{S}(\sigma)\|_2 \leq \zeta_3\) and (25) implies that the matrix \(Y + \mu_k U + \mathbf{S}(\sigma)\) has rank \(n_k\). Therefore, the space of solutions \(w(\sigma)\) of (20) has dimension one. Equation (25) shows that

\[
\tilde{w} := v + \kappa(\mathbf{S}(\sigma))
\]

is a solution of (20). It follows that all solutions \(w(\sigma)\) of (20) are of the form \(\gamma \tilde{w}\), with \(\gamma \in C\). Since all these solutions lead to the same approximant (\(\gamma\) cancels out), the approximants are defined by \(\tilde{w}\).

Equation (25) shows that \(\delta v = \kappa(\mathbf{S}(\sigma))\) is such that \(|\delta v| \leq \zeta_3\) and leads to \(\theta(\delta \nu)\) satisfying (27). Since (27) is equivalent to \(q_\nu(\xi(\sigma)) = 0\) for \(\xi(\sigma) := \alpha_k + \theta(\delta \nu)\), we have verified the last condition in (17). Moreover, \(|\xi(\sigma) - \alpha_k| = |\theta(\delta \nu)| < \zeta_1 < \tau_k\).

Consider \(\varepsilon\) with \(|\varepsilon| < r_h\), with \(r_h\) in (13) and \(f_h\) in (14). Since \(|\sigma_h| \leq \rho_k \tau_h\) for \(h > k\), equations (15) and (29) show that \(\delta f(h) := f(z; \sigma) - f_h(z)\) satisfies

\[
|\delta f(h)| = |f(z; \sigma) - f_h(z)| \leq \sum_{h=k+1}^{\infty} |\sigma_h| r_h \sum_{m=0}^{n_k} |d_{p,m}| r_m^h \leq
\]

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\[ \leq \rho_k \sum_{h=k+1}^{\infty} \tau_h r_h^{2n_h} \| d_h \|_1 \leq \rho_k (1 + n_k) \chi \leq \zeta_3. \]

Therefore \( |\delta f_m| = |\delta f_k(t_{m,k})| \leq \zeta_3 \) for \( 0 \leq m \leq n_k \). Since, for \( f(z; \sigma) \) in (14),

\[ f(t_{k,m}) = f_k(n_{m}; \sigma) + \delta f_k(t_{m,k}), \]
equation (26) shows that \( p_n(\xi(\sigma)) = p_n(\alpha_k + \theta(\delta v)) \neq 0 \). Therefore, we have verified the first condition in (17) and we are done. \( \square \)

4 Lemmas

Lemma 1 For \( R > 0 \), suppose that \( \sum_{m=0}^{\infty} \left| c_m \right| R^m < \infty \) and consider distinct points \( x_{n,0}, \ldots, x_{n,n} \) with \( 0 < |x_{n,m}| < R \). The functions \( p_n(z) \) and \( q_n(z) \) in (1) yield the \( n \)-th degree barycentric Padé approximant for \( f(z) = \sum_{m=0}^{\infty} c_m z^m \) if and only if, for \( 0 \leq i < n \),

\[ \sum_{j=0}^{n} \left( \sum_{k=-n-i}^{\infty} c_k x_{n,j}^{k-i+n} \right) w_{n,j} = 0. \] (30)

Lemma 2 For the coefficients \( c_m \) in (10)–(12), there are matrices \( Y, U \) and \( S(\sigma) \) with dimension \( n_k \times (1 + n_k) \) such that \( w \in \mathbb{C}^{1+n_k} \) satisfies (30) if and only if

\[ (Y + \sigma_k U + S(\sigma)) w = 0, \]

and

(1) \( Y \) does not depend on \( \sigma_m \) for \( m \geq k \).

(2) All the entries in the first row of \( Y \) are equal to zero, i.e., \( y_{0,j} = 0 \).

(3) \( U \) has entries

\[ u_{i,j} = \frac{t_{k,j}^i}{\alpha_k - t_{k,j}}, \]

and, in particular \( u_{0,j} = \frac{1}{\alpha_k - t_{k,j}} \). (31)

(4) If \( \sigma_m \leq \varepsilon \tau_m \) for all \( m > k \) and \( \{\tau_m, \, m \in \mathbb{N}\} \) satisfies (15) then

\[ \| S(\sigma) \|_2 \leq \varepsilon (1 + n_k) \chi. \]

Lemma 3 Let \( U \) be as in Lemma 2. For every matrix \( M \) with dimension \( n_k \times (1 + n_k) \), there exists a finite set \( E \) such that if \( \varepsilon \notin E \) then the matrix \( M + \varepsilon U \) has rank \( n_k \) and there exists a vector \( \nu(\varepsilon) \) such that \((M + \varepsilon U) \nu(\varepsilon) = 0\), and for \( 0 \leq m \leq n_k \),

(1) \( \nu_m(\varepsilon) \neq 0 \) and \( \nu_m(\varepsilon) \) is a rational function of \( \varepsilon \).

(2) For the vector \( \Lambda_k \) with entries \( \lambda_{k,m} \) in (5) and \( B_k \) in (7),

\[ \lim_{\varepsilon \to \infty} \nu(\varepsilon) = B_k^{-1} \Lambda_k. \] (32)
Lemma 4. For $d_k$ in (9), consider a constant $\kappa \in C$ and the polynomial
\[
g_k(z) := z^n \sum_{j=0}^{n_k} d_{k,j} z^j,
\]
(33)
If $\alpha_k$ satisfies (16) and $\psi(\varepsilon)$ is a vector whose coordinates are rational functions of $\varepsilon$ and satisfy (32), then there exists a finite set $\mathcal{E}$ such that if $\varepsilon \notin \mathcal{E}$ then
\[
\sum_{m=0}^{n_k} \frac{v_m(\varepsilon)}{(\alpha_k - \ell_{k,m})} z \neq 0 \quad \text{and} \quad \varepsilon \sum_{m=0}^{n_k} \frac{v_m(\varepsilon) g_k(\ell_{k,m})}{(\alpha_k - \ell_{k,m})} \neq \kappa.
\]
(34)

5. Proofs of the lemmas

Proof of Lemma 1. If $|z| < \min_{0 \leq j \leq n} |x_{n,j}|$ then equation (11) yields
\[
p_n(z) = \sum_{j=0}^{n} \frac{w_{n,j}}{z - x_{n,j}} \sum_{k=0}^{\infty} c_k x_{n,j}^k = \sum_{k=0}^{\infty} c_k \left( \sum_{j=0}^{n} \frac{w_{n,j} x_{n,j}^k}{z - x_{n,j}} \right) =
\]
\[
= \sum_{k=0}^{\infty} c_k \left( \sum_{j=0}^{n} w_{n,j} x_{n,j}^k \frac{1}{x_{n,j} - 1} \right)
\]
\[
- \sum_{k=0}^{\infty} c_k \left( \sum_{j=0}^{n} \frac{w_{n,j} x_{n,j}^k}{x_{n,j} - 1} \right) = - \sum_{k=0}^{\infty} c_k \left( \sum_{j=0}^{n} w_{n,j} x_{n,j}^{k-1} \right) z^h.
\]
(35)
Moreover,
\[
q_n(z) = \sum_{j=0}^{n} \frac{w_{n,j}}{z - x_{n,j}} = \sum_{j=0}^{n} w_{n,j} x_{n,j} - 1 = - \sum_{j=0}^{n} w_{n,j} x_{n,j}^{h_0} x_{n,j}^{h_1} =
\]
\[
= - \sum_{h=0}^{\infty} \left( \sum_{j=0}^{n} w_{n,j} x_{n,j}^{-(h+1)} \right) z^h.
\]
(36)
Equation (36) shows that
\[
f(z) q_n(z) = - \sum_{h=0}^{n-1} \left( \sum_{k=0}^{h} c_k \left( \sum_{j=0}^{n} w_{n,j} x_{n,j}^{k-1} \right) \right) z^h + O(z^n).
\]
(37)
Combining (35) with (37) we get that $f(z) q_n(z) = p_n(z) + O(z^n)$ if and only if, for $0 \leq h < n$,
\[
\sum_{k=0}^{\infty} c_k \left( \sum_{j=0}^{n} w_{n,j} x_{n,j}^{k-1} \right) = \sum_{k=0}^{h} c_k \left( \sum_{j=0}^{n} w_{n,j} x_{n,j}^{k-1} \right).
\]
Subtracting the right-hand-side from the left-hand-side in this equation we obtain
\[
\sum_{j=0}^{n} \left( \sum_{k=h+1}^{\infty} c_k x_{n,j}^{k-1} \right) w_{n,j} = 0,
\]
and replacing $h$ by $n - i - 1$ in the equation above we obtain (30).
Proof of Lemma 2. Equations (4) and (9) show that
\[
\sum_{m=0}^{n_i} d_{k,m} f_{k,j}^m = (V_k d_k)_j = a_{k,j}.
\] (38)

Given \(0 \leq i < n_k\), we write \(P(z) = \sum_{h=0}^{n_k-1} p_h z^h\) and define
\[
\begin{align*}
A_i & := \{0 \leq h < n_0\} \cap \{h \geq n_k - i\}, \\
B_i & := \{n_0 \leq h < n_k\} \cap \{h \geq n_k - i\},
\end{align*}
\]
\[\gamma_h := p_h \text{ for } h \in A_i \text{ and } \gamma_h := c_h \text{ for } h \in B_i.\]
Equations (10)–(12), (31) and (38) show that, for \(0 \leq i < n_k\), we have
\[
\sum_{h=n_k-i}^{n_k} c_{h_i} h^{-n_k+i} = y_{i,j} + \hat{u}_{i,j} + s_{i,j}(\sigma),
\]
with
\[
\begin{align*}
y_{i,j} & := \sum_{h \in A \cup B_i} \gamma_{h_i} h^{-n_k+i}, \\
\hat{u}_{i,j} & := \sum_{h=n_k}^{2n_k} c_{h_i} h^{-n_k+i} = \sigma_{l_{k_i}} \sum_{m=0}^{n_k} d_{m} f_{k,j}^m = \sigma_{l_{k_i}} a_{k,j} = \sigma_{l_{k_i}} u_{i,j}, \\
s_{i,j}(\sigma) & := \sum_{h=2n_k+1}^{\infty} c_{h_i} h^{-n_k+i} = \sum_{l=k+1}^{\infty} \sigma_{l} \sum_{m=0}^{n_k} d_{m} f_{k,j}^{l-n_k+i+m}.
\end{align*}
\] (39)

Therefore, the system of equations (31) can be written as \((Y + \sigma_{l_i} U + S(\sigma)) w = 0\), for the matrices \(Y\) and \(S(\sigma)\) with entries \(y_{i,j}\) and \(s_{i,j}(\sigma)\) above and \(u_{i,j}\) in (31).

Note that \(y_{i,j}\) does not depend on \(\sigma_{m}\) for \(m \leq k\). When \(i = 0\) we have \(A_i \cup B_i = \emptyset\) and, as a result, \(y_{0,j} = 0\). Thus, the \(y_{i,j}\) in (39) satisfy items (1) and (2) in Lemma 2. Moreover, if \(0 \leq \sigma_m \leq \varepsilon \tau_m\), then, for \(0 \leq i < n_k\), (44) and (45) imply that
\[
|s_{i,j}(\sigma)| \leq \sum_{l=k+1}^{\infty} \sigma_{l} \sum_{m=0}^{n_k} d_{l,m} \left|t_{k,l}\right| r_{k,j}^{-n_k+i+m} \leq \varepsilon \sum_{l=k+1}^{\infty} \tau_l \left\|d_{l}\right\|_1 r_{k,j}^{2n_k} \leq \varepsilon \chi.
\]
Therefore,
\[
\left\|S(\sigma)\right\|_2 \leq \sqrt{\sum_{0 \leq i,j \leq n_k} \left|s_{i,j}(\sigma)\right|^2} \leq \varepsilon (1 + n_k) \chi
\]
and we are done. \(\square\)

Proof of Lemma 3. Let \(\tilde{M}\) be the matrix obtained by adding a null \(n_k\)-th row to \(M\) and let \(\tilde{U}\) the matrix we obtain by adding to \(U\) the \(n_k\)-th row with entries
\[
u_{n_k,j} = \frac{t_{R_k,j}}{a_k - t_{k,j}}.
\]
(40)
The matrix \(\tilde{U}\) can be recast as
\[
\tilde{U} = V_k^{n_k} B_k.
\]
for \(V_k\) in (3) and \(B_k\) in (7). Thus, \(\tilde{U}\) is non-singular and the determinant of the matrix \(N(\varepsilon) := \tilde{M} + \varepsilon \tilde{U}\) is a polynomial \(Q(\varepsilon)\). This polynomial is not identically zero, because
the non-singularity of $\tilde{U}$ implies that $\lim_{\varepsilon \to +\infty} |Q(\varepsilon)| = +\infty$. Therefore, there exists only a finite set of $\varepsilon$s for which $N(\varepsilon)$ is singular. We define $\mathcal{E}_{-1}$ as the union of this finite set with $\{0\}$. Given $\varepsilon \not\in \mathcal{E}_1$ and $e_k$ in (38), the vector

$$v(\varepsilon) := \varepsilon N(\varepsilon)^{-1} e_k$$

satisfies $(M + \varepsilon U) v(\varepsilon) = 0$ and its coordinates are rational functions of $\varepsilon$. Moreover,

$$v(\varepsilon) = \varepsilon \left( \tilde{M} + \varepsilon \tilde{U} \right)^{-1} e_k = \left( \frac{1}{\varepsilon} \tilde{M} + \tilde{U} \right)^{-1} e_k.$$  

Therefore, (40) yields

$$\lim_{\varepsilon \to +\infty} v(\varepsilon) = \tilde{U}^{-1} e_k = \psi := B_k^{-1} V_k^{-1} e_k.$$  

Cramer’s rule, Laplace’s expansion and equation (8) show that

$$\bar{v}_m = (-1)^{n+m} (\alpha_k - t_{k,j}) \frac{\det(W_{k,m})}{\det(V_k)},$$  

(41)

where $W_{k,m}$ is the matrix obtained by the removal of the $m$-th row and last column of $V_k$. $V_k$ and $W_{k,m}$ are Vandermonde matrices and, therefore,

$$\det(V_k) = \prod_{0 \leq i < j \leq n} (t_{k,j} - t_{k,i}) \quad \text{and} \quad \det(W_{k,m}) = \prod_{0 \leq i < j \leq n, i,j \neq m} (t_{k,j} - t_{k,i}).$$

The equation above and equations (5) and (41) imply that

$$\bar{v}_m = \frac{(\alpha_k - t_{k,j}) (-1)^{n+m}}{\left( \prod_{0 \leq j < m} (t_{k,j} - t_{k,m}) \right) \left( \prod_{m < i \leq n} (t_{k,m} - t_{k,i}) \right)} = \frac{\alpha_k - t_{k,j}}{\prod_{j \neq m} (t_{k,j} - t_{k,m})} = \frac{\alpha_k - t_{k,j}}{\lambda_{k,j}},$$

and we have verified (32).

Finally, for every $0 \leq i \leq n_k$, $v_i(\varepsilon)$ is a rational function of $\varepsilon$ and the last paragraph shows that this rational function does not vanish for large $\varepsilon$. This implies that there exists a finite set $\mathcal{E}_i$ such that if $\varepsilon \not\in \mathcal{E}_i$ then $v_i(\varepsilon) \neq 0$. We complete this proof by taking $\mathcal{E} := \bigcup_{i=1}^{n_k} \mathcal{E}_i$. \hfill $\square$

**Proof of Lemma 4** Let us show that there exist a finite set $\mathcal{E}_1$ such that if $\varepsilon \not\in \mathcal{E}_1$ then the first inequality in (34) holds. Equations (16) and (32) show that the rational function of $\varepsilon$ given by

$$\mu(\varepsilon) := \sum_{m=0}^{n_k} \frac{v_m(\varepsilon)}{(\alpha_k - t_{k,m})^2}$$

satisfies

$$\lim_{\varepsilon \to +\infty} \mu(\varepsilon) = \sum_{m=0}^{n_k} \frac{\lambda_{k,m}}{\alpha_k - t_{k,m}} \neq 0.$$  

This implies that the finite set $\mathcal{E}_1$ mentioned above exists.
We now prove that there exist a finite set \( \mathcal{E}_2 \) such that if \( \varepsilon \not\in \mathcal{E}_2 \) then the second inequality in (34) holds. The definitions of \( V_k, a_k \) and \( d_k \) in (4), (6), (9) and (33) yield

\[
g_k(t_k, m) = t_k^{n_k} \sum_{j=0}^{n_k} d_k,j t_k,j m = t_k^{n_k} \sum_{j=0}^{n_k} t_k,j m d_k,j = \frac{t_k^{n_k}}{\alpha_k - t_k,m}.
\]

This implies that the vector \( h \) with coordinates \( h_m = g_k(t_k,m) \) satisfies

\[
h = B_k V_k e_k,
\]

for \( B_k \) in (7) and \( e_k \) in (8). Consider the function

\[
\gamma(\varepsilon) := \frac{1}{\varepsilon} \left( \sum_{m=0}^{n_k} v_m(\varepsilon) h_k(t_k,m) \alpha_k - t_k,m \right) - \kappa = \frac{1}{\varepsilon} \left( \varepsilon v(\varepsilon) \right)^T B_k h - \kappa.
\]

Equations (16), (32) and (42) imply that

\[
\lim_{\varepsilon \to \infty} \gamma(\varepsilon) = \Lambda_k^T B_k V_k e_k = \sum_{m=0}^{n_k} \frac{\alpha_k - t_k,m}{\alpha_k - t_k,m} \neq 0,
\]

and Lemma 4 follows from the observation that \( \gamma(\varepsilon) \) is rational function of \( \varepsilon \). \( \square \)

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