The Incomplete statistics and black holes thermodynamics

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Abstract

Incomplete statistics, an important extension of the Boltzmann-Gibbs statistical mechanics, was proposed some years back by Q. A. Wang. The main subject-matter is that the formalism adopts an incomplete normalization condition given by $\sum_{i=1}^{W} p_i^q = 1$, where $q$ is a positive real parameter and $p_i$ is the probability of determined microstate. In this paper, we have used the Incomplete statistics approach to describe the thermodynamics of black holes. We have obtained an equipartition theorem and, after that we have derived the heat capacity. Depending on the values of $q$ and mass $M$ then the black hole can become thermally stable.

Keywords: Incomplete statistics, Tsallis statistics, black holes thermodynamics

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1. INTRODUCTION

Hawking’s discovery [1] of thermal radiation from a black hole (BH) was unexpected to the majority of specialists altogether. Even though the existence of quite a few demonstrations of a restrict connection between BH physics and thermodynamics had appeared before his paper. Bekenstein [2] realized that the properties of one BH features, its area, for example, resemble the ones from entropy. In fact, the Hawking’s area theorem [1] means that the area $A$ does not diminish in any classical scenarios. Namely, it behaves just like entropy does. It was found, as a matter of fact, that the similarity of BH physics to thermodynamics is quite general. It deals with *gedanken* mental exercises where we have specific thermodynamic devices and with the general laws of thermodynamics. Both have an analogue in BH physics. An arbitrary BH, like a well established thermodynamic system, enters in an equilibrium (stationary) state after the relaxation processes are finished.

Having said that, the so-called incomplete statistics (IS) [3, 4], analogously to Tsallis thermostatistics formalism [5] and Kaniadakis statistics [6], generalizes the usual Boltzmann-Gibbs (BG) statistics since this last one is not adequate to deal with more complicated physical phenomena. Among then we can mention for example fractal and self-similar frameworks, long-range interactions, long-duration memory, anomalous diffusion phenomena and Loop Quantum Gravity. The IS entropy can be directly used to obtain a new equation for the probability distribution. The intention is to overcome problems in BG probability distribution. These statistical formalisms constitute the main part of the so-called nonextensive statistical mechanics, and we know that the most difficult physical systems are frequently or usually nonextensive.

Some applications related to IS can be found in references [7, 8]. A normalization condition adopted in IS is

$$\sum_{i=1}^{W} p_i^q = 1,$$

where $p_i$ is the probability of the system to be in a $i$-microstate, $W$ is the total number of configurations and $q$ is known as the $q$-parameter, that measures the nonextensivity of the system. At the limit $q \to 1$ we must recover the usual normalization condition which is given by $\sum_{i=1}^{W} p_i = 1$. 

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Let us begin with the Tsallis formalism, the nonextensive entropy is given by

\[ S_q = -k_B \sum_{i=1}^{W} p_i^q \ln_q p_i \]

\[ = -k_B \sum_{i=1}^{W} p_i^q \frac{p_i^{1-q} - 1}{1 - q}, \tag{2} \]

where

\[ \ln_q f = \frac{f^{1-q} - 1}{1 - q}. \tag{3} \]

and it can be shown that at the limit \( q \to 1 \) we recover \( \ln f \). So, this limit recovers the extensive formalism, i.e., the BG entropy. The \( q \)-entropy in Eq. (2), satisfies the properties concerning nonnegativity, concavity and pseudo-additivity. Adopting the Wang condition, which is given in Eq. (1), the entropy in Eq. (2) can be written as

\[ S_q = k_B \left[ 1 - \sum_{i=1}^{W} p_i \right], \tag{4} \]

where \( q > 0 \) is required by the incomplete normalization in Eq. (1). Using the microcanonical ensemble definition, where all the states have the same probability and consequently, due to the normalization condition in Eq. (1), we have that \( p_i^q = 1/W \) and the IS entropy reduces to

\[ S_q = k_B \frac{W^{(q-1)/q} - 1}{q - 1}, \tag{5} \]

where at the limit \( q \to 1 \), we must recover the usual BG entropy formula, i.e., \( S = k_B \ln W \).

2. BLACK HOLES THERMODYNAMICS

It is well known that the thermodynamics of BHs is based on the concepts of both entropy and temperature of a BH [1, 2]. The temperature of a BH horizon is directly proportional to its surface gravity. In Einstein gravitation theory, the horizon entropy of a BH is proportional to its horizon area, i.e., the entropy area law of a BH. From now on we will use that \( \hbar = c = k_B = 1 \).

Our first issue [9] will be the Schwarzschild BH entropy which is written as

\[ S_{BH} = 4\pi GM^2, \tag{6} \]
where $G$ is the gravitational constant and $M$ is the mass of BH. The temperature is given by

$$\frac{1}{T} = \frac{\partial S(M)}{\partial M}, \quad (7)$$

and using Eq. (6) we have that

$$\frac{1}{T} = \frac{\partial S_{BH}(M)}{\partial M} = 8\pi GM. \quad (8)$$

Moreover, the number $N$ of degrees of freedom (DF) in the horizon can be given by assuming the relation \[10\]

$$N = 4S, \quad (9)$$

where $S$ is an specific entropy describing the horizon. So, using Eq. (9) in our initial case, we have

$$N = 16\pi GM^2. \quad (10)$$

Combining Eqs. (8) and (10) and making some algebra then we can derive the usual equipartition theorem \[9, 11–13\]

$$M = \frac{1}{2}NT, \quad (11)$$

which corresponds to the horizon energy.

To establish the physical coherence a standard test is to calculate the heat capacity of the model. The sign of the heat capacity can support us in determining the stability of BHs. Namely, a positive heat capacity is meaningful. On the other hand, a negative heat capacity in such system shows a thermodynamical unstableness.

The heat capacity can be computed from the expression

$$C = -\frac{[S'_{BH}(M)]^2}{S''_{BH}(M)}, \quad (12)$$

where the prime means a single derivative relative to $M$. So, substituting the entropy of Eq. (6) into Eq. (12) we have that

$$C_{BH} = -8\pi GM^2. \quad (13)$$

which means that due to the negative value of Eq. (13) the Schwarzschild BH is thermally unstable.
3. THE INCOMPLETE STATISTICS AND BLACK HOLES THERMODYNAMICS

Our scheme begins by initially considering that the BG entropy, \( S = \ln W \), describes the BH entropy, Eq. (6) \[14, 15\]

\[
\ln W = 4\pi GM^2 \quad \Rightarrow \quad W = \exp \left( 4\pi GM^2 \right).
\]

(14)

Now, using Eq. (14) into (5) we find

\[
S_q = \frac{1}{q-1} \left[ \exp \left( \frac{4\pi (q-1)GM^2}{q} \right) - 1 \right].
\]

(15)

Making use of Eqs. (7) and (9) we can derive, respectively, the Hawking temperature and the number \( N \) of DF as

\[
\frac{1}{T} = \frac{8\pi GM}{q} \exp \left( \frac{4\pi (q-1)GM^2}{q} \right),
\]

(16)

and

\[
N = \frac{4}{q-1} \left[ \exp \left( \frac{4\pi (q-1)GM^2}{q} \right) - 1 \right].
\]

(17)

Combining Eqs. (16) and (17) and after some algebra then we can obtain an equipartition theorem for the BH mass in the IS as

\[
M = \frac{q^2}{2\pi G} \frac{1}{[4 + (q-1)N]} T.
\]

(18)

When we make \( q = 1 \) in (18) we recover the usual relation for temperature and mass in the Schwarzschild BH which is Eq. (8). An alternative way would be from Eq. (16) where we have that

\[
q = 8\pi GM T e^{\delta M^2},
\]

(19)

where, conveniently and momentarily, \( \delta = 4\pi G(q-1)/q \). Notice that \( q \to 1 \) is equivalent to \( \delta \to 0 \). Let us show that we have the standard equipartition law in the \( q \to 1 \) limit.

From Eq. (17) we can write that

\[
q - 1 = \frac{4}{N} \left( e^{\delta M^2} - 1 \right).
\]

(20)
If we subtract Eq. (19) from Eq. (20) we obtain that
\[ \left( 8\pi GMT - \frac{4}{N} \right) e^{\delta M^2} = 1 - \frac{4}{N}. \]  
(21)

If we carry out the expansion of the exponential and, for simplicity, only the first two terms we write the general mass equation
\[ M^3 - \frac{1}{2\pi GTN} M^2 + \frac{1}{\delta} M - \frac{1}{8\pi GT\delta} = 0, \]  
(22)
and a possible solution for this system is
\[ M_1 = \frac{1}{2} q NT, \]
\[ M_2 = \frac{2}{(4\pi GT)^2 N (q - 1)} \left( A \pm \sqrt{A^2 - 4B} \right), \]  
(23)
\[ M_3 = \frac{1}{2} \left( A \pm \sqrt{A^2 - 4B} \right), \]

where
\[ A = \frac{1}{2} \left( 1 - \pi Gq N^2 T^2 \right), \]
\[ B = \frac{1}{(4\pi GT)^2 N (q - 1)}, \]  
(24)

and some conditions are quite obvious such as \( A^2 > 4B \) and \( A > \sqrt{A^2 - 4B} \), if we wish to take the negative, real values of the roots, since \( M \) is a positive quantity. However, the limit \( q \to 1 \) introduced a divergence in \( B \), which turns \( M_2 \) and \( M_3 \) not physically viable. Hence, we have that \( M_1 = NT/2 \) when \( q \to 1 \), which is the standard equipartition law. Therefore, Eqs. (16), (17) and (18) represent the thermodynamical parts of the equipartition law in Eq. (18) brought from IS.

The heat capacity, using Eqs. (15) and (12), is
\[ C_{IS} = -\frac{8\pi GM^2}{q} e^{4\pi GM^2 \left( \frac{q-1}{q} \right)} \frac{1}{1 + 8\pi GM^2 \left( \frac{q-1}{q} \right)}. \]  
(25)
When we make \( q = 1 \) in (25) we recover the usual value of the Schwarzschild BH heat capacity which is Eq. (13).

From Eq. (16) we can derive a temperature variational condition [14] such that
\[ \left. \frac{\partial T}{\partial M} \right|_{M=M_{ext}} = 0 \Rightarrow 8\pi GM_{ext}^2 = \frac{q}{1-q}. \]  
(26)
So, using Eq. (26) in (16) we can write the Hawking temperature as

\[ T = \frac{(1-q) M_{\text{ext}}}{M} e^{\frac{1}{2} \frac{M^2}{M_{\text{ext}}^2}}. \]  

(27)

In Fig. 1 the temperature, Eq. (27), has been plotted as a function of the mass \( M \) for \( q = 1/2 \), and from Eq. (26) and making \( G = 1 \) we have \( M_{\text{ext}} = 0.2 \).

![Graph of temperature vs. mass](image)

FIG. 1: Black hole temperature, Eq. (27), as a function of the mass \( M \) for \( q = 1/2 \) and \( M_{\text{ext}} = 0.2 \).

From Fig. 1 we can observe that for \( M = 0.2 \) we have the minimum value of the temperature.

On the other hand, Eq. (26) together with the normalization condition, Eq. (1), determines that the allowed values for the \( q \)-parameter, since the term \( 8\pi GM_{\text{ext}}^2 \) is always positive, is

\[ 0 < q < 1. \]  

(28)

Due to Eq. (16), the temperature is always positive, as it should be, if the condition (28) is obeyed. So, using Eq. (26) in (25) we can write the heat capacity as

\[ C_{\text{ext}} = \frac{M^2}{(q-1)(M_{\text{ext}}^2 - M^2)} \exp \left( -\frac{M^2}{2M_{\text{ext}}^2} \right). \]  

(29)
From Eqs. (29) and (28) we can observe that for $M < M_{\text{ext}}$ the heat capacity of system is negative. Consequently the BH is unstable. For $M > M_{\text{ext}}$ the heat capacity of system is positive and the BH is thermally stable. In Fig. 2, the heat capacity, Eq. (29), has been plotted as a function of the mass $M$ for $q = 1/2$ and $M_{\text{ext}} = 0.2$. So, we can observe that for $M = M_{\text{ext}} = 0.2$ the heat capacity diverges. This result can indicate a possible phase transition between the thermally unstable phase and the thermally stable phase of the BH in the IS theory.

![Graph of heat capacity vs mass](image)

**FIG. 2:** Black hole heat capacity, Eq. (29), as a function of the mass $M$. We have used $q = 1/2$ and $M_{\text{ext}} = 0.2$.

4. **CONCLUSIONS**

In this work we have investigated the effect of IS in the framework of BH thermodynamics. The procedure consists of using the microstates number $W$, Eq. (14), in the microcanonical IS entropy, Eq. (5). An equipartition theorem for the mass of BH was obtained, which is compatible with IS approach in Eq. (18). Then, in the expression in Eq. (25) a BH heat capacity was computed. Using a variational condition for the temperature function, Eq. (26), we could write the BH temperature and the heat capacity in more suitable forms which are Eqs. (27) and (29). This last equation exhibits a divergent
point when $M = M_{ext}$ for a fixed $q$. This result means that it is possible for a phase transition to occur, a fact that can not be noticed when we use BG theory. Consequently we can have a BH thermally stable phase. Without a doubt, this is an interesting result that points out the relevance of using non-Gaussian IS entropies and among them, for example, we can mention again Tsallis, Kaniadakis and IS entropies, in the analysis of BH thermodynamics.

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[1] S. W. Hawking, Commun. Math. Phys. 43 (1975) 199.
[2] J. D. Bekenstein, Phys. Rev. D 7 (8) (1973) 2333.
[3] Q. A. Wang, Chaos, Soliton and Fractals 12 (2001) 1431.
[4] J. A. S. Lima, J. R. Bezerra and R. Silva, Chaos, Solitons and Fractals 19 (2004) 1095.
[5] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[6] G. Kaniadakis, Physica A 296 (2001) 405.
[7] M. Pezeril, A. L. Méhauté and Q. A. Wang, Physica A 340 (2004) 117;
    Q. A. Wang, Eur. Phys. J. B 26 (2002) 357;
    Q. A. Wang, Eur. Phys. J. B 31 (2003) 75;
    A. L. Méhauté and Q. A. Wang, Chaos, Solitons and Fractals 15 (2003) 537;
    L. Nivanen, M. Pezeril, Q. A. Wang and A. L. Méhauté, Chaos, Solitons and Fractals 24 (2005) 1337.
[8] E. M. C. Abreu, J. Ananias Neto, E. M. Barboza and B. B. Soares, EPL 127 (2019) 10006.
[9] E. M. C. Abreu, J. Ananias Neto, E. M. Barboza, A. C. R. Mendes and B. B. Soares, Mod. Phys. Lett. A, vol 35 (2020) 2050266-1.
[10] N. Komatsu, Eur. Phys. J. C, 77 (2017) 229.
[11] E. M. C. Abreu, J. Ananias Neto and E. M. Barboza, EPL 130 (2020) 4, 40005.
[12] E. M. C. Abreu and J. Ananias Neto, Eur. Phys. J. C, 80 8 (2020) 776.
[13] E. M. C. Abreu and J. Ananias Neto, Phys. Lett. B 807 (2020) 135602.
[14] K. Mejhrhit and S.-E. Ennadifi, Phys. Lett. B 794 (2019) 45.

[15] H. Moradpour, A. H. Ziaie and M. Kord Zangeneh, Eur. Phys. J. C, 80 8 (2020) 732.