ON THE CUSPIDAL SUPPORT OF DISCRETE SERIES FOR P-ADIC QUASISPLIT
$Sp(N)$ AND $SO(N)$
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Abstract. Zelevinsky’s classification theory of discrete series of $p$-adic general linear groups has been well
known. Mœglin and Tadić gave the same kind of theory for $p$-adic classical groups, which is more compli-
cated due to the occurrence of nontrivial structure of $L$-packets. Nonetheless, their work is independent of
the endoscopic classification theory of Arthur (also Mok in the unitary case), which concerns the structure
of $L$-packets in these cases. So our goal in this paper is to make more explicit the connection between these
two very different types of theories. To do so, we reprove the results of Mœglin and Tadić in the case of
quasisplit symplectic groups and orthogonal groups by using Arthur’s theory.

1. Introduction

Let $F$ be a $p$-adic field and $G$ be a quasisplit connected reductive group over $F$. For simplicity we will
also denote $G(F)$ by $G$, which should not cause any confusion in the context. We consider pairs $(M, \pi_{\text{cusp}})$
for $G$, where $M$ is a Levi subgroup of $G$ and $\pi_{\text{cusp}}$ is an irreducible supercuspidal representation of $M$.
Such pairs carry an action of $G$ by conjugation, i.e.
$$(M, \pi_{\text{cusp}})g = (M^g, \pi_{\text{cusp}}^g),$$
where $M^g = g^{-1}Mg$, and $\pi_{\text{cusp}}^g(m) = \pi_{\text{cusp}}(gmg^{-1})$ for $m \in M^g$.
For any pair $(M, \pi_{\text{cusp}})$, let $P = MN$ be any parabolic subgroup containing $M$, and we have the normalized parabolic induction $\text{Ind}_G^P(\pi_{\text{cusp}} \otimes 1_N)$.
For simplicity we always abbreviate this to $\text{Ind}_G^P(\pi_{\text{cusp}})$, and we have the following facts about the parabolic
induction.

(1) $\text{Ind}_G^P(\pi_{\text{cusp}})$ is a smooth admissible representation of finite length, i.e., the semi-simplification
$s.s.\text{Ind}_G^P(\pi_{\text{cusp}})$ of $\text{Ind}_G^P(\pi_{\text{cusp}})$ is a direct sum of finitely many irreducible admissible representations.

(2) For $g \in G$, $s.s.\text{Ind}_G^P(\pi_{\text{cusp}}) = s.s.\text{Ind}_{P^g}^G(\pi_{\text{cusp}}^g)$ for any parabolic subgroup $P'$ containing $M^g$.

It is a theorem of Bernstein and Zelevinsky [BZ77] that all irreducible admissible representations of $G$
can be constructed by parabolic induction from supercuspidal representations.

Theorem 1.1 (B-Z). For any irreducible admissible representation $\pi$ of $G$, there exists a unique pair
$(M, \pi_{\text{cusp}})$ up to conjugation by $G$ such that
$$\pi \subseteq s.s.\text{Ind}_G^P(\pi_{\text{cusp}}).$$
Moreover, one can always find $P'$ containing $M$ such that
$$\pi \hookrightarrow \text{Ind}_{P'}^G(\pi_{\text{cusp}})$$
as a subrepresentation.

Remark 1.2. The $G$-conjugacy class of pairs $(M, \pi_{\text{cusp}})$ in this theorem is called the cuspidal support
of $\pi$. For our later purposes, we would like to fix a Borel subgroup $B$ of $G$ together with a maximal
torus $T \subseteq B$, and we have the standard parabolic subgroups $P = MN$, i.e., $P \supseteq B, M \supseteq T$. Then this
theorem implies that for any irreducible admissible representation $\pi$ of $G$, one can always find a standard
parabolic subgroup $P = MN$ with a supercuspidal representation $\pi_{\text{cusp}}$ of $M$ such that $\pi \hookrightarrow \text{Ind}_P^G(\pi_{\text{cusp}})$
as a subrepresentation.

Based on this theorem, it is natural to ask the following questions.
Question 1.3. How to determine the cuspidal support of any irreducible admissible representation of $G$?
Question 1.4. How can one classify the irreducible unitary representations of $G$ in terms of their cuspidal supports?

Question 1.5. How can one classify the irreducible discrete series representations of $G$ in terms of their cuspidal supports?

Question 1.3 is properly the most difficult one, and we are not able to say much about it here. Question 1.4 is often referred to as the unitary dual problem, and it has been solved for $GL(n)$ [Tad86]. For the classical groups, Tadić and Mučić have done many works (see [Tad09], [MT11]), and again we will not say anything about it here. Our main interest is in Question 1.5, and it has the most complete theories for both $GL(n)$ (see [Zel80]) and classical groups (see [Mœg02], [MT02]). Our goal is to present the results for the quasisplit symplectic groups and special orthogonal groups. To be more precise about what we want to show, we consider the following two examples.

If $G = GL(n)$, let us take $B$ to be the group of upper-triangular matrices and $T$ to be the group of diagonal matrices, then the standard Levi subgroup $M$ can be uniquely identified with $GL(n_1) \times \cdots \times GL(n_r)$ in a canonical way, with respect to a partition of $n = n_1 + \cdots + n_r$. So an irreducible supercuspidal representation $\pi_{\text{cusp}}$ of $M$ can be written as $\pi_{\text{cusp}} = \pi_1 \otimes \cdots \otimes \pi_r$ where $\pi_i$ is an irreducible supercuspidal representation of $GL(n_i)$ for $1 \leq i \leq r$. For simplicity, we denote the normalized induction $\text{Ind}_T^G(\pi_{\text{cusp}})$ by $\pi_1 \times \cdots \times \pi_r$. An irreducible supercuspidal representation $\pi$ of $GL(n)$ can always be written in a unique way as $\rho|^x = \rho \otimes |\det(\cdot)|^x$ for an irreducible unitary supercuspidal representation $\rho$ and a real number $x$. To fix notations, we will always denote by $\rho$ the unitary irreducible supercuspidal representations of $GL(d_{\rho})$. Now for a finite length arithmetic progression of real numbers of common length 1 or $-1$

$$x, \ldots, y$$

and an irreducible unitary supercuspidal representation $\rho$ of $GL(d_{\rho})$, it is a general fact that

$$\rho|^x \times \cdots \times \rho|^y$$

has a unique irreducible subrepresentation, denoted by $< \rho; x, \ldots, y >$ or $< x, \ldots, y >$. If $x \geq y$, it is called Steinberg representation; if $x < y$, it is called Speh representation. Such sequence of ordered numbers is called a segment (cf. Appendix B). In particular, when $x = -y > 0$, we can let $a = 2x + 1 \in \mathbb{Z}$ and write

$$\text{St}(\rho, a) := \langle \frac{a-1}{2}, \ldots, -\frac{a-1}{2} >,$$

which is an irreducible admissible representation of $GL(ad_{\rho})$. In fact it is a discrete series representation by Zelevinsky’s classification theorem.

Theorem 1.6 (Zelevinsky [Zel80]). All irreducible discrete series representations of $GL(n)$ can be obtained in a unique way as $\text{St}(\rho, a)$ for certain irreducible unitary supercuspidal representation of $GL(d_{\rho})$ and integer $a$ so that $n = ad_{\rho}$.

If $G = Sp(2n)$, let us take $B$ to be subgroup of upper-triangular matrices in $G$ and $T$ to be subgroup of diagonal matrices in $G$, then the standard Levi subgroup $M$ can be uniquely identified with $GL(n_1) \times \cdots \times GL(n_r) \times G_-$ in a canonical way, where $G_- = Sp(2n_-)$ and $n = n_1 + \cdots + n_r + n_-$. Note $n_-$ can be 0, in which case we simply write $Sp(0) = 1$. So an irreducible supercuspidal representation $\pi_{\text{cusp}}$ of $M$ can be written as $\pi_{\text{cusp}} = \pi_1 \otimes \cdots \otimes \pi_r \otimes \sigma$ where $\pi_i$ is an irreducible supercuspidal representation of $GL(n_i)$ for $1 \leq i \leq r$ and $\sigma$ is an irreducible supercuspidal representation of $G_-$. For simplicity, we denote $\text{Ind}_T^G(\pi_{\text{cusp}})$ by $\pi_1 \times \cdots \times \pi_r \times \sigma$. Note that $\sigma$ is always unitary. The discussion here can be easily extended to special orthogonal groups, the only thing that one has to be careful is in the case of $SO(2n)$, there are always two standard Levi subgroups, which can be canonically identified with the same group of the form $GL(n_1) \times \cdots \times GL(n_r)$ for $n = n_1 + \cdots + n_r$, and they are conjugate to each other by $O(2n)$. In this case, we will only identify the one contained in the standard Levi subgroup of $GL(2n)$ with $GL(n_1) \times \cdots \times GL(n_r)$.

Finally for any irreducible discrete series representation $\pi$ of a symplectic group or special orthogonal group $G$, our goal is to find unitary supercuspidal representations $\rho_i$ of $GL(d_{\rho_i})$ for $1 \leq i \leq r$ together
with real numbers \(x_1, \ldots, x_r\), and a supercuspidal representation \(\sigma\) of \(G_-\) which is of the same type as \(G\), such that

\[
\pi \lor \pi^{\theta_0} \hookrightarrow \rho_1 \boxtimes \cdots \boxtimes \rho_r \boxtimes \sigma
\]
as a subrepresentation. Here \(\theta_0\) is an automorphism of \(SO(2n)\) induced by the conjugate action of the nonconnected component of \(O(2n)\).

The approach that we are going to take will highly rely on Arthur’s endoscopic classification theory for symplectic and orthogonal groups [Art13], especially the structure of tempered Arthur packets (or \(L\)-packets). It is different from the original approaches of Mœglin and Tadić (see [Mœg02], [MT02]), where although possibly motivated by the structure of \(L\)-packets, they do not need to use it in their arguments. There are two reasons for us to adopt the new approach. One is there are certain reducibility assumptions (see Proposition 3.2) taken in the works of Mœglin and Tadić that could be cleared under Arthur’s work, so it would be very natural to start with Arthur’s theory at the first place. The other reason is the endoscopic theory is “hided” in their works, but we want to see how it could play a role in this kind of classification theory, to be more precise, the interplay of endoscopy theory with the theory of Jacquet modules (see Section 3).

Acknowledgements: This work comes out from the author’s talks in the automorphic forms seminar at Fields institute in 2014. The original goal of the talks is to simply summarize various results of Mœglin on Arthur packets for \(p\)-adic classical groups, but in the end we end up with this new approach. So the author wants to thank the organizer Chung Pang Mok for initially asking him to do these talks, otherwise this paper would not appear. The author also wants to thank all the participants in the seminar for their interests and supports. The writing up of the current version of this paper is supported by the National Science Foundation number agreement No. DMS-1128155 and DMS-1252158. Any opinions, findings and conclusions or recommendations expressed in this paper are those of the author and do not necessarily reflect the views of the National Science Foundation.

2. Tempered Arthur packet

Let \(F\) be a \(p\)-adic field and \(G\) be a quasisplit symplectic group or special orthogonal group. We define the local Langlands group as \(L_F = W_F \times SL(2, \mathbb{C})\), where \(W_F\) is the usual Weil group. We write \(\Gamma_F = \Gamma_{F/F}\) for the absolute Galois group over \(F\). Let \(\hat{G}\) be the complex dual group of \(G\), and \(L\) be the Langlands dual group of \(G\). A tempered (or generic) Arthur parameter of \(G\) is a \(\hat{G}\)-conjugacy class of admissible homomorphisms \(\phi : L_F \rightarrow \hat{G}\), such that \(\phi|_{W_F}\) is bounded. Denote by \(\Phi_{\text{bdd}}(G)\) the set of tempered Arthur parameters. Here we can simplify the Langlands dual groups as in the following table:

| \(G\) | \(L\) |
|-------|------|
| \(Sp(2n)\) | \(SO(2n + 1, \mathbb{C})\) |
| \(SO(2n + 1)\) | \(Sp(2n, \mathbb{C})\) |
| \(SO(2n, \eta)\) | \(SO(2n, \mathbb{C}) \times \Gamma_{E/F}\) |

In the last case, \(\eta\) is a quadratic character associated with a quadratic extension \(E/F\) and \(\Gamma_{E/F}\) is the associated Galois group. \(SO(2n, \eta)\) is the outer form of the split \(SO(2n)\) with respect to \(\eta\) and the outer automorphism from the conjugation of the nonconnected component of \(O(2n)\), and we can view \(SO(2n, \mathbb{C}) \times \Gamma_{E/F} \cong O(2n, \mathbb{C})\) (In the case of \(SO(8)\), there is another outer form which will not be considered in this paper). So in either of these cases, there is a natural embedding \(\xi_N\) of \(L\) into \(GL(N, \mathbb{C})\)-up to \(GL(N, \mathbb{C})\)-conjugate, where \(N = 2n + 1\) if \(G = Sp(2n)\) or \(N = 2n\) otherwise. Under such an embedding, we can view the parameter \(\phi\) as an equivalence class of \(N\)-dimensional self-dual representations of \(L_F\), i.e., \(\phi^T = \phi\). Let \(\pi_\phi\) be the self-dual representation of \(GL(N)\) associated with \(\phi\) under the local Langlands correspondence (cf. [HT01], [Hen00], and [Sch13]). If we decompose \(\phi\) into equivalence classes of irreducible subrepresentations, we get

\[
\phi = \bigoplus_{i=1}^{q} l_i \phi_i,
\]
where \( \phi_i \) is an equivalence class of irreducible representations of \( L_F \) and \( l_i \) is the multiplicity. Since \( L_F \) is a product of \( W_F \) and \( SL(2, \mathbb{C}) \), we can further decompose \( \phi_i \) as an tensor product

\[
\phi_i = \phi_{cusp,i} \otimes \nu_{a_i},
\]

where \( \phi_{cusp,i} \) is an equivalence class of irreducible representations of \( W_F \) and \( \nu_{a_i} \) is the \((a_i - 1)\)-th symmetric power representation of \( SL(2, \mathbb{C}) \). Now we have obtained all the combinatorial data needed from \( \phi \) in the paper. To put it in a nice way, we first identify the set of equivalence classes of irreducible unitary supercuspidal representations of \( GL(d) \) with equivalence classes of \( d \)-dimensional irreducible representations of \( L_F \) through the local Langlands correspondence for \( GL(d) \), and denote by \( \rho_i \) the corresponding representation for \( \phi_{cusp,i} \). Also notice the representation \( \nu_{a_i} \) is completely determined by its dimension. So altogether we can simply write \( \phi_i = \rho_i \otimes [a_i] \) formally. After this discussion we can define the multi-set of Jordan blocks for \( \phi \) as follows,

\[
\text{Jord}(\phi) := \{ (\rho_i, a_i) \text{ with multiplicity } l_i : 1 \leq i \leq q \},
\]

and

\[
\text{Jord}_{\rho}(\phi) := \{ a_i \text{ with multiplicity } l_i : \rho = \rho_i \}.
\]

To parametrize the discrete series representations, we need to introduce a subset \( \Phi_2(G) \) of \( \Phi_{bdd}(G) \). Define

\[
\Phi_2(G) := \{ \phi \in \Phi_{bdd}(G) : \phi = \bigoplus_{i=1}^q \phi_i, \phi_i^\vee = \phi_i \}.
\]

It is clear that the defining condition for \( \Phi_2(G) \) is equivalent to requiring \( \text{Jord}(\phi) \) is multiplicity free and \( \text{Jord}_{\rho}(\phi) \) is empty unless \( \rho \) is self-dual. Moreover, for certain parity reason (see Section 3), the integers in \( \text{Jord}_{\rho}(\phi) \) must be all odd or all even when \( \phi \in \Phi_2(G) \). Besides, there is another description of \( \Phi_2(G) \). For \( \phi \in \Phi_{bdd}(G) \), we fix a representative \( \hat{\phi} \). Let us define

\[
S_{\hat{\phi}} = \text{Cent}(\text{Im} \, \hat{\phi}, G),
\]

\[
\tilde{S}_{\hat{\phi}} = S_{\hat{\phi}}/Z(G)^{\Gamma_F},
\]

\[
S_{\hat{\phi}} = \tilde{S}_{\hat{\phi}}/\tilde{S}_{\hat{\phi}}^0 = S_{\hat{\phi}}/S_{\hat{\phi}}^0 Z(G)^{\Gamma_F}.
\]

Then we have the following fact.

**Lemma 2.1.** For \( \phi \in \Phi_{bdd}(G) \), \( \phi \in \Phi_2(G) \) if and only if \( \tilde{S}_{\hat{\phi}} \) is finite.

This lemma can be shown by computing the group \( S_{\hat{\phi}} \) explicitly (see [Art\`{e}b, 1.4]). In particular, one can show \( S_{\hat{\phi}} \) is abelian.

To state Arthur’s classification theory of tempered representations of quasisplit symplectic and orthogonal groups, we need to introduce some more notations. We will fix an outer automorphism \( \theta_0 \) of \( G \) preserving an \( F \)-splitting. If \( G \) is symplectic or special odd orthogonal, we let \( \theta_0 = id \). If \( G \) is special even orthogonal, we let \( \theta_0 \) be induced from the conjugate action of the nonconnected component of the full orthogonal group. Let \( \theta_0 \) be the dual automorphism of \( \theta_0 \). We write \( \Sigma_0 = < \theta_0 >, G^{\Sigma_0} = G \rtimes < \theta_0 >, \) and let \( \omega_0 \) be the character of \( G^{\Sigma_0}/G \), which is nontrivial when \( G \) is special even orthogonal. So in the special even orthogonal case, \( G^{\Sigma_0} \) is isomorphic to the full orthogonal group. If \( G \) has rank \( n \), we write \( G = G(n) \). Let \( G(0) = G(0)^{\Sigma_0} = 1 \). Also for the trivial representation of \( G(0) \), we require formally \( 1^{\theta_0} \equiv 1 \) if \( G(0) = SO(0) \), and \( 1^{\theta_0} \equiv 1 \) otherwise.

Let \( \Pi_{\text{temp}}(G) \) (resp. \( \Pi_2(G) \)) be the set of equivalence classes of irreducible tempered representations (resp. discrete series representations) of \( G \). \( \Sigma_0 \) acts on these sets, and we denote the set of \( \Sigma_0 \)-orbits in \( \Pi_{\text{temp}}(G) \) (resp. \( \Pi_2(G) \)) by \( \Pi_{\text{temp}}(G) \) (resp. \( \Pi_2(G) \)). \( \Sigma_0 \) also acts on \( \Phi_{bdd}(G) \) (resp. \( \Phi_2(G) \)) through \( \theta_0 \), and we denote the corresponding set of \( \Sigma_0 \)-orbits by \( \Phi_{bdd}(G) \) (resp. \( \Phi_2(G) \)). It is clear that for \( \phi \in \Phi_{bdd}(G) \), \( \text{Jord}(\phi) \) only depends on its image in \( \Phi_{bdd}(G) \). It is because of this reason, we will also denote the elements in \( \Phi_{bdd}(G) \) by \( \phi \). Moreover, through the natural embedding \( \xi_N \), we can view \( \Phi_{bdd}(G) \) as a subset of equivalence classes of \( N \)-dimensional self-dual representations of \( L_F \).
Theorem 2.2 (Arthur). (1) For \( \phi \in \Phi_{\text{bsd}}(G) \), one can associate a finite set \( \Pi_{\phi} \) of \( \Pi_{\text{temp}}(G) \), determined by \( \pi_{\phi} \) through the theory of twisted endoscopy (cf. Section 4). And for a fixed Whittaker datum, there is a canonical bijection between \( \Pi_{\phi} \) and characters \( \hat{S}_{\phi} \) of \( S_{\phi} \).

\[ \Pi_{\phi} \longrightarrow \hat{S}_{\phi} \]

\[ [\pi] \longrightarrow \langle \cdot, \pi \rangle_{\phi} \]

(2) There are decompositions

\[ \Pi_{\text{temp}}(G) = \bigsqcup_{\phi \in \Phi_{\text{bsd}}(G)} \Pi_{\phi}, \]

\[ \Pi_{2}(G) = \bigsqcup_{\phi \in \Phi_{2}(G)} \Pi_{\phi}. \]

We will denote the characters of \( S_{\phi} \) by \( \varepsilon \), and denote the corresponding \( \Sigma_{0} \)-orbit \( [\pi] \) of irreducible representations by \( \pi(\phi, \varepsilon) \). Let us define \( \Pi_{\Sigma_{0}}^{\phi} \) to be set of irreducible representations of \( G_{\Sigma_{0}} \) whose restriction to \( G \) belong to \( \Pi_{\phi} \). We call an irreducible representation \( \pi_{\Sigma_{0}}^{\phi} \) of \( G_{\Sigma_{0}} \) is a discrete series if its restriction to \( G \) are discrete series representations. We also define \( S_{\Sigma_{0}}^{\phi} \), \( \tilde{S}_{\Sigma_{0}}^{\phi} \) and \( S_{\Sigma_{0}}^{\phi} \) as before simply by taking \( \tilde{G}_{\Sigma_{0}} \) in place of \( \tilde{G} \). The following theorem asserts \( \Pi_{\Sigma_{0}}^{\phi} \) can be parametrized by the characters of \( S_{\Sigma_{0}}^{\phi} \).

Theorem 2.3 (Arthur). Suppose \( \phi \in \Phi_{\text{bsd}}(G) \), for a fixed \( \Sigma_{0} \)-stable Whittaker datum there is a canonical bijection between \( \Pi_{\Sigma_{0}}^{\phi} \) and characters \( \hat{S}_{\Sigma_{0}}^{\phi} \) of \( S_{\Sigma_{0}}^{\phi} \).

\[ \Pi_{\Sigma_{0}}^{\phi} \longrightarrow \hat{S}_{\Sigma_{0}}^{\phi} \]

\[ \pi_{\Sigma_{0}} \longrightarrow \langle \cdot, \pi_{\Sigma_{0}} \rangle_{\phi}, \]

such that

\[ \langle \cdot, \pi_{\Sigma_{0}} \rangle_{\phi} |_{S_{\phi}} = \langle \cdot, \pi \rangle_{\phi} \]

where \( \pi \in \pi_{\Sigma_{0}}^{\phi} |_{G} \).

We denote the characters of \( S_{\Sigma_{0}}^{\phi} \) by \( \varepsilon \), and denote the corresponding representations by \( \pi_{\Sigma_{0}}^{\phi}(\phi, \varepsilon) \). We also denote the image of \( \varepsilon \) in \( \tilde{S}_{\phi} \) by \( \tilde{\epsilon} \). Then this theorem implies

\[ \pi_{\Sigma_{0}}^{\phi}(\phi, \varepsilon \varepsilon_{0}) \cong \pi_{\Sigma_{0}}^{\phi}(\phi, \varepsilon) \otimes \omega_{0}. \]

Therefore, if \( S_{\Sigma_{0}}^{\phi} \neq S_{\phi} \), then \( \pi(\phi, \varepsilon) \) is a representation of \( G \) satisfying \( \pi(\phi, \varepsilon)^{\delta_{0}} \cong \pi(\phi, \tilde{\varepsilon}) \). In the rest of this paper, we will always fix a \( \Sigma_{0} \)-stable Whittaker datum of \( G \).

3. Parameters of supercuspidal representations

We keep the notations from the previous section.

Proposition 3.1. Suppose \( \pi \) is a supercuspidal representation of \( G \), let \( [\pi] \in \Pi_{\phi} \) for some \( \phi \in \Phi_{2}(G) \).

Then if \( (\rho, a) \in \text{Jord}(\phi) \), one must have \( (\rho, a - 2) \in \text{Jord}(\phi) \) as long as \( a - 2 > 0 \).
Proof. Let $\rho$ be a unitary irreducible supercuspidal representation of $GL(d_{\rho})$. We can view $GL(d_{\rho}) \times G$ as the Levi component $M_+$ of a standard maximal parabolic subgroup $P_+$ of $G_+$, where $G$ and $G_+$ are of the same type. Let $\pi_{M_+} = \rho \otimes \pi$, and $w$ is the unique non-trivial element in the relative Weyl group $W(M_+, G_+)$, which acts on $GL(d_{\rho})$ as an outer automorphism. Let $\pi_{M_+, \lambda} = |\rho|^\lambda \otimes \pi$ for $\lambda \in \mathbb{C}$. It is a result of Arthur (see [Art13, 2.3]) that for any representative $\tilde{w}$ of $w$, the standard intertwining operator between $\text{Ind}_{P_+}^G(\pi_{M_+, \lambda})$ and $\text{Ind}_{P_+}^G(\tilde{w} \pi_{M_+, \lambda})$, i.e.,

$$J_{P_+}(\tilde{w}, \pi_{M_+, \lambda})h(g) = \int_{N_{P_+} \cap wN_{P_+}w^{-1}\backslash N_{P_+}} h(w^{-1}ng)dn \quad h \in \text{Ind}_{P_+}^G(\pi_{M_+, \lambda}),$$

and the standard intertwining operator $J_{P_+}(\tilde{w}^{-1}, \tilde{w} \pi_{M_+, \lambda})$ between $\text{Ind}_{P_+}^G(\tilde{w} \pi_{M_+, \lambda})$ and $\text{Ind}_{P_+}^G(\pi_{M_+, \lambda})$ can be normalized by meromorphic functions $r_{P_+}(w, \phi_{M_+, \lambda})$ and $r_{P_+}(w^{-1}, w \phi_{M_+, \lambda})$ respectively, i.e.,

$$R_{P_+}(\tilde{w}, \pi_{M_+, \lambda}) := r_{P_+}(w, \phi_{M_+, \lambda})^{-1}J_{P_+}(\tilde{w}, \pi_{M_+, \lambda}),$$
$$R_{P_+}(\tilde{w}^{-1}, \tilde{w} \pi_{M_+, \lambda}) := r_{P_+}(w^{-1}, w \phi_{M_+, \lambda})^{-1}J_{P_+}(\tilde{w}^{-1}, \tilde{w} \pi_{M_+, \lambda}),$$

so that

$$R_{P_+}(\tilde{w}^{-1}, \tilde{w} \pi_{M_+, \lambda})R_{P_+}(\tilde{w}, \pi_{M_+, \lambda}) = \text{Id}. \quad (3.1)$$

Here $\phi_{M_+, \lambda}$ denotes the Langlands parameter for $\pi_{M_+, \lambda}$, and

$$r_{P_+}(w, \phi_{M_+, \lambda}) \sim \frac{L(\lambda, \rho \times \pi_\phi)L(2\lambda, \rho, R)}{L(1 + \lambda, \rho \times \pi_\phi)L(1 + 2\lambda, \rho, R)}$$

where $R$ is either a symmetric square $(S^2)$ or a skew-symmetric square $(\wedge^2)$ representation of $GL(d_{\rho}, \mathbb{C})$ and “$\sim$” means equal up to a non-vanishing holomorphic function of $\lambda$ (that is given by the $\epsilon$-factors here). Note $R = \wedge^2$ if $G$ is $Sp(2n), SO(2n, \eta)$ or $R = S^2$ if $G = SO(2n + 1)$. Similarly we have

$$r_{P_+}(w^{-1}, w \phi_{M_+, \lambda}) \sim \frac{L(-\lambda, \rho^\vee \times \pi_\phi)L(-2\lambda, \rho^\vee, R)}{L(1 - \lambda, \rho^\vee \times \pi_\phi)L(1 - 2\lambda, \rho^\vee, R)}.$$ 

Then we can rewrite (3.1) as

$$J_{P_+}(\tilde{w}^{-1}, \tilde{w} \pi_{M_+, \lambda})J_{P_+}(\tilde{w}, \pi_{M_+, \lambda}) \sim \frac{L(\lambda, \rho \times \pi_\phi)L(2\lambda, \rho, R)L(-\lambda, \rho^\vee \times \pi_\phi)L(-2\lambda, \rho^\vee, R)}{L(1 + \lambda, \rho \times \pi_\phi)L(1 + 2\lambda, \rho, R)L(1 - \lambda, \rho^\vee \times \pi_\phi)L(1 - 2\lambda, \rho^\vee, R)}. \quad (3.2)$$

Since $\pi_{M_+}$ is supercuspidal, it follows from a theorem of Harish-Chandra ([Sil79, Theorem 5.4.2.1] and [Sha81, Lemma 2.2.5]) that both $J_{P_+}(\tilde{w}, \pi_{M_+, \lambda})$ and $J_{P_+}(\tilde{w}^{-1}, \tilde{w} \pi_{M_+, \lambda})$ are holomorphic for Re $\lambda \neq 0$. So now we will assume $\lambda \in \mathbb{R}$ and $\lambda > 1/2$. Since $L(s, \rho \times \pi_\phi)$ does not have a pole (non-vanishing is clear) if $s > 0$, and $L(s, \rho, R)$ does not have a pole (non-vanishing is clear) for Re $s \neq 0$, we have

$$J_{P_+}(\tilde{w}^{-1}, \tilde{w} \pi_{M_+, \lambda})J_{P_+}(\tilde{w}, \pi_{M_+, \lambda}) \sim \frac{L(-\lambda, \rho^\vee \times \pi_\phi)}{L(1 - \lambda, \rho^\vee \times \pi_\phi)} \quad (\lambda > 1/2). \quad (3.3)$$

Finally, we learn from the definition of $L(s, \rho^\vee \times \pi_\phi)$ that it has a pole at $s = -(a-1)/2$ if and only if $\rho \cong \rho^\vee$ and $(\rho, a) \in \text{Jord}(\phi)$ (see Appendix[A]). So if we know $(\rho, a) \in \text{Jord}(\phi)$ for $a > 2$, then $L(-\lambda, \rho^\vee \times \pi_\phi)$ has a pole at $\lambda = -(a-1)/2 > 1/2$. By the holomorphy of standard intertwining operators on the left hand side of (3.3), $L(1 - \lambda, \rho^\vee \times \pi_\phi)$ must also have a pole at $\lambda = -(a-1)/2$, i.e., $1 - \lambda = 1 - (a-1)/2 = -(a-3)/2$ is a pole of $L(s, \rho^\vee \times \pi_\phi)$. So $(\rho, a - 2) \in \text{Jord}(\phi)$.\n
$\square$

If $\rho$ is a self-dual unitary irreducible supercuspidal representation of $GL(d_{\rho})$, we know from Appendix[A] that

$$L(s, \rho \times \rho) = L(s, \rho, \wedge^2)L(s, \rho, S^2)$$

has a pole at $s = 0$. We call $\rho$ is of $\textbf{symplectic type}$ if $L(s, \rho, \wedge^2)$ has a pole at $s = 0$, and we call $\rho$ is of $\textbf{orthogonal type}$ if $L(s, \rho, S^2)$ has a pole at $s = 0$. Moreover, for any positive integer $a$, the pair $(\rho, a)$ is called having orthogonal type if $\rho$ is of orthogonal type and $a$ is odd, or $\rho$ is of symplectic type.
type and \( a \) is even. Otherwise \((\rho, a)\) is called having symplectic type. Next we are going to prove a very important reducibility result, which is named “Basic Assumption” in [Mœg02], [MT02]. Those careful readers may notice there is a slight difference between our statement below and the original one. The reason is they consider the group \( G^{\Sigma_0} \) rather than \( G \), nonetheless one can translate between these two statements without difficulty (see Corollary 9.1).

**Proposition 3.2.** Suppose \( \pi \) is a supercuspidal representation of \( G \) and \([\pi] \in \bar{\Pi}_\phi \) for some \( \phi \in \bar{\Phi}_2(G) \). Then for any unitary irreducible supercuspidal representation \( \rho \) of \( GL(d_\rho) \), the parabolic induction

\[
\rho^{|\pm(a_\rho+1)/2} \rtimes \pi
\]

reduces exactly for

\[
a_\rho = \begin{cases} 
\max \text{Jord}_d(\phi), & \text{if Jord}_d(\phi) \neq 0, \\
0, & \text{if Jord}_d(\phi) = 0, \rho \text{ is self-dual and is of opposite type to } \tilde{G}, \\
-1, & \text{otherwise, provided } d_\rho \text{ is even or } \pi \cong \pi^{\theta_0}.
\end{cases}
\]

**Proof.** We will follow the proof of the previous proposition. By the theory of Langlands’ quotient and also the holomorphy of standard intertwining operators in (3.2), we know \( \rho^{|\lambda} \rtimes \pi \) for \( \lambda > 0 \) reduces only when

\[
\mathcal{P}_+(\tilde{w}^{-1}, \tilde{w} \pi_{M^+,\lambda}) \mathcal{P}_+(\tilde{w}, \pi_{M^+,\lambda}) = 0.
\]

Let us first assume \( \lambda > 1/2 \), then from (3.3) it is enough to see when

\[
\frac{L(-\lambda, \rho^\vee \rtimes \pi_\phi)}{L(1-\lambda, \rho^\vee \rtimes \pi_\phi)} = 0,
\]

i.e., \( L(1-\lambda, \rho^\vee \rtimes \pi_\phi) \) has a pole, but \( L(-\lambda, \rho^\vee \rtimes \pi_\phi) \) does not. From our discussion in the previous proof we know this can only happen when \( \rho = \rho^\vee \) and \( \lambda = (a_\rho + 1)/2 \), where \( a_\rho \) is max \( \text{Jord}_d(\phi) \). Next we assume \( 0 < \lambda \leq 1/2 \), it follows from (3.2) that

\[
\mathcal{P}_+(\tilde{w}^{-1}, \tilde{w} \pi_{M^+,\lambda}) \mathcal{P}_+(\tilde{w}, \pi_{M^+,\lambda}) \sim \frac{L(-\lambda, \rho^\vee \rtimes \pi_\phi)}{L(1-2\lambda, \rho^\vee, R)}.
\]

And the right hand side can be zero only when \( L(1-2\lambda, \rho^\vee, R) \) has a pole, but \( L(-\lambda, \rho^\vee \rtimes \pi_\phi) \) does not. So necessarily \( \rho = \rho^\vee \) and \( \lambda = 1/2 \). By our assumption on the representation \( R \), we know \( L(s, \rho, R) \) has a pole at \( s = 0 \) if and only if \( \rho \) is of opposite type to \( \tilde{G} \). And the requirement that \( L(s, \rho \rtimes \pi_\phi) \) does not have a pole at \(-1/2\) implies \( \text{Jord}_d(\phi) = 0 \).

For \( \lambda < 0 \), one just needs to notice \( s.s.(\rho^|s| \rtimes \pi) = s.s.w(\rho^|s| \rtimes \pi) = s.s.(\rho^\vee)^{|s|} \rtimes \pi^{\theta} \) for some \( \theta \in \Sigma_0 \) and \([\pi^{\theta}] \in \bar{\Pi}_\phi \), so one can apply the same argument to \( \rho^\vee^{|s|} \rtimes \pi^{\theta} \).

In both cases above, we have only shown the reducibility condition (3.4) is necessary. To see it is also sufficient, we need to use more knowledge about tempered Arthur packet and this will be done in the end of Section 8.

Finally, we consider \( \lambda = 0 \), where our previous criterion does not work. However the reducibility of \( \rho \rtimes \pi \) follows from the standard theory of representation theoretic \( R \)-groups. In Arthur’s theory these groups have been shown to be isomorphic to \( R \)-groups defined by parameters, which can be computed explicitly. So our reducibility condition in this case will follow from there.

\[\square\]

Suppose \( \pi \) is an irreducible supercuspidal representation of \( G \) and \([\pi] \in \bar{\Pi}_\phi \) for some \( \phi \in \bar{\Phi}_2(G) \). We know from Proposition 4.1 that \( \text{Jord}(\phi) \) should be in a certain shape, and in view of Theorem 2.2 we would also like to know what kind of character \( \bar{\varepsilon} \) of \( \tilde{S}_\phi \) will parametrize \([\pi]\). To give a description of such characters, we have to first make an identification between \( \tilde{S}_\phi \) with \( \mathbb{Z}_2 \)-valued functions over \( \text{Jord}(\phi) \). To be more precise, let us assume

\[
\phi = \phi_1 \oplus \cdots \oplus \phi_r,
\]
where \( \phi \) are self-dual irreducible representations of dimension \( n_i \). By Shur’s Lemma,

\[
\text{Cent}(\phi, GL(N, \mathbb{C})) \cong \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times,
\]

where each \( \mathbb{C}^\times \) acts on the corresponding representation space of \( \phi \). So

\[
\text{Cent}(\phi, \hat{G}) \cong \{ s = (s_i) \in \mathbb{Z}_2^n : \prod_i (s_i)^{n_i} = 1 \}.
\]

Note \( S_{\Sigma_0} = \hat{S}_{\phi} \) in this case. Then \( S_{\Sigma_0} \cong \mathbb{Z}_2 / \langle -1, \cdots, -1 \rangle \). If \( G \) is special even orthogonal,

\[
S_{\phi} \cong \{ s = (s_i) \in \mathbb{Z}_2^n : \prod_i (s_i)^{n_i} = 1 \}/ \langle -1, \cdots, -1 \rangle
\]

which is a subgroup of \( S_{\Sigma_0} \) of index 1 or 2. Let us denote by \( S_{\phi} \) (resp. \( S^\Sigma_{\phi} \)) the corresponding quotient space of \( \mathbb{Z}_2 \)-valued functions on \( \text{Jord}(\phi) \) such that \( S_{\phi} \cong \hat{S}_{\phi} \) (resp. \( S^\Sigma_{\phi} \cong \hat{S}^\Sigma_{\phi} \)) under these isomorphisms.

Let us define the characters of \( \mathbb{Z}_2^n / \langle -1, \cdots, -1 \rangle \) to be \( \mathbb{Z}_2 \)-valued functions \( \varepsilon = (\varepsilon_i) \) such that \( \prod_i \varepsilon_i = 1 \). Moreover, for \( s \in \mathbb{Z}_2^n \), we define \( \varepsilon(s) = \prod_i (\varepsilon_i * s_i) \), where

\[
\varepsilon_i * s_i = \begin{cases} 
-1, & \text{if } \varepsilon_i = s_i = -1 \\
1, & \text{otherwise}.
\end{cases}
\]

So \( \hat{S}^\Sigma_{\phi} = \{ \varepsilon = (\varepsilon_i) \in \mathbb{Z}_2^n : \prod_i \varepsilon_i = 1 \} \). In particular, when \( G \) is special even orthogonal, we define \( \varepsilon_0 = (\varepsilon_{0,i}) \in \hat{S}^\Sigma_{\phi} \) satisfying \( \varepsilon_{0,i} = 1 \) if \( n_i \) is even, and \( \varepsilon_{0,i} = -1 \) if \( n_i \) is odd, then

\[
\hat{S}_{\phi} = \{ \varepsilon = (\varepsilon_i) \in \mathbb{Z}_2^n : \prod_i \varepsilon_i = 1 \}/ \langle \varepsilon_0 \rangle
\]

In general, let \( \varepsilon_0 = 1 \) if \( G \) is not special even orthogonal.

Now we can formulate the theorem for parametrizing supercuspidal representations inside tempered Arthur packets.

**Theorem 3.3** (Moeglin). The \( \Sigma_0 \)-orbits of supercuspidal representations of \( G \) can be parametrized by \( \phi \in \Phi_2(G) \) and \( \varepsilon \in \hat{S}_{\phi} \) satisfying the following properties:

1. if \( (\rho, a) \in \text{Jord}(\phi) \), then \( (\rho, a - 2) \in \text{Jord}(\phi) \) as long as \( a - 2 > 0 \);
2. if \( (\rho, a), (\rho, a - 2) \in \text{Jord}(\phi) \), then \( \varepsilon(\rho, a)\varepsilon(\rho, a - 2) = -1 \);
3. if \( (\rho, 2) \in \text{Jord}(\phi) \), then \( \varepsilon(\rho, 2) = -1 \).

The proof that we are going to give makes use of the (twisted) endoscopic character identities and explicit computation of Jacquet modules. So we will first review these two subjects in the next two sections.

4. **Endoscopy: based on examples**

The endoscopy theory can be stated for any connected reductive groups over a local field of characteristic zero, and is still conjectural in general. Here we would rather restrict to the cases that we are primarily concerned with, i.e., \( G \) is a quasisplit symplectic group or special orthogonal group. Suppose \( \phi \in \Phi_2(G) \) and \( s \in S_{\phi} \), then \( \phi \) will factor through \( \text{Cent}(s, L^G) \). In our case, there is a quasisplit reductive group \( H \) with the property that \( \text{Cent}(s, \hat{G}) \cong \hat{H} \), and the isomorphism extends to an embedding \( \xi : L^H \to L^G \) such that \( \phi \) factors through \( L^H \). So from \( \phi \) we get a parameter \( \phi_{\hat{H}} \in \Phi_{bdl}(H) \). In fact it is easy to show \( S_{\phi_{\hat{H}}} \) is also finite, so \( \phi_{\hat{H}} \) is a special even orthogonal group. We say \( (H, \phi_{\hat{H}}) \) corresponds to \( (\phi, s) \), and denote this relation by \( (H, \phi_{\hat{H}}) \to (\phi, s) \). Such \( H \) is called an **elliptic endoscopic group** of \( G \). Here we give the most important examples in this paper.
Example 4.1. (1) If $G = Sp(2n)$, then $L^G = SO(2n + 1, \mathbb{C})$. For $\phi \in \tilde{\Phi}_2(G)$, let us write $\phi = \phi_1 \oplus \cdots \oplus \phi_r$ as in (3.6). Then $S_\phi = \mathbb{Z}_2^r \times <-1, \cdots, -1>$, and for any $s = (s_i) \in S_\phi$, it gives a partition on $\text{Jord}(\phi)$, i.e.,

$$\phi = \left(\oplus_{s_i = 1} \phi_i\right) \oplus \left(\oplus_{s_j = -1} \phi_j\right).$$

Without loss of generality, let us assume $\sum_{s_i = 1} n_i = 2n_I + 1$, and $\sum_{s_j = -1} n_j = 2n_{II}$. Note $n = n_I + n_{II}$. Let $G_I = Sp(2n_I)$, and $G_{II} = SO(2n_{II}, \eta')$, where $\eta' = \prod_{s_j = -1} n_j$ with $n_j$ being the central character of $\pi_{\phi_j}$. Then $\phi_I := \left(\oplus_{s_i = 1} \phi_i\right) \otimes \eta' \in \tilde{\Phi}_2(G_I)$ and $\phi_{II} := \oplus_{s_j = -1} \phi_j \in \tilde{\Phi}_2(G_{II})$. In this case $H = G_I \times G_{II}$ and $\phi_H = \phi_I \times \phi_{II}$.

(2) If $G = SO(2n + 1)$, then $L^G = Sp(2n, \mathbb{C})$. For $\phi \in \tilde{\Phi}_2(G)$, let us write $\phi = \phi_1 \oplus \cdots \oplus \phi_r$ as in (3.6). Then $S_\phi = \mathbb{Z}_2^r \times \{-1, \cdots, -1, 1\}$, and for any $s = (s_i) \in S_\phi \subseteq S_{\Sigma_0}$, it gives a partition on $\text{Jord}(\phi)$, i.e.,

$$\phi = \left(\oplus_{s_i = 1} \phi_i\right) \oplus \left(\oplus_{s_j = -1} \phi_j\right).$$

We can assume $\sum_{s_i = 1} n_i = 2n_I$, and $\sum_{s_j = -1} n_j = 2n_{II}$. Note $n = n_I + n_{II}$. Let $G_I = SO(2n_I + 1)$, and $G_{II} = SO(2n_{II} + 1)$. Then $\phi_I := \oplus_{s_i = 1} \phi_i \in \tilde{\Phi}_2(G_I)$ and $\phi_{II} := \oplus_{s_j = -1} \phi_j \in \tilde{\Phi}_2(G_{II})$. In this case $H = G_I \times G_{II}$ and $\phi_H = \phi_I \times \phi_{II}$.

(3) If $G = SO(2n, \eta)$, then $L^G = SO(2n, \mathbb{C}) \times \Gamma_{E/F}$. For $\phi \in \tilde{\Phi}_2(G)$, let us write $\phi = \phi_1 \oplus \cdots \oplus \phi_r$ as in (3.6). Then $S_\phi = \mathbb{Z}_2^r \times \{-1, \cdots, -1, 1\}$, and for any $s = (s_i) \in S_\phi \subseteq S_{\Sigma_0}$, it gives a partition on $\text{Jord}(\phi)$, i.e.,

$$\phi = \left(\oplus_{s_i = 1} \phi_i\right) \oplus \left(\oplus_{s_j = -1} \phi_j\right).$$

By our description of $S_\phi$, we can assume $\sum_{s_i = 1} n_i = 2n_I$, and $\sum_{s_j = -1} n_j = 2n_{II}$. Note $n = n_I + n_{II}$. Let $G_I = SO(2n_I, \eta')$, and $G_{II} = SO(2n_{II}, \eta')$, where $\eta' = \prod_{s_j = -1} n_j$ with $n_j$ being the central character of $\pi_{\phi_j}$. Then $\phi_I := \oplus_{s_i = 1} \phi_i \in \tilde{\Phi}_2(G_I)$ and $\phi_{II} := \oplus_{s_j = -1} \phi_j \in \tilde{\Phi}_2(G_{II})$. In this case $H = G_I \times G_{II}$ and $\phi_H = \phi_I \times \phi_{II}$.

In the examples above, we can define $\tilde{\Phi}_2(H) = \tilde{\Phi}_2(G_I) \times \tilde{\Phi}_2(G_{II})$ (resp. $\tilde{\Phi}_{\text{bdd}}(H) = \tilde{\Phi}_{\text{bdd}}(G_I) \times \tilde{\Phi}_{\text{bdd}}(G_{II})$), then $\phi_H \in \tilde{\Phi}_2(H)$. For $s \in S_\phi$, we still say $(H, \phi_H)$ corresponds to $(\phi, s)$, and denote this relation by $(H, \phi_H) \rightarrow (\phi, s)$. Note in part (3), it is possible to also choose $s \in S_{\Sigma_0}$ but not in $S_\phi$, and then we get a partition on $\text{Jord}(\phi)$, i.e.,

$$\phi = \left(\oplus_{s_i = 1} \phi_i\right) \oplus \left(\oplus_{s_j = -1} \phi_j\right),$$

such that $\sum_{s_i = 1} n_i = 2n_I + 1$ and $\sum_{s_j = -1} n_j = 2n_{II} + 1$, where $n = n_I + n_{II} + 1$. Let $G_I = Sp(2n_I)$, $G_{II} = Sp(2n_{II})$, and $\eta_I = \prod_{s_i = 1} n_i$, $\eta_{II} = \prod_{s_j = -1} n_j$. Then $\phi_I := (\oplus_{s_i = 1} \phi_i) \otimes \eta_I \in \tilde{\Phi}_2(G_I)$, and $\phi_{II} := (\oplus_{s_j = -1} \phi_j) \otimes \eta_{II} \in \tilde{\Phi}_2(G_{II})$. Let $H = G_I \times G_{II}$ and $\phi_H = \phi_I \times \phi_{II}$. In this case, $H$ is called a twisted elliptic endoscopic group of $G$, and we again denote this relation by $(H, \phi_H) \rightarrow (\phi, s)$.

In this paper, we also want to consider the twisted elliptic endoscopic groups of $GL(N)$, but we will only need the simplest case here. Recall for $\phi \in \tilde{\Phi}_{\text{bdd}}(G)$, we can view $\phi$ as a self-dual $N$-dimensional representation through the natural embedding $\xi_N : L^G \rightarrow GL(N, \mathbb{C})$, and in this way we get a self-dual parameter for $GL(N)$. We fix an outer automorphism $\theta_N$ of $GL(N)$ preserving an $F$-splitting, and let $\tilde{\theta}_N$ be the dual automorphism on $GL(N, \mathbb{C})$, then $\xi_N(L^G) \subseteq \text{Cent}(s, GL(N, \mathbb{C}))$ and $\tilde{G} = \text{Cent}(s, GL(N, \mathbb{C}))^0$ for some $s \in GL(N, \mathbb{C}) \times \tilde{\theta}_N$. So as before $G$ can be called a twisted elliptic endoscopic group of $GL(N)$.

What lies in the heart of the endoscopy theory is a transfer map on the spaces of smooth compactly supported functions from $G$ to its (twisted) elliptic endoscopic group $H$ (similarly from $GL(N)$ to its twisted elliptic endoscopic group $G$). The existence of the transfer map is quite deep, and here we will only mention its final establishment relies on the celebrated Fundamental Lemma, which has been proved by Ngo \cite{Ngo10}. Let us denote such transfers by

\[ C_c^\infty(G) \longrightarrow C_c^\infty(H) \]

\[ f \longmapsto f^H \]
and similarly

\[(4.2)\]  
\[C^\infty_c(GL(N)) \longrightarrow C^\infty_c(G)\]

\[f \longrightarrow f^G\]

We should point out these transfer maps are only well defined after we pass to the space of (twisted) \(\text{orbital integrals}\) on the source and the space \(\text{stable orbital integrals}\) on the target. Note the space of (twisted) (resp. stable) orbital integrals are dual to the space of (twisted) (resp. stable) invariant distributions \(G\) as linear functionals of the space of (twisted) (resp. stable) orbital integrals. So dual to these transfer maps, the stable invariant distributions on \(H\) (resp. \(G\)) will map to the (twisted) invariant distributions on \(G\) (resp. \(GL(N)\)). We call this map the (twisted) spectral endoscopic transfer.

If \(\pi\) is an irreducible admissible representation of \(G\), then it defines an invariant distribution on \(G\) by the trace of \(\pi(f) = \int_G f(g)\pi(g)dg\) for \(f \in C^\infty_c(G)\). We call this the character of \(\pi\) and denote it by \(f_G(\pi)\). For any irreducible representation \(\pi_{\Sigma_0}\) of \(G_{\Sigma_0}\), which contains \(\pi\) in its restriction to \(G\), we define a twisted invariant distribution on \(G\) by the trace of \(\pi_{\Sigma_0}(f) = \int_{G \times \theta_0} f(g)\pi_{\Sigma_0}(g)dg\) for \(f \in C^\infty_c(G \times \theta_0)\). We call this the twisted character of \(G\), and denote it by \(f_G(\pi_{\Sigma_0})\). We can also define the twisted characters for \(GL(N)\) similarly, but we will write it in a slightly different way. Let \(\pi\) be a self-dual irreducible admissible representation of \(GL(N)\), we can define a twisted invariant distribution on \(GL(N)\) by taking the trace of \(\pi(f) \circ A_{\pi}(\theta_N)\) for \(f \in C^\infty_c(GL(N))\), where \(A_{\pi}(\theta_N)\) is an intertwining operator between \(\pi\) and \(\pi^{\theta_N}\). We call this the twisted character of \(\pi\) and denote it by \(f_{N^\theta}(\pi)\).

Since the (twisted) elliptic endoscopic groups \(H\) in our case are all products of quasisplit symplectic and special orthogonal groups, we can define a group of automorphisms of \(G\) by taking the product of \(\Sigma_0\) on each factor, and we denote this group again by \(\Sigma_0\). Let \(\tilde{\mathcal{H}}(G)\) (resp. \(\tilde{\mathcal{H}}(H)\)) be the subspace of \(\Sigma_0\)-invariant functions in \(C^\infty_c(G)\) (resp. \(C^\infty_c(H)\)). Then it follows from a simple property of the transfer map (which we will not explain here) that we can restrict both \(C^\infty_c\) and \(C^\infty_c\) to \(\tilde{\mathcal{H}}(G)\) and \(\tilde{\mathcal{H}}(H)\). Now we are ready to state a more precise version of Theorem 2.2.

**Theorem 4.2 (Arthur).**

1. Suppose \(\phi \in \tilde{\Phi}_2(G)\), the sum of characters in \(\tilde{\Pi}_\phi\)

\[f(\phi) = \sum_{[\pi] \in \tilde{\Pi}_\phi} f_G(\pi)\]

defines a stable invariant distribution for \(f \in \tilde{\mathcal{H}}(G)\). Moreover it is determined uniquely by \(\pi_\phi\) through

\[(4.3)\]  
\[f_G(\phi) = f_{N^\theta}(\pi_\phi) \quad f \in C^\infty_c(GL(N))\]

after we normalize the Haar measures on \(G\) and \(GL(N)\) in a compatible way.

2. Suppose \(\phi \in \tilde{\Phi}_2(G)\), and \((H, \phi_H) \rightarrow (\phi, s)\) for \(s \in S_\phi\). If we define a stable invariant distribution \(f(\phi_H)\) for \(\tilde{\mathcal{H}}(H)\) as in (1), then after we normalize the Haar measures on \(G\) and \(H\) in a compatible way the following identity holds

\[(4.4)\]  
\[f_H(\phi_H) = \sum_{[\pi] \in \tilde{\Pi}_\phi} <s, \pi > f_G(\pi) \quad f \in \tilde{\mathcal{H}}(G)\]

where

\[<\cdot, \pi > := <\cdot, \pi >\]

under the isomorphism \(S_\phi \cong S_\tilde{\phi}\).
Remark 4.3. Although we only state the theorem for discrete parameters, these statements are also true for tempered parameters (once we extend the definition \((H, \phi_H) \to (\phi, s)\) appropriately). The two identities (4.3) and (4.4) are the ones we call (twisted) endoscopic character identities in the end of Section 3 and they are also often referred to as (twisted) character relations. There are some ambiguities that we need to clarify in such identities. On one hand, in the definition of \(f_{\Sigma^0}(\pi_\phi)\) we need to choose a normalization of the intertwining operator \(A_{\pi_\phi}(\theta_N)\). In this theorem, we require \(A_{\pi_\phi}(\theta_N)\) to fix some Whittaker functional for \(\pi_\phi\). On the other hand, in the definition of the transfer maps there is also a normalization issue. To resolve that, we need to fix certain (resp. \(\theta_N\)-stable) Whittaker datum for \(G\) (resp. \(GL(N)\)), and we will take the so-called Whittaker normalization on the transfer maps.

When \(G\) is special even orthogonal, we have an additional character identity. To state it, we need to identify \(C_c^\infty(G \times \theta_0)\) with \(C_c^\infty(G)\) by sending \(g \times \theta_0\) to \(g\), so the twisted transfer map on \(C_c^\infty(G)\) can also be translated to \(C_c^\infty(G \times \theta_0)\).

**Theorem 4.4.** Suppose \(\phi \in \tilde{\Phi}_2(G)\), and \((H, \phi_H) \to (\phi, s)\) for \(s \in \mathcal{S}^\Sigma_0\) but not in \(\mathcal{S}_\phi\). Then after we normalize the Haar measures on \(G\) and \(H\) in a compatible way the following identity holds

\[
(4.5) \quad f^H(\phi_H) = \sum_{[\pi] \in \Pi_\phi} <s, \pi^{\Sigma_0}> f_G(\pi^{\Sigma_0}) \quad f \in C_c^\infty(G \times \theta_0)
\]

where \(\pi^{\Sigma_0}|_G = \pi\) and

\[
<s, \pi^{\Sigma_0}> = <\cdot, \pi^{\Sigma_0}>_{\phi}
\]

under the isomorphism \(\mathcal{S}_\phi^{\Sigma_0} \cong \mathcal{S}_\phi^{\Sigma_0}\).

Again this theorem also holds for \(\phi \in \tilde{\Phi}(G)\) (once we extend the definition \((H, \phi_H) \to (\phi, s)\) appropriately), and we have taken the Whittaker normalization on the transfer maps with respect to the fixed \(\Sigma_0\)-stable Whittaker datum in Theorem 2.3. We will only need this theorem in Section 9.

5. **Jacquet modules**

First let us assume \(G\) is any connected reductive group over \(F\), and let \(\text{Rep}(G)\) be the category of finite-length admissible representations of \(G\). If \(M\) is the Levi component of a parabolic subgroup \(P\) of \(G\), then the normalized parabolic induction defines a functor from \(\text{Rep}(M)\) to \(\text{Rep}(G)\). The normalized Jacquet module is its left adjoint functor, i.e.,

\[(5.1) \quad \text{Hom}_M(\text{Jac}_P \pi, \sigma) \cong \text{Hom}_G(\pi, \text{Ind}_P^G \sigma),\]

for \(\pi \in \text{Rep}(G)\) and \(\sigma \in \text{Rep}(M)\). This relation (5.1) is usually referred to as **Frobenius reciprocity**. One can see easily from (5.1) and Theorem 1.1 that \(\pi \in \text{Rep}(G)\) is supercuspidal if and only if \(\text{Jac}_P \pi = 0\) for all standard parabolic subgroups \(P\) of \(G\). In fact this is one of the equivalent definitions of supercuspidal representations. The next two lemmas state some general facts about Jacquet modules, and we refer the interested readers to ([MT02], Section 3) for their proofs.

**Lemma 5.1.** If \(\pi\) is irreducible, and \(\sigma\) is an irreducible supercuspidal constituent of \(\text{Jac}_P \pi\), then there is an inclusion

\[\pi \hookrightarrow \text{Ind}_P^G \sigma.\]

**Lemma 5.2.** Suppose \(\pi\) is irreducible, and \(M = M_1 \times M_2\). Let \(\tau_1\) be an irreducible representation of \(M_1\) and \(\tau_2\) be a finite-length representation of \(M_2\). If

\[\pi \hookrightarrow \text{Ind}_P^G (\tau_1 \otimes \tau_2),\]

then there exists an irreducible constituent \(\tau'_2\) in \(\tau_2\) such that

\[\pi \hookrightarrow \text{Ind}_P^G (\tau_1 \otimes \tau'_2).\]
Now let us restrict to the case when $G$ is a quasisplit symplectic or special orthogonal group. We would like to define a modified Jacquet functor. For this we first fix a unitary irreducible supercuspidal representation $\rho$ of $GL(d_\rho)$, and we assume $M = GL(d_\rho) \times G_-$ is the the Levi component of a standard maximal parabolic subgroup $P$ of $G$. In case $G_- = 1$ and $G$ is special even orthogonal, we require $P$ to be contained in the standard parabolic subgroup of $GL(2n)$. Then for $\pi \in \text{Rep}(G)$,

$$s.s.Jac_P(\pi) = \bigoplus \tau_i \otimes \sigma_i,$$

where $\tau_i \in \text{Rep}(GL(d_\rho))$ and $\sigma_i \in \text{Rep}(G_-)$, both of which are irreducible. We define $Jac_x \pi$ for any real number $x$ to be

$$Jac_x(\pi) = \bigoplus_{\tau_i=|\rho|^x} \sigma_i.$$

Note unlike $Jac_P \pi$, in our definition $Jac_x \pi$ is always semisimple. If we have an ordered sequence of real numbers $\{x_1, \ldots, x_s\}$, we can define

$$Jac_{x_1,\ldots,x_s} \pi = Jac_{x_s} \circ \cdots \circ Jac_{x_1} \pi.$$

It is not hard to see $Jac_x$ can be defined for $GL(n)$ in a similar way by replacing $G_-$ by $GL(n_-)$. Furthermore, we can define $Jac^o_P$ analogous to $Jac$ but with respect to $\rho^\vee$ and the standard Levi subgroup $GL(n_-) \times GL(d_{\rho^\vee})$. So let us define $Jac^{o}_{x} = Jac \circ Jac^{op}_{-x}$ for $GL(n)$. Next we want to give some properties of this modified Jacquet functor.

**Lemma 5.3.** If $\pi \in \text{Rep}(G)$ is irreducible, and $Jac_{x_1,\ldots,y} \pi = \sigma$ for $\sigma \in \text{Rep}(G_-)$. Then there exists an irreducible constituent $\sigma'$ in $\sigma$ so that we get an inclusion

$$\pi \hookrightarrow \rho||^x \times \cdots \times \rho||^y \times \sigma'.$$

**Proof.** By Theorem 1.1 there exists a standard parabolic subgroup $P_-$ of $G_-$ with an irreducible supercuspidal representation $\pi_{M_-}$ on the Levi component $M_-$ such that there is a nontrivial equivariant homomorphism from $\sigma$ to $\text{Ind}^{G_-}_{P_-} \pi_{M_-}$. Then by Frobenius reciprocity, $\pi_{M_-}$ is in $s.s.Jac_{P_-} \sigma$. In particular, we can take $M = GL(d_\rho) \times \cdots \times GL(d_\rho) \times M_-$ with $P$ being the corresponding standard parabolic subgroup of $G$, and take $\pi_M = \rho||^x \times \cdots \times \rho||^y \otimes \pi_{M_-}$ to be an irreducible supercuspidal representation of $M$. Then $\pi_M$ is in $s.s.Jac_P \pi$. By Lemma 5.1 we know

$$\pi \hookrightarrow \rho||^x \times \cdots \times \rho||^y \otimes \text{Ind}^{G_-}_{M_-}(\pi_{M_-}).$$

So by Lemma 5.2 there exists an irreducible constituent $\sigma'$ in $\text{Ind}^{G_-}_{M_-}(\pi_{M_-})$ such that

$$\pi \hookrightarrow \rho||^x \times \cdots \times \rho||^y \times \sigma'.$$

Finally by Frobenius reciprocity again, we know $\sigma'$ is in $Jac_{x_1,\ldots,y} \pi = \sigma$. This finishes the proof. □

As a special case of this lemma, we have the following corollary.

**Corollary 5.4.** If $\pi \in \text{Rep}(G)$ is irreducible, and $Jac_{x_1,\ldots,y} \pi = \sigma$ for $\sigma \in \text{Rep}(G_-)$, which is also irreducible. Then there is an inclusion

$$\pi \hookrightarrow \rho||^x \times \cdots \times \rho||^y \times \sigma.$$

**Remark 5.5.** The Lemma 5.3 and Corollary 5.4 are also valid in the case of general linear group, and the proofs are the same.

**Lemma 5.6.** If $\pi \in \text{Rep}(G)$ and $|x-y| \neq 1$, then $Jac_{x,y} \pi = Jac_{y,x} \pi$.

**Proof.** We take the standard parabolic subgroup $P = MN$ of $G$ with $M = GL(2d_\rho) \times G_-$. If

$$Jac_P \pi = \bigoplus \tau_i \otimes \sigma_i,$$
then \( \sigma_i \) is in \( \text{Jac}_{x,y}\pi \) if and only if \( \text{Jac}_{x,y}\tau_i \neq 0 \). Let us assume \( \text{Jac}_{x,y}\tau_i \neq 0 \), by Corollary 5.4 (also see Remark 5.5) we have \( \tau_i \leftrightarrow \rho||^x \times \rho||^y \). Since \( |x - y| \neq 1 \), \( \rho||^x \times \rho||^y \cong \rho||^y \times \rho||^x \) is irreducible (see Appendix B), so we must have \( \tau_i \cong \rho||^x \times \rho||^y \). Hence

\[
\text{Jac}_{x,y}\pi = \bigoplus_{\tau_i \cong \rho||^x \times \rho||^y} (\text{Jac}_{x,y}\tau_i) \otimes \sigma_i.
\]

By the same argument, we have

\[
\text{Jac}_{y,x}\pi = \bigoplus_{\tau_i \cong \rho||^y \times \rho||^x} (\text{Jac}_{y,x}\tau_i) \otimes \sigma_i.
\]

Therefore, \( \text{Jac}_{x,y}\pi = \text{Jac}_{y,x}\pi \).

\[
\square
\]

**Lemma 5.7.** Suppose \( \pi \) is an irreducible constituent of

\[
\rho||^a \times \cdots \times \rho||^b
\]

for a segment \( \{a, \cdots , b\} \), and \( \text{Jac}_{x}\pi = 0 \) unless \( x=a \), then \( \pi=\langle a, \cdots , b > \).}

**Proof.** It is clear that \( \text{Jac}_{x}\pi = 0 \) unless \( x \in \{a, \cdots , b\} \). Suppose \( \{a, \cdots , y \} \subseteq \{a, \cdots , b\} \) is the longest segments such that

\( \text{Jac}_{a,\cdots , y}\pi \neq 0 \).

If \( y \neq b \), then we can find \( z \in \{a, \cdots , b\}\backslash \{a\} \) such that \( |x-z| > 1 \) for all \( x \in \{a, \cdots , y \} \) and \( \text{Jac}_{a,\cdots , y,z}\pi \neq 0 \). By Lemma 5.6

\[
\text{Jac}_{z,a,\cdots , y}\pi = \text{Jac}_{a,\cdots , y,z}\pi \neq 0.
\]

This means \( \text{Jac}_{z}\pi \neq 0 \), and we get a contradiction. So we can only have \( y=b \), and by Corollary 5.4 we have

\( \pi \leftrightarrow \rho||^a \times \cdots \times \rho||^b \).

Hence \( \pi=\langle a, \cdots , b > \).}

\[
\square
\]

There are some explicit formulas for computing the Jacquet modules in the case of classical groups and general linear groups (cf. [MT02], Section 1), and we want to recall some of them here.

For \( GL(n) \), we know the irreducible discrete series representations are given by

\[
\text{St}(\rho', a)=\langle \rho'; \frac{a-1}{2}, \cdots , \frac{a-1}{2} > .
\]

More generally we have irreducible representations \( \langle \rho'; \zeta a, \cdots , \zeta b > \) attached to any decreasing segment \( \{a, \cdots , b\} \) (cf. Section 1) for \( \zeta = \pm 1 \). If we fix \( \rho \) as before, then we have the following formulas for their Jacquet modules.

\[
\text{Jac}_x < \rho'; \zeta a, \cdots , \zeta b > = \begin{cases} < \rho'; \zeta(a-1), \cdots , \zeta b >, & \text{if } x = \zeta a \text{ and } \rho' \cong \rho, \\ 0, & \text{otherwise}; \end{cases}
\]

and

\[
\text{Jac}_x^{op} < \rho'; \zeta a, \cdots , \zeta b > = \begin{cases} < \rho'; \zeta a, \cdots , \zeta(b+1) >, & \text{if } x = \zeta b \text{ and } \rho' \cong \rho^y, \\ 0, & \text{otherwise}. \end{cases}
\]

If \( \pi_i \in \text{Rep}(GL(n_i)) \) for \( i = 1 \) or \( 2 \), we have

\[
\text{Jac}_x(\pi_1 \times \pi_2) = \text{Jac}_x\pi_1 \oplus \text{Jac}_x\pi_2,
\]

and

\[
\text{Jac}_x^{op}(\pi_1 \times \pi_2) = \text{Jac}_x^{op}\pi_1 \oplus \text{Jac}_x^{op}\pi_2.
\]

For \( G \), suppose \( \pi \in \text{Rep}(G) \) and \( \tau \in \text{Rep}(GL(d)) \). If \( G \) is symplectic or special odd orthogonal, then

\[
\text{Jac}_x (\tau \times \pi) = (\text{Jac}_x \tau) \times \pi \oplus (\text{Jac}_x^{op} \tau) \times \pi \oplus \tau \times \text{Jac}_x \pi.
\]
If $G = SO(2n, \eta)$, the situation is more complicated, and we would like to divide it into three cases.

1. When $n \neq d_\rho$ or 0,
   \[
   \text{Jac}_x(\tau \times \pi) = \tau \times \text{Jac}_x\pi \oplus \begin{cases} 
   (\text{Jac}_x\tau) \times \pi \oplus (\text{Jac}_{-x}^{op}\tau) \times \pi & \text{if } d_\rho \text{ is even} \\
   (\text{Jac}_x\tau) \times \pi \oplus (\text{Jac}_{-x}^{op}\tau) \times \pi^{\theta_0} & \text{if } d_\rho \text{ is odd}
   \end{cases}
   \]

2. When $n = d_\rho$,
   \[
   \text{Jac}_x(\tau \times \pi) = \tau \times \text{Jac}_x\pi \oplus (\tau \times \text{Jac}_x\pi^{\theta_0})^{\theta_0} \oplus \begin{cases} 
   (\text{Jac}_x\tau) \times \pi \oplus (\text{Jac}_{-x}^{op}\tau) \times \pi & \text{if } d_\rho \text{ is even} \\
   (\text{Jac}_x\tau) \times \pi \oplus (\text{Jac}_{-x}^{op}\tau) \times \pi^{\theta_0} & \text{if } d_\rho \text{ is odd}
   \end{cases}
   \]

3. When $n = 0$ and $d \neq d_\rho$,
   \[
   \text{Jac}_x(\tau \times 1) = \begin{cases} 
   (\text{Jac}_x\tau) \times 1 \oplus (\text{Jac}_{-x}^{op}\tau) \times 1 & \text{if } d_\rho \text{ is even} \\
   (\text{Jac}_x\tau) \times 1 \oplus (\text{Jac}_{-x}^{op}\tau \times 1)^{\theta_0} & \text{if } d_\rho \text{ is odd}
   \end{cases}
   \]

4. When $n = 0$ and $d = d_\rho$,
   \[
   \text{Jac}_x(\tau \times 1) = \begin{cases} 
   (\text{Jac}_x\tau) \times 1 \oplus (\text{Jac}_{-x}^{op}\tau) \times 1 & \text{if } d_\rho \text{ is even} \\
   (\text{Jac}_x\tau) \times 1 & \text{if } d_\rho \text{ is odd}
   \end{cases}
   \]

The formulas for special even orthogonal groups here are deduced from [Jan06]. At last we define

\[
\text{Rep}(G) \quad \text{to be the category of finite-length } \tilde{\text{H}}(G)\text{-modules. For } [\pi] \in \text{Rep}(G), \text{ let us define}
\]
\[
\tau \times [\pi] := [\tau \times \pi] \quad \text{and } \text{Jac}_x[\pi] := [\text{Jac}_x\pi].
\]

Then we can combine all cases into the following uniform formula

\[
(5.2) \quad \text{Jac}_x(\tau \times [\pi]) = (\text{Jac}_x\tau) \times [\pi] \oplus (\text{Jac}_{-x}^{op}\tau) \times [\pi] \oplus \tau \times \text{Jac}_x[\pi].
\]

Finally, we would like to extend the discussion of this section to the category $\text{Rep}(G^{\Sigma_0})$ of finite-length representations of $G^{\Sigma_0}$. Let $P = MN$ be a standard parabolic subgroup of $G$, and suppose

\[
M = GL(n_1) \times \cdots \times GL(n_r) \times G_-
\]

Let

\[
M^{\Sigma_0} = GL(n_1) \times \cdots \times GL(n_r) \times G_-^{\Sigma_0},
\]

Suppose $\sigma^{\Sigma_0} \in \text{Rep}(M^{\Sigma_0})$, $\pi^{\Sigma_0} \in \text{Rep}(G^{\Sigma_0})$.

1. If $M^{\Sigma_0} \neq M$, we define the normalized parabolic induction $\text{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0}$ to be the extension of the representation $\text{Ind}_{P}^{G}(\sigma^{\Sigma_0}|_{M})$ by an induced action of $\Sigma_0$, and we define the normalized Jacquet module $\text{Jac}_{P^{\Sigma_0}}^{\Sigma_0} \pi^{\Sigma_0}$ to be the extension of the representation $\text{Jac}_{P}^{\Sigma_0} \pi^{\Sigma_0}|_{G}$ by an induced action of $\Sigma_0$.

2. If $M^{\Sigma_0} = M$, we define the normalized parabolic induction $\text{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0}$ to be $\text{Ind}_{G}^{G^{\Sigma_0}} \text{Ind}_{P}^{G}(\sigma^{\Sigma_0}|_{M})$, and we define the normalized Jacquet module $\text{Jac}_{P^{\Sigma_0}}^{\Sigma_0} \pi^{\Sigma_0}$ to be $\text{Jac}_{P}^{\Sigma_0} \pi^{\Sigma_0}|_{G}$.

It follows

\[
(\text{Jac}_{P^{\Sigma_0}}^{\Sigma_0} \pi^{\Sigma_0})|_{M} = \text{Jac}_{P}^{\Sigma_0} \pi^{\Sigma_0}|_{G}.
\]

And

\[
(\text{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0})|_{G} = \text{Ind}_{P}^{G}(\sigma^{\Sigma_0}|_{M})
\]

unless $G$ is special even orthogonal and $M^{\Sigma_0} = M$, in which case

\[
(\text{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0})|_{G} = \text{Ind}_{P}^{G}(\sigma^{\Sigma_0}|_{M}) \oplus (\text{Ind}_{P}^{G}(\sigma^{\Sigma_0}|_{M}))^{\theta_0}.
\]
Let us define
\[ \text{Jac}_P = \begin{cases} \text{Jac}_P + \text{Jac}_P \circ \theta_0, & \text{if } G = SO(2n) \text{ and } M^{\Sigma_0} = M, \\ \text{Jac}_P, & \text{otherwise.} \end{cases} \]

Then we have
\[ (\text{Jac}_{P^{G_0}} \text{Ind}_{P^{G_0}} G^{\Sigma_0})|_M = \text{Jac}_P \text{Ind}_{P} (\sigma_0^{\Sigma_0})|_M, \]
and
\[ [(\text{Ind}_{P^{G_0}} \text{Jac}_{P^{G_0}} \sigma_0^{\Sigma_0})|_G] = \text{Ind}_{P} \text{Jac}_P [\sigma_0^{\Sigma_0}|_G]. \]

The Frobenius reciprocity still holds in this case, i.e.,
\[ \text{Hom}_{M^{G_0}}(\text{Jac}_{P^{G_0}}, \sigma_0^{\Sigma_0}) \cong \text{Hom}_{G^{G_0}}(\sigma_0^{\Sigma_0}, \text{Ind}_{P^{G_0}} G^{\Sigma_0}). \]

Moreover, the results of this section can be carried out similarly for representations of \( G^{\Sigma_0} \) with respect to the parabolic induction and Jacquet modules that we have defined. In particular, for \( \tau \in \text{Rep}(GL(d)) \) we have
\[ \text{Jac}_x(\tau \times \pi^{\Sigma_0}) = (\text{Jac}_x \tau) \times \pi^{\Sigma_0} \oplus (\text{Jac}_x^{op} \tau) \times \pi^{\Sigma_0} \oplus \tau \times \text{Jac}_x \pi^{\Sigma_0}. \]

6. Compatibility of Jacquet modules with endoscopic transfer

As normalized parabolic induction is compatible with endoscopic transfer, the normalized Jacquet module is also compatible with endoscopic transfer. Since the Jacquet module is originally defined on representations, we need to first extend it to the Grothendieck group of representations, and then to the space of (twisted) invariant distributions. If \( G \) is any quasisplit connected reductive group and \( \theta \) is an automorphism of \( G \) preserving an \( F \)-splitting, we denote the space of (resp. twisted) invariant distributions on \( G \) by \( \hat{I}(G) \) (resp. \( \hat{I}(G^\theta) \)), and denote the space of stable invariant distributions on \( G \) by \( \hat{S}\hat{I}(G) \). In particular when \( G = GL(N) \), we simply write \( \hat{I}(N^\theta) \) for the space of twisted invariant distributions on \( GL(N) \). In the following discussion we will assume \( G \) is a quasisplit symplectic or special orthogonal group.

If \( H \) is an elliptic endoscopic group of \( G \), we know from Section [4] that \( H = G_I \times G_{II} \), and there is an embedding
\[ \xi : L H \hookrightarrow LG. \]

We fix \( \Gamma \)-splittings \( (B_H, \mathcal{T}_H, \{X_{\alpha_H}\}) \) and \( (B_G, \mathcal{T}_G, \{X_{\alpha}\}) \) for \( \hat{H} \) and \( \hat{G} \) respectively. By taking certain \( \hat{G} \)-conjugate of \( \xi \), we can assume \( \xi(\mathcal{T}_H) = \mathcal{T}_G \) and \( \xi(B_H) \subseteq B_G \). Then we can view the Weyl group \( W_H = W(\hat{H}, \mathcal{T}_H) \) as a subgroup of \( W_G = W(\hat{G}, \mathcal{T}_G) \). Let \( S \) be a \( \Gamma_F \)-invariant torus in \( \mathcal{T}_H \), it gives a Levi subgroup \( M \) of \( G \), where \( \hat{M} := \text{Cent}(\xi(S), \hat{G}) \). We also denote the \( W_G \)-conjugacy class of \( S \) in \( \mathcal{T}_H \) by \( \{S\}_G \), then each \( W_H \)-conjugacy class \( \{S'\}_H \) in \( \{S\}_G \) corresponds to an \( H \)-conjugacy class of Levi subgroup \( M' = M(S') \) of \( H \), where \( \hat{M}' := \text{Cent}(S', \hat{H}) \). In fact, \( M' \) are endoscopic groups of \( M \) (see [KS99]). We fix a parabolic subgroup \( P \supseteq M \) with an embedding \( L P \hookrightarrow LG \), which extends \( L M \hookrightarrow LG \). Then the embedding \( \xi_{M'} : L M' \hookrightarrow L M \) can be given by any element \( g_0 \in \hat{G} \) such that \( \text{Int}(g_0)(\xi(S')) = \xi(S) \), i.e., the following diagram commutes
\[
\begin{array}{ccc}
L M' & \xrightarrow{\xi_{M'}} & L M \\
\downarrow & & \downarrow \\
L H & \xrightarrow{\xi} & LG \\
& \downarrow{\text{Int}(g_0)} & \downarrow{\text{Int}(g_0)} \\
& L G & \xrightarrow{\text{Int}(g_0)} LG.
\end{array}
\]

We denote the set of all such embeddings by \( \{\xi_{M'}\} \). For \( (g, h) \in \text{Norm}(\xi(S), \hat{G}) \times \text{Norm}(S', \hat{H}) \), we define another embedding \( (g, h) \ast \xi_{M'} \) by changing \( g_0 \) to \( g_0 g \xi(h) \). In this way, we get a transitive action of \( \text{Norm}(\xi(S), \hat{G}) \times \text{Norm}(S', \hat{H}) \) on \( \{\xi_{M'}\} \). For each \( \xi' = (g, h) \ast \xi_{M'} \in \{\xi_{M'}\} \), we can associate it with a parabolic subgroup \( P' \supseteq M' \) such that \( \xi(P') = \text{Int}(g_0 g \xi(h))^{-1}(P) \cap \xi(H) \). Then we have
Subgroups of $H$ and the horizontal maps correspond to spectral endoscopic transfers with respect to $\xi$ on the top and $\xi'$ on the bottom. In particular, the pair $(S', \xi')$ corresponds to a unique $H$-conjugacy class of parabolic subgroups of $H$, so we can always take $P'$ to be standard in this diagram for application.

Suppose $S$ has rank one, and $M \cong GL(m) \times G_-$, then the Levi subgroups $M(S')$ of $H$ appearing in (6.1) are of the form $M_I \times M_{II}$, where $M_I \cong GL(m_I) \times G_{I-}$ is a Levi subgroup of $G_I$, and $M_{II} \cong GL(m_{II}) \times G_{II-}$ is a Levi subgroup of $G_{II}$, and $m = m_I + m_{II}$. The spectral endoscopic transfer sends $\widehat{SI}(G_{I-} \times G_{II-})$ to $\widehat{I}(G_-)$, and it also sends $\widehat{SI}(GL(m_I) \times GL(m_{II}))$ to $\widehat{I}(GL(m))$, which is equivalent to parabolic induction. Now we fix a unitary irreducible supercuspidal representation $\rho$ of $GL(d_\rho)$, and let $m = d_\rho$. Let $P = MN$ be a standard parabolic subgroup of $G$. We would like to restrict (6.1) to distributions of $M$ such that on $GL(d_\rho)$ they are given by $\rho|^{\mathbb{F}}$, then the relevant Levi subgroups of $H$ will satisfy $m_I = 0$ or $m_{II} = 0$. After we choose the relevant $P'$ to be the standard parabolic subgroup of $H$, then we can further choose $\xi'$ to be identity on $GL(d_\rho, \mathbb{C})$ by taking certain $\widehat{M}$-conjugate. Let us write

$$M(S_I) = GL(d_\rho) \times H_{I-} := GL(d_\rho) \times G_{I-} \times G_{II},$$

and

$$M(S_{II}) = GL(d_\rho) \times H_{II-} := GL(d_\rho) \times G_{I} \times G_{II-}.$$ 

We also keep the notations in Example 4.1 in particular when $G$ is symplectic, $G_I$ is symplectic and $G_{II}$ is special even orthogonal. Let $\theta_i = \theta_0$ with respect to $G_i$ for $i = I, II$.

1. If $G$ is symplectic, then $M(S_I)$, $M(S_{II})$ and $M(S_{II})^{\theta_{II}}$ are the relevant standard Levi subgroups of $H$. Note $M(S_{II}) = M(S_{II})^{\theta_{II}}$ if and only if $G_{II-} \neq 1$. And we get a modified diagram of (6.1) as follows.

2. If $G$ is special odd orthogonal, then $M(S_I)$, $M(S_{II})$ are the only relevant standard Levi subgroups of $H$, and we get a modified diagram of (6.1) as follows.

3. If $G$ is special even orthogonal, then $M(S_I)$, $M(S_I)^{\theta_I}$, $M(S_{II})$ and $M(S_{II})^{\theta_{II}}$ are the relevant standard Levi subgroups of $H$. Note $M(S_i) = M(S_i)^{\theta_i}$ if and only if $G_{i-} \neq 1$ for $i = I, II$. And we get a modified diagram of (6.1) as follows.
Here the sum is over all \( \xi \) of parabolic subgroups of \( G \), and if we let \( \hat{M} := \text{Cent}(\xi(N(S), GL(N, \mathbb{C})) \) of \( G \), then there is an embedding
\[
\xi_N : L G \hookrightarrow LGL(N)
\]
(see Section \[\text{III}].) We also fix a \( \hat{\theta}_N \)-stable \( \Gamma \)-splitting \( (\mathcal{B}_N, \mathcal{T}_N, \{\chi_N\}) \) of \( GL(N, \mathbb{C}) \). And by taking certain \( GL(N, \mathbb{C}) \)-conjugate of \( \xi_N \) we can assume \( \xi_N(T_G) = (\mathcal{T}_N^\theta)^0, \xi_N(B_G) \subseteq B_N \). Then we can view the Weyl group \( W_G = W(\hat{G}, \mathcal{T}_G) \) as a subgroup of \( W_{N^\theta} := W(GL(N, \mathbb{C}), \mathcal{T}_N)^{\mathcal{D}_N} \). Let \( S \) be a \( \Gamma \)-invariant torus in \( T_G \), it gives a Levi subset \( M \times \theta_N \) of \( GL(N) \times \theta_N \), where \( \hat{M} = \text{Cent}(\xi_N(S), GL(N, \mathbb{C})) \). Here being a Levi subset means \( M \) is the Levi factor of some \( \theta_N \)-stable parabolic subgroup \( P \). We denote the \( W_{N^\theta} \)-conjugacy class of \( S \) in \( T_G \) by \( \{S\}_N \). Then each \( W_G \)-conjugacy class \( \{S\}'_G \) in \( \{S\}_N \) corresponds to a \( G \)-conjugacy class of Levi subgroup \( M' = M(S') \) of \( G \), where \( \hat{M}' := \text{Cent}(S', \hat{G}) \). As before, \( M' \) are twisted endoscopic groups of \( M \) (see [\text{KS99}]). We fix a \( \theta_N \)-stable parabolic subgroup \( P \supseteq M \) with an embedding \( L P \hookrightarrow GL(N, \mathbb{C}) \), which extends \( L M \hookrightarrow GL(N, \mathbb{C}) \). Then the embedding \( \xi_{M'} : L M' \hookrightarrow L M \) can be given by any element \( g_0 \in GL(N, \mathbb{C}) \) such that \( \text{Int}(g_0)(\xi_N(S')) = \xi_N(S) \), i.e., the following diagram commutes
\[
\begin{array}{c}
L M' \downarrow \xi_{M'} \downarrow \xi_N \downarrow \text{Int}(g_0) \downarrow \text{Jac}_P \\
L M \downarrow \text{Jac}_P \\
L G \downarrow GL(N, \mathbb{C}) \\
GL(N, \mathbb{C}).
\end{array}
\]
We denote the set of all such embeddings by \( \{\xi_{M'}\} \). For \( (g_N, g) \in \text{Norm}(\xi_N(S), GL(N, \mathbb{C})) \times \text{Norm}(S', \hat{G}) \), we define another embedding \( (g_N, g) * \xi_{M'} \) by changing \( g_0 \) to \( g g_0 \xi_N(g) \). In this way, we get a transitive action of \( \text{Norm}(\xi_N(S), GL(N, \mathbb{C})) \times \text{Norm}(S', \hat{G}) \) on \( \{\xi_{M'}\} \). For each \( \xi' = (g_N, g) * \xi_{M'} \in \{\xi_{M'}\} \), we can associate it with a parabolic subgroup \( P' \supseteq M' \) such that \( \xi_N(\hat{P}') = \text{Int}(g_N g_0 \xi_N(g))^{-1} \hat{P} \cap \xi_N(\hat{G}) \). Then we have
\[
\xi_{M'} : \hat{I}(M(S')) \longrightarrow \hat{I}(M^\theta).
\]
Here the sum is over all \( W_G \)-conjugacy classes \( \{S\}'_G \) in \( \{S\}_N \), and \( \hat{M} \times \text{Norm}(S', \hat{G}) \)-orbits \( \{\xi'\} \subseteq \{\xi_{M'}\} \). And the horizontal maps correspond to spectral endoscopic transfers with respect to \( \xi_N \) on the top and \( \xi' \) on the bottom. As in the non-twisted case, the pair \( (S', \xi') \) corresponds to a unique \( G \)-conjugacy class of parabolic subgroups of \( G \), so we can always take \( P' \) to be standard in application.

We again fix a unitary irreducible supercuspidal representation \( \rho \) of \( GL(d_\rho) \), and take \( P = MN \) to be a standard \( \theta_N \)-stable parabolic subgroup of \( GL(N) \) such that
\[
M = GL(d_\rho) \times GL(N_-) \times GL(d_{\rho'}). \]
We take \( P' \) to be standard, then there is exactly one standard Levi subgroups of \( G \) appearing in \( (6.5) \), i.e.,
\[
M(S) = GL(d_\rho) \times G_-,
\]
unless \( G \) is special even orthogonal and \( N_- = 0 \); in which case, there are two standard Levi subgroups of \( G \) appearing in \( (6.5) \), and if we let \( M(S) = GL(d_\rho) \) then the other will be \( M(S') = M(S)^{\theta_0} \). Again we
can choose $\xi'$ to be identity on $GL(d_\rho, \mathbb{C})$. By restricting (6.5) to twisted distribution of $M$, which are contributed from representations having $\rho|_x$ on $GL(d_\rho)$, we get a modified diagram as follows

\[
\begin{align*}
\widehat{SI}(G) & \xrightarrow{\text{Jac}_x} \widehat{I}(N^\theta) \\
\widehat{SI}(G_-) & \xrightarrow{\text{Jac}_x^\theta} \widehat{I}(N_\theta).
\end{align*}
\]

At last, when $G$ is special even orthogonal, there is a twisted version of the diagram (6.4), which can be derived as in the case of $GL(N)$ (also see Appendix C for the general case). Here we will only state the result, and we keep the notations from Section 4 and the non-twisted case. We assume $G_- \neq 1$.

\[
\begin{align*}
\widehat{SI}(H) & \xrightarrow{\text{Jac}_x^I \oplus \text{Jac}_x^{I^I}} \widehat{I}(G^\theta_0) \\
\widehat{SI}(H_1\cdot\cdot\cdot\cdot\cdot H_{II}) & \xrightarrow{\text{Jac}_x} \widehat{I}(G^\theta_0),
\end{align*}
\]

where $\text{Jac}_x^I$ is with respect to $\rho \otimes \eta_i$ for $i = I, II$.

Both diagrams (6.1) and (6.5) can be established by using Casselman’s formula [Cas77] and its twisted version for relating the (twisted) characters of representations with that of their Jacquet modules. We will give the proof of the general case in Appendix C. In the next section, we are going to prove Theorem 3.3 by applying (6.6) (resp. (6.2), (6.3) and (6.4)) to the (twisted) endoscopic identity (4.3) (resp. (4.4)). We will only need (6.7) in Section 9.

7. Proof of Theorem 3.3

In the following sections we will always assume $G$ is a quasisplit symplectic group or special orthogonal group. Before we start the proof, we would like to make explicit the effects of Jacquet modules on the (twisted) endoscopic character identities (4.3) and (4.4). So let us fix a self-dual irreducible unitary supercuspidal representation $\rho$ of $GL(d_\rho)$, and let $\phi \in \Phi_2(G)$. We define $\phi_- \in \Phi_{bdd}(G_-)$ by its Jord($\phi_-)$ as follows.

\[
\text{Jord}(\phi_-) = \text{Jord}(\phi) \cup \{ (\rho, 2x - 1) \} \setminus \{ (\rho, 2x + 1) \},
\]

if $(\rho, 2x + 1) \in \text{Jord}(\phi)$ and $x > 0$, or $\emptyset$ otherwise. And we set $\pi_{\phi_-} = 0$ if $\text{Jord}(\phi_-) = \emptyset$. Then it is clear that

\[
\pi_{\phi_-} = \text{Jac}_x^\theta \pi_\phi
\]

by our explicit formulas. So after applying (6.6) to the twisted endoscopic identity (4.3), we have

\[
f_{G^-}(\sum_{[\pi] \in \Pi_\phi} \text{Jac}_x \pi) = f_{N^\theta}(\pi_{\phi_-})
\]

for $f \in C_c^\infty(GL(N_-))$. Since Theorem 4.2 is also valid for all tempered parameters (see Remark 4.3), the left hand side of (7.1) has to be $f_{G^-}(\phi_-)$, i.e.,

\[
\Pi_{\phi_-} = \text{Jac}_x \Pi_\phi.
\]

This also implies

\[
\text{Jac}_x \Pi_\phi = 0 \text{ if } (\rho, 2x + 1) \notin \text{Jord}(\phi).
\]

Suppose $\text{Jord}(\phi_-) \neq \emptyset$, then $x > 0$ and there are two possibilities, i.e., $\phi_- \in \Phi_2(G_-)$, or

\[
\phi_- = 2\phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_r \in \Phi_{bdd}(G_-),
\]

\[
\Pi_{\phi_-} = \text{Jac}_x \Pi_\phi.
\]

This also implies

\[
\text{Jac}_x \Pi_\phi = 0 \text{ if } (\rho, 2x + 1) \notin \text{Jord}(\phi).
\]

Suppose $\text{Jord}(\phi_-) \neq \emptyset$, then $x > 0$ and there are two possibilities, i.e., $\phi_- \in \Phi_2(G_-)$, or

\[
\phi_- = 2\phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_r \in \Phi_{bdd}(G_-),
\]

\[
\Pi_{\phi_-} = \text{Jac}_x \Pi_\phi.
\]
where \( \phi_1 = \rho \otimes [2x - 1] \). In the first case we have \((\rho, 2x - 1) \notin \text{Jord}(\phi)\). If \( x \neq 1/2 \), then there is a canonical isomorphism \( \mathcal{S}_\phi \cong \mathcal{S}_{\phi_-} \) after identifying \( \text{Jord}(\phi) \) with \( \text{Jord}(\phi_-) \) by sending \((\rho, 2x + 1)\) to \((\rho, 2x - 1)\). If \( x = 1/2 \), we have a projection from \( \mathcal{S}_\phi \) to \( \mathcal{S}_{\phi_-} \) by restricting \( \mathbb{Z}_2 \)-valued functions on \( \text{Jord}(\phi) \) to \( \text{Jord}(\phi_-) \). And hence we get an exact sequence

\[
\begin{array}{c}
1 \longrightarrow < s > \longrightarrow \mathcal{S}_\phi \longrightarrow \mathcal{S}_{\phi_-} \longrightarrow 1,
\end{array}
\]

where \( s(\cdot) = 1 \) over \( \text{Jord}(\phi) \) except for \( s(\rho, 1/2) = -1 \).

In the second case we can also identify \( \mathcal{S}_{\phi_-} \) and its characters \( \widehat{\mathcal{S}}_{\phi_-} \) with certain quotient space of \( \mathbb{Z}_2 \)-valued functions on \( \text{Jord}(\phi_-) \) (here we forget the multiplicities in \( \text{Jord}(\phi_-) \)), as in the case of discrete parameters. Note

\[
\text{Cent}(\widehat{\phi}_-, GL(N_-, \mathbb{C})) \cong \underbrace{GL(2, \mathbb{C}) \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_r,
\]

and then

\[
\text{Cent}(\widehat{\phi}_-, \hat{G}_-) \cong \{ s = (s_i) \in O(2, \mathbb{C}) \times \mathbb{Z}_2^{r-1} : \det(s_1)^{n_1} \cdot \prod_{i \neq 1} (s_i)^{n_i} = 1 \}.
\]

We write \( z \) for the nontrivial central element of \( O(2, \mathbb{C}) \). Then \( \widehat{\mathcal{S}}_{\phi_-}^{\Sigma_0} \cong O(2, \mathbb{C}) \times \mathbb{Z}_2^{r-1} / < z, -1, \cdots, -1 > \), and hence \( \mathcal{S}_{\phi_-}^{\Sigma_0} \cong \mathbb{Z}_2^r / < 1, -1, \cdots, -1 > \). If \( G \) is special even orthogonal,

\[
\mathcal{S}_{\phi_-} \cong \{ s = (s_i) \in \mathbb{Z}_2^r : \prod_i (s_i)^{n_i} = 1 \} / < 1, -1, \cdots, -1 >
\]

which is a subgroup of \( \mathcal{S}_{\phi_-}^{\Sigma_0} \) of index 1 or 2. Let us denote by \( \mathcal{S}_{\phi_-} \) (resp. \( \mathcal{S}_{\phi_-}^{\Sigma_0} \)) the corresponding quotient space of \( \mathbb{Z}_2 \)-valued functions on \( \text{Jord}(\phi_-) \) (forgetting multiplicities) such that \( \mathcal{S}_{\phi_-} \cong \mathcal{S}_{\phi_-} (\text{resp.} \mathcal{S}_{\phi_-}^{\Sigma_0} \cong \mathcal{S}_{\phi_-}^{\Sigma_0}) \) under these isomorphisms. There is a projection from \( \mathcal{S}_\phi \to \mathcal{S}_{\phi_-} \) (resp. \( \mathcal{S}_\phi^{\Sigma_0} \to \mathcal{S}_{\phi_-}^{\Sigma_0} \)) by sending \( s \) to \( s_- (\cdot) = s(\cdot) \) over \( \text{Jord}(\phi) \backslash \{(\rho, 2x + 1), (\rho, 2x - 1)\} \) and \( s_- (\rho, 2x - 1) = s(\rho, 2x + 1) s(\rho, 2x - 1). \)

And hence there is a short exact sequence

\[
\begin{array}{c}
1 \longrightarrow < s > \longrightarrow \mathcal{S}_\phi \longrightarrow \mathcal{S}_{\phi_-} \longrightarrow 1
\end{array}
\]

( resp. \( 1 \longrightarrow < s > \longrightarrow \mathcal{S}_\phi^{\Sigma_0} \longrightarrow \mathcal{S}_{\phi_-}^{\Sigma_0} \longrightarrow 1 \) )

where \( s(\cdot) = 1 \) over \( \text{Jord}(\phi) \) except for \( s(\rho, 2x + 1) = s(\rho, 2x - 1) = -1 \). For the characters of \( \mathcal{S}_{\phi_-} \), we have

\[
\widehat{\mathcal{S}}_{\phi_-}^{\Sigma_0} = \{ \epsilon = (\epsilon_i) \in \mathbb{Z}_2^r : \prod_i \epsilon_i = 1 \},
\]

and if \( G \) is special even orthogonal,

\[
\widehat{\mathcal{S}}_{\phi_-} = \{ \epsilon = (\epsilon_i) \in \mathbb{Z}_2^r : \prod_i \epsilon_i = 1 \} / < \epsilon_0 >,
\]

where \( \epsilon_0 = (\epsilon_{0,i}) \in \widehat{\mathcal{S}}_{\phi_-}^{\Sigma_0} \) satisfies \( \epsilon_{0,i} = 1 \) if \( n_i \) is even, and \( \epsilon_{0,i} = -1 \) if \( n_i \) is odd. So \( \epsilon_0 \) is trivial when restricted to \( \mathcal{S}_{\phi_-} \). In general, let \( \epsilon_0 = 1 \) if \( G \) is not special even orthogonal. At last we want to point out in this case \( \phi_- \) factors through \( \phi_{M_-} \in \widehat{\Phi}_2(M_-) \), for a Levi subgroup \( M_- = GL(n_1) \times G' \) and \( \phi_{M_-} = \phi_1 \times \phi' \) such that \( \phi' = \phi_2 \oplus \cdots \oplus \phi_r \). Since

\[
\mathcal{S}_{M_-}^{\Sigma_0} \to \mathcal{S}_{\phi_-}^{\Sigma_0} \text{ and } \mathcal{S}_{M_-}^{\Sigma_0} \cong \mathcal{S}_{\phi_-}^{\Sigma_0},
\]
we can get an inclusion $S_{q'}^{\Sigma_0} \hookrightarrow S_{\phi'}^{\Sigma_0}$, which in fact just extends $s'(\cdot) \in \mathbb{Z}_2^{Jord(\phi')}$ trivially to $Jord(\phi_-)$ (forgetting multiplicities). So on the dual side, there is a projection

$$\widetilde{S}_{\phi_-}^{\Sigma_0} \rightarrow \widetilde{S}_{\phi'}^{\Sigma_0}$$

given by restricting $\varepsilon(\cdot)$ to $Jord(\phi')$. Taking quotient by $< \varepsilon_0 >$, we get $\widetilde{S}_{\phi_-} \rightarrow \widetilde{S}_{\phi'}$. It follows from Arthur’s theory (i.e., Theorem 2.2, 2.3, 4.2 and 4.4) that

$$(7.6) \quad \Pi_{\phi_-} = \pi_{\phi_1} \rtimes \Pi_{\phi'} \quad \text{(resp. } \Pi_{\phi_-}^{\Sigma_0} = \pi_{\phi_1} \rtimes \Pi_{\phi'}^{\Sigma_0})$$

Moreover,

$$\pi_{\phi_1} \rtimes \pi(\phi', \varepsilon') = \oplus_{\varepsilon \in \widehat{S}_{\phi_-}} \pi(\phi_-, \varepsilon)$$

$$(\text{resp. } \pi_{\phi_1} \rtimes \pi(\phi', \varepsilon') = \oplus_{\varepsilon \in \widehat{S}_{\phi_-}^{\Sigma_0}} \pi(\phi_-, \varepsilon)).$$

We will need this description of $\Pi_{\phi_-}$ (resp. $\Pi_{\phi_-}^{\Sigma_0}$) in Section 8 (resp. Section 9).

In all the above cases, we can canonically identify $\widetilde{S}_{\phi_-}$ (resp. $\widetilde{S}_{\phi_-}^{\Sigma_0}$) with a subgroup of $\widetilde{S}_{\phi}$ (resp. $\widetilde{S}_{\phi}^{\Sigma_0}$) of index 1 or 2, so later on we will always view $\varepsilon \in \widehat{S}_{\phi_-}$ as functions on $Jord(\phi)$.

**Lemma 7.1.** Suppose $\phi \in \Phi_2(G)$, and $(\rho, 2x+1) \in Jord(\phi)$.

1. If $x > 1/2$ and $(\rho, 2x-1) \notin Jord(\phi)$, then $\pi(\phi_-, \varepsilon) = \tilde{Jac}_{\varepsilon} \pi(\phi, \varepsilon)$ for all $\varepsilon \in \widehat{S}_{\phi_-} \cong \widetilde{S}_{\phi}$.
2. If $x > 1/2$ and $(\rho, 2x-1) \in Jord(\phi)$, then $\tilde{Jac}_{\varepsilon} \pi(\phi, \varepsilon) = 0$ unless $\varepsilon \in \widehat{S}_{\phi_-}$, i.e.,

$$\varepsilon(\rho, 2x+1)\varepsilon(\rho, 2x-1) = 1,$$

in which case $\pi(\phi_-, \varepsilon) = \tilde{Jac}_{\varepsilon} \pi(\phi, \varepsilon)$.
3. If $x = 1/2$, then $\tilde{Jac}_{1/2} \pi(\phi, \varepsilon) = 0$ unless $\varepsilon \in \widehat{S}_{\phi_-}$, i.e.,

$$\varepsilon(\rho, 2) = 1,$$

in which case $\pi(\phi_-, \varepsilon) = \tilde{Jac}_{\varepsilon} \pi(\phi, \varepsilon)$.

**Proof.** First we know from (7.2) that $\tilde{Jac}_J \Pi_{\phi} = \Pi_{\phi}$, so in particular $\tilde{Jac}_J \pi$ do not have common irreducible constituents with each other for $[\pi] \in \Pi_{\phi}$. Next for $s \in S_{\phi}$, suppose $(H, \phi_H) \rightarrow (\phi, s)$, then we have

$$f^H(\phi_H) = \sum_{[\pi] \in \Pi_{\phi}} \varepsilon(\pi) \varepsilon(\phi_H) = \sum_{\varepsilon \in \widehat{S}_{\phi}} \varepsilon(s) \varepsilon(\phi(\phi, \varepsilon)),$$

for $f \in \mathcal{H}(G)$. If $H = G_J \times G_J$ and $\phi_H = \phi_J \times \phi_J$, we can assume without loss of generality that $(\rho, 2x+1) \notin Jord(\phi_J)$. By (7.3), $\tilde{Jac}_J \Pi_{\phi_J} = 0$. We let $H_- = H_{-J}$ (see Section 6), and define $\phi_{H_-}$ in the same way as $\phi_-$. So after applying (6.2), (6.3) and (6.4) accordingly to (7.8), we get

$$f^{H_-}(\phi_{H_-}) = \sum_{[\pi] \in \Pi_{\phi_-}} \varepsilon(s) \varepsilon(\phi_{H_-}(\tilde{Jac}_J \pi(\phi, \varepsilon))),$$

for $f \in \mathcal{H}(G_-)$. On the other hand note $(H_-, \phi_{H_-}) \rightarrow (\phi_-, s_-)$ where $s_-$ is the image of $s$ under the projection $S_{\phi} \rightarrow S_{\phi_-}$, so we have

$$f^{H_-}(\phi_{H_-}) = \sum_{[\pi] \in \Pi_{\phi_-}} \varepsilon(s_-) \varepsilon(\phi_{H_-}(\pi(\phi_-, \varepsilon))).$$

for $f \in \mathcal{H}(G_-)$. Combining this identity with (7.9), we get

$$\sum_{\varepsilon \in \widehat{S}_{\phi}} \varepsilon(s) \varepsilon(\phi_{H_-}(\tilde{Jac}_J \pi(\phi, \varepsilon))) = \sum_{\varepsilon' \in \widehat{S}_{\phi_-}} \varepsilon'(s_-) \varepsilon(\phi_{H_-}(\pi(\phi_-, \varepsilon'))).$$

(7.10)
By the linear independence of characters, \( \pi(\phi, \xi) \) is in \( \Jac_{\chi} \pi(\phi, \xi) \) only when \( \xi(s) = \xi'(s) \) for all \( s \in S_{\psi} \), i.e., \( \xi' = \xi \). This implies \( \Jac_{\chi} \pi(\phi, \xi) = 0 \) for \( \xi \notin S_{\psi} \). Then after a little thought, one can see \( \pi(\phi, \xi) = \Jac_{\chi} \pi(\phi, \xi) \) for all \( \xi \in S_{\psi} \).

Now we are in the position to prove Theorem 7.2. For the convenience of readers we will restate the theorem here.

**Theorem 7.2** (Moeglin). The \( \Sigma_{0} \)-orbits of irreducible supercuspidal representations of \( G \) can be parametrized by \( \phi \in \Phi_{2}(G) \) and \( \xi \in S_{\psi} \) satisfying the following properties:

1. If \( (\rho, a) \in \Jord(\phi) \), then \( (\rho, a - 2) \in \Jord(\phi) \) as long as \( a - 2 > 0 \);
2. If \( (\rho, a), (\rho, a - 2) \in \Jord(\phi) \), then \( \varepsilon(\rho, a)(\rho, a - 2) = -1 \);
3. If \( (\rho, 2) \in \Jord(\phi) \), then \( \varepsilon(\rho, 2) = -1 \).

**Proof.** Let \( \pi \) be an irreducible discrete series representation of \( G \) and we can always assume \( |\pi| = \pi(\phi, \xi) \) for some \( \phi \in \Phi_{2}(G) \) and \( \xi \in S_{\psi} \). It is not hard to see that \( \pi \) is a supercuspidal if and only \( \Jac_{\chi} \pi(\phi, \xi) = 0 \) for any supercuspidal representation \( \rho \) of \( GL(d_{\rho}) \) and any real number \( x \). Then by (7.3), it is enough to consider the cases when \( (\rho, 2x + 1) \in \Jord(\phi) \). Notice each of the conditions in this theorem excludes exactly one situation in Lemma 7.1. And it is easy to check one by one that these conditions are both necessary and sufficient.

**Remark 7.3.** The necessity of condition (1) has already been established by Proposition 3.1 but in this proof we do not need to know that result.

8. Cuspidal support of discrete series

In this section we are going to characterize the cuspidal supports of discrete series representations of \( G \). Let \( \phi \in \Phi_{2}(G) \), for any \( (\rho, a) \in \Jord(\phi) \), we denote by \( a_{-} \) the biggest positive integer smaller than \( a \) in \( \Jord_{\rho}(\phi) \). And we would also like to write \( a_{-} \) for the minimum of \( \Jord_{\rho}(\phi) \).

**Proposition 8.1.** Suppose \( \phi \in \Phi_{2}(G) \), and \( \varepsilon \in S_{\psi}^{\Sigma_{0}} \).

1. If \( \varepsilon(\rho, a)(\rho, a_{-}) = -1 \), then

\[
\pi(\phi, \xi) \leftrightarrow (a_{-} - 1)/2, \cdots, (a_{-} + 3)/2 \geq \pi(\phi', \xi')
\]

as the unique irreducible \( \mathcal{H}(G) \)-submodule, where

\[
\Jord(\phi') = \Jord(\phi) \cup \{(\rho, a_{-} + 2)\} \setminus \{(\rho, a)\},
\]

and

\[
\varepsilon'(\cdot) = \varepsilon(\cdot) \text{ over } \Jord(\phi) \setminus \{(\rho, a)\}, \quad \varepsilon'(\rho, a_{-} + 2) = \varepsilon(\rho, a).
\]

2. If \( \varepsilon(\rho, a)(\rho, a_{-}) = 1 \), then

\[
\pi(\phi, \xi) \leftrightarrow (a_{-} - 1)/2, \cdots, -(a_{-} - 1)/2 \geq \pi(\phi', \xi'),
\]

where

\[
\Jord(\phi') = \Jord(\phi) \setminus \{(\rho, a), (\rho, a_{-})\},
\]

and \( \varepsilon'(\cdot) \) is the restriction of \( \varepsilon(\cdot) \). In particular, suppose \( \varepsilon_{1} \in S_{\psi}^{\Sigma_{0}} \) satisfying \( \varepsilon_{1}(\cdot) = \varepsilon(\cdot) \) over \( \Jord(\phi') \) and

\[
\varepsilon_{1}(\rho, a) = -\varepsilon(\rho, a), \quad \varepsilon_{1}(\rho, a_{-}) = -\varepsilon(\rho, a_{-}).
\]

If \( \varepsilon_{1} = \xi \), then the induced \( \mathcal{H}(G) \)-module in (8.2) has a unique irreducible submodule. Otherwise, it has two irreducible submodules, namely

\[
\pi(\phi, \xi) \oplus \pi(\phi, \xi_{1}).
\]
(3) If $\varepsilon(\rho, a_{\min}) = 1$ and $a_{\min}$ is even, then

$$\pi(\phi, \varepsilon) \leftrightarrow \langle (a_{\min} - 1)/2, \ldots, 1/2 \rangle \rtimes \pi(\phi', \varepsilon')$$

as the unique irreducible $\mathcal{H}(G)$-submodule, where

$$\text{Jord}(\phi') = \text{Jord}(\phi) \setminus \{(\rho, a_{\min})\},$$

and $\varepsilon'(\cdot)$ is the restriction of $\varepsilon(\cdot)$.

**Proof.** The proofs of part (1) and part (3) are almost the same, so here we will only give the proof of part (1). We start by considering the Jacquet module $\text{Jac}_{(a-1)/2, \ldots, (a + 3)/2} \pi(\phi, \varepsilon)$, and by applying Lemma 7.1 multiple times we have

$$\text{Jac}_{(a-1)/2, \ldots, (a + 3)/2} \pi(\phi, \varepsilon) = \pi(\phi', \varepsilon').$$

It follows from Corollary 5.4 that

$$\pi(\phi, \varepsilon) \leftrightarrow \rho \left| \frac{a - 1}{2} \right| \times \cdots \times \rho \left| \frac{a + 3}{2} \right| \rtimes \pi(\phi', \varepsilon').$$

By Lemma 5.2 we can take an irreducible constituent $\tau$ in $\rho \left| \frac{a - 1}{2} \right| \times \cdots \times \rho \left| \frac{a + 3}{2} \right|$, such that

$$\pi(\phi, \varepsilon) \leftrightarrow \tau \rtimes \pi(\phi', \varepsilon').$$

So it is enough to show $\tau = \langle \frac{a - 1}{2}, \ldots, \frac{a + 3}{2} \rangle$. If this is not the case, we know from Lemma 5.7 that $\text{Jac}_{x} \tau \neq 0$ for some $(a + 3)/2 \leq x < (a - 1)/2$. So $\tau \leftrightarrow \rho \left| \frac{x}{2} \right| \times \tau'$ for some irreducible representation $\tau'$, and

$$\pi(\phi, \varepsilon) \leftrightarrow \rho \left| \frac{x}{2} \right| \times \tau' \times \pi(\phi', \varepsilon').$$

By Frobenius reciprocity, $\text{Jac}_{x} \pi(\phi, \varepsilon) \neq 0$. However, $(\rho, 2x + 1) \notin \text{Jord}(\phi)$ under our assumption, so we get a contradiction (see (7.3)).

To see the induced $\mathcal{H}(G)$-module in (8.1) has a unique irreducible submodule, we can compute its Jacquet module under $\text{Jac}_{(a-1)/2, \ldots, (a + 3)/2}$. By applying the formula (5.2), we find the Jacquet module consists of

$$\text{Jac}_{X_{1}} < (a - 1)/2, \ldots, (a + 3)/2 > \rtimes \text{Jac}_{X_{2}}^{\text{op}} < (a - 1)/2, \ldots, (a + 3)/2 > \rtimes \text{Jac}_{X_{3}} \pi(\phi', \varepsilon'),$$

where

$$\{(a - 1)/2, \ldots, (a + 3)/2\} = X_{1} \sqcup X_{2} \sqcup X_{3},$$

and $X_{i}$ inherits the order from $\{(a - 1)/2, \ldots, (a + 3)/2\}$. Note $\text{Jac}_{X_{1}} < (a - 1)/2, \ldots, (a + 3)/2 > \neq 0$ only if $X_{1}$ is a segment $\{(a - 1)/2, \ldots, x_{1}\}$. Similarly, $\text{Jac}_{X_{2}}^{\text{op}} < (a - 1)/2, \ldots, (a + 3)/2 > \neq 0$ only if $X_{2}$ is a segment $\{- (a + 3)/2, \ldots, x_{2}\}$. Since $-(a + 3)/2 \notin \{(a - 1)/2, \ldots, (a + 3)/2\}$, $X_{2}$ has to be empty. Therefore the Jacquet module can only contain terms like

$$\text{Jac}_{(a - 1)/2, \ldots, x_{1}} < (a - 1)/2, \ldots, (a + 3)/2 > \rtimes \text{Jac}_{x_{1} - 1, \ldots, (a + 3)/2} \pi(\phi', \varepsilon').$$

But from our definition of $\text{Jord}(\phi')$, we see $\{a, \ldots, (a + 4)\}$ has no intersection with $\text{Jord}_{\rho}(\phi')$, so

$$\text{Jac}_{x_{1} - 1, \ldots, (a + 3)/2} \pi(\phi', \varepsilon') = 0$$

by (7.3). Hence we can only have

$$\text{Jac}_{(a - 1)/2, \ldots, (a + 3)/2} \left( < (a - 1)/2, \ldots, (a + 3)/2 > \rtimes \pi(\phi', \varepsilon') \right) = \pi(\phi', \varepsilon').$$

Note this implies $< (a - 1)/2, \ldots, (a + 3)/2 > \rtimes \pi(\phi', \varepsilon')$ has a unique irreducible $\mathcal{H}(G)$-submodule.

For part (2), we will first consider $\text{Jac}_{(a-1)/2, \ldots, (a + 1)/2} \pi(\phi, \varepsilon)$, and again by applying Lemma 7.1 multiple times we have

$$\text{Jac}_{(a-1)/2, \ldots, (a + 1)/2} \pi(\phi, \varepsilon) = \pi(\phi, \varepsilon'),$$

where $\text{Jord}(\phi) = \text{Jord}(\phi) \cup \{(\rho, a_{\min})\} \setminus \{(\rho, a)\}$, and $\varepsilon(\cdot)$ is the restriction of $\varepsilon(\cdot)$ to $\text{Jord}(\phi)$ (forgetting multiplicities). As in part (1), we can show from here that

$$\pi(\phi, \varepsilon) \leftrightarrow < (a - 1)/2, \ldots, (a + 1)/2 > \rtimes \pi(\phi, \varepsilon').$$
Note $\Pi_{\phi_\epsilon} = St(\rho, a_\epsilon) \rtimes \Pi_{\phi'}$ (see (7.8)), so
\[
\pi(\phi_\epsilon, \epsilon) \hookrightarrow St(\rho, a_\epsilon) \rtimes \pi(\phi', \epsilon') = \langle (a_\epsilon - 1)2, \cdots, -(a_\epsilon - 1)/2 > \times \pi(\phi', \epsilon'),
\]
and hence
\[
(8.5) \quad \pi(\phi, \epsilon) \hookrightarrow \langle (a_\epsilon - 1)/2, \cdots, (a_\epsilon + 1)/2 > \times \langle (a_\epsilon - 1)/2, \cdots, -(a_\epsilon - 1)/2 > \times \pi(\phi', \epsilon').
\]
By Lemma 5.2 we can take an irreducible constituent $\tau$ in
\[
\langle (a_\epsilon - 1)/2, \cdots, (a_\epsilon + 1)/2 > \times \langle (a_\epsilon - 1)/2, \cdots, -(a_\epsilon - 1)/2 >,
\]
such that
\[
\pi(\phi, \epsilon) \hookrightarrow \tau \rtimes \pi(\phi', \epsilon').
\]
Therefore it suffices to show $\tau = \langle (a_\epsilon - 1)/2, \cdots, -(a_\epsilon - 1)/2 >$. If this is not the case, then by Theorem 3.1
\[
\tau \hookrightarrow \langle (a_\epsilon - 1)/2, \cdots, -(a_\epsilon - 1)/2 > \times \langle (a_\epsilon - 1)/2, \cdots, (a_\epsilon + 1)/2 >.
\]
And by Frobenius reciprocity, we have
\[
\overline{Jac}_{(a_\epsilon - 1)/2, \cdots, -(a_\epsilon - 1)/2} \neq 0.
\]
But this is impossible, because one can check
\[
\overline{Jac}_{(a_\epsilon - 1)/2, \cdots, -(a_\epsilon - 1)/2} \pi_\phi = 0.
\]
At last, we still need to show the irreducible submodules of the induced $\mathcal{H}(G)$-module in (8.2) are either $\pi(\phi, \epsilon)$ or $\pi(\phi, \epsilon) \oplus \pi(\phi, \epsilon_1)$ depending on whether $\epsilon$ and $\epsilon_1$ are equal or not. Note we can show in the same way as in part (1) that $\pi(\phi, \epsilon)$ is the unique irreducible submodule of the induced $\mathcal{H}(G)$-module in (8.4). And the same is true for $\pi(\phi, \epsilon_1)$. Since $\epsilon = \epsilon_1$ if and only if $\epsilon = \epsilon_1$, where $\epsilon_1(\cdot)$ is again the restriction of $\epsilon_1(\cdot)$ to $Jord(\phi_\epsilon)$ (forgetting multiplicities), let us assume $\epsilon \neq \epsilon_1$ first. Then by (7.7)
\[
\pi(\phi_\epsilon, \epsilon_\epsilon) \oplus \pi(\phi_\epsilon, \epsilon_1) = St(\rho, a_\epsilon) \rtimes \pi(\phi', \epsilon),
\]
and hence the irreducible submodule of the induced $\mathcal{H}(G)$-module in (8.5) are exactly $\pi(\phi, \epsilon) \oplus \pi(\phi, \epsilon_1)$. So we only need to show the induced $\mathcal{H}(G)$-modules in (8.2) and (8.5) have the same irreducible submodules. One direction is clear, i.e., the irreducible submodules of
\[
\langle (a_\epsilon - 1)/2, \cdots, (a_\epsilon + 1)/2 > \times \langle (a_\epsilon - 1)/2, \cdots, -(a_\epsilon - 1)/2 > \times \pi(\phi', \epsilon')
\]
contain that of
\[
\langle (a_\epsilon - 1)/2, \cdots, -(a_\epsilon - 1)/2 > \times \pi(\phi', \epsilon')
\]
And from what we have shown, it is clear that $\pi(\phi, \epsilon) \oplus \pi(\phi, \epsilon_1)$ are in $\langle (a_\epsilon - 1)/2, \cdots, -(a_\epsilon - 1)/2 > \times \pi(\phi', \epsilon')$, so they have to contain the same irreducible $\mathcal{H}(G)$-submodules. Now if $\epsilon = \epsilon_1$, we have by (7.7)
\[
\pi(\phi_\epsilon, \epsilon_\epsilon) = St(\rho, a_\epsilon) \rtimes \pi(\phi', \epsilon),
\]
and the rest of the argument is the same.

As a consequence of this theorem, we can now complete the proof of Proposition 3.2.

**Proof.** Let $\pi$ be a supercuspidal representation of $G$, and we assume $[\pi] = \pi(\phi, \epsilon)$ for some $\phi \in \hat{\Phi}_2(G)$. Recall we still need to show for a self-dual irreducible unitary supercuspidal representation $\rho$,
\[
\rho^{[\epsilon(a_\epsilon + 1)/2]} \rtimes \pi
\]
reduces for $a_\rho$ being max $Jord_\rho(\phi)$ if $Jord_\rho(\phi) \neq \emptyset$, or zero if $Jord_\rho(\phi) = \emptyset$ and $\rho$ is of opposite type to $\hat{G}$. 

□
It is equivalent to consider the reducibility of $\rho|^{(a,\tau)/2} \times \pi(\phi, \bar{\epsilon})$. Moreover, $s.s.\rho|^{(a,\tau)/2} \times \pi(\phi, \bar{\epsilon}) = s.s.\rho|^{- (a,\tau)/2} \times \pi(\phi, \bar{\epsilon})$, so it suffices to show $\rho|^{(a,\tau)/2} \times \pi(\phi, \bar{\epsilon})$ is reducible under those conditions. In the first case, we have
\[ \pi(\phi, \bar{\epsilon}) \hookrightarrow \rho|^{(a,\tau)/2} \times \pi(\phi, \bar{\epsilon}) \]
as the unique irreducible $\mathcal{H}(G)$-submodule, where $\phi_+$ is obtained from $\phi$ by changing $(\rho, a_\rho)$ to $(\rho, a_\rho + 2)$ and $\epsilon_+(\cdot)$ is extended from $\epsilon(\cdot)$ with $\epsilon(\rho, a_\rho + 2) = \epsilon(\rho, a_\rho)$. From (7.3), we see $\text{Jac}_{ (a,\tau)/2} \pi(\phi, \bar{\epsilon}) = 0$. On the other hand, $\text{Jac}_{ (a,\tau)/2} (\rho|^{(a,\tau)/2} \times \pi(\phi, \bar{\epsilon})) = \pi(\phi, \bar{\epsilon})$. So $\pi(\phi, \bar{\epsilon}) \neq \rho|^{(a,\tau)/2} \times \pi(\phi, \bar{\epsilon})$, and hence $\rho|^{(a,\tau)/2} \times \pi(\phi, \bar{\epsilon})$ is reducible. The second case can be proved almost in the same way. The only difference is $\phi_+$ is obtained from $\phi$ by adding $(\rho, 2)$ and $\epsilon_+(\cdot)$ is extended from $\epsilon(\cdot)$ with $\epsilon(\rho, 2) = 1$.

\[ \Box \]

9. Remarks on even orthogonal groups

The previous results of this paper can also be extended to representations of $G^\Sigma_0$. Note the only nontrivial case here is when $G$ is special even orthogonal. First, we will extend Proposition 3.2

Corollary 9.1. Suppose $\pi$ is a supercuspidal representation of $G$ and $[\pi] \in \overline{\Pi}_\phi$ for some $\phi \in \Phi_2(G)$. Let $\pi^\Sigma_0$ be any irreducible representation of $G^\Sigma_0$, whose restriction to $G$ contains $\pi$. Then for any unitary irreducible supercuspidal representation $\rho$ of $GL(d_\rho)$, the parabolic induction
\[ \rho|^{(a,\tau)/2} \times \pi^\Sigma_0 \]
reduces exactly for
\[ a_\rho = \begin{cases} \max \text{Jord}_\rho(\phi), & \text{if } \text{Jord}_\rho(\phi) \neq \emptyset, \\ 0, & \text{if } \text{Jord}_\rho(\phi) = \emptyset, \rho \text{ is self-dual and is of opposite type to } \hat{G}, \\ -1, & \text{otherwise}. \end{cases} \]

Proof. We can assume $G$ is special even orthogonal. First, we would like to give the relation of reducibility between an irreducible representation $\pi$ of $G$ and an irreducible representation $\pi^\Sigma_0$ of $G^\Sigma_0$ which contains $\pi$ in its restriction to $G$. For any irreducible representation $\tau$ of $GL(d)$, it is easy to show the following fact:

- If $\pi \cong \pi^\theta_0$, $\tau \times \pi^\Sigma_0$ is irreducible if and only if $\tau \times \pi$ is irreducible and $(\tau \times \pi)^\theta_0 \cong \tau \times \pi$.
- If $\pi \cong \pi^\theta_0$, $\tau \times \pi$ is irreducible if and only if $\tau \times \pi^\Sigma_0$ is irreducible and $\tau \times \pi^\Sigma_0 \cong (\tau \times \pi^\Sigma_0) \odot \omega_0$.

Let $\tau = \rho|^{(a,\tau)/2}$ and $\pi$ be supercuspidal. Note the condition (3.3) implies (9.1). To see the necessity of the condition (9.1), we need to show if it is not satisfied, then $\tau \times \pi^\Sigma_0$ is irreducible. Since $\tau \times \pi$ is irreducible in this case, it suffices to consider $\pi \cong \pi^\theta_0$, and we would like to show $(\tau \times \pi)^\theta_0 \cong \tau \times \pi$. Since $\tau$ and $\pi$ are both supercuspidal, this is also equivalent to show there does not exit a Weyl group element of $G$ sending $\tau \times \pi^\theta_0$ to $\tau \times \pi$, i.e., $\tau \not\cong \tau^\vee$ or $d$ is even. Suppose $\tau \cong \tau^\vee$ and $d$ is odd, then $a_\rho = -1$ and $\rho$ is necessary of orthogonal type, hence one can only have $\text{Jord}_\rho(\phi) \neq \emptyset$ in view of (9.1). This implies $\pi \cong \pi^\theta_0$ and we get a contradiction.

To see the reducibility condition (9.1) is also sufficient, one notices when $\pi \cong \pi^\theta_0$, the condition (9.1) becomes the same as (3.3). If (9.1) is satisfied, then $\tau \times \pi$ reduces. Suppose $\tau \times \pi^\Sigma_0$ is irreducible, then $\tau \times \pi^\Sigma_0 \cong (\tau \times \pi^\Sigma_0) \odot \omega_0$, and hence
\[ (\tau \times \pi^\Sigma_0)|_G \cong (\tau \times \pi^\Sigma_0)|_{G}$
\[ \cong \tau \times \pi \cong \pi_+ \oplus \pi_+^\theta_0, \]
where $\pi_+ \not\cong \pi_+^\theta_0$. By the theory of Langlands quotient, one must have $\tau \cong \rho$. Define $\phi_+$ by
\[ \text{Jord}(\phi_+) := \text{Jord}(\phi) \cup \{ (\rho, 1) \text{ with multiplicity } 2 \}. \]

Then $\tau \times \pi \cong \overline{\Pi}_{\phi_+}$, and hence $\pi_+^\theta_0 \cong \pi_+$. This is a contradiction.

At last, we can assume $\pi \not\cong \pi^\theta_0$, and it suffices for us to show if $\tau \times \pi^\Sigma_0$ is irreducible, then (9.1) is not satisfied. In this case $\tau \times \pi$ is irreducible and $(\tau \times \pi)^\theta_0 \not\cong \tau \times \pi$. In particular, (3.3) is not satisfied. So we only need to exclude the case that $a_\rho = -1$ and $d_\rho$ is odd. By the previous discussion, we have $(\tau \times \pi)^\theta_0 \cong \tau \times \pi$ in this case, which again leads to a contradiction. This finishes the proof.
Next, we would like to extend Lemma 4.11.

**Lemma 9.2.** Suppose \( \phi \in \Phi_2(G) \), and \( (\rho, 2x + 1) \in \text{Jord}(\phi) \) with \( x > 0 \). Let \( \phi_- \in \Phi_{bad}(G_-) \) such that \( \text{Jord}(\phi_-) = \text{Jord}(\phi) \cup \{(\rho, 2x - 1)\} \backslash \{(\rho, 2x + 1)\} \).

Then we have the following facts:

1. If \( x > 1/2 \) and \( (\rho, 2x - 1) \notin \text{Jord}(\phi) \), then \( \pi^\Sigma_0(\phi_-, \varepsilon) = \text{Jac}_x \pi^\Sigma_0(\phi, \varepsilon) \) for all \( \varepsilon \in \mathcal{S}_{\phi_-}^\Sigma_0 \cong \mathcal{S}_{\phi}^\Sigma_0 \).
2. If \( x > 1/2 \) and \( (\rho, 2x - 1) \in \text{Jord}(\phi) \), then \( \text{Jac}_x \pi^\Sigma_0(\phi, \varepsilon) = 0 \) unless \( \varepsilon \in \mathcal{S}_{\phi_-}^\Sigma_0 \), i.e.,
   \[ \varepsilon(\rho, 2x + 1) = 0 \]
   in which case \( \pi^\Sigma_0(\phi_-, \varepsilon) = \text{Jac}_x \pi^\Sigma_0(\phi, \varepsilon) \).
3. If \( x = 1/2 \), then \( \text{Jac}_{1/2} \pi^\Sigma_0(\phi, \varepsilon) = 0 \) unless \( \varepsilon \in \mathcal{S}_{\phi_-}^\Sigma_0 \), i.e.,
   \[ \varepsilon(\rho, 2) = 1 \]
   in which case \( \pi^\Sigma_0(\phi_-, \varepsilon) = \text{Jac}_x \pi^\Sigma_0(\phi, \varepsilon) \).

**Proof.** Let \( G = G(n) \). If \( n = d_\rho \), then it suffices to assume \( \text{Jord}(\phi) = \{(\rho, 2)\} \). In this case, we necessarily have \( \varepsilon = 1 \) and

\[ [(\text{Jac}_{1/2}\pi^\Sigma_0(\phi, \varepsilon))|_G] = [\text{Jac}_{1/2}(\pi^\Sigma_0(\phi, \varepsilon))|_G] = 1. \]

So now we can assume \( n \neq d_\rho \). We claim \( \text{Jac}_x \pi^\Sigma_0(\phi, \varepsilon) = 0 \) unless \( \varepsilon \in \mathcal{S}_{\phi_-}^\Sigma_0 \), in which case,

\[ \text{Jac}_x \pi^\Sigma_0(\phi, \varepsilon) = \pi^\Sigma_0(\phi_-, \varepsilon) \text{ or } \pi^\Sigma_0(\phi_-, \varepsilon=0). \]

Suppose \( \mathcal{S}_{\phi_-}^\Sigma_0 \neq \mathcal{S}_\phi \), then

\[ [(\text{Jac}_x \pi^\Sigma_0(\phi, \varepsilon))|_G] = [\text{Jac}_x (\pi^\Sigma_0(\phi, \varepsilon))|_G] = \text{Jac}_x (\pi(\phi, \varepsilon)) = 0 \text{ or } \pi(\phi_-, \varepsilon), \]

and it is nonzero only when \( \varepsilon \in \mathcal{S}_{\phi_-}^\Sigma_0 \). Suppose \( \mathcal{S}_{\phi_-}^\Sigma_0 = \mathcal{S}_\phi \), then

\[ [(\text{Jac}_x \pi^\Sigma_0(\phi, \varepsilon))|_G] = [\text{Jac}_x (\pi^\Sigma_0(\phi, \varepsilon))|_G] = 2 \text{Jac}_x (\pi(\phi, \varepsilon)) = 0 \text{ or } 2 \pi(\phi_-, \varepsilon), \]

and it is again nonzero only when \( \varepsilon \in \mathcal{S}_{\phi_-}^\Sigma_0 \). So the claim is clear and it also suffices to show the lemma when \( \mathcal{S}_{\phi_-}^\Sigma_0 \neq \mathcal{S}_\phi \), i.e., \( \varepsilon_0 \neq 1 \). Let us choose \( s^* \in \mathcal{S}_{\phi_-}^\Sigma_0 \) such that \( \varepsilon_0(s^*) = -1 \). Then \( s^* \notin \mathcal{S}_\phi \). Suppose \( (H, \phi_H) \to (\phi, s^*) \), then we have from (4.3)

\[ f^H(\phi_H) = \sum_{[\pi] \in \Pi_\phi} \varepsilon(s^*) f_G(\pi^\Sigma_0) = \sum_{\varepsilon \in \mathcal{S}_{\phi_-}} \varepsilon(s^*) f_G(\pi^\Sigma_0(\phi_-, \varepsilon)) \]

for \( f \in C_c^\infty(G \times \theta_0) \). If \( H = G_I \times G_{II} \) and \( \phi_H = \phi_I \times \phi_{II} \), we can assume without loss of generality that \( (\rho \otimes \eta_{II}, 2x + 1) \notin \text{Jord}(\phi_{II}) \), and hence by (7.3), \( \text{Jac}_{H}^{\Sigma_0 \Pi_{\phi_II}} = 0 \). We let \( H_- = H_{II} \) (see Section 4), and define \( \phi_{H_-} \) in the same way as \( \phi_- \). So after applying (6.7) to (9.4), we get

\[ f^{H_-}(\phi_{H_-}) = \sum_{\varepsilon \in \mathcal{S}_{\phi_-}} \varepsilon(s^*) \sum_{\varepsilon \in \mathcal{S}_{\phi_-}} \varepsilon(s^*) f_G(\pi^\Sigma_0(\phi_-, \varepsilon')), \]

for \( f \in C_c^\infty(G_- \times \theta_0) \), where \( \varepsilon' = \varepsilon \) or \( \varepsilon' = \varepsilon \varepsilon_0 \). Since \( (H_- \cup S_{H_-}) \rightarrow (\phi_-, s^*) \), where \( s^* \) is the image of \( s^* \) under the projection \( \mathcal{S}_{\phi_0} \rightarrow \mathcal{S}_{\phi_-} \) and \( s^* \notin \mathcal{S}_{\phi_-} \), we also have

\[ f^{H_-}(\phi_{H_-}) = \sum_{\varepsilon \in \mathcal{S}_{\phi_-}} \varepsilon(s^*) f_G(\pi^\Sigma_0(\phi_-, \varepsilon)), \]

for \( f \in C_c^\infty(G_- \times \theta_0) \). Combining this identity with (9.3), we get
By the linear independence of twisted characters, we have
\[ \varepsilon(s^*) f_{G_\wedge}(\pi_{\Sigma_0}(\phi_-, \varepsilon')) = \varepsilon(s^*) f_{G_\wedge}(\pi_{\Sigma_0}(\phi_-, \varepsilon)) \]
and hence
\[ f_{G_\wedge}(\pi_{\Sigma_0}(\phi_-, \varepsilon')) = f_{G_\wedge}(\pi_{\Sigma_0}(\phi_-, \varepsilon)). \]
This implies \( \pi_{\Sigma_0}(\phi_-, \varepsilon') = \pi_{\Sigma_0}(\phi_-, \varepsilon), \) so \( \varepsilon = \varepsilon'. \)
\[ \square \]

As a consequence of this lemma, we can extend Proposition 8.1.

**Proposition 9.3.** Suppose \( \phi \in \Phi_2(G) \), and \( \varepsilon \in \hat{S}_{\phi}^{\Sigma_0}. \)

1. If \( \varepsilon(\rho, a) \varepsilon(\rho, a_-) = -1 \), then
\[ \pi_{\Sigma_0}(\phi, \varepsilon) \rightarrow (a - 1)/2, \cdots , (a_- + 3)/2 > \pi_{\Sigma_0}(\phi', \varepsilon') \]
as the unique irreducible subrepresentation, where
\[ \text{Jord}(\phi') = \text{Jord}(\phi) \cup \{(\rho, a_- + 2)\} \backslash \{(\rho, a)\}, \]
and
\[ \varepsilon'(\cdot) = \varepsilon(\cdot) \text{ over } \text{Jord}(\phi) \backslash \{(\rho, a)\}, \quad \varepsilon'(\rho, a_- + 2) = \varepsilon(\rho, a). \]

2. If \( \varepsilon(\rho, a) \varepsilon(\rho, a_-) = 1 \), then
\[ \pi_{\Sigma_0}(\phi, \varepsilon) \rightarrow (a - 1)/2, \cdots , -(a_- - 1)/2 > \pi_{\Sigma_0}(\phi', \varepsilon'), \]
where
\[ \text{Jord}(\phi') = \text{Jord}(\phi) \backslash \{(\rho, a), (\rho, a_-)\}, \]
and \( \varepsilon'(\cdot) \) is the restriction of \( \varepsilon(\cdot) \). In particular, suppose \( \varepsilon_1 \in \hat{S}_{\phi}^{\Sigma_0} \) satisfying \( \varepsilon_1(\cdot) = \varepsilon(\cdot) \text{ over } \text{Jord}(\phi') \) and
\[ \varepsilon_1(\rho, a) = -\varepsilon(\rho, a), \quad \varepsilon_1(\rho, a_-) = -\varepsilon(\rho, a_-). \]
Then the induced representation in \( \mathfrak{S}_2 \) two irreducible subrepresentations, namely
\[ \pi_{\Sigma_0}(\phi, \varepsilon) \oplus \pi_{\Sigma_0}(\phi, \varepsilon_1). \]

3. If \( \varepsilon(\rho, a_{\text{min}}) = 1 \) and \( a_{\text{min}} \) is even, then
\[ \pi_{\Sigma_0}(\phi, \varepsilon) \rightarrow (a_{\text{min}} - 1)/2, \cdots , 1/2 > \pi_{\Sigma_0}(\phi', \varepsilon') \]
as the unique irreducible subrepresentation, where
\[ \text{Jord}(\phi') = \text{Jord}(\phi) \backslash \{(\rho, a_{\text{min}})\}, \]
and \( \varepsilon'(\cdot) \) is the restriction of \( \varepsilon(\cdot) \).

The proof of this proposition is almost the same as Proposition 8.1 so we omit it here.
10. Classification of discrete series

Now we want to characterize the irreducible discrete series representations of \(G^{\Sigma_0}\) in terms of their cuspidal supports. For any irreducible discrete series representation \(\pi^{\Sigma_0}_c(\phi, \varepsilon)\) of \(G^\Sigma\), we can associate a triple \((\text{Jord}, \pi^{\Sigma_0}_c, \Delta)\). Here \(\text{Jord} = \text{Jord}(\phi)\) and \(\pi^{\Sigma_0}_c\) is a supercuspidal representation of \(G^{\Sigma_0}\) which is in the cuspidal support of \(\pi^{\Sigma_0}_c\). Let us assume \(\pi^{\Sigma_0}_c = \pi^{\Sigma_0}_c(\phi_{\text{cusp}}, \varepsilon_{\text{cusp}})\). Finally, \(\Delta\) is a \(\mathbb{Z}_2\)-valued function defined on a subset of

\[\text{Jord} \sqcup (\text{Jord} \times \text{Jord}),\]

i.e., \(\Delta\) is not defined on \((\rho, a) \in \text{Jord}\) with \(a\) being odd and \(\text{Jord}_\rho(\phi_{\text{cusp}}) \neq \emptyset\); \(\Delta\) is not defined on pairs \((\rho, a), (\rho', a') \in \text{Jord}\) with \(\rho \neq \rho'\). Moreover, we require \(\Delta\) to satisfy the following properties:

1. \(\Delta(\rho, a)\Delta(\rho, a')^{-1} = \Delta(a, \rho; a, a')\),
2. \(\Delta(a, \rho; \rho', a') = \Delta(a, a'; \rho, \rho)\),
3. \(\Delta(a, a; \rho, \rho') = \Delta(a, a'; \rho, \rho)\).

In our case, we can define

\[\Delta(a, a; \rho, \rho') = \varepsilon(\rho, a)\]

for \((\rho, a) \in \text{Jord}\) with \(a\) being even and \(\text{Jord}_\rho(\phi_{\text{cusp}}) = \emptyset\); and

\[\Delta(a, a; \rho', a') = \varepsilon(\rho, a)\varepsilon(\rho', a')^{-1}\]

for \((\rho, a), (\rho', a') \in \text{Jord}\) with \(\rho = \rho'\); otherwise \(\Delta\) is not defined.

In general, we can consider all triples \((\text{Jord}, \pi^{\Sigma_0}_c, \Delta)\) such that \(\text{Jord} = \text{Jord}(\phi)\) for some \(\phi \in \hat{\Phi}_2(G)\), \(\pi^{\Sigma_0}_c\) is some supercuspidal representation of \(G^{\Sigma_0}\) which is of the same type as \(G^{\Sigma_0}\), and \(\Delta\) satisfies the property that we have mentioned above. Let \(\text{Jord}_\rho = \text{Jord}_\rho(\phi)\). Next we will introduce the concept of admissibility for such pairs. Let

\[\text{Jord}_\rho^+(\phi_{\text{cusp}}) = \begin{cases} \text{Jord}_\rho(\phi_{\text{cusp}}) \cup \{0\}, & \text{if } a_{\min} = \min \text{Jord}_\rho \text{ is even and } \Delta(\rho, a_{\min}) = 1, \\ \text{Jord}_\rho(\phi_{\text{cusp}}), & \text{otherwise}. \end{cases}\]

Then \((\text{Jord}, \pi^{\Sigma_0}_c, \Delta)\) is called an admissible triple of alternated type if

1. \(\Delta(\rho, a; \rho, a_{\min}) = -1\), if \(a_{\min}\) is the biggest positive integer smaller than \(a\) in \(\text{Jord}_\rho\).
2. \(|\text{Jord}_\rho^+(\phi_{\text{cusp}})| = |\text{Jord}_\rho|\).

We say \((\text{Jord}', \pi^{\Sigma_0}_c, \Delta')\) is subordinated to \((\text{Jord}, \pi^{\Sigma_0}_c, \Delta)\) if \(\text{Jord}' = \text{Jord} \setminus \{a, a_{\min}\}\), where \(\Delta(\rho, a; \rho, a_{\min}) = 1\), and \(\Delta'\) is the restriction of \(\Delta\). Then \((\text{Jord}, \pi^{\Sigma_0}_c, \Delta)\) is called an admissible triple of mixed type if there exists a sequence of triples \((\text{Jord}_i, \pi^{\Sigma_0}_c, \Delta_i)\) for \(1 \leq i \leq k\) such that

1. \((\text{Jord}, \pi^{\Sigma_0}_c, \Delta) = (\text{Jord}_1, \pi^{\Sigma_0}_c, \Delta_1)\),
2. \((\text{Jord}_{i+1}, \pi^{\Sigma_0}_c, \Delta_{i+1})\) is subordinated to \((\text{Jord}_i, \pi^{\Sigma_0}_c, \Delta_i)\) for \(1 \leq i \leq k - 1\),
3. \((\text{Jord}_k, \pi^{\Sigma_0}_c, \Delta_k)\) is an admissible triple of alternated type.

It follows from Theorem 3.3 and Proposition 9.3 that the triples we associate with irreducible discrete series representations are admissible. On the other hand, from any admissible triple \((\text{Jord}, \pi^{\Sigma_0}_c, \Delta)\) with \(\text{Jord} = \text{Jord}(\phi)\) for some \(\phi \in \hat{\Phi}_2(G)\) and \(\pi^{\Sigma_0}_c = \pi^{\Sigma_0}_c(\phi_{\text{cusp}}, \varepsilon_{\text{cusp}})\), we can always extend \(\varepsilon_{\text{cusp}}(\cdot)\) in a unique way to \(\varepsilon(\cdot) \in S_{\phi^{\Sigma_0}}\) such that the triple is associated with \(\pi^{\Sigma_0}_c(\phi, \varepsilon)\). Therefore we have shown the following theorem due to Mœglin and Tadić.

**Theorem 10.1.** There is a one to one correspondence between irreducible discrete series representations of \(G^{\Sigma_0}\) and admissible triples \((\text{Jord}, \pi^{\Sigma_0}_c, \Delta)\).

One can also see how to construct irreducible discrete series representations from admissible triples according to Proposition 9.3 If \((\text{Jord}, \pi^{\Sigma_0}_c, \Delta)\) is an admissible triple of alternated type, let

\[l_\rho : \text{Jord}_\rho \rightarrow \text{Jord}_\rho^+(\phi_{\text{cusp}})\]
be the monotone bijection. Then the corresponding irreducible discrete series representation \( \pi^{\Sigma_0} \) can be viewed as the unique irreducible subrepresentation of

\[
\left( \prod_{\rho} \left( \prod_{a \in \text{Jord}_\rho} (a - 1)/2, \ldots, (l_\rho(a) + 1)/2 \right) \right) \times \pi^{\Sigma_0}_{\text{cusp}},
\]

where the product over \( \text{Jord}_\rho \) is in the increasing order. If \((\text{Jord}, \pi^{\Sigma_0}_{\text{cusp}}, \Delta)\) is an admissible triple of mixed type, we can assume \((\text{Jord}', \pi^{\Sigma_0}_{\text{cusp}}, \Delta')\) is subordinated to \((\text{Jord}, \pi^{\Sigma_0}_{\text{cusp}}, \Delta)\), where \(\text{Jord}' = \text{Jord}_\rho \setminus \{a, a_-\}\).

Suppose \(\pi^{\Sigma_0} \) corresponds to \((\text{Jord}', \pi^{\Sigma_0}_{\text{cusp}}, \Delta')\), then

\[
(a - 1)/2, \ldots, -(a_- - 1)/2 > \times \pi^{\Sigma_0}
\]

has two irreducible subrepresentations, and one will correspond to \(\pi^{\Sigma_0} \) while the other corresponds to the other extension of \(\Delta'\) to \(Jord\).

11. REMARKS ON THE ORIGINAL APPROACH OF Mœglin and TADIĆ

The original approach of Möglin and Tadić to Theorem 10.1 does not depend on Arthur’s theory, i.e., Theorem 2.2 and 4.1 So the first immediate question becomes how to associate a set of Jordan blocks to every irreducible discrete series representation of \(G^{\Sigma_0}\) without assuming Arthur’s theory. The answer can be motivated by the following result due to Arthur.

**Theorem 11.1.** Suppose \(\pi^{\Sigma_0}\) is an irreducible discrete series representation of \(G^{\Sigma_0}\), and \(\pi^{\Sigma_0} \in \Pi_\phi \Sigma_0\) for some \(\phi \in \hat{\Phi}_2(G)\). Then for any self-dual irreducible supercuspidal representation \(\rho\) of \(GL(d_\rho)\) and positive integer \(a\), \((\rho, a) \in \text{Jord}(\phi)\) if and only if \((\rho, a)\) is of the same type as \(\hat{G}\), and

\[
(11.1) \quad \text{St}(\rho, a) \times \pi^{\Sigma_0}
\]

is irreducible.

It is clear from this theorem that we can associate every irreducible discrete series representation \(\pi^{\Sigma_0}\) of \(G^{\Sigma_0}\) with a set \(\text{Jord}(\pi^{\Sigma_0})\) of Jordan blocks as follows,

\[
\text{Jord}(\pi^{\Sigma_0}) := \{ (\rho, a) \text{ of the same type as } \hat{G} : \rho \text{ is self-dual supercuspidal, } a \in \mathbb{Z}_{>0} \text{ and } (11.1) \text{ is irreducible} \}.
\]

The next question is about the construction of \(\mathbb{Z}_2\)-valued function \(\Delta\) (see Section 10). In [Mœg02], Möglin defines \(\Delta\) over a subset of

\[
\text{Jord}(\pi^{\Sigma_0}) \sqcup (\text{Jord}(\pi^{\Sigma_0}) \times \text{Jord}(\pi^{\Sigma_0})),
\]

i.e., \(\Delta\) is not defined on \((\rho, a) \in \text{Jord}(\pi^{\Sigma_0})\) with \(a\) being odd and \(\text{Jord}_\rho(\pi^{\Sigma_0}_{\text{cusp}}) \neq \emptyset\); \(\Delta\) is not defined on pairs \((\rho, a), (\rho', a') \in \text{Jord}(\pi^{\Sigma_0}_{\text{cusp}})\) with \(\rho \neq \rho'\). Moreover, \(\Delta\) satisfies those properties that we have described in Section 10. Here we will only mention how to define \(\Delta\) for pairs \((\rho, a), (\rho, a_-) \in \text{Jord}(\pi^{\Sigma_0}_{\text{cusp}})\), where \(a_-\) is the biggest positive integer in \(\text{Jord}_\rho(\pi^{\Sigma_0})\) that is smaller than \(a\), and also for \((\rho, a_{\text{min}}) \in \text{Jord}(\pi^{\Sigma_0})\) with \(a_{\text{min}} = \min \text{Jord}_\rho(\pi^{\Sigma_0})\) even. In view of Proposition 9.3, this definition is given in the reversed way, i.e.,

1. \(\Delta(\rho, a; \rho, a_-) = 1\) if and only if

\[
\pi^{\Sigma_0} \leftrightarrow (a - 1)/2, \ldots, (a_- + 1)/2 > \times \pi^{\Sigma_0}_{\text{cusp}}
\]

for some irreducible representation \(\pi^{\Sigma_0}_{\text{cusp}}\) of \(G^{\Sigma_0}_{\text{cusp}}\).

2. When \(a_{\text{min}}\) is even, \(\Delta(\rho, a_{\text{min}}) = 1\) if and only if

\[
\pi^{\Sigma_0} \leftrightarrow (a_{\text{min}} - 1)/2, \ldots, 1/2 > \times \pi^{\Sigma_0}_{\text{cusp}}
\]

for some irreducible representation \(\pi^{\Sigma_0}_{\text{cusp}}\) of \(G^{\Sigma_0}_{\text{cusp}}\).

At last, for \(G(n)\) we let \(N = 2n + 1\) if \(G\) is symplectic, and \(N = 2n\) otherwise. Then Möglin proved the following dimension equality.
Theorem 11.2 (Mœglin [Mœg14]). Suppose \( \pi^{\Sigma_0} \) is a discrete series representation of \( G^{\Sigma_0}(n) \), then

\[
\sum_{(\rho,a) \in \text{Jord}(\pi^{\Sigma_0})} ad_{\rho} = N.
\]

This theorem becomes trivial if we know Theorem 2.2 and identify \( \text{Jord}(\pi^{\Sigma_0}) = \text{Jord}(\phi) \) under Theorem 11.1. But without assuming all these results of Arthur, this theorem is far from being obvious.

Appendix A. Local L-function

In this appendix, we give explicit formulas for three different types of local L-functions, i.e., Rankin-Selberg L-function, symmetric square L-function and skew symmetric square L-function. Let \( F \) be a \( p \)-adic field, and \( q \) be the number of elements in the residue class field of \( F \).

A.1. Rankin-Selberg L-function. We follow [JPSSS83] here. Let \( \pi \) be an irreducible admissible representation of \( GL(n) \) and \( \sigma \) be an irreducible admissible representation of \( GL(m) \), the local Rankin-Selberg L-function is denoted by \( L(s, \pi \times \sigma) \) for \( s \in \mathbb{C} \). It satisfies \( L(s, \pi \times \sigma) = L(s, \sigma \times \pi) \) and \( L(s, \pi||^t \times \sigma) = L(s + t, \pi \times \sigma) \).

Cuspidal case:
Suppose both \( \pi \) and \( \sigma \) are unitary supercuspidal representations.

1. If \( n \neq m \), then \( L(s, \pi \times \sigma) = 1 \);
2. If \( n = m \), then

\[
L(s, \pi \times \sigma) = \prod_t (1 - q^{-(s+it)})^{-1}
\]

where the product is over all real numbers \( t \) such that \( \pi||^t \cong \sigma^\vee \).

Discrete series case:
We assume \( \pi \) is \( St(\rho, a) \) for an irreducible unitary supercuspidal representations \( \rho \) and integer \( a \). Similarly we assume \( \sigma \) is \( St(\rho', b) \). If \( n \geq m \), then

\[
L(s, \pi \times \sigma) = \prod_{i=1}^{b} L(s + \frac{a + b}{2} - i, \rho \times \rho').
\]

Tempered case:
Suppose \( \pi = \pi_1 \times \cdots \times \pi_l \) and \( \sigma = \sigma_1 \times \cdots \times \sigma_k \), where \( \pi_i, \sigma_j \) are discrete series representations. Then

\[
L(s, \pi \times \sigma) = \prod_{i,j} L(s, \pi_i \times \sigma_j).
\]

Non-tempered case:
Let \( \pi \) be the Langlands quotient of the induced representation \( \Pi = \pi_1||^{u_1} \times \cdots \times \pi_l||^{u_l} \) for tempered representation \( \pi_i \) and real numbers \( u_1 > \cdots > u_l \). Let \( \sigma \) be the Langlands quotient of the induced representation \( \Sigma = \sigma_1||^{v_1} \times \cdots \times \sigma_k||^{v_k} \) for tempered representation \( \sigma_j \) and real numbers \( v_1 > \cdots > v_k \). Then

\[
L(s, \pi \times \sigma) = L(s, \Pi \times \Sigma) = \prod_{i,j} L(s + u_i + v_j, \pi_i \times \sigma_j).
\]

A.2. Symmetric square and skew-symmetric square L-functions. We follow [Sha92] here. Let \( \pi \) be an irreducible admissible representation of \( GL(n) \). The symmetric square (resp. skew-symmetric square) L-function is denoted by \( L(s, \pi, S^2) \) (resp. \( L(s, \pi, \wedge^2) \)). We have \( L(s, \pi \times \pi) = L(s, \pi, S^2) L(s, \pi, \wedge^2) \), and \( L(s, \pi||^t, R) = L(s + 2t, \pi, R) \) for \( R = S^2 \) or \( \wedge^2 \).

Cuspidal case
Suppose \( \pi \) is a unitary supercuspidal representation of \( GL(n) \).


(1) $L(s, \pi, \wedge^2) = 1$ unless $n$ is even and some unramified twist of $\pi$ is self-dual. So let us suppose $n$ is even and $\pi$ is self-dual. Let $S$ be the set of real numbers $t$ modulo $\frac{\pi}{\ln q} \mathbb{Z}$, such that

$$
\int_{Sp_n(F) \backslash GL_n(F)} f(tgw^{-1}gw)dg \neq 0,$$

for some $f \in C_\infty_c(GL_n(F))$ defining a matrix coefficient of $\pi||^{it}$. Here $^tg$ is the transpose of $g$ and

$$w = \begin{pmatrix} 0 & -1 & 1 \\ 1 & \ddots & \vdots \\ -1 & 0 \end{pmatrix}.$$ (A.1)

Then

$$L(s, \pi, \wedge^2) = \prod_{t \in S} (1 - q^{-(s+2it)})^{-1}.$$

(2) $L(s, \pi, S^2) = 1$ unless some unramified twist of $\pi$ is self-dual. So let us suppose $\pi$ is self-dual.

(a) If $n$ is odd, then

$$L(s, \pi, S^2) = (1 - q^{-rs})^{-1},$$

where $r$ is the maximal integer such that $\pi \cong \pi||^{2\pi i/(r \ln q)}$.

(b) If $n$ is even,

$$L(s, \pi, S^2) = \prod_{t \in S'} (1 - q^{-(s+2it)})^{-1},$$

where $S'$ is the set of real numbers $t$ modulo $\frac{\pi}{\ln q} \mathbb{Z}$ such that $\pi||^{2it} \cong \pi$ and for any $f \in C_\infty_c(GL_n(F))$ defining a matrix coefficient of $\pi||^{it}$, one has

$$\int_{Sp_n(F) \backslash GL_n(F)} f(tgw^{-1}gw)dg = 0.$$

Here $w$ is again given by (A.1) and $^tg$ is the transpose of $g$.

**Discrete series case:**

We assume $\pi$ is $St(\rho, a)$ for an irreducible unitary supercuspidal representations $\rho$ and integer $a$. Set $\pi_i = \rho||^{(a+1)/2-i}$ for $1 \leq i \leq a$.

(1) Suppose $a$ is even, then

$$L(s, \pi, \wedge^2) = \prod_{i=1}^{a/2} L(s, \pi_i, \wedge^2)L(s, \pi_i||^{-1/2}, S^2),$$

$$L(s, \pi, S^2) = \prod_{i=1}^{a/2} L(s, \pi_i, S^2)L(s, \pi_i||^{-1/2}, \wedge^2).$$

(2) Suppose $a$ is odd, then

$$L(s, \pi, \wedge^2) = \prod_{i=1}^{(a+1)/2} L(s, \pi_i, \wedge^2) \prod_{i=1}^{(a-1)/2} L(s, \pi_i||^{-1/2}, S^2),$$

$$L(s, \pi, S^2) = \prod_{i=1}^{(a+1)/2} L(s, \pi_i, S^2) \prod_{i=1}^{(a-1)/2} L(s, \pi_i||^{-1/2}, \wedge^2).$$
Tempered case:
Suppose \( \pi = \pi_1 \times \cdots \times \pi_l \), where \( \pi_i \) are discrete series representations. Then
\[
L(s, \pi, \wedge^2) = \prod_{i=1}^{l} L(s, \pi_i, \wedge^2) \prod_{1 \leq i < j \leq l} L(s, \pi_i \times \pi_j),
\]
\[
L(s, \pi, S^2) = \prod_{i=1}^{l} L(s, \pi_i, S^2) \prod_{1 \leq i < j \leq l} L(s, \pi_i \times \pi_j).
\]

Non-tempered case:
Let \( \pi \) be the Langlands quotient of the induced representation \( \Pi = \pi_1 |^{| u_1 \times \cdots \times \pi_l |^{| u_l} \) for tempered representation \( \pi_i \) and real numbers \( u_1 \succ \cdots \succ u_l \). Then
\[
L(s, \pi, \wedge^2) = L(s, \Pi, \wedge^2) = \prod_{i=1}^{l} L(s + 2u_i, \pi_i, \wedge^2) \prod_{1 \leq i < j \leq l} L(s + u_i + u_j, \pi_i \times \pi_j),
\]
\[
L(s, \pi, S^2) = L(s, \Pi, S^2) = \prod_{i=1}^{l} L(s + 2u_i, \pi_i, S^2) \prod_{1 \leq i < j \leq l} L(s + u_i + u_j, \pi_i \times \pi_j).
\]

Appendix B. Reducibility for Some Induced Representations of \( GL(n) \)

We define a segment to be a finite length arithmetic progression of real numbers with common difference 1 or \(-1\), it is completely determined by its endpoints \( x, y \), and hence we denote a segment by \( [x, y] \) or \( \{x, \cdots, y\} \). Let \( \rho \) be a unitary irreducible supercuspidal representation of \( GL(d_\rho) \). The normalized induction
\[
\rho|^x \times \cdots \times \rho|^y
\]
has a unique irreducible subrepresentation, which is denoted by \( < \rho; x, \cdots, y >= \). If \( x \succ y \), this is called Steinberg representation; if \( x < y \), this is called Speh representation.

For any two segments \( [x, y] \) and \( [x', y'] \) such that \( (x - y)(x' - y') \geq 0 \), we say they are linked if as sets \( [x, y] \not\subseteq [x', y'], [x', y'] \not\subseteq [x, y] \), and \( [x, y] \cup [x', y'] \) can form a segment after imposing the same order. The following theorem is fundamental in determining the reducibility of an induced representation of \( GL(n) \).

Theorem B.1 (Zelevinsky \[Zel80\]). For unitary irreducible supercuspidal representations \( \rho, \rho' \) of general linear groups, and segments \( [x, y], [x', y'] \) such that \( (x - y)(x' - y') \geq 0 \),
\[
< \rho; x, \cdots, y > \times < \rho'; x', \cdots, y' >
\]
is reducible if and only if \( \rho \cong \rho' \) and \([x, y], [x', y'] \) are linked. In case it is reducible, it consists of the unique irreducible subrepresentations of
\[
< x, \cdots, y > \times < x', \cdots, y' > \quad \text{and} \quad < x', \cdots, y' > \times < x, \cdots, y > .
\]

Remark B.2. In fact, Zelevinsky proved this theorem only when both \( x - y \geq 0 \) and \( x' - y' \geq 0 \). Nonetheless, the Aubert duality convolution functor on the Grothendieck group of finite length representations of \( G \) will send
\[
< \rho; x, \cdots, y > \times < \rho'; x', \cdots, y' > \quad \text{to} \quad < \rho; y, \cdots, x > \times < \rho'; y', \cdots, x' >
\]
up to a sign, and it preserves irreducibility (see \[Aub95\]). So one can easily extend the result of Zelevinsky to this theorem.

It is natural to ask for the notion of “link” for two segments \( [x, y] \) and \( [x', y'] \) such that \( (x - y)(x' - y') < 0 \). To do so, we need to first generalize the notion of “segment”. We define a generalized segment to be a matrix
\[
\begin{bmatrix}
x_{11} & \cdots & x_{1n} \\
\vdots & & \vdots \\
x_{m1} & \cdots & x_{mn}
\end{bmatrix}
\]
such that each row is a decreasing (resp. increasing) segment and each column is an increasing (resp. decreasing) segment. The normalized induction
\[ x_{i\in [1,n]} < \rho; x_{i1}, \cdots, x_{in} > \]
has a unique irreducible subrepresentation, and we denote it by \(< \rho; \{x_{ij}\}_{m \times n} >\). Moreover,
\[ < \rho; \{x_{ij}\}_{m \times n} > \cong < \rho; \{x_{ij}\}_{T_{m \times n}} > \]
where \(\{x_{ij}\}_{T_{m \times n}}\) is the transpose of \(\{x_{ij}\}_{m \times n}\).

For any two generalized segments \(\{x_{ij}\}_{m \times n}\) and \(\{y_{ij}\}_{m' \times n'}\) with the same monotone properties for the rows and columns, we say they are linked if \([x_{m1}, x_{1n}], [y_{m'1}, y_{1n'}]\) are linked, and the four sides of the rectangle formed by \(\{x_{ij}\}_{m \times n}\) do not have inclusion relations with the corresponding four sides of the rectangle formed by \(\{y_{ij}\}_{m' \times n'}\) (e.g., \([x_{11}, x_{1n}] \not\subseteq [y_{11}, y_{1n'}]\) and \([x_{11}, x_{1n}] \not\supset [y_{11}, y_{1n'}]\), etc). It is easy to check that if \(\{x_{ij}\}_{m \times n}\) and \(\{y_{ij}\}_{m' \times n'}\) are linked, then \(\{x_{ij}\}_{T_{m \times n}}\) and \(\{y_{ij}\}_{T_{m' \times n'}}\) are also linked. So for generalized segments \(\{x_{ij}\}_{m \times n}\) and \(\{y_{ij}\}_{m' \times n'}\) with different monotone properties for the rows and columns, we say they are linked if \(\{x_{ij}\}_{T_{m \times n}}\) and \(\{y_{ij}\}_{T_{m' \times n'}}\) are linked, or equivalently \(\{x_{ij}\}_{m \times n}\) and \(\{y_{ij}\}_{m' \times n'}\) are linked. One can check this notion of “link” is equivalent to the one in [MWS9].

**Example B.3.** For any two segments \([x, y]\) and \([x', y']\) such that \((x - y)(x' - y') < 0\), we can view them as generalized segments by taking them as rows, and note they have different monotone properties. So we take
\[ [x, y]^T = \begin{bmatrix} x \\ \vdots \\ y \end{bmatrix} \quad \text{and} \quad [x', y'] = [x' \cdots y'] . \]

It follows that \([x, y]\) and \([x', y']\) are linked if and only if \([y, x], [x', y']\) are linked, and \(x, y \not\in [x', y']\) and \(x', y' \not\in [x, y]\).

The next theorem generalizes Theorem B.1 to the case of generalized segments.

**Theorem B.4** (Mœglin-Waldspurger [MWS9]). For unitary irreducible supercuspidal representations \(\rho, \rho'\) of general linear groups, and generalized segments \(\{x_{ij}\}_{m \times n}, \{y_{ij}\}_{m' \times n'}\),
\[ < \rho; \{x_{ij}\}_{m \times n} > \times < \rho; \{y_{ij}\}_{m' \times n'} > \]
is irreducible unless \(\rho \cong \rho'\) and \(\{x_{ij}\}_{m \times n}, \{y_{ij}\}_{m' \times n'}\) are linked.

Let \(a, b\) be integers, we define \(Sp(St(\rho, a), b)\) to be the unique irreducible subrepresentation of
\[ St(\rho, a)^{(b-1)/2} \times St(\rho, a)^{(b-3)/2} \times \cdots \times St(\rho, a)^{-(b-1)/2} . \]
By the definition one can see \(Sp(St(\rho, a), b)\) is given by the following generalized segment
\[ \begin{bmatrix} (a - b)/2 & \cdots & 1 - (a + b)/2 \\ \vdots \\ (a + b)/2 - 1 & \cdots & -(a - b)/2 \end{bmatrix} \]
The following result is a reinterpretation of Theorem B.4.

**Corollary B.5.** For unitary irreducible supercuspidal representations \(\rho, \rho'\) of general linear groups, and integers \(a, b, a', b'\), and real number \(s\),
\[ Sp(St(\rho, a), b)^{s} \times Sp(St(\rho, a'), b') \]
is irreducible unless \(\rho \cong \rho'\), \((a + b + a' + b')/2 + s\) is an integer and
\[ |(a - a')/2| + |(b - b')/2| < |s| \leq |(a + a' + b + b')/2| - 1. \]
Appendix C. Casselman’s Formula and Application

Let $F$ be a $p$-adic field, and $G$ be a quasisplit connected reductive group over $F$. Let $\theta$ be an automorphism of $G$ preserving an $F$-splitting, and we assume $\theta$ has order $l$. Suppose $\pi$ is an irreducible admissible representation of $G$ such that $\pi \cong \pi^\theta$. Let $A_\pi(\theta)$ be an intertwining operator between $\pi$ and $\pi^\theta$, then we can define the twisted character of $\pi$ to be

$$f_{G^\theta}(\pi) := \text{trace} \int_G f(g)\pi(g)dg \circ A_\pi(\theta)$$

for $f \in C_c^\infty(G)$. It follows from results of Harish-Chandra [HC99] in the non-twisted case and Lemaire [Lem13] in the twisted case, that there exists a locally integrable function $\Theta^G_{\pi^\theta}$ on $G$ such that

$$f_{G^\theta}(\pi) = \int_G f(g)\Theta^G_{\pi^\theta}(g)dg.$$

We also call this function the twisted character of $\pi$. If $P = MN$ is a $\theta$-stable parabolic subgroup of $G$, we denote by $\pi_N$ the unnormalized Jacquet module of $\pi$ with respect to $P$, then $\text{Jac}P\pi = \pi_N \otimes \delta_P^{-1/2}$, where $\delta_P$ is the usual modulus character. Note $\pi_N \cong \pi^\theta_N$ and $A_\pi(\theta)$ induces an intertwining operator on $\pi_N$.

For a strongly $\theta$-regular $\theta$-semisimple element $g$ in $G$, let $h = \prod_{i=1}^l \theta^i(g)$, and one can associate it with a $\theta$-stable parabolic subgroup $P_h = M_hN_h$ by the construction in Casselman’s formula.

Theorem C.1 ([Cas77], Theorem 5.2). Suppose $\pi$ is an irreducible admissible representation of $G$ such that $\pi \cong \pi^\theta$, and $g$ is a strongly $\theta$-regular $\theta$-semisimple element in $G$. Let $h = \prod_{i=1}^l \theta^i(g)$. Then

$$\Theta^G_{\pi^\theta}(g) = \Theta^M_{\pi_N^\theta}(g).$$

Casselman proved this theorem only for $\theta = id$, but one can extend his proof to the twisted case without difficulty. What we are going to use is the following corollary of this theorem.

Corollary C.2. Let $P = MN$ be a $\theta$-stable parabolic subgroup of $G$. Suppose $\pi$ is an irreducible admissible representation of $G$ such that $\pi \cong \pi^\theta$, and $m$ is a strongly $\theta$-regular $\theta$-semisimple element in $G$, which is also contained in $M$. Then one can choose $z_M \in A_M$ (the maximal split component in the centre of $M$) with $|\alpha(z_M)|$ sufficiently small (depending on $m$) for all roots $\alpha$ in $N$, such that

$$\Theta^G_{\pi^\theta}(z_M m) = \Theta^M_{\pi_N^\theta}(z_M m).$$

Proof. Let $g = z_M m$, and $h = \prod_{i=1}^l \theta^i(g)$. It is not hard to check from the definition of $P_h$ that $P \supseteq P_h$. Let $P_h^M = M \cap P_h$, it is the parabolic subgroup of $M$ associated with $h$, and it has Levi component $M_h$. Let $N_h^M = M \cap N_h$. Then

$$\Theta^G_{\pi^\theta}(g) = \Theta^M_{\pi_N^\theta}(g) = \Theta^M_{\pi_N^\theta}(g) = \Theta^M_{\pi_N^\theta}(g).$$

This finishes the proof.

As an application of this corollary, we are going to establish the diagrams (6.1) and (6.5). First, let us recall the general setup of these diagrams. Let $H$ be a twisted endoscopic group of $G$, and we assume there is an embedding

$$\xi: L \to L \tilde{G},$$

and $\xi(LH) \subseteq \text{Cent}(s, L \tilde{G})$ and $\tilde{H} \cong \text{Cent}(s, \tilde{G})^0$ for some $s \in \tilde{G} \ltimes \tilde{\theta}$. We fix $(\tilde{\theta}^\text{-stable}) \Gamma_F$-splittings $(B_H, T_H, \{X_{\alpha}H\})$ and $(\tilde{B}_G, \tilde{T}_G, \{X_{\alpha}\})$ for $H$ and $\tilde{G}$ respectively. By taking certain $\tilde{G}$-conjugate of $\xi$, we can assume $s \in T_G \ltimes \tilde{\theta}$ and $\xi(T_H) = (T^0_H)$ and $\xi(B_H) \subseteq B_G$. Let $W_H = W(\tilde{H}, T_H)$ and $W_{G^\theta} = W(\tilde{G}, T_G)^\tilde{\theta}$, then $W_H$ can be viewed as a subgroup of $W_{G^\theta}$ through $\xi$. Let $S$ be a $\Gamma_F$-invariant torus in $T_H$, it gives a Levi subgroup $M$ of $G$, where $\tilde{M} := \text{Cent}(\xi(S), \tilde{G})$. We also denote the $W_{G^\theta}$-conjugacy class of $S$ in $T_H$ by $\{S\}_G$, then each $W_H$-conjugacy class $\{S\}_H$ in $\{S\}_G$ corresponds to an $H$-conjugacy class of Levi...
subgroup $M' = M(S')$ of $H$, where $\hat{M}' := \text{Cent}(S', \hat{H})$. And $M'$ are endoscopic groups of $M$. We fix a $\theta$-stable parabolic subgroup $P \supseteq M$ with an embedding $L_P \hookrightarrow L_G$, which extends $L_M \hookrightarrow L_G$. Then the embedding $\xi_{M'}: L_M' \hookrightarrow L_M$ can be given by any element $g_0 \in \hat{G}$ such that $\text{Int}(g_0)(\xi(S')) = \xi(S)$, i.e., the following diagram commutes

$$
\begin{array}{ccc}
L_M' & \xrightarrow{\xi_{M'}} & L_M \\
\downarrow & & \downarrow \\
L_H & \xrightarrow{\xi} & L_G \\
\end{array}
L_P \hookrightarrow L_G \xrightarrow{\text{Int}(g_0)} L_G.
$$

We denote the set of all such embeddings by $\{\xi_{M'}\}$. For $(g, h) \in \text{Norm}(\xi(S), \hat{G}) \times \text{Norm}(S', \hat{H})$, we define another embedding $(g, h) \ast \xi_{M'}$ by changing $g_0$ to $g_0g\xi(h)$. In this way, we get a transitional action of $\text{Norm}(\xi(S), \hat{G}) \times \text{Norm}(S', \hat{H})$ on $\{\xi_{M'}\}$. For each $\xi' = (g, h) \ast \xi_{M'} \in \{\xi_{M'}\}$, we can associate it with a parabolic subgroup $P' \supseteq M'$ such that $\xi(P') = \text{Int}(g_0g\xi(h))^{-1}(P) \cap \xi(H)$. Then we claim the following diagram commutes

$$(C.1)
\begin{array}{ccc}
\hat{SI}(H) & \overrightarrow{\oplus_{(s', \xi')} \text{Jac}_{P'}} & \hat{I}(G^\theta) \\
\downarrow & \downarrow \text{Jac}_P & \\
\oplus_{(s', \xi')} \hat{SI}(M(S')) & \rightarrow & \hat{I}(M^\theta).
\end{array}
$$

Here the sum is over all $W_H$-conjugacy classes $\{S'\}_H$ in $\{S\}_G$, and $\hat{M} \times \text{Norm}(S', \hat{H})$-orbits $\{\xi'\} \subseteq \{\xi_{M'}\}$. And the horizontal maps correspond to spectral endoscopic transfers with respect to $\xi$ on the top and $\xi'$ on the bottom.

To apply Corollary [C2], we need to give another description of the spectral endoscopic transfer. With respect to the embedding $\xi$, there is a map from the semisimple $\hat{F}$-conjugacy classes of $H$ to the $\theta$-twisted semisimple $\hat{F}$-conjugacy classes of $G$ (see [KS99]). If $\Theta^{G^\theta}$ is a finite linear combination of twisted characters of $G$ and $\Theta^H$ is a stable finite linear combination of characters of $H$, then we say $\Theta^H$ transfers to $\Theta^{G^\theta}$ if for any strongly $\theta$-regular $\theta$-semisimple element $\gamma_G$ in $G$

$$(C.2)\quad \Theta^{G^\theta}(\gamma_G) = \sum_{\gamma_H \mapsto \gamma_G} \frac{D_H(\gamma_H)^2}{D_{G^\theta}(\gamma_G)^2} \Delta_{G, H}(\gamma_H, \gamma_G) \Theta^H(\gamma_H)$$

where the sum is over $\hat{F}$-conjugacy classes of $\gamma_H$ in $H$ that maps to the $\hat{F}$-conjugacy class of $\gamma_G$ in $G$. In this formula $\Delta_{G, H}(\cdot, \cdot)$ is the transfer factor (see [KS99]), and it is built into the transfer map introduced in Section II $D_H(\cdot)$ and $D_{G^\theta}(\cdot)$ are the (twisted) Weyl discriminants. Now we can fix $\gamma_G = \gamma_M$ contained in $M$. If the $\hat{F}$-conjugacy class of $\gamma_H$ in $H$ maps to the $\theta$-twisted $\hat{F}$-conjugacy class of $\gamma_G$ in $G$, then there exists a Levi subgroup $M' = M(S')$ of $H$ and embedding $\xi'_H: L_M' \hookrightarrow L_M$ such that some $\hat{F}$-conjugate $\gamma_{M'}$ of $\gamma_H$ is contained in $M'$ and the $\hat{F}$-conjugacy class of $\gamma_{M'}$ in $M'$ maps to the $\theta$-twisted $\hat{F}$-conjugacy class of $\gamma_M$ in $M$ with respect to $\xi'$. It is not hard to see the $\hat{F}$-conjugacy classes of such $\gamma_H$ in $H$ can be parametrized by $W_H$-conjugacy classes $\{S'\}_H$ in $\{S\}_G$, and $\hat{M} \times \text{Norm}(S', \hat{H})$-orbits $\{\xi'\} \subseteq \{\xi_{M'}\}$. So we can rewrite the right hand side of (C2) as

$$\sum_{S'} \sum_{\xi'} \sum_{\gamma_{M'} \mapsto \gamma_M} \frac{D_H(\gamma_{M'})^2}{D_{G^\theta}(\gamma_M)^2} \Delta_{G, H}(\gamma_{M'}, \gamma_M) \Theta^H(\gamma_{M'}).$$

So the next step is write the summands in terms of $M$ and $M'$. First, it is easy to check from the definition of transfer factors that

$$\frac{D_H(\gamma_{M'})}{D_{G^\theta}(\gamma_M)} \Delta_{G, H}(\gamma_{M'}, \gamma_M) = \frac{D_{M'}(\gamma_{M'})}{D_{M^\theta}(\gamma_M)} \Delta_{M, M'}(\gamma_{M'}, \gamma_M)$$
Secondly, there is a natural homomorphism from $A_M$ to $A_{M'}$ with respect to $\xi'$ (see [KS99]). For $z_M \in A_M$, let $z_{M'}$ be its image in $A_{M'}$, then the $\tilde{F}$-conjugacy class of $z_{M'}\gamma_{M'}$ in $M'$ maps to the $\tilde{F}$-conjugacy class of $z_M\gamma_M$ in $M$. Moreover, $\sup|\alpha'(z_{M'})|$ for roots $\alpha'$ in $N'$ is less than $\sup|\alpha(z_M)|$ for roots $\alpha$ in $N$.

Let us denote by $\Theta^{M_\theta}$ (resp. $\Theta^{M'}$) the (resp. stable) finite linear combination of (twisted) characters of $M$ (resp. $M'$) obtained from $\Theta^{G_\theta}$ (resp. $\Theta^H$) by applying the unnormalized Jacquet modules. If we take $z_M \in A_M$ so that $\sup|\alpha(z_M)|$ is sufficiently small, then by Corollary [C.2]

$$\Theta^{G_\theta}(z_M\gamma_M) = \Theta^{M_\theta}(z_M\gamma_M),$$

and

$$\Theta^{H}(z_{M'}\gamma_{M'}) = \Theta^{M'}(z_{M'}\gamma_{M'}).$$

Besides, it is easy to verify

$$D_{\gamma}(z_M\gamma_M) = \delta_P(z_M\gamma_M)^{-1/2} \cdot D_{M_\theta}(z_M\gamma_M),$$

and

$$D_{H}(z_{M'}\gamma_{M'}) = \delta_P(z_{M'}\gamma_{M'})^{-1/2} \cdot D_{M'}(z_{M'}\gamma_{M'}).$$

Putting all these together, we get

$$\delta_P(z_M\gamma_M)^{-1/2} \Theta^{M_\theta}(z_M\gamma_M) = \sum_{S'} \sum_{\xi'} \sum_{\gamma_{M'} \rightarrow \gamma_M} \frac{D_{M'}(z_{M'}\gamma_{M'})}{D_{M_\theta}(z_M\gamma_M)} \Delta_{M,M'}(z_{M'}\gamma_{M'}, \z_M\gamma_M) \delta_P(z_{M'}\gamma_{M'})^{-1/2} \Theta^{M'}(z_{M'}\gamma_{M'}).$$

Since $z_M, z_{M'}$ are in the centres of $M$ and $M'$ respectively, we have

$$\Delta_{M,M'}(z_{M'}\gamma_{M'}, \z_M\gamma_M) = \chi'(z_M) \Delta_{M,M'}(\gamma_{M'}, \gamma_M),$$

$$\delta_P(z_M\gamma_M)^{-1/2} \Theta^{M_\theta}(z_M\gamma_M) = \zeta_M(z_M) \delta_P(\gamma_M)^{-1/2} \Theta^{M_\theta}(\gamma_M),$$

$$\delta_P(z_{M'}\gamma_{M'})^{-1/2} \Theta^{M'}(z_{M'}\gamma_{M'}) = \zeta_{M'}(z_{M'}) \delta_P(\gamma_{M'})^{-1/2} \Theta^{M'}(\gamma_{M'}),$$

$$D_{M_\theta}(z_M\gamma_M) = D_{M_\theta}(\gamma_M),$$

$$D_{M'}(z_{M'}\gamma_{M'}) = D_{M'}(\gamma_{M'}).$$

where $\chi', \zeta_M, \zeta_{M'}$ are the corresponding central characters. Hence

$$(\delta_P(\gamma_M)^{-1/2} \Theta^{M_\theta}(\gamma_M)) \cdot \zeta_M(z_M) = \sum_{S'} \sum_{\xi'} \sum_{\gamma_{M'} \rightarrow \gamma_M} \frac{D_{M'}(\gamma_{M'})}{D_{M_\theta}(\gamma_M)} \Delta_{M,M'}(\gamma_{M'}, \gamma_M) \delta_P(\gamma_{M'})^{-1/2} \Theta^{H}(\gamma_{M'}) \cdot \chi'(z_M) \zeta_{M'}(z_{M'}).$$

Let

$$a_M = \delta_P(\gamma_M)^{-1/2} \Theta^{M_\theta}(\gamma_M),$$

and

$$b_{S', \xi'} = \frac{D_{M'}(\gamma_{M'})}{D_{M_\theta}(\gamma_M)} \Delta_{M,M'}(\gamma_{M'}, \gamma_M) \delta_P(\gamma_{M'})^{-1/2} \Theta^{H}(\gamma_{M'}).$$

We also write $\chi_M(z_M) = \zeta_M(z_M)$ and $\chi_{S', \xi'}(z_M) = \chi'(z_M) \zeta_{M'}(z_{M'})$. Then we can get a short expression

$$a_M \cdot \chi_M(z_M) = \sum_{S'} \sum_{\xi'} b_{S', \xi'} \cdot \chi_{S', \xi'}(z_M),$$

and it suffices for us to show this holds when $z_M = 1$, i.e.,

$$a_M = \sum_{S'} \sum_{\xi'} b_{S', \xi'}.$$

In fact, we can choose $z_M \in F^\times \hookrightarrow A_M$ such that (C.3) holds provided $|z_M| < q_F^{-k}$ for some positive integer $k$, where $q_F$ is the order of the residue field of $F$. Then it is enough to have the following lemma.
Lemma C.3. For quasicharacters $\chi_i$ of $F^\times$ and complex numbers $a_i$, if
\[
\sum_{i=1}^{r} a_i \chi_i(z) = 0
\]
provided $|z| < q_F^{-k}$ for some positive integer $k$, then
\[
\sum_{i=1}^{r} a_i = 0.
\]

Proof. Suppose $\chi_i$ are distinct, we claim $a_i = 0$ for $1 \leq i \leq r$. It is clear that this lemma will follow from our claim. So next we will show the claim by induction on $r$. When $r = 1$, there is nothing to show. In general, let us first assume all $\chi_i$ are unramified. We choose $z_0 \in F^\times$ such that $|z_0| < q_F^{-k}$, and denote $\chi_i(z_0)$ by $C_i$. Then
\[
\sum_{i} a_i C_i^j = \sum_{i} a_i \chi_i(z_0)^j = 0
\]
for any positive integer $j$. In particular, $\{a_i\}$ forms a solution of the linear system of equations defined by the matrix $\{C_i^j\}^{r \times r}_r$. Since $|\det(\{C_i^j\}^{r \times r})| = |\prod_i C_i | \cdot \prod_{i,j} |C_i - C_j| \neq 0$, then $a_i$ have to be all zero. Now suppose some $\chi_i$ is ramified, we can replace $\chi_i$ by $\chi'_i := \chi_i/\chi_1$. If $\chi'_i$ are all unramified, then we are back to the previous case. If $\chi'_i$ is ramified for some $i_0 > 1$, then we can choose some unit element $u$ of $F^\times$ such that $\chi'_{i_0}(u) \neq 1$. By subtracting $\chi_1(u) \sum_i a_i \chi_i(z)$ from $\sum_i a_i \chi_i(uz)$, we get
\[
\sum_{i > 1} a_i (\chi_i(uz) - \chi_1(uz)) \chi_i(z) = 0
\]
provided $|z| < q_F^{-k}$. By induction, we have $a_i (\chi_i(uz) - \chi_1(uz)) = 0$ for $i > 1$. Since $\chi'_{i_0}(u) - \chi_1(u) \neq 0$, by our assumption, this implies $a_{i_0} = 0$. Hence
\[
\sum_{i \neq i_0} a_i \chi_i(z) = 0
\]
provided $|z| < q_F^{-k}$. By induction again, we have $a_i = 0$ for $i \neq i_0$. This finishes the proof of the claim. \qed

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