Rediscovering GF Becker’s early axiomatic deduction of a multiaxial nonlinear stress–strain relation based on logarithmic strain

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**Abstract**

We discuss a completely forgotten work of the geologist GF Becker on the ideal isotropic nonlinear stress–strain function (*Am J Sci* 1893; 46: 337–356). Due to the fact that the mathematical modelling of elastic deformations has evolved greatly since the original publication we give a modern reinterpretation of Becker’s work, combining his approach with the current framework of the theory of nonlinear elasticity.

Interestingly, Becker introduces a multiaxial constitutive law incorporating the logarithmic strain tensor, more than 35 years before the quadratic Hencky strain energy was introduced by Heinrich Hencky in 1929. Becker’s deduction is purely axiomatic in nature. He considers the finite strain response to applied shear stresses and spherical stresses, formulated in terms of the principal strains and stresses, and postulates a principle of superposition for principal forces which leads, in a straightforward way, to a unique invertible constitutive relation, which in today’s notation can be written as

\[ T^{\text{Biot}} = 2 G \cdot \text{dev}_3 \log(U) + K \cdot \text{tr}[\log(U)] \cdot \mathbb{I} \]

where \( T^{\text{Biot}} \) is the Biot stress tensor, \( \log(U) \) is the principal matrix logarithm of the right Biot stretch tensor \( U = \sqrt{F^TF} \), \( \text{tr} X = \sum_{i=1}^{3} X_{ii} \) denotes the trace and \( \text{dev}_3 X = X - (1/3) \text{tr}(X) \cdot \mathbb{I} \) denotes the deviatoric part of a matrix \( X \in \mathbb{R}^{3 \times 3} \).

Here, \( G \) is the shear modulus and \( K \) is the bulk modulus. For Poisson’s number \( \nu = 0 \) the formulation is hyperelastic and the corresponding strain energy

\[ W^{\text{Becker}}(U) = 2 G \left[ < U, \log(U) - \mathbb{I} > + 3 \right] \]

has the form of the maximum entropy function.

**Keywords**

Hencky strain, logarithmic strain, nonlinear elasticity, matrix logarithm
I. Introduction

1.1. Some reflections on constitutive assumptions in nonlinear elasticity

The question of proper constitutive assumptions in nonlinear elasticity has puzzled many generations of researchers. The problem of finding simple enough constitutive assumptions which are sufficient to characterize physically plausible behaviour of ‘completely elastic’ materials was even called ‘das ungelöste Hauptproblem der endlichen Elastizitätstheorie’ (the unsolved main problem of finite elasticity theory) by Truesdell [1]. While such assumptions can only lead to idealized material behaviour, the merits of such an ideal model were already described by H Hencky in his 1928 article On the form of the law of elasticity for ideally elastic materials [2,3]:

Like so many mathematical and geometric concepts, it is a useful ideal, because once its deducible properties are known it can be used as a comparative rule for assessing the actual elastic behaviour of physical bodies. […] While it is certainly a matter of empirical observation to determine how actual materials compare to the ideally elastic body, the law itself acts as a measuring instrument which is extended into the realm of the intellect, making it possible for the experimental researcher to make systematic observations.

The range of applicability of such an idealized response, however, must necessarily be restricted to minute strains, perhaps in the order of 1%, for otherwise we are to expect interference with non-elastic effects like plastic deformations, microstructural instabilities or bifurcations. Nonetheless, it should be formulated tensorially correct for arbitrarily large strains. It is also clear that the restriction to small elastic strains does not imply that one can use linear elasticity theory, nor that the ideal elastic response for larger stresses or strains is arbitrary. On the contrary, our idealization should work, as an ideal model, for arbitrarily large strains.

In the past, a large number of possible basic assumptions for elastic materials have been suggested in order to respond to the idealization described above. Among the most commonly accepted are:

- Hyperelasticity: the existence of a strain energy function $W$.
- Homogeneity: the strain energy $W$ does not depend on the position in the body.
- Simple material: the strain energy $W = W(F)$ depends only on the first deformation gradient $F$.
- Objectivity: $W(Q \cdot F) = W(F)$ for all $Q \in SO(3)$.
- Isotropy: $W(F \cdot Q) = W(F)$ for all $Q \in SO(3)$.
- Unique (up to rotations) stress-free reference state $U = \sqrt{F^T F} = I$.
- Linearity: consistent with linear elasticity theory at the reference state.
- Well-posedness of the corresponding linear elasticity model in statics and dynamics.
- Correct stress response for extreme strains: $\sigma \to \infty$ as $V = \sqrt{F^T F} \to \infty$ as well as $\sigma \to -\infty$ as $\det(V) \to 0$, where $\sigma$ denotes the Cauchy stress tensor.
- Second-order behaviour in agreement with Bell’s experimental observations [4], that is, the instantaneous elastic modulus $E$ decreases for tension and increases in the case of compression (see Figure 1).
- Correct energetic behaviour for extreme strains in order to ensure invertibility of the deformation gradient $F$: $W \to \infty$ for $\|F\| \to \infty$ as well as $W \to \infty$ for $\det(F) \to 0$.
- Polyconvexity [5–10], quasiconvexity [8,11].
- Legendre–Hadamard ellipticity [12].
- Baker–Ericksen inequalities [13].

Apart from these conditions there are several properties which, while not generally viewed as necessary for an elasticity model, may be considered as constitutive assumptions for an idealized material as well:

- Superposition principle: $T(V_1 \cdot V_2) = T(V_1) + T(V_2)$ for all coaxial stretches $V_1$ and $V_2$ and some corresponding stress tensor $T$.
- Invertible stress–strain relation: the mapping $E \mapsto T(E)$ is invertible for some stress tensor $T$ and a corresponding work-conjugate strain tensor $E$ (if $T$ is the Cauchy stress tensor, then this invertibility condition is satisfied for example for a variant of the compressible neo-Hookean energy [15]; if $T$ is the Biot stress tensor, then this condition is Truesdell’s invertible force stretch (IFS) relation [16, p. 156]).
- Tension–compression symmetry: $T(V^{-1}) = -T(V)$ for some stress tensor $T$ (note that the classical hyperelastic tension–compression symmetry $W(F) = W(F^{-1})$ is equivalent to $\tau(V^{-1}) = -\tau(V)$ for the Kirchhoff stress $\tau$ and the left stretch tensor $V$).
Figure 1. Nonlinear behaviour of incompressible elastic materials in a simple tensile stress test, drawing the Biot stress $T^{\text{Biot}}$ versus the principal stretch $\lambda$.

- Plausible behaviour under simple homogeneous finite stresses (similar to linear elasticity):
  - Pure shear stresses of the form
    \[
    T = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
    \]
    should induce stretches of the form (c.f. Figure 2)
    \[
    V = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}
    \]
    with
    \[
    \det \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix} = 1.
    \]
  - Spherical stresses of the form
    \[
    T = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}
    \]
    should induce volumetric stretches of the form (c.f. Figure 3)
    \[
    V = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}
    \]

- Ordered stresses (‘greater stress corresponds to greater stretch’): $(T_i - T_j) \cdot (\lambda_i - \lambda_j) > 0$ for all $\lambda_i \neq \lambda_j$ and some stress tensor $T$, where $T_k$ are the principal stresses, that is, the principal values of $T$, and $\lambda_k$ are the principal stretches (note that the Baker–Ericksen inequality can be stated as $(\sigma_i - \sigma_j) \cdot (\lambda_i - \lambda_j) > 0$ for all $\lambda_i \neq \lambda_j$ where $\sigma$ is the Cauchy stress tensor).
- Simple volumetric–isochoric decoupling to ensure a suitable formulation of the incompressibility constraint.
- Minimal number of physically motivated and experimentally identifiable constitutive coefficients, for example only the two isotropic Lamé constants.
- Clear physical interpretation of Poisson’s number $\nu$ for finite deformations: $\nu = 1/2$ enforces exact incompressibility (det $F = 1$) and $\nu = 0$ implies no lateral contraction under uniaxial tension (as in linear elasticity).
Figure 2. Pure shear stress should induce pure shear stretch, preserving the area $A$.

Figure 3. Spherical stress should induce purely volumetric stretch.

- Greatest possible extent of elastic determinacy [3, p. 19]: the stress response should not depend on a specific reference state or previously applied deformations; a similar condition was proposed by Murnaghan [17, 18], who argued that the dependence of the stress response on a specific ‘position of zero strain’ was tantamount to an ‘action at a distance’ and should therefore be avoided.

Several attempts to propose such an idealized model of elasticity can be found in the literature. Becker’s deduction can be seen as an early example of such an attempt [19].

1.2. A modern interpretation of Becker’s development

Becker, in his development of a nonlinear law of elasticity, rejects many of his contemporaries’ approaches to the problem of finite elasticity. He starts his introduction with a description of Hooke’s law, stating that apart from its original formulation (‘Strain is proportionate to the load, or the stress initially applied to an unstrained mass’ [19, p. 337]) it is often interpreted in a different way (‘Strain is proportional to the final stress required to hold a strained mass in equilibrium’ [19, p. 337]). These two different interpretations of Hooke’s law for finite deformations can be expressed as

$$T^\text{Biot} = 2G \cdot \text{dev}_3(U - \mathbb{I}) + K \cdot \text{tr}(U - \mathbb{I}) \cdot \mathbb{I}$$

$$= 2G \cdot (U - \mathbb{I}) + \Lambda \cdot \text{tr}(U - \mathbb{I}) \cdot \mathbb{I}$$  \hspace{1cm} (1)$$

and

$$\sigma = 2G \cdot \text{dev}_3(V - \mathbb{I}) + K \cdot \text{tr}(V - \mathbb{I}) \cdot \mathbb{I}$$

$$= 2G \cdot (V - \mathbb{I}) + \Lambda \cdot \text{tr}(V - \mathbb{I}) \cdot \mathbb{I}$$  \hspace{1cm} (2)$$

respectively, where $U = \sqrt{F^\text{T}F}$ is the right Biot stretch tensor, $V = \sqrt{FF^\text{T}}$ is the left Biot stretch tensor, $\sigma$ is the Cauchy stress tensor, $T^\text{Biot}$ is the Biot stress tensor, $\text{dev}_3 X = X - (1/3) \text{tr}(X) \cdot \mathbb{I}$ denotes the deviatoric part of $X \in \mathbb{R}^{3 \times 3}$, $K$ is the bulk modulus and $G$, $\Lambda$ are the Lamé constants. According to Becker, it was already ‘universally acknowledged that either law [(1), (2)] is applicable only to strains so small that their squares are negligible’ [19, p. 337]. He gives a number of reasons for this rejection of Hooke’s law as a model for finite deformations, including the fact that it allows for infinite distortions ($\det F = 0$) under finite stresses [19, p. 337]. In addition, Becker states that Hooke’s law ‘rests entirely upon experiment’, that is, that there is no
underlying framework necessitating the linearity of the stress–strain relation. However, Becker also rejects the idea that the stress response could be discovered by ‘any process of pure reason’ alone,2 an approach which he attributes to Barré de Saint-Venant. In fact, in Becker’s time the elasticity models that had been developed through purely geometrical considerations generally implied the so-called Cauchy relations [20–22], in other words, they determined the lateral contraction independent of the specific material, corresponding to a fixed value $\nu = 1/4$ for Poisson’s number $\nu$. As Becker points out in a footnote later on [19, p. 348], this value for $\nu$ should only be regarded as a special case and not as a general law.3

Instead, his approach to describe the deformation of an ideally elastic body can be summarized as follows: motivated by geometric considerations he postulates a connection between shear stresses and shear strains as well as between volumetric stresses and dilational strains. He then shows that every homogeneous finite deformation can be decomposed into two shear stretches and a purely dilational deformation. Finally Becker assumes that a law of superposition holds for all coaxial finite strains, allowing him to reduce the problem of a general stress–stretch relation to shears and dilations only.

Thus Becker makes a number of assumptions about the stress–stretch relation from which he then deduces a law of elasticity. His final result is a stress–stretch relation which in today’s notation can be written as

$$T^\text{Biot} = 2G \cdot \text{dev} \log(U) + K \cdot \text{tr}[\log(U)] \cdot \mathbb{I}$$

$$= 2G \cdot \log(U) + \Lambda \cdot \text{tr}[\log(U)] \cdot \mathbb{I},$$

where $\log(U)$ is the principal matrix logarithm of the right Biot stretch tensor $U = \sqrt{F^TF}$.

In order to reproduce Becker’s approach in a more modern framework of elasticity theory we will therefore interpret Becker’s implicit assumptions as axioms for a law of ideal elasticity. While Sections 2 and 3 summarize Becker’s motivation for these axioms as well as some of his computations, a generalized deduction of Becker’s law of elasticity will be given in Section 4. Finally, we will investigate some basic properties of the resulting stress–stretch relation in Section 5. The notation employed throughout the article can be found in Appendix A.

While the axioms are in fact sufficient to completely characterize an isotropic stress–stretch relation uniquely up to two material parameters, it turns out that some of them may be weakened considerably without changing the result.

2. Becker’s assumptions

In order to understand Becker’s approach it is important to distinguish his (often implicitly stated) assumptions from his deduced results. Like Becker, we consider a unit cube in the reference configuration $\Omega_0$ the edges of which are aligned with an orthogonal coordinate system $e_1, e_2, e_3$. Unless indicated otherwise, all matrix representations of linear mappings are given with respect to this coordinate system.

2.1. Basic assumptions

Becker’s most basic assumptions are that the stress–stretch relation is an analytic function [19, p. 342] as well as isotropic4 [19, p. 338] and that it is possible to ‘regard strains as functions of load’ [19, p. 341], that is, that the strain-load mapping is invertible. Note that here and throughout we will interpret Becker’s ‘initial stress’ (which, for a unit cube, is equal to the load) as the Biot stress tensor $\sigma^B$ [23]

$$\sigma^B = U \cdot S_2 = J \cdot R^T \cdot \sigma^0 \cdot F^{-T},$$

where $F$ denotes the deformation gradient, $F = RU$ is the polar decomposition [24] of $F$ with $R \in \text{SO}(3)$ and $U = \sqrt{F^TF} \in \text{PSym}(3)$, $J = \det F = \det U$ is the Jacobian determinant, $\sigma$ is the Cauchy stress tensor and $S_2$ is the symmetric second Piola–Kirchhoff stress tensor. A justification of this interpretation can be found in Appendix B.2. Note that in the isotropic case, $T^\text{Biot}$ is a symmetric tensor as well [25].

2.2. Pure finite shear

One of Becker’s main assumptions is that ‘a simple finite shearing strain must result from the action of two equal loads or initial stresses of opposite signs at right angles to one another’ [19, p. 339]. This assumption is
most motivated by geometric considerations: if the deformation is a homogeneous pure shear of the form

\[
F = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \alpha > 1,
\]  

(3)

then the so-called planes of no distortion have a number of properties connected to the quantity \( \alpha \). Becker then relates these properties of strain to certain properties of stress, thereby establishing that the stress corresponding to the above shear deformation must be a pure shear stress of the form

\[
T^{Biot} = \begin{pmatrix}
s & 0 & 0 \\
0 & -s & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad s \in \mathbb{R}.
\]  

(4)

His arguments are largely based on the assumption that ‘[in] the particular case of a shear (or a pure shear) there are two sets of planes on which the stresses are purely tangential, for otherwise there could be no planes of zero distortion’ [19, p. 338].

If a Biot shear stress of the form (4) corresponds to a pure shear deformation of the form (3), then the corresponding Cauchy stress tensor \( \sigma \) computes to

\[
\sigma = \frac{1}{\det U} \cdot U^{-1} \cdot F \cdot T^{Biot} \cdot F^T = 1 \cdot F^{-1} \cdot F \cdot T^{Biot} \cdot F = \begin{pmatrix}
\alpha s & 0 & 0 \\
0 & -\frac{s}{\alpha} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]  

(5)

The principal Cauchy stresses are therefore \( \sigma_1 = \alpha s \), \( \sigma_2 = -s/\alpha \) and \( \sigma_3 = 0 \). Note carefully that Becker’s shear deformation is oriented differently: in his considerations, the contractile axis (along the eigenvector to the smaller eigenvalue \( 1/\alpha \)) is aligned with the \( e_1 \)-axis. This difference is reflected in Becker’s formula \(- \sigma_1 \alpha = \sigma_2 / \alpha\) [19, p. 338]. For our choice of axes, the corresponding equality reads

\[
-\sigma_2 \alpha = \frac{\sigma_1}{\alpha}.
\]

A more detailed geometric description of this relation will be given in Section 3. More recent discussions of shear stresses and shear strains can be found in [26] and [27]. The so-called planes of no distortion also play an important role in Becker’s treatment of the rupture of rocks [28] as well as his later works on schistosity and slaty cleavage [29, 30], where properties of the planes of no distortion are linked to failure criteria for deformations beyond the range of elastic deformations. A summary of Becker’s work on yield criteria for rocks can be found in [31], while the concept of planes of no distortion and its relation to the tangential shear strain is described in a more detailed form in [32].

### 2.2.1. The planes of no distortion

Let \( F \in \text{GL}^+(3) \) denote an invertible linear mapping. We call a plane \( E \subset \mathbb{R}^3 \) through the origin an initial plane of no distortion if the restriction of \( F \) to \( E \) is a rotation, which is the case if and only if \( \langle Fx, Fy \rangle = \langle x, y \rangle \) for all \( x, y \in E \) or, equivalently, if \( \|Fx\| = \|x\| \) for all \( x \in E \), where \( \langle \cdot, \cdot \rangle \) denotes the
Euclidean inner product on \( \mathbb{R}^3 \). Furthermore, we call \( F \subset \mathbb{R}^3 \) a **final plane of no distortion** if \( F \) is the image under \( F \) of an initial plane of no distortion, which is the case if and only if \( \|F^{-1}x\| = \|x\| \) for all \( x \in F \). Since Becker’s considerations of such planes are confined to the deformed (or final) configuration of a homogeneous deformation, we will often refer to \( F \) simply as a plane of no distortion.

The following basic existence property can be found in [32].

**Proposition 2.1.** Let \( F \in \text{GL}^+(3) \). Then there exists a plane of no distortion for \( F \) if and only if \( \lambda_2 = 1 \), where \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \) denote the singular values of \( F \). If \( \lambda_1 > \lambda_2 = 1 > \lambda_3 \), then there exist exactly two such planes.

**Remark 2.2.** If \( \lambda_2 \neq 1 \) for the second singular value \( \lambda_2 \) of a linear mapping \( F \in \text{GL}^+(3) \), then instead of a plane there exists a **surface of no distortion** in the form of an elliptical cone [32, p. 133] instead of a plane of no distortion. If, however, we generalize the term to mean any plane \( F \) such that the restriction of \( F \) to \( F \) is a **dilated rotation** (also called a **conformal mapping**) of the form \( \lambda \cdot Q \) with \( \lambda \in \mathbb{R}^+ \) and \( Q \in \text{SO}(3) \), then Proposition 2.1 shows that such a plane exists for \( F \in \text{GL}^+(3) \) if and only if the dilational factor \( \lambda \) is the second singular value of \( F \).

**Remark 2.3.** It was shown by Ball and James [33, Proposition 4] that the equality \( \lambda_2 = 1 \) holds if and only if the right Cauchy-Green tensor \( C = F^T F \) corresponding to \( F \) is expressible in the form

\[
C = (\mathbb{I} + \xi \otimes \eta) \cdot (\mathbb{I} + \eta \otimes \xi) = (\mathbb{I} + \eta \otimes \xi)^T \cdot (\mathbb{I} + \eta \otimes \xi)
\]

with \( \xi, \eta \in \mathbb{R}^3 \), where \( \xi \otimes \eta \in \mathbb{R}^{3 \times 3} \) denotes the tensor product of \( \xi \) and \( \eta \). Thus there exists a plane of no distortion for \( F \in \text{GL}^+(3) \) if and only if there exists a rank-one tensor \( H \in \mathbb{R}^{3 \times 3} \) with

\[
F^T F = C = (\mathbb{I} + H)^T \cdot (\mathbb{I} + H).
\]

Since Becker only considers planes of no distortion for pure shear deformations, we will assume from now on that \( F \) has the form

\[
F = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha^2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

with \( \alpha > 1 \). Then clearly \( \lambda_2 = 1 \), hence there are two distinct planes of no distortion which can be determined by direct computation: for \( x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \) with \( \|x\| = 1 \) we can use the equalities

\[
x_1^2 + x_2^2 + x_3^2 = \|x\|^2 \quad \Rightarrow \quad x_3^2 = \|x\|^2 - x_1^2 - x_2^2
\]

to find

\[
\|Fx\|^2 = \alpha^2 x_1^2 + \frac{1}{\alpha^2} x_2^2 + x_3^2 = (\alpha^2 - 1) x_1^2 + \left( \frac{1}{\alpha^2} - 1 \right) x_2^2 + \|x\|^2.
\]

Thus the equality \( \|Fx\| = \|x\| \) is equivalent to

\[
\|x\|^2 = (\alpha^2 - 1) x_1^2 + \left( \frac{1}{\alpha^2} - 1 \right) x_2^2 + \|x\|^2
\]

\[
\iff \quad \left( 1 - \frac{1}{\alpha^2} \right) x_2^2 = (\alpha^2 - 1) x_1^2 \iff x_2^2 = \alpha^2 x_1^2,
\]

**Figure 5.** An initial plane of no distortion is only rotated by the linear mapping \( F \), preserving the angles between two vectors as well as their lengths.
hence every \( x \in \mathbb{R}^3 \) with \( \|Fx\| = \|x\| \) is of the form \( x = (s, \pm \alpha s, t)^T \) with \( s, t \in \mathbb{R} \). Since the final directions of no distortion are the images of those vectors, they in turn have the form \( y = Ax = (\pm \alpha s, s, t)^T \). Therefore, for every shear deformation of the form (6) with \( \alpha > 1 \), there are two initial planes of no distortion,

\[
E^+ = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_2 = \alpha \cdot x_1 \right\} \quad \text{and} \quad E^- = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_2 = -\alpha \cdot x_1 \right\},
\]

as well as two final planes of no distortion

\[
F^+ = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = \alpha \cdot x_2 \right\} \quad \text{and} \quad F^- = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = -\alpha \cdot x_2 \right\}.
\]

To verify Becker’s claim that [19, p. 339] ‘[in] a finite shearing strain of ratio \( \alpha \), it is easy to see that the normal to the planes of no distortion makes an angle with the contractile axis of shear the cotangent of which is \( \alpha’ \), we consider the direction of the contractile axis along the eigenvector \((0, 1, 0)^T = e_2\) corresponding to the smallest eigenvalue \(1/\alpha\). Then the cotangent of the angle between this axis and a vector \( \hat{n} = (y_1, y_2, y_3)^T \) with \( y_1 \neq 0 \) is given by \( y_2/y_1 \). Since

\[
n^+ = (1, -\alpha, 0)^T \quad \text{and} \quad n^- = (1, \alpha, 0)^T
\]

are normal vectors to the planes of no distortion \( F^+ \) and \( F^- \), respectively, the cotangent of the angle between these normals and the contractile axis is indeed given by \( \pm \alpha \). Similarly, it is easy to see that the normals to the initial planes of no distortion \( E^+ \) and \( E^- \) form angles of cotangent \( \pm 1/\alpha \) with the contractile axis.

This definition of the planes of no distortion is also consistent with the definition by means of the shear ellipsoid given by Leith \(^6\) in Structural geology [34]:

In a strain ellipsoid with three unequal principal axes there are only two cross-sections which are circular in outline [...] These planes [...] are called ‘planes of no distortion’ because they preserve a circular cross-section similar to a section of the original sphere...

The strain ellipsoid of the deformation with principal stretches \( \alpha, \alpha^{-1} \) and 1 is defined by the equation

\[
\frac{x_1^2}{\alpha^2} + \frac{x_2^2}{\alpha^{-2}} + x_3^2 = 1,
\]

where \( x_1 \) and \( x_3 \) denote coordinates with respect to the tensile axis and the contractile axis respectively. The planes \( F^+ \) and \( F^- \) are characterized by the equation \( x_1 = \pm \alpha x_2 \), thus we can verify that their intersections with the ellipsoid are indeed circles of radius 1 centred at the origin \((0, 0, 0)^T\):

\[
\frac{x_1^2}{\alpha^2} + \frac{x_2^2}{\alpha^{-2}} + x_3^2 = 1 \quad \land \quad x_1 = \pm \alpha x_2
\]

\[
\Rightarrow \quad \| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \| = x_1^2 + x_2^2 + x_3^2 = x_1^2 + x_2^2 + 1 - \frac{x_1^2}{\alpha^2} - \alpha^2 x_2^2 = x_1^2 + x_2^2 + 1 - x_2^2 - x_1^2 = 1.
\]

2.2.2. Different characterizations of the planes of no distortion. The initial planes of no distortion can also be characterized in a number of different ways, for example by means of the cofactor matrix \( \text{Cof} F \): if the restriction of \( F \) to a plane is a rotation, then

\[
\langle Fx, FY \rangle = \langle x, y \rangle
\]

for all \( x, y \) in the plane. Then for all \( x, y \) in the plane with \( \langle x, y \rangle = 0 \) we find

\[
\|Fx \times FY\|^2 = \|Fx\|^2 \cdot \|FY\|^2 - 2 \cdot \langle Fx, FY \rangle = \|x\|^2 \cdot \|y\|^2 - 2 \cdot \langle x, y \rangle = \|x\|^2 \cdot \|y\|^2.
\]

Now let \( n \) denote a unit normal vector to the plane of no distortion. Then \( n \) can be represented as \( n = x \times y \) with unit vectors \( x, y \) in the plane and \( \langle x, y \rangle = 0 \). Since, in general,

\[
(\text{Cof} F)(x \times y) = Fx \times FY,
\]
we find
\[ \| \text{Cof} F \| n \|^2 = \| \text{Cof} (x \times y) \|^2 = \| Fx \times Fy \|^2 = \| x \|^2 \cdot \| y \|^2 = 1. \]
Therefore the equality \( \| \text{Cof} F \| n \| = 1 \) is a necessary condition for a unit normal vector \( n \) to be the normal to an initial plane of no distortion. We compute
\[
\text{Cof} F = \text{det}(F) \cdot F^{-T} = \frac{1}{\alpha} \cdot \left( \alpha \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\alpha} \\ 0 & 1 \end{pmatrix} \right)^{-1} = \left( \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\alpha} & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix} \right)
\]
as well as
\[
\| \text{Cof} F \| n \|^2 = \frac{n_1^2}{\alpha^2} + \alpha^2 n_2^2 + n_3^2 = \frac{n_1^2}{\alpha^2} + \alpha^2 n_2^2 + 1 - n_1^2 - n_2^2 \\
= 1 + \left( \frac{1}{\alpha^2} - 1 \right) n_1^2 + (\alpha^2 - 1) n_2^2 = 1 + \left( \frac{1}{\alpha^2} - 1 \right) (n_1^2 - \alpha^2 n_2^2),
\]
thus
\[
\| \text{Cof} F \| n \| = 1 \iff 0 = \left( \frac{1}{\alpha^2} - 1 \right) (n_1^2 - \alpha^2 n_2^2) \iff n_1^2 = \alpha^2 n_2^2
\]
for \( \alpha > 1 \), showing that the cotangent of the angle between \( n \) and the contractile axis is \( \pm 1/\alpha \). Therefore the equality \( \| \text{Cof} F \| n \| = 1 \) holds only if \( n \) is a unit normal vector to a plane of no distortion. Note that this equality also implies
\[
0 = \langle (\text{Cof} F) n, (\text{Cof} F) n \rangle - \langle n, n \rangle = \langle (\text{Cof} F)^T (\text{Cof} F) n - n, n \rangle = \langle (\text{Cof} B - \text{I}) n, n \rangle,
\]
where \( B = FF^T \) denotes the left Cauchy-Green deformation tensor.

Becker gives another important characterization of the planes of no distortion: ‘the planes of no distortion […] are also the planes of maximum tangential strain’ [19, p. 339]. In Becker’s *Finite homogeneous strain, flow and rupture of rocks* [28], the tangential strain of \( x \in \mathbb{R}^3 \) is defined as the tangent of the angle between \( x \) and \( Fx \). Accordingly, the *plane of maximum tangential strain* of a pure shear deformation \( F \) is defined as the plane containing the undistorted axis as well as the line for which the tangential strain is maximal, c.f. Figure 6.

We will show that the planes of maximum tangential strain and the initial planes of no distortion are, in fact, identical. Let \( x \) be of the form \( x = (x_1, x_2, 0)^T \) with \( x_1^2 + x_2^2 = 1 \); in other words, we assume that \( x \) is a unit vector orthogonal to the undistorted \( e_3 \)-axis. Then the angle \( \tilde{\vartheta} \) between \( x \) and \( Fx \) is given by
\[
\cos \vartheta = \frac{\langle x, Fx \rangle}{\| x \| \cdot \| Fx \|} = \frac{\langle x, Fx \rangle}{\| Fx \|}.
\]
The direction \( x \) of maximum tangential strain is characterized by the angle \( \vartheta \) for which \( \tan \vartheta \) is maximal. In order to find \( x \) it is therefore sufficient to maximize
\[
\vartheta = \arccos \frac{\langle x, Fx \rangle}{\| Fx \|} = \arccos \frac{\alpha x_1^2 + \frac{1}{\alpha} x_2^2}{\sqrt{\alpha^2 x_1^2 + \frac{1}{\alpha^2} x_2^2}},
\]

**Figure 6.** The shear ellipsoid of a deformation \( F \), showing an arbitrary tangential strain \( \tan \tilde{\vartheta} \) (left) and the maximum tangential strain \( \tan \vartheta \) (right), which is realized for the plane of no distortion.
Another visualization of the tangential strain: the shift between two parallel vertical lines under the deformation is maximal along the plane of no distortion.

Figure 7. Another visualization of the tangential strain: the shift between two parallel vertical lines under the deformation is maximal along the plane of no distortion.

since \( \vartheta \mapsto \tan \vartheta \) is monotone. Using the equality \( x_1^2 = 1 - x_2^2 \), we obtain

\[
\arccos \frac{x_1 F x}{\|F x\|} = \arccos \frac{\alpha (1 - x_1^2) + \frac{\alpha}{\alpha} x_2^2}{\sqrt{\alpha^2 (1 - x_1^2) + \frac{\alpha}{\alpha} x_2^2}}.
\]

In order to find \( t \in [-1, 1] \) for which the function

\[
f : [-1, 1] \to \mathbb{R}, \quad f(t) = \arccos \frac{\alpha (1 - t^2) + \frac{1}{\alpha} t^2}{\sqrt{\alpha^2 (1 - t^2) + \frac{1}{\alpha} t^2}}
\]

attains its maximum, we compute the first derivative of \( f \) to be

\[
\frac{d}{dt} f(t) = \frac{(\alpha^2 t^2 - 1) + t^2}{\alpha t (t^2 - 1)} \cdot \frac{\sqrt{\frac{(\alpha^2 - 1) t^2}{\alpha^2 (t^2 - 1) - t^2}}}{\sqrt{\alpha^2 - \frac{(\alpha^2 - 1)^2}{\alpha^2}}}
\]

Thus the possible extremal points of \( f \) are 0, ±1 and \( \pm \alpha/\sqrt{1 + \alpha^2} \). Since

\[
f \left( \frac{\alpha}{\sqrt{1 + \alpha^2}} \right) = f \left( - \frac{\alpha}{\sqrt{1 + \alpha^2}} \right)
\]

as well as \( f(-1) = f(0) = f(1) = 0 \) and \( f(t) \geq 0 \) for all \( t \in [-1, 1] \), the global maxima of \( f \) are given by \( t = \pm \alpha/\sqrt{1 + \alpha^2} \). Applying this result to our original problem, we find that the maximum tangential strain is attained for the directions

\[
\hat{x}_\pm = \left( \sqrt{1 - \frac{\alpha^2}{1 + \alpha^2}}, \pm \frac{\alpha}{\sqrt{1 + \alpha^2}}, 0 \right)^T
\]

Finally, in order to verify Becker’s claim that ‘the planes of no distortion […] are also the planes of maximum tangential strain’ [19, p. 339], we observe that

\[
\|F \hat{x}_\pm\|^2 = \left\| \left( \alpha \cdot \sqrt{1 - \frac{\alpha^2}{1 + \alpha^2}} \right) \right\|^2 = \alpha^2 \cdot \left( 1 - \frac{\alpha^2}{1 + \alpha^2} \right) + \frac{1}{1 + \alpha^2}
\]

\[
= \alpha^2 + \frac{1 - \alpha^4}{1 + \alpha^2} = \alpha^2 + 1 - \alpha^2 = 1 = \|\hat{x}_\pm\|.
\]

Thus the plane of maximum tangential strain is indeed the initial plane of no distortion.
Figure 8. Simple glide deformation with shear angle $\tan \varphi = \gamma / 1$; horizontal lines slide relative to each other, vertical lines tilt to accommodate; originally right angles are distorted, the shear strain measures the change of angles.

2.2.3. Simple glide deformation. A homogeneous deformation $F$ is called a simple glide if it has the form [35]

$$F = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\gamma \in \mathbb{R}, \gamma > 0$. Since $F \cdot (t, 0, s)^T = (t, 0, s)^T$ for all $t, s \in \mathbb{R}$, the restriction of $F$ to the $e_1, e_3$-plane is the identity function, showing that it is the initial plane of no distortion as well as the final plane of no distortion. We examine the right Cauchy-Green deformation tensor of a simple glide, which is given by

$$C = F^T F = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1 + \gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

The principal stretches $\lambda_i$ are the singular values of $F$ which, in turn, are the square roots of the eigenvalues of $C$,

$$\lambda_1 = \frac{1}{2} \left( \gamma + \sqrt{\gamma^2 + 4} \right), \quad \lambda_2 = \frac{1}{2} \left( \gamma - \sqrt{\gamma^2 - 4} \right), \quad \lambda_3 = 1,$$

and the principal axes are given by the corresponding eigenvectors of $C$:

$$v_1 = \begin{pmatrix} \gamma + \sqrt{\gamma^2 + 4} \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 0 \\ \sqrt{\gamma^2 - 4} \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

Since $1 < \lambda_1 = 1/\lambda_2$ as well as $\lambda_3 = 1$, we can interpret the simple glide as a rotated pure shear with shear ratio $\alpha = \lambda_1$, where the direction of the tensile axis is $v_1$ and the contractile axis is given by $v_2$. The cotangent of the angle $\vartheta$ between $v_2$ and the $e_1$-axis is

$$\cot (\vartheta) = \frac{-2}{\gamma + \sqrt{\gamma^2 - 4}} = \frac{-2}{2 \cdot \lambda_2} = -\frac{1}{\lambda_2} = -\alpha.$$

Hence the angle between $(1, 0, 0)^T = e_1$ and the contractile axis is

$$\cot (-\vartheta) = -\cot (\vartheta) = \alpha,$$

yielding another angular characterization of the shear ratio $\alpha$.

2.3. Dilation

Similarly to the connection between shear stress and shear stretches, Becker further assumes that ‘dilational forces acting positively and equally in all directions’ [19, p. 339] are ‘[the loads] effecting dilation’ [19, p. 342], in other words, that purely volumetric initial stresses correspond to purely dilational stretches. More precisely, this assumption can be stated in the following way: if the principal axes remain fixed, every Biot stress of the form $T^{\text{Biot}} = a \cdot \mathbb{I}$ with $a \in \mathbb{R}$ corresponds to a deformation of the form $F = \lambda \cdot \mathbb{I}$ with $\lambda > 0$. Again, this assumption is stated only implicitly.
2.4. Superposition

Becker’s most important assumption is that of a *law of superposition* for coaxial deformations: ‘the load sums correspond to the products of the strain ratios’ [19, p. 341]. For homogeneous, coaxial stretch tensors $U_1, U_2$ this law can be stated as

$$T_{\text{Biot}}(U_1 \cdot U_2) = T_{\text{Biot}}(U_1) + T_{\text{Biot}}(U_2),$$

where $T_{\text{Biot}}(U)$ denotes the Biot stress tensor corresponding to the right Biot stretch tensor $U$. Here, $U_1$ and $U_2$ are called coaxial if their principal axes coincide.

2.4.1. The decomposition of stresses and strains. An important application of the law of superposition involves the additive decomposition of stresses and the corresponding multiplicative decomposition of stretches. Consider an initial load (i.e. a Biot stress tensor) of the form

$$T_{\text{Biot}}^1 = \begin{pmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the case of isotropic materials, the corresponding stretches are identical for all directions orthogonal to the $x_1$-axis for symmetry reasons, which is the case if and only if the stretch tensor corresponding to $T_{\text{Biot}}^1$ has the form

$$U_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$$

for some $a, b \in \mathbb{R}^+$ with respect to the principal axes. Then, for $h_1 = (ab^2)^{1/3}$ and $p = (\frac{a}{b})^{1/3}$, we find

$$U_1 = \left( \begin{array}{ccc} \left( ab^2 \cdot \frac{a^2}{p^2} \right)^{\frac{1}{3}} & 0 & 0 \\ 0 & \left( ab^2 \cdot \frac{b}{a} \right)^{\frac{1}{3}} & 0 \\ 0 & 0 & \left( ab^2 \cdot \frac{b}{a} \right)^{\frac{1}{3}} \end{array} \right) = h_1 \cdot \begin{pmatrix} p^2 & 0 & 0 \\ 0 & \frac{1}{p} & 0 \\ 0 & 0 & \frac{1}{p} \end{pmatrix} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_1 \end{pmatrix} \cdot \begin{pmatrix} p & 0 & 0 \\ 0 & \frac{1}{p} & 0 \\ 0 & 0 & \frac{1}{p} \end{pmatrix}.$$

In a similar way we can obtain the stretch tensors corresponding to the stresses

$$T_{\text{Biot}}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_{\text{Biot}}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R \end{pmatrix}$$

respectively:

$$U_2 = h_2 \cdot \begin{pmatrix} \frac{1}{q} & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & \frac{1}{q} \end{pmatrix} = \begin{pmatrix} h_2 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{q} & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & \frac{1}{q} \end{pmatrix},$$

$$U_3 = h_3 \cdot \begin{pmatrix} \frac{1}{r} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & \frac{1}{r} \end{pmatrix} = \begin{pmatrix} h_3 & 0 & 0 \\ 0 & h_3 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{r} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{pmatrix}. $$
for some \( h_2, h_3, q, r \in \mathbb{R}^+ \). Therefore every homogeneous deformation corresponding to a uniaxial load can be decomposed into a dilation and two shear deformations with perpendicular axes. Finally, for the general case of arbitrary stresses

\[
T^{\text{Biot}} = \begin{pmatrix}
P & 0 & 0 \\
0 & Q & 0 \\
0 & 0 & R
\end{pmatrix} = \begin{pmatrix}
P & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & Q & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & R
\end{pmatrix}
\]

the law of superposition yields the general formula

\[
U = U_1 \cdot U_2 \cdot U_3 = h_1 h_2 h_3 \cdot \begin{pmatrix}
p^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & q^2 & 0 \\
0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & r^2
\end{pmatrix} = h_1 h_2 h_3 \cdot \begin{pmatrix}
p^2 & 0 & 0 \\
0 & q^2 & 0 \\
0 & 0 & r^2
\end{pmatrix}
\]

for the stretch tensor \( U \) corresponding to \( T^{\text{Biot}} \). Then \( U \) can be decomposed into a dilation and two shears as well:

\[
U = \begin{pmatrix}
h_1 h_2 h_3 & 0 & 0 \\
0 & h_1 h_2 h_3 & 0 \\
0 & 0 & h_1 h_2 h_3
\end{pmatrix} \cdot \begin{pmatrix}
p^2 & 0 & 0 \\
0 & q^2 & 0 \\
0 & 0 & r^2
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & p & 0 \\
0 & 0 & p
\end{pmatrix}.
\]

Furthermore we can find an additive decomposition

\[
T^{\text{Biot}} = \begin{pmatrix}
P+Q+R & 0 & 0 \\
0 & P+Q+R & 0 \\
0 & 0 & P+Q+R
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & Q+R-2p & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & P+Q-2r \\
0 & 0 & -P+Q-2r
\end{pmatrix}
\]

of \( T^{\text{Biot}} \) into a spherical stress and two pure shear stresses. Note that the decomposition of \( U \) can be found in Becker’s second table [19, p. 340] while the decomposition of \( T^{\text{Biot}} \) can be found in the third table [19, p. 340].

In decomposing the strains \( U_1, U_2 \) and \( U_3 \) into a dilation and two shear strains, the planes of shear were chosen arbitrarily for every strain (or, more precisely, such that the two resulting shear ratios were identical). However, we may also choose two fixed planes (or, equivalently, fixed axes of tension and contraction) and decompose all strains into shears along the same axes:

\[
U_1 = h_1 \cdot \begin{pmatrix}
p^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = h_1 \cdot \begin{pmatrix}
p^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & p & 0 \\
0 & 0 & p
\end{pmatrix},
\]

\[
U_2 = h_2 \cdot \begin{pmatrix}
0 & 0 & 0 \\
0 & q^2 & 0 \\
0 & 0 & 1
\end{pmatrix} = h_2 \cdot \begin{pmatrix}
0 & 0 & 0 \\
0 & q^2 & 0 \\
0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & q & 0 \\
0 & 0 & q
\end{pmatrix},
\]

\[
U_3 = h_3 \cdot \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & r^2 \\
0 & 0 & r^2
\end{pmatrix} = h_3 \cdot \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & r^2 \\
0 & 0 & r^2
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Note that the axes chosen here are identical to those chosen for the decomposition of \( U \). Using this approach we obtain a modified version of Becker’s first table, see Table 1.
Table 1. Modified version of Becker's first table.

| Active force | P | Q | R |
|--------------|---|---|---|
| Axis of strain | x | y | z | x | y | z | x | y | z |
| Dilation ear | $h_1$ | $h_1$ | $h_1$ | $h_2$ | $h_2$ | $h_2$ | $h_3$ | $h_3$ | $h_3$ |
| Shear | $p^2$ | $\frac{1}{p}$ | 1 | $\frac{1}{q}$ | $q$ | 1 | $\frac{1}{r}$ | $r$ | 1 |

Figure 9. Load and deformation of the uniaxial case can be decomposed into two finite shear and a dilational mode.

2.4.2. The uniaxial case. Consider, again, a uniaxial initial stress of the form

$$T^{\text{Biot}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Then $T^{\text{Biot}}$ can be decomposed into

$$T^{\text{Biot}} = \frac{Q}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{Q}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{Q}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

By Becker’s assumption the two shear stresses $T_1^{\text{Biot}}$ and $T_2^{\text{Biot}}$ correspond to shear strains

$$U_1 = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix}$$

respectively, while the volumetric stress $T_3^{\text{Biot}}$ corresponds to a dilational strain

$$U_3 = \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix}.$$ 

Then, according to the law of superposition, the strain corresponding to $T^{\text{Biot}}$ is given by

$$U = U_1 \cdot U_2 \cdot U_3 = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & h \alpha^2 & 0 \\ 0 & 0 & \frac{h}{\alpha} \end{pmatrix}.$$
The resulting principal stretch \( h \alpha^2 \) in the direction of the applied load is referred to by Becker as the ‘length of the strained mass’ [19, p. 343] in his further computations for the uniaxial case.

Now we consider another uniaxial load given by \( \hat{T}_{\text{Biot}} \)

\[
\hat{T}_{\text{Biot}} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Then \( \hat{T}_{\text{Biot}} \) can be decomposed in a number of different ways. For example we could choose, for symmetry reasons, a decomposition similar to that of \( T_{\text{Biot},1} \), that is,

\[
\hat{T}_{\text{Biot}} = \frac{P}{3} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{P}{3} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{P}{3} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

yielding the strain

\[
\hat{U} = \begin{pmatrix}
\hat{\alpha} & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
\hat{\alpha} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{\alpha}
\end{pmatrix} \cdot \begin{pmatrix}
\hat{h} & 0 & 0 \\
0 & 0 & \hat{h} \\
0 & \hat{h} & 0
\end{pmatrix} = \begin{pmatrix}
\hat{h} \hat{\alpha}^2 & 0 & 0 \\
0 & \frac{\hat{h}}{\alpha} & 0 \\
0 & 0 & \frac{\hat{h}}{\alpha}
\end{pmatrix}.
\]

It is also possible to decompose \( \hat{T}_{\text{Biot}} \) into shears coaxial to \( T_{\text{Biot},1} \) and \( T_{\text{Biot},2} \), that is,

\[
\hat{T}_{\text{Biot}} = -\frac{2P}{3} \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{P}{3} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix} + \frac{P}{3} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Note that the resulting strain is independent of this choice: since

\[
-\frac{2P}{3} \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \frac{P}{3} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} + \frac{P}{3} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
\]

the law of superposition yields

\[
\hat{U} = \begin{pmatrix}
\hat{\alpha} & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
\hat{\alpha} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{\alpha}
\end{pmatrix} \cdot \begin{pmatrix}
\hat{h} & 0 & 0 \\
0 & 0 & \hat{h} \\
0 & \hat{h} & 0
\end{pmatrix} = \begin{pmatrix}
\hat{h} \hat{\alpha}^2 & 0 & 0 \\
0 & \frac{\hat{h}}{\alpha} & 0 \\
0 & 0 & \frac{\hat{h}}{\alpha}
\end{pmatrix},
\]

in this case as well.

2.4.3. Decomposition along fixed axes. The additive decomposition of the stress tensor into a volumetric stress and two shears along fixed axes can also be expressed in basic algebraic terms. We will identify the set of all diagonal matrices in \( \mathbb{R}^{3 \times 3} \) with the Euclidean space \( \mathbb{R}^3 \) in the canonical way. Since the set

\[
\mathcal{B} = \left\{ \begin{pmatrix}
-1 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
-1
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix} \right\},
\]

which corresponds to two shear stresses and a volumetric stress in diagonal form, is a basis of \( \mathbb{R}^3 \), every

\[
\begin{pmatrix}
P \\
Q \\
R
\end{pmatrix} \in \mathbb{R}^3
\]
can be written as
\[
\begin{bmatrix}
P \\ Q \\ R
\end{bmatrix} = \begin{bmatrix}
-1 & 0 & 1 \\ 0 & 1 & -1
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 1 \\ 1 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
-1 & 0 & 1 \\ 1 & 1 & 1
\end{bmatrix} \cdot \begin{bmatrix}
A \\ B \\ C
\end{bmatrix}.
\]
(12)

For given \(P, Q, R \in \mathbb{R}\) we can therefore find the coefficients \(A, B, C\) of the decomposition
\[
\begin{bmatrix}
P \\ Q \\ R
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1
\end{bmatrix}.
\]

by rewriting equation (12) to read
\[
\begin{bmatrix}
P \\ Q \\ R
\end{bmatrix} = \begin{bmatrix}
-1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1
\end{bmatrix}^{-1} \cdot \begin{bmatrix}
P \\ Q \\ R
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
-2 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1
\end{bmatrix} \cdot \begin{bmatrix}
P \\ Q \\ R
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
-2P + Q + R \\ P + Q - 2R \\ P + Q + R
\end{bmatrix}.
\]

The decomposition obtained in this way is the same as given in (11).

3. Geometry and static analysis of finite shear deformation

In this section we investigate the geometric and static motivation of the relation between shear deformations and shear stresses assumed by Becker. The unit cube \(\Omega_0\) aligned to the orthogonal coordinate system \(e_1, e_2, e_3\) in the reference configuration is considered again. We distinguish six variants of orthogonal deformation along the edges of the unit cube such that one edge preserves its length; see Figure 10.

Specifying the deformation \(\Phi_\alpha : \Omega_0 \rightarrow \Omega_1\) as pure finite shear with ratio \(\alpha\), the volume in the actual configuration \(\Omega_1\) is invariant. Thus, stretching one edge in the plane of deformation by \(\alpha\) must result in a contraction \(1/\alpha\) of the corresponding edge; see also Figure 11. The volume invariance holds for any intermediate configuration \(\Omega_i\) and goes along with the multiplicative and nonlinear behaviour of finite deformation. We consider the intermediate configuration defined by the symmetry condition
\[
\Phi_\sqrt{\alpha} : \Omega_0 \rightarrow \Omega_i, \quad \Phi_\sqrt{\alpha} : \Omega_i \rightarrow \Omega_1.
\]
(13)
Let us inscribe a circle and a square into $\Omega_0$ as drawn on the left-hand side of Figure 11. Then $\Phi_{\sqrt{\alpha}}$ deforms the circle into an ellipse and the square into a rhombus. Increasing values for $\alpha$ decrease the angle $\psi$ between the normal vector

$$\tilde{n} = \frac{1}{\sqrt{\alpha + 1/\alpha}} \left( \frac{1}{\sqrt{\alpha}} \right)$$

and the horizontal direction $e_1$; see Figure 12.

Becker denotes the scalar product $\langle n, e_i \rangle$ between the normal $n$ of a plane and the coordinate axis $e_i$ as ‘direction cosines’ $n_i$ of the plane \[19, \text{p. 338}\]. For the normal $\tilde{n}$ we find

$$n_1 = \langle \tilde{n}, e_1 \rangle = \sqrt{\alpha^2 + 1}, \quad n_2 = \langle \tilde{n}, e_2 \rangle = \sqrt{1 + 1/\alpha^2} \quad \Rightarrow \quad n_1 = \alpha n_2;$$

thus the inclination $\psi$ of the normal $\tilde{n}$ is given by

$$\cot \psi = \frac{n_1}{n_2} = \alpha.$$  

Since here the direction $e_1$ is the contractile axis, it follows from the calculations in Section 2.2.1 that $\tilde{n}$ is normal to the plane of no distortion. The rhombus in Figure 12 therefore describes the two directions of no distortion. We explain this fact geometrically for the line perpendicular to $\tilde{n}$ in Figure 13. Drawing this line onto the deformed body $\Omega_1$ and then releasing the deformation leads to a rotated but undistorted line in $\Omega_0$. Thus, the intermediate configuration $\Omega_i$ displays an important geometrical aspect in a pure shear deformation.

These properties, Becker argues, suggest that the normal component of the Cauchy stress in the directions of no distortion has to vanish, that is, $N = 0$, ‘for otherwise there could be no planes of zero distortion’ \[8, \text{p. 338}\]. Combining this assumption with the equality $n_1/n_2 = \alpha$ from (15), it follows that the components of the principal Cauchy stresses need to fulfil

$$-\sigma_1 \alpha = \frac{\sigma_2}{\alpha}, \quad \sigma_3 = 0,$$
Figure 13. The plane of no distortion preserves its length after release of the finite shear deformation.

Figure 14. Static analysis in the actual configuration along the cut of a plane of no distortion.

where \(-\sigma_1 \alpha\) and \(\sigma_2 / \alpha\) are the resultant loads; note that here, \(e_1\) is the contractile axis. From the principal Cauchy stresses in equation (17) he infers that the action of ‘two equal loads […] of opposite signs at right angles to one another’ [19, p. 339] must correspond to a simple finite shear.⁸

Next, we discuss one of the six variants in Figure 10 without loss of generality. The deformation in Figure 14 results in principal stresses of horizontal and vertical direction. Defining the Cauchy stress as load per actual area, the quantity \(-\sigma_1 \alpha\) represents a compressive force in the horizontal axis and \(\sigma_2 / \alpha\) is a tensional force in the vertical direction. Equation (17) yields that these forces are of the same amount, say \(Q\). Then, for a static analysis, we cut the body in \(\Omega_1\) by a plane of no distortion. The lower (respectively, upper) part of the body is in an equilibrium balanced by

\[
T = Q \begin{pmatrix} -1/\alpha \\ 1 \end{pmatrix}; \text{ respectively } T = Q \begin{pmatrix} 1 - 1 + 1/\alpha \\ -1 \end{pmatrix} = Q \begin{pmatrix} 1/\alpha \\ -1 \end{pmatrix}.
\]  

(18)

The scalar product of \(T\) with \(\vec{n}\) from equation (14) vanishes and thus, the equilibrium is free of normal components in the plane of no distortion as required; see Figure 14.

Becker concludes that the six variants of simple finite shearing in Figure 10 are caused by pairwise forces as sketched in Figure 15. This is in concordance with one of five possible invariance conditions for the definition of pure shear [36], namely that the shear tensor is a planar deviator. Note that in the case of isotropic material it is obvious to claim that \(P = Q = R\).

3.1. Mohr’s stress circle for finite shear

In this section we discuss the plane of no distortion in the context of Mohr’s stress circle.⁹ The Mohr circle is a graphical method to determine the stress components acting on a plane rotated against the coordinate system.
Figure 15. Six variants of finite pure shear and the corresponding loads.

Given any symmetric stress tensor in $\mathbb{R}^{2\times2}$, Mohr’s stress circle allows for a graphical solution of the spectral decomposition. Vice versa, knowing Mohr’s stress circle and its spectral decomposition, the tensor components are graphically given for arbitrary but orthogonal coordinate systems, for example for a coordinate system in alignment with the plane of no distortion.

It is important to note that the plane of maximal shear stress $\rho_{\text{max}}$ does not coincide with the plane of no distortion for finite shear. Considering the loading in Figure 14 together with equation (17) results in the principal stresses

$$\sigma_1 = -\frac{Q}{\alpha}, \quad \sigma_2 = Q\alpha. \quad (19)$$

Therefore, Mohr’s stress circle for the finite pure shear loading is drawn in Figure 16.

Becker introduces the quantity $2s = \alpha - 1/\alpha$ as the ‘amount of the shear’ [28]. Thus, the centre of the circle is shifted from the origin of the axis by

$$\sigma_m = \frac{\sigma_1 + \sigma_2}{2} = \frac{Q}{2} \left( \alpha - \frac{1}{\alpha} \right) = Qs. \quad (20)$$

For $\alpha > 1$ the angle $\vartheta = \pi/4$ pointing to the plane of maximum shear stress $\rho_{\text{max}}$ does not coincide with the angle $\psi$ describing the plane of no distortion. Demanding $\sigma_{\xi} = 0$ in the plane of no distortion determines the rotation angle of the $\xi, \eta$ coordinate system by

$$\psi = \frac{1}{2} \arccos \left( \frac{\alpha^2 - 1}{\alpha^2 + 1} \right) = \arccot(\alpha), \quad (21)$$

which is in accordance with equation (16). For $\psi = \arccot(\alpha)$ the stress components are given by

$$\sigma_{\xi} = 0, \quad \sigma_\eta = Q \frac{\alpha^2 - 1}{\alpha}, \quad \sigma_{\xi\eta} = \rho_0 = Q. \quad (22)$$
Thus, in the plane of no distortion the shear stress \( \rho_0 \) is a simple function of the loading \( Q \) and independent of \( \alpha \).

### 3.2. Tangential force and failure criteria in finite shear

To discuss the tangential force in the spirit of continuum mechanics, we analyse the equilibrium of stresses in the vicinity of a point. In case of simple finite shear the surrounding of a point is a circle in \( \Omega_0 \) (respectively, a shear ellipse in \( \Omega_1 \)); see Figure 11. Considering the diameter of the circle to be 1, the diameter \( r \) of the ellipse is given in terms of \( \alpha \) and the direction cosines \( n_1, n_2 \) from equation (15) by

\[
\frac{1}{r^2} = \alpha^2 n_2^2 + \frac{n_1^2}{\alpha^2}. \tag{23}
\]

Considering the finite shear loading from Figure 14 with principal Cauchy stresses from equations (19) gives a resultant force \( \mathcal{R} \) with constant magnitude \( Q \) on any line cutting the ellipsis through the mid-point

\[
\mathcal{R}^2 = r^2 \| \sigma n \|^2 = r^2 \left\| \left( -\frac{Q}{\alpha} n_1, \frac{n_2}{\alpha} \right) \right\|^2 = r^2 Q^2 \left( \frac{n_1^2}{\alpha^2} + \alpha^2 n_2^2 \right) = Q^2. \tag{24}
\]

The normal \( n \) of the line defines the direction cosines \( n_1, n_2 \) as explained in equation (15). Next, we investigate the magnitude of \( \mathcal{R} \) normal to the line, which is given by

\[
\mathcal{N}^2 = r^2 \langle \sigma n, n \rangle ^2 = r^2 \left( \left( -\frac{Q}{\alpha} n_1 \right), \left( \frac{n_2}{\alpha} \right) \right)^2 = r^2 Q^2 \left( \frac{n_1^2}{\alpha^2} + \alpha n_2^2 \right)^2. \tag{25}
\]

The magnitude of stress \( T \) tangential to the plane follows from the Pythagorean theorem by

\[
T^2 = \mathcal{R}^2 - \mathcal{N}^2. \tag{26}
\]

Thus the plane of maximal tangential force is due to \( \mathcal{N}^2 = 0 \) and we conclude from equation (25)

\[
\frac{-n_1^2}{\alpha} + \alpha n_2^2 = 0 \iff n_1^2 = n_2^2 \alpha^2. \tag{27}
\]

The planes of no distortion fulfil equation (27) and attend Becker’s discussion on failure criteria:11 ‘Rupture by shearing is determined by maximum tangential load, not [Cauchy] stress’ [19, p. 339]; see Appendix B.4. In the present example the maximum shear stress appears in a cut inclined at \( \psi = 45^\circ \) to the horizontal axis which does not, however, maximize the tangential load. The situation is illustrated in Figure 17.
Figure 17. Equilibrium and resultant $R$ (respectively $\overline{R}$) on lines cutting the finite shear ellipsis; note that $R^2 > T^2$.

Note that the shear stress $\rho_0 = Q/1 = Q$ in the plane of no distortion is lower than the Tresca and the von Mises stresses for this kind of loading:

$$\sigma^{\text{Tresca}}_y = Q(\alpha + 1/\alpha), \quad \sigma^{\text{Mises}}_y = Q\sqrt{\alpha^2 + 1 + 1/\alpha^2}, \quad \sigma_{\xi\eta} = \sigma^{\text{Becker}}_y = Q.$$  \hspace{1cm} (28, 29, 30)

Thus, the inequality $\sigma^{\text{Becker}}_y < \sigma^{\text{Mises}}_y < \sigma^{\text{Tresca}}_y$ for $\alpha > 0$ shows that the tangential force in the plane of no distortion represents a more conservative lower bound as failure criterion.

It is worth mentioning that the Tresca and the von Mises stresses would account for both the loading $Q$ and the deformation $\alpha$. However, decoupling the failure criteria from the deformation suggests a basic model, which is also simple from an experimental point of view.

4. The axiomatic approach

Our aim in this section is to formulate Becker’s stress–stretch relation for ideally elastic, isotropic materials in terms of the Biot stress tensor $T^\text{Biot}$ and the right Biot stretch tensor $U = \sqrt{F^TF}$, in other words, to find a stress response function $U \mapsto T^\text{Biot}(U)$ or an inverse response function $T^\text{Biot} \mapsto U(T^\text{Biot})$ which fulfils Becker’s assumptions listed in Section 2. In order to deduce such a law of ideal elasticity we will introduce a number of axioms corresponding to Becker’s assumptions to uniquely characterize this stress–stretch relationship. In this we will, at first, closely follow Becker’s approach, utilizing all of the given axioms. However, we will later show that the same results can be deduced with only a subset of those assumptions. Furthermore, in order to obtain a more general result, we will formulate our computations in terms of an arbitrary stress tensor instead of only the Biot stress.

4.1. The basic axioms for an isotropic stress–stretch relation

Let $T : \text{PSym}(3) \rightarrow \text{Sym}(3)$, $\mathcal{E} \mapsto T(\mathcal{E})$ denote a matrix function which maps the set of positive definite symmetric matrices $\text{PSym}(3)$ to the set of symmetric matrices $\text{Sym}(3)$. In accordance with our interpretation of $T$ as a stress response function relation we will refer to the argument $\mathcal{E}$ as the stretch and to $T(\mathcal{E})$ as the stress tensor. Furthermore, if the mapping is invertible we will, for given $\bar{T} \in \text{Sym}(3)$, often write $\mathcal{E}(\bar{T})$ to denote the unique $\mathcal{E} \in \text{PSym}(3)$ with $\bar{T} = T(\mathcal{E})$.

Of course we also assume that $T$ fulfils the axioms stated in this section.

The first two axioms are common postulates for an arbitrary stress–stretch relation.

\textbf{Axiom 0.1: Continuous stress response function}

The function $T : \text{PSym}(3) \rightarrow \text{Sym}(3)$, $\mathcal{E} \mapsto T(\mathcal{E})$ is continuous.
Note that this is a weakened version of Becker’s assumption that the function relating stretch and stress is analytic.

**Axiom 0.2: Unique stress-free reference configuration**

The equivalence

\[
T(\varepsilon) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \iff \varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}
\]

holds.

Axiom 0.2 states that the undeformed (and possibly rotated) reference configuration is the only stress-free configuration. Again, this is a weakened version of one of Becker’s assumptions, namely that the stress response function is invertible.

Another basic assumption is that of isotropy: the response of many materials can be idealized to be independent of the direction of applied stresses.

**Axiom 0.3: Isotropy**

The equality

\[
T^{\text{Biot}}(Q^T \varepsilon Q) = Q^T T^{\text{Biot}}(\varepsilon) Q
\]

holds for all \(Q \in O(3)\) and \(\varepsilon \in \text{PSym}(3)\).

Note that isotropy is sometimes defined for \(Q \in SO(3)\) only. However, since \(-Q \in SO(3)\) for \(Q \in O(3) \setminus SO(3)\), even under this narrower definition we find

\[
T(Q^T \varepsilon Q) = T((-Q)^T \varepsilon (-Q)) = (-Q)^T T(\varepsilon) (-Q) = Q^T T(\varepsilon) Q
\]

for all \(Q \in O(3) \setminus SO(3)\) as well.

Since the assumption of isotropy implies that \(\varepsilon\) and \(T(\varepsilon)\) are coaxial [37, Theorem 4.2.4] for all \(\varepsilon \in \text{PSym}(3)\), Axiom 0.3 restricts the possible choices for the considered stretch–stress pair \(\varepsilon, T\) to coaxial pairs of tensors. However, this axiom allows us to formulate the other axioms regarding strains and stresses in terms of the principal stretches and principal stresses only [38]:

For an isotropic nonlinear elastic solid, the principal directions of Cauchy stress \(\sigma\) must coincide with the axes of the Eulerian strain ellipsoid. Also, to fully specify the state of strain in a material element we need only know the three principal stretches \(\lambda_i\) relative to some reference configuration and the principal directions of strain. Thus, the constitutive law is completely determined once the relations between the principal components of Cauchy stress \(\sigma_i\) and principal stretches \(\lambda_i\) are known.

Since any given \(\varepsilon \in \text{PSym}(3)\) is unitarily diagonalizable we can write \(\varepsilon\) as

\[
\varepsilon = Q^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} Q
\]

with \(Q \in O(3)\), where \(\lambda_i\) denote the (positive) eigenvalues of \(\varepsilon\). Then \(T(\varepsilon)\) is given by

\[
T(\varepsilon) = Q^T \cdot T(Q \varepsilon Q^T) \cdot Q = Q^T \cdot T\left( \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \right) \cdot Q.
\]

This allows us to focus on stretches given in a diagonal representation, that is, stretches of the form

\[
\varepsilon = \text{diag}(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}
\]
with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+$. 

### 4.2. Becker's three main axioms

The first two main axioms for our law of ideal isotropic elasticity refer to two special cases of deformation, pure shear and pure volumetric dilation.

**Axiom 1: Pure shear stresses correspond to pure shear stretches**

For every $\alpha \in \mathbb{R}^+$ there exists $s \in \mathbb{R}$ such that

$$E = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \iff T(E) = \begin{pmatrix} s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $T$ is an isotropic tensor function (Axiom 0.3), then Axiom 1 can also be stated in a more general way. Let

$$E = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with

$$\det\begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix} = 1.$$

Then the two eigenvalues of

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix}$$

are mutually reciprocal, and thus $E$ can be diagonalized to

$$E = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix} \in \mathbb{O}(2)$$

and $\alpha \in \mathbb{R}^+$. Using Axiom 1 and Axiom 0.3 we compute

$$T(E) = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \cdot T\left(\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with

$$\text{tr}\begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} = \text{tr}\begin{pmatrix} s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$
In the isotropic case, Axiom 1 may therefore be equivalently stated as follows (see Figure 2 on p. 5): for every
\[
\begin{pmatrix}
B_{11} & B_{12} \\
B_{12} & B_{22}
\end{pmatrix} \in \text{SL}(2)
\]
there exists
\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{12} & A_{22}
\end{pmatrix} \in \text{sl}(2)
\]
such that
\[
E = \begin{pmatrix}
B_{11} & B_{12} & 0 \\
B_{12} & B_{22} & 0 \\
0 & 0 & 1
\end{pmatrix} \iff T(E) = \begin{pmatrix}
A_{11} & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
where \( \text{SL}(n) \) denotes the group of all \( X \in \text{GL}(n) \) with \( \det X = 1 \), and \( \text{sl}(n) \) is the corresponding Lie algebra of all trace-free matrices in \( \mathbb{R}^{n \times n} \).

To further understand the relation between shear stress and shear stretch constituted by Axiom 1 we consider two examples. First, assume that the stress tensor \( T \) corresponding to a stretch tensor \( E \) is a trace-free pure shear stress of the form
\[
T(E) = \begin{pmatrix}
0 & s & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
The eigenvalues of \( T(E) \) are the principal stresses \( T_1 = s, \ T_2 = -s, \ T_3 = 0 \), and thus \( T(E) \) can be diagonalized to
\[
T(E) = Q^T \begin{pmatrix}
s & 0 & 0 \\
0 & -s & 0 \\
0 & 0 & 0
\end{pmatrix} Q
\]
with \( Q \in \text{SO}(3) \). If \( E \mapsto T(E) \) is an isotropic mapping satisfying Axiom 1, then
\[
Q T(E) Q^T = \begin{pmatrix}
s & 0 & 0 \\
0 & -s & 0 \\
0 & 0 & 0
\end{pmatrix} \implies T(Q E Q^T) = \begin{pmatrix}
s & 0 & 0 \\
0 & -s & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
\[
\implies Q E Q^T = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} \implies E = Q^T \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} Q
\]
for some \( \alpha \in \mathbb{R}^+ \). Thus the stretch \( E \) corresponding to a pure shear stress is a pure shear of the form
\[
E = Q^T \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} Q = \begin{pmatrix}
B_{11} & B_{12} & 0 \\
B_{12} & B_{22} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
with
\[
\det \begin{pmatrix}
B_{11} & B_{12} \\
B_{12} & B_{22}
\end{pmatrix} = 1.
\]

Now assume that a deformation gradient \( F \) is a simple glide of the form
\[
F = \begin{pmatrix}
1 & \gamma & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Then the right Biot stretch tensor \( U \) has the form
\[
U = \sqrt{F^T F} = \begin{pmatrix}
1 & \gamma & 0 \\
\gamma & \gamma^2 + 1 & 0 \\
0 & 0 & 1
\end{pmatrix}^{\frac{1}{2}}.
\]
Since 1 is an eigenvalue of $U$ and $\det U = \det F = 1$, the remaining eigenvalues of $U$ must be of the form $\lambda_1 = \alpha$ and $\lambda_2 = 1/\alpha$. Thus the principal stretches of the deformation are $\lambda_1 = \alpha$, $\lambda_2 = 1/\alpha$, $\lambda_3 = 1$ for some $\alpha \in \mathbb{R}^+$ and $U$ can be diagonalized to

$$U = Q^T \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} Q$$

with $Q \in O(3)$. Then, if the function $U \mapsto T(U)$ mapping $U$ to a stress tensor $T$ is isotropic and satisfies Axiom 1, we can compute

$$T(U) = T \left( Q^T \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} Q \right) = Q^T T \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) Q = Q^T \begin{pmatrix} s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0 \end{pmatrix} Q$$

for some $s \in \mathbb{R}$. Therefore the stress tensor $T$ corresponding to a simple glide is a pure shear stress.

The second axiom relates spherical stresses, that is, purely normal stresses with the same magnitude in each direction, to volumetric stretches, that is, uniform stretches in all directions.

**Axiom 2: Spherical stresses correspond to volumetric stretches**

For every $\lambda \in \mathbb{R}^+$ there exists $a \in \mathbb{R}$ such that

$$\mathcal{E} = \lambda \cdot \mathbb{I} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \iff \quad T(\mathcal{E}) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = a \cdot \mathbb{I}.$$

This relation between spherical stresses and volumetric stretches for isotropic elastic materials is highly intuitive: if the initial load is equal in every direction the resulting deformation should be equal in all directions as well and vice versa. However, whether this feature is true for all magnitudes of applied loads depends on the chosen constitutive framework. It is known that Axiom 2 is not satisfied for a number of well-known isotropic nonlinear elastic formulations, such as the Mooney–Rivlin energy and the Ogden type energy [26]. The loss of uniqueness of the symmetric solution is encountered in Rivlin's cube problem [39].

While the first two axioms refer only to stresses and stretches of a specific diagonal form, our third and final axiom states a law of superposition that holds for all coaxial stress–stretch pairs. Recall that we call symmetric matrices $A, B \in \text{Sym}(3)$ coaxial if their principal axes coincide, which is the case if and only if $A$ and $B$ are simultaneously diagonalizable as well as if and only if $A$ and $B$ commute.

**Axiom 3: Law of superposition**

Let $\mathcal{E}_1, \mathcal{E}_2 \in \text{PSym}(3)$ be coaxial. Then

$$T(\mathcal{E}_1 \cdot \mathcal{E}_2) = T(\mathcal{E}_1) + T(\mathcal{E}_2).$$

Note that, for an invertible stress–stretch relation, the third axiom could equivalently be stated as

$$\mathcal{E}(T_1 + T_2) = \mathcal{E}(T_1) \cdot \mathcal{E}(T_2)\quad(31)$$

for all coaxial $T_1, T_2 \in \text{Sym}(3)$.

This law of superposition can be summarized as follows: the (multiplicative) concatenation of stretch tensors should effect the (additive) superposition of the corresponding stress tensors. This nonlinear connection is closely related to a modern approach [40] involving the theory of Lie groups: the deformation tensors correspond to the (multiplicative) group $\text{GL}^+(3)$ while the stress tensors can be represented by the (linear) Lie algebra $\mathfrak{gl}(3)$. By focusing on symmetric stresses $T$ as well as on positive-definite symmetric stretches $\mathcal{E}$ we can
also relate the stretch to the subset $\text{PSym}(3) \subset \text{GL}^+(3)$ and the stress to the subspace $\text{Sym}(3) \subset \text{gl}(3)$. Since the canonical homomorphism mapping the additive structure of $\text{Sym}(3)$ to the multiplicative group structure of $\text{PSym}(3)$ in the way described by equation (31) is exponential function

$$\exp : \text{Sym}(3) \to \text{PSym}(3),$$

which is invertible with its inverse given by principal logarithm

$$\log : \text{PSym}(3) \to \text{Sym}(3),$$

it is to be expected that the resulting stress–stretch relation $E \mapsto T(E)$ is, in turn, logarithmic in nature.

This approach is also closely related to the later deduction of the quadratic Hencky strain energy by Heinrich Hencky, who employed a similar law of superposition to obtain a logarithmic stress response function. However, Hencky considered the superposition of stresses in the deformed configuration: the Cauchy stress $\sigma$ in his 1928 article [2] as well as the Kirchhoff stress $\tau$ in his 1929 article [41] (an English translation of both papers can be found in [3]). Becker on the other hand assumes a law of superposition for initial loads: his law refers to the Biot stress tensor $T^{\text{Biot}}$. A more detailed comparison of Becker’s and Hencky’s work can be found in Appendix B.1.

4.3. Deduction of the general stress–stretch relation from the axioms

We will now show that the general stress–stretch relation is determined by the given axioms up to only two constitutive parameters. Since our law of elasticity is isotropic by assumption we will mostly consider deformations given in the diagonal form

$$F = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Recall from Section 4.1 that the stress–stretch relation is uniquely determined by the stress response to such deformations.

4.3.1. Basic properties. Before we explicitly compute the stress–stretch relation from the axioms, we will first deduce some basic properties. The following properties of symmetry and invertibility follow directly from the law of superposition and the uniqueness of the stress-free reference state.

**Lemma 4.1.** Let $E \in \text{PSym}(3)$. Then $T(E^{-1}) = -T(E)$.

**Proof.** Since $E$ and $E^{-1}$ obviously commute, the law of superposition implies

$$T(E) + T(E^{-1}) = T(E \cdot E^{-1}) = T(\mathbb{I}) = 0,$$

where the last equality is due to Axiom 0.2. ■

**Remark 4.2.** The symmetry property given in Lemma 4.1 is generally not equivalent to the symmetric tension–compression symmetry in hyperelasticity, which is defined by the equality

$$W(F^{-1}) = W(F)$$

for the energy function $W$ and all $F \in \text{GL}^+(3)$. A hyperelastic stress–stretch relation is tension–compression-symmetric if and only if

$$\tau(V^{-1}) = -\tau(V)$$

for all $V \in \text{PSym}(3)$, where $\tau$ denotes the Kirchhoff stress tensor.

**Lemma 4.3.** The mapping $E \mapsto T(E)$ is injective, that is, $T : \text{PSym}(3) \to \text{range}(T)$, $E \mapsto T(E)$ is invertible.
Proof. Let \( \mathcal{E}_1, \mathcal{E}_2 \in \text{PSym}(3) \) with \( T(\mathcal{E}_1) = T(\mathcal{E}_2) \). Then using Lemma 4.1 we find

\[
0 = T(\mathcal{E}_1) - T(\mathcal{E}_2) = T(\mathcal{E}_1) + T(\mathcal{E}_2^{-1}) = T(\mathcal{E}_1 \cdot \mathcal{E}_2^{-1}),
\]

and thus Axiom 0.2 yields \( \mathcal{E}_1 \cdot \mathcal{E}_2^{-1} = \mathbb{I} \) and therefore \( \mathcal{E}_1 = \mathcal{E}_2 \).

Remark 4.4. We will denote the inverse of the stress response by writing \( \mathcal{E}(\hat{T}) \) for \( \hat{T} \in \text{range}(T) \) to denote the unique \( \mathcal{E} \in \text{PSym}(3) \) with \( \hat{T} = T(\mathcal{E}) \).

Combined with the continuity of the stress–stretch relation, the law of superposition allows us to compute the stress response to arbitrary powers of stretches. For a further discussion of non-rational powers of matrices as well as primary matrix functions in general we refer to [42].

Lemma 4.5. Let \( \mathcal{E} \in \text{PSym}(3) \). Then

\[
T(\mathcal{E}^r) = r \cdot T(\mathcal{E})
\]

for all \( r \in \mathbb{R} \).

Proof. Since

\[
T((Q \cdot \mathcal{E} \cdot Q^T)^r) = T(Q \cdot \mathcal{E}^r \cdot Q^T) = Q \cdot T(\mathcal{E}^r) \cdot Q^T
\]

we will assume without loss of generality that \( \mathcal{E} \) is in diagonal form, that is, \( \mathcal{E} = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \) with \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+ \). Then \( \mathcal{E}^r = \text{diag}(\lambda_1^r, \lambda_2^r, \lambda_3^r) \) for each \( r \in \mathbb{R} \).

For \( n \in \mathbb{N} \) we can use the law of superposition to compute

\[
T(\mathcal{E}_r) = T \left( \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix} \right) = T \left( \prod_{k=1}^{n} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \right) = \sum_{k=1}^{n} T \left( \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \right) = n \cdot T \left( \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \right).
\]

Furthermore we find

\[
0 = T(\mathbb{I}) = T(\mathcal{E}^n \cdot \mathcal{E}^{-n}) = n \cdot T(\mathcal{E}) + T(\mathcal{E}^{-n}) \quad \Rightarrow \quad T(\mathcal{E}^{-n}) = -n \cdot T(\mathcal{E}),
\]

and thus \( T(\mathcal{E}^r) = z \cdot T(\mathcal{E}) \) for all \( z \in \mathbb{Z} \). Similarly we compute

\[
z \cdot T(\mathcal{E}) = T(\mathcal{E}^z) = T \left( (\mathcal{E}^{\frac{1}{n}})^n \right) = n \cdot T(\mathcal{E}^{\frac{z}{n}}) \quad \Rightarrow \quad T(\mathcal{E}^{\frac{z}{n}}) = \frac{z}{n}
\]

for \( z \in \mathbb{Z} \) and \( n \in \mathbb{N} \), and therefore \( T(\mathcal{E}^q) = q \cdot T(\mathcal{E}) \) for all \( q \in \mathbb{Q} \). Finally, we simply point to the continuity of \( T \), which follows from Axiom 0.1, to conclude \( T(\mathcal{E}^r) = r \cdot T(\mathcal{E}) \) for all \( r \in \mathbb{R} \).

Remark 4.6. Using our notation \( T \mapsto \mathcal{E}(T) \) for the inverse stress–stretch relation this proposition may equivalently be stated as

\[
\mathcal{E}(r \cdot T) = \mathcal{E}(T)^r \quad \forall r \in \mathbb{R}, \ T \in \text{range}(T).
\]

It is obvious that in a one-dimensional setting, Lemma 4.5 would already characterize the stress response as the logarithm to a fixed base. However, this is not immediately clear in the general case since not every stretch \( \mathcal{E} \in \text{PSym}(3) \) can be written as the real power of a single fixed matrix.

Note also that the assumption of continuity is in fact necessary for the proof of Lemma 4.5.
4.3.2. Spherical stresses. While Axiom 2 relates dilations to purely spherical stresses, no assumption about the amount of stress is made. By using Lemma 4.5, however, it is easy to give an explicit formula for \( T(\mathcal{E}) \) for arbitrary pure dilations \( \mathcal{E} \).

**Lemma 4.7.** There exists \( d \in \mathbb{R} \) such that

\[
T(\lambda \cdot \mathbb{I}) = d \cdot \log(\lambda \cdot \mathbb{I}) = d \cdot \log(\lambda) \cdot \mathbb{I}
\]

for all \( \lambda \in \mathbb{R}^+ \).

**Proof.** Choose \( \lambda_0 \in \mathbb{R}^+ \) with \( \lambda_0 \neq 1 \). Then, according to Axiom 2, the stress response to \( \lambda_0 \cdot \mathbb{I} \) is given by

\[T(\lambda_0 \cdot \mathbb{I}) = a_0 \cdot \mathbb{I}\]

for some \( a_0 \in \mathbb{R} \), and we define \( d = a_0 / \log \lambda_0 \). Then

\[
\lambda = \frac{\log \lambda}{\log \lambda_0}
\]

and

\[
T(\lambda \cdot \mathbb{I}) = T \left( \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right) = \frac{\log \lambda}{\log \lambda_0} \cdot T \left( \begin{pmatrix} \frac{\log \lambda}{\log \lambda_0} & 0 & 0 \\ 0 & \frac{\log \lambda}{\log \lambda_0} & 0 \\ 0 & 0 & \frac{\log \lambda}{\log \lambda_0} \end{pmatrix} \right) = \frac{\log \lambda}{\log \lambda_0} \cdot \begin{pmatrix} \frac{\log \lambda}{\log \lambda_0} & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix}
\]

Finally, if \( \lambda = 1 \), then Axiom 0.2 implies \( T(\lambda \cdot \mathbb{I}) = \lambda(\mathbb{I}) = 0 = c \cdot \log(\mathbb{I}) \).

**Remark 4.8.** Note that this proposition can equivalently be stated as

\[
\mathcal{E}(a \cdot \mathbb{I}) = \exp \left( \frac{1}{d} \cdot a \cdot \mathbb{I} \right) = e^{\frac{a}{d}} \cdot \mathbb{I} \quad \forall a \in \mathbb{R}.
\]

4.3.3. Shear stresses. Let us now consider a pure shear stretch of the form

\[
\mathcal{E} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1/\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

with the ratio of shear \( \alpha \in \mathbb{R}^+ \). Again, while Axiom 1 only provides a general relation between shear stretches and shear stresses, the law of superposition yields a quantitative result for the case of pure shears.

**Lemma 4.9.** There exists \( c \in \mathbb{R} \) such that

\[
T \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1/\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = c \cdot \log \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1/\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = c \cdot \begin{pmatrix} \log(\alpha) & 0 & 0 \\ 0 & -\log(\alpha) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (32)
\]

for all \( \alpha \in \mathbb{R}^+ \).

**Proof.** The proof is analogous to that of Lemma 4.7: choose \( \alpha_0 \in \mathbb{R}^+ \) with \( \alpha_0 \neq 1 \). Then there exists \( s_0 \) such that

\[
T \left( \begin{pmatrix} \alpha_0 & 0 & 0 \\ 0 & 1/\alpha_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} s_0 & 0 & 0 \\ 0 & -s_0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
according to Axiom 1, and we define $c = s_0 / \log \alpha_0$.

Now let $\alpha \in \mathbb{R}^+$. Again we can use Axiom 0.2 and the equality $\log(1) = 0$ to show that the equality obviously holds for $\alpha = 1$, hence we can assume $\alpha \neq 1$ without loss of generality. Then $\alpha = \alpha_0^{(\log\alpha) / (\log\alpha_0)}$ and

$$
T \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = T \left( \begin{pmatrix} \frac{\log\alpha_0}{\log\alpha_0} & 0 & 0 \\ 0 & \frac{1}{\alpha_0} & 0 \\ 0 & 0 & \frac{1}{\log\alpha_0} \end{pmatrix} \right) = \frac{\log\alpha}{\log\alpha_0} \cdot T \left( \begin{pmatrix} \alpha_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \frac{\log\alpha}{\log\alpha_0} \cdot \begin{pmatrix} s_0 & 0 & 0 \\ 0 & -s_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{s_0}{\log\lambda_0} \cdot \begin{pmatrix} \log\alpha & 0 & 0 \\ 0 & -\log\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} = c \cdot \log \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).
$$

\[ \blacksquare \]

**Remark 4.10.** Again, this proposition can also be stated as

$$
\mathcal{E} \left( \begin{pmatrix} s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \exp \left( \frac{1}{c} \cdot \begin{pmatrix} s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} e^{s} & 0 & 0 \\ 0 & e^{-s} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \forall s \in \mathbb{R}.
$$

Although Lemma 4.9 refers only to deformations without strain along the $x_3$-axis, the following corollary shows that a similar property holds for shear deformations along the other principal axes as well.

**Corollary 4.11.** Let $\alpha \in \mathbb{R}^+$. Then

$$
T \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix} \right) = c \cdot \log \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix}
$$

as well as

$$
T \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix} \right) = c \cdot \log \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix}
$$

with $c \in \mathbb{R}$ as given in Lemma 4.9.

**Proof.** Let

$$
Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in O(3).
$$

Then

$$
T \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix} \right) = T \left( Q^T \cdot \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot Q \right) = Q^T \cdot T \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \cdot Q
$$

$$
= Q^T \cdot c \cdot \log \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot Q = c \cdot \log \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

\[ \blacksquare \]
proving (33). To show (34) we let
\[ Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \text{SO}(3) \]
and find
\[
T \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix} \right) = T \left( Q^T \cdot \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot Q \right) = Q^T \cdot T \left( \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right) \cdot Q
\]
\[
= Q^T \cdot c \cdot \log \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot Q = c \cdot \log \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot Q.
\]

4.3.4. The general case. Finally we consider the general case of an arbitrary stretch tensor \( \mathcal{E} \).

**Proposition 4.12.** A stress response function \( \mathcal{E} \mapsto T^{\text{Biot}}(\mathcal{E}) \) fulfils Axioms 0.1 to 0.3 and Axioms 1 to 3 if and only if there exist constants \( G, \Lambda \in \mathbb{R}, G \neq 0, 3 \Lambda + 2 G \neq 0 \) such that
\[
T(\mathcal{E}) = 2 G \cdot \log(\mathcal{E}) + \Lambda \cdot \text{tr}[\log(\mathcal{E})] \cdot \mathbb{I}
\]
or, equivalently, constants \( G, K \in \mathbb{R} \setminus \{0\} \) with
\[
T(\mathcal{E}) = 2 G \cdot \text{dev}_3 \log(\mathcal{E}) + K \cdot \text{tr}[\log(\mathcal{E})] \cdot \mathbb{I}
\]
for all \( \mathcal{E} \in \text{PSym}(3) \), where \( \log : \text{PSym}(3) \to \text{Sym}(3) \) is the principal matrix logarithm and \( \text{dev}_3 X = X - (1/3) \text{tr}(X) \cdot \mathbb{I} \) denotes the deviatoric part of \( X \in \mathbb{R}^{3 \times 3} \).

**Remark 4.13.** A justification for the use of the Lamé constants \( G, \Lambda \) and the bulk modulus \( K \) in these formulae will be given by means of linearization in Section 5.1.

**Proof.** First we consider a stretch tensor \( \mathcal{E} \) in the diagonal form
\[
\mathcal{E} = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix}.
\]

Then \( \mathcal{E} \) can be decomposed multiplicatively into three stretches \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_3 \):
\[
\mathcal{E} = \left( \begin{pmatrix} p & q & r \end{pmatrix} ^{1/3} \cdot 0 \cdot 0 \right) \cdot \left( \begin{pmatrix} p & q & r \end{pmatrix} ^{1/3} \cdot 0 \cdot 0 \right) \cdot \left( \begin{pmatrix} p & q & r \end{pmatrix} ^{1/3} \cdot 0 \cdot 0 \right)
\]
\[
= \left( \begin{pmatrix} p & q & r \end{pmatrix} ^{1/3} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \cdot \left( \begin{pmatrix} p & q & r \end{pmatrix} ^{1/3} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \cdot \left( \begin{pmatrix} p & q & r \end{pmatrix} ^{1/3} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)
\]
\[
\mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3
\]

Using the law of superposition we find \( T(\mathcal{E}) = T(\mathcal{E}_1) + T(\mathcal{E}_2) + T(\mathcal{E}_3) \). Lemma 4.7 allows us to compute
\[
T(\mathcal{E}_1) = T \left( \begin{pmatrix} p & q & r \end{pmatrix} ^{1/3} \cdot \mathbb{I} \right) = d \cdot \log \left( \begin{pmatrix} p & q & r \end{pmatrix} ^{1/3} \right) \cdot \mathbb{I}
\]
\[
= \frac{d}{3} \cdot \log(p q r) \cdot \mathbb{I} = \frac{d}{3} \cdot \log(\det(\mathcal{E})) \cdot \mathbb{I} = \frac{d}{3} \cdot \text{tr}[\log(\mathcal{E})] \cdot \mathbb{I}
\]
with a constant $d \in \mathbb{R}$, while Lemma 4.9 and Corollary 4.11 simply yield
\[ T(\mathcal{E}_2) = c \cdot \log(\mathcal{E}_2), \quad T(\mathcal{E}_3) = c \cdot \log(\mathcal{E}_3) \]
with $c \in \mathbb{R}$. Therefore
\[
T(\mathcal{E}_2) + T(\mathcal{E}_3) = c \cdot \log(\mathcal{E}_2) + c \cdot \log(\mathcal{E}_3)
\]
\[
= c \cdot \begin{pmatrix}
\log \left( \frac{\epsilon^2}{\eta^2} \right) \\
0 \\
0
\end{pmatrix} + c \cdot \begin{pmatrix}
0 \\
\log \left( \frac{\epsilon^2}{\eta^2} \right) \\
0
\end{pmatrix}
\]
\[
= c \cdot \begin{pmatrix}
\log \left( \frac{\epsilon^2}{\eta^2} \right) \\
0 \\
0
\end{pmatrix}
\]
\[
= c \cdot \begin{pmatrix}
\log(p) \\
0 \\
0
\end{pmatrix} + c \cdot \begin{pmatrix}
0 \\
\log(q) \\
\log(r)
\end{pmatrix}
\]
\[
= c \cdot \log(\mathcal{E}) - \frac{c}{3} \cdot \log(\det \mathcal{E}) \cdot \mathbf{I}
\]
\[
= c \cdot (\log(\mathcal{E}) - \frac{1}{3} \cdot \text{tr} \log(\mathcal{E}) \cdot \mathbf{I}) = c \cdot \text{dev}_3 \log(\mathcal{E}).
\]
Thus $T(\mathcal{E})$ computes to
\[
T(\mathcal{E}) = T(\mathcal{E}_2) + T(\mathcal{E}_3) + T(\mathcal{E}_4) = c \cdot \text{dev}_3 \log(\mathcal{E}) + d \cdot \text{tr} \log(\mathcal{E}) \cdot \mathbf{I}.
\]
Finally, for arbitrary $\mathcal{E} \in \text{PSym}(3)$ we can choose $Q \in \text{O}(3)$ and a diagonal matrix $D$ such that $\mathcal{E} = Q^T D Q$.
Utilizing the isotropy property as well as the above computations we find
\[
T(\mathcal{E}) = T(Q^T D Q) = Q^T T^\text{Biot}(D) Q
\]
\[
= Q^T \left( c \cdot \text{dev}_3 \log(D) + d \cdot \text{tr} \log(D) \cdot \mathbf{I} \right) \cdot Q
\]
\[
= c \cdot \text{dev}_3 \log(Q^T D Q) + d \cdot \text{tr} \log(\mathcal{E}) \cdot (Q^T Q) = c \cdot \text{dev}_3 \log(\mathcal{E}) + d \cdot \text{tr} \log(\mathcal{E}) \cdot \mathbf{I},
\]
and thus we obtain equation (35) with $G = c/2$ and $K = d/3$. It is also easy to see that the restrictions $G \neq 0$ and $K \neq 0$ follow directly from the injectivity of the response function. Furthermore, with $\Lambda = K - 2G/3$ we obtain the equivalent representation

$$
T(\mathcal{E}) = 2G \cdot \left[ \log \mathcal{E} - \frac{1}{3} \text{tr}[\log \mathcal{E}] \cdot \mathbb{I} \right] + K \cdot \mathbb{I}
$$

$$
= 2G \cdot \log(\mathcal{E}) + \left( K - \frac{2G}{3} \right) \text{tr}[\log \mathcal{E}] \cdot \mathbb{I} = 2G \cdot \log(\mathcal{E}) + \Lambda \cdot \text{tr}[\log \mathcal{E}] \cdot \mathbb{I}.
$$

It remains to show that the stress response function (35) does indeed satisfy all our axioms. Since the matrix log-arithmetic and the trace operator are continuous functions on $\text{PSym}(3)$, Axiom 0.1 obviously holds. The isotropy of the matrix logarithm immediately implies

$$
2G \cdot \log((Q^T \mathcal{E} Q)) + \Lambda \cdot \text{tr}(Q^T \log(Q \mathcal{E} Q)) \cdot \mathbb{I} = 2G \cdot Q^T \log(Q \mathcal{E} Q) + \Lambda \cdot \text{tr}(Q^T (\log(Q \mathcal{E} Q)) \cdot \mathbb{I}
$$

$$
= 2G \cdot Q^T \log(Q \mathcal{E} Q) + \Lambda \cdot \text{tr}(\log(Q \mathcal{E} Q) \cdot \mathbb{I}) \cdot Q,
$$

and thus Axiom 0.3 holds as well. To show Axiom 0.2 we employ the equivalent representation formula (36): for $G, K \neq 0$ we first note that the mapping

$$
X \mapsto 2G \cdot \text{dev}_3 X + K \cdot \text{tr}[X] \cdot \mathbb{I}
$$

is an isomorphism from $\text{Sym}(3)$ onto itself. Thus

$$
2G \cdot \text{dev}_3 \log(\mathcal{E}) + K \cdot \text{tr}[\log(\mathcal{E})] \cdot \mathbb{I} = 0
$$

if and only if $\log \mathcal{E} = 0$, which is the case if and only if $\mathcal{E} = \mathbb{I}$.

We will now consider the remaining three axioms.

**Axiom 1:** For $\mathcal{E} = \text{diag}(\alpha, 1/\alpha, 1)$ we directly compute

$$
2G \cdot \text{dev}_3 \log(\mathcal{E}) + K \cdot \text{tr}[\log(\mathcal{E})] \cdot \mathbb{I} = 2G \cdot \text{dev}_3 \left( \begin{array}{ccc} \log \alpha & 0 & 0 \\ 0 & -\log \alpha & 0 \\ 0 & 0 & 0 \end{array} \right) + K \cdot \left[ \log \alpha + \log \frac{1}{\alpha} \right] \cdot \mathbb{I}
$$

$$
= 2G \cdot \left( \begin{array}{ccc} \log \alpha & 0 & 0 \\ 0 & -\log \alpha & 0 \\ 0 & 0 & 0 \end{array} \right),
$$

and thus Axiom 1 is fulfilled with $s = 2G \log \alpha$.

**Axiom 2:** For $\mathcal{E} = \lambda \cdot \mathbb{I}$ we compute

$$
2G \cdot \text{dev}_3 \log(\mathcal{E}) + K \cdot \text{tr}[\log(\mathcal{E})] \cdot \mathbb{I} = 2G \cdot \text{dev}_3 \left( \begin{array}{ccc} \log \lambda & 0 & 0 \\ 0 & \log \lambda & 0 \\ 0 & 0 & \log \lambda \end{array} \right) + K \cdot \text{tr}[3 \cdot \log \lambda] \cdot \mathbb{I}
$$

$$
= K \cdot [3 \log \lambda] \cdot \mathbb{I},
$$

and thus Axiom 2 is fulfilled with $a = 3K \log \alpha$.

**Axiom 3:** First assume that $\mathcal{E}_1, \mathcal{E}_2 \in \text{PSym}(3)$ have the diagonal forms

$$
\mathcal{E}_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \mathcal{E}_2 = \begin{pmatrix} \hat{\lambda}_1 & 0 & 0 \\ 0 & \hat{\lambda}_2 & 0 \\ 0 & 0 & \hat{\lambda}_3 \end{pmatrix}.
$$
Then
\[
\log(\mathcal{E}_1 \cdot \mathcal{E}_2) = \log \begin{pmatrix}
\lambda_1 & \hat{\lambda}_1 & 0 & 0 \\
0 & \lambda_2 & \hat{\lambda}_2 & 0 \\
0 & 0 & \lambda_3 & \hat{\lambda}_3 \\
0 & 0 & 0 & \lambda_3 & \hat{\lambda}_3
\end{pmatrix} = \begin{pmatrix}
\log(\lambda_1) & \hat{\lambda}_1 & 0 & 0 \\
0 & \log(\lambda_2) & \hat{\lambda}_2 & 0 \\
0 & 0 & \log(\lambda_3) & \hat{\lambda}_3 \\
0 & 0 & 0 & \log(\lambda_3) & \hat{\lambda}_3
\end{pmatrix}
\]

and therefore
\[
T(\mathcal{E}_1 \cdot \mathcal{E}_2) = 2G \cdot \log(\mathcal{E}_1 \cdot \mathcal{E}_2) + \Lambda \cdot \text{tr}[\log(\mathcal{E}_1 \cdot \mathcal{E}_2)] \cdot \mathbb{I}
\]
\[
= 2G \cdot [\log(\mathcal{E}_1) + \log(\mathcal{E}_2)] + \Lambda \cdot \text{tr}[\log(\mathcal{E}_1) + \log(\mathcal{E}_2)] \cdot \mathbb{I}
\]
\[
= 2G \cdot \log(\mathcal{E}_1) + 2G \cdot \log(\mathcal{E}_2) + \Lambda \cdot \text{tr}[\log(\mathcal{E}_1)] \cdot \mathbb{I} + \Lambda \cdot \text{tr}[\log(\mathcal{E}_2)] \cdot \mathbb{I}
\]
\[
= T(\mathcal{E}_1) + T(\mathcal{E}_2).
\]

Now let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) denote arbitrary coaxial matrices. Then \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) can be simultaneously diagonalized, in other words, there exist diagonal matrices \( D_1 \) and \( D_2 \) as well as \( Q \in O(3) \) with
\[
\mathcal{E}_1 = Q^T D_1 Q, \quad \mathcal{E}_2 = Q^T D_2 Q.
\]

Then
\[
T(\mathcal{E}_1 \cdot \mathcal{E}_2) = T(Q^T D_1 D_2 Q) = Q^T \cdot T(D_1 D_2) \cdot Q
\]
\[
= Q^T \cdot [T(D_1) + T(D_2)] \cdot Q = Q^T T(D_1) Q + Q^T T(D_2) Q
\]
\[
= T(Q^T D_1 Q) + T(Q^T D_2 Q) = T(\mathcal{E}_1) + T(\mathcal{E}_2).
\]

While Becker assumed Axioms 1 and 2 to hold, we will now show that they are, in fact, not necessary to characterize Becker’s law of elasticity but can be deduced from Axioms 0.1–0.3 and Axiom 3 alone.

**Lemma 4.14.** If Axioms 0.1, 0.2, 0.3 and 3 hold, then Axioms 1 and 2 hold as well.

**Proof.** First, for \( \alpha = 1 \) and \( \lambda = 1 \), we find
\[
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
\hat{\lambda} & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \hat{\lambda}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \mathbb{I}
\]
and thus
\[
T \left( \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} \right) = T \left( \begin{pmatrix}
\hat{\lambda} & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \hat{\lambda}
\end{pmatrix} \right) = T(\mathbb{I}) = 0
\]
due to Axiom 0.2. We can therefore assume without loss of generality that \( \lambda \neq 1 \neq \alpha \).

Axiom 1: Let \( \alpha \in \mathbb{R}^+ \) with \( \alpha \neq 1 \) and \( \mathcal{E} = \text{diag}(\alpha, 1/\alpha, 1) \). Then, because the principal axes of \( T(\mathcal{E}) \) and \( \mathcal{E} \) coincide, \( T(\mathcal{E}) \) is in diagonal form as well:
\[
T(\mathcal{E}) = T \left( \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} \right) = \begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{pmatrix}
\]
for some \( a, b, c \in \mathbb{R} \). With
\[
Q = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \in O(3),
\]
the property of isotropy allows us to compute
\[
T \left( \begin{pmatrix}
\frac{1}{\alpha} & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & 1
\end{pmatrix} \right) = T(Q^T \cdot E \cdot Q) = Q^T \cdot T(E) \cdot Q = \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}
\]
and using the law of superposition we find
\[
T(\Pi) = T \left( \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} \right) = Q^T \cdot \left( \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} \right) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} + \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a+b & 0 & 0 \\ 0 & a+b & 0 \\ 0 & 0 & 2c \end{pmatrix}.
\]
Since \(T(\Pi) = 0\) we conclude \(b = -a\) as well as \(c = 0\), and thus \(T(E)\) has the form
\[
T \left( \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix} \right) = \begin{pmatrix} s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0
\end{pmatrix}
\] (37)
with \(s = a\). As was shown in the proof of Lemma 4.3, a function satisfying Axioms 3 and 0.2 is injective, hence
\[
T(E) = \begin{pmatrix} s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0
\end{pmatrix} \iff E = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Axiom 2: Now let \(\lambda \in \mathbb{R}^+\) with \(\lambda \neq 1\) and \(E = \text{diag}(\lambda, \lambda, \lambda) = \lambda \cdot \Pi\). Then
\[
T(E) = T \left( \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{pmatrix} \right) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c
\end{pmatrix}
\]
for some \(a, b, c \in \mathbb{R}\). We let
\[
Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0
\end{pmatrix} \in \text{SO}(3)
\]
to find
\[
T(E) = T(\lambda \cdot \Pi) = T(Q^T \cdot (\lambda \cdot \Pi) \cdot Q)
\]
\[
= Q^T \cdot T(\lambda \cdot \Pi) \cdot Q = Q^T \cdot \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c
\end{pmatrix} \cdot Q = \begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & a
\end{pmatrix}.
\]
Therefore \(a = b\) and \(b = c\), and hence \(T(E)\) has the form
\[
T(\lambda \cdot \Pi) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a
\end{pmatrix} = \lambda \cdot \Pi
\] (38)
with \(a \in \mathbb{R}\). Then the injectivity of \(T\) yields
\[
T(E) = a \cdot \Pi \iff E = \lambda \cdot \Pi,
\]
concluding the proof.
From this lemma and Proposition 4.12, it immediately follows that the reduced set of axioms is sufficient to characterize the stress response function. This result is summarized in the following proposition.

**Proposition 4.15.** Let \( T : \text{PSym}(3) \to \text{Sym}(3) \) be a continuous isotropic tensor function with

\[
T(\mathcal{E}) = 0 \iff \mathcal{E} = \mathbb{I}
\]

and

\[
T(\mathcal{E}_1 \cdot \mathcal{E}_2) = T(\mathcal{E}_2) + T(\mathcal{E}_2)
\]

for all \( \mathcal{E}_1, \mathcal{E}_2 \in \text{PSym}(3) \). Then there exist constants \( G, \Lambda \in \mathbb{R}, G \neq 0, 3 \Lambda + 2 G \neq 0 \) such that

\[
T(\mathcal{E}) = 2 G \cdot \log(\mathcal{E}) + 3 \Lambda \cdot \text{tr}[\log \mathcal{E}] \cdot \mathbb{I}
\]  

(39)

or, equivalently, constants \( G, K \in \mathbb{R} \setminus \{0\} \) with

\[
T(\mathcal{E}) = 2 G \cdot \text{dev}_3 \log(\mathcal{E}) + K \cdot \text{tr}[\log \mathcal{E}] \cdot \mathbb{I}
\]  

(40)

for all \( \mathcal{E} \in \text{PSym}(3) \).

If we assume beforehand that the stress response function is invertible, we can also deduce this general law in terms of the inverse stress–stretch relation: let \( T \) denote a given stress tensor of the form

\[
T = \begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{pmatrix}.
\]

Then \( T \) can be written in form of the additive decomposition

\[
T = \frac{P + Q + R}{3} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{Q + R - 2P}{3} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{P + Q - 2R}{3} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

into two pure shear stresses and one spherical stress. Using Remarks 4.8 and 4.10 we compute

\[
\mathcal{E}(T_1) = e^{\frac{P+Q+R}{3}} \cdot \mathbb{I}
\]

as well as

\[
\mathcal{E}(T_2) = \begin{pmatrix} e^{\frac{Q+R-2P}{3}} & 0 & 0 \\ 0 & e^{\frac{Q+R-2P}{3}} & 0 \\ 0 & 0 & e^{\frac{Q+R-2P}{3}} \end{pmatrix}
\]

and

\[
\mathcal{E}(T_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{P+Q-2R}{3}} & 0 \\ 0 & 0 & e^{\frac{P+Q-2R}{3}} \end{pmatrix}.
\]
Therefore the law of superposition yields
\[
\mathcal{E}(T) = \mathcal{E}(T_1) \cdot \mathcal{E}(T_2) \cdot \mathcal{E}(T_3)
\]
\[
= e^{\frac{Q + Q}{2}} \cdot \left( \begin{array}{ccc} e^{-\frac{Q + Q}{2}} & 0 & 0 \\ 0 & e^{\frac{Q + Q}{2}} & 0 \\ 0 & 0 & 1 \end{array} \right) \cdot \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & e^{\frac{Q + Q}{2}} & 0 \\ 0 & 0 & e^{-\frac{Q + Q}{2}} \end{array} \right)
\]
\[
= e^{\frac{Q}{2}} \cdot \left( \begin{array}{ccc} e^{-\frac{Q + Q}{2}} & 0 & 0 \\ 0 & e^{\frac{Q + Q}{2}} & 0 \\ 0 & 0 & e^{\frac{Q + Q}{2}} \end{array} \right)
\]
\[
= e^{\frac{Q}{2}} \cdot \exp \left( \begin{array}{ccc} \frac{2P-Q-R}{3} & 0 & 0 \\ 0 & \frac{2Q-P-R}{3} & 0 \\ 0 & 0 & \frac{2R-P-Q}{3} \end{array} \right) = e^{\frac{Q}{2}} \cdot \exp \left( \frac{1}{3} \cdot \text{dev}_3 T \right).
\]

With constants \( G = c/2 \) and \( K = d/3 \) our law of ideal elasticity can therefore be stated as
\[
\mathcal{E}(T) = \exp \left( \frac{1}{2G} \cdot \text{dev}_3 T + \frac{1}{9K} \cdot \text{tr}(T) \cdot \mathbb{I} \right) = e^{\frac{Q}{2}} \cdot \exp \left( \frac{1}{3} \cdot \text{dev}_3 T \right). \tag{41}
\]

### 4.3.5. Becker’s stress response function.

Finally, since Becker assumed Axioms 0.1 to 0.3 and 1 to 3 to hold for the Biot stress \( T = T_{Biot} \) and the right Biot stretch tensors \( \mathcal{E} = U \), we conclude that Becker’s law of elasticity is given by
\[
T_{Biot}(U) = 2G \cdot \log(U) + \Lambda \cdot \text{tr}[\log U] \cdot \mathbb{I} \tag{42}
\]
or, equivalently, by
\[
T_{Biot}(U) = 2G \cdot \text{dev}_3 \log(U) + K \cdot \text{tr}[\log U] \cdot \mathbb{I} \tag{43}
\]
\[
= \frac{E}{1 + \nu} \cdot \text{dev}_3 \log(U) + \frac{E}{3(1-2\nu)} \cdot \text{tr}[\log U] \cdot \mathbb{I} \tag{44}
\]
with Young’s modulus
\[
E = \frac{9K G}{3K + G}
\]
and Poisson’s number
\[
\nu = \frac{3K - 2G}{2(3K + G)}.
\]

### 4.4. Application to other stresses and stretches

If we apply Proposition 4.15 to other coaxial stress–stretch pairs, the resulting law of elasticity will, in general, differ from that given by Becker. Two examples of such combinations are especially important: the left stretch tensor \( \mathcal{E} = V = \sqrt{FF^T} \) with the Cauchy stress tensor \( T = \sigma \) as well as the left stretch with the Kirchhoff stress tensor \( T = \tau \). Those cases were considered by Heinrich Hencky in 1928 and 1929, respectively [2, 41]. His approach was remarkably similar to Becker’s: from the assumption of a law of superposition for these two stresses he deduced two laws of idealized elasticity.

**Corollary 4.16.** If the Cauchy stress \( \sigma \) is a continuous isotropic function of the left stretch tensor \( V \) with
\[
\sigma(V) = 0 \iff V = \mathbb{I}
\]
and
\[
\sigma(V_1 \cdot V_2) = \sigma(V_2) + \sigma(V_2)
\]
for all $V_1, V_2 \in \text{PSym}(3)$, then there exist constants $G, K \in \mathbb{R} \setminus \{0\}$ such that
\[
\sigma(V) = 2G \cdot \text{dev} \log(V) + K \cdot \text{tr}[\log V] \cdot \mathbb{I}
\] (45)

for all $V \in \text{PSym}(3)$.

If the Kirchhoff stress $\tau$ is a continuous isotropic function of the left Biot stretch tensor $V$ with
\[
\tau(V) = 0 \iff V = \mathbb{I}
\]
and
\[
\tau(V_1 \cdot V_2) = \tau(V_2) + \tau(V_2)
\]
for all $V_1, V_2 \in \text{PSym}(3)$, then there exist constants $G, K \in \mathbb{R} \setminus \{0\}$ such that
\[
\tau(V) = 2G \cdot \text{dev} \log(V) + K \cdot \text{tr}[\log V] \cdot \mathbb{I}
\] (46)

for all $V \in \text{PSym}(3)$.

It was shown by Hencky that only the latter of these two stress response functions constitutes a hyperelastic law of elasticity: the stress–stretch relation in equation (46) can be obtained from the quadratic Hencky strain energy
\[
W(V) = G \|\text{dev} \log V\|^2 + \frac{K}{2} [\text{tr}(\log V)]^2.
\] (47)

This energy function has also been given another rigorous justification, based on purely differential geometric reasoning, as the geodesic distance of the deformation gradient $F$ to the special orthogonal group $\text{SO}(3)$ with respect to the canonical left-invariant metric on $\text{GL}^+(3)$ [40, 43, 44]. The continuing application of the Hencky strain energy or modifications thereof is described in [12]; see also [45].

4.5. The axioms in the linear case of infinitesimal elasticity

Some of Becker’s assumptions seem to have been adapted from simple results for the linearized theory of elasticity. To explain his motivation it is insightful to discover some of these parallels. The most general stress–strain relation for isotropic homogeneous materials in the case of linear elasticity is
\[
\sigma = 2 G \varepsilon + \Lambda \tr(\varepsilon) \cdot \mathbb{I} = 2 G \text{dev} \varepsilon + K \tr(\varepsilon) \cdot \mathbb{I} = \frac{E}{1 + \nu} \text{dev} \varepsilon + \frac{E}{3(1 - 2\nu)} \tr(\varepsilon) \cdot \mathbb{I},
\]
where $\sigma$ is the linearized stress tensor and $\varepsilon = \text{sym} \nabla u = (1/2)(\nabla u + \nabla u^T)$ is the linearized strain tensor of the deformation $\varphi(x) = x + u(x)$ with the displacement $u : \Omega_0 \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Note that this linear relation is invertible with
\[
\varepsilon = \frac{1}{2G} \text{dev} \sigma + \frac{1}{9K} \tr(\sigma) \cdot \mathbb{I},
\]
similar to the first equality in equation (41).

Since the trace operator is the linear approximation of the determinant at $\mathbb{I}$, that is, $\det(\mathbb{I} + H) = 1 + \tr(H) + \mathcal{O}(\|H\|^2)$, the first-order approximation
\[
\det(\nabla \varphi) = \det(\mathbb{I} + \nabla u) \approx 1 + \tr(\nabla u)
\]
holds for sufficiently small $\|\nabla u\|$. Therefore, the condition
\[
\det(U) = \det(\nabla \varphi) = 1
\]

can be linearized to the equation
\[
\tr(\nabla u) = \tr \varepsilon = 0.
\]

For $\Lambda \neq -2G/3$, this is the case if and only if
\[
0 = \tr \sigma = (2G + 3\Lambda) \cdot \tr \varepsilon,
\]
thus (linearized) isochoric deformations always correspond to trace-free stress tensors $\sigma$ and vice versa, analogously to Axiom 1 for the nonlinear case. Similarly, volumetric stresses occur if and only if the strain is (linearly) volumetric (Axiom 2):

$$\varepsilon = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \iff \sigma = \begin{pmatrix} (2G + 3\Lambda)a & 0 & 0 \\ 0 & (2G + 3\Lambda)a & 0 \\ 0 & 0 & (2G + 3\Lambda)a \end{pmatrix}.$$  

Furthermore it is easy to see that the linearized shear strain

$$\varepsilon = \text{sym} \left[ \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \mathbb{I} \right] = \begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

corresponds to the shear stress (Axiom 1)

$$\sigma = \begin{pmatrix} 0 & G \cdot \gamma & 0 \\ G \cdot \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Finally, we consider two deformation gradients $\nabla \varphi_1 = \mathbb{I} + \nabla u_1$ and $\nabla \varphi_2 = \mathbb{I} + \nabla u_2$ with the corresponding strain tensors $\varepsilon_1$ and $\varepsilon_2$. Then

$$\nabla \varphi_1 \cdot \nabla \varphi_2 = (\mathbb{I} + \nabla u_1) \cdot (\mathbb{I} + \nabla u_2) = \mathbb{I} + \nabla u_1 + \nabla u_2 + \nabla u_1 \cdot \nabla u_2.$$  

By omitting the higher-order term $\nabla u_1 \cdot \nabla u_2$, we find the linear approximation

$$\nabla \varphi_1 \cdot \nabla \varphi_2 \approx \mathbb{I} + \nabla u_1 + \nabla u_2,$$

and hence the strain tensor $\varepsilon$ corresponding to $\nabla \varphi_1 \cdot \nabla \varphi_2$ has the linear approximation

$$\varepsilon \approx \text{sym}(\nabla u_1 + \nabla u_2) = \varepsilon_1 + \varepsilon_2.$$  

Thus, in the linear case, the multiplicative superposition of deformation gradients corresponds to an additive composition of the strain tensors. The law of superposition (Axiom 3) can therefore be linearized to

$$\sigma(\varepsilon_1 + \varepsilon_2) = \sigma(\varepsilon_1) + \sigma(\varepsilon_2),$$

which obviously holds for all $\varepsilon_1, \varepsilon_2 \in \text{Sym}(3)$.

The linear analogies of the three main axioms can therefore be summarized as follows.

### Axiom 1, linear version:

The equivalence

$$\varepsilon = \begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \iff \sigma(\varepsilon) = \begin{pmatrix} 0 & G \cdot \gamma & 0 \\ G \cdot \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

holds for all $\gamma \in \mathbb{R}$.

### Axiom 2, linear version:

The equivalence

$$\varepsilon = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \iff \sigma(\varepsilon) = \begin{pmatrix} (2G + 3\Lambda)a & 0 & 0 \\ 0 & (2G + 3\Lambda)a & 0 \\ 0 & 0 & (2G + 3\Lambda)a \end{pmatrix}$$

holds for all $a \in \mathbb{R}$. 


Axiom 3, linear version:
The equality
\[ \sigma(\varepsilon_1 + \varepsilon_2) = \sigma(\varepsilon_1) + \sigma(\varepsilon_2) \]
holds for all \( \varepsilon_1, \varepsilon_2 \in \text{Sym}(3) \).

Note that many of the properties listed in Section 1.1 have linearized counterparts which are satisfied by the general linear model, for example the (linearized) tension–compression symmetry \( \sigma(-\varepsilon) = -\sigma(\varepsilon) \).

4.5.1. Linearized shear. Becker’s comments on the finite shear response show similarities to the linear case as well. We consider the linearized shear stress
\[ \sigma = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \]
with the corresponding linear shear strain\(^{13}\)
\[ \varepsilon = \begin{pmatrix} 0 & \frac{\gamma}{2} \\ \frac{\gamma}{2} & 0 \end{pmatrix} \]
in the two-dimensional case. Then for given \( n = (n_1, n_2)^T \in \mathbb{R}^2 \) with \( \|n\| = 1 \) we can compute
\[ \|\sigma n\| = \| \begin{pmatrix} s n_2 \\ s n_1 \end{pmatrix} \| = s \cdot \|n\| = s. \]

In order to find a direction of maximum tangential linearized stress (not the tangential load), we decompose the resultant traction \( \sigma n \) in the direction of a given unit normal vector \( n \) into a normal and a tangential part:
\[ \|\sigma n\|^2 = \langle \sigma n, n \rangle^2 + \langle \sigma n, n_\perp \rangle^2, \]
where \( n_\perp \) is a unit vector normal to \( n \). Since for such a stress the resultant \( \|\sigma n\|^2 = s^2 \) is constant,\(^{14}\) that is, independent of the unit normal \( n \), the amount of tangential stress
\[ \langle \sigma n, n_\perp \rangle^2 = s^2 - \langle \sigma n, n \rangle^2 \]
assumes its maximum among all \( n \in \mathbb{R}^2 \) with \( \|n\| = 1 \) if and only if \( \langle \sigma n, n \rangle^2 \) attains its minimum. We find
\[ \langle \sigma n, n \rangle^2 = \begin{pmatrix} \frac{s}{n_2} \\ \frac{s}{n_1} \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = s^2 n_1 n_2, \]
which is minimal if and only if \( n_1 = 0 \) or \( n_2 = 0 \). Since the directions of the principal axes are given by the eigenvectors \((1, 1)^T\) and \((1, -1)^T\) of \( \varepsilon \), the vectors \( n_1 \) and \( n_2 \) cut these axes at angles of 45°.

5. Applications and properties of Becker’s law of elasticity

5.1. Infinitesimal deformations
For small deformations the linear approximation
\[
T(\mathbb{I} + \varepsilon) = 2G \cdot \log(\mathbb{I} + \varepsilon) + \Lambda \cdot \text{tr}[\log(\mathbb{I} + \varepsilon)] \cdot \mathbb{I}
\]
\[
= 2G \cdot (\varepsilon + \mathcal{O}(\|\varepsilon\|^2)) + \Lambda \cdot (\text{tr}[\varepsilon + \mathcal{O}(\|\varepsilon\|^2)]) \cdot \mathbb{I}
\]
\[
= 2G \cdot \varepsilon + \Lambda \cdot \text{tr}(\varepsilon) \cdot \mathbb{I} + \mathcal{O}(\|\varepsilon\|^2)
\]
shows that the stress–stretch relation is compatible with the model of linear elasticity if and only if $G$ and $\Lambda$ are the two Lamé constants. In this case the additional constraints

$$G > 0, \quad K > 0$$

follow from the uniform positivity of the linear strain energy density

$$W_{\text{lin}}(\varepsilon) = G \| \text{dev}_3 \text{sym} \varepsilon \|^2 + \frac{K}{2} [\text{tr}(\varepsilon)]^2 = G \| \text{sym} \varepsilon \|^2 + \frac{\Lambda}{2} [\text{tr}(\varepsilon)]^2.$$ 

5.2. Uniaxial stresses

If the initial load is given by a Biot stress tensor of the form

$$T^{\text{Biot}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we can give an explicit formula for the stretch tensor $U$ corresponding to $T^{\text{Biot}}$: since $\text{tr} T^{\text{Biot}} = Q$ and $\text{dev}_3 T^{\text{Biot}} = T^{\text{Biot}} - \frac{1}{3} \text{tr}[T^{\text{Biot}}] \cdot \mathbb{I}$, we can use equation (41) to find

$$U(T^{\text{Biot}}) = e^{\frac{Q}{9K}} \cdot \begin{pmatrix} e^{-\frac{Q}{2K}} & 0 & 0 \\ 0 & e^{\frac{Q}{2K}} & 0 \\ 0 & 0 & e^{-\frac{Q}{2K}} \end{pmatrix} = e^{\frac{Q}{9K}} \cdot \begin{pmatrix} e^{-\frac{Q}{2\tau}} & 0 & 0 \\ 0 & e^{\frac{Q}{2\tau}} & 0 \\ 0 & 0 & e^{-\frac{Q}{2\tau}} \end{pmatrix}.$$ 

In particular, the deformation along the axis of stress is given by

$$\lambda_2 = e^{\frac{Q}{9K}} \cdot e^{\frac{Q}{3\tau}} = e^{\frac{Q}{9K}} \cdot \frac{1}{e^{\frac{Q}{2\tau}}} = e^{\frac{Q}{9K}} \cdot \frac{1}{e^{\frac{Q}{2\tau}}}, \quad (48)$$

while the deformation along the axes orthogonal to the stress axis is

$$\lambda_1 = \lambda_3 = e^{\frac{Q}{9K}} \cdot e^{-\frac{Q}{3\tau}} = e^{\frac{Q}{9K}} \cdot \frac{1}{e^{\frac{Q}{2\tau}}} = e^{-\frac{Q}{9K}}, \quad (49)$$

The factors $e^{Q/9K}$ and $e^{Q/3\tau}$ appearing in equation (48) are the dilational stretch and the shear stretch, respectively, as given by Becker [19, equation (5) on page 345] as his main result. Furthermore, equation (49) shows that in the case $9K = 6G$, which corresponds to $\nu = 0$ for Poisson’s ratio $\nu$, the stretch along the unstressed axes is 1. Therefore, as in the linear model, there is no lateral contraction in Becker’s model for $\nu = 0$. A similar result holds for Hencky’s elastic law [46, 47].

5.2.1. Application to incompressible materials. To apply the uniaxial stress response to incompressible materials we will now consider the limit $K \to \infty$, that is, we approximate the incompressible case through the nearly incompressible case with a sufficiently large ratio $K/G$. From equations (48) and (49) we readily obtain

$$\lim_{K \to \infty} \lambda_2 = e^{\frac{Q}{9K}},$$

as well as

$$\lim_{K \to \infty} \lambda_1 = \lim_{K \to \infty} \lambda_3 = e^{-\frac{Q}{3\tau}}.$$
and thus in our theory uniaxial stress induces deformations of the form

\[
U = \begin{pmatrix}
\frac{1}{\sqrt{\lambda}} & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \frac{1}{\sqrt{\lambda}}
\end{pmatrix} = \begin{pmatrix}
e^{-\frac{Q}{G}} & 0 & 0 \\
0 & e^{\frac{Q}{G}} & 0 \\
0 & 0 & e^{-\frac{Q}{G}}
\end{pmatrix}
\]

in the incompressible case. Going to the inverse we obtain the formula

\[
Q = 3 \, G \cdot \log(\lambda) = E \cdot \log(\lambda),
\]

where \( E = 3 \, G \) denotes Young’s modulus for incompressible materials.

Equation (50) is identical to the uniaxial stress–stretch relation given by Imbert in 1880 as a phenomenological model for the deformation of vulcanized rubber bands under tension [48, p. 53]. Similarly, in 1893 Hartig applied the same logarithmic law to describe the uniaxial tension and compression of rubber [50]. A comparison of Becker’s results for very large strain to experimental data by Jones and Treloar for the uniaxial deformation of vulcanized rubber [14] as well as the corresponding stress responses for the quadratic Hencky energy and Ogden’s elasticity model [50] are shown in Figure 18. Another possible way to apply Becker’s law to incompressible materials is described in Section 5.5.1.

5.3. Becker’s law of elasticity for simple shear

Consider a simple glide deformation of the form

\[
F = \begin{pmatrix}
1 & \gamma & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

with \( \gamma > 0 \); see Section 2.2.3. Then the polar decomposition of \( F = R \cdot U \) into the right Biot stretch tensor \( U = \sqrt{F^T F} \) of the deformation and the orthogonal polar factor \( R = FU^{-1} \) is given by

\[
U = \frac{1}{\sqrt{\gamma^2 + 4}} \cdot \begin{pmatrix}
\gamma^2 + 2 & 0 & 0 \\
0 & \gamma & 0 \\
0 & 0 & \sqrt{\gamma^2 + 4}
\end{pmatrix}, \quad R = \frac{1}{\sqrt{\gamma^2 + 4}} \cdot \begin{pmatrix}
\gamma & 0 & 0 \\
\frac{2}{\gamma} & \gamma & 0 \\
0 & 0 & \sqrt{\gamma^2 + 4}
\end{pmatrix}
\]
Further, $U$ can be diagonalized to
\[
U = L \cdot \begin{pmatrix}
1 & \frac{1}{2}(\sqrt{\gamma^2 + 4} + \gamma) & 0 \\
0 & 0 & \frac{1}{2}(\sqrt{\gamma^2 + 4} - \gamma) \\
0 & 0 & 0
\end{pmatrix} \cdot L^{-1} = L \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \frac{1}{\lambda_1}
\end{pmatrix} \cdot L^{-1},
\]
where
\[
L = \begin{pmatrix}
0 & 2 & \frac{-2}{\sqrt{\gamma^2 + 4} + \gamma} \\
0 & \sqrt{\gamma^2 + 4} + \gamma & \sqrt{\gamma^2 + 4} - \gamma \\
1 & 0 & 0
\end{pmatrix}
\]
and $\lambda_1 = (1/2)(\sqrt{\gamma^2 + 4} + \gamma)$ denotes the first eigenvalue of $U$, and the principal logarithm of $U$ is
\[
\log U = L \cdot \log \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \frac{1}{\lambda_1}
\end{pmatrix} \cdot L^{-1} = L \cdot \begin{pmatrix}
0 & 0 & 0 \\
0 & \log(\lambda_1) & 0 \\
0 & 0 & -\log(\lambda_1)
\end{pmatrix} \cdot L^{-1}
\]
\[
= \frac{1}{\sqrt{\gamma^2 + 4}} \cdot \begin{pmatrix}
-\gamma & 2 \log(\lambda_1) & 0 \\
-\gamma & 2 \log(\lambda_1) & 0 \\
0 & \gamma \log(\lambda_1) & 0
\end{pmatrix}
\]
Then according to Becker’s law of elasticity, the first Piola–Kirchhoff stress tensor $S_1$ corresponding to $F$ computes to
\[
S_1(F) = R \cdot T^{\text{Biot}}(U) = R \cdot (2G \cdot \log(U) + \Lambda \cdot \log(\det U) \cdot \mathbb{I}) = 2G \cdot R \cdot \log U
\]
\[
= \frac{2G}{\gamma^2 + 4} \begin{pmatrix}
2 & \gamma & 0 \\
-\gamma & 2 & 0 \\
0 & 0 & \sqrt{\gamma^2 + 4}
\end{pmatrix} \cdot \begin{pmatrix}
-\gamma & 2 \log(\lambda_1) & 0 \\
-\gamma & 2 \log(\lambda_1) & 0 \\
0 & \gamma \log(\lambda_1) & 0
\end{pmatrix}
\]
\[
= \frac{2G}{\gamma^2 + 4} \begin{pmatrix}
0 & (4 + \gamma^2) \log(\lambda_1) & 0 \\
0 & 0 & \log(\lambda_1)
\end{pmatrix}
\]
\[
= 2G \cdot \log \left(\frac{1}{2}(\sqrt{\gamma^2 + 4} + \gamma)\right) \cdot \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
Finally, the Cauchy stress $\sigma$ for the simple glide deformation $F$ is
\[
\sigma = \frac{1}{\det F} \cdot S_1(F) \cdot F^T = 2G \cdot \log \left(\frac{1}{2}(\sqrt{\gamma^2 + 4} + \gamma)\right) \cdot \begin{pmatrix}
\gamma & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
Note that $\sigma$ is independent of the Lamé constant $\Lambda$. In particular, the simple shear stress $\sigma_{12}$ corresponding to the amount of shear $\gamma$ is given by
\[
\sigma_{12} = \log \left(\frac{1}{2}(\sqrt{\gamma^2 + 4} + \gamma)\right)
\]
for Becker’s law of elasticity. Figure 19 shows a comparison of the simple shear stress resulting from different constitutive laws with experimental data measured by Jones and Treloar [14] for shear deformations of vulcanized rubber.

5.4. A comparison of Becker’s and Hencky’s laws of elasticity

The stress–stretch relation corresponding to the quadratic Hencky strain energy is, in terms of the Kirchhoff stress $\tau$, the left Biot stretch tensor $V$ and the Finger tensor $B$, given by
\[
\tau_{\text{H}} = 2G \log V + \Lambda \tr(\log V) \cdot \mathbb{I} = G \log B + \frac{\Lambda}{2} \tr(\log B) \cdot \mathbb{I},
\]
(51)
while Becker’s stress–stretch relation, expressed in terms of the Biot stress $T^{\text{Biot}}$, the right stretch tensor $U$ and the right Cauchy-Green deformation tensor $C$, is

$$T^{\text{Biot}}_B = 2 G \log U + \Lambda \text{tr}(\log U) \cdot \mathbb{I} = G \log C + \frac{\Lambda}{2} \text{tr}(\log C) \cdot \mathbb{I}.$$ 

Since, in general,

$$T^{\text{Biot}} = U \cdot S_2 = \det(U) \cdot U^{-1} \cdot \sigma \cdot F^{-T}$$

and

$$\tau = \det(U) \cdot \sigma,$$

where $S_2$ is the symmetric second Piola–Kirchhoff stress and $\sigma$ is the Cauchy stress tensor, we find

$$\sigma = \det(U)^{-1} \cdot F \cdot U^{-1} \cdot T^{\text{Biot}} \cdot F^T \implies \tau = \underbrace{F \cdot U^{-1} \cdot T^{\text{Biot}} \cdot F^T}_{=R} = \underbrace{V^{-1} \cdot F \cdot T^{\text{Biot}} \cdot F^T}_{=R},$$

where the last equality follows directly from the polar decomposition $F = RU = VR$ of the deformation gradient $F$. We employ the identity $C = F^T F = F^{-1} F^T F = F^{-1} BF$ to compute

$$\tau_B = V^{-1} \cdot F \cdot T^{\text{Biot}}_B \cdot F^T = V^{-1} F \cdot \left[ G \log C + \frac{\Lambda}{2} \text{tr}(\log C) \cdot \mathbb{I} \right] \cdot F^T$$

$$= V^{-1} F \cdot \left[ G \log(F^{-1}BF) + \frac{\Lambda}{2} \text{tr}(\log(F^{-1}BF)) \cdot \mathbb{I} \right] \cdot F^T$$

$$= V^{-1} F \cdot \left[ G F^{-1}(\log B) F + \frac{\Lambda}{2} \text{tr}(F^{-1}(\log B)F) \cdot \mathbb{I} \right] \cdot F^T$$

$$= G V^{-1} F F^{-1}(\log B) F F^T + \frac{\Lambda}{2} \text{tr}(\log B) \cdot V^{-1} F F^T_{=B}$$

$$= G V^{-1}(\log B) B + \frac{\Lambda}{2} \text{tr}(\log B) \cdot V^{-1} B$$

$$= V^{-1} \cdot \left[ G(\log B) + \frac{\Lambda}{2} \text{tr}(\log B) \cdot \mathbb{I} \right] \cdot B = \underbrace{V^{-1} \tau_H B}_{=B}.$$

The symmetric tensors $V^{-1}$, $\tau_H$ and $B$ commute because their principal axes coincide, therefore

$$\tau_B = V^{-1} \tau_H B = V^{-1} B \cdot \tau_H = RF^{-1} FF^T \cdot \tau_H = RF^T \cdot \tau_H = V \cdot \tau_H.$$

(52)
Figure 20. Energy (- - -) and Biot stress (— ) according to Becker’s law for extreme volumetric stretches \( \lambda \); the tangent of \( W^{\nu=0}_{\text{Becker}} \) in 0 is vertical.

This identity allows us to obtain an upper estimate for the difference between the Kirchhoff stress corresponding to the Hencky energy and the one given by Becker’s stress–stretch relation:

\[
\tau_B = V \cdot \tau_H = \tau_H + (V - \mathbb{I}) \cdot \tau_H \implies \|\tau_B - \tau_H\| \leq \|V - \mathbb{I}\| \cdot \|\tau_H\|,
\]

where \( \| \cdot \| \) denotes the Frobenius matrix norm on \( \mathbb{R}^{3 \times 3} \). Thus, for very small elastic strains \( \|V - \mathbb{I}\| \ll 1 \), the corresponding Kirchhoff tensors \( \tau_B \) and \( \tau_H \) coincide to lowest order.

5.5. Hyperelasticity

Unlike Hencky’s logarithmic stress–stretch relation, Becker’s idealized response is generally not hyperelastic for arbitrary parameters \( G \) and \( K \). Incidentally, this result was also shown by Carroll [51], who gave an explicit example of a cycle of loading and unloading without conservation of energy, showing that the elastic behaviour modelled, in fact, by Becker’s law \(^{16} \) is not path-independent.

Proposition 5.1. The stress–stretch relation

\[
T^{\text{Biot}}(U) = 2G \cdot \log(U) + \Lambda \cdot \text{tr} \log(U) \cdot \mathbb{I}
\]

is hyperelastic if and only if \( \Lambda = 0 \) or, equivalently, \( \nu = 0 \) for Poisson’s number \( \nu \). In this case \(^{17} \) the energy is given by

\[
W^{\nu=0}_{\text{Becker}}(U) = 2G \left[ <U, \log(U) - \mathbb{I}> + 3 \right] = 2G \left[ <\exp(\log U), \log U - \mathbb{I}> + 3 \right],
\]

which is the maximum entropy function.

However, Becker’s law is, of course, Cauchy-elastic for all admissible choices of parameters since the Cauchy stress depends only on the state of deformation.

Note that the elastic energy \( W^{\nu=0}_{\text{Becker}} \) given by equality (53) does not fulfil some of the constitutive properties listed in Section 1.1. For example, \( \det F \to 0 \) does not generally imply \( W^{\nu=0}_{\text{Becker}}(F) \to \infty \); in fact, \( W^{\nu=0}_{\text{Becker}}(F) \) remains finite even for \( F = 0 \). However, the implication

\[
\det F \to 0 \implies T^{\text{Biot}}(F) \to \infty
\]

holds true for Becker’s elastic law, even in the case \( \nu = 0 \), c.f. Figure 20.
Figure 21. Becker’s law for the uniaxial deformation of incompressible materials, obtained via the limit case $K \to \infty$ (---) and by applying the incompressibility restraint $\det F = 1$ to the energy function $W^{ν=0}_{\text{Becker}}$ (—).

5.5.1. Comparison to the Valanis–Landel energy. In terms of the principal stretches $\lambda_1, \lambda_2, \lambda_3$, that is, the eigenvalues of a stretch tensor $U$, the energy function $W^{ν=0}_{\text{Becker}}$ can be expressed as

$$W^{ν=0}_{\text{Becker}}(U) = 2G \left[ <U, \log(U) - \mathbb{I}> + 3 \right]$$

$$= 2G \cdot \left[ \sum_{i=1}^{3} \lambda_i \cdot (\log(\lambda_i) - 1) \right] + 6G =: \hat{W}^{ν=0}_{\text{Becker}}(\lambda_1, \lambda_2, \lambda_3).$$

This energy function is identical to the Valanis–Landel energy, which was introduced in 1967 by KC Valanis and RF Landel [52]. However, the Valanis–Landel energy is used as a model for incompressible hyperelastic materials exclusively, while $W^{ν=0}_{\text{Becker}}$ is only applicable to Becker’s law of elasticity in the (compressible) case $\nu = 0$ or, equivalently, for $\Lambda = 0$. Since Becker only considers compressible materials, it is not clear how to extend his constitutive law to the incompressible case. One possible way, involving the limit $K \to \infty$, was discussed in Section 5.2.1. Another possibility, however, is to directly apply the incompressibility condition $\det F = 1$, $F$ the deformation gradient, to the hyperelastic model induced by the energy $W^{ν=0}_{\text{Becker}}$. This approach leads to a different result for uniaxial deformations: using the general formula [50]

$$\dot{t} = \frac{d}{d\lambda} \hat{W} \left( \lambda, \frac{1}{\sqrt[3]{\lambda}}, \frac{1}{\sqrt[3]{\lambda}} \right),$$

where $\dot{t}$ is the (uniaxial) load, $\lambda$ is the stretch and $\hat{W}$ is an energy function of an incompressible hyperelastic material expressed in the principal stretches, we find

$$\dot{t}_{\text{Becker}} = \frac{d}{d\lambda} \hat{W}^{ν=0}_{\text{Becker}} \left( \lambda, \frac{1}{\sqrt[3]{\lambda}}, \frac{1}{\sqrt[3]{\lambda}} \right)$$

$$= 2G \cdot \frac{d}{d\lambda} \left[ \lambda \cdot (\log(\lambda) - 1) + 2\lambda^{-1/2} \cdot (\log(\lambda^{-1/2}) - 1) \right] = G \cdot \log(\lambda) \cdot (2 + \lambda^{-3/2}).$$

Figure 21 shows the Biot stress response for uniaxial deformations, computed from the two different applications (50) and (54) of Becker’s law to the incompressible case.
5.6. Becker’s law of elasticity in terms of other stresses and stretches

In Section 5.4 we already established the relation between the left Biot stretch tensor $V$ and the Kirchhoff stress tensor $\tau$ for Becker’s law of elasticity: combining equations (52) and (51) we find

$$\tau(V) = V \cdot (2G \log V + \Lambda \text{tr}(\log V) \cdot \mathbb{I}) = 2G \cdot V \cdot \log V + \Lambda \text{tr}(\log V) \cdot V. \quad (55)$$

Since $\tau = \det(V) \cdot \sigma$ in general, the Cauchy stress tensor $\sigma$ can be expressed as

$$\sigma(V) = \frac{2G}{\det(V)} \cdot V \cdot \log V + \frac{\Lambda}{\det(V)} \cdot \text{tr}(\log V) \cdot V. \quad (56)$$

Furthermore, we can obtain a representation of the symmetric second Piola–Kirchhoff stress tensor $S_2$ from the general formula $T_{\text{Biot}} = U \cdot S_2$:

$$S_2(U) = U^{-1} \cdot T_{\text{Biot}}(U) = 2G \cdot U^{-1} \cdot \log(U) + \Lambda \cdot \text{tr}[\log U] \cdot U^{-1} \quad (57)$$

5.7. Constitutive inequalities

5.7.1. Invertibility of the force-stretch relation. The invertibility of the force-stretch relation, also known as Truesdell’s IFS condition [16, p. 156], is fulfilled by a stress-stretch relation if and only if the mapping $U \mapsto T_{\text{Biot}}(U)$ is invertible. Becker’s law of elasticity satisfies this condition, as was shown in Section 4.

5.7.2. The M-condition. Since the principal matrix logarithm $\log : \text{PSym}(3) \to \text{Sym}(3)$ is strictly monotone [53], the Krawietz M-condition [54]

$$\langle T_{\text{Biot}}(U_1) - T_{\text{Biot}}(U_2), U_1 - U_2 \rangle > 0 \quad \forall U_1, U_2 \in \text{PSym}(3), \ U_1 \neq U_2, \quad (58)$$

where $\langle X, Y \rangle = \text{tr}(Y^T X)$ denotes the canonical inner product on $\mathbb{R}^{3 \times 3}$, is satisfied by the stress-stretch relation $T_{\text{Biot}}(U) = 2G \cdot \log(U)$, that is, in the special case $\Lambda = 0$. However, it is not satisfied in the general case

$$T_{\text{Biot}}(U) = 2G \cdot \log(U) + \Lambda \cdot \text{tr}[\log U] \cdot \mathbb{I}$$

for sufficiently large $K > 0$: choosing

$$U_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad U_2 = \mathbb{I}$$

we find

$$T(U_1) = 2G \cdot \log(U_1) + \Lambda \cdot \text{tr}[\log U] \cdot \mathbb{I}$$

$$= 2G \cdot \begin{pmatrix} \log 2 & 0 & 0 \\ 0 & \log \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \Lambda \cdot \log(\det U_2) \cdot \mathbb{I}$$

$$= 2G \cdot \begin{pmatrix} \log 2 & 0 & 0 \\ 0 & -\log 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \Lambda \cdot \log \left(\frac{1}{2}\right) \cdot \mathbb{I} = 2G \cdot \begin{pmatrix} \log 2 & 0 & 0 \\ 0 & -\log 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \Lambda \cdot \log(2) \cdot \mathbb{I}$$
as well as $T(U_2) = T(\mathbb{I}) = 0$ and thus

$$
\langle T^{\text{Biot}}(U_1) - T^{\text{Biot}}(U_2), U_1 - U_2 \rangle = 2 G \cdot \begin{pmatrix} \log 2 & 0 & 0 \\ 0 & -\log 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \Lambda \cdot \log(2) \cdot \mathbb{I},
\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
= 2 G \cdot \left[ \log(2) + \left(- \log(4)\right) \cdot \left(-\frac{3}{4}\right) \right] - \Lambda \cdot \log(2) \cdot \left[ 1 - \frac{3}{4} \right]
$$

$$
= 2 G \cdot \left[ \log(2) + \frac{3 \cdot \log(2^2)}{4} \right] - \Lambda \cdot \log(2) \cdot \frac{4}{4}
$$

$$
= 2 G \cdot \left[ \frac{4 \cdot \log(2)}{4} + \frac{6 \cdot \log(2)}{4} \right] - \Lambda \cdot \log(2)
$$

$$
= \frac{\log(2)}{4} \cdot \left[ 20 G - \Lambda \right] < 0
$$

for $\Lambda > 20 G$.

5.7.3. Hill’s inequality. Since the energy function of any hyperelastic law satisfying the M-condition is convex in terms of the right Biot stretch tensor $U$, it follows from Section 5.7.2 that the mapping

$$
U \mapsto W^{\nu=0}_{\text{Becker}}(U) = 2 G \left[ < U, \log(U) > - \mathbb{I} > +3 \right]
$$

is convex on PSym(3). However, the mapping $X \mapsto (\exp(X), X - \mathbb{I})$ is not convex on Sym(3). Therefore $W^{\nu=0}_{\text{Becker}}$ does not satisfy Hill’s inequality [55], which holds for an energy function $W : \text{PSym}(3) \to \mathbb{R}$, $U \mapsto W(U)$ if and only if the mapping

$$
\tilde{W} : \text{Sym}(3) \to \mathbb{R}, \quad \tilde{W}(X) = W(\exp(X)),
$$

is convex. This condition is often restated as the convexity of the mapping $\log U \mapsto \tilde{W}(\log U)$ for $\log U \in \text{Sym}(3)$. This inequality is independent of the rank-one convexity of the energy: for example, while it is easy to see that the quadratic Hencky strain energy fulfils Hill’s inequality, it is not rank-one convex [56, 57].

5.7.4. The Baker–Ericksen inequality. A stress–stretch relation fulfils the Baker–Ericksen inequality if

$$(\sigma_i - \sigma_j) \cdot (\lambda_i - \lambda_j) > 0 \quad \text{for all } \lambda_i \neq \lambda_j,$$

where $\lambda_k$ denotes the $k$th principal stretch and $\sigma_k$ denotes the corresponding principal Cauchy stress, that is, the corresponding eigenvalue of the Cauchy stress tensor $\sigma$.

**Proposition 5.2.** The stress–stretch relation

$$
\sigma(V) = \frac{2 G}{\det(V)} \cdot V \cdot \log V + \frac{\Lambda}{\det(V)} \cdot \text{tr}(\log V) \cdot V
$$

does not satisfy the Baker–Ericksen inequality for any $G > 0$, $3 \Lambda + 2 G > 0$.

**Proof.** We assume without loss of generality that $V$ is in the diagonal form $V = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1, \lambda_2, \lambda_3 > 0$. Then

$$
\log(V) = \log \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \log \lambda_1 & 0 & 0 \\ 0 & \log \lambda_2 & 0 \\ 0 & 0 & \log \lambda_3 \end{pmatrix}
$$

and

$$
\sigma(V) = \frac{2 G}{\lambda_1 \lambda_2 \lambda_3} \cdot V \cdot \log V + \frac{\Lambda}{\lambda_1 \lambda_2 \lambda_3} \cdot \text{log}(\lambda_1 \lambda_2 \lambda_3) \cdot V
$$

$$
= \frac{2 G}{\lambda_1 \lambda_2 \lambda_3} \cdot \begin{pmatrix} \lambda_1 \cdot \log \lambda_1 & 0 & 0 \\ 0 & \lambda_2 \cdot \log \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \cdot \log \lambda_3 \end{pmatrix} + \frac{\Lambda}{\lambda_1 \lambda_2 \lambda_3} \cdot \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.
$$
The principal stresses \( \sigma_k \) are the diagonal entries of \( \sigma \), thus

\[
\sigma_k = \frac{\lambda_k}{\lambda_1 \lambda_2 \lambda_3} \cdot (2G \log \lambda_k + \Lambda \log(\lambda_1 \lambda_2 \lambda_3)).
\] (59)

We let \( \lambda_1 = 1/e, \lambda_2 = 1/e^2 \) and \( \lambda_3 = e^3 \) to find

\[
\sigma_1 = \frac{1}{e} \cdot \frac{1}{e-1} \cdot e^3 \cdot \left(2G \log \left(\frac{1}{e}\right) + \Lambda \log \left(\frac{1}{e} \cdot \frac{1}{e^2} \cdot e^3\right)\right) = \frac{2G}{e} \cdot \log \left(\frac{1}{e}\right) = -\frac{2G}{e}
\]

as well as

\[
\sigma_2 = \frac{1}{e} \cdot \frac{1}{e-1} \cdot e^3 \cdot \left(2G \log \left(\frac{1}{e^2}\right) + \Lambda \log \left(\frac{1}{e} \cdot \frac{1}{e^2} \cdot e^3\right)\right) = \frac{2G}{e^2} \cdot \log \left(\frac{1}{e^2}\right) = -\frac{4G}{e^2}.
\]

Since \( \lambda_1 > \lambda_2 \) we find

\[
(\sigma_1 - \sigma_2) \cdot (\lambda_1 - \lambda_2) > 0 \iff \sigma_1 > \sigma_2 \iff -\frac{2G}{e} > -\frac{4G}{e^2} \iff 1 < \frac{2}{e},
\]

showing that the Baker–Ericksen inequality does not hold in this case.

Therefore Becker’s law does not satisfy the rank-one convexity condition either, since a rank-one convex stress–stretch relation always fulfils the Baker–Ericksen inequality. In contrast, Hencky’s elastic law (see (46)) does fulfil the Baker–Ericksen inequality [12].

5.7.5. The \textit{ordered force inequalities}. An isotropic stress–stretch relation satisfies the ordered force inequalities (or ‘OF inequalities’) if

\[
(T_i - T_j) \cdot (\lambda_i - \lambda_j) \geq 0 \quad \text{for all } i, j \in \{1, 2, 3\}, \ i \neq j,
\] (60)

where \( \lambda_i, \lambda_j \) are the principal stretches of a deformation and \( T_i, T_j \) are the corresponding principal forces, that is, the eigenvalues of the Biot stretch tensor \( T^{\text{Biot}} \). To show that Becker’s law fulfils the OF inequalities for all \( G > 0, 3 \Lambda + 2G > 0 \), we assume without loss of generality that a given stretch tensor \( U = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \) and compute

\[
T^{\text{Biot}} = 2G \cdot \log(U) + \Lambda \cdot \text{tr}[\log(U)] \cdot \mathbf{I}
\]

\[
= 2G \cdot \begin{pmatrix}
\log(\lambda_1) & 0 & 0 \\
0 & \log(\lambda_2) & 0 \\
0 & 0 & \log(\lambda_3)
\end{pmatrix} + \Lambda \cdot \text{tr} \begin{pmatrix}
\log(\lambda_1) & 0 & 0 \\
0 & \log(\lambda_2) & 0 \\
0 & 0 & \log(\lambda_3)
\end{pmatrix} \cdot \mathbf{I}
\]

\[
= \begin{pmatrix}
2G \log(\lambda_1) & 0 & 0 \\
0 & 2G \log(\lambda_2) & 0 \\
0 & 0 & 2G \log(\lambda_3)
\end{pmatrix} + \Lambda \log(\lambda_1 \lambda_2 \lambda_3) \cdot \mathbf{I}.
\]

The eigenvalues of \( T_i \) of \( T^{\text{Biot}} \) corresponding to the principal stretches \( \lambda_i \) are therefore

\[
\lambda_i = 2G \log(\lambda_i) + \Lambda \log(\lambda_1 \lambda_2 \lambda_3),
\]

and thus (60) can be written as

\[
\left(2G \log(\lambda_i) + \Lambda \log(\lambda_1 \lambda_2 \lambda_3) - 2G \log(\lambda_j) + \Lambda \log(\lambda_1 \lambda_2 \lambda_3)\right) \cdot (\lambda_i - \lambda_j) \geq 0
\]

\[
\iff 2G \cdot (\log(\lambda_i) - \log(\lambda_j)) \cdot (\lambda_i - \lambda_j) \geq 0
\] (61)

Due to the monotonicity of the natural logarithm, (61) holds for all \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+ \) and all \( G > 0 \).
5.8. Existence results

The following proposition represents a basic existence result by Ciarlet [58, Theorem 6.7-1] for solutions to the so-called pure displacement problem in nonlinear elasticity.

**Proposition 5.3.** Let $\Omega \subset \mathbb{R}^3$ be a domain with a boundary $\Gamma$ of class $C^2$, and let

$$E = \frac{1}{2}(C - \mathbb{I}) = \frac{1}{2}((\mathbb{I} + \nabla u)^T(\mathbb{I} + \nabla u) - \mathbb{I})$$

denote the Green–Lagrange strain tensor of a deformation $\varphi(x) = x + u(x)$. Moreover, assume that the constitutive law is of the form

$$S_2(E) = \Lambda \cdot \text{tr}(E) \cdot \mathbb{I} + 2GE + \mathcal{O}(\|E\|^2)$$

with $\Lambda, G > 0$, where $S_2$ denotes the second Piola–Kirchhoff stress tensor. Then for each number $p > 3$ there exists a neighbourhood $Z^p$ of the origin in the space $U^p(\Omega)$ and a neighbourhood $U^p$ of the origin in the subspace $V^p(\Omega) = \{v \in W^{2,p}(\Omega) \mid v = 0 \text{ on } \Gamma\}$ of the Sobolev space $W^{2,p}(\Omega)$ such that for each $f \in Z^p$, the boundary value problem

$$\begin{align*}
-\text{div} S_1(F) &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \Gamma
\end{align*}$$

has exactly one solution $u$ in $U^p$.

To show that Becker’s stress–stretch relation fulfils the conditions of Theorem 5.3 we compute

$$S_2 = U^{-1} \cdot T^{\text{Biot}} = [2G \cdot \log(U) + \Lambda \cdot \text{tr}[\log U] \cdot \mathbb{I}] \cdot U^{-1}$$

$$= [2G \cdot \log(\sqrt{C}) + \Lambda \cdot \text{tr}[\log(\sqrt{C})] \cdot \mathbb{I}] \cdot \sqrt{C}^{-1}$$

$$= \left[G \cdot \log(C) + \frac{\Lambda}{2} \cdot \text{tr}[\log C] \cdot \mathbb{I}\right] \cdot \sqrt{C}^{-1}.$$  \hspace{1cm} (62)

For small enough $\|C - \mathbb{I}\|$, we can employ the series expansion

$$\log(C) = (C - \mathbb{I}) - \frac{1}{2} \cdot (C - \mathbb{I})^2 + \ldots$$

of the matrix logarithm to find

$$S_2 = \left[G \cdot \log(C) + \frac{\Lambda}{2} \cdot \text{tr}[\log C] \cdot \mathbb{I}\right] \cdot \sqrt{C}^{-1}$$

$$= \left[G \cdot (C - \mathbb{I} + \mathcal{O}(\|C - \mathbb{I}\|^2)) + \frac{\Lambda}{2} \cdot \text{tr}[(C - \mathbb{I}) + \mathcal{O}(\|C - \mathbb{I}\|^2)] \cdot \mathbb{I}\right] \cdot \sqrt{C}^{-1}$$

$$= \left[G \cdot (C - \mathbb{I}) + \frac{\Lambda}{2} \cdot \text{tr}[C - \mathbb{I}] \cdot \mathbb{I} + \mathcal{O}(\|C - \mathbb{I}\|^2)\right] \cdot \sqrt{C}^{-1}.$$  \hspace{1cm} (62)

Since

$$\sqrt{C}^{-1} = \mathbb{I} - \frac{1}{2} \cdot (C - \mathbb{I}) + \mathcal{O}(\|C - \mathbb{I}\|^2)$$

for small $\|C - \mathbb{I}\|$, (62) can be expressed as

$$S_2 = \left[G \cdot (C - \mathbb{I}) + \frac{\Lambda}{2} \cdot \text{tr}[C - \mathbb{I}] \cdot \mathbb{I} + \mathcal{O}(\|C - \mathbb{I}\|^2)\right] \cdot \left[\mathbb{I} - \frac{1}{2} \cdot (C - \mathbb{I}) + \mathcal{O}(\|C - \mathbb{I}\|^2)\right]$$

$$= G \cdot (C - \mathbb{I}) + \frac{\Lambda}{2} \cdot \text{tr}[(C - \mathbb{I}) \cdot \mathbb{I} + \mathcal{O}(\|C - \mathbb{I}\|^2)] = \Lambda \cdot \text{tr}(E) \cdot \mathbb{I} + 2GE + \mathcal{O}(\|E\|^2).$$

Proposition 5.3 can therefore be directly applied to Becker’s law of elasticity.
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Notes

1. The different possibilities of a Hookean law for finite deformations have been discussed in [74] and [75].
2. As a proponent of the works of Immanuel Kant [76], it is consequential that Becker rejects the purely empiricist approach as well the rationalist one.
3. This model, with a stress–strain law of the form
   \[ \sigma = G(V - \mathbb{I}) + \frac{\nu}{2} \text{tr}(V - \mathbb{I}) \cdot \mathbb{I} = G [(V - \mathbb{I}) + \frac{1}{2} \text{tr}(V - \mathbb{I}) \cdot \mathbb{I}] \]
   is also called the *rari-constant* theory of isotropic elasticity. The elastic behaviour of many materials, including metal, cannot be described accurately by this one-parameter model.
4. By isotropy, Becker means the absence of any directional information. He does not have a representation theorem for isotropic tensor functions at his disposal; note that Finger’s influential monograph on isotropic nonlinear elasticity in terms of the three principal invariants [77] was not published until 1894.
5. The shear ellipsoid is also discussed by Becker in the context of hyperbolic functions [78, p. xxxii].
6. Equivalent but more implicit definitions are given by Becker [28] and Griggs [32].
7. The equation of the shear ellipse is also mentioned by Becker [19, p. 339].
8. A more detailed description of the decomposition of the Cauchy stress Becker employs to arrive at this result can be found in Appendix B.3.
9. Mohr [79] published his work on the representation and transformation of two-dimensional stresses by means of a circle in 1882.
10. In the context of beams, Culmann [80] published a similar idea in 1866, using a different proof.
11. Becker uses only $Q/3$ as loading in his footnote on rupture, page 65.
12. The role of the plane of no distortion in failure criteria is further discussed by Becker in [29] and [30].
13. The linear shear strain appears in the linearization of a simple shear deformation, which can be written as
   \[ \left( \begin{array}{c} 1 \\ \gamma \\ 1 \end{array} \right) = \mathbb{I} + \left( \begin{array}{c} 0 \\ \frac{\gamma}{2} \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ -\frac{\gamma}{2} \\ 0 \end{array} \right) = \mathbb{I} + \varepsilon + W, \]
   where $\varepsilon \in \text{Sym}(3)$ is a linear shear strain and $W \in \text{sof}(3)$ corresponds to an infinitesimal rotation.
14. Note carefully that this is no longer true in the nonlinear case: $\|\sigma n\|/\|n\|$ is not constant for $\sigma = \text{diag}(s\alpha, -s\alpha^{-1}, 1)$ and $\alpha > 1$, that is, for a (nonlinear) Cauchy stress tensor $\sigma$ corresponding to a pure shear load $T^\text{Biot} = \text{diag}(s, -s, 0)$ and a shear deformation $\mathcal{F} = \text{diag}(\alpha, 1/\alpha, 1)$.
15. More details on the conversion of the material parameters can be found in Appendix B.5.
16. Although the stress response considered by Carroll is identical to the one deduced by Becker, Carroll seems not to be aware of Becker’s work.
17. For example, the parameter $\nu = 0$ is used to model the elastic behaviour of cork.
18. While Imbert’s contributions are also mentioned by Truesdell, he only cites a summary by Mehmke [81], who in turn refers to Hartig [49] instead of Imbert’s original paper.
19. Ludwik arrived at the logarithmic strain measure through the integral $\int_{l_0}^{l} dl/l = \log(l/l_0)$ over the instantaneous strain $dl/l$ for uniaxial elongations.
20. Although the matrix logarithm had already been investigated in 1892 by Metzler [82], Becker, like Imbert and Ludwik, only considers the (scalar) logarithm of individual stretches instead of the logarithm function in a tensorial setting. The efficient computation of the matrix logarithm is still an open field of research [83, 84].
21. Truesdell and Toupin [62, p. 270] also attribute the deduction of the logarithmic measure of strain from a law of superposition to Richter [69], although Becker and Hencky used the same approach much earlier. Furthermore, Truesdell [85, p. 144] claims that ‘Hencky himself did not give a systematic treatment’ when introducing the logarithmic strain measure.
22. Hencky’s fundamental view of the natural sciences and their relation to mathematics are laid out in his philosophical article ‘Über die Beziehungen der Philosophie des „Als Ob“ zur mathematischen Naturbeschreibung’ [86].
23. Prandtl [66] calls a system in which ‘already occurring prestresses have no significant influence on the stresses induced by additional loads, i.e. in which the stresses simply superimpose’ elastically determinate.

24. Note carefully that we have switched $n_1$ and $n_2$ to fit the orientation of our coordinate system as explained in Section 2.2.

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A. Notation

The following notation is employed throughout the article:

\[
\mathbb{I} = \text{diag}(1,1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

identity matrix

\[\Omega_0 \subset \mathbb{R}^3\]

reference configuration

\[\varphi : \Omega_0 \rightarrow \mathbb{R}^3\]

deformation mapping

\[F = \nabla \varphi(x) \in \mathbb{R}^{3 \times 3}\]
deformation gradient

\[U = \sqrt{\text{det}(F)}\]

right Biot stretch tensor

\[C = F^T F = U^2\]

right Cauchy-Green deformation tensor

\[V = \sqrt{\text{det}(F)} F^T\]

left Biot stretch tensor

\[B = FF^T = V^2\]

left Cauchy-Green deformation tensor

\[R = FU^{-1} = V^{-1}F \in SO(3)\]

orthogonal polar factor of the deformation gradient

\[\sigma\]

Cauchy stress tensor, ‘true stress’

\[\tau = \text{det}(F) \cdot \sigma\]

Kirchhoff stress tensor

\[S_1 = \text{det}(F) \cdot \sigma F^{-T}\]

first Piola–Kirchhoff stress tensor, ‘nominal stress’

\[S_2 = \text{det}(F) \cdot F^{-1} \sigma F^{-T}\]

symmetric second Piola–Kirchhoff stress tensor

\[T^\text{Biot} = US_2 = R^T S_1\]

Biot stress tensor

\[G, \Lambda\]

Lamé constants

\[K\]

bulk modulus

\[E\]

Young’s modulus

\[\nu\]

Poisson’s ratio

B. Appendix

B.1. A brief history of logarithmic strain measures

Becker was not the first to consider a law of elasticity based on the logarithm of the principal stretches. In 1880, Imbert proposed a logarithmic stress response function as a model for the uniaxial tension of vulcanized rubber [48], while Hartig applied a similar law to the uniaxial deformation of rubber [49] in 1893. However, both of these approaches are purely phenomenological: neither Imbert nor Hartig consider a theoretical framework or state underlying reasons for the use of a logarithmic strain measure; they merely employ the logarithm to give an approximation of data obtained through (uniaxial) experiments.

Although the present article by Becker was summarized in a review article in Beiblätter zu Wiedemanns Annalen der Physik [59] (by a reviewer only identified as ‘G Lübeck, Berlin’) and cited in Lueger’s Lexikon der
gesamten Technik [60], Becker’s work appears to have gathered little attention outside the field of geology. The introduction of the logarithmic strain measure to the theory of elasticity is therefore often attributed to Ludwik, for example by Hencky [61, p. 175] or Truesdell and Toupin [62, p. 254]. However, the earliest mention of the logarithmic strain by Ludwik appears in his 1909 monograph Elemente der technologischen Mechanik [63] on plastic deformations, while Becker derived a detailed connection between stresses and the logarithm of the principal stretches in 1893. This error of attribution seems to originate from Hencky who, in a 1931 article [61], referred to a brief section on plastic deformations in Hütte’s Des Ingenieurs Taschenbuch [64] where Ludwik is cited. The same misattribution to Ludwik is given by Truesdell and Toupin [62], who do not mention Becker at all.

Becker’s work can be seen as an early attempt to find an idealized law of nonlinear elasticity for finite deformations through deduction from a number of simple assumptions for the behaviour of an ideally elastic material, predating a remarkably similar approach by Hencky, who deduced a logarithmic law of elasticity from the assumption of a law of superposition in his 1928 article ‘Über die Form des Elastizitätsgesetzes bei ideal elastischen Stoffen’ [2, 3]. Unlike Becker, however, Hencky gave an explicit motivation for his assumed law of superposition, which he later expanded upon in his 1929 article ‘Das Superpositionsgesetz eines endlich deformierten relaxationsfähigen elastischen Kontinuums und seine Bedeutung für eine exakte Ableitung der Gleichungen für die zähe Flüssigkeit in der Eulerschen Form’ [3, 65]: referring to Prandtl’s distinction between ‘elastically determinate’ and ‘elastically indeterminate constructs’ [66], Hencky assumes that a law of elasticity for an ideally elastic body should provide ‘elastic determinacy to the greatest extent for epistemological reasons’ [3, p. 19], a requirement motivated by Dingler [67]. From this he concludes that the multiplicative composition of coaxial stretches must effect the additive composition of the respective Cauchy stresses \( \sigma \), leading to the stress response function

\[
\sigma(V) = 2G \cdot \text{dev}_3 \log(V) + K \cdot \text{tr} \log(V) \cdot \mathbb{I},
\]

as described in Corollary 4.16. In a later 1929 article [3, 41], however, Hencky corrected his statements, proposing then that the law of superposition must hold for the Kirchhoff stress tensor \( \tau \) instead of the Cauchy stress. Although his reasoning for this correction is based on Brillouin’s suggestion [68] that the Cauchy stress ‘is not a true tensor of weight 0 but a tensor density’ as well as a ‘lack of group properties for pure deformations in the general case’ [3, p. 20], the fact that the stress–stretch relation

\[
\tau(V) = 2G \cdot \text{dev}_3 \log(V) + K \cdot \text{tr} \log(V) \cdot \mathbb{I}
\]

resulting from this new approach with respect to the Kirchhoff stress \( \tau \) is hyperelastic with the corresponding strain energy

\[
W(V) = G \| \text{dev}_3 \log(V) \|^2 + \frac{K}{2} [\text{tr} \log(V)]^2
\]

can be seen as a motivating factor as well, especially since Hencky in his 1928 article explicitly computed that the stress response (63) does not lead to a path-independent energy potential and is therefore not hyperelastic.

Although his deductions of the stress–stretch relations (63) and (64) from the respective laws of superposition are correct (c.f. Corollary 4.16), Hencky does not provide explicit computations for either one. A proof for a generalized version of this deduction from the law of superposition was later given by Richter [69], who did extensive work on the matrix logarithm in finite elasticity [70–73].

More information on the historical development of nonlinear elasticity theory and logarithmic strain measures in particular as well as related articles by Becker, Hencky, Richter and other authors can be found at http://www.uni-due.de/mathematik/ag_neff/neff_hencky.

B.2. The stress tensors

Throughout his work Becker refers to ‘initial stresses’ as well as ‘final stresses’. Since Becker only considers homogeneous deformations along fixed axes, there is some ambiguity as to which stress tensors are represented by these terms. However, Becker’s remark that ‘[in] a shear of ratio \( \alpha \) with a tensile axis in the direction of \( oy \), minus \( N \alpha \) is the negative stress acting in the direction of the \( x \) axis into the area \( \alpha \) on which it acts’ [19, p. 339] allows us to infer that ‘final stress’ refers to the force per area in the deformed configuration. Furthermore, the terms ‘load’ and ‘initial stress’ are often used interchangeably (e.g. [19], p. 339). Since Becker considers
the deformation of a unit cube [19, p. 339], the load is the force acting on an area of size 1 in the undeformed configuration, hence his equating load and ‘initial stress’ strongly suggests that the latter should be interpreted as ‘force per unit area of the undeformed configuration’.

Note that this information is not sufficient to completely characterize the two stress tensors: since Becker only considers the case of fixed principal axes, the principal directions of the stress tensors are undetermined. However, the assumption of isotropy ensures that the resulting law of elasticity only depends on the principal stress response to deformations along fixed axes. Thus the choice of tensorial directions is irrelevant to the resulting stress–stretch relation.

To simplify the resulting expressions we will therefore interpret the term ‘final stress’ as the Cauchy stress tensor $\sigma$ and the ‘initial stress’ as the Biot stress tensor $T^{\text{Biot}}$.

### B.3. The basic decomposition of traction by Cauchy stress quadrics

Let $\sigma$ denote the symmetric Cauchy stress tensor here and throughout. With respect to its principal axes, $\sigma$ has the diagonal representation

$$
\sigma = \begin{pmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{pmatrix},
$$

(66)

where $\sigma_i$ denotes the $i$th principal stress. Then for a given plane in the deformed configuration, the traction $t$ in direction $n$ is given by

$$
t = \sigma \cdot n,
$$

where $n$ is the unit normal vector of the plane. If $n = (n_1, n_2, n_3)^T$ is the representation of $n$ with respect to the principal axes of $\sigma$, the traction $t$ computes to

$$
t = \sigma \cdot n = \begin{pmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{pmatrix} \cdot \begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix} = \begin{pmatrix}
\sigma_1 n_1 \\
\sigma_2 n_2 \\
\sigma_3 n_3
\end{pmatrix}.
$$

Therefore the magnitude $R$ of the traction, which is also called the resultant stress on the plane by Becker [19, p. 338], is given by

$$
R^2 = \|t\|^2 = \|\sigma n\|^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2.
$$

By decomposing the traction $t = t_N + t_T$ into a tangential part $t_T$ parallel to the plane and a normal part $t_N$ we obtain the magnitude of normal stress $N$ via

$$
N = \langle t, n \rangle = \langle \sigma n, n \rangle = \begin{pmatrix}
\sigma_1 n_1 \\
\sigma_2 n_2 \\
\sigma_3 n_3
\end{pmatrix} \cdot \begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2,
$$

as well as the magnitude of tangential stress $T$: since $R^2 = T^2 + N^2$ by Pythagoras’ theorem we obtain

$$
T^2 = R^2 - N^2
$$

$$
= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2
$$

$$
- (\sigma_1^4 n_1^4 + \sigma_2^4 n_2^4 + \sigma_3^4 n_3^4 + 2 \sigma_1 \sigma_2 n_1^2 n_2^2 + 2 \sigma_1 \sigma_3 n_1^2 n_3^2 + 2 \sigma_2 \sigma_3 n_2^2 n_3^2)
$$

$$
= \sigma_1^2 n_1^2(1 - n_1^2) + \sigma_2^2 n_2^2(1 - n_2^2) + \sigma_3^2 n_3^2(1 - n_3^2)
$$

$$
- 2(\sigma_1 \sigma_2 n_1^2 n_3^2 + \sigma_1 \sigma_3 n_1^2 n_3^2 + \sigma_2 \sigma_3 n_2^2 n_3^2)
$$

$$
= \sigma_1^2 n_1^2 n_2^2 + \sigma_2^2 n_2^2 n_3^2 + \sigma_3^2 n_3^2 n_1^2
$$

$$
- 2(\sigma_1 \sigma_2 n_1^2 n_3^2 + \sigma_1 \sigma_3 n_1^2 n_3^2 + \sigma_2 \sigma_3 n_2^2 n_3^2)
$$

$$
= n_1^2 n_2^2 (\sigma_1^2 - 2 \sigma_1 \sigma_2 + \sigma_2^2) + n_2^2 n_3^2 (\sigma_2^2 - 2 \sigma_1 \sigma_3 + \sigma_3^2) + n_3^2 n_1^2 (\sigma_3^2 - 2 \sigma_2 \sigma_3 + \sigma_1^2)
$$

$$
= (\sigma_1 - \sigma_2)^2 n_1^2 n_2^2 + (\sigma_1 - \sigma_3)^2 n_1^2 n_3^2 + (\sigma_2 - \sigma_3)^2 n_2^2 n_3^2.
$$

(67)

Note that the tangential Cauchy stress $T$ is not the tangential load Becker refers to as a failure criterion [19, p. 339]. In the case of a pure shear, the tangential load is maximal if $n$ is normal to the plane of no distortion (see Appendix B.4), while the tangential stress $T$ attains its maximum if $n$ is normal to ‘planes making angles of 45° with the axes’ [19, p. 339].
B.4. Becker’s computations of the directions of maximum tangential stress

As was discussed in Section 3, the plane of no distortion is the plane of maximum tangential load in Becker’s model. According to Becker ([19], in the footnote on page 339), the tangential load acting on a plane with unit normal \( n = (n_1, n_2, 0)^T \) is \( T r \), where \( 2 \alpha^2 n_1^2 + 2 \alpha^2 n_2^2 \). His computation of the plane of maximum tangential load depends on his assumption that a pure shear deformation \( F \) corresponds to a pure shear stress tensor \( T_{\text{Biot}} \) (Axiom 1). In this case we can compute the Cauchy stress tensor:

\[
F = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \alpha > 1, \quad T_{\text{Biot}} = \begin{pmatrix}
s & 0 & 0 \\
0 & -s & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad s \in \mathbb{R}
\]

\[
\Rightarrow \quad \sigma = \frac{1}{\det U} \cdot U^{-1} \cdot F \cdot T_{\text{Biot}} \cdot F^T = F^{-1} \cdot F \cdot T_{\text{Biot}} \cdot F = \begin{pmatrix}
\alpha s & 0 & 0 \\
0 & -\frac{1}{\alpha} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Then, for a unit vector \( n = (n_1, n_2, 0)^T \), we find

\[
R^2 = \| \sigma n \|^2 = s^2 \cdot (\alpha^2 n_1^2 + \alpha^2 n_2^2)
\]

as well as

\[
N^2 = \langle \sigma n, n \rangle^2 = s^2 \cdot (\alpha n_1^2 - \alpha^{-1} n_2^2)^2.
\]

As Becker states [19, p. 339], the resultant load

\[
R^2 r^2 = \frac{1}{\alpha^2 n_1^2 + \alpha^2 n_2^2} \cdot s^2 \cdot (\alpha^2 n_1^2 + \alpha^2 n_2^2) = s^2
\]

is independent of \( n \). In order to find the normal \( n \) to the plane of maximum tangential load, that is, \( n \) such that \( T^2 r^2 = R^2 r^2 - N^2 \) is maximal, it is therefore sufficient to minimize

\[
N^2 r^2 = r^2 \cdot s^2 \cdot (\alpha n_1^2 - \alpha^{-1} n_2^2)^2.
\]

Since the term is non-negative, the minimum is attained if \( N^2 r^2 = 0 \), which is the case if \( n_2^2 = \alpha^2 n_1^2 \). As we have seen in (10) in Section 2.2.1, this equation characterizes the normals to the plane of no distortion, showing again that they are indeed the planes of maximum tangential load under Becker’s assumptions.

B.5. Conversion of the moduli

Throughout his article, Becker refers to the modulus of cubical dilation (or bulk modulus) \( K \), the modulus of distortion (or shear modulus) \( G \) and Young’s modulus \( E \). His equation [19, p. 343]

\[
Q \left( \frac{1}{9K} + \frac{1}{3G} \right) = \frac{Q}{E}
\]

follows directly from the well-known conversion formula \( E = (9K G)/(3K + G) \) for these moduli:

\[
\frac{1}{9K} + \frac{1}{3G} = \frac{G + 3K}{9 \cdot K \cdot G} = \frac{1}{E}
\]

Similarly, with \( \nu = (3K - 2G)/(2(3K + G)) \) denoting Poisson’s ratio, we find

\[
\frac{1}{9K} - \frac{1}{6G} = \frac{2G - 3K}{18 K G} = -\frac{3K - 2G}{2(3K + G)} \cdot \frac{2(3K + G)}{18 K G} = -\nu \cdot \frac{3K + G}{9 K G} = -\nu \cdot \frac{E}{E}.
\]