Lattice determination of $f_{B_d}$, $f_{B_s}$, and $\xi$

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The search for new physics is very often limited by the size of the theoretical uncertainties, mainly due to the errors affecting the hadronic quantities. This is especially true in the $B$-physics area, where in many cases the errors are largely dominated by quantities like heavy-light decay constants or bag parameters, which are obtained by lattice simulations. If the recent progress of the lattice community in the light quark sector is very impressive, lattice simulations around the $b$-quark mass are still difficult (recent lattice progress has been reviewed at this conference in [1]). In this talk, I summarize the recent lattice determinations of the decay constants and of the bag parameters of the heavy-light and heavy-strange neutral mesons. In the next section, I remind the reader where lattice computations enter in neutral $B$ meson phenomenology. In section 2, I briefly describe the different lattice techniques for the quark $b$, and in section 3, I present and comment some recent lattice results, and discuss the main advantages and the disadvantages of the various methods used to simulate the quarks.
Figure 1: Example of a box diagram which dominates $B_{(q)} - \bar{B}_{(q)}$ mixing ($q = d, s$).

1 Reminders

The physical eigenstate $B^L,H_q$ ($q \in \{d, s\}$) are related to the flavor eigenstate $B_q, \bar{B}_q$ by (see for example [2])

$$B^L,H_q = \alpha B_q \pm \beta \bar{B}_q, \quad |\alpha|^2 + |\beta|^2 = 1 .$$

(1)

Experimentally, by measuring the frequencies of the $B$ oscillations, one can access to the differences of mass

$$\Delta m_q = m_{B^H_q} - m_{B^L_q} .$$

(2)

In the standard model this quantity is dominated by box diagrams with $t$-quark and $W$ exchanges (see fig 1). After performing an OPE to separate the long distance physics from the short distance one, one finds

$$\Delta m_q = C m_{B_q} f^2_{B_q} B_{B_q} |V_{tq} V_{tb}^*|$$

(3)

where $C$ is a (known) numerical constant which contains the short distance effects, and $V_{ij}$ are CKM matrix elements. The non-perturbative part is factorized in the mass $m_{B_q}$, the decay constant $f_{B_q}$ and the bag parameter $B_{B_q}$ of the heavy-light neutral meson $B_q$ :

$$\langle 0 | A_0 | B_q \rangle = f_{B_q} m_{B_q} ,$$

(4)

$$\langle B_q | (\bar{b} \gamma^\mu q)(\bar{b} \gamma^\mu q) | B_q \rangle = \frac{8}{3} B_{B_q} m_{B_q}^2 f^2_{B_q} .$$

(5)

The unitarity of the CKM matrix implies twelve distinct complex relations among the matrix elements, including

$$V_{ud} V^*_ub + V_{cd} V^*_cb + V_{td} V^*_tb = 0 ,$$

(6)

which is traditionally represented by a triangle in the complex plane $(\bar{\rho}, \bar{\eta})$, where $\bar{\rho}$ and $\bar{\eta}$ are Wolfenstein parameters. Hence, setting $q = d$ in eq.(3) one sees that a computation of $f^2_{B_d} B_{B_d}$ implies a constraint on $V_{td} V^*_tb$ (so on the length of one side of the triangle). In the next section I will discuss some difficulties of such a computation.
Because it is expected that some uncertainties cancel in the ratio of bag parameters, one can compute on the lattice the $SU(3)$ breaking ratio $\xi = \frac{B_{Bs,f} f_{Bs}^2}{B_{B_d,f} f_{B_d}^2}$, and then use the experimental value of $\Delta m_s$ in the relation $\frac{\Delta m_s}{\Delta m_d} = \frac{m_{Bs}}{m_{B_d}} \xi^2 \left| \frac{V_{ts}}{V_{td}} \right|^2$. Finally we note that the decay constants and bag parameters of the $B$ mesons also appear in other quantities, like the decay rate differences $\Delta \Gamma$ and the semileptonic CP asymmetries, which also play an important role in the search for new physics (see eg [3]).

2 Lattice computation

Lattice QCD provides a natural framework for the computation of heavy-light decay constant and bag parameters. However, such computations are not that easy for different reasons. First one has to compute the matrix element of an operator (a four quark-operator for the bag parameters) and this matrix element has to be renormalized, non-perturbatively if possible. This renormalization can be complicated if the discretization of the quarks breaks chiral symmetry, which is the case for example if Wilson-type fermions are employed. But the main difficulty comes from the heavy-light nature of the particle considered. On the one hand, in order to keep the discretization effects under control when one simulates a heavy quark of mass $m_Q$, one needs a fine lattice spacing $a$, typically $a \ll m_Q^{-1} \sim 0.04$ fm for a b quark. On the other hand the simulated volume should be large enough to contain the light degrees of freedom, for example one would like the space extent $L$ of the lattice to be larger than the Compton wavelength of a pion $L \gg m_\pi$. Putting these two constraints together, one see that a very large number of points $(L/a)^4$ is needed.

Various strategies have been developed in the literature to circumvent this problem (see e.g. [4, 5] for recent lattice review). First, one can use an effective theory for the b-quark, like non-relativistic QCD (NRQCD) [6], or heavy quark effective theory (HQET) [7]. In that case one replaces the QCD Lagrangian in the heavy sector by an expansion in the inverse quark mass and in the velocity. For the heavy-light systems, one difference between these effective theories is the way $1/m_Q$ terms (or higher order), like for example the kinetic term $(D_\perp^2/(2m))$, are treated. In HQET, only the leading order (where the $b$ quark is infinitely heavy) is kept in the exponent of the path integral, and corrections in inverse power of the heavy quark mass are inserted into the static Green function. In NRQCD, all the terms (up to a given order in the heavy quark expansion) are kept in the Dirac operator. An important consequence (see e.g. [8]) is that the continuum limit $a \to 0$ cannot be taken in NRQCD, in contrast with HQET where the continuum limit exists if the theory is non-perturbatively renormalized.
Another way to simulate heavy quark on the lattice is given by the Fermilab formulation [9], where one uses on-shell Symanzik improvement, treating both $a$ and $1/m_Q$ as short distances, in such a way that the theory still makes sense when $am_Q > 1$. By construction the theory reproduces an effective theory in the heavy quark region by a matching to HQET or NRQCD, and is still well-defined in the limit $a \to 0$. Probably inspired by the Fermilab action, other relativistic heavy quark formulations have been proposed in the literature [10, 11] where one advantage of the latter is the fact that the matching can be done non-perturbatively [12] (a study of the heavy-light and heavy-strange decay constant along this line has been presented very recently [13]).

It is also possible to use the effective theory as a guide, as done for example, in [14] by the Alpha collaboration in a (quenched) computation of the decay constant $f_{B_s}$. They first compute a decay constant at several “heavy” quark masses around the charm, then they redo the computation with an infinitely heavy (static) $b$ quark, and they interpolate to the mass of the $b$ to obtain $f_{B_s}$. Finally let us mention a proposal by ETMC [15], which is somehow a variant of the previous method. The authors construct ratios of quantities possessing a known static limit, evaluate them at various pairs of heavy quark masses around the charm, and interpolate to the physical point.

3 Recent Lattice results

The lattice simulations are now systematically taking into account the sea quark effects (or at least the dominant ones). Here I discuss some results obtained with $n_f = 2$ or $n_f = 2 + 1$ (meaning a doublet of degenerate quarks for the $u$ and the $d$ and a heavier quark for the $s$) flavors of sea quarks. Various discretizations of the light quarks (both in the valence and in the sea sectors) are available on the market. Because also the light quark masses are expensive, a chiral extrapolation is in general necessary to reach the physical light quark masses. In table 1 I summarize some recent results for the decay constants and for the bag parameters, together with the chosen discretization (or the strategy) for the heavy and for the light quark, and the number of dynamical flavors. In the following I comment briefly the different aspects of the various formulations, starting by the $n_f = 2 + 1$ flavors case.

The FNAL/MILC collaborations [16, 17] employ the Fermilab formulation for the heavy quark, on the $n_f = 2 + 1$ MILC ensembles. They use three values of the lattice spacing $a \sim 0.09, 0.12, 0.15$ fm in order to extrapolate to the continuum limit. The light quarks are described by the improved-staggered fermions called Asqtad. Such fermions are numerically cheap and exhibit good chiral properties, but because they use a “rooting” procedure, one can argue that they might not fall into the right uni-
versatility class. A discussion whether or not one should employ such a formulation in a lattice simulation is beyond the scope of this review.

HPQCD [18] uses the same MILC ensembles (and then also the same discretization of the light quarks) and NRQCD for the heavy quark, implying as discussed earlier that the continuum limit cannot be taken. Nevertheless, two different values of the lattice spacing $a \sim 0.09, 0.012$ fm have been studied, and the authors fit their result to a functional form, performing then together the chiral and the continuum extrapolations.

The RBC-UKQCD[19] collaborations employ a completely different approach. The light quark masses are simulated with a Domain-Wall action (which possess an almost exact chiral-flavor symmetry), and the heavy quark mass is taken in the static limit. This is a theoretically interesting framework, since the continuum limit can be taken, the discretization effects are expected to be small and the renormalization is simplified by the good chiral properties of the light fermions. The main drawback of this approach is probably the numerical cost, because the static action is noisy by nature, and the Domain-Wall fermion numerically expensive, it is hard to obtain a good signal-over-noise ratio. For this reason, only one lattice spacing was simulated.

We turn now to the $n_f = 2$ simulations. ETMC has computed the decay constants $f_{B_d}$ and $f_{B_s}$ (but no bag parameters) using twisted-mass fermions. In this formulation, one can obtain automatic O($a$) improvement (ie no order $a$ discretization effects) without having to pay the price of a chiral formulation. For example in the case of four quark operators, one obtains the same simple renormalization pattern as in the continuum. The disadvantages of this formulation is that one breaks isospin symmetry, and that only $n_f = 2$ (or $n_f = 2 + 1 + 1$) flavors or dynamical quarks can be simulated. The two methods employed for the heavy quark rely on an interpolation between the static approximation and the charm region. Depending on the method, three or four different lattice spacings varying in the range $0.064$ fm $< a < 0.1$ fm have been simulated.

Finally the Alpha collaboration uses non-perturbative HQET, developed at the $\Lambda_{QCD}/m_b$ order. The key idea is to match HQET to QCD in a small volume in order to obtain the HQET parameters. This is theoretically advantageous, because the power divergences cancel explicitly and thus the continuum limit can be taken. Moreover one expects the remaining $(\Lambda_{QCD}/m_b)^2$ effects to be small. The light quarks are described by O($a$)-improved fermions. Even if only a preliminary result of $f_B$ is available [20], where a single (but rather fine) lattice spacing $a \sim 0.07$ fm has been considered so far, this is a very encouraging direction for the future.
Table 1: Selection of recent lattice results. I took the liberty to add in quadrature the statistical and the systematic errors. * refers to results which have been published only in proceedings (and the error budget might be incomplete). “Stat.” stands for static, “Int.” for interpolation, “DW” for Domain-Wall and “TM” for Twisted Mass.

| Group          | $f_{B_d}$ (MeV) | $f_{B_s}$ (MeV) | $\xi$      | $n_f$ | Heavy  | Light |
|----------------|-----------------|-----------------|------------|-------|--------|-------|
| FNAL/MILC      | 195(11)*        | 243(11)*        | 1.205(50)* | 2 + 1 | Fermilab | Asqtad |
| HPQCD          | 190(13)         | 231(15)         | 1.258(33)  | 2 + 1 | NRQCD  | Asqtad |
| RBC-UKQCD      |                 | 1.13(12)        |            | 2 + 1 | Stat.  | DW    |
| ETMC           | 191(14)*        | 243(13)*        |            | 2     | Stat. + Int. | TM |
| ETMC           | 194(16)         | 235(12)         |            | 2     | Stat. + Int. | TM |
| ALPHA          | 178(16)*        |                 |            | 2     | Stat+1/m | Clover |

For the decay constants, one notices that all the results are in good agreement, despite the different formulation used. Concerning the SU(3)-breaking ratio, the only non-staggered result has been obtained by RBC-UKQCD, and even if the quoted error is larger than the ones given by FNAL/MILC and by HPQCD (which are based on the same MILC ensembles), we see that also here all the results are compatible.

4 Conclusions and outlook

Lattice QCD is making very good progress, even in the heavy quark sector it becomes now possible to obtain precise results. But there is still a lot of room for improvement, for example, concerning the bag parameters, the results are largely dominated by formulations which employ rooted staggered fermions in the light sector. It is important that the lattice community gives also precise results with other light quark formulations. Different approaches to treat the heavy quark on the lattice, in particular with more solid theoretical foundations, have been developed and tested in the last years. I believe that such approaches, where one gets a much better handle on the systematics errors, are fundamental to constrain the standard model, and hopefully see the effects of new physics. Moreover, we also need a better control of the chiral extrapolations (so pion masses closer to the physical one) and of the discretization effects (eg by using three or more lattice spacings on a range $\sim 0.05 \text{ fm} - 0.1 \text{ fm}$). If the charm quark is quenched, this effect has to be quantified as well. Finally let us mention that some quantities like beyond the standard model contributions to $B$ mesons mixing are still missing a computation with dynamical fermions [*].

*Although at least one computation is on the way [21].
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