ONE DIMENSIONAL WEIGHTED RICCI CURVATURE
AND DISPLACEMENT CONVEXITY OF ENTROPIES

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ABSTRACT. In the present paper, we prove that a lower 1-weighted
Ricci curvature bound is equivalent to a convexity of entropies on
the Wasserstein space. Based on such characterization, we provide
inequalities of Brunn-Minkowski type, Prékopa-Leindler type and
several functional inequalities under the curvature bound.

1. Introduction

The aim of this article is to characterize a lower 1-weighted Ricci
curvature bound in terms of a convexity of entropies on the Wasser-
stein space. Such characterization enables us to produce inequalities
of Brunn-Minkowski type, Prékopa-Leindler type and some functional
inequalities under the lower 1-weighted Ricci curvature bound.

For \( n \geq 2 \), let \((M, d, m)\) be an \( n \)-dimensional weighted Riemannian
manifold, namely, \( M = (M, g) \) is an \( n \)-dimensional complete Riemann-
ian manifold (without boundary), \( d \) is the Riemannian distance on \( M \),
and \( m := e^{-f} \text{vol} \) for some smooth \( f : M \to \mathbb{R} \), where \( \text{vol} \) denotes the
Riemannian volume measure on \( M \). For \( N \in (-\infty, \infty) \), the associated
\( N \)-weighted Ricci curvature \( \text{Ric}^N \) is defined as follows \((2, 8)\):

\[
\text{Ric}^N_f := \text{Ric}_g + \text{Hess} f - \frac{df \otimes df}{N - n},
\]

where \( \text{Ric}_g \) is the Ricci curvature determined by \( g \), and \( df \) and \( \text{Hess} f \)
are the differential and the Hessian of \( f \), respectively. For \( \mathcal{F} : M \to \mathbb{R} \),
we mean by \( \text{Ric}^N_{f,M} \geq \mathcal{F} \) for every \( x \in M \), and for every unit tangent
vector \( v \) at \( x \) we have \( \text{Ric}^N_f (v) \geq \mathcal{F}(x) \). Traditionally, the parameter \( N \)
has been chosen from \([n, \infty]\), and in that case, we have already known
many geometric and analytic properties (see e.g., \([9, 19, 25]\)). On the
other hand, very recently, in the complementary case of \( N \in (-\infty, n) \),
various properties have begun to be studied (see e.g., \([6, 7, 13, 15, 16, 17, 18, 26, 27]\)).
It is well-known that lower $N$-weighted Ricci curvature bounds can be characterized by convexities of entropies on the Wasserstein space via optimal transport theory. Let us consider a curvature condition

\begin{equation}
\text{Ric}^N_{f,M} \geq K
\end{equation}

for $K \in \mathbb{R}$. In the traditional case of $N \in [n, \infty]$, the characterization of the curvature condition \((1.2)\) is due to von Renesse and Sturm \[20\], and Sturm \[21\] for $N = \infty$, and Sturm \[22, 23\], and Lott and Villani \[10, 11\] for $N \in [n, \infty)$. Based on such characterization results, for general metric measure spaces, Sturm \[22, 23\], and Lott and Villani \[10, 11\] have independently introduced the so-called \textit{curvature-dimension-condition} $\text{CD}(K,N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty]$ that is equivalent to the condition \((1.2)\) when $N \in [n, \infty]$ on weighted Riemannian manifolds. Metric measure spaces satisfying the curvature-dimension condition or more restricted version called \textit{Riemannian-curvature-dimension condition} introduced by Ambrosio, Gigli and Savaré \[4\], and Erbar, Kuwada and Sturm \[5\] have been widely studied from various perspectives.

In the complementary case of $N \in (-\infty, n)$, Ohta \[15\] has recently characterized the condition \((1.2)\) for $N \in (-\infty,0)$, and formulated the curvature-dimension condition $\text{CD}(K,N)$ for $K \in \mathbb{R}$ and $N \in (-\infty,0)$ (see also earlier works done by Ohta and Takatsu \[17, 18\]). Ohta \[16\] has also extended this program to the case of $N = 0$.

Now, we are concerned with the characterization problem of lower $N$-weighted Ricci curvature bounds in the case of $N \in (0, n)$. We focus on the case of $N = 1$, especially a curvature condition

\begin{equation}
\text{Ric}^1_{f,M} \geq (n-1) \kappa e^{-\frac{4f}{n-1}}
\end{equation}

for $\kappa \in \mathbb{R}$ introduced by Wylie and Yeroshkin \[27\] from the view point of the study of weighted affine connections. Wylie and Yeroshkin \[27\] have observed that the curvature condition \((1.3)\) is equivalent to a lower Ricci curvature bound by $(n-1)\kappa$ with respect to some weighted affine connection. They further established comparison geometry under the condition \((1.3)\) (more precisely, see Subsection 2.1). In this paper, we will prove that the curvature condition \((1.3)\) can be characterized by a convexity of entropies on the Wasserstein space. Using the equivalence, we conclude inequalities of Brunn-Minkowski type, Prékopa-Leindler type and several functional inequalities under the condition \((1.3)\).

1.1. \textbf{Main result.} To state our main theorem, we introduce a convexity property of entropies on the Wasserstein space. Let $(M,d,m)$ be an $n$-dimensional weighted Riemannian manifold, where $m = e^{-f} \text{vol}$ for
some smooth function \( f : M \to \mathbb{R} \). Let \( \mathcal{P}_{ac}(M) \) be the set of all compactly supported Borel probability measures on \( M \) that are absolutely continuous with respect to \( m \).

Let \( P_2(M) \) denote the set of all Borel probability measures \( \mu \) on \( M \) satisfying \( \int_M d(x,x_0)^2 \, d\mu(x) < \infty \) for some \( x_0 \in M \). Let \( \mathcal{DC} \) stand for the set of all continuous convex functions \( U : [0, \infty) \to \mathbb{R} \) such that a function \( \varphi_U : (0, \infty) \to \mathbb{R} \) defined by \( \varphi_U(r) := r^n U(r^{-n}) \) is convex. For \( U \in \mathcal{DC} \), a functional \( U_m \) on \( P_2(M) \) is defined by

\[
U_m(\mu) := \int_M U(\rho) \, dm,
\]

where \( \rho \) is the density of the absolutely continuous part in the Lebesgue decomposition of \( \mu \) with respect to \( m \). For a function \( H \in \mathcal{DC} \) defined by \( H(r) := n r (1 - r^{-\frac{1}{n}}) \), the functional \( H_m \) on \( P_2(M) \) defined as (1.4) is called the Rényi entropy.

In order to introduce our convexity property of entropies, we need to define a twisted coefficient. For \( t \in [0, 1] \), we consider two lower semi-continuous functions \( d_{f,t}, d_f : M \times M \to \mathbb{R} \) by

\[
d_{f,t}(x,y) := \inf_{\gamma} \int_0^t d_f(x,y) \, e^{-\frac{df(\gamma(\xi))}{n-1}} \, d\xi, \quad d_f := d_{f,1},
\]

where the infimum is taken over all unit speed minimal geodesics \( \gamma : [0,d(x,y)] \to M \) from \( x \) to \( y \). The function \( d_f \) has been called the re-parametrize distance in [27] (cf. Subsection [23]). In the unweighted case of \( f = 0 \), we have \( d_{f,t} = td \). Notice that for \( t \in (0, 1] \), the function \( d_{f,t} \) is not necessarily distance since the triangle inequality does not hold in general. We also remark that for \( t \in (0, 1) \), the function \( d_{f,t} \) is not always symmetric. For \( \kappa \in \mathbb{R} \), let \( s_\kappa(t) \) be a unique solution of the Jacobi equation \( \psi''(t) + \kappa \psi(t) = 0 \) with \( \psi(0) = 0, \psi'(0) = 1 \). For \( t \in [0,1] \), we define a twisted coefficient \( \beta_{\kappa,f,t} : M \times M \to \mathbb{R} \cup \{\infty\} \) by

\[
\beta_{\kappa,f,t}(x,y) := \left( \frac{s_\kappa(d_{f,t}(x,y))}{ts_\kappa(d_f(x,y))} \right)^{n-1} \quad \text{if} \quad d_f(x,y) \in [0,C_\kappa); \quad \text{otherwise,} \quad \beta_{\kappa,f,t}(x,y) := \infty,
\]

where \( C_\kappa \) denotes the diameter of the space form of constant curvature \( \kappa \). Furthermore, let \( \overline{\beta}_{\kappa,f,t} \) denote a function on \( M \times M \) defined as

\[
\overline{\beta}_{\kappa,f,t}(x,y) := \beta_{\kappa,f,t}(y,x).
\]

We introduce the following notion:

**Definition 1.1.** For \( \kappa \in \mathbb{R} \), we say that \( (M,d,m) \) has \( \kappa \)-twisted curvature bound if for every pair \( \mu_0, \mu_1 \in \mathcal{P}_{ac}(M) \), there are an optimal
coupling $\pi$ of $(\mu_0, \mu_1)$, and a minimal geodesic $(\mu_t)_{t \in [0, 1]}$ in the $L^2$-Wasserstein space from $\mu_0$ to $\mu_1$ such that for all $U \in \mathcal{DC}$ and $t \in (0, 1)$,

$$U_m(\mu_t) \leq (1 - t) \int_{M \times M} U \left( \frac{\rho_0(x)}{\beta_{\kappa,f,1-t}(x,y)} \frac{\beta_{\kappa,f,1-t}(x,y)}{\rho_0(x)} d\pi(x,y) \right) + t \int_{M \times M} U \left( \frac{\rho_1(y)}{\beta_{\kappa,f,t}(x,y)} \frac{\beta_{\kappa,f,t}(x,y)}{\rho_1(y)} d\pi(x,y) \right),$$

where $\rho_i$ is the density of $\mu_i$ with respect to $m$ for each $i = 0, 1$.

We also introduce the following weaker version:

**Definition 1.2.** For $\kappa \in \mathbb{R}$, we say that $(M, d, m)$ has $\kappa$-weak twisted curvature bound if for every pair $\mu_0, \mu_1 \in \mathcal{P}^{ac}(M)$, there exist an optimal coupling $\pi$ of $(\mu_0, \mu_1)$, and a minimal geodesic $(\mu_t)_{t \in [0, 1]}$ in the $L^2$-Wasserstein space from $\mu_0$ to $\mu_1$ such that for $H \in \mathcal{DC}$ defined as $H(r) := nr(1-r^{-\frac{1}{n}})$, and for every $t \in (0, 1)$ the inequality (1.8) holds.

**Remark 1.1.** In the unweighted case where $f = 0$, the notion of the $\kappa$-twisted curvature bound coincides with that of the curvature-dimension condition $CD((n - 1)\kappa, n)$ in the sense of Lott and Villani [10], [11]. Similarly, the notion of the $\kappa$-weak twisted curvature bound coincides with that of the curvature-dimension condition $CD((n - 1)\kappa, n)$ in the sense of Sturm [22], [23].

Our main result is the following characterization theorem:

**Theorem 1.1.** Let $(M, d, m)$ be an $n$-dimensional weighted Riemannian manifold, where $m := e^{-f} \text{vol}$ for some smooth function $f : M \to \mathbb{R}$. Let $\kappa \in \mathbb{R}$. Then the following statements are equivalent:

1. $\text{Ric}_f^1(M) \geq (n - 1)\kappa e^{\frac{-nf}{n-1}}$;
2. $(M, d, m)$ has $\kappa$-twisted curvature bound;
3. $(M, d, m)$ has $\kappa$-weak twisted curvature bound.

For $K \in \mathbb{R}$ and $N \in [n, \infty]$, Lott and Villani [11] have characterized the curvature condition (1.2) by a convexity of entropies on the Wasserstein space (see Theorem 4.22 in [11]). The Lott-Villani theorem in a special case where $f = 0$, $K = (n - 1)\kappa$ and $N = n$ states that the statements (1) and (2) in Theorem 1.1 are equivalent when $f = 0$.

For $K \in \mathbb{R}$ and $N \in [n, \infty)$, Sturm [23] has characterized a condition that $\text{Ric}_M \geq K$ and $n \leq N$ (see Theorem 1.7 in [23]), where $\text{Ric}_M \geq K$ means that for every $x \in M$, and for every unit tangent vector $v$ at $x$ we have $\text{Ric}_g(v) \geq K$. The Sturm theorem in the special case where
\[ K = (n-1)\kappa \text{ and } N = n \text{ tells us that the statements (1) and (3) in Theorem 1.1 are equivalent when } f = 0. \]

One of the key ingredients of the proof of Theorem 1.1 is to obtain inequalities for Jacobians of optimal transport maps that are associated with \( \text{Ric}_f^1 \). We first show an inequality of Riccati type (see Lemma 3.3). From the inequality of Riccati type, we derive an inequality concerning the concavity of the Jacobians under the curvature condition (1.3) (see Proposition 3.1). By using the concavity, we prove that the curvature condition (1.3) implies the convexity of entropies.

1.2. Organization. In Section 2, we review the works done by Wylie and Yeroshkin [27], and also recall basics of the optimal transport theory. In Section 3, we show key inequalities for the proof of Theorem 1.1. In Section 4, we prove Theorem 1.1. Furthermore, under the curvature condition (1.3), we conclude inequalities of Brunn-Minkowski type and Prékopa-Leindler type (see Corollaries 4.4 and 4.5). In Section 5, we present some applications of Theorem 1.1 (see Corollaries 5.4, 5.5, 5.6).

2. Preliminaries

Hereafter, for \( n \geq 2 \), let \((M,d,m)\) denote an \( n \)-dimensional weighted Riemannian manifold, namely, \( M = (M,g) \) is an \( n \)-dimensional complete Riemannian manifold (without boundary), \( d \) is the Riemannian distance on \( M \), and \( m := e^{-f} \text{vol} \) for some smooth function \( f : M \to \mathbb{R} \), where \( \text{vol} \) is the Riemannian volume measure on \( M \).

2.1. Geometric analysis on 1-weighted Ricci curvature. In this subsection, we briefly recall the work done by Wylie and Yeroshkin [27] concerning the curvature condition (1.3).

Wylie and Yeroshkin [27] have suggested a new approach to investigate geometric properties of weighted manifolds. Let \( \alpha \) be a 1-form on \( M \). The basic tool in [27] was a torsion free affine connection

\[ \nabla^\alpha \nabla := \nabla \nabla - \alpha(\nabla) \nabla - \alpha(\nabla) \nabla, \]

where \( \nabla \) denotes the Levi-Civita connection induced from \( g \). They have studied weighted manifolds in view of this weighted affine connection.

They have examined the relation between the 1-weighted Ricci curvature and the Ricci curvature induced from \( \nabla^\alpha \). The \( \nabla^\alpha \)-curvature tensor and the \( \nabla^\alpha \)-Ricci tensor are defined as

\[ R^{\nabla^\alpha}(\mathcal{U}, \mathcal{V}) \mathcal{W} := \nabla^\alpha_{\hat{\alpha}} \nabla^\alpha_{\mathcal{V}} \mathcal{W} - \nabla^\alpha_{\hat{\alpha}} \nabla^\alpha_{\mathcal{U}} \mathcal{W} - \nabla^\alpha_{[\mathcal{U}, \mathcal{V}]} \mathcal{W}, \]

(2.1)

\[ \text{Ric}^{\nabla^\alpha}(\mathcal{U}, \mathcal{V}) := \sum_{i=1}^{n} g(\text{Ric}^{\nabla^\alpha}(e_i, \mathcal{U}) \mathcal{V}, e_i), \]
where \( \{e_i\}_{i=1}^n \) is an orthonormal basis with respect to \( g \). Let us consider a closed 1-form \( \alpha_f \) on \( M \) defined by

\[
\alpha_f := \frac{df}{n - 1}.
\]

The first key observation in [27] is that \( \text{Ric}^{\nabla \alpha_f} \) coincides with the 1-weighted Ricci tensor \( \text{Ric}^1_f \) defined as (1.1) (see Proposition 3.3 in [27]).

They also investigated geodesics for \( \nabla \alpha \). For \( x \in M \), we denote by \( U_xM \) the unit tangent sphere at \( x \). For \( v \in U_xM \), let \( \gamma_v : [0, \infty) \to M \) be the \( (\nabla-)\)geodesic with initial conditions \( \gamma_v(0) = x \) and \( \gamma_v'(0) = v \).

We now define a function \( s_{f,v} : [0, \infty) \to [0, S_{f,v,\infty}] \) by

\[
s_{f,v}(t) := \int_0^t e^{-\frac{2f(\gamma_v(\xi))}{n-1}} d\xi, \quad S_{f,v,\infty} := \int_0^{\infty} e^{-\frac{2f(\gamma_v(\xi))}{n-1}} d\xi.
\]

Let \( t_{f,v} : [0, S_{f,v,\infty}] \to [0, \infty] \) be the inverse function of \( s_{f,v} \). The second key observation in [27] is that a curve \( \hat{\gamma}_{f,v} : [0, S_{f,v,\infty}) \to M \) defined as

\[
\hat{\gamma}_{f,v} := \gamma_v \circ t_{f,v}
\]

is a \( \nabla \alpha_f \)-geodesic (see Proposition 3.1 in [27]).

Summarizing the above two key observations, Wylie and Yeroshkin [27] have concluded the following interpretation of the curvature condition (1.3) in terms of the \( \nabla \alpha_f \)-Ricci curvature \( \text{Ric}^{\nabla \alpha_f} \) defined as (2.1):

**Proposition 2.1** ([27]). For \( \kappa \in \mathbb{R} \), the following are equivalent:

1. \( \text{Ric}^1_f(\gamma'_v(t)) \geq (n-1)\kappa e^{-\frac{2f(\gamma_v(t))}{n-1}} \) for all \( v \in U_xM \) and \( t \in [0, \infty) \);
2. \( \text{Ric}^{\nabla \alpha_f}(\hat{\gamma}_{f,v}(s)) \geq (n-1)\kappa \) for all \( v \in U_xM \) and \( s \in [0, S_{f,v,\infty}] \).

Keeping in mind Proposition 2.1, Wylie and Yeroshkin [27] has developed comparison geometry under the curvature condition (1.3). Before the work of them, Wylie [26] has obtained a splitting theorem of Cheeger-Gromoll type under the condition \( \text{Ric}^N_{f,M} \geq 0 \) for \( N \in (-\infty, 1] \). After that Wylie and Yeroshkin [27] have proved a Laplacian comparison for the distance function from a single point, a diameter comparison of Bonnet-Myers type for the deformed metric \( e^{-\frac{nf}{n-1}}g \), and a volume comparison of Bishop-Gromov type for the weighted volume measure \( e^{-\frac{n+1}{n-1}}f \) vol under the condition (1.3).

For later convenience, we will review the diameter comparison. For \( x \in M \), we denote by \( d_x : M \to \mathbb{R} \) the distance function from \( x \) defined as \( d_x(y) := d(x, y) \). For \( v \in U_xM \) we set

\[
\tau_x(v) := \sup \{ t > 0 \mid d_x(\gamma_v(t)) = t \}, \quad \tau_{f,x}(v) := s_{f,v}(\tau_x(v)).
\]

Wylie and Yeroshkin [27] have obtained the following comparison for the re-parametrized distance \( d_f \) (see Theorem 2.2 in [27]):
Theorem 2.2 ([27]). For \( \kappa > 0 \), if \( \text{Ric}_f^1 \geq (n - 1) \kappa e^{-\frac{\kappa}{n-1}} \), then for all \( x \in M \) and \( v \in U_x M \) we have

\[
\tau_{f,x}(v) \leq \frac{\pi}{\sqrt{\kappa}}.
\]

Moreover, for the re-parametrized distance \( d_f \) defined as [15], we have

\[
\sup_{x,y \in M} d_f(x,y) \leq \frac{\pi}{\sqrt{\kappa}}.
\]

2.2. Optimal transport. We review some basic facts of the optimal transport theory in our setting. We refer to [4], [12] (see also [14], [24]).

Let \( P(M) \) be the set of all Borel probability measure on \( M \). We denote by \( c : M \times M \to \mathbb{R} \) a cost function defined as \( c(x,y) := d(x,y)^2/2 \). For \( \mu, \nu \in P(M) \) we consider a value \( \inf_F \int_M c(x,F(x)) \, d\mu(x) \), where the infimum is taken over all Borel measurable maps \( F : M \to M \) such that the pushforward measure \( F#\mu \) of \( \mu \) by \( F \) coincides with \( \nu \). A Borel measurable map \( F \) is said to be an optimal transport map from \( \mu \) to \( \nu \) if it attains the infimum.

For \( \mu, \nu \in P(M) \) a Borel probability measure \( \pi \) on \( M \times M \) is said to be a coupling of \( (\mu, \nu) \) if \( \pi(X \times M) = \mu(X) \) and \( \pi(M \times X) = \nu(X) \) for all Borel subsets \( X \subset M \). Let \( \Pi(\mu, \nu) \) denote the set of all couplings of \( (\mu, \nu) \). Let us consider a value \( \inf_{\pi \in \Pi(\mu, \nu)} \int_{M \times M} c(x,y) \, d\pi(x,y) \). A coupling \( \pi \in \Pi(\mu, \nu) \) is called optimal if it attains the infimum.

Let \( X, Y \subset M \) be compact, and let \( \phi : X \to \mathbb{R} \cup \{ -\infty \} \) be a function that is not identically \( -\infty \). The c-transformation \( \phi^c : Y \to \mathbb{R} \cup \{ -\infty \} \) of \( \phi \) relative to \( (X,Y) \) is defined as \( \phi^c(y) := \inf_{x \in X} \{ c(x,y) - \phi(x) \} \).

The function \( \phi \) is said to be \( c \)-concave relative to \( (X,Y) \) if \( \phi = \psi^c \) for some \( \psi : Y \to \mathbb{R} \cup \{ -\infty \} \) with \( \psi \neq -\infty \). If \( \phi \) is \( c \)-concave relative to \( (X,Y) \), then it is Lipschitz, and \( t \phi \) is also \( c \)-concave for every \( t \in [0,1] \).

We recall the following Brenier-McCann theorem:

Theorem 2.3 ([3], [12]). Let \( \mu \in P^c(M) \), and let \( \nu \in P(M) \) be compactly supported. Let \( \text{supp} \mu \) and \( \text{supp} \nu \) denote their supports. Take compact subsets \( X, Y \subset M \) with \( \text{supp} \mu \subset X \), \( \text{supp} \nu \subset Y \). Then there is a \( c \)-concave function \( \phi \) relative to \( (X,Y) \) such that a map \( F \) on \( X \) defined by \( F(z) := \exp_z(-\nabla \phi(z)) \) gives a unique optimal transport map from \( \mu \) to \( \nu \), where \( \exp_z \) is the exponential map at \( z \), and \( \nabla \phi \) is the gradient of \( \phi \). Moreover, for the identity map \( \text{Id}_M \) of \( M \), the pushforward measure \( (\text{Id}_M \times F)#\mu \) is a unique optimal coupling of \( (\mu, \nu) \).

For a relatively compact, open subset \( V \subset M \), let \( X \) be its closure \( \overline{V} \). Let \( Y \subset M \) be compact, and let \( \phi \) be a \( c \)-concave function relative to \( (X,Y) \). It is well-known that the function \( \phi \) is twice differentiable
m-almost everywhere on $V$ due to the Alexandorov-Bangert theorem. For a map $F$ on $V$ defined as $F(z) := \exp_z(-\nabla \phi(z))$, if $\phi$ is twice differentiable at $x \in V$, then $F(x)$ does not belong to the cut locus $\text{Cut }x$ of $x$, and the differential $(dF)_x$ of $F$ at $x$ is well-defined.

We further recall the following:

**Theorem 2.4** ([1]). Let $\mu, \nu \in P_c^\text{ac}(M)$. Take relatively compact, open subsets $V, W \subset M$ with $\text{supp }\mu \subset V$, $\text{supp }\nu \subset W$. Assume that a $c$-concave function $\phi$ relative to $(V, W)$ satisfies $F_\#\mu = \nu$, where $F$ is a map on $V$ defined as $F(z) := \exp_z(-\nabla \phi(z))$. Then for $\mu$-almost every $x \in V$ the following hold:

1. $\phi$ is twice differential at $x$;
2. for every $t \in [0, 1]$ the determinant $\det (dF_t)_x$ of $dF_t$ at $x$ is positive, where $F_t$ is a map on $V$ defined by $F_t(z) := \exp_z(-t\nabla \phi(z))$;
3. $\rho_0(x) = \rho_1(F(x))e^{-f(F(x)) - f(x)}\det(dF)_x$, where $\rho_0$ and $\rho_1$ are the densities of $\mu$ and of $\nu$ with respect to $m$, respectively.

Let $(Z, d_Z)$ be a metric space. A curve $\gamma : [0, l] \to Z$ is said to be a minimal geodesic if there exists $a \geq 0$ such that for all $t_0, t_1 \in [0, l]$ we have $d_Z(\gamma(t_0), \gamma(t_1)) = a |t_0 - t_1|$. Moreover, if $a = 1$, then $\gamma$ is called a unit speed minimal geodesic.

Let $W_2 : P_2(M) \times P_2(M) \to \mathbb{R}$ denote a function defined as

$$W_2(\mu, \nu) := \inf_{\pi \in \mathcal{M}(\mu, \nu)} \left( \int_{M \times M} d(x, y)^2 d\pi(x, y) \right)^{\frac{1}{2}}.$$ 

It is well-known that $(P_2(M), W_2)$ is a complete separable metric space, and the metric space is called the $L^2$-Wasserstein space over $M$.

We summarize some well-known facts for interpolants:

**Proposition 2.5.** Let $\mu \in P_c^\text{ac}(M)$, and let $\nu \in P(M)$ be compactly supported. Take relatively compact, open subsets $V, W \subset M$ with $\text{supp }\mu \subset V$, $\text{supp }\nu \subset W$. Let $\phi$ be a $c$-concave function relative to $(V, W)$. For each $t \in [0, 1]$ we put $\mu_t := (F_t)_\#\mu$, where $F_t$ is a map on $V$ defined as $F_t(z) := \exp_z(-t\nabla \phi(z))$. Then the map $F_t$ gives a unique optimal transport map from $\mu$ to $\mu_t$. Moreover, if $\nu \in P_c^\text{ac}(M)$ and $\mu_1 = \nu$, then $(\mu_t)_{t \in [0, 1]}$ is a unique minimal geodesic in $(P_2(M), W_2)$ from $\mu$ to $\nu$, and it lies in $P_c^\text{ac}(M)$.

### 3. Key Inequalities

In the present section, we will prove the following key inequality for the proof of Theorem 1.1.
Proposition 3.1. Let $V,W \subset M$ be relatively compact, open subsets, and let $\phi$ be a $c$-concave function relative to $(V,W)$. Fix a point $x \in V$. Assume that $\phi$ is twice differentiable at $x$, and let $\det (dF_t)_x > 0$ for every $t \in [0,1]$, where $F_t$ is a map on $V$ defined as $F_t(z) := \exp_z (-t \nabla \phi(z))$. For each $t \in [0,1]$ we put
\begin{equation}
J_t(x) := e^{-f(F_t(x)) + f(x)} \det (dF_t)_x.
\end{equation}
For $\kappa \in \mathbb{R}$, if $\text{Ric}_{f,M}^1 \geq (n-1) \kappa e^{-\frac{M}{t}}$, then for every $t \in (0,1)$
\begin{equation}
J_t(x)^{\frac{1}{n}} \geq (1-t) \frac{\beta_{\kappa,f,1-t}}{\beta_{\kappa,f,t}}(x,F_t(x))^{\frac{1}{n}} J_0(x)^{\frac{1}{n}} + t \beta_{\kappa,f,t}(x,F_t(x))^{\frac{1}{n}} J_{1}(x)^{\frac{1}{n}},
\end{equation}
where $\beta_{\kappa,f,t}$ and $\beta_{\kappa,f,1-t}$ are defined as (1.6) and (1.7), respectively.

Throughout this section, as in Proposition 3.1 let $V,W \subset M$ denote relatively compact, open subsets, and let $\phi$ denote a $c$-concave function relative to $(V,W)$. Moreover, for a fixed point $x \in V$, we assume that $\phi$ is twice differentiable at $x$, and $\det (dF_t)_x > 0$ for all $t \in [0,1]$.

3.1. Riccati inequalities. Define a curve $\gamma_x : [0,1] \to M$ by $\gamma_x(t) := F_t(x)$, and choose an orthonormal basis $\{e_i\}_{i=1}^n$ of the tangent space at $x$ with $e_n = \gamma_x'(0)/\|\gamma_x'(0)\|$, where $\| \cdot \|$ is the canonical norm induced from $g$. For each $i = 1,\ldots,n$, we define a Jacobi field $E_i$ along $\gamma_x$ by $E_i(t) := (dF_t)_x(e_i)$. For each $t \in [0,1]$ let $A(t) = (a_{ij}(t))$ be an $n \times n$ matrix determined by
\begin{equation}
E'_i(t) = \sum_{j=1}^n a_{ij}(t) E_j(t).
\end{equation}
We define a function $h_x : [0,1] \to \mathbb{R}$ by
\begin{equation}
h_x(t) := \log \det (dF_t)_x - \int_0^t a_{nn}(\xi) d\xi.
\end{equation}
It is well-known that the function $h_x$ satisfies the following inequality of Riccati type (see e.g., [23], and Chapter 14 in [24]):

Lemma 3.2. For every $t \in (0,1)$ we have
\begin{equation}
h''_x(t) \leq -\frac{h'_x(t)^2}{n-1} - \text{Ric}_g(\gamma'_x(t)).
\end{equation}
We define a function $l_x : [0,1] \to \mathbb{R}$ by
\begin{equation}
l_x(t) := h_x(t) - f(\gamma_x(t)) + f(x).
\end{equation}
For distance functions, Wylie and Yeroshkin [27] have obtained an inequality of Riccati type that is associated with $\text{Ric}_f^1$ (see Lemma 4.1 in [27]). By using the same method, we have the following:
**Lemma 3.3.** For every $t \in (0, 1)$ we have
\[
\left( e^{\frac{2f(\gamma_x(t))}{n-1}} l'_x(t) \right)' \leq -e^{\frac{2f(\gamma_x(t))}{n-1}} \left( \frac{l'_x(t)^2}{n-1} + \text{Ric}_f(\gamma'_x(t)) \right).
\]

**Proof.** Put $f_x := f \circ \gamma_x$. From Lemma 3.2 we deduce
\[
l''_x(t) = h''_x(t) - f''_x(t) \leq -\frac{h'_x(t)^2}{n-1} - (\text{Ric}_y(\gamma'_x(t)) + f''_x(t))
= -\frac{l'_x(t)^2}{n-1} - \frac{2l'_x(t) f'_x(t)}{n-1} - \text{Ric}_f(\gamma'_x(t)).
\]
Hence we have
\[
e^{-\frac{2f_x(t)}{n-1}} \left( e^{\frac{2f_x(t)}{n-1}} l'_x(t) \right)' = l''_x(t) + \frac{2l'_x(t) f'_x(t)}{n-1} \leq -\frac{l'_x(t)^2}{n-1} - \text{Ric}_f(\gamma'_x(t)).
\]
This proves the desired inequality.

\[\square\]

3.2. **Jacobian inequalities.** We recall the following elementary comparison argument (see e.g., Theorem 14.28 in [24]):

**Lemma 3.4.** For $a > 0$, let $D : [0, a] \to \mathbb{R}$ be a non-negative continuous function that is $C^2$ on $(0, a)$. Take $\kappa \in \mathbb{R}$ and $d \geq 0$. Assume $\kappa d^2 \in (-\infty, a^{-2}\pi^2)$. Then $D'' + \kappa d^2 D \leq 0$ on $(0, a)$ if and only if for all $s_0, s_1 \in [0, a]$ and $\lambda \in [0, 1]$,\[
D((1 - \lambda)s_0 + \lambda s_1) \geq \frac{s_\kappa((1 - \lambda)|s_0 - s_1|d)}{s_\kappa(|s_0 - s_1|d)} D(s_0) + \frac{s_\kappa(\lambda |s_0 - s_1|d)}{s_\kappa(|s_0 - s_1|d)} D(s_1).
\]

We define a function $D_x : [0, 1] \to \mathbb{R}$ by
\[
D_x(t) := \exp \left( \frac{l_x(t)}{n-1} \right).
\]

Lemmas 3.3 and 3.4 yield the following concavity of the function $D_x$:

**Lemma 3.5.** For $\kappa \in \mathbb{R}$, if $\text{Ric}_f^t \geq (n - 1) \kappa e^{-\frac{at}{n-1}}$, then for every $t \in (0, 1)$ we have
\[
D_x(t) \geq s_\kappa(\frac{d_{f,1-t}(F_1(x), x)}{s_\kappa(d_f(F_1(x), x))}) D_x(0) + \frac{s_\kappa(d_{f,t}(x, F_1(x)))}{s_\kappa(d_f(x, F_1(x)))} D_x(1),
\]
where $d_{f,t}$ and $d_f$ are defined as (1.5).

**Proof.** We define a function $s_{f,x} : [0, 1] \to \mathbb{R}$ by
\[
s_{f,x}(t) := \int_0^t e^{-\frac{2f(\gamma_x(t))}{n-1}} d\xi.
\]
Put $a := s_{f,x}(1)$, and let $t_{f,x} : [0, a] \rightarrow [0, 1]$ be the inverse function of $s_{f,x}$. We define functions $\hat{l}_x, \hat{D}_x : [0, a] \rightarrow \mathbb{R}$ by

$$\hat{l}_x := l_x \circ t_{f,x}, \quad \hat{D}_x := D_x \circ t_{f,x},$$

where $l_x$ is defined as (3.3). For each $s \in (0, a)$ we see

$$\tag{3.4} (n - 1) \frac{\hat{D}_x''(s)}{\hat{D}_x(s)} = \frac{\hat{l}_x'(s)}{n - 1} + \frac{\hat{l}_x'(s)^2}{(n - 1)^2}.$$

We also define functions $L_x : [0, 1] \rightarrow \mathbb{R}$ and $\tilde{L}_x : [0, a] \rightarrow \mathbb{R}$ by

$$L_x(t) := e^{\frac{2f(t)}{n - 1}} \hat{l}_x(t), \quad \tilde{L}_x := L_x \circ t_{f,x}.$$  

Note that $\hat{l}_x(s) = \tilde{L}_x(s)$. From Lemma 3.3 it follows that

$$\tag{3.5} \hat{l}_x''(s) = \tilde{L}_x'(s) = t_{f,x}'(s) L_x'(t_{f,x}(s)) \leq -e^{\frac{4f(t)}{n - 1}} \left( \frac{L_x'(t_{f,x}(s))^2}{n - 1} + \text{Ric}_f^1(\gamma_x(t_{f,x}(s))) \right)$$

$$= -\frac{\hat{l}_x(s)^2}{n - 1} - e^{\frac{4f(t)}{n - 1}} \text{Ric}_f^1(\gamma_x(t_{f,x}(s))).$$

Combining (3.4) and (3.5), we obtain

$$\tag{3.6} (n - 1) \frac{\hat{D}_x''(s)}{\hat{D}_x(s)} \leq -e^{\frac{4f(t)}{n - 1}} \text{Ric}_f^1(\gamma_x(t_{f,x}(s))) \leq -(n - 1) \kappa d(x, y)^2,$$

where $y := F_1(x)$. Therefore, $\hat{D}_x'' + \kappa d(x, y)^2 \hat{D}_x \leq 0$ on $(0, a)$.

Since the $c$-concave function $\phi$ is twice differentiable at $x$, the curve $\gamma_x$ lies in the complement of $\text{Cut} x$. In particular, $\gamma_x$ is a unique minimal geodesic from $x$ to $y$, and hence

$$a \ d(x, y) = d_f(x, y) < \tau_{f,x} \left( \frac{\gamma_x'(0)}{\|\gamma_x'(0)\|} \right),$$

where $\tau_{f,x}$ is defined as (2.2). By Theorem 2.2, $\kappa d(x, y)^2 \in (-\infty, a^{-2}\pi^2)$. Lemma 3.4 implies that for all $s_0, s_1 \in [0, a]$ and $\lambda \in [0, 1]$

$$\tag{3.6} \hat{D}_x ((1 - \lambda)s_0 + \lambda s_1) \geq \frac{s_\kappa((1 - \lambda) |s_0 - s_1| d(x, y))}{s_\kappa(|s_0 - s_1| d(x, y))} \hat{D}_x(s_0)$$

$$+ \frac{s_\kappa(\lambda |s_0 - s_1| d(x, y))}{s_\kappa(|s_0 - s_1| d(x, y))} \hat{D}_x(s_1).$$

For every $s \in (0, a)$ we obtain

$$\hat{D}_x(s) \geq \frac{s_\kappa((a - s) d(x, y))}{s_\kappa(a d(x, y))} \hat{D}_x(0) + \frac{s_\kappa(s d(x, y))}{s_\kappa(a d(x, y))} \hat{D}_x(a)$$
by letting $s_0 \to 0$, $s_1 \to a$ and $\lambda \to s/a$ in (3.6). For every $t \in (0,1)$
\[
D_x(t) \geq \frac{s_\kappa((a - s f \cdot x(t))d(x,y))}{s_\kappa(a d(x,y))} D_x(0) + \frac{s_\kappa(s f \cdot x(t)d(x,y))}{s_\kappa(a d(x,y))} D_x(1).
\]
From the uniqueness of the geodesic $\gamma_x$, for every $t \in [0,1]$ we see
\[
(a - s f \cdot x(t))d(x,y) = d_{f,1-t}(y,x), \quad s f \cdot x(t)d(x,y) = d_{f,t}(x,y).
\]
This completes the proof. 

We define a function $\overline{D}_x : [0,1] \to \mathbb{R}$ by
\[
\overline{D}_x(t) := \exp \left( \int_0^t a_{nn}(\xi) \, d\xi \right),
\]
where $a_{nn}$ is determined by (3.2). Notice that for every $t \in (0,1)$
\begin{equation}
(3.7)
J_t(x) = D_x(t)^{n-1} \overline{D}_x(t),
\end{equation}
where $J_t(x)$ is defined as (3.1).

The following concavity of the function $\overline{D}_x$ is well-known (see e.g., [23], and Chapter 14 in [24]):

**Lemma 3.6.** For every $t \in (0,1)$ we have
\[
\overline{D}_x(t) \geq (1 - t) \overline{D}_x(0) + t \overline{D}_x(1).
\]

Now, we prove Proposition 3.1.

**Proof of Proposition 3.1.** For $\kappa \in \mathbb{R}$, we assume $\text{Ric}^1_{f,M} \geq (n-1)\kappa e^{-\frac{4}{n-4}}$. From (3.7) we deduce that for every $t \in (0,1)$
\[
J_t(x)^{\frac{1}{n}} = D_x(t)^{1-\frac{1}{n}} \overline{D}_x(t)^{\frac{1}{n}}.
\]
By Lemmas 3.5 and 3.6 and by the Hölder inequality, we obtain
\[
J_t(x)^{\frac{1}{n}} \geq (1 - t)^{\frac{1}{n}} \left( \frac{s_\kappa((d_{f,1-t}(F_1(x),x)))}{s_\kappa(d_{f}(F_1(x),x))} \right)^{1-\frac{1}{n}} J_0(x)^{\frac{1}{n}}
+ t^{\frac{1}{n}} \left( \frac{s_\kappa(d_{f,t}(x,F_1(x)))}{s_\kappa(d_{f}(x,F_1(x)))} \right)^{1-\frac{1}{n}} J_1(x)^{\frac{1}{n}}.
\]
The right hand side is equal to that of the desired one. Therefore, we conclude the proposition. 

4. **Displacement convexity**

In this section, we prove Theorem 1.1 by using Proposition 3.1 and the same argument as that in the proof of Theorem 1.7 in [23].
4.1. Curvature bounds imply displacement convexity. First, we prove the following part of Theorem 1.1.

**Proposition 4.1.** For $\kappa \in \mathbb{R}$, if $\text{Ric}^1_{f,M} \geq (n-1)\kappa e^{-\frac{4t}{f}}$, then $(M,d,m)$ has $\kappa$-twisted curvature bound.

Proof. Fix $\mu_0, \mu_1 \in P_{ac}^c(M)$. Take relatively compact, open subsets $V,W \subset M$ with $\text{supp} \mu_0 \subset V$, $\text{supp} \mu_1 \subset W$. Due to Theorem 2.3 there exists a $c$-concave function $\phi$ relative to $(V,W)$ such that a map $F$ on $V$ defined as $F(z) := \exp_z(-\nabla \phi(z))$ gives a unique optimal transport map from $\mu_0$ to $\mu_1$. Moreover, $\pi := (\text{Id}_M \times F)_\# \mu_0$ is a unique optimal coupling of $(\mu_0, \mu_1)$. For each $t \in [0,1]$ we define a map $F_t$ on $V$ as $F_t(z) := \exp_z(-t\nabla \phi(z))$, and put $\mu_t := (F_t)_\# \mu_0$. By Lemma 2.5, $F_t$ is a unique optimal transport map from $\mu_0$ to $\mu_t$, and $\mu_t \in P_{ac}^c(M)$.

We fix $t \in (0,1)$. By Theorem 2.4, for $\mu_0$-almost every $x \in V$ the following hold: (1) $\phi$ is twice differential at $x$; (2) $\det (dF_u)_x > 0$ for all $u \in [0,1]$; (3) the Jacobian equations

$$J_t(x) \frac{\rho_0(x)}{\rho_0(x)} = \rho_1(F(x)) J_1(x) = \rho_t(F_t(x)) J_t(x)$$

hold, where $J_1(x)$ and $J_t(x)$ are defined as (3.1). Proposition 3.1 implies that for $\mu_0$-almost every $x \in V$ we have

$$J_t(x) \frac{\rho_0(x)}{\rho_0(x)} \geq (1-t) \beta_{\kappa,f,1-t}(x,F(x)) \frac{\rho_0(x)}{\rho_0(x)} J_0(x) \frac{\rho_0(x)}{\rho_0(x)} + t \beta_{\kappa,f,t}(x,F(x)) \frac{\rho_0(x)}{\rho_0(x)} J_1(x) \frac{\rho_0(x)}{\rho_0(x)}.$$

Fix $U \in DC$. By using $\mu_t = (F_t)_\# \mu_0$ and (4.1), we see

$$U_m(\mu_t) = \int_M U \left( \frac{\rho_0(x)}{\rho_0(x)} J_t(x) \rho_0(x) \right) d\mu_0(x)$$

$$= \int_M \varphi_U \left( \frac{J_t(x)}{\rho_0(x)} \right)^\frac{1}{\kappa} d\mu_0(x).$$

where $\varphi_U$ denotes the function defined as $\varphi_U(x) := r^n U(r^{-n})$. Notice that $\varphi_U$ is non-increasing and convex. From (4.2) we derive

$$U_m(\mu_t) \leq (1-t) \int_M \varphi_U \left( \beta_{\kappa,f,1-t}(x,F(x)) \frac{\rho_0(x)}{\rho_0(x)} \right)^\frac{1}{\kappa} \left( \frac{J_0(x)}{\rho_0(x)} \right)^\frac{1}{\kappa} d\mu_0(x)$$

$$+ t \int_M \varphi_U \left( \beta_{\kappa,f,t}(x,F(x)) \frac{\rho_0(x)}{\rho_0(x)} \right)^\frac{1}{\kappa} \left( \frac{J_1(x)}{\rho_0(x)} \right)^\frac{1}{\kappa} d\mu_0(x).$$
Using the Jacobian equation (4.1) again, we obtain

\[
U_m(\mu_t) \leq (1 - t) \int_M \varphi_U \left( \left( \frac{\beta_{\kappa,f,1-t}(x,F(x))}{\rho_0(x)} \right)^{\frac{1}{n}} \right) \, d\mu_0(x)
\]
\[
+ t \int_M \varphi_U \left( \left( \frac{\beta_{\kappa,f,t}(x,F(x))}{\rho_1(F(x))} \right)^{\frac{1}{n}} \right) \, d\mu_0(x).
\]

Since \( \pi = (\text{Id}_M \times F)_\# \mu_0 \), the right hand side of (4.3) is equal to that of (4.8). We complete the proof.

4.2. Displacement convexity implies curvature bounds. For subsets \( X,Y \subset M \) and \( t \in [0,1] \), let \( Z_t(X,Y) \) be the set of all points \( \gamma(t) \), where \( \gamma : [0,1] \to M \) is a minimal geodesic with \( \gamma(0) \in X, \gamma(1) \in Y \).

Let us show that the weak twisted curvature bound implies the following inequality of Brunn-Minkowski type:

Lemma 4.2. Let \( X, Y \subset M \) denote two bounded Borel subsets with \( m(X), m(Y) \in (0,\infty) \). For \( \kappa \in \mathbb{R} \), if \( (M,d,m) \) has \( \kappa \)-weak twisted curvature bound, then for every \( t \in (0,1) \) we have

\[
m(Z_t(X,Y))^{\frac{1}{n}} \geq (1 - t) \left( \inf_{(x,y) \in X \times Y} \beta_{\kappa,f,1-t}(x,y)^{\frac{1}{n}} \right) m(X)^{\frac{1}{n}}
\]
\[
+ t \left( \inf_{(x,y) \in X \times Y} \beta_{\kappa,f,t}(x,y)^{\frac{1}{n}} \right) m(Y)^{\frac{1}{n}}.
\]

Proof. Let \( 1_X \) and \( 1_Y \) be the characteristic functions of \( X \) and of \( Y \), respectively. We set
\[
\rho_0 := \frac{1_X}{m(X)}, \quad \mu_0 := \rho_0 m, \quad \rho_1 := \frac{1_Y}{m(Y)}, \quad \mu_1 := \rho_1 m.
\]

By Proposition 2.5, there exists a unique minimal geodesic \( (\mu_t)_{t \in [0,1]} \) in \( (P_2(M),W_2) \) from \( \mu_0 \) to \( \mu_1 \), and it lies in \( P_{ac}(M) \). For each \( t \in (0,1) \) let \( \rho_t \) stand for the density of \( \mu_t \) with respect to \( m \). From the Jensen inequality one can derive

\[
m(Z_t(X,Y))^{\frac{1}{n}} \geq \int_M \rho_t(x)^{1 - \frac{1}{n}} \, dm(x).
\]

Since \( (M,d,m) \) has \( \kappa \)-weak twisted curvature bound, we have

\[
\int_M \rho_t(x)^{1 - \frac{1}{n}} \, dm(x) \geq (1 - t) \int_{M \times M} \rho_0(x)^{-\frac{1}{n}} \beta_{\kappa,f,1-t}(x,y)^{\frac{1}{n}} \, d\pi(x,y)
\]
\[
+ t \int_{M \times M} \rho_1(y)^{-\frac{1}{n}} \beta_{\kappa,f,t}(x,y)^{\frac{1}{n}} \, d\pi(x,y),
\]
where $\pi$ is a unique optimal coupling of $(\mu_0, \mu_1)$. The coupling $\pi$ is supported on $X \times Y$. Hence, the right hand side of (4.5) is bounded from below by

$$(1 - t) \left( \inf_{(x,y) \in X \times Y} \beta_{\kappa, f, 1-t}(x, y)^{\frac{1}{n}} \right) \int_M \rho_0(x)^{1-\frac{1}{n}} \, dm(x)$$

$$+ t \left( \inf_{(x,y) \in X \times Y} \beta_{\kappa, f, t}(x, y)^{\frac{1}{n}} \right) \int_M \rho_1(y)^{1-\frac{1}{n}} \, dm(y)$$

that is equal to the right hand side of (4.4). This proves the lemma. $\Box$

We next prove the following part of Theorem 1.1:

**Proposition 4.3.** For $\kappa \in \mathbb{R}$, if $(M, d, m)$ has $\kappa$-weak twisted curvature bound, then $\text{Ric}_{f, M}^1 \geq (n - 1) \kappa \, e^{-\frac{4f}{n}}$.

**Proof.** Fix $x \in M$ and $v \in U_x M$. For $\epsilon > 0$, let $\gamma : (-\epsilon, \epsilon) \to M$ be the geodesic with $\gamma(0) = x$, $\gamma'(0) = v$. Take $\delta \in (0, \epsilon)$ and $\eta \in (0, \delta)$. For $y \in M$, we denote by $B_\eta(y)$ the open geodesic ball of radius $\eta$ centered at $y$. We set $X := B_\eta(\gamma(-\delta))$ and $Y := B_\eta(\gamma(\delta))$. From Lemma 4.2 we deduce

$$m \left( Z_{\frac{1}{2}}(X, Y) \right)^{\frac{1}{n}} \geq \frac{1}{2} \left( \inf_{(x, y) \in X \times Y} \beta_{\kappa, f, \frac{1}{2}}(x, y)^{\frac{1}{n}} \right) m(X)^{\frac{1}{n}}$$

$$+ \frac{1}{2} \left( \inf_{(x, y) \in X \times Y} \beta_{\kappa, f, \frac{1}{2}}(x, y)^{\frac{1}{n}} \right) m(Y)^{\frac{1}{n}},$$

where the functions $\beta_{\kappa, f, \frac{1}{2}}$ and $\beta_{\kappa, f, \frac{1}{4}}$ are defined as (1.6) and (1.7), respectively. By letting $\eta \to 0$ in the above inequality,

$$\liminf_{\eta \to 0} \left( \frac{m \left( Z_{\frac{1}{2}}(X, Y) \right)}{\omega_n \eta^n} \right)^{\frac{1}{n}} \geq \frac{1}{2} \left( e^{-f(\gamma(-\delta))} \beta_{\kappa, f, \frac{1}{2}}(\gamma(-\delta), \gamma(-\delta)) \right)^{\frac{1}{n}}$$

$$+ \frac{1}{2} \left( e^{-f(\gamma(\delta))} \beta_{\kappa, f, \frac{1}{2}}(\gamma(\delta), \gamma(\delta)) \right)^{\frac{1}{n}},$$

where $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. 
Let us recall that the function $d_{f,t}$ is defined as (1.5). By the definition of $d_{f,t}$ we see
\begin{align*}
    d_{f,\frac{1}{2}}(\gamma(\delta), \gamma(-\delta)) &= \int_0^\delta e^{-\frac{2f(\gamma(\xi))}{n-1}} \, d\xi, \\
    d_{f,\frac{1}{2}}(\gamma(-\delta), \gamma(\delta)) &= \int_0^{-\delta} e^{-\frac{2f(\gamma(\xi))}{n-1}} \, d\xi, \\
    d_f(\gamma(-\delta), \gamma(\delta)) &= \int_{-\delta}^\delta e^{-\frac{2f(\gamma(\xi))}{n-1}} \, d\xi.
\end{align*}

Hence, the Taylor series of $\beta_{\kappa,f,\frac{1}{2}}(\gamma(-\delta), \gamma(\delta))$ and $\beta_{\kappa,f,\frac{1}{2}}(\gamma(-\delta), \gamma(\delta))$ with respect to $\delta$ at 0 are given as follows:
\begin{align*}
    \beta_{\kappa,f,\frac{1}{2}}(\gamma(-\delta), \gamma(\delta)) &= 1 - g((\nabla f)_x, v) \delta + (n - 1) \kappa e^{-\frac{4f(v)}{n-1}} \frac{\delta^2}{2} \\
    &\quad + \frac{n-2}{n-1} g((\nabla f)_x, v)^2 \frac{\delta^2}{2} + O(\delta^3), \\
    \beta_{\kappa,f,\frac{1}{2}}(\gamma(-\delta), \gamma(\delta)) &= 1 + g((\nabla f)_x, v) \delta + (n - 1) \kappa e^{-\frac{4f(v)}{n-1}} \frac{\delta^2}{2} \\
    &\quad + \frac{n-2}{n-1} g((\nabla f)_x, v)^2 \frac{\delta^2}{2} + O(\delta^3).
\end{align*}

On the other hand,
\begin{align*}
    e^{-f(\gamma(-\delta))+f(x)} &= 1 + g((\nabla f)_x, v) \delta \\
    &\quad + (g((\nabla f)_x, v)^2 - \text{Hess } f(v, v)) \frac{\delta^2}{2} + O(\delta^3), \\
    e^{-f(\gamma(\delta))+f(x)} &= 1 - g((\nabla f)_x, v) \delta \\
    &\quad + (g((\nabla f)_x, v)^2 - \text{Hess } f(v, v)) \frac{\delta^2}{2} + O(\delta^3).
\end{align*}

Substituting these series into (4.6), we have
\begin{align*}
    \liminf_{\eta \to 0} \frac{m(Z_{\frac{1}{2}}(X,Y))}{\omega_n \eta^n} &\geq e^{-f(x)} \left( 1 + (n - 1) \kappa e^{-\frac{4f(v)}{n-1}} \frac{\delta^2}{2} \right) \\
    &\quad + e^{-f(x)} \left( - \text{Hess } f(v, v) + \frac{g((\nabla f)_x, v)^2}{1 - n} \right) \frac{\delta^2}{2} + O(\delta^3).
\end{align*}

We recall the following fundamental inequality (see e.g., [23]):
\begin{align*}
    \limsup_{\eta \to 0} \frac{m(Z_{\frac{1}{2}}(X,Y))}{\omega_n \eta^n} &\leq e^{-f(x)} \left( 1 + \text{Ric}_g(v) \frac{\delta^2}{2} \right) + O(\delta^3).
\end{align*}
Comparing (4.7) with (4.8), we obtain
\[ \text{Ric}_g(v) \geq (n-1) \kappa e^{-\frac{4f(v)}{n-1}} - \text{Hess } f(v,v) + \frac{g((\nabla f)_x,v)^2}{1-n}; \]
in particular, \( \text{Ric}_f(v) \geq (n-1) \kappa e^{-\frac{4f(v)}{n-1}}. \) This completes the proof.  

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Propositions 4.1 and 4.3, we complete the
proof of Theorem 1.1.

4.3. Brunn-Minkowski and Prékopa-Leindler inequalities. Due
to Theorem 1.1 and Lemma 4.2, we obtain the following inequality of
Brunn-Minkowski type under the curvature condition (1.3):

**Corollary 4.4.** Let \( X, Y \subset M \) denote two bounded Borel subsets with
\( m(X), m(Y) \in (0, \infty). \) For \( \kappa \in \mathbb{R}, \) if \( \text{Ric}_{f,M}^1 \geq (n-1) \kappa e^{-\frac{4f}{n-1}}, \) then for
every \( t \in (0, 1) \) we have

\[
m(Z_t(X,Y))^{\frac{1}{n}} \geq (1-t) \left( \inf_{(x,y) \in X \times Y} \beta_{\kappa,f,1-t}(x,y)^{\frac{2}{n}} \right) m(X)^{\frac{1}{n}} + t \left( \inf_{(x,y) \in X \times Y} \beta_{\kappa,f,t}(x,y)^{\frac{2}{n}} \right) m(Y)^{\frac{1}{n}}.
\]

Let \( t \in (0, 1) \) and \( a, b \in [0, \infty). \) For \( p \in \mathbb{R} \setminus \{0\} \) we define

\[ M_p^t(a, b) := ((1-t) a^p + t b^p)^{\frac{1}{p}}, \]
if \( ab \neq 0, \) and \( M_p^t(a, b) := 0 \) if \( ab = 0. \) As the limits, we further define

\[ M_p^t(a, b) := a^{1-t} b^t, \quad M_p^{-\infty}(a, b) := \min\{a, b\}. \]

We also have the following inequality of Prékopa-Leindler type (cf.
Corollary 1.1 in [4], Corollary 9.4 in [14] and Theorem 19.18 in [24]):

**Corollary 4.5.** For \( i = 0, 1, \) let \( \psi_i : M \to \mathbb{R} \) be non-negative, com-
pactly supported, integrable functions. We take relatively compact, open
subsets \( V, W \subset M \) with \( \text{supp } \psi_0 \subset V, \text{supp } \psi_1 \subset W, \) and put \( X := \overline{V} \)
and \( Y := \overline{W}. \) Let \( \psi : M \to \mathbb{R} \) be a non-negative function. For \( t \in (0, 1) \)
and \( p \geq -1/n, \) we assume that for all \((x, y) \in X \times Y \) and \( z \in Z_t(X,Y), \)

\[ \psi(z) \geq M_p^t \left( \frac{\psi_0(x)}{\beta_{\kappa,f,1-t}(x,y)}, \frac{\psi_1(y)}{\beta_{\kappa,f,t}(x,y)} \right). \]

For \( \kappa \in \mathbb{R}, \) if \( \text{Ric}_{f,M}^1 \geq (n-1) \kappa e^{-\frac{4f}{n-1}}, \) then we have

\[ \int_M \psi \, dm \geq M_p^{t+np} \left( \int_M \psi_0 \, dm, \int_M \psi_1 \, dm \right). \]
Theorem 19.18 in [24] states that for \( K \in \mathbb{R} \) and \( N \in [n, \infty) \), the curvature condition (1.2) implies an inequality of Prékopa-Leindler type. One can prove Corollary 4.5 only by replacing the role of Theorem 19.4 in [24] with that of Proposition 3.1 in the proof. We omit the proof.

5. Applications

In this last section, as applications of Theorem 1.1, we present several functional inequalities under the curvature condition (1.3). Throughout this section, we always assume that \( M \) is compact, and the function \( f : M \to \mathbb{R} \) satisfies \( \int_M e^{-f} \text{d}vol = 1 \); in particular, \( m \in P_2(M) \).

5.1. Derivatives of entropies. For \( \kappa \in \mathbb{R} \), let us define a function \( \tilde{\beta}_{\kappa,f} : M \times M \to \mathbb{R} \cup \{ \infty \} \) by

\[
(5.1) \quad \tilde{\beta}_{\kappa,f}(x, y) := \left( \frac{e^{-\frac{2f(x)}{n-1}} d(x, y)}{s_\kappa(d_f(x, y))} \right)^{n-1} \quad \text{if } d_f(x, y) \in [0, C_\kappa); \quad \text{otherwise, } \tilde{\beta}_{\kappa,f}(x, y) := \infty.
\]

Let \( c_\kappa := s'_\kappa \). We also define a function \( \tilde{\beta}_{\kappa,f} : M \times M \to \mathbb{R} \cup \{ \infty \} \) by

\[
(5.2) \quad \tilde{\beta}_{\kappa,f}(x, y) := \frac{n-1}{n} \left( \frac{e^{-\frac{2f(x)}{n-1}} d(x, y) c_\kappa(d_f(x, y))}{s_\kappa(d_f(x, y))} - 1 \right) \quad \text{if } d_f(x, y) \in [0, C_\kappa); \quad \text{otherwise, } \tilde{\beta}_{\kappa,f}(x, y) := \infty.
\]

We check the following basic properties of \( \tilde{\beta}_{\kappa,f} \) and \( \tilde{\beta}_{\kappa,f} \).

Lemma 5.1. Let \( \kappa \in \mathbb{R} \). We take \( x, y \in M \) with \( d_f(x, y) \in [0, C_\kappa) \). We assume \( y \notin \text{Cut } x \). Then by letting \( t \to 0 \) we have

\[
(5.3) \quad \beta_{\kappa,f,t}(x, y) \to \tilde{\beta}_{\kappa,f}(x, y),
\]

\[
(5.4) \quad \frac{1 - \beta_{\kappa,f,1-t}(x, y)^{\frac{1}{n}}}{t} \to \tilde{\beta}_{\kappa,f}(x, y).
\]

Proof. First, we show (5.3). Since the point \( y \) does not belong to \( \text{Cut } x \), there exists a unique minimal geodesic \( \gamma : [0, 1] \to M \) from \( x \) to \( y \). We define a function \( s_f : [0, 1] \to M \) by

\[
s_f(t) := \int_0^t e^{-\frac{2f(\xi)}{n-1}} \text{d}\xi.
\]

The uniqueness of \( \gamma \) tells us that for every \( t \in [0, 1] \) we have \( d_{f,t}(x, y) = s_f(t) d(x, y) \). This implies

\[
\frac{s_\kappa(d_{f,t}(x, y))}{t s_\kappa(d_f(x, y))} \to \frac{s'_f(0) d(x, y)}{s_\kappa(d_f(x, y))} = \frac{e^{-\frac{2f(x)}{n-1}} d(x, y)}{s_\kappa(d_f(x, y))}.
\]
as \( t \to 0 \). We obtain (5.3).

We next show (5.4). For a unique minimal geodesic \( \gamma : [0, 1] \to M \) from \( y \) to \( x \), we define a function \( \bar{s}_f : [0, 1] \to M \) as
\[
\bar{s}_f(t) := \int_0^t e^{-\frac{2f(\xi)}{n-1}} d\xi.
\]
The uniqueness of \( \gamma \) implies that \( d_{f,t}(y, x) = \bar{s}_f(t) d(y, x) \) for every \( t \in [0, 1] \). Define a function \( G_f : [0, 1] \to \mathbb{R} \) by
\[
G_f(t) := \left( \frac{\bar{s}_f(t) d_{f,t}(y, x)}{\sigma_k(d_f(y, x))} - 1 \right) = \tilde{\beta}_{\kappa,f}(x, y).
\]
From direct computations we deduce
\[
G_f'(1) = \frac{n-1}{n} \left( \bar{s}_f(1) d(x, y) c_k(d_f(x, y)) - 1 \right) = \tilde{\beta}_{\kappa,f}(x, y).
\]
This proves (5.4). \( \square \)

For a positive Lipschitz function \( \rho \) on \( M \) with \( \int_M \rho \, dm = 1 \), we put \( \mu := \rho m \). The generalized Fisher information \( I_m(\mu) \) of \( \mu \) is defined as
\[
I_m(\mu) := \int_M \left\| \nabla \rho^{1-\frac{1}{n}} \right\|^2 \rho \, dm.
\]

Recall the following fact concerning the derivative of the Rényi entropy \( H_m \) defined as (1.4) for \( H \in DC \) (see e.g., Theorem 20.1 in [24]):

**Proposition 5.2.** For \( i = 0, 1 \), let \( \rho_i : M \to \mathbb{R} \) be positive Lipschitz functions with \( \int_M \rho_i \, dm = 1 \). We put \( \mu := \rho_0 m \) and \( \nu := \rho_1 m \). Then for a unique minimal geodesic \( (\mu_t)_{t \in [0, 1]} \) in \( (P_2(M), W_2) \) from \( \mu \) to \( \nu \),
\[
\liminf_{t \to 0} \frac{H_m(\mu_t) - H_m(\mu)}{t} \geq -\sqrt{I_m(\mu)} W_2(\mu, \nu).
\]

Using Theorem 1.1 and Proposition 5.2, we prove the following:

**Proposition 5.3.** For \( i = 0, 1 \), let \( \rho_i : M \to \mathbb{R} \) be positive Lipschitz functions with \( \int_M \rho_i \, dm = 1 \). We put \( \mu := \rho_0 m \) and \( \nu := \rho_1 m \). For \( \kappa \in \mathbb{R} \), if \( \text{Ric}_f^{1, M} \geq (n-1)\kappa e^{\frac{-4f}{n-4}} \), then we have
\[
H_m(\mu) \leq \sqrt{I_m(\mu)} W_2(\mu, \nu) + n \int_{M \times M} \rho_0(x)^{-\frac{1}{n}} \tilde{\beta}_{\kappa,f}(x, y) \, d\pi(x, y)
\]
\[
- n \int_{M \times M} \rho_1(y)^{-\frac{1}{n}} \left( \tilde{\beta}_{\kappa,f}(x, y) \frac{1}{n} - 1 \right) \, d\pi(x, y)
\]
\[
- n \int_{M \times M} \left( \rho_1(y)^{-\frac{1}{n}} - 1 \right) \, d\pi(x, y),
\]
where \( \pi \) is a unique optimal coupling of \( (\mu, \nu) \).
Proof. By Theorem 1.1, \((M, d, m)\) has \(\kappa\)-weak twisted curvature bound. It follows that

\[
H_m(\mu_t) \leq n - (1 - t)n \int_{M \times M} \rho_0(x)^{-\frac{1}{n}} \beta_{\kappa, f, 1-t}(x, y)^{\frac{1}{n}} d\pi(x, y)
- t n \int_{M \times M} \rho_1(y)^{-\frac{1}{n}} \beta_{\kappa, f, t}(x, y)^{\frac{1}{n}} d\pi(x, y),
\]

where \((\mu_t)_{t \in [0, 1]}\) is a unique minimal geodesic in \((P_2(M), W_2)\) from \(\mu\) to \(\nu\). This leads to

\[
\frac{H_m(\mu_t) - H_m(\mu)}{t} \leq n \int_{M \times M} \rho_0(x)^{-\frac{1}{n}} \left(1 - \frac{1}{n} \beta_{\kappa, f, 1-t}(x, y)^{\frac{1}{n}}\right) d\pi(x, y)
+ n \int_{M \times M} \rho_0(x)^{-\frac{1}{n}} \left(\beta_{\kappa, f, 1-t}(x, y)^{\frac{1}{n}} - 1\right) d\pi(x, y)
- n \int_{M \times M} \rho_1(y)^{-\frac{1}{n}} \left(\beta_{\kappa, f, t}(x, y)^{\frac{1}{n}} - 1\right) d\pi(x, y)
- n \int_{M \times M} \left(\rho_1(y)^{-\frac{1}{n}} - 1\right) d\pi(x, y) - H_m(\mu).
\]

We remark that for a unique optimal transport map \(F\) from \(\mu\) to \(\nu\), and for \(\mu\)-almost every \(x \in M\) we have \(F(x) \notin \text{Cut} x\); in particular, Theorem 2.2 implies \(d_f(x, F(x)) \in [0, C_n]\). Therefore, by using \(\pi = (\text{Id}_M \times F)_{\#}\mu\) and Lemma 5.1, we see

\[
\limsup_{t \to 0} \frac{H_m(\mu_t) - H_m(\mu)}{t} \leq n \int_{M \times M} \rho_0(x)^{-\frac{1}{n}} \beta_{\kappa, f}(x, y) d\pi(x, y)
- n \int_{M \times M} \rho_1(y)^{-\frac{1}{n}} \left(\beta_{\kappa, f}(x, y)^{\frac{1}{n}} - 1\right) d\pi(x, y)
- n \int_{M \times M} \left(\rho_1(y)^{-\frac{1}{n}} - 1\right) d\pi(x, y) - H_m(\mu).
\]

Comparing this inequality with (5.5), we arrive at the desired one. \(\Box\)

5.2. Functional inequalities. We formulate three functional inequalities under the curvature condition \((1.3)\).

Let \(\mu \in P_2(M)\) be absolutely continuous with respect to \(m\). We say that \(m\) is \(\mu\)-constant if for a unique optimal transport map \(F\) from \(\mu\) to \(m\), it holds that \(d_f(x, F(x)) = e^{-\frac{2f(x)}{n}} d(x, F(x))\) on \(M\).

We first prove the following HWI inequality under the curvature condition \((1.3)\) (cf. Theorem 20.10 in [24]):

**Corollary 5.4.** Let \(\rho : M \to \mathbb{R}\) denote a positive Lipschitz function with \(\int_M \rho dm = 1\), and put \(\mu := \rho m\). We assume that \(f \leq (n - 1)\delta\),
and that \( m \) is \( \mu \)-constant. For \( \kappa > 0 \), if \( \text{Ric}^1_{f,M} \geq (n - 1)\kappa e^{\frac{-4\delta}{n}} \), then
\[
H_m(\mu) \leq \sqrt{I_m(\mu)} W_2(\mu, m) \\
- \frac{(n - 1)\kappa e^{-4\delta}}{6} \max\{1, \sup \rho\}^{\frac{1}{n}} W_2(\mu, m)^2.
\]

**Proof.** Let \( F \) be a unique optimal transport map from \( \mu \) to \( m \), and let \( \pi \) be a unique optimal coupling of \( (\mu, m) \). Since \( \pi = (\text{Id}_M \times F)_{\#}\mu \), and since \( m \) is \( \mu \)-constant, on the support of \( \pi \),
\[
\tilde{\beta}_{\kappa,f}(x,y) = \left( \frac{d_f(x,y)}{s_\kappa(d_f(x,y))} \right)^{n-1},
\]
\[
\tilde{\beta}_{\kappa,f}(x,y) = \frac{n - 1}{n} \left( \frac{d_f(x,y)}{c_\kappa(d_f(x,y))} - 1 \right),
\]
where \( \tilde{\beta}_{\kappa,f} \) and \( \tilde{\beta}_{\kappa,f} \) are defined as (5.1) and as (5.2), respectively. We recall the following elementary estimates (see e.g., Lemma 5.13 in [10]):
\[
\left( \frac{\alpha}{\sin \alpha} \right)^{1 - \frac{1}{n}} - 1 \geq \frac{n - 1}{n} \frac{\alpha^2}{6}, \quad 1 - \frac{\alpha \cos \alpha}{\sin \alpha} \geq \frac{\alpha^2}{3}
\]
for all \( \alpha \in [0, \pi] \). By this elementary estimates and \( f \leq (n - 1)\delta \),
\[
-\left( \tilde{\beta}_{\kappa,f}(x,y)^{\frac{1}{n}} - 1 \right) \leq -\frac{(n - 1)\kappa}{6} d_f(x,y)^2 \leq -\frac{(n - 1)\kappa e^{-4\delta}}{6n} d(x,y)^2,
\]
\[
\tilde{\beta}_{\kappa,f}(x,y) \leq -\frac{(n - 1)\kappa}{3n} d_f(x,y)^2 \leq -\frac{(n - 1)\kappa e^{-4\delta}}{3n} d(x,y)^2.
\]
Applying Proposition [5.3] to \( \rho_0 = \rho \) and \( \rho_1 = 1 \), and using the estimates for \( \tilde{\beta}_{\kappa,f} \) and \( \tilde{\beta}_{\kappa,f} \), we see that \( H_m(\mu) \) is at most
\[
\sqrt{I_m(\mu)} W_2(\mu, m) - \frac{(n - 1)\kappa e^{-4\delta}}{3} \int_{M \times M} \rho(x)^{\frac{1}{\kappa}} d(x,y)^2 d\pi(x,y)
\]
\[
- \frac{(n - 1)\kappa e^{-4\delta}}{6} \int_{M \times M} d(x,y)^2 d\pi(x,y),
\]
and hence
\[
H_m(\mu) \leq \sqrt{I_m(\mu)} W_2(\mu, m) \\
- \frac{(n - 1)\kappa e^{-4\delta}}{6} \max\{1, \sup \rho\}^{\frac{1}{n}} \int_{M \times M} d(x,y)^2 d\pi(x,y).
\]
By the optimality of \( \pi \), the right hand side of the above inequality is equal to that of the desired one. We conclude Corollary 5.4. \( \square \)

We further show the following Logarithmic Sobolev inequality under our curvature condition (cf. Theorem 21.7 in [24]):
Corollary 5.5. Let $\rho : M \to \mathbb{R}$ denote a positive Lipschitz function with $\int_M \rho \, dm = 1$, and put $\mu := \rho \, m$. We assume that $f \leq (n - 1)\delta$, and that $m$ is $\mu$-constant. For $\kappa > 0$, if $\text{Ric}^1_{f, M} \geq (n - 1)\kappa e^{-\frac{4f}{n}}$, then

$$H_m(\mu) \leq \frac{3 \max\{1, \sup \rho\}^{\frac{1}{n}}}{2(n - 1)\kappa e^{-\frac{4f}{n}}} I_m(\mu).$$

Proof. For all $a, b \in \mathbb{R}$ and $K > 0$ we see

$$ab \leq Ka^2 + \frac{b^2}{2K}.$$

Using this elementary inequality, we have

$$\sqrt{I_m(\mu)} W_2(\mu, m) \leq \frac{3 \max\{1, \sup \rho\}^{\frac{1}{n}}}{2(n - 1)\kappa e^{-\frac{4f}{n}}} I_m(\mu) + \frac{(n - 1)\kappa e^{-\frac{4f}{n}}}{6} \max\{1, \sup \rho\}^{-\frac{1}{n}} W_2(\mu, m)^2.$$

From Corollary 5.4 one can derive Corollary 5.5. \hfill \square

Finally, we conclude the following finite dimensional transport energy inequality (cf. Theorem 22.37 in [24]):

Corollary 5.6. Let $\rho : M \to \mathbb{R}$ be positive, Lipschitz and $\int_M \rho \, dm = 1$. We put $\mu := \rho \, m$. For $\kappa > 0$, if $\text{Ric}^1_{f, M} \geq (n - 1)\kappa e^{-\frac{4f}{n}}$, then we have

$$n \int_{M \times M} \tilde{\beta}_{\kappa, f}(x, y)^{\frac{1}{n}} \left(1 - \rho(y)^{-\frac{1}{n}}\right) d\pi(x, y)$$

$$\geq \int_{M \times M} n \left(\frac{e^{-2f(x)} d(x, y)}{s_\kappa(df(x, y))}\right)^{1-\frac{1}{n}} d\pi(x, y)$$

$$- \int_{M \times M} \left((n - 1) e^{-2f(x)} \frac{d(x, y) c_\kappa(df(x, y))}{s_\kappa(df(x, y))} + 1\right) d\pi(x, y),$$

where $\pi$ is a unique optimal coupling of $(m, \mu)$.

Proof. We apply Proposition 5.3 to $\rho_0 = 1$ and $\rho_1 = \rho$. From $H_m(m) = 0$ and $I_m(m) = 0$ we deduce

$$0 \leq \int_{M \times M} \left(\tilde{\beta}_{\kappa, f}(x, y) - \rho(y)^{-\frac{1}{n}} \tilde{\beta}_{\kappa, f}(x, y)^{\frac{1}{n}}\right) d\pi(x, y) + 1.$$

Hence, the left hand side of the desired inequality is at least

$$n \int_{M \times M} \left(\tilde{\beta}_{\kappa, f}(x, y)^{\frac{1}{n}} - \tilde{\beta}_{\kappa, f}(x, y) - 1\right) d\pi(x, y).$$
By substituting (5.1) and (5.2), we obtain the desired inequality. Thus, we complete the proof of Corollary 5.6.

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