Abstract. Let
\[ S(N) = \sum_{n \sim N} \lambda_f(n) \chi(n), \]
where \( \lambda_f(n) \)'s are Fourier coefficients of Hecke-eigen form, and \( \chi \) is a primitive character of conductor \( p^r \). In this article we prove a sub-Weyl strength bounds for \( S(N) \). Indeed, we obtain
\[ S(N) \ll N^{5/9} p^{4r/13}, \]
provided that \( p^{3r/20} \leq N \leq p^{4r/5} \). Note that the above bound for \( S(N) \) is non-trivial if \( N \geq (p^r)^{2/3 - 1/60} \).

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1. Introduction

Let \( f \) be a holomorphic Hecke-eigen cusp form on \( SL(2, \mathbb{Z}) \) with normalized Fourier coefficients \( \lambda_f(n) \). Let \( \chi \) be a Dirichlet character of conductor \( p^r \). In this article we consider the character sum
\[ S_{f, \chi}(N) = \sum_{n=1}^{\infty} \lambda_f(n) \chi(n) W \left( \frac{n}{N} \right), \]
where $W$ is a bump function supported on the interval $[1, 2]$ and satisfies $W^{(j)}(x) \ll_j 1$. One can use information (bounds for $L(1/2, f \times \chi)$) about $L$-values $L(1/2, f \times \chi)$ to show cancellations in the sum $S_{f, \chi}(N)$. Indeed, by Mellin inversion we have that

$$S_{f, \chi}(N) = \frac{1}{2\pi i} \int_{(\sigma)} N^s \tilde{W}(s) L(s, f \times \chi) \, ds, \quad \sigma > 1.$$ 

First shift the contour to 1/2-line and estimate trivially to get

$$S_{f, \chi}(N) \ll N^{1/2} |L(1/2, f \times \chi)| N^\epsilon,$$

where we have used the fact that $\tilde{W}(s)$ decays rapidly as $\Im(s) \to \infty$. We fix the form $f$. Then the analytic conductor of the $L$ value

$$L(1/2, f \times \chi)$$

becomes $p^{2r}$. The convexity bound $L(1/2, f \times \chi) \ll p^{r/2}$ would imply that

$$S_{f, \chi}(N) \ll N^{1/2} p^{r/2} N^\epsilon,$$

which is non-trivial if $N > p^r$. The best known bound for $L(1/2, f \times \chi)$ is the Weyl bound $L(1/2, f \times \chi) \ll p^{r/3}$, due to D. Miličević and V. Blomer [3], and R. Munshi and S. Singh [5], which would then imply that

$$S_{f, \chi}(N) \ll N^{1/2} p^{r/3} N^\epsilon,$$

which is non-trivial if $N > p^{2r/3}$. Currently we do not know how to obtain sub-Weyl bounds for $L(1/2, f \times \chi)$. But the sub-Weyl type bounds $L(1/2, f \times \chi) \ll (p^r)^{1/2 - \eta}$ for some $\eta > 0$ would give non-trivial bounds for $S_{f, \chi}(N)$ whenever $N > (p^r)^{1/2 - 2\eta}$. It is needless to mention that Lyndelőf hypothesis would give non-trivial bounds for $L(1/2, f \times \chi)$ if $N > p^\epsilon$.

Let $K(n)$ be a trace function modulo prime $p$. In [6], E. Fouvry, E. Kowalski and P. Michel showed that

$$\sum_n \lambda_f(n) K(n) W \left( \frac{n}{N} \right) \ll N^{1/2} p^{3/8} N^\epsilon.$$ 

This is Burgess type bound which gives non-trivial bounds for above sums if $N > p^{3/4}$. Recently, in [1] the first author showed that we have cancellation in the above sums if $N > p^{2/3}$ (Weyl strength) in the case when the trace function $K(n)$ is taken to be a Dirichlet character. But for general trace functions $K(n)$ the bound of Fouvry, Kowalski, and Michel is the best.

We are interested in showing that sub-Weyl strength cancellations in $S_{f, \chi}(N)$ when $\chi$ is Dirichlet character of conductor $p^r$ and $r \to \infty$ (depth aspect). This will be a counterpart result to the that of R. Holowinsky, R. Munshi, and Z. Qi [4] where they showed sub Weyl strength cancellations in analytic twist of $\lambda_f(n)$.

Our aim in this article is to establish the following theorem.

**Theorem 1.** Let $p$ be odd prime such that $p > 5$. Then we have

$$S_{f, \chi}(N) \ll N^{\frac{\epsilon}{2}} p^{\frac{3r}{20}} N^\epsilon,$$

where implied constant depends on the prime $p$, and provided $p^{13r/20} \leq N \leq p^{4r/5}$.

**Remark 1.** We need condition $p > 5$ on the prime $p$ to apply $p$-adic exponent pair $(1/30, 13/15)$. 
Remark 2. we can get same kind of bounds for \( S_{f, \chi}(N) \) even if we consider Fourier coefficients \( \lambda_f(n) \) of Hecke Maass cusp form \( f \) as we only require Ramanujan bound for the coefficients in \( L^2 \) sense.

1.1. Method of the proof. We take the path of circle method to bound the sum \( S_{f, \chi}(n) \), especially the approach of R. Munshi. First we separate the oscillations \( \lambda_f(n) \) and \( \chi(n) \) using the delta symbol. While separating these oscillations we introduce extra additive harmonic in the sum which serve as conductor lowering in the delta method. Once these get separated we apply Voronoi and Poisson summation formulas accordingly. We then remove the Fourier coefficients by applying Cauchy-Schwartz inequality. In the resulting expression we open the absolute square and again we employ Poisson summation. In zero frequency we can not do much other than evaluating trivially. But in non-zero frequency we end up with sums of the form

\[
\sum_{R \leq m \leq 2R} e \left( \frac{f(m)}{p^{r}} \right)
\]

with \( R \leq p^{r}/N \) and “nice” phase function \( f \). In this sum we seek to get some cancellations which we can achieve by using \( p \)-adic exponent pair \( (1/15, 13/15) \). The main novelty of this paper is to apply \( p \)-adic analogue exponent pair is developed by D. Milićević in [2].

1.2. Notations. We write \( p^{s} \parallel m \) to denote that \( p^{s} | m \) and \( p^{s+1} \not| m \).

2. Application of the circle method

We separate the oscillations \( \lambda_f(n) \) and \( \chi(n) \) in the sum \( S_{f, \chi}(N) \) by using the delta symbol \( \delta \) which is defined on the set of integers by \( \delta(0) = 1 \) and \( \delta(m) = 0 \) if \( m \neq 0 \). We have the following expression for \( \delta \) which is due to Duke, Friedlander and Iwaniec [7]. Let \( L \geq 1 \) be a large number. For \( n \in [-2L, 2L] \), we have

\[
\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sum_{a \mod q} e \left( \frac{na}{q} \right) \int_{\mathbb{R}} g(q, x) e \left( \frac{nx}{qQ} \right) dx,
\]

where \( Q = 2L^{1/2} \), and the function \( g(q, x) \) satisfies the following properties (see, [8, Lemma 5,]):

- \( g(q, x) = 1 + O \left( \frac{Q}{q} \left( \frac{q}{Q} + |x| \right)^A \right) \), \( g(q, x) \ll |x|^{-A} \) for any \( A > 1 \).
- \( x^j \frac{\partial}{\partial x} g(q, x) \ll Q \min \{ \frac{Q}{q}, \frac{1}{(|x|)} \} \).
- \( \int_{\mathbb{R}} (|g(q, x)| + |g(q, x)|^2) dx \ll Q^3 \).

Indeed, we have

\[
S_{f, \chi}(N) = \sum_{m} \sum_{n=1}^{\infty} \lambda_f(n) \chi(m) \delta \left( \frac{n-m}{p^{r}} \right) W \left( \frac{n}{N} \right) V \left( \frac{m}{N} \right)
\]

with the condition that \( p^{r} \leq N \) and \( \ell \leq r \). Now by writing the expression for \( \delta \), with the choice \( Q = \sqrt{N/p^{r}} \), in the above sum we arrive at
Lemma 3.1. We have
$$S_f(N; a, b, q, x) = \sum_{m=1}^{\infty} \chi(m) e \left( \frac{-aq}{pq} \right) e \left( \frac{-mx}{pq} \right),$$
where
$$S_f(N; a, b, q, x) = \sum_{n=1}^{\infty} \lambda_f(n) e \left( \frac{(a + bq)n}{pq} \right) e \left( \frac{xn}{pq} \right) W \left( \frac{n}{N} \right),$$
and
$$S_f(N; a, b, q; x) = \sum_{m=1}^{\infty} \chi(m) e \left( \frac{-aq}{pq} \right) e \left( \frac{-mx}{pq} \right) V \left( \frac{m}{N} \right).$$

3. Application of summation formulas

3.1. Applying Poisson summation formula. We shall apply the Poisson summation formula to the sum over $m$ in equation (3) to get the following lemma.

Lemma 3.1. We have
$$S_f(N; a, b, q, x) = \frac{N}{pq} \sum_{m \in \mathbb{Z}} C(a, b, q, m) \mathcal{I}(x, q, m) + O \left( N^{-2022} \right),$$
with $M_0 := \frac{p^2q}{N} N^\epsilon$, where
$$C(a, b, q, m) = \sum_{\beta \pmod{pq}} \chi(\beta) e \left( -\frac{(a + bq)\beta}{pq} + \frac{m\beta}{pq} \right),$$
and
$$\mathcal{I}(x, q, m) = \int_{\mathbb{R}} V(z)e \left( \frac{-Nxz}{pq} \right) e \left( \frac{Nmz}{pq} \right) dz.$$

Proof. We split the $m$-sum in (3) into congruence classes modulo $p^r q$. Indeed, we write $m = \beta + cp^r q$ with $\beta \pmod{p^r q}$, and $c \in \mathbb{Z}$ to get
$$S_f(N; a, b, q, x) = \sum_{\beta \pmod{p^r q}} \chi(\beta) e \left( -\frac{(a + bq)\beta}{pq} \right) \sum_{c \in \mathbb{Z}} V \left( \frac{\beta + cp^r q}{N} \right) e \left( -\frac{(\beta + cp^r q)x}{pq} \right)$$
$$= \sum_{\beta \pmod{p^r q}} \chi(\beta) e \left( -\frac{(a + bq)\beta}{pq} \right) \sum_{c \in \mathbb{Z}} \int_{\mathbb{R}} V \left( \frac{\beta + cp^r q}{N} \right) e \left( -\frac{(\beta + cp^r q)x}{pq} \right) e(-my)dy,$$
the second equality follows by applying Poisson summation formula. We now substitute the change of variable $(\beta + yp^r q)/N = z$ to obtain the value of $S_f(N; a, b, q, x)$ to be
$$\frac{N}{p^r q} \sum_{m \in \mathbb{Z}} \left\{ \sum_{\beta \pmod{p^r q}} \chi(\beta) e \left( -\frac{(a + bq)\beta}{pq} + \frac{m\beta}{pq} \right) \right\} \int_{\mathbb{R}} V(z)e \left( \frac{-Nxz}{pq} \right) e \left( \frac{-Nmz}{pq} \right) dz$$
$$= \frac{N}{p^r q} \sum_{m \in \mathbb{Z}} C(a, b, q, m) \mathcal{I}(x, q, m).$$
Lemma 3.2. Let $\mathcal{C}(a, b, q, m)$, $\mathcal{I}(x, q, m)$ are given as above. We see, by repeated integration by parts, that
\[
\mathcal{I}(x, q, m) \ll_j \left(1 + \frac{N|x|}{p^r q N} \right)^j \left(\frac{p^r q}{N m}\right)^j,
\]
for any $j \geq 0$. Thus, $\mathcal{I}(x, q, m)$ is negligibly small unless
\[
|m| \leq M_0 := \frac{p^r Q}{N} N^\epsilon.
\]

We now first evaluate the character sum in the following subsection.

3.2. Evaluation of the character sum. We have the following lemma.

Lemma 3.2. Let $q = p^r q'$ with $(p, q') = 1$ (i.e., $p^r \parallel q$). Then we have
\[
\mathcal{C}(a, b, q, m) = \begin{cases} 
q \chi(q') \times \left(\frac{m-(a+bq)p^{r-\ell}}{p^r}\right) \tau \chi & \text{if } a \equiv m p^{r-\ell} \mod q', \text{ and } p^r \parallel m \\
0 & \text{otherwise},
\end{cases}
\]
where $\tau \chi$ denotes the Gauss sum.

Proof. Since $q = p^r q'$ with $(p, q') = 1$, the character sum $\mathcal{C}(a, b, q, m)$ is given by
\[
\sum_{\beta \parallel p^{r+1} q'} \chi(\beta) e\left(\frac{-(a + bq)\beta}{p^{r+1} q'} + \frac{m\beta}{p^{r+1} q'}\right).
\]

By writing $\beta = \alpha_1 q' q + \alpha_2 p^{r+1} r + \beta_1 q'$, where $\alpha_1$ mod $p^{r+1}$ and $\alpha_2$ mod $q'$, we see that the above character sum changes to
\[
\sum_{\alpha_1(p^{r+1})} \chi(\alpha_1) e\left(\frac{-(a + bq)\alpha_1 q'}{p^{r+1} q'} + \frac{m\alpha_1 q'}{p^{r+1} q'}\right) \sum_{\alpha_2(q')} e\left(\frac{-(a + bq)\alpha_2 p^{r+1} p^{r-\ell}}{q'} + \frac{m\alpha_2 p^{r+1}}{q'}\right).
\]

Again, by writing $\alpha_1 = \beta_1 q' + \beta_2$, where $\beta_2$ is modulo $p^r$ and $\beta_1$ modulo $p^r$, the above sum becomes
\[
\sum_{\beta_2(p^r)} \chi(\beta_2) e\left(\frac{-(a + bq)p^{r-\ell} \beta_2 q'}{p^{r+1} q'} + \frac{m\beta_2 q'}{p^{r+1} q'}\right) \sum_{\beta_1(p^{r+1})} e\left(\frac{-(a + bq)\beta_1 q' p^{r-\ell}}{p^r} + \frac{m\beta_1 q'}{p^r}\right)
\times \sum_{\alpha_2(q')} e\left(\frac{-(a + bq)\alpha_2 p^{r+1} p^{r-\ell}}{q'} + \frac{m\alpha_2 p^{r+1}}{q'}\right).
\]

We execute sums over $\beta_1$ and $\alpha_2$ to transfer the above sum to
\[
q \Pi_{(m-ap^{r-\ell} \equiv 0(\mod p^r))} \Pi_{(m-ap^{r-\ell} \equiv 0(\mod q'))} \sum_{\beta_2(p^r)} \chi(\beta_2) e\left(\frac{(m-(a+bq)p^{r-\ell}) \beta_2 q'}{p^{r+1}}\right).
\]

We have $N \leq p^r$. Therefore, we have the inequality $p^r \leq q \leq Q = \sqrt{N/p^r} \leq p^{(r-\ell)/2} < p^{r-\ell}$. Thus, $\min\{r_1, r - \ell\} = r_1$. Therefore the congruence $m - ap^{r-\ell} \equiv 0(\mod p^r)$ is same as $p^r \mid m$.

Note that, since $\chi$ is a primitive character modulo $p^r$, the sum over $\beta_2$ is Gauss sum which vanishes unless
\[
\left(\frac{m-(a+bq)p^{r-\ell}}{p^r}, p\right) = 1 \iff (m/p^r, p) = 1,
\]
where $\mathcal{C}(a, b, q, m)$, $\mathcal{I}(x, q, m)$ are given as above. We see, by repeated integration by parts, that
\[
\mathcal{I}(x, q, m) \ll_j \left(1 + \frac{N|x|}{p^r q N} \right)^j \left(\frac{p^r q}{N m}\right)^j,
\]
for any $j \geq 0$. Thus, $\mathcal{I}(x, q, m)$ is negligibly small unless
\[
|m| \leq M_0 := \frac{p^r Q}{N} N^\epsilon.
\]
as \( r_1 < r - \ell \). In this case we have

\[
\sum_{\beta_2(p^r)} \chi(\beta_2) e \left( \frac{(m - (a + bq)p^{r-\ell}) \beta_2 q}{p^{r_1}} \right) = \chi(q') \chi \left( \frac{(m - (a + bq)p^{r-\ell})}{p^{r_1}} \right) \sum_{\beta_2(p^r)} \chi(\beta_2) e \left( \frac{\beta_2}{p^r} \right),
\]

Note that the last sum over \( \beta_2 \) is the Gauss sum. Which completes the proof of the lemma.

After the Poisson summation formula, the sum \( S_{f,\chi}(N) \) is given by

\[
S_{f,\chi}(N) = \frac{1}{Qp^f} \int_{\mathbb{R}} \sum_{r_1=0}^{\lfloor \frac{\log Q}{\log p} \rfloor} \sum_{1 \leq q \leq Q/p^{r_1}} g(p^{r_1}q, x) \sum_{a_1 \mod p^{r_1}} \sum_{b \mod p^f} \sum_{m' \leq M_0/p^{r_1}} \chi (m' - (a + bq)p^{r-\ell-r_1}) I(x, p^{r_1}q', p^{r_1}m') \chi(q') N \sum_{n=1}^{\infty} \frac{\lambda_f(n) e \left( \frac{(a + bq)n}{p^f q} \right) e \left( \frac{xn}{p^f qQ} \right) W \left( \frac{n}{N} \right)}{p^f} \, dx + O_A \left( N^{-A} \right),
\]

for any real \( A > 0 \), where \( a \mod q \) is determined in terms of \( a_1 \mod p^{r_1} \) and \( m' \).

Indeed we have \( a \equiv m'p^{2r_1}p^{r-\ell+r_1} + a_1qq' \mod p^{r_1}q' \). And \( q = p^{r_1}q' \).

We now split the above expression for \( S_{f,\chi}(N) \) as follows

\[
S_{f,\chi}(N) = S_{f,\chi}(N; r_1 = 0 \text{ contribution}) + S_{f,\chi}(N; r_1 \geq 1 \text{ contribution}).
\]

Since \( r_1 \geq 1 \) implies \( (a + bq, p^f q) = 1 \), and then we can apply Voronoi summation formula directly. But, if \( r_1 = 0 \), then \( a + bq \) may not be coprime to \( p^f \). Therefore we can not directly apply Voronoi summation. So we need to work with these two different situations. Note that these cases can be dealt in a similar fashion.

From now on we only focus on estimation of \( S_{f,\chi}(N; r_1 = 0 \text{ contribution}) \). In a similar way we can estimate \( S_{f,\chi}(N; r_1 \geq 1 \text{ contribution}) \) and even get better estimates in this case.

\[
S_{f,\chi}(N; r_1 = 0 \text{ contri}) = \frac{1}{Qp^f} \int_{\mathbb{R}} \sum_{1 \leq q \leq Q} g(q, x) \sum_{b \mod p^f} \sum_{m \leq M_0} \chi (m - (a + bq)p^{r-\ell}) I(x, q, m) \chi(q') N \sum_{n=1}^{\infty} \frac{\lambda_f(n) e \left( \frac{(a + bq)n}{p^f q} \right) e \left( \frac{xn}{p^f qQ} \right) W \left( \frac{n}{N} \right)}{p^f} \, dx,
\]

where \( a \equiv m \mod q \). Note that \( (m, q) = 1 \).
3.3. **Application of Voronoi summation formula.** Now we appeal to an application of Voronoi summation formula on the n-sum in (1). Recall from (2) that \( S_f(N; a, b, q, x) \) is same as this n-sum. The application of Voronoi summation formula leads to the following lemma.

**Lemma 3.3.** Let \( (a + bq, p^\ell) = p^{\ell_1} \) for some \( 0 \leq \ell_1 \leq \ell \). Then we have

\[
S_f(N; a, b, q, x) = \frac{2\pi i^k N^{3/4}}{p^{(\ell-\ell_1)/2}q^{1/2}} \sum_{\epsilon \in \{\pm\}} \sum_{1 \leq n \ll N_0} \frac{\lambda_f(n)}{n^{1/4}} e\left(-\frac{(a + bq)/p^{\ell_1}n}{p^{\ell-\ell_1}q}\right) J(\epsilon, q, x, n),
\]

where

\[
J(\epsilon, q, x, n) = \int W_{1,\epsilon}(y) e\left(\frac{xNy}{p^{\ell-\ell_1}q}\right) e\left(\frac{\epsilon 2\sqrt{nNy}}{p^{\ell-\ell_1}q}\right) dy,
\]

and \( N_0 = p^{\ell-2\ell_1} N^\epsilon \).

**Proof.** An application of Voronoi summation transfers the \( S_f(N; a, b, q, x) \) into

\[
\frac{2\pi i^k}{p^{\ell-\ell_1}q} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1/4}} e\left(-\frac{(a + bq)/p^{\ell_1}n}{p^{\ell-\ell_1}q}\right) \int \Re W(y) e\left(\frac{xNy}{p^{\ell-\ell_1}q}\right) J_{k-1} e\left(\frac{\epsilon 2\sqrt{nNy}}{p^{\ell-\ell_1}q}\right) dy,
\]

where \( k \) is the weight of the holomorphic Hecke-eigenform \( f \) and \( J_{k-1}(x) \) is the Bessel function. We make a change of variables \( y/N \mapsto z \) in the above integration and use the expression

\[
J_{k-1}(x) = \frac{1}{\sqrt{2\pi}} \sum_{\epsilon \in \{\pm\}} W_{k,\epsilon}(x) e^{\epsilon i x},
\]

where \( x^j W_{k,\epsilon}^{(j)}(x) \ll_{k, j} 1 \) for \( x \gg 1 \), for Bessel function to get

\[
\frac{N^{3/4} p^{(\ell-\ell_1)/2} q^{1/2}}{n^{1/4}} \sum_{\epsilon \in \{\pm\}} \int W_{1,\epsilon}(y) e\left(\frac{xNy}{p^{\ell-\ell_1}q}\right) e\left(\frac{\epsilon 2\sqrt{nNy}}{p^{\ell-\ell_1}q}\right) dy,
\]

where \( W_{1,\epsilon}(y) = W(y) W_{k,\epsilon} \left(\frac{xNy}{p^{\ell-\ell_1}q}\right) \) which satisfies \( y^{j} W_{1,\epsilon}^{(j)}(y) \ll_{k, j} 1 \). We see , repeated integration by parts, that the above integral in negligibly small unless

\[
1 \leq n \ll N_0 = p^{\ell-2\ell_1} N^\epsilon.
\]

Thus the lemma follows. \( \square \)

By noting the following identities

\[
\frac{(a + bq)/p^{\ell_1}}{p^{\ell-\ell_1}q} = \frac{\bar{a}p^{\ell_1}p^{\ell-\ell_1}q}{p^{\ell-\ell_1}q} + \frac{(a + bq)/p^{\ell_1}q}{p^{\ell-\ell_1}q}
\]

\[
= \frac{(a + bq)/p^{\ell_1}q}{p^{\ell-\ell_1}q} + \frac{mp^{2(\ell-\ell_1)}q}{q},
\]

in the second equality we have used the fact that where \( a \equiv m p^{\ell-\ell_1} \mod q \), and by arranging all terms we get the following proposition.
Proposition 1. We have

$$S_{f, \chi}(N; r_1 = 0 \text{ contr}) = \frac{\tau_s N^{7/4} 2\pi \ell^k}{Q p^{r_1 + \frac{7}{2}}} \sum_{\varepsilon \in \{\pm\}} \sum_{\ell_1 = 0}^{\ell} p^{\ell_1/2} T(\varepsilon, \ell_1, N)$$

where

$$T(\varepsilon, \ell_1, N) = \sum_{1 \leq n \leq N_0} \frac{\lambda_f(n)}{n^{1/4}} \sum_{1 \leq q \leq Q} \frac{\chi(q)}{q^{3/2}} \sum_{\substack{m \leq M_0 \beta \mod p^{\ell_1} \\in \{\pm\}}} \sum_{(m, p) = 1}^{\varepsilon} \int \mathfrak{N}(m - (a + bq)p^{r_1}) e \left( -\frac{(a + bq)/p^{\ell_1}}{p^{r_1}} \right) \mathcal{I}(\varepsilon, q, n, m),$$

where $b = -a\bar{q} + \beta p^{\ell_1}$, the symbol $\dagger$ on the $b$-sum means that $((a + bq)/p^{\ell_1}, p) = 1$, and

$$\mathcal{I}(\varepsilon, q, n, m) = \int_{\mathbb{R}} g(q, x) \mathcal{J}(\varepsilon, q, x, n) \mathcal{I}(q, x, m) \, dx.$$

3.4. Bounds for integral $\mathcal{I}(\varepsilon, q, n, m)$. In this subsection we give bounds for the integral $\mathcal{I}(\varepsilon, q, n, m)$ which is useful when we will deal with small $q$ (note that phase functions appear in $\mathcal{I}(\varepsilon, q, n, m)$ oscillates when $q$ is small).

Lemma 3.4. We have

$$\mathcal{I}(\varepsilon, q, n, m) \ll \frac{p^{l_q} Q}{N} N^\epsilon.$$

Proof. Recall that $\mathcal{I}(\varepsilon, q, n, m)$ is give by

$$\mathcal{I}(\varepsilon, q, x, n) = \int_{\mathbb{R}} g(q, x) \mathcal{J}(\varepsilon, q, x, n) \mathcal{I}(q, x, m) \, dx.$$

The function $q(q, x)$ is negligible unless $|x| \leq N^\epsilon$. Therefore we have

$$\mathcal{I}(\varepsilon, q, x, n) = \int_{|x| \leq N^\epsilon} g(q, x) \int_{\mathbb{R}} W_{1, \varepsilon}(y) e \left( \frac{x N y}{p^l q Q} \right) e \left( \frac{\varepsilon 2 \sqrt{n N y}}{p^{r_1} q} \right) \int_{\mathbb{R}} V(z) e \left( \frac{-N x z}{p^l q Q} \right) e \left( \frac{-N m z}{p^l q} \right) \, dz \, dy \, dx + O \left( N^{-2022} \right).$$

Now we consider the $z$ integral. By repeated integration by parts we see that $z$ integral is negligible unless

$$\left| \frac{N x}{p^l q Q} + \frac{N m}{p^l q} \right| \ll N^\epsilon \iff \left| x + \frac{m Q}{p^l} \right| \ll \frac{p^{l_q} Q}{N} N^\epsilon.$$

We know, by properties of the function $g(q, x)$, see Section 2 that

$$g(q, x) = 1 + O \left( N^{-2022} \right),$$

if $q \leq Q^{1-\epsilon}$ or $|x| \leq N^{-\epsilon}$. Therefore we divide $x$-integral into two parts. Indeed, we write

$$\mathcal{I}(\varepsilon, q, n, m) = \left( \int_{|x| \leq N^{-\epsilon}} + \int_{|x| \geq N^{-\epsilon}} \right) g(q, x) \{\ldots\} \, dx \, dy \, dz + O \left( N^{-2022} \right).$$
In the first integral we can replace \( g(q, x) \) by 1 up to a negligible error term. We treat everything else trivially in the first \( x \)-integral to get this integral to be
\[
\ll \frac{p' q Q}{N} N^\varepsilon.
\]

In the second \( x \)-integral, we have the condition that \( N^{-\varepsilon} \leq |x| \leq N^\varepsilon \). In this case we consider \( y \)-integral in (5). In this integral we make change of variable \( y \to y^2 \), then the resulting expression of this integral is given by
\[
\int_{\mathbb{R}} 2y W_{1, x}(y^2) e(f(y)) \, dy,
\]
where the phase function
\[
f(y) = \frac{x N y^2}{p' q Q} + \varepsilon 2\sqrt{n N y}.
\]
The stationary point \( y_0 \) of \( f(y) \) is given by \( y_0 = \varepsilon \sqrt{n Q p' \ell_1} / x \sqrt{N} \), and we have
\[
f^{(n)}(y_0) = \frac{2 x N}{p' q Q}
\]
Thus we have
\[
\frac{1}{\sqrt{|f^{(n)}(y_0)|}} \ll \sqrt{\frac{p' q Q}{N} N^\varepsilon}.
\]
Therefore, by stationary method this \( y \)-integral is at most
\[
\ll \sqrt{\frac{p' q Q}{N} N^\varepsilon}.
\]
Thus, the second \( x \)-integral is at most
\[
\ll \sqrt{\frac{p' q Q}{N} N^\varepsilon} \int_{N^{-\varepsilon} \leq |x| \leq N^\varepsilon} |q(q, x)| \, dx \ll \sqrt{\frac{p' q Q}{N} N^\varepsilon}
\]
\[
\ll \frac{p' q Q}{N} N^\varepsilon \int_{\mathbb{R}} |g(q, x)|^2 \, dx
\]
\[
\ll \frac{p' q Q}{N} N^\varepsilon.
\]
We have used the \( L^2 \)-bound for the function \( g(q, x) \) from the Section 2.

4. CAUCHY AND POISSON

An application of the Cauchy’s inequality on the \( n \) sum in \( T(\varepsilon, \ell_1, N) \) along with Ramanujan bound for the Fourier coefficients \( \lambda_f(n) \) gives that
\[
T(\varepsilon, \ell_1, N) \ll N_0^{1/4} \Theta^{1/2}
\]
is negligibly small unless upto a negligible error term.

\[ T = \sum_{n} W_2 \left( \frac{n}{N_0} \right) \sum_{1 \leq q \leq Q} \frac{\chi(q)}{q^{3/2}} \sum_{m \equiv M_0 \beta \pmod{p^{\ell-1}}} \sum_{(m,p) = 1} \sum_{\ell=1}^{\dagger} \chi \left( m - (a + bq) p^{-\ell} \right) e \left( -\frac{(a + bq)/p^{\ell-1}}{q} \right) \mathcal{J}(...) \],

where \( W_2 \) is smooth bump function supported in \([1, 2]\). After opening the absolute square and interchanging sums we arrive at

\[ \Theta = \sum_{1 \leq q_1, q_2 \leq Q} \frac{\chi(q_1 q_2)}{q_1^{3/2} q_2^{3/2}} \sum_{m_1, m_2 \equiv M_0 \beta_1 \beta_2 \pmod{p^{\ell-1}}} \sum_{(m_1, m_2, p) = 1} \sum_{\ell=1}^{\dagger} \chi \left( m_1 - (a_1 + b_1 q_1) p^{-\ell} \right) \chi \left( m_2 - (a_2 + b_2 q_2) p^{-\ell} \right) T(...), \]

where

\[ T(m_1, m_2, q_1, q_2, b_1, b_2) = \sum_{n \in \mathbb{Z}} W_2 \left( \frac{n}{N_0} \right) e \left( \frac{(a_1 + b_1 q_1)/p^{\ell-1} q_1 n - ((a_2 + b_2 q_2)/p^{\ell-1} q_2 n)}{p^{\ell-1}} \right) \mathcal{J}(\varepsilon, q_1, n, m_1, \bar{\mathcal{J}}(\varepsilon, q_2, n, m_2)). \]

First we split the sum over \( n \) into congruence classes modulo \( p^{\ell-1} q_1 q_2 \). That is, for any \( \alpha \) modulo \( p^{\ell-1} q_1 q_2 \) we write \( n = \alpha + kp^{\ell-1} q_1 q_2 \) with \( k \in \mathbb{Z} \). Then by applying Poisson summation on \( k \) variable we arrive at the expression

\[ \frac{N_0}{p^{\ell-1} q_1 q_2} \sum_{n \in \mathbb{Z}} \sum_{\alpha \equiv \frac{p^{\ell-1} q_1 q_2}{\alpha} \pmod{p^{\ell-1} q_1 q_2}} e \left( \frac{(a_1 + b_1 q_1)/p^{\ell-1} q_1 q_2 - ((a_2 + b_2 q_2)/p^{\ell-1} q_2 q_2)}{p^{\ell-1} q_1 q_2} \right) \mathcal{J}_1(n, q_1, m_1, \varepsilon), \]

where

\[ \mathcal{J}_1(n, q_1, m_1, \varepsilon) = \int W_2(y) \mathcal{J}(\varepsilon, q_1, N_0 y, m_1) \mathcal{J}(\varepsilon, q_2, N_0 y, m_2) e \left( -\frac{n N_0 y}{p^{\ell-1} q_1 q_2} \right) dy, \]

for \( T(m_1, m_2, q_1, q_2, b_1, b_2) \). Note, by repeated integration by parts, that \( \mathcal{J}_1(n, q_1, m_1, \varepsilon) \) is negligibly small unless

\[ |n| \leq \frac{q_1 q_2 p^{\ell-1}}{N_0} N^\varepsilon. \]

Therefore after executing the sum over \( \alpha \), the value of \( T(...) \) is given by

\[ N_0 \sum_{|n| \leq \frac{q_1 q_2 p^{\ell-1}}{N_0} N^\varepsilon} \mathcal{J}_1(...), \]

\[ \left( \frac{(a_1 + b_1 q_1)/p^{\ell-1} q_1 q_2 - ((a_2 + b_2 q_2)/p^{\ell-1} q_2 q_2)}{p^{\ell-1} q_1 q_2} \right) \bar{q}_2 q_1 q_2 + p^{\ell-1} p^{\ell-1} p^{\ell-1} q_1 q_2 - p^{\ell-1} q_1 q_2 \equiv 0 \pmod{p^{\ell-1} q_1 q_2} \]

upto a negligible error term.
The above congruence relation gives that
\[ \frac{(a_1 + b_1q_1)}{p^{\ell_1}} q_2 - \frac{(a_2 + b_2q_2)}{p^{\ell_1}} q_1 + n \equiv 0 \mod p^{\ell - \ell_1}, \]
and
\[ p^{r - \ell + \ell_1} \bar{m}_1 q_2 - p^{r - \ell + \ell_1} \bar{m}_2 q_1 + n \equiv 0 \mod q_1 q_2. \]

After changing the variables \((a_1 + b_1q_1)/p^{\ell_1} \mapsto \alpha_1\) and \((a_2 + b_2q_2)/p^{\ell_1} \mapsto \alpha_2\), we see that \(\Theta\) is given by

\begin{equation}
\Theta = N_0 \sum_{1 \leq q_1, q_2 \leq Q} \frac{\chi(q_1 q_2)}{q_1^{3/2} q_2^{3/2}} \sum_{m_1, m_2 \in M_0 \mod \omega} \sum_{\alpha_1 \mod p^{\ell - \ell_1}} \sum_{\alpha_2 \mod p^{\ell - \ell_1}} \sum_{|n| \leq 2q_1 q_2 p^{\ell - \ell_1}} \sum_{\alpha_1 q_2 - \alpha_2 q_1 + n \equiv 0 \mod p^{\ell - \ell_1}} \sum_{\bar{m}_1 q_2 - \bar{m}_2 q_1 + n \equiv 0 \mod q_1 q_2} \chi \left( \frac{m_1 - \alpha_1 p^{r - \ell + \ell_1}}{p^{r - \ell + \ell_1} \bar{m}_1 q_2 - p^{r - \ell + \ell_1} \bar{m}_2 q_1 + n} \right) \chi \left( \frac{m_2 - \alpha_2 p^{r - \ell + \ell_1}}{p^{r - \ell + \ell_1} \bar{m}_1 q_2 - p^{r - \ell + \ell_1} \bar{m}_2 q_1 + n} \right) \mathcal{J}_1(n, q_1, m_1, \epsilon). \tag{7} \end{equation}

4.1. Zero frequency \(n = 0\): We write \(\Theta_{\text{zero}}\) for the contribution of the zero frequency to \(\Theta\). In the following lemma we give estimates for \(\Theta_{\text{zero}}\).

**Lemma 4.1.** We have
\[ \Theta_{\text{zero}} \ll \frac{p^{r + \frac{3}{2} - 3\ell_1}}{N^{3/2}} N^\varepsilon, \]
provided that \(N \geq p^{r - (\ell - \ell_1)}\).

**Proof.** For \(n = 0\), the congruence conditions in (7) becomes
\[ \alpha_1 q_2 - \alpha_2 q_1 \equiv 0 \mod p^{\ell - \ell_1}, \] and
\[ \bar{m}_1 q_2 - \bar{m}_2 q_1 \equiv 0 \mod q_1 q_2. \]

From the second congruence we infer that \(q_1 | q_2\) and \(q_2 | q_1\) which implies that \(q_1 = q_2 = q\), and we also have that \(q | m_1 - m_2\). Then from the first congruence we immediately see that \(\alpha_1 \equiv \alpha_2 \mod p^{\ell - \ell_1}\). Therefore \(\Theta_{\text{zero}}\) is given by

\begin{equation}
N_0 \sum_{1 \leq q \leq Q} \frac{\chi(q)}{q^3} \sum_{m_1, m_2 \in M_0 \mod \omega} \sum_{\alpha_1 \mod p^{\ell - \ell_1}} \mathcal{J}_1(n, q, m_1, \epsilon). \tag{8} \end{equation}

For \(m_1 \neq m_2\), we sum over \(\alpha_1\). To this end note that we have \(\chi(1 + z p^{r - (\ell - \ell_1)}) = e \left( -A_1 p^{2r - 2(\ell - \ell_1) - 2} - A_2 p^{r - (\ell - \ell_1) - 2} \right)\), for some integers \(A_1\) and \(A_2\) which are coprime to \(p\), as our choice of \(\ell\) satisfies the condition \(r - \ell \geq r/3\) which is same as \(\ell \leq 2r/3\) (see [2, Lemma 13]). Thus, the \(\alpha_1\) sum is given by
\[ \sum_{\alpha \mod p^{\ell - \ell_1}} e \left( \frac{Y_1 \alpha^2 + Y_2 \alpha}{p^r} \right), \]
where
\[ Y_1 = A_1 p^{2r - 2(\ell - \ell_1)} (\bar{m}_2^2 - \bar{m}_1^2), \]
and
\[ Y_2 = A_2 p^{r - (\ell - \ell_1)} (\bar{m}_1 - \bar{m}_2). \]
Note that this character sum is same as
\[
\sum_{\alpha \mod p^{\ell_1}} e\left( A_1 p^{-\ell_1} (\bar{m}^2 - \bar{m}^2) \alpha^2 + A_2 (\bar{m} - \bar{m}) \alpha \right) + O(\ell).
\]
This sum is quadratic Gauss sum which vanishes unless
\[ p^{\ell_1} \mid (m_1 - m_2). \]
Thus the value of above quadratic Gauss is
\[ p^{\ell_1} \mathbb{I}_{m_1 \equiv m_2 \mod p^{\ell_1}}. \]
By substituting the bound for the character sum over \( \alpha \) and the bound for the integral
\[ I_1(n, q, m_i, \varepsilon) \ll \frac{p^{2r} q^2 Q^2}{N^2} N^{\varepsilon} \]
in \( \Theta_{\text{zero}} \), we see that
\[
\Theta_{\text{zero}} \ll N_0 \left\{ p^{\ell_1} \sum_{1 \leq q \leq Q} \sum_{(q,p)=1} \frac{1}{q^3} \sum_{m_1, m_2 \in M_0} \sum_{q | m_1 - m_2} \frac{p^{2r} q^2 Q^2}{N^2} \mathbb{I}_{m_1 \equiv m_2 \mod p^{\ell_1}} \right. \\
\left. + \sum_{1 \leq q \leq Q} \sum_{(q,p)=1} \frac{1}{q^3} \sum_{m_1, m_2 \in M_0} \sum_{q | m_1 - m_2} \frac{p^{2r} q^2 Q^2}{N^2} \right\} N^{\varepsilon} \\
\ll \frac{p^{r+\frac{3r}{2}-3\ell_1}}{N^{3/2}} N^{\varepsilon} + \frac{p^{2r+\frac{3r}{2}-2\ell_1}}{N^{5/2}} N^{\varepsilon} \\
\ll \frac{p^{r+\frac{3r}{2}-3\ell_1}}{N^{3/2}} N^{\varepsilon},
\]
where in the second inequality the first term corresponds to the contribution of \( m_1 = m_2 \), and second one corresponds to the contribution of \( m_1 \neq m_2 \), and in the last inequality we have used the assumption that \( N \geq p^{r-(\ell-\ell_1)} \). This concludes the proof the lemma.
\[\square\]

Let \( T_0(\varepsilon, \ell_1, N) \) and \( S_{f,\chi}(N; r_1 = 0 \text{ contri}, \Theta_{\text{zero}}) \) denote the contribution of \( \Theta_{\text{zero}} \) to \( T(\varepsilon, \ell_1, N) \) and \( S_{f,\chi}(N; r_1 = 0 \text{ contri}) \) respectively. Then we have that
\[
T_0(\varepsilon, \ell_1, N) \ll \frac{p^{r+3r-4\ell_1}}{N^{3/4}} N^{\varepsilon}
\]
and consequently we have that
\[
S_{f,\chi}(N; r_1 = 0 \text{ contri}, \Theta_{\text{zero}}) \ll \sqrt{N} p^{\ell/2} N^{\varepsilon},
\]
provided \( N \geq p^{r-\ell} \). We record this as the following proposition.

**Proposition 2.** We have
\[
S_{f,\chi}(N; r_1 = 0 \text{ contri}, \Theta_{\text{zero}}) \ll \sqrt{N} p^{\ell/2} N^{\varepsilon},
\]
provided \( N \geq p^{r-\ell} \).
4.2. Non-zero frequency \( n \neq 0 \): Assume that \( n \neq 0 \). In this case we determined \( \alpha_2 \mod p^\ell \) and write \( m_1, m_2 \) in terms of \( q_1, q_2 \) and \( n \) modulo \( q_1, q_2 \) respectively using congruences. Indeed,
\[
\alpha_2 \equiv q_1(\bar{\alpha}q_2 + n) \mod p^{\ell - \ell_1},
\]
and
\[
m_1 \equiv -\bar{n}p^{r - (\ell - \ell_1)} q_2 \mod q_1, \quad m_2 \equiv \bar{n}p^{r - (\ell - \ell_1)} q_1 \mod q_2.
\]
By writing \( m_1 = -\bar{n}p^{r - (\ell - \ell_1)} q_2 + r_1 q_1 \) and \( m_2 = \bar{n}p^{r - (\ell - \ell_1)} q_1 + r_2 q_2 \), we see that

\[
\Theta_{\text{non-zero}} = N_0 \sum_{q_1/q_2, p = 1} \frac{\chi(q_1 q_2)}{q_1 q_2} \sum_{0 < |n| < 2q_2 \nu_0^{\ell - \ell_1} N_0} \sum_{|r_1| \leq \frac{\nu_0^{\ell - \ell_1} N_0}{p^\ell}} \sum_{|r_2| \leq \frac{\nu_0^{\ell - \ell_1} N_0}{p^\ell}} C(r_1, r_2, q_1, q_2, n) \mathcal{H}_1(n, q_1, m_1, z),
\]
where \( C(\ldots) \) is given by
\[
\sum_{\alpha \mod p^{\ell - \ell_1}} \mathcal{X}(-\bar{n}p^{r - (\ell - \ell_1)} q_2 + r_1 q_1 - \alpha p^{r - (\ell - \ell_1)}) \chi\left(\frac{\bar{n}p^{r - (\ell - \ell_1)} q_1 + r_2 q_2 - q_1(\bar{\alpha}q_2 + n)p^{r - (\ell - \ell_1)}}{p^{\ell - \ell_1}}\right).
\]

4.3. Evaluation of sum over \( \alpha \). The \( \alpha \) sum is given by
\[
C(\ldots) = \sum_{\alpha \mod p^{\ell - \ell_1}} \mathcal{X}(r_1 q_1 + (-\alpha - \bar{n}q_2) p^{r - (\ell - \ell_1)}) \chi\left(\frac{r_2 q_2 + (-q_1(\bar{\alpha}q_2 + n) + \bar{n}q_1)p^{r - (\ell - \ell_1)}}{p^{\ell - \ell_1}}\right).
\]
Note that \( \chi(1 + zp^{r - (\ell - \ell_1)}) = e\left(-\frac{A_1 p^{2(\ell - \ell_1) - 2} + A_2 p^{\ell - (\ell - \ell_1)} z}{p}\right) \) for some integers \( A_1, A_2 \) which are coprime to \( p \), as our choice of \( \ell \) satisfies the condition \( r - \ell \geq r/3 \) which is same as \( \ell \leq 2r/3 \). Thus, the character sum \( C(\ldots) \) is same as

\[
\mathcal{X}(r_1 q_1) \chi(r_2 q_2) \left. \begin{array}{c}
\frac{A_1 \bar{n}(r_1 q_1 q_2 + r_2 q_2 q_1) + (-A_2 \bar{n} q_2^2 \bar{q}_1^2 q_2^2 + A_2 \bar{n} q_2^2 \bar{q}_1^2 q_1^2) p^{r - (\ell - \ell_1)}}{p^{\ell - \ell_1}} \\
\sum_{\alpha \mod p^{\ell - \ell_1}} \sum_{\bar{\alpha}q_2 + n \neq 0 \mod p^{\ell - \ell_1}} e\left(\frac{A_1 r_1 q_1 - 2A_2 \bar{n} q_2^2 \bar{q}_1^2 q_2 p^{r - (\ell - \ell_1)}}{p^{\ell - \ell_1}} \alpha - \left(A_1 \bar{n} q_2 q_1 + 2A_2 \bar{n} q_2^2 \bar{q}_1^2 q_1 p^{r - (\ell - \ell_1)}\right) \left(\bar{\alpha}q_2 + n\right)\right) \\
\times \left. e\left(-A_2 \bar{n} q_1^2 q_2^2 p^{r - (\ell - \ell_1)} \alpha^2 + A_2 \bar{n} q_2^2 \bar{q}_1^2 q_1^2 p^{r - (\ell - \ell_1)}(\bar{\alpha}q_2 + n)^2\right) \right|_{p^{\ell - \ell_1}}.
\end{array} \right.
\]

The above \( \alpha \) sum is equals to

\[
\sum_{\alpha \mod p^{\ell - \ell_1}} \sum_{\alpha + 1 \neq 0 \mod p^{\ell - \ell_1}} e\left(\frac{A_1 \bar{n} r_1 q_1 q_2 - 2A_2 \bar{n} q_2^2 \bar{q}_1^2 q_2 p^{r - (\ell - \ell_1)}}{p^{\ell - \ell_1}} \bar{\alpha} + \left(A_1 \bar{n} q_2 q_1 + 2A_2 \bar{n} q_2^2 \bar{q}_1^2 q_1 p^{r - (\ell - \ell_1)}\right) (\alpha + 1)\right) \\
\times \left. e\left(-A_2 \bar{n} q_1^2 q_2^2 p^{r - (\ell - \ell_1)} \bar{\alpha}^2 + A_2 \bar{n} q_2^2 \bar{q}_1^2 q_1^2 p^{r - (\ell - \ell_1)}(\alpha + 1)^2\right) \right|_{p^{\ell - \ell_1}}.
\]

We assume that \( (\ell - \ell_1) \) is an even positive integer to make exposition simpler and to keep ideas clear. We now evaluate the above sum by splitting the \( \alpha \) variable. Indeed, we write
\[
\alpha = \alpha_1 + \alpha_2 \frac{p^{(\ell - \ell_1)/2}}{2}, \quad \alpha_1 (\neq 0, \neq -1) \mod p^{(\ell - \ell_1)/2}, \quad \alpha_2 \mod p^{(\ell - \ell_1)/2}.
\]
Thus the $\alpha$ sum can be written as

$$
\sum^{*}_{\alpha_1 \mod p^{(\ell-\ell_1)/2} \atop \alpha_1+1 \not\equiv 0 \mod p^{(\ell-\ell_1)/2}} \sum^{*}_{\alpha_2 \mod p^{(\ell-\ell_1)/2}} e \left( \frac{X_1 \bar{\alpha}_1 + X_2 (\alpha_1 + 1) + X_3 \bar{\alpha}_1^2 + X_4 (\alpha_1 + 1)^2}{p^{\ell-\ell_1}} \right) \\
\times e \left( \frac{X_1 \bar{\alpha}_1 + X_2 (\alpha_1 + 1) + X_3 \bar{\alpha}_1^2 + X_4 (\alpha_1 + 1)^2}{p^{\ell-\ell_1}} \right),
$$

where

$$
X_1 = A_1 \bar{\mu} r_1 q_1 - 2 A_2 \bar{\mu}^2 r_1^2 q_1^2 q_2 p^{r-(\ell-\ell_1)},
$$

$$
X_2 = - (A_1 \bar{\mu} r_2 q_2 q_1 + 2 A_2 \bar{\mu}^2 r_2^2 q_2^2 q_1 q_2 p^{r-(\ell-\ell_1)}),
$$

$$
X_3 = -A_2 \bar{r}_1^2 q_1^2 p^{r-(\ell-\ell_1)},
$$

and

$$
X_4 = A_2 \bar{r}_2^2 q_2^2 q_1^2 p^{r-(\ell-\ell_1)}.
$$

Note that $X_1 \equiv A_1 \bar{\mu} r_1 q_1 \mod p^{(\ell-\ell_1)/2}$, and $X_2 \equiv -A_1 \bar{\mu} r_2 q_2 q_1 \mod p^{(\ell-\ell_1)/2}$ as $r-(\ell-\ell_1) \geq (\ell-\ell_1)/2$.

Thus this $\alpha$ sum is given by

$$
p^{(\ell-\ell_1)/2} e \left( \frac{(A_1 \bar{\mu} r_1 q_1 - 2 A_2 \bar{\mu}^2 r_1^2 q_1^2 q_2 p^{r-(\ell-\ell_1)}) (r_2 \bar{r}_1)^{1/2} - 1}{p^{\ell-\ell_1}} \right) \\
\times e \left( \frac{(A_1 \bar{\mu} r_2 q_2 q_1 + 2 A_2 \bar{\mu}^2 r_2^2 q_2^2 q_1 q_2 p^{r-(\ell-\ell_1)}) (1 - (r_2 \bar{r}_1)^{1/2})}{p^{\ell-\ell_1}} \right) \\
\times e \left( \frac{-A_2 \bar{r}_1^2 q_1^2 p^{r-(\ell-\ell_1)} ((r_2 \bar{r}_1)^{1/2} - 1)^2 - A_2 \bar{r}_2^2 q_2^2 p^{r-(\ell-\ell_1)} (1 - (r_2 \bar{r}_1)^{1/2})^2}{p^{\ell-\ell_1}} \right),
$$

if $r_2 \bar{r}_1 \equiv \square \mod p^{(\ell-\ell_1)/2}$, other wise this $\alpha$ sum is zero. Note that $r_2 \bar{r}_1$ is square modulo $p^{(\ell-\ell_1)/2}$ if and only if $r_2 \bar{r}_1$ is square modulo $p$. Any $m$ modulo $p$ be such that $r_2 \bar{r}_1 \equiv m^2 \mod p$ can be uniquely extended to modulo $p^{(\ell-\ell_1)/2}$ with the property that $r_2 \bar{r}_1 \equiv m^2 \mod p^{(\ell-\ell_1)/2}$. 
Therefore, we have
\[
C(\ldots) = p^{(\ell - \ell_1)/2} \mathbb{I}_{r_2 r_1 \equiv \square \mod p} \chi(r_1 q_1) \chi(r_2 q_2) e \left( \frac{A_1 \tilde{n} (r_1 q_1 + r_2 q_2)}{p^{\ell - \ell_1}} \right) \times e \left( \frac{-A_2 \tilde{n}^2 q_1 r_1^2 + A_2 \tilde{n} q_2 q_1^2 p^{\ell - (\ell - \ell_1)}}{p^{\ell - \ell_1}} \right) \times e \left( \frac{(A_1 \tilde{n} r_1 q_1 - 2A_2 \tilde{n}^2 q_1^2 q_2 p^{\ell - (\ell - \ell_1)}) (r_2 \tilde{r}_1)^{1/2} - 1}{p^{\ell - \ell_1}} \right) \times e \left( \frac{-A_2 r_1 q_1^2 p^{\ell - (\ell - \ell_1)} (r_2 \tilde{r}_1)^{1/2} - 1 - A_2 r_1 q_1^2 p^{\ell - (\ell - \ell_1)} (1 - (r_2 \tilde{r}_1)^{1/2})}{p^{\ell - \ell_1}} \right).
\]

4.4. The sum over \( r_2 \). We now consider the \( r_2 \) sum which is given by
\[
\Delta(n, q_1, r_1, N, \varepsilon) = \sum_{\substack{|r_2| \leq \sqrt{N}/N \leq 2 \frac{\ell^{1/6}}{p^{\ell - \ell_1}} \chi(r_2) e \left( \frac{g(r_2)}{p^{\ell - \ell_1}} \right) \mathcal{J}_1(n, q_1, r_1, r_2, \varepsilon),
\]
where
\[
g(r_2) = A_1 \tilde{n} q_2 q_1 \tilde{r}_2 + A_2 \tilde{n}^2 q_2 q_1^2 p^{\ell - (\ell - \ell_1)} \tilde{r}_2^2 + (A_1 \tilde{n} r_1 q_1 - 2A_2 \tilde{n}^2 q_1^2 q_2 p^{\ell - (\ell - \ell_1)}) (r_2 \tilde{r}_1)^{1/2} - (A_1 \tilde{n} r_2 q_2 q_1 - 2A_2 \tilde{n}^2 r_2 q_2 q_1^2 p^{\ell - (\ell - \ell_1)}) (1 - (r_2 \tilde{r}_1)^{1/2}) - A_2 \tilde{r}_1 q_1^2 p^{\ell - (\ell - \ell_1)} (r_1 \tilde{r}_2 - 2(r_2 \tilde{r}_1)^{1/2}).
\]

By taking dyadic sub-division we see that this sum is at most
\[
\Delta(n, q_1, r_1, N, \varepsilon) \ll N^\varepsilon \sup_{R \leq R^2 \leq 2R} |T(R)|,
\]
where
\[
T(R) = \sum_{\substack{R \leq r_2 \leq 2R \leq \frac{p^{\ell - \ell_1}}{p^{\ell - \ell_1}} \chi(r_2) e \left( \frac{g(r_2)}{p^{\ell - \ell_1}} \right) \mathcal{J}_1(n, q_1, r_1, r_2, \varepsilon).}}
\]

**Remark 3.** *Note that we have*
\[
\frac{\partial}{\partial r_2} \mathcal{J}_1(n, q_1, r_1, r_2, \varepsilon) \ll \frac{N}{p^r} \frac{p^{2\ell^2} Q^2}{N^2} N^\varepsilon,
\]
*so we can ignore the integral \( \mathcal{J}_1(\ldots) \) while estimating \( T(R) \).*

In the following lemma we give estimate for \( T(R) \).

**Lemma 4.2.** *We have*
\[
T(R) \ll p^{14/15} R^{1/5} N^\varepsilon,
\]
*for any \( \varepsilon > 0 \).*
Proof. Let $\kappa$ be a large positive integer but fixed. Then we have that
\[
T(R) = \frac{1}{2} \sum_{1 \leq m \leq p} \sum_{R \leq r_2 \leq 2R} \sum_{r_1 \equiv r_1 \mod p} \chi(r_2) e\left( \frac{g(r_2)}{pf} \right)
\]
\[
= \frac{1}{2} \sum_{1 \leq m \leq p} \chi(r_1 m^2) \sum_{R-\frac{1}{p}m^2 \leq t \leq 2R-\frac{1}{p}m^2} \chi\left( 1 + \bar{r}_1 \bar{m}^2 t \right) e\left( \frac{g(r_1 m^2 + tp)}{pf} \right)
\]
where $f(t) = a_0 \log_p (1 + p^r_1 m^2 t) + p^{-(\ell-1)} g(r_1 m^2 + tp)$. Note that
\[
f'(t) = p^r a_0 r_1 \bar{m}^2 (1 + p^r_1 \bar{m}^2 t)^{-1} + p^{-(\ell-1)} h(t),
\]
where $h(t) = p \bar{g}'(r_1 m^2 + pt)$. Our phase function $f$ is in the class $F(\kappa, 1, \lambda, u)$ for arbitrarily large positive $\lambda$ and positive integer $u$ but fixed (See [2, Section 3], and page number 871 of [2]) so that we can apply $p$-adic exponent pair $(1/30, 13/15)$, when $p \neq 2, 3, 5$, to the above inner sum to get
\[
T(R) \ll_p \left( \frac{p^{\gamma - 2\kappa}}{R} \right)^{1/30} R^{13/15} N^\epsilon
\]
\[
\ll_p p^\frac{r}{30} R^{1/5} N^\epsilon,
\]
where absolute constant depends on prime $p$. Which concludes the lemma. \qed

The consequence of the above lemma we have
\[
\Delta(n, q_i, r_1, N, \epsilon) \ll p^{13r_1} N^{-5/6} N^\epsilon.
\]

In the following lemma we estimate $\Theta_{\text{non-zero}}$.

Lemma 4.3. We have
\[
\Theta_{\text{non-zero}} \ll p^{28r_1 + \ell - \frac{3\ell_1}{2}} N^{\epsilon}.
\]

Proof. We have
\[
\Theta_{\text{non-zero}} \ll \frac{p^{r+\ell - \frac{3\ell_1}{2}}}{\sqrt{N}} \sup_{q_i \leq Q} \sup_{|r_1| \leq \frac{N}{p}} \sup_{|n| \leq \frac{N}{Q}} |\Delta(n, q_i, r_1, N, \epsilon)| N^\epsilon.
\]
By substituting the bound for $\Delta(\ldots)$ from equation (9) in the above inequality we get
\[
\Theta_{\text{non-zero}} \ll \frac{p^{28r_1 + \ell - \frac{3\ell_1}{2}}}{N^{4/3}} N^\epsilon.
\]
\qed
Let $T_{\neq 0}(\varepsilon, \ell_1, N)$ and $S_{f,\chi}(N; r_1 = 0 \text{ contri}, \Theta_{\text{non-zero}})$ denote the contribution of $\Theta_{\text{non-zero}}$ to $T(\varepsilon, \ell_1, N)$ and $S_{f,\chi}(N; r_1 = 0 \text{ contri})$ respectively. Then we have that

$$T_{\neq 0}(\varepsilon, \ell_1, N) \ll \frac{p^{14r/15} p^{3r-54}}{N^{2/3}} N^{\varepsilon}$$

and consequently we have that

$$S_{f,\chi}(N; r_1 = 0 \text{ contri}, \Theta_{\text{non-zero}}) \ll \frac{p^{13r/30} N^{7/12}}{p^{r/4}} N^{\varepsilon}.$$ 

Thus we have the following proposition.

**Proposition 3.** We have

$$S_{f,\chi}(N; r_1 = 0 \text{ contri}, \Theta_{\text{non-zero}}) \ll \frac{p^{13r/30} N^{7/12}}{p^{r/4}} N^{\varepsilon}.$$ 

5. CONCLUSION

In this section we complete the proof of our main Theorem. We have

$$S_{f,\chi}(N; r_1 = 0 \text{ contri}) = S_{f,\chi}(N; r_1 = 0 \text{ contri}, \Theta_{\text{zero}}) + S_{f,\chi}(N; r_1 = 0 \text{ contri}, \Theta_{\text{non-zero}}).$$

From propositions 2 and 3 we infer that

$$S_{f,\chi}(N; r_1 = 0 \text{ contri}) \ll \left( \sqrt{N} p^{\ell/2} + \frac{p^{13r/30} N^{7/12}}{p^{r/4}} \right) N^{\varepsilon},$$

provided $\max\{p^{\ell}, p^{r-\ell}\} \leq N$, and $\ell \leq 2r/3$. By equating two terms in parenthesis we get the value of $\ell$ which is given by

$$\ell = \frac{26r}{45} + \frac{1}{9} \log_p N.$$ 

Note that this choice of $\ell$ satisfies above conditions $p^{\ell} \neq N$ and $\ell \leq 2r/3$ provided that $N \geq p^{13r/20}$ and $N \leq p^{4r/5}$ respectively. Therefore we conclude that

$$S_{f,\chi}(N) \ll N^{\frac{7}{9}} p^{\frac{13r}{45}} N^{\varepsilon},$$

provided $p^{13r/20} \leq N \leq p^{4r/5}$ and absolute value may depend on prime $p$. This concludes the proof of Theorem 1.

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