A DEFORMATION OF PENNER’S SIMPLICIAL COORDINATE

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Abstract

We find a one-parameter family of coordinates \( \{ \Psi_h \}_{h \in \mathbb{R}} \) which is a deformation of Penner’s simplicial coordinate of the decorated Teichmüller space of an ideally triangulated punctured surface \((S, T)\) of negative Euler characteristic. If \( h \geq 0 \), the decorated Teichmüller space in the \( \Psi_h \) coordinate becomes an explicit convex polytope \( P(T) \) independent of \( h \), and if \( h < 0 \), the decorated Teichmüller space becomes an explicit bounded convex polytope \( P_h(T) \) so that \( P_h(T) \subset P_{h'}(T) \) if \( h < h' \). As a consequence, Bowditch-Epstein and Penner’s cell decomposition of the decorated Teichmüller space is reproduced.

1. Introduction

Decorated Teichmüller space of a punctured surface was introduced by Penner in [P1] as a fiber bundle over the Teichmüller space of complete hyperbolic metrics with cusp ends. He also gave a cell decomposition of the decorated Teichmüller space invariant under the mapping class group action. To give the cell decomposition, Penner used the convex hull construction and introduced the simplicial coordinate \( \Psi \) in which the cells can be easily described. In [BE], Bowditch-Epstein obtained the same cell decomposition using the Delaunay construction. The corresponding results for the Teichmüller space of a surface with geodesic boundary have also been obtained. Using Penner’s convex hull construction, Ushijima [U] found a mapping class group invariant cell decomposition, and following the approach of Bowditch-Epstein [BE], Hazel [Ha] obtained a natural cell decomposition of the Teichmüller space of a surface with fixed geodesic boundary lengths. As a counterpart of Penner’s simplicial coordinate \( \Psi \), Luo [L1] introduced a coordinate \( \Psi_0 \) on the Teichmüller space of an ideally triangulated surface with geodesic boundary, and Mondello [M] pointed out that the \( \Psi_0 \) coordinate gave a natural cell decomposition of the Teichmüller space.

In [L2], Luo deformed the \( \Psi_0 \) coordinate to a one-parameter family of coordinates \( \{ \Psi_h \}_{h \in \mathbb{R}} \) of the Teichmüller space of a surface with geodesic boundary, and proved that, for \( h \geq 0 \), the image of \( \Psi_h \) is an explicit open convex polytope independent of \( h \). For \( h < 0 \), Guo [G] proved that the image of \( \Psi_h \) is an explicit bounded open polytope. It is then a natural question to ask if there is a corresponding deformation of Penner’s simplicial coordinate \( \Psi \). The purpose of this paper is to provide an affirmative answer to this question. We give a one-parameter family of coordinates \( \{ \Psi_h \}_{h \in \mathbb{R}} \) of the decorated Teichmüller space of an ideally triangulated punctured surface so that \( \Psi_0 \) coincides with Penner’s simplicial coordinate \( \Psi \) (Theorem 1.1). We also describe the image of \( \Psi_h \) (Theorem 1.2) and show that \( \Psi_h \) is the unique possible deformation of \( \Psi \) (Theorem 5.1). As an application, Bowditch-Epstein and Penner’s cell decomposition of the decorated Teichmüller space is reproduced using the \( \Psi_h \) coordinate (Corollary 1.4). The main results of this paper can be considered as a counterpart of the work of [G], [L2] and [GL2].

To be precise, let \( \mathcal{T} \) be a triangulation of a closed surface \( \mathcal{S} \) and let \( V, E \) and \( F \) be the set of vertices, edges and triangles of \( \mathcal{T} \) respectively. We call \( T = \{ \sigma - V \mid \sigma \in F \} \) an ideal
triangulation of the punctured surface $S = \overline{S} - V$, and $V$ the set of ideal vertices (or cusps) of $S$. As a convention in this paper, $S$ is assumed to have negative Euler characteristic. Let $T_c(S)$ be the Teichmüller space of complete hyperbolic metrics with cusp ends on $S$. According to Penner [P1], a decorated hyperbolic metric $(d, r) \in T_c(S) \times \mathbb{R}^V_0$ on $S$ is the equivalence class of a hyperbolic metric $d$ in $T_c(S)$ such that each cusp $v$ is associated with a horodisk $B_v$ centered at $v$ so that the length of $\partial B_v$ is $r_v$. The space of decorated hyperbolic metrics $T_c(S) \times \mathbb{R}^V_0$ is the decorated Teichmüller space.

Let us recall Penner’s simplicial coordinate $\Psi$. Let $(d, r) \in T_c(S) \times \mathbb{R}^V_0$ be a decorated hyperbolic metric and let $e$ be an edge of $T$. If $a$ and $a'$ are the generalized angles (see Section 2) facing $e$, and $b, b', c$ and $c'$ are the generalized angles adjacent to $e$, then Penner’s simplicial coordinate $\Psi: T_c(S) \times \mathbb{R}^V_0 \to \mathbb{R}^E$ is defined by

$$\Psi(d, r)(e) = \frac{b + c - a}{2} + \frac{b' + c' - a'}{2}.$$  

Figure 1: Penner’s simplicial coordinate.

An edge path $(t_0, e_1, t_1, \ldots, e_n, t_n)$ in a triangulation $T$ is an alternating sequence of edges $e_i$ with $e_i \neq e_{i+1}$ for $i = 1, \ldots, n-1$ and triangles $t_i$ so that adjacent triangles $t_{i-1}$ and $t_i$ share the same edge $e_i$ for any $i = 1, \ldots, n$. An edge loop is an edge path with $t_n = t_0$. A fundamental edge path is an edge path so that each edge in the triangulation appears at most once, and a fundamental edge loop is an edge loop so that each edge in the triangulation appears at most twice. In [P1], Penner proved that for any vector $z \in \mathbb{R}^E_{\geq 0}$ such that $\sum_{i=1}^{k} z(e_i) > 0$ for any fundamental edge loop $(e_1, t_1, \ldots, e_k, t_k)$, there exists a unique decorated complete hyperbolic metric $(d, r)$ on $S$ so that $\Psi(d, r) = z$. By using a variational principle on decorated ideal triangles, Guo and Luo [GL1] were able to prove that Penner’s simplicial coordinate $\Psi: T_c(S) \times \mathbb{R}^V_0 \to \mathbb{R}^E$ is a smooth embedding with image the convex polytope

$$P(T) = \{ z \in \mathbb{R}^E \mid \sum_{i=1}^{k} z(e_i) > 0 \text{ for any fundamental edge loop } (e_1, t_1, \ldots, e_k, t_k) \}.$$  

Let $(S, T)$ be an ideally triangulated punctured surface. To deform Penner’s simplicial coordinate, we define for each $h \in \mathbb{R}$ a map $\Psi_h: T_c(S) \times \mathbb{R}^V_0 \to \mathbb{R}^E$ by

$$\Psi_h(d, r)(e) = \int_0^{b+c-a} e^{ht^2} dt + \int_0^{b'+c'-a'} e^{ht^2} dt,$$  

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where \(a\) and \(a'\) are the generalized angles facing \(e\), and \(b, b', c\) and \(c'\) are the generalized angles adjacent to \(e\) as in Figure 1. The main theorems of this paper are the following

**Theorem 1.1** Suppose that \((S, T)\) is an ideally triangulated punctured surface. Then for all \(h \in \mathbb{R}\), the map \(\Psi_h : T_c(S) \times \mathbb{R}^{V_0}_+ \to \mathbb{R}^E\) is a smooth embedding.

**Theorem 1.2** For \(h \in \mathbb{R}\) and an ideally triangulated punctured surface \((S, T)\), let \(P_h(T)\) be the set of points \(z \in \mathbb{R}^E\) such that

(a) \(z(e) < 2 \int_0^{+\infty} e^{ht^2} dt\) for each edge \(e \in E\),

(b) \(\sum_{i=1}^n z(e_i) > -2 \int_0^{+\infty} e^{ht^2} dt\) for each fundamental edge path \((t_0, e_1, t_1, \ldots, e_n, t_n)\),

(c) \(\sum_{i=1}^n z(e_i) > 0\) for each fundamental edge loop \((e_1, t_1, \ldots, e_n, t_n)\).

Then we have \(\Psi_h(T_c(S) \times \mathbb{R}^V_0) = P_h(T)\). Furthermore, if \(h \geq 0\), then conditions (a) and (b) become trivial, and the image of \(\Psi_h\) is the open convex polytope \(P(T)\), hence independent of \(h\); and if \(h < 0\), then the image \(P_h(T)\) is a bounded open convex polytope so that \(P_h(T) \subset P_{h'}(T)\) if \(h < h'\).

Clearly \(\Psi_0\) coincides with Penner’s simplicial coordinate \(\Psi\) and \(\Psi_h\) is a deformation of \(\Psi\). Theorem 1.1 is proved in Section 2 using the strategy of Guo-Luo [GL1]. We set up a variational principle from the derivative cosine law of decorated ideal triangles whose energy function \(V_h\) is strictly concave. For \(i = 1, \ldots, |E|\), each variable of \(V_h\) is a smooth monotonic function of the edge length \(l_i\) in the decorated hyperbolic metric \((d, r)\), and \(\Psi_h\) is the gradient of \(V_h\), hence is a smooth embedding. We study various degenerations of decorated ideal triangles in Section 3 with which we will prove Theorem 1.2 in Section 4. We will also prove that \(\{\Psi_h\}_{h \in \mathbb{R}}\) is the unique possible deformation of Penner’s simplicial coordinate by using a variational principle (Theorem 5.1).

The Delaunay cell decomposition of a decorated hyperbolic surface will be reviewed in Section 6 and we will prove the following

**Theorem 1.3** Suppose \((S, T)\) is an ideally triangulated punctured surface, and \((d, r) \in T_c(S) \times \mathbb{R}^V_0\) is a decorated hyperbolic metric so that the horodisks associated to the ideal vertices do not intersect. Then for all \(h \in \mathbb{R}\), the corresponding Delaunay decomposition \(\Sigma_{d,r}\) coincides with the ideal triangulation \(T\) if and only if \(\Psi_h(d, r)(e) > 0\) for each \(e \in E\).

Bowditch-Epstein [BE] and Penner [P1] showed that there is a natural cell decomposition of the decorated Teichmüller space \(T_c(S) \times \mathbb{R}^V_0\) invariant under the mapping class group action. One interesting consequence of Theorems 1.1, 1.2 and 1.3 is the following. Let \(A(S) - A_{\infty}(S)\) be the fillable arc complex as in [H], and let \(|A(S) - A_{\infty}(S)|\) be its underlying space. Penner [P1] provided a mapping class group equivariant homeomorphism

\[
\Pi : T_c(S) \times \mathbb{R}^V_0 \to |A(S) - A_{\infty}(S)| \times \mathbb{R}^+_0
\]

so that the restriction of \(\Pi\) to each simplex of maximum dimension is given by the simplicial coordinate \(\Psi\). Using Penner’s method, we have the following

**Corollary 1.4** Suppose that \(S\) is a punctured surface of negative Euler characteristic.
(a) For all $h > 0$, there is a homeomorphism

$$\Pi_h : T_c(S) \times \mathbb{R}_{>0}^V \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

equivariant under the mapping class group action so that the restriction of $\Pi_h$ to each simplex of maximum dimension is given by the $\Psi_h$ coordinate.

(b) The cell structures for various $h > 0$ are the same as Penner’s.

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2. A variational principle on decorated ideal triangles

Let $(S, T)$ be an ideally triangulated punctured surface with a set of ideal vertices $V$ and a set of edges $E$. We assume that $S$ has negative Euler characteristic. The proof of Theorem 1.1 goes as follows. By Penner [P1], there is a smooth parametrization of the decorated Teichmüller space $T_c(S) \times \mathbb{R}_{>0}^V$ by $\mathbb{R}^E$ using the edge lengths. From the cosine law of decorated ideal triangles [P1], we construct for each $h \in \mathbb{R}$ a smooth strictly convex function $V_h$ on a convex subset of $\mathbb{R}^E$ so that its gradient is $\Psi_h$. By a variational principle, for each $h \in \mathbb{R}$, the map $\Psi : T_c(S) \times \mathbb{R}_{>0}^V \to \mathbb{R}^E$ is a smooth embedding. This variational principle, whose proof is elementary, is: If $X$ is an open convex set in $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ is smooth strictly concave, then the gradient $\nabla f : X \to \mathbb{R}^n$ is injective. Furthermore, if the Hessian of $f$ is negative definite for all $x \in X$, then $\nabla f$ is a smooth embedding.

A decorated ideal triangle $\Delta$ in the hyperbolic plane $\mathbb{H}^2$ is an ideal triangle such that each ideal vertex $v$ is associated with a horodisk $B_v$ centered at $v$. If $e_1$ and $e_2$ are two edges adjacent to an ideal vertex $v$ of $\Delta$, then the generalized angle of $\Delta$ at $v$ is defined to be the length of the intersection of $\partial B_v$ and the cusp region enclosed by $e_1$ and $e_2$. (In [P1], Penner called the generalized angles the $h$-lengths of a decorated ideal triangle, and in [GL1], Guo and Luo defined the generalized angle to be twice of the generalized angle defined here.) If $e$ is an edge of $\Delta$ with ideal vertices $u$ and $v$, then the generalized edge length (or edge length for simplicity) of $e$ in $\Delta$ is the signed hyperbolic distance between the intersection of $e$ and $\partial B_u$ and the intersection of $e$ and $\partial B_v$ (Figure 2(a)). Note that if $B_u \cap B_v \neq \emptyset$, then the generalized edge length of $e$ is either zero or negative (Figure 2(b)). In a decorated hyperbolic metric $(d, r) \in T_c(S) \times \mathbb{R}_{>0}^V$, each triangle $\sigma$ in $T$ is isometric to an ideal triangle and the decoration $r \in \mathbb{R}_{>0}^V$ induces a decoration on $\sigma$. If $e \in E$ is an edge and $\sigma$ is an ideal triangle adjacent to $e$, then the generalized edge length $l_{d,r}(e)$ of $e$ is defined to be the generalized edge length of $e$ in $\sigma$. It is clear that $l_{d,r}(e)$ does not depend on the choice of $\sigma$.

Penner [P1] defined the length parametrization

$$L : T_c(S) \times \mathbb{R}_{>0}^V \to \mathbb{R}^E$$

$$(d, r) \mapsto l_{d,r}$$
and showed that $L$ is a diffeomorphism. (The exponential of half of the generalized edge length, which is called the $\lambda$-length in [P1], is sometimes called Penner’s coordinate in the literature.) Penner also proved the following cosine law of decorated ideal triangles. Suppose that $\Delta$ is a decorated ideal triangle with edge lengths $l_1$, $l_2$ and $l_3$ and opposite generalized angles $\theta_1$, $\theta_2$ and $\theta_3$. For $i,j,k = 1, 2, 3$,

$$\theta_i = e^{\frac{l_i-l_j-l_k}{2}} \text{ and } e^{li} = \frac{1}{\theta_j \theta_k}. \tag{1}$$

As a consequence, there is the sine law of decorated triangles:

$$\frac{\theta_1}{e^{l_1}} = \frac{\theta_2}{e^{l_2}} = \frac{\theta_3}{e^{l_3}}. \tag{2}$$

For $i,j,k = 1, 2, 3$ and $x_i = \frac{\theta_i + \theta_k - \theta_j}{2}$, let $\mu(x_i) = \int_0^{x_i} e^{ht^2} dt$ and $u_i = \int_0^1 e^{-he^{-t}} dt$. Denote by $U \subset \mathbb{R}^3$ the set of all possible values of $u = (u_1, u_2, u_3)$.

**Lemma 2.2** For each $h \in \mathbb{R}$, the differential 1-form $\omega_h = \sum_{i=1}^3 \mu(x_i) du_i$ is closed in $U$ and the function $F_h$ defined by the integral $F_h(u) = \int_0^u \omega_h$ is strictly concave in $U$. Furthermore,

$$\frac{\partial F_h}{\partial u_i} = \int_0^{x_i} e^{ht^2} dt.$$

**Proof:** Consider the matrix $H = [\frac{\partial \mu(x_i)}{\partial x_j}]_{3 \times 3}$. The closedness of $\omega_h$ is equivalent to that $H$ is symmetric, and the strict concavity of $F_h$ will follow from the negative definiteness of $H$. It follows from the partial derivatives of (1) that $\frac{\partial x_i}{\partial u_i} = -\frac{x_i + x_j + x_k}{2}$ and $\frac{\partial x_i}{\partial u_j} = \frac{x_k}{2}$. We have

$$\frac{\partial \mu(x_i)}{\partial u_i} = \frac{e^{hx_i^2}}{e^{-he^{-t}}}, \quad \frac{\partial x_i}{\partial u_i} = -\frac{x_i + x_j + x_k}{2} e^{h(\frac{\theta^2 + \theta_j^2 + \theta_k^2}{4} + \frac{3\theta_k \theta_j - \theta_j \theta_k - \theta_i \theta_j}{2})},$$

and for $i \neq j$, we have

$$\frac{\partial \mu(x_i)}{\partial u_j} = \frac{e^{hx_i^2}}{e^{-he^{-t}}}, \quad \frac{\partial x_i}{\partial u_j} = \frac{x_k}{2} e^{h(\frac{\theta^2 + \theta_j^2 + \theta_k^2}{4} + \frac{3\theta_k \theta_j - \theta_j \theta_k - \theta_i \theta_j}{2})},$$

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from which we see that $H$ is symmetric. Let $c = \frac{1}{2} e^{h(\theta_1^2 + \theta_2^2 + \theta_3^2)} > 0$ and let $D$ be the diagonal matrix whose $(i, i)$-th entry is $e^{h\theta_i^2}$. The matrix $H$ can be written as $cDMD$, where

$$M = \begin{bmatrix}
-x_1 + x_2 + x_3 & x_3 & x_2 \\
x_3 & -x_1 + x_2 + x_3 & x_1 \\
x_2 & x_1 & -(x_1 + x_2 + x_3)
\end{bmatrix}.$$ 

The negative definiteness of $H$ is equivalent to that of $M$, i.e., the positive definiteness of $-M$. This follows from the direct calculation that each leading principal minor is positive using Sylvester’s criterion.

\textbf{Proof of Theorem 1.1:} For a decorated hyperbolic metric $(d, r) \in T_c(S) \times \mathbb{R}^V$, let $l_{d, r} \in \mathbb{R}^E$ be its length parameter. The integral $u(e) = \int_{0}^{l_{d, r}(e)} e^{-he^{-t}} dt$ is a smooth monotonic function of $l_{d, r}(e)$, and the possible values of $u$ form an open convex cube $U$ in $\mathbb{R}^E$. With $u_i = u(e_i)$, the energy function $V_h: U \rightarrow \mathbb{R}$ is defined by

$$V_h(u) = \sum_{\{e_i, e_j, e_k\}} F_h(u_i, u_j, u_k),$$

in which the summation is taken over all of the decorated ideal triangles. By Lemma 2.2, $V_h$ is smooth and strictly concave in $U$ and

$$\frac{\partial V_h}{\partial u_i} = \Psi_h(e_i),$$

i.e., $\nabla V_h = \Psi_h$. By the variational principle, the map $\Psi_h = \nabla V_h: U \rightarrow \mathbb{R}^E$ is a smooth embedding.

\textbf{q.e.d}

3. Degenerations of decorated ideal triangles

To describe the image of $\Psi_h$, we study degenerations of decorated ideal triangles. Suppose $\Delta$ is a decorated ideal triangle with edge lengths $l_1, l_2$ and $l_3$ and opposite generalized angles $\theta_1, \theta_2$ and $\theta_3$.

\textbf{Lemma 3.1}

(I) If $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, c_2, c_3)$ with $c_2, c_3 \in (-\infty, +\infty)$, then $\{\theta_1\}$ converges to 0, and we can take a subsequence so that at least one of $\{\theta_2\}$ and $\{\theta_3\}$ converges to $+\infty$.

(II) If $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, -\infty, c_3)$ with $c_3 \in (-\infty, +\infty)$, then $\{\theta_3\}$ converges to $+\infty$, and we can take a subsequence so that at least one of $\{\theta_1\}$ and $\{\theta_2\}$ converges to a finite number.

(III) If $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, -\infty, -\infty)$, then we can take a subsequence such that at least two of $\{\theta_1\}$, $\{\theta_2\}$ and $\{\theta_3\}$ converge to $+\infty$.

\textbf{Proof:} For (I), if $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, c_2, c_3)$, then $\{\frac{l_1 - l_2 - l_3}{2}\}$ converges to $-\infty$. By cosine law, $\{\theta_1\} = \{\frac{l_1 - l_2 - l_3}{2}\}$ converges to 0. Let $a_2 = \frac{l_2 - l_1 - l_3}{2}$ and $a_3 = \frac{l_3 - l_1 - l_2}{2}$, so $\{a_2 + a_3\} = \{-l_1\}$ converges to $+\infty$. Thus, by taking a subsequence if necessary, at
least one of \( \{a_2\} \) and \( \{a_3\} \), say \( \{a_2\} \), converges to \(+\infty\), and \( \{\theta_2\} = \{e^{a_2}\} \) converges to \(+\infty\). For (II), if \( \{(l_1, l_2, l_3)\} \) converges to \((-\infty, -\infty, c_3)\), then \( \{\frac{x_1 - l_1}{2}\} \) converges to \(+\infty\), and \( \{\theta_3\} = \{e^{\frac{x_1 - l_1}{2}}\} \) converges to \(+\infty\). Letting \( a_1 = \frac{4-x_1-l_3}{2} \) and \( a_2 = \frac{2-l_1-\theta_1}{2} \), we have \( \{a_1 + a_2\} = \{-l_3\} \) converges to \(-c_3\). Thus, either both \( \{a_1\} \) and \( \{a_2\} \) converge to a finite number, or by taking a subsequence if necessary, at least one of \( \{a_1\} \) and \( \{a_2\} \), say \( \{a_1\} \), converges to \(-\infty\). In the former case, both \( \{\theta_1\} = \{e^{a_1}\} \) and \( \{\theta_2\} = \{e^{a_2}\} \) converge to a finite number, and in the latter case, \( \{\theta_1\} = \{e^{a_1}\} \) converges to \(0\). For (III), we have by cosine law that \( \{\theta_1\theta_2\} = \{e^{-l_3}\} \) converges to \(+\infty\). Thus, by taking a subsequence if necessary, at least one of \( \{\theta_1\} \) and \( \{\theta_2\} \), say \( \{\theta_1\} \), converges to \(+\infty\). Since \( \{\theta_2\theta_3\} = \{e^{-l_1}\} \) converges to \(+\infty\) as well, by taking a subsequence, at least one of \( \{\theta_2\} \) and \( \{\theta_3\} \) converges to \(+\infty\). q.e.d

We call a converging sequence of decorated ideal triangles in (I), (II) and (III) of Lemma 3.1 a degenerated decorated ideal triangle of type I, II and III respectively. If \( a \) is the generalized angle facing an edge \( e \) in a decorated triangle \( \Delta \), and \( b \) and \( c \) are the generalized angles adjacent to \( e \), then we call \( x(e) = \frac{b+c-a}{2} \) the \( x \)-invariant of \( e \) in \( \Delta \).

**Corollary 3.2** If \( \Delta \) is a degenerated decorated ideal triangle of type I, II or III, then by taking a subsequence if necessary, there is an edge \( e \) of \( \Delta \) such that \( l(e) \) converges to \(-\infty\) and \( x(e) \) converges to \(+\infty\).

**Proof:** If \( \Delta \) is of type I and \( \{l_1\} \) converges to \(-\infty\), then by Lemma 3.1 (I), \( \{x_1\} = \{\theta_2+\theta_3-\theta_1\} \) converges to \(+\infty\). If \( \Delta \) is of type II and \( \{(l_1, l_2, l_3)\} \) converges to \((-\infty, -\infty, c_3)\), then by Lemma 3.1 and taking a subsequence if necessary, at least one of \( \{\theta_1\} \) and \( \{\theta_2\} \), say \( \{\theta_1\} \), converges to a finite number, and \( \{\theta_3\} \) converges to \(+\infty\). Thus, \( \{l_1\} \) converges to \(-\infty\) and \( \{x_1\} = \{\theta_2+\theta_3-\theta_1\} \) converges to \(+\infty\). If \( \Delta \) is of type III, then there are at least two of \( \{\theta_1\} \), \( \{\theta_2\} \) and \( \{\theta_3\} \) that converge to \(+\infty\). Suppose \( \{\theta_3\} \) is one of the two that converge to \(+\infty\). Since \( \{x_1 + x_2\} = \{\theta_3\} \) converges to \(+\infty\), by taking a subsequence if necessary, at least one of \( \{x_1\} \) and \( \{x_2\} \), say \( \{x_1\} \), converges to \(+\infty\). Thus, \( \{l_1\} \) converges to \(-\infty\) and \( \{x_1\} \) converges to \(+\infty\).

q.e.d

We call an edge \( e \) as in Corollary 3.2 where \( l(e) \to -\infty \) and \( x(e) \to +\infty \) a bad edge of \( \Delta \), and otherwise, \( e \) is a good edge. Note that there may be more than one bad edge in a degenerated ideal triangle.

**Lemma 3.3** Let \( \{\Delta^{(m)}\} \) be a sequence of decorated ideal triangles that converges to a degenerated decorated ideal triangle \( \Delta \) of type I, II or III. Then we can take a subsequence so that for \( m \) sufficiently large, the length of each bad edge of \( \Delta^{(m)} \) is strictly less than the length of each good edge.

**Proof:** If \( \Delta \) is of type I, then by Lemma 3.1, the length of the only bad edge converges to \(-\infty\) and the length of other two edges converge to a finite number. For \( m \) sufficiently large, the length of the bad edge is less than the lengths of the good edges.

If \( \Delta \) is of type II, we may assume that \( \{(l_1^{(m)}, l_2^{(m)}, l_3^{(m)})\} \) converges to \((-\infty, -\infty, c)\) with \( c \in (-\infty, +\infty) \). By Lemma 3.1, there are two cases to be considered (Figure 3).

**Case 1.** Suppose that \( \theta_3^{(m)} \) converges to \(+\infty\) and both \( \theta_1^{(m)} \) and \( \theta_2^{(m)} \) converge to a finite number. In this case, both \( l_1 \) and \( l_2 \) are bad and converge to \(-\infty\). The only good edge length
Figure 3: Type II.

$l_3$ converges to $c \in (-\infty, +\infty]$. Hence for $m$ sufficiently large, $l_1^{(m)} < l_3^{(m)}$ and $l_2^{(m)} < l_3^{(m)}$.

**Case 2.** Suppose that $\theta_3^{(m)}$ converges to $+\infty$, and one of $\theta_1^{(m)}$ and $\theta_2^{(m)}$, say $\theta_2^{(m)}$, converges to $+\infty$ and $\theta_1^{(m)}$ converges to a finite number. In this case $l_1$ is bad. If $l_2$ is also bad, then both $l_1$ and $l_2$ converge to $-\infty$, and $l_3$ converges to $c \in (-\infty, +\infty]$. Hence for $m$ sufficiently large, $l_1^{(m)} < l_3^{(m)}$ and $l_2^{(m)} < l_3^{(m)}$. If $l_2$ is good, then $\theta_1^{(m)} < \theta_2^{(m)}$ for $m$ sufficiently large, since $\theta_1^{(m)}$ converges to a finite number and $\theta_2^{(m)}$ converges to $+\infty$. By sine law (2), $l_1^{(m)} < l_2^{(m)}$.

Figure 4: Type III.

If $\Delta$ is of type III, then by Lemma 3.1, we also consider two cases (Figure 4).

**Case 1.** Two of $\theta_1^{(m)}$, $\theta_2^{(m)}$ and $\theta_3^{(m)}$, say $\theta_1^{(m)}$ and $\theta_2^{(m)}$ converge to $+\infty$, and $\theta_3^{(m)}$ converges to a finite number. In this case, $l_3$ is bad. Since $\theta_3^{(m)} < \theta_1^{(m)}$ and $\theta_3^{(m)} < \theta_2^{(m)}$ for $m$ sufficiently large, by sine law (2), $l_3^{(m)} < l_1^{(m)}$ and $l_3^{(m)} < l_2^{(m)}$. If one of $l_1$ and $l_2$, say $l_2$, is also bad, then $x_2^{(m)} = \frac{\theta_2^{(m)} - \theta_3^{(m)} - \theta_1^{(m)}}{2}$ converges to $+\infty$. Since $\theta_3^{(m)}$ converges to a finite number, $\theta_2^{(m)} < \theta_1^{(m)}$ for $m$ sufficiently large. By sine law (2), $l_2^{(m)} < l_1^{(m)}$. 8
Case 2. All of $\theta_1^{(m)}$, $\theta_2^{(m)}$ and $\theta_3^{(m)}$ converge to $+\infty$. In this case, since $x_i^{(m)} + x_j^{(m)} = \theta_k^{(m)}$ converges to $+\infty$, by taking a subsequence if necessary, at least two of $x_1^{(m)}$, $x_2^{(m)}$ and $x_3^{(m)}$, say $x_1^{(m)}$ and $x_2^{(m)}$, converge to $+\infty$. Therefore, $l_3$ is the only possible good edge length, and $x_3^{(m)}$ converges to a finite number. For $m$ sufficiently large, $\theta_1^{(m)} = x_2^{(m)} + x_3^{(m)} < x_1^{(m)} + x_2^{(m)} = \theta_3^{(m)}$ and $\theta_2^{(m)} = x_1^{(m)} + x_3^{(m)} < x_1^{(m)} + x_2^{(m)} = \theta_3^{(m)}$. By sine law (2), $l_1^{(m)} < l_3^{(m)}$ and $l_2^{(m)} < l_3^{(m)}$.

d.e.d

Lemma 3.4

(a) If $\{(l_1, l_2, l_3)\}$ converges to $(+\infty, f_2, f_3)$ with $f_2, f_3 \in \mathbb{R}$, then $\{\theta_1, \theta_2, \theta_3\}$ converges to $(+\infty, 0, 0)$.

(b) If $\{(l_1, l_2, l_3)\}$ converges to $(+\infty, +\infty, f_3)$ with $f_3 \in \mathbb{R}$, then $\{\theta_3\}$ converges to 0.

(c) If $\{(l_1, l_2, l_3)\}$ converges to $(+\infty, +\infty, +\infty)$, then we can take a subsequence such that at least two of $\{\theta_1\}$, $\{\theta_2\}$ and $\{\theta_3\}$ converge to 0.

Figure 5: Type IV and other types.

We call a converging sequence of decorated ideal triangles in (a) of Lemma 3.4 a degenerated decorated ideal triangle of type IV (Figure 5).

Proof: For (a), if $\{(l_1, l_2, l_3)\}$ converges to $(+\infty, f_2, f_3)$, then by cosine law (1), $\{\theta_1\} = \{e^{\frac{l_1-l_2-l_3}{2}}\}$ converges to $+\infty$, $\{\theta_2\} = \{e^{\frac{l_2-l_1-l_3}{2}}\}$ converges to 0, and $\{\theta_3\} = \{e^{\frac{l_3-l_1-l_2}{2}}\}$
converges to 0. For (b), if \( \{l_1, l_2, l_3\} \) converges to \((+\infty, +\infty, f_3)\), then \( \{l_{i-1} - l_{i-2}\} \) converges to \(-\infty\), and \( \{\theta_3\} = \{e^{-l_{i-1} - l_{i-2}}\} \) converges to 0. For (c), if \( \{l_1, l_2, l_3\} \) converges to \((+\infty, +\infty, +\infty)\), then we have by cosine law that \( \{\theta_2\} = \{e^{-l_3}\} \) converges to 0. Thus, by taking a subsequence if necessary, at least one of \( \{\theta_1\} \) and \( \{\theta_2\} \), say \( \{\theta_1\} \), converges to 0. Since \( \{\theta_2\} = \{e^{-l_3}\} \) converges to 0 as well, by taking a subsequence, at least one of \( \{\theta_2\} \) and \( \{\theta_3\} \) converges to 0.

4. The image of \( \Psi_h \)

The image of \( \Psi_h \) is described in Theorem 1.2. The main task of this section is to give a proof of this theorem. To show that the image of \( \Psi_h \) is indeed \( P_h(T) \), we make use of the following propositions which are proved in this section.

**Proposition 4.1** \( \Psi_h(T_c(S) \times \mathbb{R}^V_\geq 0) \subset P_h(T) \) for all \( h \in \mathbb{R} \).

**Proposition 4.2** For all \( h \in \mathbb{R} \), the image \( \Psi_h(T_c(S) \times \mathbb{R}^V_\geq 0) \) is closed in \( P_h(T) \).

**Proof of Theorem 1.2:** Let \( P(T) \) be defined as in Theorem 1.2. For \( h \geq 0 \), \( P(T) = P_h(T) \) is determined by finitely many strict linear inequalities corresponding to the fundamental edge loops and hence is an open convex polytope independent of \( h \). For \( h < 0 \), \( P_h(T) \) is likewise determined by fundamental edge loops and fundamental edge paths. Moreover, since each edge \( e \) can be regarded as a fundamental edge path, conditions (a) and (b) imply that \(-2 \int_0^{+\infty} e^{ht^2} dt < z(e) < 2 \int_0^{+\infty} e^{ht^2} dt \) for each \( e \in E \). Thus, \( P_h(T) \) is bounded. The monotonicity of the function \( f(h) = \int_0^{+\infty} e^{ht^2} dt \) implies that \( P_h(T) \subset P_h(T) \) if \( h < h' \), and the fact that \( \lim_{h \to -\infty} f(h) = \lim_{h \to -\infty} \sqrt{-\frac{\pi}{2h}} = 0 \) implies that \( \bigcap_{h \in \mathbb{R} < 0} P_h(T) = \emptyset \). By Theorem 1.1 and the Invariance of Domain Theorem, \( \Psi_h(T_c(S) \times \mathbb{R}^V_\geq 0) \) is open in \( P_h(T) \). By Proposition 4.2, \( \Psi_h(T_c(S) \times \mathbb{R}^V_\geq 0) \) is closed in \( P_h(T) \). Connectedness of \( P_h(T) \) therefore implies that \( \Psi_h(T_c(S) \times \mathbb{R}^V_\geq 0) = P_h(T) \).

The following Lemma 4.3 will be used in the proof of Propositions 4.1 and 4.2.

**Lemma 4.3** If \( r \in \mathbb{R} \) and \( x > 0 \), then

(a) for each \( h \in \mathbb{R} \),

\[
\int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt > 0,
\]

(b) for each \( h \geq 0 \),

\[
\int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt \geq 2 \int_0^{x} e^{ht^2} dt.
\]

**Proof:** For (a), let \( f(x) = \int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt \). Since \( f'(x) = e^{h(x+r)^2} + e^{h(x-r)^2} > 0 \), the function \( f \) is strictly increasing, hence \( f(x) > f(0) = 0 \) for \( x > 0 \). For (b), let \( g(x) = \int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt - 2 \int_0^{x} e^{ht^2} dt \). We have that \( g(0) = 0 \) and \( g'(x) = e^{h(x+r)^2} + e^{h(x-r)^2} - 2e^{hx^2} \geq 0 \). The last inequality follows from the convexity of the function \( F(t) = e^{ht^2} \) for \( h \geq 0 \). Since \( g \) is increasing, \( g(x) \geq g(0) = 0 \) for \( x > 0 \).

**q.e.d**

**Proof of Proposition 4.1:** For \( h \geq 0 \), fix a decorated hyperbolic metric \((d, r) \in T_c(S) \times \mathbb{R}^V_\geq 0\). For any fundamental edge loop \((e_1, t_1, \ldots, e_k, t_k)\), let \( a_i \) be the generalized angle adjacent to...
Let denote by $a$ and $c_{i+1}$ (where $e_{k+1} = e_1$). Let the generalized angles of $t_i$ facing $e_i$ and $e_{i+1}$ respectively be $b_i$ and $c_i$. By definition, the contribution of $\sum_{i=1}^{k} z(e_i)$ from $t_i$ is
\[
\int_{0}^{a_i + b_i - c_i} e^{ht^2} dt + \int_{0}^{a_i + c_i - b_i} e^{ht^2} dt,
\]
which is strictly larger than 0 from Lemma 4.3 (a) since $a_i > 0$.

For $h < 0$, let $e$ be any edge in the ideal triangulation $T$, and let $a$ and $a'$ be the generalized angles facing $e$. Let $b$, $c$, $b'$ and $c'$ be the generalized angles adjacent to $e$. Then
\[
\Psi_h(d, r)(e) = \int_{0}^{b + c - a} e^{ht^2} dt + \int_{0}^{b' + c' - a'} e^{ht^2} dt < 2 \int_{0}^{+\infty} e^{ht^2} dt.
\]
Thus, condition (a) in the definition of $P_h(T)$ is satisfied. Given a fundamental edge path $(t_0, e_0, t_1, \ldots, e_n, t_n)$, let $\theta_i$ be the generalized angle in $t_i$ adjacent to $e_i$ and $e_{i+1}$ for $i = 1, \ldots, n - 1$, and let $\beta_i$ and $\gamma_i$ respectively be the generalized angles of $t_i$ facing $e_i$ and $e_{i+1}$. Denote by $a_0$ the generalized angle of $t_0$ facing $e_0$, and by $a_n$ the generalized angle of $t_n$ facing $e_n$. Let $b_0$ and $c_0$ be the generalized angles of $t_0$ adjacent to $e_0$, and let $b_n$ and $c_n$ be the generalized angles of $t_n$ adjacent to $e_n$. We have
\[
\sum_{i=1}^{n} \Psi_h(d, r)(e_i) = \int_{0}^{b_0 + c_0 - a_0} e^{ht^2} dt + \sum_{i=1}^{n-1} \left( \int_{0}^{\theta_i + \gamma_i - \beta_i} e^{ht^2} dt + \int_{0}^{\theta_i + \beta_i - \gamma_i} e^{ht^2} dt \right) + \int_{0}^{b_n + c_n - a_n} e^{ht^2} dt
\]
\[> \int_{0}^{b_0 + c_0 - a_0} e^{ht^2} dt + \int_{0}^{b_n + c_n - a_n} e^{ht^2} dt
\]
\[> - 2 \int_{0}^{+\infty} e^{ht^2} dt,
\]
where the first inequality is by Lemma 4.3 (a). Thus, condition (b) is satisfied. Given a fundamental edge loop $(e_1, t_1, \ldots, e_n, t_n)$ with $e_{n+1} = e_1$, let $\theta_i$ for $i = 1, \ldots, n$ be the generalized angle in $t_i$ adjacent to $e_i$ and $e_{i+1}$, and let $\beta_i$ (resp. $\gamma_i$) be the generalized angle in $t_i$ facing $e_i$ (resp. $e_{i+1}$). Again by Lemma 4.3 (a),
\[
\sum_{i=1}^{n} \Psi_h(d, r)(e_i) = \sum_{i=1}^{n} \left( \int_{0}^{\theta_i + \gamma_i - \beta_i} e^{ht^2} dt + \int_{0}^{\theta_i + \beta_i - \gamma_i} e^{ht^2} dt \right) > 0.
\]
Thus, condition (c) is satisfied, and $\Psi_h(Tc(S) \times \mathbb{R}^V_{>0}) \subset P_h(T)$. q.e.d

To prove Proposition 4.2, we use Penner’s length parametrization. For each sequence $\{l^{(m)}\}$ in $\mathbb{R}^E$ such that $\{\Psi_h(l^{(m)})\}$ converges to a point $z \in P(T)$, we claim that $\{l^{(m)}\}$ contains a subsequence converging to a point in $\mathbb{R}^E$. Let $\theta^{(m)}$ be the generalized angles of the decorated ideal triangles in $(S, T)$ in the decorated hyperbolic metric $l^{(m)}$. By taking a subsequence if necessary, we may assume that $\{l^{(m)}\}$ converges in $[-\infty, +\infty]^E$ and that for
the decorated ideal triangle $\Delta$ opposite to $\theta$. Must be zero. Suppose without loss of generality that $\lim_{m \to \infty} \theta_i^{(m)} = 0$. Let $e_2$ and $e_3$ be the edges adjacent to $\theta_1$ in the triangle $t_1$, and $\theta_2$ and $\theta_3$ respectively be the generalized angles facing $e_2$ and $e_3$. Take a fundamental edge loop $(e_{n_1}, t_{n_1}, \ldots, e_{n_k}, t_{n_k})$ containing $(e_2, t_1, e_3)$. By Lemma 4.3, we have

$$
\sum_{i=1}^k z(e_{n_i}) = \lim_{m \to \infty} \sum_{i=1}^k \Psi_h(l^{(m)})(e_{n_i})
\geq \lim_{m \to \infty} \left( \int_0^{\frac{\theta_1^{(m)} + \theta_2^{(m)} - \theta_3^{(m)}}{2}} e^{ht^2} dt + \int_0^{\frac{\theta_2^{(m)} + \theta_3^{(m)} - \theta_1^{(m)}}{2}} e^{ht^2} dt \right)
\geq \lim_{m \to \infty} 2 \int_0^{\frac{\theta_1^{(m)}}{2}} e^{ht^2} dt
= +\infty.
$$

This contradicts the assumption that $z \in P(T)$. q.e.d

Proof of Proposition 4.2: For $h \geq 0$, by taking a subsequence of $\{l^{(m)}\}$, we may assume that $\lim_{m \to \infty} l^{(m)} = l \in [-\infty, +\infty]^E$. If $l$ were not in $\mathbb{R}^E$, then there would exist an edge $e \in E$ so that $l(e) = \pm \infty$. Let $\Delta$ be a decorated ideal triangle adjacent to $e$, and let $\theta_1^{(m)}$ and $\theta_2^{(m)}$ be the generalized angles in $\Delta$ adjacent to $e$ in the metric $l^{(m)}$. By [1],

$$e^{(m)}(e) = \frac{1}{\theta_1^{(m)} \theta_2^{(m)}},$$

and $\theta_i^{(m)} \in (0, +\infty)$ for $i = 1, 2$.

Case 1 If $l(e) = -\infty$, then $e^{l(e)} = 0$. By the identity above, one of $\lim_{m \to \infty} \theta_i^{(m)}$ for $i = 1, 2$ must be $+\infty$. This contradicts Lemma 4.4.

Case 2 If $l(e) = +\infty$, then $e^{l(e)} = +\infty$. By the identity above, one of $\lim_{m \to \infty} \theta_i^{(m)}$ for $i = 1, 2$ must be zero. Suppose without loss of generality that $\lim_{m \to \infty} \theta_1^{(m)} = 0$. Let $e_1$ be the edge in the decorated ideal triangle $\Delta$ opposite to $\theta_2$, and let $\theta_3$ be the generalized angle in $\Delta$ facing $e$. By [1], we have

$$e^{l(e_1)} = \frac{1}{\theta_1^{(m)} \theta_3^{(m)}}.$$

By Lemma 4.4, $\theta_3^{(m)}$ is bounded above, hence $l(e_1) = +\infty$. For any decorated ideal triangle $\Delta$ adjacent to $e$ with $l(e) = +\infty$, we have an edge $e_1$ in $\Delta$ and a generalized angle $\theta_1$ adjacent to $e$ and $e_1$ so that $l(e_1) = +\infty$ and $\lim_{m \to \infty} \theta_1^{(m)} = 0$. Applying this logic to $e_1$ and the decorated ideal triangle $\Delta_1$ adjacent to $e_1$ other than $\Delta$, we obtain the next angle $\theta_2$ and edge $e_2$ in $\Delta_1$ so that $l(e_2) = +\infty$ and $\lim_{m \to \infty} \theta_2^{(m)} = 0$. Since there are only finitely many
edges and triangles, this yields a fundamental edge loop \((e_k, \Delta_k, \ldots, e_n, \Delta_n)\) in \(T\) such that 
\(l(e_i) = +\infty\) for \(i = k, \ldots, n\) and 
\(\lim_{m \to \infty} \theta_i^{(m)} = 0\), where \(\theta_i\) is the generalized angle in \(\Delta_{i-1}\) adjacent to \(e_{i-1}\) and \(e_i\). Denote respectively by \(\beta_i\) and \(\gamma_i\) the generalized angles of \(\Delta_{i-1}\) facing \(e_{i-1}\) and \(e_i\), and let 
\(\bar{\beta}_i = \lim_{m \to \infty} \beta_i^{(m)}\) and 
\(\bar{\gamma}_i = \lim_{m \to \infty} \gamma_i^{(m)}\). By Lemma 4.4, both \(\bar{\beta}_i\) and 
\(\bar{\gamma}_i\) are finite real numbers, and we have

\[
\sum_{i=k}^{n} z(e_i) = \lim_{m \to \infty} \sum_{i=k}^{n} \Psi_h(l^{(m)}(e_i)) = \lim_{m \to \infty} \sum_{i=k}^{n} \left( \int_{0}^{\theta_i^{(m)} + \frac{\beta_i^{(m)} - \gamma_i^{(m)}}{2}} e^{ht^2} dt + \int_{0}^{\theta_i^{(m)} - \frac{\beta_i^{(m)} - \gamma_i^{(m)}}{2}} e^{ht^2} dt \right) = \sum_{i=k}^{n} \left( \int_{0}^{\bar{\beta}_i - \bar{\gamma}_i} e^{ht^2} dt + \int_{0}^{\bar{\gamma}_i - \bar{\beta}_i} e^{ht^2} dt \right) = 0.
\]

This contradicts the assumption that \(z \in P(T)\).

For \(h < 0\) and each sequence \(\{l^{(m)}\}\) in \(\mathbb{R}^E\) so that \(\{\Psi_h(l^{(m)})\}\) converges to a point 
\(z \in P_h(T)\), we claim that \(\{l^{(m)}\}\) contains a subsequence converging to a point in \(\mathbb{R}^E\). By 
taking a subsequence if necessary, we may assume that \(\{l^{(m)}\}\) converges to \(l \in [-\infty, +\infty]^E\).
If \(l\) were not in \(\mathbb{R}^E\), there would exist an edge \(e\) so that \(l(e) = ±\infty\).

**Case 1.** If \(l(e) = -\infty\) for some \(e \in E\), then there is a degenerated decorated ideal triangle \(\Delta\) of type I, II or III. By Corollary 3.2, there is a bad edge \(e_1\) in \(\Delta\). Let \(\Delta_1\) be the other decorated ideal triangle adjacent to \(e_1\), and let \(x_0\) and \(x_1\) respectively be the \(x\)-invariants of 
\(e_1\) in \(\Delta\) and \(\Delta_1\). If \(e_1\) is bad in \(\Delta_1\), then

\[
z(e_1) = \lim_{m \to \infty} \Psi_h(l^{(m)}(e_1)) = \lim_{m \to \infty} \left( \int_{0}^{x_0^{(m)}} e^{ht^2} dt + \int_{0}^{x_1^{(m)}} e^{ht^2} dt \right) = 2 \int_{0}^{+\infty} e^{ht^2} dt,
\]

which contradicts the assumption that \(z \in P_h(T)\). Therefore \(e_1\) has to be a good edge in \(\Delta_1\). Since 
\(l(e_1) = -\infty\), the decorated triangle \(\Delta_1\) is degenerated of type I, II or III. By Corollary 3.2, there is a bad edge \(e_2\) in \(\Delta_1\). For the same reason, \(e_2\) has to be good in the other decorated ideal triangle \(\Delta_2\) adjacent to \(e_2\), and there is a bad edge \(e_3\) in \(\Delta_2\). Serially applying this logic and using that there are finitely many edges, we find an edge loop \((e_k, \Delta_k, \ldots, e_n, \Delta_n)\) with 
\(e_{n+1} = e_k\) so that for each \(i = k, \ldots, n\) the edge \(e_i\) is good in \(\Delta_i\) and the edge \(e_{i+1}\) is bad in 
\(\Delta_{i+1}\). By Lemma 3.3, we can take a subsequence so that \(l^{(m)}(e_i) > l^{(m)}(e_{i+1})\) for \(m\) sufficiently large. Thus, we have \(l^{(m)}(e_k) > l^{(m)}(e_{n+1})\), which contradicts that \(e_{n+1} = e_k\).

In light of Case 1, we may assume that \(l \in (-\infty, +\infty]^E\).

**Case 2.** If \(l(e) = +\infty\) for some \(e \in E\), let \(\Delta_1\) be a decorated ideal triangle adjacent to \(e\). If 
\(\Delta_1\) is not of type IV, then by Lemma 3.4, there is an edge \(e_1\) of \(\Delta_1\) and an generalized 
angle \(\theta_1\) adjacent to \(e\) and \(e_1\) so that \(l(e_1) = +\infty\) and \(\lim_{m \to \infty} \theta_1^{(m)} = 0\) (see Figure 5). The 
other decorated ideal triangle \(\Delta_2\) adjacent to \(e_1\) is either of type IV or contains an edge \(e_2\)
and a generalized angle $\theta_2$ adjacent to $e_1$ and $e_2$ so that $l(e_2) = +\infty$ and $\lim_{m \to \infty} \theta_2^{(m)} = 0$. Again, the serial application of this procedure terminates with an edge $e_p$ and a decorated ideal triangle $\Delta_{p+1}$ adjacent to $e_p$ so that $l(e_p) = +\infty$ and $\Delta_{p+1}$ is of type IV, or since there are only finitely many edges, produces a fundamental edge loop $(e_k, \Delta_k, \ldots, e_n, \Delta_n)$ such that $l(e_i) = +\infty$ for $i = k, \ldots, n$ and $\lim_{m \to \infty} \theta_i^{(m)} = 0$, where $\theta_i$ is the generalized angle in $\Delta_i$ adjacent to $e_i$ and $e_{i+1}$. If it yields such a fundamental edge loop $(e_k, \Delta_k, \ldots, e_n, \Delta_n)$, denote by $\beta_i$ (resp. $\gamma_i$) the generalized angle in $\Delta_i$ facing $e_i$ (resp. $e_{i+1}$) for $i = k, \ldots, n$. Let $\beta_i = \lim_{m \to \infty} \beta_i^{(m)}$ and $\gamma_i = \lim_{m \to \infty} \gamma_i^{(m)}$, so that

\[
\sum_{i=k}^{n} z(e_i) = \lim_{m \to \infty} \sum_{i=1}^{k} \Psi_h(l^{(m)})(e_i)
= \lim_{m \to \infty} \sum_{i=1}^{k} \left( \int_{0}^{\theta_i^{(m)} + \beta_i^{(m)} - \gamma_i^{(m)}} e^{ht^2} dt + \int_{0}^{\theta_i^{(m)} + \gamma_i^{(m)} - \beta_i^{(m)}} e^{ht^2} dt \right)
= \sum_{i=1}^{k} \left( \int_{0}^{\beta_i - \gamma_i} e^{ht^2} dt + \int_{0}^{\gamma_i - \beta_i} e^{ht^2} dt \right)
= 0,
\]

which contradicts the assumption that $z \in P_h(T)$. If it terminates with $e_p$ and $\Delta_{p+1}$ of type IV, then we consider the other decorated ideal triangle $\Delta_0$ adjacent to $e$. If $\Delta_0$ is not of type IV, then it contains an edge $e_{-1}$ and a generalized angle $\theta_0$ adjacent to $e_{-1}$ and $e$ so that $l(e_{-1}) = +\infty$ and $\lim_{m \to \infty} \theta_0^{(m)} = 0$. As before, either there is a fundamental edge loop, contradicting the assumption that $z \in P_h(T)$, or the procedure terminates with an edge $e_{-q}$ and a decorated ideal triangle $\Delta_{-q}$ adjacent to $e_{-q}$ so that $l(e_{-q}) = +\infty$ and $\Delta_{-q}$ is of type IV. If the procedure stops at $e_{-q}$ and $\Delta_{-q}$ of type IV, we get a fundamental edge path $(\Delta_{-q}, e_{-q}, \ldots, e_p, \Delta_{p+1})$, where $e_0 = e$, such that $\Delta_{-q}$ and $\Delta_p$ are of type IV with $l(e_{-q}) = +\infty$ and $l(e_p) = +\infty$, and $\lim_{m \to \infty} \theta_i^{(m)} = 0$, where $\theta_i$ is the generalized angle of $\Delta_i$ adjacent to $e_{i-1}$ and $e_i$ for $i = 1 - q, \ldots, p$. Denote by $a_{-q}$ the generalized angle of $\Delta_{-q}$ facing $e_{-q}$, and by $a_p$ the generalized angle of $\Delta_{p+1}$ facing $e_p$. Let $b_{-q}$ and $c_{-q}$ be the generalized angles of $\Delta_{-q}$ adjacent to $e_{-q}$, and let $b_p$ and $c_p$ be the generalized angles of $\Delta_{p+1}$ adjacent to $e_p$. We find

\[
\sum_{i=-q}^{p} z(e_i) = \lim_{m \to \infty} \sum_{i=-q}^{p} \Psi_h(l^{(m)})(e_i)
= \lim_{m \to \infty} \left( \int_{0}^{b_{-q} + c_{-q} - a_{-q}} e^{ht^2} dt + \int_{0}^{b_{-q} + a_{-q} - c_{-q}} e^{ht^2} dt \right)
+ \sum_{i=1-q}^{p} \left( \int_{0}^{b_i + c_i - a_i} e^{ht^2} dt + \int_{0}^{b_i + a_i - c_i} e^{ht^2} dt \right)
= \int_{0}^{\infty} e^{ht^2} dt + \int_{0}^{\infty} e^{ht^2} dt + \sum_{i=1-q}^{p} \left( \int_{0}^{\beta_i - \gamma_i} e^{ht^2} dt + \int_{0}^{\gamma_i - \beta_i} e^{ht^2} dt \right)
= -2 \int_{0}^{\infty} e^{ht^2} dt,
\]
which contradicts the assumption that $z \in P_h(T)$.

q.e.d

5. Uniqueness of the energy function

Let $\Delta$ be a decorated ideal triangle with edge lengths $l_1, l_2, l_3$ with opposite generalized angles $\theta_1, \theta_2, \theta_3$ and set $x_i = \frac{\theta_i + \theta_j - \theta_k}{2}$ for $i, j, k = 1, 2, 3$. The following theorem shows that $\Psi_h$ is the unique possible deformation of Penner’s simplicial coordinate by using the variational principle stated in Section 2.

**Theorem 5.1** Let $\mu$ and $u$ be two non-constant smooth functions. Up to an overall scale, there is a unique closed 1-form $\omega = \sum_{i=1}^{3} \mu(x_i) du(l_i)$ which is given by

$$w_h = \sum_{i=1}^{3} \int^{x_i} e^{ht^2} dt d(l_i) e^{-he^{-t} dt}$$

for some $h \in \mathbb{R}$.

The proof of Theorem 5.1 makes use of the following lemma.

**Lemma 5.2** Let $f$ and $g$ be two non-constant smooth functions on $\mathbb{R}$. If $\frac{f(x_i)}{g(l_j)}$ is symmetric in $i, j = 1, 2$, then there are constants $h, c_1$ and $c_2$ so that

$$f(t) = e^{ht^2+c_1} \quad \text{and} \quad g(t) = e^{-he^{-t}+c_2}.$$

**Proof:** By taking $\frac{\partial}{\partial x_k}$ in the equality $\frac{f(x_i)}{g(l_j)} = f(x_i) \frac{\partial x_i}{g(l_j)}$, we have $\frac{f'(x_i)}{g(l_j)} \frac{\partial x_i}{g(l_j)} = f(x_i) \frac{\partial x_i}{g(l_j)}$ for $i, j, k = 1, 2, 3$. We deduce from (1) that $\frac{\partial x_i}{\partial x_j} = \frac{x_i}{2}$, so $\frac{f'(x_i)}{g(l_j)} \frac{\partial x_i}{g(l_j)} = f'(x_i) \frac{x_i}{2}$. Thus, $\frac{f'(x_i)}{f(x_i)} = \frac{g(l_j)}{g(l_i)} = f(x_i)$, which implies $\frac{f'(x_i)}{f(x_i)} \frac{1}{x_i} = \frac{f'(x_i)}{f(x_i)} \frac{1}{x_j}$ and $f'(t) \frac{1}{t} = 2h_1$ for some $h_1 \in \mathbb{R}$. Solving this ordinary differential equation for $f$, we find

$$f(t) = e^{ht^2+c_1}$$

for some $c_1 \in \mathbb{R}$. By taking $\frac{\partial}{\partial x_k}$ in the equality $\frac{g(l_j)}{f(x_i)} = \frac{g(l_j)}{f(x_i)}$, we have $\frac{g'(l_j)}{f(x_i)} \frac{\partial x_i}{g(l_j)} = \frac{g'(l_j)}{f(x_i)} \frac{\partial x_i}{g(l_j)}$ for $i, j, k = 1, 2, 3$. From (1) again, we deduce that $\frac{\partial x_i}{\partial x_j} = -\frac{1}{x_k}$, so $-\frac{g'(l_j)}{f(x_i)} \frac{1}{x_k} = -\frac{g'(l_j)}{f(x_i)} \frac{1}{x_k}$. Thus,

$$\frac{g'(l_j)}{g(l_j)} \frac{\partial l_j}{\partial x_j} = f(x_j) \frac{\partial x_i}{g(l_j)} = g(l_i) \frac{\partial x_i}{g(l_j)}$$

which implies $g'(l_j) \frac{\partial l_j}{\partial x_j} = g'(l_i) \frac{\partial l_i}{\partial x_j}$ and $g'(t) \frac{1}{t} = h_2$ for some $h_2 \in \mathbb{R}$. Solving this ordinary differential equation for $g$, we find

$$g(t) = e^{-ht^2+c_1}$$

for some $c_1 \in \mathbb{R}$. From $f(t) = e^{ht^2+c_1}$ and the equality $\frac{f(x_i)}{g(l_j)} = f(x_i) \frac{\partial x_i}{g(l_j)}$, we conclude that $h_1 = h_2$.

q.e.d

*Proof of Theorem 5.1:* The differential 1-form $\omega = \sum_{i=1}^{3} \mu(x_i) du(l_i)$ is closed if and only if $\frac{\partial \mu(x_i)}{\partial u(l_j)} = \frac{\partial \mu(x_j)}{\partial u(l_i)}$ is symmetric in $i$ and $j$. Since $\frac{\partial x_i}{\partial x_j} = \frac{\partial x_j}{\partial x_i} = \frac{x_i}{2}$, $\omega$ is closed if and only if $\mu'(x_i) \frac{\partial l_i}{\partial x_j}$ is symmetric in $i$ and $j$. By Lemma 5.2, if $\mu(x_i) \frac{\partial l_i}{\partial x_j}$ is symmetric in $i$ and $j$, then $\mu'(x_i) = e^{ht^2+c_1}$ and $\mu'(l_i) = e^{-he^{-t}+c_2}$ for some constants $h, c_1$ and $c_2$.

q.e.d

6. $\Psi_h$ and the Delaunay decomposition
We first review the construction of the Delaunay decomposition associated to a decorated hyperbolic metric following Bowditch-Epstein [BE]. Suppose $S$ is a punctured surface with a set of ideal vertices $V$, and let $(d,r)$ be a decorated hyperbolic metric on $S$ so that the horodisks associated to the ideal vertices do not intersect. Let $B_v$ be the horodisks associated to the ideal vertex $v$, and let $B = \bigcup_{v \in V} B_v$. The spine $\Gamma_{d,r}$ of $S$ is the set of points in $S$ which have at least two distinct shortest geodesics to $\partial B$. The spine $\Gamma_{d,r}$ is shown [BE] to be a graph whose edges are geodesic arcs on $S$.

Let $e_1^*, \ldots, e_N^*$ be the edges of $\Gamma_{d,r}$. By construction each interior point of an edge $e_i^*$ has exactly two distinct shortest geodesics to $\partial B$. For each edge $e_i^*$, there are two horodisks $B_1$ and $B_2$ (possibly coincide) so that points in the interior of $e_i^*$ have precisely two shortest geodesics to $\partial B_1$ and $\partial B_2$. Let $e_i$ be the shortest geodesic from $\partial B_1$ to $\partial B_2$. It is known that $e_i$ intersects $e_i^*$ perpendicularly, and $\{e_1, \ldots, e_N\}$ are disjoint. The components of $S \setminus \{e_1, \ldots, e_N\}$ consists of decorated polygons (ideal polygons with horodisks associated to the ideal vertices) which are the 2-cells of the Delaunay decomposition $\Sigma_{d,r}$. The 1-cells of $\Sigma_{d,r}$ consist of the edges $\{e_1, \ldots, e_N\}$ and the arcs on $\partial B$ which are the intersection of $\partial B$ with the ideal polygons. For a generic decorated hyperbolic metric $(d,r)$, each 2-cell of $\Sigma_{d,r}$ is a decorated ideal triangle, and $\Sigma_{d,r}$ is a decorated ideal triangulation of $S$.

Let $D$ be a 2-cell of $\Sigma_{d,r}$. We call the hyperbolic circle on $S$ tangent to all arcs of $D \cap \partial B$ the inscribed circle of $D$. By the construction of the Delaunay decomposition, for each 2-cell $D$ of $\Sigma_{d,r}$, there is exactly one vertex $v^*$ of the spine $\Gamma_{d,r}$ lying in the interior of $D$. Moreover, $v^*$ is of equal distance to all arcs of $D \cap \partial B$, hence is the center of the inscribed circle of $D$. Thus, the center of the inscribed circle of each 2-cell $D$ of the Delaunay decomposition is in the interior of $D$. We need the following proposition of Penner [P3] whose proof is included here to the convenience of the readers.

**Lemma 6.1 (P3)** Suppose $\Delta$ is a decorated ideal triangle with edge lengths $l_i > 0$ and opposite generalized angles $\theta_i$ for $i = 1, 2, 3$. Then $x_i = \frac{\theta_i - \theta_{i+1} - \theta_i}{2} > 0$ for $i = 1, 2, 3$ if and only if the center of the inscribed circle of $\Delta$ is in the interior of $\Delta$.

**Proof:** For $i = 1, 2, 3$ let $B_i$ be the horodisks associated to the ideal vertices of $\Delta$, and let $Z_i$ be the point of tangency of the inscribe circle of $\Delta$ and $\partial B_i$. Label the intersection of the horodisks and the edges of $\Delta$ by $X_1, Y_1, X_2, Y_2, X_3$ and $Y_3$ cyclically as in Figure 6(a). For two points $A$ and $B$ in the hyperbolic plane $\mathbb{H}^2$, let $AB$ be the geodesic segment connecting $A$ and $B$, and let $|AB|$ the length of $AB$. If the center $v$ of the inscribed circle is in the interior of $\Delta$, then $x_i = |X_iZ_{i+1}| > 0$ for $i = 1, 2, 3$. If $v$ is on $X_iY_i$, or $v$ and $\Delta$ are on different sides of $X_iY_i$ for some $i \in \{1, 2, 3\}$, then $x_i = -|X_iZ_{i+1}| \leq 0$. See Figure 6(b). q.e.d

**Proof of Theorem 1.3:** Let $(d,r)$ be a decorated hyperbolic metric so that the associated Delaunay decomposition $\Sigma_{d,r}$ is a decorated ideal triangulation of $S$. For each edge $e$ of $\Sigma_{d,r}$, let $\Delta$ and $\Delta'$ be the decorated ideal triangles adjacent to $e$, and let $\theta_1$ and $\theta'_1$ respectively be the generalized angles of $\Delta$ and $\Delta'$ facing $e$, and $\theta_2$, $\theta_3$, $\theta'_2$ and $\theta'_3$ be the generalized angles adjacent to $e$. Let $x(e) = \frac{\theta_2 + \theta_3 - \theta_1}{2}$ and $x'(e) = \frac{\theta'_2 + \theta'_3 - \theta'_1}{2}$. From Lemma 6.1 and the fact that the center of the inscribed circle of each 2-cell of the Delaunay decomposition is in the interior of the 2-cell, we conclude that $x(e)$ and $x'(e)$ are positive, and

$$
\Psi_{h}(d,r)(e) = \int_0^{x(e)} e^{ht^2} dt + \int_0^{x'(e)} e^{ht^2} dt > 0.
$$
On the other hand, if $T$ is an ideal triangulation of $S$ such that $\Psi_h(d, r)(e) \leq 0$ for some edge $e$, then at least one of $x(e)$ and $x'(e)$, say $x(e)$, is less than or equal to zero. By Lemma 6.1, the center of the inscribed circle of $\Delta$ is not in the interior of $\Delta$. Since the center of the inscribed circle of each 2-cell of the Delaunay decomposition has to be in the interior of the 2-cell, $T$ cannot be the Delaunay decomposition $\Sigma_{d,r}$ of $S$. q.e.d

7. Further questions

1. Suppose $\Delta$ is a decorated ideal triangle with edge lengths $l_1$, $l_2$ and $l_3$ and opposite generalized angles $\theta_1$, $\theta_2$ and $\theta_3$. For each $h \neq -1$, the differential 1-form $\omega_h = \sum_{i=1}^{3} \theta_{i+1}^{-1} d(e^{-(h+1)l_i})$ is closed in $\mathbb{R}^3$. However, the primitive $F_h(u) = \int_0^u \omega_h$ is not strictly concave on $\mathbb{R}^3$. Let $(S, T)$ be an ideally triangulated punctured surface. For each $h \neq -1$, we define a map $\Phi_h : T_c(S) \times \mathbb{R}_{>0} \to \mathbb{R}^E$ by

$$\Phi_h(d, r)(e) = \theta_{h+1} + \theta'_{h+1},$$

where $\theta$ and $\theta'$ are the generalized angles facing $e$. To the best of the author’s knowledge, there is no counterexample to the following

**Conjecture 7.1** The map $\Phi_h : T_c(S) \times \mathbb{R}_{>0} \to \mathbb{R}^E$ is a smooth embedding, and the image of $\Phi_h$ is a convex polytope.

The motivation of this conjecture is as follows. Penner’s simplicial coordinate $\Psi$ and its deformation $\Psi_h$ are in some sense analogues to Colin de Vedière’s invariant [CV] for circle packings in a different setting, and the quantities $\Phi_h$ are the corresponding analogues to Rivin’s invariant [R] for the polyhedra surfaces in this setting, see also [BS] and [L].

2. By Corollary 1.4, for each $h \geq 0$, there is a homeomorphism

$$\Pi_h : T_c(S) \times \mathbb{R}_{>0} \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

equivariant under the mapping class group action. If $h \neq h'$, then $\Pi_h^{-1} \Pi_{h'}$ is a self-homeomorphism of the decorated Teichmüller space equivariant under the mapping class group action. These
self-homeomorphisms deserve a further study. We do not know yet if these self-homeomorphisms are smooth on the decorated Teichmüller space. As suggested by the referee of this article, it also seems natural to ask if these self-homeomorphisms have bounded distortion.

3. The Weil-Petersson Kähler form on the Teichmüller space was computed in the length coordinates in [P2]. How to express the Weil-Petersson symplectic form on the decorated Teichmüller space in terms of the simplicial coordinate \( \Psi \) and in terms of the \( \Psi_h \) coordinate, and how to relate the \( \Psi_h \) coordinate to the quantum Teichmüller space are interesting problems ([B], [BL], [M] and [P3]).

References

[BS] Bobenko, A. I.; Springborn, B. A., Variational principles for circle patterns and Koebe’s theorem. Trans. Amer. Math. Soc. 356 (2004) 659-689.

[B] Bonahon, Francis, Shearing hyperbolic surfaces, bending pleated surfaces and Thurston’s symplectic form. Ann. Fac. Sci. Toulouse Math. (6) 5 (1996), 233-297.

[BL] Bonahon, Francis; Liu, Xiaobo, Representations of the quantum Teichmüller space and invariants of surface diffeomorphisms. Geom. Topol. 11 (2007), 889-937.

[BE] Bowditch, Brian H.; Epstein, David B. A., Natural triangulations associated to a surface. Topology 27 (1988), 91-117.

[CV] Colin de Verdière, Yves, Un principe variationnel pour les empilements de cercles. Invent. Math. 104 (1991) 655-669.

[G] Guo, Ren, On parameterizations of Teichmüller spaces of surfaces with boundary. J. Differential Geom. 82 (2009), 629-640.

[GL1] Guo, Ren; Luo, Feng, Rigidity of polyhedral surfaces. II. Geom. Topol. 13 (2009), 1265-1312.

[GL2] ——–, Cell decompositions of Teichmüller spaces of surfaces with boundary. Preprint, arXiv:1006.3119

[H] Harer, John L., The virtual cohomological dimension of the mapping class group of an orientable surface. Invent. Math. 84 (1986), 157-176.

[Ha] Hazel, Graham P., Triangulating Teichmüller space using the Ricci flow. Ph.D. thesis, University of California, San Diego. 2004.

[L] Leibon, Gregory, Characterizing the Delaunay decomposition of compact hyperbolic surfaces. Geom. Topol. 6 (2002) 361-391.

[L1] Luo, Feng, On Teichmüller spaces of surfaces with boundary. Duke Math. J. 139 (2007), 463-482.

[L2] ——–, Rigidity of polyhedral surfaces. Preprint, arXiv:math/0612714

[M] Mondello, Gabriele, Triangulated Riemann surfaces with boundary and the Weil-Petersson Poisson structure. J. Differential Geom. 81 (2009), 391-436.

[P1] Penner, Robert C, The decorated Teichmüller space of punctured surfaces. Comm. Math. Phys. 113 (1987), 299-339.
[P2] ———, Weil-Petersson volumes. J. Differential Geom. 35 (1992), no. 3, 559-608.

[P3] ———, An arithmetic problem in surface geometry. In: The moduli space of curves (Texel Island, 1994), 427-466, Progr. Math., 129, Birkhauser 1995.

[R] Rivin, Igor, Euclidean structures on simplicial and hyperbolic volume. Ann. of Math. (2) 139 (1994) 553-580.

[U] Ushijima, Akira, A canonical cellular decomposition of the Teichmüller space of compact surface with boundary. Comm. Math. Phys. 201 (1990), 305-326.

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