Event-triggered Consensus Control of Multi-agent Systems with Nonuniform Communication Delays via Reduced-Order Observers

Qiuzhen Wang · Jiangping Hu · Yiyi Zhao
Bijoy Kumar Ghosh

Abstract: This paper studies a consensus problem for linear multi-agent systems (MASs) over directed communication networks with nonuniform time-varying delays. To overcome the limited computing and storage resources, a distributed control scheme is designed for each agent by using the event-triggered strategy. At the same time, a reduced-order observer is put forward in the controller design when only the relative output measurement is available. The communication network model with nonuniform time-varying delays is more challenging than the fixed delays or non-delays in the literature. Theoretical analysis is provided to show that the proposed control scheme can guarantee the consensus of MASs, with Zeno-behavior excluded and the upper bound of time delay obtained. A numerical example is provided to illustrate the feasibility and effectiveness of the theoretical results.

Keywords: Consensus control, multi-agent system, distributed event-triggered strategy, reduced-order observer, nonuniform time-varying delay.

1. INTRODUCTION

Consensus problems of MASs have been investigated extensively since Degroot (1974) developed a consensus algorithm in an opinion pooling problem from the perspective of the probability theory. Up to now, consensus control strategies have been widely applied in many fields, such as formation control, containment control, and distributed optimization and learning (see Hu and Feng (2011); Hu et al. (2013); Chen et al. (2018); Peng et al. (2019); Yuan et al. (2019), etc.

Event-triggered control strategies seem more applicable for cooperative control of MASs when the communication, computation and storage resources are limited. Generally, the event-triggered strategies can be divided into two categories: state-dependent and state-independent strategies. The early works mainly adopted state-dependent strategies. For example, a distributed event-triggered control strategy was firstly proposed for a first-order MAS in Dimarogonas and Johansson (2009), in which centralized and decentralized schemes were studied. Hu et al. (2011) made the first attempt to propose a distributed event-triggered control for leader-follower MASs. An extension was further presented to consider a tracking problem of second-order leader-follower MASs in Hu et al. (2015). Another class of event-triggered control schemes often adopt state-independent strategies, which can be found in Seyboth et al. (2013); Yang et al. (2016). The key point is that the threshold function is independent of the state information of neighbors. Very recently, event-triggered strategy was further extended to some practical scenarios such as quasi-containment control in Yuan et al. (2019), consensus control with input saturation in Yi et al. (2019).

It is noted that the aforementioned works related to event-triggered consensus control strategies seldom concerned observer-type protocols. However, the state information of agents may not be fully measured in practice, so the protocols based on observer type seem to have more practical significance. As far as we know, observer based control protocols can be divided into full-order and reduced-order state observers. In the case of full-order observer protocols, some observer-based consensus controls were developed in Li, Duan, et al. (2010); Li, Soh, et al. (2017). If the agent dynamics are of high order or the number of agents is large, the full-order observer design

Copyright lies with the authors 3292
may result in computational redundancy and need large storage capacity. To this end, Li et al. (2011) presented a new algorithm to design reduced-order observers in consensus control. Li et al. (2019) proposed two kinds of reduced-order output-feedback consensus controls with adaptive gain laws based on edge and node respectively. A new $Kx$-functional observer-based output feedback event-based control was proposed in Jian et al. (2019a,b).

In this paper, we present the first attempt to address the event-based consensus problem for MASs under directed graph with reduced-order observer and nonuniform time-varying delays, a topic that remains challenging. The main contributions of this paper are threefold. First, in order to effectively reduce the storage space, a dimension reduction method is introduced in the reduced-order observer design. Second, an event-triggered strategy based on state-independent threshold is proposed and thus the strategy avoids to compute the threshold with the information from the neighbors. Third, nonuniform time-varying delays are considered in the event-triggered consensus of MASs.

The rest of paper is organized as follows. Section 2 gives some preliminaries and the consensus control problem is formulated. Section 3 presents the main results. Some simulation results are given in Section 4. Conclusions are given in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Some preliminaries

By convention, $\mathbb{R}^{m \times n}$ and $\mathbb{C}^{m \times n}$ are the set of $m \times n$ real and complex matrices respectively. $Re(s)$ denotes the real part of $s \in \mathbb{C}$. $\mathbf{1}_n = (1, \ldots , 1)^T \in \mathbb{R}^n$. $x^T$ denotes the transpose of vector $x$. The conjugate transpose of matrix $A$ is represented by $A^H$. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ represent the minimum and maximum eigenvalues of the matrix $A$, respectively. $\| \cdot \|$ represents the Euclidean norm. $\otimes$ denotes the Kronecker product. A matrix is said to be Hurwitz stable if all of its eigenvalues have positive real parts.

A directed communication network can be modeled as a digraph $G = (V, E, A)$, where $V = \{v_1, \ldots , v_N\}$ represents the set of $N$ agents, $E = \{(e_{ij} \mid (v_i, v_j) \in V \times V\}$ is the set of edges, and $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the adjacency matrix of $G$ defined by $a_{ii} = 0$, $a_{ij} > 0$ if $(v_j, v_i) \in E$, and $a_{ij} = 0$ otherwise. The set of all neighbors of node $v_i$ can be defined by $N_i = \{v_j \in V \mid e_{ij} \in E\}$. The degree matrix $D = \{d_i, \ldots , d_N\} \in \mathbb{R}^{N \times N}$ of $G$ is a diagonal matrix with diagonal elements $d_i = \sum_{j \in N_i} a_{ij}$. The Laplacian matrix of $G$ is defined as $L = D - A$, which satisfies $L\mathbf{1}_N = 0$. The Laplacian matrix has the following property.

**Lemma 1.** (Ren and Beard (2005)) Zero is an eigenvalue of $L$ with $\mathbf{1}_N$ and a nonnegative vector $r \in \mathbb{R}^N$ as the corresponding right and left eigenvectors, respectively, that is, $r^T L = 0$, $r^T \mathbf{1}_N = 1$. Moreover, all other nonzero eigenvalues have positive real parts. Furthermore, zero is a simple eigenvalue of $L$ if and only if the graph $G$ has a directed spanning tree.

2.2 Problem formulation

Consider a general linear MAS, where the agent dynamics is given by

$$
\begin{align*}
\dot{x}_i(t) &= Ax_i(t) + Bu_i(t), \\
y_i(t) &= Cx_i(t),
\end{align*}
$$

where $x_i(t) \in \mathbb{R}^n$ is the state, $u_i(t) \in \mathbb{R}^m$ is the input, $y_i(t) \in \mathbb{R}^q$ is the measured output, and $A, B, C$ are constant matrices with compatible dimensions.

The objective of this study is to design a suitable event-triggered consensus control scheme such that all the agents can achieve consensus, i.e., \( \lim_{t \to \infty} (x_i(t) - x_j(t)) = 0 \) for all $i, j = 1, \ldots , N$, when nonuniform delays exist in the communication links, and at the same time, Zeno behavior is excluded.

Throughout this study, the following assumptions are adopted.

**Assumption 1.** For the high-order MAS (1), $(A, B)$ is controllable, $(A, C)$ is observable, and $C$ is of full row rank.

**Assumption 2.** The directed graph $G$ has a spanning tree.

The following lemma will be used in the consensus analysis of the MAS (1).

**Lemma 4.** (Yang et al. (2016)) Suppose that $A \in \mathbb{R}^{n \times n}$ is Hurwitz. Then there exists a nonsingular matrix $P_A$ such that $P_A^{-1}AP_A = J_A$ with $J_A$ being the Jordan canonical form of $A$ and $\|cA^T\| \leq \|PA\|\|P_A^{-1}\|e^{-a_A t}$, where $c_A$ is a positive constant determined by $A$, and $0 < a_A < -\max\{\lambda_i(A)\})$.

3. MAIN RESULTS

3.1 Reduced-order observer based consensus control

For each agent $i$, there exists a series of event time instants $\tau_{ik}$ $(k = 0, 1, \ldots)$ determined by an event-triggered threshold function. To reach the consensus of MAS (1) without using any global information, a distributed event-triggered control scheme together with a reduced-order observer is proposed for $t \in [\tau_k, \tau_{k+1})$:

$$
\begin{align*}
\dot{v}_i(t) &= Fv_{i}(t) + Gy_{i}(t) + TBu_{i}(t), \\
u_i(t) &= cKQ_1 \sum_{j \in N_i} a_{ij} [y_i(t_k) - \tau_{ij}(t)] - y_j(t_k) - \tau_{ij}(t)]) + cKQ_2 \sum_{j \in N_i} a_{ij} [v_i(t_k) - \tau_{ij}(t)] - v_j(t_k) - \tau_{ij}(t)])].
\end{align*}
$$

where $y_i(t) \in \mathbb{R}^{n-q}$ is the observer state, $c > 0$ is the coupling strength, $\tau_{ij}(t)$ represents the communication delay from agent $j$ to agent $i$, $K \in \mathbb{R}^{p \times n}$ is a control gain matrix, and $t_k$ denotes the last event instant of agent $j$. $F \in \mathbb{R}^{(n-q) \times (n-q)}$ is Hurwitz and has no common eigenvalues with matrix $A$, $G \in \mathbb{R}^{(n-q) \times q}$, $T \in \mathbb{R}^{(n-q) \times n}$ is the only solution to the Sylvester equation $TA - FT = GC$ and $[C \ T \ T]^T$ is nonsingular, $Q_1 \in \mathbb{R}^{n-q \times q}$ and $Q_2 \in \mathbb{R}^{n \times (n-q)}$ are given by $[Q_1, Q_2] = \left\{ [C \ T \ T]^T \right\}^{-1}$. 3293
3.2 Network model with nonuniform time-varying delays

Assume that the nonuniform time-varying delays are asymmetrical and uniformly bounded. For convenience, let \( \Psi = \{ \tau_\sigma(t) = \tau_\sigma : \sigma \in \{1, \ldots, m\} \ (m \leq N(N-1)) \) be the collection of independent time-varying delays affecting the communication links. There exists a set of communication topologies \( \{ G_1, \ldots, G_m \} \) such that the network \( G_\sigma \) contains only a delay \( \tau_\sigma \).

To illustrate the topology decomposition technique related to nonuniform delays, we consider a MAS having six agents, with the communication topology \( G \) shown in Fig. 1. The MAS has four different delays, i.e., \( \tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_6 \), \( \tau_2 = \tau_3 = \tau_4 = \tau_5 \), \( \tau_3 = \tau_5 \), and \( \tau_4 = \tau_2 \), and thus \( G \) is decomposed to four subgraphs \( G_1, G_2, G_3 \) and \( G_4 \), as shown in Fig. 2.

Fig. 1. A communication network with nonuniform delays

\begin{align*}
& \text{(a) } G_1 \\
& \text{(b) } G_2 \\
& \text{(c) } G_3 \\
& \text{(d) } G_4
\end{align*}

Fig. 2. The subgraphs of \( G \) associated with the delays \( \tau_1, \tau_2, \tau_3 \) and \( \tau_4 \).

Suppose that the Laplacian matrix corresponding to \( G_\sigma \) is \( L_\sigma \). It is clear that \( L_\sigma 1_N = 0, \forall \sigma \in \{1, \ldots, m\} \) and \( \sum_{\sigma=1}^{m} L_\sigma = L \).

Lemma 5. \( L_\sigma \) is a Laplacian matrix of the subgraph of \( G \). Then there exists a non-singular matrix \( S \) such that

\[ S^{-1} L S = J_\sigma = \begin{bmatrix} 0 & 0 \\ 0 & J_\sigma \end{bmatrix}, \]

where \( L_\sigma \in \mathbb{C}^{(N-1)\times(N-1)}, J_\sigma \in \mathbb{C}^{(N-1)\times(N-1)} \).

Proof. According to Assumption 2 that \( G \) contains a directed spanning tree, assuming that there exists a constant \( \alpha \neq 0 \) and non-singular matrices \( S = [a_1 1_N] S_1 \),

\[ S^{-1} = \begin{bmatrix} 1 \\ \alpha S_2 \end{bmatrix}, \]

where \( r \in \mathbb{R}^N \) is a nonnegative vector such that \( r^T L = 0 \) and \( r^T 1_N = 1 \). From the Jordan decomposition of \( L \), we have \( S^{-1} L S = J_1 \in \mathbb{C}^{(N-1)\times(N-1)} \), of which the Jordan matrix \( J_1 \) is an upper triangular matrix with diagonal line consist with the nonzero eigenvalue of \( L \) (see ?). From the foreshadowing above \( L_\sigma 1_N = 0, \sigma \in \{1, \ldots, m\} \). Hence,

\[ S^{-1} L S = \begin{bmatrix} 1 - T \alpha & 0 \\ 0 & S_2 \end{bmatrix}, \]

and

\[ S^{-1} L_\sigma S = \begin{bmatrix} 1 - T \alpha & 0 \\ 0 & S_2 \end{bmatrix}. \]

The proof is thus completed.

3.3 Consensus analysis

For each agent \( i \), we define two measurement error vectors as \( e_i(t) = x_i(t) - x_i(t), e_i(t) = v_i(t) - v_i(t) \). Let \( u_i(t) = \begin{bmatrix} x_i^T(t), v_i^T(t) \end{bmatrix}, \eta(t) = \begin{bmatrix} \eta_1(t), \ldots, \eta_N(t) \end{bmatrix} \) and \( \xi_i(t) = \begin{bmatrix} \xi_1^T(t), \ldots, \xi_N^T(t) \end{bmatrix} \). Then, from (1) and (2), the closed-loop system is given by

\[ \dot{\eta}(t) = (I_N \otimes M) \eta(t) + \sum_{\sigma=1}^{m} \begin{bmatrix} c_{L_\sigma} \otimes R \end{bmatrix} [\eta(t - \tau_\sigma(t)) + \xi(t - \tau(t))], \]

where \( M = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix}, R = \begin{bmatrix} BKQ_1 C \\ TBKQ_2 C \end{bmatrix} \).

Next, we will show that the event-triggered control (2) can guarantee consensus of the MAS under a threshold function given by

\[ f_i(t) = \sum_{\sigma=1}^{m} \begin{bmatrix} c_{L_\sigma} \otimes R \end{bmatrix} [\eta(t - \tau_\sigma(t)) + \xi(t - \tau(t))], \]

for some \( c_1 > 0 \) and \( \alpha \) is a positive constant to be determined. Thus, the event triggered times are given by \( t_k = \inf \{ t : t > t_k, f_i(t) > 0 \} \).

Theorem 6. Under Assumptions 2 and 3, consensus of the MAS can be achieved under the event-triggered control scheme (2).

Proof. From Lemma 1, zero is a simple eigenvalue of \( L \) and all other eigenvalues have positive real parts. By Lemma 5, there exists a coordinate transformation \( x(t) = (S^{-1} \otimes I_{2n-\sigma}) \eta(t) \). Let \( \xi(t) = \begin{bmatrix} \xi_1^T(t), \ldots, \xi_N^T(t) \end{bmatrix} \), where \( \xi_i(t) = \begin{bmatrix} \xi_1^T(t), \xi_2^T(t) \end{bmatrix} \). Then, system (3) can be rewritten as follow:

\[ \dot{\xi}(t) = (S^{-1} \otimes I_{2n-\sigma}) \eta(t) \]

\[ = (I_N \otimes M) \xi(t) + \sum_{\sigma=1}^{m} \begin{bmatrix} c_{L_\sigma} \otimes R \end{bmatrix} \xi(t - \tau_\sigma(t)) \]

\[ + \sum_{\sigma=1}^{m} \begin{bmatrix} c_{L_\sigma} \otimes R \end{bmatrix} \xi(t - \tau_\sigma(t)). \]

Define \( \xi_{2-N}(t) = \begin{bmatrix} \xi_1^T(t), \ldots, \xi_N^T(t) \end{bmatrix} \), then system (5) can be divided into the following two subsystems:

\[ \dot{\xi}_1(t) = M \xi_1(t) + \sum_{\sigma=1}^{m} \begin{bmatrix} c_{L_\sigma} \otimes R \end{bmatrix} \xi_{2-N}(t - \tau_\sigma(t)) \]

\[ + \sum_{\sigma=1}^{m} \begin{bmatrix} c_{L_\sigma} \otimes R \end{bmatrix} \xi_1(t - \tau_\sigma(t)), \]

and
\[ \dot{\varepsilon}_{2-N}(t) = (I_{N-1} \otimes M)\varepsilon_{2-N}(t) + \sum_{\sigma=1}^{m} (cJ_{1\sigma} \otimes R) \varepsilon_{2-N}(t - \tau_{\sigma}) + \sum_{\sigma=1}^{m} (cJ_{1\sigma} S_{2} \otimes R) \xi_{2-N}(t - \tau_{\sigma}), \] (7)

By Newton-Leibniz formula, one has \( \varepsilon_{2-N}(t - \tau_{\sigma}) = \varepsilon_{2-N}(t) - \int_{t - \tau_{\sigma}}^{t} \dot{\varepsilon}_{2-N}(s) ds \), then the subsystem (7) can be rewritten as

\[ \dot{\varepsilon}_{2-N}(t) = (I_{N-1} \otimes M)\varepsilon_{2-N}(t) + \sum_{\sigma=1}^{m} (cJ_{1\sigma} \otimes R) \int_{t - \tau_{\sigma}}^{t} \dot{\varepsilon}_{2-N}(s) ds + \sum_{\sigma=1}^{m} (cJ_{1\sigma} S_{2} \otimes R) \xi_{2-N}(t - \tau_{\sigma}), \] (8)

where \( M = I_{N-1} \otimes M + cJ_{1} \otimes R, C_{1\sigma} = cJ_{1\sigma} \otimes R \), and \( C_{2\sigma} = cJ_{1\sigma} S_{2} \otimes R \). The solution of (8) is given by

\[ \varepsilon_{2-N}(t) = e^{M(t-t_{0})}\varepsilon_{2-N}(t_{0}) + \int_{t_{0}}^{t} e^{M(t-s)} \sum_{\sigma=1}^{m} (-C_{1\sigma}) \int_{s - \tau_{\sigma}}^{t} \dot{\varepsilon}_{2-N}(\theta) d\theta d\theta, \] (9)

In order to check whether \( M \) is Hurwitz, it is equivalent to analyze the stability of

\[ \dot{F} = \begin{bmatrix} A + c\lambda_{1} BKQ_{1}C & c\lambda_{1} BKQ_{2} \\ GC + c\lambda_{1} TBKQ_{1}C & F + c\lambda_{1} TBKQ_{2} \end{bmatrix}, \]

Let matrix \( \dot{F} \) be multiplied by \( \tilde{T} = \begin{bmatrix} I_{n} & 0 \\ 0 & -T \end{bmatrix} \) and \( \tilde{T}^{-1} \), then we have

\[ \tilde{T}^{-1} = \begin{bmatrix} I_{n} & 0 \\ T^{-1} & I_{n} \end{bmatrix}, \]

\[ \tilde{T}^{-1}(A + c\lambda_{1} BKQ_{1}C) \tilde{T}^{-1} = \begin{bmatrix} A + c\lambda_{1} BK & c\lambda_{1} BKQ_{2} \\ GC + c\lambda_{1} TBKQ_{1}C & F + c\lambda_{1} TBKQ_{2} \end{bmatrix}, \]

Selecting the coupling coefficient \( c \geq \frac{1}{2 \max_{\lambda_{i} \in \sigma_{1}(\text{Re}(\lambda_{i}(L)))}} \), then there exists a \( P > 0 \) which satisfies the following algebraic Riccati equation:

\[ (A + c\lambda_{1} BK)^{T}P + P(A + c\lambda_{1} BK) = A^{T}P + PA - 2c\text{Re}(\lambda_{i}(L)) PBB^{T}P \leq -Q, \]

Hence, \( A + c\lambda_{1} BK \) is a stable matrix. Hence, if \( F \) is Hurwitz, \( \dot{F} \) is Hurwitz as well. Then, from Lemma 4, one has, for \( t \geq t_{0} \),

\[ \|e^{M(t-t_{0})}\| \leq k_{1} e^{-\gamma(t-t_{0})}, \]

where \( k_{1} = \|P_{M}\| \|P_{M}^{-1}\| c_{M}, P_{M} \) is a nonsingular matrix such that \( P_{M}^{-1}MP_{M} = J_{M} \), \( J_{M} \) is the Jordan canonical form of \( M \), \( c_{M} > 0 \) is a positive constant determined by \( M \), and \( 0 < \gamma < -\max \{\text{Re}(\lambda_{i}(M))\} \).

Assume that there exist \( \alpha, \lambda \in (0, \gamma) \) such that

\[ k_{1}(m_{1} + m_{2} \sum_{\sigma=1}^{m} e^{\gamma_{\sigma}} ) (e^{\lambda_{\max} - 1}) < 1, \]

\[ \chi = \frac{\sqrt{N - 1} \alpha k_{1} mc_{1} (e^{\lambda_{\max} - 1}) + ak_{2} k_{2}}{\alpha - \alpha k_{1} (m_{1} + m_{2} \sum_{\sigma=1}^{m} e^{\gamma_{\sigma}} ) (e^{\lambda_{\max} - 1})}, \]

(12)

where \( \alpha_{1} = \|C_{1\max}||I_{N-1} \otimes M||, \alpha_{2} = \|C_{2\max}\|^{2} \), and \( \alpha_{3} = \|C_{1\max}\| \|C_{2\max}\| \) with \( C_{1\max} = \max_{C_{1\sigma}} \), \( C_{2\max} = \max_{C_{2\sigma}} \), \( k_{2} = \|C_{2\max}\| \sqrt{N - 1} \alpha k_{1} m_{2} \sum_{\sigma=1}^{m} e^{\gamma_{\sigma}} \), then the following inequality holds for \( t \geq t_{0} \):

\[ \|\varepsilon_{2-N}(t)\| < k_{1} \|\varepsilon_{2-N}(t_{0})\| e^{-\gamma(t-t_{0})} + \chi e^{-\alpha(t-t_{0})}. \]

First, we show that \( \lambda \) exists. Define \( f(\lambda) = k_{1}(m_{1} + m_{2} \sum_{\sigma=1}^{m} e^{\gamma_{\sigma}} ) (e^{\lambda_{\max} - 1}) - \lambda(\gamma - \lambda) \). Obviously, \( f(0) = 0 \) and \( f'(0) = m_{1} \gamma_{\max} + m^{2} \alpha_{1} \gamma_{\max} - \gamma < 0 \) when \( \gamma_{\max} < \frac{1}{k_{1}(m_{1} + m_{2} \sum_{\sigma=1}^{m} e^{\gamma_{\sigma}} )} \). Thus the constant \( \lambda \) satisfies (11) can be set up.

Next, we prove that inequality (13) is true. Define \( k_{1} \|\varepsilon_{2-N}(t_{0})\| e^{-\lambda(t-t_{0})} + \chi e^{-\alpha(t-t_{0})} = \omega(t) \). If equation (13) does not hold for any \( t \in (t_{0} - \tau_{\max}, t^{*}) \), then there must exist a \( t^{*} > t_{0} \) such that \( \|\varepsilon_{2-N}(t^{*})\| = \omega(t^{*}) \), and \( \|\varepsilon_{2-N}(t)\| < \omega(t) \). Define \( D = \frac{k_{1}}{\lambda} \|\varepsilon_{2-N}(t_{0})\| (m_{1} + m_{2} \sum_{\sigma=1}^{m} e^{\gamma_{\sigma}} ) (e^{\lambda_{\max} - 1}) \), and \( E = \frac{1}{\alpha} (m_{1} + m_{2} \sum_{\sigma=1}^{m} e^{\gamma_{\sigma}} ) (e^{\lambda_{\max} - 1}) \).

\[ \|C_{1\max} \| \sum_{\sigma=1}^{m} e^{\gamma_{\sigma}} \|e^{\lambda_{\max} - 1}\| \leq k_{1} e^{-\alpha(t-t_{0})}. \]

From above and (9), (13), we have

\[ \omega(t^{*}) = \|\varepsilon_{2-N}(t^{*})\| < k_{1} e^{-\gamma(t^{*}-t_{0})} \|\varepsilon_{2-N}(t_{0})\| + \int_{t_{0}}^{t^{*}} k_{1} e^{-\gamma(t^{*})} \left[ D e^{-\lambda(t^{*}-t)} + (E + k_{2}) e^{-\alpha(t^{*}-t)} \right] d\theta \]

\[ < k_{1} \|\varepsilon_{2-N}(t_{0})\| e^{-\gamma(t^{*}-t_{0})} + \frac{k_{1}(E + k_{2})}{\gamma - \alpha} \left( e^{-\alpha(t^{*}-t)} - e^{-\gamma(t^{*}-t_{0})} \right) \]

\[ < k_{1} \|\varepsilon_{2-N}(t_{0})\| e^{-\gamma(t^{*}-t_{0})} + \chi e^{-\alpha(t^{*}-t_{0})} \geq \omega(t^{*}). \]
The contradictory result shows that (13) holds under the conditions (11) and (12). Thus, (13) implies \( \lim_{t \to \infty} \varepsilon_{2-N}(t) = 0 \), that is, \( \lim_{t \to \infty} \hat{x}_i(t) = 0 \), and \( \lim_{t \to \infty} \hat{v}_i(t) = 0 \), \( i = 2, \ldots, N \).

On the other hand, since \( \eta(t) = (S \otimes I_{2n-q})\varepsilon(t) = 1_N \otimes \varepsilon(1) + (S_1 \otimes I_{2n-q})\varepsilon_{2-N}(t) \), we have \( \eta(t) - 1_N \otimes \varepsilon(1) \rightarrow 0 \) as \( t \to \infty \). Moreover, the variable \( \varepsilon_1(t) \) evolves according to system (6). Then, from the definitions of \( \eta(t) \) and \( \varepsilon_i(t) \), we have \( x_i(1-t) - \hat{x}_i(t) \rightarrow 0, v_i(t) - \hat{v}_i(t) \rightarrow 0, i = 1, \ldots, N \), as \( t \to \infty \), which shows that consensus is reached for the MAS. The proof is thus completed.

**Theorem 7.** Zeno behavior can be avoided in the close-loop system (3) under event-triggered control scheme (2).

**Proof.** Since \( \xi_i(t) = \eta_i(t) - \eta_i(t) \), thus the upper right-hand Dini derivative of \( \xi_i(t) \) over interval \([t_k, t_{k+1}]\) is given by

\[
D^+ \| \xi_i(t) \| \leq \| \tilde{\xi}_i(t) \| \leq \| \eta_i(t) \| \leq \| \hat{\eta}_i(t) \|
\]

\[
= \left\| (I_N \otimes M) \eta(t) + \sum_{\sigma=1}^{m} \{ (cL_\sigma \otimes R)[\eta(t - \tau_\sigma) + \xi(t - \tau_\sigma)] \right\|
\]

\[
\leq \| I_N \otimes M \| \| \eta(t) \| + \sum_{\sigma=1}^{m} \| cL_\sigma \otimes R \| \| \| \eta(t - \tau_\sigma) \| + \| \xi(t - \tau_\sigma) \| \|\|]
\]

Since \( \| \eta(t) \| \leq \alpha_4 e^{-\lambda(t-t_0)} + \alpha_5 e^{-\alpha(t-t_0)} \), where \( \alpha_4 = k_1 \| S \| \| I_{2n-q} \| \| \xi(t_0) \| , \alpha_5 = \chi \| S \| \| I_{2n-q} \| . \) Thus, we have

\[
\| \xi_i(t) \| \leq \| I_N \otimes M \| \left[ \alpha_4 e^{-\lambda(t-t_0)} + \alpha_5 e^{-\alpha(t-t_0)} \right]
\]

\[
+ \sum_{\sigma=1}^{m} \| cL_\sigma \otimes R \| \left[ \alpha_4 e^{-\lambda(t-t_0 - \tau_\sigma)} + \alpha_5 e^{-\alpha(t-t_0 - \tau_\sigma)} \right]
\]

\[
= \alpha_6 e^{-\lambda(t-t_0)} + \alpha_7 e^{-\alpha(t-t_0)} \leq \varphi(t),
\]

where \( \alpha_6 = \alpha_4 \left( \| I_N \otimes M \| + mc \| L_{max} \otimes R \| \sum_{\sigma=1}^{m} e^{\lambda \tau_\sigma} \right) \),

\( \alpha_7 = \| I_N \otimes M \| \alpha_5 + mc \sum_{\sigma=1}^{m} e^{\alpha \tau_\sigma} \| L_{max} \otimes R \| \alpha_5 + \sqrt{N} \alpha_1 \). During the interval \([t_k, t_{k+1}]\), it is not difficult to obtain that \( \| \xi_i(t) \| \leq \int_{t_k}^{t_{k+1}} \phi(s) ds \). From the threshold function given by (4), the next event time of agent \( i \) will not be triggered before \( f_i(t, \xi_i(t)) \geq \theta \) or equivalently \( \| \xi_i(t) \| \leq c_1 e^{-\alpha(t-t_0)} \). Hence, the next event time is not triggered before \( \int_{t_k}^{t} \phi(s) ds \geq c_1 e^{-\alpha(t-t_0)} \). Let \( \tau = t - t_k \) be the time length between the two triggered events. Thus, \( \tau \) is greater than or equal to the solution to the implicit equation \( \alpha_6 e^{-\lambda(t-t_0)} + \alpha_7 e^{-\alpha(t-t_0)} \tau = c_1 e^{-\alpha(t-t_0)} \), or equivalently, \( \alpha_6 e^{-\lambda(t-t_0)} + \alpha_7 e^{-\alpha(t-t_0)} \tau = c_1 e^{-\alpha(t-t_0)} \). Since \( 0 < \lambda \leq \alpha \), we know that \( \alpha_6 e^{-\lambda(t-t_0)} + \alpha_7 \) is bounded by \( \alpha_6 + \alpha_7 \). Thus, the solution to the implicit equation is greater than or equal to the solution to \( (\alpha_6 + \alpha_7) \tau = c_1 e^{-\alpha(t-t_0)} \). Thus, if there exist \( c_1 > 0 \) and \( 0 < \alpha < \lambda < \gamma \), there is a positive lower bound \( \tau \) on the inter-event time for agent \( i \). Therefore, Zeno behavior is avoided. The proof is completed.
5. CONCLUSION

This paper addressed an event-triggered consensus problem of a general linear MAS over directed communication network. A distributed consensus control scheme has been proposed by using reduced-order observer. Moreover, state-independent threshold function has been presented for each agent to achieve consensus under the proposed control scheme. Some sufficient conditions have been established for the consensus of MAS, with the upper bound of nonuniform time-varying delays obtained. Additionally, it has been shown that Zeno behavior can be avoided.

REFERENCES

Chen, W., Ding, D., Ge, X., Han, Q.L., and Wei, G. (2018). $\mathcal{H}_\infty$ containment control of multiagent systems under event-triggered communication scheduling: The finite-horizon case. IEEE Transactions on Cybernetics, 49(8), 1688–1697.

Degroot, M. (1974). Reaching a consensus. Journal of the American Statistical Association, 69(345), 118–121.

Dimarogonas, D. and Johansson, K. (2009). Event-triggered control for multi-agent systems. In 48th IEEE Conference on Decision and Control (CDC), 7131–7136.

Hu, J., Chen, G., and Li, H. (2011). Distributed event-triggered tracking control of leader-follower multi-agent systems with communication delays. Kybernetika, 47(4), 630–643.

Hu, J. and Feng, G. (2011). Quantized tracking control for a multi-agent system with high-order leader dynamics. Asian Journal of Control, 13 (6), 988–997.

Hu, J., Geng, J., and Zhu, H. (2015). An observer-based consensus tracking control and application to event-triggered tracking. Communications in Nonlinear Science and Numerical Simulation, 20(2), 559–570.

Hu, J., Xiao, Z., et al. (2013). Formation control over antagonistic networks. in 2013 the 32nd Chinese Control Conference, 6879–6884.

Jian, L., Hu, J., Wang, J., and Shi, K. (2019a). Distributed event-triggered protocols with $\mathcal{L}_\infty$-functional observer for leader-following multi-agent systems. Physica A: Statistical Mechanics and its Applications, 535, 122457.

Jian, L., Hu, J., Wang, J., and Shi, K. (2019b). New event-based control for sampled-data consensus of multi-agent systems. International Journal of Control, Automation and Systems, 17(5), 1107–1116.

Li, X., Liu, F., Buss, M., and Hirche, S. (2019). Fully distributed consensus control for linear multi-agent systems: A reduced-order adaptive feedback approach. IEEE Transactions on Control of Network Systems, DOI: 10.1109/TCNS.2019.2930916, 1-10.

Li, X., Soh, Y., and Xie, L. (2017). Output-feedback protocols without controller interaction for consensus of homogeneous multi-agent systems: A unified robust control view. Automatica, 81, 37–45.

Li, Z., Duan, Z., and Chen, G. (2010). Consensus of multi-agent systems and synchronization of complex networks: A unified viewpoint. IEEE Transactions on Circuits and Systems I: Regular Papers, 57(1), 213–224.

Li, Z., Liu, X., Lin, P., and Ren, W. (2011). Consensus of linear multi-agent systems with reduced-order observer-based protocols. Systems & Control Letters, 60(7), 510–516.

Peng, Z., Zhao, Y., Hu, J., and Ghosh, B. (2019). Data-driven optimal tracking control of discrete-time multi-agent systems with two-stage policy iteration algorithm. Information Sciences, 481, 189–202.

Ren, W. and Beard, R. (2005). Consensus seeking in multi-agent systems under dynamically changing interaction topologies. IEEE Transactions on Automatic Control, 50(5), 655–661.

Seyboth, G., Dimarogonas, D., and Johansson, K. (2013). Event-based broadcasting for multi-agent average consensus. Automatica, 49(1), 245–252.

Yang, D., Ren, W., Liu, X., and Chen, W. (2016). Decentralized event-triggered consensus for linear multi-agent systems under general directed graphs. Automatica, 69, 242–249.

Yi, X., Yang, T., Wu, J., and Johansson, K. (2019). Distributed event-triggered control for global consensus of multi-agent systems with input saturation. Automatica, 100, 1–9.

Yuan, X., Mo, L., and Yu, Y. (2019). Observer-based quasi-containment of fractional-order multi-agent systems via event-triggered strategy. International Journal of Systems Science, 50(3), 517–533.