ON INTEGRAL MODULAR CATEGORIES OF FROBENIUS-PERRON DIMENSION $pq^n$

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ABSTRACT. We prove that integral modular categories of Frobenius-Perron dimension $pq^2$ are group-theoretical, where $p, q$ are distinct prime numbers. Combining this with previous results in the literature, integral modular categories of Frobenius-Perron dimension $pq^i$, $0 \leq i \leq 5$, are group-theoretical. We also prove a sufficient and necessary condition for integral modular categories of Frobenius-Perron dimension $pq^n$ being group-theoretical, under the restriction that $p < q$, where $n$ is a positive integer.

1. INTRODUCTION

Classifying all fusion categories of a given dimension plays an important role in the study of fusion categories, and many results have been obtained. For example, integral fusion categories of Frobenius-Perron dimension $p^n$ and $pqr$ were proved to be group-theoretical [8, 4], fusion categories of Frobenius-Perron dimension $p^aq^b$ were proved to be solvable [8] and a complete classification of integral fusion categories of Frobenius-Perron dimension $pq^2$ was obtained in [12], where $p, q, r$ are distinct prime numbers, $a, b$ and $n$ are nonnegative integers.

One recent classification result shows that integral modular categories of Frobenius-Perron dimension $pq^4$ are group-theoretical [2], where $p, q$ are distinct prime numbers. So, such categories can be explicitly described by finite group data [20]. In particular, they are equivalent to fusion subcategories of $\text{Rep}(D^\omega(G))$, where $G$ is a finite group and $\omega$ is a 3-cocycle on $G$ [20, 19]. On the other hand, the work of Naidu and Rowell [18] shows that integral modular categories of Frobenius-Perron dimension $pq^2$ and $pq^3$ are pointed. Together with the obvious fact that integral modular categories of Frobenius-Perron dimension $p$ and $pq$ are pointed, all integral modular categories of Frobenius-Perron dimension $pq^i$, $0 \leq i \leq 4$, are group-theoretical. So, it is natural to ask the following question:

Question 1.1. Is any integral modular category of Frobenius-Perron dimension $pq^n$ group-theoretical?

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In this paper we give a partial answer to this question; that is, we prove Theorem 1.2 below.

**Theorem 1.2.** Let \( C \) be an integral fusion category of Frobenius-Perron dimension \( pq^5 \), where \( p, q \) are distinct prime numbers. Then \( C \) is group-theoretical.

If \( C \) be an integral braided nilpotent fusion category, then \( C \) is group-theoretical [4, Theorem 6.10]. This motivates us to think whether the converse is true. We prove the theorem below.

**Theorem 1.3.** Let \( C \) be an integral modular category of Frobenius-Perron dimension \( pq^n \), where \( p < q \) are prime numbers and \( n \) is a positive integer. Then \( C \) is group-theoretical if and only if it is nilpotent.

As an immediate consequence, an integral modular category of Frobenius-Perron dimension \( 2q^n \) is group-theoretical if and only if it is nilpotent.

The paper is organized as follows. In Section 2, we give some basic definitions and results which will be used below. In Section 3, we study the group-theoreticality of integral modular categories of Frobenius-Perron dimension \( pq^n \). In Section 4, we study nilpotency of integral modular categories of Frobenius-Perron dimension \( pq^n \).

## 2. Preliminaries

### 2.1. Results from group theory.

Let \( G \) be a finite group and \( S \subseteq G \) be a subset. The normal closure of \( S \) in \( G \) is the intersection of all normal subgroups containing the subset \( S \). This definition will be used in Section 4.

The following lemma is the Ore Theorem on finite groups (see e.g. [21, Exercise 3(b)]). In fact, this theorem has been generalized to the Hopf algebra setting by Kobayashi and Masuoka [14].

**Lemma 2.1.** Let \( G \) be a finite group and \( K \subseteq G \) be a subgroup. If the index \([G : K]\) is the smallest prime divisor of the order of \( G \) then \( K \) is normal in \( G \).

The following lemma is the well known Sylow D-Theorem (see e.g. [11, Theorem 1.14]).

**Lemma 2.2.** Let \( P \) be a \( p \)-subgroup of a finite group \( G \). Then \( P \) is contained in some Sylow \( p \)-subgroup of \( G \).

### 2.2. Fusion categories.

Let \( C \) be a fusion category over an algebraically closed field \( k \) of characteristic 0. We will use \( \text{Irr}(C) \) to denote the set of isomorphism classes of simple objects of \( C \), and use \( \text{Irr}_a(C) \) to denote the set of isomorphism classes of simple objects of Frobenius-Perron dimension \( a \), where \( a \) is a positive real number.

A simple object \( X \) is invertible if and only if its Frobenius-Perron dimension is 1. Let \( G(C) \) denote the set of isomorphism classes of invertible simple
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objects of \( C \). That is, \( \text{Irr}_1(C) = G(C) \). \( G(C) \) is a group with multiplication given by tensor multiplication.

A fusion category is pointed if every simple object has Frobenius-Perron dimension 1. Such a category is the category of vector spaces graded by some finite group \( G \) with associativity defined by a cohomology class \( \omega \in H^3(G, k^\times) \). We denote such category by \( \text{Vec}(G, \omega) \). The group \( G(C) \) generates a fusion subcategory \( C_{\text{pt}} \) of \( C \). This is the unique largest pointed fusion subcategory of \( C \).

A full tensor subcategory \( D \subseteq C \) is a fusion subcategory if every object of \( C \) isomorphic to a direct summand of an object in \( D \) is contained in \( D \). If \( D \) is a fusion subcategory of \( C \) then the quotient \( \text{FPdim}(C)/\text{FPdim}(D) \) is an algebraic integer \([7, \text{Proposition } 8.15]\), where \( \text{FPdim}(C) \) denotes the Frobenius-Perron dimension of the category \( C \). So the order of \( G(C) \) coincides with the Frobenius-Perron dimension of \( C_{\text{pt}} \) and hence it divides \( \text{FPdim}(C) \).

There is an action of the group \( G(C) \) on the set \( \text{Irr}(C) \) by left tensor multiplication. This action preserves Frobenius-Perron dimensions. For every \( X \in \text{Irr}(C) \), we use \( G[X] \) to indicate the stabilizer of \( X \) in \( G(C) \).

**Lemma 2.3.** Let \( C \) be a fusion category such that the order of \( G(C) \) is a power of a prime number \( p \). Suppose that the cardinality of \( \text{Irr}_a(C) \) is relatively prime to \( p \), for some \( a > 1 \). Then there is a simple object \( X \in \text{Irr}_a(C) \) such that \( G[X] = G(C) \).

**Proof.** Consider the action of \( G(C) \) on the set \( \text{Irr}_a(C) \) by left tensor multiplication. \( \text{Irr}_a(C) \) is a union of disjoint orbits under this action. Every orbit has length 1 or a power of \( p \). Since the cardinality of \( \text{Irr}_a(C) \) is relatively prime to \( p \), there exists at least one orbit with length 1, which implies our result. \( \square \)

2.3. **Extensions.** Let \( G \) be a finite group and \( e \in G \) be the identity element. A fusion category \( C \) is said to have a \( G \)-grading if there is a direct sum of full Abelian subcategories \( C = \bigoplus_{g \in G} C_g \), such that \( C_g^* = C_{g^{-1}} \) and the tensor product \( \otimes : C \otimes C \to C \) maps \( C_g \times C_h \) to \( C_{gh} \). The neutral component \( C_e \) of the grading is a fusion subcategory of \( C \). The grading \( C = \bigoplus_{g \in G} C_g \) is called faithful if \( C_g \neq 0 \) for all \( g \in G \). The fusion category \( C = \bigoplus_{g \in G} C_g \) is said to be a \( G \)-extension of \( D \) if the grading is faithful and \( C_e = D \). If \( C \) is a \( G \)-extension of a fusion category \( D \) then the Frobenius-Perron dimensions of \( C_g \) are equal for all \( g \in G \), and \( \text{FPdim}(C) = |G| \text{FPdim}(D) \) \([7, \text{Proposition } 8.20]\).

The adjoint subcategory \( C_{\text{ad}} \) of \( C \) is the fusion subcategory of \( C \) generated by the objects \( X \otimes X^* \), for all \( X \in \text{Irr}(C) \). By \([10]\), every fusion category \( C \) has a canonical faithful grading \( C = \bigoplus_{g \in \mathcal{U}(C)} C_g \) with neutral component \( C_e = C_{\text{ad}} \). The group \( \mathcal{U}(C) \) is called the universal grading group of \( C \).
2.4. Braided fusion categories. A braiding $c$ of a fusion category $C$ is a natural isomorphism $c_{X,Y}: X \otimes Y \to Y \otimes X$ for all $X,Y \in \text{Irr}(C)$, satisfying the hexagon axioms (see e.g. [13, Chapter XIII]). A fusion category $C$ is braided if it admits a braiding.

A braided fusion category $C$ is called symmetric if $c_{Y,X}c_{X,Y} = \text{id}_X \otimes Y$ for all objects $X,Y \in C$. A classical example of a symmetric fusion category is the category $\text{Rep}(G)$ of representations of a finite group $G$ equipped with its standard symmetric braiding $c_{X,Y}(x \otimes y) = y \otimes x$. A symmetric fusion category $C$ is said to be Tannakian if there exists a finite group $G$ such that $C$ is equivalent to $\text{Rep}(G)$ as a braided fusion category.

Let $D$ be a fusion subcategory of $C$. The M"uger centralizer of $D$ in $C$ is given by $D' = \{Y \in C | c_{Y,X}c_{X,Y} = \text{id}_X \otimes Y \text{ for all } X \in D\}$. The centralizer $D'$ is a fusion subcategory of $C$. The M"uger center of $C$ is the M"uger centralizer $Z_2(C) := C'$ of $C$ in itself. Clearly, $Z_2(C)$ is a symmetric fusion category. A braided fusion category $C$ is called nondegenerate if its M"uger center $Z_2(C)$ is trivial.

A braided fusion category is called premodular if it has a spherical structure. A modular category is a nondegenerate premodular category. Hence, an integral fusion category is modular if and only if it is nondegenerate, by [7, Proposition 8.23, 8.24]. A fusion category $C$ is integral if $\text{FPdim}(X)$ is a positive integer for any object $X \in C$. We only study integral fusion categories in this paper.

A braided fusion category is called slightly degenerate if its M"uger center $Z_2(C)$ is equivalent to the category $\text{SuperVec}$ of super vector spaces. The following lemma is taken from [15, Lemma 5.4].

**Lemma 2.4.** Let $C$ be a braided fusion category. Suppose that the M"uger center $Z_2(C)$ contains $\text{SuperVec}$. Let $g$ be the invertible object generating $\text{SuperVec}$, and let $X$ be any simple object of $C$. Then $g \otimes X$ is not isomorphic to $X$.

2.5. Morita equivalency. Let $C$ be a fusion category and let $M$ be an indecomposable right $C$-module category. Then the category $C_M^*$ of $C$-module endomorphisms of $M$, called the dual of $C$ with respect to $M$, is a fusion category. Two fusion categories $C$ and $D$ are Morita equivalent if $D$ is equivalent to $C_M^*$ for some indecomposable right $C$-module category $M$.

Let $C^{(0)} = C, C^{(1)} = C_{ad}$ and $C^{(i)} = (C^{(i-1)}_{ad})_{ad}$ for every integer $i \geq 1$. If the sequence $C^{(0)} \supseteq C^{(1)} \supseteq \cdots \supseteq C^{(i)} \supseteq \cdots$ converges to $\text{Vec}$ then $C$ is called nilpotent. Equivalently, $C$ is nilpotent if there is a sequence of fusion categories $C_0 = \text{Vec}, C_1, \cdots, C_n = C$ and a sequence of finite groups $G_1, \cdots, G_n$ such that $C_i$ is obtained from $C_{i-1}$ by a $G_i$-extension. If the groups $G_i$ can be chosen to be cyclic of prime order then $C$ is called cyclically nilpotent.
A fusion category $\mathcal{C}$ is called group-theoretical if and only if it is Morita equivalent to a pointed category. A fusion category is called solvable if it is Morita equivalent to a cyclically nilpotent fusion category.

**Lemma 2.5.** Let $\mathcal{C} = \sum_{g \in G} \mathcal{C}_g$ be a nilpotent fusion category. Suppose that $\text{FPdim}(\mathcal{C}_e)$ is square-free. Then $\mathcal{C}$ is pointed.

**Proof.** Let $X$ be a simple object of $\mathcal{C}$. By [10, Corollary 5.3], the square of $\text{FPdim}(X)$ divides $\text{FPdim}(\mathcal{C}_{\text{ad}})$. Since any faithful grading comes from a group epimorphism $U(\mathcal{C}) \to G$, we have that $\mathcal{C}_e \supseteq \mathcal{C}_{\text{ad}}$. It follows from the assumption that $\text{FPdim}(\mathcal{C}_{\text{ad}})$ is also square-free, which implies that $\text{FPdim}(X)$ must be 1. Thus $\mathcal{C}$ is pointed. □

2.6. **Equivariantizations and de-equivariantizations.** Let $\mathcal{C}$ be a fusion category and let $G$ be a group acting on $\mathcal{C}$ by tensor autoequivalences. A $G$-equivariant object in $\mathcal{C}$ is a pair $(X, (u_g)_{g \in G})$, where $X$ is an object of $\mathcal{C}$ and $u_g : T_g(X) \to X$ is an isomorphism such that

$$
\gamma_{g,h}(X) \downarrow \downarrow
\begin{array}{c}
T_{gh}(X) \\
\downarrow u_{gh} \\
X
\end{array}
\downarrow
\begin{array}{c}
T_g(T_h(X)) \\
\downarrow T_g(u_h) \\
T_g(X)
\end{array}
\downarrow
\begin{array}{c}
u_g \\
\uparrow
\end{array}
\gamma_{g,h}(X)
$$

commutes for all $g, h \in G$, where $\gamma_{g,h} : T_g(T_h(X)) \to T_{gh}(X)$ is the natural isomorphism associated to the group action. The morphisms of equivariant objects are defined to be the morphisms in $\mathcal{C}$ commuting with $u_g, g \in G$.

Equivariant objects in $\mathcal{C}$ form a fusion category. This category is called the $G$-equivariantization of $\mathcal{C}$ and is denoted by $\mathcal{C}^G$.

There is a procedure opposite to equivariantization. Let $\mathcal{C}$ be a fusion category and let $\text{Rep}(G) \subseteq Z(\mathcal{C})$ be a Tannakian subcategory which embeds into $\mathcal{C}$ via the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$, where $Z(\mathcal{C})$ is the Drinfeld center of $\mathcal{C}$. Let $A = \text{Fun}(G)$ be the algebra of function on $G$. Let $\mathcal{C}_G$ be the category of left $A$-modules in $\mathcal{C}$. Then $\mathcal{C}_G$ is a fusion category and is called the de-equivariantization of $\mathcal{C}$ by $\text{Rep}(G)$.

There are canonical equivalences $(\mathcal{C}_G)^G \cong \mathcal{C}$ and $(\mathcal{C}^G)_G \cong \mathcal{C}$. Moreover, we have

$$
\text{FPdim}(\mathcal{C}^G) = |G| \text{FPdim}(\mathcal{C}) \quad \text{and} \quad \text{FPdim}(\mathcal{C}_G) = \frac{\text{FPdim}(\mathcal{C})}{|G|}.
$$

The reader is directed to [1] [5] [16] for a detailed study of equivariantizations and de-equivariantizations.

Let $\mathcal{C}$ be a braided fusion category, and $\text{Rep}(G) \subseteq \mathcal{C}$ be a Tannakian subcategory. The de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ by $\text{Rep}(G)$ is a braided $G$-crossed fusion category. This implies that there is (not necessarily faithful) grading $\mathcal{C}_G = \oplus_{g \in G} (\mathcal{C}_G)_g$. The category $\mathcal{C}_G$ is not braided in general. But the neutral component $(\mathcal{C}_G)_e$ of the associated $G$-grading of $\mathcal{C}_G$ is braided.
By [3] Proposition 4.56, \( C \) is nondegenerate if and only if \((C_G)_e\) is nondegenerate and the associated grading of \( C_G \) is faithful.

**Lemma 2.6.** [17] Theorem 7.2] Let \( \mathcal{C} \) be a braided fusion category. Then \( \mathcal{C} \) is group-theoretical if and only if it contains a Tannakian subcategory \( \text{Rep}(G) \) such that the de-equivariantization \( C_G \) by \( \text{Rep}(G) \) is pointed.

### 3. Group-theoreticality

Let \( p, q \) be prime numbers, and \( a, b \) be nonnegative integers. It was shown in [8] Theorem 1.6] that any fusion category of Frobenius-Perron dimension \( p^a q^b \) is solvable. In this section, we will extend this result under some restrictions. We first recall the classification of symmetric fusion categories [3].

Let \( G \) be a finite group and let \( u \in G \) be a central element such that \( u^2 = 1 \). Then the category \( \text{Rep}(G) \) has a braiding \( c_{X,Y}^u \) defined as follows:

\[
c_{X,Y}^u(x \otimes y) = (-1)^{mn} y \otimes x \text{ if } x \in X, y \in Y \text{ and } ux = (-1)^m x, uy = (-1)^n y.
\]

Let \( \text{Rep}(G,u) \) be the fusion category \( \text{Rep}(G) \) equipped with the braiding \( c_{X,Y}^u \). Deligne proved that any symmetric fusion category is equivalent to some \( \text{Rep}(G,u) \) [3].

**Lemma 3.1.** [5] Corollary 2.50] Let \( \mathcal{C} = \text{Rep}(G,u) \) be a symmetric fusion category. Then either \( \mathcal{C} \) is Tannakian or \( \text{Rep}(G/\langle u \rangle) \) is a Tannakian subcategory of Frobenius-Perron dimension \( \frac{1}{2} \text{FPdim}(\mathcal{C}) \). In particular, if \( \text{FPdim}(\mathcal{C}) \) is odd then \( \mathcal{C} \) is Tannakian.

Let \( \mathcal{C} \) be a modular category, and \( \mathcal{D} \subseteq \mathcal{C} \) be a fusion subcategory. Then \( \text{FPdim}(\mathcal{D}) \text{FPdim}(\mathcal{D}') = \text{FPdim}(\mathcal{C}) \). Let \( \mathcal{C} \) be a modular category. Then the universal grading group \( \mathcal{U}(\mathcal{C}) \) is isomorphic to the group \( G(\mathcal{C}) \) [10] Theorem 6.2]. In particular, \( \text{FPdim}(\mathcal{C}_{pt}) = |\mathcal{U}(\mathcal{C})| \). Let \( \mathcal{C} \) be a modular category of Frobenius-Perron dimension \( p^a q^b \). Then the universal grading group \( \mathcal{U}(\mathcal{C}) \) is not trivial by [8] Theorem 1.6 and Proposition 4.5(iii)].

Let \( \mathcal{C} \) be an integral modular category of Frobenius-Perron dimension \( pq^n \), where \( p, q \) are distinct prime numbers and \( n \) is a positive integer. Let \( X \) be a simple object of \( \mathcal{C} \). By [5] Lemma 1.2], the square of \( \text{FPdim}(X) \) divides \( pq^n \). Hence, \( \text{FPdim}(X) \) can only be \( q^i \), \( i = 0, 1, \cdots, m \), where we write \( n = 2m \) if \( n \) is even and \( n = 2m + 1 \) if \( n \) is odd.

Let \( a_i, i = 0, 1, \cdots, m \), be the number of nonisomorphic simple objects of \( \mathcal{C} \) of Frobenius-Perron dimension \( q^i \). We then have an equation

\[
\sum_{i=0}^{m} a_i q^{2i} = pq^n.
\]
It follows that \( q^2 \) divides \( a_0 = \text{FPdim}(C_{pt}) = |\mathcal{U}(C)| \). Hence, the universal grading group \( \mathcal{U}(C) \) is not trivial. Let 
\[
\mathcal{C} = \bigoplus_{g \in \mathcal{U}(C)} C_g
\]
be the universal grading of \( \mathcal{C} \). For every \( g \in \mathcal{U}(C) \), we denote \( a^{g}_i, \ i = 0, 1, \cdots, m \), the number of nonisomorphic simple objects of Frobenius-Perron dimension \( q^i \) in \( C_g \). Since integral modular categories of dimension \( pq, pq^2 \) and \( pq^3 \) are pointed [15], we assume that \( n \geq 4 \) in the following context.

**Proposition 3.2.** With the above notations, one of the following holds:

1. \( \mathcal{C} \) is group-theoretical;
2. \( \mathcal{C} \) has a Tannakian subcategory of Frobenius-Perron dimension \( q^i \), where \( i \geq 2 \). In particular, if \( q \) is odd then the largest pointed fusion subcategory of \( \mathcal{C}_{ad} \) is Tannakian.

**Proof.** If \( \text{FPdim}(C_{pt}) = q^n \) then \( \text{FPdim}((C_{pt})') = p \). Then \( (C_{pt})' \) is pointed, and hence it is contained in \( C_{pt} \). It is impossible since \( p \) does not divide \( q^n \).

If \( \text{FPdim}(C_{pt}) = pq^{n-1} \) then every component \( C_g \) in the universal grading of \( \mathcal{C} \) has Frobenius-Perron dimension \( q \). It is impossible since \( C_g \) can not contain a noninvertible object of \( \mathcal{C} \).

If \( \text{FPdim}(C_{pt}) = q^{n-1} \) then every component \( C_g \) in the universal grading of \( \mathcal{C} \) has Frobenius-Perron dimension \( pq \). We have an equation
\[
\sum_{i=0}^{m} a^{g}_i q^{2i} = pq.
\]
This implies that \( a^{g}_0 \neq 0 \) and \( q \) divides \( a^{g}_0 \) for every \( g \in \mathcal{U}(C) \). This means that every \( C_g \) contains at least \( q \) nonisomorphic invertible objects. Therefore, there are at least \( q^n \) nonisomorphic invertible objects in \( \mathcal{C} \), a contradiction.

If \( \text{FPdim}(C_{pt}) = pq^i, \ i = 2, \cdots, n-2 \), then \( \text{FPdim}(C_{ad}) = q^{n-i} \). By the classification of fusion categories [7, Theorem 8.28], \( C_{ad} \) is a nilpotent fusion category, so is \( \mathcal{C} \) itself. It follows from [4] Theorem 6.1 that \( \mathcal{C} \) is group-theoretical.

In the rest of the proof, we consider the case when \( \text{FPdim}(C_{pt}) = q^i, \ i = 2, \cdots, n-2 \). In this case \( \text{FPdim}(C_{ad}) = pq^{n-i} \). Write \( \mathcal{D} := (C_{ad})_{pt} \).

Then \( \text{FPdim}(\mathcal{D}) = q^i \), where \( 2 \leq j \leq i, n-i \). From \( \mathcal{D} \subseteq C_{pt} \) and \( C_{ad} = (C_{pt})' \) [10, Corollary 6.8], we have that \( \mathcal{D} \subseteq \mathcal{D}' \). Hence, \( \mathcal{D} \) is a symmetric fusion subcategory. We shall prove that either \( \mathcal{D} \) is a Tannakian subcategory of \( \mathcal{C} \) or \( \mathcal{D} \) contains a Tannakian subcategory of Frobenius-Perron dimension \( q^i \), where \( i \geq 2 \).

If \( q \) is odd then \( \mathcal{D} \) is Tannakian, so we are done. We assume below that \( q = 2 \) and \( \mathcal{D} \) contains the category of super vectors \( \mathcal{E} \). Let \( 1 \neq g \in \mathcal{E} \) be the unique invertible object which generates \( \mathcal{E} \) as a symmetric category. Then \( g \otimes X \not\cong X \) for every simple object of \( C_{ad} \).
Let $a_i^l$, $l = 0, 1, \cdots, m$, be the number of nonisomorphic simple objects of Frobenius-Perron dimension $2^i$ in $\mathcal{C}_{ad}$. Then

$$\sum_{l=0}^{m} a_i^l 2^l = 2^{n-i}p. \tag{3.3}$$

This implies that $2^{2k}$ divides $\text{FPdim}(\mathcal{D})$, where $k$ is the smallest index $l$ such that $a_i^l \neq 0$.

Let $G$ be a finite group of order $\text{FPdim}(\mathcal{D})$ and $u \in G$ be a central element of order 2 such that $\mathcal{D} \cong \text{Rep}(G, u)$ as symmetric categories. If $\text{FPdim}(\mathcal{D}) \geq 2^3$ then $\text{FPdim}(\text{Rep}(G/\langle u \rangle)) \geq 4$ is a Tannakian subcategory of $\mathcal{C}$. So we are one. Now we may assume that $\text{FPdim}(\mathcal{D}) = 4$. In this case $a_1^0 \neq 0$ and equation (3.3) shows that $a_1^1$ is odd. By Lemma 2.3 there exists a simple object $Y$ of Frobenius-Perron dimension 2 such that $h \otimes Y \cong Y$ for all $h \in G(\mathcal{C}_{ad})$. This is a contradiction. \hfill $\square$

**Theorem 3.3.** Let $\mathcal{C}$ be an integral fusion category of Frobenius-Perron dimension $pq^i$, where $p, q$ are distinct prime numbers, $i = 0, 1, 2, 3, 4, 5$. Then $\mathcal{C}$ is group-theoretical.

**Proof.** Obviously, if $i = 0, 1$ then $\mathcal{C}$ is pointed. By [18], if $i = 3, 4$ then $\mathcal{C}$ is pointed [18]. By [2], if $i = 4$ then $\mathcal{C}$ is group-theoretical. So it is enough to consider the case when $i = 5$ in the following context.

By the proof of Proposition 3.2 it suffices to consider the case when $\text{FPdim}(\mathcal{C}_{pt}) = q^2$ or $q^3$. A direct calculation shows that $\text{FPdim}(\mathcal{C}_{ad})_{pt} = q^2$ in both cases. So the question is reduced to considering the case when $(\mathcal{C}_{ad})_{pt} \cong \text{Rep}(G)$ is a Tannakian subcategory of $\mathcal{C}$, where $G$ is an Abelian group of order $q^2$.

Let $\mathcal{C}_G$ be the de-equivariantization of $\mathcal{C}$ by $\text{Rep}(G)$. Then $\mathcal{C}_G$ is a $G$-extension of a fusion category $\mathcal{F}$. Since $\text{FPdim}(\mathcal{F}) = pq$ and $\mathcal{F}$ is modular, $\mathcal{F}$ must be a pointed. So $\mathcal{C}_G$ is nilpotent. Further, Lemma 2.5 shows that $\mathcal{C}_G$ is pointed. Therefore, $\mathcal{C}$ is group-theoretical by Lemma 2.6. \hfill $\square$

### 4. Nilpotency

Recall from Subsection 2.1 that a pointed fusion category has the form $\text{Vec}(G, \omega)$, where $G$ is a finite group and $\omega \in H^3(G, k^\times)$ is a cohomology class. Let $H \subseteq G$ be a subgroup such that $\omega|_{H \times H \times H}$ is cohomologically trivial. Let $\psi \in C^2(H, k^\times)$ be a 2-cochain such that $\omega|_{H \times H \times H} = d\psi$. The twisted group algebra $k^\psi(H)$ is an associative unital algebra in $\text{Vec}(G, \omega)$. Denote $\mathcal{C} = \mathcal{C}(G, \omega, H, \psi)$ the category of $k^\psi(H)$-bimodules in $\text{Vec}(G, \omega)$. Then $\mathcal{C}$ is a fusion category with tensor product $\otimes_{k^\psi(H)}$ and unit object $k^\psi(H)$. A fusion category is group-theoretical if and only if it is of the form $\mathcal{C}(G, \omega, H, \psi)$ [7].
Proposition 4.1. Let $\mathcal{C}$ be a group-theoretical fusion category of Frobenius-Perron dimension $pq^n$, where $p < q$ are prime numbers and $n$ is a positive integer. Suppose that the order of $G(\mathcal{C})$ is a power of $q$. Then $\mathcal{C}$ is nilpotent.

Proof. Since $\mathcal{C}$ is group-theoretical, there exist a finite group $G$ of order $pq^n$, a subgroup $H \subseteq G$, a 3-cocycle $\omega \in H^3(G, k^\times)$ and a 2-cochain $\psi \in C^2(H, k^\times)$ such that $\mathcal{C}$ is equivalent to the category $\mathcal{C}(G, \omega, H, \psi)$ of $k^\psi(H)$-bimodules in the pointed fusion category $\text{Vec}(G, \omega)$.

Let $\hat{H} := \text{Hom}(H, k^\times)$ be the dual group of $H$. As explained in \cite[Theorem 5.2]{9}, there is an embedding $\hat{H} \hookrightarrow G(\mathcal{C})$, and hence subgroup $H$ is a $q$-group of $G$ by assumption. By Lemma 2.2, $H$ is contained in a Sylow $q$-subgroup $K$. Since $p < q$, $K$ is normal in $G$ by Lemma 2.1. Hence, the normal closure of $H$ in $G$ is a $q$-subgroup of $G$ which must be nilpotent. Thus $\mathcal{C}$ is nilpotent by \cite[Corollary 4.3]{9}.

\qed

Corollary 4.2. Let $\mathcal{C}$ be an integral modular category of Frobenius-Perron dimension $pq^n$, where $p < q$ are prime numbers and $n$ is a positive integer. Then $\mathcal{C}$ is group-theoretical if and only if it is nilpotent. In particular, an integral modular category of Frobenius-Perron dimension $2q^n$ is group-theoretical if and only if it is nilpotent.

Proof. One direction follows from \cite[Theorem 6.10.]{4}, and the other direction follows from Proposition 3.2 and Proposition 4.1.

\qed

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