A SURVEY ON THE SPECIAL FUNCTION

ANDREA OSSICINI

Abstract. The purpose of the work is to furnish a complete study of a discrete and special function, discovered by the author and named with the Arabian letter ﷾ (Shin) [1].

It includes three other papers, published in the international journal "Kragujevac Journal of Mathematics".

The methods, the techniques and the style of the demonstration inside such a work, are all related to Leonhard Euler, the very great Swiss mathematician.

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1 The letter ﷾ is the thirteenth letter of the Arabian alphabet.
THE SPECIAL FUNCTION ﺷ.

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Abstract. The purpose of the work is to furnish a study of a discrete and special function, discovered by the author and named with the Arabian letter ﺷ (Shin), produced by a family of functions. A fundamental theorem is enunciated: it connects the rational values of this family with a natural number; to this aim two rational-value functions will be created, with the characteristic, in the field of real positive numbers, to be piecewise continuous. We state not only the existence of a separation element, but we prove that this element is just formed by one only integer constant function, the value of which is equal to 2. We point out a hypothetical, subtle connection among the special function ﺷ, the Eulerian function Gamma and the second-order Eulerian numbers. It is finally proved that ﺷ is completely monotonic: this characteristic is peculiar for the functions that have considerable applications in different fields of pure and applied Mathematics.

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\(^{2}\) The letter ﺷ is the thirteenth letter of the Arabian alphabet.
1. FROM THE CONTINUOUS TO THE DISCRETE

Let’s consider the three following transcendental functions, determined by the letter $\theta$:

$$\theta[1] = \left(1 + \frac{1}{3k}\right)^{2k+1}$$
$$\theta[2] = \left(1 + \frac{1}{3k-1}\right)^{2k+1}$$
$$\theta[3] = \left(1 + \frac{1}{3k-2}\right)^{2k+1}$$

and where the variable $k$ will be restricted only in the real positive values, except some values, explained later.

The analytic study of the three functions permits to identify their progress, for the presence, in each of them, of two asymptotes: a horizontal asymptote, got by the limit for $k$ going to infinity, represented by the horizontal straight line, of height $e^{2/3} \approx 1.94734041$

A vertical asymptote, specific for every function, got by the following limits:

$$\lim_{k \to 0^+} \theta[1] = \infty ; \lim_{k \to \frac{1}{3}^+} \theta[2] = \infty ; \lim_{k \to \frac{2}{3}^+} \theta[3] = \infty$$

Besides by the calculus of the first derivative of each function and the study of its sign, it’s possible to verify that the $\theta$ functions are decreasing in their whole field of existence, more precisely, if we consider for the variable $k$ the whole positive real axis, we must exclude at least for the second and third function, respectively the intervals $\left(0, \frac{1}{3}\right]$ and $\left(0, \frac{2}{3}\right]$. In fact, if we derive the $\theta$ functions we get respectively:

$$\frac{d}{dk} \theta[1] = \theta[1] \cdot \left[2 \cdot \log \left(1 + \frac{1}{3k}\right) - \frac{2k+1}{k(3k+1)}\right] < 0 \quad \forall \ k \in \mathbb{R}^+$$
$$\frac{d}{dk} \theta[2] = \theta[2] \cdot \left[2 \cdot \log \left(1 + \frac{1}{3k-1}\right) - \frac{2k+1}{k(3k-1)}\right] < 0 \quad \forall \ k \in \mathbb{R}^+ \quad \text{and} \quad k > \frac{1}{3}$$
$$\frac{d}{dk} \theta[3] = \theta[3] \cdot \left[2 \cdot \log \left(1 + \frac{1}{3k-2}\right) - \frac{3(2k+1)}{(3k-1)(3k-2)}\right] < 0 \quad \forall \ k \in \mathbb{R}^+ \quad \text{and} \quad k > \frac{2}{3}$$

In the Fig. 1 we have represented three functions where, among other things, it’s evident that all of them have only one point of intersection with the horizontal straight line of height 2.

\(^3\) $e$ represents the Euler’s number.
That being stated, let’s proceed in the passage from the continuous to the discrete, by considering for \( k \), the only integer positive values; under these hypotheses it’s possible to verify:

\[
\begin{align*}
\text{for } 1 \leq k \leq 8 & \quad \Rightarrow \quad \text{for } 1 \leq k \leq 8 \\
\text{for } 9 \leq k \leq 16 & \quad \Rightarrow \quad \text{for } 9 \leq k \leq 16 \\
\text{for } 17 \leq k \leq 25 & \quad \Rightarrow \quad \text{for } 17 \leq k \leq 25 \\
\end{align*}
\]

\[\Rightarrow I_2; \quad \Rightarrow I_3\]

and therefore, in the discrete it’s possible to define some intervals\(^4\) \( I_\ell \) of integer values of the variable \( k \), in order to characterize some limitations of the values that have the above stated functions; in fact it’s possible to verify that for appropriate intervals \( I_\ell; \ell = 2, 3 \) the following are valid (see Fig. 2):

\[
\begin{align*}
\text{for } 9 \leq k \leq 16 & \quad \Rightarrow \quad I_2; \\
\text{for } 17 \leq k \leq 25 & \quad \Rightarrow \quad I_3
\end{align*}
\]

\(^4\) \( \ell \) shows the “ordinal number” of the interval \( I_\ell \).
In conclusion it’s possible to introduce a family of \( \mathcal{f} \) functions, by the definition of appropriate arcs, whose separation element is the horizontal straight line of height 2.

The construction-algorithm of the above stated family is therefore describable in the following way: we begin from the first algebraic expression of the \( \mathcal{f} \) function, that is 
\[
\left( 1 + \frac{1}{3k} \right)^{2k+1}
\]
starting to calculate by growing values of the integer positive variable \( k \), the corresponding rational values of \( \mathcal{f}[1] \); for the first 8 integer values of \( k \), the function has rational values greater than 2 and it’s therefore possible to associate to such values a bounded arc of the same function, represented in the discrete field by a sequence of rational numbers, each of them greater than 2.

After that we decrease of a unity the value of the denominator of the fraction inside the \( \mathcal{f} \) function, consequently we’ll get the algebraic expression of a new function, that is 
\[
\left( 1 + \frac{1}{3k-1} \right)^{2k+1}
\]
that we have previously identified with \( \mathcal{f}[2] \) and that we can define as the following of \( \mathcal{f}[1] \).
We’ll repeat, for it too, the same procedure and therefore we’ll calculate by growing values of the integer variable \( k \), but greater than the previous 8, the corresponding rational values of \( \frac{1}{AH} [2] \); also in this case for 8 integer values of \( k \), the function has rational values greater than 2 and therefore it’s possible to associate to such values a bounded arc of the same function, represented in the discrete field by a sequence of rational numbers, everyone greater than 2.

Now, for the same 8 integer values of \( k \), utilized for \( \frac{1}{AH} [2] \), it’s besides possible to verify that the \( \frac{1}{AH} [1] \) function has, on the contrary, rational values smaller than 2 and it’s therefore possible to associate to such values a bounded arc of the same function, represented in the discrete field by a sequence of rational numbers, all smaller than 2.

If we repeat the procedure and then we decrease, as usual, the value of the denominator of the fraction inside the \( \frac{1}{AH} \) function, we can build the \( \frac{1}{AH} [3] \) function, that so results the following of \( \frac{1}{AH} [2] \).

In this case, differently from the first two, exactly for 9 integer values of \( k \), greater than the previous 8, the function has rational values greater than 2 and therefore it’s possible to associate to such values a bounded arc of the same function, represented in the discrete field by a sequence of rational numbers, all of them greater than 2.

Here, for the same 9 integer values of \( k \), it’s possible to verify that the \( \frac{1}{AH} [2] \) function has, on the contrary, rational values smaller than 2, too, and it’s so possible to associate to such values a bounded arc of the same function, represented, in the discrete field, by a sequence of rational numbers, each of them smaller than 2.

Consequently about what described, if we consider the second interval of 8 integer values of \( k \), that is \( k =9,10,11,12,13,14,15,16 \), we can build two arcs, represented by two sequences of rational numbers: the first sequence formed by numbers greater than 2, because belonging to the arc of the \( \frac{1}{AH} [2] \) function, the second sequence formed by numbers smaller than 2, because belonging to the arc of the \( \frac{1}{AH} [1] \) function (see Fig. 2).

What shown is repeatable and it’s possible to experiment, while the integer variable \( k \) grows, the determination of two appropriate arcs, belonging to two following \( \frac{1}{AH} \) functions.

To simplify the use and the control of the described algorithm a vector \( \frac{1}{AH} \) function has been defined by the software product DERIVE\(^5\) Version 6 for WINDOWS, which allows the display of 11 consecutive values of a generic \( \frac{1}{AH} \) function, by two only parameters: an integer value of \( k \) and a further integer value, corresponding to the value of the interval (decreased of a unity) that we wish to study.

In APPENDIX it’s given, besides the macro function, which identifies the vector function, the result of a display got by its use.

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\(^5\) DERIVE is a powerful instrument of CAS (Computer Algebra System), spread by Texas Instruments.
2. THE FUNDAMENTAL THEOREM OF THE FUNCTION

Generalizing what described in the first paragraph and using the same method to build a couple of functions, identifiable by a precise interval \( I_\ell \), it’s possible therefore to enunciate the following **Fundamental Theorem**:

Let’s consider \( k, \ell \) natural numbers different from zero, \( I_\ell \) an interval of integer values of the variable \( k \), \( \Omega(I_\ell) \) an auxiliary integer function, then for each interval \( I_\ell \) it’s always possible the construction of a couple of functions with rational values, exclusively depending on \( k \) and that we denote with \( \mathfrak{F} \), so as the following boundary is always valid:

\[
\mathfrak{F}[k, \Omega(I_\ell - 1)] < 2 < \mathfrak{F}[k, \Omega(I_\ell)] \quad \text{with} \quad \Omega(I_\ell - 1) = \Omega(I_\ell) - 1 ; \forall I_\ell , k, \ell \in \mathbb{N}
\]

and for the generic \( \mathfrak{F} \) function it’s valid:

\[
\mathfrak{F}[k, \Omega(I_\ell)] = \left(1 + \frac{1}{3k - \Omega(I_\ell)}\right)^{2k+1} \quad (1)
\]

The **auxiliary** integer function \( \Omega(I_\ell) \), that really represents a growing “step function”, is defined, for the intervals of 8 or 9 following values of \( k \), in the following way:

- \( \Omega(I_1) = 0 \) for \( k=1, \ldots , 8 \)
- \( \Omega(I_2) = 1 \) for \( k=9, \ldots , 16 \); \( \Omega(I_3) = 2 \) for \( k=17, \ldots , 25 \); \( \Omega(I_4) = 3 \) for \( k=26, \ldots , 34 \)
- \( \Omega(I_5) = 4 \) for \( k=35, \ldots , 43 \); \( \Omega(I_6) = 5 \) for \( k=44, \ldots , 51 \); \( \Omega(I_7) = 6 \) for \( k=52, \ldots , 60 \)
- \( \Omega(I_8) = 7 \) for \( k=61, \ldots , 69 \); \( \Omega(I_9) = 8 \) for \( k=70, \ldots , 78 \); \( \Omega(I_{10}) = 9 \) for \( k=79, \ldots , 86 \)
- \( \Omega(I_{11}) = 10 \) for \( k=87, \ldots , 95 \); \( \Omega(I_{12}) = 11 \) for \( k=96, \ldots , 104 \). ; etc.

The sequence of growing values, defined for the **auxiliary** function \( \Omega(I_\ell) \), is therefore obtained to allow the individuation of two bounded arcs, belonging to two **following** \( \mathfrak{F} \) functions, one above the line of height 2 and the other below it.

The set of the 11 specific intervals, by which the function \( \Omega(I_\ell) \) is always positive and growing, represents a dominant characteristic of the family of \( \mathfrak{F} \) functions.

For a better precision we write the **numerical series** that identifies the extent of such intervals, in terms of consecutive values of the variable \( k \):

\[8, 9, 9, 9, 8, 9, 9, 8, 9, 9\] for a sum of 96 values of the variable \( k \).
As we can see, the first interval of 8 values has been neglected: in it the function $\Omega(I_1)$ has value zero, but such interval has a peculiarity, as exactly at interval 31 the above mentioned series is interrupted and an interval of 8 values takes place of one of 9, giving origin again to the same sequence: the following 11 values of such function are determined by the extent of intervals, typical of the series.

Now this event is regularly repeated and precisely every 40 and 51 intervals: for instance the first substitutions happen at the intervals: 31, 71, 122, 162, 213, 253, 293, 344, 384, 435, 475, 526, 566, 617, 657, 697, 748, 788, 839, 879, 930, 970, 1021, 1061, 1112, 1152 and, all of them, in advantage of the 8 value intervals.

The first real effect of this phenomenon on a procedure that allows to calculate exactly the last value of $k$, present in a determined interval, foreseeing a constant and complete repetition of the numerical series of 11 intervals, is exactly quantifiable at the interval 122, where it is practically possible to verify that the last value of $k$ inside it, results lower of a unity: this means that we could obtain the same result if at the interval 122, excluding the initial one, there were 120 integer intervals and one reduced of a unity, because of one only interruption.

Growing $k$, the substitutions immediately determinate a further but stable effect, for instance, at the interval 617 it’s possible to verify that the last value of $k$, present in the interval is of two unitsies inferior than the computable one and, going further it’s at the interval 1112, that it is possible to verify that the last value of $k$, present in the interval, is of three unities inferior than the computable one, and nothing short, for many following intervals it’s possible to observe that such reduction of a unity is noticeable exactly every 495 intervals for indeed 10 times (first case) and every 484 intervals only once (second case).

In the first case we can get the same results by hypothesizing the constant presence of 494 integral intervals and one, reduced of a unity and this because 11 substitutions produce inside 495 intervals 38 complete series of 96 values of $k$ and 11 series, reduced to only 61 values of $k$; under these conditions it’s immediate to verify that we obtain 4319 values of $k$, that are inferior of one only unity in relation with the possible values (4320) inside 495 intervals, in case there were exclusively 45 numeric complete series.

In the second case we can get the same results by assuming the constant presence of 483 integral intervals and one, reduced of a unity and this because 11 substitutions produce inside 484 intervals 37 complete series of 96 values of $k$ and 11 series, reduced to only 61 values of $k$; under these conditions it’s immediate to verify that we obtain 4223 values of $k$, that are inferior of one only unity in relation with the possible values (4224) inside 484 intervals, in case there were exclusively 44 numeric complete series.

\[\text{For precision it is the last 9, belonging to the second group of three consecutive 9, that is of the eighth term of the series.}\]
The peculiarity of the first interval of 8 values, for which the function $\Omega(I_\ell)$ is worthless, consists therefore in the fact that, as it was determined before the birth of the sequence, characteristic of the series, it is possible to think that at the origin, the number of integer intervals corresponds to 1.

We specify that such characteristics, even if relevant, are minor, both in relation to the continuous and repetitive presence of the above described series and in relation with the largeness of the single intervals, which never descend under 8 values for very great values of $k$.

The ratio of the number of integers inside the numeric series with the 11 intervals corresponds to $8.\overline{72}$ and this value diminishes in a little meaningful way while $k$ tends to infinity, if we consider the ratio of the value of a very great $k$ with its own belonging interval $I_\ell$: a sufficiently precise value is obtainable with the following expression $\frac{96}{11} - \frac{1}{494}$, where we deduce by the denominator of the second fraction the importance of the number 494.

Successively we give (Fig. 3) the graphs, related both to the family of functions, or better, to the set of the arcs belonging to them, and to the auxiliary function $\Omega(I_\ell)$ and successively in particular the development of the graph of a couple of functions, characteristic of a precise interval $I_\ell$ (Fig. 4) and of the pointers than put into evidence the behaviour, growing $k$.

This last behaviour results particularly evident, by examining the various displays, produced by DERIVE, in relation with the first 11 intervals, typical of the standard series, while for the greater values of $k$ it’s necessary to outdistance in an appropriate way the intervals on which to do a comparison to have a further confirmation of such behaviour.

Now if $k$ tends to infinity it’s possible to compute the limit towards which the generic function, that in this case represents the fusion, to infinity, of two arcs of following functions, that refer themselves to a hypothetical and extreme interval $I_\ell$.

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9 This means that every interval $I_\ell$ will never be empty.
Keeping in mind what noticed in the first and second case, previously described we can, first of all, calculate the two following values, by “excess” and “defect” of the ratio of a value of $k$, with the value of the integer function $\Omega(I_\ell)$, determined as a function of the $I_\ell$ interval, containing the same value of $k$:

1\textsuperscript{st} case: $\frac{k}{\Omega(I_\ell)} < \frac{4319}{495} = 8,725$

2\textsuperscript{nd} case: $\frac{k}{\Omega(I_\ell)} > \frac{4223}{484} = 8,725206611570247933884297$

These values, by considering the frequency of the two cases (10 times the first case and once the second case), permit to the approximate the ratio with:

$$\frac{k}{\Omega(I_\ell)} \approx \frac{10 \times 8,725 + 8,725206611570247933884297}{11} = 8,7252483512814091326488 =$$

$$= \frac{96}{11} - \frac{1}{493,9793814432989690721649} \approx \frac{96}{11} - \frac{1}{494}$$

In this way, according to the analysis of the progress of the various and following functions, we can furnish only an esteem of the ratio $\frac{k}{\Omega(I_\ell)}$.

If we want to determinate the effective value of such ratio, it’s necessary to compare the values of the two quantities for sufficiently great values of $k$.

Keeping in mind the algorithm described at the end of paragraph 1, which puts in evidence the continuous oscillation of the rational values of the following functions, in proximity of the integer number 2, we will compare the values of the ratio $\frac{k}{\Omega(I_\ell)}$ as $k$ grows with the value: $(3 - 2/\log 2)^{-1}$.

Fig. 4
More precisely, considering the values raised to the 18\textsuperscript{th} power \( (k_1 = 10^{18}, \text{ that is one quintillion}) \), raised to the 33\textsuperscript{rd} power \( (k_2 = 10^{33}, \text{ that is one decillion}) \) and indeed raised to the 63\textsuperscript{rd} power \( (k_3 = 10^{63}, \text{ that is one vigintillion}) \) it’s possible to observe that the real value of the ratio \( \Psi (k) = \frac{k}{\Omega (I_l)} \) tends to the previously shown value; in fact in the three various cases it’s possible to determinate in order what follows:

\[
\Rightarrow \Psi (k_1) = \frac{k_1}{\Omega (I_l)} - (3 - 2/\log 2)^{-1} < 5 \cdot 10^{-19};
\Rightarrow \Psi (k_2) = (3 - 2/\log 2)^{-1} < 5 \cdot 10^{-34};
\Rightarrow \Psi (k_3) = (3 - 2/\log 2)^{-1} < 5 \cdot 10^{-64}
\]

Definitely, going to the limit for \( k \), which tends to infinity we can expect that:

\[
\lim_{k \to \infty} \left( \frac{k}{\Omega (I_l)} \right) = (3 - 2/\log 2)^{-1}
\]

This result has obviously an immediate consequence in the calculation of the following limit:

\[
\lim_{k \to \infty} \left[ \text{Sh} [k, \Omega (I_l)] \right] = \lim_{k \to \infty} \left( 1 + \frac{1}{3k - \Omega (I_l)} \right)^{2k+1} = \lim_{k \to \infty} e^{\frac{2k+1}{3k-\Omega (I_l)}} = e^{\lim_{k \to \infty} \frac{2k+1}{3k-\frac{2}{\log 2}}} = e^{\log 2} = 2
\]

In conclusion, it happened that the natural number 2 (by the integer constant function \( \text{Y}=2 \)) can represent the separation element of two arcs, belonging to two \textit{following} \( \text{Sh} \) functions, which therefore can be identified with two contiguous classes\textsuperscript{10}, which tend to approach indefinitely.

Even if we observe that the validity of the \textit{fundamental} theorem is actually included in the construction algorithm\textsuperscript{11} of the family of \( \text{Sh} \) functions, keeping obviously in mind the characteristic of monotonicity (see previous paragraph and the following) of the generic \( \text{Sh} \) function, by a more appropriate notation, due to Iverson, further we will give the elements to get a strict proof of the theorem in the modern sense of the term.

Let’s extend the dependence of the integer step function \( \Omega (I_l) \) to the real field and let’s use the following definition:

\[
\Omega (x) = \min \{ k \in \mathbb{N}: S_{k+1} (x) \geq 2 \}; \quad x \in \mathbb{R}^+ \quad \text{and where} \quad S_k (x) = \left( 1 + \frac{1}{3x - k + 1} \right)^{2x+1}
\]

\textsuperscript{10} The contiguous classes are meant represented by two groups of rational non-integer numbers, greater and smaller than 2, separated by the rational integer number 2, which obviously doesn’t belong to any of the two classes.

\textsuperscript{11} Let’s observe that the algorithm determinates for the generic \( \text{Sh} \) function an inferior limit: it’s established an inferior extreme which corresponds to the integer number 2.
By simple algebraic passages we have that:

$$
\Omega (x) = \left\lceil 3x - \frac{1}{\frac{1}{2x+1} - 1} \right\rceil
$$

(2)

where \(\lceil x \rceil\) means the smallest integer, greater than \(x\) or equal to it.

So, from such a particular definition of \(\Omega (x)\) the fundamental theorem immediately derives.

In fact the function possesses the following explicit formula:

$$
S(x) = \left(1 + \frac{1}{3x-\Omega(x)}\right)^{2x+1} = \left(1 + \frac{1}{3x - \left\lceil 3x - \frac{1}{\frac{1}{2x+1} - 1} \right\rceil}\right)^{2x+1}
$$

and passing to the sequence we have:

$$
S(n) = \left(1 + \frac{1}{\frac{1}{\frac{1}{2n+1} - 1}}\right)^{2n+1}
$$

where \(\lfloor x \rfloor\) shows the greatest integer smaller or equal to \(x\).

From now on we will use the locution of “special function” for the discrete function \(\mathcal{F}[k, \Omega(\ell)]\), defined by the formula (1).

Extending the field of definition of the variable \(k\) to the real positive numbers, it’s possible to notice that such function, being represented by the union of continuous arcs (all above the straight line of height 2) is actually assimilable to a piecewise continuous function.

3. AN APPROACH OF THE SPECIAL FUNCTION \(\mathcal{F}\) WITH THE EULERIAN \(\Gamma\)

One of the fundamental characteristics of the Eulerian gamma function is a certain condition of monotonicity, which is the fact that the function, which can be intended as the most spontaneous extension of the factorial \(n! = 1 \cdot 2 \cdot 3 \cdots n\), out the field of the natural numbers, is logarithmically convex.

\(^{12}\) Chapter II, [5].
If we consider the expression (1) and we calculate its first derivative, we have:

\[
\frac{d}{dk} \Psi[k, \Omega(I_\ell)] = \Psi[k, \Omega(I_\ell)] \cdot \left[ 2 \log \left( 1 + \frac{1}{(3k - \Omega(I_\ell))} \right) - \frac{3 \cdot (2k + 1)}{(3k - \Omega(I_\ell)) (3k - \Omega(I_\ell) + 1)} \right] \tag{3}
\]

∀ \( k \in \mathbb{N} \) and \( k > \frac{\Omega(I_\ell)}{3} \) it is always negative, therefore the function results monotonic and decreasing, and besides it’s interesting to observe that, growing the \( k \) value, its absolute value diminishes.

That being stated we can also verify that the special function \( \Psi \) has the same characteristic of monotonicity of the Eulerian function \( \Gamma \): it’s sufficient in fact to verify that the second derivative of the logarithm of the same \( \Psi \) function is positive.

Such result is immediate, in fact for the (3), being:

\[
\frac{d^2}{dk^2} \log \Psi[k, \Omega(I_\ell)] = \frac{d}{dk} \cdot \left( \frac{d}{dk} \Psi[k, \Omega(I_\ell)] / \Psi[k, \Omega(I_\ell)] \right)
\]

we have that:

\[
\frac{d^2}{dk^2} \log \Psi[k, \Omega(I_\ell)] = \frac{d}{dk} \left[ 2 \log \left( 1 + \frac{1}{(3k - \Omega(I_\ell))} \right) - \frac{3 \cdot (2k + 1)}{(3k - \Omega(I_\ell)) (3k - \Omega(I_\ell) + 1)} \right] = \left[ \frac{12 \cdot (3k \cdot \Omega(I_\ell) - \Omega(I_\ell)^2) + 6 \cdot (6k - \Omega(I_\ell)) + 9}{(3k - \Omega(I_\ell))^2 \cdot (3k - \Omega(I_\ell) + 1)^2} \right]
\]

Since, for definition we have \( k > \Omega(I_\ell) \), ∀ \( k \in \mathbb{N} \), we have in conclusion:

\[
\frac{d^2}{dk^2} \log \Psi[k, \Omega(I_\ell)] \succ 0
\]

This demonstrates that also the special function \( \Psi \) is logarithmically convex.

The special function \( \Psi \) is actually an exponential general function; that being stated, considering its base, defined by the analysis of its behaviour to infinity, we consider the following two particular expressions:

\[
\prod_{k=1}^{n} \left( 1 + \frac{\log 2}{2k} \right) \quad \text{and} \quad \sum_{k=1}^{n} \left( \frac{\log 2}{2k + \log 2} \right)
\]

In the previous paragraph we have practically put into evidence that for great values of \( k \) the rational non-integer term of the base of the special function \( \Psi \) tends (∼) to:

\[
\frac{1}{3k - \Omega(I_\ell)} \sim \frac{\log 2}{2k} \tag{4}
\]
consequently, as there is the following relation \( \Omega (I_\ell - 1) = \Omega (I_\ell) - 1 \), we have:

\[
\frac{1}{3k - \Omega (I_\ell - 1)} \sim \frac{\log 2}{2k + \log 2}
\]

The value \(^{13}\) if fixed for the base of the generic \( \Psi \) function, is very well fit to “interpolate” in the continuous field the union of the various arcs of the family of \( \Psi \) functions, got in the discrete field, above the straight line of height 2.

By this last value it’s possible to calculate and verify\(^{13}\) the following sizeable expression:

\[
\prod_{k=1}^{n} \left(1 + \frac{\log 2}{2k}\right) = \frac{\Gamma \left(n + 1 + \frac{\log 2}{2}\right)}{\Gamma (n + 1) \cdot \Gamma \left(1 + \frac{\log 2}{2}\right)}
\]

The value \(^{14}\), on the contrary, if fixed for the base of the generic \( \Psi \) function, is very well fit to “interpolate” in the continuous field the union of the various arcs of the family of \( \Psi \) functions, got in the discrete field, below the straight line of height 2.

By it, it’s possible to define the following partial sum:

\[
\sum_{k=1}^{n} \frac{\log 2}{2k + \log 2} = \frac{\log 2}{2} \sum_{k=1}^{n} \frac{1}{k + \log 2}
\]

From here, keeping in mind the formulae of recurrence of the logarithmic derivative of the gamma, function named digamma:

\[
\psi(x + 1) = \psi(x) + \frac{1}{x}, \quad \psi(x + n) = \psi(x) + \frac{1}{x} + \frac{1}{x + 1} + \frac{1}{x + 2} + \cdots + \frac{1}{x + n - 1}
\]

we have, giving x the value \(1 + \frac{\log 2}{2}\), the following identity:

\[
\sum_{k=1}^{n} \frac{\log 2}{2k + \log 2} = \frac{\log 2}{2} \left[ \psi \left(n + 1 + \frac{\log 2}{2}\right) - \psi \left(1 + \frac{\log 2}{2}\right) \right]
\]

Now, going to the limit for \(n \to \infty\), for the known properties of the digamma function, we have that the following series is divergent, that is:

\[
\sum_{k=1}^{\infty} \frac{\log 2}{2k + \log 2} = \infty
\]

That being stated, the importance of the result \(^{15}\) must be evaluated above all according to the following identity:

\[
\left(1 + \frac{\log 2}{2k}\right) = \left(1 - \frac{\log 2}{2k + \log 2}\right)^{-1}
\]

\(^{13}\) \( \Gamma (x + n) = x \cdot (x + 1) \cdot (x + 2) \cdots (x + n - 1) \cdot \Gamma (x) \) with \(n \in \mathbb{N}\) and \(x > 0\)
From this last identity we can deduce:

\[
\prod_{k=1}^{n} \left(1 - \frac{\log 2}{2k + \log 2}\right) = \frac{\Gamma (n+1) \cdot \Gamma \left(1 + \frac{\log 2}{2}\right)}{\Gamma \left(n + 1 + \frac{\log 2}{2}\right)} \tag{9}
\]

and considering the infinite product:

\[
\prod_{k=1}^{\infty} \left(1 - \frac{\log 2}{2k + \log 2}\right)
\]

diverges \[\Box\] to zero.

Such conclusion can be immediately verified also by applying the result (8), modified in the sign and considering the following:

**Theorem:** Supposed that \(-1 < a_n \leq 0\) and the series \(\sum a_n\) is divergent, then the infinite product \(\prod (1 + a_n)\) diverges to zero.

**Proof**

Assumed \(b_n = -a_n \Rightarrow \ 0 \leq b_n < 1\), as for \(0 \leq x < 1\) results \(1 - x \leq e^{-x}\), we can write:

\[
0 < P_n = \prod_{r=1}^{n} \left(1 + a_r\right) \leq e^{-(b_1 + b_2 + \ldots + b_n)} \tag{10}
\]

then if the series \(\sum a_n\) is not convergent and necessarily diverges to \(-\infty\), from (10) we have that \(P_n \to 0\) that is the infinite product diverges to zero.

Besides, keeping in mind that, supposed \(a\) and \(b\) non negative, the following relation is valid:

\[
\prod_{r=1}^{\infty} \frac{r \cdot (r + a + b)}{(r + a) \cdot (r + b)} = \frac{\Gamma (1 + a) \cdot \Gamma (1 + b)}{\Gamma (1 + a + b)}
\]

verifiable, applying the known formula by Euler:

\[
\Gamma (x) = \lim_{n \to \infty} \frac{n^x \cdot n!}{x \cdot (x + 1) \cdot (x + 2) \cdot \ldots \cdot (x + n)}
\]

and considering the limit for \(n \to \infty\) of the development of the following finished product:

\[
\prod_{r=1}^{n+1} \frac{r \cdot (r + a + b)}{(r + a) \cdot (r + b)} = \frac{(1 + a + b) \cdot (2 + a + b) \cdot \ldots \cdot (1 + a + b + n)}{(n^{1+a+b}) \cdot n!} \cdot \frac{(n^{1+a}) \cdot n!}{(1 + a) \cdot (2 + a) \cdot \ldots \cdot (1 + a + n)} \cdot \frac{(n^{1+b}) \cdot n!}{(1 + b) \cdot (2 + b) \cdot \ldots \cdot (1 + b + n)} \cdot \frac{n + 1}{n}
\]

\[\Box\] Chapter II, pag. 33, [6].
we can state that:

\[
\frac{\Gamma (1 + n) \cdot \Gamma \left(1 + \frac{\log 2}{2}\right)}{\Gamma \left(1 + n + \frac{\log 2}{2}\right)} = \prod_{r=1}^{\infty} \frac{r \cdot \left(r + n + \frac{\log 2}{2}\right)}{(r + n) \cdot \left(r + \frac{\log 2}{2}\right)}
\]

By exploiting the properties of the gamma function, or resorting to the definition of the famous hypergeometric series, it’s also possible to verify the following:

\[
\frac{\Gamma (1 + n) \cdot \Gamma \left(1 + \frac{\log 2}{2}\right)}{\Gamma \left(1 + n + \frac{\log 2}{2}\right)} = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{\log 2}{2}\right) \cdot \left(\frac{\log 2}{2} - 1\right) \cdots \left(\frac{\log 2}{2} - r\right)}{(1 + n + r) \cdot r!}
\]

In fact, adopting the classic symbolism for the hypergeometric function\(^\text{15}\):

\[
F(a, b; c; x) = 1 + \frac{a \cdot b}{c \cdot 1!} \cdot x + \frac{a \cdot (a + 1) \cdot b \cdot (b + 1)}{c \cdot (c + 1) \cdot 2!} \cdot x^2 + \cdots
\]

we can exploit the possibility to express it in terms of gamma functions, considering the hypergeometric integral, that is:

\[
F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b) \cdot \Gamma(c - b)} \cdot \int_0^1 t^{b-1} \cdot (1 - t)^{c-b-1} \cdot (1 - t \cdot x)^{-a} \, dx \quad (\Re c > \Re b > 0)
\]

From this, considering the limit for \(x \to 1^-\) (Abel’s theorem) and exploiting the properties of the Eulerian Beta function, we obtain the important relation of the hypergeometric Gauss’s series:

\[
F(a, b; c; 1) = \frac{\Gamma(c) \cdot \Gamma(c - a - b)}{\Gamma(c - a) \cdot \Gamma(c - b)} \quad (c \neq 0, -1, -2, \ldots, \Re (c - a - b) > 0)
\]

With the positions \(a = -\frac{\log 2}{2}, b = n\) and \(c = n + 1\), as all the required limitations are satisfied for the parameters, we easily reach the (11).

In conclusion, starting from the characteristic base of the generic \(\mathcal{J}\) function, defined by the analysis of its behaviour to infinity, we have also stated the following sizeable relation between an infinite product and a numeric series:

\[
F\left(-\frac{\log 2}{2}, n; n + 1; 1\right) = \prod_{r=1}^{\infty} \frac{r \cdot \left(r + n + \frac{\log 2}{2}\right)}{(r + n) \cdot \left(r + \frac{\log 2}{2}\right)} = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{\log 2}{2}\right) \cdot \left(\frac{\log 2}{2} - 1\right) \cdots \left(\frac{\log 2}{2} - r\right)}{(1 + n + r) \cdot r!}
\]

In the real field such result is graphically represented by Fig. 5, shown below, where it is evident that the \(x\) axis represents a horizontal asymptote for \(x \to +\infty\).

\(^{15}\) Chapter III, [5].
Besides, in the field of the non-negative real numbers, both the infinite product and the series are absolutely and uniformly convergent.

Fig. 5

4. THE SPECIAL FUNCTION $\mathcal{F}$ AND THE SECOND-ORDER EULERIAN NUMBERS

In the second paragraph we have shown that one of the most important characteristics in the construction of the family of $\mathcal{F}$ functions is related with the repetitive presence of 11 specific intervals $I_k$ of the integer variable $k$: we have indeed seen that the extension of such 11 intervals is always characterized by the following numeric series:

$$8, 9, 9, 8, 9, 9, 8, 9, 9$$

for a sum of 96 values of the variable $k$.

But it also happened that such series undergoes inside a precise number of integer intervals some interruptions, as 8 value intervals substitute some 9.

More precisely this event is regularly repeated every 40 and 51 intervals and it’s possible to observe that such intervals are aggregable so that to give origin to two groups: the first one formed by 484 intervals, produced by 7 interruptions at a distance of 40 intervals and 4 interruptions at a distance of 51 intervals, the second one formed by 495 intervals, produced by 6 interruptions at a distance of 40 intervals and 5 interruptions at a distance of 51 intervals.

\[16\] pag. 247-251,[3].
For each group of 484 intervals we besides note always 10 groups of 495 intervals: this corresponds on average to a number of 494 intervals for all the groups.

In conclusion the characteristics of such phenomenon, keeping also in mind what explained in the initial part of the second paragraph, allow correctly some estimates for the integer values of $k$, referred to an interval $I_\ell$, if we suppose the effective presence, even if virtual, of the following sequence of integer intervals:

$$1, 120, 494, 494, \ldots, 494, \approx 494^{17}, 494, \ldots, 494, \ldots, 494, \ldots, \approx 494 \text{ (ad infinitum).}$$

This interpretation is besides confirmed by the check carried out on the limit of the ratio $k/\Omega (I_\ell)$.

Now, as these numbers are closely connected to the nature of the complete and periodic sequence of the numerical series shown above, we have supposed that the numbers 1, 120 and 494 can belong to a category of special numbers; particularly a research pointed to verify this hypothesis, has implied the following curious discovery even if it is always a conjecture on an almost light and remote connection: the numbers 1, 120 and 494 belong to the family of the so called “second-order Eulerian numbers”, which result important for the tight connection that they have with Stirling’s numbers.

They satisfy similar recurrence to the characteristic one of the “ordinary Eulerian numbers” , which are useful, above all because they give a connection among ordinary powers and consecutive binomial coefficients.

To be clearer we show below the recurrence, which characterizes the second-order Eulerian numbers, showing how it’s possible, by it, to produce the three special numbers, typical of the special function $\mathcal{A}H$.

In fact we have:

$$\left\langle \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle \right\rangle = (k + 1) \left\langle \left\langle \begin{array}{c} n - 1 \\ k \end{array} \right\rangle \right\rangle + (2n - 1 - k) \left\langle \left\langle \begin{array}{c} n - 1 \\ k - 1 \end{array} \right\rangle \right\rangle ;$$

$$\left\langle \left\langle \begin{array}{c} n \\ 0 \end{array} \right\rangle \right\rangle = 1 \quad \forall n \neq 0 ; \quad \left\langle \left\langle \begin{array}{c} n \\ n \end{array} \right\rangle \right\rangle = 0 \text{ for } n \neq 0$$

$^{17} \approx 494$: with such notation we want to put into evidence that for great values of $k$ the estimable number of integer intervals is, in some very near case, to such value, in fact it’s as if every 495,000 intervals, two others of them, on average, undergo a reduction of one unity (from 9 to 8), but on the whole this further phenomenon is absolutely neglectable.
At the end, to be complete, we give besides the general formula of the second-order Eulerian numbers, which puts into evidence the connection with the binomial coefficients and with Stirling’s numbers, a representation of the second order Eulerian triangle (Fig. 6):

\[
\langle\langle n \rangle\rangle = \sum_{k=0}^{m} \binom{2n+1}{k} \left\{ \binom{n+m+1-k}{m+1-k} \right\} \cdot (-1)^k \quad \text{for } n > m \geq 0
\]

where for the binomial coefficients it is valid: \( \binom{n}{k} = \frac{n^k}{k! (n-k)!} \).
and for the Stirling’s numbers:

\[ \binom{n}{k} = k \cdot \binom{n-1}{k} + \binom{n-1}{k-1}. \]

| N  | \( \frac{N}{K} \) |
|----|------------------|
| 0  | 1                |
| 1  | 1 0 0 0 0 0 0 0 0 |
| 2  | 1 2 6 0 0 0 0 0 0 |
| 3  | 1 8 6 0 0 0 0 0 0 |
| 4  | 1 22 56 24 0 0 0 0 0 |
| 5  | 1 52 326 444 120 0 0 0 0 |
| 6  | 1 114 1451 4400 3708 720 0 0 0 |
| 7  | 1 240 5618 32120 56140 33984 5040 0 0 |
| 8  | 1 494 19950 195800 644020 785304 341136 40920 0 |
| 9  | 1 1004 67260 1062500 5766500 12440064 10262596 3733920 362880 |

**Fig. 6: Second Order Eulerian Triangle**

If we consider, for fundamental of the special function, the number \( \left\langle \left\langle \frac{8}{1} \right\rangle \right\rangle \) from \( \text{[12]} \) we have:

\[ \left\langle \left\langle \frac{8}{1} \right\rangle \right\rangle = \left[ \begin{array}{c} 17 \\ 0 \end{array} \right] \left\{ \begin{array}{c} 10 \\ 2 \end{array} \right\} - \left[ \begin{array}{c} 17 \\ 1 \end{array} \right] \left\{ \begin{array}{c} 9 \\ 1 \end{array} \right\} \]

but keeping in mind that for Stirling’s numbers are valuable the following identities for \( n > 0 \):

\[ \left\{ \begin{array}{c} n \\ 2 \end{array} \right\} = 2^{n-1} - 1 \quad \text{and} \quad \left\{ \begin{array}{c} n \\ 1 \end{array} \right\} = 1 \]

we finally have:

\[ \left\langle \left\langle \frac{8}{1} \right\rangle \right\rangle = 1 \cdot \left( 2^9 - 1 \right) - 17 \cdot 1 = 511 - 17 = 494 \]

5. FROM DISCRETE TO COMPLEX FIELD

In the previous paragraphs we have discussed on arguments by infinitesimal, asymptotic, numerical and combinatorial analysis to characterize the special function \( \mathcal{F} \).

It is known that in these fields the completely monotonic functions play a fundamental role.

We recall that a function \( f : I \to \mathbb{R} \) is said to be completely monotonic (c.m.) on a real interval \( I \), if \( f \) has derivatives of all orders on \( I \) which alternate successively in sign, that is:

\[ (-1)^n \cdot f^{(n)}(x) \geq 0 \quad \forall x \in I \quad \text{and} \quad \forall n \geq 0 \quad \text{with} \quad n = 0, 1, 2, 3, \ldots \]
In the recent past, various authors showed that numerous functions, which are defined in terms of gamma, polygamma and other special functions, as the hypergeometrical ones, are completely monotonic and used this fact to derive many interesting new inequalities.

We shall confine ourselves only to prove, in very simple way, that the special function also, even if it is for definition a piecewise continuous function, in the real field, it possesses the same property (c.m.).

Lemma. If \( f(x) \) and \( g(x) \) are c.m., then \( a \cdot f(x) + b \cdot g(x) \), where \( a \) and \( b \) are non-negative constants and \( f(x) \cdot g(x) \) are also c.m.

The proof of the first thesis is obvious, the second one is then easily seen from the Leibniz formula:

\[
\frac{d^n}{dx^n} \cdot [f(x) \cdot g(x)] = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n-k)}(x)
\]

**Theorem 1.** The special function \( [k; \Omega (I_\ell)] \) is completely monotonic in each \( I_\ell \).

**Proof:** The base function \( f(k) = 1 + \frac{1}{3k - \Omega (I_\ell)} \) is c.m. in each \( I_\ell \); in fact the n-th derivative of this function is:

\[
\frac{d^n}{dk^n} f(k) = \frac{(-1)^n \cdot 3^n \cdot n!}{[3k - \Omega (I_\ell)]^{n+1}} \Rightarrow (-1)^n \cdot f^{(n)}(k) \geq 0
\]

Thus by the Lemma, in case \( k \) is an integer, it’s obvious that also the special function \( [k; \Omega (I_\ell)] \) is c.m. in each \( I_\ell \); therefore it remains to prove Theorem 1 for the case when \( k \) is a real and positive number.

To obtain this it is necessary to the use the following obvious Theorem 2, which is a consequence of the Lemma, for composed functions.

**Theorem 2:** Let \( y = f(x) \) c.m. and let the power series \( \varphi(y) = \sum_{j=0}^{\infty} a_j y^j \) converge for all \( y \) in the range of the function \( y = f(x) \). If \( a_j \geq 0 \) for all \( j = 0, 1, 2, 3, \ldots \) then \( \varphi(f(x)) \) is c.m..

**Corollary:** If \( f(x) \) is c.m., then \( e^{f(x)} \) is c.m..

In particular, as the special function \( [k; \Omega (I_\ell)] \) is equal:

\[
[k; \Omega (I_\ell)] = e^{(2k+1-\log(1+\frac{1}{3k-\Omega (I_\ell)})}
\]

it is c.m. in each \( I_\ell \).

---

18 pag. 445-460, [1].
19 We must also remember the (2) without giving up the characterization, determined for each interval \( I_\ell \).
An interesting exposition of the main results on completely monotonic functions is given in Widder’s work.\(^{20}\)

That being stated, with some limitations due to the nature of the special function $\phi$ in the real field, we can describe a method for estimating the same function in the complex field with an important improper integral.

The $\phi$ function results to be a piecewise continuous function because of the presence of the step function $\Omega (I_\ell)$, that is discontinuous, and actually its complete monotonicity has been proved for all the closed intervals $I_\ell$.

Passing to the interval $I = [0, \infty)$ the continuity is not guaranteed and therefore the application of the following Hausdorff-Bernstein-Widder\(^{21}\) Theorem, must be done carefully, or it can be limited to characterize the behaviour of the special function $\phi$ at the origin and infinity.

**Theorem 3:** A necessary and sufficient condition for the function $f (s)$ in order to be completely monotonic in the interval $I = [0, \infty)$ is that:

$$f (s) = L_s [F (t)] = \int_0^\infty e^{-st} dF (t) \quad (13)$$

where $F (t)$ is non-decreasing and the integral converges in the interval $I = [0, \infty)$.

The \(^{13}\) represents the transformation of Laplace-Stieltjes of a locally and absolutely continuous function, with real values, in the interval $I = [0, \infty)$.

Now, keeping in mind the relation existing in such case between Laplace-Stieltjes transform $L_s$ and *ordinary* \(^{22}\) Laplace transform $L$ (observing that we can suppose $F (0) = 0$):

$$L_s [F (t)] = s \cdot \int_0^\infty e^{-st} F (t) \cdot dt = s \cdot L [F (t)] \quad (14)$$

we can determine the expression of the function $F (t)$ by the inverse Laplace transform.

From \(^{13}\) and \(^{14}\) we get $\Phi (s) = \frac{f (s)}{s} = L [F (t)]$ and successively we need to face the calculus of integral of the type \(^{23}\):

$$F (t) = L^{-1} [\Phi (s)] = \text{v.p.} \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} e^{ts} \Phi (s) \cdot ds \quad (14.1)$$

with $t \succ 0$, $\Phi (s)$ holomorphic function in the half-plane $\Re (s) \succ 0$ and $x_0$ arbitrary real positive number.

\(^{20}\) Chapter IV, pag. 160-161, [7].

\(^{21}\) Chapter IV, pag. 168, [2].

\(^{22}\) Chapter IV, [2].

\(^{23}\) pag. 168, [2].
In general the calculus of Bromwich’s integral (14.1), that is defined as a *Cauchy principal value*, we can only do it numerically, applying quadrature formulae, but in our case \( f(s) = \Phi(s) \), being the \( \Phi(s) \) function *piecewise analytic*, we could prove by the direct calculus of the considered integral (see the following paragraph), that \( F(t) = 2 \cdot \Theta[t] \) and \( \Theta[t] \) represents the unit step function or Heaviside’s function \( \Theta[t < 0] = 0; \Theta[t > 0] = 1 \) and in (13) we’ll have \( dF(t) = 2 \cdot d\Theta[t] = 2 \cdot \delta(t) \, dt \), with \( \delta(t) \) that is the distribution of Dirac.

6. THE BEHAVIOUR OF THE SPECIAL FUNCTION \( \Phi \) AT THE ORIGIN AND INFINITY

In the previous paragraph we have stated the following approximation “\( \approx \)”, in terms of Laplace transform:

\[
\Phi(s) = \frac{\Phi(s)}{s} \approx L[F(t)] = L[2 \cdot \Theta(t)] = \int_0^\infty e^{-st}2 \cdot \Theta(t) \, dt = \frac{2}{s}
\]

The approximation is essentially originated by neglecting the point of discontinuities of the first kind of the special function \( \Phi \), between an interval \( I_\ell \) and the following \( I_{\ell+1} \) as far as the interval \( I [0, \infty) \).

Said that to calculate the Bromwich’s integral (14.1), we consider the path of integration rightly deformed, as we can see in the Fig. 7.

\[\text{Fig. 7: The path of integration}\]

\[\text{\[24\] pag. 30-35, [2].}\]
\[\text{\[25\] pag. 491-492, [4].}\]
Φ (s) is a multiple-valued analytic function; in the interest of obtaining a single value mapping we consider the principal branch of the power in \( \mathcal{H} (s) \), that allows us to treat a holomorphic branch of the same Φ (s) function.

In its field of single-values, the function Φ (s) possesses a simple pole at \( s = 0 \) and two branch points at \( s = -1/3 \) and \( s = -2/3 \); let’s observe, in fact, that in the interval \([-1, 0]\) the special function \( \mathcal{H} (s) \) is:

\[
\eta (s) = \left( 1 + \frac{1}{3s+1} \right)^{2s+1} = e^{(2s+1) \cdot \log(1+\frac{1}{3s+1})}
\]

A cut, joining the two branch points, would prevent \( s \) to circulate around them, and the special function \( \mathcal{H} (s) \) can be treated as a piecewise holomorphic function.

In conclusion, defined with \( C \) the boundary (shown in Fig. 7), with \( C_R \) the circular arc of radius \( R \), with \( B \) the vertical line with \( \Re (s) = x_0 \), with \( L \) the boundary of the cut branch, formed by \( D_\varepsilon \) and \( E_\varepsilon \), that is the semi-circles of radius \( \varepsilon \) capping the ends of the branch cut and by \( L^+ \) and \( L^- \), lines above and below, we’ll have:

\[
\frac{1}{2\pi i} \int_{x_0+i\infty}^{x_0-i\infty} e^{i\Phi (s)} \cdot ds = \frac{1}{2\pi i} \int_C \cdots - \int_{C_R} \cdots - \int_{L^+} \cdots - \int_{L^-} \cdots - \int_{D_\varepsilon} \cdots - \int_{E_\varepsilon} \cdots \] (15)

The first integral in the second member of (15), by the residue theorem, is \( 2 \cdot \Theta (t) \), as the first order pole, at the origin, gives:

\[
\lim_{s \to 0} \eta (s) = 2
\]

The second integral with \( s = R \cdot e^{i\theta} \) is:

\[
\int_{C_R} e^{i\Phi (s)} \cdot ds = \int_{\pi/2}^{\pi/2 - \delta} d\theta + \int_{\pi/2 + \delta}^{\pi/2} d\theta + \int_{\delta}^{3\pi/2} d\theta
\]

and therefore it vanishes: in fact the first and third integral in the second member of (16) vanish as \( R \to \infty \) by the maximum modulus bound and the second integral vanished by Jordan’s Lemma.

The third and fourth integral in the second member of (15) cancel each other along the paths \( L^+ \) and \( L^- \): in fact their values, calculated along their opposite paths, eliminate each other.

In the end, the last two integrals of (15), by the maximum modulus bound, vanish as \( \varepsilon \to 0 \).

Now, from the asymptotic behaviour of the function \( F (t) \), so determined, we can therefore deduce asymptotic properties of the correspondent Laplace transform, that is to use the following Abelian theorems (initial and final value theorem):

**Theorem:** Let \( F \) be a transformable function and let’s suppose that the \( \lim_{t \to \infty} F (t) \) exists, then the \( \lim_{s \to 0} [ s \cdot \Phi (s) ] \) exists, too (let’s suppose, for convenience \( s \in \Re \)) and is:

\[
\lim_{s \to 0} [ s \cdot \Phi (s) ] = \lim_{t \to \infty} F (t)
\]

If the \( \lim_{t \to 0^+} F (t) \) exists, then the \( \lim_{s \to \infty} s \cdot \Phi (s) \) exists, too and is (with \( s \in \Re \)):

\[
\lim_{s \to \infty} [ s \cdot \Phi (s) ] = \lim_{t \to 0^+} F (t)
\]
In our case we have seen that \( F(t) = 2 \cdot \Theta(t) \), with \( \Theta(t) \), that is Heaviside’s function, and therefore we have that:

\[
\lim_{t \to \infty} F(t) = \lim_{t \to \infty} 2 \cdot \Theta(t) = 2
\]

and also:

\[
\lim_{t \to 0^+} F(t) = \lim_{t \to 0^+} 2 \cdot \Theta(t) = 2
\]

The existence and the calculus of such limits, in (17) and (18), give the following results (between them a further confirm of the fundamental theorem):

\[
\lim_{s \to 0} \mathcal{H}(s) = 2 ; \lim_{s \to \infty} \mathcal{H}(s) = 2
\]

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APPENDIX

**DERIVE Version 6**: the vector function: \( S(k, \Omega) := \lim_{t \to k} \left(1 + \frac{1}{3t-\Omega}\right)^{2t+1} \)

\#1: \( V(k, \Omega) := \text{VECTOR}([t, o, S(t, \Omega)], t, k, k + 10) \)

*a display result:*

| V( k, Ω ) | VECTOR |
|-----------|--------|
| 09 1      | 2.0484148121729077984789748528464903812082646338028 |
| 10 1      | 2.0379259208387064562838079920238964441117176933744 |
| 11 1      | 2.029416167223667191636908626945714916029532064944 |
| 12 1      | 2.0223737073469397533461445484297949184415534016560 |
| 13 1      | 2.0164491799135882361365114303236301480590762568587 |
| 14 1      | 2.01139597471896635944584365668068886371655211482846 |
| 15 1      | 2.0070350457364044054984268130457357906298862812402 |
| 16 1      | 2.0032332566108411453651981972386971733090123101528 |
| 17 1      | 1.999889552662455165696859397607876311933058370586 |
| 18 1      | 1.9969258468076576081148471529242828805292201418873 |
| 19 1      | 1.9942808454379732420411582337304540085201659981375 |
Abstract. We describe a method for estimating the special function $\phi$, in the complex cut plane $A = \mathbb{C} \setminus (-\infty, 0]$, with a Stieltjes transform, which implies that the function $\phi$ is logarithmically completely monotonic. To be complete, we find a nearly exact integral representation. At the end, we also establish that $1/\phi(x)$ is a complete Bernstein function and we give the representation formula which is analogous to the Lévy-Khintchin formula.

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Keywords: special functions; completely monotonic functions; integral transforms; Bernstein functions.

1. INTRODUCTION

In [8] the author introduces a new special function, named with the Arabian letter $\phi$, and proves that this is logarithmically convex and completely monotonic for all the closed real intervals $I_\ell$ with $\ell = 1, 2, 3, \ldots$.

The explicit formula of the special function $\phi$, in the discrete field, is:

$$\phi[k, \Omega (I_\ell)] = \left(1 + \frac{1}{3k - \Omega (I_\ell)}\right)^{2k+1}$$

(19)

where is always valid the following boundary:

$$\phi[k, \Omega (I_\ell - 1)] < 2 < \phi[k, \Omega (I_\ell)] \quad \text{with} \quad \Omega (I_\ell - 1) = \Omega (I_\ell) - 1 ; \forall I_\ell, k, \ell \in \mathbb{N}$$

$\phi$ is the thirteenth letter of the Arabian alphabet.

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The auxiliary integer function $\Omega(I_\ell)$, that really represents a growing “step function”, is defined, for the intervals of 8 or 9 following values of $k$, in the following way:

- $\Omega(I_1) = 0$ for $k=1,\ldots,8$
- $\Omega(I_2) = 1$ for $k=9,\ldots,16$; $\Omega(I_3)=2$ for $k=17,\ldots,25$; $\Omega(I_4)=3$ for $k=26,\ldots,34$
- $\Omega(I_5) = 4$ for $k=35,\ldots,43$; $\Omega(I_6)=5$ for $k=44,\ldots,51$; $\Omega(I_7)=6$ for $k=52,\ldots,60$
- $\Omega(I_8) = 7$ for $k=61,\ldots,69$; $\Omega(I_9)=8$ for $k=70,\ldots,78$; $\Omega(I_{10})=9$ for $k=79,\ldots,86$
- $\Omega(I_{11}) = 10$ for $k=87,\ldots,95$; $\Omega(I_{12})=11$ for $k=96,\ldots,104$. etc.

Successively we give (Fig. 1) the graphs, related to the families of $\mathcal{H}$ functions, that are $\mathcal{H}[k,\Omega(I_\ell)]$ and $\mathcal{H}[k,\Omega(I_\ell-1)]$, or better, to the set of the arcs belonging to them, and to the auxiliary function $\Omega(I_\ell)$.

Let’s extend the dependence of the integer step function $\Omega(I_\ell)$ to the real field and let’s use the following definition:

$$\Omega(x) = \min \{ k \in \mathbb{N} : S_{k+1}(x) \geq 2 \} ; \ x \in \mathbb{R}^+ \text{ and where } S_k(x) = \left(1 + \frac{1}{3x-k+1}\right)^{2x+1}$$
By simple algebraic passages we have, by a more appropriate notation, due to Iverson, that:

\[ \Omega (x) = \left\lceil 3x - \frac{1}{2^{2x+1} - 1} \right\rceil \]  

(20)

where \( \lceil x \rceil \) means the smallest integer, greater than \( x \) or equal to it.

Therefore the special function \( \mathcal{S} \) possesses the following explicit formula in the real field:

\[
\mathcal{S}(x) = \left(1 + \frac{1}{3x - \Omega(x)}\right)^{2x+1} = \left(1 + \frac{1}{3x - \left\lceil 3x - \frac{1}{2^{2x+1} - 1} \right\rceil}\right)^{2x+1}
\]

Extending the field of definition of the variable \( k \) to the real positive numbers, it’s possible to notice that such function, being represented by the union of continuous arcs (all above the straight line of height 2, see Fig.1) is actually assimilable to a piecewise continuous function.

That being stated, an important subclass of completely monotonic functions consists of the Stieltjes transforms defined as the class of functions \( f: (0, \infty) \rightarrow \mathbb{R} \) of the form:

\[
f(x) = a + \int_0^\infty \frac{d\mu(t)}{x + t}
\]

(21)

where \( a \geq 0 \) and \( \mu(t) \) is a nonnegative measure on \([0, \infty)\) with \( \int_0^\infty \frac{d\mu(t)}{1+t} \leq \infty \), see [2].

In the Addenda and Problems in ([1], p.127), it is stated that if a function \( f \) is holomorphic in the cut plane \( A = \mathbb{C} \setminus (-\infty, 0] \) and satisfies the following conditions:

(i) \( \Im f(z) \leq 0 \) for \( \Im(z) > 0 \)

(ii) \( f(x) \geq 0 \) for \( x > 0 \)

then \( f \) is a Stieltjes transform.
2. THE REPRESENTATION AS A STIELTJES TRANSFORM

In ([8], §6) the author characterizes the holomorphy (piecewise analytic) of the special function \( \text{sh}(z) \) in the cut plane \( B = \mathbb{C} \setminus \left[ -\frac{2}{3}, -\frac{1}{3} \right] \) and proves a remarkable result that implies:

\[
\lim_{|z| \to \infty} \text{sh}(z) = 2 \quad (z \in B) \quad (22)
\]

To prove that the harmonic function \( \Im(\text{sh}) \) satisfies \( \Im(\text{sh}) \leq 0 \) for \( \Im(z) > 0 \), we use the maximum principle for subharmonic functions, that can be found in ([4], p. 20), and show that \( \lim \sup \) of \( \Im(\text{sh}) \) at all boundary points including infinity is less than or equal to 0.

From (22) we conclude that this is true at infinity.

How, for definition \( \text{sh}(x) > 0 \) for \( x > 0 \); these last statements imply the result (21).

The constant \( a \) in (21) is given by:

\[
a = \lim_{x \to \infty} \text{sh}(x)
\]

and therefore for the fundamental theorem of the special function \( \text{sh} \) we have, see ([8], §2):

\[
a = \lim_{x \to \infty} \text{sh}(x) = 2
\]

In (21) \( \mu(t) \) is the limit in the vague topology of measures

\[
d\mu(t) = \lim_{y \to 0^+} -\frac{1}{\pi} \Im f(-t+iy) \, dt
\]

For \( z \in B = \mathbb{C} \setminus \left[ -\frac{2}{3}, -\frac{1}{3} \right] \) we have in the close interval \([-1, 0] \):

\[
\text{sh}(z) = \left(1 + \frac{1}{3z+1} \right)^{2z+1} = \exp \left( (2z+1) \cdot \text{Log} \left(1 + \frac{1}{3z+1} \right) \right)
\]

where \( \text{Log} \) denotes the principal branch of the logarithm.
Let $t \in \mathbb{R}$ and $z \in \mathbb{C}$ with $\Im(z) > 0$.

If $z$ tends to $t$, then for (19) and (23), results (with $\ell \in \mathbb{Z}$)

$$\mathcal{A}H(\mathcal{D}) = \begin{cases} 
(1 + \frac{1}{3\pi - \Omega(t)})^{2t+1} & \text{if } t > 0 \\
2 & \text{if } t = 0 \\
\exp\left[(2t + 1) \cdot \log\left(\frac{3t+2}{3t}\right) - k \cdot i \pi (2t + 1)\right] & \text{with } k = 1 \text{ if } -\frac{2}{3} < t < -\frac{1}{3} \\
& \text{and with } k = 0 \text{ if } -1 < t < -\frac{2}{3} \text{ or } -\frac{1}{3} < t < 0 \\
2 & \text{if } t = -1 \\
(1 + \frac{1}{3\pi - \Omega(t)})^{2t+1} & \text{if } t < -1 
\end{cases}$$

In particular then we obtain, if $y$ tends to $0^+$, for $t \in \mathbb{R}$:

$$-\frac{1}{\pi} \Im f(-t + iy) = \begin{cases} 
0 & \text{if } t \leq \frac{1}{3} \text{ or } t \geq \frac{2}{3} \\
\frac{1}{2\pi} (\frac{(3t-1)^2}{(3t-2)^2})^t \cdot \sin (2\pi t) \cdot \left\{\frac{3t-2}{3t-1} - \frac{3t-2}{3t-1}\right\} & \text{if } \frac{1}{3} < t < \frac{2}{3} 
\end{cases}$$

and using the identity (the Euler reflection formula):

$$\Gamma(\alpha) \cdot \Gamma(1 - \alpha) = \frac{\pi}{\sin(\alpha\pi)}$$

we are now in a position to determine the following nearly exact integral representation (Stieltjes transform):

$$\mathcal{A}^*(x) \approx 2 + \frac{1}{2} \cdot \int_{1/3}^{2/3} \frac{1}{\Gamma(2 \cdot t) \cdot \Gamma(1 - 2 \cdot t)} \cdot \left[\left(\frac{3t-2}{3t-1}\right) - \left(\frac{3t-2}{3t-1}\right)\right] \cdot \frac{dt}{(x + t)} \quad (24)$$

The approximation is essentially originated by neglecting the point of discontinuities of the first kind of the special function $\mathcal{A}^*$, in the real field, between an interval $I_\ell$ and the following $I_{\ell+1}$ as far as the interval $I = [0, \infty)$.

---

$^{28}$ $\mathbb{Z}$ denotes the relative integer set.

$^{29}$ In the discontinuity points $1/3$ and $2/3$ we respectively compute the limits of the real variable $t$ on the left and on the right (see Fig. 2).
Successively we give (Fig. 2: $x \rightarrow t$) the graphs (red color) related to the function $\bullet (t)$ in the real interval $I[-1,1]$: for $t = \frac{1}{2} \Rightarrow \bullet (t) = 0$ and this point is a flex point with oblique tangent.

![Graph of $\bullet (t)$]

**Fig. 2: the graph of $\bullet (t)$**

That being stated, we denote the set of completely monotonic functions with $C$.

Now, we also recall that a function $f: ]0, \infty[ \rightarrow ]0, \infty[$ is said to be logarithmically completely monotonic [5], if it is $C^\infty$ and

$$(-1)^k \cdot [\log f(x)]^{(k)} \geq 0 \quad for \ k = 1, 2, 3, ...$$

To simplify we denote the class of logarithmically completely monotonic functions by $\mathcal{L}$ and the set of Stieltjes transforms by $\mathcal{S}$.

In order to prove that the special function $\mathcal{G}(x)$ is logarithmically completely monotonic, we need the following lemma:

$$\mathcal{S} \setminus \{0\} \subset \mathcal{L}$$

---

$\bullet (t) = \frac{du(t)}{dt}$
This lemma is a consequence of the following result, established by Horn [6], that allows also to characterize the class of logarithmically completely functions as the infinitely divisible completely monotonic functions:

**Theorem 1:** For a function \( f : [0, \infty) \to [0, \infty) \) the following are equivalent:

1. \( f \in \mathcal{L} \);
2. \( f^\alpha \in \mathcal{C} \) for all \( \alpha > 0 \) and \( \alpha \in \mathbb{R} \);
3. \( \sqrt[n]{f} \in \mathcal{C} \) for all \( n = 1, 2, 3, \ldots \)

In fact, let \( f \in S \ (S \subset \mathcal{C}) \) and non-zero and let \( \alpha > 0 \), by Theorem 1 it is immediate to prove that \( f^\alpha \in \mathcal{C} \).

Now, writing \( \alpha = n + a \) with \( n = 0, 1, 2, \ldots \) and \( 0 \leq a < 1 \) we have \( f^\alpha = f^n \cdot f^a \), and using the stability of \( \mathcal{C} \) under multiplication and that \( f^a \in S \Rightarrow S \setminus \{0\} \subset \mathcal{L} \).

In conclusion for (24) also the special function \( \Theta(x) \in \mathcal{L} \).

3. THE CLASS OF BERNSTEIN FUNCTIONS

There is an important relation between the set \( S \) of Stieltjes transforms and the class \( \mathcal{B} \) of Bernstein functions.

We recall that a function \( f : (0, \infty) \to [0, \infty) \) is called a Bernstein function, if \( f \) has derivatives of all orders and \( f' \) is completely monotonic.

Now, if \( f \) is non-zero Stieltjes transform, then \( 1/f \) is a Bernstein function ([3], Prop. 1.3).

The special function \( \Theta(x) \in S \setminus \{0\} \) and this fact implies that \( 1/\Theta(x) \) is a Bernstein function.

In addition, using the identity:

\[
1/\Theta(x) = x/x \cdot \Theta(x)
\]

and remembering the following definition:

A Bernstein function \( \phi \) is called a special Bernstein function if the function \( \lambda/\phi(\lambda) \) is also a Bernstein function.

we can conclude that \( x \cdot \Theta(x) \) is a special Bernstein function.

The family of special Bernstein functions is very large, and it contains in particular the family of complete Bernstein functions (also known as operator-monotone functions, see [7], for instance).
Recall that a function $\phi : (0, \infty) \to \mathbb{R}$ is called a complete Bernstein function if there exists a Bernstein function $\eta$ such that:

$$\phi(\lambda) = \lambda^2 L[\eta(\lambda)], \quad \lambda > 0$$

where $L$ stands for the Laplace transform.

Now, using the main results about the special function $\mathcal{E}(x)$ ([8], §5 and §6) it is immediate to establish that $x \cdot \mathcal{E}(x)$ is a complete Bernstein function.

Note also that a function $f(x)$ is called a complete Bernstein function if, and only if,

$$f(x) = a + bx + \int_{0+}^{\infty} \frac{x}{t+x} \rho(dt)$$

where $a, b \geq 0$ and $\rho$ is a Radon measure on $(0, \infty)$ such that $\int_{0+}^{\infty} (1/(1+t)) \rho(dt) < \infty$.

From this one, we may deduce that the function $x \to f(x)/x$ is a Stieltjes transform [2].

This result was actually already obtained with the representation (24) of Stieltjes of the special function $\mathcal{E}$.

At the end, recall that the following conditions are equivalent:

(i) $\phi$ is a complete Bernstein function;

(ii) $\lambda/\phi(\lambda)$ is a complete Bernstein function.

This result implies also that the function $1/\mathcal{E}(x)$ is a complete Bernstein function and, remembering the standard form (25) and that the functions $1/[x \cdot \mathcal{E}(x)]$ and $1/\mathcal{E}(1/x)$ are Stieltjes transforms, it is easy and immediate to establish that the constant $a$ (killing rate) and $b$ (drift coefficient) are given by:

$$a = \lim_{x \to \infty} 1/\mathcal{E}(1/x) = 1/\mathcal{E}(0) = \frac{1}{2}$$

$$b = \lim_{x \to \infty} 1/[x \cdot \mathcal{E}(x)] = 0$$

and the following representation formula which is analogous to the Lévy-Khinchin formula:

$$1/\mathcal{E}(x) \approx \frac{1}{2} + \frac{1}{2} \cdot \int_{0+}^{\infty} \frac{1}{\Gamma(2 \cdot t) \cdot \Gamma(1 - 2 \cdot t)} \left(\frac{(3t - 2)^2}{(3t - 1)^2}\right)^t \left[\frac{3t - 1}{3t - 2} - \frac{3t - 1}{3t - 2}\right] \frac{x \cdot t}{(t + x)} dt$$
For the interplay between complete Bernstein functions and Stieltjes transforms we refer also to [9].

Finally, with an Euler-Venn diagram, we give the most important analytic properties of the special function \( \mathfrak{F} \):  

\begin{center}
Completely Monotonic Functions \( \sim \) vs \( \sim \) Bernstein Functions
\end{center}

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THE SPECIAL FUNCTION $\phi$, III.

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Abstract. We prove the following exact symbolic formula of the special function $\phi$, in the entire s-complex plane with the negative real axis (including the origin) removed, with a double Laplace transform:

$$\phi(s) = L\{2 \cdot \delta(t) + L\left\{\frac{1}{2\pi i}\left[\phi(t \cdot e^{-i\pi}) - \phi(t \cdot e^{i\pi})\right]\right\}\}$$

where $\delta(t)$ stands for the distribution of Dirac and $e$ represents the Euler’s number.

1. THE EXACT SYMBOLIC FORMULA

In [4] and [5] the author introduces and characterizes the discrete and special function $\phi$, in the complex field.

We recall that the explicit formula of the special function $\phi$, in the discrete field, is:

$$\phi[k, \Omega(I_{\ell})] = \left(1 + \frac{1}{3k - \Omega(I_{\ell})}\right)^{2k+1}$$

where it is always valid the following boundary:

$$\phi[k, \Omega(I_{\ell} - 1)] \prec 2 \prec \phi[k, \Omega(I_{\ell})] \quad \text{with} \quad \Omega(I_{\ell} - 1) = \Omega(I_{\ell}) - 1 \; \forall \; I_{\ell}, \; k, \ell \in \mathbb{N}$$

\[31\] The boundary can include the sign “=” if the integer variable $k$ goes towards zero or the infinity.
The auxiliary integer function \( \Omega(I_\ell) \), that really represents a growing “step function”, is defined, for the intervals of 8 or 9 following values of \( k \), in the following way:

- \( \Omega (I_1) = 0 \) for \( k=1,\ldots,8 \)
- \( \Omega (I_2) = 1 \) for \( k=9,\ldots,16 \); \( \Omega (I_3) = 2 \) for \( k=17,\ldots,25 \); \( \Omega (I_4) = 3 \) for \( k=26,\ldots,34 \)
- \( \Omega (I_5) = 4 \) for \( k=35,\ldots,43 \); \( \Omega (I_6) = 5 \) for \( k=44,\ldots,51 \); \( \Omega (I_7) = 6 \) for \( k=52,\ldots,60 \)
- \( \Omega (I_8) = 7 \) for \( k=61,\ldots,69 \); \( \Omega (I_9) = 8 \) for \( k=70,\ldots,78 \); \( \Omega (I_{10}) = 9 \) for \( k=79,\ldots,86 \)
- \( \Omega (I_{11}) = 10 \) for \( k=87,\ldots,95 \); \( \Omega (I_{12}) = 11 \) for \( k=96,\ldots,104 \); etc.

In particular the author proves in ([4], §§5-6) that the special function in the real field is completely monotonic for all the closed intervals \( I_\ell \) and therefore the following first approximation “≈” in the complex field:

\[
\mathcal{S}(s) \approx L_s[F(t)] = L_s[2 \cdot \Theta(t)] = \int_0^\infty e^{-st}d[2 \cdot \Theta(t)] = 2
\]  

(26)

where \( L_s \) denotes the Laplace-Stieltjes transform and \( \Theta[t] \) represents the unit step function or Heaviside’s function (\( \Theta[t<0] = 0 \); \( \Theta[t>0] = 1 \))

Now, we observe that the Laplace-Stieltjes transform is closely related to other integral transforms, including the Fourier transform and the Laplace transform.

In particular if \( g \) has derivative \( g’ \), then the Laplace-Stieltjes transform of \( g \) is the Laplace transform of \( g’ \).

Consequently, considering the derivative of Heaviside step function, from (26) we have:

\[
\mathcal{S}(s) \approx L[2 \cdot \delta(t)] = \int_0^\infty e^{-st}2\delta(t) \, dt = 2
\]  

(27)

where \( L \) denotes the ordinary Laplace transform and \( \delta(t) \) stands for distribution of Dirac.

In ([5], §2) the author find practically the following nearly exact integral representation:

\[
\mathcal{S}(x) \approx 2 + \frac{1}{2} \int_0^\infty \left\{ \frac{1}{\Gamma(2 \cdot t) \cdot \Gamma(1 - 2 \cdot t)} \left( \frac{(3t - 1)^2}{(3t - 2)^2} \right)^t \left[ \frac{3t - 2}{3t - 1} - \frac{3t - 2}{3t - 1} \right] \right\} \frac{dt}{(x+t)}
\]  

(28)

where \( \Gamma \) denotes the Eulerian gamma function (or the Eulerian integral of second kind).
The approximation is essentially originated by neglecting the point of discontinuities of the first kind of the special function $\mathcal{A}$, in the real field, between an interval $I_\ell$ and the following $I_{\ell+1}$ as far as the interval $[0, \infty)$. We recall that extending the field of definition of the integer variable $k$ of the discrete and special function $\mathcal{A}$ to the real positive number, it’s possible to notice that such function is actually assimilable to a piecewise continuous function.

On the contrary of equation (27), in the equation (28) there is a Stieltjes transform, that arises naturally as an iteration of the ordinary Laplace transform. In fact, if

$$f(x) = \int_0^\infty e^{-xt} \varphi(t) \, dt$$

where

$$\varphi(x) = \int_0^\infty e^{-xt} \psi(t) \, dt$$

then, changing the order of integrations in the double integral by appealing to Fubini’s Theorem, we have formally ([1], p. 127 and [8], p. 335):

$$f(x) = L\{L[\psi(t)]\} = \int_0^\infty e^{-xu} \, du \int_0^\infty e^{-ut} \psi(t) \, dt$$

$$= \int_0^\infty \psi(t) \, dt \int_0^\infty e^{-u(x+t)} \, du$$

Hence

$$f(x) = \int_0^\infty \frac{\psi(t)}{x+t} \, dt$$

This last equation we refer to as Stieltjes transform, or from another point view as the Stieltjes integral equation.

However, we shall usually be concerned with the more general case in which the integral equation (30) is replaced by a Stieltjes integral:

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t}$$

and in this form the equation was considered by T.J. Stieltjes in connection with his work on continued fractions [6].
In [28] the Stieltjes integral, that is analogous to (30), converges and then we have (see [8], Theorem 7b, p. 340):

\[
\lim_{\eta \to 0^+} \frac{f(-\xi - i \eta) - f(-\xi + i \eta)}{2\pi i} = \frac{\psi(\xi+) + \psi(\xi-)}{2}
\]  

(31)

for any positive \( \xi \) at which \( \psi(\xi+) \) and \( \psi(\xi-) \) exist.

For, simple computation gives:

\[
\frac{f(-\xi - i \eta) - f(-\xi + i \eta)}{2\pi i} = \frac{1}{\pi} \int_0^\infty \frac{\eta \psi(t)}{(t - \xi)^2 + \eta^2} dt
\]

The integral is known as Poisson’s integral for the half-plane or as Cauchy’s singular integral ([7], p. 30).

The result (31) may also be written symbolically as

\[
\frac{f(x \cdot e^{-i\pi}) - f(x \cdot e^{i\pi})}{2\pi i} = \psi(x)
\]  

(32)

In fact it is sufficient to compare the equation 11.8.4 in ([7], p. 318).

That being stated in ([4], §6) the author characterizes the holomorphy (piecewise analytic) of the special function \( \mathcal{H}(s) \) in the cut plane \( A = \mathbb{C} \setminus [-\frac{2}{3}, -\frac{1}{3}] \).

In the close interval \([-1, 0]\) results:

\[
\mathcal{H}(s) = \left(1 + \frac{1}{3s + 1}\right)^{2s+1} = \exp\left((2s + 1) \cdot \log\left(1 + \frac{1}{3s + 1}\right)\right)
\]  

(33)

where \( \log \) denotes the principal branch of the logarithm in the interest of obtaining a single value mapping.

In effects the special function \( \mathcal{H} \) possesses two branch points at \( s = -\frac{1}{3} \) and \( s = -\frac{2}{3} \) and therefore a cut, joining the two branch points, would prevent \( s \) to circulate around them, and the special function \( \mathcal{H}(s) \) can be treated as a holomorphic function.

We observe that, in our particular case, the results (32) and (31) with (33) can be written and compute in the following way:

\[
\frac{\mathcal{H}(x \cdot e^{-i\pi}) - \mathcal{H}(x \cdot e^{i\pi})}{2\pi i} = \lim_{y \to 0^+} \frac{\mathcal{H}(-x - iy) - \mathcal{H}(-x + iy)}{2\pi i}
\]
\[
\lim_{y \to 0^+} \left\{ \frac{\exp\left(2(-x-iy)+1\right) \cdot \log\left(1 + \frac{1}{3(-x-iy)+1}\right) - \exp\left(2(-x+iy)+1\right) \cdot \log\left(1 + \frac{1}{3(-x+iy)+1}\right)}{2\pi i} \right\}
\]

\[
= \frac{\sin \left(2\pi t\right)}{2\pi} \cdot \left(\frac{(3t-1)^2}{(3t-2)^2}\right) \cdot \left\{ \frac{3t-2}{3t-1} - \frac{3t-2}{3t-1} \right\}
\]

Finally, with (27), (28), (29), (34) and using the identity (the Euler reflection formula):

\[
\Gamma (\alpha) \cdot \Gamma (1 - \alpha) = \frac{\pi}{\sin (\alpha \pi)}
\]

we can prove, for the linearity property of the Laplace transformation, the following exact symbolic formula of the special function \(\mathcal{H}\), in the cut plane \(A = \mathbb{C} \setminus (-\infty, 0)\):

\[
\mathcal{H}(s) = L \left\{ 2 \cdot \delta(t) + L \left\{ \frac{1}{2\pi i} \cdot \left[ \mathcal{H}(t \cdot e^{-i\pi}) - \mathcal{H}(t \cdot e^{i\pi}) \right] \right\} \right\}
\]

where \(\delta(t)\) stands for the distribution of Dirac and \(e\) represents the Euler's number.

In conclusion we give also the three-dimensional graph of absolute value of the special function \(\mathcal{H}\) of a complex variable \(s = x + iy\) (Fig. 1) from DERIVE computer algebra system and in APPENDIX we explain the choice of the Arabian letter \(\mathcal{H}\).
2. ADDITIONAL ANALYTIC REMARKS

The beautiful symbolic formula (35) is in practice a consequence of the fact that Laplace-Stieltjes transform are often written as ordinary Laplace transform involving the distribution of Dirac, sometimes referred to as the symbolic impulse function $\delta(t)$.

Besides that the construction-algorithm ([4], §1) of the discrete and special function $\delta(t)$ had already evidenced the necessity of an asymmetric impulsive force in order to maintain the set of the points belonging to them all above the horizontal straight line of height 2.

Now, we observe that the asymmetrical impulse function $\delta_+(t)$ is more suitable for use in connection with the one-sided Laplace transformation that the symmetrical impulse function $\delta(t)$.

In effects in the formula (35) the presence of the symbolic impulse function $\delta_+(t)$ would be more appropriated, also because if one applies the Laplace transformation to the “definition” of the impulse function $\delta_+(t)$, one obtains the formal result:

$$L[\delta_+(t)] = 1$$

We recall that the asymmetrical impulse function $\delta_+(t)$ is defined ([3], see Sec. 21.9-6) by:

$$\int_{a+0}^{b} f(\xi) \delta_+(\xi - x) \, d\xi = \begin{cases} 0 & \text{if } x \prec a \text{ or } x \geq b \\ f(x+0) & \text{if } a \leq x \prec b \end{cases} \quad (a < b)$$

It is possible to write:

$$\delta_+(t) = \frac{d}{dx} U_+(x)$$

where $U_+(x)$ denotes the asymmetrical unit-step function:

$$U_+(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Successively we give the cartesian diagram (Fig. 2) of the unit-step function $U_+(x)$ and of the one related to approximation of the impulse function $\delta_+(t)$. 
In the papers [4], [5] and in this paper it has emerged all the importance of the tools like the Laplace transformation and the Stieltjes integral with certain facts from the theory of functions of a complex variable in order to characterizing the analytic properties of the completely monotonic functions.

In particular we recall that the Stieltjes integral is much used in Mechanics and Probability, since it unifies the treatment of the continuous and discrete (and mixed) distributions of mass or probability.

If \( \alpha(x) \) is piecewise differentiable, then \( d\alpha(x) = \alpha'(x) \, dx \), and the Stieltjes integral is simply in the following form (reduction of a Stieltjes integral to a Riemann integral):

\[
\int_{a}^{b} f(x) \, \alpha'(x) \, dx
\]

For instance a real and similar case to ours is the following: if \( \alpha(x) \) is a Heaviside step function, with point masses \( m_i \) at \( x = x_i \), then

\[
d\alpha(x_i) = \lim_{\varepsilon \downarrow 0} [\alpha(x_i + \varepsilon) - \alpha(x_i - \varepsilon)] = m_i, \quad \int_{a}^{b} f(x) \, d\alpha(x) = \sum_{i} m_i f(x_i)
\]

In this case the integration by parts is usual:

\[
\int_{a}^{b} f(x) \, dg(x) = f(x) \cdot g(x) \Big|_{a}^{b} - \int_{a}^{b} g(x) \, df(x)
\]
Suppose $\alpha(0) = 0$, $\alpha(x) = o(e^{cx})$ as $x \to \infty$, and that $\Re s \geq c$, then

$$\int_0^\infty e^{-sx} d\alpha(x) = s \int_0^\infty \alpha(x) e^{-sx} dx$$

(36)

The integral on the left side represents a Laplace-Stieltjes transform, while the integral on the right side is an ordinary Laplace transform.

More precisely (see [4], compare the equation 14) the second member of (36) is called $s$-multiplied Laplace transform.

However, we also recall the following result, due to S. Bernstein ([2], pp. 439-440), who was the starting point of many researches in the Probability theory:

**Theorem:** A function $\psi(s)$ on $(0, \infty)$ is the Laplace transform of a probability distribution $F(x)$:

$$\psi(s) = \int_0^\infty e^{-sx} dF(x)$$

if and only if it is completely monotone in $(0, \infty)$ with $\psi(0+) = 1$.

In particular, an immediate example for this beautiful theorem is the following function:

$$\psi(s) = \frac{\zeta(s)}{2}$$

Notice that the Dirac delta function may be interpreted as a probability density function and that the cumulative distribution function is the Heaviside step function.

It is known that if $X$ is a random variable, the corresponding probability distribution assigns to the interval $[a, b]$ the probability $\Pr[a \leq X \leq b]$; for example the probability that the variable $X$ will take a value in the interval $[a, b]$.

Now, the probability distribution of the variable $X$ can be uniquely described by its cumulative distribution function $F(x)$, which is defined by:

$$F(x) = \Pr[X \leq x]$$

for any $x \in \mathbb{R}$ and where the right-hand side represent the probability that the variable $X$ takes on a value less than or equal to $x$.

We observe that, in our case, the Heaviside step function is the cumulative distribution function of a random variable which is almost surely 0.
APPENDIX

The choice of the Arabian letter ١ is simply a consequence of the following four reasons:

• the exhaustion of Latin, Greek, Gothic, Jewish and other letters to name a new special function;

• the reference to the strokes of continuous curves (small arcs) that characterize the same function ([4], see Fig. 3);

• the three dots over the letter that one by one remember the three second-order Eulerian numbers: 1, 120, 494, inside the Eulerian triangle ([4], see Fig. 6);

• the meaning of such a letter, that in Al Karaji’s algebra (about 1000 A.D.) was declared as the unknown “par excellence”, that is absolutely comparable to our “x”, but also with the characteristic to form the only conjunction-“ring” between the world of the unknown (algebra) and the world of the known (arithmetics).

Actually it seems that the origin of the symbol ١, that is pronounced shin, is in the Ancient Egypt.

In fact the hieroglyph 𓊀𓊀𓊀, that is the Egyptian syllable sha, is similar and it represents papyrus plants along the Nile.

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