On the well-posedness of the inviscid SQG equation

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September 29, 2016

Abstract

In this paper we consider the inviscid SQG equation on the Sobolev spaces $H^s(\mathbb{R}^2)$, $s > 2$. Using a geometric approach we show that for any $T > 0$ the corresponding solution map, $\theta(0) \mapsto \theta(T)$, is nowhere locally uniformly continuous.

1 Introduction

The initial value problem for the inviscid SQG equation is given by

$$\partial_t \theta + (u \cdot \nabla) \theta = 0, \quad \theta(0) = \theta_0$$  \hspace{1cm} (1)

where $\theta : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a scalar function and $u$ is the velocity of the flow given by

$$u = \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) = \left( \begin{array}{c} -\mathcal{R}_2 \theta \\ \mathcal{R}_1 \theta \end{array} \right)$$

Here we denote by $\mathcal{R}_1, \mathcal{R}_2$ the Riesz transforms

$$\mathcal{R}_k = \partial_k(-\Delta)^{-1/2}, \quad k = 1, 2$$

Our main interest in this equation is because of its similarities with the incompressible Euler equation – take a look at [3, 4] for this relation and the physics of (1).

Because of the special structure of $u$, the flow is incompressible. One can prove local well-posedness of (1) in $H^s(\mathbb{R}^2)$ for $s > 2$ using the same techniques as for the incompressible Euler equation – see e.g. [13]. We will establish this using a geometric approach.
Theorem 1.1. The inviscid SQG equation is locally well-posed in the Sobolev spaces $H^s(\mathbb{R}^2), s > 2$.

In the following we denote for $s$ fixed and $T > 0$, the set $U_T \subseteq H^s(\mathbb{R}^2)$ to be the set of those initial values $\theta_0 \in H^s(\mathbb{R}^2)$ for which the solution of (11) exists longer than time $T$. With this our main result reads as

Theorem 1.2. The solution map $U_T \to H^s(\mathbb{R}^2), \theta_0 \mapsto \theta(T)$ is nowhere locally uniformly continuous.

The same result was established in [10] for the incompressible Euler equation and in [11] for the Holm-Staley $b$-family of equations. To establish Theorem 1.2 we will use the same techniques as in [10, 11]. The idea is to use some sort of ”gliding hump”. If we denote by $\varphi$ the flow of $u$, i.e.

$$\partial_t \varphi = u \circ \varphi, \quad \varphi(0) = \text{id}$$

then we have

$$\theta(T) \circ \varphi(T) = \theta_0 \quad \text{or} \quad \theta(T) = \theta_0 \circ \varphi(T)^{-1}$$

which is the key to produce the ”gliding hump”. To accomplish that one needs to control $\varphi$. Here the geometric formulation comes into play, which is nothing other than the formulation of (11) in the Lagrangian variable $\varphi$.

2 Geometric formulation

In this section we describe the equation (11) in a geometric way. The principle is not new – see e.g. [1, 6]. It works quite for a lot of equations. For the incompressible Euler equation [10], for the Holm-Staley $b$-family of equations [11], for the Burgers equation and so on. All these equations can be written in the form

$$\partial_t u + (u \cdot \nabla)u = F(u)$$

where on the right hand side there is no loss of regularity. Now consider the flow map of $u$ which we denote by $\varphi$. Using $\varphi_t = u \circ \varphi$ and taking the derivative of this expression one gets

$$\varphi_{tt} = u_t \circ \varphi + [(u \cdot \nabla)u] \circ \varphi$$
Hence \( \phi_{tt} = F(\phi_t \circ \phi^{-1}) \circ \phi \). This is a second order equation in \( \phi \). Let us establish this for equation (1). Applying \(-R_2^2\) to (1) we get

\[
\partial_t u_1 + (u \cdot \nabla) u_1 = R_2 \left( (u \cdot \nabla) \theta - (u \cdot \nabla) \theta \right) = [u \cdot \nabla, -R_2] \theta
\]

where \([A, B] = AB - BA\) denotes the commutator of the operators \(A, B\). Similarly applying \(R_1\) to (1) we get

\[
\partial_t u_2 + (u \cdot \nabla) u_2 = [u \cdot \nabla, R_1] \theta
\]

Replacing \(\theta = R_2 u_1 - R_1 u_2\) and recasting both equations we get

\[
\partial_t u + (u \cdot \nabla) u = \left( \left[ u \cdot \nabla, -R_2 \right] (R_2 u_1 - R_1 u_2) \right) =: B(u, u) \quad (2)
\]

with \(B\) the quadratic expression in \(u\) on the right. Introducing the variables \((\phi, v)\) where \(\phi\) is the flow map of \(u\) and \(v\) is \(\phi_t\) we can rewrite (2) as an equation on \(D^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2; \mathbb{R}^2)\)

\[
\partial_t \begin{pmatrix} \phi \\ v \end{pmatrix} = \begin{pmatrix} v \\ B(v \circ \phi^{-1}, v \circ \phi^{-1}) \circ \phi \end{pmatrix} \quad (3)
\]

where the function space \(D^s(\mathbb{R}^2)\) is defined for \(s > 2\) as

\[
D^s(\mathbb{R}^2) = \{ \phi : \mathbb{R}^2 \to \mathbb{R}^2 \mid \phi - \text{id} \in H^s(\mathbb{R}^2; \mathbb{R}^2), \det(d_x \phi) > 0 \quad \forall x \in \mathbb{R}^2 \}
\]

This space consists of diffeomorphisms of \(\mathbb{R}^2\) which are perturbations of the identity map. It is connected and has a differential structure by considering it as an open subset of \(H^s(\mathbb{R}^2; \mathbb{R}^2)\). Furthermore it is a topological group under composition of maps. For details one can consult [7]. Note that the quadratic nature of \(B\) makes (3) to a geodesic equation on \(D^s(\mathbb{R}^2)\). One of the main difficulties is to prove the regularity of the equation (3). We have

**Proposition 2.1.** Let \(s > 2\). Then the map

\[
D^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2; \mathbb{R}^2) \to H^s(\mathbb{R}^2; \mathbb{R}^2), \quad (\phi, v) \mapsto B(v \circ \phi^{-1}, v \circ \phi^{-1}) \circ \phi
\]

is real analytic.

The proof is in the Appendix. An immediate consequence of this proposition is that we get by Picard-Lindelöf local solutions to (3) which are unique.
In the following we show that (2) is an equivalent formulation of (1) and in the next section the equivalence of (3) and (2). But first we make for (1) the notion of solution precise. For $s > 2$ we say that $\theta \in C([0, T]; H^s(\mathbb{R}^2))$ is a solution to (1) on $[0, T]$ if we have

$$\theta(t) = \theta_0 + \int_0^t -(u \cdot \nabla)\theta \, ds \quad \forall \ 0 \leq t \leq T$$

as an equality in $H^{s-1}$. Note that $u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) \in C([0, T]; H^s(\mathbb{R}^2; \mathbb{R}^2))$ and so the integral lies in $C([0, T]; H^{s-1}(\mathbb{R}^2; \mathbb{R}^2))$ by the Banach algebra property of $H^{s-1}$.

**Lemma 2.2.** Let $s > 2$ and $T > 0$. For $u \in C([0, T]; H^s(\mathbb{R}^2; \mathbb{R}^2))$ a solution to (2) with divergence-free initial value, i.e. $\text{div} u(0) = 0$ we have

$$\text{div} u(t) = 0 \quad \forall \ 0 \leq t \leq T$$

**Proof.** We denote $u_0 = u(0)$. By (2) we have

$$u(t) = u_0 + \int_0^t B(u, u) - (u \cdot \nabla)u \, ds \quad \forall \ 0 \leq t \leq T$$

Let $\Phi = \mathcal{R}_1 u_1 + \mathcal{R}_2 u_2$. Note that as $u_0$ is divergence free, we have $\Phi(0) = 0$. Applying $\mathcal{R}_1$ to the first component and $\mathcal{R}_2$ to the second component one gets

$$\partial_t \Phi = \mathcal{R}_1 ([u \cdot \nabla, -\mathcal{R}_2](\mathcal{R}_2 u_1 - \mathcal{R}_1 u_2)) + \mathcal{R}_2 ([u \cdot \nabla, \mathcal{R}_1](\mathcal{R}_2 u_1 - \mathcal{R}_1 u_2))$$

We consider the terms separately. We have

$$\mathcal{R}_1 ([u \cdot \nabla, -\mathcal{R}_2](\mathcal{R}_2 u_1 - \mathcal{R}_1 u_2)) = -\mathcal{R}_1 u_1 \mathcal{R}_2^2 \partial_1 u_1 + \mathcal{R}_1 \mathcal{R}_2 u_1 \mathcal{R}_2 \partial_1 u_1$$

$$-\mathcal{R}_1 u_2 \mathcal{R}_2^2 \partial_2 u_1 + \mathcal{R}_1 \mathcal{R}_2 u_2 \mathcal{R}_2 \partial_2 u_1$$

$$+ \mathcal{R}_1 u_1 \mathcal{R}_1 \mathcal{R}_2 \partial_1 u_2 - \mathcal{R}_1 \mathcal{R}_2 u_1 \mathcal{R}_1 \partial_1 u_2$$

$$+ \mathcal{R}_1 u_2 \mathcal{R}_1 \mathcal{R}_2 \partial_2 u_2 - \mathcal{R}_1 \mathcal{R}_2 u_2 \mathcal{R}_1 \partial_2 u_2$$

Similarly we have

$$\mathcal{R}_2 ([u \cdot \nabla, \mathcal{R}_1](\mathcal{R}_2 u_1 - \mathcal{R}_1 u_2)) = \mathcal{R}_2 u_1 \mathcal{R}_1 \mathcal{R}_2 \partial_1 u_1 - \mathcal{R}_1 \mathcal{R}_2 u_1 \mathcal{R}_2 \partial_1 u_1$$

$$+ \mathcal{R}_2 u_2 \mathcal{R}_1 \mathcal{R}_2 \partial_2 u_1 - \mathcal{R}_1 \mathcal{R}_2 u_2 \mathcal{R}_2 \partial_2 u_1$$

$$- \mathcal{R}_2 u_1 \mathcal{R}_1 \partial_1 u_2 + \mathcal{R}_1 \mathcal{R}_2 u_1 \mathcal{R}_1 \partial_1 u_2$$

$$- \mathcal{R}_2 u_2 \mathcal{R}_1 \partial_2 u_2 + \mathcal{R}_1 \mathcal{R}_2 u_2 \mathcal{R}_1 \partial_2 u_2$$

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Finally we have

\[- \mathcal{R}_1((u \cdot \nabla)u_1) - \mathcal{R}_2((u \cdot \nabla)u_2) = -\mathcal{R}_1 u_1 \partial_1 u_1 - \mathcal{R}_1 u_2 \partial_2 u_1 - \mathcal{R}_2 u_1 \partial_1 u_2 - \mathcal{R}_2 u_2 \partial_2 u_2\]

Using the identity \(- \mathcal{R}_1^2 - \mathcal{R}_2^2 = \text{id}\) we rewrite this as

\[- \mathcal{R}_1((u \cdot \nabla)u_1) - \mathcal{R}_2((u \cdot \nabla)u_2) = - \mathcal{R}_1 u_1 (-\mathcal{R}_1^2 - \mathcal{R}_2^2) \partial_1 u_1 - \mathcal{R}_1 u_2 (-\mathcal{R}_1^2 - \mathcal{R}_2^2) \partial_2 u_1 - \mathcal{R}_2 u_1 (-\mathcal{R}_1^2 - \mathcal{R}_2^2) \partial_1 u_2 - \mathcal{R}_2 u_2 (-\mathcal{R}_1^2 - \mathcal{R}_2^2) \partial_2 u_2\]

Adding up we find

\[\partial_t \Phi = \mathcal{R}_1 u_1 \mathcal{R}_2^2 \partial_1 u_1 + \mathcal{R}_1 u_2 \mathcal{R}_2^2 \partial_2 u_1 + \mathcal{R}_1 u_1 \mathcal{R}_1 \mathcal{R}_2 \partial_1 u_2 + \mathcal{R}_1 u_2 \mathcal{R}_1 \mathcal{R}_2 \partial_2 u_2 + \mathcal{R}_2 u_1 \mathcal{R}_1 \mathcal{R}_2 \partial_1 u_1 + \mathcal{R}_2 u_2 \mathcal{R}_1 \mathcal{R}_2 \partial_1 u_2 + \mathcal{R}_2 u_1 \mathcal{R}_2^2 \partial_1 u_2 + \mathcal{R}_2 u_2 \mathcal{R}_2^2 \partial_2 u_2\]

\[= (\mathcal{R}_1 u_1 \mathcal{R}_1 \partial_1 + \mathcal{R}_1 u_2 \mathcal{R}_1 \partial_2 + \mathcal{R}_2 u_1 \mathcal{R}_2 \partial_1 + \mathcal{R}_2 u_2 \mathcal{R}_2 \partial_2)(\mathcal{R}_1 u_1 + \mathcal{R}_2 u_2)\]

Hence we have

\[\partial_t \frac{1}{2} \langle \Phi, \Phi \rangle_{L^2} = \langle (\mathcal{R}_1 u_1 \mathcal{R}_1 \partial_1 + \mathcal{R}_1 u_2 \mathcal{R}_1 \partial_2 + \mathcal{R}_2 u_1 \mathcal{R}_2 \partial_1 + \mathcal{R}_2 u_2 \mathcal{R}_2 \partial_2) \Phi, \Phi \rangle_{L^2}\]

Using integration by parts the righthand side is

\[\langle (\mathcal{R}_1 u_1 \mathcal{R}_1 \partial_1 + \mathcal{R}_1 u_2 \mathcal{R}_1 \partial_2 + \mathcal{R}_2 u_1 \mathcal{R}_2 \partial_1 + \mathcal{R}_2 u_2 \mathcal{R}_2 \partial_2) \Phi, \Phi \rangle_{L^2} - \langle \Phi, \partial_1 (\mathcal{R}_1 u_1 \mathcal{R}_1 \Phi) + \partial_2 (\mathcal{R}_1 u_2 \mathcal{R}_1 \Phi) + \partial_1 (\mathcal{R}_2 u_1 \mathcal{R}_2 \Phi) + \partial_2 (\mathcal{R}_2 u_2 \mathcal{R}_2 \Phi) \rangle_{L^2} - \langle \Phi, (\mathcal{R}_1 u_1 \mathcal{R}_1 \partial_1 + \mathcal{R}_1 u_2 \mathcal{R}_1 \partial_2 + \mathcal{R}_2 u_1 \mathcal{R}_2 \partial_1 + \mathcal{R}_2 u_2 \mathcal{R}_2 \partial_2) \Phi \rangle_{L^2}\]

Thus we have

\[\partial_t \frac{1}{2} \langle \Phi, \Phi \rangle_{L^2} = -\frac{1}{2} \langle \Phi, \partial_1 (\mathcal{R}_1 u_1 \mathcal{R}_1 \Phi) + \partial_2 (\mathcal{R}_1 u_2 \mathcal{R}_1 \Phi) + \partial_1 (\mathcal{R}_2 u_1 \mathcal{R}_2 \Phi) + \partial_2 (\mathcal{R}_2 u_2 \mathcal{R}_2 \Phi) \rangle_{L^2}\]

Using the Sobolev imbedding \(H^{s-1} \hookrightarrow L^\infty\) we have on \([0,T]\) therefore the estimate

\[\partial_t \|\Phi\|_{L^2}^2 \leq C \|\Phi\|_{L^2}^2\]

As \(\Phi(0) = 0\) we get by Gronwall’s inequality \(\Phi \equiv 0\) showing that \(\text{div} \, u = 0\). \(\square\)
A consequence of Lemma 2.2 is

**Proposition 2.3.** Let $s > 2$ and $T > 0$. For $\theta_0 \in \mathcal{H}^s(\mathbb{R}^2)$ let $\theta \in C([0, T]; \mathcal{H}^s(\mathbb{R}^2))$ be a solution to (1). Then $u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta)$ is a solution to (2) on $[0, T]$. On the other hand if $u \in C([0, T]; \mathcal{H}^s(\mathbb{R}^2))$ is a solution to (2) with initial value $u(0) = (-\mathcal{R}_2 \theta_0, \mathcal{R}_1 \theta_0)$, then $\theta = \mathcal{R}_2 u_1 - \mathcal{R}_1 u_2$ is a solution to (1) on $[0, T]$ with $\theta(0) = \theta_0$.

**Proof.** The first part was already shown in the derivation of (2). To show the second part we take a solution $u$ to (2) with $u(0) = (-\mathcal{R}_2 \theta_0, \mathcal{R}_1 \theta_0)$ on $[0, T]$. As $\text{div} \ u(0) = 0$ we have by Lemma 2.2 for all $t \in [0, T]$

$$\mathcal{R}_1 u_1 + \mathcal{R}_2 u_2 = 0 \quad (4)$$

We have to show that $\theta := \mathcal{R}_2 u_1 - \mathcal{R}_1 u_2$ is a solution to (1). By (4) we have

$$-\mathcal{R}_2 \theta = -\mathcal{R}_2^2 u_1 + \mathcal{R}_1 \mathcal{R}_2 u_2 = (-\mathcal{R}_2^2 - \mathcal{R}_1^2) u_1 = u_1$$

Similarly we have $\mathcal{R}_1 \theta = u_2$. Applying $-\mathcal{R}_2$ to the first equation in (2), $\mathcal{R}_1$ to the second equation and sum up we get

$$\partial_t \theta - \mathcal{R}_2((u \cdot \nabla) u_1) + \mathcal{R}_1((u \cdot \nabla) u_2) = -\mathcal{R}_2([u \cdot \nabla, -\mathcal{R}_2] \theta) + \mathcal{R}_1([u \cdot \nabla, \mathcal{R}_1] \theta)$$

Simplifying we arrive at

$$\partial_t \theta + (u \cdot \nabla) \theta = 0$$

$\square$

### 3 Local well-posedness

The goal of this section is to establish the equivalence of (1) and (3). By Proposition 2.3 we just have to prove the equivalence between (2) and (3).

**Lemma 3.1.** Let $s > 2$ and $T > 0$. Assume that $\varphi$ is a solution of (3) on $[0, T]$ for the initial values $\varphi(0) = \text{id}$ and $v(0) = u_0 \in \mathcal{H}^s(\mathbb{R}^2; \mathbb{R}^2)$. Then $u$ given by

$$u(t) := \varphi(t) \circ \varphi(t)^{-1}$$

is a solution to (2).
Proof. It is to prove that we have

\[ u(t) = u_0 + \int_0^t B(u(s), u(s)) - (u(s) \cdot \nabla)u(s) \, ds \quad \forall \, 0 \leq t \leq T \]

Note that by Proposition 2.1 we have \( \varphi \in C^\infty([0, T]; D^s(\mathbb{R}^2)) \). Therefore by the properties of the composition (see [7]) \( u \in C([0, T]; H^s(\mathbb{R}^2; \mathbb{R}^2)) \). By the Sobolev imbedding \( H^s \hookrightarrow C^1 \) we also have \( u = \varphi_t \circ \varphi^{-1} \in C^1([0, T] \times \mathbb{R}^2; \mathbb{R}^2) \). Thus we have pointwise

\[ \varphi_{tt} = (u_t + (u \cdot \nabla)u) \circ \varphi \]

As \( \varphi \) is a solution to (3) we conclude pointwise

\[ (u_t + (u \cdot \nabla)u) \circ \varphi = B(u, u) \circ \varphi \]

or \( u_t + (u \cdot \nabla)u = B(u, u) \). Rewriting this we get

\[ u(t) = u_0 + \int_0^t B(u(s), u(s)) - (u(s) \cdot \nabla)u(s) \, ds \quad \forall \, 0 \leq t \leq T \]

By the Banach algebra property of \( H^{s-1} \) and the imbedding \( H^{s-1} \hookrightarrow C^0 \) the integral is also an identity for \( H^{s-1} \) functions.

The reverse is

**Lemma 3.2.** Let \( s > 2 \) and \( T > 0 \). If \( u \in C([0, T]; H^s(\mathbb{R}^2; \mathbb{R}^2)) \) is a solution to (2) then its flow map \( \varphi \) is a solution to (3).

*Proof.* We know (see [3]) that for \( u \) there is a unique \( \varphi \in C^1([0, T]; D^s(\mathbb{R}^2)) \) with

\[ \varphi_t = u \circ \varphi \quad \varphi(0) = \text{id} \]

By the integral relation \( u(t) = u_0 + \int_0^t B(u, u) - (u \cdot \nabla)u \, ds \) we see that \( u \in C^1([0, T] \times \mathbb{R}^2; \mathbb{R}^2) \). Taking the derivative we get pointwise

\[ \varphi_t = (u_t + (u \cdot \nabla)u) \circ \varphi = B(u, u) \circ \varphi \]

This means

\[ \varphi(t) = \varphi(0) + \int_0^t B(\varphi_t \circ \varphi^{-1}, \varphi_t \circ \varphi^{-1}) \circ \varphi \, ds \quad \forall \, 0 \leq t \leq T \]

which is also an identity in \( H^s \). Therefore \( \varphi \in C^2([0, T]; D^s(\mathbb{R}^2)) \) and \( \varphi \) is a solution to (3).
Lemma 3.1 and Lemma 3.2 establish the equivalence of (2) and (3). The solutions of (3) can be described by an exponential map as follows: Consider (3) with initial values $\varphi(0) = \text{id}$ and $v(0) \in H^s(\mathbb{R}^2; \mathbb{R}^2)$. Further we denote by $U \subseteq H^s(\mathbb{R}^2; \mathbb{R}^2)$ those initial values $v(0) = u_0$ for which we have a solution on $[0, 1]$. With this we define

$$\exp : U \to \mathcal{D}^s(\mathbb{R}^2), \quad u_0 \mapsto \varphi(1; u_0)$$

where $\varphi(1; u_0)$ is the time one value of the solution $\varphi$ corresponding to the initial values $\varphi(0) = \text{id}$ and $v(0) = u_0$. By Proposition (2.1) we know that $\exp$ is real analytic because we have analytic dependence on the initial value $u_0$. The $\varphi$-solution can be totally described by the exponential map. For $u_0$ the corresponding $\varphi$ is just given by

$$\varphi(t) = \exp(tu_0)$$

for all $t$ for which $tu_0$ lies in $U$. Furthermore the derivative of $\exp$ at 0 is the identity map. For details on the exponential map one can consult [12]. We end this section by giving a proof of Theorem 1.1.

**Proof of Theorem 1.1.** Take $\theta_0 \in H^s(\mathbb{R}^2)$ and define for $u_0 = (-R_2\theta_0, R_1\theta_0)$

$$\varphi(t) = \exp(tu_0) \quad \text{and} \quad u(t) = \varphi(t) \circ \varphi(t)^{-1}$$

for all $t \geq 0$ with $tu_0$ in the domain of definition of $\exp$. By the properties of the composition map – see [7] – we know that $u \in C([0, T]; H^s(\mathbb{R}^2; \mathbb{R}^2))$ for some $T > 0$. With this we define

$$\theta(t) = R_2u_1(t) - R_1u_2(t)$$

which solves (1). Furthermore the dependence on $\theta_0$ is continuous. Uniqueness of solutions follows from the uniqueness of solutions to ODEs. More precisely assume two solutions $\theta$ and $\tilde{\theta}$ with the same initial value $\theta_0$. Define the corresponding $u = (-R_2\theta, R_1\theta)$ and $\tilde{u} = (-R_2\tilde{\theta}, R_1\tilde{\theta})$. By Lemma 3.2 their flows $\varphi$ resp. $\tilde{\varphi}$ are solutions to (3). So they have to be equal which implies that $\theta \equiv \tilde{\theta}$. \hfill \Box

4 Non-uniform dependence

Throughout this section we assume $s > 2$. In this section we prove Theorem 1.2. We introduce the notation $\Phi_T$ for the time $T$-solution map, $T > 0$, i.e.
\( \Phi_T(\theta_0) \) denotes the value of the solution to (1) with initial value \( \theta_0 \) at time \( T \). As already introduced we denote by \( U_T \subseteq H^s(\mathbb{R}^2) \) the domain of definition of \( \Phi_T \). In the case of \( T = 1 \) we use \( \Phi := \Phi_T \) and \( U := U_T \). By the scaling property of (1) we have
\[
\Phi_T(\theta_0) = \frac{1}{T} \Phi(T \theta_0) \quad \text{and} \quad U_T = \frac{1}{T} U
\]
So to prove Theorem 1.2 it suffices to give a proof for the special case \( T = 1 \).

**Proposition 4.1.** The map
\[
\Phi : U \to H^s(\mathbb{R}^2), \quad \theta_0 \mapsto \Phi(\theta_0)
\]
is nowhere locally uniformly continuous.

Before proving Proposition 4.1 we have to make some preparation.

**Lemma 4.2.** For any \( x \in \mathbb{R}^2 \) there is a \( \theta \in H^s(\mathbb{R}^2) \) such that \( u(x) \neq 0 \) where
\[
u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta)
\]
Proof. Recall the integral representation for the Riesz transforms \( \mathcal{R}_k, k = 1, 2 \)
\[
\mathcal{R}_k \theta(x) = \frac{1}{\sqrt{2\pi}} \text{p.v.} \int_{\mathbb{R}^2} \frac{x_k - y_k}{|x - y|^3} \theta(y) \, dy, \quad k = 1, 2
\]
in the principal value sense. For the given \( x \in \mathbb{R}^2 \) we can just choose a smooth positive \( \theta \) with compact support lying on the left-down of \( x \). We therefore have trivially \( \mathcal{R}_k \theta(x) > 0 \) for \( k = 1, 2 \).

A consequence of this lemma is the following technical lemma.

**Lemma 4.3.** There is a dense subset \( S \subseteq U \) consisting of functions with compact support such that each function \( \theta_0 \in S \) has the following property: There is \( x \in \mathbb{R}^2 \) and \( \theta \in H^s(\mathbb{R}^2) \) depending on \( \theta_0 \) such that
\[
\text{dist}(x, \text{supp} \theta_0) \geq 2
\]
and
\[
(d_{u_0} \exp(u))(x) \neq 0
\]
where \( u_0 = (-\mathcal{R}_2 \theta_0, \mathcal{R}_1 \theta_0), u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) \) and \( d_{u_0} \exp \) denotes the differential of \( \exp \) at the point \( u_0 \).
Proof. Take \( \theta_0 \in U \) with compact support. Take an arbitrary \( x \in \mathbb{R}^2 \) which has a distance of more than 2 to the support of \( \theta_0 \). By Lemma 4.2 there is \( \theta \in H^s(\mathbb{R}^2) \) with \( u(x) \neq 0 \). Consider now the analytic function

\[
t \mapsto (d_{t_0 \theta_0} \exp(u))(x)
\]

At \( t = 0 \) this equals to \( u(x) \neq 0 \). Therefore there is a sequence \( 0 \leq t_n \uparrow 1 \) with

\[
(d_{t_n \theta_0} \exp(u))(x) \neq 0, \quad \forall n \geq 1
\]

We put all \( t_n \theta_0 \) into \( S \). Doing that for all \( \theta_0 \) with compact support we get our desired result, as the compactly supported functions are dense in \( H^s(\mathbb{R}^2) \).

In the following we will use inequalities for functions with disjoint compact support of the type

\[
||f + g|| \geq C(||f|| + ||g||)
\]

for Sobolev norms. More precisely, given \( s' \geq 0 \) and fixed disjoint compact sets \( K_1, K_2 \subseteq \mathbb{R}^2 \) there is a constant \( C > 0 \) such that we have

\[
||f + g||_{s'} \geq C(||f||_{s'} + ||g||_{s'})
\]

for all \( f, g \in H^{s'}(\mathbb{R}^2) \) with \( f, g \) supported in \( K_1 \) resp. \( K_2 \). We have a similar situation if the geometry of the supports is in a fixed ratio. We will use it as follows: There is a constant \( C > 0 \) such that for \( x, y \in \mathbb{R}^2 \) with

\[
0 < r := \frac{|x - y|}{4} < 1
\]

we have

\[
||f + g||_{s'} \geq C(||f||_{s'} + ||g||_{s'})
\]

for all \( f, g \in H^{s'}(\mathbb{R}^2) \) with \( f \) supported in \( B_r(x) \) and \( g \) supported in \( B_r(y) \). Here \( B_r(z) \) denotes the ball around \( z \) with radius \( r \). For the details one can look at the Appendix in [11].

Now we prove Proposition 4.1.

Proof of Proposition 4.1. For a given \( \theta_0 \in U \) we will show that there is \( R_* > 0 \) such that \( \Phi \) is not uniformly continuous on \( B_R(\theta_0) \) for all \( 0 < R \leq R_* \). We will choose \( R_* \) in several steps. It is enough to show that for \( \theta_0 \) in the dense
subset $S \subseteq U$. So take an arbitrary $\theta_0$ in $S$. To make the notation easier we introduce the analytic map $\tilde{\exp}$

$$\tilde{\exp} : U \rightarrow D^*(\mathbb{R}^2), \quad \theta \mapsto \exp((-R_2 \theta, R_1 \theta))$$

In particular we then have for a solution $\theta$ of (1)

$$\theta(1) = \theta(0) \circ (\tilde{\exp}(\theta(0)))^{-1}$$

(7)

Furthermore we fix by Lemma 4.3 a $v \in H^s(\mathbb{R}^2; \mathbb{R}^2), v \neq 0$, and $x^* \in \mathbb{R}^2$ with $\text{dist}(x^*, \supp \theta_0) \geq 2$ and

$$|(d_{\theta_0} \tilde{\exp}(v))(x^*)| \geq m ||v||_s$$

for some $m > 0$. By the Sobolev imbedding we have

$$||f||_{C^1} \leq C_1 ||f||_s, \quad \forall f \in H^s(\mathbb{R}^2)$$

(8)

for some $C_1 > 0$. We choose an $R_1 > 0$ such that for some $C_2 > 0$ we have

$$\frac{1}{C_2} ||f||_s \leq ||f \circ \varphi^{-1}||_s \leq C_2 ||f||_s$$

(9)

for all $f \in H^s(\mathbb{R}^2)$ and $\varphi \in \tilde{\exp}(B_{R_1} (\theta_0))$. That this is indeed possible follows from the continuity of the composition map and the linearity in $f$ – see [7].

We try to get a situation as described in (5). Let $\varphi_0 = \tilde{\exp}(\theta_0)$. Then as $x^*$ is enough away from the support of $\theta_0$ we have

$$d := \text{dist} \left( \varphi_0(\supp \theta_0), \varphi_0(B_1(x^*)) \right) > 0$$

We choose $K_1, K_2$ as

$$K_1 := \{ x \in \mathbb{R}^2 | \text{dist}(x, \varphi_0(\supp \theta_0)) \leq d/4 \}$$

resp.

$$K_2 := \{ x \in \mathbb{R}^2 | \text{dist}(x, \varphi_0(B_1(x^*))) \leq d/4 \}$$

By choosing $0 < R_2 \leq R_1$ we can ensure by the Sobolev imbedding (8)

$$|\varphi(x) - \varphi(y)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}^2 \quad \text{and} \quad ||\varphi - \overline{\varphi}||_{L^\infty} \leq \min\{d/4, 1\}$$

for all $\varphi, \overline{\varphi} \in \tilde{\exp}(B_{R_2}(\theta_0))$. So the second condition ensures

$$\varphi(\supp \theta_0) \subseteq K_1 \quad \text{and} \quad \varphi(B_1(x^*)) \subseteq K_2$$
for all $\varphi \in \tilde{\exp}(B_{R_2}(\theta_0))$. Consider the Taylor expansion
\[
\tilde{\exp}(\theta + h) = \tilde{\exp}(\theta) + d_\theta \tilde{\exp}(h) + \int_0^1 (1 - t) d_{\theta + th} \tilde{\exp}(h) \, dt
\]
We need to estimate the terms appearing in this expansion. We can choose $0 < R_3 \leq R_2$ in such a way that we have for some constant $K > 0$
\[
||d_\theta^2 \tilde{\exp}(h_1, h_2)||_s \leq K||h_1||_s||h_2||_s
\]
and
\[
||d_\theta^2 \tilde{\exp}(h_1, h_2) - d_\theta^2 \tilde{\exp}(h_1, h_2)||_s \leq K||\theta_1 - \theta_2||_s||h_1||_s||h_2||_s
\]
for all $\theta, \theta_1, \theta_2 \in B_{R_3}(\theta_0)$ and $h_1, h_2 \in H^s(\mathbb{R}^2)$. Finally we choose $0 < R_* \leq R_3$ with
\[
\max\{C_1KR_*^2, C_1KR_*\} < m/8 \tag{10}
\]
Now take an arbitrary $0 < R \leq R_*$. We will construct two sequences of initial values

\[
(\theta_0^{(n)})_{n \geq 1}, (\tilde{\theta}_0^{(n)})_{n \geq 1} \subseteq B_{R}(\theta_0)
\]
with $\lim_{n \to \infty} ||\theta_0^{(n)} - \tilde{\theta}_0^{(n)}||_s = 0$ but

\[
\limsup_{n \geq 1} ||\Phi(\theta_0^{(n)}) - \Phi(\tilde{\theta}_0^{(n)})||_s > 0
\]
showing the claim. The first sequence will be chosen in the form

\[
\theta_0^{(n)} = \theta_0 + w^{(n)}
\]
where we take $w^{(n)} \in H^s(\mathbb{R}^2)$ arbitrarily with $||w^{(n)}||_s = R/2$ and having its support in $B_{r_n}(x^*)$ where

\[
r_n = \frac{m}{8nL}||v||_s
\]
Thus the mass of $w^{(n)}$ is constant whereas its support shrinks to $x^*$. The second sequence is a perturbation of the first one so as to get a shift in the supports. We take it as

\[
\tilde{\theta}_0^{(n)} = \theta_0^{(n)} + \frac{1}{n}v = \theta_0 + w^{(n)} + \frac{1}{n}v
\]
We will use the notation \( v^{(n)} := \frac{1}{n}v \). Taking \( N \) large enough we clearly have
\[
\theta_0^{(n)}, \tilde{\theta}_0^{(n)} \in B_R(\theta_0) \text{ and } r_n \leq 1, \quad \forall n \geq N
\]
By construction we have
\[
\lim_{n \to \infty} \|\theta_0^{(n)} - \tilde{\theta}_0^{(n)}\|_s = \lim_{n \to \infty} \|v^{(n)}\|_s = 0
\]
We introduce for \( n \geq N \)
\[
\varphi^{(n)} = \widehat{\exp}(\theta_0^{(n)}) \quad \text{resp.} \quad \tilde{\varphi}^{(n)} = \widehat{\exp}(\tilde{\theta}_0^{(n)})
\]
By the conservation law \((7)\) we have
\[
\Phi(\theta_0^{(n)}) = \theta_0^{(n)} \circ (\varphi^{(n)})^{-1} \quad \text{resp.} \quad \Phi(\tilde{\theta}_0^{(n)}) = \tilde{\theta}_0^{(n)} \circ (\tilde{\varphi}^{(n)})^{-1}
\]
We will use these expressions to evaluate \( \|\Phi(\theta_0^{(n)}) - \Phi(\tilde{\theta}_0^{(n)})\|_s \). Plugging in the expressions we get
\[
\|\Phi(\theta_0^{(n)}) - \Phi(\tilde{\theta}_0^{(n)})\|_s = \|(\theta_0 + w^{(n)}) \circ (\varphi^{(n)})^{-1} - (\theta_0 + w^{(n)} + v^{(n)}) \circ (\tilde{\varphi}^{(n)})^{-1}\|_s
\]
\[
\geq \|(\theta_0 + w^{(n)}) \circ (\varphi^{(n)})^{-1} - (\theta_0 + w^{(n)}) \circ (\tilde{\varphi}^{(n)})^{-1}\|_s - \|v^{(n)} \circ (\tilde{\varphi}^{(n)})^{-1}\|_s
\]
By \((9)\) the last expression vanishes if we take the \( \lim \sup \). So we just have to look at the first term on the right
\[
\|((\theta_0 + w^{(n)}) \circ (\varphi^{(n)})^{-1} - (\theta_0 + w^{(n)}) \circ (\tilde{\varphi}^{(n)})^{-1})\|_s = \|((\theta_0 \circ (\varphi^{(n)})^{-1} - \tilde{\theta}_0 \circ (\tilde{\varphi}^{(n)})^{-1}) + (w^{(n)} \circ (\varphi^{(n)})^{-1} - w^{(n)} \circ (\tilde{\varphi}^{(n)})^{-1})\|_s
\]
The first two terms in the latter expression have their support in \( K_1 \) and the other two in \( K_2 \). By \((5)\) it will be enough to establish
\[
\lim_{n \to \infty} \|w^{(n)} \circ (\varphi^{(n)})^{-1} - w^{(n)} \circ (\tilde{\varphi}^{(n)})^{-1}\|_s > 0
\]
We will do that by showing that the supports of these two expressions are disjoint in a way that we can apply \((6)\). To do that we will estimate the distance \( |\varphi^{(n)}(x^*) - \tilde{\varphi}^{(n)}(x^*)| \) using the Taylor expansion of \( \widehat{\exp} \). So we have
\[
\varphi^{(n)} = \widehat{\exp}(\theta_0 + w^{(n)})
\]
\[
= \widehat{\exp}(\theta_0) + d_{\theta_0} \widehat{\exp}(w^{(n)}) + \int_0^1 (1 - t)d_{\theta_0 + tw^{(n)}}^2 \widehat{\exp}(w^{(n)}) dt
\]
Similarly
\[ \tilde{\varphi}^{(n)} = \exp(\theta_0 + w^{(n)} + v^{(n)}) = \exp(\theta_0 + d_{\theta_0} \exp(w^{(n)} + v^{(n)})) \]
\[ + \int_0^1 (1 - t) d_{\theta_0 + t(w^{(n)} + v^{(n)})}^2 \exp(w^{(n)} + v^{(n)}, w^{(n)} + v^{(n)}) \, dt \]

So the difference reads as
\[ \varphi^{(n)} - \tilde{\varphi}^{(n)} = -d_{\theta_0} \exp(v^{(n)}) + I_1 + I_2 + I_3 \]

where
\[ I_1 = \int_0^1 (1 - t) \left( d_{\theta_0 + tw^{(n)}}^2 \exp(w^{(n)}, w^{(n)}) - d_{\theta_0 + tw^{(n)} + v^{(n)}}^2 \exp(w^{(n)}, w^{(n)}) \right) \, dt \]
and
\[ I_2 = -2 \int_0^1 (1 - t) d_{\theta_0 + tw^{(n)} + v^{(n)}}^2 \exp(v^{(n)}, w^{(n)}) \, dt \]
and
\[ I_3 = -\int_0^1 (1 - t) d_{\theta_0 + tw^{(n)} + v^{(n)}}^2 \exp(v^{(n)}, v^{(n)}) \, dt \]

Using the estimates for the second derivatives from above we get
\[ ||I_1||_s \leq K ||v^{(n)}||_s ||w^{(n)}||^2_s = \frac{K}{4n} ||v||_s R^2 \]
and
\[ ||I_2||_s \leq 2K ||v^{(n)}||_s ||w^{(n)}||_s = \frac{K}{n} ||v||_s R \]
and
\[ ||I_3||_s \leq \frac{K}{n} ||v||_s \frac{||v||_s}{n} \leq \frac{KR}{n} ||v||_s \]
where the last inequality holds for \( n \geq N \) by enlarging \( N \) if necessary. Thus we see by the Sobolev imbedding that the value at \( x^* \) can be estimated by
\[ |I_1(x^*)| + |I_2(x^*)| + |I_3(x^*)| \leq \frac{C_1 KR^2}{4n} ||v||_s + \frac{C_1 KR}{n} ||v||_s + \frac{C_1 KR}{n} ||v||_s \]

By the choice for \( R \) it follows from (10)
\[ |I_1(x^*)| + |I_2(x^*)| + |I_3(x^*)| \leq \frac{m}{2n} ||v||_s \]
Using this inequality we arrive at

\[ |\varphi^{(n)}(x^*) - \tilde{\varphi}^{(n)}(x^*)| \geq |d_{\delta_0} \exp(v^{(n)})(x^*)| - \frac{m}{2n}||v||_s \]

Hence

\[ |\varphi^{(n)}(x^*) - \tilde{\varphi}^{(n)}(x^*)| \geq \frac{1}{n} m ||v||_s - \frac{m}{2n} ||v||_s = \frac{m}{2n} ||v||_s \]

By the Lipschitz property for \(\varphi^{(n)}, \tilde{\varphi}^{(n)}\) we have

\[ \varphi^{(n)}(B_{R_n}(x^*)) \subseteq B_{R_n}(\varphi^{(n)}(x^*)) \]

with \(R_n = L \frac{m}{8nL} ||v||_s = \frac{m}{8n} ||v||_s\). Similarly

\[ \tilde{\varphi}^{(n)}(B_{R_n}(x^*)) \subseteq B_{R_n}(\tilde{\varphi}^{(n)}(x^*)) \]

This means \(w^{(n)} \circ (\varphi^{(n)})^{-1}\) is supported in \(B_{R_n}(\varphi^{(n)}(x^*))\) and \(w^{(n)} \circ (\tilde{\varphi}^{(n)})^{-1}\) is supported in \(B_{R_n}(\tilde{\varphi}^{(n)}(x^*))\). So we are in a situation where we can apply (6) since the distance between the centers of support is larger than \(\frac{m}{2n} ||v||_s\) and the radii of the supports are \(\frac{m}{8n} ||v||_s\). Thus we have

\[ \|w^{(n)} \circ (\varphi^{(n)})^{-1} - w^{(n)} \circ (\tilde{\varphi}^{(n)})^{-1}\|_s \geq C(\|w^{(n)} \circ (\varphi^{(n)})^{-1}\|_s + \|w^{(n)} \circ (\tilde{\varphi}^{(n)})^{-1}\|_s) \geq \frac{C}{C_2} R/2 \]

where we used (9). Thus we have

\[ \limsup_{n \to \infty} \|\Phi(\theta^{(n)}_0) - \Phi(\tilde{\theta}^{(n)}_0)\|_s \geq \tilde{C} R \]

with \(\tilde{C}\) independent of \(0 < R \leq R_*\) whereas \(\lim_{n \to \infty} \|\theta^{(n)}_0 - \tilde{\theta}^{(n)}_0\|_s = 0\). As this holds for every \(0 < R \leq R_*\) we are done. \(\square\)

### A Proof of Proposition 2.1

In this section we will prove Proposition 2.1. The ideas we will use are inspired by [2], [5] and [9]. Throughout this section we assume \(s > n/2 + 1\). We introduce the operator

\[ \Lambda = (-\Delta)^{1/2} \]
So the Fourier transform of $\Lambda f$ is given by $|\xi|\hat{f}(\xi)$ where $\hat{f}$ denotes the Fourier transform of $f$. In the following we will also use the definition of $\Lambda$ in terms of a principal value integral

$$\Lambda f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+1}} \, dy = \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \frac{f(x) - f(y)}{|x - y|^{n+1}} \, dy$$

We will use also the following regularization of the above singular integral – see [14] for the technical details – for $f$ regular enough

$$\Lambda f(x) = \int_{\mathbb{R}^n} \frac{f(x) - f(y) + (x - y) \cdot \nabla f(x)}{|x - y|^{n+1}} e^{-|x-y|^2} \, dy + \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+1}} (1 - e^{-|x-y|^2}) \, dy$$

From [5] we can deduce that the analytic functions $K_1(y), K_2(y)$ satisfy the estimates

$$|\partial^\alpha K_1(y)| \leq C^{(|\alpha|!)} |y|^{n+1+|\alpha|} e^{-|y|^2/2}$$

resp.

$$|\partial^\alpha K_2(y)| \leq C^{(|\alpha|!)} \min\{ \frac{1}{|y|^{n+1+|\alpha|}}, \frac{1}{|y|^{n+1+|\alpha|}} \}$$

for all $\alpha \in \mathbb{N}^n$ and some universal constant $C > 0$. In the following we will also often use the algebra property of Sobolev spaces. Making the above $C$ larger if necessary we have the Kato-Ponce inequality

$$||f \cdot g||_{s-1} \leq C(||f||_{s-1}||g||_{\infty} + ||f||_{\infty}||g||_{s-1})$$

and also (can be deduced from (13))

$$||f \cdot g||_{s-1} \leq C||f||_{s-1}||g||_{s-1}$$

We prove Proposition 2.1 in several steps.

**Lemma A.1.** The map

$$D^s(\mathbb{R}^n) \to L(H^s(\mathbb{R}^n); H^{s-1}(\mathbb{R}^n)), \quad \varphi \mapsto [f \mapsto R_\varphi \Lambda R_\varphi^{-1} f]$$

is real analytic.

For the concept of analyticity in Hilbert spaces one can consult [9].
Proof. The goal is to establish a power series expansion of

\[(Af(\varphi^{-1}(x))) \circ \varphi(x)\]

in terms of \(g = (g_1, \ldots, g_n)\) where \(\varphi = \text{id} + g\). We split this expression according to the above regularization as

\[R_{\varphi}AR_{\varphi}^{-1}f = \mathcal{K}_1(\varphi) + \mathcal{K}_2(\varphi)\]

We first treat the easier case \(\mathcal{K}_2(\varphi)\). We have

\[\mathcal{K}_2(\varphi) = \int_{\mathbb{R}^n} K_2(\varphi(x) - \varphi(y))(f(x) - f(y))J_\varphi(y) \, dy\]

where \(J_\varphi\) is the determinant of the Jacobian \(d\varphi\). Note that \(J_\varphi\) is a fixed polynomial in the first derivatives of \(g\). We use \(\varphi(x) - \varphi(y) = (x - y) + (g(x) - g(y))\) and expand into the Taylor series of \(K_2\)

\[\mathcal{K}_2(\varphi) = \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha K_2(x - y)(g(x) - g(y))^\alpha(f(x) - f(y))J_\varphi(y) \, dy\]

If we separate the monomials in \(J_\varphi\) we see that the individual terms are multilinear expressions in \(g\). Taking the \(H^{s-1}\)-norm in the individual summand and using the Banach algebra properties we can dominate it by

\[\int_{\mathbb{R}^n} \frac{1}{\alpha!} |\partial^\alpha K_2(y)|C^{2|\alpha|} ||g(x) - g(x - y)||^{|\alpha|}_{s-1} 2||f||^{|\alpha|}_{s-1} K(1 + ||g||^n_s) \, dy\]

for some fixed \(K > 0\). Using \(||g(x) - g(x - y)||_{s-1} \leq ||g||_s |y|\) and \((12)\) we can estimate this by

\[\frac{|\alpha|!}{\alpha!} 2C^{2|\alpha|}K||g||^{|\alpha|}_s ||f||^{|\alpha|}_s (1 + ||g||^n_s) \int_{\mathbb{R}^n} \min\{\frac{1}{|y|^{n-1}}, \frac{1}{|y|^{n+1}}\} \, dy\]

Summing over all \(\alpha\) with \(|\alpha| = k\) for a fixed \(k \in \mathbb{N}\) we have an upper bound

\[\tilde{C}^k||g||^{k}_s ||f||^{k}_s (1 + ||g||^n_s)\]
which is the general term in a convergent series for small $||g||_s$, i.e. for $\varphi$ near to the identity map $id$. Now consider $\mathcal{K}_1(\varphi)$. We have

$$\mathcal{K}_1(\varphi) = \int_{\mathbb{R}^n} K_1(\varphi(x)-\varphi(y))(f(x)-f(y)+ (\varphi(x)-\varphi(y))^T [d\varphi^{-1}(x)]^T \nabla f(x)) J_\varphi(y) \, dy$$

Developing into the Taylor series of $K_1$ as above we have

$$\int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha K_1(x-y)(g(x)-g(y))^\alpha (f(x)-f(y)+ (\varphi(x)-\varphi(y))^T [d\varphi^{-1}(x)]^T \nabla f(x)) J_\varphi(x-y) \, dy$$

By pulling out $1/J_\varphi(x)$ in front of the integral one sees by the formula for the inverse of a matrix that the individual terms under the integral are polynomials in $g$. Note that $1/J_\varphi$ depends analytically on $\varphi$ – see [9]. A change of variables leads to

$$\int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha K_1(y)(g(x)-g(x-y))^\alpha (f(x)-f(x-y)+ (\varphi(x)-\varphi(x-y))^T [d\varphi^{-1}(x)]^T \nabla f(x)) J_\varphi(x-y) \, dy$$

In order to get integrability of the kernels we need to replace $g(x)-g(x-y)$ by terms which are of higher order than 1. Therefore we write

$$g(x)-g(x-y) = g(x)-g(x-y) + dg(x) \cdot y - dg(x) \cdot y = R_g(x,y) - dg(x) \cdot y$$

The $R_g(x,y)$ term is convenient since we have

$$g(x)-g(x-y) + dg(x) \cdot y = \int_0^1 (dg(x) - dg(x-ty)) \cdot y \, dt$$

Thus we have the estimates

$$||R_g(x,y)||_{H^{s-1}(dx)} \leq 2||g||_s|y|$$

and

$$||R_g(x,y)||_{L^\infty(dx)} \leq C||g||_s|y|^{1+\varepsilon}$$

for some $0 < \varepsilon < 1$ because of the Sobolev imbedding $H^{s-1} \hookrightarrow C^\varepsilon$. We have similarly

$$R_{f,\varphi}(x,y) := (f(x)-f(x-y)+ (\varphi(x)-\varphi(x-y))^T [d\varphi^{-1}(x)]^T \nabla f(x)) = (f \circ \varphi^{-1})(\varphi(x)) - (f \circ \varphi^{-1})(\varphi(x-y)) + (\varphi(x)-\varphi(x-y))^T \nabla (f \circ \varphi^{-1})(\varphi(x))$$
Now if we restrict $\varphi$ to a small ball we can assume – see [7]
\[
||f \circ \varphi||_s \leq C||f||_s \quad \text{and} \quad |\varphi(x) - \varphi(x - y)| \leq C|y|
\]
for all $f \in H^s$ and $\varphi$ in this ball. With that we get the same estimates
\[
||R_{f,\varphi}(x, y)||_{H^{-1}(dx)} \leq C||f||_s|y|
\]
resp.
\[
||R_{f,\varphi}(x, y)||_{L^\infty(dx)} \leq C||f||_s|y|^{1+\varepsilon}
\]
Using this notation the individual term in the integral looks like
\[
\int_{\mathbb{R}^n} \frac{1}{\alpha!} \partial^n K_1(y)(R_g(x, y) - dg(x) \cdot y)^{\alpha} R_{f,\varphi}(x, y) J_{\varphi}(x - y) dy
\]
Expanding the bracket we see that $2^{[\alpha]} - 1$ terms appear with at least two $R$ terms and one with one $R$ term. Using (13) and (11) one can estimate the $H^{s-1}$ norm of the $2^{[\alpha]} - 1$ integrals as
\[
\int_{\mathbb{R}^n} \frac{|\alpha|!}{\alpha!} C^{[\alpha]} \frac{1}{|y|^{n+1+|\alpha|}} 2^{[\alpha]} e^{-|y|^2/2} C^{[\alpha]} |y|^{1+|\alpha|+\varepsilon} ||g||_s^{[\alpha]} ||f||_s (1 + ||g||_s^n) dy
\]
Summing over $\alpha$ with $|\alpha| = k$ we have the bound
\[
\hat{C}^k ||g||_s^k ||f||_s (1 + ||g||_s^n) \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\varepsilon}} e^{-|y|^2/2} dy
\]
which is the general term for a convergent series for $||g||_s$ small, i.e. for $\varphi$ near id. The remaining term we have to consider is
\[
\int_{\mathbb{R}^n} \frac{1}{\alpha!} \partial^n K_1(y)(-dg(x) \cdot y)^{\alpha} R_{f,\varphi}(x, y) J_{\varphi}(x - y) dy
\]
Expanding the bracket gives $n^{[\alpha]}$ terms of the form
\[
(-1)^{[\alpha]} \partial_{k_1} g^{m_1}(x) \cdots \partial_{k_{[\alpha]}} g^{m_{[\alpha]}}(x) \int_{\mathbb{R}^n} \frac{1}{\alpha!} \partial^n K_1(y) y_{k_1} \cdots y_{k_{[\alpha]}} R_{f,\varphi}(x, y) J_{\varphi}(x - y) dy
\]
for certain $1 \leq k_1, \ldots, k_{[\alpha]}, m_1, \ldots, m_{[\alpha]} \leq n$. Let us introduce $\tilde{K}_j(y) = \partial^n K_j(y) y_{k_1} \cdots y_{k_{[\alpha]}}, j = 1, 2$ and $\tilde{K} = \tilde{K}_1 + \tilde{K}_2$. So we have to examine after a change of variables
\[
\int_{\mathbb{R}^n} \tilde{K}_1(x - y)(f(x) - f(y) + (\varphi(x) - \varphi(y))^\top \cdot [d\varphi^{-1}(x)]^\top \cdot \nabla f(x)) J_{\varphi}(y) dy
\]
We claim that
\[ \text{p.v.} \int_{\mathbb{R}^n} \tilde{K}_1(x-y)(f(x) - f(y))J_\varphi(y) \, dy \]
resp.
\[ \text{p.v.} \int_{\mathbb{R}^n} \tilde{K}_1(x-y)((\varphi(x) - \varphi(y))^\top \cdot [d\varphi^{-1}(x)]^\top \cdot \nabla f(x))J_\varphi(y) \, dy \]
exist separately. Let’s consider the first one. By (12) we see from the considerations regarding \( \mathcal{K}_2(\varphi) \) that
\[
|| \int_{\mathbb{R}^n} \tilde{K}_2(x-y)(f(x) - f(y))J_\varphi(y) \, dy ||_{H^{s-1}(dx)} \leq |\alpha||C^{(\alpha)}||f||_s(1 + ||g||^s) \]
holds. Therefore it’s enough to consider
\[ \text{p.v.} \int_{\mathbb{R}^n} \tilde{K}(x-y)(f(x) - f(y))J_\varphi(y) \, dy \]
But as \( J_\varphi(y) = 1 + j(y) \) with some \( j \in H^{s-1} \) (note that \( \varphi(y) = y + g(y) \)) we can apply Lemma \( \text{(A.2)} \) for \( \tilde{K} \) to get
\[
|| \int_{\mathbb{R}^n} \tilde{K}(x-y)(f(x) - f(y))J_\varphi(y) \, dy ||_{H^{s-1}(dx)} \leq |\alpha||C^{(\alpha)}||f||_s(1 + ||g||^s) \]
We split the second principal value integral into
\[
\left( \text{p.v.} \int_{\mathbb{R}^n} \tilde{K}_1(x-y)(x-y)^\top J_\varphi(y) \, dy \right) \cdot [d\varphi^{-1}(x)]^\top \cdot \nabla f(x) \]
and
\[
\left( \text{p.v.} \int_{\mathbb{R}^n} \tilde{K}_1(x-y)(g(x) - g(y))^\top J_\varphi(y) \, dy \right) \cdot [d\varphi^{-1}(x)]^\top \cdot \nabla f(x) \]
One can handle the second part exactly as above, now \( f \) replaced by \( g \). To handle the first part note that by the Leibniz rule the only term which is critical is the one where all the derivatives fall on the \( 1/|y|^{n+1} \) term in the expression
\[
\tilde{k}(y) := \partial^\alpha \left( \frac{1}{|y|^{n+1}} \right) e^{-|y|^2} y_{k_1} \cdots y_{k_{|\alpha|}} y
\]
The others are integrable and can be estimated altogether by

\[ C^{|\alpha|} |\alpha|! (1 + ||g||_s^n) \int_{\mathbb{R}^n} \frac{1}{|y|^{n-1}} e^{-|y|^2/2} dy \]

So the only remaining integral is

\[ \text{p.v.} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n+1}} e^{-|x-y|^2} (x-y)_{k_1} \cdots (x-y)_{k_{|\alpha|}} (x-y) (1 + j(y)) dy \]

But from [2] we know that for any sphere \( S \) around 0 we have

\[ \int_S \partial^\alpha \frac{1}{|y|^{n+1}} y_{k_1} \cdots y_{k_{|\alpha|}} y dS(y) = 0 \]

Therefore the 1 term in \( 1 + j(y) \) vanishes. Finally consider

\[ \text{p.v.} \int_{\mathbb{R}^n} (\partial^\alpha \frac{1}{|x-y|^{n+1}}) e^{-|x-y|^2} (x-y)_{k_1} \cdots (x-y)_{k_{|\alpha|}} (x-y) j(y) dy \]

The Fourier transform of the kernel above is

\[ \mathcal{F}[\tilde{k}(y)] * F[e^{-|y|^4}](\xi) = \int_{\mathbb{R}^n} \mathcal{F}[\tilde{k}](\xi - \eta) e^{-|\eta|^2} d\eta \]

which by the calculations of [A.2] is seen to be bounded by

\[ |\mathcal{F}[\tilde{k}(y)] * F[e^{-|y|^4}](\xi)| \leq C^{l(|\alpha|)} |\alpha|! \]

Thus the principal value integral is just a Fourier multiplicator operator with a bounded multiplier acting on \( j(y) \). Thus

\[ ||\text{p.v.} \int_{\mathbb{R}^n} (\partial^\alpha \frac{1}{|x-y|^{n+1}}) e^{-|x-y|^2} (x-y)_{k_1} \cdots (x-y)_{k_{|\alpha|}} (x-y) j(y) dy||_{H^{s-1}(dx)} \leq C^{l(|\alpha|)} ||\alpha||_s |||j||_{s-1} \leq C^{l(|\alpha|)} ||\alpha||_s ||g||_s^n \]

So far we have proved that

\[ \mathcal{D}(\mathbb{R}^n) \to L(H^{s}(\mathbb{R}^n); H^{s-1}(\mathbb{R}^n)), \quad \varphi \mapsto R_\varphi \Lambda R_{\varphi}^{-1} \]

is analytic around the identity map \( \text{id} \), i.e. we have a power series expansion in \( g = \varphi - \text{id} \)

\[ R_\varphi \Lambda R_{\varphi}^{-1} = \sum_{k \geq 0} P_k(g, \ldots, g) \]
where $P_k$ is a continuous homogeneous polynomial of degree $k$ with values in $L(H^s(\mathbb{R}^n); H^{s-1}(\mathbb{R}^n))$. The series has a radius of convergence $R > 0$ which means
\[ \sup_{k \geq 0} ||P_k|| r^k < \infty, \quad \forall 0 \leq r < R \]
where $||P_k||$ is the norm given by
\[ \sup_{|g|_s \leq 1, |f|_s \leq 1} ||P_k(g, \ldots, g)(f)||_{s-1} \]
We have to prove that $R_\varphi \Lambda R_\varphi^{-1}$ is analytic around any $\varphi_\bullet \in D^s(\mathbb{R}^n)$. We do the calculations first in the smooth category (e.g. $H^\infty$). Taking the derivative at $\varphi_\bullet$ in direction of $g$ we get
\[ R_\varphi [g \circ \varphi^{-1}, \Lambda] \nabla (f \circ \varphi^{-1}) = R_\varphi P_1(g \circ \varphi^{-1})(f \circ \varphi^{-1}) \]
Similarly the higher derivatives look like
\[ k! R_\varphi P_k(g \circ \varphi^{-1}, \ldots, g \circ \varphi^{-1})(f \circ \varphi^{-1}) \]
These are polynomials which can be extended continuously to all $g, f \in H^s$. Now we have in the smooth category for $\varphi = \varphi_\bullet + g$ the identity
\[ R_\varphi \Lambda R_\varphi^{-1} f - R_\varphi P_1(g \circ \varphi^{-1})(f \circ \varphi^{-1}) = \int_0^t R_\varphi P_1(g \circ \varphi^{-1})(f \circ \varphi^{-1}) \, d\tau \]
where $\varphi_\tau = \varphi_\bullet + \tau g, 0 \leq \tau \leq 1$. By continuity we can extend this to all $g, f \in H^s$ and $\varphi, \varphi_\bullet \in D^s$. So one can conclude that $\varphi \mapsto R_\varphi \Lambda R_\varphi^{-1} f$ is $C^1$ with derivative $R_\varphi P_1(g \circ \varphi^{-1})(f \circ \varphi^{-1})$. Inductively one then shows that it is actually $C^\infty$ with the corresponding derivatives. That $\varphi \mapsto R_\varphi \Lambda R_\varphi^{-1}$ is smooth follows now from general principles – see [12]. Choosing $C > 0$ with $||h \circ \varphi_\bullet||_s \leq C ||h||_s, ||h \circ \varphi^{-1}||_s \leq C ||h||_s$ for all $h \in H^s$ gives
\[ ||R_\varphi P_k(g \circ \varphi^{-1}, \ldots, g \circ \varphi^{-1})(f \circ \varphi^{-1})||_{s-1} \leq C^{k+2} ||P_k|| ||g||_s^k ||f||_s \]
This shows the convergence of the Taylor series. Thus $\varphi \mapsto R_\varphi \Lambda R_\varphi^{-1}$ is analytic.

**Lemma A.2.** Let $K$ be the function
\[ K : y = (y_1, \ldots, y_n) \mapsto (\partial_y^n \frac{1}{|y|^{n+1}}) y^\beta \]
with \( \alpha, \beta \in \mathbb{N}^n, |\alpha| = |\beta| = k \). There is \( C > 0 \) independent of \( k \) with

\[
\| \text{p.v.} \int_{\mathbb{R}^n} K(x - y)(f(x) - f(y)) \, dy \|_{H^{s-1}(dx)} \leq k! C^k \| f \|_s
\]

and

\[
\| \text{p.v.} \int_{\mathbb{R}^n} K(x - y)(f(x) - f(y))g(y) \, dy \|_{H^{s-1}(dx)} \leq k! C^k \| f \|_s \| g \|_{s-1}
\]

for all \( f \in H^s(\mathbb{R}^n), g \in H^{s-1}(\mathbb{R}^n) \).

**Proof.** The Fourier Transform of \( K \) is given by

\[
\hat{K}(\xi) = \frac{1}{(1 - i|\xi|)^{n+1}} \frac{1}{y^{n+1}}(\xi) = \frac{i^{|\alpha|}}{(1 - i|\xi|)^{n+1}} \partial_{\xi}^{|\alpha|} (\xi^{|\alpha|} |\xi|) = (-1)^k \partial_{\xi}^{|\alpha|} (\xi^{|\alpha|} |\xi|)
\]

Thus one has with the same derivation as for (11) and (12)

\[
|\hat{K}(\xi)| \leq k! C^k |\xi|
\]

(14)

Adjusting \( C \) one has in a similar fashion

\[
|\nabla_{\xi} \hat{K}(\xi)| \leq k! C^k
\]

This implies in particular \( |K(\xi) - K(\eta)| \leq k! C^k |\xi - \eta| \) which will be used later. Now note that the first principal value integral is nothing other than the Fourier multiplier \( K(D) \) acting on \( f \). Thus using (14) we can estimate this integral by

\[
\| K(D)f \|_{s-1} = \|(1 + |\xi|^2)^{(s-1)/2} \hat{K}(\xi) \hat{f}(\xi)\|_{L^2} \leq k! C^k \| f \|_s
\]

The second integral is equal to the commutator \([f, K(D)]g\) which can be seen using the integral representation of \( K(D) \) above. Taking the Fourier transform we get

\[
\mathcal{F}[[f, K(D)]g](\xi) = \mathcal{F}[f \cdot K(D)g](\xi) - K(\xi) \mathcal{F}[fg](\xi) = \int_{\mathbb{R}^n} f(\xi - \eta)(K(\eta) - K(\xi)) \cdot g(\eta) \, d\eta
\]

Thus using \( |K(\eta) - K(\xi)| \leq k! C^k |\xi - \eta| \) we can bound

\[
|\mathcal{F}[[f, K(D)]g](\xi)| \leq k! C^k |\hat{f}| \ast |\hat{g}|(\xi)
\]
where \( f' \) is defined by \( \hat{f}'(\xi) = |\xi|\hat{f}(\xi) \). Taking the inverse Fourier transform we see that this convolution is a multiplication of \( H^{s-1} \) functions. Therefore we can bound the second principal value integral by

\[
||[f, K(D)]g||_{s-1} \leq Ck!C^k||f||_s||g||_{s-1}
\]

which is the desired result after adjusting \( C \).

\[\square\]

**Corollary A.1.** For \( 1 \leq k \leq n \) the map

\[
D^s(\mathbb{R}^n) \rightarrow L(H^s(\mathbb{R}^n); H^s(\mathbb{R}^n)), \quad \phi \mapsto [f \mapsto R_\phi R_k R_\phi^{-1} f]
\]

is real analytic.

**Proof.** Denote by \( \chi(\xi) \) the indicator function of the unit ball in \( \mathbb{R}^n \) and by \( \chi(D) \) the corresponding Fourier multiplier. We write

\[
R_k = \chi(D)R_k + (1 - \chi(D))R_k
\]

We consider these two parts seperately. First consider

\[
R_\phi(1 - \chi(D))R_k R_\phi^{-1} = R_\phi(1 - \chi(D))\partial_k \Lambda^{-1} R_\phi^{-1}
\]

We claim that

\[
\phi \mapsto R_\phi(\chi(D) + (1 - \chi(D))\Lambda)R_\phi^{-1}
\]

is real analytic. For the analyticity of \( \phi \mapsto R_\phi \chi(D) R_\phi^{-1} \) one can consult [9]. So one concludes by Lemma A.1 the analyticity of

\[
\phi \mapsto R_\phi \chi(D) R_\phi^{-1} + R_\phi(1 - \chi(D)) R_\phi^{-1} R_\phi \Lambda R_\phi^{-1}
\]

as a map \( D^s(\mathbb{R}^n) \rightarrow L(H^s(\mathbb{R}^n); H^{s-1}(\mathbb{R}^n)) \). As inversion is an analytic process (see Neumann series) one has also by taking the inverse in \( L(H^s(\mathbb{R}^n); H^{s-1}(\mathbb{R}^n)) \) that

\[
\phi \mapsto R_\phi \chi(D) R_\phi^{-1} + R_\phi(1 - \chi(D)) R_\phi^{-1} R_\phi \Lambda R_\phi^{-1}
\]

is real analytic. In particular

\[
\phi \mapsto R_\phi(1 - \chi(D)) \Lambda^{-1} R_\phi^{-1}
\]

is real analytic. Further we have

\[
R_\phi \partial_k R_\phi^{-1} f = df[df]^{-1}
\]

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which is a polynomial expression in the first derivatives of $\varphi$ divided by $\det(d\varphi)$ hence analytic in $\varphi$—see [9] for the division by $\det(d\varphi)$. Thus

$$R_\varphi(1 - \chi(D))R_k R_{\varphi}^{-1} = R_\varphi \partial_k R_{\varphi}^{-1} R_\varphi (1 - \chi(D)) R_{\varphi}^{-1}$$

is real analytic in $\varphi$. Now consider the first part of the splitting of $R_k$. This is treated in [9]. There it is shown that expressions of the form

$$R_\varphi \chi(D) R_k R_{\varphi}^{-1}$$

are analytic in $\varphi$. In the same manner it follows that

$$R_\varphi \chi(D) R_k R_{\varphi}^{-1}$$

is real analytic in $\varphi$. This concludes the proof.

Finally we can give the proof of Proposition 2.1

**Proof of Proposition 2.1.** Consider the map

$$\varphi \mapsto R_\varphi R_k R_{\varphi}^{-1} f$$

which by Corollary 4.1 is analytic. So is its derivative. We take the derivative in direction $w \in H^s(\mathbb{R}^n; \mathbb{R}^n)$ and get

$$R_\varphi (R_k d(f \circ \varphi^{-1})) w \circ \varphi^{-1} - R_\varphi (R_k [d(f \circ \varphi^{-1})] w \circ \varphi^{-1})$$

or using the commutator notation

$$R_\varphi [R_k, (w \circ \varphi^{-1}) \cdot \nabla] (f \circ \varphi^{-1})$$

If we plug in the analytic expression $R_\varphi R_j R_{\varphi}^{-1} g$ for $f$ we see that the expressions appearing in $B(v \circ \varphi^{-1}, v \varphi^{-1}) \circ \varphi$ are analytic expressions of $\varphi$ which proves the proposition.

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