Interactive Mechanical Systems Using Mathematica

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Abstract. This work shows a new proposal for teaching mechanical systems (SM) modelled through differential equations (SED) using the Mathematica package. The adopted strategy considers four stages. In the first, simple SMS, such as the mass-spring system on a plane or the simple pendulum, are analysed. The Lagrangian of each system is determined and the Euler-Lagrange equations are used to obtain the equations of motion. In other cases, the Hamiltonian of the systems is determined and the Hamilton equations are used. Afterwards, the resulting EDS are solved analytically. During the second stage, SED solution programs in Mathematica are developed by using classical numerical methods such as a fourth order Runge-Kutta method (Rk4) and a Runge-Kutta-Feldberg method of order 4-5 (RKF45). The programs are tested with the examples from the previous stage. In the next stage, a set of programs is elaborated in order to determine the equations of motion for any system by using the Euler-Lagrange equations (Hamilton) from the Lagrangian (Hamiltonian) of the system. In the last stage, complex problems such as the movement of planets in the solar system or the movement of a particle over a conical surface are solved. The programs developed during stages 2 and 3 are used in order to obtain a numerical solution to these examples. Finally, interactive graphical interfaces are constructed for each proposed system, and are used to study and analyse the physical phenomena. As a result, engineering students who use and build graphical interfaces have improved their understanding of SED and their use in classical mechanics. In addition, they change their previous ideas about the scope of mechanics and SEDs, they gain confidence in their knowledge and they are able to find solutions to dynamical systems through the use of simple numerical techniques.

1. Introduction

Differential equations courses at Tecnológico de Monterrey are very robust and they seek to develop and boost mathematical modelling and technological skills of students, besides the improvement in their algorithmic skills. Several studies have concluded that students that take this course, are not able to analyse problematic situations within the corresponding context, which is majorly due for two reasons: the poor development of skills for the solution of problems, and the excessive use of algorithmic methods for the solution of differential equations [1]. As a result, training students in mathematical modelling is lost as well as the analysis of complex situations, in order to spend more time for the study of different algorithms. Recent papers [2] pretend to develop the necessary skills on mathematical modelling by using interactive simulators. Several alternatives to study DE have been implemented by using technological tools. In Ref. [3] proposals are made to use computational applications that support learning through the use of graphics, numerical and algebraic computations in order to enhance the ability to solve problems through DE. In Ref. [2] proposals are developed to
use mobile technology in order to build mathematical interactive labs which allow the analysis of different physical phenomena. The author’s proposal is based on developing simulators, and students are asked to answer questions on what they observe during the learning experience and at simultaneously, they build widgets and systems.

Mathematical modelling on teaching differential equations, allows students to develop analysis, problem solution and technology usage skills [3, 4, 5]. It is very common to study mechanical systems on differential equations courses, such as two point masses attached by springs, the dynamics of epidemics and electrical circuit analysis. All of these examples are modelled through systems of linear differential equations, where in most cases it is possible to determine exact analytical solutions. However, the study of more complex systems requires a technique that allows the analysis of different possibilities. For example, a double pendulum where the approximation for small oscillations can not be applied has to be modelled through a system of two non-linear second order differential equations or the equivalent system of four non-linear first order differential equations.

Most professors are used to teach differential equations by incorporating a structured scheme where they show the concepts and analytical methods and finally problems and examples are solved. It is considered that in order to improve the learning skills of students when using a model based on differential equations it is necessary to use computational tools, which allow students to simulate various physical phenomena of their interest.

In this work, we show a simple scheme that incorporates the analysis and study of mechanical systems by using classical numerical methods such as RK4, for the solution of systems of differential equations. The numerical code was written in Mathematica, where it is also possible to develop generic widgets for computational exploration.

2. Methodology

2.1. Proposal

The proposed methodology has been developed in four different stages. Through the first stage, simple mechanical systems are solved analytically. Later, a series of programs are developed to solve systems of differential equations through numerical methods. During the third stage, a general program that builds the system of the equations of motion, is written. Finally, students are asked to generate a graphical user interface to analyse their solutions.

As an example, during the first stage, two simple mechanical systems are analysed. The first system consists of a simple pendulum with small oscillations and another system consisting of two point masses attached by three springs constrained to one-dimensional motion along the x-axis, as shown in figure 1. The two masses are located at $x_1$, $x_2$, their velocities are given by $v_1$ and $v_2$; therefore, the linear momentum of each mass is given by $p_1 = m_1 v_1$ and $p_2 = m_2 v_2$.

![Figure 1. Two masses attached by linear springs.](image)

The springs have elastic constants $k_1$, $k_2$ and $k_3$ and therefore, the Hamiltonian of the system is given by

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2 (x_2 - x_1)^2 + \frac{1}{2}k_3 x_2^2$$

The equations of motion of each mass, are determined through the Hamilton equations [6] as follows
\[ \dot{x}_1 = \frac{\partial H}{\partial P_1}; \quad \dot{x}_2 = \frac{\partial H}{\partial P_2}; \quad \dot{P}_1 = -\frac{\partial H}{\partial x_1}; \quad \dot{P}_2 = -\frac{\partial H}{\partial x_2} \]

The following equations are obtained

\[ \begin{align*}
\dot{x}_1 &= p_1/m_1; \\
\dot{x}_2 &= p_2/m_2; \\
\dot{p}_1 &= -k_1x_1 + k_2(x_2 - x_1); \\
\dot{p}_2 &= -k_3x_2 - k_2(x_2 - x_1)
\end{align*} \]

The solutions when \( x_1(0) = a, \ x_2(0) = b, \ P_1(0) = 0, \ P_2(0) = 0 \) is are given by

\[ \begin{align*}
x_1(t) &= \frac{1}{2}[(a - b) \cos(t) + (a + b) \cos(2t)]; \\
x_2(t) &= \frac{1}{2}[(a - b) \cos(t) - (a + b) \cos(2t)] \\
\end{align*} \]

\[ \begin{align*}
p_1(t) &= -\frac{1}{2} \sin(t) (4(a + b) \cos(t) + a - b); \\
p_2(t) &= \frac{1}{2} \sin(t) (4(a + b) \cos(t) + a + b) \\
\end{align*} \]

The potential \( U(t) \), kinetic \( K(t) \) and total \( E \) energies are given by

\[ \begin{align*}
U(t) &= \frac{1}{4}[(a - b)^2 \cos^2(t) + 4(a + b)^2 \cos^2(2t)] \\
K(t) &= \frac{1}{4}[(a - b)^2 \sin^2(t) + 4(a + b)^2 \sin^2(2t)] \\
E &= \frac{1}{4}(a - b)^2 + (a + b)^2 = \frac{1}{4}(5a^2 + 6ab + 5b^2) \\
\end{align*} \]

Another way for obtaining the classical equations of motion is by using the Lagrangian of the system shown in figure 1 and using the Euler-Lagrange equations. Later, the system can be solved analytically and the solutions to the energies can be plotted for their analysis.

During the second stage several programs based on the Euler, RK4 and RK5 methods for the numerical solution of the resulting equations of motion, are written in Mathematica [7]. The corresponding code for the RK4 method is shown in table 1. Through this code, the numerical and analytical solutions to the mechanical system shown in figure 1, are compared. During the third stage, more general programs are developed, which starting from the Hamiltonian or Lagrangian of the mechanical system, are able to produce numerical solutions. Table 2 shows the code in Mathematica for the solution of the Hamilton equations of motion.

**Table 1. Runge Kutta code**

```mathematica
rk4s[func_, vars_, ci_, {n_, h_}] := Module[{m, vals, funciones, datos, regla, k, i, k1, k2, k3, k4},
m = Length[vars];
vals = ci;
funciones = Prepend[func, 1];
datos = Table[
    regla = Table[vars[[k]] -> vals[[k]], {k, 1, m}];
k1 = h*funciones /. regla;
    regla = Table[vars[[k]] -> vals[[k]] + k1[[k]]/2, {k, 1, m}];
k2 = h*funciones /. regla;
    regla = Table[vars[[k]] -> vals[[k]] + k2[[k]]/2, {k, 1, m}];
k3 = h*funciones /. regla;
    regla = Table[vars[[k]] -> vals[[k]] + k3[[k]], {k, 1, m}];
k4 = h*funciones /. regla;
    vals = vals + (k1 + 2 k2 + 2 k3 + k4)/6, {i, 1, n}];
    datos = Prepend[datos, ci]]
```
Table 2. Hamilton equations code

```mathematica
hamiltonsol[ham_, {q_, p_}, {q0_, p0_}, {n_, h_}] :=
Module[{qp, qa, pa, m, ec1, ec2},
  m = Length[q];
  ec1 = Table[D[ham, p[[i]]], {i, 1, m}];
  ec2 = Table[-D[ham, q[[i]]], {i, 1, m}];
  rk4s[Flatten[{ec1, ec2}], Flatten[{t, q, p}], Flatten[{0, q0, p0}], {n, h}]]
```

Finally, interactive interfaces or widgets are developed by using the Manipulate script of the Mathematica software. In figure 2 an image of this interface is shown. Through this widget, the mass and spring constants can be changed and the time evolution of physical observables such as the kinetic and potential energy and motion through the phase space, are shown.

2.2. Classroom Experience
The development of widgets used to model mechanical systems was incorporated in the course of differential equations during the fall semester of 2016. Twenty eight students were enrolled on this course and they were organized in teams of four members during the analysis and solutions of different mechanical systems such as the motion of a point mass over a conical shaped surface, a simple double and triple pendulum or the three body problem (restricted), with a point mass in a binary solar system. The didactic technique was based on 1) the four stages previously described; 2) Workshop on widget development with Mathematica; 3) Project assignment and 4) Oral presentations. This work was developed in four sessions of one hour and a halve. Additionally, at the end of the project, the students were asked to evaluate the activity through a survey where the students are asked about the design of the activity and technology usage.

![Figure 2. Widget for the system of two interacting masses](image-url)
3. Results
The results from the written reports handed out by students are shown in figure 3. These results show that in general, there is a good mathematical statement of the model assigned to the students (Strat: Solution-Strategy) and the use of the Mathematica Software (Tec: Technology); however, there is a lack of deeper understanding and analysis of the proposed solution (An: Solution Analysis).

![Figure 3. Results based on the graphical interface reports](image)

The results based on the student’s perception on this kind of activities are shown in figure 4. The students consider that the way for obtaining the model and numerical algorithm for its solution is simple. (Alg: Use of algorithms); however, they need a more complete user guide for building the Widgets (Su). Additionally, the students consider that the use of the Mathematica software (Math) is handy, even if the assigned problems are very complex and adequate support was not available.

![Figure 4. Results base on the course evaluation](image)

4. Conclusions
The use of technology is changing the teaching and learning experience in college courses, where new ways of communication are available to the teacher and students. The usage of technological tools allows students to reach a deeper understanding of mathematical concepts involved in each situation. Even more, the use of graphical user interfaces and the programming experience offered by the software Mathematica makes possible for students to reach an understanding of certain physical phenomena that are present in classical mechanics. Technological tools used in courses of mathematics makes possible for students to explore concepts and reach conclusions. However, it is necessary that students gain a deeper understanding of laws in Physics through the creation of their own tools. In this proposal, students are able to understand the importance of models based on differential equations and their applicability in the study of mechanical
systems. Finally, it is found that students change their perception on the scope of classical mechanics and differential equations. Students enhance self-confidence on their own knowledge and are able to solve dynamical and complex systems by using simple numerical techniques.

5. References
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