New methods to reconstruct $X_{\text{max}}$ and the shower energy with high accuracy in large wide-field gamma-ray observatories

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Abstract Novel methods to reconstruct the slant depth of the maximum of the longitudinal profile ($X_{\text{max}}$) of high-energy showers initiated by gamma-rays as well as their energy ($E_0$) are presented. An estimator of $X_{\text{max}}$ is obtained, event-by-event, from its correlation with the distribution of the arrival time of the particles at the ground, or the signal at the ground for lower energies. An estimator of $E_0$ is obtained, event-by-event, using a parametrization that has as inputs the total measured energy at the ground, the amount of energy contained in a region near to the shower core and the estimated $X_{\text{max}}$.

Resolutions about 40 (20) g/cm$^2$ and about 30 (20)% for, respectively, $X_{\text{max}}$ and $E_0$ at 1 (10) TeV energies are obtained, considering vertical showers. These remarkable results clearly validate these new approaches and encourage their applicability in large wide field-of-view gamma-ray observatories. The dependence of the resolutions with experimental conditions is discussed.

Keywords High Energy gamma rays · Wide field observatories · Depth of the shower maximum · energy distribution at the ground · Primary energy reconstruction resolution

1 Introduction

High energy cosmic and gamma rays entering the Earth atmosphere originate Extensive Air Showers (EAS) which may be characterised by the distributions of the number of shower particles $N$ as a function of the traversed atmospheric slant depth $X$ (longitudinal profiles) and/or by the distributions of the particles arriving at the ground level as a function of the distance to the shower core (Lateral Density Function - LDF).

The longitudinal development of gamma-ray initiated showers was historically described by Rossi and Greisen diffusion equations [1] being the well known Greisen [2] and Gaisser-Hillas [3] functions approximate solutions. It can be demonstrated that these functions lead to a quasi-universal shape [4]. This universality can be shown by representing the shower longitudinal profile in the plane ($N' = N/N_{\text{max}}$, $X' = X - X_{\text{max}}$), where $X_{\text{max}}$ is the slant depth of the maximum of the profile and $N_{\text{max}}$ is the number of the shower particles at that depth. In this reference frame, the profile may be seen as a slightly asymmetric Gaussian with variable width and is essentially insensitive to variations induced by the depth of the first interaction [5]. However, at TeV energies or below, the fraction of events where the longitudinal profile do not follow the quasi Gaussian shape may not be negligible. A few of the profiles will have a slower decrease after the $X_{\text{max}}$ or even having a double peak structure. These anomalous shower profile structures are associated with interactions where particles travel several radiation/interaction lengths before interacting, or when one of the sub-products of the interaction takes nearly all of the available energy.

Imaging Air Cherenkov Telescopes (IACTs) collect Cherenkov light produced by the EAS and are able, from the size, intensity and orientation of the projected image in the camera focal plane, to reconstruct the energy and the direction of the primary gamma-ray. Typical energy resolution of 15% – 20% are often quoted for TeV gammas and zenith angles of about 20°, for instance, the resolution of MAGIC-II was measured to be 15% [6].

Ground-based gamma-ray observatories sample the particles (mainly electrons and photons) arriving at the
ground level and from their time and position distributions can determine, with reasonable accuracy, the shower core position and the direction of the primary gamma-ray. The determination of the shower energy has, however, a large uncertainty. Indeed, energy resolutions of the order of several tens of percent are often quoted. For instance, the HAWC collaboration has recently quoted an improved energy resolution of 40% at energies of 10 TeV using a neural network analysis [7].

One of the main limiting factors in the reconstruction of the primary energy for ground arrays is the uncertainty on the position of the first interaction in the atmosphere. Contrary to IACT arrays, there is for ground arrays no direct measurement of the contents of the EAS in the region of the shower maximum, and therefore the shower development stage is unknown. In fact, for showers induced by gamma-rays, with the same energy and zenith angle, the number of particles at the ground is expected to increase with $X_{\text{max}}$. The previous statement is not absolute as the width of the longitudinal profile is also an important factor to determine not only the total energy at the ground but also the fraction of this energy present in the region of the shower core. A larger shower profile width will, for the same $X_{\text{max}}$, have larger energy at the ground and a more substantial fraction of energy in the region near the core. Indeed, it is possible to establish, at fixed primary energy, a correlation between the fraction of the energy in the region near the core and the total energy at the ground ($S_{\text{em}}$).

The energy measured by a detector array with electromagnetic calorimetric capabilities, like Water Cherenkov Detectors (WCD), is by definition highly correlated with $S_{\text{em}}$. Such correlation is explored to build an estimator of $S_{\text{em}}$ (section 2).

$X_{\text{max}}$ is not easily estimated at ground gamma-ray observatories. However, at very high energies in cosmic rays experiments, correlations between $X_{\text{max}}$ and the distribution of the arrival times of the particles at the ground in each event, have been established [8]. Such correlations are exploited to build an estimator of $X_{\text{max}}$ (section 3).

An innovative method for the reconstruction of the primary energy of each event, having as inputs the total measured energy at the ground, the fraction of this energy measured in the region near the shower core and the estimated $X_{\text{max}}$, is then presented (section 4).

Finally, the applicability of such method in real large wide-field gamma-ray observatories is discussed (section 5).

All the present results were obtained using CORSIKA [9] to simulate vertical gamma-ray showers assuming an observatory at an altitude of 5200 m a.s.l, able to make a calorimetric measurement of the incoming particles with detectors placed in a grid of 4 m side. A full study including different zenith angles, detailed simulations of realistic detectors, a wider energy range and also its application to hadronic induced showers, is out of the scope of this work. The main focus of this article is just the explanation of the newly proposed method to achieve an accurate energy reconstruction of EAS between several hundreds of GeV and a few TeV, which is the most challenging energy region. A first glance on its potential is given.

2 Energy distribution at the ground

The density of the shower particles arriving at the ground, as a function of the distance to the shower core, is steeper in the region near the core and flatter at larger distances. This distribution is usually parametrized using the NKG (Nishimura-Kamata-Greisen) formula [10]. The particle density at a given distance from the air shower axis is often used to obtain an estimator of the primary energy, trying to find a region less sensitive to the fluctuations in the shower development, to the primary nature and to array sampling effects. For instance, in the Tibet Air Shower Array, this distance was found to be 50 m [11].

The energy distribution at the ground as a function of the distance $r$ to the core position and its cumulative function, $F(r)$, are shown in figure 1 for an event with $S_{\text{em}} = 96.5$ GeV, $E_0 = 1165.9$ GeV and $X_{\text{max}} = 334$ g cm$^{-2}$. (right) It is shown the respective cumulative function $F(r)$.

![Figure 1](image-url)

Figure 1 (left) The energy distribution at the ground for one event with $S_{\text{em}} = 96.5$ GeV, $E_0 = 1165.9$ GeV and $X_{\text{max}} = 334$ g cm$^{-2}$. (right) It is shown the respective cumulative function $F(r)$. 

The strategy followed in this article is different. Instead of using the shower particle density at the ground at some optimized distance from the shower core, the aim is to characterize the shower development through two variables that will be then used to predict, event by event, the calibration factor between the gamma-ray energy ($E_0$) and the electromagnetic energy arriving at the ground ($S_{\text{em}}$).
$X_{\text{max}}$, whose estimator will be discussed in the next section, is naturally one of these variables. The other, $f_{20}$, which will be the main responsible for the improvement of the energy resolution reached in this article, is defined as the ratio between the energy at the ground collected at a distance lower than 20 m from the shower core and the total energy at the ground. The rationale of this second variable was already introduced in the previous section. For a given $E_0$ and $X_{\text{max}}$, the development of the shower between the $X_{\text{max}}$ region and the ground level will strongly determine $f_{20}$.

An estimator of $S_{\text{em}}$, designated as $A_0$, may be obtained using the correlation between $S_{\text{em}}$ and $F_{50} \equiv F(r_0)$, being $r_0$ a reference distance. In any case, $r_0$ should be greater than 20 m to ensure a good correlation, and lower than some tens of meters to ensure a high number of events where the event footprint, with $r < r_0$, is fully contained within the compact array region of the observatory. For the purpose of this article $r_0$ is set to 50 m.

The correlation between $S_{\text{em}}$ and $F_{50}$ is shown in figure 2 and $A_0$ is parametrized as:

$$A_0 = F_{50} + G F_{50}^\delta$$

where $G$ and $\delta$ are free positive parameters. This parametrization ensures, by construction, that $A_0$ is always greater than $F_{50}$. With $A_0$ and $F_{50}$ in GeV, the best values found to $G$ and $\delta$ were respectively $1.63 \text{GeV}^{0.28}$ and 0.72. The result is shown as a red curve in figure 2.

The obtained resolutions and bias of $A_0$ are summarized in figure 3 as a function of $S_{\text{em}}$. As a reference a primary energy of 1 TeV and 10 TeV corresponds, to a mean value of $S_{\text{em}}$ of 115 GeV and 3 TeV, respectively (see figure 10). Thus, resolutions of about 12% and 5% were found respectively at primaries energies of 1 TeV and 10 TeV while the bias was consistently of just a few %.

In principle, a better estimator of $S_{\text{em}}$ can be obtained using all the measured $F(r)$ and not just the value at a given point $r_0$. With this purpose, $F(r)$, which is a smooth and continuous function, was parametrized as:

$$F(r) = A_1 \left[ 1 - \exp \left( -\frac{k_1 r^{\alpha_1}}{1 + k_2 r^{\alpha_2}} \right) \right].$$

The parameter $A_1$ is the $S_{\text{em}}$ estimator while the terms $k_i r^{\alpha_i}$ describe the steepness of the function, being $k_1 > 0$, $\alpha_1 > 0$ and $0 < k_2 \leq 1$, $0 \leq \alpha_2 < \alpha_1$.

In the limits $r \to 0$ and $r \to \infty$ this parametrization becomes respectively:

$$F(r)/A_1 = [1 - \exp (-k_1 r^{\alpha_1})];$$

$$F(r)/A_1 = \left[ 1 - \exp \left( -\frac{k_1}{k_2} r^{(\alpha_1 - \alpha_2)} \right) \right].$$

which have the form of Weibull cumulative distribution functions.

Indeed, as an example, it is shown in figure 3 that, in the plane $(\ln(-\ln(1 -(F(r)/A_1))), \ln(r))$ and assuming...
A1 = S_{em}, the cumulative function distribution of the event shown in figure 1 (blue points) is well described by the above parametrization.

In this plane a pure Weibull function is just a straight line while to describe this event two straight lines are needed, being the transition between the two regimes in the region 20 m − 30 m (ln(r) = 3 - 3.5). This transition region matches the known behaviour of the energy distribution at the ground with a higher concentration of energy in the core region. Similar fits to a large number of events and whose cumulative functions correspond to quite different steepness in the core region are equally very good.

In a real event S_{em} is not known and the parameter A1, as well as the parameters k1, α1, k2 and α2, have to be fitted. However, it was found that the convergence of this fit is not trivial, as A1 is highly correlated with combinations of the other parameters, and thus elaborated fit strategies will have to be defined, what is beyond the scope of the present article. In these terms A0 will be the S_{em} estimator used hereafter.

The variable f_20 is then defined as F_{20}/A_0. The choice of 20 m for the definition of this variable is a compromise which should be optimized for each specific experiment. Nevertheless, its value should be typically between 15 m and 30 m. Lower values will conflict with the possible experimental resolutions on the shower core, higher values will enter in the region where the cumulative function has a slower increase and also where, for events with the core nearer to the border of the compact region of the array, there will be no direct measurement of the cumulative function.

3 X_{max} reconstruction and resolution

A first order estimation of X_{max} may be obtained observing that the mean value of X_{max} increases with the increase of the electromagnetic energy arriving at the ground (S_{em}), reflecting the increase of the shower size with the primary energy. This correlation is demonstrated in figure 5 where X_{max} is represented as a function of S_{em}. It is then possible to parametrize X_{max} as a function of S_{em} as:

\[ X_{\text{max}}^0 = B_0 + \gamma_0 \log(S_{\text{em}}/\text{GeV}). \] (5)

with B_0 and \gamma_0 parameters tuned to describe the mean behaviour. The best achieved parameterization is shown in figure 5 as a red filled curve (corresponding to \( B_0 = 237.1 \text{ g cm}^{-2} \) and \( \gamma_0 = 62.3 \text{ g cm}^{-2} \)).

A more precise estimate of X_{max} may be obtained exploring the fact that the shower front at the ground is a curved surface. Ideally, if the shower particles were originated in a single point located along the shower axis, for instance at the X_{max}, this surface will be spherical, assuming that all the particles travel basically with the speed of light. The arrival time in each
Figure 6  Arrival time as a function of the distance to the core for an event with $X_{\text{max}} = 339 \text{ g cm}^{-2}$ and $E_0 = 1.3 \text{ TeV}$. The line correspond to a quadratic fit as explained in the text with its $\chi^2/\text{ndf} = 0.93$.

The surface station would then change accordingly as a function of the distance to the shower core and with the primary particle direction. Using a simple geometrical fit, the $X_{\text{max}}$ position would be reconstructed straightforwardly, with an accuracy that would depend on the time resolution of the stations.

In reality, the geometry is more complex, but nevertheless, it is possible to establish a clear correlation between $X_{\text{max}}$ and the arrival time distribution of particles at the ground. As an example in figure 6 this correlation is shown for an event with $X_{\text{max}} = 339 \text{ g cm}^{-2}$ and $E_0 = 1.3 \text{ TeV}$. It was found that most of the events can be described by a quadratic polynomial of the form,

$$t = a + b r + c r^2.$$  \hfill (6)

In fact, the application of the above equation to the time profiles as a function of the distance to the shower core leads to a well behaved $\chi^2/\text{ndf}$ distribution with the distribution maximum peaking at $\sim 1.2$.

The parameter of the quadratic term of the polynomial, $c$, is strongly correlated with $X_{\text{max}}$ (figure 7). The parameter $b$ is nearly independent of $X_{\text{max}}$, and $a$ is associated with the event initial time, $T_0$, usually set to zero when the shower front reaches the shower core position. The dependence of $c$ with $X_{\text{max}}$ can be understood if one assumes that most of the particles produced in a shower come from $X_{\text{max}}$ and the shower particles propagate as spherical front. This is of course an approximation but figure 7 supports it and it helps to build some intuition.

Hence, it is possible to parametrize $X_{\text{max}}$ as a function of $c$ using:

$$X_{\text{max}}^1 = B_1 + \gamma_1 c.$$  \hfill (7)

$B_1$ and $\gamma_1$ are parameters tuned to describe the profile shown in figure 7. The best achieved parametrization is shown by the red curve, with $B_1 = 11.2 \text{ g cm}^{-2}$ and $\gamma_1 = 2.28 \times 10^6 \text{ g s}^{-1}$.

$X_{\text{max}}^1$ does not show any relevant bias even for low $X_{\text{max}}$, as shown in figure 8. Nevertheless, in a few cases, particularly at lower energies where the number of particles arriving at the ground is small, the fit may converge to $c$ values leading to non-physical values of $X_{\text{max}}$. In practice, to be safe, whenever the estimation of $X_{\text{max}}$ from the fit indicates values lower than 300 g cm$^{-2}$, the first order estimation $X_{\text{max}}^0$ is used.

The obtained resolutions as a function of $S_{\text{em}}$, both for $X_{\text{max}}^0$ and $X_{\text{max}}^1$, are summarized in figure 9. Resolutions of about 40 g/cm$^2$ and 20 g/cm$^2$ were found respectively for primaries energies of 1 TeV and 10 TeV.

The two resolutions are similar in the region $A_0^{\text{crX}} \approx 400 - 600$ GeV. To be on the safe side, avoiding possible tail effects we will set $A_0^{\text{crX}} = 600$ GeV. Therefore, the estimator of $X_{\text{max}}$, designated as $X_{\text{max}}^R$, is defined as:

$$X_{\text{max}}^R = \begin{cases} X_{\text{max}}^1 & \text{if } A_0 > A_0^{\text{crX}} \\ X_{\text{max}}^0 & \text{and } X_{\text{max}}^1 > 300 \text{ g cm}^{-2} \\ X_{\text{max}}^0 & \text{otherwise} \end{cases}$$ \hfill (8)
4 Energy reconstruction and resolution

In electromagnetic showers, the production of muons, either via the photo-production of mesons or by the direct creation of muon pairs is quite small [14] and thus can be neglected in the global accounting of the shower energy.

On the other hand, the logarithm of the energy deposited in the atmosphere \(E_0 - S_{\text{em}}\) is linearly correlated with the logarithm of the energy deposited at the Earth surface \(S_{\text{em}}\), as shown in figure [10].

It is then possible to parametrize \(E_0\) as a function of \(S_{\text{em}}\) as:

\[ E_0^{(1)} = S_{\text{em}} + C (S_{\text{em}})^\beta, \tag{9} \]

where \(C\) and \(\beta\) are free positive parameters. This parametrization ensure, by construction, that \(E_0^{(1)}\) is always greater than \(S_{\text{em}}\). The best values found to \(C\) and \(\beta\) were respectively 37.2 GeV^{0.36} and 0.64 and the result is shown as a red curve in figure [10].

Using the above parametrization and \(A_0\) (see section [2], as the estimator of \(S_{\text{em}}\), it is possible to make a first energy reconstruction considering an ideal detector. An energy resolution of about 40% is obtained at 1 TeV.

The coefficient \(C\), in this first calibration attempt, is a constant. However, \(C\) can be shown to be correlated with \(f_{20}\), \(X_{\text{max}}\) and \(S_{\text{em}}\). Indeed, it is shown in figure [11] a striking correlation between \(f_{20}\) and \(C = (E_0 - S_{\text{em}})/(S_{\text{em}})^3\), for events with \(S_{\text{em}} \in [100; 250]\) GeV and \(X_{\text{max}} \in [330; 385]\) g cm\(^{-2}\). The line in the figure is the best linear parameterization imposing that for \(C = 0\) (no energy deposited in the atmosphere), \(f_{20} = 1\) (all the deposited at the ground is at a distance lower than 20 m from the shower core position).
The set of the correlation lines \((f_{20}, C)\) for several \(X_{\text{max}}\) ranges and \(S_{\text{em}} \sim 200\,\text{GeV}\), are shown in figure 12. There is a linear monotonous decrease of the slope \(m\) of these lines with the increase of \(X_{\text{max}}\). In fact, in figure 13 the obtained \(m\) are represented as a function of \(X_{\text{max}}\) for different bins of \(S_{\text{em}}\) together with the best linear parametrization for each \(S_{\text{em}}\) bin.

The extrapolation of these lines for \(X_{\text{max}} = 0\) points to a non-physical small positive value of \(m\) \((b_m \sim 0.011\,\text{GeV}^{-0.36})\), which means that this linear model is no longer valid for \(X_{\text{max}} < 200\,\text{g}\,\text{cm}^{-2}\), which is far below the relevant \(X_{\text{max}}\) region for this article. So we will keep the linear approximation using \(b_m = 0.011\,\text{GeV}^{-0.36}\) for all \(S_{\text{em}}\).

Finally, the slope \(s_m\) of the lines represented in figure 12 are shown in figure 14 as a function of \(\log(S_{\text{em}})\). A linear correlation is found between \(s_m\) and \(\log(S_{\text{em}})\). As such, one can write,

\[
f_{20} = 1 + m(X_{\text{max}}, S_{\text{em}})C(f_{20}, X_{\text{max}}, S_{\text{em}}); \quad (10)
\]

and,

\[
m(X_{\text{max}}, S_{\text{em}}) = b_m + [s_{m0} + s_{m1}\log(S_{\text{em}}/\text{GeV})]X_{\text{max}} \quad (11)
\]

The best achieved parametrization with the above equation is shown in figure 14 with parameters \(s_{m0} = -1.1 \times 10^{-4}\,\text{GeV}^{-0.36}\) \(\text{g}^{-1}\text{cm}^2\) and \(s_{m1} = 1.87 \times 10^{-5}\,\text{GeV}^{-0.36}\) \(\text{g}^{-1}\text{cm}^2\).

Finally,

\[
C(f_{20}, X_{\text{max}}, S_{\text{em}}) = \frac{1 - f_{20}}{-(b_m + [s_{m0} + s_{m1}\log(S_{\text{em}})]X_{\text{max}})} \quad (12)
\]

and,

\[
E_q^{(2)} = S_{\text{em}} + C(f_{20}, X_{\text{max}}, S_{\text{em}})(S_{\text{em}})^3, \quad (13)
\]

which is our best estimator for the primary energy.

Using in the above parametrization, \(A_0\) (see section 2) as the estimator of \(S_{\text{em}}\) and \(X_{\text{max}}^R\) (see section 3) as
the estimator of $X_{\text{max}}$, an energy resolution below 30% and 20% were obtained at 1 TeV and 10 TeV, respectively.

Using instead, the real $X_{\text{max}}$ and $S_{\text{em}}$, these energy resolutions improves to about 8% and 4% . These results may be considered as the ultimate resolutions. The difference between the estimated resolutions and the ultimate resolutions are driven, in the TeV region, mainly by the resolution of the $S_{\text{em}}$ estimator. Indeed, a resolution of 12% on the estimation of the total energy at the ground, setting $X_{\text{max}}$ to the simulated value, will lead to a resolution of about 22%.

In figure 15 are shown (full red thick line) the estimated energy resolution as a function of the primary energy. For comparison, the equivalent resolutions obtained applying the constant $C$ calibration (full red thin line), defined by equation [9] or using systematically the $X_{\text{max}}^0$ estimator (dashed red thin line) are also shown, as well as, the resolutions that would be obtained using the simulated value of $X_{\text{max}}$ (dashed blue line), or the simulated value of $S_{\text{em}}$ (pointed blue line) or finally using both simulated values (blue thin line).

These results were obtained considering an ideal detector. The degradation factors due to the detector effects are briefly discussed in the next section.

5 Discussion and Conclusions

In this article, new and innovative methods for the determination of the total electromagnetic energy at the ground, of the slant depth of the maximum of the longitudinal profile and of the primary energy were proposed. In particular, as much as we know, it is the first time that the steepness in the core region of the cumulative function of the energy arriving at the ground is used as a determinant factor to obtain the energy calibration constants.

The obtained results are impressive and open new physics avenues. The improvement on the energy resolution has, as a direct consequence, a meaningful increase of the sensitivity of the Observatories. The obtained resolutions on $X_{\text{max}}$ will allow very interesting studies on the development of the shower, namely on the characterization of the primary mass composition of hadronic cosmic rays showers.

These results, even if obtained considering an ideal detector, are quite robust as they rely only on the estimation of the energy at the ground, $S_{\text{em}}$, and of the slant depth of the maximum of the longitudinal profile, $X_{\text{max}}$, which may be, in a first-order, also obtained from the $S_{\text{em}}$. Typically, a region of a few tens of meters around the core and an energy at the ground of several tens of GeV have to be measured to be able to efficiently apply the proposed algorithms.

The estimation of $S_{\text{em}}$ is thus the critical factor to be able to achieve a good resolution on the reconstruc-
tion of the primary gamma-ray energy. Further progress may be envisaged using more sophisticated $S_{em}$ estimators, like the one suggested by the cumulative function parametrization described in equation (2) or applying, for instance, machine learning techniques. All the present results were obtained using vertical showers. Inclined showers would imply lower energies at the ground. For instance, at 1 TeV an inclined 30° shower would deposit at the ground around less 20% than a vertical shower of the same energy (see, for instance [15]). Therefore, at lower energies, the energy resolutions will be worst, scaling with $S_{em}$. However, the ability to measure higher energies may improve as the probability of the depth of shower maximum to be above the ground surface will increase.

A detailed study of the performance of the new reconstruction methods for a specific Wide Field of View Gamma Ray Observatory is out of the scope of this work. Nevertheless, it is important to discuss briefly possible factors that would contribute to the degradation of the results obtained in the previous sections. Most of these effects are mainly critical for primary energies below a few hundreds of GeV, corresponding to energies at the ground of a few tens of GeV, where a detailed design and simulation of the detectors would be mandatory.

The following factors were considered:

- **The amount of signal (p.e.) generated in each station**
  The electromagnetic energy deposited in electrons (p.e.) using as conversion factors $n_{photon} \sim 40 F$ p.e./GeV where $F$ is, following [16], a scale factor. $F = 1$ corresponds to the existing performance at HAWC. Much higher values of $F$ were found in LATTES end-to-end simulations ([17]). Fluctuations may then be generated using a Poisson distribution. Anyway, for energies at 1 TeV or above the total number of generated p.e. per event is above several thousands of p.e., even for the more conservative $F$ scale factors. The uncertainty in the measurements of $F_{50}$ and $F_{20}$ is then, at lower energies of the order of just a few % and at higher energies negligible, which in any case will be translated at most in a minor increase of the obtained energies resolutions.

- **The precision on the location of the shower core**
  The reconstructed shower core position should, for each event, be smeared by a few meters following the resolutions quoted in [17] (3 m at 1 TeV, < 1 m at 10 TeV). This would introduce distortions in the cumulative functions which would correspond to an increase of at most a few % in the value of the resolution of the $A(S_{em})$ estimator.

- **The finite dimension of the compact array region of the observatory**
  The distance of the core position to the border of the compact array region of the observatory will determine the fraction of the event footprint at the ground that would be measured. However, it is possible to imagine more sophisticated methods, using, for instance, neuronal networks, to make a reasonably accurate estimation of the $S_{em}$ even in the case where there is only a partial containment of the shower footprint in the required region around the core.

- **The time resolution at each detector station**
  Time resolutions better than 2 ns are crucial for good angular resolutions ([18]). Assuming that the detectors would comply with a 2 ns time resolution, the particle arrival times should be smeared and the $X_{max}$ estimator (see section 3) re-computed. Once again, this effect should be mainly important for lower energies; at higher energies, above a few TeV, there will be a high number of hit stations and the curvature fit will be more robust. Nevertheless, giving up the possibility to compute, for each event, the curvature of the shower front, the systematic uncertainty on the particle arrival times should be smeared and the $X_{max}$ estimator would have only impact at energies above a few TeV degrading the energy resolutions at 10 TeV from 20% to 25%, as shown by the dashed red line in figure [15].

The results obtained, representing such a considerable improvement concerning the presently quoted energy resolutions of the existing or planned Wide Field of View Gamma-Ray Observatories, clearly will encourage detailed simulations and studies on the applicability of the proposed methods.

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