A REVERSE HILBERT-LIKE OPTIMAL INEQUALITY

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Abstract. We prove an inequality on positive real numbers, that looks like a reverse to the well-known Hilbert inequality, and we use some unusual techniques from Fourier analysis to prove that this inequality is optimal.

1. Introduction and Notation

This research was initiated by a proposed problem to the American Mathematical Monthly [2], where it was asked to prove that

\[(\sum_{j=1}^{n} a_j b_j)^2 - 2 \left( \sum_{j,k=1}^{n} \frac{a_j a_k}{(b_j + b_k)^2} \right)^2 \leq 2 \left( \sum_{j,k=1}^{n} \frac{a_j a_k}{(b_j + b_k)} \sum_{l,m=1}^{n} a_l a_m (b_l + b_m)^3 \right)^{1/2} \]

for positive real numbers \(a_1, \ldots, a_n\) and \(b_1, \ldots, b_n\). Our aim is not to prove or to discuss this inequality, but to notice that its form suggests the possibility of a typographic error in the denominator of the second term on left, should it be \((b_j + b_k)^2\) instead of \((b_j + b_k)^2?\). In this note we show that the rectified version of this inequality does not hold, but rather another one with a larger constant on the right side, and we will show this constant is the best possible. So, let us fix some notation and describe this work.

For a positive integer \(n\) and two vectors \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\) of positive real numbers we consider the quantities

\[T_{a,b} = \sum_{k=1}^{n} \frac{a_k}{b_k},\] (1)

and

\[S_{a,b}^{(m)} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{a_k a_l}{(b_k + b_l)^m} \text{ for } m = 1, 2, 3.\] (2)

In Proposition 2.1 we prove that, for every positive integer \(n\) and every vectors \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\) of positive real numbers, we have

\[(T_{a,b})^2 \leq 2S_{a,b}^{(2)} + 2\sqrt{2} \sqrt{S_{a,b}^{(1)} S_{a,b}^{(3)}}\] (3)

The difficulty does not reside in the proof of (3) but, in fact, it resides in showing that it is optimal in the sense that \(2\sqrt{2}\) is the best possible constant. Precisely, we will prove in Theorem 2.5 that if for every positive integer \(n\) and every vectors \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\) of positive real numbers, we have \((T_{a,b})^2 \leq 2S_{a,b}^{(2)} + \lambda \sqrt{S_{a,b}^{(1)} S_{a,b}^{(3)}}\) then \(\lambda \geq 2\sqrt{2}.

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This appears a difficult task, and requires tools from approximation theory and Fourier analysis. Indeed, we will prove in Proposition 2.4 that, for every $h > 0$ there exists two families of positive numbers $(a_j(h))_{j \in \mathbb{Z}}$ and $(b_j(h))_{j \in \mathbb{Z}}$ such that

$$\forall t \geq 0, \quad \left| \frac{1}{(1 + t)^2} - \sum_{j \in \mathbb{Z}} a_j(h)e^{-b_j(h)t} \right| \leq \frac{\delta(h)}{(1 + t)^2}$$

with $\lim_{h \to 0^+} \delta(h) = 0$, and this will be exploited in proving the announced optimality result.

2. The Main Results

In the next proposition, we give a proof of (3).

**Proposition 2.1.** For every positive integer $n$ and every vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of positive real numbers, we have

$$(T_{a,b})^2 \leq 2S_{a,b}^{(2)} + 2\sqrt{2}S_{a,b}^{(1)}S_{a,b}^{(3)}.$$  

**Proof.** Consider the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, defined by

$$f(t) = \sum_{j=1}^n a_j e^{-b_j t}$$

Using the Cauchy-Schwarz inequality, we have

$$\left( \int_0^\infty f(t)dt \right)^2 = \left( \int_0^\infty \frac{1}{1 + t} (1 + t)f(t)dt \right)^2$$

$$\leq \left( \int_0^\infty \frac{dt}{(1 + t)^2} \right) \left( \int_0^\infty f^2(t)dt + 2 \int_0^\infty tf^2(t)dt + \int_0^\infty t^2 f^2(t)dt \right)$$

$$= 2 \int_0^\infty t^2 f^2(t)dt + \int_0^\infty f^2(t)dt + \int_0^\infty t^2 f^2(t)dt.$$  \hspace{1cm} (4)

Noting that $\int_0^\infty t^m e^{-bt}dt = \frac{m!}{b^{m+1}}$ for $m = 0, 1, 2$, we obtain

$$\int_0^\infty f(t)dt = T_{a,b}, \quad \int_0^\infty t^m f^2(t)dt = m! S_{a,b}^{(m)},$$

and the (4) becomes

$$(T_{a,b})^2 \leq S_{a,b}^{(1)} + 2S_{a,b}^{(2)} + 2S_{a,b}^{(3)}.$$  \hspace{1cm} (5)

Applying (5) to $\lambda a = (\lambda a_1, \ldots, \lambda a_n)$ and $\lambda b = (\lambda b_1, \ldots, \lambda b_n)$ for some $\lambda > 0$, we obtain

$$(T_{a,b})^2 \leq \lambda S_{a,b}^{(1)} + 2S_{a,b}^{(2)} + \frac{2}{\lambda} S_{a,b}^{(3)}$$

and the desired inequality follows by choosing $\lambda = \sqrt{2S_{a,b}^{(3)} / S_{a,b}^{(1)}}$. \hfill \Box
Analyzing the preceding proof, we see that in order to prove the optimality of (3), and to have equality we need the function $t \mapsto f(t)$ to be proportional to $t \mapsto 1/(1 + t)^2$, but this is impossible since the first has an exponential decay at $+\infty$. This remark holds the idea of what we will do next! We will look for “almost” equality by approximating $t \mapsto 1/(1 + t)^2$ by a linear combination of decreasing exponentials with positive coefficients. The next Proposition 2.4 provides us with the desired conclusion. But before we proceed, we will need the next two technical lemmas.

**Lemma 2.2.** The necessary and sufficient condition, on the positive parameter $\lambda$, for the following inequality to hold, for $x \in \mathbb{R}$,

$$\frac{\pi x (1 + x^2)}{\sinh(\pi x)} \leq \frac{1}{\cosh^2(\lambda x)}$$

is that $\lambda \leq \lambda_0 \overset{\text{def}}{=} \sqrt{\frac{\pi^2}{6} - 1} \approx 0.803078$.

**Proof.** Suppose that the proposed inequality is satisfied for some $\lambda > 0$ then we must have

$$\frac{(1 + x^2) \cosh^2(x) - 1}{x^2} \leq \frac{1}{x^2} \left( \frac{\sinh(\pi x)}{\pi x} - 1 \right)$$

for every nonzero $x$. Letting $x$ tend to 0 we obtain $1 + \lambda^2 \leq \frac{\pi^2}{6}$.

Conversely, let $\lambda_0 = \sqrt{\frac{\pi^2}{6} - 1}$, and consider the function

$$f(x) = \frac{\sinh(\pi x)}{\pi x} - (1 + x^2) \cosh^2(\lambda_0 x) = \frac{\sinh(\pi x)}{\pi x} - \frac{1}{2}(1 + x^2)(1 + \cosh(2\lambda_0 x)).$$

The power series expansion of $f$ is given by

$$f(x) = \sum_{n=2}^{\infty} (1 - a_n) \left( \frac{\pi x}{2n+1} \right)^{2n}$$

where

$$a_n = (2n + 1) \left( \frac{2\lambda_0}{\pi} \right)^{2n-2} \left( \frac{1}{3} + \frac{2n^2 - n - 2}{\pi^2} \right)$$

Now,

$$\frac{a_{n+1}}{a_n} = \left( \frac{2}{3} - \frac{4}{\pi^2} \right) \left( 1 + \frac{2}{2n+1} \right) \left( 1 + \frac{12n + 3}{6n^2 - 3n + \pi^2 - 6} \right)$$

From this, it is straightforward to see that the sequence $\left( \frac{a_{n+1}}{a_n} \right)_{n \geq 2}$ is decreasing, and that $\frac{a_2}{a_2} \approx 0.8177 < 1$. Thus, $a_n \leq a_2 \approx 0.96531 < 1$ for every $n \geq 2$. This proves that $f(x) \geq 0$ for every real number $x$, and the proposed inequality follows for $\lambda \in [0, \lambda_0]$. \qed

**Lemma 2.3.** For $t \geq 0$, let $f_t : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$f_t(x) = e^{2x - (1+t)e^x}.$$

Then the Fourier transform $\hat{f}_t = \int_{\mathbb{R}} f_t(x)e^{ix(x)} \, dx$ of $f_t$ satisfies

$$|\hat{f}_t(w)| = \frac{1}{(1+t)^2} \sqrt{\frac{\pi w(1 + w^2)}{\sinh(\pi w)}} \leq \frac{1}{(1+t)^2} \cdot \frac{1}{\cosh(\lambda_0 w)}.$$


Proof. Indeed we have
\[ \hat{f}(w) = \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx = \int_{-\infty}^{\infty} e^{(2-iw)x} e^{-(1+t)e^x} \, dx, \quad \text{setting } s \leftarrow (1+t)e^x, \]
\[ = \frac{1}{(1+t)^{2-iw}} \int_0^\infty s^{1-iw} e^{-s} \, ds = \frac{\Gamma(2-iw)}{(1+t)^2}, \]
where \( \Gamma \) is the well-known Eulerian Gamma function \(^{[4]}\). Therefore
\[ \left| \hat{f}(w) \right|^2 = \frac{\Gamma(2-iw)}{(1+t)^{2-iw}} \cdot \frac{\Gamma(2+iw)}{(1+t)^{2+iw}} \]
\[ = \frac{1}{(1+t)^4} \cdot (1-iw)(1+iw)iw\Gamma(1-iw)\Gamma(iw) \]
\[ = \frac{1}{(1+t)^4} \cdot (1+iw) \cdot \frac{i\pi w}{\sin(i\pi w)} \]
\[ = \frac{1}{(1+t)^4} \cdot (1+iw) \cdot \frac{\pi w}{\sinh(\pi w)}. \]
Here we used Euler’s reflection formula for the Gamma function: \( \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \) (see \(^{[4]}\) Chapter 6, formula 6.1.17). Finally
\[ \left| \hat{f}(w) \right| = \frac{1}{(1+t)^2} \sqrt{\frac{\pi w(1+w^2)}{\sinh(\pi w)}}, \]
and the proposed inequality follows from Lemma 2.2. \( \Box \)

In the next proposition we prove the announced approximation result. In fact, the approach consists of approximating the function \( t \mapsto \frac{1}{(1+t)^2} \), written as an integral, with of a positive function of exponential type, using the trapezoidal quadrature rule, making use of Poisson’s formula to yield a good control on the committed error. For more details on this approach, we refer the reader to \(^{[3]}\) and the references therein. The details of the proof are provided for the convenience of the reader.

**Proposition 2.4.** For \( h > 0 \) and \( n \in \mathbb{Z} \), let
\[ a_n(h) = h \exp{(2nh - e^{nh})}, \quad b_n(h) = e^{nh}, \]
then
\[ \forall t \geq 0, \quad \left| \frac{1}{(1+t)^2} - \sum_{n \in \mathbb{Z}} a_n(h)e^{-b_n(h)} \right| \leq \frac{\delta(h)}{(1+t)^2} \]
with
\[ \delta(h) = \frac{4}{\exp{(\frac{2\lambda_0}{k})}} - 1, \]
where \( \lambda_0 \) was defined in Lemma 2.2.

**Proof.** Noting that, for \( t \geq 0 \) we have
\[ \frac{1}{(1+t)^2} = \int_0^\infty u e^{-(1+tu)} \, du = \int_{-\infty}^{\infty} f_t(x) \, dx \]
where \( f_t \) is the positive function defined in Lemma 2.2. The function \( f_t \) is super-exponentially decreasing for positive \( x \) and exponentially decreasing for negative \( x \). A simple upper bound for \( f_t \) is obtained as follows, for \( x \geq 0 \) we have

\[
2x - (1 + t)e^x \leq 2x - e^x \leq 2e^{x-1} - e^x = (2 - e)e^{x-1} \leq (2 - e)x
\]  

since \( x \leq e^{x-1} \) for every real \( x \). And, for \( x < 0 \), we have

\[
2x - (1 + t)e^x < 2x < (e - 2)x
\]  

Combining (8) and (9) we see that \( f_t(x) \leq e^{(2-e)|x|}, \) for \( x \in \mathbb{R} \).

This simple upper bound shows that the series \( \sum_{n \in \mathbb{Z}} f_t(x + nh) \) is uniformly convergent on every compact subset of \( \mathbb{R} \). Therefore, we define an \( h \)-periodic continuous function \( F_t \) by the formula

\[
F_t(x) = \sum_{n \in \mathbb{Z}} f_t(x + nh).
\]  

Moreover, the exponential Fourier coefficients \( (C_m(F_t))_{m \in \mathbb{Z}} \) of \( F_t \) are given by

\[
C_m(F_t) = \frac{1}{h} \int_0^h F_t(x)e^{-2\pi mx/h}dx = \frac{1}{h} \sum_{n \in \mathbb{Z}} \int_0^h f_t(x + nh)e^{-2\pi mx/h}dx = \frac{1}{h} \sum_{n \in \mathbb{Z}} \int_{nh}^{(n+1)h} f_t(x)e^{-2\pi mx/h}dx = \frac{1}{h} \int_{-\infty}^{\infty} f_t(x)e^{-2\pi mx/h}dx = \frac{1}{h} \hat{f_t} \left( \frac{2\pi m}{h} \right)
\]  

where \( \hat{f_t} \) is the Fourier transform of \( f_t \). In particular, according to Lemma 2.3 the Fourier series of \( F_t \) is normally convergent, and consequently it is equal to \( F_t \). Taking the value at \( x = 0 \) we get

\[
h \sum_{n \in \mathbb{Z}} f_t(nh) = \sum_{m \in \mathbb{Z}} \hat{f_t} \left( \frac{2\pi m}{h} \right)
\]  

Using (7) and Lemma 2.3 we get

\[
\left| \frac{1}{(1 + t)^2} - h \sum_{n \in \mathbb{Z}} f_t(nh) \right| \leq 2 \sum_{m = 1}^{\infty} \left| \hat{f_t} \left( \frac{2\pi m}{h} \right) \right| \leq \frac{2}{(1 + t)^2} \sum_{m = 1}^{\infty} \cosh(2\pi \lambda_0 m / h) \leq \frac{4}{(1 + t)^2} \sum_{m = 1}^{\infty} \exp \left( -\frac{2\pi \lambda_0 m}{h} \right) = \frac{\delta(h)}{(1 + t)^2},
\]  

and the proposition follows.

Now, we have what we need to prove the next result.
Theorem 2.5. Consider a positive real constant \( \lambda \) such that, for every positive integer \( n \) and every vectors \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) of positive real numbers, we have
\[
(T_{a,b})^2 \leq 2S_{a,b}^{(2)} + \lambda \sqrt{S_{a,b}^{(1)}S_{a,b}^{(3)}},
\]
(13)
Then \( \lambda \geq 2\sqrt{2} \).

Proof. Consider \( h > 0 \) and let the families \( (a_n(h))_{n \in \mathbb{Z}} \) and \( (b_n(h))_{n \in \mathbb{Z}} \) be defined as in Proposition 2.4. Accordingly we have
\[
1 - \delta(h) \leq \frac{\sum_{n \in \mathbb{Z}} a_n(h)e^{-b_n(h)t}}{1 + t} \leq \frac{1 + \delta(h)}{(1 + t)^2},
\]
we conclude that
\[
1 - \delta(h) \leq \int_0^\infty \left( \sum_{n \in \mathbb{Z}} a_n(h)e^{-b_n(h)t} \right) dt = \sum_{n \in \mathbb{Z}} \frac{a_n(h)}{b_n(h)}
\]
(14)
and, for \( m = 0, 1, 2, \)
\[
\int_0^\infty t^m \left( \sum_{n \in \mathbb{Z}} a_n(h)e^{-b_n(h)t} \right)^2 dt \leq (1 + \delta(h))^2 \int_0^\infty \frac{t^m}{(1 + t)^4} dt
\]
(15)
This yields
\[
\sum_{(k,l) \in \mathbb{Z}^2} \frac{a_k(h)a_l(h)}{b_k(h) + b_l(h)} \leq \frac{(1 + \delta(h))^2}{3},
\]
(16)
\[
\sum_{(k,l) \in \mathbb{Z}^2} \frac{a_k(h)a_l(h)}{b_k(h) + b_l(h)}^2 \leq \frac{(1 + \delta(h))^2}{6},
\]
(17)
\[
\sum_{(k,l) \in \mathbb{Z}^2} \frac{a_k(h)a_l(h)}{b_k(h) + b_l(h)}^3 \leq \frac{(1 + \delta(h))^2}{6}
\]
(18)
Now, according to (14) there is a positive integer \( \nu \) such that
\[
1 - 2\delta(h) \leq \sum_{n = -\nu}^\nu \frac{a_n(h)}{b_n(h)}
\]
(19)
Taking \( n = 2\nu + 1, a = (a_n(h))_{-\nu \leq n \leq \nu} \), and \( b = (b_n(h))_{-\nu \leq n \leq \nu} \), we obtain using (16 – 19):
\[
1 - 2\delta(h) \leq T_{a,b},
\]
\[
S_{a,b}^{(1)} \leq \frac{(1 + \delta(h))^2}{3}, \quad S_{a,b}^{(2)} \leq \frac{(1 + \delta(h))^2}{6}, \quad S_{a,b}^{(3)} \leq \frac{(1 + \delta(h))^2}{6}
\]
and from (13) we conclude that
\[
(1 - 2\delta(h))^2 \leq \frac{(1 + \delta(h))^2}{3} \left( 1 + \frac{\lambda}{\sqrt{2}} \right).
\]
Letting \( h \) tend to 0 and recalling that \( \lim_{h \to 0} \delta(h) = 0 \) we obtain \( \lambda \geq 2\sqrt{2} \). □
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