Elementary Proof for Asymptotics of Large Haar-Distributed Unitary Matrices

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Abstract

We provide an elementary proof for a theorem due to Petz and Réffy which states that for a random $n \times n$ unitary matrix with distribution given by the Haar measure on the unitary group $U(n)$, the upper left (or any other) $k \times k$ submatrix converges in distribution, after multiplying by a normalization factor $\sqrt{n}$ and as $n \to \infty$, to a matrix of independent complex Gaussian random variables with mean 0 and variance 1.

MSC(2000): 15A52; 60B10. Key words: random matrices, Haar measure on the unitary group, Gaussian matrices.

1 Introduction

The aim of this paper is to give an alternative, elementary proof of a theorem first established by Petz and Réffy in [5], concerning the joint distribution of the upper left $k \times k$ entries of a random unitary $n \times n$ matrix in the limit $n \to \infty$ and formulated as Theorem 1 below. This theorem is of particular interest in quantum statistical mechanics, where one often studies the behavior of a small system (corresponding to dimension $k$) coupled to a heat bath—a much larger system corresponding to dimension $n$. Specifically, Theorem 1 can be used for studying the distribution of the conditional wave function of a system coupled to a heat bath in the relevant limit (in which

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the size of the heat bath tends to infinity). As we show in [4], this distribution typically converges, as a consequence of Theorem 1, to the so-called “GAP” measure [3], which can thus be regarded as the thermal equilibrium distribution of the conditional wave function. We explain this application further in Section 2.

We fix some notation and terminology. Let \( P \) denote probability and \( E \) expectation, \( U(n) \) the group of unitary \( n \times n \) matrices, and Haar(\( U(n) \)) the (normalized) Haar measure on this group, representing the “uniform” probability distribution over \( U(n) \). We write \( (a_{ij}) \) for the matrix with entries \( a_{ij} \). The relevant notion of convergence of probability distributions is weak convergence, also known as “convergence in distribution” of the random variables [1, Sec. 25]. By a complex Gaussian random variable \( G \) with mean 0 and variance \( \sigma^2 \) we mean \( G = X + iY \), where \( X \) and \( Y \) are independent real Gaussian random variables with means \( E_X = 0 \) and \( E_Y = 0 \) and variances \( E X^2 = \sigma^2 / 2 \) and \( E Y^2 = \sigma^2 / 2 \).

**Theorem 1.** If \( (U_{ij}) \) is Haar(\( U(n) \)) distributed, then the upper left (or, in fact, any) \( k \times k \) submatrix, multiplied by a normalization factor \( \sqrt{n} \), converges in distribution, as \( n \to \infty \), to a random \( k \times k \) matrix \( (G_{ij}) \) whose entries \( G_{ij} \) are independent complex Gaussian random variables with mean 0 and variance \( E|G_{ij}|^2 = 1 \).

To understand the factor \( \sqrt{n} \), note that a column of a unitary \( n \times n \) matrix is a unit vector, and thus a single entry should be of order \( 1/\sqrt{n} \). A random \( k \times k \) matrix such as \( (G_{ij}) \), consisting of independent complex Gaussian variables with mean 0 and variance 1, is also called “\( \sqrt{k} \) times a standard non-selfadjoint Gaussian matrix.”

Theorem 1 is a generalization of the familiar fact that the first \( k \) entries of a random unit vector in \( \mathbb{R}^n \) (with uniform probability distribution over the unit sphere), multiplied by a normalization factor \( \sqrt{n} \), converge in distribution to a vector whose \( k \) entries are independent real Gaussian random variables with mean 0 and variance 1

This fact (with \( \mathbb{R}^n \) replaced by \( \mathbb{C}^n \)) is contained in Theorem 1 by specializing to just the first columns of the matrices \( (U_{ij}) \) and \( (G_{ij}) \).

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1 As a physical interpretation of this fact, consider \( N \) classical particles without interaction in a box \( \Lambda \subseteq \mathbb{R}^3 \); a given energy corresponds to a surface in phase space \( \Lambda^N \times \mathbb{R}^{3N} \) given by \( \Lambda^N \times \mathcal{S} \), where \( \mathcal{S} \) is the sphere of appropriate radius \( \propto \sqrt{N} \) in momentum space \( \mathbb{R}^{3N} \); assuming a random phase point with micro-canonical distribution (i.e., uniform on \( \Lambda^N \times \mathcal{S} \)), the marginal distribution of the momentum of the first particle is, in the limit \( N \to \infty \), Gaussian. This fact is part of the justification of Maxwell’s law of the Gaussian distribution of momenta.
The proof of Petz and Réffy is based on the convergence of the joint
distribution of the eigenvalues of a \( k \times k \) submatrix of an unitary matrix
to the corresponding distribution for a \( k \times k \) Gaussian matrix. Our proof,
in contrast, is based on the geometric properties of Gaussian random ma-
trices. While it involves some more cumbersome estimates, it employs only
elementary methods.

2 Application to Typicality of GAP Measures

We briefly describe the application of Theorem 1 in quantum statistical
mechanics.

Consider a quantum system entangled to its environment, so that the
composite has a wave function \( \psi \in \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}} \), with \( \mathcal{H}_{\text{sys}} \) and \( \mathcal{H}_{\text{env}} \) the
Hilbert spaces of the system and the environment. Suppose \( \mathcal{H}_{\text{sys}} \) has dimension \( k \), while \( \mathcal{H}_{\text{env}} \) has very large dimension \( n \). According to the Schmidt
decomposition, every \( \psi \in \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}} \) can be written as

\[
\psi = \sum_{i=1}^{k} c_i \chi_i \otimes \phi_i
\]  

with coefficients \( c_i \in \mathbb{C} \), an orthonormal basis \( \{\chi_1, \ldots, \chi_k\} \) of \( \mathcal{H}_{\text{sys}} \) and an
orthonormal system \( \{\phi_1, \ldots, \phi_k\} \) in \( \mathcal{H}_{\text{env}} \). Relative to any fixed orthonormal
basis \( \{b_1, \ldots, b_n\} \) of \( \mathcal{H}_{\text{env}} \), the coefficients \( U_{ij} = \langle b_j | \phi_i \rangle \) of the \( \phi_i \) form the
first \( k \) rows of an \( n \times n \) unitary matrix, and the uniform distribution over all
\( \psi \)'s with a given reduced density matrix

\[
\rho_{\text{sys}} = \sum_i |c_i|^2 |\chi_i\rangle \langle \chi_i|
\]  

gives rise to (the appropriate marginal of) the Haar measure on \( (U_{ij}) \).

For reasons we explain below, it is of interest to consider, for a fixed but
typical \( \psi \), a random column of \( (U_{ij}) \), or, equivalently, the random vector
(arising from a random choice of \( j \))

\[
\psi_{\text{sys}} = \sum_i c_i U_{ij} \chi_i = \langle b_j | \psi \rangle_{\text{env}} \in \mathcal{H}_{\text{sys}},
\]  

where the scalar product is a partial scalar product. By Theorem 1, in the
limit \( n \to \infty \), each column of \( (U_{ij}) \) has a Gaussian distribution, and any
two columns are independent; as a consequence, by the law of large num-
bers, for typical \( \psi \) the empirical distribution of \( \psi_{\text{sys}} \) approximates a Gaussian distribution on \( \mathcal{H}_{\text{sys}} \) with covariance \( \rho_{\text{sys}} \).

This fact is significant for the proof that the thermal equilibrium distribution of the conditional wave function is the GAP measure, a particular probability distribution on the unit sphere of Hilbert space. Let us explain.

The notion of conditional wave function [2] is a precise mathematical version of the concept of collapsed wave function. Conditional on the state \( b_j \) of the environment, the conditional wave function \( \psi_{\text{sys}} \) of the system is given by the expression (3) (times a normalizing factor). Now replace \( j \) by a random variable \( J \) with the quantum theoretical probability distribution \( \mathbb{P}(J = j) = \left\langle b_j|\psi\right\rangle_{\text{env}}^2 \).

The resulting random vector \( \psi_{\text{sys}} \) is called the conditional wave function. For example, a system after a quantum measurement is still entangled with the apparatus, but its collapsed wave function is a conditional wave function.

Now consider a system kept in thermal equilibrium at a temperature \( 1/\beta \) by a coupling to a large heat bath. Even if we assume that \( \psi \in \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}} \) (with the environment being the heat bath) is non-random, the conditional wave function \( \psi_{\text{sys}} \) is random, and for typical \( \psi \) within the microcanonical ensemble (i.e., for most \( \psi \) relative to the uniform distribution over the subspace corresponding to a narrow energy interval), the distribution of \( \psi_{\text{sys}} \) is a universal distribution that depends only on \( \beta \) (but neither on the details of the heat bath nor on the basis \( \{b_j\} \)). As conjectured in [3] and proven using Theorem 1 in [4], this distribution, the thermal equilibrium distribution of the conditional wave function, is the Gaussian-adjusted-projected (GAP) measure associated with the canonical density matrix of temperature \( 1/\beta \),

\[
\rho_{\beta} = \frac{1}{Z} e^{-\beta H}, \quad Z = \text{Tr} e^{-\beta H}.
\]

For any density matrix \( \rho \), the measure \( \text{GAP}(\rho) \) is defined as follows. Let \( G(\rho) \) be the Gaussian measure on Hilbert space with covariance \( \rho \); multiply \( G(\rho) \) by the density function \( \|\cdot\|^2 \) (adjustment factor) to obtain the measure \( \text{GA}(\rho) \); project \( \text{GA}(\rho) \) to the unit sphere in Hilbert space to obtain \( \text{GAP}(\rho) \).

We now turn to the proof of Theorem 1.

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\(^2\)This fact is similar to Maxwell’s law in the classical setting of Footnote 1: For a typical phase point on \( \Lambda^N \times S \), the empirical distribution of the momenta (over all \( N \) particles) approximates a Gaussian distribution on \( \mathbb{R}^3 \) as \( N \to \infty \). This follows using the law of large numbers from the fact, described in Footnote 1, that the momentum of each particle is Gaussian-distributed, and that the momenta of different particles are independent.
3 First Part of the Proof: Construction of $U_{ij}$

We write $M_j$ for the $j$-th column of any $n \times n$ matrix $(M_{ij})$ and

$$\langle M_j | M_\ell \rangle = \sum_{i=1}^n M_{ij}^* M_{i\ell}, \quad \|M_j\|^2 = \sum_{i=1}^n |M_{ij}|^2.$$  (6)

For $i, j = 1, \ldots, n$ let $G_{ij}$ be i.i.d. complex Gaussian random variables with mean 0 and variance 1. To the $n$ columns of the matrix $(G_{ij})$ apply the Gram–Schmidt orthonormalization procedure, and call the resulting matrix $(U_{ij})$. That is,

$$U_{ij} = \frac{G_{ij} - \Delta_{ij}}{\|G_j - \Delta_j\|}$$  (7)

with

$$\Delta_{ij} = \sum_{\ell=1}^{j-1} \langle G_j | U_\ell \rangle U_{i\ell}.$$  (8)

The procedure fails if the columns of $(G_{ij})$ are linearly dependent, but this event has probability 0. Then, as also remarked in [5], $(U_{ij})$ is Haar($U(n)$) distributed because its first column is uniformly distributed over the unit sphere in $\mathbb{C}^n$, the distribution of the second column conditional on the first column is uniform over the unit sphere in the orthogonal complement of the first column, ..., the distribution of the $j + 1$-st column conditional on the first $j$ columns is uniform over the unit sphere in the orthogonal complement of the first $j$ columns—and this is exactly the Haar measure.

Our method of proof is to show that $|\sqrt{n}U_{ij} - G_{ij}|$ is in fact small if $n$ is large. More precisely, we show that for every $\epsilon > 0$,

$$\mathbb{P}\left(\sum_{i,j=1}^k |\sqrt{n}U_{ij} - G_{ij}| < \epsilon\right) \to 1$$  (9)

as $n \to \infty$. This is called convergence in probability, and to obtain the claim of the theorem we use the known fact [1, Theorem 25.2, p. 284] that convergence in probability implies weak convergence (of the joint distribution of $\sqrt{n}U_{ij}$ for $i, j = 1, \ldots, k$), provided that all random variables are defined on the same probability space. Here, we can assume that for all $i, j \in \mathbb{N}$, the $G_{ij}$ are defined on the same probability space.

4 Second Part of the Proof: Probable Geometry

The proof of (9) is based on the following observations:
• Any two different columns of \((G_{ij})\) tend to be nearly orthogonal.

• Every column of \((G_{ij})\) tends to have norm close to \(\sqrt{n}\).

• The size of every single entry, \(|G_{ij}|\), stays bounded as \(n\) grows.

These statements are to be understood in the sense that they are fulfilled with high probability for sufficiently large \(n\). We now make them precise.

Fix a (small) \(\delta > 0\). Choose \(R > 0\) so large that
\[
P\left(|G_{ij}| < R\right) \geq 1 - \delta .
\]
(10)

Define the following events corresponding to the three bullets above:
\[
A_{n}^{j\ell} := \left\{ \left| \langle G_{j} | G_{\ell} \rangle \right| < \sqrt{\frac{n}{\delta}} \right\}
\]
(11)

\[
B_{j}^{n} := \left\{ \left| \|G_{j}\|^{2} - 1 \right| < \sqrt{\frac{2}{n\delta}} \right\}
\]
(12)

\[
C_{ij}^{n} := \{|G_{ij}| < R\}
\]
(13)

for \(i, j, \ell \leq k\). (\(C_{ij}^{n}\) actually does not depend on \(n\), but never mind.) Each of these events has at least probability \(1 - \delta\): \(A_{n}^{j\ell}\) and \(B_{j}^{n}\) by Chebyshev’s inequality and \(C_{ij}^{n}\) by (10). Thus, the event
\[
D^{n} := \bigcap_{j, \ell = 1}^{k} A_{n}^{j\ell} \cap \bigcap_{j = 1}^{k} B_{j}^{n} \cap \bigcap_{i,j = 1}^{k} C_{ij}^{n}
\]
(14)

has at least probability \(1 - 2k^{2}\delta\), as \(2k^{2}\) is the number of intersecting sets.

We now show that for sufficiently large \(n\), \(D^{n} \subseteq E^{n}\), where
\[
E^{n} := \left\{ \sum_{i,j = 1}^{k} \left| \sqrt{n}U_{ij} - G_{ij} \right| < \varepsilon \right\}
\]
(15)

is the event in the brackets of (9). Since \(\delta\) was arbitrary, this fact implies (9). The remainder of the proof makes no reference to probabilities, but concerns only the inclusion \(D^{n} \subseteq E^{n}\), which can be regarded as an inclusion between subsets of \(\mathbb{C}^{n^{2}}\). Also \(A_{n}^{j\ell}\), \(B_{j}^{n}\), and \(C_{ij}^{n}\) will from now on be regarded as subsets of \(\mathbb{C}^{n^{2}}\). (Now the upper index \(n\) in the notation \(C_{ij}^{n}\) becomes useful.) We thus regard \(G_{ij}\) as fixed numbers, and assume that the matrix \((G_{ij})\) lies in the set \(D^{n}\). When we refer to “the condition \(A_{n}^{j\ell}\)” we mean the condition that the \(n \times n\) matrix \((G_{im})\) lies in the set \(A_{n}^{j\ell}\).
We proceed to show, by induction over $j \in \{1, \ldots, k\}$, that for sufficiently large $n$ we have that for all $(G_{ij}) \in D^n$, and for all $i = 1, \ldots, k$,

$$|\sqrt{n}U_{ij} - G_{ij}| < \frac{\varepsilon}{k^2}$$  \hfill (16)

and there are constants $C_1, \ldots, C_k > 0$ such that for sufficiently large $n$

$$\|\sqrt{n}U_j - G_j\| < C_j.$$  \hfill (17)

From (16) we see that $(G_{ij}) \in E^n$, which is what we need to show. This induction is the contents of the next, and last, section.

5 Third Part of the Proof: Estimates

For $j = 1$, note that $U_1 = G_1/\|G_1\|$. By conditions $B^n_1$ and $C^n_{i1}$,

$$|\sqrt{n}U_{1i} - G_{1i}| = \left|\frac{\sqrt{n}}{\|G_1\|} - 1\right| |G_{1i}| < \frac{2}{\sqrt{\delta n}}R < \frac{\varepsilon}{k^2}$$  \hfill (18)

for sufficiently large $n$. By condition $B^n_1$,

$$\|\sqrt{n}U_1 - G_1\| = \left|\frac{\sqrt{n}}{\|G_1\|} - 1\right| \|G_1\| < \frac{2}{\sqrt{\delta n}}2\sqrt{n} = 4\sqrt{\delta} =: C_1.$$  \hfill (19)

We now collect four estimates. For $\ell < j$ and sufficiently large $n$ we find

$$|\langle G_j | \sqrt{n}U_\ell \rangle| \leq |\langle G_j | \sqrt{n}U_\ell - G_\ell \rangle| + |\langle G_j | G_\ell \rangle| \leq$$

$$\leq \|G_j\| \|\sqrt{n}U_\ell - G_\ell\| + \sqrt{n}/\delta < 2\sqrt{n}C_\ell + \sqrt{n}/\delta =: C'_{\ell} \sqrt{n}$$  \hfill (21)

where we have used the Cauchy–Schwarz inequality, $A^n_{j\ell}$, $B^n_j$, and the induction hypothesis (17). As the next estimate, for $i \leq k$,

$$|\Delta_{ij}| \leq \sum_{\ell=1}^{j-1} |\langle G_j | \sqrt{n}U_\ell \rangle| |\sqrt{n}U_{i\ell}| \leq$$

$$\leq \sum_{\ell=1}^{j-1} C'_{\ell} \frac{1}{\sqrt{n}} \left( |\sqrt{n}U_{i\ell} - G_{i\ell}| + |G_{i\ell}| \right) <$$

$$< \left( \sum_{\ell=1}^{j-1} C'_{\ell} \right) \left( \frac{\varepsilon}{k^2} + R \right) \frac{1}{\sqrt{n}} =: \frac{C''_j}{\sqrt{n}}$$  \hfill (24)
using (21), the induction hypothesis (16), and $C_{it}^n$. As the third estimate, for $j \leq k$
\begin{equation}
\|\Delta_j\|^2 = \sum_{i=1}^{n} |\Delta_{ij}|^2 \leq \frac{1}{n} \sum_{i=1}^{n} \sum_{\ell=1}^{j-1} |(G_j|\sqrt{n}U_\ell)|^2 |U_{i\ell}|^2 < (25)
\end{equation}
\begin{equation}
< \frac{1}{n} \sum_{\ell=1}^{j-1} (C'_\ell)^2 n U_{i\ell}^2 = \sum_{\ell=1}^{j-1} (C'_\ell)^2 =: C''_j
\end{equation}
using (21) and the fact that $U_\ell$ is a unit vector. As the last estimate,
\begin{equation}
\left| \frac{\sqrt{n}}{\|G_j - \Delta_j\|} - 1 \right| < \left( 2 \sqrt{\frac{2}{\delta}} + 2 \sqrt{C''_j} \right) \frac{1}{\sqrt{n}} =: \tilde{C}_j
\end{equation}
which is easily obtained from $\|G_j\| - \|\Delta_j\| \leq \|G_j - \Delta_j\| \leq \|G_j\| + \|\Delta_j\|$ and (26) in the following way:
\begin{equation}
\frac{\sqrt{n}}{\|G_j - \Delta_j\|} - 1 \leq \frac{\sqrt{n}}{\|G_j\| - \|\Delta_j\|} - 1 < (28)
\end{equation}
\begin{equation}
< \frac{\sqrt{n}}{\|G_j\|} - 1 = \frac{1}{\|G_j\|/\sqrt{n} - \sqrt{C''_j}/n} - 1.
\end{equation}
Since, using $B^n_j$, $\|G_j\|/\sqrt{n} > \sqrt{1 - \sqrt{2/\delta}} > 1 - \sqrt{2/\delta}$, and since
\begin{equation}
\frac{1}{1-x} < 1 + 2x
\end{equation}
for sufficiently small $x > 0$, we obtain that, for sufficiently large $n$,
\begin{equation}
\frac{\sqrt{n}}{\|G_j - \Delta_j\|} - 1 < \frac{1}{1 - \sqrt{2/\delta} - \sqrt{C''_j}/n} - 1 < 2 \sqrt{\frac{2}{\delta}} + 2 \sqrt{\frac{C''_j}{n}}.
\end{equation}
Together with an (even narrower) lower bound obtained by similar arguments, this yields (27).
From these four estimates, the first induction claim (16) follows for $j \leq k$ because, for sufficiently large $n$,
\begin{equation}
|\sqrt{n}U_{ij} - G_{ij}| = |\frac{\sqrt{n}}{\|G_j - \Delta_j\|}(G_{ij} - \Delta_{ij}) - G_{ij}| \\
\leq |\frac{\sqrt{n}}{\|G_j - \Delta_j\|} - 1| |G_{ij}| + \frac{\sqrt{n}}{\|G_j - \Delta_j\||\Delta_{ij}| <
\end{equation}
where we have used (27), $C_{ij}^n$, (27) with $\tilde{C}_j/\sqrt{n} < 1$, and (24). The second induction claim (17) follows from

$$< \tilde{C}_j \sqrt{n} R + 2 \frac{C_j''}{\sqrt{n}} < \frac{\varepsilon}{k^2},$$

(34)

This completes the proof.

We close with a remark on the parenthesis in Theorem 1: “the upper left (or, in fact, any) $k \times k$ submatrix.” We elucidate the meaning of “any.” To select a $k \times k$ submatrix means to select $k$ rows and $k$ columns. This selection must be deterministic (i.e., non-random, or at least independent of the $U_{ij}$) but may depend on $n$. Indeed, if the selection depended on the $U_{ij}$, one could, for example, select those rows and columns where $U_{ij}$ happens to be exceptionally close to zero, which would lead to a different asymptotic distribution. On the other hand, for a selection depending on $n$, Theorem 1 remains true: to see this, recall that for a compact group such as $U(n)$, the Haar measure is both left-invariant and right-invariant; as a consequence, Haar$(U(n))$ is invariant under any (non-random) permutation of either the rows or the columns, and thus all $k \times k$ submatrices have the same distribution.

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