Killing–Yano equations with torsion, worldline actions and $G$-structures

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Abstract
We determine the geometry of the target spaces of supersymmetric non-relativistic particles with torsion and magnetic couplings, and with symmetries generated by the fundamental forms of $G$-structures for $G = U(n), SU(n), Sp(n), Sp(1) \cdot Sp(n), G_2$ and $Spin(7)$. We find that the Killing–Yano equation, which arises as a condition for the invariance of the worldline action, does not always determine the torsion coupling uniquely in terms of the metric and fundamental forms. We show that there are several connections with skew-symmetric torsion for $G = U(n), SU(n)$ and $G_2$ that solve the invariance conditions. We describe all these compatible connections for each of the $G$-structures and explain the geometric nature of the couplings.

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1. Introduction

It is well known that worldvolume actions of particles and strings admit $W$-type of symmetries generated by spacetime forms [1]. In string theory, such forms are parallel with respect to a metric connection. So such symmetries exist provided that the spacetime is a manifold with special holonomy and therefore special $G$-structure. These symmetries are part of the chiral $W$-algebra of the worldvolume theory [2, 3] and so they are instrumental in the investigation of quantum theory.

On the other hand for particles, the spacetime forms that generate the symmetries are not always parallel. Instead they satisfy the Killing–Yano (KY) equation with respect to a metric connectionootnote{The connection depends on the choice of couplings in the action.}. This is a weaker condition than that which arises for strings, and first observed in the construction of worldline actions with more than one supersymmetries [4–6]. The existence of such symmetries does not necessarily imply the reduction of either the holonomy of the connection or the $G$-structure of spacetime. Nevertheless a large class of examples can be found by making an identification of the forms that generate the symmetries...
with the fundamental forms of a $G$-structure. Such an analysis has been done for a particle action with only a metric coupling in [7]. In such case, the KY equations are respect to the Lévi-Civitá connection and the structure groups considered are

$$ U(n)(2n), \ SU(n)(2n), \ Sp(n)(4n), \ Sp(n) \cdot Sp(1)(4n), \ G_2(7), \ Spin(7)(8), $$

where in parenthesis is the dimension of the associated manifold$^2 M$. It has been found that apart from the $U(n), \ SU(n)$ and $G_2$ cases, the KY equations imply that the fundamental forms are parallel and so the holonomy group of the Lévi-Civitá connection reduces to $G$. For the $U(n), \ SU(n)$ and $G_2$ $G$-structures, the KY equations do not always imply that the fundamental forms are parallel but rather restrict the $G$-structure of the spacetime, i.e. some of the Gray–Hervella type of classes [8], or a linear combination of them, must vanish. An extension of these results to other $G$-structures from those stated in (1.1) has been given in [9].

In this paper, we shall extend the analysis of the symmetries to particle actions which apart from the metric $g$ also have a 3-form torsion $c$ and magnetic $A$ couplings. The presence of torsion modifies the conditions that are required for the invariance of the action under symmetries generated by spacetime forms. In particular, one finds that

$$ \hat{\nabla}_{j_1}L_{j_2...j_{k+1}} = \hat{\nabla}_{[j_1}L_{j_2...j_{k+1}]} $$

and

$$ (-1)^{k-1}\partial_{j_1}(L^m_{j_2...j_k}c_{j_{k+2}j_{k+3}...j_{k+m}}) - \frac{(k+1)}{6}L^m_{[j_1...j_{k-1}}dA_{j_{k+2}j_{k+3}...j_{k+m}]} = 0, $$

where $L$ is the form that generates the symmetries and $\hat{\nabla}$ is the metric connection with torsion $c$. The magnetic coupling $A$ does not restrict the geometry but there is a condition on $dA$ which will appear in section 2. The first condition (1.2) is the KY equation with respect to the connection $\hat{\nabla}$. This generalizes the standard KY equation which is taken with respect to the Lévi-Civitá connection [10], and has found applications in the integrability of geodesic flows and Klein–Gordon and Dirac equations on curved manifolds [11–17]. In a similar context, the KY equation with respect to $\hat{\nabla}$ has also been considered before in [18, 19]. The second condition is an additional restriction on $c$ which does not arise for particle systems with just a metric coupling.

The main aim of this paper is to solve both invariance conditions (1.2), (1.3) assuming that the forms $L$ that generate the symmetries are the fundamental forms of the $G$-structures$^3$ stated in (1.1). We shall find that for the $G$-structures $U(n), \ SU(n)$ and $G_2$ and for the symmetries generated by some of the fundamental forms, there are several connections with skew-symmetric torsion that solve both (1.2) and (1.3). As a result, the couplings of the particle action are not uniquely determined in terms of the metric and the fundamental forms. This is unlike the string case where all the worldvolume couplings are given in terms of the metric and fundamental forms of the $G$-structures (1.1) that generate the symmetries. We identify all connections which are compatible with both invariance conditions (1.2) and (1.3). In the remaining cases, we show that the KY equation implies that the fundamental forms are parallel with respect to the connection with skew-symmetric torsion $\hat{\nabla}$. Such a connection is unique and so all the couplings of the action are determined in terms of the metric and fundamental forms of the $G$-structure.

Furthermore, we identify the restrictions on the $G$-structures such that KY equation (1.2) admits a solution. Typically, these are expressed as the vanishing conditions of some of the

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$^2$ We assume that the spacetime is a product $\mathbb{R}^4 \times M$ and we shall focus on the particle dynamics on the Riemannian manifold $M$.

$^3$ There are many other $G$-structures that can be considered but the focus will be on those given in (1.1) as they are related to irreducible Riemannian manifolds; see e.g. [9].
Gray–Hervella type of classes, or a linear combination of them. We show that in all cases, the restrictions on the Gray–Hervella type of classes are the same as those required for $M$ to admit a connection $\hat{\nabla}$ with skew-symmetric torsion compatible with the $G$-structure. Therefore on all manifolds that $(1.2)$ has a solution, there is a connection with skew-symmetric torsion $\hat{\nabla}$, which may be different from $\hat{\nabla}$, such that $\hat{\nabla} L = 0$ where $L$ are the fundamental forms of a $G$-structure.

This paper is organized as follows. In section two, we present a derivation of the invariance conditions $(1.2)$ and $(1.3)$ as well as a refinement for the special case that $\hat{\nabla}$ is compatible with the corresponding $G$-structure. In section three, we present the solution of $(1.2)$ and $(1.3)$ for the $U(n)$ structure. In section four, we solve the invariance conditions for the $SU(n)$ structure, and similarly in section five for the $Sp(n)$ and $Sp(n) \cdot Sp(1)$ structures. The analysis of the $G_2$ and $Spin(7)$ cases are given in sections six and seven, respectively.

2. Particle actions and their symmetries

2.1. Invariance of action

Consider particle with $N = 1$ supersymmetry propagating in a manifold $M$. The particle positions are maps $X$ from $\mathbb{R}^{1|1}$ superspace into $M$. The most general action with up to two derivative terms written in superspace can be given \cite{20} as

$$I = -\frac{1}{2} \int dt \, d\theta \left( g_{ij} DX^i \partial_t X^j - \frac{i}{6} c_{ijk} DX^i DX^j DX^k + A_i DX^i \right), \tag{2.1}$$

where $g$ and $c$ is a metric and a 3-form in $M$, respectively, while the magnetic gauge potential $A$ is a locally defined 1-form on $M$. Moreover,

$$D^2 = i\partial_t, \quad D = \partial_\theta + i\theta \partial_t, \tag{2.2}$$

where $(t, \theta)$ are the coordinates of $\mathbb{R}^{1|1}$.

Suppose that this action is invariant under the transformation

$$\delta X^i = L^i_{j_1 \ldots j_{k-1}} DX^{j_1} \ldots DX^{j_{k-1}}, \tag{2.3}$$

where $L$ is a $k$-form on $M$. In such a case, the conditions for invariance of the action are given\footnote{In our conventions, $\nabla_i X_j = \partial_i X_j - \frac{1}{2} c_{ij} X_k$.} in $(1.2)$, $(1.3)$ and

$$dA_{i[j_1} L^i_{|j_2 \ldots j_{k}]}} = 0. \tag{2.4}$$

Observe that $(1.3)$ can be rewritten as

$$\mathcal{L}_L c + (-1)^{k-1} \frac{k - 1}{2} i_L dc = 0, \tag{2.5}$$

where $\mathcal{L}_L = d i_L + (-1)^{k-1} i_L d$ is the Lie derivative with respect to $L$, see e.g. [1]. Therefore, $c$ is invariant provided that either $k = 1$ or $dc = 0$. Similarly, we have that $(2.4)$ can be expressed as

$$i_L dA = 0. \tag{2.6}$$

The invariance conditions $(1.2)$ and $(1.3)$ for the action are different from those that have appeared in 2-dimensional sigma models related to the propagation of strings in curved spaces, see [1, 21]. In particular, the invariance condition in two-dimensional models implies that $L$ is parallel with respect $\hat{\nabla}$ together with $dc = 0$. The commutators of symmetries $(2.3)$
are similar to those investigated in [1, 21] for similar transformations in (1,0)-supersymmetric two-dimensional sigma models and they will not be repeated here.

Before we proceed to investigate the conditions (1.2) and (1.3) for various $G$-structures, let us consider the special case where

$$\hat{\nabla}[L_{i_1...i_k} = 0. (2.7)$$

Clearly this condition implies (1.2). Moreover, it simplifies (1.3). In particular using the integrability condition of (2.7) and a Bianchi identity, one finds that

$$L_{ic} = 0, (2.8)$$

and so (1.3) can be re-expressed as

$$i_{Xc} = 0. (2.9)$$

or equivalently in components

$$L^m [j_1...j_k d c_{j_{k+1}j_{k+2}...j_m} = 0. (2.10)$$

It turns out that for many $G$-structures, the KY equation (1.2) implies (2.7). In such cases, it is simpler to consider (2.9) rather than (1.3).

### 2.2. 1-form symmetries

The simplest case to consider is that for which the symmetry is generated by a 1-form $X$. Then

$$\nabla(X) = 0, \quad L_X c = 0, \quad i_X dA = 0. (2.7)$$

Thus, there are such symmetries provided that $M$ admits a Killing vector field and $c$ is invariant under the action of isometries.

It is instructive to compare these conditions with those that arise by taking $\hat{\nabla}X = 0$ as in (2.7). In this case, one finds that (2.7), (2.9) and (2.4) imply that

$$\nabla(X) = 0, \quad dX = i_{Xc}, \quad i_X dc = 0, \quad i_X dA = 0. (2.12)$$

Clearly (2.12) implies (2.11) but not conversely.

### 2.3. Solution of the invariance equations for general fundamental forms

The solution of (1.2), (1.3) and (2.4) is simplified whenever $L$ is identified with a fundamental form of a $G$-structure. This is because all such forms are invariant under the action of $G$. To solve (1.2) observe that for all $G$-structures in (1.1), the Lie algebra of $G$, $g$, is included in the space of 2-forms, $\Lambda^2(F)$, of the typical fibre $F$ of the tangent bundle of $M$. So, one can write

$$\Lambda^2(F) = g \oplus g^\perp. Using this decomposition, one can decompose $\nabla$ as

$$\nabla = \pi(\hat{\nabla}) \oplus \sigma(\hat{\nabla}). (2.13)$$

Clearly, $\pi(\hat{\nabla})L = 0$ and so (1.2) turns into an equation for $\sigma(\hat{\nabla})$ which is an element of $F \otimes g^\perp$. Note that $\sigma(\hat{\nabla})$ may not be a 3-form, though it will turn out to be the case for all $G$-structures that we shall examine. This decomposition is identical to that used in [7] to solve the KY equation associated with the Lévi-Civitá connection. One of the advantages of this observation is that the results of [7] can be used to determine $\sigma(\hat{\nabla})$ and a new calculation is not needed. However, the presence of torsion leads to weaker conditions on the $G$-structures required for the existence of solutions to the KY equation (1.2) than those we have found in [7] for the existence of solutions to the KY equation associated with the Lévi-Civitá connection.
To solve (1.3) and (2.4), one decomposes $\Lambda^4 (F)$ and $\Lambda^2 (F)$ in irreducible representations of $G$, respectively. The restrictions imposed by (1.3) and (2.4) can be expressed as the vanishing conditions of some of these irreducible representations.

In what follows, the conventions that we shall use for $G$-structures, included the choice of representatives for fundamental forms, are those given in [22]. Moreover, a collection of the expressions for the torsion $\hat{\nabla}$ of the compatible connections $\nabla$ to the $G$-structures in (1.1) in terms of the metric and fundamental forms can also be found in [22].

3. $U(n)$ structure

The form that generates the symmetry is the Hermitian 2-form $\omega(X, Y) = g(X, Y)$ of an almost complex structure $I$, where the metric $g$ is Hermitian with respect to $I$, i.e. $g(IX, IY) = g(X, Y)$. There are two cases to consider depending on whether the almost complex structure $I$ is integrable or not. In the integrable case, this symmetry is identified with a second supersymmetry of the system. In the non-integrable case, the symmetry is again associated with a second supersymmetry but an additional charge should be included in the supersymmetry algebra generated by the Nijenhuis tensor [23]. First, we shall consider the integrable case and then we shall extend our results to manifolds with non-integrable almost complex structures.

3.1. Integrable complex structures

3.1.1. Solution of the conditions. It is convenient to do the analysis of the conditions (1.2) and (1.3) in complex coordinates. For this, decompose $c$ with respect to the complex structure $I$ as

$$c = c^{3,0} + c^{2,1} + c^{1,2} + c^{0,3},$$

where $c^{1,2}$ and $c^{0,3}$ are complex conjugate to $c^{2,1}$ and $c^{3,0}$, respectively. Next decomposing (1.2) in (3,0) and (2,1) parts, one finds that

$$c^{2,1} = -i\partial \omega,$$

where $\partial$ is the holomorphic exterior derivative. The (3,0) part of $c$ is not restricted by (1.2) and so it can be arbitrary.

Next repeating the procedure for (1.3), one finds that from the vanishing of (4,0) component one gets

$$\partial c^{3,0} = 0.$$

The remaining components are identically zero. Thus the full content of both (1.2) and (1.3) equations are the conditions given in (3.2) and (3.3). The above calculation has been made in [24] reaching a similar conclusion. It remains to solve (2.4). This implies that $dA$ is a (1,1) form, i.e. $dA^{3,0} = 0$.

3.1.2. Geometry. It is clear from (3.2) and (3.3) the 3-form $c$ that appears in the action is not uniquely determined in terms of the metric and the Hermitian form $\omega$. This is because the (3,0) component can be any $\partial$-closed (3,0)-form on $M$. A consequence of this is that there is not a unique connection $\hat{\nabla}$ that solves the KY equation (1.2).

A special case is to take $c^{3,0} = 0$ and denote the remaining non-vanishing components with $\hat{c}$. In such case, $\hat{c}$ is entirely given in (3.2) and it is uniquely expressed in terms of the
metric and $\omega$. Moreover, $I$ is parallel with respect to the connection $\hat{\nabla}$ with skew-symmetric torsion $\hat{\tau}$, $\hat{\nabla} I = 0$. $\hat{\nabla}$ coincides with the Bismut connection.

In conclusion, for a fixed metric and complex structure the KY equation (1.2) admits many solutions on any Hermitian manifold. These solutions are parameterized by (3,0)-forms on $M$. If in addition (1.3) is also imposed, then the (3,0)-forms are restricted to be $\delta$-closed.

A physical consequence of the above geometric results is that the 3-form coupling of the action (2.1) for models with two supersymmetries is not uniquely determined in terms of the metric and complex structure.

3.2. Non-integrable complex structures

Next suppose that the almost complex structure $I$ is non-integrable. To investigate the consequences of (1.2) and (1.3) on the geometry, it is convenient to introduce a compatible frame to the $U(n)$ structure as
\[ ds^2 = 2\delta_{\alpha\bar{\beta}} e^\alpha \bar{e}^{\bar{\beta}}, \quad \omega = -i\delta_{\alpha\bar{\beta}} e^\alpha \wedge \bar{e}^{\bar{\beta}}. \] (3.4)

In this frame, the (3,0) component of KY equation (1.2) implies that
\[ \Omega_{\alpha\beta\gamma} = \Omega_{[\alpha,\beta\gamma]}, \] (3.5)

where $\Omega$ is the frame Lévi-Civita connection. This is a geometric restriction which is equivalent to requiring that the Nijenhuis tensor of the almost complex structure is skew-symmetric in all three indices. This is not always the case for every $2n$-dimensional almost Hermitian manifold. In fact, it is required that one of the Gray–Hervella classes [8] must vanish, $W_2 = 0$.

Next the (2,1) part of (1.2) implies that
\[ c_{\alpha\beta\gamma} = 2\Omega_{\alpha\beta\gamma}. \] (3.6)

This is not a geometric condition. It simply expresses some of the components of $c$ in terms of the metric and almost complex structure of $M$.

To proceed with solving (1.3), it is convenient to observe that the geometric condition (3.5) implies that there is another metric connection $\hat{\nabla}$ on $M$ with skew-symmetric torsion $\hat{\tau}$ such that
\[ \hat{\nabla} \omega = 0. \] (3.7)

In particular, we have that
\[ \hat{\tau}_{\alpha\beta\gamma} = 2\Omega_{\alpha\beta\gamma}, \quad \hat{\tau}_{\alpha\beta\gamma} = 2\Omega_{\alpha\beta\gamma}. \] (3.8)

In fact an explicit expression for $\hat{\tau}$ in terms of the $I$ and $g$ can be found in [25]; see also [22]. Writing
\[ \hat{\nabla} = \hat{\nabla} + \hat{S}, \] (3.9)

we observe that
\[ S^{2,1} = 0 \] (3.10)

and
\[ c^{3,0} = \hat{c}^{3,0} + S^{3,0}. \] (3.11)

This decomposition is the same as that in (2.13) for $G = U(n)$ with $\pi(\hat{\nabla}) = \hat{\nabla}$ and $\sigma(\hat{\nabla}) = S$. 

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3.2.1. \( S = 0 \). In this case \( c = \hat{c} \). As a result, (1.3) can be rewritten as (2.9). In turn this implies that
\[
d\hat{c}^{4,0} = d\hat{c}^{3,1} = 0, \tag{3.12}
\]
but the (2,2) part of \( dc \) remains unconstrained.

The Bianchi identities of \( \hat{R} \) together with the integrability condition of (3.7) allow for further conditions on \( \hat{c} \). In particular, one finds that
\[
c^m_{[\alpha_1\alpha_2]S^{\alpha_3\alpha_4]} = 0, \quad \delta\hat{c}^{1,2} = \delta\hat{c}^{3,1}. \tag{3.13}
\]
Moreover, the second condition in (3.12) can be rewritten as
\[
\hat{\nabla}_\beta \hat{c}_{\alpha_1\alpha_2\alpha_3} = 0. \tag{3.14}
\]
Unlike the case with an integrable complex structure, it is not straightforward to solve (3.12). Nevertheless, the conditions (3.12) can be easily checked for particular examples of almost Hermitian manifolds.

3.2.2. \( S \neq 0 \). The investigation of (1.3) can be organized in different ways. One way is to observe that \( C_I \) coincides with the exterior derivative with respect to \( I, d_I \). Then, using \( d_I \hat{c} = 0 \) and that \( S \) is \((3,0) \) and \((0,3) \) form, we can write (1.3) as
\[
d_I S - \frac{1}{2} dS - \frac{1}{2} i_I d\hat{c} = 0. \tag{3.15}
\]
Using again that \( S \) is a \((3,0) \) and \((0,3) \) form, it is easy to prove that the \((4,0) \) part of the above equation gives
\[
d\hat{c}^{4,0} + 2 d\hat{c}^{4,0} = 0. \tag{3.16}
\]
Similarly, the \((3,1) \) component gives
\[
d\hat{c}^{3,1} = 0, \tag{3.17}
\]
and the \((2,2) \) component implies
\[
(d_I S)^{2,2} = 0. \tag{3.18}
\]
Using that \( S = S^{3,0} + S^{0,3} \), the latter can be expressed as the algebraic condition
\[
\hat{c}^\gamma_{\alpha_1\alpha_2} S^\gamma_{\beta_1\beta_2} = \hat{c}^\gamma_{\alpha_1\alpha_2} S^\gamma_{\beta_1\beta_2} = 0. \tag{3.19}
\]
This concludes the analysis of (1.3) for this case. It is also straightforward to see that (2.4) implies that \( dA \) is a \((1,1) \) form on \( M \).

3.2.3. Geometry. It is clear that the KY equation (1.2) can be solved for a family of connections parameterized by a \((3,0) \) and \((0,3) \) form \( S \) subject to the condition that the Nijenhuis tensor of the almost complex structure is a 3-form (3.5). If \( S = 0 \), then there is a unique metric connection \( \hat{\nabla} \) with skew-symmetric torsion \( \hat{c} \) such that the almost complex structure is parallel \( \hat{\nabla} I = 0 \) subject again to the same geometric condition. \( \hat{c} \) is uniquely determined in terms of the metric and the Hermitian form \( \omega \) [25, 22].

The second condition (1.3) imposes additional restrictions on both \( \hat{c} \) and \( S \). These are given in (3.16), (3.17) and (3.19). Only (3.16) and (3.19) restrict \( S \). Both of these have a solution
\[
S^{3,0} = -2\hat{c}^{3,0} \tag{3.20}
\]
that may not be necessarily unique. It is therefore clear that there is more than one connection with skew-symmetric torsion which solves both (1.2) and (1.3) for manifolds with a \( U(n) \)-structure.
4. $SU(n)$ structure

The fundamental forms are the Hermitian form $\omega$ and the (n,0) form $\epsilon$. So symmetries are generated\(^5\) by $\omega$ and $\epsilon$. In an adapted basis, we have

$$ds^2 = 2\delta_{\hat{a}\hat{b}} e^{\hat{a}} e^{\hat{b}}, \quad \omega = -i\delta_{\hat{a}\hat{b}} e^{\hat{a}} \wedge e^{\hat{b}}, \quad \epsilon = \frac{1}{n!} e_{\alpha_1 \ldots \alpha_n} e^{\alpha_1} \wedge \ldots \wedge e^{\alpha_n}, n \geq 3. \quad (4.1)$$

It is apparent that the analysis of the conditions for invariance of the action under the symmetries generated by $\omega$ is the same as that we have presented for the $U(n)$-structures. So we shall solve (1.2), (1.3) and (2.4) for the symmetries generated by $\epsilon$.

First let us consider the KY equation (1.2). Up to a complex conjugation, there are four different arrangements of the indices that this conditions does not vanish identically and so imposes some restriction on the couplings. Performing the calculation in the frame adapted in (4.1), we find that the $n+1, 0$ component gives

$$\hat{\Omega}_{a, \hat{b}} = 0, \quad (4.2)$$

the $(n, 1)$ component gives

$$n\hat{\Omega}_{\hat{b}, a} + \hat{\Omega}_{a, \hat{b}} = 0, \quad (4.3)$$

while for two different arrangements of the anti-holomorphic indices the $(n-1, 2)$ components give

$$n\hat{\Omega}_{\hat{b}, \hat{c}} + \hat{\Omega}_{\hat{c}, \hat{b}} = 0, \quad \hat{\Omega}_{\hat{b}, \hat{c}} \hat{\Omega}_{\hat{c}, \hat{b}} = 0. \quad (4.4)$$

As we have explained all components of the frame connection $\hat{\Omega}$ of $\hat{\nabla}$ that appear in the above expressions belong to $\sigma(\hat{\nabla})$.

The above conditions can be solved to express some of the components of the flux in terms of the geometry and also to find the conditions on the geometry implied by (1.2). In particular, one has

$$c_{\beta a} = 2\hat{\Omega}_{\hat{b}, a}, \quad c_{a \beta y} = 2\Omega_{[a, \beta y]} \quad (4.5)$$

and

$$\hat{\Omega}_{a, \hat{b}} = \hat{\Omega}_{\hat{b}, a}, \quad \Omega_{a, \beta y} = \Omega_{[a, \beta y]} \quad (4.6)$$

In (4.5) some of the components of $c$ are expressed in terms of the geometry. Observe that the $(2,1)$ and $(1,2)$ and traceless component of $c$ is not restricted.

Both conditions in (4.6) are restrictions on the geometry of $M$. The first can be expressed as the vanishing condition of a linear combination of the fourth and fifth Gray–Hervella classes [26] of an $SU(n)$ structure while the latter implies that the Nijenhuis tensor of $M$ is a 3-form. Such conditions have appeared before\(^6\) in the analysis of geometries with $SU(n)$ structure compatible with a connection with skew-symmetric torsion [27]. The geometric conditions (4.6) are significant and imply that there is a metric connection $\hat{\nabla}$ with skew-symmetric torsion $\hat{\epsilon}$ such that

$$\hat{\nabla}_\omega = \hat{\nabla}_\epsilon = 0, \quad (4.7)$$

i.e. the holonomy of $\hat{\nabla}$ is contained in $SU(n)$. In addition, $\hat{\epsilon}$ is unique and it is determined in terms of the metric and the fundamental form $\omega$ as in the $U(n)$ case, see also [25, 22].

\(^5\) Symmetries can be generated by either the real or imaginary components of $\epsilon$ separately. However, we shall take that both the real and imaginary parts generate symmetries simultaneously. If $n > 3$, the two different cases give the same conditions.

\(^6\) One difference is that $M$ is almost complex while in [27] is complex. Observe that the class $W_1$ expressed in terms of the Nijenhuis tensor does not vanish.
To continue observe that the solution of (1.2) for \( L = \epsilon \) can be written as
\[
c = \hat{\epsilon} + S, \tag{4.8}
\]
where now
\[
S_{\alpha \beta \gamma} = 0, \quad S_{\beta \alpha} = 0, \tag{4.9}
\]
but in general \( S^{2,1} \neq 0 \). \( S \) is a 3-form as it is the difference of two connections with skew-symmetric torsion. It is worth comparing the non-vanishing components of \( S \) with those of the \( U(n) \) case in (3.10).

It remains to solve (1.3). For this, we observe that
\[
i_\epsilon S = 0, \tag{4.10}
\]
as a consequence of (4.9). Furthermore, we also have as a consequence of the Bianchi identity that \( L_\epsilon \hat{\epsilon} = 0 \). Thus (2.5) can be rewritten as
\[
i_\epsilon dS + \frac{n-1}{n+1} i_\epsilon d\hat{\epsilon} = 0. \tag{4.11}
\]
In turn this gives that
\[
dS^{1,0} + \frac{n-1}{n+1} d\hat{\epsilon}^{1,0} = 0, \\
\]
\[
dS_{\alpha \beta \gamma} + \frac{n-1}{n+1} d\hat{\epsilon}_{\alpha \beta \gamma} = 0. \tag{4.12}
\]
The first equation can be rewritten as an algebraic equation on \( S \) because \( S \) is (2,1) and (1,2) form on \( M \). It is not straightforward to solve these equations for \( S \). Nevertheless, one can easily evaluate them for particular examples.

The third invariance equation (2.4) can be easily solved. It is easy to see that
\[
dA^\alpha_\epsilon = 0. \tag{4.13}
\]
The other components are not restricted.

4.1. Geometry

4.1.1. \( \epsilon \) symmetries. The solution to the KY equation for symmetries generated by \( \epsilon \) require that \( M \) is geometrically restricted by (4.6). These are precisely the conditions for \( M \) to admit a metric connection \( \hat{\nabla} \) with skew-symmetric torsion \( \hat{\epsilon} \) such that the holonomy of \( \hat{\nabla} \) is included in \( SU(n) \). The torsion \( \hat{\epsilon} \) is determined in terms of the metric and Hermitian form \( \omega \). So if the metric and almost complex structure are fixed, \( \hat{\nabla} \) is unique. However, the connection \( \hat{\nabla} \) that solves the KY equation is not unique. There is a family of solutions parameterized by the (2,1) and (1,2) and traceless form \( S \) as in (4.8). This is the full content of the KY equation.

The second invariance condition (1.3), or equivalently (2.5), imposes additional conditions. These are given in (4.12). If \( S = 0 \), the resulting conditions can be viewed as further restrictions on the geometry as \( \hat{\epsilon} \) is determined in terms of the metric and Hermitian form. However, if \( S \neq 0 \), they can be viewed as equations for \( S \). It is not apparent that these determine \( S \) uniquely. For example, \( S \) is specified up to the exterior derivative of a co-closed 2-form provided that it is a (2,1) and (1,2) form.

4.1.2. \( \omega \) and \( \epsilon \) symmetries. So far we have investigated the conditions for either \( \omega \) or \( \epsilon \) to generate a symmetry. In the \( SU(n) \) case, there is the possibility that both these fundamental forms generate symmetries. An inspection of the conditions (3.11), (3.10) (4.8) and (4.9) reveals that in such case it is required that
\[
c = \hat{\epsilon}, \tag{4.14}
\]
and so $\hat{\nabla} = \hat{\nabla}$. As a result, the holonomy of $\hat{\nabla}$ is contained in $SU(n)$ and both fundamental tensors $\omega$ and $\varepsilon$ are parallel.

The second symmetric conditions requires that

$$d\hat{c}^{3,0} = d\hat{c}^{3,1} = 0,$$

i.e. the only non-vanishing component of $d\hat{c}$ is the (2,2). This is a condition on the geometry as $\hat{c}$ is expressed in terms of the metric and Hermitian form. It is straightforward to write this condition in terms of $\omega$ using the expression for $\hat{c}$ in [25, 22].

5. $Sp(n)$ and $Sp(n) \cdot Sp(1)$ structures

5.1. $Sp(n)$ structure

A manifold with a $Sp(n)$ structure admits an almost hyper-complex structure, i.e. three almost complex structures $(I, J, K)$ satisfying the algebra of imaginary unit quaternions, $I^2 = J^2 = -1, IJ = -JI = K$. The metric $g$ is Hermitian with respect to all three complex structures and so there are associated Hermitian forms $(\omega_I, \omega_J, \omega_K)$.

The three Hermitian forms generate three additional anti-commuting symmetries for the worldline action (2.1). If the complex structures are integrable, the symmetries satisfy the standard supersymmetry algebra in one dimension. Therefore, the action (2.1) admits four supersymmetries. If the almost complex structures are not integrable, the additional symmetries are again supersymmetries but now the closure of the algebra requires the addition of new generators associated with symmetries generated by the Nijenhuis tensors.

To solve the KY equation (1.2), one can easily adapt the calculation\(^7\) in [7] to reveal that that all three almost complex structures $I, J$ and $K$ are parallel with respect to $\hat{\nabla}$,

$$\hat{\nabla}I = \hat{\nabla}J = \hat{\nabla}K = 0.$$  \hspace{1cm} (5.1)

Therefore the holonomy of $\hat{\nabla}$ is in $Sp(n)$ and $M$ is an almost HKT manifold [28]. Fixing the almost hyper-complex structure and the metric, $\hat{\nabla}$ and so $c$ are unique. The torsion can be expressed in terms of the metric and Hermitian forms as in the $U(n)$ case.

The invariance condition (1.3) can be expressed as (2.9) which in turn implies that

$$i_\lambda dc = i_\omega dc = i_K dc = 0,$$

i.e. $dc$ is (2,2) form with respect to all almost complex structures. Similarly, the third invariance condition (2.4) implies that $dA$ is (1, 1) form with respect to all three almost complex structures and so lies in $sp(n)$.

5.2. $Sp(n) \cdot Sp(1)$ structure

Manifolds with a $Sp(n) \cdot Sp(1)$ structure admit an almost quaternionic structure which can be locally represented by a basis of three almost complex structures $(I, J, K)$ satisfying the algebra of imaginary unit quaternions. These are compatible with a metric and so there are three associated Hermitian forms $(\omega_I, \omega_J, \omega_K)$. In terms of these, the fundamental form which generates the symmetry is

$$\lambda = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$  \hspace{1cm} (5.3)

Observe that $\lambda$ is a 4-form on $M$ because it is invariant under local $SO(3)$ patching conditions which rotate the three almost complex structures of the basis.

\(^7\) We have not given details. The calculation is lengthy but straightforward. The KY equation implies that the component of the connection $\hat{\nabla}$ which lies in the complement of the subspace $sp(n)$ in the space of 2-forms vanishes.
An investigation reveals that for $n > 1$ the KY equation (1.2) implies that $\lambda$ is parallel. The proof is based on a lengthy but straightforward calculation and is similar to that given for the case without torsion in [7]. Therefore, we have that
\[
\hat{\nabla}\lambda = 0. \tag{5.4}
\]
Thus $M$ is an almost QKT manifold [29] and $c$ is uniquely determined in terms of the fundamental forms and the metric [30].

Since the KY equation implies that $\lambda$ is $\hat{\nabla}$-parallel, the second invariance equation (1.3) can be reexpressed as (2.5) and so we have that
\[
\iota_\lambda dc = 0. \tag{5.5}
\]
In addition a direct calculation reveals that the third invariance condition (2.4) implies that $dA$ lies in $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$.

6. $G_2$ structure

6.1. $\varphi$ symmetry

The fundamental forms are a 3-form $\varphi$ and its dual 4-form $\star \varphi$. To solve the KY equation (1.2), we write $\hat{\nabla} = \pi(\hat{\nabla}) + \sigma(\hat{\nabla})$ following the decomposition of 2-forms $\Lambda^2(\mathbb{R}^7) = \mathfrak{g}_2 + \Lambda_7$ in $G_2$ representations, see section 2 where the general procedure is described. We have that $\pi(\hat{\nabla}) \varphi = 0$ and $\sigma(\hat{\nabla})$ lies in the seven-dimensional representation $\Lambda_7$. Thus, one can set
\[
\sigma(\hat{\nabla})_{i j k} = L_{\varphi} \varphi_{i j k} \tag{6.1}
\]
for some $L$ tensor.

Suppose now that $\varphi$ generates a symmetry. The KY equation (1.2) depends only on the $\sigma(\hat{\nabla})$. A similar calculation to that in [7] implies that $L_{ij} = \beta \delta_{ij}$ for some function $\beta$. Thus, we have that $\hat{\nabla} = \pi(\hat{\nabla}) + \beta \varphi$.

It remains to specify $\pi(\hat{\nabla})$. Since $\sigma(\hat{\nabla}) = \beta \varphi$ is a 3-form, $\sigma(\hat{\nabla})$ is again a metric connection with skew-symmetric torsion which in addition satisfies $\pi(\hat{\nabla}) \varphi = 0$. Such a connection is unique and it exists provided that $G_2$ structure on $M$ is restricted as $d \star \varphi = \theta \varphi \wedge \star \varphi$, \hspace{1cm} (6.2)
where $\theta \varphi$ is the Lee form of $\varphi$. Following the notation of the previous sections we write $\pi(\hat{\nabla}) = \hat{\nabla}$. The associated 3-form torsion $\hat{c}$ is uniquely determined in terms of the metric and $\varphi$, and it is given in [25], see also [22]. Thus there is a family of solutions
\[
\hat{\nabla} = \hat{\nabla} + \beta \varphi \tag{6.3}
\]
to the KY equation (1.2) labeled by $\beta$.

The second invariance condition (2.9) gives
\[
-22 \delta \beta \wedge \star \varphi + \iota_\varphi \, dc = 0. \tag{6.4}
\]
To see this substitute $c = \hat{c} + \beta \varphi$ in (2.9) using $L_\varphi \hat{c} = 0$ and the geometric conditions (6.2) to get
\[
-22 \delta \beta \wedge \star \varphi - 6 \beta \theta \varphi \wedge \star \varphi + 2 \beta \iota_\varphi \, d\varphi + \iota_\varphi \, dc = 0, \tag{6.5}
\]
where $\iota_\varphi \varphi = -6 \star \varphi$. Furthermore, observe that the Kernel of $\iota_\varphi$ in the space of 4-forms is $\Lambda_1 \oplus \Lambda_7$. Thus $\iota_\varphi \varphi$ depends only on the $\Lambda_7$ component of $d\varphi$. In particular, one can show that $\iota_\varphi \varphi = 3 \varphi \wedge \star \varphi$, and so (1.3) gives (6.4).

It remains to solve (2.4). Using the decomposition $\Lambda^2(\mathbb{R}^7) = \mathfrak{g}_2 + \Lambda_7$, one can easily show that $\iota_\varphi \, dA = 0$ implies that $dA$ lies in $\mathfrak{g}_2$. 

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6.2. $\star \phi$ symmetry

A similar analysis to that presented in the previous section reveals that the KY equation (1.2) associated with the symmetry generated by $\star \phi$ implies that $\star \phi$ is $\hat{\nabla}$-parallel. As a result, the only solution to the KY equation is

$$\hat{\nabla} = \hat{\nabla}$$  \hspace{1cm} (6.6)

where $\hat{\nabla}$ is defined as in the previous section. Thus, the solution in this case is unique.

Furthermore, the second invariance condition for $c = \hat{c}$ implies that

$$i_{\phi} d\hat{c} = 0.$$  \hspace{1cm} (6.7)

Using the decomposition of $\Lambda^4(\mathbb{R}^7) = \Lambda_1 \oplus \Lambda_7 \oplus \Lambda_{27}$ under $G_2$, one finds that (6.7) implies that $d\hat{c}$ lies in $\Lambda_1 \oplus \Lambda_{27}$. As $\hat{c}$ is determined in terms of the metric and $\varphi$, (6.7) becomes a condition on the geometry of $M$. Again it is not apparent how to solve it in general. Nevertheless, it will be straightforward to verify it for particular examples. The third invariance condition $i_{\varphi} dA = 0$ gives that $dA$ lies in $g_2$.

6.3. $\varphi$ and $\star \phi$ symmetries

Combining the results of the previous two section, we find that the KY equation has a unique solution given by the connection with skew-symmetric torsion and holonomy contained in $G_2$, $\hat{\nabla} = \hat{\nabla}$. Furthermore, the second condition for the invariance of the action (1.3) implies that

$$i_{\phi} d\hat{c} = i_{\phi} d\hat{c} = 0.$$  \hspace{1cm} (6.8)

Since $\hat{\nabla} = \hat{\nabla}$, the second invariance condition (1.3) is given in terms of (2.9). So this can be written as

$$i_{\varphi} d\hat{c} = 0.$$  \hspace{1cm} (7.1)

To find the restriction that this equation imposes on $d\hat{c}$, one uses the decomposition $\Lambda^4(\mathbb{R}^8) = \Lambda_1 \oplus \Lambda_7 \oplus \Lambda_{27} \oplus \Lambda_{35}$ under $Spin(7)$, where $\Lambda_{27}$ is the symmetric traceless and $\Lambda_{35}$ is the 3-form representation of $SO(7)$, respectively. The above condition implies that the $\Lambda_7$ component of $d\hat{c}$ must vanish.

The third invariance condition (2.4) is straightforward to see that $i_{\phi} dA = 0$ implies that $dA$ lies in $su(4)$ fundamental forms.

8 This calculation is most easily done by writing the fundamental form $\varphi$ in terms of $SU(4)$ fundamental forms.
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