Tracing sharing in an imperative pure calculus
(extended version)

Paola Giannini\textsuperscript{a}, Tim Richter\textsuperscript{b}, Marco Servetto\textsuperscript{c}, Elena Zucca\textsuperscript{d}

\textsuperscript{a}Università del Piemonte Orientale, Italy
\textsuperscript{b}Universität Potsdam, Germany
\textsuperscript{c}Victoria University of Wellington, New Zealand
\textsuperscript{d}Università di Genova, Italy

Abstract

We introduce a type and effect system, for an imperative object calculus, which infers sharing possibly introduced by the evaluation of an expression, represented as an equivalence relation among its free variables. This direct representation of sharing effects at the syntactic level allows us to express in a natural way, and to generalize, widely-used notions in literature, notably uniqueness and borrowing. Moreover, the calculus is pure in the sense that reduction is defined on language terms only, since they directly encode store. The advantage of this non-standard execution model with respect to a behaviourally equivalent standard model using a global auxiliary structure is that reachability relations among references are partly encoded by scoping.

Keywords: imperative calculi, sharing, type and effect systems
2010 MSC: 68N15, 68Q55

1. Introduction

In the imperative programming paradigm, sharing is the situation when a portion of the store can be accessed through more than one reference, say $x$ and $y$, so that a change to $x$ affects $y$ as well. Unwanted sharing relations are common bugs: unless sharing is carefully maintained, changes through a reference might propagate unexpectedly, objects may be observed in an inconsistent state, and conflicting constraints on shared data may inadvertently invalidate invariants. Preventing such errors is even more important in increasingly ubiquitous multi-core and many-core architectures.

For this reason, there is a huge amount of literature on type systems for controlling sharing and interference, notably using type annotations to restrict
the usage of references, see Sect. 7 for a survey. In particular, it is very useful for a programmer to be able to rely on the following properties of a reference $x$.

- **Capsule reference**: $x$ denotes an isolated portion of store, that is, the subgraph reachable from $x$ cannot be reached through other references. This allows programmers (and static analysis) to identify state that can be safely handled by a thread. In this paper we will use the name capsule for this property, to avoid confusion with many variants in the literature [1, 2, 3, 4, 5, 6].

- **Lent reference** [7, 8], also called *borrowed* [9, 10]: the subgraph reachable from $x$ can be manipulated by a client, but no sharing can be introduced through $x$. Typically, borrowing can be employed to ensure that the capsule guarantee is not broken.

In this paper, we propose a type and effect system which provides, in our opinion, a very powerful, yet natural, way to express sharing. Notably, the two above mentioned notions are smoothly included and generalized.

The distinguishing features are the following:

1. Rather than declaring type annotations, the type system *infers* sharing possibly introduced by the evaluation of an expression.
2. Sharing is *directly represented at the syntactic level*, as an equivalence relation among the free variables of the expression.
3. The calculus is *pure* in the sense that reduction is defined on language terms only, rather than requiring an auxiliary structure.

We now describe these three features in more detail.

Given an expression $e$, the type system computes a *sharing relation* $\mathcal{S}$ which is an equivalence relation on a set containing the free variables of $e$ and an additional, distinguished variable $\text{res}$ denoting the result of $e$. That two variables, say $x$ and $y$, are in the same equivalence class in $\mathcal{S}$ means that the evaluation of $e$ can possibly introduce sharing between $x$ and $y$, that is, connect their reachable object graphs, so that a modification of (a subobject of) $x$ could affect $y$ as well, or conversely. For instance, evaluating the expression $x.f = y; z.f$ introduces connections

- between $x$ and $y$,
- between $\text{res}$ (the result) and $z$.

The *capsule* notion becomes just a special case: an expression is a capsule iff its result will be disjoint from any free variable (formally, $\text{res}$ is a singleton in $\mathcal{S}$). For instance, the expression $x.f = y; \text{new} C(\text{new} D()).f$ is a capsule, whereas the previous expression is not.\footnote{Note that our notion is related to the whole reachable object graph. For instance, a doubly-linked list whose elements can be arbitrarily aliased can be externally unique and properly follow an owners as dominator strategy, but is not a capsule.}

The *lent* notion also becomes a special case: a variable $x$ is used as lent in an expression if the evaluation of the expression will neither connect $x$ to
any other variable, nor to the result (formally, \( x \) is a singleton in \( S \)). In other words, the evaluation of the expression does not introduce sharing between \( x \) and other variables (including \texttt{res}). For instance \( x \) is lent in \( x.f_1=x.f_2; z.f \). In our type system, this notion is generalized from singletons to arbitrary sets of variables: for instance, in the previous example \( x.f=y; z.f \), the set \( \{x, y\} \) is an equivalence class in \( S \) since the evaluation of the expression does not introduce sharing between this set and other variables (including \texttt{res}).

Altogether, this direct representation at the syntactic level allows us to express sharing in a natural way. Moreover, execution is modeled by a pure calculus, where store is encoded directly in language terms, rather than by an auxiliary structure. Formally, this is achieved by the block construct, introducing local variable declarations, which play the role of store when evaluated. This operational semantics will be informally introduced in Sect. 2 and formalized in Sect. 5.

A preliminary presentation of the approach presented in this paper has been given in [12, 13].

The rest of the paper is organized as follows: in Sect. 2 we provide syntax and an informal execution model, in Sect. 3 the type system, and in Sect. 4 some examples. The operational semantics of the calculus is presented in Sect. 5 and the main results and proofs in Sect. 6. In Sect. 7 we discuss related work, and in Sect. 8 we draw some conclusion and highlight future work. Appendix A contains a (rather complex) type derivation. The proofs omitted from the main paper are in Appendix B.

2. Language

The syntax of the language is given in Fig. 1. We assume sets of variables \( x, y, z \), class names \( C, D \), field names \( f \), and method names \( m \). We adopt the convention that a metavariable which ends in \( s \) is implicitly defined as a (possibly empty) sequence, for example, \( ds \) is defined by \( ds ::= \epsilon | d ds \), where \( \epsilon \) denotes the empty sequence.

The calculus is designed with an object-oriented flavour, inspired by Featherweight Java [14]. This is only a presentation choice: all the ideas and results of the paper could be easily rephrased in a different imperative calculus, e.g.,

\[
\begin{align*}
\text{e} & ::= x | e.f | e.f=e' | \text{new } C(es) | \{ds e\} | e.m(es) & \text{expression} \\
\text{d} & ::= T x=e; & \text{declaration} \\
T & ::= C^\mu | \text{int} & \text{declaration type} \\
\mu & ::= \epsilon | a & \text{optional modifier}
\end{align*}
\]

Figure 1: Syntax

2Which is, of course, expected to be behaviorally equivalent to the conventional semantics where store is a global flat auxiliary structure, as we plan to formally state and prove in further work.
Figure 2: Example of reduction

```
D z = new D(0); C x = new C(z, z); C y = x;
D w = new D(y.f1.f + 1); x.f2 = w; x
```

supporting data type constructors and reference types. For the same reason, we omit features such as inheritance and late binding, which are orthogonal to our focus.

An expression can be a variable (including the special variable `this` denoting the receiver in a method body), a field access, a field assignment, a constructor invocation, a block consisting of a sequence of declarations and a body, or a method invocation. In a block, a declaration specifies a type, a variable and an initialization expression. We assume that in well-formed blocks there are no multiple declarations for the same variable, that is, `ds` can be seen as a map from variables to expressions.

A declaration type is a class name with an optional modifier `a`, which, if present, indicates that the variable is affine. We also include `int` as an example of primitive type, but we do not formally model related operators used in the examples, such as integer constants and sum. An affine variable can occur at most once in its scope, and should be initialized with a capsule, that is, an isolated portion of store. In this way, it can be used as a temporary reference, to “move” a capsule to another location in the store, without introducing sharing. In the examples, we generally omit the brackets of the outermost block, and abbreviate `{ T x = e; e' } by e; e' when x does not occur free in e'.

We turn our attention to the operational semantics now. Fig.2 shows an example of a reduction sequence in the calculus.

The main idea is to use variable declarations to directly represent the store. That is, a declared (non affine) variable is not replaced by its value, as in standard `let`, but the association is kept and used when necessary, as it happens, with different aims and technical problems, in cyclic lambda calculi [12, 10].

In the figure, we emphasize at each step the declarations which can be seen as the store (in grey) and the redex which is reduced (in a box).

Assuming a program (class table) where class `C` has two fields `f1` and `f2` of type `D`, and class `D` has an integer field `f`, in the initial term in Fig.2 the first two declarations can be seen as a store which associates to `z` an object of class `D` whose field contains `0`, and to `x` an object of class `C` whose two fields contains (a reference to) the previous object. The first reduction step eliminates an alias, by replacing occurrences of `y` by `x`. The next three reduction steps compute `x.f1.f + 1`, by performing two field accesses and one sum. The last step performs a field assignment, modifying the current store. Finally, in the last line we have
a term which can no longer be reduced, consisting of a store and the expression \( x \) which denotes a reference in the store, taken as entry point. In other words, the final result of the evaluation is an object of class \( C \) whose fields contain (references to) two objects of class \( D \), whose fields contain 0 and 1, respectively.

As usual, references in the store can be mutually recursive\(^3\) as in the following example, where we assume a class \( B \) with a field of type \( B \).

\[
B \ x = \text{new } B(y); \ B \ y = \text{new } B(x); \ y
\]

Again, this is a term which can no longer be reduced, consisting of a store and the reference \( y \) as entry point. In other words, this term can be seen as an object of class \( B \) whose field contains (a reference to) another object of class \( B \), whose field contains (a reference to) the original object.

In the examples until now, store is flat, as it usually happens in models of imperative languages. However, in our calculus, we are also able to represent a hierarchical store, as shown in the example below, where we assume a class \( A \) with two fields of type \( B \) and \( D \), respectively.

\[
D \ z = \text{new } D(0);
A \ w = \{
B \ x = \text{new } B(y);
B \ y = \text{new } B(x);
A \ u = \text{new } A(x, z);
u\};
w
\]

Here, the store associates to \( w \) a block introducing local declarations, that is, in turn a store. The advantage of this representation is that it models in a simple and natural way constraints about sharing among objects, notably:

- the fact that an object is not referenced from outside some enclosing object is directly modeled by the block construct: for instance, the object denoted by \( y \) can only be reached through \( w \)
- conversely, the fact that an object does not refer to the outside is modeled by the fact that the corresponding block is closed (that is, has no free variables): for instance, the object denoted by \( w \) is not closed, since it refers to the external object \( z \).

In other words, our calculus smoothly integrates memory representation with shadowing and \( \alpha \)-conversion. However, there is a problem which needs to be handled to keep this representation correct: reading (or, symmetrically, updating) a field could cause scope extrusion. For instance, the term

\[
C \ y = \{D \ z = \text{new } D(0); \ C \ x = \text{new } C(z, z); \ x\}; \ y.f1
\]

under a naive reduction strategy would reduce to the ill-formed term

\[
C \ y = \{D \ z = \text{new } D(0); \ C \ x = \text{new } C(z, z); \ x\}; \ z
\]

\(^3\)However, mutual recursion is not allowed between declarations which are not evaluated, e.g., \( B \ x = \text{new } B(y.f); \ B \ y = \text{new } B(x.f); \ y \) is ill-formed.
To avoid this problem, the above reduction step is forbidden. However, reduction is not stuck, since we can transform the above term into an equivalent term where the inner block has been flattened, and get the following correct reduction sequence:

\[
\begin{align*}
C & y = \{D \ z= \text{new} \ D(0); \ C \ x= \text{new} \ C(z,z); \ x\} \ y.f1 \ \cong \\
D & z= \text{new} \ D(0); \ C \ x= \text{new} \ C(z,z); \ C y= \ x; \ y.f1 \ \rightarrow \\
D & z= \text{new} \ D(0); \ C x= \text{new} \ C(z,z); \ x.f1 \ \rightarrow \\
D & z= \text{new} \ D(0); \ C x= \text{new} \ C(z,z); \ z \ \cong \\
D & z= \text{new} \ D(0); \ z
\end{align*}
\]

Formally, in addition to the reduction relation which models actual computation, our operational semantics is defined by a congruence relation \(\cong\), which captures structural equivalence, as in \(\pi\)-calculus \([17]\). Note also that in the final term the declaration of \(x\) can be removed (again by congruence), since it is useless.

Moving declarations from a block to the directly enclosing block is not always safe. For instance, in the following variant of the previous example

\[
C^a \ y= \{D \ z= \text{new} \ D(0); \ C \ x= \text{new} \ C(z,z); \ x\}; \ y.f1
\]

the affine variable is required to be initialized with a capsule, and this is the case indeed, since the right-hand side of the declaration is a closed block. However, by flattening the term:

\[
D \ z= \text{new} \ D(0); \ C x= \text{new} \ C(z,z); \ C^a \ y= \ x; \ y.f1
\]

this property would be lost, and we would get an ill-typed term. Indeed, these two terms are not considered equivalent in our operational model. Technically, this is obtained by detecting, during typechecking, which local variables will be connected to the result of the block, as \(z\) in the example, and preventing to move such declarations from a block which is the initialization expression of an affine variable.

In this case, reduction proceeds by replacing the (unique) occurrence of the affine variable by its initialization expression, as shown below.

\[
\begin{align*}
C^a \ y & = \{D \ z= \text{new} \ D(0); \ C \ x= \text{new} \ C(z,z); \ x\} \ y.f1 \ \rightarrow \\
\{D \ z= \text{new} \ D(0); \ C \ x= \text{new} \ C(z,z); \ x\}.f1 \ \cong \\
D & z= \text{new} \ D(0); \ C \ x= \text{new} \ C(z,z); \ x.f1 \ \rightarrow \\
D & z= \text{new} \ D(0); \ C \ x= \text{new} \ C(z,z); \ z \ \cong \\
D & z= \text{new} \ D(0); \ z
\end{align*}
\]

3. Type system

In this section we introduce the type and effect system for the language. We use \(X, Y\) to range over sets of variables.

A sharing relation \(\mathcal{S}\) on a set of variables \(X\) is an equivalence relation on \(X\). As usual \([x]_\mathcal{S}\) denotes the equivalence class of \(x\) in \(\mathcal{S}\). We will call the elements \((x, y)\) of a sharing relation connections, and say that \(x\) and \(y\) are connected. The intuitive meaning is that, if \(x\) and \(y\) are connected, then their reachable graphs
in the store are possibly shared (that is, not disjoint), hence a modification of
the reachable graph of \( x \) could affect \( y \) as well, or conversely.

We use the following notations on sharing relations:

- A sequence of subsets of \( X \), say, \( X_1 \ldots X_n \), represents the smallest equiva-
  lence relation on \( X \) containing the connections \( \langle x, y \rangle \), for all \( x, y \) belonging
to the same \( X_i \). So, \( \epsilon \) represents the identity relation on any set of variables.
  Note that this representation is deliberately ambiguous as to the
domain of the defined equivalence: any common superset of the \( X_i \) will
do.

- \( S_1 + S_2 \) is the smallest equivalence relation containing \( S_1 \) and \( S_2 \). It is
easy to show that sum is commutative and associative. With \( S + X \) we
denote the sum of \( S \) with the sharing relation containing the connections
\( \langle x, y \rangle \), for all \( x, y \in X \).

- \( S \setminus \) is the sharing relation obtained by “removing” \( X \) from \( S \), that is,
the smallest equivalence relation containing the connections \( \langle x, y \rangle \), for all
\( \langle x, y \rangle \in S \) such that \( x, y \notin X \). \( S \setminus \) stands for \( S \setminus \{ y \} \). It is easy to see
that \( S \setminus (X \cup Y) = (S \setminus X) \setminus Y \).

- \( S[y/x] \) is the sharing relation obtained by “replacing” \( x \) by \( y \) in \( S \), that is,
the smallest equivalence relation containing the connections:
\( \langle z, z' \rangle \), for all \( \langle z, z' \rangle \in S \), \( z \neq x, z' \neq x \)
\( \langle y, z \rangle \), for all \( \langle x, z \rangle \in S \).

- \( S_1 \) has less (or equal) sharing effects than \( S_2 \), dubbed \( S_1 \sqsubseteq S_2 \), if, for all
\( \langle x, y \rangle \), \( [x]_{S_1} \subseteq [x]_{S_2} \).

The following proposition asserts some properties of sharing relations.

Proposition 1.

1. Let \( x \neq y \), \( \langle x, y \rangle \in \sum_{i=1}^{n} S_i \) if and only if there are sequences \( i_1 \ldots i_{k-1} \)
   (\( 1 \leq i_h \leq n \) for all \( h \)) and \( z_1 \ldots z_k \) (\( k > 1 \)) such that \( x = z_1 \) and \( y = z_k \)
   and \( \langle z_j, z_{j+1} \rangle \in S_{i_j} \) and \( i_j \neq i_{j+1} \) and \( z_j \neq z_{j+1} \) for \( 1 \leq j \leq (k-1) \).

2. \( S_1 \sqsubseteq S_2 \) implies \( S + S_1 \sqsubseteq S + S_2 \) for all \( S \).

3. \( S_1 \sqsubseteq S_2 \) implies \( S_1 \setminus X \sqsubseteq S_2 \setminus X \) for all \( X \).

4. If \( S_1 \setminus X = S_1 \), then \( (S_1 + S_2) \setminus X = S_1 \setminus X + S_2 \setminus X \).

5. If \( y \in [x]_S \), then \( S[y/x] = S \setminus x \).

Since \( S + \epsilon = S \) and \( \epsilon \sqsubseteq S \) for all \( S \), from 2. we have that \( S \sqsubseteq S + S' \) for all
\( S \) and \( S' \).

Proof.

1. From the fact that \( \sum_{i=1}^{n} S_i \) is the transitive closure of \( \bigcup_{1 \leq i \leq n} \{ \langle x, y \rangle \mid \langle x, y \rangle \in S_i \} \).
2. From 1. and the fact that for all \( z \) and \( z' \), if \( \langle z, z' \rangle \in S_1 \) then \( \langle z, z' \rangle \in S_2 \).
3. Let \( \langle z, z' \rangle \in S_1 \setminus X \) with \( z \neq z' \). Then \( \langle z, z' \rangle \in S_1 \) and \( z, z' \notin X \). From \( S_1 \subseteq S_2 \), then \( \langle z, z' \rangle \in S_2 \) and so also \( \langle z, z' \rangle \in S_2 \setminus X \).

Let \( \langle z, z' \rangle \) be such that \( \langle z, z' \rangle \in S_1[y/x] \) and \( z \neq z' \). Then \( z \neq x \) and \( z' \neq x \). If \( \langle z, z' \rangle \in S_1 \), then \( \langle z, z' \rangle \in S_2 \) and so also \( \langle z, z' \rangle \in S_2[y/x] \). If \( \langle z, z' \rangle \notin S_1 \), then there are pairs \( \langle z, x \rangle \) and \( \langle z', y \rangle \) such that \( \langle z, x \rangle \in S_1 \) and \( \langle y, z' \rangle \in S_1 \). Therefore \( \langle z, x \rangle \in S_2 \) and \( \langle y, z' \rangle \in S_2 \) and so \( \langle z, z' \rangle \in S_2[y/x] \).

4. From 2. and 3. we have that \( S_1 \setminus X \subseteq (S_1 + S_2) \setminus X \) and \( S_2 \setminus X \subseteq (S_1 + S_2) \setminus X \).

By definition of + we derive \( S_1 \setminus X + S_2 \setminus X \subseteq (S_1 + S_2) \setminus X \).

We now prove that \( (S_1 + S_2) \setminus X \subseteq S_1 \setminus X + S_2 \setminus X \). Let \( \langle x, y \rangle \in (S + S') \setminus X \) with \( x \neq y \). Then \( \langle x, y \rangle \in S + S' \) and \( x, y \notin X \). By 1., there are sequences \( i_1, \ldots, i_{k-1} \) and \( z_1, \ldots, z_k \) \((k > 1)\) such that \( x = z_1 \) and \( y = z_k \) and \( \langle z_j, z_{j+1} \rangle \in S_{i_j} \) and \( i_j \neq i_{j+1} \) and \( z_j \neq z_{j+1} \) for \( 1 \leq j \leq (k-1) \). The fact \( i_j \neq i_{j+1} \) implies that the sequence \( i_1, \ldots, i_{k-1} \) alternates between 1 and 2.

So for any \( j \), \( 2 \leq j \leq (k-1) \), \( \langle z_{j-1}, z_j \rangle \in S_1 \) or \( \langle z_j, z_{j+1} \rangle \in S_1 \). Since \( S_1 = S_1 \setminus X \), in either cases \( z_j \notin X \). So no element of \( z_1, \ldots, z_k \) is in \( X \), thus for any \( j \), \( 1 \leq j \leq (k-1) \), \( \langle z_j, z_{j+1} \rangle \in S_{i_j} \) implies \( \langle z_j, z_{j+1} \rangle \in S_1 \setminus X \). By 1., we have that \( \langle x, y \rangle \in S_1 \setminus X + S_2 \setminus X \).

5. Let \( \langle z, z' \rangle \in S \setminus X \) with \( z \neq z' \), if \( z \neq x \) and \( z' \neq x \), then \( \langle z, z' \rangle \in S[y/x] \).

So \( S \setminus x \subseteq S[y/x] \).

To show \( S[y/x] \subseteq S \setminus x \), first observe that there cannot be \( \langle z, z' \rangle \in S[y/x] \) such that \( z \neq z' \) and either \( z = x \) or \( z' = x \).

Let \( \langle z, z' \rangle \in S[y/x] \) with \( z \neq z' \). If \( z \neq y \) and \( z' \neq y \), then, by definition of \( S[y/x] \) we get \( \langle z, z' \rangle \in S \setminus x \). If \( z = y \) there are 2 cases: either \( \langle y, z' \rangle \in S \), or \( \langle x, z' \rangle \in S \). In the first case \( \langle y, z' \rangle \in S \setminus x \). In the second, from \( y \in [x]_S \) we get that \( \langle x, z' \rangle \in S \) and \( \langle y, z' \rangle \in S \) and so also \( \langle y, z' \rangle \in S \setminus x \).

Similar if \( z' = y \). Therefore \( S[y/x] \subseteq S \setminus x \).
An affine variable will never be connected to another, nor to \texttt{res}, since it is initialized with a capsule and used only once. Analogously, a variable of a primitive type will never be connected to another.

Moreover, during typechecking expressions are annotated. The syntax of annotated expressions is given by:

\[
e ::= x | e.f | e.m(e_1, \ldots, e_n) | e.f=e' | \texttt{new } C(es) | \{X \text{ ds } e\}
\]

where \(X \subseteq \text{dom}(ds)\). We use the same metavariable of source expressions for simplicity. As we can see, the only difference is that blocks are annotated by a set \(X\) of variables. In an annotated block obtained as output of typechecking, \(X\) will be the local variables declared in the block (possibly) connected with the result of the body, see rule \((\pi\text{-block})\). Such annotations, as we will see in the next section, are used to define the congruence relation among terms. Notably, we can move local store from a block to the directly enclosing block, or conversely, as it happens with rules for \textit{scope extension} in the \(\pi\)-calculus \[17\]. However, this is not allowed if such block initializes an affine variable declaration, and we would move outside variables possibly connected to the result of the block. Indeed, this would make the term ill-typed, as shown in the last example of Sect[2]

The class table is abstractly modeled by the following functions:

- \texttt{fields}(C) gives, for each declared class \(C\), the sequence \(T_1 f_1 \ldots T_n f_n\) of its fields declarations, with \(T_i\) either class name or primitive type.

- \texttt{meth}(C, m) gives, for each method \(m\) declared in class \(C\), the tuple \(\langle T | S, \mu, T_1 x_1 \ldots T_n x_n, e \rangle\) consisting of its return type paired with the resulting sharing relation, optional a modifier for \texttt{this}, parameters, and body.

We assume a well-typed class table, that is, method bodies are expected to be well-typed with respect to method types. Formally, if \(\texttt{meth}(C, m) = (T | S, \mu, T_1 x_1 \ldots T_n x_n, e)\), then it should be

\[
\begin{align*}
\Gamma \vdash e : T | S \rightsquigarrow e', & \quad \text{with} \\
\Gamma = \texttt{this}:C^\mu, x_1:T_1, \ldots, x_n:T_n. & \end{align*}
\]

Note that the \(S\) effects in the return type of a method can be inferred by typechecking the body for a non-recursive method. Recursion could be handled by a global fixed-point inference to find effects across methods. Alternatively, and also to support interfaces, (some) effect annotations in method return types could be supplied by the programmer, likely in a simpler form, e.g., using the \textit{capsule} modifier. In this case, typechecking the body should check conformance to its declared interface. Still, the fixed-point inference scheme would be useful.

\[5\]That is, the \texttt{a} modifier, denoting a temporary reference, makes no sense for fields.
in porting over code-bases, and might help to identify how effective the type system is in practice. We leave this matter to further work.

The typing rules are given in Fig. 3.

In rule (t-var), the evaluation of a variable (if neither affine nor of a primitive type) connects the result of the expression with the variable itself. In rule (t-affine-var), the evaluation of an affine variable does not introduce any connection, so the resulting sharing relation is the identity relation. Indeed, affine variables are temporary references and will be substituted with capsules. The same happens for variables of primitive types.

In rule (t-field-access), the connections introduced by a field access are those introduced by the evaluation of the receiver.

In rule (t-field-assign), the connections introduced by a field assignment are those introduced by the evaluation of the two expressions (S₁ and S₂). Since both S₁ and S₂ contain the variable res, the equivalence class of this variable in the resulting sharing relation is, as expected, the (transitive closure of the) union of the two equivalence classes. For instance, given the assignment e.f = e', if the evaluation of e connects y with z and x with its result, and the evaluation of e' connects y' with z' and x' with its result, then the evaluation of the field

### Figure 3: Typing rules

| Rule | Description |
|------|-------------|
| (t-var) | \[ \Gamma \vdash x : C \{ x, \text{res} \} \Rightarrow x \] \[ \Gamma(x) = C \] \[ \Gamma(x) = T \] \[ T = C^* \mid \text{int} \] |
| (t-affine-var) | \[ \Gamma \vdash e : C \{ S \Rightarrow e' \} \] \[ \text{fields}(C) = T_{f_1} \ldots T_{f_n} \] |
| (t-field-access) | \[ \Gamma \vdash e : C \{ S \Rightarrow e' \} \] \[ \Gamma \vdash e.f : T_i \{ S \Rightarrow e' \} \] \[ i \in 1 \ldots n \] |
| (t-field-assign) | \[ \Gamma \vdash e_1 : C S \Rightarrow e'_1 \] \[ \Gamma \vdash e_2 : T_i S \Rightarrow e'_2 \] \[ \text{fields}(C) = T_{f_1} \ldots T_{f_n} \] |
| (t-new) | \[ \Gamma \vdash e_i : T_i S \Rightarrow e'_i \] \[ 1 \leq i \leq n \] \[ \text{fields}(C) = T_{f_1} \ldots T_{f_n} \] |
| (t-block) | \[ \Gamma \vdash \{ T_1 x_1 = e_1_1 \ldots T_n x_n = e_{n_1} ; e \} : T \{ S \Rightarrow \text{dom}(\Gamma') \Rightarrow S' \} \] \[ T_{i_1} x_1 = e'_1 \ldots T_{i_n} x_n = e'_{n_1} ; e' \] |
| (t-inv) | \[ \Gamma \vdash e_0 : C \{ S_0 \Rightarrow e'_0 \} \] \[ 0 \leq i \leq n \] |
assignment connects $y$ with $z$, $y'$ with $z'$, and both $x$ and $x'$ with the result.

In rule (t-new), the connections introduced by a constructor invocation are those introduced by the evaluation of the arguments. As for field assignment, the equivalence class of $\text{res}$ in the resulting sharing relation is, as expected, the (transitive closure of the) union of the equivalence classes of $\text{res}$ in the sharing relations of the arguments of the constructor.

In rule (t-block), the initialization expressions and the body of the block are type-checked in the current type environment, enriched by the association to local variables of their declaration types. We denote by $\Gamma[\Gamma']$ the type environment which to a variable $x$ assigns $\Gamma'(x)$ if this is defined, and $\Gamma(x)$ otherwise. If a local variable is affine, then its initialization expression is required to denote a capsule. Moreover, the variable can occur at most once in its scope, as abbreviated by the side condition “$x_i$ affine”\footnote{In our case the affinity requirement can be simply expressed as syntactic well-formedness condition, rather than by context rules, as in affine type systems.}

The connections introduced by a block are obtained modifying those introduced by the evaluation of the initialization expressions ($S_i, 1 \leq i \leq n$) plus those introduced by the evaluation of the body $S'$. More precisely, for each declared variable, the connections of the result of the initialization expression are transformed in connections to the variable itself. Finally, we remove from the resulting sharing relation the local variables. The block is annotated with the subset of local variables which are in the sharing relation $S$ with the result of the block.

In rule (t-invk), the typing of $e_0. m(e_1, \ldots, e_n)$ is similar to the typing of the block \{$C \mu \text{this}=e_0; \ T_1 x_1=e_1; \ldots \ T_n x_n=e_n; \ e$\}. For instance, assume that method $m$ has parameters $x$ and $y$, and the evaluation of its body possibly connects $x$ with $\text{this}$, and $y$ with the result, i.e., the sharing relation associated to the method is $S' = \{x.\text{this}\}{y, \text{res}}$. Then, the evaluation of the method call $z. m(x', y')$, possibly connects $x'$ with $z$, and $y'$ with the result of the expression, i.e., has sharing effects $\{x', z\}{y', \text{res}}$.

Finally, note that primitive types are used in the standard way. For instance, in the premise of rule (t-new) the types of constructor arguments could be primitive types, whereas in rule (t-meth-call) the type of receiver could not.

The following proposition formalizes some properties of the typing judgment. Notably, if two different variables are in sharing relation, then they must have a reference type and cannot be affine. This is true also for variables in sharing relation with the result of an expression. So affine variables are always singletons in the sharing relation. In the following proposition we omit the annotations of terms, which are irrelevant.

**Proposition 2.** Let $\Gamma \vdash e : D | S$. If $(x, y) \in S$ and $x \neq y$, then

- if $x \neq \text{res}$ and $y \neq \text{res}$, then $\Gamma(x) = C$ and $\Gamma(y) = C'$ (for some $C$ and $C'$) and $x, y \in \text{FV}(e)$.
- if $y = \text{res}$ or $x = \text{res}$, then $\Gamma(x) = C$ or $\Gamma(y) = C$ (for some $C$) and $x \in \text{FV}(e)$ or $y \in \text{FV}(e)$.
Proof. The proof is by induction on the type derivation $\Gamma \vdash e : D \mid S$. Consider the last typing rule used in the type derivation.

Rule (T-Var). In this case $e = x$, $\Gamma(x) = D$, and $\text{FV}(e) = \{x\}$. The only nontrivial equivalence class is $\{x, \text{res}\}$. Therefore the result holds.

Rule (T-Affine-Var). In this case $e = x$ and $S$ is the identity. Therefore there is no $(x, y) \in S$ such that $x \neq y$ and the result holds trivially.

Rule (T-Field-Access). In this case $e = e_1.f$ and the result derives by induction hypothesis on $e_1$.

Rule (T-Field-Assign). In this case $e = e_1.f = e_2$ and $S = S_1 + S_2$ and $\Gamma \vdash e_i : C \mid S_i$ for $i = 1, 2$. By induction hypotheses on (c), we have that for all $\langle yi \rangle \in S_i$ and $z \neq z'$

1. if $z \neq \text{res}$ and $z' \neq \text{res}$, then $\Gamma(z) = C$ and $\Gamma(z') = C'$ (for some $C$ and $C'$) and $\{z, z'\} \subseteq \text{FV}(e_i)$ for $1 \leq h \leq 2$

2. if either $z = \text{res}$ or $z' = \text{res}$, then $\Gamma(z') = C$ or $\Gamma(z) = C$ (for some $C'$) and $z' \in \text{FV}(e_i)$ or $z \in \text{FV}(e_i)$ for $1 \leq h \leq 2$

If $(x, y) \in S$ and $x \neq y$, then, by Proposition [III], there are sequences $i_1 \ldots i_{k-1}$ and $z_1 \ldots z_k$ ($k > 1$) such that $x = z_1$ and $y = z_k$ and $(z_j, z_{j+1}) \in S_{i_j}$ and $i_j \neq i_{j+1}$ and $z_j \neq z_{j+1}$ for $1 \leq j \leq (k-1)$. The fact $i_j \neq i_{j+1}$ implies that the sequence $i_1 \ldots i_{k-1}$ alternates between 1 and 2. So for any $j$, $1 \leq j \leq (k-1)$, $(z_j, z_{j+1}) \in S_1$ or $1 \leq j \leq (k-1)$, $(z_j, z_{j+1}) \in S_2$. In both cases, by inductive hypotheses (1) and (2) on $S_1$ or $S_2$, we have that for all $i$, $1 \leq i \leq (k-1)$

- if $z_i \neq \text{res}$ and $z_{i+1} \neq \text{res}$, then $\Gamma(z_i) = C$ and $\Gamma(z_{i+1}) = C'$ (for some $C$ and $C'$) and $\{z_i, z_{i+1}\} \subseteq \text{FV}(e_h)$ ($h = 1$ or $h = 2$)

- if $z_{i+1} = \text{res}$ or $z_i = \text{res}$, then $\Gamma(z_i) = C$ or $\Gamma(z_{i+1}) = C$ (for some $C$) and $z_i \in \text{FV}(e_h)$ or $z_{i+1} \in \text{FV}(e_h)$ ($h = 1$ or $h = 2$)

By transitivity of equality we have that, for all $i$, $1 \leq i \leq k$, if $z_i \neq \text{res}$, then $\Gamma(z_i) = C$ and if there is $j$, $1 \leq j \leq k$, such that $z_j = \text{res}$ also $C = D$. Moreover, $\{z_i \mid z_i \neq \text{res} \ 1 \leq i \leq k\} \subseteq \text{FV}(e_1) \cup \text{FV}(e_2) = \text{FV}(e)$. Therefore the result holds.

Rule (T-Block). In this case $e = \{T_1.x_1 = e_1; \ldots ; T_n.x_n = e_n; e_0\}$.

Let $\Gamma' = \Gamma|_{x_1:T_1, \ldots , x_n:T_n}$ we have that

(a) $S = (\sum_{i=0}^{n} S'_i) \setminus X$ where $X = \text{dom}(\Gamma')$

(b) $S'_i = S_i[x_i/\text{res}]$ ($1 \leq i \leq n$)

(c) $\Gamma' \vdash e_i : T_i | S_i$ ($1 \leq i \leq n$)

(d) $\Gamma' \vdash e_0 : T | S'_0$

(e) if $T_i = C_i^n$, then $[\text{res}|_{S_i} = \{\text{res}\}$ ($1 \leq i \leq n$)

By induction hypotheses on (c), we have that for all $i$, $1 \leq i \leq n$, if $(x, y) \in S_i$ and $z \neq z'$

1. if $z \neq \text{res}$ and $z' \neq \text{res}$, then $\Gamma'(z) = C$ and $\Gamma'(z') = C'$ (for some $C$ and $C'$) and $\{z, z'\} \subseteq \text{FV}(e_i)$

2. if either $z = \text{res}$ or $z' = \text{res}$, then $\Gamma'(z') = C$ or $\Gamma'(z) = C$ (for some $C'$) and $z' \in \text{FV}(e_i)$ or $z \in \text{FV}(e_i)$
By induction hypotheses on (d), we have that if \( \langle z, z' \rangle \in S'_0 \) and \( z \neq z' \)

(3) if \( z \neq \text{res} \) and \( z' \neq \text{res} \), then \( \Gamma'(z) = C \) and \( \Gamma'(z') = C' \) (for some \( C \) and \( C' \)) and \( \{ z, z' \} \subseteq \text{FV}(e_0) \)

(4) if either \( z = \text{res} \) or \( z' = \text{res} \), then \( \Gamma'(z') = C \) or \( \Gamma'(z) = C \) (for some \( C \))

and \( z' \in \text{FV}(e_0) \) or \( z \in \text{FV}(e_0) \).

Observe that if for all \( i, 1 \leq i \leq n \) if \( z \neq z' \), if \( z \neq z' \) and \( \langle z, z' \rangle \in S'_i \), then \( z, z' \neq \text{res} \) and either \( \langle z, z' \rangle \in S_i \) or \( \langle z, x_i \rangle \in S_i \) and \( \langle z', \text{res} \rangle \in S_i \). Therefore by (1) and (2) we have that

(5) if \( z \neq z' \) and \( \langle z, z' \rangle \in S'_i \), then \( z, z' \neq \text{res} \) and \( \Gamma'(z) = C \) and \( \Gamma'(z) = C' \)

(for some \( C \) and \( C' \)) and \( \{ z, z' \} \subseteq \text{FV}(e_i) \).

Let \( x \neq y \) and \( \langle x, y \rangle \in S \), then \( \langle x, y \rangle \in \bigcup_{i=0}^n S'_i \) and \( x, y \notin X \). By Proposition 11 there are sequences \( i_1 \ldots i_{k-1} \) (\( 0 \leq i_n \leq n \) for all \( h \)) and \( z_1 \ldots z_k \) (\( k > 1 \)) such that \( x = z_1 \) and \( y = z_k \).

(B) \( \langle z_j, z_{j+1} \rangle \in S'_{i_j} \) and \( i_j \neq i_{j+1} \) and \( z_j \neq z_{j+1} \) for \( 1 \leq j \leq (k-1) \).

By (5) and (3) (the induction hypothesis on \( S'_0 \)) for all \( i, 1 \leq i \leq k-1 \)

(A) if \( z_i \neq \text{res} \) and \( z_{i+1} \neq \text{res} \), then \( \Gamma(z_i) = C \) and \( \Gamma(z_{i+1}) = C' \) (for some \( C \) and \( C' \)) and \( \{ z_i, z_{i+1} \} \subseteq \text{FV}(e_i) \).

By (5) and (4) (the induction hypothesis on \( S'_0 \)) for all \( i, 1 \leq i \leq k-1 \)

(B) if \( z_{i+1} = \text{res} \) or \( z_i = \text{res} \), then \( \Gamma(z_i) = C \) or \( \Gamma(z_{i+1}) = C \) (for some \( C \)) and \( z_i \in \text{FV}(e_i) \) or \( z_{i+1} \in \text{FV}(e_i) \).

Finally, by transitivity of equality we have that, for all \( i, 1 \leq i \leq k \), if \( z_i \neq \text{res} \), then \( \Gamma(z_i) = C \) and if there is \( j, 1 \leq j \leq k \), such that \( z_j = \text{res} \) also \( C = D \). Moreover \( \{ z_i \mid z_i \neq \text{res} \} \leq \bigcup_{1 \leq i \leq n} (\text{FV}(e_i) \setminus X) = \text{FV}(e) \). Therefore, if \( \langle x, y \rangle \in S \) and \( x \neq y \) we get

- if \( x \neq \text{res} \) and \( y \neq \text{res} \), then \( \Gamma(x) = C \) and \( \Gamma(y) = C' \) (for some \( C \) and \( C' \)) and \( x, y \in \text{FV}(e) \),

- if \( y = \text{res} \) or \( x = \text{res} \), then \( \Gamma(x) = C \) or \( \Gamma(y) = C \) (for some \( C \)) and \( x \in \text{FV}(e) \) or \( y \in \text{FV}(e) \).

The proofs for rules \((\text{T-Inv})\) and \((\text{T-New})\) are similar.

4. Examples

In this section we illustrate the expressiveness of the type system by programming examples, and we show a type derivation.

Example 3. Assume we have a class \( D \) with a field \( f \) of type \( D \), and a class \( C \) with two fields of type \( D \). Consider the following closed expression \( e \):
The inner block (right-hand side of the declaration of \(z\)) refers to the external variables \(x\) and \(y\), that is, they occur free in the block. In particular, the execution of the block has the sharing effect of connecting \(x\) and \(y\). However, these variables will not be connected to the final result of the block, since the result of the assignment will be only connected to a local variable which is not used to build the final result, as more clearly shown by using the sequence abbreviation: \(\{D \text{ } z2= \text{ new } D(z2); \text{ } y.f= x; \text{ } \text{ new } C(z2,z2)\}\). Indeed, as will be shown in the next section, the block reduces to \(\{D \text{ } z2= \text{ new } D(z2); \text{ } \text{ new } C(z2,z2)\}\) which is a closed block. In existing type systems supporting the capsule notion this example is either ill-typed [6], or can be typed by means of a rather tricky swap typing rule [7, 8] which, roughly speaking, temporarily changes, in a subterm, the set of variables which can be freely used.

**Example 4.** As a counterexample, consider the following ill-typed term

\[
D \text{ } y= \text{ new } D(y);
D \text{ } x= \text{ new } D(x);
C^a \text{ } z= \{D \text{ } z2= \text{ new } D(z2); \text{ } D \text{ } z1= (y.f= x); \text{ new } C(z2,z2)\};
\]

Here the inner block is not a capsule, since the local variable \(z1\) is initialized as an alias of \(x\), hence connected to both \(x\) and \(y\). Indeed, the block reduces to \(\text{ new } C(x,x)\) which is not closed.

**Type derivation for Example 4.** Let \(\Gamma_1 = y:D, x:D, z:C^a\), and \(\Gamma_2 = z2:D, z1:D\). In Fig.4 we give the type derivation that shows that expression \(e\) of Example 3 is well-typed. To save space we omit the annotated expression produced by the derivation, and show the annotations for the blocks at the bottom of the figure.

Consider the type derivation \(D_2\) for the expression \(e^i\) which initializes \(z\) (inner block). The effect of the declaration of \(z2\) is the sharing relation \(\{z2, res\}\) and the effect of the declaration of \(z1\) is the sharing relation \(\{z1, x, y, res\}\). Before joining these sharing relations, we remove \(res\) from their domains, since the results of the two expressions are not connected with each other. So the resulting sharing relation is represented by \(\{z1, x, y\}\) (\(z2\) is only connected with itself). The effect of the evaluation of the body is the sharing relation represented by \(\{z2, res\}\). Therefore, before removing the local variables \(z1\) and \(z2\) from the sharing relations we have the sharing relation \(\{z1, x, y\}\) \{\(z2, res\}\}. The block is annotated with \(\{z2\}\) (the local variable in the equivalence class of \(res\)). Since, after removing the local variables, \([res] = \{res\}\), \(e^i\) denotes a capsule, and may be used to initialize an affine variable.

\(\mathcal{D}\) is the type derivation for the whole expression \(e\). Note that, since \(z\) is an affine variable, the sharing relation of \(z\), which is the body of the block, is the identity. In particular, \([res] = \{res\}\), so the annotation of the block \(e\) is \(\emptyset\).
Example 5. We provide now a more realistic programming example, assuming a syntax enriched by usual programming constructs. See the Conclusion for a discussion on how to extend the type system to such constructs.

The class `CustomerReader` below models reading information about customers out of a text file formatted as shown in the example:

```
Bob 1 500 2 1300
Mark 42 8 99 100
```

In odd lines we have customer names, in even lines we have a shop history: a sequence of product codes. The method `CustomerReader.read` takes a `Scanner`, assumed to be a class similar to the one in Java, for reading a file and extracting different kinds of data.

```java
class CustomerReader {
  static Customer read (Scanner s) { /*S = \epsilon*/
    Customer c = new Customer (s.nextLine ());
    while (s.hasNextNum ()) {
      c.addShopHistory (s.nextNum ());
    }
    return c;
  }
}
class Scanner {

```

Figure 4: Type derivation for Example 3
Here and in the following, we insert after method headers, as comments, their sharing effects. In a real language, a library should declare sharing effects of methods by some concrete syntax, as part of the type information available to clients. In this example, `CustomerReader.read` uses some methods of class `Scanner`. Such methods have no sharing effects, as represented by the annotation $S = \epsilon$. Note that for the last two methods this is necessarily the case since they have no explicit parameters and a primitive return type. For the first method, the only possible effect could have been to mix this with the result. A `Customer` object is read from the file, and then its shop history is added. Since methods invoked on the scanner introduce no sharing, we can infer that the same holds for method `CustomerReader.read`. In other words, we can statically ensure that the data of the scanner are not mixed with the result.

The following method `update` illustrates how we can “open” capsules, modify their values and then recover the original capsule guarantee. The method takes a customer, which is required to be a capsule by the fact that the corresponding parameter is affine, and a scanner as before.

```java
class CustomerReader {
    // as before
    static Customer update ( Customer old , Scanner s ) /*$S = \epsilon$/*/ {
        Customer c = old // we open the capsule 'old'
        while ( s.hasNextNum () ) {
            c.addShopHistory ( s.nextNum () )
        }
        return c
    }
}
```

A method which has no sharing effects can use the pattern illustrated above: one (or many) affine parameters are opened (that is, assigned to local variables) and, in the end, the result is guaranteed to be a capsule again. This mechanism is not possible in [2, 1, 5] and relies on destructive reads in [6].

A less restrictive version of method `update` could take a non affine `Customer old` parameter, that is, not require that the old customer is a capsule. In this case, the sharing effects would be $S = \{\text{res, old}\}$. Hence, in a call `Customer.update(c1, s)`, the connections of `c1` would be propagated to the result of the call. In other words, the method type is “polymorphic” with respect to sharing effects. Notably, the method will return a capsule if invoked on a parameter which is a capsule.

**Example 6.** The following method takes two teams $t_1, t_2$. Both teams want to add a reserve player from their respective lists $p_1$ and $p_2$, assumed to be sorted with best players first. However, to keep the game fair, the two reserve players can only be added if they have the same skill level.

```java
static void addPlayer ( Team t1 , Team t2 , Players p1 , Players p2 ) /*$S = \{t1.p1, t2.p2\}$/*/ {
```
while (true) {
    // could use recursion instead
    if (p1.isEmpty() || p2.isEmpty()) { /* error */
        if (p1.top().skill == p2.top().skill) {
            t1.add(p1.top());
            t2.add(p2.top());
            return;
        } else {
            removeMoreSkilled(p1, p2);
        }
    }
    The sharing effects express the fact that each team is only mixed with its list of reserve players.
}

Example 7. Finally, we provide a more involved example which illustrates the expressive power of our approach. Assume we have a class C as follows:

class C {
    C f;
    C clone() /* S = \epsilon */ { ...
    C mix(C x) /* S = \{x, this, res\} */ { ...
}
}

The method clone is expected to return a deep copy of the receiver. Indeed, this method has no parameters, apart from the implicit non-affine parameter this, and returns an object of class C which is not connected to the receiver, as specified by the fact that the sharing relation is the identity, represented by \( \epsilon \), where res is not connected to this. Note that a shallow clone method would be typed C clone() /* S = \{this, res\} */.

The method mix is expected to return a “mix” of the receiver with the argument. Indeed, this method has, besides this, a parameter x of class C, both non affine, returns an object of class C and its effects are connecting x with this, and both with the result.

Consider now the following closed expression e:

C c1 = new C(c1);
C outer = {
    C c2 = new C(c2);
    C inner = {
        C c3 = new C(c3);
        C r = c2.mix(c1).clone()
            r.mix(c3));
        inner.mix(c2);
    inner.mix(c2);
}
outer
The key line in this example is C r=c2.mix(c1).clone(). Thanks to the fact that clone returns a capsule, we know that r will not be connected to the external variables c1 and c2, hence also the result of the block

\[7\] We assume this to be non-affine unless explicitly indicated, e.g., by inserting a as first element of the list of parameters.
5. The calculus

The calculus has a simplified syntax, defined in Fig. 5, where we assume that, except from right-hand sides of declarations and bodies of blocks, subterms of a compound expression are only values. This simplification can be easily obtained by a (type-driven) translation of the syntax of Fig. 1 generating for each subterm which is not a value a local declaration of the appropriate type. Moreover we omit primitive types. Finally, the syntax describes runtime terms, where blocks are annotated as described in the previous section.

A value is the result of the reduction of an expression, and is either a variable (a reference to an object), or a block where the declarations are evaluated (hence, correspond to a local store) and the body is in turn a value, or a constructor call where argument are evaluated.

A sequence of evaluated declarations plays the role of the store in conventional models of imperative languages, that is, each can be seen as an association of a right-value to a reference.

As anticipated in Sect. 2 mutual recursion among evaluated declarations is allowed, whereas we do not allow references to variables on the left-hand side of forward unevaluated declarations. E.g., is allowed, whereas is not.

That is, our calculus supports recursive object initialization, whereas, e.g., in Java, we cannot directly create two mutually referring objects as in the allowed example above, but we need to first initialize their fields to null. However, to make recursive object initialization safe, we should prevent access to an object which is not fully initialized yet, as in the non-allowed example. Here we take

| $e$ ::= $x$ | $v.f$ | $v.m(vs)$ | $v.f=v$ | $\texttt{new} \ C(vs)$ | expression |
| $d$ ::= $T \ x=e$; | declaration |
| $T$ ::= $C^\mu$ | declaration type |
| $v$ ::= $x$ | $\{X \ dv \ v\}$ | $\texttt{new} \ C(vs)$ | value |
| $dv$ ::= $C \ x=\texttt{new} \ C(vs)$; | evaluated declaration |
a simplifying assumption, just requiring initialization expressions to be values. More permissive assumptions can be expressed by a sophisticated type system as in [18].

Semantics is defined by a congruence relation, which captures structural equivalence, and a reduction relation, which models actual computation, similarly to what happens, e.g., in π-calculus [17].

The congruence relation, denoted by $\equiv$, is defined as the smallest congruence satisfying the axioms in Fig. 6. We write $\text{FV}(\text{ds})$ and $\text{FV}(e)$ for the free variables of a sequence of declarations and of an expression, respectively, and $X[y/x]$, $\text{ds}[y/x]$, and $e[y/x]$ for the capture-avoiding variable substitution on a set of variables, a sequence of declarations, and an expression, respectively, all defined in the standard way.

| Rule          | Description                                                                 |
|---------------|-----------------------------------------------------------------------------|
| (alpha)       | $\{X \ \text{ds} \ T \equiv; \ \text{ds'} \ e'\} \equiv \{X[y/x] \ \text{ds}[y/x] \ T[y/x] \equiv; \ \text{ds'')[y/x] \ e'[y/x]\}$ |
| (reorder)     | $\{X \ \text{ds} C \equiv; \ \text{ds'} \ e\} \equiv \{X \ C \equiv; \ \text{ds} \ \text{ds'} \ e\}$                              |
| (new)         | $\text{new } C(\text{ds}) \equiv \{x\} \ C \equiv; \ x\}$                   |
| (block-elim)  | $\{x\} \equiv e$                                                           |
| (dec)         | $\{Y \ \text{ds} C \equiv; \ \text{ds'} \ e\} \equiv \{Y' \ \text{ds} \ \text{ds'} \ C \equiv; \ \text{ds'} \ e\}$ |
| (body)        | $\{Y \ \text{ds} \ X \equiv; \ \text{ds'} \ e\} \equiv \{Y' \ \text{ds} \ \text{ds'} \ X \equiv; \ \text{ds'} \ e\}$ |
| (val-ctx)     | $\text{V} \{X \ \text{ds} \ v\} \equiv \{Y \ \text{ds} \ v\}$            |

Figure 6: Congruence rules

Rule (alpha) is the usual $\alpha$-conversion. The condition $x, y \not\in \text{dom}(\text{ds} \ \text{ds'})$ is implicit by well-formedness of blocks.

Rule (reorder) states that we can move evaluated declarations in an arbitrary order. Note that, instead, $\text{ds}$ and $\text{ds'}$ cannot be swapped, because this could change the order of side effects.

In rule (new), a constructor invocation can be seen as an elementary block where a new object is allocated.

Rule (block-elim) states that a block with no declarations is equivalent to its body. With the remaining rules we can move a sequence of declarations from a block to the directly enclosing block, or conversely, as it happens with rules for scope extension in the π-calculus [17].

In rules (dec) and (body), the inner block is the body, or the right-hand side of a declaration, respectively, of the enclosing block. The first two side conditions
ensure that moving the declarations outside the block does cause neither scope extrusion nor capture of free variables. More precisely: the first prevents moving outside declarations which depend on local variables of the inner block. The second prevents capturing free variables of the enclosing block. Note that the second condition can be obtained by \( \alpha \)-conversion of the inner block, but the first cannot. Finally, the third side condition of rule (dec) prevents, in case the block initializes an affine variable, to move outside declarations of variables that will be possibly connected to the result of the block. Indeed, in this case we would get an ill-typed term. In case of a non-affine declaration, instead, this is not a problem.

Rule (val-ctx) handles the cases when the inner block is a subterm of a field access, method invocation, field assignment or constructor invocation. Note that in this case the inner block is necessarily a (block) value. To express all such cases in a compact way, we define value contexts \( \mathcal{V} \) in the following way:

\[
\mathcal{V} ::= [ ] | \mathcal{V}.f | \mathcal{V}.f = v | v.f = \mathcal{V} | \text{new } C(\mathcal{V}, x, vs')
\]

For instance, if \( \mathcal{V} = \text{new } C(\mathcal{V}, [ ], vs') \), we get

\[
\text{new } C(\mathcal{V}, \{X dvs_1 dvs_2 v\}, \mathcal{V}) \equiv \{Y dvs_1 \text{new } C(\mathcal{V}, \{X' dvs_2 v\}, \mathcal{V})\}
\]

As for rules (dec) and (body), the first side condition prevents moving outside a declaration in \( dvs_1 \) which depends on local variables of the inner block, and the second side condition prevents capturing free variables of \( \mathcal{V} \), defined in the standard way.

The following definition introduces a simplified syntactical form for values and evaluated declarations. In this canonical form, a sequence of evaluated declarations (recall that its order is immaterial) can be seen as a store which associates to references object states of shape \( \text{new } C(xs) \), where fields contain in turn references, and a value is a variable (a reference to an object) possibly enclosed in a local store.

**Definition 8.**

1. A sequence of evaluated declarations \( dvs \) is in canonical form if, for all \( dv \) in \( dvs \), \( dv = \text{C } x = \text{new } C(xs) \); for some \( C \), \( x \) and \( xs \).
2. A value \( v \) is in canonical form if either \( v = x \) for some \( x \), or \( v = \{X dvs x\} \) for some \( X \), \( dvs \), and \( x \), with \( dvs \neq \epsilon \) in canonical form.

The following proposition shows that congruence allows us to assume that values are in canonical form.

**Proposition 9.** If \( v \) is a value, then there exists \( v' \) such that \( v \cong v' \) and \( v' \) is in canonical form.

**Proof.** By structural induction on values, and for blocks by induction on the number of declarations that are not in canonical form. The full proof is in Appendix B.
From now on, unless otherwise stated, we assume that values and evaluated declarations are in canonical form. Moreover, we also need to characterize values which are garbage-free, in the sense that they do not contain useless store. To this end, we first inductively define $X \xrightarrow{ds} x$, meaning that $x$ is (transitively) used by $X$ through $ds = T_1 x_1 = e_1; \ldots; T_n x_n = e_n;$, by:

$$X \xrightarrow{ds} x \text{ if } x \in X$$

$$X \xrightarrow{ds} x \text{ if } x \in \text{FV}(e_i), \text{ for some } i \in 1..n,$$

and $X \xrightarrow{ds} x_i$.

Then, we write $ds|_X$ for the subsequence of $ds$ (transitively) used by $X$, defined by: for all $i \in 1..n$, $T_i x_i = e_i; \in ds|_X$ if $X \xrightarrow{ds} x_i$.

Finally, we define $\text{gc}((X \text{ dvs } x)) = (X \cap \text{dom}(dvs)) \text{ dvs}|_x x$, and we say that a value $v$ is garbage-free if either $v = x$ or $v = \text{gc}(v)$.

**Evaluation contexts**, defined below, express standard left-to-right evaluation.

$$\mathcal{E} ::= [ ] | \{ X \text{ dvs } T x = \mathcal{E}; \hspace{1mm} ds \hspace{1mm} e \} | \{ X \text{ dvs } \mathcal{E} \}
$$

In the evaluation context $\{ X \text{ dvs } T x = \mathcal{E}; \hspace{1mm} ds \hspace{1mm} e \}$ we assume that no declaration in $ds$ is evaluated. This can always be achieved by the congruence rule (\text{REORDER}.

We introduce now some notations which will be used in reduction rules. We write $\text{dvs}(x)$ for the \textit{declaration} of $x$ in $\text{dvs}$, if any (recall that in well-formed blocks there are no multiple declarations for the same variable). We write $\text{HB}(\mathcal{E})$ for the \textit{hole binders} of $\mathcal{E}$, that is, the variables declared in blocks enclosing the context hole, defined by:

- if $\mathcal{E} = \{ \text{dvs } T x = \mathcal{E}'; \hspace{1mm} ds \hspace{1mm} e \}$, then $\text{HB}(\mathcal{E}) = \text{dom}(\text{dvs}) \cup \{x\} \cup \text{HB}(\mathcal{E}') \cup \text{dom}(ds)$
- if $\mathcal{E} = \{ \text{dvs } \mathcal{E}' \}$, then $\text{HB}(\mathcal{E}) = \text{dom}(\text{dvs}) \cup \text{HB}(\mathcal{E}')$

We write $\mathcal{E}_x$ and $\text{dec}(\mathcal{E}, x)$ for the \textit{sub-context declaring $x$} and the \textit{evaluated declaration of $x$} extracted from $\mathcal{E}$, defined as follows:

- if $\text{dvs}(x) = dv$ and $x \notin \text{HB}(\mathcal{E}')$, then $\mathcal{E}_x = \{ \text{dvs } T y = [ ]; \hspace{1mm} ds \hspace{1mm} e \}$ and $\text{dec}(\mathcal{E}, x) = dv$
- else $\mathcal{E}_x = \{ \text{dvs } T y = \mathcal{E}_x'; \hspace{1mm} ds \hspace{1mm} e \}$ and $\text{dec}(\mathcal{E}, x) = \text{dec}(\mathcal{E}', x)$

- if $\text{dvs}(x) = dv$ and $x \notin \text{HB}(\mathcal{E}')$, then $\mathcal{E}_x = \{ \text{dvs } [ ] \}$ and $\text{dec}(\mathcal{E}, x) = dv$
- else $\mathcal{E}_x = \{ \text{dvs } \mathcal{E}_x' \}$, and $\text{dec}(\mathcal{E}, x) = \text{dec}(\mathcal{E}', x)$.

Note that $\mathcal{E}_x$ and $\text{dec}(\mathcal{E}, x)$ are not defined if there is no evaluated declaration for $x$ in some block enclosing the context hole.

Reduction rules are given in Fig[7].

Rule (\text{congr}) can be used to reduce a term which otherwise would be stuck, as it happens for the $\alpha$-rule in lambda calculus.

In rule (\text{field-access}), given a field access of shape $x.f$, the first enclosing declaration for $x$ is found (through the auxiliary function $\text{dec}$). The fields of the class $C$ of $x$ are retrieved from the class table. If $f$ is actually the name of a field of $C$, say, the $i$-th, then the field access is reduced to the reference $x_i$ stored in this field. In the last side condition, $\mathcal{E}_x$ is the (necessarily defined) sub-context
containing the first enclosing declaration for \( x \), and the condition \( x_i \not\in \text{HB}(E') \) ensures that there are no declarations for \( x_i \) in inner blocks (otherwise \( x_i \) would be erroneously bound). This can always be obtained by rule \((\text{alpha})\) of Fig.6.

For instance, assuming a class table where class \( A \) has an \text{int} field, and class \( B \) has an \( A \) field \( f \), without this side condition, the term (without annotations):

\[
A \ a = \text{new} \ A(0); \ B \ b = \text{new} \ B(a); \ \{ A \ a = \text{new} \ A(1); \ b.f \}
\]

would reduce to

\[
A \ a = \text{new} \ A(0); \ B \ b = \text{new} \ B(a); \ \{ A \ a = \text{new} \ A(1); \ a \}
\]

whereas this reduction is forbidden, and by rule \((\text{alpha})\) the term is instead reduced to

\[
A \ a = \text{new} \ A(0); \ B \ b = \text{new} \ B(a); \ \{ A \ a1 = \text{new} \ A(1); \ a \}
\]

For this example: \( \mathcal{E}_b = \{ A \ a = \text{new} \ A(0); \ B \ b = \text{new} \ B(a) \ \mathcal{E}' \} \) and \( \mathcal{E}' = \{ A \ a = \text{new} \ A(1) \ [ ] \} \).

In rule \((\text{invk})\), the class \( C \) of the receiver \( v \) is found through the auxiliary function \( \text{class} \) defined by

\[
\text{class}(E, x) = C \ \text{if dec}(E, x) = C \ \text{and new}(x_1, \ldots, x_n);
\]

\[
\text{class}(E, \{ X \ dv = y; \ ds e \}) = C \ \text{if dv}(x) = C \ \text{and new}(x_1, \ldots, x_n, e)
\]

and method \( m \) of \( C \), if any, is retrieved from the class table. The call is reduced to a block where declarations of the appropriate type for \( \text{this} \) and the parameters are initialized with the receiver and the arguments, respectively, and the body is the method body.
In rule (field-assign), given a field assignment of shape \( x.f = y \), the first enclosing declaration for \( x \) is found (through the auxiliary function \( \text{dec} \)). If \( f \) is actually the name of a field of \( C \), say, the \( i \)-th, then this first enclosing declaration is updated, by replacing the \( i \)-th constructor argument by \( y \), obtaining \( C\ x = \text{new} \ C(x_1, x_{i-1}, y, x_{i+1}, \ldots, x_n) \), as expressed by the notation \( \varepsilon^{x.1=y} \) (the obvious formal definition of which is omitted). As for rule (field-access) we have the side condition that \( y \not\in \text{HB}(\varepsilon') \). This side condition, requiring that there are no inner declarations for the reference \( y \), prevents scope extrusion, since if \( y \in \text{HB}(\varepsilon') \), \( \varepsilon^{x.1=y} \) would take \( y \) outside the scope of its definition. The congruence rules (dec) and (body) of Fig.6 can be used to correctly move the declaration of \( y \) outside its declaration block, as previously described. For example, without the side condition, the term (without annotations)

\[
A \ a = \text{new} \ A(0); \ B \ b = \text{new} \ B(a); \ {A \ a1 = \text{new} \ A(1); \ b.f = a1}
\]

would reduce to

\[
A \ a = \text{new} \ A(0); \ B \ b = \text{new} \ B(\ a1); \ {A \ a1 = \text{new} \ A(1); \ a1}
\]

The previous term is congruent to

\[
A \ a = \text{new} \ A(0); \ B \ b = \text{new} \ B(a); \ A \ a1 = \text{new} \ A(1); \ b.f = a1
\]

by applying rule (body), and then (block-elim). This term reduces correctly to

\[
A \ a = \text{new} \ A(0); \ B \ b = \text{new} \ B(\ a1); \ A \ a1 = \text{new} \ A(1); \ a1
\]

The last two rules eliminate evaluated declarations from a block. In rule (alias-elim), a (non-affine) variable \( x \) which is initialized as an alias of another reference \( y \) is eliminated by replacing all its occurrences. In rule (affine-elim), an affine variable is eliminated by replacing its unique occurrence with the value associated to its declaration from which we remove garbage.

We conclude this section by briefly discussing how the reduction relation could actually be computed. Indeed, the definition in Fig.7 is not fully algorithmic, since rule (congr) can always be applied in a non-deterministic way, analogously to what happens, e.g., with \( \alpha \)-conversion in lambda calculus or structural rules in \( \pi \)-calculus. However, again as is usually done for analogous cases, congruence is expected to be applied only when needed (since otherwise reduction would be stuck). All our congruence rules except for (alpha) and (reorder) are meant to be applied from left to right (as a reduction). This is witnessed by the fact that values in canonical form do not match the left-hand side of any of the previously mentioned congruence rules and Proposition 9 ensures that any value may be reduced in this form.

6. Results

In this section we present the main formal results on our calculus. First, we show a canonical form theorem describing constraints on free variables of well-typed garbage-free values. Then we prove subject reduction, stating that reduction preserves the type, and may reduce the sharing effects. In addition,
reduction preserves an invariant on the store that allows us to prove that lent and capsule references have the expected behaviour.

Finally, we prove progress, i.e., that well-typed expressions do not get “stuck”. First of all we extend the typing judgment to annotated expressions, and to (annotated) sequences of declarations, as follows.

Definition 10.

- Given an annotated expression $e$, $e^-$ is the expression obtained by erasing the annotations from $e$, and $\Gamma \vdash e : C \mid S$ if $\Gamma \vdash e^- : C \mid S \leadsto e$.

- Given the annotated expression $e_1$ and $e_2$, we say that $e_1$ and $e_2$ are equal up to annotations, dubbed $e_1 \approx^- e_2$ if $e_1^- = e_2^-$. 

- Given $ds = C_1^{\mu_1} x_1 = e_1; \ldots C_n^{\mu_n} x_n = e_n$; an (annotated) sequence of declarations, $\Gamma \vdash ds : S$ if

  - $\Gamma \vdash e_i : C_i \mid S_i$, for some $S_i$ (1 ≤ $i$ ≤ $n$),
  - $S = \sum_{i=1}^{n} (S_i[x_i / \text{res}])$, and
  - if $\mu_i = a$ then capsule($S_i$) (1 ≤ $i$ ≤ $n$).

Note that in the $S$ derived for a sequence of declarations the equivalence class of res is a singleton, according to the fact that a sequence of declarations has no “result”. As we can see from the typing rules of Fig.3 if the non-annotated expression $e$ is typable, then there is a unique annotated $e'$ such that $\Gamma \vdash e' : C \mid S$ for some $C$ and $S$.

Canonical form theorem. In this subsection we state a theorem describing constraints on free variables of well-typed garbage-free values. Notably, such free variables are either affine or connected to its result. Therefore, a garbage-free capsule value can contain only affine free variables.

In the following we use the underscore _ for a type, when the specific type is irrelevant. Moreover, we will say that $x$ is affine/non-affine in $\Gamma$ if $\Gamma(y) = C^a$ or $\Gamma(y) = C$, respectively.

Theorem 11. If $\Gamma \vdash v : C \mid S$ where $v$ is garbage-free and $y \in FV(v)$, then:

1. if $y$ is non-affine in $\Gamma$, then $\langle y, \text{res} \rangle \in S$
2. if capsule($S$), then $y$ is affine in $\Gamma$.

Proof. 1. By cases on the shape of canonical values.

- If $v = x$, then the only free variable in $v$ is $x$ itself. Since $x$ is non-affine in $\Gamma$, then the judgment $\Gamma \vdash x : C \mid S$ is derived by rule (t-var), hence $\langle x, \text{res} \rangle \in S$.

\[\text{Recall that values are assumed to be in canonical form.}\]
If \( v = \{ X \text{dvs} x \} \), since \( \text{dvs} \) is in canonical form and garbage-free we have
\[
d\text{v} = C_1 x_1 \text{new} C_1 (x_{s_1}) ; \ldots ; C_n x_n \text{new} C_n (x_{s_n}) ; \text{and } x \xrightarrow{\text{dvs}} x_i \text{ for all } i \in 1..n.
\]
The judgment \( \Gamma \vdash v : \_ \mid S \) is derived by rule \((\text{T-block})\) with premises derived by \((\text{T-new})\) for each declaration (which in turn have premises that are derived with rules \((\text{T-var})\) or \((\text{T-affine-var})\)) and rule \((\text{T-var})\) to derive a type for the body \((x = x_i \text{ for some } 1 \leq i \leq n)\). Hence, letting \( \Gamma' = x_1 : C_1, \ldots, x_n : C_n \), we have
\[
- \Gamma[\Gamma'] \vdash \text{new} C_i (x_{s_i}) : C_i | \{x_{s_i}', \text{res}\} \text{ (1 \leq i \leq n)} \text{ where } x_{s_i}' \text{ are the non affine variables in } x_{s_i}
- \Gamma[\Gamma'] \vdash x : C | \{x, \text{res}\}
- S_i = \{x_{s_i}', x_i\}
- S = S' \setminus \text{dom}(\Gamma') \text{, with } S' = \sum_{i=1}^{n} S_i + \{x, \text{res}\}
\]
From Proposition [11] and \( x \xrightarrow{\text{dvs}} x_i \text{ for all } i \in 1..n \), we have that \( S' \) has a unique equivalence class \( \bigcup_{1 \leq i \leq n} \{x_{s_i}', x_i\} \cup \{\text{res}\} \). If \( y \in \text{FV}(e) \), then \( y \in x_{s_j} \text{ for some } j \in \{1, \ldots, n\} \) and \( y \notin \{x_1, \ldots, x_n\} \). From the fact that \( y \) is not affine we have that \( y \in x_{s_j}' \). Therefore \( \langle y, \text{res} \rangle \in S \).

2. If \( y \) is free in \( v \) and \( y \) is non-affine in \( \Gamma \), then by the previous point we would have \( \langle y, \text{res} \rangle \in S \), contradicting \text{capsule}(S). Hence, \( \Gamma(y) = C^a \).

The following lemma is a corollary of the canonical form theorem.

**Lemma 12.** If \( \Gamma \vdash v : C \mid S \) where \( v \) is garbage-free, and \( \text{capsule}(S) \), then \( S = \epsilon \).

**Proof.** Assume \( S \neq \epsilon \). Then, from Proposition [2] there would be a free variable in \( v \) non-affine in \( \Gamma \), but this is impossible by Theorem [11].

**Subject reduction.** To show subject reduction, we need some preliminary lemmas.

The following lemma states that typing essentially depends only on the free variables of the expression. We denote by \( \Gamma \setminus x \) the type environment obtained by removing the type association for \( x \) from \( \Gamma \), if any.

**Lemma 13.** (Weakening) Let \( \Gamma \vdash e : C \mid S \). If \( x \notin \text{FV}(e) \), then

1. \( \Gamma[x:T] \vdash e : C \mid S + \{x\} \text{ for all } T \), and
2. \( \Gamma \setminus x \vdash e : C \mid S \setminus x \).

**Proof.** By induction on derivations.

The following lemma states the dependency between the type and sharing relation derived for a block and the ones derived for its declarations and body.

**Lemma 14.** (Inversion for blocks) If \( \Gamma \vdash \{X \text{dvs} e\} : C \mid S \), then
• \( \Gamma \vdash ds : S_{ds} \) for some \( S_{ds} \)

• \( \Gamma \vdash e : C \mid S_e \) for some \( S_e \) such that

\[ S = (S_{ds} + S_e) \setminus \text{dom}(ds) \] and \( X = \text{res}(S_{ds} + S_e) \cap \text{dom}(ds) \).

**Proof.** By rule `(T-Block)` and definition of the type judgement for declarations. \(\square\)

The following lemma asserts that congruent expressions have the same type and sharing effects. Regarding annotations, which are uniquely determined by the type derivation, if one of the two expressions is well-typed, then the annotations of the other are also uniquely determined.

**Lemma 15.** (Congruence preserves types) Let \( e_1 \) and \( e_2 \) be annotated expressions. If \( \Gamma \vdash e_1 : C \mid S \) and \( e_1 \equiv e_2 \), then \( \Gamma \vdash e'_2 : C \mid S \) for some \( e'_2 \) such that \( e'_2 \approx e_2 \).

**Proof.** The proof is in Appendix B. \(\square\)

In the following when we have \( \Gamma \vdash e_1 : C \mid S \) and \( e_1 \equiv e_2 \) we assume \( \Gamma \vdash e_2 : C \mid S \), that is, we picked the term with the right annotations.

The **type environment** extracted from \( ds \), denoted \( \Gamma_{ds} \), is defined by:

\[ \Gamma_{ds} = x_1 : T_1, \ldots, x_n : T_n \mid ds \text{ if } ds = T_1 x_1 = e_1 \ldots T_n x_n = e_n \].

Given an evaluation context \( \mathcal{E} \), the **type environment** extracted from \( \mathcal{E} \), denoted \( \Gamma_{\mathcal{E}} \), is defined by:

- \( \Gamma[\ ] \) is the empty type environment,
- \( \Gamma\{d vs \overline{x} \cdot : T \mid ds \} = (\Gamma\{d vs ds\})[x:T][\Gamma_{\mathcal{E}}] \) and
- \( \Gamma\{d \} = \Gamma\{d\} \cdot \Gamma_{\mathcal{E}} \).

The following lemma asserts that subexpressions of typable expressions are themselves typable, and may be replaced with expressions that have the same type and the same or possibly less sharing effects. Annotations may change by effect of the reduced sharing relations since the equivalence class of \( \text{res} \) in the reduced sharing relations may contain less variables.

**Lemma 16.** (Context) Let \( \Gamma \vdash \mathcal{E}[e] : C \mid S \), then

1. \( \Gamma[\Gamma_{\mathcal{E}}] \vdash e : D \mid S_1 \) for some \( D \) and \( S_1 \),
2. if \( \Gamma[\Gamma_{\mathcal{E}}] \vdash e' : D \mid S_2 \) where \( S_2 \subseteq S_1 \) (\( S_2 = S_1 \)), then \( \Gamma \vdash \mathcal{E}'[e'] : C \mid S' \) for some \( \mathcal{E}' \) such that \( \mathcal{E}'[e'] \approx \mathcal{E}[e'] \) and \( S' \subseteq S \) (\( S' = S \)).

**Proof.** The proof is in Appendix B. \(\square\)
The following lemma is used to prove that the elimination rules, namely (alias-elim) and (affine-elim), do not introduce sharing. In particular, for (alias-elim) a non-affine variable $x$ is substituted with a non-affine variable $y$ which is in the equivalence class of $x$ in the sharing relation $S$, so that there is no newly produced connection. For (affine-elim), an affine variable is substituted with a capsule value, so also in this case there is no newly produced connection.

**Lemma 17.** (Substitution)

1. If $\Gamma, x:D, y:D \vdash e : C \mid S$, then $\Gamma \vdash e[y/x] : C \mid S\setminus x$.
2. Let $\Gamma, x:D \vdash e : C \mid S$. If $\Gamma \vdash v : D \mid \epsilon$, then $\Gamma \vdash e[v/x] : C \mid S$.

**Proof.** By induction on type derivation. For point 1. we use Proposition 1.5.

The previous lemma can be easily extended to the type checking judgement for declarations $\Gamma \vdash ds : S$.

The following lemma asserts that the sharing relation of a subexpression is finer than the sharing relation of the expression that contains it.

**Lemma 18.** Let $\Gamma \vdash E[e] : C \mid S$ and $\Gamma \vdash e : D \mid S'$. If $(x, y) \in S'$ with $x, y \notin HB(E)$ and $x, y \neq res$, then $(x, y) \in S$.

**Proof.** The proof is in Appendix B.

The following lemma states a technical property needed to prove that sharing relations are preserved when we reduce a field access redex $x.f$ to the reference $y$ stored in the field. Recall that a set of variables $X$ stands for the sharing relation where $X$ is an equivalence class and the others are singletons.

**Lemma 19.** If $\Gamma \vdash E[e_1] : C \mid S_1$, $\Gamma \vdash E[e_2] : C \mid S_2$, $\Gamma \vdash e_1 : D \mid \{x, res\}$ and $\Gamma \vdash e_2 : D \mid \{y, res\}$ with $\{x, y\} \cap HB(E) = \emptyset$. Then $S_1 + \{x, y\} = S_2 + \{x, y\}$.

**Proof.** The proof is in Appendix B.

As already mentioned, the subject reduction theorem states that in a reduction step $e_1 \rightarrow e_2$:

1. $e_2$ preserves the type of $e_1$, and has less or equal sharing effects.
2. For each variable declaration $x \equiv e$; occurring in $e_1$ and reduced to $x \equiv e'$; in $e_2$, $e'$ preserves the type of $e$. Moreover, “$e'$ inside $e_2$” has less or equal sharing effects than “$e$ inside $e_1””, where such sharing effects are those of the initialization expression, plus the connections existing in the store (sequence of evaluated declarations) currently available in the enclosing expression.

Invariant (2) corresponds, in a sense, to the invariant on store which we would have in a conventional calculus, and allows us to express and prove the expected properties of lent and capsule references. Note that there is no guarantee that the sole sharing effects of the initialization expression are reduced, since a new free variable could be introduced in the expression as an effect of field access.

To formally express invariant (2), we need some notations and definitions. First of all, we need to trace the reduction of right-hand sides of declarations.
To simplify the notation, we assume in the following that expressions contain at most one declaration for a variable (no shadowing, as can be always obtained by alpha-conversion). We define declaration contexts $\mathcal{D}_{\mu x}$ by:

$$
\mathcal{D}_{\mu x} ::= \{X \text{ dvs } C^\mu x \in [ ]; \text{ ds } e \} | \{X \text{ dvs } T \text{ y=}D_{\mu x}; \text{ ds } e \} | \{X \text{ dvs } D_{\mu x}\}
$$

That is, in $\mathcal{D}_{\mu x}[e]$ the expression $e$ occurs as the right-hand side of the (unique) declaration for reference $x$, which has qualifier $\mu$. Since declaration contexts are a subset of evaluation contexts, the same assumptions and definitions hold.

To lighten the notation we write simply $\mathcal{D}_x$ when the modifier is not relevant, and $\mathcal{D}$ when not even the variable is relevant.

We define now the sharing relation $\mathcal{S}(\mathcal{D}_x)$ induced by the store (sequence of evaluated declarations) enclosing the hole in $\mathcal{D}_x$. To this end, we first inductively define $\text{store}(\mathcal{D}_x)$:

- $\text{store}(\{X \text{ dvs } C^\mu x \in [ ]; \text{ ds } e \}) = \text{dvs}$
- $\text{store}(\{X \text{ dvs } D_{\mu x}\}) = \text{store}(\{X \text{ dvs } T \text{ y=}D_{\mu x}; \text{ ds } e \}) = \text{dvs } \text{store}(\mathcal{D}_x)$

Then, if $\text{store}(\mathcal{D}_x) = \mathcal{C}_1 x_1=\text{neu} \mathcal{C}_1(x_1); \cdots \mathcal{C}_n x_n=\text{neu} \mathcal{C}_n(x_n)$; we define $\mathcal{S}(\mathcal{D}_x) = X_1 + \cdots + X_n$ where $X_i = \{x_i, x_1\}$ ($1 \leq i \leq n$).

To prove invariant (2) in the subject reduction we first need to show that it holds for the congruence relation.

**Lemma 20.** If $\Gamma \vdash \mathcal{D}_x[e] : C \mid \mathcal{S}$ and $\Gamma[\Gamma_{\mathcal{D}_x}] \vdash e : D \mid \mathcal{S}_x$ and $\mathcal{D}_x[e] \equiv e_1$ where $e_1 = \mathcal{D}_x'[e']$ for some $\mathcal{D}_x'$ and $e'$, then $\Gamma[\Gamma_{\mathcal{D}_x}] \vdash e' : D \mid \mathcal{S}_x'$ and $\mathcal{S}(\mathcal{D}_x) + \mathcal{S}_x = \mathcal{S}(\mathcal{D}_x') + \mathcal{S}_x'$.

**Proof.** By induction on $\mathcal{D}_x$.

Case $\{X \text{ dvs } T \text{ x=}e; \text{ ds } e_0\}$. Then $\Gamma \vdash \{X \text{ dvs } T \text{ x=}e; \text{ ds } e_0\} : C \mid \mathcal{S}$ and $\Gamma_{\mathcal{D}_x} = \Gamma_{\text{dvs } x:T}. \Gamma_{\mathcal{D}_x}$. Let $\mathcal{D}_x[e] \equiv e_1$, by Lemma 15 we have $\Gamma \vdash e_1 : C \mid \mathcal{S}$.

Consider how $e_1$ could be obtained from $\mathcal{D}_x[e]$ by applying congruence rules to its subexpressions.

Rule (reorder) applied to the block $\{X \text{ dvs } T \text{ x=}e; \text{ ds } e_0\}$ does not modify $e$ or $\mathcal{S}_{\text{dvs}}$. In particular, since no declaration following $[ ]$ can be evaluated, no declaration of $\text{dvs}$ can be moved after the one of $x$, and no declaration in $\text{ds}$ can be moved. So $e_1 = \{X \text{ dvs}' T \text{ x=}e; \text{ ds } e_0\}$ where $\text{dvs}'$ is a reordering of $\text{dvs}$. So the result is obvious.

Rule (body) applied to the block $\{X \text{ dvs } T \text{ x=}e; \text{ ds } e_0\}$ can only modify the structure of the block by “inserting curly brackets” between declaration. For example $e_1$ could be $\{X \text{ dvs} \{Y \text{ dvs}_2 T \text{ x=}e; \text{ ds } e_0\}\}$ where $\text{dvs} = \text{dvs}_1 \text{ dvs}_2$ and $\text{FV}(\text{dvs}_1) \cap \text{dom}(\text{dvs}_2) = \emptyset$. Again in this case $e_1 = \mathcal{D}_x'[e']$ for some $\mathcal{D}_x'$ such that $\mathcal{S}(\mathcal{D}_x) = \mathcal{S}(\mathcal{D}_x')$ and therefore the result holds.

If $e_1$ is obtained from $\mathcal{D}_x[e]$ by applying the congruence rules in $\text{ds}$ or $e_0$ or $e$, then $e_1 = \{X \text{ dvs } T \text{ x=}e'; \text{ ds}' e'_0\}$, so $e_1 = \mathcal{D}_x'[e'] = \{X \text{ dvs } T \text{ x=}e'; \text{ ds}' e'_0\}$ where $e \equiv e'$. By Lemma 15 and $\Gamma[\Gamma_{\mathcal{D}_x}] \vdash e : D \mid \mathcal{S}_x$ we have $\Gamma[\Gamma_{\mathcal{D}_x}] \vdash e' : D \mid \mathcal{S}_x$. It is easy to see that $\text{FV}(e) = \text{FV}(e')$. (Congruent expressions have the same set of free variables.) So again the result holds.
Since the congruence has to produce an expression of the shape of $D'_x$ no rule can be applied to $dvs$.

Finally, we may have that $D'_x[e']$ is obtained by application of rule $(\text{dec})$ to the declaration of $x$. There are two cases

1. $D_x[e] = \{ y \ dvs \ T \ x = \{ x \ dvs_1 \ ds_2 \ e_b \}; \ ds' \ e_b' \}$ and $D'_x[e'] = \{ y' \ dvs \ dvs_1 \ T \ x = \{ x' \ dvs_2 \ e_b \}; \ ds' \ e_b' \}$ or
2. $D_x[e] = \{ y' \ dvs \ dvs_1 \ T \ x = \{ x' \ dvs_2 \ e_b \}; \ ds' \ e_b' \}$ and $D'_x[e'] = \{ y \ dvs \ T \ x = \{ x \ dvs_1 \ ds_2 \ e_b \}; \ ds' \ e_b' \}$

where in both cases

3. $\text{FV}(dvs_2) \cap \text{dom}(ds_2) = \emptyset$ and $\text{FV}(dvs \ ds \ ds' \ e_b') \cap \text{dom}(dvs_1) = \emptyset$ and
4. $\mu = a$ implies $\text{dom}(dvs_1) \cap X = \emptyset$

From (3) we have that, if $\Gamma' = \Gamma[\Gamma_{dvs}, \Gamma_{ds}] | \Gamma_{dvs_1}, \Gamma_{ds_2}$ and $\Gamma'' = \Gamma[\Gamma_{dvs}, \Gamma_{ds'}, \Gamma_{dvs_1}] | \Gamma_{ds_2}$, then $\Gamma' = \Gamma''$. From $\Gamma \vdash D_x[e] : C \mid S$ and Lemma 13 we have that

(a) $\Gamma[\Gamma_{dvs}, \Gamma_{ds}] | \Gamma_{dvs_1}, \Gamma_{ds_2} \vdash dvs : S_d$ where
(b) $S_x = S_1 + S_2 + S_e$
(c) $\Gamma' \vdash dvs_1 : S_1$ and $\Gamma' \vdash ds_2 : S_2$ and $\Gamma' \vdash e_b : S_e$

From $D_x[e] \cong D'_x[e']$ and Lemma 15 we have that $\Gamma \vdash D'_x[e'] : C \mid S$. From Lemma 13

(d) $\Gamma[\Gamma_{dvs}, \Gamma_{ds'}, \Gamma_{dvs_1}] | \Gamma_{dvs_2} \vdash \{ x' \ dvs_2 \ e_b \} : D \mid S'_c$ and
(e) $S'_x = S'_2 + S'_e$
(f) $\Gamma'' \vdash ds_2 : S'_2$ and $\Gamma'' \vdash e_b : S'_e$

By Lemma 13 1 and 2 we have that $S'_1 = S_1$ and $S'_2 = S_2$ and $S'_e = S_e$ and $S'_d = S_d$.

From the fact that forward references can be done only to evaluated declarations and there are none in $ds$ we have that in $dvs$ and $dvs_1$ there cannot free variable in $\text{dom}(ds)$. Therefore $S_d = S_{d_{ds}}$. Moreover, from (3) in $dvs_1$ there cannot free variable in $\text{dom}(dvs_2)$ and $S_1 = S_{d_{dvs_1}}$. So we have that

$S(D_x) + S_x = S_{dvs} + (S_1 + S_2 + S_e)$ by definition of $S(D_x)$ and $S_x$ $= S_{dvs} + S_{dvs_1} + (S_2 + S_e)$ $= S(D'_x) + (S_2 + S_e)$ by definition of $S(D'_x)$ $= S(D'_x) + S'_e$ by definition of $S'_x$

This proves the result for both cases (1) and (2)

The cases $\{ x \ dvs \ T \ y = dvs_{2}; \ ds \ e_b \}$ and $\{ x \ dvs \ dvs_{2} \}$ are proved using the inductive hypothesis and a case analysis on the congruence used for the block as for the previous case.

\[ \square \]

**Theorem 21.** (Subject reduction) If $\Gamma \vdash e_1 : C \mid S$ and $e_1 \rightarrow e_2$, then
1. $\Gamma \vdash e'_1 : C \mid S'$ where $e'_2 \simeq e_2 S' \sqsubseteq S$, and

2. if $e_1 = D_x[e]$ and Lemma 16.1, we have that $\Gamma \vdash e'' : D \mid S'' \subseteq (S'(D'_x) + S'_y)$. By induction hypothesis $\Gamma \vdash e'' : D \mid S'' \subseteq (S'(D'_x) + S'_y)$. Therefore $\Gamma \vdash e'' : D \mid S'' \subseteq (S'(D'_x) + S'_y)$.

Proof. By induction on the derivation of $e_1 \rightarrow e_2$ with a case analysis on the last rule of Fig[7] used for $E[\rho] \rightarrow E'[\rho']$. We show the two most interesting cases, which are 

Rule (field-access).

1. By induction hypothesis on $e'_1$ we have that $\Gamma \vdash e'_1 : C \mid S$.

2. If $e_1 = D_y[e]$ and $e_2 = D'_y[e']$ for some $D'_y$. Let $e_2 = D'_y[e']$, from Lemma 19, we have that $\Gamma \vdash e'' : D \mid S''$ and (S(D'_y) + S'_y) = (S(D'_y) + S'_y). By induction hypothesis $\Gamma \vdash e'' : D \mid S'' \subseteq (S(D'_y) + S'_y)$. Therefore $(S(D'_y) + S'_y) \subseteq (S(D'_y) + S'_y)$.

Rule (field-access).

1. In this case

2. If $e_1 = D_y[e]$ and $e_2 = D'_y[e']$, then $e'_1 = D'_y[e']$, from Lemma 19, we have that $\Gamma \vdash e'' : D \mid S''$ and (S(D'_y) + S'_y) = (S(D'_y) + S'_y). By induction hypothesis $\Gamma \vdash e'' : D \mid S'' \subseteq (S(D'_y) + S'_y)$. Therefore $(S(D'_y) + S'_y) \subseteq (S(D'_y) + S'_y)$.

Rule (field-access).

1. In this case

2. If $e_1 = D_y[e]$ and $e_2 = D'_y[e']$, then $e'_1 = D'_y[e']$, from Lemma 19, we have that $\Gamma \vdash e'' : D \mid S''$ and (S(D'_y) + S'_y) = (S(D'_y) + S'_y). By induction hypothesis $\Gamma \vdash e'' : D \mid S'' \subseteq (S(D'_y) + S'_y)$. Therefore $(S(D'_y) + S'_y) \subseteq (S(D'_y) + S'_y)$.
1. In this case

\[ \Gamma[\epsilon_x] \vdash \{ \text{inv} \} T z = \epsilon_1[x_i] ; \; \text{ds} e_b ; \; C' \mid S_1 \]

and from Lemma [16]2 we derive \( \Gamma \vdash \epsilon'[\epsilon''_y] : C \mid S \) where \( \epsilon'[\epsilon''_y] \approx \epsilon'[\epsilon''_y] \).

2. Let \( \epsilon_1 = Dy[e] \), \( \epsilon_2 = D_y[e] \), and \( \epsilon_1 = \epsilon[x.f_i] \rightarrow \epsilon_2 = \epsilon'[x] \). If the redex \( x.f_i \) is not a subexpression of \( e \) then \( e = e' \), and since \( \Gamma_D x = \Gamma_D' y \), the result is obvious. If \( x.f_i \) is a subexpression of \( e \), then from Lemma 16.2 for some \( \epsilon'' \) we have that \( \epsilon = \epsilon'[x.f_i] \) and \( \epsilon' = \epsilon''[x] \). From \( \Gamma_D x \vdash \epsilon''[x.f_i] : D \mid S_\epsilon x \) and Lemma [16]1 we have that \( \Gamma[\Gamma_D y][\epsilon''_x] \vdash x, f_i : D_1 \{ x, \epsilon \text{res} \} \) and \( \Gamma[\Gamma_D y][\epsilon'][x] : D_1 \{ x_i, \epsilon \text{res} \} \). If \( dv_x \in \text{store}(\epsilon''_y) \) then \( S_\epsilon = S'_x \), otherwise \( dv_x \in \text{store}(D_y) \). In both cases \( S(D_y) + S_\epsilon = S(D_y) + S'_x \).

Rule (field-assign).

1. In this case

- \( \mathcal{E} = \epsilon_x[\epsilon_1] \) for some \( \epsilon_1 \),
- \( \mathcal{E}' = \epsilon_x.x = y[\epsilon_1] \) since \( x \not\in \text{HB}(\epsilon_1) \)
- \( \text{dec}(\mathcal{E}, x) = dv_x = D x = \text{new} D(x_1, \ldots, x_n) \);
- \( \rho = x.f_i = y \) and \( \epsilon' = y \) with \( y \not\in \text{HB}(\epsilon_1) \).

As for the case of field update \( \epsilon_x \) has either shape (1) or (2) with \( y \not\in \text{HB}(\epsilon_1) \). We consider only case (1).

From Lemma [16]1 we have that \( \Gamma[\Gamma_x][\epsilon_x] \vdash \{ \text{ds} e_b \} : C' \mid S_1 \) for some \( C' \) and \( S_1 \). From typing rule \( \text{(T-block)} \) we have that \( \Gamma[\Gamma_x][\epsilon_x] \vdash \epsilon_1[x.f_i = y] : D' \mid S_2 \) for some \( D' \) and \( S_2 \). From Lemma [16]1, typing rule \( \text{(T-field-assign)} \) and \( \text{(T-var)} \) we have that \( \Gamma[\Gamma_x][\epsilon_x] \vdash x, f_i : D_1 \{ x, \epsilon \text{res} \}, \Gamma[\Gamma_x][\epsilon'][x] : y : D_1 \{ y, \epsilon \text{res} \}, \Gamma[\Gamma_x][\epsilon'] \vdash x, f_i = y : D_1 \{ x, y, \epsilon \text{res} \} \) where \( \text{fields}(D) = D_1 f_1 \ldots D_n f_n \) and \( \{ y, \epsilon \text{res} \} \subseteq \{ x, y, \epsilon \text{res} \} \).

From \( \Gamma[\Gamma_x] \vdash \{ \text{ds} e_b \} : C' \mid S_1 \) and Lemma [14] we have that

(a) \( \Gamma[\Gamma_x][\epsilon_x] \vdash T z = \epsilon_1[x.f_i = y] ; : S_3 \),
(b) \( \Gamma[\epsilon_x][\epsilon_x] \vdash dv_x \mid S_x \mid \epsilon = \{ x, x_1, \ldots, x_n \} \),
(c) \( \Gamma[\epsilon_x][\epsilon_x] \vdash \text{ds} e_b \mid S_4 \mid \epsilon \),
(d) \( \epsilon : C' \mid S_5 \).

Let \( dv_{x'} = D x = \text{new} D(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \); and \( dv'' = dv'_{x} T z = \epsilon_1[y] ; \; \text{ds} e_b \).

We have that \( \Gamma_{d_{x'}} = \Gamma_{d_{x''}} \).

From typing rules \( \text{(T-field-assign)} \) and \( \text{(T-var)} \), and Lemma [19] we get \( S_3 + \{ x, y \} = S'_3 + \{ x, y \} \). Moreover, from (a), typing rule \( \text{(T-field-assign)} \), and Lemma [14] we have that \( x \) and \( y \) are in the same equivalence class in \( S_3 \), i.e., \([x]_{S_3} = [y]_{S_3} \). Therefore, \( S'_3 = S' \mid S_3 + S' \mid S_3 \). From (A) \( (D) \), typing rule \( \text{(T-block)} \) and Lemma [16]2 we get

\[ \Gamma[\epsilon_x] \vdash \{ \text{inv} \} T z = \epsilon_1[y] ; \; \text{ds} e_b ; : C' \mid S'_1 \]
Lent and capsule properties. We can now formally express the lent and capsule properties.

Let \( S' \subseteq S \). From Lemma 16 we derive that \( \Gamma \vdash \mathcal{E}_2(\{dv \mapsto x; z; \mathcal{E}_1[y]; \text{ds} e_k\} : C \mid S') \) where \( S' \subseteq S \). Consider \( \mathcal{E}_2^{x,y}(\mathcal{E}_1) = \mathcal{E}_2(\{dv \mapsto x; z; \mathcal{E}_1[y]; \text{ds} e_k\}) \), we have that \( \Gamma \vdash \mathcal{E}_2(x,y)(\mathcal{E}_1[y]) : C \mid S' \), which proves the result.

2. Let \( e_1 = D_y[e] \), \( e_2 = D_y'[e'] \), and \( e_1 = \mathcal{E}[x.f=y] \rightarrow e_2 = \mathcal{E}'[y] \). If the redex is not a subexpression of \( e \) then \( e = e' \), and since \( \Gamma_{D_y} = \Gamma_{D_y}' \), the result is obvious. If \( x.f=y \) is a subexpression of \( e \), then from Lemma 25 for some \( \mathcal{E}'' \) we have that \( e = \mathcal{E}''[x.f=y] \) and \( e' = \mathcal{E}'''[y] \). From \( \Gamma_{D_y} \vdash \mathcal{E}''[x.f=y] : D \mid S_x \), \( \Gamma_{D_y} \vdash \mathcal{E}'''[y] : D \mid S_x' \), and Lemma 16 \( \Gamma \vdash \mathcal{E}_2(x,y)(\mathcal{E}_1[y]) : C \mid S' \), which proves the result.

Rule (alias-elim). Clause 1. is proved using Lemma 17.1. Clause 2. is proved as in the case of (field-assign).

Rule (affine-elim). Clause 1. is proved using Lemma 17 and 12. Clause 2. is proved as in the case of (field-assign).

Lent and capsule properties. We can now formally express the lent and capsule notions, informally described in the introduction.

Informally, a reference \( x \) is (used as) lent if no sharing can be introduced through \( x \). Formally, if \( \Gamma \vdash e : \_ \mid S \), then \( x \) is lent in \( e \) if \( [x]_S = \{x\} \). The notion can be generalized to a set of references \( xs \), that is, \( xs \) is lent in \( e \) if, for each \( x \in xs \), \( [x]_S \subseteq xs \).

Consider now an expression in a declaration context \( D_x[e] \), and a reference \( y \). The portion of store connected to \( y \) before the evaluation of \( e \) is \( [y]_{S(D_x)} \). Now, if the expression \( e \) can access such portion of store only through lent references, then the two following properties are ensured by the evaluation of \( e \):

1. such portion of store remains isolated from others
2. such portion of store cannot be part of the final result of \( e \).

This is formally expressed by the following theorem.

**Theorem 22.** (Lent) If \( \vdash D[e] \), \( \Gamma_D \vdash e : \_ \mid S \), and \( ys = [y]_{S(D)} \) is lent in \( e \), then:

1. if \( D[e] \rightarrow D'[e'] \) then \( \Gamma_{D'} \vdash e'' : \_ \mid S' \) where \( e'' \approx e' \) and \( [y]_{(S(D)+S')} \subseteq ys \)
2. if \( D[e] \rightarrow* D'[v] \), then \( y \notin FV(gc(v)) \).

**Proof.**

1. Since \( ys \) is lent in \( e \), \( [y]_{S} \subseteq ys \), hence \( [y]_{(S(D)+S)} = [y]_{S(D)} \).

From Theorem 21, since \( \Gamma_D \vdash e : \_ \mid S \), we have that \( \Gamma_{D'} \vdash e'' : \_ \mid S' \) where \( e'' \approx e' \) and \( (S(D') + S') \subseteq (S(D) + S) \), hence \( [y]_{(S(D') + S')} \subseteq [y]_{S(D)} = ys \).

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2. By induction on the number $n$ of steps of the reduction $D[e] \rightarrow^* D'[v]$. For $n = 0$, we have $\Gamma_D \vdash e : \_ | S$. From Theorem 11.1, it should be either $(y, \text{res}) \in S$ or $y$ affine in $\Gamma_D$. However, $(y, \text{res}) \in S$ contradicts the hypothesis that $y$s are lent in $e$, and it is easy to see from the definition that in $D$ there are no affine variable declarations. Let $D[e] \rightarrow D'[e']$ and $D'[e''] \rightarrow^* D''[v]$, where $e'' \approx e'$, in $n$ steps. By point 1. of this theorem we have $\Gamma_{D'} \vdash e' : \_ | S'$ and $[y]_{S'(D')} \subseteq y$, hence $[y]_{S''} \subseteq y$s, that is, $y$s is lent in $e''$ (note that to be lent does not depend on the annotations), and by inductive hypothesis on $\Gamma_{D'} \vdash e'' : \_ | S'$ we derive the thesis.

Informally, a capsule is a reachable object graph which is an isolated portion of store, that is, it does not contain nodes reachable from the outside. In our calculus, a reachable object subgraph is a value $v$, nodes reachable from the outside are free variables, hence the condition to be a capsule can be formally expressed by requiring that $v$ has no free variables. The following theorem states that the right-hand side of a capsule declaration actually reduces to a closed portion of store.

**Theorem 23.** (Capsule) If $\vdash D_{ax}[e]$, $\Gamma_{D_s} \vdash e : \_ | S$ with $\text{capsule}(S)$, and $D_{ax}[e] \rightarrow^* D'_{ax}[v]$, then $\text{FV}(\text{gc}(v)) = \emptyset$.

**Proof.** By induction on the number $n$ of steps of the reduction $D_{ax}[e] \rightarrow^* D'_{ax}[v]$. For $n = 0$, we have $\Gamma_D \vdash e : \_ | S$. From Theorem 11.2, $\text{capsule}(S)$ implies $y$ affine in $\Gamma_D$ for each $y \in \text{FV}(v)$. However, there are no affine variable declarations in $D_{ax}$.

Let $D_{ax}[e] \rightarrow D'_{ax}[e']$ and $D'_{ax}[e''] \rightarrow^* D''_{ax}[v]$, where $e'' \approx e'$, in $n$ steps. From Theorem 21.2, since $\Gamma_D \vdash e : \_ | S$, and $[\text{res}]_S = \emptyset$, we have $\Gamma_{D'} \vdash e'' : \_ | S'$, and $[\text{res}]_{S'} = \emptyset$. By induction hypothesis on $\Gamma_{D'} \vdash e'' : \_ | S'$ we derive the result.

Progress. Closed expressions are not “stuck”, that is, they either are values or can be reduced.

To prove the theorem we introduce the set of redexes, and we show that expressions can be decomposed in an evaluation context filled with a redex. Therefore an expression either is a value or it matches the left-hand side of exactly one reduction rule.

**Definition 24.** Redexes, $\rho$, are defined by:

$$\rho ::= x.f | v.m(\text{vs}) | x.f \equiv y | \{x \text{ dvs } C x \equiv y; \text{ ds } e\} | \{x \text{ dvs } C^x x \equiv v; \text{ ds } e\}$$

**Lemma 25.** (Decomposition) If $e$ is not a value, then there are $E$ and $\rho$ such that $e \cong E[\rho]$.

**Proof.** The proof is in Appendix B

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We write $e \rightarrow$ for $e \rightarrow e'$ for some $e'$, $\vdash e : C$ for $\emptyset \vdash e : C \mid e$, and $\vdash e : C$ for some $C$ (note that an expression with no free variables has the identity as sharing effects).

**Theorem 26.** (Progress) If $\vdash e$, and $e$ is not a value, then $e \rightarrow$.

**Proof.** By Lemma \textcircled{26} if $e$ is not a value, then $e \cong E[\rho]$. By rule (congr), it is enough to show the thesis for $E[\rho]$. For all $\rho$, except field access and field update, we have that the corresponding reduction rule is applicable, since either there are no side conditions (cases (alias-elim) and (affine-elim)), or the side conditions can be easily shown to hold (case (nvnk)).

In the proof for field access and field update, we use the following auxiliary notation. Given an evaluation context $E$, the context $B_E$, the outermost block of the evaluation context, is defined by

- $B_E = \{ \text{dvs } T y=[ ]; \text{ ds } e \}$ if $E = \{ \text{dvs } T y=\varepsilon' ; \text{ ds } e \}$ and
- $B_E = \{ \text{dvs } [ ] \}$ if $E = \{ \text{dvs } \varepsilon' \}$.

If $\rho$ is $x.f_i$, from Lemma \textcircled{101}, rule (t-field-access), and rule (t-var) of Fig \textcircled{3} we have that $\Gamma_E \vdash x.f_i : C_i \mid \{ x, \text{res} \}$ where $\text{fields}(C) = C_1 f_1 \ldots C_n f_n$. So, we have that $\text{dec}(\varepsilon, x) = C x=\text{new } C(x_1, \ldots, x_n); \ E_x$ is defined, and, for some $\varepsilon'$, $\varepsilon = \varepsilon_x[\varepsilon']$. If $x_i \notin \text{HB}(\varepsilon')$, then rule (field-access) is applicable. Otherwise, since $x_i \in \text{HB}(\varepsilon')$, $\varepsilon_x'$ is defined, and $\varepsilon'[x.f_i] = \varepsilon_1[B_{E_{x_i}}[\varepsilon_2[x.f_i]]]$ for some $\varepsilon_1$ and $\varepsilon_2$ such that $B_{E_{x_i}}[\varepsilon_2[x.f_i]] = \{ X \text{ dvs } C_1 x_1=v; \text{ ds } e \}$. Using rule (alpha) of Fig \textcircled{3} we have that

$$\{ X \text{ dvs } C_1 x_1=v; \text{ ds } e \} \cong \{ X[y/x_1] \text{ dvs } y/x_1 \ T y=v[y/x_1]; \text{ ds}'[y/x_1] \ e[y/x_1] \}$$

where $y$ can be chosen such that $y \notin \text{HB}(\varepsilon_x)$. Therefore $\varepsilon_x[\varepsilon'[x.f_i]] \cong \varepsilon_x[\varepsilon_2[x.f_i]]$, where $x_i \notin \text{HB}(\varepsilon_3)$. So $\varepsilon_x[\varepsilon_3[x.f_i]] \rightarrow e_2$ by applying rule (field-access).

If $\rho$ is $x.f_i=y$, from Lemma \textcircled{101}, rule (t-field-assign), and rule (t-var) of Fig \textcircled{3} we have that $\Gamma_E \vdash x.f_i=y : C_i \mid \{ x, y, \text{res} \}$ where $\text{fields}(C) = C_1 f_1 \ldots C_n f_n$. So, we have that $\varepsilon_x$ is defined, and $\varepsilon = \varepsilon_x[\varepsilon']$ for some $\varepsilon'$. Therefore, for some $\varepsilon_1'$, $\varepsilon = \varepsilon_1'[B_{E_1} \varepsilon']$. If $y \notin \text{HB}(\varepsilon')$, then rule (field-assign) is applicable. Otherwise, since $y \in \text{HB}(\varepsilon')$, $\varepsilon_1'$ is defined, and $\varepsilon'[x.f_i=y] = \varepsilon_1[B_{E_2} \varepsilon_2[x.f_i=y]]$ for some $\varepsilon_1$ and $\varepsilon_2$ such that $B_{E_2}[\varepsilon_2[x.f_i=y]] = \{ \text{dvs } C_1 y=v; \text{ ds } e \}$ are all the declarations connected to the free variables of $v$, hence to be extruded together with the declaration of $y$.

By induction on the number $n > 0$ of blocks from which we have to extrude the declaration of $y$. Let $\text{dvs}_1 = \text{dvs } C_i y=v$. If $n > 1$, then for some $\varepsilon'' \neq [ ]$, either

(a) $B_{E_2} \varepsilon''[\rho] = \{ \text{dvs' } C x=v'; \varepsilon''[\text{dvs}_1 \text{ ds } e] \}$ or
(b) $B_{E_2} \varepsilon''[\rho] = \{ \text{dvs' } C x=v'; \ T z=\varepsilon''[\{ X \text{ dvs } C \text{ dvs } e \}]; \text{ ds'} e' \}$.

For (a), by induction hypothesis we have that

$$\{ X \text{ dvs } C x=v'; \varepsilon''[\text{dvs}_1 \text{ ds } e] \} \cong \{ \text{dvs' } C x=v'; \{ \text{dvs}_1 \text{ ds'} e' \} \}$$

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for some \( ds' \), \( e' \), and \( ds'_1 \) containing \( C; y = v; \), which are the declarations that have been extruded. Let \( ds' = ds_2 ds'_2 \) where \( ds_2 \) are not evaluated declarations and let \( ds_2 = ds'_2 ds'_2 \) where \( ds'_2 = \text{dom}(ds'_2) = \emptyset \). Since we cannot have forward reference to unevaluated variables \( \text{FV}(ds'_2, ds'_2) \cap \text{dom}(ds'_2) = \emptyset \). Therefore we can apply rule (bool) of Fig\[8\] Applying rule (boolv) of Fig\[8\] we have that

\[
\{ ds' \ C x = v'; \ { ds_1 \ ds' \} \} \cong \{ ds' \ C x = v'; \ { ds'_1 \ ds'_2 \} \{ ds' \ e' \} \}.
\]

For (b), by induction hypothesis we have that

\[
\{ ds' \ C x = v'; \ { ds_1 \ ds' \} \} \cong \{ ds' \ C x = v'; \ { ds'_1 \ ds' \} \; ds'' \ e'' \}
\]

for some \( ds' \), \( ds'' \), \( e' \), \( e'' \), \( Y \) and \( ds'_1 \) containing \( C; y = v; \), which are the declarations that have been extruded. Let \( ds' = ds_2 ds'_2 \) where \( ds_2 \) are not evaluated declarations and let \( ds_2 = ds'_2 ds'_2 \) where \( ds'_2 = \text{dom}(ds'_2) = \emptyset \). Since we cannot have forward reference to unevaluated variables \( \text{FV}(ds'_2, ds'_2) \cap \text{dom}(ds'_2) = \emptyset \). From the fact that the term is well typed, we have that \( Y = [\text{res}]_S \cap (\text{dom}(ds'_2) \cup \text{dom}(ds'_2)) \) for some \( S \) such that \( (\text{dom}(ds'_2) \cup \text{dom}(ds'_2)) \subseteq [y]_S \) and from Lemma [13] and \( T x = f = y : C; \{ x, y, \text{res} \} \) also \( x \in [y]_S \). Moreover, let \( Z = \text{dom}(ds'_2) \cup \text{dom}(ds'), S \backslash Z \) is the sharing relation associated to the inner block.

In order to apply congruence rule (acc) we have to prove that if \( T = D^\mu \) and \( \mu = a \), then \( (\text{dom}(ds'_2) \cup \text{dom}(ds'_2)) \cap Y = \emptyset \).

If \( y \notin [\text{res}]_S \), then \([\text{res}]_S \cap (\text{dom}(ds'_2) \cup \text{dom}(ds'_2)) = \emptyset \).

If \( y \in [\text{res}]_S \), then, since \( x \in [y]_S \) we have that \( x \in [\text{res}]_S \). Therefore \([\text{res}]_S \backslash Z \neq \emptyset \) and \( \mu \) cannot be \( a \).

Applying rule (Dec) of Fig\[8\] we get

\[
\{ ds' \ T x = v'; \ { ds'_1 ds' \} \} \cong \{ ds' \ T x = v'; \ { ds'_1 ds'_2 \} \; ds'' \ e'' \}
\]

Therefore \( E_3[E'_3[x.f = y]] \cong E'_3[E_3[x.f = y]] \) for some \( E_3 \) such that \( y \notin \text{HB}(E_3) \). So \( E'_3[B_E.E_3[x.f = y]] \to e_2 \) applying rule (Field-Assign).

7. Related work

Capsule and lent notions. As mentioned, the capsule property has many variants in the literature, such as isolated [8], uniqueness [19] and external uniqueness [1], balloon [2, 3], island [5]. The fact that aliasing can be controlled by using lent (borrowed) references is well-known [9, 10]. However, before the work in [6], the capsule property was only detected in simple situations, such as using a primitive deep clone operator, or composing subexpressions with the same property.

The important novelty of the type system in [8] has been recovery, that is, the ability to detect properties (e.g., capsule or immutability) by keeping into account not only the expression itself but the way the surrounding context
is used. In [6] an expression which does not use external mutable references is recognized to be a capsule. However, expressions which do use external mutable references, but do not introduce sharing between them and the final result, are not recognized to be capsules. For instance, Example 3 and Example 7 in Sect. 4 would be ill-typed in [6]. Our type system generalizes recovery by precisely computing sharing effects.

Capabilities. In other proposals [20, 21, 22, 23, 24] types are compositions of one or more capabilities, and expose the union of their operations. The modes of the capabilities in a type control how resources of that type can be aliased. By using capabilities it is possible to obtain an expressivity which looks similar to our type system, even though with different sharing notions and syntactic constructs. For instance, the full encapsulation notion in [20] is apart from the fact that sharing of immutable objects is not allowed, equivalent to the guarantee of our a modifier. Their model has a higher syntactic/logic overhead to explicitly track regions. As for all work before [6], objects need to be born @unique (that is, with the capsule property) and the type system permits to manipulate data preserving their uniqueness. With recovery [6] (as with our type and effect system), instead, we can forget about uniqueness, use normal code designed to work on conventional shared data, and then recover the aliasing encapsulation property.

Ownership. A closed stream of research is that on ownership (see an overview in [25]) which, however, offers an opposite approach. The ownership invariant, which can be expressed and proved, is, however, expected to be guaranteed by defensive cloning. In our approach, instead, the capsule concept models an efficient ownership transfer. In other words, when an object $x$ is “owned” by another object $y$, it remains always true that $y$ can only be accessed through $x$, whereas the capsule notion is more dynamic: a capsule can be “opened”, that is, assigned to a standard reference and modified, and then we can recover the original capsule guarantee. Cloning, if needed, becomes a responsibility of the client. Other work in the literature supports ownership transfer, see for example [26, 1]. In the literature it is however applied to uniqueness/external uniqueness, thus not the whole reachable object graph is transferred.

Full, deep and shallow interpretation. The literature on sharing control distinguishes three interpretations for properties of objects.

- Full: the whole reachable object graph respects that property.
- Shallow: only the object itself respects that property.
- Deep: the reachable object graph is divided in 2 parts: The first part is the one that is logically “owned” by the object itself, while the second part is just “referenced”.

In our approach, as in [2, 8, 6, 8, 5], properties have the full interpretation, in the sense that they are propagated to the whole reachable object graph. In a deep

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9 This paper includes a very good survey of work in this area, notably explaining the difference between external uniqueness [1] and full encapsulation.

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interpretation, instead, as in \[13, 14, 27\], it is possible, for instance, to reach a mutable object from an immutable object. In this sense, approaches based on ownership, or where it is somehow possible to have any form of “internal mutation” are (only) deep, as in \[23, 24, 28, 29\]. This also includes \[1\], where an unique object can point to arbitrarily shared objects, if they do not, in turn, point back to the unique object itself.

The advantage of the full interpretation is that libraries can declare strong intentions in a coherent and uniform way, independently of the concrete representation of the user input (that, with the use of interfaces, could be unknown to the library). On the other side, providing (only) full modifiers means that we do not offer any language support for (as an example) an immutable list of mutable objects.

**Destructive reads.** Uniqueness can be enforced by destructive reads, i.e., assigning a copy of the unique reference to a variable and destroying the original reference, see \[6, 19\]. Traditionally, borrowing/fractional permissions \[10\] are related to uniqueness in the opposite way: a unique reference can be borrowed, it is possible to track when all borrowed aliases are buried \[8\], and then uniqueness can be recovered. These techniques offer a sophisticate alternative to destructive reads. We also wish to avoid destructive reads. In our work, we ensure uniqueness by linearity, that is, by allowing at most one use of an a reference.

**Alias analysis.** Alias analysis is a fundamental static analysis, used in compilers and code analysers. Algorithms such as Steensgaard’s algorithm, \[30\], infer equivalence classes that may alias. In \[31\] is presented a refined version of such algorithm, performing a uniqueness analysis, similar to our detection of “capsule” values. However, the aim of our work is to design a language in which annotations, such as the affine modifier, can be used by the user to enforce properties of its code. Then the inference system checks that such annotations are correctly used.

**Calculus.** Finally, an important distinguishing feature of our work is that sharing can be directly represented at the syntactic level as a relation among free variables, thanks to the fact that the calculus is pure. Models of the imperative paradigm as pure calculi have been firstly proposed in \[32, 33\].

8. Conclusion

We have presented a type and effect system which infers sharing possibly introduced by the evaluation of an expression. Sharing is directly represented at the syntactic level as a relation among free variables. This representation allows us to express in a natural way, and to generalize, widely-used notions in literature, providing great expressivity, as shown by the examples of Sect. 4.

We adopt a non standard operational model, where store is encoded directly in the language terms. In this model, sharing properties are directly expressed at the syntactic level. Notably, in a subterm \(e\) of a program, objects reachable from other parts of the program are simply those denoted by free variables in \(e\), whereas local objects are those denoted by local variables declared in \(e\). In our
opinion, this offers a very intuitive and simple understanding of the portion of memory only reachable from \( e \), since it is encoded in \( e \) itself. Another advantage is that, the store being encoded in terms, most proofs can be done by structural induction on terms, as we have exploited in the Coq implementation mentioned below.

On the other hand, a disadvantage of our model may be that it is possibly more esoteric for people used to the other one. Moreover, since isolation is encoded by scoping, some care is needed to avoid scope extrusion during reduction. For this reason, reduction is defined on typechecked terms, where blocks have been annotated with the information of which local variables will be connected to the result. In this way, it is possible to define a rather sophisticated notion of congruence among terms, which allows to move declarations out of a block only when this preserves well-typedness.

Besides standard type soundness, we proved that uniqueness and lent properties inferred by the type system actually ensure the expected behaviour.

To focus on novelties and allow complete formal proofs, we illustrate our type and effect system on a minimal language, only supporting the features which are strictly necessary for significance. However, we expect that the approach could be smoothly extended to typical constructs of imperative/object-oriented languages, in the same way as other type systems or type and effect systems (e.g., when effects are possibly thrown exceptions [34]). We briefly discuss two key features below.

**Inheritance** For a method redefined in a heir class, the returned sharing relation should be *contained* in that of the parent, that is, less connections should be possibly caused by the method, analogously to the requirement that possibly thrown exceptions should be a subset (modulo subtyping) of those of the parent.

**Control flow** As it is customary in type systems, control flow constructs are handled by taking the “best common approximation” of the types obtained by the different paths. For instance, in the case of if-then-else we would get the union of the sharing relations of the two branches (the same happens for possibly thrown exceptions).

Note that including control flow constructs we would get, again as it is customary in type systems, which are never complete for exactly this reason, examples which are ill-typed but well-behaved at runtime. For instance, an expression of shape:

```plaintext
D y = new D(y);
C z = {
   D x = new D(x);
   if (...) x.f = x else x.f = y;
   x
};
z
```
will be ill-typed, since the sharing relation computed for the right-hand side of
the declaration of $z$ is \{y, res\}, that is, the type and effect system conservatively
assumes that this expression does not denote a capsule. However, if the guard
of the conditional evaluates to true, then the right-hand side of the declaration
reduces to a capsule.

We have implemented in Coq the type and effect system and the reduction
rules. The current code can be found at //github.com/paola-giannini/sharing. We
did not include the construct of method call since, as we can see from the
typing and reduction rules, it can be translated into a block. To complete the
implementation of the operational semantics, we need an oriented version of the
congruence relation on terms to be applied before reduction steps, as explained
at the end of Sect 5. Then, we plan to mechanize the full proof of soundness.
As mentioned before, we argue that the Coq implementation nicely illustrates
the advantages of our purely syntactic model, since proofs can be carried out
inductively, without requiring more complicated techniques such as, e.g., bisim-
ulation.

In further work, in addition to completing the formalization in Coq of the
proof of soundness, we will enrich the type system to also handle immutable
references. A preliminary presentation of this extension is in [35].

We also plan to formally state and prove behavioural equivalence of the
calculus with the conventional imperative model. Intuitively, the reachability
information which, in our calculus, is directly encoded in terms, should be re-
constructable from the dependencies among references in a conventional model
with a flat global memory.

As a long term goal, it would be interesting to investigate (a form of) Hoare
logic on top of our model. We believe that the hierarchical structure of our
memory representation should help local reasoning, allowing specifications and
proofs to mention only the relevant portion, similarly to what is achieved by
separation logic [27].

Acknowledgement. We are indebted to the anonymous SCP referees for the
thorough job they did reviewing our paper and for their valuable suggestions.
We also thank the OOPS’17 and FTfJP’17 referees for their helpful comments
on preliminary versions. Finally, we thank Isaac Oscar Gariano for his careful
reading.

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Appendix A. Type derivation for Example 7

In Fig. A.8 we give the type derivation that shows that the expression $e$ of Example 7 is well-typed, where $\Gamma_1 = c1:C, outer:C$, $\Gamma_2 = c2:C, inner:C^a$, and $\Gamma_3 = c3:C, r:C$ are the type contexts corresponding to the top level, outer, and inner block, respectively.

Derivations $D_1$ and $D_2$ end with an application of rule (T-Invk). Consider $D_2$. The method mix produces sharing between its receiver, parameter, and result. Then the call of mix with receiver $r$ and parameter $c3$ returns a result connected with both these variables. The call of mix with receiver $c2$ and parameter $c1$ in the derivation $D_1$ does the same with $c2$ and $c1$. Instead, the call of clone does not produce any connection from its receiver and result. So the call of clone with receiver $c2.mix(c1)$ in the derivation $D_1$ does not cause connections for $res$.

The type derivation $D_3$ justifies the judgment $\Gamma_1[\Gamma_2] \vdash e^i : C | \{c1, c2\}$ where

$D_1$ yields $e^i = \{(r, c3) C3 = new C(c3); Cr = c2.mix(c1).clone(); r.mix(c3)\}$

$D_4$ yields $e^a = \{(c2) C2 = new C(c2); C^a inner = e_i; inner.mix(c2)\}$

$D$ yields $e' = \{(outer) C1 = new C(c1); Couter = e^i; outer\}$
e^i is the inner block (the initialization expression of inner). The effects of the evaluation of initialization expressions and body are to mix the external variables c1 and c2, and the local variables r and c3. Moreover, the result is only connected with the two local variables. Hence, the inner block denotes a capsule, so it can be used as initialization expression of the affine variable inner. The sharing relation resulting from the evaluation of the declarations and the body (before removing the local variables r and c3) is represented by \{c1, c2\} \{c3, r, res\}. The annotation for the block, i.e., the set of local variables connected to the result, is \{r, c3\}, so, when applying the congruence relation, the variables r and c3 cannot be moved outside this block.

The type derivation \(D_4\) justifies the judgment \(\Gamma_1 \vdash e^o : C|\{c1, res\}\), where \(e^o\) is the outer block (the initialization expression of outer). The effects of the evaluation of initialization expressions and body are to mix the external variable c1 with the local variable c2 (this effect propagates from the inner block). Moreover, the result is connected with variable c2. Hence, the result turns out to be connected with the external variable c1 as well. Therefore, this block is not a capsule, and could not be used to initialize an affine variable. Note that the variable inner, being affine, is not in the domain of the sharing relation. Indeed, it will be substituted with the result of the evaluation of \(e^i\) and so it will disappear.

Finally, \(D\) is the derivation for the expression e. The block is a closed expression, and closed expressions are capsules. The block is annotated with the local variable outer, which is connected with its result.

Appendix B. Proofs

**Proposition 9.** If v is a value, then there exists v' such that v \(\equiv v'\) and v' is in canonical form.

**Proof.** By structural induction on values.

Consider \(\text{new } C(\text{vs})\). By inductive hypotheses on vs we have that \(\text{new } C(\text{vs}) \equiv \text{new } C(\text{vs}')\) for some vs' in canonical form.

By induction on the number n of \(v \in \text{vs}'\) such that \(v\) is not a variable, i.e., is a block value, we show that \(\text{new } C(\text{vs}') \equiv \{x^i \text{ new } C(\text{vs}')\} x\) where x is a fresh variable using congruence rule (new).

If \(n = 0\), then \(\text{new } C(\text{vs}') \equiv \{x^i \text{ new } C(\text{vs}')\} x\) where x is a fresh variable using congruence rule (new).

If \(n > 0\), then \(\text{new } C(\text{vs}') = \text{new } C(\text{xs}, \{x^i \text{ new } C(\text{vs}')\} vs'')\) where \(\text{vs}''\) is in canonical form.

We may assume, by \(\alpha\)-renaming, that \((\text{FV}(\text{vs}') \cup \{x\}) \cap \text{dom}(\text{vs}') = \emptyset\). Using rule (val-ctx) we have that \(\text{new } C(\text{vs}') \equiv \{x^i \text{ new } v'\} v' = \text{new } C(\text{xs}, \text{vs}')\). Since \(v'\) has \(n - 1\) arguments which are not variables by inductive hypothesis we have that \(\text{new } C(\text{xs}, \text{vs}') \equiv \{Y \text{ new } C(\text{xs}, \text{ys})\}\).

Therefore \(\text{new } C(\text{vs}') \equiv \{x^i \text{ new } v'\} \{Y \text{ new } C(\text{xs}, \text{ys})\}\).

We may assume, by \(\alpha\)-renaming, that \((\text{FV}(\text{vs}') \cup \text{dom}(\text{vs}') \cap \text{dom}(\text{vs}) = \emptyset). Using congruence rule (body), we get \(\text{new } C(\text{vs}') \equiv \{x^i \cup Y \text{ new } C(\text{xs}, \text{ys})\}.\)
Applying congruence rule (new) followed by (body) we have that $\text{new } C(\upsilon') \cong \{X \cup \Gamma(x)\} \text{ des } \upsilon' \text{ if } C \text{ is new } C(x, y, \upsilon'); z \}$ with $z$ a fresh variable.

Consider $\{X \text{ des } \upsilon\}$. By induction hypothesis on values $\{X \text{ des } \upsilon\} \cong \{X \text{ des } \upsilon'\}$ with $\upsilon'$ in canonical form.

By induction on the number $n$ of declarations $C x = \text{new } C(\upsilon); \in \text{ des}$ which are not in canonical form we show that $\{X \text{ des } \upsilon'\} \cong \{Y \text{ des } \upsilon'\}$ where all the $\upsilon'$ are in canonical form.

If $n = 0$, then $\text{des}$ is in canonical form. We have two cases $\upsilon' = x$ or $\upsilon' = \{Y \text{ des } y\}$ with $\upsilon'$ in canonical form. In the first case $\{X \text{ des } \upsilon\} \cong \{X \text{ des } x\}$ and we are done. In the second case $\{X \text{ des } \upsilon\} \cong \{X \text{ des } \{Y \text{ des } y\}\}$. We may assume, by $\alpha$-renaming, that $\text{dom(\upsilon')} \cap (\text{FV(\upsilon)} \cup \text{dom(\upsilon)}) = \emptyset$, so congruence rule (body) can be applied to obtain $\{X \text{ des } \upsilon\} \cong \{X \cup Y \text{ des } \upsilon' \{\emptyset \ y\}\}$ and with rule (block-elim) we obtain $\{X \text{ des } \upsilon\} \cong \{X \cup Y \text{ des } \upsilon' \{\emptyset \ y\}\}$.

If $n > 0$, then We may assume that $\text{des} = \text{ds}_1 C z = \text{new } C(\upsilon); \text{des}_2$ with $\text{des}_1$ in canonical form, the values in $\upsilon$ in canonical form, and some $\upsilon \in \upsilon'$ not a variable. As for the proof for constructor values we can prove that $\text{new } C(\upsilon) \cong \{Y \text{ des' new } C(\upsilon)\}$ for some $Y$, $ys$ and $\upsilon'$ in canonical form. (We just omit the application of the congruence rule (new) from the proof.) Therefore $\{X \text{ des } \upsilon\} \cong \{X \text{ des}_1 C z = \{Y \text{ des' new } C(\upsilon)\}; \text{des}_2 \upsilon\}$. We may assume, by $\alpha$-renaming, that $\text{dom(\upsilon')} \cap (\text{FV(\upsilon)} \cup \text{dom(\upsilon)}) = \emptyset$. Therefore, by applying congruence rule (dec) and (block-elim) we get

$$\{X \text{ des}_1 C z = \{Y \text{ des' new } C(\upsilon)\}; \text{des}_2 \upsilon\} \cong \upsilon'$$

where $\upsilon' = \{X \text{ des}_1 \upsilon' \text{ C z = new } C(\upsilon) ; \text{des}_2 \upsilon\}$. Since $\upsilon'$ is in canonical form, the number of declarations which are not in canonical form in $\upsilon'$ is $n - 1$, hence the thesis holds by the inductive hypothesis.

The following lemma shows that scope extrusion preserves types. In particular, annotations on blocks associated to capsule variables prevent extrusion of declarations that may be connected to the result of the block. The lemma is the main result needed to prove that congruence preserves typability.

**Lemma 27.** Let

(i) $d = C^\mu x = \{X \text{ des}_1 \upsilon \text{ e}\}$; and

(ii) $\text{ds} = \text{des}_1 C^\mu x = \{Y \text{ des}_2 \upsilon \text{ e}\}$; and

(iii) $\text{dom(\upsilon)} \cap \text{FV(\upsilon)} = \emptyset$ and

(iv) if $\mu = a$, then $D_1 \cap X = \emptyset$.

$\Gamma[x: C^\mu] \vdash d : S$ if and only if $\Gamma[x: C^\mu, \Gamma_{\text{des}_1}] \vdash \text{ds} : S'$ where $S = S' \setminus D_1$.

**Proof.** Let $D_1 = D_1$ and $D_2 = \text{dom(\upsilon)}$.

We first show the “only if” implication. Let $\Gamma[x: C^\mu] \vdash d : S$ and $\Gamma_1 = \Gamma[x: C^\mu] \Gamma_{\text{des}_1}$. From Lemma [? we have that

1. $S = S_x[x/\text{res}]$ where $S_x = (S_1 + S_2 + S_x) \setminus (D_1 \cup D_2)$
(2) $\Gamma_1 \vdash ds_1 : S_1$ and $\Gamma_1 \vdash ds_2 : S_2$ and $\Gamma_1 \vdash e : C | S_e$

(3) $X = [\text{res}]_{(S_1 + S_2 + S_e)} \cap (D_1 \cup D_2)$

(4) if $\mu = a$, then capsule($S_x$)

By $\alpha$-renaming we may assume that $x \notin (D_1 \cup D_2)$ so let $\Gamma'_1 = \Gamma[x: C^\mu, \Gamma_{ds_1}] | \Gamma_{ds_2}$ we have $\Gamma_1 = \Gamma'_1$. From (2) we derive

(a) $\Gamma'_1 \vdash ds_2 : S_2$ and $\Gamma'_1 \vdash e : C | S_e$

and applying rule (T-BLOCK) to (a)

(b) $\Gamma[x: C^\mu, \Gamma_{ds_1}] \vdash \{x \in D_2 \}\vdash x : C | S_x'$ where

(c) $S_x' = (S_2 + S_e) \setminus D_2$

(d) $Y = [\text{res}]_{(S_2 + S_e)} \cap D_2$

If $\mu = a$, from (iv) we get $D_1 \cap ([\text{res}]_{(S_1 + S_2 + S_e)} \cap (D_1 \cup D_2)) = \emptyset$. Therefore, $[\text{res}]_{S_1} = [\text{res}]_{(S_1 + S_2 + S_e)} \setminus (D_1 \cup D_2) = [\text{res}]_{(S_1 + S_2 + S_e)} \setminus D_2$. From (d) $[\text{res}]_{S_x'} = [\text{res}]_{(S_2 + S_e)} \cap D_2 \subseteq [\text{res}]_{(S_1 + S_2 + S_e)} \setminus D_2$. Therefore from (4), capsule($S_x'$) and we get that capsule($S_x$).

(e) $\Gamma[x: C^\mu, \Gamma_{ds_1}] \vdash C^\mu \ C^\mu \ x = \Upsilon \ d s_2 \ : \ S_x'[x/\text{res}]$

From (iii) and Lemma 13.2 and (2)

(f) $\Gamma[x: C^\mu, \Gamma_{ds_1}] \vdash ds_1 : S_1$

Therefore

$\Gamma[x: C^\mu, \Gamma_{ds_1}] \vdash ds_1 C^\mu \ x = \Upsilon \ d s_2 \ : \ S' = S_1 + S_e'[x/\text{res}]$. Let $S'' = S_2 + S_e$. From (iii) and Proposition 2 we have that $S_1 \setminus D_2 = S_1$ and so by Proposition 13

(g) $(S_1 + S'') \setminus D_2 = S_1 \setminus D_2 + S'' \setminus D_2 = S_1 + S'' \setminus D_2$

Therefore

$S = ((S_1 + S'') \setminus (D_1 \cup D_2))[x/\text{res}]

= (((S_1 + S'') \setminus D_2) \setminus D_1)[x/\text{res}]

= ((S_1 + S'' \setminus D_2)[x/\text{res}]

= ((S_1 + S'') \setminus D_2)[x/\text{res}]

= S_1 + (S'' \setminus D_2)[x/\text{res}]

= S_1 + S'' \setminus D_2$

We now show the “if” implication.

Let $\Gamma[x: C^\mu, \Gamma_{ds_1}] \vdash ds : S'$ and $\Gamma_1 = \Gamma[x: C^\mu, \Gamma_{ds_1}] | \Gamma_{ds_2}$. From Lemma 13 we have that

(1) $S' = S_1 + S_e[x/\text{res}]$ where $S_e = (S_2 + S_e) \setminus D_2$

(2) $\Gamma[x: C^\mu, \Gamma_{ds_1}] \vdash ds_1 : S_1$

(3) $\Gamma_1 \vdash ds_2 : S_2$ and $\Gamma_1 \vdash e : C | S_e$

(4) $Y = [\text{res}]_{(S_2 + S_e)} \cap D_2$

(5) if $\mu = a$, then capsule($S_x$)
By α-renaming we may assume that \( x \not\in D_2 \) so let \( \Gamma_1' = \Gamma[x:C\mu][\Gamma_{\text{dvs}_1}, \Gamma_{\text{dvs}_2}] \) we have \( \Gamma_1 = \Gamma_1' \). From (3) we derive

(a) \( \Gamma_1' \vdash d_{\text{dvs}_2} : S_2 \) and \( \Gamma_1' \vdash e : C \mid S_e \)

and, from (iii) and Lemma 13.2 also

(b) \( \Gamma_1' \vdash d_{\text{dvs}_1} : S_1 \)

Applying rule (T-block) to (a) and (b) we have

(c) \( \Gamma'[x:C\mu] \vdash \{X\ d_{\text{dvs}_1} d_{\text{dvs}_2} e\} : C \mid S'_x \) where

(d) \( S'_x = (S_1 + S_2 + S_e) \setminus (D_1 \cup D_2) \) and

(e) \( X = [\text{res}]_{S_1 + S_2 + S_e} \cap (D_1 \cup D_2) \)

From (iii) and Proposition 2 there are no \( \langle y, y' \rangle \in S_2 \) with \( y \neq y' \) such that either \( y \in D_1 \) or \( y' \in D_1 \). Therefore \( \langle z, \text{res} \rangle \in (S_1 + S_2 + S_e) \setminus (D_1 \cup D_2) \) and \( z \in D_2 \) implies \( \langle z, \text{res} \rangle \in (S_2 + S_e) \setminus D_1 \). If \( \mu = a \), then from (5) we have that \( \text{capsule}(S'_y) \). From Definition 10 we derive

(f) \( \Gamma'[x:C\mu] \vdash C\mu x\{X\ d_{\text{dvs}_1} d_{\text{dvs}_2} e\} : S'_x[x/\text{res}] \)

As for the “only if” proof we can show that \( S = S' \setminus D_1 \).

Lemma 15 (Congruence preserves types) Let \( e_1 \) and \( e_2 \) be annotated expressions. If \( \Gamma \vdash e_1 : C \mid S \) and \( e_1 \equiv e_2 \), then \( \Gamma \vdash e'_2 : C \mid S \) for some \( e'_2 \) such that \( e'_2 \equiv e_2 \).

Proof. By cases on the congruence rule used. We do the case for rules (dec) and (val-ctx) with new that are the most significative and show how scope extrusion preserves typing. The other cases are similar and simpler. In both cases we first show that typability of the left-side of \( \equiv \) implies typability of the right-side with the same type and sharing relation and then the viceversa.

Rule (dec). Let

- \( e_1 = \{Y\ d_{\text{dvs}_1} C\mu x\{X\ d_{\text{dvs}_2} e\} : d's \ 'e' \) and

- \( e_2 = \{Y'\ d_{\text{dvs}_1} C\mu x\{X'\ d_{\text{dvs}_2} e\} : d's \ 'e' \).

where

1. \( \text{FV}(d_{\text{dvs}_1}) \cap \text{dom}(d_{\text{dvs}_2}) = \emptyset \)
2. \( \text{FV}(d_{\text{dvs}_1}) \cap \text{dom}(d_{\text{dvs}_2}) = \emptyset \)
3. if \( \mu = a \) then \( \text{dom}(d_{\text{dvs}_1}) \cap X = \emptyset \).

Let \( D_1 = \text{dom}(d_{\text{dvs}_1}) \).

We show that \( \Gamma \vdash e_1 : C \mid S \) implies \( \Gamma \vdash e_2 : C \mid S \) for some \( X' \) and \( Y' \).

Let \( \Gamma \vdash e_1 : C \mid S \), define \( \Gamma_1 = \Gamma[d_{\text{dvs}_1}, x:C\mu, \Gamma_{d_{\text{dvs}_2}}] \), and \( Z_d = \text{dom}(d_{\text{dvs}}) \cup \text{dom}(d_{\text{dvs}}') \cup \{x\} \). From Lemma 14 we have that

(a) \( S = (S_d + S_{x} + S_e) \setminus Z_d \)

(b) \( Y = [\text{res}]_{(S_d + S_{x} + S_e)} \cap Z_d \)

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From (d), (1), (3) and Lemma 27, letting $\Gamma_1' = \Gamma[\Gamma_{\text{res}}, x: C^\mu, \Gamma_{\text{ds}}, \Gamma_{\text{ds}'}]$, we have that

$$\Gamma_1' \vdash \text{ds}_1 C^\mu x\{x^d_{\text{ds}1} \text{ ds}_2\} : S_x'$$

where $S_x = S_x' \setminus D_1$

From (2), (c) and Lemma 13.1 we have that

$$\Gamma_1' \vdash \text{ds}_1 C^\mu x\{x^d_{\text{ds}1} \text{ ds}_2\} : S_x'$$

From (e), (f) and rule (T-BLOCK)

$$\Gamma \vdash \{\text{e}' \text{ ds}_1 C^\mu x\{x^d_{\text{ds}1} \text{ ds}_2\} : d's' e'\} : C \mid S'$$

where $S' = (S_d + S_e + S_c) \setminus (Z_d \cup D_1)$ and $Y' = [\text{res}]_{S_d + S_e + S_c} \cap (Z_d \cup D_1)$.

From (2), Proposition 2 and Proposition 14 we have that

$$\Gamma_1' \vdash \text{ds}_1 C^\mu x\{x^d_{\text{ds}1} \text{ ds}_2\} : S_x'$$

Therefore

$$\begin{align*}
S' &= \left((S_x' + S_d + S_c) \setminus D_1\right) \setminus Z_d \quad \text{by definition of } \setminus \\
&= (S_x + S_d + S_c) \setminus Z_d \quad \text{from (g)} \\
&= S
\end{align*}$$

We show that $\Gamma \vdash e_2 : C \mid S$ implies $\Gamma \vdash e_1 : C \mid S$ for some $X$ and $Y$.

Let $\Gamma \vdash e_2 : C \mid S$, define $\Gamma_1 = \Gamma[\Gamma_{\text{res}}, \Gamma_{\text{ds}1}, x: C^\mu, \Gamma_{\text{ds}'}]$, and $Z_d = \text{dom}(\text{ds}') \cup \text{dom}(\text{ds}_1) \cup \{x\}$. From Lemma 13 we have that

(a) $S = (S_d + S_1 + S_2 + S_c) \setminus (Z_d \cup D_1)$
(b) $Y' = [\text{res}]_{S_d + S_1 + S_2 + S_c} \cap (Z_d \cup D_1)$
(c) $\Gamma_1 \vdash e' : C \mid S_e$ and $\Gamma_1 \vdash \text{ds}_1 : S_d$
(d) $\Gamma_1 \vdash \text{ds}_1 : S_1$
(e) $\Gamma_1 \vdash C^\mu x\{x^d_{\text{ds}1} \text{ ds}_2\} : S_x''$

From (d), (e) and Definition 19, letting $S_x'' = S_1 + S_2''$, we have that $\Gamma_1 \vdash \text{ds}_1 C^\mu x\{x^d_{\text{ds}1} \text{ ds}_2\} : S_x''$. Therefore from Lemma 27 letting $\Gamma_1' = \Gamma[\Gamma_{\text{res}}, x: C^\mu, \Gamma_{\text{ds}'}]$, we have that

(f) $\Gamma_1' \vdash C^\mu x\{x^d_{\text{ds}1} \text{ ds}_2\} : S_x$ where $S_x = S_x'' \setminus D_1$ and $X' \setminus D_1 = X'$.

From (2), (e) and Lemma 13.2 we have that

$$\Gamma \vdash \text{e}' : C \mid S_e$$

and $\Gamma_1' \vdash \text{ds}_1 : S_d$. From rule (T-BLOCK) we have that

$$\Gamma \vdash \{\text{e}' \text{ ds}_1 C^\mu x\{x^d_{\text{ds}1} \text{ ds}_2\} : d's' e'\} : C \mid S'$$

where, $Y = [\text{res}]_{S_1 + S_2 + S_c} \cap Z_d$ and $S' = S_x + S_e + S_d \setminus Z_d$. The prof that $S' = S$ is as for the previous implication.

\textbf{Rule (VAL-CTX) with new.} Let

- $e_1 = \text{new } C(v_1, \ldots, v_n, \{x^d_{\text{ds}1} \text{ ds}_2\}, v_{n+1}, \ldots, v_{n+m})$ and
\[ e_2 = \{ Y \mathit{dus}_1 \mathbf{new} C (v_1, \ldots, v_n, \{ X' \mathit{dus}_2 \mathit{v} \}, v_{n+1}, \ldots, v_{n+m}) \}. \]

where

1. \( \text{FV}(\mathit{dus}_1) \cap \text{dom}(\mathit{dus}_2) = \emptyset \)
2. \( \text{FV}(\mathit{vs} \mathit{vs'}) \cap \text{dom}(\mathit{dus}_1) = \emptyset \)

Let \( D_1 = \text{dom}(\mathit{dus}_1) \) and \( D_2 = \text{dom}(\mathit{dus}_2) \).

We show that \( \Gamma \vdash e_1 : C \mid S \) implies \( \Gamma \vdash e_2 : C \mid S \) for some \( X' \) and \( Y \).

Let \( \Gamma \vdash e_1 : C \mid S \), define \( \Gamma_1 = \Gamma [ \mathit{dus}_1, \Gamma_1 \mathit{dus}_2 ] \). From rule (T-New) Lemma 14 we have that

\[ (a) \quad S = S_v + (S'_1 + S'_2 + S') \setminus (D_1 \cup D_2) \text{ where } S_v = \sum_{i=1}^{n+m} S_i \]

\[ (b) \quad \Gamma \vdash v_i : T_i \mid S_i \text{ for all } i, 1 \leq i \leq n + m \]

\[ (c) \quad \Gamma' \vdash \mathit{dus}_1 : S'_1 \]

\[ (d) \quad \Gamma' \vdash \mathit{dus}_2 : S'_2 \text{ and } \Gamma' \vdash v : T' \mid S' \] and

\[ (e) \quad X = [\text{res}]_{(S'_1 + S'_2 + S')} \cap (D_1 \cup D_2) \]

where \( \text{fields}(C) = T_1 f_1 \ldots T_n f_n T' f'_n T_{n+1} f_{n+1} \ldots T_{n+m} f_{n+m} \).

By wellformedness of blocks \( D_1 \cap D_2 = \emptyset \). Therefore \( \Gamma_1 = \Gamma [\mathit{dus}_1] [\Gamma_1 \mathit{dus}_2] \). From Lemma 13.2 and (d) we get \( \Gamma [\mathit{dus}_2] \vdash \mathit{dus}_2 : S'_2 \) and \( \Gamma [\mathit{dus}_2] \vdash v : T' \mid S' \) and by rule (T-Block)

\[ (A) \quad \Gamma [\mathit{dus}_1] \vdash \{ X' \mathit{dus}_2 \mathit{v} \} : T' \mid (S'_2 + S') \setminus D_2 \text{ where } X' = [\text{res}]_{(S'_1 + S')} \cap D_2 \]

From (1), (c) and Lemma 13.2 we get

\[ (B) \quad \Gamma [\mathit{dus}_1] \vdash \mathit{dus}_1 : S'_1 \]

From (b), (2) and Lemma 13.1 we get

\[ (C) \quad \Gamma [\mathit{dus}_1] \vdash v_i : T_i \mid S_i \text{ for all } i, 1 \leq i \leq n + m \]

From (A), (B), (C) and rules (T-New) and (T-Block) we get

\[ \Gamma \vdash e_2 : C \mid ((S'_1 + S_v) + (S'_2 + S')) \setminus D_1 \]

and \( Y = [\text{res}]_{((S'_1 + S_v) + (S'_2 + S'))} \setminus D_1 \). By \( \alpha \)-congruence we may assume that \( \text{FV}(\mathit{vs} \mathit{vs'}) \cap (D_1 \cup D_2) = \emptyset \) so

\[ (D) \quad S_v + (S'_1 + S'_2 + S') \setminus (D_1 \cup D_2) = (S_v + S'_1 + S'_2 + S') \setminus (D_1 \cup D_2) \]

By (1) and \( \text{FV}(\mathit{vs} \mathit{vs'}) \cap D_2 = \emptyset \) and Proposition 2 we have that \((S_v + S'_1) \setminus D_2 = S_v + S'_1\), so from Proposition 14 we have that

\[ (E) \quad (S_v + S'_1 + S'_2 + S') \setminus D_2 = (S_v + S'_1) + ((S'_2 + S') \setminus D_2) \]

Therefore we have that

\[ S = (S_v + S'_1 + S'_2 + S') \setminus (D_1 \cup D_2) \text{ by (D)} \]

\[ = ((S_v + S'_1 + S'_2 + S') \setminus D_2) \setminus D_1 \text{ by definition of } \setminus \]

\[ = ((S_v + S'_1) + (S'_2 + S') \setminus D_2) \setminus D_1 \text{ by (E)} \]

which proves that the typing of \( e_1 \) and \( e_2 \) produce the same sharing relation.

We show that \( \Gamma \vdash e_2 : C \mid S \) implies \( \Gamma \vdash e_1 : C \mid S \) for some \( X \).

Let \( \Gamma \vdash e_2 : C \mid S \), define \( \Gamma_1 = \Gamma [\mathit{dus}_1] [\Gamma_1 \mathit{dus}_2] \). From rule (T-New) Lemma 14 we have that
(a) $S = S_3 \setminus D_1$ where $S_3 = S_v + S'_1 + (S'_2 + S'') \setminus D_2$ and $S_v = \sum_{i=1}^{n+m} S_i$

(b) $\Gamma|_{\text{des}_1} \vdash v_i : T_i | S_i$ for all $i$, $1 \leq i \leq n + m$

(c) $\Gamma|_{\text{des}_1} \vdash \text{dvs}_1 : S'_1$

(d) $\Gamma_1 \vdash \text{dvs}_2 : S'_2$ and $\Gamma_1 \vdash v : T' | S'$ and

(e) $X' = \text{res}_{(S'_1 + S'_2 + S')} \cap (D_2 \cup D_1)$

where fields$(C) = T_1 f_1 \ldots T_n f_n T'_1 f'_1 T_{n+1} f_{n+1} \ldots T_{n+m} f_{n+m}$.

By wellformedness of blocks $D_1 \cap D_2 = \emptyset$. Therefore, letting $\Gamma_2 = \Gamma|_{\text{des}_1, \text{dvs}_2}$ we have that $\Gamma_1 = \Gamma_2$. From Lemma 13.2 and (c) and (d) we get $\Gamma_2 \vdash \text{dvs}_2 : S'_2$ and $\Gamma_2 \vdash v : T' | S'$ and $\Gamma_2 \vdash \text{des}_1 : S'_1$ and by rule (T-BLOCK)

(A) $\Gamma|_{\text{des}_1} \vdash \{ x \text{ dvs}_2 v \} : T' | (S'_1 + S'_2 + S') \setminus (D_2 \cup D_1)$ where $X' = \text{res}_{(S'_1 + S'_2 + S')} \cap (D_2 \cup D_1)$

From (1), (b) and Lemma 13.2 we get

(B) $\Gamma \vdash v_i : T_i | S_i$ for all $i$, $1 \leq i \leq n + m$

From (A), (B) and rule (T-NEW) we get

$$\Gamma \vdash e_1 : C | S_v + (S'_1 + S'_2 + S') \setminus (D_1 \cup D_2)$$

and $X = \text{res}_{(S'_1 + S'_2 + S')} \cap (D_1 \cup D_2)$.

The proof that $S = S_v + (S'_1 + S'_2 + S') \setminus (D_1 \cup D_2)$ is as for the previous implication.

The following lemma asserts that subexpressions of typable expressions are themselves typable, and may be replaced with expressions that have the same type and the same or possibly less sharing effects.

**Lemma 16.** (Context) Let $\Gamma \vdash E[e] : C | S$, then

1. $\Gamma|_{\text{e}} \vdash e : D | S_1$ for some $D$ and $S_1$.

2. if $\Gamma|_{\text{e}} \vdash e' : D | S_2$ where $S_2 \subseteq S_1 (S_2 = S_1)$, then $\Gamma \vdash E'[e'] : C | S'$ for some $S'$ such that $E'[e'] \approx E[e']$ and $S' \subseteq S$ ($S' = S$).

**Proof.** 1. Easy, by induction on evaluation contexts.

2. Let $\Gamma \vdash E[e] : C | S$. By point 1. of this lemma $\Gamma|_{\text{e}} \vdash e : D | S_1$ for some $D$ and $S_1$. By induction on evaluation contexts.

If $E = [ \ ]$, then $C = D$ and $S_1 = S$. The result is immediate.

If $E = \{ x \text{ dvs } T x \equiv E'; \; ds \; e_b \}$, then $\Gamma \vdash \{ x \text{ dvs } T x \equiv E'; \; ds \; e_b \} : C | S$.

Let $\Gamma' = \Gamma|_{\text{dvs}, x:T, \Gamma|_{\text{dvs}}}$, then from Lemma 13 we have

1. $S = (S_1 + S_2 + S_3) \setminus \text{dom} \Gamma'$

2. $\Gamma|_{\text{e}} \vdash T x \equiv E'[e'] : S_1$ where $S_1 = S_1[x/\text{res}]$ and $T = C_\mu$ and $\Gamma|_{\text{e}} \vdash E'[e'] : C_\mu | S_1$.

3. $\Gamma|_{\text{e}} \vdash dvs ds : S_2$.

4. $\Gamma|_{\text{e}} \vdash e_b : | S_3$ and

5. $X = \text{res}_{S_1 + S_2 + S_3} \cap (\text{dom (dvs)} \cup \text{dom (ds)} \cup \{ x \})$
From (2) and point 1. of this lemma $\Gamma[\Gamma']|\Gamma_E| \vdash e : D | S_4$ for some $D$ and $S_4$. Let $\Gamma[\Gamma']|\Gamma_E| \vdash e' : D | S'_4$ where $S'_4 \subseteq S_4$. From (2), by induction hypothesis on $E'$, we have that $\Gamma[\Gamma'] \vdash E'[e'] : D | S'_{e'}$ where $E'[e'] \approx E''[e']$ and $S'_{e'} \subseteq S_{e'}$. Moreover, from $\Gamma \vdash \{ \forall x \text{ dvs } T \exists x \in E'[e'] ; ds e_b \} : C | S$, if $\mu = a$, then capsule($S_e$), and so also capsule($S'_{e'}$). Therefore

(a) $\Gamma[\Gamma'] \vdash T \exists x \in E''[e'] ; ds e_b : C | S''$ where $S'' = (S'_1 + S_2 + S_3) \setminus \text{dom}(\Gamma')$ and $X' = [\text{res}]_1 + S_2 + S_3 \cap (\text{dom}(ds) \cup \text{dom}(ds - e))$. From Proposition 1.2 and 3 we derive that $S'' \subseteq S$. The case with equality is similar.

If $E = \{ \forall x \text{ dvs } E \}$, then the proof is similar to the previous one.

\[\square\]

**Lemma 18.** Let $\Gamma \vdash E[e] : C | S$ and $\Gamma \vdash e : D | S$. If $\langle x, y \rangle \in S'$ with $x, y \notin \text{HB}(E)$ and $x, y \neq \text{res}$, then $\langle x, y \rangle \in S$.

**Proof.** By induction on $E$.

\begin{enumerate}
\item Let $E = \{ \forall x \text{ dvs } C \}$. Then $S = (S'_1 + S_2 + S_3) \setminus \text{dom}(\Gamma')$ where $S'_1 = (S_1 + \{ z, \text{res} \}) \setminus \text{res}$.
\end{enumerate}

Assume that, $\langle x, y \rangle \in S'$ with $x, y \notin \text{HB}(E)$ and $x, y \neq \text{res}$. From $\text{HB}(E') \subseteq \text{HB}(E)$ and $\Gamma \vdash e : D | S'$, by induction hypothesis on $E'$ we derive that $\langle x, y \rangle \in S_1$. Since $(\{ z \} \cup \text{dom}(\Gamma')) \subseteq \text{HB}(E)$ we have that $x, y \notin (\{ z \} \cup \text{dom}(\Gamma'))$. Therefore $\langle x, y \rangle \in S$.

Similar (and simpler) for $E = \{ \forall x \text{ dvs } E' \}$.

\[\square\]

**Lemma 19.** If $\Gamma \vdash E[e_1] : C | S_1$, $\Gamma \vdash e_2 : D | S_2$ and $\Gamma \vdash e_1 : D | \{ xs, \text{res} \}$ and $\Gamma \vdash e_2 : D | \{ ys, \text{res} \}$ with $\{ xs, ys \} \cap \text{HB}(E) = \emptyset$. Then $S_1 + \{ xs, ys \} = S_2 + \{ xs, ys \}$.

**Proof.** Let $E = \{ \}$ Then $\{ xs, \text{res} \} + \{ xs, ys \} = \{ xs, ys, \text{res} \}$ and $\{ ys, \text{res} \} + \{ xs, ys \} = \{ xs, ys, \text{res} \}$

Let $E = \{ \forall x \text{ dvs } D^\mu z \in E' ; ds e \} \vdash \{ \forall x \text{ dvs } D^\mu z \in E'[e_1] ; ds e \} : C | S_1$, $\Gamma \vdash e_1 : C | S_2$, $\Gamma \vdash e_1 : C | \{ xs, \text{res} \}$, and $\Gamma \vdash e_2 : C | \{ ys, \text{res} \}$. Let $\Gamma' = \Gamma_{\text{dvs}, z : T}, \Gamma_{\text{dvs}, e_b}$, from Lemma 14.

(a) $\Gamma[\Gamma'] \vdash E'[e_1] : C | S_3$ and $\Gamma[\Gamma'] \vdash E'[e_2] : C | S_4$

(b) $\Gamma[\Gamma'] \vdash ds e : S_d$
Theorem 21 (Subject reduction) If \( \Gamma \vdash e : C | S \) and \( e_1 \rightarrow e_2 \), then

1. \( \Gamma \vdash e'_2 : C | S' \) for \( e'_2 \approx e_2 \) and \( S' \subseteq S \), and
2. for all \( x \) such that \( e_1 = D_x[e] \), \( e'_2 = D'_x[e'] \), and \( \Gamma \vdash e : D \mid S_\omega \) we have that: \( \Gamma \vdash e : D \mid S'_\omega \) and \( (S'_\omega + S_{\{x\}}) \subseteq (S_\omega + S_{\{x\}}) \) where \( S_{\{x\}} = S \mid x \). From Lemma 10,1 we have that \( \Gamma[\Gamma'] \vdash e_0 : D \mid S'' \) for some \( S'' \). From typing rule \( \text{T-NVK} \)

(1) \( \forall 0 \leq i \leq n \) \( \mu_i = a \implies \text{capsule}(S_i) \)
(2) \( S'_0 = S_0 \mid \text{this}/\text{res} \)
(3) \( S'_0 = S_0 \mid \text{this} \)
(4) \( S'_i = S_i \mid x_i \mid \text{res} \) \( 0 \leq i \leq n \)
(5) \( S'' = \sum_{i=1}^{n} S'_i + S_\omega \mid \{ \text{this}, \ldots, x_n \} \)

From the fact that the class table is well-typed we have that \( \Gamma' \vdash e_0 : D \mid S_\mu \) where \( \Gamma' = \{ \text{this}:C_\mu, x_1:C_1^{\mu_1}, \ldots, x_n:C_n^{\mu_n} \} \). Moreover, since we may assume that \( \{ \text{this}, x_1, \ldots, x_n \} \cap \text{dom}(\Gamma[\Gamma']) = \emptyset \), from Lemma 10,2 we have that

(a) \( \Gamma[\Gamma'][\Gamma'] \vdash e_0 : D \mid S_{\mu} \)
(b) \( \Gamma[\Gamma'][\Gamma'] \vdash e_0 : C_i \mid S_i \ (0 \leq i \leq n) \).

Therefore by typing rule \( \text{T-BLOCK} \), (1) \( \vdash (5) \), (a) and (b) we have that \( \Gamma[\Gamma'][\Gamma'] \vdash e' : D \mid S'' \). From Lemma 10,2 we derive \( \Gamma \vdash E'[e''] : D \mid S \) where \( E'[e''] \approx E[e'] \).

2. The result is proved as in the case of \( \text{FIELD-ASSIGN} \) just replacing \( S'_\omega \subseteq S_x \) with \( S'_\omega = S_x \) since the sharing relation of the redex is equal to the one of the block to which it reduces.
1. In this case

1. In this case

(2) \( e' = \{X \setminus \{x\}\} \text{dus} \ ds[v/x] \ e_b[v/x] \}\}

From Lemma 14 we have that \( \Gamma[i] \vdash \{X ds' e_b\} : D \mid S_1 \) for some \( S_1 \).

Therefore from Lemma 14 we have that

(a) \( \Gamma[i] \mid[i] \vdash \text{dus} : S_2 \) for some \( S_2 \),
(b) \( \Gamma[i] \mid[i] \vdash C_1 x = y \mid \{x, y\} \),
(c) \( \Gamma[i] \mid[i] \vdash ds : S_3 \) for some \( S_3 \),
(d) \( \Gamma[i] \mid[i] \vdash e_b : D \mid S_4 \) for some \( S_4 \),

\( \text{dom}(ds') \).

Since \( x \) cannot be free in \( \text{dus} \), from Lemma 13.2 and (a) we derive

(A) \( \Gamma[i] \mid[i] \vdash \text{dus} : S_2 \setminus x \).

From Lemma 17 and (c) and (d) we have that

(C) \( \Gamma[i] \mid[i] \vdash \text{ds}[y/x] : S_3 \setminus x \) and

(D) \( \Gamma[i] \mid[i] \vdash e_b[y/x] : S_4 \setminus x \).

Moreover,

(E) let \( S'' = \sum_{i=2}^{4} (S_i \setminus x) \),

from (A), (C) \( \vdash (E) \) and rule (T-BLOCK) we have that

\[ \Gamma[i] \mid[i] \vdash \{Y \setminus dom(ds) \} \vdash \text{dus} : S_2 \setminus x \] \( Y = [\text{res}], [\text{res}]_i', \cap \text{dom(ds)} \).

Therefore from Lemma 16 we derive \( S_2' \subseteq S_2 \).

2. The result is proved as in the case of (FIELD-ASSIGN) since from \( S_2 \subseteq S_1 \) by Lemma 14 we derive \( S_2' = S_2 \).

Rule (AFFINE-ELIM).

1. In this case

(1) \( \rho = \{ X ds' e_b\} \where ds' = \text{dus} \ C_1 x = y \mid ds \)

(2) \( e' = \{X \setminus \{x\}\} \text{dus} \ ds[v/x] \ e_b[v/x] \}\}

From Lemma 14 we have that \( \Gamma[i] \vdash \{X ds' e_b\} : D \mid S_1 \) for some \( S_1 \).

Therefore from Lemma 14 we have that

(a) \( \Gamma[i] \mid[i] \vdash \text{dus} : S_2 \) for some \( S_2 \),
(b) \( \Gamma[i] \mid[i] \vdash v : C_1 \mid \text{capsule}(S_v) \), therefore also \( \Gamma[i] \mid[i] \vdash C_1 x = v \mid \text{capsule}(S_v) \),
(c) \( \Gamma[i] \mid[i] \vdash ds : S_3 \) for some \( S_3 \),
(d) \( \Gamma[i] \mid[i] \vdash e_b : D \mid S_4 \) for some \( S_4 \),

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(e) $S_1 = S_1 \setminus \text{dom}(d's')$ where $S_1' = \sum_{i=2}^{4} S_i + S_e$ and $X = [\text{res}]_{S_1' \cap \text{dom}(d's')}$. Let

(B) $\Gamma[\Gamma x][\Gamma d's'] \vdash \text{gc}(v) : C_1 | S'_e$

we also have capsule($S'_e$).

From (B), the fact that $\Gamma d's'(x) = C_1$, and Lemma[12] we have that $S'_e = \epsilon$. Since we do not have forward references to unevaluated variables, $x$ cannot be free in $d's$ and from Lemma[13]2 and (a) we derive

(A) $\Gamma[\Gamma x][\Gamma d's'] \vdash \text{dvs} : S_2 \setminus x$.

From (B) with $S'_e = \epsilon$, Lemma[17]2 and (c) and (d) we have that

(C) $\Gamma[\Gamma x][\Gamma d's'] \vdash \text{dvs} \cup \text{gc}(v) : S_3 \setminus x$ and

(D) $\Gamma[\Gamma x][\Gamma d's'] \vdash \text{dvs} \cup e_b \text{gc}(v) : S_4 \setminus x$.

Moreover,

(E) let $S'' = \sum_{i=2}^{4} (S_i \setminus x)$,

from (A), (C)-(E) and rule (T-BLOCK) we have that

$$\Gamma[\Gamma x][\Gamma d's'] \vdash \{Y \text{dvs} v/x \text{e}_b[v/x] : D | S'' \cup \text{dom}(d's \text{dvs})$$

where $Y = [\text{res}]_{S'' \cap \text{dom}(d's \text{dvs})}$. Since $[x]_{S_1} = \{x\}$, we have that $x \notin [\text{res}]_{S_1}$. Moreover $\text{dom}(d's \text{dvs}) \cup \{x\} = \text{dom}(d's')$. Therefore we have that $X = X' \setminus x = Y$. From $S_i \setminus x = S_i$ (2 ≤ i ≤ 4) and Proposition[12] we get $S'' \subseteq S_3'$. 2. The result is proved as in the case of (FIELD-ASSIGN) since from $S_2 \subseteq S_1$ by Lemma[10] we derive $S'_x = S_x$.

Lemma[25] (Decomposition) If $e$ is not a value, then there are $E$ and $\rho$ such that $e \equiv E[\rho]$.

Proof. By structural induction on expressions.

If $v.f = v'$, then from Proposition[3] we have that $v = x$ or $v = \{X \text{dvs} x\}$ and $v' = y$ or $v' = \{Y \text{dvs} y\}$. If $v = \{X \text{dvs} x\}$ and $v' = \{Y \text{dvs} y\}$, we may assume, by $\alpha$-renaming, that $\text{dom}(dvs) \cap \text{dom}(dvs') = \emptyset$. From rule (VAL-CTX) (applied twice) $v.f = v' \equiv \{X \cup Y \text{dvs} d's' x.f = y\}$. So $v.f = v' \equiv E[x.f = y]$ where $E = \{X \cup Y \text{dvs} d's' [\text{ }]\}$. The other cases are easier.

For field access the proof is similar.

Method call is a redex so the result holds with $E = [\text{ }]$.

If $\{X \text{dvs} e\}$ is not a value, then either

1. $d's = d's T x.e_1$; $d's_1$ where $e_1$ is not new $C(xs)$ for some $C$ and $xs$ or
2. $d's = d's_1$ and $e$ is not a value.

In case (1), either $e_1$ is not a value or $e_1$ is a value but not new $C(xs)$ for some $C$ and $xs$.

In case (2), $e_1$ is not a value, by induction hypothesis, there are $E$, and $\rho$ such that $e_1 = E[\rho]$. Applying congruence rule (REORDER) of Fig[6] we have that

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\[
\{X \text{dvs } dvs_1 \Gamma T x = e_1; \ ds_2 \ e\} \equiv \{X \text{dvs } T x = e_1; \ ds_1 \ e\}
\]
where \(dvs_1\) are all the evaluated declarations of \(ds_1\). Therefore \(\{X \ ds \ e\} \equiv \mathcal{E}'[\rho]\) where \(\mathcal{E}' = \{\{X \text{dvs } dvs_1 \ T x = \mathcal{E}; \ ds_2 \ e\}\}\).

In case \(e_1\) is a value but not \new C(x) for some \(C\) and \(xs\), by Proposition \( \Box \) either \(e_1 \equiv y\) or \(e_1 \equiv \{\ Y \text{dvs}' y\} \). If \(T = D\) for some \(D\), then the block is a redex, else either \(e_1 \equiv y\) or \(e_1 \equiv \{Y \text{dvs}' y\}\).

If \(e_1 \equiv y\), then \(\{Y \ ds \ e\}\) is congruent to \(\{X \text{dvs } T x = y; \ ds_1 \ e\}\), which is a redex.

If \(e_1 \equiv \{X \text{dvs}' y\}\), then \(\{X \text{dvs } T x = e_1; \ ds_1 \ e\} \equiv \{X \text{dvs } T x = \{X \text{dvs}' y\}; \ ds_1 \ e\}\)

Since \(T = D\) for some \(D\), with \(\alpha\)-renaming of variables in \(\text{dom}(dvs')\), applying congruence rule (Dec) we have

\[
\{X \text{dvs } T x = e_1; \ ds_1 \ e\} \equiv \{X \text{dvs } ds' T x = y; \ ds_1 \ e\}
\]
and the expression on the right is a redex.

In case (2), by induction hypothesis, there are \(\mathcal{E}\), and \(\rho\) such that \(e \equiv \mathcal{E}[\rho]\). Therefore \(\{X \ ds \ e\} \equiv \mathcal{E}'[\rho]\) where \(\mathcal{E}' = \{X \text{dvs } \mathcal{E}\}\)