CROSSING VELOCITIES FOR AN ANNEALED RANDOM WALK IN A RANDOM POTENTIAL

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Abstract. We consider a random walk in an i.i.d. non-negative potential on the $d$-dimensional integer lattice. The walk starts at the origin and is conditioned to hit a remote location $y$ on the lattice. We prove that the expected time under the annealed path measure needed by the random walk to reach $y$ grows only linearly in the distance from $y$ to the origin. In dimension one we show the existence of the asymptotic positive speed.

1. Introduction

Model description and main results. Let $V(z, \omega)$, $z \in \mathbb{Z}^d$, be i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which represent a random potential on $\mathbb{Z}^d$. We assume that

$$V(0, \omega) \in [0, \infty] \text{ a.s., } \mathbb{P}(V(0, \omega) = 0) < 1, \text{ and } \underset{\omega \in \Omega}{\text{essinf}} V(0, \omega) = 0. \quad (1.1)$$

Remark 1.1. The last equality is not needed for any of our results and could have been simply replaced by the condition $\mathbb{P}(V(0, \omega) = \infty) < 1$. In the case when the potential $V$ is bounded away from zero, Theorem 1.1 below becomes very simple (see Section 2.2 of [Zy09]). The last assumption makes the situation much more delicate, and we would like to emphasize this from the beginning. A good example to have in mind is when $V(0) \in \{0, 1, \infty\}$, $\mathbb{P}(V(0) = 0) > 0$, and $\mathbb{P}(V(0) = 1) < 1$.

Let $P^x$ be the measure on the space of nearest-neighbor paths on $\mathbb{Z}^d$, which corresponds to a simple symmetric random walk $(S_n)_{n \geq 0}$ that starts at $x \in \mathbb{Z}^d$. The expectation with respect to $P^x$ will be denoted by $E^x$. Let us fix $y \in \mathbb{Z}^d$, $y \neq x$, and set $\tau_y = \inf\{n \geq 0 : S_n = y\}$. For $\omega \in \Omega$ such that

$$Z_y^{\omega,x} = E^x \left( \mathbb{1}_{\{\tau_y < \infty\}} e^{-\sum_{n=0}^{\tau_y-1} V(S_n, \omega)} \right) > 0$$

define the quenched path measure by

$$Q_y^{\omega,x}(A) := (Z_y^{\omega,x})^{-1} E^x \left( \mathbb{1}_{\{\tau_y < \infty\}} \mathbb{1}_A e^{-\sum_{n=0}^{\tau_y-1} V(S_n, \omega)} \right). \quad (1.2)$$

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The annealed path measure $Q^x_y$ is then given by
\begin{equation}
Q^x_y(A) := (Z^x_y)^{-1} \mathbb{E} E^x \left( \mathbb{1}_{\{\tau_y < \infty\}} \mathbb{1}_A e^{-\sum_{n=0}^{\tau_y-1} V(S_n, \omega)} \right) 
\end{equation}
where $Z^x_y = \mathbb{E} Z^{\omega,x}_y$. These measures have a natural interpretation in terms of the “killed random walk”, which we recall in the next subsection.

In the continuous setting, namely, for Brownian motion in a Poissonian potential, the above path measures were introduced and studied by A.-S. Sznitman (see [Sz98] and references therein), T. Povel ([Po97]), M. Wüthrich ([Wu98]). In the context of random walks, various aspects of these measures were addressed in, for example, [Ze98], [Fl08], [Zy09], [IV10a].

The rates of decay of the quenched and annealed partition functions,
\begin{align}
\alpha_V(h) &:= -\lim_{r \to \infty} \frac{1}{r} \log Z^{\omega,0}_{[r, h]} \quad (\mathbb{P}-\text{a.s.}) \quad \text{and} \\
\beta_V(h) &:= -\lim_{r \to \infty} \frac{1}{r} \log Z^0_{[r, h]}, \quad h \in \mathbb{R}^d,
\end{align}
known also as the quenched and annealed Lyapunov exponents respectively, are well defined (non-random) norms on $\mathbb{R}^d$ (see [Ze98], [Fl08], and [KMZ10]; for the existence of $\alpha_V(\cdot)$ it is sufficient to assume that $\mathbb{E} V < \infty$). Moreover, by Jensen’s inequality, $\beta_V(h) \leq \alpha_V(h)$ for all $h \in \mathbb{R}^d$.

In this paper we consider a random walk under the annealed path measures $Q^0_y$ and address the question whether it is ballistic in the sense that the average time it takes the walk to hit $y$, $E_{Q^0_y} \tau_y$, grows linearly in $\|y\|$ as $\|y\| \to \infty$. The same question regarding the quenched path measures for Brownian motion in a Poissonian potential was positively resolved in ([Sz95]). Our main results are contained in the following two theorems.

**Theorem 1.1.** Let $V(z, \omega)$, $z \in \mathbb{Z}^d$ be i.i.d. under $\mathbb{P}$ and satisfy (1.1). If $d = 1$, assume, in addition, that
\begin{equation}
\mathbb{P}(V(0, \omega) \in (0, \infty)) > 0.
\end{equation}
Then there exists a constant $C \in (0, \infty)$ such that
\begin{equation}
\limsup_{\|y\| \to \infty} \frac{E_{Q^0_y} \tau_y}{\|y\|} \leq C.
\end{equation}

**Remark 1.2.** The condition (1.6) is necessary: if $d = 1$ and our potential can take only two values, 0 and $\infty$, both with strictly positive probability, then it can be shown that under the annealed measure as $y \to \infty$, the process $y^{-1} S_{\lfloor sy^2 \rfloor} \mathbb{1}_{\{sy^2 < \tau_y\}}$, $s \geq 0$, converges in law to a Brownian excursion from 0 to 1, killed upon arriving at 1. In particular, $E_{Q^0_y} \tau_y/y$ converges to infinity as $y \to \infty$. This example runs counter to the “natural assumption” that the larger the potential the faster the random walk will achieve its target.

Theorem 1.1 readily leads to the following bound (the proof is given in Section 2).
Corollary 1.1. For every unit vector \( s \in \mathbb{R}^d \)
\[
\frac{d\beta_{\lambda+V}(s)}{d\lambda} \bigg|_{\lambda=0^+} \leq C.
\]

Remark 1.3. The existence of the above derivative follows from the concavity of the function \( \lambda \mapsto \beta_{\lambda+V}(h) \) (see [Fl07, Theorem A(b)]).

In one dimension we can say more.

Theorem 1.2. Let \( d = 1 \) and \( V(z, \omega), \ z \in \mathbb{Z}^d \), satisfy the assumptions of Theorem 1.1.
Then there exists a constant \( v \in (0, 1) \) such that
\[
\lim_{y \to \infty} \frac{E_{Q^y_0}(\tau_y)}{y} = \frac{1}{v}.
\]
Moreover,
\[
\frac{d\beta_{\lambda+V}(1)}{d\lambda} \bigg|_{\lambda=0^+} = \frac{1}{v}.
\]

“Killed random walk” description of the model. Consider the following Markov chain (“killed random walk”) on \( \mathbb{Z}^d \cup \dagger \), where \( \dagger \) is an absorbing state. If the walk is at \( z \in \mathbb{Z}^d \) then with probability \( 1 - e^{-V(z)} \) it goes to \( \dagger \) and otherwise goes to one of the \( 2d \) nearest-neighbor sites with equal probabilities. We denote by \( \bar{P}^{\omega,x} \) the measure, corresponding to this Markov chain starting from \( x \) in a fixed environment \( V(z, \omega), \ z \in \mathbb{Z}^d \). Averaging over the environments gives the averaged measure, \( \bar{P}^x(\cdot) := \mathbb{E}\bar{P}^{\omega,x}(\cdot) \).

Let us record the following obvious relations:
\[
Q^x_y(\cdot) = \bar{P}^{\omega,x}(\cdot | \tau_y < \infty), \quad Z^x_y = \bar{P}^{\omega,x}(\tau_y < \infty);
Q^x_y(\cdot) = \bar{P}(\cdot | \tau_y < \infty), \quad Z^x_y = \bar{P}(\tau_y < \infty);
Q^x_y(A|B) = \bar{P}^{\omega,x}(A | B \cap \{\tau_y < \infty\}).
\]

The last equality will allow us to use the Markov property of the “killed random walk” to do computations under \( Q^x_y \). Throughout the paper, when the starting point of a random walk is 0 we shall often drop the superscript indicating the starting point.

Motivation and open problems. There are several connections that motivate our interest and make us believe that ballisticity is an important issue.

Recently, several works ([Fl08], [Zy09], [IV10a]) addressed the question about the equality of quenched and annealed Lyapunov exponents for small perturbations of a constant potential in dimensions four and higher. In particular, it was shown that when \( d \geq 4 \) then under mild conditions on the potential for every \( \lambda > 0 \) there is a \( \gamma^* > 0 \) such that for all \( \gamma \in (0, \gamma^*) \)
\[
\beta_{\lambda+\gamma V}(\cdot) \equiv \alpha_{\lambda+\gamma V}(\cdot).
\]
Recall the already mentioned fact that for \( \lambda > 0 \) the random walk under \( Q_y \) is ballistic. Paper [IV10a], Theorem A, proves a stronger result under even weaker conditions but still under the restriction that \( \lambda > 0 \). It is certainly an interesting question whether (1.10) and its stronger version can be extended up to \( \lambda = 0 \) and whether \( \gamma^* \) is locally
uniform in $\lambda$ on $[0, \infty)$. Such an extension, which is important in its own right, would also help to clarify the relationship between the quenched and annealed large deviations rate functions for random walks in a random potential. This is the next connection that we would like to briefly discuss.

Random walks in a random potential are more often considered under the condition that they survive up to (a large) time $n \in \mathbb{N}$ (see, for example, [Si95], [AZ96], [Kh96] and references therein). The corresponding quenched and annealed measures with the starting point 0 are

$$Q^\omega_n(\cdot) := \hat{P}^\omega(\cdot | \tau^\dagger > n); \quad Q_n(\cdot) := \hat{P}(\cdot | \tau^\dagger > n).$$

It is known ([Ze98], [Fl07]) that random walks under each of these measures satisfy a full large deviation principle and the large deviations rate functions, $I(\cdot)$ and $J(\cdot)$ respectively, are given by the relations

$$I(h) = \sup_{\lambda \geq 0} (\alpha_{\lambda + V}(h) - \lambda);$$

$$J(h) = \sup_{\lambda \geq 0} (\beta_{\lambda + V}(h) - \lambda).$$

Corollary 1.1 implies, in particular, that for small $\|h\|$ we have $J(h) = \beta_V(h)$. A similar result holds for $I(h)$ if the right derivative of $\alpha_{\lambda + V}(s)$ with respect to $\lambda$ at $\lambda = 0^+$ is bounded uniformly in $s$, $\|s\| = 1$ (see [Sz95, Corollary 2.3], for the quenched result in a continuous setting). If (1.10) were shown to hold also for $\lambda = 0$ then we would immediately conclude that for $d \geq 4$ and sufficiently small $\gamma$ the large deviations rate functions $I$ and $J$ coincide in some neighborhood of the origin.

For further details, connections with polymer measures, and open problems we refer to the review [IV10b].

Remark 1.4. After this paper was submitted for publication, we learned about the concurrent and completely independent work [IV11]. The authors consider $d \geq 2$ and employ a different method, which allows them not only to show that the walk under $Q^y$ is ballistic (our Theorem 1.1 for $d \geq 2$) but also to obtain the corresponding law of large numbers and central limit theorem (see Theorem C of [IV11]). For dimension one, Theorem 1.1 can seemingly be also derived from the large deviations approach of [GdH92]. We believe, nonetheless, that our treatment is more direct and less technical.

Organization of the paper. In Section 2 we prove Theorem 1.1 but only for $d > 1$. The argument given does not seem to be adaptable to one dimension. However, since Theorem 1.2 implies Theorem 1.1 for $d = 1$, we just need to prove the former. This is done in Section 3 modulo several technical results (Lemmas 3.2, 3.3, 3.4, and 3.5). The crucial result among these is Lemma 3.3. Its proof, as well as proofs of other listed above lemmas, are given in Section 5 after the key exponential estimate Theorem 4.1 is established in Section 4. Several elementary auxiliary results are collected in the Appendix.
2. Proof of Theorem [1.1] for dimension higher than one

The quenched case in a continuous setting was investigated in [Sz95]. The argument given there applies to the quenched discrete random walk with minor modification. Though we deal with the annealed case, the basic division of space into occupied and unoccupied cubes (see below) and the exploitation of lattice animal bounds are lifted from [Sz95].

Let \( d > 1 \) and \( y \in \mathbb{Z}^d \setminus \{0\} \) be the “target point”. For \( A \subset \mathbb{Z}^d \) define \( \tau(A) = \inf\{n \geq 0 : S_n \in A\} \). Fix a large even \( L \) and for \( q \in \mathbb{Z}^d \) let \( B(q) = (Lq + [-L/2, L/2]^d) \cap \mathbb{Z}^d \). The set of these cubes, \( \{B(q), \ q \in \mathbb{Z}^d\} \), forms a partition of \( \mathbb{Z}^d \). Choose some \( \kappa \in (0, 1) \) so that \( \mathbb{P}(V(0) \geq \kappa) > 0 \). Given an environment \( \omega \in \Omega \) and \( A \subset \mathbb{Z}^d \) we shall say that

\[
A \text{ is occupied if } \max_{A \setminus \{y\}} V(x, \omega) \geq \kappa \text{ and empty otherwise.}
\]

Denote by \( \mathcal{O} = \mathcal{O}(\omega) \) the union of all occupied cubes in our partition and by \( \mathcal{O}^c \) the union of all empty cubes.

**Step 1.** We shall estimate the time spent by our random walk in \( \mathcal{O} \). This is not difficult, since from every point in \( \mathcal{O} \) there is a path of length at most \( d(L-1) \) to a point where the potential is at least \( \kappa \). This observation essentially provides the proof of the following lemma, which is very much analogous to Theorem 1.1 of [Sz95].

**Lemma 2.1.** There exists constant \( C_1 = C_1(L, \kappa) \) such that for all \( y \in \mathbb{Z}^d \setminus \{0\} \)

\[
E_{Q_y} \left( \sum_{n=0}^{\tau_y-1} \mathbb{I}_{\{S_n \in \mathcal{O}\}} \right) \leq C_1 \|y\|.
\]

**Proof.** We shall show that there is an \( \varepsilon > 0 \) and \( n_0 = n_0(\varepsilon, \kappa) \) such that for all \( n \geq n_0 \),

\[
Q_y \left( \frac{1}{dL\|y\|} \sum_{t=0}^{\tau_y-1} \mathbb{I}_{\{S_t \in \mathcal{O}\}} > n \right) \leq 2e^{-\varepsilon \kappa (\|y\|-2)/2}.
\]

This will immediately imply the statement of the lemma.

Define the stopping times \( \sigma_m, \ m \in \mathbb{N}, \) by

\[
\sigma_1 = \inf\{n \geq 0 : S_n \in \mathcal{O}\}, \quad \sigma_{m+1} = \inf\{n \geq \sigma_m + dL : S_n \in \mathcal{O}\}.
\]

The probability that during the time interval \([\sigma_m, \sigma_m + dL]\) a simple symmetric random walk hits a point with the potential at least \( \kappa \) and does not hit \( y \) is greater than \( (2d)^{-dL} \).

Using the Markov property of the killed random walk we get that for \( \varepsilon \in (0, 1) \) and all \( m \geq 2 \)

\[
\mathbb{E}\tilde{P}^{\omega,0} (\sigma_m < \tau_y < \infty) \leq e^{-(m-1)\varepsilon \kappa} + P(Y < (m-1)\varepsilon),
\]

where \( Y \) is a binomial random variable with parameters \((m-1)\) and \((2d)^{-dL}\). We choose \( \varepsilon \in (0, (2d)^{-dL}) \) sufficiently small to ensure that \( P(Y < (m-1)\varepsilon) < e^{-(m-1)\varepsilon} \) for all \( m \) large. By (1.3) we know that there is \( \beta_0 \in (0, \infty) \) such that \( Z_y \geq e^{-\beta_0 \|y\|} \) for all \( y \in \mathbb{Z}^d \setminus \{0\} \). Dividing (2.2) by \( Z_y \) and using (1.9) and the last inequality we get (recall that \( \kappa \in (0, 1) \))

\[
Q_y(\sigma_m < \tau_y) \leq 2e^{-(m-1)\varepsilon \kappa} e^{\beta_0 \|y\|}.
\]
This completes the proof, since the set in the left hand side of (2.1) is contained in \( \{ \sigma_{n}\|y\| < \tau_{y} \} \).

\[ \square \]

**Step 2.** To get a bound on the time spent by the walk in empty cubes we need some information about sizes of connected components of \( \mathcal{O}^{c} \) under \( Q_{y} \) (considered as a measure on environments).

Given an \( \omega \in \Omega \), we shall say that \( x_{1} \) and \( x_{2} \), both in \( \mathbb{Z}^{d} \), are connected in \( \mathcal{O}^{c} \) if there is a simple random walk path from \( x_{1} \) to \( x_{2} \) entirely contained in \( \mathcal{O}^{c} \). This defines a partition of \( \mathcal{O}^{c} \) into connected components. If we consider a site percolation on \( \mathbb{Z}^{d} \) where the site \( q \) is open if and only if \( B(q) \) is empty then standard percolation results (see e.g. \([Gr99]\)) imply that for sufficiently large \( L \) all connected components of \( \mathcal{O}^{c} \) are finite \( \mathbb{P} \)-a.s.. Since \( Q_{y} \) is absolutely continuous with respect to \( \mathbb{P} \), the same conclusion is true for \( Q_{y} \)-a.e. \( \omega \). From now on we suppose that \( L \) is sufficiently large so that the above holds.

We shall need the following notation. Let \( D(x) = B(q) \) if \( x \in B(q) \cap \mathcal{O} \) and let \( D(x) \) be equal to the connected component of \( \mathcal{O}^{c} \) that contains \( x \) if \( x \in \mathcal{O}^{c} \). Set \( |D(x)| = \# \{ q \in \mathbb{Z}^{d} : B(q) \subset D(x) \} \). Notice that \( |D(x)| = 1 \) for every \( x \in \mathcal{O} \). For \( D(x) \subset \mathcal{O}^{c} \) define the outer boundary \( \text{Ad} \, D(x) \) as the union of all cubes in \( \mathcal{O} \) which are adjacent to \( D(x) \), i.e.

\[
\text{Ad} \, D(x) = \{ z \in \mathcal{O} : \exists x_{1} \in D(x), \exists z_{1} \in D(z) \text{ such that } \|x_{1} - z_{1}\| = 1 \}.
\]

When \( x \in \mathcal{O} \) we set \( \text{Ad} \, D(x) = \emptyset \). The usual internal boundary of \( D(x) \) will be denoted by \( \partial D(x) \), i.e.

\[
\partial D(x) = \{ z \in D(x) : \exists z_{1} \notin D(x) \text{ such that } \|z - z_{1}\| = 1 \}.
\]

Consider a sequence of stopping times \( (\rho_{i})_{i \geq 0} \) and an increasing sequence of sets \( (A_{i})_{i \geq 0} \) given by \( \rho_{0} = -1 \), \( A_{0} = \emptyset \) and for \( i \in \mathbb{N} \),

\[
\rho_{i} = \inf \{ n > \rho_{i-1} : S_{n} \notin A_{i-1} \}, \quad A_{i} = A_{i-1} \cup D(S_{\rho_{i}}) \cup \text{Ad} \, D(S_{\rho_{i}}).
\]

Note that \( A_{i-1} \cap D(S_{\rho_{i}}) = \emptyset \). Finally, we introduce the “discovery” filtration \( (\mathcal{G}_{i})_{i \geq 1} \) where \( \mathcal{G}_{i} \) is the sigma field generated by \( (S_{n \wedge \rho_{i}})_{n \geq 0} \) and \( (V(x))_{x \in A_{i-1}} \).

**Lemma 2.2.** There exist strictly positive constants \( c_{2} \) and \( C_{2} \) not depending on \( L \) so that for all \( i \geq 1 \), \( N \in \mathbb{N} \), and all sufficiently large \( L \)

\[
Q_{y}(|D(S_{\rho_{i}})| = N, D(S_{\rho_{i}}) \subset \mathcal{O}^{c} | \mathcal{G}_{i}) \leq C_{2}e^{-c_{2}NL^{d}} \tag{2.3}
\]

**Proof.** Define \( q_{i} \) by the relation \( S_{\rho_{i}} \in B(q_{i}) \). We first note that there are less than \( (3^{d})^{2N} \) distinct connected sets in \( \mathbb{Z}^{d} \) of cardinality \( N \) containing \( q_{i} \) (see e.g. p. 1009 of \([Sz95]\)). For each such set \( \mathfrak{A}_{N}, q_{i} \in \mathfrak{A}_{N} \), define \( D_{N} = \cup_{q \in \mathfrak{A}_{N}} B(q) \). It is sufficient to show that there are strictly positive constants \( c_{3} \) and \( C_{3} \), not depending on \( L \) or \( N \), such that for every \( D_{N} \) with \( |D_{N}| = N \) and containing \( S_{\rho_{i}} \)

\[
Q_{y}(D(S_{\rho_{i}}) = D_{N}, D_{N} \subset \mathcal{O}^{c} | \mathcal{G}_{i}) \leq C_{3}e^{-c_{3}NL^{d}}.
\]
From this point on we fix $D_N$ (and so $\mathfrak{A}_N$). We suppose that $y$ is not in $D_N$ and leave it to the reader to make the minor modifications for the case when $y \in D_N$.

Given $\mathcal{G}_i$ and $D_N$, denote by $\Omega_{i,N}$ all environments which agree with $(V(x))_{x \in A_{i-1}}$ and have $D_N$ as a connected component of $\mathcal{O}^c$ containing $S^*_\rho_i$. This means, in particular, that $\text{Ad}_D N \subset \mathcal{O}$. We need to get an upper bound on $Q_y(\Omega_{i,N} \mid \mathcal{G}_i)$. We shall compare this probability with the probability of the following modified set of environments. Let $B'(q) = Lq + [-L/4, L/4]^d$ and $D'_N = \cup_{q \in \mathfrak{A}_N} B'(q)$, $q_i \in \mathfrak{A}_N$. Denote by $\Omega'_{i,N}$ all environments which can be obtained from those in $\Omega_{i,N}$ by changing the potential only on $D'_N$ so that each center cube $B'(q)$, $q \in \mathfrak{A}_N$, becomes occupied. A suitable upper bound on

$$Q_y(\Omega_{i,N} \mid \mathcal{G}_i) = \frac{\mathbb{E}(\bar{P}^{\omega, S^*_\rho_i}(\tau_y < \infty) 1_{\Omega_{i,N}} \mid \mathcal{G}_i)}{\mathbb{E}(\bar{P}^{\omega, S^*_\rho_i}(\tau_y < \infty) 1_{\Omega'_{i,N}} \mid \mathcal{G}_i)}$$

will complete the proof of this lemma. Let

$$M^\omega_i = \max_{x \in D_N} E_x^\omega \left( e^{-\sum_{n=1}^{\nu_0} V(S_n, \omega)} 1_{\{\tau(D_N) > \tau_y\}} 1_{\{\nu_k < \tau_y\}} \right).$$

The expression we maximize can be non-zero only at $x \in \partial D_N$ (or at $x$ neighboring $y$ if $y \in D_N$). Note that $M^\omega_i$ does not depend on the values of $V$ in $D_N$.

To bound the numerator in (2.4) we first observe that replacing the potential by 0 in $D_N$ can only increase the expectation. Let $\nu_0 = 0$ and $\nu_{i+1} = \inf\{n > \nu_i : S_n \in \partial D_N\}$. Then, given $\mathcal{G}_i$, for $\omega \in \Omega_{i,N}$ we have

$$\bar{P}^{\omega, S^*_\rho_i}(\tau_y < \infty) \leq \sum_{k=0}^{\infty} E^{S^*_\rho_i} \left( e^{-\sum_{n=0}^{\nu_k} V(S_n)} 1_{\{\tau(D_N) > \tau_y\}} 1_{\{\nu_k < \tau_y\}} \right) \leq \sum_{k=0}^{\infty} M^\omega_i E^{S^*_\rho_i} \left( e^{-\sum_{n=0}^{\nu_k} V(S_n)} 1_{\{\nu_k < \tau_y\}} \right) = M^\omega_i \sum_{k=0}^{\infty} \bar{P}^{S^*_\rho_i}(\nu_k < \tau_y).$$

For any point in $\partial D_N$ which is not adjacent to $y$, we have a uniform strictly positive lower bound, $(2d)^{-dL}$, of hitting a site $z$ with $V(z) \geq \chi$ before returning to $D_N$. But to return to $D_N$ from $z$ the walk has to survive. It follows easily from the Markov property that

$$\bar{P}^{S^*_\rho_i}(\nu_k < \tau_y) \leq (1 - (2d)^{-dL}(1 - e^{-\chi}))^k.$$
there is a path of length at most $dL(N + 1)$ within this set. In particular, there is such a path from $S_{\rho_i}$ to $x_0$. Thus we have

$$
\mathbb{E}(\tilde{P}^{\omega, \rho_i}(\tau_y < \infty) \mathbbm{1}_{\Omega'_{i,N}^L} \mid G_i) \geq (2d)^{-dL(N+1)} e^{-\kappa dL(N+1)} \mathbb{E}(M_i^{\omega} \mid G_i) \mathbb{P}(\Omega'_{i,N} \mid G_i).
$$

Since $M_i^{\omega}$ does not depend on the potential in $D_N$, we can now conclude that

$$
\frac{Q_y(\Omega_{i,N} \mid G_i)}{Q_y(\Omega'_{i,N} \mid G_i)} \leq \frac{(2d)^{dL(N+2)} e^{\kappa dL(N+1)}}{(1 - e^{-\kappa})} \frac{\mathbb{P}(\Omega_{i,N} \mid G_i)}{\mathbb{P}(\Omega'_{i,N} \mid G_i)}
$$

The ratio of probabilities is bounded above by

$$
\frac{\mathbb{P}(V(x) < \kappa)^{N(L/2)^d}}{(1 - \mathbb{P}(V(x) < \kappa)^{(L/2)^d})^N} \leq e^{-c_4NL^d}
$$

for all sufficiently large $L$. The last two bounds imply that there are positive $c_3$ and $C_3$ not dependent on $L$ such that

$$
Q_y(D(S_{\rho_i}) = D_N \mid G_i) = Q_y(\Omega_{i,N} \mid G_i) \leq C_3 e^{-c_3NL^d}
$$

as claimed. \hfill \square

**Step 3.** We now wish to show that, given $G_i$, the expected amount of time spent inside the (unknown) new component $D(S_{\rho_i})$, by the random walk before $\tau_y$ is of order one. As Lemma 2.2 gives very strong bounds on the size of this new component, all we need is a crude upper bound on this expectation in terms of the size of $D(S_{\rho_i})$.

We use the following lemma, which is basically an $h$-process result (see [Do01] for a general exposition).

**Lemma 2.3.** Consider a domain $D \subset \mathbb{Z}^d$ of cardinality $N$. Suppose that the environment is such that for every (internal) boundary point $b \in \partial D$ there exists a path from $b$ to a point $z$ with $V(z) \geq \kappa$ which is entirely in the complement of $D \setminus \{b\}$ and of length less than $dL$. Then for some universal $C_4 = C_4(\kappa, L)$ and all $x \in D$ such that $Z_y^{\omega,x} > 0$

$$
(2.6) \quad E_{Q_y^{\omega,x}} \left( \sum_{n=0}^{\nu_{y-1}} \mathbbm{1}_{\{S_n \in D\}} \right) \leq C_4 N^{2/d} \log \left( 1 + \sup_{u,v \in D} \frac{Z_y^{\omega,u}}{Z_y^{\omega,v}} \right).
$$

**Proof.** Again we suppose that $y \notin D$ and leave the remaining case to the reader. Consider the stopping times $(\nu_k)_{k \geq 0}$ where $\nu_0 = 0$ and

$$
\nu_{k+1} = \inf\{n > \nu_k : S_n \in \partial D\}.
$$

We have as in Lemma 2.2 that for any initial $x \in D$ and $k \in \mathbb{N}$,

$$
(2.7) \quad E_x^x \left( e^{-\sum_{n=0}^{\nu_k + dL} V(S_n) \mathbbm{1}_{\{\nu_k + dL < \tau_y\}}} \right) \leq (1 - (2d)^{-dL}(1 - e^{-\kappa}))^k.
$$

If $D$ has cardinality $N$, then for some constants $c_5$ and $C_5$ (depending on $d$) and all $x \in D$ we have $P_x(\tau(D^c) > t) \leq C_5 e^{-c_5 t N^{-2/d}}$. This follows, since by the local central
limit theorem (see e.g. [Du05]), there exist universal and nontrivial constants $c$ and $C$
so that $P^x(S_{CN^{2/d}} \in D) < c < 1$ uniformly over $x \in D$. Therefore,

\begin{equation}
(2.8) \quad P^x \left( \sum_{n=0}^{\nu_k + dL} \mathbb{1}_{\{S_n \in D\}} > C_6 N^{2/d} k, \nu_k + dL < \tau_y \right) \leq e^{-c_6 k}
\end{equation}

for some universal $c_6, C_6 \in (0, \infty)$. Let $T = T(x, k)$ be the stopping time when the
number of steps in $D$ numbers more than $C_6 N^{2/d} k$. Then for any $x$ and any $k$,

\begin{equation}
Q^x,\omega \left( \sum_{n=0}^{\tau_y - 1} \mathbb{1}_{\{S_n \in D\}} > C_6 N^{2/d} k \right) = \frac{E^x \left( e^{-\sum_{n=0}^{T-1} V(S_n)} \mathbb{1}_{\{T < \tau_y\}} Z_y^{\omega, S_T} \right)}{Z_y^{\omega, x}}.
\end{equation}

Partitioning the path space into the event $\left\{ \sum_{n=0}^{\nu_k + dL} \mathbb{1}_{\{S_n \in D\}} > C_6 N^{2/d} k \right\}$ and its complement and using (2.7) and (2.8) we get that the last ratio is dominated by

\begin{equation}
(2.9) \quad \left( e^{-c_6 k} + (1 - (2d)^{-dL}(1 - e^{-\kappa}))^k \right) \sup_{u, v \in D} \frac{Z_y^{\omega, u}}{Z_y^{\omega, v}}.
\end{equation}

If we choose now

\[ k = \left\lfloor C_7 m \log \left( 1 + \sup_{u, v \in D} \frac{Z_y^{\omega, u}}{Z_y^{\omega, v}} \right) \right\rfloor, \]

then it is easy to see that for sufficiently large $C_7$ the expression in (2.9) will be less than $e^{-c_7 m}$ for some strictly positive $c_7$ depending only on our choice of $C_7$. This immediately implies (2.6).

**Step 4.** Now we can estimate the time spent by the random walk in $D(S_{\rho_i}) \cap \mathcal{O}^c$. First, we notice that for some $c_8, C_8 \in (0, \infty)$ and all $D(S_{\rho_i}) \subset \mathcal{O}^c$

\[ \sup_{u, v \in D(S_{\rho_i})} \frac{Z_y^{\omega, u}}{Z_y^{\omega, v}} \leq C_8 e^{c_8 \cdot |D(S_{\rho_i})|dL} \]

uniformly over all environments for which $Z_y^{\omega, S_{\rho_i}} > 0$. The above bound is obtained simply by forcing the walk that starts at $v$ first to go to $u$. For this we can choose a path, which is entirely contained in $D(S_{\rho_i})$ and has length less than $|D(S_{\rho_i})|dL$. Then (2.6) and (2.9) give us that there is $C_9 = C_9(\omega, L)$ such that

\begin{equation}
(2.10) \quad E_{Q_y} \left( \sum_{n=0}^{\tau_y - 1} \mathbb{1}_{\{S_n \in D(S_{\rho_i}) \cap \mathcal{O}^c\}} \mid G_t \right) \leq C_9.
\end{equation}

**Step 5.** This step will complete the proof of Theorem 1.1. We have

\begin{equation}
(2.11) \quad E_{Q_y}(\tau_y) = E_{Q_y} \left( \sum_{n=0}^{\tau_y - 1} \mathbb{1}_{\{S_n \in \mathcal{O}\}} \right) + E_{Q_y} \left( \sum_{n=0}^{\tau_y - 1} \mathbb{1}_{\{S_n \in \mathcal{O}^c\}} \right).
\end{equation}

Lemma 2.1 takes care of the first term in the right hand side. We have a uniform bound on the expected time spent in every connected component of $\mathcal{O}^c$ visited by the random walk prior to $\tau_y$. The only question we have to answer is how many of these components it visited. Notice that in the time interval $[\rho_i, \rho_{i+1})$ the walk necessarily visits a new
occupied cube. We define stopping times \((\beta_i)_{i \geq 0}\) by \(\beta_0 = 0, \beta_{i+1} = \inf\{n > \beta_i : S_n \in O \setminus D(S_{\beta_i})\}\). Then, clearly, \(\rho_i \geq \beta_i\). Arguing as in the proof of Lemma 2.2 we have for some positive \(c_0\) that \(Q_y(\beta_i, \parallel y \parallel < \tau_y) < e^{-c_0 \parallel y \parallel}\). The last term of (2.11) is equal to

\[
E_Q y \left( \sum_{i=0}^{\infty} \mathbb{1}_{\{\rho_i < \tau_y\}} \sum_{n=0}^{\tau_y-1} \mathbb{1}_{\{S_n \in D(S_{\rho_i}) \cap O\}} \right) \leq E_Q y \left( \sum_{i=0}^{\infty} \mathbb{1}_{\{\rho_i < \tau_y\}} \right) \leq C_9 \sum_{i=0}^{\infty} Q_y(\beta_i, \parallel y \parallel < \tau_y) \leq C_{10} \parallel y \parallel.
\]

for some \(C_{10}\). The proof is now complete.

In the remainder of the section we show how to derive Corollary 1.1 from Theorem 1.1. The following simple lemma holds in all dimensions. Its proof is very similar to the proof of Corollary 2.3 in [Sz95].

**Lemma 2.4.** For every unit vector \(s \in \mathbb{R}^d\)

\[
\frac{d}{d\lambda} \frac{\beta_{\lambda+V}(s)}{\mathbb{P}} \bigg|_{\lambda=0+} \leq \limsup_{\parallel y \parallel \to \infty} \frac{E_Q y(\tau_y)}{\parallel y \parallel}.
\]

**Proof.** Fix any unit vector \(s \in \mathbb{R}^d\) and let \(y = [rs]\). Observe that

\[
- \frac{d}{d\lambda} \left( \liminf_{r \to \infty} \frac{1}{r} \log E_Q y(e^{-\lambda \tau_y}) \right) \bigg|_{\lambda=0+} = \frac{d}{d\lambda} \beta_{\lambda+V}(s) \bigg|_{\lambda=0+}, \quad \text{and}
\]

\[
- \liminf_{r \to \infty} \frac{1}{r} \left( \frac{d}{d\lambda} \log E_Q y(e^{-\lambda \tau_y}) \right) \bigg|_{\lambda=0+} = \limsup_{r \to \infty} \frac{1}{r} \frac{E_Q y(\tau_y e^{-\lambda \tau_y})}{E_Q y(e^{-\lambda \tau_y})} \bigg|_{\lambda=0+}.
\]

\[
= \limsup_{\parallel y \parallel \to \infty} \frac{E_Q y(\tau_y)}{\parallel y \parallel}.
\]

The statement of the lemma is an easy direction of the above exchange of limits. Since \(\beta_{\lambda+V}\) is an increasing concave function of \(\lambda\) on \([0, \infty)\) (see [Fl07, Theorem A(b)]), it is enough to show that for each \(\lambda > 0\)

\[
\limsup_{\parallel y \parallel \to \infty} \frac{E_Q y(\tau_y)}{\parallel y \parallel} \geq \frac{\beta_{\lambda+V}(s) - \beta_{V}(s)}{\lambda}.
\]

Let \(b(0, y, V) := -\log \mathbb{E} \mathbb{E}^0 e^{-\sum_{n=0}^{\tau_y-1} V(S_n)}\). Then \(b(0, y, \lambda + V)\) is a concave increasing function of \(\lambda\) on \([0, \infty)\) and

\[
E_Q y(\tau_y) = \frac{d}{d\lambda} \left( -\log E_Q y(e^{-\lambda \tau_y}) \right) \bigg|_{\lambda=0} = \lim_{\lambda \to 0+} \frac{b(0, y, \lambda + V) - b(0, y, V)}{\lambda}.
\]
By concavity, for each \( \lambda > 0 \),
\[
\lim_{\|y\| \to \infty} \frac{E_{Q_\lambda} y \tau_y}{\|y\|} \geq \lim_{\|y\| \to \infty} \frac{b(0, y, \lambda + V) - b(0, y, V)}{\|y\| \lambda} = \frac{1}{\lambda} \lim_{\|y\| \to \infty} \frac{b(0, y, \lambda + V) - b(0, y, V)}{\|y\|} = \frac{\beta_{\lambda + V}(s) - \beta_V(s)}{\lambda}. \]

\[\square\]

3. Asymptotic speed in one dimension

We start by introducing some notation. For \( x \in \mathbb{Z} \) define
\[
\tau^{(1)}_x := \tau_x, \quad \text{and for } m \in \mathbb{N}
\]

\[\tau^{(m+1)}_x := \inf\{n > \tau^{(m)}_x : S_n = x\}; \quad (3.1)\]

\[\ell_y(x) := \#\{n \in \{0, 1, \ldots, \tau_y - 1\} : S_n = x\}. \quad (3.2)\]

In addition to \( Q^\omega_y \) and \( Q_y \) we shall need the following measures and partition functions:

\[Q_{0, r}^\omega(\cdot) = \bar{P}^\omega(\cdot | \tau_r < \tau_0^2, \tau_r < \infty), \quad Z_{0, r}^\omega = \bar{P}^\omega(\tau_r < \tau_0^2, \tau_r < \infty);\]

\[Q_{0, r}(\cdot) = \bar{P}(\cdot | \tau_r < \tau_0^2, \tau_r < \infty), \quad Z_{0, r} = \bar{P}(\tau_r < \tau_0^2, \tau_r < \infty);\]

\[Q_{0, r}(\cdot) = \bar{P}(\cdot | \tau_r < \tau_0^2, \tau_r < \infty, \ell_r(x) \geq 2, x \in \{1, 2, \ldots, r - 1\});\]

\[Z_{0, r} = \bar{P}(\tau_r < \tau_0^2, \tau_r < \infty, \ell_r(x) \geq 2, x \in \{1, 2, \ldots, r - 1\}).\]

Denote by \( X_y \) the smallest non-negative integer in \([0, y]\), which is visited by the walk at most once up to the time \( \tau_y \), i.e.
\[
(3.3) \quad X_y = \min\{x \in \{0, 1, \ldots, y\} : \ell_y(x) \leq 1\}.
\]

We shall refer to \( X_y \) and all points between 0 and \( y \) inclusively that were visited at most once up to time \( \tau_y \) as “renewal points”. We use “at most once” instead of “exactly once” just to include \( y \) in the set of renewal points. The main idea of our proof is to obtain (1.7) using renewal theory.

The main ingredient of the proof of (1.7) is the following proposition.

**Proposition 3.1.** There is a constant \( v \in (0, 1) \) such that
\[
\lim_{y \to \infty} \frac{E_{Q_{0, y}}(\tau_y)}{y} = \frac{1}{v}. \]

We now indicate how this implies (1.7). The formal argument will be given after the proof of Proposition 3.1. We have
\[
(3.4) \quad \frac{E_{Q_y}(\tau_y)}{y} = \frac{E_{Q_y}(\tau_{X_y})}{y} + \frac{1}{y} E_{Q_y} \left[ E_{Q_y}(\tau_y - \tau_{X_y} | X_y) \right]
\]

\[= \frac{E_{Q_y}(\tau_{X_y})}{y} + \frac{1}{y} E_{Q_y} \left[ E_{Q_{0, y - X_y}}(\tau_{y - X_y} | X_y) \right]. \]

The following lemma, whose proof is postponed until Section 5, takes care of the first term in the right-hand side of (3.4).
Lemma 3.2. \( \lim_{y \to \infty} \frac{E_{Q_y}(\tau_{X_y})}{y} = 0. \)

The second term in \((3.4)\) will converge to \(1/y\) by Proposition 3.1, provided that we have sufficient control on \(X_y\).

Proof of Proposition 3.1. The proof relies on two technical lemmas. We shall state them as needed and supply proofs in Section 5.

Notice that 0 is the first renewal point under \(Q_{0,y}\). Decomposition of the path space over all possible renewal points in \([0, y]\) gives

\[
E_{Q_{0,y}}(\tau_y) = \frac{E_p(\tau_y; \tau_y < \tau_0^{(2)}, \tau_y < \infty)}{\hat{P}(\tau_y < \tau_0^{(2)}, \tau_y < \infty)}
\]

\[
= \sum_{y=1}^{\infty} \sum_{k=1}^{\infty} \prod_{j=1}^{y} \hat{Z}_{0,x_j-x_{j-1}} \sum_{i=1}^{k} \frac{E_{Q_{0,x_i-x_{i-1}}}(\tau_{x_i-x_{i-1}})}{\prod_{j=1}^{y} \hat{Z}_{0,x_j-x_{j-1}}}
\]

(3.5)

If the weights \(\hat{Z}_{0,r}, r \geq 1\), formed a probability distribution then the denominator would simply be the probability that \(y\) is a renewal point of a renewal sequence with this distribution, and the numerator would be the expectation of some function of the renewal lengths up to \(y\) restricted to the set where \(y\) is a renewal point. But, as it turns out, these weights do not add up to 1. We shall have to make an adjustment that is based on the following important fact.

Lemma 3.3. Let \(\beta := \beta_y(1)\) and \(q(r) := e^{\beta r} \hat{Z}_{0,r}\). Then there is an \(\epsilon > 0\) and \(r_0 > 0\) such that for all \(r \geq r_0\)

\[
q(r) \leq e^{-\epsilon r}.
\]

Moreover, \(\sum_{r=1}^{\infty} q(r) = 1.\)

Remark 3.1. A result analogous to \(\sum_{r=1}^{\infty} q(r) = 1\) is essentially shown in [ZY09].

Assume the above lemma. Multiplying and dividing \((3.5)\) by \(e^{\beta y}\) and writing \(e^{\beta y}\) as \(\prod_{j=1}^{k} e^{\beta(x_j-x_{j-1})}\) we get

\[
E_{Q_{0,y}}(\tau_y) = \frac{\sum_{k=1}^{y} \sum_{0=x_0 < x_1 < \cdots < x_k = y}^{\infty} \prod_{j=1}^{y} e^{\beta(x_j-x_{j-1})} \hat{Z}_{0,x_j-x_{j-1}} \sum_{i=1}^{k} E_{Q_{0,x_i-x_{i-1}}}(\tau_{x_i-x_{i-1}})}{\sum_{k=1}^{y} \sum_{0=x_0 < x_1 < \cdots < x_k = y}^{\infty} \prod_{j=1}^{y} e^{\beta(x_j-x_{j-1})} \hat{Z}_{0,x_j-x_{j-1}}}
\]

(3.6)

\[
= \frac{\sum_{k=1}^{y} \sum_{0=x_0 < x_1 < \cdots < x_k = y}^{\infty} \prod_{j=1}^{y} q(x_j-x_{j-1}) \sum_{i=1}^{k} E_{Q_{0,x_i-x_{i-1}}}(\tau_{x_i-x_{i-1}})}{\sum_{k=1}^{y} \sum_{0=x_0 < x_1 < \cdots < x_k = y}^{\infty} \prod_{j=1}^{y} q(x_j-x_{j-1})}
\]

(3.7)
If we denote by $(X_i)_{i \geq 0}$ the renewal sequence with $X_0 = 0$ corresponding to the probability kernel $q(\cdot)$, by $A_y$ the event that $y$ is a renewal point, and set $g(r) = E_{Q_0,r}(\tau_r)$, then the above expression is equal to

$$E_{Q_0,y} \left( \sum_{i : X_i \leq y} g(X_i - X_{i-1}) \big| A_y \right).$$

The next lemma provides a bound on the growth of $g(r)$. This bound is not optimal but it will be sufficient for our purposes as all we need to know is that $g(r)$ has subexponential growth.

**Lemma 3.4.** There are constants $M_1$ and $M_2$ such that for all $r \geq 1$

$$E_{Q_0,r}(\tau_r) \leq M_1 r^3 \quad \text{and} \quad E_{Q_0,r}(\tau_r) \leq M_2 r^3.$$

**Remark 3.2.** The last claim is not needed at this point. It will be used only in Section 5.

The law of large numbers and the renewal theorem tell us that, as $y \to \infty$,

$$\frac{1}{y} \sum_{i : X_i \leq y} g(X_i - X_{i-1}) \xrightarrow{Q_0,a.s.} \frac{\sum_{r=1}^{\infty} g(r)q(r)}{\sum_{r=1}^{\infty} rq(r)}, \quad \text{and} \quad Q_0,y(A_y) \to \frac{1}{\sum_{r=1}^{\infty} rq(r)}.$$

The above relations together with Lemma 3.4 and (3.6) allow us to conclude that

$$\lim_{y \to \infty} \frac{E_{Q_0,y}(\tau_y)}{y} = \frac{\sum_{r=1}^{\infty} g(r)q(r)}{\sum_{r=1}^{\infty} rq(r)} =: \frac{1}{v} < \infty. \quad \square$$

We note that the fact that $q(r) > 0$ for $r > 2$ and that for such $r$, $g(r) > r$ implies that $v$ is strictly less than 1.

To proceed with the proof of (1.7) we need one more auxiliary result.

**Lemma 3.5.** There are $c, C \in (0, \infty)$ such that $Q_y(X_y > r) \leq Ce^{-cr}$ for all $0 \leq r < y$.

**Proof of (1.7) in Theorem 1.2.** By Lemma 3.2 we only need to estimate the last term in (3.4). We fix an arbitrary $\varepsilon > 0$ and consider separately the expectation restricted to $\{X_y > \varepsilon y\}$ and to its complement. By Lemmas 3.4 and 3.5 we get

$$E_{Q_y} \left[ E_{Q_0,y-X_y}(\tau_{y-X_y} \mid X_y)1_{\{X_y > \varepsilon y\}} \right] \leq M_2 y^3 Q_y(X_y > \varepsilon y) \leq C M_2 y^3 e^{-cy} \to 0 \quad \text{as} \quad y \to \infty.$$

Consider now the expectation over $\{X_y \leq \varepsilon y\}$. We have

$$(3.8) \quad \frac{1}{y} E_{Q_y} \left[ E_{Q_0,y-X_y}(\tau_{y-X_y} \mid X_y)1_{\{X_y \leq \varepsilon y\}} \right] \leq E_{Q_y} \left[ E_{Q_y}(\tau_{y-X_y} \mid X_y) \frac{1}{y-X_y} 1_{\{X_y \leq \varepsilon y\}} \right].$$

By Proposition 3.1 for every $\varepsilon_1 > 0$ there is $r_0$ such that for all $y \geq r_0/(1 - \varepsilon)$ and $x \leq \varepsilon y$

$$(3.9) \quad \left| \frac{Q_{0,y-x}(\tau_{y-x})}{y-x} - \frac{1}{v} \right| \leq \varepsilon_1.$$
Thus, (3.8) is bounded above by \( v^{-1} + \varepsilon_1 \) for all \( y \geq r_0/(1 - \varepsilon) \). On the other hand, by (3.9) and Lemma 3.5 for all sufficiently large \( y \)
\[
\frac{1}{y} E_{Q_y} \left[ E_{Q_{0,y-X_y}}(\tau_y-X_y | X_y) \mathbb{I}(x_y \leq \varepsilon y) \right] \geq \frac{1}{y} \left( \frac{1}{v} - \varepsilon_1 \right) E_{Q_y} \left( (y - X_y) \mathbb{I}(x_y \leq \varepsilon y) \right) 
\geq \left( \frac{1}{v} - \varepsilon_1 \right) (1 - \varepsilon) \mathbb{P}(X_y \leq \varepsilon y) \geq \left( \frac{1}{v} - \varepsilon_1 \right) (1 - \varepsilon)^2.
\]

Since \( \varepsilon \) and \( \varepsilon_1 \) were arbitrary, this finishes the proof. \( \square \)

We close this section with a proof of (1.8). (See [Ze00] for a related result for random walks in random environment.)

**Proof of (1.8).** Lemma 2.4 implies that
\[
\frac{d\beta_{\lambda + V}(1)}{d\lambda} \bigg|_{\lambda=0} = \frac{1}{y}.
\]
It remains to show the converse inequality. We fix \( \varepsilon > 0 \). It will be enough to show that for \( \lambda \) positive and sufficiently small
\[
\frac{\beta_{\lambda + V}(1) - \beta_V(1)}{\lambda} \geq \frac{1 - \varepsilon}{v}.
\]
By (1.3) it is therefore sufficient to show that for such \( \lambda \) fixed and all large \( y \)
\[
\frac{1}{\lambda y} \left( - \log E \left( e^{-\sum_{r=0}^{s-1} (V(S_r) + \lambda)} \right) + \log E \left( e^{-\sum_{r=0}^{s-1} V(S_r)} \right) \right) \geq \frac{1 - \varepsilon}{v}.
\]
By Lemma 5.5 this reduces to proving that
\[
- \frac{1}{\lambda y} \log E_{Q_{0,y}}(e^{-\lambda \tau_y}) = \frac{1}{\lambda y} \left( - \log E \left( e^{-\sum_{r=0}^{s-1} (V(S_r) + \lambda)} \mathbb{I}(\tau_y > \tau_0) \right) 
\right)
\geq \frac{1 - \varepsilon}{v}.
\]
Thus, we need to show that for \( \lambda \) small and all sufficiently large \( y \)
\[
E_{Q_{0,y}}(e^{-\lambda \tau_y}) \leq \exp \left( -\lambda y (1 - \varepsilon)/v \right).
\]
Conditioning on the number and locations of renewal points we get
\[
E_{Q_{0,y}}(e^{-\lambda \tau_y}) = \sum_{k=1}^{y} \sum_{0 < x_1 < \cdots < x_{k} = y} \prod_{j=1}^{k} q(x_j - x_{j-1}) E_{Q_{0,x_j-x_{j-1}}}^y(e^{-\lambda \tau_{x_j-x_{j-1}}}).
\]
The dominated convergence theorem implies that for each \( r \in \mathbb{N} \), \( \varepsilon_1 > 0 \), and all sufficiently small \( \lambda \)
\[
E_{Q_{0,r}}(\frac{1 - e^{-\lambda \tau}}{\lambda}) > (1 - \varepsilon_1) E_{Q_{0,r}}^y(\tau_r)
\]
and, thus,
\[
E_{Q_{0,r}}(e^{-\lambda \tau}) < 1 - \lambda (1 - \varepsilon_1) E_{Q_{0,r}}(\tau_r) \leq e^{-\lambda (1 - \varepsilon_1) E_{Q_{0,r}}(\tau_r)}.
\]
Next, we observe that for a renewal sequence based on the kernel \( q(\cdot) \) and conditioned on having \( y \) as a renewal point, \( 0 < x_1 < \cdots < x_k = y \), there exists \( M < \infty \) and \( \varepsilon_2 > 0 \) not depending on \( y \), such that for all sufficiently large \( y \) with probability at least \( 1 - e^{-y\varepsilon_2} \)

\[
\sum_{j=1}^{k} \mathbb{1}_{\{x_j - x_{j-1} \leq M\}} E_{Q_0,x_j-x_{j-1}} (\tau_{x_j-x_{j-1}}) \geq \frac{y(1 - \varepsilon_1)}{\nu}.
\]

This statement follows from Lemma 3.3 and large deviations bounds on i.i.d. random variables conditioned on an event of probability bounded away from zero.

Now let us consider \( \lambda > 0 \) sufficiently small and such that (3.12) holds for each \( r \in \{1, 2, \ldots, M\} \). Then (3.11) is bounded above by

\[
\sum_{k=1}^{y} \sum_{0 < x_1 < \cdots < x_k = y} \prod_{j=1}^{k} q(x_j - x_{j-1}) e^{-\lambda(1-\varepsilon_1)1_{\{x_j - x_{j-1} \leq M\}}} E_{Q_0,x_j-x_{j-1}} (\tau_{x_j-x_{j-1}})
\]

\[
\leq \sum_{k=1}^{y} \sum_{0 < x_1 < \cdots < x_k = y} \prod_{j=1}^{k} q(x_j - x_{j-1}) e^{-y(1-\varepsilon_1)^2/\nu} + e^{-\varepsilon_2y}.
\]

We conclude that

\[
- \log \left( E(e^{-\sum_{r=0}^{y} -1(V(S_r)+\lambda) \mathbb{1}_{\{\tau_0 > \tau_y\}}}) \right) + \log \left( E(e^{-\sum_{r=0}^{y} -1(V(S_r) \mathbb{1}_{\{\tau_0 > \tau_y\}}}) \right)
\]

\[
\geq - \log \left( e^{-\varepsilon_2y} + e^{-y(1-\varepsilon_1)^2/\nu} \right).
\]

This gives the desired inequality for \( \lambda \) small, and we are done. \( \square \)

4. The key environment estimate

A simple but important observation is that \( Q_y \) and \( Q_{0,y} \) can be considered as measures not only on paths but also on the product space of paths and environments. Theorem 4.1 below provides key estimates on the environment under these measures. It is crucial for proofs of technical results that we used in Section 3.

Letters \( a, b, x, z, x_i \) (\( i \in \mathbb{N} \cup \{0\} \)) will always denote integers. Due to (1.7) there exist \( \varpi, K \in (0, \infty) \) such that \( \mathbb{P}(V(0, \omega) \in [\varpi, K]) > 0 \). Given an environment \( \omega \), a site \( x \in \mathbb{Z} \) will be called “reasonable” if \( V(x, \omega) \in [\varpi, K] \).

Let \( I \subset [a, b] \) be an interval and \( x_i \in I \), \( i = 1, 2, \ldots, m \), \( x_1 < x_2 < \cdots < x_m \). Define an “environment event”

\[
\Omega_I(x_1, x_2, \ldots, x_m) = \{ \omega \in \Omega | V(x_i, \omega) \in [\varpi, K] \forall i \in \{1, 2, \ldots, m\} \text{ and } V(x, \omega) \notin [\varpi, K] \forall x \in I \setminus \{x_1, x_2, \ldots, x_m\} \}.
\]

Observe that \( \Omega_{(a,b)} \) is just the event that \( V(x, \omega) \) is not reasonable for every site \( x \in (a, b) \). We shall also need measures

\[
Q_{x,y}(\cdot) := \bar{P}^x(\cdot | \tau_x^{(2)} > \tau_y, \tau_y < \infty), \quad 0 \leq x < y.
\]
Theorem 4.1. There exist constants $M_1$, $M_2$ and $\theta \in (0,1)$ not depending on $a$, $b$, $y$, or $x_i$, $i = 1, 2, \ldots, m$, $0 \leq a = x_0 < x_1 < \cdots < x_m < x_{m+1} = b \leq y$, so that

$$Q_{0,y}(\Omega_{(a,b)}(x_1, x_2, \ldots, x_m)) \leq \prod_{j=0}^{m} (M_1 \theta^{x_{j+1} - x_j});$$

(4.1)

$$Q_y(\Omega_{(a,b)}(x_1, x_2, \ldots, x_m)) \leq \prod_{j=0}^{m} (M_2 \theta^{x_{j+1} - x_j}).$$

(4.2)

This theorem is a consequence of Theorem 4.2 and Lemma 4.3 below.

Theorem 4.2. There exist constants $M_3$, $M_4 \geq 1$ and $\theta \in (0,1)$ not depending on $x$, $a$, $b$, and $y$, $0 \leq x \leq a < b \leq y$, such that

$$Q_{x,y}(\Omega_{(a,b)}) \leq M_3 \theta^{b-a};$$

(4.3)

$$Q_y(\Omega_{(a,b)}) \leq M_4 \theta^{b-a}.$$

(4.4)

Lemma 4.3. Let $0 \leq x < y$ and

$$E_{x-} \in \sigma(\{V(z, \omega)\}, z < x), \ E_{x+} \in \sigma(\{V(z, \omega)\}, z > x).$$

Then

$$Q_{0,y}(E_{x-} \cap \{V(x) \in [\kappa, K]\} \cap E_{x+}) \leq \frac{1}{1 - e^{-\kappa}} Q_{0,x}(E_{x-}) Q_{x,y}(E_{x+});$$

(4.5)

$$Q_y(E_{x-} \cap \{V(x) \in [\kappa, K]\} \cap E_{x+}) \leq \frac{1}{1 - e^{-\kappa}} Q_x(E_{x-}) Q_{x,y}(E_{x+}).$$

(4.6)

Lemma 4.3 follows easily from the decomposition of the path space according to the number of visits to a reasonable site $x$. The details are given in the Appendix. The proof of Theorem 4.2 is the main content of this section. For now we assume both statements and show how they imply Theorem 4.1.

Proof of Theorem 4.1. The proofs of (4.1) and (4.2) are identical, and we show only (4.4).

$$Q_y(\Omega_{(a,b)}(x_1, x_2, \ldots, x_m))$$

$$= Q_y(\Omega_{(a,x_m)}(x_1, \ldots, x_{m-1}) \cap \{V(x_m) \in [\kappa, K]\} \cap \Omega_{(x_m,b)})$$

$$\leq \frac{1}{1 - e^{-\kappa}} Q_{x_m}(\Omega_{(a,x_m)}(x_1, \ldots, x_{m-1})) Q_{x_m,y}(\Omega_{(x_m,b)})$$

$$\leq \left(\frac{1}{1 - e^{-\kappa}}\right)^m Q_{x_1}(\Omega_{(a,x_1)}) \prod_{j=1}^{m} Q_{x_j,x_{j+1}}(\Omega_{(x_j,x_{j+1})})$$

$$\leq \prod_{j=0}^{m} \left(\frac{M_3}{1 - e^{-\kappa}} \theta^{x_{j+1} - x_j}\right). \quad \square$$

We turn now to the proof of Theorem 4.2. It is a consequence of several simple lemmas. We derive only (4.4), the proof of (4.3) being practically the same.
Lemma 4.4. Let $0 \leq a = x_0 < x_1 < \cdots < x_m = b \leq y$ and $\delta = \mathbb{P}(V(0) \in [x, K]) \in (0, 1)$. Then

$$Q_b(\Omega_{(a,b)}(x_1, x_2, \ldots, x_{m-1})) \geq \frac{1}{2m} \left( \frac{e^{-K\delta}}{1 - \delta} \right)^{m-1} \prod_{j=1}^m \frac{1}{x_j - x_{j-1}}. \quad (4.7)$$

We postpone the proof of this lemma to record its immediate corollary.

**Corollary 4.1.** Set

$$C(a, b, \delta, K) := e^K \frac{1}{\delta} \sum_{m=1}^\infty \sum_{a=x_0 < x_1 < \cdots < x_m = b} \prod_{j=1}^m \left( \frac{e^{-K\delta}}{2(1 - \delta)} \right) \frac{1}{x_j - x_{j-1}}.$$

Then $Q_b(\Omega_{(a,b)}) \leq (C(a, b, \delta, K))^{-1}$.

The next lemma from renewal theory shows that the above inequality actually gives an exponential bound on $Q_b(\Omega_{(a,b)})$.

**Lemma 4.5.** Choose $\theta \in (0, 1)$ so that

$$\frac{e^{-K\delta}}{2(1 - \delta)} \sum_{k=1}^\infty \frac{\theta^k}{k} = 1.$$

Then there is a constant $c > 0$ such that for all $a \leq b$, $C(a, b, \delta, K) \geq c \theta^{-(b-a)}$.

**Proof.** Let $(\xi_n)_{n \geq 1}$ be random variables such that

$$f_k = P(\xi_1 = k) = \frac{e^{-K\delta}}{2(1 - \delta)} \frac{\theta^k}{k}, \quad k \in \mathbb{N},$$

and define the renewal times $T_0 = 0$, $T_m = \sum_{j=1}^m \xi_j$, $m \in \mathbb{N}$. Denote by $u_n$ the probability that $n$ is a renewal time. Then $u_0 = 1$ and

$$u_n = \sum_{k=1}^n f_k u_{n-k} > 0, \quad n \geq 1.$$

On the other hand,

$$\sum_{m=1}^\infty \sum_{a=x_0 < x_1 < \cdots < x_m = b} \prod_{j=1}^m \left( \frac{e^{-K\delta}}{2(1 - \delta)} \right) \frac{1}{x_j - x_{j-1}}$$

$$= \theta^{-(b-a)} \sum_{m=1}^\infty \sum_{a=x_0 < x_1 < \cdots < x_m = b} \prod_{j=1}^m \left( \frac{e^{-K\delta}}{2(1 - \delta)} \right) \frac{\theta^{x_j-x_{j-1}}}{x_j - x_{j-1}} = \theta^{-(b-a)} u_{b-a}.$$

By the renewal theorem ([Fe68], Ch. 13, Sec. 11), $u_n \to \mu^{-1}$ as $n \to \infty$, where $\mu = \sum_{n=1}^\infty n f_n < \infty$. This implies that $\min_{a \in \mathbb{N}} u_n > 0$, and the claim follows. \hfill \Box

**Proof of Lemma 4.4**. Denote the right hand side of (4.7) by $C_m(\bar{x})$. Let $U$ be any potential on $(a, b) \setminus \{x_1, x_2, \ldots, x_{m-1}\}$ such that $U(\bar{x}) \notin [x, K]$. Then it is enough to show that conditional on $V = U$ on $(a, b) \setminus \{x_1, x_2, \ldots, x_{m-1}\}$

$$Q_b(\Omega_{(a,b)}(x_1, x_2, \ldots, x_{m-1}) \mid U) \geq C_m(\bar{x}) Q_b(\Omega_{(a,b)} \mid U).$$
From now on assume that
\begin{equation}
\frac{E E^0 \left( e^{-\sum_{n=0}^{\tau_a} V(S_n)} \mid \Omega_{(a,b)(x_1, x_2, \ldots, x_{m-1})} \right)}{E E^0 \left( e^{-\sum_{n=0}^{\tau_a} V(S_n)} \mid U \right)} \geq C_m(\bar{x}).
\end{equation}

From now on assume that $V(x) = U(x)$ for all $x \in (a, b) \setminus \{x_1, x_2, \ldots, x_{m-1}\}$ and drop the conditioning from the notation.

Restricting the random walk expectation to those paths which on their way to $b$ hit every $x_i, i \in \{0, 1, \ldots, m - 1\}$, only once, we obtain a lower bound on the numerator of (4.8):
\[
\mathbb{E} E^0 \left( e^{-\sum_{n=0}^{\tau_a} V(S_n)} \left( \cap_{i=1}^m \{\tau_i^{(2)} > \tau_{x_i}\} \right) \cap \Omega_{(a,b)(x_1, x_2, \ldots, x_{m-1})} \right) =
\mathbb{E} E^0 \left( e^{-\sum_{n=0}^{\tau_a} V(S_n)} \left( (e^{-V(0)}; V(0) \in [x, K]) \right)^{m-1} \times \prod_{i=1}^m E^{x_{i-1}} \left( e^{-\sum_{n=\tau_{x_{i-1}} + 1}^{\tau_i^{(2)}} U(S_n)} \right) \right) \geq
\mathbb{E} E^0 \left( e^{-\sum_{n=0}^{\tau_a} V(S_n)} \left( \delta \frac{1}{eK} \right)^{m-1} \prod_{i=1}^m E^{x_{i-1}} \left( e^{-\sum_{n=\tau_{x_{i-1}} + 1}^{\tau_i^{(2)}} U(S_n)} \right) \right) =
\mathbb{E} E^0 \left( e^{-\sum_{n=0}^{\tau_a} V(S_n)} \left( \delta \frac{1}{eK} \right)^{m-1} \prod_{i=1}^m \frac{1}{2(x_i - x_{i-1})} \times \prod_{i=1}^m E^{x_{i-1}} \left( e^{-\sum_{n=\tau_{x_{i-1}} + 1}^{\tau_i^{(2)}} U(S_n)} \right) \right) \geq
\mathbb{E} E^0 \left( e^{-\sum_{n=0}^{\tau_a} V(S_n)} \left( 1 - \delta \right)^{m-1} \prod_{i=1}^m E^{x_{i-1}} \left( e^{-\sum_{n=\tau_{x_{i-1}} + 1}^{\tau_i^{(2)}} U(S_n)} \right) \right).
\]

To estimate the denominator of (4.8), we first define the following random times:

\[\sigma_i = \sup \{n \leq \tau_b : S_n = x_{i-1} \}, \quad i \in \{1, 2, \ldots, m\};\]
\[\rho_i = \inf \{n > \sigma_i : S_n = x_i \}, \quad i \in \{1, 2, \ldots, m\}.\]

The denominator of (4.8) is clearly bounded above by
\[
\mathbb{E} E^0 \left( e^{-\sum_{n=0}^{\tau_a} V(S_n)} \right) \mathbb{E} E^0 \left( e^{-\sum_{n=1}^{\rho_i} \sum_{n=\tau_{x_{i-1}} + 1}^{\tau_i^{(2)}} U(S_n)} \right) \leq
\mathbb{E} E^0 \left( e^{-\sum_{n=0}^{\tau_a} V(S_n)} \left( 1 - \delta \right)^{m-1} \prod_{i=1}^m E^{x_{i-1}} \left( e^{-\sum_{n=\tau_{x_{i-1}} + 1}^{\tau_i^{(2)}} U(S_n)} \right) \right).
\]

Dividing the lower bound on the numerator by the upper bound on the denominator of (4.8) we obtain the statement of the lemma. \qed

We summarize the results of Corollary 4.1 and Lemma 4.5.
Corollary 4.2. There are constants $C > 0$ and $\theta \in (0, 1)$ such that for all $0 \leq a < b \leq y$

$$Q_b(\Omega_{(a,b)}) \leq C \theta^{b-a}.$$  

Corollary 4.2 gives us (4.4) in the case when $b = y$.

Proof of Theorem 4.2. We would like to get a bound on $Q_y(\Omega_{(a,b)})$. Let $X = \min\{x \geq b : V(x) \in [\mathcal{X}, K]\}$. Since $Q_y$ (restricted to environment events) is absolutely continuous with respect to $\mathbb{P}$, $Q_y(X = \infty) = 0$. Using Lemma 4.3 and Corollary 4.2 we get

$$Q_y(\Omega_{(a,b)}) = \sum_{m=0}^{\infty} Q_y(\Omega_{(a,b)} \cap \{X = m\})$$

$$= \sum_{m=b}^{y-1} Q_y(\Omega_{(a,m)} \cap \{V(m) \in [\mathcal{X}, K]\}) + \sum_{m=y}^{\infty} Q_y(\Omega_{(a,b)} \cap \{X = m\})$$

$$\leq \frac{1}{1 - e^{-\lambda}} \sum_{m=b}^{y-1} Q_m(\Omega_{(a,m)}) + \sum_{m=y}^{\infty} Q_y(\Omega_{(a,y)}) \delta(1 - \delta)^{m-y}$$

$$\leq \frac{C \theta^{b-a}}{(1 - e^{-\lambda})(1 - \theta)} + C \theta^{y-a} \leq M_3 \theta^{b-a}$$

for some constant $M_3$. \hfill \Box

5. Proofs of technical lemmas

Theorem 4.1 gives us good control on environments under probability measures $Q_y$ and $Q_{0,y}$ and we are now in a position to prove Lemmas 3.5, 3.3, and 3.2.

We start with two auxiliary statements, Lemmas 5.1 and 5.2. Recall that, given $\omega \in \Omega$, a site $x \in \mathbb{Z}$ is called “reasonable” if $V(x, \omega) \in [\mathcal{X}, K]$, where $0 < \mathcal{X} < K < \infty$. In Lemma 5.1 we argue that outside of an event of exponentially small (in $y$) probability with respect to $Q_y$ or $Q_{0,y}$ there are of order $y$ reasonable sites in $(0, y)$. Lemma 5.2 shows that having so many reasonable sites in $(0, y)$ and not having a renewal in $(0, y)$ is very unlikely. These two facts easily imply Lemmas 3.5 and 3.3. Lemma 3.2 requires additional steps, and its proof takes the rest of the section.

Denote by $R_y$ the set of all reasonable sites in $\{0, 1, \ldots, y\}$ and by $|R_y|$ the number of elements in $R_y$.

Lemma 5.1. There exist $M_5$, $M_6$ and $\nu_1 \in (0, 1)$ so that for all $y > 0$ and all intervals $I \subset (0, y)$

$$(5.1) \quad Q_{0,y}(|R_y \cap I| \leq \nu_1 |I|) \leq M_5 e^{-\nu_1 |I|};$$

$$(5.2) \quad Q_y(|R_y \cap I| \leq \nu_1 |I|) \leq M_6 e^{-\nu_1 |I|}.$$  

Proof. We shall prove only (5.1). Let $I = (a, b)$, $0 \leq a < b \leq y$, and $r = b - a - 1 = |I|$. Notice that

$$\{|R_y \cap I| \leq \nu_1 r\} = \bigcup_{k=0}^{\lfloor \nu_1 r \rfloor} \bigcup_{a < x_1 < \cdots < x_k < b} \Omega_I(x_1, x_2, \cdots, x_k),$$
where for \( k = 0 \) the second union reduces to a set \( \Omega_I \). Thus, by Theorem 4.1
\[
Q_{0,y}(|R_y \cap I| \leq \nu_1|I|) \leq \sum_{k=0}^{[\nu_1 r]} \binom{r}{k} M_1^{k+1} \theta^{r+1} \leq 2 \left( \frac{r}{[\nu_1 r]} \right) M_1^{\nu_1+1} \theta^{r+1}
\]
for \( \nu_1 \) small. Applying Stirling’s formula to this bound, we have that for some universal \( C \),
\[
Q_{0,y}(|R_y \cap I| \leq \nu_1|I|) \leq CM_1 \theta^r \left( \frac{1}{1 - \nu_1} \right)^{(1-\nu_1)r} \left( \frac{M_1}{\nu_1} \right)^{\nu_1 r} < CM_1 \left( \frac{1 + \theta}{2} \right)^r
\]
for all \( r \) provided that \( \nu_1 \) is chosen sufficiently small to ensure that
\[
(1 - \nu_1)|\log(1 - \nu_1)| + \nu_1|\log \nu_1| + \nu_1 \log M_1 < \log \frac{1 + \theta}{2\theta}.
\]

For \( 0 \leq a < b \leq y \) we define \( B(a,b,y) \) to be the event that there are no renewal points in interval \((a,b)\) up to time \( \tau_y \). The event \( K(a,b,\ell) \) contains all environments with at least \( \ell \) reasonable sites in interval \((a,b)\).

**Lemma 5.2.** There exist nontrivial constants \( C_2, \nu_2 \) so that uniformly over \( \ell \) and \( 0 \leq a < b \leq y \),
\[
Q_y(K(a,b,\ell) \cap B(a,b,y)) \leq C_2 e^{-\nu_2 \ell}.
\]

Our proof is based on a coupling with a simple asymmetric random walk. We shall use the following two elementary lemmas.

**Lemma 5.3.** Let \( Y_n, n \geq 0, \) be a simple asymmetric random walk with the rightward probability \( p \in (1/2, 1) \). Let \( B(0,m) \) be the event that there are no renewal points in \((0,m)\). There exists \( c = c(p) > 0 \) so that for each \( m > 1 \),
\[
P(B(0,m)|Y_0 = 0) \leq e^{-cm}.
\]

This statement follows from much more general arguments in [Sz00] (Lemma 1.2) and [Zy09] (Proposition 4.3). Since the proof in our case is basic, we give it in the Appendix for completeness.

Denote the first \( \ell \) reasonable sites in \((a,b)\) by \( x_i, i \in \{1,2,\ldots, \ell \} \). It is easily seen that we may suppose without loss of generality that the sequence \( x_i \) extends to infinity in interval \([b, \infty)\).

**Lemma 5.4.** Let \( r \in \mathbb{N}, x_i, x_{i+1}, \ldots, x_{i+r} \) be reasonable points, and \( x_i < x_{i+1} < \cdots < x_{i+r} < y \). Then for all \( \omega \in \{ Z_{y,x_i}^{\omega,0} > 0 \} \)
\[
(a) \ Q_{y}^{\omega, x_i}(\tau_{x_{i+r}} > \tau_y^{(2)}(x_i)) \leq e^{-xr}; \quad (b) \ Q_{y}^{\omega, x_{i+r}}(\tau_{x_i} < \tau_y) \leq e^{-xr}.
\]

The proof is given in the Appendix.

**Proof of Lemma 5.2.** Fix \( r \) so that \( e^{-xr} \leq 1/3 \). We say that a reasonable site \( x_i \) is alive if \( \tau_{x_{i+r}} < \tau_y^{(2)}(x_i) \). By Lemma 5.3 and our choice of \( r \), the probability that the random walk returns to an alive site prior to \( \tau_y \) is less than 1/3. Of course, the events \( \{ x_i \text{ is alive} \} \) and
Proof of Lemma 3.3. We have by (5.1) and Lemma 5.2 that there is
\[ \lambda_0 = 0, \lambda_{i+1} = \inf\{ j \geq \lambda_i + r : \tau^{(2)}_{x_{j+r}} < \tau^{(2)}_{x_j} \} . \]

Points of \( J \) are called good points. They are our candidates for renewal points. Again we can extend good points infinitely outside interval \((a, b)\).

Let \( C(h, a, b) \) be the event that interval \((a, b)\) contains less than \( h \) good points. Denote by \( N \) a binomial random variable with parameters \([\ell/r]\) and \( 1 - e^{-\kappa} \). We first note that by the strong Markov property, given that \( \omega \) is in \( K(a, b, \ell) \cap \{ Z_0^\omega, 0 > 0 \} \),
\[ Q^0_0(0, \ell/a, b) \leq P \left( N < \frac{(1 - e^{-\kappa})\ell}{2r} \right) < e^{-\varepsilon_2 \ell}, \]
for some small universal \( \varepsilon_2 \). We now suppose that event \( C((1 - e^{-\kappa})\ell/(2r), a, b) \) does not occur, so that there are at least \( (1 - e^{-\kappa})\ell/(2r) \) good points.

By Lemma 5.4 and our choice of \( \ell \) we can couple our process \( S \) and a simple random walk, \( Y \), with rightward probability \( p = 2/3 \) so that (even though \( \tau_{x_{\lambda_i}} \) is not a stopping time) for each \( i > 0 \),
\[ \{ j \leq i : x_{\lambda_j} \text{ is visited in } (\tau^S_{x_{\lambda_i}}, \tau^S_{x_{\lambda_{i+1}}}) \text{ by } S \} \]
is a subset of
\[ \{ j \leq i : j \text{ is visited in } (\tau^Y_{\tau_i}, \tau^Y_{\tau_{i+1}}) \text{ by } Y \} \]
This coupling necessarily entails that for \( 1 \leq i \leq (1 - e^{-\kappa})\ell/(2r) \)
\[ \{ i \text{ is a renewal point for } Y \} \subset \{ x_{\lambda_i} \text{ is a renewal point for } S \} . \]
The result follows since \( K(a, b, \ell) \cap B(a, b, y) \) is contained in the union of events \( C((1 - e^{-\kappa})\ell/(2r), a, b) \) and \( \{ Y \text{ has no renewal points in } (0, (1 - e^{-\kappa})\ell/(2r)) \} \).

Now we can easily derive Lemma 3.3.

Proof of Lemma 3.3 Let \( 0 \leq r < y \). Then for some fixed \( \nu_1 \in (0, 1) \) by Lemma 5.2 and (5.2) we get
\[ Q_y(X_y > r) = Q_y(X_y > r; K(0, r, \nu_1 r)) + Q_y(X_y > r; K^c(0, r, \nu_1 r)) \leq Q_y(K(0, r, \nu_1 r) \cap B(0, r, y)) + Q_y(K^c(0, r, \nu_1 r)) \leq C_2 e^{-\nu_1 r} + M_6 e^{-\nu_1 r} . \]

We are also ready to prove Lemma 3.3.

Proof of Lemma 3.3 We have by (5.1) and Lemma 5.2 that there is \( \nu_1 \in (0, 1) \) such that
\[ Z_{0,r} = \tilde{Z}_{0,r}(K^c(0, r, \nu_1 r)) \tilde{Z}_{0,r} + \tilde{Q}_{0,r}(K(0, r, \nu_1 r)) \tilde{Z}_{0,r} \leq Q_{0,r}(K^c(0, r, \nu_1 r)) Z_{0,r} + Q_r(K(0, r, \nu_1 r) \cap B(0, r, r)) Z_r \leq (M_6 e^{-\nu_1 r} + C_2 e^{-\nu_1 r}) Z_r . \]
This proves (3.6) as \( (\log Z_r)/r \to -\beta \) as \( r \to \infty \).

We also have the following fact, which we prove in the Appendix.
Lemma 5.5. \( \lim_{y \to \infty} \frac{1}{y} \log Z_{0,y} = -\beta. \)

This result immediately implies that the power series \( \sum_{y=1}^{\infty} s^y e^{\beta y} Z_{0,y} \) has radius of convergence equal to 1. But decomposing \( Z_{0,y} \) according to its renewal points (see the denominator in (3.7)) gives

\[
\sum_{y=1}^{\infty} s^y e^{\beta y} Z_{0,y} = \sum_{k=1}^{\infty} \left( \sum_{r=1}^{\infty} q(r) s^r \right)^k
\]

from which the conclusion that \( \sum_{r=1}^{\infty} q(r) = 1 \) is immediate.

\[\square\]

Proof of Lemma 3.2. We have

\[
(5.3) \quad \frac{1}{y} E_{Q_y}(\tau_{X_y}) = \frac{1}{y} E_{Q_y} \left( \sum_{x \leq -y^{1/4}} \ell_y(x) \right) + \frac{1}{y} E_{Q_y} \left( \sum_{-y^{1/4} < x \leq X_y} \ell_y(x) \right).
\]

We shall need the following two inequalities. Their proofs can be found in the Appendix.

Lemma 5.6. Let \( B \in \sigma(\{V(x, \omega), x < 0\}) \). Then

\[
Q_y(B) \leq 2y \mathbb{P}(B).
\]

Lemma 5.7. For every \( z \leq x \leq y \), \( m \in \mathbb{Z} \), and \( \mathbb{P}-a.e. \, \omega \in \{Z^\omega_x > 0\} \)

\[
Q_y^\omega(\ell_x(z) > m) \leq P^0(\ell_x(z) > m).
\]

Now we can estimate the first term in the right hand side of (5.3).

Lemma 5.8. \( \lim_{y \to \infty} E_{Q_y} \left( \sum_{x \leq -y^{1/4}} \ell_y(x) \right) = 0. \)

Proof. First, we note that by Lemma A.3 it is enough to show that for \( R \) fixed

\[
\lim_{y \to \infty} E_{Q_y} \left( \sum_{n=0}^{\tau_y-1} 1_{\{ -Ry \leq S_n \leq -y^{1/4} \}} \right) = 0.
\]

We introduce the event

\[
A_{\delta,y} = \{ \text{# of sites in } (-y^{1/4}, 0) \text{ that are reasonable is less than } \delta y^{1/4}/2 \}.
\]

We have immediately that

\[
(5.4) \quad E_{Q_y} \left( \sum_{n=0}^{\tau_y-1} 1_{\{ -Ry \leq S_n \leq -y^{1/4} \}} \right) = E_{Q_y} \left( \sum_{n=0}^{\tau_y-1} 1_{\{ -Ry \leq S_n \leq -y^{1/4} \} A_{\delta,y}} \right) + E_{Q_y} \left( \sum_{n=0}^{\tau_y-1} 1_{\{ -Ry \leq S_n \leq -y^{1/4} \} A_{\delta,y}^c} \right).
\]
By Lemma 5.7 the first term is bounded by

\[ Q_y(A_{\delta;y})E^0\left(\sum_{n=0}^{\tau_y-1}\mathbb{1}_{\{-Ry \leq S_n \leq -y^4/4\}}\right) \]

for a universal \( C \). In the last line we used standard large deviations bounds for Bernoulli random variables (see e.g. [DZ98]) and the fact that for every \(-Ry < x < y\) the local time \( \ell_y(x) \) is stochastically dominated by a geometric random variable of parameter \((2y(R + 1))^{-1}\). Therefore, it remains to deal with the last term in (5.3). But by part (b) of Lemma 5.4 and Lemma 5.7 we have that \( E_{Q_y}(\ell_y(z)) \leq e^{-dy^4/4}/2(R + 1)y \) for each \( \omega \in A_{\delta;y} \) and each \( z, -Ry \leq z \leq -y^4/4 \).

Finally, we shall deal with the last term in (5.3).

**Lemma 5.9.** \( \lim_{y \to \infty} \frac{1}{y} E_{Q_y} \left( \sum_{-y^4/4 \leq z < X_y} \ell_y(z) \right) = 0. \)

**Proof.** Set \( T = \sum_{-y^4/4 \leq z < X_y} \ell_y(z) \). Then for each \( x \geq y^{3/4} \),

\[ Q_y(T \geq x) \leq Q_y\left(\sum_{n=0}^{\tau_y/3}\mathbb{1}_{\{S_n \geq -y^4/4\}} \geq x\right) + Q_y(X_y > x^{1/3}). \]

The first probability is dominated by the corresponding probability for an unconditioned simple random walk by Lemma 5.7 and so is bounded by \( Ce^{-cx^{1/3}} \) for suitable nontrivial \( c \) and \( C \), while by (b) and Lemma 5.2, the second term is similarly bounded.

Lemmas 5.8 and 5.9 imply Lemma 3.2 and we are done.

**APPENDIX A.**

**Proof of Lemma 5.4.** We shall give a proof of the first statement. The second one is even simpler. Recall that \( E^0 \) denotes the expectation with respect to the simple symmetric random walk measure \( P^0 \). Since \( P^0(\tau_r < \infty) = 1 \), we shall drop \( \mathbb{1}_{\{\tau_r < \infty\}} \) when appropriate. For each \( C > 0 \)

\[ E_{\bar{Q}_0,r}(\tau_r) = E_{\bar{Q}_0,r}(\tau_r; \tau_r \leq Cr^3) + E_{\bar{Q}_0,r}(\tau_r; \tau_r > Cr^3) \]

\[ \leq Cr^3 + \frac{E^0(\tau_r\mathbb{1}_{\{\tau_r > Cr^3, \tau_r < r^3(2)\}})}{\mathbb{E}E^0\left(e^{-\sum_{n=0}^{r^3-1}V(S_n,\omega)}\mathbb{1}_{\{r^3(2) > \tau_r\}}\prod_{x=1}^{r^3-1}\mathbb{1}_{\{\ell_x(2) \geq 2\}}\right)} \]

\[ \leq Cr^3 + \frac{E^0(\tau_r\mathbb{1}_{\{\tau_r > Cr^3, \tau_r < r^3\}})}{2^{-2r-1}e^{-L(2)r^2}e^{-L(3)+\Lambda(2)}}. \]

In the transition from the second to the third line, the lower bound on the denominator was obtained by choosing a particular path, which visits every \( x, 0 < x < r \), exactly
twice before hitting \( r \) except in the case where \( r \) is even when \( r - 2 \) is visited three times. The numerator in the third line is equal to

\[
E^0 \left[ (\tau_r \land \tau_{-r}) \mathbb{1}_{\{\tau_r > C r^3, \tau_{-r} < \tau_r \}} \right] \leq \left( E^0[(\tau_r \land \tau_{-r})^2] P^0(\tau_r \land \tau_{-r} > C r^3) \right)^{1/2} \leq C_0 r^2 e^{-C r}.
\]

In the last line we used two basic facts about a simple symmetric random walk: (a) \( E^0[(\tau_r \land \tau_{-r})^2] \leq C_0 r^4 \) and (b) the probability that the exit time from the strip \((-r, r)\) exceeds \(C r^3\) is bounded by \(e^{-2C r} \), where \(C' \to \infty\) as \(C \to \infty\) (this follows from the invariance principle and a compactness argument). Choosing large enough \(C\) we can ensure that \(C' > 2 \log 2 + \Lambda(2)\).

**Proof of Lemma 4.3.** We shall prove (4.5). The proof of (4.6) is the same. Let \( \mathcal{E} = \mathcal{E}_{x-} \cap \{V(x) \in [x, k] \} \cap \mathcal{E}_{x+} \), then

\[
Q_{t_0, y}(\mathcal{E}) = \frac{\mathbb{E} \mathbb{E}^0 \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_0^{(2)} > \tau_y \}} \mathbb{1}_{\{t_0(x) = m \}} \mathbb{1}_y} \right)}{\mathbb{E} \mathbb{E}^0 \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_0^{(2)} > \tau_y \}} \mathbb{1}_y} \right)} = \frac{I}{II}.
\]

We start with the numerator. Decomposing the path space according to the number of visits to \( x \) and applying the strong Markov property we get

\[
I = \sum_{m=1}^{\infty} \mathbb{E} \mathbb{E}^0 \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_0^{(2)} > \tau_y \}} \mathbb{1}_{\{t_0(x) = m \}} \mathbb{1}_y} \right)
\]

\[
= \sum_{m=1}^{\infty} \mathbb{E} \mathbb{E}^0 \left( e^{-\sum_{n=0}^{x(m)-1} V(S_n) \mathbb{1}_{\{t_0^{(2)} > \tau_{x(m)} \}} E^x \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_2^{(2)} > \tau_y \}} \mathbb{1}_y} \right)} \right)
\]

\[
\leq \sum_{m=1}^{\infty} e^{-(m-1)\kappa} \mathbb{E} \mathbb{E}^0 \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_0^{(2)} > \tau_x \}} E^x \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_2^{(2)} > \tau_y \}} \mathbb{1}_y} \right)} \right)
\]

\[
= \frac{1}{1 - e^{-\kappa}} \mathbb{E} \mathbb{E}^0 \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_0^{(2)} > \tau_x \}} \mathbb{1}_y} \right) E^x \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_2^{(2)} > \tau_y \}} \mathbb{1}_y} \right).
\]

We also have that

\[
II \geq \mathbb{E} \mathbb{E}^0 \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) - \sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_0^{(2)} > \tau_x \}} \mathbb{1}_{\{t_2^{(2)} > \tau_y \}}} \right)
\]

\[
= \mathbb{E} \mathbb{E}^0 \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_0^{(2)} > \tau_x \}}} \right) \mathbb{E} E^x \left( e^{-\sum_{n=0}^{\tau_{x}-1} V(S_n) \mathbb{1}_{\{t_2^{(2)} > \tau_y \}}} \right).
\]

Taking the ratio of the above two estimates completes the proof. \hfill \Box

**Proof of Lemma 5.3.** Consider a terminating renewal sequence \((L_i)_{i \geq 1}\) defined as follows:

\[
L_0 = 1, \quad L_{i+1} = 1 + \max_{\tau_{L_i} \leq n \leq \tau_{L_i}^{(2)}} Y_n, \quad i \geq 0.
\]

The kernel of this sequence, \(q(r)\), has a non-trivial mass at infinity, \(q(\infty) = 2p - 1\), which is equal to the probability of a random walk to never return to its current location. Moreover, it is elementary to compute that \(q(r) < e^{-cr}, \quad r \in \mathbb{N}\), for some positive \(c\).
Let $N = \min\{i \geq 0 : L_{i+1} = \infty\}$. Then $N$ has a geometric distribution with parameter $2p - 1$ and for $m > 1$

$$P(B(0, m)) = \sum_{j=1}^{\infty} P(L_j \geq m \mid N = j)P(N = j)$$

$$\leq e^{-\varepsilon(m-1)} \sum_{j=1}^{\infty} E(e^{\varepsilon(L_j - L_0)} \mid N = j)P(N = j) \leq e^{-\varepsilon(m-1)} \sum_{j=1}^{\infty} (K(\varepsilon))jP(N = j),$$

where $K(\varepsilon) := E(e^{\varepsilon(L_1 - L_0)} \mid L_1 < \infty) \to 1$ as $\varepsilon \to 0$. This implies the statement of the lemma.

\textbf{Lemma A.1.} Let $z \in \mathbb{Z}$ and $\omega \in \{V(z, \omega) \geq \varkappa\} \cap \{Z_y^\omega > 0\}$. Then

$$Q_y^\omega(\ell_y(z) > 1) \leq e^{-\varkappa}.$$

\textbf{Proof.} This is an obvious statement, which says that to return to $z$ it is necessary not to get absorbed when leaving $z$ after the first visit. Formally, using the strong Markov property of the killed random walk we get

$$Q_y^\omega(\ell_y(z) > 1) = \hat{P}_0^\omega(\ell_y(z) > 1 \mid \tau_y < \infty) \leq \frac{P_0^\omega(\tau_y^{(2)} < \tau_y \mid \tau_y < \infty, \tau_y < \infty) = \hat{P}_z^\omega(\tau_y^{(2)} < \tau_y \mid \tau_y < \infty)}{P_z^\omega(\tau_y < \infty)} \leq e^{-\varkappa}. \quad \Box$$

\textbf{Proof of Lemma 5.4.} Part (a) follows from Lemma A.1. For (b), by the Markov property we have

$$Q_y^\omega(x_{i+1} < \tau_y) = \hat{P}_0^\omega(x_{i+1} < \tau_y) \frac{P_0^\omega(x_{i+1} < \infty \mid \tau_y < \infty)}{\hat{P}_0^\omega(x_{i+1} < \tau_y)} \leq e^{-\varkappa}.$$

\textbf{Proof of Lemma 5.5.} Since $Z_{0,y} \leq Z_y$, we only need to show that

$$\liminf_{y \to \infty} \frac{\log Z_{0,y}}{y} \geq -\beta.$$ 

By comparison with a simple symmetric random walk (see Lemma 5.1), we have that for $\mathbb{P}$-a.e. $\omega \in \{Z_y^\omega > 0\}$

$$Q_y^\omega(\tau_y < \tau_0^{(2)}) \geq P^0(\tau_y < \tau_0^{(2)}) = \frac{1}{2y}.$$ 

Therefore,

$$Z_{0,y} = Q_y(\tau_y < \tau_0^{(2)})Z_y = E(Q_y^\omega(\tau_y < \tau_0^{(2)})Z_y^\omega) \geq \frac{Z_y}{2y}.$$ 

This finishes the proof. \Box
Proof of Lemma 5.7. This is a consequence of the strong Markov property of the killed random walk.

\[ Q_y(\ell_x(z) > m) = \]
\[ (Z_y^\omega)^{-1} E^0 \left( e^{-\sum_{n=0}^{\tau_y-1} V(S_n, \omega)} \mid \tau_x^{(m+1)} < \tau_x \right) P^0(\tau_x^{(m+1)} < \tau_x) \leq \]
\[ (Z_y^\omega)^{-1} E^0 \left( e^{-\sum_{n=0}^{\tau_y-1} V(S_n, \omega)} E^0 \left( e^{-\sum_{n=\tau_x^{(m+1)}}^{\tau_y-1} V(S_n, \omega)} \mid \tau_x^{(m+1)} < \tau_x \right) \right) \]
\[ \times P^0(\ell_x(z) > m) = P^0(\ell_x(z) > m). \] 

The following follows from basic properties of simple random walks.

Lemma A.2. Let \( 0 \leq x_1 < x_2 \leq y \) and \( \zeta = \max\{n : 0 \leq n \leq \tau_y, S_n = x_1\} \). Then
\[ (A.1) \]
\[ E^0 \left( e^{-\sum_{n=x_2}^{x_1-1} V(S_n, \omega)} \right) = E^{x_1} \left( e^{-\sum_{n=x_2}^{x_1-1} V(S_n, \omega)} \mid \tau_x^{(2)} > \tau_x \right). \]

Proof of Lemma 5.6. Let \( \zeta = \max\{n : 0 \leq n \leq \tau_y, S_n = 0\} \). Then
\[ Q_y(B) = \frac{E \left( E \left( e^{-\sum_{n=x_1}^{\tau_y-1} V(S_n, \omega)} \mid \tau \right) \right) \mathbb{P}(B)}{E \left( E \left( e^{-\sum_{n=x_1}^{\tau_y-1} V(S_n, \omega)} \mid \tau \right) \right) \mathbb{P}(B)} = \frac{A \mathbb{1}}{2y \mathbb{P}(B)}. \]

Finally, we turn to the expected (with respect to \( Q_y \)) total time spent by the random walk below \( -R \). Whenever the event \( A \) depends only on the random walk, we can rewrite \( Q_y(A) \) as follows:
\[ (A.2) \]
\[ Q_y(A) = \frac{E^0 \left( \mathbb{1}_A e^{-\sum_{z>y} \Lambda_V(\ell_y(z))} \right)}{E^0 \left( e^{-\sum_{z>y} \Lambda_V(\ell_y(z))} \right)}, \] where \( \Lambda_V(t) := -\log \mathbb{E} e^{-t V(0)}. \)

Lemma A.3. For each \( R > \beta_V(1)/\Lambda_V(1) - 1 \) there is a \( c > 0 \) such that for all sufficiently large \( y \)
\[ (A.3) \]
\[ E_{Q_y} \left( \sum_{z<-Ry} \ell_y(z) \right) \leq e^{-cy}. \]
Proof. Let \( R > \beta_V(1)/\Lambda_V(1) - 1 \). Choose \( \varepsilon > 0 \) so that \( \Lambda_V(1)(R + 1) > \beta_V(1) + \varepsilon \). Since \( Z_y \geq e^{-(\beta_V(1)+\varepsilon/2)y} \) for all sufficiently large \( y \), we get

\[
E_{Q_y}\left( \sum_{z < -Ry} \ell_y(z) \right) = \sum_{z < -Ry} \sum_{m=1}^{\infty} Q_y(\ell_y(z) \geq m)
= \frac{1}{Z_y} \sum_{z < -Ry} \sum_{m=1}^{\infty} E^0 \left( 1_{\{\ell_y(z) \geq m\}} e^{-\sum_{x<y} \Lambda_V(\ell_y(x))} \right)
\leq e^{(\beta_V(1)+\varepsilon/2)y} \sum_{z < -Ry} \sum_{m=1}^{\infty} E^0 \left( 1_{\{\ell_y(z) \geq m\}} e^{-\sum_{x<y} \Lambda_V(\ell_y(x))} \right)
\leq e^{(\beta_V(1)+\varepsilon/2)y} \sum_{z < -Ry} e^{-\Lambda_V(1)(|z|+y-1)} \sum_{m=1}^{\infty} P^0(\ell_y(z) \geq m) e^{-\Lambda_V(m)}.
\]

Since for \( z < 0 \)

\[
P^0(\ell_y(z) \geq m) = \left( -\frac{1}{2(y + |z|)} \right)^{m-1} \frac{1}{y + |z|},
\]

we obtain

\[
\sum_{m=1}^{\infty} P^0(\ell_y(z) \geq m) e^{-\Lambda_V(m)} = \frac{y}{y + |z|} \sum_{m=1}^{\infty} E e^{-mV} \left( 1 - \frac{1}{2(y + |z|)} \right)^{m-1}
= \frac{y}{y + |z|} E \left( \frac{e^{-V}}{1 - e^{-V}} \frac{e^{-1}}{1 - \frac{1}{2(y + |z|)}} \right) = E \left( \frac{2y}{2(e^V - 1)(y + |z|) + 1} \right) \leq 2y.
\]

Therefore,

\[
E_{Q_y}\left( \sum_{z < -Ry} \ell_y(z) \right) \leq 2y e^{(\beta_V(1)+\varepsilon/2)y} \sum_{z < -Ry} e^{-\Lambda_V(1)(|z|+y-1)}
= 2y e^{(\beta_V(1)+\varepsilon/2-\Lambda_V(1)(R+1))y} \sum_{x=0}^{\infty} e^{-\Lambda_V(1)x}
= 2y e^{(\beta_V(1)+\varepsilon/2-\Lambda_V(1)(R+1))y} \frac{1}{1 - e^{-\Lambda_V(1)}}
\leq e^{-(\Lambda_V(1)(R+1)-\beta_V(1)-\varepsilon)y}
\]

for all sufficiently large \( y \). \( \square \)

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