Bi-orthogonal Polynomials and the Five parameter
Asymmetric Simple Exclusion Process

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Abstract

We apply the bi-moment determinant method to compute a representation of the
matrix product algebra – a quadratic algebra satisfied by the operators \( d \) and \( e \) – for
the five parameter \((\alpha, \beta, \gamma, \delta \) and \( q \)) Asymmetric Simple Exclusion Process. This
method requires an \( LDU \) decomposition of the “bi-moment matrix”. The decom-
position defines a new pair of basis vectors sets, the ‘boundary basis’. This basis is
defined by the action of polynomials \( \{P_n\} \) and \( \{Q_n\} \) on the quantum oscillator basis
(and its dual). Theses polynomials are orthogonal to themselves (ie. each satisfy a
three term recurrence relation) and are orthogonal to each other (with respect to the
same linear functional defining the stationary state). Hence termed ‘bi-orthogonal’.
With respect to the boundary basis the bi-moment matrix is diagonal and the rep-
resentation of the operator \( d + e \) is tri-diagonal. This tri-diagonal matrix defines
another set of orthogonal polynomials very closely related to the the Askey-Wilson
polynomials (they have the same moments).

Keywords: Askey-Wilson polynomials, bi-orthogonal polynomials, orthogonal poly-
nomials, totally asymmetric simple exclusion process, \( LDU \)-decomposition, diffusion
algebra, quadratic algebra.

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1 Introduction

The ASEP is a continuous time Markov process defined by particles hopping along a line of $L$ sites – see Figure 1. Particles hop on to the line on the left (resp. right) with rates $\alpha$ (resp. $\delta$), off at the right (resp. left) with rate $\beta$ (resp. $\gamma$) and they hop to neighbouring sites to the left with rate $q$ and rate one to the right with the constraint that only one particle can occupy a site.

The matrix product Ansatz [1] expresses the stationary distribution of a given state as an inner product on a certain product of matrices $D$ and $E$ which satisfy the relation

$$DE - qED - D - E = 0$$

and requires two vectors $\langle W \mid$ and $\mid V \rangle$ which satisfy

$$\langle \beta D - \delta E - 1 \rangle \mid V \rangle = 0,$$
$$\langle W \mid (\alpha E - \gamma D - 1) = 0.$$

We will call $\langle W \mid$ and $\mid V \rangle$ the boundary vectors. The vectors $\langle W \mid$ and $\mid V \rangle$ are used to define a linear functional which maps any (non-commutative) polynomial, $p(D, E)$ in the matrices $D$ and $E$ to the set of (commuting) polynomials, $\mathbb{Z}[\alpha, \beta, \gamma, \delta, q]$, via

$$\langle W [p(D, E)] V \rangle \in \mathbb{Z}[\alpha, \beta, \gamma, \delta, q].$$

The representations of the matrices $D$ and $E$ fall into three natural cases;

Two parameter: $\alpha, \beta; \quad q = \gamma = \delta = 0$,
Three parameter: $q, \alpha, \beta; \quad \gamma = \delta = 0$,
Five parameter: $q, \alpha, \beta, \gamma, \delta$.

The two parameter case is (algebraically) simple. The three parameter case has been studied in [2]. In this paper we apply the method introduced in [2] to the five parameter case.
case. This generalisation is not a simple extension of the three parameter case – several new difficulties appear. The more important ones are discussed in the Concluding Remarks section.

For the five parameter case, new parameters are defined from the five hopping parameters, leading to the following change in the set of parameters

\[
a = \frac{1}{2\alpha} (1 - q - \alpha + \gamma + \sqrt{(1 - q - \alpha + \gamma)^2 + 4\alpha\gamma}),
\]

\[
c = \frac{1}{2\alpha} (1 - q - \alpha + \gamma - \sqrt{(1 - q - \alpha + \gamma)^2 + 4\alpha\gamma}),
\]

\[
b = \frac{1}{2\beta} (1 - q - \beta + \delta + \sqrt{(1 - q - \beta + \delta)^2 + 4\beta\delta}),
\]

\[
d = \frac{1}{2\beta} (1 - q - \beta + \delta - \sqrt{(1 - q - \beta + \delta)^2 + 4\beta\delta}).
\]

This change is motivated by the parameters that occur in the Askey-Wilson polynomials \[3\] discussed further below.

Rather than using \(D\) and \(E\) the algebra is simplified by working with the standard shifted variables,

\[
d = q'D - 1,
\]

\[
e = q'E - 1,
\]

where \(q' = 1 - q\). In these variables the commutation relations of \(d\) and \(e\) become \[4\],

\[
\text{Two parameter: } \quad de = 1,
\]

\[
\text{Three and Five parameter: } \quad de - q'ed = q'.
\]

and (1.1) can be written in the form,

\[
(d + bde - (b + d)1)|V\rangle = 0,
\]

\[
(W)|e + acd - (a + c)1\rangle = 0.
\]

Each matrix representation of \(d\) and \(e\) is associated with a basis for the vector space upon which the matrices act. The standard quantum oscillator basis is the set \(\{ |n\rangle : n \geq 0 \}\). If the linear operators \(d\) and \(e\) in the respective cases are defined by their action on the basis vectors \(|n\rangle\) by:

\[
\text{Two parameter: } |n + 1\rangle = e|n\rangle,
\]

\[
d|n\rangle = |n - 1\rangle, \quad d|0\rangle = 0.
\]

\[
\text{Three and Five parameter: } |n + 1\rangle = e|n\rangle,
\]

\[
d|n\rangle = (1 - q^n)|n - 1\rangle, \quad d|0\rangle = 0.
\]

then it is simple to show that \(de = 1\) (two parameter) and \(de - q'ed = q'\) (three and five parameter respec.). Thus the basis \(\{|n\rangle\}\), in conjunction with the action of \(d\)
and e above, gives the standard, \[5\], matrix representation for d and e which satisfy the appropriate commutation relations. In this representation the matrix \(d + e\) is tri-diagonal and for the three and five parameter cases gives a three term recurrence related to \(q\)-Hermite polynomials \[6\].

To find the vector \(|V\rangle\) (respec. \(\langle W|\)) there are (at least) two approaches. The first is to express \(|V\rangle\) (respec. \(\langle W|\)) as a linear combination of the standard basis vectors ie. \(|V\rangle = \sum_n a_n |n\rangle\) (respec. \(\langle W| = \sum_n b_n \langle n|\)), and then compute the coefficient \(a_n\) (respec. \(b_n\)). For example, in the three parameter case this leads to

\[
|V\rangle = a_0 \sum_{n \geq 0} \frac{(q'/\beta - 1)^n}{\prod_{k=1}^n (1 - q^k)} |n\rangle.
\]

The second approach (used in this paper) is to find a new pair of bases \(\{|V_n\rangle : n \geq 0\}\) and \(\{\langle W_n| : n \geq 0\}\) such at \(|V\rangle = |V_0\rangle\) and \(\langle W| = \langle W_0|\). This pair of bases are constructed such that the “bimoment matrix”, \(B\), is diagonal. The matrix elements of \(B\) are defined by a linear functional \(L\) (see Definition \[1\]) via

\[
B_{n,m} = L(d^n e^m).
\]

We will call \(\{|V_n\rangle\}\) and \(\{\langle W_n|\}\) the boundary basis. This method of finding a basis (and hence representation) reduces to computing determinants and ultimately to finding an \(LDU\) decomposition.

Representations of the \(d\) and \(e\) matrices for the five parameter model can be found in \[6\] (and references therein), with one of those representations reproduced in \[1.17a]. If the matrices associated with a given representation have sufficiently simple structure (eg. bi- or tri-diagonal) then they can be usefully interpreted as transfer matrices for lattice path models \[7\]. This leads to combinatorial methods for computing the inner product (or linear functional, \(L\)).

The primary objective of this paper is to the find the change of basis associated with the five parameter model representation obtained by Uchiyama et. al. \[6\] where the tri-diagonal matrix \(d + e\) gives a three term recurrence related to the Askey-Wilson polynomials. As will be shown this change of basis is affected by the action of sequences of polynomials (with matrix argument) acting on the boundary vectors.

The Askey-Wilson polynomials play a prominent role in the representation of the \(d\) and \(e\) matrices. The polynomials also motivate the \(a, b, c\) and \(d\) choice of parameters (rather than the hopping rates) defined above and several other choices defined below. We thus briefly discuss the Askey-Wilson polynomials.

The Askey-Wilson polynomials \[8\] are ‘\(q\)-orthogonal’ polynomials with four parameters, \(a, b, c\) and \(d\) (and \(q\)). They are at the top of the Askey-scheme of \(q\)-orthogonal, one variable polynomials. The basic hypergeometric functions, \(r\phi_s\), give a compact expression for the Askey-Wilson polynomial, \(W_n(x) = W_n(x; a, b, c, d|q), n > 0\), which is given by

\[
W_n(x) = a^{-n}(ab, ac, ad; q)_n 4\phi_3 \left[\begin{array}{c}
q^{-n}, & q^{n-1}abcd, & ae^{i\theta}, & ae^{-i\theta} \\
ab, & ac, & ad & ; q, q
\end{array}\right]
\]

(1.9)
with \( x = \cos \theta \) and the basic hypergeometric function is

\[
\phi_r \left[ a_1, \ldots, a_r; b_1, \ldots, b_s; q, z \right] = \sum_{k=0}^\infty \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s, q; q)_k} \left( (-1)^k q^{\binom{k}{2}} \right)^{1+s-r} z^k
\]  

(1.10)

where the \( q \)-shifted factorial is

\[
(a_1, a_2, \ldots, a_s; q)_n = \prod_{r=1}^{s} \prod_{k=0}^{n-1} (1 - a_r q^k).
\]  

(1.11)

The Askey-Wilson polynomial satisfies a three-term recurrence relation

\[
A_n W_{n+1}(x) + B_n W_n(x) + C_n W_{n-1}(x) = 2x W_n(x),
\]  

(1.12)

with \( W_0(x) = 1 \), \( W_{-1}(x) = 0 \) and

\[
A_n = \frac{1 - q^{n-1}abcd}{(1 - q^{2n-1}abcd)(1 - q^{2n}abcd)},
\]  

(1.13)

\[
B_n = \frac{q^{n-1}}{(1 - q^{2n-2}abcd)(1 - q^{2n}abcd)} [(1 + q^{2n-1}abcd)(qs + abcds') - q^{n-1}(1 + q)abcd(s + qs')],
\]  

(1.14)

\[
C_n = \frac{(1 - q^n)(1 - q^{n-1}ab)(1 - q^{n-1}ac)(1 - q^{n-1}ad)}{(1 - q^{2n-1}abcd)} \times \frac{(1 - q^{n-1}bc)(1 - q^{n-1}bd)(1 - q^{n-1}cd)}{(1 - q^{2n-2}abcd)}
\]  

(1.15)

and

\[
s = a + b + c + d, \quad s' = a^{-1} + b^{-1} + c^{-1} + d^{-1}.
\]  

(1.16)

Uchiyama et al. [6] found a representation of \( d \) and \( e \) related to the Askey-Wilson polynomials. The matrices are tridiagonal and given by

\[
d = \begin{pmatrix}
  d_0^0 & d_0^0 & 0 & \cdots \\
  d_0^r & d_1^r & d_1^r & \cdots \\
  0 & d_2^r & d_2^r & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]  

and

\[
e = \begin{pmatrix}
  e_0^0 & e_0^0 & 0 & \cdots \\
  e_0^r & e_1^r & e_1^r & \cdots \\
  0 & e_2^r & e_2^r & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]  

(1.17a)
where

\begin{align*}
\frac{d_n'}{q^{n-1}} &= \frac{1}{(1 - q^{2n-2}abcd)(1 - q^{2n}abcd)} \\
&\cdot [bd(a + c) + (b + d)q - abcd(b + d)q^{n-1} - \{bd(a + c) + abcd(b + d)\}q^n \\
&- bd(a + c)q^{n+1} + ab^2cd^2(a + c)q^{2n-1} + abcd(b + d)q^{2n}], \\
\frac{e_n'}{q^{n-1}} &= \frac{1}{(1 - q^{2n-2}abcd)(1 - q^{2n}abcd)} \\
&\cdot [ac(b + d) + (a + c)q - abcd(a + c)q^{n-1} - \{ac(b + d) + abcd(a + c)\}q^n \\
&- ac(b + d)q^{n+1} + a^2bc^2d(b + d)q^{2n-1} + abcd(a + c)q^{2n}],
\end{align*}

(1.17b)

\begin{align*}
A_n &= \left[ \frac{(1 - abcdq^{n-1})(1 - q^{n+1})(1 - abq^n)(1 - bcq^n)(1 - acl)n)(1 - cdq^n)}{(1 - abcdq^{2n-1})(1 - abcdq^{2n})^2(1 - abcdq^{2n+1})} \right]^{1/2}, \quad (1.17d)
\end{align*}

and

\begin{align*}
\frac{d_n^*}{1 - q^nac} &= A_n, \quad \frac{e_n^*}{1 - q^nac} = A_n, \\
\frac{d_n^*}{1 - q^nbd} &= -A_n, \quad \frac{e_n^*}{1 - q^nbd} = A_n.
\end{align*}

(1.17e)

We have introduced the parameters, \(d_n^*, d_n^*, e_n^*, e_n^*\) as they will reoccur in computations in the rest of this paper.

2 The Linear Functional

In this section we set up the tensor algebra used to represent the ASEP [11]. Let \(\mathcal{R}\) be the ring of integer coefficient commutative polynomials, \(\mathbb{Z}[\alpha, \beta, \gamma, \delta, q]\) and \(\mathcal{M}\) the \(\mathcal{R}\)-module (or tensor algebra)

\[ \mathcal{M} = \bigoplus_{n \geq 0} \mathcal{V}_2^\otimes n \]  

(2.1)

where \(\mathcal{V}_2\) is a free rank two \(\mathcal{R}\)-module with generators \(d\) and \(e\). Here \(\mathcal{V}_2^\otimes n\) denotes the ring \(\mathcal{R}\) of the module and \(\mathcal{V}_2^\otimes n = \mathcal{V}_2 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_2\) \((n\ \text{factors})\). The homogeneous submodule \(\mathcal{V}_2^\otimes n\), of degree \(n\), is generated by the standard monomial basis elements \(e_i \otimes e_i \otimes \cdots \otimes e_i\) where \(e_i \in \{d, e\}\). For brevity we will frequently omit the tensor product symbol, thus \(d^m e^n\) denotes \(d^m \otimes e^n\) etc.

We use the five parameter version of the original matrix Ansatz algebra equations of Derrida et al. [1] as modified in [12]. The latter form allows for arbitrary monomial pre- and post-factors \((u\ \text{and} v\ \text{in the equations below})\). The original algebra (in [1]) was stated in terms of matrices and vectors. Here we give a slightly more abstract version by using a linear functional in terms of \(d\) and \(e\).
Definition 1. Let \( u, v \) be any monomial basis elements of \( \mathcal{M} \). The \( \mathcal{R} \)-module homomorphism \( \mathcal{L} : \mathcal{M} \to \mathcal{R} \) is defined by the following equations:

\[
\mathcal{L}(u \otimes (d \otimes e - q e \otimes d - q') \otimes v) = 0, \\
\mathcal{L}(u \otimes (d + bde - (b + d)d)1) = 0, \\
\mathcal{L}((e + acd - (a + c)d) \otimes v) = 0,
\]

where \( a, b, c \) and \( d \) are defined in (1.3), \( \mathcal{L}(1) = 1 \) and extended linearly to other elements of \( \mathcal{M} \).

The matrix product Ansatz of [1] for the stationary state can now be (trivially) restated using the linear functional \( \mathcal{L} \).

Theorem 1 (Derrida, Evans, Hakim and Pasquier [1]). The stationary state probability distribution, \( f(\tau) \), of the five parameter ASEP for the system in state \( \tau = (\tau_1, \ldots, \tau_L) \), is given by

\[
f(\tau) = \frac{1}{Z_L} \mathcal{L} \left( \prod_{i=1}^{L} (\tau_i d + (1 - \tau_i)e) \right)
\]

where

\[
Z_L = \mathcal{L}((d + e)^L)
\]

and \( \tau_i = 1 \) if site \( i \) is occupied and zero otherwise.

3 Bi-Orthogonal Pair of Polynomial Sequences

In this section we define a pair of polynomials sequences. These polynomials are then used to construct the boundary basis which leads to a matrix representation of \( d \) and \( e \).

Consider the pair of sequences,

\[
\{P_n(d)\}_{n \geq 0} \quad \text{and} \quad \{Q_n(e)\}_{n \geq 0}
\]

of monic polynomials where \( P_n \) and \( Q_n \) are degree \( n \). We wish to determine if it is possible to find such a pair which are orthogonal with respect to \( \mathcal{L} \) (as defined in Definition 1), that is \( \mathcal{L}(P_n Q_m) = \Lambda_n \delta_{n,m}, \Lambda_n \neq 0? \)

In order to show such a pair of sequences does indeed exist we consider the infinite dimensional ‘bimoment matrix’, \( \mathbf{B} \), whose matrix elements are defined to be

\[
\mathbf{B}_{n,m} = \mathcal{L}(d^n e^m), \quad n, m \geq 0.
\]

The bimoment matrix elements satisfy a pair of partial difference equations as given in the following theorem.
Theorem 2. The bimoment matrix elements, (3.2), satisfy the recursions

\[ B_{i,j} = (1 - q^i)B_{i-1,j-1} + (a + c)q^iB_{i,j-1} - acq^iB_{i+1,j-1}, \]  
\[ B_{i,j} = (1 - q^i)B_{i-1,j-1} + (b + d)q^jB_{i-1,j} - bdq^jB_{i,j+1}, \]  
\[ B_{i,j} = (1 - q^i)B_{i-1,j-1} + (a + c)q^iB_{i,j-1} - acq^iB_{i+1,j-1}, \]  
for \( i, j > 0 \) with boundary values \( B_{0,j} \) and \( B_{i,0} \), \( i, j \geq 0 \),

\[ B_{i,0} = \frac{(b + d - bd(a + c)q^{i-1})B_{i-1,0} - bd(1 - q^{i-1})B_{i-2,0}}{1 - abcdq^{i-1}}, \]  
\[ B_{0,j} = \frac{(a + c - ac(b + d)q^{j-1})B_{0,j-1} - ac(1 - q^{j-1})B_{0,j-2}}{1 - abcdq^{j-1}} \]

and \( B_{0,0} = 1 \).

Note, the bimoment matrix elements satisfy both (3.3a) and (3.3b), however, to generate the matrix elements it is sufficient to use only one of (3.3a) or (3.3b) together with both boundary recurrences. The reason both (3.3a) and (3.3b) are stated is that it makes explicit a symmetry of the matrix which we will use below.

Proof. The idea of the proof is similar to [2]. However, eliminating an \( e \) (resp. \( d \)) from the left (resp. right) side of is more complicated due to the more complicated boundary vector equations (1.6). Using (2.2c) (resp. (2.2b)), an \( e \) (resp. \( d \)) on the left (resp. right) side of a monomial can be removed giving,

\[ \mathcal{L}(ed^n e^{m-1}) = (a + c)\mathcal{L}(d^n e^{m-1}) - ac\mathcal{L}(d^{n+1} e^{m-1}), \]  
\[ \mathcal{L}(d^{n-1} e^m d) = (b + d)\mathcal{L}(d^{n-1} e^m) - bd\mathcal{L}(d^{n-1} e^{m+1}). \]

Similar to the proof in [2], commuting an \( e \) all the way to the left and then eliminating the \( e \) on the left using (3.5) gives the following recurrence (3.3a),

\[ \mathcal{L}(d^n e^m) = (1 - q^n)\mathcal{L}(d^{n-1} e^{m-1}) + q^n\mathcal{L}(ed^n e^{m-1}) \]
\[ = (1 - q^n)\mathcal{L}(d^{n-1} e^{m-1}) + (a + c)q^n\mathcal{L}(d^n e^{m-1}) \]
\[ - acq^n\mathcal{L}(d^{n+1} e^{m-1}). \]  
\[ \mathcal{L}(d^n e^m) = (1 - q^n)\mathcal{L}(d^{n-1} e^{m-1}) + q^n\mathcal{L}(d^{n-1} e^m d) \]
\[ = (1 - q^n)\mathcal{L}(d^{n-1} e^{m-1}) + (b + d)q^n\mathcal{L}(d^{n-1} e^m) \]
\[ - bdq^n\mathcal{L}(d^{n-1} e^{m+1}). \]  

By rearranging (2.2b) and (2.2c), an \( e \) can be eliminated from the left and a \( d \) eliminated from the right, as distinct from the 3-parameter case, giving,

\[ \mathcal{L}(d^n e) = \frac{1}{bd} \left((b + d)\mathcal{L}(d^n) - \mathcal{L}(d^{n+1})\right), \]  
\[ \mathcal{L}(de^m) = \frac{1}{ac} \left((a + c)\mathcal{L}(e^m) - \mathcal{L}(e^{m+1})\right). \]
Finally, using the recursions (3.3a) and (3.3b) for \( m = 1 \) and \( n = 1 \) gives the following expression in terms of the boundary values,

\[
\mathcal{L}(d^n e) = (1 - q^n) \mathcal{L}(d^{n-1}) + (a + c)q^n \mathcal{L}(d^n) - acq^n \mathcal{L}(d^{n+1}),
\]

(3.11)

\[
\mathcal{L}(de^n) = (1 - q^n) \mathcal{L}(e^{n-1}) + (b + d)q^n \mathcal{L}(e^n) - bdq^n \mathcal{L}(e^{n+1}).
\]

(3.12)

Combining these results gives a recurrence for the boundary value terms.

The existence of the pair of polynomial sequences (3.1) requires that the determinant of the \((n + 1) \times (n + 1)\) sub-matrix \( B(n) = (B_{i,j})_{0 \leq i,j \leq n} \) be non-zero for all \( n \geq 0 \) (see [2] for further details).

In the case of the three parameter model the corresponding determinant was evaluated using theorems from [13] and [14]. In this five parameter case we have been unable to find any similar theorems and thus attempted an \( LDU \) decomposition directly.

For small values of \( n \) the determinant, \( \det B(n) \) can be found by computer by iterating the recurrence relations (3.3a) to construct \( B \). From these values a product form for \( \det B(n) \) (stated below in (3.27)) can be conjectured. It is similarly possible to conjecture the \( LDU \) decomposition of \( B \), that is, find upper and lower triangular matrices, \( U \) and \( L \) respectively, such that

\[
B = LDU
\]

(3.13)

(with \( D \) diagonal). The product of the first \( n \) diagonal elements of \( D \) then gives the determinants, \( \det B(n) \). These small \( n \) computations lead us to define a lower triangular matrix \( L \) via a recurrence relation for the matrix elements \( L_{i,j} \) given by

\[
L_{i,j} = \begin{cases} 
L_{i-1,j-1} + d'_j L_{i-1,j} - bdq^j g_j L_{i-1,j+1} & \text{for } i, j \geq 0 \text{ and } i \geq j, \\
1 & \text{for } i = j = 0, \\
0 & \text{otherwise}, 
\end{cases}
\]

(3.14)

where

\[
g_j = \frac{(1 - abcdq^{j-1})(1 - q^{j+1})(1 - abq^j)(1 - bcq^j)(1 - adq^j)(1 - cdq^j)}{(1 - abcdq^{2j-1})(1 - abcdq^{2j+1})}. \tag{3.15}
\]

For small values of \( n \) this matrix (and a similar one for \( U \)) give the \( LDU \) decomposition of \( B \), but unfortunately we have not been able to prove the decomposition for arbitrary \( n \). Thus we make the following conjecture.

**Conjecture 1.** The bimoment matrix, (3.2), has an \( LDU \) decomposition with lower triangular matrix, \( L \), given by (3.14).

We make two remarks: first, one of the final results of this conjecture is a representation for \( d \) and \( e \). Having obtained a candidate representation it is then straightforward
to verify that is a representation by substituting back into the defining algebra – Definition\[1\]. This has been done and hence the representation verified. Assuming the logic of the calculation can be reversed, that would prove the conjecture. However, one of the primary aims of this paper is to derive the representation and thus from this perspective is more satisfactory if the conjecture be proved directly.

Secondly, it is only necessary to conjecture $L$ as, assuming (3.14) is valid, we can compute the corresponding recurrence relations for the upper triangular matrix $U$ using a symmetry of $B$. The bimoment matrix is invariant when taking the transpose and performing the substitutions $a \leftrightarrow b, c \leftrightarrow d$ or $a \leftrightarrow d, b \leftrightarrow c$. Under this action, the equations (3.3a) and (3.3b) swap as do the equations (3.4a) and (3.4b). Thus the upper triangular matrix can be obtained from the lower triangular matrix by performing these substitutions.

**Corollary 1.** Assuming Conjecture 1 is true. The the upper triangular matrix elements $U_{i,j}$ of the $LDU$ decomposition of the bimoment matrix are given by the recurrence relation

\[
U_{i,j} = \begin{cases} 
U_{i-1,j-1} + e_j'U_{i,j-1} - acq_jg_iU_{i+1,j-1} & \text{for } i, j \geq 0 \text{ and } i \leq j, \\
1 & \text{for } i = j = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

(3.16)

The diagonal matrix $D$ of the $LDU$ decomposition can be calculated using the inverse of the $L$ matrix.

**Corollary 2.** Assuming Conjecture 1 is true. Let $L$ be the lower triangular matrix of the $LDU$ decomposition of the bimoment matrix $B$. Then the elements $L_{i,j}^{-1}$ of the inverse of $L$ satisfy

\[
L_{i,j}^{-1} = \begin{cases} 
L_{i-1,j-1}^{-1} - d_i'_{i-1}L_{i-1,j}^{-1} + bdq_{i-2}g_iL_{i-2,j}^{-1} & \text{for } i, j \geq 0 \text{ and } i \geq j, \\
1 & \text{for } i = j = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

(3.17)

Let $U$ be the upper triangular matrix of the $LDU$ decomposition of the bimoment matrix $B$. Then the elements $U_{i,j}^{-1}$ of the inverse of $U$ satisfy

\[
U_{i,j}^{-1} = \begin{cases} 
U_{i-1,j-1}^{-1} - e_j'_{i-1}U_{i,j-1}^{-1} + acq_jg_{i-2}U_{i+1,j}^{-1} & \text{for } i, j \geq 0 \text{ and } i \leq j, \\
1 & \text{for } i = j = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

(3.18)

**Proof.** We will show that $L^{-1}L = 1$. Thus, substituting (3.17) and then using (3.14) gives,

\[
(L^{-1}L)_{i,j} = \sum_k L_{i,k}^{-1}L_{k,j} = (L^{-1}L)_{i-1,j-1} + (d_i'_{i-1} - d_i'_{i-2})(L^{-1}L)_{i-1,j} - bdq_jg_i(L^{-1}L)_{i-2,j} + bdq_jg_{i-2}(L^{-1}L)_{i-2,j}.
\]

(3.19)
All entries above the main diagonal are zero since the matrix is lower triangular. On
the diagonal (3.19) gives,
\[(L^{-1}L)_{i,i} = (L^{-1}L)_{i-1,i-1} = (L^{-1}L)_{0,0} = 1. \tag{3.20}\]
All that needs to be shown is that all the other diagonals contain only zeros. The
recurrence (3.19) gives a matrix element in terms of elements from its own diagonal and
the two diagonals above it. The only non-zero elements above the first off-diagonal are
in the main diagonal. Therefore,
\[(L^{-1}L)_{i,i-1} = (L^{-1}L)_{i-1,i-2} = (L^{-1}L)_{0,-1} = 0. \tag{3.21}\]
Similarly for the second off-diagonal,
\[(L^{-1}L)_{i,i-2} = (L^{-1}L)_{i-1,i-3} = (L^{-1}L)_{1,-1} = 0. \tag{3.22}\]
Since the first two off-diagonals are zero, we get for \(c > 0\)
\[(L^{-1}L)_{i,i-c} = (L^{-1}L)_{i-1,i-1-c} = (L^{-1}L)_{c-1,-1} = 0. \tag{3.23}\]
The proof for \(U^{-1}\) follows similarly.

We can now compute the elements of the diagonal matrix \(D\).

**Theorem 3.** Assuming Conjecture 1 is true. The diagonal matrix elements \(D_n\) of the
matrix \(D\) of (3.13) satisfy a first order recurrence relation giving
\[
D_n = \prod_{i=0}^{n-1} g_i \tag{3.24}
\]
for \(n \geq 1\), where \(g_i\) is given by (3.15) and \(D_0 = 1\).

**Proof.** This proof follows similarly to the proof of \(L^{-1}\). Assuming the conjecture is true,
\(L^{-1}B\) is an upper triangular matrix. Using (3.17), the recurrence for \(L^{-1}\), gives,
\[
(L^{-1}B)_{n,m} = \sum_{i=0}^{n} L_{n,i}^{-1}B_{i,m} = \sum_{i=0}^{n} L_{n-1,i-1}^{-1}B_{i,m} - d_{n-1}^{-1}(L^{-1}B)_{n-1,m} \\
+ bdq^{n-2}g_{n-2}(L^{-1}B)_{n-2,m}. \tag{3.25}
\]
Using (3.3b), the recurrence for the bimoment matrix, gives
\[
(L^{-1}B)_{n,m} = (1 - q^n)(L^{-1}B)_{n-1,m-1} + ((b + d)q^n - d_{n-1}^{-1})(L^{-1}B)_{n-1,m} \\
- bdq^{n}g_{m}(L^{-1}B)_{n-1,m+1} + bdq^{n-2}g_{n-2}(L^{-1}B)_{n-2,m}. \tag{3.26}
\]
with \(n = m + 2\) and the fact that the matrix product is upper triangular gives the stated
result.
The value of the determinant, \( \text{det } B^{(n)} \), is simple to calculate from the \( LDU \)-decomposition of the bimoment matrix it being the product of the elements of the diagonal matrix.

**Theorem 4.** Assuming Conjecture 1 is true. Let \( B^{(n)} = (B_{i,j})_{0 \leq i,j \leq n} \) be the truncated \((n+1) \times (n+1)\) bimoment matrix whose elements are defined by Theorem 3. Then

\[
\text{det } B^{(n)} = \prod_{i=1}^{n} \frac{(abcd/q, q, ab, bc, ad, cd; q)_i}{(abcd/q, abcd, abcd, abcdq; q^2)_i}.
\]

(3.27)

We now use the bimoment matrix to show the existence and uniqueness of the polynomials sequences (3.1). For \( n, m \geq 0 \) we require the bi-orthogonality condition

\[
\mathcal{L}(P_n(d) Q_m(e)) = \Lambda_n \delta_{n,m}
\]

(3.28)

where \( \Lambda_n \) is a sequence of non-zero normalisation factors determined by \( \mathcal{L} \) and the monic constraint.

If this bi-orthogonality is translated into the form of the original matrix product Ansatz, then the equation is asking the question: Does there exist polynomials \( P_n(d) \) and \( Q_m(e) \) in the matrices \( d \) and \( e \) such that

\[
\langle W | P_n(d) Q_m(e) | V \rangle = \Lambda_n \delta_{n,m}
\]

(3.29)

for the boundary vectors \( |V\rangle \) and \( \langle W| \). If the sequences exist we get a new pair of basis vectors \( \hat{V}_n \rangle \rangle_{n \geq 0} \) and their orthonormal (with respect to \( \mathcal{L} \)) duals \( \langle \hat{W}_n| \rangle_{n \geq 0} \), given by

\[
\langle \hat{W}_n| = \langle W | P_n(d) \frac{1}{\sqrt{\Lambda_n}} \rangle \text{ and } \hat{V}_n \rangle = \frac{1}{\sqrt{\Lambda_n}} Q_n(e) | V \rangle \ .
\]

(3.30)

where \( |\hat{V}_0\rangle = |V\rangle \) and \( \langle \hat{V}_0| = \langle W| \). We normalise so that \( \langle W|V \rangle = 1 \). From these basis vectors, we get matrix representations for \( d \) and \( e \) by computing

\[
d_{n,m} = \langle \hat{W}_n|d|\hat{V}_m \rangle \text{ and } e_{n,m} = \langle \hat{W}_n|e|\hat{V}_m \rangle .
\]

(3.31)

Returning to the question of the existence of bi-orthogonal polynomials we have the following theorem stating a unique pair of sequences exists.

**Theorem 5.** Assuming Conjecture 1 is true. Let \( \{P_n(d)\}_{n \geq 0} \) and \( \{Q_n(e)\}_{n \geq 0} \) be a pair of sequences of monic polynomials satisfying

\[
\mathcal{L}(P_n Q_m) = \Lambda_n \delta_{n,m}
\]

(3.32)

where the linear functional \( \mathcal{L} \) is defined by equations (2.2). Then \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \) exist and are unique with

\[
\Lambda_n = \mathbf{D}_n
\]

(3.33)

for \( n \geq 0 \)

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The proof is exactly the same as that which appears in [2] (except for the different value of \( \Lambda_n \)) and thus we omit it.

To find the explicit form of the polynomials we need to evaluate two determinants (see [2] for their derivation),

\[
P_n(d) = \frac{1}{\det B^{(n-1)}} \det \begin{pmatrix}
B_{0,0} & B_{0,1} & \cdots & B_{0,n-1} & 1 \\
B_{1,0} & B_{1,1} & \cdots & B_{1,n-1} & d \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
B_{n-1,0} & B_{n-1,1} & \cdots & B_{n-1,n-1} & d^{n-1} \\
B_{n,0} & B_{n,1} & \cdots & B_{n,n-1} & d^n
\end{pmatrix},
\]

(3.34)

and

\[
Q_n(e) = \frac{1}{\det B^{(n-1)}} \det \begin{pmatrix}
B_{0,0} & B_{0,1} & \cdots & B_{0,n-1} & B_{0,n} \\
B_{1,0} & B_{1,1} & \cdots & B_{1,n-1} & B_{1,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
B_{n-1,0} & B_{n-1,1} & \cdots & B_{n-1,n-1} & B_{n-1,n} \\
1 & e & \cdots & e^{n-1} & e^n
\end{pmatrix}.
\]

(3.35)

The two determinants can be evaluated by \( LDU \) decomposition of the two matrices leading to the following result.

**Theorem 6.** Assuming Conjecture 1 is true. The pair of sequences of monic polynomials \( \{P_n(d)\}_{n \geq 0} \) and \( \{Q_n(e)\}_{n \geq 0} \) satisfy

\[
d^n = \sum_{k=0}^{n} L_{n,k} P_k(d),
\]

(3.36a)

\[
e^n = \sum_{k=0}^{n} Q_k(e) U_{k,n},
\]

(3.36b)

where \( L_{n,k} \) and \( U_{k,n} \) are the matrix elements of the lower triangular matrix \( L \) and upper triangular matrix \( U \) given by (3.14) and (3.16) respectively.

**Proof.** Since the two matrices are very similar to the bimoment matrix thus, once the \( LDU \) decomposition of the the bimoment matrix is know that for (3.34) and (3.35) are readily obtained. Thus, assuming Conjecture 1, we get the following:

\[
P_n(d) = \det \begin{pmatrix}
L_{0,0} & 0 & \cdots & 0 & 1 \\
L_{1,0} & L_{1,1} & \cdots & 0 & d \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
L_{n-1,0} & L_{n-1,1} & \cdots & L_{n-1,n-1} & d^{n-1} \\
L_{n,0} & L_{n,1} & \cdots & L_{n,n-1} & d^n
\end{pmatrix}
\]

(3.37)
and
\[
Q_n(e) = \det \begin{pmatrix}
U_{0,0} & U_{0,1} & \cdots & U_{0,n-1} & U_{0,n} \\
0 & U_{1,1} & \cdots & U_{1,n-1} & U_{1,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & U_{n-1,n-1} & U_{n-1,n} \\
1 & e & \cdots & e^{n-1} & e^n
\end{pmatrix}.
\] (3.38)

Expanding (3.37) using the bottom row leaves a sub-matrix determinant which reduces down to a \(k \times k\) determinant of the same form as (3.37) but with \(n = k\) and hence is \(P_k(d)\). Thus we get (3.36a). Similarly for (3.36b).

**Corollary 3.** Assuming Conjecture 1 is true. The pair of sequences of monic polynomials \(\{P_n(d)\}_{n \geq 0}\) and \(\{Q_n(e)\}_{n \geq 0}\) can be expressed as
\[
P_n(d) = \sum_{k=0}^{n} L^{-1}_{n,k} d^k, \quad \text{(3.39a)}
\]
\[
Q_n(e) = \sum_{k=0}^{n} e^k U^{-1}_{k,n}, \quad \text{(3.39b)}
\]
where \(L^{-1}_{n,k}\) and \(U^{-1}_{k,n}\) are the matrix elements of the inverse lower triangular \(L^{-1}\) and inverse upper triangular \(U^{-1}\) given by (3.17) and (3.18) respectively.

We now use (3.39a) and (3.39b) to find a recursion formulation for \(P_n\) and \(Q_n\).

**Theorem 7.** Assuming Conjecture 1 is true. The pair of sequences of monic polynomials \(\{P_n(d)\}_{n \geq 0}\) and \(\{Q_n(e)\}_{n \geq 0}\) are given by the three-term recurrences
\[
dP_n(d) = P_{n+1}(d) + d'_n P_n(d) - bdq^{n-1}g_{n-1}P_{n-1}(d) \quad \text{(3.40)}
\]
and
\[
eQ_n(e) = Q_{n+1}(e) + e'_n Q_n(e) - acq^{n-1}g_{n-1}Q_{n-1}(e) \quad \text{(3.41)}
\]
with \(P_0 = Q_0 = 1, P_{-1} = Q_{-1} = 0\) and the coefficients given by (1.17).

**Proof.** We prove the recurrence relation of the bi-orthogonal polynomials by using the recurrence relations (3.17) and (3.18) satisfied by the inverses of the upper and lower triangular matrix elements respectively. Multiplying (3.39a) by \(d\) we get
\[
dP_n(d) = \sum_k L^{-1}_{n,k} d^{k+1} \quad \text{(3.42)}
\]
\[
= \sum_k (L^{-1}_{n+1,k+1} + d'_n L^{-1}_{n,k+1} - bdq^{n-1}g_{n-1}L^{-1}_{n-1,k+1})d^{k+1} \quad \text{(3.43)}
\]
\[
= P_{n+1}(d) + d'_n P_n(d) - bdq^{n-1}g_{n-1}P_{n-1}(d). \quad \text{(3.44)}
\]
The proof of (3.41) follows similarly. □
4 Matrix representation and the boundary basis

The recurrence relations for $P_n$ and $Q_n$ in Theorem 7 can be used to compute the following two moments (which lead to a matrix representation of $d$ and $e$). This gives the following theorem.

**Theorem 8.** Assuming Conjecture 1 is true. Let $P_n$ and $Q_n$ be the polynomials of Theorem 7. The two first moments

$$X_{n,m} = \mathcal{L}(P_n \, d \, Q_m),$$

$$Y_{n,m} = \mathcal{L}(P_n \, e \, Q_m),$$

for $n, m \geq 0$, are given by

$$X_{n,m} = \Lambda_{n+1} \delta_{n+1,m} + d'_n \Lambda_n \delta_{n,m} - bdq^{n-1} \Lambda_n \delta_{n-1,m},$$

$$Y_{n,m} = \Lambda_{m+1} \delta_{n,m+1} + e'_m \Lambda_m \delta_{n,m} - acq^{m-1} \Lambda_m \delta_{n,m-1},$$

for $n, m \geq 0$.

To obtain a representation we need to use the orthonormal (non-monic) versions of the polynomials:

$$\hat{P}_n(d) = \frac{P_n(d)}{\sqrt{\Lambda_n}}, \quad \text{and} \quad \hat{Q}_m(e) = \frac{1}{\sqrt{\Lambda_m}} Q_m(e).$$

**Theorem 9.** Assuming Conjecture 1 is true. The matrices $d$ and $e$ with matrix elements

$$d_{n,m} = \mathcal{L}(\hat{P}_n \, d \, \hat{Q}_m) = X_{n,m}/\sqrt{\Lambda_n \Lambda_m},$$

$$e_{n,m} = \mathcal{L}(\hat{P}_n \, e \, \hat{Q}_m) = Y_{n,m}/\sqrt{\Lambda_n \Lambda_m},$$

for $n, m \geq 0$, give a matrix representation of (2.2).

The theorem is proved by direct verification that the matrices (4.4) satisfy the quotient relation $de - qed = q1$.

Using (4.2a) we see that $d$ and $e$ have a tri-diagonal structure

$$d = \begin{pmatrix}
 d'_0 & \sqrt{g_0} & 0 & \cdots \\
 -bd\sqrt{g_0} & d'_1 & \sqrt{g_1} & \cdots \\
 0 & -bdq\sqrt{g_1} & d'_2 & \cdots \\
 \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

and

$$e = \begin{pmatrix}
 e'_0 & -ac\sqrt{g_0} & 0 & \cdots \\
 \sqrt{g_0} & e'_1 & -acq\sqrt{g_1} & \cdots \\
 0 & \sqrt{g_1} & e'_2 & \cdots \\
 \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$
These matrices are similar to those obtained by Sasamoto [15]. Clearly the sum, \( R = d + e \), also has a tri-diagonal form,

\[
\begin{pmatrix}
  d_0' + e_0' & (1 - ac)\sqrt{g_0} & 0 & \cdots \\
  (1 - bd)\sqrt{g_0} & d_1' + e_1' & (1 - acq)\sqrt{g_1} & \cdots \\
  0 & (1 - bdq)\sqrt{g_1} & d_2' + e_2' & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

and thus \( R \) defines a sequence of orthogonal polynomials, \( \{T_n(x)\}_{n \geq 0} \), via the three term recurrence relation obtained from the rows,

\[
R_{n,n-1}T_{n-1} + (R_{n,n} - 2x)T_n + R_{n,n+1}T_{n+1} = 0
\]

where

\[
R_{n,n+1} = (1 - acq^n)\sqrt{g_n},
\]

\[
R_{n,n} = d_n' + e_n',
\]

\[
R_{n,n-1} = (1 - bdq^{n-1})\sqrt{g_{n-1}}
\]

and initial values \( T_0 = 1 \) and \( T_{-1} = 0 \). This three term recurrence is not exactly that for Askey-Wilson polynomials (cf. (1.12)) however, the middle coefficient \( R_{n,n} \) is the same as the Askey-Wilson middle recurrence coefficient and the product of the first and last coefficients \( R_{n,n+1}R_{n+1,n} \) is the same as the product of the first and last coefficients of the the Askey-Wilson recurrence. This means that the polynomials \( \{T_n(x)\}_{n \geq 0} \) have the same moments as the Askey-Wilson polynomials.

5 Concluding Remarks

In many cases finding a matrix representation of an algebra (eg. the equations of Definition 1) can be achieved by the ‘method of verification’: conjecture the matrix elements and then prove they indeed form a representation by showing the requisite matrix sums and products satisfy the algebra. This is the apparent ‘method’ used by Littlewood [5] in presenting a representation for the Weyl algebra \( (xp - px = 1) \) – closely related to (2.2a)), in the original matrix product ASEP paper [1] (the most general representation is a four parameter representation \( \alpha, \beta, \gamma, \delta \) with \( q = 1 \)) and in the five parameter ASEP paper [6].

One of the motivations for the previous three parameter paper [2] was to try and find a systematic algebraic method for computing the matrix elements of the representations of \( d \) and \( e \). The method presented there essentially reduces the determination of the matrix elements of the representation of \( d \) and \( e \) to the calculation of the determinant of the bi-moment matrix (3.2) (the determinant gives \( \Lambda_n \) and hence via (4.2a) and (4.4) the matrix elements).

In this paper we have applied the method of [2] to the five parameter case. Going from three to five parameters has a dramatic affect on the complexity of the calculations. This can be seen at several places. It begins rather subtly, in that the boundary
equations (2.2a) and (2.2b), are now a pair of coupled functional equations which cannot be simply solved as a pair of ‘simultaneous linear equations’ (which would have made the calculation only a little more complex than the three parameter case) – the actual method required is more complex and detailed in the proof of Theorem 2.

Another, more significant impact, is on the complexity of calculating the determinant of the bi-moment matrix. To compute a determinant one usually has the matrix elements explicitly, however, the matrix elements of the bi-moment matrix $B_{i,j}$ are not given explicitly but indirectly via the partial $q$-difference equations stated in Theorem 2. In the case of the three parameter model [2] the boundary matrix elements $B_{i,0}$ and $B_{i,0}$ (required to solve the $B_{i,j}$ partial difference equation) are given explicitly (as simple monomials), however in the five parameter model the boundary matrix elements are given implicitly by three term recurrence relations and hence are themselves related to the value of yet another set of non-trivial $q$-orthogonal polynomials. Thus, if one wanted to use a determinant evaluation method that required explicit expressions for the matrix elements $B_{i,j}$, one has to solve a partial $q$-difference equation whose boundary values are given implicitly as the values of $q$-orthogonal polynomials. Once these two tasks have been achieved one can then try to compute the determinant.

To circumvent the difficulty of computing the matrix elements explicitly we turned to $LDU$ decomposition of the bi-moment matrix. This method somewhat mitigates the task of evaluating the matrix elements explicitly by translating the $B_{i,j}$ partial difference equation into partial difference equations for the $L$ and $U$ matrix elements (our primary conjecture – (1)). Given triangular $L$ and $U$ we get a diagonal bi-moment matrix. Once $B$ is diagonalised the determinant evaluation is straightforward – see (3.24).

If $B$ is interpreted as a linear operator, $B : V_1 \rightarrow V_2$, between two infinite dimensional vectors spaces then clearly the matrix elements of $B$ are determined by the choice of basis for $V_1$ and $V_2$. If the matrix elements of $B$ are defined by (1.8) then clearly $B$ is not diagonal in the basis implicitly used for $V_1$ and $V_2$.

This brings us to the primary significance (for this calculation) of the set of ‘boundary bases’ vectors, $(\hat{W}_n)_{n \geq 0}$ and $(\hat{V}_n)_{n \geq 0}$ constructed in this paper (see (3.30)). In this basis the $U$ and $L$ matrices are triangular and hence by choosing the boundary basis the linear map $B$ has a diagonal matrix representation and hence its determinant becomes a product of the diagonal elements (3.24). The ‘boundary bases’ vectors have the further significance in that it is in this basis that the matrix representation of $d+e$ is tri-diagonal and hence defines a three term recurrence relation (see (4.6)). It is this recurrence relation that is related to the Askey-Wilson orthogonal polynomials. Whilst the Askey-Wilson polynomials (or their moments) do not appear to have any particular physical significance it is an open question as to whether or not the boundary basis vectors have any physical interpretation.

It would be very interesting to determine if the method of this paper and the new basis it defines has implications (e.g. new representations) for other ASEP (or similar) related work associated with finding representations such as the finite representations of Mallick and Sandow [16], Sandow [17], the MacDonald and Koornwinder polynomials that appear in Cantini et. al. [18] and Finn and Vanicat [19] as well as the connection
to Schur polynomials that appear in Crampe et. al. [20].

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