SPACES OF ABELIAN DIFFERENTIALS AND HITCHIN’S SPECTRAL COVERS

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Abstract

Using the embedding of the moduli space of generalized $GL(n)$ Hitchin’s spectral covers to the moduli space of meromorphic abelian differentials we study the variational formulæ of the period matrix, the canonical bidifferential, the prime form and the Bergman tau function. This leads to residue formulæ which generalize the Donagi-Markman formula for variations of the period matrix. Computation of second derivatives of the period matrix reproduces the formula derived in [2] using the framework of topological recursion.

CONTENTS

1. Introduction 2
2. Spaces of generalized spectral covers 3
3. Variational formulæ and Bergman tau function on moduli spaces of meromorphic Abelian differentials 6
4. Variational formulæ on spaces of generalized Hitchin’s covers 8
4.1. Variations of Abelian differentials 12
4.2. Variations of prime-form and canonical bidifferential 13
4.3. The Bergman tau-function on spaces of spectral covers 14
5. Higher order derivatives on $\mathcal{H}_\theta$ and $\mathcal{M}_H^n$ 14
5.1. Space $\mathcal{H}_\theta$ 14
5.2. The space $\mathcal{M}_H^n$ 16
References 20

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1. Introduction

Geometry of spaces of Abelian differentials on Riemann surfaces has attracted interest in relationship with the theory of Teichmüller flow [16, 17, 7]. Methods inspired by the theory of integrable systems were applied to study these spaces in [14, 15, 12] where an appropriate version of deformation theory of Riemann surfaces and the formalism of tau functions was developed. In particular, variations of moduli and of various canonical objects associated to a Riemann surface were computed in [14] (holomorphic case) and in [12] (meromorphic case). The Bergman tau function introduced in [14] is a natural generalization of Dedekind’s eta-function to higher genus.

The origin of Hitchin’s spectral covers and their moduli spaces is the dimensional reduction of self-dual Yang-Mills equations on a four-dimensional space represented as the product of a Riemann surface and $\mathbb{R}^2$ [10]. Such a dimensional reduction gives a family of completely integrable systems associated to families of Riemann surfaces of arbitrary genus [11]. Hamiltonians of such integrable systems (we consider here only the $GL(n)$ gauge group) are encoded in the $n$-sheeted spectral cover of a Riemann surface. The moduli space of spectral covers for a base Riemann surface of given genus was also intensively studied (see [1, 6]). In particular, the Donagi-Markman cubic describes variations of the period matrix of the spectral cover for fixed base, answering the question posed in [1]. Variations of the canonical meromorphic bi-differential on these spaces were derived in [2] using the formalism developed in [9].

The space of Hitchin’s spectral covers admits a natural embedding in a space of abelian differentials; this embedding was used in [19] to define a natural version of Bergman tau functions on spaces of spectral covers (with variable or fixed base) and find the class of the locus of degenerate covers (the universal Hitchin’s discriminant) in the Picard group of the universal moduli space of spectral covers.

In this paper we further exploit this embedding to show how variational formulæ for the period matrix, the canonical bidifferential and the prime form on the moduli spaces of generalized Hitchin’s systems (when the coefficients of the equation defining the spectral cover are allowed to be meromorphic differentials) can be deduced from variational formulæ on moduli spaces of meromorphic abelian differentials derived in [14, 12]. In the special case of regular Hitchin’s systems we reproduce residue formulæ for the canonical bidifferential obtained in [2] and for the period matrix (given by the Donagi-Markman cubic [6]). We also derive residue formulæ for variations of Bergman tau function of spaces of spectral covers for the holomorphic case.

Formulas for second derivatives of the period matrix (in holomorphic case) found in our formalism coincide with expressions derived in [2] using the formalism of topological recursion of [8]. These formulæ are rather cumbersome in contrast to analogous formulæ on spaces of Abelian differentials. This suggest a possibility of existence of a natural simple structure on spaces of abelian differentials which underlie the topological recursion framework on spaces of spectral covers.
2. Spaces of Generalized Spectral Covers

Denote by $C$ a Riemann surface of genus $g$, with $N$ marked points $y_1, \ldots, y_N$ on $C$ and associated corresponding multiplicities $k_1, \ldots, k_N$, $k_j \geq 1$. We consider a $GL(n)$ Higgs field $\Phi$ with poles at the $y_j$’s of the corresponding order $k_j$, $j = 1, \ldots, m$. We also assume a generic form of the polar expansion of $\Phi$ near these poles. The spectral curve $\hat{C}$ is defined by the equation $\det(\Phi - vI) = 0$, which can be written as

$$v^n + Q_1 v^{n-1} + \cdots + Q_n = 0$$

where $Q_\ell$ is a meromorphic $k$-differential on $C$ with pole of order $\ell k_j$ at the point $y_j$ thanks to the genericity assumption. Denote by $\pi$ the projection $\hat{C} \to C$. Assuming that the branch points of $\hat{C}$ do not coincide with $y_j$ we have

$$\pi^{-1}(y_j) = \{y_j^{(k)}\}_{k=1}^n.$$

The meromorphic abelian differential $v$ has on $\hat{C}$ poles of order $k_j$ at all $y_j^{(k)}$. Denote by $\chi_j$ a local coordinate on $C$ near $y_j$; since we have assumed that $y_j$ is a not a branch point of $\hat{C}$ we can use $\chi_j$ also as local coordinate near each $y_j^{(k)}$ for $k = 1, \ldots, n$. Consider the singular parts of $v$ at $y_j^{(s)}$:

$$v(\zeta_j) = \frac{C_j^{(s),k_j}}{\chi_j^{k_j}} + \frac{C_j^{(s),k_j-1}}{\chi_j^{k_j-1}} + \cdots + \frac{C_j^{(s),1}}{\chi_j} + O(1).$$

The discriminant $W$ of equation (2.1) is a meromorphic $n(n-1)$ differential on $C$ which has pole of order $n(n-1)k_j$ at $y_j$. Therefore, the total degree of poles of $W$ is $n(n-1)\sum_{j=1}^m k_j$ and the number of its zeros (i.e. the number of branch points of $\hat{C}$) is

$$p = n(n-1) \left(2g - 2 + \sum_{j=1}^m k_j\right).$$

It follows from Riemann–Hurwitz formula that the genus of $\hat{C}$ equals

$$\hat{g} = n^2(g-1) + 1 + \frac{n(n-1)}{2} \sum_{j=1}^m k_j.$$

The degree of the divisor of zeroes for the abelian differential $v$ on $\hat{C}$ is

$$r = 2\hat{g} - 2 + n \sum_{j=1}^m k_j.$$

The dimension of $M_H^n[k]$ equals to the sum of dimensions of spaces of coefficients of (2.1), which is computed as

$$\left(\sum k_j - 1 + g\right) + \left(2 \sum k_j + 3(g-1)\right) + \cdots + \left(n \sum k_j + (2n-1)(g-1)\right).$$
Assuming that at least one \( k_j > 0 \) the above gives

\[
\dim \mathcal{M}_H^k[n] = \frac{n(n + 1)}{2} \sum_{j=1}^{m} k_j + n^2(g - 1) = \hat{g} + n \sum_{j=1}^{m} k_j - 1.
\]

On the moduli space \( \mathcal{M}_H^k[n] \) we introduce the following coordinates:

\[
\{A_{\alpha} \}_{\alpha=1}^{\hat{g}}, \{C_{j}^{(s),l} \}, j = 1, \ldots, m, s = 1, \ldots, n, l = 1, \ldots, k_j, (j, s, l) \neq (1, 1, 1)
\]

where \( C_{j}^{(s),l} \) are coefficients in singular parts of near \( y_j^{(k)} \) (2.2) (these coefficients of course depend on the choice of local coordinates \( \chi_j \) near \( y_j \) on \( C \)), and \( A_{\alpha} \) are \( \alpha \)-periods of \( v \) under an arbitrary choice of Torelli marking:

\[
A_{\alpha} = \int_{a_{\alpha}} v.
\]

The coefficient \( C_{1}^{(1),1} \) is not an independent coordinate since sum of residues of \( v \) on \( \hat{C} \) vanishes:

\[
\sum_{j=1}^{m} \sum_{s=1}^{k_j} C_{j}^{(s),1} = 0.
\]

We observe that the number of coordinates (2.7) coincides with the dimension (2.6) of \( \mathcal{M}_H^k[n] \).

Subordinate to the Torelli marking we also define the first-kind Abelian differentials (holomorphic) \( v_{\alpha} \) with the normalizing property

\[
\oint_{a_{\alpha}} v_{\alpha} = \delta_{\alpha\beta}.
\]

We also define alongside them the normalized second-kind differentials \( w_{j}^{(s),l} \) on \( \hat{C} \) and having the following singular parts:

\[
\oint_{a_{\alpha}} w_{j}^{(s),l} = 0, \quad w_{j}^{(s),l}(x) = \left( \frac{1}{\chi_j} + O(1) \right) d\chi_j, \quad x \sim y_j^{(s)}, \quad l = 2, \ldots, k_j
\]

and the normalized differentials of the third kind \( u_{j}^{(s),l}(x) \) on \( \hat{C} \) which have a simple poles at \( y_1^{(1)} \) and \( y_j^{(s)} \) with residue \(-1, +1\), respectively.

Since the moduli of the base curve \( C \) are kept constant, we can define unambiguously the derivative with respect to the moduli of our space for any abelian differential \( w \) on \( \hat{C} \). To wit, we fix a local chart \( D \) on \( C \) with a local coordinate \( \xi \) and lift \( D \) to all sheets of \( \hat{C} \). Then in any connected component of \( \pi^{-1}(D) \) we can use \( \xi \) as a local coordinate. We express the differential \( w \) in such coordinate \( w = f(\xi)d\xi \) and define

\[
\frac{dw}{dz_k} = \frac{df(\xi)}{dz_k}d\xi.
\]
where the coordinate \( \xi \) remains fixed under differentiation. Clearly, the definition (2.12) is independent of the choice of the local coordinate \( \xi \) because the moduli of the base curve are kept constant. Keeping this in mind we formulate the following proposition.

**Proposition 2.1.** The following variational formulae of \( v \) with respect to coordinates (2.7) on \( M^N_{nH}[^k] \) hold:

\[
\frac{\partial v}{\partial A_\alpha} = v_\alpha ,
\]

\[
\frac{\partial v}{\partial C^{(s)}_j,l} = w^{(s),l}_j, \quad l = 2, \ldots, k_j
\]

where \( w^{(s),l}_j \) are (normalized by conditions \( \int_{A_i} w^{(s),l}_j = 0 \)) differentials of second kind defined by (2.11) and

\[
\frac{\partial v}{\partial C^{(s),l}_j} = u^{(s)}_j
\]

where \( j = 1, \ldots, m \) and \( s = 1, \ldots, n; u^{(s)}_j(x) \) are the normalized differentials of the third kind on \( \hat{C} \) defined after (2.11).

**Proof.** First notice that the differential \( v \) vanishes at all branch points \( x_j \) of \( \hat{C} \); generically these zeros are of first order. This is due the fact that a coefficient \( Q_k \) of equation (2.1) is a \( k \)-differential on \( C \). Being lifted from \( C \) to \( \hat{C} \), it gains a zero of order \( k \) at each branch point since near the ramification point \( x_j \) the local coordinate on \( \hat{C} \) is given by \( (\xi - \xi_j)^{1/2} \) where \( \xi \) is the local coordinate on \( C \) near \( \pi(x_j) \) (\( \xi \) is assumed to be independent of coordinates (2.7)) and \( \xi_j = \xi(\pi(x_j)) \). In particular, the \( n \)-differential \( Q_n \), being lifted to \( \hat{C} \), has zeros of order \( n \) at all branch points (as well as zeros lifted to \( \hat{C} \) from its zeros on \( C \)).

Therefore, locally near \( x_j \) we have

\[
v(\xi) = (\xi - \xi_j)^{1/2}(a_0 + a_1(\xi - \xi_j)^{1/2} + \ldots) d(\xi - \xi_j)^{1/2} = \frac{1}{2}(a_0 + a_1(\xi - \xi_j)^{1/2} + \ldots) d\xi .
\]

Although \( \xi \) is independent of the moduli coordinates (2.7), the coordinate \( \xi_j \) of the branch point \( \pi(x_j) \) does depend on them, and differentiation with respect to any coordinate \( z \) from the list (2.7) gives

\[
\frac{\partial v}{\partial z} = \frac{1}{4} \left( -\frac{a_1(\xi_j)_z}{(\xi - \xi_j)^{1/2}} + O(1) \right) d\xi = \frac{1}{2}(a_1(\xi_j)_z + o(1)) d\sqrt{\xi - \xi_j}
\]

which is holomorphic (although generically non-vanishing) at \( x_j \).

Therefore all the differentials \( \partial v/\partial z \) are holomorphic at the branch points, and can have poles only at the \( y^{(s)}_j \)'s.

The differentials \( \partial v/\partial A_j \) are holomorphic since the coefficients of the singular parts of \( v \) near all \( y^{(s)}_j \) are independent of \( A_j \). Moreover, all a-periods of \( \partial v/\partial A_j \) vanish except for the period over \( a_j \), which equals 1. Therefore, we deduce to (2.13).
Consider $\partial v / \partial C^{(s),l}_j$ for $l \geq 2$. The only singularity of this differential is at $y^{(s)}_j$ and its singular part there coincides with the one of $w^{(s)}_j$. Moreover, since the $A_\alpha$ and the $C^{(s),l}_j$ coordinates are independent of each other, all $a$-periods of $\partial v / \partial C^{(s),l}_j$ vanish; thus this differential coincides with $w^{(s)}_j$.

Similarly, one verifies that the differential $\partial v / \partial C^{(s),1}_j$ coincides with the third kind differential $u^{(s)}_j$.

We are going to combine this proposition with the variational formulæ on moduli spaces of meromorphic Abelian differentials obtained in [14, 12] which we discuss next.

3. VARIATIONAL FORMULÆ AND BERGMAN TAU FUNCTION ON MODULI SPACES OF MEROMORPHIC ABELIAN DIFFERENTIALS

Denote by $\mathcal{H}_g[d_1, \ldots, d_k]$ the moduli space of pairs $(\hat{C}, v)$ where $\hat{C}$ is a Riemann surface of genus $\hat{g}$ and $v$ is a meromorphic differential on $\hat{C}$ with $k$ poles $y_1, \ldots, y_k$ of orders $d_1, \ldots, d_k$, respectively, and simple zeros $x_1, \ldots, x_r$ where $r = 2\hat{g} - 2 + \sum_{i=1}^k d_i$. The notations $\hat{C}$ and $\hat{g}$ are now used in agreement with the previous discussion. The dimension of $\mathcal{H}_g[d_1, \ldots, d_k]$ is the sum of: $3\hat{g} - 3$ moduli parameters of $\hat{C}$, $k$ positions of the singularities, $\sum_{j=1}^k d_k - 1$ coefficients of the singular parts and $\hat{g}$ additional moduli corresponding to the addition of an arbitrary holomorphic differential to $v$. Altogether, we get

\[(3.1)\quad \dim \mathcal{H}_g[d_1, \ldots, d_k] = 4\hat{g} - 4 + k + \sum_{j=1}^k d_j\]

The dimension of $\dim \mathcal{H}_g[d_1, \ldots, d_k]$ coincides with the dimension of the relative homology group

\[(3.2)\quad H_1(\hat{C} \setminus \{y_j\}_{j=1}^k, \{x_i\}_{i=1}^r)\]

A set of generators of this group can be chosen as follows:

\[(3.3)\quad \{s_i\}_{i=1}^{\dim \mathcal{H}_g[d_1, \ldots, d_k]} = \{ \{a_\alpha, b_\alpha\}_{\alpha=1}^{\hat{g}}, \{c_i\}_{i=2}^k, \{l_i\}_{i=1}^{r-1} \}\]

where $\{a_\alpha, b_\alpha\}$ form a Torelli marking on $\hat{C}$, $c_i$ are small counter-clockwise contours around $y_i$ and each contour $l_i$ connects $x_r$ with $x_i$.

The homology group dual to (3.2) is

\[(3.4)\quad H_1(\hat{C} \setminus \{x_i\}_{i=1}^r, \{y_j\}_{j=1}^k)\]

and the set of generators dual to the set (3.3) with the intersection index

\[s_i^* \circ s_j = \delta_{ij}\]

is given by

\[(3.5)\quad s_i^* = \{-b_\alpha, a_\alpha\}_{\alpha=1}^{\hat{g}}, \{-\tilde{l}_i\}_{i=2}^k, \{\tilde{c}_i\}_{i=1}^{r-1}\]
where $\tilde{l}_i$ is the contour connecting the pole $y_k$ with $y_i$; $\tilde{c}_i$ is a small counter-clockwise contour around $x_i$.

The set of homological, or period coordinates on $H_{\beta}[d_1, \ldots, d_k]$ is given by integrals of $v$ over the basis $\{s_i\}$ (3.3):

\begin{equation}
\mathcal{P}_i = \int_{s_i} v, \quad i = 1, \ldots, \dim H_{\beta}[d_1, \ldots, d_k].
\end{equation}

Introduce the following objects on $\hat{\mathcal{C}}$: the prime-form $E(x, y)$, canonical bidifferential $B(x, y)$, abelian differentials $v_\alpha$ normalized via $\int_{a_\alpha} v_\beta = \delta_{\alpha\beta}$ and the period matrix $\Omega_{\alpha\beta} = \int_{b_\alpha} v_\beta$.

Choose a fundamental polygon of $\hat{\mathcal{C}}$ with vertex at $x_r$ and dissected along paths connecting $x_r$ with poles $y_j$ (having only $x_r$ as common point); denote the resulting simply connected domain by $\tilde{\mathcal{C}}$; on it we define the "flat" coordinate

\begin{equation}
z(x) = \int_{x_r}^{x} v
\end{equation}

which can be used as local coordinate on $\hat{\mathcal{C}}$ outside of zeros and poles of $v$.

**Proposition 3.1.** [14, 12] The following variational formulæ for the period matrix $\Omega$ on the space $H_{\beta}[d_1, \ldots, d_k]$ hold:

\begin{equation}
\frac{\partial \Omega_{\alpha\beta}}{\partial \mathcal{P}_j} = \int_{s_j^*} v_\alpha v_\beta v.
\end{equation}

To present variational formulæ for $v_\alpha, B$ and $E$ we need to define their variations: for $v_\alpha$ we define

\begin{equation}
\frac{\partial v_\alpha}{\partial \mathcal{P}_j}(x) = \frac{\partial}{\partial \mathcal{P}_j} \left( \frac{v_\alpha(x)}{v(x)} \right) \bigg|_{z(x) = \text{const}} v(x).
\end{equation}

The result is a differential in $\hat{\mathcal{C}}$ with discontinuities across all the dissecting cuts of $\hat{\mathcal{C}}$ where the discontinuity is the addition of a constant depending which boundary component of the dissection we are crossing. Analogously we define variations of $B(x, y)$ and $E(x, y)$ in $\mathcal{P}_j$.

**Proposition 3.2.** [14, 12] The following variational formulæ on the space $H_{\beta}[d_1, \ldots, d_k]$ hold

\begin{equation}
\frac{\partial v_\alpha(x)}{\partial \mathcal{P}_i} = \frac{1}{2\pi i} \int_{t \in s_i^*} \frac{v_\alpha(t) B(x, t)}{v(t)},
\end{equation}

\begin{equation}
\frac{\partial B(x, y)}{\partial \mathcal{P}_i} = \frac{1}{2\pi i} \int_{t \in s_i^*} \frac{B(x, t) B(t, y)}{v(t)},
\end{equation}

\begin{equation}
\frac{\partial}{\partial \mathcal{P}_i} \ln \left( E(x, y) \sqrt{v(x) v(y)} \right) = -\frac{1}{4\pi i} \int_{t \in s_i^*} \frac{1}{v(t)} \left[ d_t \ln \frac{E(x, t)}{E(y, t)} \right]^2.
\end{equation}
In the next section we show to deduce variational formulæ on spaces of spectral covers by restriction of the above ones.

On the subspace of $\mathcal{H}^0_{\hat{g}}[d_1, \ldots, d_k]$ of $\mathcal{H}_{\hat{g}}[d_1, \ldots, d_k]$ defined by the vanishing of the residues of $v$ we define the Bergman tau-function via the system of differential equations [14, 12]:

\[
\frac{\partial}{\partial P_j} \ln \tau_B(\hat{C}, v) = \int_{s_j} B_{reg}(x, x) \frac{B^v(x, x)}{v(x)}
\]

where

\[
B_{reg}^v(x, x) = \left( B(x, y) - \frac{v(x)v(y)}{(\int_x y)^2} \right) \bigg|_{x=y}.
\]

We refer to [14, 12] for explicit formula for $\tau_B$ and to [15, 12] for its properties and applications.

4. VARIATIONAL FORMULÆ ON SPACES OF GENERALIZED HITCHIN’S COVERS

We start from discussing variations of the period matrix $\hat{\Omega}$ of $\hat{C}$ on the moduli space $M_H^n[k]$ of spectral covers: these formulæ are obtained by pullback of the variational formulæ on the space $\mathcal{H}_{\hat{g}}(d)$ of abelian differentials on Riemann surfaces of genus $\hat{g}$ where vector $d$ is given by

\[
d = (k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_n, \ldots, k_n)
\]

where each $k_i$ is repeated $n$ times. Thus in the context of previous section we have $k = nm$, and the set of poles $\{y_j\}$ coincides with the set $\{y_{(s)}^j\}, j = 1, \ldots, m, s = 1, \ldots, n$.

Assume that the branch points of $\hat{C}$ i.e. zeros of $W$ are also simple. We have $(v) = D_{br} + D_0$ where $D_{br}$ is the divisor of ramification points of $\hat{C}$; projection of $D_{br}$ on $C$ coincides with the divisor of the discriminant $W$: $\pi(D_{br}) = (W)$. The projection of $D_0$ on $C$ coincides with the divisor of the $n$-differential $Q_n$: $\pi(D_0) = (Q_0)$; $\deg D_0 = n(2g-2) + n \sum_{j=1}^n k_j$ i.e. $\deg D_{br} + \deg D_0 = m$ as expected. Let

\[
D_{br} = \{x_i\}_{i=1}^{\deg D_{br}}, \quad D_0 = \{x_i\}_{i=\deg D_{br}+1}^m.
\]

We start from considering the easier case of variations of period matrix.

The map of $M_H^n[k]$ to $\mathcal{H}_{\hat{g}}(d_1, \ldots, d_{mn})$ is defined by assigning to a point of $M_H^n[k]$ the pair $(\hat{C}, v)$; for generic point of $M_H^n[k]$ all zeros of $v$ are simple.

Theorem 4.1. The variations of the period matrix $\Omega$ with respect to coordinates (2.7) on $M_H^n[k]$ are given by:

\[
\frac{\partial \Omega_{\alpha\beta}}{\partial A_\gamma} = -2\pi i \sum_{x_i \in D_{br}} \frac{v_\gamma}{d\ln(v/d\xi)}(x_i) \res_{x_i} \frac{v_\alpha v_\beta}{v},
\]
\[
\frac{\partial \Omega_{\alpha\beta}}{\partial C_j^{(s),l}} = -2\pi i \sum_{x_i \in D_{br}} \frac{w_j^{(s),l}(x_i)}{d \ln(v/d\xi)}(x_i) \left. \frac{v_\alpha v_\beta}{v} \right|_{x_i},
\]
\[
\frac{\partial \Omega_{\alpha\beta}}{\partial C_j^{(s),1}} = -2\pi i \sum_{x_i \in D_{br}} \frac{u_j^{(s)}(x_i)}{d \ln(v/d\xi)}(x_i) \left. \frac{v_\alpha v_\beta}{v} \right|_{x_i},
\]
where in these formulae \( \xi \) denotes a local coordinate on \( C \) near \( x_i \); the right-hand side of (4.5) is independent on the choice of these coordinates near \( x_i \).

The formula (4.2) can be written alternatively in the following more familiar form:
\[
\frac{d \Omega_{\alpha\beta}}{d A_\gamma} = -2\pi i \sum_{x_i \in D_{br}} \left. \frac{v_\alpha v_\beta v_\gamma}{d \xi d(v/d\xi)} \right|_{x_i},
\]
and analogous versions of (4.3) and (4.3) where \( v_j \) is replaced by \( w_j^{(s),l} \) and \( u_j^{(s)} \), respectively.

On the submanifold \( M_H^n[k] \) of \( \mathcal{H}_d(d) \) we use the set of independent coordinates given by (2.7) so that the period coordinates (3.3) on \( M_H^n[k] \) become functions of (2.7) defined implicitly by the condition that the moduli of the base curve \( C \) are constants.

For the proof of Theorem 4.1 we need the following Lemma.

**Lemma 4.2.** Denote by \( s_i \) a contour from the list (3.3) which does not coincide with \([x_r, x_k]\) with \( x_k \in D_{br} \) (a branchpoint). The derivatives of integrals of \( v \) over the basis (3.3) with respect to the coordinates (2.7) are then given by
\[
\frac{\partial (\int_{x_k}^{x_r} v)}{\partial \kappa_j} = \int_{s_i} \frac{\partial v}{\partial \kappa_j},
\]
where \( z_j \) is any coordinate from the list (2.7) and the periods of the right-hand side are given by standard formulæ taking into account (2.13), (2.14), (2.15).

If \( x_k \) is a branch point then the derivatives have the following additional contributions:
\[
\frac{\partial (\int_{x_k}^{x_r} v)}{\partial A_\alpha} = \int_{x_r}^{x_k} \frac{v_\alpha}{d \ln(v/d\xi)}(x_k),
\]
\[
\frac{\partial (\int_{x_k}^{x_r} v)}{\partial C_j^{(s),l}} = \int_{x_r}^{x_k} \frac{w_j^{(s),l}}{d \ln(v/d\xi)}(x_k),
\]
\[
\frac{\partial (\int_{x_k}^{x_r} v)}{\partial C_j^{(s),1}} = \int_{x_r}^{x_k} \frac{u_j^{(s)}}{d \ln(v/d\xi)}(x_k),
\]
coordinate \( \xi \) is assumed to be invariant under the deformation. Expressions (4.7)-(4.9) are independent of the choice of local coordinate \( \xi \) on \( C \).

\(^3\)We did not carry in the notation the dependence on \( i \) for brevity of notation.
Proof. We start from (4.6): if the contour $s$ is closed (i.e. coincides with one of $a$- or $b$-cycles or a small contour around one of $y_i(s)$) then the differentiation commutes with integration. If $s$ connects $x_r$ with another zero $x_j$ which is not a brach point of $\hat{C}$ then $s$ can be projected on $C$, and in a local coordinate on $C$ the integrand vanishes at both endpoints. Therefore, the differentiation commutes with integration in this case, too.

The only case when the dependence of the endpoint on the differentiation variable gives a non-trivial contribution is the case when $s$ connects $x_r$ with one of branch points $x_k$ of $\hat{C}$. Below we prove (4.7); proof of (4.8) and (4.9) is almost identical.

Let $x_k \in \hat{C}$ be a ramification point of $\hat{C}$ and $\xi_k = \xi(\pi(x_k)) \in C$ be the corresponding critical value in some local coordinate $\xi$ on $C$ which remains fixed under deformation of $\hat{C}$; let $\zeta = \xi - \xi_k$ be a coordinate on $C$ vanishing at $\pi(x_k)$ (the coordinate $\zeta$ deforms when $\hat{C}$ varies). A suitable local coordinate on $\hat{C}$ near $x_k$ can then be chosen to be

$$\hat{\zeta}(x_k) = \frac{\zeta}{2}. $$

Then the differentiation with respect to $A_\alpha$ of the endpoint also gives a contribution to $\partial z_k/\partial A_\alpha$ and we get

$$\frac{\partial(\int_{x_k}^{x_r} v)}{\partial A_\alpha} = \int_{x_r}^{x_k} v_\alpha + \frac{\partial \xi_k}{\partial A_\alpha} \frac{v}{d\xi}(\xi_k) $$

for $k = 1, \ldots, \deg D_{br}$.

To compute the derivative $\partial \xi_k/\partial A_\alpha$ we follow [3] and we write $v(\xi)$ near $\xi_k$ in the form

$$v = (a + b \sqrt{\xi - \xi_k} + \ldots) d\xi$$

(recall that $v$ has simple zero in the local parameter $\sqrt{\xi - \xi_k}$ and $\delta \xi$ has already a simple zero). Thus

$$b = \frac{d(v/d\xi)}{d\sqrt{\xi - \xi_k}} \bigg|_{\xi = \xi_k};$$

and

$$v_\alpha = \frac{\partial v}{\partial A_\alpha} = \left( a A_\alpha - \frac{\xi_k A_\alpha}{2\sqrt{\xi - \xi_k}} b + \ldots \right) d\xi .$$

Therefore,

$$- \frac{v_\alpha}{d\sqrt{\xi - \xi_k}} \bigg|_{\xi = \xi_k} = b \frac{\partial \xi_k}{\partial A_\alpha}$$

and,

$$\frac{\partial \xi_k}{\partial A_\alpha} = - \frac{v_j/d\hat{\zeta}_k(x_k)}{(v/d\xi)_{\hat{\zeta}}}(x_k).$$

Now (4.10) takes the form

$$\frac{\partial \int_{x_k}^{x_r} v}{\partial A_\alpha} = \int_{x_{2j-2}}^{x_k} v_j - \frac{v_j/d\hat{\zeta}_k(x_k)}{[\ln(v/d\xi)]_{\hat{\zeta}}}(x_k)$$

for $k = 1, \ldots, \deg D_{br}$. This proves the lemma. ■
Proof of Theorem 4.1. Let us prove (4.2); the proofs of (4.3) and (4.4) are parallel.

Applying the chain rule we get

\[
\frac{d\Omega_{\alpha\beta}}{dA_\gamma} = \frac{\partial \Omega_{\alpha\beta}}{\partial A_\gamma} + \sum_{\delta=1}^{g} \frac{\partial \Omega_{\alpha\beta}}{\partial B_\delta} \frac{\partial B_\delta}{\partial A_\gamma} + \sum_{k=1}^{r-1} \frac{\partial \Omega_{\alpha\beta}}{\partial \left( \int_{x_k}^{x_r} v \right)} \frac{\partial \left( \int_{x_k}^{x_r} v \right)}{\partial A_\gamma}
\]

(since \( A_\alpha \) and residues of \( v \) are independent coordinates we omit the term involving these derivatives). Using (2.13), (4.7) together with variational formulæ (3.8)

\[
\frac{\partial \Omega_{\alpha\beta}}{\partial A_\gamma} = -\int_{b_\gamma} v_{\alpha\beta} \frac{dC}{v} \quad , \quad \frac{\partial \Omega_{\alpha\beta}}{\partial B_\gamma} = \int_{a_\gamma} v_{\alpha\beta} \frac{dC}{v} \quad , \quad \frac{\partial \Omega_{\alpha\beta}}{\partial \left( \int_{x_k}^{x_r} v \right)} = 2\pi i \text{ res}_{x_k} v_{\alpha\beta} (v)
\]

(where \( x_k \) runs through the set of all zeros of \( v \)) we rewrite (4.17) as follows:

\[
\frac{\partial \Omega_{\alpha\beta}}{\partial A_\gamma} = \sum_{\delta=1}^{g} \left[ -\left( \int_{a_\delta} v_\gamma \right) \left( \int_{b_\delta} v_{\alpha\beta} \frac{dC}{v} \right) + \left( \int_{b_\delta} v_\gamma \right) \left( \int_{a_\delta} v_{\alpha\beta} \frac{dC}{v} \right) \right]
\]

\[
+ 2\pi i \sum_{k=1}^{r-1} \left( \int_{x_k}^{x_r} v_\gamma \right) \left( \text{res}_{x_k} v_{\alpha\beta} \frac{dC}{v} \right) - 2\pi i \sum_{k=1}^{\deg D_{br}} \frac{v_\gamma}{d \ln (v/d_\zeta)} (x_k) \text{res}_{x_k} v_{\alpha\beta} (v)
\]

Due to the Riemann bilinear identity the sum of first three terms in (4.19) vanishes. The remaining terms give (4.2).

Formulas (4.3) and (4.4) are obtained in a similar way by applying Riemann bilinear identities to pairs the \((u^{(s)}_j, \frac{v_{\alpha\beta}}{v})\) and \((u^{(s)}_j, \frac{v_{\alpha\beta}}{v})\), respectively.

We give below the computation leading to (4.4); the proof of (4.3) requires only minimal modifications. Taking into account (4.9) we get (recall that all \( a \)-periods of \( u^{(s)}_j \) vanish)

\[
\frac{d\Omega_{\alpha\beta}}{dC^{(s),1}_j} = \frac{\partial \Omega_{\alpha\beta}}{\partial C^{(s),1}_j} + \sum_{\delta=1}^{g} \left[ \frac{\partial \Omega_{\alpha\beta}}{\partial A_\delta} \frac{\partial A_\delta}{\partial C^{(s),1}_j} + \frac{\partial \Omega_{\alpha\beta}}{\partial B_\delta} \frac{\partial B_\delta}{\partial C^{(s),1}_j} \right] + \sum_{k=1}^{r-1} \frac{\partial \Omega_{\alpha\beta}}{\partial \left( \int_{x_k}^{x_r} v \right)} \frac{\partial \left( \int_{x_k}^{x_r} v \right)}{\partial C^{(s),1}_j}.
\]

We have \( \partial A_\delta / \partial C^{(s),1}_j = 0 \) since all \( a \)-periods of \( u^{(s)}_j \) vanish; according to (3.8),

\[
\frac{\partial \Omega_{\alpha\beta}}{\partial C^{(s),1}_j} = -2\pi i \left( \int_{y^{(s)}_1}^{y^{(s)}_j} v_{\alpha\beta} \frac{dC^{(s),1}_j}{v} \right),
\]

which gives

\[
\frac{d\Omega_{\alpha\beta}}{dC^{(s),1}_j} = -2\pi i \left( \int_{y^{(s)}_1}^{y^{(s)}_j} v_{\alpha\beta} \frac{dC^{(s),1}_j}{v} \right) + \sum_{\delta=1}^{g} \left[ \left( \int_{y^{(s)}_1}^{y^{(s)}_j} v_{\alpha\beta} \frac{dC^{(s),1}_j}{v} \right) \right]
\]

\[
+ 2\pi i \sum_{k=1}^{r-1} \left( \text{res}_{x_k} v_{\alpha\beta} \frac{dC^{(s),1}_j}{v} \right) \left( \int_{x_k}^{x_r} u^{(s)}_j \right) - 2\pi i \sum_{k=1}^{\deg D_{br}} \frac{u^{(s)}_j}{d \ln (v/d_\zeta)} (x_k) \text{res}_{x_k} v_{\alpha\beta} (v) .
\]
Again, the Riemann bilinear identities applied to the pair of differentials of third kind $u^{(s)}_j$ and $v_\alpha u_j^g$ prove the vanishing of the sum of all terms except the last one, leading to (4.4) (we notice that these two differentials have different positions of poles).

4.1. Variations of Abelian differentials. Here we are going to use variational formulæ (3.10)-(3.12) on moduli spaces of abelian differentials to derive the following analogs of Theorem 4.1.

**Theorem 4.3.** The variations of canonical differentials $v_\alpha$ with respect to coordinates (2.7) on $\mathcal{M}_H^n[k]$ are expressed by the following formulæ:

\[
\frac{\partial v_\alpha(x)}{\partial A_\gamma} = - \sum_{x_i \in D_{br}} \frac{v_\gamma}{d \ln(v/d\xi)}(x_i) \text{res}_{t=x_i} v_\alpha(t)B(t,x) v(t) \tag{4.23}
\]

\[
\frac{\partial v_\alpha(x)}{\partial C_j^{(s),l}} = - \sum_{x_i \in D_{br}} \frac{w_j^{(s),l}}{d \ln(v/d\xi)}(x_i) \text{res}_{t=x_i} v_\alpha(t)B(t,x) v(t) \tag{4.24}
\]

\[
\frac{\partial v_\alpha(x)}{\partial C_j^{(s),1}} = - \sum_{x_i \in D_{br}} \frac{v_j^{(s)}}{d \ln(v/d\xi)}(x_i) \text{res}_{t=x_i} v_\alpha(t)B(t,x) v(t) \tag{4.25}
\]

where $\xi$ is a local coordinate on $C$ near $x_i$ as in Theorem 4.1. The right-hand side of (4.5) is independent of the choice of these coordinates near $x_r$.

**Proof.** Let us show how to derive (4.23) from variational formulæ (3.10). In comparison with the variational formulæ for $\Omega$ proven above it is essential to carefully consider the dependence of $v_\alpha$ on the point of $\tilde{C}$, since the latter is deforming. Moreover, the variation of $v_\alpha$ with respect to $P_j$ used in (3.10) is defined by (3.9) where the “flat” coordinate is kept fixed, while in (4.23) the differentiation is performed according to the rule (2.12) where $\xi$ is a local parameter lifted to $\tilde{C}$ from $C$ which is assumed to be independent of moduli coordinates on $\mathcal{M}_H^n[k]$.

Taking into account these differences, one can compute the left-hand side of (4.23) as follows. Let $f_\alpha(x) = v_\alpha/v$: then the left-hand side of (4.23) is rewritten as

\[
\frac{\partial v_\alpha(x)}{\partial A_\gamma} = \frac{\partial (vf_\alpha(x))}{\partial A_\gamma} = \frac{\partial f_\alpha(x)}{\partial A_\gamma} v(x) + f_\alpha(x) \frac{\partial v(x)}{\partial A_\gamma} \big|_{\xi(x)}
\]

\[
= \frac{\partial f_\alpha(x)}{\partial A_\gamma} v(x) + f_\alpha(x) \frac{\partial v(x)}{\partial A_\gamma} \big|_{\xi(x)} + f_\alpha(x) v_\gamma(x)
\]

\[
= \frac{\partial f_\alpha(x)}{\partial A_\gamma} v(x) + \mathcal{A}_\gamma(x) d\left(\frac{v_\alpha}{v}\right) + \frac{v_\alpha v_\gamma(x)}{v}
\]

where $\mathcal{A}_\gamma(x) = \int_{x_r} x, v_\gamma$ is the component $\gamma \in \{1, \ldots, g\}$ of the Abel map.

The computation of the first term in (4.26) can then be performed in complete analogy to (4.19) with the differential $\frac{v_\alpha v_\gamma(x)}{v}$ replaced by the differential $\frac{1}{2\pi i} \frac{v_\alpha(t)B(t,x)}{v(t)}$. Applying
the Riemann bilinear relations to the differentials \(v_\gamma\) and 
\[
\frac{1}{2\pi i} \frac{v_\alpha(t)B(x,t)}{v(t)}
\]
we obtain the sum of terms entering the right-hand side of (4.23) minus the residue of 
\[
\frac{1}{2\pi i} \frac{v_\alpha(t)B(x,t)}{v(t)} \int_{v_x} v_\gamma \text{ at } t = x.
\]
This residue is equal to the sum of the last two terms in (4.26) with the opposite sign. This gives (4.23). The proofs of the formulæ (4.24) and (4.25) are parallel.

4.2. Variations of prime-form and canonical bidifferential. Variational formulæ for \(E(x,y)\) and \(B(x,y)\) can be proven in parallel to Th.4.3.

As in the case of normalized canonical differential, we define the derivative of \(B(x,y)\) and \(E(x,y)\) with respect to any coordinate \(z_i\) on \(\mathcal{M}_H^n[k]\) as

\[
\frac{\partial B(x,y)}{\partial z_i} = \frac{\partial}{\partial z_i} \left( \frac{B(x,y)}{d\xi(x)d\xi(y)} \right) d\xi(x)d\xi(y)
\]

\[
\frac{\partial E(x,y)}{\partial z_i} = \frac{\partial}{\partial z_i} \left( E(x,y)[d\xi(x)d\xi(y)]^{1/2} \right) [d\xi(x)d\xi(y)]^{-1/2}
\]

where \(\xi(x)\) and \(\xi(y)\) are local coordinates lifted to \(\hat{C}\) from moduli-independent local coordinates on \(C\), and these coordinates remain fixed under differentiation.

**Theorem 4.4.** The variations of the canonical bidifferential \(B(x,y)\) with respect to the coordinates (2.7) on \(\mathcal{M}_H^n[k]\) are given by:

\[
\frac{\partial B(x,y)}{\partial A_{\gamma}} = - \sum_{x_i \in \mathcal{D}_{br}} \frac{v_\gamma}{d\ln(v/d\xi)}(x_i) \text{ res}_{t=x_i} \frac{B(x,t)B(t,y)}{v(t)}
\]

\[
\frac{\partial B(x,y)}{\partial C_{s,j}} = - \sum_{x_i \in \mathcal{D}_{br}} \frac{u_{s,j}}{d\ln(v/d\xi)}(x_i) \text{ res}_{t=x_i} \frac{B(x,t)B(t,y)}{v(t)}
\]

\[
\frac{\partial B(x,y)}{\partial C_{s,j}^{(s),l}} = - \sum_{x_i \in \mathcal{D}_{br}} \frac{u_{s,j}^{(s),l}}{d\ln(v/d\xi)}(x_i) \text{ res}_{t=x_i} \frac{B(x,t)B(t,y)}{v(t)}
\]

where \(\xi\) is a local coordinate on \(C\) near \(x_i\); the right-hand side of (4.5) is independent on the choice of these coordinates near \(x_i\).

**Theorem 4.5.** The variations of the prime-form with respect to coordinates (2.7) on \(\mathcal{M}_H^n[k]\) are given by:

\[
\frac{\partial \ln E(x,y)}{\partial A_{\gamma}} = - \frac{1}{2} \sum_{x_i \in \mathcal{D}_{br}} \frac{v_\gamma}{d\ln(v/d\xi)}(x_i) \text{ res}_{t=x_i} \frac{1}{v(t)} \left[ d_t \ln \frac{E(x,t)}{E(y,t)} \right]^2
\]

\[
\frac{\partial \ln E(x,y)}{\partial C_{s,j}^{(s),l}} = - \frac{1}{2} \sum_{x_i \in \mathcal{D}_{br}} \frac{w_{s,j}^{(s),l}}{d\ln(v/d\xi)}(x_i) \text{ res}_{t=x_i} \frac{1}{v(t)} \left[ d_t \ln \frac{E(x,t)}{E(y,t)} \right]^2
\]
\[
\frac{\partial \ln E(x, y)}{\partial C_j(x, y)} = -\frac{1}{2} \sum_{x_i \in D_{br}} \frac{u_j^{(s)}}{d \ln(v/d\xi)}(x_i) \text{res}_{t=x_i} \left\{ \frac{1}{v(t)} \left[ \frac{d_t \ln E(x, t)}{E(y, t)} \right]^2 \right\}
\]

where \(\xi\) is a local coordinate on \(C\) near \(x_i\); the right-hand side of (4.5) is independent on the choice of these coordinates near \(x_i\).

4.3. The Bergman tau-function on spaces of spectral covers. The Bergman tau function on the moduli spaces of abelian differentials is a natural higher genus analog of the Dedekind’s eta-function [18, 14, 15]. One can define two natural tau functions associated to the moduli space of spectral covers; in the case of holomorphic \(v\) these tau functions were introduced in [19] and used to study the Picard group of the moduli spaces (in [19] we considered the tau functions on universal spaces of spectral covers i.e. we allowed the base curve \(C\) to vary).

Here we restrict ourselves to the case of holomorphic \(v\), namely, to moduli space \(\mathcal{M}_n^H\) of spectral covers of the ordinary Hitchin systems. In this case the equations for the Bergman tau functions take a similar form to the variational formulae for the canonical objects considered above.

Denote the moduli space of ordinary Hitchin’s spectral covers by \(\mathcal{M}_n^H\); in this case all coefficients \(Q_k\) of the equation (2.1) are holomorphic \(k\)-differentials, the genus of the spectral cover is \(\hat{g} = n^2(g - 1) + 1\), the number of branch points is \(p = n(n - 1)(2g - 2)\) and the total number of zeros of \(v\) is \(r = 2\hat{g} - 2 = p + 2n(g - 1)\). The differential \(v\) is holomorphic, and the local coordinates on \(\mathcal{M}_n^H\) are given by the \(a\)-periods \(A_\gamma = \int_{\alpha_\gamma} v\).

Considering \(\mathcal{M}_n^H\) as a subspace of the space of holomorphic Abelian differentials with simple zeros \(\mathcal{H}_\hat{g}\) we define the Bergman tau function on \(\mathcal{M}_n^H\) by restriction of the Bergman tau function (3.13) on \(\mathcal{H}_\hat{g}\).

The resulting equations for \(\tau_B\) (this tau function is defined by the formula (4.3) of [19]) as function of periods \(A_\gamma\) can be derived from (3.13) in complete analogy to the proof of (4.2):

\[
\frac{\partial \ln \tau_B}{\partial A_\gamma} = -2\pi i \sum_{x_i \in D_{br}} \frac{v_\gamma}{d \ln(v/d\xi)}(x_i) \text{res}_{x_i} \frac{B_{reg}^\gamma}{v}
\]

where the regularization of \(B(x, y)\) on the diagonal is defined by (3.14).

5. Higher order derivatives on \(\mathcal{H}_\hat{g}\) and \(\mathcal{M}_n^H\)

5.1. Space \(\mathcal{H}_\hat{g}\). Higher order derivatives of the canonical objects with respect to moduli on the space \(\mathcal{H}_\hat{g}\) can be obtained by a simple iteration of first derivatives.

We start from multiple derivatives for the Bergman tau function.

Using the coordinates \(P_i = \int_{s_i} v\) where \(s_i \in H_1(\tilde{C}, \{x_i\}_{i=1}^r)\) we get from (3.13) and (3.11):

\[
\frac{\partial^2}{\partial P_i \partial P_j} \ln \tau_B = \frac{1}{2\pi i} \int_{s_i} \int_{s_j} \frac{B^2(x, y)}{v(x)v(y)}
\]
Further differentiation using (3.11) gives
\[
\frac{\partial^3}{\partial \mathcal{P}_i \partial \mathcal{P}_j \partial \mathcal{P}_k} \ln \tau_B = \frac{2}{(2\pi i)^2} \int_{x \in s^*_1} \int_{y \in s^*_j} \int_{t \in s^*_k} \frac{B(x, y)B(x, t)B(t, y)}{v(x)v(y)v(t)}
\]
The $n$th derivatives of $\tau_B$ are given by
\[
\frac{\partial^{(n)}}{\partial \mathcal{P}_{i_1} \ldots \partial \mathcal{P}_{i_n}} \ln \tau_B = \frac{1}{(2\pi i)^{n-1}} \int_{s^*_1} \cdots \int_{s^*_n} Q_n(z_{i_1}, \ldots, z_{i_n})
\]
where completely symmetric multi-differential $Q_n$ is given by
\[
Q_n(z_1, \ldots, z_n) = 2 \sum_{s} \frac{\prod_{j=1}^{n} B(z_{k_j}, z_{k_{j+1}})}{\prod_{j=1}^{n} v(z_j)}
\]
The sum runs over all $(n-1)!/2$ permutations $\Gamma = (k_1, \ldots, k_n)$ of $z_1, \ldots, z_n$ which form a cycle of length $n$ (two such permutations are considered equivalent if they are related by cyclic permutation i.e. we do not distinguish between $(1234)$ and $(2341)$; $k_{n+1}$ is identified with $k_1$.

The multi-differentials $Q_n(z_1, \ldots, z_n)$ satisfy the relations
\[
\frac{\partial}{\partial \mathcal{P}_i} Q_n(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{1 \in s^*_i} Q_n(z_1, \ldots, z_n, t).
\]

Another natural hierarchy of multi-differentials (although no longer completely symmetric) which are given by combinations of $B(x, y)$ can be obtained by differentiation of $B(x, y)$ itself. Namely, using the variational formula (3.11) on the space $\mathcal{H}_g$ we get
\[
\frac{\partial^n}{\partial \mathcal{P}_{i_1} \ldots \partial \mathcal{P}_{i_n}} B(x, y) = \frac{1}{(2\pi i)^n} \int_{s^*_1} \cdots \int_{s^*_n} R_{n+2}(x, z_{i_1}, \ldots, z_{i_n}, y)
\]
where the multi-differentials $R_n$ with $n$ arguments are given by
\[
R_n(z_1, \ldots, z_n) = \sum_{\text{all } \Gamma} \frac{\prod_{j=1}^{n-1} B(z_{k_j}, z_{k_{j+1}})}{\prod_{j=2}^{n} v(z_j)}
\]
where in all products entering this sum, the indices $k_1, k_n$ are given by $k_1 = 1$ and $k_n = n$; the sum goes over all $(n-2)!$ paths $\Gamma$ connecting $x_1$ with $x_n$ which go only once through every vertex representing other arguments $x_2, \ldots, x_{n-1}$.

The multi-differentials $R_n$ are symmetric with respect to the interchange of the intermediate arguments $x_2, \ldots, x_{n-1}$, but not fully symmetric, in contrast to $Q_n$.

The families of multi-differentials $Q_n$ and $R_n$ as well as their variational formulæ resemble the structures arising in the framework of topological recursion of [8] (the genus of the base curve $C$ equals zero in constructions of [8]). However, all multi-differentials considered in [8] are completely symmetric while $R_n$ are not (although the $Q_n$’s are).
Moreover, both $Q_n$’s and $R_n$’s have second order poles when any two arguments coincide while the multidifferentials $W_n$ of [8] have poles only at the ramification points of the cover.

The formula (5.6) implies the following expression for multiple derivatives of the period matrix $\Omega$ of $\hat{C}$ on the space $\mathcal{H}_g$:

$$\frac{\partial^{(n)}}{\partial P_{i_1} \cdots \partial P_{i_n}} \Omega_{\alpha\beta} = \frac{1}{(2\pi i)^{n-1}} \int_{s^*_{i_1}} \cdots \int_{s^*_{i_n}} \mathcal{R}_{n^\alpha\beta}(z_{i_1}, \ldots, z_{i_n})$$

where

$$\mathcal{R}_{n^\alpha\beta}(z_{i_1}, \ldots, z_{i_n}) = \int_{x \in b_{\alpha}} \int_{y \in b_{\beta}} R_{n+2}(x, z_{i_1}, \ldots, z_{i_n}, y)$$

or

$$\mathcal{R}_{n^\alpha\beta}(z_{i_1}, \ldots, z_{i_n}) = v_\alpha(z_{i_1})v_\beta(z_{i_n}) \sum_{\text{all } \tilde{\Gamma}} \prod_{j=1}^{n-1} B(z_{k_j}, z_{k_{j+1}}) \prod_{j=1}^{n} v(z_{i_j})$$

where, as before, in all products entering this sum $k_1 = 1$ and $k_n = n$; the sum goes over all $(n - 2)!$ paths $\tilde{\Gamma}$ connecting $x_1$ with $x_n$ which go only once through every vertex representing other arguments $x_2, \ldots, x_{n-1}$.

5.2. The space $\mathcal{M}_{H}^n$. On spaces of spectral covers $\mathcal{M}_{H}^n$ multi-differentials $Q_n$ are related to $Q_{n+1}$ by formulæ which can be derived from (5.5) in parallel to the proof of (4.23):

$$\frac{\partial}{\partial A^\gamma} Q_n(z_{i_1}, \ldots, z_n) = - \sum_{x_i \in D_{br}} \frac{v_\gamma}{d \ln(v/d\xi)}(x_i) \res_{t=x_i} \left\{ \frac{1}{v(t)} Q_{n+1}(z_{i_1}, \ldots, z_n, t) \right\}$$

Similarly, the multi-differentials $R_n$ and $R_{n+1}$ are related by

$$\frac{\partial}{\partial A^\gamma} R_n(z_{i_1}, \ldots, z_n) = - \sum_{x_i \in D_{br}} \frac{v_\gamma}{d \ln(v/d\xi)}(x_i) \res_{t=x_i} \left\{ \frac{1}{v(t)} R_{n+1}(z_{i_1}, \ldots, z_{n-1}, t, z_n) \right\}$$

Integrating (5.10) over two $b$-cycles with respect to $z_1$ and $z_2$ we get similar formulæ for $R_{n^\alpha\beta}$ (5.9).

While higher derivatives of the period matrix, tau-function and canonical bidifferential on the space $\mathcal{H}_3$ are given by a simple formulæ (5.3), (5.6) and (5.8) their restriction to the space $\mathcal{M}_{H}^n$ is much less trivial. As an example of such computation we find below the second derivatives of the period matrix.

5.2.1. Second derivatives of $\Omega_{\alpha\beta}$. The period matrix on the space $\Omega_{\alpha\beta}$ is known to be given by second derivatives of a single function (the "prepotential")

$$F_0 = \frac{1}{2} \sum_{\gamma=1}^{\delta} A^\gamma B_{\gamma}.$$
We recall the proof of (5.12). Using the relation \( \partial B_\gamma / \partial A_\alpha = \Omega_{\alpha \beta} \) we get

\[
\frac{\partial F_0}{\partial A_\alpha} = \frac{1}{2} \left( B_\alpha + \sum_{\gamma=1}^{g} A_\gamma \Omega_{\alpha \gamma} \right)
\]

and

\[
\frac{\partial^2 F_0}{\partial A_\alpha \partial A_\beta} = \Omega_{\alpha \beta} + \frac{1}{2} \sum_{\gamma=1}^{g} A_\gamma \partial \Omega_{\alpha \gamma} \partial A_\beta.
\]

The last sum in this formula equals zero since, due to \((\alpha, \beta, \gamma)\)-symmetry of the formula (4.5) for \( \partial \Omega_{\alpha \beta} / \partial A_\gamma \), we have

\[
\sum_{\gamma=1}^{g} A_\gamma \partial \Omega_{\alpha \gamma} \partial A_\beta = \left( \sum_{\gamma=1}^{g} A_\gamma \partial \Omega_{\alpha \gamma} \right) \Omega_{\alpha \beta}.
\]

The last expression vanishes because it is the action of the scaling operator \( E = \sum_{\gamma=1}^{g} A_\gamma \partial \Omega_{\alpha \gamma} \partial A_\beta \) generating the map \( \nu \mapsto \lambda \nu (\lambda \in \mathbb{C}^*) \) and the period matrix is clearly invariant under such rescaling.

Due to (5.12) all higher derivatives of \( \Omega_{\alpha \beta} \) in \( A_\gamma \)'s are also completely symmetric with respect to all indices.

It is convenient to use the following notation:

\[
y = \frac{v}{d\xi}
\]

which is a function defined on the union of small disks on \( \hat{C} \) around ramification points \( x_i \) depending on the choice of local parameter \( \xi \) on \( C \) near each branch point \( x_i \). Since \( v \) has a simple zero at \( x_i \), in a neighbourhood of \( x_i \) \( y \) is a holomorphic function of the corresponding local parameter \( \hat{\xi}_i = \sqrt{\xi - \xi(x_i)} \).

To compute second derivatives of \( \Omega_{\alpha \beta} \) on \( M_H^n \) one can differentiate the formula (4.5)

\[
\frac{\partial \Omega_{\alpha \beta}}{\partial A_\gamma} = -2\pi i \sum_{x_i \in DBr} \text{res}_{x_i} \frac{v_\alpha v_\beta v_\gamma}{d\xi dy} d\xi dy
\]

with respect to coordinate \( A_\delta \) using (2.13) and (4.23). Then due to (4.23) we have

\[
\frac{\partial(dy)}{\partial A_\delta} = d \left( \frac{v_\delta}{d\xi} \right)
\]

which has second order pole at \( x_i \). We have then

\[
\frac{\partial^2 \Omega_{\alpha \beta}}{\partial A_\delta \partial A_\gamma} = 2\pi i \sum_{x_i, x_j \in Br} \text{res}_{t=x_i} \text{res}_{t=x_j} \left\{ B(t, \tilde{t}) \frac{v_\delta(t) v_\gamma(t) v_\alpha(t) v_\beta(t) + (\text{perm of } (\alpha, \beta, \gamma))}{(dy d\xi)(t) (dy d\xi)(\tilde{t})} \right\}
\]
and also from first order zero of $d\xi$ at $x_i$. Namely, we have $d\xi = 2\xi d\xi$ near each $x_i$ and using the notation $v_\alpha(x_i) = v_\alpha/d\xi(x_i)$ we have

$$\text{res}_{i=x_i} B(t, \xi) v_\delta(t) v_\gamma(t) = \frac{1}{2} B(t, x_i) v_\delta(x_i) v_\gamma(x_i)$$

where the notation $v_\gamma(x_i)$ is used to denote $(v_\gamma/d\xi)(x_i)$; prime denotes the derivative in $\xi_i$.

The resulting expression has third order pole at $x_i$ (it arises from double pole of $B(t, x_i)$ and also from first order zero of $d\xi$):

$$\text{res}_{i=x_i} \left\{ \frac{B(t, x_i)}{(dyd\xi)(t)} v_\alpha(t) v_\beta(t) \right\} = \frac{1}{2} B_{\text{reg}}^d(x_i) \frac{v_\alpha v_\beta}{y''(x_i)} + \frac{1}{2} \left( \frac{v_\alpha v_\beta}{y'} \right)''$$

where

$$B_{\text{reg}}^d(x_i) = \left( B(x, y) - \frac{d\xi(x)d\xi(y)}{(\xi(x) - \xi(y))^2} \right)_{x=y=x_i}$$

is equal to 1/6 of the Bergman projective connection computed at $x_i$ in coordinate $\xi_i$.

To compute the last residue in (5.15) we notice that the corresponding expression has a pole of third order at $x_i$. Starting from

$$v_\delta = \left( v_\delta(x_i) + v_\delta'\xi_i + \frac{v_\delta''}{2}\xi_i^2 + \ldots \right) d\xi_i \quad d\xi = 2\xi_i d\hat{\xi}_i$$

and

$$\frac{1}{d\xi} d\left( \frac{v_\delta}{d\xi} \right) = \frac{-v_\delta}{4\xi_i^3} + \frac{v_\delta''}{8\xi_i^2} + \ldots$$

the last term in (5.15) can be computed as follows:

$$\text{res}_{i=x_i} \left\{ \frac{v_\alpha v_\beta v_\gamma}{(dy)^2} \right\} = -\frac{1}{8} \left( \frac{v_\alpha v_\beta v_\gamma}{(y')^2} \right)^{''} v_\delta + \frac{1}{8} v_\alpha v_\beta v_\gamma v_\delta''.$$ 

Now the formula (5.15) can be written as

$$\frac{1}{2\pi i} \sum_{x_i \in Br} \left\{ B(x_i, x_j) v_\delta(x_i) v_\gamma(x_i) v_\alpha(x_j) v_\beta(x_j) + \left( \text{cycl of } (\alpha, \beta, \gamma) \right) \right\}$$

$$+ \frac{1}{4} \sum_{x_i \in Br} B_{\text{reg}}^d v_\delta + \left( \text{cycl}(\alpha, \beta, \gamma) \right) (x_i) + \frac{1}{8} \sum_{x_i \in Br} \left( \frac{v_\alpha v_\beta}{y'} \right)^{''} (x_i) + \left( \text{cycl}(\alpha, \beta, \gamma) \right)$$
Using the complete symmetry of this expression in indices $(\alpha, \beta, \gamma, \delta)$ we can compute various terms as follows:

- The coefficient of $\frac{y''}{y^2}(v'_\alpha v_\beta v_\gamma v_\delta + \text{cycl}(\alpha, \beta, \gamma, \delta))$ is $(1/8)(1/4)2(-2 \cdot 3 + (3 \cdot 2)) = 0$
- The coefficient of $(1/y^2)B_{\text{reg}}^\delta v_\alpha v_\beta v_\gamma v_\delta$ is $3/4$
- The coefficient of $(1/y^2)(v'_\alpha v'_\beta v_\gamma v_\delta + \text{cycl}(\alpha, \beta, \gamma, \delta))$ is $3(1/4)(1/4) - (1/8)(1/4) \cdot 3 = 0$
- The coefficient of $(y''/y^3)v_\alpha v_\beta v_\gamma v_\delta$ is $(-1/8) \cdot 3 + (-1/8)(-2) = -1/8$.
- The coefficient of $(1/y^2)(v''_\alpha v_\beta v_\gamma v_\delta + \text{cycl}(\alpha, \beta, \gamma, \delta))$ is $(1/8)(2 \cdot 3/4 - 3/4 + 1/4) = 1/8$.

Therefore we get the following proposition:

**Proposition 5.1.**

\[
\frac{1}{2\pi i} \frac{\partial^2 \Omega_{\alpha \beta}}{\partial A_\delta \partial A_{\gamma}} = \frac{1}{4} \sum_{x_i \neq x_j \in Br} \left[ B(x_i, x_j) v_\delta(x_i) v_\alpha(x_i) v_\beta(x_j) + \text{cycl}(\alpha, \beta, \gamma) \right] \frac{y'(x_i) y'(x_j)}{y(x_i) y(x_j)}
\]

\[
+ \frac{1}{8} \sum_{x_i \in Br} \left[ \left( \frac{6B_{\text{reg}}^\delta}{y^2} - \frac{y''}{y^3} \right) v_\alpha v_\beta v_\gamma v_\delta(x_i) + \frac{1}{y^2} (v''_\alpha v_\beta v_\gamma v_\delta + \text{cycl}(\alpha, \beta, \gamma)) \right].
\]

The formula (5.16) coincides with the expression obtained in Theorem 7.5 of [2] using the framework of topological recursion of [8] (notice that $6B_{\text{reg}}^\delta$ nothing but the Bergman projective connection $S_B$ which enters the formula (7.4) of [2].

Concluding, we have shown that the deformation calculus on spaces of Hitchin’s spectral covers can be naturally induced from a much more transparent deformation theory on the moduli space of holomorphic or meromorphic Abelian differentials on Riemann surfaces. Considering a close relationship between deformations of spectral covers and the theory of topological recursion of [8] it is natural to expect that the topological recursion itself could be a manifestation of a much less involved structure associated to moduli spaces of abelian differentials.

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