On determination of perfectly plastic disturbed rock mass state on condition of full plasticity

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Abstract. A solution is given for determining the stress state of a compressible elastic-plastic space with a spherical cavity. In the problem statement, there is no pressure inside the cavity and mutually perpendicular forces are applied at infinity. The problem is solved with the use of the small parameter method, in the spherical coordinate system, in dimensionless length units. Zero values of stresses in elastic and plastic regions, zero approximation of the elastoplastic zone boundary for the spherical cavity case as well as first approximations of stress components in the plastic region. The obtained results can be used in the mining industry, structural mechanics and other adjacent fields.

1. Introduction

In the mining industry, structural mechanics and other adjacent fields determination of the stressed and deformed state of rock mass around cavities is of great importance. To accurately characterize the rock mass properties, both elastic and plastic behavior of the rock need to be studied. Those properties can be temperature and/or strain-rate dependent in some cases.

The present paper investigates the stress state of a perfectly plastic rock mass weakened by a cavity. The limit equilibrium theory relations originating from Coulomb's works are used to determine the stress state in the plastic region. The development of this theory is associated with the names of A.Yu. Ishlinsky [1], who developed the original theory of dynamic stability, D.D. Ivlev [2], L.V. Ershov [3], who devoted a series of his works to the linearized problems of the elastoplastic state of bodies.

In the given problem a perfectly plastic stress state is determined by boundary conditions on the cavity surface. The solution in the elastic region mates with the existing solution for the elastic hollow sphere.

Initially, the problem of the elastic sphere equilibrium was examined by G. Lame [4]. W. Thomson in solving the problem of the equilibrium of a continuous and hollow sphere [5] proceeded from the representation of the solution of the equations of the theory of elasticity in displacements through three harmonic functions, which he sought in the form of series with respect to the spatial harmonic polynomials \( \phi_n(x, y, z) \). A.I. Lurie [6], while following Tomson's method, relying on expressing...
stress tensor components in terms of harmonic functions suggested by P.F. Popkovitch, solved the general problem of the elastic sphere equilibrium. B.V. Galerkin [7] gave solutions relating to the problem of the hollow symmetrically-loaded sphere and introduced construction of a class of solutions that can be used to solve problems of equilibrium of an elastic body bounded by two concentric spheres and shears along conical surfaces with the top in the sphere center. It should be noted that the hollow sphere problem was also examined by E. Sternberg, R. Enbanes and M. Sadowsky [8]. Also, Sadowsky and Sternberg examined the problem of stress state in the neighborhood of an ellipsoidal cavity [9]. Lurie [10] solved this problem in Cartesian coordinates.

For the first time, the problem of triaxial tension of incompressible elastoplastic space with a spherical cavity was investigated by T.D. Semykina [11]. Earlier A.N. Maksimov and V.G. Efremova [12, 13] considered spaces with spherical and ellipsoidal cavities in the case of an incompressible elastoplastic material.

2. Formulation of the problem

The problem is solved in a spherical coordinate system, in dimensionless units of length (all quantities having the dimension of length are divided to spherical cavity radius \( \rho_0 \)) by the method of a small parameter (the solution is sought in the form of a series expansion (6) in powers of a small parameter \( \delta = 1 \)). The method is developed in the works of D.D. Ivlev and allows to obtain solutions for a number of axisymmetric, spatial problems.

A mass of loose medium that has internal friction and cohesion properties is examined. The loose medium limit state is determined as [2]:

\[
f(\sigma'_i) = k_0 + a\sigma,
\]

where \( \sigma'_i \) is stress deviator components, \( k_0 \) is cohesion coefficient, \( a = \tan \alpha \) is internal friction coefficient, \( \alpha \) is internal friction angle.

To solve the problem in the spherical coordinate system, we use the equilibrium equation [2]:

\[
\frac{\partial \sigma}{\partial \rho} + \frac{1}{\rho} \frac{\partial \tau_{\rho\theta}}{\partial \theta} + \frac{1}{\rho \sin \theta} \frac{\partial \tau_{\rho\phi}}{\partial \phi} + \frac{1}{\rho} (2\sigma_{\rho} - \sigma_{\theta} - \sigma_{\phi} + \tau_{\rho\phi} \cot \theta) = 0,
\]

\[
\frac{\partial \tau_{\rho\theta}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \sigma_{\theta}}{\partial \theta} + \frac{1}{\rho \sin \theta} \frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{1}{\rho} ((\sigma_{\theta} - \sigma_{\phi}) \cot \theta + 3\tau_{\rho\phi}) = 0,
\]

\[
\frac{\partial \tau_{\rho\phi}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \sigma_{\phi}}{\partial \theta} + \frac{1}{\rho \sin \theta} \frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{1}{\rho} (3\tau_{\rho\theta} + 2\tau_{\theta\phi} \cot \theta) = 0.
\]

Tresca-Saint-Venant plasticity conditions [2] with account for (1):

\[
\begin{align*}
\left( \sigma_{\rho} - \sigma + \frac{2}{3}(k_0 + a\sigma) \right)
& \left( \sigma_{\theta} - \sigma + \frac{2}{3}(k_0 + a\sigma) \right) - \tau^2_{\rho\phi} = 0, \\
\left( \sigma_{\theta} - \sigma + \frac{2}{3}(k_0 + a\sigma) \right)
& \left( \sigma_{\phi} - \sigma + \frac{2}{3}(k_0 + a\sigma) \right) - \tau^2_{\phi\theta} = 0, \\
\left( \sigma_{\phi} - \sigma + \frac{2}{3}(k_0 + a\sigma) \right)
& \left( \sigma_{\rho} - \sigma + \frac{2}{3}(k_0 + a\sigma) \right) - \tau^2_{\rho\phi} = 0,
\end{align*}
\]

as well

\[
\begin{align*}
\left( \sigma_{\rho} - \sigma + \frac{2}{3}(k_0 + a\sigma) \right) \tau_{\rho\theta} &= \tau_{\rho\phi} \tau_{\rho\phi}, \\
\left( \sigma_{\phi} - \sigma + \frac{2}{3}(k_0 + a\sigma) \right) \tau_{\rho\phi} &= \tau_{\rho\theta} \tau_{\rho\theta},
\end{align*}
\]
\[
\left(\sigma_\theta - \sigma + \frac{2}{3}(k_0 + a\sigma)\right)\tau_{\rho\phi} = \tau_{\rho\phi}\tau_{\theta\phi}.
\]

Boundary conditions:
\[
\sigma_{\rho}, l + \tau_{\rho\phi}m + \tau_{\phi\rho}n = \rho_m, \tau_{\rho\phi}, l + \sigma_{\rho\phi}m + \tau_{\phi\rho}n = \rho_m, \tau_{\rho\phi}, l + \tau_{\phi\rho}m + \sigma_{\rho\phi}n = \rho_m,
\]
where \(\sigma_{\rho}, \tau_{\rho\phi}, \ldots\) are stress deviator components, \(l, m, n\) are normal direction cosines, \(\rho_m, \rho_m, \rho_m\) are force projections on the axis \(\rho, \theta, \phi\), \(\sigma = (\sigma_{\rho} + \sigma_{\theta} + \sigma_{\phi})/3\) is mean pressure.

We represent stress components as small parameter series \(\delta\) (\(\delta = 1\)):
\[
\sigma_{\rho} = \sigma_0^\rho + \delta\sigma_{\rho}', \sigma_{\theta} = \sigma_0^\theta + \delta\sigma_{\theta}', \sigma_{\phi} = \sigma_0^\phi + \delta\sigma_{\phi}',
\]
\[
\tau_{\rho\phi} = \tau_0^\rho + \delta\tau_{\rho\phi}', \tau_{\rho\phi} = \tau_0^\rho + \delta\tau_{\rho\phi}', \tau_{\theta\phi} = \tau_0^\theta + \delta\tau_{\theta\phi}'.
\]
Plasticity conditions (3) and (4) can be met in three cases:
\[
\sigma_{\rho}^0 - \sigma_0^0 + \frac{2}{3}(k_0 + a\sigma_0^0) = 0, \quad \sigma_{\theta}^0 - \sigma_0^0 + \frac{2}{3}(k_0 + a\sigma_0^0) \neq 0, \quad \sigma_{\phi}^0 - \sigma_0^0 + \frac{2}{3}(k_0 + a\sigma_0^0) = 0. (7)
\]
\[
\sigma_{\rho}^0 - \sigma_0^0 + \frac{2}{3}(k_0 + a\sigma_0^0) = 0, \quad \sigma_{\theta}^0 - \sigma_0^0 + \frac{2}{3}(k_0 + a\sigma_0^0) = 0, \quad \sigma_{\phi}^0 - \sigma_0^0 + \frac{2}{3}(k_0 + a\sigma_0^0) \neq 0. (8)
\]
\[
\sigma_{\rho}^0 - \sigma_0^0 + \frac{2}{3}(k_0 + a\sigma_0^0) \neq 0, \quad \sigma_{\theta}^0 - \sigma_0^0 + \frac{2}{3}(k_0 + a\sigma_0^0) = 0, \quad \sigma_{\phi}^0 - \sigma_0^0 + \frac{2}{3}(k_0 + a\sigma_0^0) = 0. (9)
\]

3. Determination of zero stress components

Let's consider Case (7). When solving (7) in combination with (3), we have:
\[
\sigma_0^0 = \sigma_0^\phi, \tau_0^0 = \tau_0^\rho = \tau_0^\phi = 0. \quad (10)
\]
Equation (10) corresponds to the spherical cavity.

Separate positions of the case of a sphere are given in the works of A.N. Maksimov and N.N. Pushkarenko [14], [15]. Two other cases (8), (9) correspond to the solution of analytic problems and are considered in the works of A.N. Maksimov, N.N. Pushkarenko, E.A. Derevyannikh, N.V. Khramova [16], [17].

Then (6) with account for (10) will be as follows:
\[
\sigma_0 = (\sigma_{\rho}^0 + 2\sigma_{\phi}^0)/3. \quad (11)
\]
When solving (7) in combination with (11), we will have:
\[
\sigma_{\rho}^0 = \sigma_{\phi}^0 / A + D, \quad (12)
\]
where \(A = (3 + 4a) / (3 - 2a), D = -6k_0 / (3 + 4a)\).

The equilibrium equations (2) with account for (10) will be as follows:
\[
\frac{\partial\sigma_0^0}{\partial\rho} + \frac{2}{\rho}(\sigma_0^\rho - \sigma_0^0) = 0, \quad \frac{\partial\sigma_0^\theta}{\partial\theta} = \frac{\partial\sigma_0^\phi}{\partial\phi} = 0. \quad (13)
\]

If we substitute (12) in the first equation (13) and take into consideration that there is no pressure inside the spherical cavity, we will get for normal stress components in zero approximation in the plastic region:
\[
\sigma_{\rho}^0 = k_0(\rho^{-12a}_{3 + 4a} - 1) / a, \quad \sigma_{\phi}^0 = \sigma_{\phi}^0 = k_0(\rho^{-12a}_{3 + 4a} - 1) / aA + D. \quad (14)
\]

To determine stress components in zero approximation in the elastic region, we use: equilibrium equation (11), incompressibility equation:
\[
\varepsilon_{\rho}^0 + \varepsilon_{\theta}^0 + \varepsilon_{\phi}^0 = 0, \quad (15)
\]
where \(\varepsilon_{\rho}, \varepsilon_{\theta}, \varepsilon_{\phi}\) is deformations along axes \(\rho, \theta, \phi\) in the elastic region, geometrical equations:
\[
\varepsilon^{0e}_\rho = \partial U / \partial \rho, \varepsilon^{0e}_\theta = \varepsilon^{0e}_\phi = U / \rho,
\]
where \( U \) is displacement along axis \( \rho \), physical equations (Hooke's law), taking into account that Poisson's ratio for incompressible material is \( \mu = 1/2 \):
\[
\varepsilon^{0e}_\rho = (\sigma^{0e}_\rho - \sigma^{0e}_\theta) / E, \varepsilon^{0e}_\theta = \varepsilon^{0e}_\phi = (\sigma^{0e}_\theta - \sigma^{0e}_\rho) / 2E,
\]
where \( E \) is elasticity modulus.
Solving together (15) and (16), we obtain:
\[
\varepsilon^{0e}_\theta = \varepsilon^{0e}_\phi = c / \rho^3,
\]
where \( c \) is the constant to be found.
Solving together (17) and (18), we obtain:
\[
\sigma^{0e}_\rho - \sigma^{0e}_\theta = -2Ec / \rho^3,
\]
Substituting (19) into the first equation (13), and taking equation \( \left( \sigma^{0e}_\rho \right)_{\rho=\infty} = -p_0 \) into account, we obtain:
\[
\sigma^{0e}_\rho = -4Ec / 3 \rho^3 - p_0.
\]
Solving together (20) and (19), we obtain:
\[
\sigma^{0e}_\theta = 2Ec / 3 \rho^3 - p_0.
\]
Using the conjugation of the solutions on the boundary of the elastoplastic region \( \left( \sigma^{0e}_\rho \right)_{\rho=\rho_0} = \sigma^{0p}_\rho, \sigma^{0e}_\theta = \sigma^{0p}_\theta \), we obtain:
\[
c = \frac{D}{2E} \beta_0^{3+4a}, \beta_0 = \left( \frac{D}{2(p_0a - k_0)} \right)^{\frac{3+4a}{12a}}.
\]
Substituting the first equation (22) into (20) and (21), we obtain:
\[
\sigma^{0e}_\rho = \frac{2D}{3 \rho^3} \beta_0^{3+4a} - p_0, \sigma^{0e}_\theta = \sigma^{0e}_\phi = \frac{D}{3 \rho^3} \beta_0^{3+4a} - p_0.
\]

4. Determination of the first approximations of stress components

Let's find the stress components in the first approximation in the plastic region.

By linearizing plasticity conditions (3), (4) and taking into account (7), (8), we obtain:
\[
\sigma^\prime_\rho = A\sigma^\prime,
\]
where \( \sigma^\prime = \sigma^\prime_\theta = \sigma^\prime_\phi. \)
\[
\tau^\prime_{\theta\phi} = 0.
\]

The equilibrium equation (2) with account for (24), (25) will be as follows:
\[
A \frac{\partial \sigma^\prime}{\partial \rho} + \frac{1}{\rho} \frac{\partial \tau^\prime_{\rho\theta}}{\partial \rho} + \frac{1}{\rho \sin \theta} \frac{\partial \tau^\prime_{\rho\phi}}{\partial \phi} + \frac{1}{\rho} \frac{\partial \tau^\prime_{\rho\phi}}{\partial \rho} + \frac{3}{\rho} \tau^\prime_{\rho\phi} \tan \theta = 0,
\]
\[
\frac{\partial \tau^\prime_{\rho\theta}}{\partial \rho} + \frac{1}{\rho \sin \theta} \frac{\partial \sigma^\prime}{\rho \sin \theta} + \frac{3}{\rho} \tau^\prime_{\rho\theta} = 0,
\]
\[
\frac{\partial \tau^\prime_{\rho\phi}}{\partial \rho} + \frac{1}{\rho \sin \theta} \frac{\partial \sigma^\prime}{\rho \sin \theta} + \frac{3}{\rho} \tau^\prime_{\rho\phi} = 0.
\]

To solve (26), the function \( U(\rho, \theta, \phi) \) is introduced so that equations would be satisfied:
\[
\sigma^\prime = -\frac{1}{\rho} \frac{\partial U}{\partial \rho}, \tau^\prime_{\rho\theta} = \frac{1}{\rho^2} \frac{\partial U}{\partial \theta}, \tau^\prime_{\rho\phi} = \frac{1}{\rho^2 \sin \theta} \frac{\partial U}{\partial \phi}.
\]
Then two last equations (27) are identically satisfied, and the first one will be as follows:

\[-Ap^2 \frac{\partial^2 U}{\partial \rho^2} + 2p \frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial \varphi^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 U}{\partial \varphi^2} + \cot \theta \frac{\partial U}{\partial \theta} = 0. \quad (28)\]

The \(U = (\rho, \varphi, \varphi)\) function will be as follows:

\[U = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( C_1 \rho^{n+3} + C_2 \rho^{n+3} \right) (a_{mn} \cos m \varphi + b_{mn} \sin m \varphi) P_n^m(\cos \theta), \quad (29)\]

where \(a_{mn}, b_{mn}\) are Fourier coefficients determined:

\[a_{mn} = \frac{1}{N_{mn}} \int_0^{2\pi} \rho_1(\theta, \varphi) P_n^m(\cos \theta) \cos m \varphi \cdot \sin \theta \cdot d\theta \cdot d\varphi,\]

\[b_{mn} = \frac{1}{N_{mn}} \int_0^{2\pi} \rho_1(\theta, \varphi) P_n^m(\cos \theta) \sin m \varphi \cdot \sin \theta \cdot d\theta \cdot d\varphi,\]

\[N_{mn} = \frac{2\pi E_m (n+m)!}{(2n+1)(n-m)!}, E_m = 2(m = 0), 1(m > 0),\]

\(P_n^m(\cos \theta)\) is associated Legendre function; \(\rho_1(\theta, \varphi)\) determines the cavity equation in first approximation, \(\chi_{21} = \left( \frac{1}{A} + \frac{1}{2} \right) \pm \sqrt{\left( \frac{1}{A} + \frac{1}{2} \right)^2 - \frac{n(n+1)}{A}}\).

By substituting (29) in (27), we get for the first approximation of stress components in the plastic region:

\[\sigma^\prime_\varphi = \sigma^\prime_\rho = \sigma^\prime = -\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( C_1 \chi_1 \rho^{n+3} + C_2 \chi_2 \rho^{n+3} \right) (a_{mn} \cos m \varphi + b_{mn} \sin m \varphi) P_n^m(\cos \theta),\]

\[\sigma^\prime_\rho = -A \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( C_1 \chi_1 \rho^{n+3} + C_2 \chi_2 \rho^{n+3} \right) (a_{mn} \cos m \varphi + b_{mn} \sin m \varphi) P_n^m(\cos \theta),\]

\[\tau^\prime_{\rho\varphi} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( C_1 \rho^{n+3} + C_2 \rho^{n+3} \right) (a_{mn} \cos m \varphi + b_{mn} \sin m \varphi) \frac{\partial P_n^m(\cos \theta)}{\partial \theta}, \tau^\prime_{\rho\varphi} = 0,\]

\[\tau^\prime_{\rho\rho} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sin \theta \left( C_1 \rho^{n+3} + C_2 \rho^{n+3} \right) (b_{mn} \cos m \varphi - a_{mn} \sin m \varphi) P_n^m(\cos \theta),\]

where \(C_1 = \frac{D(\chi_2 - 2)}{A(\chi_1 - \chi_2)}, C_2 = \frac{D(\chi_1 - 2)}{A(\chi_1 - \chi_2)}\).

5. Conclusions

The zero stress values in the elastic (23) and plastic (12) regions, the zero approximation of the boundary of the elastoplastic zone (22), as well as the first approximations of the stress components in the plastic region (31) are determined. The obtained results can be used in the mining industry, construction mechanics and other related fields. Similarly, sufficient convergence of approximations obtained by the method of small parameters can serve as confirmation of the truth of classical solutions obtained by other methods.

The next step in my research will be to determine the stress state of an ideally plastic space of compressible material weakened by an ellipsoidal cavity.

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