Exotic continuous quantum phase transition between $Z_2$ topological spin liquid and Néel order

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Recent numerical simulations with different techniques have all suggested the existence of a continuous quantum phase transition between the $Z_2$ topological spin liquid phase and a conventional Néel order. Motivated by these numerical progresses, we propose a candidate theory for such $Z_2$–Néel transition. We first argue on general grounds that, for a SU(2) invariant system, this transition cannot be interpreted as the condensation of spinons in the $Z_2$ spin liquid phase. Then we propose that such $Z_2$–Néel transition is driven by proliferating the bound state of the bosonic spinon and vison excitation of the $Z_2$ spin liquid, i.e. the so-called $(e,m)$--type excitation. Universal critical exponents associated with this exotic transition are computed using $1/N$ expansion. This theory predicts that at the $Z_2$–Néel transition, there is an emergent quasi long range power law correlation of columnar valence bond solid order parameter.

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I. INTRODUCTION

Thanks to the rapid development of numerical techniques, more and more candidates of exotic liquid states have been identified in frustrated spin models$^{3,4}$, hardcore quantum boson model$^{5,6}$, or even Hubbard model$^{11}$. All these phases that are identified numerically are fully gapped liquid phases with short range correlation between both spin order parameters and also valence bond solid (VBS) order parameters. The simplest fully gapped spin liquid state is the $Z_2$ topological liquid state, which has the same topological order as the toric code model$^{12}$. In addition to the fully gapped spectrum, the computation of critical exponents at the order-disorder transition of these models$^{7-10}$, and the computation of topological entanglement entropy$^{11-13}$ both convincingly proved that the spin liquid states of some of these models (such as the $J_1-J_2$ model on the square lattice, and the extended Bose-Hubbard model on the Kagome lattice) are indeed the $Z_2$ topological liquid. In some other models (such as the spin-$1/2$ Heisenberg model on the Kagome lattice$^{13}$, and the Hubbard model on the honeycomb lattice$^{12}$), although an accurate topological entanglement entropy computation is still demanded, it is broadly believed that the spin liquid state is indeed the $Z_2$ liquid state, or a similar ($Z_2$)$^n$ liquid state.

Besides the spin liquid state itself, the quantum phase transitions of these models are equally interesting. For example, continuous quantum phase transitions between Néel order and a fully gapped spin liquid phase have been found in the honeycomb lattice Hubbard model$^{14,15}$, and the $J_1-J_2$ spin-$1/2$ Heisenberg model on the square lattice$^{12}$. In terms of the Landau-Ginzburg (LG) theory, this transition should be an ordinary O(3) transition, and the $Z_2$ liquid phase is identified as the disordered phase, while the Néel phase is the ordered phase. However, because the $Z_2$ liquid phase has a nontrivial topological order and topological degeneracy$^{16-18}$, it cannot be adiabatically connected to the trivial direct product state, thus it should not be identified as the trivial disordered phase in the classical case. Thus if the $Z_2$–Néel transition exists, it means that the quantum disordering of the Néel order and the emergence of the $Z_2$ topological order happen simultaneously at one point, this unusual fact implies that this $Z_2$–Néel quantum critical point (QCP) must be an unconventional one that is beyond the LG paradigm. The goal of this paper is to understand this unconventional QCP.

II. FAILURE OF THE SPINON THEORIES

We first argue on general grounds that such a continuous $Z_2$–Néel transition cannot be understood using an ordinary spinon theory. We stress that we will only consider SU(2) invariant systems here.

First of all, if this $Z_2$ spin liquid phase has a gapped fermionic spinon excitation $f_\alpha$, then a Néel order parameter can in principle be represented as $\vec{N} \sim (-1)^i f_i^\alpha \vec{\sigma} \otimes f_i^\beta$. Thus it appears that we can interpret this $Z_2$–Néel transition as the disorder-order transition of the vector $\vec{N}$ using an ordinary Landau-Ginzburg theory. However, this theory is incorrect because the vector $\vec{N}$ does not carry any gauge charge, thus the order parameter does not immediately suppress the $Z_2$ topological order. This implies that between the $Z_2$ spin liquid and the Néel order with nonzero $\langle \vec{N} \rangle$, there must be an intermediate state with the coexistence of both Néel order and $Z_2$ topological order, and it is usually called the Néel$^*$ state. Thus a direct continuous transition between $Z_2$ and Néel order cannot be obtained this way without fine-tuning.

In order to suppress the $Z_2$ topological order, the usual wisdom is to condense a topological excitation that carries the $Z_2$ gauge charge. Then after the topological excitations are condensed, the $Z_2$ gauge field is Higgsed, and the topological order disappears. Along with sup-
pressing the topological order, if we want to induce spin order simultaneously, then the excitation that condenses must also carry certain representation of the spin SU(2) symmetry group, in addition to the $Z_2$ gauge charge. Let us call this gauge-charged spin excitation as *spinon* in general. Then the nature of the spin order and the universality class of this transition both depend on the particular spin representation of spinon.

The smallest representation of SU(2) is spin-1/2 representation, and there is no consistent “fractional” representation of SU(2) group that is smaller than spin-1/2. Thus let us first assume the spinon is a spin-1/2 boson, which is described by a two component complex boson field $z_\alpha = (z_1, z_2)^i$, and $z_\alpha$ is subject to the constraint $|z_1|^2 + |z_2|^2 = 1$. Then $z_\alpha$ is coupled to a $Z_2$ gauge field in the following way:

$$H = \sum_{i,\mu} \sum_\alpha -t\sigma_{i,\mu}^\alpha z^*_\alpha z_\alpha, i + \mu + H.c. + \cdots$$ \hspace{1cm} (1)

where the ellipsis stands for higher order interaction terms. $\sigma_{i,\mu}^\alpha$ is the $Z_2$ gauge field that is defined on the link $(i, \mu)$ of the lattice, and Eq. (1) is invariant under the gauge transformation

$$z_{i,\alpha} \rightarrow \eta_i z_{i,\alpha}, \quad \sigma_{i,\mu}^\alpha \rightarrow \eta_i \sigma_{i,\mu}^\alpha \eta_i$$ \hspace{1cm} (2)

where $\eta_i = \pm 1$ is an arbitrary Ising function defined on the sites of the lattice. The condensed phase of $z_\alpha$ is the spin ordered phase, and because $z_\alpha$ is coupled to the $Z_2$ gauge field, the $Z_2$ topological order is automatically destroyed due to the Higgs mechanism in the condensate of $z_\alpha$. The gapped phase of $z_\alpha$ is the deconfined $Z_2$ topological phase.

Since $z_\alpha$ has in total two complex bosonic fields, i.e. four real fields, then with the constraint $|z_1|^2 + |z_2|^2 = 1$, the entire configuration of $z_\alpha$ is equivalent to a three dimensional sphere $S^3$. Since the spinon field $z_\alpha$ is coupled to a $Z_2$ gauge field, then the physical configuration of the condensate of $z_\alpha$ is $S^3/Z_2$, which is mathematically equivalent to the group manifold SO(3). Since $z_\alpha$ itself is not a physical observable, inside the condensate of $z_\alpha$ the physical observables are the three following vectors:

$$\vec{N}_1 = \text{Re}[z^i i\sigma^\alpha \vec{\sigma}^\alpha], \quad \vec{N}_2 = \text{Im}[z^i i\sigma^\alpha \vec{\sigma}^\alpha], \quad \vec{N}_3 = z^i \vec{\sigma}^\alpha.$$ \hspace{1cm} (3)

A simple application of the Fierz identity $\sum_\sigma \sigma_\alpha^\sigma \sigma_\gamma^\gamma = 2\delta_\alpha^\gamma \delta_\beta^\beta - \delta_\alpha^\beta \delta_\gamma^\gamma$ proves that these three vectors are orthogonal with each other. Since the first homotopy group of SO(3) is $\pi_1[SO(3)] = Z_2$, inside this spin ordered phase there are vortex-like topological defects. Two of these vortices can annihilate each other.

The spin-1/2 boson field $z_\alpha$ can be viewed as the low energy mode of the usual Schwinger boson $b_\alpha$, but our argument is more general, and it is independent of the microscopic origin of $z_\alpha$. If we identify one of the three vectors $\vec{N}_i$ as the Néel vector, then this phase must have two other spin vector orders that are perpendicular to the Néel vector. The condensation transition of $z_\alpha$ while coupled to a $Z_2$ gauge field is usually called the O(4)$^*$ transition[23].

Now let us assume the spinon of the $Z_2$ topological phase carries a spin-1 representation. A spin-1 representation is a vector representation of SU(2), i.e. it can be parametrized as a unit real vector $\vec{n}$, $|\vec{n}|^2 = 1$. Now the coupling between the spinon and $Z_2$ gauge theory reads

$$H = \sum_{i,\mu} \sum_a -t\sigma_{i,\mu}^a n_a^\alpha n_a^\alpha + \cdots$$ \hspace{1cm} (4)

Again, since $\vec{n}$ couples to a $Z_2$ gauge field, it is not a physical observable: $\vec{n}$ and $-\vec{n}$ are physically equivalent. If vector $\vec{n}$ condenses, the condensate is in fact a spin nematic, or quadrupole order, with physical order parameter

$$Q^{ab} = n^a n^b - \frac{1}{3} \delta_{ab}.$$ \hspace{1cm} (5)

This spin order has manifold $S^2/Z_2$, which also supports vortex excitation since $\pi_1[S^2/Z_2] = Z_2$. One example state of this type is the spin quadrupolar state that has been observed in the spin-1 material NiGa$_2$S$_4$.14

We have discussed two types of unconventional QCPs between $Z_2$ liquid phase and spin orders. In either case, the spin ordered phase is different from the ordinary collinear Néel order, because a Néel order should have ground state manifold (GSM) $S^2$. In particular, in both cases we have considered, the spin ordered phase must have a nontrivial homotopy group $\pi_1$, which corresponds to the vison excitation of the $Z_2$ gauge field. Generalization of our analysis to higher spin representations is straightforward, but the conclusion is unchanged.

As we already discussed, in Ref.2 and Ref.11, a *continuous* quantum phase transition between a fully gapped spin liquid phase and a Néel order was reported. If the fully gapped spin liquid discovered in these numerical works is indeed a $Z_2$ spin liquid as we expected, then such continuous quantum phase transition is beyond the spinon theory discussed in this section. In order to understand the continuous transition between the gapped spin liquid and Néel order reported in the phase diagram of the Hubbard model on the honeycomb lattice, in Ref.13–17 the authors had to introduce extra “hidden” order parameters in the Néel phase, which change the GSM of the Néel phase completely.

In this section we argued on general grounds that the $Z_2$–Néel transition cannot be interpreted as the condensation of an ordinary spinon. Our argument is independent of specific spin model or lattice structure. However, this argument can only be applied to SU(2) invariant systems. For a system with U(1) symmetry, for example the hard-core Boson model on the Kagome lattice discussed in Ref.15, the transition between $Z_2$ topological phase and the superfluid phase can be understood as the condensation of a fractionalized “half-boson” that couples to the $Z_2$ gauge field, and this transition is the so-called 3d XY$^*$ transition.
In this field theory, there are two types of matter fields, $z\alpha$ and $v\alpha$, and they are interacting with each other through a mutual Chern-Simons (CS) theory, which grants them a mutual semionic statistics i.e. when $v\alpha$ adiabatically encircles $z\alpha$ through a closed loop, the system wavefunction acquires a minus sign. This is one of the key properties of the $Z_2$ topological phase. Here $z\alpha$ corresponds to the electric ($e$–type) excitation of the $Z_2$ liquid, and $v\alpha$ corresponds to the magnetic ($m$–type) excitation. $v\alpha$ is usually called the vison excitation.

The minimal field theory Eq. 6 has symmetry $\text{SU}(N_z)\times\text{SU}(N_v)$. However, depending on the details of the microscopic model, the higher order interactions between matter fields can break this symmetry down to its subgroups. We will first ignore this higher order symmetry breaking effects, and focus on the case with $N_z = 2$, and $N_v = 1$. In Ref.\textsuperscript{20}, the authors used the model Eq. 6 with $N_z = 2$, $N_v = 1$ to describe the global phase diagram of spin-1/2 quantum magnets on a distorted triangular lattice, which is a very common structure in many materials. The same theory can be applied to the square and honeycomb lattice as well, and in this paper we will take the square lattice as an example. Here $z\alpha$ is a bosonic spin-1/2 spinon, and $v$ is the low energy mode of a vison, and it corresponds to the expansion of the vison at two opposite momenta $\pm \vec{Q}$:

$$\tau \sim ve^{i\vec{Q}\cdot\vec{r}} + v^* e^{-i\vec{Q}\cdot\vec{r}}.$$  \hfill (7)

Thus $v$ is a complex scalar field. On the square lattice or distorted triangular lattice, there is a $Z_2$ anisotropy on $v$, that is allowed by the symmetry of the lattice\textsuperscript{20,21}. This anisotropy is highly irrelevant in the quantum critical region, and it will be ignore throughout the paper.

The phase diagram of this model is tuned by two parameters: $s_z$ and $s_v$, and depending on the sign of these two parameters, there are in total four different phases (Fig. 1):

**Phase 1.** This is the phase with $s_z > 0$, $s_v > 0$. In this phase, both matter fields $z\alpha$ and $v$ are gapped, and they have a topological statistic interaction through the mutual CS theory. Since all the matter fields are gapped, the low energy properties of phase 1 is described by the mutual CS theory only. The mutual CS theory defined on a torus has a four-fold degenerate ground state, thus this phase is precisely the gapped $Z_2$ topological phase\textsuperscript{22}.

**Phase 2.** $s_v > 0$, $s_z < 0$. When vison $v$ is gapped, integrating out $v$ induces a Maxwell term for gauge field $b\mu$, which implies that the flux of $b\mu$ is condensed. In other words the flux-creation operator (denoted as $M_b$) acquires a nonzero expectation value. $M_b$ corresponds to a Dirac monopole configuration of $b\mu$ in the space-time. Due to the mutual CS coupling between gauge fields $a\mu$ and $b\mu$, the condensate of $M_b$ breaks $a\mu$ to a $Z_2$ gauge field. Thus after we integrate out $v$ and $b\mu$, the spinon $z\alpha$ is only coupled to a $Z_2$ gauge fields. Thus when $N_z = 2$, the condensate of $z\alpha$ has GSM SO(3) as was discussed in the previous section. An example of this phase is the spiral spin density wave phase. Once we assume $s_v > 0$, Eq. 6 precisely reduces to the previously studied O(4)\textsuperscript{+} theory for the transition between $Z_2$ spin liquid and spiral spin order\textsuperscript{13}.

**Phase 3.** $s_v < 0$, $s_z > 0$. This is a phase where $v$ condenses while $z\alpha$ is gapped out. This phase is the four fold degenerate columnar VBS phase that breaks the reflection and translation symmetry of the lattice. The columnar VBS order parameter can be written as $\nu^2 M_a$, where $M_a$ is the monopole operator of gauge field $a\mu$, which creates a $2\pi$ flux of $a\mu$. When $s_z > 0$, spinon $z\alpha$ is gapped, and it leads to a Maxwell term for $a\mu$. This implies that $M_a$ is condensed, and it breaks $b\mu$ to a $Z_2$ gauge field. In this case the low energy effective theory that describes phase 3 is a complex field $v$ that couples to a $Z_2$ gauge field, thus our theory reduces to the pure
vison theory that was thoroughly discussed in Ref.\textsuperscript{23}.

Phase 4. \( s_v < 0, s_z < 0 \). This is a phase where both \( z_o \) and \( v \) condense. Because in this phase the only gauge invariant order parameter that condenses is \( \tilde{N} \sim z^\dagger \tilde{\sigma} z \), this is precisely the collinear Néel phase with GSM \( S^2 \). In fact, when \( v \) is condensed, the gauge field \( b_\mu \) acquires a mass term \( b_\mu^2 \) due to the Higgs mechanism. Then integrating out \( v \) and \( b_\mu \) leads to a Maxwell term for gauge field \( a_\mu \), due to the mutual CS coupling. Thus the spinon \( z_\alpha \) is coupled to a dynamical gapless U(1) gauge field \( a_\alpha \). Then the GSM of the condensate of \( z_\alpha \) is \( S^3/U(1) = S^2 \), which is equivalent to the collinear Néel order. Thus under the assumption \( s_v < 0 \), Eq. \( \text{[6]} \) reduces to the CP(1) model that describes the deconfined QCP between Néel and VBS order \textsuperscript{23,24}.

We have shown that the mutual CS formalism Eq. \( \text{[6]} \) unifies many previously discussed exotic states and exotic phase transitions. A more detailed discussion of the phase diagram can be found in Ref.\textsuperscript{20}.

B. \( Z_2 \)–Néel transition driven by \((e,m)\) excitation

Now we are ready to discuss our theory for the direct continuous transition between \( Z_2 \) liquid phase and Néel phase. In a \( Z_2 \) topological phase, using the standard notation, there are three types of topological excitations: the electric excitation \( e \), the magnetic excitation \( m \), and their bound state \((e,m)\). In Eq. \( \text{[6]} \) the spinon field \( z_\alpha \) is the \( e \)--type excitation, while the vison field \( v \) is the \( m \)--type excitation. Eq. \( \text{[6]} \) is based on the assumption that inside the \( Z_2 \) liquid phase the \( e \)--type and \( m \)--type excitations have lower energy than \((e,m)\), thus in the global phase diagram Fig. 1 the Néel and \( Z_2 \) topological phases are separated by a multicritical point \( s_z = s_v = 0 \). However, if we consider the opposite possibility, namely the \((e,m)\)--type excitation has the lowest energy in the \( Z_2 \) spin liquid, then a different quantum phase transition can occur by condensing the \((e,m)\)-type excitation.

Let us first take the simplest Toric code model \( \text{[5]} \) as an example: \( H = \sum_i -\sigma_i^z \sigma_{i-\pm x}^\dagger \sigma_{i,y}^\dagger \sigma_{i+y,x}^\dagger - \sigma_i^z \sigma_{i,y}^\dagger \sigma_{i+y,x}^\dagger \sigma_{i-x,y}^\dagger \). The “condensation” of an excitation simply means that the system enters a phase where the kinetic energy of this excitation dominates. It is well-known that in the Toric code model the condensation of the \( e \)--excitation is driven by a magnetic field \( h_z \sigma_i^z \mu \), while the condensation of \( m \)--excitation is driven by field \( h_x \sigma_i^x \mu \), because these two fields enable the hopping of \( e \) and \( m \) excitations respectively. In order to “condense” the \((e,m)\) excitation, we simply need to turn on field \( h_y \sigma_i^y \mu \), which hops the \((e,m)\) excitation along the diagonal directions of the square lattice. When any of the three excitations is condensed, the system enters a trivial polarized state without any topological degeneracy. Generally speaking, in the topological phase, starting from one of the topological sectors on the torus, the other sectors can be generated by locally creating a pair of topological excitations, and annihilating them after adiabatically moving one excitation of the pair around the torus. Because all three types of topological excitations are mutual semions, condensing one of the three excitations will lead to a strong local flux fluctuation for the other two excitations, thus the other two excitations are confined in this condensate, i.e. the system no longer has topological degeneracy.

In our current case, both \( e \) and \( m \) excitations carry extra global symmetries besides their gauge charges. In order to describe the \((e,m)\) excitation in our situation, let us define new complex bosonic fields \( \phi_\alpha \) and \( \psi_\alpha \):

\[
\phi_\alpha = z_\alpha v, \quad \psi_\alpha = z_\alpha v^*, \quad (8)
\]

\( \phi_\alpha \) and \( \psi_\alpha \) carry the quantum number of \((e,m)\) excitation. Because \( v \) is a complex variable, fields \( \phi_\alpha \) and \( \psi_\alpha \) are independent from each other, and they interact with each other as follows:

\[
\mathcal{L} = \sum_\alpha \left[ (\partial_\mu - ia_\mu - ib_\mu)\phi_\alpha \right]^2 + \left[ (\partial_\mu - ia_\mu + ib_\mu)\psi_\alpha \right]^2 + r |\phi_\alpha|^2 + |\psi_\alpha|^2 + \frac{i}{\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu b_\rho + g(|\phi|^2)^2 + g(|\psi|^2)^2 + u |\phi|^2 |\psi|^2 - \omega \phi^\dagger \bar{\sigma} \phi \cdot \psi^\dagger \bar{\sigma} \psi. (9)
\]

Notice that \( \phi_\alpha \) and \( \psi_\alpha \) carry gauge charges of both gauge fields \( a_\mu \) and \( b_\mu \). In order to understand the QCP at \( r = 0 \) more quantitatively, it is more convenient to define new gauge field \( A_\mu^\pm = a_\mu \pm b_\mu \), then the Lagrangian reads:

\[
\mathcal{L} = \sum_\alpha [ (\partial_\mu - iA_\mu^+) \phi_\alpha ]^2 + [ (\partial_\mu - iA_\mu^-) \psi_\alpha ]^2
\]
Simons-Higgs model: this case this transition is described by the simple Chern-
completely decoupled at the transition
\[ T \]
the ordinary spinon theory is illustrated in Fig. 2.
order. The difference between this new transition and
state of
from each other, \( i.e \), they are only coupled through the quartic terms \( u \) and \( w \). The mass gaps \( r \) for \( \phi_\alpha \) and \( \psi_\alpha \) are equal, because the vison modes \( v \) and \( v^* \) are guar-
teed to be degenerate by the symmetry of the square lattice. \( \phi_\alpha \) and \( \psi_\alpha \) are introduced as bosonic fields, but
gauge fields \( A^\mu_\alpha \) and \( A^-_\mu \) make them fermionic fields af-
after the standard flux attachment, due to the existence of the Chern-Simons terms in this Lagrangian. In our for-
mulation, fields \( \phi_\alpha \) and \( \psi_\alpha \) can still condense by tuning parameter \( r \) in Eq. [10] After \( \phi_\alpha \) and \( \psi_\alpha \) both condense simultane-
ously, \( A^\mu_\alpha \) and \( A^-_\mu \) are both Higgsed, and in the
Higgs phase the only gauge invariant operators are
\[
\phi^\dagger \tilde{\sigma} \phi, \quad \psi^\dagger \tilde{\sigma} \psi. \tag{11}
\]
Since these two vectors both carry the same quantum
number as the Néel order parameter \( z^I \tilde{\sigma} \), in Eq. [10] \( w \) is
is naturally positive, thus these two vectors are aligned
parallel with each other, so the condensate of \( \phi_\alpha \) and \( \psi_\alpha \) has
a manifold \( S^2 \), \( i.e \), it is the standard collinear Néel
order. The difference between this new transition and
the ordinary spinon theory is illustrated in Fig. 2.

What is the universality class of this transition? The
simplest possibility is that, both \( u \) and \( w \) are irrelevant at
the transition, although they are relevant in the conden-
sate of \( \phi_\alpha \) and \( \psi_\alpha \). Under this assumption \( \phi_\alpha \) and \( \psi_\alpha \) are
decompletely decoupled at the transition \( r = 0 \). Then in
this case this transition is described by the simple Chern-
Simons-Higgs model:
\[
\mathcal{L} = \sum_{\alpha=1}^{N} |(\partial_\mu - iA_\mu)\phi_\alpha|^2 + r|\phi_\alpha|^2 + g(\sum_\beta |\phi_\beta|^2)^2
+ \frac{iN}{8\theta} \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{12}
\]
Here we have generalized the equation to have \( N \) flavors of
matter fields \( \phi_\alpha \), and introduced a statistical angle
\( \theta \). Our physical situation corresponds to \( N = 2 \) and
\( \theta = \pi \). Notice that Eq. [12] explicitly breaks the
time-reversal symmetry due to the Chern-Simons term. But
the complete theory Eq. [10] is time-reversal invariant,
because under time-reversal transformation \( \phi_\alpha \) and \( \psi_\alpha \) are
exchanged, the two gauge fields \( A^\mu_\alpha \) and \( A^-_\mu \) are also
exchanged.

The critical exponents of this transition can be com-
puted using a systematic 1/N expansion. Ref. 22 has com-
puted the critical exponent \( \nu \) defined as \( \xi \sim |r|^{-\nu}, \) here
we will focus on the scaling dimension of \( \phi^\dagger T^a \phi \) at the
QCP, where \( T^a \) is the SU(N) generator. To the first order
1/N expansion, this scaling dimension reads
\[
\Delta[\phi^\dagger T^a \phi] = 1 + \frac{4}{3\pi^2} \left( \frac{N}{N-1} - \frac{\theta^2/4}{N + \theta^2/64} \right). \tag{13}
\]
Let us briefly comment how we obtain this result. Similar
calculation without the \( \theta \) term was obtained before. See
Fig.3 and Fig.4 of the previous work 25 for necessary Feyn-
man diagrams. First, we need to evaluate wave function
renormalization of \( \phi \) from both gauge fluctuation and the
density fluctuation, which contain the factor(1/N).
Then, using the standard operator insertion method,
one can calculate renormalization function of the corre-
spanding vertex. We note that the traceless condition,
(Tr(T_\nu) = 0), reduces one diagram compared with the
calculation of the scaling dimension of \( |\phi|^2 \) and simplify
our calculation.

In the limit of \( \theta \rightarrow \infty \), \( i.e \), the CS term is
effectively zero, these results converges to the ordinary
CP(N − 1) results computed in Ref. 22. In the limit of
\( \theta \rightarrow 0 \), the gauge fluctuation is totally frozen by the
CS term, and the universality class of this quantum critical
point only acquires corrections from the short range
self-interaction between field \( \phi_\alpha \), thus it is equivalent to
an O(2N) transition of the O(2N) bosonic vector field
\( \{\text{Re}[\phi_1], \cdots, \text{Re}[\phi_N], \text{Im}[\phi_1], \cdots, \text{Im}[\phi_N]\} \).
Scaling dimension of the “Néel” type operator \( \Delta[\phi^\dagger T^a \phi] \) in our theory is larger than that in the CP(N − 1) theory with large
\( N \), \( i.e \), at the Z_2−Néel QCP, the anomalous dimension of
the Néel order parameter is predicted to be larger than
that of the deconfined QCP between the Néel and VBS
order. This prediction can be tested in the future by a
careful comparison between the critical exponents of the
\( J_1 − J_2 \) model and the \( J − Q \) model 26,28.

It is pretty clear that at least in the large−N limit, the
perturbation of \( u \) in Eq. [10] is irrelevant, because in
this limit the scaling dimension \( \Delta[|\phi|^2] = \Delta[|\psi|^2] = 2 \),
\( i.e \), \( \Delta[u] = -1 \). Higher order 1/N or \( \epsilon \) expansion is
demanded to determine whether \( w \) is relevant or not at this
transition.

Assuming at the QCP \( r = 0 \) both \( u \) and \( w \) are irrele-
vant, then besides the Néel order parameter, some other
physical order parameters also have power-law corre-
lation. For example, the columnar VBS order parameter
can be written as
\[
\text{VBS} \sim \psi_\alpha^\dagger \phi_\alpha M_\alpha \sim v^2 M_\alpha, \tag{14}
\]
where \( M_\alpha \) is the monopole operator for gauge field \( a_\mu \).
When \( \phi_\alpha \) and \( \psi_\alpha \) both have a large \( N \) component, the
scaling dimension of \( M_\alpha \) is proportional with \( N \). Thus
with large \( N \) the VBS order parameter is expected to
have a much larger scaling dimension compared with the
Néel order parameter at the \( Z_2 \)−Néel QCP. We stress that the
VBS order parameter has short-range correlation in the
\( Z_2 \) spin liquid and the Néel phase, its emergent
quasi long range correlation occurs only at the QCP. This
result has already been confirmed numerically in Ref. 22,
and it was demonstrated that the scaling dimension of
the VBS order parameter is indeed larger than that of
Néel order at the QCP 24.

In \( 2+1 \) dimension, the entanglement entropy of a con-
formal field theory can in general be written as
\( S = c\ell - \beta \), where the first term is the nonuniversal area law
contribution, while the second term is a universal constant. In Ref. 19, it was argued that at a QCP where a bosonic field condenses while coupling to a discrete gauge field, the universal entanglement entropy is a direct sum of the contribution from the bosonic matter field and the contribution from the discrete gauge field: \( \beta = \beta_0 + \beta_{\text{gauge}} \). This conclusion is based on the assumption that the matter field dynamics is not affected by the discrete gauge field in the infrared limit, and this is indeed true for the XY* transition observed in Ref. 19. However, at the exotic \( Z_2 \)–Néel transition discussed here where the \((e, m)\)-type excitations condense, the bosonic matter fields \( \phi_\alpha \) and \( \psi_\alpha \) are indeed strongly affected by the gauge field, thus at this QCP the universal entanglement entropy \( \beta \) is no longer a direct sum of the two different degrees of freedom of the system. The universal entanglement entropy of field theory Eq. 12 in the large-\( N \) limit can be found in Ref. 20.

**C. A Toy model with \( N = 1 \)**

Now let us discuss a toy model with \( N = 1 \). This is actually the case where the critical theories can be all understood exactly. This field theory with \( N = 1 \) can be applied to the following extended Toric-code model:

\[
H = \sum_i K_i \sigma_i^x \sigma_i^{-x} + \sigma_i^y \sigma_i^{-y} + K_z \sigma_i^z \sigma_i^{-z} + \sum_{i,\mu} h_\pm \sigma_i^{\pm \mu} + \cdots. \tag{15}
\]

Here the \( e \)-type (\( m \)-type) excitation is the end of a string product of \( \sigma^x \) (\( \sigma^z \)). The \( e \) and \( m \)-type excitations view \( \sigma^z \) and \( \sigma^y \) as \( Z_2 \) gauge fields respectively, and the \( h_x \) and \( h_z \) terms enable the hopping of these excitations. Unlike the standard toric-code model, here we keep \( K_x, K_z > 0 \). When \( K_z, K_z > 0 \), both \( \sigma^z \) and \( \sigma^x \) have a \( \pi \)-flux in the ground state. Then the dynamics of both \( e \) and \( m \) type of excitations are frustrated, and both excitations have two different minima \( \pm \vec{Q} \) in their band structure. As a result, the low energy dynamics of \( e \) and \( m \) excitations are described by complex scalar fields \( z \) and \( v \) expanded at momentum \( \vec{Q} \). If one of these two fields condenses while the other one remains gapped, the \( Z_2 \) topological order is destroyed, and the system must spontaneously break the lattice translation symmetry as well. The condensates of \( e \) and \( m \)-type excitations physically corresponds to the valence bond solid phase of \( \sigma^x \) and \( \sigma^z \) respectively.

When \( N = 1 \), if \( e \) and \( m \) type of excitations condense separately, then the phases in Fig. 1 would be \( Z_2 \) liquid, VBS order of \( \sigma^y \), trivial phase, and VBS order of \( \sigma^z \) (counted counterclockwise around the multicritical point \( s_z = s_v = 0 \)). On the other hand, if the bound state \((e, m)\) has the lowest energy in the \( Z_2 \) liquid phase, then again we can introduce two independent complex fields \( \phi \) and \( \psi \) as \( \phi = zv, \psi = zv^\ast \). Then the transition driven by the condensation of \( \phi \) and \( \psi \) is described by Eq. 12 with \( N = 1 \) and \( \theta = \pi/2 \).

What kind of transition is this? If in Eq. 12 the complex field \( \phi \) is also coupled to an external \( U(1) \) gauge field \( A_{\mu}^{\text{ext}} \), then we can see that in the disordered phase of \( \phi \), after integrating out the massive \( \phi \) and dynamical gauge field \( A_\mu \), the lowest order contribution to the effective Lagrangian of \( A_{\mu}^{\text{ext}} \) is still a Maxwell term: \( \mathcal{L}_{\text{eff}} \sim (\partial A_{\text{ext}})^2 + e\phi (\partial^2 A_{\text{ext}}) (\partial A_{\text{ext}}) + \cdots \). While in the condensate of \( \phi \), the effective Lagrangian of \( A_{\mu}^{\text{ext}} \) acquires a Chern-Simons term at level 1. This analysis implies that this transition is equivalent to a topological transition between a trivial insulator and a Chern insulator with Chern number 1. The universality class of this type of topological transition of Chern insulator is very well-understood, it can be simply described by a 2+1d Dirac fermion:

\[
\mathcal{L} = \bar{\psi} \gamma_\mu \partial_\mu \psi + m \bar{\psi} \psi, \tag{16}
\]

here the trivial insulator and Chern insulator correspond to \( m > 0 \) and \( m < 0 \) respectively, and \( m = 0 \) corresponds to the quantum critical point at \( r = 0 \) in Eq. 12 with \( N = 1 \). Thus we conjecture that when \( N = 1 \) and \( \theta = \pi/2 \), the critical point in Eq. 12 is dual to a massless free Dirac fermion. In Ref. 21 a similar conjecture was made that the 3D XY transition is dual to a massless Dirac fermion coupled to a noncompact \( U(1) \) gauge field.

**IV. SUMMARY AND DISCUSSION**

In this work we have discussed a possible theory for the direct continuous transition between the \( Z_2 \) liquid phase and the Néel order, and this is a candidate theory for the liquid-Néel transition observed in Ref. 21. We have taken the square lattice as an example, but results discussed in this paper can also be applied to the honeycomb lattice after straightforward generalization.

In our theory, we exploited the fact that in two dimension, the \( e \) and \( m \)-type excitations are both point like defects, thus their nontrivial statistics can be described well with a mutual Chern-Simons theory. By contrast, in a three dimensional \( Z_2 \) liquid phase, there is a mutual semion statistics between the point particle like \( e \)-excitation and loop like \( m \)-type excitation. Thus the effective field theory for the three dimensional \( Z_2 \) liquid phase is the so-called BF theory \( \mathcal{L}_{\text{eff}} \sim \frac{1}{8} \epsilon_{\mu\nu\rho\tau} \phi_\mu \partial_\nu \psi_\rho \psi_\tau \), where \( \phi_\mu \) is the \( U(1) \) gauge field that couples to the \( e \)-type point particle, and \( \psi_\mu \) is an antisymmetric rank-2 antisymmetric tensor gauge field that couples to the \( m \)-type loop excitation. The global phase diagram around the three dimensional \( Z_2 \) liquid phase is another interesting subject, and we will leave it to future studies.
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