Semiflat Orbifold Projections

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In Memory of Mom

ABSTRACT. We compute the semiflat positive cone $K_0^{+SF}(A_0^\sigma)$ of the $K_0$-group of the irrational rotation orbifold $A_0^\sigma$ under the noncommutative Fourier transform $\sigma$ and show that it is determined by classes of positive trace and the vanishing of two topological invariants. The semiflat orbifold projections are 3-dimensional and come in three basic topological genera: $(2,0,0)$, $(1,1,2)$, $(0,0,2)$. (A projection is called semiflat when it has the form $h + \sigma(h)$ where $h$ is a flip-invariant projection such that $h\sigma(h) = 0$.) Among other things, we also show that every number in $(0,1) \cap (2\mathbb{Z} + 2\mathbb{Z}\theta)$ is the trace of a semiflat projection in $A_\theta$. The noncommutative Fourier transform is the order 4 automorphism $\sigma : V \to U \to V^{-1}$ (and the flip is $\sigma^2 : U \to U^{-1}$, $V \to V^{-1}$), where $U, V$ are the canonical unitary generators of the rotation algebra $A_\theta$ satisfying $UV = e^{2\pi i \theta}UV$.

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1. Introduction

For each irrational number \( \theta \) in \((0, 1)\) the irrational rotation \( C^* \)-algebra \( A_\theta \) is the universal (and unique) \( C^* \)-algebra generated by unitaries \( U, V \) satisfying the Heisenberg relation

\[ VU = e^{2\pi i \theta} UV. \]

The noncommutative Fourier transform is the canonical order four automorphism \( \sigma \) of \( A_\theta \) defined by the equations

\[ \sigma(U) = V^{-1}, \quad \sigma(V) = U. \]

The flip automorphism \( \Phi = \sigma^2 \) of the rotation \( C^* \)-algebra \( A_\theta \) is defined by \( \Phi(U) = U^{-1}, \Phi(V) = V^{-1} \), and it was studied extensively in [1] [2] [3] [9] [10].

If one represents the unitaries \( U, V \) on the Hilbert space \( L^2(\mathbb{R}) \) in the canonical way as complex phase multiplication and translation operators, the automorphism \( \sigma \) corresponds exactly to the classical Fourier transform on \( L^2 \), hence the name.

In this paper we show that the semiflat positive cone \( K_0^{+SF}(A^\sigma_\theta) \) of the \( K_0 \)-group of the irrational rotation orbifold \( A^\sigma_\theta \) (a sort of noncommutative sphere [12]) is determined by classes of positive trace and the vanishing of two topological invariants (Theorem 1.4). We also determine semiflat and flat projections by their canonical traces and topological invariants up to Fourier-invariant unitary equivalence (Theorems 1.11 and 1.12): namely, Fourier invariant projections as well as projections that are orthogonal to their transform (of which there are two kinds). First, let us define these.

**Definition 1.1.** By a cyclic (or \( \sigma \)-cyclic) projection in \( A_\theta \) we mean a projection \( g \) that is orthogonal to its \( \sigma \)-orbit – that is, \( g, \sigma(g), \sigma^2(g), \sigma^3(g) \) are mutually orthogonal. The associated \( \sigma \)-invariant projection

\[ f = \sigma^+(g) := g + \sigma(g) + \sigma^2(g) + \sigma^3(g) \]

will be called flat (or \( \sigma \)-flat). We refer to \( g \) as a cyclic projection for \( f \).

Flat projections have been used in [16] to obtain the K-inductive structure of the Fourier transform \( \sigma \).

**Definition 1.2.** A projection \( h \) is semicyclic (with respect to \( \sigma \)) if \( h\sigma(h) = 0 \) and \( \sigma^2(h) = h \) (i.e., \( h \) is flip invariant). The associated Fourier invariant projection \( f = h + \sigma(h) \) is called semiflat. Two projections are \( \sigma \)-unitarily equivalent when they are unitarily equivalent by a unitary that is \( \sigma \)-invariant.

The sum of two orthogonal flat (resp., semiflat) projections is flat (resp., semiflat). If \( g \) is a cyclic projection, then \( g + \sigma^2(g) \) is semicyclic.
We will see that the trace and the topological genus of semiflat projections determines them up to $\sigma$-unitary equivalence. (The meaning of topological genus is given in Definition 2.1.)

**Definition 1.3.** The semiflat positive cone, denoted $K^+_0(A_\theta^\sigma)$, consists of the nonzero classes in positive cone $K^+_0(A_\theta^\sigma)$ given by semiflat projections in $A_\theta^\sigma$.

(We excluded the zero class for simplicity, although we could have included it.)

**Theorem 1.4.** (Main Theorem.) Let $\theta$ be irrational. The semiflat positive cone $K^+_0(A_\theta^\sigma)$ consists of the $K_0$-classes $x$ of positive trace $\tau(x)$ and

$$\psi_{10}(x) = \psi_{11}(x) = 0.$$  

Further, the semiflat projections come in three basic topological genera: $(2,0,0)$, $(1,1,2)$, $(0,0,2)$.

Thus, in general, a semiflat projection has genus that is an integral linear combination of these three basic genus types.

**Corollary 1.5.** Let $\theta$ be irrational and let $f, h$ be two semiflat projections in $A_\theta^\sigma$. Then $f$ and $h$ are $\sigma$-unitarily equivalent if and only if they have the same trace and same genus.

It is well known that the $K_0$-group of $A_\theta$ is $\mathbb{Z}^2$, and that the unique tracial state $\tau$ of $A_\theta$ induces a group isomorphism $\tau_* : K_0(A_\theta) \to \mathbb{Z} + \mathbb{Z} \theta$ when $\theta$ is irrational. (This is a classic theorem of Pimsner and Voiculescu [6], and Rieffel [8] from 1980-81.) Further, it is also known that each number in $(0,1) \cap (\mathbb{Z} + \mathbb{Z} \theta)$ is the trace of a projection in $A_\theta$, namely a Powers-Rieffel projection [8]. For our purposes here, we prove the following related results for cyclic, flat, and semiflat projections.

**Theorem 1.6.** Let $\theta$ be irrational. Each number in $(0,1) \cap (\mathbb{Z} + \mathbb{Z} \theta)$ is the trace of a cyclic projection in $A_\theta$.

**Theorem 1.7.** Let $\theta$ be irrational. Each number in $(0,1) \cap (4\mathbb{Z} + 4\mathbb{Z} \theta)$ is the trace of a flat projection in $A_\theta$.

The analogous results hold for semicyclic and semiflat projections.

**Theorem 1.8.** Let $\theta$ be irrational. Each number in $(0,1) \cap (\mathbb{Z} + \mathbb{Z} \theta)$ is the trace of a semicyclic projection in $A_\theta$.

**Theorem 1.9.** Let $\theta$ be irrational. Each number in $(0,1) \cap (2\mathbb{Z} + 2\mathbb{Z} \theta)$ is the trace of a semiflat projection in $A_\theta$.

The next result shows that the vanishing of all the topological invariants of a Fourier invariant projection means that it must be flat.

**Theorem 1.10.** Let $e$ be a Fourier invariant projection in $A_\theta$ where $\theta$ is irrational. If the topological invariants of $e$ vanish (i.e., $\psi_{**}(e) = 0$), then $e$ is flat.
Proof. By Lemma 3.6 the trace of $e$ has to be in $4\mathbb{Z} + 4\mathbb{Z}\theta$. By Theorem 1.7, $\tau(e)$ would also be the trace of some flat projection $f$. Since the Connes-Chern character invariant $T_4$ (mentioned below) of $e$ and $f$ are equal, they are unitarily equivalent by a $\sigma$-invariant unitary, and hence $e$ is flat also.

This characterizes what one might call the flat positive cone $K^+_0(A^\sigma_0)$, the nonzero classes in the positive cone $K^+_0(A^\sigma_0)$ represented by flat projections in $A^\sigma_0$.

We also obtain a result that identifies cyclic and flat projections up to Fourier invariant unitary equivalence simply by means of the trace.

**Theorem 1.11.** Let $\theta$ be any irrational number, and let $g_1$ and $g_2$ be two cyclic projections in $A_\theta$.

1. Then $g_1$ and $g_2$ are $\sigma$-unitarily equivalent iff $g_1$ and $g_2$ have the same trace.
2. Two flat projections $f_1 = \sigma^*(g_1)$ and $f_2 = \sigma^*(g_2)$ are $\sigma$-unitarily equivalent iff they have the same trace iff $g_1$ and $g_2$ are $\sigma$-unitarily equivalent.

We also have the corresponding result for semiflat projections and their semicyclic components.

**Theorem 1.12.** Let $\theta$ be any irrational number, and let $g$ and $h$ be two semicyclic projections in $A^\sigma_0$.

1. Then $g$ and $h$ are $\sigma$-unitarily equivalent iff they are $\Phi$-unitarily equivalent iff $T_2(g) = T_2(h)$.
2. Two semiflat projections $f = g + \sigma(g)$ and $f' = h + \sigma(h)$ are $\sigma$-unitarily equivalent iff $\tau(g) = \tau(h)$ and $\phi_{00}(g) = \phi_{00}(h), \quad \phi_{11}(g) = \phi_{11}(h), \quad \phi_{01}(g) + \phi_{10}(g) = \phi_{01}(h) + \phi_{10}(h)$.

Lastly, we show that all possible trace values are realized by Fourier invariant projections.

**Theorem 1.13.** (See Theorem 5.7) Let $\theta$ be irrational. Each number in $(0, 1) \cap (\mathbb{Z} + \mathbb{Z}\theta)$ is the trace of a Fourier invariant projection in $A_\theta$.

**2. Topological Invariants**

In this section we recall the topological invariants for the flip and the Fourier transform associated with the “twisted” unbounded traces on the canonical smooth dense $\ast$-subalgebra $A^\infty_\theta$.

The flip automorphism $\Phi$ has associated unbounded $\Phi$-traces defined on the basic unitaries $U^mV^n$ by

\[
\phi_{ij}(U^mV^n) = e\left(-\frac{\theta}{2\pi}mn\right)\delta_i^m - \delta_j^n
\]

(2.1)
for \(ij = 00, 01, 10, 11\), \(m, n \in \mathbb{Z}\), where \(\delta^b_a\) is the divisor delta function defined to be 1 when \(a\) divides \(b\), and 0 otherwise. (See [9] or [10].) These are (unbounded) linear functionals defined on the canonical smooth dense *-subalgebra \(A^\infty_\theta\) which are \(\Phi\)-invariant and satisfy the \(\Phi\)-trace condition

\[\phi_{ij}(xy) = \phi_{ij}(\Phi(y)x)\]

for all \(x, y\) in \(A^\infty_\theta\). In addition, they are Hermitian maps: they are real on Hermitian elements. Clearly, on the fixed point subalgebra \(A^\infty_\theta\) of \(A^\infty_\theta\) under the flip they give rise to (unbounded) trace functionals. Together with the canonical trace \(\tau\) one has the Connes-Chern character

\[T_2 : K_0(A^\Phi_\theta) \rightarrow \mathbb{R}^5, \quad T_2(x) = (\tau(x); \phi_{00}(x), \phi_{01}(x), \phi_{10}(x), \phi_{11}(x))\] (2.2)

the injectivity of which was shown in [9] (Proposition 3.2) for irrational \(\theta\). We may sometimes refer to \(T_2(x)\), or simply the \(\phi_{ij}(x)\), as the \(\Phi\)-topological invariant(s) of the class. For the identity element one has \(T_2(1) = (1; 1, 0, 0, 0)\).

For the Fourier transform \(\sigma\), one has five basic unbounded twisted trace functionals defined on the canonical smooth *-subalgebra \(A^\infty_\theta\) of \(A_\theta\), defined as follows on generic unitary elements:

\[\psi_{10}(U^mV^n) = e(-\frac{\theta}{2}(m+n)^2)\delta^m_2\delta^n_2, \quad \psi_{20}(U^mV^n) = e(-\frac{\theta}{2}mn)\delta^m_2\delta^n_2,\] (2.3)

\[\psi_{11}(U^mV^n) = e(-\frac{\theta}{2}(m+n)^2)\delta^{m-n}_2, \quad \psi_{21}(U^mV^n) = e(-\frac{\theta}{2}mn)\delta^{m-n}_2\delta^n_2,\] (2.4)

\[\psi_{22}(U^mV^n) = e(-\frac{\theta}{2}mn)\delta^{m-n}_2\delta^{m-n}_2.\] (2.5)

(See [11]2.) These maps were calculated in [11] and were used in [12], [13]. The Fourier Connes-Chern character is the group homomorphism

\[T_4 : K_0(A^\sigma_\theta) \rightarrow \mathbb{C}^6, \quad T_4(x) = (\tau(x); \psi_{10}(x), \psi_{11}(x); \psi_{20}(x), \psi_{21}(x), \psi_{22}(x))\]

where \(\tau\) is the canonical trace on \(A^\sigma_\theta\), the (orbifold) fixed point subalgebra with respect to the Fourier transform. For irrational \(\theta\) the map \(T_4\) is injective, so defines a complete invariant for projections in the fixed point algebra \(A^\sigma_\theta\) (up to Fourier invariant unitary equivalence)3 For the identity one has \(T_4(1) = (1; 1, 0, 1, 0, 0)\).

**Definition 2.1.** By the *topological genus* (or simply *genus*) of a semiflat projection \(f\) we mean the triple \((\psi_{20}(f), \psi_{21}(f), \psi_{22}(f))\).

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1In [9] we worked with the crossed product algebra \(A_\theta \times_\delta \mathbb{Z}_2\), but since this is strongly Morita equivalent to the fixed point algebra, the injectivity follows.

2In [11] our Fourier transform was the inverse of the one used in this paper, so the unbounded traces in [11] are “conjugate” to those above. See also the proof of Lemma 4.1.

3When \(\theta\) is rational, one needs to include the Chern number arising from Connes’ cyclic 2-cocycle to the \(T_4\) invariant to ensure injectivity – however, for our purposes, this is not necessary.
The injectivity of the Fourier Connes-Chern map $T_4$ was shown in [12] for a dense $G_δ$ set of $θ$'s, but later it was shown in [7], and independently in [5], that $K_0(A^θ_0) \cong \mathbb{Z}^0$ for all $θ$, which gives the injectivity of $T_4$ for all irrational $θ$. This allows us to conclude that since $A^θ_0$ has the cancellation property for any irrational $θ$, two projections $e$ and $e'$ in $A^θ_0$ are $σ$-unitarily equivalent if and only if $T_4(e) = T_4(e')$.

One easily checks the following relations between the $Φ$ and $σ$ unbounded traces
\[
ψ_{20} = φ_{00}, \quad ψ_{21} = φ_{11}, \quad ψ_{22} = φ_{01} + φ_{10}.
\] (2.6)

We will need to use the parity automorphism $γ$ of $A_θ$ defined by
\[
γ(U) = -U, \quad γ(V) = -V
\]
which will be useful because it commutes with the Fourier transform and has the property of switching the signs of the topological maps $ψ_{11}, ψ_{22}$ (while preserving the others). It also has the useful property
\[
φ_{00}γ = φ_{00}, \quad φ_{11}γ = φ_{11}, \quad φ_{01}γ = -φ_{01}, \quad φ_{10}γ = -φ_{10}.
\] (2.7)

The topological numbers of $σ$-invariant projections are quantized. Indeed, in view of [11] and [12], the $ψ_{10}, ψ_{11}$ invariants of such projections take values in the lattice subgroup $\mathbb{Z} + \mathbb{Z}(\frac{1}{2}i)$ of $\mathbb{C}$; the $ψ_{20}, ψ_{21}$ invariants take values in $\frac{1}{2}\mathbb{Z}$, and $ψ_{22}$ in $\mathbb{Z}$.

Analogous results can probably be established for the Cubic and Hexic transforms studied in [4] and [15]. For example, for the Hexic transform $ρ$ (the canonical order 6 automorphism), there are three kinds of ‘flat’ projections: $g + ρ(g)$ (where $g$ is $ρ^2$-invariant), $g + ρ(g) + ρ^2(g)$ (where $g$ is $ρ^3$-invariant, $ρ^3$ being the flip), and $g + ρ(g) + \cdots + ρ^5(g)$ (where $g$ is $ρ$-cyclic).

### 3. Density of Topological Types

In this section we establish key lemmas needed for the proof of the main theorem. For the reader’s convenience, we quote the part of Lemma 3.1 from [9] (p. 594) that is relevant to the proofs below.

**Lemma 3.1.** Let $α = rθ + s$ be irrational in the interval $(\frac{1}{2}, 1)$ where $r, s$ are integers. With $U^r$ and $V$ being unitaries satisfying $VU^r = e^{2πiα}U^rV$, there exists a Powers-Rieffel projection
\[
e = Vg(U^r) + f(U^r) + g(U^r)V^{-1}
\]
of trace $\alpha$ that is flip-invariant, where $f, g$ are certain smooth functions. Further, if $r$ is even, then
\[
\phi_{ij}(f(U^r)) = 0, \quad \phi_{ij}(g(U^r)V^{-1}) = \begin{cases} 
0 & \text{if } s \text{ is even}, \\
\frac{1}{2}\delta_2^{i-1}\delta_2^j & \text{if } s \text{ is odd}.
\end{cases}
\]

If $r$ is odd, one has
\[
\phi_{ij}(f(U^r)) = \frac{1}{2}(-1)^i\delta_2^j, \quad \phi_{ij}(g(U^r)V^{-1}) = \frac{1}{4}(-1)^{(s+1)}\delta_2^{j-1}.
\]

(N.B., this slightly more simplified version of the lemma was obtained by setting “$p = q = 0$” in the notation of Lemma 3.1 of [2].)

**Lemma 3.2.** Let $\theta$ be irrational. There are flip-invariant Powers-Rieffel projections $e, e', e''$ in $A_\theta$ with $\Phi$-invariants
\[
T_2(e) = (\tau(e); 0, 1, 0, 0), \quad T_2(e') = (\tau(e'); \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \quad T_2(e'') = (\tau(e''); 1, 0, 0, 0)
\]
such that the set of traces of each type is dense in $(0, \frac{1}{2})$.

**Proof.** Using Lemma 3.1 with $\alpha = r\theta + s$ in $(\frac{1}{2}, 1)$ where $r$ is even and $s$ odd, the Powers-Rieffel projection
\[
e_1 = Vg(U^r) + f(U^r) + g(U^r)V^{-1}
\]
has invariant $T_2(e_1) = (r\theta + s; 0, 1, 0, 0)$. Indeed, in this case $\phi_{ij}(f(U^r)) = 0$ and
\[
\phi_{ij}(e_1) = 2\phi_{ij}(g(U^r)V^{-1}) = \delta_2^{i-1}\delta_2^j.
\]
Thus, $\phi_{01}(e_1) = 1$ and the other $\phi_{ij}(e_1) = 0$.

Now let us take any other irrational $\alpha' = r'\theta + s'$ in $(\frac{1}{2}, 1)$ but this time with both $r'$ and $s'$ even. The corresponding Powers-Rieffel projection (Lemma 3.1)
\[
e_2 = Vg(U^{r'}) + f(U^{r'}) + g(U^{r'})V^{-1}
\]
has $\phi_{ij}(e_2) = 0$ and $T_2(e_2) = (\alpha'; 0, 0, 0, 0)$. Since the sets of such $\alpha$ and $\alpha'$ are dense in $(\frac{1}{2}, 1)$, choosing $1 > \alpha > \alpha' > \frac{1}{2}$ we get a dense set of traces $\{\alpha - \alpha'\}$ in $(0, \frac{1}{2})$. Upon picking a flip-invariant unitary $w$ such that $we_2w^* \leq e_1$, we obtain the flip-invariant projection
\[
e = e_1 - we_2w^*
\]
with invariant $T_2(e) = (\tau(e); 0, 1, 0, 0)$ and traces dense in $(0, \frac{1}{2})$, giving us the projections $e'$ in the statement of the lemma.

Note that
\[
T_2(1 - e_2) = (1; 1, 0, 0, 0) - (\alpha'; 0, 0, 0, 0) = (1 - \alpha'; 1, 0, 0, 0)
\]
whose traces are dense in $(0, \frac{1}{2})$, which gives us the projections $e''$ in the statement of the lemma.
Now let’s suppose that \( r \) and \( s \) are odd and let \( e_0 \) be the associated Powers-Rieffel projection of trace \( \beta = r\theta + s \in (\frac{1}{2}, 1) \). Then \( \phi_{ij}(e_0) = \frac{1}{2}(-1)^i\delta_2^j + \frac{1}{2}\delta_2^{-j-1} \) and
\[
\text{T}_2(e_0) = (\beta; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})
\]
which in view of (2.7) gives
\[
\text{T}_2(\gamma e_0) = (\beta; \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}).
\]
Subtracting from \( \gamma e_0 \) subprojections equivalent to \( e_2 \)’s of traces \( \alpha' \) less than \( \beta \) (as done previously), we obtain flip-invariant projections \( e''' \) such that
\[
\text{T}_2(e''') = \text{T}_2(\gamma e_0) - \text{T}_2(e_2) = (\beta - \alpha'; \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})
\]
where the set of traces \( \{\beta - \alpha'\} \) is dense in \((0, \frac{1}{2})\). Adding projections \( \Phi \) unitarily equivalent to \( e \) orthogonally to \( e''' \) one gets flip-invariant projections \( e' \) such that
\[
\text{T}_2(e') = (\tau(e'); \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})
\]
with traces \( \tau(e') \) is dense in \((0, \frac{1}{2})\). \( \square \)

In view of Theorem 1.9, the projections \( e, e', e'' \) in Lemma 3.2 can be conjugated by suitable flip-invariant unitaries \( u \) so that, for instance, \( h = u e u^* \) is under a semicyclic projection \( g \) of some suitably larger trace. (Recall that this means \( g\sigma(g) = 0 \) where \( g \) is flip invariant.) Clearly, the \( T_2 \) invariants of \( e \) and \( h \) are the same (since \( u \) is flip-invariant), with the difference that \( h \) is now a semicyclic projection. Therefore, we obtain the following.

**Corollary 3.3.** Let \( \theta \) be irrational. There are semicyclic projections \( h, h', h'' \) in \( A_0 \) with invariants
\[
\text{T}_2(h) = (\tau(h); 0, 1, 0, 0), \quad \text{T}_2(h') = (\tau(h'); \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \quad \text{T}_2(h'') = (\tau(h''); 1, 0, 0, 0)
\]
such that the set of traces of each type is dense in \((0, \frac{1}{2})\).

Now consider the semiflat projection \( f = h + \sigma(h) \) associated to the first type of projection \( h \) in this corollary. We have
\[
\psi_{20}(f) = 2\psi_{20}(h) = 2\phi_{00}(h) = 0
\]
likewise \( \psi_{21}(f) = 2\phi_{11}(h) = 0 \), and
\[
\psi_{22}(f) = 2\psi_{22}(h) = 2\phi_{01}(h) + 2\phi_{10}(h) = 2.
\]
Therefore \( f \) is semiflat of topological genus \((0, 0, 2)\).\(^4\) In addition, the traces of such \( f \) form a dense set in \((0, 1)\).

In the same way, from \( h' \) and \( h'' \) we obtain semiflat projections \( f' \) and \( f'' \) with respective topological genera \((1, 1, 2)\) and \((2, 0, 0)\). We will say that a class of projections has *trace density* when the set of its traces is dense in \((0, 1)\). We have therefore addressed part of the following result.

\(^4\)The author had some difficulty finding semiflat projections of genus \((0, 0, 2)\).
Lemma 3.4. Let $\theta$ be irrational. Each triple

$$(1,1,2), \quad (2,0,0), \quad (0,0,2), \quad (0,2,0),$$

$$(-1,-1,-2), \quad (-2,0,0), \quad (0,0,-2), \quad (0,-2,0),$$

is the topological genus of semiflat projections in $A^g_\theta$ with dense traces in $(0,1)$.

Applying the automorphism $\gamma$ to semiflats of genus $(0,0,2)$ we obtain semiflat projections of genus $(0,0,-2)$ with trace density.

Since $(0,2,0) = 2(1,1,2) + (-2,0,0) + 2(0,0,-2)$ is a positive linear combination of genera with trace density, we immediately get semiflat projections of genus $(0,2,0)$ with trace density. To complete the proof Lemma 3.4 we must deal with the remaining genera:

$$(-2,0,0), \quad (-1,-1,-2), \quad (0,-2,0).$$

These, however, follow from the lemma.

Lemma 3.5. For each semiflat projection of genus $(a,b,c)$, there is a semiflat projection of genus $(-a,-b,-c)$. If the former has trace density, so does the latter.

Proof. Let $p = h + \sigma(h)$ be a semiflat projection of genus $(a,b,c)$ (with trace density). We show how to construct semiflat projections of its negative genus $(-a,-b,-c)$ (with trace density). By Theorem 1.7, choose a flat projection

$$k = g + \sigma(g) + \sigma^2(g) + \sigma^3(g)$$

where $g$ is a cyclic projection such that $2\tau(h) = \tau(p) < \tau(k) = 4\tau(g)$. Choose a flip-invariant unitary $u$ such that $uhu^* \leq g + \sigma^2(g)$, where $g + \sigma^2(g)$ is flip-invariant and semicyclic. The difference projection

$$\tilde{h} := g + \sigma^2(g) - uhu^*$$

is semicyclic as well (and is flip invariant) and its associated semiflat projection $\tilde{p} = \tilde{h} + \sigma(\tilde{h})$ has genus $(-a,-b,-c)$ (since the $\psi_{ij}$ invariants of $g$ and $k$ vanish). Further, the traces

$$\tau(\tilde{p}) = 2\tau(\tilde{h}) = 4\tau(g) - 2\tau(h) = \tau(k) - \tau(p)$$

are dense in $(0,1)$ (since the set of traces $\tau(k)$ and $\tau(p)$ are each dense in $(0,1)$).

This completes the proof of Lemma 3.4.

Lemma 3.6. Let $x$ be a class in $K_0(A^g_\theta)$. If $\psi_{10}(x) = \psi_{11}(x) = 0$, then $\tau(x) \in 2\mathbb{Z} + 2\mathbb{Z}\theta$. If all $\psi_{ij}(x) = 0$, then $\tau(x) \in 4\mathbb{Z} + 4\mathbb{Z}\theta$.

The proof of this lemma is contained in the proof of Theorem 1.4 given in Section 4 below (see paragraph following equation 4.3 below).
4. **Proof of Main Theorem**

In this section we prove the main theorem on the determination of the semiflat positive cone of $K_0$ of the Fourier orbifold $A_\theta^\sigma$. Before doing so we state a lemma that is essentially a paraphrase of a result from [11], which was originally stated for crossed products, for our fixed point subalgebra situation.

**Lemma 4.1.** The range of the homomorphism $T_4 : K_0(A_\theta^\sigma) \rightarrow \mathbb{C}^6$ is spanned by the nine vectors

\[
V_1 = (2; 0, 0; 2, 0, 0) \\
V_2 = (2; 1 + i, 0; 0, 0, 0) \\
V_3 = (1; 1, 0; 1, 0, 0) \\
V_4 = (2; 0, 0; 0, 2, 0) \\
V_5 = (2; 0, 1 + i; 0, 0, 0) \\
V_6 = (1; 0, 1; 0, 1, 0) \\
V_7 = (\theta; \frac{1}{2} - \frac{1}{2}i; \frac{1}{2} - \frac{1}{2}i; \frac{1}{2}, \frac{1}{2}, 1) \\
V_8 = (\theta; -\frac{1}{2} - \frac{1}{2}i; -\frac{1}{2} - \frac{1}{2}i; -\frac{1}{2}, -\frac{1}{2}, -1) \\
V_9 = (\theta; -\frac{1}{2} + \frac{1}{2}i; -\frac{1}{2} + \frac{1}{2}i; \frac{1}{2}, \frac{1}{2}, 1).
\]

**Proof.** These can be obtained from the range of the associated homomorphism calculated for the crossed product $A_\theta \rtimes_{\sigma} \mathbb{Z}_4$ in [11] (see character table on page 645). The only difference that we need to take into account is that the “Fourier transform” used in [11] was the inverse of the one used in the current paper – and this has the effect of taking the complex conjugates of the $\psi_{10}, \psi_{11}$ values obtained in [11], which correspond, respectively, to the values of the maps “$T_{10}$” and “$\lambda^{1/4}T_{11}$” used in the character table therein. Further, we need to multiply all entries in that character table by 4 in view of the normalizations used for the unbounded traces in [11]. Once these are taken into account, we obtain the above 9 vectors from those in [11] in view of the canonical isomorphism $K_0(A_\theta^\sigma) \cong K_0(A_\theta \rtimes_{\sigma} \mathbb{Z}_4)$.

We are now ready to prove the Main Theorem [1.4].

**Proof.** (Proof of Theorem [1.4]) Fix a class $x$ in $K_0(A_\theta^\sigma)$ such that $\tau(x) > 0$ and $\psi_{10}(x) = \psi_{11}(x) = 0$. If $\tau(x) > 1$, by Theorem [1.9] we can subtract from $x$ the sum of a finite number of $K_0$-classes of semiflat projections so that the difference has positive trace less than 1. Further, the fact that $\tau(x) \neq 1$ will follow from the computation below which show that the vanishing of $\psi_{10}(x)$ and $\psi_{11}(x)$ implies that the trace of $x$ is a multiple of 2 – see, for example, equation (4.1) below. Therefore, with no loss of generality we may assume that $0 < \tau(x) < 1$. 


Write $T_4(x)$ as an integral linear combination of the nine vectors in Lemma 4.1:

$$T_4(x) = \sum_{j=1}^{9} N_j V_j$$

for some integers $N_j$. Reading off the $\psi_{10}$ and $\psi_{11}$ coordinates of $x$, we have

$$\psi_{10}(x) = N_2(1 + i) + N_3 + N_7(\frac{1}{2} - \frac{1}{4}i) + N_8(\frac{1}{2} - \frac{1}{4}i) + N_9(-\frac{1}{2} + \frac{1}{4}i) = 0$$

and

$$\psi_{11}(x) = N_5(1 + i) + N_6 + N_7(\frac{1}{2} - \frac{1}{4}i) + N_8(\frac{1}{2} - \frac{1}{4}i) + N_9(-\frac{1}{2} + \frac{1}{4}i) = 0.$$ 

Solving these gives

$$N_6 = N_3, \quad N_5 = N_2, \quad N_7 = N_9 - N_3, \quad N_8 = 2N_2 + N_3.$$ 

In terms of the integers $N_1, N_2, N_3, N_4, N_9$, the total trace is

$$\tau(x) = \sum_{j} N_j \tau(V_j) = 2N_1 + 4N_2 + 2N_3 + 2N_4 + (2N_2 + 2N_9) \theta$$

which is a multiple of 2 as we had noted at the beginning of the proof. Simplifying, one gets the $\psi_{2k}$ invariants

$$\psi_{20}(x) = 2N_1 - N_2 + N_9$$
$$\psi_{21}(x) = 2N_4 - N_2 + N_9$$
$$\psi_{22}(x) = 2N_9 - 2N_2 - 2N_3.$$ 

Therefore one gets

$$T_4(x) = (2N_1 + 4N_2 + 2N_3 + 2N_4 + (2N_2 + 2N_9) \theta; \quad 0,0;$$

$$2N_1 - N_2 + N_9, \quad 2N_4 - N_2 + N_9, \quad 2N_9 - 2N_2 - 2N_3)$$

$$= N_1(2; 0,0; 2,0,0) + N_2(4 + 2\theta; 0,0; -1,-1,-2) + N_3(2; 0,0; 0,0,0)$$

$$+ N_4(2; 0,0; 0,2,0) + N_9(2\theta; 0,0; 1,1,2).$$

We digress momentarily to note that in view of this calculation, the condition $\psi_{10}(x) = \psi_{11}(x) = 0$ has yielded the conclusion that $\tau(x)$ is in $2\mathbb{Z} + 2\mathbb{Z}\theta$ (as can be seen from (4.2)), thus establishing the first assertion of Lemma 3.6. If, in addition, the remaining invariants $\psi_{2k}(x)$ vanish, then it is easy to check that $\tau(x)$ is in $4\mathbb{Z} + 4\mathbb{Z}\theta$, which establishes the second assertion of Lemma 3.6. Thus, in particular, if all the topological invariants of a $K_0$-class $x$ vanish, then its trace is a multiple of 4 - and we know from Theorem 1.7 that any such number is the trace of a flat projection.

Returning to our current proof, in view of Lemma 3.4 we can pick semiflat projections $f_1, f_2, f_3, f_4, f_9$ with the respective topological genera appearing in the
last equality in (4.3), and of appropriately small trace, such that
\[ T_4 \left( x - \sum_{i=1,2,3,4,9} |N_i| |f_i| \right) = (\alpha; 0, 0, 0, 0, 0) \]
where \( \alpha < 1 \) is some positive number in \( \mathbb{Z} + \mathbb{Z} \theta \). We now have a class with all its topological invariants vanishing, so by Lemma 3.6, \( \alpha = 4\alpha' \) for some \( \alpha' \in \mathbb{Z} + \mathbb{Z} \theta \). By Theorem 1.7 there is flat projection \( f \) of trace \( 4\alpha' \) and \( T_4[f] = (4\alpha'; 0, 0, 0, 0, 0) \). This gives
\[ T_4(x) = T_4 \left( [f] + \sum_{i=1,2,3,4,9} |N_i| |f_i| \right) \]
and by the injectivity of \( T_4 \), the class \( x \) is a finite non-negative integral linear combination of classes of semiflat projections (at least one of them nonzero). Since \( \tau(x) < 1 \), Lemma 5.1 allows us to write \( x = [e] \) for a single semiflat projection \( e \) (since the projections \( f, f_i \) could all be unitarily combined into a sum of orthogonal semiflat projections, which is also semiflat).

5. Proofs of Trace Results

In this section we prove Theorems 1.7 and 1.9 stated in the Introduction. To do this we begin with two lemmas.

**Lemma 5.1.** Let \( e, e_1, \ldots, e_n \) be Fourier invariant projections in \( A_\theta \) such that
\[ t := \tau(e_1) + \cdots + \tau(e_n) < \tau(e). \]
There are \( \sigma \)-invariant unitaries \( w_1, \ldots, w_n \) such that
\[ w_1e_1w_1^* + \cdots + w_ne_nw_n^* \]
is a Fourier invariant subprojection of \( e \) of trace \( t \).

**Proof.** Since the order structure on \( K_0(A_\theta^\sigma) \) is determined by the canonical trace \( \tau \), the hypothesis implies that there is a projection \( Q \) in \( M_m(A_\theta^\sigma) \) such that
\[ [e_1 \oplus \cdots \oplus e_n \oplus Q] = [e] = [e \oplus O_r] \]
in \( K_0(A_\theta^\sigma) \), where \( r = n + m - 1 \) and \( O_r \) is the zero \( r \times r \) matrix. By the cancellation property of \( A_\theta^\sigma \), there is a unitary \( W \) in \( M_{r+1}(A_\theta^\sigma) \) such that
\[ W(e_1 \oplus \cdots \oplus e_n \oplus Q)W^* = e \oplus O_r. \]
From this, one obtains a set \( f_1, \ldots, f_n \) of pairwise orthogonal subprojections of \( e \) such that \( [f_j] = [e_j] \) for each \( j \), which by cancellation again gives unitaries \( w_j \) in \( A_\theta^\sigma \) such that \( f_j = w_je_jw_j^* \). One therefore gets the projection
\[ w_1e_1w_1^* + \cdots + w_ne_nw_n^* \]
contained in \( e \) and with trace \( t \). 

We cite the following lemma from [14].

**Lemma 5.2.** (See Theorem 1.6 of [14].) Let \( \theta \) be irrational, \( p/q \) a rational (in reduced form) approximant of \( \theta \) such that \( 0 < q|q\theta - p| < 1 \). Then for each positive integer \( k \) such that \( k|q\theta - p| < \frac{1}{2} \), there exists a cyclic projection in \( A_\theta \) of trace \( k|q\theta - p| \).

(Of course, one would then have the corresponding flat projection whose trace is \( 4k|q\theta - p| \).)

**Proposition 5.3.** Let \( \theta \) be any irrational number in \((0, 1)\). Then each number in \((0, 1) \cap (4\mathbb{Z} + 4\mathbb{Z}\theta)\) is the trace of a \( \sigma \)-flat projection.

**Proof.** Fix \( t \in (0, 1) \cap (4\mathbb{Z} + 4\mathbb{Z}\theta) \). With no loss of generality we can assume \( t = 4k(n\theta - m) \) where \( k,n \geq 1 \) and \( m \geq 0 \). (If \( t = 4\ell(r - s\theta) \) where \( r,s \geq 0 \), then \( t = 4\ell[s(1 - \theta) - (s - r)] \) so that we could obtain a \( \sigma \)-flat projection in \( A_{1 - \theta} \) of this trace which maps to \( A_\theta \) via a Fourier compatible isomorphism that gives one a \( \sigma \)-flat projection of trace \( t \).)

Since \( 0 \leq \frac{m}{n} < \theta \), one can choose a pair of consecutive convergents \( \frac{p}{q}, \frac{p'}{q'} \) of \( \theta \) such that \( 0 < \frac{m}{n} < \frac{p}{q} < \theta < \frac{p'}{q'} \) where \( p'q - pq' = 1 \). We can write

\[
\theta = p'(q\theta - p) + p(p' - q'\theta), \quad 1 = q'(q\theta - p) + q(p' - q'\theta)
\]

each as sums of positive terms. Thus

\[
t = 4k[np'(q\theta - p) + np(p' - q'\theta) - mq'(q\theta - p) - mq(p' - q'\theta)] = 4a(q\theta - p) + 4b(p' - q'\theta)
\]

where \( a = k(np' - mq') \) and \( b = k(np - mq) \) are positive integers. Now we are in the situation of Lemma 5.2 which gives us \( \sigma \)-flat projections \( f_1 \) and \( f_2 \) in \( A_\theta \) with respective traces \( 4a(q\theta - p) \) and \( 4b(p' - q'\theta) \). Using Lemma 5.1 there exists a Fourier invariant unitary \( w \) such that the orthogonal sum \( f_1 + wf_2w^* \) is a flat projection with trace \( t \).

**Corollary 5.4.** Let \( \theta \) be irrational. Then each number in \((0, \frac{1}{2}) \cap (\mathbb{Z} + \mathbb{Z}\theta)\) is the trace of a cyclic projection.

**Theorem 5.5.** Let \( \theta \) be irrational. Then each number in \((0, \frac{1}{2}) \cap (\mathbb{Z} + \mathbb{Z}\theta)\) is the trace of a semicyclic projection.

**Proof.** First, note that each number \( 2x \) in \((2\mathbb{Z} + 2\mathbb{Z}\theta) \cap (0, \frac{1}{2})\) is the trace of a semicyclic projection. Since \( x < \frac{1}{4} \), the preceding corollary gives a cyclic projection \( g \) of trace \( x \). The projection \( g + \sigma^2(g) \) is then semicyclic of trace \( 2x \).

Now fix \( t \in (0, \frac{1}{2}) \cap (\mathbb{Z} + \mathbb{Z}\theta) \). By density, pick \( 2x \in (2\mathbb{Z} + 2\mathbb{Z}\theta) \cap (0, \frac{1}{2}) \) such that \( t < 2x \). By the claim just proved, there exists a semicyclic projection \( h \) of trace
Lemma 5.6. Let $\theta \in (0,1)$ be irrational. Then for each pair of integers $m,n$ there exists a Fourier invariant projection of trace $(m^2 + n^2)\theta \mod 1$.

Proof. By Theorem 1.1 of [13] for any rational $p/q$ such that $0 < q|q\theta - p| < 1$, there is a Fourier invariant projection of trace $q|q\theta - p|$. Applying this with $p = 0, q = 1$, one obtains a Fourier invariant projection of trace $\theta$. Fix $m,n$ and let

$$\theta' = (m^2 + n^2)\theta \mod 1.$$ 

The unitaries

$$\tilde{U} = e(\frac{1}{2}mn\theta)V^{-n}U^m, \quad \tilde{V} = e(\frac{1}{2}mn\theta)U^nV^m$$

are easily checked to satisfy

$$\tilde{V}\tilde{U} = e(\theta')\tilde{U}\tilde{V}, \quad \sigma(\tilde{U}) = \tilde{V}^{-1}, \quad \sigma(\tilde{V}) = \tilde{U}$$

so that $\sigma$ induces the Fourier transform on the rotation C*-subalgebra $A_{\theta'}$ of $A_\theta$ generated by $\tilde{U}$ and $\tilde{V}$. Therefore, by what was just noted, there exists a Fourier invariant projection in $A_{\theta'}$ (hence in $A_\theta$) of trace $\theta'$, as required. 

Theorem 5.7. Let $\theta$ be irrational in $(0,1)$. Each $t \in (0,1) \cap (\mathbb{Z} + \mathbb{Z}\theta)$ is the trace of a Fourier invariant projection.

Proof. Write $t = m\theta - n < 1$, and assume, with no loss of generality, that $m > 0$. By Lagrange’s Theorem, write $m = m_1^2 + m_2^2 + m_3^2 + m_4^2$ as a sum of four integer squares. By Lemma 5.6 there are Fourier invariant projections $e$ of trace $(m_1^2 + m_2^2)\theta - n_1 < 1$ and $f$ of trace $(m_3^2 + m_4^2)\theta - n_2 < 1$ for some nonnegative integers $n_1, n_2$. The class $[e] + [f]$ in $K_0(A_\theta^G)$ has positive trace

$$(m_1^2 + m_2^2 + m_3^2 + m_4^2)\theta - n_1 - n_2 = t + k < 2$$ (5.1)

where $k = n - n_1 - n_2$ is either 0 or 1. If $k = 0$, then since the sum of the traces of $e$ and $f$ is less than 1, Lemma 5.1 implies that there are unitaries $u,v$ in $A_\theta^G$ such that $ueu^* + vfv^*$ is a Fourier invariant projection of trace $t$. If $k = 1$, then $[e] + [f] + [1] = [e] - [1 - f]$ has positive trace $t$ (from (5.1)), which means that $[e] > [1 - f]$ in $K_0(A_\theta^G)$, so that $1 - f$ is unitarily equivalent to a subprojection of $e$ by a unitary $w$ in $A_\theta^G$: $w(1 - f)w^* \leq e$. One therefore gets the Fourier invariant projection $e - w(1 - f)w^*$ of trace $t$. 

6. Determination of Flat and Semiflat Projections

In this section we prove Theorems 1.11 and 1.12.

**Theorem 6.1.** Let \( \theta \) be any irrational number, and let \( g_1 \) and \( g_2 \) be two cyclic projections in \( A_\theta \).

1. Then \( g_1 \) and \( g_2 \) are unitarily equivalent by a unitary in \( A_\theta^\sigma \) if and only if \( g_1 \) and \( g_2 \) have equal traces.

2. Two flat projections \( f_1 = \sigma^*(g_1) \) and \( f_2 = \sigma^*(g_2) \) are \( \sigma \)-unitarily equivalent iff they have the same trace iff \( g_1 \) and \( g_2 \) \( \sigma \)-unitarily equivalent.

**Proof.** We prove (2) first. If \( g_1 \) and \( g_2 \) are \( \sigma \)-unitarily equivalent then clearly so are \( f_1 \) and \( f_2 \). So we start with two \( \sigma \)-unitarily equivalent flat projections \( f_1, f_2 \). Since the projections \( g_1, g_2 \) have the same trace, they are Murray von Neumann equivalent in \( A_\theta \) (by a theorem of Rieffel). Let \( v \in A_\theta \) be a partial isometry such that \( vv^* = g_1, v^*v = g_2 \) and \( g_1v = v = vg_2 \). As the projections \( g_1 \) and \( g_2 \) are cyclic, one has

\[
v^*\sigma^j(v) = 0, \quad v\sigma^j(v^*) = 0
\]

for \( j = 1, 2, 3 \). The first of these follows by replacing \( v \) by \( g_1v \) and likewise the second by replacing \( v \) by \( vg_2 \). This lends us the Fourier invariant element

\[w = v + \sigma(v) + \sigma^2(v) + \sigma^3(v)\]

which acts as a partial isometry between the flat projections

\[ww^* = f_1, \quad w^*w = f_2.\]

Further, one has

\[g_1w = vv^*[v + \sigma(v) + \sigma^2(v) + \sigma^3(v)] = v,\]

\[wg_2 = [v + \sigma(v) + \sigma^2(v) + \sigma^3(v)]v^*v = v\]

where we noted that \( vv^*v = v \). These give \( f_1w = w = w f_2 \).

As the complements \( 1 - f_1 \) and \( 1 - f_2 \) are also equivalent projections in \( A_\theta^\sigma \) (having same \( T_4 \)'s), there is a \( \sigma \)-invariant partial isometry \( x \) such that \( xx^* = 1 - f_1 \) and \( x^*x = 1 - f_2 \), as well as \( (1 - f_1)x = x = x(1 - f_2) \). Since one has \( wx^* = 0 = w^*x \), the element \( w + x \) is a \( \sigma \)-invariant unitary that is easily checked to satisfy

\[g_1(w + x) = (w + x)g_2.\]

This proves (2).

To see (1), assume \( g_1 \) and \( g_2 \) have equal trace. Since their corresponding flat projections \( f_1 = \sigma^*(g_1) \), \( f_2 = \sigma^*(g_2) \) also have equal traces, we have \( T_4(f_1) = T_4(f_2) = (\text{trace; } 0, 0; 0, 0, 0) \) (recalling that the topological invariants \( \psi \) of flat projections all vanish). By the injectivity of \( T_4 \), it follows that \( f_1 \) and \( f_2 \) are \( \sigma \)-unitarily equivalent, and the result follows from (2). \( \blacksquare \)
**Remark 6.2.** In view of (2), it also follows that $g_1$ is $\sigma$-unitarily equivalent to $\sigma^j(g_2)$ for any $j$.

**Theorem 6.3.** Let $\theta$ be any irrational number, and let $g$ and $h$ be two semicyclic projections in $A_\theta^\Phi$.

1. Then $g$ and $h$ are $\sigma$-unitarily equivalent iff they are $\Phi$-unitarily equivalent iff $T_2(g) = T_2(h)$.

2. Two semiflat projections $f = g + \sigma(g)$ and $f' = h + \sigma(h)$ are equivalent in $A_\theta^\sigma$ iff $\tau(g) = \tau(h)$ and

$$\phi_{00}(g) = \phi_{00}(h), \quad \phi_{11}(g) = \phi_{11}(h), \quad \phi_{01}(g) + \phi_{10}(g) = \phi_{01}(h) + \phi_{10}(h),$$

**Proof.** For (1) it is enough to assume that $g$ and $h$ are $\Phi$-unitarily equivalent (since it is already known from [9] that this is equivalent to $T_2(g) = T_2(h)$). Let $u$ be a partial isometry in $A_\theta^\Phi$ such that $uu^* = g, u^*u = h$ (and also $gu = u = uh$). The orthogonalities $g\sigma(g) = h\sigma(h) = 0$ give $u^*\sigma(u) = 0 = u\sigma(u^*)$. Letting $w = u + \sigma(u)$, we get

$$ww^* = g + \sigma(g) =: f, \quad w^*w = h + \sigma(h) =: f'$$

and $f w = w = w f'$ so that $w$ is a $\sigma$-invariant partial isometry. Further, $gw = u = wh$ (as $uu^*u = u$). Since $1 - f$ and $1 - f'$ are also equivalent projections in $A_\theta^\sigma$ (since they have equal $T_4$'s), there is a $\sigma$-invariant partial isometry $w'$ such that $w'w'^* = 1 - f$ and $w'^*w' = 1 - f'$. One then forms the $\sigma$-invariant unitary

$$W = w + w'$$

which satisfies $gW = Wh$.

For statement (2), if $f = g + \sigma(g)$ and $f' = h + \sigma(h)$ are equivalent in $A_\theta^\sigma$ then in view of (2.6) one has the assertion stated on the $\phi_{ij}$ values (and the trace). The converse follows likewise since the $T_4$ invariants of $f$ and $f'$ are equal in view of the hypothesis. □

**Remark 6.4.** We emphasize that the semicyclic projections $g, h$ in (2) of the preceding theorem are not necessarily equivalent even in the flip fixed point algebra $A_\theta^\Phi$. Indeed, given any $g$ one can consider a semicyclic projection $h$ arising from the equation

$$T_2(h) = T_2(g) + (0; 0, n, -n, 0)$$

for any integer $n$ (where $h$ is obtained in the same manner that lead to Corollary [3.3]). Such $h$ gives rise to a semiflat projection $f' = h + \sigma(h)$ equivalent to $f = g + \sigma(g)$ (in $A_\theta^\sigma$), but $h$ is neither equivalent to $g$ nor to $\sigma(g)$. This contrasts with statement (2) of Theorem [1.11]

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