Transport equations of nonlinear geometric optics in stratified media exhibiting mixed nonlinearity

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Abstract

In this paper, we concerned with the propagation of sound waves through stratified media. Transport equation of nonlinear geometric optics in media with mixed nonlinearity, in the case of spatially varying density and entropy fields, is derived; this equation contains a cubic nonlinear term in addition to the quadratic nonlinearity. We consider an atmosphere where thermodynamic quantities depend on the height only. Effects of real gas parameters, density, and entropy attenuation parameters on the breakdown of the solution are investigated numerically.

Keywords:
Hyperbolic system, Mixed nonlinearity, Stratified media,

1. Introduction

The efficacy of the geometric acoustics method has lead many researchers to extend the study and develop the underlying ideas to wave propagation problems (see, for example Cramer and Sen [1], Hunter and Keller[2], Kluwick and Cox[3], Majda and Rosales [5]); the technique requires the introduction of fast and slow variables and the phase functions. Krylov and Bogoliubov [7] have employed these methods in the context of ODEs as a substitute for the method employed by Poincaré and Lindstedt in the problems of celestial mechanics. The exact scaling of the fast variable with respect to the slow variable may vary depending on the problem under consideration.

In certain systems, having singular thermodynamic behavior, where the fundamental derivative is small, nonlinear distortions are observed over a time scale longer by an order of magnitude; hence, it is necessary to use a higher order, i.e., \(O(\epsilon^{-2})\) fast variables. Kluwick and Cox [3] have used this methodology to the equations of gasdynamics and found that the evolution equation governing the asymptotic behavior contains a quadratic nonlinear term besides a cubic nonlinearity.

In this chapter, transport equation of nonlinear geometric acoustics in media with mixed nonlinearities following Kluwick and Cox [3] is derived in the case of stratified fluid and gravitational source terms. The characteristic feature is that cubic nonlinearities arise in addition to delayed nonlinear distortions. We provide singular correction terms to the transport equations that result when the quasilinear equations are manipulated through multiplication by polynomial nonlinearities. Effectively, any problem of the acoustics occurs in the presence of the gravitational field, and as a result, the unperturbed state ceases to be uniform. For problems considering the propagation of the sound waves over a large distance, such as in the ocean or in the atmosphere, these effects may be significantly important and generate rarefaction and amplification of sound waves.
Considering a quiet and steady atmosphere where thermodynamic quantities depend on the height only and varying according to the exponential laws, a simplified form of the evolution equation is obtained. It is shown that the solution of the Cauchy problem for the evolution equation exhibits a breakdown of the continuous solution on the expansive phase of the wave profile, which is monotonic increasing, in the sense that the Jacobian of the transformation vanishes after a finite time. This behaviour is due to the presence of the cubic nonlinearity term in the flux function and is quite different from the quadratic nonlinearity case where the solution is always continuous.

The work is organized as follows: Basic equations and formulation of the problem are given in Section 2. In Section 3, using the ideas of Kluwick and Cox [3], a detailed derivation of transport equations is given. The expressions for quadratic and cubic nonlinear terms are obtained in the Section 4. A brief discussion of the atmospheric model is considered in Section 5. In Section 6, the effects of real gas parameters and atmospheric parameters are seen on the breakdown of the solution. Finally, we conclude this chapter with a discussion of our results in Section 7.

2. Basic equations and short wave limit

Equations describing the propagation of acoustics waves through a stratified fluids may be expressed in the form

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \rho^{-1} \nabla p = - \mathbf{G}, \\
\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho + \rho (\nabla \cdot \mathbf{u}) = 0, \\
\frac{\partial s}{\partial t} + (\mathbf{u} \cdot \nabla) s = 0,
\]

where \( \rho \) is the density of the fluid, \( p = p(\rho, s) \) the pressure, \( s \) the entropy, \( t \) the time, \( \nabla \) the gradient operator with respect to the space coordinates \((x_1, x_2, x_3)\), and \( \mathbf{G} \) the forcing function, which balances the initial conditions; a comma followed by the letter \( t \) denotes partial differentiation with respect to time, \( t \). The governing system (1) can be rewritten into the form

\[
\frac{\partial \mathbf{U}}{\partial t} + A^k(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial x_k} + \mathbf{F}(\mathbf{U}) = 0, \quad k = 1, 2, 3,
\]

representing a quasilinear hyperbolic system of equations with source terms which are attributed to the influences of temperature and gravity. Here \( \mathbf{U} \) and \( \mathbf{F} \) are column vectors defined as \( \mathbf{U} = (u_1, u_2, u_3, \rho, s)^{tr} \) and \( \mathbf{F} = (f_1, f_2, f_3, 0, 0)^{tr} \), respectively; \( u_k \) are the fluid velocity components and \( f_k \) the known function of \( \mathbf{U} \), and \( A^k \) are \( 5 \times 5 \) matrices with entries \( A^k_{mn} \), \( 1 \leq m, n \leq 5 \), defined as

\[
A^k_{11} = A^k_{22} = A^k_{33} = A^k_{44} = A^k_{55} = u_k, \\
A^k_{12} = A^k_{21} = A^k_{13} = A^k_{31} = A^k_{23} = A^k_{32} = A^k_{45} = A^k_{54} = A^k_{5j} = 0, \\
A^k_{ij} = \rho \delta_{jk} A^k_{i4} = \frac{a^2 \delta_{ik}}{\rho}, \quad A^k_{i5} = \frac{\partial P \delta_{ik}}{\partial s \rho},
\]

where \( 1 \leq i, j, k \leq 3 \), \( a^2 = (\partial p/\partial \rho)|_s \) is the speed of sound, and \( \delta \) is the Kroneker delta.

We take the unperturbed solution to be \( \mathbf{U}_0 = (0, 0, 0, \rho_0(x), s_0(x))^{tr} \), where the subscript 0 is used to characterize the unperturbed fluid in equilibrium.
We non-dimensionalize the system (2) by introducing the dimensionless variables which are starred, defined as
\[ x^* = x/L, \quad t^* = t \sqrt{gH/L}, \quad u^* = u/\sqrt{gH}, \quad \rho^* = \rho/\rho_c, \quad p^* = p/\rho_c g H, \quad a^* = a/\sqrt{gH}, \]
where \( L \) is the length of the disturbed region, \( H \) is the scale height of stratification defined as the typical value of \( \rho |\nabla \rho_0|^{-1} \), \( g \) is the acceleration due to gravity, and \( \rho_c \) and \( s_c \) are constants representing convenient reference density and entropy, respectively. When \( L \) is much smaller than the scale height \( H \) of stratification, the wave is characterised as a short wave; such waves are common in the atmosphere and ocean.

Using nondimensionalize variables in (2) gives,
\[ \frac{\partial U}{\partial t} + A^k(U) \frac{\partial U}{\partial x_k} + \nu(\epsilon) F(U) = 0, \quad k = 1, 2, 3, \]
(3)

Where \( \nu(\epsilon) = L/H \) with \( \epsilon \ll 1 \), so that (3) describes short waves in the interior of a stably stratified fluid. In (3) and hereafter, we drop the star on the nondimensional variables. In the short wave limit, we have assumed that the perturbations caused by the waves are of size \( O(\epsilon) \), and they depend significantly on the fast characteristic variable \( \xi = \phi(x, t)/\epsilon^2 \), where \( \phi \) is the phase function to be determined. Using the method of multiple scales, for this change of variables and replaced partial derivatives \( \frac{\partial}{\partial X} \to \frac{\partial}{\partial X} + \epsilon^{-2} \left( \frac{\partial \phi}{\partial X} \right) \frac{\partial}{\partial \xi} \), \( X \) being either \( t \) or \( x_k \) the system (3) becomes
\[ \epsilon^2 \left[ \frac{\partial U}{\partial t} + A^k(U) \frac{\partial U}{\partial x_k} + \nu(\epsilon) F(U) \right] + \left( \frac{\partial \phi}{\partial t} I + A^k(U) \frac{\partial \phi}{\partial x_k} \right) \frac{\partial U}{\partial \xi} = 0, \]
(4)

where \( I \) is the 5 × 5 unit matrix.

3. Evolution equation

We seek for small amplitude high frequency wave solutions of (2) with an asymptotic approximation as \( \epsilon \to 0 \) of the form,
\[ U = U_0(x) + \epsilon U^{(1)}(\xi, x, t) + \epsilon^2 U^{(2)}(\xi, x, t) + \epsilon^3 U^{(3)}(\xi, x, t) + O(\epsilon^4), \]
(5)

which corresponds to the propagation of waves of small amplitude with a disturbed region of small extent. Here \( \epsilon \) is a small parameter measuring the wave amplitude, \( U_0 = [0, 0, 0, \rho_0(x), s_0(x)]^T \), the known background state, is a solution of
\[ U_{0,t} + A^0(U_0) U_{0,x_k} = 0, \]
(6)

(recall that \( \nu = 0 \) when \( \epsilon = 0 \)) and \( U^{(1)}, U^{(2)} \) and \( U^{(3)} \) are first, second and third order perturbations of the undisturbed state \( U_0^{(0)} \), respectively. We now expand \( A^k(U) \) and \( F(U) \) as a power series in \( \epsilon \) about \( U = U_0 \) and use them in (4) together with (5) to obtain the resulting asymptotic expansion in the form
\[ \epsilon Z_1 + \epsilon^2 Z_2 + \epsilon^3 Z_3 + \ldots = 0; \]
(7)

where \( Z_1 = 0 \) is the linearized system associated with (4) that admits five families of characteristic surfaces two of which represent waves propagating with speeds \( \pm a_0 \) through the background state \( U = U_0 \) and the remaining three form a set of coincident characteristics representing entropy waves or particle paths; here we shall be concerned with the
propagation of a right running acoustic wave \( \phi(x,t) = \text{constant} \), and so the linear rays of (3) are characteristic curves (bicharacteristics of (2)) of

\[
\frac{\partial \phi}{\partial t} + a_0|\nabla \phi| = 0,
\]

We denote the left and right eigenvectors of \( \sum_k (\partial_k \phi)A^k \) associated with the eigenvalue \( a_0|\nabla \phi| \) by \( \mathbf{l} \) and \( \mathbf{r} \), respectively; these are given by

\[
\mathbf{l} = \left( n_1, n_2, n_3, \frac{a_0}{\rho_0}, (\rho_0 a_0)^{-1} \left( \frac{\partial p}{\partial s} \right)_0 \right), \\
\mathbf{r} = \left( n_1, n_2, n_3, \rho_0 a_0, 0 \right),
\]

where \( \mathbf{n} = \nabla \phi/|\nabla \phi| \). Furthermore, it follows from \( Z_1 = 0 \) that \( U^{(1)} = \sigma(\xi, \mathbf{x}, t) \mathbf{r} \), where \( \sigma \) is the scaler amplitude function will be determine at the next order. In the present context we refer to the work of Kluwick and Cox [3] and Cramer and Sen [1], who treat the wave propagation problem when the nonlinear effects are noticeable over times of order \( O(\epsilon^{-2}) \) rather than \( O(\epsilon^{-1}) \); their main result focus on the case when the quadratic nonlinearity parameter \( \Gamma \) defined as

\[
\Gamma = \frac{\partial \phi}{\partial x_k} (1 \mathbf{r})^{-1} \mathbf{l} [\mathbf{r} \cdot (\nabla U A^k)_0] \mathbf{r},
\]

is of order \( O(\epsilon) \), where \( \nabla U \) is the gradient operator with respect to vector \( \mathbf{U} \) and

\[
\mathbf{r} \cdot (\nabla U A^k)_0 = r_m \frac{\partial A^k}{\partial U_m} |_{\mathbf{U} = \mathbf{U}_0}.
\]

Computation of \( \Gamma \) using the definition of \( A^k_0 \) yield

\[
\Gamma = \left( 1 + \frac{\rho_0}{a_0} \frac{\partial a}{\partial \rho} \right) |\nabla \phi|,
\]

which is of order \( O(\epsilon) \). In order to account for a small but nonzero \( \Gamma \), it would be convenient to write (7) as

\[
\epsilon Z_1 + \epsilon^2 (Z_2 - \mu) + \epsilon^3 (Z_3 + \bar{\mu}) + \ldots = 0,
\]

where \( \mu = \Gamma \sigma \frac{\partial \sigma}{\partial \xi} \mathbf{r} \) and \( \bar{\mu} = \Gamma \mathbf{r} \sigma \frac{\partial \sigma}{\partial \xi} / \epsilon \), noting that \( \Gamma/\epsilon = O(1) \). Thus to the second order, we obtain \( Z_2 - \mu = 0 \) or equivalently,

\[
\left( \frac{\partial \phi}{\partial t} I + A^k(\mathbf{U}) \frac{\partial \phi}{\partial x_k} \frac{\partial \mathbf{U}}{\partial \xi}^{(2)} \right) = - \left\{ \frac{\partial \mathbf{r} \cdot (\nabla U A^k)_0 \mathbf{r} - \Gamma \mathbf{r}}{\partial x_k} \right\} \sigma \frac{\partial \sigma}{\partial \xi},
\]

Thus the solvability condition for \( \mathbf{U}^{(2)} \), which requires that the right hand side of (14) be orthogonal to \( \mathbf{l} \), is satisfied automatically in view of the definition of \( \Gamma \). In order to have a solution which exhibit the character of a progressive wave in which both nonlinear and source terms are present, we must get, to the third order, a new equation involving the source terms; for this one must choose \( \nu(\epsilon) \) of the order \( O(\epsilon) \) to keep the source term in \( Z_3 + \bar{\mu} = 0 \). Thus, choosing \( \nu(\epsilon) = \epsilon \), we obtain to the third order;
\[
\left( \frac{\partial \phi}{\partial t} + A^k(U) \frac{\partial \phi}{\partial x_k} \right) \frac{\partial U^{(3)}}{\partial \xi} = \frac{\partial (\sigma r)}{\partial t} - A_0^k \frac{\partial (\sigma r)}{\partial x_k} - [r, (\nabla_U A^k)_0] \frac{\partial U^{(0)}}{\partial x_k} + \left( \frac{\partial \phi}{\partial x_k} [U^{(2)}, (\nabla_U A^k)_0] r \sigma + \frac{1}{2} \frac{\partial \phi}{\partial x_k} [r, (\nabla_U A^k)_0] r \sigma^2 \right) \frac{\partial \sigma}{\partial \xi} \right.
\]
\[ - \left. \left( \frac{\partial \phi}{\partial x_k} [U^{(2)}, (\nabla_U A^k)_0] r \sigma + \frac{1}{2} \frac{\partial \phi}{\partial x_k} [r, (\nabla_U A^k)_0] r \sigma^2 \right) \frac{\partial \sigma}{\partial \xi} \right] + \Gamma \sigma \left[ (\nabla_U F)_0 \right] \sigma - F_0 - \left[ r, (\nabla_U F)_0 \right] \sigma, \quad (15) \]

where \( rr : (\nabla_U \nabla_U A^k)_0 = \sum_{m,n=1}^5 r_m r_n \frac{\partial^2 A^k}{\partial t \partial x_m} |u=u_0 \), and \( \hat{\Gamma} = \Gamma / \epsilon = O(1) \).

Solvability condition for \( U^{(3)} \) yields the desired evolution equation for \( \sigma \), namely
\[
\frac{\partial \sigma}{\partial \tau} + \left( \Gamma + \frac{E}{2} \sigma \right) \frac{\partial \sigma}{\partial \xi} + (M, U^{(2)}) \frac{\partial \sigma}{\partial \xi} + (N, \frac{\partial U^{(2)}}{\partial \xi}) \sigma + \chi \sigma + \frac{1}{2} (\frac{\partial F_0}{\partial \xi} + \frac{1}{2} \Gamma (l, U^{(2)}) \sigma) = 0, \quad (16)
\]

where \( \partial / \partial \tau = \partial / \partial t + a_0 \mathbf{n} \nabla \) is the ray derivative with \( \nabla \) as the gradient operator with respect to the space variable \( x_k \), and \( E, b, c \) and \( \chi \) are given by
\[
E = \frac{1}{2} \frac{\partial \phi}{\partial x_k} [r, (\nabla_U \nabla_U A^k)_0] r, \quad (17)
\]
\[
M = \frac{1}{2} \left\{ \frac{\partial \phi}{\partial x_k} [l, (\nabla_U A^k)_0] r - \Gamma I \right\}, \quad (18)
\]
\[
N = \frac{1}{2} \left\{ \frac{\partial \phi}{\partial x_k} [l, (r, \nabla_U A^k)_0] - \Gamma I \right\}, \quad (19)
\]
\[
\chi = \frac{1}{2} \left\{ \Gamma A^k_0 \frac{\partial r}{\partial x_k} + 1 [r, (\nabla_U A^k)_0] \frac{\partial U^{(0)}}{\partial x_k} + 1 \left[ r, (\nabla_U F)_0 \right] \right\}. \quad (20)
\]

It may be notice that in (16), it will be essential to have a knowledge of the second order approximation, \( U^{(2)} \) and \( U^{(2)}_\xi \); this can be achieved by noting that (14) can be integrated to yield
\[
A. U^{(2)} = - \frac{1}{2} \left( \frac{\partial \phi}{\partial x_k} \left[ r, (\nabla_U A^k)_0 \right] r - \Gamma r \right) \sigma^2, \quad (21)
\]
where \( U^{(2)} = 0 \) when \( U^{(1)} = 0 \), and \( B = [\frac{\partial \phi}{\partial t} I + A^k_0 \frac{\partial \phi}{\partial x_k}] \). In view of Eqs. (12), (18), and (19), we find that \( M \) and \( N \) are orthogonal to \( r \) and hence they lie in the linearly independent row space of \( B \). Hence, we can write
\[
M = \omega_\alpha \ B_\alpha, \quad N = \delta_\alpha \ B_\alpha, \quad (22)
\]
where \( \mathbf{B}_\alpha (\alpha = 1, 2, 3, 4) \) being the linearly independent rows of \( \mathbf{B} \) then, the terms \( M, U^{(2)} \) and \( N, U^{(2)}_\xi \) in Eq. (16) can be written as
\[
M, U^{(2)} = - \frac{\omega}{2} \left( \frac{\partial \phi}{\partial x} \left[ r, (\nabla_U A^k)_0 \right] - \Gamma I \right) r \sigma^2, \quad (23)
\]
\[
N, U^{(2)}_\xi = - \delta \left( \frac{\partial \phi}{\partial x} \left[ r, (\nabla_U A^k)_0 \right] - \Gamma I \right) r \sigma \sigma_\xi, \quad (24)
\]
Further, since

\[ \frac{\partial \sigma}{\partial \tau} + \left( \hat{\Gamma} + \frac{\Lambda}{2} \sigma \right) \frac{\partial \sigma}{\partial \xi} + \chi \sigma + \frac{n.F_0}{2} = 0, \]  

(25)

where

\[ \Lambda = E - (\omega + 2\delta) \left( \frac{\partial \phi}{\partial x} \left[ r.(\nabla U A^k_1)_{0} \right] - \Gamma I_1 \right) r. \]  

(26)

We now turn to the calculation of coefficients \( \hat{\Lambda} \), \( \Gamma \) and \( \chi \). If we define \( \nabla_i A^k_{ij} \) and \( \nabla_{lm} A^k_{ij} = \left( \frac{\partial^2 A^k_{ij}}{\partial t_{lm}} \right) \) \( |\nabla=0, \) we find that

\[ \nabla_k A^k_{4k} = 1, \quad \nabla_4 A^k_{4k} = -\frac{a_0^2}{\rho_0^2} + \frac{2a_0}{\rho_0} a_{\rho 0}, \quad \nabla_5 A^k_{5k} = \frac{2a_0}{\rho_0} a_{s0}, \]  

(27)

\[ \nabla_{5k} A^k_{5k} = \frac{p_{s0}}{\rho_0}, \quad \nabla_4 A^k_{5k} = -\frac{p_{s0}}{\rho_0} + \frac{2a_0}{\rho_0} a_{\rho 0}, \quad \nabla_4 A^k_{4k} = 1; \quad 1 \leq i \leq 5, \quad k = 1, 2, 3, \]  

(28)

and hence,

\[ \Gamma = \left( 1 + \frac{\rho_0}{a_0} a_{\rho 0} \right) |\nabla=0, \]  

(29)

As \( \Gamma = O(\epsilon) \), the term containing \( \Gamma \) and \( \Gamma^2 \) can be neglected for evaluating \( \Lambda \) and \( \chi \). Further, since \( \nabla_{lm} A^k_{ij} \) in \( E \) appears in the evolution equation as a combination \( r_j, r_l \) and \( r_m \), it is easily verified that all the terms \( \nabla_{lm} A^k_{ij} \) vanish except \( \nabla_{44} A^k_{4k} \) which is given by

\[ \nabla_{44} A^k_{4k} = \frac{2a_0^2}{\rho_0^2} (6 + \Omega). \]  

(30)

Hence, \( E = (6 + \Omega) |\nabla=0, \omega = \frac{\rho_{0}^2}{a_0} \Sigma \left( \rho_0, s_0 \right) = O(1) \) and \( \Sigma = \frac{1}{\rho} \frac{\partial (a_\rho)}{\partial \rho} \). From (18) and (19), the vectors \( \mathbf{M} \) and \( \mathbf{N} \) are obtained as follows:

\[ \mathbf{M} = \mathbf{N} = |\nabla=0, \left( n_1, n_2, n_3, -\frac{a_0}{\rho_0} a_{s0} \right) \]  

(31)

Using (22), we find that

\[ \omega_{\alpha} = \delta_{\alpha} = \frac{n_\alpha}{a_0} \left( \frac{a_0 \rho_0 a_{s0}}{p_{s0}} \right), \quad \alpha = 1, 2, 3; \quad \omega_4 = \delta_4 = \frac{1}{\rho_0} \left( \frac{a_0 \rho_0 a_{s0}}{p_{s0}} + 1 \right), \]  

(32)

and hence, \( \omega + 2\delta \) \( \left( \phi_k \left[ r.(\nabla U A^k_1)_{0} \right] - \Gamma I_1 \right) r = 6 |\nabla=0, \omega = 0; \) taking the vector \( \mathbf{g} \) as the acceleration due to gravity and using the foregoing results, we find that

\[ \Lambda = |\nabla=0, \omega = 0, \quad \chi = (a_0 \nabla_\mathbf{n} + a_{s0} s_0 k_0) / 2, \]  

(33)

where \( \nabla_\mathbf{n} \) is the mean curvature of the wavefront. It is noticeable that the quadratic nonlinearity coefficient \( \hat{\Gamma} \) in (25) is Lax [4] genuine nonlinearity coefficient, whereas the cubic nonlinearity coefficient \( \Lambda \), indicates the degree of material nonlinearity; the source term \( \chi \sigma \) in (25) corresponds to the changes in wave amplitude \( \sigma \) attributable to the wave interactions with the changing medium ahead and the wavefront curvature as the wave moves along the rays.
4. Real gas parameters

For the van der Waals gas, where the proper volume of the gas molecule is reduced by an amount $\beta$ and the gas pressure is reduced with respect to the ideal pressure due to the attractive interaction of the molecule, the resulting equation of the state is [8]

$$ (p + \alpha \rho^2)(1 - \beta \rho) = \rho RT $$

where $T$ is the temperature, $R$ the gas constant, and the constants $\alpha$ and $\beta$ depends on a particular gas. The expression for the entropy can be obtained from the first law of thermodynamics as $\rho^3 \exp((s - s_0)/c_v) = (p + \alpha \rho^2)(1 - \beta \rho)^\gamma$, where $c_v$ is the specific heat at constant volume and $\gamma$ is the specific heat ratio. The speed of sound $a^2 = \left(\frac{\partial p}{\partial \rho}\right)^{(1/2)}$ is thus given by $a^2 = \frac{\gamma(p + \alpha \rho^2)}{\rho(1 - \beta \rho)} - 2 \rho \alpha$. Then the quadratic nonlinearity parameter $\Gamma$ turns out to be

$$ \Gamma = \left(1 + \frac{\rho_0}{a_0^2 a_\phi}\right) |\nabla \phi| = \frac{(\gamma + 1)}{2} \left(\frac{1}{(1 - \beta \rho_0)} - \frac{2 \alpha \rho_0 (2 - \gamma - 3 \beta \rho_0)}{a_0^2(\gamma + 1)(1 - \beta \rho_0)}\right) |\nabla \phi|, \quad (35) $$

Further if we choose $\alpha$ and $\beta$ such that $\left(\frac{1}{1 - \beta \rho_0} - \frac{2 \alpha \rho_0 (2 - \gamma - 3 \beta \rho_0)}{a_0^2(\gamma + 1)(1 - \beta \rho_0)}\right) |\nabla \phi| = \epsilon$, then $\Gamma = O(\epsilon)$ whilst $\hat{\Gamma} = \Gamma/\epsilon = (\gamma + 1)/2 = O(1)$. Similarly we can get the expression of cubic nonlinearity parameter, neglecting terms of the form $\Gamma$ and $\Gamma^2$ we get,

$$ \Omega = \frac{\rho_0^2}{a_0} \frac{\partial}{\partial \rho} \left(\frac{\partial (a \rho)}{\rho \partial \rho}\right) (\rho_0, s_0) = - \left(\frac{3(1 + \gamma)}{2(1 - \beta \rho_0)} - \frac{3 \alpha \beta \rho_0^2}{(1 - \beta \rho_0)a_0^2}\right) = O(1) $$

neglecting $O(\beta^2)$ terms the expression for $\Lambda$ can be found from the relation

$$ \Lambda = \frac{\Omega |\nabla \phi|}{a_0} = - \left(\frac{3(1 + \gamma)}{2a_0}(1 + \beta \rho_0) - \frac{3 \alpha \beta \rho_0^2}{a_0^2}\right) |\nabla \phi|, \quad (36) $$

and an expression used in the calculation of $\chi$ can also be found $\left(\frac{a_{s0}}{p_{s0}}\right) = \gamma a_0 \rho_0 (1 - \beta \rho_0)^{-1}$.

5. Atmospheric model

We now consider the wave propagations in the troposphere region of the earth’s atmosphere (approximately $0 - 12$ km above the earth’s surface) with a quiet and steady atmosphere with thermodynamic quantities depending on height only, i.e., they depend on one coordinate only, say $x_3$, and satisfy the exponential laws based on the U.S. Standard Atmosphere (1966 supplement) ([6])

$$ \rho_0 = \exp(-\theta x_3) \quad \text{and} \quad a_0 = \exp(-\omega x_3), \quad (37) $$

where $\theta, \omega \geq 0$ are attenuation rates for the density and sound speed, respectively. The dependence of the entropy on $x_3$ can be obtained from the equilibrium condition $\nabla p|_{s0} = 0$, so that

$$ \left. \frac{\partial a}{\partial s}\right|_{s_{0,k} n_k} = \left(\frac{a_{s0}}{p_{s0}}\right) (-a_0^2 \rho_0) = \frac{\gamma \theta \exp(-\omega x_3)}{2(1 - \beta \exp(-\theta x_3))}, \quad (38) $$

where prime denotes derivative with respect to $x_3$; (38) will be needed in the computation of $\chi$ which appears in the transport equation (25). If we choose our initial disturbance
on the horizontal plane \( x_3 = 0 \), the characteristic surface (or wavefront) at any time \( t \) can be obtained by solving the eikonal equation (8), indeed the characteristic rays are the solution of the ODE’s

\[
\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = \frac{\nabla \phi}{|\nabla \phi|} a_0(x_3), \quad \frac{d(\partial \phi / \partial x_i)}{d\tau} = -|\nabla \phi| a_0'(x_3),
\]

(39)

let \( \tau = 0 \), at \( t = 0 \), so that (19) implies that \( \tau = t \) along the rays. Since \( \nabla \phi \big|_{t=0} = (0, 0, 1) \), it follows from (37) and (38), that at any time \( t \) the vector \( \nabla \phi \) and the location of the wave front is given by

\[
\nabla \phi = (0, 0, 1 + \omega t) \quad x_3 = \omega^{-1} \ln(1 + \omega t),
\]

(40)

which describes an ascending wave as \( x_3 \) is an increasing function of time. The source term \( F \) in (2) is indeed \( F = (0, 0, g, 0, 0) \), which in view of the dimensionless quantities assume the form \( F = (0, 0, 1, 0, 0) \) and thus the coefficient \( \hat{\Gamma}, \Lambda, \chi \) and the inhomogeneous term \( n_0 F \) in (25) are now explicitly known thus finally the evolution equation for the amplitude \( \sigma \) describes the propagation of an acoustic wave in a stratified medium becomes,

\[
\frac{\partial \sigma}{\partial t} + \left( \frac{\gamma + 1}{2} \right) (1 + \omega t) \sigma \frac{\partial \sigma}{\partial \xi} - \frac{3}{4} (\gamma + 1)(1 + \omega t)^2 \left[ 1 + \beta (1 + \omega t)^{(1 - \theta/\omega)} - 2\alpha \beta (1 + \omega t)^{(2 - \theta/\omega)} \right] \sigma^2 + \frac{\gamma \theta (1 + \omega t)^{(\theta/\omega) - 1} \sigma}{4((1 + \omega t)\theta - \beta)} + \frac{1}{2} = 0,
\]

(41)

by the method of characteristics, the solution along the characteristics

\[
\frac{d\xi}{dt} = \left( \frac{\gamma + 1}{2} \right) (1 + \omega t)\sigma - \frac{3}{4} (\gamma + 1)(1 + \omega t)^2 \left[ 1 + \beta (1 + \omega t)^{(1 - \theta/\omega)} - 2\alpha \beta (1 + \omega t)^{(2 - \theta/\omega)} \right] \sigma^2
\]

(42)

is given by

\[
\frac{d\sigma}{dt} = -\frac{\gamma \theta (1 + \omega t)^{(\theta/\omega) - 1} \sigma}{4((1 + \omega t)\theta - \beta)} - \frac{1}{2},
\]

(43)

which is an ODE and can be solved to obtain

\[
\sigma(\xi, t) = \left( \sigma_0 + \frac{2(1 - u^{1 - \frac{\theta}{\omega}})}{(\gamma \theta + 4\omega)} \right) \frac{u^{-\frac{\theta}{\omega}}}{\omega} + \frac{\gamma \beta}{4} \left( \sigma_0 u^{-\frac{\theta}{\omega}} (u^{-\frac{\theta}{\omega}} - 1) - \frac{2(u^{1 - \frac{\theta}{\omega}} - u^{-\frac{\theta}{\omega}(1 + \frac{\gamma}{4})})}{4\omega + \gamma \theta} + \frac{2(u^{1 - \frac{\theta}{\omega}} - u^{-\frac{\theta}{\omega}})}{4\omega + \theta(\gamma - 4)} \right)
\]

(44)

where \( \sigma_0 \) is the initial value of \( \sigma \) and \( u = (1 + \omega t) \) using this value of \( \sigma \) in characteristic equation we can get a condition for the Jacobian. If we choose initial data at time \( t = t_0 \) as

\[
\sigma(\xi, t_0) = \begin{cases} 
\sin(\xi) & \xi \in (0, \pi/2), \\
0 & \text{otherwise}
\end{cases}
\]

(45)
expression for the Jacobian can be written as

\[
\xi_\eta = \frac{(u^6 - \theta/\omega(2 + \gamma/4) - 1)}{24\omega - \theta(8 + \gamma)} \left( \frac{-24\alpha\beta \cos(\eta)}{(\gamma\theta + 4\omega)} \right) + \frac{(u^5 - \theta/\omega(2 + \gamma/4) - 1)}{20\omega - \theta(8 + \gamma)} \left( \frac{24\alpha\beta \cos(\eta)}{(\gamma\theta + 4\omega)} + 12\alpha\beta \cos(\eta)\sin(\eta) \right)
\]

Now we consider two cases for seeing the effect of parameter \( \theta \) and \( \omega \) on breaking of solution.

- If we consider the case when density is constant, i.e., \( \theta = 0 \) and taking the initial condition as above the expression for jacobian reduces into the following form

\[
t^6[(-\omega/2)\alpha\beta \cos(\eta)] + t^5[(6/5)(\omega \cos(\eta) - 2)\omega^3 \alpha\beta] + t^4[((1 + \gamma)(1 + \beta)/8) - (3/2)\alpha\beta + 2\sin(\eta)\alpha\beta\omega^3 \cos(\eta)] + t^3[(1 + \gamma)(1 + \beta)(1 - \omega \sin(\eta)) - 4\alpha\beta(1 - 3\omega \sin(\eta))] \omega \cos(\eta) + t^2 [3(1 + \gamma)(1 + \beta)(1/4 - \omega \sin(\eta)) - 3\alpha\beta((1/2) - 4\omega \sin(\eta) + \omega(1 + \gamma)/2) \cos(\eta)] + t[6\sin(\eta)\alpha\beta - (1 + \gamma)(1 + \beta)/2] + (1 + \gamma) \cos(\eta) + 2 = \xi_\eta.
\]

- Similarly when we consider the case when speed of sound parameter is zero, i.e., \( \omega \to 0 \) and considering the expression \( \lim \omega \to 0(1 + \omega t)^{-\gamma/4\omega} = e^{-(\gamma/4\omega)t} \) and taking above initial condition the expression of jacobian takes following form

\[
\xi_\eta = \frac{(1 - e^{-\theta/(2 + \gamma/4)t})\beta \cos(\eta)}{\theta(8 + \gamma)} \left[ 4\alpha \left( \frac{2}{\gamma \theta} (1 - \beta + \sin(\eta)) - (1 + \gamma) \beta \left( \frac{\theta}{\gamma \theta - 4} - \frac{2(2 - \gamma)}{\gamma \theta} \right) \right) \right]
\]

5.1. Bounds on parameters

In our case we have chosen \( \Gamma \) such that

\[
\Gamma = \frac{(\gamma + 1)}{2} \left( \frac{1}{(1 - \beta_0)} - \frac{2\alpha_0(2 - \gamma - 3\beta_0)}{a_0^2(\gamma + 1)(1 - \beta_0)} \right) |\nabla \phi| = O(\epsilon),
\]
also substituting the corresponding expression for $\rho_0$ and $a_0$ the expression of $\Gamma$ can be rewritten into the form

$$
\frac{(\gamma + 1)}{2} \left( 1 - \frac{2\alpha (1 + \omega t)^{2-\theta/\omega} (2 - \gamma - 3\beta (1 + \omega t)^{-\theta/\omega})}{a_0^2 (\gamma + 1)} \right) \frac{[\nabla \phi]}{(1 - \beta (1 + \omega t)^{-\theta/\omega})}.
$$

(50)

In order to choose $\Gamma = O(\epsilon)$ along with the conditions $\alpha, \beta, \omega, \theta$ all are positive also $t \geq 0$ and $1 < \gamma \leq 5/3$. If we assume $\theta = \omega$, then we obtained the following expression relating $\alpha, \beta$, and $t$

$$
t = \frac{1}{\omega} \left[ (\frac{\gamma + 1}{2\alpha} + 3\beta) \frac{1}{(2 - \gamma)} - 1 \right],
$$

(51)

which in view of $t \geq 0$ gives the relation $\frac{(\gamma + 1)}{2\alpha} + 3\beta \geq (2 - \gamma)$. We have taken $\gamma = 1.01$ and $\omega = 0.1$ and hence the above condition reduces to

$$
\left( \frac{67}{66 - 200\beta} \right) \geq \alpha \quad \text{and} \quad t_0 = 10 \left[ \left( \frac{67}{200\alpha + \beta} \right) \frac{100}{33} - 1 \right],
$$

(52)

we have used these relation in our numerical calculation for finding the values of $\alpha$ and $\beta$, and the initial time $t_0$ for corresponding $\Gamma = O(\epsilon)$.

6. Effects of parameters on breaking of solution

In this section we have shown the effect of parameters $\alpha, \beta, \theta, \omega$ on breaking of solution. We have taken initial data

$$
\sigma(\xi, t_0) = \begin{cases} 
\sin(\xi) & \xi \in (0, \pi), \\
0 & \text{otherwise}
\end{cases}
$$

(53)

where $t_0$ is the value obtain of time obtained from the restriction $\Gamma = O(\epsilon)$.

6.0.1. Effects of $\beta$ on breakdown of solution:

To discuss the effect of $\beta$ on the breaking of solution we have varied the value of $\beta$ while keeping all the other parameters as constant with values $\alpha = 0.15$, $\gamma = 1.01$, $\theta = 0.1$, $\omega = 0.1$, $t = 1.4$, and plotted the Jacobian $\xi, \eta$ against $\eta$ and found that with the increase in beta nonlinear effect serve to expedite as seen in the Figure 1.

6.0.2. Effects of $\alpha$ on breakdown of solution:

All the parameter with same values are taken as in the last case except $\alpha$ and the effects of the variation in $\alpha$ are noticed keeping all other parameters as constants. In contrast to the last case, with the increase in $\alpha$ nonlinear effect serves to delay as shown in Figure 2.

6.0.3. Effects of $\theta$ and $\omega$ on breakdown of solution:

Finally, the effects of the atmospheric parameters, i.e., density variation parameter $\theta$ and sound speed variation parameter $\omega$ are observed. It is find out that, increase in the density parameter helps in the breakdown whereas speed of sound variation parameter delays breaking of solution as displayed in Figs. 3, 4, respectively.
6.1. Evolution of waves in a van der Waals fluid

In this section, to discuss the effects of van der Waals parameters ($\alpha, \beta$) in the case when flux function of evolution equation has quadratic as well cubic nonlinearity, we present numerical solution of evolution equation (41) with the following initial data:

$$\sigma(\xi, t_0) = \begin{cases} 
\sin(\xi) & \xi \in (0, \pi), \\
0 & \text{otherwise.}
\end{cases}$$  \hspace{2cm} (54)
For various values of \((\alpha, \beta)\), other parameters are so chosen such that the condition (49), i.e. \(\Gamma = O(\epsilon)\) is satisfied, we observe that the breaking of solution delays with an increase in \(\alpha\) as is exhibited in Figure 5; however, an increase in \(\beta\) has just the opposite effect, i.e., the breaking of solutions gets expedited as seen in the Figure 6. Here we notice that it is the cubic nonlinearity in the flux function that is responsible for the breakdown of solution on an expansion phase of the wave profile, which is quite different from the quadratic flux case, where there is no breakdown on the expansion phase if the initial
Figure 5: Numerical solutions of (41) and (53) with $\gamma = 1.01$, $\beta = 0.06$, and $\theta = 0.1$, $\omega = 0.1$, at time $t = 1.4$; for $\alpha = 0.15$, $\alpha = 0.25$, and $\alpha = 0.35$.

datum is monotonic increasing.
7. Conclusions

We have studied, using perturbation methods, propagation of high frequency waves with mixed nonlinearity in a stratified atmosphere with van der Waals equation of state. A transport equation for the wave amplitude is derived; which exhibits both quadratic and cubic nonlinearities. A quiet and steady atmosphere with thermodynamic quantities, depending only on one spatial coordinate (height) with varying density and sound speed, is considered. It is shown that the Cauchy problem exhibits a breakdown of the continuous solution on the expansive phase of the wave profile, which is monotonic increasing, in the sense that the Jacobian of the transformation vanishes after a finite time. This behaviour is due to the presence of the cubic nonlinearity term in the flux function and is quite different from the quadratic nonlinearity case where the solution is always continuous. Effects of the influence of van der Waals parameters $\alpha$, $\beta$ on the breaking of solution is displayed in Figs. 1, 2, respectively, while effects of atmospheric parameters $\theta$, $\omega$ was observed in Figs. 2 and 3. Indeed, the effect of the van der Waals parameter $\alpha$ is to delay the onset of singularity in the solution, whereas the effect of $\beta$ is to hasten the process of singularity formation in the solution as shown in Figs. 5, 6.

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