HORIZONTAL PATTERNS FROM FINITE SPEED DIRECTIONAL QUENCHING

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Abstract. In this paper we study the process of phase separation from directional quenching, considered as an externally triggered variation in parameters that changes the system from monostable to bistable across an interface (quenching front); in our case the interface moves with speed $c$ in such a way that the bistable region grows. According to results from [9, 10], several patterns exist when $c \geq 0$, and here we investigate their persistence for finite $c > 0$. We find existence and nonexistence results of multidimensional horizontal striped patterns, clarifying the selection mechanism relating their existence to the speed $c$ of the quenching front. We further illustrate our results by applying them to those of [9], hence obtaining the existence of a family of single interface patterns displaying different contact angles between their nodal lines and the quenching front; the existence of these patterns was known for small speeds $c > 0$ and here we show that they also exist in the range $0 < c < 2$.

1. Introduction. In the theory of reaction diffusion, the interplay between stable and unstable mechanisms can give rise to spatial patterns, i.e., stationary nonhomogeneous structures. In the presence of controllable external parameters the existence and persistence of these patterns are worth investigating, both for their mathematical interest and industrial applications; see [17] and the survey [4]. In this paper we are interested in a directional quenching scenario, where a planar interface (also called the quenching front) moves with constant speed $c$, across which a phase separation process takes place: ahead of the interface the system is monostable, while in its wake it is bistable. We study this phenomenon in the scalar model

$$
\partial_t u = \Delta_{\xi,y} u + \mu(\xi - ct)u - u^3,
$$

where $\Delta_{\xi,y} := \partial^2_{\xi} + \partial^2_{y}$ and $\mu(\xi - ct) \geq 0 = \mp 1$. Equation (1) is a particular case of the Allen-Cahn model, which describes the behavior of a heterogeneous binary mixture: the unknown $u(\xi, y; t)$ denotes the relative concentration of one of the two

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metallic components of the alloy at time \( t \in \mathbb{R}^+ := [0, \infty) \) and point \((\xi, y) \in \mathbb{R}^2\). The stationary problem in a moving frame \((x, t) = (\xi - ct, t)\) is written
\[
-c \partial_x u = \Delta_{x,y} u + \mu(x) u - u^3. \tag{2}
\]
The most physically relevant scenario to be considered consists of the case \( c \geq 0 \). According to results in \cite{10}, whenever the speed \( c \) of the quenching front is small the equation \eqref{2} admits a rich family of patterns, as we now briefly describe.

**Pure phase selection:** \((1 \rightarrow 0)^{(c)} \) fronts. This is the simplest, one dimensional case, when \eqref{2} reads
\[
-c \partial_x u(x) = \partial_x^2 u(x) + \mu(x) u(x) - u(x)^3, \quad x \in \mathbb{R}. \tag{3}
\]
The quenching trigger generates a pattern \( \theta^{(c)}(\cdot) \) solving \eqref{3} and satisfying spatial asymptotic conditions \( \lim_{x \to -\infty} \theta^{(c)}(x) = 1 \) and \( \lim_{x \to +\infty} \theta^{(c)}(x) = 0 \) in the wake and ahead of the quenching front, respectively; see Figure 1.

**Horizontal patterns:** \( \mathcal{H}_\kappa, \; \pi < \kappa \leq \infty \). In this scenario the patterns sought are truly two-dimensional; furthermore, we can take advantage of the odd nonlinearity to reduce the study of \eqref{2} to a problem in \( \mathbb{R} \times [0, \kappa] \). Whenever \( \kappa < \infty \) the solution \( \Xi^{(c)}(\cdot, \cdot) \) has boundary conditions \( \Xi^{(c)}(x, y) \bigg|_{y=0,\kappa} = 0 \) and spatial asymptotic conditions
\[
\lim_{x \to -\infty} \Xi^{(c)}_\kappa(x, y) = 0, \quad \text{and} \quad \lim_{x \to +\infty} \left| \Xi^{(c)}_\kappa(x, y) - \bar{u}(y; \kappa) \right| = 0, \tag{4}
\]
where \( \bar{u}(y; \kappa) \) is a parametrized family of periodic solutions to
\[
\partial_y^2 \bar{u} + \bar{u} - \bar{u}^3 = 0, \quad \bar{u}(y; \kappa) = -\bar{u}(y + \kappa; \kappa) = -\bar{u}(-y; \kappa) \neq 0, \quad \text{for} \quad y \in \mathbb{R}. \tag{5}
\]
with half-periods \( \pi < \kappa < \infty \), and normalization \( \partial_y \bar{u}(0; \kappa) > 0, \; \bar{u}(0; \kappa) = 0 \). In the limiting case \( \kappa = \infty \) a single interface is created, describing what we call \( \mathcal{H}_\infty \)-pattern (see Fig. 2); asymptotically we have
\[
\lim_{y \to +\infty} \Xi^{(c)}_\infty(x, y) = \theta^{(c)}(x), \quad \lim_{x \to +\infty} \Xi^{(c)}_\infty(x, y) = 0, \quad \text{and} \quad \lim_{x \to -\infty} \Xi^{(c)}_\infty(x, y) = \bar{u}(y; \infty), \tag{6}
\]
where \( \bar{u}(\cdot; \infty) := \tanh \left( \frac{\cdot}{\sqrt{2}} \right) \) and \( \theta^{(c)}(\cdot) \) is a \((1 \rightarrow 0)^{(c)}\)-front.

**Figure 1.** Sketches of solutions for pure phase selection \(1 \rightarrow 0\); solution \( \theta^{(c)}(x) \) (left) and contour plot for \((x, y) \in \mathbb{R}^2 \) (right).

**Figure 2.** Sketches of solutions for horizontal patterns; \( \mathcal{H}_\alpha \) pattern (left) and \( \mathcal{H}_\infty \) pattern (right).
We summarize below the properties of these solutions in the regime $c \geq 0$ as given in [10]. Their approach is based on a continuation argument from the case in which quenching front has zero speed ($c = 0$), somehow explaining the nature of the smallness on $c$ in their results.

**Proposition 1 ([10, Theorems 1.1 and 1.5]; (1 \to 0)(c) problem).** For any $c \geq 0$ sufficiently small there exists a function $\theta^{(c)}(\cdot) \in C^{(1,\alpha)}(\mathbb{R}; [0,1])$, $\forall \alpha \in [0,1]$, that solves the $(1 \to 0)(c)$ problem, i.e., $\theta^{(c)}$ solves the boundary value problem

$$
\begin{cases}
\partial_x^2 \theta^{(c)}(x) + c \partial_x \theta^{(c)}(x) + \mu(x) \theta^{(c)}(x) - (\theta^{(c)}(x))^3 = 0 \quad \text{in the sense of distributions} \\
0 < \theta^{(c)}(x) < 1, \quad \theta^{(c)}(-\infty) = 1, \quad \theta^{(c)}(+\infty) = 0,
\end{cases}
$$

where the boundary conditions are attained in the limit sense. Furthermore, the mapping $x \mapsto \theta^{(c)}(x)$ is non-increasing. The solutions found for $c = 0$ can be continued smoothly to $c > 0$ for sufficiently small $c$. More precisely, there exist families of solutions $\theta^{(c)}(x)$ to (3) for $0 < c < \delta_1$, satisfying the same limiting conditions as the solutions at $c = 0$ for $x \rightarrow \pm \infty$. Moreover, the solutions depend smoothly on $c$, uniformly in $x$.

**Proposition 2 ([10, Thm. 1 and Proposition 1.4]; $\mathcal{H}_\kappa$ problem, $\pi < \kappa \leq \infty$).** For any $c \geq 0$ sufficiently small equation (2) admits a family of solutions (in the sense of distributions) $\Xi^{(c)}(\cdot, \cdot) \in C^{(1,\alpha)}(\mathbb{R}^2; \mathbb{R})$, $\forall \alpha \in [0,1]$, $\kappa \in (\pi, \infty]$. In the case $\pi < \kappa < \infty$ for the solution $\Xi^{(c)}(\cdot, \cdot)$ has the symmetries $\Xi^{(c)}(x, y) = -\Xi^{(c)}(x, -y) = -\Xi^{(c)}(x, y + \kappa)$ and satisfies the asymptotic spatial conditions (4). Moreover, the convergence is exponential, uniformly in $y$. On the other hand, whenever $\kappa = \infty$ the solution $\Xi^{(c)}(\cdot, \cdot)$ has the symmetries $\Xi^{(c)}(x, y) = -\Xi^{(c)}(x, -y)$ and satisfies the asymptotic spatial conditions (6), where convergence is exponential and uniform.

In this paper we give a deeper understanding of the range of validity of these continuations in $c$.

**Remark 1.** Besides the patterns described above, oblique and vertical structures with respect to the quenching front were also studied in [10], where they were shown to not exist as solutions to (2) when $c > 0$. Therefore, the only patterns of relevance for us are the ones described above. Throughout this paper we add sub and superscripts to the patterns found in [10] under the inconvenience of disagreeing with that paper’s notation; this is done because the classification of patterns has become more involved and richer. In this way our notation highlights the dependence of the solutions on the quenching speed $c$ and on the $\kappa$-periodicity in the $y$-direction as $x \rightarrow -\infty$ (in the multidimensional case; see Fig. 2).

1.1. **Main results.** Initially we study the problem in one dimensional setting; although less physically relevant, it stands as one of the cornerstones of the construction of the 2D patterns $-\mathcal{H}_\kappa$ and $\mathcal{H}_\infty$ (see for instance, [10, §2 and 3]).

**Theorem 1.1** ($(1 \to 0)(c)$-patterns). The solutions $\theta^{(0)}(\cdot)$ defined in Prop. 1 when $c = 0$ can be continued smoothly in $c > 0$ in an unique way within the range $c \in [0, 2)$ to solutions $\theta^{(c)}(\cdot) \in C^{(1,\alpha)}(\mathbb{R}; (0, 1))$, $\forall \alpha \in [0,1]$ satisfying

$$
\begin{cases}
\partial_x^2 \theta^{(c)}(x) + c \partial_x \theta^{(c)}(x) + \mu(x) \theta^{(c)}(x) - (\theta^{(c)}(x))^3 = 0 \quad \text{in the sense of distributions} \\
0 < \theta^{(c)}(x) < 1, \quad \lim_{x \to -\infty} \theta^{(c)}(x) = 1, \quad \lim_{x \to \infty} \theta^{(c)}(x) = 0,
\end{cases}
$$

(7)
where the convergence to its asymptotic end states takes place at exponential rate. Furthermore, we have that \( (\theta^{(c)})'(x) < 0 \) for all \( x \in \mathbb{R} \). No solution to \( (7) \) exists when \( c \geq 2 \).

In the higher dimensional setting, both the magnitude of the speed \( c \) of the quenching front and the \( y \)-period \( \kappa \) of the end-state \( \bar{u}(\cdot; \kappa) \) play important roles in the analysis.

**Figure 3.** Existence diagram for parameters \( c \geq 0 \) (speed of the front) and \( \kappa > \pi \) (\( y \)-periodicty of the patterns); the dashed curve represents the critical case \( P(c, \kappa) = 1 \) (see Thm. 1.2).

**Theorem 1.2** (\( H_\kappa \), \( H_\infty \) patterns). Let \( \pi < \kappa \leq \infty \) be a fixed number. Define the quantity

\[
P(c; \kappa) := \begin{cases} 
\frac{c^2}{4} + \frac{\pi^2}{\kappa^2}, & \text{for } \pi < \kappa < \infty; \\
\frac{c^2}{4}, & \text{for } \kappa = \infty.
\end{cases}
\]  

(i) **(Existence)** The solutions \( \Xi^{(0)}_{c}(\cdot, \cdot) \) defined in Prop. 2 when \( c = 0 \) can be continued smoothly in \( c > 0 \) in an unique way within the range \( P(c; \kappa) < 1 \) to solutions \( \Xi^{(c)}_{c}(\cdot, \cdot) \) solving \( (2) \) in the sense of distributions and satisfying the asymptotic spatial condition \( (4) \) (resp. \( (6) \)) when \( \pi < \kappa < \infty \) (resp., when \( \kappa = \infty \)). The convergence to their spatial asymptotic states takes place at exponential rate, uniformly. Furthermore, for any \( \kappa > \pi \) the mapping \( x \mapsto \Xi^{(c)}_{c}(x, y) \) is non-increasing for any fixed \( y \in [0, \kappa] \) and \( 0 < \Xi^{(c)}_{c}(x, y) < \bar{u}(y; \kappa) \) in \( (x, y) \in \mathbb{R} \times (0, \kappa) \), where \( \bar{u}(\cdot, \cdot) \) is given by \( (5) \). The solutions have the symmetries \( \Xi^{(c)}_{c}(x, y) = -\Xi^{(c)}_{c}(x, -y) = -\Xi^{(c)}_{c}(x, y + \kappa) \) (resp., \( \Xi^{(c)}_{\infty}(x, y) = -\Xi^{(c)}_{\infty}(x, -y) \)).

(ii) **(Nonexistence)** Whenever \( \kappa \in (\pi, \infty) \) and \( P(c; \kappa) > 1 \) (resp., \( \kappa = \infty \) and \( P(c; \kappa = \infty) > 1 \)) no solution to \( (2) \) satisfying \( 0 < \Xi^{(c)}_{c}(x, y) < \bar{u}(y; \kappa) \) in \( (x, y) \in \mathbb{R} \times (0, \kappa) \) and the asymptotic spatial condition \( (4) \) (resp. \( (6) \)) can be found.

One can see from the previous result that whenever \( c \geq 0 \) the region \( P(c; \kappa) < 1 \) (resp. \( P(c; \kappa) > 1 \)) corresponds to \( c < 2 \sqrt{1 - \frac{\pi^2}{\kappa^2}} \) (resp., \( c > 2 \sqrt{1 - \frac{\pi^2}{\kappa^2}} \)), namely, the linear spreading speed obtained from the linearization of \( (2) \) about \( u \equiv 0 \) on the region \( x \leq 0 \). Overall, we point out that the dependence of the critical spreading speed curve on the parameter \( \kappa \) is a true manifestation of the multidimensionality...
of the $H_\kappa$-patterns; the quantity $P(\cdot; \cdot)$ describes the maximal speed of spreading of the bistable region and its dependence on the $y$-period of the pattern away from the quenching interface.

**Remark 2** (Uniqueness results). It is worthwhile to point out that the uniqueness result in Theorems 1.1 and 1.2 allow us to compare the solutions constructed in [10] using perturbation methods for $c \gtrsim 0$ with those obtained here, namely, the solutions we construct in this paper agree with those obtained by perturbation methods in [10].

**Critical cases:** $P(c; \kappa) = 1$. To the best of the author’s knowledge, the case $P(c; \kappa) = 1$ for the $H_\kappa$-problem ($\pi < \kappa < \infty$) is open. See Sec. 5.

A summary of our results is given in the Table 1.

| $(1 \leadsto 0)^{(c)}$ and $H_\infty$ problems | $H_\kappa$ problem ($\pi < \kappa < \infty$) |
|-----------------------------------------------|------------------------------------------|
| $0 \leq c < 2$ | $P(c; \kappa) < 1$ |
| Yes | Thms. 1.1 & 1.2 |
| No | Thm. 1.1 & Obs. 4.1 |
| $c \geq 2$ | $P(c; \kappa) = 1$ |
| Yes | Thm. 1.2(i) |
| No | Thm. 1.2(ii) |
| $P(c; \kappa) > 1$ | Not known |

**Table 1.** A grasshopper’s guide to the existence and nonexistence of patterns $(1 \leadsto 0)^{(c)}$ and $H_\kappa$ ($\pi < \kappa \leq \infty$). Critical quantity $P(c; \kappa)$ defined in (8), where $c$ denotes the speed of the quenching front and $2\kappa$ denotes the $y$-period of the pattern (see Fig. 2).

**Single interfaces with contact angle.** The $H_\infty$-patterns obtained in Theorem 1.2 are odd functions with respect to the $y$ variable, and as so they satisfy $\Xi^{(c)}(x, 0) = 0$. Thus, the patterns present a nodal set at the negative part of the $x$-axis that forms a right angle ($\phi = \frac{\pi}{2}$) with respect to the quenching front located at a line $x - ct = 0$ (parallel to the $y$ axis). This observation motivates the question of how extra terms added to equation (2) could deform this nodal set. More precisely, we consider the equation

$$\Delta_{x,y} u + c_x \partial_x u + c_y \partial_y u + \mu(x) u - u^3 + \alpha g(x, u) = 0, \quad (x, y) \in \mathbb{R}^2, \quad (9)$$

where $\mu(x) = \pm 1$ when $x \leq 0$ and $g(x, u) = \begin{cases} g_l(u), & x < 0 \\ g_r(u), & x > 0 \end{cases}$, for $g_{l,r}(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^2\mathbb{R})$. Notice that when $c_y = 0$, $\alpha = 0$, the function $u(\cdot, \cdot) = \Xi^{(c_x)}_{\infty}(\cdot, \cdot)$ solves (9).

With regards to (9), we remark that two mechanisms are in play: the growth of the bistable region and new, “unbalancing terms”, that break the odd symmetry of the solutions.

**Definition 1.3** (Contact angle). We say (9) possesses a solution $u$ with contact angle $\phi_*$ if $u$ possesses the limits

$$\lim_{x \to +\infty} u(x, y) = 0, \quad \lim_{x \to -\infty} u(x, \cot(\phi)x) = \begin{cases} u_+, & \phi > \phi_* \\ u_-, & \phi < \phi_* \end{cases}, \quad (10)$$
for all $0 < \varphi < \pi$.

For instance, when $\alpha = 0$ and $c_y = 0$ the function $\Xi_{\infty}(c_x)(\cdot, \cdot)$ solves (9) satisfying the limit (10) for $\varphi^* = \frac{\pi}{2}$ and $u_{\pm} = \pm 1$. It was shown in [9] that for any $c_x > 0$ fixed, the function $\Xi_{\infty}(c_x)(\cdot, \cdot)$ can be continued in $\varphi^*$ as a solution to (9) for all $|\varphi^* - \frac{\pi}{2}|$ sufficiently small. One of the most important properties used in their proof concerns to the strict monotonicity of the mapping $y \mapsto \Xi_{\infty}(c_x)(x, y)$ for any $x \in \mathbb{R}$, namely,

$$\partial_y \Xi_{\infty}(c_x)(x, y) > 0, \quad (x, y) \in \mathbb{R}^2.$$ \hspace{1cm} (11)

where $\Xi_{\infty}(c_x)(\cdot, \cdot)$ is given in Theorem 1.2(i). An inconvenient fact in their analysis is that the patterns $\Xi(c_x)_{\infty}(x, \cdot)$ for $c_x > 0$ were obtained through perturbation methods (in [10, §5]), hence some qualitative information on the patterns are not immediately available. Nevertheless, the authors managed to prove (11) for $c_x \geq 0$ sufficiently small (see [9, Prop. 4.1]). Our construction readily gives the validity of (11) for all $0 < c_x < 2$, thus we can make use of the analysis in [9] to conclude the following result:

**Corollary 1** (Unbalanced patterns). For $0 < c_x < 2$, there exists $\alpha_0(c_x) > 0$ such that for all $|\alpha| < \alpha_0(c_x)$ there exist a speed $c_y(\alpha)$ and a solution $u(x, y; \alpha)$ to (9) with contact angle $\varphi(\alpha)$. We have that $u(x, y; 0) = \Xi_{\infty}(c_x)(\cdot, \cdot)$, the latter given by Theorem 2. Moreover, $c_y(\alpha)$ and $\varphi(\alpha)$ are smooth with $\varphi(0) = \pi/2$, $c_y(0) = 0$, and $u(x, y; \alpha)$ is smooth in $\alpha$ in a locally uniform topology, that is, considering the restriction $u|_{B_R(0)}$ to an arbitrary large ball.

![Figure 4. Sketch of an unbalanced pattern with a contact angle; see Def. 1.3 or [9].](image)

1.2. **Outline.** In Section 2 we focus on the $(1 \rightarrow 0)^{(c)}$ problem, where we prove Theorem 1.1. Section 3 is devoted to Theorem 1.2 and $\mathcal{H}_\kappa$ patterns ($\pi < \kappa < \infty$), while the study of the $\mathcal{H}_{\infty}$-pattern is left to Section 4. A brief discussion and further extensions brings the paper to an end in Section 5.

**Notation.** In this paper we write $B \subset \subset \tilde{B}$ whenever both $B$ and $\tilde{B}$ are open sets and the closure of $B$ is compact and contained in $\tilde{B}$. We also write $\mathcal{C}^k(X; Y)$, $\mathcal{C}_c^k(X; Y)$ and $\mathcal{C}^{(k, \alpha)}(X; Y)$ to denote respectively, the space of $k$-times continuously differentiable functions, the space of $k$ times continuously differentiable functions with compact support in $X$, the space of $(k, \alpha)$-Hölder continuously differentiable functions from $X$ to $Y$. We define $|w|_{\mathcal{C}^k(K)} := \sum_{0 \leq l \leq k} \sup_{x \in K} |\partial_x^l w(x)|$. We denote
the Sobolev spaces over an open set $\Omega$ by $W^{(k,p)}(\Omega)$; whenever $p = 2$ we write $W^{(k,2)}(\Omega) = H^k(\Omega)$. The inner product of elements in a Hilbert space $\mathcal{H}$ is written as $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Norms on a Banach space $\mathcal{B}$ are denoted as $\| \cdot \|_{\mathcal{B}}$. For a given operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset X \to Y$ we write $\text{Ker} (\mathcal{L}) := \{u \in \mathcal{D}(\mathcal{L}) | Lu = 0\}$ and $\text{Rg}(\mathcal{L}) := \{f \in Y | \exists u \in \mathcal{D}(\mathcal{L}), Lu = f\}$. A distribution $T \in \mathcal{D}'(\Omega)$ satisfies $T \geq 0$ in the sense of distributions if $T(\phi) \geq 0$ for any $\phi(\cdot) \in C_0^\infty(\Omega; [0, \infty))$. We define a $C^\infty(\mathbb{R}; [0, 1])$ partition of unity \{\(\chi^\pm(\cdot)\)\} of $\mathbb{R}$, of the form
\[
\chi^-(x) + \chi^+(x) = 1,
\]
where $\chi^-(x) = 1$ for $x \leq -2$, and $\chi^-(x) = 0$ for $x \geq -1$. Last, we denote the Implicit Function Theorem by IFT.

2. One dimensional directional quenching: $(1 \to 0)^c$ problem, $c > 0$. The construction of the patterns $(1 \to 0)^c(\cdot)$ follows ideas from [8] and [10]: initially we solve a family of similar problems in truncated, bounded intervals; later, as we enlarge these intervals and exhaust $\mathbb{R}$, we show that these functions converge to a solution of problem (3). We begin by setting up a truncated $(1 \to 0)^c$ problem:

\[
\begin{align*}
\partial^3_x u(x) + c\partial_x u(x) + \mu(x)u(x) - u^3(x) &= 0, \\
u(-M) = 1, u(L) = 0,
\end{align*}
\]
for $0 < M,L$, with continuity of $u$ and $u_x$ at $x = 0$. We show the existence of a unique solution $\theta^{(c)(-M,L)}$; later on in the section we let $M \to \infty$ and, subsequently, $L \to \infty$. Roughly speaking, the $(1 \to 0)^c$ front $\theta^{(c)}(\cdot)$ will be given by
\[
\theta^{(c)}(\cdot) = \sup_{L > 0} \left\{ \inf_{M > 0} \theta^{(c)(-M,L)}(\cdot) \right\}.
\]
The qualitative properties of the function $\theta^{(c)}(\cdot)$ are proved in this section, where we also show that $\theta^{(c)}(x)$ converges to 1 and 0 as $x \to -\infty$ and $x \to \infty$, respectively. We finalize with a proof of Theorem 1.1. A substantial part of the techniques and proofs are similar to those in [10]; whenever possible we skip details and refer to that article for full proofs.

2.1. The $(1 \to 0)^c$ truncated problem.

Lemma 2.1 (Existence and uniqueness). The truncated problem (13) has a unique solution $\theta^{(c)(-M,L)}(\cdot) \in C^{(1,\alpha)}([-M,L]; [0, 1])$, $\forall \alpha \in [0, 1)$. Furthermore, $\theta^{(c)(-M,L)}(x) > 0$ whenever $x \in (-M,L)$.

Proof. Throughout the proof we write $(\cdot)'$ to denote $\partial_x$. To establish the existence of a solution we define an iterative scheme,

\[
\begin{align*}
-U''_{n+1}(x) - cU''_{n+1}(x) + 5U_{n+1}(x) &= (5 + \mu(x))U_n(x) - U^3_n(x), \\
U_{n+1}(-M) = 1, \\
U_{n+1}(L) = 0.
\end{align*}
\]

We choose $U_0$ from the class
\[
\Psi^{(1 \to 0)^c} := \left\{ U \in C^{(1,\alpha)}([-M,L]) \left| U(-M) = 1, U(L) \geq 0, \forall x \in [0, 1] \text{ for all } x \right. \right\}
\]
\cap \{U''(x) + cU''(x) + \mu(x)U(x) - U^3(x) \leq 0\}
\cap \{U''(x) + cU''(x) + \mu(x)U(x) - U^3(x) \neq 0\},
\]
where the statement in the last set should be understood in the sense of distributions. Observe that $1 \in \Psi_{(1-\epsilon)}$, thus the latter set is non-empty. We mostly follow the reasoning in [10, §2.2] thus some of the proofs are only outlined. First, it is shown that $\{U_n\}_{n \in \mathbb{N}}$ is so that $0 < U_{n+1}(x) < U_n(x)$ in $x \in (-M, L)$. Hence, $\theta^{(c)}_{(-M, L)}(x) := \inf_{n \in \mathbb{N}} U_n(x)$ is a well-defined non-negative element of $W^{2,p}$ for any $p < \infty$, and consequently it is a function in $\mathcal{C}^{(1,\alpha)}([-M, L]; [0, 1]), \forall \alpha \in [0, 1)$, thanks to the Sobolev embedding. Using the maximum principle (as in [5, §8, Thm. 8.19]) one can show that $\theta^{(c)}_{(-M, L)}(x) > 0$ for $x \in (-M, L)$; see further details in [10, Lem. 2.1]. The uniqueness proof is a bit different due to the transport term $c \partial x$ and we give it here for completeness: assume the existence of two solutions, $\theta(\cdot), \tilde{\theta}(\cdot)$ so that $\theta(\cdot) \neq \tilde{\theta}(\cdot)$. Define the set $\mathcal{S} = \{x \in [-M, L] | \theta(x) \neq \tilde{\theta}(x)\}$; this set is open due to continuity of $\theta, \tilde{\theta}$. As an open subset of the real line, we can assume without loss of generality that $\mathcal{S} = (a, b)$ where $\theta(x) > \tilde{\theta}(x)$, $x \in (a, b)$, and $\theta(x) = \tilde{\theta}(x)$, $x \in \{a, b\}$. Now, since both $\theta$ and $\tilde{\theta}$ are solutions, we can integrate against test functions $e^{c\theta x}(x)$ and $e^{c\tilde{\theta} x}(x)$ on the interval $(a, b)$:

$$
\int_a^b e^{c\theta x}(\theta' \tilde{\theta} - \theta \tilde{\theta}')(x)dx + c \int_a^b e^{c\theta x}(\theta' \tilde{\theta} - \theta \tilde{\theta}')(x)dx
= \int_a^b e^{c\theta x}(\theta^2 - \tilde{\theta}^2)(x)\theta(x)\tilde{\theta}(x)dx.
$$

Integration by parts gives

$$
e^{c\theta x}(\theta' \tilde{\theta} - \theta \tilde{\theta}')(x)|_a^b = \int_a^b e^{c\theta x}(\theta^2 - \tilde{\theta}^2)(x)\theta(x)\tilde{\theta}(x)dx.
$$

The term on the left hand side is non-positive, since $\theta > \tilde{\theta}$ in $(a, b)$, $\theta(x) = \tilde{\theta}(x)$ for $x \in \{a, b\}$. On the right hand side, the term $\theta \tilde{\theta}$ is strictly positive, thanks to the strict positivity of solutions in $(-M, L)$ (in particular, in $(a, b) \subset (-M, L)$). Using that $\theta > \tilde{\theta}$ in $(a, b)$ we conclude that the integral on the right hand side is positive. This contradiction proves the result.

In order to compare the families of solutions as $M, L$ vary, we construct trivial extensions of functions $u$ defined on an interval $(-M, L)$ given by the operator $\mathcal{E}$,

$$
\mathcal{E}[u](x) = \begin{cases} 
 u(x), & \text{for } x \in (-M, L), \\
 1, & \text{for } x \leq -M, \\
 0, & \text{for } x \geq L.
\end{cases}
$$

By construction, $x \mapsto \mathcal{E}\left[\theta^{(c)}_{(-M, L)}\right](x)$ is a continuous function, and this extension is one of the main tools to make (14) meaningful and rigorous. Indeed, thanks to the positivity and strict monotonicity of the mappings $x \mapsto \theta^{(c)}_{(-M, L)}(x)$ we can use a sliding type of argument to compare $\mathcal{E}\left[\theta^{(c)}_{(-M, L)}\right](\cdot)$ for different values of $M$ and $L$. In fact, subsolutions or supersolutions to problem (13) can be manufactured by considering the functions $\mathcal{E}\left[\theta^{(c)}_{(-M, L)}\right](\cdot)$ and either restricting them to smaller intervals $I \subset (-M, L)$ or appropriately translating them. These ideas are deeply

---

1Recall that distributions of finite order (say, order $k$) can be extended to the space of $\mathcal{C}_0^k$ functions (cf. [7, §2]; see also [10, Lem. 2.2]).
exploited in [10, §2] and the next result, whose proof we omit and refer to the latter article, relies on them.

**Lemma 2.2** (Properties of solutions to the truncated problem). The following properties of \( \mathcal{E} [\theta^{(c)}_{(-M,L)}] (\cdot) \) hold.

(i) (Monotonicity of \( \mathcal{E} \)) We have \( 0 \leq \mathcal{E} [\theta^{(c)}_{(-M,L)}] (\cdot) \leq 1 \). Furthermore, for \( w \) defined on a subset \( A \), \( -M, L \subset A \subset (-\infty, L) \) with \( 0 \leq w(\cdot) \leq 1 \) and \( 0 \leq w(\cdot) \leq \theta^{(c)}_{(-M,L)}(\cdot) \) in \( (-M, L) \), we have \( 0 \leq \mathcal{E} [w] (\cdot) \leq \mathcal{E} [\theta^{(c)}_{(-M,L)}] (\cdot) \) on \( \mathbb{R} \).

(ii) (Monotonicity in \( M \)) Let \( 0 \leq M < \tilde{M} \) and \( L \geq 0 \) be fixed. Then

\[
\mathcal{E} [\theta^{(c)}_{(-M,L)}] (x) \leq \mathcal{E} [\theta^{(c)}_{(-M,L)}] (x).
\]

(iii) (Monotonicity in \( L \)) Let \( 0 \leq L < \tilde{L} \) and \( M \geq 0 \) be fixed. Then

\[
\mathcal{E} [\theta^{(c)}_{(-M,L)}] (x) \leq \mathcal{E} [\theta^{(c)}_{(-M,L)}] (x).
\]

(iv) (Monotonicity in \( x \)) For every fixed \( M \) and \( L \) the mapping \( x \mapsto \mathcal{E} [\theta^{(c)}_{(-M,L)}] (x) \) is non-increasing.

(v) (Continuous dependence of \( \theta^{(c)}_{(-M,L)}(\cdot) \) on \( M \)) Let \( 0 < M < \infty \), \( 0 < L < \infty \). The mappings \( L \mapsto \mathcal{E} [\theta^{(c)}_{(-M,L)}] (\cdot) \) and \( M \mapsto \mathcal{E} [\theta^{(c)}_{(-M,L)}] (\cdot) \) are continuous in the sup norm.

### 2.2. Passing to the limit

We are now ready to pass to the limit \( M = \infty \) as a first step towards the proof of Prop. 1. Define

\[
\theta^{(c)}_{(-\infty,L]}(x) := \inf_{M>0} \mathcal{E} [\theta^{(c)}_{(-M,L)}] (x) = \lim_{M \to \infty} \mathcal{E} [\theta^{(c)}_{(-M,L)}] (x),
\]

where the last equality is a consequence of Lem. 2.2(i). We remark that, as the infimum of functions defined on the whole real line, \( \theta^{(c)}_{(-\infty,L)}(\cdot) \) also has \( \mathbb{R} \) as its domain of definition; reasoning as in Lemma 2.1 shows that \( \theta^{(c)}_{(-\infty,L)}(\cdot) \in \mathcal{C}^{(1,\alpha)}((-\infty, L)), \forall \alpha \in [0,1) \). The following proposition highlights the role of the front speed \( c \); roughly speaking it says that stretching procedure \( M \to \infty \) we designed “loses mass” whenever \( c \geq 2 \), i.e., \( \theta^{(c)}_{(-\infty,L]}(\cdot) \equiv 0 \) when \( c \geq 2 \). Although zero is a (trivial) solution to the \((1 \mapsto 0)\) truncated problem, one might wonder about the usefulness of the minimax construction we developed in (14), for it seems to be not good enough to obtain nontrivial solutions to (13) in \((-\infty, L)\). It turns out that the limitation is not on the method, but on the nature of the problem: no solution to (7) exists when \( c \geq 2 \), as we will show afterwards in Lem. 2.3.

**Proposition 3** (Dichotomy \( c < 2, c \geq 2 \)). The family \( \theta^{(c)}_{(-\infty,L]}(\cdot) \) is uniformly bounded in \( L \geq 0 \), namely, \( \sup_{L \geq 0} \| \theta^{(c)}_{(-\infty,L]}(\cdot) \|_{L^{\infty}(\mathbb{R})} \leq 1 \). In particular, for any fixed \( c > 0 \) we verify

\[
\theta^{(c)}_{(-\infty,L]}(\cdot) \neq 0, \quad \text{whenever } \ c < 2; \quad \theta^{(c)}_{(-\infty,L]}(\cdot) \equiv 0, \quad \text{whenever } \ c \geq 2.
\]

Furthermore, whenever \( 0 \leq c < 2 \) and \( L > 0 \), the family \( \theta^{(c)}_{(-\infty,L]}(\cdot) \) has the following properties:

(i) The mapping \( L \mapsto \theta^{(c)}_{(-\infty,L]}(\cdot) \) is continuous in the sup norm on \( 0 \leq L \leq \infty \).
(ii) (Monotonicity) The functions \( x \mapsto \theta^{(c)}_{(-\infty,L)}(x) \) are defined for every \( x \in \mathbb{R} \).

The mapping \( L \mapsto \theta^{(c)}_{(-\infty,L)}(x) \) is non-decreasing for any fixed \( x \). Furthermore, the mapping \( x \mapsto \theta^{(c)}_{(-\infty,L)}(x) \) is non-increasing for any fixed \( L \).

(iii) The function \( \theta^{(c)}_{(-\infty,L)}(x) \) solves \((13)\) on \((-\infty,L)\) and satisfies \( \lim_{x \to -\infty} \theta^{(c)}_{(-\infty,L)}(x) = 1 \). It admits an upper bound \( \bar{w}(\cdot) > 0 \) in \( x \geq 0 \) that is uniform in \( L \geq 0 \), namely, \( \theta^{(c)}_{(-\infty,L)}(x) \leq \bar{w}(x) \) in \( x \geq 0 \), for all \( L \geq 0 \). Moreover, \( \bar{w}(x) \) converges exponentially fast to zero as \( x \to \infty \).

Proof. We readily obtain uniform boundedness of \( \theta^{(c)}_{(-\infty,L)}(\cdot) \) in \( L \) as a direct consequence of its construction and of Lemma 2.2(i). Pointwise convergence and uniform boundedness imply convergence in the sense of distributions to a weak solution \( \theta^{(c)}_{(-\infty,L)}(\cdot) \), which solves the ODE in \((13)\) on \((-\infty,L)\) and satisfies the boundary condition at \( x = L \). With regards to \((15)\) we deal first with the case \( c < 2 \): from ODE theory (cf. [3, §4.4]) there exists a solution \( w(\cdot) \in C^\infty(\mathbb{R};[-1,1]) \) to \( \partial_x^2 w + c \partial_x w + w - w^3 = 0 \) satisfying \( \lim_{x \to -\infty} w(x) = 1 \) and so that \( w(x) \) is oscillatory as \( x \to +\infty \) whenever \( 0 \leq c < 2 \); see Fig. 5. Translation invariance of solutions to this ODE allow us to assume without loss of generality that \( 0 = w(0) < w(x) < 1 \) for \( x < 0 \).

![Figure 5. Sketch of solutions to $\partial_x^2 w(x) + c \partial_x w(x) + w(x) - w^3(x) = 0$ for $0 < c < 2$ (left) and $c \geq 2$ (right) satisfying $\lim_{x \to -\infty} w(x) = 1$ and $\lim_{x \to \infty} w(x) = 0$.](image)

Applying classical comparison principles to the problem \((13)\) on the interval \([-M,0]\) we conclude that \( w(x) \leq \theta^{(c)}_{(-M,0)}(x) \) hence \( w(x) \leq \mathcal{E} \left[ \theta^{(c)}_{(-M,0)} \right](x) \leq \mathcal{E} \left[ \theta^{(c)}_{(-M,L)} \right](x) \) for \( M > 0 \), thanks to Lem. 2.2(i) and to the monotonicity Lem. 2.2(ii). Taking the infimum in \( M > 0 \) we conclude that \( \theta^{(c)}_{(-\infty,L)}(\cdot) \neq 0 \), which proves the first part of \((15)\). As a byproduct we obtain the limit in (iii) using a squeezing property, for

\[
1 = \lim_{x \to -\infty} w(x) \leq \liminf_{x \to -\infty} \theta^{(c)}_{(-\infty,L)}(x) \leq 1.
\]

To conclude the proof of (iii) we use the function \( \bar{w}(x) := \frac{\text{csch}(x + x_0)}{\sqrt{2}} \), which is appropriately shifted so that \( \bar{w}(0) = 1 \); notice that \( \bar{w} \) satisfies \( \partial_x^2 \bar{w} - \bar{w} - (\bar{w})^3 = 0 \) and is monotonic, i.e., \( \partial_x \bar{w}(\cdot) \leq 0 \). Hence, \( \bar{w}(\cdot) \) is a supersolution on any interval \([0,L]\) and classical comparison principles imply that

\[
\theta^{(c)}_{(0,L)}(x) \leq \bar{w}(x), \quad \text{on} \quad x \in [0,L], \quad \forall L \geq 0.
\]
Thus, $\theta^{(c)}_{(-\infty,L)}(x) \leq \bar{w}(x)$ using Lem. 2.2(ii), and the result is finally obtained once we observe that $\theta^{(c)}_{(-\infty,L)}(x) \geq 0$ and that $\bar{w}(x) \to 0$ exponentially fast as $x \to \infty$.

In order to prove the monotonicity of the solution as asserted in (ii), we use Lem. 2.2(iv): the mapping $x \mapsto \theta^{(c)}_{(-\infty,L)}(x)$ is monotonic in $x$ as the sup of monotonic functions, i.e., $\partial_x \theta^{(c)}_{(-\infty,L)} \leq 0$. In fact, one obtains $\partial_x \theta^{(c)}_{(-\infty,L)}(\cdot) < 0$ in $x < L$ by applying the Hopf Lemma and the maximum principle (notice that the discontinuity of the control parameter $\mu(\cdot)$ plays no role here since, by classical regularity theory, we know that $\theta^{(c)}_{(-\infty,L)}(\cdot)$ is in fact smooth away from the quenching front).

Monotonicity of $\theta^{(c)}_{(-\infty,L)}(\cdot)$ in the parameter $L$ is a consequence of its construction, allied to Lem. 2.2. Item (i) is a direct consequence of Lem. 2.2(v).

Last, arguing by contradiction, we prove nonexistence for $c \geq 2$, which we now fix. By construction we must have $\theta^{(c)}_{(-\infty,L)}(\cdot) \geq 0$. Assume that $\theta^{(c)}_{(-\infty,L)}(\cdot) \neq 0$. Thanks to the monotonicity properties of the mapping $x \mapsto \theta^{(c)}_{(-\infty,L)}(x)$ and the uniform boundedness of $\theta^{(c)}_{(-\infty,L)}(\cdot)$ for all $L \geq 0$, the quantity $\lim_{x \to -\infty} \theta^{(c)}_{(-\infty,L)}(x)$ must exist. Because $\theta^{(c)}_{(-\infty,L)}(x) \neq 0$ and solves (3) we must have $\lim_{x \to -\infty} \theta^{(c)}_{(-\infty,L)}(x) = 1$.

Whenever $c \geq 2$ there exists, up to translations, a unique non-constant bounded solution $w(\cdot)$ satisfying

$$\partial^2_x w(x) + c\partial_x w(x) + w(x) - (w(x))^3 = 0,$$

and so that $\lim_{x \to -\infty} w(x) = 1$ (we also know that $w(\cdot)$ is positive and $\lim_{x \to -\infty} w(x) = 0$; cf. [3, §4]). We exploit this property by translating $w(\cdot)$ in such a way that it coincides with $\theta^{(c)}_{(-\infty,L)}(\cdot)$ in $x \leq 0$, where the latter is also a solution to (16) converging to 1 as $x \to -\infty$. Now define $z(\cdot) := w(\cdot) - \theta^{(c)}_{(-\infty,L)}(\cdot)$; clearly, $z(x) = 0$ when $x \leq 0$. Thus,

$$\partial^2_x z(x) + c\partial_x z(x) + \mu(x) z(x) - f[\theta^{(c)}_{(-\infty,L)}(x), \nu(x)] z(x) = |\mu(x) - 1| w(x) \leq 0,$$

in $x \in (-\infty, L)$, where $f[a,b] := \frac{a^3 - b^3}{a - b}$ whenever $a \neq b$ and $3a^2$ otherwise. Hence, $z(\cdot)$ is a supersolution on $x \in (0, L)$ satisfying $z(0) = 0$ and $z(L) = w(L) - \theta^{(c)}_{(-\infty,L)}(L) > 0$. Due to classical maximum principles we conclude that $z(x) > 0$ when $x \in (0, L)$ (cf. [5, §3]). Now, an application of Hopf Lemma gives $\partial_x z(0) > 0$, which is absurd since $\partial_x z(0) = 0$ due to its regularity. Therefore we must have $\theta^{(c)}_{(-\infty,L)} \equiv 0$ and the proof is complete. 

**Lemma 2.3 (Existence/nonexistence; (1 → 0)\(^{(c)}\) problem).** There exists a solution $\theta^{(c)}(\cdot) \in C^{1,\alpha}(\mathbb{R})$ to problem (7) whenever $0 < c < 2$. Moreover, the mapping $x \mapsto \theta^{(c)}(x)$ is strictly decreasing, positive and satisfies the spatial asymptotic limits $\lim_{x \to -\infty} \theta^{(c)}(x) = 1$ and $\lim_{x \to \infty} \theta^{(c)}(x) = 0$. No solution satisfying (7) exists when $c \geq 2$.

**Proof.** We begin by proving existence when $c < 2$, following the ideas in [10, §2] which we refer to for further details: define $\theta^{(c)}(x) = \sup_{L > 0} \theta^{(c)}_{(-\infty,L)}(x) = \lim_{L \to -\infty} \theta^{(c)}_{(-\infty,L)}(x)$. The asymptotic behavior as $x \to \pm \infty$ is a consequence of Prop. 2.2(ii)-2.2(iii), since the mapping $L \mapsto \theta^{(c)}_{(-\infty,L)}(\cdot)$ is monotonic and the upper bound $\bar{w}(\cdot)$ is uniform in $L \geq 0$. Monotonicity in $x$ follows from the properties
of the function \( \theta^{(c)}_{(-\infty,L^2)}(\cdot) \), and it can be further improved to strict monotonicity in a standard fashion by using the Hopf Lemma.

To prove nonexistence when \( c \geq 2 \) we closely follow the analysis of Prop. 3. Assume that \( \theta^{(c)}(\cdot) \neq 0 \). We can choose \( w(\cdot) \) satisfying (16) in such a way that \( w(x) = \theta^{(c)}(x) \) for all \( x \leq 0 \). The function \( z(\cdot) := w(\cdot) - \theta^{(c)}(\cdot) \) verifies (17) for all \( x \in \mathbb{R} \). Since \( z(0) = \lim_{x \to -\infty} z(x) = 0 \) we can argue as in Prop. 3 to conclude that \( z(x) = 0 \) for all \( x > 0 \) due to the maximum principle and the Hopf Lemma. However, \( w(\cdot) \) cannot solve both (7) and (17). This contradiction finishes the proof. \( \square \)

From the properties of the supersolution used in the previous proof we readily derive the next result:

**Lemma 2.4** (Exponential convergence; \( (1 \rightarrow 0)^{(c)} \) problem). For any \( \theta^{(c)}(\cdot) \) solving (7) when \( c < 2 \) there exists a \( M, C, \delta > 0 \) independent of \( x \) such that
\[
|\theta^{(c)}(x) - 1| \leq Ce^{-\delta|x|}, \quad \text{for } x \leq -M, \quad \text{and } |\theta^{(c)}(x)| \leq Ce^{-\delta|x|}, \quad \text{for } x \geq M.
\]

To finalize this section we show that the solutions obtained in [10] for small \( c \geq 0 \) through perturbation methods agree with those constructed here. In passing we show their continuity in the parameter \( c \).

**Lemma 2.5** (Uniqueness of the continuation in \( c; (1 \rightarrow 0)^{(c)} \) fronts). There exists a unique continuation \( \theta^{(c)}(\cdot) \) solving the problem (3) whenever \( c \in (0, 2) \). Moreover, the mappings \( c \mapsto \theta^{(c)}(\cdot) : [0, 2] \to L^\infty(\mathbb{R}; \mathbb{R}) \) are smooth.

**Proof.** Initially, define
\[
\mathcal{L}_{\theta^{(d)}}[u] = \partial_2^2 u + d\partial_x u + \mu(x) u - 3(\theta^{(d)})^2 u, \quad \mathcal{D}(\mathcal{L}_{\theta^{(d)}}) = H^2(\mathbb{R}).
\]
Plugging \( \theta^{(d)} + u \) in (2) allow us to rewrite the latter equation as
\[
\mathcal{L}_{\theta^{(d)}}[u] = \mathcal{A}[\theta^{(d)}, u] + (d - c)\partial_x \theta^{(d)} + (d - c)\partial_x u.
\]
Notice that the right hand side is of the form \( \mathcal{O}(|u|^2 + |d - c|\partial_x u + |d - c|\partial_x \theta^{(d)}) \), hence in \( L^2(\mathbb{R}) \), thanks to the Sobolev embedding \( H^2(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \) (cf. [1, §8 & 9]); consequently the nonlinearity is smooth in \( H^2(\mathbb{R}) \). The proof consists in showing that for any \( d \in [0, 2] \) the linear operator (18) is a boundedly invertible from \( H^2(\mathbb{R}) \) to \( L^2(\mathbb{R}) \). Indeed, assuming the latter is true the result will follow from an application of the IFT, hence we immediately obtain existence and uniqueness of solutions in a neighborhood of \( (\theta^{(d)}(\cdot), d) \). Keeping these arguments in mind, we devote the rest of the proof to showing the invertibility of operator \( \mathcal{L}_{\theta^{(d)}} \).

To begin with, consider \( H^2_0(\mathbb{R}) := e^{\frac{\mathcal{A}}{4}}H^2(\mathbb{R}) \), where \( \|u\|_{H^2_0} = \|e^{-\frac{\mathcal{A}}{4}}u\|_{H^2} \).

Clearly, the mapping \( \mathcal{I} : H^2(\mathbb{R}) \rightarrow H^2_0(\mathbb{R}) \), \( \mathcal{I}[u] = e^{\frac{\mathcal{A}}{4}}u \) is an isometry. We write the conjugate operator \( \mathcal{L}_{\theta^{(d)}}[\cdot] := \mathcal{I} \circ \mathcal{L}_{\theta^{(d)}} \circ \mathcal{I}^{-1}[\cdot] = e^{\frac{\mathcal{A}}{4}}\mathcal{L}_{\theta^{(d)}}[e^{-\frac{\mathcal{A}}{4}} \cdot] \):
\[
\mathcal{L}_{\theta^{(d)}}[\tilde{v}] = \partial_2^2 \tilde{v} + \left( \mu(x) - \frac{d^2}{4} \right) \tilde{v} - 3(\theta^{(d)})^2 \tilde{v},
\]
\[
\tilde{v} \in \mathcal{D}\left(\mathcal{L}_{\theta^{(d)}}\right) = e^{\frac{\mathcal{A}}{4}}H^2(\mathbb{R}) =: H^2_0(\mathbb{R}).
\]

This conjugation implies that \( \mathcal{L}_{\theta^{(d)}} \) is invertible on \( H^2(\mathbb{R}) \) if and only if \( \mathcal{L}_{\theta^{(d)}} \) is invertible on \( H^2_0(\mathbb{R}) \). For a moment, consider the operator \( \mathcal{L}_{\theta^{(d)}}[\cdot] := \mathcal{L}_{\theta^{(d)}}[\cdot] \) with domain \( \mathcal{D}\left(\mathcal{L}_{\theta^{(d)}}\right) = H^2(\mathbb{R}) \); the analysis in [10, §5] shows that this is a self-adjoint, Fredholm operator of index 0, and its essential spectrum is contained in
\{ z \in \mathbb{C} | \text{Re}(z) < 0 \}; therefore, in order to show invertibility it suffices to show that this operator has a trivial kernel. This is proved as follows: the properties of the operator \( \widetilde{L}_1 \) imply that the \( \sigma \left( \widetilde{L}_1 \right) \cap \{ x \in \mathbb{R} | x \geq 0 \} \) is either empty or consists of point spectrum only. It is straightforward to show that this set is bounded, therefore assume that there exists a \( \lambda_0 \geq 0 \) maximal eigenvalue, with corresponding eigenfunction \( u_0(\cdot) \). In the referred paper it was also proven that \( u_0(\cdot) \) is spatially localized, namely,

\[
|\partial_x u_0(x)| + |u_0(x)| \leq C e^{-\delta |x|}, \quad x \in \mathbb{R} \quad \text{a.e.,}
\]

whenever \( u_0 \in \text{Ker} \left( \widetilde{L}_1 - \lambda_0 I \right) \). In fact, we know that we can take \( \delta = \frac{d}{4} \), thanks to the results in [9, \S 4]. From the self-adjoint properties of \( \widetilde{L}_1 \) we know that \( u_0(\cdot) \) is a ground state, hence it has a sign which we assume to be \( \text{sign} \). We observe that the spatial localization of \( \widetilde{L}_1 \) implies that \( \lambda_0 \geq 0 \) almost everywhere (cf. [12, XII.12]). We can write the eigenvalue equation \( \widetilde{L}_1[u_0] = \lambda_0 u_0 \) as

\[
\widetilde{L}_1[u_0] = \partial_x^2 u_0(x) + \left( \mu(x) - \frac{d^2}{4} \right) u_0(x) - 3e^{-d x (\theta^{(d)}(x))} u_0(x) = \lambda_0 u_0(x) \tag{21}
\]

Setting \( \theta^{(d)}(\cdot) = e^{\frac{d}{2}x} \theta^{(d)}(\cdot) \) and using the properties of the function \( \theta^{(d)}(\cdot) \), we have

\[
\partial_x^2 \theta^{(d)}(x) + \left( \mu(x) - \frac{d^2}{4} \right) \theta^{(d)}(x) - e^{-d x (\theta^{(d)}(x))^3(x)} = 0; \tag{22}
\]

asymptotic theory of ODEs (cf. [2, Chap. 3, Sec. 8]) implies that \( \lim_{|x| \to \infty} \theta^{(d)}(x) = 0 \).

Now, multiply (21) by \( \theta^{(d)}(\cdot) \) and (22) by \( u_0(\cdot) \) subtract both equations and integrate in \( \mathbb{R} \) to find

\[
\int \theta^{(d)}(x) \partial_x^2 u_0(x) - u_0(x) \partial_x^2 \theta^{(d)}(x) \, dx - 2 \int e^{-d x (\theta^{(d)}(x))} \theta^{(d)}(x) u_0(x) \, dx = \lambda_0 \int \theta^{(d)}(x) u_0(x) \, dx
\]

Integration by parts shows that the first integral vanishes, thanks to the decay estimates for \( \theta^{(d)}(\cdot) \) and \( u_0(\cdot) \). We are left with

\[
-2 \int e^{-d x (\theta^{(d)}(x))} \theta^{(d)}(x) u_0(x) \, dx = -2 \int \left( \theta^{(d)}(x) \right)^2 \theta^{(d)}(x) u_0(x) \, dx = \lambda_0 \int \theta^{(d)}(x) u_0(x) \, dx.
\]

We observe that the spatial localization of \( u_0(\cdot) \) as asserted in (20) and the fact that \( \theta^{(d)}(x) = O(e^{-\frac{d}{4}x}) \) as \( x \to \infty \) imply that both integrals are finite. Now, recall the strict positivity of the pattern \( \theta^{(d)}(\cdot) \) (or equivalently, that of \( \theta^{(d)}(\cdot) \)). These sign considerations, allied to \( u_0(\cdot) \geq 0 \) a.e., show that the right-hand side is non-positive while the left-hand is non-negative (since \( \lambda_0 \geq 0 \)), therefore the integral on the left is zero. We conclude that \( u_0(\cdot) \equiv 0 \) almost everywhere, which contradicts the fact that \( u_0(\cdot) \) is an eigenfunction. Therefore, no eigenvalue can be found on \( \{ z \in \mathbb{C} | \text{Re}(z) \geq 0 \} \); in other words, the operator \( \widetilde{L}_1 \) is boundedly invertible.

An intermediate step is necessary in order to go back to the operator \( \widetilde{L}_1 \): first, define the family of weighted Sobolev spaces \( H_{(\delta, \delta)}(\mathbb{R}) = e^{d x} H^2(\mathbb{R}) \), where
<x> := \sqrt{1 + x^2}. The action of the operator \( \mathcal{L}_1 \) on these spaces can be studied by the operators \( \mathcal{L}_1^{(\delta)}[\cdot] = e^{-\delta<x>}\mathcal{L}_1[e^{\delta<x>\cdot}] \), defined as \( H^2(\mathbb{R}) \to L^2(\mathbb{R}) \) mappings; we point out that the mapping \( \delta \to \mathcal{L}_1^{(\delta)} \) is continuous in the operator norm. Standard Fourier analysis shows that the far field operators \( \mathcal{L}_1^{(\delta;\pm\infty)} = \lim_{x \to \pm\infty} \mathcal{L}_1^{(\delta)} \) are boundedly invertible operators from \( H^2(\mathbb{R}) \) to \( L^2(\mathbb{R}) \) for any \( \delta \leq \frac{d}{2} \), hence for any \( \delta \) in this range the operators \( \mathcal{L}_1^{(\delta)} \) are Fredholm (cf. [9, Prop. 4.3 and Rem. 4.7]). Continuity in \( \delta \) implies that these operators also have index 0. Therefore, proving invertibility is equivalent to showing that the kernel is trivial. In that regard, observe that we have a scale of Banach spaces, i.e., \( H^2(\theta',\delta')(\mathbb{R}) \subset H^2(\theta,\delta)(\mathbb{R}) \) whenever \( \theta' > \theta \). Hence, one can invoke [10, Lem. 5.3] or [9, Lem. 4.6] to derive the persistence of elements in the kernel, namely, the equality \( \text{Ker} \left( \mathcal{L}_1^{(\delta)} \right) = \text{Ker} \left( \mathcal{L}_1 \right) = \{0\} \) holds for any \( |\delta| \leq \frac{d}{2} \); being Fredholm operators of index 0 this property is equivalent to their invertibility.

The previous paragraph is all we need to finalize our proof. Clearly,

\[
H^2_d(\mathbb{R}) = e^{\frac{d}{2}}H^2(\mathbb{R}) \subset H^2_{(d,d)}(\mathbb{R}) = D \left( \mathcal{L}_1^{(\delta)} \right) |_{\delta = \frac{d}{2}}.
\]

Since \( \text{Ker} \left( \mathcal{L}_1^{(\delta)} \right) |_{\delta = \frac{d}{2}} \) is trivial, the same is also true of the kernel of \( \mathcal{L}_1 \) taken with domain \( H^2_d(\mathbb{R}) \), which corresponds to the operator \( \mathcal{L}_1^{(\delta)}[\cdot] \). The bounded invertibility of \( \mathcal{L}_1^{(\delta)} \) is immediately obtained, for it is Fredholm operator with index 0 and trivial kernel, and we are done. \( \square \)

**Proof.** [of Theorem 1.1] Combine the proofs of Lem. 2.3, 2.4 and 2.5. \( \square \)

### 3. Two-dimensional quenched patterns – periodic horizontal interfaces: \( \mathcal{H}_\kappa \) patterns, \( \pi < \kappa < \infty \)

In this section we prove Theorem 1.2 in the case \( \pi < \kappa < \infty \). As mentioned before, it is important in our approach that the nonlinearity is odd so that we can restrict the study of equation (2) to the stripe \((x, y) \in \mathbb{R} \times [0, \kappa] \); any solution \( U(\cdot, \cdot) \) in \( \mathbb{R} \times [0, \kappa] \) is extended to the whole plane \( \mathbb{R}^2 \) by successive reflections \( U(x, -y) = -U(x, y) \) and \( U(x, \kappa + y) = -U(x, \kappa - y) \). The method of proof is similar to the one used in the \( (1 \sim 0)^{(c)} \) problem, although the construction of subsolutions is more involved; we mostly follow the arguments in [10, §3 & 4] by truncating \( \mathcal{H}_\alpha \) problem (with parameter \( \pi < \kappa < \infty \)) to a strip \( \mathcal{S}(-M, L) := (-M, L) \times (0, \kappa) \), with Dirichlet boundary conditions:

\[
\begin{align*}
\Delta_x y U + c \partial_x U + \mu(x) U - U^3 &= 0, \quad (x, y) \in \mathcal{S}(-M, L), \\
U - h_{(-M, L)} &\quad = 0, \quad (x, y) \in \partial \mathcal{S}(-M, L),
\end{align*}
\]

where \( h_{(-M, L)}(x, y) := \theta^{(c)}_{(\cdot, -M, L)}(x) \cdot \bar{u}(y; \kappa) \), for \( \theta^{(c)}_{(\cdot, -M, L)}(\cdot) \) solution to the 1D problem 13 and \( \bar{u}(\cdot, \kappa) \) given in (5). We obtain a unique solution \( \Xi^{(c,\kappa)}_{(-M, L)} \in \mathcal{C}^{(1,\alpha)}(\mathcal{S}(-M, L); [0, 1]) \) by an iteration scheme

\[
\begin{align*}
- \Delta_x y \Xi_{n+1} - c \partial_x \Xi_{n+1} + 5 \Xi_{n+1} &= (5 + \mu(x)) \Xi_n - \Xi_n^3 \\
(\Xi_{n+1} - h_{(-M, L)}) |_{\partial \mathcal{S}(-M, L)} &= 0
\end{align*}
\]

(24)
where $\Xi_0(\cdot)$ is chosen in the class
\[ \Psi_{\mathcal{H}_\kappa} := \left\{ \Xi \in \mathcal{C}^{(1,\alpha)}(\partial S_{(-M,L)}) \mid (\Xi - h_{(-M,L)})|_{\partial S_{(-M,L)}} \geq 0, \, \Xi(x) \in [0,1] \text{ for all } x, \, y \right\} \]
\[ \cap \{ \Delta_{x,y} \Xi + c \partial_x \Xi + \mu(x) \Xi - \Xi^3 \leq 0 \} \]
\[ \cap \{ \Delta_{x,y} \Xi + c \partial_x \Xi + \mu(x) \Xi - \Xi^3 \neq 0 \text{ in the sense of distributions} \} \]
The latter is a non-empty set, for it contains the function $\Xi_0(\cdot, \cdot) \equiv 1$. Throughout this section, we fix $\kappa \in (\pi, \infty)$. As in the previous section, proofs that are similar to those in [10] are only outlined and details are referred to that paper.

**Proposition 4** (Existence and uniqueness; truncated $\mathcal{H}_\kappa$ problem). Problem (23) has a unique solution $\Xi^{(c)}(\cdot, \cdot) \in \mathcal{C}^{(1,\alpha)}([-M,L] \times [0,\kappa]; [0,1]), \forall \alpha \in [0,1)$.

**Proof.** The existence is obtained as in [10, Lem. 3.1] using the iterative scheme (24); uniqueness follows as in [10, Prop. 3.2] and integration by parts, as in Lem. (2.1). The stated regularity is derived using classical results in elliptic theory, as shown in [10, §3].

We define extension operators in order to compare solutions for different values of $M, L$, namely,
\[ \mathcal{E}^{(c)} \left[ \Xi^{(c)}_{(-M,L)} \right](x, y) = \begin{cases} \Xi^{(c)}_{(-M,L)}(x, y), & \text{for } (x, y) \in S_{(-M,L)}, \\ \bar{u}(y; \kappa), & \text{for } (x, y) \in (-\infty, -M) \times [0, \kappa], \\ 0, & \text{for } (x, y) \in (L, \infty) \times [0, \kappa], \end{cases} \]
where $\bar{u}(\cdot; \kappa)$ is given in (5). We use the same symbols for the one- and two-dimensional extension operators, slightly abusing notation, distinguishing between the two through the domain of definition of the function $\mathcal{E}$ is applied to. The proofs of the following Lem. 3.1 and Prop. 5 are obtained as in [10, §3]:

**Lemma 3.1** (Comparison principles; $\mathcal{H}_\kappa$-problem). Let $\Xi^{(c)}_{(-M,L)}(\cdot, \cdot)$ be the solution from Prop. 4.

(i) (2D supersolutions) If $v$ satisfies, in the sense of distributions,
\[ \Delta_{x,y}v + c \partial_x v + \mu(x)v - v^3 \leq 0, \, (x, y) \in S_{(-M,L)}, \]
\[ (v - h_{(-M,L)})|_{\partial S_{(-M,L)}} \geq 0, \, 0 \leq v \leq 1, \]
then $v \geq \Xi^{(c)}_{(-M,L)}$ in $S_{(-M,L)}$. In particular, $v \geq \Xi^{(c)}_{(-M,L)}$ in $S_{(-M,L)}$ for any solution of
\[ \Delta_{x,y}v + c \partial_x v + \mu(x)v - v^3 = 0, \, (x, y) \in S_{(-M,L)}, \]
\[ (v - h_{(-M,L)})|_{\partial S_{(-M,L)}} \geq 0, \, 0 \leq v \leq 1. \]

(ii) (2D subsolutions) If $v$ satisfies, in the sense of distributions,
\[ \Delta_{x,y}v + c \partial_x v + \mu(x)v - v^3 \geq 0, \, (x, y) \in S_{(-M,L)}, \]
\[ (v - h_{(-M,L)})|_{\partial S_{(-M,L)}} \leq 0, \, 0 \leq v \leq 1, \]
then \( v \leq \Xi^{(c)}_{(-M, L)} \) in \( S_{(-M, L)} \). In particular, \( v \leq \Xi^{(c)}_{(-M, L)} \) in \( S_{(-M, L)} \) for any solution of

\[
\Delta_{x,y}v + \alpha \partial_x v + \mu(x)v - v^3 = 0, \quad (x,y) \in S_{(-M, L)},
\]

\[
(v - h_{(-M, L)}) \big|_{\partial S_{(-M, L)}} \leq 0, \quad 0 \leq v \leq 1.
\]

(iii) The functions \( \bar{u}(\cdot; \kappa) \) and \( \theta_{(-M, L)}^{(c)}(x) \) given by (5) and Theorem 1.1, respectively, are supersolutions to (23) in \( S_{(-M, L)} \). Consequently, for any \( L, M > 0 \) we have that

\[
\Xi^{(c)}_{(-M, L)}(x,y) \leq \min\{\theta^{(c)}_{(-M, L)}(x), \bar{u}(y; \kappa)\}.
\]

**Proposition 5** (Properties of the extension operator, \( \mathcal{H}_\kappa \)-problem). The following properties of \( \mathcal{H}_\kappa \) hold:

(i) (Monotonicity of \( \mathcal{H}_\kappa \)) We have \( 0 \leq \mathcal{H}_\kappa \left[ \Xi^{(c)}_{(-M, L)} \right](\cdot, \cdot) \leq 1 \). Furthermore, if \( w \) is only defined in a subset \( A \subset \mathbb{R}^2 \) so that \( S_{(-M, L)} \subset A \subset S_{(-\infty, L)} \), \( 0 \leq w(\cdot) \leq 1 \) and \( w(\cdot) \leq \Xi^{(c)}_{(-M, L)}(\cdot, \cdot) \), then \( 0 \leq \mathcal{H}_\kappa [w](\cdot) \leq \mathcal{H}_\kappa \left[ \Xi^{(c)}_{(-M, L)}(\cdot, \cdot) \right](\cdot) \) in \( \mathbb{R}^2 \).

(ii) (Monotonicity in \( M \)) Let \( 0 \leq M < \bar{M} \) and \( L \geq 0 \) be fixed. Then \( M < \bar{M} \implies \mathcal{H}_\kappa \left[ \Xi^{(c)}_{(-M, L)} \right](x,y) \leq \mathcal{H}_\kappa \left[ \Xi^{(c)}_{(-M, L)} \right](x,y) \).

(iii) (Monotonicity in \( L \)) Let \( 0 \leq L < \bar{L} \) and \( M \geq 0 \) be fixed. Then \( L < \bar{L} \implies \mathcal{H}_\kappa \left[ \Xi^{(c)}_{(-M, L)} \right](x,y) \leq \mathcal{H}_\kappa \left[ \Xi^{(c)}_{(-M, L)} \right](x,y) \).

(iv) (Monotonicity in \( x \)) Let \( L, M, y \) be fixed. Then the mapping

\[
x \rightarrow \mathcal{H}_\kappa \left[ \Xi^{(c)}_{(-M, L)} \right](x,y)
\]

is non-increasing.

### 3.1. Passing to the limit.

We are now ready to pass to the limit \( M = \infty \). Define

\[
\Xi^{(c)}_{(-\infty, L)}(x,y) := \inf_{M > 0} \mathcal{H}_\kappa \left[ \Xi^{(c)}_{(-M, L)} \right](x,y) = \lim_{M \to +\infty} \mathcal{H}_\kappa \left[ \Xi^{(c)}_{(-M, L)} \right](x,y),
\]

where the last equality holds due to monotonicity of the mapping \( M \mapsto \Xi^{(c)}_{(-M, L)}(x,y) \), Prop. 3.1(ii).

In this section we verify monotonicity properties and limits at spatial infinity of the limits \( \Xi^{(c)}_{(-\infty, L)}(\cdot, \cdot) \) constructed in (27).

**Lemma 3.2** (Monotonicity of \( \Xi^{(c)}_{(-\infty, L)}(\cdot, \cdot) \)). The following properties hold:

(i) The function \( \Xi^{(c)}_{(-\infty, L)}(\cdot, \cdot) \) solves the problem (23) in \( S_{(-\infty, L)} := \{ (x,y) \in \mathbb{R}^2 \mid x < L, y \in [0, \kappa] \} \).

(ii) The function \( \Xi^{(c)}_{(-\infty, L)}(\cdot, \cdot) \) is non-increasing in \( x \) and non-decreasing in \( L \).

(iii) The inequality \( \Xi^{(c)}_{(-\infty, L)}(x,y) \leq \min\{\theta_{(-\infty, L)}^{(c)}(x), \bar{u}(y; \kappa)\} \) holds for all \( (x,y) \in S_{(-M, L)} \), where \( \theta_{(-\infty, L)}^{(c)}(\cdot) \) is given by (1.1) and \( \bar{u}(\cdot; \kappa) \) by Prop. 5. In particular, we have

\[
\sup_{L > 0} \left( \Xi^{(c)}_{(-\infty, L)}(x,y) \right) \leq \min\{\theta^{(c)}(x), \bar{u}(y; \kappa)\}.
\]
Proof. Assertions (i) and (ii) follow as in [10, Lem. 3.6]. The inequalities in (iii) are derived from Prop. 3.1(iii), by passing to the limit \( L = \infty \), using the fact that for all \( x \in \mathbb{R} \) the mapping \( L \mapsto \theta^{(c)}_{(-\infty,L)}(x) \) is non-decreasing and last, invoking the results of Lem. 2.3.

\[ \square \]

Remark 3. One can readily conclude from Lem. 3.2(iii) that \( \Xi^{(c)}_{(-\infty,L)}(\cdot,\cdot) \equiv 0 \) whenever \( c > 2, L > 0 \). In fact, the nonexistence of nontrivial solutions happens in a wider range for the parameter \( c \), as we show next.

3.2. Existence for the \( \mathcal{H}_\kappa \)-problem: case \( \frac{c^2}{4} + \frac{\pi^2}{\kappa^2} < 1 \). In order to prove the existence of solutions we construct appropriate subsolutions with the help of the next lemmas:

Lemma 3.3 (Properties of the family of periodic solutions \( \bar{u}(\cdot,\kappa) \)). Let \( \bar{u}(\cdot,\kappa) \) be a solution to (5) and \( \kappa > \pi \). The following two properties hold:

(i) The quantity \( \bar{M} := \sup_{y \in [0,\kappa]} \bar{u}(y;\kappa) \) satisfies \( \bar{M} \leq \sqrt{\left(1 - \frac{\pi^2}{\kappa^2}\right)} \):

(ii) For any \( 0 \leq \alpha \leq \bar{M} \) we have \( v(y) := \alpha \sin \left(\frac{\pi y}{\kappa}\right) \leq \bar{u}(y;\kappa) \), for all \( y \in [0,\kappa] \).

Proof. To prove the estimate in (i) we use the elliptic integral that gives the relation between amplitude and spatial period given in [10][Lem. 4.1, equation (4.4)], [6, §V],

\[
\kappa = \kappa(\mathcal{M}) := 2\sqrt{2} \int_0^1 \frac{dv}{\sqrt{[(1-v^2)(1-\mathcal{M}^2)(1+v^2)]}} = \frac{2\sqrt{2}\gamma}{\mathcal{M}} \int_0^1 \frac{dv}{\sqrt{[(1-v^2)(1-\gamma v^2)]}},
\]

(28)

for \( \gamma^2 = \frac{\pi^2}{2-\mathcal{M}^2} \). Notice that \( 0 \leq \gamma < 1 \). We find a lower bound to the integral on the right hand side:

\[
\kappa(\mathcal{M}) > \frac{2\sqrt{2}\gamma}{\mathcal{M}} \int_0^1 \frac{dv}{\sqrt{[(1-v^2)(1-\gamma^2)]}} = \frac{2\sqrt{2}\gamma}{\mathcal{M}} \int_0^1 \frac{dv}{\sqrt{(1-v^2)}} = \frac{\pi\sqrt{2}\gamma}{\mathcal{M}\sqrt{1-\gamma^2}}.
\]

Squaring both sides and plugging in \( \gamma \) we obtain \( \kappa^2 > \frac{\pi^2}{1-\mathcal{M}^2} \iff \mathcal{M}^2 < 1 - \frac{\pi^2}{\kappa^2} \), which finishes the proof of (i). In order to prove (ii) we exploit the structure of this ODE in (5), whose Hamiltonian is \( \mathcal{H}(u, \partial_y u) = (\partial_y u)^2 + u^2 - \frac{u^4}{2} \), cf. [10, §2.1]. Indeed, considering \( \bar{u}(\cdot,\kappa) \) a periodic orbit with period \( \kappa \) and maximal amplitude \( \mathcal{M} \), we readily obtain that \( \mathcal{H}(\bar{u}, \partial_y \bar{u}) = \mathcal{M}^2(2-\mathcal{M}^2) \). Let \( v(y) := \alpha \sin \left(\frac{\pi y}{\kappa}\right) \) and \( z(x) := \bar{u}(y;\kappa) - v(y) \). Whenever \( 0 \leq \alpha \leq \mathcal{M} \) one can see that

\[
\mathcal{H}(v, \partial_y v) \leq \mathcal{H}(\bar{u}, \partial_y \bar{u}).
\]

(29)

As \( \partial_y \bar{u}(0;\kappa) > \partial_y v(0) \) and \( \bar{u}(0;\kappa) = v(0) = 0 \) it is clear that \( z(x) > 0 \) for all \( x > 0 \) sufficiently small. By translation invariance of the solutions to the ODE (5), reversibility of the solutions \( \bar{u} \) with respect to \( x \mapsto -x \), and the fact that the mapping \( y \mapsto \sin(y + \pi/2) \) is even it suffices to show that \( z(y) \geq 0 \) for \( 0 \leq y \leq \frac{\pi}{4} \), a fact that we shall prove by contradiction. To begin with, assume that there exists a
$0 < x_0 < \frac{\kappa}{2}$ such that $z(x_0) = 0$. The function $z(\cdot) \geq 0$ solves the elliptic differential equation, hence we can find an $A > 0$ sufficiently large so that

$$\partial_y^2 z(y) + (1 - f[u, v](y) - A)z(y) \leq \partial_y^2 z(y) + (1 - f[u, v](y))z(y) = 0,$$

where $f[a, b] := \frac{a^3 - b^3}{a - b}$ whenever $a \neq b$ and $3a^2$ otherwise. On the interval $[0, x_0]$ the function $z(\cdot)$ assumes its first zero at $x_0$; hence we can apply the Hopf Lemma to conclude that $\partial_y z(x_0) < 0$, which is absurd, since the inequality (29) prevents it from happening. Therefore $z(y) \geq 0$, hence $v(y) \leq \bar{u}(y; \kappa)$ for $y \in [0, \frac{\kappa}{2}]$, and by symmetry, for $y \in [0, \kappa]$.

**Lemma 3.4** (Sub and supersolutions). Choose $d \in (c, 2)$ so that $\frac{c^2}{4} + \frac{\pi^2}{\kappa^2} < \frac{d^2}{4} < 1$ and let $w(\cdot) \in C^\infty([\kappa; \infty, 1])$ be a solution to $\partial_y^2 w + d\partial_x w + w - w^3 = 0$ satisfying $w(-\infty) = 1$ and so that $0 = w(0) < w(x) < 1$ for $x < 0$ (see. Prop. 3 and Fig. 5). Let $\alpha$ and $\bar{u}(\cdot; \kappa)$ be given as in Lem. 3.3. Define

$$V(x, y) := e^{-(c-d)x}w(x) \cdot v_\kappa(y), \quad \text{where} \quad v_\kappa(y) := \alpha \sin \left(\frac{\pi y}{\kappa}\right).$$

Then,

$$V(x, y) \leq \Xi^{(c)}_{(-\infty, L)}(x, y) \quad \text{in} \quad (-\infty, 0] \times [0, \kappa]. \quad (30)$$

Furthermore, whenever $\theta^{(c)}(\cdot)$ is given by (1.1) and $\bar{u}(\cdot; \kappa)$ by Prop. 5, the following inequality holds

$$\Xi^{(c)}_{(-\infty, L)}(x, y) \leq \min\{\theta^{(c)}(x), \bar{u}(y; \kappa)\}. \quad (31)$$

**Proof.** Inequality (31) is readily derived from Lem. 3.1(iii), taking infimum in $M > 0$, using the definition of $\Xi^{(c)}_{(-\infty, L)}(\cdot, \cdot)$ (see (27)) and the monotonicity of the mapping $L \mapsto \theta^{(c)}_{(-\infty, L)}(\cdot)$.

Similar monotonicity arguments are used in the proof of (30): according to Lem. 3.1(ii) it suffices to show that $V(\cdot, \cdot) \leq \Xi^{(c)}_{(-M, 0)}(\cdot, \cdot)$ in $S_{(-M, 0)} = [-M, 0] \times [0, \kappa]$ for any fixed $M > 0$. Thanks to Lem. 3.3(ii) and the choice of $d$ it is straightforward to verify that on the boundary $\partial S_{(-M, 0)}$ the inequality $V(x, y) \leq \Xi^{(c)}_{(-M, 0)}(x, y)$ holds. Now we show that $V$ satisfies

$$\Delta_{x,y} V + c\partial_x V + V - V^3 \geq 0,$$

in $S_{(-M, 0)} = [-M, 0] \times [0, \kappa]$. Indeed, a direct calculation shows that

$$\Delta_{x,y} V + c\partial_x V + V - V^3 = V \left[ \frac{d^2}{4} - \frac{c^2}{4} - \frac{\pi^2}{\kappa^2} \right] + e^{-\frac{(c-d)x}{2}} w^3(x) \left[ v_\kappa(y) - e^{-\frac{(c-d)x}{2}} v^3_\kappa(y) \right] \geq 0,$$

since $V(\cdot, \cdot)$ is non-negative, $x \leq 0$ and $d > c$. It follows that $V(x, y) \leq \Xi^{(c)}_{(-M, 0)}(x, y)$.

We obtain (30) by invoking the monotonicity of the mapping $L \mapsto \Xi^{(c)}_{(-M, L)}(\cdot, \cdot)$ and using the definition (27).

**Lemma 3.5** (Existence; $\mathcal{H}_\alpha$-problem, $\pi < \kappa < \infty$). Equation (2) has a solution $\Xi^{(c)}_{\kappa}(\cdot, \cdot)$ in $\mathcal{C}^{(1, \alpha)}([\mathbb{R} \times [0, \kappa]; \mathbb{R}])$, for any $0 \leq \alpha < 1$ and defined as

$$\Xi^{(c)}_{\kappa}(x, y) := \lim_{L \to \infty} \Xi^{(c)}_{(-\infty, L)}(x, y) \quad (32)$$
Furthermore, it satisfies the spatial asymptotic conditions \( \lim_{x \to -\infty} \Xi^{(c)}_\kappa(x, y) = \bar{u}(y; \kappa) \) and \( \lim_{x \to \infty} \Xi^{(c)}_\kappa(x, y) = 0 \).

**Proof.** Most of the proof goes as in the paper [10, Prop. 3.11]. The monotonicity properties of the functions \( \Xi^{(c)}_{(-\infty,L)}(\cdot, \cdot) \) show that the definition (32) makes sense and Lem. 3.4 shows that \( \Xi^{(c)}_{(-\infty,L)}(\cdot, \cdot) \neq 0 \) whenever \( \frac{c^2}{4} + \frac{\kappa^2}{\pi^2} < 1 \). As \( 0 \leq \Xi^{(c)}_{(-\infty,L)}(\cdot, \cdot) \leq \Xi^{(c)}_\kappa(\cdot, \cdot) \) we conclude that \( \Xi^{(c)}_\kappa(\cdot, \cdot) \) is also nontrivial. The monotonicity of the mapping \( L : \Xi^{(c)}_{(-\infty,L)}(\cdot, \cdot) \) implies that \( \Xi^{(c)}_\kappa(\cdot, \cdot) \) exists in the pointwise sense. Furthermore, using the Lebesgue Dominated Convergence Theorem we conclude that this sequence converges in \( L^1_{\text{loc}} \), hence in the sense of distribution, hence \( \Xi^{(c)}_\kappa(\cdot, \cdot) \) solves the equation (2) in the domain \( \mathbb{R} \times [0, \kappa] \). Now it remains to show that the asymptotic limits are satisfied, namely, that

\[
\lim_{x \to -\infty} \Xi^{(c)}_\kappa(x, y) = \bar{u}(y; \kappa), \quad \lim_{x \to \infty} \Xi^{(c)}_\kappa(x, y) = 0.
\]

The limit on the right follows easily from inequality (31), for \( \lim_{x \to \infty} \theta^{(c)}(x) = 0 \). The proof of the limit on the left is more involved, and our analysis has some similarities to those of [15] and [16, Thm. 1]. Indeed, monotonicity results derived in Lem. 5 allow us to conclude that

\[
v_L(y) := \lim_{x \to -\infty} \Xi^{(c)}_{(-\infty,L)}(x, y) \leq \liminf_{x \to -\infty} \Xi^{(c)}_\kappa(x, y) \leq \limsup_{x \to -\infty} \Xi^{(c)}_\kappa(x, y) \leq \bar{u}(y; \kappa),
\]

where the first limit is known to exists thanks to the monotonicity in \( x \) of \( \Xi^{(c)}_{(-\infty,L)}(\cdot, \cdot) \).

According to Lem. 3.2(i) we know that \( \Xi^{(c)}_{(-\infty,L)}(\cdot, \cdot) \) solves the equation (2). Thus, \( v_L(\cdot) \) satisfies

\[
\partial_y^2 v_L(y) + v_L(y) - (v_L(y))^3 = 0
\]

in the sense of distributions in \([0, \kappa]\), hence in the classical sense. As \( v_L(y) \rvert_{y=0,\kappa} = 0 \) we conclude that either \( v \equiv 0 \) or \( v \) is a periodic solution with period \( \tau \) so that \( 2\kappa/\tau \in \mathbb{N} \). We can readily exclude the first possibility, since the mapping \( x \mapsto \Xi^{(c)}_{(-\infty,L)}(x, y) \) is non-increasing (as the infimum of non-increasing functions) and Lem. 3.4 provides a nontrivial positive subsolution \( V(\cdot, \cdot) \) satisfying (30). The same inequality also implies that \( \tau = 2\kappa \), i.e., \( \bar{u}(\cdot; \kappa) \) and \( v_L(\cdot) \) have the same period therefore and obey the same normalization, therefore \( v_L(\cdot) \equiv \bar{u}(\cdot; \kappa) \), and the result follows from the equality of (33). \( \square \)

In fact, one can show by following the steps in the proof of Lem. 2.5 that the operator \( \mathcal{L}_{\Xi^{(c)}_\kappa} \) in (34) is boundedly invertible from \( H^2(\mathbb{R} \times [0, \kappa]) \) to \( L^2(\mathbb{R} \times [0, \kappa]) \).

Once more, using the IFT, we conclude the following result.

**Lemma 3.6** (Uniqueness of the continuation in \( c; \mathcal{H}_\kappa \) problem, \( \pi < \kappa < \infty \)). Recall the definition of \( \mathcal{P}(c; \kappa) \) given in (8). For any fixed \( \kappa \in (\pi, \infty) \) the following properties hold:

(i) there exists a unique solution \( \Xi^{(c)}_\kappa(\cdot, \cdot) \) to the \( \mathcal{H}^{(c)}_\kappa \) problem;

(ii) the mapping \( c \mapsto \Xi^{(c)}_\kappa(\cdot) : \{ c \geq 0 \mid \mathcal{P}(c; \kappa) < 1 \} \to L^\infty(\mathbb{R} \times [0, \kappa]; \mathbb{R}) \) is smooth.
Proof. The analysis is analogous to that of Lem. 2.5 and is outlined below, where we point out the necessary modifications. Fix $\kappa \in (\pi, \infty)$. Initially we define the linearized operator about the solutions $\Xi_{\kappa}$

$$\mathcal{L}_{\Xi_{\kappa}}^{(c)}[v] = \Delta_{x,y}v + c\partial_x v + \mu(x)v - 3(\Xi_{\kappa}^{(c)}))^2 v,$$

$$v \in D(\mathcal{L}_{\Xi_{\kappa}}^{(c)}) = H^2(\mathbb{R} \times [0,\kappa]) \cap H^1_0(\mathbb{R} \times [0,\kappa]).$$

Writing $v = e^{-\frac{\pi}{2}x} u$ we rewrite the operator above in a “self-adjoint” form,

$$\tilde{\mathcal{L}}_{\Xi_{\kappa}}^{(c)}[u] = \Delta_{x,y} u + \left(\mu(x) - \frac{c^2}{4}\right) u - 3(\Xi_{\kappa}^{(c)})^2 u,$$

$$u \in D(\tilde{\mathcal{L}}_{\Xi_{\kappa}}^{(c)}) = e^{\frac{\pi}{2}x} (H^2(\mathbb{R} \times [0,\kappa]) \cap H^1_0(\mathbb{R} \times [0,\kappa])).$$

Notice that the mapping $u(\cdot) \mapsto \mathcal{I}[u] := e^{\frac{\pi}{2}x} u(\cdot)$ is an isometry between the spaces $H^2(\mathbb{R} \times [0,\kappa]) \cap H^1_0(\mathbb{R} \times [0,\kappa])$ and $\mathcal{Y} := e^{\frac{\pi}{2}x} (H^2(\mathbb{R} \times [0,\kappa]) \cap H^1_0(\mathbb{R} \times [0,\kappa]))$ where $||u||_{\mathcal{Y}} = ||e^{-\frac{\pi}{2}x} u||_{\mathcal{Y}^2}$, therefore it suffices to study the invertibility of $\tilde{\mathcal{L}}_{\Xi_{\kappa}}^{(c)}$ only. At this point we define $\tilde{\mathcal{L}}_{\Xi_{\kappa}}^{(c)}(\delta)$ as the action of the operator $\tilde{\mathcal{L}}_{\Xi_{\kappa}}^{(c)}$ on the scale of Banach spaces $e^{\delta c x} H^2(\mathbb{R} \times [0,\kappa])$. Continuity of the Fredholm index in the weight $\delta$ shows in the range $|\delta| \leq \frac{\pi}{2}$ we have that $\tilde{\mathcal{L}}_{\Xi_{\kappa}}^{(c)}(\delta)$ are Fredholm operators and have index zero, with the same kernel. As $\tilde{\mathcal{L}}_{\Xi_{\kappa}}^{(c)}(\delta = 0)$ is invertible one can argue as in Lem. 2.5 to obtain the bounded invertibility of $\tilde{\mathcal{L}}_{\Xi_{\kappa}}^{(c)}$ in $e^{\frac{\pi}{2}x} (H^2(\mathbb{R} \times [0,\kappa]) \cap H^1_0(\mathbb{R} \times [0,\kappa]))$. The results are then obtained from an application of the IFT.

Unlike the previous case, it is not directly clear that $\lim_{x \to -\infty} \Xi_{\kappa}^{(c)}(x,y) = \bar{u}(y;\kappa)$ has exponential rate of convergence. Our next result implies that.

**Lemma 3.7** (Exponential convergence; $\mathcal{H}_\kappa$-problem, $\pi < \kappa < \infty$). There exists constants $M$, $C$, $\delta > 0$ independent of $x$ and $y$ such that

$$|\Xi_{\kappa}^{(c)}(x,y) - \bar{u}(y;\kappa)| \leq Ce^{-\delta|x|}, \quad \text{for} \quad x \leq -M,$$

$$\text{and} \quad |\Xi_{\kappa}^{(c)}(x,y)| \leq Ce^{-\delta|x|}, \quad \text{for} \quad x \geq M.$$

**Proof.** Initially we show exponential rate of convergence to the far-field as $|x| \to \infty$. Recall the partition of unity $\chi^\pm(\cdot)$ defined in (12). The result follows once we prove that $\chi^\pm(x)\partial_x(\Xi_{\kappa}^{(c)}) \in e^{-\delta c x} H^2(\mathbb{R} \times [0,\kappa])$ for some $\delta > 0$ and $< x > := \sqrt{1 + x^2}$. Indeed, as we know from Lem. 3.5, $\lim_{x \to -\infty} \Xi_{\kappa}^{(c)}(x,y) = \bar{u}(y;\kappa)$; using the Sobolev embedding $H^2(\mathbb{R} \times [0,\kappa]) \hookrightarrow L^\infty(\mathbb{R} \times [0,\kappa])$ we have

$$|\Xi_{\kappa}^{(c)}(x,y) - \bar{u}(y;\kappa)| \leq \int_{-\infty}^x \partial_x(\Xi_{\kappa}^{(c)})(s,y)ds \lesssim \int_{-\infty}^x e^{\delta x} ds \lesssim e^{\delta x}, \quad \text{for} \quad x \leq -2,$$

which gives the result. The proof requires several tools of Fredholm theory for elliptic operators. The linearization of the equations (2) at $\Xi_{\kappa}^{(c)}(\cdot,\cdot)$ gives

$$\mathcal{L}_{\Xi_{\kappa}}^{(c)}[v] = \Delta_{x,y}v + c\partial_x v + \left[\mu(x) - 3(\Xi_{\kappa}^{(c)}(x,y))^2\right] v,$$

with domain of definition $D(\mathcal{L}_{\Xi_{\kappa}}^{(c)}) = H^2(\mathbb{R} \times [0,\kappa]) \cap H^1_0(\mathbb{R} \times [0,\kappa])$. The analysis on Lem. 3.6 shows that the operator $\mathcal{L}_{\Xi_{\kappa}}^{(c)}$ is invertible in $H^2(\mathbb{R} \times [0,\kappa]) \cap H^1_0(\mathbb{R} \times [0,\kappa])$. Now, notice that $v^{-}(x,y) = \chi^{-}(x)\partial_x \Xi_{\kappa}^{(c)}(x,y) \in C^\infty(\mathbb{R} \times [0,\kappa]; \mathbb{R})$
solves a problem of the form \( \mathcal{L}^{[\xi,\psi]} [v^r] = f \), where \( f \in L^2(\mathbb{R} \times [0,\kappa]) \) is spatially localized. Arguing as in [10, Cor. 5.5] we conclude that \( e^{\delta \sigma x} v \in H^2(\mathbb{R} \times [0,\kappa]) \cap H^1_0(\mathbb{R} \times [0,\kappa]) \) for all \( \delta > 0 \) sufficiently small. A similar analysis can be done by considering \( w(x,y) = \chi^+(x) \partial_x \Xi_k^{(c)}(x,y) \), whence exponential rate of convergence to the far field as \( |x| \to \infty \) is derived. This finishes the proof. \( \square \)

3.3. nonexistence for the \( \mathcal{H}_\kappa \)-problem: case \( \frac{c^2}{4} + \frac{\pi^2}{k^2} > 1 \). In this section we prove the nonexistence of patterns when \( \frac{c^2}{4} + \frac{\pi^2}{k^2} > 1 \). The method is standard: roughly speaking, assuming an existing solution \( \Xi_k^{(c)}(\cdot,\cdot) \), we can obtain a supersolution \( V(\cdot,\cdot) \) that is above \( \Xi_k^{(c)}(\cdot,\cdot) \) and which, under certain conditions, can touch the solution in at least one point. The function \( Z(\cdot,\cdot) := \Xi_k^{(c)}(\cdot,\cdot) - V(\cdot,\cdot) \) solves an elliptic problem and has an interior maximum, which contradicts the maximum principle and the Hopf Lemma. We make these words more precise in what follows.

We begin with the main ingredients in the construction of the subsolution.

**Lemma 3.8** (nonexistence; \( \frac{c^2}{4} + \frac{\pi^2}{k^2} > 1 \)). No solution to the \( \mathcal{H}_\kappa \) problem exists when \( \mathcal{P}(c, \kappa) := \frac{c^2}{4} + \frac{\pi^2}{k^2} > 1 \).

**Proof.** The first step on our proof is the construction of a supersolution \( V(x,y) \geq \Xi_k^{(c)}(x,y) \) on \( x \leq 0, y \in [0,\kappa] \). Set \( V(x,y) = e^{-\lambda(x-\tau)} \sin \left( \frac{\pi y}{\kappa} \right) \), where \( \tau \in \mathbb{R} \) will be defined later. We claim that we can choose \( \lambda \) appropriately so that \( V \) is a supersolution, namely

\[
\Delta_{x,y} V + c \partial_x V + \mu(x) V - V^3 = \left( \lambda^2 - \frac{\pi^2}{k^2} - c\lambda + 1 \right) V + (\mu(x) - 1) V - V^3 \leq 0,
\]

holds. Indeed, the mapping \( \lambda \mapsto \left( \lambda^2 - \frac{\pi^2}{k^2} - c\lambda + 1 \right) \) has a minimum

\[
\left. \lambda^2 - \frac{\pi^2}{k^2} - c\lambda + 1 \right|_{\lambda=\frac{c^2}{4}} = 1 - c^2 - \frac{\pi^2}{k^2} = 1 - \mathcal{P}(c, \kappa) < 0,
\]

hence \( V(x,y) := e^{-\frac{c^2}{4}(x-\tau)} \sin \left( \frac{\pi y}{\kappa} \right) \) is a supersolution. This verifies the claim.

Our second step is to shown that \( \tau \) can be chosen so that

\[
\Xi_k^{(c)}(x,y) - V(x,y) \leq 0, \text{ whenever } x \in \mathbb{R}, \quad y \in [0,\kappa] \tag{35}
\]

To begin with, define \( Z(x,y) := \Xi_k^{(c)}(x,y) - V(x,y) \). We first consider the case \( x \leq 0 \): as the derivatives of \( \tilde{u}_k(\cdot;\kappa) \) are finite, we can certainly choose \( \tau \) so that \( e^{\frac{c^2}{4}\tau} \sin \left( \frac{\pi y}{\kappa} \right) \geq \tilde{u}_k(y;\kappa) \) whenever \( y \in [0,\kappa] \). Exploiting the fact that \( c > 0 \) and the monotonicity of the mapping \( x \mapsto \Xi_k^{(c)}(x,\cdot) \) we get that \( Z(x,y) \leq 0 \) whenever \( x \leq 0 \). In fact, \( \tau \) can be taken sufficiently large so that \( Z(x,y) \leq 0 \) is an equality in at least one point in \( \{x \leq 0\} \times [0,\kappa] \), for \( V(x,y) \downarrow 0 \) pointwise when \( \tau \to -\infty \).

We turn to the case \( x > 0 \): the function \( Z(\cdot,\cdot) \) satisfies

\[
\Delta_{x,y} Z + c \partial_x Z + \mu(x) Z - f[\Xi_k^{(c)}, V] Z \geq 0,
\]

where \( f[a,b] := \frac{a^3-b^3}{a-b} \) whenever \( a \neq b \) and \( 3a^2 \) otherwise. Moreover, whenever \( x > 0 \) we have that \( \mu(x) - f[\Xi_k^{(c)}, V] < 0 \), thus classical maximum principles can be applied. Notice then that \( Z(x,y) \to 0 \) as \( x \to \infty \). In particular, \( Z(x,y) \leq 0 \) whenever \( (x,y) \in \{x > 0\} \times [0,\kappa] \) converges to the boundary of this latter set. We conclude that \( Z(\cdot,\cdot) \) cannot assume a positive value in \( \{x > 0\} \times [0,\kappa] \), hence \( Z(x,\cdot) \leq 0 \) for all \( x > 0 \), and this finally establishes (35).
The rest of the proof goes as in Prop. 3: as $Z(\cdot, \cdot) \leq 0$ in $\mathbb{R} \times [0, \kappa]$ with equality in at least one point, the maximum principle implies that $Z(\cdot, \cdot) \equiv 0$. This contradiction leads to the nonexistence of solutions.

Finally, we put all these auxiliary results together and prove the main result of this section:

**of Theorem 1.2; case $\pi < \kappa < \infty$.** Combine the above discussion on the nonexistence of solutions with the results of Lemmas 3.5, 3.6, 3.7, and 3.8.

4. **Two-dimensional quenched patterns – single horizontal interfaces; $\mathcal{H}_\infty$ problem.** In this section, we shall prove Theorem 1.2 in the case $\kappa = \infty$. To be consistent with the notation introduced in Section 2 we exploit the fact that the nonlinearity in (2) is odd to solve the problem in the half space $\mathbb{R} \times (-\infty, 0]$. Further symmetries of the equation are also exploited: we solve an equivalent $\mathcal{H}_\infty$-problem, seeking for a solution $\Xi(c)(\cdot, \cdot)$ to (2) in $\mathbb{R}^2$, satisfying

$$
\lim_{x \to \infty} \Xi(c)(x, y) = -\tanh \left( \frac{y}{\sqrt{2}} \right), \quad \lim_{y \to \infty} \Xi(c)(x, y) = \mp \theta(c)(x),
$$

and

$$
\lim_{x \to \infty} \Xi(c)(x, y) = 0,
$$

where $\theta(c)(\cdot)$ is the one-dimensional solution to the $(1 \sim 0)(c)$-problem. Notice that the odd symmetry solutions to (2) with respect to $(x, y) \mapsto (y, \Xi(c)(x, y))$ readily gives the pattern with the properties stated in Theorem 1.2. In this fashion, restricting the problem to the upper half-plane gives a zero boundary condition at $y = 0$ that removes the non-uniqueness of solutions induced by $y$-translation invariance.

Thanks to the results of Sec. 2 related to the 1D problem the following observation is readily available.

**Observation 4.1** (Restriction to the case $c < 2$). It is clear from (36) that the problem above is meaningless when $c \geq 2$ for the patterns $\theta(c)(\cdot)$ do not exist in this range. We can readily say that no solution to this problem exists when $c \geq 2$, immediately restricting our study to the range $0 \leq c < 2$.

In fact, in this section we prove that for all quenching fronts speeds in the range $c \in [0, 2)$ there exists a unique $\mathcal{H}_\infty$-pattern (up to translations in the $y$ direction), which corresponds to the statement of Theorem 1.2. The strategy goes as in Sections 2 and 3: first, by reducing the problem to a half plane and truncating it, restricting the problem to a rectangle $\Omega_{(-M, L)} := (-M, L) \times (-M, 0)$. Then we let $M \to \infty$ and, subsequently, we let $L \to \infty$.

The truncated $\mathcal{H}_\infty$-problem is set up as

$$
\begin{cases}
\Delta_{x,y} u + cu_x + \mu(x)u - u^3 = 0, & (x, y) \in \Omega_{(-M, L)}, \\
u = g(-M, L), & (x, y) \in \partial \Omega_{(-M, L)},
\end{cases}
$$

where $g(-M, L)(x, y) := \theta(s,-M,L)(x) \theta^0(-M,0)(y)$ and $\theta(c)(-M, \cdot)$ are the solutions to the truncated one-dimensional problem (13) on the interval $(-M, L)$. Similar to the results of Prop. 4 and [10, §3], we construct unique solutions to this truncated problem using an iterative scheme. The solution $\Theta(c)(-M, L)(\cdot, \cdot) : \Omega_{(-M, L)} \to [0, 1]$ is shown to be unique; Furthermore, exploiting that $0 \leq u \leq 1$ and Agmon-Douglis-Nirenberg
regularity we readily conclude that \( u \) and derivatives are Hölder continuous across \( x = 0 \), and, in fact, \( \Theta^{(c)}_{(-M,L)}(\cdot,\cdot) \in C^{(1,\alpha)}(\tilde{\Omega}_{(-M,L)}) \), for all \( 0 \leq \alpha < 1 \). Following the method in Sec. 3, we extend these functions to the whole plane \( \mathbb{R}^2 \):

\[
\mathcal{E} \left[ \Theta^{(c)}_{(-M,L)} \right](x,y) = \begin{cases} 
\Theta^{(c)}_{(-M,L)}(x,y), & \text{for } (x,y) \in \Omega_{(-M,L)}, \\
\mathcal{E} \left[ \Theta^{(c)}_{(-M,L)} \right](x) \cdot \mathcal{E} \left[ \Theta^{(0)}_{(-M,0)} \right](y), & \text{for } (x,y) \in \mathbb{R}^2 \setminus \Omega_{(-M,L)}. 
\end{cases}
\]

We summarize the main properties of the functions \( \mathcal{E} \left[ \Theta^{(c)}_{(-M,L)} \right](\cdot,\cdot) \) in the following Proposition, whose proof is similar to that of [10, Prop. 3.4]:

**Proposition 6** (Properties of the extension operator, \( \mathcal{H}_\infty \)-problem). The following properties of \( \mathcal{E}[\cdot] \) hold.

(i) (Monotonicity of \( \mathcal{E} \)) We have \( 0 \leq \mathcal{E} \left[ \Theta^{(c)}_{(-M,L)} \right](\cdot,\cdot) \leq 1 \). Furthermore, if \( w \) is only defined in a subset \( A \subset \mathbb{R}^2 \) so that \( \Omega_{(-M,L)} \subset A \subset \Omega_{(-\infty,L)} \), \( 0 \leq w(\cdot) \leq 1 \) and \( w(\cdot) \leq \Theta^{(c)}_{(-M,L)}(\cdot,\cdot) \), then \( 0 \leq \mathcal{E}[w](\cdot) \leq \mathcal{E} \left[ \Theta^{(c)}_{(-M,L)} \right](\cdot) \) in \( \mathbb{R}^2 \).

(ii) (Monotonicity in \( M \)) Let \( 0 \leq M < \tilde{M} \) and \( L \geq 0 \) be fixed. Then \( M < \tilde{M} \implies \mathcal{E} \left[ U_{(-M,L)} \right](x,y) \leq \mathcal{E} \left[ \Theta^{(c)}_{(-M,L)} \right](x,y) \).

(iii) (Monotonicity in \( L \)) Let \( 0 \leq L < \tilde{L} \) and \( M \geq 0 \) be fixed. Then \( L < \tilde{L} \implies \mathcal{E} \left[ \Theta^{(c)}_{(-M,L)} \right](x,y) \leq \mathcal{E} \left[ U_{(-M,L)} \right](x,y) \).

(iv) (Monotonicity in \( x \)) Let \( L, M, y \) be fixed. Then the mapping

\[
x \to \mathcal{E} \left[ \Theta^{(c)}_{(-M,L)} \right](x,y)
\]

is non-increasing.

(v) (Monotonicity in \( y \)) Let \( L, M, x \) be fixed. Then the mapping

\[
y \to \mathcal{E} \left[ \Theta^{(c)}_{(-M,L)} \right](x,y)
\]

is non-increasing.

When allied to the results in [9, §4], one derives Corollary 1 as a direct consequence of the next Lemma.

**Lemma 4.2** (Existence; \( \mathcal{H}_\infty \) problem). There exists a solution \( \Xi_{\infty}^{(c)}(\cdot,\cdot) \) to (2) in \( \mathbb{R}^2 \) satisfying the limits (36). Furthermore, \( \partial_y \Xi_{\infty}^{(c)}(\cdot,\cdot) < 0 \).

**Proof.** We shall construct a solution \( \Xi_{\infty}^{(c)}(\cdot,\cdot) \) in \( (x, y) \in \mathbb{R} \times (-\infty, 0) \), extending it to the whole plane \( \mathbb{R}^2 \) by reflection, that is, \( \Xi_{\infty}^{(c)}(x, -y) = -\Xi_{\infty}^{(c)}(x, y) \). Similarly to Lem. 3.5, we set

\[
\Xi_{\infty}^{(c)}(\cdot,\cdot) := \sup_{L \geq 0} \Theta^{(c)}_{(-\infty,L)}(\cdot,\cdot) \quad \text{where} \quad \Theta^{(c)}_{(-\infty,L)}(\cdot,\cdot) := \inf_{M \geq 0} \Theta^{(c)}_{(-M,L)}(\cdot,\cdot).
\]

It is standard to show that \( \Xi_{\infty}^{(c)}(\cdot,\cdot) \) solves (2) in the sense of distributions. With regards to its asymptotic properties, most of them are obtained as a consequence of the limits

\[
\Theta^{(c)}_{(-\infty,L)}(x) = \lim_{y \to \infty} \Theta^{(c)}_{(-\infty,L)}(x,y), \quad \text{and} \quad \Theta^{(0)}_{(-\infty,0)}(y) = \lim_{x \to -\infty} \Theta^{(c)}_{(-\infty,L)}(x,y),
\]

\[ (38) \]
the first one, attained uniformly in \( x \in \mathbb{R} \); the second, attained uniformly in \( y \in (-\infty, 0] \). We shall assume (38) for a moment and show how it implies the result. Indeed, by construction and monotonicity we readily obtain

\[
\Theta^{(c)}_{(-\infty, L)}(x, y) \leq \Xi^{(c)}_{\infty}(x, y) \leq \min\{\theta^{(c)}(x), \theta^{(0)}_{(-\infty, 0)}(y)\}. \tag{39}
\]

Thanks to the construction of \( \theta^{(c)}(\cdot) \) given by Theorem (1.1), we can use once more the monotonicity of the mappings \( L \mapsto \Theta^{(c)}_{(-\infty, L)}(\cdot, \cdot) \) and \( L \mapsto \theta^{(c)}_{(-\infty, L)}(\cdot) \) to conclude that \( \Xi^{(c)}_{\infty}(\cdot, \cdot) \) verifies (36), i.e., it achieves the asymptotic limits

\[
\lim_{y \to -\infty} \Xi^{(c)}_{\infty}(x, y) = \theta^{(c)}(x), \quad \lim_{x \to +\infty} \Xi^{(c)}_{\infty}(x, y) = 0,
\]

\[
\text{and} \quad \lim_{x \to -\infty} \Xi^{(c)}_{\infty}(x, y) = \theta^{(0)}_{(-\infty, 0)}(y),
\]

the first one, uniformly in \( x \in \mathbb{R} \); the latter two, uniformly in \( y \in (-\infty, 0] \). The exponential decay of \( \Xi^{(c)}_{\infty}(\cdot, \cdot) \) to its end states is derived as in Lem. 2.3 (see also [10, §5]). The inequality \( \partial_\beta \Xi^{(c)}_{\infty}(\cdot, \cdot) \leq 0 \) is a consequence of Prop. 4.1(v), because the mapping \( y \mapsto \Xi^{(c)}_{\infty}(x, y) \) is non-increasing; the strict inequality \( \partial_\beta \Xi^{(c)}_{\infty} < 0 \) a.e. then follows using a Harnack inequality (cf. [5, Thm. 9.22]; see also [9, Prop. 4.1]). Hence, (38) implies all the statements announced.

We finally prove (38). Let \( L \geq 0 \) be fixed. Initially, we invoke the monotonicity of \( \Theta^{(c)}_{(-\infty, L)}(\cdot, \cdot) \) in its arguments and its uniform boundedness for all \( L \geq 0 \) to legitimate the construction of

\[
H_L(x) := \lim_{y \to -\infty} \Theta^{(c)}_{(-\infty, L)}(x, y), \quad \text{and} \quad V_L(y) := \lim_{x \to -\infty} \Theta^{(c)}_{(-\infty, L)}(x, y).
\]

We wish to show that \( H_L(\cdot) = \theta^{(c)}_{(-\infty, L)}(\cdot) \) and \( V_L(\cdot) = \theta^{(0)}_{(-\infty, 0)}(\cdot) = -\tanh \left( \frac{\cdot}{\sqrt{2}} \right) \). We first prove that \( H_L(\cdot) \) satisfies (3) in the sense of distributions. To see this, we note that \( H_L(\cdot) \) is a well-defined pointwise limit of \( \Theta^{(c)}_{(-\infty, L)}(\cdot, \cdot) \). Now, we exploit the fact that \( \Theta^{(c)}_{(-\infty, L)}(\cdot, \cdot) \) is a solution to (37) in the sense of distributions and then apply it to test functions of the type \( \phi(\cdot, y) = \phi_1(x)\phi_2(y + \tau) \), where \( \phi_1, \phi_2 \in C^\infty_0(\mathbb{R}^2) \), subsequently taking \( \tau \to \infty \). A similar reasoning shows that \( V_L(\cdot) \) satisfies \( \partial_\beta^2 V_L + V_L - V_L^3 = 0 \) in the sense of distributions.

In order to study \( H_L(\cdot) \) and \( V_L(\cdot) \) in detail, we use the subdistributions constructed in Lem. 3.4. First, since \( c < 2 \), we can fix \( \kappa_1 > 0 \) sufficiently large in order to define a quantity \( d = d(\kappa) > 0 \) that verifies \( \frac{d^2}{4} = \frac{c^2}{4} + \frac{\pi^2}{\kappa^2} < 1 \), for all \( \kappa > \kappa_1 \). Now, let \( w(\cdot) := \theta^{(c)}_{(0, 0)}(\cdot) \); recall that \( w(\cdot) \in C^\infty(\mathbb{R}; [-1, 1]) \) satisfies \( \partial_\beta^2 w + c\partial_\beta w + w - w^3 = 0 \), and \( 0 = w(0) < w(x) < \lim_{s \to -\infty} w(s) = 1 \) for \( x < 0 \); furthermore, \( w(x) \) is decreasing in \( x \leq 0 \). Define

\[
W(x, y) := e^{-\left(\frac{\kappa^2}{\kappa^2}\right)} w(x) \cdot v_\kappa(y),
\]

where \( v_\kappa(y) := -\beta \sin \left( \frac{\pi y}{\kappa} \right) \), \( (x, y) \in (-\infty, L) \times (-\kappa, 0) \).

The parameter \( \beta \) is chosen in two steps: first, in such a way that the inequality \( \partial_\beta^2 v_\kappa + \kappa^2 v_\kappa - v_\kappa^3 \geq 0 \) holds, which a straightforward calculation shows to be true for
any \( \beta \in \left[0, \sqrt{1 - \frac{\pi^2}{\kappa^2}}\right]. \) Now one makes use of the Hamiltonian structure of the equation \( \partial_y^2 v + v - v^3 = 0, \) which conserves the quantity \( \mathcal{H}(v, \partial_y v) = (\partial_y v)^2 + v^2 - \frac{v^4}{2} \)
along trajectories, to finally pick \( \beta = \beta(\kappa) := \sqrt{1 - \frac{\pi^2}{\kappa^2}}, \) for which
\[
\sup_{y \in \left[-\frac{\pi}{2}, 0\right]} \{ \mathcal{H}(v_\kappa(y), \partial_y v_\kappa(y)) \} = \frac{1}{2} \left( 1 - \frac{\pi^4}{\kappa^4} \right) < \mathcal{H} \left( \Theta^{(0)}_{(-\infty, 0)}(\cdot), \partial_y \Theta^{(0)}_{(-\infty, 0)}(\cdot) \right) = \frac{1}{2}.
\]

Reasoning as in Lem. 3.3, we combine these arguments with the Hopf Lemma to conclude that \( v_\kappa(y) \leq \Theta^{(0)}_{(-\infty, 0)}(y) \) in \( y \in \left(-\frac{\kappa}{2}, 0\right). \) The latter inequality also implies that \( v_\kappa(y) \leq \Theta^{(0)}_{(-M, 0)}(y), \) because \( \Theta^{(0)}_{(-M, 0)}(y) \downarrow \Theta^{(0)}_{(-\infty, 0)}(y) \) as \( M \to \infty \) (thanks to Lem. 2.2(ii) and the construction of the patterns \((1 \to 0)^c)\). Using comparison principles one concludes that \( W(x, y) \leq \Theta^{(c)}_{(-\infty, L)}(x, y) \) in \((x, y) \in \left[-\frac{\kappa}{2}, 0\right] \times \left[-\frac{\kappa}{2}, 0\right]. \)

Exploiting the monotonicity of the mapping \( M \mapsto \Theta^{(c)}_{(-M, L)} \) we invoke the maximum principle and the Hopf Lemma to extend this result for all \( M > \frac{\kappa}{2}, \) namely, \( W(x, y) \leq \Theta^{(c)}_{(-M, L)}(x, y) \) in \((x, y) \in (-M, 0) \times (-M, 0). \) In particular, this inequality holds whenever \((x, y) \in (-M, 0) \times \left[-\frac{\kappa}{2}, 0\right], \) yielding
\[
W(x, y) \leq \Theta^{(c)}_{(-\infty, L)}(x, y), \quad \text{for } (x, y) \in (-\infty, 0) \times \left[-\frac{\kappa}{2}, 0\right], \tag{40}
\]

once we pass to the limit \( M \to \infty. \) Due to the properties of \( w(\cdot) \) and our choice of \( d = d(\kappa), \) for any given \( \epsilon > 0, \) we can find a \( \kappa_2 > \kappa_1 \) such that
\[
1 - \epsilon \leq e^{\frac{(c-d)x}{\kappa^2}} w(-\kappa) \leq \sup_{x \leq 0} \left\{ e^{-\frac{(c-d)x}{\kappa^2}} w(x) \right\}, \quad \text{whenever } \kappa > \kappa_2,
\]

since \( \lim_{\kappa \to \infty} (c-d) = 0. \) Consequently, we can use (40) and the monotonicity of the mapping \( x \mapsto \Theta^{(c)}_{(-\infty, L)}(x, y) \) to obtain the existence of a constant \( C_* = C(\kappa_2) > 0 \) such that
\[
(1 - \epsilon) v_\kappa(y) \leq \Theta^{(c)}_{(-\infty, L)}(x, y) \leq V_L(y), \quad \text{for } y \in \left[-\frac{\kappa}{2}, 0\right], \quad x \leq -C_*.
\]

Due to our definition of \( v_\kappa(\cdot) \) and choice of \( \beta(\kappa), \) these inequalities imply that
\[
(1 - \epsilon) \sqrt{1 - \frac{\pi^2}{\kappa^2}} \leq \Theta^{(c)}_{(-\infty, L)}(x, y) \leq V_L(y) \text{ whenever } y \leq -\frac{\kappa}{2} \text{ and } x \leq -C_*, \text{ for the mapping } y \mapsto \Theta^{(c)}_{(-\infty, L)}(x, y) \text{ is monotonic.}
\]
Furthermore, once we combine the latter inequality with (39), it follows that \( \lim_{y \to -\infty} V_L(y) = 1. \) Consequently, arguing as in Prop. 3, we must have \( V_L(y) = \Theta^{(0)}_{(-\infty, 0)}(y + \delta) \) for some \( \delta \in \mathbb{R}. \) Thanks to (40) we have that \( V_L(y) > 0 \) for all \( y < 0; \) using (39) we conclude that \( \delta = 0, \) namely,
\[
V_L(y) = \Theta^{(0)}_{(-\infty, 0)}(y), \text{ and this proves that limit on the right of (38) holds. Finally,}
\]

invoking the monotonicity of the \( \Theta^{(c)}_{(-\infty, L)}(\cdot, \cdot) \) in its arguments and regularity of \( V_L(\cdot), \) we can use Dini’s Theorem (cf. [13, §9]) to obtain uniform convergence in compact subsets of \((-\infty, 0]. \) When allied to the previous estimates, this shows that the limit is attained uniformly in \( y \in (-\infty, 0]. \)
We now prove the limit asserted on the left of (38). Looking again at (40), we get

\[ W\left(x, -\frac{\kappa}{2}\right) = \beta(\kappa)\theta^{(c)}_{(-\infty,0)}(x) \leq \Theta^{(c)}_{(-\infty,L)}\left(x, -\frac{\kappa}{2}\right), \quad \text{for } x \leq 0. \quad (41) \]

Monotonicity implies that \( \Theta^{(c)}_{(-\infty,L)}(x, y) \leq \theta^{(c)}_{(-\infty,L)}(x) \). Passing to the limit \( \kappa \to \infty \) in (41) and making use of \( H_L(\cdot) \)'s definition, we derive

\[ 0 \leq \theta^{(c)}_{(-\infty,0)}(x) \leq H_L(x) \leq \theta^{(c)}_{(-\infty,L)}(x), \quad \text{for } x \leq 0, \quad (42) \]

since \( \beta(\kappa) \to 1 \) as \( \kappa \to \infty \). Now we define a quantity \( \tilde{L} \), chosen as the minimum \( \tilde{L} \geq 0 \) such that

\[ \inf_{x \in [0, \tilde{L}]} \{ H_L(x) - \theta^{(c)}_{(-\infty,L)}(x) \} = 0. \]

Notice that \( 0 \leq H_L(x) - \theta^{(c)}_{(-\infty,L)}(x) \leq 0 = H_L(L) - \theta^{(c)}_{(-\infty,L)}(L) \). Thus, \( \tilde{L} \) exists, and \( 0 \leq \tilde{L} \leq L \).

We claim that \( 0 < \tilde{L} \leq L \). Indeed, as a solution to (3) we must have \( H_L(x) > 0 \) when \( x < L \), due to the Hopf Lemma and non-negativity of the solution. According to Lemma 2.2(v), the mapping \( \tilde{L} \to \theta^{(c)}_{(-\infty,L)}(\cdot) \) is continuous in the sup norm. These observations show that \( \tilde{L} > 0 \).

It turns out that \( \tilde{L} = L \). However, before we embark in the proof, we make a minor digression concerning the unique family of non-constant bounded solutions \( w(\cdot) \) solving (16) and “hitting" \((1,0)\) as \( x \to -\infty \), that is \( \lim_{x \to -\infty} (w(x), \partial_x w(x)) = (1,0) \):

for any two solutions \( w_1(\cdot) \) and \( w_2(\cdot) \) solving (16) in \( x \leq 0 \), if there exists an \( \tilde{x} \leq 0 \) for which \( w_1(\tilde{x}) = w_2(\tilde{x}) \) then \( w_1(x) = w_2(x) \) for all \( x \leq \tilde{x} \).

We now start the proof of \( \tilde{L} = L \), arguing by contradiction: assume that \( 0 < \tilde{L} < L \). Denote by \( x_L \in [0, \tilde{L}] \) the smallest point for which \( H_L(x_L) = \theta^{(c)}_{(-\infty,L)}(x_L) \).

It must hold that \( x_L < \tilde{L} \), because \( H_L(\cdot) > 0 \) in \((-\infty, L)\). By definition of \( \tilde{L} \) and \( x_L \), we have that \( z(\cdot) := H_L(\cdot) - \theta^{(c)}_{(-\infty,L)}(\cdot) \geq 0 \) in \([0, \tilde{L}]\), with \( z(x_L) = 0 \).

Note that \( z(\cdot) \) also satisfies (17), hence it is a supersolution in the interval \([0, \tilde{L}]\); a combination of both the maximum principle and the Hopf Lemma shows that we cannot have \( x_L \) as an interior point. We must then have \( x_L = 0 \), for \( x_L < \tilde{L} \). From \( z(0) = z(x_L) = 0 \) we derive the equality \( H_L(x) = \theta^{(c)}_{(-\infty,L)}(x) \) for all \( x \leq 0 \), due to the properties of the solution \( w(\cdot) \) of (16) as previously discussed. Now, another application of the maximum principle and the Hopf Lemma on the interval \([0, \tilde{L}]\) implies that \( H_L(x) = \theta^{(c)}_{(-\infty,L)}(x) \) in \([0, \tilde{L}]\). As \( H_L(x) > 0 \) when \( x < L \) we obtain that \( \tilde{L} = L \). This concludes the proof of (38).

The monotonic convergence of \( \Xi^{(c)}(x, y) \) to its continuous spatial limits as \( x \to \pm \infty \) imply the uniformity of the limits in compact sets, once more thanks to Dini’s Theorem. Using (42) we get that \( \theta^{(c)}_{(-\infty,0)}(x) \leq \lim_{y \to \infty} \Xi^{(c)}(x, y) \leq \theta^{(c)}(x) \). Hence, the asymptotic properties of \( \theta^{(c)}_{(-\infty,0)}(\cdot) \) and \( \theta^{(c)}(\cdot) \) allow us to extend this result to uniform convergence in \( x \in \mathbb{R} \), that is, \( \lim_{y \to -\infty} \Xi^{(c)}(x, y) = \theta^{(c)}(x) \) uniformly in \( x \in \mathbb{R} \). This establishes Lemma 4.2.

Lemma 4.3 (Exponential convergence; \( H_\infty \) problem). The limits in (36) take place at exponential rate, i.e., there exists constants \( C, \delta > 0 \) independent of \( x \) and \( y \) such that
whenever $\phi(x) + \tanh \left(\frac{y}{\sqrt{2}}\right) \leq Ce^{-\delta|x|}$, for $x \leq -M_x$;

$|\Xi^{(c)}_\infty(x, y)| \leq Ce^{-\delta|x|}$, for $x \geq M_x$;

where $M_x > 0$ is independent of $y$. Likewise, $|\Xi^{(c)}_\infty(x, y) \pm \theta^{(c)}(x)| \leq Ce^{-\delta|y|}$, for $|y| \geq M_y$, for an $M_\theta$ is independent of $x$.

**Proof.** Some aspects of the proof are similar to those in Lem. 3.7. However, neither $\partial_x \Xi^{(c)}(\cdot, \cdot)$ nor $\partial_y \Xi^{(c)}(\cdot, \cdot)$ is in $H^2(\mathbb{R}^2) \cap \mathcal{H}_0^1(\mathbb{R}^2)$, thus a different approach has to be adopted. As before, we first trace the essential spectrum of the linearized operator (2) at $\Xi^{(c)}(\cdot, \cdot)$,

$$\mathcal{L}_{\Xi^{(c)}}[v] := \Delta_{x,y} v + c \partial_x v + \left[\mu(x) - 3 \left(\Xi^{(c)}_\infty(x, y)\right)^2\right] v,$$

with domain of definition $\mathcal{D}(\mathcal{L}_{\Xi^{(c)}}) = H^2_{\text{odd}}(\mathbb{R}^2) := \{ w \in H^2(\mathbb{R}^2) : w(x, y) = -w(x, -y)\}$. We remark that [10, Lem. 5.3] still applies: indeed, we can define the asymptotic operators

$$\mathcal{M}^- [v] := \partial^2_y v(-\cdot) + \left[1 - 3 \tanh^2 \left(\frac{y}{\sqrt{2}}\right)\right] v(-\cdot), \quad \mathcal{M}^+ [v] := \partial^2_y v(\cdot) - v(\cdot),$$

with domain $\mathcal{D}(\mathcal{M}^\pm) = H^2_{\text{odd}}(\mathbb{R}) \cap H^0_\text{loc}(\mathbb{R})$. We claim that these operators are invertible: indeed, coercivity implies that $\text{Ker}(\mathcal{M}^+) = \{0\}$; the same holds in the case of $\mathcal{M}^-$, for in $H^2$ its kernel is given by $\{ \partial_y \tanh \left(\frac{y}{\sqrt{2}}\right)\}$ which is an even simple eigenfunction, since it has no nodal points (see [5, Thm. 8.38]). Therefore, in the space of odd functions, we have that both operators $\mathcal{M}^\pm$ are invertible. In our next step, we argue as in [10, Lem. 5.1], describing limiting operators associated with $\mathcal{L}_{\Xi^{(c)}}[\cdot]$: 

$$\mathcal{L}^{(x \to +\infty)}_{\Xi^{(c)}}[v] = (\partial_x^2 + c \partial_x + \mathcal{M}^+) [v], \quad \mathcal{L}^{(x \to -\infty)}_{\Xi^{(c)}}[v] = (\partial_x^2 + c \partial_x + \mathcal{M}^-) [v], \quad \mathcal{L}^{(y \to +\infty)}_{\Xi^{(c)}}[v] = \left(\Delta_{x,y} + c \partial_x + \mu(x) - 3(\theta^{(c)})^2\right) [v].$$

Fourier transforming the operators (44a)-(44b) in $x$, and (44c) in $y$, shows that these operators are boundedly invertible. It follows that we can find an $\epsilon > 0$ such that all the operators

$$\mathcal{L}^{(x \to +\infty)}_{\Xi^{(c)}} + \mathcal{P}, \quad \mathcal{L}^{(x \to -\infty)}_{\Xi^{(c)}} + \mathcal{P}, \quad \text{and} \quad \mathcal{L}^{(y \to +\infty)}_{\Xi^{(c)}} + \mathcal{P},$$

are invertible for any bounded operator $\mathcal{P}$ satisfying $\|\mathcal{P}\|_{H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \leq \epsilon$.

A few remarks are necessary before we continue the proof. Following the approach on [10, §3.1], one can use standard elliptic regularity techniques to show that $\Xi^{(c)}(\cdot, \cdot) \in W_\text{loc}^{1,2,p}(\mathbb{R}^2)$. Furthermore, as this solution is uniformly bounded on $\mathbb{R}^2$ we conclude that the same bounds hold on arbitrary translations of any given compact set. We remark that the function $\Xi^{(c)}_\infty(\cdot, \cdot)$ is infinitely differentiable away from the discontinuity of $\mu(\cdot)$ at $\{x = 0\}$ and satisfies (2). In fact, whenever $x \neq 0$ the coefficients of the latter equation are constant, thus we get $\mathcal{L}_{\Xi^{(c)}}[\partial_x \Xi^{(c)}_\infty] = 0$ upon differentiation of (2) with respect to $x$. In particular, $\phi(x) \mathcal{L}_{\Xi^{(c)}}[\partial_x \Xi^{(c)}_\infty] = 0$ whenever $\phi(\cdot) \in C_0^\infty(\mathbb{R})$ is supported away from $\{x = 0\}$. 

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We shall use the previous consideration to prove the exponential rate of decay as \( x \to -\infty \); the other cases are derived through similar reasoning. Recall the partition function of the real line \( \chi^\pm(\cdot) \) given by (12) and define a smooth function \( \tilde{\chi}(\cdot) \) such that \( \tilde{\chi}(y) = 1 \) whenever \( |y| \leq 1 \), \( \tilde{\chi}(y) = 0 \) whenever \( |y| \geq 2 \). Let \( \tau_x, \tau_y \in \mathbb{R} \) to be chosen later. Define \( v^-(x,y) = \tilde{\chi}(y + \tau_y) \chi^-(x + \tau_x) \partial_x \Xi^{(c)}(x,y) \). Notice that \( \mathcal{L}_{\Xi}^{(x \to -\infty)}[\partial_x \Xi^{(c)}] = 0 \) on the support of \( \chi^-(\cdot + \tau_x) \) for all \( \tau_x \geq 0 \). Thus,

\[
\mathcal{L}_{\Xi}^{(x \to -\infty)}[v^-] = \mathcal{L}_{\Xi}^{(x \to -\infty)}[\partial_x \Xi^{(c)}] - \mathcal{L}_{\Xi}^{(x \to -\infty)}[v^-] = \tilde{\chi}(y + \tau_y) \chi^-(x + \tau_x) \partial_x \Xi^{(c)}\bigg|_{x=0} + 3 \left( \Xi^{(c)}(x,y) - \tanh^2\left( \frac{y}{\sqrt{2}} \right) \right) v^- + O \left( \sum_{0 \leq |\alpha| \leq 2} |\tilde{\chi}| ||\partial_x^{(\alpha)} \chi^-|| \partial_x^{(3-\alpha)} \Xi^{(c)} || + \sum_{1 \leq |\beta| \leq 2} |\tilde{\chi}| ||\partial_x^{(\beta-1)} \chi^-|| \partial_x^{(3-\beta)} \Xi^{(c)} || \right) + O \left( \sum_{0 \leq |\gamma| \leq 2} |\chi^-||\partial_y^{(\gamma)} \tilde{\chi}|| \partial_y^{(2-\gamma)} \partial_x \Xi^{(c)} || \right).
\]

Since \( \chi^-(\cdot) \chi^-(\cdot - 2) = \chi^- \), we can write \( v^-(x,y) = \chi^-(x + \tau_x - 2) v^- (x,y) \), which allow us to rearrange the previous expression as

\[
\mathcal{L}_{\Xi}^{(x \to -\infty)}[v^-] - 3 \left( \Xi^{(c)}(x,y) - \tanh^2\left( \frac{y}{\sqrt{2}} \right) \right) v^- (x + \tau_x - 2) = O \left( \sum_{0 \leq |\alpha| \leq 2} |\tilde{\chi}| ||\partial_x^{(\alpha)} \chi^-|| \partial_x^{(3-\alpha)} \Xi^{(c)} || \right) + O \left( \sum_{1 \leq |\beta| \leq 2} |\chi^-||\partial_x^{(\beta-1)} \chi^-|| \partial_x^{(3-\beta)} \Xi^{(c)} || \right) + O \left( \sum_{0 \leq |\gamma| \leq 2} |\chi^-||\partial_y^{(\gamma)} \tilde{\chi}|| \partial_y^{(2-\gamma)} \partial_x \Xi^{(c)} || \right) =: f^-.
\]

Now, fix \( \tau_x \geq 0 \) sufficiently large so that the mapping

\[
w \mapsto \mathcal{P}[w] := -3 \left( \Xi^{(c)}(x,y) - \tanh^2\left( \frac{y}{\sqrt{2}} \right) \right) \chi^- \chi^-(x + \tau_x - 2) \]

is a bounded operator from \( L^2(\mathbb{R}^2) \) to itself with the property that \( ||\mathcal{P}||_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \leq \epsilon \), which can be achieved thanks to the asymptotic properties of \( \Xi^{(c)}(\cdot, \cdot) \); the boundedness of the embedding \( H^2(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2) \) immediately implies that \( \mathcal{P} \) can be constructed in such a way that all the operators in (45) are invertible. Taking into account the discussion following (45), we conclude that the operator on the left hand side is invertible. We finally have

\[
\left( \mathcal{L}_{\Xi}^{(x \to -\infty)} + \mathcal{P} \right) [v^-] = f^-,
\]

where the right hand side is spatially localized. Furthermore, thanks to the remarks regarding to the \( W^{2,1}_{\text{loc}}(\mathbb{R}^2) \) norm of \( \Xi^{(c)} \) we see that the norm of \( f^- \) does not depend on the parameter \( \tau_y \), for the latter only shifts in the y-direction the box that contains the support of \( f^- \). It follows from the same reasoning as in Lem. 3.7 that \( e^{\delta \cdot x \cdot \tau_y} v^- \in \)
\( H^2(\mathbb{R}^2) \cap H^1_0(\mathbb{R}^2) \), with \( \delta > 0 \) independent of \( \tau_y \). Now, using the Sobolev embedding Lemma we obtain
\[
|e^{\delta x} v^-(x,y)| \lesssim \| e^{\delta x} v^-(x,y) \|_{H^2(\mathbb{R}^2)} \leq C, \quad \implies |v^-(x,y)| \leq C e^{-\delta |x|}. 
\]
By definition of \( v^-(\cdot, \cdot) \), we have \( |\chi(x + \tau_y) \chi^-(x + \tau_x) \partial_x \Xi_\infty^{(c)}(x,y)| \leq C e^{-\delta |x|} \), thus
\[
|\partial_x \Xi_\infty^{(c)}(x,y)| \leq C e^{-\delta |x|}, \quad \text{for} \quad y \in (\tau_y - 1, \tau_y + 1), \quad x \leq -\tau_x - 1. 
\]
As \( \tau_y \) can be chosen arbitrarily, we obtain the result uniformly on \( y \in \mathbb{R} \). The rest of the analysis is similar to that used in the proof of Lem. 3.7, for
\[
\left| \Xi_\kappa^{(c)}(x,y) - \tanh \left( \frac{y}{\sqrt{2}} \right) \right| \leq \left| \int_{-\infty}^{x} \partial_x \Xi_\kappa^{(c)}(s,y) \, ds \right| 
\leq \int_{-\infty}^{x} e^{\delta s} \, ds 
\leq e^{\delta x}, \quad \text{for} \quad x \leq -\tau_x - 1.
\]
\[ \square \]
**Remark 4** (Exponential decay for first order derivatives). It turns out the previous result can also be shown for first derivatives. For instance, there exists a \( C, \delta > 0 \) independent of \( x \) and \( y \) such that
\[
\lim_{y \to \pm \infty} \left| \partial_x^{\alpha_x} \partial_y^{\alpha_y} \left( \Xi_{\kappa}^{(c)}(x,y) - \theta^{(c)}(x) \right) \right| \leq C e^{-\delta |y|},
\]
whenever \( \alpha_x + \alpha_y = 1 \); the same also holds for the other limits in Lem. 4.3. The idea here is standard and relies on a bootstrap argument, cf. \[10, Lem. 5.3\]. As the function \( \left[ \Xi_{\kappa}^{(c)}(x,y) - \theta^{(c)}(x) \right] \) is uniformly bounded on the whole plane \( \mathbb{R}^2 \) and solves an elliptic PDE, standard elliptic estimates imply that \( \left[ \Xi_{\kappa}^{(c)}(x,y) - \theta^{(c)}(x) \right] \in W^{(2,p)}(\mathbb{R}^2) \) for all \( p < \infty \). Moreover,
\[
\| \Xi_{\kappa}^{(c)}(x,y) - \theta^{(c)}(x) \|_{W^{(2,p)}(B)} \lesssim \| \Xi_{\kappa}^{(c)}(x,y) - \theta^{(c)}(x) \|_{L^p(B)} 
\lesssim \| \Xi_{\kappa}^{(c)}(x,y) - \theta^{(c)}(x) \|_{L^p(\tilde{B}) \text{meas}(\tilde{B})},
\]
for any arbitrary bounded balls \( B \subset \subset \tilde{B} \), where \( \text{meas}(\cdot) \) denotes the Lebesgue measure. Using the continuous embedding \( W^{(2,p)}(\mathbb{R}^2) \hookrightarrow C^{(1,\alpha)}(\mathbb{R}^2), \alpha \in [0, 1) \) and the exponential decay of \( \Xi_{\kappa}^{(c)}(\cdot, \cdot) \) to its far-field, we get
\[
\| \Xi_{\kappa}^{(c)}(x,y) - \theta^{(c)}(x) \|_{C^{(1,\alpha)}(B)} \lesssim \| \Xi_{\kappa}^{(c)}(x,y) - \theta^{(c)}(x) \|_{W^{(2,p)}(B)} 
\lesssim \| \Xi_{\kappa}^{(c)}(x,y) - \theta^{(c)}(x) \|_{L^p(\tilde{B})},
\]
\[
\lesssim e^{-\delta |y|}.
\]
Similar reasoning implies that \( |\partial_x \theta^{(c)}(x)| \lesssim e^{-\eta |x|} \), for some \( \eta > 0 \).

**Lemma 4.4** (Uniqueness of the continuation in \( c; \mathcal{H}_\kappa \) problem, \( \pi < \kappa < \infty \)). The following properties hold:

(i) there exists an unique solution \( \Xi_{\kappa}^{(c)}(\cdot, \cdot) \) to the \( \mathcal{H}_\kappa \) problem;
(ii) the mapping \( c \mapsto \Xi_{\kappa}^{(c)}(\cdot) : [0, 2) \to L^\infty(\mathbb{R}^2; \mathbb{R}) \) is smooth.
The linearization of the equation (2) at $\Xi$ as before, we exploit the symmetries of the problem to reduce the analysis to the half-space $H$ of the form $u = \chi(x,y) = \chi(x)\cdot\chi(y)$ from $H$ to $L^2(\mathbb{R} \times [0,\infty))$. However, the result does not readily follow, because $\Xi$.

We circumvent this issue with a far-field core decomposition, which we now explain: for a fixed $d \in [0, 2)$, we look for a solution to the equation
$$\Delta_{x,y}u + d\partial_xu + \mu(x)u - u^3 = 0, \quad (46)$$
of the form $u(x,y;d) = w(x,y;d) + \Xi_\infty(x,y) + \chi^{-}(y) [\theta^{(d)}(x) - \theta^{(c)}(x)]$, where $w \in H^2(\mathbb{R} \times [0,\infty))$. Plugging this Ansatz into (46) we get
$$\mathcal{L}_\Xi[w] := \Delta_{x,y}w + c\partial_xw + \mu(x)w - 3(\Xi_\infty)^2w = \mathcal{N}(w,\Xi_\infty,\theta^{(d)}(x),\theta^{(c)}(x)).$$
where
$$\mathcal{N}(w,\Xi_\infty,\theta^{(d)}(x),\theta^{(c)}(x))
= (c - d)\partial_x \left[ \Xi_\infty - \chi^{-}(y)\theta^{(c)}(x) \right]
+ (c - d)\partial_x w + \mathcal{O}(w^2, w^3) - \partial_y^2 \chi^{-}(y) \left( \theta^{(d)}(x) - \theta^{(c)}(x) \right)
+ \left( \Xi_\infty + \chi^{-}(y) \left[ \theta^{(d)}(x) - \theta^{(c)}(x) \right] \right)^3
- \left( \Xi_\infty \right)^3 - \chi^{-}(y) \left[ \left( \theta^{(d)}(x) \right)^3 - \left( \theta^{(c)}(x) \right)^3 \right]. \quad (47)$$

Notice that whenever $d = c$ and $w(\cdot,\cdot;d)|_{d=c} = 0$, our Ansatz $u(\cdot,\cdot;d)|_{d=c}$ solves the $H_\infty$ problem.

We claim that the $\mathcal{N}(w,\Xi_\infty,\theta^{(d)}(x),\theta^{(c)}(x)) \in L^2(\mathbb{R} \times [0,\infty))$. Indeed, all the terms on the right hand side of (47) that depend on $w$ are clearly integrable, due to the embedding $H^2(\mathbb{R} \times [0,\infty)) \hookrightarrow L^\infty(\mathbb{R} \times [0,\infty))$. The terms on the last line of (47) are also in $L^2(\mathbb{R} \times [0,\infty))$, thanks to the exponential decay of $\Xi_\infty$ to its end states. The only terms to be concerned with are in $(c - d)\partial_x \left[ \Xi_\infty - \chi^{-}(y)\theta^{(c)}(x) \right]$, but we handle them with the help of Rem. 4: we initially note that
$$\partial_x \left( \Xi_\infty - \chi^{-}(y)\theta^{(c)}(x) \right) \lesssim e^{-\delta|x|};$$
a further application of the same Remark implies that
$$\partial_x \left( \Xi_\infty - \chi^{-}(y)\theta^{(c)}(x) \right) \lesssim e^{-\delta|x|}$$
(possibly for a smaller $\delta > 0$). In the end, an $L^2$-upper bound
$$\partial_x \left( \Xi_\infty(x,y) - \chi^{-}(y)\theta^{(c)}(x) \right) \lesssim e^{-\delta(x|+|y|)}$$
is obtained. Consequently, the nonlinearity (47) in $L^2(\mathbb{R} \times [0,\infty))$, and we can further assert that it is a smooth mapping from $H^2(\mathbb{R} \times [0,\infty)) \cap H^1_0(\mathbb{R} \times [0,\infty))$ to $L^2(\mathbb{R} \times [0,\infty))$ (in $w$). As the IFT provides a unique characterization of the local
solution around, we use the fact that \( \Xi^{(d)}_\infty \) also solves the equation (46) with the imposed asymptotic condition to assert the equivalence

\[
\Xi^{(d)}(x,y) = w(x,y;d) + \Xi^{(c)}_\infty(x,y) + \chi^{-}(y) \left[ \theta^{(d)}(x) - \theta^{(c)}(x) \right]
\]

for all \( |d-c| \) sufficiently small. Now, we rely on the results of Lem. 3.6 and Sobolev embeddings to conclude that \( \Xi^{(d)}(x,y) - \Xi^{(c)}_\infty \) is continuous in \( L^\infty(\mathbb{R}^2) \).

Proof. [of Theorem 1.2; case \( \kappa = \infty \)] Combine the results of Lemmas 4.2, 4.3, and 4.4.

5. Discussion. Among several possible directions of further investigation, we would like to mention the following:

**Metastability of patterns.** As addressed by the numerical studies of [4, §VI], defining the parameter regions of metastability for creation of patterns (either perpendicular or parallel to the quenching front) is a challenging and interesting direction of investigation. From a broader perspective, a numerical, if possible analytical, description of parameter curves on the boundary of different morphological states would be valuable in applications.

**Selection mechanisms.** What are the crucial mechanisms involved in the wavenumber selection in the wake of the front? How relevant are the nonlinearity and the speed of the quenching front in this selection? We refer to [11, §3.3] for a general discussion about wavenumber selection.

**Critical cases;** \( \mathcal{P}(c;\kappa) = 1 \) (\( \pi < \kappa < \infty \)). The behavior of the patterns in this critical scenario possibly requires a different approach, since one can see in the proofs of Theorem 1.1 and 1.2 that the speed of the quenching front has to be away from the critical case. The result would be interesting and add valuable knowledge in the classification of patterns obtained from directional quenching.

**Non-planar quenching fronts and oblique stripes.** Is it possible to control the contact angle of the \( H_\kappa \)-patterns? Although it was shown in [10] that oblique patterns do not exist in (2), these patterns can still exist in the case of the unbalanced equation (9). It is worth to mention that the result of [9] describes a family of solutions displaying one single (almost) horizontal interface whose contact angle with the quenching front can be varied by modification of the chemical potential parameters across the quenching interface. We refer to [14] for physics motivation and a more detailed discussion on the chirality of helicoidal patterns in the context of recurrent precipitation.

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