A group method solving many-body systems in intermediate statistical representation

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Abstract – The exact solution of the interacting many-body system is important and is difficult to solve. In this paper, we introduce a group method to solve the interacting many-body problem using the relation between the permutation group and the unitary group. We prove a group theorem first, then using the theorem, we represent the Hamiltonian of the interacting many-body system by the Casimir operators of unitary group. The eigenvalues of Casimir operators could give the exact values of energy and thus solve those problems exactly. This method maps the interacting many-body system onto an intermediate statistical representation. We give the relation between the conjugacy-class operator of permutation group and the Casimir operator of unitary group in the intermediate statistical representation, called the Gentile representation. Bose and Fermi cases are two limitations of the Gentile representation. We also discuss the representation space of symmetric and unitary group in the Gentile representation and give an example of the Heisenberg model to demonstrate this method. It is shown that this method is effective to solve interacting many-body problems.

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Introduction. – To give the exact solution of the interacting many-body problem is always difficult in physics. Few low-dimensional systems could be solved exactly, such as the one-dimensional Ising model \([1,2]\). Usually, some simplified methods or one and two transformations are used to get the approximate solution, for example the mean-field method which considers the far particle-particle interaction as a field effect, and the Jordan-Wigner transformation which maps the interaction of particles onto a fermion representation \([1,2]\). Intermediate statistics is a hot topic in physics these decades, e.g., in topological quantum computation \([3–7]\), quantum material \([8,9]\), and quantum information \([10]\). If the interacting many-body system is an intermediate statistical system, getting the approximate solution or even the low-dimensional exact solution becomes more difficult. Gentile statistics is a kind of intermediate statistics which is named after Gentile \([11]\). The maximum occupation number of Gentile statistics is a finite number \(n\) \([11–13]\). Gentile statistics needs two different kinds of creation and annihilation operators to realize the angular momentum algebra just like the Schwinger representation. If only one kind of creation and annihilation operators is introduced, like the Holstein-Primakoff transformation, the constraint of the occupation number must be considered. It is not a pure bosonic realization. In Gentile statistics, the states are labelled by the practical occupation number, which makes it easy to research. It can be used as an effective tool to solve the interacting many-body problems. Gentile statistics goes back to Bose and Fermi statistics in limit cases. The Gentile system is not just a theoretical model, more and more real systems are found obeying Gentile statistics, such as conjugate annulenes \([14]\) and spin waves of magnetic systems \([15–22]\).

We deal with the intermediate statistical system or normal many-body spin system in the Gentile representation. It is a Gentile realization. For instance, the many-body system is an anyon system. Anyon statistics could give the wave function an additional phase by braiding two different types of anyons \([23–25]\). This phase depends on the winding number and the statistical parameter. The system of anyons is always complicated. In this case, we can convert the problem of the winding number...
representation [26] of the anyon statistics. In the occupation number representation of Gentile statistics, the many-body problem is easier to solve in certain cases. At the same time, in this paper, we give a Gentile realization of the Heisenberg spin model. This model describes the long-range interaction.

The group method is useful to deal with many-body interacting systems [27,28]. In this paper, we construct the relation between the permutation group and the unitary group in the Gentile representation. According to the relation between the Casimir operator of the unitary group and the conjugacy-class operator of the permutation group in Gentile statistics presented in the paper, it is easy to solve the Hamiltonian of the many-body interaction system.

This paper is organized as follows. In the next section, we introduce the relation between permutation group and unitary group in the Gentile representation. In the third section, the representation space of permutation group and unitary group in the Gentile representation is discussed. In the fourth section, we solve the Heisenberg model in the Gentile representation as an example. Finally, in the last section, we conclude our result and make some discussions.

The relation between permutation group and unitary group in the Gentile representation. — Any finite group is isomorphic to a certain subgroup of the permutation group, any compact Lie group is isomorphic to a certain subgroup of the unitary group, and all infinite groups are isomorphic to the general linear groups. The relation between the permutation group and the unitary group is important and useful. The conjugacy-class operator consists of the generators of the permutation group and commutes with all group elements. Also, the Casimir operator consists of the generators of the unitary group and commutes with each group element [27,28]. In order to rewrite the Hamiltonian of an interacting many-body system, taking the Heisenberg model as an example, using the Casimir operator, we prove a relation between the permutation group and the unitary group. Using the theorem, we can solve exact solutions of the interacting many-body problem.

The conjugacy-class operator is the sum of group elements over a conjugate class [27,28]. \( \langle a \rangle = (a_1, \cdots, a_m) \) with \( a_1 \geq \cdots \geq a_m \geq 0 \) denotes an integer partition of integer \( a \) and \( a_1 \) is the element of this integer partition [29–31]. The length of \( \langle a \rangle \) is \( l_1(a) = m \). For example, \( \langle a \rangle = (4) \). The partitions for \( \langle a \rangle \) are \((4, 3, 1, 2), (2, 2), (2, 1^2), (1^4) \). Each integer partition of the permutation group gives a conjugacy-class operator [29–31]. The conjugacy-class operator \( P(2, 1^{\nu-2}) \) denotes the exchange of any two particles when the spin system includes \( \nu \) particles, and \( (2, 1^{\nu-2}) \) is the integer partition. For example the system consists of \( \nu \) particles \( \{1, 2, \cdots, i \cdots, j \cdots, \nu \} \). Only the \( i \)-th and the \( j \)-th particles are exchanged.

The operators which commute with all operators of a certain group are called the Casimir operators of this group [27]. The number of the Casimir operators equals to the rank of the group. The eigenvalues of the Casimir operators represent the irreducible representations [27,28]. The unitary group \( U(m) \) has \( m \) linear independent Casimir operators. The Casimir operator of order \( p \) reads [27,28]

\[
C_p = \sum_{k_1 \cdots k_p=1}^m E_{k_1 k_2 \cdots E_{k_p k_1}},
\]

where \( E_{k_1} \) is the generator of the unitary group \( U(m) \). The eigenvalue of the Casimir operator of order \( p \) can be expressed as

\[
S_p = \sum_{i=1}^m [(a_i + m - i)^p - (m - i)^p],
\]

where \( a_i \) are \( m \) non-negative integers that represent the irreducible representation of \( U(m) \) with \( a_1 \geq \cdots \geq a_m \geq 0 \) [27,29–31].

**Theorem:** The conjugacy-class operator \( P(2, 1^{\nu-2}) \) of the permutation group \( S_\nu \) and the Casimir operators \( C_2 \) and \( C_3 \) (order one and two) of the unitary group \( U(m) \) satisfy

\[
Re[e^{\pm i \pi/2} P(2, 1^{\nu-2})] + m \sum_{k=1}^{m} \sum_{i=1}^{\nu} J(N_k^{(i)}) = \frac{1}{2} C_2 - \frac{m}{2} C_1,
\]

where

\[
J(N_k^{(i)}) = -2 \csc^2 \left( \frac{\pi}{n+1} \right) \sin \left( \frac{\pi}{2(n+1)} \right) \times \sin \left( \frac{N_k^{(i)}}{n+1} \sin \left( \frac{2(N_k^{(i)} + n) \pi}{2(n+1)} \right) \right),
\]

\( n \) is the maximum occupation number of intermediate statistics, \( \nu \) is the practical particle number of one quantum state, and \( N_k^{(i)} \) is the particle number operator of the representation (or energy) \( k \) and particle (or position) \( i \). When \( n \to \infty \) (the Bose case) and \( n = 1 \) (the Fermi case), \( J(N_k^{(i)}) = -N_k^{(i)} \). In these two limitations, the relation becomes

\[
\pm P(2, 1^{\nu-2}) - m \sum_{k=1}^{m} \sum_{i=1}^{\nu} N_k^{(i)} = \frac{1}{2} C_2 - \frac{m}{2} C_1,
\]

where \( \pm \) correspond to bosons and fermions, respectively.

**Proof.** For the Gentile statistics, we use two sets of creation and annihilation operators \( a^\dagger, a, b^\dagger, b \) and \( a = b^\dagger \). This is because one set of creation and annihilation operators is not enough to realize Gentile statistics. Let \( a^{(i)} \) with superscript ranging from 1 to \( \nu \) and subscript ranging from 1 to \( m \) represent the creation of the \( i \)-th particle.
of state \(k\) (or creating a particle of energy \(k\) at position \(i\)). The relation of creation and annihilation operators follows the \(n\)-bracket [12,13]
\[
\left[ b^{(i)}_k, a^{(j)}_l \right]_n = b^{(i)}_k a^{(j)}_l - e^{i\pi/(n+1)} a^{(i)}_k b^{(j)}_l = \delta_{ij}\delta_{kl},
\]
where \(\delta_{ij}\) is the Kronecker symbol.

At the same time, we have the operator relations
\[
e^{i\pi/(n+1)} b^{(i)}_k a^{(j)}_l = a^{(j)}_l b^{(i)}_k,
\]
\[
e^{i\pi/(n+1)} a^{(i)}_k b^{(j)}_l = a^{(j)}_l b^{(i)}_k.
\]
The generator \(\tau_{ij}\) (exchanging \(i\)-th and \(j\)-th particles) of the permutation group \(S_N\) can be expressed in terms of creation and annihilation operators as
\[
\tau_{ij} = \sum_{k,l=1}^{m} (a^{(i)}_k b^{(j)}_l + e^{i\pi/(n+1)} a^{(i)}_k b^{(j)}_l).
\]

The form of \(\tau_{ij}\) is an obvious exchange of two particles (please see the next section). What calls for special attention is that the exchange of two particles happens both in the Hilbert space and its conjugate space. We will discuss in details in the next section. So the conjugacy-class operator \(P(2,1^{\nu-2})\) of permutation group \(S_N\) has the form
\[
P(2,1^{\nu-2}) = \sum_{i<j=1}^{\nu} \tau_{ij} = \sum_{i<j=1}^{\nu} \sum_{k,l=1}^{m} (a^{(i)}_k b^{(j)}_l + e^{i\pi/(n+1)} a^{(i)}_k b^{(j)}_l).
\]

We also construct the generator of the unitary group \(U(m)\) as
\[
E_{kl} = \sum_{i=1}^{\nu} (a^{(i)}_k b^{(i)}_l + b^{(i)}_k a^{(i)}_l).
\]

In this case, the generator commutation relation of the unitary group \(U(m)\) satisfies
\[
[E_{kl}, E_{pq}] = \delta_{lp} E_{kq} - \delta_{kq} E_{pl} + 2Re \sum_i \delta_{ip} a^{(i)}_k a^{(i)}_l f(N^{(i)}_k) a^{(i)}_q
- 2Re \sum_i \delta_{iq} a^{(i)}_p f(N^{(i)}_k) a^{(i)}_l,
\]
where
\[
f(N^{(i)}_k) = a^{(i)}_k a^{(i)}_k - a^{(i)}_k a^{(i)}_k
= \csc \frac{\pi}{n+1} \left( \cos \frac{\pi}{n+1} - 1 \right) \sin \frac{N^{(i)}_k \pi}{n+1} + \cos \frac{N^{(i)}_k \pi}{n+1}.
\]

Here we use \(a^{(i)}_k f(N^{(i)}_k) a^{(i)}_k = (b^{(i)}_k f(N^{(i)}_k) b^{(i)}_k)^*\). For bosons \(n \to \infty\) and fermions \(n = 1\), eq. (12) turns to a familiar one
\[
[E_{kl}, E_{pq}] = \delta_{lp} E_{kq} - \delta_{kq} E_{pl}.
\]

Equation (14) verifies that the construction equation (11) in the Gentile statistics represents the Lie algebra of the unitary group \(U(m)\).

The Casimir operator always relates to some invariances. This generator ensures that the Casimir operator of the unitary group \(U(m)\) is Hermitian. In this case, we have the Casimir operators of order one and two,
\[
C_2 = \sum_{k,l=1}^{m} E_{kl} E_{lk} = \sum_{k,l=1}^{m} \left[ \sum_{i=1}^{\nu} (a^{(i)}_k b^{(i)}_l + b^{(i)}_k a^{(i)}_l) \right] \times \left[ \sum_{j=1}^{\nu} (a^{(j)}_l b^{(j)}_k + b^{(j)}_k a^{(j)}_l) \right],
\]
\[
C_1 = \sum_{l=1}^{m} E_{li} = \sum_{l=1}^{m} \sum_{i=1}^{\nu} (a^{(i)}_l b^{(i)}_l + b^{(i)}_l a^{(i)}_l).
\]

The creation and annihilation operators in the Gentile statistics (intermediate statistics) satisfy
\[
\langle a^\dagger |\nu\rangle_n = \sqrt{\nu + 1} \langle\nu + 1\rangle_n,
\]
\[
\langle b^\dagger |\nu\rangle_n = \sqrt{(\nu + 1)\nu} \langle\nu + 1\rangle_n,
\]
\[
\langle b |\nu\rangle_n = \sqrt{\nu} \langle\nu - 1\rangle_n,
\]
\[
\langle a |\nu\rangle_n = \sqrt{\nu + 1} \langle\nu - 1\rangle_n,
\]
where
\[
\langle\nu\rangle_n = \frac{1 - e^{i\pi/\nu}}{1 - e^{i\pi}}
\]
and \(N\) is the particle number operator
\[
N |\nu\rangle_n = \nu |\nu\rangle_n.
\]

We also have
\[
[a^{(i)}_k, b^{(j)}_l] = [a^{(i)}_k, a^{(j)}_l] = f(N^{(i)}_k)\delta_{ij}\delta_{kl},
\]
and
\[
[a^{(i)}_k, b^{(i)}_k] = [a^{(i)}_k, b^{(i)}_k] = [a^{(i)}_k, b^{(i)}_k] = [b^{(i)}_k, b^{(i)}_k] = [b^{(i)}_k, b^{(i)}_k] = 0.
\]

According to eqs. (9)–(19), we prove eqs. (3) and (4).

**The representation space of permutation group and unitary group.** – To seek the exact solution, we discuss the representation space of the interacting many-body system. The structure of the group space is realized in the language of creation and annihilation operators. The representation space of the permutation group and the unitary group is usually expressed as the direct product of permutation and unitary group \(S_N \otimes U(m)\), which is called dual structure [27,28,32]. In the Gentile representation, we also have this result with a little difference.
Because according to eqs. (9) and (11), we have the relation of generators

\[ \tau_{ij}E_{st} = 0. \]  

(22)

**Proof.**

\[
\tau_{ij}E_{st} = \sum_{kl}(a^+_k a^+_l b^{(i)}_l b^{(j)}_k + e^{-i\epsilon_{ij}} a^+_k a^+_l b^{(j)}_l b^{(i)}_k)
\times \sum_r (a^{(r)}_r a^+_r + b^{(r)}_r b^+_r),
\]

(23)

\[
E_{st} \tau_{ij} = \sum_r (a^{(r)}_r b^+_r + b^{(r)}_r a^+_r)
\times \sum_{kl}(a^+_k a^+_l b^{(i)}_l b^{(j)}_k + e^{-i\epsilon_{ij}} a^+_k a^+_l b^{(j)}_l b^{(i)}_k).
\]

(24)

We explain the first term of eq. (23) \[a^+_k a^+_l b^{(i)}_l b^{(j)}_k \] as an example. When the operator acts on the many-body system, the system first annihilates the \(r\)-th particle in the state \(t\). And then the \(r\)-th particle in the state \(s\) is created. Then the system annihilates the \(i\)-th particle in the state \(l\) after the annihilation of the \(j\)-th particle in the state \(k\). After that the \(j\)-th particle in the state \(l\) and the \(i\)-th particle in the state \(k\) are created sequentially. The first term of eq. (24) describes the same process. So do other terms of these two equations. It can be summarized as

\[
\begin{pmatrix}
  i & j & r \\
  l & k & t \\
  k & l & s
\end{pmatrix}.
\]

(25)

The labels of the first line \(i, j, r\) denote the numbers (or the positions) of the particles. The second line shows the original states and the third line shows the new states of the particles. So as to the following three terms of eqs. (23) and (24). We have \(\begin{pmatrix} l & k & t^*\end{pmatrix} \cdot \begin{pmatrix} l & k^* & t \end{pmatrix} \) and \(\begin{pmatrix} l & k^* & t^*\end{pmatrix} \cdot \begin{pmatrix} l & k & t \end{pmatrix}\). The superscript \(*\) means the state in the conjugate space.

According to the basic assumption \(a = b^*\), the space of a Gentile many-body system is the direct product of the Hilbert space and its complex conjugate space. The space of the Gentile many-body system can be expressed as \(S_N \otimes S_N^* \otimes U(m) \otimes U^*(m)\).

In the permutation group space \(S_N \otimes S_N^*\), the generator in eq. (9) shows the exchanges in \(S_N\) and \(S_N^*\), respectively. These two exchanges read

\[
\begin{pmatrix} l & k \end{pmatrix} + \begin{pmatrix} l & k^* \end{pmatrix}.
\]

(26)

The irreducible representation of a single particle is written as \(\varepsilon_m \equiv \{1\}\) of the \(U(m)\) group. The creation of particle goes to

\[
a^+_k |0\rangle_m = |\varepsilon_m, k\rangle, \quad k = 1, \ldots, m,
\]

\[
b^+_k |0\rangle_m = |\varepsilon^*_m, k\rangle, \quad k = 1, \ldots, m.
\]

(27)

The many-body Gentile system can be expressed as an \(N\) times single space \(\varepsilon_m^N = \varepsilon_m \otimes \cdots \otimes \varepsilon_m\) of \(U_N(m) = U(m) \otimes \cdots \otimes U(m)\). Also we have

\[
a^+_1 \cdots a^+_N |0\rangle_m = |\varepsilon^N_m, k_1, \ldots, k_N\rangle, \quad 1 \leq k_1, \ldots, k_N \leq m.
\]

(28)

We label each irreducible representation by partition \(\lambda(S_N \otimes S_N^*) \otimes \lambda'(U(m) \otimes U^*(m))\), \(\langle \lambda \rangle = \langle \lambda_1 \cdots \lambda_m, \lambda_1^* \cdots \lambda_m^* \rangle\) and \(\lambda = \lambda^*\) in the space \(S_N \otimes S_N^* \otimes U(m) \otimes U^*(m)\), where \(\lambda_1 + \cdots + \lambda_m = \lambda_1^* + \cdots + \lambda_m^* = N\) and \(\lambda_1 \geq \cdots \geq \lambda_m \geq 0, \lambda_1^* \geq \cdots \geq \lambda_m^* \geq 0\). The space \(S_N \otimes S_N^* \otimes U(m) \otimes U^*(m)\) can be rewritten as

\[
|\varepsilon^N_m, k_1, \ldots, k_N\rangle = \sum |\varepsilon^N_m, \lambda_{i_{1^*}}, \lambda_{i_1'}, \lambda_{i_{2^*}}, \lambda_{i_2'}, \ldots, \varepsilon^N_m, \lambda_{i_{N^*}}, \lambda_{i_N'}\rangle, \]

(29)

where \(|\varepsilon^N_m, \lambda_{i_{1^*}}, \lambda_{i_1'}, \lambda_{i_{2^*}}, \lambda_{i_2'}, \ldots, \varepsilon^N_m, \lambda_{i_{N^*}}, \lambda_{i_N'}\rangle\) is called the Schur-Weyl basis [32], \(\lambda_{i_{1^*}}\) is the basis of the irreducible representation of \(S_N \otimes S_N^*\), and \(\lambda_{i_{1'}}\) is the basis of the irreducible representation of \(U(m) \otimes U^*(m)\). When \(\tau\) and \(E\) act on this state,

\[
\tau \times E |\varepsilon^N_m, \lambda_{i_{1^*}}, \lambda_{i_1'}, \lambda_{i_{2^*}}, \lambda_{i_2'}, \ldots, \varepsilon^N_m, \lambda_{i_{N^*}}, \lambda_{i_N'}\rangle = \lambda(\tau)^{i_{1^*}'} \cdot \lambda(E)^{i_{1^*}'} |\varepsilon^N_m, \lambda_{i_{1^*}'}, \lambda_{i^*_{1'}, \lambda_{i_{2^*}}, \lambda_{i_{2^*}'}, \ldots, \lambda_{i_{N^*}}, \lambda_{i_{N^*}'}\rangle, \]

(30)

where \(\lambda(\tau)^{i_{1^*}'}\) and \(\lambda(E)^{i_{1^*}'}\) are the elements of the irreducible representation matrix \(\lambda\).

**The Heisenberg model of many-body system in Gentile representation.** In this section, we give an example for demonstrating how to solve an interacting many-body system using the group method in the occupation number representation of Gentile statistics. The Heisenberg model is an useful model to describe the interaction between spins. It usually can be used to research the phase transition and the critical phenomenon of magnetic systems and strongly correlated electronic systems. The Hamiltonian is

\[
H = \sum_{ij} \frac{1}{2} \mathbf{S}_i \cdot \mathbf{S}_j,
\]

(31)

where \(\mathbf{S}\) is the spin and \(\sum_{ij}\) indicate the summation over all spin pairs. The model we discussed here is a long-range interaction. The interactions are the same between any pair of particles. There are two kinds of long-range interactions. One likes the Coulomb interaction, the longer the distance, the weaker the interaction is. The other likes the oscillator potential, the longer the distance, the stronger the interaction is. The interaction we discussed is the long-range interaction between the Coulomb interaction and the oscillator interaction.

It is easy to check that this Hamiltonian can be rewritten as (for example, a system only has two spins: \(H = 1/2\mathbf{S}_1 \cdot \mathbf{S}_2 + 1/2\mathbf{S}_2 \cdot \mathbf{S}_1\); this is a permutation of two pairs; for multipartite system, the two-particle interactions of the hamiltonian give the following representation)

\[
H = \sum_{ij} \tau_{ij} + \text{const} = P(2, 1^{n-2}) + \text{const},
\]

(32)

and the constant can be ignored.
According to the theorem in the second section, eq. (3) shows the relation between the Casimir operator of the unitary group and the conjugacy-class operator of the permutation group in the Gentile statistical representation. We substitute the Casimir operator of the unitary group for the conjugacy-class operator of the permutation group,

\[ H = \cos^{-1} \left( \frac{2m}{n+1} \right) \left( \frac{1}{2} C_2 - C_1 - 2 \sum_{k_1} J(N_k^{(i)}) \right). \]  

(33)

Here, we have \( m = 2 \). This relation tells us that the energy spectrum of the many-body intermediate statistical system is related to the maximum occupation number \( n \) and the particle number of the system. In the Gentile statistical representation, it is easy to calculate the energy spectrum of a many-body system. Reference [27] gives the eigenvalues of \( C_1 \) and \( C_2 \) as \( \langle C_1 \rangle = S_1 \) and \( \langle C_2 \rangle = S_2 - (m-1)S_1 \), where

\[ S_i = \sum_{i=1}^{m} [(a_i + m - i)^l - (m - i)^l], \]  

(34)

and \( a_i \) is the partition of the irreducible representation of \( U(m) \) [27–31].

When \( n \to \infty \) and \( n = 1 \), it returns the Bose system and Fermi system, respectively:

\[ H = \frac{1}{2} C_2 - C_1, \quad \text{Bose case}, \]  

(35)

\[ H = -(\frac{1}{2} C_2 - C_1), \quad \text{Fermi case}. \]

The exact value of energy spectrum and degeneracy are discussed in ref. [33].

**Discussion and conclusion.** – The exactly solvable models are very important in physics, but they are rare. Various approximation methods are introduced to solve the problems such as the mean-field approximation and the Jordan-Wigner transformation. A group is a useful framework to simulate the quantum computer using the intermediate statistical representation. Our method provides a new approach to simulate the quantum computer using the intermediate statistical system. At the same time, this method makes it possible to deal with the intermediate statistical system by quantum computer.

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