Local Unitary Invariants of Generic Multi-qubit States

Naihuan Jing,1,2 Shao-Ming Fei,3,4 Ming Li,5 Xianqing Li-Jost,4 and Tinggui Zhang6
1School of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China
2Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA
3School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
4Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany
5Department of Mathematics, China University of Petroleum, Qingdao, Shandong 266555, China
6School of Mathematics and Statistics, Hainan Normal University, Haikou, Hainan 571158, China

We present a complete set of local unitary invariants for generic multi-qubit systems, which gives necessary and sufficient conditions for two states being local unitary equivalent. These invariants are canonical polynomial functions in terms of the generalized Bloch representation of the quantum states. In particular, we prove that there are at most 12 polynomial local unitary invariants for two-qubit states and at most 90 polynomials for three-qubit states. Comparison with Makhlin’s 18 local unitary invariants is given for two-qubit systems.

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Local unitary equivalence is a foundational concept in quantum entanglement and quantum information, as it provides the key symmetry in classifying quantum entangled states of physical systems [1]. Two quantum states are of the same nature in implementing quantum information processing if they are equivalent under a local unitary (LU) transformation, and many crucial properties such as the degree of entanglement [2–5], maximal violations of Bell inequalities [6–11], and the teleportation fidelity [12, 13] remain invariant under LU transformations. Moreover, quantum entanglement in multipartite qubits has also figured prominently in many quantum information processing such as one-way quantum computing, quantum error correction and quantum secret sharing [2–8]. For this reason, it has been a key problem to find a complete and operational procedure to distinguish two quantum states under LU transformations.

In [14], Makhlin presented a complete set of 18 polynomial LU invariants for classifying two-qubit states. There are numerous results on LU invariants for three qubits states [15], some general mixed states [16–19, 24], tripartite pure and mixed states [20]. A theoretical method to reduce the problem to pure n-qubit states was proposed in [21], and later generalized to arbitrary dimensions in [22]. From a different viewpoint, [23] gave a procedure to find the LU operator for multi-qubits using the core tensor method. Very recently a method to judge LU equivalence for multi-qubits [20] was also proposed and more generally SLOCC invariants for multi-particle states are found [27]. Nevertheless, it remains a wild problem to find a complete set of invariants to answer the LU question except for two qubit cases. Even for two partite cases it is also desirable to find an alternative set of invariants to judge LU equivalence, as the original Makhlin invariants contain some nontrivial tensor vectors.

In this article, we propose a brand new method to quantify polynomial LU invariants for multi-qubit systems in an operational way. For the special case of two-qubit systems, our method is more efficient and needs fewer invariants than that in [14] in general. In fact, we show that many invariants in [14] are consequences of other invariants, and there are at most 12 invariants to determine the LU equivalence for two-qubit states. We prove for the first time that there are at most 90 invariants for generic mixed 3-qubit states. We also propose an operational method to derive a list of polynomial invariants for generic multi-qubit states. We remark that the invariants can not be derived from [23] as the latter aimed to compute the LU operator for two equivalent multi-qubits, while our current work takes a different strategy to seek a complete set of polynomial invariants.

We start our discussion to express an N-qubit state \( \rho \) in terms of Pauli matrices \( \sigma_\alpha \), \( \alpha = 1, 2, 3, \)

\[
\rho = \frac{1}{2^N} I^\otimes N + \sum_{j_1=1}^{N} \sum_{\sigma_1}^{3} \sum_{j_2=1}^{N} \sum_{\sigma_2}^{3} \cdots \sum_{j_M=1}^{N} \sum_{\sigma_M}^{3} T_{j_1j_2\cdots j_M}^{\alpha_1\alpha_2\cdots \alpha_M} \sigma_{\alpha_1} j_1 \sigma_{\alpha_2} j_2 \cdots \sigma_{\alpha_M} j_M \tag{1}
\]

where \( I \) is the \( 2 \times 2 \) identity matrix, \( \sigma_{\alpha_k} \otimes I \otimes \cdots \otimes I \) with \( \sigma_{\alpha_k} \) at the \( j_k \)-th position and

\[
T_{j_1j_2\cdots j_M}^{\alpha_1\alpha_2\cdots \alpha_M} = \frac{1}{2^N} \text{Tr}[\rho \sigma_{\alpha_1} j_1 \sigma_{\alpha_2} j_2 \cdots \sigma_{\alpha_M} j_M] , \quad M \leq N , \tag{2}
\]

are real coefficients. In particular, \( T_j = (T_1^j, T_2^j, T_3^j) \), \( j = 1, \ldots, N \), are three dimensional vectors, \( T_{jk}^j = (T_1^j \sigma_{\alpha_2}), \)

\( 1 \leq j < k \leq N \), are \( 3 \times 3 \) matrices. Generally, \( T_{j_1j_2\cdots j_M} = (T_{j_1j_2\cdots j_M}^{\alpha_1\alpha_2\cdots \alpha_M}) \) are tensors.

*Electronic address: jing@math.ncsu.edu
Let $\rho$ and $\rho'$ be two $N$-qubit mixed states. They are called local unitary equivalent if

$$\rho' = (U_1 \otimes \ldots \otimes U_N)\rho(U_1 \otimes \ldots \otimes U_N)^\dagger$$  \hspace{1cm} (3)

for some unitary operators $U_i \in SU(2)$, $i = 1, 2, \ldots, N$, where $\dagger$ denotes transpose and conjugate.

**Lemma 1** Two mixed states $\rho$ and $\rho'$ are local unitary equivalent if and only if there are special orthogonal matrices $O_1, \ldots, O_N \in SO(3)$ such that

$$(O_j \otimes \ldots \otimes O_k)T_{j_1\ldots j_k} = T_{i_1\ldots i_k}$$  \hspace{1cm} (4)

for any $1 \leq j_1 < \cdots < j_k \leq N$, $k = 1, 2, \ldots, N$.

**Proof.** The group $SU(2)$ acts on the real vector space spanned by $\sigma_i$, $i = 1, 2, 3$ via $27$:

$$U_i\sigma_k U_i^\dagger = \sum_{j=1}^3 O_ki\sigma_1,$$  \hspace{1cm} (5)

where $O = (O_{ki})$ belongs to $SO(3)$. From $(1, 3, 4)$ and $5$, one gets the tensor relation $4$. Note that this action realizes the well-known double-covering map $SU(2) \to SO(3)$. The sufficiency then follows from the fact that $SU(2)$ is the universal double covering of $SO(3)$.

**Two-qubit states:** To derive explicitly the invariants under the transformation $(3)$, we first consider the two-qubit case. From $(1)$ a two-qubit state is given by the 3-dimensional real column vectors $T_1, T_2$, and the real $3 \times 3$-matrix $T_{12}$. Two states $\rho$ and $\rho'$ are local unitary equivalent if and if there are $SO(3)$ operators $O_1$ and $O_2$ such that

$$T'_1 = O_1T_1, \quad T'_2 = O_2T_2, \quad T'_{12} = (O_1 \otimes O_2)T_{12} = O_1T_{12}O_2^\dagger,$$  \hspace{1cm} (6)

where $t$ denotes the transpose of a matrix.

We introduce the following sets of 3-dimensional real column vectors:

$$\langle O_1 \rangle = \{T_1, T_1T_2, T_2T_1, T_2T_1T_2, T_1^3, \ldots \} \subset \mathbb{R}^3$$

$$\langle O_2 \rangle = \{T_2, T_2T_1, T_1T_2, T_1T_2T_1, T_2^3, \ldots \} \subset \mathbb{R}^3,$$

which are respectively generated by the $(T_{12}T_{12})$-orbit of $\{T_1, T_1T_2\}$ and the $(T_{12}T_{12})$-orbit of $\{T_2, T_2T_1\}$. Here $(g)$ denotes the cyclic group generated by $g$. By the Cayley-Hamilton theorem the minimal polynomials of $T_{12}T_{12}$ and $T_{12}T_{12}$ have degree $\leq 3$, therefore it is enough to use elements in the orbits up to the quadratic powers. It is straightforward to verify that all the vectors in $\langle O_1 \rangle$ are transformed to $O_1\langle O_1 \rangle$ under the transformation $4$, while all the vectors in $\langle O_2 \rangle$ are transformed into $O_2\langle O_2 \rangle$.

Moreover, there are at most three linear independent vectors in $\langle O_1 \rangle$, $dim\langle O_1 \rangle \leq 3$, $i = 1, 2$. We say that a two-qubit state is generic if $dim\langle O_1 \rangle = dim\langle O_2 \rangle = 3$. For simplicity, we only deal with generic cases in the following. The non-generic (degenerate) cases can be studied in details too, see remarks after the proof of Theorem 1.

Let $\{\mu_1, \ldots, \mu_6\}$ and $\{\nu_1, \ldots, \nu_6\}$ denote the first six (spanning) vectors in $\langle O_1 \rangle$ and $\langle O_2 \rangle$, respectively. We first give a general result using all the spanning vectors.

**Theorem 1** Two generic two-qubit states are local unitary equivalent if and only if they have the same values of the following invariant polynomials:

$$\langle O_{\mu_i} \rangle, \quad \langle O_{\nu_j} \rangle, \quad i = 1, 2, \ldots, 6.$$

$$tr(T_{12}T_{12})^\alpha, \quad \alpha = 1, 2, 3.$$  \hspace{1cm} (7)

**Proof.** By using the relations in $(5)$, it is direct to verify that the quantities given in $(7)$ are invariants under local unitary transformations.

For generic states, the matrix $T_{12}T_{12}$ is nonsingular, so is $T_{12}T_{12}$ by trace property. Thus $T_{12}$ and $T_{12}$ are also nonsingular. We notice that $\langle O_1 \rangle$ and $\langle O_2 \rangle$ have the same values of the invariant polynomials $(7)$, namely, the inner products of any two vectors in $\langle O_1 \rangle$ are invariant under $\rho \to \rho'$, one has that there must exist an orthogonal matrix $O_i$ such that

$$O_i\langle O_1 \rangle = \langle O_i' \rangle.$$

In particular $O_iT_i = T'_i$. Then we can build the following commutative diagram:

$$\begin{array}{ccc}
\langle O_1 \rangle & \xrightarrow{O_i} & \langle O_i' \rangle \\
\downarrow T_{12} & & \downarrow T'_{12} \\
\langle O_2 \rangle & \xrightarrow{O_2} & \langle O'_2 \rangle
\end{array}$$

Consequently $T'_{12}O_1 = O_2T'_{12}$ in $End(\mathbb{R}^3)$, or $T'_{12} = O_1T_{12}O_2^\dagger$. Therefore, $\rho$ and $\rho'$ are local unitary equivalent.

**Remark.** In the above discussions we are only concerned with the generic case. For degenerate cases, one needs to analyze case by case. For instance, let us consider the case $T_1 = T_2 = 0$. In this case, $dim\langle O_1 \rangle = 0$, $i = 1, 2$. The only invariants left are $tr(T_{12}T_{12})^\alpha$, $\alpha = 1, 2, 3$. Note that

$$p_\alpha = tr(T_{12}T_{12})^\alpha = \sum_{i=1}^3 \lambda_i^\alpha$$  \hspace{1cm} (8)

is the $\alpha$-th power sum of the eigenvalues of $T_{12}T_{12}$. A well-known result of symmetric polynomials implies that any $p_\alpha$ ($\alpha \geq 4$) is an algebraic function of $p_1, p_2, p_3$.

For example, $p_4 = \frac{7}{3}p_3 - p_2^2 + \frac{8}{3}p_1p_3$. Hence $tr(T_{12}T_{12})^\alpha$ are invariants for any $\alpha \geq 1$. By $(28)$ if two states $\rho$ and $\rho'$ have the same values of $tr(T_{12}T_{12})^\alpha$, there exists a unitary matrix $U$ such that $T'_{12}T'_{12} = UT_{12}T_{12}U^\dagger$, which means that $T_{12}T_{12}$ and $T'_{12}T'_{12}$ have...
identical eigenvalues. Both $T_{12}^T$ and $T_{12}^{T_2}$ are similar to $\text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Then there exists an $O_1 \in \text{SO}(3)$ such that $T_{12}^T O_1 = T_{12}^{T_2} O_1$. Similarly there exists $O_2$ such that $T_{12}^T O_2 = T_{12}^{T_2} O_2$. Subsequently $T_{12}^T = O_1 T_{12} O_2^T$ for some $O_1$ and $O_2$, so $\rho$ and $\rho'$ are local unitary equivalent.

We now sharpen the result of Theorem 1. Since there are at most three linearly independent 3-dimensional vectors of $\mu_i$ and $\nu_i$ in (7) respectively, one can apply Theorem 1 to the basis vectors. The standard Gaussian elimination on the matrix $[\mu_1, \cdots, \mu_6]$ can pare down the column vectors into a basis $\{\mu_i, \mu_{ij}, \mu_{ijk}\}$ of $\langle O_1 \rangle$, where $\{i_1, i_2, i_3\} \subset \{1, 2, \cdots, 6\}$. This means that the number of independent invariants that are used to judge the local unitary equivalence of two generic two-qubit states is at most 15 in general (instead of 33 as is Theorem 1). In fact, further analysis can reduce the number to at most 12 polynomial invariants.

**Theorem 2** Two generic two-qubit states are local unitary equivalent if and only if they have the same values for the following 12 invariants:

\[
\langle T_1, (T_{12} T_{12}^T)^{\beta} T_1 \rangle, \quad \langle T_2, (T_{12} T_{12}^T)^{\beta} T_2 \rangle, \quad \langle T_1, (T_{12} T_{12}^T)^{\beta} T_2 T_2 \rangle, \quad \beta = 0, 1, 2, \quad \text{tr}(T_{12} T_{12}^T)^{\alpha}, \quad \alpha = 1, 2, 3. \tag{9}
\]

**Proof.** The set $\langle O_1 \rangle$ is a union of two orbits $(T_{12} T_{12}^T) \cdot T_1$ and $(T_{12} T_{12}^T) \cdot T_2$. The independent inner products given in Theorem 1 are $\langle T_1, (T_{12} T_{12}^T)^{\beta} T_1 \rangle$, $\langle T_1, (T_{12} T_{12}^T)^{\beta} T_1 T_2 \rangle$ for $\beta = 0, 1, 2, 3$, due to the Cayley-Hamilton theorem and the fact that $\langle T_{12} u, v \rangle = \langle u, T_{12}^T v \rangle$ for any vectors $u, v$ ($T_{12}$ is a real matrix). Similarly the orbit $\langle O_2 \rangle$ will only contribute the remaining independent inner products $\langle T_2, (T_{12} T_{12}^T)^{\beta} T_2 \rangle$, $\beta = 0, 1, 2, 3$.

We claim that the 3 invariants with $\beta = 3$ are not needed if the traces (11) are known. The Cayley-Hamilton theorem says that

\[
(T_{12} T_{12}^T)^{\beta} = e_1 (T_{12} T_{12}^T)^{\beta - 1} - e_2 (T_{12} T_{12}^T)^{\beta - 2} + e_3 I, \tag{12}
\]

where $e_i$ are the elementary symmetric polynomials in the eigenvalues $\lambda_i$. By the fundamental theorem of symmetric polynomials, the $e_i$ can be expressed as classical polynomials in the traces $p_\alpha$, i.e. $e_i$ are classical invariant polynomials of the density matrix:

\[
e_1 = p_1, \quad e_2 = \frac{1}{2}(p_1^2 - p_2), \tag{13}
\]

\[
e_3 = \frac{1}{6}(p_1^3 - 3p_2 p_1 + 2p_3). \tag{14}
\]

Plugging (12) into the three invariants $(T_1, (T_{12} T_{12}^T)^{\beta} T_1)$ etc., we see that they are given by linear combinations of the invariants (9) with fixed coefficients of the classical invariant polynomials (13, 14) of the density matrix, so they are redundant.

As we commented above if we use the Gaussian elimination we also worry about just 12 invariants, i.e., if we add $\beta = 3$ in the first set of invariants (9) for the 3 basis vectors we can waive the trace identities. Hence the number of invariants is at most 12 either way. We still include the trace identities (11) for the sake of general (non-generic) cases.

**Multi-qubit case:** To simplify presentation, we introduce the following notation: $\langle T_{ij} \rangle = T_{ij}^T$. We say that a word of $T_i$, $T_{ij}$ is admissible if the adjacent subindices match. For example, $T_{12} T_2, T_{12} T_{21} T_1 T_{12}$ are admissible ones.

We first consider the three-qubit case to present our general results. In this case, corresponding to (1), a quantum state has the form:

\[
\rho = \frac{1}{8} I + \sum_{i=1}^{3} T_1 \sigma^{(i)} + \sum_{i<j}^{3} T_{ij} \sigma^{(i)} \sigma^{(j)} + T_{23} \sigma^{(1)} \sigma^{(2)} \sigma^{(3)}. \tag{15}
\]

If two states $\rho$ and $\rho'$ are local unitary equivalent, then there are orthogonal matrices $O_i \in \text{SO}(3)$ such that

\[
T_{ij}' = O_1 T_i, \quad T_{ij}' = O_2 T_2, \quad T_{ij}' = O_3 T_3. \tag{16}
\]

\[
T_{12}' = O_1 T_{12} O_2', \quad T_{13}' = O_1 T_{12} O_3', \quad T_{23}' = O_2 T_{23} O_3'. \tag{17}
\]

\[
T_{123}' = (O_1 \otimes O_2 \otimes O_3) T_{123}. \tag{18}
\]

It is known (10) that the last relation (18) is equivalent to either of the following two relations:

\[
T_{12} = O_1 T_{123} (O_2 \otimes O_3)^T, \quad T_{12}' = (O_1 \otimes O_2) T_{123} O_3^T. \tag{19}
\]

Here $T_{123}$ is understood as the bipartition $T_{123} (\text{resp. } T_{123})$ in the first (resp. 2nd) equation of (19). To state our results we introduce two subsets of vectors:

\[
\langle O_1 \rangle_{1|23} = \{T_1, T_{123} (T_{23} T_{23})^{\beta} T_{23}, T_{123} T_{12} T_{12} T_{23} T_{123} T_{23} T_{123} T_{123} \} \subset \mathbb{R}^3
\]

\[
\langle O_2 \otimes O_3 \rangle_{1|23} = \{T_{23}, T_{123} T_{123} (T_{23} T_{23})^{\beta} T_{23}, T_{123} T_{123} T_{123} T_{123} T_{123} T_{123} \} \subset \mathbb{R}^9 \simeq \mathbb{R}^3 \otimes \mathbb{R}^3.
\]
where \( \beta = 0, \ldots, 3 \), which are respectively the 
\((T_{123}T_{123}^{\dagger})\)-orbit of \( (T_1, T_{123}(T_1T_{23}T_{12}T_{23}^{\dagger})T_2, \beta = 0, 1, 2, 3) \) and 
the \((T_{123}T_{123}^{\dagger})\)-orbit of \((T_3T_{123}T_{123}^{\dagger})T_1, T_{123}T_2, T_3, T_1, \beta = 0, 1, 2, 3 \). Here \( T_{23} \) is taken as its (column) vector re-alignment in \( \mathbb{R}^9 \) and \( T_{123} \) is folded as a \( 3 \times 9 \)-matrix, by viewing \( T_{123} \) as the bipartition \( 123 \) and \( T_{123}^{\dagger} \) is the transpose with respect to such partition. As before we also use the same symbols for the corresponding real subspaces.

Similarly, by permuting the indices we define \( (O_2) := (O_2)_{23} \mid \beta \rangle = (O_1)_{3} \mid \beta \rangle = (O_3)_{12} \mid \beta \rangle \) to be the \((T_{23}T_{23}^{\dagger})\)-orbit of \((T_2, T_{231}T_{123}T_{12}T_{23}^{\dagger})T_2, T_{231}T_{123}, T_2, T_1, \beta = 0, 1, 2, 3 \) respectively. Here the \( 3 \times 9 \)-matrix \( T_{231} \) is the realignment of \( T_{123} \) with respect to the partition of \( \{123\} \) into \( \{23\} \) (resp. \( \{3\} \)). Let \( O_1 = \{ \mu_1, \mu_2, \mu_3 \}, O_2 = \{ \nu_1, \nu_2, \nu_3 \} \) and \( O_3 = \{ \lambda_1, \lambda_2, \lambda_3 \} \);
\[ O_2 \otimes O_3 \mid 123 \rangle = \{ \alpha_1, \alpha_2, \ldots, \alpha_9 \}, O_3 \otimes O_1 \mid 312 \rangle = \{ \beta_1, \beta_2, \ldots, \beta_9 \}, \text{and } O_1 \otimes O_2 \otimes O_3 \mid 123 \rangle = \{ \gamma_1, \gamma_2, \ldots, \gamma_9 \} \).

**Theorem 3** A three-qubit state \( \rho \) is local unitary equivalent to a three-qubit state \( \rho' \) if and only if the respective invariant polynomials are equal:
\[
\begin{align*}
\langle \mu_1, \mu_2 \rangle & = \langle \mu_1', \mu_2' \rangle, \\
\langle \lambda_1, \lambda_2 \rangle & = \langle \lambda_1', \lambda_2' \rangle, \\
\langle \alpha_k, \alpha_l \rangle & = \langle \alpha_{k'}, \alpha_{l'} \rangle, \quad 1 \leq k \leq l, \quad (20)
\end{align*}
\]

**Proof.** By the result of two-qubit case, the invariance of inner products of vectors in \( O_1 \) implies the existence of orthogonal matrices \( O_i, i = 1, 2, 3 \) such that Eqs. **L7** hold. Thus we are left to show that the orthogonal matrices \( O_i \) also satisfy Eq. **L8** or equivalently Eq. **L9**.

We use a similar method of Theorem 1 to show this by viewing the three-state \( \rho \) as a bi-partite one on \( \mathbb{C}^3 \otimes \mathbb{C}^3 \) and partition the hyper-matrix \( T_{123} \) as a rectangular matrix \( T_{123} \). Then the \( 3 \times 9 \)-matrix \( T_{123} \) maps the subset \( O_2 \otimes O_3 \mid 123 \rangle \) into the subset \( O_1 \mid 123 \rangle \) by left multiplication.

We have already seen that there exists an orthogonal matrix \( O_i \) such that
\[ O_i (O_i) = (O_i') \]
and Eqs. **L7** hold. Then we can directly verify that the following diagram is commutative:
\[
\begin{array}{ccc}
\langle O_2 \otimes O_3 \rangle_{O_2} & \xrightarrow{O_2 \otimes O_3} & \langle O_2' \otimes O_3' \rangle \\
\downarrow T_{123} & & \downarrow T_{123}' \\
\langle O_1 \rangle_{123} & \xrightarrow{O_1} & \langle O_1' \rangle_{123}
\end{array}
\]

Consequently, \( T_{123}'(O_2 \otimes O_3) = O_1 T_{123} \) in \( \text{End}(\mathbb{R}^3 \otimes \mathbb{R}^3) \), or \( T_{123}' = O_1 T_{123} (O_2 \otimes O_3)' \).

The following result shows that there are at most 90 invariants to judge LU equivalence for two-three qubits.

**Theorem 4** Two generic three-qubit states are local unitary equivalent if and only if they have the same values of the following invariants:
\[
\begin{align*}
&\langle T_1, (T_{12}T_{12}^{\dagger})\rangle, \quad \langle T_2, (T_{23}T_{23}^{\dagger})\rangle, \\
&\langle T_3, (T_{31}T_{31}^{\dagger})\rangle, \\
&\langle T_4, (T_{42}T_{42}^{\dagger})\rangle, \text{ and } \langle T_{123}, (T_{123}T_{123}^{\dagger})\rangle,
\end{align*}
\]

where \( \alpha = 0, 1, 2; \beta = 1, 2, 3 \) and \( k = 0, 1, \ldots, 8; l = 1, \ldots, 9 \).

The above criteria can be generalized to multi-qubits. Define \( O_i = \langle T_{i1}, T_{i2}, \ldots, T_{in}, \ldots \rangle \subset \mathbb{R}^{3^n} \) as the \( (T_{in}, T_{in}^{\dagger}) \)-orbit, where \( i = 1 \cdots n \) means the index \( i \) is absent. In general for any strict sequence \( i = (i_1 \cdots i_k) \) (i.e. distinct \( i_j \)'s), we define the \( (T_{in}, T_{in}^{\dagger}) \)-orbit \( \langle O_i \rangle = \langle T_{i1}, \ldots \rangle \), where the admissible generating words have only \( i \) when crossing out redundant strings. e.g., \( T_{312}T_{12}T_1 = \text{word of indices } 3,1 \text{ when crossing out } 12 \).

Then we have the following result. Let \( O_i = \{ \mu_1, \mu_2, \ldots, \mu_m \}, O_2 = \{ \nu_1, \nu_2, \ldots, \nu_n \}, \ldots \), \( O_N = \{ \lambda_1, \lambda_2, \ldots \} \), and more generally, for any strict sequence \( i \), let \( O_i = \{ \tau_1, \ldots, \tau_n \} \), where \( n = n(i) \). Let's list these \( O_i \) as \( O_i = \{ \tau_1^{(j)}, \ldots, \tau_n^{(j)} \}, j = 1, \ldots, M \).

**Theorem 5** Two generic multi-qubits states \( \rho \) and \( \rho' \) are local unitary equivalent if and only if the respective invariant polynomials are equal:
\[
\begin{align*}
&\langle \tau_1^{(1)}, \tau_2^{(1)} \rangle = \langle \tau_1^{(1)'}, \tau_2^{(1)'} \rangle, \\
&\langle \tau_1^{(i)}, \tau_2^{(i)} \rangle = \langle \tau_1^{(i)'}, \tau_2^{(i)'} \rangle, \quad (22)
\end{align*}
\]

where each pair of indices \( (i, j) \) are such that \( 1 \leq i, j \leq m(i) \) for the sequences \( i_1, \ldots, i_M \).

**Proof.** We use induction on \( n \) to reduce the problem to \( (n-1) \)-partite qubits. Note that for any sequence \( i \) of indices for \( n \)-partite state, we can view the elements in \( O_i \) as \( O_i \otimes O_j \) where \( i' \) is obtained by realignment of the Block matrix with respect to the index \( j \), and \( i' \) is obtained from \( i \) after the realignment. Then we can use the similar commutative diagram
\[
\begin{array}{ccc}
\langle O_2 \otimes O_3 \cdots O_n \rangle_{O_2} & \xrightarrow{O_2 \otimes O_3 \cdots O_n} & \langle O_2' \otimes O_3' \cdots O_n' \rangle \\
\downarrow T_{12\cdots n} & & \downarrow T_{12\cdots n}' \\
\langle O_1 \rangle_{12\cdots n} & \xrightarrow{O_1} & \langle O_1' \rangle_{12\cdots n}
\end{array}
\]
to get \( T^T_{12\cdots n}(O_2 \otimes O_3 \cdots)_n = O_1 T_{12\cdots n} \) in \( \text{End}(R^3 \otimes R^{3(n-2)}) \), or \( T^T_{12\cdots n} = O_1 T_{12\cdots n}(O_2 \otimes O_3 \cdots)^T \). Here \( O_3 \cdots_n \) is an orthogonal matrix in the bigger orthogonal group. Then we use the induction to argue further for the matrix \( T_{12\cdots n} \), viewed as a reduced matrix for \((n-1)\)-partite state to get the final result.

Conclusions and Remarks: It is a basic and fundamental question to classify quantum states under local unitary operations. The problem has been figured out in \[21, 22\] for pure multipartite quantum states. However, it is much more difficult to classify mixed quantum states under LU transformations. Operational methods have been presented only for non-degenerate bipartite states. Although the authors in \[25\] have shown that the problem of mixed states can be reduced to one of pure states in terms of the purification of mixed states mathematically, the protocol is far from being operational. We have provided an operational way to verify and classify quantum states by using the generalized Bloch representation in terms of the generators of \( SU(2) \). We remark that \[23\] gives a practical procedure to compute the LU operator for two equivalent multi-qubits, but it cannot derive the polynomial invariants from the procedure, as it is based on a different strategy. In our current approach we set our goal to write down a set of simple invariants with which two states can be easily checked if they are LU equivalent. Since the coefficients (tensors) in the representation can be determined directly by measuring some local quantum mechanical observables-Pauli operators, the method is experimentally feasible. Our criterion is both sufficient and necessary for generic multi-qubit quantum systems, thus gives rise to a complete classification of multi-qubit generic quantum states under LU transformations.

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