Multipliers for Hardy spaces of Dirichlet series

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Abstract

We characterize the space of multipliers from the Hardy space of Dirichlet series $H_p$ into $H_q$ for every $1 \leq p, q \leq \infty$. For a fixed Dirichlet series, we also investigate some structural properties of its associated multiplication operator. In particular, we study the norm, the essential norm, and the spectrum for an operator of this kind. We exploit the existing natural identification of spaces of Dirichlet series with spaces of holomorphic functions in infinitely many variables and apply several methods from complex and harmonic analysis to obtain our results. As a byproduct we get analogous statements on such Hardy spaces of holomorphic functions.

1 Introduction

A Dirichlet series is a formal expression of the type $D = \sum a_n n^{-s}$ with $(a_n)$ complex values and $s$ a complex variable. These are one of the basic tools of analytic number theory (see e.g., [3, 24]) but, over the last two decades, as a result of the work initiated in [14] and [16], they have been analyzed with techniques coming from harmonic and functional analysis (see e.g. [19] or [9] and the references therein). One of the key point in this analytic insight on Dirichlet series is the deep connection with power series in infinitely many variables. We will use this fruitful perspective to study multipliers for Hardy spaces of Dirichlet series. We begin by recalling some standard definitions of these spaces.

The natural regions of convergence of Dirichlet series are half-planes, and there they define holomorphic functions. To settle some notation, we consider the set $\mathbb{C}_\sigma = \{ s \in \mathbb{C} : \text{Re} \, s > \sigma \}$, for $\sigma \in \mathbb{R}$. With this, Queffélec [18] defined the space $H_\infty$ as that consisting of Dirichlet series that define a bounded, holomorphic function on the half-plane $\mathbb{C}_0$. Endowed with the norm $\|D\|_\infty := \sup_{s \in \mathbb{C}_0} |\sum \frac{a_n}{n^s}| < \infty$ it becomes a Banach space, which together with the product $(\sum a_n n^{-s}) \cdot (\sum b_n b^{-s}) = \sum_{n=1}^\infty (\sum_{k,j=n} a_k \cdot b_j) n^{-s}$ results a Banach algebra.

The Hardy spaces of Dirichlet series $H_p$ were introduced by Hedenmalm, Lindqvist and Seip [14] for $p = 2$, and by Bayart [5] for the remaining cases in the range $1 \leq p < \infty$.

*Supported by CONICET-PIP 11220200102336
†Supported by PICT 2018-4250.
‡Supported by MINECO and FEDER Project MTM2017-83262-C2-1-P and by GV Project AICO/2021/170

Keywords: Multipliers, Spaces of Dirichlet series, Hardy spaces, Infinite dimensional analysis

2020 Mathematics subject classification: Primary: 30H10,46G20,30B50. Secondary: 47A10
A way to define these spaces is to consider first the following norm in the space of Dirichlet polynomials (i.e., all finite sums of the form \( \sum_{n=1}^{N} a_n n^{-s} \), with \( N \in \mathbb{N} \)),

\[
\left\| \sum_{n=1}^{N} a_n n^{-s} \right\|_{\mathcal{H}_p} := \lim_{R \to \infty} \left( \frac{1}{2R} \int_{-R}^{R} \left| \sum_{n=1}^{N} a_n n^{-it} \right|^p dt \right)^{\frac{1}{p}},
\]

and define \( \mathcal{H}_p \) as the completion of the Dirichlet polynomials under this norm. Each Dirichlet series in some \( \mathcal{H}_p \) (with \( 1 \leq p < \infty \)) converges on \( \mathbb{C}_{1/2} \), and there it defines a holomorphic function.

The Hardy space \( \mathcal{H}_p \) with the function product is not an algebra for \( p < \infty \). Namely, given two Dirichlet series \( D, E \in \mathcal{H}_p \), it is not true, in general, that the product function \( D \cdot E \) belongs to \( \mathcal{H}_p \). Nevertheless, there are certain series \( D \) that verify that \( D \cdot E \in \mathcal{H}_p \) for every \( E \in \mathcal{H}_p \). Such a Dirichlet series \( D \) is called a multiplier of \( \mathcal{H}_p \) and the mapping \( M_D : \mathcal{H}_p \to \mathcal{H}_p \), given by \( M_D(E) = D \cdot E \), is referred as its associated multiplication operator.

In [5] (see also [9, 14, 19]) it is proved that the multipliers of \( \mathcal{H}_p \) are precisely those Dirichlet series that belong to the Banach space \( \mathcal{H}_\infty \). Moreover, for a multiplier \( D \) we have the following equality:

\[
\|M_D\|_{\mathcal{H}_p \to \mathcal{H}_p} = \|D\|_{\mathcal{H}_\infty}.
\]

Given \( 1 \leq p, q \leq \infty \), we propose to study the multipliers of \( \mathcal{H}_p \) to \( \mathcal{H}_q \); that is, we want to understand those Dirichlet series \( D \) which verify that \( D \cdot E \in \mathcal{H}_q \) for every \( E \in \mathcal{H}_p \).

For this we use the relation that exists between the Hardy spaces of Dirichlet series and the Hardy spaces of functions. The mentioned connection is given by the so-called Bohr lift \( \mathcal{L} \), which identifies each Dirichlet series with a function (both in the polytorus and in the polydisk; see below for more details).

This identification allows us to relate the multipliers in spaces of Dirichlet series with those of function spaces. As consequence of our results, we obtain a complete characterization of \( \mathcal{M}(p, q) \), the space of multipliers of \( \mathcal{H}_p \) into \( \mathcal{H}_q \). It turns out that this set coincides with the Hardy space \( \mathcal{H}_{pq/(p-q)} \) when \( 1 \leq q < p \leq \infty \) and with the null space if \( 1 \leq p < q \leq \infty \). Precisely, for a multiplier \( D \in \mathcal{M}(p, q) \) where \( 1 \leq q < p \leq \infty \) we have the isometric correspondence

\[
\|M_D\|_{\mathcal{H}_p \to \mathcal{H}_q} = \|D\|_{\mathcal{H}_{pq/(p-q)}}.
\]

Moreover, for certain values of \( p \) and \( q \) we study some structural properties of these multiplication operators. Inspired by some of the results obtained by Vukotić [26] and Demazeyux [11] for spaces of holomorphic functions in one variable, we get the corresponding version in the Dirichlet space context. In particular, when considering endomorphisms (i.e., \( p = q \)), the essential norm and the operator norm of a given multiplication operator coincides if \( p > 1 \). In the remaining cases, that is \( p = q = 1 \) or \( 1 \leq q < p \leq \infty \), we compare the essential norm with the norm of the multiplier in different Hardy spaces.

We continue by studying the structure of the spectrum of the multiplication operators over \( \mathcal{H}_p \). Specifically, we consider the continuum spectrum, the radial spectrum and the approximate spectrum. For the latter, we use some necessary and sufficient conditions regarding the associated Bohr lifted function \( \mathcal{L}(D) \) (see definition below) for which the multiplication operator \( M_D : \mathcal{H}_p \to \mathcal{H}_p \) has closed range.
2 Preliminaries on Hardy spaces

2.1 Of holomorphic functions

We note by \( D_N = D \times D \times \cdots \) the cartesian product of \( N \) copies of the open unit disk \( D \) with \( N \in \mathbb{N} \cup \{ \infty \} \) and \( D_2^\infty \) the domain in \( \ell_2 \) defined as \( \ell_2 \cap D_2^\infty \) (for coherence in the notation we will sometimes write \( D_2^N \) for \( D_N \) also in the case \( N \in \mathbb{N} \)). We define \( \mathbb{N}_0^{(N)} \) as consisting of all sequences \( \alpha = (\alpha_n)_n \) with \( \alpha_n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) which are eventually null. In this case we denote \( \alpha! := \alpha_1! \cdots \alpha_M! \) whenever \( \alpha = (\alpha_1, \ldots, \alpha_M, 0, 0, 0, \ldots) \).

A function \( f : D_2^\infty \to \mathbb{C} \) is holomorphic if it is Fréchet differentiable at every \( z \in D_2^\infty \), that is, if there exists a continuous linear functional \( x^* \) on \( \ell_2 \) such that

\[
\lim_{h \to 0} \frac{f(z + h) - f(z) - x^*(h)}{\|h\|} = 0.
\]

We denote by \( H_\infty(D_2^\infty) \) the space of all bounded holomorphic functions \( f : D_2^\infty \to \mathbb{C} \). For \( 1 \leq p < \infty \) we consider the Hardy spaces of holomorphic functions on the domain \( D_2^\infty \) defined by

\[
H_p(D_2^\infty) := \{ f : D_2^\infty \to \mathbb{C} : f \text{ is holomorphic and } \|f\|_{H_p(D_2^\infty)} := \sup_{M\in\mathbb{N}_0} \sup_{0<r<1} \left( \int_{\mathbb{T}_M} |f(r\omega, 0)|^p \, d\omega \right)^{1/p} < \infty \}.
\]

The definitions of \( H_\infty(D_2^N) \) and \( H_p(D_2^N) \) for finite \( N \) are analogous (see [9, Chapters 13 and 15]).

For \( N \in \mathbb{N} \cup \{ \infty \} \), each function \( f \in H_p(D_2^N) \) defines a unique family of coefficients \( c\alpha(f) = \frac{\partial^\alpha f(0)}{\alpha!} \) (the Cauchy coefficients) with \( \alpha \in \mathbb{N}_0^N \) having always only finitely many non-null coordinates. For \( z \in D_2^N \) one has the following monomial expansion [9, Theorem 13.2]

\[
f(z) = \sum_{\alpha \in \mathbb{N}_0^N} c\alpha(f) \cdot z^\alpha,
\]

with \( z^\alpha = z_1^{\alpha_1} \cdots z_M^{\alpha_M} \) whenever \( \alpha = (\alpha_1, \cdots, \alpha_M, 0, 0, 0, \ldots) \).

Let us note that for each fixed \( N \in \mathbb{N} \) and \( 1 \leq p \leq \infty \) we have \( H_p(D_2^N) \hookrightarrow H_p(D_2^\infty) \) by doing \( f \leadsto [z = (z_n)_n \in D_2^\infty \leadsto f(z_1, \ldots, z_N)] \). Conversely, given a function \( f \in H_p(D_2^\infty) \), for each \( N \in \mathbb{N} \) we define \( f_N(z_1, \ldots, z_N) = f(z_1, \ldots, z_N, 0, 0, \ldots) \) for \( (z_1, \ldots, z_N) \in D_2^N \). It is well known that \( f_N \in H_p(D_2^N) \).

An important property for our purposes is the so-called Cole-Gamelin inequality (see [9, Remark 13.14 and Theorem 13.15]), which states that for every \( f \in H_p(D_2^N) \) and \( z \in D_2^N \) (for \( N \in \mathbb{N} \cup \{ \infty \} \)) we have

\[
|f(z)| \leq \left( \prod_{j=1}^N \frac{1}{1 - |z_j|^2} \right)^{1/p} \|f\|_{H_p(D_2^N)}.
\]
For functions of finitely many variable this inequality is optimal in the sense that if \( N \in \mathbb{N} \) and \( z \in \mathbb{D}^N \), then there is a function \( f_z \in H_p(\mathbb{D}_2^N) \) given by
\[
f_z(u) = \left( \prod_{j=1}^{N} \frac{1 - |z_j|^2}{(1 - \overline{z}_j u_j)^2} \right)^{1/p},
\]
such that \( \|f_z\|_{H_p(\mathbb{D}_2^N)} = 1 \) and \( |f_z(z)| = \left( \prod_{j=1}^{N} \frac{1}{1 - |z_j|^2} \right)^{1/p} \).

### 2.2 On the polytorus

On \( \mathbb{T}^\infty = \{\omega = (\omega_n)_n; |\omega_n| = 1, \text{ for every } n\} \) consider the product of the normalized Lebesgue measure on \( \mathbb{T} \) (note that this is the Haar measure). For each \( F \in L_1(\mathbb{T}^\infty) \) and \( \alpha \in \mathbb{Z}^{(N)} \), the \( \alpha \)-th Fourier coefficient of \( F \) is defined as
\[
\hat{F}(\alpha) = \int_{\mathbb{T}^N} f(\omega) \cdot \omega^\alpha d\omega
\]
where again \( \omega^\alpha = \omega_1^{\alpha_1} \cdots \omega_M^{\alpha_M} \) if \( \alpha = (\alpha_1, \ldots, \alpha_M, 0, 0, 0, \ldots) \). The Hardy space on the polytorus \( H_p(\mathbb{T}^\infty) \) is the subspace of \( L_p(\mathbb{T}^\infty) \) given by all the functions \( F \) such that \( \hat{F}(\alpha) = 0 \) for every \( \alpha \in \mathbb{Z}^{(N)} - \mathbb{N}^{(N)}_0 \). The definition of \( H_p(\mathbb{T}^N) \) for finite \( N \) is analogous (note that these are the classical Hardy spaces, see [20]). We have the canonical inclusion \( H_p(\mathbb{T}^N) \hookrightarrow H_p(\mathbb{T}^\infty) \) by doing \( F \sim \{\omega = (\omega_n)_n \in \mathbb{T}^\infty \hookrightarrow F(\omega_1, \ldots, \omega_N)\} \).

Given \( N_1 < N_2 \leq \infty \) and \( F \in H_p(\mathbb{T}^{N_2}) \), then the function \( F_{N_1} \), defined by \( F_{N_1}(\omega) = \int_{\mathbb{T}^{N_2-N_1}} F(\omega, u) du \) for every \( \omega \in \mathbb{T}^{N_1} \), belongs to \( H_p(\mathbb{T}^{N_1}) \). In this case, the Fourier coefficients of both functions coincide: that is, given \( \alpha \in \mathbb{N}_0^{N_1} \) then
\[
\hat{F}_{N_1}(\alpha) = \hat{F}(\alpha_1, \alpha_2, \ldots, \alpha_{N_1}, 0, 0, \ldots).
\]
Moreover,
\[
\|F\|_{H_p(\mathbb{T}^{N_2})} \geq \|F_{N_1}\|_{H_p(\mathbb{T}^{N_1})}.
\]
Let \( N \in \mathbb{N} \cup \{\infty\} \), there is an isometric isomorphism between the spaces \( H_p(\mathbb{D}_2^N) \) and \( H_p(\mathbb{T}^N) \). More precisely, given a function \( f \in H_p(\mathbb{D}_2^N) \) there is a unique function \( F \in H_p(\mathbb{T}^N) \) such that \( c_\alpha(f) = \hat{F}(\alpha) \) for every \( \alpha \) in the corresponding indexing set and \( \|f\|_{H_p(\mathbb{D}_2^N)} = \|F\|_{H_p(\mathbb{T}^N)} \). If this is the case, we say that the functions \( f \) and \( F \) are associated. In particular, by the uniqueness of the coefficients, \( f_M \) and \( F_M \) are associated to each other for every \( 1 \leq M \leq N \). Even more, if \( N \in \mathbb{N} \), then
\[
F(\omega) = \lim_{r \to 1^-} f(r\omega),
\]
for almost all \( \omega \in \mathbb{T}^N \).

We isolate the following important property which will be useful later.

**Remark 1.** Let \( F \in H_p(\mathbb{T}^\infty) \). If \( 1 \leq p < \infty \), then \( F_N \to F \) in \( H_p(\mathbb{T}^\infty) \) (see e.g [9, Remark 5.8]). If \( p = \infty \), the convergence is given in the \( w(L_{\infty}, L_1) \)-topology. In particular, for any \( 1 \leq p \leq \infty \), there is a subsequence so that \( \lim_k F_{N_k}(\omega) = F(\omega) \) for almost \( \omega \in \mathbb{T}^\infty \) (note that the case \( p = \infty \) follows directly from the inclusion \( H_\infty(\mathbb{T}^\infty) \subset H_2(\mathbb{T}^\infty) \)).
2.3 Bohr transform

We previously mentioned the Hardy spaces of functions both on the polydisk and on the polytorus and the relationship between them based on their coefficients. This relation also exists with the Hardy spaces of Dirichlet series and the isometric isomorphism that identifies them is the so-called Bohr transform. To define it, let us first consider $p = (p_1, p_2, \cdots)$ the sequence of prime numbers. Then, given a natural number $n$, by the prime number decomposition, there are unique non-negative integer numbers $\alpha_1, \ldots, \alpha_M$ such that $n = p_1^{\alpha_1} \cdots p_M^{\alpha_M}$. Therefore, with the notation that we already defined, we have that $n = \mathbf{p}^\alpha$ with $\alpha = (\alpha_1, \cdots, \alpha_M, 0, 0, \ldots)$. Then, given $1 \leq p \leq \infty$, the Bohr transform $B_{\mathbb{D}^\infty_2}$ on $H_p(\mathbb{D}^\infty_2)$ is defined as follows:

$$B_{\mathbb{D}^\infty_2}(f) = \sum_n a_n n^{-s},$$

where $a_n = c_\alpha(f)$ if and only if $n = \mathbf{p}^\alpha$. The Bohr transform is an isometric isomorphism between the spaces $H_p(\mathbb{D}^\infty_2)$ and $\mathcal{H}_p$ (see [9, Theorem 13.2]).

We denote by $\mathcal{H}^{(N)}$ the set of all Dirichlet series $\sum a_n n^{-s}$ that involve only the first $N$ prime numbers; that is $a_n = 0$ if $p_i$ divides $n$ for some $i > N$. We write $\mathcal{H}^{(N)}_p$ for the space $\mathcal{H}^{(N)} \cap \mathcal{H}_p$ (endowed with the norm in $\mathcal{H}_p$). Note that the image of $H_p(\mathbb{D}^N)$ (seen as a subspace of $H_p(\mathbb{D}^\infty_2)$ with the natural identification) through $B_{\mathbb{D}^\infty_2}$ is exactly $\mathcal{H}_p^{(N)}$.

The inverse of the Bohr transform, which sends the space $\mathcal{H}_p$ into the space $H_p(\mathbb{D}^\infty_2)$, is called the Bohr lift, which we denote by $L_{\mathbb{D}^\infty_2}$.

With the same idea, the Bohr transform $B_{\mathbb{T}^\infty}$ on the polytorus for $H_p(\mathbb{T}^\infty)$ is defined; that is,

$$B_{\mathbb{T}^\infty}(F) = \sum_n a_n n^{-s},$$

where $a_n = \hat{F}(\alpha)$ if and only if $n = \mathbf{p}^\alpha$. It is an isometric isomorphism between the spaces $H_p(\mathbb{T}^N)$ and $\mathcal{H}_p$. Its inverse is denoted by $L_{\mathbb{T}^\infty}$.

In order to keep the notation as clear as possible we will carefully use the following convention: we will use capital letters (e.g., $F$, $G$, or $H$) to denote functions defined on the polytorus $\mathbb{T}^\infty$ and lowercase letters (e.g., $f$, $g$ or $h$) to represent functions defined on the polydisk $\mathbb{D}^\infty_2$. If $f$ and $F$ are associated to each other (meaning that $c_\alpha(f) = \hat{F}(\alpha)$ for every $\alpha$), we will sometimes write $f \sim F$. With the same idea, if a function $f$ or $F$ is associated through the Bohr transform to a Dirichlet series $D$, we will write $f \sim D$ or $F \sim D$.

3 The space of multipliers

As we mentioned above, our main interest is to describe the multipliers of the Hardy spaces of Dirichlet series. Let us recall again that a holomorphic function $\varphi$, defined on $C_{1/2}$ is a $(p, q)$-multiplier of $\mathcal{H}_p$ if $\varphi \cdot D \in \mathcal{H}_q$ for every $D \in \mathcal{H}_p$. We denote the set of all such functions by $\mathfrak{M}(p, q)$. Since the constant 1 function belongs to $\mathcal{H}_p$ we have that, if $\varphi \in \mathfrak{M}(p, q)$, then necessarily $\varphi$ belongs to $\mathcal{H}_q$ and it can be represented by a Dirichlet series. So, we will use that the multipliers of $\mathcal{H}_p$ are precisely Dirichlet series. The set $\mathfrak{M}^{(N)}(p, q)$ is defined in the obvious way, replacing $\mathcal{H}_p$ and $\mathcal{H}_q$ by $\mathcal{H}_p^{(N)}$ and $\mathcal{H}_q^{(N)}$. The
The set \( \mathcal{M}(p, q) \) is clearly a vector space. Each Dirichlet series \( D \in \mathcal{M}(p, q) \) induces a multiplication operator \( M_D \) from \( \mathcal{H}_p \) to \( \mathcal{H}_q \), defined by \( M_D(E) = D \cdot E \). By the continuity of the evaluation on each \( s \in C_{1/2} \) (see e.g. [9, Corollary 13.3]), and the Closed Graph Theorem, \( M_D \) is continuous. Then, the expression

\[
\|D\|_{\mathcal{M}(p, q)} := \|M_D\|_{\mathcal{H}_p \rightarrow \mathcal{H}_q},
\]

defines a norm on \( \mathcal{M}(p, q) \). Note that

\[
\|D\|_{\mathcal{H}_q} = \|M_D(1)\|_{\mathcal{H}_q} \leq \|M_D\|_{\mathcal{H}_p \rightarrow \mathcal{H}_q} \cdot \|1\|_{\mathcal{H}_q} = \|D\|_{\mathcal{M}(p, q)},
\]

and the inclusions that we presented above are continuous. A norm on \( \mathcal{M}^{(N)}(p, q) \) is defined analogously.

Clearly, if \( p_1 < p_2 \) or \( q_1 < q_2 \), then

\[
\mathcal{M}(p_1, q) \subseteq \mathcal{M}(p_2, q) \text{ and } \mathcal{M}(p, q_2) \subseteq \mathcal{M}(p, q_1),
\]

for fixed \( p \) and \( q \).

Given a Dirichlet series \( D = \sum a_n n^{-s} \), we denote by \( D_N \) the ‘restriction’ to the first \( N \) primes (i.e., we consider those \( n \)’s that involve, in its factorization, only the first \( N \) primes). Let us be more precise. If \( n \in \mathbb{N} \), we write \( gpd(n) \) for the greatest prime divisor of \( n \). That is, if \( n = p_{\alpha_1}^{a_1} \cdots p_{\alpha_N}^{a_N} \) (with \( \alpha_N \neq 0 \)) is the prime decomposition of \( n \), then \( gpd(n) = p_N \). With this notation, \( D_N := \sum_{gpd(n) \leq p_N} a_n n^{-s} \).

**Proposition 2.** Let \( D = \sum a_n n^{-s} \) be a Dirichlet series and \( 1 \leq p, q \leq \infty \). Then \( D \in \mathcal{M}(p, q) \) if and only if \( D_N \in \mathcal{M}^{(N)}(p, q) \) for every \( N \in \mathbb{N} \) and \( \sup_N \|D_N\|_{\mathcal{M}^{(N)}(p, q)} < \infty \).

**Proof.** Let us begin by noting that, if \( n = jk \), then clearly \( gpd(n) \leq p_N \) if and only if \( gpd(j) \leq p_N \) and \( gpd(k) \leq p_N \). From this we deduce that, given any two Dirichlet series \( D \) and \( E \), we have \( (DE)_N = D_N E_N \) for every \( N \in \mathbb{N} \).

Take some Dirichlet series \( D \) and suppose that \( D \in \mathcal{M}(p, q) \). Then, given \( E \in \mathcal{H}_p^{(N)} \) we have \( DE \in \mathcal{H}_q \), and \( (DE)_N \in \mathcal{H}_p^{(N)} \). But \( (DE)_N = D_N E_N = D_N E \) and, since \( E \) was arbitrary, \( D_N \in \mathcal{M}^{(N)}(p, q) \) for every \( N \). On the other hand, if \( E \in \mathcal{H}_q \), then \( E_N \in \mathcal{H}_q^{(N)} \) and \( \|E_N\|_{\mathcal{H}_q} \leq \|E\|_{\mathcal{H}_q} \) (see [9, Corollary 13.9]). This gives \( \|D_N\|_{\mathcal{M}^{(N)}(p, q)} \leq \|D\|_{\mathcal{M}(p, q)} \) for every \( N \).

Suppose now that \( D \) is such that \( D_N \in \mathcal{M}^{(N)}(p, q) \) for every \( N \) and \( \sup_N \|D_N\|_{\mathcal{M}^{(N)}(p, q)} < \infty \) (let us call it \( C \)). Then, for each \( E \in \mathcal{H}_p \) we have, by [9, Corollary 13.9],

\[
\|(DE)_N\|_{\mathcal{H}_p} = \|D_N E_N\|_{\mathcal{H}_p} \leq \|D_N\|_{\mathcal{M}^{(N)}(p, q)} \|E_N\|_{\mathcal{H}_p} \leq C \|E\|_{\mathcal{H}_p}.
\]

Since this holds for every \( N \), it shows (again by [9, Corollary 13.9]) that \( DE \in \mathcal{H}_p \) and completes the proof. \( \square \)

We are going to exploit the connection between Dirichlet series and power series in infinitely many variables. This leads us to consider spaces of multipliers on Hardy spaces of functions. If \( U \) is either \( \mathbb{T}^N \) or \( \mathbb{D}_2^N \) (with \( N \in \mathbb{N} \cup \{\infty\} \)) we consider the corresponding Hardy spaces \( H_p(U) \) (for \( 1 \leq p \leq \infty \)), and say that a function \( f \) defined on \( U \) is a
Let \((p,q)\)-multiplier of \(H_p(U)\) if \(f \cdot g \in H_q(U)\) for every \(f \in H_p(U)\). We denote the space of all such functions by \(\mathcal{M}_U(p,q)\). The same argument as before with the constant 1 function shows that \(\mathcal{M}_U(p,q) \subseteq H_q(U)\). Also, each multiplier defines a multiplication operator \(M : H_p(U) \to H_q(U)\) which, by the Closed Graph Theorem, is continuous, and the norm of the operator defines a norm on the space of multipliers, as in (3).

Our first step is to see that the identifications that we have just shown behave ‘well’ with the multiplication, in the sense that whenever two pairs of functions are identified to each other, then so also are the products. Let us make a precise statement.

**Theorem 3.** Let \(D, E \in \mathcal{K}_1\), \(f, g \in H_1(\mathbb{D}_2^\infty)\) and \(F, G \in H_1(\mathbb{T}^\infty)\) so that \(f \sim F \sim D\) and \(g \sim G \sim E\). Then, the following are equivalent

\[ a)\ DE \in \mathcal{K}_1\]
\[ b)\ fg \in H_1(\mathbb{D}_2^\infty)\]
\[ c)\ FG \in H_1(\mathbb{T}^\infty)\]

and, in this case \(DE \sim fg \sim FG\).

The equivalence between b) and c) is based in the case for finitely many variables.

**Proposition 4.** Fix \(N \in \mathbb{N}\) and let \(f, g \in H_1(\mathbb{D}^N)\) and \(F, G \in H_1(\mathbb{T}^N)\) so that \(f \sim F\) and \(g \sim G\). Then, the following are equivalent

\[ a)\ f g \in H_1(\mathbb{D}^N)\]
\[ b)\ FG \in H_1(\mathbb{T}^N)\]

and, in this case, \(fg \sim FG\).

**Proof.** Let us suppose first that \(fg \in H_1(\mathbb{D}^N)\) and denote by \(H \in H_1(\mathbb{T}^N)\) the associated function. Then, since

\[ F(\omega) = \lim_{r \to 1^-} f(\alpha) \quad \text{and} \quad G(\omega) = \lim_{r \to 1^-} g(\alpha) \]

for almost all \(\omega \in \mathbb{T}^N\), we have

\[ H(\omega) = \lim_{r \to 1^-} (fg)(\alpha) = F(\omega)G(\omega) \]

for almost all \(\omega \in \mathbb{T}^N\). Therefore \(FG = H \in H_1(\mathbb{T}^N)\), and this yields b).

Let us conversely assume that \(FG \in H_1(\mathbb{T}^N)\), and take the associated function \(h \in H_1(\mathbb{D}^N)\). The product \(fg : \mathbb{D}^N \to \mathbb{C}\) is a holomorphic function and \(fg - h\) belongs to the Nevanlinna class \(\mathcal{N}(\mathbb{D}^N)\), that is

\[ \sup_{0 < r < 1} \int_{\mathbb{T}^N} \log^+ |f(\alpha)g(\alpha) - h(\alpha)| d\alpha < \infty \]

where \(\log^+(x) := \max\{0, \log x\}\) (see [21, Section 3.3] for a complete account on this space). Consider \(H(\omega)\) defined for almost all \(\omega \in \mathbb{T}^N\) as the radial limit of \(fg - h\). Then by [21,
Theorem 3.3.5] there are two possibilities: either \( \log |H| \in L_1(\mathbb{T}^N) \) or \( fg - h = 0 \) on \( \mathbb{D}^N \).

But, just as before, we have

\[
\lim_{r \to 1^-} f(r\omega)g(r\omega) = F(\omega)G(\omega) = \lim_{r \to 1^-} h(r\omega)
\]

for almost all \( \omega \in \mathbb{T}^N \), and then necessarily \( H = 0 \). Thus \( fg = h \) on \( \mathbb{D}^N \), and \( fg \in H_1(\mathbb{D}^N) \).

This shows that b) implies a) and completes the proof.

For the general case we need the notion of the Nevanlinna class in the infinite dimensional framework. Given \( \mathbb{D}_1^{\infty} := \ell_1 \cap \mathbb{D}^{\infty} \), a function \( u : \mathbb{D}_1^{\infty} \to \mathbb{C} \) and \( 0 < r < 1 \), the mapping \( u_{[r]} : \mathbb{T}^{\infty} \to \mathbb{C} \) is defined by

\[
uu_{[r]}(\omega) = (r\omega_1, r^2\omega_2, r^3\omega_3, \cdots).
\]

The Nevanlinna class on infinitely many variables, introduced recently in [13] and denoted by \( \mathcal{N}(\mathbb{D}_1^{\infty}) \), consists on those holomorphic functions \( u : \mathbb{D}_1^{\infty} \to \mathbb{C} \) such that

\[
\sup_{0 < r < 1} \int_{\mathbb{T}^{\infty}} \log^+ |u_{[r]}(\omega)|d\omega < \infty.
\]

We can now prove the general case.

**Proof of Theorem 3.** Let us show first that a) implies b). Suppose that \( D = \sum a_n n^{-s}, E = \sum b_n n^{-s} \in \mathcal{H}_1 \) are so that \( (\sum a_n n^{-s})(\sum b_n n^{-s}) = \sum c_n n^{-s} \in \mathcal{H}_1 \). Let \( h \in H_1(\mathbb{D}_2^{\infty}) \) be the holomorphic function associated to the product. Recall that, if \( \alpha \in \mathbb{N}_0^{(N)} \) and \( n = p^\alpha \in \mathbb{N} \), then

\[
\begin{align*}
\alpha(f) &= \alpha_n, \quad \alpha(g) = b_n \quad \text{and} \quad \alpha(h) = c_n = \sum_{j=0}^{\infty} a_j b_j.
\end{align*}
\]

On the other hand, the function \( f \cdot g : \mathbb{D}_2^{\infty} \to \mathbb{C} \) is holomorphic and a straightforward computation shows that

\[
\alpha(fg) = \sum_{\beta + \gamma = \alpha} \alpha(\beta), \quad \alpha(\gamma)(g).
\]

for every \( \alpha \). Now, if \( jk = n = p^\beta \) for some \( \beta, \gamma \in \mathbb{N}_0^{(N)} \), then there are \( \beta, \gamma \in \mathbb{N}_0^{(N)} \) so that \( j = p^\beta, k = p^\gamma \) and \( \beta + \gamma = \alpha \). This, together with (6) and (7) shows that \( \alpha(h) = \alpha(fg) \) for every \( \alpha \) and, therefore \( f = h \in H_1(\mathbb{D}_2^{\infty}) \). This yields our claim.

Suppose now that \( fg \in H_1(\mathbb{D}_2^{\infty}) \) and take the corresponding Dirichlet series \( \sum a_n n^{-s}, \sum b_n n^{-s}, \sum c_n n^{-s} \in \mathcal{H}_1 \) (associated to \( f, g \) and \( fg \) respectively). The same argument as above shows that

\[
\sum_{j=0}^{\infty} a_j b_j = \sum_{j=0}^{\infty} \alpha(fg) = \sum_{\beta + \gamma = \alpha} \alpha(\beta), \quad \alpha(\gamma)(g).
\]

hence \( (\sum a_n n^{-s})(\sum b_n n^{-s}) = \sum c_n n^{-s} \in \mathcal{H}_1 \), showing that b) implies a).

Suppose now that \( fg \in H_1(\mathbb{D}_2^{\infty}) \) and let us see that c) holds. Let \( H \in H_1(\mathbb{T}^{\infty}) \) be the function associated to \( fg \). Note first that \( \hat{f}_N \sim \hat{F}_N, \hat{g}_N \sim \hat{G}_N \) and \( \hat{(fg)}_N \sim \hat{H}_N \) for every \( N \). A straightforward computation shows that \( \hat{(fg)}_N = \hat{f}_N \hat{g}_N \), and then this product is in \( H_1(\mathbb{D}^N) \). Then Proposition 4 yields \( \hat{f}_N \hat{g}_N \sim \hat{f}_N \hat{g}_N \), therefore

\[
\hat{H}_N(\alpha) = \overline{(\hat{F}_N \hat{G}_N)}(\alpha)
\]
for every $\alpha \in \mathbb{N}_0^{(N)}$ and, then, $H_N = F_N G_N$ for every $N \in \mathbb{N}$. We can find a subsequence in such a way that

$$\lim_k F_{N_k}(\omega) = F(\omega), \lim_k G_{N_k}(\omega) = G(\omega), \text{ and } \lim_k H_{N_k}(\omega) = H(\omega)$$

for almost all $\omega \in \mathbb{T}^\infty$ (recall Remark 1). All this gives that $F(\omega)G(\omega) = H(\omega)$ for almost all $\omega \in \mathbb{T}^\infty$. Hence $FG = H \in H_1(\mathbb{T}^\infty)$, and our claim is proved.

Finally, if $FG \in H_1(\mathbb{T}^\infty)$, we denote by $h$ its associated function in $H_1(\mathbb{D}_2^\infty)$. By [13, Propostions 2.8 and 2.14] we know that $H_1(\mathbb{D}_2^\infty)$ is contained in the Nevanlinna class $\mathcal{N}(\mathbb{D}_1^\infty)$, therefore $f, g, h \in \mathcal{N}(\mathbb{D}_1^\infty)$ and hence, by definition, $f \cdot g - h \in \mathcal{N}(\mathbb{D}_2^\infty)$. On the other hand, [13, Theorem 2.4 and Corollary 2.11] tell us that, if $u \in \mathcal{N}(\mathbb{D}_1^\infty)$, then the radial limit $u^*(\omega) = \lim_{r \to 1^{-}} u_{[r]}(\omega)$ exists for almost all $\omega \in \mathbb{T}^\infty$. Even more, $u = 0$ if and only if $u^*$ vanishes on some subset of $\mathbb{T}^\infty$ with positive measure. The radial limit of $f, g$ and $h$ coincide a.e. with $F, G$ and $F \cdot G$ respectively (see [1, Theorem 1]). Since

$$(f \cdot g - h)^*(\omega) = \lim_{r \to 1^{-}} f_{[r]}(\omega) \cdot g_{[r]}(\omega) - h_{[r]}(\omega) = 0,$$

for almost all $\omega \in \mathbb{T}^\infty$, then $f \cdot g = h$ on $\mathbb{D}_1^\infty$. Finally, since the set $\mathbb{D}_1^\infty$ is dense in $\mathbb{D}_2^\infty$, by the continuity of the functions we have that $f \cdot g \in H_1(\mathbb{D}_2^\infty)$.

As an immediate consequence of Theorem 3 we obtain the following.

**Proposition 5.** For every $1 \leq p, q \leq \infty$ we have

$$\mathcal{M}(p, q) = \mathcal{M}_{\mathbb{D}_2^\infty}(p, q) = \mathcal{M}_{\mathbb{T}^\infty}(p, q),$$

and

$$\mathcal{M}^{(N)}(p, q) = \mathcal{M}_{\mathbb{D}_2}(p, q) = \mathcal{M}_{\mathbb{T}^\infty}(p, q),$$

for every $N \in \mathbb{N}$, by means of the Bohr transform.

Again (as in Proposition 2), being a multiplier can be characterized in terms of the restrictions (this follows immediately from Proposition 2 and Proposition 5).

**Proposition 6.**

1. $f \in \mathcal{M}_{\mathbb{D}_2^\infty}(p, q)$ if and only if $f_N \in \mathcal{M}_{\mathbb{D}_2}(p, q)$ for every $N \in \mathbb{N}$ and $\sup_N \|M_{f_N}\| < \infty$.

2. $F \in \mathcal{M}_{\mathbb{T}^\infty}(p, q)$, then, $F_N \in \mathcal{M}_{\mathbb{T}^\infty}(p, q)$ for every $N \in \mathbb{N}$ and $\sup_N \|M_{F_N}\| < \infty$.

The following statement describes the spaces of multipliers, viewing them as Hardy spaces of Dirichlet series. A result of similar flavour for holomorphic functions in one variable appears in [23].

**Theorem 7.** The following assertions hold true

1. $\mathcal{M}(\infty, q) = \mathcal{H}_q$ isometrically.

2. If $1 \leq q < p < \infty$ then $\mathcal{M}(p, q) = \mathcal{H}_{pq/(p-q)}$ isometrically.

3. If $1 \leq p \leq \infty$ then $\mathcal{M}(p, p) = \mathcal{H}_\infty$ isometrically.
d) If \(1 \leq p < q \leq \infty\) then \(\mathcal{M}(p, q) = \{0\}\).

The same equalities hold if we replace in each case \(\mathcal{M}\) and \(\mathcal{H}\) by \(\mathcal{M}^{(N)}\) and \(\mathcal{H}^{(N)}\) (with \(N \in \mathbb{N}\)) respectively.

**Proof.** To get the result we use again the isometric identifications between the Hardy spaces of Dirichlet series and both Hardy spaces of functions, and also between their multipliers given in Proposition 5. Depending on each case we will use the most convenient identification, jumping from one to the other without further notification.

a) We already noted that \(\mathcal{M}^\infty(\infty, q) \subset H_q(\mathbb{T}^N)\) with continuous inclusion (recall (4)). On the other hand, if \(D \in \mathcal{H}_q\) and \(E \in \mathcal{H}_\infty\), then \(D \cdot E\) a Dirichlet series in \(\mathcal{H}_q\). Moreover,

\[
\|M_D(E)\|_{\mathcal{H}_q} \leq \|D\|_{\mathcal{H}_q} \|E\|_{\mathcal{H}_\infty}.
\]

This shows that \(\|M_D\|_{\mathcal{M}^\infty(\infty, q)} \leq \|D\|_{\mathcal{H}_q}\), providing the isometric identification.

b) Suppose \(1 \leq q < p < \infty\) and take some \(f \in H_{pq/(p-q)}(\mathbb{D}_2^\infty)\) and \(g \in H_p(\mathbb{D}_2^\infty)\), then \(f \cdot g\) is holomorphic on \(\mathbb{D}_2^\infty\). Consider \(t = \frac{p}{p-q}\) and note that \(t\) is the conjugate exponent of \(\frac{p}{q}\) in the sense that \(\frac{q}{p} + \frac{1}{t} = 1\). Therefore given \(M \in \mathbb{N}\) and \(0 < r < 1\), by Hölder inequality

\[
\left(\int_{\mathbb{T}^M} |f \cdot g(r, 0)|^q \, d\omega\right)^{1/q} \leq \left(\int_{\mathbb{T}^M} |f(r, 0)|^{qt} \, d\omega\right)^{1/qt} \left(\int_{\mathbb{T}^M} |g(r, 0)|^{qp/q} \, d\omega\right)^{q/pq}.
\]

Since this holds for every \(M \in \mathbb{N}\) and \(0 < r < 1\), then \(f \in \mathcal{M}_{\mathbb{D}_2^\infty}(p, q)\) and furthermore \(\|M_f\|_{\mathcal{M}_{\mathbb{D}_2^\infty}(p, q)} \leq \|f\|_{H_{pq/(p-q)}(\mathbb{D}_2^\infty)}\). Thus \(H_{pq/(p-q)}(\mathbb{D}_2^\infty) \subseteq \mathcal{M}_{\mathbb{D}_2^\infty}(p, q)\). The case for \(\mathbb{D}_N^\infty\) with \(N \in \mathbb{N}\) follows with the same idea.

To check that the converse inclusion holds, take some \(F \in \mathcal{M}_N^\infty(p, q)\) (where \(N \in \mathbb{N} \cup \{\infty\}\)) and consider the associated multiplication operator \(M_F : H_p(\mathbb{T}^N) \to H_q(\mathbb{T}^N)\) which, as we know, is continuous. Let us see that it can be extended to a continuous operator on \(L_q(\mathbb{T}^N)\). To see this, take a trigonometric polynomial \(Q\), that is a finite sum of the form

\[
Q(z) = \sum_{|\alpha| \leq k} a_\alpha z^\alpha,
\]

and note that

\[
Q = \left(\prod_{j=1}^M z_j^k\right) \cdot P,
\]

where \(P\) is the polynomial defined as \(P := \sum_{0 \leq \beta \leq 2k} b_\beta z^\beta\) and \(b_\beta = a_\alpha\) whenever \(\beta = \alpha + (k, \cdots, k, 0)\). Then,

\[
\left(\int_{\mathbb{T}^N} |F \cdot Q(\omega)|^q \, d\omega\right)^{1/q} = \left(\int_{\mathbb{T}^N} |F \cdot P(\omega)|^{q \prod_{j=1}^M |\omega_j|^{-kq}} \, d\omega\right)^{1/q} = \left(\int_{\mathbb{T}^N} |F \cdot P(\omega)|^q \, d\omega\right)^{1/q}.
\]

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\[ \left( \int_{\mathbb{T}^N} |P(\omega)|^p \prod_{j=1}^{M} |\omega_j|^{-k_j} d\omega \right)^{1/p} \]
\[ \leq C \|P\|_{L^p(\mathbb{T}^N)} = C \left( \int_{\mathbb{T}^N} |P(\omega)|^p \prod_{j=1}^{M} |\omega_j|^{-k_j} d\omega \right)^{1/p} \]
\[ = C \|Q\|_{L^p(\mathbb{T}^N)}. \]

Consider now an arbitrary \( H \in L_p(\mathbb{T}^N) \) and, using [9, Theorem 5.17] find a sequence of trigonometric polynomials \((Q_n)_n\) such that \( Q_n \to H \) in \( L_p \) and also a.e. on \( \mathbb{T}^N \) (taking a subsequence if necessary). We have
\[ \|F \cdot Q_n - F \cdot Q_m\|_{H_q(\mathbb{T}^N)} = \|F \cdot (Q_n - Q_m)\|_{H_q(\mathbb{T}^N)} \leq C \|Q_n - Q_m\|_{H_p(\mathbb{T}^N)} \to 0 \]
which shows that \((F \cdot Q_n)_n\) is a Cauchy sequence in \( L_q(\mathbb{T}^N) \). Since \( F \cdot Q_n \to F \cdot H \) a.e. on \( \mathbb{T}^N \), then this proves that \( F \cdot H \in L_q(\mathbb{T}^N) \) and \( F \cdot Q_n \to F \cdot H \) in \( L_q(\mathbb{T}^N) \). Moreover,
\[ \|F \cdot H\|_{H_q(\mathbb{T}^N)} = \lim_{n} \|F \cdot Q_n\|_{H_q(\mathbb{T}^N)} \leq C \lim_{n} \|Q_n\|_{H_p(\mathbb{T}^N)} = C \|H\|_{H_p(\mathbb{T}^N)}, \]
and therefore the operator \( M_F : L_p(\mathbb{T}^N) \to L_q(\mathbb{T}^N) \) is well defined and bounded. In particular, \( |F|^q \cdot |H|^q \in L_1(\mathbb{T}^N) \) for every \( H \in L_p(\mathbb{T}^N) \).

Now, consider \( H \in L_p/_{q} (\mathbb{T}^N) \) then \( |H|^{1/q} \in L_1(\mathbb{T}^N) \) and \( |F|^q \cdot |H| \in L_1(\mathbb{T}^N) \) or, equivalently, \( |F|^q \cdot H \in L_1(\mathbb{T}^N) \). Hence
\[ |F|^q \in L_{p/q}(\mathbb{T}^N)^* = L_{p/(p-q)}(\mathbb{T}^N), \]
and therefore \( F \in L_{p/(p-q)}(\mathbb{T}^N) \). To finish the argument, since \( \hat{F}(\alpha) = 0 \) whenever \( \alpha \in \mathbb{Z}^N \setminus \mathbb{N}_0^N \) then \( F \in H_{p/q/(p-q)}(\mathbb{T}^N) \). We then conclude that
\[ H_{p/q/(p-q)}(\mathbb{T}^N) \subseteq \mathcal{M}_{\mathbb{T}^N}(p, q). \]

In order to see the isometry, given \( F \in H_{p/q/(p-q)}(\mathbb{T}^N) \) and let \( G = |F|^r \in L_p(\mathbb{T}^N) \) with \( r = q/(p-q) \) then \( F \cdot G \in L_q(\mathbb{T}^N) \). Let \( Q_n \) a sequence of trigonometric polynomials such that \( Q_n \to G \) in \( L_p(\mathbb{T}^N) \), since \( M_F : L_p(\mathbb{T}^N) \to L_q(\mathbb{T}^N) \) is continuous then \( F \cdot Q_n = M_F(Q_n) \to F \cdot G \). On the other hand, writing \( Q_n \) as \( (8) \) we have for each \( n \in \mathbb{N} \) a polynomial \( P_n \) such that \( \|F \cdot Q_n\|_{L_q(\mathbb{T}^N)} = \|F \cdot P_n\|_{L_q(\mathbb{T}^N)} \) and \( \|Q_n\|_{L_p(\mathbb{T}^N)} = \|P_n\|_{L_p(\mathbb{T}^N)} \). Then we have that
\[ \|F \cdot G\|_{L_q(\mathbb{T}^N)} = \lim_n \|F \cdot Q_n\|_{L_q(\mathbb{T}^N)} = \lim_n \|F \cdot P_n\|_{L_q(\mathbb{T}^N)} \leq \lim_n \|M_F\|_{\mathcal{M}_{\mathbb{T}^N}(p,q)} \|P_n\|_{L_p(\mathbb{T}^N)} \]
\[ = \lim_n \|M_F\|_{\mathcal{M}_{\mathbb{T}^N}(p,q)} \|Q_n\|_{L_p(\mathbb{T}^N)} = \|M_F\|_{\mathcal{M}_{\mathbb{T}^N}(p,q)} \|G\|_{L_p(\mathbb{T}^N)}. \]
Now, since
\[ \|F\|_{L_{p/(p-q)}(\mathbb{T}^N)}^{p/(p-q)} = \|F^{r+1}\|_{L_q(\mathbb{T}^N)} = \|F \cdot G\|_{L_q(\mathbb{T}^N)} \]
and
\[ \|F\|_{L_{p/(p-q)}(\mathbb{T}^N)}^{q/(p-q)} = \|F^r\|_{L_p(\mathbb{T}^N)} = \|G\|_{L_p(\mathbb{T}^N)} \]
than
\[ \|M_F\|_{\mathcal{M}_{\mathbb{T}^N}(p,q)} \geq \|F\|_{L_{p/(p-q)}(\mathbb{T}^N)} = \|F\|_{H_{p/q/(p-q)}(\mathbb{T}^N)}, \]
as we wanted to show.
c) was proved in [5, Theorem 7].

We finish the proof by seeing that d) holds. On one hand, the previous case and (5) immediately give the inclusion
\[ \{0\} \subseteq \mathcal{M}_{\mathbb{T}^N}(p,q) \subseteq H_\infty(\mathbb{T}^N). \]

We now show that \( \mathcal{M}_{\mathbb{D}_2^N}(p,q) = \{0\} \) for any \( N \in \mathbb{N} \cup \{\infty\} \). We consider in first place the case \( N \in \mathbb{N} \). For \( 1 \leq p < q < \infty \), we fix \( f \in \mathcal{M}_{\mathbb{D}_2^N}(p,q) \) and \( M_f \) the associated multiplication operator from \( H_p(\mathbb{D}^N) \) to \( H_q(\mathbb{D}^N) \). Now, given \( g \in H_p(\mathbb{D}_2^N) \), by (1) we have
\[
|f \cdot g(z)| \leq \left( \prod_{j=1}^{N} \frac{1}{1 - |z_j|^2} \right)^{1/q} \|f \cdot g\|_{H_q(\mathbb{D}_2^N)} \leq \left( \prod_{j=1}^{N} \frac{1}{1 - |z_j|^2} \right)^{1/q} C\|g\|_{H_p(\mathbb{D}_2^N)}. \tag{9}
\]

Now since \( f \in H_\infty(\mathbb{D}_2^N) \) and
\[
\|f\|_{H_\infty(\mathbb{D}_2^N)} = \lim_{r \to \infty} \sup_{z \in \mathbb{D}_2^N} |f(z)| = \lim_{r \to \infty} \sup_{z \in \mathbb{D}_2^N} |f(z)|,
\]
then there is a sequence \( (u_n)_n \subseteq \mathbb{D}^N \) such that \( \|u_n\|_{\infty} \to 1 \) and
\[
|f(u_n)| \to \|f\|_{H_\infty(\mathbb{D}_2^N)}. \tag{10}
\]

For each \( u_n \) there is a non-zero function \( g_n \in H_p(\mathbb{D}^N) \) (recall (2)) such that
\[
|g_n(u_n)| = \left( \prod_{j=1}^{N} \frac{1}{1 - |u_n|^2} \right)^{1/p} \|g_n\|_{H_p(\mathbb{D}^N)}.
\]

From this and (9) we get
\[
|f(u_n)| \left( \prod_{j=1}^{N} \frac{1}{1 - |u_n|^2} \right)^{1/p} \|g_n\|_{H_p(\mathbb{D}^N)} \leq \left( \prod_{j=1}^{N} \frac{1}{1 - |u_n|^2} \right)^{1/q} C\|g_n\|_{H_p(\mathbb{D}^N)}.
\]

Then,
\[
|f(u_n)| \left( \prod_{j=1}^{N} \frac{1}{1 - |u_n|^2} \right)^{1/p-1/q} \leq C.
\]

Since \( 1/p - 1/q > 0 \) we have that \( \left( \prod_{j=1}^{N} \frac{1}{1 - |u_n|^2} \right)^{1/p-1/q} \to \infty \), and then, by the previous inequality, \( |f(u_n)| \to 0 \). By (10) this shows that \( \|f\|_{H_\infty(\mathbb{D}^N)} = 0 \) and this gives the claim for \( q < \infty \). Now if \( q = \infty \), by noticing that \( H_\infty(\mathbb{D}^N) \) is contained in \( H_t(\mathbb{D}^N) \) for every \( 1 \leq p < t < \infty \) the result follows from the previous case. This concludes the proof for \( N \in \mathbb{N} \).

To prove that \( \mathcal{M}_{\mathbb{D}_2^N}(p,q) = \{0\} \), fix again \( f \in \mathcal{M}_{\mathbb{D}_2^N}(p,q) \). By Proposition 6, for every \( N \in \mathbb{N} \) the truncated function \( f_N \in \mathcal{M}_{\mathbb{D}_2^N}(p,q) \) and therefore, by what we have shown before, is the zero function. Now the proof follows using that \((f_N)_N\) converges pointwise to \( f \). \( \square \)
4 Multiplication operator

Given a multiplier \( D \in \mathcal{M}(p, q) \), we study in this section several properties of its associated multiplication operator \( M_D : \mathcal{H}_p \to \mathcal{H}_q \). In [26] Vukotić provides a very complete description of certain Toeplitz operators for Hardy spaces of holomorphic functions of one variable. In particular he studies the spectrum, the range and the essential norm of these operators. Bearing in mind the relation between the sets of multipliers that we proved above (Proposition 5), it is natural to ask whether similar properties hold when we look at the multiplication operators on the Hardy spaces of Dirichlet series.

In our first result we characterize which operators are indeed multiplication operators. These happen to be exactly those that commute with the monomials given by the prime numbers.

**Theorem 8.** Let \( 1 \leq p, q \leq \infty \). A bounded operator \( T : \mathcal{H}_p \to \mathcal{H}_q \) is a multiplication operator if and only if \( T \) commutes with the multiplication operators \( M_{p_i^{-s}} \) for every \( i \in \mathbb{N} \).

The same holds if we replace in each case \( \mathcal{H} \) by \( \mathcal{H}^{(N)} \) (with \( N \in \mathbb{N} \)), and considering \( M_{p_i^{-s}} \) with \( 1 \leq i \leq N \).

**Proof.** Suppose first that \( T : \mathcal{H}_p \to \mathcal{H}_q \) is a multiplication operator (that is, \( T = M_D \) for some Dirichlet series \( D \)) and for \( i \in \mathbb{N} \), let \( p_i^{-s} \) be a monomial, then

\[
T \circ M_{p_i^{-s}}(E) = D \cdot p_i^{-s} \cdot E = p_i^{-s} \cdot D \cdot E = M_{p_i^{-s}} \circ T(E).
\]

That is, \( T \) commutes with \( M_{p_i^{-s}} \).

For the converse, suppose now that \( T : \mathcal{H}_p \to \mathcal{H}_q \) is a bounded operator that commutes with the multiplication operators \( M_{p_i^{-s}} \) for every \( i \in \mathbb{N} \). Let us see that \( T = M_D \) with \( D = T(1) \). Indeed, for each \( p_i^{-s} \) and \( k \in \mathbb{N} \) we have that

\[
T((p_i^k)^{-s}) = T((p_i^{-s})^k) = T(M_{p_i^{-s}}^k(1)) = M_{p_i^{-s}}^k(T(1)) = (p_i^{-s})^k \cdot D = (p_i^{-s})^k \cdot D,
\]

and then given \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{N}_0^{(N)} \) such that \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \)

\[
T(n^{-s}) = T \left( \prod_{j=1}^{k} (p_i^{\alpha_j})^{-s} \right) = T(M_{p_1^{\alpha_1}}^{-s} \circ \cdots \circ M_{p_k^{\alpha_k}}^{-s})(1) = M_{p_1^{\alpha_1}}^{-s} \circ \cdots \circ M_{p_k^{\alpha_k}}^{-s}(T(1)) = (n^{-s}) \cdot D.
\]

This implies that \( T(P) = P \cdot D \) for every Dirichlet polynomial \( P \). Take now some \( E \in \mathcal{H}_p \) and choose a sequence of polynomials \( P_n \) that converges in norm to \( E \) if \( 1 \leq p < \infty \) or weakly if \( p = \infty \) (see [9, Theorems 5.18 and 11.10]). In any case, if \( s \in \mathbb{C}_{1/2} \), the continuity of the evaluation at \( s \) (see again [9, Corollary 13.3]) yields \( P_n(s) \to E(s) \). Since \( T \) is continuous, we have that

\[
T(E) = \lim_n T(P_n) = \lim_n P_n \cdot D
\]

(where the limit is in the weak topology if \( p = \infty \)). Then for each \( s \in \mathbb{C} \) such that \( \text{Re } s > 1/2 \), we have

\[
T(E)(s) = \lim_n P_n \cdot D(s) = E(s)D(s).
\]

Therefore, \( T(E) = D \cdot E \) for every Dirichlet series \( E \). In other words, \( T \) is equal to \( M_D \), which concludes the proof. \( \square \)
Given a bounded operator \( T : E \rightarrow F \) the essential norm is defined as
\[
\|T\|_{\text{ess}} = \inf\{\|T - K\| : K : E \rightarrow F \text{ compact}\}.
\]
This norm tells us how far from being compact \( T \) is.

The following result shows a series of comparisons between essential norm of \( M_D : \mathcal{H}_p \rightarrow \mathcal{H}_q \) and the norm of \( D \), depending on \( p \) and \( q \). In all cases, as a consequence, the operator is compact if and only if \( D = 0 \).

**Theorem 9.**

a) Let \( 1 \leq q < p < \infty \), \( D \in \mathcal{H}_{pq/(p-q)} \) and \( M_D \) its associated multiplication operator from \( \mathcal{H}_p \) to \( \mathcal{H}_q \). Then
\[
\|D\|_{\mathcal{H}_q} \leq \|M_D\|_{\text{ess}} \leq \|M_D\| = \|D\|_{\mathcal{H}_{pq/(p-q)}}.
\]

b) Let \( 1 \leq q < \infty \), \( D \in \mathcal{H}_q \) and \( M_D : \mathcal{H}_\infty \rightarrow \mathcal{H}_q \) the multiplication operator. Then
\[
\frac{1}{2} \|D\|_{\mathcal{H}_q} \leq \|M_D\|_{\text{ess}} \leq \|M_D\| = \|D\|_{\mathcal{H}_q}.
\]

In particular, \( M_D \) is compact if and only if \( D = 0 \). The same equalities hold if we replace \( \mathcal{H} \) by \( \mathcal{H}^{(N)} \) (with \( N \in \mathbb{N} \)).

We start with a lemma based on [6, Proposition 2] for Hardy spaces of holomorphic functions. We prove that weak-star convergence and uniformly convergence on half-planes are equivalent on Hardy spaces of Dirichlet series. We are going to use that \( \mathcal{H}_p \) is a dual space for every \( 1 \leq p < \infty \). For \( 1 < p < \infty \) this is obvious because the space is reflexive. For \( p = 1 \) in [10, Theorem 7.3] it is shown, for Hardy spaces of vector valued Dirichlet series, that \( \mathcal{H}_1(X) \) is a dual space if and only if \( X \) has the Analytic Radon-Nikodym property. Since \( \mathbb{C} \) has the ARNP, this gives what we need. We include here an alternative proof in more elementary terms.

**Proposition 10.** The space \( \mathcal{H}_1 \) is a dual space.

**Proof.** Denote by \( (B_{H_1}, \tau_0) \) the closed unit ball of \( H_1(\mathbb{D}^\infty_2) \), endowed with the topology \( \tau_0 \) given by the uniform convergence on compact sets. Let us show that \( (B_{H_1}, \tau_0) \) is a compact set. Note first that, given a compact \( K \subseteq \ell_2 \) and \( \varepsilon > 0 \), there exists \( j_0 \in \mathbb{N} \) such that \( \sum_{j \geq j_0} |z_j|^2 < \varepsilon \) for all \( z \in K \) [12, Page 6]. Then, from Cole-Gamelin inequality (1), the set
\[
\{f(K) : f \in B_{H_1}\} \subset \mathbb{C}
\]
is bounded for each compact set \( K \). By Montel’s theorem (see e.g. [9, Theorem 15.50]), \( (B_{H_1}, \tau_0) \) is relatively compact. We now show that \( (B_{H_1}, \tau_0) \) is closed. Indeed, suppose now that \( (f_n) \subset B_{H_1} \) is a net that converges to \( B_{H_1} \) uniformly on compact sets, then we obviously have
\[
\int_{\mathbb{T}^N} |f(r\omega, 0, 0, \cdots)|d\omega \leq \int_{\mathbb{T}^N} |f(r\omega, 0, 0, \cdots) - f_\circ(r\omega, 0, 0, \cdots)|d\omega + \int_{\mathbb{T}^N} |f_\circ(r\omega, 0, 0, \cdots)|d\omega.
\]
Since the first term tends to 0 and the second term is less than or equal to 1 for every \( N \in \mathbb{N} \) and every \( 0 < r < 1 \), then the limit function \( f \) belongs to \( B_{H_1} \). Thus, \( (B_{H_1}, \tau_0) \) is
compact. We consider now the set of functionals

\[ \{ev_z : H_1(\mathbb{D}_2^\infty) \to \mathbb{C} : z \in \mathbb{D}_2^\infty \} \]

Note that the weak topology \( w(H_1, E) \) is exactly the topology given by the pointwise convergence. Thus, since a priori \( \tau_0 \) is clearly a stronger topology than \( w(H_1, E) \) we have that \( (B_{H_1}, w(H_1, E)) \) is also compact. Since \( E \) separates points, by [15, Theorem 1], \( H_1(\mathbb{D}_2^\infty) \) is a dual space and hence, using the Bohr transform, \( \mathcal{H}_1 \) also is a dual space. \( \square \)

**Lemma 11.** Let \( 1 \leq p < \infty \) and \( (D_n) \subseteq \mathcal{H}_p \) then the following statements are equivalent

a) \( D_n \to 0 \) in the weak-star topology.

b) \( D_n(s) \to 0 \) for each \( s \in \mathbb{C}_{1/2} \) and \( \|D_n\|_{\mathcal{H}_p} \leq C \) for some \( C < 0 \).

c) \( D_n \to 0 \) uniformly on each half-plane \( \mathbb{C}_{\sigma} \) with \( \sigma > 1/2 \) and \( \|D_n\|_{\mathcal{H}_p} \leq C \) for some \( C < 0 \).

**Proof.** The implication a) then b) is verified by the continuity of the evaluations in the weak-star topology, and because the convergence in this topology implies that the sequence is bounded.

Let us see that b) implies c). Suppose not, then there exists \( \varepsilon > 0 \), a subsequence \( (D_{n_j})_j \) and a half-plane \( \mathbb{C}_{\sigma} \) with \( \sigma > 1/2 \) such that \( \sup_{s \in \mathbb{C}_{\sigma}} |D_{n_j}(s)| \geq \varepsilon \). Since \( D_{n_j} = \sum_m a_m m^{-s} \) is uniformly bounded, by Montel’s theorem for \( \mathcal{H}_p \) (see [8, Theorem 3.2]), there exists \( D = \sum m a_m m^{-s} \in \mathcal{H}_p \) such that

\[ \sum_m \frac{a_{n_j}}{m^\delta} m^{-s} \to \sum_m \frac{a_m}{m^\delta} m^{-s} \text{ in } \mathcal{H}_p \]

for every \( \delta > 0 \). Given \( s \in \mathbb{C}_{1/2} \), we write \( s = s_0 + \delta \) with \( \delta > 0 \) and \( s_0 \in \mathbb{C}_{1/2} \), to have

\[ D_{n_j}(s) = \sum_m \frac{a_{n_j}}{m^\delta} m^{-(s_0+\delta)} = \sum_m \frac{a_{n_j}}{m^\delta} m^{-s_0} \to \sum_m \frac{a_m}{m^\delta} m^{-s_0} = D(s_0 + \delta) = D(s). \]

We conclude that \( D = 0 \) and by Cole-Gamelin inequality for Dirichlet series (see [9, Corollary 13.3]) we have

\[ \varepsilon \leq \sup_{Re s > 1/2 + \sigma} |D_{n_j}(s)| = \sup_{Re s > 1/2 + \sigma/2} |D_{n_j}(s + \sigma/2)| \]

\[ = \sup_{Re s > 1/2 + \sigma/2} \left\| \sum_m \frac{a_{n_j}}{m^{\sigma/2}} m^{-s} \right\| \leq \zeta(2 \text{ Re } s)^{1/p} \left\| \sum_m \frac{a_{n_j}}{m^{\sigma/2}} m^{-s} \right\| \mathcal{H}_p \]

\[ \leq \zeta(1 + \sigma)^{1/p} \left\| \sum_m \frac{a_{n_j}}{m^{\sigma/2}} m^{-s} \right\| \mathcal{H}_p \rightarrow 0, \]

for every \( \sigma > 0 \), which is a contradiction.

To see that c) implies a), let \( B_{\mathcal{H}_p} \) denote the closed unit ball of \( \mathcal{H}_1 \). Since for each \( 1 \leq p < \infty \) the space \( \mathcal{H}_p \) is a dual space, by Alaoglu’s theorem, \( (B_{\mathcal{H}_p}, w^*) \) (i.e. endowed
with the weak-star topology) is compact. On the other hand \((B_{\mathcal{H}_p}, \tau_0)\) (that is, endowed with the topology of uniform convergence on compact sets) is a Hausdorff topological space. If we show that the identity \(Id: (B_{\mathcal{H}_p}, w^*) \rightarrow (B_{\mathcal{H}_p}, \tau_0)\) is continuous, then it is a homeomorphism and the proof is completed. To see this let us note first that \(\mathcal{H}_p\) is separable (note that the set of Dirichlet polynomials with rational coefficients is dense in \(\mathcal{H}_p\)) and then \((B_{\mathcal{H}_p}, w^*)\) is metrizable (see [7, Theorem 5.1]). Hence it suffices to work with sequences. If a sequence \((D_n)\) converges in \(w^*\) to some \(D\), then in particular \((D_n - D)_n\) \(w^*\)-converges to 0 and, by what we just have seen, it converges uniformly on compact sets. This shows that \(Id\) is continuous, as we wanted. 

Now we prove Theorem 9. The arguments should be compared with [11, Propositions 4.3 and 5.5] where similar statements have been obtained for weighted composition operators for holomorphic functions of one complex variable.

**Proof of Theorem 9.** a) By definition \(\|M_D\|_{\text{ess}} \leq \|M_D\| = \|D\|_{\mathcal{H}_{pq}/(p,q)}\). In order to see the lower bound, for each \(n \in \mathbb{N}\) consider the monomial \(E_n = (2^n)^{-\delta} \in \mathcal{H}_p\). Clearly \(\|E_n\|_{\mathcal{H}_p} = 1\) for every \(n\), and \(E_n(s) \rightarrow 0\) for each \(s \in \mathbb{C}_{1/2}\). Then, by Lemma 11, \(E_n \rightarrow 0\) in the weak-star topology.

Take now some compact operator \(K: \mathcal{H}_p \rightarrow \mathcal{H}_q\) and note that, since \(\mathcal{H}_p\) is reflexive, we have \(K(E_n) \rightarrow 0\), and hence

\[
\|M_D - K\| \geq \limsup_{n \to \infty} \|M_D(E_n) - K(E_n)\|_{\mathcal{H}_q} 
\]

\[
\geq \limsup_{n \to \infty} \|D \cdot E_n\|_{\mathcal{H}_q} - \|K(E_n)\|_{\mathcal{H}_q} = \|D\|_{\mathcal{H}_q}.
\]

b) Let \(K: \mathcal{H}_p \rightarrow \mathcal{H}_q\) be a compact operator, and take again \(E_n = (2^n)^{-\delta} \in \mathcal{H}_p\) for each \(n \in \mathbb{N}\). Since \(\|E_n\|_{\mathcal{H}_p} = 1\) then there exists a subsequence \((E_{n_j})\) such that \((K(E_{n_j}))\) converges in \(\mathcal{H}_q\). Given \(\epsilon > 0\) there exists \(m \in \mathbb{N}\) such that if \(j, l \geq m\) then

\[
\|K(E_{n_j}) - K(E_{n_l})\|_{\mathcal{H}_q} < \epsilon.
\]

On the other hand, if \(D = \sum a_k k^{-s}\) then \(D \cdot E_{n_j} = \sum a_k (k \cdot 2^n)^{-s}\) and by [9, Proposition 11.20] the norm in \(\mathcal{H}_q\) of

\[
(D \cdot E_{n_j} \delta) = \sum \frac{a_k}{(k \cdot 2^n)^{\delta}} (k \cdot 2^n)^{-s}
\]

tends increasingly to \(\|D \cdot E_{n_j}\|_{\mathcal{H}_q} = \|D\|_{\mathcal{H}_q}\) when \(\delta \to 0\). Fixed \(j \geq m\), there exists \(\delta > 0\) such that

\[
\|(D \cdot E_{n_j} \delta)\|_{\mathcal{H}_q} \geq \|D\|_{\mathcal{H}_q} - \epsilon.
\]

Given that \(\|E_{n_j} - E_{n_l}\|_{\mathcal{H}_p} = 1\) for every \(j \neq l\) then

\[
\|M_D - K\| \geq \|(M_D - K) \frac{E_{n_j} - E_{n_l}}{2}\|_{\mathcal{H}_q} 
\]

\[
\geq \frac{1}{2} \|D \cdot E_{n_j} - D \cdot E_{n_l}\|_{\mathcal{H}_q} - \frac{1}{2} \|K(E_{n_j}) - K(E_{n_l})\|_{\mathcal{H}_q} 
\]

\[
\geq \frac{1}{2} \|(D \cdot E_{n_j} \delta)\|_{\mathcal{H}_q} - \|D \cdot E_{n_l} \delta\|_{\mathcal{H}_q} - \frac{1}{2} \|K(E_{n_j}) - K(E_{n_l})\|_{\mathcal{H}_q} - \epsilon 
\]

\[
\geq \frac{1}{2} \|D\|_{\mathcal{H}_q} - \frac{1}{2} \|D \cdot E_{n_j} \delta\|_{\mathcal{H}_q} - \epsilon.
\]
Finally, since
\[ \| (D \cdot E_n) \delta \|_{\mathcal{H}_q} \leq \| D \delta \|_{\mathcal{H}_q} \| (E_n) \delta \|_{\mathcal{H}_\infty} \leq \| D \delta \|_{\mathcal{H}_q} \left( \frac{2^n}{2n \delta} \right)^s \| \mathcal{H}_{\infty} = \| D \delta \|_{\mathcal{H}_q} \cdot \frac{1}{2n \delta}, \]
and the latter tends to 0 as \( l \to \infty \), we finally have \( \| M_D - K \| \geq \frac{1}{2} \| D \|_{\mathcal{H}_q} \). \( \square \)

In the case of endomorphism, that is \( p = q \), we give the following bounds for the essential norms.

**Theorem 12.** Let \( D \in \mathcal{H}_\infty \) and \( M_D : \mathcal{H}_p \to \mathcal{H}_p \) the associated multiplication operator.

\( a) \) If \( 1 < p \leq \infty \), then
\[ \| M_D \|_{\text{ess}} = \| M_D \| = \| D \|_{\mathcal{H}_\infty}. \]

\( b) \) If \( p = 1 \), then
\[ \max \left\{ \frac{1}{2} \| D \|_{\mathcal{H}_\infty}, \| D \|_{\mathcal{H}_1} \right\} \leq \| M_D \|_{\text{ess}} \leq \| M_D \| = \| D \|_{\mathcal{H}_\infty}. \]

In particular, \( M_D \) is compact if and only if \( D = 0 \). The same equalities hold if we replace \( \mathcal{H} \) by \( \mathcal{H}^{(N)} \), with \( N \in \mathbb{N} \).

The previous theorem will be a consequence of the Proposition 14 which we feel is independently interesting. For the proof we need the following technical lemma in the spirit of [6, Proposition 2]. It relates weak-star convergence and uniform convergence on compact sets for Hardy spaces of holomorphic functions. It is a sort of ‘holomorphic version’ of Lemma 11.

**Lemma 13.** Let \( 1 \leq p < \infty \), \( N \in \mathbb{N} \cup \{ \infty \} \) and \( (f_n) \subseteq H_p(\mathbb{D}_2^N) \) then the following statements are equivalent

\( a) \) \( f_n \to 0 \) in the weak-star topology,

\( b) \) \( f_n(z) \to 0 \) for each \( z \in \mathbb{D}_2^N \) and \( \| f_n \|_{H_p(\mathbb{D}_2^N)} \leq C \)

\( c) \) \( f_n \to 0 \) uniformly on compact sets of \( \mathbb{D}_2^N \) and \( \| f_n \|_{H_p(\mathbb{D}_2^N)} \leq C \).

**Proof.** \( a) \Rightarrow b) \) and \( c) \Rightarrow a) \) are proved with the same arguments used in Lemma 11. Let us see \( b) \Rightarrow c) \). Suppose not, then there exists \( \varepsilon > 0 \), a subsequence \( f_{n_j} \) and a compact set \( K \subseteq \mathbb{D}_2^\infty \) such that \( \| f_{n_j} \|_{H_\infty(K)} \geq \varepsilon \). Since \( f_{n_j} \) is bounded, by Montel’s theorem for \( H_p(\mathbb{D}_2^N) \) (see [25, Theorem 2]), we can take a subsequence \( f_{n_{j_i}} \) and \( f \in H_p(\mathbb{D}_2^N) \) such that \( f_{n_{j_i}} \to f \) uniformly on compact sets. But since it tends pointwise to zero, then \( f = 0 \) which is a contradiction. \( \square \)

**Proposition 14.** Let \( 1 \leq p < \infty \), \( f \in H_\infty(\mathbb{D}_2^\infty) \) and \( M_f : H_p(\mathbb{D}_2^\infty) \to H_p(\mathbb{D}_2^\infty) \) the multiplication operator. If \( p > 1 \) then
\[ \| M_f \|_{\text{ess}} = \| M_f \| = \| f \|_{H_\infty(\mathbb{D}_2^\infty)}. \]

If \( p = 1 \) then
\[ \| M_f \| \geq \| M_f \|_{\text{ess}} \geq \frac{1}{2} \| M_f \|. \]

In particular \( M_f : H_p(\mathbb{D}_2^\infty) \to H_p(\mathbb{D}_2^\infty) \) is compact if and only if \( f = 0 \). The same equalities hold if we replace \( \mathbb{D}_2^\infty \) by \( \mathbb{D}^N \), with \( N \in \mathbb{N} \).
Proof. The inequality \( \|M_f\|_{\text{ess}} \leq \|M_f\| = \|f\|_{H_0(\mathbb{D}_2^N)} \) is already known for every \( N \in \mathbb{N} \cup \{\infty\} \). It is only left, then, to see that

\[
\|M_f\| \leq \|M_f\|_{\text{ess}}.
\]

We begin with the case \( N \in \mathbb{N} \). Assume in first place that \( p > 1 \), and take a sequence \((z^{(n)})_n \subseteq \mathbb{D}^N\), with \( \|z^{(n)}\|_\infty \rightarrow 1 \), such that \( |f(z^{(n)})| \rightarrow \|f\|_{H_0(\mathbb{D}^N)} \). Consider now the function given by

\[
h_{z^{(n)}}(u) = \left( \prod_{j=1}^{N} \frac{1 - |z_j^{(n)}|^2}{(1 - z_j^{(n)}u_j)^2} \right)^{1/p},
\]

for \( u \in \mathbb{D}^N \). Now, by the Cole-Gamelin inequality (1)

\[
|f(z^{(n)})| = |f(z^{(n)}) \cdot h_{z^{(n)}}(z^{(n)})| \left( \prod_{j=1}^{N} \frac{1}{1 - |z_j^{(n)}|^2} \right)^{-1/p} \leq \|f \cdot h_{z^{(n)}}\|_{H_p(\mathbb{D}_2^N)} \leq \|f\|_{H_0(\mathbb{D}_2^N)},
\]

and then \( \|f \cdot h_{z^{(n)}}\|_{H_p(\mathbb{D}_2^N)} \rightarrow \|f\|_{H_0(\mathbb{D}_2^N)} \).

Observe that \( \|h_{z^{(n)}}\|_{H_p(\mathbb{D}_2^N)} = 1 \) and that \( h_{z^{(n)}}(u) \rightarrow 0 \) as \( n \rightarrow \infty \) for every \( u \in \mathbb{D}^N \). Then Lemma 13 \( h_{z^{(n)}} \) tends to zero in the weak-star topology and then, since \( H_p(\mathbb{D}_2^N) \) is reflexive (recall that \( 1 < p < \infty \)), also in the weak topology. So, if \( K \) is a compact operator on \( H_p(\mathbb{D}_2^N) \) then \( K(h_{z^{(n)}}) \rightarrow 0 \) and therefore

\[
\|M_f - K\| \geq \limsup_{n \rightarrow \infty} \|f \cdot h_{z^{(n)}} - K(h_{z^{(n)})}\|_{H_p(\mathbb{D}_2^N)} \\
\geq \limsup_{n \rightarrow \infty} \|f \cdot h_{z^{(n)}}\|_{H_p(\mathbb{D}_2^N)} - \|K(h_{z^{(n)})\|_{H_p(\mathbb{D}_2^N)} = \|f\|_{H_0(\mathbb{D}_2^N)}.
\]

Thus, \( \|M_f - K\| \geq \|f\|_{H_0(\mathbb{D}_2^N)} \) for each compact operator \( K \) and hence \( \|M_f\|_{\text{ess}} \geq \|M_f\| \) as we wanted to see.

The proof of the case \( p = 1 \) follows some ideas of Demazeux in [11, Theorem 2.2]. First of all, recall that the \( N \)-dimensional Féjer’s Kernel is defined as

\[
K_n^N(u) = \sum_{|x_1|, \ldots, |x_N| \leq N} \prod_{j=1}^{N} \left( 1 - \frac{|x_j|}{n+1} \right) u^x,
\]

for \( u \in \mathbb{D}_2^N \). With this, the \( n \)-th Féjer polynomial with \( N \) variables of a function \( g \in H_p(\mathbb{D}_2^N) \) is obtained by convoluting \( g \) with the \( N \)-dimensional Féjer’s Kernel, in other words

\[
\sigma_n^N g(u) = \frac{1}{(n+1)^N} \sum_{l_1, \ldots, l_N=1}^{n} \sum_{|x_j| \leq l_j} \hat{g}(x) u^x. \tag{12}
\]

It is well known (see e.g. [9, Lemmas 5.21 and 5.23]) that \( \sigma_n^N : H_1(\mathbb{D}_2^N) \rightarrow H_1(\mathbb{D}_2^N) \) is a contraction and \( \sigma_n^N g \rightarrow g \) on \( H_1(\mathbb{D}_2^N) \) when \( n \rightarrow \infty \) for all \( g \in H_1(\mathbb{D}_2^N) \). Let us see how \( R_n^N = I - \sigma_n^N \) gives a first lower bound for the essential norm.

Let \( K : H_1(\mathbb{D}_2^N) \rightarrow H_1(\mathbb{D}_2^N) \) be a compact operator, since \( \|\sigma_n^N\| \leq 1 \) then \( \|R_n^N\| \leq 2 \) and hence

\[
\|M_f - K\| \geq \frac{1}{2} \|R_n^N \circ (M_f - K)\| \geq \frac{1}{2} \|R_n^N \circ M_f\| - \frac{1}{2} \|R_n^N \circ K\|.
\]
On the other side, since $R_n^N \to 0$ pointwise, $R_n^N$ tends to zero uniformly on compact sets of $H_1(\mathbb{D}^N)$. In particular on the compact set $K(B_{H_1}(\mathbb{D}^N))$, and therefore $\|R_n^N \circ K\| \to 0$. We conclude then that $\|M_f\|_{\text{ess}} \geq \frac{1}{2} \lim_{n \to \infty} \|R_n^N \circ M_f\|$.

Our aim now is to obtain a lower bound for the right-hand-side of the inequality. To get this, we are going to see that

$$\|\sigma_n^N \circ M_f(h_z)\|_{H_1(\mathbb{D}^N)} \to 0 \text{ when } \|z\|_\infty \to 1,$$

where $h_z$ is again defined, for each fixed $z \in \mathbb{D}^N$, by

$$h_z(u) = \prod_{j=1}^N \frac{1 - |z_j|^2}{(1 - z_j u_j)^2}.$$

To see this, let us consider first, for each $z \in \mathbb{D}^N$, the function $g_z(u) = \prod_{j=1}^N \frac{1}{(1 - z_j u_j)^2}$. This is clearly holomorphic and, hence, has a development $a$ as Taylor series

$$g_z(u) = \sum_{\alpha \in \mathbb{N}_0^N} c_{\alpha}(g_z) u^\alpha$$

for $u \in \mathbb{D}^N$. Our first step is to see that the Taylor coefficients up to a fixed degree are bounded uniformly on $z$. Recall that $c_{\alpha}(g_z) = \frac{1}{\alpha!} \frac{\partial^{\alpha} g(0)}{\partial u^\alpha}$ and, since

$$\frac{\partial^{\alpha} g_z(u)}{\partial u^\alpha} = \prod_{j=1}^N \frac{(\alpha_j + 1)!}{(1 - z_j u_j)^{2+\alpha_j}} (z_j)^{\alpha_j},$$

we have

$$c_{\alpha}(g_z) = \frac{1}{\alpha!} \frac{\partial^{\alpha} g_z(0)}{\partial u^\alpha} = \frac{1}{\alpha!} \prod_{j=1}^N (\alpha_j + 1)! (z_j)^{\alpha_j} = \left( \prod_{j=1}^N (\alpha_j + 1) \right) z^\alpha.$$

Thus $|c_{\alpha}(g_z)| \leq (M + 1)^N$ whenever $|\alpha| \leq M$.

On the other hand, for each $\alpha \in \mathbb{N}_0^N$ (note that $h_z(u) = g_z(u) \prod_{j=1}^N (1 - |z_j|)$ for every $u$) we have

$$c_{\alpha}(f \cdot h_z) = \left( \prod_{j=1}^N (1 - |z_j|^2) \right) \sum_{\beta + \gamma = \alpha} \hat{f}(\beta) \hat{g}_z(\gamma).$$

Taking all these into account we finally have (recall (12)), for each fixed $n \in \mathbb{N}$

$$\|\sigma_n^N \circ M_f(h_z)\|_{H_1(\mathbb{D}^N)} \leq \left( \prod_{j=1}^N (1 - |z_j|^2) \right) \frac{1}{(n + 1)^N} \sum_{l_1, \ldots, l_N = 1}^N \sum_{|\alpha_j| \leq l_j} \sum_{\beta + \gamma = \alpha} \|\hat{f}(\beta)\hat{g}_z(\gamma)\| u^\alpha\|_{H_1(\mathbb{D}^N)}$$

$$\leq \left( \prod_{j=1}^N (1 - |z_j|^2) \right) \frac{1}{(n + 1)^N} \sum_{l_1, \ldots, l_N = 1}^N \sum_{|\alpha_j| \leq l_j} \sum_{\beta + \gamma = \alpha} \|f\|_{H_{\infty}(\mathbb{D}^N)} (N + 1)^N,$$

which immediately yields (13). Once we have this we can easily conclude the argument. For each $n \in \mathbb{N}$ we have
\[ \| R_n^N \circ M_f \| = \| M_f - \sigma_n^N \circ M_f \| \geq \| M_f(h_z) - \sigma_n^N \circ M_f(h_z) \|_{H_1(\mathbb{D})} \geq \| M_f(h_z) \|_{H_1(\mathbb{D})} - \| \sigma_n^N \circ M_f(h_z) \|_{H_1(\mathbb{D})}, \]

and since the last term tends to zero if \( \| z \|_\infty \to 1 \), then

\[ \| R_n^N \circ M_f \| \geq \lim \sup_{\| z \| \to 1} \| M_f(h_z) \|_{H_1(\mathbb{D})} \geq \| f \|_{H_\infty(\mathbb{D})}, \]

which finally gives

\[ \| M_f \|_{\text{ess}} \geq \frac{1}{2} \| f \|_{H_\infty(\mathbb{D})} = \frac{1}{2} \| M_f \|, \]

as we wanted.

To complete the proof we consider the case \( N = \infty \). So, what we have to see is that

\[ \| M_f \| \geq \| M_f \|_{\text{ess}} \geq C \| M_f \|, \quad (14) \]

where \( C = 1 \) if \( p > 1 \) and \( C = 1/2 \) if \( p = 1 \). Let \( K : H_p(\mathbb{D}_2^\infty) \to H_p(\mathbb{D}_2^\infty) \) be a compact operator, and consider for each \( N \in \mathbb{N} \) the continuous operators \( J_N : H_p(\mathbb{D}) \to H_p(\mathbb{D}_2^\infty) \) given by the inclusion and \( J_N : H_p(\mathbb{D}_2^\infty) \to H_p(\mathbb{D}) \) defined by \( J(g)(u) = g(u_1, \ldots, u_N, 0) = g_N(u) \) then \( K_N = J_N \circ K \circ J_N : H_p(\mathbb{D}_2^\infty) \to H_p(\mathbb{D}) \) is compact. On the other side we have that \( J_N \circ M_f \circ J_N(g) = f_n \cdot g = M_{j_N}(g) \) for every \( g \), furthermore given any operator \( T : H_p(\mathbb{D}_2^\infty) \to H_p(\mathbb{D}_2^\infty) \) and defining \( T_N \) as before we have that

\[ \| T \| = \sup_{\| g \|_{H_p(\mathbb{D}_2^\infty)} \leq 1} \| T(g) \|_{H_p(\mathbb{D}_2^\infty)} \geq \sup_{\| g \|_{H_p(\mathbb{D}_2^\infty)} \leq 1} \| T(g) \|_{H_p(\mathbb{D}_2^\infty)} \geq \sup_{\| g \|_{H_p(\mathbb{D}_2^\infty)} \leq 1} \| T_M(g) \|_{H_p(\mathbb{D}_2^\infty)} = \| T_N \|, \]

and therefore

\[ \| M_f - K \| \geq \| M_{J_N} - K_N \| \geq \| M_{f_N} \|_{\text{ess}} \geq C \| f_N \|_{H_{\infty}(\mathbb{D}_2^\infty)}. \]

Since \( \| f_N \|_{H_{\infty}(\mathbb{D}_2^\infty)} \to \| f \|_{H_{\infty}(\mathbb{D}_2^\infty)} \) when \( N \to \infty \) we have (14), and this completes the proof. \( \square \)

We can now prove Theorem 12.

**Proof of Theorem 12.** Since for every \( 1 \leq p < \infty \) the Bohr lift \( \mathcal{L}_D^N : \mathcal{F}_p^N \to H_p(\mathbb{D}_2^N) \) and the Bohr transform \( \mathcal{B}_D^N : H_p(\mathbb{D}_2^N) \to \mathcal{F}_p^N \) are isometries, then an operator \( K : \mathcal{F}_p^N \to \mathcal{F}_p^N \) is compact if and only if \( K_h = \mathcal{L}_D^N \circ K \circ \mathcal{B}_D^N : H_p(\mathbb{D}_2^N) \to H_p(\mathbb{D}_2^N) \) is a compact operator. On the other side \( f = \mathcal{L}_D^N(D) \) hence \( M_f = \mathcal{L}_D^N \circ M_D \circ \mathcal{B}_D^N \) and therefore

\[ \| M_D - K \| = \| \mathcal{L}_D^{-1} \circ (M_f - K_h) \circ \mathcal{L}_D^N \| = \| M_f - K_h \| \geq C \| f \|_{H_{\infty}(\mathbb{D}_2^N)} = C \| D \|_{\mathcal{F}_p^N}, \]

where \( C = 1 \) if \( p > 1 \) and \( C = 1/2 \) if \( p = 1 \). Since this holds for every compact operator \( K \) then we have the inequality that we wanted. The upper bound is clear by the definition of essential norm.
On the other hand, if \( p = 1 \) and \( N \in \mathbb{N} \cup \{ \infty \} \). Let \( 1 < q < \infty \) an consider \( M^q_D : \mathcal{H}^{(N)}_q \to \mathcal{H}^{(N)}_1 \) the restriction. If \( K : \mathcal{H}^{(N)}_1 \to \mathcal{H}^{(N)}_1 \) is compact then its restriction \( K^q : \mathcal{H}^{(N)}_q \to \mathcal{H}^{(N)}_1 \) is also compact and then
\[
\|M^q_D - K\|_{\mathcal{H}^{(N)}_1 \to \mathcal{H}^{(N)}_1} = \sup_{\|E\|_{\mathcal{H}^{(N)}_1} \leq 1} \|M^q_D(E) - K(E)\|_{\mathcal{H}^{(N)}_1} \\
\geq \sup_{\|E\|_{\mathcal{H}^{(N)}_q} \leq 1} \|M^q_D(E) - K(E)\|_{\mathcal{H}^{(N)}_1} \\
= \|M^q_D - K^q\|_{\mathcal{H}^{(N)}_q \to \mathcal{H}^{(N)}_1} \geq \|M^q_D\|_{\text{ess}} \geq \|D\|_{\mathcal{H}^{(N)}_1}.
\]
Finally, the case \( p = \infty \) was proved in [17, Corollary 2.4].

5 Spectrum of Multiplication operators

In this section, we provide a characterization of the spectrum of the multiplication operator \( M_D \), with respect to the image of its associated Dirichlet series in some specific half-planes. Let us first recall some definitions of the spectrum of an operator. We say that \( \lambda \) belongs to the spectrum of \( M_D \), that we note \( \sigma(M_D) \), if the operator \( M_D - \lambda I : \mathcal{H}_p \to \mathcal{H}_p \) is not invertible. Now, a number \( \lambda \) can be in the spectrum for different reasons and according to these we can group them into the following subsets:

- If \( M_D - \lambda I \) is not injective then \( \lambda \in \sigma_p(M_D) \), the point spectrum.
- If \( M_D - \lambda I \) is injective and the \( \text{Ran}(A - \lambda I) \) is dense (but not closed) in \( \mathcal{H}_p \) then \( \lambda \in \sigma_c(M_D) \), the continuous spectrum of \( M_D \).
- If \( M_D - \lambda I \) is injective and its range has codimension greater than or equal to 1 then \( \lambda \) belongs to \( \sigma_r(M_D) \), the radial spectrum.

We are also interested in the approximate spectrum, noted by \( \sigma_{ap}(M_D) \), given by those values \( \lambda \in \sigma(M_D) \) for which there exist a unit sequence \( (E_n)_n \subseteq \mathcal{H}_p \) such that \( \|M_D(E_n) - \lambda E_n\|_{\mathcal{H}_p} \to 0 \).

Vukotić, in [26, Theorem 7], proved that the spectrum of a Multiplication operator, induced by function \( f \) in the one dimensional disk, coincides with \( \overline{f(D)} \). In the case of the continuous spectrum, the description is given from the outer functions in \( H_\infty(\mathbb{D}) \). The notion of outer function can be extended to higher dimensions. If \( N \in \mathbb{N} \cup \{ \infty \} \), a function \( f \in H_\rho(\mathbb{D}^N_2) \) is said to be outer if it satisfies
\[
\log |f(0)| = \int_{\mathbb{T}^N} \log |F(\omega)|d\omega,
\]
with \( f \sim F \). A closed subspace \( S \) of \( H_\rho(\mathbb{D}^N_2) \) is said to be invariant, if for every \( g \in S \) it is verified that \( z_1 \cdot g \in S \) for every monomial. Finally, a function \( f \) is said to be cyclic, if the invariant subspace generated by \( f \) is exactly \( H_\rho(\mathbb{D}^N_2) \). The mentioned characterization comes from the generalized Beurling’s Theorem, which affirms that \( f \) is a cyclic vector if and only if \( f \) is an outer function. In several variables, there exist outer functions which fail to be cyclic (see [21, Theorem 4.4.8]). We give now the aforementioned characterization of the spectrum of a multiplication operator.
Theorem 15. Given $1 \leq p < \infty$ and $D \in \mathcal{H}_\infty$ a non-zero Dirichlet series with associated multiplication operator $M_D : \mathcal{H}_p \to \mathcal{H}_p$. Then

a) $M_D$ is onto if and only if there is some $c > 0$ such that $|D(s)| \geq c$ for every $s \in \mathbb{C}_0$.

b) $\sigma(M_D) = \overline{D(\mathbb{C}_0)}$.

c) If $D$ is not constant then $\sigma_c(M_D) \subseteq \overline{D(\mathbb{C}_0)} \setminus D(\mathbb{C}_{1/2})$. Even more, if $\lambda \in \sigma_c(M_D)$ then $f - \lambda = D_{D_2}(D) - \lambda$ is an outer function in $H_\infty(\mathbb{D}_2^\infty)$.

The same holds if we replace in each case $\mathcal{H}$ by $\mathcal{H}^{(N)}$ (with $N \in \mathbb{N}$).

Proof. a) Because of the injectivity of $M_D$, and the Closed Graph Theorem, the mapping $M_D$ is surjective if and only if $M_D$ is invertible and this happens if and only if $M_{D^{-1}}$ is well defined and continuous, but then $D^{-1} \in \mathcal{H}_\infty$ and [19, Theorem 6.2.1] gives the conclusion.

b) Note that $M_D - \lambda I = M_{D-\lambda}$; this and the previous result give that $\lambda \notin \sigma(M_D)$ if and only if $|D(s) - \lambda| > \epsilon$ for some $\epsilon > 0$ and all $s \in \mathbb{C}_0$, and this happens if and only if $\lambda \notin \overline{D(\mathbb{C}_0)}$.

c) Let us suppose that the range of $M_D - \lambda = M_{D-\lambda}$ is dense. Since polynomials are dense in $\mathcal{H}_p$ and $M_{D-\lambda}$ is continuous then $A := \{(D - \lambda) : P : P \text{ Dirichlet polynomial}\}$ is dense in the range of $M_{D-\lambda}$. By the continuity of the evaluation at $s_0 \in \mathbb{C}_{1/2}$, the set of Dirichlet series that vanish in a fixed $s_0$, which we denote by $B(s_0)$, is a proper closed set (because $1 \notin B(s_0)$). Therefore, if $D - \lambda \in B(s_0)$ then $A \subseteq B(s_0)$, but hence $A$ cannot be dense in $\mathcal{H}_p$. So we have that if $\lambda \in \sigma_c(M_D)$ then $D(s) - \lambda \neq 0$ for every $s \in \mathbb{C}_{1/2}$ and therefore $\lambda \notin \overline{D(\mathbb{C}_0)}$.

Finally, since $\sigma_c(M_D) = \sigma_c(M_f)$ then $\lambda \in \sigma_c(M_D)$ if and only if $M_{f-\lambda}(H_p(\mathbb{D}_2^\infty))$ is dense in $H_p(\mathbb{D}_2^\infty)$. Consider $S(f - \lambda)$ the smallest closed subspace of $H_p(\mathbb{D}_2^\infty)$ such that $z_i \cdot (f - \lambda) \in S(f - \lambda)$ for every $i \in \mathbb{N}$. Take $\lambda \in \sigma_c(M_f)$ and note that

$$\{(f - \lambda) : P : P \text{ polynomial}\} \subseteq S(f - \lambda) \subseteq H_p(\mathbb{D}_2^\infty).$$

Since the polynomials are dense in $H_p(\mathbb{D}_2^\infty)$, and $S(f - \lambda)$ is closed, we obtain that $S(f - \lambda) = H_p(\mathbb{D}_2^\infty)$. Then $f - \lambda$ is a cyclic vector in $H_\infty(\mathbb{D}_2^\infty)$ and therefore the function $f - \lambda \in H_\infty(\mathbb{D}_2^\infty)$ is an outer function (see [13, Corollary 5.5]).

Note that, in the hypothesis of the previous Proposition, if $D$ is non-constant, then $\sigma_p(M_D)$ is empty and therefore, $\sigma_c(M_D) = \sigma(M_D) \setminus \sigma_c(M_D)$. As a consequence, $\sigma_c(M_D)$ must contain the set $D(\mathbb{C}_{1/2})$.

Note that a value $\lambda$ belongs to the approximate spectrum of a multiplication operator $M_D$ if and only if $M_D - \lambda I = M_{D-\lambda}$ is not bounded from below. If $D$ is not constant and equal to $\lambda$ then, $M_{D-\lambda}$ is injective. Therefore, being bounded from below is equivalent to having closed ranged. Thus, we need to understand when does this operator have closed range. We therefore devote some lines to discuss this property.

The range of the multiplication operators behaves very differently depending on whether or not it is an endomorphism. We see now that if $p \neq q$ then multiplication operators never have closed range.

Proposition 16. Given $1 \leq q < p \leq \infty$ and $D \in \mathcal{H}_p$, with $t = pq / (p - q)$ if $p < \infty$ and $t = q$ if $p = \infty$, then $M_D : \mathcal{H}_p \to \mathcal{H}_q$ does not have a closed range. The same holds if we replace $\mathcal{H}$ by $\mathcal{H}^{(N)}$ (with $N \in \mathbb{N}$).
Proof. Since $M_D : \mathcal{H}_p \to \mathcal{H}_q$ is injective, the range of $M_D$ is closed if and only if there exists $C > 0$ such that $C\|E\|_{\mathcal{H}_p} \leq \|D \cdot E\|_{\mathcal{H}_q}$ for every $E \in \mathcal{H}_p$. Suppose that this is the case and choose some Dirichlet polynomial $P \in \mathcal{H}_1$ such that $\|D - P\|_{\mathcal{H}_1} < \frac{C}{2}$. Given $E \in \mathcal{H}_p$ we have

$$\|P \cdot E\|_{\mathcal{H}_q} = \|D \cdot E - (D - P) \cdot E\|_{\mathcal{H}_q} \geq \|D \cdot E\|_{\mathcal{H}_q} - \|(D - P) \cdot E\|_{\mathcal{H}_q} \geq C\|E\|_{\mathcal{H}_p} - \|D - P\|_{\mathcal{H}_p} \|E\|_{\mathcal{H}_q} \geq \frac{C}{2}\|E\|_{\mathcal{H}_p}.$$  

Then $M_P : \mathcal{H}_p \to \mathcal{H}_q$ has closed range. Let now $(Q_n)_n$ be a sequence of polynomials converging in $\mathcal{H}_q$ but not in $\mathcal{H}_p$, then

$$C\|Q_n - Q_m\|_{\mathcal{H}_p} \leq \|P \cdot (Q_n - Q_m)\|_{\mathcal{H}_q} \leq \|P\|_{\mathcal{H}_\infty} \|Q_n - Q_m\|_{\mathcal{H}_q},$$

which is a contradiction. \hfill \square

As we mentioned before, the behaviour of the range is very different when the operator is an endomorphism, that is, when $p = q$. Recently, in [2, Theorem 4.4], Antenaza, Carando and Scotti have established a series of equivalences for certain Riesz systems in $L_2(0, 1)$. Within the proof of this result, they also characterized those Dirichlet series $D \in \mathcal{H}_\infty$, for which their associated multiplication operator $M_D : \mathcal{H}_p \to \mathcal{H}_p$ has closed range. The proof also works for $\mathcal{H}_q$. In our aim to be as clear and complete as possible, we develop below the arguments giving all the necessary definitions.

A character is a function $\gamma : \mathbb{N} \to \mathbb{C}$ that satisfies

- $\gamma(mn) = \gamma(m)\gamma(n)$ for all $m, n \in \mathbb{N}$,
- $|\gamma(n)| = 1$ for all $n \in \mathbb{N}$.

The set of all characters is denoted by $\Xi$. Given a Dirichlet series $D = \sum a_n n^{-s}$, each character $\gamma \in \Xi$ defines a new Dirichlet series by

$$D^\gamma(s) = \sum a_n \gamma(n)n^{-s}. \quad (15)$$

Each character $\gamma \in \Xi$ can be identified with an element $\omega \in \mathbb{T}^\infty$, taking $\omega = (\gamma(p_1), \gamma(p_2), \cdots)$, and then we can rewrite (15) as

$$D^\omega(s) = \sum a_n \omega(n)^{\alpha(n)}n^{-s},$$

being $\alpha(n)$ such that $n = p^{\alpha(n)}$.

Note that if $\mathcal{L}_{\mathbb{T}^\infty}(D)(u) = F(u) \in H_\infty(\mathbb{T}^\infty)$, then by comparing coefficients we have that $\mathcal{L}_{\mathbb{T}^\infty}(D^\omega)(u) = F(\omega \cdot u) \in H_\infty(\mathbb{T}^\infty)$. By [9, Lemma 11.22], for all $\omega \in \mathbb{T}^\infty$ the limit

$$\lim_{\sigma \to 0} D^\omega(\sigma + it),$$

exists for almost all $t \in \mathbb{R}$.

Using [22, Theorem 2], we can choose a representative $\tilde{F} \in H_\infty(\mathbb{T}^\infty)$ of $F$ which satisfies

$$\tilde{F}(\omega) = \begin{cases} \lim_{\sigma \to 0^+} D^\omega(\sigma) & \text{if the limit exists;} \\ 0 & \text{otherwise.} \end{cases}$$
To see this, consider
\[ A := \{ \omega \in \mathbb{T}^{\infty} : \lim_{\sigma \to 0} D^{\omega} (\sigma) \text{ exists} \}, \]
and let us see that |A| = 1. To that, take \( T_t : \mathbb{T}^{\infty} \to \mathbb{T}^{\infty} \) the Kronecker flow defined by \( T_t(\omega) = (p^{-it} \omega) \), and notice that \( T_t(\omega) \in A \) if and only if \( \lim_{\sigma \to 0} D^{T_t(\omega)} (\sigma) \) exists. Since
\[ D^{T_t(\omega)} (\sigma) = \sum a_n (p^{-it} \omega)^{\sigma(n)} n^{-\sigma} = \sum a_n \omega^{\sigma(n)} n^{(\sigma+it)} = D^{\omega} (\sigma + it), \]
then for all \( \omega \in \mathbb{T}^{\infty} \) we have that \( T_t(\omega) \in A \) for almost all \( t \in \mathbb{R} \). Finally, since \( \chi_A \in L^1 (\mathbb{T}^{\infty}) \), applying the Birkhoff Theorem for the Kronecker flow [19, Theorem 2.2.5], for \( \omega_0 = (1, 1, 1, \ldots) \) we have
\[ |A| = \int_{\mathbb{T}^{\infty}} \chi_A (\omega) d\omega = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \chi_A (T_t(\omega_0)) dt = 1. \]

Then \( \hat{F} \in H_{\infty} (\mathbb{T}^{\infty}) \), and to see that \( \hat{F} \) is a representative of \( F \) it is enough to compare their Fourier coefficients (see again [22, Theorem 2]). From now to the end \( F \) is always \( \hat{F} \).

Fixing the notation
\[ D^{\omega} (it_0) = \lim_{\sigma \to 0} D^{\omega} (\sigma + it), \]
then taking \( t_0 = 0 \), we get
\[ F(\omega) = D^{\omega} (0) \]
for almost all \( \omega \in \mathbb{T}^{\infty} \). Moreover, given \( t_0 \in \mathbb{R} \) we have
\[ D^{\omega} (it_0) = \lim_{\sigma \to 0^+} D^{\omega} (\sigma + it_0) = \lim_{\sigma \to 0^+} D^{T_{t_0}(\omega)} (\sigma) = F(T_{t_0}(\omega)). \tag{16} \]

From this identity one has the following.

**Proposition 17.** The followings conditions are equivalent.

a) There exists \( \tilde{t}_0 \) such that \( |D^{\omega} (it_\tilde{t}_0)| \geq \varepsilon \) for almost all \( \omega \in \mathbb{T}^{\infty} \).

b) For all \( t_0 \) there exists \( B_{t_0} \subset \mathbb{T}^{\infty} \) with total measure such that \( |D^{\omega} (it_0)| \geq \varepsilon \) for all \( \omega \in B_{t_0} \).

**Proof.** If a), holds take \( t_0 \) and consider
\[ B_{t_0} = \{ p^{-i(-t_0+\tilde{t}_0)} \cdot \omega : \omega \in B_{\tilde{t}_0} \}, \]
which is clearly a total measure set. Take \( \omega' \in B_{t_0} \) and choose \( \omega \in B_{\tilde{t}_0} \) such that \( \omega' = p^{-i(-t_0+\tilde{t}_0)} \cdot \omega \), then by (16) we have that
\[ |D^{\omega'} (it_0)| = |F(T_{t_0}(\omega))| \geq \varepsilon, \]
and this gives b). The converse implications holds trivially. \( \square \)

We now give an \( \mathcal{H}_p \)-version of [2, Theorem 4.4.].

**Theorem 18.** Let \( 1 \leq p < \infty \), and \( D \in \mathcal{H}_{\infty} \). Then the following statements are equivalent.

a) There exists \( m > 0 \) such that \( |F(\omega)| \geq M \) for almost all \( \omega \in \mathbb{T}^{\infty} \);
b) The operator $M_D : \mathcal{H}_p \to \mathcal{H}_p$ has closed range; 

c) There exists $m > 0$ such that for almost all $(\gamma, t) \in \mathbb{R} \times \mathbb{R}$ we have 

$$|D^\gamma(it)| \geq m.$$ 

Even more, in that case, 

$$\inf \left\{ \|M_D(E)\|_{\mathcal{H}_p} : E \in \mathcal{H}_p, \|E\|_{\mathcal{H}_p} = 1 \right\} = \text{ess inf} \left\{ |F(\omega)| : \omega \in \mathbb{T}^\infty \right\} = \text{ess inf} \left\{ |D^\gamma(it)| : (\gamma, t) \in \mathbb{R} \times \mathbb{R} \right\}.$$ 

Proof. a) $\Rightarrow$ b) $M_D$ has closed range if and only if the range of $M_F$ is closed. Because of the injectivity of $M_F$ we have, by Open Mapping Theorem, that $M_F$ has closed range if and only if there exists a positive constant $m > 0$ such that 

$$\|M_F(G)\|_{H_p(\mathbb{T}^\infty)} \geq m \|G\|_{H_p(\mathbb{T}^\infty)},$$ 

for every $G \in H_p(\mathbb{T}^\infty)$. If $|F(\omega)| \geq m$ a.e. $\omega \in \mathbb{T}^\infty$, then for $G \in H_p(\mathbb{T}^\infty)$ we have that 

$$\|M_F(G)\|_{H_p(\mathbb{T}^\infty)} = \|F \cdot G\|_{H_p(\mathbb{T}^\infty)} = \left( \int_{\mathbb{T}^\infty} |FG(\omega)|^p d\omega \right)^{1/p} \geq m \|G\|_{H_p(\mathbb{T}^\infty)}.$$ 

b) $\Rightarrow$ a) Let $m > 0$ be such that $\|M_F(G)\|_{H_p(\mathbb{T}^\infty)} \geq m \|G\|_{H_p(\mathbb{T}^\infty)}$ for all $G \in H_p(\mathbb{T}^\infty)$. Let us consider 

$$A = \{ \omega \in \mathbb{T}^\infty : |F(\omega)| < m \}.$$ 

Since $\chi_A \in L^p(\mathbb{T}^\infty)$, by the density of the trigonometric polynomials in $L^p(\mathbb{T}^\infty)$ (see [9, Proposition 5.5]) there exist a sequence $(P_k)_k$ of degree $n_k$ in $N_k$ variables (in $\mathbb{Z}$ and $\overline{\mathbb{Z}}$) such that 

$$\lim_k P_k = \chi_A \text{ in } L^p(\mathbb{T}^\infty).$$ 

Therefore 

$$m^p |A| = m^p \|\chi_A\|_{L_p(\mathbb{T}^\infty)}^p = m^p \lim_k \|P_k\|_{L_p(\mathbb{T}^\infty)}^p = m^p \lim_k \left\| z_1^{n_k} \cdots z_{N_k}^{n_k} P_k \right\|_{L_p(\mathbb{T}^\infty)}^p \leq \lim_k \left\| M_F(z_1^{n_k} \cdots z_{N_k}^{n_k} P_k) \right\|_{L_p(\mathbb{T}^\infty)}^p \leq \|F \cdot \chi_A\|_{L_p(\mathbb{T}^\infty)}^p = \int_A |F(\omega)|^p d\omega.$$ 

Since $|F(\omega)| < m$ for all $\omega \in A$, this implies that $|A| = 0$.

b) $\Rightarrow$ c) By the definition of $F$ we have $m \leq |F(\omega)| = \lim_{\sigma \to 0^+} |D^\omega(\sigma)|$ for almost all $\omega \in \mathbb{T}^\infty$. Combining this with Remark 17 we get that the $t$–sections of the set 

$$C = \{ (\omega, t) \in \mathbb{T}^\infty \times \mathbb{R} : |D^\omega(it)| < \varepsilon \},$$ 

have zero measure. As a corollary of Fubini’s Theorem we get that $C$ has measure zero. The converse c) $\Rightarrow$ b) also follows from Fubini’s Theorem. The last equality follows from the proven equivalences. \qed
In the case of polynomials, using the continuity of the polynomials and Kronecker’s theorem (see e.g. [9, Proposition 3.4]), the condition of Theorem 18 is restricted to the image of the polynomial on the line of complex with zero real part. As a consequence, one can extend this characterization to the Dirichlet series belonging to $\mathcal{H}(\mathbb{C}_0)$, that is the closed subspace of $\mathcal{H}_{\infty}$ given by the Dirichlet series that are uniformly continuous on $\mathbb{C}_0$ (see [4, Definition 2.1]).

**Corollary 19.** Let $1 \leq p < \infty$ then

a) Let $P \in \mathcal{H}_{\infty}$ be a Dirichlet polynomial. Then $M_P : \mathcal{H} \to \mathcal{H}$ has closed range if and only if there exists a constant $m > 0$ such that $|P(it)| \geq m$ for all $t \in \mathbb{R}$.

b) Let $D \in \mathcal{A}(\mathbb{C}_0)$, then $M_D : \mathcal{H} \to \mathcal{H}$ has closed range if and only if there exists a constant $m > 0$ such that $|D(it)| \geq m$ for all $t \in \mathbb{R}$.

Even more, in each case

$$\inf\{|MD(E)| : E \in \mathcal{H}, \|E\|_{\mathcal{H}_p} = 1\} = \inf\{|D(it)| : t \in \mathbb{R}\}.$$  

**Proof.** a) Let $F = \mathcal{L}_{\mathbb{T}^\circ}(P)$ then, by Theorem 18, $M_P$ has close range if and only if there exists a constant $m > 0$ such that $|F(\omega)| \geq m$ a.e. $\omega \in \mathbb{T}^\circ$. Since $F(\omega) = \sum a_n \omega^n$ is continuous and by Kronecker’s theorem

$$\{(p_1^{-it}, \ldots, p_N^{-it}, \omega) : t \in \mathbb{R}, \omega \in \mathbb{T}^\circ\}$$

is dense in $\mathbb{T}^\circ$, then $M_P$ has closed range if and only if $|F(p_1^{-it}, \ldots, p_N^{-it}, \omega)| \geq m$ for every $t \in \mathbb{R}$ and $\omega \in \mathbb{T}^\circ$. The result is concluded from the fact that

$$F(p_1^{-it}, \ldots, p_N^{-it}, \omega) = \sum a_n p_1^{-ita_1} \cdots p_N^{-ita_N} = \sum a_n n^{-it} = P(it).$$

b) Since $D$ is uniformly continuous on $\mathbb{C}_0$ then $D$ admits a uniformly continuous extension to the half-plane $\{s \in \mathbb{C} : \text{Re } s \geq 0\}$. By [4, Theorem 2.3], $D$ is the uniform limit on $\mathbb{C}_0$ of a sequence of Dirichlet polynomials $P_n$. Let $\mathcal{A}(\mathbb{T}^\circ)$ be the closed subspace of $H_{\infty}(\mathbb{T}^\circ)$ given by the Bohr transform of $\mathcal{A}(\mathbb{C}_0)$. If $\mathcal{L}_{\mathbb{T}^\circ}(D) = F \in \mathcal{A}(\mathbb{T}^\circ)$, since it is the uniform limit of polynomials, then $F$ is continuous. Then, given $t \in \mathbb{R}$ we have that

$$|F(p^{-it})| = \lim_n |B_{\mathbb{T}^\circ}(P_n)(p^{-it})| = \lim_n |P_n(it)| = |D(it)|. \quad (17)$$

Let us suppose first that the range of $M_D : \mathcal{H} \to \mathcal{H}$ is closed and let $m > 0$ be such that $\|M_D(E)\|_{\mathcal{H}_p} \geq \|E\|_{\mathcal{H}_p}$. Given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|D - P_n\|_{\mathcal{H}_\infty} < \varepsilon$ for every $n_0 \leq n$. Therefore, if $E \in \mathcal{H}$ we have that

$$\|M_{P_n}(E)\|_{\mathcal{H}_p} \geq \|M_D(E)\|_{\mathcal{H}_p} - \|M_{D - P_n}(E)\|_{\mathcal{H}_p} \geq (m - \varepsilon)\|E\|_{\mathcal{H}_p}.$$ 

Then by a), $|P_n(it)| \geq m - \varepsilon$ for every $n \geq n_0$ and $t \in \mathbb{R}$ and hence, by (17), $|D(it)| \geq m - \varepsilon$ for every $\varepsilon > 0$.

Let us suppose now that there exists a constant $m > 0$ such that $|D(it)| \geq m$ for every $t \in \mathbb{R}$, then from (17) we have that $|F(p^{-it})| \geq m$ for all $t \in \mathbb{R}$. Since $F$ is continuous, and by Kronecker’s theorem, $|F(\omega)| \geq m$ for all $\omega \in \mathbb{T}^\circ$. Therefore, by Theorem 18, $M_D$ has closed range. \qed
For what was said above, in the non trivial case, a value $\lambda$ belongs to the approximate spectrum of $M_D$ if and only if the range of $M_{D-\lambda}$ is not closed. Then, Theorem 18 and Proposition 19 give us a characterization of the approximate spectrum. For this, we need the definition of the essential range of the function $[(\gamma, t) \rightsquigarrow D^\gamma(it)]$. That is,

$$\left\{ \lambda \in \mathbb{C} : \text{ for all } \varepsilon > 0, \mu\{(\gamma, t) : |D^\gamma(it) - \lambda| < \varepsilon \} > 0 \right\},$$

where $\mu$ stands for the Haar measure in $\Xi \times \mathbb{R}$.

**Theorem 20.** Let $1 \leq p < \infty$

a) If $D \in \mathcal{H}_\infty$, then $\sigma_{ap}(M_D) = \text{essran}[(\gamma, t) \rightsquigarrow D^\gamma(it)]$.

b) If $D \in \mathcal{A}(C_0)$, then $\sigma_{ap}(M_D) = \{D(it) : t \in \mathbb{R}\}$.

**Proof.** a) A value $\lambda$ belongs to $\sigma_{ap}(M_D)$ if and only if the range of $M_{D-\lambda}$ is not closed; and by Theorem 18, if and only if

$$\text{essinf}\{|D^\gamma(it) - \lambda| : (\gamma, t) \in \Xi \times \mathbb{R}\} =\text{essinf}\{|(D - \lambda)^\gamma(it)| : (\gamma, t) \in \Xi \times \mathbb{R}\} = 0,$$

but that is equivalent to say that the measure of $\{|D^\gamma(it) - \lambda| < \varepsilon : (\gamma, t) \in \Xi \times \mathbb{R}\}$ is bigger than zero for every $\varepsilon > 0$. In other words, $\lambda$ belongs to the essential range of $[(\gamma, t) \rightsquigarrow D^\gamma(it)]$.

b) Following the same arguments used in a) and using Corollary 19 we have that $\lambda \in \sigma_{ap}(M_D)$ if and only if $\inf\{|D(it) - \lambda| : t \in \mathbb{R}\} = 0$, if and only if $\lambda \in \overline{\{D(it) : t \in \mathbb{R}\}}$. □

**References**

[1] A. Aleman, J.-F. Olsen, and E. Saksman. Fatou and brothers Riesz theorems in the infinite-dimensional polydisc. *J. Anal. Math.*, 137(1):429–447, 2019. doi:10.1007/s11854-019-0006-x.

[2] J. Antezana, D. Carando, and M. Scotti. Splitting the Riesz basis condition for systems of dilated functions through Dirichlet series. *J. Math. Anal. Appl.*, 507(1):Paper No. 125733, 20, 2022. doi:10.1016/j.jmaa.2021.125733.

[3] T. M. Apostol. *Introduction to analytic number theory*. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.

[4] R. M. Aron, F. Bayart, P. M. Gauthier, M. Maestre, and V. Nestoridis. Dirichlet approximation and universal Dirichlet series. *Proc. Amer. Math. Soc.*, 145(10):4449–4464, 2017. doi:10.1090/proc/13607.

[5] F. Bayart. Hardy spaces of Dirichlet series and their composition operators. *Monatsh. Math.*, 136(3):203–236, 2002. doi:10.1007/s00605-002-0470-7.

[6] L. Brown and A. L. Shields. Cyclic vectors in the Dirichlet space. *Trans. Amer. Math. Soc.*, 285(1):269–303, 1984. doi:10.2307/1999483.

[7] J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.
[8] A. Defant, T. Fernández Vidal, I. Schoolmann, and P. Sevilla-Peris. Fréchet spaces of general Dirichlet series. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 115(3):Paper No. 138, 34, 2021. doi:10.1007/s13398-021-01074-8.

[9] A. Defant, D. García, M. Maestre, and P. Sevilla-Peris. *Dirichlet Series and Holomorphic Functions in High Dimensions*, volume 37 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2019. doi:10.1017/9781108691611.

[10] A. Defant and A. Pérez. Hardy spaces of vector-valued Dirichlet series. *Studia Math.*, 243(1):53–78, 2018. doi:10.4064/sm170303-26-7.

[11] R. Demazeux. Essential norms of weighted composition operators between Hardy spaces $H^p$ and $H^q$ for $1 \leq p, q \leq \infty$. *Studia Math.*, 206(3):191–209, 2011. doi:10.4064/sm206-3-1.

[12] J. Diestel. *Sequences and series in Banach spaces*, volume 92 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984. doi:10.1007/978-1-4612-5200-9.

[13] K. Guo and J. Ni. Dirichlet series and the Nevanlinna class in infinitely many variables. *arXiv preprint arXiv:2201.01993*, 2022.

[14] H. Hedenmalm, P. Lindqvist, and K. Seip. A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$. *Duke Math. J.*, 86(1):1–37, 1997. doi:10.1215/S0012-7094-97-08601-4.

[15] S. Kaijser. A note on dual Banach spaces. *Math. Scand.*, 41(2):325–330, 1977. doi:10.7146/math.scand.a-11725.

[16] S. V. Konyagin and H. Queffélec. The translation $\frac{1}{2}$ in the theory of Dirichlet series. *Real Anal. Exchange*, 27(1):155–175, 2001/02.

[17] P. Lefèvre. Essential norms of weighted composition operators on the space $\mathcal{H}^\infty$ of Dirichlet series. *Studia Math.*, 191(1):57–66, 2009. doi:10.4064/sm191-1-4.

[18] H. Queffélec. H. Bohr’s vision of ordinary Dirichlet series; old and new results. *J. Anal.*, 3:43–60, 1995.

[19] H. Queffélec and M. Queffélec. *Diophantine approximation and Dirichlet series*, volume 80 of *Texts and Readings in Mathematics*. Hindustan Book Agency, New Delhi; Springer, Singapore, 2020. Second edition. doi:10.1007/978-981-15-9351-2.

[20] W. Rudin. *Fourier analysis on groups*. Interscience Tracts in Pure and Applied Mathematics, No. 12. Interscience Publishers (a division of John Wiley and Sons), New York-London, 1962.

[21] W. Rudin. *Function theory in polydiscs*. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[22] E. Saksman and K. Seip. Integral means and boundary limits of Dirichlet series. *Bull. Lond. Math. Soc.*, 41(3):411–422, 2009. doi:10.1112/blms/bdp004.
[23] M. Stessin and K. Zhu. Generalized factorization in Hardy spaces and the commutant of Toeplitz operators. *Canad. J. Math.*, 55(2):379–400, 2003. doi:10.4153/CJM-2003-017-1.

[24] G. Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 46 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Translated from the second French edition (1995) by C. B. Thomas.

[25] T. F. Vidal, D. Galicer, and P. Sevilla-Peris. A Montel-type theorem for Hardy spaces of holomorphic functions. *arXiv preprint arXiv:2004.10511*, 2020.

[26] D. Vukotić. Analytic Toeplitz operators on the Hardy space $H^p$: a survey. *Bull. Belg. Math. Soc. Simon Stevin*, 10(1):101–113, 2003. URL: http://projecteuclid.org/euclid.bbms/1047309417.

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