Compactness and Index of Ordinary Central Configurations for the Curved N-Body Problem

Shuqiang Zhu1*

1School of Economic Mathematics, Southwestern University of Finance and Economics, 611130 Chengdu, China

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Abstract—For the curved n-body problem, we show that the set of ordinary central configurations is away from singular configurations in $\mathbb{H}^3$ with positive momentum of inertia, and away from a subset of singular configurations in $S^3$. We also show that each of the $n!/2$ geodesic ordinary central configurations for $n$ masses has Morse index $n-2$. Then we get a direct corollary that there are at least $\frac{(3n-4)(n-1)!}{2}$ ordinary central configurations for given $n$ masses if all ordinary central configurations of these masses are nondegenerate.

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1. INTRODUCTION

The curved n-body problem studies the motion of particles interacting under the cotangent potential in a 3-dimensional sphere and a 3-dimensional hyperbolic sphere. It is a natural extension of the Newtonian n-body problem in $\mathbb{R}^3$. It has its roots in the research of Bolyai and Lobachevsky. For the history and recent advances, the reader can be referred to Arnold et al. [2], Borisov et al. [3], and Diacu [6]. There has been a large body of research in this area over the past two decades: on the Kepler problem, the two-body problem, relative equilibria, stability of periodic orbits, etc.

The curved n-body problem is a Lagrangian mechanical system. A solution of the Euler-Lagrange equation in the form $A(t)q$ is called a relative equilibrium if $A(t)$ is a 1-parameter subgroup of the symmetry group. The topic of relative equilibria has received much attention recently (see [4, 6–8, 12, 13, 20, 24, 27] among others). Unlike the set of relative equilibria of the Newtonian n-body problem in $\mathbb{R}^3$, that of the curved n-body problem is divided into five classes, see Section 2.3. Diacu, Stoica and Zhu introduce a unified criterion for relative equilibria of the curved n-body problem. The configurations of relative equilibria are characterized as critical points of a single function [9]. The criterion simplifies the job of finding relative equilibria [10]. The configurations are called central configurations and the set of central configurations is divided into the ordinary ones and the special ones. A counting problem of ordinary central configurations is also proposed, see Section 2.4.

In the Newtonian n-body problem, the celebrated problem of the finiteness of configurations of relative equilibria is still unsolved for $n>5$ [1, 23, 25]. On the other hand, the collinear case is clear. For any $n$ masses, there are exactly $n!/2$ collinear configurations of relative equilibria (Moulton [16]) and their Morse index is $n-2$ [17]. In general the set of normalized configurations of relative equilibria is known to be compact (Shub [21]) in the configuration space. Moreover, Palmore [18] proved that there are at least $\frac{(3n-4)(n-1)!}{2}$ configurations of relative equilibria for given $n$ masses if all configurations of relative equilibria of these masses are nondegenerate.

*E-mail: zhusq@swufe.edu.cn

236
The purpose of this paper is to extend the results mentioned above to the curved $n$-body problem. More precisely, we show that the set of ordinary central configurations is away from most singular configurations in $\mathbb{H}^3$, and away from a subset of singular configurations in $\mathbb{S}^3$. We show that there are $n!/2$ “collinear” ordinary central configurations and their Morse index is $n-2$ in $\mathbb{H}^3$, which also holds in $\mathbb{S}^3$ provided that some conditions are satisfied. Furthermore, Palmore’s estimation also holds.

The paper is organized as follows. In Section 2, we review the $n$-body problem in the three manifolds, $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, the criterion for relative equilibria and the counting of ordinary central configurations. Then we state our main results. In Section 3, we prove the results on compactness of the set of ordinary central configurations. In Section 4, we prove the results on the number and Morse index of the geodesic ordinary central configurations. In the Appendix, we discuss association between the central configurations and the motions of relative equilibria.

2. RELATIVE EQUILIBRIA OF THE CURVED $n$-BODY PROBLEM AND MAIN RESULTS

In this section, we first briefly review the $n$-body problem on the three manifolds, $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, criterion for relative equilibria, then state the main results of this paper. Vectors are all column vectors, but written as row vectors in the text. The masses $m_1, \ldots, m_n$ are always positive.

2.1. Relative Equilibria of the $n$-body Problem in $\mathbb{R}^3$

The Newtonian $n$-body problem in $\mathbb{R}^3$ studies the motion of $n$ particles in $\mathbb{R}^3$ with masses $m_1, \ldots, m_n$ under the gravitational interaction. It is a Lagrangian mechanical system with Lagrangian function

$$L = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{q}_i \cdot \dot{q}_i + U_0(q),$$

where $q = (q_1, \ldots, q_n)$, $q_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, and $U_0 = \sum \frac{m_i m_j}{\|q_i - q_j\|}$ is the potential defined on the configuration space $(\mathbb{R}^3)^n - \Delta$, $\Delta = \cup_{1 \leq i < j \leq n} \{q \in (\mathbb{R}^3)^n | q_i = q_j\}$.

A relative equilibrium is an integral curve of the system in the form $A(t)q$, with $A(t)$ being a uniform rotation in $SO(3)$. The corresponding configuration $q$ is called a configuration of relative equilibria. Every uniform rotation in $SO(3)$ has a fixed axis. Assume that the $z$-axis is the rotation axis. Then it is well known that the configurations of relative equilibria must be on a plane perpendicular to the $z$-axis. Without loss of generality, we assume that they are on the plane $\{z = 0\}$, then they are critical points of $U_0 - \lambda I_0$ for some $\lambda \in \mathbb{R}$, where $I_0(q) = \sum m_i(x_i^2 + y_i^2)$.

Two configurations of relative equilibria are said to be in one class if one can be deduced from the other by some rotation in $SO(2)$ and nonzero scalar multiplication. The finiteness problem on configurations of relative equilibria is: given $n$ masses $m_1, \ldots, m_n$, is the number of classes of configurations of relative equilibria finite? In other words, is the number of configurations of relative equilibria in $\{q \in (\mathbb{R}^2)^n - \Delta | I_0(q) = 1\}/S^1$ finite? For the history and advance of this problem, the reader can be referred to [1, 23, 25] and references therein.

2.2. The Curved $n$-body Problem in $\mathbb{S}^3$ and $\mathbb{H}^3$

The curved $n$-body problem studies the motion of $n$ particles interacting under the so-called cotangent potential in $\mathbb{S}^3$ and $\mathbb{H}^3$. The two manifolds can be parameterized in many ways. The Cartesian coordinates are convenient in many cases. That is, $\mathbb{S}^3$ (resp. $\mathbb{H}^3$) is the unit sphere in $\mathbb{R}^4$ (resp. $\mathbb{R}^{3,1}$). Recall that the “inner product” in those 4-dimensional linear space is $q_1 \cdot q_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 + \sigma w_1 w_2$, where $q_i = (x_i, y_i, z_i, w_i)$, $\sigma = 1$ for $\mathbb{R}^4$ and $\sigma = -1$ for $\mathbb{R}^{3,1}$. Then $\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 + z^2 + w^2 = 1 \}$, $\mathbb{H}^3 = \{(x, y, z, w) \in \mathbb{R}^{3,1} | x^2 + y^2 + z^2 - w^2 = -1, w \geq 1 \}$. The Riemannian metrics on $\mathbb{S}^3$ and $\mathbb{H}^3$ are induced from the “inner product”. The
distance between two point masses $m_i$ and $m_j$, $d_{ij} = d(q_i, q_j)$, is computed by $\cos d_{ij} = q_i \cdot q_j$ on $S^3$ and $\cosh d_{ij}(q) = -q_i \cdot q_j$ on $\mathbb{H}^3$.

The curved $n$-body problem in $S^3$ is a Lagrangian mechanical system with Lagrangian function

$$L = \frac{1}{2} \sum_{i=1}^{n} m_i q_i \cdot \dot{q}_i + U_1(q),$$

where $q = (q_1, \ldots, q_n)$, $q_i \in S^3$. Let $\Delta_+$ be the singular set $\Delta_+ = \cup_{1 \leq i < j \leq n} \{ q \in (S^3)^n \mid q_i = \pm q_j \}$. The potential is $U_1 = \sum m_i m_j \cot d_{ij}$ defined on the configuration space $(S^3)^n - \Delta_+$. The equations of motion are [6, 9]

$$\begin{cases} m_i \ddot{q}_i = \sum_{j=1, j \neq i}^{n} \frac{m_i m_j [q_j - \cos d_{ij} q_i]}{\sin^3 d_{ij}} - \sigma m_i (q_i \cdot \dot{q}_i) q_i, \\ q_i \cdot q_i = \sigma, \quad i = 1, \ldots, n. \end{cases} \quad (2.1)$$

Recall that $\sigma = 1$ for $S^3$ and $\sigma = -1$ for $\mathbb{H}^3$.

Likewise, the curved $n$-body problem in $\mathbb{H}^3$ is a Lagrangian mechanical system with Lagrangian function

$$L = \frac{1}{2} \sum_{i=1}^{n} m_i q_i \cdot \dot{q}_i + U_{-1}(q),$$

where $q = (q_1, \ldots, q_n)$, $q_i \in \mathbb{H}^3$. Let $\Delta_-$ be the singular set $\Delta_- = \cup_{1 \leq i < j \leq n} \{ q \in (\mathbb{H}^3)^n \mid q_i = q_j \}$. The potential is $U_{-1}(q) = \sum m_i m_j \coth d_{ij}$ defined on the configuration space $(\mathbb{H}^3)^n - \Delta_-$. Replacing the trigonometrical functions by the hyperbolic ones and putting $\sigma = -1$, Eqs. (2.1) become the equations of motion for the curved $n$-body problem in $\mathbb{H}^3$.

### 2.3. Relative Equilibria in $S^3$ and $\mathbb{H}^3$

A simple mechanical system with symmetry in the terminology of Smale is a Lagrangian mechanical system on a manifold $M$ in the form $L = K(q) + U(q)$, where $K$ is a Riemannian metric on $M$ and there is a Lie group $G$ acting on $M$ preserving $K$ and $U$ smoothly. A solution of the Euler–Lagrange equation in the form $A(t)q$ is called a relative equilibrium if $A(t)$ is a 1-parameter subgroup of the group $G$. It is well known that the corresponding configuration $q$ is a critical point of the augmented potential

$$U_{\xi}(q) = U(q) + K(\xi_M(q)),$$

where $\xi$ belongs to the Lie algebra of $G$ and $\xi_M(q) = \frac{d}{ds}|_{s=0} \exp(s\xi)q$ is the vector field on $M$ generated by $\xi$ [15, 22].

The curved $n$-body problem in $S^3$ (resp. $\mathbb{H}^3$) is a simple mechanical system with symmetry $O(4)$ (resp. $O(3, 1)$), the set of matrices that keeps the inner product in $\mathbb{R}^4$ (resp. $\mathbb{R}^{3,1}$). Let $\xi$ be some element in the Lie algebra of $O(4)$ ($O(3, 1)$). Then the 1-parameter subgroup of $O(4)$ ($O(3, 1)$) takes the form of $\exp(t\xi)$, and the corresponding vector field on the configuration space is $(\xi q_1, \ldots, \xi q_n)$. The augmented potential takes the form

$$U_1 + \frac{1}{2} \sum_{i=1}^{n} m_i \xi q_i \cdot \xi q_i, \quad \text{or} \quad U_{-1} + \frac{1}{2} \sum_{i=1}^{n} m_i \xi q_i \cdot \xi q_i.$$

This coordinate-free way is adopted in the study of relative equilibria in the Newtonian $n$-body problem in higher dimensions, see [5, 19].
It is convenient to use coordinates for our problem. Note that each 1-parameter subgroup is conjugate to

\[
A_{\alpha,\beta}(t) = \begin{bmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & \cos \beta & -\sin \beta \\
0 & 0 & \sin \beta & \cos \beta
\end{bmatrix}
\text{ in } O(4) \quad \text{and,}
\]

\[
B_{\alpha,\beta}(t) = \begin{bmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & \cosh \beta t & \sinh \beta t \\
0 & 0 & \sinh \beta t & \cosh \beta t
\end{bmatrix}
\text{ in } O(3,1).
\]

We have neglected the 1-parameter subgroups of \(SO(3,1)\) that represent the parabolic rotations since they do not lead to relative equilibria of the curved \(n\)-body problem [6, 7]. Following Diacu [6], in \(S^3\), we call relative equilibria elliptic if only one of \(\alpha\) and \(\beta\) is nonzero, and elliptic-elliptic if \(\alpha \beta \neq 0\); in \(\mathbb{H}^3\), we call relative equilibria elliptic if \(\alpha \neq 0, \beta = 0\), hyperbolic if \(\alpha = 0, \beta \neq 0\), and elliptic-hyperbolic if \(\alpha \beta \neq 0\).

The Lie algebra elements corresponding to \(A_{\alpha,\beta}(t)\) and \(B_{\alpha,\beta}(t)\) are

\[
\begin{bmatrix}
0 & -\alpha & 0 & 0 \\
\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & -\beta \\
0 & 0 & \beta & 0
\end{bmatrix}
\text{ and,}
\begin{bmatrix}
0 & -\alpha & 0 & 0 \\
\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & \beta \\
0 & 0 & \beta & 0
\end{bmatrix}
\]

respectively. Hence, the function \(K(\xi_{q_1}, \ldots, \xi_{q_n})\) for the \(S^3\) case is \(\alpha^2 \sum_{i=1}^{n} m_i(x_i^2 + y_i^2)/2 + \beta^2 \sum_{i=1}^{n} m_i(z_i^2 + w_i^2)/2\). Note that \(q_i \cdot q_i = 1\). The function reduces to

\[
\frac{\alpha^2 - \beta^2}{2} \sum_{i=1}^{n} m_i(x_i^2 + y_i^2) + \frac{\beta^2}{2} \sum_{i=1}^{n} m_i.
\]

Similarly, the function \(K(\xi_{q_1}, \ldots, \xi_{q_n})\) for the \(\mathbb{H}^3\) case reduces to

\[
\frac{\alpha^2 + \beta^2}{2} \sum_{i=1}^{n} m_i(x_i^2 + y_i^2) + \frac{\beta^2}{2} \sum_{i=1}^{n} m_i.
\]

Let \(I_1(q) = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2)\) (resp. \(I_{-1}(q) = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2)\)) be the momentum of inertia for configurations \(q\) in \(S^3\) (resp. \(\mathbb{H}^3\)). Let

\[
S^+_c = I_1^{-1}(c) - \Delta^+_c, \quad S^-_c = I_{-1}^{-1}(c) - \Delta^-_c.
\]

We have the following criterion of relative equilibria.

**Theorem 1 ([9])**. For the curved \(n\)-body problem, \(A_{\alpha,\beta}(t)q \ (B_{\alpha,\beta}(t)q)\) is a relative equilibrium if and only if the configuration \(q\) is a critical point of

\[
U_1(q) + \frac{\alpha^2 - \beta^2}{2} \sum_{i=1}^{n} m_i(x_i^2 + y_i^2), \text{ or } U_{-1}(q) + \frac{\alpha^2 + \beta^2}{2} \sum_{i=1}^{n} m_i(x_i^2 + y_i^2).
\]
Thus, all relative equilibria, no matter whether they are elliptic, elliptic-elliptic, hyperbolic, or elliptic-hyperbolic, can be obtained by finding configurations that are critical points of one function

\[ U_1 - \lambda I_1, \text{ or } U_1|_{S_2^x}, \text{ (resp. } U_{-1} - \lambda I_{-1}, \text{ or } U_{-1}|_{S_2^-}). \]  

(2.3)

This unity is obtained by restricting the relative equilibria to those in the form \( A_{\alpha,\beta}(t) q \) (resp. \( B_{\alpha,\beta}(t) q \)) and by studying relative equilibria in the 3-dimensional space. For example, in the hyperbolic case, if we study the relative equilibria on a 2-dimensional physical space, then the augmented potentials for the elliptic relative equilibria and the hyperbolic ones are different, see [7, 11].

Let \( q \) be a critical point of (2.3). Then it leads to infinitely many relative equilibria \( A_{\alpha,\beta}(t) q \) with \( \lambda \) equal to \(-\frac{\alpha^2-\beta^2}{2}\) (resp. \( B_{\alpha,\beta}(t) q \) with \( \lambda = -\frac{\alpha^2+\beta^2}{2} \)), which may be elliptic or elliptic-elliptic (resp. elliptic, hyperbolic, or elliptic-hyperbolic). So to study the relative equilibria, it is convenient to start with the associated configurations. We will discuss the relationship between the critical points of (2.3) and relative equilibria in the Appendix.

In \( S^3 \), the function \( U_1 \) has critical points. Besides relative equilibria, such configurations also lead to equilibria of the system \( q(t) = q \). Let \( q \) be such a critical point. Then it is a critical point of \( U_1 - \lambda I_1 \) with \( \lambda = 0 \), or with \( \lambda \) being any real value if it is one configuration that lies on \( S_{\beta}^y \cup S_{\beta}^w \), two circles to be defined in Section 2.5 [6, 9, 26]. The property of these configurations and the other critical points of (2.3) are different in many ways. This motivates the following definition.

**Definition 1** ([9]). A configuration \( q \) is called a *central configuration* if it is a critical point of (2.3). If it is a critical point of \( U_1 \), it is a *special central configuration* or an *equilibrium configuration*; otherwise, it is an *ordinary central configuration*. The value \( \lambda \) in (2.3) is the *multiplier*.

The central configurations defined above are deduced from the relative equilibria, and may not have some nice properties possessed by the well-known central configurations of the Newtonian \( n \)-body problem. The reader can be referred to [9] for more of their properties. We only discuss the ordinary central configurations in the main part of this paper.

### 2.4. The Counting of Ordinary Central Configurations

Thanks to Theorem 1, we can discuss the counting of ordinary central configurations. Note that the set of ordinary central configurations in \( S^3 \) (resp. \( \mathbb{H}^3 \)) is invariant under the action of \( SO(2) \times SO(2) \) (resp. \( SO(2) \times SO^+(1,1) \)) on the configuration space. The group \( SO^+(1,1) \) is \( \{ \begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix} | s \in \mathbb{R} \} \), the identity component of \( SO(1,1) \). The symmetry acts on the configuration space in the following way. For instance, in \( \mathbb{H}^3 \), let \( \chi = (\chi_1, \chi_2) \in SO(2) \times SO^+(1,1) \). Then

\[ \chi q = (\chi q_1, \ldots, \chi q_n), \quad \chi q_i = (\chi_1(x_i, y_i), \chi_2(z_i, w_i)) \].

The quotient of the set of ordinary central configurations under \( SO(2) \times SO(2) \) (resp. \( SO(2) \times SO^+(1,1) \)) will be called the set of *classes of ordinary central configurations*.

The major difference between the ordinary central configurations and the configurations of relative equilibria in the Newtonian case is the lack of homothety symmetry. For the Newtonian \( n \)-body problem, by the homothety symmetry, the set of configurations of relative equilibria on \( I_0^{-1}(c) \) is equivalent to that on \( I_0^{-1}(1) \). So it is enough to do the counting just on \( I_0^{-1}(1) \). For the curved \( n \)-body problem, the structure of the set of ordinary central configurations depends on the value of \( I_{\pm 1}(q) \) in an essential way. For instance, for two given masses in \( S^3 \), the number of ordinary central configurations varies as the value of \( I_1 \) varies [9]. It is also easy to see the existence of critical points of \( U_1|_{S_2^x} \) and \( U_{-1}|_{S_2^-} \) for each \( c \) in some interval.

**Corollary 1** ([9]). For any given \( n \geq 2 \) masses, there are infinitely many classes of ordinary central configurations in the curved \( n \)-body problem in \( S^3 \) and \( \mathbb{H}^3 \).
Thus, to make the counting problem reasonable, we propose to count ordinary central configurations on $S^+_c$ ($S^-_c$) for the value of $c$. To imitate the finiteness problem on configurations of relative equilibria of the Newtonian $n$-body problem [23], we ask: Are there always only finitely many classes of ordinary central configurations on $S^+_c$ ($S^-_c$) for the curved $n$-body problem for almost all choices of masses $(m_1, \ldots, m_n)$? The answer is negative for some choices of masses. For example, for two masses $m_1 = m_2$ in $S^3$, there are infinitely many classes of ordinary central configurations on $S^+_m$; see Section 10 of [9].

2.5. Main Results

We are interested in the investigation of the set of the ordinary central configurations. We first consider the compactness, then focus on the counting of geodesic ordinary central configurations and their Morse index. We postpone the proofs of Proposition 1, Theorems 2 and 3 to Section 3, and the proofs of Theorems 7 and 8 to Section 4.

In $\mathbb{H}^3$, similar to the singular set of the Newtonian $n$-body problem, a point in $\Delta_-$ can be written as

$$X = (q'_1, \ldots, q'_k, q'^{k+1}_1, \ldots, q'^{k+1}_2, \ldots, q'^{k-1+1}_k, \ldots, q'^{k+1}_k),$$

where we have grouped the equal terms $q'_1 = \ldots = q'_k, q'^{k+1}_1 = \ldots = q'^{k+1}_2, \ldots, q'^{k-1+1}_k = \ldots = q'^{k+1}_k, k = n$. We call each group of particles a cluster of $X$. Denote by $\Lambda_i$ the index set of the $i$th cluster, i.e., $\Lambda_i = \{k_{i-1} + 1, \ldots, k_i\}, 1 \leq i \leq s$. Let $|\Lambda_1| = k_1, \ldots, |\Lambda_i| = k_i - k_{i-1}, \ldots, |\Lambda_s| = n - k_{s-1}$. Let $q(l) = (q_{1(l)}, \ldots, q_{n(l)}), l = 1, \ldots, \infty$, be a sequence of ordinary central configurations that converges to $X$. Let $q_{i(l)} = (x_{i(l)}', y_{i(l)}', z_{i(l)}', w_{i(l)}')$. Denote by $\lambda(l)$ the multiplier of $q(l)$.

**Proposition 1.** Given $n$ masses in $\mathbb{H}^3$, if there is a sequence of ordinary central configurations that converges to some point $X \in \Delta_-$, then the sequence of multipliers approaches $-\infty$.

The set of ordinary central configurations is not compact. For example, consider the following regular polygonal configuration formed by $n$ equal masses:

$$q_i = (\sinh \theta \cos \frac{i2\pi}{n}, \sinh \theta \sin \frac{i2\pi}{n}, 0, \cosh \theta), \; i = 1, \ldots, n.$$  

It is easy to check that it is an ordinary central configuration for any $\theta \in (0, \infty)$. These ordinary central configurations are not in one class since there is no homothety symmetry in the set of ordinary central configurations. As $\theta \to 0$, the configuration converges to a singular configuration. Note that the momentum of inertia of that singular point is 0.

**Theorem 2.** Given $n$ masses in $\mathbb{H}^3$ and any point $X \in \Delta_-$ with $I_{-1}(X) = c > 0$, there is a neighborhood of $X$ in which there is no ordinary central configuration.

**Remark 1.** Consider the subset of ordinary central configurations on $\mathbb{H}^2_{x=0}$, the intersection of $\mathbb{H}^3$ and the hyperplane $z = 0$, with the property that all particles lie on the same plane perpendicular to the $w$-axis and the value of the multiplier is fixed. Tibboel [24] proved that this subset is compact in the configuration space. Our result is stronger.

In $S^3$, a point in $\Delta_+$ can be written as

$$X = (q'_1, \ldots, q'_k, q'^{k+1}_1, \ldots, q'^{k+1}_2, \ldots, q'^{k-1+1}_k, \ldots, q'^{k+1}_k),$$

where we have grouped the equal and antipodal terms $q'_1 = \ldots = q'_k, q'^{k-1+1}_1 = \ldots = q'^{k-1+1}_k, q'^{k+1}_1 = q'^{k+1}_k, \ldots, q'^{k-1+1}_k = q'^{k+1}_k, -q'_1 = -q'_k, \ldots, -q'^{k-1+1}_1 = -q'^{k-1+1}_k, q'^{k+1}_2 = n$. If $|\Lambda_k| > 1$, particles in the $k$th cluster form a collision singular configuration. If $|\Lambda_{2i-1}| = |\Lambda_{2i}| = 1$, particles in the two clusters form an antipodal singular configuration. If $|\Lambda_{2i-1}| \geq 2$ and $|\Lambda_{2i}| \geq 1$, particles in the two clusters form a collision-antipodal singular configuration.
As in the case of $\mathbb{H}^3$, the set of ordinary central configurations is not compact. For instance, consider the regular polygonal configurations formed by $n$ equal masses at position
\[ q_i = \left( \sin \theta \cos \frac{i2\pi}{n}, \sin \theta \sin \frac{i2\pi}{n}, \cos \theta, 0 \right), \; i = 1, \ldots, n. \]

They approach a collision singular configuration with momentum of inertia $0$ as $\theta \to 0$. The situation is more complicated than that in $\mathbb{H}^3$. By the transform $\tau \in O(4)$ of Theorem 5, we get a 1-parameter family of ordinary central configurations approaching a singular point with momentum of inertia $n$. As another example, consider three masses $m_1 = m, m_2 = m_3 = M$ at position
\[ q_1 = (1, 0, 0, 0), \; q_2 = (-\cos \theta, 0, \sin \theta, 0), \; q_3 = (-\cos \theta, 0, -\sin \theta, 0). \]

It is an ordinary central configuration for any $\theta \neq 0$ by Eq. (3.1). As $\theta \to 0$, the configuration approaches a collision-antipodal singular configuration. Let $S^1_{xy} := \{(x, y, z, w) \in S^3 : z = w = 0\}$, $S^1_{zw} := \{(x, y, z, w) \in S^3 : x = y = 0\}$. Note that the singular configurations of the three examples all lie on the union of two circles, $S^1_{xy} \cup S^1_{zw}$.

We consider only a subset, denoted by $\mathcal{A}$, of $\Delta_+$ with the following two properties: 1. if $X \in \mathcal{A}$, then not all particles of $X$ lie on $S^1_{xy} \cup S^1_{zw}$; 2. $X$ contains a collision singular subconfiguration or an antipodal singular subconfiguration, that is, there is some $i$ such that $|\Lambda_{2i-1}| \geq 2, |\Lambda_{2i}| = 0$, or $|\Lambda_{2i-1}| = |\Lambda_{2i}| = 1$.

**Theorem 3.** Given $n$ masses in $S^3$ and any point $X \in \mathcal{A}$, there is a neighborhood of $X$ in which there is no ordinary central configuration.

We now consider the geodesic configurations. Let us introduce some notation. A geodesic central configuration is one for which all particles lie on the same geodesic. A 2-dimensional central configuration is one for which all particles lie on the same 2-dimensional great sphere, but not on the same geodesic. Denote by $S^2_{xyz}$ (resp. $S^2_{xzw}$) the 2-dimensional great sphere intersected by $S^3$ and the hyperplane $w = 0$ (resp. $y = 0$). Denote by $\mathbb{H}^2_{xyw}$ the intersection of $\mathbb{H}^3$ and the hyperplane $z = 0$. Let $S^1_{xz} := \{(x, y, z, w) \in S^3 : y = w = 0\}$, $\mathbb{H}^1_{xw} := \{(x, y, z, w) \in \mathbb{H}^3 : y = z = 0\}$. We have some related preliminary results.

**Theorem 4 ([9]).** Let $q = (q_1, \ldots, q_n)$, $q_i = (x_i, y_i, z_i, w_i)$, $i = 1, \ldots, n$, be an ordinary central configuration in $\mathbb{H}^3 (S^3)$. Then we have the relationships
\[ \sum_{i=1}^{n} m_i x_i z_i = \sum_{i=1}^{n} m_i x_i w_i = \sum_{i=1}^{n} m_i y_i z_i = \sum_{i=1}^{n} m_i y_i w_i = 0. \]

**Remark 2.** The above relationships have been found in [11, 12] for two-body ordinary central configurations, where they read as $m_1 \sin 2\theta_1 = m_2 \sin 2\theta_2$ or $m_1 \sinh 2\theta_1 = m_2 \sinh 2\theta_2$. Recall that configurations of relative equilibria in $\mathbb{R}^2$ have their center of mass at the origin, i.e., $\sum_{i=1}^{n} m_i x_i = \sum_{i=1}^{n} m_i y_i = 0$. Theorem 4 can be viewed as an analogy of that fact.

**Theorem 5 ([9]).** In $S^3$, each geodesic (resp. 2-dimensional) ordinary central configuration is equivalent to one on $S^1_{xz}$ (resp. $S^2_{xyz}$ or $S^2_{xzw}$). Any ordinary central configuration with multiplier $\lambda$ is mapped to one ordinary central configuration with multiplier $-\lambda$ by $\tau \in O(4)$, where $\tau(x, y, z, w) = (z, w, x, y)$.

**Theorem 6 ([9, 27]).** In $\mathbb{H}^3$, each ordinary central configuration is equivalent to one on $\mathbb{H}^2_{xyw}$. Each geodesic ordinary central configuration is equivalent to one on $\mathbb{H}^1_{xw}$.

Thus, for geodesic (resp. 2-dimensional) ordinary central configurations, it is enough to study the ones on $S^1_{xz}$ and $\mathbb{H}^1_{xw}$ (resp. $S^2_{xyz}$ and $\mathbb{H}^2_{xyw}$). We will call those special submanifolds $S^1$ and $\mathbb{H}^1$ (resp. $S^2$ and $\mathbb{H}^2$).
Theorem 8 gives an upper bound of the value of $m_1(q)$ (resp. $m_{-1}(q)$) restricted to $S^+_c \cap (S^2)^n$ (resp. $S^-_c \cap (S^2)^n$). With a slight abuse of notation, we still call them $S^+_c$ and $S^-_c$. Furthermore, the classes of ordinary central configurations in $I_{1}^{-1}(c)$ (resp. $I_{-1}^{-1}(c)$) correspond in a 1-1 manner to the critical points of $U_1(q)$ (resp. $U_{-1}(q)$) restricted to $S^+_c / S^1$ (resp. $S^-_c / S^1$). Recall that, for the Newtonian n-body problem, the classes of configurations of relative equilibria correspond in a 1-1 manner to the critical points of $U_0(q)$ restricted to $S / S^1$, where

$$S = \{q \in (\mathbb{R}^2)^n - \Delta \mid I_0(q) = 1\}.$$

For the $\mathbb{H}^2$ case, the set $S^-_c / S^1 (c > 0)$ is obviously diffeomorphic to $S / S^1$. Thus, $S^-_c / S^1$ is a smooth $(2n - 2)$-dimensional manifold.

**Theorem 7.** Given $n$ masses on $\mathbb{H}^1$ and any positive value of $c$, there are exactly $n! / 2$ geodesic ordinary central configurations in $S^-_c$, one for each ordering of the masses along $\mathbb{H}^1$. At each of them, the Hessian of $U_{-1}|_{S^-_c / S^1}$ has inertia $(0, n, n) = (0, n, n - 2)$.

For the $S^2$ case, the set $S^+_c / S^1$ is more complicated. Assume that $m_1$ is the smallest mass.

**Proposition 2.** For given $n$ masses, the critical values of the function $I_1(q)$ are $\{\sum \epsilon_i m_i | \epsilon_i = 0 \text{ or } 1\}$. Assume that $m_1$ is the smallest mass. If $c < m_1$, then $S^+_c$ is a smooth $(2n - 2)$-dimensional manifold with $2^n$ components.

**Proof.** Let $f(x, y, z) = x^2 + y^2$, $(x, y, z) \in S^2$. Obviously, the critical points of $f$ consist of the equator and the two poles. Thus, the critical points of $I_1(q)$ are

$$\{q \in (S^2)^n | q_i = (0, 0, \pm 1) \text{ or } (\cos \varphi, \sin \varphi, 0), \ i = 1, \ldots, n\},$$

which gives the set of critical values. If $c$ is less than the first critical value $m_1$, no particle of configurations in $S^+_c$ can lie on the equator. Then each $z_i$ must be either positive or negative, which implies that there are $2^n$ components of $S^+_c$. \hfill $\boxdot$

We will only consider geodesic ordinary central configurations on the following component of $S^+_c$:

$$\mathcal{M}_c = \{q \in S^+_c | z_i > 0, 1 = 1, \ldots, n\}, \ 0 < c < m_1.$$

Note that $\mathcal{M}_c / S^1$ is diffeomorphic to $S / S^1$. The multiplier of ordinary central configurations on $\mathcal{M}_c$ must be negative, see Proposition 5.

**Theorem 8.** Given $n$ masses on $S^1$ and any value of $c \in (0, m_1/2)$, there are exactly $n! / 2$ geodesic ordinary central configurations on $\mathcal{M}_c$, one for each ordering of the masses. Provided that $0 < c < m_1/4$, at each of them, the Hessian of $U_1|_{\mathcal{M}_c / S^1}$ has inertia $(0, n, n) = (0, n, n - 2)$.

The restriction of the value of $c$ might not be sharp, but it is necessary. In the two-body case, if $m_1 < m_2$, then there is no ordinary central configuration on $\mathcal{M}_c$ for $c \in [m_1, m_2]$. If $m_1 = m_2 = m$, there is a continuum of ordinary central configurations on $\mathcal{M}_m$, and each of them has inertias $(0, n, n) = (1, 1, 0)$, see Section 10 of [9].

Recall that the inertia of each collinear configuration of relative equilibria of the Newtonian n-body problem is also $(0, n, n - 2)$ if we study the Hessian of $U_0$ on $\{q : I_0(q) = 1\} / S^1$ instead of $\{q : I_0(q) = 1, \sum m_i q_i = 0\} / S^1$, [14]. So Theorems 7 and 8 confirm the following general belief: many properties of configurations of relative equilibria of the Newtonian n-body problem also hold for ordinary central configurations of the curved n-body problem provided that the value of $I_{-1}(q)$ is small enough. For the number and inertia of geodesic ordinary central configurations, Theorem 7 indicates that the restriction of the small value of $I_{-1}(q)$ is not necessary on $\mathbb{H}^2$; Theorem 8 gives an upper bound of the value of $I_1(q)$ such that the corresponding results hold on $S^2$.

The three manifolds $S / S^1$, $S^-_c / S^1$ and $\mathcal{M}_c / S^1 (0 < c < m_1)$ are diffeomorphic and so share the same Poincaré polynomial. The number of geodesic ordinary central configurations on $\mathcal{M}_c / S^1$
(\mathbb{S}^1) and Morse index of them are the same as that on \mathbb{S}/\mathbb{S}^1 of the Newtonian n-body problem. By Theorems 2 and 3, the set of ordinary central configurations on \mathcal{M}_c/\mathbb{S}^1 (\mathbb{S}_c^1/\mathbb{S}^1) is compact. Thus, we can apply the argument of Palmore [18, 23] to obtain the following estimation on the number of critical points of \mathcal{U}_1(q) (resp. \mathcal{U}_1(q)) restricted to \mathcal{M}_c (resp. \mathbb{S}_c).

**Corollary 2.** Suppose that for a certain choice of masses of the curved n-body problem on \mathbb{S}^2 (resp. \mathbb{H}^2) all ordinary central configurations are nondegenerate critical points of \mathcal{U}_1|_{\mathcal{M}_c/\mathbb{S}^1} (resp. \mathcal{U}_1|_{\mathcal{S}_c^1/\mathbb{S}^1}). Then in \mathcal{M}_c (0 < c < \frac{m_1}{2}) (resp. \mathbb{S}_c^1 (c > 0)) there are at least \frac{(3\pi - 4)(n-1)!}{2} ordinary central configurations of which at least \frac{(2\pi - 4)(n-1)!}{2} are nongeodesic.

### 3. PROOF OF THEOREMS 2 AND 3

Theorems 2 and 3 are analogous to Shub’s lemma in the Newtonian n-body problem. It was first proved by Shub [21]. Moeckel gave a shorter proof in [14]. The idea of Moeckel is applicable if we use the following Cartesian coordinate system for \((\mathbb{S}^3)^n (\mathbb{H}^3)^n\).

Recall that we have written the Euler–Lagrange equation in Section 2.2 with the Cartesian coordinates. For each particle, we use the four coordinates \((x_i, y_i, z_i, w_i)\) to represent its position. That coordinate system is redundant. Three coordinates are enough to represent the positions of each particle. For instance, if \(x_i \neq 0\), then \(q_i\) is in an open region of \(\mathbb{S}^3\) for \((y, z, w)\) to serve as a local chart. Then \(\partial(U_1-\lambda M_1)/\partial q_i = 0\) is equivalent to \(\partial(U_1-\lambda M_1)/\partial y_i = \partial(U_1-\lambda M_1)/\partial z_i = \partial(U_1-\lambda M_1)/\partial w_i = 0\). Since \(x_i^2 + y_i^2 = 1 - z_i^2 - w_i^2\), we have

\[
\frac{\partial I}{\partial y_i} = 0, \quad \frac{\partial I}{\partial z_i} = -2m_i z_i, \quad \frac{\partial I}{\partial w_i} = -2m_i w_i.
\]

By \(\cos d_{ij} = \frac{x_i x_j + y_i y_j + z_i z_j + w_i w_j}{\sqrt{1 - x_i^2 - y_i^2 - z_i^2 - w_i^2}}\), we obtain

\[
\frac{\partial \cot d_{ij}}{\partial y_i} = -\frac{1}{\sin^2 d_{ij}} \frac{\partial d_{ij}}{\partial y_i} = \frac{1}{\sin^3 d_{ij}} \left( \frac{\partial x_i}{\partial y_i} x_j + \frac{\partial y_i}{\partial y_i} y_j \right) = \frac{1}{\sin^3 d_{ij}} \left( y_j - \frac{x_j}{x_i} y_i \right).
\]

Similarly, we have \(\partial \cot d_{ij} = \frac{1}{\sin^3 d_{ij}} \left( z_j - \frac{x_j}{x_i} z_i \right)\), \(\partial \cot d_{ij} = \frac{1}{\sin^3 d_{ij}} \left( w_j - \frac{x_j}{x_i} w_i \right)\). Thus, the central configuration equations for \(q_i\) can be written as

\[
\sum_{j \neq i} m_j m_j 
\frac{v_j - \frac{x_j}{x_i} v_i}{\sin^3 d_{ij}} = -\lambda 2m_i (0, z_i, w_i) \quad \text{if} \quad x_i \neq 0, \quad (3.1)
\]

where \(v_i = (y_i, z_i, w_i)\). Similarly, if \(w_i \neq 0\), the central configuration equation for \(q_i\) can be written as

\[
\sum_{j \neq i} m_j m_j 
\frac{u_j - \frac{w_j}{w_i} u_i}{\sin^3 d_{ij}} = \lambda 2m_i (x_i, y_i, 0) \quad \text{if} \quad w_i \neq 0, \quad (3.2)
\]

where \(u_i = (x_i, y_i, z_i)\). Similarly, the equations can be written in other forms if \(y_i \neq 0\) or \(z_i \neq 0\). In \(\mathbb{H}^3\), \((x, y, z)\) serve as a global chart. Thus, the central configuration equations in \(\mathbb{H}^3\) can be written as

\[
\sum_{j \neq i} m_j m_j 
\frac{u_j - \frac{w_j}{w_i} u_i}{\sinh^3 d_{ij}} = \lambda 2m_i (x_i, y_i, 0), \quad i = 1, \ldots, n. \quad (3.3)
\]

**Proof (of Proposition 1).** View the two sides of the central configuration equations (3.3) as vectors in \(\mathbb{R}^3\), and multiply on both sides by \(u_i\). We obtain

\[
2\lambda m_i (x_i^2 + y_i^2) = \sum_{j \neq i} \frac{u_j \cdot u_i - w_j/w_i (w_i^2 - 1)}{\sinh^3 d_{ij}} = \sum_{j \neq i} \frac{w_j/w_i - \cosh d_{ij}}{\sinh^3 d_{ij}}.
\]

\[\text{REGULAR AND CHAOTIC DYNAMICS Vol. 26 No. 3 2021}\]
Assume that $|\Lambda_1| = k_1 \geq 2$. Denote by $q(l)$ the sequence of ordinary central configurations that converges to $X$. For each $l$, assume that $q(l)$ of the first cluster has the largest value of $w$, i.e., $w_i(l) \geq w_j(l)$ for any $1 \leq j(l) \leq k_1$. The above equality for $q(l)$ is

$$2\lambda(l)m_i(l)(x_i^2(l) + y_i^2(l)) = \sum_{j(l) \neq i(l), j(l) \in \Lambda_1} \frac{w_i(l) - \cosh d_{ij(l)}}{\sinh^3 d_{ij(l)}} + O(1).$$

As $l \to \infty$, each term in the above sum approaches $-\infty$ since

$$\frac{w_i(l) - \cosh d_{ij(l)}}{\sinh^3 d_{ij(l)}} \leq 1 - \cosh d_{ij(l)} \leq -\frac{d_{ij(l)^2}}{2} + O(d_{ij(l)^3}) \to -\infty.$$

The value of $x_{i(l)}^2 + y_{i(l)}^2$ is obviously bounded above. Hence, the multiplier of the sequence of ordinary central configurations approaches $-\infty$.

**Proof (of Theorem 2).** Since $I_{-1}(X) = c > 0$, not all clusters of $X$ are at $(0,0,0,1)$. Assume that the first cluster is not at $(0,0,0,1)$. Note that the central configuration equations can be written as $\sum_{j \neq i} \frac{(w_i - w_j)u_j}{\sinh d_{ij}} = 2\lambda m_i w_i(x_i, y_i, 0), i = 1, \ldots, n$. Then adding the equations corresponding to particles in the first cluster, we obtain

$$\sum_{i \in \Lambda_1} \sum_{j \neq i} \frac{w_i u_j - w_j u_i}{\sinh^3 d_{ij}} = \sum_{i \in \Lambda_1} \sum_{j \notin \Lambda_1} \frac{w_i u_j - w_j u_i}{\sinh^3 d_{ij}} = 2\lambda \sum_{i \in \Lambda_1} m_i w_i(x_i, y_i, 0).$$

Assume that there is a sequence of ordinary central configurations $q(l)$ that converges to $X$. The above equality reads $O(1) = \infty$. This contradiction shows that there is a neighborhood of $X$ in which there is no ordinary central configuration.

**Proof (of Theorem 3).** Let $X \in A$. Assume that the first cluster of $X$ does not lie on $S^1_{xy} \cup S^1_{zw}$, i.e., $q' \notin S^1_{xy} \cup S^1_{zw}$. Since $z_1^2 + w_1^2 \neq 0$, we can assume that $w_1' \neq 0$. Let $q$ be an ordinary central configuration close to $X$. Then the central configuration equations for particles in the first two clusters can be written as

$$\sum_{j \neq i, j \in \Lambda_1 \cup \Lambda_2} \frac{w_i u_j - w_j u_i}{\sinh^3 d_{ij}} + O(1) = 2\lambda m_i w_i(x_i, y_i, 0), i \in \Lambda_1 \cup \Lambda_2,$$

where the $O(1)$ term corresponds to interactions between $q_i$ and particles of the other $2s - 2$ clusters. Adding those equations, we obtain

$$\sum_{i \in \Lambda_1 \cup \Lambda_2} \left( \sum_{j \neq i, j \in \Lambda_1 \cup \Lambda_2} \frac{w_i u_j - w_j u_i}{\sinh^3 d_{ij}} + O(1) \right) = O(1) = 2\lambda \sum_{i \in \Lambda_1 \cup \Lambda_2} m_i w_i(x_i, y_i, 0).$$

Assume that there is a sequence of ordinary central configurations $q(l)$ that converges to $X$. Since $X$ contains a subconfiguration which is collision singular or antipodal singular, we can prove that the absolute value of multiplier $|\lambda(l)|$ goes to infinity by an argument similar to that in the proof of Proposition 1. Note that

$$\sum_{i \in \Lambda_1 \cup \Lambda_2} m_i w_i(x_i, y_i, 0) \to k_1 \sum_{i=1}^{k_1} m_i(x_i w_i', y_i w_i', 0) + \sum_{i=k_1+1}^{k_2} m_i(x_i w_i', y_i w_i', 0) \neq 0$$

since $x_i^2 + y_i^2 \neq 0$ and $w_i' \neq 0$. The above equality reads $O(1) = \infty$. This contradiction shows that there is a neighborhood of $X$ in which there is no ordinary central configuration.
4. PROOF OF THEOREMS 7 AND 8

In the Newtonian $n$-body problem, the number of collinear configurations of relative equilibria was first found to be $\frac{n}{2}$ by Moulton [16], then Smale gave a shorter proof [22]. The index of them is $n-2$ [17]. The idea in [17], due to Conley, is applicable if we use an angle coordinate system for $(S^2)^n$ ($(\mathbb{H}^2)^n$). Theorem 7 (the $\mathbb{H}^2$ case) is proved in the first subsection, divided into two parts. Theorem 8 (the $S^2$ case) is proved in a similar way with a minor modification in the last subsection.

4.1. Geodesic Ordinary Central Configurations on $\mathbb{H}^2$

**Proposition 3.** Let $q$ be an ordinary central configuration on $\mathbb{H}^2$. Then the multiplier is negative.

**Proof.** Assume $w_i \leq w_n$ for $i = 1, \ldots, n$. Then $w_n > 1$. Recall that the central configuration equation (3.3) for $q_n$ can be written as $\sum_{j \neq n} \frac{u_j \cdot w_n}{\sinh d_{jn}} = 2\lambda w_n u_n$. Here $u_i = (x_i, y_i)$ since the configuration is on $\mathbb{H}^2$. Multiplying $u_n$ on both sides, the right-hand side becomes $2\lambda w_n (w_n^2 - 1)$, and the left-hand side becomes

$$\sum_{j \neq n} \frac{u_j \cdot u_n - w_j w_n}{\sinh^3 d_{jn}} (w_n^2 - 1) - \sum_{j \neq n} \frac{w_j}{w_n} \cosh d_{jn} < 0.$$

This shows that the multiplier $\lambda$ is negative.

We use an angle coordinate system for $\mathbb{H}^2$, $(\theta, \varphi)$, $\theta, \varphi \in \mathbb{R}$. The relationship between Cartesian coordinates $(x, y, w)$ and $(\theta, \varphi)$ is

$$(x, y, w) = (\sinh \theta, \cosh \theta \sinh \varphi, \cosh \theta \cosh \varphi).$$

Then $\mathbb{H}^1 = \mathbb{H}^1_{xw}$ is parameterized by $(\theta, 0)$ and $(\mathbb{H}^2)^n$ is parameterized by $(\theta_1, \ldots, \theta_n, \varphi_1, \ldots, \varphi_n)$. The momentum of inertia is

$$I_{-1}(q) = \sum_{i=1}^n m_i (x_i^2 + y_i^2) = \sum_{i=1}^n m_i (\sinh^2 \theta_i + \cosh^2 \theta_i \sinh^2 \varphi_i).$$

In the above angle coordinates, a configuration $(\theta_1, \ldots, \theta_n, \varphi_1, \ldots, \varphi_n)$ is an ordinary central configuration if

$$\frac{\partial U_{-1}}{\partial \theta_i} = \lambda \frac{\partial I_{-1}}{\partial \theta_i}, \quad \frac{\partial U_{-1}}{\partial \varphi_i} = \lambda \frac{\partial I_{-1}}{\partial \varphi_i}, \quad i = 1, \ldots, n.$$

Restricting to $\mathbb{H}^1$, we can just use the theta coordinates $(\theta_1, \ldots, \theta_n)$ to parameterize the configurations. Then

$$q_i = (\sinh \theta_i, 0, \cosh \theta_i), \quad \quad d_{ij} = |\theta_i - \theta_j|,$$

$$U(q) = \sum_{1 \leq i < j \leq n} m_i m_j \coth d_{ij}, \quad I(q) = \sum_{i=1}^n m_i \sinh^2 \theta_i.$$

A configuration $(\theta_1, \ldots, \theta_n)$ is an ordinary central configuration if $\frac{\partial U_{-1}}{\partial \theta_i} = \lambda \frac{\partial I_{-1}}{\partial \theta_i}$, or explicitly,

$$\sum_{j \neq i} \frac{m_i m_j \sinh(\theta_j - \theta_i)}{\sinh^3 d_{ij}} = \lambda m_i \sinh 2\theta_i, \quad i = 1, \ldots, n. \quad (4.1)$$
Proof (of Theorem 7, Part I). The numbers of geodesic ordinary central configurations are proved to be $\frac{n!}{2}$ in [9]. The argument is similar to that of the Newtonian $n$-body problem [22]. We briefly repeat the idea here.

Restrict the function $U_-$ to the set $S_c^- \cap (\mathbb{H}^1)^n = \{ q \in (\mathbb{H}^1)^n - \Delta_- | L_- (q) = c \}$, which has $n!$ components. Each component is homeomorphic to an open ball. On each component, there is at least one minimum. All critical points are minima since the Hessian of $U_-$ at each of them is positive definite. Thus, there are $n!$ critical points, and the number of ordinary central configurations is $\frac{n!}{2}$ by the $SO(2)$ symmetry.

For completeness, we repeat the proof of positive definiteness. Let $\bar{q} = (\bar{\theta}_1, \ldots, \bar{\theta}_n)$ be one critical point of $U_-|_{S_c^-}$. Assume that $\bar{\theta}_1 < \ldots < \bar{\theta}_n$ and that the multiplier is $\bar{\lambda}$. Then the Hessian of $U_-|_{S_c^- \cap (\mathbb{H}^1)^n}$ equals $D^2(U_- - \bar{\lambda}I_-)|_{\bar{q}}$ as quadratic forms on $T_{\bar{q}}S_c^- \cap (\mathbb{H}^1)^n$, the tangent space of $S_c^- \cap (\mathbb{H}^1)^n$ at $\bar{q}$. By straightforward computations, we obtain

$$D^2(U_- - \bar{\lambda}I_-)|_{\bar{q}} = 2 \begin{bmatrix}
\frac{m_1 m_1 \cosh d_{1j}}{\sinh^2 d_{1j}} & -\frac{m_1 m_2 \cosh d_{12}}{\sinh^2 d_{12}} & \cdots & -\frac{m_1 m_n \cosh d_{1n}}{\sinh^2 d_{1n}} \\
-\frac{m_2 m_1 \cosh d_{12}}{\sinh^2 d_{12}} & \frac{m_2 m_2 \cosh d_{22}}{\sinh^2 d_{22}} & \cdots & -\frac{m_2 m_n \cosh d_{2n}}{\sinh^2 d_{2n}} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{m_n m_1 \cosh d_{1n}}{\sinh^2 d_{1n}} & -\frac{m_n m_2 \cosh d_{2n}}{\sinh^2 d_{2n}} & \cdots & \frac{m_n m_n \cosh d_{nn}}{\sinh^2 d_{nn}}
\end{bmatrix} - 2\bar{\lambda} \begin{bmatrix}
\cosh 2\theta_1 & 0 & \cdots & 0 \\
0 & m_2 \cosh 2\theta_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & m_n \cosh 2\theta_n
\end{bmatrix}.$$

The second part is obviously positive definite. For the first part, let us take any nonzero vector $v = (v_1, \ldots, v_n) \in T_{\bar{q}}S_c^- \cap (\mathbb{H}^1)^n$. We obtain

$$v^T (D^2U_-) v = \sum_{i=1}^{n} \sum_{j=1}^{n} (D^2U_-)_{ij} v_i v_j = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{m_i m_j \cosh d_{ij}}{\sinh^2 d_{ij}} v_i^2 v_j^2 - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{m_i m_j \cosh d_{ij}}{\sinh^2 d_{ij}} v_i v_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{m_i m_j \cosh d_{ij}}{\sinh^2 d_{ij}} (v_i - v_j)^2 \geq 0.$$

Hence, we conclude that the Hessian of $U_-|_{S_c^- \cap (\mathbb{H}^1)^n}$ is positive definite. \hfill \Box

We have seen that system (4.1) has $n!/2$ solutions. Each is a minimum of $U_-|_{S_c^- \cap (\mathbb{H}^1)^n}$. We are going to study the Hessian on $S_c^-$ at those critical points.

Let $\bar{q} = (\bar{\theta}_1, \ldots, \bar{\theta}_n, 0, \ldots, 0)$ be one of the $n!/2$ geodesic ordinary central configurations. Assume that $\bar{\theta}_1 < \ldots < \bar{\theta}_n$ and that the multiplier is $\bar{\lambda}$. Then $\bar{q}$ is also a critical point of $f = U_- - \bar{\lambda}I_-$. Denote by $\mathcal{H}(U_-|_{S_c^-}, \bar{q})$ (resp. $\mathcal{H}(f, \bar{q})$) the Hessian of $U_-|_{S_c^-}$ (resp. $f$) at the critical point $\bar{q}$. Then we have

$$\mathcal{H}(f, \bar{q}) = \mathcal{H}(U_-|_{S_c^-}, \bar{q})$$

as quadratic forms on $T_{\bar{q}}S_c^-$, the tangent space of $S_c^-$ at $\bar{q}$.
The set $S_c^-$ is a hypersurface in $(\mathbb{H}^2)^n$ with codimension 1. The tangent space of $(\mathbb{H}^2)^n$ at $q$, $T_q(\mathbb{H}^2)^n$, is spanned by $\frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n}, \frac{\partial}{\partial \varphi_1}, \ldots, \frac{\partial}{\partial \varphi_n}$. The normal of $S_c^-$ at $q$ is $\nabla I_{-1}|_q = \sum_{i=1}^n m_i \sinh 2\theta_i \frac{\partial}{\partial \theta_i}$, which is in the subspace spanned by $\frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n}$. Thus, we see that $T_q S_c^- = V_1 \oplus V_2$,

$$V_1 = \langle \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n} \rangle / \langle \nabla I_{-1}|_q \rangle, \quad V_2 = \langle \frac{\partial}{\partial \varphi_1}, \ldots, \frac{\partial}{\partial \varphi_n} \rangle,$$

where $\langle b_1, \ldots, b_n \rangle$ is the linear space spanned by the vectors $b_1, \ldots, b_n$.

We claim that $\mathcal{H}(f, \bar{q})$ is block-diagonal with respect to this splitting of $T_q S_c^-$. Direct computation leads to $\frac{\partial^2 U_{-1}}{\partial \varphi_i \partial \theta_j}|_q = 0$ for all pairs of $(i, j)$. We claim that $\frac{\partial^2 U_{-1}}{\partial \theta_i \partial \theta_j}|_q = 0$ for all pairs of $(i, j)$. Denote by $q + h_i$ the coordinate $(\theta_1, \ldots, \theta_i + h_i, \ldots, \theta_n, \varphi_1, \ldots, \varphi_n)$ and by $q + k_j$ the coordinate $(\theta_1, \ldots, \theta_n, \varphi_1, \ldots, \varphi_j + k_j, \ldots, \varphi_n)$. Then

$$\frac{\partial^2 U_{-1}}{\partial \varphi_i \partial \theta_j}|_q = \lim_{k \to 0} \frac{1}{2k} (\frac{\partial U_{-1}}{\partial \theta_i}|_{q + k_j} - \frac{\partial U_{-1}}{\partial \theta_i}|_{q - k_j})$$

$$= \lim_{(h, k) \to (0, 0)} \frac{1}{2hk} (U_{-1}(q + k_j + h_i) - U_{-1}(q + k_j) - U_{-1}(q - k_j + h_i) + U_{-1}(q - k_j))$$

$$= 0.$$

Here, we use the symmetry $U_{-1}(q + k_j) = U_{-1}(q - k_j)$, $U_{-1}(q + k_j + h_i) = U_{-1}(q - k_j + h_i)$. Thus, $\mathcal{H}(f, \bar{q})$ is diagonal,

$$\mathcal{H}(f, \bar{q})|_q = D^2 U_{-1} - \lambda D^2 I_{-1} = \text{diag}\{[\frac{\partial^2 U}{\partial \theta_i \partial \theta_j} - \lambda \frac{\partial^2 I_{-1}}{\partial \varphi_i \partial \varphi_j}], [\frac{\partial^2 U}{\partial \theta_i \partial \theta_j} - \lambda \frac{\partial^2 I_{-1}}{\partial \varphi_i \partial \varphi_j}]\}.$$

Denote by $\mathcal{H}_1, \mathcal{H}_2$ the two $n \times n$ blocks, respectively. If $b \in T_q S_c^-$ has decomposition $b = b_1 + b_2$, then

$$\mathcal{H}(f, \bar{q})(b) = \mathcal{H}_1(b_1) + \mathcal{H}_2(b_2). \quad (4.2)$$

We move on to find the inertia of $\mathcal{H}_1$ on $V_1$ and that of $\mathcal{H}_2$ on $V_2$. The first one has already been found in the above proof. Indeed, it is the Hessian of $U_{-1}|_{S_c^-(\mathbb{H}^2)^n}$. So $\mathcal{H}_1$ on $V_1$ has inertia $(n_0, n_+, n_-) = (0, n - 1, 0)$.

**Proof (of Theorem 7, Part II).** By the above argument, it only remains to show that $\mathcal{H}_2$ on $V_2$ has inertia $(n_0, n_+, n_-) = (1, 1, n - 2)$.

Direct computation shows that $\mathcal{H}_2$ is

$$-2\lambda \begin{bmatrix} m_1 \cosh^2 \bar{\theta}_1 & 0 & \cdots & 0 \\ 0 & m_2 \cosh^2 \bar{\theta}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & m_n \cosh^2 \bar{\theta}_n \end{bmatrix}.$$
Let \( C := \text{diag}\{ \cosh \theta_1, \cdots, \cosh \theta_n \} \), \( M := \text{diag}\{ m_1, \cdots, m_n \} \), and

\[
A := \begin{bmatrix}
    \sum_{j=1, j \neq 1}^{n} \frac{-m_j \cosh \theta_j}{\cosh \theta_1 \sinh^3 d_{1j}} & \frac{m_2}{\sinh^3 d_{12}} & \cdots & \frac{m_n}{\sinh^3 d_{1n}} \\
    \frac{m_1}{\sinh^3 d_{12}} & \sum_{j=1, j \neq 2}^{n} \frac{-m_j \cosh \theta_j}{\cosh \theta_2 \sinh^3 d_{2j}} & \cdots & \frac{m_n}{\sinh^3 d_{2n}} \\
    \cdots & \cdots & \cdots & \cdots \\
    \frac{m_1}{\sinh^3 d_{1n}} & \cdots & \cdots & \sum_{j=1, j \neq n}^{n} \frac{-m_j \cosh \theta_j}{\cosh \theta_n \sinh^3 d_{nj}} 
\end{bmatrix}.
\]

Then it is easy to check that \( \mathcal{H}_2 = CM(A - 2\bar{\lambda})C \). Let \( D = CM \). Note that \( CM, C, D \) are all positive definite and diagonal. Since

\[
\mathcal{H}_2 = [(D^{1/2})^T D^{1/2} (A - 2\bar{\lambda}) D^{1/2} D^{1/2}] C
\]

\[
= C^{1/2} C^{1/2} [(D^{1/2})^T D^{1/2} (A - 2\bar{\lambda}) D^{1/2} D^{1/2}] C^{1/2} (C^{1/2})^T,
\]

the inertia of \( \mathcal{H}_2 \) equals that of \( A - 2\bar{\lambda} \) by Sylvester’s law of inertia.

For the inertia of \( A - 2\bar{\lambda} \), we will study the eigenvalues of \( A \) and compare them with the negative constant \( 2\bar{\lambda} \). First, there are two obvious eigenvectors of \( A \):

\[
c_1 = (\cosh \theta_1, \cdots, \cosh \theta_n), \quad Ac_1 = 0c_1,
\]

\[
c_2 = (\sinh \theta_1, \cdots, \sinh \theta_n), \quad Ac_2 = 2\bar{\lambda}c_2.
\]

The first vector \( c_1 \) can be obtained by inspecting the matrix, and the second vector \( c_2 \) can be seen from Eq. (4.1).

Note that the matrix \( CMAC \) is symmetric, the sum of each row is zero, the diagonal is negative and the other elements are positive. By the argument used in the proof of Theorem 7 (Part I), we conclude that all other eigenvalues of \( A \) are strictly negative.

We claim that all other eigenvalues of \( A \) are smaller than \( 2\bar{\lambda} \). The idea is to consider the linear vector field \( Y = Au \) in \( \mathbb{R}^n \), \( u = (u_1, \cdots, u_n)^T \in \mathbb{R}^n \). Introduce an inner product in \( \mathbb{R}^n \), \( \langle u, v \rangle = u^T M v \). Then \( A \) is symmetric with respect to this inner product and all eigenvectors of \( A \) are mutually orthogonal. Then the line \( tc_1 \) consists of all fixed points of the flow, and the \((n - 1)\)-dimensional space \( c_1^\perp \) is invariant under the flow. In particular, each of the other eigenvectors corresponds to a stable manifold of the flow. Conley observed that showing that all other eigenvalues of \( A \) are smaller than \( 2\bar{\lambda} \) is equivalent to showing that the line \( tc_2 \) attracts the flow lines. It is enough to find a cone \( K \subset c_1^\perp \) around \( tc_2 \) that is carried strictly inside itself by the flow.

![Fig. 1. The linear flow in \( c_1^\perp \).](image)

Define the polyhedral cone in \( c_1^\perp \) as

\[
K = \left\{ u \in \mathbb{R}^n \mid \sum_{i=1}^{n} m_i \cosh \theta_i u_i = 0, \quad \frac{u_1}{\cosh \theta_1} \leq \frac{u_2}{\cosh \theta_2} \leq \cdots \leq \frac{u_n}{\cosh \theta_n} \right\}.
\]
We verify that \( c_2 \in K \). First, \( c_2 \perp c_1 \) since \( c_2 \) is also an eigenvector of \( A \). Or, the identity \( \sum_{i=1}^{n} m_i x_i w_i = 0 \) of Theorem 4 reads (\( c_1, c_2 \) = 0 in this case. Second, the inequalities in the definition read \( \frac{\sinh \bar{\theta}_i}{\cosh \theta_i} < \frac{\sinh \bar{\theta}_j}{\cosh \theta_j} \leq \cdots \leq \frac{\sinh \bar{\theta}_n}{\cosh \theta_n} \), which is true since \( \bar{\theta}_1 < \bar{\theta}_2 < \cdots < \bar{\theta}_n \).

We verify that \( c_2 \) is the only eigenvector of \( A \) in \( K \). Since \( \bar{\theta}_1 < \bar{\theta}_2 < \cdots < \bar{\theta}_n \) and \( \sum_{i=1}^{n} m_i \sinh 2\bar{\theta}_i = 0 \), we assume that there is some \( I \in (1, n) \) such that \( \bar{\theta}_1 < \cdots < \bar{\theta}_I \leq 0 < \bar{\theta}_{I+1} < \cdots < \bar{\theta}_n \). Then by the definition of \( K \), for any \( u = (u_1, \ldots, u_n)^T \in K \), we have

\[
\begin{cases}
\frac{u_i}{\cosh \theta_i} m_i \sinh 2\bar{\theta}_i \geq \frac{u_{I+1}}{\cosh \theta_{I+1}} m_i \sinh 2\bar{\theta}_i, & 1 \leq i \leq I; \\
\frac{u_i}{\cosh \theta_i} m_i \sinh 2\bar{\theta}_i \geq \frac{u_{I+1}}{\cosh \theta_{I+1}} m_i \sinh 2\bar{\theta}_i, & I + 1 \leq i \leq n.
\end{cases}
\]

Then for any point \( u \in K \) we have

\[
2(u, c_2) = 2 \sum_{i=1}^{n} u_i m_i \sinh \bar{\theta}_i = \sum_{i=1}^{n} \frac{u_i}{\cosh \theta_i} m_i \sinh 2\bar{\theta}_i
\]

\[
= \sum_{i=1}^{I} \frac{u_i}{\cosh \theta_i} m_i \sinh 2\bar{\theta}_i + \sum_{i=I+1}^{n} \frac{u_i}{\cosh \theta_i} m_i \sinh 2\bar{\theta}_i
\]

\[
\geq \frac{u_{I+1}}{\cosh \theta_{I+1}} \sum_{i=1}^{I} m_i \sinh 2\bar{\theta}_i + \frac{u_{I+1}}{\cosh \theta_{I+1}} \sum_{i=I+1}^{n} m_i \sinh 2\bar{\theta}_i
\]

\[
\geq \left( \frac{u_{I+1}}{\cosh \theta_{I+1}} - \frac{u_{I+1}}{\cosh \theta_I} \right) \sum_{i=I+1}^{n} m_i \sinh 2\bar{\theta}_i\geq 0.
\]

Moreover, the product is zero only if \( \frac{u_1}{\cosh \theta_1} = \cdots = \frac{u_n}{\cosh \theta_n} \), which is impossible. Hence, \((u, c_2) > 0\), and there are no other eigenvectors than \( c_2 \) of \( A \) in \( K \).

The boundary \( \partial K \) consists of points for which one or more equalities hold. However, except for the origin, at least one inequality must hold (otherwise \( u = tc_1 \)). Consider a boundary point with

\[
\frac{u_1}{\cosh \theta_1} \leq \cdots \leq \frac{u_i}{\cosh \theta_i} = \cdots = \frac{u_j}{\cosh \theta_j} \leq \cdots \leq \frac{u_n}{\cosh \theta_n}.
\]

Let \( g(u) = \frac{u_j}{\cosh \theta_j} - \frac{u_i}{\cosh \theta_i} \). Then \( g = 0 \) at this point, and \( g \) is positive in \( K \). To prove that at this point the flow is pointing inwards (see Fig. 1), we show that \( L_\gamma g = \frac{u_j}{\cosh \theta_j} - \frac{u_i}{\cosh \theta_i} > 0 \). Direct computation shows that \( \frac{u_i}{\cosh \theta_i} - \frac{u_k}{\cosh \theta_k} \) is

\[
\sum_{k=1, k \neq j}^{n} \frac{m_k}{\cosh \theta_j \sinh^3 d_{kj}} \left( u_k - \frac{u_j \cosh \theta_k}{\cosh \theta_j} \right) - \sum_{k=1, k \neq i}^{n} \frac{m_k}{\cosh \theta_i \sinh^3 d_{ki}} \left( u_k - \frac{u_i \cosh \theta_k}{\cosh \theta_i} \right)
\]

\[
= \sum_{k=1, k \neq i, j}^{n} \frac{m_k}{\sinh^3 d_{ij} \cosh \theta_j} \left( u_i - \frac{u_j \cosh \theta_i}{\cosh \theta_j} \right) - \sum_{k=1, k \neq i, j}^{n} \frac{m_k}{\sinh^3 d_{ij} \cosh \theta_i} \left( u_j - \frac{u_i \cosh \theta_j}{\cosh \theta_i} \right)
\]

Since \( \frac{u_i}{\cosh \theta_i} = \frac{u_j}{\cosh \theta_j} \), the last two terms are zero, and the expression \( \frac{u_i \cosh \theta_k}{\cosh \theta_i \cosh \theta_j \sinh^3 d_{kj}} \) in the first part can be written as \( \frac{u_i \cosh \theta_k}{\cosh \theta_i \cosh \theta_j \sinh^3 d_{kj}} \). Then \( L_\gamma g \) can be written as

\[
\sum_{k=1, k \neq i, j}^{n} \frac{m_k}{\sinh^3 d_{kj} \cosh \theta_j} \left( u_k - \frac{u_i \cosh \theta_k}{\cosh \theta_i \cosh \theta_j} \right) \left( \frac{1}{\sinh^3 d_{kj} \cosh \theta_j} - \frac{1}{\sinh^3 d_{ki} \cosh \theta_i} \right).
\]
Every term in this expression is nonnegative by Proposition 4. Proposition 4 will be given after this proof:

\[
\begin{align*}
\text{If } k &< i, \quad \text{then } u_k - \frac{u_i \cosh \theta_k}{\cosh \theta_i} \leq 0, \quad \frac{1}{\sinh^3 d_{kj} \cosh \theta_j} - \frac{1}{\sinh^3 d_{ki} \cosh \theta_i} < 0. \\
\text{If } i \leq k \leq j, \quad \text{then } u_k - \frac{u_i \cosh \theta_k}{\cosh \theta_i} = 0. \\
\text{If } j < k, \quad \text{then } u_k - \frac{u_i \cosh \theta_k}{\cosh \theta_i} \geq 0, \quad \frac{1}{\sinh^3 d_{kj} \cosh \theta_j} - \frac{1}{\sinh^3 d_{ki} \cosh \theta_i} > 0.
\end{align*}
\] (4.3)

Moreover, at least one term is strictly positive since at least one inequality in the definition of the cone must hold. Thus, we have proved that on the boundary of the cone the flow is pointing inwards, or, all the other eigenvalues of \( A \) are smaller than \( 2\lambda \).

Hence, the eigenvalues of \( A - 2\lambda \) are \(-2\lambda > 0, 0, \lambda_3 < 0, \ldots, \lambda_n < 0\). Then the inertia of \( A - 2\lambda \) is \((n_0, n_+, n_-) = (1, 1, n - 2)\), so is the inertia of \( \mathcal{H}_2 \), on the space \( V_2 \). On the space \( V_2/S^1 \), obviously, the inertia is \((n_0, n_+, n_-) = (0, 1, n - 2)\). Combined with the inertia of \( \mathcal{H}_1 \), we conclude that the inertia of \( U_{-1}|_{S^2_+}/S^1 \) is

\[
(0, n - 1, 0) + (0, 1, n - 2) = (0, n, n - 2).
\]

This completes the proof of Theorem 7. \( \square \)

**Proposition 4.** If \( \theta_1 < \theta_2 < \cdots < \theta_n \), then the following inequalities hold.

1) If \( k < i < j \), then \( \sinh^3(\theta_j - \theta_k) \cosh \theta_j - \sinh^3(\theta_i - \theta_k) \cosh \theta_i > 0 \).

2) If \( i < j < k \), then \( \sinh^3(\theta_k - \theta_i) \cosh \theta_i - \sinh^3(\theta_k - \theta_j) \cosh \theta_j > 0 \).

**Proof.** Let \( h(x) = \sinh^3(x - \theta_k) \cosh x - \sinh^3(\theta_i - \theta_k) \cosh \theta_i \) defined for \( x \geq \theta_i \). Then \( h(\theta_i) = 0 \), and

\[
h'(x) = \sinh^2(x - \theta_k)[\cosh(2x - \theta_k) + 2 \cosh(x - \theta_k) \cosh x] > 0.
\]

This implies the first inequality. We omit the proof of the second one since it is similar. \( \square \)

**4.2. The \( S^2 \) Case**

We omit the proof of the following result since it is similar to that of Proposition 3.

**Proposition 5.** Let \( q \) be an ordinary central configuration on \( S^2 \). If \( z_i > 0 \) for all particles, then the multiplier is negative.

We use an angle coordinate system for \( \{(x, y, z) \in S^2 | z > 0\} \), \((\theta, \varphi)\), with \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}, -\frac{\pi}{2} < \varphi < \frac{\pi}{2} \). The relationship between Cartesian coordinates \((x, y, z)\) and \((\theta, \varphi)\) is

\[
(x, y, z) = (\sin \theta, \cos \theta \sin \varphi, \cos \theta \cos \varphi).
\]

Then \( S^1 = S^1_{x_2} \) is parameterized by \((\theta, 0)\) and \((S^2)^n \) is parameterized by \((\theta_1, \ldots, \theta_n, \varphi_1, \ldots, \varphi_n)\), and the momentum of inertia is

\[
I_1(q) = \sum m_i(\sin^2 \theta + \cos^2 \theta \sin \varphi).
\]

**Proof (of Theorem 8).** For the number of geodesic ordinary central configurations, we apply the argument used in the proof of Theorem 7. All arguments run well except showing that the Hessian
of $U_1$ restricted to $\mathcal{M}_c \cap (\mathbb{S}^1)^n$ at each critical point is positive definite. By direct computation, we obtain the Hessian

$$
2 \begin{bmatrix}
    \sum_{j=1,j\neq 1}^{n} \frac{m_1 m_j \cos d_{ij}}{\sin^3 d_{ij}} & -\frac{m_1 m_2 \cos d_{12}}{\sin^3 d_{12}} & \cdots & -\frac{m_1 m_n \cos d_{1n}}{\sin^3 d_{1n}} \\
    -\frac{m_2 m_1 \cos d_{12}}{\sin^3 d_{12}} & \sum_{j=1,j\neq 2}^{n} \frac{m_2 m_j \cos d_{2j}}{\sin^3 d_{2j}} & \cdots & -\frac{m_2 m_n \cos d_{2n}}{\sin^3 d_{2n}} \\
    \cdots & \cdots & \cdots & \cdots \\
    -\frac{m_n m_1 \cos d_{1n}}{\sin^3 d_{1n}} & \cdots & \cdots & \sum_{j=1,j\neq n}^{n} \frac{m_n m_j \cos d_{jn}}{\sin^3 d_{jn}} \\
    \frac{m_1 \cos 2\theta_1}{\sin^3 \theta_1} & 0 & \cdots & 0 \\
    0 & \frac{m_2 \cos 2\theta_2}{\sin^3 \theta_2} & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \frac{m_n \cos 2\theta_n}{\sin^3 \theta_n}
\end{bmatrix} - 2\lambda
$$

Restrict $c < \frac{1}{4}m_1$. Then $m_i \sin^2 \theta < \frac{1}{4}m_1$, $-\frac{\pi}{6} < \theta_i < \frac{\pi}{6}$ for all $i$. Hence, $\cos 2\theta_i > 0, d_{ij} < \frac{\pi}{6}$. The second part is positive definite since $\lambda < 0$. Each element of the first matrix that is not on the diagonal is negative. By the argument used in the proof of Theorem 7 (Part I), we see that the first matrix is positive semidefinite. Thus, the Hessian of $U_1|_{\mathcal{M}_c \cap (\mathbb{S}^1)^n}$ is positive definite and there are exactly $n!/2$ geodesic ordinary central configurations on $\mathcal{M}_c$ provided that $c < \frac{m_1}{2}$.

For the Morse index of the geodesic ordinary central configurations, we need to restrict further $c < \frac{1}{4}m_1$, which leads to $m_i \sin^2 \theta < \frac{1}{4}m_1$, $-\frac{\pi}{6} < \theta_i < \frac{\pi}{6}$ for all $i$. Then all the arguments used in the proof of Theorem 7 work if we replace the hyperbolic functions with the trigonometrical ones. In particular, inequalities (4.3) are replaced by the following inequalities.

**Proposition 6.** If $-\frac{\pi}{6} < \theta_1 < \cdots < \theta_n < \frac{\pi}{6}$, then the following inequalities hold.

1) If $k < i < j$, then $\sin^3(\theta_j - \theta_k) \cos \theta_j - \sin^3(\theta_i - \theta_k) \cos \theta_i > 0$.

2) If $i < j < k$, then $\sin^3(\theta_k - \theta_i) \cos \theta_i - \sin^3(\theta_k - \theta_j) \cos \theta_j > 0$.

**Proof.** We only prove the first inequality. Let $h(x) = \sin^3(x - \theta_k) \cos x - \sin^3(\theta_i - \theta_k) \cos \theta_i$ defined for $x \geq \theta_i$. Then $h(\theta_i) = 0$, and

$$
  h'(x) = \sin^2(x - \theta_k) [\cos(2x - \theta_k) + 2 \cos(x - \theta_k) \cos x] > 0.
$$

This proves the first inequality. \hfill \Box

This completes the proof of Theorem 8.

---

**APPENDIX. THE RELATIVE EQUILIBRIA AND CENTRAL CONFIGURATIONS**

Recall that $\mathbb{S}^1_{xy} := \{(x, y, z, w) \in \mathbb{S}^3 : z = w = 0\}$, $\mathbb{S}^1_{zw} := \{(x, y, z, w) \in \mathbb{S}^3 : x = y = 0\}$. Recall that the critical points of $U_1$ are special central configurations, the critical points of $U_1 - \lambda I_1, U_{-1} - \lambda I_{-1}$ that are not special central configurations are ordinary central configurations. The motions in the form $q(t) = q(0)$ are called *equilibria*, and the motions in the form $\exp(t\xi)q$ ($\xi \neq 0$) are called *relative equilibria*.

**Proposition 7 ([9]).** Let $q = (q_1, \ldots, q_n) \in (\mathbb{H}^3)^n$ be an ordinary central configuration with multiplier $\lambda$. Then the associated relative equilibria are $B_{\alpha, \beta}(t)q$ with $\alpha = \sqrt{-2\lambda} \cos s, \beta = \sqrt{-2\lambda} \sin s, s \in (0, 2\pi]$. 

---
• Let \( \mathbf{q} = (q_1, \ldots, q_n) \in (S^3)^n \) be a special central configuration. Then the associated equilibrium is \( \mathbf{q}(t) = \mathbf{q} \). The associated relative equilibria are

\[
- A_{\alpha,\beta}(t)\mathbf{q} \text{ with } \alpha, \beta \in \mathbb{R} \text{ if all particles are on } S^1_{xy} \cup S^1_{zw};
- A_{\alpha,\beta}(t)\mathbf{q} \text{ with } \beta = \pm \alpha, \alpha \in \mathbb{R} \text{, if not all particles are on } S^1_{xy} \cup S^1_{zw}.
\]

• Let \( \mathbf{q} = (q_1, \ldots, q_n) \in (S^3)^n \) be an ordinary central configuration. The associated relative equilibria are

\[
- A_{\alpha,\beta}(t)\mathbf{q} \text{ with } \alpha = \sqrt{2\lambda} \sinh s, \beta = \sqrt{2\lambda} \cosh s, s \in \mathbb{R}, \text{ if } \lambda > 0;
- A_{\alpha,\beta}(t)\mathbf{q} \text{ with } \alpha = -\sqrt{2\lambda} \cosh s, \beta = -\sqrt{2\lambda} \sinh s, s \in \mathbb{R}, \text{ if } \lambda < 0.
\]

For one ordinary central configuration, among the set of associated relative equilibria, there are periodic ones (in \( H^3, \beta = 0 \); in \( S^3, k + \beta = 0 \) for some \( k \in \mathbb{Z} \), and quasi-periodic ones (in \( H^3 \), none; in \( S^3, k + \beta \neq 0 \) for any \( k \in \mathbb{Z} \)).

In the Newtonian \( n \)-body problem, the relative equilibrium is always planar, while in the curved \( n \)-body problem the set of relative equilibria has richer structure. We divide them into three classes: geodesic, 2-dimensional, and 3-dimensional ones. A geodesic relative equilibrium is one with all particles on the same geodesic for all \( t \); a 2-dimensional relative equilibrium is one with all particles on the same 2-dimensional great sphere for all \( t \), but not on the same geodesic; the others are 3-dimensional relative equilibria.

If a \( k \)-dimensional relative equilibrium is associated with an \( m \)-dimensional configuration, then \( k \geq m \). Let \( Q(t)\mathbf{q} \) be one relative equilibrium. Then it is a geodesic one if \( \mathbf{q} \) is on a geodesic and \( Q(t) \) keeps that geodesic; it is a 2-dimensional one if \( \mathbf{q} \) is on a 2-dimensional great sphere (\( \mathbf{q} \) may be a geodesic one in that sphere), and \( Q(t) \) keeps that 2-dimensional great sphere.

Let \( G_1 = SO(2) \times SO^+(1,1) \) and \( G_2 = SO(2) \times SO(2) \). Assume that \( \mathbf{q} \) is an ordinary central configuration in \( \mathbb{H}^3 \) (resp. \( S^3 \)). Then \( g\mathbf{q} \) is an ordinary central configuration with the same multiplier if \( g \) is in \( G_1 \) (resp. \( G_2 \)). If \( B_{\alpha,\beta}(t)\mathbf{q} \) (resp. \( A_{\alpha,\beta}(t)\mathbf{q} \)) are relative equilibria associated with \( \mathbf{q} \), then the relative equilibria associated with \( g\mathbf{q} \) are

\[
B_{\alpha,\beta}(t)g\mathbf{q} = gB_{\alpha,\beta}(t)\mathbf{q}, \text{ (resp. } A_{\alpha,\beta}(t)g\mathbf{q} = gA_{\alpha,\beta}(t)\mathbf{q}).
\]

Let \( \tau \) be the isometry in \( O(4), \tau(x, y, z, w) = (z, w, x, y) \). By Theorem 5, if \( \mathbf{q} \) is an ordinary central configuration in \( S^3 \) with multiplier \( \lambda \), then the multiplier of \( \tau \mathbf{q} \) is \( -\lambda \). If \( A_{\alpha,\beta}(t)\mathbf{q} \) are relative equilibria associated with \( \mathbf{q} \), then the solutions associated with \( \tau \mathbf{q} \) are

\[
A_{\beta,\alpha}(t)\tau\mathbf{q} = \tau A_{\alpha,\beta}(t)\mathbf{q}.
\]

Thus, thanks to Theorems 5 and 6, to find geodesic and 2-dimensional relative equilibria, it is enough to assume that the associated ordinary central configuration lies on \( H^2_{xyz} \) for the \( \mathbb{H}^3 \) case, and on \( S^2_{xyz} \) with negative multiplier for the \( S^3 \) case.

**Proposition 8.** Consider the curved \( n \)-body problem in \( \mathbb{H}^3 \). Let \( G_1 = SO(2) \times SO^+(1,1) \).

• There is no geodesic relative equilibrium.

• Any 2-dimensional relative equilibria must be in one of the following three forms:

\[
- gB_{\pm \sqrt{-2\lambda},0}(t)\mathbf{q} \text{ and } gB_{0,\pm \sqrt{-2\lambda}}(t)\mathbf{q} \text{ for } \mathbf{q} \text{ being a geodesic ordinary central configuration on } H^1_{xyw}, \text{ with multiplier } \lambda;
- gB_{\pm \sqrt{-2\lambda},0}(t)\mathbf{q} \text{ for } \mathbf{q} \text{ being a 2-dimensional ordinary central configuration on } H^2_{xyz} \text{ with multiplier } \lambda,
\]

where \( g \) is some isometry in \( G_1 \).

• Any other relative equilibrium is 3-dimensional.
Proof. Let \( q \) be an ordinary central configuration on \( \mathbb{H}^1_{xw} \) with multiplier \( \lambda \). The 1-parameter subgroup \( B_{\alpha,\beta}(t) \) keeps the geodesic \( \mathbb{H}^1_{xw} \) only if \( \alpha = \beta = 0 \), which is impossible since \( \lambda < 0 \) by Proposition 3 and \( 2\lambda = -(\alpha^2 + \beta^2) \). So there is no geodesic relative equilibrium. Obviously, the 1-parameter subgroup \( B_{\alpha,\beta}(t) \) keeps a 2-dimensional great sphere containing \( \mathbb{H}^1_{xw} \) only if \( \alpha = 0 \) or \( \beta = 0 \). If \( \beta = 0 \) (resp. \( \alpha = 0 \)), the associated 2-dimensional relative equilibrium is \( B_{\pm\sqrt{-2\lambda},0}(t)q \) (resp. \( B_{0,\pm\sqrt{-2\lambda}}(t)q \)).

Let \( q \) be a 2-dimensional ordinary central configuration on \( \mathbb{H}^2_{xyw} \). Obviously, the 1-parameter subgroup \( B_{\alpha,\beta}(t) \) keeps \( \mathbb{H}^2_{xyw} \) only if \( \beta = 0 \). So 2-dimensional relative equilibria associated with \( q \) are \( B_{\pm\sqrt{-2\lambda},0}(t)q \). By the discussion before Proposition 8, the proof is complete. \( \square \)

Recall that a relative equilibrium \( B_{0,\beta}(t)q \) is hyperbolic. In [20], Pérez-Chavela and Sánchez-Cerritos consider 2-dimensional hyperbolic relative equilibria. They show that, if the masses are equal, the configuration of such relative equilibria cannot be a regular polygon. In fact, those motions can be characterized as follows:

**Proposition 9.** All 2-dimensional hyperbolic relative equilibria must be associated with geodesic ordinary central configurations. Given \( n \) masses, there are exactly \( n! \) families of 2-dimensional hyperbolic relative equilibria, one for each ordering of the masses along the geodesic.

**Proof.** The first part is from the second statement of Proposition 8. We can prove it directly. Since the motion is 2-dimensional, we use the Poincaré half-plane model: \( H, (x, y), y > 0, ds^2 = \frac{dx^2 + dy^2}{y^2} \).

Then the kinetics is \( K = \frac{1}{2} \sum m_i \frac{x_i^2 + y_i^2}{y_i^2} \). In this model, the hyperbolic 1-parameter subgroup acts on \( H \) by \( (x, y) \mapsto e^{it}(x, y) \) [7]. Thus, the vector field on \( H^n \) generated by the hyperbolic 1-parameter subgroup is \( \xi_{H^n}(q) = \alpha(x_1, y_1, \ldots, x_n, y_n) \). Then the augmented potential is

\[
U + K(\xi_{H^n}(q)) = U + \frac{\alpha^2}{2} \sum_{i=1}^{n} m_i \frac{x_i^2}{y_i^2} + \frac{\alpha^2}{2} \sum_{i=1}^{n} m_i.
\]

If \( q(t) = e^{it}q(0) \) is a hyperbolic relative equilibrium on \( H \), then \( q(0) \) is a critical point of the augmented potential. That is, \( q(0) \) must satisfy the equation

\[
\nabla_q U = -\frac{\alpha^2}{2} \nabla_q \sum_{i=1}^{n} m_i \frac{x_i^2}{y_i^2} = -\alpha^2 m_1 \frac{1}{y_1} (x_1 y_1, -x_1^2).
\]  

(A.1)

We claim that the critical points of this potential must be geodesic configurations. Recall that the geodesics on \( H \) are straight lines and circles perpendicular to the \( x \)-axis. Assume that the particles of \( q(0) \) are distributed on several circular geodesics \( x^2 + y^2 = R_j^2, j = 1, \ldots, p \) and that \( R_j \leq R_p \) if \( 1 \leq j \leq p \). Consider Eq. (A.1) for one particle, say \( q_1 \), on the largest circle. Note that the right-hand side of (A.1) is a vector tangent to the largest circle, but the left-hand side, the force exerted on \( q_1 \), is pointing inwards since there are particles on some smaller circle. This contradiction shows that the critical points of the augmented potential have to be geodesic configurations.

The second part is from Theorem 7. \( \square \)

Note that, for all 2-dimensional hyperbolic relative equilibria, the velocities are orthogonal to the geodesic containing the configuration. This is true for any relative equilibrium associated to a geodesic configuration. By Proposition 8, we may assume that the geodesic is \( \mathbb{H}^1_{xw} \). Then the velocity of the 1st particle is \( (0, \alpha x_1, \beta w_1, 0) \) for some \( \alpha, \beta \), which is orthogonal to the geodesic.

**Proposition 10.** For the curved \( n \)-body problem in \( \mathbb{S}^3 \), consider the relative equilibria associated with ordinary central configurations. Let \( \tau \) be the isometry \( \tau(x, y, z, w) = (z, w, x, y) \) and \( G_2 = SO(2) \times SO(2) \).

- **There is no geodesic relative equilibrium.**
Any 2-dimensional relative equilibria must be in one of the following four forms:

\[-gA_{\pm\sqrt{2}\lambda,0}(t)q \quad \text{and} \quad \tau g A_{\pm\sqrt{2}\lambda,0}(t)q \quad \text{for} \quad q \ \text{being a geodesic ordinary central configuration on} \ S^1_{xy} \ \text{with multiplier} \ \lambda < 0.\]

\[-gA_{\pm\sqrt{2}\lambda,0}(t)q \quad \text{and} \quad \tau g A_{\pm\sqrt{2}\lambda,0}(t)q \quad \text{for} \quad q \ \text{being a 2-dimensional ordinary central configuration on} \ S^2_{xyz} \ \text{with multiplier} \ \lambda < 0,\]

where \(g\) is some isometry in \(G_2\).

Any other relative equilibrium is 3-dimensional.

We omit the proof since it is similar to that of Proposition 8. Note that the ordinary central configurations on \(S^2_{xyz}\) with multiplier \(\lambda > 0\) lead to 3-dimensional relative equilibria. Similar to the hyperbolic case, the velocity of any relative equilibrium associated to a geodesic configuration is orthogonal to the geodesic containing the configuration.

We now turn to the special central configurations. In this case, the velocity of relative equilibrium associated to a geodesic configuration may or may not be along the geodesic containing the configuration. There is no need to discuss the dimension of equilibrium solutions.

**Proposition 11.** For the curved \(n\)-body problem in \(S^3\), consider the relative equilibria associated with special central configurations.

- Any geodesic relative equilibrium must be \(A_{\alpha,0}(t)q\), or \(\tau A_{\alpha,0}(t)q, \alpha \neq \mathbb{R}\) for \(q\) being a special central configuration on \(S^1_{xy}\).

- There is no 2-dimensional relative equilibrium.

- Any other relative equilibrium is 3-dimensional.

**Proof.** The symmetry group for special central configurations is \(O(4)\). Any geodesic special central configuration is in the form \(gq\), where \(q\) is on \(S^1_{xy}\) and \(g \in O(4)\). If \(gq\) is on \(S^1_{xy}\) (resp. \(S^1_{zw}\)), by Proposition 7, the associated relative equilibria are \(A_{\alpha,\beta}(t)q\) for any \(\alpha, \beta \in \mathbb{R}\), which are the same as \(A_{\alpha,0}(t)q\) (resp. \(\tau A_{\alpha,0}(t)q\)). If \(gq\) is not on \(S^1_{xy}\) nor on \(S^1_{zw}\), by Proposition 7, the associated relative equilibria are \(A_{\alpha,\pm0}(t)q\), \(\alpha \in \mathbb{R}\). The relative equilibrium is geodesic only if \(\alpha = 0\), and it is not geodesic nor 2-dimensional otherwise.

Let \(q\) be a 2-dimensional special central configuration. We claim that \(q\) cannot be within \(S^1_{xy} \cup S^1_{zw}\). Note that the particles on one of the two circles must be collinear since the configuration is contained in a 3-dimensional hyperplane. Assume that the particles on \(S^1_{xy}\) are collinear. The number of particles on \(S^1_{xy}\) is one, otherwise \(q \in \Delta_+\). Then all particles of \(q\) are within one 2-dimensional hemisphere, which is impossible, see Section 12.3 of [6] or [28]. The contradiction proves the claim. Hence, the associated relative equilibria must be \(A_{\alpha,\pm0}(t)q, \alpha \in \mathbb{R}\). The relative equilibrium is 2-dimensional only if \(\alpha = 0\).

To study the 2-dimensional relative equilibria of the curved \(n\)-body problem, instead of restricting to a 2-dimensional physical space directly, i.e., to \(T((S^2)^n - \Delta_+)\) or \(T((H^2)^n - \Delta_-)\), it seems convenient to start with the configurations and to use the augmented potentials introduced in Theorem 1 in some proper coordinates. Moreover, this restriction would make counting the configurations clumsy.

Let us finish the discussion with one concrete example. In [8], Diacu and Sergiu consider 3-dimensional relative equilibria of three bodies in \(S^3\) with the following property: the configuration is not geodesic and the three mutual distances are the same. They show that the three masses must be equal. This result can be obtained quickly as follows:

The associated central configuration must be 2-dimensional since there are only three bodies and the configuration is not geodesic. It cannot be a special central configuration since it is not
geodesic [28]. We may assume that it is on $S^2_{xyz}$, i.e., $q_i = (x_i, y_i, z_i)$. According to Diacu and Zhu [10], a three-body configuration on $S^2_{xyz}$ is an ordinary central configuration if and only if

$$\sum m_i z_i x_i = \sum m_i z_i y_i = 0, \quad (z_1, z_2, z_3) = k(\sin^3 d_{23}, \sin^3 d_{13}, \sin^3 d_{12}).$$

Since the three mutual distances are the same, we find immediately that the three masses are the same.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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