Cohomology foundations of one-loop amplitudes in pure spinor superspace

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We describe a pure spinor BRST cohomology framework to compactly represent ten-dimensional one-loop amplitudes involving any number of massless open- and closed-string states. The method of previous work to construct scalar and vectorial BRST invariants in pure spinor superspace signals the appearance of the hexagon gauge anomaly when applied to tensors. We study the systematics of the underlying BRST anomaly by defining the notion of pseudo-cohomology. This leads to a rich network of pseudo-invariant superfields of arbitrary tensor rank whose behavior under traces and contractions with external momenta is determined from cohomology manipulations. Separate papers will illustrate the virtue of the superfields in this work to represent one-loop amplitudes of the superstring and of ten-dimensional super-Yang–Mills theory.

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1. Introduction

Pure spinors are known to facilitate the superspace description of ten-dimensional super-Yang–Mills (SYM) [1,2] which descends from the pure spinor superstring [3]. As will be explained below, ten-dimensional pure spinor superspace allows to take advantage of BRST symmetry to provide valuable guidance for the construction of scattering amplitudes in both string- and field-theory.

1.1. Amplitudes as expressions in pure spinor superspace

The prescription to compute multiloop superstring amplitudes in the pure spinor formalism [3,4,5] is considerably simpler than in the Ramond–Neveu–Schwarz (RNS) [6] and Green–Schwarz (GS) [7] formulations of superstring theory. Unlike in the RNS, spacetime supersymmetry is manifest and there is no need to sum over spin structures since there are no worldsheet spinors. And in contrast to the GS, the amplitudes are computed in a manifestly super-Poincaré covariant manner. These two features combined allow to bypass the technical challenges associated with amplitude computations in the RNS and GS. However, there is another feature in the pure spinor setup which is not as prominently stressed but is of equal importance: The result of amplitude computations belongs to the BRST cohomology of pure spinor superspace expressions at ghost number three.

Pure spinor superspace\(^1\) is defined in terms of the standard ten-dimensional superspace variables \((x^m, \theta^\alpha)\) and the pure spinor \(\lambda^\alpha\) (of ghost number one) satisfying \(\lambda^\alpha \gamma^m \lambda^\beta = 0\) [8]. As will be explained below, it turns out that the kinematic factors\(^2\) of multiloop amplitudes can be written as pure spinor superspace expressions of the form

\[
K = \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha \beta \gamma}(x, \theta),
\]

where \(f_{\alpha \beta \gamma}(x, \theta)\) represents a function of ten-dimensional superspace and includes the dependence on polarizations and momenta. This novel type of superspace was shown in [3] to encode the results of tree-level string amplitudes and proven to be supersymmetric and gauge invariant when \(K\) is in the cohomology of the pure spinor BRST charge.

\(^1\) The superspace defined here is the minimal pure spinor superspace associated with the original formulation in [3]. The non-minimal superspace appropriate in the context of the non-minimal formalism of [5] also contains \(\lambda^\alpha\) variables and is not the subject of the present paper.

\(^2\) Kinematic factors are understood as the polarization-dependent parts of amplitudes accompanying a basis of worldsheet integrals.
Furthermore, in order to extract the precise contractions of polarizations and momenta from a superspace expression $K$, one computes its pure spinor bracket $\langle K \rangle$ defined by $\langle (\lambda \gamma^m \theta)(\lambda \gamma^n \theta)(\lambda \gamma^p \theta)(\theta \gamma_{mnp} \theta) \rangle = 1$ [3]. Since component expansions are straightforward to evaluate and can be automated [9,10], the real challenge in computing string scattering amplitudes consists of obtaining their corresponding superspace expressions in the BRST cohomology, which motivates the studies presented in this paper.

1.2. Multiloop amplitude prescription and pure spinor superspace

The prescription to compute multiloop amplitudes in the pure spinor formalism was presented in [4,5] and integrates out all eleven pure spinor components $\lambda^\alpha$ and all sixteen $\theta^\alpha$ variables. Performing those integrations leads to awkward expressions which are hard to manipulate. As observed at one-loop [11] and emphasized in the two-loop computations of [12], one can equivalently rewrite those complicated-looking expressions by reinstating three pure spinors $\lambda^\alpha$ in such a way as to obtain the same type of tree-level pure spinor superspace expressions (1.1) discussed above. To illustrate the above point, the two-loop kinematic factor of [13] after performing the path integral over all variables is written as

$$(T^{-1})^{\alpha\beta\gamma}_{\rho_1...\rho_{11}} \epsilon^{\rho_1...\rho_{16}} \frac{\partial}{\partial \theta^{\rho_{12}}} \cdots \frac{\partial}{\partial \theta^{\rho_{16}}} (\gamma^{mnpqr})_{\alpha\beta} \gamma^s_{\rho} F^1_{mn}(\theta) F^2_{pq}(\theta) F^3_{rs}(\theta) W^4(\theta).$$

(1.2)

The superfields $F_{mn}(\theta)$ and $W^\delta(\theta)$ represent the gauge multiplet of ten-dimensional SYM, and the tensor $(T^{-1})^{\alpha\beta\gamma}_{\rho_1...\rho_{11}}$ is proportional to a complicated combination of gamma matrices, $\epsilon^{\rho_1...\rho_{16}} (\gamma^m)_{\kappa \rho_{12}} (\gamma^n)_{\sigma \rho_{13}} (\gamma^p)_{\tau \rho_{14}} (\gamma_{mnp})_{\rho_{15} \rho_{16}} (\delta^\eta_{\kappa \sigma \tau} - \frac{1}{40} \gamma^q (\alpha \beta \gamma^s_{\tau})).$ However, the kinematic factor (1.2) can be equivalently written as the tree-level pure spinor superspace expression $\langle K \rangle$ after reinstating three pure spinors, where

$$K = (\lambda \gamma^{mnpqr} \lambda)(\lambda \gamma^s W^4) F^1_{mn} F^2_{pq} F^3_{rs}.$$  

(1.3)

In writing the kinematic factor (1.2) as (1.3), its BRST invariance becomes easier to prove; using standard manipulations of gamma matrices, the pure spinor constraint and SYM equations of motion for $D_\alpha W^\delta$ and $D_\alpha F^{mn}$, it follows that

$$Q[\langle (\lambda \gamma^{mnpqr} \lambda)(\lambda \gamma^s W^4) F^1_{mn} F^2_{pq} F^3_{rs} \rangle] = 0.$$  

(1.4)

Furthermore, one can also show that the kinematic factor (1.3) is in the cohomology of the BRST charge. The compact nature of pure spinor superspace expressions as exemplified by (1.3) compared to (1.2) and the observation (1.4) constitute the central pillars in the study of multiloop string scattering amplitudes as objects in the BRST cohomology of tree-level pure spinor superspace.

3 A superspace proof that (1.3) is not BRST exact requires a combination of the identities in [14] and section 9 of this paper.
1.3. BRST cohomology considerations as a method to simplify computations

Following observations based on the BRST structure of explicit lower-point results it was suggested in [15] that the field-theory amplitudes at tree-level could be uniquely obtained as pure spinor superspace expressions in the BRST cohomology. Indeed, the color-ordered $N$-point tree amplitude of SYM can be compactly written as [16]

$$A(1,2,\ldots,N) = \langle V_1 E_{23\ldots N} \rangle,$$

where $E_{23\ldots N}$ is a superfield in the BRST cohomology. The pursuit of the general expression of the SYM tree amplitude as the solution of a cohomology problem in pure spinor superspace led to the discovery of interesting mathematical objects such as the BRST blocks and supersymmetric Berends–Giele currents reviewed in section 2. These BRST-covariant objects also played an essential role in the derivation of the general $N$-point open superstring tree-level amplitude in [17]. And as a byproduct of the BRST-covariant organization of the string tree-level amplitudes, the worldsheet integrals conspire to a particularly symmetric form which was later exploited to find interesting patterns in their $\alpha'$ expansion [18–23].

1.3.1. Challenges at one-loop

Studying the one-loop open superstring amplitudes as a BRST cohomology problem was firstly put forward in [24]. Using the multiloop prescription of [4] as a guide to obtain the patterns of zero-mode saturation, the kinematic factors could be expressed in pure spinor superspace. Furthermore, integration by parts identities among the worldsheet integrals built up BRST-closed linear combinations of those kinematic factors, denoted $C_{i|A,B,C}$ and reviewed in section 2.4.

This cohomology setup led to a general and manifestly BRST-invariant expression for the $N$-point amplitude. For example, the six-point amplitude of open superstrings was found to be a worldsheet integral over\footnote{The objects $X_{ij}$ are related to the one-loop worldsheet Green function $G(z)$, and the Koba–Nielsen factor $\prod_{i<j} e^{\alpha'(k_i \cdot k_j)G(z_i-z_j)}$ is suppressed [24].}

$$X_{23}X_{34}\langle C_{1|234,5,6} \rangle + X_{23}X_{45}\langle C_{1|23,45,6} \rangle + \text{permutations}.$$  

However, there is one subtlety in the one-loop BRST cohomology program outlined above, the bare one-loop amplitudes are in general anomalous and therefore not BRST invariant.
The cancelation of the anomaly as described in [25] involves a sum over amplitudes with different worldsheet topologies, but the composing amplitudes are still anomalous when the number of external particles is six or higher. The BRST-invariant expression (1.6) could not be the whole story since it is non-anomalous\(^5\).

It is clear that in order to study the missing pieces of the one-loop amplitudes associated to the anomaly in a BRST cohomology setup one needs to relax the condition of BRST invariance. So in this work, among other things, we introduce the notion of a pseudo BRST cohomology which meets this criterion. The essential idea behind the pseudo BRST cohomology goes back to the pure spinor analysis of the gauge anomaly in [26]. It was shown that the gauge variation of the six-point amplitude w.r.t particle one is proportional to the pure spinor superspace expression,

\[
\langle (\lambda \gamma^m W^2)(\lambda \gamma^n W^3)(\lambda \gamma^p W^4)(W^5 \gamma_{mnp} W^6) \rangle,
\]

whose component expansion correctly reproduces the known form of the anomaly, \(\epsilon_{10} F^5\).

As discussed in section 3, one can recursively construct objects whose BRST variation is proportional to the anomalous superfield (1.7). It will be shown in a subsequent work that these pseudo BRST invariants correctly capture the anomalous parts of the one-loop amplitudes which were not considered in [24].

So as the main focus of this work, we will study the (pseudo-)cohomology properties of various superfields expected to appear in one-loop amplitudes of open- and closed-strings. We introduce a grid of superspace kinematic factors which naturally describe the BRST cohomology properties of one-loop amplitudes. The axes of this grid are set by the number of free vector indices \(m, n, p, \ldots\) and the number of multiparticle slots \(A, B, \ldots, G\) which are interpreted as representing external tree-level subdiagrams. This leads to the arrangement in fig. 1, and we will derive recursion relations whose flow is indicated by the diagonal arrows. The tensorial superfields therein play an important role in two different contexts:

\(^5\) We thank Michael Green for insisting on a clarification of this point.
(i) Closed string amplitudes involving five and more external legs allow for vector contractions between left- and right-moving degrees of freedom, see e.g. [27,28]. They originate from the zero modes of the worldsheet fields $\partial x^m$ and $\bar{\partial} x^m$. In a manifestly BRST-invariant representation of the five- and six-point torus amplitude, the left-right contractions enter in the form $C_{1|A,B,C,D,E,F}$, details will be elaborated in [29]. Accordingly, scattering of $N$ closed strings requires tensors of rank $r \leq N - 4$.

(ii) The field theory limit of open and closed string amplitudes reproduces $n$-gon Feynman integrals [30] where the loop momentum $\ell^m$ may contract kinematic factors. In a manifestly BRST invariant form of the five-point amplitude, this loop momentum dependence enters in the form $\ell_m C_{1|2,3,4,5}^m$. At six-points, the significance of the tensor hexagon $\ell_m \ell_n C_{1|2,3,4,5,6}^{mn}$ for the gauge anomaly of SYM will be clarified in [29]. More generally, the systematic association of tensorial Feynman integrals with the superfields in fig. 1 is discussed in [31].

The present work is devoted to the cohomology foundations of one-loop amplitudes in string- and field-theory. The key definitions and results are formulated in generality to describe any number of external legs. Applications to six-point string amplitudes and to field-theory amplitudes at multiplicity $\leq 7$ are given in upcoming work [29,31], and the generalization to arbitrary multiplicity is left for the future.
1.4. The anatomy of one-loop amplitudes

One can embed the pseudoinvariants listed in fig. 1 into a broader context. BRST cohomology methods are of crucial importance to decompose the computation of scattering amplitudes into smaller and more manageable problems. As we will see in various places of this work, pseudo-invariance of the superfields $C^m_{1|...}$, $P^m_{1|...}$ in fig. 1 requires BRST-covariant substructures, which in turn furnish systematic arrangements of smaller constituents. The different hierarchy levels of this decomposition are made more specific in fig. 2. The figure applies universally to one-loop scattering amplitudes involving SYM or supergravity states in maximally supersymmetric string- and field-theory.

These classes of one-loop amplitudes are claimed to have a beautiful representation in terms of pseudoinvariants $C^m_{1|...}$ and $P^m_{1|...}$ (or their holomorphic squares in case of supergravity and closed-string amplitudes). Their composition rules in terms of integrals over a loop momentum or over worldsheet moduli are the subject of upcoming work [29,31]. As detailed in the first six sections, pseudoinvariants are built from Berends–Giele currents $M_B$ (whose trilinears exhaust tree amplitudes of SYM [16] and the open superstring [17]) and ghost-number-two superfields $J$ which are specific to the one-loop order. These $J$ superfields in turn encompass various numbers of further Berends-Giele superfields $\mathcal{K}_B$ with kinematic poles in external momenta [32]. Both $M_B$ and $\mathcal{K}_B$ represent external tree subamplitudes which can be expanded in terms of (products of) external propagators. Their numerators are multiparticle superfields of SYM $K_B \in \{A^B_\alpha, A^m_B, W^\alpha_B, F^{mn}_B\}$ which have been recursively constructed in [32]. They encompass the degrees of freedom of several standard superfields $A^i_\alpha$, $A^m_i$, $W_\alpha_i$, $F^{mn}_i$ describing a single particle $i$. Finally, the components of the supersymmetry multiplet – a gluon with polarization vector $e_i$ and a gaugino with spinor
wavefunction $\chi_i$ – furnish the lowest hierarchy level. They are incorporated into the expansion of the superfields in terms of the Grassmann coordinate $\theta$ of pure spinor superspace [33], e.g.,

$$A_\alpha(x, \theta) = \left( \frac{1}{2} e_m(\gamma^m \theta)_\alpha - \frac{1}{3}(\chi \gamma_m \theta)(\gamma^m \theta)_\alpha - \frac{1}{16} k_m e_n(\gamma_p \theta)_\alpha(\theta \gamma^{mnp} \theta) + \cdots \right) e^{ik \cdot x} \quad (1.8)$$

As the number of external legs increases, every intermediate structure in fig. 2 reduces the complexity of amplitudes by more and more orders of magnitude. And it should be stressed that the four lower hierarchy levels in fig. 2 – from Berends–Giele superfields $M_B, K_B$ to the components $e_i, \chi_i$ – are expected to play a universal role at any loop order.

1.5. Outline

The main body of this work begins with a review of multiparticle SYM superfields [32] in section 2. This sets the stage to define the notion of anomalous superfields and BRST pseudo-cohomology in section 3. The introductory examples are then generalized to arbitrary tensor rank in section 4. The resulting tensor traces are shown to involve several constituents (indicated by $P$ in fig. 1) which are separately BRST pseudoinvariant, see sections 5 and 6. This completes the construction of the pseudo cohomology at ghost number three which is visualized in fig. 1.

In section 7, we point out a close parallel between the superfields in anomalous BRST variations and the previously-constructed pseudoinvariants. The more abstract viewpoint on this connection is opened up in section 8 and rewarded by manifold relations between superfields at different rank, see sections 9 and 10. The same approach leads to the proof in section 11 that – up to anomaly subtleties – the span of the pseudoinvariants in fig. 1 is independent on the choice of reference leg 1 which descends from the choice of unintegrated vertex operator $V_1$ in the one-loop string amplitude prescription [4].

Some appendices supplement the discussion by examples or serve to outsource technical aspects from the main body. For example, appendix A displays the expansions of superfields in fig. 1 at higher multiplicity, and appendix B provides the prerequisites to extract anomalous gauge variations of pseudoinvariants from their BRST transformations given in section 7.
2. Review and conventions

This section provides a brief review of multiparticle SYM superfields introduced in [32] as well as their simplest applications to one-loop kinematic factors. It also introduces notation and conventions used in the rest of this work.

2.1. Diagrammatic introduction of BRST blocks

Linearized super-Yang–Mills (SYM) theory in ten dimensions can be described using the superfields\(^6\) \(A_i^\alpha(x, \theta), A_m^i(x, \theta), W_\alpha^i(x, \theta)\) and \(F_{mn}^i(x, \theta)\) encoding the on-shell degrees of freedom of one external particle \(i\).

They satisfy equations of motion [34,35]

\[
2D_{(\alpha A^i_\beta)} = \gamma^{m\beta}_{\alpha} A^i_m \\
D_\alpha A^i_m = (\gamma_m W_i)_\alpha + k^i m A^i_\alpha \\
D_\alpha F^i_{mn} = 2k^i_m (\gamma_n) W^i_\alpha \\
D_\alpha W^\beta_i = \frac{1}{4} (\gamma^{mn})^\beta_{\alpha} F^i_{mn}
\]

(2.1)

with light-like momentum \(k^i\) and gauge transformations \(\delta_i A^i_\alpha = D_\alpha \omega_i\) as well as \(\delta_i A^i_m = k^i_m \omega_i\) for some scalar superfield \(\omega_i\). The fermionic operator

\[
D_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2} k^m (\gamma_m \theta)_\alpha
\]

(2.2)

denotes the standard superspace covariant derivative. Note that if a superfield \(K(x, \theta)\) depends only on the zero-modes of \(\theta\), the action of the pure spinor BRST charge \(Q\) is given by a covariant derivative:

\[
Q \equiv \oint \lambda^\alpha d_\alpha \quad \Rightarrow \quad QK = \lambda^\alpha D_\alpha K.
\]

(2.3)

This fact allows one to use the equations of motion (2.1) in combination with BRST cohomology manipulations to simplify expressions considerably.

In previous work [32], these superfields were promoted to multiparticle versions

\[
K_i \in \{A^i_\alpha, A^i_m, W^\alpha_i, F^i_{mn}\} \rightarrow K_B \in \{A^B_\alpha, A^B_m, W^B_\alpha, F^B_{mn}\},
\]

(2.4)

where the multiparticle label \(B = b_1 b_2 \ldots b_p\) describes \(|B| \equiv p\) external particles attached to a tree subdiagram as shown in fig. 3. The off-shell leg indicated by the \(\ldots\) in the figure reflects that the overall momentum \(k^m_B \equiv \sum_{i=1}^p k^m_{b_i}\) is no longer lightlike in general, \(k^2_B \neq 0\).

\(^6\) It is customary to use a calligraphic letter for the superfield field-strength. However in this paper calligraphic letters will denote the Berends–Giele currents associated to the superfields, see section 4.
Fig. 3 Four superfield realizations $K_B \in \{A_\alpha^B, A_m^B, W^\alpha_B, F_{mn}^B\}$ of cubic tree graphs $B = b_1 b_2 \ldots b_p$.

As detailed in [32], the diagrammatic interpretation is supported by the Lie symmetries of $K_B$ matching with the color representative of the diagram in fig. 3,

$$K_{1234\ldots p} \leftrightarrow f^{12a_2} f^{a_23a_3} f^{a_34a_4} \ldots f^{a_{p-1}p} ,$$

subject to antisymmetry $f^{abc} = f^{[abc]}$ and Jacobi identities $f^{[abc]} f^{[def]} = 0$. For example,

$$0 = K_{12} + K_{21} , \quad 0 = K_{123} + K_{231} + K_{312} \quad (2.6)$$

$$0 = K_{1234} - K_{1243} + K_{3412} - K_{3421} \quad (2.7)$$

furnish the kinematic analogue of $f^{12a} = -f^{21a}$ and Jacobi identities among permutations of $f^{12a} f^{3a} b$ and $f^{12a} f^{3b} f^{4c}$.

2.2. Recursive construction of BRST blocks

A recursive procedure was described in [32] to construct BRST blocks at arbitrary multiplicity from the elementary SYM superfields. The definition of the multiparticle fields (2.4) is inspired by the OPE among integrated massless vertex operators in the pure spinor formalism [3]. For two particles, this directly leads to

$$A^{12}_\alpha = -\frac{1}{2} \left[ A^1_\alpha (k^1 \cdot A^2) + A^1_m (\gamma^m W^2)_{\alpha} - (1 \leftrightarrow 2) \right] \quad (2.8)$$

$$A^{12}_m = \frac{1}{2} \left[ A^1_p F_{pm}^2 - A^1_m (k^1 \cdot A^2) + (W^1 \gamma_m W^2) - (1 \leftrightarrow 2) \right]$$

$$W^{12}_\alpha = \frac{1}{4} (\gamma^{mn} W^2)^\alpha F^{1}_{mn} + W^{\alpha}_2 (k^2 \cdot A^1) - (1 \leftrightarrow 2)$$

$$F^{12}_{mn} = k^m M^{12}_n - k^1 M^{12}_m - (k^1 \cdot k^2) (A^1_m A^2_n - A^1_n A^2_m) ,$$

compatible with antisymmetry $K_{12} = -K_{21}$. Remarkably, the equations of motion for these two-particle superfields take the same form as their single-particle equations (2.1)
with the addition of contact terms,

\[ 2D_{(\alpha}A^1_{\beta)} = \gamma^m_{\alpha\beta} A^1_m + (k^1 \cdot k^2)(A^1_{\alpha} A^2_{\beta} + A^1_{\beta} A^2_{\alpha}) \]  

\[ D_{\alpha}A^1_m = (\gamma_m W^{12})_\alpha + k^m_{\alpha} A^1_{12} + (k^1 \cdot k^2)(A^1_{\alpha} A^m_{12} - A^2_{\alpha} A^2_{12}) \]  

\[ D_{\alpha}W^\beta_{12} = \frac{1}{4}(\gamma^m_{\alpha\beta} F^1_{mn} + (k^1 \cdot k^2)(A^1_{\alpha} W^\beta_{2} - A^2_{\alpha} W^\beta_{1})) \]  

\[ D_{\alpha}F^1_{mn} = k^m_{\alpha}(\gamma_n W^{12})_\alpha - k^n_{\alpha}(\gamma_m W^{12})_\alpha + (k^1 \cdot k^2)(A^1_{\alpha} F^2_{mn} - A^2_{\alpha} F^1_{mn}) + (k^1 \cdot k^2)(A^1_{n}(\gamma_m W^{2})_\alpha - A^2_{n}(\gamma_m W^{1})_\alpha) - A^1_{m}(\gamma_n W^{2})_\alpha + A^2_{m}(\gamma_n W^{1})_\alpha). \]

Starting from multiplicity three, application of the recursion (2.8) yields superfields

\[ \hat{A}^{123}_\alpha = \frac{1}{2} [A^1_{m}(k^1 \cdot A^3) + A^2_{m}(\gamma^m W^{3})_\alpha - (12 \leftrightarrow 3)] \]  

\[ \hat{A}^{12}_m = \frac{1}{2} [A^p_{m} F^3_{pm} - A^m_{12}(k^1 \cdot A^3) + (W^{12} \gamma_m W^3) - (12 \leftrightarrow 3)] \]

which require redefinitions

\[ A^{123}_m = \hat{A}^{123}_m - k^m_{\alpha} H^{123}, \quad A^{123}_\alpha = \hat{A}^{123}_\alpha - D_{\alpha}H^{123} \]  

\[ H^{123} = \frac{1}{6} [(A^1 \cdot A^{23}) - (k^p \cdot k^3) A^p_{m}(A^2 \cdot A^3) + \text{cyclic}(123)] \]

by some scalar superfield \( H_{ijk} \) before they satisfy the Lie symmetries in (2.6) and qualify as BRST blocks. The three-particle set of BRST blocks \( K_{123} \in \{A^{123}_\alpha, A^{123}_m, W^{123}_m, F^{123}_{mn}\} \) is completed by field strengths

\[ W^{123}_m = \left[ \frac{1}{4}(\gamma^{rs} W^{3})^\alpha F^1_{rs} + W^\alpha_3(k^3 \cdot A^{12}) - (12 \leftrightarrow 3) \right] + \frac{1}{2}(k^1 \cdot k^2)[W^\alpha_2(A^1 \cdot A^3) - (1 \leftrightarrow 2)] \]  

\[ F^{123}_{mn} = k^{123}_m A^{123}_m - k^{123}_m A^{123}_m - (k^1 \cdot k^2)[2A^1_{m} A^{23}_n - (1 \leftrightarrow 2)] - (k^{12} \cdot k^3)2A^1_{m} A^{3}_n. \]

As shown in [32], the equations of motion for the \( K_{123} \) reproduce the universal structure of (2.1) and (2.9) and incorporate a richer set of contact terms \( \sim (k^1 \cdot k^2) \) and \( (k^{12} \cdot k^3) \):

\[ 2D_{(\alpha}A^{123}_{\beta)} = \gamma^m_{\alpha\beta} A^{123}_m + (k^1 \cdot k^3)[A^2_{\alpha} A^3_{\beta} - (12 \leftrightarrow 3)] \]  

\[ D_{\alpha}A^{123}_m = (\gamma_m W^{123})_\alpha + k^{123}_m A^{123}_m + (k^1 \cdot k^3)(A^1_{\alpha} A^2_{12} - A^2_{\alpha} A^3_{12}) + (k^1 \cdot k^2)[A^1_{\alpha} A^2_{m} + A^3_{\alpha} A^2_{m} - A^2_{\alpha} A^2_{m} - A^3_{\alpha} A^3_{m}] \]  

\[ D_{\alpha}W^\beta_{123} = \frac{1}{4}(\gamma^{mn}_{\alpha\beta} F^{123}_{mn} + (k^1 \cdot k^3)[A^{12}_{m} W^\beta_{3} - (12 \leftrightarrow 3)] + (k^1 \cdot k^2)[A^1_{m} W^\beta_{2} + A^3_{m} W^\beta_{2} - (1 \leftrightarrow 2)] \]  

\[ D_{\alpha}F^{123}_{mn} = 2k^{123}_{m}(\gamma_n W^{123})_\alpha + (k^1 \cdot k^3)[A^1_{m} F^{23}_{mn} - (12 \leftrightarrow 3)] + (k^1 \cdot k^2)[A^1_{m} F^{23}_{mn} + A^3_{m} F^{23}_{mn} - (1 \leftrightarrow 2)] + (k^1 \cdot k^2)[2A^1_{m}(\gamma_m W^{23})_\alpha + 2A^3_{m}(\gamma_m W^{23})_\alpha - (1 \leftrightarrow 2)]. \]
Fig. 4 From cubic diagrams $K_A$ to Berends–Giele currents $K_A$.

The starting point towards BRST blocks at higher multiplicity $p$ is the recursion

$$\hat{A}_{\alpha}^{12\ldots p} = -\frac{1}{2} \left[ A_{\alpha}^{12\ldots p-1}(k^{12\ldots p-1} \cdot A^p) + A_m^{12\ldots p-1}(\gamma^m W^p)_\alpha - (12\ldots p-1 \leftrightarrow p) \right]$$

(2.15)

with manifest Lie symmetries (2.5) in the first $p-1$ labels. Then, an algorithmic redefinition along the lines of (2.11) enforces the remaining symmetry (2.5) involving the last label $p$, see [32] for details.

The equations of motion at any multiplicity combine the single-particle structure from (2.1) with a growing tail of contact terms $\sim (k^{12\ldots j-1} \cdot k^j)$ generalizing the three-particle example (2.14). Their explicit forms can be found in [32].

2.3. Berends–Giele currents

BRST blocks $K_B$ of multiplicity $|B|$ are diagrammatically interpreted as off-shell cubic graphs shown in fig. 3. This suggests to assemble diagrams to a color-ordered SYM $(|B|+1)$-point tree amplitude where one of the legs is off-shell, as schematically depicted in fig. 4. The precise form of this diagrammatic construction was explained in the appendix A of [32], and the result is the promotion of BRST blocks $K_B$ to Berends–Giele currents $K_B$,

$$K_B \in \{ A_B^B, A_B^m, W_B^\alpha, F_B^{mn} \} \rightarrow K_B \in \{ A_B^B, A_B^m, W_B^\alpha, F_B^{mn} \},$$

(2.16)

The name goes back to Berends and Giele who recursively constructed gluonic currents which were then used to compute tree-level amplitudes [36]. From now on the ordered subsets $B = b_1 b_2 \ldots b_{|B|}$ of external particle labels which appear along with Berends–Giele currents $K_B$ will be denoted “words”.

The first examples of Berends–Giele currents $K_B$ are given by,

$$K_{12} = \frac{K_{12}}{s_{12}}, \quad K_{123} = \frac{K_{123}}{s_{12}s_{123}} + \frac{K_{321}}{s_{23}s_{2323}},$$

$$K_{1234} = \frac{1}{s_{1234}} \left( \frac{K_{1234}}{s_{12}s_{123}} + \frac{K_{3214}}{s_{23}s_{2323}} + \frac{K_{3421}}{s_{34}s_{34234}} + \frac{K_{3241}}{s_{23}s_{23234}} + \frac{2K_{12[34]}}{s_{12}s_{34}} \right),$$

(2.17)

(2.18)
where the conventions for the generalized Mandelstam invariants is

\[ s_{12\ldots p} = \sum_{1 \leq i < j}^{p} (k_i \cdot k_j) = \frac{1}{2} k_{12\ldots p}^2. \]  

(2.19)

It turns out that they enjoy simplified BRST variations compared to their corresponding BRST blocks \( K_B \). In particular, the Berends–Giele version of the unintegrated\(^7\) multiparticle vertex \( V_B \equiv \lambda^\alpha A^B_\alpha \),

\[ M_B \equiv \lambda^\alpha A^B_\alpha , \]  

(2.20)
satisfies

\[ QM_B = \sum_{XY=B} M_X M_Y = \sum_{j=1}^{\left| B \right| - 1} M_{b_1b_2\ldots b_j} M_{b_{j+1}\ldots b_p} . \]  

(2.21)

This has been exploited in [16,17] to construct tree amplitudes of ten-dimensional SYM and of the open superstring. Throughout this paper the notation \( XY = B \) (as in (2.21)) denotes a sum over all deconcatenations of the word \( B \) into smaller (non-empty) words \( X = b_1b_2\ldots b_j \) and \( Y = b_{j+1}\ldots b_p \) with \( j = 1, 2, \ldots, \left| B \right| - 1 \).

As partially used in [24] and generalized in [32], the equations of motion for the remaining \( K_B \) representatives are given by,

\[ QA^m_B = (\lambda^m \gamma^B_W) + k^m_B M_B + \sum_{XY=B} (M_X A^m_Y - M_Y A^m_X) \]

\[ QW^m_B = \frac{1}{4} (\lambda^m \gamma_{mn})^\alpha F^{mn}_B + \sum_{XY=B} (M_X W^m_Y - M_Y W^m_X) \]  

(2.22)

\[ QF^{mn}_B = 2k_B^m (\lambda^m \gamma^n W_B) + \sum_{XY=B} (M_X F^{mn}_Y - M_Y F^{mn}_X) \]

\[ + \sum_{XY=B} 2[A^m_X (\lambda^m \gamma^n W_Y) - A^m_Y (\lambda^m \gamma^n W_X)], \]

where the contact terms present in \( QK_B \) are traded by deconcatenations as the result of the Berends–Giele map (2.16). At general multiplicity, the transformation matrix between BRST blocks and their Berends–Giele currents was identified in [21] to be the momentum kernel [37,38], see [32] for further details.

\(^7\) According to the calligraphic-letter convention of (2.16) the Berends–Giele current associated to \( V_B \) would be denoted \( V_B \). However, the definition (2.20) is used for historic reasons.
According to their definition as off-shell SYM amplitudes, the Lie symmetries of the BRST-blocks $K_B$ translate into Kleiss–Kuijf relations [39] among their Berends–Giele counterparts $\mathcal{K}_B$ [16,17]. Up to multiplicity four, these are

\[
0 = \mathcal{K}_{12} + \mathcal{K}_{21}, \quad 0 = \mathcal{K}_{123} - \mathcal{K}_{321} = \mathcal{K}_{123} + \mathcal{K}_{231} + \mathcal{K}_{312}
\]

\[
0 = \mathcal{K}_{1234} + \mathcal{K}_{4321} = \mathcal{K}_{1234} + \mathcal{K}_{2134} + \mathcal{K}_{2314} + \mathcal{K}_{3124},
\]

and higher multiplicity generalizations are most compactly written as\(^8\)

\[
\mathcal{K}_{B1A} = (-1)^{|B|}\mathcal{K}_{1(A\omega B^T)}.
\]

The superscript along with $B^T$ denotes the reversal of the word $B$ in external particles $b_j$ such as $(b_1b_2\ldots b_{|B|})^T = (b_{|B|}\ldots b_2b_1)$, and $\uparrow\downarrow$ denotes the shuffle product.

### 2.4. One–loop building blocks

The saturation of fermionic zero modes in the pure spinor formalism [3] imposes tight constraints on contributions to loop amplitudes, see e.g. [4,40]. As argued in [24], the one-loop prescription in the minimal version of the formalism requires zero modes $d_\alpha d_\beta N^{mn}$ from the external vertices, leaving behind $(\lambda\gamma^m)_\alpha(\lambda\gamma^n)_\beta$. This effective rule leads to the BRST-closed expression $(\lambda\gamma^m W^i)(\lambda\gamma^n W^j)F_{mn}^k$ in the four-point amplitude [4] and motivates the following higher-point generalization [24,32]

\[
T_{A,B,C} \equiv \frac{1}{3}(\lambda\gamma_m W_A)(\lambda\gamma_n W_B)F_{mn}^C + (C \leftrightarrow A, B),
\]

as well as its associated Berends–Giele current

\[
M_{A,B,C} \equiv \frac{1}{3}(\lambda\gamma_m W_A)(\lambda\gamma_n W_B)F_{mn}^C + (C \leftrightarrow A, B).
\]

From now on, the Berends–Giele versions of various superfield combinations will be emphasized since explicit results for BRST (pseudo-)invariants and amplitudes simplify in this basis. Furthermore, their BRST-block counterparts such as $T_{A,B,C}$ can always be trivially recovered by using the superfields $A_B, W_B, F_B$ instead of $A_B, W_B, F_B$.

\(^8\) We follow the convention $\mathcal{K}_{\ldots A\omega B \ldots} \equiv \sum_{C \in A\omega B} \mathcal{K}_{\ldots C \ldots}$. 

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The universal form (2.22) of $Q\mathcal{W}_B^{m}$ and $Q\mathcal{F}_B^{mn}$ gives rise to the BRST-covariant transformation [24,32]

$$QM_{A,B,C} = \sum_{XY=A} \left( M_{X}M_{Y,B,C} - M_{Y}M_{X,B,C} \right) + (A \leftrightarrow B, C), \quad (2.27)$$

governed by deconcatenations of the multiparticle labels. Note that $QM_{1,2,3} = 0$ and that $M_{A,B,C}$ is totally symmetric in $A$, $B$ and $C$.

In closed-string amplitudes of multiplicity higher than four, additional zero-mode contributions can arise from the $\Pi^m$ fields in the external vertices. In the simplest case at five points [28], this leads to a single vector index contraction among left- and right-movers. The aforementioned $d_\alpha d_\beta N^{mn} \rightarrow (\lambda\gamma^m)_\alpha (\lambda\gamma^n)_\beta$ prescription identifies contributions of the form $A^m_m M_{B,C,D}$ to the left/right-contracting part of the closed-string amplitude.

However, as pointed out in [28] and generalized in [32], a separate $b$-ghost contribution proportional to $\Pi^m d_\alpha d_\beta$ leads to an additional kinematic factor

$$\mathcal{W}_B^{m}_{A,B,C,D} \equiv \frac{1}{12} (\lambda\gamma^m A)(\lambda\gamma^m_B)(\mathcal{W}_C \gamma^{mnp})\mathcal{W}_D + (A, B| A, B, C, D) \quad (2.28)$$

$$Q\mathcal{W}_B^{m}_{A,B,C,D} = \sum_{XY=A} \left( M_{X}\mathcal{W}_Y^{m}_{B,C,D} - M_{Y}\mathcal{W}_X^{m}_{B,C,D} \right) - (\lambda\gamma^m A)M_{B,C,D} + (A \leftrightarrow B, C, D). \quad (2.29)$$

The notation $(A_1, \ldots, A_p | A_1, \ldots, A_n)$ will also be used in later sections and instructs to sum over all possible ways to choose $p$ elements $A_1, A_2, \ldots, A_p$ out of the set $\{A_1, \ldots, A_n\}$, for a total of $\binom{n}{p}$ terms. This yields the following vector building block,

$$M^m_{A,B,C,D} \equiv [A^m_A M_{B,C,D} + (A \leftrightarrow B, C, D)] + \mathcal{W}_A^{m}_{B,C,D} \quad (2.30)$$

$$QM^m_{A,B,C,D} = \sum_{XY=A} \left( M_{X}M_{Y,B,C,D} - M_{Y}M_{X,B,C,D} \right) + k^m_{A} M_{A}M_{B,C,D} + (A \leftrightarrow B, C, D), \quad (2.31)$$

which is totally symmetric in $A, B, C, D$. Apart from the last line, the BRST covariant transformation (2.31) stems from deconcatenation terms in $Q\mathcal{K}_B$. This goes back to cancellations between (2.29) and the first term of $QA^m_B = (\lambda\gamma^m \mathcal{W}_B) + \ldots$, see (2.22).

---

9 Due to the tensor $(\lambda\gamma^m)_\alpha (\lambda\gamma^n)_\beta$ in (2.26), the pure spinor constraint projects out all terms in (2.22) with an explicit appearance of $\lambda^\alpha$, regardless of the words $A, B$ and $C$. 

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2.5. Scalar and vector one–loop cohomology

The simplest kinematic expressions compatible with the one-loop amplitude prescription [4] are \( M_{AM_{B,C,D}} \) and \( M_{AM_{B,C,D,E}} \) where the Berends–Giele current \( M_A \) stems from OPE contractions of the unintegrated vertex \( V_i \) with its integrated counterparts. Their covariant BRST transformations motivate to combine them to BRST-closed expressions. Following the experience with scalars [24], BRST invariants are classified in [32] by a “leading term” where a reference leg \( i \) is represented through a single-particle unintegrated vertex \( M_i = V_i \)

\[
C_{i|A,B,C} = M_i M_{A,B,C} + \sum_{E \neq \emptyset} M_{iE} \ldots 
\]

\[
C_{i|A,B,C,D} = M_i M_{A,B,C,D} + \sum_{E \neq \emptyset} M_{iE} \ldots .
\]

Apart from the explicit leading term, the singled-out label \( i \) always enters in a multiparticle Berends–Giele current. This is formally represented by a sum over (non-empty) words \( E \) of external particles which join the reference leg \( i \) in \( M_iE \). The \( \ldots \) in \( C_{i|A,B,C} \) (\( C_{i|A,B,C,D}^m \)) represent linear combinations of \( M_{A,B,C} \) (\( M_{A,B,C,D}^m \) and \( M_{A,B,C,D} k_D^m \)) such that

\[
QC_{i|A,B,C} = QC_{i|A,B,C,D}^m = 0.
\]

In later sections, we will encounter plenty of further examples where a leading term \( M_i \ldots \) is combined with a BRST completion made of multiparticle currents \( M_{iE} \) (with \( E \neq \emptyset \)) and ghost-number-two objects. Note that \( C_{i|A,B,C} \) and \( C_{i|A,B,C,D}^m \) are totally symmetric in \( A, B, C \) and \( A, B, C, D \) which follows a general convention used throughout this work: Whenever multiparticle slots \( A, B \) in a subscript are separated by a comma rather than by a vertical bar as in \( \ldots , A|B, \ldots \), then the parental object is understood to be symmetric in \( A \leftrightarrow B \).

In [32], the following two observations were exploited to set up a recursive construction of BRST invariants in (2.32) and (2.33):

(i) Nilpotency \( Q^2 = 0 \) implies that also \( Q M_{A,B,C} \) and \( Q M_{A,B,C,D}^m \) as given by (2.27) and (2.31) are BRST closed. By promoting each \( M_i M_{A,B,C} \) and \( M_i M_{A,B,C,D}^m \) therein

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to the corresponding invariant $C_{i|A,B,C}$ and $C_{i|A,B,C,D}^m$, one arrives at an alternative form of the BRST transformations\(^{10}\),

\[
QM_{A,B,C} = C_{a_1|a_2...a_{|A|},B,C} - C_{a_{|A|}|a_1...a_{|A|-1},B,C} + (A \leftrightarrow B, C) \quad (2.35)
\]

\[
QM_{A,B,C,D}^m = C_{a_1|a_2...a_{|A|},B,C,D}^m - C_{a_{|A|}|a_1...a_{|A|-1},B,C,D}^m
+ \delta_{|A|,1}k_{a_1}^mC_{a_1|B,C,D} + (A \leftrightarrow B, C, D) . \quad (2.36)
\]

Note that $\delta_{|A|,1}$ is equal to one when the word $A$ represents a single particle (i.e. $|A| = 1$) and zero otherwise.

(ii) We define a linear concatenation operator

\[
M_B \otimes_{a_1} M_{a_1a_2...a_{|A|}} = M_{Ba_1a_2...a_{|A|}} , \quad (2.37)
\]

acting on $M_A$ which does not interfere with ghost-number-two superfields such as $M_{A,B,C}$ or $M_{A,B,C,D}^m$. The deconcatenation formula (2.21) for $QM_A$ and (2.34) imply

\[
Q(M_i \otimes_j C_{j|A,B,C}) = M_iC_{j|A,B,C} \quad (2.38)
\]

\[
Q(M_i \otimes_j C_{j|A,B,C,D}^m) = M_iC_{j|A,B,C,D}^m \quad , \quad (2.39)
\]

see subsection 3.3 for a more detailed and general derivation.

On these grounds, one can show that the following recursions generate BRST invariants for arbitrary multiplicity:

\[
C_{i|A,B,C} = M_iM_{A,B,C} + M_i \otimes [C_{a_1|a_2...a_{|A|},B,C} - C_{a_{|A|}|a_1...a_{|A|-1},B,C} + (A \leftrightarrow B, C)]
\]

\[
C_{i|A,B,C,D}^m = M_iM_{A,B,C,D}^m + M_i \otimes [C_{a_1|a_2...a_{|A|},B,C,D}^m - C_{a_{|A|}|a_1...a_{|A|-1},B,C,D}^m
+ \delta_{|A|,1}k_{a_1}^mC_{a_1|B,C,D} + (A \leftrightarrow B, C, D)] . \quad (2.40)
\]

Once the leading terms $Q(M_iM_{A,B,C})$ and $Q(M_iM_{A,B,C,D}^m)$ are evaluated via (i), they are easily seen to cancel the BRST variations (ii) of the concatenated terms.

In (2.40) as well as later equations in this paper, the subscript $j$ of the concatenation $\otimes_j$ in (2.37) is suppressed and understood to match the reference leg $j$ of subsequent kinematic factor such as $C_{j|...}$ or $C_{j|...}^m$. In principle, $\otimes$ without further specification does

\(^{10}\) At this point, uniqueness of the BRST completions in (2.32) and (2.33) is assumed. We don’t have a rigorous argument to prove this in full generality but rely on “experimental” evidence at finite multiplicities.
not preserve the Kleiss–Kuijf symmetries (2.24) of the Berends–Giele currents\textsuperscript{11}. However, this slight ambiguity does not matter in the recursive formulas (2.40) as long as the objects generated by the recursion are directly used in the next steps without any prior symmetry manipulations.

The simplest instances of scalar and vector invariants following from the recursions in (2.40) are

\[
C_{1|2,3,4} \equiv M_1 M_{2,3,4} \tag{2.41}
\]

\[
C_{1|23,4,5} \equiv M_1 M_{23,4,5} + M_1 \otimes [C_{2|3,4,5} - C_{3|2,4,5}] = M_1 M_{23,4,5} + M_{12} M_{3,4,5} - M_{13} M_{2,4,5}
\]

\[
C_{1|2,3,4,5}^m \equiv M_1 M_{2,3,4,5}^m + M_1 \otimes [k^m_2 C_{2|3,4,5} + (2 \leftrightarrow 3, 4, 5)] = M_1 M_{2,3,4,5}^m + [k^m_2 M_{12} M_{3,4,5} + (2 \leftrightarrow 3, 4, 5)].
\]

The corresponding six-point examples are expanded in appendix A, see (A.1). Seven- and eight-point examples of $C_{i|A,B,C}$ can be found in [24].

The families of scalar and vector invariants $C_{i|A,B,C}$, $C_{i|A,B,C,D}^m$ as well as their recursive construction in (2.40) furnish the first two cells from the left in fig. 1.

3. Towards a BRST pseudo-cohomology

In this section, we investigate the applicability of the BRST program in sections 2.4 and 2.5 to tensorial building blocks. The one-loop prescription of the pure spinor formalism [4] suggests a natural two-tensor generalization $M_{\ldots}^{mn}$ of the $M_{A,B,C}$ and $M_{A,B,C,D}^m$ above, but its BRST variation turns out to involve new classes of superfields. This obstruction is closely related to the pure spinor superspace description of the hexagon anomaly [26]. It leads us to define a pseudo-cohomology as an extension of the standard cohomology in order to systematically study the multiparticle superfields which play a role in the gauge anomaly of open superstring amplitudes and its cancellation [25].

\textsuperscript{11} For example, $M_{132} \neq -M_{123}$ implies that $M_1 \otimes M_{32} \neq -M_1 \otimes M_{23}$ even though $M_{32} = -M_{23}$.
3.1. Tensorial building blocks $M^{mn}$

Higher-point loop amplitudes in the closed-string allow for an arbitrary number of $\Pi^m$ zero mode contractions between left- and right-movers. This motivates the study of higher-rank tensors generalizing (2.30) such as

$$M^{mn}_{A,B,C,D,E} \equiv 2[A^{(m}_{A}A^{n)}_{B}M_{C,D,E} + (A, B| A, B, C, D, E)] + 2[A^{(m}_{A}W^{n)}_{B,C,D,E} + (A \leftrightarrow B, C, D, E)] = A^{m}_{A}W^{n}_{B,C,D,E} + A^{n}_{A}M^{m}_{B,C,D,E} + (A \leftrightarrow B, C, D, E),$$

firstly relevant for the six-point amplitude. Its first term $\sim A^{(m}_{A}A^{n)}_{B}$ stems from the $\Pi^m \Pi^n d_{\alpha}d_{\beta}N_{pq}$ zero-mode coefficient and its second term $\sim A^{(m}_{A}W^{n)}_{B,C,D,E}$ originates from the $b$-ghost sector linear in $\Pi^m$. The BRST variations (2.22), (2.29) and (2.31) for its constituents imply that

$$QM^{mn}_{A,B,C,D,E} = \delta^{mn}(\mathcal{Y}_{A,B,C,D,E} + \left[ \sum_{XY=A} (M_{X}M^{mn}_{Y,B,C,D,E} - M_{Y}M^{mn}_{X,B,C,D,E}) \right]) + 2k^{(m}_{A}M^{n)}_{A,B,C,D,E} + (A \leftrightarrow B, C, D, E),$$

where the first term is a shorthand for

$$\mathcal{Y}_{A,B,C,D,E} \equiv \frac{1}{2}(\lambda^{\gamma}_{p}W_{A})(\lambda^{\gamma}_{q}W_{B})(\lambda^{\gamma}_{r}W_{C})(\lambda^{\gamma}_{s}W_{D})\delta^{\gamma\gamma}_{mn}W_{E}.$$

The superfield $\mathcal{Y}_{A,B,C,D,E}$ has ghost-number three and is totally symmetric in $A, B, C, D, E$ due to the pure spinor constraint. It stems from the term $(\lambda^{\gamma}_{p}W_{A})W^{n}_{B,C,D,E}$ in $Q(A^{(m}_{A}W^{n)}_{B,C,D,E})$ where a group-theoretic analysis\textsuperscript{12} has been used to replace

$$(\lambda^{\gamma}_{p})^{[\alpha_{1}}(\lambda^{\gamma}_{q})^{\alpha_{2}}(\lambda^{(m)}_{\alpha_{3}}\gamma^{n}_{\alpha_{4}\alpha_{5}}) = \frac{1}{10}\delta^{mn}(\lambda^{\gamma}_{p})^{[\alpha_{1}}(\lambda^{\gamma}_{q})^{\alpha_{2}}(\lambda^{\gamma}_{r})^{\alpha_{3}}\gamma^{pqr}_{\alpha_{4}\alpha_{5}}).$$

Apart from the extra term $\mathcal{Y}_{A,B,C,D,E}$, the BRST variation (3.2) of $M^{mn}_{A,B,C,D,E}$ is a direct rank-two generalization of $QM^{mn}_{A,B,C,D}$ given in (2.31).

\textsuperscript{12} The spinors indices in $(\lambda^{\gamma}_{p})^{[\alpha_{1}}(\lambda^{\gamma}_{q})^{\alpha_{2}}(\lambda^{(m)}_{\alpha_{3}}\gamma^{n}_{\alpha_{4}\alpha_{5}})$ fall into the tensor product of $[0, 0, 0, 0, 3] \equiv \lambda^{3}$ and $[0, 0, 0, 1, 1]^{5} = [0, 0, 0, 3, 0] \oplus [1, 1, 0, 1, 0] \equiv W^{5}$. The LiE program \cite{41} identifies one scalar $[0, 0, 0, 0, 0]$ but no symmetric and traceless $[2, 0, 0, 0, 0]$ component in $[0, 0, 0, 3] \otimes [0, 0, 0, 0, 0]^{5}$. Hence, only the trace with respect to vector indices $m, n$ contributes. We are using standard Dynkin label notation $[a_{1}, a_{2}, a_{3}, a_{4}, a_{5}]$ for SO(10) irreducibles and denote an antisymmetrized $k^{th}$ tensor power by $[a_{1}, a_{2}, a_{3}, a_{4}, a_{5}]^{k}$.\textsuperscript{20}
3.2. Pseudo BRST cohomology

The pure spinor analysis of the hexagon gauge anomaly [25] showed that the gauge variation of the open superstring six-point amplitude (w.r.t. leg one) is proportional to [26],

\[ \mathcal{Y}_{2,3,4,5,6} = \frac{1}{2} (\lambda \gamma^m W_2)(\lambda \gamma^n W_3)(\lambda \gamma^p W_4)(W_5 \gamma_{mnp} W_6). \]  

Since gauge invariance is related to BRST invariance of the kinematic factors in the amplitudes (see appendix B), terms of the form (3.3) are expected to describe the BRST anomaly of the amplitudes. They represent obstructions in finding elements in the cohomology. Therefore it will be convenient to define a pseudo BRST cohomology in which the variation of pseudo BRST-closed elements vanish up to anomalous superfields such as (3.3). These objects give rise to gauge transformations with bosonic components proportional to the \( \epsilon_{10} \) tensor, see appendix B.5, so they are suitable to describe the parity odd gauge anomaly of the open superstring. Indeed, it will be shown in [29] that the scalar pseudo BRST cohomology for six particles discussed in the next section correctly describes the anomaly terms of the six-point one-loop amplitude which were not included in the discussion of [24].

**Definition 1.** A superfield \( \mathcal{Y} \) of ghost-number three or four is called anomalous if it contains a factor of \( \mathcal{Y}_{A,B,C,D,E} \) as in (3.3) with some multiparticle labels \( A, B \ldots, E \).

**Definition 2.** Superfields of ghost-number two and three are called pseudo-invariant if their BRST variation is entirely anomalous. The space of pseudo-invariants is referred to as the pseudo-cohomology.

3.3. Tools for constructing BRST pseudo-invariants

The subsequent sections are concerned with a systematic construction of BRST pseudo-invariants. As a driving force for this endeavor, we generalize the recursions of section 2.5 to situations with anomalous BRST variations.

**Lemma 1.** Let \( \mathcal{J} \) denote any superfield unaffected by the operation \( \otimes \) defined in (2.37). Then, concatenations of \( M_A \mathcal{J} \) satisfy

\[ Q(M_i \otimes M_A \mathcal{J}) = M_i M_A \mathcal{J} + M_i \otimes Q(M_A \mathcal{J}). \]  

**Proof.** By the deconcatenation formula (2.21) for \( QM_A \), one can identify

\[ Q(M_A \mathcal{J}) = \sum_{XY=A} M_X M_Y \mathcal{J} - M_A Q \mathcal{J} \]
in the third line of

\[ Q(M_i \otimes M_A \mathcal{J}) = Q(M_{iA} \mathcal{J}) \]
\[ = \left\{ M_i M_A + \sum_{XY=A} M_i X M_Y \right\} \mathcal{J} - M_{iA} Q \mathcal{J} \]
\[ = M_i M_A \mathcal{J} + M_i \otimes \left\{ \sum_{XY=A} M_X M_Y \mathcal{J} - M_A Q \mathcal{J} \right\} \]
\[ = M_i M_A \mathcal{J} + M_i \otimes Q(M_A \mathcal{J}) \].

**Corollary 1.** Let \( C \) denote a BRST-invariant superspace expression

\[ C = \sum_k M_{Ak} \mathcal{J}_k, \quad QC = 0 \] (3.7)

where \( \otimes \) acts trivially on the \( \mathcal{J}_k \), then

\[ Q(M_i \otimes C) = M_i C \] (3.8)

**Proof.** Upon applying Lemma 1 to \( M_{Ak} \mathcal{J}_k \), the second term \( M_i \otimes Q(M_{Ak} \mathcal{J}_k) \) in (3.6) builds up \( M_i \otimes QC \) by linearity of \( \otimes \) which vanishes by the assumption (3.7).

Note that (2.38) and (2.39) are special cases of (3.8) with \( C \to C_{j|A,B,C} \) and \( C \to C_{j|A,B,C,D}^m \), respectively.

**Corollary 2.** Let \( P \) denote a BRST-pseudoinvariant superspace expression

\[ P = \sum_k M_{Ak} \mathcal{J}_k, \quad QP = \sum_l M_{Bl} \mathcal{Y}_l \] (3.9)

where the \( \mathcal{Y}_l \) are anomalous and \( \otimes \) does not act on \( \mathcal{J}_k \) or \( \mathcal{Y}_l \), then the right-hand side of

\[ Q(M_i \otimes P) = M_i P + M_i \otimes QP \] (3.10)

is anomalous up to the first term \( M_i P \).

**Proof.** Again, apply Lemma 1 to \( M_{Ak} \mathcal{J}_k \), and the second term \( M_i \otimes Q(M_{Ak} \mathcal{J}_k) \) in (3.6) builds up the expression \( M_i \otimes QP \). The latter is anomalous by (3.9) since \( M_i \otimes \) action does not alter the anomaly nature of the \( M_{Bl} \mathcal{Y}_l \) in \( QP \).
3.4. Rank-two example of pseudo-cohomology

As a first example of BRST pseudo-invariants, we derive a rank-two analogue of the recursions in (2.40) for scalar and vectorial BRST invariants. According to the anomalous transformation (3.2) of the tensorial building block $M_{mn}$ in (3.1), the resulting tensors $C^m_{i|A,B,C,D,E}$ can at best be pseudo-invariant in the sense of Definition 2.

BRST pseudo-completions of rank-two tensors originate from an ansatz

$$C^m_{i|A,B,C,D,E} = M_i M^m_{A,B,C,D,E} + \sum_{F \neq \emptyset} M_i F \cdots$$

(3.11)

similar to (2.32) and (2.33). Apart from the leading term $M_i M^m_{A,B,C,D,E}$, particle $i$ always appears in a multiparticle word $iF$, and the ellipsis represents tensor superfields of the form $M_{A,B,C,D,E} \cdot k^m_{A} M^n_{B,C,D,E}$ or $k^m_{A} k^n_{B} M_{C,D,E}$.

Similar to the expressions (2.35) and (2.36) for $Q M_{A,B,C}$ and $Q M^m_{A,B,C,D}$, one can express $Q M^m_{A,B,C,D,E}$ given in (3.2) in terms of pseudo-invariants: Each term containing a factor of $M_i$ in (3.2) signals the leading term of a (pseudo-)invariant, hence:

$$Q M^m_{A,B,C,D,E} = \delta^{mn} Y_{A,B,C,D,E} + [C^m_{a_1|a_2 \ldots a_{|A|},B,C,D,E} - C^m_{a|A||a_1 \ldots a_{|A|-1},B,C,D,E}$$

$$+ \delta_{|A|,1} (k^m_{a_1} C^m_{a|B,C,D,E} + k^n_{a_1} C^m_{a|B,C,D,E}) + (A \leftrightarrow B, C, D, E)] \cdot (A, B, C, D, E)$$

(3.12)

This motivates the following recursion for pseudo-invariants $C^m_{i|A,B,C,D,E}$:

$$C^m_{i|A,B,C,D,E} = M_i M^m_{A,B,C,D,E} + M_i \otimes [C^m_{a_1|a_2 \ldots a_{|A|},B,C,D,E} - C^m_{a|A||a_1 \ldots a_{|A|-1},B,C,D,E}$$

$$+ \delta_{|A|,1} (k^m_{a_1} C^m_{a|B,C,D,E} + k^n_{a_1} C^m_{a|B,C,D,E}) + (A \leftrightarrow B, C, D, E)] \cdot (A, B, C, D, E)$$

(3.13)

Their BRST variation is purely anomalous by (3.12) and (3.10) at $P \rightarrow C^m_{2|3,4,5,6}$. The simplest example occurs at six points and uses the expression (2.41) for $C^m_{2|3,4,5,6}$.

$$C^m_{1|2,3,4,5,6} = M_1 M^m_{2,3,4,5,6} + [M_1 \otimes (k^m_{2} C^m_{2|3,4,5,6} + k^n_{2} C^m_{2|3,4,5,6}) + (2 \leftrightarrow 3, 4, 5, 6)]$$

$$= M_1 M^m_{2,3,4,5,6} + [k^m_{2} M_{1} M^m_{3,4,5,6} + k^n_{2} M_{1} M^m_{3,4,5,6}] + (3 \leftrightarrow 4, 5, 6)]$$

(3.14)

$$+ [(k^m_{2} k^m_{3} + k^n_{2} k^n_{3}) (M_{1} + M_{12}) M_{4,5,6} + (2, 3|2, 3, 4, 5, 6)] \cdot (A, B, C, D, E)$$

The seven-point analogue $C^m_{1|23,4,5,6,7}$ is displayed in appendix A, see (A.2).

One can explicitly check their BRST pseudo-invariant nature at low orders,

$$Q C^m_{1|2,3,4,5,6} = - \delta^{mn} M_1 Y_{2,3,4,5,6}$$

(3.15)

$$Q C^m_{1|23,4,5,6,7} = - \delta^{mn} (M_1 Y_{23,4,5,6,7} + M_{12} Y_{34,5,6,7} - M_{13} Y_{24,5,6,7})$$

(3.16)
The general structure of $QC_{i|A,B,C,D,E}^{mn} = - \delta^{mn}(\ldots)$ will be discussed in section 7.2. Note that traceless components are BRST-closed,

$$Q\left(C_{i|A,B,C,D,E}^{mn} - \frac{1}{10} \delta^{mn} C_{i|A,B,C,D,E}^{pp}\right) = 0. \quad (3.17)$$

The family of two-tensor pseudo-invariants constructed via (3.13) furnishes the third cell in the leading diagonal of the overview grid in fig. 1. As we will see in the next section, the tools of this section enable to address an infinite tower of higher-rank generalizations to complete this diagonal.

4. Tensor pseudo-cohomology

This section introduces a recursive method to construct higher-rank generalizations of the scalar, vector and two-tensor BRST (pseudo-)invariants discussed in the previous sections. As shown below, $C_{i|A_1,\ldots,A_{r+3}}^{m_1\ldots m_r}$ for $r \geq 2$ is a pseudo-invariant according to Definition 2.

4.1. Higher-rank building blocks and anomaly blocks

Following the logic behind the two-tensor in (3.1), we define a building block of arbitrary rank $r$ by extracting zero modes of $\Pi^{m_1} \ldots \Pi^{m_r} d_\alpha d_\beta N_{pq}$ from $r+3$ integrated vertex operators in their multiparticle Berends–Giele version. Similarly, the $b$-ghost sector proportional to the zero-mode of $\Pi^m$ gives rise to a second sort of superfield with a factor of $W_{A,B,C,D}^m$:

$$M_{B_1, B_2, \ldots, B_{r+3}}^{m_1 \ldots m_r} \equiv r! \left[ M_{B_1, B_2, B_3} A_{B_4}^{m_1} A_{B_5}^{m_2} \ldots A_{B_{r+3}}^{m_r} + (B_1, B_2, B_3; B_1, B_2, \ldots, B_{r+3}) \right]$$

$$+ r! \left[ W_{B_1, B_2, B_3, B_4}^{m_1} A_{B_5}^{m_2} \ldots A_{B_{r+3}}^{m_r} + (B_1, \ldots, B_4; B_1, B_2, \ldots, B_{r+3}) \right] \quad (4.1)$$

In order to get a recursive handle on the combinatorics in (4.1), it is convenient to define higher-rank versions of $W_{A,B,C,D}^m$ in (2.28),

$$W_{B_1, B_2, \ldots, B_{r+3}}^{m_1 \ldots m_r} \equiv A_{B_1}^{m_1} W_{B_2, \ldots, B_{r+3}}^{m_2 \ldots m_{r-1} m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{r+3}) \quad (4.2)$$

Then, the rank-$r$ building block $M_{B_1, B_2, \ldots, B_{r+3}}^{m_1 \ldots m_r}$ in (4.1) can be written recursively as

$$M_{B_1, \ldots, B_{r+3}}^{m_1 \ldots m_r} = A_{B_1}^{m_1} M_{B_2, \ldots, B_{r+3}}^{m_2 \ldots m_r} + A_{B_1}^{m_r} W_{B_2, \ldots, B_{r+3}}^{m_1 \ldots m_{r-1} m_2} + (B_1 \leftrightarrow B_2, B_3, \ldots, B_{r+3})$$

$$= A_{B_1}^{m_1} M_{B_2, \ldots, B_{r+3}}^{m_2 \ldots m_r} + (B_1 \leftrightarrow B_2, B_3, \ldots, B_{r+3}) + W_{B_1, B_2, \ldots, B_{r+3}}^{m_1 m_2 \ldots m_{r-1} m_r} \quad (4.3)$$

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for example (3.1) at rank two and
\[
M^{mnp}_{B_1,\ldots,B_6} = A^m_{B_1} M^{np}_{B_2,\ldots,B_6} + A^p_{B_1} W^{m|n}_{B_2,\ldots,B_6} + (B_1 \leftrightarrow B_2, \ldots, B_6) \tag{4.4}
\]
\[
M^{mnpq}_{B_1,\ldots,B_7} = A^m_{B_1} M^{npq}_{B_2,\ldots,B_7} + A^p_{B_1} W^{m|n}_{B_2,\ldots,B_7} + (B_1 \leftrightarrow B_2, \ldots, B_7)
\]
at \( r = 3,4 \). Note that \( W^{m_1,\ldots,m_{r-1}|m_r}_{B_1,B_2,\ldots,B_{r+3}} \) defined in (4.2) is symmetric in all its slots \( B_i \) but
only in its first \( r - 1 \) vector indices \( m_i \). That explains the notation \( \ldots m_{r-1}|m_r \) in (4.2).

Also the scalar anomaly building block \( \gamma_{A,B,C,D,E} \) defined in (3.3) has a natural higher-rank generalization. It can be defined explicitly in analogy to (4.1),
\[
\gamma^{m_1,\ldots,m_r}_{B_1,B_2,\ldots,B_{r+5}} \equiv r! \gamma_{B_1,\ldots,B_5} A^{(m_1} A^{m_2} \ldots A^{m_r)}_{B_6} + (B_1,\ldots,B_5|B_1,\ldots,B_{r+5}) , \tag{4.5}
\]
or recursively like (4.3),
\[
\gamma^{m_1,\ldots,m_r}_{B_1,B_2,\ldots,B_{r+5}} \equiv A^{m_1}_{B_1} \gamma^{m_2,\ldots,m_r}_{B_2,\ldots,B_{r+5}} + (B_1 \leftrightarrow B_2, B_3, \ldots, B_{r+5}) . \tag{4.6}
\]
Even though the recursion (4.6) for anomaly blocks resembles (4.2) for \( W^{m_1,\ldots,m_{r-1}|m_r}_{B_1,B_2,\ldots,B_{r+3}} \), the vector indices of the latter are not entirely carried by \( A^m_B \) superfields. That is why only \( \gamma^{m_1,\ldots,m_r}_{B_1,B_2,\ldots,B_{r+5}} \) is totally symmetric in both \( B_i \) and \( m_i \).

### 4.2. Anomalous BRST variations at higher rank

Both expressions (4.1) and (4.3) for higher-rank building blocks serve as a starting point to determine their BRST variation
\[
QM^{m_1,\ldots,m_r}_{B_1,B_2,\ldots,B_{r+3}} = \binom{r}{2} \delta^{(m_1 m_2} \gamma^{m_3,\ldots,m_r)}_{B_1,B_2,\ldots,B_{r+3}} + \left[ rM^{k_{B_1} m_1 m_2,\ldots,m_r}_{B_2,B_3,\ldots,B_{r+3}} + \sum_{XY = B_1} (M_X M^{m_1,\ldots,m_r}_{Y,B_2,\ldots,B_{r+3}} - M_Y M^{m_1,\ldots,m_r}_{X,B_2,\ldots,B_{r+3}}) + (B_1 \leftrightarrow B_2, \ldots, B_{r+3}) \right] . \tag{4.7}
\]
The \( \delta^{mn} \) tensors in the anomalous part are due to the group-theory identity (3.4). The rank-two example has been given in (3.2), and ranks three and four give rise to
\[
QM^{mnp}_{B_1,B_2,\ldots,B_6} = 3 \delta^{(mnp} \gamma^{p)}_{B_1,B_2,\ldots,B_6} + \left[ 3M^{k_{B_1} m_{B_2,B_3,\ldots,B_6}}^{mnp} + \sum_{XY = B_1} (M_X M^{mnp}_{Y,B_2,\ldots,B_6} - M_Y M^{mnp}_{X,B_2,\ldots,B_6}) + (B_1 \leftrightarrow B_2, \ldots, B_6) \right] . \tag{4.8}
\]
\[
QM^{mnpq}_{B_1,B_2,\ldots,B_7} = 6 \delta^{(mnpq} \gamma^{pq)}_{B_1,B_2,\ldots,B_7} + \left[ 4M^{k_{B_1} m_{B_2,B_3,\ldots,B_7}}^{mnpq} + \sum_{XY = B_1} (M_X M^{mnpq}_{Y,B_2,\ldots,B_7} - M_Y M^{mnpq}_{X,B_2,\ldots,B_7}) + (B_1 \leftrightarrow B_2, \ldots, B_7) \right] . \tag{4.9}
\]
The recursive approach makes use of the BRST variation

\[ QW_{B_1, B_2, \ldots, B_{r+3}}^{m_1, \ldots, m_r | m_r} = (r - 1) \delta^{m_r} (m_1) Y_{B_1, B_2, \ldots, B_{r+3}}^{m_2, \ldots, m_{r-1}} \]

\[ + \left( (r - 1) M_B A_{B_1}^{m_1} W_{B_2, B_3, \ldots, B_{r+3}}^{m_2, \ldots, m_{r-1}} - (\lambda \gamma^{m_r} W_B) M_{B_2, B_3, \ldots, B_{r+3}}^{m_1, \ldots, m_{r-1}} \right) 

\[ + \sum_{XY = B_1} (M_Y W_{X, B_2, \ldots, B_{r+3}} - M_Y W_{X, B_2, \ldots, B_{r+3}} + (B_1 \leftrightarrow B_2, \ldots, B_{r+3})) \],

which generalizes the rank-one variation (2.29) and specializes as follows at rank \( r \leq 3 \),

\[ QW_{B_1, B_2, \ldots, B_6}^{m_1, n_1} = \delta^{m_1, n_1} Y_{B_1, B_2, \ldots, B_5} + \left[ k_B A_{B_1}^{m_2} W_{B_2, B_3, B_4, B_5} - (\lambda \gamma^{n_1} W_B) M_{B_2, B_3, B_4, B_5}^{m_1} \right] 

\[ + \sum_{XY = B_1} (M_Y W_{X, B_2, \ldots, B_5} - M_Y W_{X, B_2, \ldots, B_5} + (B_1 \leftrightarrow B_2, B_3, B_4, B_5)) \]  \( (4.11) \)

\[ QW_{B_1, B_2, \ldots, B_6}^{m_1, n_1 | p} = 2\lambda^{p(m_1)} Y_{B_1, B_2, \ldots, B_6} + \left[ 2k_B A_{B_1}^{m_2} W_{B_2, \ldots, B_6} - (\lambda \gamma^{p} W_B) M_{B_2, B_3, B_4, B_5}^{m_1} \right] 

\[ + \sum_{XY = B_1} (M_Y W_{X, B_2, \ldots, B_5} - M_Y W_{X, B_2, \ldots, B_5} + (B_1 \leftrightarrow B_2, \ldots, B_6)) \].  \( (4.12) \)

### 4.3. Recursion for higher rank pseudoinvariants

The construction of general BRST pseudo-invariants

\[ C_{i | A_1, A_2, \ldots, A_{r+3}}^{m_1, \ldots, m_r} = M_i M_{A_1, A_2, \ldots, A_{r+3}}^{m_1, \ldots, m_r} + \sum_{B \neq \emptyset} M_{i B} \cdots \] \( (4.13) \)

generalizes the scalars (2.32), vectors (2.33) and two-tensors (3.11) to arbitrary rank. As before, the leading term \( M_i M_{A_1, A_2, \ldots, A_{r+3}} \) is the only instance where the reference leg \( i \) enters through a single-particle vertex operator \( M_i \). The ellipsis along with multiparticle \( M_{i B} \) takes the form \( k_{A_1}^{m_1} \ldots k_{A_j}^{m_j} M_{A_{j+1}, \ldots, A_{r+3}} \) with \( j = 0, 1, \ldots, r \). The role of \( M_i \) as defining the pseudoinvariant (4.13) leads to the following alternative form of (4.7):

\[ QM_{A_1, A_2, \ldots, A_{r+3}}^{m_1, \ldots, m_r} = \left( r \right) \delta^{m_1, m_2} Y_{A_1, A_2, \ldots, A_{r+3}}^{m_3, \ldots, m_{r-1}} + \left( r \delta_{A_1}^{A_1} \right) k_{a_1}^{m_1} C_{a_1 | A_2, \ldots, A_{r+3}}^{m_2, \ldots, m_{r-1}} \]

\[ + \left\{ r \delta_{A_1}^{A_1} \right\} k_{a_1}^{m_1} C_{a_1 | A_2, \ldots, A_{r+3}}^{m_2, \ldots, m_{r-1}} + \left\{ r \delta_{A_1}^{A_1} \right\} k_{a_1}^{m_1} C_{a_1 | A_2, \ldots, A_{r+3}}^{m_2, \ldots, m_{r-1}} \] \( (4.14) \)

This in turn gives rise to a recursion for the pseudo-invariants \( C_{i | A_1, A_2, \ldots, A_{r+3}}^{m_1, \ldots, m_r} \) in terms of lower-multiplicity representatives of rank \( r \) and \( r - 1 \),

\[ C_{i | A_1, A_2, \ldots, A_{r+3}}^{m_1, \ldots, m_r} = M_i M_{A_1, A_2, \ldots, A_{r+3}}^{m_1, \ldots, m_r} + M_i \otimes \left\{ r \delta_{A_1}^{A_1} \right\} k_{a_1}^{m_1} C_{a_1 | A_2, \ldots, A_{r+3}}^{m_2, \ldots, m_{r-1}} + \left\{ r \delta_{A_1}^{A_1} \right\} k_{a_1}^{m_1} C_{a_1 | A_2, \ldots, A_{r+3}}^{m_2, \ldots, m_{r-1}} \] \( (4.15) \)
which reduces to (2.40) and (3.13) for \( r \leq 2 \). BRST pseudoinvariance follows from (3.10) at \( \mathcal{P} \to C_{i_1 \ldots i_r}^{m_1 \ldots m_r} \). The anomalous BRST variations entirely reside in trace components \( \sim \delta^{m_i m_j} \) and will be systematically discussed in section 7.2. Similar as before, the traceless components are BRST invariant, e.g.

\[
Q \left( C_{i|A,B,C,D,E,F}^{mnp} - \frac{1}{4} \delta^{mn} C_{i|A,B,C,D,E,F}^{pq} \right) = 0 .
\] (4.16)

The simplest pseudoinvariant of rank greater than two is \( C_{i|A,B,C,D,E,F}^{mnp} \); its expansion is displayed in appendix A, see (A.3).

The recursion (4.15) for pseudo-invariants of arbitrary rank completes the leading diagonal of the overview grid in fig. 1. In the next sections we explore the building blocks and recursions governing the subleading diagonals.

5. Towards a refined pseudo-cohomology

The discussion of BRST invariance of the closed-string five-point amplitude in [28] naturally led to consider the following combination of superfields\(^{13}\)

\[
k_1^m V_1 T_{2,3,4,5}^m + \left[ V_{12} T_{3,4,5} + (2 \leftrightarrow 3, 4, 5) \right] + Y_{1,2,3,4,5}
\] (5.1)

which was shown to be BRST exact in the appendix B of [28]. Given the appearance of the anomalous superfield \( Y_{1,2,3,4,5} \), it will not be surprising to discover that this particular combination (5.1) signals a much deeper family of pseudo cohomology elements which will play an important role in the discussion of anomalous terms in the one-loop open superstring amplitudes [29].

\(^{13}\) In various places of this section, we will encounter the local representatives \( V_A, T_{A,B,C}, W_{A,B,C,D}, T_{A,B,C,D}^m \) and \( Y_{A,B,C,D,E} \) of the more frequently-used Berends–Giele superfields \( M_A, M_{A,B,C}, W_{A,B,C,D}^m, M_{A,B,C,D}^m \) and \( Y_{A,B,C,D,E} \) as defined by (2.20), (2.26), (2.28), (2.30) and (3.3). They follow by trading any \( K_B \in \{ A_B^\alpha, A_B^m, W_B^\alpha, F_B^{mn} \} \) in these definitions for the standard BRST blocks \( K_B \in \{ A_B^\alpha, A_B^m, W_B^\alpha, F_B^{mn} \} \), see section 2.3. Some of their BRST variations are displayed in appendix C.
5.1. Refined currents

It turns out that to extend and generalize the discussion of (5.1) it is convenient to define the following superfield

\[ \mathcal{J}_{A|B,C,D,E} = \frac{1}{2} (A^m_A T^m_{B,C,D,E} + A^m_A W^m_{B,C,D,E}) , \]  

(5.2)

symmetric in \( B, C, D, E \). In view of the special role of the first slot \( A \), we refer to such objects as refined currents. Accordingly, any slot \( A | \ldots \) on the left of the vertical bar of the subscript will be referred to as refined. It is not hard to check that the simplest case \( \mathcal{J}_{1|2,3,4,5} \) gives rise to (5.1) under \( Q \) variation and that higher-multiplicity currents satisfy

\[
Q \mathcal{J}_{1|2,3,4,5} = k_1^m V_1 T^m_{2,3,4,5} + \left[ V_{12} T_{3,4,5} + (2 \leftrightarrow 3, 4, 5) \right] + Y_{1,2,3,4,5} \quad (5.3)
\]

\[
Q \mathcal{J}_{12|3,4,5,6} = k_2^m V_{12} T^m_{3,4,5,6} + \left[ \tilde{V}_{123} T_{4,5,6} + (3 \leftrightarrow 4, 5, 6) \right] + Y_{12,3,4,5,6}
\]

\[ + s_{12} (V_1 \mathcal{J}_{2|3,4,5,6} - V_2 \mathcal{J}_{1|3,4,5,6}) \quad (5.4)
\]

\[
Q \mathcal{J}_{1|23,4,5,6} = k_1^m V_1 T^m_{23,4,5,6} - \tilde{V}_{231} T_{4,5,6} + \left[ V_{14} T_{23,5,6} + (4 \leftrightarrow 5, 6) \right]
\]

\[ + Y_{1,23,4,5,6} + s_{23} (V_2 \mathcal{J}_{1|3,4,5,6} - V_3 \mathcal{J}_{1|2,4,5,6}) . \]  

(5.5)

Both \( V_{12} \) in (5.3) and the hatted superfields \( \tilde{V}_A \) in (5.4) and (5.5) build up through the recursions (2.8) and (2.10) for \( A^m_{12} \) and \( A^m_{123} \). More generally, the recursion (2.15) relates the multiparticle spinor superpotential \( A^B_\alpha \) to BRST blocks \( K_C \) at lower multiplicity \( |C| < |B| \) which are generated by \( Q \mathcal{J}_{A|B,C,D,E} \). However, the direct output \( \mathcal{J}^B_\alpha \) of the recursion requires redefinitions by BRST trivial components \( H_{12 \ldots p} \equiv H_{[12 \ldots p-1, p]} \) in order to yield the BRST block \( A^B_\alpha \) subject to Lie symmetries, see (2.11) and [32]. The appearance of \( \tilde{V}_B = \lambda^\alpha \tilde{A}^B_\alpha \) in (5.4) and (5.5) suggests to redefine \( \mathcal{J} \) by the tensors \( H_{ijk} \equiv H_{[ij,k]} \) in (2.12) such that their \( Q \) variation can be expressed in terms of the BRST block \( V_B = \lambda^\alpha A^B_\alpha \), e.g.

\[
J_{1|2,3,4,5} = \mathcal{J}_{1|2,3,4,5}
\]

\[
J_{12|3,4,5,6} = \mathcal{J}_{12|3,4,5,6} - \left[ H_{[12,3]} T_{4,5,6} + (3 \leftrightarrow 4, 5, 6) \right],
\]

(5.6)

\[
J_{1|23,4,5,6} = \mathcal{J}_{1|23,4,5,6} + H_{[23,1]} T_{4,5,6}.
\]

Generalizations \( H_{[A,B]} \) of the redefining superfields are explained in appendix D. They give rise to

\[
J_{A|B,C,D,E} \equiv \mathcal{J}_{A|B,C,D,E} - \left[ H_{[A,B]} T_{C,D,E} + (A \leftrightarrow B, C, D, E) \right],
\]

(5.7)
with the understanding that $H_{[A,B]} = 0$ for $|A| = |B| = 1$ [32]. After the redefinition (5.7), the BRST transformation of $J_{A|B,C,D,E}$ contains BRST blocks $V_X$ rather than $\hat{V}_X$,

$$Q J_{A|B,C,D,E} = k_A^m V_A T_{B,C,D,E}^m + V_{[A,B]} T_{C,D,E} + V_{[A,C]} T_{B,D,E}$$

$$+ V_{[A,D]} T_{B,C,E} + V_{[A,E]} T_{B,C,D} + Y_{A,B,C,D,E} + O(k_i \cdot k_j).$$

(5.8)

Appendix C displays the four inequivalent seven-point examples of (5.8), including the contact terms $O(k_i \cdot k_j)$, see (C.4). The latter represent the generalization of $s_{12}(V_1 \hat{J}_2|3,4,5,6 - V_2 \hat{J}_1|3,4,5,6)$ in (5.4) and $s_{23}(V_2 \hat{J}_1|3,4,5,6 - V_3 \hat{J}_1|2,4,5,6)$ in (5.5) which simplifies once the $J_{A|B,C,D,E}$ in (5.7) are converted to Berends–Giele currents $J_{A|B,C,D,E}$.

The bracket notation $V_{[A,B]}$ has been explained in the appendix A of [32] and can be diagrammatically understood from figure fig. 5. A few explicit examples are as follows

$$V_{[1,2]} = V_{12}, \quad V_{[12,3]} = V_{123}, \quad V_{[12...p-1,p]} = V_{12...p}$$

$$V_{[1,23]} = V_{123} - V_{132} = -V_{231}, \quad V_{[12,34]} = V_{1234} - V_{1243} = -V_{3412} + V_{3421}.$$ (5.9)

5.2. Berends–Giele version of refined currents

The Berends–Giele version $J_{A|B,C,D,E}$ of refined currents $J_{A|B,C,D,E}$ in (5.7) can be obtained by applying the Berends–Giele map discussed in section 2.3 to each of its five slots. The resulting definition incorporates the Berends–Giele version $H_{[A,B]}$ of the above superfields $H_{[A,B]}$, see appendix D and in particular (D.11) for examples. The contact terms in $Q J_{A|B,C,D,E}$ translate into deconcatenations in $Q J_{A|B,C,D,E}$ in the same way as contact terms in $Q V_B$ are mapped into the deconcatenation formula (2.21) for $Q M_B$. Moreover, $k_A^m V_A T_{B,C,D,E}^m$ and $Y_{A,B,C,D,E}$ on the right-hand side of (5.8) can be straightforwardly replaced by $k_A^m M_A M_{B,C,D,E}^m$ and $Y_{A,B,C,D,E}$, respectively. Only the four permutations of $V_{[A,B]} T_{C,D,E}$ require closer inspection since their expansion in terms of $M_X M_{C,D,E}$ will introduce explicit Mandelstam variables.

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14 As explained in [32], the multiparticle label $B = b_1 b_2 \ldots b|_B|$ in a BRST block $K_B$ satisfies Lie symmetries. They can be elegantly incorporated by writing $B = [...] [[b_1, b_2], b_3], b_4], \ldots, b|_B|$ and using Jacobi identities for iterated brackets. In particular $B = 23$ translates into $B = [2, 3]$. Furthermore, the translation from bracketed to non-bracketed labels is given by $K[...[[[1,2],[3],4],...]|_B] = K_{123...|_B}$, for example $K_{[1,2],[3]} = K_{123}$.

15 As will become clear in later sections, $J_{A|B,C,D,E}$ should really be denoted $M_{A|B,C,D,E}$ since many formulas would acquire a more natural interpretation. However, we use the notation of (5.10) for hysterical reasons.
5.2.1. The $S[A, B]$ map

At the superfield level, the recursive definition of BRST blocks in [32] has the structure of a commutator; $V_{[A,B]} \rightarrow [V_A, V_B]$. At the level of diagrams, $V_{[A,B]}$ can be interpreted as connecting the off-shell legs in the subdiagrams represented by $V_A$ and $V_B$ through a cubic vertex, see [32] and fig. 5. Expanding any $V_C$ in terms of Berends–Giele currents $M_C$ gives rise to a similar diagrammatic interpretation shown in fig. 5, i.e. if two Berends–Giele currents $M_A$ and $M_B$ are attached to a cubic vertex, the resulting diagram is a linear combination of currents $M_C$ at overall multiplicity $|C| = |A| + |B|$. We denote this linear combination by $M_{S[A,B]}$, where the letter $S$ reminds of a factor of $s_{ij}$ which enters on dimensional grounds. In other words, the $S[A, B]$ map captures the difference of applying the Berends–Giele map as described in section 2.3 to the multiparticle label $C$ as a whole as compared to applying it simultaneously and individually to $A$ and $B$, where $|C| = |A| + |B|$, $V_{[A,B]} = [V_A, V_B] \implies M_{S[A,B]} = [M_A, M_B] \, . \quad (5.11)$

For example, converting both sides of $V_{12} = [V_1, V_2]$ to Berends–Giele currents leads to $s_{12}M_{12} = [M_1, M_2]$ and therefore $M_{S[1,2]} = s_{12}M_{12}$. Similarly, converting both sides of $V_{123} = [V_{12}, V_3]$ to Berends–Giele currents gives $s_{12}(s_{23}M_{123} - s_{13}M_{213}) = [s_{12}M_{12}, M_3] \implies M_{S[12,3]} = s_{23}M_{123} - s_{13}M_{213} \, . \quad (5.12)$

To find $S[1, 23]$ one repeats the analysis with $[V_1, V_{23}] = -[V_{23}, V_1]$ and uses the antisymmetry $M_{S[A,B]} = -M_{S[B,A]}$ due to (5.11). Following this procedure one obtains,

$$M_{S[1,2]} = s_{12}M_{12}$$
$$M_{S[1,23]} = s_{12}M_{123} - s_{13}M_{132}$$
$$M_{S[1,234]} = s_{12}M_{1234} - s_{13}(M_{1324} + M_{1342}) + s_{14}M_{1432}$$
$$M_{S[12,34]} = -s_{13}M_{2134} + s_{14}M_{2143} + s_{23}M_{1234} - s_{24}M_{1243} \, .$$
It turns out that the general formula for $M_{S[A,B]}$ reads,

$$M_{S[A,B]} \equiv \sum_{i=1}^{\mid A \mid} \sum_{j=1}^{\mid B \mid} (-1)^{i-j+\mid A \mid-1} s_{a_i b_j} M(a_1 a_2 \ldots a_{i-1} | A | \{ a_i | A | a_{i+1} \ldots a_{|A|} b_j | B | b_{j+1} \ldots b_{|B|}) \cdot$$

(5.14)

The $S[A,B]$ map has been investigated in appendix B of [32] in a different context – it facilitates the expansion of $C_{i[A,B,C}$ in terms of SYM tree subamplitudes.

5.2.2. The BRST variation of refined currents

Let us compare the $M_{S[A,B]}$ in (5.13) with the BRST transformations of various $J_{A|B,C,D,E}$, starting with the trivial five point case,

$$QJ_{1[2,3,4,5} = k_1^m M_1 M_{2,3,4,5}^m + \left[ s_{12} M_1 M_{3,4,5} + (2 \leftrightarrow 3, 4, 5) \right] + \mathcal{Y}_{1,2,3,4,5} \cdot$$

(5.15)

At six points there are two inequivalent partitions of legs in $\mathcal{J}$ satisfying

$$QJ_{12[3,4,5,6} = k_1^m M_1 M_{2,3,4,5,6}^m + \left[ (s_{23} M_{123} - s_{24} M_{124}) M_{3,4,5,6} + (3 \leftrightarrow 4, 5, 6) \right]$$
$$+ \mathcal{Y}_{12,3,4,5,6} + M_1 J_{2[3,4,5,6} - M_2 J_{1[3,4,5,6}$$

(5.16)

$$QJ_{1[23,4,5,6} = k_1^m M_1 M_{23,4,5,6}^m + (s_{12} M_{123} - s_{13} M_{132}) M_{4,5,6} + \left[ s_{14} M_{14} M_{23,5,6} + (4 \leftrightarrow 5, 6) \right]$$
$$+ \mathcal{Y}_{1,23,4,5,6} + M_2 J_{1[3,4,5,6} - M_3 J_{1[2,4,5,6} \cdot$$

(5.17)
The four inequivalent seven-point examples are displayed in appendix C, see (C.5).

Using the $S[A,B]$ map in (5.14), we can write down a general formula for the BRST variation of refined currents,

$$QJ_{A|B,C,D,E} = \mathcal{Y}_{A,B,C,D,E} + k_A^m M_A M_{B,C,D,E} = \sum_{X \neq A} (M_X J_{Y,B,C,D,E} - M_Y J_{X,B,C,D,E})$$

$$+ M_{S[A,B]} C_{D,E} + \sum_{X \neq B} (M_X J_{Y,C,D,E} - M_Y J_{X,C,D,E}) + (B \leftrightarrow C, D, E)$$

(5.18)

It is amusing that the anomaly building block $\mathcal{Y}_{A,B,C,D,E}$ is completely symmetric in $A, B, C, D, E$ whereas $J_{A|B,C,D,E}$ has a reduced symmetry in $B, C, D, E$ due to the refined slot $A$. Note that the non-anomalous part of the right-hand side contains the same kind of terms as they appear in $QM_{A,B,C}$ and $QM_{A,B,C,D}$, see (2.27) and (2.31). This allows to assemble superfields $J_{A|B,C,D,E}$, $k_F^m M_A M_{B,C,D,E}$ and $s_{ij} M_A M_{B,C,D}$ into BRST-pseudoinvariants, the details are worked out in following section.

5.3. Scalar pseudo cohomology

Scalar BRST pseudoinvariants involving refined currents $J_{A|B,C,D,E}$ are defined along the lines of the pseudoinvariants $C_{m_i|A_1,\ldots,A_{r+3}}$ in (4.13)$^{16}$,

$$P_{i|A|B,C,D,E} \equiv M_i J_{A|B,C,D,E} + \sum_{F \neq 0} M_i F \ldots$$

(5.19)

The leading term $M_i J_{A|B,C,D,E}$ furnishes the only instance of a single-particle vertex $M_i$ and therefore defines the reference leg $i$ as well as the multiparticle labels of the pseudoinvariant $P_i|A|B,C,D,E$. The suppressed terms in the $\ldots$ along with a multiparticle $M_i F$ follow from demanding $QP_{i|A|B,C,D,E}$ to be purely anomalous. We determine them by writing (5.18) in terms of vector invariants from section 2.5 and pseudoinvariants (5.19),

$$QJ_{A|B,C,D,E} = \mathcal{Y}_{A,B,C,D,E} + \delta_{|A|,1} k_A^m C_{1|B,C,D,E}$$

$$+ P_{a_1|a_{2\ldots a_{|A|-1}|B,C,D,E} - P_{a_1|a_{2\ldots a_{|A|-1}|B,C,D,E}$$

$$+ [P_{b_1|B|b_2\ldots b_{|B|-1}|C,D,E} - P_{b_1|B|b_2\ldots b_{|B|-1}|C,D,E} + (B \leftrightarrow C, D, E)]$$

(5.20)

$^{16}$ As will become clear in later sections, $P_i|A|B,C,D,E$ should really be denoted $C_i|A|B,C,D,E$ since many formulas would acquire a more natural interpretation. However, we use the notation of (5.19) for hysterical reasons.
As usual, any $M_i$ in (5.18) has been identified as a leading term of some $C_{k_{ij}|A,B,C,D}^m$ or $P_{k_{ij}|A,B,C,D,E}$, see (2.33) and (5.19). Furthermore, equation (5.20) can be verified once the explicit form of the pseudo-invariants $P_{k_{ij}|A,B,C,D,E}$ to be determined below is plugged in and the result compared to (5.18). By (3.10), the non-anomalous terms in (5.20) drop out (A.5). For these simple cases, pseudoinvariance is still easy to check explicitly, and the two inequivalent seven-point analogues are displayed in appendix A, see (A.4) and (A.5). For these simple cases, pseudoinvariance is still easy to check explicitly,

$$\delta_{k_{ij}|A,B,C,D} = M_1 J_{2|3,4,5,6} + M_1 \otimes C_{k_{ij}|3,4,5,6}^m + \left[ P_{a_1|A}a_2...a_{|A|-1}B_{C,D,E} = P_{a_1|A}a_2...a_{|A|-1}B_{C,D,E} \left( \begin{array}{c} \delta_{k_{ij}|A,B,C,D,E} \\ (5.21) \end{array} \right. \right] .$$

When applied to the simplest six-point example, the recursion yields

$$P_{k_{ij}|A,B,C,D,E} = M_1 J_{2|3,4,5,6} + M_1 \otimes C_{k_{ij}|3,4,5,6}^m + \left[ P_{a_1|A}a_2...a_{|A|-1}B_{C,D,E} = P_{a_1|A}a_2...a_{|A|-1}B_{C,D,E} \left( \begin{array}{c} \delta_{k_{ij}|A,B,C,D,E} \\ (5.22) \end{array} \right. \right] .$$

and the two inequivalent seven-point analogues are displayed in appendix A, see (A.4) and (A.5). For these simple cases, pseudoinvariance is still easy to check explicitly,

$$QP_{k_{ij}|A,B,C,D,E} = -M_1 J_{2|3,4,5,6}$$

$$QP_{k_{ij}|A,B,C,D,E} = -M_1 J_{2|3,4,5,6}$$

A general discussion of $QP_{k_{ij}|A,B,C,D,E}$ is given in the later section 7.4. Note that the $P_{k_{ij}|A,B,C,D,E}$ furnish the first cell of the subleading diagonal in the overview grid in fig. 1.

5.4. Scalar pseudoinvariants versus tensor traces

The definition (5.10) of the refined current $J_{A|B,C,D,E}$ exhibits a strong similarity to the trace of the two tensor $M_{A,B,C,D,E}^{mn}$ in (3.1). Only the redefinitions by $H_{[A,B]} = -H_{[B,A]}$ terms might pose an obstruction, but their antisymmetry implies that the difference between $\hat{J}_{A|B,C,D,E}$ and $J_{A|B,C,D,E}$ in (5.7) drops out upon symmetrization in $A, B, C, D, E$,

$$\hat{J}_{A|B,C,D,E} - J_{A|B,C,D,E} + (A \leftrightarrow B, C, D, E) = 0 .$$

Similarly, the $H_{[A,B]}$ corrections in (5.10) cancel out when symmetrizing their corresponding Berends–Giele currents and one gets

$$\delta_{mn}M_{A,B,C,D,E}^{mn} = 2J_{A|B,C,D,E} + (A \leftrightarrow B, C, D, E) .$$

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After multiplication with $M_i$, (5.25) can be viewed as relating the leading terms of $\delta_{mn} C_{i|A,B,C,D,E}^{mn}$ and permutations of (5.19), leading to

$$\delta_{mn} C_{i|A,B,C,D,E}^{mn} = 2 P_{i|A|B,C,D,E} + (A \leftrightarrow B, C, D, E).$$

In other words, scalar pseudoinvariants $P_{i|A|B,C,D,E}$ describe the tensor trace of $C_{i|A,B,C,D,E}^{mn}$ in terms of more fundamental objects. Similarly, it will be shown in the next section that traces of higher-rank pseudo-invariants $\delta_{m_1 m_2} C_{i|A_1,\ldots,A_{r+3}}^{m_1 \ldots m_r}$ decompose into tensorial generalizations of $P_{i|A|B,C,D,E}$. Starting from rank two, the latter give rise to traces by themselves (corresponding to double traces of $C_{i|A_1,\ldots,A_{r+3}}^{m_1 \ldots m_r}$), and one can anticipate an infinite number of all-rank families of pseudoinvariants. These are the different diagonals in the overview grid in fig. 1 where contractions with $\delta_{mn}$ move any tensorial object downwards to the next diagonal. Since an individual $P_{i|A|B,C,D,E}$ contains more information than the trace $\delta_{mn} C_{i|A,B,C,D,E}^{mn}$, we refer to the former as belonging to the refined pseudo-cohomology.

### 6. Generalizing the refined pseudo-cohomology

In this section, we generalize the refined currents (5.10) in two directions: firstly by defining tensorial counterparts and secondly by increasing the number of refined slots such as the distinguished word $A$ in $J_{A|B,C,D,E}$. Each of these currents gives rise to a pseudoinvariant which can be recursively constructed along the lines of sections 4.3 and 5.3.

#### 6.1. Higher-rank refined currents and their anomaly

We define the higher-rank version of the scalar refined current (5.10) by

$$J_{A|B_1,\ldots,B_{r+4}}^{m_1 \ldots m_r} \equiv \frac{1}{2} \mathcal{A}^p A \left[ M_{B_1,\ldots,B_{r+4}}^{p m_1 \ldots m_r} + W_{B_1,\ldots,B_{r+4}}^{m_1 \ldots m_r | p} \right]$$

$$- \left[ H_{[A,B_1]} M_{B_2,\ldots,B_{r+4}}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, B_3, \ldots, B_{r+4}) \right]$$

in terms of higher-rank building blocks $M_{B_1,\ldots,B_{r+4}}^{p m_1 \ldots m_r}$ and $W_{B_1,\ldots,B_{r+4}}^{m_1 \ldots m_r | p}$ defined in (4.3) and (4.2). The redefinition by superfields $H_{[A,B_1]}$ as in appendix D is necessary to trade the $\hat{V}_A$ in its BRST variation for BRST blocks $V_A$. As before, it vanishes whenever both slots are of single-particle type, i.e. $|A| = |B_i| = 1$. 

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At rank \( r \geq 2 \), the BRST variation of \( J^{m_1 \ldots m_r}_{A|B_1, \ldots, B_{r+4}} \) turns out to involve anomalous traces in the same way as \( QM^{m_1 \ldots m_r}_{B_1, \ldots, B_{r+3}} \) given by (4.7). They are accompanied by anomalous counterparts of the refined current (6.1),

\[
\gamma^{m_1 \ldots m_r}_{A|B_1, \ldots, B_{r+6}} \equiv \frac{1}{2} A^p \gamma^{p m_1 \ldots m_r}_{A|B_1, \ldots, B_{r+6}} - [\mathcal{H}_{[A,B_1]} \gamma^{m_1 \ldots m_r}_{B_2, \ldots, B_{r+6}} + (B_1 \leftrightarrow B_2, \ldots, B_{r+6})], \tag{6.2}
\]

whose corrections \( \sim \mathcal{H}_{[A,B]} \) are analogous to (6.1) and ensure a BRST variation in terms of \( V_A \) rather than \( \hat{V}_A \), see section 5.1.

Equipped with the definitions above, we can write down the general BRST variation of higher-rank refined currents,

\[
QJ^{m_1 \ldots m_r}_{A|B_1, \ldots, B_{r+4}} = \binom{r}{2} \delta^{(m_1 m_2} \gamma^{m_3 \ldots m_r)}_{A|B_1, \ldots, B_{r+4}} + \gamma^{m_1 \ldots m_r}_{A,B_1, \ldots, B_{r+4}} + k^p A_M M^{p m_1 \ldots m_r} + 2k^p B_1 J^{m_1 \ldots m_r}_{A|B_2, \ldots, B_{r+4}}
\]

\[
+ \sum_{X \neq Y = A} (M_X J^{m_1 \ldots m_r}_{A|Y,B_2, \ldots, B_{r+4}} - M_Y J^{m_1 \ldots m_r}_{A|X,B_2, \ldots, B_{r+4}}) + (B_1 \leftrightarrow B_2, \ldots, B_{r+4}) \]

see (5.14) for the map \( S[A,B] \). For example, in the case of vectors and two-tensors the general formula (6.3) yields

\[
QJ^m_{A|B_1,B_2,B_3,B_4,B_5} = \gamma^m_{A,B_1,B_2,B_3,B_4,B_5} + k^p A_M M^{p m}_{B_1,B_2,B_3,B_4,B_5} + \sum_{X \neq Y = A} (M_X J^m_{A|Y,B_2,B_3,B_4,B_5} - M_Y J^m_{A|X,B_2,B_3,B_4,B_5}) + (B_1 \leftrightarrow B_2, \ldots, B_5) \]

\[
+ \sum_{X \neq Y} (M_X J^m_{Y,B_1,B_2,B_3,B_4,B_5} - M_Y J^m_{X,B_1,B_2,B_3,B_4,B_5}) \tag{6.4}
\]

\[
QJ^m_{A|B_1,B_2,B_3,B_4,B_5,B_6} = \delta^{mn} \gamma^m_{A|B_1,B_2,B_3,B_4,B_5} + \gamma^m_{A,B_1,\ldots,B_6} + k^p A M^{p m} + \sum_{X \neq Y} (M_X J^m_{A|Y,B_2,\ldots,B_6} - M_Y J^m_{A|X,B_2,\ldots,B_6}) + (B_1 \leftrightarrow B_2, \ldots, B_6) \]

\[
+ \sum_{X \neq Y} (M_X J^m_{Y,B_1,\ldots,B_6} - M_Y J^m_{X,B_1,\ldots,B_6}) \tag{6.5}
\]

6.2. Recursion for refined higher-rank pseudoinvariants

Each of the tensorial refined currents in (6.1) can be regarded as the leading term of a tensorial refined pseudoinvariant,

\[
P^{m_1 \ldots m_r}_{i|A,B_1,\ldots,B_{r+4}} \equiv M_i J^{m_1 \ldots m_r}_{A|B_1,\ldots,B_{r+4}} + \sum_{C \neq \emptyset} M_i C \ldots \tag{6.6}
\]
The BRST pseudo-completion through multiparticle $M_{iC}$ along with momenta and ghost number two objects $J_{A|B_1,\ldots,B_{r+4}}$, $M_{B_1,\ldots,B_{r+3}}$ follows the same logic as explained below (4.13) and (5.19). The recursive construction of the $P_{i|A|B_1,\ldots,B_{r+4}}$ relies on an alternative form of the BRST variation (6.3),

$$Q J_{A|B_1,\ldots,B_{r+4}} = \left( \frac{r}{2} \right) \delta^{m_1 \ldots m_r} J_{A|B_1,\ldots,B_{r+4}} + \delta^{m_1 \ldots m_r} \gamma_{A,B_1,\ldots,B_{r+4}} + \delta^{m_1 \ldots m_r} k^p_{a_1} C_{A|B_1,\ldots,B_{r+4}} + \delta^{m_1 \ldots m_r} a_1 a_2 \ldots a_{|A|} B_{r+4}$

$$+ P_{a_1 a_2 \ldots a_{|A|}} B_1,\ldots,B_{r+4} - P_{a_1 a_2 \ldots a_{|A|}-1} B_{r+4}$

$$+ P_{b_1 b_2 \ldots b_{|B|}} B_1,\ldots,B_{r+4} - P_{b_1 b_2 \ldots b_{|B|}-1} B_{r+4}$

As usual, this follows from isolating the single-particle $M_i$ in (6.3) and promoting them to a (pseudo-)invariant $C_{i|A_1,\ldots,A_{r+3}}$ or $P_{i|A_1,\ldots,A_{r+4}}$. By (3.10) and (6.7), the following recursion eliminates any non-anomalous contribution from $Q P_{i|A_1,\ldots,A_{r+4}}$,

$$P_{i|A_1,\ldots,A_{r+4}} = M_i J_{A_1,\ldots,A_{r+4}} + M_i \otimes \left\{ \delta^{m_1 \ldots m_r} \gamma_{A_1,\ldots,A_{r+4}} + P_{a_1 a_2 \ldots a_{|A|}} B_1,\ldots,B_{r+4}$

$$- P_{a_1 a_2 \ldots a_{|A|}-1} B_{r+4}$

$$+ P_{b_1 b_2 \ldots b_{|B|}} B_1,\ldots,B_{r+4} - P_{b_1 b_2 \ldots b_{|B|}-1} B_{r+4}$

This completes the subleading diagonal in the overview grid in fig. 1. The anomalous BRST variations of (6.8) are discussed in section 7.4.

At rank one, the general recursion (6.7) reduces to

$$P_{i|A,B,C,D,E,F} = M_i J_{A,B,C,D,E,F} + M_i \otimes \left\{ \delta^{m_1 \ldots m_r} \gamma_{A,B,C,D,E,F} + P_{a_1 a_2 \ldots a_{|A|}} B,C,D,E,F$

$$- P_{a_1 a_2 \ldots a_{|A|}-1} B,C,D,E,F$ + $P_{b_1 b_2 \ldots b_{|B|}} B,C,D,E,F$

$$- P_{b_1 b_2 \ldots b_{|B|}-1} B,C,D,E,F$ + $(B \leftrightarrow C, D, E, F)$ \right\}$

and the simplest vectorial pseudoinvariant $P_{i|A,B,C,D,E,F}$ is displayed in appendix A.

6.3. Higher-refinement building blocks

The definition (6.1) of refined building blocks can be endowed with a recursive structure which allows to successively increase the number $d$ of refined slots. In order to do that, first define the generalization of the tensor $\mathcal{W}_{B_1,\ldots,B_{r+4}}$ in (4.2) for any number of specialized legs,

$$\mathcal{W}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}} = \frac{1}{2} A^p_{A_1} \mathcal{W}_{A_2,\ldots,A_d|B_1,\ldots,B_{d+r+3}}$

$$- \mathcal{H}_{[A_1,B_1]} \mathcal{W}_{A_2,\ldots,A_d|B_1,\ldots,B_{d+r+3}} + (B_1 \leftrightarrow B_2,\ldots,B_{d+r+3})$$.  

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Then the recursion for refined currents $J_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ of arbitrary refinement can be immediately written down\footnote{We keep both notations for $M_{B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} = J_{B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ and $C_{i|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} = P_{i|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ to make the source of anomalous BRST transformations more transparent in the scattering amplitudes presented in [29,31].},

\begin{equation}
J_{B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} = M_{B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} \\
J_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} = \frac{1}{2} A^p A_1 \left[ J_{A_2,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{p m_1\ldots m_r} + W_{A_2,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r p} \right] \\
- \left[ H_{A_1,B_1} J_{A_2,\ldots,A_d|B_2,\ldots,B_{d+r+3}}^{m_1\ldots m_r} + (B_1 \leftrightarrow B_2,\ldots,B_{d+r+3}) \right].
\end{equation}

Even though it is not manifest from their definitions \((6.10)\) and \((6.11)\), the objects $\mathcal{W}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ and $J_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ are totally symmetric under exchange of refined slots $A_i \leftrightarrow A_j$. Moreover, symmetry in $B_i \leftrightarrow B_j$ is obviously inherited from $M_{B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ and $\mathcal{W}_{B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ in the first step \((6.1)\) of the recursion.

The BRST variation of \((6.11)\) involves anomaly building blocks of higher refinement which are defined in analogy to \((6.10)\),

\begin{equation}
\mathcal{W}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} = \frac{1}{2} A^p A_1 \mathcal{W}_{A_2,\ldots,A_d|B_1,\ldots,B_{d+r+5}}^{p m_1\ldots m_r} \\
- \left[ H_{A_1,B_1} \mathcal{W}_{A_2,\ldots,A_d|B_2,\ldots,B_{d+r+5}}^{m_1\ldots m_r} + (B_1 \leftrightarrow B_2,\ldots,B_{d+r+5}) \right].
\end{equation}

These definitions give rise to the following formula for the most general case,

\begin{align*}
Q J_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} &= \binom{r}{2} \delta^{m_1 m_2} \mathcal{W}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_3\ldots m_r} \\
&+ \left[ \mathcal{W}_{A_2,\ldots,A_d|A_1,B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} + p^p A_1 M_{A_1} J_{A_2,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} + (A_1 \leftrightarrow A_2,\ldots,A_d) \right] \\
&+ \left[ r k_{B_1}^{m_1 m_2} M_{B_1} J_{A_2,\ldots,A_d|B_2,\ldots,B_{d+r+3}}^{m_1\ldots m_r} + (B_1 \leftrightarrow B_2,\ldots,B_{d+r+3}) \right] \\
&+ \left[ M_{S[A_1,B_1]} J_{A_2,\ldots,A_d|B_2,\ldots,B_{d+r+3}}^{m_1\ldots m_r} + (A_1 \leftrightarrow A_2, A_3,\ldots,A_d) \right] \\
&+ \left[ \sum_{XY=A_1} (M_X J_{Y,A_2,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} - M_Y J_{X,A_2,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}) + (A_1 \leftrightarrow A_2,\ldots,A_d) \right] \\
&+ \left[ \sum_{XY=B_1} (M_X J_{Y,A_2,\ldots,A_d|X,B_2,\ldots,B_{d+r+3}}^{m_1\ldots m_r} - M_Y J_{X,A_2,\ldots,A_d|X,B_2,\ldots,B_{d+r+3}}^{m_1\ldots m_r}) \right. \\
&+ (B_1 \leftrightarrow B_2,\ldots,B_{d+r+3}) \right].
\end{align*}

Any term of $Q J_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+4}}^{m_1\ldots m_r}$ as given by \((6.3)\) has a counterpart in \((6.13)\) at higher degree $d$ of refinement. Moreover, the three classes of terms $\mathcal{W}_{A_1,B_1,\ldots,B_{d+r+4}}^{m_1\ldots m_r}$, $k^p A_M M_{B_1,\ldots,B_{d+r+4}}^{m_1\ldots m_r}$ and
$M_{[A,B]}M_{B_2,...,B_{r+4}}^{m_1...m_r}$ in (6.3) which release $A$ from the refined slot of $\mathcal{J}_{A|B_1,...,B_{r+4}}^{m_1...m_r}$ have multiple images in $Q\mathcal{J}_{A_1,...,A_d|B_1,...,B_{d+r+3}}$ according to $A_1 \leftrightarrow A_2, \ldots, A_d$.

For scalars of refinement $d = 2$, for instance,

\[
Q \mathcal{J}_{A,B|C,D,E,F,G} = \mathcal{J}_{A,B|C,D,E,F,G} + \mathcal{J}_{A,B|C,D,E,F,G} + M_{A}k_{A}^{m} \mathcal{J}_{B|C,D,E,F,G}^{m} + \sum_{\text{X,Y} = \text{C}} (M_{X} \mathcal{J}_{A,B|Y,D,E,F,G} - M_{Y} \mathcal{J}_{A,B|X,D,E,F,G}) + (C \leftrightarrow D, E, F, G)
\]

\[+ \sum_{\text{X,Y} = \text{A}} (M_{X} \mathcal{J}_{Y,B|C,D,E,F,G} - M_{Y} \mathcal{J}_{X,B|C,D,E,F,G}) + \sum_{\text{X,Y} = \text{B}} (M_{X} \mathcal{J}_{A,Y|C,D,E,F,G} - M_{Y} \mathcal{J}_{A,X|C,D,E,F,G}) . \tag{6.14}\]

6.4. The general recursion for pseudoinvariants

The refined current (6.11) of arbitrary rank $r$ and refinement $d$ can be promoted to a pseudoinvariant via

\[
P_{i|A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1...m_r} \equiv M_{i} \mathcal{J}_{A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1...m_r} + \sum_{\text{C} \neq \emptyset} M_{i} C \ldots , \tag{6.15}\]

which generalizes (6.6) to $d > 1$ and by convention reduces to $P_{i|B_1,...,B_{r+3}}^{m_1...m_r} \equiv C_{i|B_1,...,B_{r+3}}^{m_1...m_r}$ at $d = 0$. The suppressed companions of the multiparticle $M_{i} C$ are further instances of momenta and $\mathcal{J}_{A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1...m_r}$ which have to be chosen such that $Q P_{i|A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1...m_r}$ is purely anomalous. These contributions are determined by the following rewriting of (6.13):

\[
Q \mathcal{J}_{A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1...m_r} = \left(\begin{array}{c}
r \\
2 \end{array}\right) \delta^{m_1 m_2, \ldots, m_r} \mathcal{J}_{A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1 m_2, \ldots, m_r} + \mathcal{J}_{A_2,...,A_d|A_1,B_1,...,B_{d+r+3}}^{m_1 m_2, \ldots, m_r} + \sum_{\text{C} \neq \emptyset} \mathcal{J}_{A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1 m_2, \ldots, m_r} . \tag{6.16}\]

As usual, (3.10) allows to derive a recursion from (6.16):

\[
P_{i|A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1...m_r} = M_{i} \mathcal{J}_{A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1...m_r} + \sum_{\text{C} \neq \emptyset} M_{i} C \ldots . \tag{6.17}\]
This is the most general pseudoinvariant presented in this work, it completes the overview grid in fig. 1. Its anomalous BRST variation will be discussed in section 7.4.

In the \( d = 2 \) example of \( J_{A,B|C,D,E,F,G} \), the variation in (6.14) can be rewritten as

\[
QJ_{A,B|C,D,E,F,G} = \mathcal{V}_{A|B,C,...,G} + \mathcal{V}_{B|A,C,...,G} + \delta_{A|1} k_{A|B|C,...,G} + \delta_{B|1} k_{B|A|C,...,G} \\
+ [P_{a_1|a_2} a_3 ... A|B|C,...,G - P_{A|A|a_1 ... a_{|A| - 1}} - B|C,...,G + (A \leftrightarrow B)] \\
+ [P_{c_1|A,B|c_2} c_3 ... C|D,...,G - P_{c|C|A,B|c_1 ... c_{|C| - 1}} - D,...,G + (C \leftrightarrow D, \ldots, G)] 
\]

and converted to the recursion

\[
P_{i|A,B|C,D,E,F,G} = M_i J_{A,B|C,D,E,F,G} + M_i \otimes \left\{ \delta_{A|1} k_{A|B|C,...,G} + \delta_{B|1} k_{B|A|C,...,G} \\
+ [P_{a_1|a_2} a_3 ... A|B|C,...,G - P_{A|A|a_1 ... a_{|A| - 1}} - B|C,...,G + (A \leftrightarrow B)] \\
+ [P_{c_1|A,B|c_2} c_3 ... C|D,...,G - P_{c|C|A,B|c_1 ... c_{|C| - 1}} - D,...,G + (C \leftrightarrow D, \ldots, G)] \right\} .
\]

The simplest example \( P_{1|2,3|4,5,6,7,8} \) is displayed in appendix A, see (A.7).

6.5. Trace relations among pseudoinvariants

In section 5.4, we have discussed the relation between tensor traces \( \delta_{mn} M_{A,B,C,D,E}^{mn} \), \( \delta_{mn} C_{i|A,B,C,D,E}^{mn} \) and the refined objects, \( J_{A|B,C,D,E}, P_{i|A,B,C,D,E} \). The trace relations (5.25) and (5.26) are now generalized to higher rank \( r \) and refinement \( d \).

**Lemma 2.** The following is true,

\[
\delta_{np} W_{A_1,...,A_d|B_1,...,B_{d+r+5}}^{m_1,...,m_{r-1}|m_r} = 2W_{A_1,...,A_d|B_1,...,B_{d+r+5}}^{m_1,...,m_{r-1}|m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+5}) , \\
\delta_{np} \mathcal{J}_{A_1,...,A_d|B_1,...,B_{d+r+5}}^{m_1,...,m_r} = 2\mathcal{J}_{A_1,...,A_d|B_1,...,B_{d+r+5}}^{m_1,...,m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+5}) .
\]

**Proof.** To prove this inductively, first assume that (6.20) is true for \( d - 1 \),

\[
\delta_{np} W_{A_1,...,A_{d-1}|B_1,...,B_{d+r+4}}^{m_1,...,m_{r-1}|m_r} = 2W_{A_1,...,A_{d-1}|B_1,...,B_{d+r+4}}^{m_1,...,m_{r-1}|m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+4}) .
\]

From the definition (6.10) it follows that,

\[
\delta_{np} W_{A_1,...,A_d|B_1,...,B_{d+r+5}}^{m_1,...,m_{r-1}|m_r} = \frac{1}{2} A^t_{A_d} W_{A_1,...,A_{d-1}|B_1,...,B_{d+r+5}}^{tppm1,...,m_{r-1}|m_r} \\
- [H_{A_d|B_1}] W_{A_1,...,A_{d-1}|B_1,...,B_{d+r+5}}^{ppm1,...,m_{r-1}|m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+5}) ,
\]
and therefore (6.22) leads to,

\[
\delta_{\eta p} \mathcal{W}^{m_1 \ldots m_{r-1}}_{A_1 \ldots , A_d | B_1 , \ldots , B_{d+r+5}} = \frac{1}{2} A^t_{A_d} 2 \mathcal{W}^{t m_1 \ldots m_{r-1}}_{A_1 \ldots , A_d-1, B_1 | B_2 , \ldots , B_{d+r+5}} - 2 \left[ \mathcal{H}_{A_d, B_1} \left( \mathcal{W}^{m_1 \ldots m_{r-1}}_{A_1 \ldots , A_d-1, B_2 | B_3 , \ldots , B_{d+r+5}} + (B_2 \leftrightarrow B_3, \ldots , B_{d+r+5}) \right) \right]
\]

where in the last line one uses that,

\[
\mathcal{H}_{A_d, B_1} \left[ \mathcal{W}^{m_1 \ldots m_{r-1}}_{A_1 \ldots , A_d-1, B_2 | B_3 , \ldots , B_{d+r+5}} + (B_2 \leftrightarrow B_3, \ldots , B_{d+r+5}) \right] + (B_1 \leftrightarrow B_2, \ldots , B_{d+r+5})
\]

Furthermore, it is easy to show that when \( d = 0 \),

\[
\delta_{\eta p} \mathcal{W}^{m_1 \ldots m_{r-1}}_{B_2 , \ldots , B_{d+r+5}} = 2 \mathcal{W}^{m_1 \ldots m_{r-1}}_{B_2 , \ldots , B_{d+r+5}} + (B_1 \leftrightarrow B_2, \ldots , B_{d+r+5}) ,
\]

finishing the proof of (6.20).

To show (6.21) one proceeds similarly by first assuming that it holds for \( d - 1 \),

\[
\delta_{\eta p} \mathcal{J}^{m_1 \ldots m_r}_{A_1 \ldots , A_{d-1} | B_1 , \ldots , B_{d+r+4}} = 2 \mathcal{J}^{m_1 \ldots m_r}_{A_1 \ldots , A_{d-1} | B_2 , \ldots , B_{d+r+4}} + (B_1 \leftrightarrow B_2, \ldots , B_{d+r+4}) .
\]

A direct application of the definition (6.11) leads to

\[
\mathcal{J}^{m_1 \ldots m_r}_{A_1 \ldots , A_d | B_1 , B_2 , \ldots , B_{d+r+5}} = \frac{1}{2} A^q_{A_d} \left[ \mathcal{J}^{m_1 \ldots m_r q}_{A_1 \ldots , A_{d-1} B_1 B_2 , \ldots , B_{d+r+5}} + \mathcal{W}^{m_1 \ldots m_r q}_{A_1 \ldots , A_{d-1} | B_1 , B_2 , \ldots , B_{d+r+5}} \right]
\]

Now one rewrites (6.26) using (6.20) together with the assumption (6.25),

\[
\mathcal{J}^{m_1 \ldots m_r}_{A_1 \ldots , A_d | B_1 , B_2 , \ldots , B_{d+r+5}} = \frac{1}{2} A^q_{A_d} \left[ 2 \mathcal{J}^{m_1 \ldots m_r q}_{A_1 \ldots , A_{d-1} B_1 B_2 , \ldots , B_{d+r+5}} + 2 \mathcal{W}^{m_1 \ldots m_r q}_{A_1 \ldots , A_{d-1} | B_1 , B_2 , \ldots , B_{d+r+5}} \right]
\]

and therefore (6.27) leads to,

\[
\mathcal{J}^{m_1 \ldots m_r}_{A_1 \ldots , A_d | B_1 , B_2 , \ldots , B_{d+r+5}} = 2 \mathcal{J}^{m_1 \ldots m_r}_{A_1 \ldots , A_d | B_2 , B_3 , \ldots , B_{d+r+5}} + (B_1 \leftrightarrow B_2, B_3, \ldots , B_{d+r+5}) ,
\]
where a relation analogous to (6.23) has been used to arrive at (6.28). When \( d = 0 \) (6.21) can be easily verified using the definitions (6.11) and (4.3) since the commutator drops out due to the sum over permutations,

\[
\mathcal{J}_{B_1, B_2, \ldots, B_{r+5}}^{ppm_1 \ldots m_r} = M_{B_1, B_2, \ldots, B_{r+5}}^{ppm_1 \ldots m_r p} = A_{B_1}^B M_{B_2, B_3, \ldots, B_{r+5}}^{m_1 \ldots m_r p} + A_{B_1}^B W_{B_2, B_3, \ldots, B_{r+5}}^{m_1 \ldots m_r |p} + (B_1 \leftrightarrow B_2, B_3, \ldots, B_{r+5})
\]

\[
= 2\mathcal{J}_{B_1, B_2, B_3, \ldots, B_{r+5}}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, B_3, \ldots, B_{r+5}).
\]

The above manipulations make use of the symmetry properties \( M_{ppm_1 \ldots m_r} = M_{pm_1 \ldots m_r p} \) and \( W_{m_r m_{r-1} \ldots m_1 |p} = W_{m_1 \ldots m_r |p} \).

After multiplication by \( M_i \), (6.29) and (6.21) relate the leading terms of pseudoinvariants, so we can directly promote them to their BRST pseudo-completion:

\[
\delta_{np} C_{i|B_1, \ldots, B_{r+5}}^{ppm_1 \ldots m_r} = 2P_{i|B_1, B_2, \ldots, B_{r+5}}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{r+5})
\]

\[
\delta_{np} P_{i|A_1, \ldots, A_d|B_1, \ldots, B_{d+r+5}}^{ppm_1 \ldots m_r} = 2P_{i|A_1, \ldots, A_d, B_1 \ldots, B_{d+r+5}}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+5}).
\]

This demonstrates that the family of pseudoinvariants defined in (6.15) and recursively constructed in (6.17) is closed under the trace operation. This is particularly relevant for their contractions with loop momenta in one-loop amplitudes, see [31].

### 7. Anomalous BRST invariants

This section is devoted to BRST variations of anomaly blocks such as \( \mathcal{Y}_{A,B,C,D,E} \) given by (3.3) as well as its generalization to higher rank and refinement, see (4.6) and (6.12). We are led to BRST-invariant ghost-number-four objects built from \( M_{C} \mathcal{Y}_{A_1, \ldots, A_d|B_1, \ldots, B_{d+r+5}}^{m_1 \ldots m_r} \) and momenta. They turn out to share the grid structure of pseudoinvariants in fig. 1, see fig. 6 for an overview and the subsequent sections for the notation therein.

These anomaly invariants capture the systematics of anomalous BRST variations of pseudoinvariants. Moreover, we point out close analogies between the \( Q \) action on \( \mathcal{J}_{A_1, \ldots, A_d|B_1, \ldots, B_{d+r+3}}^{m_1 \ldots m_r} \) and \( \mathcal{Y}_{A_1, \ldots, A_d|B_1, \ldots, B_{d+r+5}}^{m_1 \ldots m_r} \). This firstly allows to recycle a lot of results from previous sections and secondly motivates a more abstract viewpoint on the recursion for pseudoinvariants which will prove essential for the subsequent sections.
7.1. BRST variation of unrefined anomaly blocks

Let us firstly analyze the BRST variations of unrefined anomaly building blocks $\gamma_{A_1,\ldots,A_{r+5}}^{m_1\ldots m_r}$. In the scalar case (3.3),

$$Q\gamma_{A,B,C,D,E} = \sum_{XY=A} (M_X \gamma_{Y,B,C,D,E} - M_Y \gamma_{X,B,C,D,E}) + (A \leftrightarrow B, C, D, E) \quad (7.1)$$

has the same structure as $QM_{A,B,C}$ in (2.27), and in particular $\gamma_{1,2,3,4,5}$ is BRST closed. The pure spinor constraint guarantees that the first term in $QW_B$ given by (2.22) does not contribute. Starting from the vector building block as in (4.6), we additionally need the group-theoretic identity\(^{18}\) [4],

$$(\lambda \gamma^m)_{[\alpha_1}( \lambda \gamma_{p})_{\alpha_2}( \lambda \gamma_{q})_{\alpha_3}( \lambda \gamma_{r})_{\alpha_4}\gamma^{pqr}_{\alpha_5\alpha_6]} = 0 \quad (7.2)$$

to prove that

$$Q\gamma_{A,B,C,D,E,F}^{m} = \sum_{XY=A} (M_X \gamma_{Y,B,C,D,E,F}^{m} - M_Y \gamma_{X,B,C,D,E,F}^{m})$$
$$+ k_{A}^{m} M_{A} \gamma_{B,C,D,E,F} + (A \leftrightarrow B, C, D, E, F), \quad (7.3)$$

i.e. the first term of $Q\gamma_{A,B,C,D,E,F}^{m}$ drops out from $Q\gamma_{A,B,C,D,E,F}^{m}$. Note the direct correspondence of (7.3) with $QM_{A,B,C,D}^{m}$ given by (2.31). Accordingly, higher-tensor

\(^{18}\) This is a consequence of having no vector representation $[1, 0, 0, 0, 0]$ in the decomposition $[0, 0, 0, 0, 4] \otimes [0, 0, 0, 0, 1]^6$. 

Fig. 6 Overview of anomaly invariants. The arrows indicate whenever superfields of different type enter the recursion for the invariants on their right.
generalizations of $Q\mathcal{Y}^m_{A,B,C,D,E,F}$ can be almost literally borrowed from $Q\mathcal{M}^m_{B_1,...,B_{r+3}}$ given by (4.7) except for one simplification: There is no anomaly analogue of the $\mathcal{W}^m_{B_1,...,B_{r+3}}$ tensor in (4.6) which prevents trace contributions in the following expression,

$$Q\mathcal{Y}^m_{B_1,B_2,...,B_{r+5}} = \sum_{XY=B_1} (M_X\mathcal{Y}^m_{Y,B_2,B_3,...,B_{r+5}} - M_Y\mathcal{Y}^m_{X,B_2,B_3,...,B_{r+5}}) + rM_B k^{(m)}_{B_1} \mathcal{Y}^m_{B_2,B_3,...,B_{r+5}} + (B_1 \leftrightarrow B_2, ..., B_{r+5}). \quad (7.4)$$

### 7.2. Unrefined anomaly invariants

We repeat the steps of section 4.3 to recursively construct tensorial BRST invariants $\Gamma_i^m$ at ghost-number four from anomaly blocks. They are defined by a leading term $\sim M_i$,

$$\Gamma_i^m_{A_1,A_2,...,A_{r+5}} \equiv M_i\mathcal{Y}^m_{A_1,A_2,...,A_{r+5}} + \sum_{B\neq\emptyset} M_iB \ldots. \quad (7.5)$$

The suppressed terms ... along with multiparticle $M_iB$ are also anomalous and can be found from a recursion relation. To see this, firstly rewrite (7.4) as follows

$$Q\mathcal{Y}^m_{A_1,A_2,...,A_{r+5}} = r\delta_{\vert A_1\vert,1} k^{(m)}_{a_1} \Gamma^m_{A_2,...,A_{r+5}} + \gamma_{a_1a_2...a_{\vert A_1\vert},A_2,...,A_{r+5}} - \gamma_{a_1a_2...a_{\vert A_1\vert-1},A_2,...,A_{r+5}} + (A_1 \leftrightarrow A_2, ..., A_{r+5}), \quad (7.6)$$

which resembles (4.14) for $Q\mathcal{M}^m_{A_1,...,A_{r+3}}$. Together with (3.8), this implies BRST invariance of the recursively-generated objects

$$\Gamma_i^m_{A_1,A_2,...,A_{r+5}} = M_i\mathcal{Y}^m_{A_1,A_2,...,A_{r+5}} + M_i \otimes \left[ r\delta_{\vert A_1\vert,1} k^{(m)}_{a_1} \Gamma^m_{A_2,...,A_{r+5}} + \gamma_{a_1a_2...a_{\vert A_1\vert},A_2,...,A_{r+5}} - \gamma_{a_1a_2...a_{\vert A_1\vert-1},A_2,...,A_{r+5}} + (A_1 \leftrightarrow A_2, ..., A_{r+5}) \right], \quad (7.7)$$

see (4.15) for the non-anomalous counterpart. For example,

$$\Gamma^m_{1|2,3,4,5,6} = M_1\mathcal{Y}^m_{2,3,4,5,6} \quad (7.8)$$

$$\Gamma^m_{1|23,4,5,6,7} = M_1\mathcal{Y}^m_{23,4,5,6,7} + M_1 \otimes \left[ \Gamma^m_{2|3,4,5,6,7} - \Gamma^m_{3|2,4,5,6,7} \right] = M_1\mathcal{Y}^m_{23,4,5,6,7} + M_{12}\mathcal{Y}^m_{3,4,5,6,7} - M_{13}\mathcal{Y}^m_{2,4,5,6,7}$$

$$\Gamma^m_{1|2,3,4,5,6,7} = M_1\mathcal{Y}^m_{2,3,4,5,6,7} + M_1 \otimes \left[ k^m_2 \Gamma^m_{2|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7) \right] = M_1\mathcal{Y}^m_{2,3,4,5,6,7} + k^m_2 M_{12}\mathcal{Y}^m_{3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)$$

furnish the anomaly counterparts of $C_{1|2,3,4}, C_{1|23,4,5} \text{ and } C_{1|2,3,4,5}$ given in (2.41).
The anomaly invariants in (7.7) allow to concisely describe the anomalous BRST variations of pseudoinvariants at ghost-number three:

\[
QC_{\text{m}1\ldots\text{m}r}^{\text{m}1\ldots\text{m}r} = -\left(\frac{r}{2}\right)^3 \delta^{(m_1 m_2 \Gamma_{\text{m}1\ldots\text{m}r})}_{\text{m}1\ldots\text{m}r}. \tag{7.9}
\]

The anomaly counterpart of this statement is simply

\[
Q\Gamma_{\text{m}1\ldots\text{m}r}^{\text{m}1\ldots\text{m}r} = 0, \tag{7.10}
\]

which follows from \(Q^2 = 0\). To justify (7.9), it is sufficient to study the anomalous BRST variation of the leading term \(M_i M_{\text{m}1\ldots\text{m}r}^{\text{m}1\ldots\text{m}r}\) in \(C_{\text{m}1\ldots\text{m}r}^{\text{m}1\ldots\text{m}r}\) and to promote the resulting \(M_i \mathcal{Y}_{\text{m}1\ldots\text{m}r}^{\text{m}1\ldots\text{m}r}\) to their BRST invariant completion in (7.5).

7.3. BRST variation of general anomaly blocks

The close parallels between BRST manipulations of building blocks \(M_{\text{A}1\ldots\text{A}r+3}^{\text{m}1\ldots\text{m}r}\) and their anomaly counterparts \(\mathcal{Y}_{\text{A}1\ldots\text{A}r+3}^{\text{m}1\ldots\text{m}r}\) propagate to their refined versions. This can be seen by comparing the recursions (6.11) and (6.12) for \(\mathcal{J}_{\text{A}1\ldots\text{A}r+5}^{\text{m}1\ldots\text{m}r}|_{B_1\ldots B_{d+r+3}}\) and \(\mathcal{Y}_{\text{A}1\ldots\text{A}d|B_1\ldots B_{d+r+3}}^{\text{m}1\ldots\text{m}r}\), respectively. The absence of the \(\mathcal{W}_{\text{A}1\ldots\text{A}d|B_1\ldots B_{d+r+3}}^{\text{m}1\ldots\text{m}r-1}\) tensor in the anomalous case implies that there is no higher anomaly image of the \(\mathcal{Y}\) contribution to \(Q\mathcal{J}\). In particular, the expression (6.3) for \(Q\mathcal{J}_{\text{A}1\ldots\text{A}4}^{\text{m}1\ldots\text{m}r}|_{B_1\ldots B_{r+6}}\)

More generally, (6.13) for \(Q\mathcal{J}_{\text{A}1\ldots\text{A}d|B_1\ldots B_{d+r+3}}^{\text{m}1\ldots\text{m}r}\) leads to

\[
Q\mathcal{Y}_{\text{A}1\ldots\text{A}d|B_1\ldots B_{d+r+5}}^{\text{m}1\ldots\text{m}r} = \left[ k_A^p M_A \mathcal{Y}_{\text{A}2\ldots\text{Ad}|B_1\ldots B_{d+r+5}}^{\text{m}1\ldots\text{m}r} + (A_1 \leftrightarrow A_2, \ldots, A_d) \right]
+ \sum_{XY = A_1} \left[ k_B^p M_B \mathcal{Y}_{\text{A}1\ldots\text{Ad}|B_2\ldots B_{d+r+5}}^{\text{m}1\ldots\text{m}r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+5}) \right]
+ \sum_{XY = A_2} \left[ k_B^p M_B \mathcal{Y}_{\text{A}1\ldots\text{Ad}|B_2\ldots B_{d+r+5}}^{\text{m}1\ldots\text{m}r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+5}) \right]. \tag{7.12}
\]

Recall that the \(S[A, B]\) map entering \(M_{\text{S}[A, B]}\) is explained in section 5.2 and defined in (5.14). The \(\mathcal{H}_{[A, B]}\) corrections in the recursion (6.12) for \(\mathcal{Y}_{\text{A}1\ldots\text{Ad}|B_1\ldots B_{d+r+5}}^{\text{m}1\ldots\text{m}r}\) ensure that any \(M_{\text{S}[A, B]}\) in (7.11) and (7.12) is built from BRST blocks \(V_C\) rather than \(\tilde{V}_C\).
7.4. Refined anomaly invariants

Similar to (7.5), we introduce refined anomaly invariants with a more general leading term,

$$\Gamma_{i|A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} \equiv M_i \eta_{A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} + \sum_{C \neq \emptyset} M_i C \ldots \quad (7.13)$$

The BRST completion ... along with multiparticle $M_i C$ is built from $\eta_{A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r}$ and momenta to ensure that $Q \Gamma_{i|A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} = 0$. This is the anomaly counterpart of the pseudo-invariant $P_{i|A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ given by (6.15). The definition (7.13) leads to the following rewriting of (7.12),

$$Q \eta_{A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} = \left[ \delta_{A_1|1} k_{A_1} \Gamma_{i|A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} \right. \right.$$

$$+ \Gamma_{a_1|A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} + \delta_{A_1|a_2,\ldots,a_{|A_1|-1},A_2,\ldots,A_{d}|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} (A_1 \leftrightarrow A_2, \ldots, A_d)$$

$$+ \left. \Gamma_{b_1|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} + \left( B_1 \leftrightarrow B_2, \ldots, B_{d+r+5} \right) \right] . \quad (7.14)$$

This in turn suggests a recursion for the most general anomaly invariant in (7.13),

$$\Gamma_{i|A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} = M_i \eta_{A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} \quad (7.15)$$

For example

$$\Gamma_{1|2^3,4,5,6,7,8} = M_1 \eta_{3,4,5,6,7,8} + k_2^2 M_1 \otimes \eta_{2^3,4,5,6,7,8} \quad (7.16)$$

Note that (7.14) and (7.15) resemble the derivation of pseudo-invariants $P_{i|A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ via (6.16) and (6.17), and the example (7.16) is the anomaly counterpart of $P_{1|2^3,4,5,6,7,8}$ given in (5.22).

The ghost-number-four invariants (7.15) describe the anomalous BRST transformation

$$Q P_{i|A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} = - \left( \begin{array}{c} r \end{array} \right) \delta_{r|A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} \quad (7.17)$$

$$- \left[ \Gamma_{i|A_1,\ldots,A_{d}|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r} + (A_1 \leftrightarrow A_2, \ldots, A_d) \right]$$
with anomaly counterpart
\[
Q \Gamma_{i|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}}^{m_1 \ldots m_r} = 0. 
\]  
\[7.18\]

The former can be see from the BRST variation of the leading term \( M_i \mathcal{J}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1 \ldots m_r} \) in \( P_{i|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1 \ldots m_p} \) where any \( M_i \mathcal{Y}_{A_1,\ldots,A_d|B_1,\ldots,B_{p+q+5}}^{m_1 \ldots m_p} \) is identified as a leading term in (7.13) and promoted to its completion \( \Gamma_{i|A_1,\ldots,A_d|B_1,\ldots,B_{p+q+5}}^{m_1 \ldots m_p} \).

7.5. Anomaly trace relations

The analysis of trace relations in section 6.5 straightforwardly carries over to anomalous building blocks. As before, the \( \mathcal{H}_{[A,B]} \) corrections in the definition (6.12) of \( \mathcal{Y}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}}^{m_1 \ldots m_r} \) drop out in the combinations on the right-hand side of
\[
\delta_{np} \mathcal{Y}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+7}}^{mpm_1 \ldots m_r} = 2 \mathcal{Y}_{A_1,\ldots,A_d,B_1,\ldots,B_{d+r+7}}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+7}). \quad (7.19)
\]

The inductive proof for the non-anomalous counterpart (6.21) fits to the present setting after trivial adjustments – adding two extra slots and suppressing the \( \mathcal{W}_{m_1 \ldots m_r} \) contribution. Similar to (6.30), one can uplift \( M_i \) times (7.19) to the BRST completions,
\[
\delta_{np} \Gamma_{i|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+7}}^{mpm_1 \ldots m_r} = 2 \Gamma_{i|A_1,\ldots,A_d,B_1,\ldots,B_{d+r+7}}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+7}), \quad (7.20)
\]
consistent with the BRST variations in (7.17).

8. Generalizing the recursion scheme

In sections 4 to 6, we have built up a grid of BRST pseudo-invariant objects of ghost number three whose structure is summarized in fig. 1. As we have seen in section 7 and in particular fig. 6, the grid of pseudoinvariants has a straightforward extension to the anomaly sector at ghost-number four with two further slots. Given the almost identical recursion relations (6.17) and (7.15) for the BRST (pseudo-)invariants \( P_{i|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1 \ldots m_r} \) and \( \Gamma_{i|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}}^{m_1 \ldots m_r} \), it is natural to embed these two cases into a unified framework.

We will do so in section 8.1 by promoting (6.17) and (7.15) to a “master recursion”. The latter points towards further special cases besides \( P_{i|\ldots}^{m_1 \ldots} \) and \( \Gamma_{i|\ldots}^{m_1 \ldots} \). In sections 8.2 and 8.3, we are led to two families of ghost-number-two objects, and their anomalous counterparts at ghost-number three are discussed in subsection 8.4 and 8.5. Each of these four cases exhibits a grid structure almost identical to fig. 1 and fig. 6. The superficial
disparity in the number of unrefined slots $B_i$ is taken care of as an integer parameter of the master recursion.

As a major benefit of the ghost-number-two families described in sections 8.2 and 8.3, their BRST variation generates a rich network of relations among pseudoinvariants, beyond the trace identities in section 6.5,

$$Q(\text{ghost-number-two object}) = \text{ghost-number-three relation}. \quad (8.1)$$

These BRST-exact relations turn out to connect momentum contractions $k_{B_i}^p B_i^{pm_1\ldots m_r}$ with pseudoinvariants of lower rank. For example, the five-point combinations

$$k_1^m C_{1|2,3,4,5}^m + k_2^m C_{1|2,3,4,5}^m + s_{23} C_{1|23,4,5}^m + s_{24} C_{1|24,3,5}^m + s_{25} C_{1|25,3,4}^m \quad (8.2)$$

will be identified as BRST-exact if momentum conservation $k_{12345}^m = 0$ holds. As will be detailed in sections 9 and 10, the master recursion in section 8.1 systematically constructs the required ghost-number-two superfields which generate meaningful relations via (8.1).

### 8.1. The master recursion

The purpose of this section is to unify the almost identical recursions (6.17) and (7.15) for the BRST (pseudo-)invariants $P_{i|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ and anomaly invariants $\Gamma_{i|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r}$. The superficial difference set by the number three and five of unrefined slots in the simplest constituents $M_{A,B,C}$ and $\mathcal{Y}_{A,B,C,D,E}$ is described by an integer parameter. This amounts to replacing the superfields by an abstract symbol $U_{A_1,\ldots,A_N}$ with a variable number $N$ of slots.

In the same way as $M_{A,B,C}$ and $\mathcal{Y}_{A,B,C,D,E}$ have been generalized to arbitrary rank $r$ and refinement $d$, we introduce formal symbols at all values of $d$ and $r$,

$$\mathcal{J}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}, \quad \mathcal{Y}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r} \rightarrow U_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+N}}^{m_1\ldots m_r} \quad (8.3)$$

They are defined to be symmetric in $m_i$, $A_i$, $B_i$ but not under exchange of $A_i \leftrightarrow B_j$, so they may be identified with $\mathcal{J}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+3}}^{m_1\ldots m_r}$ and $\mathcal{Y}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}}^{m_1\ldots m_r}$ if $N = 3$ and $N = 5$, respectively. In terms of the standard Berends–Giele currents $M_{A}$ and the symbol in (8.3), we recursively define abstract tensors

$$R_{i|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+N}}^{(N),m_1\ldots m_r} = M_{i} U_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+N}}^{m_1\ldots m_r}$$

$$+ M_i \otimes \left\{ \delta_{i|A_1} k_{a_1}^{m_1} R_{a_1|A_2,\ldots,A_d|B_1,\ldots,B_{d+r+N}}^{(N),m_1\ldots m_r} + R_{a_1|A_2,\ldots,a_{i|A_1}|A_2,\ldots,A_d|B_1,\ldots,B_{d+r+N}}^{(N),m_1\ldots m_r} \right.$$

$$- R_{a_i|A_2,\ldots,a_{i|A_1}|A_2,\ldots,A_d|B_1,\ldots,B_{d+r+N}}^{(N),m_1\ldots m_r} + (A_1 \leftrightarrow A_2, \ldots, A_d)]$$

$$+ [r \delta|B_1] k_{b_1}^{m_1} R_{b_1|A_2,\ldots,A_d|B_2,\ldots,B_{d+r+N}}^{(N),m_1\ldots m_r} + R_{b_1|A_2,\ldots,A_d|b_2,\ldots,b_{i|B_1}|B_2,\ldots,B_{d+r+N}}^{(N),m_1\ldots m_r}$$

$$- R_{b_i|B_1}^{(N),m_1\ldots m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+N}) \right\}. \quad (8.4)$$
The following two specializations reproduce the known recursions (6.17) and (7.15):

\[
P^{m_1...m_r}_{i|A_1,...,A_d|B_1,...,B_{d+r+3}} = R^{(N=3), m_1...m_r}_{i|A_1,...,A_d|B_1,...,B_{d+r+3}} [U^{i...} \to J^{i...}] \quad (8.5)
\]

\[
\Gamma^{m_1...m_r}_{i|A_1,...,A_d|B_1,...,B_{d+r+5}} = R^{(N=5), m_1...m_r}_{i|A_1,...,A_d|B_1,...,B_{d+r+5}} [U^{i...} \to Y^{i...}] \quad (8.6)
\]

The simplest examples for the generalized symbol, leading to new families of superfields whose notation and schematic form is summarized by

\[
R^{(N)}_{1|2,3,...,N+1} = M_1 U_{2,3,...,N+1} \quad (8.7)
\]

\[
R^{(N)}_{1|23,4,...,N+2} = M_1 U_{23,4,...,N+2} + M_2 U_{3,4,...,N+2} - M_3 U_{2,4,...,N+2} \quad (8.8)
\]

\[
R^{(N)}_{1|2,3,...,N+2} = M_1 U_{2,3,...,N+2} + \left[ k_2^m M_2 U_{3,4,...,N+2} + (2 \leftrightarrow 3, 4, \ldots, N + 2) \right] \quad (8.9)
\]

\[
R^{(N)}_{1|23,4,...,N+3} = M_1 U_{23,4,...,N+3} + M_2 k_2^m U_{3,4,...,N+3} + \left[ s_{23} M_2 U_{4,...,N+3} + (3 \leftrightarrow 4, 5, \ldots, N + 3) \right] \quad (8.10)
\]

The right-hand sides obviously specialize to familiar expressions such as

- (2.41) and (5.22) for \( C_{1|2,3,4}, C_{1|23,4,5}, C_{1|2,3,4,5} \) and \( P_{1|2|3,4,5,6} \) under (8.5)

- (7.8) and (7.16) for \( \Gamma_{1|2,3,4,5,6,7}, \Gamma_{1|23,4,5,6,7} \) and \( \Gamma_{1|23,4,5,6,7,8} \) under (8.6).

In the following sections, we consider the abstract tensors \( R^{(N)}_{i|...} \) in (8.4) at values \( N = 2, 4, 6 \). In order to accommodate this with the number of slots of \( U \in \{ J, Y \} \), we have to eliminate the Berends–Giele currents \( M_A \) and adjoin the word \( A \) to the slots of the accompanying symbol. In the non-anomalous case \( U = J \), this gives rise to ghost-number-two objects, and the anomalous choice \( U = Y \) yields ghost number three. Moreover, the word \( A \) from the eliminated \( M_A \) can become either a refined or a non-refined slot of the symbol, leading to \( U_A,...,|... \) or \( U,...,A_{...} \). These two independent choices yield a total of four new families of superfields whose notation and schematic form is summarized by

\[
D^{i...}_{i|...} \equiv R^{(N=2),...}_{i|...} [M_A U_{i(B_j)}|C_j] \to J_{i(B_j)|A_{...}C_j} \quad (8.11)
\]

\[
L^{i...}_{i|...} \equiv R^{(N=4),...}_{i|...} [M_A U_{i(B_j)}|C_j] \to J_{A_{...}(B_j)|C_j} \quad (8.12)
\]

\[
\Delta^{i...}_{i|...} \equiv R^{(N=4),...}_{i|...} [M_A U_{i(B_j)}|C_j] \to Y_{i(B_j)|A_{...}C_j} \quad (8.13)
\]

\[
\Lambda^{i...}_{i|...} \equiv R^{(N=6),...}_{i|...} [M_A U_{i(B_j)}|C_j] \to Y_{A_{...}(B_j)|C_j} \quad (8.14)
\]

The precise definitions and simplest examples are given in the following subsections.
8.2. The $D$-superfields at ghost-number two

As a first avenue towards BRST generators at ghost number two, we consider the tensors $R^{(N)-}_{i|...}$ in (8.4) at $N = 2$ and convert the word $A$ associated with $M_A$ to a non-refined slot of the associated symbol $U \rightarrow J$. As sketched in (8.11), this gives rise to the definition

$$ D_{i|B_1,...,B_d|C_1,...,C_{d+r+2}}^{m_1,...m_r} = R^{(N=2),m_1,...m_r}_{i|B_1,...,B_d|C_1,...,C_{d+r+2}} [M_A U_{\{\bar{F}_j\}|\{G_j\}} \rightarrow J_{\{\bar{F}_j\}|\{G_j\}}] \quad (8.15) $$

The replacement rule in (8.15) converts the formal objects (8.7), (8.8), (8.9) and (8.10) to

$$ D_{1|2,3} = M_{1,2,3}, \quad (8.16) $$
$$ D_{1|23,4} = M_{12,3,4} + M_{1,23,4} + M_{31,2,4}, $$
$$ D_{1|2,3,4}^m = M_{1,2,3,4}^m + k_2^m M_{12,3,4} + k_3^m M_{13,2,4} + k_4^m M_{14,2,3}, $$
$$ D_{1|23,4,5} = J_{2|1,3,4,5} + k_2^m M_{12,3,4,5} + [s_{23} M_{123,4,5} + (3 \leftrightarrow 4, 5)]. $$

In the next section 9, these ghost-number-two objects and their generalizations are shown to serve as powerful BRST generators in the sense of (8.1).

8.3. The $L$-superfields at ghost-number two

The $N = 4$ version of the $R^{(N)-}_{i|...}$ in (8.4) allows to convert the word $A$ associated with $M_A$ to a refined slot of the associated symbol $U \rightarrow J$. The precise form of the definition sketched in (8.12) is

$$ L_{i|B_1,...,B_d|C_1,...,C_{d+r+4}}^{m_1,...m_r} = R^{(N=4),m_1,...m_r}_{i|B_1,...,B_d|C_1,...,C_{d+r+4}} [M_A U_{\{\bar{F}_j\}|\{G_j\}} \rightarrow J_{\{\bar{F}_j\}|\{G_j\}}] \quad (8.17) $$

Starting from the examples in (8.7) to (8.10), the prescription (8.17) yields

$$ L_{1|2,3,4,5} = J_{1|2,3,4,5} \quad (8.18) $$
$$ L_{1|23,4,5,6} = J_{1|23,4,5,6} + J_{12|3,4,5,6} - J_{13|2,4,5,6} $$
$$ L_{1|2,3,4,5,6}^m = J_{1|2,3,4,5,6}^m + [k_2^m J_{12|3,4,5,6} + (2 \leftrightarrow 3, 4, 5, 6)] $$
$$ L_{1|23,4,5,6,7} = J_{12|3,4,5,6,7} + k_2^m J_{12|3,4,5,6,7}^m + [s_{23} J_{123|4,5,6,7} + (3 \leftrightarrow 4, 5, 6, 7)]. $$

These superfields of ghost-number two serve as another family of BRST generators, see section 10.
8.4. The $\Delta$-superfields at ghost-number three

Another specialization of the $R_{i|\ldots}^{(N)}$ in (8.4) to $N = 4$ generates anomalous superfields. In this case, the word $A$ associated with $M_A$ is adjoined to the non-refined slots of the associated symbol $U \rightarrow \mathcal{Y}$. As sketched in (8.13), this gives rise to the definition

$$\Delta_{i|B_1,\ldots,B_d|C_1,\ldots,C_{d+r+4}}^{m_1\ldots m_r} \equiv R_{i|B_1,\ldots,B_d|C_1,\ldots,C_{d+r+4}}^{(N=4), m_1\ldots m_r} [M_A U_{i|\tilde{F}_j}|\{G_j\} \rightarrow \mathcal{Y}_{i|\tilde{F}_j}|A,\{G_j\}]$$

(8.19)

which can be viewed as the anomaly counterparts of $D_{i|B_1,\ldots,B_d|C_1,\ldots,C_{d+r+2}}^{m_1\ldots m_r}$ in (8.15).

Applying the replacement rule (8.19) to the examples in (8.7) to (8.10), one arrives at

$$\Delta_{1|2,3,4,5} = \mathcal{Y}_{1,2,3,4,5}$$

(8.20)

$$\Delta_{1|2,3,4,5,6} = \mathcal{Y}_{1,2,3,4,5,6} + \mathcal{Y}_{12,3,4,5,6} - \mathcal{Y}_{13,2,4,5,6}$$

(8.21)

$$\Delta_{1|2,3,4,5,6} = \mathcal{Y}_{12,3,4,5,6} + [k^m_2 \mathcal{Y}_{12,3,4,5,6} + (2 \leftrightarrow 3, 4, 5, 6)]$$

(8.22)

$$\Delta_{1|2,3,4,5,6,7} = \mathcal{Y}_{2|1,3,4,5,6,7} + k^m_2 \mathcal{Y}_{12,3,4,5,6,7} + [s_{23} \mathcal{Y}_{123,4,5,6,7,9} + (3 \leftrightarrow 4, 5, 6, 7)]$$

(8.23)

which can be easily recognized as the anomaly analogues of (8.16).

8.5. The $\Lambda$-superfields at ghost-number three

Finally, there is a $N = 6$ version of the $R_{i|\ldots}^{(N)}$ in (8.4) where the word $A$ associated with $M_A$ becomes a refined slot of the symbol $U \rightarrow \mathcal{Y}$. The resulting anomalous superfields were sketched in (8.14) and are more cleanly defined as

$$\Lambda_{i|B_1,\ldots,B_d|C_1,\ldots,C_{d+r+6}}^{m_1\ldots m_r} \equiv R_{i|B_1,\ldots,B_d|C_1,\ldots,C_{d+r+6}}^{(N=6), m_1\ldots m_r} [M_A U_{i|\tilde{F}_j}|\{G_j\} \rightarrow \mathcal{Y}_{i|\tilde{F}_j}|A,\{G_j\}]$$

(8.24)

This is the anomaly counterpart of the objects $L_{i|B_1,\ldots,B_d|C_1,\ldots,C_{d+r+4}}^{m_1\ldots m_r}$ in (8.17).

Under the prescription in (8.24), the examples in (8.7) to (8.10) are mapped to

$$\Lambda_{1|2,3,4,5,6,7} = \mathcal{Y}_{1|2,3,4,5,6,7}$$

(8.25)

$$\Lambda_{1|2,3,4,5,6,7,8} = \mathcal{Y}_{1|2,3,4,5,6,7,8} + \mathcal{Y}_{12|3,4,5,6,7,8} - \mathcal{Y}_{13|2,4,5,6,7,8}$$

(8.26)

$$\Lambda_{1|2,3,4,5,6,7,8} = \mathcal{Y}_{12|3,4,5,6,7,8} + [k^m_2 \mathcal{Y}_{12|3,4,5,6,7,8} + (2 \leftrightarrow 3, \ldots, 8)]$$

(8.27)

$$\Lambda_{1|2,3,4,5,6,7,8,9} = \mathcal{Y}_{12|3,4,5,6,7,8,9} + k^m_2 \mathcal{Y}_{12|3,4,5,6,7,8,9} + [s_{23} \mathcal{Y}_{123|4,5,6,7,8,9} + (3 \leftrightarrow 4, \ldots, 9)]$$

(8.28)

They can be quickly seen to furnish the anomaly counterparts of (8.18).
9. Pseudoinvariant relations for $k_B$ momentum contractions

The main concern in this paper is to systematically study the properties of and relations among the pseudoinvariants $P^{m_1 \ldots m_i \ldots}$ which carry the polarization dependence of one-loop amplitudes. In the previous section, we have constructed two families of ghost-number-two superfields whose BRST variations will be demonstrated to generate relations among the $P^{m_1 \ldots m_i \ldots}$.

Recall that pseudoinvariants $P^{m_1 \ldots m_i \ldots}$ single out a reference leg $i$ which always enter through a Berends–Giele current of type $M_{i \ldots}$ and which is represented by an unintegrated vertex $V_i$ in the one-loop amplitude prescription [4]. The ghost-number-two generators of relations among pseudoinvariants in (8.1) must be carefully chosen in order to avoid admixtures of pseudoinvariants $P^{m_1 \ldots m_k \ldots}$ with a different reference leg $k \neq i$.

It turns out that both the $D$ superfields from section 8.2 and the $L$ superfields from section 8.3 satisfy this criterion, see (8.15) and (8.17) for their precise definitions. BRST variations of type $QD$ are systematically analyzed in the present section, and section 10 is devoted to $QL$. We will see how the ghost-number-three expression for $QD_i^{m_1 \ldots m_r} |A,B,C,D|$ relates momentum contractions $k_{B_j} P^{m_2 \ldots m_r}_{i|A_1 \ldots A_d|B_1 \ldots B_{d+r+2}}$ to pseudoinvariants at lower rank. These relations are crucial to translate the SYM one-loop amplitudes presented in [31] into worldline parametrization and to make contact with their string theory ancestors.

9.1. BRST-exactness versus momentum phase space

As a starting point, we investigate the $Q$ action on unrefined $D$ superfields $D^{m_1 \ldots m_r}_{i|A_1 \ldots A_{r+2}}$ such as the simplest examples given in (8.16). For scalars and vectors, we find

$$QD_i^{m} |A,B \equiv 0$$

(9.1)

$$QD_i^{m} |A,B,C = k^{m}_{iABC} C_i^{A,B,C}$$

(9.2)

with overall momentum $k^{m}_{iABC} \equiv k_i^m + k_A^m + k_B^m + k_C^m$. This can be verified case by case using the BRST variations (2.35) and (2.36) for each $M_{A,B,C}$ and $M^{m}_{A,B,C,D}$ occurring in $D_i^{A,B}$ and $D_i^{m} |A,B,C$, respectively. Analogous methods are used in all the subsequent cases when $Q$ variations are computed.

Contracting (9.2) with any momentum, one can solve for $C_i^{A,B,C}$,

$$C_i^{A,B,C} = Q \left[ \frac{k_i^m D_i^{m} |A,B,C}{(k_i \cdot k_{ABC})} \right],$$

(9.3)
i.e. $C_{i|A,B,C}$ is BRST exact unless $k_i \cdot k_{iABC} = 0$. Note, however, that momentum conservation $k_i^{\mu}k_{iABC} = 0$ in an $n$-point amplitude (with $n = 1 + |A| + |B| + |C|$) implies $k_i \cdot k_{iABC} = 0$ and renders the right-hand side of (9.3) ill-defined. Hence, momentum phase space constraints for $n$ massless particles save $C_{i|A,B,C}$ from being BRST exact and preserve its cohomological nature in $n$ point amplitudes.

This is analogous to the superspace representation $\sum_{j=1}^{n-2} M_{12\ldots j}M_{j+1\ldots n-1}V_n$ of color ordered SYM tree amplitudes [17]. This expression can be rewritten as $Q(M_{12\ldots n-1}V_n)$ as long as the overall propagator $M_{12\ldots n-1} \sim s_{12\ldots n-1}^{-1}$ does not diverge. Again, $n$ particle momentum conservation implying $s_{12\ldots n-1} = 0$ is essential to avoid BRST exactness of the tree amplitude.

In both cases, the cohomology nature of BRST-closed kinematic factors crucially depends on vanishing conformal weight $h \sim s_{12\ldots n}$. Recall that in a topological conformal field theory where $Qb_0 = L_0$, the cohomology at non-zero conformal weight is empty since every BRST-closed operator would also be BRST-exact [42],

$$Q\phi = 0 , \quad L_0\phi = h\phi , \quad h \neq 0 \implies \phi = Q\left(\frac{b_0\phi}{h}\right) . \quad (9.4)$$

Starting from rank two, the $Q$ transformations of $D_{i|A_1\ldots,A_{r+2}}^{m_1m_2\ldots m_r}$ additionally give rise to anomalous superfields $\Delta_{i|A_1\ldots,A_{p+4}}^{m_1\ldots m_p}$ defined in (8.19), e.g.

$$QD_{i|A,B,C,D}^{mn} = \delta^{mn}\Delta_{i|A,B,C,D} + 2k_{iABC,D}C_{i|A,B,C,D}^{mn} , \quad (9.5)$$

and more generally,

$$QD_{i|A_1\ldots,A_{r+2}}^{m_1m_2\ldots m_r} = \binom{r}{2}\delta^{m_1m_2}\Delta_{i|A_1\ldots,A_{r+2}}^{m_3m_4\ldots m_r} + rk_{iA_1\ldots A_{r+2}}^{m_1}C_{i|A_1\ldots,A_{r+2}}^{m_2m_3\ldots m_r} . \quad (9.6)$$

As a consequence, the identification of BRST-exact quantities crucially depends on the momentum phase space. In case of momentum conservation $k_{iA_1\ldots A_{r+2}}^{m} = 0$, (9.6) implies\footnote{At rank $r = 2$ and $r = 3$, BRST exactness of $\Delta_{i|A,B,C,D}$ and $\Delta_{i|A,B,C,D,E}^m$ immediately follows from single traces of (9.6) at $k_{iA_1\ldots A_{r+2}}^{m} = 0$. Higher rank $r \geq 4$ requires combinations of multiple $\delta_{m_im_j}$ contractions in order to identify the BRST generator of $\Delta_{i|A_1\ldots A_{r+4}}^{m_1\ldots m_p}$ at any rank.} $Q$ exactness of the anomalous superfield $\Delta_{i|A_1\ldots,A_{p+4}}^{m_1\ldots m_p}$, hence the latter does not contribute to physical amplitudes at multiplicity $1 + \sum_{j=1}^{p-4} |A_j|$. However, it is important to stress that the hexagon gauge anomaly superfield $\Delta_{2|3,4,5,6} = \mathcal{Y}_{2,3,4,5,6}$ in the one-loop six-point
amplitude [26] is not BRST exact for the momentum phase space of six particles since 
\( k_{23456}^m \) is not zero.

On the other hand, generic momentum configurations with 
\( k_{i|A_1...A_{r+2}}^m \neq 0 \) render the traceless components of pseudoinvariants 
\( C_{i|A_1,...,A_{r+3}}^{m_1m_2...m_r} \) BRST exact. This can be seen from the traceless projection of (9.6), see (E.2) in appendix E for the explicit form of the BRST generator for 
\( C_{i|A,B,C,D}^m \).

The correspondence between the pseudoinvariants 
\( C_{i|A_1,...,A_{r+3}}^{m_1m_2...m_r} \) and the anomaly invariants 
\( \Gamma_{i|A_1,...,A_{r+5}}^{m_1m_2...m_r} \) described in section 7 and formalized in section 8.1 allows to immediately write down the anomaly correspondent of (9.6):

\[
Q\Delta_{i|A_1,...,A_{r+4}}^{m_1m_2...m_r} = r k_{i|A_1...A_{r+4}}^{(m_1} \Gamma_{i|A_1,...,A_{r+4}}^{m_2m_3...m_r)} .
\]

We exploit that \( \Delta_{i|A_1,...,A_{r+4}}^{m_1m_2...m_r} \) is the anomaly counterpart of \( D_{i|A_1,...,A_{r+2}}^{m_1m_2...m_r} \) which, loosely speaking, does not have a higher anomaly image. Of course, (9.7) confirms that momentum conservation 
\( k_{i|A_1...A_{r+4}}^m = 0 \) implies BRST closure of \( \Delta_{i|A_1,...,A_{r+4}}^{m_1m_2...m_r} \), in lines with the discussion of BRST exactness along with (9.6).

### 9.2. Momentum contractions of unrefined pseudoinvariants

Refined versions of the \( D \) superfields turn out to generate a much richer set of ghost number three relations than their unrefined counterparts studied in section 9.1. We start by exploring the case of minimal refinement \( d = 1 \) and will find that 
\( QD_{i|A_1,...,A_{r+3}}^{m_1m_2...m_r} \) relates momentum contractions \( \sim k_{A_j} \) of 
\( C_{i|A_1,...,A_{r+3}}^{m_1...m_r} \) to pseudoinvariants at lower rank.

As a first example, consider the inequivalent cases at five- and six-points,

\[
QD_{1|2|3,4,5} = \Delta_{1|2,3,4,5} + k_{1|2,3,4,5}^m C_{1|2,3,4,5}^m + \left[ s_{23} C_{1|23,4,5} + (3 \leftrightarrow 4, 5) \right]
\]

\[
QD_{1|23|4,5,6} = \Delta_{1|23,4,5,6} + P_{1|23,4,5,6} - \Delta_{1|2,3,4,5,6} + k_{23}^m C_{1|23,4,5,6} + \left[ s_{34} C_{1|234,5,6} - s_{24} C_{1|324,5,6} + (4 \leftrightarrow 5, 6) \right]
\]

\[
QD_{1|4|23,5,6} = \Delta_{1|23,4,5,6} + k_{4}^m C_{1|234,5,6} + s_{24} C_{1|324,5,6} - s_{34} C_{1|234,5,6} + s_{45} C_{1|234,5,6} + s_{46} C_{1|234,6,5},
\]

where (9.8) underpins the second example in (8.2) provided that momentum conservation 
\( k_{123456}^m = 0 \) renders \( \Delta_{1|2,3,4,5} \) BRST exact.

The combinations of \( s_{ij} C_{1|A,B,C} \) can be neatly described using the \( S[A,B] \) map in (5.14), see in particular (5.13) for examples. The seven-point instances of \( QD_{1|A|B,C,D} \)
displayed in (E.3) to (E.6) support this pattern and help to identify the appearance of $P_{i|A|B,C,D,E}$ as deconcatenations. These observations lead to the following generalization,

$$QD_{i|A|B,C,D} = \Delta_{i|A|B,C,D} + k_A^m C_i^m_{|A|B,C,D} + \sum_{XY=A} (P_{i|Y|X,B,C,D} - P_{i|X|Y,B,C,D}) + C_i[S_{A,B},C,D] + C_i[S_{A,C},B,D] + C_i[S_{A,D},B,C] .$$

(9.9)

Note that $QD_{i|A|B,C,D}$ always generates relations for contractions of $C_i^m_{|A|B,C,D}$ with the entire momentum $k_A^m = \sum_j |A_i|^j k_a^m$ in the slot $A = a_1 a_2 \ldots a_{|A|}$. This method does not provide any information on contractions with partial slot momenta, e.g. $k_a^m C_i^m_{|A|B,C,D}$ with $|A| \geq 2$.

It is natural to repeat the BRST manipulations for vectorial $D$ superfields such as

$$QD_{1|2|3,4,5,6}^m = \Delta_{1|2,3,4,5,6} + k^2_{2P} C_{1|2,3,4,5,6}^m + (k^m_{123456} - k^m_2) P_{1|2,3,4,5,6}$$

(9.10)

$$QD_{1|2|3,4,5,6}^m = \Delta_{1|2,3,4,5,6} + k^p_{2P} C_{1|2,3,4,5,6}^m + (k^m_{123456} - k^m_{23}) P_{1|2,3,4,5,6}$$

$$QD_{1|2|3,4,5,6}^m = \Delta_{1|2,3,4,5,6} + k^p_{2P} C_{1|2,3,4,5,6}^m + (k^m_{123456} - k^m_{23}) P_{1|2,3,4,5,6}$$

$$QD_{1|2}^m = \Delta_{1|2,3,4,5,6} + k^p_{2P} C_{1|2,3,4,5,6}^m + (k^m_{123456} - k^m_{23}) P_{1|2,3,4,5,6}$$

which can be summarized by a general formula similar to the scalar case in (9.9),

$$QD_{1|2|3,4,5,6}^m = \Delta_{1|2,3,4,5,6} + k^p_{2P} C_{1|2,3,4,5,6}^m + (k^m_{123456} - k^m_A) P_{1|2,3,4,5,6}$$

(9.11)

Tensorial generalizations at rank $r \geq 2$ additionally involve refined versions of the anomalous $\Delta$ superfields in (8.19). The simplest example occurs at seven points,

$$QD_{1|2,3,4,5,6}^m = \Delta_{1|2,3,4,5,6} + \delta_{1|2,3,4,5,6} + k^p_{2P} C_{1|2,3,4,5,6}^m + (k^m_{123456} - k^m_{23}) P_{1|2,3,4,5,6}$$

(9.12)

where $\Delta_{1|2,3,4,5,6}$ is given by (8.23). The structure of the scalar and vector cases (9.9) and (9.11) inspires the following generalization of (9.12) to multiparticle slots:

$$QD_{1|A|B,C,D,E,F}^m = \Delta_{1|A|B,C,D,E,F} + \delta_{1|A|B,C,D,E,F} + k^p_{A} C_{1|A,B,C,D,E,F}^m + 2(k^m_{123456} - k^m_{A}) P_{1|2,3,4,5,6} + (3 \leftrightarrow 4, 5, 6, 7) ,$$

(9.13)
This in turn allows to infer the BRST variation of $D_{i|A|...}^{m_1...}$ superfields at generic rank:

\[
QD_{i|A|B_1,...,B_{r+3}}^{m_1m_2...m_r} = \Delta_{i|A|B_1,...,B_{r+3}}^{m_1m_2...m_r} + \left(\frac{r}{2}\right)\delta_{i|A|B_1,...,B_{r+3}}^{m_1m_2...m_r} + k^p_A C_{i|A|B_1,...,B_{r+3}}^{pm_1...m_r} \\
+ r(k^{(m_1}_{i|AB_1,...,B_{r+3}} C^{m_2...m_r}_{i|A|B_1,...,B_{r+3}} + [C^{m_1...m_r}_{i|S[A,B_1],B_2,...,B_{r+3}} + (B_1 \leftrightarrow B_2, \ldots, B_{r+3})] \\
+ \sum_{XY=A} (P^{m_1...m_r}_{i|Y|X,B_1,...,B_{r+3}} - P^{m_1...m_r}_{i|X|Y,B_1,...,B_{r+3}}) ,
\]

(9.14)

Recall that momentum conservation $k^{m}_{iA_1...A_{r+4}} = 0$ implies BRST exactness of the unrefined representatives $\Delta_{i|A_1,...,A_{r+4}}^{m_1...m_r}$ of the anomalous $\Delta$ superfields. Hence, the latter do not contribute when the relations (9.9), (9.11), (9.13) and (9.14) are applied to physical BRST closed, regardless of momentum phase space constraints, so $Q$ exactness can be clearly ruled out. Starting from seven points, refined anomaly superfields $\Delta_{i|A|B_1,...,B_{r+5}}^{m_1...m_r}$ at ghost-number three cannot be discarded in the discussion of one-loop amplitudes, see [31].

The ghost-number-four expression for $Q\Delta_{i|A|B_1,...,B_{r+5}}^{m_1...m_r}$ can be inferred by analogy with (9.9) to (9.14). Since (8.19) identifies $\Delta_{i|A|B_1,...,B_{r+5}}^{m_1...m_r}$ to be the anomaly counterpart of $D_{i|A|B_1,...,B_{r+3}}^{m_1m_2...m_r}$, the BRST transformation of the former follows from (9.9) and (9.14) upon discarding anomalous terms and converting $C_{i|...}^{m_1...}, P_{i|...}^{m_1...} \rightarrow \Gamma_{i|...}$:

\[
Q\Delta_{i|A|B,...,F} = k^{m}_{iA_1}, \Gamma_{i|A,B,...,F}^{m} + \sum_{XY=A} (\Gamma_{i|Y|X,B,...,F} - \Gamma_{i|X|Y,B,...,F}) \\
+ [\Gamma_{i|S[A,B],C,D,E,F} + (B \leftrightarrow C, D, E, F)]
\]

(9.15)

\[
Q\Delta_{i|A|B_1,...,B_{r+5}}^{m_1m_2...m_r} = k^p_A P_{i|A,B_1,...,B_{r+5}}^{pm_1...m_r} + r(k^{(m_1}_{i|AB_1,...,B_{r+5}} C^{m_2...m_r}_{i|A|B_1,...,B_{r+5}} + [C^{m_1...m_r}_{i|S[A,B_1],B_2,...,B_{r+5}} + (B_1 \leftrightarrow B_2, \ldots, B_{r+5})] \\
+ \sum_{XY=A} (\Gamma_{i|Y|X,B_1,...,B_{r+5}} - \Gamma_{i|X|Y,B_1,...,B_{r+5}}) .
\]

(9.16)

Together with expressions for $QC_{i|...}^{m_1...}$ and $QP_{i|...}^{m_1...}$ in (7.9) and (7.17), one can check BRST closure of the right-hand side of (9.9) and (9.14).
9.3. Momentum contractions of refined pseudoinvariants

The procedure from the previous section is now extended to higher refinement. In the simplest scalar cases at seven- and eight-points, we find

\[ QD_{1[2,3|4,5,6,7} = \Delta_{1[2|3,4,5,6,7} + \Delta_{1|2|3,4,5,6,7} + k^m_{3} P^m_{1[2|3,4,5,6,7} + k^m_{2} P^m_{1|2|3,4,5,6,7} + [s_{34} P^m_{1[2|3,4,5,6,7} + s_{24} P^m_{1|2|3,4,5,6,7} + (4 \leftrightarrow 5, 6, 7)] \]

(9.17)

\[ QD_{1[2,3|4,5,6,7} = \Delta_{1[2|3,4,5,6,7} + \Delta_{1|2|3,4,5,6,7} + k^m_{3} P^m_{1|2|3,4,5,6,7} + k^m_{2} P^m_{1|2|3,4,5,6,7} + [s_{35} P^m_{1|2|3,4,5,6,7} + s_{25} P^m_{1|2|3,4,5,6,7} + s_{45} P^m_{1|2|3,4,5,6,7} + (5 \leftrightarrow 6, 7, 8)]
\]

- \( P_{1[2,3|4,5,6,7} = [s_{35} P^m_{1|2|3,4,5,6,7} + s_{25} P^m_{1|2|3,4,5,6,7} + s_{45} P^m_{1|2|3,4,5,6,7} + (5 \leftrightarrow 6, 7, 8)] \)

signaling the general rule

\[ QD_{i[A, B]|C, D, E, F} = \Delta_{i[A, B]|C, D, E, F} + \Delta_{i|B|[A, C, D, E, F} + k^m_{A} P^m_{i|B|[A, C, D, E, F} + k^m_{B} P^m_{i|A|[B, C, D, E, F} + \sum_{XY = A} (P_{1|Y, B|[X, C, D, E, F} - P_{1|X, B|[Y, C, D, E, F}) + \sum_{XY = B} (P_{1|Y, A|[X, C, D, E, F} - P_{1|X, A|[Y, C, D, E, F}). \]

(9.18)

Given the appearance of two different momentum contractions \( k^m_{A} P^m_{i|B|[A, ...,} \) and \( k^m_{B} P^m_{i|A|[B, ...,} \), (9.18) can be viewed as a weaker result in comparison to the relations in section 9.2 for a single \( k^m_{A} C_{i|A, ...,}^{p,m} \).

Recall that tensorial superfields \( D^{m,...}_{i[A|B,...} \) give rise to additional terms \( \sim k^m, \delta^{mn} \) absent in the scalar case, see (9.11), (9.13) and (9.14). The same kind of contributions appear in the vector and tensor generalization of (9.18), e.g.

\[ QD_{1[2,3|4,5,6}, 8 = \Delta_{1[2|3,4,5,6}, 8 + \Delta_{1|2|3,4,5,6}, 8 + k^m_{3} P^m_{1[2|3,4,5,6}, 8 + k^m_{2} P^m_{1|2|3,4,5,6}, 8 + [s_{34} P^m_{1[2|3,4,5,6}, 8 + s_{24} P^m_{1|2|3,4,5,6}, 8 + (4 \leftrightarrow 5,..., 8)]\]

(9.19)

\[ QD_{1[2,3|4,5,6}, 9 = \Delta_{1[2|3,4,5,6}, 9 + \Delta_{1|2|3,4,5,6}, 9 + \delta^{mn} \Delta_{1[2,3|4,5,6}, 9 + k^m_{3} P^m_{1|2|3,4,5,6}, 9 + [s_{34} P^m_{1[2|3,4,5,6}, 9 + s_{24} P^m_{1|2|3,4,5,6}, 9 + (4 \leftrightarrow 5,..., 9)]\]

+ 2(k^m_{12345678} - k^m_{23} P^m_{1[2,3|4,...,9}.

(9.20)
This allows to anticipate the multiparticle version at general rank,

\[ QD_{i|[A,B|C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} = \Delta_{i|[A,B|C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} + \Delta_{i|[B,A|C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} + \left( \begin{array}{c} r \\ 2 \end{array} \right) \delta_{i|[A,B|C_1,\ldots,C_{r+4}}^{m_1 m_2} \Delta_{i|[A,B|C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} \right. \]

\[ + \left[ P_{i|[A|S|B,C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} + P_{i|[B|S|A,C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} + (C_1 \leftrightarrow C_2, \ldots, C_{r+4}) \right] \]

\[ + r(k^{m_1}_{i A B C_1,\ldots,C_{r+4}} - k^{m_1}_{A B}) P_{i|[A,B|C_1,\ldots,C_{r+4}}^{m_2\ldots m_r} \]

\[ + k^p_{A} P_{i|[A,B|C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} + k^p_{B} P_{i|[A,B|C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} \]

\[ - \sum_{XY=A} (P_{i|[X,B|Y,C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} - P_{i|[Y,B|X,C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} \]

\[ - \sum_{XY=B} (P_{i|[A,X,Y,C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} - P_{i|[A,Y,X,C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} \right) , \] (9.21)

where the deconcatenation terms \( \sim \sum_{XY=A,B} \) follow by analogy with (9.18). In comparison to the counterpart (9.14) of lower refinement, terms of the form \( \Delta_{i|[A,B,C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} \), \( r k^{m_1}_{i A B C_1,\ldots,C_{r+4}} \), \( k^p_{A} P_{i|[A,B|C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} \), \( P_{i|[A,B|C_1,\ldots,C_{r+4}}^{m_1\ldots m_r} \), and \( \sum_{XY=A \ldots} \) are doubled in (9.21). This suggests the following BRST variation for \( D \) superfields at general refinement,

\[ QD_{i|[A_1,\ldots,A_d|B_1,\ldots,B_{r+d+4}}^{m_1\ldots m_r} = \left[ \Delta_{i|[A_2,\ldots,A_d|A_1,B_1,\ldots,B_{r+d+4}}^{m_1\ldots m_r} + (A_1 \leftrightarrow A_2, \ldots, A_d) \right] \]

\[ + \left( \begin{array}{c} r \\ 2 \end{array} \right) \delta_{i|[A_2,\ldots,A_d|A_1,B_1,\ldots,B_{r+d+4}}^{m_1 m_2} \Delta_{i|[A_2,\ldots,A_d|A_1,B_1,\ldots,B_{r+d+4}}^{m_1\ldots m_r} \right. \]

\[ + r k^{m_1}_{A_1 B_1 B_2 \ldots B_{r+d+2}} P_{i|[A_1,\ldots,A_d|B_1,\ldots,B_{r+d+2}}^{m_2\ldots m_r} \]

\[ + \left( k^p_{A_1} P_{i|[A_2,\ldots,A_d|A_1,B_1,\ldots,B_{r+d+2}}^{m_1\ldots m_r} + \right. \]

\[ - \sum_{XY=A_1} (P_{i|[X,B_2,\ldots,B_{r+d+2}]}^{m_1\ldots m_r} - P_{i|[Y,B_2,\ldots,B_{r+d+2}]}^{m_1\ldots m_r} + (A_1 \leftrightarrow A_2, \ldots, A_d) \] (9.22)

Again, we can directly infer the BRST variation of the anomalous counterparts \( \Delta_{i|[A_1,\ldots,A_d|B_1,\ldots,B_{r+d+4}}^{m_1\ldots m_r} \) by discarding their appearance in the right-hand side of (9.22) and replacing the remaining terms via \( C_{i|[\ldots]}^{m_1\ldots m_r} \to \Gamma_{i|[\ldots]}^{m_1\ldots m_r} \)

\[ Q\Delta_{i|[A_1,\ldots,A_d|B_1,\ldots,B_{r+d+4}}^{m_1\ldots m_r} = r k^{m_1}_{A_1 B_1 B_2 \ldots B_{r+d+4}} \Gamma_{i|[A_1,\ldots,A_d|B_1,\ldots,B_{r+d+4}}^{m_2\ldots m_r} \]

\[ + \left( k^p_{A_1} \Gamma_{i|[A_2,\ldots,A_d|A_1,B_1,\ldots,B_{r+d+4}}^{m_1\ldots m_r} + \right. \]

\[ - \sum_{XY=A_1} (\Gamma_{i|[X,A_2,\ldots,A_d|Y,B_1,\ldots,B_{r+d+4}}^{m_1\ldots m_r} - \Gamma_{i|[Y,A_2,\ldots,A_d|X,B_1,\ldots,B_{r+d+4}}^{m_1\ldots m_r} + (A_1 \leftrightarrow A_2, \ldots, A_d) \] (9.23)

Using (9.23) and the \( Q \) variation (7.17) of the pseudoinvariants, one can verify BRST closure of the right-hand side of (9.22). This is a strong consistency check since it requires every single term in (9.22) to conspire.
10. Pseudoinvariant relations for $k_i$ momentum contractions

In the previous section, $D$ superfields defined in (8.15) were shown to generate relations for $k_{B_j}$ contractions of pseudoinvariants. We shall now investigate the second family of ghost-number-two objects, the $L$ superfields defined in (8.17). It turns out that their BRST variations relate contractions of $P_{i|\ldots}^{m_1\ldots}$ with the momentum $k_i$ of the reference leg $i$ to pseudoinvariants of lower rank.

10.1. $k_i$ contractions of unrefined pseudoinvariants

This section is devoted to the unrefined superfields $L_{i|A_1,\ldots,A_{\nu+4}}^{m_1\ldots m_{\nu}}$, see (8.17). The BRST variations of the simplest scalars are given by

$$QL_{1|2,3,4,5} = \Delta_{1|2,3,4,5} + k_{1}^{m} C_{1|2,3,4,5}$$

$$QL_{1|23,4,5,6} = \Delta_{1|23,4,5,6} + k_{1}^{m} C_{1|23,4,5,6} + P_{1|2|3,4,5,6} - P_{1|3|2,4,5,6}$$

$$QL_{1|234,5,6,7} = \Delta_{1|234,5,6,7} + k_{1}^{m} C_{1|234,5,6,7}$$

$$+ P_{1|2|34,5,6,7} + P_{1|2|34,5,6,7} - P_{1|34|2,5,6,7} - P_{1|4|23,5,6,7}$$

$$QL_{1|2345,6,7} = \Delta_{1|2345,6,7} + k_{1}^{m} C_{1|2345,6,7}$$

$$+ P_{1|2|345,6,7} - P_{1|32,45,6,7} + P_{1|4|23,5,6,7} - P_{1|5|23,4,6,7},$$

and thereby provide relations for $k_{i}^{m} C_{i|A,B,C,D}^{m}$. The explicit form of $L_{1|2,3,4,5}$ and $L_{1|23,4,5,6}$ can be found in (8.18), and the former underpins the first example in (8.2) provided that momentum conservation $k_{12345}^{m} = 0$ renders $\Delta_{1|2,3,4,5}$ BRST exact.

The examples in (10.1) suggest the multiparticle pattern,

$$QL_{i|A,B,C,D} = \Delta_{i|A,B,C,D} + k_{i}^{m} C_{i|A,B,C,D}^{m}$$

$$+ \left[ \sum_{XY=A} (P_{i|X|Y,B,C,D} - P_{i|Y|X,B,C,D}) + (A \leftrightarrow B, C, D) \right],$$

where the anomalous $\Delta_{i|A,B,C,D}$ are defined by (8.19) and also appear in the relations (9.9) for different contractions $k_{A}^{m} C_{i|A,B,C,D}^{m}$.

The simplest $Q$ variations of ghost number two vectors $L_{i|A,B,C,D,E}^{m}$ read

$$QL_{1|2,3,4,5,6} = \Delta_{1|2,3,4,5,6} + k_{1}^{m} C_{1|2,3,4,5,6}^{m} + \left[ k_{2}^{m} P_{1|2|3,4,5,6} + (2 \leftrightarrow 3, 4, 5, 6) \right]$$

$$QL_{1|23,4,5,6,7} = \Delta_{1|23,4,5,6,7} + k_{1}^{m} C_{1|23,4,5,6,7}^{m} + \left[ k_{4}^{m} P_{1|4|23,5,6,7} + (4 \leftrightarrow 5, 6, 7) \right]$$

$$+ k_{23}^{m} P_{1|23|4,5,6,7} + P_{1|2|3,4,5,6,7} - P_{1|3|2,4,5,6,7},$$
see (8.18) for the expansion of $L_{i|2,3,4,5,6}^{m}$. The novel class of terms $\sim k^{n}$ in (10.3) and (10.4) are reproduced by the general formula,

$$QL_{i|A,B,C,D,E}^{m} = \Delta_{i|A,B,C,D,E}^{m} + k^{n}C_{i|A,B,C,D,E}^{m} + \left[k^{m}P_{i|A|B,C,D,E} + \sum_{XY=A} \left(P_{i|X,Y,B,C,D,E}^{m} - P_{i|Y,X,B,C,D,E}^{m}\right) + (A \leftrightarrow B,C,D,E)\right].$$  (10.5)

As the last explicit example in this section, consider the two-tensor relation,

$$QL_{i|2,3,4,5,6,7}^{mn} = \Delta_{i|2,3,4,5,6,7}^{mn} + \delta_{i|2,3,4,5,6,7}^{mn} + k^{p}C_{i|2,3,4,5,6,7}^{mnp}$$

$$+ 2\left[k^{(m}P_{i|2,3,4,5,6,7}^{n)} + (2 \leftrightarrow 3, 4, 5, 6, 7)\right],$$  (10.6)

subject to an anomalous trace with $\Lambda_{i|2,3,4,5,6,7}$ given by (8.25). Its generalization $\Lambda_{i|A_{1}, A_{2}, \ldots, A_{r+6}}^{m_{1} \ldots m_{r}}$ to multiparticle slots and higher rank is defined in (8.24) and finds appearance in the general rank-two relation,

$$QL_{i|A,B,C,D,E,F}^{mn} = \Delta_{i|A,B,C,D,E,F}^{mn} + \delta_{i|A,B,C,D,E,F}^{mn} + k^{p}C_{i|A,B,C,D,E,F}^{mnp}$$

$$+ \left[\sum_{XY=A} \left(P_{i|X,Y,B,C,D,E,F}^{mn} - P_{i|Y,X,B,C,D,E,F}^{mn}\right) + 2k^{(m}P_{i|A|B,C,D,E,F}^{n)} + (A \leftrightarrow B, \ldots, F)\right].$$  (10.7)

The expressions (10.2), (10.5) and (10.7) for $QL_{i|A_{1}, A_{2}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$ at rank $r = 0, 1, 2$ lead to a natural generalization for higher ranks,

$$QL_{i|A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}} = \Delta_{i|A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}} + \binom{r}{2}\delta_{i|A_{1}, \ldots, A_{r+4}}^{m_{1}m_{2}}\Lambda_{i|A_{1}, \ldots, A_{r+4}}^{m_{3} \ldots m_{r}} + k^{p}C_{i|A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$$

$$+ \left[\sum_{XY=A_{1}} \left(P_{i|X,Y,A_{2}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}} - P_{i|Y,X,A_{2}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}\right) + \sum_{A_{1}} k^{(m_{1}}P_{i|A|A_{2}, \ldots, A_{r+4}}^{m_{2} \ldots m_{r})} + (A_{1} \leftrightarrow A_{2}, \ldots, A_{r+4})\right].$$  (10.8)

Similar to (9.14) for $k^{p}C_{i|A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$ as derived from $QD_{i|\ldots}^{m_{1} \ldots}$, two classes of anomalous terms appear in (10.8). The unrefined $\Delta_{i|A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$ common to both relations are BRST exact under momentum conservation, see (9.6), and can be discarded in the context of amplitudes. However, the second anomalous term $\Lambda_{i|A_{1}, \ldots, A_{r+6}}^{m_{1} \ldots m_{r}}$ in the trace of (10.8) is not even BRST closed, regardless of the momentum phase space. Its non-vanishing Q variation follows as the anomaly analogue of (10.2) and (10.8), i.e. by dropping the anomalous...
contributions from the latter and replacing the rest according to $C_{i|...}^{m_1...} \rightarrow P_{i|...}^{m_1...} \rightarrow \Gamma_{i|...}^{m_1...}$:

$$QL_{i[A,B,C,D,E,F} = k_i^m \Gamma_{i]A,B,C,D,E,F}$$

$$+ \left[ \sum_{XY = A} (\Gamma_{i]X|Y,B,C,D,E,F} - \Gamma_{i]Y|X,B,C,D,E,F} + (A \leftrightarrow B, C, D, E, F) \right]$$

$$QL_{i|A_1,A_2,...,A_{r+6}}^{m_{1...m_r}} = k_i^m \Gamma_{i|A_1,...,A_{r+6}}^{m_{1...m_r}}$$

$$+ \left[ r k_i^m \Gamma_{i]A_1,...,A_{r+6}}^{m_{1...m_r}} + \sum_{XY = A} (\Gamma_{i]X|A_2,...,A_{r+6}}^{m_{1...m_r}} - \Gamma_{i]Y|A_2,...,A_{r+6}}^{m_{1...m_r}} + (A_1 \leftrightarrow A_2, \ldots, A_{r+6}) \right]$$

Together with the BRST transformations (7.9) and (7.17) of pseudoinvariants, (10.9) and (10.10) allow to check BRST closure of (10.8) and furnish a strong consistency check on the results in this section.

### 10.2. $k_i$ contractions of refined pseudoinvariants

We next proceed to refined versions $L_{i|A_1,...,A_d|B_1,...}^{m_1...}$ of the ghost-number-two objects under discussion. In the simplest scalar cases,

$$QL_{i|23|3,4,5,6,7} = \Delta_{i|23|3,4,5,6,7} + \Lambda_{i|23,3,4,5,6,7} + k_i^m P_{i|23|3,4,5,6,7}$$

$$+ \left[ s_{23} P_{i|23|3,4,5,6,7} + (3 \leftrightarrow 4, 5, 6, 7) \right]$$

$$QL_{i|23|4,5,6,7,8} = \Delta_{i|23|4,5,6,7,8} + \Lambda_{i|23,4,5,6,7,8} + k_i^m P_{i|23|4,5,6,7,8}$$

$$+ \left[ s_{34} P_{i|23|4,5,6,7,8} - s_{24} P_{i|32|4,5,6,7,8} + (4 \leftrightarrow 5, 6, 7, 8) \right]$$

$$QL_{i|4|23,5,6,7,8} = \Delta_{i|4|23,5,6,7,8} + \Lambda_{i|23,4,5,6,7,8} + k_i^m P_{i|4|23,5,6,7,8}$$

$$+ s_{24} P_{i|32|4,5,6,7,8} - s_{34} P_{i|23|4,5,6,7,8} + \left[ s_{45} P_{i|45|23,6,7,8} + (5 \leftrightarrow 6, 7, 8) \right]$$

$$+ P_{i|24,3,5,6,7,8} - P_{i|34,2,5,6,7,8}$$

where the expansion of $L_{i|23|3,4,5,6,7}$ is given in (8.18). This generalizes to

$$QL_{i[A,B,C,D,E,F} = \Delta_{i|A|B,C,D,E,F} + \Lambda_{i|A,B,C,D,E,F} + k_i^m P_{i|A|B,C,D,E,F}$$

$$+ \left[ P_{i|S|A,B|C,D,E,F} + \sum_{XY = B} (P_{i|A,X,Y,C,D,E,F} - P_{i|A,Y,X,C,D,E,F}) + (B \leftrightarrow C, D, E, F) \right]$$

Interestingly, (10.12) contains a contraction of $P_{i|...}^m$ with the combined momentum $k_i^m = k_i^m + k_A^m$ including the refined slot $A$ and does not isolate its $k_i$ contraction. This is analogous to the shortcoming of the relation (9.18) to address the two-term combinations $k_A^m P_{i|B|A,C,D,E,F} + k_B^m P_{i|A|B,C,D,E,F}$ instead of the individual terms.
Vectorial and tensorial BRST variations exhibit novel terms $\sim k^m, \delta^{mn}$ which are absent for the scalars (10.12), e.g.

$$QL_{1[2]3,...,8}^m = \Lambda_{1[2]3,...,8}^m + \Lambda_{1[2]3,...,8}^m + k_{12}^m P_{1[2]3,...,8}^m$$
$$\quad + \left[ s_{23} P_{1[2]3,...,8}^m + k_{3}^n P_{1[2]3,...,8}^m + (3 \leftrightarrow 4, \ldots, 9) \right], \quad (10.13)$$

$$QL_{1[2]3,...,9}^{mn} = \Lambda_{1[2]3,...,9}^{mn} + \Lambda_{1[2]3,...,9}^{mn} + \delta^{mn} \Lambda_{1[2]3,...,9}^{mn} + k_{12}^m P_{1[2]3,...,9}^{pmn}$$
$$\quad + \left[ s_{23} P_{1[2]3,...,9}^{mn} + 2k_{3}^n P_{1[2]3,...,9}^{m} + (3 \leftrightarrow 4, \ldots, 9) \right]. \quad (10.14)$$

Experience with the scalar counterpart (10.12) suggests that only the non-refined multiparticle slots $B_j$ give rise to deconcatenation terms. This leads to the following all-rank generalization:

$$QL_{i[A]B_1,...,B_{r+5}}^{m_1...m_r} = \Lambda_{i[A]B_1,...,B_{r+5}}^{m_1...m_r} + \Lambda_{i[A]B_1,...,B_{r+5}}^{m_1...m_r} + \left( \frac{r}{2} \right) \delta^{(m_1 m_2)} \Lambda_{i[A]B_1,...,B_{r+5}}^{m_3...m_r}$$
$$\quad + k_{iA}^m P_{i[i|A][B_1,...,B_{r+5}]}^{pm_1...m_r} + \left[ P_{i|i[S[A,B_1]]B_2,...,B_{r+5}}^{m_1...m_r} + r k_{iB_1}^m P_{i[A,B_i][B_2,...,B_{r+5}]}^{m_2...m_r} \right.$$  
$$\quad + \sum_{XY=B_1} \left( P_{i[A,Y|X,B_2,...,B_{r+5}]}^{m_1...m_r} - P_{i[A,Y|X,B_2,...,B_{r+5}]}^{m_1...m_r} \right) + (B_1 \leftrightarrow B_2, \ldots, B_{r+5}) \right] \quad (10.15)$$

For the extension of (10.15) to $L$ superfields of higher refinement $d$, one can expect that three classes of terms $\Lambda_{i[A,B_1,...,B_{r+5}]}^{m_1...m_r} k_{iA}^m P_{i|i[A][B_1,...,B_{r+5}]}^{pm_1...m_r}$ and $P_{i|i[S[A,B_1]]B_2,...,B_{r+5}}^{m_1...m_r}$ have to be symmetrized in $A_1 \leftrightarrow A_2, \ldots, A_d$. We therefore propose the following expression for the most general case:

$$QL_{i[A_1,...,A_d][B_1,...,B_{r+d+4}]}^{m_1...m_r} = \Lambda_{i[A_1,...,A_d][B_1,...,B_{r+d+4}]}^{m_1...m_r} + \left( \frac{r}{2} \right) \delta^{(m_1 m_2)} \Lambda_{i[A_1,...,A_d][B_1,...,B_{r+d+4}]}^{m_3...m_r}$$
$$\quad + \left[ \Lambda_{i[A_1,...,A_d][A_1,B_1,...,B_{r+d+4}]}^{m_1...m_r} + (A_1 \leftrightarrow A_2, \ldots, A_d) \right] + k_{iA_1 A_2,...,A_d}^m P_{i[i[A_1,...,A_d][B_1,...,B_{r+d+4}]}^{pm_1...m_r}$$
$$\quad + \left[ P_{i[A,...,A_d,S[A_1,B_1]]B_2,...,B_{r+d+4}]}^{m_1...m_r} + (A_1 \leftrightarrow A_2, \ldots, A_d) \right] + r k_{iB_1}^m P_{i[A_1,...,A_d][B_1,...,B_{r+d+4}]}^{m_2...m_r}$$
$$\quad + \sum_{XY=B_1} \left( P_{i[A_1,...,A_d,Y|X,B_2,...,B_{r+d+4}]}^{m_1...m_r} - P_{i[A_1,...,A_d,Y|X,B_2,...,B_{r+d+4}]}^{m_1...m_r} \right) + (B_1 \leftrightarrow B_2, \ldots, B_{r+d+4}) \right]. \quad (10.16)$$

Similar to the correspondence between (10.8) and (10.10), one can infer the BRST variation of $\Lambda_{i[...]}^{m_1...m_r}$ by trading the constituents of (10.16) for their anomaly counterparts,

$$QA_{i[...]}^{m_1...m_r} = k_{iA_1 A_2,...,A_d}^m \Gamma^{pm_1...m_r}_{i[A_1,...,A_d][B_1,...,B_{r+d+6}]}$$
$$\quad + \left[ \gamma_{i[A_1,...,A_d,S[A_1,B_1]]B_2,...,B_{r+d+6}]}^{m_1...m_r} + (A_1 \leftrightarrow A_2, \ldots, A_d) \right] + r k_{iB_1}^m \Gamma^{m_2...m_r}_{i[A_1,...,A_d,B_1][B_2,...,B_{r+d+6}]}$$
$$\quad + \sum_{XY=B_1} \left( \gamma_{i[A,...,A_d,Y|X,B_2,...,B_{r+d+6}]}^{m_1...m_r} - \gamma_{i[A,...,A_d,Y|X,B_2,...,B_{r+d+6}]}^{m_1...m_r} \right) + (B_1 \leftrightarrow B_2, \ldots, B_{r+d+6}) \right]. \quad (10.17)$$
It turns out that this relation is not pseudoinvariants \( P \) split amplitudes. Hence, it is desirable to identify relations among these anomalous admixtures, whereas refined \( \Delta \) superfields and any \( \Lambda \) superfield cannot be discarded in scattering amplitudes. Hence, it is desirable to identify relations among these anomalous admixtures.

10.3. Trace relations and anomaly bookkeeping

The purpose of the above BRST-exact relations is to express momentum contractions of pseudoinvariants in terms of simpler pseudoinvariants at lower rank. However, two classes of obstructions arose, set by anomalous superfields \( \Delta \) and \( \Lambda \). Only the unrefined special case \( \Delta_{i|A_1,\ldots,A_r}^{m_1\ldots m_r} \) was shown to be BRST trivial under momentum conservation, see (9.6), whereas refined \( \Delta \) superfields and any \( \Lambda \) superfield cannot be discarded in scattering amplitudes. Hence, it is desirable to identify relations among these anomalous admixtures.

In particular, one might wonder if the trace relations (6.30) and (7.20) found among pseudoinvariants \( P_{i|\ldots}^{m_1\ldots} \) and anomaly invariants \( \Gamma_{i|\ldots}^{m_1\ldots} \) carry over to their counterparts \( D, L \) and \( \Delta, \Lambda \) at different ghost-number. Even though they all originate from the same master recursion (8.4), there are subtleties under the slot rearrangements \( M_AU_{\{B_j\}}\{\{C_j\} \to U_{\{B_j\}}\{\{C_j\} \) and \( M_AU_{\{B_j\}}\{\{C_j\} \to U_{\{B_j\}}\{\{C_j\} \) entering the definitions (8.11) to (8.14).

Since the formal symbols \( U_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+N}} \) are eventually identified with either \( J \) or \( Y \), it is safe to impose their trace relations (6.21) and (7.19) on the \( U \).

\[
\delta_{np}U_{B_1,\ldots,B_d|C_1,\ldots,C_{d+r+N}}^{npm_1\ldots m_{r-2}} = 2U_{B_1,\ldots,B_d,C_1|C_2,\ldots,C_{d+r+N}}^{m_1\ldots m_{r-2}} + (C_1 \leftrightarrow C_2, \ldots, C_{d+r+N}). \tag{10.18}
\]

It turns out that this relation is not preserved under the slot rearrangement \( M_AU_{\{B_j\}}\{\{C_j\} \to U_{\{B_j\}}\{\{C_j\} \) relevant for \( D \) and \( \Delta \) (followed by multiplication with \( M_A \)) since

\[
\delta_{np}U_{B_1,\ldots,B_d|A,C_1,\ldots,C_{d+r+N}}^{npm_1\ldots m_{r-1}} = 2U_{B_1,\ldots,B_d,C_1\ldots,C_{d+r+N}}^{m_1\ldots m_{r-1}} + (C_1 \leftrightarrow C_2, \ldots, C_{d+r+N}). \tag{10.19}
\]

The missing term to restore (10.18) is easily seen to be \( 2U_{B_1,\ldots,B_d,A|C_1,\ldots,C_{d+r+N}}^{m_1\ldots m_{r-1}} \) which in turn originates from the alternative slot rearrangement \( M_AU_{\{B_j\}}\{\{C_j\} \to U_{A,\{B_j\}}\{\{C_j\} \). Since this is the defining map for \( L \) and \( \Lambda \), we are led to the following trace relation:

\[
\delta_{np}D_{i|\ldots|B_1,\ldots,B_{d+r+4}}^{npm_1\ldots m_r} = 2\delta_{i|\ldots|B_1,\ldots,B_{d+r+4}}^{m_1\ldots m_r} + 2\delta_{i|\ldots|B_1,\ldots,B_{d+r+4}}^{m_1\ldots m_r} \tag{10.20}
\]

\[+ 2\delta_{i|\ldots|B_1,\ldots,B_{d+r+4}}^{m_1\ldots m_r} \] .
The mixing between $L$ and $D$ superfields propagates to their anomalous counterparts $(D, L) \to (\Delta, \Lambda)$:

$$
\delta_{np} \Delta_{i[A_1, \ldots, A_d]}^{npm_1 \ldots m_r} = 2 \Lambda_{i[A_1, \ldots, A_d]}^{m_1 \ldots m_r} + 2 \left[ \Delta_{i[A_1, \ldots, A_d]B_1B_2 \ldots B_{d+r+6}}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+6}) \right].
$$

(10.21)

Note that (10.21) and the BRST variation of (10.20) are consistent with the expressions (9.22) and (10.16) for $Q D_{i[\ldots}^{m_1 \ldots}$ and $Q L_{i[\ldots}^{m_1 \ldots}$.

The slot rearrangement $M_{A} U_{\{B_j\}} \{C_j\} \to U_{\{A\}} \{B_j\} \{C_j\}$ entering the definition of $L$ and $\Lambda$ preserves the trace relations (10.18) and bypasses the subtlety in (10.19). Hence, traces of $L$ and $\Lambda$ fall into the same pattern found for $P_{i[\ldots}^{m_1 \ldots}$ and $\Gamma_{i[\ldots}^{m_1 \ldots}$ in (6.30) and (7.20),

$$
\delta_{np} L_{i[A_1, \ldots, A_d]}^{npm_1 \ldots m_r} = 2 \left[ L_{i[A_1, \ldots, A_d]B_1B_2 \ldots B_{d+r+6}}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+6}) \right]
$$

(10.22)

$$
\delta_{np} \Lambda_{i[A_1, \ldots, A_d]}^{npm_1 \ldots m_r} = 2 \left[ \Lambda_{i[A_1, \ldots, A_d]B_1B_2 \ldots B_{d+r+8}}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{d+r+8}) \right].
$$

(10.23)

Again, one can verify consistency of (10.23) and the BRST variation of (10.22) by means of the expression (10.16) for $Q L_{i[\ldots}^{m_1 \ldots}$.

As a main benefit of this discussion, the trace relations (10.21) and (10.23) are useful in manipulating anomalous ghost-number-three contributions to scattering amplitudes. In particular, we can take advantage of the decoupling of unrefined objects $\Delta_{i[A_1, \ldots, A_{r+4}]i}^{m_1 \ldots m_r}$ as well as its traces and discard the following right-hand sides:

$$
\frac{1}{2} \delta_{np} \Delta_{i[B_1, \ldots, B_{r+6}]}^{npm_1 \ldots m_r} = \Lambda_{i[B_1, \ldots, B_{r+6}]}^{m_1 \ldots m_r} + \left[ \Delta_{i[B_1, B_2, \ldots, B_{r+6}]}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{r+6}) \right]
$$

$$
\frac{1}{4} \delta_{npqr} \Delta_{i[B_1, \ldots, B_{r+8}]}^{npm_1 \ldots m_r} = \left[ \Delta_{i[B_1, B_2, \ldots, B_{r+8}]}^{m_1 \ldots m_r} + (B_1 \leftrightarrow B_2, \ldots, B_{r+8}) \right] + \left[ \Delta_{i[B_1, B_2, B_3, \ldots, B_{r+8}]}^{m_1 \ldots m_r} + (B_1, B_2 | B_1, B_2, \ldots, B_{r+8}) \right].
$$

(10.24)

Generalizations to multitraces of $\Delta_{i[A_1, \ldots, A_{r+4}]i}^{m_1 \ldots m_r}$ are straightforward.
10.4. The web of relations between ghost-number two and four

In the last sections we have constructed a variety of superfields and derived a rich set of relations among them. The recursions which led to pseudoinvariants $P$ in section 6 and to anomaly invariants $\Gamma$ in section 7 were unified to a master recursion (8.4) in section 8. As detailed in sections 8.2 to 8.5, the master recursion points towards four further replicae $D$, $L$, $\Delta$ and $\Lambda$ of the family of $P$ and $\Gamma$ which can be visualized in grids similar to fig. 1 and fig. 6.

The BRST-exact relations presented in this section and section 9 mediate between the six families of superfields as visualized in fig. 7. Since BRST action increases the ghost number by one, we arrange the superfields according to their ghost number. As a second coordinate for the roadmap of superfields, we take the number $N$ of multiparticle slots in the scalar and unrefined representatives, see section 8.1.

In fig. 7, the BRST variations of all the six families are represented by solid lines, the arrows pointing towards the image of higher ghost number. The underlying expressions are given in

- (9.22), (10.16) and (7.17) for BRST action on the non-anomalous fields $D, L$ and $P$
- (9.23) and (10.17) for their anomalous counterparts $\Delta$ and $\Lambda$

whereas $Q\Gamma = 0$. Horizontal dashed arrows additionally remind of the mixing of $D \leftrightarrow L$ as well as $\Delta \leftrightarrow \Lambda$ under the trace relations (10.20) and (10.21).
11. Canonicalizing pseudoinvariants

In the previous sections, we have systematically derived two families of relations among pseudoinvariants \( P_{i_1 \ldots i}^{m_1 \ldots} \) at fixed reference leg \( i \). This section is devoted to a different class of relations which mediates between different choices of the reference particle \( i \). It will be demonstrated that any \( P_{k_1 \ldots}^{m_1 \ldots} \) with \( k \neq i \) can be expressed in terms of \( P_{i_1 \ldots}^{m_1 \ldots} \), up to a constrained set of anomaly terms. This rearrangement will be referred to as canonicalization.

The anomalous admixtures in the canonicalization process reflect the property of the hexagon gauge anomaly in both field and string theory to break the permutation symmetry of one-loop amplitudes at multiplicity \( n \geq 6 \). Apart from this anomaly subtlety, however, we learn that the set of pseudoinvariants \( P_{i_1 \ldots i_r}^{m_1 \ldots m_r} \) at fixed reference leg \( i \) spans the same space of kinematic factors as would be obtained by any other choice of reference leg \( k \neq i \). This is a necessary condition for the independence of string amplitudes on the choice of the unintegrated vertex \( i \).

The methods parallel the procedure in section 9 and 10, i.e. we identify suitable BRST generators at ghost-number two to derive relations among ghost-number-three objects. The \( Q \) generators to trade different choices of the reference leg \( i \) in \( P_{i_1 \ldots}^{m_1 \ldots} \) turn out to be minor modifications of the ghost-number-two superfields of types \( D \) and \( L \), see section 8.

We start by discussing the canonicalization of scalar invariants in detail. This serves as a motivation of certain operations which capture the structure of the canonicalization process and easily carry over to more general pseudoinvariants.

11.1. Canonicalizing scalar invariants

As pointed out in [24] through a few examples at low multiplicity, any scalar invariant \( C_{k|A,B,C} \) as given in (2.40) can be cast into a basis of \( C_{i|D,E,F} \) with \( i \neq k \). We shall now present the general solution for arbitrary \( A, B, C \) and thereby develop the maps and notation to extend the procedure to higher tensors and to refined pseudoinvariants.

The trial and error method in [24] to canonicalize \( C_{2|A,B,C} \) towards \( C_{1|D,E,F} \) leads to the following expressions at multiplicity \( n \leq 6 \) (suppressing the laborious case \( C_{2|34,56,1} \)):

\[
\begin{align*}
C_{2|1,3,4} &= -Q(M_{12,3,4}) + C_{1|2,3,4} \\
C_{2|13,4,5} &= -Q(M_{132,4,5}) + C_{1|32,4,5} \\
C_{2|1,34,5} &= -Q(M_{12,34,5} + M_{123,4,5} - M_{124,3,5}) + C_{1|2,34,5} + C_{1|23,4,5} - C_{1|24,3,5} \\
C_{2|34,5,6} &= -Q(M_{1342,5,6}) + C_{1|342,5,6}
\end{align*}
\]
\[ C_{2|13,45,6} = -Q(M_{132,45,6} + M_{1324,5,6} - M_{1325,4,6}) + C_{1|32,45,6} + C_{1|324,5,6} - C_{1|325,4,6} \quad (11.5) \]
\[ C_{2|1,345,6} = -Q(M_{12,345,6} + M_{1234,5,6} + M_{1254,3,6} - M_{1235,4,6} - M_{12345,6} + M_{123456} + M_{12543,6}) + C_{1|2,345,6} + C_{1|234,5,6} + C_{1|254,3,6} - C_{1|235,4,6} - C_{1|2345,6} + C_{1|23456} + C_{1|2543,6} \quad (11.6) \]

These examples can be easily verified using the form \( QM_{A,B,C} \). The following three observations on (11.1) to (11.6) guide the way towards a general solution for both the BRST ancestor and the set of \( C_{1|D,E,F} \) which appear in the canonicalization of \( C_{2|...} \):

(i) Each term of the form \(-QM_{1,D,E,F}\) in the BRST generator is accompanied by a corresponding invariant \( C_{1|D,E,F} \). By assuming this pattern to hold in general, knowledge of the BRST generator already determines the canonicalization in terms of \( C_{1|D,E,F} \).

(ii) Suppose particle 1 appears in the left hand side as \( C_{2|1,A,B,C} \) (where \( A \) can be empty), then each term of the BRST generator is of the form \( M_{1AD,E,F} \). In other words, the entire slot 1A of the desired reference leg is concatenated with a pattern of \( M_{...D,E,F} \).

(iii) This pattern of \( M_{...D,E,F} \) must contain the information on the remaining labels 2, B, C. The superfield \( D_{2|B,C} \) as defined in (8.15) is the natural object to do so, and indeed, its concatenation (2.37) with \( M_{1A} \) reproduces the above BRST generators, e.g.

\[ M_1 \otimes D_{2|34,5} = M_{123,4,5} + M_{1234,5,6} - M_{1243,5} \quad (11.7) \]
\[ M_{13} \otimes D_{2|45,6} = M_{132,45,6} + M_{1324,5,6} - M_{1325,4,6} \quad (11.8) \]
\[ M_1 \otimes D_{2|345,6} = M_{1234,5,6} + M_{12345,6} + M_{1254,3,6} - M_{12354,6} - M_{1253,4,6} + M_{12345,6} + M_{12543,6} \quad (11.9) \]

for (11.3), (11.5) and (11.6), respectively. The \( D \) superfields in the first two cases are given in (8.16) whereas \( D_{2|345,6} \) can be inferred from \( C_{1|234,5,6} \) in (A.1).

So one can promote the sample canonicalizations of \( C_{2|1,A,B,C} \) in (11.1) to (11.6) to the following general formula\(^{20}\),

\[ C_{k|iA,B,C} = (\varphi_i - Q)(M_{iA} \otimes D_{k|B,C}) \ . \quad (11.10) \]

\(^{20}\) To support the plausibility of the canonicalization prescription in (11.10), note that any term in the concatenation product \( M_{iA} \otimes D_{k|B,C} \) takes the form \( M_{iA,...,D,E} \). BRST action then generates one term \( C_{i|A,...,D,E} \) and up to five others, see (2.35). The map \( \varphi_i \) makes sure that the \( C_{i|A,...,D,E} \) contribution in \(-QM_{iA,...,D,E}\) is compensated. Other invariants of the form \( C_{i|\neq i|F,G,H} \) largely cancel thanks to the fine-tuned arrangement of slots governed by the recursive origin (8.4) of \( D_{k|B,C} \). Only one term \( C_{k|iA,B,C} \) with reference leg \( \neq i \) will emerge from the \( Q \) action on the leading term \( D_{k|B,C} \rightarrow M_{k,B,C} \), as required by the left-hand side of (11.10).
The “pseudoinvariantization” map \( \varphi_i \) in (11.10) is defined to transform \( M_{iA,B,C} \to C_{i|A,B,C} \) according to observation (i) and, more generally,

\[
\begin{align*}
\varphi_i(J_{B_1,...,B_d|A,C_2,...,C_{d+r+3}}^{m_1,...,m_r}) & \equiv P_{i|B_1,...,B_d|A,C_2,...,C_{d+r+3}}^{m_1,...,m_r} \quad (11.11) \\
\varphi_i(J_{A,B_2,...,B_d|C_1,...,C_{d+r+3}}^{m_1,...,m_r}) & \equiv P_{i|A,B_2,...,B_d|C_1,...,C_{d+r+3}}^{m_1,...,m_r}. \quad (11.12)
\end{align*}
\]

Loosely speaking, \( \varphi_i \) in (11.11) and (11.12) removes particle \( i \) from the leading position of a word in \( J_{B_1,...,B_d|C_1,...,C_{d+r+3}}^{m_1,...,m_r} \) and converts it into the reference leg of a pseudoinvariant. Its remaining labels are those of the above current with particle \( i \) removed.

Further applications of (11.10) can be found in appendix F, see (F.1) to (F.6).

11.2. Canonicalizing unrefined pseudoinvariants

In order to canonicalize tensorial pseudoinvariants \( C_{k|A,B_2,...,B_{r+3}}^{m_1,...,m_r} \) to the form \( C_{i|A,B_2,...,B_{r+3}}^{m_1,...,m_r} \), it is tempting to simply replace \( D_{k|B,C} \to D_{k|B_2,...,B_{r+3}} \) in the scalar prescription (11.10). However, \( D \) superfields at rank \( r \geq 2 \) additionally generate anomalous terms such as \( \Delta_{i|A_1,...,A_4}^{m_1,...,m_r} \) in (9.5) and (9.6). In order to have a well-defined notion of \( M_{iA} \otimes \Delta_{k|B_1,...,B_{r+4}}^{m_1,...,m_r} \) and \( M_{iA} \otimes J_{k|B_1,...,B_{r+2}}^{m_1,...,m_r} \) (needed in the subsequent), we extend the concatenation operation to anomaly building blocks and refined currents,

\[
\begin{align*}
M_{iA} \otimes J_{k,B_2,...,B_d|C_1,...,C_{d+r+3}}^{m_1,...,m_r} & \equiv J_{iA,k,B_2,...,B_d|C_1,...,C_{d+r+5}}^{m_1,...,m_r} \quad (11.13) \\
M_{iA} \otimes J_{k,B_1,...,B_d|C,C_2,...,C_{d+r+5}}^{m_1,...,m_r} & \equiv J_{iA,k,B_1,...,B_d|iA,k,C,C_2,...,C_{d+r+5}}^{m_1,...,m_r} \quad (11.14) \\
M_{iA} \otimes J_{k,B_2,...,B_d|C_1,...,C_{d+r+3}}^{m_1,...,m_r} & \equiv J_{iA,k,B_2,...,B_d|C_1,...,C_{d+r+3}}^{m_1,...,m_r} \quad (11.15) \\
M_{iA} \otimes J_{k,B_1,...,B_d|k,C,C_2,...,C_{d+r+3}}^{m_1,...,m_r} & \equiv J_{iA,k,B_1,...,B_d|iA,k,C,C_2,...,C_{d+r+3}}^{m_1,...,m_r}. \quad (11.16)
\end{align*}
\]

As before, the instruction to concatenate the word \( iA \) with \( kB \) and \( kC \) is clear from the reference leg \( k \) of the parental \( D_{k|...}^{m_1,...} \). The anomaly concatenations \( M_{iA} \otimes \Delta_{k|B_1,...,B_{r+4}}^{m_1,...,m_r} \) serve to compensate for the anomalous part of \( Q(M_{iA} \otimes D_{k|...}^{m_1,...}) \):

\[
C_{k|A,B_2,...,B_{r+3}}^{m_1,m_2,...,m_r} = (\varphi_i - Q)(M_{iA} \otimes D_{k|B_2,B_3,...,B_{r+3}}^{m_1,m_2,...,m_r}) + \binom{r}{2} \delta^{m_1 m_2} (M_{iA} \otimes \Delta_{k|B_2,B_3,...,B_{r+3}}^{m_3 m_4,...,m_r}) \quad (11.17)
\]

The cancellation of anomalous superfields on the right-hand side can be understood as follows: The anomalous term in \( QM_{B_1,...,B_{r+3}}^{m_1,...,m_r} = \binom{r}{2} \delta^{m_1 m_2} Y_{B_1,...,B_{r+3}}^{m_3 m_4,...,m_r} + \ldots \) preserves the structure of the slots \( B_k \). In a truncation to anomaly building blocks, BRST action and
the concatenation through \( M_{iA} \otimes \) commute. Since \( \delta^{(m_1 m_2)} \Delta^{m_3 m_4 \ldots m_r}_{k|B_2, B_3, \ldots, B_{r+3}} \) in the second line of (11.17) can be traced back to \( Q D^{m_1 m_2 \ldots m_r}_{k|B_2, B_3, \ldots, B_{r+3}}, \) see (9.6), any anomalous contribution effectively originates from a “commutator” of the operations \( M_{iA} \otimes \) and \( Q \) acting on \( D^{m_1 m_2 \ldots m_r}_{k|B_2, B_3, \ldots, B_{r+3}}. \)

Two simple examples of the general procedure in (11.17) are given by,

\[
C^m_{2|1,3,4,5} = (\varphi_1 - Q) (M_1 \otimes D^m_{2|3,4,5}) \\
= -Q \left( M^m_{12,3,4,5} + \left[ k^m_3 M_{123,4,5} + (3 \leftrightarrow 4, 5) \right] \right) \\
+ C^m_{1|2,3,4,5} + \left[ k^m_3 C_{1|23,4,5} + (3 \leftrightarrow 4, 5) \right] \\
(11.18)
\]

\[
C^{mn}_{2|1,3,4,5,6} = (\varphi_1 - Q) (M_1 \otimes D^{mn}_{2|3,4,5,6}) + \delta^{mn} (M_1 \otimes \Delta_{2|3,4,5,6}) \\
= -Q \left( M^{mn}_{12,3,4,5,6} + \left[ k^m_3 (M_{123,4,5,6} + (3 \leftrightarrow 4, 5, 6)) \right] \\
+ [k^m_3 k^n_4 (M_{1234,5,6} + M_{1243,5,6} + (3, 4|3, 4, 5, 6))] \right) \\
+ \delta^{mn} Y_{12,3,4,5,6} + C^{mn}_{1|2,3,4,5,6} + \left[ k^m_3 C^n_{1|23,4,5,6} + (3 \leftrightarrow 4, 5, 6) \right] \\
+ [k^m_3 k^n_4 (C_{1|234,5,6} + C_{1|243,5,6}) + (3, 4|3, 4, 5, 6)] . \\
(11.19)
\]

Further examples can be found in appendix F, see (F.7).

11.3. Canonicalizing non-refined slots in refined pseudoinvariants

Only minor modifications are required to generalize the above canonicalization rules to refined pseudoinvariants \( P^{m_1 m_2 \ldots m_r}_{k|A_1, \ldots, A_d|B_1, \ldots, B_{d+r+3}} \), as long as the preferred reference leg \( i \) resides in a unrefined slot \( B_j \). By analogy with the first line of (11.17), it is natural to expect a BRST generator of the form \( M_{iA} \otimes D^{m_1 m_2 \ldots m_r}_{k|B_1, \ldots, B_d|C_2, \ldots, C_{d+r+3}} \).

Similar to the unrefined tensors, the BRST variation of refined \( D \) superfields incorporates anomalous \( \Delta \) superfields, see (9.22). In order to address them, recall that the anomalous part of \( Q J^{m_1 m_2 \ldots m_r}_{A_1, \ldots, A_d|B_1, \ldots, B_{d+r+3}} \) given in (6.16) has an unmodified slot structure. Hence, we can neglect the concatenation with \( M_{iA} \) and compensate the anomalous part of \( Q(M_{iA} \otimes D^{m_1 m_2 \ldots m_r}_{k|B_1, \ldots, B_d|C_2, \ldots, C_{d+r+3}}) \) using the corresponding terms of \( M_{iA} \otimes Q D^{m_1 m_2 \ldots m_r}_{k|B_1, \ldots, B_d|C_2, \ldots, C_{d+r+3}} \).

This reasoning motivates the last two lines of

\[
P^{m_1 m_2 \ldots m_r}_{k|B_1, \ldots, B_d|iA, C_2, \ldots, C_{d+r+3}} = (\varphi_i - Q)(M_{iA} \otimes D^{m_1 m_2 \ldots m_r}_{k|B_1, \ldots, B_d|C_2, \ldots, C_{d+r+3}}) \\
+ M_{iA} \otimes \left\{ \left( \begin{array}{c} r \\ 2 \end{array} \right) \delta^{(m_1 m_2} \Delta^{m_3 m_4 \ldots m_r)}_{k|B_1, \ldots, B_d|C_2, C_3, \ldots, C_{d+r+3}} \\
+ \left[ \Delta^{m_1 m_2 \ldots m_r}_{k|B_2, \ldots, B_d|B_1, C_2, C_3, \ldots, C_{d+r+3}} + (B_1 \leftrightarrow B_2, \ldots, B_d) \right] \right\} \\
(11.20)
\]
and guarantees that they cancel the anomalous contributions from the first line.

As the simplest application of (11.20),

\[
P_{2[3]1,4,5,6} = (\varphi_1 - Q)(M_1 \otimes D_{2[3]4,5,6}) + M_1 \otimes \Delta_{2[3]4,5,6}
\]

\[
= -Q \left( J_{3[12,4,5,6]} + k_3^m M_{1[23,4,5,6]}^m + [s_{34} M_{1234,5,6} + (4 \leftrightarrow 5, 6)] \right) + \nu_{12,3,4,5,6} + P_{1[3]2,4,5,6} + k_3^m C_{1[23,4,5,6]}^m + [s_{34} C_{1234,5,6} + (4 \leftrightarrow 5, 6)],
\]

and more involved cases are displayed in appendix F, see (F.8) to (F.11).

11.4. Canonicalizing refined slots in refined pseudoinvariants

A different canonicalization procedure is needed when the preferred reference label \( i \) resides in a refined slot \( A_j \) of a pseudoinvariant \( P_{k[A_1, \ldots, A_d | B_1, \ldots, B_{d+r+3}} \) at \( d \neq 0 \). In order to gain intuition for suitable BRST generators, consider the following examples:

\[
P_{2[1]3,4,5,6} = -(\varphi_1 - Q)(J_{12})_{3,4,5,6} + \nu_{12,3,4,5,6} + P_{1[2]3,4,5,6}
\]

\[
P_{2[1]34,5,6,7} = -(\varphi_1 - Q)(J_{13})_{34,5,6,7} + \nu_{12,34,5,6,7} + P_{1[2]34,5,6,7}
\]

\[
P_{2[1]34,5,6,7} = -(\varphi_1 - Q)(J_{12})_{34,5,6,7} + (J_{13})_{34,5,6,7} - (J_{12})_{35,6,7} + \nu_{12,34,5,6,7} + \nu_{12,35,6,7} - P_{1[2]35,6,7}.
\]

The appearance of anomalous contributions on the right-hand side is not surprising in view of the examples (11.19) and (11.21). In the present cases, however, the BRST generators are entirely built from refined building blocks \( J \) at \( d \neq 0 \). This is a defining property of the \( L \) superfields defined in (8.17). Indeed, (11.22) is consistently described by

\[
P_{k[iA|B,C,D,E} = (\varphi_i - Q)(M_{iA} \otimes L_{k[B,C,D,E)}) + M_{iA} \otimes \Delta_{k[B,C,D,E}.}
\]

Similar to (11.17) and (11.20), the anomalous part of the BRST generator (i.e. the first term of \( QL_{k[B,C,D,E} = \Delta_{k[B,C,D,E} + \ldots \) in (10.2)) is manually compensated by the last term in (11.23), see the arguments in the previous sections 11.2 and 11.3.

In the tensorial generalization of (11.23), the anomalous traces in the expression (10.8) for \( QL_{k[\cdots} = \delta \cdots \Lambda_{k[\cdots} + \ldots \) have to be taken into account. This leads to the second line of

\[
P_{k[iA|B_1, \ldots, B_{r+4}} = (\varphi_i - Q)(M_{iA} \otimes L_{k[B_1, \ldots, B_{r+4}}^{m_1 \cdots m_r})
\]

\[
+ M_{iA} \otimes \left\{ \left( \frac{r}{2} \right) \delta^{(m_1 m_2 \Lambda_{k[B_1, \ldots, B_{r+4}}^{m_3 \cdots m_r} + \Delta_{k[B_1, \ldots, B_{r+4}}^{m_1 \cdots m_r}} \right).}
\]

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The structure of the following vector and tensor examples resembles the canonicalization of $C^m_{2|1,3,4,5}$ and $C^mn_{2|1,3,4,5,6}$ performed in (11.18) and (11.19), respectively:

\[
P^m_{2|1,3,4,5,6,7} = (\varphi_1 - Q)(M_1 \otimes L^m_{2|3,4,5,6,7}) + M_1 \otimes \Delta^m_{2|3,4,5,6,7}
\]

\[
= - Q \left( J^m_{12|3,4,5,6,7} + [k^m_3 J_{123|4,5,6,7} + (3 \leftrightarrow 4, 5, 6, 7)] \right)
\]

\[
+ J^m_{1,2,3,4,5,6,7} + [k^m_3 Y_{1234,5,6,7} + (3 \leftrightarrow 4, 5, 6, 7)]
\]

\[
+ P^m_{1|2,3,4,5,6,7} + [k^m_3 P^m_{1|23,4,5,6,7} + (3 \leftrightarrow 4, 5, 6, 7)]
\]  

(11.25)

\[
P^mn_{2|1,3,4,5,6,7,8} = (\varphi_1 - Q)(M_1 \otimes L^mn_{2|3,4,5,6,7,8}) + M_1 \otimes (\delta^mn_{2|3,4,...,8} + \Delta^mn_{2|3,4,...,8})
\]

\[
= - Q \left( J^mn_{12|3,4,5,6,7,8} + [2k^m_3 J^mn_{123|4,5,6,7,8} + (3 \leftrightarrow 4, 5, 6, 7, 8)] \right)
\]

\[
+ \delta^mn_{12|3,4,5,6,7,8} + J^mn_{123|4,5,6,7,8} + [2k^m_3 J^mn_{1234|5,6,7,8} + (3 \leftrightarrow 4, 5, 6, 7, 8)]
\]

\[
+ [k^m_3 P^mn_{1|23|4,5,6,7,8} + (3 \leftrightarrow 4, 5, 6, 7, 8)]
\]

\[
+ [k^m_3 P^mn_{1|234|5,6,7,8} + (3, 4|3, 4, 5, 6, 7, 8)]
\]  

(11.26)

Finally, the canonicalization prescription (11.24) can be generalized to higher refinement. We follow the usual logic and manually remove the anomalous contributions from the natural BRST generator, see (10.16) for $QL^m_{k|B_2,...,B_d|C_1,...,C_{d+r+3}}$:

\[
P^m_{k|A,B_2,...,B_d|C_1,...,C_{d+r+3}} = (\varphi_i - Q)(M_{iA} \otimes L^m_{k|B_2,...,B_d|C_1,...,C_{d+r+3}})
\]

\[
+ M_{iA} \otimes \left\{ \binom{r}{2} \delta^{m_1m_2}_{k|B_2,...,B_d|C_1,...,C_{d+r+3}} \Lambda^{m_3...r}_{k|B_2,...,B_d|C_1,...,C_{d+r+3}}
\]

\[
+ \left[ \Lambda^{m_1...r}_{k|B_3,...,B_d|B_2|C_1,...,C_{d+r+3}} + (B_2 \leftrightarrow B_3,...,B_d) \right] \right\}.
\]  

(11.27)

The simplest application occurs at eight-points,

\[
P_{2|1,3|4,5,6,7,8} = (\varphi_1 - Q)(M_1 \otimes L_{2|3|4,5,6,7,8}) + M_1 \otimes (\Delta_{2|3|4,5,6,7,8} + \Lambda_{2|3,4,5,6,7,8})
\]

\[
= - Q \left( J_{12,3|4,5,6,7,8} + k^m_3 J^m_{123|4,5,6,7,8} + [s_{34} J_{1234|5,6,7,8} + (4 \leftrightarrow 5, 6, 7, 8)] \right)
\]

\[
+ J_{123|4,5,6,7,8} + J_{34|12,4,5,6,7,8} + k^m_3 Y^m_{1234,5,6,7,8}
\]

\[
+ [s_{34} Y_{12345,6,7,8} + (4 \leftrightarrow 5, 6, 7, 8)]
\]  

(11.28)

\[
+ P^m_{1|2,3|4,5,6,7,8} + k^m_3 P^m_{1|23|4,5,6,7,8} + [s_{34} P^m_{1|234|5,6,7,8} + (4 \leftrightarrow 5, 6, 7, 8)]
\].

With the most general canonicalization prescriptions (11.20) and (11.27), any pseudovariant $P^m_{k|...}$ with reference leg $k \neq i$ can be rewritten in terms of $P^m_{i|...}$. The anomalous extra terms built from concatenations of superfields $\Delta$ and $\Lambda$ signal the breakdown of permutation symmetry in anomalous one-loop amplitudes, see [29,31].

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12. Conclusion and outlook

As explained in the Introduction, the result of a multiloop superstring scattering amplitude in the pure spinor formalism can be written in terms of pure spinor superspace expressions in the cohomology of the BRST charge. This realization was the guiding principle which led us to consider the general structures for one-loop amplitudes presented in this work. The claim is that the kinematic factors considered in the previous sections form a convenient and complete set of building blocks which make manifest the BRST cohomology properties of one-loop amplitudes in pure spinor superspace.

Saturation of zero-modes in the pure spinor prescription implies that the external vertices in the four-point amplitude contribute through a term proportional to $d_\alpha d_\beta N^{mn}$ [4]. The measures defined in [4] summarize the net effect of zero-mode integrations by the following rule,

$$d_\alpha d_\beta N^{mn} \rightarrow (\lambda\gamma^m)_\alpha (\lambda\gamma^n)_\beta.$$  \hspace{1cm} (12.1)

The resulting kinematic factor of the open superstring four-point amplitude [4] is written as $V_1 T_{2,3,4}$ in the notation of equation (2.25) and can be checked to be in the BRST cohomology. Its multiparticle generalization found in [24] incorporates the contributions from OPE singularities through the basic structure $V_A T_{B,C,D}$ which is most elegantly described using the BRST blocks from [32]. The reduction of the associated worldsheet integrals [24] organizes these superfields into BRST-invariants such as the scalar

$$C_{1|23,4,5} = M_1 M_{23,4,5} + M_{12} M_{3,4,5} - M_{13} M_{2,4,5}$$ \hspace{1cm} (12.2)

in the five-point amplitude. The trial-and-error construction of scalar BRST-invariants up to multiplicity eight [24] was improved to a systematic and recursive procedure in [32], see section 2. Since their origin is ultimately related to the one-loop zero-mode pattern (12.1) from the one-loop amplitude prescription, these BRST invariants encode the manifestly gauge-invariant pieces of the $N$-point one-loop superstring amplitudes of [24].

However, each topology of open superstring amplitudes at genus one is anomalous for $N \geq 6$ external legs, and the cancellation of the anomaly relies on an interplay between the cylinder and the Möbius strip [25]. Therefore the manifestly gauge-invariant form of the amplitudes in [24] could not be the complete answer. Finding the missing anomalous terms from their BRST properties was one of the main goals of this paper and led to the concept of pseudo-cohomology introduced in section 3.
As discussed in the main body, a more general class of superfields $J$ extending the prescription (12.1) gives rise to a recursive procedure to construct anomalous superfields $P_{i|mnp\ldots}$, called BRST pseudoinvariants. As a defining property, their BRST variation takes the form $V_A(\lambda \gamma^m W_B)(\lambda \gamma^n W_C)(\lambda \gamma^p W_D)(W_E \gamma_{mnp} W_F)$ which generalizes the hexagon anomaly $\epsilon_{10} F^5$ to superspace and to higher number of external particles. The bosonic components of several pseudoinvariants can be downloaded from the website [43].

Therefore, BRST cohomology considerations point towards superfields with the correct properties to describe the anomalous parts of one-loop open superstring amplitudes which were not considered in [24]. The methods to generate these pseudoinvariants are natural extensions of the well-tested recursion [32] for scalar BRST-invariants [24]. In an upcoming work [29] these pseudoinvariants will be assembled into six-point one-loop amplitudes of the open and closed superstring, the analogous treatment of higher multiplicity is left for the future.

The field theory limit of superstring amplitudes is composed of scalar and tensorial Feynman integrals. The underlying degeneration limit of the worldsheet reorganizes the scalar kinematic factors of the superstring such that loop momenta contract tensorial BRST pseudoinvariants. The recursive construction of this work naturally includes superfields of arbitrary tensor rank and motivates kinematic companions for loop momenta. Their precise appearance in one-loop amplitudes of SYM will be detailed in upcoming work [31].

The matching of worldsheet and momentum space representations of one-loop amplitudes requires a precise control of the momentum contractions of pseudoinvariants. As shown in sections 8 to 10, this problem is addressed by cohomology considerations which will allow to identify the difference of the two representations as BRST exact [31].

Tensorial pseudoinvariants also play an essential role for closed string amplitudes and capture their contributions beyond the naive doubling of open string worldsheet correlators. As will be demonstrated in [29], the tensorial kinematic factors in this work provide a compact description of the interactions between left- and right-moving degrees of freedom. From a field-theory perspective, this points towards a squaring relation between the numerators of Feynman integrals in SYM and supergravity amplitudes. It would be interesting to realize the BCJ duality between color and kinematics [44] through the pseudoinvariants of this work.

After finding the recursive formulas for pseudoinvariants of arbitrary orders, the natural question to ask is how these pseudoinvariants can be derived from the pure spinor
multiloop amplitude prescription\textsuperscript{21} [4]. This is a challenge for the future. We suspect that the solution involves a careful treatment of OPE contractions between the b-ghost and the external vertices. The combinatorics must be such that spurious OPE singularities combine to local functions on the worldsheet (which are regular for all values of $z_i - z_j$ and denoted by $f_{ij}$ in [31]). This might bypass the subtleties related to b-ghost singularities pointed out in [45]. Given that the b-ghost is a source of technical difficulties in amplitude calculations with the pure spinor superstring, a first principles explanation for the pseudoinvariants constructed in this paper might shed new light into this difficult corner of the formalism.

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Appendix A. Examples of BRST pseudoinvariants

This appendix gathers recursively generated expansions of various (pseudo-)invariants. Their component expansions can be found at the website [43].

At six points, the recursions (2.40) for scalar and vector invariants yield

\[
C_{1|234,5,6} = M_1 M_{234,5,6} + M_1 \otimes [C_{2|34,5,6} - C_{4|23,5,6}]
\]

\[
= M_1 M_{234,5,6} + M_{12} M_{34,5,6} + M_{123} M_{4,5,6} - M_{124} M_{3,5,6}
\]

\[
- M_{14} M_{23,5,6} - M_{142} M_{3,5,6} + M_{143} M_{2,5,6}
\tag{A.1}
\]

\[
C_{1|234,5,6} = M_1 M_{234,5,6} + M_1 \otimes [C_{2|34,5,6} - C_{3|45,2,6} + C_{4|23,5,6} - C_{5|23,4,6}]
\]

\[
= M_1 M_{234,5,6} + M_{12} M_{45,3,6} - M_{13} M_{45,2,6} + M_{14} M_{23,5,6} - M_{15} M_{23,4,6}
\]

\[
+ M_{124} M_{3,5,6} - M_{134} M_{2,5,6} + M_{142} M_{3,5,6} - M_{152} M_{3,4,6}
\]

\textsuperscript{21} Some terms in the scalar $P_{i|...}$ can be explained as the leftovers of the partial fraction manipulations described in [24]. This applies to both the single-pole contribution from iterated OPEs and to spurious double-pole singularities which can be removed via integration by parts.
\[-M_{125}M_{3,4,6} + M_{135}M_{2,4,6} - M_{143}M_{2,5,6} + M_{153}M_{2,4,6}\]

\[C_{1|23,4,5,6}^{mn} = M_1 M_{23,4,5,6}^{mn} + M_1 \otimes \left[ C_{2|3,4,5,6}^{mn} - C_{3|2,4,5,6}^{mn} + \{ k_m^C 4_{23,5,6,7} + (4 \leftrightarrow 5, 6) \}\right] \]

\[= M_1 M_{23,4,5,6}^{mn} + M_{12} M_{3,4,5,6}^{mn} - M_{13} M_{2,4,5,6}^{mn} + k_3^m M_{123} M_{4,5,6}^{mn} - k_2^m M_{132} M_{4,5,6}^{mn} \]

\[+ \left[ k_4^m M_{14} M_{23,5,6} + (M_{124} + M_{142}) M_{3,5,6} - (M_{134} + M_{143}) M_{2,5,6} + (4 \leftrightarrow 5, 6) \right].\]

The five-point invariants \(C_{1|23,4,5,6}^{mn}\) and \(C_{1|3,4,5,6}^{mn}\) entering (A.1) are given by (2.41).

The recursions (3.13) and (4.15) for tensorial pseudoinvariants gives rise to

\[C_{1|23,4,5,6,7}^{mn} = M_1 M_{23,4,5,6,7}^{mn} + M_1 \otimes \left[ C_{2|3,4,5,6,7}^{mn} - C_{3|2,4,5,6,7}^{mn} + 2\{ k_4^m C_{4|23,5,6,7}^{mn} + (4 \leftrightarrow 5, 6, 7) \}\right] \]

\[= M_1 M_{23,4,5,6,7}^{mn} + M_{12} M_{3,4,5,6,7}^{mn} - M_{13} M_{2,4,5,6,7}^{mn} + 2(k_3^m M_{123} - k_2^m M_{132}) M_{4,5,6,7}^{mn} \]

\[+ 2(k_4^m k_5^m (M_{145} + M_{154}) M_{23,5,6,7} + (M_{1245} + \text{sym}(2, 4, 5)) M_{3,6,7} \]

\[- (M_{1345} + \text{sym}(3, 4, 5)) M_{2,6,7} \} + (4, 5|4, 5, 6, 7) \]

\[+ 2k_4^m \{ M_{14} M_{23,5,6,7} + (M_{124} + M_{142}) M_{3,5,6,7} \]

\[- (M_{134} + M_{143}) M_{2,5,6,7} - k_2^m (M_{1432} + M_{1342} + M_{1234}) M_{5,6,7} \]

\[+ k_3^m (M_{1423} + M_{1243} + M_{1324}) M_{5,6,7} \} + (4 \leftrightarrow 5, 6, 7) \}

\[C_{1|2,3,4,5,6,7}^{mn} = M_1 M_{2,3,4,5,6,7}^{mn} + M_1 \otimes \left[3k_2^m C_{2|3,4,5,6,7}^{mn} + (2 \leftrightarrow 3, 4, 5, 6, 7) \right] \]

\[= M_1 M_{2,3,4,5,6,7}^{mn} + \left[3k_2^m M_{12} M_{3,4,5,6,7}^{np} \right. \]

\[+ \left. 6k_2^m k_3^m (M_{123} + M_{132}) M_{4,5,6,7}^{np} + (2, 3|2, 3, 4, 5, 6, 7) \right] \]

\[+ \left. 6k_2^m k_3^m k_4^m (M_{134} + \text{sym}(2, 3, 4)) M_{5,6,7} + (2, 3, 4|2, 3, 4, 5, 6, 7) \right \].

where \(C_{1|2,3,4,5,6}^{mn}\) is given by (3.14).

One can extract the following seven-point pseudoinvariants from the recursion in (5.21) where the expansion of \(P_{1|2|3,4,5,6}\) is given by (5.22):

\[P_{1|23|4,5,6,7} = M_1 J_{23|4,5,6,7} + M_1 \otimes \left[ P_{2|3|4,5,6,7} - P_{3|2|4,5,6,7} \right] \]

\[= M_1 J_{23|4,5,6,7} + M_{12} J_{3|4,5,6,7} - M_{13} J_{2|4,5,6,7} + k_3^m M_{123} M_{4,5,6,7} \]

\[- k_2^m M_{132} M_{4,5,6,7} + \left[(s_{34} M_{1234} - s_{24} M_{1324}) M_{5,6,7} + (4 \leftrightarrow 5, 6, 7) \right] \}

\[P_{1|23|4,5,6,7} = M_1 J_{23|4,5,6,7} + M_1 \otimes \left[ P_{3|2|4,5,6,7} - P_{4|2|3,5,6,7} + k_2^m C_{2|3,4,5,6,7} \right] \]

\[= M_1 J_{23|4,5,6,7} + M_{13} J_{2|4,5,6,7} - M_{14} J_{2|3,5,6,7} \]

\[- s_{23}(M_{1243} + M_{1423}) M_{5,6,7} + s_{24}(M_{1234} + M_{1324}) M_{5,6,7} \]

\[+ k_3^m (M_{12} M_{3,4,5,6,7} + (M_{123} + M_{132}) M_{4,5,6,7} - (M_{124} + M_{142}) M_{3,5,6,7} \]

\[+ \left[(s_{25} M_{12534} + M_{1325} + M_{1253}) M_{4,6,7} \right. \]

\[- (M_{1425} + M_{1245} + M_{1254}) M_{3,6,7} \} + (5 \leftrightarrow 6, 7) \].
The recursion (6.9) for vector pseudoinvariants yields the following seven-point example:

\[ P^m_{1[2,3|4,5,6,7,8]} = M_1 J^m_{2|3,4,5,6,7} + M_1 \otimes \left\{ k_2^p C^p_{2|3,4,5,6,7} + [k_3^m P^m_{3|2|4,5,6,7} + (3 \leftrightarrow 4, 5, 6, 7)] \right\} \]

\[ = M_1 J^m_{2|3,4,5,6,7} + [k_3^m \{ M_{123} J^m_2|4,5,6,7 + (M_{123} + M_{132})k_2^p M^p_{4,5,6,7} \} + (3 \leftrightarrow 4, 5, 6, 7)] \]

\[ + k_2^p M_{123} J^m_{3|4,5,6,7} + [s_{23} \{ M_{123} M^m_{4,5,6,7} + k_2^m (M_{1234} + M_{1243} + M_{1423}) M_{4,5,6,7} \]

\[ + k_2^m (M_{1235} + M_{1253} + M_{1523}) M_{4,5,6,7} + k_2^m (M_{1236} + M_{1263} + M_{1623}) M_{4,5,6,7} \]

\[ + k_2^m (M_{1237} + M_{1273} + M_{1723}) M_{4,5,6,7} \} + (3 \leftrightarrow 4, 5, 6, 7) \] . \quad \text{(A.6)}

The simplest pseudoinvariant of refinement \( d > 1 \) is generated by the recursion (6.19):

\[ P^m_{1[2,3|4,5,6,7,8]} = M_1 J^m_{2,3|4,5,6,7,8} + M_1 \otimes \left[ k_2^m P^m_{2|3,4,5,6,7,8} + k_3^m P^m_{3|2|4,5,6,7,8} \right] \]

\[ + [s_{24} M_{1243} J^m_{3|5,6,7,8} + s_{34} (M_{1234} J^m_{2,5,6,7,8} + s_{34} (M_{1234} + M_{1324} + M_{1432}) k_2^m M^m_{5,6,7,8} \]

\[ + s_{24} (M_{1234} + M_{1324} + M_{1243}) k_2^m M^m_{5,6,7,8} + (4 \leftrightarrow 5, 6, 7, 8)] \]

\[ + [s_{25} s_{35} (M_{12345} + M_{12354} + M_{12345} + M_{12354} + M_{13254} + M_{13254}) M_{6,7,8} \]

\[ + s_{25} s_{34} (M_{12354} + M_{13254} + M_{13245} + M_{13254} + M_{13245} + M_{13254} M_{6,7,8} \]

\[ + (4,5|4,5,6,7,8) \] . \quad \text{(A.7)}

\section*{Appendix B. Gauge transformations versus BRST transformations}

The purpose of this appendix is to clarify the relation between gauge transformations and BRST variations. As mentioned below (2.1), the response of the superfields in ten-dimensional SYM to a gauge transformation \( \delta_i \) in particle \( i \) is given by

\[ \delta_i A_i^\alpha = D_\alpha \omega_i, \quad \delta_i A_i^{\dagger m} = k_i^{m\dagger} \omega_i, \quad \delta_i W_i^\alpha = \delta_i F_i^{\dagger mn} = 0 , \quad \text{(B.1)} \]

with some scalar superfield \( \omega_i \). In the following, we infer the gauge transformation of multiparticle superfields from (B.1) using their recursive definition presented in [32] and reviewed in section 2.2. This in turn determines the action of \( \delta_i \) on the complete set of building blocks for one-loop amplitudes as well as their (pseudo-)invariant combinations. In particular, we will arrive at a dictionary to translate anomalous BRST variations at ghost-number four to the corresponding anomalous gauge variations at ghost-number three. This is a convenient approach to component expansions of the hexagon gauge anomaly in one-loop amplitudes of multiplicity \( n \geq 6 \).
B.1. Gauge variations of multiparticle superfields

The recursive definitions of the rank-two superfields $K_{12} \in \{A_1^2, A_{12}^m, W_{12}^\alpha, F_{12}^{mn}\}$ in (2.8) allows to infer their gauge variation from (B.1),

$$
\begin{align*}
\delta_1 A_1^2 &= D_\alpha \omega_{1|2} + (k_1 \cdot k_2) \omega_1 A_1^2 \\
\delta_1 W_{12}^\alpha &= (k_1 \cdot k_2) \omega_1 W_2^\alpha \\
\delta_1 A_{12}^m &= k_{12}^m \omega_{1|2} + (k_1 \cdot k_2) \omega_1 A_2^m \\
\delta_1 F_{12}^{mn} &= (k_1 \cdot k_2) \omega_1 F_{2}^{mn}.
\end{align*}
$$

(B.2)

We have introduced a shorthand for the multiparticle gauge scalar,

$$
\omega_{1|2} = -\frac{1}{2} \omega_1 (k_1 \cdot A_2),
$$

(B.3)

to unify the two-particle expressions in $\delta_1 A_1^2$ and $\delta_1 A_{12}^m$. The two-particle gauge transformations (B.2) reproduce the single particle pattern (B.1) with $\omega_1 \rightarrow \omega_{1|2}$ and enrich it by contact terms $\sim (k_1 \cdot k_2)$. This closely mimics the appearance of contact terms in the two-particle equations of motion (2.9).

It is straightforward to work out the multiparticle gauge transformation at $|B| > 2$ using the recursion for $K_B$ as described in section 2.2. The structure of contact terms in gauge transformations up to multiplicity three is captured by the following variations for the unintegrated vertex $V_B \equiv \lambda^i A_B^i$,

$$
\begin{align*}
\delta_1 V_1 &= Q \omega_1, \\
\delta_1 V_{12} &= Q \omega_{1|2} + (k_1 \cdot k_2) \omega_1 V_2 \\
\delta_1 V_{123} &= Q \omega_{1|23} + (k_1 \cdot k_2) (\omega_1 V_{23} + \omega_{1|3} V_2) + (k_{12} \cdot k_3) \omega_{1|2} V_3 \\
\delta_1 V_{231} &= Q \omega_{23|1} + (k_2 \cdot k_3) (\omega_{1|3} V_2 - \omega_{1|2} V_3) - (k_1 \cdot k_{23}) \omega_1 V_{23}.
\end{align*}
$$

(B.4)

(B.5)

(B.6)

This parallels the contact terms in (2.14) and defines additional multiparticle gauge scalars

$$
\begin{align*}
\omega_{1|23} &\equiv -\frac{1}{2} \omega_1 (k_{12} \cdot A_3) - \frac{1}{6} \left[ \omega_1(k^1 \cdot A^{23}) + \omega_{1|3}(k^3 \cdot A^2) - \omega_{1|2}(k^2 \cdot A^3) \right], \\
\omega_{23|1} &\equiv \frac{1}{2} \omega_1 (k_1 \cdot A_{23}) - \frac{1}{6} \left[ \omega_1(k^1 \cdot A^{23}) + \omega_{1|3}(k^3 \cdot A^2) - \omega_{1|2}(k^2 \cdot A^3) \right],
\end{align*}
$$

(B.7)

where the terms proportional to $\frac{1}{6}$ come from the corrections $H_{ijk}$ of (2.11). As a consistency check of (B.7) one can show that $\delta_1 (V_{123} + V_{231} + V_{312}) = 0$ after using $\delta_1 V_{312} = -\delta_1 V_{132}$ due to the rank-two Lie symmetry in the first two labels.
B.2. Gauge variations of Berends–Giele currents $M_A$

As detailed in section 2.3, a convenient basis of multiparticle fields $K_B$ is furnished by Berends–Giele currents $K_B$, represented by calligraphic letters. In a cubic graph interpretation of multiparticle fields $K_B$ shown in fig. 3, Berends–Giele currents $K_B$ assemble the diagrams of a color-ordered SYM tree including $|B| - 1$ propagators. The dictionary up to multiplicity four is given in (2.17) and (2.18).

For the Berends–Giele current $M_B = \lambda^\alpha A^B_\alpha$ associated with the unintegrated vertex $V_B$, (B.4) to (B.6) translate into

$$\delta_1 M_1 = Q \Omega_1, \quad \delta_1 M_{12} = Q \Omega_{1|2} + \Omega_1 M_2$$

$$\delta_1 M_{123} = Q \Omega_{1|23} + \Omega_{1|2} M_3 + \Omega_1 M_{23}$$

with Berends–Giele gauge scalars

$$\Omega_1 \equiv \omega_1, \quad \Omega_{1|2} \equiv \frac{\omega_{1|2}}{s_{12}}, \quad \Omega_{1|23} \equiv \frac{\omega_{1|23}}{s_{12}s_{123}} - \frac{\omega_{23|1}}{s_{23}s_{123}}.$$  \hspace{1cm} (B.10)

With suitable multiparticle generalizations $\Omega_{1|23...p}$ of (B.10), one can directly write down a closed formula for the gauge transformations of $M_B$,

$$\delta_1 M_{12...p} = \sum_{j=1}^{p-1} \Omega_{1|23...j} M_{j+1...p} + Q \Omega_{1|23...p}.$$  \hspace{1cm} (B.11)

With this form of $\delta_1 M_{12...p}$, the superspace representation $\sum_{j=1}^{n-2} M_{12...j} M_{j+1...n-1} M_n$ of the SYM tree amplitude [17] can be easily checked to be gauge invariant up to BRST exact terms,

$$\delta_1 \left( \sum_{j=1}^{n-2} M_{12...j} M_{j+1...n-1} M_n \right) = Q \left( \sum_{j=1}^{n-2} \Omega_{1|2...j} M_{j+1...n-1} M_n \right).$$  \hspace{1cm} (B.12)

B.3. Gauge variations of ghost number two building blocks

Similarly to $\delta_1 M_{12...p}$ in (B.11), the Berends–Giele currents $A^m_B, W^\alpha_B$ and $F^{mn}_B$ give rise to gauge transformations

$$\delta_1 A^m_{12...p} = k^m_{12...p} \Omega_{1|23...p} + \sum_{j=1}^{p-1} \Omega_{1|23...j} A^m_{j+1...p}$$  \hspace{1cm} (B.13)

$$\delta_1 W^\alpha_{12...p} = \sum_{j=1}^{p-1} \Omega_{1|23...j} W^\alpha_{j+1...p}$$  \hspace{1cm} (B.14)

$$\delta_1 F^{mn}_{12...p} = \sum_{j=1}^{p-1} \Omega_{1|23...j} F^{mn}_{j+1...p}$$  \hspace{1cm} (B.15)
which resemble their BRST variations (2.22). With (B.13) to (B.15), one can straightforwardly compute the gauge variations of all the building blocks \( M_{B_1,...,B_{r+3}}^{m_1,...,m_r} \) and \( J_{A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1,...,m_r} \) introduced in (4.3) and (6.11), respectively. The simplest examples \( M_{A,B,C} \) and \( M_{A,B,C,D}^{m} \) are defined in (2.26) and (2.30), and their gauge variation

\[
\delta_1 M_{12...p,B,C} = \sum_{j=1}^{p-1} \Omega_{1|23...j} M_{j+1...p,B,C}
\]  

(B.16)

\[
\delta_1 M_{12...p,B,C,D}^{m} = k_{12...p}^m \Omega_{1|23...p} M_{B,C,D} + \sum_{j=1}^{p-1} \Omega_{1|23...j} M_{j+1...p,B,C,D}^{m}
\]  

(B.17)

resembles the contributions from a single slot to the BRST variations (2.27) and (2.31). The variation of tensors or refined building blocks

\[
\delta_1 M_{12...p,B,C,D,E}^{mn} = 2k_{12...p}^m \Omega_{1|23...p} M_{B,C,D,E}^{n} + \sum_{j=1}^{p-1} \Omega_{1|23...j} M_{j+1...p,B,C,D,E}^{mn}
\]  

(B.18)

\[
\delta_1 J_{12...p|B,C,D,E} = k_{12...p}^m \Omega_{1|23...p} M_{B,C,D,E} + [\Omega_{S|12...p,B} M_{C,D,E} + (B \leftrightarrow C, D, E)]
\]

\[
+ \sum_{j=1}^{p-1} \Omega_{1|23...j} J_{j+1...p|B,C,D,E}
\]  

(B.19)

\[
\delta_1 J_{B|12...p,C,D,E} = -\Omega_{S|12...p,B} M_{C,D,E} + \sum_{j=1}^{p-1} \Omega_{1|23...j} J_{B|j+1...p,C,D,E}
\]  

(B.20)

does not reproduce the anomalous terms \( Y_{A,B,C,D,E} \) present in the BRST variations (3.2) and (5.18). The \( S[A,B] \) map is defined in (5.14) and \( \Omega_{S|12...p,B} \) is understood to be arranged in the form \( \Omega_{1|...} \).

As general rule, \( \delta_i J_{A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1,...,m_r} \) can be reconstructed from those terms in \( Q_J_{A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1,...,m_r} \) given in (6.13) where particle \( i \) appears in a current \( M_{iC} \). The gauge variation follows by replacing \( M_{iC} \rightarrow \Omega_{i|C} \) and discarding any other term in the BRST variation. The same prescription applies to anomaly building blocks \( Y_{A_1,...,A_d|B_1,...,B_{d+r+5}}^{m_1,...,m_r} \) which are recursively defined by (6.12), see (3.3) for the simplest scalar \( Y_{A,B,C,D,E} \) and (7.12) for the general BRST transformation.

\[B.4.\text{ Gauge variations of pseudoinvariants}\]

The above gauge variations of \( M_B \) and \( J_{A_1,...,A_d|B_1,...,B_{d+r+3}}^{m_1,...,m_r} \) are sufficient to study the anomalous gauge transformations of pseudoinvariants. This in turn allows to probe the
hexagon anomaly in field theory and string theory, see [31, 29] for details. All the pseudoinvariants $P_{\mathcal{I}i\ldots}^{m_1\ldots}$ constructed by the recursion (6.17) are combinations of $M_iA\mathcal{J}$ where $\mathcal{J}$ represents any ghost number two building block $\mathcal{J}_{A_1\ldots A_d|B_1\ldots B_{d+r+3}}^{m_1\ldots m_r}$, possibly adjoined by momenta. This makes the gauge variation in the reference leg $i$ particularly convenient to study: By (B.11), we have

$$\delta_i M_i A \mathcal{J} = \sum_{j=0}^{|A|-1} \Omega_{i|a_1a_2\ldots a_j} M_{a_{j+1}\ldots a_{|A|}} \mathcal{J} - \Omega_{i|A} Q \mathcal{J} + Q(\Omega_{i|A} \mathcal{J}) \quad (B.21)$$

Up to the last BRST-exact term, this is closely related to the BRST transformation

$$Q M_i A \mathcal{J} = \sum_{j=0}^{|A|-1} M_{i|a_1a_2\ldots a_j} M_{a_{j+1}\ldots a_{|A|}} \mathcal{J} - M_i A Q \mathcal{J} \quad (B.22)$$

upon interchanging $\Omega_{i|B} \leftrightarrow M_{i|B}$ for any $|B| = 0, 1, \ldots, |A|$, i.e.

$$\delta_i M_i A \mathcal{J} = (QM_i A \mathcal{J})|_{M_i B \rightarrow \Omega_{i|B}} + Q(\Omega_{i|A} \mathcal{J}) \quad (B.23)$$

This can be applied term by term to the pseudoinvariants $P_{\mathcal{I}i\ldots}^{m_1\ldots}$ obtained from (6.17),

$$\delta_i P_{\mathcal{I}i|A_1\ldots A_d|B_1\ldots B_{d+r+3}}^{m_1\ldots m_r} = (QP_{\mathcal{I}i|A_1\ldots A_d|B_1\ldots B_{d+r+3}}^{m_1\ldots m_r})|_{M_i C \rightarrow \Omega_{i|C}} + Q\left(P_{\mathcal{I}i|A_1\ldots A_d|B_1\ldots B_{d+r+3}}^{m_1\ldots m_r}|_{M_i C \rightarrow \Omega_{i|C}}\right) \quad (B.24)$$

where the BRST transformations are given by (7.17). For example, the anomalous $Q$ variations of the simplest pseudoinvariants at six and seven points

$$QP_{1|2|3,4,5,6} = - M_1 \mathcal{Y}_{2,3,4,5,6} \quad (B.25)$$
$$QP_{1|2|3,4,5,6,7} = - M_1 \mathcal{Y}_{3,4,5,6,7} - M_{12} \mathcal{Y}_{2,3,4,5,6,7} + M_{13} \mathcal{Y}_{2,4,5,6,7} \quad (B.26)$$
$$QP_{1|2|3,4,5,6,7}^{m} = - M_1 \mathcal{Y}_{2,3,4,5,6,7} - \left[k_2^m M_{12} \mathcal{Y}_{3,4,5,6,7} + (2 \leftrightarrow 3, 4, 5, 6, 7)\right] \quad (B.27)$$

translate into gauge variations

$$\delta_1 P_{1|2|3,4,5,6} = - \Omega_1 \mathcal{Y}_{2,3,4,5,6} + Q(\ldots) \quad (B.28)$$
$$\delta_1 P_{1|2|3,4,5,6,7} = - \Omega_1 \mathcal{Y}_{2,3,4,5,6,7} - \Omega_{1|2} \mathcal{Y}_{3,4,5,6,7} + \Omega_{1|3} \mathcal{Y}_{2,4,5,6,7} + Q(\ldots) \quad (B.29)$$
$$\delta_1 P_{1|2|3,4,5,6,7}^{m} = - \Omega_1 \mathcal{Y}_{2,3,4,5,6,7} - \left[k_2^m \Omega_{1|2} \mathcal{Y}_{3,4,5,6,7} + (2 \leftrightarrow 3, \ldots, 7)\right] + Q(\ldots) \quad (B.30)$$
Gauge variations $\delta_k P_{[i|...}^{m1...}$ beyond the reference leg $i$ can still be obtained by trading any $M_{kB}$ in the BRST variation $QP_{[i|...}^{m1...}$ for $\Omega_k|B$, e.g.

$$
\delta_2 P_{1[23|4,5,6,7} = Q(\ldots)
$$

$$
\delta_2 P_{1[23|4,5,6,7} = \Omega_{2|1} Y_{3,4,5,6,7} + Q(\ldots) \quad (B.31)
$$

$$
\delta_2 P_{1[23|4,5,6,7} = \Omega_{2|1} k_2^m Y_{3,4,5,6,7} + Q(\ldots) .
$$

These expressions of ghost number three can be evaluated in components using the methods in [9]. As shown in [26], the bosonic components of (B.28) yield a Levi-Civita contraction of five gluon field strengths, i.e. $\sim \epsilon_{m_1 n_1...m_5 n_5} k_2^m e_1^{n_1} ... k_6^m e_6^{n_5}$. We will argue in the next section that any anomalous superfield has parity odd bosonic components.

B.5. Parity odd nature of multiparticle anomaly tensors

The component evaluation of $(\langle (\lambda \gamma^m W_2)(\lambda \gamma^n W_3)(\lambda \gamma^p W_4)(W_5 \gamma_{mnp} W_6) \rangle)$ using the prescription $\langle \lambda^3 \theta^5 \rangle = 1$ [3] is particularly simple for five external bosons: The lowest bosonic component in the superfields occurs at order $W_i^\alpha \rightarrow -\frac{1}{4}(\gamma_{mn}\theta)^\alpha f_i^{mn}$ (with $f_i^{mn} = 2k_i^m e_i^n$) in terms of the gluon polarization vector $e_i^m$, so the contribution from five factors of $W_i$ to the order $\theta^5$ is unique. The single-particle instance $Y_{2,3,4,5,6}$ of the anomaly superfields in (3.3) therefore reduces to the correlator

$$
\langle (\lambda \gamma^m \gamma^{a_1 b_1} \theta)(\lambda \gamma^n \gamma^{a_2 b_2} \theta)(\lambda \gamma^p \gamma^{a_3 b_3} \theta)(\theta \gamma^{a_4 b_4} \gamma_{mnp} \gamma^{a_5 b_5} \theta) \rangle = \frac{1}{45} \epsilon^{a_1 b_1 a_2 b_2...a_5 b_5} , \quad (B.32)
$$

which has been evaluated in [26] and shown to flip sign under spacetime parity.

It turns out that the same correlator (B.32) governs the bosonic components of a generic $Y_{A,B,C,D,E}$ built from multiparticle superfields $W_A^\alpha$. This follows from the two-form nature of the lowest bosonic component in the $\theta$-expansion of the BRST blocks,

$$
W_A^\alpha = -\frac{1}{4}(\gamma_{mn}\theta)^\alpha f_A^{mn} + O(\theta^3) , \quad F_A^{mn} = f_A^{mn} + O(\theta^2) , \quad (B.33)
$$

where the two-particle instance of the bosonic field strength $f_A^{mn}$ is given by,

$$
\frac{1}{2} f_{12}^{mn} \equiv k_{12}^m e_2^n (e_1 \cdot k_2) - k_{12}^m e_1^n (e_2 \cdot k_1) - k_1^m k_2^n (e_1 \cdot e_2) - e_1^m e_2^n (k_1 \cdot k_2) . \quad (B.34)
$$

The appearance of $f_A^{mn}$ in both superfields in (B.33) is an inevitable consequence of the multiparticle equation of motion for $D_0 W_A^\beta$, see (2.9) and (2.14) for $|A| = 2, 3$ and (2.22) for the Berends–Giele version at general multiplicity. The contact terms $\sim A_A^\beta W_C^\beta$ in the

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multiparticle equations of motion do not contribute at zero\’th order in $\theta$ since both factors
are fermionic with lowest gluon contributions at order $\theta^1$.

The correlator (B.32) and the leading $\theta$ behavior in (B.33) for the bosonic part of $W^\alpha_A$
imply the gluon component

$$
\langle \mathcal{Y}_{A,B,C,D,E} \rangle = \frac{1}{45} \left( -\frac{1}{4} \right)^5 \epsilon_{a_1 b_1 a_2 b_2 \ldots a_5 b_5} f_{A}^{a_1 b_1} f_{B}^{a_2 b_2} \ldots f_{E}^{a_5 b_5}
$$

(B.35)

for the scalar and unrefined anomaly superfield $\mathcal{Y}_{A,B,C,D,E}$. Its generalization to higher
rank or refinement simply adjoins superspace factors of $A_{B}^{m}$, see (4.6) and (6.12). The
latter can only contribute through their $\theta = 0$ component since $\mathcal{Y}_{A,B,C,D,E}$ has a minimum
contribution of five thetas for external bosons. The same is true for the $\Omega_{i|C}$ superfields due
to gauge transformations in particle $i$. Hence, the gluon components of an anomalous gauge
transformation $\langle \Omega_{i|C}\mathcal{Y}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}} \rangle$ are proportional to the $\epsilon_{10}$ tensor generated by
the correlator (B.32).

Appendix C. BRST variations of miscellaneous superfields

In this appendix we display explicit BRST variations of various superfields that were
omitted from the main text.

C.1. BRST variations before the Berends–Giele map

Even though we emphasized the simpler BRST transformations of the Berends–Giele ver-
sion of the various building blocks in the main body of this work, it is still convenient
to know the explicit $Q$ variations of those building blocks prior to the application of the
Berends–Giele map in (2.17) and (2.18).

The precursor of the Berends–Giele recursion (4.3) for $M^{m_1 \ldots m_r}_{B_1,\ldots,B_{r+3}}$ is based on the
expression (2.25) for $T_{A,B,C}^m$ as well as

$$
W^{m}_{A,B,C,D} = \frac{1}{12} (\lambda_n \gamma_n W_A)(\lambda_p \gamma_p W_B)(W_C \gamma^{mnp} W_D) + (A,B|A,B,C,D).
$$

(C.1)

For the higher rank generalizations

$$
W^{m_1 \ldots m_r}_{B_1,B_2,\ldots,B_{r+3}} = A_{B_1}^{m_1} W^{m_2 \ldots m_r}_{B_2,\ldots,B_{r+3}} + (B_1 \leftrightarrow B_2,\ldots,B_{r+3})
$$

(C.2)

$$
T^{m_1 \ldots m_r}_{B_1,B_2,\ldots,B_{r+3}} = A_{B_1}^{m_1} T^{m_2 \ldots m_r}_{B_2,\ldots,B_{r+3}} + A_{B_1}^{m_r} W^{m_{r-1} \ldots m_2}_{B_2,\ldots,B_{r+3}} + (B_1 \leftrightarrow B_2, B_3,\ldots,B_{r+3}).
$$
one can show that
\[ QT_{1,2,3,4}^{m} = k_1^m V_1 T_{2,3,4} + (1 \leftrightarrow 2, 3, 4) \]  
\[ QT_{12,3,4,5}^{m} = [k_1^m V_{12} T_{3,4,5} + (12 \leftrightarrow 3, 4, 5)] + (k^1 \cdot k^2)(V_1 T_{2,3,4,5} - V_2 T_{1,3,4,5}) \]  
\[ QT_{123,4,5,6}^{m} = [k_1^m V_{123} T_{4,5,6} + (123 \leftrightarrow 4, 5, 6)] + (k^1 \cdot k^2)[V_1 T_{2,3,4,5,6} + V_{13} T_{2,4,5,6} - (1 \leftrightarrow 2)] + (k^{12} \cdot k^3)[V_2 T_{3,4,5,6} - (12 \leftrightarrow 3)] \]  
\[ QT_{1,2,3,4,5}^{mn} = [2k_1^{(m} V_{12} T_{3,4,5}^{n}) + (1 \leftrightarrow 2, 3, 4, 5)] + \delta^{mn} Y_{1,2,3,4,5} \]  
\[ QT_{12,3,4,5,6}^{mn} = [2k_1^{(m} V_{12} T_{3,4,5,6}^{n}) + (12 \leftrightarrow 3, 4, 5, 6)] + \delta^{mn} Y_{12,3,4,5,6} \]  
\[ QT_{123,4,5,6}^{mp} = 3\delta^{(mn} Y_{1,2,3,4,5,6}^{p) \leftrightarrow} + [3V_1 k_1^{(m} T_{2,3,4,5,6}^{np}) + (1 \leftrightarrow 2, 3, 4, 5, 6)] . \]

Similarly, the BRST variations of refined currents (5.7) can be computed to be
\[ QJ_{1|23,45,6,7}^{m} = k_1^m V_{123} T_{23,45,6,7}^{m} + [V_{123} T_{45,6,7} + (23 \leftrightarrow 45, 6, 7)] + Y_{1,23,45,6,7} \]  
\[ QJ_{12|34,5,6,7}^{m} = k_1^m V_{12} T_{34,5,6,7}^{m} + [V_{12} T_{34,5,6,7} + (34 \leftrightarrow 5, 6, 7)] + Y_{12,34,5,6,7} \]  
\[ QJ_{123|4,5,6,7}^{m} = k_1^m V_{123} T_{4,5,6,7}^{m} + [V_{123} T_{5,6,7} + (4 \leftrightarrow 5, 6, 7)] + Y_{123,4,5,6,7} \]  
\[ QJ_{1234|5,6,7}^{m} = k_1^m V_{1234} T_{5,6,7}^{m} + [V_{1234} T_{5,6,7} + (12 \leftrightarrow 5, 6, 7)] + Y_{1234,5,6,7} \]  
\[ QJ_{12345|6,7}^{m} = k_1^m V_{12345} T_{6,7}^{m} + [V_{12345} T_{6,7} + (12 \leftrightarrow 3, 4, 5, 6, 7)] + Y_{12345,6,7} \]  
\[ QJ_{123456|7}^{m} = k_1^m V_{123456} T_{7}^{m} + [V_{123456} T_{7} + (23 \leftrightarrow 45, 6, 7)] + Y_{123456,7} \]

\[ C.2. \text{BRST variations after the Berends–Giele map} \]

After applying the Berends–Giele map to the refined currents from (C.4), their BRST variations become
\[ QJ_{1|23,45,6,7} = k_1^m M_1 M_{23,45,6,7} \]  
\[ + (s_{12} M_{123} - s_{13} M_{132}) M_{45,6,7} + (s_{14} M_{145} - s_{15} M_{154}) M_{23,45,6,7} \]  
\[ 83 \]
As mentioned in section 2.2 and detailed in [32], the recursive construction of BRST blocks situations where both \( \hat{\text{L}} \) for the redefinition of refined currents. We define

\[ D.1. \]

This appendix provides a general definition for the superfields \( H_{[A,B]} \) relevant for the redefinition of refined currents \( J_{A_1,\ldots,A_d;B_1,\ldots,B_d+r+3} \).

\[ \text{Appendix D. The } H \text{ superfields in the redefinition of refined currents} \]

The following relations are useful to derive (C.5) from (C.4):

\[ (C.6) \]

\[ (C.7) \]

\[ (C.8) \]

\[ \text{Appendix D. The } H_{[A,B]} \text{ tensors from BRST blocks } V_C \]

As mentioned in section 2.2 and detailed in [32], the recursive construction of BRST blocks \( (A^B_{\alpha}, A^m_B, W^m_B, F^m_B) \) requires redefinitions by BRST trivial quantities to maintain the Lie symmetries. We define \( \hat{V}_{[A,B]} \) through a generalization of the recursion in (2.15) to situations where both \(|A| \neq 1\) and \(|B| \neq 1\):

\[ \hat{V}_{[A,B]} \equiv - \frac{1}{2} \left[ V_A(k_A \cdot A_B) + A^A_m(\lambda^m W_B) - (A \leftrightarrow B) \right] \]
There are two obstructions to express $\hat{V}_{[A,B]}$ as a linear combination of BRST blocks $V_C$ at multiplicity $|C| = |A| + |B|$: 

(i) Generic contributions to $Q\hat{V}_{[A,B]}$ have the form $s_{ij}V_C\hat{V}_{[D,E]}$. They can be corrected to $s_{ij}V_CV_{[D,E]}$ by subtracting combinations of $s_{ij}V_CH_{[D,E]}$ for some scalar superfields $H_{[D,E]}$ to be defined in the next step.

(ii) After the above subtraction, the modified $\hat{V}_{[A,B]}$ must still be shifted by a BRST exact quantity $QH_{[A,B]}$ before it can be expressed in a basis of BRST blocks $V_C$. If $B = b_1$, i.e. $|B| = 1$, this amounts to enforcing the Lie symmetries at multiplicity $|A| + 1$ by adding $QH_{[A,b_1]}$. The latter was denoted by $H_{[A,b_1]} \equiv H_{a_1a_2...a_{|A|}b_1}$ in [32].

Let us illustrate the recursive nature of these points through the simplest examples at multiplicity $|A| + |B| \leq 5$. Cases with $|B| = 1$ have been discussed in [32],

\[
\begin{align*}
\hat{V}_{[12,3]} &= V_{[12,3]} + QH_{[12,3]} \quad \text{(D.2)} \\
\hat{V}_{[123,4]} &= V_{[123,4]} + (k_{12} \cdot k_3)H_{[12,4]}V_3 + (k_1 \cdot k_2)(H_{[13,4]}V_2 - H_{[23,4]}V_1) + QH_{[123,4]} \quad \text{(D.3)} \\
\hat{V}_{[1234,5]} &= V_{[1234,5]} + (k_{123} \cdot k_4)H_{[123,5]}V_4 + (k_{12} \cdot k_3)(H_{[14,5]}V_3 + H_{[12,5]}V_5 - H_{[34,5]}V_2) \\
&\quad + (k_1 \cdot k_2)(H_{[13,5]}V_2 + H_{[13,5]}V_2 - H_{[14,5]}V_3 - H_{[24,5]}V_1) \\
&\quad + QH_{[1234,5]} \quad , \quad \text{(D.4)}
\end{align*}
\]

where the corrections by $QH_{[12...p−1,p]}$ are explained in (ii) and the remaining terms are due to step (i), see [32] for a closed formula. Requiring $V_{[12...p−1,p]} = V_{12...p}$ to satisfy the Lie symmetries at rank $p$ turns (D.2) to (D.4) into a recursive procedure to determine $H_{[12...p−1,p]} = H_{12...p}$ and $V_{12...p}$ [32].

Cases with $|B| \neq 1$ introduce new classes of corrections $H_{[A,B]}$:

\[
\begin{align*}
\hat{V}_{[12,34]} &= V_{[12,34]} + (k_1 \cdot k_2)(H_{[34,2]}V_1 - H_{[34,1]}V_2) \\
&\quad + (k_3 \cdot k_4)(H_{[12,3]}V_4 - H_{[12,4]}V_3) + QH_{[12,34]} \quad \text{(D.5)} \\
\hat{V}_{[123,45]} &= V_{[123,45]} + (k_{12} \cdot k_3)(H_{[12,45]}V_3 - H_{[3,45]}V_2) \\
&\quad + (k_1 \cdot k_2)(H_{[13,45]}V_2 + H_{[14,5]}V_3 - H_{[24,5]}V_1) \\
&\quad + (k_4 \cdot k_5)(H_{[123,4]}V_5 - H_{[123,5]}V_4) + QH_{[123,45]} \quad , \quad \text{(D.6)}
\end{align*}
\]

The bracket notation $V_{[A,B]}$ on the right-hand side represents linear combinations of BRST blocks such as $V_{[12,34]} = V_{1234} - V_{1243}$ or $V_{[123,45]} = V_{12345} - V_{12354}$, see appendix A of [32] for more details. They follow by identifying $V_{[A,B]}$ with the cubic diagram depicted in fig. 5 and expanding the latter in terms of a multiperipheral basis in fig. 3. The required
Fig. 8  Triplet of subdiagrams whose color representatives sum to zero by virtue of the Jacobi identity $f^e[ab]f^e[de] = 0$. The expansion of the above $V_{[A,B]}$ in a basis of BRST blocks $V_C$ can be understood using the same vanishing statement for triplets of diagrams. The latter allows to expand the diagrammatic representative for $V_{[A,B]}$ shown in fig. 5 in terms of multiperipheral trees depicted in fig. 3 and described by $V_C$.

diagram manipulations are depicted in fig. 8 and can be though of as the kinematic dual of the Jacobi identity $f^e[ab]f^e[de] = 0$ among color tensors along the lines of [44]. The $S[A,B]$ map defined in (5.14) will efficiently address the conversion of $V_{[A,B]} \rightarrow V_C$ once the participating superfields are transformed into a basis of Berends–Giele currents.

With this understanding of the $V_{[A,B]}$ on the right-hand side of (D.2) to (D.6), the redefining tensors $QH_{[A,B]}$ with $|B| \neq 1$ can be obtained recursively. In order to bypass the inconvenience of “inverting” the BRST charge, we next present a setup to determine the $H_{[A,B]}$ directly.

D.2. The $H_{[A,B]}$ tensors from BRST blocks $A_C^m$

Also the BRST block $\hat{A}_{12\ldots p}^m$ in its hatted version before $H_{[B,C]}$ modifications is defined recursively in [32]. The expression given for $\hat{A}_{12\ldots p}^m \equiv \hat{A}_{12\ldots p-1,p}^m$ can be straightforwardly generalized to $\hat{A}_{[B,C]}^m$ with $|C| \neq 1$:

$$\hat{A}_{[B,C]}^m \equiv \frac{1}{2} \left[ A_B^p F_C^{pm} + A_C^m (k_C \cdot A_B) - (B \leftrightarrow C) \right] + (W_B \gamma^m W_C).$$

As shown in [32] for $|C| = 1$, the redefinitions of $\hat{A}_{[B,C]}^m$ and $\hat{V}_{[B,C]}$ are mapped into each other by exchanging the relevant BRST blocks $V_D \leftrightarrow A_B^m$ and trading $Q \leftrightarrow k_{BC}^m = k_B^m + k_C^m$. Up to multiplicity four, this converts (D.2), (D.3) and (D.5) into

$$\hat{A}_{[12,3]}^m = A_{[12,3]}^m + k_{123}^m H_{[12,3]},$$

$$\hat{A}_{[12,43]}^m = A_{[12,43]}^m + (k_{12} \cdot k_3) H_{[12,4]} A_3^m$$
$$+ (k_1 \cdot k_2) (H_{[13,4]} A_2^m - H_{[23,4]} A_1^m) + k_{1234}^m H_{[12,34]}.$$

$$\hat{A}_{[12,34]}^m = A_{[12,34]}^m + (k_1 \cdot k_2) (H_{[34,2]} A_1^m - H_{[34,1]} A_2^m)$$
$$+ (k_3 \cdot k_4) (H_{[12,3]} A_4^m - H_{[12,4]} A_3^m) + k_{1234}^m H_{[12,34]}.$$

As emphasized in [32], knowledge of $k_{BC}^m H_{[B,C]}$ is a more convenient starting point to solve for the scalar $H_{[B,C]}$ as compared to $QH_{[B,C]}$. 

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D.3. The $\mathcal{H}_{[A,B]}$ Tensors from Berends–Giele Currents $\mathcal{K}_C$

A convenient basis of multiparticle SYM fields $\mathcal{K}_C$ to construct BRST (pseudo-)invariants is furnished by the Berends–Giele currents, see section 2.3. The contact terms in the above formulae turn out to simplify once we transform the superfields involved according to

$$
\mathcal{H}_{[12,3]} = \frac{H_{[12,3]}}{s_{12}}, \quad \mathcal{H}_{[12,34]} = \frac{H_{[12,34]}}{s_{12}s_{34}}, \quad \mathcal{H}_{[123,4]} = \frac{H_{[123,4]}}{s_{12}s_{13}} + \frac{H_{[321,4]}}{s_{23}s_{12}}, \\
\mathcal{H}_{[123,456]} = \frac{1}{s_{123}s_{456}} \left( \frac{H_{[123,456]}}{s_{12}s_{45}} + \frac{H_{[321,456]}}{s_{23}s_{45}} + \frac{H_{[123,654]}}{s_{12}s_{56}} + \frac{H_{[321,654]}}{s_{23}s_{56}} \right). 
$$

(D.11)

This amounts to applying the map in (2.17) and (2.18) separately to $B$ and $C$ in $\mathcal{H}_{[B,C]}$. The resulting $\mathcal{H}_{[B,C]}$ are the natural superfields to describe the redefinitions of

$$
\hat{V}_{[B,C]} \equiv -\frac{1}{2} \left[ M_B (k_B \cdot A_C) + A^B_m (\lambda^m \gamma W_C) - (B \leftrightarrow C) \right] 
$$

(D.12)

$$
\hat{A}^m_{[B,C]} \equiv \frac{1}{2} \left[ A^p_B F^m_{B,C} + A^m_C (k_C \cdot A_B) - (B \leftrightarrow C) \right] + (\gamma^m W_B). 
$$

(D.13)

In order to obtain the Berends–Giele images of the BRST blocks $V_D$ and $A^m_D$ with Lie symmetries, we have to modify $\hat{V}_{[A,B]}$ and $\hat{A}^m_{[B,C]}$ via

$$
\hat{V}_{[B,C]} \equiv M_{S[B,C]} + \sum_{XY=B} (\mathcal{H}_{[X,C]} M_Y - \mathcal{H}_{[Y,C]} M_X) \\
+ \sum_{XY=C} (\mathcal{H}_{[B,X]} M_Y - \mathcal{H}_{[B,Y]} M_X) + Q \mathcal{H}_{[B,C]} 
$$

(D.14)

$$
\hat{A}^m_{[B,C]} \equiv A^m_{S[B,C]} + \sum_{XY=B} (\mathcal{H}_{[X,C]} A^m_Y - \mathcal{H}_{[Y,C]} A^m_X) \\
+ \sum_{XY=C} (\mathcal{H}_{[B,X]} A^m_Y - \mathcal{H}_{[B,Y]} A^m_X) + k^m_{BC} \mathcal{H}_{[B,C]}.
$$

(D.15)

Note that $\mathcal{H}_{[B,C]} = 0$ whenever $|B| = |C| = 1$. Comparison of (D.14) and (D.15) with the above examples (say (D.6) or (D.10)) reveals two benefits of the basis of Berends–Giele currents: Firstly, the pattern of BRST blocks on the right-hand side without an accompanying factor of $\mathcal{H}_{[B,C]}$ can be described by the $S[B,C]$ map defined in (5.14). Secondly, the contact terms in (D.6) or (D.10) are converted to simple deconcatenations. Since $M_{S[B,C]}$ and $A^m_{S[B,C]}$ are known in terms of BRST blocks $V_D$ and $A^m_D$ of multiplicity $|D| = |B| + |C| [32]$, one can view (D.15) as a constructive definition of $\mathcal{H}_{[B,C]}$. 

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Appendix E. On BRST exact relations among pseudoinvariants

E.1. BRST generator of \( C^m_{i|A,B,C,D} \)

According to the discussion in section 9.1 and in particular (9.6), the traceless components of \( C^{m_1\ldots m_r}_{i|A_1\ldots A_{r+3}} \) are BRST exact. However, it is difficult to extract the BRST generators, so we will explicitly carry out the analysis for vectors \( C^m_{i|A,B,C,D} \).

When contracting (9.5) with momenta \( k_r^m k_q^n \) of any particle \( r = i \) or \( r \in A, B, C, D \), the on-shell condition \( k_r^2 \) decouples the anomalous term \( \sim \delta^{mn} \), and we obtain the BRST generator for the corresponding momentum contraction,

\[
Q \left[ \frac{k_r^m k_q^n D_{i|A,B,C,D}^{mn}}{2(k_r \cdot k_{i|ABCD})} \right] = k_r^m C^m_{i|A,B,C,D} .
\] (E.1)

Plugging this back into the \( k_i^n \) contraction of (9.5) with its trace subtracted:

\[
C^m_{i|A,B,C,D} = Q \left[ \frac{1}{(k_i \cdot k_{i|ABCD})} \right] \left[ k_i^p D_{i|A,B,C,D}^{pm} - \frac{1}{10} k_i^m \delta_{np} D_{i|A,B,C,D}^{np} \right]
- \frac{k_{i|ABCD}^m k^n_{i|ABCD} D_{i|A,B,C,D}^{np}}{2(k_i \cdot k_{i|ABCD})} + \frac{1}{10} k_i^m \sum_{r \in i,A,B,C,D} \frac{k_r^m k^n_{r|ABCD} D_{i|A,B,C,D}^{np}}{(k_r \cdot k_{i|ABCD})} .
\] (E.2)

Similar to (9.3), the right-hand side is ill-defined if momentum conservation \( k_{i|ABCD}^m = 0 \) is imposed, so the vector invariant \( C^m_{i|A,B,C,D} \) is not BRST-exact in the momentum phase space of \( 1 + |A| + |B| + |C| + |D| \) massless particles. The BRST generator for traceless tensors of rank \( r \geq 2 \) can be found by the same method.

E.2. Seven-point momentum contractions of \( C^m_{i|A,B,C,D} \)

The general formula (9.9) for \( QD_{i|A,B,C,D} \) specializes to the following BRST exact relations at seven-points:

\[
QD_{1|23456,7} = \Delta_{1|234,5,6,7} + k_{234}^m C_{1|234,5,6,7}^m - P_{1|2|34,5,6,7} - P_{1|23|4,5,6,7}
+ P_{1|34|2,5,6,7} + P_{1|4|23,5,6,7} + \left[ -s_{25}C_{1|5234,6,7} - s_{45}C_{1|5432,6,7} + s_{35}(C_{1|5324,6,7} + C_{1|5342,6,7}) + (5 \leftrightarrow 6, 7) \right]
\] (E.3)

\[
QD_{1|5|234,6,7} = \Delta_{1|234,5,6,7} + k_{5234}^m C_{1|234,5,6,7}^m + s_{56}C_{1|234,5,6,7} + s_{57}C_{1|234,5,7,6}
+ [s_{25}C_{1|5234,6,7} - s_{35}(C_{1|5324,6,7} + C_{1|5342,6,7}) + s_{45}C_{1|5432,6,7}] \] (E.4)

\[
QD_{1|23|45,6,7} = \Delta_{1|234,5,6,7} + k_{234}^m C_{1|234,5,6,7}^m - P_{1|2|3|4,5,6,7} + P_{1|3|2|4,5,6,7}
+ [s_{25}C_{1|3254,6,7} - s_{24}C_{1|3245,6,7} - s_{35}C_{1|2354,6,7} + s_{34}C_{1|2345,6,7}
+ [s_{36}C_{1|2364,5,7} - s_{36}C_{1|2364,5,7} + (6 \leftrightarrow 7)] \] (E.5)

\[
QD_{1|6|234,5,7} = \Delta_{1|234,5,6,7} + k_{6234}^m C_{1|234,5,6,7}^m + (s_{56}C_{1|3264,5,7} - s_{56}C_{1|456,23,7} + s_{57}C_{1|2364,5,7}) \] (E.6)
Appendix F. Examples of the canonicalization procedure

This appendix gathers further applications of the canonicalization procedure in section 11. We suppress the BRST generators since they can be reconstructed from the right-hand side and do not contribute to amplitudes.

The canonicalization prescription (11.10) for scalar invariants implies that

\[
C_{2|1,34,56} = C_{1|2,34,56} + C_{1|23,56,4} - C_{1|24,56,3} + C_{1|25,34,6} - C_{1|26,34,5} \\
+ C_{1|235,6,4} - C_{1|236,5,4} - C_{1|245,6,3} + C_{1|246,5,3} + C_{1|253,4,6} \\
- C_{1|254,3,6} - C_{1|263,4,5} + C_{1|264,3,5} + Q(\ldots) \tag{F.1}
\]

\[
C_{2|1345,6,7} = C_{1|3452,6,7} + Q(\ldots) \tag{F.2}
\]

\[
C_{2|1,3456,7} = C_{1|2,3456,7} + C_{1|23,456,7} - C_{1|26,345,7} + C_{1|234,56,7} - C_{1|236,45,7} - C_{1|263,45,7} \\
+ C_{1|265,34,7} + C_{1|2345,6,7} - C_{1|2364,5,7} - C_{1|2365,4,7} - C_{1|2364,5,7} \\
+ C_{1|2635,4,7} + C_{1|2653,4,7} - C_{1|2654,3,7} + Q(\ldots) \tag{F.3}
\]

\[
C_{2|13456,7} = C_{1|3425,6,7} + C_{1|3425,6,7} - C_{1|3426,5,7} + Q(\ldots) \tag{F.4}
\]

\[
C_{2|13456,7} = C_{1|32,45,67} + C_{1|32,45,67} - C_{1|3265,4,7} + C_{1|3245,6,7} \\
- C_{1|3246,5,7} - C_{1|3264,5,7} + C_{1|3265,4,7} + Q(\ldots) \tag{F.5}
\]

\[
C_{2|13456,7} = C_{1|32,45,67} + C_{1|32,45,67} - C_{1|3256,7,4} + C_{1|3264,5,7} - C_{1|327,45,6} \\
+ C_{1|3246,7,5} - C_{1|3247,6,5} - C_{1|3256,7,4} + C_{1|3257,6,4} + C_{1|3264,5,7} \\
- C_{1|3265,4,7} - C_{1|3274,5,6} + C_{1|3275,4,6} + Q(\ldots) \tag{F.6}
\]

Except for the more laborious \(C_{2|3456,7,1}\), (F.1) to (F.6) and the opening examples of section 11.1 cover all canonicalizations of scalars \(C_{2|A,B,C}\) up to multiplicity seven.

Next, we apply the canonicalization rule (11.17) to vectors and tensors:

\[
C_{2|134,5,6}^m = C_{1|32,4,5,6}^m + \left[k_4^m C_{1|324,5,6} + (4 \leftrightarrow 5, 6) \right] + Q(\ldots) \tag{F.7}
\]

\[
C_{2|1,34,5,6}^m = C_{1|2,34,5,6}^m + C_{1|23,5,6} - C_{1|24,3,5,6} + k_4^m C_{1|234,5,6} - k_3^m C_{1|243,5,6} + Q(\ldots) + \left[k_5^m (C_{1|25,34,6} + C_{1|25,4,6} + C_{1|253,4,6} - C_{1|245,3,6} - C_{1|254,3,6}) + (5 \leftrightarrow 6) \right]
\]

\[
C_{2|134,5,6}^m = C_{1|32,4,5,6}^m + \left[k_5^m C_{1|3245,6,7} + (5 \leftrightarrow 6, 7) \right] + Q(\ldots) \tag{F.8}
\]

\[
C_{2|134,5,6}^m = C_{1|32,4,5,6}^m + C_{1|32,4,5,6} + C_{1|325,4,6} - C_{1|324,5,6} + k_5^m C_{1|3245,6,7} - k_4^m C_{1|3254,6,7} + Q(\ldots) + \left[k_6^m (C_{1|3264,5,7} + C_{1|3264,5,7} + C_{1|3264,5,7} - C_{1|3265,4,7} - C_{1|3264,4,7}) + (6 \leftrightarrow 7) \right]
\]

\[
C_{2|134,5,6}^m = C_{1|32,4,5,6}^m + 6^m \gamma_{132,4,5,6,7} + 2\left(k_4^m C_{1|324,5,6,7} + (4 \leftrightarrow 5, 6, 7) \right)\]

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\[ C_{2|1,34,5,6,7}^{mn} = C_{2|1,34,5,6,7}^{1m} + C_{2|1,34,5,6,7}^{m1} \]
\[ = C_{2|1,34,5,6,7}^{1m} + C_{2|1,34,5,6,7}^{m1} - C_{2|1,34,5,6,7}^{1m} + C_{2|1,34,5,6,7}^{m1} + \delta^{mn} (Y_{12345,6,7} + Y_{12345,6,7} - Y_{12435,6,7}) 
+ 2\left[ k_5^{(m)} (C_{1|2534,6,7} + C_{1|2534,6,7} - C_{1|2534,6,7}) + (5 \leftrightarrow 6, 7) \right] 
+ 2k_4^{(m)} C_{1|2435,6,7} - 2k_3^{(m)} C_{1|2435,6,7} + 2\left[ k_5^{(m)} \left( k_4^{(m)} C_{1|2345,6,7} + C_{1|2345,6,7} + C_{1|2345,6,7} \right) \right] 
- k_3^{(m)} (C_{1|2435,6,7} + C_{1|2435,6,7} + C_{1|2435,6,7}) + (5 \leftrightarrow 6, 7) \right] 
+ 2\left[ k_5^{(m)} k_6^{(n)} \left( C_{1|2563,4,7} + C_{1|265,3,4,7} + (C_{1|2356,4,7} + \text{symm}(3, 5, 6)) \right) \right] + (5, 6|5, 6, 7) \right] + Q(\ldots) \]

The more laborious vectors \( C_{2|1,34,5,6,7}^{m} \) and \( C_{2|1,34,5,6,7}^{m} \) at multiplicity seven are omitted.

Finally, the following pseudoinvariants are canonicalized using (11.20):

\[ P_{2|3|14,5,6,7}^{m} = P_{2|3|14,5,6,7} + Y_{14235,6,7} + k_3^{(m)} C_{1|4235,6,7} 
+ \left[ s_{35} C_{1|4235,6,7} + (5 \leftrightarrow 6, 7) \right] + Q(\ldots) \] (F.8)

\[ P_{2|3|14,5,6,7}^{m} = P_{2|3|14,5,6,7} + P_{2|4|23,5,6,7} - P_{2|1|3|24,5,6,7} + Y_{12345,6,7} + Y_{12345,6,7} 
- Y_{12435,6,7} + \left[ s_{45} C_{1|2345,6,7} - s_{35} C_{1|2345,6,7} + (5 \leftrightarrow 6, 7) \right] 
+ k_4^{(m)} C_{1|2345,6,7} - k_3^{(m)} C_{1|2345,6,7} + Q(\ldots) \] (F.9)

\[ P_{2|3|14,5,6,7}^{m} = P_{2|3|14,5,6,7} + P_{2|3|14,5,6,7} - P_{2|3|24,5,6,7} + Y_{12345,6,7} + Y_{12435,6,7} 
- Y_{12534,6,7} + \left[ s_{35} (C_{1|2345,6,7} + C_{1|2345,6,7}) - (4 \leftrightarrow 5) \right] 
+ k_3^{(m)} (C_{1|2345,6,7} + C_{1|2345,6,7} - C_{1|2345,6,7} + C_{1|2345,6,7} + C_{1|2345,6,7} - C_{1|2345,6,7} \right] 
+ \left[ s_{36} (C_{1|2364,5,7} + C_{1|2346,5,7} + C_{1|2346,5,7} + C_{1|2346,5,7} + C_{1|2346,5,7} \right] 
- C_{1|2356,4,7} - C_{1|2365,4,7} + C_{1|2356,4,7} + (6 \leftrightarrow 7) \right] + Q(\ldots) \] (F.10)

\[ P_{2|3|14,5,6,7}^{m} = P_{2|3|14,5,6,7} + Y_{12345,6,7} + \left[ k_4^{(m)} (P_{2|3|24,5,6,7} + Y_{12435,6,7}) + (4 \leftrightarrow 5, 6, 7) \right] 
+ \left[ s_{34} C_{1|2345,6,7} + k_4^{(m)} k_3^{(p)} (C_{1|2345,6,7} + C_{1|2345,6,7}) + (4 \leftrightarrow 5, 6, 7) \right] 
+ k_3^{(m)} C_{1|2345,6,7} + \left[ s_{34} \left( k_5^{(m)} (C_{1|2345,6,7} + C_{1|2345,6,7} + C_{1|2345,6,7}) \right) \right] 
+ k_6^{(m)} (C_{1|2346,5,7} + C_{1|2346,5,7} + C_{1|2346,5,7}) 
+ k_7^{(m)} (C_{1|2347,5,6} + C_{1|2374,5,6}) + (4 \leftrightarrow 5, 6, 7) \right] + Q(\ldots) \] (F.11)
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