Modulation instability of discrete angular momentum in coupled fiber rings

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Received 9 February 2019, revised 30 March 2019
Accepted for publication 12 April 2019
Published 28 May 2019

Abstract
We present an analysis of temporal modulation instability in a ring array of coupled optical fibers. Continuous-wave signals are shown to be unstable to perturbations carrying discrete angular momenta, both for normal and anomalous group velocity dispersion. We find the frequency spectrum of modulation instability is different for each perturbation angular momentum and depends strongly on the coupling strength between fibers in the ring. Twisting the ring array also allows the frequency spectra to be tuned through the induced tunnelling Peierls phase.

Keywords: orbital angular momentum of light, modulation instability, fibre optics

Some nonlinear optical properties of fiber rings have been examined previously, particularly in the context of \(\mathcal{PT}\) -symmetry breaking [16], optical switching [17] and the stability of modes in fibers with only a few cores [18, 19]. Here, we show that plane wave supermodes of fiber arrays as shown in figure 1 can be unstable in the presence of perturbations and that the gain spectra of these perturbations depends on their angular momenta. This temporal MI of discrete angular momentum signals has been little explored compared to the well-known azimuthal instability pointed out above. The generation of angular momentum modes over a wide range of frequencies is of practical interest for multiplexing in optical communications [20–22]. We find a rich spectral structure in their instability gain spectra. A Peierls phase introduced by coiling the ring around its propagation axis adds a further degree of depth and potential control to these spectra, but it is by no means necessary to observe this novel MI.

We first outline our derivation of MI in a general fiber ring array with \(N\) cores. Then we exhibit some calculated MI gain spectra for perturbations carrying angular momentum in straight six-core rings. Lastly, we examine how twisting the array can introduce (or suppress) unstable modes, depending on the coupling between neighbouring cores in the ring.
We model light propagation along the spatial axis $z$ of the $N$-core fiber array over time $t$ with $N$ coupled $(1+1)$ dimensional nonlinear Schrödinger equations; the equation for the electric field $E_n$ in the $n$th fiber is

$$i\partial_t E_n = \frac{\beta_2}{2} \partial_z^2 E_n - \gamma |E_n|^2 E_n - \Delta (\exp(-i\phi) E_{n+1} + \exp(i\phi) E_{n-1})$$

(1)

in which $\beta_2$ is the group velocity dispersion (GVD), $\gamma$ is the nonlinear coefficient, $\Delta$ the coupling strength between nearest-neighbour fiber cores and $+(-)\phi$ the Peierls’ phase acquired by photons tunnelling between cores in the direction of (against) the array twist. This Peierls phase is analogous to the phase acquired by charged particles travelling along a magnetic vector potential through the Aharonov–Bohm effect [23]. $\Delta$ is proportional to the overlap between guided modes in adjacent cores [14, 15]; as such it is extremely sensitive to the distance separating cores and the extent of their fundamental modes’ evanescent fields. Careful engineering of these degrees of freedom should allow $\Delta$ to be tuned over several orders of magnitude. Here, we have neglected fiber losses for simplicity’s sake, but their inclusion in this model would be straightforward. Following the standard MI derivation in Agrawal [1], we consider a plane wave solution of (1) with optical power $P_0$ incident in each core (indexed by the integer $n$ running from 1 to $N$) which carries angular momentum with winding number $m$

$$E_n = \sqrt{P_0} \exp(i2\pi mn/N + ik_0z)$$

(2)

as a pump with wavenumber

$$k_0 = \gamma P_0 + 2\Delta \cos(2\pi m/N - \phi).$$

(3)

This is perturbed by a pair of weak excitations, with (possibly complex) wavenumber $k$ and frequency detuning $\Omega$ from the pump,

$$a_g(z,t) = a_S \exp\left(i(Kz + 2\pi ln/N - \Omega t)\right) + a_I \exp\left(-i(Kz - 2\pi qn/N - \Omega t)\right)$$

(4)

which are denoted as signal ($\propto a_S$) and idler ($\propto a_I$), defining the idler angular momentum $q = 2m - l$. This parameterises the lowest order nonlinear mixing processes in (1); the non-linearity couples each perturbation to itself through $|E_n|^2 a_n$ and the other through $E_n^* a_n^*$, hence the idler’s angular momentum is twice the pump’s minus the signal’s and vice versa. The finite number of cores restricts the signal winding number $l$ to a limited set of integers, $l \in [-N/2 + 1, N/2]$, as $N$ even or $l \in [-N - 1/2, N - 1/2]$ for $N$ odd. These perturbations will grow exponentially if $\Omega$ has an imaginary component, which requires $\Omega$ to lie between two critical frequencies, $\Omega_{c1}$ and $\Omega_{c2}$:

$$\Omega_{c1} = -\frac{2}{\beta_2} (2\gamma P_0 + \Delta_{l,m} + \Delta_{q,m})$$

$$\Omega_{c2} = -\frac{2}{\beta_2} (\Delta_{l,m} + \Delta_{q,m})$$

(5)

(assuming all square root arguments are positive). That is, the relative frequency detuning $\Omega$ of the perturbations from the pump occupies two bands ($\Omega_{c2}, \Omega_{c1}$) and ($-\Omega_{c1}, -\Omega_{c2}$) symmetric about $\Omega = 0$. If this holds, then the perturbations grow exponentially in power at a rate

$$G(\Omega, \Delta) = 2\beta_2 \Omega_0^2 - \left(\frac{\beta_2 \Omega_0^2}{2} + \gamma P_0 + \Delta_{l,m} + \Delta_{q,m}\right)^2,$$

(6)

where we have defined

$$\Delta_{l,m} \equiv \Delta (\cos(2\pi x/N - \phi) - \cos(2\pi m/N - \phi)).$$

(7)

To clarify, the gain rate $G(\Omega, \Delta) \equiv 2\beta_\Omega^2(K)$, meaning the perturbation intensity increases with propagation distance $z$ as $|a_{l,m}^0| \propto \exp(Gz)$ [1]. The frequency where this gain is maximised is given by

$$\Omega_{\text{max}} = \pm \sqrt{-\frac{2}{\beta_2} (\gamma P_0 + \Delta_{l,m} + \Delta_{q,m})}.$$  

(8)

Example plots of gain spectra in an untwisted ($\phi = 0$) six-core fiber ring are shown in figure 2, given pump and signal angular momentum $m = 0$ and $l = 1$ respectively. In contrast to standard MI, here we see MI can occur for both normal and anomalous GVD. If $\beta_2 < 0$, perturbations with non-zero $l$ see suppressed instability above a threshold coupling strength. With $\beta_2 > 0$, gain initially grows stronger with increasing coupling, until it forks into two sidebands. Higher-order angular momenta experience similar instabilities, with their sidebands narrowed and shifted to higher $|\Omega|$. Whenever $l = m, \Delta_{l,m} = \Delta_{q,m} = 0$ meaning that the gain is insensitive to the coupling parameters and the standard results for MI in a single fiber apply. In general, the peak gain is capped at $G_{\text{max}} = 2\gamma P_0$. 

Figure 1. Six core fiber ring over a single twist period $\Lambda$. Light couples between cores through the overlap of the evanescent fields of their fundamental guided modes (not shown).
To verify these analytical results, we numerically simulate propagation of a plane wave of power $P_0$ in each core and no angular momentum ($m = 0$) in a strongly-coupled straight six-fiber array with the full coupled NLSE (1). At the start of propagation $z = 0$ we add complex noise $f_d(t)$ to each fiber with average power $\approx 10^{-8} P_0$ to mimic fluctuations from which spontaneous MI may emerge. We decompose the resulting field into its angular momentum $l$ and frequency $\Omega$ spectrum via the projection

$$
\tilde{E}_l(\Omega, z) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \exp\left(-i \frac{2\pi ln}{N}\right) \times \int_{-\infty}^{\infty} dt \exp(-i\Omega t) E_n(t, z). \tag{9}
$$

Results are shown in figure 3, which compares the gain sidebands for $|l| = 1, 2, 3$ (a) predicted by equation (6) for a normally-dispersive, untwisted six-core fiber ring with the spectra in a single core (b) obtained by numerically integrating equation (1). We also calculate the time-averaged power in each angular momentum channel as

$$
P_l(z) = \frac{1}{NT} \int_{-T/2}^{T/2} dt \sum_{n=1}^{N} \exp\left(-i \frac{2\pi ln}{N}\right) E_n(t, z)^2 \tag{10}
$$

given that the length of the simulation’s temporal window is $T = 2 \text{ ps}$. In total, $0.2\%$ of the pump’s power is converted in modes with different angular momenta over $z = 80 \text{ cm}$ propagation. However, this conversion happens at an exponential rate due to the parametric nature of the gain. We determine a mean gain coefficient

$$
\bar{G}_l = \frac{1}{z} \log\left( \frac{P_l(z)}{P_l(0)} \right) \tag{11}
$$

which we find to be approximately $0.1 \text{ cm}^{-1}$ for each $l$. Hence, we anticipate that the average power in each $l = m$ channel will be $10\%$ of the total input power (giving $50\%$ combined conversion to non-pump OAM) after propagating $180 \text{ cm}$. By this point, higher-order nonlinear processes that fall outside the scope of our theory become relevant and our results should not be extrapolated beyond here.

We can make some general observations in the case of normal dispersion; in the strong coupling limit $\Delta > \gamma P_0$, increasing $\Delta$ simultaneously shifts $\Omega_{\text{max}}$ and shrinks the bandwidth $\Omega_{c1} - \Omega_{c2}$, while in the weak coupling regime $\Delta < \gamma P_0$, $\Omega_{\text{max}} = 0$ and decreasing $\Delta$ reduces both the bandwidth and magnitude of the gain. Besides occurring in the time-domain, the azimuthal MI observed here is distinct from that found for vortices in continuous self-focusing media; there it is triggered with vortex collapse [3]. Hence, focusing dynamics along the radial degree of freedom should play an important role in the instability gain, which are themselves influenced by the vortex’s angular momentum [4]. As such the gain dependence on the angular momentum is substantially more complicated than in the fiber array presented here. Radial confinement and discrete angular symmetry are key differences distinguishing this form of MI from those previously discovered.

Twisting the fiber ring adds an additional degree of freedom to the MI gain spectrum by way of the Peierls phase $\phi$, which is related to the fiber twist period $\Lambda$ as [24]

$$
\phi = \frac{8\pi^3 n_r r_0^2}{N\lambda\Lambda}, \tag{12}
$$

where $n_r$ is the substrate refractive index, $r_0$ approximately the ring radius, $N$ the number of cores and $\lambda$ the central pump wavelength. When the twisted array is pumped with $m = 0$ light, the gain spectrum is simple to characterise as only the real part $\Delta \cos(\phi)$ of the complex tunnelling coefficient $\Delta \exp(i\phi)$ is relevant. This is not the case when $m \neq 0$, however the same spectral structure emerges, centred around a different $\phi \neq 0$. We can calculate the twist phase about which the gain spectra are centred for particular $m$ by finding

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**Figure 2.** Gain spectra $G(\Omega, \Delta)$ for perturbations carrying angular momentum $|l| = 1$ as a function of the detuning $\Omega$ and the fiber coupling strength $\Delta$, in an untwisted array ($\phi = 0$). (a) Anomalous GVD $\beta_2 = -1$; (b) Normal GVD $\beta_2 = 1$. Here $\gamma P_0 = 1$. 

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Figure 3. (a) Predicted MI gain for perturbations carrying different angular momenta $|l| = 1, 2, 3$ in an homogeneously pumped six-core fiber ring with $\beta_2 = 1 \text{ps}^2 \text{km}^{-1}$, $\Delta = 1 \text{cm}^{-1}$, $\gamma P_0 = 0.1 \text{cm}^{-1}$. (b) Spectrum $|E_0(\Omega, z)|^2$ in a single core of the ring from a numerical simulation of equation (1). Blue, magenta and red dashed lines indicate the MI critical frequencies for $|l| = 1, 2, 3$, verifying the accuracy of equation (6). (c) Average power in each angular momentum channel relative to the total input power $6P_0$ after $z = 80 \text{ cm}$ propagation. The $l = 0$ bar extends to unity on this scale (not shown).

$\phi_0$ which gives the greatest $\Omega_{\text{max}}$:

$$
\phi_0 = \begin{cases} 
\frac{2\pi m}{N}, & \beta_2 > 0 \\
\frac{2\pi m}{N} + \pi, & \beta_2 < 0
\end{cases}
$$

(13)

Otherwise the spectra with common winding number differences $|l - m|$ are identical for varying $m$. Figures 4 and 5 show the gain spectrum’s dependence on $\phi$ for $|l - m| = 1, 3$ in a six-core array pumped with light carrying no angular momentum $m = 0$, for normal and anomalous dispersion respectively. As might be expected, the effect of varying $\phi$ is more noticeable at higher coupling strengths. Further, the smaller the coupling is the less distinction there is between MI spectra for different $l$; in the limit $\Delta \to 0$ they are equivalent as the fibre cores decouple. Comparing normal and anomalous dispersion results in the strong coupling limit $\Delta = 100$, we see very similar spectral structures centred around different $\phi_0$ as described by (13). As seen in the untwisted ring, a large coupling also leads to higher peak frequencies $\Omega_{\text{max}}$ and a reduced bandwidth. MI can be suppressed by choosing $\phi = \phi_0 \pm \pi$, given any $\Delta$ for $\beta_2 > 0$, or $\Delta \geq \gamma P_0/(1 - \cos(2\pi/N))$ for $\beta_2 < 0$, assuming $l \approx m$. Under these conditions, the square root arguments of $\Omega_{c1}$ and $\Omega_{c2}$ are always negative, meaning no MI spectral bands exist. We have verified numerically that repeating the simulation whose spectrum is shown in figure 3(b) with an additional Peierls phase $\phi = \pi$ prevents the growth of MI frequency bands. In practice, $\phi$ will be limited to small values, as strong twisting will also reduce $\Delta$ since it will push light within cores to their outer edges through an effective ‘centrifugal’ force [15]. However, it is not impossible that these detrimental effects may be countered by fabrication innovations, allowing for larger $\phi$ to be achieved through twisting. Alternative schemes for realising the Peierls phase may also be possible. As noted earlier, it can be equivalently described due to a synthetic magnetic field for photons, oriented along the $z$ axis of the fiber ring. Artificial gauge fields for light is a research area making steady progress at present [23, 25, 26–28].

In summary, we have developed a general theory of time-domain MI of angular momenta in ring array fibers with an arbitrary number of cores. It fully explains the influences of twisting the array on the MI spectra, though we stress that this instability is clearly present in untwisted arrays. In every $N$ core array with fixed material and coupling parameters, there are always $N/2 + 1$ distinct MI spectra for $N$ even or $(N + 1)/2$ for $N$ odd, due to the relative difference in winding number from the pump. Adjusting the strength of or imparting
Figure 4. Gain spectra $G(\Omega, \phi)$ at various coupling strengths $\Delta$ for perturbations carrying angular momentum $l = \pm 1$ (left column) and $l = \pm 3$ (right column), given normal dispersion $\beta_2 = 1$ and a $m = 0$ pump.
Figure 5. Gain spectra $G(\Omega, \phi)$ at various coupling strengths $\Delta$ for perturbations carrying angular momentum $l = \pm 1$ (left column) and $l = \pm 3$ (right column), given anomalous dispersion $\beta_2 = -1$ and a $m = 0$ pump.
a Peierls phase to evanescent coupling between fiber cores in the ring allows for a great degree of control over these spectra. This is a promising first step towards wider knowledge of time-domain nonlinear optics in fiber rings, which may enable novel, tunable sources of broadband light carrying angular momentum.

Acknowledgments

C M acknowledges studentship funding from EPSRC under CM-CDT Grant No. EP/L015110/1. F B acknowledges support from the German Max Planck Society for the Advancement of Science (MPG), in particular the IMPP partnership between Scottish Universities and MPG.

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