ON THE SINGULARITY OF THE DEMJANENKO MATRIX OF QUOTIENTS OF FERMAT CURVES

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Abstract. Given a prime $\ell \geq 3$ and a positive integer $k \leq \ell - 2$, one can define a matrix $D_{k,\ell}$, the so-called Demjanenko matrix, whose rank is equal to the dimension of the Hodge group of the Jacobian $\text{Jac}(C_{k,\ell})$ of a certain quotient of the Fermat curve of exponent $\ell$. For a fixed $\ell$, the existence of $k$ for which $D_{k,\ell}$ is singular (equivalently, for which the rank of the Hodge group of $\text{Jac}(C_{k,\ell})$ is not maximal) has been extensively studied in the literature. We provide an asymptotic formula for the number of such $k$ when $\ell$ tends to infinity.

1. Introduction

For a prime $\ell \geq 3$ and a positive integer $k \leq \ell - 2$, define the set

$$M_{k,\ell} := \{ j \in (\mathbb{Z}/\ell\mathbb{Z})^* \mid (kj)_{\ell} + (j)_{\ell} < \ell \},$$

where, for $j \in (\mathbb{Z}/\ell\mathbb{Z})^*$, we denote by $(j)_{\ell}$ the unique integer representative of $j$ modulo $\ell$ in the range $1, \ldots, \ell - 1$. This is a set of cardinality $(\ell - 1)/2$. Using some results of Koblitz and Rohrlich [KR78, Theorem 2] one can show that the subgroup

$$W_{k,\ell} := \{ w \in (\mathbb{Z}/\ell\mathbb{Z})^* \mid wM_{k,\ell} = M_{k,\ell} \}$$

of elements stabilizing $M_{k,\ell}$ has cardinality 3 or 1 depending on whether the parameter $k$ is a primitive cubic root of unity modulo $\ell$ or not; see [FGL14, Lemma 2.7]. The Demjanenko matrix is then defined as

$$D_{k,\ell} := \left( E_{k,\ell}(-c^{-1}a) - \frac{1}{2} \right)_{c,a \in M_{k,\ell}/W_{k,\ell}},$$

where

$$E_{k,\ell}(a) := \begin{cases} 0 & \text{if } a \in M_{k,\ell}, \\ 1 & \text{if } a \notin M_{k,\ell}. \end{cases}$$

Consider now the curve

$$C_{k,\ell} : V^\ell = U(U + 1)^{\ell-k-1}.$$ 

This is a curve of genus $(\ell - 1)/2$ that may be obtained as a quotient of the Fermat curve $F_\ell : Y^\ell = X^\ell + 1$ by a certain subgroup of automorphisms of $F_\ell$; we refer to [FGL14] for details. The fact that the rank of $D_{k,\ell}$ coincides with the dimension of the Hodge group of the Jacobian of $C_{k,\ell}$ has been exploited in [FGL14] to determine the distribution of Frobenius traces attached to $C_{k,\ell}$ when $D_{k,\ell}$ is non-singular.
Let $K_{\ell}$ denote the set of positive integers $k \leq \ell - 2$ for which $D_{k,\ell}$ is singular. It is easy to see that for every $\ell \equiv 2 \pmod{3}$, the set $K_{\ell}$ is empty (see Lemma 6). Lenstra has shown (see [Gre80, p. 354]) that $K_{\ell}$ is non-empty for every sufficiently large $\ell \equiv 7 \pmod{12}$. In this note, we give an asymptotic formula for the cardinality of $K_{\ell}$, which, in particular, shows that $K_{\ell}$ is non-empty for an overwhelming majority of primes $\ell \equiv 1 \pmod{3}$.

**Theorem 1.** Let $\ell - 1 = 2^{\alpha}3^{\beta}m$ for some integers $\alpha > 0$, $\beta \geq 0$ and $m$ with $\gcd(m,6) = 1$. Then

$$\left|\#K_{\ell} - \frac{1}{2^{2\alpha+2}} \left(1 - \frac{1}{3^{2\beta}}\right)\ell\right| \leq 4\beta^2\sqrt{\ell} + \frac{33}{16}.$$ 

The key result to prove Theorem 1 is the characterisation of the non-singularity of $D_{k,\ell}$ in terms of certain conditions on the multiplicative orders of $k$ and $k^2 + k$ modulo $\ell$ obtained in [FGL14] (see Lemma 6 below).

**Corollary 2.** Let $\ell - 1 = 2^{\alpha}3^{\beta}m$ for some integers $\alpha, \beta > 0$ and $m$ with $\gcd(m,6) = 1$. If $\ell > 441 \cdot 2^\alpha \beta^4$, then $\#K_{\ell} > 0$.

The previous result can be verified by direct calculations for $\ell \leq 7$, and for $\ell > 11$ it follows from the inequalities

$$\frac{1}{2^{2\alpha+2}} \left(1 - \frac{1}{3^{2\beta}}\right)\ell \geq \frac{1}{2^{2\alpha-19}}\ell$$

and

$$4\beta^2\sqrt{\ell} + \frac{33}{16} < \left(4 + \frac{33}{16\sqrt{11}}\right)\beta^2\sqrt{\ell} \leq \left(4 + \frac{2}{3}\right)\beta^2\sqrt{\ell}.$$ 

We can now obtain an explicit form of the observation of Lenstra.

**Corollary 3.** For every prime $\ell \equiv 7 \pmod{12}$ distinct from 7 and 19 we have $\#K_{\ell} > 0$.

This can be deduced from Corollary 2 in the following way. First note that $\alpha = 1$. Observe that for

$$m \geq m_\beta := \lceil 441 \cdot 2^3 \cdot 3^{-\beta} \beta^4 \rceil,$$

we have that

\begin{equation} \tag{1}
\ell = 2 \cdot 3^{\beta} m
\end{equation}

satisfies the hypothesis of Corollary 2 and thus $\#K_{\ell} > 0$. Since for $\beta \geq 18$, one has $m_\beta = 1$, we can limit our search for primes $\ell \equiv 7 \pmod{12}$ with $\#K_{\ell} = 0$ among the finite set of primes $\ell$ of the form (1) with

$$\beta \in \{1, \ldots, 17\} \quad \text{and} \quad m \leq m_\beta - 1 \text{ with } \gcd(m,6) = 1.$$ 

A computer search establishes that the only primes of this form are

$$7, 19, 163, 487, 1459, 39367, 86093443, 258280327.$$ 

Among the above primes, we have $\#K_{\ell} = 0$ only for $\ell = 7, 19$.

As we have mentioned, Lemma 6 below immediately implies that if $\ell \not\equiv 1 \pmod{3}$, then $K_{\ell} = \emptyset$. This is consistent with the vanishing of the main term of Theorem 1 for $\beta = 0$. We also use Corollary 2 to derive a bound on the density of primes $\ell \equiv 1 \pmod{3}$ with $K_{\ell} = \emptyset$. 
Theorem 4. For $x \geq 2$ there are at most $O(x^{3/4}\log x)$ primes $\ell \equiv 1 \pmod{3}$ with $\ell \leq x$ and $\#K_\ell = 0$.

We cannot answer the question of whether there exist infinitely many primes $\ell \equiv 1 \pmod{3}$ with $\#K_\ell = 0$. However, we provide a reason to believe so. Indeed, standard heuristic arguments suggest that for any $\beta \geq 1$ and $m \geq 1$ with $\gcd(m,6) = 1$ there are infinitely many primes of the form $\ell = 2\alpha^3\beta m + 1$, with $\alpha > 0$, and we now show that $\#K_\ell = 0$ for most such primes. To this aim, for fixed integers $\beta \geq 0$ and $m \geq 1$ with $\gcd(m,6) = 1$, we define $L_{\beta,m}$ to be the set of primes of the form $\ell = 2\alpha^3\beta m + 1$, for some $\alpha > 0$, such that $\#K_\ell > 0$. Then we have the following finiteness result.

Theorem 5. For any fixed $\beta \geq 0$ and $m \geq 1$ such that $(m,6) = 1$, the set $L_{\beta,m}$ is finite. More precisely, if $\beta = 0$, then $\#L_{\beta,m} = 0$, and we have

$$\#L_{\beta,m} = O\left(3^{2\beta}m^2/\beta\right)$$

for $\beta \geq 1$.

2. Preparations

Let $\ord_\ell k$ denote the multiplicative order of $k$ modulo $\ell$. Also for a prime $p$ and an integer $m$ we denote by $\nu_p(m)$ the $p$-adic order of $m$, that is, the largest integer $\nu$ with $p^\nu | m$. Our main tool is the following characterisation of the elements of $K_\ell$ given in [FG14].

Lemma 6. For a prime $\ell \geq 3$ and a positive integer $k \leq \ell - 2$, we have $k \in K_\ell$ if and only if the three following conditions hold:

(i) $\ord_\ell k \neq 3$;
(ii) $\nu_2(\ord_\ell k) = \nu_2(\ord_\ell (k^2 - k)) = 0$;
(iii) $\nu_3(\ord_\ell k) > \nu_3(\ord_\ell (k^2 + k))$.

Now, let $X_\ell$ denote the group of multiplicative characters modulo $\ell$. Furthermore, let $X_{\ell,d}$ denote the set of characters of order dividing $d$, that is, the set of characters $\chi \in X_\ell$ such that $\chi^d = \chi_0$, where $\chi_0$ is the principal character; see [IK04, Chapter 3] for a background on characters. We also use $X_{\ell,d}^*$ to denote the set of non-principal characters of $X_{\ell,d}$. Given $\chi \in X_\ell$, we extend it to $\mathbb{F}_\ell$ in the following way: if $\chi = \chi_0$ is principal, then set $\chi_0(0) := 1$. Otherwise, set $\chi_0(0) := 0$.

Since $X_\ell$ is dual to the multiplicative group $\mathbb{F}_\ell^*$ of the finite field of $\ell$ elements, for any divisor $t | \ell - 1$ and $u \in \mathbb{F}_\ell^*$, for $d = (\ell - 1)/t$ we have

$$\frac{1}{d} \sum_{\chi \in X_{\ell,d}} \chi(u) = \begin{cases} 1, & \text{if } u^t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we recall the following special case of the Weil bound of character sums (see [IK04, Theorem 11.23]).

Lemma 7. For any polynomial $Q(X) \in \mathbb{F}_\ell[X]$ with $N$ distinct zeros in the algebraic closure $\overline{\mathbb{F}}_\ell$ of $\mathbb{F}_\ell$ and which is not a perfect $s$th power in $\overline{\mathbb{F}}_\ell[X]$ for an integer $s \geq 2$, and a non-principal character $\chi \in X_\ell^*$ of order $s$, we have

$$\left| \sum_{k \in \mathbb{F}_\ell} \chi(Q(k)) \right| \leq (N - 1)\ell^{1/2}.$$
3. Proof of Theorem 1

Since condition (i) of Lemma 6 fails to hold for at most two integers \( k \in [1, \ell - 2] \) we have

\[
|\#\mathcal{K}_\ell - \#\mathcal{K}_\ell^*| \leq 2,
\]

where \( \mathcal{K}_\ell^* \) is the set of integers \( k \in [1, \ell - 2] \) satisfying conditions (ii) and (iii) of Lemma 6.

Let \( \zeta(u) \) be the characteristic function of the condition \( \nu_2(\text{ord}_\ell u) = 0 \). This is equivalent to

\[
\text{ord}_\ell u \mid 3^\beta m = (\ell - 1)/2^\alpha.
\]

So, we see from (2) that

\[
\zeta(u) = \frac{1}{2^\alpha} \sum_{\chi \in \mathcal{X}_{\ell,2^\alpha}} \chi(u) = \frac{1}{2^\alpha} + \frac{1}{2^\alpha} \sum_{\chi \in \mathcal{X}_{\ell,2^\alpha}} \chi(u).
\]

Furthermore, for a non-negative integer \( h \), let \( \eta_h(u) \) be the characteristic function of the condition \( \nu_3(\text{ord}_\ell u) = h \). This is equivalent to

\[
\text{ord}_\ell u \mid 2^\alpha 3^h m = \frac{\ell - 1}{3^\beta - h} \quad \text{and} \quad \text{ord}_\ell u \nmid 2^\alpha 3^{h-1} m = \frac{\ell - 1}{3^\beta - h+1}.
\]

So, we see from (2) that

\[
\eta_h(u) = \frac{1}{3^\beta - h} \sum_{\chi \in \mathcal{X}_{\ell,3^\beta - h}} \chi(u) - \frac{1}{3^\beta - h+1} \sum_{\chi \in \mathcal{X}_{\ell,3^\beta - h+1}} \chi(u)
\]

\[= 2 + \vartheta_h + \frac{1}{3^\beta - h} \sum_{\chi \in \mathcal{X}_{\ell,3^\beta - h}} \chi(u) - \frac{1}{3^\beta - h+1} \sum_{\chi \in \mathcal{X}_{\ell,3^\beta - h+1}} \chi(u),
\]

where in the case \( h = 0 \) we define \( \mathcal{X}_{\ell,3^\beta - 1}^* = \emptyset \) and we also set \( \vartheta_0 = 1 \) and \( \vartheta_h = 0 \) for \( h \geq 1 \).

Then we have

\[
\#\mathcal{K}_\ell^* = \sum_{k=1}^{\ell-2} B_{k,\ell} = \sum_{k \in \mathcal{F}_\ell} B_{k,\ell} - B_{0,\ell} - B_{-1,\ell},
\]

where

\[
B_{k,\ell} := \zeta(k)\zeta(-k^2 - k) \sum_{r=1}^{\beta} \sum_{s=0}^{r-1} \eta_r(k)\eta_s(k^2 + k).
\]

Examining the expressions (4) and (5) we conclude that, after expanding, each product \( \zeta(k)\zeta(-k^2 - k)\eta_r(k)\eta_s(k^2 + k) \) contains the constant term

\[
\frac{1}{2^{2\alpha}} \cdot \frac{2 + \vartheta_r}{3^{\beta - r - 1}} \cdot \frac{2 + \vartheta_s}{3^{\beta - s - 1}} = \frac{1}{2^{2\alpha}} \cdot \frac{2}{3^{\beta - r - 1}} \cdot \frac{2}{3^{\beta - s - 1}}
\]

(provided that \( r \geq 1 \) in our settings), which does not depend on \( k \), and also several terms with products of the form

\[
\chi_1(k)\chi_2(-k^2 - k)\chi_3(k)\chi_4(k^2 + k)
\]

with some characters \( \chi_1, \chi_2 \in \mathcal{X}_{\ell,2^\alpha}, \chi_3, \chi_4 \in \mathcal{X}_{\ell,3^\beta - h} \cup \mathcal{X}_{\ell,3^\beta - h+1} \) such that at least one of them is non-principal. Since multiplicative characters form a cyclic group.
(see [IK04 Chapter 3]), we see that for some character \( \chi \) of order \( \ell - 1 \) and integers \( f, g, h \) with \( 0 \leq f, g < \ell - 1, f + g > 0, h = 0, 1 \) we have
\[
\chi_1(k)\chi_2(-k^2 - k)\chi_3(k)\chi_4(k^2 + k) = \chi \left( k^f (k^2 + k)^g (-1)^h \right).
\]

Consider the polynomial
\[
P(X, Y, Z, T, U, V) := P_1(X) P_2(Y) \sum_{r=1}^{\beta} \sum_{s=0}^{r-1} P_{3, r}(Z, T) P_{4, r, s}(U, V),
\]
where
\[
P_1(X) := \frac{1}{2^\alpha} + \frac{1}{2^\alpha} \sum_{i=1}^{2^{\alpha}-1} X_i, \quad P_2(Y) := \frac{1}{2^\alpha} + \frac{1}{2^\alpha} \sum_{i=1}^{2^{\alpha}-1} Y_i,
\]
\[
P_{3, r}(Z, T) := \frac{2}{3^{\beta-r+1}} + \frac{1}{3^{\beta-r}} \sum_{i=1}^{3^{\beta-r}-1} Z_{i, r} - \frac{1}{3^{\beta-r+1}} \sum_{i=1}^{3^{\beta-r+1}-1} T_{i, r},
\]
\[
P_{4, r, s}(U, V) := \frac{2 + \beta_s}{3^{\beta-s+1}} + \frac{1}{3^{\beta-s}} \sum_{i=1}^{3^{\beta-s}-1} U_{i, r, s} - \frac{1}{3^{\beta-s+1}} \sum_{i=1}^{3^{\beta-s+1}-1} V_{i, r, s},
\]
and where \( X, Y, Z, T, U, V \) are vector indeterminates given by
- \( X := (X_i) \) and \( Y := (Y_i) \) for \( 1 \leq i \leq 2^\alpha - 1 \);
- \( Z := (Z_{i, r}) \) and \( T := (T_{i, r}) \) for \( 1 \leq i \leq 3^{\beta-r} - 1 \) and \( 1 \leq r \leq \beta \);
- \( U := (U_{i, r, s}) \) and \( V := (V_{i, r, s}) \) for \( 1 \leq i \leq 3^{\beta-s} - 1, 0 \leq s \leq r - 1, \) and \( 1 \leq r \leq \beta \).

Let \( a_0, a_1, \ldots, a_N \) be the set of coefficients of the polynomial \( P \) with \( a_0 \) denoting the constant term. One observes that
\[
\sum_{k \in \mathbb{F}_\ell} B_{k, \ell} = \ell a_0 + \sum_{i=1}^{N} a_i \sum_{k \in \mathbb{F}_\ell} \chi_i(k^f (k^2 + k)^g (-1)^h),
\]
where for every \( i = 1, \ldots, \ell - 1 \) we have that \( 0 \leq f_i, g_i < \ell - 1, f_i + g_i > 0 \) are integers, \( h_i = 0, 1 \), and \( \chi_i \) are characters of order \( \ell - 1 \).

Note that, on the one hand, we have
\[
a_0 = \frac{1}{2^{2\alpha}} \left( \sum_{r=1}^{\beta} \frac{2}{3^{2\beta-r+2}} + \sum_{r=1}^{\beta} \sum_{s=0}^{r-1} \frac{4}{3^{2\beta-r-s+2}} \right)
\]
\[
= \frac{1}{2^{2\alpha}} \left( \sum_{r=1}^{\beta} \frac{2}{3^{2\beta-r+2}} + \sum_{r=1}^{\beta} \frac{4}{3^{2\beta-r+2}} \cdot \frac{3^r - 1}{2} \right)
\]
\[
= \frac{1}{2^{2\alpha}} \left( \sum_{r=1}^{\beta} \frac{2}{3^{2\beta-r+2}} + \sum_{r=1}^{\beta} \left( \frac{2}{3^{2\beta-2r+2}} - \frac{2}{3^{2\beta-r+2}} \right) \right)
\]
\[
= \frac{1}{2^{2\alpha}} \cdot \frac{2}{3^{2\beta+2}} \cdot \sum_{r=1}^{\beta} 9^r = \frac{1}{2^{2\alpha}} \cdot \frac{2}{3^{2\beta+2}} \cdot \frac{9^{\beta+1} - 9}{8}.
\]

Hence
\[
a_0 = \frac{1}{2^{2\alpha+2}} \left( 1 - \frac{1}{3^{2\beta}} \right).
\]
On the other hand, it is clear that the sum of the absolute values of the coefficients of $P$ is equal to the sum over $r$ and $s$ of the products of the sums of the absolute values of the coefficients of the polynomials $P_1$, $P_2$, $P_{3,r}$, and $P_{4,r,s}$. Note that the sum of the absolute values of the coefficients of $P_1$ or $P_2$ is 1, whereas the sum of the absolute values of the coefficients of $P_{3,r}$ or $P_{4,r,s}$ is bounded by 2. This yields the bound

$$\sum_{i=1}^{N} |a_i| \leq \sum_{i=0}^{N} |a_i| \leq 1 \cdot 1 \cdot 2 \cdot 2 = 4 \beta^2.$$  

Putting (7), (8), and (9) together, it follows from Lemma 7 that

$$\left| \sum_{k \in \mathbb{F}_\ell} B_{k,\ell} - \frac{\ell}{2^{2\alpha+2}} \left( 1 - \frac{1}{2^\beta} \right) \right| \leq 4 \beta^2 \sqrt{\ell}.$$

It is immediate that

$$B_{0,\ell} = a_0.$$

Furthermore, observe that

$$B_{-1,\ell} = \left( \frac{1}{2^\alpha} + \frac{1}{2^\alpha} \sum_{\chi \in \mathbb{F}_\ell^*} \chi(-1) \right) \frac{1}{2^\alpha} \sum_{r=1}^{2^\beta-2} \sum_{s=0}^{r-1} \frac{2 + \vartheta_s}{3^2 \beta - r + 2} \left( 2 + \vartheta_r + 3 \sum_{\chi \in \mathbb{F}_\ell^*} \chi(-1) - \sum_{\chi \in \mathbb{F}_\ell^*} \chi(-1) \right) = (2^\alpha - 2)a_0.$$

For the last equality we have used that $\chi(-1) = 1$ if the order of $\chi$ is a power of 3, and the equality

$$\sum_{\chi \in \mathbb{F}_\ell^*} \chi(-1) = 2^\alpha - 3.$$

Combining (6) and (10), (11) and (12), we obtain

$$\left| \# K_\ell^* - \frac{\ell - 2^\alpha + 1}{2^{2\alpha+2}} \left( 1 - \frac{1}{2^\beta} \right) \right| \leq 4 \beta^2 \sqrt{\ell}.$$

Recalling (3) we obtain

$$\left| \# K_\ell - \frac{\ell}{2^{2\alpha+2}} \left( 1 - \frac{1}{2^\beta} \right) \right| \leq 4 \beta^2 \sqrt{\ell} + 2 + \frac{2^\alpha - 1}{2^{2\alpha+2}} \left( 1 - \frac{1}{3^2 \beta} \right).$$

Since

$$2 + \frac{2^\alpha - 1}{2^{2\alpha+2}} \left( 1 - \frac{1}{3^2 \beta} \right) \leq \frac{33}{16}$$

for $\alpha = 1, 2, \ldots$, the result now follows.
4. PROOF OF THEOREM 4

We see from Corollary 2 that if $\# K_\ell = 0$, then $1/2^\alpha = O(\beta / \ell^{1/4})$. So for $m$ in the representation $\ell - 1 = 2^\alpha 3^\beta m$ we have

$$m = O\left(\frac{\ell}{2^\alpha 3^\beta}\right) = O\left(\frac{\ell^{3/4} \beta}{3^\beta}\right) = O(x^{3/4}).$$

The total number $L$ of such primes $\ell - 1 = 2^\alpha 3^\beta m$ can be estimated as

$$J \leq \sum_{\substack{\alpha, \beta = 1 \\ 2^\alpha 3^\beta < x}} \pi(C 2^\alpha 3^\beta x^{3/4}, 2^\alpha 3^\beta, 1),$$

for some absolute constant $C > 0$ (that corresponds to the implied constant in (13)), where, as usual, for integers $a$ and $q \geq 1$, we use $\pi(z, q, a)$ to denote the number of primes $p \leq z$ in the arithmetic progression $p \equiv a \pmod{q}$.

We now recall that by the Brun-Titchmarsh Theorem (see [IK04, Theorem 6.6]) we have

$$\pi(z, q, a) = O\left(\frac{z}{\varphi(q) \log(z/q)}\right),$$

uniformly over integers $a$ and $q$ with $z > q \geq 1$, where, as usual, $\varphi(q)$ denotes the Euler function of $q$. Hence

$$\pi(C 2^\alpha 3^\beta x^{3/4}, 2^\alpha 3^\beta, 1) = O\left(\frac{C 2^\alpha 3^\beta x^{3/4}}{\varphi(2^\alpha 3^\beta) \log(x^{3/4})}\right) = O\left(\frac{x^{3/4}}{\log x}\right).$$

Substituting this bound in (14) and summing it over $O((\log x)^2)$ pairs of non-negative integers $(\alpha, \beta)$ with $2^\alpha 3^\beta \leq x$, we obtain the result.

5. PROOF OF THEOREM 5

Let us assume that $\beta \geq 1$ (otherwise the statement is immediate). Suppose that there exists $k \in K_\ell$. Then Lemma 3 implies that

$$\text{ord}_\ell k = 3^\beta d, \quad \text{ord}_\ell (-k^2 - k) = 3^\beta e,$$

where $1 \leq a \leq \beta$, $0 \leq b \leq a - 1$, and $d, e$ are divisors of $m$. Note that $d$ must be non-trivial if $a = 1$ and $e$ must be non-trivial if $b = 0$. Let $\Phi_m(X)$ denote the $m$th cyclotomic polynomial. Then $k$ is simultaneously a root of

$$p_{a,d}(X) := \Phi_{3^\beta d}(X) \quad \text{and} \quad q_{b,e}(X) := \Phi_{3^\beta e}(-X^2 - X)$$

modulo $\ell$. This means that $\ell$ divides the resultant

$$R_{a,b,d,e} := \text{Res}(p_{a,d}, q_{b,e}).$$

Note that $p_{a,d}(X)$ and $q_{b,e}(X)$ have no roots in common. Indeed, let $\zeta$ be a root of $p_{a,d}(X)$ (and so a root of unity of order $3^\beta d$) that is also a root of $q_{b,e}(X)$. Then $\zeta$ must satisfy that $-\zeta^2 - \zeta = \eta$ or, equivalently,

$$\zeta + 1 = -\frac{\eta}{\zeta},$$

where $\eta$ is root of unity of order $3^\beta e$. Note that if a root of unity plus 1 is again a root of unity, then this root of unity is a primitive cubic root of unity. This is a contradiction with the fact that we cannot have $a = 1$ and $d = 1$ simultaneously. Hence $R_{a,b,d,e} \neq 0$. 
Furthermore, since all roots $\zeta$ of $p_{a,d}(X)$ have absolute value 1, we have
\[ |R_{a,b,d,e}| = \prod_{\zeta: p_{a,d}(\zeta) = 0} |q_{b,e}(\zeta)| = \exp \left( O(3^{a+b}\varphi(d)\varphi(e)) \right), \]
where the product runs over the $\zeta \in \mathbb{C}$ such that $p_{a,d}(\zeta) = 0$ and, as before, $\varphi(q)$ denotes the Euler function of $q$. Note that if $\omega(t)$ is the number of distinct prime divisors of an integer $t \geq 2$, then one has the inequality $\omega(t)! \leq t$. Using Stirling’s formula, we derive $\omega(t) = O(\log t / \log(1 + \log t))$. Hence $R_{a,b,d,e}$ has at most
\[ \omega(R_{a,b,d,e}) = O \left( \frac{3^{a+b}\varphi(d)\varphi(e)}{a+b} \right) \]
distinct prime divisors. Hence
\[
\#L_{\beta,m} = O \left( \sum_{a=1}^{\beta} \sum_{b=0}^{a-1} \sum_{d|m} \sum_{e|m} 3^{a+b}\varphi(d)\varphi(e) \right) / \beta
\]
\[ = O \left( \sum_{a=1}^{\beta} \sum_{b=0}^{a-1} m^2 \frac{3^{a+b}}{a} \right) = O \left( \sum_{a=2}^{\beta} \frac{3^{2a}}{a} m^2 \right) = O \left( \frac{3^{2\beta} m^2}{\beta} \right). \]

6. Comments

In addition to $\#L_{0,m} = 0$ of Theorem 5, we also note that $L_{1,1} = L_{2,1} = L_{3,1} = \emptyset$. Indeed, for $L_{1,1}$ the statement is immediate. We now let $L_{a,b,d,e}$ denote the set of primes dividing $R_{a,b,d,e}$. One computes
\[ L_{2,1,1,1} = \{3\}, \quad L_{3,2,1,1} = \{3, 271\}, \quad L_{3,1,1,1} = \{3, 271\}. \]
It remains to note that $K_3 = 0$ and that 271 is not of the form $2^a 3^3 + 1$ for any $\alpha$.

We now define $\ell_s$ as the smallest prime $\ell \equiv 1 \pmod{3}$ with $K_\ell = 0$ and $\omega(\ell-1) \geq s$ (if such prime exists), where, as before, $\omega(t)$ denotes the number of distinct prime divisors of an integer $t \geq 2$. From Theorem 5, we expect that in fact $\ell_s$ exists for any $s \geq 2$. In Table 1, we present some computational results which characterise the growth $\ell_s$.

**Table 1. Values $\ell_s$ with $3 \leq s \leq 6$**

| $s$ | $\ell_s$ | Factorization of $\ell_s - 1$ |
|-----|----------|------------------------------|
| 3   | 31       | $2 \cdot 3 \cdot 5$         |
| 4   | 3121     | $2^4 \cdot 3 \cdot 5 \cdot 13$ |
| 5   | 127681   | $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 19$ |
| 6   | 25858561 | $2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 37$ |

We have not found $\ell_7$, but our computation shows that if $\ell_7$ exists, then $\ell_7 > 31 \cdot 10^6$. On the other hand, combining the bound of Theorem 5 with the standard heuristic on the distribution on primes, one can derive a heuristic upper bound on $\ell_s$.

We remark that it is shown in [FGL14] that if $k$ satisfies the conditions of Lemma 6, that is, $k \in K_\ell$, then the rank $\text{rk}(D_{k,\ell})$ of the corresponding Demjanenko matrix satisfies
\[ \text{rk}(D_{k,\ell}) = \frac{\ell - 1}{2} \left( 1 - \frac{2}{M(k,\ell)} \right), \]
where
\[ M(k, \ell) := \text{lcm}\{\text{ord}_\ell(-k^2 - k), \text{ord}_\ell(k)\}. \]
The resultant argument of [FGL14] shows that
\[ \min_{k \in \mathbb{K}_\ell} M(k, \ell) \to \infty \]
as \( \ell \to \infty \). This can easily be sharpened as
\[ \min_{k \in \mathbb{K}_\ell} M(k, \ell) \geq c\sqrt{\log \ell} \]
for an absolute constant \( c > 0 \). In fact the same argument shows that for any real function \( \psi(z) \) with \( \psi(z) \to 0 \) as \( z \to \infty \), all but \( o(x/\log x) \) primes \( \ell \leq x \), we have
\[ \min_{k \in \mathbb{K}_\ell} M(k, \ell) \geq \psi(\ell)\ell^{1/3}. \]
Finally, we remark that our approach allows us to study the distribution of the values of \( M(k, \ell) \) for every \( \ell \).

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REFERENCES

[FGL14] Francesc Fité, Josep González, Joan-Carles Lario, Frobenius distribution for quotients of Fermat curves of prime exponent, Canad. J. Math, to appear.
[Gre80] Ralph Greenberg, On the Jacobian variety of some algebraic curves, Compositio Math. 42 (1980/81), no. 3, 345–359. MR607375 (82j:14036)
[IK04] Henryk Iwaniec and Emmanuel Kowalski, Analytic number theory, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004. MR2061214 (2005k:11005)
[KR78] Neal Koblitz and David Rohrlich, Simple factors in the Jacobian of a Fermat curve, Canad. J. Math. 30 (1978), no. 6, 1183–1205, DOI 10.4153/CJM-1978-099-6. MR511556 (80d:14022)

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