LIE SYMMETRIES OF THE CHOW GROUP OF A JACOBIAN AND THE TAUTOLOGICAL SUBRING

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Abstract. Let \( J \) be the Jacobian of a smooth projective curve. We define a natural action of the Lie algebra of polynomial Hamiltonian vector fields on the plane, vanishing at the origin, on the Chow group \( \text{CH}(J)_{\mathbb{Q}} \). Using this action we obtain some relations between tautological cycles in \( \text{CH}(J)_{\mathbb{Q}} \).

Introduction

Let \( J \) be the Jacobian of a smooth projective curve \( C \) of genus \( g \geq 2 \). We fix a point \( x_0 \in C \) and consider the corresponding embedding

\[ \iota : C \to J \]

mapping a point \( x \in C \) to the isomorphism class of the line bundle \( \mathcal{O}_C(x-x_0) \). We always consider \( C \) as a subvariety in \( J \) via this embedding. We define the tautological subring \( \mathcal{T} \subset \text{CH}(J)_{\mathbb{Q}} \) in the Chow ring of \( J \) with coefficients in \( \mathbb{Q} \) as the smallest subring containing the class \( [C] \) of the curve and closed under taking pull-backs with respect to the natural isogenies \( [n] : J \to J \) and under the Fourier transform \( S : \text{CH}(J)_{\mathbb{Q}} \to \text{CH}(J)_{\mathbb{Q}} \) (defined in [2]). The corresponding subring in the quotient of \( \text{CH}(J)_{\mathbb{Q}} \) modulo algebraic equivalence was considered by Beauville in [4] and by the author in [13]. It is known that modulo algebraic equivalence this subring is generated by the characteristic classes of the Picard bundle on \( J \). Also, a number of nontrivial relations between these generators (still modulo algebraic equivalence) was described in [13]. In the present paper we will show how to lift these relations to the Chow ring (after adding some more generators). This is achieved using the action of a certain Lie algebra on \( \text{CH}(J)_{\mathbb{Q}} \) extending the well action of \( \mathfrak{s}_2 \) associated with the natural polarization of \( J \) (see [9]). The construction of this action may be of independent interest.

To state the results precisely we need to introduce some more notation. Recall that the Chow ring of \( J \) with rational coefficients admits a decomposition

\[ \text{CH}(J)_{\mathbb{Q}} = \bigoplus_{p,s} \text{CH}^p_s(J), \]

where \( \text{CH}^p_s \) consists of \( c \in \text{CH}^p(J)_{\mathbb{Q}} \) such that \( [n]^*c = n^{2p-s}c \) (see [3]). For every class \( c \in \text{CH}^p(J)_{\mathbb{Q}} \) we denote by \( c_s \in \text{CH}^p_s(J) \) its components with respect to the above decomposition. We also denote by \( \theta \in \text{CH}^1(J)_{\mathbb{Q}} \) the class of a symmetric theta divisor.

Let us define two families of classes in \( \text{CH}(J)_{\mathbb{Q}} \) by setting

\[ p_n = S([C]_{n-1}), \ n \geq 1, \]
\[ q_n = S(\theta \cdot [C]_n), \ n \geq 0. \]

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Note that $p_1 = -\theta$ and $q_0 = g \cdot [J]$. It is easy to see that all the classes $(p_n)$ and $(q_n)$ belong to the tautological subring $T \text{CH}(J)_Q$. All the classes $q_n$ for $n \geq 1$ are algebraically equivalent to zero.

**Theorem 0.1.** There exists a family of operators $(X_{m,n}, Y_{m,n})$ on $\text{CH}(J)_Q$, where $m,n \in \mathbb{Z}$, such that $X_{m,n} = 0$ unless $m,n \geq 0$ and $m+n \geq 2$ (resp., $Y_{m,n} = 0$ unless $m,n \geq 0$), satisfying the commutation relations

$$[X_{m,n}, X_{m',n'}] = (nm' - mn')X_{m+m'-1, n+n'-1}, \quad (0.1)$$

$$[X_{m,n}, Y_{m',n'}] = (nm' - mn')Y_{m+m'-1, n+n'-1}, \quad (0.2)$$

$$[Y_{m,n}, Y_{m',n'}] = 0$$

and such that

$$\frac{1}{n!} X_{0,n}(a) = p_{n-1} \cdot a,$$

$$\frac{1}{n!} Y_{0,n}(b) = q_n \cdot a.$$ 

Furthermore, one has

$$SX_{m,n}S^{-1} = (-1)^n X_{n,m}, \quad (0.3)$$

$$SY_{m,n}S^{-1} = (-1)^n Y_{n,m}. \quad (0.4)$$

**Remark.** In fact, one can easily see from the proof that the above operators on the Chow group are induced by endomorphisms of a $\mathbb{Q}$-motive of $J$ and the relations are satisfied already for this motive action.

Explicit formulas for operators $X_{m,n}$ and $Y_{m,n}$ will be given in section 2 (see (2.3) and (2.6)). We will also show that the tautological subring $T \text{CH}(J)_Q$ is closed under all operators $X_{m,n}$ and $Y_{m,n}$. Note that the commutation relation (0.1) is the defining relation for the Lie algebra of polynomial Hamiltonian vector fields on the plane with the standard symplectic form (see e.g., [8], ch. 1, §1). The restriction $m+n \geq 2$ that we imposed for our generators $X_{m,n}$ corresponds to considering the subalgebra of vector fields vanishing at the origin. Also, note that the operators $(X_{2,0}/2, X_{1,1}, X_{0,2}/2)$ generate the well known action of $\mathfrak{sl}_2$ on $\text{CH}(J)_Q$ (see [9]).

As an application of the above Lie action we prove the following result.

**Theorem 0.2.** (i) The ring $T \text{CH}(J)_Q$ is generated by the classes $(p_n)$ and $(q_n)$. Furthermore, let us consider the following differential operator

$$\mathcal{D} = \frac{1}{2} \sum_{m,n \geq 1} \binom{m+n}{n} p_{m+n-1} \partial_{p_n} \partial_{p_m} + \sum_{m \geq 1, n \geq 1} \binom{m+n-1}{n} q_{m+n-1} \partial_{q_m} \partial_{p_n} - \sum_{n \geq 1} q_{n-1} \partial_{p_n},$$

where $(\partial_{p_n})$ (resp., $(\partial_{q_m})$) are partial derivatives with respect to $(p_n)$ (resp., $(q_n)$). Then the space of polynomial relations between $(p_n, q_n)$ in $\text{CH}(J)_Q$ is stable under the action of $\mathcal{D}$. 

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(ii) The operators $X_{m,n}$ and $Y_{m,n}$ preserve the subspace $\mathcal{T} \, \text{CH}(J)_Q$ and act on it via the following differential operators (for $m \geq 1$):

\[
(-1)^m \frac{1}{m!} X_{m,n} |_{\mathcal{T} \, \text{CH}(J)_Q} = \sum_{i_1, \ldots, i_m \geq 1} \frac{(n + i_1 + \ldots + i_m)!}{i_1! \ldots i_m!} p_{n+i_1+\ldots+i_m-1} \partial_{p_{i_1}} \cdots \partial_{p_{i_m}} \\
+ \frac{1}{(m-1)!} \sum_{i_1, \ldots, i_m-1 \geq 1} \frac{(n + i_1 + \ldots + i_m-1 + j-1)!}{i_1! \ldots i_m-1!(j-1)!} q_{n+i_1+\ldots+i_m-1+j-1} \partial_{q_1} \cdots \partial_{q_{j-1}} \\
- \frac{1}{(m-1)!} \sum_{i_1, \ldots, i_m \geq 1} \frac{(n + i_1 + \ldots + i_m-1)!}{i_1! \ldots i_m-1!} q_{n+i_1+\ldots+i_m-1} \partial_{p_1} \cdots \partial_{p_{i_m-1}},
\]

where for $m = 1$ the last term should be understood as $-n!q_{n-1}$.

It is easy to check that the above differential operators in independent variables $(p_n, q_n)$ satisfy relations (0.1) and (0.2). Since $\mathcal{D}$ corresponds to the action of $\mathbb{X}_{2,0}/2$, it follows that any $\mathcal{D}$-invariant ideal in $Q[p, q]$ is also invariant under all the other differential operators above. Since the operator $\mathcal{D}$ lowers the degree by 1 (where $\deg p_n = \deg q_n = n$), starting from vanishing of polynomials of degree $g + 1$ and applying powers of $\mathcal{D}$ we get nontrivial relations in $\text{CH}(J)_Q$ (see section 4 for some examples). Since all the classes $q_n$ for $n \geq 1$ are algebraically equivalent to zero, we recover the relations between $(p_n)$ modulo algebraic equivalence proved in [13]. The above theorem also has the following corollary closely related to the work of Beauville [1]. Consider the group of 0-cycles $\text{CH}^0(J)_Q$ equipped with the Pontryagin product. Let $K$ be the canonical class on the curve $C$. Then we have a special 0-cycle $\iota_n K \in \text{CH}^0(J)_Q$. The proof of the following corollary will be given in section 1.

**Corollary 0.3.** The intersection $\mathcal{T} \, \text{CH}(J)_Q \cap \text{CH}^0(J)_Q$ coincides with the $Q$-subalgebra with respect to the Pontryagin product generated by the classes $[n] \iota_n K$, where $n \in \mathbb{Z}$.

**Remarks.** 1. It is easy that the classes of the subvarieties of special divisors $W_d \subset J$ belong to $\mathcal{T} \, \text{CH}(J)_Q$. More precisely, we will show in section 4 that they can be expressed as universal polynomials in classes $(p_n - q_n)$.

2. Of course, the ring $\mathcal{T} \, \text{CH}(J)_Q$ depends on a choice of a point $x_0 \in C$. For example, if $(2g-2)x_0 = K$ in Pic($C)_Q$ then all $q_n$ vanish. In fact, using Abel's theorem it is easy to see that $(2g-2)x_0 = K$ iff $q_1 = 0$. The vanishing of other classes $q_n$ in this case follows also from the formula

\[ q_n = \mathcal{D}(q_1p_n) + q_1q_{n-1}. \]

**Notation.** We use the convention $\binom{n}{m} = 0$ for $m < 0$ and for $n < m$.

1. Preliminaries

Let $\Theta \subset J$ be a symmetric theta divisor (corresponding to some choice of a theta characteristic on $C$), so that $\theta = c_1(\Theta)$. Consider the line bundle on $J \times J$ given by
$\mathcal{L} = \mathcal{O}_{J \times J}(p_1^{-1}\Theta + p_2^{-1}\Theta - m^{-1}\Theta)$ where $p_1, p_2 : J \times J \to J$ are the natural projections and $m = p_1 + p_2 : J \times J \to J$ is the group law. It is easy to see that $\mathcal{L}|_{C \times C} \simeq \mathcal{O}_{C}(\Delta_C - x_0 \times C - C \times x_0)$, where $\Delta_C \subset C \times C$ is the diagonal. Indeed, it suffices to check that $\mathcal{L}|_{C \times J} \simeq \mathcal{P}_C$, where $\mathcal{P}_C$ is the universal family of degree 0 line bundles on $C$ trivialized at $x_0$. This in turn follows from the fact that $\mathcal{L}^{-1}$ corresponds to the normalized Poincaré line bundle on $J \times \hat{J}$ under the principal polarization isomorphism $\phi : J \overset{\sim}{\longrightarrow} \hat{J}$ and from the equality $\phi \circ \iota = -a$, where $a : C \to \hat{J}$ is the embedding induced by $\mathcal{P}_C$ (see [12], 17.3). We denote by $S$ the Fourier transform on $\text{CH}(J)_Q$ defined by

$$S(c) = p_{2*}(\exp(c_1(\mathcal{L})) \cdot p_1^*c).$$

We refer to [2] for the detailed study of this transform. In particular we will use the following properties:

$$S^2 = (-1)^g[-1]^*,$$
$$S(\text{CH}^p(J)) \subset \text{CH}^*[−p−s](J),$$
$$S(a \ast b) = S(a) \cdot S(b),$$

where $a \ast b$ denotes the Pontryagin product on $\text{CH}(J)_Q$.

It is easy to see that $S([C]) = \sum_{n \geq 1} p_n$ is exactly the decomposition of the class $S([C])$ into components of different codimensions, so that $p_n \in \text{CH}^n_{n-1}(J)$. Similarly, $q_n \in \text{CH}^n_{n}(J)$. It is also well known that $p_1 = -\theta$ (see e.g. [4], Prop. 2.3, or [12], 17.2 and 17.3).

Recall that we have defined the tautological subring $\mathcal{T} \subset \text{CH}(J)_Q$ as the smallest subring containing $[C]$ and closed under $S$ and under all the pull-back operations $[n]^*$. Equivalently, this is the smallest subring closed under $S$ and containing all classes $[C_n]$. This immediately implies that all classes $p_n$ and $q_n$ belong to $\mathcal{T} \subset \text{CH}(J)_Q$.

Let us consider the element

$$\eta := \iota_*K/2 + [0] \in \text{CH}^q(J)_Q.$$

From the Riemann’s Theorem we get

$$\eta = \theta \cdot [C]$$

(see e.g. [12], Thm. 17.4). Hence, $\theta \cdot [C]_n = \eta_n$ and we have

$$q_n = S(\eta_n).$$

Note that for every point $x \in J$ we have

$$S([x]) = \exp(c_1(\mathcal{L}_x)) = \exp(\theta_x - \theta),$$

where $\mathcal{L}_x = \mathcal{L}|_{J \times x}$ and $\theta_x = [\Theta + x]$. Hence, we can rewrite the definition of $q_n$ for $n \neq 0$ as follows:

$$q_n = \frac{1}{2} \sum_{i=1}^{2g-2} c_1(\mathcal{L}_{x_i})^n,$$

where $(x_i)$ are points on $C$ such that $K = x_1 + \ldots + x_{2g-2}$. In particular,

$$q_1 = \frac{1}{2} c_1(\mathcal{L}_\kappa),$$

where $\kappa \in J$ is the point corresponding to $K(-(2g-2)x_0)$.
Proof of Corollary 0.3. Theorem 0.2 implies that $T \text{CH}(J)_Q$ is generated with respect to the Pontryagin product by the classes $([C]_n)$ and $(\eta_n)$. Therefore, the group of tautological 0-cycles is generated with respect to this product by the classes $(\eta_n)$, or equivalently, by the classes $([n]_* \eta)$. □

The action of $\mathfrak{sl}_2$ on $\text{CH}(J)_Q$ (in fact, on the motive of $J$) mentioned in the introduction is generated by the operators

$$e(a) = p_1 \cdot a = -\theta \cdot a,$$
$$f(a) = -[C]_0 * a,$$
$$h(a) = (2n - s - g)a$$

for $a \in \text{CH}_n$ (the operators $e$ and $f$ differ from those of [9] by the sign). In fact, this action is induced by an algebraic action of the group $\text{SL}_2$, so that the Fourier transform corresponds to the action of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(see [11], Thm. 5.1). This leads to the formula

$$S = \exp(e) \exp(-f) \exp(e)$$

that can also be checked directly (see [4], (1.7)).

2. Commutation relations

Let us consider the following family of binary operations on $\text{CH}(J)$:

$$a *_n b = (p_1 + p_2)_* (c_1(L)^n \cdot p_1^* a \cdot p_2^* a), \ n \geq 0,$$

where $a, b \in \text{CH}(J)$. Note that $a *_0 b = a \ast b$ is the usual Pontryagin product.

Lemma 2.1. One has

$$a *_{n+1} b = (\theta \cdot a) *_n b + a *_n (\theta \cdot b) - \theta \cdot (a *_n b).$$

Proof. This follows immediately from the identity $c_1(L) = p_1^* \theta + p_2^* \theta - (p_1 + p_2)^* \theta$. □

Lemma 2.2. If $a \in \text{CH}_p^{s_1}, b \in \text{CH}_p^{s_2}$, then $a *_n b \in \text{CH}_p^{s_1+s_2-n-g}$.

Proof. Since $\theta \cdot \text{CH}_p^s \subset \text{CH}_p^{s+1}$, the assertion follows from Lemma 2.1 by induction in $n$. □

For every $a \in \text{CH}(J)$ and $n \geq 0$ let us denote by $A_n(a)$ the operator $b \mapsto a *_n b$ on $\text{CH}(J)$. For $n < 0$ we set $A_n(a) = 0$. Note that Lemma 2.1 is equivalent to the following identity

$$[e, A_n(a)] = A_{n+1}(a) - A_n(\theta \cdot a), \quad (2.1)$$

where $e$ is the operator of the $\mathfrak{sl}_2$-action (see section 1).

Lemma 2.3. For every $s \geq 0$ one has $A_n(\eta_s) = 0$ for $n > s$ and $A_n([C]_s) = 0$ for $n > s+2$. 5
Proof. We start by observing that the operator \( f = -A_0([C]_0) \) commutes with \( A_0([C]_s) \) and with \( A_0(\eta_s) \). Also,

\[
[h, A_0([C]_s)] = (-s - 2)A_0([C]_s), \quad [h, A_0(\eta_s)] = -sA_0(\eta_s).
\]

Hence, \( A_0([C]_s) \) and \( A_0(\eta_s) \) are lowest weight vectors with respect to the adjoint action of \( \mathfrak{sl}_2 \) on \( \text{End}(\text{CH}(J)) \) of weights \(-(s + 2)\) and \(-s\), respectively. It follows that

\[
\text{ad}(e)^n(A_0([C]_s)) = 0
\]

for \( n > s + 2 \) and

\[
\text{ad}(e)^n(A_0(\eta_s)) = 0
\]

for \( n > s \). Using equalities (2.1) and (1.1) we find by induction in \( n \) that

\[
\text{ad}(e)^n(A_0(\eta_s)) = A_n(\eta_s),
\]

\[
\text{ad}(e)^n(A_0([C]_s)) = A_n([C]_s) - nA_{n-1}(\eta_s).
\]

The first equality implies that \( A_n(\eta_s) = 0 \) for \( n > s \). Together with the second equality this implies that \( A_n([C]_s) = 0 \) for \( n > s + 2 \).

\[\square\]

**Lemma 2.4.** One has

\[
A_s(\eta_s)(x) = s! \cdot q_s \cdot x,
\]

\[
A_{s+2}([C]_s)(x) = (s + 2)! \cdot p_{s+1} \cdot x.
\]

Proof. As we have seen in the previous proof, the operator \( A_0([C]_s) \) (resp., \( A_0(\eta_s) \)) is a lowest weight vector of weight \(-(s + 2)\) (resp., \(-s\)) with respect to the \( \mathfrak{sl}_2 \)-action. Since the Fourier transform \( S \) is given by the action of \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) in \( \text{SL}_2 \), it follows that

\[
\text{ad}(e)^{s+2}(A_0([C]_s)) = \lambda_s \cdot S A_0([C]_s) S^{-1},
\]

\[
\text{ad}(e)^s(A_0(\eta_s)) = \mu_s \cdot S A_0(\eta_s) S^{-1}
\]

for some nonzero constants \( \lambda_s, \mu_s \). But \( \text{ad}(e)^s(A_0(\eta_s)) = A_s(\eta_s) \) and

\[
\text{ad}(e)^{s+2}(A_0([C]_s)) = A_{s+2}([C]_s) - (s + 2)A_{s+1}(\eta_s) = A_{s+2}([C]_s)
\]

as we have seen in the proof of Lemma 2.3. Hence,

\[
A_s(\eta_s)(a) = \mu_s \cdot S(\eta_s) \cdot a,
\]

\[
A_{s+2}([C]_s)(a) = \lambda_s \cdot S([C]_s) \cdot a.
\]

Setting \( a = 1 \) we get

\[
(p_1 + p_2)_s(c_1(\mathcal{L})^s \cdot p^*_s \eta_s) = \mu_s \cdot S(\eta_s),
\]

\[
(p_1 + p_2)_s(c_1(\mathcal{L})^{s+2} \cdot p^*_1[C]_s) = \lambda_s \cdot S([C]_s).
\]

Making the change of variables \((x, y) \mapsto (x, x + y)\) in the formula defining the Fourier transform and using the theorem of the cube we get

\[
S(a) = (p_1 + p_2)_s(\exp(c_1(\mathcal{L}) + p^*_1\Delta c_1(\mathcal{L})) \cdot p^*_1 a) = (p_1 + p_2)_s(\exp(c_1(\mathcal{L}) + 2p^*_1 \theta) \cdot p^*_1 a),
\]

(2.2)

where \( a \in \text{CH}(J) \). Applying this to \( a = \eta_s \) and keeping in mind that \( \theta \cdot \eta_s = 0 \) we get

\[
S(\eta_s) = (p_1 + p_2)_s(\exp(c_1(\mathcal{L})) \cdot p^*_1 \eta_s).
\]
Since \( S(\eta_s) \in \text{CH}^{g-s} \), this implies that
\[
S(\eta_s) = (p_1 + p_2)_s(\frac{c_1(L)^s}{s!} \cdot p_1^s(\eta_s)),
\]
so \( \mu_s = s! \). Similarly, applying (2.2) to \( a = [C]_s \) and using the fact that \( \theta^2 \cdot [C]_s = 0 \) and \( S([C]_s) \in \text{CH}^{g-s-1} \) we obtain
\[
S([C]_s) = (p_1 + p_2)_s(\frac{c_1(L)^{s+2}}{(s+2)!} \cdot p_2^s[C]_s) + 2(p_1 + p_2)_s(\frac{c_1(L)^{s+1}}{(s+1)!} \cdot p_1^s(\theta \cdot [C]_s)).
\]
It remains to observe that the second term is proportional to \( A_{s+1}(\eta_s)(1) \), hence it vanishes by Lemma 2.3.

**Lemma 2.5.** For \( n_1, n_2 \geq 0 \) and \( a_1, a_2, b \in \text{CH}(J)_Q \) one has
\[
[A_{n_1}(a_1), A_{n_2}(a_2)](b) = \sum_{i \geq 1} (p_1 + p_2 + p_3)_s \left( \binom{n_1}{i} p_{13}^s c_1(L)^{n_1-i} p_{23}^s c_1(L)^{n_2} - \binom{n_2}{i} p_{13}^s c_1(L)^{n_1} p_{23}^s c_1(L)^{n_2-i} \right)
\cdot p_1^s c_1(L)^i \cdot p_1^s a_1 \cdot p_2^s a_2 \cdot p_3^s b,
\]
where \( p_{ij} : J \times J \times J \to J \times J \) and \( p_i : J \times J \times J \to J \) are the natural projections.

**Proof.** Using the projection formula we find
\[
a_1 \ast_{n_1} (a_2 \ast_{n_2} b) = (p_1 + p_2 + p_3)_s \left( (p_1, p_2 + p_3)^s c_1(L)^{n_1} \cdot p_{23}^s c_1(L)^{n_2} \cdot p_1^s a_1 \cdot p_2^s a_2 \cdot p_3^s b \right).
\]
Similarly,
\[
a_2 \ast_{n_2} (a_1 \ast_{n_1} b) = (p_1 + p_2 + p_3)_s \left( (p_1, p_2 + p_3)^s c_1(L)^{n_2} \cdot p_{23}^s c_1(L)^{n_1} \cdot p_1^s a_2 \cdot p_2^s a_1 \cdot p_3^s b \right)
\cdot (p_1 + p_2 + p_3)_s \left( (p_2, p_1 + p_3)^s c_1(L)^{n_2} \cdot p_{13}^s c_1(L)^{n_1} \cdot p_2^s a_1 \cdot p_3^s a_2 \cdot p_3^s b \right).
\]
It remains to use the equalities
\[
(p_1, p_2 + p_3)^s c_1(L) = p_{12}^s c_1(L) + p_{13}^s c_1(L), \quad (p_2, p_1 + p_3)^s c_1(L) = p_{12}^s c_1(L) + p_{23}^s c_1(L).
\]

Note that from the above lemma (or directly from the definition) one can immediately see that for \( a_1, a_2 \in \text{CH}(J)_Q \) the operators \( A_{n_1}(a_1) \) and \( A_{n_2}(a_2) \) commute. Hence, \([A_{n_1}(\eta_{s_1}), A_{n_2}(\eta_{s_2})] = 0.\)

**Theorem 2.6.** One has the following commutation relations
\[
[A_{n_1}([C]_{s_1}), A_{n_2}([C]_{s_2})] = \left( \binom{n_1}{1} \left( \frac{s_1 + s_2 - n_1 - n_2 + 3}{s_1 - n_1 + 2} \right) - n_2 \cdot \left( \frac{s_1 + s_2 - n_1 - n_2 + 3}{s_2 - n_2 + 2} \right) \right) A_{n_1+n_2-1}([C]_{s_1+s_2}) -\]
\[
2 \cdot \left( \binom{n_1}{2} \left( \frac{s_1 + s_2 - n_1 - n_2 + 3}{s_1 - n_1 + 2} \right) - \binom{n_2}{2} \left( \frac{s_1 + s_2 - n_1 - n_2 + 2}{s_2 - n_2 + 2} \right) \right) A_{n_1+n_2-2} (\eta_{s_1+s_2}).
\]
\[ [A_{n_1}([C]_{s_1}), A_{n_2}(\eta_{s_2})] = \]
\[ (n_1 \left( s_1 + s_2 - n_1 - n_2 + 1 \right) - n_2 \left( s_1 + s_2 - n_1 - n_2 + 1 \right)) A_{n_1+n_2-1}(\eta_{s_1+s_2}). \]

**Proof.** Since \([m]_s[C] = \sum_{s \geq 0} m^{s+2}[C]_s\) (resp., \([m]_*\eta_s = \sum_{s \geq 0} m^s\eta_s\)), the first (resp., the second) commutator is the coefficient with \(m_1^{s_1+2}m_2^{s_2+2}\) (resp., \(m_1^{s_1+2}m_2^{s_2}\)) in the commutator

\[ [A_{n_1}([m]_*[C]), A_{n_2}([m_2]_*[C])]\) (resp., \([A_{n_1}([m_1]_*[C]), A_{n_2}([m_2]_*\eta)]\)).

Let us apply Lemma 2.5 to compute these commutators. Taking into account the formula for \(L_{C\times C}\) we find

\[ [A_{n_1}([m_1]_*[C]), A_{n_2}([m_2]_*[C])] (b) = \]
\[ \sum_{i \geq 1} (p_1 + p_2 + p_3)_* \left( \binom{n_1}{i} p_{13}^* c_1(L)^{n_1-i} p_{23}^* c_1(L)^{n_2} - \binom{n_2}{i} p_{13}^* c_1(L)^{n_1} p_{23}^* c_1(L)^{n_2-i} \right) \cdot m_1^i m_2^i p_{12}^* \left( [m_1] \times [m_2] \right)_* (i \times i)_* ([\Delta C] - [x_0 \times C] - [C \times x_0]) \cdot p_3^* b. \]

Therefore, the sum has only two terms, \(T_1\) and \(T_2\), corresponding to \(i = 1\) and \(i = 2\). We have

\[ T_1 = (p_1 + p_2 + p_3)_* \left( (m_1 p_{13}^* c_1(L)^{n_1-1} p_{23}^* c_1(L)^{n_2} - n_2 p_{13}^* c_1(L)^{n_1} p_{23}^* c_1(L)^{n_2-1}) m_1 m_2 p_{12}^* \right) (\eta_{s_1} [C] - [x_0 \times C] - [C \times x_0]) \cdot p_3^* b. \]

Note that since we are only interested in the coefficient with \(m_1^{s_1+2}m_2^{s_2+2}\) we can discard the terms linear in \(m_1\) or \(m_2\). Therefore, we can replace \(T_1\) with

\[ T_1' = (m_1 + m_2) p_1 + p_2)_* \left( m_1 m_2 (m_1 m_1^{n_1-1} m_2^{n_2} - n_2 m_1^{n_1} m_2^{n_2-1}) c_1(L)^{n_1+n_2-1} \cdot p_1^*[C] \cdot p_2^* b. \right) \]

Using the formula

\[ c_1(L) = \frac{([m_1] + [m_2]) \times \text{id}}{m_1 + m_2} c_1(L) \]

we obtain

\[ T_1' = \frac{m_1 n_1 m_2^{n_1+n_2-1} - n_2 m_1^{n_1+1} m_2^{n_2}}{(m_1 + m_2)^{n_1+n_2-2}} (p_1 + p_2)_* \left( c_1(L)^{n_1+n_2-1} \cdot p_1^*[m_1 + m_2]_* [C] \cdot p_2^* b \right) = \]
\[ (m_1 m_2^{n_1+n_2+1} - n_2 m_1^{n_1+1} m_2^{n_2}) \sum_s (m_1 + m_2)^{s-n_1-n_2+3} A_{n_1+n_2-1}([C]_s)(b). \]

Let us observe that by Lemma 2.3 we can restrict the summation to \(s\) such that \(n_1 + n_2 - 1 \leq s + 2\), i.e., \(s \geq n_1 + n_2 - 3\).

On the other hand, using the formula \([\Delta C]^2 = -\Delta_* K\) we obtain

\[ (i \times i)_* ([\Delta C] - [x_0 \times C] - [C \times x_0]) = 2 \Delta_* \eta. \]

Hence,

\[ T_2 = -2 (p_1 + p_2 + p_3)_* \left( \binom{n_1}{2} p_{13}^* c_1(L)^{n_1-2} p_{23}^* c_1(L)^{n_2} - \frac{n_2}{2} p_{13}^* c_1(L)^{n_1} p_{23}^* c_1(L)^{n_2-2} \right) \]
\[ \cdot m_1^2 m_2^2 p_{12}^* (m_1, m_2)_* \eta \cdot p_3^* b. \]
We can rewrite this as
\[ -T_2 = 2 \frac{(n_1)}{2} \frac{m_1^{n_1} m_2^{n_2}}{m_1 + m_2} (p_1 + p_2) \cdot (c_1(L)^{n_1 + n_2 - 2} \cdot p^*_1 (m_1 + m_2) \cdot p^*_2 b) = \\
2 \left( \left( \frac{n_1}{2} \right) m_1^{n_1} m_2^{n_2} + \left( \frac{n_2}{2} \right) m_1^{n_1 + 2} m_2^{n_2} \right) \sum_s (m_1 + m_2)^{s - n_1 - n_2 + 2} A_{n_1 + n_2 - 2}(\eta_s)(b) \]

Again by Lemma 2.3 we can restrict the summation to \( s \geq n_1 + n_2 - 2 \). Now the required formula for \([A_{n_1}([C]_{s_1}), A_{n_2}([C]_{s_2})]\) follows easily by considering the coefficients with \( m_1^{s_1 + 2} m_2^{s_2 + 2} \) in \( T_1 \) and \( T_2 \).

Following similar steps we can write
\[
[A_{n_1}([m_1], [C]), A_{n_2}([m_2], \eta)](b) = \\
(p_1 + p_2 + p_3) \cdot [(n_1 p^*_1) c_1(L)^{n_1} - n_2 p^*_2 c_1(L)^{n_2} - n_2 p^*_1 c_1(L)^{n_1} p^*_2 c_1(L)^{n_2 - 1}) \\
\cdot m_1 m_2 p^*_2 ((m_1, m_2), \eta - [0] \times [m_2], \eta) \cdot p^*_2 b].
\]

Since we are interested in the coefficient with \( m_1^{s_1 + 2} m_2^{s_2} \), we can discard the term linear in \( m_1 \). Hence, \([A_{n_1}([C]_{s_1}), A_{n_2}([m_2], \eta)]\) is equal to the coefficient with \( m_1^{s_1 + 2} m_2^{s_2} \) in
\[
\frac{n_1 m_1^{n_1} m_2^{n_2 + 1} - n_2 m_1^{n_1 + 1} m_2^{n_2}}{(m_1 + m_2)^{n_1 + n_2 - 1}} (p_1 + p_2) \cdot (c_1(L)^{n_1 + n_2 - 1} \cdot p^*_1 (m_1 + m_2) \cdot p^*_2 b) = \\
(n_1 m_1^{n_1} m_2^{n_2 + 1} - n_2 m_1^{n_1 + 1} m_2^{n_2}) \sum_s (m_1 + m_2)^{s - n_1 - n_2 + 1} A_{n_1 + n_2 - 1}(\eta_s)(b),
\]
where the summation can be restricted to \( s \geq n_1 + n_2 - 1 \) by Lemma 2.3. This immediately implies the result.

Setting
\[
\tilde{X}_{k,n} = k! \cdot A_n([C]_{k+n-2}), \\
Y_{k,n} = k! \cdot A_n(\eta_{k+n})
\]
for \( n \geq 0, k \geq 0 \) we see that these operators satisfy the commutation relations
\[
[\tilde{X}_{k,n}, \tilde{X}_{k',n'}] = (nk' - n'k) \tilde{X}_{k+k'-1,n+n'-1} - 4 \cdot \binom{n}{2} \binom{k'}{2} - \binom{n'}{2} \binom{k}{2} Y_{k+k'-2,n+n'-2},
\]
\[
[\tilde{X}_{k,n}, Y_{k',n'}] = (nk' - n'k) Y_{k+k'-1,n+n'-1},
\]
where we set \( \tilde{X}_{k,n} = Y_{k,n} = 0 \) for \( k < 0 \) (note that this convention agrees with Lemma 2.3).

Also, by Lemma 2.4 we have
\[
\frac{1}{n!} Y_{0,n}(a) = q_n \cdot a,
\]
\[
\frac{1}{n!} \tilde{X}_{0,n}(a) = p_{n-1} \cdot a.
\]
Lemma 2.7. One has
\[ \tilde{X}_{2,0} = -2f, \]
\[ \tilde{X}_{1,1} = -h + g \cdot \text{id}, \]
\[ \tilde{X}_{0,2} = 2e. \]

Proof. The first equality holds by the definition of \( f \). The third equality follows from (2.5). It remains to use the relation
\[ \frac{1}{4} \cdot [\tilde{X}_{0,2}, \tilde{X}_{2,0}] = \tilde{X}_{1,1} - Y_{0,0} = \tilde{X}_{1,1} - g \cdot \text{id}. \]

Lemma 2.8. One has the following relations
\[ \text{ad}(e)^n(Y_{k,0}) = \frac{k!}{(k-n)!} \cdot Y_{k-n,n}, \]
\[ \text{ad}(f)^n(Y_{0,k}) = \frac{k!}{(k-n)!} \cdot Y_{n,k-n}, \]
\[ \text{ad}(e)^n(\tilde{X}_{k,0}) = \frac{k!}{(k-n)!} \cdot \left( \tilde{X}_{k-n,n} - n(k-n)Y_{k-n-1,n-1} \right), \]
\[ \text{ad}(f)^n(\tilde{X}_{0,k}) = \frac{k!}{(k-n)!} \cdot \left( \tilde{X}_{n,k-n} - n(k-n)Y_{n-1,k-n-1} \right). \]

Proof. We have
\[ [e, Y_{k,n}] = \frac{1}{2} [\tilde{X}_{0,2}, Y_{k,n}] = kY_{k-1,n+1}, \]
\[ [e, \tilde{X}_{k,n}] = \frac{1}{2} [\tilde{X}_{0,2}, \tilde{X}_{k,n}] = k\tilde{X}_{k-1,n+1} - k(k-1)Y_{k-2,n}. \]
From this one can easily deduce the first and the third formulas by induction in \( n \). The other two are proved in the same way since our relations are skew-symmetric with respect to switching \( \tilde{X}_{k,n} \) with \( \tilde{X}_{n,k} \) and \( Y_{k,n} \) with \( Y_{n,k} \). □

Finally, we set
\[ X_{k,n} = \tilde{X}_{k,n} - knY_{k-1,n-1} = k! \cdot A_n([C]_{k+n-2}) - (k-1)! \cdot A_{n-1}(\eta_{k+n-2}). \]

(2.6)

Proof of Theorem 0.1. An easy computation shows that \( X_{k,n} \) and \( Y_{k,n} \) satisfy relations (0.1) and (0.2). It remains to check (0.3) and (0.4). We have
\[ SY_{m,0}S^{-1} = m!SA_0(\eta_m)S^{-1} = m!q_m = A_m(\eta_m) = Y_{0,m}, \]
\[ S\tilde{X}_{m,0}S^{-1} = m!SA_0([C]_{m-2})S^{-1} = m!p_{m-1} = A_m([C]_{m-2}) = \tilde{X}_{0,m}. \]
Since \( SeS^{-1} = -f \) we immediately derive (0.4) and (0.3) from Lemma 2.8. □
3. Proof of Theorem 0.2.

(i) Let us denote by $T'\, CH(J) \subset CH(J)$ the subring generated by the classes $(p_n)$ and $(q_n)$. Consider the operator $f = -SeS^{-1} = -\tilde{X}_{2,0}/2 = -X_{2,0}/2$ on $CH(J)$. We are going to show that it preserves $T'\, CH(J)$. Note that

$$[[f, p_n], p_m] = \frac{1}{(m+1)!n!}[X_{1,n-1}, X_{0,m+1}] = -\frac{1}{m!n!}X_{0,m+n} = -\binom{m+n}{m}p_{m+n-1},$$

$$[[f, p_n], q_m] = \frac{1}{n!m!}[X_{1,n}, Y_{0,m}] = -\frac{1}{n!(m-1)!}Y_{0,m+n-1} = -\binom{m+n-1}{m-1}q_{m+n-1},$$

and $[[f, q_n], q_m] = 0$. (3.3)

On the other hand, from the definition of $q_n$ we derive

$$q_n = -Se([C]_n) = fS([C]_n) = f(p_{n+1}).$$

Since $f(1) = 0$, this gives

$$[f, p_n](1) = f(p_n) = q_{n-1}. \quad (3.4)$$

Also,

$$0 = -Se(\eta_n) = fS(\eta_n) = f(q_{n}),$$

so

$$[f, q_n](1) = 0. \quad (3.5)$$

We claim that these relations imply that for any polynomial $F$ in $(p_n)$ and $(q_n)$ one has

$$f(F) = -D(F),$$

where $D$ is the differential operator defined in the formulation of the theorem. Indeed, from relations (3.1)-(3.3) we see that

$$[[f, x], y](F) = -[[D, x], y](F),$$

where $x$ and $y$ are any of the classes $(p_n)$ or $(q_n)$. By induction in the degree this implies that

$$[f, p_n](F) = -[D, p_n](F)$$

and

$$[f, q_n](F) = -[D, q_n](F),$$

where the base of induction follows from relation (3.4) and (3.5). Finally, another induction in degree proves (3.6).

Thus, we proved the operator $f \in sl_2$ preserves $T'\, CH(J)$ and acts on it by the differential operator $-D$. Since $T'\, CH(J)$ is also closed under the operator $e \in sl_2$, it follows that $T'\, CH(J)$ is closed under the Fourier transform $S = \exp(e) \exp(-f) \exp(e)$. Therefore, $T'\, CH(J)$ coincides with the tautological subring $T\, CH(J)$. 


(ii) For \( m = 0 \) the operator \( X_{0,n} \) (resp., \( Y_{0,n} \)) acts as multiplication by \( n!p_{n-1} \) (resp., \( n!q_n \)). In particular, they preserve \( \mathcal{T} \text{CH}(J)_Q \). The general case follows from this by induction in \( m \) using commutation relations

\[
[X_{2,0}, X_{m,n}] = -2nX_{m+1,n-1}, \quad [X_{2,0}, Y_{m,n}] = -2nY_{m+1,n-1}
\]

together with the fact that \( X_{2,0}/2 \) acts on \( \mathcal{T} \text{CH}(J)_Q \) via the operator \( \mathcal{D} \).

\[
\Box
\]

4. Some relations in \( \mathcal{T} \text{CH}(J)_Q \)

Let us denote by \( \mathbb{Q}[q] \subset \mathcal{T} \text{CH}(J)_Q \) the subring generated by the classes \( (q_n) \). First, we collect some general observations in the following

**Proposition 4.1.**

(i) \( \oplus_s \mathcal{T} \text{CH}(J)_s^* = \mathbb{Q}[q] \).

(ii) \( \mathcal{T} \text{CH}(J)_s^* = p_1^{g-s} \cdot \mathcal{T} \text{CH}(J)_s^* = p_1^{g-s} \cdot (\text{CH}_s \cap \mathbb{Q}[q]) \).

(iii) \( \mathcal{T} \text{CH}(J)_g^* = 0 \).

**Proof.**

(i) This follows from the fact that \( q_n \in \text{CH}(J)_n \) and \( p_n \in \text{CH}(J)_{n-1}^n \).

(ii) Since \( f \) acts on \( \mathcal{T} \text{CH}(J)_Q \) by the operator \(-\mathcal{D}\), it preserves the subring \( \mathbb{Q}[p_1, q] \) generated by \( p_1 \) and \( (q_n) \). Hence, the Fourier transform \( S \) also preserves this subring. But \( \mathcal{T} \text{CH}(J)_g^* = S(\text{CH}_g^*) \), so the assertion follows from (i).

(iii) It is enough to prove that \( q_{n_1} \ldots q_{n_k} = 0 \) for \( n_1 + \ldots + n_k = g \). We can use induction in \( k \). The base of induction follows from

\[
q_g = -\mathcal{D}(p_{g+1}) = 0.
\]

Assume the assertion holds for \( k - 1 \). Then for \( n_1 + \ldots + n_k = g \) we have

\[
0 = -\mathcal{D}(q_{n_1} \ldots q_{n_{k-1}} p_{n_{k+1}}) = q_{n_1} \ldots q_{n_{k-1}} q_{n_k},
\]

since all the other terms vanish by the induction assumption.

Part (ii) of the above proposition implies that for every \( n_1 + \ldots + n_k + m_1 + \ldots + m_t = g \) we have a relation of the form

\[
p_{n_1} \ldots p_{n_k} q_{m_1} \ldots q_{m_t} = p_1^k f(q).
\]

The simplest example of such a relation is

\[
p_g = p_1 q_{g-1}
\]

obtained by applying \( \mathcal{D} \) to \( p_1 p_g = 0 \). Similarly, applying \( \mathcal{D} \) to \( p_1 p_{q_{g-i}} = 0 \) we get

\[
p_{g-i} q_{g_i} = p_1 q_{g_i - 1} q_{g_i} - \binom{g-1}{i} p_1 q_{g-1}.
\]

**Proposition 4.2.** The ring \( \mathcal{T} \text{CH}(J)_Q \) is generated over \( \mathbb{Q} \) by the classes \( (p_n)_{n < g/2 + 1} \) and \( (q_n)_{n < 2g+1/2} \). Furthermore, for \( n \geq g/2 + 1 \) the class \( p_n \) belongs to the ideal generated by \( (q_i)_{i \geq 1} \).

**Proof.**

First, let us prove that for \( n \geq \frac{g+1}{2} \) the class \( q_n \) can be expressed in terms of \( (q_i)_{i < n} \). The idea is to represent \( n \) in the form \( n = n_1 + \ldots + n_k - k \), where \( n_i \geq 2 \) for all \( i \) and \( n_1 + \ldots + n_k \geq g + 1 \). Then it is enough to check \( 0 = \mathcal{D}^k(p_{n_1} \ldots p_{n_k}) \) is a polynomial of degree \( n \) in \( (q_i)_{i \leq n} \) (where \( \text{deg } q_i = i \)) that has a nonzero coefficient with \( q_n \). Note that \( \mathcal{D} \)
preserves the subring generated by all the classes \((p_i)\), where \(i \geq 2\), together with all the classes \((q_i)\) and acts on this subring as the operator \(\mathcal{D}' = \mathcal{D}'_0 - \mathcal{D}'_1\), where

\[
\mathcal{D}'_0 = \frac{1}{2} \sum_{m,n \geq 2} \binom{m+n}{n} p_{m+n-1} \partial_{p_n} \partial_{p_m} + \sum_{m \geq 1, n \geq 2} \binom{m+n-1}{n} q_{m+n-1} \partial_{q_n} \partial_{p_n},
\]

\[
\mathcal{D}'_1 = \sum_{n \geq 2} q_{n-1} \partial_{p_n}.
\]

Let us consider two more gradings \(\deg_p\) and \(\deg_q\) on the algebra of polynomials in \((p_i)\) and \((q_i)\) such that \(\deg_p(p_i) = \deg_q(q_i) = 1\) and \(\deg_p(p_i) = \deg_q(p_i) = 0\). Since \(\mathcal{D}\) decreases \(\deg_p\) by \(1\), we obtain that \(\mathcal{D}^k(p_{n_1} \ldots p_{n_k})\) is a polynomial in \((q_i)\). Furthermore, since \(\mathcal{D}'_0\) preserves \(\deg_q\) and \(\mathcal{D}'_1\) raises it by \(1\), we have

\[
\mathcal{D}^k(p_{n_1} \ldots p_{n_k}) = (\mathcal{D}')^k(p_{n_1} \ldots p_{n_k}) = (\mathcal{D}'_0)^k(p_{n_1} \ldots p_{n_k}) - \lambda \cdot q_n + f(q_1, \ldots, q_{n-1}),
\]

where

\[
\lambda \cdot q_n = \sum_{i=1}^{k} (\mathcal{D}'_0)^{i-1} \mathcal{D}'_1 (\mathcal{D}'_0)^{k-i}(p_{n_1} \ldots p_{n_k}).
\]

Now it is clear from the explicit form of \(\mathcal{D}'_0\) and \(\mathcal{D}'_1\) that \((\mathcal{D}'_0)^k(p_{n_1} \ldots p_{n_k}) = 0\) and that \(\lambda > 0\).

Next, let us show that for \(n \geq g/2 + 1\) the class \(p_n\) belongs to the ideal generated by \((q_i)_{i \geq 1}\) in the subring generated by \((p_i)_{i < n}\) over \(\mathbb{Q}[q]\). To this end we represent \(n\) in the form \(n = n_1 + \ldots + n_k - k + 1\), where \(n_i \geq 2\) for all \(i\) and \(n_1 + \ldots + n_k \geq g\). Now let us consider the class

\[
a = \mathcal{D}^{k-1}(p_{n_1} \ldots p_{n_k}) = (\mathcal{D}')^{k-1}(p_{n_1} \ldots p_{n_k}).
\]

Note that if \(n_1 + \ldots + n_k = g\) then we have \(p_{n_1} \ldots p_{n_k} \in \mathcal{C}_{g-k}^g(J)\). Hence, by Proposition 4.1(ii) the class \(p_{n_1} \ldots p_{n_k}\) belongs to the ideal generated by \((q_i)\) in the subring \(\mathbb{Q}[p_1, q] \subset \mathfrak{T} \mathcal{C}(J)_{\mathbb{Q}}\). Therefore the same is true about the class \(a\) (and if \(n_1 + \ldots + n_k > g\) then \(a = 0\)). On the other hand, we can write \(\mathcal{D}' = \mathcal{D}'_p + \mathcal{D}'_q\), where

\[
\mathcal{D}'_p = \frac{1}{2} \sum_{m,n \geq 2} \binom{m+n}{n} p_{m+n-1} \partial_{p_n} \partial_{p_m},
\]

\[
\mathcal{D}'_q = \sum_{m \geq 1, n \geq 2} \binom{m+n-1}{n} q_{m+n-1} \partial_{q_n} \partial_{p_n} - \sum_{n \geq 2} q_{n-1} \partial_{p_n}.
\]

Since \(\deg_p(a) = 1\) and since the image of \(\mathcal{D}'_q\) is contained in the ideal generated by \((q_i)\), we obtain

\[
a = (\mathcal{D}')^{k-1}(p_{n_1} \ldots p_{n_k}) = \mu \cdot p_n + a',
\]

where

\[
\mu \cdot p_n = (\mathcal{D}'_p)^{k-1}(p_{n_1} \ldots p_{n_k})
\]

and \(a'\) is a linear combination of \(f_i(q)p_{n-i}\) with \(i \geq 1\) and \(f_i(q) \in \mathbb{Q}[q]\). It is clear from the formula for \(\mathcal{D}'_p\) that \(\mu > 0\), so our assertion follows.

For example, if \(g = 2\) then \(\mathfrak{T} \mathcal{C}(J)_{\mathbb{Q}}\) is generated by \(p_1\) and \(q_1\). In fact, in this case we have \(q_2 = q_1^2 = 0\) (by Proposition 4.1) and \(p_2 = p_1q_1\) (by (4.1)). For \(g = 3\) the above
proposition states that $\mathcal{T}\operatorname{CH}(J)_\mathbb{Q}$ is generated by $p_1$, $p_2$ and $q_1$. Indeed, first we see that $q_3 = q_1 q_2 = q_1^3 = 0$ and $p_3 = p_1 q_2$. Also, applying $\mathcal{D}^2$ to $p_2^2 = 0$ we derive the relation $q_2 = q_1^2/4$ (and hence $p_3 = p_1 q_1^2/4$). In addition, (4.2) gives $p_2 q_1 = \frac{3}{2} p_1 q_1^2$.

Finally, let us show that the classes of the subvarieties of special divisors $W_d \subset J$ can be expressed as universal polynomials in $(p_n - q_n)$. Recall that $W_d$ is the image of the natural map $C^{[d]} \to J : D \to \mathcal{O}_C(D - dx_0)$. Let us set $w_i = [W_{g-i}] \in \operatorname{CH}^i(J)$.

**Proposition 4.3.** One has $-p_k + q_k = N^k(w)$, where $N^k(w)$ is the $k$-th Newton polynomial in the classes $(w_i)$: $N^k(w) = \frac{1}{k!} \sum_{i=1}^g \lambda_i^k$, where $(\lambda_i)$ are roots of $\lambda^g - w_1 \lambda^{g-1} + \ldots + (\lambda q_1^2 = 4p_1 q_2$. 

We will use the formula $\delta^* E_d = \delta \mathcal{O}_C(dx_0)$ and $\delta : J \to J$ is the involution $x \mapsto \kappa - x$, where $\kappa \in J$ corresponds to the line bundle $K(-2g - 2)x_0$. (see [12], sec. 19.5). On the other hand, using Grothendieck-Riemann-Roch formula one can easily find that

$$\operatorname{ch}(E_d) = p_i - q_i$$

for $i > 0$ (where $\operatorname{ch}_i$ is the component of the Chern character of codimension $i$). Hence,

$$\operatorname{ch}_1(\delta^* E_d) = \delta_*(p_i - q_i) = [\kappa] * [-1]_*(p_i - q_i) = (-1)^i - 1 [\kappa] * (p_i + q_i).$$

Next, we observe that

$$S([\kappa]) = \exp(c_1(\mathcal{L}_\kappa)) = \exp(2q_1).$$

Using Theorem 0.1 we obtain that

$$[\kappa] * a = \exp(2Y_{1,0})(a)$$

for $a \in \operatorname{CH}^*(J)_\mathbb{Q}$. Now the formula for $Y_{1,0}$ from Theorem 0.2(ii) implies that

$$[\kappa] * q_i = q_i, \quad [\kappa] * p_i = p_i - 2q_i.$$

Therefore,

$$\operatorname{ch}_i(\delta^* E_d) = (-1)^i - 1 [\kappa] * (p_i + q_i) = (-1)^i - 1 (p_i - q_i).$$

Combining this with (4.3) we immediately derive our assertion. \qed

Thus, we have $p_1 - q_1 = -w_1$, $p_2 - q_2 = w_2 - w_1^2/2$, $p_3 - q_3 = w_3/2 + w_2w_1/2 - w_1^3/6$, etc. Hence, we can express the classes $(w_i)$ as polynomials in $(p_i - q_i)$.

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