New results from a number operator interpretation of the compositeness of bound and resonant states

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Abstract

A novel theoretical approach to the problem of the compositeness ($X$) of a resonance or bound state is developed on the basis of the expectation values of the number operators of the free particles in the continuum. This formalism is specially suitable for effective field theories in which the bare elementary states are integrated out but that give rise to resonance and bound states when implemented in non-perturbative calculations. We demonstrate that $X = 1$ for finite-range energy-independent potentials, either regular or singular. A non-trivial example for an energy-dependent potential is discussed where it is shown that $X$ is independent of any type of cutoff regulator employed. The generalization of these techniques to relativistic states is developed. We also explain how to obtain a meaningful compositeness with respect to the open channels for resonances, even if it is complex in a first turn, by making use of suitable phase-factor transformations. Defining elementariness as $X = 0$, we derive a new universal criterion for the elementariness of a bound state. Along the same lines, a necessary condition for a resonance to be qualified as elementary is given. The application of the formalism here developed might be of considerable practical interest.
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1 Introduction and basic definitions

The problem we consider is to discern whether a bound state or a resonance is elementary or composite with respect to the asymptotic states in the theory. E.g. a Hydrogen atom is a composite state of a proton and an electron in which these two particles are typically separated by a much larger distance than its intrinsic sizes (this is a simple example of what is usually referred to as a “molecular” state). The Hydrogen wave function can be fully expressed as a Fourier transform in terms of free plane waves of a proton and an electron. In turn, the proton appears as a bound state in the $P$-wave of $n\pi^+$ scattering. However, the proton, with a charge radius less than 0.9 fm, cannot be certainly qualified as a “molecular” state of a $n\pi^+$ but rather the opposite. Another example in this direction is the baryon Λ, a resonance that decays by weak interactions into $p\pi^-$ and $n\pi^0$, and despite this splitting the Λ baryon is not a composite object made up by protons and pions. Our work here is a continuation in the historical effort to quantify the weight of the asymptotic states in a bound or resonance state.

Since the earlier seminal papers treating the problem of compositeness/elementariness of a state in terms of the asymptotic particles [1–7], it has come clear that this set up cannot answer all the interesting questions on this respect. We have in mind here the clear example of Quantum Chromodynamics (QCD), which appeared later than all these papers [8]. According to this theory the proton is indeed a composite object of three valence quarks, though the quarks are not asymptotic states because of the phenomenon of color confinement in QCD [9]. Therefore, the possible impact of underlying degrees of freedom to the actual observed spectrum leaves open the question whether experiment can decide what sort of elementary particles exist [4]. Nonetheless, it is a definitively settled matter once it is demonstrated that a bound or resonance state is not elementary in terms of the asymptotic degrees of freedom.

Let us discuss first the case of a bound state within non-relativistic quantum mechanics (NRQM) as a prototypical example of the compositeness relation. We will discuss later its relativistic generalization within Quantum Field Theory (QFT). We follow at this stage the basic set up discussed in Refs. [4,5], and split the full Hamiltonian $H$ in an unperturbed free-particle part $H_0$ plus an interaction $V$,

\[ H = H_0 + V. \]  

(1)

The spectrum of the full Hamiltonian consists of the continuum states

\[ H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle, \]  

(2)

and it might contain also discrete bound states $|\psi_n\rangle$

\[ H|\psi_n\rangle = E_n|\psi_n\rangle. \]  

(3)

The continuum eigenstates of $H$ are normalized to Dirac delta functions and the discrete ones to Kronecker deltas.

In turn the free-particle Hamiltonian $H_0$ also contains the continuum spectrum and, in addition, there may be discrete states (or bare elementary ones). To fix the notation,

\[ H_0|\varphi_\alpha\rangle = E_\alpha|\varphi_\alpha\rangle, \]  

(4)

\[ H_0|\varphi_n\rangle = E_n|\varphi_n\rangle. \]

The eigenstates of $H_0$ fulfill the completeness relation

\[ I = \int d\alpha|\varphi_\alpha\rangle\langle\varphi_\alpha| + \sum_n |\varphi_n\rangle\langle\varphi_n|. \]  

(5)
with $H$ and $H_0$ sharing the same spectrum [10,11]. Given a bound state $|\psi_B\rangle$ of $H$ with energy $E_B$

$$H|\psi_B\rangle = E_B|\psi_B\rangle,$$  

we express it in terms of the eigenstates of $H_0$ as

$$|\psi_B\rangle = \int d\alpha \langle \phi_\alpha | \psi_B \rangle |\phi_\alpha\rangle + \sum_n \langle \phi_n | \psi_B \rangle |\phi_n\rangle.$$  

(7)

Since $|\psi_B\rangle$ is normalized to unity, it follows that

$$\langle \psi_B | \psi_B \rangle = 1 = \int d\alpha |\langle \phi_\alpha | \psi_B \rangle|^2 + \sum_n |\langle \phi_n | \psi_B \rangle|^2 = Z + X,$$  

(8)

where

$$X = \int d\alpha |\langle \phi_\alpha | \psi_B \rangle|^2,$$  

$$Z = \sum_n |\langle \phi_n | \psi_B \rangle|^2.$$  

These quantities are usually called compositeness ($X$) and elementariness ($Z$).

Making use of the Schrödinger equation, written in the form $(H_0 + V)|\psi_B\rangle = E_B|\psi_B\rangle$, we can express $X$ from Eq. (6) as [5]

$$X = 1 - Z = \int d\alpha |\langle \phi_\alpha | V | \psi_B \rangle|^2 \frac{1}{(E_\alpha - E_B)^2}.$$  

(11)

Notice that the integrand in the previous equation is just the modulus square of the continuum part of the bound-state wave function $\psi_B(\alpha)$. The latter is given by

$$\psi_B(\alpha) = \frac{\langle \phi_\alpha | V | \psi_B \rangle}{E_\alpha - E_B}.$$  

(12)

This equation is well-known in ordinary quantum mechanics for energy-independent local potentials [12,13]. Within a more general scenario, Eq. (11) expresses the fact that it might not be normalized to 1 when there are elementary states ($Z \neq 0$).

## 2 A different perspective on the compositeness of a bound state

We now offer a reinterpretation of the concept of compositeness $X$ introduced in the previous section. This allows one to calculate $X$ by focusing entirely on the free particle spectrum, which is certainly the one always accessible in scattering/production experiments, without the need to introduce the bare-elementary state contribution to the normalization to 1 of the bound (resonance) state.

Our main motivation here lies in the fact that in many applications within effective field theory (EFT), according to our own experience, the bare elementary discrete states are typically integrated out and do not appear explicitly in the Lagrangian of the theory (which is written in terms of “low-energy” effective degrees of freedom). Nonetheless, one can still generate bound states and resonances after complementing the perturbative calculations in the corresponding EFT with nonperturbative techniques. Some examples in this respect can be found e.g. in Refs. [14–20]. In particular, a near-threshold bare elementary discrete
state can be mimicked by including a Castillejo-Dalitz-Dys on pole in the scattering amplitude of the free continuum states. Explicit examples are worked out in Refs. [17, 21]. It is also the case that \( H \) might be expressed in terms of degrees of freedom that are not asymptotically free, as it occurs in Quantum Chromodynamics in ordinary conditions. Therefore, trying to calculate the wave-function renormalization factor \( Z \) is not practical in such situations and we better derive results from the knowledge of the scattering operator \( T \) among the effective degrees of freedom. Of course, in the situations that fit the scheme presented in Sec. 1 one could calculate \( Z \) as explained there or pose the problem in the terms that we expose next.

The new perspective on \( X \) heavily relies on the number operator for a given particle species, a basic concept in QFT [11]. For definiteness, let us take two particle species \( A \) and \( B \) whose annihilation/creation operators are denoted by \( a_\alpha/a_\alpha^\dagger \) and \( b_\beta/b_\beta^\dagger \), respectively. In terms of them \( H_0 \) reads

\[
H_0 = \int d\alpha E_\alpha a_\alpha^\dagger a_\alpha + \int d\beta E_\beta b_\beta^\dagger b_\beta + \sum_n E_n |\varphi_n \rangle \langle \varphi_n|.
\]

(13)

The decomposition of the bound state in eigenstates of \( H_0 \), Eq. (7), reads now

\[
|\psi_B \rangle = \int d\gamma \langle AB_\gamma |\psi_B \rangle |AB_\gamma \rangle + \sum_n \langle \varphi_n |\psi_B \rangle |\varphi_n \rangle.
\]

(14)

For a given particle species \( A \) its number operator is denoted by \( N^A_D \) and defined by

\[
N^A_D = \int d\alpha a_\alpha^\dagger a_\alpha.
\]

(15)

Here the subscript \( D \) refers to the Dirac or interaction image. Notice that since \( N_D \) and \( H_0 \) obviously commute then

\[
N^A_D(t) = e^{iH_0t}N^A_D(0)e^{-iH_0t} = N_D.
\]

(16)

Based on the number operators of \( A \) and \( B \) we define the compositeness \( X \) of the bound state \( |\psi_B \rangle \) as

\[
X = \frac{1}{2}\langle \psi_B |N^A_D + N^B_D|\psi_B \rangle.
\]

(17)

That is, \( X \) is the expectation value of the number operators of the free-particle constituents in the eigenstate \( |\psi_B \rangle \) of \( H \) divided by their nominal number, which in this case is 2.

We can see that the new definition of \( X \) is equivalent to the original one of Eq. (9) because \( (N^A_D + N^B_D)|AB_\gamma \rangle = 2|AB_\gamma \rangle \) and the annihilation operators \( a_\alpha \) and \( b_\beta \) destroy the bare elementary discrete states present in Eq. (14). It then follows that \( X \), as defined in Eq. (17), reads

\[
X = \int d\gamma |\langle AB_\gamma |\psi_B \rangle|^2,
\]

(18)

as in Eq. (9).

In general, if we are applying NRQM to a bound state \( |\psi_B \rangle \) of \( n \) particles corresponding to \( m \) particle species \( A_1, \ldots, A_m \), the compositeness is defined by a straightforward generalization of the two-body case of Eq. (17) as

\[
X = \frac{1}{n}\langle \psi_B |\sum_{i=1}^m N^A_i|\psi_B \rangle.
\]

(19)
To simplify the notation in the following the sum over the number operators is denoted simply by \( N_D \)

\[
N_D = \sum_{i=1}^{m} N^D_{Ai} .
\]  

(20)

It is worth stressing that for a given total Hamiltonian \( H \) with regular interactions \( V \) (it is sufficient that it fulfills in momentum space the Eqs. (74) and (75) below \[22\]) the compositeness \( X \) is an observable (this is similarly expressed at the end of Sec.IV of Ref. \[4\]). Accordingly to the postulates of NRQM this is clear from its new interpretation in Eq. (19) as the expectation value of a linear self-adjoint operator. The same comment also applies over partial compositeness coefficients.

However, for more singular interactions a nonperturbative regularization process is required, which is an issue that is not fully settled (we elaborate more on this point in Sec. 3). In this regards, so as to appreciate the limitation of straightforward extrapolations of perturbative results in renormalization theory to nonperturbative calculations, it is written in Ref. \[5\] that generally \( Z^{-1} \) is divergent in (relativistic) QFT calculations, but it seems reasonable to expect that this is a failure of perturbation theory, and not that \( Z \) is really zero for all particles. We show in Sec. 4.2 that the total compositeness \( X \) is one for a general finite-range energy-independent potential, irrespectively of whether it is regular or singular. We also give two interesting examples in which \( X \) is independent of the type of cutoff regularization employed for energy-dependent potentials.

3 Quantum Field Theory calculation of \( X \)

An interesting consequence of the new definition for \( X \), Eq. (19), is that it is amenable to a direct computation within NR Quantum Field Theory (QFT). To show it let us consider the Dirac or interaction picture and introduce the interaction adiabatically

\[
V \rightarrow V e^{-\varepsilon |t|}
\]

(21)

with \( \varepsilon \rightarrow 0^+ \). At time \( t \) the states \( |\varphi(t)\rangle \) in the Dirac picture are related to the states \( |\psi\rangle \) in the Heisenberg picture by

\[
|\varphi(t)\rangle = e^{iH_0 t} e^{-iHt} |\psi\rangle = U_D(t,0) |\psi\rangle ,
\]

\[
|\varphi(0)\rangle = |\psi\rangle .
\]

(22)

The time evolution operator for the Dirac states is denoted by \( U_D(t_2, t_1) \) and it corresponds to \( \hat{U}_D(t_2, t_1) = e^{iH_0 t_2} e^{-iH(t_2-t_1)} e^{-iH_0 t_1} \). In particular, the bound state \( |\psi_B\rangle = |\varphi_B(0)\rangle \) can be expressed by the time evolution from the asymptotic bare elementary discrete state \( |\varphi_B\rangle \) as

\[
|\psi_B\rangle = U_D(0, -\infty) |\varphi_B\rangle ,
\]

\[
|\psi_B\rangle = U_D(0, +\infty) |\varphi_B\rangle .
\]

(23)

In this way

\[
X = \frac{1}{n} \langle \varphi_B | U_D(+\infty,0) N_D U_D(0, -\infty) | \varphi_B \rangle ,
\]

(24)

where we have used that the Möller matrix \( U_D(0, +\infty) \) satisfies that \( U_D(0, +\infty)^\dagger U_D(0, +\infty) = I \) \[23\]. The previous matrix element can be written in a time-ordered way by introducing an extra time evolution from 0 to \( t \). For that let us notice that

\[
|\varphi_B(t)\rangle = U_D(t,0) |\psi_B\rangle = e^{iH_0 t} e^{-iHt} |\psi_B\rangle = e^{iH_0 t} e^{-iE_B t} U_D(t,\pm\infty) |\varphi_B\rangle .
\]

(25)
Therefore, equating the last step with the second one, we can express

\[ U_D(0, -\infty) \vert \varphi_B \rangle = e^{iE_B t}e^{-iH_0 t}U_D(t, 0) \vert \psi_B \rangle = e^{iE_B t}e^{-iH_0 t}U_D(t, -\infty) \vert \varphi_B \rangle \]  

(26)

and similarly,

\[ \langle \varphi_B \vert U(+\infty, 0) = \langle \varphi_B \vert U_D(+\infty, t)e^{iH_0 t} \]  

(27)

Next, we replace Eqs. (26) and (27) into Eq. (24) which then reads

\[ X = \frac{1}{n} \langle \varphi_B \vert U_D(+\infty, t)e^{iH_0 t}N_D e^{iE_B t}e^{-iH_0 t}U_D(t, 0) \vert \psi_B \rangle \]  

(28)

for arbitrary \( t \). The phase factors \( e^{\pm iE_B t} \) cancel out while

\[ e^{iH_0 t}N_D e^{-iH_0 t} = N_D(t) = N_D, \]  

(29)

recall Eq. (16). In this way, after averaging in \( t \), the Eq. (24) becomes

\[ X = \frac{1}{n} \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} dt \langle \varphi_B \vert U_D(+\infty, t)N_D(t)U_D(t, -\infty) \vert \varphi_B \rangle, \]  

(30)

which is the form that we are seeking for. The factor \( 1/T \) in the previous equation cancels in the limit \( T \to +\infty \) with the Dirac delta function of total energy conservation (times \( 2\pi \)).

It might be advantageous to express the number operator in terms of NR fields in Eq. (30), e.g. in order to apply Feynman diagrams for its calculation. For a generic scalar particle species \( A_i \) of physical mass \( m_{A_i} \) we have the free field

\[ \psi_{A_i}(x) = \int \frac{d^3q}{(2\pi)^3} a_i(q)e^{-i\tilde{q}x}, \]  

(31)

where \( q^0 = q^2/2m_{A_i}, \tilde{q} = (q^0, q) \) and \( x = (t, \mathbf{x}) \). It is then straightforward to show that

\[ N_D = \sum_i \int d^3x \psi_{A_i}^\dagger(x)\psi_{A_i}(x). \]  

(32)

Inserting this expression into Eq. (30) it reads

\[ X = \frac{1}{n} \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} dt \int d^3x \langle \varphi_B \vert U_D(+\infty, t) \sum_i \psi_{A_i}^\dagger(x)\psi_{A_i}(x)U_D(t, -\infty) \vert \varphi_B \rangle, \]  

\[ = \frac{1}{n} \lim_{T \to +\infty} \frac{1}{T} \int d^4x \langle \varphi_B \vert P \left[ e^{-i \int^{+\infty}_{-\infty} dt' V_D(t') \sum_i \psi_{A_i}^\dagger(x)\psi_{A_i}(x) \right] \vert \varphi_B \rangle. \]  

(33)

Here we denote the time-ordered product by \( P \) and \( V_D(t) \) is the interaction in the Dirac picture.\(^1\) The extension to particles with other spin is straightforward.

\(^1\)In Eq. (33) only the connected diagrams should be considered [24].
Figure 1: Feynman diagrams for the calculation of $X$ within NR QFT for the two-particle case. The insertion of the number operators for the particles $A$ and $B$ is indicated by the double dots.

For the two-particle case, with particles of types $A$ and $B$, the evaluation of $X$ according to Eq. (30) corresponds to the calculation of the diagrams in Fig. 1. Its evaluation is straightforward and in the $\ell S$ basis (with $\ell$ the orbital angular momentum and $S$ the total spin) we have

$$X = \sum_{\ell,S} X_{\ell S}, \quad (34)$$

$$X_{\ell S} = \frac{1}{2\pi^2} \int_0^\infty dk k^2 \frac{g_{\ell S}^2(k^2)}{(k^2/2\mu - E_B)^2}. \quad (35)$$

In this equation, $\mu$ is the reduced mass of particles $A$ and $B$, and $g_{\ell S}^2(k^2)$ is the coupling squared of the bound state, $(AB\ell S|V|\psi_B)^2$. This equation is in agreement with Eq. (11). We discuss explicitly in Appendix A the angular momentum algebra needed to express $X$ as the diagonal sum over the compositeness in a partial wave $(X_{\ell S})$, Eq. (34).

The coupling can be calculated by taking into account the Lippmann-Schwinger (LS) equation in partial waves

$$T(E) = V + \frac{1}{E - H_0} T(E) \quad (36)$$

for the off-shell $T$ matrix, with matrix elements $T(k',k;E)$ (if several partial waves mix the previous LS equation is still valid in a matrix notation). The $T$ matrix has a pole at $E = E_B$ and then, by taking the limit $E \to E_B$ in the LS equation, it follows that $g(k)$ satisfies a homogeneous integral equation for $k \in [0, \infty]$ (again a matrix notation should be employed if appropriate)

$$g(k) = \frac{1}{2\pi^2} \int_0^\infty dk' k'^2 V(k,k') \frac{1}{E_B - k'^2/2\mu} g(k'). \quad (37)$$

From Eq. (37) and the fact that $V(-k,k') = (-1)^\ell V(k,k')$ (parity conservation) one concludes that the coupling squared only depends on $k^2$, as already expressed in Eq. (35).

The global normalization factor in Eq. (37) is fixed by the requirement that $g(k)$ matches the residue of the $T$ matrix at the pole position

$$g^2(x^2) = \lim_{E \to E_B} (E - E_B)T(x,x;E). \quad (38)$$

$^2$One could make use of standard Feynman rules in relativistic QFT, integrate over the temporal component of the loop momentum (taking into account that the coupling squared does not depend on this integration variable in the NR case) and proceed with the non-relativistic reduction of the kinematics. The necessary steps in QFT from the Feynman diagrams in Fig. 1 are given in the Appendix A.
Here, \( \kappa = \sqrt{2\mu E_B} \) with \( \text{Im}\kappa > 0 \) [1st or physical Riemann sheet (RS)].

Taking advantage of the fact that the integrand in Eq. (35) is an even function of \( k \) one can symmetrize it and rewrite Eq. (35) as

\[
X = \left( \frac{\mu}{\pi} \right)^2 \int_{-\infty}^{+\infty} dk k^2 \frac{g^2(k^2)}{(k^2 - \kappa^2)^2},
\]

(39)

where for briefing the writing we have suppressed the subscript \( \ell S \).

## 4 Calculations of \( X \) in NR QFT

We stress that there are no further contributions beyond Eq. (35) for \( X_{\ell S} \). The absence of tadpole like contributions within an appropriate regularization procedure in NR QFT drives to Eq. (35) as the final expression without any possible counterterm contributions. As shown in Eq. (11) this expression also follows from the Schrödinger equation [5]. Thus, \( X \) is a fully derived quantity from the knowledge of the (full or half) off-shell \( T \) matrix which allows also to determine the coupling function \( g^2(k^2) \). This is an expected result because the \( T \) matrix must comprise all the spectroscopical information of the corresponding quantum system.

The analytical properties of \( g(k) \) in the \( k \)-complex plane can be deduced from Eq. (37). In the case of a separable potential the deduction is straightforward. For more involved potentials one could use the techniques derived in Refs. [25, 26] for the more complicated problem of establishing the LS equation for complex momenta (which are suitable for potentials that can be given by a spectral decomposition with spectral functions analytic in \( E \)). This latter case will be explained in more detail in Ref. [26], and then \( g(k) \) is analytic in \( k \) without cuts. As a result, a non-constant \( g(k) \) is not bounded for \( k \to \infty \) in the \( k \) complex plane because of the Liouville’s theorem in complex analysis.

### 4.1 Zero-range potentials

We consider first the case in which the wavelengths of the two scattering particles are large compared with the range of their interaction. In configuration space the potential is then approximated as the sum of delta functions at the origin and derivatives of them [27, 28]. We treat the potential given as exact and a full nonperturbative solution of the LS equation is worked out in partial waves. The solution obtained is also valid for energy-dependent potentials.

Despite the fact that solutions for the on-shell \( T \) matrix of the LS equation differ between cutoff and dimensional regularization when implemented nonperturbatively [27], we are able to show that, regardless of the regularization method employed, \( X = 1 \) for zero-range energy-independent potentials.

For a given set of quantum numbers the potential for \( n \) two-body coupled channels (here we do not distinguish between channel and partial wave since they can be treated on the same foot) is given by the
sum
\[ v_{\alpha\beta}(k_\alpha, p_\beta) = k_\alpha^{\ell_\alpha} p_\beta^{\ell_\beta} \sum_{i,j} N_{ij} v_{\alpha\beta;ij} k_{2i}^{\ell_\alpha} p_{2j}^{\ell_\beta}, \] (40)

where the channels are numbered by employing Greek letters. The factor in front of the sum is driven by the threshold-behavior of the different partial waves, being \( k_\alpha \) and \( p_\beta \) the three-momenta of the corresponding channels. The coefficients \( v_{\alpha\beta;ij} \) could be energy dependent but they do not depend on the three-momenta. These coefficients are grouped as the matrix elements of the \( N \times N \) matrices \([v]\) which, in turn, are the block matrix elements of the potential matrix \([v]\)

\[
[v] = \begin{pmatrix}
[v_{11}] & [v_{12}] & \ldots & [v_{1n}] \\
[v_{21}] & [v_{22}] & \ldots & [v_{2n}] \\
\vdots & \vdots & \ddots & \vdots \\
[v_{n1}] & [v_{n2}] & \ldots & [v_{nn}]
\end{pmatrix}.
\] (41)

We also introduce the \( Nn \) column vectors \([k_\alpha]\) as

\[
[k_\alpha]^T = (0, \ldots, 0, k^{\ell_\alpha}, k^{\ell_\alpha+2}, \ldots, k^{\ell_\alpha+2N}, 0, \ldots, 0),
\] (42)

such that \( v_{\alpha\beta}(k_\alpha, p_\beta) \) in Eq. (40) can be conveniently written in matrix notation as

\[
v_{\alpha\beta}(k_\alpha, p_\beta) = [k_\alpha]^T \cdot [v] \cdot [p_\beta].
\] (43)

In this way the solution for the LS equation Eq. (36), \( t_{\alpha\beta}(k, p; E) \), is also written in matrix notation as

\[
t_{\alpha\beta}(k_\alpha, p_\beta; E) = [k_\alpha]^T \cdot [t(E)] \cdot [p_\beta],
\] (44)

where we have introduced the scattering matrix \([t(E)]\). In order to satisfy the LS equation the latter is required to fulfill

\[
[t(E)] = [v(E)] - [v(E)] \cdot [G(E)] \cdot [t(E)],
\] (45)

where the block-diagonal matrix of unitarity one-loop functions \([G(E)]\) is defined as

\[
[G(E)] = \sum_\alpha [G_\alpha(E)].
\] (46)

The \([G_\alpha(E)]\) is an \( Nn \times Nn \) matrix

\[
[G_\alpha(E)] = \frac{m_\alpha}{\pi^2} \int_0^\infty dq \frac{q^2}{q^2 - 2m_\alpha E} [q_\alpha] \cdot [q_\alpha]^T,
\] (47)

with \( m_\alpha \) the reduced mass of the \( \alpha \) channel. The algebraic solution of Eq. (45) is

\[
[t(E)] = [D(E)]^{-1},
\] (48)

\[
[D(E)] = [v(E)]^{-1} + [G(E)],
\] (49)

Of course the matrices \([t(E)]\) and \([D(E)]\) have a block matrix from analogous to that of \([v]\) in Eq. (41). One can work out several expressions for calculating \( g_\alpha^2(k^2) \), which is the square of the coupling function
of the bound state to the channel \( \alpha \). We derive here two of them making use of the half-off-shell \( T \) matrix (with \( E_B = -\gamma^2/2m_\alpha \) for all the channels) that will be used below. In these expressions the determinant of the matrix \([D]\) cancels.

We write \([D]^{-1}\) in terms of the adjoint matrix of \([D]\) ([\(d = \text{adj}[D]\)] and its determinant \(\Delta\),

\[
[D]^{-1} = \frac{[d]}{\Delta}. \tag{50}
\]

Next, taking the limit \(E \to E_B\) (we assume that the zero of \(\Delta(E)\) at \(E = E_B\) is of order 1) the residue of the half-off-shell \( T \) matrix \((p_\beta = i\gamma_\beta)\) provides us with the coupling functions

\[
g_\alpha(k_\alpha)g_\beta(p_\beta) = \lim_{E \to E_B} (E - E_B)t_{\alpha\beta}(k_\alpha, p_\beta; E) = \frac{[k_\alpha]^T \cdot [d] \cdot [p_\beta]}{\Delta'} |_{E=E_B, p_\beta=i\gamma_\beta}, \tag{51}
\]

\[
\Delta' = \left. \frac{\partial \Delta}{\partial E} \right|_{E=E_B}.
\]

Therefore, by squaring the previous expression we have for \(g_\alpha^2(k_\alpha^2)\)

\[
g_\alpha^2(k_\alpha^2) = \frac{1}{(\Delta')^2} \left( [k_\alpha]^T \cdot [d] \cdot [p_\beta] \right)^2 \bigg|_{E=E_B, p_\beta=i\gamma_\beta}. \tag{52}
\]

Another expression is obtained by taking the derivative of \(t_{\alpha\beta}(k_\alpha, p_\beta; E)\) with respect to \(E\) and then moving to the pole position, such that \(E \to E_B\) and \(p_\beta \to i\gamma_\beta\). We make use here of the result

\[
\frac{\partial [D]^{-1}}{\partial E} = -[D]^{-1} \cdot \frac{\partial [D]}{\partial E} \cdot [D]^{-1}, \tag{53}
\]

that follows trivially by taking the derivative of \([D][D]^{-1} = I\). We end with a double pole whose coefficient is

\[
g_\alpha(k_\alpha)g_\beta(p_\beta) = \frac{1}{(\Delta')^2} \left[ [k_\alpha]^T \cdot [d] \cdot \frac{\partial [D]}{\partial E} \cdot [d] \cdot [p_\beta] \right] |_{E=E_B, p_\beta=i\gamma_\beta}. \tag{54}
\]

We combine Eqs. (52) and (54) (the latter particularized on-shell, \(k_\alpha \to p_\alpha = i\gamma_\alpha\), and express \(g_\alpha^2(k_\alpha^2)\) as

\[
g_\alpha^2(k_\alpha^2) = \frac{([k_\alpha]^T \cdot [d] \cdot [p_\alpha])^2}{[p_\alpha]^T \cdot [d] \cdot \frac{\partial [D]}{\partial E} \cdot [d] \cdot [p_\alpha]} |_{E=E_B, p_\alpha=i\gamma_\alpha}. \tag{55}
\]

in which the factors \(\Delta'\) have cancelled out.

We are now ready to calculate the compositeness \(X_\alpha\), Eq. (35), which now becomes

\[
X_\alpha = \frac{m_\alpha/\pi^2}{[p_\alpha]^T \cdot [d] \cdot \frac{\partial [D]}{\partial E} \cdot [d] \cdot [p_\alpha]} \left. \frac{\partial}{\partial E} \int_0^\infty dk \frac{k^2}{k^2 - 2m_\alpha E} \left[ [k_\alpha]^T \cdot [d] \cdot [p_\alpha] \left[ [k_\alpha]^T \cdot [d] \cdot [p_\alpha] \right)^2 \right] \right|_{E=E_B, p_\alpha=i\gamma_\alpha} \tag{56}
\]

Let us assume that \([v]\) is energy independent, \(\partial[v]/\partial E = 0\). Then, we have that

\[
\frac{\partial [D]}{\partial E} = \frac{\partial [G]}{\partial E}, \tag{57}
\]
which is a block-diagonal matrix with

$$\frac{\partial [G_\alpha]}{\partial E} = \frac{m_\alpha}{\pi^2} \frac{\partial}{\partial E} \int_0^\infty dk \frac{k^2}{k^2 - 2m_\alpha E} [k_\alpha] : [k_\alpha]^T$$  \hspace{1cm} (58)

For $\partial[v]/\partial E = 0$ the total compositeness $X = \sum_\alpha X_\alpha = 1$, as can be seen by performing the following steps in Eq. (56):

First, we rewrite the denominator in the first fraction of this equation by taking into account Eq. (58) as

$$\sum_\beta [p_\alpha]^T : [d] : \frac{m_\beta}{\pi^2} \frac{\partial}{\partial E} \int_0^\infty dk \frac{k^2}{k^2 - 2m_\beta E} [k_\beta] : [k_\beta]^T : [d] : [p_\alpha]$$  \hspace{1cm} (59)

Second, we make use of Eq. (51) and rewrite the factors $[p_\alpha]^T : [d] : [k_\beta]$ as $\Delta'(E_B)g_\alpha(p_\alpha)g_\beta(k_\beta)$, and similarly for $[k_\beta]^T : [d] : [p_\alpha]$.

$$\sum_\alpha X_\alpha = \sum_{\alpha=1}^n \frac{g_\alpha^2(p_\alpha^2)}{g_\beta^2(p_\beta^2)} \frac{m_\alpha}{\pi^2} \frac{\partial}{\partial E} \int_0^\infty dk \frac{k^2}{k^2 - 2m_\alpha E} g_\alpha^2(k_\alpha) \bigg|_{E=E_B,p_\alpha=i\gamma_\alpha} \bigg|_{E=E_B,p_\beta=i\gamma_\beta}$$

$$= 1.$$ \hspace{1cm} (60)

Now that we have shown that the total compositeness is one for zero-range energy-independent potentials it is illustrative to explicitly calculate $X$ employing a particularly simple regularization method. In this way we also show the emergence of other contributions to $X$, beyond the prototypical Weinberg’s result of Ref. [5] in the limit of vanishing binding energy. A simple way to treat with the power-like divergences that emerge when employing a potential like that in Eq. (40) is to regularize the potential as

$$V(k', k) \rightarrow V(k', k)e^{i\epsilon(k+k')}.$$ \hspace{1cm} (61)

with $\epsilon \rightarrow 0^+$. The use of the convergent factor $e^{i\epsilon(k+k')} \epsilon(k+k')$ removes all the power-like divergences at the same time that it preserves the right analytical properties. Indeed, it gives the same results as dimensional regularization for three dimensions (as it is our case here). The redefinition of $V(k', k)$ in Eq. (61) transforms $g(k)$ as

$$g(k) \rightarrow g(k)e^{i\epsilon k},$$ \hspace{1cm} (62)

as it is clear from Eq. (37). The presence of $e^{i\epsilon k}$ in the coupling function allows us to close the integration contour of the integral in Eq. (39) along the upper half plane of the $k$-complex plane with a semicircle of infinite radius, as shown in Fig. 2. The calculation is straightforward by applying the Cauchy’s integration theorem with the result

$$X = \frac{2i\mu^2}{\pi} \frac{\partial}{\partial k} \left[ \frac{k^2g^2(k^2)}{(k + \gamma)2} \right]_{k=\gamma}$$

$$= g^2(\gamma^2) \frac{\mu^2}{2\pi \gamma} + \frac{\mu^2}{2\pi} \frac{\partial g^2(-\gamma^2)}{\partial \gamma} \bigg|_{\gamma=\gamma}.$$ \hspace{1cm} (63)

where $\gamma = -ik$ and $\gamma = -i\gamma$. The first term on the right hand side (r.h.s.) of this equation is a well-known contribution [5,29–32]. It is model independent in the sense that it is fixed once the pole position and the

\footnote{Because of time reversal symmetry $[d]$ is a symmetric matrix.}
When substituting Eqs. (which are calculated following the same procedure as explained with regards Eq. (g residue of the on-shell T matrix at the pole position is known (which in principle can be be fixed from experiment). The second term of Eq. (63) is an extra contribution, which cannot be fixed directly from the knowledge of the on-shell T matrix and depends on the interaction V(k', k). For its evaluation one needs first to solve the integral equation for g(k), Eq. (37). The next step is to proceed by analytical continuation and evaluate the derivative of g(k) at k = ∞. This extra term is sensitive to the threshold dependence g_{\ell S}^2(k^2) \propto k^{2\ell}.

The first explicit example is a pure S-wave potential given by

\[ V(k', k) = \left[ v_0 + v_2(k^2 + k'^2) \right] e^{i\epsilon(k+k')} , \tag{64} \]

where \( v_0 \) and \( v_2 \) are constants. For our purposes of calculating g(k) it is enough to work out the half-off-shell T matrix, with E = k^2/2\( \mu \). The latter can be solved in the form, cf. Eq. (44),

\[ T(E)(k', k) = \left[ t_0(E) + t_2(E)(k^2 + k'^2) \right] e^{i\epsilon(k+k')} . \tag{65} \]

When substituting Eqs. (64) and (65) in the LS equation, Eq. (36), one encounters the integrals

\[ \lim_{\varepsilon \to 0^+} \frac{\mu}{2\pi^2} \int_{-\infty}^{\infty} dq \frac{q^{2+n} e^{2i\epsilon q}}{k^2 + i\varepsilon - q^2} = -\frac{i\mu k^{n+1}}{2\pi} , \tag{66} \]

which are calculated following the same procedure as explained with regards Eq. (63). We then find

\[ T(k', k; E) = \frac{v_0 + v_2(k^2 + k'^2)}{D(E)} , \tag{67} \]

\[ D(E) = 1 + \frac{i\mu\sqrt{2\mu E}}{2\pi}[v_0 + 4\mu E v_2] . \]

From the residue of the T matrix at the pole position one can calculate straightforwardly the coupling function g(k). Its square is

\[ g^2(k^2) = \frac{2\pi\gamma/\mu^2}{1 - 6\gamma^2 v_2/v_0} \frac{(1 + (k^2 - \gamma^2)v_2/v_0)^2}{1 - 2\gamma^2 v_2/v_0} . \tag{68} \]

For energy-dependent \( v_0 \) or \( v_2 \) the expression for the T-matrix in Eq. (67) is still valid, since the energy E enters only parametrically in the LS equation. However, the formula for \( g^2(k^2) \) would be different.

By replacing Eq. (68) for \( g^2(k^2) \) into Eq. (63) we find that both terms in the right-hand side of the equation give rise to non-zero contributions which sum is 1, as they should. More specifically the partial contributions of the first and second terms are

\[ g^2(\kappa^2) \frac{\mu^2}{2\pi\gamma} = \frac{1 - 2\gamma^2 v_2/v_0}{1 - 6\gamma^2 v_2/v_0} , \tag{69} \]

\[ \frac{\mu^2}{2\pi} \frac{\partial g^2(-\gamma^2)}{\partial\gamma} \bigg|_{\gamma=\gamma} = -\frac{4\gamma^2 v_2/v_0}{1 - 6\gamma^2 v_2/v_0} . \]

In this case, since \( g^2(k^2) \) is not zero for k = 0 the last contribution in Eq. (69) is suppressed by a factor \( \gamma^2 v_2/v_0 \sim \gamma R \). The last step is based on the relation between \( v_0 \) and \( v_2 \) with the effective range parameters. Being \( a \) the scattering length and \( r \) the effective range we have

\[ v_0 = \frac{2\pi a}{\mu} , \tag{70} \]

\[ v_2 = \frac{\pi a^2 r}{2\mu} . \]
as can be easily worked out. Therefore, \( v_2/v_0 = ra/4 \) and then \( \gamma^2|v_2/v_0| \sim \gamma R / 4 \). Here, we take into account that for standard situations \( r = O(R) \) [33], a exception would be a zero of the partial wave close enough to threshold, and then for a shallow bound state it follows that \( \gamma = 1/a + O(\gamma R) \).

Next, we work out another example in which the second term on the right-hand-side of Eq. (63) is not suppressed compared to the first one for shallow bound states. This occurs when \( g^2(k^2) \) is zero at \( k^2 = 0 \), in which case the derivative of \( g^2(-\gamma^2) \) with respect to \( \gamma \) gives rise to a term that counts as \( g^2(-\gamma^2)/\gamma \).

As a specific example let us take a potential projected with orbital angular momentum \( \ell \) which reads

\[
V(k', k) = v_\ell k^\ell k'^\ell e^{i\epsilon(k+k')}, \tag{71}
\]

where \( v_\ell \) is a constant (this potential is separable). The solution of the LS equation \( T(k', k; E) \) and the coupling function is

\[
T(k', k; E) = \frac{v_\ell k^\ell k'^\ell}{D(E)}, \tag{72}
\]

\[
D(E) = 1 + i\mu(\sqrt{2\mu E})^{2\ell+1}2\pi v_\ell, \tag{73}
\]

\[
g^2(-\gamma^2) = \frac{\gamma^2\ell 2\pi}{\mu^2(2\ell + 1)\gamma^{2\ell-1}}. \tag{74}
\]

The two terms that sum up \( X = 1 \) in Eq. (63) are, in order,

\[
1 = \frac{1}{2\ell + 1} + \frac{2\ell}{2\ell + 1}. \tag{75}
\]

We see that both contributions count on the same footing (though as \( \ell \) increases the 2nd one becomes indeed dominant).

### 4.2 Regular and singular potentials

Analogously as in the previous section we consider a partial-wave projected potential for the coupling of \( n \) two-body coupled channels with a given set of quantum numbers. The difference is that now we do not assume a zero-range interaction, as in Eq. (40), but a general finite-range potential in coupled channels \( v_{\alpha\beta}(k_\alpha, p_\beta) \). It is not necessary that this potential be local. When the following requirements are satisfied

\[
\int_0^\infty \int_0^\infty dk dp |v_{\alpha\beta}(k, p)|^2 < \infty \tag{76}
\]

and

\[
\int_0^\infty dp |v_{\alpha\beta}(k, p)|^2 < M, \tag{77}
\]

with \( M \) a bound independent of \( k, \alpha \) and \( \beta \), the potential can be approximated with arbitrary precision as a separable potential of rank \( N \), with \( N \) arbitrarily large [22, 34]. The potential is qualified as regular then. If this is not the case one should introduce a regularization method (e.g. some sort of cutoff regularization) such that Eqs. (76) and (77) are fulfilled with the regularized potential, denoted in the

\[\text{footnote text}\]
following as \( \omega_{\alpha\beta}(k_{\alpha}, p_{\beta}) \), and the potential is qualified as singular. Note that if the potential has a finite range the integrals in Eqs. (74) and (75) are finite in the lower limit of integration.

We consider a complete set of orthonormal linearly independent real functions \( \{ f_s(k) \} \) in \([0, \infty)\) (we could relax the condition of being real functions and allow also complex ones, but then the writing would be more cumbersome). Had we regularized \( v_{\alpha\beta} \) with a sharp cutoff \( \Lambda \), such that \( \omega_{\alpha\beta}(k_{\alpha}, p_{\beta}) = \theta(\Lambda - p_{\alpha})\theta(\Lambda - k_{\beta})v_{\alpha\beta}(k_{\alpha}, p_{\beta}) \), it would be enough that these functions be complete in \([0, \Lambda)\).\(^5\) The potential \( \omega_{\alpha\beta}(k_{\alpha}, p_{\beta}) \) is expanded in this basis of functions

\[
\omega_{\alpha\beta}(k_{\alpha}, p_{\beta}) = \sum_{s,s'=1}^{\infty} f_s(k_{\alpha})\omega_{\alpha\beta;ss'}f_{s'}(p_{\beta}) ,
\]

with the coefficients given by

\[
\omega_{\alpha\beta;ss'} = \int_{0}^{\infty} \int_{0}^{\infty} dk dp f_s(k)\omega_{\alpha\beta}(k, p)f_{s'}(p)
\]

Then, we approximate \( \omega_{\alpha\beta}(k_{\alpha}, p_{\beta}) \) by a separable potential of rank \( N \) \([22]\), \( \omega^{(N)}_{\alpha\beta}(k_{\alpha}, p_{\beta}) \), given by the truncation of the previous series in Eq. (76)

\[
\omega^{(N)}_{\alpha\beta}(k_{\alpha}, p_{\beta}) = \sum_{s,s'=1}^{N} f_s(k_{\alpha})\omega_{\alpha\beta;ss'}f_{s'}(p_{\beta}) .
\]

The solutions of the LS equation for the truncated potential \( \omega^{(N)}_{\alpha\beta}(k_{\alpha}, p_{\beta}) \) is denoted by \( t^{(N)}_{\alpha\beta}(k_{\alpha}, p_{\beta}; E) \), which fulfills

\[
t^{(N)}_{\alpha\beta}(k_{\alpha}, p_{\beta}; E) = \omega^{(N)}_{\alpha\beta}(k_{\alpha}, p_{\beta}) + \sum_{\gamma} \frac{m_{\gamma}}{\pi^2} \int_{0}^{\infty} dq \frac{q^2}{q^2 - 2m_{\gamma}E} \omega^{(N)}_{\alpha\gamma}(k_{\alpha}, q)t^{(N)}_{\gamma\beta}(q, p_{\beta}; E) .
\]

Let us show that \( t^{(N)}_{\alpha\beta}(k_{\alpha}, p_{\beta}; E) \) is given by the truncated series expansion of \( t_{\alpha\beta}(k_{\alpha}, p_{\beta}; E) \), namely,

\[
t^{(N)}_{\alpha\beta}(k_{\alpha}, p_{\beta}; E) = \sum_{s,s'=1}^{N} f_s(k_{\alpha})t_{\alpha\beta;ss'}(E)f_{s'}(p_{\beta}) ,
\]

\[
t_{\alpha\beta;ss'}(E) = \int_{0}^{\infty} \int_{0}^{\infty} dk dp f_s(k)t_{\alpha\beta}(k, p; E)f_{s'}(p) .
\]

Implementing the expansions of \( t^{(N)} \) and \( \omega^{(N)} \) in the LS equation of Eq. (79), and recalling the orthonormal character of functions \( \{ f_s(k) \} \), we have the following algebraic equation for the coefficients \( t_{\alpha\beta;ss'}(E) \),

\[
t_{\alpha\beta;ss'}(E) = \omega_{\alpha\beta;ss'} + \sum_{\gamma=1}^{n} \sum_{s''=1}^{N} \omega_{\alpha\gamma;ss''} \frac{m_{\gamma}}{\pi^2} \int_{0}^{\infty} dq \frac{q^2}{q^2 - 2m_{\gamma}E} f_{s''}(q)f_{s'''}(q)t_{\gamma\beta;ss'''}(E) .
\]

We now settle an analogous matrix notation to that of Sec. 4.1. The coefficients \( \omega_{\alpha\beta;ss'} \) are collected in the \( N \times N \) matrices \( [\omega_{\alpha\beta}] \) which are the block matrix elements of the matrix \( [\omega] \),

\[
[\omega] = \begin{pmatrix}
[\omega_{11}] & [\omega_{12}] & \cdots & [\omega_{1n}] \\
[\omega_{21}] & [\omega_{22}] & \cdots & [\omega_{2n}] \\
\cdots & \cdots & \cdots & \cdots \\
[\omega_{n1}] & [\omega_{n2}] & \cdots & [\omega_{nn}]
\end{pmatrix} .
\]

\(^5\)In this case we could use e.g. the Legendre polynomials \( \{ P_n(x) \} \) with \( x = k/\Lambda \).
Similarly to Eq. (42) we also introduce the $Nn$ column vector $[f(k_\alpha)]$ as

$$[f(k_\alpha)]^T = (0, \ldots, 0, f_1(k_\alpha), f_2(k_\alpha), \ldots, f_N(k_\alpha), 0, \ldots, 0)$$  \hspace{1cm} (83)

The unitarity loop functions are gathered in the block diagonal matrix $[G(E)]$, cf. Eq. (46), but now $[G_\alpha(E)]$ is given by

$$[G_\alpha(E)] = \frac{m_\alpha}{\pi^2} \int_0^\infty dq \, \frac{q^2}{q^2 - 2m_\alpha E} [f_\alpha(q)] \cdot [f_\alpha(q)]^T .$$  \hspace{1cm} (84)

Using this nation we then have

$$\omega_{\alpha\beta}(k_\alpha, p_\beta) = [f(k_\alpha)]^T \cdot [\omega] \cdot [f(p_\beta)] ,$$  \hspace{1cm} (85)

$$t_{\alpha\beta}(k_\alpha, p_\beta; E) = [f(k_\alpha)]^T \cdot [t(E)] \cdot [f(p_\beta)] .$$

The LS equation reduces to the algebraic Eq. (45), whose solution is the same as the one in Eq. (48) but now with $[k_\alpha]$ and $[p_\beta]$ replaced by $[f(k_\alpha)]$ and $[f(p_\beta)]$, in order. Indeed we can also perform the same replacements in Eq. (56) to obtain the expression for $X_\alpha$,

$$X_\alpha = \frac{m_\alpha/\pi^2}{[f(p_\alpha)]^T \cdot [d] \cdot \frac{\partial [D]}{\partial E} \cdot [d] \cdot [f(p_\alpha)]} \frac{\partial}{\partial E} \int_0^\infty dk \, \frac{k^2}{k^2 - 2m_\alpha E} [f(k_\alpha)]^T \cdot [d] \cdot [f(p_\alpha)]$$

$$\times [f(k_\alpha)]^T \cdot [d] \cdot [f(p_\alpha)] \bigg|_{E=E_B, p_\alpha=i\gamma_\alpha}$$  \hspace{1cm} (86)

$$\bigg| [f(k_\alpha)]^T \cdot [d] \cdot [f(p_\alpha)] \bigg|_{E=E_B, p_\alpha=i\gamma_\alpha}$$  \hspace{1cm} (87)

For an energy-independent potential, $\partial v_{\alpha\beta}(k, p)/\partial E = 0$, we can follow analogous steps as those in Eqs. (57)–(60) to end with

$$\sum_{\alpha=1}^n X_\alpha = 1 .$$  \hspace{1cm} (88)

We conclude that $X = 1$ for regular or singular energy-independent potentials since this result is always the same independently of how large $N$ and $\Lambda$ are [the latter needed for a singular finite-range potential to satisfy Eqs. (74) and (75)]. This demonstrates that in these cases the right normalization of the bound-state wave function is to one. Physically our derivation means that the total number of asymptotic particles in the continuum of any sort involved is two. Of course, this demonstration could also be used in the case of zero-range potentials, but we have preferred to be more specific for them because of its intrinsic importance at the practical and conceptual level.

### 4.3 Exchange of a bare elementary particle

Let us assume that the free Hamiltonian $H_0$ has an elementary particle eigenstate $|0\rangle$,

$$H_0 |0\rangle = E_0 |0\rangle ,$$  \hspace{1cm} (89)

$$\langle 0 | 0 \rangle = 1 .$$

As in Ref. [4] we express the full $T(E)$ matrix in terms of a “proper” $T$-operator $T_1(E)$ defined as what $T(E)$ would be if the elementary particle were omitted in sums over intermediate states. The relation
between them is rather simple and intuitive \[4\]
\[
T(E) = T_1(E) + T_1(E)|0\rangle\Delta(E)|0\rangle T_1(E),
\]
\[
\Delta(E) = [E - E_0 - \Pi(E)]^{-1},
\]
\[
\Pi(E) = \langle 0|T_1(E)|0 \rangle.
\]

Here we see that the total \( T \)-matrix is \( T_1(E) \) plus an extra term coming from the exchange of the elementary particle with a fully dressed propagator \( \Delta(E) \), being \( \Pi(E) \) the corresponding self-energy. Notice also that \( T_1(E)|0 \rangle \) is the complete vertex that converts the virtual elementary particle into the outgoing particles.

At the pole position of the assumed bound state \( E_B \) the full propagator \( \Delta(E) \) vanishes, which implies the equation
\[
E_B - E_0 - \Pi(E_B) = 0,
\]
that gives the relation between the unrenormalized mass \( E_0 \) and the physical one \( E_B \). The residue of the \( T \)-matrix between particle states in the continuum gives us the coupling functions
\[
g_\alpha(k_\alpha)g_\beta(p_\beta) = Z \langle k_\alpha,\alpha|T_1(E_B)|0\rangle\langle 0|T_1(E_B)|p_\beta,\beta \rangle.
\]

Since \( E_B < 0 \) the last factor in the previous equation is the same as \( \langle p_\beta,\beta|T_1(E_B)|0 \rangle \). In Eq. (92) we denote by \( Z \) the wave function renormalization of the bare elementary field, which is the residue of \( \Delta(E) \) at the pole position
\[
Z = \left[ 1 - \frac{\partial \Pi(E)}{\partial E} \right]^{-1}|_{E=E_B}.
\]

For a two-body system with the quantum numbers of the elementary state, we introduce the “bare” coupling constant by \( \tilde{g}_\alpha(k_\alpha) = \langle k_\alpha,\alpha|T_1(E_B)|0 \rangle = Z^{-1/2}g_\alpha(k_\alpha) \). Then the self-energy \( \Pi(E) \) is given by
\[
\Pi(E) = - \sum_{\beta} \frac{m_\beta}{\pi^2} \int_0^\infty \frac{dk}{k^2 + \gamma_\beta^2} \tilde{g}_\beta^2(k^2),
\]
and its derivative by
\[
\frac{\partial \Pi(E)}{\partial E} = - \sum_{\beta} \frac{2m_\beta^2}{\pi^2} \int_0^\infty \frac{dk}{(k^2 + \gamma_\beta^2)^2} \tilde{g}_\beta^2(k^2).
\]

Given Eqs. (35) and (92) we have for \( X_\alpha \),
\[
X_\alpha = \frac{1}{1 + \sum_{\beta} \frac{2m_\beta^2}{\pi^2} \int_0^\infty \frac{dk}{(k^2 + \gamma_\beta^2)^2} \tilde{g}_\beta^2(k^2)} \frac{2m_\alpha^2}{\pi^2} \int_0^\infty \frac{dk}{(k^2 + \gamma_\alpha^2)^2} \tilde{g}_\alpha^2(k^2).
\]

From the previous expression it follows the basic relation, cf. Eq. (11),
\[
X = \sum_\alpha X_\alpha = 1 - Z.
\]

The simplest example for this scenario is that with a constant bare coupling,
\[
\langle 0|V|k_\alpha,\alpha \rangle = \langle k_\alpha,\alpha|V|0 \rangle = \tilde{g}_\alpha,
\]
with all the other matrix elements involving particles in the continuum being zero. For this example Eq. (96) becomes

$$X_\alpha = \frac{\tilde{g}_\alpha^2 m_\alpha^2 / (2\pi \gamma_\alpha)}{1 + \sum_\beta \tilde{g}_\beta^2 m_\beta^2 / (2\pi \gamma_\beta)}.$$  \hspace{1cm} (99)

This value is independent of regulator. This can be seen by performing the renormalization of the on-shell \(T\)-matrix, from which the value of the bare coupling constant can be obtained. E.g. for the one-channel case (to simply matters) we have, Eq. (48),

$$T(k, p; E) = \left[ \frac{1}{\tilde{g}^2} (E - E_0) + \frac{m}{\pi^2} \int_0^\infty dk \frac{k^2}{k^2 - 2mE - i\varepsilon} \right]^{-1}. $$ \hspace{1cm} (100)

Thus, \(E_0/\tilde{g}^2\) absorbs the divergence of the unitarity integral (which is finite after a subtraction is done) by renormalizing \(E_0\), while \(\tilde{g}^2\) can be determined by the energy dependence of the phase shifts. Notice that if we match with the effective range expansion then the effective range resulting from Eq. (100) should be negative because \(\tilde{g}^2 \geq 0\), cf. Eqs. (107) and (108) below.

An example which explicitly gives rise to diverging integrals for \(X\) and \(Z\) is the same as before but with the bare coupling function squared proportional to \(k^2\),

$$\tilde{g}(k)^2 = \lambda k^2.$$ \hspace{1cm} (101)

In this way, had we used straightforwardly this bare coupling function in the calculation of \(X\) then the integration

$$\frac{2m}{\pi^2} \int_0^\infty dk \frac{\lambda k^4}{(k^2 + \gamma_\alpha^2)^2},$$ \hspace{1cm} (102)

would be divergent. However, the correct calculation of \(X\) requires the complete coupling function squared, for which determination we need to implement nonperturbative regularization and renormalization. We show below that once this is accomplished the compositeness \(X\) has a value independent of the type of cutoff regularization employed in the limit \(\Lambda \to \infty\).

To calculate the \(T\)-matrix we apply Eq. (41) with

$$[v] = [v_{11}] = \frac{1}{E - E_0} \begin{pmatrix} 0 & v_{12} \\ v_{12} & 0 \end{pmatrix}. $$ \hspace{1cm} (103)

The unitarity loops in \([G_1(E)]\) are

$$I_{n+1} = \frac{m}{\pi^2} \int_0^\infty dq q^n \frac{q^2 q^n}{q^2 - 2mE - i\varepsilon}$$ \hspace{1cm} (104)

with \(n = 1, 3\) or 5 in the present case. These integrals are divergent so that regularization and renormalization are necessary. The divergences can be identified to arise from the simpler integrals

$$L_{n+1} = \int_0^\infty dq q^n = \theta_n \Lambda^{n+1},$$ \hspace{1cm} (105)

with \(\Lambda\) some sort of cutoff, whose precise type fixes the value of the numbers \(\theta_n\). E.g. \(\theta_n = 1/(n + 1)\) for a sharp cutoff regularization. In the case of dimensional regularization all of them vanish, \(\theta_n = 0\) and \(L_{n+1} = 0\). Employing this notation, the matrix \([G_1(E)]\), with \(k = \sqrt{2mE}\), reads

$$[G_1(E)] = \frac{m}{\pi^2} \begin{pmatrix} L_1 + i\frac{\pi}{2} k \\
L_3 + k^2 L_1 + i\frac{\pi}{2} k^3 \\
L_5 + k^2 L_3 + L_1 k^4 + i\frac{\pi}{2} k^5 \end{pmatrix}. $$ \hspace{1cm} (106)
We match the on-shell \( T \)-matrix with the effective range expansion in powers of \( k^2 \) around \( k = 0 \), which reads
\[
\frac{1}{T(k,k)} = \left[ \alpha + \frac{1}{2} r k^2 + \mathcal{O}(k^4) + i \frac{m k}{2 \pi} \right]^{-1} .
\] (107)
The relation with the standard scattering length \( a_s \) and effective range \( r_s \) is
\[
a_s = \frac{m}{2 \pi \alpha} ,
\]
\[
r_s = -\frac{2 \pi}{m} r .
\] (108)
For cutoff regularization in the limit \( \Lambda \to \infty \) we obtain
\[
\frac{1}{T(k,k)} = \alpha + \frac{1}{2} r k^2 + i \frac{m k}{2 \pi} .
\] (109)
Notice that here there is not expansion in \( k^2 \), so that the previous result is the limit \( \Lambda \to \infty \) for the on-shell \( T \)-matrix once the bare parameters \( E_0 \) and \( v_{12} \) are expressed as a function of \( \alpha, r \) and \( \Lambda \):
\[
E_0 = v_{12} \left( L_3 + \epsilon \sqrt{L_5(L_1-\alpha)} \right) ,
\]
\[
v_{12} = \frac{2 \epsilon \sqrt{L_5(L_1-\alpha)}}{m \left( rL_5 - 2L_3(L_1-\alpha) - 4 \epsilon \sqrt{L_5(L_1-\alpha)^{3/2}} \right)} ,
\] (110)
with \( \epsilon = \pm 1 \). We note that the potential of Eq. (103) can give rise to \( r_s \) of either sign while keeping real values for the bare parameters \( E_0 \) and \( v_{12} \). This is not possible for the energy-independent potential \( v = v_{11} + v_{12}(k^2 + p^2) \) because the bare parameter \( v_{12} \) becomes complex for \( r_s > 0 \), as shown in Ref. [27]. Nonetheless, we derive below that the cutoff regularized result in the limit \( \Lambda \to \infty \) is inconsistent for \( r_s > 0 \) because the requirement \( 0 \leq X \leq 1 \) does not hold.

For the half-off-shell \( T \)-matrix, \( T(k,p) \) we have in the same limit
\[
\frac{T(k,p)}{T(k,k)} = 1 + (k^2 - p^2) \rho_\Lambda \Lambda^{-2} + \mathcal{O}(\Lambda^{-3}) ,
\] (111)
where \( \rho_\Lambda \) depends on the type of cutoff regularization method employed.

In the case of dimensional regularization we obtain for the on-shell \( T \)-matrix
\[
\frac{1}{T(k,k)_{DR}} = \frac{k^2 - 2mE_0}{4mv_{12}k^2} + i \frac{mk}{2 \pi} ,
\] (112)
that can only be matched with the effective range expansion if \( E_0 = 0 \), in which case we are left with only the scattering-length approximation
\[
\frac{1}{T(k,k)_{DR}} = \alpha + i \frac{mk}{2 \pi} .
\] (113)
This simple example shows that dimensional and cutoff regularizations might give rise to different on-shell \( T \)-matrices in a nonperturbative calculation. This is another instance of this issue (involving now an energy-dependent potential), which is discussed in depth in Ref. [27] for energy-independent potentials. The differences in the results are shown in this reference to be due to causality (Wigner bound) that has
a clear impact on cutoff regularization, but it is not so clear how it reflects on dimensional regularization (though these two methods agree in perturbative QFT calculations).

We can also shed light on the difficulties that dimensional regularization could have when applied in nonperturbative calculations by evaluating the compositeness $X$. For that we need the half-off-shell $T$ matrix, which in dimensional regularization is

$$T(k,p)_{DR} = \frac{(k^2 + p^2)/(2k^2)}{\alpha + \frac{mk}{2\pi}},$$

from where we obtain for the coupling function squared

$$g^2_{DR}(p^2) = \left(\frac{p^2 - \gamma^2}{2\gamma^2}\right)^2 \frac{2\pi\gamma}{m^2}.$$  

By applying Eq. (35) we can calculate straightforwardly the compositeness in dimensional regularization $X_{DR}$ [it gives the same value for the integrals as Eq. (66)], with the result

$$X_{DR} = \frac{1}{4\gamma^4} \frac{\partial}{\partial k} k(k^2 - \gamma^2)^2 \bigg|_{k = i\gamma} = 3.$$  

This is certainly a nonsense because from general principles we know that $0 \leq X \leq 1$. This calculation then shows the potential problems of applying dimensional regularization to non-perturbative calculations.

Let us now calculate $X$ in an arbitrary type of cutoff regularization, $X_\Lambda$. We have from Eq. (111) for the coupling function $g^2_\Lambda(p^2)$,

$$g^2_\Lambda(p^2) = \left(1 - (\gamma^2 + p^2)\frac{\rho_\Lambda}{\Lambda^2} + \mathcal{O}(\Lambda^{-3})\right) g^2_\Lambda(-\gamma^2),$$

$$g^2_\Lambda(-\gamma^2) = \frac{2\pi/m^2\gamma}{1 - \gamma r_s}.$$  

In terms of it we have for $X_\Lambda$,

$$X_\Lambda = g^2_\Lambda(-\gamma^2) 2 \left(\frac{m}{\pi}\right)^2 \int_0^\infty dp \frac{p^2}{(p^2 + \gamma^2)^2} \left(1 - (\gamma^2 + p^2)\frac{\rho_\Lambda}{\Lambda^2} + \mathcal{O}(\Lambda^{-3})\right).$$  

Now, after regularizing the divergent integral one has that in the limit $\Lambda \to \infty$ the contribution to $X_\Lambda$ from terms suppressed by $\Lambda^{-2}$ and higher inverse power of $\Lambda$ do not contribute because the resulting integration is only linearly divergent in $\Lambda$. This is the slowest degree of vanishing because there are no higher powers of $p^2$ in the half-off-shell amplitude $T_\Lambda(k,p;E)$ [as we have worked out explicitly from the general solution from Eq. (48)]. Thus, in the limit $\Lambda \to \infty$ we have

$$X_\Lambda = \frac{1}{1 - \gamma r_s}.$$  

This result, which is independent of the cut-off regularization method employed, gives $0 \leq X_\Lambda \leq 1$ for $r_s \leq 0$. Working out the explicit expression of $\gamma$ as a function of $a_s$ and $r_s$ from Eq. (109) one has

$$\gamma = \frac{1}{r_s} \left(1 \pm \sqrt{1 - \frac{2r_s}{a_s}}\right).$$  

(120)
Thus, for \( r_s \leq 0 \) we only have the branch in Eq. (120) with the minus sign (\( \gamma \geq 0 \)), which implies that \( r_s/a_s \leq 0 \) and then \( a_s > 0 \). The Eq. (119) simplifies to

\[
X_\Lambda = \frac{1}{\sqrt{1 - 2r_s/a_s}} \leq 1 , \ r_s \leq 0 , \ a_s > 0 . \tag{121}
\]

The issue of having a positive effective range when using cutoff regularization for an energy-independent potential (while requiring it to be Hermitian) [27], as well as for the energy-dependent potential of Eq. (98), has another manifestation here. For \( r_s > 0 \) the branch in Eq. (120) with the plus sign is the one possible for \( a_s < 0 \), while the two branches of \( \gamma \) are allowed for \( a_s > 0 \) and \( a_s/r_s \geq 2 \). Despite that the potential in this case keeps its real character the compositeness becomes larger than 1, which is unacceptable. Therefore, having \( r_s > 0 \) is not either compatible with the potential of Eq. (103).

In summary, a detailed analysis of the regularization and renormalization process is required for energy-dependent potentials between particle states in the continuum in order to conclude whether the result for \( X \) is independent of the regularization method used. This is in contrast with the the general results for energy-independent potentials, Eqs. (60) and (88), as well as for the general relation of Eq. (97). We have studied the potential of Eq. (103) for which the on-shell \( T \) matrix and \( X \) are different between cutoff and dimensional regularization, with \( X \) having an absurd value for the later. This is an extra deficiency of dimensional regularization when used in some nonperturbative calculations, in addition to those already analyzed in Ref. [27] for energy-independent potentials. However, in all the examples considered here the result for the compositeness is the same for any sort of cutoff regularization employed in the limit \( \Lambda \to \infty \), similarly as happens for the on-shell \( T \)-matrix. As indicated, we are not able to provide a proof that this is always the case within a NR QFT calculation involving a singular potential, for which the calculation of physical results requires regularization and renormalization. A general nonperturbative analysis is still lacking, though we think on physical grounds that the compositeness of a bound state would come out as a derived quantity from the knowledge of the \( S \)-matrix, which should contain the spectroscopical information of the quantum mechanical problem.

## 5 Relativistic bound state

Up to the best of our knowledge there is no a general criterion for a relativistic bound state to be qualified as elementary. In the relativistic case one generally relies on the study of the wave-function renormalization and there is a series of results within specific models, like the Lee model [2] or Yukawa type of interactions [3,7]. For these cases Refs. [2,3,7] conclude that a bound state with \( Z = 0 \) is purely composite. Relativistic models with Yukawa-like interaction have been revisited frequently in the recent literature, e.g. in Refs. [29,31,36]. The property \( 0 \leq Z \leq 1 \) can be obtained from the Källen-Lehmann representation, if the interaction Lagrangian does not involve field derivatives and the integral of the spectral function is finite, see e.g. Refs. [10,37].

The straightforward extrapolation of the definition of \( X \) in Eq. (19) cannot be given because contributions of eigenstates of \( H_0 \) belonging to the continuum spectrum with different number of asymptotic particles can be generated by the standard conversion of energy into matter. In this way, Eq. (14) for the representation of \( |\psi_B\rangle \) in terms of eigenstates of \( H_0 \) generalizes to

\[
|\psi_B\rangle = \int d\gamma C_\gamma |AB\gamma\rangle + \int d\eta D_\eta |AAB\eta\rangle + \int d\mu \delta_\mu |ABB_\mu\rangle + \ldots + \int d\eta F_\nu |CD_\nu\rangle + \ldots + \sum_n C_n |\varphi_n\rangle + \sum_n \int d\alpha \delta_n |A\alpha \varphi\rangle + \ldots + \sum_{n,m} C_{nm} |\varphi_n \varphi_m\rangle + \ldots \tag{122}
\]
with quite an obvious notation.

Nonetheless, we can still take advantage of the use of the number operators which are defined in the relativistic case as in NR QFT, cf. Eq. (15). E.g. the average number of asymptotic particles of type $A$ in $|\psi_B\rangle$ as given by the decomposition in Eq. (122) is

$$\langle \psi_B|N_D^A|\psi_B\rangle = \int d\gamma |C_{\gamma}|^2 + 2 \int d\eta |D_{\eta}|^2 + \sum_n \int d\alpha |C_{n\alpha}|^2 + \ldots$$  \hspace{1cm} (123)

In this way we can deduce the following universal criterion for a bound state to be considered as elementary with respect to the particles in the continuum spectrum, applicable both in the relativistic and NR cases:

$$\langle \psi_B|N_D^A|\psi_B\rangle = 0 \ , \ \forall A \ .$$  \hspace{1cm} (124)

Strictly speaking we have another extra condition in addition to Eq. (124) for relativistic systems in order to avoid the possibility that $|\psi_B\rangle$ had components of states made by several bare elementary discrete states, as the last contribution shown in Eq. (122). Denoting by $N_D^E$ the sum of the number operators for the bare elementary discrete states ($N_D^B$), $N_D^E = \sum_n N_D^B$, one also has to discard that

$$\langle \psi_B|N_D^E|\psi_B\rangle > 1 \ .$$  \hspace{1cm} (125)

Another consequence that can be extracted by evaluating the expectation value of the number operators in $|\psi_B\rangle$, Eq. (123), is the following. Let us consider that for a particle species $A$ one has that

$$\langle \psi_B|N_D^A|\psi_B\rangle = x_A$$  \hspace{1cm} (126)

with $x_A \geq m$ and $m \geq 0$ a natural number. In such circumstances we can conclude that the free-particle states containing $m$ or more asymptotic particles of type $A$ are relevant in the bound state $|\psi_B\rangle$.

6 Calculation of $\langle \psi_B|N_D^A|\psi_B\rangle$ in relativistic QFT

Let us now discuss the calculation of the expectation value $\langle \psi_B|N_D^A|\psi_B\rangle$ in relativistic QFT. We follow the same steps as introduced in Sec. 3 for NR QFT, since many of them are equally valid in the relativistic case. Being specific, Eqs. (22), (23) can be used also now and then instead of Eq. (24) for $X$ we have the analogous expression

$$\langle \psi_B|N_D^A|\psi_B\rangle = \langle \varphi_B|U(+\infty,0)N_D^A U(0,-\infty)|\varphi_B\rangle \ .$$  \hspace{1cm} (127)

Again the extra time evolution from 0 to $t$ in Eqs. (25) and (28) can be equally applied in the relativistic case (of course, here also $[N_D^A, H_0] = 0$ [11]). We then arrive to the time-ordered expression for $\langle \psi_B|N_D^A|\psi_B\rangle$ ready to be applied in QFT:

$$\langle \psi_B|N_D^A|\psi_B\rangle = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} dt \langle \varphi_B|U_D(+\infty,t)N_D^A(t)U_D(t,-\infty)|\varphi_B\rangle \ .$$  \hspace{1cm} (128)

The infinite factor $T$ in the denominator of this equation cancels with the Dirac delta function of total energy conservation.

We can also express the number operator $N_D^A$ in terms of free fields, analogously as done in the non-relativistic case. Let us consider a scalar particle $A$ and define the free fields

$$\psi^+(x) = \int \frac{d^3q}{(2\pi)^3} a(q)e^{-iqx} ,$$  \hspace{1cm} (129)

$$\psi^-(x) = \int \frac{d^3q}{(2\pi)^3} a^\dagger(q)e^{iqx} = \psi^+(x)^\dagger ,$$
so that

$$N_D^A(t) = -2i \int d^3x \dot{\psi}(-)(x) \psi(+) (x) .$$

(130)

with \( \dot{\psi}(-)(x) = \frac{\partial \psi(-)(x)}{\partial t} \). The following expression for the expectation value \( \langle \psi_B | N_D^A | \psi_B \rangle \) results

$$\langle \psi_B | N_D^A | \psi_B \rangle = -2i \lim_{T \to +\infty} \frac{1}{T} \int d^4x \langle \varphi_B | P \left[ e^{-i \int d^4x' \mathcal{H}_D(x')} \dot{\psi}(-)(x') \psi(+) (x') \right] | \varphi_B \rangle .$$

(131)

where the interaction have been written in terms of an interaction-Hamiltonian density \( \mathcal{H}_D(x) \) in the Dirac picture.

The evaluation of Eqs. (128) or (131) involves (infinite) more diagrams than in NR QFT. The set of Feynman diagrams involved can be schematically represented as in Fig. 3, where the shaded circle represents any set of connected vertices without any insertion of the number operator which is indicated by the double dot. However, in order to apply Feynman rules to the calculation of Eq. (131) one has to take into account that the two internal lines in Fig. 3 ending in any of the double dots is not a standard Feynman propagator. According to the Wick theorem for a neutral scalar field they correspond to

$$\langle \varphi_0 | P \left[ \psi(x_1) \psi(+) (x_2) \right] | \varphi_0 \rangle = i \int \frac{dk}{(2\pi)^4 2E_k (k^0 - E_k + i\varepsilon)} \ e^{ik(x_1-x_2)} ,$$

(132)

$$\langle \varphi_0 | P \left[ \psi(x_1) \psi(-) (x_2) \right] | \varphi_0 \rangle = i \int \frac{dk}{(2\pi)^4 2E_k (k^0 - E_k + i\varepsilon)} \ e^{ik(x_2-x_1)} ,$$

(133)

$$\psi(x_1) = \psi(+) (x_1) + \psi(-) (x_1) .$$

Here, \( | \varphi_0 \rangle \) is the non-interacting vacuum and \( E_k = \sqrt{m_A^2 + k^2} \), with \( m_A \) the physical mass. As a result, any of the two internal lines explicitly shown in Fig. 3 correspond to

$$\frac{i}{2E_k (k^0 - E_k + i\varepsilon)} ,$$

(134)

instead of a standard Feynman propagator for a scalar field. The arrows in the same lines in Fig. 3 refer to the momentum flow according to Eqs. (132) and (133), such that a line ends at \( \psi(+) (x) \) and another one leaves at \( \psi(-) (x) \).
Two limit cases are worth pointing out. In the case in which the bound state occurs nearby a two-body threshold the new formalism reduces to the NR case again.\textsuperscript{6} Furthermore, if one can conclude that only two-body channels dominate then one expects that the more important Feynman diagrams are those of Fig. 1.

For a given total Hamiltonian $H$ the expectation values of $N^A_D$ are invariant under unitary transformations and field reparametrizations. This is a direct consequence of Eq. (123). Nonetheless, its evaluation is nonperturbative and it is beyond the scope of the present manuscript to study the possible regulator independence of $X$ (beyond the discussions given above for NR QFT, cf. Sec. 4).

7 Resonances

In this section we discuss the generalization of many of the results given in Secs. 2–6 to the case of resonance states. The latter correspond to poles of the $T$ matrix in an unphysical RS that can be reached by the analytical extrapolation, typically in the complex energy or $s$ plane for NR and relativistic cases, respectively (with $s$ the usual Mandelstam variable). We assume in the following that the pole is of order one.

7.1 Definitions and QFT formalism

An approximate way to afford the problem of evaluating $Z$ in the non-relativistic case for an unstable particle near a two-body threshold was considered in Refs. [38, 39]. The approach is based on integrating the spectral density of a bare elementary discrete state around the resonance signal region, in such a way that if this integral is small the state is mostly composite while if it is close to 1 then it is mostly elementary. These results have also a clear connection with the counting pole rule of Morgan [40] and with the possible presence of near Castillejo-Dalitz-Dyson poles [21].

Let us continue here with our interpretation of the compositeness $X$ based on the definition of Eq. (19) in NRQM. We derive our results for resonance states by the analytical continuation of the expressions from the physical energy axis. In this respect, let us first discuss which is the matrix element that one should extrapolate analytically in order to reach the resonance pole.

The most straightforward option would be to calculate the expectation value of the operator number $N^A_D$ in an in state, $|\psi^+_\alpha\rangle$. For definiteness let us take a two-body in state of particles $A$ and $B$. In the same way that it is demonstrated that $\langle \psi^+_\alpha | \psi^+_\alpha \rangle = \langle \varphi_\alpha | \varphi_\alpha \rangle$ [10] one concludes that

$$\langle \psi^+_\alpha | N^A_D + N^B_D | \psi^+_\alpha \rangle = 2 \langle \varphi_\alpha | \varphi_\alpha \rangle .$$

However, this matrix element cannot be analytically continued to the resonance pole at $E_R = M_R - i\Gamma / 2$ in the 2nd RS. The reason is because of the bra $\langle \psi^+_\alpha |$, which obeys the equation (it can be derived from $|\psi^+_\alpha\rangle = U(0,-\infty)|\varphi_\alpha\rangle$, see e.g. Ref. [10])

$$\langle \psi^+_\alpha | = \langle \varphi_\alpha | + \int d\gamma \frac{T_{\gamma\alpha}(E_\alpha + i\varepsilon)^\dagger}{E_\alpha - i\varepsilon - E_\gamma} \langle \varphi_\gamma | + \sum_n \frac{T_{\alpha n}(E_\alpha + i\varepsilon)^\dagger}{E_\alpha - i\varepsilon - E_n} \langle \varphi_n | ,$$

$$(\ref{eq:136})$$

For a relativistic version of the integral equation for the coupling of the bound-state with the continuum asymptotic states the reader is referred to Ref. [3], which employs the Bethe-Salpeter equation instead of the LS equation.
Figure 4: Deformation of the integration contour along the physical energy in Eq. (137) needed to reach the resonance pole at $E_R = M_R - i\Gamma/2$.

Here we have taken into account that $T(E \pm i\varepsilon)\dagger = T(E \mp i\varepsilon)$ as follows from the LS equation, Eq. (36). The analytically continuation of Eq. (136) to $E = E_R$ must be done in the 1st RS because the imaginary part of the energy is already negative.

In order to reach the resonance pole we have to use the bra of an out state and the ket of an in state, as it is the case when evaluating the $S$-matrix elements. For the bra of the out state instead of Eq. (136) we have

$$
\langle \psi^-_\alpha | = \langle \varphi_\alpha | + \int d\gamma \frac{T_{\alpha\gamma}(E_\alpha - i\varepsilon)}{E_\alpha + i\varepsilon - E_\gamma} \langle \varphi_\gamma | + \sum_n \frac{T_{\alpha n}(E_\alpha - i\varepsilon)}{E_\alpha + i\varepsilon - E_n} \langle \varphi_n |
$$

and now its analytical extrapolation to $E = E_R$ requires to cross the unitarity cut ($E > 0$) to reach energy values with $\text{Im}E < 0$, and then one moves to the 2nd RS. The analytical continuation of the previous equation requires to deform the integration contour along the physical axis of energy as shown in Fig. 4.

In this way, instead of the expectation value of Eq. (135) one should consider the matrix element

$$
\langle \psi^\pm_\alpha | N_D | \psi^\pm_\alpha \rangle , \tag{138}
$$

with $N_D$ defined in Eq. (20), and extrapolate it to the resonance pole position. The previous matrix element has a double pole at the resonance pole position, because of the initial and final state interactions. The residue of this double pole divided by the coupling squared is the expectation value of the operator $N_D$ in the resonance states [41].

This limit process can be avoided if we use an analogous formalism to that explained in Sec. 3, but now for resonance states. We express the in/out resonance state $|\psi^\pm_R\rangle$ by evolving the bare one $|\varphi_R\rangle$ from asymptotic times

$$
|\psi^+_R\rangle = U_D(0, -\infty)|\varphi_R\rangle , \tag{139}
$$

$$
\langle \psi^-_R | = \langle \varphi_R | U_D(+\infty, 0) .
$$

Thus,

$$
X = \frac{1}{n} \langle \varphi_R | U(+\infty, 0) N_D U(0, -\infty) |\varphi_R \rangle , \tag{140}
$$

To talk about the bare resonance state might be qualified as an abuse of language. Nonetheless, it is consistent since $H_0|\varphi_R\rangle = E_R|\varphi_R\rangle$ as follows by relating $|\psi^\pm_R\rangle$ with $|\varphi_R\rangle$ with analogous equations to Eqs. (136) and (137). Additionally one ends with the same results as obtained by proceeding with the analytical continuation of Eq. (138) as already discussed.
Next, we introduce the extra time evolution from 0 to \( t \). For that let us notice that
\[
U_{D}(t, 0)|\psi_{R}^{+}\rangle = e^{iH_{0}t}e^{-iH_{I}t}|\psi_{R}^{+}\rangle = e^{-(i\mu + \frac{E}{2})t}e^{iH_{0}t}U_{D}(0, -\infty)|\varphi_{R}\rangle ,
\]
\[
\langle \psi_{R}^{-}|U_{D}(0, t) = \langle \psi_{R}^{-}|e^{iH_{I}t}e^{-iH_{0}t} = \langle \varphi_{R}|U_{D}(+\infty, 0)e^{-iH_{0}t}e^{(iMR + \frac{E}{2})t} .
\]

The factors \( e^{(iMR + \frac{E}{2})t} \) cancel between them in Eq. (140) while \( e^{iH_{0}t}N_{D}e^{-iH_{0}t} = N_{D}(t) = N_{D} \). As a result we can write
\[
X = \lim_{n \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} dt \langle \varphi_{R}|U_{D}(+\infty, t)N_{D}(t)U_{D}(t, -\infty)|\varphi_{R}\rangle ,
\]
\[(142)\]

The same steps as in Eqs. (31)–(33) allow us to re-express \( N_{D} \) in terms of NR fields \( \psi_{i}(x) \). Therefore, for scalar particles \( A_{i} \) we end with the following expression for \( X \)
\[
X = \lim_{n \to +\infty} \frac{1}{T} \int d^{4}x \langle \varphi_{R}|P \left[ e^{-i\int_{-\infty}^{+\infty} dt V_{D}(t)} \sum_{i} \psi_{i}^{+}(x)\psi_{i}(x) \right]|\varphi_{R}\rangle .
\]
\[(143)\]

analogous to Eq. (33).

In the case of two-particle asymptotic states the calculation of \( X \) can be done by evaluating the Feynman diagrams of Fig. 1. Performing the corresponding partial-wave decomposition as in Eq. (34) one has the following expression for \( X_{\ell S} \)
\[
X_{\ell S} = \frac{1}{2\pi^{2}} \int_{0}^{\infty} dk \frac{g_{\ell S}^{2}(k^{2})}{(k^{2}/2\mu - E_{R})^{2}} + \frac{i\mu^{2}}{\pi} \frac{\partial k g_{\ell S}^{2}(k^{2})}{\partial k} \bigg|_{k=\infty} .
\]
\[(144)\]

Compared with Eq. (35) there is an extra term due to the deformation of the analytical contour for integration, as shown in Fig. 4. Because of the same reason, the homogeneous integral equation satisfied by \( g(k) \) (a matrix notation should be used if several partial waves mix) is now
\[
g(k) = \frac{\mu}{\pi^{2}} \int_{0}^{\infty} dk \frac{k^{2}g(k')}{(\pi^{2} - k^{2})^{2}} \left[ V(k, k') - \frac{i\mu \pi V(k, \kappa) V(\kappa, k')/\pi}{1 + i\mu \pi V(\kappa, \kappa)/\pi} \right] .
\]
\[(145)\]

From this equation it is also clear that \( g_{\ell S}(-k) = (-1)^{\ell} g_{\ell S}(k) \) and then \( g_{\ell S}^{2} \) is a function of \( k^{2} \), as reflected in Eq. (144). We can also determine the coupling function \( g(k) \) by calculating the residue of the (half-)off-shell \( T \) matrix at the resonance pole position,
\[
g(k)g(p) = \lim_{E \to E_{R}} (E - E_{R})T(k, p; E) .
\]
\[(146)\]

At the resonance pole position the on-shell three-momentum is denoted by \( \kappa \) which is defined in the 2nd RS as \( \kappa = \sqrt{2\mu E_{R}} \), where \( \sqrt{2} \) is defined in the 2nd RS with \( argz \in [2\pi, 4\pi] \).

Although Eq. (144) is not explicitly real and positive, we can show that \( X = 1 \) for an energy-independent potential. We follow analogous steps as performed in Sec. 4.1 for a zero-range potential and in Sec. 4.2 for a regular or singular finite-range potential. In both cases the only change concerns the analytical extrapolation of the full-off-shell \( T \)-matrix, cf. Eq. (44), from the 1st to the 2nd RS by taking into account the deformation of the integration contour of Fig. 4. This analytical extrapolation only affects the matrices \( [G_{\alpha}(E)] \). We give explicit expressions for the zero-range potential case, since the same expressions are valid for finite-range potentials under the exchange \( [g_{\alpha}] \to [f_{\alpha}(g_{\alpha})] \), once they are approximated with
arbitrary precision by a separable potential of rank \( N \), Eq. (78) (for singular potentials this would require regularization, as discussed in Sec. 4.2).

Instead of \([G_\alpha(E)]\) we have now its analytical extrapolation in the 2nd RS, \([G^{\text{II}}_\alpha(E)]\), given by

\[
[G^{\text{II}}_\alpha(E)] = \frac{m_\alpha}{\pi^2} \int_0^\infty dq \frac{q^2}{q^2 - 2m_\alpha E} [q_\alpha] \cdot [q_\alpha]^T + \frac{im_\alpha}{\pi} i\sqrt{2m_\alpha E} \left[ i\sqrt{2m_\alpha E} \right]^T .
\]

(147)

Then, the \( T \) matrix in the 2nd RS

\[
\frac{\partial t^{\text{II}}_\alpha(k_\alpha, p_\beta; E)}{\partial E} = [k_\alpha]^T \cdot [d^{\text{II}}] \cdot [p_\beta],
\]

(148)

can be calculated making use of an analogous equation to Eq. (48),

\[
[t^{\text{II}}(E)] = [D^{\text{II}}(E)]^{-1},
\]

(149)

\[
[D^{\text{II}}(E)] = [v(E)]^{-1} + [G^{\text{II}}(E)].
\]

(150)

In terms of it we can calculate the coupling functions from the residue of the \( T \) matrix at the resonance pole position

\[
g_\alpha(k_\alpha)g_\beta(p_\beta) = \lim_{E \to E_R} (E - E_R)t^{\text{II}}_\alpha(k_\alpha, p_\beta; E) = \frac{[k_\alpha]^T \cdot [d^{\text{II}}] \cdot [p_\beta]}{\Delta^{\text{II}'}},
\]

(151)

\[
\Delta^{\text{II}'} = \frac{\partial \Delta^{\text{II}}(E)}{\partial E} \bigg|_{E = E_R},
\]

where \( \Delta^{\text{II}}(E) \) is the determinant of \([D^{\text{II}}(E)]\), \([d^{\text{II}}(E)]\) is its adjoint matrix and \( \kappa_\beta = i\sqrt{2m_\beta E_R} \). An analogous formula for \( g^2_\alpha(k_\alpha)^2 \) to that already obtained in the bound-state case results

\[
g^2_\alpha(k_\alpha)^2 = \frac{([k_\alpha]^T \cdot [d^{\text{II}}] \cdot [p_\beta])^2}{[p_\alpha]^T \cdot [d^{\text{II}}] \cdot \frac{\partial[D^{\text{II}}]}{\partial E} \cdot [d^{\text{II}}] \cdot [p_\alpha] \big|_{E = E_R, p_\beta = \kappa_\beta}.}
\]

(152)

The compositeness \( X_\alpha \) for the resonance state then reads, cf. Eq. (144),

\[
X_\alpha = \left( [p_\alpha]^T \cdot [d^{\text{II}}] \cdot \frac{\partial[D^{\text{II}}]}{\partial E} \cdot [d^{\text{II}}] \cdot [p_\alpha] \right)^{-1} \times \frac{\partial}{\partial E} \left( \frac{m_\alpha}{\pi^2} \int_0^\infty dk \frac{k^2}{k^2 - 2m_\alpha E} [k_\alpha]^T \cdot [d^{\text{II}}] \cdot [k_\alpha] \cdot [d^{\text{II}}] \cdot [p_\alpha] \right)
\]

\[
+ i\frac{m_\alpha}{\pi} \sqrt{2m_\alpha E} \left[ i\sqrt{2m_\alpha E} \right]^T \cdot [d^{\text{II}}] \cdot [p_\alpha] \right|_{E = E_R, p_\beta = \kappa_\beta}.}
\]

(153)

For an energy-independent potential \( \partial[D^{\text{II}}]/\partial E = \partial[G^{\text{II}}]/\partial E \). Following them completely analogous steps as in Eqs. (59)–(60) we have also for a resonance that the total compositeness is 1,

\[
\sum_{\alpha=1}^n X_\alpha = 1 .
\]

(154)

This result is valid as well for a finite-range energy-independent singular or regular potential by following the same steps as in Sec. 4.2, replacing \([G(E)]\) by \([G^{\text{II}}(E)]\), with

\[
[G^{\text{II}}_\alpha(E)] = \frac{m_\alpha}{\pi^2} \int_0^\infty dq \frac{q^2}{q^2 - 2m_\alpha E} [f_\alpha(q)] \cdot [f_\alpha(q)]^T + \frac{im_\alpha}{\pi} i\sqrt{2m_\alpha E} \left[ f_\alpha(i\sqrt{2m_\alpha E}) \right]^T .
\]

(155)
To avoid being too repetitive we refrain from reproducing them explicitly. The same conclusion was deduced in Ref. [22] for an energy-independent regular potential making use of the analytical extrapolation of the Schrödinger equation in the 2nd RS of the complex energy plane. However, our demonstration (based on the use of the LS equation) allows to treat singular potentials too. In addition, the normalization to one of the resonance state within our formalism is a consequence with a clear physical picture behind (the number of asymptotic particles in the state is 2), while in Ref. [22] it relies on a pure mathematical basis. Our conclusion that the total compositeness is 1 for a finite-range energy-independent potential implies that a resonance is then a purely composite state. However, the compositeness \( X \) is in general a complex number for \( \partial v(E)/\partial E \neq 0 \) and we discuss below in Sec. 7.2 how one can still give sense to \( X \).

As in Sec. 4.1 for a bound state, it is illustrative to write a closed formula for \( X \) in the case of a resonance with an energy-independent zero-range potential, for which \( X = 1 \) as just shown above. We again employ the regularization method based on including a convergent factor in Eq. (144) (which gives the same results for the integrals as dimensional regularization in three dimensions). Since the resonance pole lies in the 2nd RS, with \( \text{Im} k < 0 \), now we close the integration contour along the lower half plane in the \( k \)-complex plane. In this way, instead of Eq. (61) we now regularize the potential as

\[
V(k', k) \rightarrow V(k', k)e^{-i\varepsilon(k+k')},
\]

including an extra minus sign in the exponent of the convergent factor. It is now straightforward to obtain

\[
X_{\ell S} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk k^2 \frac{g_{\ell S}^2(k^2)e^{-i\varepsilon k}}{(k^2/2\mu - E_R)^2} + \frac{i\mu^2}{\pi \varepsilon} \frac{\partial g_{\ell S}^2(k^2)}{\partial k} \bigg|_{k = \varepsilon}.
\]

The first term is already well-known while the latter is a new contribution. By direct computation Eq. (157) can also be expressed as

\[
X_{\ell S} = \frac{2\mu^2}{\pi^2} \int_{0}^{\infty} dk k^2 \frac{i/\sqrt{k^2 + i\varepsilon} - k}{(k^2 - \varepsilon^2)^2} g_{\ell S}^2(k^2).
\]

Notice that \( i/\sqrt{k^2 + i\varepsilon} = -i/\sqrt{k^2 - i\varepsilon} = -k \). This equation is entirely equivalent to that for a bound state, Eq. (35), but written in the 2nd RS as corresponds to a resonance state. It is also interesting to realize about the presence of the factor \( g_{\ell S}^2(k^2)/(k^2 - \varepsilon^2)^2 \) and not of its modulus, as it corresponds to a Gamow state [22].

For the relativistic case we can evaluate the matrix elements of the operator numbers \( N_D^A \) between resonances states. As in the bound state case we can directly export the equations derived within NR QFT and use them also for relativistic QFT. Namely, we are referring to Eqs. (139) and (141). In this way, we can write the matrix element of \( N_D^A \) between in/out resonance states in relativistic QFT as

\[
\langle \psi_R^+(t)|N_D^A|\psi_R^-(t)\rangle = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} dt \langle \psi_R(t)|U_D(+\infty, t)N_D^A(t)U_D(t, -\infty)|\psi_R(t)\rangle.
\]

As in the relativistic bound state case we can express the number operators \( N_D^A(t) \) as bilinear operators of relativistic fields. For \( A \) being a scalar field we can use the result of Eq. (130) and write the matrix element of \( N_D^A \) between resonance states as

\[
\langle \psi_R^-(t)|N_D^A|\psi_R^+(t)\rangle = -2i \lim_{T \to +\infty} \frac{1}{T} \int d^4x \langle \phi_R(t)|P \left[ e^{-i \int d^4x' \phi_D(x') \psi_R^-(x) \psi_R^+(x)} \right] |\phi_R\rangle.
\]
The set of Feynman diagrams is represented in Fig. 3, with the obvious replacement of $|\psi_B\rangle$ by $|\psi_R^\pm\rangle$ to the right and left, respectively. Together with Eq. (160) one also has to keep in mind the meaning of the internal lines joining the field bilinear associated to the number operator, as explained in Sec. 6, cf. Eqs. (132), (133) and (134). For other particle species the expression of $N^A_D$ in terms of a field bilinear can be worked out straightforwardly. Another convenient way to proceed is to re-express the right hand side of Eq. (160) by including an interpolation field for the resonance state and then applying standard reduction techniques in QFT \cite{41}. Another possibility is to consider the scattering between particles in the presence of the number operator density and then extract the residue of the scattering amplitude at the double resonance pole. This idea was applied to calculate the scalar form factor of the $f_0(500)$ in Ref. \cite{41} by evaluating $\pi\pi$ scattering in the presence of a scalar source.

It is obvious that a necessary condition for a resonance being elementary is the that the expectation value of the number operators of the asymptotic free particles of any species be zero,

$$\langle \psi_R^- | N^A_D | \psi_R^+ \rangle = 0 \ , \ \forall A \ .$$

(161)

In practical application it would be enough that $|\langle \psi_R^- | N^A_D | \psi_R^+ \rangle| \ll 1 \ , \ \forall A$. As a clarification remark why we cannot state it as a sufficient condition as well, let us consider a decomposition of a resonance state as in Eq. (122) (with $|\psi_B\rangle$ replaced by $|\psi_R^+\rangle$). Now, by taking the expectation value in Eq. (161) we would only pick up contributions from those basis states including free particles of type $A$. However, as follows from the NR QFT analysis for a resonance, cf. Eq. (158), one should not expect to have the sum of the modules squared of the coefficients in the linear decomposition (as in Eq. (122) for a bound state) but rather the coefficients squared (at least for those channels that are open at the resonance mass) because of the analytical extrapolation to the resonance pole in the 2nd RS. Therefore, we have in general the sum of several complex numbers which can be zero even though they are not separately.

### 7.2 Phase-factor transformations

The main point of Ref. \cite{42} is to establish the existence of transformations at the level of the partial-wave projected $S$ matrix such that

$$S \rightarrow \Theta S \Theta^T$$

(162)

$$\Theta \Theta^\dagger = I \ .$$

(163)

In order not to modify the modulus of the residues at the resonance pole the unitary matrix $\Theta$ is taken diagonal. This can be probed to be the case for a narrow resonance\footnote{A resonance lying above threshold but with vanishing width. The width of its signal is fully reflected in the physical energy axis.} by invoking unitarity and the physical requirement that the module of every coupling to an open channel properly determines its branching decay ratio or partial width, so that it should not be modified.\footnote{This also applies to closed channels whose thresholds are much closer to the resonance mass than the width of the resonance \cite{42}, e.g. the $f_0(980)$ and the $K\bar{K}$ channel.} However, its phase is quite arbitrary and it is determined by the smooth non-resonant contributions. For more details see Ref. \cite{42}.

Because of this result from Ref. \cite{42} we can then properly choose the phase of the coupling to a partial wave so that its compositeness is $|X_{ES}|$. We then have the following criterion for the elementariness of a narrow resonance with respect to the explicit channel considered in the NR treatment [of course, there could be several partial waves and this fact is properly taking by the sum over them, cf. Eq. (164)]

$$|X| \ll 1 \ .$$

(164)
with $X$ calculated as in Eq. (144). This criterion cannot be strictly extended to a relativistic narrow resonance because the expectation value of an operator number $N^A_R$ counts all the particles of type $A$ present in any possible open or closed channel.

The outlined procedure for the narrow resonance case was generalized in the same reference [42] to a finite width resonance whose pole lies in the Riemann sheet that connects continuously with the physical axis between two consecutive channels. In NR QFT one should require that $E_{th,n} < M_R < E_{th,n+1}$, while in the relativistic case one should use the $s$ variable and write $s_{th,n} < Res_R < s_{th,n+1}$, with $s_R = (M_R - iΓ/2)^2$, and the thresholds for the channels $n$ and $n + 1$ are indicated with an obvious notation. The point of this requirement is that the Laurent series around the resonance pole can match with the physical axis within some energy interval, so that the modules of the residue at the resonance poles have still physical meaning as couplings. In order to apply safely this requirement one should ascertain a physical process in which the non-resonant terms would play little role and then the resonance signal becomes well manifest. A good example of this is the $f_0(500)$ resonance or $σ$ which can hardly be seen in isoscalar scalar $ππ$ scattering while it is manifest in the pion scalar form factor which is the one that drives the low-energy part of the decays of $D^+ \to π^−π^+π^+$ [43], as discussed in Ref. [44].

We can give another thought (a more “microscopic” one) for the origin of such phase transformation of the couplings stemming from Eq. (162) and introduced in Ref. [42]. To accomplish this aim let us consider energy-dependent transformations in the partial-wave projected in/out states. These are driven by a function $η_i(E)$, which at least has a unitarity cut and satisfy the Schwarz reflection principle $η_i(E ± iε) = η_i(E = iε)^*$, as it is the case for partial-wave scattering amplitudes. Here the subscript $i$ refers to any partial wave to which the resonance couples. The transformation in question is

\[
|ψ^+_{α}| \to e^{iη_i(E_α + iε)}|ψ^+_{α}| , \tag{165}
\]

\[
⟨ψ^-_{α}| \to ⟨ψ^-_{α}| e^{iη_i(E_α - iε)^*} = ⟨ψ^-_{α}| e^{iη_i(E_α + iε)} . \tag{166}
\]

In this way when performing the analytical extrapolation to the 2nd RS to reach the resonance pole at $E_R$ we have to cross the unitarity cut and enter in this unphysical sheet so that first

\[
η_i(E_α + iε) \to η_i^H(E_α - iε) , \tag{167}
\]

and from here, with $E$ having already negative imaginary part, reach $E_R$ with the value $η_i^H(M_R - iΓ/2)$. Let us stress that this transformation has no analogue for a bound state. As a result of this transformation the couplings change as

\[
g^2_i(k^2) \to g^2_i(k^2)e^{2Imη_i^H(E_R)} . \tag{168}
\]

For the case of a narrow resonance we can write a plausible dispersion relation for the smooth function $η_i(E)$ around the resonance region as

\[
η_i(E) = \frac{1}{π} \int dE' \frac{Imη_i(E')}{E' - M_R - iΓ/2} \approx \frac{1}{π} \int dE' \frac{Imη_i(E')}{E' - M_R} + i Imη_i(M_R) . \tag{169}
\]

Since $Imη_i(E')$ is nearly constant around the narrow-resonance mass, its Cauchy principal value around the latter should be very small and the dominant contribution in Eq. (169) is its imaginary part. Therefore in this case we recover the results of Ref. [42] and we have the change in the coupling by a phase factor

\[
g^2_i(k^2) \to g^2_i(k^2)e^{2Imη_i^H(E_R)} . \tag{170}
\]

This derivation also shows that for a finite width resonance is not so clear that $η_i(E)$ is just a purely imaginary number. However, in the lines of the discussion above, for a resonance that is manifest on
the physical real energy axis the modules of its residues can be interpreted as physical couplings and the corrections on them (if any) would be relatively small and a transformation like that in Eq. (170) should be reasonable.

In summary, for a narrow resonance lying above threshold in NR QFT we can calculate its compositeness on an open partial wave by taking the absolute value of $X_{\ell S}$. For a finite-width resonance we can say that this is also a reasonable calculation if the resonance is manifest at some interval along the physical energy axis.

In the relativistic case the situation is a priori less clear since one cannot exclude contributions from closed channels containing particles of type $A$ in the evaluation of the expectation value of a number operator $N^A_D$. Therefore, the change of phase in the couplings of only the open channels is not of general usage. Nonetheless, in practical applications within models that incorporate only a few coupled channels and with expected suppression of extra multi-particle components, one could still apply these changes of phase in the couplings for the open channels and give physically reasonable results.

8 Conclusions

We have given a new perspective to the problem of the compositeness/elementariness of a bound state or a resonance by considering the expectation values in the state of the number operators of the free particle species. This new formalism is an important step forward for this relevant problem.

At the fundamental level there are important examples in which the Hamiltonian is not expressed in terms of the asymptotic degrees of freedom, e.g. Quantum Chromodynamics (QCD). It is then clear that answering the question whether a bound or resonance state generated in such theories is elementary or a composite of the asymptotic states in the continuum might be particularly demanding. It could be also the case that the bare elementary states are integrated out in the effective field theory, so that in relativistic QFT one cannot address then the issue of compositeness of the dynamically-generated bound states and resonances in terms of the traditional language based on the wave-function renormalization of the bare elementary field. Let us stress that one can address both important questions on the composite or elementary nature of a bound or resonance state with respect to the states in the continuum by evaluating the expectation values of the operator numbers of the free particle states in QFT, as developed in this work. These questions correspond indeed to common situations in hadron physics.

We have discussed first the non-relativistic case and developed suitable expressions for its computation within QFT, e.g. by using Feynman diagrams. In terms of them we have provided a new closed equation for the compositeness of a non-relativistic bound state in the scattering of two particles with large wavelengths compared with the typical range of their interaction. This equation has allowed us to conclude that $X = 1$ for zero-range energy-independent potentials. This conclusion has been also be extended for any finite-range energy-independent potential, being regular or singular. The equation for the calculation of the expectation values of the number operators in a bound state within relativistic QFT can be easily derived from its NR QFT counterparts. In this way a universal criterion for the elementary character of a bound state, both in NR and relativistic QFT, has been given for the first time. It is also shown that $X$ is independent under unitarity transformations and field redefinitions. We also offered a non-trivial example in the NR QFT case for an energy-dependent potential between the asymptotic particles in the continuum in which it is shown that $X$ is independent of any type of cutoff regulator employed, once its nonperturbative calculation is undertaken. This case also illustrates the difficulties that dimensional regularization might have in nonperturbative calculations (complementing with more examples those already given in Ref. [27]).

Next, we have moved on to the resonance case. The equations for the calculation of the expectation values of the number operators in a resonance state have been given within NR and relativistic QFT.
We have deduced as well a universal necessary condition for a resonance being qualified as elementary. Within non-relativistic scattering theory we have derived that $X = 1$ for finite-range energy-independent potentials, similarly as for bound states. We have also introduced suitable phase-factor transformation that are closely related to the $S$-matrix transformations first given in Ref. [42]. In terms of them one can end with real positive values for the compositeness of a narrow resonance in NR QFT with respect to the open channels. This result can be also extended with quite confidence to the case of finite-width resonances following the same methods. For the relativistic case, the use of unitary transformations only upon the couplings to the open channels is not enough to derive meaningful positive real values of these expectation values in the general case. Nonetheless, one should stress that in many practical examples (e.g. when just a few coupled-channels are included in the model) they are of interest.

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A Decomposition in partial waves in the $\ell S$ basis

Let us follow the steps indicated in the footnote 2 to calculate the diagrams in Fig. 1. In QFT we have that their sum is

$$M = i \int \frac{d^4k}{(2\pi)^4} f(k)^2 \left\{ \frac{(2M_A)^2(2M_B)}{[(P - K)^2 - M_A^2][k^2 - M_B^2]} + \frac{(2M_A)(2M_B)^2}{(k^2 - M_A^2)[(P - k)^2 - M_B^2]} \right\}, \quad (A.1)$$

where $f(k)^2$ is the coupling squared, that only depends on the three-momentum in the non-relativistic case, $P = 0$ and $P^0 = M_A + M_B + E_B$ (of course, the masses of the particles $A$ and $B$, $M_A$ and $M_B$, in order, are much bigger than $|E_B|$). We have also included the right mass factors multiplying the relativistic propagators so as to end with the standard NR reduction, corresponding to the standard normalization to $(2\pi)^3\delta(p - q)$ for NR plane-wave states. We next perform the integration over $k^0$ in Eq. (A.1), e.g. by closing the $k^0$ integration contour along the upper half plane. Let us consider the first term on the r.h.s. of the previous equation and introduce the notation $w_i = \sqrt{M_i^2 + k^2}$. We have

$$I_1 = \int \frac{dk^0}{2\pi} \frac{(2M_A)^2(2M_B)}{[(P^0 - k^0)^2 - w_A^2 + i\varepsilon]^2(k^0 - w_B + i\varepsilon)} \cdot \quad (A.2)$$

This integral has two poles for $\text{Im} \ k^0 > 0$, a double one at $k^0 = P^0 - w_A + i\varepsilon$ and a simple pole at $k^0 = -w_B + i\varepsilon$. Calculating the residues of these poles we have

$$I_1 = i(2M_A)^2(2M_B) \left[ \frac{\partial}{\partial k^0} \left\{ \frac{1}{[(k^0 - P^0 - w_A)^2(k^0 - w_B + i\varepsilon)(k^0 + w_B)]} \right\} \right]_{k^0 = P^0 - w_A + i\varepsilon} \quad (A.3)$$

$$- i \frac{1}{(w_A + w_B + P^0)^2(w_A - w_B - P^0)^2w_B}.$$
We now proceed with the NR reduction of every factor in the denominators
\[ k^0 = P^0 - w_A + i \varepsilon \rightarrow M_B + \frac{\kappa^2}{2\mu} - \frac{k^2}{2M_A} + i\varepsilon , \quad (A.4) \]
\[ k^0 - P^0 - w_A \rightarrow -2M_A , \]
\[ k^0 - w_B + i \varepsilon \rightarrow \frac{\kappa^2}{2\mu} - \frac{k^2}{2\mu} + i\varepsilon , \]
\[ k^0 + w_B \rightarrow 2M_B , \]
\[ w_A + w_B + P^0 \rightarrow 2(M_A + M_B) , \]
\[ w_A - w_B - P^0 \rightarrow -2M_B , \]
\[ 2w_B \rightarrow 2M_B . \quad (A.5) \]

As a result
\[ I_1 \rightarrow -\frac{i}{(\kappa^2/2\mu - k^2/2\mu + i\varepsilon)^2} . \quad (A.6) \]

For the second term on the r.h.s. of Eq. (A.1) we have the same result. Summing both contributions we obtain
\[ X = \int \frac{d^3k}{(2\pi)^3} \frac{f(k)^2}{(\kappa^2/2\mu - k^2/2\mu + i\varepsilon)^2} . \quad (A.7) \]

For a bound state the \(+i\varepsilon\) can be dropped because \(\kappa^2 < 0\).

The coupling of a resonance with angular momentum \(J\) to a two-particle channel with orbital angular momentum \(\ell\) and total spin \(S\) is
\[ \sqrt{4\pi} g_{\ell S} \sum_{m,M} (\sigma_1 \sigma_2 M|s_1 s_2 S)(mM|\ell S)Y^m_\ell(\hat{p}) \quad (A.8) \]

When using this decomposition of the coupling into the equation for calculating \(X\), which depends on the coupling squared, the diagonal sum over \(X_{\ell S}\) in Eq. (34) results once the angular integration and the sum over the \(\sigma_i, i = 1, 2\), are performed. Here one has to use the orthogonality properties of the spherical harmonics and the Clebsch-Gordan coefficients.

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