Topology Change
in (2+1)-Dimensional Gravity

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Abstract

In (2+1)-dimensional general relativity, the path integral for a manifold \(M\) can be expressed in terms of a topological invariant, the Ray-Singer torsion of a flat bundle over \(M\). For some manifolds, this makes an explicit computation of transition amplitudes possible. In this paper, we evaluate the amplitude for a simple topology-changing process. We show that certain amplitudes for spatial topology change are nonvanishing—in fact, they can be infrared divergent—but that they are infinitely suppressed relative to similar topology-preserving amplitudes.

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1. Introduction

The path integral in general relativity is a sum over geometries, and it is natural to ask whether this sum should be extended to include topologies as well [1]. This question can take two forms: (1) should path integrals include sums over intermediate *spacetime* topologies (“spacetime foam”); and (2) should we allow transitions between different *spatial* topologies? These issues are closely related—it is hard to imagine a formalism that permits spatial topology change while forbidding sums over intermediate topologies—but they are distinct. In this paper, we focus on the second.

Topology change, if it occurs, is expected to be a quantum mechanical process [2]. Unfortunately, we do not yet have a viable quantum theory of gravity with which to compute topology-changing amplitudes. We can, however, look for models that give us hints about what to expect in the full theory.

One particularly interesting model is (2+1)-dimensional gravity, standard general relativity in two spatial dimensions plus time. The underlying conceptual issues of quantum gravity, and some of the technical aspects as well, are identical in 2+1 and 3+1 dimensions. But the elimination of one dimension vastly simplifies the theory, making many computations possible for the first time. Moreover, general relativity in 2+1 dimensions is renormalizable (it is in fact finite), allowing us to avoid the difficult problems of interpreting path integrals in (3+1)-dimensional gravity.

Indeed, as first noted by Witten [3,4], the path integral for (2+1)-dimensional gravity can be evaluated explicitly in terms of a standard topological invariant, the Ray-Singer torsion. In first-order form, the action may be written schematically as

\[
S = \int_M e^a R_a, \quad R_a = d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \omega^c
\]  

(see the next section for details). The Lagrangian is cubic in the fields, but transition amplitudes can still be computed nonperturbatively: the triad e acts as a Lagrange multiplier, giving a delta functional that absorbs the remaining integral over \( \omega \). With careful gauge-fixing, one obtains a combination of determinants of Laplacians known as the Ray-Singer torsion, a well-known topological invariant of a flat bundle over \( M \). This torsion, in turn, is equivalent to the Reidemeister torsion, a combinatorial invariant that can be relatively simple to compute.

In reference [3], Witten points out that this equality makes the explicit computation of topology-changing amplitudes possible. He obtains a set of selection rules relating holonomies of \( \omega \) on the initial and final boundaries of \( M \). In reference [4], Amano and Higuchi obtain a much more stringent set of selection rules by demanding that the initial and final boundaries be spacelike. They further write down a set of three-manifolds, expressed in terms of Heegaard decompositions, that interpolate between all topology changes allowed by their selection rules among surfaces of genus greater than one.

The initial goal of this paper was to compute the Reidemeister torsion for the simplest of these interpolating three-manifolds, thus obtaining an explicit topology-changing amplitude. We do so in section 6. In the course of the computation, however, we found it necessary to
reanalyze Witten’s expression for the path integral, treating boundary conditions somewhat more carefully. To our surprise, we discovered that the path integrals for the Amano-Higuchi manifolds typically involve zero-modes of the triad $e$, leading to infrared divergences of the type discussed by Witten in “classical” spacetimes \cite{4}.

To make sense of our topology-changing amplitudes, we must therefore add an infrared cutoff and compare the results to similar topology-preserving amplitudes. Unfortunately, the meaning of the word “similar” is not entirely clear. As noted above, once we permit spatial topology change, we ought to also allow sums over intermediate spacetime topologies as well. Such sums are typically badly divergent \cite{6}, and it is not clear how to normalize the total amplitude. This is a deep conceptual question, which goes beyond the issue of divergences: a topology-changing amplitude relates states in different Hilbert spaces, and we do not yet have a “Fock space” that includes multiple topologies.

We shall not attempt to address this issue here. Instead, we take the less ambitious route of normalizing our amplitude relative to a topology-preserving amplitude coming from a three-manifold with a related Heegaard decomposition. We shall show that the effect of the infrared divergences discussed above is to infinitely suppress the topology-changing amplitude. In the absence of a clear prescription for normalization, we cannot yet claim that quantum gravity prohibits topology change, but we find this result rather suggestive.

2. Action, States, and Boundary Conditions

We begin with a brief review of (2+1)-dimensional gravity in the first-order formalism \cite{3}. Let $M$ be a three-dimensional spacetime. As our fundamental variables, we choose a triad $e^a_\mu$ —a section of the bundle of orthonormal frames on $M$— and an SO(2,1) connection on the same bundle, which we describe by a connection one-form $\omega^a_\mu$. The standard Einstein-Hilbert action is

$$I_{\text{grav}}[M] = \int_M e^a \wedge \left( d\omega_a + \frac{1}{2} \epsilon^{abc} \omega_b \wedge \omega_c \right),$$

where $e^a = e^a_\mu dx^\mu$ and $\omega^a = \frac{1}{2} \epsilon^{abc} \omega_{\mu bc} dx^\mu$. The equations of motion coming from this action are easily derived: they are

$$de^a + \epsilon^{abc} \omega_b \wedge e_c = 0 \quad (2.2)$$

and

$$R^a = d\omega^a + \frac{1}{2} \epsilon^{abc} \omega_b \wedge \omega_c = 0. \quad (2.3)$$

The first of these is the condition of vanishing torsion, which ensures that the connection $\omega$ is compatible with the metric $g_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu$. The second is then equivalent to the ordinary vacuum field equations of general relativity. Note that (2.3) can also be interpreted as the requirement that the connection $\omega$ be flat, a feature peculiar to 2+1 dimensions that goes a long way towards explaining the model’s simplicity.

The action (2.1) is invariant under local SO(2,1) transformations,

$$\delta e^a = \epsilon^{abc} e_b \tau_c,$$

$$\delta \omega^a = d\tau^a + \epsilon^{abc} \omega_b \tau_c, \quad (2.4)$$
as well as “local translations,”

\[ \delta e^a = d \rho^a + \epsilon^{abc} \omega_b \rho_c \]
\[ \delta \omega^a = 0. \]  

(2.5)

These transformations together form an ISO(2,1) algebra, and Witten has shown that the one-forms \( e^a \) and \( \omega^a \) constitute an ISO(2,1) connection. \( I_{\text{grav}} \) is also invariant under diffeomorphisms, but this is not an independent symmetry: when the triad \( e_\mu^a \) is invertible, diffeomorphisms in the connected component of the identity are equivalent to transformations of the form (2.4)–(2.5). We must still account for the “large” diffeomorphisms—the mapping class group of \( M \)—but for most of this paper, these will not play an important role.

Since we are interested in topology change, we shall use path integral techniques to quantize the action (2.1). But because we are dealing with manifolds with boundary, we must first determine the appropriate boundary conditions. A simple heuristic argument is as follows. Let us start by picking boundary values for either \( e \) or \( \omega \) on \( \partial M \). This amounts to choosing Dirichlet boundary conditions for (say) \( \omega \). Now, the kinetic term in the action (2.1) can be written as \( \langle e, * d \omega \rangle \), where \( * \) is the Hodge star operator and \( \langle, \rangle \) is the corresponding inner product. But if \( \omega \) obeys Dirichlet boundary conditions, \( * d \omega \) obeys Neumann boundary conditions, so we ought to require the same of \( e \) in the inner product. We thus expect opposite boundary conditions for our two fields.

To make this argument more rigorous, let us first examine the Hilbert space of (2+1)-dimensional quantum gravity. Canonical quantization of this theory on a manifold \( \mathbb{R} \times \Sigma \) has been discussed by a number of authors; see [7] for a summary. A key feature is that the classical reduced phase space is a cotangent bundle, whose base space is the space of flat connections \( \omega \) on a slice \( \Sigma \) modulo SO(2,1) gauge transformations and large diffeomorphisms. In the simplest approach to quantization, the “connection representation” [8], states are therefore gauge-invariant functionals \( \Psi[\omega_i^a] \) of the spatial part of the connection, subject to the constraint that \( \omega \) be flat on \( \Sigma \).

The corresponding boundary conditions for the path integral therefore require us to to fix a flat connection \( \omega_i^a \) on \( \partial M \). More precisely, let \( I: \partial M \hookrightarrow M \) be the inclusion map. We can then freely specify the induced connection one-form \( I^* \omega \) on \( \partial M \), as long as the induced curvature \( I^* R \) vanishes. SO(2,1) gauge invariance of the resulting amplitude is formally guaranteed by the functional integral over the normal component of \( \omega \): at \( \partial M \), \( \omega \perp a \) is a Lagrange multiplier for the constraint

\[ N^a = \frac{1}{2} \epsilon^{ij} \left( \partial_i e_j^a - \partial_j e_i^a + \epsilon^{abc} (\omega_{ib} e_{jc} - \omega_{ic} e_{jb}) \right) \]  

(2.6)

that generates SO(2,1) transformations of \( I^* \omega \) [3, 9]. Observe that we must integrate over \( \omega \perp \) at the boundary to enforce this constraint—that is, we must not fix \( \omega \perp \) as part of the boundary data. This is in accord with the canonical theory, in which wave functionals depend only on the tangential components of \( \omega \).

Note that the flat connection \( I^* \omega \) is completely determined by its holonomies, that is, by a group homomorphism

\[ H: \pi_1(\partial M) \to \text{SO}(2,1). \]  

(2.7)
As we shall see later, the transition amplitude can be described rather explicitly in terms of these holonomies.

The specification of $I^*\omega$ is not quite sufficient to give us a well-defined path integral. To obtain an additional boundary condition, it is useful to decompose $\omega$ into a background field $\bar{\omega}$ that satisfies the classical field equations and a fluctuation $\Omega$:

$$\omega = \bar{\omega} + \Omega, \quad d\omega^a + \frac{1}{2}\epsilon^{abc}\bar{\omega}_b \wedge \bar{\omega}_c = 0. \quad (2.8)$$

Note that if there is no classical field with our chosen boundary values, then the transition amplitude is zero: the integral over $e$ gives a delta functional $\delta[R^a]$ that vanishes everywhere. The requirement of existence of a classical solution leads to Witten’s selection rules for topology change [4]. Assuming now that $\bar{\omega}$ exists, the boundary condition $I^*\Omega = 0$ can be recognized as a part of the standard Dirichlet, or relative, boundary conditions for a one-form [10, 11],

$$I^*\Omega = 0 = \ast \bar{D} \ast \Omega. \quad (2.9)$$

Here, $\ast$ is the Hodge star operator with respect to an auxiliary Riemannian metric $h$ that we introduce in order to define a direction normal to the boundary, while $\bar{D}$ is the covariant exterior derivative coupled to the background connection $\bar{\omega}$,

$$\bar{D} \beta^a = d\beta^a + \epsilon^{abc}\bar{\omega}_b \wedge \beta_c.$$ 

These boundary conditions make the operators $\bar{D}$ and $\ast \bar{D} \ast$ hermitian conjugates, and guarantee that the Laplacian $\Delta = \bar{D} \ast \bar{D} + \ast \bar{D} \ast \bar{D}$ is hermitian. We shall see below that the new condition on the derivatives of $\Omega$, which can be written in component form as $\bar{D}_\mu \Omega^\mu_{\mid \partial M} = 0$, is actually a gauge condition. Note that (2.9) depends on the nonphysical metric $h$; we must check that the final transition amplitudes are independent of $h$.

We next turn to the boundary conditions for the triad $e$. $I^*e$ and $I^*\omega$ are conjugate variables, so we cannot expect to specify them simultaneously. On the other hand, we must not integrate over the normal component of $e$ at the boundary. Indeed, $e_{\perp a}$ acts as a Lagrange multiplier for the constraint

$$\bar{N}^a = \frac{1}{2} \epsilon^{ij} \left( \partial_i \omega_j^a - \partial_j \omega_i^a + \epsilon^{abc}\bar{\omega}_b \omega_j^c \right), \quad (2.10)$$

and would lead to a delta functional $\delta[\bar{N}^a] = \delta[I^*R^a]$ at the boundary. But we have already required that $I^*\omega$ be flat, so such a delta functional would diverge. We avoid this redundancy by fixing $e_{\perp}$ at $\partial M$. We shall see below that transition amplitudes do not depend on the specific value of $e_{\perp}$, so this does not contradict the canonical picture.

As with $\omega$, we can obtain additional boundary conditions by decomposing $e$ into a classical background field and a fluctuation

$$e = \bar{e} + E, \quad de^a + \epsilon^{abc}\bar{\omega}_b \wedge \bar{e}_c = 0 \quad (2.11)$$

where $E_{\perp}$ vanishes, i.e., $I^*(\ast E) = 0$. This restriction on $E$ is a part of the standard Neumann, or absolute, boundary conditions for a one-form,

$$I^*(\ast E) = 0 = I^*(\ast \bar{D} E). \quad (2.12)$$

Once again, these conditions make the Laplacian $\Delta$ hermitian.
3. Path Integrals and Ray-Singer Torsion

We are interested in path integrals of the form
\[ Z_M[I^*\omega] = \int [d\omega][de] \exp \{iI_{\text{grav}}[M] \}, \tag{3.1} \]
where \( M \) is a manifold whose boundary
\[ \partial M = \Sigma_1 \amalg \Sigma_2 \tag{3.2} \]
is the disjoint union of an “initial” surface \( \Sigma_1 \) and a “final” surface \( \Sigma_2 \). (\( \Sigma_1 \) and \( \Sigma_2 \) need not be connected surfaces.) Our first step is to choose gauge conditions to fix the transformations (2.4)–(2.5). To do so, we employ the auxiliary Riemannian metric \( h \) introduced in the last section, and impose the Lorentz gauge conditions
\[ *D*E^a = *D*\Omega^a = 0. \tag{3.3} \]
For later convenience, we use the covariant derivative \( D \) coupled to the full connection \( \omega \) rather than \( \bar{\omega} \) in our gauge-fixing condition. \( D \) and \( \bar{D} \) agree at the boundary, however, so the gauge condition on \( \Omega \) reduces to the second equation of (2.9) on \( \partial M \), as promised.

To impose (3.3) in the path integral, we introduce a pair of three-form Lagrange multipliers \( u_a \) and \( v_a \), and add a term
\[ I_{\text{gauge}} = -\int_M (u_a \wedge *D*E^a + v_a \wedge *D*\Omega^a) \tag{3.4} \]
to the action. It is not hard to see that for the path integral to be well-defined, \( u \) should obey relative boundary conditions \( (I^*(D*u) = 0) \), while \( v \) should obey absolute boundary conditions \( (*v = 0 \text{ on } \partial M) \). The latter restriction again has a rather straightforward interpretation: since we are already imposing the gauge condition (2.9) on \( \Omega \) at the boundary, we do not need the added delta functional \( \delta[\ast D \Omega] \) that would come from integrating over \( v \) at \( \partial M \).

As usual, the process of gauge-fixing leads to a Faddeev-Popov determinant, which can be incorporated by adding a ghost term
\[ I_{\text{ghost}} = -\int_M \left( \bar{f} \wedge *D*Df + \bar{g} \wedge *D*Dg \right), \tag{3.5} \]
where \( f, \bar{f}, g, \) and \( \bar{g} \) are anticommuting ghost fields. We must again be careful about boundary conditions: corresponding to restrictions (2.9) and (2.12) on \( \Omega \) and \( E \), we choose \( f \) and \( \bar{f} \) to satisfy relative boundary conditions and \( g \) and \( \bar{g} \) to satisfy absolute boundary conditions. The full gauge-fixed action is then
\[ I = I_{\text{grav}} + I_{\text{gauge}} + I_{\text{ghost}} \]
\[ = \int_M \left[ E^a \wedge \left( \bar{D}\Omega_a + \frac{1}{2} \epsilon_{abc} \Omega^b \wedge \Omega^c + *D*u_a \right) \right. \]
\[ + \frac{1}{2} \epsilon_{abc} \bar{e}^a \wedge \Omega^b \wedge \Omega^c - v^a \wedge *D* \Omega_a - \bar{f} \wedge *D*Df - \bar{g} \wedge *D*Dg \right]. \tag{3.6} \]

Much of this section is a summary and elaboration of Witten’s work in [4].

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\( E \) and \( v \) occur linearly in (3.6), so following Witten [4], we shall first integrate over these fields to obtain delta functionals. There is one subtlety here: certain modes of \( E \) do not contribute to the action, and must be treated separately in the integration measure. The issue is most easily understood in the case of a linear action, for instance

\[
I = \int_M d^3x \sqrt{g} \alpha \Delta \beta.
\]

If we expand \( \alpha \) and \( \beta \) in terms of orthonormal modes \( \phi_n \) of the Laplacian, \( \alpha = \sum a_n \phi_n \), \( \beta = \sum b_n \phi_n \), it is easy to see that \( I = \sum' \lambda_n a_n b_n \), where \( \lambda_n \) are the eigenvalues of the Laplacian and the sum automatically excludes the coefficients of the zero-modes of \( \Delta \). The path integral measure \([d\alpha]\) thus splits into an integral over modes for which \( \lambda_n \neq 0 \)—giving a delta function \( \prod' \lambda_n^{-1} \delta(b_n) \)—and an integral \( \int [d\alpha_0] \) over zero-modes.

Unfortunately, the action (3.6) is not linear in \( \Omega \), and such a mode decomposition is much more difficult. However, we can argue as follows. The integral over the “nonzero-modes” of \( E \) will give a delta functional of \( \bar{\partial} \Omega a + \frac{1}{2} \epsilon_{abc} \Omega^b \wedge \Omega^c + *D*u_a \). The zeros of this expression form a surface \( (\tilde{\Omega}(s), \tilde{u}(s)) \) in the space of fields, and if we expand the action around these zeros, only those fields infinitesimally close to this surface should contribute to the path integral. Writing \( \Omega = \tilde{\Omega} + \delta \Omega \), we easily find that the relevant zero-modes of \( E \) are those \( \tilde{E} \) for which

\[
D_{\omega + \tilde{\Omega}} \tilde{E} = 0. \tag{3.7}
\]

(Note that the \( \tilde{E} \) depend on \( \tilde{\Omega} \), so the order of integration below cannot be changed.) Performing the integration over \( E \) and \( v \), we obtain

\[
\int [d\Omega][du][dE][dv] e^{iI} = \int [d\Omega][du][d\tilde{E}] \delta[\bar{D}\Omega_a + \frac{1}{2} \epsilon_{abc} \Omega^b \wedge \Omega^c + *D*u_a] \delta[*D*\Omega_a]. \tag{3.8}
\]

The argument of the first delta functional vanishes only when \( D*D*u_a = 0 \); assuming that the connection \( \omega \) is irreducible, this implies that \( u_a = 0 \). The delta functional then imposes the condition

\[
\bar{D}\Omega_a + \frac{1}{2} \epsilon_{abc} \Omega^b \wedge \Omega^c = 0, \tag{3.9}
\]

which can be recognized as the requirement that \( \omega = \bar{\omega} + \Omega \) be a flat connection. This, in turn, allows us to eliminate the term

\[
\int_M \frac{1}{2} \epsilon_{abc} e^a \wedge \Omega^b \wedge \Omega^c = -\int_M e^a \wedge \tilde{D}\Omega_a = \int_M \tilde{D}e^a \wedge \Omega_a = 0
\]
in (3.6).

We can now use the delta functionals to perform the remaining integration over \( \Omega \). It is straightforward to show that

\[
[d\Omega] \delta \left[ \bar{D}\Omega_a + \frac{1}{2} \epsilon_{abc} \Omega^b \wedge \Omega^c + *D*u_a \right] \delta[*D*\Omega_a] = [d\tilde{\omega}] [det' \bar{L}_{rel}]^{-1}, \tag{3.10}
\]

1See [12] for a related treatment of the Abelian \( B-F \) system, in which this nonlinearity is not an issue.
where $\tilde{\omega} = \tilde{\omega} + \tilde{\Omega}$ ranges over flat connections with our specified boundary values and $\tilde{L}_-^{\text{rel}} = *D\tilde{\omega} + D\tilde{\omega}*$ maps a one-form plus a three-form $(\alpha, \beta)$ obeying relative boundary conditions to a one-form plus a three-form $(*D\tilde{\omega}\alpha + D\tilde{\omega} * \beta, D\tilde{\omega} * \alpha)$ obeying absolute boundary conditions. Performing the ghost integrals, we finally obtain

$$Z_M[I^*\omega] = \int [d\tilde{\omega}][d\tilde{E}] \frac{\det' \tilde{\Delta}^{\text{rel}}_{(0)} \det' \tilde{\Delta}^{\text{abs}}_{(0)}}{|\det' \tilde{L}^{\text{rel}}_0|},$$  \hspace{1cm} (3.11)$$

where $\tilde{\Delta}_{(k)}$ is the Laplacian $*D\tilde{\omega} * D\tilde{\omega} + D\tilde{\omega} * D\tilde{\omega} *$ acting on $k$-forms.

(Strictly speaking, the determinants $\det' \tilde{\Delta}$ are not well-defined for a noncompact gauge group like SO(2,1). But this is a minor problem, whose solution has been discussed in [10] and [14], and it does not affect our final expression for amplitudes in terms of Reidemeister torsion [14].)

Now, by expanding one-forms and three-forms in modes of $L^F L_-$, one may easily show that

$$|(\det' \tilde{L}_-)(\det' \tilde{L}_-^F)| = \det' \tilde{\Delta}^{\text{rel}}_{(1)} \det' \tilde{\Delta}^{\text{rel}}_{(3)} = \det' \tilde{\Delta}^{\text{abs}}_{(1)} \det' \tilde{\Delta}^{\text{abs}}_{(3)}$$  \hspace{1cm} (3.12)$$

Moreover, $\det' \tilde{\Delta}^{\text{rel}}_{(k)} = \det' \tilde{\Delta}^{\text{abs}}_{(3-k)}$, since the Hodge star operator maps any eigenfunction $\alpha$ of $\tilde{\Delta}^{\text{rel}}_{(k)}$ to an eigenfunction $*\alpha$ of $\tilde{\Delta}^{\text{abs}}_{(3-k)}$ with the same eigenvalue. Some simple manipulation then shows that

$$Z_M[I^*\omega] = \int [d\tilde{\omega}][d\tilde{E}] T[\tilde{\omega}],$$

$$T[\tilde{\omega}] = \frac{(\det' \tilde{\Delta}^{(3)}^{\text{rel}})^{3/2}(\det' \tilde{\Delta}^{(1)}_{(3)}^{\text{rel}})^{1/2}}{(\det' \tilde{\Delta}^{(2)}_{(3)}^{\text{rel}})} = \frac{(\det' \tilde{\Delta}^{(3)}_{(3)}^{\text{abs}})^{3/2}(\det' \tilde{\Delta}^{(1)}_{(3)}^{\text{abs}})^{1/2}}{(\det' \tilde{\Delta}^{(2)}_{(3)}^{\text{abs}})}.$$  \hspace{1cm} (3.13)$$

The combination of determinants $T[\tilde{\omega}]$ may be recognized as the Ray-Singer torsion [10]; the equality of relative and absolute torsions on odd-dimensional manifolds is shown in [13]. A similar transition amplitude for Abelian $B$-$F$ theories has been discussed by Wu [16].

We can now return to the question of whether our amplitude $Z_M$ depends on the choice of auxiliary metric $h$. When there are no zero-modes, $T[\tilde{\omega}]$ is known to be independent of $h$ [10,17]. When zero-modes are present, $T[\tilde{\omega}]$ depends on $h$, but so does the volume element $[d\tilde{\omega}][d\tilde{E}]$. If, as we have assumed, there are no ghost zero-modes—that is, if $H^0(M; V_{\tilde{\omega}}) = H^0(M, \partial M; V_{\tilde{\omega}}) = 0$—then the combination $[d\tilde{\omega}][d\tilde{E}] T[\tilde{\omega}]$ is again independent of $h$ [18,19]. If ghost zero-modes are present, they must be included in the integral (3.13); when they are, the amplitude is once again independent of $h$.

We conclude this section with a discussion of the range of integration in (3.13). As noted above, $\tilde{\omega}$ ranges over the space of gauge-fixed flat SO(2,1) connections—i.e., the moduli space of flat connections modulo gauge transformations—with specified boundary values. In general, this space is rather complicated. We can at least determine its dimension, however,

\footnote{For the remainder of this paper, we shall treat this space as if it were a manifold. When $M$ has the topology $\mathbb{R} \times \Sigma$, this is essentially correct [20]; for more complicated topologies, we do not know whether this assumption is justified.}
by linearizing (3.3) and (3.9): solutions $\delta \Omega$ of

$$D_\omega \delta \Omega^a = 0 = *D_\omega * \delta \Omega^a$$  \hspace{1cm} (3.14)

satisfying the boundary conditions (2.9) are cotangent vectors to the moduli space of flat connections at $\tilde{\omega}$.

These conditions have a natural cohomological interpretation. Since the connection $\tilde{\omega}$ is flat, $D_\omega^2 = 0$, so we can construct a de Rham cohomology $H^*(M; V_{\tilde{\omega}})$ on the complex of forms on $M$ with values in the flat bundle $V_{\tilde{\omega}}$ determined by $\tilde{\omega}$. Then (3.14) is equivalent to the condition

$$\delta \Omega \in H^1(M, \partial M; V_{\tilde{\omega}}),$$  \hspace{1cm} (3.15)

where the boundary conditions determine the use of relative cohomology. The range of $\tilde{E}$ has a similar interpretation. From (3.3) and (3.7) and the boundary conditions (2.12), it follows that

$$\tilde{E} \in H^1(M; V_{\tilde{\omega}}).$$  \hspace{1cm} (3.16)

Because of the change in boundary conditions, $\tilde{E}$ and $\delta \Omega$ do not lie in the same cohomology groups; indeed, by Poincaré-Lefschetz duality,

$$H^1(M; V_{\tilde{\omega}}) \cong H^2(M, \partial M; V_{\tilde{\omega}}).$$  \hspace{1cm} (3.17)

Unlike (3.15), equation (3.16) determines the entire space of fields $\tilde{E}$, not merely its tangent space. Moreover, the integrand $T[\tilde{\omega}]$ in (3.13) is independent of $\tilde{E}$. This means that if the cohomology group (3.17) is nontrivial, the amplitude $Z_M [I^* \omega]$ diverges. This is the infrared divergence cited by Witten as an indication of classical behavior [4]. In contrast to the cases discussed by Witten, however, our boundary conditions allow this divergence to appear even when the moduli space of flat connections $\tilde{\omega}$ consists of isolated points. This will be the case in specific examples we discuss below.

4. From Ray-Singer to Reidemeister Torsion

In principle, the integral (3.13) determines the transition amplitude for an arbitrary topology change in 2+1 dimensions. In practice, however, the determinants in $T[\tilde{\omega}]$ are usually impossible to evaluate. We must therefore take one further step, and relate the Ray-Singer torsion to the combinatorial, or Reidemeister, torsion.

We begin with a brief description of the Reidemeister torsion. (For more details, see [10], [21], or [22].) It is instructive to start with a concrete description of the flat bundle $V_{\tilde{\omega}}$. Let $\pi_1$ be the fundamental group of $M$, and let $\tilde{M}$ denote the universal covering space of $M$, so

$$M \approx \tilde{M}/\pi_1.$$  \hspace{1cm} (4.1)

As in (2.7), the flat connection $\tilde{\omega}$ is determined up to gauge transformations by its holonomy group

$$H : \pi_1 \to \text{SO}(2,1).$$  \hspace{1cm} (4.2)
We can now define our flat bundle as

\[ V_\varphi = \left( \tilde{M} \times \text{so}(2,1) \right) / \pi_1, \]

where \( \pi_1 \) acts on \( \tilde{M} \) as in (4.1) and on the Lie algebra \( \text{so}(2,1) \) by the adjoint action of the holonomy group.

We now repeat this construction in a slightly different form. Let us treat \( M \) as a CW complex, with \( k \)-cells \( \{ \alpha_k \} \). \( M \) can be viewed as a fundamental domain embedded in \( \tilde{M} \), which then has a corresponding cell decomposition in terms of the translates \( \{ g \alpha_k \}, g \in \pi_1 \} \).

The chain groups \( C_k(\tilde{M}) \) of the universal covering space \( \tilde{M} \) thus become modules over the real group algebra \( \mathbb{R}[\pi_1] \), with the \( \{ \alpha_k \} \) constituting a preferred basis.\footnote{Note that if \( \alpha_k \) is a cell in \( M \), its boundary will not, in general, lie in \( M \), but will rather be a sum of translates of \( (k-1) \)-cells \( \sigma_k \) by elements of \( \pi_1 \). Relative to our preferred basis, the boundary operator can thus be viewed as a matrix with elements in \( \mathbb{R}[\pi_1] \).}

We can now construct the twisted chain complex

\[ \mathcal{C}(\tilde{M}; V_\varphi) = \mathcal{C}(\tilde{M}) \otimes_{\mathbb{R}[\pi_1]} \text{so}(2,1), \]

where, as in (4.3), \( \pi_1 \) acts on \( \mathcal{C}(\tilde{M}) \) by translation and on \( \text{so}(2,1) \) by the adjoint action. A preferred basis for \( \mathcal{C}(\tilde{M}; V_\varphi) \) consists of the elements \( \{ \alpha_k \} \otimes J^a \), where the \( J^a \) are an orthonormal set of generators of \( \text{so}(2,1) \). We shall abbreviate these basis elements by \( c(k) \) below. The boundary operator for this twisted complex can be viewed as a real matrix: if \( \partial \alpha_k = g \alpha_{k-1} + \ldots \), then

\[ \partial (\alpha_k \otimes J^a) = e^\beta_{(k-1)} \otimes \text{ad}(g) J^a + \ldots = e^\beta_{(k-1)} \otimes S[g]^a_b J^b + \ldots, \]

where \( S[g]^a_b \) are the matrix elements of \( g \) in the adjoint representation.

The Reidemeister torsion of \( M \) is now defined as follows. We have chosen a preferred basis \( c(k) \) for each of the chain groups \( C_k(\tilde{M}; V_\varphi) \). We also have a preferred basis for each homology group \( H_k(\tilde{M}; V_\varphi) \), determined from the harmonic forms of the previous section by the de Rham isomorphism. Let us select a set \( \tilde{B}_k \in Z_k \) to represent these basis elements. We next choose an arbitrary basis \( b(k) \) for each \( B_k \), and a set \( \tilde{b}(k) \in C_k \) such that \( \partial \tilde{b}(k) = b(k-1) \).

It is easy to see that the set \( (b(k), \tilde{B}_k, \tilde{b}(k-1)) \) forms a new basis—call it \( \tilde{c}(k) \)—for \( C_k \). Let \( T(k) \) denote the matrix representing the change of basis from \( c(k) \) to \( \tilde{c}(k) \), that is, \( \tilde{c}(k) = T(k)c(k) \). The Reidemeister torsion is then defined as

\[ \tau(M; V_\varphi) = \frac{\det T(0) \det T(2)}{\det T(1) \det T(3)}. \]

The relative Reidemeister torsion \( \tau(M, \partial M; V_\varphi) \) is obtained by the same construction with the relative chain complex. Since cellular decompositions of manifolds are often rather simple, this invariant can sometimes be calculated quite explicitly.

*As usual, we denote chain groups by \( C_k \), the kernel of \( \partial \) in \( C_k \) by \( Z_k \), and the image of \( C_{k+1} \) in \( Z_k \) by \( B_k \); the homology groups are \( H_k = Z_k/B_k \).*
For a manifold without boundary, the Reidemeister torsion is, remarkably, equal to the Ray-Singer torsion $T[\tilde{\omega}]$ of equation (3.13) [22,23]. When a boundary is present, this equality no longer holds, but only a small correction is needed [15]:

$$T[\tilde{\omega}] = 2^{3\chi(\partial M)/4} \tau(M, \partial M; V_{\tilde{\omega}}),$$

(4.7)

where $\chi(\partial M)$ is the Euler characteristic of the boundary. (The factor of three in the exponent is the dimension of $\text{so}(2,1)$.) We can therefore determine the transition amplitude (3.13) by evaluating the Reidemeister torsion for $M$.

5. Topology-Changing Manifolds

To compute a topology-changing amplitude, it remains for us to specify the manifold $M$ that mediates the transition. This choice is not trivial—as Amano and Higuchi have shown [5], the requirement that the initial and final boundaries be spacelike strongly restricts the allowed topologies. We begin with $M_2$, the simplest of the interpolating manifolds of reference [5], which describes a transition from a genus three surface to two genus two surfaces. This manifold, shown in figure 1, can be described as follows:

Let $V$ be a genus four handle-body, and let $\bar{V}$ denote a reflected copy of $V$. Remove from the interior of $V$ two genus two handle-bodies, as indicated by the shaded areas in figure 1, to obtain a manifold $W_1$ with boundary $\partial W_1 \approx \partial V \cup \Sigma_2 \cup \Sigma', \Sigma_2$ and $\Sigma'$ are genus two surfaces. Next, remove from the interior of $\bar{V}$ a single genus three handle-body, as shown, to obtain a manifold $W_2$ with boundary $\partial W_2 \approx \partial \bar{V} \cup \Sigma_3$, where $\Sigma_3$ is a genus three surface. Now identify $W_1$ and $W_2$ along their common boundary $\partial V \approx \partial \bar{V}$ to obtain our desired manifold $M_2$, which clearly has a boundary $\partial M_2 \approx \Sigma_2 \cup \Sigma_2' \cup \Sigma_3$. More general amplitudes can be obtained by combining manifolds $M_n$ that mediate between a genus $n+1$ surface and $n$ genus two surfaces; we refer the reader to reference [5] for details.

Figure 1 also indicates a set of generators $\rho_1, \ldots, \rho_8$ for $\pi_1(M_2)$, which obey the relations

$$[\rho_2, \rho_1^{-1}][\rho_3^{-1}, \rho_4^{-1}] = [\rho_6, \rho_5^{-1}][\rho_7^{-1}, \rho_8^{-1}] = 1,$$

$$\rho_3 = \rho_5,$$

(5.1)

where $[\sigma, \tau] = \sigma \tau \sigma^{-1} \tau^{-1}$ is the commutator. Apart from a slightly unusual normalization that will be useful later in describing the cell decomposition, the relation obeyed by $\rho_1, \ldots, \rho_4$ can be recognized as that of a Fuchsian group that uniformizes a genus two surface [24]. The generators $\rho_5, \ldots, \rho_8$ determine a similar group.

It will be helpful to compare the amplitude coming from $M_2$ to a related topology-preserving amplitude. We obtain the latter from a new manifold $P_2$ constructed by attaching two copies of $W_2$ along their common boundary $\partial V$. This manifold mediates transitions from $\Sigma_3$ to another genus three surface $\Sigma'_3$. Figure 2 shows $P_2$, along with a set of generators $\sigma_1, \ldots, \sigma_8$ of $\pi_1(P_2)$, which satisfy the relations

$$\sigma_4[\sigma_2, \sigma_1^{-1}][\sigma_3^{-1}, \sigma_4^{-1}]\sigma_4^{-1} = \sigma_6^{-1}[\sigma_8^{-1}, \sigma_7^{-1}][\sigma_3^{-1}, \sigma_6]\sigma_6 = \sigma_5.$$
To compute the Reidemeister torsion, we need a cell decomposition for the universal covering space $\tilde{M}_2$. The first step in obtaining such a decomposition is to dissect $M_2$ into a simply connected region $U_M$ that can serve as a fundamental domain upon which $\pi_1(M_2)$ acts by translation. We do so by cutting $M_2$ open along a set of surfaces transverse to the generators of the fundamental group.

Since this process is somewhat difficult to visualize, let us begin with a simple example. Let $\Sigma_{1,1}$ be a genus one surface with a single hole, and consider a thickening of $\Sigma_{1,1}$ into a three-manifold $N$, one of the elementary building blocks of $M_2$. Figure 3 shows a dissection of $N$ into a simply connected fundamental domain. It is clear that $N$ can be recovered by identifying the boundaries of this figure; this identification represents the action of $\pi_1(N)$.

Figure 4 shows the corresponding dissection of $M_2$. The cuts on the top half of the figure are evidently analogous to those of figure 3; they are then extended through the boundary $\partial V$ into the bottom half of the figure, where they are continued until they reach the inner boundary $\Sigma_3$.

The corresponding construction for $P_2$ is illustrated in figure 5. We start with two copies of the bottom half of figure 4. However, a new noncontractible loop now appears, representing the element $\sigma_5$ of $\pi_1(P_2)$, that was not present in figure 4. To obtain a simply connected fundamental domain $U_P$ for $P_2$, we therefore need one more cut transverse to this loop. This final cut, required by the larger fundamental group of $P_2$, will play a critical role in the suppression of topology change—it will lead to the existence of extra zero-modes, and thus more highly divergent infrared behavior, in the topology-preserving amplitude.

To obtain cell decompositions of $\tilde{M}_2$ and $\tilde{P}_2$, it now suffices to find cell decompositions of $U_M$ and $U_P$ compatible with the action of their fundamental groups—that is, decompositions for which the process of gluing (say) $U_M$ back together to obtain $M_2$ identifies like cells. The final results are shown in figures 4 and 5. We have labeled a basis $e^{(k)}$ of $k$-cells in each figure; the remaining cells are translates of these basis elements by elements of the appropriate fundamental group.

6. An Explicit Computation

We are finally ready to calculate the amplitude $Z_{M_2}$ for tunneling from the genus three surface $\Sigma_3$ to the genus two surfaces $\Sigma_2$ and $\Sigma'_2$. We shall proceed in two steps, first obtaining some general information about the $\tilde{\omega}$ and $\tilde{E}$ integrals in (3.13) and then computing the Reidemeister torsion for a particular choice of flat connection.

We begin by showing that the topology-preserving path integral $Z_{P_2}$ has at least six more zero-modes than $Z_{M_2}$. This is in itself enough to indicate a suppression of topology change—each $\tilde{E}$ mode leads to an infrared divergence, so this mismatch shows that $Z_{P_2}$ is more divergent than $Z_{M_2}$. We caution the reader, however, that the choice of $P_2$ to “normalize” $M_2$ is somewhat arbitrary; until we understand more about the overall normalization of amplitudes, we can draw no firm conclusions about absolute probabilities for topology change.
Observe first that the twisted Euler characteristics \[25\]
\[
\sum (-1)^i \dim H_i(M_2, \partial M_2; V_\omega) = \chi(M_2, \partial M_2; V_\omega) = \frac{1}{2} \chi(\partial M_2; V_\omega) = 12
\]
\[
\sum (-1)^i \dim H_i(P_2, \partial P_2; V_\omega) = \chi(P_2, \partial P_2; V_\omega) = \frac{1}{2} \chi(\partial P_2; V_\omega) = 12
\] (6.1)
serve as constraints on the dimensions of the twisted homology groups, placing a lower limit of twelve on the total number of zero-modes. In the case of \(M_2\), this limit is realized. Indeed, the twisted cell complex associated with \(M_2\) is
\[
\begin{align*}
C_0(M_2, \partial M_2; V_\omega) & \xrightarrow{\partial_2} C_1(M_2, \partial M_2; V_\omega) \xrightarrow{\partial_3} C_2(M_2, \partial M_2; V_\omega) \xrightarrow{\partial_4} C_3(M_2, \partial M_2; V_\omega). \\
0 & \xrightarrow{\partial_2} \mathbb{R} \oplus \ldots \oplus \mathbb{R} \xrightarrow{\partial_3} \mathbb{R} \oplus \ldots \oplus \mathbb{R} \xrightarrow{\partial_4} \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}
\end{align*}
\] (6.2)
If we can show that \(\partial_2\) is an epimorphism and that \(\partial_3\) is a monomorphism, it will follow that \(H_0(M_2, \partial M_2; V_\omega) = H_1(M_2, \partial M_2; V_\omega) = H_3(M_2, \partial M_2; V_\omega) = 0\), and hence that \(\dim H_2(M_2, \partial M_2; V_\omega) = 12\).

Given a set of generators \(\{J^0, J^1, J^2\}\) for SO(2,1), the adjoint action described in section 4 determines a matrix representation of \(\pi_1(M_2)\),
\[
S : \pi_1(M_2) \to \text{GL}(3, \mathbb{R}).
\]
To show that \(\partial_2\) is an epimorphism, consider the following six boundary operations, which can be read off figure 4:
\[
\partial_2(e_{(2)}^1 \otimes J^a) = e_{(1)}^1 \otimes (1 - S[\rho_3])J^a,
\]
\[
\partial_2(e_{(2)}^3 \otimes J^a) = e_{(1)}^1 \otimes S[\rho_2](1 - S[\rho_1])S[\rho_3^{-1} \rho_4^{-1} \rho_3], J^a.
\] (6.3)
The \(S[\rho_3]\) are nondegenerate matrices which each stabilize a one-dimensional subspace of \(\mathbb{R}^3\), so \(1 - S[\rho_3]\) is a rank two matrix that is zero on the subspace stabilized by \(S[\rho_3]\). Similarly, \(S[\rho_3^{-1} \rho_4 \rho_3^{-1} \rho_4^{-1} \rho_3]\) is a rank two matrix that is zero on the subspace stabilized by \(S' = S[\rho_3^{-1} \rho_4 \rho_3^{-1} \rho_4^{-1} \rho_3]\). But in order for the boundary \(\partial M_2\) to be spacelike, the commutator \([\rho_3, \rho_3^{-1} \rho_4 \rho_3 \rho_3^{-1} \rho_4^{-1} \rho_3]\), which represents a loop on \(\partial M_2\), must be nontrivial. Hence \(S[\rho_3]\) and \(S'\) stabilize different one-dimensional subspaces, and the images of \(1 - S[\rho_3]\) and \(1 - S'\) span all of \(\mathbb{R}^3\). This, in turn, means that the image under \(\partial_2\) of the space spanned by \(\{e_{(2)}^1 \otimes J^a, e_{(2)}^3 \otimes J^a\}\) contains \(\{e_{(1)}^1 \otimes \text{so}(2, 1)\}\). A similar argument shows that the image of the space spanned by \(\{e_{(2)}^2 \otimes J^a, e_{(2)}^4 \otimes J^a\}\) contains \(\{e_{(2)}^1 \otimes \text{so}(2, 1)\}\), thus establishing that \(\partial_2\) is an epimorphism.

An analogous argument, starting with the boundary operations
\[
\partial_3(e_{(3)} \otimes J^a) = e_{(2)}^1 \otimes (1 - S[\rho_3^{-1}])J^a - e_{(2)}^1 \otimes (1 - S[\rho_3])J^a + \ldots
\] (6.4)
and the nontriviality of the commutator \([\rho_3^{-1}, \rho_3]\), shows that \(\partial_3\) is a monomorphism. We have thus established that \(H_2(M_2, \partial M_2; V_\omega)\) is twelve-dimensional. This conclusion has been
checked by making explicit choices for the holonomies \{S[\rho_i]\} and identifying generators of the twisted homology groups.

We can repeat the same analysis for \(P_2\), but let us instead take a shortcut. It is easy to see that \(H_0(P_2, \partial P_2; V_{\bar{\omega}}) = 0\), so the constraint (6.1) implies that

\[
\dim H_2(P_2, \partial P_2; V_{\bar{\omega}}) \geq 12 + \dim H_1(P_2, \partial P_2; V_{\bar{\omega}}).
\]

So if we can show the existence of three generators of \(H_1(P_2, \partial P_2; V_{\bar{\omega}})\), we will have proven that \(Z_{P_2}\) has at least six more zero-mode integrations—three \(\bar{\omega}\) modes and three \(\bar{E}\) modes—than \(Z_{M_2}\).

But this is apparent from figure 5: no combination of \(e_3^{(1)} \otimes J^a\) and \(e_4^{(1)} \otimes J^a\) lies in the image of \(\partial_2\), and

\[
\partial_1 \left( e_3^{(1)} \otimes J^a + e_4^{(1)} \otimes J^a \right) = 0. \tag{6.5}
\]

Hence \(\{e_3^{(1)} + e_4^{(1)}\} \otimes J^a\) represent three generators of \(H_1(P_2, \partial P_2; V_{\bar{\omega}})\), as claimed.

We conclude this section by describing an explicit computation of the Reidemeister torsion for \(M_2\). This requires a choice of flat connection \(\bar{\omega}\), which may be determined by its holonomies around the curves \(\rho_i\) that generate \(\pi_1(M_2)\). Note, however, that each of the \(\rho_i\) can be deformed to a curve on one of the boundary components of \(M_2\). The connection \(\bar{\omega}\) is therefore fixed by its holonomies on \(\Sigma_2 \cup \Sigma_3 \cup \Sigma_3\), and hence by the boundary data \(\Gamma^*\omega\). This means that the relevant moduli space of flat connections is a single point, and that the integral over \(\bar{\omega}\) disappears from (3.13), as expected from the vanishing of \(H_1(M_2, \partial M_2; V_{\bar{\omega}})\).

Following reference [3], we choose the \(S[\rho_i]\) as follows. \(S[\rho_1]\) through \(S[\rho_4]\) are the generators of an arbitrary Fuchsian group uniformizing a genus two surface. Such groups form a six-parameter family, which can be written down from Fenchel-Nielsen coordinates by using the results of [27]. A convenient two-parameter family \(S[\rho_i](k, r)\) is given in the appendix. We then take

\[
S[\rho_5] = S[\rho_3], \quad S[\rho_6] = S[\rho_4],
\]

\[
S[\rho_7] = S[\rho_4 \rho_2 \rho_1 \rho_2^{-1} \rho_4^{-1}], \quad S[\rho_8] = S[\rho_4 \rho_2 \rho_4^{-1}]. \tag{6.6}
\]

This expression differs slightly from that of reference [3], but only because of our different choice of generators of \(\pi_1(M_2)\). Amano and Higuchi show that with these choices, the boundaries of \(M_2\) are spacelike and nonsingular.

As described in section 4, calculating the torsion requires computing the determinants of the matrices \(T(k)\) that give the change of basis from \(c(k)\) to \((b(k), \tilde{h}(k), \tilde{b}(k-1))\). The total measure appearing in the amplitude (3.13) is independent of the choice of basis \((b(k), \tilde{h}(k), \tilde{b}(k-1))\), but the determinants (4.6) by themselves are not—we must make an explicit choice of the homology basis \(\tilde{h}(k)\) for a numerical value of the torsion \(\tau\) to have meaning. For now, we choose the simple but rather arbitrary basis \(\tilde{h}(k)\) described in the appendix. A tedious but routine calculation then gives a Reidemeister torsion as a complicated rational function of \(k\) and \(r\). A plot is shown in figure 6. The torsion falls off very rapidly for large values of \(k\) and \(r\); for example,

\[
\tau(r = 10, k = 1000) \approx 7 \times 10^{-52}.
\]
To interpret these results, we must understand the remaining $\tilde{E}$ integrals in (3.13). The measure $d\tilde{E}$ is determined by the homology basis $\tilde{h}_{(2)}$. We first select an orthonormal basis $\tilde{E}^\alpha$ such that

$$\int_{\tilde{h}_{(2)}^\alpha} \tilde{E}^\beta = \delta_\beta^\alpha,$$

(6.7)

where $*\tilde{h}_{(2)}^\alpha$ is the union of one-cells dual to $\tilde{h}_{(2)}^\alpha$ and the integral includes a trace over the generators of $so(2,1)$ (see [10] for details). If we then decompose an arbitrary zero-mode as $\tilde{E} = \sum c_\alpha \tilde{E}_\alpha$, the integral (3.13) is $\int dc_\alpha$, and the dependence of $c_\alpha$ on the choice of homology basis cancels the dependence of the torsion $\tau$.

If we wish to cut off the infrared divergent $\tilde{E}$ integrals, however, this basis dependence reappears—the range of integration will, in general, depend on the choice of $\tilde{h}_{(2)}$. This dependence can be translated into a dependence on the auxiliary metric $h$ introduced in section 3 to fix the gauge. The difficulty appears to be a typical regularization problem—we do not know how to regulate our integrals in a way that respects the invariance of the original theory.

There is some hope for a physical resolution of this problem. Witten has suggested that the infrared divergences of transition amplitudes in (2+1)-dimensional gravity reflect the appearance of infinite-volume “classical” spacetimes. If we could give a concrete geometrical meaning to the limits of integration, cutting off at some (observable) scale, we might be able to define a basis-independent regularization. It is also possible that the addition of matter to our vacuum theory might regulate the divergences, again by limiting maximum lengths. We leave these questions for future investigation.

7. Conclusion

We have now seen that path integrals representing spatial topology change in (2+1)-dimensional general relativity need not vanish. Starting with any initial data in our two-parameter family—or, by a long but straightforward generalization, any other admissible initial geometry—we can explicitly compute the torsion $T[I^*\omega]$, and thus the amplitude (3.13). Indeed, we have seen that such topology-changing amplitudes may diverge, thanks to the existence of zero-modes $\tilde{E}^\alpha$ of the triad $e^a$. These divergences presumably reflect the appearance of “classical” spacetimes, in which distances measured with the metric $g_{\mu\nu} = e_\mu^a e_{\nu a}$ become arbitrarily large [4].

Nevertheless, our results may be interpreted as providing evidence that topology change is suppressed. We have seen that while the topology-changing amplitude mediated by $M_2$ is infrared divergent, the closely related topology-preserving amplitude mediated by $P_2$ is even more divergent. Evidently, no firm conclusions about topology change can be drawn without a much better understanding of the overall normalization of amplitudes in (2+1)-dimensional gravity. This is a difficult problem: we must not only consider an infinite number of possible interpolating manifolds, but must also find a sensible way to regulate infrared divergences without breaking the symmetries of the original theory. Clearly, much work remains to be done.
One important step would be to find an easier and more intuitive method for computing the degree of divergence for an arbitrary interpolating manifold without requiring a full cell decomposition. We do not have a complete answer to this problem, but we offer the following observations. As we have seen, \( H_1(M, \partial M; V_\omega) \) counts the dimension of the moduli space of flat connections on \( M \) with specified boundary data. In general, a flat connection is determined up to gauge transformations by its holonomies, that is, by an assignment of an element of \( \text{SO}(2,1) \) to each independent generator of \( \pi_1(M) \). For our manifold \( M_2 \), it is evident from figure 1 that every generator of the fundamental group can be deformed to the boundary; thus, the connection is completely determined by boundary data, and \( \dim H_1(M_2, \partial M_2; V_\omega) = 0 \). For \( P_2 \), on the other hand, one independent generator—\( \sigma_4 \), for example—cannot be deformed to the boundary, and accounts for our three generators of \( H_1(M_2, \partial M_2; V_\omega) \). We do not know whether this argument can be made rigorous when generalized to an arbitrary three-manifold, but such an extension may be possible. Similarly, it may be possible to obtain information about \( \dim H_2(M, \partial M; V_\omega) = \dim H_1(M; V_\omega) \) by looking at connections on the double of \( M \). Note that paths connecting boundary components of \( M \) become closed loops on the double, and may contribute to the dimensions \( \dim H_*(M, \partial M; V_\omega) \). Finally, the Euler characteristic constraint (6.1) holds for any three-manifold, and places a useful constraint on the number of divergent integrals.

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**Appendix. Computational Details**

In this appendix, we briefly describe some of the intermediate steps in the computation of the torsion plotted in figure 6. We begin with a choice of holonomies, which we give in the two-dimensional representation of \( \text{SL}(2, \mathbb{R}) \):

\[
\begin{align*}
\rho_1 & \mapsto \begin{pmatrix}
\frac{5+\sqrt{5}}{2} & -\frac{(1-\sqrt{5})kr^2}{2} \\
-3 \frac{1+\sqrt{5}}{2} & -\frac{5+\sqrt{5}}{2}
\end{pmatrix}, & \rho_2 & \mapsto \begin{pmatrix}
\frac{(1+\sqrt{5})(2+3k^2)}{2} & \frac{(1+k^2)r^2}{2} \\
-\frac{9k^2+4}{k^2r^2} & \frac{(1-\sqrt{5})(2+3k^2)}{2}
\end{pmatrix}, \\
\rho_3 & \mapsto \begin{pmatrix}
\frac{(1+\sqrt{5})(2+3k^2)}{2} & -9 - \frac{1}{k^2r^2} \\
\frac{1}{1+k^2} & \frac{(1-\sqrt{5})(2+3k^2)}{2k}
\end{pmatrix}, & \rho_4 & \mapsto \begin{pmatrix}
\frac{-5+\sqrt{5}}{2} & \frac{3(1+\sqrt{5})}{k} \\
\frac{(1-\sqrt{5)}k}{2} & \frac{5+\sqrt{5}}{2}
\end{pmatrix}.
\end{align*}
\]

(A.1)

These describe a genus two surface consisting of two identical one-holed tori (\( \Sigma_{1,1} \) of figure 3) joined with a relative twist; \( r \) parametrizes the twist, while \( k \) parametrizes the length of a closed geodesic on each copy of \( \Sigma_{1,1} \).
We next describe the choice of homology basis used in the computation of the torsion $\tau$. Using the cell structure shown in Figure 4, we begin by selecting basis elements $\tilde{b}(1)$ and $\tilde{b}(2)$ as defined in section 4:

\[
\begin{align*}
\tilde{b}_1 &= e_1 \otimes J^0, \\
\tilde{b}_2 &= e_2 \otimes J^0, \\
\tilde{b}_3 &= e_3 \otimes J^0, \\
\tilde{b}_4 &= e_4 \otimes J^0, \\
\tilde{b}_5 &= e_5 \otimes J^0, \\
\tilde{b}_6 &= e_6 \otimes J^0, \\
\tilde{b}_7 &= e_7 \otimes J^0, \\
\tilde{b}_8 &= e_8 \otimes J^0, \\
\tilde{b}_9 &= e_9 \otimes J^0, \\
\tilde{b}_{10} &= e_{10} \otimes J^0, \\
\tilde{b}_{11} &= e_{11} \otimes J^0, \\
\tilde{b}_{12} &= e_{12} \otimes J^0,
\end{align*}
\]

(A.2)

\[
\tilde{h}_1 = e_1 \otimes J^2 + \sum K^1_{\alpha} \tilde{b}_1, \\
\tilde{h}_2 = e_2 \otimes J^2 + \sum K^2_{\alpha} \tilde{b}_2, \\
\tilde{h}_3 = e_3 \otimes J^2 + \sum K^3_{\alpha} \tilde{b}_3, \\
\tilde{h}_4 = e_4 \otimes J^2 + \sum K^4_{\alpha} \tilde{b}_4, \\
\tilde{h}_5 = e_5 \otimes J^2 + \sum K^5_{\alpha} \tilde{b}_5, \\
\tilde{h}_6 = e_6 \otimes J^2 + \sum K^6_{\alpha} \tilde{b}_6, \\
\tilde{h}_7 = e_7 \otimes J^2 + \sum K^7_{\alpha} \tilde{b}_7, \\
\tilde{h}_8 = e_8 \otimes J^2 + \sum K^8_{\alpha} \tilde{b}_8, \\
\tilde{h}_9 = e_9 \otimes J^2 + \sum K^9_{\alpha} \tilde{b}_9, \\
\tilde{h}_{10} = e_{10} \otimes J^2 + \sum K^{10}_{\alpha} \tilde{b}_{10}, \\
\tilde{h}_{11} = e_{11} \otimes J^2 + \sum K^{11}_{\alpha} \tilde{b}_{11}, \\
\tilde{h}_{12} = e_{12} \otimes J^2 + \sum K^{12}_{\alpha} \tilde{b}_{12}.
\]

(A.3)

Our homology basis is then

\[
\begin{align*}
\tilde{h}_1 &= e_1 \otimes J^2 + \sum K^1_{\alpha} \tilde{b}_1, \\
\tilde{h}_2 &= e_2 \otimes J^2 + \sum K^2_{\alpha} \tilde{b}_2, \\
\tilde{h}_3 &= e_3 \otimes J^2 + \sum K^3_{\alpha} \tilde{b}_3, \\
\tilde{h}_4 &= e_4 \otimes J^2 + \sum K^4_{\alpha} \tilde{b}_4, \\
\tilde{h}_5 &= e_5 \otimes J^2 + \sum K^5_{\alpha} \tilde{b}_5, \\
\tilde{h}_6 &= e_6 \otimes J^2 + \sum K^6_{\alpha} \tilde{b}_6, \\
\tilde{h}_7 &= e_7 \otimes J^2 + \sum K^7_{\alpha} \tilde{b}_7, \\
\tilde{h}_8 &= e_8 \otimes J^2 + \sum K^8_{\alpha} \tilde{b}_8, \\
\tilde{h}_9 &= e_9 \otimes J^2 + \sum K^9_{\alpha} \tilde{b}_9, \\
\tilde{h}_{10} &= e_{10} \otimes J^2 + \sum K^{10}_{\alpha} \tilde{b}_{10}, \\
\tilde{h}_{11} &= e_{11} \otimes J^2 + \sum K^{11}_{\alpha} \tilde{b}_{11}, \\
\tilde{h}_{12} &= e_{12} \otimes J^2 + \sum K^{12}_{\alpha} \tilde{b}_{12},
\end{align*}
\]

where the coefficients $K^i_{\alpha}$, whose exact values are not needed for the computation of the torsion $\tau$, are uniquely determined by the requirement that $\partial \tilde{h}_2 = 0$.

Of course, this choice of homology basis depends on the boundary operator $\partial$, and therefore on the holonomies $\rho_i$. The choice $\{A.4\}$ appears to be valid for generic values of the holonomies, but there are points at which linear combinations of our $\tilde{h}_2$ lie in the image of $\partial_3$. At these points, (A.4) is no longer a valid basis, and our computed value of the Reidemeister torsion $\tau$ is not correct—in fact, it appears to go to zero. This behavior is an artifact of our basis choice, and does not affect the integral (3.13).

We do not include the final functional form of $\tau(r, k)$—it would require three pages—but it is available from the authors.

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Figure 1. The manifold $M_2$ is formed by identifying $W_1$ and $W_2$ along their common genus four boundary. A basis of loops for $\pi_1(M_2)$ is shown.

Figure 2. The manifold $P_2$ and a basis for its fundamental group.
Figure 3. A dissection of the thickened one-holed torus into a simply connected fundamental domain.
Figure 4. A dissection of $M_2$. A basis of cells $e^\alpha_{(k)}$ is shown.

Figure 5. A dissection of $P_2$ and a basis of cells.
Figure 6. A two-parameter family of Reidemeister torsions $\tau(r,k)$ for different choices of the flat connection on $\partial M_2$; the range shown is $1.1 < r < 12$, $0.3 < k < 1.2$. 