Free-stream preserving metrics and Jacobian for the conservative finite difference method on curvilinear grids

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Abstract

For the compressible flows simulations, the conservative finite difference based on the upwind schemes, i.e. the linear upwind and WENO, is widely used for their simplicity and conservative property. However, this method loses the geometric conservation law (GCL) identity due to the upwind dissipation when applied on the curvilinear grids. In this paper, we suggest a free-stream preserving metrics and Jacobian in the upwind dissipation to maintain the free-stream preserving identity. This technique avoids destroying the accuracy and the standard forms of the upwind schemes as far as possible. Therefore, this technique is convenient to operate in conservative finite difference method. Some verifications are conducted to show the accuracy in the smooth flow filed and the robustness in the discontinuous regions of the present free-stream preserving method, such as the isotropic vortex problem, the double Mach reflection problem, the transonic flow past NACA0012 airfoil and ONERA M6 wing, etc..

Keywords: Conservative finite difference, geometric conservative law, free-stream preserving, linear-upwind scheme, WENO scheme

1. Introduction

The geometric conservation law (GCL) [1], including volume conservation law (VCL) and surface conservation law (SCL) [2, 3, 4], is very important in computational fluid dynamics, especially for the high-accuracy simulations. Unlike the finite volume method (FVM) and finite element method (FEM), the conservative finite difference method (FDM) binds the physical quantities and geometric metrics together during the process of flux reconstruction in the computational space such that the GCL is not easy to be satisfied due to the discretization errors of grid metrics for the upwind dissipation. The violation of the GCL will yield large errors, oscillations and instabilities for the simulations [1, 5, 6], and even lead to the non-conservation of the physical quantities.

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of the governing equations [7, 4]. On the stationary grids, the VCL identity is satisfied naturally but the SCL is not. Therefore, some efforts should be made to the numerical method to achieve the SCL identity.

Many achievements have been put forward to maintain the free-stream preservation, especially for the low order schemes, as concluded in Ref. [8, 9]. In the high-accuracy FDM, the symmetry conservative metric method (SCMM) [10], inspired by the conservative metrics in Ref. [1], have been an efficient technique to fulfill this identity under the sufficient condition of evaluating the derivatives of the grid metrics and the convection fluxes with the unique scheme given by Deng et al. [11] and Abe et al. [2]. However, the sufficient condition is only acceptable for linear central schemes to guarantee the free-stream preservation. For the upwind schemes which are very important in the numerical simulations of compressible flows, it is not easy to be achieved due to the inconsistent differential operators applied for the grid metrics and fluxes [8]. At present, there are mainly two ideas to deal with the free-stream conservation problem for high-order upwind schemes. The first one is to consider the independent interpolation for flow variables and metrics, such as WCNS [12, 8, 11], alternative finite-difference form of WENO (AWENO) [13, 14, 15], etc.. In those schemes, the dissipation is handled by the finite volume method and then obtain their derivatives by central schemes. The second method is to separate the central part from the standard upwind schemes and employ the high-order central schemes to it. Then, employs the finite-volume-like schemes, i.e. freezing metrics either for the entire stencil [9] or for the local difference form partially [16], or replacing the transformed conservative variables with the original one [17] to the dissipative part. Among the above schemes, either the standard WENO scheme is modified or its metrics are frozen in the local stencil, which results in more or less additional complications or metrics frozen errors.

In this study, we propose a simple, efficient and non-frozen high-order technique to achieve the free-stream preserving identity for the standard linear upwind and WENO schemes. The present method replaces the discretized metrics and Jacobian with a free-stream preserving one in the dissipation part. This technique possesses at least two advantages, that is, it destroys the accuracy and the standard forms of the standard linear upwind and WENO schemes as less as possible. As a result, it is convenient to operate this technique in the standard upwind schemes. The outline is organized as follows. In section 2, we introduce the governing equations, SCL, and the upwind schemes in conservative finite difference method. In Section 3, the free-stream preserving metrics and Jacobian are explained in detail. Next, several validations and numerical examples are given in Section 4. Finally, a brief conclusion is given in Section 5.
2. Governing equations and metrics on stationary curvilinear coordinates

2.1. Navier-Stokes equations

The compressible Navier-Stokes equations on curvilinear grids are given by

\[
\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial}{\partial \xi} \left( \xi_x F + \xi_y G + \xi_z H \right) + \frac{\partial}{\partial \eta} \left( \eta_x F + \eta_y G + \eta_z H \right) + \frac{\partial}{\partial \zeta} \left( \zeta_x F + \zeta_y G + \zeta_z H \right) = 0
\]

with

\[
\mathbf{Q} = \begin{pmatrix} \rho & \rho u_1 & \rho u_2 & \rho u_3 & \rho E \end{pmatrix}^T, \\
\mathbf{F} = \begin{pmatrix} \rho u_1 & \rho u_1 u_1 + p & \rho u_2 u_1 & \rho u_3 u_1 \end{pmatrix} \begin{pmatrix} \rho E + p \end{pmatrix}, \\
\mathbf{G} = \begin{pmatrix} \rho u_2 & \rho u_1 u_2 & \rho u_2 u_2 + p & \rho u_3 u_2 \end{pmatrix} \begin{pmatrix} \rho E + p \end{pmatrix}, \\
\mathbf{H} = \begin{pmatrix} \rho u_3 & \rho u_1 u_3 & \rho u_2 u_3 & \rho u_3 u_3 + p \end{pmatrix} \begin{pmatrix} \rho E + p \end{pmatrix}, \\
\mathbf{F}_v = \begin{pmatrix} 0 & \tau_{11} & \tau_{12} & \tau_{13} & u_i \tau_{1i} - \dot{q}_1 \end{pmatrix}^T, \\
\mathbf{G}_v = \begin{pmatrix} 0 & \tau_{21} & \tau_{22} & \tau_{23} & u_i \tau_{2i} - \dot{q}_2 \end{pmatrix}^T, \\
\mathbf{H}_v = \begin{pmatrix} 0 & \tau_{31} & \tau_{32} & \tau_{33} & u_i \tau_{3i} - \dot{q}_3 \end{pmatrix}^T
\]

where \(\xi, \eta, \zeta\) are the transformed coordinates on a uniform computational domain, and \(J\) is the transformed Jacobian. \(u_i, i = 1, 2, 3\) are the velocity components. \(\mathbf{F}, \mathbf{G}, \mathbf{H}\) and \(\mathbf{F}_v, \mathbf{G}_v, \mathbf{H}_v\) represent the inviscid and viscous flux vectors in x, y and z direction, respectively. \(\rho, p\) and \(E\) are the density, pressure and the total specific energy. \(t\) is the physical time. \(\tau_{ij}\) is the shear stress tensor

\[
\tau_{ij} = 2\mu(S_{ij} - \delta_{ij} \frac{S_{kk}}{3}),
\]

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),
\]

and \(\dot{q}_i\) is the heat flux in direction \(i\)

\[
\dot{q}_i = -\lambda \frac{\partial T}{\partial x_i},
\]

where \(\mu\) and \(\lambda\) is the shear viscosity and thermal conductivity.

The equation of state for ideal gas is

\[
p = (\gamma - 1) \rho e,
\]

where the specific heat ratio \(\gamma = 1.4\). Generally, the fluxes in computational space are written as

\[
\tilde{Q} = \frac{Q}{J}
\]
are expressed as conservative metrics are unique with that of the fluxes discretization. The symmetric conservative metrics under the sufficient condition of Deng et al. [11] and Abe et al. [2] that the operators within the symmetric satisfied while the discretized errors of the metrics can easily destroy this identity.

These equations are regarded as the SCL by Zhang et al. [4] because they delineate the consistence of vectorized computational cell surfaces in finite volume method [18]. Theoretically, Eq. (16) are strictly satisfied while the discretized errors of the metrics can easily destroy this identity.

2.2. Metrics and SCL

Imposing the free-stream condition to the Navier-Stokes equations gives

\[ I_x = \left( \frac{\xi_x}{\eta} \right) + \left( \frac{\eta_x}{\xi} \right) + \left( \frac{\zeta_x}{\zeta} \right) = 0, \]
\[ I_y = \left( \frac{\xi_y}{\eta} \right) + \left( \frac{\eta_y}{\xi} \right) + \left( \frac{\zeta_y}{\zeta} \right) = 0, \]
\[ I_z = \left( \frac{\xi_z}{\eta} \right) + \left( \frac{\eta_z}{\xi} \right) + \left( \frac{\zeta_z}{\zeta} \right) = 0. \]

These equations are regarded as the SCL by Zhang et al. [4] because they delineate the consistence of vectorized computational cell surfaces in finite volume method [18]. Theoretically, Eq. (16) are strictly satisfied while the discretized errors of the metrics can easily destroy this identity.

The SCMM is widely used to satisfy the SCL in high accuracy finite difference numerical simulations under the sufficient condition of Deng et al. [11] and Abe et al. [2] that the operators within the symmetric conservative metrics are unique with that of the fluxes discretization. The symmetric conservative metrics are expressed as

\[ \frac{\xi_x}{\eta} = \frac{1}{2} \left[ (y\zeta)_\eta - (y\xi)_\zeta + (y\zeta)_\eta - (y\zeta)_\zeta \right], \]
\[ \frac{\xi_y}{\eta} = \frac{1}{2} \left[ (x\zeta)_\eta - (x\xi)_\zeta + (x\zeta)_\eta - (x\zeta)_\zeta \right], \]
\[ \frac{\xi_z}{\eta} = \frac{1}{2} \left[ (x\eta)_\zeta - (x\xi)_\zeta + (x\eta)_\zeta - (x\eta)_\zeta \right], \]
\[ \frac{\eta_x}{\eta} = \frac{1}{2} \left[ (y\zeta)_\xi - (y\xi)_\zeta + (y\zeta)_\xi - (y\zeta)_\zeta \right], \]
\[ \frac{\eta_y}{\eta} = \frac{1}{2} \left[ (x\zeta)_\xi - (x\xi)_\zeta + (x\zeta)_\xi - (x\zeta)_\zeta \right], \]
\[ \frac{\eta_z}{\eta} = \frac{1}{2} \left[ (x\eta)_\xi - (x\xi)_\eta + (x\eta)_\xi - (x\eta)_\eta \right], \]
\[ \frac{\zeta_x}{\eta} = \frac{1}{2} \left[ (y\xi)_\eta - (y\xi)_\zeta + (y\xi)_\eta - (y\xi)_\zeta \right], \]
\[ \frac{\zeta_y}{\eta} = \frac{1}{2} \left[ (x\xi)_\eta - (x\xi)_\zeta + (x\xi)_\eta - (x\xi)_\zeta \right], \]
\[ \frac{\zeta_z}{\eta} = \frac{1}{2} \left[ (x\xi)_\eta - (x\xi)_\eta + (x\xi)_\eta - (x\xi)_\eta \right]. \]
and
\[ \frac{1}{J} = \frac{1}{3} \left[ \left( \frac{x \xi_x + y \xi_y + z \xi_z}{J} \right) \xi + \left( \frac{x \eta_x + y \eta_y + z \eta_z}{J} \right) \eta + \left( \frac{x \zeta_x + y \zeta_y + z \zeta_z}{J} \right) \zeta \right]. \] (18)

The geometrical metrics and Jacobian are usually discretized with central schemes so that it is not easy for the upwind schemes to satisfy the SCL preserving sufficient condition given by Deng et al. [11] and Abe et al. [2].

2.3. Discretization methods

The conservative finite difference method [19, 20, 21] is explained briefly to discrete the Navier-Stokes equations. The key thought of this method is to reconstruct the high-order consistent numerical fluxes at each cell-face. Without loss of generality, we choose \( \xi \) direction ordered by \( i \), shown in Fig. 1, to delineate how to reconstruct the cell-face fluxes. The fluxes \( \tilde{F}_i \) at cell \( i \) is regarded as an average of a primitive function \( \hat{H}(\xi) \)
\[ \tilde{F}_i = \frac{1}{\Delta \xi} \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \hat{H}(\xi) d\xi \] (19)
Then we can exactly obtain the derivative of \( \tilde{F}_i \),
\[ \left( \frac{\partial \tilde{F}}{\partial \xi} \right)_i = \frac{\hat{H}(i + 1/2) - \hat{H}(i - 1/2)}{\Delta \xi}. \] (20)
Therefore, the derivative of the convective fluxes can be approximated by the reconstructed cell-face fluxes
\[ \tilde{F}_{i+1/2} \]
\[ \left( \frac{\partial \tilde{F}}{\partial \xi} \right)_i = \frac{\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}}{\Delta \xi} + O(\Delta \xi^{2k-1}), \] (21)
where \( \tilde{F}_{i+1/2} \) is the approximation of the primitive function value at cell-face \( \hat{H}_{i+1/2} \), which can be reconstructed by unwind schemes from the cell fluxes \( \tilde{F}_{i-k+1}, \cdots, \tilde{F}_{i+k-1} \) to keep the \((2k-1)\)th-order accuracy, such as WENO scheme.

2.3.1. The characteristic-wise WENO scheme

For the purpose of improving the robustness of the simulations, the fluxes and conservative variables are transformed into the characteristic space and then a flux vector splitting scheme, such as local Lax-Friedrichs splitting, is applied,
\[ \tilde{F}_{m}^\pm = \frac{1}{2} L_{i+1/2} \cdot \left( \tilde{F}_{m} \pm \lambda_{i+1/2} \tilde{Q}_{m} \right), m = i - k + 1, \cdots, i + k, \] (22)
where $\lambda$ is the diagonal matrix of the eigenvalues of the local linearized Roe-average Jacobian matrix $A_{i+1/2} = \left( \partial \mathbf{F} / \partial \mathbf{Q} \right)_{i+1/2}$. $L_{i+1/2}$ is the left matrix composed of the corresponding eigenvectors of $A_{i+1/2}$. Then the cell-face fluxes are given by

$$\mathbf{F}_{i+1/2} = R_{i+1/2} \cdot (\mathbf{F}_{i+1/2}^+ + \mathbf{F}_{i+1/2}^-),$$

(23)

where $R_{i+1/2}$ is the inverse matrix of $L_{i+1/2}$.

For the smooth and continuous flow field, $\mathbf{F}_{i+1/2}^{\pm}$ can be reconstructed by the 5th-order linear upwind scheme,

$$\mathbf{F}_{i+1/2}^+ = \frac{1}{60} \left( 2 \mathbf{F}_{i-2}^+ - 13 \mathbf{F}_{i-1}^+ + 47 \mathbf{F}_i^+ + 27 \mathbf{F}_{i+1}^+ - 3 \mathbf{F}_{i+2}^+ \right)$$

$$\mathbf{F}_{i+1/2}^- = \frac{1}{60} \left( -3 \mathbf{F}_{i-1}^- + 27 \mathbf{F}_i^- + 47 \mathbf{F}_{i+1}^- - 13 \mathbf{F}_{i+2}^- + 2 \mathbf{F}_{i+3}^- \right),$$

(24)

With respect to the flow field containing noncontinuous zones, we choose the classical 5th-order WENO scheme [22] to obtain the cell-face flux by

$$\tilde{\mathbf{F}}_{i+1/2}^\pm = \sum_{k=0}^{2} \omega_k^\pm q_k^\pm,$$

(25)

where $\tilde{\mathbf{F}}_i^\pm$ denotes one of the component of $\mathbf{F}_i^\pm$. Taking the positive fluxes as an example, there 3rd-order approximations for the different sub-stencils are formulated as

$$q_0^+ = \frac{1}{3} \tilde{f}_{i-2}^+ - \frac{7}{6} \tilde{f}_{i-1}^+ + \frac{7}{6} \tilde{f}_i^+,$$

$$q_1^+ = -\frac{1}{6} \tilde{f}_{i-1}^+ + \frac{5}{6} \tilde{f}_i^+ + \frac{1}{3} \tilde{f}_{i+1}^+,$$

$$q_2^+ = \frac{1}{3} \tilde{f}_i^+ + \frac{5}{6} \tilde{f}_{i+1}^+ - \frac{1}{6} \tilde{f}_{i+2}^+.$$

(26)

The non-linear weight $\omega_k^+$ in Eq. (25) is evaluated by

$$\omega_k^+ = \frac{C_k}{(\beta_k^+ + \epsilon)^n} \sum_{r=0}^{2} \frac{C_r}{(\beta_r^+ + \epsilon)^n},$$

(27)

where $C_0 = \frac{1}{10}$, $C_1 = \frac{3}{5}$, $C_2 = \frac{3}{10}$ are the optical weights and $\epsilon = 1.0 \times 10^{-6}$, $n = 2$. The smooth indicators are determined by

$$\beta_0^+ = \frac{1}{4} \left( \tilde{f}_{i-2}^+ - 4 \tilde{f}_{i-1}^+ + 3 \tilde{f}_i^+ \right)^2 + \frac{13}{12} \left( \tilde{f}_{i-2}^+ - 2 \tilde{f}_{i-1}^+ + \tilde{f}_i^+ \right)^2,$$

$$\beta_1^+ = \frac{1}{4} \left( -\tilde{f}_{i-1}^+ + \tilde{f}_{i+1}^+ \right)^2 + \frac{13}{12} \left( \tilde{f}_{i-1}^+ - 2 \tilde{f}_i^+ + \tilde{f}_{i+1}^+ \right)^2,$$

$$\beta_2^+ = \frac{1}{4} \left( -3 \tilde{f}_{i-1}^+ + 4 \tilde{f}_{i+1}^+ - \tilde{f}_{i+2}^+ \right)^2 + \frac{13}{12} \left( \tilde{f}_{i-1}^+ - 2 \tilde{f}_{i+1}^+ + \tilde{f}_{i+2}^+ \right)^2.$$

(28)

### 3. Free-stream preserving metrics and Jacobian for the upwind schemes

In this section, the free-stream preserving metrics and Jacobian are deduced to give a novel, simple, non-frozen and high-order strategy on free-stream preserving for the upwind schemes. Without loss of generality,
the 5th-order linear upwind and WENO scheme are considered to reconstruct the cell-face fluxes with this suggested free-stream preserving method.

First, the SCMM [9] is applied to discrete the geometric metrics and Jacobian, as shown in Eq. (29). Therefore, the errors generated by the metrics discretization are effectively decreased if the unique central scheme is applied to the discretization of the fluxes, due to the sufficient condition of Deng et al. [11] and Abe et al. [2].

\[
x_{i+1/2} = \frac{1}{60} (x_{i-2} - 8x_{i-1} + 37x_i + 37x_{i+1} - 8x_{i+2} + x_{i+3}),
\]

(29)

In the following, the metrics and Jacobian in the local reconstruction stencil discretized by the SCMM with the 6th-order central scheme are denoted by \(g_{1+1/2}^*, g_{i-2}^*, \ldots, g_{i+3}^*\). The proposed free-stream preserving metrics and Jacobian are represented by \(g_{1+1/2}^*, g_{i-2}^*, \ldots, g_{i+3}^*\). Then, a sufficient condition to maintain the free-stream preserving identity which is proved in Appendix A is given as

**Theorem 1.** During the 5th-order linear upwind and WENO reconstruction procedures, if the cell-face metrics and Jacobian \(g_{1+1/2}^*\) reconstructed in each sub-stencil share the unique value, that is,

\[
\begin{align*}
\frac{1}{3}g_{i-2}^* - \frac{7}{6}g_{i-1}^* + \frac{11}{6}g_i^* &= g_{i+1/2}^*, \\
\frac{1}{6}g_{i-1}^* + \frac{5}{6}g_i^* + \frac{1}{3}g_{i+1}^* &= g_{i+1/2}^*, \\
\frac{1}{3}g_{i+1}^* + \frac{5}{6}g_{i+1}^* - \frac{1}{6}g_{i+2}^* &= g_{i+1/2}^*, \\
\frac{11}{6}g_{i+1}^* + \frac{7}{6}g_{i+2}^* + \frac{1}{3}g_{i+3}^* &= g_{i+1/2}^*,
\end{align*}
\]

(30)

the free-stream preserving identity can be satisfied for their upwind dissipations. Moreover, if \(g_{1+1/2}^*\) equals \(g_{1+1/2}\), where

\[
g_{1+1/2} = \frac{1}{60} (g_{i-2} - 8g_{i-1} + 37g_i + 37g_{i+1} - 8g_{i+2} + g_{i+3}),
\]

(31)

the free-stream preserving identity can be satisfied for their central parts.

According to this theorem, we suggest the free-stream preserving metrics and Jacobian as

\[
\begin{align*}
\frac{1}{3}g_{i-2}^* &= \frac{7}{6}g_{i-1}^* - \frac{11}{6}g_i^* + g_{i+1/2}^*, \\
\frac{1}{6}g_{i-1}^* &= \frac{5}{6}g_i^* + \frac{1}{3}g_{i+1}^* - g_{i+1/2}^*, \\
g_i^* &= g_i, \\
g_{i+1/2}^* &= g_{i+1/2}, \\
g_{i+1}^* &= g_{i+1}, \\
\frac{1}{6}g_{i+2}^* &= \frac{5}{6}g_{i+1}^* + \frac{1}{3}g_i^* - g_{i+1/2}^*, \\
\frac{1}{3}g_{i+3}^* &= \frac{7}{6}g_{i+2}^* - \frac{11}{6}g_{i+1}^* + g_{i+1/2}^*.
\end{align*}
\]

(32)
where $g_{i+1/2}$ is calculated by Eq. (31).

It is obvious to see that the proposed free-stream preserving metrics and Jacobian $g^*_m$ ($m = i + 1/2, i - 2 \cdots, i + 3$) are reconstructed by the 3rd-order scheme from the original 6th-order central one $g_m$ ($m = i, i + 1/2, i + 1$). The 3rd-order scheme is the same with the fluxes reconstruction in the sub-stencil of WENO5 scheme, shown in Eq.(26). Therefore, the proposed free-stream preserving metrics and Jacobian $g^*_i, g^*_i+1$ and $g^*_i+1/2$ maintain the 6th-order accuracy while $g^*_{i-2}, g^*_{i-1}, g^*_{i+2}$ and $g^*_{i+3}$ retain only 3rd-order accuracy, as shown in Appendix B.

Next, we adopt $g^*_m$ and $g_m$ to compute the cell-averaged fluxes and the conservative variables in transformed space, respectively. For example,

$$
\begin{align*}
\hat{F}_m^* &= F_m \left( \frac{\xi_x}{J} \right)_m^* + G_m \left( \frac{\xi_y}{J} \right)_m^* + H_m \left( \frac{\xi_z}{J} \right)_m^*, \\
\hat{Q}_m^* &= Q_m \left( \frac{1}{J} \right)_m^*,
\end{align*}
$$

and

$$
\begin{align*}
\hat{F}_m &= F_m \left( \frac{\xi_x}{J} \right)_m + G_m \left( \frac{\xi_y}{J} \right)_m + H_m \left( \frac{\xi_z}{J} \right)_m, \\
\hat{Q}_m &= Q_m \left( \frac{1}{J} \right)_m.
\end{align*}
$$

After that, we reconstruct the cell-face fluxes $\hat{F}^*_{i+1/2}$ from the cell-averaged fluxes $\hat{F}^*_m$ by the specific upwind schemes, say WENO5,

$$
\hat{F}^*_{i+1/2} = WENO5_{LF} \left( \hat{F}^*_{i-2}, \cdots, \hat{F}^*_{i+3}, \hat{Q}^*_{i-2}, \cdots, \hat{Q}^*_{i+3} \right),
$$

where $WENO5_{LF}$ stands for the operator of the characteristic WENO5 scheme coupled with Lax-Friedrichs flux splitting.

Unfortunately, the fluxes $\hat{F}^*_{i+1/2}$ only achieve a 3rd-order accuracy due to applying the 3rd-order metrics and Jacobian. Nevertheless, as proved in Appendix B, we can realize a fact that

**Theorem 2.** The 3rd-order accurate free-stream metrics and Jacobian given in Eq. (32) do not change the 5th-order accuracy of the upwind dissipations of the cell-face fluxes reconstructed by the 5th-order linear upwind or WENO scheme.

Therefore, we suggest replacing the central part of $\hat{F}^*_{i+1/2}$, denoted by $\hat{F}^{(3)}_{i+1/2}$, with the 6th-order one, denoted by $\hat{F}^{(6)}_{i+1/2}$. Specifically,

$$
\begin{align*}
\hat{F}^{(3)}_{i+1/2} &= \frac{1}{60} \left( \hat{F}^*_{i-2} - 8\hat{F}^*_{i-1} + 37\hat{F}^*_{i} - 8\hat{F}^*_{i+1} + \hat{F}^*_{i+2} \right), \\
\hat{F}^{(6)}_{i+1/2} &= \frac{1}{60} \left( \hat{F}^*_{i-2} - 8\hat{F}^*_{i-1} + 37\hat{F}^*_{i} - 8\hat{F}^*_{i+1} - \frac{5}{2}\hat{F}^*_{i+2} \right), \\
\hat{F}_{i+1/2} &= \hat{F}^*_{i+1/2} + \hat{F}^{(6)}_{i+1/2} - \hat{F}^{(3)}_{i+1/2}.
\end{align*}
$$
Finally, the new fluxes $\tilde{F}_{i+1/2}$ can approach the 5th-order accuracy. If we choose a linear upwind scheme in Eq. (35), the free-stream preserving identity can be satisfied as well without destroying the convergence order of this scheme.

4. Numerical tests on curvilinear grids

Several verifications, such as the isotropic vortex convection, the double Mach reflection problem, the transonic flow past the ONERA M6 wing, etc. are conducted to evaluate the accuracy and the capability in shock capturing of the proposed free-stream preserving method on curvilinear grids. If not otherwise specified, the local Lax-Friedrichs flux splitting and the 3rd-order TVD Runge-Kutta scheme [23] are utilized for the simulations. For the viscous issues, the 6th-order central scheme is adopted to discrete the viscous terms. In the following, WENO5/WENO7 stand for the standard 5th/7th-order WENO schemes of Shu [24], WENOZ is the standard improved 5th-order WENO scheme of Borges et al. [25] and WENO5-Present, WENO7-Present, WENOZ-Present are the free-stream preserving schemes suggested in the present paper.

4.1. Free-stream

The wavy and randomized grids, as shown in Fig. [2] are considered to verify the proposed free-stream preserving scheme. The wavy grid is defined in the domain $(x, y) \in [-10, 10] \times [-10, 10]$ by

$$
x_{i,j} = x_{min} + \Delta x_0 \left[ (i - 1) + A_x \sin \left( \frac{n_{xy}(j - 1)\Delta y_0}{L_y} \right) \right]
$$

$$
y_{i,j} = y_{min} + \Delta y_0 \left[ (j - 1) + A_y \sin \left( \frac{n_{yx}(i - 1)\Delta x_0}{L_x} \right) \right]
$$

where $L_x = L_y = 20$, $x_{min} = -L_x/2$, $y_{min} = -L_y/2$, $A_x \Delta x = 0.6$, $A_y \Delta y = 0.6$, and $n_{xy} = n_{yx} = 8$. The randomized grid is disturbed randomly with 20% grid spacing of the uniform Cartesian grid. The grid resolution of the two grids are both $21 \times 21$.

The uniform free-stream of $M = 0.5$ in x direction is given as

$$
u = 0.5, v = 0, p = 1, \rho = \gamma
$$

where $\gamma = 1.4$ is the specific heat ratio. The $L_2$-norm and $L_\infty$-norm errors of the velocity components $v$ for the two grids are estimated at $t = 20$. In Table [1] compared with the standard WENO5/WENO7 scheme, the proposed free-stream preserving method effectively decreases the geometrically induced errors, which are both below $1 \times 10^{-14}$ and close to the machine zero for the double-precision computations.

4.2. Isotropic vortex

The two-dimensional moving isotropic vortex problems on the wavy and randomized grids are investigated to evaluate the accuracy and vortex preserving capability of the present free-stream preserving schemes.
Figure 2: Illustration of the wavy and randomized grids.

Table 1: The $L_2$ and $L_\infty$ errors of the $v$ component on the wavy and randomized grids.

| Method       | Wavy grid          | Random grid        |
|--------------|--------------------|--------------------|
|              | $L_2$ error        | $L_\infty$ error  | $L_2$ error | $L_\infty$ error |
| WENO5        | $2.45 \times 10^{-2}$ | $4.72 \times 10^{-2}$ | $1.29 \times 10^{-2}$ | $4.41 \times 10^{-2}$ |
| WENOZ        | $6.53 \times 10^{-3}$ | $1.32 \times 10^{-2}$ | $4.97 \times 10^{-3}$ | $1.65 \times 10^{-2}$ |
| WENO7        | $1.03 \times 10^{-2}$ | $1.98 \times 10^{-2}$ | $1.57 \times 10^{-2}$ | $5.08 \times 10^{-2}$ |
| WENO5-Present| $6.00 \times 10^{-16}$ | $2.13 \times 10^{-15}$ | $7.88 \times 10^{-16}$ | $2.12 \times 10^{-15}$ |
| WENOZ-Present| $1.10 \times 10^{-15}$ | $3.17 \times 10^{-15}$ | $1.75 \times 10^{-15}$ | $5.01 \times 10^{-15}$ |
| WENO7-Present| $5.49 \times 10^{-16}$ | $1.74 \times 10^{-15}$ | $6.19 \times 10^{-16}$ | $1.95 \times 10^{-15}$ |
The fluid is treated as the ideal gas with the specific heat ratio $\gamma = 1.4$. An isotropic vortex centered at $(x_c, y_c) = (0, 0)$ is superposed to a uniform flow with Mach 0.5. Specifically, the perturbations of the velocity, temperature and entropy are expressed by:

\[
\begin{align*}
\delta u, \delta v &= \epsilon \tau \alpha (1 - \tau^2) (\sin \theta, -\cos \theta) \\
\delta T &= -\frac{(\gamma - 1) \epsilon^2}{4 \alpha \gamma} e^{2\alpha(1 - \tau^2)} \\
\delta S &= 0
\end{align*}
\]

where $\alpha = 0.204$, $\tau = r/r_c$ and $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$. $r_c = 1.0$, $\epsilon = 0.02$ denote the vortex core length and strength, respectively. $T = p/\rho$ is the temperature and $S = p/\rho^\gamma$ is the entropy. The periodic boundary condition is imposed and the results are estimated when the vortex moving back to the original position at $t = 40$.

The numerical vorticity magnitude contours on the two wavy and random grids at a resolution of $21 \times 21$, given in Fig. 3 and Fig. 4 indicate that the moving vortex on those inhomogeneous grid can not be resolved at all for the standard WENO5 and WENOZ schemes, while they can retain well after a periodical moving for the proposed WENO5-Present, WENOZ-Present and WENO7-Present. Totally five severe wavy grids with the resolution of $21 \times 21$, $41 \times 41$, $81 \times 81$, $161 \times 161$ and $321 \times 321$ are considered to evaluate the convergence rate of the proposed schemes. The time step $\Delta t$ respect to each grid decreases until the $L_2$ and $L_\infty$ errors are invariant to eliminate the errors by the 3rd-order time integration, as proposed in Ref. [9] and [16]. The $L_2$ and $L_\infty$ errors of the $v$ component and their corresponding convergence rates on those wavy grids, listed in Table 2 indicate that the WENO5-Present, WENOZ-Present and WENO7-Present schemes can maintain the theoretical convergence orders.

### 4.3. Double Mach reflection

The double Mach problem [26] is carried out to demonstrate the shock-capturing capability of the present free-stream preserving WENO scheme. The initial condition is

\[
\begin{align*}
(\rho, u, v, p) &= \begin{cases} (1.4, 0, 0, 1.0) & x - y\tan(\pi/6) > 1/6, \\
(8.0, 7.14, -4.125, 116.5) & x - y\tan(\pi/6) < 1/6. \end{cases}
\end{align*}
\]

The computation is conducted up to $t = 0.2$ under a CFL number of 0.5 and with a grid resolution of $961 \times 241$. The global Lax-Friedrichs dissipation is chosen for these simulations. The grids are randomized by 5% and 20% of the uniform grid spacing. As illustrated in Fig. 5 the calculated density contours of the double Mach reflection on the 5% randomized grid show that the standard WENO5, WENOZ and WENO7 schemes induce spurious oscillations due to the lack of the free-stream preserving identity. In contrast, the proposed WENO5-Present, WENOZ-Present and WENO7-Present overcome this disadvantage and resolve...
Table 2: The $L_2$ and $L_{\infty}$ errors of the $v$ component and their corresponding convergence rates on the wavy grids.

| Method      | Grid size | $L_2$ error | Convergence rate | $L_{\infty}$ error | Convergence rate |
|-------------|-----------|-------------|------------------|---------------------|------------------|
| WENO5       | 21 × 21   | 2.14 × 10^{-2} | -                | 5.06 × 10^{-2}     | -                |
|             | 41 × 41   | 2.62 × 10^{-3} | 3.03             | 9.32 × 10^{-3}     | 2.44             |
|             | 81 × 81   | 1.71 × 10^{-4} | 3.94             | 5.61 × 10^{-4}     | 4.05             |
|             | 161 × 161 | 3.00 × 10^{-6} | 5.83             | 1.89 × 10^{-5}     | 4.89             |
|             | 321 × 321 | 4.33 × 10^{-8} | 6.11             | 4.53 × 10^{-7}     | 5.38             |
| WENOZ       | 21 × 21   | 7.90 × 10^{-3} | -                | 2.43 × 10^{-2}     | -                |
|             | 41 × 41   | 8.34 × 10^{-4} | 3.24             | 4.98 × 10^{-3}     | 2.29             |
|             | 81 × 81   | 3.92 × 10^{-5} | 4.41             | 1.94 × 10^{-4}     | 4.68             |
|             | 161 × 161 | 1.23 × 10^{-6} | 4.99             | 1.05 × 10^{-5}     | 4.21             |
|             | 321 × 321 | 3.89 × 10^{-8} | 4.98             | 4.10 × 10^{-7}     | 4.68             |
| WENO5-Present | 21 × 21   | 2.29 × 10^{-3} | -                | 1.61 × 10^{-2}     | -                |
|             | 41 × 41   | 4.82 × 10^{-4} | 2.25             | 4.36 × 10^{-3}     | 1.88             |
|             | 81 × 81   | 1.66 × 10^{-5} | 4.86             | 1.47 × 10^{-4}     | 4.89             |
|             | 161 × 161 | 5.85 × 10^{-7} | 4.83             | 5.82 × 10^{-6}     | 4.66             |
|             | 321 × 321 | 3.89 × 10^{-8} | 4.90             | 2.17 × 10^{-7}     | 4.75             |
| WENOZ-Present | 21 × 21   | 2.31 × 10^{-3} | -                | 1.58 × 10^{-2}     | -                |
|             | 41 × 41   | 5.23 × 10^{-4} | 2.14             | 4.45 × 10^{-3}     | 1.83             |
|             | 81 × 81   | 1.91 × 10^{-5} | 4.78             | 1.99 × 10^{-4}     | 4.48             |
|             | 161 × 161 | 5.89 × 10^{-7} | 5.02             | 5.76 × 10^{-6}     | 5.11             |
|             | 321 × 321 | 1.96 × 10^{-8} | 4.91             | 2.17 × 10^{-7}     | 4.73             |
| WENO7-Present | 21 × 21   | 2.16 × 10^{-3} | -                | 1.49 × 10^{-2}     | -                |
|             | 41 × 41   | 4.37 × 10^{-4} | 2.31             | 3.88 × 10^{-3}     | 1.94             |
|             | 81 × 81   | 3.71 × 10^{-6} | 6.88             | 3.83 × 10^{-5}     | 6.66             |
|             | 161 × 161 | 3.71 × 10^{-8} | 6.64             | 4.65 × 10^{-7}     | 6.36             |
|             | 321 × 321 | 3.12 × 10^{-10}| 6.89             | 4.44 × 10^{-9}     | 6.71             |
Figure 3: The vorticity magnitude distributions ranging from 0.0 to 0.006 of the 2D moving vortex on the wavy and randomized grids (21 × 21) for the WENOZ scheme.

Figure 4: The vorticity magnitude distributions ranging from 0.0 to 0.006 of the 2D moving vortex on the wavy and randomized grids (21 × 21) for the WENO5 and WENO7 schemes.
the flow field smoothly. When the grid is randomized up to 20% of the uniform grid spacing, the density contours in Fig 5 indicate that the proposed free-stream preserving WENO schemes can still reduce the geometry errors. Some disturbances are observed in the result of WENO7-Present scheme, but they are essentially improved on such a highly distorted grid, compared with the standard WENO7 schemes.

4.4. A Mach 3 wind tunnel with a step

The case is from Ref [2] to demonstrate the shock-capture capabilities and high-resolution of the schemes. The length and width of the wind tunnel are 3 units and 1 unit, respectively. The step in the bottom of the wind tunnel is located at 0.6 units from the left boundary with a height of 0.2 units. The initial flow field is given by a right-going Mach 3 flow with $P = 1$ and $\rho = \gamma = 1.4$. The in-flow and out-flow boundary condition are implied to the left and right boundary, and the reflective boundary conditions are considered.
along the walls of the tunnel. The computational domain is discretized by two grids with a randomization of 5% and 20%, respectively, under a resolution of $\Delta x = 1/200$ units. The global Lax-Friedrichs dissipation is considered. The computed results at $t = 4$ in Fig. 7 indicate that either WENO5-Present or WENO7-Present achieve the free-stream preserving identity on the severely randomized grids. The reflective shocks around the wall of the wind tunnel are captured correctly and the vortexes generated in the shear layer are resolved significantly, which show that the present schemes have been improved a lot compared with the standard WENO schemes.

4.5. Supersonic flow past a cylinder

The supersonic flow past a cylinder \cite{22} is solved to examine the shock capturing capability of the free-stream preserving schemes on the inhomogeneous curvilinear grid. The $M = 2$ supersonic flow moves towards the cylinder and the slip wall boundary condition is imposed to the wall and supersonic inflow and outflow boundary condition are assigned to the left boundary and others, respectively. The grid is given by:

$$
\begin{align*}
 x &= (R_x - (R_x - 1)\eta') \cos (\theta(2\xi' - 1)) \\
 y &= (R_y - (R_y - 1)\eta') \sin (\theta(2\xi' - 1)) \\
 \xi' &= \frac{\xi - 1}{i_{\max} - 1}, \xi = i + 0.2\phi_i \\
 \eta' &= \frac{\eta - 1}{j_{\max} - 1}, \eta = j + 0.2\sqrt{1 - \phi_i^2}
\end{align*}
$$

(41)

where $\theta = 5\pi/12$, $R_x = 3$, $R_y = 6$ and $\phi_i$ is a random number distributed between $[0, 1]$. The resolution of the grid is $i_{\max} = 61$ and $j_{\max} = 81$. The free stream pressure and density are $p = 1$ and $\rho = \gamma$, respectively. The computational results with the global Lax-Friedrichs dissipation are evaluated after $t = 25$. In Fig. 8,
the pressure distributions around the cylinder calculated by the standard WENO schemes are significantly disturbed by the unphysical oscillations. However, the results from the present free-stream preserving WENO schemes are very smooth and the pressure distributions are well resolved.

4.6. Transonic flow past a NACA0012 airfoil

In this section, the inviscid transonic flows past a NACA0012 airfoil with Mach number $M = 0.8$ and angle of attack $AOA = 1.25^\circ$ (case 1) and Mach number $M = 0.85$ and angle of attack $AOA = 1.0^\circ$ (case 2) are simulated by the present free-stream preserving scheme. A coarse grid discretized with $160 \times 32$ cells in circumferential and radial, respectively, is chosen to demonstrate the accuracy of the present scheme. The reference simulations are conducted by the FVM (ROE scheme coupled with 2nd-order MUSCL reconstruction) on this coarse grid and the fine grid with a resolution of $1280 \times 177$. As shown in Fig. 9, the sharper shock patterns are captured by the WENO5-Present scheme and the calculated pressure coefficient distributions along the airfoil are closer to the results of the fine grid. The calculated numerical Mach contours around the NACA0012 airfoil in Fig. 10 exhibit a rather smooth and continuous flow field.
Figure 8: The pressure distributions from 1.2 to 5.4 of the supersonic flow past a cylinder.

Figure 9: The pressure coefficient distributions along the wall of the NACA0012 airfoil.
4.7. Transonic flow pass the ONERA M6 wing

The three-dimensional transonic flow pass the ONERA M6 wing is considered in this test case. The geometry is very simple but the transonic flow features are complicated. The simulation is conducted at a Mach number $M = 0.84$ and an angle of attack $AOA = 3.06^\circ$ with a Reynold number of $Re_l = 1.172 \times 10^7$ based on the mean aerodynamic chord of $l = 0.64607m$. The computational grid consists of 12 blocks and 294912 cells in total, as illustrated in Fig. 11. In this case, the viscous effects are taken into consideration and the SA turbulence model [27] is adopted. The lower-upper symmetric-Gauss-Seidel implicit method (LUSGS) is employed for the time marching. The numerical pressure contours around the surface of M6 wing and the symmetry plane drawn in Fig. 12 show that the transonic flows around the 3D wing can be resolved smoothly by the WENO5-Present scheme. Fig. 13 compares the simulated pressure coefficient distributions with experimental data at six spanwise stations. They are in good agreement with the experimental results except at $y/b = 0.8$, which is because the ideal symmetry boundary of the middle plane in simulation can not exactly reproduce the flow physics of the half-wing in wind tunnel [28].

5. Concluding remarks

In this paper, we give a sufficient condition on preserving the free-stream identity for the upwind dissipations. Based on this sufficient condition, the free-stream preserving metrics and Jacobian are proposed for the upwind dissipation of the linear upwind and WENO schemes. Coupled with the high-order accurate central part, this technique avoids destroying the accuracy and the forms of the standard upwind schemes as
Figure 11: The grid illustration of ONERA M6 wing.

Figure 12: The pressure contours around the surface and the symmetry pane of the ONERA M6 wing. Totally 61 contours from 130 Kpa to 490 Kpa.
Figure 13: The pressure coefficient distributions at six stations along the wall of the ONERA M6 wing.
far as possible. For example, the suggested WENO5-Present retains the 5th-order accuracy in the smooth non-critical points regions. Therefore, the present technique is convenient to operate in the conservative finite difference scheme and easy to extend to the others. Some verifications are conducted to demonstrate the accuracy and robustness of the present free-stream preserving method, such as the isotropic vortex problem, the double Mach reflection problem, the transonic flow past NACA0012 airfoil and ONERA M6 wing, etc. The simulated results indicate that the present method indeed maintain the free-stream preserving identity on the curvilinear grids with high-order accuracy.

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Appendix A

The standard WENO5 scheme can be divided into a central part and a dissipation part [11,16],

\[
\tilde{F}_{i+1/2} = \tilde{F}_{i+1/2}^+ + \tilde{F}_{i+1/2}^-
\]

\[
= \sum_s R_{i+1/2}^s f_{i+1/2}^s + \sum_s R_{i+1/2}^s f_{i+1/2}^s
\]

\[
= \frac{1}{60} \left( \tilde{F}_{i-2} - 8\tilde{F}_{i-1} + 37\tilde{F}_i + 37\tilde{F}_{i+1} - 8\tilde{F}_{i+2} + \tilde{F}_{i+3} \right)
\]

\[
- \frac{1}{60} \sum_s R_{i+1/2}^s \left[ (20\omega_0 - 1) \tilde{f}_{i+1}^s + (10\omega_0 + 10\omega_1 - 5) \tilde{f}_{i+1/2}^s + \tilde{f}_{i,3}^s \right] 
\]

\[
+ \frac{1}{60} \sum_s R_{i+1/2}^s \left[ (20\omega_0 - 1) \tilde{f}_{i-1}^s - (10\omega_0 + 10\omega_1 - 5) \tilde{f}_{i-1/2}^s + \tilde{f}_{i,-3}^s \right]
\]

(42)

where

\[
\tilde{f}_{i+r+1}^s = \tilde{f}_{i+r+1}^s - 3\tilde{f}_{i+r+2}^s + 3\tilde{f}_{i+r+1}^s - \tilde{f}_{i+r-2}^s, \quad r = 0, 1, 2
\]

\[
= \frac{1}{2} L_{i+1/2}^s \left( \tilde{F}_{i+r+1} - 3\tilde{F}_{i+r+2} + 3\tilde{F}_{i+r+1} - \tilde{F}_{i+r-2} \right) + \frac{1}{2} \lambda^s L_{i+1/2}^s \left( \tilde{Q}_{i+r+1} - 3\tilde{Q}_{i+r+2} + 3\tilde{Q}_{i+r+1} - \tilde{Q}_{i+r-2} \right)
\]

(43)

\[
\tilde{f}_{i+r+1}^s = \tilde{f}_{i+r+1}^s - 3\tilde{f}_{i+r+2}^s + 3\tilde{f}_{i+r+1}^s - \tilde{f}_{i-r}^s, \quad r = 0, 1, 2
\]

\[
= \frac{1}{2} L_{i+1/2}^s \left( \tilde{F}_{i+r+3} - 3\tilde{F}_{i+r+2} + 3\tilde{F}_{i+r+1} - \tilde{F}_{i-r} \right) - \frac{1}{2} \lambda^s L_{i+1/2}^s \left( \tilde{Q}_{i+r+3} - 3\tilde{Q}_{i+r+2} + 3\tilde{Q}_{i+r+1} - \tilde{Q}_{i-r} \right)
\]

(44)

First, taking the dissipation terms \( D^+ \) as examples, the upwind dissipations are rearranged as

\[
D^+ = D_F^+ + D_Q^+,
\]

(45)
In the following discussions, we mainly focus on $D_F^+$ where the linear dissipation part can be reformulated as

$$
D_F^+ = -\frac{1}{60} \sum_s R^I_{i+1/2} \left[ \frac{1}{2} (20\omega_0^+ - 1) L^s_{i+1/2} \left( \tilde{F}_{i+1} - 3\tilde{F}_{i+2} - 3\tilde{F}_{i-1} + \tilde{F}_{i-2} \right) 
- \frac{1}{2} (10\omega_0^+ + 10\omega_1^+ - 5) L^s_{i+1/2} \left( \tilde{F}_{i+2} - 3\tilde{F}_{i+1} - 3\tilde{F}_{i-1} + \tilde{F}_{i-2} \right) 
+ \frac{1}{2} L^s_{i+1/2} \left( \tilde{F}_{i+3} - 3\tilde{F}_{i+2} + 3\tilde{F}_{i+1} - \tilde{F}_i \right) \right],
$$

and

$$
D_Q^+ = -\frac{1}{60} \sum_s R^I_{i+1/2} \lambda_s \left[ \frac{1}{2} (20\omega_0^+ - 1) L^s_{i+1/2} \left( \tilde{Q}_{i+1} - 3\tilde{Q}_{i+2} + \tilde{Q}_{i-1} - \tilde{Q}_{i-2} \right) 
- \frac{1}{2} (10\omega_0^+ + 10\omega_1^+ - 5) L^s_{i+1/2} \left( \tilde{Q}_{i+2} - 3\tilde{Q}_{i+1} + 3\tilde{Q}_{i-1} - \tilde{Q}_{i-2} \right) 
+ \frac{1}{2} L^s_{i+1/2} \left( \tilde{Q}_{i+3} - 3\tilde{Q}_{i+2} + 3\tilde{Q}_{i+1} - \tilde{Q}_{i} \right) \right].
$$

In the following discussions, we mainly focus on $D_F^+$ because $D_Q^+$ has the similar form. After simplifying $D_F^+$, we obtain

$$
D_F^+ = -\frac{1}{120} \sum_s R^I_{i+1/2} \left[ \begin{aligned}
&- \tilde{F}_{i-2} + 5\tilde{F}_{i-1} - 10\tilde{F}_i + 10\tilde{F}_{i+1} - 5\tilde{F}_{i+2} + \tilde{F}_{i+3} \\
&\frac{1}{3} \tilde{F}_{i-2} - \frac{7}{6} \tilde{F}_{i-1} + \frac{11}{6} \tilde{F}_i \\
&- \frac{1}{6} \tilde{F}_{i-1} + \frac{5}{6} \tilde{F}_i + \frac{1}{3} \tilde{F}_{i+1} \\
&\frac{1}{3} \tilde{F}_i + \frac{5}{6} \tilde{F}_{i+1} - \frac{1}{6} \tilde{F}_{i+2} \\
&\end{aligned} \right],
$$

with the linear dissipation part can be reformulated as

$$
\begin{aligned}
&- \tilde{F}_{i-2} + 5\tilde{F}_{i-1} - 10\tilde{F}_i + 10\tilde{F}_{i+1} - 5\tilde{F}_{i+2} + \tilde{F}_{i+3} \\
&= -3 \left( \frac{1}{3} \tilde{F}_{i-2} - \frac{7}{6} \tilde{F}_{i-1} + \frac{11}{6} \tilde{F}_i \right) + 9 \left( -\frac{1}{6} \tilde{F}_{i-1} + \frac{5}{6} \tilde{F}_i + \frac{1}{3} \tilde{F}_{i+1} \right) \\
&+ 9 \left( \frac{1}{3} \tilde{F}_i + \frac{5}{6} \tilde{F}_{i+1} - \frac{1}{6} \tilde{F}_{i+2} \right) + 3 \left( \frac{11}{6} \tilde{F}_{i+1} - \frac{7}{6} \tilde{F}_{i+2} + \frac{1}{3} \tilde{F}_{i+3} \right).
\end{aligned}
$$

Eq. (48) and Eq. (49) can be extended to $D_Q^+$ directly. They indicate that the positive dissipations are composed of the linear part and the non-linear part reconstructed in the three sub-stencils. Furthermore,
whether the linear part or the non-linear part is a combination of the 3rd-order reconstruction in the sub-stencil. Therefore, when the free-stream condition is imposed, the upwind dissipations can not satisfy the free-stream preserving identity due to conflicting with the sufficient condition of Deng et al. [11] and Abe et al. [2].

However, if the Eqs. (30) in Theorem 2 are satisfied, ignoring the flow variables because they are all constant vectors under the free-stream condition, the values of the metrics and Jacobian reconstructed in the four sub-stencils in Eq. (49) and (48) are unique. Then, it is obvious to see that the linear dissipations cancel each other in Eq. (49), and the accumulation of the three non-linear parts in Eq. (48) is zero as well under the relations

\[ C_0 + C_1 + C_2 = \omega_0 + \omega_1 + \omega_2 = 1. \] (50)

Finally, the central part can be rearranged as

\[
\frac{1}{60} \left( \tilde{F}_{i-2} - 8\tilde{F}_{i-1} + 37\tilde{F}_i + 37\tilde{F}_{i+1} - 8\tilde{F}_{i+2} + \tilde{F}_{i+3} \right)
\]

\[
+ \frac{1}{60} \sum_s R^s_{i+1/2} L^s_{i+1/2} \left( \tilde{Q}_{i-2} - 5\tilde{Q}_{i-1} + 10\tilde{Q}_i - 10\tilde{Q}_{i+1} + 5\tilde{Q}_{i+2} - \tilde{Q}_{i+3} \right). \]

Similarly, ignoring the flow variables, if the Eqs. (30) in Theorem 2 are satisfied, the values of the metrics and Jacobian reconstructed in the four sub-stencils and their combination in Eq. 51 are all \( g_{i+1/2} \) which means that it equals to obtaining the fluxes by the 6th-order central scheme. As a result, the free-stream preserving identity is satisfied because of the sufficient condition given by Deng et al. [11] and Abe et al. [2].

For the 5th-order linear upwind scheme, as proposed in Ref. [10], it can be written as

\[
\tilde{F}_{i+1/2} = \frac{1}{60} \left( \tilde{F}_{i-2} - 8\tilde{F}_{i-1} + 37\tilde{F}_i + 37\tilde{F}_{i+1} - 8\tilde{F}_{i+2} + \tilde{F}_{i+3} \right)
\]

\[
+ \frac{1}{60} \sum_s R^s_{i+1/2} L^s_{i+1/2} \left( \tilde{Q}_{i-2} - 5\tilde{Q}_{i-1} + 10\tilde{Q}_i - 10\tilde{Q}_{i+1} + 5\tilde{Q}_{i+2} - \tilde{Q}_{i+3} \right). \]

(51)
Similarly, we rearrange this form to
\[
\tilde{F}_{i+1/2} = \frac{1}{60} \left( \tilde{F}_{i-2} - 8\tilde{F}_{i-1} + 37\tilde{F}_{i} + 37\tilde{F}_{i+1} - 8\tilde{F}_{i+2} + \tilde{F}_{i+3} \right) \\
+ \frac{1}{60} \sum_{s} R_{i+1/2}^{s} \lambda \ast L_i^{s}/2 \left[ 3 \left( \tilde{Q}_{i-2} - \frac{7}{6} \tilde{Q}_{i-1} + \frac{11}{6} \tilde{Q}_{i} \right) + 9 \left( -\frac{1}{6} Q_{i-1} + \frac{5}{6} \tilde{Q}_{i} + \frac{1}{3} \tilde{Q}_{i+1} \right) \right]
\]

\text{sub-stencil 1}
\]
\]
\]
\]
\]

\text{sub-stencil 2}
\]
\]
\]
\]

\text{sub-stencil 3}
\]
\]
\]
\]

\text{sub-stencil 4}
\]
\]
\]
\]

Obviously, the above analyses can be applied to the 5th-order linear upwind scheme to verify its free-stream preserving identity.

**Appendix B**

*The 5th-order linear upwind scheme*

The proposed free-stream preserving metrics and Jacobian \(g_{i-2}^{*}, g_{i-1}^{*}, g_{i+2}^{*}\) and \(g_{i+3}^{*}\) are computed by the 3rd-order reconstruction from three 6th-order ones \(g_{i}, g_{i+1/2}, g_{i+1}\). Therefore, they maintain 3rd-order accuracy at least. In smooth regions, Taylor expansion of Eq. (32) gives, respectively,

\[
\Delta g_{i-2}^{*} = g_{i-2}^{*} - g_{i-2} = \frac{5}{2} R''' \Delta \xi^{3} - \frac{7}{20} R^{(4)} \Delta \xi^{4} + \frac{103}{240} R^{(5)} \Delta \xi^{5} + O \left( \Delta \xi^{6} \right),
\]

\[
\Delta g_{i-1}^{*} = g_{i-1}^{*} - g_{i-1} = \frac{1}{2} R''' \Delta \xi^{3} + \frac{1}{20} R^{(4)} \Delta \xi^{4} + \frac{11}{240} R^{(5)} \Delta \xi^{5} + O \left( \Delta \xi^{6} \right),
\]

\[
\Delta g_{i+1}^{*} = g_{i+1}^{*} - g_{i+1} = 0,
\]

\[
\Delta g_{i+2}^{*} = g_{i+2}^{*} - g_{i+2} = \frac{-1}{2} R''' \Delta \xi^{3} - \frac{9}{20} R^{(4)} \Delta \xi^{4} + \frac{59}{240} R^{(5)} \Delta \xi^{5} + O \left( \Delta \xi^{6} \right),
\]

\[
\Delta g_{i+3}^{*} = g_{i+3}^{*} - g_{i+3} = \frac{-5}{2} R''' \Delta \xi^{3} - \frac{57}{20} R^{(4)} \Delta \xi^{4} - \frac{487}{240} R^{(5)} \Delta \xi^{5} + O \left( \Delta \xi^{6} \right),
\]

where \(R''' = R'''(\xi)\) and \(R^{(4)} = R^{(4)}(\xi)\) are the third and fourth derivatives at \(i\) of the primary function \(R(\xi)\),

\[
g_{i} = \frac{1}{\Delta \xi} \int_{i-1/2}^{i+1/2} R(\xi) d\xi.
\]

If the free-stream preserving metrics and Jacobian are adopted to the dissipation part of the 5th-order linear upwind scheme shown in Eq. (32), we can obtain

\[
\tilde{Q}_{m}^{*} = \tilde{Q}_{m}^{*} + \left( \tilde{Q}_{m}^{*} - \tilde{Q}_{m} \right), m = i - 2, i + 3
\]

\[
= \tilde{Q}_{m} + Q_{m} \Delta g_{m}^{*}.
\]
and

\[ D_{i+1/2} = \left( \tilde{Q}_{i-2}^* - 5\tilde{Q}_{i-1}^* + 10\tilde{Q}_{i}^* - 10\tilde{Q}_{i+1}^* + 5\tilde{Q}_{i+2}^* - \tilde{Q}_{i+3}^* \right) \]

\[ = \left( \tilde{Q}_{i-2} - 5\tilde{Q}_{i-1} + 10\tilde{Q}_{i} - 10\tilde{Q}_{i+1} + 5\tilde{Q}_{i+2} - \tilde{Q}_{i+3} \right) + \left( 10R''\tilde{Q}_{i+1/2}^* + 5R^{(4)}\tilde{Q}_{i+1/2}^* + R^{(5)}\tilde{Q}_{i+1/2} \right) \Delta \xi^5 + O(\Delta \xi^6), \]

where \( g \) refers to the Jacobian \( 1/J \). Similarly,

\[ D_{i-1/2} = \left( \tilde{Q}_{i-3} - 5\tilde{Q}_{i-2} + 10\tilde{Q}_{i-1} - 10\tilde{Q}_{i} + 5\tilde{Q}_{i+1} - \tilde{Q}_{i+2} \right) \]

\[ + \left( 10R''\tilde{Q}_{i-1/2}^* + 5R^{(4)}\tilde{Q}_{i-1/2}^* + R^{(5)}\tilde{Q}_{i-1/2} \right) \Delta \xi^5 + O(\Delta \xi^6). \]

The additional terms retain 5th-order accuracy in the conservative finite difference scheme because

\[
\frac{D_{i+1/2} - D_{i-1/2}}{\Delta \xi} = \left[ \left( \tilde{Q}_{i+1/2}^{(5)} - \tilde{Q}_{i-1/2}^{(5)} \right) + 10R''\tilde{Q}_{i+1/2} - Q_{i+1/2}' \right] + 5R^{(4)}\left( \tilde{Q}_{i+1/2}' - Q_{i+1/2}' \right) + R^{(5)}\left( \tilde{Q}_{i+1/2} - Q_{i+1/2} \right) \Delta \xi^4 + O(\Delta \xi^5)
\]

\[
= \left( \tilde{Q}_i + 10R''\tilde{Q}_i'' + 5R^{(4)}\tilde{Q}_i' + R^{(5)}\tilde{Q}_i \right) \Delta \xi^5 + O(\Delta \xi^5) \]

Obviously, even the 3rd-order metrics and Jacobian are applied to the dissipation of the 5th-order linear upwind scheme, only the extra \( O(\Delta \xi^5) \) terms are added to the standard terms such that the analytic convergence order of the proposed upwind dissipation still maintains 5th-order accuracy. With this dissipation, if the central part in Eq. \( (62) \) is obtained by the 6th-order metrics and Jacobian \( g_m \), then the linear upwind scheme achieves the 5th-order accuracy.

The 5th-order WENO scheme

According to Ref. \( 22 \), the WENO5 scheme is a convex combination of the 3rd-order reconstruction of all the candidate sub-stencils

\[ \tilde{f}_{i+1/2} = q^2(\tilde{f}_{i-2}, \ldots, \tilde{f}_{i+2}) + \sum_{k=0}^{2} \left( \omega_k - C_k \right) q_k^2 \left( \tilde{f}_{i+k-2}, \tilde{f}_{i+k-1}, \tilde{f}_{i+k} \right), \]

and under the condition of Eq. \( (50) \). As given by Borges et al. \( 25 \), the conservative finite difference scheme maintains the 5th-order accuracy if the non-linear weights \( \omega_k \) and \( q_k \) in sub-stencils satisfy the followings,

\[ \omega_k = C_k + O(\Delta \xi^2), \]

\[ q_k = \tilde{h}_{i+1} + O(\Delta \xi^3), \]

\[ \sum_{k=0}^{2} A_k \left( \omega_k - \omega_k' \right) = O(\Delta \xi^4), \]

where \( \omega_k' \) denotes the non-linear weight for \( \tilde{f}_{i-1/2} \). It should be noted that Eq. \( (63) \) is ignored in the standard WENO5 scheme of Jiang and Shu \( 22 \) due to the large \( \epsilon = 1 \times 10^{-6} \).
Therefore, the reconstructed cell-face fluxes $\tilde{q}^{\ast}$ and apply the fluxes splitting, respectively, we can obtain the relation between $\tilde{F}^{\ast \ast}$ and $\tilde{F}^{\ast}$ as

$$\tilde{F}^{\ast \ast} = \tilde{F}^{\ast} + L \left\{ F \left[ \left( \frac{\xi_{i}}{J} \right)^{\ast} - \left( \frac{\xi_{i+1}}{J} \right)^{\ast} \right] + G \left[ \left( \frac{\xi_{i}}{J} \right)^{\ast} - \left( \frac{\xi_{i+2}}{J} \right)^{\ast} \right] \right\}$$

$$+ H \left[ \left( \frac{\xi_{i}}{J} \right)^{\ast} - \left( \frac{\xi_{i+1}}{J} \right)^{\ast} \right] + \lambda Q \left[ \left( \frac{1}{J} \right)^{\ast} - \left( \frac{1}{J} \right) \right]\right\}$$

$$= \tilde{F}^{\ast} + O(\Delta \xi^{3}).$$

(64)

Therefore, the reconstructed cell-face fluxes $q^{\ast \ast}$ in the sub-stencil by $\tilde{f}_{m}^{\ast}$ which is one of the component of $\tilde{F}^{\ast \ast}$ can still maintain 3rd-order accuracy because of Eq. 64. To simplify notation, we drop the superscript $\ast$ for $\tilde{f}^{\ast}, q^{\ast}$ and $\beta^{\ast}$ in the followings. In details,

$$q_{0}^{\ast} = \frac{1}{3} \tilde{f}^{\ast}_{i-2} + \frac{7}{6} \tilde{f}^{\ast}_{i-1} + \frac{11}{6} \tilde{f}^{\ast}_{i}$$

$$= \left( \frac{1}{3} \tilde{f}^{\ast}_{i-2} - \frac{7}{6} \tilde{f}^{\ast}_{i-1} + \frac{11}{6} \tilde{f}^{\ast}_{i} \right) + O(\Delta \xi^{3})$$

$$= \tilde{h}_{i+1/2} + O(\Delta \xi^{3})$$

(65)

Similarly,

$$q_{1}^{\ast} = \tilde{h}_{i+1/2} + O(\Delta \xi^{3})$$

(66)

$$q_{2}^{\ast} = \tilde{h}_{i+1/2} + O(\Delta \xi^{3}).$$

(67)

To investigate the accuracy of the non-linear weights, we define

$$\tilde{f}^{\ast}_{m} = \tilde{f}_{m} + \left( f^{\ast}_{m} - \tilde{f}_{m} \right)$$

$$= \tilde{f}_{m} + f_{m} \Delta g^{\ast}_{m}, \quad m = i - 2, \ldots, i + 2,$$

which represents for the difference between $\tilde{f}^{\ast}_{m}$ and $\tilde{f}_{m}$. Then, the smoothness indicators can be given by

$$\beta^{\ast}_{0} = \frac{1}{4} \left\{ \left( \tilde{f}_{i-2} - 4 \tilde{f}_{i-1} + 3 \tilde{f}_{i} \right) + \left( f_{i-2} \Delta g_{i-2}^{\ast} - 4 f_{i-1} \Delta g_{i-1}^{\ast} + 3 f_{i} \Delta g_{i}^{\ast} \right) \right\}^{2}$$

$$+ \frac{13}{12} \left\{ \left( \tilde{f}_{i-2} - 2 \tilde{f}_{i-1} + \tilde{f}_{i} \right) + \left( f_{i-2} \Delta g_{i-2}^{\ast} - 2 f_{i-1} \Delta g_{i-1}^{\ast} + f_{i} \Delta g_{i}^{\ast} \right) \right\}^{2},$$

$$\beta^{\ast}_{1} = \frac{1}{4} \left\{ \left( \tilde{f}_{i-1} - \tilde{f}_{i+1} \right) + \left( f_{i-1} \Delta g_{i-1}^{\ast} - f_{i+1} \Delta g_{i+1}^{\ast} \right) \right\}^{2}$$

$$+ \frac{13}{12} \left\{ \left( \tilde{f}_{i-1} - 2 \tilde{f}_{i} + \tilde{f}_{i+1} \right) + \left( f_{i-1} \Delta g_{i-1}^{\ast} - 2 f_{i} \Delta g_{i}^{\ast} + f_{i+1} \Delta g_{i+1}^{\ast} \right) \right\}^{2},$$

$$\beta^{\ast}_{2} = \frac{1}{4} \left\{ \left( 3 \tilde{f}_{i} - 4 \tilde{f}_{i+1} + \tilde{f}_{i+2} \right) + \left( 3 f_{i} \Delta g_{i}^{\ast} - 4 f_{i+1} \Delta g_{i+1}^{\ast} + f_{i+2} \Delta g_{i+2}^{\ast} \right) \right\}^{2}$$

$$+ \frac{13}{12} \left\{ \left( \tilde{f}_{i} - 2 \tilde{f}_{i+1} + \tilde{f}_{i+2} \right) + \left( f_{i} \Delta g_{i}^{\ast} - 2 f_{i+1} \Delta g_{i+1}^{\ast} + f_{i+2} \Delta g_{i+2}^{\ast} \right) \right\}^{2}.$$
In smooth regions, Taylor expansion of Eq. (70) at $i$ gives,

\[
\beta_0^* = \tilde{f}^2 \Delta \xi^2 + \left( \frac{13}{12} \tilde{f}'' - \frac{2}{3} \tilde{f}^5 + \frac{1}{2} \tilde{f}'^{} R''' f \right) \Delta \xi^4
- \left( \frac{13}{6} \tilde{f}'' \tilde{f}'''' - \frac{1}{2} \tilde{f}^5 \tilde{f}'''' + \frac{13}{4} \tilde{f}'' R''' f + 3 \tilde{f}^{} R'' f' + \frac{11}{20} \tilde{f}^{} R(4) f \right) \Delta \xi^5 + O(\Delta \xi^6),
\] (73)

\[
\beta_1^* = \tilde{f}^2 \Delta \xi^2 + \left( \frac{13}{12} \tilde{f}'' + \frac{1}{3} \tilde{f}^5 \tilde{f}''' - \frac{1}{2} \tilde{f}^{} R''' f \right) \Delta \xi^4
+ \left( \frac{13}{12} \tilde{f}'' R''' f + \frac{1}{2} \tilde{f}'' R''' f' - \frac{1}{20} \tilde{f}^{} R(4) f \right) \Delta \xi^5 + O(\Delta \xi^6),
\] (74)

\[
\beta_2^* = \tilde{f}^2 \Delta \xi^2 + \left( \frac{13}{12} \tilde{f}'' - \frac{2}{3} \tilde{f}^5 \tilde{f}''' + \frac{1}{2} \tilde{f}^{} R''' f \right) \Delta \xi^4
+ \left( \frac{13}{6} \tilde{f}'' \tilde{f}''' - \frac{1}{2} \tilde{f}^5 \tilde{f}'''' - \frac{13}{4} \tilde{f}^{} R''' f + \tilde{f}^{} R''' f' + \frac{9}{20} \tilde{f}^{} R(4) f \right) \Delta \xi^5 + O(\Delta \xi^6).
\] (75)

Therefore, applying the 3rd-order metrics and Jacobian to calculate the smoothness indicators $\beta_k^*$ does not violate the convergence orders of them. Furthermore, it is straightforward to see that

(1) for the WENO5-Present scheme, we obtain

\[
\beta_k^* = \begin{cases} 
(\tilde{f}' \Delta \xi)^2 (1 + O(\Delta \xi^2)) & \tilde{f}' \neq 0 \\
\frac{13}{12} (\tilde{f}' \Delta \xi)^2 (1 + O(\Delta \xi)) & \tilde{f}' = 0,
\end{cases}
\] (76)

with $k = 0, 1, 2$, which is the same with the standard WENO5 scheme proposed by Jiang and Shu [22];

(2) for WENOZ-Present scheme, we obtain

\[
\tau_5^* = |\beta_0^* - \beta_2^*| = \left( \frac{13}{3} \tilde{f}'' \tilde{f}'''' - \tilde{f}^5 \tilde{f}'''' + \frac{13}{3} \tilde{f}'' R''' f_3 + 4 \tilde{f}^{} R''' f_2 + \tilde{f}^{} R(4) f \right) \Delta \xi^5 + O(\Delta \xi^6),
\] (77)

whose truncation error is the same order with the standard WENOZ scheme suggested by Borges et al. [25].

Then, Eqs. (76) and (77) indicate that the orders of the non-linear weights are retained by applying the present free-stream preserving metrics and Jacobian.

Finally, we conclude that the sufficient conditions given in Eqs. (61) ~ (63) are all satisfied in the present free-stream preserving schemes. Considering that the present linear upwind scheme achieves 5th-order accuracy as well, therefore, the dissipation parts of the present 5th-order WENO reconstruction retain 5th-order accuracy in the non-critical points as the same as the standard WENO5 and WENOZ scheme.
Appendix C

For the WENO7-Present scheme, the free-stream preserving metrics and Jacobian are represented as $g_{1+1/2}^*, g_{i-3}^*, \ldots, g_{i+4}^*$. We define the free-stream preserving metrics and Jacobian as

$$\begin{aligned}
\frac{3}{12} g_{i-3}^* &= \frac{13}{12} g_{i-2}^* - \frac{23}{12} g_{i-1}^* + \frac{25}{12} g_i^* - g_{i+1/2}^*, \\
\frac{1}{12} g_{i-2}^* &= \frac{5}{12} g_{i-1}^* - \frac{13}{12} g_i^* - \frac{3}{12} g_{i+1}^* + g_{i+1/2}^*, \\
\frac{1}{12} g_{i-1}^* &= g_{i-1}, \\
\frac{1}{12} g_i^* &= g_i, \\
\frac{1}{12} g_{i+1}^* &= g_{i+1}, \\
\frac{1}{12} g_{i+2}^* &= g_{i+2}, \\
\frac{1}{12} g_{i+3}^* &= -\frac{3}{12} g_{i+1}^* - \frac{13}{12} g_{i+2}^* + \frac{5}{12} g_{i+3}^* + g_{i+1/2}^*, \\
\frac{3}{12} g_{i+4}^* &= \frac{25}{12} g_{i+2}^* - \frac{13}{12} g_{i+3}^* + \frac{13}{12} g_{i+4}^* - g_{i+1/2}^*,
\end{aligned}$$

where

$$g_{i+1/2}^* = \frac{1}{12} \left( -g_{i-1} + 7g_i + 7g_{i+1} - g_{i+2} \right).$$

Specially, the above $g_{i+1/2}^*$ is calculated by the 4th-order scheme rather than the 8th-order one, shown in Eq. (29). Therefore, it only makes the upwind dissipation rather than the central part satisfying the free-stream preserving identity. This special treatment is to reduce the approximation from the 6th-order cell-averaged metrics and Jacobian to the 4th-order ones. Specifically, only $g_{i-4}$, $g_{i-3}$, $g_{i+3}$, $g_{i+4}$ need to be approximated by the 4th-order $g_{i-4}^*$, $g_{i-3}^*$, $g_{i+3}^*$, $g_{i+4}^*$, while $g_{i-1}$, $g_i$, $g_{i+1}$ and $g_{i+2}$ are unnecessary to be replaced.

The free-stream preserving identity and the accuracy of the central part is achieved by the followings. We replace the central part of $\tilde{F}_{i+1/2}$ with a 8th-order one

$$\begin{aligned}
\tilde{F}_{i+1/2}^{(4)} &= \frac{1}{840} \left( -3\tilde{F}_{i-3}^* + 29\tilde{F}_{i-2}^* - 139\tilde{F}_{i-1}^* + 533\tilde{F}_i^* + 533\tilde{F}_{i+1}^* - 139\tilde{F}_{i+2}^* + 29\tilde{F}_{i+3}^* - 3\tilde{F}_{i+4}^* \right), \\
\tilde{F}_{i+1/2}^{(8)} &= \frac{1}{840} \left( -3\tilde{F}_{i-3}^* + 29\tilde{F}_{i-2}^* - 139\tilde{F}_{i-1}^* + 533\tilde{F}_i^* + 533\tilde{F}_{i+1}^* - 139\tilde{F}_{i+2}^* + 29\tilde{F}_{i+3}^* - 3\tilde{F}_{i+4}^* \right), \\
\tilde{F}_{i+1/2} &= \tilde{F}_{i+1/2}^{(4)} + \tilde{F}_{i+1/2}^{(8)} - \tilde{F}_{i+1/2}^{(4)},
\end{aligned}$$

where $\tilde{F}_m^*$ and $\tilde{F}_m$ are the cell-averaged fluxes calculated by the 4th- and 8th-order metrics and Jacobian, respectively. Finally, the new cell-face fluxes $\tilde{F}_{i+1/2}$ can obtain the high-order accuracy and maintain the free-stream preserving identity at the same time.
References

[1] P. D. Thomas, C. K. Lombard, Geometric conservation law and its application to flow computations on moving grids, AIAA Journal 17 (10) (1979) 1030–1037.
[2] Y. Abe, N. Iizuka, T. Nonomura, K. Fujii, Conservative metric evaluation for high-order finite difference schemes with the GCL identities on moving and deforming grids, Journal of Computational Physics 232 (1) (2013) 14 – 21.
[3] Y. Abe, T. Nonomura, N. Iizuka, K. Fujii, Geometric interpretations and spatial symmetry property of metrics in the conservative form for high-order finite-difference schemes on moving and deforming grids, Journal of Computational Physics 260 (2014) 163–203.
[4] H. Zhang, M. Reggio, J. Trepanier, R. Camarero, Discrete form of the gcl for moving meshes and its implementation in CFD schemes, Computers & Fluids 22 (1) (1993) 9–23.
[5] M. R. Visbal, D. V. Gaitonde, On the use of higher-order finite-difference schemes on curvilinear and deforming meshes, Journal of Computational Physics 181 (1) (2002) 155 – 185.
[6] T. Nonomura, N. Iizuka, K. Fujii, Freestream and vortex preservation properties of high-order weno and wcns on curvilinear grids, Computers & Fluids 39 (2) (2010) 197–214.
[7] T. Pulliam, J. Steger, On implicit finite-difference simulations of three-dimensional flow, AIAA Paper 7810.
[8] X. Deng, M. Mao, G. Tu, H. Liu, H. Zhang, Geometric conservation law and applications to high-order finite difference schemes with stationary grids, Journal of Computational Physics 230 (4) (2011) 1100 – 1115.
[9] T. Nonomura, D. Terakado, Y. Abe, K. Fujii, A new technique for freestream preservation of finite-difference WENO on curvilinear grid, Computers & Fluids 107 (2015) 242–255.
[10] M. Vinokur, H. Yee, Extension of efficient low dissipation high order schemes for 3-d curvilinear moving grids, in: Frontiers of Computational Fluid Dynamics 2002, World Scientific, 2002, pp. 129–164.
[11] X. Deng, Y. Min, M. Mao, H. Liu, G. Tu, H. Zhang, Further studies on geometric conservation law and applications to high-order finite difference schemes with stationary grids, Journal of Computational Physics 239 (2013) 90 – 111.
[12] X. Deng, H. Zhang, Developing high-order weighted compact nonlinear schemes, Journal of Computational Physics 165 (1) (2000) 22 – 44.
[13] Y. Jiang, C. W. Shu, M. Zhang, An alternative formulation of finite difference weighted eno schemes with lax–wendroff time discretization for conservation laws, Siam Journal on Scientific Computing 35 (2) (2013) A1137–A1160.
[14] A. J. Christlieb, X. Feng, Y. Jiang, Q. Tang, A high-order finite difference weno scheme for ideal magnetohydrodynamics on curvilinear meshes, SIAM Journal on Scientific Computing 40 (4) (2018) A2631–A2666.
[15] Y. Yu, Y. Jiang, M. Zhang, Free-stream preserving finite difference schemes for ideal magnetohydrodynamics on curvilinear meshes, Journal of Scientific Computing 82 (1) (2020) 23.
[16] Y. Zhu, X. Hu, Free-stream preserving linear-upwind and WENO schemes on curvilinear grids, Journal of Computational Physics 399 (2019) 108907.
[17] Q. Li, D. Sun, P. Liu, Further study on errors in metric evaluation by linear upwind schemes with flux splitting in stationary grids, Communications in Computational Physics 22 (01) (2017) 64–94.
[18] M. Vinokur, An analysis of finite-difference and finite-volume formulations of conservation laws, Journal of Computational Physics 81 (1) (1989) 1 – 52.
[19] C. W. Shu, S. Osher, Efficient implementation of essentially non-oscillatory shock-capturing schemes, Journal of Computational Physics 77 (2) (1988) 439–471.
[20] C.-W. Shu, S. Osher, Efficient implementation of essentially non-oscillatory shock-capturing schemes, II, Journal of Computational Physics 83 (1) (1989) 32–78.
[21] B. Merriman, Understanding the shu-oster conservative finite difference form, Journal of Scientific Computing 19 (1/3) (2003) p.309–322.

[22] G.-S. Jiang, C.-W. Shu, Efficient Implementation of Weighted ENO Schemes, Journal of Computational Physics 126 (1) (1996) 202–228.

[23] S. Gottlieb, C.-W. Shu, Total variation diminishing runge-kutta schemes, Mathematics of computation of the American Mathematical Society 67 (221) (1998) 73–85.

[24] C.-W. Shu, Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws, in: Advanced numerical approximation of nonlinear hyperbolic equations, Springer, 1998, pp. 325–432.

[25] R. Borges, M. Carmona, B. Costa, W. S. Don, An improved weighted essentially non-oscillatory scheme for hyperbolic conservation laws, Journal of Computational Physics 227 (6) (2008) 3191–3211.

[26] P. Woodward, P. Colella, The numerical simulation of two-dimensional fluid flow with strong shocks, Journal of Computational Physics 54 (1) (1984) 115–173.

[27] P. Spalart, S. Allmaras, A one-equation turbulence model for aerodynamic flows, in: 30th Aerospace Sciences Meeting and Exhibit, AIAA, 1992.

[28] J. Mayeur, A. Dumont, D. Destarac, V. Gleize, Reynolds-averaged navierstokes simulations on naca0012 and onera-m6 wing with the onera elsa solver, AIAA Journal 54 (9) (2016) 2671–2687.