A new renormalization approach to the Dirichlet Casimir effect for $\phi^4$ theory in (1+1) dimensions

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The next to the leading order Casimir effect for a real scalar field, within $\phi^4$ theory, confined between two parallel plates is calculated in one spatial dimension. Here we use the Green's function with the Dirichlet boundary condition on both walls. In this paper we introduce a systematic perturbation expansion in which the counterterms automatically turn out to be consistent with the boundary conditions. This will inevitably lead to nontrivial position dependence for physical quantities, as a manifestation of the breaking of the translational invariance. This is in contrast to the usual usage of the counterterms, in problems with nontrivial boundary conditions, which are either completely derived from the free cases or at most supplemented with the addition of counterterms only at the boundaries. We obtain finite results for the massive and massless cases, in sharp contrast to some of the other reported results. Secondly, and probably less importantly, we use a supplementary renormalization procedure in addition to the usual regularization and renormalization programs, which makes the usage of any analytic continuation techniques unnecessary.

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I. INTRODUCTION

During the last fifty years many papers have been written on the calculation of the Casimir energy. In this paper we introduce a new approach in regards to the renormalization program. We found it most suitable to introduce our approach in the simplest example possible, i.e. a real scalar field confined between two parallel plates in 1+1 dimensions, with $\phi^4$ self-interaction. As we shall see, our results for the next to leading order term (NLO) are finite for both the massive and massless cases and this differs significantly from what exists in the literature. It is therefore suitable to start at the beginning. In 1948 H.B.G. Casimir found a simple yet profound explanation for the retarded van der Waals interaction \[ L_4 \]. During the last fifty years many papers have been written on the calculation of the Casimir energy. In this paper we introduce a new approach in regards to the renormalization program. We found it most suitable to introduce our approach in the simplest example possible, i.e. a real scalar field confined between two parallel plates in 1+1 dimensions, with $\phi^4$ self-interaction. As we shall see, our results for the next to leading order term (NLO) are finite for both the massive and massless cases and this differs significantly from what exists in the literature. It is therefore suitable to start at the beginning. In 1948 H.B.G. Casimir found a simple yet profound explanation for the retarded van der Waals interaction \[ L_4 \]. After a short time, he and D. Polder related this effect to the change in the zero point energy of the quantum fields due to the presence of nontrivial boundary conditions \[ L_4 \]. This energy has since been called the Casimir energy.

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The zero-order energy in perturbation theory has been calculated for various fields (see for example \[ L_4 \]). Also the NLO correction, which is usually called the first-order effect, has been computed for various fields. For the electromagnetic field this correction is said to be due to the following Feynmann diagram \[ \otimes \otimes \], and has been computed first by Bordag and collaborators \[ L_4 \]. However, note that this correction is a two loop correction in this case and is $O(\epsilon^2)$. Moreover the two-loop radiative corrections for some effective field theories have been investigated in \[ L_4 \]. Next, in the case of a real massive scalar field NLO correction to the energy has been computed in \[ L_4 \]. Moreover, N. Graham et al used new approaches to this problem by utilizing the phase shift of the scattering states \[ L_4 \], or replacing the boundary conditions by an appropriate potential term \[ L_4 \]. However, the authors use the free counterterms, by which we mean the ones that are relevant to the free cases with no nontrivial boundary conditions, and are obviously position independent. Only in Ref. \[ L_4 \] the author notes that in certain cases, counterterms can depend on the distance between the plates. The first use of nontrivial boundary conditions for the renormalization programs in problems of this sort seems to be due to Fosco and Svaiter \[ L_4 \]. These authors use free counterterms in the space between the plates and place additional surface counterterms at the boundaries. Later on various authors proposed the use of exactly the same renormalization procedure for various physical problems \[ L_4 \]. The first calculation for the NLO of Casimir energy for the massive scalar field using this renormalization program is done in Ref. \[ L_4 \]. We should note that their results for the massless limit in 1+1 dimensions, like those of \[ L_4 \], is infinite. It is also worth mentioning that all the papers on the analogous calculations of the NLO corrections to the mass of solitons, that we are aware, of use free counterterms (see for example \[ L_4 \]). In references \[ L_4 \] the authors used the mode number cutoff introduced by R.F. Dashen (1974) \[ L_4 \] to calculate the NLO Casimir energy due to the presence of solitons.

In this paper, we present a systematic approach to the renormalization program for problems which are
ameable to renormalized perturbation theory, and con-
tain either nontrivial boundary conditions or nontrivial (position dependent) backgrounds, e.g. solitons, or both. Obviously all the n-point functions of the theory will have in general nontrivial position dependence in the coordinate representation. This is one of the manifestations of the breaking of the translational symmetry. The procedure to deduce the counterterms from the n-point functions in a renormalized perturbation theory is standard and has been available for over half a century. Using this, as we shall show, we will inevitably obtain position dependent counterterms. Therefore, the radiative corrections to all the input parameters of the theory, including the mass, will be in general position dependent. Therefore, we believe the information about the non-trivial boundary conditions or position dependent backgrounds are carried by the full set of n-point functions, the resulting counterterms, and the renormalized parameters of the theory. Our preliminary investigations have revealed that our position dependent counterterms approach the free ones when the distance between the plates is large. Their main difference is for positions which are about a Compton wavelength away from the walls, although it is also nontrivial at other places. Here we use this procedure to compute the first-order radiation correction to the Casimir energy for a real scalar field in 1+1 dimensions. We compute this correction for both a massive and a massless scalar fields and show that the massless limit of the massive case exactly corresponds to the massless case.

In addition, up to now all the papers on the Casimir effect, that we are aware of, use some from of analytic continuation. We share the point of view with some authors such as the ones in [14, 18] that the analytic continuation techniques are not always completely justified physically. Moreover, like the first of the aforementioned authors, we have found counterexamples, which we point out in this paper and elsewhere [29]. The counterexamples show that it alone might not yield correct physical results, and sometimes even gives infinite results [30]. Therefore, we prefer to use a completely physical approach by enclosing the whole system in a box of linear size $L$, which eventually can go to infinity, and calculating the difference between the zero point energies of two different configurations. The main idea of this method is actually due to T.H. Boyer [31], who used spheres instead of boxes. This we shall call the “box renormalization scheme” and can be used as a supplementary part of other usual regularization or renormalization programs. This box renormalization scheme, has the following advantages:

1. Use of this procedure removes all of the ambiguities associated with the appearance of the infinities, and we use the usual prescription for removing the infinities in the regulated theory, as explained in Sec. III B. This is all done without resorting to any analytic continuation schemes.

2. The infrared divergences which generically appear in these problems in 1+1 dimensions automatically cancel each other.

3. In order to calculate the Casimir energy we subtract two physical configuration of similar nature, e.g. both confined within finite regions, and not one confined and the other in an unbounded region.

4. This method can be used as a check for the cases where analytic continuation yields finite results, and more importantly, can be used to obtain finite results when the former yields infinite results.

we should mention that some authors believe that use of box regularization or renormalization procedures, in which the size of the box eventually goes to infinity could be avoided by using appropriate boundary conditions on the fields at spatial infinity [32].

In Section III we calculate the leading order term for the Casimir energy in $d$ space dimensional case. We do this first of all to explain more completely the physical content of the problem and set up our notations. Secondly this computation is just about as easy as to do in $d$ dimensions as is in the one dimensional case. In Section III we compute the first order radiative correction to this energy. In order to do this we first state the renormalization conditions, and then derive expressions for the first order radiative corrections for both the massive and massless cases. We show that the results for the massless limit of the massive case and the massless case are equal. In Section IV we summarize our results and state our conclusions.

II. THE LEADING TERM OF THE CASIMIR EFFECT

The lagrangian density for a real scalar field with $\phi^4$ self-interaction is:

$$\mathcal{L}(x) = \frac{1}{2} (\partial_{\mu} \phi(x))^2 - \frac{1}{2} m_0^2 \phi(x)^2 - \frac{\lambda_0}{4!} \phi(x)^4,$$  \hspace{1cm} (1)

where $m_0$ and $\lambda_0$ are the bare mass and bare coupling constant, respectively. Here we calculate the leading term for the Casimir energy in $d$ spatial dimensions. Obviously the leading term, in contrast to the higher order corrections, is independent of the form of the self-interaction. The Casimir energy is in general equivalent
to the work done on the system for bringing two parallel plates from $\pm \infty$ to $\pm a/2$. As mentioned before, part of our renormalization procedure is to enclose the whole system in a $d$ dimensional cubical box of sides $L$. To compute this leading term, we first compare the energies in two different configurations: when the plates are at $\pm a/2$ as compare to $\pm b/2$. We name the axis perpendicular to the plates the $z$ axis. To keep the expressions symmetrical, we choose the coordinates so that the edges of the confining box are at $\pm L/2$ in any direction.

![Diagram](image)

**FIG. 1:** The geometry of the two different configurations whose energies are to be compared. The labels $a_1, \text{etc.}$ denote the appropriate sections in each configuration separated by the plates.

The total zero point energy of the upper configuration in figure 1 will be called $E_b$ and of the lower one $E_a$. In our box renormalization scheme we need to define the Casimir energy as follows

$$E_{\text{Cas}} = \lim_{b \to a \to \infty} \left( \lim_{L \to \infty} (E_a - E_b) \right),$$

where,

$$E_a = E_{a_1} + 2E_{a_2}, \quad E_b = E_{b_1} + 2E_{b_2}. \quad (3)$$

Here we choose the Dirichlet boundary condition on the plates. Then we can expand the field operator $\varphi$ in the eigenstate basis appropriate to this boundary condition, and its explicit second quantized form, for example in region $a_1$ becomes

$$\varphi_{a_1}(x) = \int \frac{d^d k}{(2\pi)^d} \sum_{n=1}^{\infty} \left( \frac{1}{\omega_{a_1,n}} \right)^{1/2} \left\{ e^{-i(\omega_{a_1,n} t - k^+ \cdot x^+)} \sin \left[ k_{a_1,n}(z + \frac{a}{2}) \right] a_n + e^{i(\omega_{a_1,n} t - k^+ \cdot x^+)} \sin \left[ k_{a_1,n}(z + \frac{a}{2}) \right] a_n^\dagger \right\}, \quad (4)$$

where,

$$\omega_{a_1,n}^2 = m^2 + k_-^2 + k_{a_1,n}^2,$$

$$k_{a_1,n} = \frac{n\pi}{a} \quad \text{and} \quad n = 1, 2, \ldots \quad (5)$$

Here $k^\perp$ and $k_{a_1,n}$ denote the momenta parallel and perpendicular to plates (in $z$-direction), respectively. Also $a_n^\dagger$ and $a_n$ are creation and annihilation operators obeying the usual commutation relations:

$$[a_n, a_{n'}^\dagger] = \delta_{n,n'}, \quad [a_n, a_{n'}] = [a_n^\dagger, a_{n'}^\dagger] = 0,$$

and $a(0)$ defines the vacuum state in the presence of boundary conditions. Using the above equations one can easily obtain

$$E_{a_1}^{(0)} = \int d^d x (0|H^{(0)}|0) = L^{d-1} \int \frac{d^d-1 k^\perp}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \frac{\omega_{a_1,n}}{2},$$

$$= \frac{L^{d-1}}{2} \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \int_0^{\infty} dk k^{d-2} \sum_{n=1}^{\infty} \omega_{a_1,n}, \quad (6)$$

where $H^{(0)}$ denotes the usual free Hamiltonian density, easily obtained from the Lagrangian density, and the superscript (0) denotes the zero (or leading) order term of this energy. Also $k = |k^\perp|$, and $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ is the solid angle in $d$-dimensions. Therefore,

$$E_a^{(0)} - E_b^{(0)} = \frac{L^{d-1}}{2} \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \int_0^{\infty} dk k^{d-2} \sum_n g(n), \quad (7)$$

where,

$$g(n) = \omega_{a_1,n} + 2\omega_{a_2,n} - \omega_{b_1,n} - 2\omega_{b_2,n}.$$

Now we are allowed to use the Abel-Plana summation formula, since we now expect the summand to satisfy the strict conditions for the validity of this formula. That is, we expect any reasonable renormalization program for calculating any measurable physical quantity to yield finite results. The Abel-Plana summation formula gives

$$E_a^{(0)} - E_b^{(0)} = \frac{L^{d-1}}{2} \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \int_0^{\infty} dk k^{d-2} \left[ \frac{-g(0)}{2} + \int_0^{\infty} g(x) dx + i \int_0^{\infty} \frac{g(it) - g(-it)}{e^{2\pi t} - 1} dt \right], (8)$$

where $g(0)$ vanishes in this case due to our box renormalization. The second term in the bracket, using suitable changes of variables, becomes
\[
- \frac{a}{\pi} \int_0^\infty \, dk \, (m_0^2 + k^2 + \kappa^2)^{1/2} + 2 \frac{L - a}{2\pi} \int_0^\infty \, dk \, (m_0^2 + k^2 + \kappa^2)^{1/2}
- \frac{b}{\pi} \int_0^\infty \, dk \, (m_0^2 + k^2 + \kappa^2)^{1/2} - 2 \frac{L - b}{2\pi} \int_0^\infty \, dk \, (m_0^2 + k^2 + \kappa^2)^{1/2} = 0,
\]

where \( \kappa \) for example in the first term denotes \( \frac{n\pi}{a} \), obviously treated as a continuous variable. The above calculation shows that this term is exactly zero. Therefore, only the branch-cut term (the last term in Eq. (8)) gives nonzero contribution and the final result is
\[
E_a^{(0)} - E_b^{(0)} = -2 \frac{L^{d-1} - m_0^{(d+1)/2}}{(4\pi)^{(d+1)/2}} \sum_{j=1}^{\infty} \frac{1}{j^{(d+1)/2}} \times \left\{ \frac{K_{(d+1)/2} (2a jm_0)}{a^{(d-1)/2}} - \frac{K_{(d+1)/2} (2b jm_0)}{b^{(d-1)/2}} + \frac{2K_{(d+1)/2} [(L - a) jm_0]}{(L - a)^{(d-1)/2}} - \frac{2K_{(d+1)/2} [(L - b) jm_0]}{(L - b)^{(d-1)/2}} \right\}
\]

where \( K_n(x) \) denotes the modified Bessel function of order \( n \). Using Eq. (2) for the Casimir energy and noting that \( K_n(x) \) is strongly damped as \( x \) goes to infinity, only the first term remains when the limits are taken, and the result is
\[
E_{\text{Cas}}^{(0)} = - \frac{2L^{d-1} - m_0^{(d+1)/2}}{(4\pi)^{(d+1)/2}} \frac{K_{(d+1)/2} (2a jm_0)}{a^{(d-1)/2}},
\]

If we set \( d = 3 \), we have
\[
E_{\text{Cas}}^{(0)} = - \frac{2L^2 - m_0^2}{8\pi^2 a} \sum_{j=1}^{\infty} \frac{K_2 (2a jm_0)}{j^2},
\]

with the following limits,
\[
E_{\text{Cas}}^{(0)} \to \left\{ \begin{array}{l}
- \frac{L^2}{8\pi^2 a} \sum_j \frac{1}{2a^2 j^4} \frac{1}{1440 a^3} \quad \text{as } m_0 \to 0 \\
- \frac{L^2}{8\pi^2 a} \frac{m_0}{4\pi a} e^{-2a m_0} \frac{1}{16} \quad \text{as } a m_0 \gg 1.
\end{array} \right.
\]

The results are in agreement with what exists in literature (see for instance \[3, 18\]). It is interesting to note that if we solve the massless case exactly the branch-cut terms simplify to give
\[
E_{\text{Cas}}^{(0)} = -\frac{\pi}{a} \int_0^\infty \frac{t}{e^{2\pi t} - 1} \, dt = -\frac{1}{4\alpha \pi} \zeta (2) = -\frac{\pi}{24a}
\]

which is identical with our result for the massless limit, and is reported for example, in Refs. \[33\].

III. FIRST-ORDER RADIATIVE CORRECTION

Now we calculate the next to the leading order (two loop quantum correction) shift of the Casimir energy for a scalar field in \( \phi^4 \) theory using the renormalized perturbation theory in \( 1 + 1 \) dimensions. As mentioned before, the main idea of our work is that when a systematic treatment of the renormalization program is done, the counterterms needed to retain the renormalization conditions, automatically turn out to be position dependent. This, as we shall see, will have profound consequences. However, our main scheme of canceling the divergences using counterterms and a few input experimental parameters, is in complete conformity with the standard renormalization approach. To set the stage for the calculations, we shall very briefly state the renormalization procedure and conditions.

A. Renormalization Conditions

The \( \phi^4 \) Lagrangian Eq. (1), after rescaling the field \( \varphi = Z^{1/2} \varphi_r \), where \( Z \) is called the field strength renormalization, and the standard procedure for setting up the renormalized perturbation theory, becomes (see for
expression for the two-point function is, in the configuration space. For example, the standard expression for the two-point function in its renormalized form is

\[ \langle \Omega | T \{ \phi(x_1) \phi(x_2) \} | \Omega \rangle = \lim_{T \to \infty (1-\epsilon i)} \frac{\langle 0 | D \phi(x_1) \phi(x_2) e^{\int_{-T}^{T} L dt} | 0 \rangle}{\langle 0 | D \phi e^{\int_{-T}^{T} L dt} | 0 \rangle} \tag{17} \]

Since the birth of quantum field theory, as far as we know, the assertion has always been that the above expressions can be expanded systematically when the problem is amenable to perturbation theory. For example, in the context of renormalized perturbation theory, as indicated in Eq. (16), we can symbolically represent the first few terms of the perturbation expansion of Eq. (17) by

![Diagram](image)

where \( \overline{\bullet} \) refers to the appropriate counterterm. It is obvious that the above expression represents a systematic perturbation expansion, and most importantly, all of the propagators on the right hand side should be the one appropriate to the problem under consideration, that is they should have the same overall functional form as the first term. Our first renormalization condition is that the renormalized mass \( m \) should be the pole of the propagator represented by the first term in (18). This implies the second and third diagrams should cancel each other out in the lowest order, and this in turn implies the cancelation of the UV divergences in that order, and that the counterterms will in general turn out to be position dependent. The renormalized mass \( m \) will naturally turn out to be position dependent as well. However, we only need to fix the value of \( m(x) \) at one position between the plates by our renormalization condition. The exact functional dependence of \( m(x) \) will then be completely determined by the theory. That is, we insist the overall structure of the renormalization conditions such as above, and the counterterms appearing in them should be determined solely from within the theory, and not for example be imported from the free case. The equations are self deterministic and there is no need to take such actions. Obviously we still need a few experimental input parameters for the complete renormalization program, such as \( m(x) \) for some \( x \). Analogous expression and reasonings could be easily stated for the four-point function.

To one-loop order the renormalization conditions derived from Eq. (18) and its four-point counterpart, are

\[ \delta_Z(x) = 0, \quad \delta_m(x) = -\frac{i}{\lambda} \frac{\delta m}{2 \lambda} G(x, x); \]

and

\[ \delta_\lambda(x) = 0, \tag{19} \]

respectively. Here \( G(x, x') \) is the propagator of the real scalar field and \( x = (t, z) \). Obviously the counterterms automatically incorporate the boundary conditions and are position dependent, due to the dependence of the two and four-point functions on such quantities. Now, the higher order contributions to the vacuum energy in the interval \( a1 \) (i.e. \( z \in \left[ \frac{a}{2}, \frac{a}{2} \right] \)) is

\[ \Delta E_{a_1} = E_{a_1}^{(1)} + E_{a_1}^{(2)} + \cdots = \int_{-a/2}^{a/2} dz \langle \Omega | \mathcal{H}_s | \Omega \rangle \]

\[ = i \int_{-a/2}^{a/2} dz \left( \frac{1}{8} + \frac{1}{2} + \frac{1}{8} + \cdots \right) \tag{20} \]

where \( \overline{\times} = -i \delta_\lambda(x) \) and \( \overline{\o} = i \mu^2 \delta_Z(x) - \delta_m(x) \) refer to the counterterms. Accordingly, the \( O(\lambda) \) contribution to the vacuum energy is

\[ E_{a_1}^{(1)} = i \int_{-a/2}^{a/2} dz \left( \frac{1}{8} + \frac{1}{2} \right) \]

\[ = i \int_{-a/2}^{a/2} dz \left[ \frac{-i \lambda}{8} G_{a_1}(x, x) - \frac{i}{2} \delta_m(x) G_{a_1}(x, x) \right], \tag{21} \]

where \( G_{a_1}(x, x') \) is the propagator of the real scalar field.
in region \(a_1\). Using Eqs. (19) and (21), we obtain

\[
E_{a_1}^{(1)} = -\frac{\lambda}{8} \int_{-a/2}^{a/2} G_{a_1}^2(x, x) dz. \tag{22}
\]

### B. The Massive Case

As mentioned before, here we choose the Dirichlet boundary condition on the plates. Then, after the usual Wick rotation, the expression for the Green’s function in the two dimensional Euclidean space becomes

\[
G_{a_1}(x, x') = \frac{2}{a} \int \frac{d\omega}{2\pi} \sum_n \sin \left[ k_{a_1, n}(z + \frac{a}{2}) \right] \sin \left[ k_{a_1, n}(z' + \frac{a}{2}) \right]. \tag{23}
\]

Using Eq. (23) and Eq. (22) and Carrying out the integration first over the space and then over \(\omega\), one obtains

\[
E_{a_1}^{(1)} = -\frac{\lambda}{8} \left[ \frac{\pi^2}{a} \sum_n \omega_{a_1, n}^{-1} \right]^2 + \frac{\pi^2}{2a} \sum_n \omega_{a_1, n}^{-2} \right] \\
= -\frac{\lambda \pi^2}{8a} \left[ \left( \sum_{n=1}^{\infty} \frac{1}{\sqrt{\frac{\pi^2 n^2}{a^2} + m^2}} \right)^2 + \frac{am \coth am - 1}{4m^2} \right]. \tag{24}
\]

According to Eq. (2) and Eq. (3) we have for the NLO correction

\[
E_{\text{cor}}^{(1)} = \lim_{b/a \to \infty} \left[ \lim_{L/b \to \infty} \left( E_a^{(1)} - E_b^{(1)} \right) \right],
\]

where

\[
E_a^{(1)} = E_{a_1}^{(1)} + 2E_{a_2}^{(1)}, \quad E_b^{(1)} = E_{b_1}^{(1)} + 2E_{b_2}^{(1)}.
\] \tag{25}

This computation is obviously complicated and plagued with a multitude of infinities. As explained before using the usual renormalization programs in conjunction with our box renormalization scheme, should eliminate all of the infinities, as might be apparent from the above equation. However, proper regularization schemes should still be implemented and proper care taken when handling these infinite expressions. For example, the summation appearing in the squared form in the first term of the last part of Eq. (24) is infinite. We want to use the Abel-Plana formula to convert this sum into an integral. However this sum does not satisfy the stringent requirements stated in the Abel-Plana theorem for such a conversion \[40, 41\]. However our box renormalization scheme provides a solution: We first expand the square as a double sum. Then we subtract these double sums as indicated in Eq. (25). Now we can expect this new summand to satisfy the requirements for the Abel-Plana theorem. Then all the infinities actually cancel and the result for the two-loop correction reduces to (see Appendix for details):

\[
E_a^{(1)} - E_b^{(1)} = \frac{-\lambda \pi^2}{8} \left[ f(a) - f(b) + 2f\left( \frac{L - a}{2} \right) - 2f\left( \frac{L - b}{2} \right) + \frac{2}{\pi} \left( B(a) - B(b) + 2B\left( \frac{L - a}{2} \right) - 2B\left( \frac{L - b}{2} \right) \right) \int_0^\infty \frac{ds}{\sqrt{1 + s^2}} \right], \tag{26}
\]

where

\[
f(a) = B(a) \left( \frac{B(a)}{a} - \frac{1}{am} \right) + \coth(4m), \tag{27}
\]

and \(B(a)\), defined by the following expression

\[
B(a) = 2 \int_{\frac{1}{2a}}^\infty \frac{1}{dt} - 1 \sqrt{\frac{t^2}{a^2} - m^2}. \tag{28}
\]

refers to the so called branch-cut term in the Abel-Plana summation formula and is a finite quantity. Note that the last integral in Eq. (26) seems to diverge so it must be properly regularized, and this crucially depends on our box renormalization program, as we shall explain below.

However, before we engage in this calculation, we want to raise an important point: If we were to use the free counterterm in Eq. (21), as is routinely done, this term would be absent, in addition to some minor differences. Therefore, one would easily obtain finite results which we like to dispute. Now let us proceed with our calculations. We prefer to use a regularization scheme for this integral term which is analogous to the zeta function regularization for the sums. That is, in that expression we set the power of the integrand to \(-\frac{1}{2} + \alpha'\) for the first two terms and \(-\frac{1}{2} + \alpha'\) for the remaining terms. In the final stage
we let $\alpha$ and $\alpha'$ approach zero. Hence we will have
\[
2 \frac{B(a) - B(b)}{\pi} \int_0^\infty (1 + s^2)^{-\frac{1}{2} + \alpha} \, ds \\
+ 4 \frac{B(\frac{L-a}{2}) - B(\frac{L-b}{2})}{\pi} \int_0^\infty (1 + s^2)^{-\frac{1}{2} + \alpha'} \, ds
= \frac{B(a) - B(b)}{\sqrt{\Gamma(\frac{1}{2} + \alpha)}} \Gamma(-\alpha) + 2 \frac{B(\frac{L-a}{2}) - B(\frac{L-b}{2})}{\sqrt{\Gamma(\frac{1}{2} + \alpha')}} \Gamma(-\alpha').
\]

For $\alpha$ and $\alpha'$ sufficiently small, this expression becomes
\[
2 \frac{B(a) - B(b)}{\pi} \left( \frac{-1}{2\alpha} + \ln 2 \right) \\
+ 4 \frac{B(\frac{L-a}{2}) - B(\frac{L-b}{2})}{\pi} \left( \frac{-1}{2\alpha'} + \ln 2 \right). \quad (29)
\]

Now, if $\frac{\alpha'}{\alpha} = -2 \frac{B(\frac{L-a}{2}) - B(\frac{L-b}{2})}{B(a) - B(b)}$ the infinities cancel.

The cancelation of these divergent quantities without any residual finite terms is the usual prescription in regulated theories and this is what we have used. Therefore the term in question becomes,
\[
\frac{2}{\pi} \ln 2 \left( B(a) - B(b) + 2B\left(\frac{L-a}{2}\right) - 2B\left(\frac{L-b}{2}\right) \right). \quad (30)
\]

This result is obviously finite, and we believe it could not have been obtained with any regularization or analytic continuation schemes in common use, other than our box renormalization program. Thus, Eq. (29) becomes
\[
E_a^{(1)} - E_b^{(1)} = \frac{-\lambda \pi^2}{8} \left[ f'(a) - f'(b) + 2f'(\frac{L-a}{2}) - 2f'(\frac{L-b}{2}) \right], \quad (31)
\]

where $f'(a) = f(a) + \frac{2\ln 2}{\pi} B(a)$. This is the two-loop radiative correction for the work done on the plates (or two points in this case) while moving them from $(\frac{a}{2}, \frac{b}{2})$ to $(\frac{a}{2}, \frac{b}{2})$. Now, in order to compute the Casimir energy, proper limits must be taken, as indicated in Eq. (25).

Two particular limits are interesting to calculate. First is the large mass limit. To calculate this limit it is convenient to make the following expansion in the expression for $B(a)$, Eq. (28),
\[
\frac{1}{e^{2\pi t} - 1} = \sum_{j=1}^\infty e^{-2\pi t j}, \quad (32)
\]

then integration yields
\[
B(a) = \frac{2a}{\pi} \sum_{j=1}^\infty K_0(2amj) \quad \text{as } j \to \infty \quad \sqrt{\frac{\pi}{2}} \left\{ e^{-2amj} \right\}. \quad (33)
\]

Using Eq. (33) and Eq. (31), Eq. (25) gives

\[
E_{\text{Cas}}^{(1)} \to \frac{am}{4} \lim_{\lambda \to 0} \frac{-\lambda \ln 2}{4} \ln \left[ \frac{\sqrt{\pi} m e^{-2am} - \sqrt{\frac{b\pi}{m} e^{-2bm}}}{\sqrt{\pi} + \frac{b\pi}{m} e^{-(L-b) m}} \right] \\
+ 2 \sqrt{\frac{(L-a)\pi}{2m}} e^{-(L-a)m} - 2 \sqrt{\frac{(L-b)\pi}{2m}} e^{-(L-b)m} \right] \}
= \frac{-\lambda \ln 2}{4} \sqrt{\frac{\pi}{am}} e^{-2am}. \quad (34)
\]

In the small mass limit it is easier to rewrite an expression for $B(a)$, such that its integrand appears in dimensionless form,
\[
B(a) = \frac{2a}{\pi} \int_1^\infty \frac{dt}{e^{2amt} - 1} \quad (35)
\]

Then by expanding the integrand in Eq. (35) one finds,
\[
B(a) \to \frac{a}{2m} \quad \int_1^\infty \frac{dt}{\sqrt{t^2 - 1}} = \frac{1}{2m} - aS.
\]

Note the explicit appearance of infrared divergences in this equation which is a generic feature of these problems in 1+1 dimensions (35). In this limit the first term in Eq. (31) due to region a1, for example, becomes
\[
a \left[ \left( \frac{1}{2am} - S \right) \left( \frac{1}{2am} - S \right) - \frac{1}{am} + \frac{2}{\pi} \ln 2 \right] + \frac{1}{4(am)^2}
= a(S^2 - \frac{2\ln 2}{\pi} S + \frac{\ln 2}{\pi m}). \quad (36)
\]

Hence, taking into account the analogous contributions from the other regions, i.e. b1, a2 and b2, Eq. (25) gives,
\[
E_{\text{Cas}}^{(1)} \to 0 \quad \text{as } m \to 0. \quad (37)
\]

This shows that NLO for the Casimir energy in the massless limit is zero. Most importantly the infrared divergences have also cancel completely using our regularization program. This is in sharp contrast to the analogous

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1 One may argue that ambiguities always exist in problems where one has to subtract infinite quantities, and the Casimir problems certainly fall into this category. Two methods are in common use: First is the analytic continuation techniques which, although usually yield correct results, do not have a very solid physical justification and also sometimes yield infinite results. Second is the regularization schemes, which is what we have used. In the latter category when the problem is regularized, one can make a systematic expansion of the quantities in question in terms of the regulators. Then the terms which tend to infinity when the regulators are removed and the finite terms naturally appear separately. See for example Eq. (29). What is almost invariably done is to adjust the regulators so that the singular terms exactly cancel each other, i.e. without extracting any extra finite piece from the difference between the infinite quantities (see for example [13]). This is also apparent in the leading term for the Casimir energy in Eq. (3) where, as explained in the Appendix, the four changes of variables are equivalent to choosing four different cutoffs. One could have adjusted them so that as usual the infinities cancel, but any finite term would remain. However, the well known answer is obtained only when there is no remaining extra finite term in this subtraction scheme. This is the prescription that we have used. However, we do believe that this is a subject that needs further study.
result that can be extracted from Refs. [20, 25] which is infinite.

C. The Massless Case

In the massless case it is sufficient to set the pole of the propagator to zero, i.e. one can set \( m = 0 \) in the

\[
E^{(1)}(a) = \frac{-\lambda}{8a} \left( \int d\omega \frac{a\omega \coth(a\omega) - 1}{2\omega^2} \right)^2 + \frac{1}{2} \int d\omega' d\omega \frac{\omega'^2 - \omega^2 - a\omega'\omega^2 \coth(a\omega') + a\omega'^2 \omega \coth(a\omega)}{2\omega'^2 \omega^2 (\omega'^2 - \omega^2)}
\]

(38)

where,

\[
P_1 = \left( \int dp \frac{p \coth(p) - 1}{2p^2} \right)^2,
\]

\[
P_2 = \int dp' dp \frac{p'^2 - p^2 - p'^2 p \coth(p') + p^2 p \coth(p)}{4p'^2 p^2 (p'^2 - p^2)}
\]

where \( p = a\omega \) and \( p' = a\omega' \).

Therefore,

\[
E^{(1)}_{\text{Cas}} = \frac{-\lambda}{8} \left[ a - b + \frac{L - a}{2} - 2 \frac{L - b}{2} \right] P_2 = 0,
\]

(40)

since \( P_1 \) and \( P_2 \) are independent of \( a, b \) and \( L \). This result is in exact agreement with the small mass limit calculated in previous subsection. In figure 2 we show our results for the zero and first order Casimir energies for the massive and massless cases.

Note that we are explicitly assuming \( \delta_{m=0}(x) \neq 0 \). Although this does not happen to make any difference in 1+1 dimensions, using our prescription, we like to stress that this quantity should not in general be a priori set to zero. This is in contrast to the view expressed in for example Refs. [36, 37, 38, 39]. This is yet another important counterexample for the validity of analytic continuation: As is well known the massless limit of the analytic continuation of the mass counterterm in \( \phi^4 \) theory is zero for space-time dimension bigger than two. However one cannot renormalize the massless theory without the mass counterterm (see for example [34]).

IV. CONCLUSIONS

We have introduced a new concept in this paper. We have insisted that the renormalization program should completely and self-consistently take into account the boundary conditions or any possible nontrivial background which break the translational invariance of the system. We have shown that the problem is self-contained and the above program is accomplishable. To be more specific, there should be no need to import counterterms from the free theory, or even supplementing them with the attachment of extra surface counterterms, to remedy the divergences inherent in this theory. In general this breaking of the translational invariance, is reflected in the nontrivial position dependence of all the n-point functions. As we have shown this has profound consequences. For example in the case of renormalized perturbation theory, the counterterms and hence the radiative corrections to parameters of the theory, i.e. \( m \) and \( \lambda \), automatically turn out to be position dependent. In this regard we disagree with authors who use the former counterterms (see the Introduction for actual references). Obviously we still need a few experimental input param-
eters for the complete renormalization program, such as \( m(x) \) for some \( x \). However, the interesting point is that the theory then completely determines \( m(x) \).

Secondly we have used a supplementary renormalization scheme, which is originally due to T.H. Boyer \[31\], along side the usual renormalization program. In computations of these sorts, there usually appears infinities which can sometimes be removed by the usual renormalization programs that often contain some sort of analytic continuation. These procedures are sometimes ambiguous. Our scheme is simply to confine the whole physical system in a box, and to compute the difference between the values of the physical quantity in question in two different configurations. Use of this procedure removes all of the ambiguities associated with the appearance of the infinities, and we use the usual prescription for removing the infinities in the regulated theory. Moreover all of the infrared divergences cancel each other. Using our method, we have computed the zero and first order radiative correction to the Casimir energy for the massive and massless real scalar field in 1+1 dimensions. For the zero order, our results are identical with what exists in the literature. However, our first order results are markedly different from those reported in Refs. \[19, 25\]. Our results for the massive case is different from theirs due to the aforementioned conceptual differences. Moreover, we disagree with their results for the massless case obtained as the limit of the massive case, which is infinite in 1+1 dimensions. Our analogous result is zero. The authors refer to the “exact” results obtained in Refs. \[36, 37\] as the limit of the massive case, which is infinite in 1+1 dimensions. For the zero order, our results are identical with what exists in the literature. However, our first order results are markedly different from those reported in Refs. \[19, 25\]. Our results for the massive case is different from theirs due to the aforementioned conceptual differences. Moreover, we disagree with their results for the massless case obtained as the limit of the massive case, which is infinite in 1+1 dimensions. Our analogous result is zero. The authors refer to the “exact” results obtained in Refs. \[36, 37\] as a verification of their massless limits in 3+1. However, in the latter references the authors effectively set \( \delta_m \) equal to zero in their massless cases, or its equivalent. This is our second main difference in approach to the problem. As mentioned before, we believe that \( \delta_m \) should not be arbitrarily set to zero even in the massless case, since in that case the renormalization conditions can no longer be fully implemented, although the theory is still in principle renormalizable.

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APPENDIX

In this appendix we present the details of the calculations leading to Eq. \[20\]. The Abel-Plana summation formula (see for example \[41\]) is:

\[
\sum_{n=1}^{\infty} f(n) = -\frac{f(0)}{2} + \int_0^\infty f(x)dx + i \int_0^\infty \frac{f(it) - f(-it)}{e^{\pi t} - 1}dt.
\]

(A.1)

In order to obtain first-order radiative correction, just as we discussed in section \[2\] for leading term, we need to compute \( E_a^{(1)} - E_b^{(1)} \). Using Eq. \[24\], we get

\[
E_a^{(1)} - E_b^{(1)} = -\frac{\lambda \pi^2}{8} \left\{ \frac{am \coth am - 1}{4am^2} - \frac{bm \coth bm - 1}{4bm^2} + \frac{(L-a) \coth \left( \frac{L-a}{2} \right) - 1}{2(L-a)m^2} - \frac{(L-b) \coth \left( \frac{L-b}{2} \right) - 1}{2(L-b)m^2} \right. \\
+ \left. \sum_{n,n'} \left\{ \frac{1}{a} S(a, n) S(a, n') - \frac{1}{b} S(b, n) S(b, n') + \frac{4}{L-a} S\left( \frac{L-a}{2}, n \right) S\left( \frac{L-a}{2}, n' \right) - \frac{4}{L-b} S\left( \frac{L-b}{2}, n \right) S\left( \frac{L-b}{2}, n' \right) \right\} \right\},
\]

(A.2)

where \( S(a, n) = \left( m^2 + \frac{n^2 \pi^2}{a^2} \right)^{-1/2} \). Using the Abel-Plana formula Eq. \[A.1\], and simple changes of variables in the
integrals, we obtain

\[ E^{(1)}_a - E^{(1)}_b = -\frac{\lambda \pi^2}{8} \left\{ \frac{am \coth am - 1}{4am^2} - \frac{bm \coth bm - 1}{4bm^2} + \frac{(L - a) \coth \left(\frac{L-a}{2}\right)m - 1}{2(L-a)m^2} - \frac{(L - b) \coth \left(\frac{L-b}{2}\right)m - 1}{2(L-b)m^2} \right\} \]

\[ + \sum_n \left\{ - \frac{S(a, n)}{2am} + \frac{S(b, n)}{2bm} - \frac{2S(L-a, n)}{(L-a)m} + \frac{2S(L-b, n)}{(L-b)m} \right\} \]

\[ + \frac{1}{\pi} \left( S(a, n) - S(b, n) + 2S(L-a, n) - 2S(L-b, n) \right) \int_0^\infty \frac{ds}{\sqrt{m^2 + s^2}} \]

\[ + \frac{B(a)S(a, n)}{a} - \frac{B(b)S(b, n)}{b} + \frac{2B(L-a, n)}{L-a} - \frac{2B(L-b, n)}{L-b} \right\} \}

Using Eq. (A.1) again, and making appropriate changes of variables to make the integrals dimensionless, all the actual infinities cancel and we finally obtain

\[ E^{(1)}_a - E^{(1)}_b = -\lambda \pi^2 \left[ f(a) - f(b) + 2f(L-a, 2) - 2f(L-b, 2) + \frac{2}{\pi} \left( B(a) - B(b) + 2B(L-a, 2) - 2B(L-b, 2) \right) \right] \int_0^\infty \frac{ds}{\sqrt{1 + s^2}} \]

(A.3)

It is important to note that all these cancelations are easily accomplished using our supplementary box renormalization scheme. On a minor note, it is interesting to note that the changes of the variables leading to the cancelation of infinities are, surprisingly, equivalent to setting different cutoff regularizations on the upper limits of the integrals. Equation (A.3) is our main equation for the NLO Casimir energy, and appears in the text as Eq. (26), and is analyzed further there.
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