Theoretical Derivation and Experimental Study of Liquid Equilibrium Shapes under Different Rotation Modes

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ABSTRACT: Research shows that the surface shape of rotary liquid depends on the rotation mode. Mode A is that when the container wall rotates the liquid, the rotating liquid surface is paraboloid. Mode B is that when the rotor in the center of the container rotates the liquid, the rotating liquid surface is vortex. Based on the paraboloid formed by the mode A, the identity between the liquid level parameter and the wall slope $K$ ($K \neq 0$) is derived. When $K \rightarrow \infty$, with the increase of the container angular spin rate, the liquid level parameter changes are infinite, the liquid level change and volume relationship are fixed. When $K > 0$, the container is a cylinder with a large upper part and a small lower part and the liquid level parameter changes are limited, and the limit ratio between the liquid level parameters is $\sqrt{3} + 1$. In addition, through the vortex experiment by the mode B, it is concluded that the vortex curve can be regarded as composed of three parabolas: the center triggering part, the rising part, and the edge attenuation part. Different from the mode A, the liquid level change and volume relationship caused by the vortex formed by the mode B are both variables. According to the experimental results, the influences of container inner diameter, initial liquid level, rotor size, and rotor speed on the vortex characteristics are discussed in detail. At the same time, based on the experiment, the liquid level change and volume relationship caused by the formation of the vortex are deduced under the ideal condition when a stable liquid surface is formed by the vortex.

1. INTRODUCTION

The shape and characteristics of liquid rotation have been the focus of research by scholars. The shape of the equilibrium liquid surface formed by liquid rotation driven by the vessel wall is different from that of liquid rotation driven by the vessel center; the equilibrium liquid surface of the former is a regular paraboloid, on the basis of which a large number of research results have emerged: Piva1 studied the liquid rotation driven by the rotation of the bottom of a cylindrical container; Brøns et al.,2 Jacono et al.,3 and Serre and Bontoux4 used experimental and numerical methods to analyze the rotating flow with a free surface in a cylindrical vessel; Hinch et al.5 and Benilov et al.,6 respectively, studied the motion characteristics of the liquid in the rotating container and analyzed the flow field of this kind of rotation from the microscopic point of view; Yi et al.7 studied the dynamic characteristics of pellets entering a rotating liquid; Ye et al.8 derived the free-surface equation of the liquid driven by the wall theoretically and pointed out that the volume of the projectile is one-half of that of the cylinder with the same bottom and the same height; Zhou9 studied the shape of the rotating liquid in a toroidal vessel under weightlessness, and the integral equation of the liquid surface shape is derived under the condition that the extreme point of the liquid surface is known; An10 analyzed the shape of the rotating liquid surface driven by the container wall or the center of the container, respectively; Wan et al.11 analyzed the attenuation characteristics of the...
rotating liquid under the action of boundary resistance; the constitutive equation considering the roughness of the side wall and the radius of the container is established; and Chai et al.\textsuperscript{12} studied the mixing phenomenon of the stratified liquid from the perspective of spinning up and spinning down of the liquid as it rotates. Other scholars widely used the paraboloid geometry and optical properties of the rotating liquid surface in physical experiments such as measurements of gravitational acceleration and the refractive index of liquids and the study of liquid viscosity and optical reflectivity.\textsuperscript{13–17} However, most of the existing articles, experiments, and simulations only consider the liquid rotation equilibrium under the straight wall, and it is not clear whether the general law is still applicable when the wall slope changes, only Menker and Herczynski\textsuperscript{18} studied the rotating liquid in different geometric shape (cylindrical, rectangular, spherical, and conical) containers and found that there exists a critical angular velocity in the cone that can break the limit.

Meanwhile, the shape of the liquid surface formed by the stable rotation of the liquid driven by the center of the container is in the form of vortex. Liu and Li\textsuperscript{19} derived the analytical formula of the liquid level in this state from the perspective of mathematical differentiation and gave two different forms of free-liquid level. An\textsuperscript{20} also pointed out that if the liquid is rotated by the center and the outer boundary is stationary and far, the surface of the liquid is a vortex. Yamamoto\textsuperscript{20} studied the formation mechanism of surface vorticity in stirred vessels. Yang et al.\textsuperscript{21} studied the flow characteristics of water and oil in free-surface vortices with oil slicks on the water surface. Different from the paraboloid formed by the steady rotation of the container wall, the characteristics and specific laws of the vortices have not been discussed by scholars, the geometrical relations between the parameters of the liquid surface during the formation of the vortices have not been pointed out, and the previous research results are limited to theoretical deduction; there is no relevant experiment for reference or verification to form a unified, comprehensive view on the swirling liquid level.

In this article, the identity of rotation driven by a container with a wall slope $K$ is derived, and the relation of its parameters is studied in detail. The universal law is derived using theory and combined with simulation results obtained using FLUENT. Meanwhile, a magnetic stirrer is used to drive the rotor rotation in the center of the container to generate a vortex. By qualitatively changing the parameters such as the container inner diameter, the initial liquid level, the rotor size, the rotor speed, etc., the relationship between the characteristics of the vortex and these parameters is discussed, and the experimental vortex scatter plot is fitted to the mathematical function; the results show that the vortex curve can be regarded as a parabola consisting of the center triggering part, the rising part, and the edge attenuation part. Finally, on the basis of the experiment, according to the vortex formed under ideal conditions, the liquid level change and volume relationship caused by the formation of the vortex were deduced, respectively. The study is instructive to the applied experiment teaching.

## 2. RESULTS AND DISCUSSIONS

This section includes two parts. The first part is the theoretical derivation and simulation of the liquid rotation driven by the container wall; in this part, the paraboloid equation of rotating liquid is derived from a cylindrical container with an arbitrary slope $K$ ($K \neq 0$), and the identity between the liquid level parameters and the slope $K$ is obtained. Two cases of slope $K \propto$ and slope $K > 0$ are fully discussed. Finally, FLUENT software is used to further simulate the situation when the slope $K > 0$.

The second part is the experimental and theoretical derivation of liquid rotation driven by the rotor in the center of the container. The influence of the initial liquid level, inner diameter, rotor speed, and rotor size on vortex formation is summarized. It is concluded that the vortex can be seen to consist of three parabolas, and the geometrical relation of vortex formation under ideal conditions is discussed.

### 2.1. Theoretical Derivation of the Rotation Mode A.

#### 2.1.1. Derivation of the Paraboloid Equation for Rotating Liquid.

The $K$-sloped cylindrical container ($K \neq 0$) has a certain amount of liquid, and the container rotates at the center axis with an angular velocity $\omega$; due to friction and liquid viscosity, the liquid layer in contact with the container wall will rotate, so that all the liquid mass points are rotated around the shaft. After the movement is stable, each mass point has the same angular velocity, and the liquid surface forms as a funnel shape, as shown in Figure 1. Taking the lowest point of the rotating liquid level as the coordinate origin $O$ and $Ox$ as the horizontal and $Oy$ as the central symmetry axes, a right-angle coordinate system $Oxy$ is established. When the rotating liquid reaches a steady state, the angle between the liquid surface and the horizontal axis is $\theta$, and the mass unit $m$ of the liquid surface is equilibrated in the tangent direction under the action of gravity ($F_L$) and centrifugal force ($F_c$). The equation is

$$m \sin \theta = m \omega^2 x \cos \theta$$

that is,

$$\tan \theta = dy/dx = \frac{\omega^2 x}{g}$$

The integral of eq 2 is obtained:

$$\int dy = \int \frac{\omega^2}{g} x dy$$

$$y = \frac{\omega^2 x^2}{2g} + C$$

Equation 4 proves that when a cylindrical container with a $K$ slope rotates at a uniform angular velocity around the central axis, its free-liquid level is a rotating parabola, which is consistent with the law of rotation of the straight cylinder wall.

For eq 4, the slope of any point on the free-liquid level can express $K$ as

$$K = \frac{\omega^2 x}{g}$$

![Figure 1. Force analysis of the rotating free-level particle.](image)
From eq 5, it is known that the slope of a parabola increases with the increase of angular velocity, and the slope \( x_{\text{max}} \) at the edge of the parabola is the largest.

2.1.2. The Identity of the Wall Slope \( K \) Based on a Rotating Paraboloid. When the slope of the wall of the cylinder containing liquid is \( K (K \neq 0) \), the cylinder rotates around at a certain angular velocity \( \omega \), as shown in Figure 2. Assuming that the minimum rotating liquid level is always higher than the bottom of the barrel, the container height is high enough to prevent overflow. Since the \( K \) value has no effect on the derivation process of the general equation, in order to facilitate the study, only the case of \( K > 0 \) is discussed, and the relationship between the parameters is further explored.

In Figure 2, the highest liquid level is \( H \) and the initial liquid level is \( h \), the radius of the horizontal cross-section of the liquid at the origin \( O \) of the coordinates is \( r \), and the coordinates of the highest point \( m \) of the parabolic surface is \((\frac{H}{K} + r, H)\). Setting the parabolic equation as \( y = ax^2 \) (\( a \) is parameters), it is not hard to find eq 6:

\[
y = \left(\frac{H}{K} + r\right) x^2
\]  

(6)

In Figure 2, the volume \( V_p \) of a parabola above the x-axis that rotated about the y-axis can be expressed as

\[
V_p = \int_0^H \pi x^2 \, dy = \int_0^H \pi \frac{(\frac{H}{K} + r)^2}{H} \, dy = \frac{1}{2} \pi \left(\frac{H}{K} + r\right)^2 H
\]  

(7)

The volume of the platform with \( H \) height above the x-axis \( (V_H) \) is as follows:

\[
V_H = \frac{1}{3} \pi H \left(\frac{H}{K} + r\right)^2 + r^2 + \left(\frac{H}{K} + r\right) r
\]  

(8)

Volume of liquid above the x-axis before rotation \( (V_h) \) is as follows:

\[
V_h = \frac{1}{3} \pi H \left(\frac{h}{K} + r\right)^2 + r^2 + \left(\frac{h}{K} + r\right) r
\]  

(9)

As the volume of the liquid above the x-axis before and after rotation is constant, it can be obtained as

\[
V_H - V_h = V_p
\]  

(10)

The substitution of eqs 7–9 is simplified:

\[
6r^2h + \frac{H^3}{K^2} + \frac{6hr^2}{K} - 3rH = 0
\]  

(11)

Equation 11 is the derived identity. When the slope of the rotating container is \( K (K \neq 0) \), the parameters \( r, h, H, K \) always satisfy the above relationship.

When the wall slope \( K \to \infty \), the container becomes a straight wall, as shown in Figure 3. From eqs 7–9,

\[
\lim_{K \to \infty} V_p = \frac{1}{2} \pi \left(\frac{H}{K} + r\right)^2 H = \frac{1}{2} \pi r^2 H
\]  

(12)

\[
\lim_{K \to \infty} V_H = \frac{1}{3} \pi H \left(\frac{H}{K} + r\right)^2 + r^2 + \left(\frac{H}{K} + r\right) r = \pi r^2 H
\]  

(13)

\[
\lim_{K \to \infty} V_h = \frac{1}{3} \pi H \left(\frac{h}{K} + r\right)^2 + r^2 + \left(\frac{h}{K} + r\right) r = \pi r^2 h
\]  

(14)

Since the liquid volume above the x-axis is unchanged due to the rotation, the following equation is obtained:

\[
V_H - V_h = V_p
\]

that is,

\[
\pi r^2 H - \pi r^2 h = \frac{1}{2} \pi r^2 H
\]  

(15)

simplified to

\[
H = 2h
\]  

(16)

or

\[
H - h = h
\]  

(17)

The physical meaning of eqs 16 and 17 is that the distance from the lowest point to the highest point of the liquid level in the rotating parabolic liquid level in the straight cylinder wall is twice the distance from the lowest point to the initial liquid level or that the rising height and falling depth in the change of the liquid level are always equal. Wan et al.\textsuperscript{11} has indirectly reached this conclusion.

It can be obtained from eqs 12 and 13 that

\[
V_p = \frac{1}{2} V_H
\]  

(18)

That is, the volume of liquid rotating to form a rotating paraboloid is equal to half the volume of a cylinder with the same...
bottom and height. The conclusion here is consistent with the volume relationship pointed out by Ye et al.\(^8\)

From eqs 17 and 18, it is concluded that when \(K \to \infty\), on the premise that the container size is large enough and the initial liquid level is high enough, the change of the liquid level is infinite with the infinite increase of the angular velocity. The liquid level relationship always satisfies eq 17: the height of the rising liquid level is equal to the depth of the falling liquid level. The volume change always satisfies eq 18: the volume formed by the rotating paraboloid is equal to half of the volume of the cylinder with the same base and height.

The critical value of water film formation when the wall slope \(K > 0\) is calculated. The identities from eq 11 derived above are sorted out:

\[
\frac{6r^2}{h} + \frac{6r}{K} + \frac{2h}{K^2} = \frac{3r^2H}{h^2} - \frac{H^3}{h^2K^2}
\]

(19)

Since \(K > 0\), \(H > 0\), and \(r > 0\),

\[
\frac{3r^2H}{h^2} = \frac{H^3}{h^2K^2} > 0
\]

(20)

\[
3r^2 > \frac{H^2}{K^2}
\]

(21)

that is,

\[
(\sqrt{3} + 1)r > \frac{H}{K} + r \text{ or } \sqrt{3} + 1 > \frac{H}{K} + r
\]

In the formula, \(H/K + r\) is the radius of the highest liquid level when the liquid rotates and \(r\) is the radius of the lowest liquid level when the liquid rotates.

The physical meaning of eq 22 is as follows: based on the paraboloid formed by the steady rotation of the liquid, when the container with slope \(K > 0\) rotates at an angular velocity, there exists a limit value for the change (the ratio of the rising liquid level to the falling liquid level) of the liquid level; the ratio of the diameter \(2\left(\frac{H}{K} + r\right)\) at the highest level and the diameter \(2r\) at the lowest level is always less than the fixed value of \(\sqrt{3} + 1\).

By comparing eq 17 when \(K \to \infty\) with eq 22 when \(K > 0\), it is easy to draw the conclusion that when the wall slope is \(K \to \infty\), the liquid level can change unrestrictedly as the angular velocity increases because the diameter at the highest liquid level of the rotating paraboloid formed by the straight cylinder wall is equal to that at the lowest liquid level with a ratio of 1, which is always less than the fixed value derived from eq 22; therefore, there is no limit value for the liquid level change under this case.

2.1.3. Simulation of FLUENT. As shown in Figure 4, when the container angular velocity is 50 rad/s, the liquid is stably rotated and the rotating liquid inside the container is a parabolic surface.

Figure 4. \(K = 3\), \(\omega = 50\) rad/s, liquid steady rotation. (Note: The red—blue color bar on the left represents the volume fraction of the liquid phase. The red bar on the top represents 100%, and the blue bar on the bottom represents 0%.)

Figure 5. \(K = 3\), \(\omega = 60\) rad/s, the water film is formed. (Note: The container is closed in the picture, so after the water film is formed in the right picture, the water film grows along the wall of the container and accumulates in the upper part of the container.)
As shown in Figure 5, when the container angular velocity is increased, the maximum slope of the paraboloid is greater than the container wall slope \( (K_{\text{max}} > K) \), the edge of the rotating liquid will no longer be a complete parabolic, and under the joint action of the liquid extrusion from the lower level and centrifugal force, the marginal liquid climbs along the inclined wall, forming a K-sloped water layer covering the inclined wall.

The water film grows with the increase of rotational speed growth; the rotating liquid surface is divided into two parts: (1) Rotary paraboloid in the middle of the container and (2) the water film at the edge of the container on the inclined wall; the critical angular velocity in which the water film forms can be obtained by the deformation of eq 5: \( \omega_{\text{cri}} = (gK/K_{\text{max}})^{0.5} \). At the same time, due to \( g \), \( K \) is constant; as the container angular velocity increases, \( x_{\text{max}} \) is constantly decreasing, that is, the range of the rotation parabolic is continuously reduced and the water film is larger.

2.2. Experimental Analysis of the Rotation Mode B.

2.2.1. Effect of the Initial Liquid Level on the Vortex. A 1000 mL beaker is applied in this part, and the rotor speed was adjusted from 0–2000 r/min. The initial liquid level \( C \) of the container was set to 4, 6, 8, and 10 cm. The rising liquid level \( (\Delta H) \) and the falling liquid level \( (\Delta h) \) were recorded in the experiment; the experimental results are shown in Figures 6 and

![Figure 6. Rising liquid level trend chart.](image)

![Figure 7. Falling liquid level trend chart.](image)

7. We can see from these pictures that, at the same speed, with the increase of the initial liquid level, the values of the rising liquid level \( (\Delta H) \) and falling liquid level \( (\Delta h) \) are decreasing, indicating that the depth of the vortex is decreasing, which means that the vortex generated by the stable rotation tend to be flat as the initial liquid level increases; specifically, the rising height and center falling depth of the liquid level caused by the vortex will decrease. In fact, since the rotor is rotated at a certain speed of rotation, the energy provided by the system is fixed, and when the liquid level is increased, meaning the external resistance increases, then the generated vortex becomes flat.

2.2.2. Effect of Container Inner Diameter on the Vortex. The container used in the experiment is a beaker of different inner diameters listed in Table 1, and the influence of the container inner diameter on the vortex characteristics is discussed by recording the rising liquid level value \( \Delta H \) \( (\Delta H = H - C) \) at different speeds of each container. The result is shown in Figure 8. The rising level liquid value obtained in Figure 8 is an average value of different initial liquid levels and multiple experiments.

It can be seen from Figure 8 that the rising liquid level curve increases as the rotor speed increases, and this conclusion can also be obtained from Figure 6. Meanwhile, it can be seen from Figure 7 that the falling liquid level curve also increases as the rotor speed increases. The larger the inner diameter of the container is, the gentler the upward trend of the rising liquid level curve will be caused by vortices. The increase of the container inner diameter can make the vortex smooth.

2.2.3. Effect of Rotor Speed on the Vortex. In this part, a 1000 mL beaker was first used as the container, and the initial liquid level was recorded from 3 to 10 cm as shown in Table 1. Experimental data at different rotating speeds at each liquid level were recorded, and the average depth of the vortex, rising liquid level \( \Delta H \) \( (\Delta H = C - H) \), and falling liquid level \( \Delta h \) \( (\Delta h = C - h) \) were drawn into a curve, as shown in Figure 9.

To discuss the relationship between these three parameters and the speed of the rotor (C25), the following conclusions are drawn from Figure 9:

1. The depth of the vortex and the curves of the rising liquid level and falling liquid level all show an upward trend with the increase of rotational speed.
2. The falling liquid level curve rises sharply with the increase of rotor speed, and the rising level curve increases slowly with the increase of rotor speed, and the depth of the vortex mainly depends on the falling liquid level.
3. From Figure 9, it can be concluded that the value of \( \frac{\Delta h}{\Delta H} \) is always changing and increases with the rotor speed. The maximum ratio is about 19, that is, \( \frac{\Delta h}{\Delta H} \approx 19 \), which is different from the change in the liquid level as the vessel wall drives the liquid to rotate; in eq 17, the ratio of the falling liquid level to rising liquid level is 1:1.
4. It can be inferred from Figure 9 that under the premise that the initial liquid level \( C \) is large enough, if the rotor speed can further increase, the vortex depth, rising liquid level \( (\Delta H) \), and falling liquid level \( (\Delta h) \) will also continue to increase.

In order to get a better vision of the real object in the experiment, the experimental vessel is changed into a 500 mL beaker; the inner diameter of the vessel \( d = 8.24 \text{ cm} \), the initial liquid level \( (C = 8 \text{ cm}) \), and the size of the rotor (C25 model) are constant, and by only changing the rotor speed, different swirls are obtained as shown in Figure 10.

In Figure 10, as the rotational speed of the rotor decreases, resulting in less energy to provide liquid rotation, the vortex in the center of the liquid gradually decreases and flattens, and the arc in the center of the vortex is a parabola with an upward opening. When the rotor speed is high (Figure 10, A1 and A2), the vortex center is steeper and the one-sided curve is akin to a parabola with an opening downward; the curve from the center of rotation to the edge of the container is gentle and tends to level gradually. When the rotor speed decreases to a slower speed, although the rotor is still rotating, the vortices at the center of the liquid almost disappear, with only a slight disturbance on the liquid surface (Figure 10, A3).
In Figure 11, as the size of the rotor decreases, the center rotation range decreases and the vortex in the center of the liquid decreases gradually. The vortex center is steeper and the one-sided curve is like a parabola with an opening downward under the large rotor group (Figure 11, B1 and B2); the curve from the center of rotation to the edge of the vessel is slower and tends to level gradually; when the rotor type is reduced to C10, although the rotor is still rotating at its original speed, the vortex almost disappeared and only a slight disturbance on the liquid surface is observed (Figure 11, B3).

From Figures 10 and 1111, it can be seen that the law of vortex change is basically consistent with the reduction of the rotor speed and rotor size. The increases of the initial liquid level and vessel inner diameter are consistent with the decrease of rotor speed and dimension. Under the same conditions, the more energy the liquid is driven with to rotate, a more intense and more extensive vortex is formed.

2.2.5. Liquid Equilibrium Shape Fitting under Experimental Conditions. The effects of the initial liquid level, the container inner diameter, rotor speed, and rotor size on the vortices are fully discussed in the previous section, but the specific vortices’ curve is not given. However, there is no authoritative and widely recognized statement at home and abroad at this point. On the basis of experiment group B (Figure 12), setting the center of the beaker as the origin, the radius of the beaker as the x-axis, and the center axis of rotation as the y-axis to establish the plane rectangular coordinates since the vortex has symmetry, the scatter coordinates on the right side of the vortex are taken for analysis, as shown in Figure 12.

In Figure 12, it is possible to observe the effect of rotor size on the formation of vortices more directly, and not only can it drive the conclusion that the width and depth of vortices decrease

| rotors | diameter (mm) | specifications | parts | x-range (cm) | results of vortex curve fitting | fitting equation | degree of fit | types |
|--------|---------------|---------------|-------|-------------|--------------------------------|-----------------|-------------|-------|
| C35    | 34.40         | first         | 0.177−0.577 | \(y = 2.4734x^2 + 1.8543x + 0.7385\) | 0.9970 | parabola |
|        |               | second        | 0.5776−1.830 | \(y = -3.5549x^2 + 12.462x - 3.6523\) | 0.9958 | parabola |
|        |               | third         | 0.1830−4.008 | \(y = -0.2588x^2 + 2.089x + 4.4149\) | 0.9981 | parabola |
| C30    | 29.70         | first         | 0.1687−0.534 | \(y = 15.092x^2 - 5.0082x + 1.7453\) | 0.9802 | parabola |
|        |               | second        | 0.5349−1.688 | \(y = -2.8509x^2 + 9.5485x - 0.6911\) | 0.9974 | parabola |
|        |               | third         | 1.688−3.9654 | \(y = -0.1595x^2 + 1.3616x + 5.5146\) | 0.9987 | parabola |
| C25    | 24.48         | first         | 0.0695−0.4004 | \(y = 6.2189x^2 + 0.7542x + 2.8403\) | 0.9971 | parabola |
|        |               | second        | 0.4004−1.3935 | \(y = -3.8784x^2 + 9.1933x + 1.7745\) | 0.9908 | parabola |
|        |               | third         | 1.3935−4.0507 | \(y = -0.1106x^2 + 0.8687x + 6.538\) | 0.9795 | Parabola |
| C20    | 20.10         | first         | 0.1511−0.3541 | \(y = 35.22x^2 - 11.299x + 5.5286\) | 0.9913 | parabola |
|        |               | second        | 0.3541−1.360 | \(y = -2.0461x^2 + 5.0982x + 4.5015\) | 0.9912 | parabola |
|        |               | third         | 1.360−4.051 | \(y = -0.0323x^2 + 0.3074x + 7.3973\) | 0.9902 | parabola |
| C15    | 14.90         | first         | 0.052−0.195 | \(y = 33.888x^2 - 4.6041x + 6.5971\) | 0.9927 | parabola |
|        |               | second        | 0.195−0.58485 | \(y = -2.0785x^2 + 3.0916x + 6.4848\) | 0.9939 | parabola |
|        |               | third         | 058485−1.2737 | \(y = -0.474x^2 + 1.4422x + 6.9243\) | 0.9937 | parabola |

In Figure 12, it is possible to observe the effect of rotor size on the formation of vortices more directly, and not only can it drive the conclusion that the width and depth of vortices decrease
with the decrease of rotor size but it can also easily and clearly observe the shape of the vortex curve, which can be described in three parts:

1. The center triggering part (the first part): the vortex power part; under the rotation of the center rotor, the liquid center begins to form the vortex, and then the surrounding liquid begins to rotate; this part is smaller, mainly existing at the bottom of the vortex.

2. The rising part (the second part): it is the most important part of the vortex curve; the middle rising curve is usually steeper, and the main shape of the vortex curve depends on this part; when the power or rotation range of the rotation center increases, this part will also relatively increase.

3. The edge attenuation part (the third part): near the horizontal plane or the edge of the container, it is the intermediate part that rises to the horizontal plane or the edge of the container, and this part of the vortex curve rises gently until it remains horizontal; the rotational energy of the liquid decreases from the center of the vortex to the edge.

The scatter points in Figure 12 are fitted as shown in Table 1. The following conclusions are drawn from the results of the fit in Table 1.

1. The coordinates of the vortex scatters are in good agreement with the quadratic polynomial curve, and the mean fit is 99.25%, which shows that the vortex curve can be regarded as three parabolas.

2. The first part (the center triggering part) of the vortex is a parabola with an upward opening.

3. The second part (the rising part) and the third part (the edge attenuation part) of the vortex are parabolas with downward opening.

2.2.6. Geometrical Relationships of Equilibrium Shapes of Liquids under Ideal Conditions. Based on the conclusion that the vortex curve and the parabola are basically fitted, a mathematical model is established, as shown in Figure 13: a cylinder with a container radius $r$ and an initial liquid level $C$; when the liquid rotates around the central axis of the cylinder to form a balanced liquid level, the highest liquid level is $H$, the lowest liquid level is $h$, the cylinder wall is high enough in the process of rotation, and the liquid will not overflow.

For the convenience of calculation, ideal conditions are given:

1. The vortex curve is regarded as two symmetrical parabolas of the second part, and the size of the center of rotation is ignored.

2. The lowest liquid level at the center of the container is the intersection point of two parabolas, and the highest liquid level at the edge of the container is the parabola apex.

The Cartesian coordinate system of the $xy$ plane was established with the lowest liquid level as the origin. Due to symmetry, only the first quadrant curve is taken for analysis, and the parabola equation is set as $y = \alpha (x - m)^2 + n$, where $\alpha$, $m$, and $n$ are unknown parameters.

By substituting the vortex coordinate $M (R, H - h)$ and the coordinate origin $(0, 0)$, we get

$$y = -\frac{H - h}{R^2} (x - R)^2 + (H - h)$$  \hspace{1cm} (23)

Let $H - h = \Delta H$, $\Delta H/R = b$; this simplifies to

$$y = 2bx - \frac{bx^2}{R}$$ \hspace{1cm} (24)

Then, the volume enclosed by the rotating paraboloid is

Figure 11. (Group B) From left to right, the rotor specifications are C35, C30, C25, C20, C15, and C10 (Photograph courtesy of “Yi Lei”. Copyright 2021).

Figure 12. Vortex scatter plot at the same speed and different rotor sizes.

Figure 13. Center of the liquid rotates at an angular velocity $\omega$. 
$V_f = \int_{0}^{H-h} \pi x^2 \, dy = \int_{0}^{R} \pi x^2 d(2bx - bx^2/R)$

$= \frac{1}{6}\pi R^2 \Delta H$ (25)

2.2.6.1. The Volume Relationship. When the rotating liquid level is stable, the volume of the cylinder with the same base and height as the vortex curve can be expressed as

$V_0 = \pi R^2 \Delta H$ (26)

that is,

$V_f = \frac{1}{6} V_0$ (27)

Equation 27 shows that under ideal conditions, when the center of the liquid rotates in equilibrium, the volume formed by the vortexes on the rotating liquid surface is equal to one-sixth of the volume of the cylinder with the same bottom and the same height.

2.2.6.2. The Liquid Level Relationship. Since the total volume of liquid before and after rotation is constant $V_C$,

$V_H - V_f = V_C$ (28)

that is,

$\pi R^2 H - \frac{1}{6}\pi R^2 \Delta H = \pi R^2 C$ (29)

simplified to

$s(H - C) = C - h$ (30)

Equation 30 shows that under ideal conditions, when the liquid center rotates in equilibrium, the height of the liquid level increasing due to the formation of the vortex is always one-fifth of the falling depth.

Equations 27 and 30 are derived only when the ideal conditions are satisfied, and since the experimental vortex curve is actually composed of three parts, the volume relation and the liquid level change caused by the liquid rotation are not a simple proportional relation.

3. CONCLUSIONS

(1) The liquid rotation curves under different rotation modes show different shapes. The equilibrium liquid surface formed by the liquid rotation driven by the container wall is always paraboloid, and the equilibrium liquid surface formed by the liquid rotation driven by the container center is vortex-shaped, which can be seen as three paraboloid segments.

(2) The paraboloid formed by the liquid rotation driven by the cylinder wall is independent of the slope of the cylinder wall, and the changes of the liquid level, the container inner radius, and the container wall slope always satisfy an equation. When the container is a straight cylinder wall, the change of the liquid level is infinite with the increase of the angular velocity. The rising height of the liquid level is equal to the falling depth of the liquid level. The volume formed by rotating the liquid level is half of the volume of the cylinder with the same base and height. When the container has a sloped wall ($K > 0$), based on the complete paraboloid, there is a critical value of liquid level change, that is, the ratio of the highest liquid level radius to the lowest liquid level radius is always less than the fixed value of $\sqrt{3} + 1$. When the angular velocity increases to a critical value, the rotating liquid level is divided into two parts: a rotating paraboloid with a shrinking center and a water film with a growing edge.

(3) The vortex driven by the rotation of the center of the container is closely related to the initial liquid level, the container inner diameter, the rotor speed, and the rotor size; it is shown that the increase of the initial liquid level and the container inner diameter and the decrease of the rotor speed and rotor size will lead to the decrease of the depth and width of the vortex, and the more energy the liquid provides when it is driven to rotate, relatively speaking, the more intense is the vortex formed, the wider the scope of the vortex.

(4) The vortex curve formed by the rotation of the liquid center can be summarized into three parts: the center triggering part, the rising part, and the edge attenuation part. The curve of each part can be regarded as a parabola. The volume relation and liquid level change caused by the vortex are complex and variable. Only under ideal conditions, when the vortex curve is regarded as the complete second parabola, the volume enclosed by the vortex is one-sixth of the volume of the cylinder with the same base and height, and the rising height of the liquid level is one-fifth of the falling depth.

4. THEORETICAL RESEARCH AND EXPERIMENTAL DESIGN

There are two rotation modes. In rotation mode A, the container wall drives the liquid to rotate. In rotation mode B, the rotor in the center of the container drives the liquid to rotate.

Figure 14. Technical route for the theoretical study of rotating liquid under an inclined cylinder wall.

| Table 2. Beaker Sizes used in the Experiment |
|---------------------------------------------|
| beaker size (mL) | initial diameter $d$ (cm) | initial level $C$ value (cm) |
|------------------|-----------------------------|-----------------------------|
| 200              | 6.12                        | 3, 4, 5, 6, 7               |
| 300              | 7.09                        | 3, 4, 5, 6, 7, 8            |
| 500              | 8.24                        | 3, 4, 5, 6, 6, 7, 8         |
| 1000             | 10.79                       | 3, 4, 5, 6, 7, 8, 9, 10     |

4.1. Theoretical Derivation Scheme of the Rotation Mode A. It is widely discussed that the equilibrium liquid surface of rotation driven by the wall is paraboloid by many scholars. Also, a lot of physical experiments and theoretical research have been done based on the characteristics. However, these rotating containers used in these physical experiments and theoretical studies are straight-wall containers; for example, Wan et al.13 carried out theoretical force analysis on the rotating liquid particle in the straight-wall container and carried out the rotating liquid experiment; Chen14 designed an experiment to measure the gravitational acceleration based on the paraboloid formed by...
rotating liquid in the straight-wall container. However, whether the general rule is applicable in the slope—wall container had not been considered.

Therefore, this paper starts from the derivation of the liquid level equilibrium equation driven by the rotation of the liquid in a container with a slope of K. Based on the equilibrium equation of the rotating liquid, an identical relation of the container wall with a slope of K can be obtained, and on this basis, the geometric rules of the container wall with different values of slope are discussed. In this paper, two cases of the slope are mainly discussed:

1. When \( K \to \infty \), the wall of the rotating container is straight; the general law based on the identical relation should be consistent with the previous law, which can be used to verify the correctness of the identities in this paper.

2. When \( K > 0 \), the container under this condition is an inclined cylinder wall with a large top and a small bottom. The law of the rotating liquid level under the cylinder wall can be further obtained. At the same time, FLUENT software is used to discuss the critical angular velocity under this condition; the obtained law is compared and analyzed with the law under the straight cylinder wall. The technology roadmap is shown in Figure 14:

### Table 3. Rotor Sizes used in the Experiment

| model number (straight type) | C05 | C10 | C15 | C20 | C25 | C30 | C35 |
|-----------------------------|-----|-----|-----|-----|-----|-----|-----|
| length (mm)                 | 5.04| 9.90| 14.90| 20.10| 24.48| 29.70| 34.40|
| width (mm)                  | 3.10| 6.60| 6.28| 7.50| 8.06| 7.50| 8.10|
| height (mm)                 | 3.02| 6.02| 6.06| 7.06| 7.80| 6.70| 7.66|

**Figure 15.** Simulation of liquid center rotation.

**Figure 16.** Experimental Diagram.
divided into five levels that are 400, 800, 1200, 1600, and 2000 r/min; (2) container inner diameter and initial liquid level: the container is a beaker of different inner diameters; considering the size of the rotor itself, the initial liquid level C is set as an integer higher than 3 cm, as Table 2 shows; (3) the highest liquid level H is read by the coordinate paper of the outer wall of the beaker; the lowest liquid level h is read by the light of the laser pen, which can horizontally penetrate, and the data is abandoned when the minimum level is in contact with the rotor; (4) the highest liquid level H minus the initial liquid level C is defined as the rising liquid level ΔC, that is, \( \Delta C = H - C \); the initial liquid level C minus the lowest liquid level h is defined as the falling liquid level \( \Delta h \), that is, \( \Delta h = C - h \). Figure 16 shows the schematic diagram of the experiment.

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Y.L. wrote the original draft and translation. Y.L., J.W., & H.J. wrote the methodology, contributed to conceptualization, investigation, and test analysis, and reviewed & edited the paper, and H.X., J.P., & L.F. contributed to the study’s investigation.

#### Notes
The authors declare no competing financial interest. The data that support the findings of this study are available from the corresponding author upon reasonable request.

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