NON-ESCAPING POINTS OF ZORICH MAPS

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ABSTRACT

We extend results about the dimension of the radial Julia set of certain exponential functions to quasiregular Zorich maps in higher dimensions. Our results improve on previous estimates of the dimension also in the special case of exponential functions.

1. Introduction

The Julia set $J(f)$ of an entire function $f$ is the set where the iterates $f^n$ of $f$ do not form a normal family and the escaping set $I(f)$ consists of all points which tend to infinity under iteration of $f$. These sets play a fundamental role in the iteration theory of entire functions. A result of Eremenko [6] states that $J(f) = \partial I(f)$. We refer to [2], [15] for an introduction to the iteration theory of entire functions.

We consider the exponential family consisting of the functions $E_\lambda(z) := \lambda e^z$ with $\lambda \in \mathbb{C}\setminus\{0\}$. If $0 < \lambda < 1/e$, then $E_\lambda$ has an attracting fixed point. Devaney

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and Krych [5] showed that then \( J(E_\lambda) \) is equal to the complement of the attracting basin of this fixed point and \( J(E_\lambda) \) consists of uncountably many pairwise disjoint curves (called hairs) which connect a finite point (called the endpoint of the hair) with \( \infty \). Let \( C_\lambda \) be the set of endpoints of the hairs that form \( J(E_\lambda) \). The results of Devaney and Krych also yield that \( J(E_\lambda) \setminus C_\lambda \subset I(E_\lambda) \).

McMullen [13] showed that \( \dim J(E_\lambda) = 2 \). Here and in the following \( \dim X \) denotes the Hausdorff dimension of a set \( X \). In fact, McMullen showed that \( \dim I(E_\lambda) = 2 \) and \( I(E_\lambda) \subset J(E_\lambda) \). Karpińska [11] obtained the surprising result that \( \dim J(E_\lambda) \setminus C_\lambda = 1 \).

A 3-dimensional analogue of the results of Devaney and Krych, McMullen and Karpińska was obtained in [3]. Here the exponential function was replaced by a quasiregular map \( F \): \( \mathbb{R}^3 \to \mathbb{R}^3 \) introduced by Zorich [19, p. 400]. As noted in [12, §8.1], Zorich maps exist in \( \mathbb{R}^d \) for all \( d \geq 2 \), and this can be used (see [3, Remark 9] and [4]) to obtain a \( d \)-dimensional analogue of the above results.

To define a Zorich map, following [9, §6.5.4], we fix \( \rho > 0 \) and consider the cube

\[
Q := \{ x \in \mathbb{R}^{d-1} : \|x\|_\infty \leq \rho \} = [-\rho, \rho]^{d-1}
\]

and the upper hemisphere

\[
U := \{ x \in \mathbb{R}^d : \|x\|_2 = 1, x_d \geq 0 \}.
\]

Let \( h : Q \to U \) be a bi-Lipschitz map and define

\[
F : Q \times \mathbb{R} \to \mathbb{R}^d, \quad F(x_1, \ldots, x_d) = e^{x_d} h(x_1, \ldots, x_{d-1}).
\]

The map \( F \) is then extended to a map \( F : \mathbb{R}^d \to \mathbb{R}^d \) by repeated reflection at hyperplanes.

The main result of [3] states that if \( a \in \mathbb{R} \) is sufficiently large, then the map

\[
f_a : \mathbb{R}^d \to \mathbb{R}^d, \quad f_a(x) = F(x) - (0, \ldots, 0, a),
\]

has an attracting fixed point \( \xi_a \) such that the complement of the attracting basin of \( \xi_a \) consists of hairs, the set of endpoints of the hairs has dimension \( d \), but the union of the hairs without the endpoints has dimension 1.

The purpose of this paper is to extend some further results about the exponential family to the higher dimensional setting. Let \( J_{bd}(E_\lambda) \) be the set of all \( z \in J(E_\lambda) \) for which the orbit \( \{ E_\lambda^n(z) : n \in \mathbb{N} \} \) is bounded. Karpińska [10, Theorem 2] also showed that \( \dim J_{bd}(E_\lambda) > 1 \) for all \( \lambda \in (0, 1/e) \)
and

\[(1.3) \quad 1 + \frac{1}{\log \log(1/\lambda)} < \dim J_{bd}(E_\lambda) < 1 + \frac{1}{\log \log \log(1/\lambda)} \]

if \(\lambda\) is sufficiently small.

Urbański and Zdunik [16] considered the set \(J_r(E_\lambda) := J(E_\lambda) \setminus I(E_\lambda)\). We note that, in general, the notation \(J_r(f)\) is used for the radial Julia set of an entire function \(f\), but for the functions \(E_\lambda\) with \(0 < \lambda < 1/e\) this agrees with the above definition; see [14] for a discussion of radial Julia sets. Clearly \(J_r(E_\lambda) \supset J_{bd}(E_\lambda)\). Urbański and Zdunik proved [16, Theorem 4.5] that \(\dim J_r(E_\lambda) = \dim J_{bd}(E_\lambda) < 2\) for \(0 < \lambda < 1/e\). They noted that (1.3) thus yields that

\[(1.4) \quad \lim_{\lambda \to 0} \dim J_r(E_\lambda) = 1,\]

but they also gave a direct proof of this [16, Theorem 7.2].

Urbański and Zdunik also showed that the function \(\lambda \mapsto \dim J_r(E_\lambda)\) is continuous [16, Theorem 4.7] in the interval \((0, 1/e)\) and in fact real-analytic [17, Theorem 9.3]. The function \(\lambda \mapsto \dim J_r(E_\lambda)\), and in particular its behavior as \(\lambda \to 1/e\), was further studied in [8, 18].

We consider the corresponding sets for the Zorich maps. Denoting by \(\xi_a\) the attracting fixed point of \(f_a\) we thus put

\(J_{bd}(f_a) := \{ x \in \mathbb{R}^d : f_a^n(x) \not\to \xi_a \text{ and } (f_a^n(x)) \text{ is bounded} \}\)

and

\(J_r(f_a) := \{ x \in \mathbb{R}^d : f_a^n(x) \not\to \xi_a \text{ and } |f_a^n(x)| \not\to \infty \}\).

**Theorem 1.1:** If \(a\) is sufficiently large, then

\[d - 1 + \frac{1}{2} \frac{\log \log a}{\log a} - \frac{\log \log \log a}{\log a} < \dim J_{bd}(f_a) \leq \dim J_r(f_a) \leq d - 1 + \frac{\log \log a}{\log a}.\]

It follows from Theorem 1.1 that if \(0 < \eta < \frac{1}{2}\) and \(a\) is sufficiently large, then

\[(1.5) \quad d - 1 + \eta \frac{\log \log a}{\log a} < \dim J_{bd}(f_a),\]

but this does not hold for \(\eta = 1\). It remains open for which \(\eta \in [\frac{1}{2}, 1)\) the inequality (1.5) holds.

In order to compare Theorem 1.1 with (1.3) we note that for \(d = 2, \rho = \pi/2,\)

\[h : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}^2 = \mathbb{C}, \quad h(x) = (\sin x, \cos x) = \sin x + i \cos x = ie^{-ix},\]
and \( z = (x, y) = x + iy \) the Zorich map \( F \) takes the form
\[
F(z) = e^y h(x) = ie^{y-ix} = ie^{-iz}.
\]
Hence for \( a > 0 \) and \( \lambda = e^{-a} \) we have
\[
f_a(z) = F(z) - ia = i(e^{-iz} - a) = (L \circ E_\lambda \circ L^{-1})(z)
\]
with \( L(z) = i(z - a) \). Thus \( f_a \) is conjugate to \( E_\lambda \). Since \( a = \log(1/\lambda) \) we see that Theorem 1.1 not only extends the results for the functions \( E_\lambda \) to higher dimensions, but also improves (1.3) and (1.4) to
\[
d - 1 + \frac{1}{2} \frac{\log \log \log(1/\lambda)}{\log(1/\lambda)} < \dim J_{bd}(E_\lambda) \\
\leq \dim J_r(E_\lambda) \\
\leq 1 + \frac{\log \log \log(1/\lambda)}{\log(1/\lambda)}.
\]

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2. Preliminaries

We collect some results about Zorich maps which can be found in [3, 4]. As mentioned above, in [3] only the case \( d = 3 \) is treated, but the changes to handle the general case are minor. We also note that in [3, 4] only the case \( \rho = 1 \) is considered. However, it was noted already in [3, Remark 1] that one may replace the unit cube by a cube of other sidelength and in fact by a rectangular box. The reason that we do not restrict to the case \( \rho = 1 \) is that this way the exponential map is (conjugate to) a special Zorich map, as described in the introduction.

We will use \( |x| \) for the Euclidean norm of a point \( x \in \mathbb{R}^d \); that is, we write \( |x| = \|x\|_2 \). For \( c \in \mathbb{R} \) we define the half-space
\[
H_{>c} := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > c\}.
\]
The half-spaces \( H_{<c}, H_{\geq c} \) and \( H_{\leq c} \) and the hyperplane \( H_{=c} \) are defined analogously.
First we note that the derivative $DF(x)$ exists almost everywhere and that if $DF(x_1, \ldots, x_{d-1}, 0)$ exists, then

$$DF(x_1, \ldots, x_d) = e^{x_d}DF(x_1, \ldots, x_{d-1}, 0).$$

(2.1)

This implies that there exist $\alpha, m, M \in \mathbb{R}$ with $0 < \alpha < 1$ and $m < M$ such that

$$|DF(x)| := \sup_{|h|=1} |DF(x)(h)| \leq \alpha \quad \text{a.e. for } x \in H_{\leq m}$$

while

$$\ell(DF(x)) := \inf_{|h|=1} |DF(x)(h)| \geq \frac{1}{\alpha} \quad \text{a.e. for } x \in H_{\geq M}.$$  

(2.2)

We may and will assume that $M \geq 0$. It was shown in [3] that if

$$a \geq e^{M} - m,$$

(2.3)

then $f_a$ has an attracting fixed point $\xi_a \in H_{\leq m}$ and the properties mentioned in the introduction hold; that is, the complement of the basin of attraction of $\xi_a$ consists of hairs, the set of endpoints of the hairs has dimension $d$, and the union of the hairs without endpoints has dimension 1.

The following result can be considered as an analogue of the result of Karpińska [10, Theorem 2] that $\dim J_{bd}(E_{\lambda}) > 1$ for $0 < \lambda < 1/e$.

**Theorem 2.1:** If $a$ satisfies (2.3), then $\dim J_{bd}(f_a) > d - 1$.

Theorem 2.1 will be proved in Section 4 together with the lower bound in Theorem 1.1.

**Remark 2.2:** Urbański and Zdunik [16, Corollary 7.3] showed that (1.4) implies that $\dim J_{r}(E_{\lambda}) < 2$ whenever $0 < \lambda < 1/e$ (and in fact whenever $E_{\lambda}$ has an attracting fixed point). The proof uses that if two functions in the exponential family both have an attracting fixed point, then they are quasiconformally conjugate. This argument is not available in the higher-dimensional setting. We do not know whether $\dim J_{r}(f_a) < d$ whenever $a$ satisfies (2.3).

For $r = (r_1, \ldots, r_{d-1}) \in \mathbb{Z}^{d-1}$ we put

$$P(r) := \{(x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}: |x_j - 2\rho r_j| < \rho \text{ for } 1 \leq j \leq d - 1\}$$
so that $P(0)$ is the interior of $Q$. Let

$$S := \left\{ r \in \mathbb{Z}^{d-1} : \sum_{j=1}^{d-1} r_j \text{ is even} \right\}.$$ 

Then $F$ maps $P(r) \times \mathbb{R}$ onto $H_{>0}$ if $r \in S$ and onto $H_{<0}$ if $r \in \mathbb{Z}^{d-1} \setminus S$. Thus $f_a$ maps $P(r) \times \mathbb{R}$ onto $H_{>\alpha}$ if $r \in S$ and onto $H_{<-\alpha}$ if $r \in \mathbb{Z}^{d-1} \setminus S$. For $r \in S$ we put

$$T(r) := P(r) \times (M, \infty).$$

A short computation (see [3, (2.1)] or [4, (2.2)]) shows that $f_a(T(r)) \supset H \geq M$. Thus there exists a branch $\Lambda^r : H \geq M \to T(r)$ of the inverse function of $f_a$. With $\Lambda := \Lambda(0, \ldots, 0)$ we have

$$\Lambda^{(r_1, \ldots, r_{d-1})}(x) = \Lambda(x) + (2\rho r_1, \ldots, 2\rho r_{d-1}, 0)$$

for all $x \in H \geq M$ and all $r \in S$. We have

$$(2.4) \quad D\Lambda(x) = Df_a(\Lambda(x))^{-1} = DF(\Lambda(x))^{-1} \quad \text{a.e. for } x \in H \geq M.$$ 

It thus follows from (2.2) that

$$(2.5) \quad |D\Lambda(x)| \leq \alpha \quad \text{a.e. for } x \in H \geq M.$$ 

This implies that

$$|\Lambda(x) - \Lambda(y)| \leq \alpha|x - y| \quad \text{for } x, y \in H \geq M.$$ 

Noting that $Df_a(x) = DF(x)$ we deduce from (2.1) that there exist positive constants $c_1$ and $c_2$ such that

$$(2.6) \quad c_1 e^{x_d} \leq \ell(Df_a(x)) \leq |Df_a(x)| \leq c_2 e^{x_d} \quad \text{a.e.}$$

It was shown in [3, 4] that there exist positive constants $c_3$ and $c_4$ such that

$$(2.7) \quad \frac{c_3}{|x|} \leq \ell(D\Lambda(x)) \leq |D\Lambda(x)| \leq \frac{c_4}{|x|} \quad \text{a.e. for } x \in H \geq M$$

and this was used to prove that

$$(2.8) \quad |\Lambda(x) - \Lambda(y)| \leq c_4 \pi \frac{|x - y|}{\min\{|x|, |y|\}}.$$ 

We have to consider how the bounds for $\ell(D\Lambda(x))$ and $|D\Lambda(x)|$ in (2.7) depend on $a$. We will write $\overline{a} = (0, \ldots, 0, a)$ so that $f_a(x) = F(x) - \overline{a}$. 

**Lemma 2.3:** There exist constants $c_3$ and $c_4$ depending only on $F$ such that if $a$ satisfies (2.3), then
\[
\frac{c_3}{|x + a|} \leq \ell(D\Lambda(x)) \leq |D\Lambda(x)| \leq \frac{c_4}{|x + a|} \quad \text{a.e. for } x \in H_{\geq M}.
\]

**Proof.** By (2.4), (2.6), (1.1) and (1.2) we have
\[
|D\Lambda(x)| \leq \frac{1}{\ell(DF(\Lambda(x)))} \leq \frac{1}{c_1 \exp(\Lambda_d(x))} = \frac{1}{c_1 |F(\Lambda(x))|} = \frac{1}{c_1 |f_a(\Lambda(x)) + a|} = \frac{1}{c_1 |x + \overline{a}|}.
\]
The proof of the lower bound for $\ell(D\Lambda(x))$ is similar.

**Lemma 2.3** implies that (2.8) can be improved to
\[
|\Lambda(x) - \Lambda(y)| \leq c_4 \pi \frac{|x - y|}{\min\{|x + \overline{a}|, |y + \overline{a}|\}}
\]
for $x, y \in H_{\geq M}$.

### 3. Proof of the upper bound in Theorem 1.1

For $r \in S$ and $A \subset \mathbb{R}^d$, we will use the notation
\[
A^r := (2\rho r_1, \ldots, 2\rho r_{d-1}, 0) + A = \{(2\rho r_1 + x_1, \ldots, 2\rho r_{d-1} + x_{d-1}, x_d) : x \in A\}.
\]
We also write $B(x, R)$ for the closed ball of radius $R$ around a point $x \in \mathbb{R}^d$. We note that since $M > m$ it follows from (2.3) that $a > 1$.

**Lemma 3.1:** Let $r, s \in S$ and let $A \subset T(s)$ be bounded. If $a$ satisfies (2.3), then
\[
\text{diam} \Lambda(A^r) \leq c_4 \pi \frac{\text{diam} A}{\sqrt{\rho^2 |r + s|^2 + a^2}}.
\]

**Proof.** Since $A^r \subset T(r + s)$ we find that if $x = (x_1, \ldots, x_d) \in A^r$, then
\[
|x_j| \geq \max\{2\rho|r_j + s_j| - \rho, 0\} \geq \rho|r_j + s_j|
\]
for $1 \leq j \leq d - 1$ while $x_d \geq M$. Thus, recalling that $M \geq 0$, we find that

$$|x + \overline{a}| = \sqrt{\sum_{j=1}^{d-1} x_j^2 + (x_d + a)^2} \geq \sqrt{\sum_{j=1}^{d-1} \rho^2(r_j + s_j)^2 + a^2} = \sqrt{\rho^2|r + s|^2 + a^2}.$$ 

The conclusion now follows from (2.9).

**Lemma 3.2:** There exist positive constants $c_5$ and $c_6$ depending only on $d$ such that if $d - 1 < t \leq d$ and $N \geq b \geq 3\sqrt{d - 1}$, then

$$c_5 \frac{b^{d-1-t}}{t - d + 1} \left(1 - \left(\frac{N}{b}\right)^{d-1-t}\right) \leq \sum_{r \in S, |r| \leq N} \frac{1}{(|r|^2 + b^2)^{t/2}} \leq c_6 \frac{b^{d-1-t}}{t - d + 1}. \tag{3.1}$$

Moreover,

$$\sum_{r \in S, |r| \leq N} \frac{1}{(|r|^2 + b^2)^{(d-1)/2}} \geq c_5 \log \frac{N}{b}. \tag{3.2}$$

**Proof.** For $r = (r_1, \ldots, r_{d-1}) \in \mathbb{Z}^{d-1}$ let $Q_r$ be the cube with vertices at the points $(2r_1 + e_1, \ldots, 2r_{d-1} + e_{d-1})$, where $e_j \in \{-1, 1\}$ for all $j$. For $x \in Q_r$ and $b \geq \sqrt{d - 1}/7$ we then have

$$|x|^2 \leq \sum_{j=1}^{d-1} (2|r_j| + 1)^2 \leq \sum_{j=1}^{d-1} (8r_j^2 + 1) = 8|r|^2 + d - 1 \leq 8|r|^2 + 7b^2$$

and thus $|x|^2 + b^2 \leq 8|r|^2 + 8b^2$. Hence

$$\int_{Q_r} \frac{dx_1 \cdots dx_{d-1}}{(|x|^2 + b^2)^{t/2}} \geq \int_{Q_r} \frac{dx_1 \cdots dx_{d-1}}{(8|r|^2 + 8b^2)^{t/2}} = \frac{2^{d-1}8^{-t/2}}{(|r|^2 + b^2)^{t/2}} = \frac{2^{d-1-3t/2}}{(|r|^2 + b^2)^{t/2}}.$$
For $|r| \leq N$ we have $Q_r \subset B(0, 2N + \sqrt{d-1}) \subset B(0, 2N + d)$ and thus

$$
\sum_{r \in S, |r| \leq N} \frac{1}{(|r|^2 + b^2)^{t/2}} \leq 2^{3t/2 - d + 1} \sum_{r \in S, |r| \leq N} \int_{Q_r} \frac{dx_1 \cdots dx_{d-1}}{(|x|^2 + b^2)^{t/2}}
$$

(3.3)

$$
\leq 2^{3t/2 - d + 1} \int_{B(0, 2N + d)} \frac{dx_1 \cdots dx_{d-1}}{(|x|^2 + b^2)^{t/2}}
$$

$$
= 2^{3t/2 - d + 1} \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \int_0^{2N + d} \frac{u^{d-2}}{(u^2 + b^2)^{t/2}} du.
$$

Furthermore,

$$
\int_0^{2N + d} \frac{u^{d-2}}{(u^2 + b^2)^{t/2}} du = b^{d-1-t} \int_0^{(2N + d)/b} \frac{v^{d-2}}{(v^2 + 1)^{t/2}} dv
$$

(3.4)

$$
\leq b^{d-1-t} \left(1 + \int_1^{\infty} v^{d-2-t} dv\right)
$$

$$
= b^{d-1-t} \left(1 + \frac{1}{t - d + 1}\right) \leq 2 \frac{b^{d-1-t}}{t - d + 1}.
$$

The right inequality in (3.1) now follows from (3.3) and (3.4).

To prove the left inequality, we proceed similarly. For $r \in S$ let $P_r$ be the cube with vertices at the points $(2r_1 + 3e_1, \ldots, 2r_{d-1} + 3e_{d-1})$, where $e_j \in \{ -1, 1 \}$ for all $j$. Noting that $(2y - 3)^2 \geq y^2 - 3$ for $y \in \mathbb{R}$ we find that if $x \in P_r$ and $b \geq 3\sqrt{d-1}$, then

$$
|x|^2 \geq \sum_{j=1}^{d-1} (2|r_j| - 3)^2 \geq \sum_{j=1}^{d-1} (|r_j|^2 - 3) = |r|^2 - 3(d-1) \geq |r|^2 - \frac{1}{2} b^2
$$

and thus $|x|^2 + b^2 \geq \frac{1}{2} |r|^2 + \frac{1}{2} b^2$. Hence

$$
\int_{P_r} \frac{dx_1 \cdots dx_{d-1}}{(|x|^2 + b^2)^{t/2}} \leq \int_{P_r} \frac{dx_1 \cdots dx_{d-1}}{(\frac{1}{2} |r|^2 + \frac{1}{2} b^2)^{t/2}} = \frac{\int_{P_r} dx_1 \cdots dx_{d-1}}{(|r|^2 + b^2)^{t/2}}.
$$

For $x \in \mathbb{R}^{d-1}$ we can choose $r \in S$ such that $x \in P_r$. In fact, we first choose $r \in \mathbb{Z}^{d-1}$ such that $|x_j - 2r_j| \leq 1$ for all $j$, and in order to achieve that $\sum_{j=1}^{d-1} r_j$ is even we replace $r_1$ by $r_1 + 1$ if necessary. For $x \in P_r$ we
have $|r_j| \leq (|x_j| + 3)/2$ and since $N \geq 3\sqrt{d - 1}$ this implies that

$$|r|^2 = \sum_{j=1}^{d-1} r_j^2 \leq \frac{1}{4} \sum_{j=1}^{d-1} (|x_j| + 3)^2 = \frac{1}{2} \sum_{j=1}^{d-1} 2(x_j^2 + 9)$$

$$= \frac{1}{2} |x|^2 + \frac{9}{2}(d - 1) \leq \frac{1}{2} |x|^2 + \frac{1}{2} N^2.$$ 

For $|x| \leq N$ we thus have $|r| \leq N$ so that

$$\bigcup_{r \in S} |r| \leq N$$

$$P_r \supset B(0, N).$$

Instead of (3.3) we now obtain

$$\sum_{r \in S \atop |r| \leq N} \frac{1}{(|r|^2 + b^2)^{t/2}} \geq 6^{1-d} 2^{-t/2} \sum_{r \in S \atop |r| \leq N} \int_{P_r} \frac{dx_1 \cdots dx_{d-1}}{(|x|^2 + b^2)^{t/2}}$$

(3.5)

$$\geq 6^{1-d} 2^{-t/2} \int_{B(0, N)} \frac{dx_1 \cdots dx_{d-1}}{(|x|^2 + b^2)^{t/2}}$$

$$= 6^{1-d} 2^{-t/2} \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^N \frac{u^{d-2}}{(u^2 + b^2)^{t/2}} du$$

and instead of (3.4) we have

$$\int_0^N \frac{u^{d-2}}{(u^2 + b^2)^{t/2}} du = b^{d-1-t} \int_0^{N/b} \frac{v^{d-2}}{(v^2 + 1)^{t/2}} dv$$

(3.6)

$$\geq b^{d-1-t} 2^{-t/2} \int_1^{N/b} v^{d-2-t} dv$$

$$= \frac{b^{d-1-t}}{t-d+1} 2^{-t/2} \left(1 - \left(\frac{N}{b}\right)^{d-1-t}\right).$$

The left inequality in (3.1) now follows from (3.5) and (3.6).

Finally, to prove (3.2) we only have to note that for $t = d - 1$ we obtain

$$\int_0^N \frac{u^{d-2}}{(u^2 + b^2)^{d-1/2}} du = \int_0^{N/b} \frac{v^{d-2}}{(v^2 + 1)^{d-1/2}} dv$$

$$\geq 2^{(1-d)/2} \int_1^{N/b} \frac{dv}{v} = 2^{(1-d)/2} \log \frac{N}{b}$$

instead of (3.6).
**Lemma 3.3:** There exists a constant $c_7$, depending only on $F$, such that if $a$ satisfies \([2.3]\), $d - 1 < t \leq d$, $s \in S$ and $A \subset T(s)$ is bounded, then

$$
\sum_{r \in S} (\text{diam } \Lambda(A^r))^t \leq c_7 \frac{a^{d-1-t}}{t-d+1} (\text{diam } A)^t.
$$

**Proof.** Without loss of generality we may assume that $s=0$. Applying Lemma 3.1 we obtain

$$
\sum_{r \in S} (\text{diam } \Lambda(A^r))^t \leq \sum_{r \in S} \frac{(c_4 \pi \text{ diam } A)^t}{(\rho^2 |r|^2 + a^2)^{t/2}}
\leq \left( \frac{c_4 \pi \text{ diam } A}{\rho} \right)^t \sum_{r \in S} \frac{1}{(|r|^2 + (a/\rho)^2)^{t/2}}.
$$

We note that the upper bound in \([3.1]\) does not depend on $N$. Thus we may take the limit as $N \to \infty$ there, which together with \([3.7]\) yields that

$$
\sum_{r \in S} (\text{diam } \Lambda(A^r))^t \leq \left( \frac{c_4 \pi \text{ diam } A}{\rho} \right)^t \frac{c_6}{t-d+1} \left( \frac{a}{\rho} \right)^{d-1-t} (\text{diam } A)^t
= \frac{c_6 (c_4 \pi)^t}{\rho^{d-1}} \frac{a^{d-1-t}}{t-d+1} (\text{diam } A)^t.
$$

The conclusion follows. 

**Proof of the upper bound in Theorem 1.1.** Let $J(f_a) := \{ x : f_a^n(x) \to \xi_a \}$ be the complement of the attracting basin of $\xi_a$. For $R > M$ we consider the set

$$
K(R) := \{ x \in J(f_a) \cap B(0, R) : \liminf_{k \to \infty} |f_a^k(x)| < R \}.
$$

Let $c_7$ be the constant from Lemma 3.3. We show that if $a$ is sufficiently large and $d - 1 < t \leq d$ such that

$$
\tau := \frac{c_7 a^{d-1-t}}{(t-d+1)} < 1,
$$

then $\dim K(R) \leq t$. This implies that $\dim J_{\tau}(f_a) \leq t$, since

$$
J_{\tau}(f_a) = \bigcup_{n \in \mathbb{N}} K(R_n)
$$

for any sequence $(R_n)$ which tends to $\infty$. The conclusion follows from this, since for $t = d - 1 + \log \log a / \log a$ we have

$$
\frac{a^{d-1-t}}{(t-d+1)} = 1 / \log \log a.
$$
For $s \in S$, $A \subset T(s)$ and $n \in \mathbb{N}$, let $X_n(A)$ denote the set of all components of $f^{-n}(A)$ which are contained in $T(0)$. If $U \in X_n(A)$, then $f(U)$ has the form $f(U) = V^r$ for some $V \in X_{n-1}(A)$ and some $r \in S$. Equivalently, $U = \Lambda(V^r)$. In turn, if $V \in X_{n-1}(A)$ and $r \in S$, then $\Lambda(V^r) \in X_n(A)$. Together with Lemma 3.3 we thus find that

$$\sum_{U \in X_n(A)} (\text{diam } U)^t = \sum_{V \in X_{n-1}(A)} \sum_{r \in S} (\text{diam } \Lambda(V^r))^t \leq \tau \sum_{V \in X_{n-1}(A)} (\text{diam } V)^t.$$  

Induction yields that

$$\sum_{U \in X_n(A)} (\text{diam } U)^t \leq \tau^{n-1} \sum_{V \in X_1(A)} (\text{diam } V)^t = \tau^{n-1} (\text{diam } \Lambda(A))^t \leq \tau^{n-1} (\text{diam } A)^t. \tag{3.8}$$

We will apply this for

$$A^s := H \leq R \cap T(s).$$

There exists $N \in \mathbb{N}$ such that

$$J(f_a) \cap B(0, R) \subset \bigcup_{s \in S, |s| \leq N} A^s.$$  

Next we put

$$Y_n := \bigcup_{s \in S, m \geq n, |s| \leq N} X_m(A^s) \quad \text{and} \quad Z_n := \{U^s : s \in S, |s| \leq N, U \in Y_n\}.$$  

Then $Y_n$ contains all points $x \in T(0)$ for which there exists $m \geq n$ and $s \in S$ with $|s| \leq N$ such that $f_a^m(x) \in A^s$. Hence $Z_n$ contains all $x \in \bigcup_{s \in S, |s| \leq N} T(s)$ for which there exists $m \geq n$ such that

$$f_a^m(x) \in \bigcup_{s \in S, |s| \leq N} A^s.$$  

In particular, $Z_n$ contains all $x \in J(f_a) \cap B(0, R)$ for which there exists $m \geq n$ such that $|f_a^m(x)| \leq R$. Thus $K(R) \subset Z_n$ for all $n \in \mathbb{N}$.  

Let $L$ be the cardinality of $\{s \in S : |s| \leq N\}$. Then
\[
\sum_{U \in Z_n} (diam U)^t = L \sum_{U \in Y_n} (diam U)^t = L \sum_{s \in S} \sum_{m \geq n} U \in X_m(A^s) (diam U)^t \leq L \sum_{s \in S} \sum_{m \geq n} \tau^{m-1} (diam A^s)^t = L^2 \sum_{m \geq n} \tau^{m-1} (diam A^0)^t
\]
\[
= L^2 (diam A^0)^t \frac{\tau^{n-1}}{1 - \tau}
\]
by (3.8). Since the right-hand side tends to 0 as $n \to \infty$, we deduce that
\[
\dim K(R) \leq t.
\]

4. Proof of the lower bounds

The proof is based on the theory of iterated functions systems. A similar method was used in [1] to estimate the dimensions of Julia sets from below. In particular, we will use the following result [7, Proposition 9.7].

**Lemma 4.1:** Let $S_1, \ldots, S_m$ be contractions on a closed subset $K$ of $\mathbb{R}^d$ such that there exists $b_1, \ldots, b_m \in (0, 1)$ with
\[
b_j |x - y| \leq |S_j(x) - S_j(y)| \quad \text{for } x, y \in K \text{ and } 1 \leq j \leq m.
\]
Suppose that $K_0$ is a non-empty compact subset of $K$ with
\[
K_0 = \bigcup_{j=1}^m S_j(K_0)
\]
and $S_j(K_0) \cap S_k(K_0) = \emptyset$ for $j \neq k$. Let $t > 0$ with
\[
(4.1) \quad \sum_{j=1}^m b_j^t = 1.
\]
Then $\dim K_0 \geq t$.

Since the left-hand side of (4.1) is a decreasing function of $t$, it follows that if
\[
\sum_{j=1}^m b_j^t > 1,
\]
then $\dim K_0 > t$. 

Proof of Theorem 2.1 and the lower bound in Theorem 1.1  Let $N \in \mathbb{N}$ with $N \geq a/\rho$ and put $R := 8\rho N$ and $K := B(-\overline{a}, R) \cap H \geq M$. Note that if $N$ is sufficiently large, then $R > M - a$ so that $K$ is non-empty.

Next we note that if $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with $x_d > \log R$, then

$$|f(x) + a| = |f(x)| = e^{x_d} > R$$

and thus $f(x) \notin B(-\overline{a}, R)$. It follows that

$$\Lambda^r(K) \subset A^r := H_{\leq \log R} \cap T(r)$$

for $r \in S$.

Put $L := a + \log R = a + \log(8\rho N)$. For $y \in A^r$ we have

$$(4.2) \quad |y + \overline{a}|^2 \leq \sum_{j=1}^{d-1} (2|r_j| + 1)^2 \rho^2 + L^2 \leq 8\rho^2 |r|^2 + (d-1) \rho^2 + L^2 \leq 8(\rho^2 |r|^2 + L^2)$$

if $N$ and hence $L$ are large. Since $a \leq \rho N$ we have $L \leq \rho N + \log(8\rho N) \leq 2\rho N$ if $N$ and $L$ are large. Thus we see that if $|r| \leq N$ and $y \in A_r$, then

$$|y + \overline{a}|^2 \leq 8(\rho^2 N^2 + 4\rho^2 N^2) = 40\rho^2 N^2 \leq R^2.$$ 

Thus $A^r \subset K$ if $|r| \leq N$, provided $N$ is sufficiently large. Hence $\Lambda^r(K) \subset K$ if $r \in S$ and $|r| \leq N$. Together with (2.5) this implies that the $\Lambda^r$ are contractions on $K$.

It follows that the functions $S_j$ of the form $S_j = \Lambda^r \circ \Lambda^s$, where $r, s \in S$ and $|r|, |s| \leq N$, are also contractions and hence form an iterated function system on $K$. We will apply Lemma 4.1 to this iterated function system.

It follows from Lemma 2.3 that

$$\ell(D\Lambda^r(x)) = \ell(D\Lambda(x)) \geq c_3/R \quad \text{for } x \in K \text{ and } r \in S.$$ 

By (4.2) we also have

$$\ell(D\Lambda^s(y)) = \ell(D\Lambda(y)) \geq \frac{c_3}{|y + \overline{a}|} \geq \frac{c_3}{2\sqrt{2}\sqrt{\rho^2 |r|^2 + L^2}} \quad \text{for } y \in A^r.$$ 

Hence

$$\ell(D(\Lambda^s \circ \Lambda^r)(y)) \geq \ell(D\Lambda^s(\Lambda^r(x))) \cdot \ell(D\Lambda^r(x)) \geq \frac{c_3^2}{2\sqrt{2R}\sqrt{\rho^2 |r|^2 + L^2}}$$

for $x \in K$. It follows that

$$|(\Lambda^s \circ \Lambda^r)(x) - (\Lambda^s \circ \Lambda^r)(y)| \geq b_{r,s} |x - y|$$

with
Let the cube $P_r$ be defined as in the proof of Lemma 3.2. Then each $P_r$ has volume $6^{d-1}$ and the union of all $P_r$ for which $|r| \leq N$ covers $B(0, N)$ and thus has volume at least $\pi^{(d-1)/2} N^{d-1}/\Gamma((d+1)/2)$. Thus the set of all $s \in S$ for which $|s| \leq N$ has at least $\lfloor 6^{1-d} \pi^{(d-1)/2} N^{d-1}/\Gamma((d+1)/2) \rfloor$ elements. Recalling that $R = 8 \rho N$ we deduce that there exist a constant $c_8$ such that with

$$b := \frac{L}{\rho} = \frac{(a + \log(8\rho N))}{\rho}$$

we have

$$\sum_{r \in S} \sum_{s \in S} b_{r,s}^t \geq \left[ \frac{6^{1-d} \pi^{(d-1)/2} N^{d-1}}{\Gamma((d+1)/2)} \right] \frac{c_3^2}{(2\sqrt{2}R)^t} \sum_{r \in S} \sum_{s \in S} \frac{1}{(|r|^2 + b^2)^{t/2}}$$

for $d - 1 \leq t \leq d$.

For large $N$ we have $N \geq b = (a + \log(8\rho N))/\rho \geq 3\sqrt{d-1}$. Thus Lemma 3.2 is applicable.

Suppose first that $t = d - 1$. Using (3.2) we find with $c_9 := c_8 c_5$ that

$$\sum_{r \in S} \sum_{s \in S} b_{r,s}^{d-1} \geq c_9 \log \frac{N}{b} = c_9 \log \frac{N \rho}{a + \log(8\rho N)}.$$

The right-hand side of (4.5) tends to $\infty$ as $N \to \infty$. In particular, it is greater than 1 for large $N$ so that (4.3) holds for $t = d - 1$. Thus $\dim J_{bd}(f_a) > d - 1$. This proves Theorem 2.1.

Now we consider the behavior as $a \to \infty$. Let $t = d - 1 + \gamma(a)$ where

$$\gamma(a) := \frac{1}{2} \frac{\log \log a}{\log a} - \frac{\log \log \log a}{\log a}.$$
Then $d - 1 < t \leq d$ for large $a$. We also put $\beta(a) := e^{1/\gamma(a)}$ and note that $\beta(a) \to \infty$ as $a \to \infty$. We may choose $N$ such that

$$N \sim a \beta(a) \quad \text{as } a \to \infty.$$  

Since $\log \beta(a) = 1/\gamma(a)$ we find that

$$L = a + \log(8\rho N) \sim a$$

and hence $b = L/\rho \sim a/\rho$.

With $C := 2/\rho$ we thus have $Nb \leq Ca^2 \beta(a)$ and $N/b \geq \beta(a)/C$ for large $a$. By the definition of $\beta(a)$ we have $\beta(a)^{\gamma(a)} = e$. For large $a$ we also have

$$\frac{1}{2} \leq C^{\gamma(a)} \leq 2.$$  

It thus follows from (4.4) and Lemma 3.2 that

$$\sum_{r \in S} \sum_{s \in S} b_{r,s}^t \geq c_9 \left( \frac{Nb}{t - d + 1} \right)^{d-1-t} \left(1 - \left(\frac{N}{b}\right)^{d-1-t}\right)$$

(4.6)

$$\geq c_9 \left( \frac{Ca^2 \beta(a)^{-\gamma(a)}}{\gamma(a)} \right)^{\gamma(a)} \left(1 - \left(\frac{\beta(a)}{C}\right)^{-\gamma(a)}\right)$$

$$\geq c_9 \frac{a^{-2\gamma(a)}}{2e \gamma(a)} \left(1 - \frac{2e}{e}\right)$$

for large $a$. It is easy to see that $\log \gamma(a) = \log \log \log a - \log \log a + O(1)$ as $a \to \infty$. It thus follows that

$$\log \left(\frac{a^{-2\gamma(a)}}{\gamma(a)}\right) = -2\gamma(a) \log a - \log \gamma(a) = \log \log \log a + O(1)$$

as $a \to \infty$. Thus the right hand side of (4.6) tends to $\infty$ as $a \to \infty$. Hence (4.3) holds for $t = d - 1 + \gamma(a)$ and large $a$. This proves the lower bound in Theorem 1.1.

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