An improved fountain theorem and its application

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Abstract

The main aim of the paper is to prove a fountain theorem without assuming the \(\tau\)-upper semi-continuity condition on the variational functional. Using this improved fountain theorem, we may deal with more general strongly indefinite elliptic problems with various sign-changing nonlinear terms. As an application, we obtain infinitely many solutions for a semilinear Schrödinger equation with strongly indefinite structure and sign-changing nonlinearity.

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1 Introduction

Since the pioneering works of Bartsch and Willem \cite{2, 3} (see also, \cite{21}), variant fountain theorems are established and which have been used to study the existence of infinitely many solutions for various elliptic problems, see e.g., \cite{2, 3, 4, 5, 6, 12, 22} and the references therein. In order to investigate infinitely many critical points of strongly indefinite functionals, Batkam-Colin \cite{5} established a generalized fountain theorem based on the so-called \(\tau\)-topology introduced by Kryszewski-Szulkin \cite{13}. For recalling the fountain theorem proved in \cite{5}, we introduce some notations and definitions which are also often used in the following sections of the paper.

Let \(X\) be a separable Hilbert space and \(Y \subset X\) be a closed subspace of \(X\) endowed with inner product \(\langle \cdot , \cdot \rangle\) and norm \(\| \cdot \|\). Let

\[
X = Y \bigoplus Z \text{ with } Z = Y^\perp, \quad Y = \bigoplus_{j=0}^{\infty} \mathbb{R} e_j \text{ and } Z = \bigoplus_{j=0}^{\infty} \mathbb{R} f_j, \quad (1.1)
\]

where \(\{e_j\}_{j \geq 0}\) and \(\{f_j\}_{j \geq 0}\) are orthonormal bases of \(Y\) and \(Z\), respectively. Moreover, we define

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\[ Y_k := Y \bigoplus_{j=0}^{k} \mathbb{R} f_j \quad \text{and} \quad Z_k := \bigoplus_{j=k}^{\infty} \mathbb{R} f_j, \tag{1.2} \]

and let
\[ P : X \to Y, \quad Q : X \to Z \quad \text{and} \quad P_k : X \to Y_{k-1}, \quad Q_k : X \to Z_k \tag{1.3} \]
be the orthogonal projections. The \( \tau \)-topology on \( X = Y \bigoplus Z \) introduced in [13] is the topology associated to the following norm
\[ \|u\|_\tau := \max\{ \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle Pu, e_j \rangle|, \|Qu\| \}, \quad \text{for} \ u \in X. \tag{1.4} \]

By the above definition, we see that
\[ \|u\|_\tau \leq \max\{ \|Pu\|, \|Qu\| \} \leq \|u\| \quad \text{for} \ u \in X. \tag{1.5} \]

Furthermore, it follows from [13] and the appendix of [17] that, if \( \{u_n\} \subset X \) is bounded, then
\[ u_n \overset{\tau}{\to} u \quad \text{in} \ \tau \text{-topology} \iff Pu_n \overset{\tau}{\to} Pu \ \text{and} \ Qu_n \overset{\tau}{\to} Qu. \tag{1.6} \]

**Remark 1.1** Let
\[ \tilde{e}_j = \begin{cases} f_j, & j = 0, 1, \ldots, k-1 \\ e_{j-k}, & j \geq k, \end{cases} \]
and define the following norm
\[ \|u\|_{\tau_k} = \max\{ \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle P_k u, \tilde{e}_j \rangle|, \|Q_k u\| \}, \tag{1.7} \]
then, \( \| \cdot \|_{\tau_k} \) and \( \| \cdot \|_\tau \) are equivalent for all \( k \geq 1 \). In fact, it is enough to show that \( \| \cdot \|_{\tau_1} \) and \( \| \cdot \|_\tau \) are equivalent. By (1.4),
\[ \|u\|_\tau^2 \leq \left( \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle u, e_j \rangle| \right)^2 + \sum_{j=0}^{\infty} |\langle u, f_j \rangle|^2 \]
\[ = \left( \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle u, e_j \rangle| \right)^2 + |\langle u, f_0 \rangle|^2 + \sum_{j=1}^{\infty} |\langle u, f_j \rangle|^2 \]
\[ \leq 4 \left( \frac{1}{2} |\langle u, f_0 \rangle| + \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} |\langle u, e_j \rangle| \right)^2 + 4 \sum_{j=1}^{\infty} |\langle u, f_j \rangle|^2 \leq 8 \|u\|_{\tau_1}^2. \]

On the other hand,
\[ \|u\|_{\tau_1} \leq \frac{1}{2} |\langle u, f_0 \rangle| + \sum_{j=0}^{\infty} \frac{1}{2^{j+2}} |\langle u, e_j \rangle| + \left( \sum_{j=1}^{\infty} |\langle u, f_j \rangle|^2 \right)^{1/2} \]
\[ \leq \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle u, e_j \rangle| + 2^{1/2} \left( |\langle u, f_0 \rangle|^2 + \sum_{j=1}^{\infty} |\langle u, f_j \rangle|^2 \right)^{1/2} \leq 2^{3/2} \|u\|_\tau. \]
Throughout the paper, for $r_k > 0$ and $\rho_k > 0$, we always set
\[
B_k := \{u \in Y_k : \|u\| < \rho_k\} \quad \text{and} \quad N_k := \{u \in Z_k : \|u\| = r_k\}.
\]
(1.8)
Since $X$ is a Hilbert space, if $\varphi \in C^1(X, \mathbb{R})$, $\nabla \varphi$ is given by the formula
\[
\langle \nabla \varphi(u), v \rangle = \varphi'(u)v, \quad \text{for all} \quad v \in X.
\]

With the above notations, for an even functional the fountain theorem proved in [5] can be stated as follows

**Theorem 1.1** [5, Corollary 13] Let $\varphi \in C^1(X, \mathbb{R})$ be an even functional satisfying

- $\nabla \varphi$ is weakly sequentially continuous, i.e., for any $v \in X$, $\varphi'(u_n)v \overset{\text{n}}{\to} \varphi'(u)v$ if $u_n \overset{\text{n}}{\rightharpoonup} u$ weakly in $X$.
- $\varphi$ is $\tau$-upper semi-continuous, i.e., for any $c \in \mathbb{R}$, the set $\varphi_c := \{u \in X : \varphi(u) \geq c\}$ is $\tau$-closed.
- For any $c > 0$, $\varphi$ satisfies $(PS)_c$ condition, i.e., any sequence $\{u_n\} \subset X$ with $\varphi(u_n) \overset{n}{\to} c$ and $\varphi'(u_n) \overset{n}{\to} 0$ in $X'$ (the dual space of $X$)

has a convergent subsequence.

Additionally, if there exist $\rho_k > r_k > 0$ such that:

(A1) \quad $d_k := \sup_{u \in Y_k, \|u\| \leq \rho_k} \varphi(u) < \infty$, 

(A2) \quad $a_k := \sup_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \leq 0$, 

(A3) \quad $b_k := \inf_{u \in Z_k, \|u\| = r_k} \varphi(u) \to \infty$ as $k \to \infty$.

Then, $\varphi$ has an unbounded sequence of critical values.

The $\tau$-upper semi-continuity was proposed in [13] for showing a generalized linking theorem. Similar to [13], this condition is also required in the above Theorem 1.1 which is mainly used to construct a suitable vector field. Theorem 1.1 can be used to deal with some strongly indefinite elliptic problems, but the $\tau$-upper semi-continuity assumption requires that the primitive functions of the nonlinearities of the elliptic problems should be positive, see condition (f4) in [5]. It is natural to ask what would happen if the nonlinear terms of an elliptic problem change sign and lose the positivity condition (f4)? So, the main aim of the paper is to establish a variant fountain theorem without assuming the $\tau$-upper semi-continuity and then we may answer the above question, see our Theorems 1.2 and 1.3. Our proofs are motivated by the papers [5, 7, 16]. We mention that if the $\tau$-upper semi-continuity of $\varphi$ is removed, several steps in the proofs for Theorem 1.1 in [5] seem not working any more, for examples:
We cannot construct the pseudogradient vector by the same way as in [5] since the set $\varphi^{-1}(-\infty, c)$ may not be $\tau$-open now. This difficulty is overcome in this paper by using some ideas from [7].

To the authors’ knowledge, the intersection lemma used in [5] is no longer applicable since the descending flow in our paper has different behavior from that of [5]. In this paper, we use the intersection lemma given in [16] instead.

We cannot make an explicit mini-max characterization on the critical values of $\varphi$ because of the lack of $\tau$-upper semi-continuity for $\varphi$, then it is hopeless to get infinitely many different critical points of $\varphi$ by comparing their critical values as that of [5] or [2]. In this paper, we get infinitely many different critical points $\{u_n\}$ of $\varphi$ by comparing their norm $\|u_n\|$ and proving $\|u_n\| \xrightarrow{n} +\infty$.

Now, we give our improved fountain theorem:

**Theorem 1.2** Let $\varphi \in C^1(X, \mathbb{R})$ be an even functional satisfying $(PS)^c$ condition (i.e., any sequence $\{u_n\} \subset X$ with $\sup_n \varphi(u_n) \leq c$ and $\varphi'(u_n) \xrightarrow{n} 0$ having a convergent subsequence) and let $\nabla \varphi$ be weakly sequentially continuous. For any $k \in \mathbb{N}$, if there exists $\rho_k > r_k > 0$ such that, in addition to the above assumptions ($A_1$) and ($A_3$), there holds

\[(A_2)' \quad a_k := \sup_{u \in Y_k, \|u\| = \rho_k} \varphi(u) < \inf_{u \in Z_k, \|u\| \leq r_k} \varphi(u),\]

\[(A_4) \quad \sup_{\|u\| < \delta} \varphi(u) \leq C_\delta < \infty, \text{ for any } \delta > 0,\]

then, $\varphi$ has a sequence of critical points $\{u^{km}\}$ such that $\lim_{m \to \infty} \|u^{km}\| \to \infty$.

**Remark 1.2** In our Theorem 1.2, the $\tau$-upper semi-continuity is not assumed. But, we replaced condition ($A_2$) in Theorem 1.1 by ($A_2)'$ and added a new assumption ($A_4$). However, the conditions ($A_2)'$ and ($A_4$) are easily verified in the applications, see e.g., the proof of our Theorem 1.3.

With the above Theorem 1.2, we may study the existence of infinitely many solutions for the following Schrödinger equation with strongly indefinite linear part and sign-changing nonlinear term:

\[
\begin{cases}
-\Delta u + V(x)u = g(x)|u|^{q-2}u + h(x)|u|^{p-2}u, \\
u \in H^1(\mathbb{R}^N), \quad N \geq 3,
\end{cases}
\]

where $1 < q < \frac{p}{p-1} < 2 < p < 2^*$ and $2^* = \frac{2N}{N-2}$, $V(x)$, $g(x)$ and $h(x)$ are functions satisfying

$(H_1)$ $V(x) \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty$ and $0$ lies in a spectrum gap of the operator $-\Delta + V$.

$(H_2)$ $g \in L^{q_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $q_0 = \frac{2N}{2N-4N+2q}$. 
(H3) \( h \in L^{p_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) with \( p_0 = \frac{2N}{2N-pN+2p} \) and \( h(x) > 0 \) a.e. in \( \mathbb{R}^N \).

Since 0 lies in a gap of the spectrum of \(-\Delta + V\), problem (1.9) may be strongly indefinite. The nonlinearity in (1.9) has a super-linear part and a sub-linear part which is usually called concave-convex nonlinearity, some well-known results corresponding to concave-convex nonlinearities can be found in [1, 2] and the references therein. By (H2), we see that the weight function \( g(x) \) may change sign, so, the variational functional of (1.9) does not satisfy the \( \tau \)-upper semi-continuous assumption.

We mention that there are some papers on the existence of solutions for Schrödinger equations with both sign-changing potential \( V(x) \) and indefinite nonlinearities, see, e.g., [8, 9, 10, 14, 19, 20], etc. But the problems discussed in [8, 9, 10, 19, 20] do not have strongly indefinite structure, and in paper [14] the potential \( V(x) \) has to be periodic and the weight functions in nonlinear term must satisfy some additional conditions. There seems no results for problem (1.9) under the conditions (H1)-(H3). For problem (1.9), we have the following theorem.

**Theorem 1.3** If the conditions (H1)-(H3) hold, then problem (1.9) has a sequence of nontrival solutions \( \{u_n\} \subset H^1(\mathbb{R}^N) \) with \( \|u_n\|_{H^1} \to \infty \), as \( n \to \infty \).

### 2 Proof of our fountain theorem

In this section, we are going to prove our fountain theorem, that is, Theorem 1.2. For doing this, some lemmas are required.

**Lemma 2.1** Let \( Y_k \) and \( Z_k \) be defined in (1.2). \( \varphi \in C^1(X, \mathbb{R}) \) is an even functional and \( \nabla \varphi \) is weakly sequentially continuous. If there exist \( k \in \mathbb{N} \) and \( \rho_k > r_k > 0 \) such that conditions (A1) (A2)' are satisfied and there holds

\[
b_k := \inf_{\|u\|_{\tau} = r_k} \varphi(u) > \sup_{\|u\|_{\tau} < \delta} \varphi(u), \quad \text{for some} \quad \delta > 0.
\]

Then, there exists a sequence \( \{u_n^k\} \subset \varphi_{d_k+1} \) such that

\[
\inf_n \|u_n^k\|_{\tau} \geq \frac{\delta}{2} \quad \text{and} \quad \varphi'(u_n^k) u_n^k \to 0 \quad \text{in} \quad X'(\text{the dual space of} \ X).
\]

In order to prove Lemma 2.1 we need the following deformation lemma:

**Lemma 2.2** Under the assumptions of Lemma 2.1, let

\[
E = \varphi_{d_k+1} \bigcap \{u \in X : \|u\|_{\tau} \geq \frac{\delta}{2}\},
\]

with \( \varphi_{d_k+1} := \{u \in X : \varphi(u) \leq d_k + 1\} \) and \( \delta \) given in (2.1), if there exists \( \epsilon \in (0, \frac{1}{2}) \) with

\[
0 < \epsilon < b_k - \max\{a_k, \sup_{\|u\|_{\tau} < \delta} \varphi(u)\},
\]

then there exist \( \{u_n\} \subset X \) such that

\[
\inf_n \|u_n\|_{\tau} \geq \frac{\delta}{2} \quad \text{and} \quad \varphi'(u_n) u_n \to 0 \quad \text{in} \quad X'(\text{the dual space of} \ X).
\]
such that
\[ \| \varphi'(u) \| > \epsilon, \text{ for any } u \in E, \]
then, there exist \( T > 0 \) and a map \( \eta(t, u) \in C([0, T] \times B_k, X) \) with \( B_k \) given by [1.5] such that

(i) \( \eta(0, u) = u \) and \( \eta(t, -u) = -\eta(t, u) \) for any \( u \in B_k \) and \( t \in [0, T] \).

(ii) \( \varphi(\eta(t, u)) \) is non-increasing in \( t \in [0, T] \) for fixed \( u \in B_k \).

(iii) \( \eta \) is \( \tau \)-continuous (i.e., \( \eta(t_m, u_m) \xrightarrow{\tau} \eta(t, u) \) in \( \tau \)-topology, if \( t_m \xrightarrow{\tau} t \) and \( u_m \xrightarrow{\tau} u \) in \( \tau \)-topology) and \( \eta(t, \cdot) : B_k \to \eta(t, B_k) \) is a \( \tau \)-homeomorphism for any \( t \in [0, T] \).

(iv) \( \eta(T, B_k) \subset \varphi^{b_k+\epsilon} \).

(v) For any \( (t, u) \in [0, T] \times B_k \), there exist a neighborhood \( W_{(t,u)} \) of \( (t, u) \) in the \( \cdot \times \tau \)-topology such that
\[ \{ v - \eta(s, v) | (s, v) \in W_{(t,u)} \cap ([0, T] \times B_k) \} \]
is contained in a finite-dimensional subspace of \( X \).

**Proof.** For \( \epsilon > 0 \) given in [2.3], let
\[ B_R = \{ u \in X : \| u \| \leq R \}, \text{ where } R = 2(d_k - b_k + 2\epsilon)/\epsilon + \rho_k + \delta. \] (2.4)
Firstly, we claim that there exists a vector field \( \chi : \varphi^{d_k+1} \to X \) such that

(a) \( \chi \) is odd with \( \| \chi(u) \| \leq 2/\epsilon \) and \( \langle \nabla \varphi(u), \chi(u) \rangle \leq 0 \), for any \( u \in \varphi^{d_k+\epsilon} \).

(b) \( \chi(u) \) is locally Lipschitz continuous and \( \tau \)-locally Lipschitz \( \tau \)-continuous on \( \varphi^{d_k+\epsilon} \).

(c) \( \langle \nabla \varphi(u), \chi(u) \rangle < -1 \), for any \( u \in \varphi^{-1}[b_k - \epsilon, d_k + \epsilon] \cap B_R \).

(d) For any \( u \in W, W \) is given by [2.10], there exist a \( \tau \)-open neighborhood \( U_u \in N \) of \( u \) such that \( \chi(U_u) \) is contained in a finite-dimensional subspace of \( X \).

In fact, by our assumption, \( \| \varphi'(u) \| > \epsilon \) for any \( u \in E \), we may define
\[ \omega(u) = \frac{2\nabla \varphi(u)}{\| \nabla \varphi(u) \|}, \text{ for } u \in E \bigcap B_R, \]
then, there exists a \( \tau \)-neighborhood \( V_u \subset X \) of \( u \) such that
\[ \langle \nabla \varphi(v), \omega(u) \rangle > 1, \text{ for any } v \in V_u \bigcap B_R. \] (2.5)
Otherwise, if such \( V_u \) does not exist, then there exists a sequence \( \{ v_n \} \subset B_R \) such that \( v_n \xrightarrow{\tau} u \) and \( \lim_{n \to \infty} \langle \nabla \varphi(v_n), \omega(u) \rangle \leq 1 \). By [1.6] we have \( v_n \rightharpoonup u \) weakly in \( X \) and this leads to a contradiction since \( \nabla \varphi \) is weakly continuous and \( \langle \nabla \varphi(u), \omega(u) \rangle = 2 \).
Note that $B_R$ is $\tau$-closed \cite{7}, thus $X \setminus B_R$ is $\tau$-open, and

\[ \mathcal{N} = \{ V_u : u \in E \cap B_R \} \cup \{ X \setminus B_R \} \] \hspace{1cm} (2.6)

forms a $\tau$-open covering of $E$.

Since $\mathcal{N}$ is metric, hence paracompact, there exists a local finite $\tau$-open covering $\mathcal{M} = \{ M_i : i \in \Lambda \}$, where $\Lambda$ is an index set, of $E$ finer than $\mathcal{N}$. If $M_i \subset V_{u_i}$ for some $u_i \in E$, we choose $\omega_i = \omega(u_i)$ and if $M_i \subset X \setminus B_R$, we choose $\omega_i = 0$. Let $\{ \lambda_i(u) : i \in \Lambda \}$ be a $\tau$-Lipschitz continuous partition of unity subordinated to $\mathcal{M}$ and let

\[ \xi(u) = \sum_{i \in \Lambda} \lambda_i(u) \omega_i, \quad u \in \mathcal{N}. \]

Since the $\tau$-open covering $\mathcal{M}$ of $\mathcal{N}$ is local finite, each $u \in \mathcal{N}$ belongs to finite many sets $M_i$. Therefore, for every $u \in \mathcal{N}$, the sum $\xi(u)$ is only a finite sum. It follows that, for any $u \in \mathcal{N}$, there exist a $\tau$-open neighborhood $U_u \in \mathcal{N}$ of $u$ such that $\xi(U_u)$ is contained in a finite-dimensional subspace of $X$. Then, by the equivalence of norms in a finite-dimensional vector space, we know that there exists $C > 0$ such that

\[ \| \xi(v) - \xi(w) \| \leq C \| \xi(v) - \xi(w) \|_\tau, \quad \forall v, w \in U_u. \] \hspace{1cm} (2.7)

On the other hand, by the $\tau$-Lipschitz continuity of $\lambda_i$ and \cite{1,3}, we have that there exists a constant $L_u > 0$ such that

\[ \| \xi(v) - \xi(w) \|_\tau \leq L_u \| v - w \|_\tau \leq L_u \| v - w \|, \quad \forall v, w \in U_u. \] \hspace{1cm} (2.8)

Then, from (2.7) and (2.8) we know that $\xi(u)$ is locally Lipschitz continuous and $\tau$-locally Lipschitz $\tau$-continuous. Moreover, by (2.5) and the property of $\lambda_i$, we also have that

\[ \langle \nabla \varphi(u), \xi(u) \rangle > 1 \] and $\| \xi(u) \| < \frac{2}{\epsilon}$, for any $u \in E \cap B_R$.

Since $\varphi$ is even, $\mathcal{N}$ is symmetric, we define $\tilde{\xi}(u) := \frac{\xi(u) - \xi(-u)}{2}$ for $u \in \mathcal{N}$, and $\tilde{\xi}(u)$ is odd. For $\delta > 0$ given by (2.1), let $\theta \in C^\infty(\mathbb{R}, [0, 1])$ such that

\[ \theta(t) = \begin{cases} 0, & 0 \leq t \leq \frac{2\delta}{\tau}, \\ 1, & t \geq \delta. \end{cases} \]

Define the vector field $\chi : \mathcal{N} \to X$ by

\[ \chi(u) = \begin{cases} -\theta(\| u \|_\tau) \tilde{\xi}(u), & u \in \mathcal{N}, \\ 0, & \| u \|_\tau \leq \frac{2\delta}{\tau}. \end{cases} \] \hspace{1cm} (2.9)

It’s easy to see that $\chi$ is an odd vector field and also well defined on

\[ \mathcal{W} := \mathcal{N} \bigcup \{ u \in X : \| u \|_\tau < \delta \}, \] \hspace{1cm} (2.10)

satisfying

\[ \| \chi(u) \| \leq \frac{2}{\epsilon} \] and $\langle \nabla \varphi(u), \chi(u) \rangle \leq 0$, for any $u \in \mathcal{W}$. \hspace{1cm} (2.11)
Since $0 < \epsilon < \frac{1}{2}$, we have $\mathcal{W}$ covers $\varphi^{d_k+\epsilon} \cup (X \setminus B_R)$, this shows (a). By the construction of $\chi(u)$, we know that $\chi(u)$ is locally Lipschitz continuous and $\tau$-locally Lipschitz $\tau$-continuous on $\varphi^{d_k+\epsilon}$, and (b) is proved.

Moreover, by the choice of $\epsilon$ in (2.3), we have
\[
\sup_{\|u\|_{\tau} \leq \delta} \varphi(u) < b_k - \epsilon,
\]
i.e.,
\[
\{ u \in X : \|u\|_{\tau} \leq \delta \} \subset \varphi^{b_k-\epsilon}.
\]
So, by (2.4),
\[
\langle \nabla \varphi(u), \chi(u) \rangle < -1, \text{ for any } u \in \varphi^{-1}[b_k - \epsilon, d_k + \epsilon] \cap B_R, \tag{2.12}
\]
which implies (c). Then, by the definition of $\chi(u)$, i.e., (2.4), and the properties of $\xi(u)$, we see that (d) holds. So, the claim is proved.

Next, we turn to proving (i)-(v) of the lemma. For this purpose, we construct a map $\eta$ through the following Cauchy problem:
\[
\begin{cases}
\frac{d\eta}{dt} = \chi(\eta) \\
\eta(0, u) = u \in \mathcal{W}.
\end{cases}
\]
(2.13)

By the standard theory of ordinary differential equation in Banach space, we know that the initial problem has a unique solution $\eta(t, u)$ on $[0, \infty)$. Furthermore, the similar argument to the proof of [21, Lemma 6.8] yields that $\eta$ is $\tau$-continuous. Moreover, $\eta(t, \cdot) : B_k \to \eta(t, B_k)$ is a $\tau$-homeomorphism for any $t \in [0, T]$. So, part (iii) is proved.

Let $B_k$ and $B_R$ be given by (1.8) and (2.4). Taking
\[
T = d_k - b_k + 2\epsilon. \tag{2.14}
\]
Then, $\{ \eta(t, u) : 0 \leq t \leq T, u \in B_k \} \subset B_R$. Indeed, it follows from (2.13) that
\[
\eta(t, u) = u + \int_0^t \chi(\eta(s, u))ds, \text{ for } u \in B_k.
\]
By the definition of $d_k$ (see condition $(A_1)$), we know that $B_k \subset \mathcal{W}$. Then, by (2.3), (2.4) and (2.11), we have for any $u \in B_k$ and $t \in [0, T]
\[
\|\eta(t, u)\| \leq \|u\| + \int_0^t \|\chi(\eta(s, u))\|ds
\leq \|u\| + \int_0^t \frac{2}{\epsilon}ds \leq \|u\| + \frac{2T}{\epsilon} \leq R.
\]
So, (i) is obvious by the oddness of $\chi(u)$. By (a) we have
\[
\frac{d}{dt} \varphi(\eta(t, u)) = \langle \nabla \varphi(u), \chi(u) \rangle \leq 0, \text{ for any } u \in B_k,
\]
so, \( \varphi(\eta(t, u)) \) is non-increasing in \( t \in [0, T] \) for fixed \( u \in B_k \) and (ii) is proved.

Now, we claim that \( \eta(T, B_k) \subset \varphi^{b_k - \epsilon} \). Otherwise, there exists \( u \in B_k \) such that

\[
\varphi(\eta(T, u)) > b_k - \epsilon. \tag{2.15}
\]

Since \( \eta(t, u) \) is non-increasing along \( t \), we have

\[
\eta(t, u) \in \varphi^{-1}[b_k - \epsilon, d_k + \epsilon] \bigcap B_R, \quad \text{for any } t \in [0, T]. \tag{2.16}
\]

Then, using (2.12) we see that

\[
\varphi(\eta(T, u) = \varphi(\eta(0, u) + \int_0^T \langle \varphi'(\eta(s, u)), \chi(\eta(s, u)) \rangle ds
\leq \varphi(\eta(0, u) + \int_0^T -1 ds
\leq d_k + \epsilon - T = b_k - \epsilon,
\]

which contradicts to (2.15), and (iv) is proved.

Finally, by (d) and (iii), similar to the proof of Lemma 6.8 of [21], we see that (v) also holds. □

For \( \tau_k \)-norm defined in (1.7), the same as in [16] we introduce the following definition

**Definition 2.1** Let \( B_k \) and \( N_k \) be defined in (1.8). For any \( T > 0 \), the mapping \( \gamma : [0, T] \times B_k \to X \) is a \( \tau_k \)-admissible homotopy if

- \( \gamma \) is \( \tau_k \)-continuous in the sense that
  \[
  \gamma(t_m, u_m) \xrightarrow{\tau_k} \gamma(t, u) \text{ in } \tau_k\text{-topology, if } t_m \xrightarrow{m} t \text{ and } u_m \xrightarrow{m} u \text{ in } \tau_k\text{-topology.}
  \]

- For any \( (t, u) \in [0, T] \times B_k \) there exist a neighborhood \( W_{(t,u)} \) of \( (t, u) \) in the \( | \cdot | \times \tau_k \)-topology such that
  \[
  \{ v - \gamma(s, v) | (s, v) \in W_{(t,u)} \bigcap ([0, T] \times B_k) \}
  \]
  is contained in a finite-dimensional subspace of \( X \).

We remark that such \( \gamma \) does exist since the identity mapping \( I_d : I_d(t, u) \equiv u \) is a \( \tau_k \)-admissible homotopy.

Let \( Y_k \) and \( Z_k \) be defined in (1.2), \( B_k \) and \( N_k \) be defined in (1.8). The following intersection lemma is proved in [16] where the author estimated the genus of \( (\gamma(t, B_k) \bigcap N_k) \) (see also [11]).

**Lemma 2.3** [16] Proposition 7] Let \( \varphi \in C^1(X, \mathbb{R}) \) be an even functional and let \( \gamma : [0, T] \times B_k \to X \) be a \( \tau_k \)-admissible homotopy with the following properties:

- \( \gamma(0, u) = u, \) for any \( u \in B_k \),
\[\gamma(t, -u) = -\gamma(t, u),\]

- \(\varphi(\gamma(t, u))\) is non-increasing in \(t \in [0, T]\) for fixed \(u \in B_k\),

- for any \(t \in [0, T]\), \(\gamma(t, \cdot) : B_k \to \gamma(t, B_k)\) is a \(\tau_k\)-homeomorphism.

If \(\sup_{u \in Y_k, \|u\| = \rho_k} \varphi(u) < \inf_{u \in Z_k, \|u\| \leq r_k} \varphi(u)\), with \(0 < r_k < \rho_k\), then

\[\gamma(t, B_k) \cap N_k \neq \emptyset\quad \text{for any } t \in [0, T],\]

where \(B_k\) is given by (1.8).

Now we come to prove Lemma 2.1.

**Proof of Lemma 2.1**

By contradiction, if the conclusion of Lemma 2.1 is false, then there exists \(\epsilon > 0\) such that

\[\|\varphi'(u)\| > \epsilon, \text{ for any } u \in E,\]

where \(E\) is defined by (2.2). By Lemma 2.2 we know that there exists a map \(\eta(t, u) \in C([0, T] \times B_k, X)\) satisfying Lemma 2.2 (i)-(v). By Remark 1.1 the \(\tau\)-topology and \(\tau_k\)-topology are equivalent, then it is easy to see that \(\eta(t, u)\) satisfies the assumptions of Lemma 2.3 hence,

\[\eta(T, B_k) \cap N_k \neq \emptyset,\]

and the definition of \(b_k\) implies that

\[\sup_{u \in B_k} \varphi(\eta(T, u)) \geq b_k.\]

However, Lemma 2.2 (iv) shows that

\[\sup_{u \in B_k} \varphi(\eta(T, u)) \leq b_k - \epsilon,\]

which leads to a contradiction. So, the proof is complete.

**Proof of Theorem 1.2**

Taking \(\delta_1 > 0\), by \((A_1)\) we know that

\[\sup_{\|u\| < \delta_1} \varphi(u) \leq C_{\delta_1},\]

for some \(C_{\delta_1} > 0\). Then condition \((A_3)\) implies that there exists \(k_1 \in \mathbb{N}\) sufficiently large such that

\[b_{k_1} > \sup_{\|u\| < \delta_1} \varphi(u).\]

By Lemma 2.1 we know that there exists a sequence \(\{u_{n1}^{k_1}\}\) satisfies that

\[\varphi'(u_{n1}^{k_1}) \to 0 \text{ in } X', \quad \sup_{n} \varphi(u_{n1}^{k_1}) < d_{k_1} + 1 \text{ and } \inf_{n} \|u_{n1}^{k_1}\| \geq \delta_1/2.\]
Since \( \varphi \) satisfies the \((PS)^c\) condition, \( \{u_{k_1}^n\} \) has a subsequence which is convergent to a critical point \( u^{k_1} \) of \( \varphi \) with \( \|u^{k_1}\| \geq \|u^{k_1}\|_r \geq \frac{\delta_1}{r} \).

Now, we take \( \delta_2 > 2\|u^{k_1}\| \) and similar to the above there exists \( k_2 > k_1 \) large enough such that
\[
 b_{k_2} > \sup_{\|u\| < \delta_2} \varphi(u),
\]
and we can find the second critical point \( u^{k_2} \) with \( \|u^{k_2}\| \geq \frac{\delta_2}{2} > \|u^{k_1}\| \). Clearly, \( u^{k_2} \neq u^{k_1} \).

Repeat the above procedures, we get a sequence of critical points \( \{u^{k_m}\} \) with
\[
 \lim_{m \to \infty} \|u^{k_m}\| \to \infty.
\]
So, the theorem is proved. \( \square \)

3 An application

The aim of this section is to apply Theorem 1.2 to prove the existence of infinitely many solutions of problem (1.9). In this section, \( X = H^1(\mathbb{R}^N), \ N \geq 3 \) with the norm \( \|u\|_{H^1} = (\int_{\mathbb{R}^N} (|u|^2 + |\nabla u|^2)dx)^{\frac{1}{2}}. \) \( L^p(a(x), \mathbb{R}^N) \) is the Lebesgue space with positive weight \( a(x) \) endowed with the norm \( |u|_{L^p} := (\int_{\mathbb{R}^N} a(x)|u|^pdx)^{\frac{1}{p}} \), and this norm is simply denoted by \( |u|_{L^p} \) if \( a(x) \equiv 1. \) For \( r > 0, B(x, r) = \{x \in \mathbb{R}^N : |x| < r\}. \)

The variational functional of (1.9) is defined by
\[
 \varphi(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx - \frac{1}{q} \int_{\mathbb{R}^N} g(x)|u|^qdx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^pdx, \tag{3.1}
\]
for \( u \in H^1(\mathbb{R}^N). \) By \((H_1)-(H_3), \varphi(u) \in C^1(H^1(\mathbb{R}^N)) \) and
\[
 \varphi'(u)\phi = \int_{\mathbb{R}^N} \left( \nabla u \nabla \phi + V(x)u\phi \right)dx - \int_{\mathbb{R}^N} g(x)|u|^{q-2}u\phi dx - \int_{\mathbb{R}^N} h(x)|u|^{p-2}u\phi dx, \tag{3.2}
\]
for any \( \phi \in H^1(\mathbb{R}^N). \) Moreover, \( \varphi' \) is weakly sequentially continuous by \cite{21} Theorem A.2.

Let \( L := -\Delta + V(x) \) be the Schrödinger operator acting on \( L^2(\mathbb{R}^N) \) with domain \( \mathcal{D}(L) = H^2(\mathbb{R}^N) \). Since \( L \) is self-adjoint and 0 lies in a gap of the spectrum of \( L \), by the standard spectral theory we know that the space \( H^1(\mathbb{R}^N) \) can be decomposed as \( H^1(\mathbb{R}^N) = Y \bigoplus Z \) such that the quadratic form:
\[
 u \in H^1(\mathbb{R}^N) \to \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx \tag{3.3}
\]
is negative and positive definite on \( Y \) and \( Z \) respectively, and both \( Y \) and \( Z \) may be infinite-dimensional. Let \( X = \mathcal{D}(|L|^\frac{1}{2}) \) be equipped with the inner product
\[
 \langle u, v \rangle_1 := \langle |L|^\frac{1}{4}u, |L|^\frac{1}{4}v \rangle_{L^2}, \tag{3.4}
\]
bounded sequence in $H^1$ by letting $\{\}

To show that $H^1$ give a simple proof for completeness. The following embedding result which has been used in many papers (see, e.g., [18]). Here we give some useful lemmas. The first lemma is the embedding result which has been used in many papers (see, e.g., [18]). Here we give a simple proof for completeness.

**Lemma 3.1** If $1 < q < 2^*$ and $a(x) \in L^{q_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $a(x) \geq 0$ a.e. in $\mathbb{R}^N$, where $q_0 = \frac{2N}{2N - qN + 2q}$. Then, $H^1(\mathbb{R}^N) \hookrightarrow L^q(a(x), \mathbb{R}^N)$ is compact.

**Proof.** For $u \in H^1(\mathbb{R}^N)$, by the Hölder and Sobolev inequalities, we see that

$$
\int_{\mathbb{R}^N} a(x)|u|^q dx \leq |a(x)|_{L^{q_0}} \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N - 2}} dx \right)^{\frac{2N - 2q}{2N}} = |a(x)|_{L^{q_0}} \cdot |u|_{L^{2^*}(\mathbb{R}^N)}^{q} \leq C\|u\|^q,
$$

that is, $|u|_{L^q(a(x))} \leq C\|u\|$, which means that $H^1(\mathbb{R}^N) \hookrightarrow L^q(a(x), \mathbb{R}^N)$. Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$, passing to a subsequence, we may assume that, for some $u \in H^1(\mathbb{R}^N)$,

$$
u_n \rightharpoonup u \text{ weakly in } H^1(\mathbb{R}^N) \text{ and } u_n ^{\rightarrow} u \text{ in } L^q_{loc}(\mathbb{R}^N), \text{ for } 1 < q < 2^*.
$$

To show that $H^1(\mathbb{R}^N) \hookrightarrow L^q(a(x), \mathbb{R}^N)$ is compact, we need only to show that $u_n$ strongly converges to $u$ in $L^q(a(x), \mathbb{R}^N)$ for $q \in (1, 2^*)$. In fact, for any $R > 0$,

$$
\int_{\mathbb{R}^N} a(x)|u_n - u|^q dx = \int_{\mathbb{R}^N \setminus B(0,R)} a(x)|u_n - u|^q dx + \int_{B(0,R)} a(x)|u_n - u|^q dx

\leq \left( \int_{\mathbb{R}^N \setminus B(0,R)} |a(x)|^{q_0} dx \right)^{\frac{1}{q_0}} \|u_n - u\|_{L^{2^*}(\mathbb{R}^N)} + |a|_{L^{\infty}(\mathbb{R}^N)} \int_{B(0,R)} |u_n - u|^q dx

\rightarrow 0,
$$

by letting $n \rightarrow +\infty$ and then $R \rightarrow +\infty$. \qed
Lemma 3.2 Under condition \((H_3)\), let

\[
\beta_k := \sup_{u \in Z_k, \|u\|=1} |u|_{L^p_h(x)}, \text{ for any } k \in \mathbb{N},
\]

then, \(\beta_k \to 0\) as \(k \to \infty\).

**Proof.** It is clear that \(0 < \beta_{k+1} \leq \beta_k\), and then \(\beta_k \to \beta \geq 0\) as \(k \to \infty\). For every \(k \geq 0\), there exists \(u_k \in Z_k\) with \(\|u_k\|=1\) and \(|u_k|_{L^p_h(x)} \geq \frac{\beta_k}{2}\). By the definition of \(Z_k\), we have \(u_k \rightharpoonup 0\) in \(H^1(\mathbb{R}^N)\). Thus Lemma 3.1 implies that \(u_k \to 0\) strongly in \(L^q_{g(x)}\), then, \(\beta = 0\).

\[\square\]

Lemma 3.3 If \((H_1) - (H_3)\) hold, then \(\varphi\) satisfies \((PS)^c\) condition in \(H^1(\mathbb{R}^N)\) for any \(c < +\infty\).

**Proof.** Let \(\{u_n\} \subset H^1(\mathbb{R}^N)\) be any sequence satisfying

\[
\sup_n \varphi(u_n) \leq c \text{ and } \varphi'(u_n) \to 0.
\]

We claim that \(\{u_n\}\) is bounded in \(H^1(\mathbb{R}^N)\). Indeed, for \(n\) large, there holds

\[
c + 1 + \|u_n\| \geq \varphi(u_n) - \frac{1}{2} \langle \nabla \varphi(u_n), u_n \rangle
\]

\[
= \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} h(x)|u_n|^p dx + \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^N} g(x)|u_n|^q dx
\]

that is,

\[
\int_{\mathbb{R}^N} h(x)|u_n|^p dx \leq C + \|u_n\| + C \int_{\mathbb{R}^N} |g(x)||u_n|^q dx.
\]

\[
\leq C + \|u_n\| + C\|u_n\|^q.
\]

(3.6)

Let \(u_n = y_n + z_n\), with \(y_n \in Y, z_n \in Z\). For \(n\) large,

\[
\|z_n\| \geq \langle \varphi'(u_n), z_n \rangle = \|z_n\|^2 - \int_{\mathbb{R}^N} g(x)|u|^q - 2u z_n dx - \int_{\mathbb{R}^N} h(x)|u|^{p-2}u z_n dx.
\]

thus

\[
\|z_n\|^2 \leq \|z_n\| + \int_{\mathbb{R}^N} g(x)|u|^q - 1 z_n dx + \int_{\mathbb{R}^N} h(x)|u|^{p-1}z_n dx.
\]

By Hölder inequality and Sobolev embeddings, we see that, for some \(C > 0\),

\[
\|z_n\|^2 \leq \|z_n\| + |g(x)|^{\frac{q-1}{q}} |u|^{q-1}_{L^q_{g(x)}} |g(x)|^{\frac{1}{q}} z_n^{q-1}_{L^q_{g(x)}} + |h(x)|^{\frac{p-1}{p}} u^{p-1}_{h(x)} |h(x)|^{\frac{p}{p-1}} z_n^{p-1}_{L^p_{h(x)}}
\]

\[
= \|z_n\| + |u|^{q-1}_{L^q_{g(x)}} |z_n|_{L^q_{g(x)}} + \left( \int_{\mathbb{R}^N} h(x)|u|^{p-1} dx \right)^{\frac{p-1}{p}} |z_n|_{L^p_{h(x)}}
\]

\[
\leq \|z_n\| + C\|u\|^q z_n + C(1 + \|u\| + \|u\|^q) \|z_n\|, \text{ by (3.6)},
\]
\[
\leq \|u_n\| + C\|u_n\|^q + C(1 + \|u_n\| + \|u_n\|^q)^{\frac{p-1}{p}}\|u_n\|.
\]

Similarly, it follows from \(\|y_n\| \geq -\langle \phi'(u_n), y_n \rangle\) that
\[
\|y_n\|^2 \leq \|y_n\| + \|g(x)\| L^{q-1} L^{\infty} + \|h(x)\| L^{p-1} L^{\infty} + (1 + \|u_n\| + \|u_n\|^q)^{\frac{p-1}{p}}\|u_n\|.
\]
Since \(\|u_n\|^2 = \|y_n\|^2 + \|z_n\|^2\), the above conclusions show that
\[
\|u_n\|^2 \leq 2\|u_n\| + C\|u_n\|^q + C(1 + \|u_n\| + \|u_n\|^q)^{\frac{p-1}{p}}\|u_n\|.
\]

Thus, by \(q < \frac{p}{p-1}\), we know \(\{u_n\}\) is bounded in \(H^1(\mathbb{R}^N)\).

By the boundedness of \(\{u_n\}\), we may assume, up to a subsequence, that
\[
y_n \rightharpoonup y \ \text{in} \ H^1(\mathbb{R}^N) \ \text{and} \ z_n \rightharpoonup z \ \text{in} \ H^1(\mathbb{R}^N).
\]

Let \(u = y + z\), we get that
\[
\langle \nabla \varphi(u_n) - \nabla \varphi(u), y_n - y \rangle \to 0 \ \text{as} \ n \to \infty.
\]

Using the Hölder inequality, we see that
\[
y_n \overset{n}{\rightharpoonup} y \ \text{in} \ H^1(\mathbb{R}^N).
\]

Similarly,
\[
z_n \overset{n}{\rightharpoonup} z \ \text{in} \ H^1(\mathbb{R}^N).
\]

Hence,
\[
u_n \overset{n}{\rightharpoonup} u \ \text{in} \ H^1(\mathbb{R}^N).
\]

So, we proved that \(\varphi\) satisfies the \((PS)^c\) condition for any \(c < +\infty\). \(\square\)

**Lemma 3.4** Under the conditions \((H_1) - (H_3)\), for any \(\delta > 0\), there exists \(C_\delta < \infty\) such that \(\sup_{\|u\|_r < \delta} \varphi(u) < C_\delta\).
Proof. By \( u \in H^1(\mathbb{R}^N) = Y \bigoplus Z \) with \( Z = Y^\perp \) we may set \( u = y + z \) for some \( y \in Y \) and \( z \in Z \), then,
\[
\varphi(u) = \frac{1}{2}(-\|y\|^2 + \|z\|^2) - \frac{1}{q} \int_{\mathbb{R}^N} g(x)|u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx
\]
\[
\leq \frac{1}{2}(-\|y\|^2 + \|z\|^2) + \frac{1}{q} \int_{\mathbb{R}^N} |g(x)||u|^q dx
\]
\[
\leq -\frac{1}{2}\|y\|^2 + C\|y\|^q + \frac{1}{2}\|z\|^2 + C\|z\|^q.
\]
Since \( q < 2 \), \(-\frac{1}{2}\|y\|^2 + C\|y\|^q\) is bounded from above. By (1.3), we have \( \|z\| \leq \|u\|_\tau \leq \delta \), so, there exists \( C_\delta < \infty \) such that
\[
\sup_{\|u\|_\tau < \delta} \varphi(u) < C_\delta.
\]

Proof of Theorem 1.3. By Theorem 1.2 and Lemmas 3.3.3.4, in order to prove Theorem 1.3, we need only to verify the conditions \((A_1)\), \((A_2)'\) and \((A_3)\).

Clearly, \((A_1)\) is true since \( \varphi \) maps a bounded set into a bounded set.

Next, we prove \((A_2)'\) by showing that \( a_k \to -\infty \) as \( \rho_k \to \infty \). Let \( u = y + z \) with \( y \in Y \) and \( z \in Z \). For any \( u \in Y_k \), by Lemma 3.1, we see that
\[
\varphi(u) = \frac{1}{2}(-\|y\|^2 + \|z\|^2) - \frac{1}{q} \int_{\mathbb{R}^N} g(x)|u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx
\]
\[
\leq \frac{1}{2}(-\|y\|^2 + \|z\|^2) + \frac{1}{q} |u|_{L^q_{\text{avg}}(x)}^q - \frac{1}{p} |u|_{L^p_{\text{avg}}(x)}^p
\]
\[
\leq \frac{1}{2}(-\|y\|^2 + \|z\|^2) + \frac{2q-1}{q} (\|y\|_{L^q_{\text{avg}}(x)}^q + \|z\|_{L^q_{\text{avg}}(x)}^q) - \frac{1}{p} |u|_{L^p_{\text{avg}}(x)}^p
\]
\[
\leq \frac{1}{2}(-\|y\|^2 + \|z\|^2) + C_q (\|y\|^q + \|z\|^q) - \frac{1}{p} |u|_{L^p_{\text{avg}}(x)}^p.
\]

Since \( H^1(\mathbb{R}^N) \hookrightarrow L^p(h(x), \mathbb{R}^N) \), we denote by \( E_k \) the closure of \( Y_k \) in \( L^p(h(x), \mathbb{R}^N) \), then there exists a continuous projection from \( E_k \) to \( \bigoplus_{j=0}^k f_j \), thus there exists a constant \( C > 0 \) such that
\[
|z|_{L^p_{\text{avg}}(x)}^p \leq C|u|_{L^p_{\text{avg}}(x)}^p,
\]
and note that all norms are equivalent in a finite-dimensional vector space, then for any \( z \in \bigoplus_{j=0}^k f_j \), there exists \( C > 0 \) such that
\[
\|z\|^p \leq C|z|_{L^p_{\text{avg}}(x)}^p,
\]
thus
\[
\varphi(u) \leq (-\frac{1}{2}\|y\|^2 + C\|y\|^q) + (\frac{1}{2}\|z\|^2 + C\|z\|^q - C\|z\|^p).
\]
So,
\[
a_k := \sup_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \to -\infty \text{ as } \rho_k \to \infty.
\]
Finally, for any \( u \in Z_k \) with \( \|u\| = r_k \), let \( u = y + z \) with \( y \in Y \) and \( z \in Z \), then \( y = 0, z = u \). Furthermore,

\[
\varphi(u) = \frac{1}{2}\|u\|^2 - \frac{1}{q} \int_{\mathbb{R}^N} g(x)|u|^q \, dx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p \, dx \geq \frac{1}{2}\|u\|^2 - \frac{1}{q} \int_{\mathbb{R}^N} |g(x)||u|^q \, dx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p \, dx \geq \left( \frac{1}{4}\|u\|^2 - C\|u\|^q \right) + \left( \frac{1}{4}\|u\|^2 - \frac{1}{p}\beta_k \|u\|^p \right).
\]

Choose \( r_k = (\frac{4}{q})^{\frac{1}{q-2}} \frac{1}{\beta_k^{\frac{1}{p-2}}} \), we have

\[
\varphi(u) \geq \frac{1}{4}\|u\|^2 - C\|u\|^q = \frac{1}{4}|r_k|^2 - C|r_k|^q.
\]

Since we have \( r_k \to +\infty \) by \( \beta_k \to 0 \) as \( k \to \infty \),

then,

\[
b_k := \inf_{u \in Z_k, \|u\| = r_k} \varphi(u) \to +\infty \quad \text{as} \quad k \to \infty.
\]

Thus \((A_3)\) is also proved.

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