Two kinds of phase transitions in a voting model

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Received 15 March 2012, in final form 10 July 2012
Published 2 August 2012
Online at stacks.iop.org/JPhysA/45/345002

Abstract
In this paper, we discuss a voting model with two candidates, \(C_0\) and \(C_1\). We consider two types of voters—herders and independents. The voting of independents is based on their fundamental values, while the voting of herders is based on the number of previous votes. We can identify two kinds of phase transitions. One is an information cascade transition similar to a phase transition seen in the Ising model. The other is a transition of super and normal diffusions. These phase transitions coexist. We compared our results to the conclusions of experiments and identified the phase transitions in the upper limit of the time \(t\) by using the analysis of human behavior obtained from experiments.

PACS numbers: 89.65.−s, 02.50.−r

1. Introduction

In general, collective herding behavior poses interesting problems in several cross fields, such as sociology \cite{1}, social psychology \cite{2}, ethnology \cite{3, 4} and economics. Statistical physics offers effective tools to analyze these phenomena caused by collective herding behaviors, and the associated field is known as sociophysics \cite{5}. For example, in statistical physics, anomalous fluctuations in financial markets \cite{6, 7} and opinion dynamics \cite{8–10} have been related to percolation and the random field Ising model.

To estimate public perception, people observe the actions of other individuals, they then make a choice similar to that of others. Because it is usually sensible to do what other people are doing, collective herding behavior is assumed to be the result of a rational choice. This approach can sometimes lead to arbitrary or even erroneous decisions as a macro phenomenon. This phenomenon is known as an information cascade \cite{11}.

In our previous paper, we introduced a sequential voting model that is similar to a Keynesian beauty contest \cite{12–14}. At each time step \(t\), one voter votes for one of the two candidates. As public perception, the \(r\)th voter can see all previous votes, i.e. \((t−1)\) votes. There are two types of voters—herders and independents—and two candidates. Herders are also
known as copycat voters; they vote for each candidate with probabilities that are proportional to the candidates’ votes. We refer to these herders as analogue herders. We investigated a case wherein all the voters were herders [15]. In such a case, the voting process is a Pólya process, and the voting rate converges to a beta distribution in a large time limit of $t$ [16].

Next, we added independents to the analogue herders [13]. In the upper limit of $t$, the independents cause the distribution of votes to converge to a Dirac measure against herders. This model contains three phases—two super diffusion phases and a normal diffusion phase. We refer to the transition in this model as a transition of super and normal diffusions. These transitions can be seen in several fields [17]. If herders constitute the majority or even half of the total voters, the voting rate converges to a Dirac measure slower than in a binomial distribution. These two phases have different speeds of convergence that are slower than in a binomial distribution. If independents constitute the majority of the voters, the voting rate converges at the same rate as that in a binomial distribution. If the independents vote for the correct candidate rather than for the wrong candidate, the model does not include the case wherein the majority of the voters choose the wrong candidate. The herders affect only the speed of the convergence, they do not affect the voting rates for the correct candidate.

Next, we consider herders who always choose the candidate with a majority of the previous votes, which is visible to them [18]. We refer to these herders as digital herders. Digital herders exhibit stronger herd behavior than those of analogue herders. We obtained exact solutions when the voters comprised a mix of digital herders and independents. As the fraction of herders increases, the model features a phase transition beyond which a state where most voters make the correct choice coexists with one where most of them are wrong. This phase transition is referred to as an information cascade transition.

Here, we discuss a voting model with two candidates, $C_0$ and $C_1$. We set two types of voters—independents and herders. The voting of independents is based on their fundamental values. They collect information independently. On the other hand, the voting of herders is based on the number of previous votes, which is visible to them. In this study, we consider the case wherein a voter can see all previous votes.

From experiments, we observed that human beings exhibit a behavior between that of the digital and analogue herders. We obtained the probability that a herder makes a choice under the influence of his/her prior voters’ votes. The probability can be fitted by a tanh function. If the difference between numbers of voters for candidates $C_0$ and $C_1$ is small, the probability that a herder chooses the candidate receiving a majority of the previous votes increases rapidly. If the difference between numbers of voters for candidates $C_0$ and $C_1$ is large, the probability becomes constant. In this paper, we discuss rich phases of the models in which herders exhibit behavior that can be fitted to a tanh function. We identify two types of phase transitions—information cascade transition and transition between super and normal diffusions. Furthermore, we discuss the phases of models that we obtained from experiments [19].

The remainder of this paper is organized as follows. In section 2, we introduce our voting model and mathematically define the two types of voters—independents and herders. In section 3, we derive a stochastic differential equation. In section 4, we discuss information cascade transition by using the stochastic differential equation. In section 5, we discuss the phase transition between normal and super diffusions and we demonstrate the coexistence of these phase transitions. In section 6, we verify these transitions through numerical simulations. In section 7, we discuss social experiments from the viewpoint of our models. Finally, the conclusions are presented in section 8.
We model the voting of two candidates, \( C_0 \) and \( C_1 \). At time \( t \), \( C_0 \) and \( C_1 \) have \( c_0(t) \) and \( c_1(t) \) votes, respectively. In each time step, one voter votes for one candidate; the voting is sequential. Hence, at time \( t \), the \( r \)th voter votes, after which the total number of votes is \( t \). Voters are allowed to see all the previous votes for each candidate, thus, they are aware of public perception.

There are two types of voters—indepedents and herders; we assume an infinite number of voters. The independents vote for \( C_0 \) and \( C_1 \) with probabilities \( 1 - q \) and \( q \), respectively. Their votes are independent of others’ votes, i.e. their votes are based on their fundamental values.

Here, we set \( C_0 \) as the wrong candidate and \( C_1 \) as the correct candidate in order to validate the performance of the herders. We can set \( q \geq 0.5 \) because we believe that independents vote for the correct candidate \( C_1 \) rather than for the wrong candidate \( C_0 \). In other words, we assume that the intelligence of independents is virtually accurate.

On the other hand, the herders’ votes are based on the number of previous votes. We use the following functions. If the numbers of votes are \( c_0(t) \) and \( c_1(t) \) at time \( t \), a herder votes for \( C_1 \) at time \( t + 1 \) with the following probability:

\[
q_h = \frac{1}{2} \left[ \tanh \lambda \left( \frac{c_1(t)}{(c_0(t) + c_1(t))} - \frac{1}{2} \right) + 1 \right].
\]  

If \( C_1 \) receives the majority votes, i.e. \( c_1(t)/(c_0(t) + c_1(t)) > 1/2 \), the ratio of votes for \( C_1 \), denoted by \( q_h \), increases to 1 exponentially. \( \lambda \) reflects how the previous answers affect a voter’s choice. If \( \lambda \) is positive and large, the voter has high confidence in the previous votes. We use function (1) for our social experiments, and it fits well to human behaviors in the experiments [19]. In our experiments, we can estimate \( \lambda = 3.80 \). If \( c_0(t) = c_1(t) \), herders vote for \( C_0 \) and \( C_1 \) with the same probability, i.e. \( 1/2 \). This model can also be derived from Bayes’ theorem (see appendix A). In this case, \( \lambda \) is not constant and it increases as the number of votes increases, according to the central limit theorem. The case is based on the assumption that all voters are independents. Hereafter, we treat \( \lambda \) as a parameter (see figure 1).

In the upper limit of \( \lambda \), herders vote for the candidate with the majority votes. If \( c_0(t) > c_1(t) \), the herders vote for candidate \( C_0 \), whereas if \( c_0(t) < c_1(t) \), they vote for candidate \( C_1 \). These herders are known as digital herders [18]. We expand (1) as follows:

\[
q_h \sim \frac{1}{2} + \frac{\lambda}{2} \left( \frac{c_1(t)}{(c_0(t) + c_1(t))} - \frac{1}{2} \right) - \frac{\lambda^2}{6} \left( \frac{c_1(t)}{(c_0(t) + c_1(t))} - \frac{1}{2} \right)^3 + \cdots.
\]  

If \( c_1(t)/(c_0(t) + c_1(t)) \sim 1/2 \) and \( \lambda = 2 \), a herder votes for \( C_1 \) with a probability of \( c_1(t)/(c_0(t) + c_1(t)) \). These herders are known as analogue herders with \( q = 1/2 \) [13]. Therefore, the herders who vote with probability (1) are a hybrid of analogue and digital herders.

The independents and herders appear randomly and vote. We set the ratio of independents to herders as \( (1 - p)/p \). In this study, we mainly focus on the upper limit of \( t \). This refers to the voting of infinite voters.

The evolution equation for candidate \( C_1 \) is

\[
P(k, t) = pq_h P(k - 1, t - 1) + p(1 - q_h) P(k, t - 1) \\
+ (1 - p) q P(k - 1, t - 1) + (1 - p)(1 - q) P(k, t - 1).
\]  

\(^3\) The \( r \)th voter can see \( c_0(t - 1) \) and \( c_1(t - 1) \) votes at time \( t \).
Figure 1. Representation of our model. Independents vote for $C_0$ and $C_1$ with probabilities $1-q$ and $q$, respectively. Herders vote for $C_0$ and $C_1$ with probabilities $1-q_h$ and $q_h$, respectively. We set the ratio of independents to herders as $(1-p)/p$. In the limit $\lambda \to \infty$, a herder is a digital herder. In the limit $\lambda = 2$, a herder is similar to an analogue herder. In general, the herder is a hybrid between an analogue herder and a digital herder with the parameter $\lambda$.

Here, $P(k, t)$ is the distribution of the number of votes $k$ at time $t$ for a candidate $C_1$. The first and second terms of (3) denote the votes of the herders, the third and fourth terms denote the votes of the independents.

3. Stochastic differential equation

To investigate long-ranged correlations, we analyze the limit $t \to \infty$. We can rewrite (3) as

$$c_1(t) = k \to k + 1 : P_{k,t} = \frac{p}{2} \left[ \tanh \left( \frac{k}{t-1} - \frac{1}{2} \right) + 1 \right] + (1-p)q.$$  (4)

In the scaling limit $t = c_0(t) + c_1(t) \to \infty$, we define

$$\frac{c_1(t)}{t} \equiv Z.$$  (5)

$Z$ is the ratio of voters who vote for $C_1$.

We define a new variable $\Delta_t$ such that

$$\Delta_t = 2c_1(t) - t = c_1(t) - c_0(t).$$  (6)

We change the notation from $k$ to $\Delta_t$ for convenience. Then, we have $|\Delta_t| = |2k - t| < t$. Thus, $\Delta_t$ holds within $[-t, t]$. Given $\Delta_t = u$, we obtain a random walk model

$$\Delta_t = u \to u + 1 : P_{u,\Delta_t} = \frac{p}{2} \left[ \tanh \left( \frac{\lambda u}{2(t-1)} \right) + 1 \right] + (1-p)q,$$

$$\Delta_t = u \to u - 1 : Q_{u,\Delta_t} = 1 - P_{u,\Delta_t}.$$  (7)

We now consider the continuous limit $\epsilon \to 0$,

$$X_t = \epsilon \Delta_{t/\epsilon},$$

$$P(x, t) = \epsilon P(\Delta_t/\epsilon, t/\epsilon).$$  (7)
where $\tau = t/\epsilon$ and $x = \Delta t/\epsilon$. Approaching the continuous limit, we can obtain the Fokker–Planck equation (see appendix B)

$$dX_\tau = \left[ (1 - p)(2q - 1) + p \tanh \left( \frac{\lambda X_\tau}{2\tau} \right) \right] d\tau + \sqrt{\epsilon}. \tag{8}$$

Here, we change the variable $X_\tau$ to $Y_\tau$ by using the expression

$$pY_\tau = X_\tau - (1 - p)(2q - 1)\tau. \tag{9}$$

We rewrite (8) using $Y_\tau$:

$$dY_\tau = \tanh \frac{p\lambda}{2\tau} \left( Y_\tau + \frac{(2q - 1)(1 - p)}{p} \right) d\tau + \sqrt{\epsilon}. \tag{10}$$

Using (5), (6) and (9), we can obtain the relations of the variables

$$2Z - 1 = \frac{X_\infty}{\tau} = \frac{pY_\infty}{\tau} + (1 - p)(2q - 1). \tag{11}$$

### 4. Information cascade transition

In this section, we discuss the information cascade transition. We observed this transition in the case of digital herders [18]. We are interested in the behavior of the digital headers in the limit $\tau \to \infty$. We consider the solution $Y_\infty \sim \tau^\alpha$, where $\alpha \leq 1$, since $\tanh x \leq 1$. The slow solution is $Y_\infty \sim \tau^\beta$, where $\alpha < 1$ is hidden by the fast solution $\alpha = 1$ in the upper limit of $\tau$. Hence, we can assume a stationary solution as

$$Y_\infty = \bar{v}\tau, \tag{12}$$

where $\bar{v}$ is constant. Substituting (12) into (10), we can obtain

$$\Delta Y_\infty = \tanh \frac{p\lambda}{2} \left( \bar{v} + \frac{(2q - 1)(1 - p)}{p} \right) \Delta \tau = \bar{v}\Delta \tau. \tag{13}$$

The second equality is obtained from (12). Then, we obtain the following equation:

$$\bar{v} = \tanh \frac{p\lambda}{2} \left( \bar{v} + \frac{(2q - 1)(1 - p)}{p} \right). \tag{14}$$

This is the equation of state for the Ising model. The second term on the RHS of (14) corresponds to the external field. In the cases of $q = 1/2$ and $p = 1$, the external field disappears. If $\lambda > 2$, a phase transition occurs in the range $0 \leq p \leq 1$. As the number of herders increases, the model features a phase transition beyond which a state where most voters make the correct choice coexists with the one where most of them are wrong. We refer to this transition as an information cascade transition [18].

Equation (14) admits one solution below the critical point $p \leq p_c$ (see figure 2(a)) and three solutions for $p > p_c$ (see figure 2(b)). When $p \leq p_c$, we refer to the phase as the one-peak phase. When $p > p_c$, the upper and lower solutions are stable, while the intermediate solution is unstable. Then, the two stable solutions attain a good and bad equilibrium, and the distribution becomes the sum of the two Dirac measures. We refer to this phase as two-peak phase. In the Ising model with external fields, the good equilibrium is only a physical solution. This is the difference between our voting model and the Ising model. Here, we discuss two particular cases.

1. **Digital herder case:** $\lambda = \infty$. In this case, the herders are considered to be digital herders. As shown in figure 2, the tanh function rises vertically at $\bar{v} = -\frac{(2q - 1)(1 - p)}{p}$. A phase transition occurs at $p_c = 1 - \frac{1}{2q}$. When $p \leq p_c$, we can obtain only one solution $\bar{v} = 1$. 


Figure 2. Solutions of the self-consistent equation (14). It is the state equation of the Ising model. (a) \( p \leq p_c \) and (b) \( p > p_c \). Below the critical point \( p_c \), we can obtain one solution (a). We refer to this phase as the one-peak phase. In contrast, above the critical point, we obtain three solutions. Two of them are stable and one is unstable (b). We refer to this phase as the two-peak phase. In Ising model, the physical solution is the right side one only.

Using (11), we obtain the ratio of voters for \( C_1 \) as

\[
Z = p + (1 - p)q.
\]

When \( p > p_c \), we can obtain two stable solutions: \( \bar{\nu} = 1 \) and \( \bar{\nu} = -1 \). The solution \( \bar{\nu} = -\frac{(2q - 1)(1 - p)}{p} \) is unstable. Using (11), the ratio of voters for \( C_1 \) is obtained as

\[
Z = p + (1 - p)q \quad \text{and} \quad Z = q(1 - p).
\]

When \( p > p_c \), \( Z = p + (1 - p)q \) shows a good equilibrium, whereas \( Z = q(1 - p) \) shows a bad equilibrium. This conclusion is consistent with the exact solutions [18].

(2) Symmetric independent voter case: \( q = 1/2 \). In this case, the external fields are absent. The self-consistent equation (14) becomes

\[
\bar{\nu} = \tanh \frac{p\lambda}{2}(\bar{\nu}).
\]

As shown in figure 2, the tanh function rises at \( \bar{\nu} = 0 \). If \( \lambda \leq 2 \), there is only one solution \( \bar{\nu} = 0 \) in all regions of \( p \). In this case, \( Z \) has only one peak, at 0.5, which indicates the one-peak phase. When herders are analogue herders, we do not observe an information cascade transition [18]. We observe only super and normal transitions (see section 5). On the other hand, if \( \lambda > 2 \), there are two stable solutions and an unstable solution \( \bar{\nu} = 0 \) above \( p_c \). The votes’ ratio for \( C_1 \) attains a good or bad equilibrium. This is the so-called spontaneous symmetry breaking. In one sequence, \( Z \) is taken as \( \bar{\nu}p/2 + 1/2 \) in the case of a good equilibrium, or as \( -\bar{\nu}p/2 + 1/2 \) in the case of a bad equilibrium, where \( \bar{\nu} \) is the solution of (15). This indicates the two-peak phase, and the critical point is \( p_c = 2/\lambda \).

5. Phase transition of super and normal diffusion phases

In this section, we consider the phase transition of convergence. This type of transition has been studied when herders are analogue [13]. The analogue herders exhibit weaker herd behavior than that of the digital herders. Depending on the convergence behavior, there are three phases. We expand \( Y_\tau \) around the solution \( \bar{\nu} \),

\[
Y_\tau = \bar{\nu}(\tau) + W_\tau.
\]

Here, we set \( Y_\tau \approx W_\tau \). This indicates \( \tau \gg 1 \). We rewrite (10) using (16) as follows:

\[
dY_\tau = \frac{p\lambda}{2\tau} \left[ \frac{p(2q - 1)(1 - p)}{p} \bar{\nu} + \frac{1}{p} W_\tau \right] d\tau + \sqrt{\epsilon}
\]

\[
dY_\tau = \frac{p\lambda}{2\tau} \left[ \frac{p(2q - 1)(1 - p)}{p} \bar{\nu} + \frac{1}{p} W_\tau \right] d\tau + \sqrt{\epsilon}
\]
We use relation (14) and consider the first term of the expansion. Hence, we can obtain
\[ \sim \tilde{v} \, \text{d}t + \frac{p \lambda}{2} \cosh^2 \frac{\tilde{v}}{2} \left( \tilde{v} + \frac{(2q-1)(1-p)}{p} \right) \text{d}t + \sqrt{\epsilon} \]
\[ = \tilde{v} \, \text{d}t + \frac{p \lambda (1 - \tilde{v}^2)}{2} \text{d}t + \sqrt{\epsilon}. \tag{17} \]

We use relation (14) and consider the first term of the expansion. Hence, we can obtain
\[ \text{d}W_t = \frac{p \lambda (1 - \tilde{v}^2)}{2} \text{d}t + \sqrt{\epsilon}. \tag{18} \]

From appendix C, we can obtain the phase transition of convergence. The critical point \( p_{ec} \) is the solution of
\[ p_{ec} = \frac{1}{\lambda (1 - \tilde{v}^2)}. \tag{19} \]
and (14).

As shown in figure 2, at the critical point, the gradient of the tangent line at \( \tilde{v} \) is 1/2. If the gradient of the tangent line at \( \tilde{v} \) is under 1/2, the distribution converges as in a binomial distribution. We define \( \gamma \) as \( \text{Var}(Z) = \tau^{-\gamma} \), where \( \text{Var}(Z) \) is the variance of \( Z \). The voting rate for \( C_1 \) converges as \( \text{Var}(Z) = \tau^{-1} \); \( \gamma = 1 \). We refer to this phase as a normal diffusion phase.

If the gradient of the tangent line at \( \tilde{v} \) is 1/2 or above 1/2, the voting rate converges at a speed slower than that in a binomial distribution. We refer to these phases as super diffusion phases.

In one phase, the voting rate for \( C_1 \) converges to \( \log(\tau)/\tau \), and in the other, the voting rate converges to \( \gamma = p/p_{ec} - 2 \).

1. Digital herder case: \( \lambda = \infty \). In this case, the herders are considered to be digital herders.

   In the one-peak phase, where \( p \leq p_c \), the only solution is \( \tilde{v} = 1 \). Equation (18) represents the Brownian motion. The gradient of the tangent line at \( \tilde{v} \) is 0. Hence, the distribution converges as in a binomial distribution. In the two-peak phase, where \( p > p_c \), the solutions are \( \tilde{v} = \pm 1 \). The gradient of the tangent line at \( \tilde{v} \) is 0. In each case, the distribution converges as in a binomial distribution. Hence, in all regions, the distribution converges as in a binomial distribution. In this limit, the phase transition of the convergence disappears and only an information cascade transition is observed.

2. Symmetric independent voter case: \( q = 1/2 \). We consider the case \( \lambda > 2 \). In this case, we observe an information cascade transition. If \( \lambda \leq 2 \), we do not observe an information cascade transition and we can only observe a part of the phases, as described below.

   In the one-peak phase \( p \leq p_c \), the only solution is \( \tilde{v} = 0 \). \( p_c \) is the critical point of the information cascade transition. The first critical point of convergence is \( p_{ec1} = 1/\lambda \). When \( p \leq p_c \), \( p_{ec1} \) is the solution of (15) and (19). If \( 0 < p < p_{ec1} \), then the voting rate for \( C_1 \) becomes 1/2, and the distribution converges as in a binomial distribution. If \( p_c < p > p_{ec1} \), candidate \( C_1 \) gathers 1/2 of all the votes in the scaled distributions, too. However, the voting rate converges slower than that in a binomial distribution. We refer to these phases as super diffusion phases. There are two phases, \( p = p_{ec1} \) and \( p_c < p > p_{ec1} \); these phases differ in terms of their convergence speed.

   Above \( p_c \), in the two-peak phase, we can obtain two stable solutions that are not 0. At \( p_c \), \( \tilde{v} \) moves from 0 to one of these two stable solutions. In one voting sequence, the votes converge to one of these stable solutions. If \( p_c < p \leq p_{ec2} \), the voting rate for \( C_1 \) becomes \( \tilde{v}(p)/2 + 1/2 \) or \(-\tilde{v}(p)/2 + 1/2 \), and the convergence occurs at a rate slower than that in a binomial distribution. Here, \( \tilde{v} \) is the solution of (15). We refer to this phase as a super diffusion phase. \( p_{ec2} \) is the second critical point of convergence from the super to the normal diffusion phase, and it is the solution of the simultaneous equations (15) and (19).
when \( p > p_c \). We can estimate \( p_{c2} \sim \frac{2.5}{\lambda} \) by approximation\(^4\). In fact, with higher terms, we can estimate \( p_{c2} \sim \frac{2.5}{\lambda} \) (see figure 3(a)). On the other hand, if \( p > p_{c2} \), the voting rate for \( C_1 \) becomes \( \bar{v}p/2 + 1/2 \) or \(-\bar{v}p/2 + 1/2\), too. But the distribution converges as in a binomial distribution. This is a normal diffusion phase. A total of six phases can be observed.

As discussed above, when \( q = 0.5 \), there is no difference between good and bad equilibriums. However, when \( q = 0.8 \), we observe a difference. \( Z > 1/2 \) indicates the good equilibrium and \( Z < 1/2 \) indicates the bad equilibrium. Figure 3 shows the phase diagram in space \( \lambda \) and \( p \) when (a) \( q = 0.5 \) and (b) \( q = 0.8 \). \( \gamma \) is the speed of convergence. \( \gamma = 0 \) is the boundary between the one-peak phase and the two-peak phase. The region below \( \gamma = 0 \) is the one-peak phase and that above \( \gamma = 0 \) is the two-peak phase. The region below \( \gamma = 1 \) is the normal diffusion phase and that above \( \gamma = 1 \) is the super diffusion phase. The region below \( \gamma = 1 \) is the normal diffusion phase. When \( q = 0.8 \), there is a difference in the speed of convergence between good and bad equilibriums. \( Z > 1/2 \) is the good equilibrium and \( Z < 1/2 \) is the bad equilibrium. The left side of \( \gamma = 1 \), \( Z > 1/2 \) in the two-peak phase is the phase in which a good equilibrium is a super diffusion phase. The left side of \( \gamma = 1 \), \( Z < 1/2 \) is the phase in which a bad equilibrium is a super diffusion phase.

\(^4\) Using \( \tanh x \approx x - 1/3x^3 \), equation (15), and the condition of critical point, we can obtain \( p_{c2} \approx 2.5/\lambda \).
Figure 4. Convergence of the distribution in the one-peak phase and the two-peak phase for the $q = 0.5$ case. The horizontal axis represents the ratio of herders $p$, and the vertical axis represents the speed of convergence. $γ = 1$ is the normal phase, and $0 < γ < 1$ is the super diffusion phase. (a) is for the one-peak phase. The critical point of the convergence of transition from normal diffusion to super diffusion is $p_{vc1} \sim 1/λ$, and the critical point of information cascade transition is $p_c \sim 2/λ$. (b) is for the two-peak phase. The solid lines are obtained by numerical simulations. $γ, Z < 0$ is the speed of convergence of the region $Z < 0$, which indicates a bad equilibrium. The dotted line $γ_{sc}$ represents the theoretically obtained curve and $γ_{sc}, Z < 0$ represents the theoretically obtained curve for a bad equilibrium. The critical point of information cascade transition is $p_c \sim 0.5$ and the critical point of the convergence of transition is $p_{vc1} \sim 0.25$ and $p_{vc2} \sim 0.675$.

Finally, we comment on the analogue herder case. As discussed in section 2, if we set $λ = 2$ and use only the linear terms in (1), we can obtain the analogue herder case. In this limit, we can solve (14) and obtain $t = 2q - 1$. Equation (14) has only one solution in the entire region $p$. Hence, we do not observe an information cascade transition, which is consistent with previous conclusions [13]. On the other hand, in super and normal diffusion phase transitions, $p_{vc1} = 1/2$ does not depend on $q$, because the gradient of the linear term is constant.

6. Numerical simulations

To confirm the analytical results, we performed the numerical integration of the master equation (3). We perform simulations for the symmetric independent voter case, i.e. $q = 1/2$.

1) One-peak phase: $q = 1/2, λ = 4$. Figure 4(a) shows the convergence of the distribution in the one-peak phase. We integrated the master equation up to $t = 10^5$. As discussed in previous sections, the critical point of the convergence of this transition is $p_{vc1} \sim 1/λ$, and the critical point of information cascade transition is $p_c \sim 2/λ$. At the critical point of information cascade transition, the distribution splits into two and the exponent $γ$ becomes 0.5.

2) Two-peak phase: $q = 1/2, λ = 4$. Figure 4(b) shows the convergence of the distribution in the two-peak phase. We consider the case wherein $q = 0.5$ and $λ = 4$. In this phase, a sequence of voting converges to one of the peaks. We cannot determine which peak is chosen in finite time. We divide the distribution into two parts, $Z > 0$ or $Z < 0$, which correspond to two regions around the peaks.

Here, we estimate $γ$ from the slope of $\text{Var}(Z(t))$ as $γ = \log(\text{Var}(Z(t-Δt))/\text{Var}(Z(t))) / \log(t/(t-Δt))$. 

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Here, we estimate $γ$ from the slope of $\text{Var}(Z(t))$ as $γ = \log(\text{Var}(Z(t-Δt))/\text{Var}(Z(t))) / \log(t/(t-Δt))$. 

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Above the critical point of an information cascade transition $p_c \sim 2/\gamma = 0.5$, the speed of the convergence in the region $Z < 0$ is slower than that in a binomial distribution\(^6\). This indicates the super diffusion phase. In the super diffusion phase, the distance between the two peaks is short and the influence of the other peak reduces the convergence speed. After the transition of convergence to $p_{c2} = 0.675$, the speed of the convergence is as in a binomial distribution $\gamma = 1$. This indicates the normal diffusion phase.

The critical point of convergence, $p_{c2}$, in the two-peak phase is larger than that predicted by theory. We calculated the variance in the region $Z < 0$ using the data from $t = 9.99 \times 10^4$ to $t = 10^5$. The region may be too wide for use in our assumptions, which we described in the previous section (16). To fill the gap between the numerical calculations and theory, we need to perform simulations for a large $t$.

7. Social experiments

We conducted simple social experiments for our model. In 2010, we framed 100 questions, each with two choices—knowledge and no knowledge. Thirty-one participants answered these questions sequentially in one group. We performed experiments with two groups. In 2011, we framed 120 questions and 52 participants answered these questions sequentially. We again performed experiments with two groups.

First, they answered the questions without having any information about the others’ answers, i.e. their answers were based on their own knowledge. Those who knew the answers selected the correct answers. Hence, we can set $q = 1$. Those who did not know the answers selected the correct answers with a probability of 0.5. Next, the participants were allowed to see all previous participants’ answers. Those who did not know the answers referred to this information. We are interested in knowing whether they referred to this information as digital or analogue herdiers.

From the experiments, we can obtain microscopic behavior $q_h$ [19]. The voters can see all the previous votes, and we could fit the plot by the following functional form:

$$q_h = \frac{1}{2} \left[ a \tanh \lambda \left( \frac{c_1(t)}{c_0(t) + c_1(t)} - \frac{1}{2} \right) + 1 \right]. \tag{20}$$

The difference between (1) and (20) is the constant $a$. The parameter $a$ denotes the net ratio of the herder that reacts positively to the previous votes. We estimated the parameters $\lambda = 3.80$ and $a = 0.761$ form the experiments.

We can map (20) to (1) as follows:

$$P(k, t) = pq_0 + (1 - p) = \frac{1}{2} \tilde{p} \left[ a \tanh \lambda \left( \frac{c_1(t)}{c_0(t) + c_1(t)} - \frac{1}{2} \right) + 1 \right] + (1 - p)$$

$$= \frac{1}{2} \tilde{p} \left[ \tanh \lambda \left( \frac{c_1(t)}{c_0(t) + c_1(t)} - \frac{1}{2} \right) + 1 \right] + (1 - \tilde{p})\tilde{q}. \tag{21}$$

where $\tilde{p} = pa$ and $\tilde{q} = 1 - 1/2 \cdot p(1 - a)/(1 - pa)$. Using this mapping and conclusions from previous sections, we can theoretically estimate the conclusions for a large limit of $t$.

Figure 5(a) shows the experimental data and simulation results. $\gamma = 1$ is the normal and $0 < \gamma < 1$ is the super diffusion phase. In the experimental data, when the number of voters is $t = 50$, we can confirm that $\gamma$ monotonically decreases. For the simulations, we set $t = 50$, and we perform numerical integration of the master equation (3), as described in

\(^6\) In the case $q = 0.5$, the convergence in the region $Z > 0$ is identical to that in $Z < 0$ because of the symmetry. We consider only $Z < 0$ here.
Figure 5. Behavior of convergence as given by the plot of $p$ versus $\gamma$. The horizontal axis represents the ratio of herders, $p$, and the vertical axis represents the speed of convergence. $\gamma = 1$ is the normal phase, and $0 < \gamma < 1$ is the super diffusion phase. (a) is for the experimental data and simulation results at $t = 50$. (b) is for simulation results at $t = 10^6$ and theoretical results in the limit $t \to \infty$. On the basis of the theoretical results, we show the convergence of $Z$ as $\gamma_{sc}$ and that of $Z < 1/2$ as $\gamma_{sc}, Z < 1/2$. $\gamma_{sc}, Z < 1/2$ shows the theoretical convergence for a bad equilibrium. $T = 10^6$ shows the simulation curve of the convergence of $Z$. $T = 10^6, Z < 0$ shows the simulation curve for a bad equilibrium. Above the threshold $p_c = 0.934$, we observe in the two-peak phase.

The data points observed in the experiments are on the simulation curve without $p \sim 1$.

Figure 5(b) shows the simulation and theoretical results to study the asymptotic behavior of convergence. For $T = 10^6$, we obtain the simulation curve of the convergence of $Z$. In $T = 10^6$, represents a simulation curve, $\gamma$ decreases from 1 to 0 at $p_c \sim 0.9$. This indicates the phase transition from the one-peak phase to the two-peak phase.

Using the conclusions of sections 4 and 5, we can theoretically estimate the conclusions for a large $t$ limit. At $p_c = 0.934$, we observe information cascade transition. $\gamma_{sc}$ is the theoretical curve that shows information cascade transition at $p_c$. We observe a difference between the simulation curve $T = 10^6$ and the theoretical curve $\gamma_{sc}$. This difference can be reduced by performing a simulation for a large $t$.

In the one-peak phase, below $p_c$, the peak converges as normal. Above $p_c$, we observe two peaks. The good equilibrium converges as normal. On the other hand, bad equilibrium converges slower than normal. $\gamma_{sc}, Z < 0$, which represents a theoretical curve, shows the slow convergence of the bad equilibrium and phase transition. At $p_{c2} = 0.983$, we observe the phase transition of normal and super diffusions, for the bad equilibrium. Above $p_{c2}$, both peaks converge as normal. $T = 10^6, Z < 0$ is the simulation curve for the bad equilibrium. We can observe the increase in the convergence speed for the bad equilibrium. The difference between the simulation and the theoretical results has been described in the previous section; this difference may also be reduced by performing a simulation for a large $t$.

8. Concluding remarks

We investigated a voting model that is similar to a Keynesian beauty contest. In the continuous limit, we could obtain stochastic differential equations. The model has two kinds of phase transitions. One is an information cascade transition, which is similar to the phase transition of the Ising model. In fact, we showed that the stationary condition of our model is the same as
Figure 6. Distributions for (a) $q = 0.5$ and (b) $q = 0.8$, when the number of votes is $10^3$. The three axes are $Z$, $p$ and the frequency $P(Z)$. As $p$ increases, the distribution splits into two in both cases. In the region near $p_c$, both the one-peak and the two-peak phases exhibit super diffusion. In this phase, the speed of convergence is slower than that in the normal diffusion phase. We can confirm that the width of the distribution is greater than that in the region far from $p_c$.

The equation of state for the Ising model. As the herders increased, the model featured a phase transition beyond which a state where most voters make the correct choice coexists with one where most of them are wrong. In this transition, the distribution of votes changed from the one-peak phase to the two-peak phase. These two peaks were the two stable solutions out of the three solutions. In Ising model with an external field, there is only one physical solution out of the three solutions.

The other transition was the transition of the convergence between super and normal diffusions. In the one-peak phase, if herders increased, the variance converged slower than in a binomial distribution. This is the transition from normal diffusion to super diffusion. In the two-peak phase, the sequential voting converged to one of the two peaks. When $q = 0.5$, this is spontaneous symmetry breaking. In the two-peak phase, if the herders increased, the variance converged as in a binomial distribution. This is the transition from super diffusion to normal diffusion, which is opposite to the transition in the one-peak phase. In other words, the super diffusion phase is sandwiched between normal diffusion phases and is divided by an information cascade transition.

We determined the microspecific behavior from social experiments. Using this experimental data, we confirmed the two kinds of phase transitions by performing numerical simulations and conducting analytical studies in the upper $t$ limit.

If the ratio of herders is smaller than $p_c$, we can distribute correct answers to herders that do not have information by using this voting system [20]. On the other hand, if the ratio of herders is larger than $p_c$, the system is similar to a Keynesian beauty contest; many voters are herders and there is a case in which more than half the voters make the wrong decision [12]. In the case $p < p_c$, the advantage of this system is evident, and in the case $p > p_c$, the weakness is observed. These are two sides of the same coin and are brought out by the information cascade transition.

Figure 6 shows the distributions when (a) $q = 0.5$ and (b) $q = 0.8$. As $p$ increases, the distribution splits into two in both cases. We can think of this as splitting of a particle into two parts by the interactions. In the region near $p_c$, both the one-peak and the two-peak phases exhibit super diffusion. In this phase, the speed of convergence is slower than that in the normal diffusion phase. We believe that the particles are deformed by the interactions. When $q = 0.5$, a particle is divided continuously. When $q = 0.8$, a second particle appears far from the original particle and is divided discontinuously.
In [10], the authors observed that landslides occurred mostly in countries with a small number of electors in the 2008 US presidential election. This indicates that herders refer to local information and not global information. In a previous study, we analyzed the case wherein herders could see the $r$ previous votes. This indicates that voters can share information locally.

We discussed only information cascade transition and could not discuss the transition for super and normal diffusions. We consider stochastic differential equations, which are strong tools, and use them for analysis in this study. The case of several candidates remains to be investigated. In this study, we investigated the case of only two candidates. In our future studies, we shall consider these cases.

Acknowledgments

This work was supported by a grant-in-aid for Challenging Exploratory Research, no. 21654054 (SM).

Appendix A. Derivation of model from Bayes’ theorem

We estimate the probability that the candidate $C_1$ is correct by using the previous votes. We assume that the $(t+1)$th voter is a herder. The voter estimates the posterior distribution that $C_1$ is the correct candidate by using Bayes’ theorem. The voter can see $t$ votes.

Herders estimate the probability that $C_i$, where $i = 0, 1$, is the correct candidate by using the previous votes. $Pr(C_i)$ is the probability that the voter estimates that $C_i$ is the correct candidate. We set the prior distribution as $Pr(C_0) = Pr(C_1) = 1/2$. Here, we assume that the herders believe that all voters are independent and estimate the percentage of correct answers as $\hat{q}$. The posterior distribution is

$$Pr(C_1 | c_1 = k) = \frac{1}{2} \frac{t!}{k!(t-k)!} \hat{q}^k (1 - \hat{q})^{t-k},$$

$$Pr(C_0 | c_1 = k) = \frac{1}{2} \frac{t!}{k!(t-k)!} (1 - \hat{q})^k \hat{q}^{t-k},$$

(A.1)

where $c_1 = k$ indicates that the number of votes for $C_1$ is $k$ before the voter votes. We can obtain

$$\frac{Pr(C_1 | c_1 = k)}{Pr(C_0 | c_1 = k)} = \left( \frac{\hat{q}}{1 - \hat{q}} \right)^{2k-t} = e^{2(\lambda - \frac{1}{2})t},$$

(A.2)

where $\lambda = t \log \frac{1}{1 - \hat{q}}$. Here, $\lambda$ increases as $t$ increases. Hence, the herder can calculate the probability that the candidate $C_1$ is correct when the number of votes for $C_j$ is $k$:

$$Pr(C_1 | c_1 = k) = \frac{1}{2} \left[ \tanh \lambda \left( \frac{k}{t} - \frac{1}{2} \right) + 1 \right].$$

(A.3)

In the region $t \gg 1$, $\lambda \to \infty$, a voter believes the public perception, and the behavior of herders becomes similar to that of digital herders without $\hat{q} = 1/2$. When $\hat{q} = 1/2$, the herders estimate that the previous votes are not useful to estimate the correct answer and $\lambda = 0$.

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7 We have previously discussed the cases of several candidates when all voters are analogue herders [15]. The probability functions of the share of votes of the candidates obey the gamma distributions.
Appendix B. Derivation of stochastic differential equation

We use \( \delta X = X_{t+\epsilon} - X_t \) and \( \xi_t \), a standard i.i.d. Gaussian sequence; our objective is to identify the drift \( f \) and the variance \( g^2 \) such that
\[
\delta X = f(X_t)\epsilon + \sqrt{\epsilon} g(X_t) \xi_{t+\epsilon}.
\]

Given \( X_t = x \), using the transition probabilities of \( \Delta_n \), we obtain
\[
E(\delta X) = \epsilon E(\Delta_{[\frac{t+\epsilon}{\epsilon}]\epsilon} - \Delta_{[\frac{t}{\epsilon}]\epsilon}) = \epsilon (2p_{[\frac{t+\epsilon}{\epsilon}]\epsilon} - 1)
= \epsilon \left[ (1 - p)(2q - 1) + p \tanh \left( \frac{\lambda x}{2\tau} \right) \right].
\]

Then, the drift term is
\[
f(x) = (1 - p)(2q - 1) + p \tanh \left( \frac{\lambda x}{2\tau} \right).
\]

Moreover,
\[
\sigma^2(\delta X) = \epsilon^2 \left[ 12p_{[\frac{t+\epsilon}{\epsilon}]\epsilon} + (-1)^2(1 - p_{[\frac{t+\epsilon}{\epsilon}]\epsilon}) \right] = \epsilon^2,
\]

such that \( g_{\epsilon,\tau}(x) = \sqrt{\epsilon} \). We can obtain \( X \) such that it obeys a diffusion equation with small additive noise:
\[
dX = \left[ (1 - p)(2q - 1) + p \tanh \left( \frac{\lambda X}{2\tau} \right) \right] d\tau + \sqrt{\epsilon}.
\]

Appendix C. Behavior of solutions of the stochastic differential equation

We consider the stochastic differential equation
\[
dx = \left( \frac{lx}{\tau} \right) d\tau + \sqrt{\epsilon},
\]
where \( \tau \geq 1 \). If we set \( l = p\lambda(1 - \bar{v}^2)/2 \), (C.1) is identical to (18). Let \( \sigma^2_1 \) be the variance of \( x_1 \). If \( x_1 \) is Gaussian \( (x_1 \sim N(x_1, \sigma^2_1)) \) or deterministic \( (x_1 \sim \delta x_1) \), the law of \( x_\tau \) ensures that the Gaussian is in accordance with density
\[
p_\tau(x) \sim \frac{1}{\sqrt{2\pi\sigma_\tau}} e^{-\frac{(x - \mu_\tau)^2}{2\sigma_\tau^2}},
\]
where \( \mu_\tau = E(x_\tau) \) is the expected value of \( x_\tau \) and \( \sigma^2_\tau \equiv \nu_\tau \) is its variance. If \( \Phi_\tau(\xi) = \log(a^{i\xi} \xi) \) is the logarithm of the characteristic function of the law of \( x_\tau \), we have
\[
\partial_\tau \Phi_\tau(\xi) = \frac{l}{\tau} \partial_\xi \Phi_\tau(\xi) - \frac{\epsilon}{2} \xi^2,
\]
and
\[
\Phi_\tau(\xi) = i\xi \mu_\tau - \frac{\xi^2}{2} \nu_\tau.
\]
Identifying the real and imaginary parts of \( \Phi_\tau(\xi) \), we obtain the dynamics of \( \mu_\tau \) as
\[
\dot{\mu}_\tau = \frac{l}{\tau} \mu_\tau,
\]
and the solution for \( \mu_\tau \) is
\[
\mu_\tau = x_1 \tau^l.
\]
The dynamics of \( \nu_\tau \) are given by the Riccati equation
\[
\dot{\nu}_\tau = \frac{2l}{\tau} \nu_\tau + \epsilon.
\]
If \( \nu \neq 1/2 \), we obtain
\[
\nu \tau = \nu_1 \tau^{2l} + \frac{\epsilon}{1 - 2l}(\tau - \tau^{2l}).
\]
(C.8)

If \( l = 1/2 \), we obtain
\[
\nu \tau = \nu_1 \tau + \epsilon \tau \log \tau.
\]
(C.9)

We can summarize the temporal behavior of the variance as
\[
\nu \tau \sim \frac{\epsilon}{1 - 2l} \tau \quad \text{if} \quad l < \frac{1}{2},
\]
(C.10)
\[
\nu \tau \sim \left( \nu_1 + \frac{\epsilon}{2l - 1} \right) \tau^{2l} \quad \text{if} \quad l > \frac{1}{2},
\]
(C.11)
\[
\nu \tau \sim \epsilon \tau \log (\tau) \quad \text{if} \quad l = \frac{1}{2}.
\]
(C.12)

This model has three phases. If \( l > 1/2 \) or \( l = 1/2 \), \( x_\tau/\tau \) converges slower than in a binomial distribution. These phases are the super diffusion phases. If \( 0 < p < 1/2 \), \( x_\tau/\tau \) converges as in a binomial distribution. This is the normal phase [13].

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