ON THE TRIVIALITY OF DOMAINS OF POWERS AND ADJOINTS OF CLOSED OPERATORS

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Abstract. The paper is devoted to counterexamples involving the triviality of domains of products and/or adjoints of densely defined operators.

1. Introduction

Counterexamples about non-necessarily bounded operators have not stopped to impress us. The striking example due to Chernoff is well known to specialists. It states that there is a closed, unbounded, densely defined, symmetric and semi-bounded operator $A$ such that $D(A^2) = \{0\}$ (see [3]). This counterexample came in to simplify a rather complicated construction already obtained by Naimark in [10]. It is worth noticing that Schmüdgen [12] obtained almost simultaneously (as Chernoff) that every unbounded self-adjoint $T$ has two closed and symmetric restrictions $A$ and $B$ such that

$$D(A) \cap D(B) = \{0\} \text{ and } D(A^2) = D(B^2) = \{0\}.$$ 

This fascinating result by Schmüdgen (which was later generalized by Brasche-Neidhardt in [2]. See also [1]) also dealt with higher powers.

Recently, the author obtained (jointly with S. Dehimi) in [4] a fairly simple example based upon matrices of unbounded operators. Based on results from [4], we propose the following conjecture:

Conjecture 1.1. For each $n \in \mathbb{N}$, there is a closed and densely defined $T$ such that $D(T^{n-1}) \neq \{0\}$ and $D[T^{(n-1)}] \neq \{0\}$ yet

$$D(T^n) = D(T^*n) = \{0\}.$$ 

It is worth emphasizing that even though K. Schmüdgen obtained the general case of $n$ powers, here we give more explicit counterexamples and the novelty is that the counterexamples in our case concern both the powers of an operator as well as the powers of their adjoints.

The main aim of this paper is to try to give answers to the previous conjecture. It is just amazing how matrices of unbounded operators, despite their unexpected behavior in some cases, can make things fairly easy to deal with. The same approach has equally allowed us to find more interesting counterexamples on a different topic. See [9].

In the end, we assume readers are familiar with notions and results on unbounded operators and, in particular, matrices of unbounded operators. We refer readers to [11] for properties of block operator matrices. From some recent papers on
matrices of unbounded operators, we cite [5], [8] and [11]. For the general theory of unbounded operators, readers may wish to consult [13] or [15].

2. Main Counterexamples

We start with an auxiliary example which is also interesting in its own.

**Proposition 2.1.** There is an unbounded self-adjoint and positive operator $A$ and an everywhere defined bounded and self-adjoint $B$ such that $D(AB) = \{0\}$ and $D(BA) \neq \{0\}$ (in fact $D(BA) = D(A)$ is dense).

**Proof.** In fact, we have a slightly better counterexample than what is suggested. Consider the operators $A$ and $B$ defined by

$$Af(x) = e^{x^2}f(x)$$

on $D(A) = \{ f \in L^2(\mathbb{R}) : e^{x^2}f \in L^2(\mathbb{R}) \}$ and $B := \mathcal{F}^* A \mathcal{F}$ where $\mathcal{F}$ designates the usual $L^2(\mathbb{R})$-Fourier Transform. Then $D(A) \cap D(B) = \{0\}$ (see e.g. [5]). Clearly $A$ is boundedly invertible and

$$A^{-1}f(x) = e^{-x^2}f(x)$$

is defined from $L^2(\mathbb{R})$ onto $D(A)$.

Recall that $D(BA^{-1})$ is trivial if $D(B) \cap \text{ran}(A^{-1})$ is so and if $A^{-1}$ is one-to-one. That $A^{-1}$ is injective is plain. Now,

$$D(B) \cap \text{ran}(A^{-1}) = D(B) \cap D(A) = \{0\}$$

and this is already available to us. In the end,

$$D(A^{-1}B) = D(B)$$

which is evidently dense in $L^2(\mathbb{R})$. \hfill \Box

**Proposition 2.2.** There exists a densely defined operator $T$ such that $D(T^*) = D(T^2) = D(TT^*) = D(T^*T) = \{0\}$.

**Proof.** There are known examples in the literature about the case $D(T^*) = \{0\}$. For instance, Example 3.4 on Page 105 in [9] or Example 3 on Page 69 in [15]. These examples are not that straightforward. The example we are about to give is truly simple.

Consider the operators $A$ and $B$ introduced in Proposition 2.1 that is,

$$Af(x) = e^{x^2}f(x)$$

on $D(A) = \{ f \in L^2(\mathbb{R}) : e^{x^2}f \in L^2(\mathbb{R}) \}$ and $B := \mathcal{F}^* A \mathcal{F}$. We then found that $D(BA^{-1}) = \{0\}$.

Now, set $T := A^{-1}B$. Then $T$ is densely defined because $D(T) = D(B)$ as also $A^{-1} \in B(L^2(\mathbb{R}))$. Thus,

$$D(T^*) = D((A^{-1}B)^*) = D(BA^{-1}) = \{0\},$$

as needed. Hence plainly

$$D(TT^*) = \{0\}.$$ 

Now,

$$D(T^*T) = \{ f \in D(T) : Tf \in D(T^*) \} = \{ f \in D(A^{-1}B) : A^{-1}Bf = 0 \} = \{0\}$$
for $A^{-1}B$ is one-to-one. Finally,
\[ D(T^3) = D(A^{-1}BA^{-1}B) = D[(BA^{-1})B] = \{ f \in D(B) : Bf = 0 \} \]
and so $D(T^2) = \{ 0 \}$ by the injectivity of $B$. \hfill \Box

**Proposition 2.3.** There is a densely defined and closed operator $T$ such that $D(T^2) \neq \{ 0 \}$ and $D(T^2) \neq \{ 0 \}$ but
\[ D(T^3) = D(T^4) = \{ 0 \}. \]

**Proof.** Let $A$ and $B$ be self-adjoint operators such that $D(AB) = \{ 0_{L^2(\mathbb{R})} \}$ but $D(BA) \neq \{ 0_{L^2(\mathbb{R})} \}$ as in Proposition 2.1. Remember that there $A$ is one-to-one. Now, on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, set
\[ T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}. \]
Hence
\[ T^2 = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \]
and so $D(T^2) = \{ 0_{L^2(\mathbb{R})} \} \oplus D(BA) \neq \{ (0_{L^2(\mathbb{R})}, 0_{L^2(\mathbb{R})}) \}$. Finally,
\[ T^3 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & ABA \\ B & 0 \end{pmatrix}. \]
Obviously, $D(BAB) = \{ 0_{L^2(\mathbb{R})} \}$. Since
\[ D(ABA) = \{ x \in D(A) : Ax \in D(AB) = \{ 0_{L^2(\mathbb{R})} \} \} = \ker A \]
and $A$ is injective, it follows that we equally have $D(ABA) = \{ 0_{L^2(\mathbb{R})} \}$. Accordingly, $D(T^3) = \{ (0_{L^2(\mathbb{R})}, 0_{L^2(\mathbb{R})}) \}$. Finally, as
\[ T^* = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}, \]
then we may similarly show that $D(T^*) \neq \{ 0 \}$ and $D(T^*) = \{ 0 \}$, marking the end of the proof. \hfill \Box

**Proposition 2.4.** There exists a densely defined and closed operator $T$ such that $D(T^3) \neq \{ 0 \}$ and $D(T^3) \neq \{ 0 \}$ yet
\[ D(T^4) = D(T^4) = \{ 0 \}. \]

**Remark.** Obviously, $D(T^3) \neq \{ 0 \}$ will insure that $D(T^2) \neq \{ 0 \}$. The same remark applies to $D(T^*^3) \neq \{ 0 \}$ and $D(T^*^2) \neq \{ 0 \}$.

The counterexample is based on the following recently obtained result:

**Lemma 2.5.** \([4]\) There are unbounded self-adjoint operators $A$ and $B$ such that
\[ D(A^{-1}B) = D(BA^{-1}) = \{ 0 \} \]
(where $A^{-1}$ and $B^{-1}$ are not bounded).

Now, we give the proof of Proposition 2.4.

Proof. Let \( A \) and \( B \) be two unbounded self-adjoint operators such that 
\[
D(A^{-1}B) = D(BA^{-1}) = \{0\}
\]
where \( A^{-1} \) and \( B^{-1} \) are not bounded. Now, define 
\[
S = \begin{pmatrix}
0 & A^{-1} \\
B & 0
\end{pmatrix}
\]
on \( D(S) := D(B) \oplus D(A^{-1}) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \). Then \( S \) is densely defined and closed. In addition, we already know from Lemma 2.5 that 
\[
D(A^{-1}B) = D(BA^{-1}) = \{0\}
\]
and so \( D(S^2) = D(S^{*2}) = \{0\} \) (as in [4] say). Notice now that we may write 
\[
S = \begin{pmatrix}
0 & A^{-1} \\
B & 0
\end{pmatrix}
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\]
and \( S^* = DC \)
because \( C \) and \( D \) are self-adjoint and \( D^{-1} \in B(H) \) (\( D \) is even a fundamental symmetry). Now, define \( T \) on \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) by 
\[
T = \begin{pmatrix}
0 & C \\
D & 0
\end{pmatrix}
\]
where \( 0 \) is the zero matrix of operators on \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \). Then, 
\[
T^2 = \begin{pmatrix}
CD & 0 \\
0 & DC
\end{pmatrix}, \quad T^3 = \begin{pmatrix}
0 & CDC \\
DCD & 0
\end{pmatrix}
\]
and 
\[
T^4 = \begin{pmatrix}
CD & 0 \\
0 & DC
\end{pmatrix} = \begin{pmatrix}
S^2 & 0 \\
0 & S^{*2}
\end{pmatrix}.
\]
Also, since \( T^* = \begin{pmatrix}
0 & D \\
C & 0
\end{pmatrix} \), we equally have 
\[
T^{*2} = \begin{pmatrix}
DC & 0 \\
0 & CD
\end{pmatrix}, \quad T^{*3} = \begin{pmatrix}
0 & DCD \\
CDC & 0
\end{pmatrix}
\]
and 
\[
T^{*4} = \begin{pmatrix}
DC & 0 \\
0 & CD
\end{pmatrix} = \begin{pmatrix}
S^{*2} & 0 \\
0 & S^2
\end{pmatrix}.
\]
Finally, observe that 
\[
D(T^2) = D(S) \oplus D(S^*) = D(B) \oplus D(A^{-1}) \oplus D(A^{-1}) \oplus D(B) \neq \{0\}_{L^2(\mathbb{R})^4},
\]
that 
\[
D(T^3) = D(B) \oplus D(A^{-1}) \oplus \{0\} \oplus \{0\} \neq \{0\}_{L^2(\mathbb{R})^4}
\]
but 
\[
D(T^4) = D(S^2) \oplus D(S^{*2}) = \{0\}_{L^2(\mathbb{R})^4}.
\]
The corresponding relations about the domains of adjoints may be checked similarly. The proof is therefore complete. \( \Box \)

The same idea of proof may be carried over to higher powers, however, we have not been able yet to establish a general counterexample. We give further counterexamples which may inspire readers to find the coveted general counterexample. For example, we know how to deal with the case \( n = 6 \).
Proposition 2.6. There exists a densely defined and closed $T$ such that $D(T^5) \neq \{0\}$ and $D(T^{*5}) \neq \{0\}$ whilst

$$D(T^6) = D(T^{*6}) = \{0\}.$$  

Proof. We shall avoid unnecessary details as similar cases have already been treated. First, choose $A$ and $B$ as in Proposition 2.1, that is, $D(AB) = \{0\}$ and $D(BA) = D(A)$ where $A$ and $B$ are self-adjoint operators and $B$ is bounded and everywhere defined.

Now, let $C = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$ and $D = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Then $C$ and $D$ are self-adjoint (remember that $D$ is even everywhere defined and bounded). Set $S = CD = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$. Hence $S^* = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$. Finally, define $T$ on $[L^2(\mathbb{R})]^4$ by

$$T = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}$$

and so $T^* = \begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}$ with $0$ being the zero matrix of operators on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Hence

$$T^6 = \begin{pmatrix} S^3 & 0 \\ 0 & S^{*3} \end{pmatrix}$$

and $T^{*6} = \begin{pmatrix} S^{*3} & 0 \\ 0 & S^3 \end{pmatrix}$.

Accordingly, $D(T^6) = D(S^3) \oplus D(S^{*3}) = \{(0,0)\} = D(T^{*6})$ thanks to the assumptions on $A$ and $B$. However,

$$T^5 = \begin{pmatrix} 0 & CDCDC \\ CDCDC & 0 \end{pmatrix},$$

$T^{*5} = \begin{pmatrix} 0 & DCD\bar{C}D \\ C\bar{C}D\bar{C}D & 0 \end{pmatrix}$

and, as can simply be checked,

$$D(CDCDC) = \{0\}$$

but $D(DC\bar{C}D\bar{C}) \neq \{0\}$.

Consequently, $D(T^5) \neq \{0\}$. A similar reasoning yields $D(T^{*5}) \neq \{0\}$, marking the end of the proof. \hfill $\square$

The next counterexamples settles the case of powers of the type $2^n$ via what we may call "nested matrices".

Proposition 2.7. For each $n \in \mathbb{N}$, there is a densely defined and closed operator $T$ (which is an off-diagonal matrix of operators) such that $D(T^{2^{n-1}}) \neq \{0\}$ and $D(T^{*2^{n-1}}) \neq \{0\}$ whereas

$$D(T^{2^n}) = D(T^{*2^n}) = \{0\}.$$  

Proof. We use a proof by induction. The statement is true for $n = 2$ as seen before. Assume now that there is a closed $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ such that $D(T^{2^{n-1}}) \neq \{0\}$ and $D(T^{*2^{n-1}}) \neq \{0\}$ with $D(T^{2^n}) = D(T^{*2^n}) = \{0\}$. Now, write

$$T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}$$
Next, set $S = \begin{pmatrix} 0 & C \\ \tilde{B} & 0 \end{pmatrix}$ and so
\[
S^{2^{n+1}} = \begin{pmatrix} (C\tilde{B})^{2^n} & 0 \\ 0 & (\tilde{B}C)^{2^n} \end{pmatrix} = \begin{pmatrix} T^{2^n} & 0 \\ 0 & T^{2^n} \end{pmatrix}
\]
and so $D(S^{2^{n+1}}) = \{0\}$. On the other hand,
\[
S^{2^{n+1}-1} = \begin{pmatrix} 0 & (C\tilde{B})^{2^n-1}C \\ \tilde{B}(C\tilde{B})^{2^n-1} & 0 \end{pmatrix}.
\]
Since $\tilde{B}$ is everywhere bounded, it follows that
\[
D(\tilde{B}(C\tilde{B})^{2^n-1}) = D((C\tilde{B})^{2^n-1}) = D(T^{2^n-1}) \neq \{0\}
\]
which leads to $D(S^{2^{n+1}-1}) \neq \{0\}$, as wished.

The case of adjoints may be treated similarly and hence we omit it. \qed

3. Conclusion

Even though we have managed to obtain a fair amount of counterexamples, we have not been able to solve the whole conjecture. If the conjecture is false for some $n$, then proving this is not going to be an easy task. So, we invite interested readers to try to contribute towards a complete answer to the main problem.

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