Partial differential equations/Calculus of variations

Uniqueness of degree-one Ginzburg–Landau vortex in the unit ball in dimensions \( N \geq 7 \)

Unicité du tourbillon de Ginzburg–Landau de degré un dans la boule unité en dimension \( N \geq 7 \)

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A B S T R A C T

For \( \varepsilon > 0 \), we consider the Ginzburg–Landau functional for \( \mathbb{R}^N \)-valued maps defined in the unit ball \( B^N \subset \mathbb{R}^N \) with the vortex boundary data \( x \) on \( \partial B^N \). In dimensions \( N \geq 7 \), we prove that, for every \( \varepsilon > 0 \), there exists a unique global minimizer \( u_\varepsilon \) of this problem; moreover, \( u_\varepsilon \) is symmetric and of the form \( u_\varepsilon (x) = f_\varepsilon (|x|) \frac{A}{|x|} \) for \( x \in B^N \).

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R É S U M É

Nous considérons la fonctionnelle de Ginzburg–Landau pour les applications à valeurs dans \( \mathbb{R}^N \) définies dans la boule unité \( B^N \subset \mathbb{R}^N \) avec la donnée de tourbillon \( x \) au bord \( \partial B^N \). En dimension \( N \geq 7 \), nous montrons que, pour tout \( \varepsilon > 0 \), il existe un unique minimiseur global \( u_\varepsilon \) à ce problème; de plus, \( u_\varepsilon \) est symétrique de la forme \( u_\varepsilon (x) = f_\varepsilon (|x|) \frac{A}{|x|} \) pour \( x \in B^N \).

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1. Introduction and main results

In this note, we consider the following Ginzburg–Landau-type energy functional

$$E_\varepsilon(u) = \int_{\mathbb{R}^N} \left[ \frac{1}{2} |\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2) \right] \, dx,$$

where $\varepsilon > 0$, $B^N$ is the unit ball in $\mathbb{R}^N$, $N \geq 2$, and the potential $W \in C^1((-\infty, 1]; \mathbb{R})$ satisfies

$$W(0) = 0, \quad W(t) > 0 \text{ for all } t \in (-\infty, 1] \setminus \{0\}, \text{ and } W \text{ is convex.} \quad (1)$$

We investigate the global minimizers of the energy $E_\varepsilon$ in the set

$$\mathcal{A} := \{ u \in H^1(B^N; \mathbb{R}^N) : u(x) = x \text{ on } \partial B^N = \mathbb{S}^{N-1} \}.$$

The requirement that $u(x) = x$ on $\mathbb{S}^{N-1}$ is sometimes referred to in the literature as the vortex boundary condition.

We note that, in our analysis, the convexity of $W$ needs not be strict; compare [7] where strict convexity is assumed.

The direct method in the calculus of variations yields the existence of a global minimizer $u_\varepsilon$ of $E_\varepsilon$ over $\mathcal{A}$ for all range of $\varepsilon > 0$. Moreover, any minimizer $u_\varepsilon$ belongs to $C^1(B^N; \mathbb{R}^N)$ and satisfies $|u_\varepsilon| \leq 1$ and the system of PDEs (in the sense of distributions):

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon W'(1 - |u_\varepsilon|^2) \quad \text{in } B^N. \quad (2)$$

The goal of this note is to give a short proof of the uniqueness and symmetry of the global minimizer of $E_\varepsilon$ in $\mathcal{A}$ for all $\varepsilon > 0$ in dimensions $N \geq 7$. We prove that, in these dimensions, the global minimizer is unique and given by the unique radially symmetric critical point of $E_\varepsilon$ defined by

$$u_\varepsilon(x) = f_\varepsilon(|x|) \frac{x}{|x|} \quad \text{for all } x \in B^N, \quad (3)$$

where the radial profile $f_\varepsilon : [0, 1] \to \mathbb{R}_+$ is the unique solution to

$$\begin{cases}
- \varepsilon^2 f''_\varepsilon - \frac{N-1}{r} f'_\varepsilon + \frac{N-1}{r^2} f_\varepsilon = \frac{1}{\varepsilon^2} f_\varepsilon W'(1 - f_\varepsilon^2) & \text{for } r \in (0, 1),

f_\varepsilon(0) = 0, \quad f_\varepsilon(1) = 1.
\end{cases} \quad (4)$$

Moreover, $f_\varepsilon > 0$ and $f'_\varepsilon > 0$ in $(0, 1)$ (see, e.g., [5]).

**Theorem 1.** Assume that $W$ satisfies (1). If $N \geq 7$, then for every $\varepsilon > 0$, $u_\varepsilon$ given in (3) is the unique global minimizer of $E_\varepsilon$ in $\mathcal{A}$.

To our knowledge, the question about the uniqueness of minimizers/critical points of $E_\varepsilon$ in $\mathcal{A}$ for any $\varepsilon > 0$ was raised in dimension $N = 2$ in the book of Bethuel, Brézis and Hélein [1, Problem 10, page 139], and in general dimensions $N \geq 2$ and also for the blow-up limiting problem around the vortex (when the domain is the whole space $\mathbb{R}^N$ and by rescaling, $\varepsilon$ can be assumed equal to 1) in an article of Brézis [2, Section 2].

It is well known that uniqueness is present for large enough $\varepsilon > 0$ for any $N \geq 2$. Indeed, for any $\varepsilon > (W'(1)/\lambda_1)^{1/2}$ where $\lambda_1$ is the first eigenvalue of $-\Delta$ in $B^N$ with zero Dirichlet boundary condition, $E_\varepsilon$ is strictly convex in $\mathcal{A}$ and thus has a unique critical point in $\mathcal{A}$ (that is the global minimizer of our problem).

For *sufficiently small* $\varepsilon > 0$, all results regarding uniqueness question available in the literature are in the affirmative. In particular, we have:

(i) Pacard and Rivière [12, Theorem 10.2] showed in dimension $N = 2$ that, for small $\varepsilon > 0$, $E_\varepsilon$ has in fact a unique critical point in $\mathcal{A}$;

(ii) Mironescu [11] showed in dimension $N = 2$ that, when $B^2$ is replaced by $\mathbb{R}^2$ and $\varepsilon = 1$, a local minimizer of $E_\varepsilon$ subjected to a degree-one boundary condition at infinity is unique (up to translation and suitable rotation). This was generalized to dimension $N = 3$ by Millot and Pisante [10] and dimensions $N \geq 4$ by Pisante [13], also in the case of the blow-up limiting problem on $\mathbb{R}^N$ and $\varepsilon = 1$.

These results should be compared to those for the limit problem on the unit ball obtained by sending $\varepsilon \to 0$. In this limit, the Ginzburg–Landau problem ‘converges’ to the harmonic map problem from $B^N$ to $\mathbb{S}^{N-1}$. It is well known that the vortex boundary condition gives rise to a unique minimizing harmonic map $x \mapsto \frac{x}{|x|}$ if $N \geq 3$; see Brezis, Coron and Lieb [3] in dimension $N = 3$, Jäger and Kaul [8] in dimensions $N \geq 7$, and Lin [9] in dimensions $N \geq 3$ (see also [4]).

We highlight that, in contrast to the above, our result holds for *all* $\varepsilon > 0$, provided that $N \geq 7$. The method of our proof deviates somewhat from that in the aforementioned works. In fact, it is reminiscent of our recent work [7] on
the (non-)uniqueness and symmetry of minimizers of the Ginzburg–Landau functionals for $\mathbb{R}^M$-valued maps defined on $N$-dimensional domains, where $M$ is not necessarily the same as $N$. However, we note that the results in [7] do not directly apply to the present context, as in [7] it is required that $W$ be strictly convex. Furthermore, a priori, it is not clear why non-strict convexity of the potential $W$ is sufficient to ensure uniqueness of global minimizers.

We exploit the convexity of $W$ to lower estimate the ‘excess’ energy by a suitable quadratic energy that can be handled by the factorization trick à la Hardy. Indeed, the positivity of the excess energy is then related to the validity of a Hardy-type inequality, which explains our restriction of $N \geq 7$. This echoes our observation made in [7] that a result of Jäger and Kaul [8] on the minimality of the equator map (for the harmonic map problem) in these dimensions is related to a certain inequality involving the sharp constant in the Hardy inequality; see Remark 3.

We expect that our result remains valid in dimensions $2 \leq N \leq 6$, but this goes beyond the scope of this note and remains for further investigation.

2. Proof of Theorem 1

Theorem 1 will be obtained as a consequence of a stronger result on the uniqueness of global minimizers of the $\mathbb{R}^M$-valued Ginzburg–Landau functional with $M \geq N$. By a slight abuse of notation, we consider the energy functional

$$E_\varepsilon(u) = \int_{B^N} \left[ \frac{1}{2} |\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2) \right] \, dx,$$

where $u$ belongs to

$$\mathcal{A} := \{ u \in H^1(B^N, \mathbb{R}^M) : u(x) = x \text{ on } \partial B^N = S^{N-1} \subset \mathbb{R}^M \}.$$

**Theorem 2.** Assume that $W$ satisfies (1). If $M \geq N \geq 7$, then for every $\varepsilon > 0$, $u_\varepsilon$ given in (3) is the unique global minimizer of $E_\varepsilon$ in $\mathcal{A}$.

When $W$ is strictly convex, the above theorem is proved in [7]; see [7, Theorem 1.7]. The argument therein uses the strict convexity in a crucial way.

**Proof.** The proof will be done in several steps. First, we consider the difference between the energies of the critical point $u_\varepsilon$, defined in (3), and an arbitrary competitor $u_\varepsilon + v$ and show that this difference is controlled from below by some quadratic energy functional $F_\varepsilon(v)$. Second, we employ the positivity of the radial profile $f_\varepsilon$ in (4) and apply the Hardy decomposition method in order to show that $F_\varepsilon(v) \geq 0$, which proves in particular that $u_\varepsilon$ is a global minimizer of $E_\varepsilon$. Finally, we characterise the situation when this difference is zero and conclude to the uniqueness of the global minimizer $u_\varepsilon$.

**Step 1: Lower bound for energy difference.** For any $v \in H^1_0(B^N; \mathbb{R}^M)$, we have

$$E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq \int_{B^N} \left[ \frac{1}{2} |\nabla u_\varepsilon \cdot \nabla v + \frac{1}{2} |\nabla v|^2 \right] \, dx + \frac{1}{2\varepsilon^2} \int_{B^N} \left[ W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2) \right] \, dx.$$

Using the convexity of $W$, we have

$$W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2) \geq -|W'(1 - |u_\varepsilon|^2)(|u_\varepsilon + v|^2 - |u_\varepsilon|^2)|.$$

The last two relations imply that

$$E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq \int_{B^N} \left[ \frac{1}{2} |\nabla u_\varepsilon \cdot \nabla v - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2)u_\varepsilon \cdot v \right] \, dx + \int_{B^N} \left[ \frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - f_\varepsilon^2)|v|^2 \right] \, dx.$$

Moreover, by (2), we obtain

$$E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq \int_{B^N} \left[ \frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - f_\varepsilon^2)|v|^2 \right] \, dx =: \frac{1}{2} F_\varepsilon(v)$$

for all $v \in H^1_0(B^N; \mathbb{R}^M)$. (In the sequel, for simplicity, we will also write $F_\varepsilon(v)$ for scalar $v \in H^1_0(B^N; \mathbb{R})$.)

**Step 2: A rewriting of $F_\varepsilon(v)$ using the decomposition $v = f_\varepsilon w$ for every scalar test function $v \in C^\infty_c(B^N \setminus \{0\}; \mathbb{R})$.** We consider the operator

$$L_\varepsilon := \frac{1}{2} \nabla_2^2 F_{\varepsilon} = -\Delta - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2).$$
Using the decomposition
\[ v = f_\varepsilon w \]
for the scalar function \( v \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}) \), we have (see, e.g., [6, Lemma A.1]):
\[
F_\varepsilon(v) = \int_{B^N} L_\varepsilon v \cdot v \, dx = \int_{B^N} w^2 L_\varepsilon f_\varepsilon \cdot f_\varepsilon \, dx + \int_{B^N} f_\varepsilon^2 |\nabla w|^2 \, dx
\]
\[ = \int_{B^N} f_\varepsilon^2 \left( |\nabla w|^2 - \frac{N-1}{r^2} w^2 \right) \, dx, \]
because (4) yields \( L_\varepsilon f_\varepsilon \cdot f_\varepsilon = -\frac{N-1}{r^2} f_\varepsilon^2 \) in \( B^N \).

Step 3: We prove that \( F_\varepsilon(v) \geq 0 \) for every scalar test function \( v \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}) \). Within the notation \( v = f_\varepsilon w \) of Step 2 with \( v, w \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}) \), we use the decomposition
\[ w = \varphi g \]
with \( \varphi = |x|^{-\frac{N-2}{2}} \) being the first eigenfunction of the Hardy’s operator \( -\Delta - \frac{(N-2)^2}{4|x|^2} \) in \( \mathbb{R}^N \setminus \{0\} \) and \( g \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}) \). We compute
\[ |\nabla w|^2 = |\nabla \varphi|^2 g^2 + |\nabla g|^2 \varphi^2 + \frac{1}{2} \nabla(\varphi^2) \cdot \nabla(g^2). \]
As \( |\nabla \varphi|^2 = \frac{(N-2)^2}{4|x|^2} \varphi^2 \) and \( \varphi^2 \) is harmonic in \( B^N \setminus \{0\} \), integration by parts yields
\[
F_\varepsilon(v) = \int_{B^N} f_\varepsilon^2 \left( |\nabla \varphi|^2 g^2 + \frac{(N-2)^2}{4r^2} \varphi^2 g^2 - \frac{N-1}{r^2} \varphi^2 g^2 \right) \, dx - \frac{1}{2} \int_{B^N} \nabla(\varphi^2) \cdot \nabla(f_\varepsilon^2) g^2 \, dx
\]
\[ \geq \int_{B^N} f_\varepsilon^2 |\nabla \varphi|^2 g^2 \, dx + \frac{(N-2)^2}{4} - (N-1) \int_{B^N} f_\varepsilon^2 \varphi^2 g^2 \, dx
\]
\[ \geq \frac{(N-2)^2}{4} - (N-1) \int_{B^N} \frac{v^2}{r^2} \, dx \geq 0, \tag{6} \]
where we have used \( N \geq 7 \) and \( \frac{1}{2} \nabla(\varphi^2) \cdot \nabla(f_\varepsilon^2) = 2\varphi \varphi' f_\varepsilon f_\varepsilon' \leq 0 \) in \( B^N \setminus \{0\} \).

Step 4: We prove that \( F_\varepsilon(v) \geq 0 \) for every \( v \in H^1_0(B^N; \mathbb{R}^M) \), meaning that \( u_\varepsilon \) is a global minimizer of \( E_\varepsilon \) over \( \mathcal{A} \); moreover, \( F_\varepsilon(v) = 0 \) if and only if \( v = 0 \). Let \( v \in H^1_0(B^N; \mathbb{R}^M) \). As a point has zero \( H^1 \) capacity in \( \mathbb{R}^N \), a standard density argument implies the existence of a sequence \( v_k \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}^M) \) such that \( v_k \rightharpoonup v \) in \( H^1(B^N; \mathbb{R}^M) \) and a.e. in \( B^N \). On the one hand, by definition (5) of \( F_\varepsilon \), since \( W'(1 - f_\varepsilon^2) \in L^\infty \), we deduce that \( F_\varepsilon(v_k) \to F_\varepsilon(v) \) as \( k \to \infty \). On the other hand, by (6) and Fatou’s lemma, we deduce
\[
\liminf_{k \to \infty} F_\varepsilon(v_k) \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \liminf_{k \to \infty} \int_{B^N} \frac{v_k^2}{r^2} \, dx
\]
\[ \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} \, dx. \]
Therefore, we conclude that
\[ F_\varepsilon(v) \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} \, dx \geq 0, \quad \forall v \in H^1_0(B^N; \mathbb{R}^M), \]
implying by (5) that \( u_\varepsilon \) is a minimizer of \( E_\varepsilon \) over \( \mathcal{A} \). Moreover, \( F_\varepsilon(v) = 0 \) if and only if \( v = 0 \).

Step 5: Conclusion. We have shown that \( u_\varepsilon \) is a global minimizer. Assume that \( \tilde{u}_\varepsilon \) is another global minimizer of \( E_\varepsilon \) over \( \mathcal{A} \). If \( v := \tilde{u}_\varepsilon - u_\varepsilon \), then \( v \in H^1_0(B^N; \mathbb{R}^M) \) and by steps 1 and 4, we have that \( 0 = E_\varepsilon(\tilde{u}_\varepsilon) - E_\varepsilon(u_\varepsilon) \geq F_\varepsilon(v) \geq 0 \), which yields \( F_\varepsilon(v) = 0 \). Step 4 implies that \( v = 0 \), i.e. \( \tilde{u}_\varepsilon = u_\varepsilon \). \qed
Remark 3. Recall that, in the case $M \geq N \geq 7$, Jäger and Kaul [8] proved the uniqueness of global minimizer for harmonic map problem

$$\min_{u \in \mathcal{A}_{a}} \int_{B^N} |\nabla u|^2 \, dx,$$

where $\mathcal{A}_{a} = \{ u \in H^1(B^N; S^{M-1}) : u(x) = x \text{ on } \partial B^N = S^{N-1} \subset S^{M-1} \}$. This can also be seen by the method above, as observed in our earlier paper [7]. We give the argument here for readers' convenience: take a perturbation $v \in H^1_0(B^N, \mathbb{R}^M)$ of the harmonic map $u_*(x) = \frac{x}{|x|}$ such that $|u_*(x) + v(x)| = 1$ a.e. in $B^N$. Then, by [7, Proof of Theorem 5.1],

$$\int_{B^N} \left[ |\nabla (u_* + v)|^2 - |\nabla u_*|^2 \right] \, dx = \int_{B^N} \left[ |\nabla v|^2 - |\nabla u_*|^2 \right] \, dx = \int_{B^N} \left[ |\nabla v|^2 - (N-1) \frac{|v|^2}{|x|^2} \right] \, dx.$$

Using Hardy's inequality in dimension $N$, we arrive at

$$\int_{B^N} \left[ |\nabla (u_* + v)|^2 - |\nabla u_*|^2 \right] \, dx \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{|v|^2}{|x|^2} \, dx.$$

The result follows since $N \geq 7$.

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