Bianchi type-II cosmological model: some remarks

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Abstract: Within the framework of a Bianchi type-II (BII) cosmological model the behavior of matter distribution has been considered. It is shown that the non-zero off-diagonal component of the Einstein tensor implies some severe restrictions on the choice of matter distribution. In particular, for a locally rotationally symmetric Bianchi type-II (LRS BII) space-time, it is proved that the matter distribution should be strictly isotropic if the corresponding matter field possesses only non-zero diagonal components of the energy-momentum tensor.

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1. Introduction

Spatially homogeneous and anisotropic cosmological models play a significant role in the description of the large scale behavior of the Universe. In search of a realistic picture of the early Universe such models have been widely studied within a framework of General Relativity. In this note we confine our study to the scope of a Bianchi type-II (BII) space-time, which has recently been studied by a number of authors: Christodoulakis et al. [1] investigated the set of space-time general coordinate transformations which leave the line element of a generic Bianchi-type geometry quasiform invariant; Einstein's field equations for stationary BII models with a perfect fluid source were investigated by Nilsson and Ugglä [2]; Ram and Singh [3] presented analytical solutions of the Einstein-Maxwell equations for cosmological models of LRS Bianchi type-II, VIII and IX; two-fluid BII cosmological models were studied by Pant and Oli [4]; a BII cosmological model with constant deceleration parameter was considered by Singh and Kumar [5]; Belinchon [6, 7] studied a massive cosmic string within the scope of a BII model, while LRS BII cosmological models in the presence of a massive cosmic string and varying cosmological constant were studied by Pradhan et al. [8], Kumar [9] and Yadav et al. [10], respectively. Other recent work includes exact solutions for BII cosmological model in the Jordan Brans-Dicke scalar-tensor theory of gravitation were obtained in [11], study of a BII Lyttleton-Bondi Universe [12], and determination of an anisotropic BII cosmological model in the presence of source-free electromagnetic fields in Lyra’s manifold [13]. The main aim of this report is to show that one has to take into account all ten Einstein equations in order to write the correct energy-momentum tensor for the source field.

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2. The metric and field equations

We consider a homogeneous BII space-time with the line element

$$ds^2 = -dt^2 + A^2(dx - zdy)^2 + B^2dy^2 + C^2dz^2,$$

with \( A, B, C \) being the function of \( t \) only. The metric (1) possesses the following non-zero components of an Einstein tensor:

\[
\begin{align*}
G_{0}^1 &= \frac{\dot{B}}{B} + \frac{\ddot{C}}{C} + \frac{B \dot{C}}{B C} - \frac{3 A^2}{4 B^2 C^2}, \\
G_{0}^2 &= \frac{\dot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{A} \dot{B}}{A B} - \frac{1 A^2}{4 B^2 C^2}, \\
G_{0}^3 &= \frac{\dot{A}}{A} + \frac{\ddot{B}}{B} + \frac{A \dot{B}}{A B} - \frac{1 A^2}{4 B^2 C^2}, \\
G_{0}^4 &= \frac{\ddot{A}}{A} + \frac{\dot{A} \dot{B}}{A B} + \frac{1 A^2}{4 B^2 C^2}.
\end{align*}
\]

Here over-dots denote derivation with respect to time. It can be easily verified that

\[
G_{2}^1 = x\left[G_{2}^2 - G_{1}^1\right].
\] (3)

Let us now consider the Einstein system of equations

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \delta_{\mu\nu} = \kappa T_{\mu\nu}.
\] (4)

Since \( G_{\mu\nu} \) is a symmetric tensor, in 4D space-time it has 10 independent components. Depending on the concrete metric and source field, the number of equations can be reduced; in doing so it should be remembered that any equation of the system (4) can be overlooked if and only if

i) both the left and right hand sides of this concrete equation are identically zero;

ii) this equation is a linear combination of other equations.

For example, the equation

\[
G_{2}^1 = \kappa T_{2}^1,
\] (5)

can be ignored only if the equality

\[
T_{2}^1 = x\left[T_{2}^2 - T_{1}^1\right]
\] (6)

holds.

In all other cases one has to write down all the equations. Those equations give relations between the metric functions (if \( G_{1}^j \neq 0 \) and \( T_{1}^j \equiv 0 \), as in the case of a Bianchi type-VI space-time with only non-zero diagonal components of the energy-momentum tensor [14]) or material field functions (if \( G_{1}^j \equiv 0 \) and \( T_{1}^j \neq 0 \), as in the case of a Bianchi type-I space-time with electro-magnetic fields with \( A_{\mu} = (0, A_{1}, A_{2}, A_{3}) \) [15, 16]).

Let us now consider the BII cosmological model for a source field with non-zero diagonal components only, i.e., with the energy-momentum tensor given by

\[
T_{\mu\nu} = \text{diag}\left[T_{0}^0, T_{1}^1, T_{2}^2, T_{3}^3\right].
\] (7)

The corresponding Einstein field equations are then

\[
\begin{align*}
\frac{\dot{B}}{B} + \frac{\ddot{C}}{C} + \frac{B \dot{C}}{B C} - \frac{3 A^2}{4 B^2 C^2} &= \kappa T_{1}^0, \\
\frac{\dot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{A} \dot{B}}{A B} - \frac{1 A^2}{4 B^2 C^2} &= \kappa T_{2}^0, \\
\frac{\dot{A}}{A} + \frac{\ddot{B}}{B} + \frac{A \dot{B}}{A B} - \frac{1 A^2}{4 B^2 C^2} &= \kappa T_{3}^0, \\
\frac{\ddot{A}}{A} + \frac{\dot{A} \dot{B}}{A B} + \frac{1 A^2}{4 B^2 C^2} &= \kappa T_{0}^0.
\end{align*}
\]

The certain relation between the components of the energy-momentum tensor, namely \( T_{2}^1 = T_{2}^2 \) follow from (3).

We can conclude that a BII cosmological model given by the metric element (1) does not allow the situation when \( T_{2}^1 = T_{2}^2 \), i.e., in this case the cosmic string cannot be directed along either \( x \) or \( y \) axes. Moreover, in the case when \( T_{1}^1 = T_{2}^2 \), one need not take into account Eq. (8e), as well as in case of isotropic distribution of matter when \( T_{1}^1 = T_{2}^2 = T_{3}^3 \). It should be emphasized that in the case of an LRS BII model when we have \( A = C \) in (1), the equality \( G_{1}^1 = G_{2}^1 \) implies \( T_{1}^1 = T_{2}^1 \) and (6) leads to \( T_{2}^1 = T_{2}^2 \) if the matter distribution is given by (7). This means that an LRS BII model filled with a matter field given by (6) does not allow anisotropic distribution of matter.

The results obtained in this note can be generalized as follows:

For a metric element

\[
ds^2 = -dt^2 + A^2(dx_i - x_i dx_j)^2 + B^2dx^2 + C^2dx^2,
\]

\[
i \neq j \neq k, \\
i, j, k = 1, 2, 3
\]

we find

\[
G_{j} = x_{k}\left[G_{j} - G_{i}\right].
\] (10)
Regarding the metric element (9) we can formulate the following:

*If the energy-momentum tensor possesses only non-zero diagonal components, i.e., given by (7), then a Bianchi type-II space-time depending explicitly on \( t \) and \( x_k \) essentially should have \( T_{ij} = T_{ij} \) with \( i \neq j \neq k \) and \( i, j, k = 1, 2, 3 \). The cosmic sting in this case must be directed along \( x_k \)-axis. As far as LRS BII space-time is concerned, the equality \( T_{ij} = T_{ij} = T_{ik} \) must hold.*

This simple example shows that one should be very careful in choosing source fields for cosmological models.

### 3. Conclusion

Within the scope of a BII cosmological model, it is shown that the non-zero off-diagonal component of the Einstein tensor implies severe restrictions to the components of the energy-momentum tensor. In the case of LRS BII space-time the model allows only isotropic distribution of the source field.

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