Construction of $J^{th}$-stage Nonuniform Wavelets on Local Fields

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Abstract: Shah and Abdullah [Complex Analysis Operator Theory, 9 (2015), 1589-1608] have introduced a generalized notion of nonuniform multiresolution analysis (NUMRA) on local field $K$ of positive characteristic in which the translation set $\Lambda$ acting on the scaling function to generate the core space $V_0$ is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$, given by $\Lambda = \{0, u(r)/N\} + \mathbb{Z}$, where $N \geq 1$ is an integer and $r$ is an odd integer such that $r$ and $N$ are relatively prime, and $\mathbb{Z} = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct cosets of the unit disc $\mathbb{D}$ in $K^+$. In this paper, we focus on the extension of nonuniform continuous wavelets to the construction of $J^{th}$-stage nonuniform discrete wavelets on local fields. We establish some general characterizations for the $J^{th}$-stage nonuniform discrete wavelet systems to be orthonormal bases in $L^2(\Lambda)$. Moreover, we establish a relation between the continuous wavelets of $L^2(K)$ and their discrete counterparts of $l^2(\Lambda)$.

Keywords: Nonuniform multiresolution analysis. $J^{th}$-stage discrete wavelet. local field. Fourier transform.

Mathematics Subject Classification: 42C40; 42C15; 43A70; 11S85; 47A25.

1. Introduction

In recent years, there has been a considerable interest in the study of harmonic analysis and wavelet analysis over the local fields. Local fields are essentially of two types: zero and positive characteristic. Examples of local fields of characteristic zero include the $p$-adic field $\mathbb{Q}_p$ where as local fields of positive characteristic are the Cantor dyadic group and the Vilenkin $p$-groups. Despite the fact that the structures and metrics of $p$-adic fields and local fields of positive characteristic are comparable, their wavelet and MRA theory are quite different. The notion of MRA on local fields of positive characteristic was introduced by Jiang et al. [3]. In fact, they brought up a technique for constructing orthogonal wavelets on local fields and established a necessary and sufficient condition for the solution of refinement equation to generate an MRA for $L^2(K)$. Subsequently, an explicit construction of tight wavelet frames on local fields was given by Shah and Debnath [11] by adapting the extension principles on the Euclidean spaces to the local fields. On the other hand, Shah and Abdullah [5] have set up an entire portrayal of tight wavelet frames on local fields by virtue of some fundamental equations in the frequency domain and demonstrate how to build the Parseval wavelet frames for $L^2(K)$. These studies were proceeded by Shah and his associates in [4, 6, 8, 9], where they have given some algorithms for constructing periodic wavelet frames, wave packet frames, and semi-orthogonal wavelet frames on non-Archimedean local fields of positive characteristic.
In our previous work [7], we have generalized the concept of Mallat’s classic MRA on Euclidean spaces $\mathbb{R}^n$ to nonuniform MRA on local fields of positive characteristic, in which the translation set acting on the scaling function associated with the MRA to generate the core space $V_0$ is no longer a group, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$, where $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of the unit disc $\mathcal{D}$ in the locally compact Abelian group $K^+$. More precisely, this set is of the form $\Lambda = \{0, u(r)/N\} + \mathcal{Z}$, where $N \geq 1$ is an integer and $r$ is an odd integer such that $r$ and $N$ are relatively prime. We call this a nonuniform multiresolution analysis (NUMRA) on local fields of positive characteristic. As a consequence of this generalization, we obtain a necessary and sufficient condition for the existence of associated wavelets and extension of Cohen’s theorem. Recently, we have constructed the associated nonuniform wavelet packets on local fields in [10]. Indeed, we obtain a lemma on the so-called splitting trick and several theorems concerning the Fourier transform of the nonuniform wavelet packets to show that their translates form an orthonormal basis for $L^2(K)$. More results in this direction can also be found in [13] and the references therein.

Owing to the fact that the data in both physics and engineering is often discrete in nature, makes us to focus our investigation over the discrete sequence spaces on local fields of positive characteristic. The concept of an adaptive MRA structure was introduced by Han et al. [2] for more general affine-like systems which exhibits all the favorable properties of MRA structures for wavelets whereas Han [1] has independently developed a comprehensive theory of discrete framelets and wavelets using an algorithmic approach by directly studying a discrete framelet transform. The main contribution of this paper is that we extend our previous work [7] and construct a class of $J^{th}$-stage nonuniform discrete wavelet systems on local fields of positive characteristic. Different from our previous approach in the orthonormal case, our analysis of nonuniform discrete scheme is inspired by Shukla and Mittal’s approach in [12] for construction of wavelets on the spectrum. We provide some characterizations of the $J^{th}$-stage discrete wavelet systems to be orthonormal bases for the Hilbert space $l^2(\Lambda)$. Moreover, we establish a connection between a system of nonuniform wavelets of $L^2(K)$ and a first-stage nonuniform discrete wavelet system of $l^2(\Lambda)$.

The article is organized as follows. In Sect. 2, we give a necessary background about local fields including the definitions of Fourier transform, uniform MRA and nonuniform MRA on fields fields. In Sect. 3, we introduce the construction of a first-stage nonuniform discrete wavelet system and provide a characterization for such a system to be an orthonormal basis for the Hilbert space $l^2(\Lambda)$. Sect. 4, is devoted to the construction of $J^{th}$-stage nonuniform discrete wavelets for $l^2(\Lambda)$ by decomposing of its closed subspaces. Finally, we establish a relation between the continuous wavelets of $L^2(K)$ and their discrete counterparts of $l^2(\Lambda)$ in Sect. 5.

2. Fourier and Wavelet Analysis on local Fields

In this section, we present some important preliminaries and notation that will be useful in the sequel to obtain certain characterizations of $J^{th}$-stage nonuniform discrete orthonormal wavelet bases for $l^2(\Lambda)$. More precisely, we review some concepts about Fourier and wavelet analysis on local fields of positive characteristic.
2.1. local Fields

A local field $K$ is a locally compact, non-discrete and totally disconnected field. If it is of characteristic zero, then it is a field of $p$-adic numbers $\mathbb{Q}_p$ or its finite extension. If $K$ is of positive characteristic, then $K$ is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a $p$-series field, while for $c \neq 1$, it is an algebraic extension of degree $c$ of a $p$-series field. Let $K$ be a fixed local field with the ring of integers $\mathcal{O} = \{x \in K : |x| \leq 1\}$. Since $K^+$ is a locally compact Abelian group, we choose a Haar measure $dx$ for $K^+$. The field $K$ is locally compact, non-trivial, totally disconnected and complete topological field endowed with non-Archimedean norm $| \cdot | : K \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;
(b) $|xy| = |x||y|$ for all $x, y \in K$;
(c) $|x + y| \leq \max \{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the ultrametric inequality. Let $\mathcal{B} = \{x \in K : |x| < 1\}$ be the prime ideal of the ring of integers $\mathcal{O}$ in $K$. Then, the residue space $\mathcal{O}/\mathcal{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime $p$ and $c \in \mathbb{N}$. Since $K$ is totally disconnected and $\mathcal{B}$ is both prime and principal ideal, so there exist a prime element $\mathfrak{p}$ of $K$ such that $\mathcal{B} = (\mathfrak{p}) = \mathfrak{p}\mathcal{O}$. Let $\mathcal{O}^* = \mathcal{O} \setminus \mathcal{B} = \{x \in K : |x| = 1\}$. Clearly, $\mathcal{O}^*$ is a group of units in $K^*$ and if $x \neq 0$, then can write $x = \mathfrak{p}^s y$, $y \in \mathcal{O}^*$. Moreover, if $\mathcal{U} = \{m = 0, 1, \ldots, q - 1\}$ denotes the fixed full set of coset representatives of $\mathcal{B}$ in $\mathcal{O}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=0}^{\infty} c_{\ell} \mathfrak{p}^{\ell}$ with $c_{\ell} \in \mathcal{U}$. Recall that $\mathcal{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k \mathcal{O} = \{x \in K : |x| < q^{-k}\}$ is also compact and open and is a subgroup of $K^+$. We use the notation in Taibleson’s book [14]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let $\chi$ be a fixed character on $K^+$ that is trivial on $\mathcal{O}$ but non-trivial on $\mathcal{B}^{-1}$. Therefore, $\chi$ is constant on cosets of $\mathcal{O}$ so if $y \in \mathcal{B}^k$, then $\chi_y(x) = \chi(y, x), x \in K$. Suppose that $\chi_u$ is any character on $K^+$, then the restriction $\chi_u|\mathcal{O}$ is a character on $\mathcal{O}$. Moreover, as characters on $\mathcal{O}$, $\chi_u = \chi_v$ if and only if $u - v \in \mathcal{O}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathcal{O}$ in $K^+$, then, as it was proved in [14], the set $\{\chi_u(n) : n \in \mathbb{N}_0\}$ of distinct characters on $\mathcal{O}$ is a complete orthonormal system on $\mathcal{O}$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathcal{O}/\mathcal{B} \cong GF(q)$ where $GF(q)$ is a $c$-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathcal{O}^*$ such that $\text{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q, \quad n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \quad 0 \leq a_k < p, \quad \text{and} \quad k = 0, 1, \ldots, c - 1,$$

we define

$$u(n) = (a_0 + a_1 \zeta_1 + \cdots + a_{c-1} \zeta_{c-1}) \mathfrak{p}^{-1}. \quad (2.1)$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s$, $n \in \mathbb{N}_0$, $0 \leq b_k < q$, $k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1) \mathfrak{p}^{-1} + \cdots + u(b_s) \mathfrak{p}^{-s}. \quad (2.2)$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m + n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leq s < q^k$, then $u(r q^k + s) = u(r) \mathfrak{p}^{-k} + u(s)$. Further, it is also easy
to verify that \( u(n) = 0 \) if and only if \( n = 0 \) and \( \{ u(\ell) + u(k) : k \in \mathbb{N}_0 \} = \{ u(k) : k \in \mathbb{N}_0 \} \) for a fixed \( \ell \in \mathbb{N}_0 \). Hereafter we use the notation \( \chi_n = \chi_{u(n)} ; n \geq 0 \).

Let the local field \( K \) be of characteristic \( p > 0 \) and \( \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1} \) be as above. We define a character \( \chi \) on \( K \) as follows:

\[
\chi(\zeta_\mu \zeta_j^{\frac{1}{p}}) = \begin{cases} 
\exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\
1, & \mu = 1, \ldots, c - 1 \text{ or } j \neq 1.
\end{cases}
\] (2.3)

2.2. Fourier Transforms on local Fields

The Fourier transform of \( f \in L^1(K) \) is denoted by \( \hat{f}(\xi) \) and defined by

\[
\mathcal{F}\{ f(x) \} = \hat{f}(\xi) = \int_K f(x) \chi_{\xi}(x) \, dx.
\] (2.4)

It is noted that

\[
\hat{f}(\xi) = \int_K f(x) \chi_{\xi}(x) \, dx = \int_K f(x) \chi(-\xi x) \, dx.
\]

The properties of Fourier transforms on local field \( K \) are much similar to those of on the classical field \( \mathbb{R} \). In fact, the Fourier transform on local fields of positive characteristic have the following properties:

- The map \( f \rightarrow \hat{f} \) is a bounded linear transformation of \( L^1(K) \) into \( L^\infty(K) \), and \( \| \hat{f} \|_\infty \leq \| f \|_1 \).
- If \( f \in L^1(K) \), then \( \hat{f} \) is uniformly continuous.
- If \( f \in L^1(K) \cap L^2(K) \), then \( \| \hat{f} \|_2 = \| f \|_2 \).

The Fourier transform of a function \( f \in L^2(K) \) is defined by

\[
\hat{f}(\xi) = \lim_{k \to \infty} \hat{f}_k(\xi) = \lim_{k \to \infty} \int_{|x| \leq q^k} f(x) \chi_{\xi}(x) \, dx,
\] (2.5)

where \( f_k = f \Phi_{-k} \) and \( \Phi_k \) is the characteristic function of \( \mathfrak{D}^k \). Furthermore, if \( f \in L^2(\mathfrak{D}) \), then we define the Fourier coefficients of \( f \) as

\[
\hat{f}(u(n)) = \int_\mathfrak{D} f(x) \chi_{u(n)}(x) \, dx.
\] (2.6)

The series \( \sum_{n \in \mathbb{N}_0} \hat{f}(u(n)) \chi_{u(n)}(x) \) is called the Fourier series of \( f \). From the standard \( L^2 \)-theory for compact Abelian groups, we conclude that the Fourier series of \( f \) converges to \( f \) in \( L^2(\mathfrak{D}) \) and Parseval’s identity holds:

\[
\| f \|_2^2 = \int_\mathfrak{D} |f(x)|^2 \, dx = \sum_{n \in \mathbb{N}_0} |\hat{f}(u(n))|^2.
\] (2.7)

2.3. Uniform MRA on local Fields
2.4. Nonuniform MRA on local Fields

In order to able to define the concepts of uniform MRA and wavelets on local fields, we need analogous notions of translation and dilation. Since $\bigcup_{j \in \mathbb{Z}} p^{-j} \mathcal{D} = K$, we can regard $p^{-1}$ as the dilation and since $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathcal{D}$ in $K$, the set $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$ can be treated as the translation set. Note that $\Lambda$ is a subgroup of $K^+$ and unlike the standard wavelet theory on the real line, the translation set is not a group.

The following is a definition of uniform MRA on local fields of positive characteristic.

**Definition 2.1.** Let $K$ be a local field of positive characteristic $p > 0$ and $p$ be a prime element of $K$. An MRA of $L^2(K)$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(K)$;

(c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;

(d) $f(x) \in V_j$ if and only if $f(p^{-j}x) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) There exists a function $\phi \in V_0$, such that $\{\phi(x - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for $V_0$.

According to the standard scheme for construction of MRA-based wavelets, for each $j$, we define a wavelet space $W_j$ as the orthogonal complement of $V_j$ in $V_{j+1}$, i.e., $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$, where $W_j \perp V_j$, $j \in \mathbb{Z}$. It is not difficult to see that

$$f(x) \in W_j \quad \text{if and only if} \quad f(p^{-j}x) \in W_{j+1}, \quad j \in \mathbb{Z}. \tag{2.7}$$

Moreover, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(K) = \bigoplus_{j \in \mathbb{Z}} W_j = V_0 \oplus \left( \bigoplus_{j \geq 0} W_j \right). \tag{2.8}$$

As in the case of $\mathbb{R}^n$, we expect the existence of $q - 1$ number of functions $\psi_1, \psi_2, \ldots, \psi_{q-1}$ to form a set of basic wavelets. In view of (2.7) and (2.8), it is clear that if $\{\psi_1, \psi_2, \ldots, \psi_{q-1}\}$ is a set of function such that the system $\{\psi_\ell(x - u(k)) : 1 \leq \ell \leq q - 1, k \in \mathbb{N}_0\}$ forms an orthonormal basis for $W_0$, then $\{q^{j/2}\psi_\ell(p^{-j}x - u(k)) : 1 \leq \ell \leq q - 1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ forms an orthonormal basis for $L^2(K)$.

2.4. Nonuniform MRA on local Fields

For an integer $N \geq 1$ and an odd integer $r$ with $1 \leq r \leq qN - 1$ such that $r$ and $N$ are relatively prime, we define

$$\Lambda = \left\{0, \frac{u(r)}{N}\right\} + \mathcal{Z},$$

where $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$. It is easy to verify that $\Lambda$ is not a group on local field $K$, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$. Following is the definition of nonunform multiresolution analysis (NUMRA) on local fields of positive characteristic given by Shah and Abdullah [7].
Definition 2.2. For an integer \( N \geq 1 \) and an odd integer \( r \) with \( 1 \leq r \leq qN-1 \) such that \( r \) and \( N \) are relatively prime, an associated NUMRA on local field \( K \) of positive characteristic is a sequence of closed subspaces \( \{V_j : j \in \mathbb{Z}\} \) of \( L^2(K) \) such that the following properties hold:

(a) \( V_j \subset V_{j+1} \) for all \( j \in \mathbb{Z} \);

(b) \( \bigcup_{j \in \mathbb{Z}} V_j \) is dense in \( L^2(K) \);

(c) \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \);

(d) \( f(x) \in V_j \) if and only if \( f(p^{-1}Nx) \in V_{j+1} \) for all \( j \in \mathbb{Z} \);

(e) There exist a function \( \phi \) in \( V_0 \) such that the collection \( \{\phi(x-\lambda) : \lambda \in \Lambda\} \) is a complete orthonormal basis for \( V_0 \).

It is worth noticing that, when \( N = 1 \), one recovers from the definition above the definition of an MRA on local fields of positive characteristic \( p > 0 \). When, \( N > 1 \), the dilation is induced by \( p^{-1}N \) and \( |p^{-1}| = q \) ensures that \( qN\Lambda \subset \mathbb{Z} \subset \Lambda \).

As in the standard scheme, one expects the existence of \( qN-1 \) number of functions so that their translation by elements of \( \Lambda \) and dilations by the integral powers of \( p^{-1}N \) form an orthonormal basis for \( L^2(K) \).

Definition 2.3. A set of functions \( \{\psi_1, \psi_1, \ldots, \psi_{qN-1}\} \) in \( L^2(K) \) is said to be a set of basic wavelets associated with an NUMRA \( \{V_j : j \in \mathbb{Z}\} \) if the family of functions \( \left\{(qN)^{j/2}\psi_\ell \left( (p^{-1}N)^j x - \lambda \right) : 1 \leq \ell \leq qN-1, \lambda \in \Lambda\right\} \) forms an orthonormal basis for \( W_j \).

3. First-stage Discrete Wavelets on local Fields

The main content of this section is to establish a characterization of the first-stage nonuniform discrete wavelets on local fields of positive characteristic.

We regard \( z \) as a function defined on the set \( \Lambda \) and suppose that \( z = \{z(\lambda)\}_{\lambda \in \Lambda} \). We define the spaces

\[ l^2(\Lambda) = \left\{ z : \Lambda \to \mathbb{C} : \sum_{\lambda \in \Lambda} |z(\lambda)|^2 < \infty \right\}, \]

\[ L^2(\Omega) = \left\{ f : \Omega \to \mathbb{C} : \int_{\Omega} |f(\xi)|^2 d\xi < \infty \right\}, \]

where \( \Omega \) is a Lebesgue measurable subset of \( K \) with finite positive measure. These spaces are Hilbert spaces with the inner products defined by

\[ \langle z, w \rangle = \sum_{\lambda \in \Lambda} z(\lambda)\overline{w(\lambda)} \quad \text{for } z, w \in l^2(\Lambda), \]

\[ \langle f, g \rangle = \int_{\Omega} f(\xi)g(\overline{\xi}) d\xi \quad \text{for } f, g \in L^2(\Omega), \]

respectively.
Definition 3.1. The Fourier transform on $l^2(\Lambda)$ is a map $\wedge: l^2(\Lambda) \to L^2(\Omega)$ defined by

$$\hat{z}(\xi) = \sum_{\lambda \in \Lambda} z(\lambda) \overline{\chi(\xi)}, \quad z \in l^2(\Lambda) \tag{3.1}$$

and its inverse is given by

$$f^\vee(\lambda) = \langle f, \overline{\chi(\xi)} \rangle = \int_{\Omega} f(\xi) \chi(\lambda) \, d\xi, \quad f \in L^2(\Omega). \tag{3.2}$$

For all $z, w \in l^2(\Lambda)$, the Parseval and Plancherel formulae are given by

$$\langle z, w \rangle = \sum_{\lambda \in \Lambda} z(\lambda) \overline{w(\lambda)} = \langle \hat{z}, \hat{w} \rangle, \quad \text{and} \quad ||z||^2 = \sum_{\lambda \in \Lambda} |z(\lambda)|^2 = ||\hat{z}||^2.$$

For $N \geq 1, q^{-1} = |p|$ and $\lambda \in \Lambda$, the translation operator $T_{qN\lambda} : l^2(\Lambda) \to l^2(\Lambda)$ is defined by

$$T_{qN\lambda}z(\sigma) = z(\sigma - qN\lambda), \quad \forall \sigma \in \Lambda.$$

Then, for $z, w \in l^2(\Lambda)$, it can be easily verified that

$$(T_{qN\lambda}z)^\vee(\xi) = \overline{\chi(\xi)} \hat{z}(\xi) \quad \text{and} \quad \langle T_{qN\lambda}z, T_{qN\sigma}w \rangle = \langle T_{qN(\lambda-\sigma)}z, w \rangle.$$

Definition 3.2. Let $N \in \mathbb{N}$ and let $k$ be an odd integer with $1 \leq k \leq qN-1$ such that $k$ and $N$ are relatively prime. For $w_k \in l^2(\Lambda)$, we call $\mathcal{F}(W)$ a first stage nonuniform discrete wavelet system associated with $W = \{w_k : w_k \in l^2(\Lambda)\}$ if

$$\mathcal{F}(W) = \{T_{qN\lambda}w_k : w_k \in l^2(\Lambda); \lambda \in \Lambda; 0 \leq k \leq qN-1\}. \tag{3.3}$$

is a complete orthonormal set in $l^2(\Lambda)$. We shall call $w_0$ as the nonuniform father wavelet and $\{w_k : 1 \leq k \leq qN-1\}$ as the nonuniform mother wavelets.

Theorem 3.3. For $z, w \in l^2(\Lambda)$, the systems $\{T_{qN\lambda}z\}_{\lambda \in \Lambda}$ and $\{T_{qN\lambda}w\}_{\lambda \in \Lambda}$ generates orthogonal subspaces in $l^2(\Lambda)$ if and only if the following conditions hold:

\begin{align*}
\text{(a)} & \sum_{s=0}^{qN-1} \left\{ \hat{z} \left( \xi + \frac{u(s)}{p^{-1}N} \right) \hat{w} \left( \xi + \frac{u(s)}{p^{-1}N} \right) \hat{z} \left( \xi + \frac{u(s)}{p^{-1}N} + u(N) \right) \hat{w} \left( \xi + \frac{u(s)}{p^{-1}N} + u(N) \right) \right\} = 0, \\
\text{(b)} & \sum_{s=0}^{qN-1} \chi \left( \frac{r}{N} pu(s) \right) \left\{ \hat{z} \left( \xi + \frac{u(s)}{p^{-1}N} \right) \hat{w} \left( \xi + \frac{u(s)}{p^{-1}N} \right) \hat{z} \left( \xi + \frac{u(s)}{p^{-1}N} + u(N) \right) \hat{w} \left( \xi + \frac{u(s)}{p^{-1}N} + u(N) \right) \right\} = 0, \\
\text{(c)} & \sum_{s=0}^{qN-1} \chi \left( \frac{r}{N} pu(s) \right) \left\{ \hat{z} \left( \xi + \frac{u(s)}{p^{-1}N} \right) \hat{w} \left( \xi + \frac{u(s)}{p^{-1}N} \right) \hat{z} \left( \xi + \frac{u(s)}{p^{-1}N} + u(N) \right) \hat{w} \left( \xi + \frac{u(s)}{p^{-1}N} + u(N) \right) \right\} = 0.
\end{align*}
Proof. For all \( z, w \in L^2(\Lambda) \), the orthogonality of the systems \( \{T_{qN\lambda}z\}_{\lambda \in \Lambda} \) and \( \{T_{qN\lambda}w\}_{\lambda \in \Lambda} \), is equivalent to

\[
0 = \langle T_{qN\lambda}z, T_{qN\sigma}w \rangle \\
= \langle (T_{qN\lambda}z)^\wedge, (T_{qN\sigma}w)^\wedge \rangle \\
= \int_\Omega \hat{z}(\xi)\hat{w}(\xi)\chi_{p^{-1}N(\lambda-\sigma)}(\xi)d\xi \\
= \int_{pD} \left\{ \hat{z}(\xi)\hat{w}(\xi) + \hat{z}(\xi + u(N))\hat{w}(\xi + u(N)) \right\} \chi_{p^{-1}N(\lambda-\sigma)}(\xi)d\xi \quad \lambda, \sigma \in \Lambda. \tag{3.4}
\]

Taking \( \lambda = u(m) \) and \( \sigma = u(n) \) in (3.4) and setting

\[
h(\xi) = \hat{z}(\xi)\overline{\hat{w}(\xi)} + \hat{z}(\xi + u(N))\overline{\hat{w}(\xi + u(N))},
\]

we obtain

\[
0 = \int_{pD} h(\xi)\chi(\overline{(p^{-1}N)u(q(m-n))\xi})d\xi \\
= \int_{(p/q)N \in \mathbb{N}} \left\{ \sum_{s=0}^{qN-1} h(\xi + \frac{u(s)}{p^{-1}N}) \right\} \chi((p^{-1}N)u(q(m-n))\xi)d\xi.
\]

Since the above equality holds for all \( (m-n) \in \mathbb{N}_0 \), it follows that

\[
\sum_{s=0}^{qN-1} h\left( \xi + \frac{u(s)}{p^{-1}N} \right) = 0, \quad a.e. \tag{3.6}
\]

On taking \( \lambda = r/N + u(m) \) and \( \sigma = r/N + u(n) \) where \( m, n \in \mathbb{N}_0 \), we obtain the same identity (3.6). Similarly, taking \( \lambda = r/N + u(m) \) and \( \sigma = u(n) \), where \( m, n \in \mathbb{N}_0 \), in (3.3), we obtain

\[
0 = \int_{pD} h(\xi)\chi(\overline{(p^{-1}N)u(q(m-n))\xi})\chi(p^{-1}r\xi)d\xi \\
= \int_{(p/q)N \in \mathbb{N}} \left\{ \sum_{s=0}^{qN-1} \chi(\frac{r}{N}pu(s))h(\xi + \frac{u(s)}{p^{-1}N}) \right\} \chi((p^{-1}N)u(q(m-n))\xi)\chi(p^{-1}r\xi)d\xi.
\]

Thus, we conclude that

\[
\sum_{s=0}^{qN-1} h\left( \xi + \frac{u(s)}{p^{-1}N} \right) \chi(\frac{r}{N}pu(s)) = 0, \quad a.e. \tag{3.7}
\]

By taking \( \lambda = u(m) \) and \( \sigma = \frac{r}{N} + u(n) \), we have \( \lambda - \sigma = -\frac{r}{N} + u(m-n), m, n \in \mathbb{N}_0 \) and consequently, we get

\[
\sum_{s=0}^{qN-1} h\left( \xi + \frac{u(s)}{p^{-1}N} \right) \chi(\frac{r}{N}pu(s)) = 0, \quad a.e. \tag{3.8}
\]
Consequently, Eqs. (3.6)–(3.9) yield if the following conditions hold:
\[ \bar{z}(\xi) = z_1(\xi) + \chi \left( \frac{r}{N} \right) z_2(\xi), \quad \text{and} \quad \bar{w}(\xi) = w_1(\xi) + \chi \left( \frac{r}{N} \right) w_2(\xi). \]  

Therefore, equation (3.5) becomes
\[ h(\xi) = q \left( z_1(\xi) \bar{w}(\xi) + z_2(\xi) \bar{w}(\xi) \right). \]  

Consequently, Eqs. (3.6)–(3.9) yield
\[ \sum_{s=0}^{qN-1} \left\{ z_1 \left( \xi + \frac{u(s)}{p-1N} \right) w_1 \left( \xi + \frac{u(s)}{p-1N} \right) z_2 \left( \xi + \frac{u(s)}{p-1N} \right) w_2 \left( \xi + \frac{u(s)}{p-1N} \right) \right\} = 0, \]  
\[ \sum_{s=0}^{qN-1} \chi \left( \frac{r}{N} \right) p u(s) \left\{ z_1 \left( \xi + \frac{u(s)}{p-1N} \right) w_1 \left( \xi + \frac{u(s)}{p-1N} \right) z_2 \left( \xi + \frac{u(s)}{p-1N} \right) w_2 \left( \xi + \frac{u(s)}{p-1N} \right) \right\} = 0, \]  
\[ \sum_{s=0}^{qN-1} \chi \left( \frac{r}{N} \right) p u(s) \left\{ z_1 \left( \xi + \frac{u(s)}{p-1N} \right) w_1 \left( \xi + \frac{u(s)}{p-1N} \right) z_2 \left( \xi + \frac{u(s)}{p-1N} \right) w_2 \left( \xi + \frac{u(s)}{p-1N} \right) \right\} = 0. \]

**Corollary 3.4.** Suppose for \( z, w \in l^2(\Lambda) \) and equation (3.9) is satisfied. Then, the subspaces generated by the systems \( \{T_{qN}z\}_{\lambda \in \Lambda} \) and \( \{T_{qN}w\}_{\lambda \in \Lambda} \) are orthogonal in \( l^2(\Lambda) \) if and only if the equations (3.11)-(3.13) are satisfied.

**Theorem 3.5.** For \( z \in l^2(\Lambda) \), the system \( \{T_{qN}z\}_{\lambda \in \Lambda} \) is orthonormal in \( l^2(\Lambda) \) if and only if the following conditions hold:
\[ \frac{1}{q} \sum_{s=0}^{qN-1} \left\{ \left| \hat{z} \left( \xi + \frac{u(s)}{p-1N} \right) \right|^2 + \left| \hat{z} \left( \xi + \frac{u(s)}{p-1N} + u(N) \right) \right|^2 \right\} = qN, \]  
\[ \sum_{s=0}^{qN-1} \chi \left( \frac{r}{N} \right) p u(s) \left\{ \left| \hat{z} \left( \xi + \frac{u(s)}{p-1N} \right) \right|^2 + \left| \hat{z} \left( \xi + \frac{u(s)}{p-1N} + u(N) \right) \right|^2 \right\} = 0. \]

**Proof.** For \( z \in l^2(\Lambda) \), and \( \lambda, \sigma \in \Lambda \), the orthonormality of the system \( \{T_{qN}z\}_{\lambda \in \Lambda} \) in \( l^2(\Lambda) \) is equivalent to
\[ \int_{\mathbb{P}} \left\{ \left| \hat{z}(\xi) \right|^2 + \left| \hat{z}(\xi + u(N)) \right|^2 \right\} \chi \left( \frac{p-1N}{(p-1N)u(q(m-n))} \right) d\xi = \delta_{\lambda,\sigma}. \]

Proceeding in a similar way as in the proof of Theorem 3.3, we obtain the desired result. □
Corollary 3.6. Let \( z \in l^2(\Lambda) \) be such that condition (3.9) holds. Then, the system \( \{T_{qN\lambda}z\}_{\lambda \in \Lambda} \) is orthogonal in \( l^2(\Lambda) \) if and only if the following identities hold:

\[
\begin{align*}
(a) & \quad \sum_{s=0}^{qN-1} \left\{ \hat{\psi} \left( \xi + \frac{u(s)}{p^{-1}N} \right) \right\}^2 + \left\{ \hat{\phi} \left( \xi + \frac{u(s)}{p^{-1}N} \right) \right\}^2 = qN, \\
(b) & \quad \sum_{s=0}^{qN-1} \chi \left( \frac{r}{N} pu(s) \right) \left\{ \hat{\psi} \left( \xi + \frac{u(s)}{p^{-1}N} \right) \right\}^2 + \left\{ \hat{\phi} \left( \xi + \frac{u(s)}{p^{-1}N} \right) \right\}^2 = 0.
\end{align*}
\]

Theorem 3.7. If \( \mathcal{F}(W) \) is the first-stage discrete wavelet system as defined by (3.3). Then the following statements are equivalent:

(i) The set \( \mathcal{F}(W) \) is an orthonormal basis for \( l^2(\Lambda) \)

(ii) The matrix \( M(\xi) \) of order \( q^2N \times q^2N \) is unitary, when the entries \( M_{st}(\xi), 0 \leq s, t \leq q^2N - 1 \) of \( M(\xi) \) are defined as follows:

\[
M_{st}(\xi) = \frac{1}{qN} \begin{cases} 
\hat{w}_t \left( \xi + \frac{u(s)}{p^{-1}N} \right); & 0 \leq s \leq qN - 1; 0 \leq t \leq qN - 1, \\
\hat{w}_t \left( \xi + \frac{u(s - qN)}{p^{-1}N} + u(N) \right); & qN \leq s \leq q^2N - 1; 0 \leq t \leq qN - 1, \\
\chi \left( \frac{r}{N} pu(s) \right) \hat{w}_{t-qN} \left( \xi + \frac{u(s - qN)}{p^{-1}N} \right); & 0 \leq t \leq qN - 1; qN \leq s \leq q^2N - 1, \\
\chi \left( \frac{r}{N} pu(s) \right) \hat{w}_{t-qN} \left( \xi + \frac{u(s - qN)}{p^{-1}N} + u(N) \right); & qN \leq t, s \leq q^2N - 1.
\end{cases}
\]

Proof. Suppose that the system \( \mathcal{F}(W) \) defined by (3.3) is an orthonormal basis for \( l^2(\Lambda) \). Then, \( \mathcal{F}(W) \) is an orthonormal set in \( l^2(\Lambda) \) and therefore, for \( \lambda, \sigma \in \Lambda \), and \( 0 \leq \ell, k \leq qN - 1 \), we have

\[
\langle T_{qN\lambda}w_{\ell}, T_{qN\sigma}w_{k} \rangle = \delta_{\lambda,\sigma}\delta_{\ell,k}.
\]

For each \( \ell, k, 0 \leq \ell, k \leq qN - 1 \), Theorems 3.3 and 3.6 implies that

\[
\begin{align*}
\frac{1}{q} \sum_{t=0}^{qN-1} \left\{ \hat{w}_\ell \left( \xi + \frac{u(t)}{p^{-1}N} \right) \hat{w}_k \left( \xi + \frac{u(t)}{p^{-1}N} \right) \right. \\
+ \hat{w}_\ell \left( \xi + \frac{u(t)}{p^{-1}N} + u(N) \right) \hat{w}_k \left( \xi + \frac{u(t)}{p^{-1}N} + u(N) \right) \right\} = qN\delta_{\ell,k},
\end{align*}
\]

\[
R_{\ell,k}(\xi) = \sum_{t=0}^{qN-1} \chi \left( \frac{r}{N} pu(t) \right) \left\{ \hat{w}_\ell \left( \xi + \frac{u(t)}{p^{-1}N} \right) \hat{w}_k \left( \xi + \frac{u(t)}{p^{-1}N} \right) \right. \\
+ \hat{w}_\ell \left( \xi + \frac{u(t)}{p^{-1}N} + u(N) \right) \hat{w}_k \left( \xi + \frac{u(t)}{p^{-1}N} + u(N) \right) \right\} = 0.
\]
and
\[
S_{\ell,k}(\xi) = \sum_{t=0}^{qN-1} \chi\left(\frac{r}{N} pu(t)\right) \left\{ \hat{w}_\ell\left(\xi + \frac{u(t)}{p^{-1}N}\right) \hat{w}_k\left(\xi + \frac{u(t)}{p^{-1}N}\right) \\
+ \hat{w}_\ell\left(\xi + \frac{u(t)}{p^{-1}N} + u(N)\right) \hat{w}_k\left(\xi + \frac{u(t)}{p^{-1}N} + u(N)\right) \right\} = 0. \tag{3.19}
\]

Thus, it is sufficient to consider the equations (3.17) and (3.18), as \(S_{\ell,k}(\xi) = R_{\ell,k}(\xi)\). These equations give rise to the matrix \(M(\xi)\) of order \(q^2N \times q^2N\) with entries as defined in system (3.164). Moreover, the matrix \(M(\xi)\) is unitary. This follows from the identities (3.17) and (3.18) and noting that columns of \(M(\xi)\) form an orthonormal system in \(\mathbb{C}^{q^2N}\) with respect to the usual inner product, and hence form an orthonormal basis of \(\mathbb{C}^{q^2N}\).

Conversely, assume that the matrix \(M(\xi)\) is unitary. Then, it suffices to show that the set \(\mathcal{F}(W)\) is complete. For this, let \(w \in l^2(\Lambda)\) and \(Pw\) be the projection onto \(\text{span}\mathcal{F}(W)\), then we have
\[
\|Pw\|_2^2 = \sum_{\ell=0}^{qN-1} \sum_{\lambda \in \Lambda} |\langle w, T_{qN\lambda}w_\ell \rangle|^2 = \sum_{\ell=0}^{qN-1} \sum_{\lambda \in \Lambda} |\langle \hat{w}, (T_{qN\lambda}w_\ell)\rangle|^2 = \sum_{\ell=0}^{qN-1} \sum_{\lambda \in \Lambda} \left| \int_{\Omega} \hat{w}(\xi)\overline{\hat{w}_\ell(\xi)}\chi_\lambda(p^{-1}N\xi)d\xi \right|^2.
\]

Writing \(\Omega = p\mathcal{D} \cup p(N + \mathcal{D})\) and observing that \(p\mathcal{D} = \bigcup_{s=0}^{qN-1} p(s + \mathcal{D})/N\), we obtain
\[
\|Pw\|_2^2 = \frac{1}{(qN)^2} \sum_{\ell=0}^{qN-1} \sum_{\lambda \in \Lambda} \int_{p\mathcal{D}} \int_{p\mathcal{D}} \chi_\lambda(pu(t)) \left\{ \hat{w}_\ell(\zeta_t)\overline{\hat{w}_\ell(\zeta_t)} + \hat{w}(\zeta_t + u(N))\overline{\hat{w}(\zeta_t)} \right\} \chi_\lambda(\xi)d\xi \tag{3.20}
\]

where \(\zeta_t = (\xi + u(t))/p^{-1}N\). As \(\Lambda = \mathcal{Z} \cup \{\mathcal{Z} + r/N\}\), we can rewrite (3.20), by using Plancherel formula as follows
\[
\|Pw\|_2^2
= \frac{1}{(qN)^2} \sum_{\ell=0}^{qN-1} \sum_{m \in \mathbb{N}_0} \int_{\mathbb{R}^N} \sum_{s=0}^{qN-1} \left\{ \hat{w}(\zeta_s) \hat{w}_\ell(\zeta_s) + \hat{w}(\zeta_s + u(N)) \hat{w}_\ell(\zeta_s + u(N)) \right\} \chi(pu(m)\xi) d\xi
\]
\[
+ \frac{1}{(qN)^2} \sum_{\ell=0}^{qN-1} \sum_{m \in \mathbb{N}_0} \int_{\mathbb{R}^N} \sum_{s=0}^{qN-1} \chi \left( \frac{r}{N} pu(s) \right) \left\{ \hat{w}(\zeta_s) \hat{w}_\ell(\zeta_s) + \hat{w}(\zeta_s + u(N)) \hat{w}_\ell(\zeta_s + u(N)) \right\} \chi(pu(m)\xi) d\xi
\]
\[
= \frac{1}{q^3N^2} \sum_{\ell=0}^{qN-1} \left( \int_{\mathbb{R}^N} \sum_{s=0}^{qN-1} \chi \left( \frac{r}{N} pu(s) \right) \left\{ \hat{w}(\zeta_s) \hat{w}_\ell(\zeta_s) + \hat{w}(\zeta_s + u(N)) \hat{w}_\ell(\zeta_s + u(N)) \right\} d\xi \right)^2
+ \frac{1}{q^3N^2} \sum_{\ell=0}^{qN-1} \left( \int_{\mathbb{R}^N} \sum_{s=0}^{qN-1} \chi \left( \frac{r}{N} pu(s) \right) \left\{ \hat{w}(\zeta_s) \hat{w}_\ell(\zeta_s) + \hat{w}(\zeta_s + u(N)) \hat{w}_\ell(\zeta_s + u(N)) \right\} d\xi \right)^2.
\]

Since the rows of matrix \(M(\xi)\) form an orthonormal system in \(\mathbb{C}^{q^2N}\), therefore, for each \(0 \leq r, s \leq qN - 1\), we have
\[
\frac{1}{q} \sum_{\ell=0}^{qN-1} \left( 1 - \chi \left( \frac{r}{N} pu(r) - pu(s) \right) \right) \hat{w}_\ell \left( \xi + \frac{u(s)}{p^{-1}N} \right) \hat{w}_\ell \left( \xi + \frac{u(s)}{p^{-1}N} \right) = qN \delta_{r,s}
\]
\[
\frac{1}{q} \sum_{\ell=0}^{qN-1} \left( 1 - \chi \left( \frac{r}{N} pu(r) - pu(s) \right) \right) \hat{w}_\ell \left( \xi + \frac{u(s)}{p^{-1}N} + u(N) \right) \hat{w}_\ell \left( \xi + \frac{u(s)}{p^{-1}N} + u(N) \right) = qN \delta_{r,s}
\]
and
\[
\sum_{\ell=0}^{qN-1} \left( 1 - \chi \left( \frac{r}{N} pu(r) - pu(s) \right) \right) \hat{w}_\ell \left( \xi + \frac{u(s)}{p^{-1}N} \right) \hat{w}_\ell \left( \xi + \frac{u(s)}{p^{-1}N} + u(N) \right) = 0.
\]
Thus, for $0 \leq s, t \leq qN - 1$, we have

$$\|Pw\|_2^2 = \frac{1}{(qN)^2} \int_{\mathbb{P}^D} \sum_{p=0}^{qN-1} \sum_{s=0}^{qN-1} \left\{ |\hat{w}(\zeta_s)\overline{\hat{w}_t(\zeta_s)}|^2 + |\hat{w}(\zeta_s + u(N))\overline{\hat{w}_t(\zeta_s + u(N))}|^2 \right\} d\zeta$$

$$= \frac{1}{(qN)^2} \int_{\mathbb{P}^D} \sum_{s=0}^{qN-1} \left\{ |\hat{w}(\zeta_s)|^2 \sum_{t=0}^{qN-1} |\hat{w}_t(\zeta_s)|^2 + |\hat{w}(\zeta_s + u(N))|^2 \sum_{t=0}^{qN-1} |\hat{w}_t(\zeta_s + u(N))|^2 \right\} d\zeta$$

$$= \frac{1}{qN} \int_{\mathbb{P}^D} \sum_{s=0}^{qN-1} \left\{ |\hat{w}(\zeta_s)|^2 + |\hat{w}(\zeta_s + u(N))|^2 \right\} d\zeta$$

$$= \frac{1}{qN} \sum_{s=0}^{qN-1} \int_{(1+s)p\mathbb{P}} \left\{ |\hat{w}\left(\frac{p\xi}{N}\right)|^2 + |\hat{w}\left(\frac{p\xi + u(N)}{N}\right)|^2 \right\} d\xi$$

$$= \frac{1}{qN} \int_{\mathbb{P}^D} \left\{ |\hat{w}(\xi)|^2 + |\hat{w}(\xi + u(N))|^2 \right\} d\xi$$

$$= \int_{\Omega} |\hat{w}(\xi)|^2 d\xi$$

$$= \|w\|_2^2.$$

Hence, the projection $P$ is an identity map and $\overline{\text{span} \mathcal{F}(W)} = l^2(\Lambda)$. Therefore, the set $\mathcal{F}$ is an orthonormal basis for $l^2(\Lambda)$. This completes the proof of Theorem 3.7. \hfill \Box

**Corollary 3.8.** For each $\ell, 0 \leq \ell \leq qN - 1$, let $w_\ell \in l^2(\Lambda)$ such that

$$\hat{w}_\ell(\xi) = w_{t_0}(\xi) + \chi\left(\frac{p}{N}\xi\right) w_{t_1}(\xi),$$

(3.21)

for some $p$-periodic functions $w_{t_0}$ and $w_{t_1}$. Then, the system $\mathcal{F}(W)$ as defined by (3.3) is an orthonormal basis for $l^2(\Lambda)$ if and only if the matrix

$$M_{st}(\xi) = \frac{1}{qN} \begin{cases} w_{t_0} \left( \xi + \frac{u(s)}{p-1N} \right); & 0 \leq s \leq qN - 1; 0 \leq t \leq qN - 1, \\ w_{t_1} \left( \xi + \frac{u(s-qN)}{p-1N} \right); & qN \leq s \leq q^2N - 1; 0 \leq t \leq qN - 1, \\ \chi \left( \frac{p}{N} \right) w_{(t-qN)0} \left( \xi + \frac{u(s-qN)}{p-1N} \right); & 0 \leq t \leq qN - 1; \\ \chi \left( \frac{p}{N} \right) w_{(t-qN)1} \left( \xi + \frac{u(s-qN)}{p-1N} \right); & qN \leq t, s \leq q^2N - 1; \end{cases}$$

is unitary.

Assume $w_0$ in $l^2(\Lambda)$ such that it satisfies equations (3.14) and (3.15). Following the procedure of the paper of Shah and Abdullah [7], it can be easily shown that there exists functions $w_k : 1 \leq k \leq qN - 1$ satisfying conditions (3.17) and (3.18) if and only if the function $M_0$ is of the form

$$M_0(\xi) = \left| \frac{\hat{w}_0(\xi)}{q\sqrt{N}} \right|^2 + \left| \frac{\hat{w}_0(\xi + u(N))}{q\sqrt{N}} \right|^2,$$
and satisfies the following identity

\[ M_0(\xi + p^2) = M_0(\xi). \]

**Theorem 3.9.** For each \( i \in \{0, 1, \ldots, qN - 1\} \) and \( \ell \in \mathbb{N} \), let \( f_{\ell,i} \in l^2(\Lambda) \). Then, the system

\[ \mathcal{G}(H) = \left\{ T_{(qN)^\ell \lambda} h_{\ell,i} : \lambda \in \Lambda, 0 \leq i \leq qN - 1 \right\} \]  

is an orthonormal in \( l^2(\Lambda) \) if and only if for \( 0 \leq i, j \leq qN - 1 \), the following conditions are satisfied:

\[
T_{ij}(\xi) = \sum_{s=0}^{(qN)^{\ell-1}} \left\{ \hat{h}_{\ell,i} \left( \xi + \frac{u(s)}{(p-1)^{\ell}} \right) \hat{h}_{\ell,j} \left( \xi + \frac{u(s)}{(p-1)^{\ell}} + u(N) \right) \right\} = q(qN)^\ell \delta_{i,j},
\]

\[
\sum_{s=0}^{(qN)^{\ell-1}} \chi \left( \frac{r}{N} pu(s) \right) \left\{ \hat{h}_{\ell,i} \left( \xi + \frac{u(s)}{(p-1)^{\ell}} + u(N) \right) \hat{h}_{\ell,j} \left( \xi + \frac{u(s)}{(p-1)^{\ell}} + u(N) \right) \right\} = 0.
\]

**Proof.** For \( \lambda, \sigma \in \Lambda \) and \( 0 \leq i, j \leq qN - 1 \), the orthonormality of the system \( \mathcal{G}(H) \) in \( l^2(\Lambda) \) is equivalent to

\[
\langle T_{(qN)^\ell \lambda} h_{\ell,i}, T_{(qN)^\ell \sigma} h_{\ell,j} \rangle = \delta_{\lambda,\sigma} \delta_{i,j}.
\]

By setting

\[
H_{ij}(\xi) = \hat{h}_{\ell,i}(\xi)\overline{h_{\ell,j}(\xi)} + \hat{h}_{\ell,i}(\xi + u(N))\overline{h_{\ell,j}(\xi + u(N))},
\]

and taking \( \lambda = u(m) \) and \( \sigma = u(n) \), where \( m, n \in \mathbb{N}_0 \), we have

\[
\delta_{\lambda,\sigma} \delta_{i,j} = \int_{\mathbb{P}^D} H_{ij}(\xi) \chi \left( \frac{p-1}{(p-1)^{\ell}} (u(m) - u(n)) \xi \right) d\xi
\]

\[
= \int_{(p/(qN)^\ell)^D} \left\{ \sum_{s=0}^{(qN)^{\ell-1}} H_{ij} \left( \xi + \frac{u(s)}{(p-1)^{\ell}} \right) \chi \left( \frac{(p-1)^{\ell} u(q(m-n))}{(p-1)^{\ell}} \xi \right) \right\} d\xi.
\]

Now the desired result can be proved analogously to Theorem 3.3. \qed

4. \textit{J}th-stage Discrete Wavelets on local Fields

In this section, we introduce the notion of \( J^{th} \)-stage nonuniform discrete wavelet system in the Hilbert space \( l^2(\Lambda) \) and show that this space can be expressed as an orthogonal decomposition in terms of countable number of its closed subspaces.
Definition 4.1. Let $N \in \mathbb{N}$ and let $k$ be an odd integer with $1 \leq k \leq qN - 1$ such that $k$ and $N$ are relatively prime. For $h_{j,k} \in l^2(\Lambda)$, $J \in \mathbb{N}$, we call $\mathcal{H}$ a $J$th-stage nonuniform discrete wavelet system associated with $H = \{h_{j,k} : h_{j,k} \in l^2(\Lambda)\}$ if

$$
\mathcal{H} = \left\{ T_{(qN)\lambda}h_{j,k} : h_{j,k} \in l^2(\Lambda) ; 1 \leq j \leq J, 1 \leq k \leq qN - 1, \lambda \in \Lambda \right\}.
$$

is a complete orthonormal set in $l^2(\Lambda)$.

Theorem 4.2. Let the system $\{T_{qN\lambda}w_i : w_i \in l^2(\Lambda) ; \lambda \in \Lambda ; 0 \leq i \leq qN - 1\}$ be orthonormal in $l^2(\Lambda)$, where $w_i \in l^2(\Lambda)$ satisfies equation (3.21). For $\ell \in \mathbb{N}$ and $h_{(\ell-1),i} \in l^2(\Lambda)$, let the system $\{T_{qN\lambda}h_{(\ell-1),i} : \lambda \in \Lambda, 0 \leq i \leq qN - 1\}$ be orthonormal in $l^2(\Lambda)$. Consider the following relation

$$
\hat{h}_{\ell,i}(\xi) = \hat{h}_{(\ell-1),i}(\xi)\hat{w}_i((p^{-1}N)^{\ell-1}\xi),
$$

where $\hat{h}_{0,0}(\xi) = 1$ a.e. Then, the system

$$
\{T_{(qN)\ell\lambda}h_{\ell,i} : \lambda \in \Lambda ; 0 \leq i \leq qN - 1\}
$$

is orthonormal in $l^2(\Lambda)$.

Proof. For $\ell = 1$, the result follows immediately. To prove the required result for $\ell \in \mathbb{N} - \{1\}$ and $0 \leq i, j \leq qN - 1$, it is sufficient to prove the identities (3.23) and (3.24). However, we observe that $\hat{h}_{\ell,i} \in L^2(\Omega)$ since

$$
\left\| \hat{h}_{\ell,i} \right\|^2_2 = \int_{\Omega} \left| \hat{h}_{(\ell-1),i}(\xi)\hat{w}_i((p^{-1}N)^{\ell-1}\xi) \right|^2 d\xi
\leq \sup_{\xi} \left| \hat{w}_i((p^{-1}N)^{\ell-1}\xi) \right|^2 \left\| \hat{h}_{(\ell-1),i} \right\|^2_2,
$$

$$
= qN \left\| \hat{h}_{(\ell-1),i} \right\|^2_2.
$$

From (3.23), we infer that

$$
T_{\mathcal{D}_j}(\xi) = \sum_{s=0}^{(qN)^{\ell-1}-1} \left\{ \hat{h}_{\ell,i} \left( \xi + \frac{u(s)}{(p^{-1}N)^{\ell}} \right) \hat{h}_{\ell,j} \left( \xi + \frac{u(s)}{(p^{-1}N)^{\ell}} \right) \right. \\
+ \left. \hat{h}_{\ell,i} \left( \xi + \frac{u(s)}{(p^{-1}N)^{\ell}} + u(N) \right) \hat{h}_{\ell,j} \left( \xi + \frac{u(s)}{(p^{-1}N)^{\ell}} + u(N) \right) \right\}
$$

$$
= \sum_{m=0}^{qN-1} \sum_{n=0}^{(qN)^{\ell-1}-1} \left\{ \left| \hat{h}_{(\ell-1),0} \left( \xi + \frac{u(n)}{(p^{-1}N)^{\ell-1}} + \frac{u(m)}{(p^{-1}N)^{\ell}} \right) \right|^2 \\
+ \left| \hat{h}_{(\ell-1),0} \left( \xi + \frac{u(n)}{(p^{-1}N)^{\ell-1}} + \frac{u(m)}{(p^{-1}N)^{\ell}} + u(N) \right) \right|^2 \right\}
$$

$$
\times \left\{ \hat{w}_i \left( (p^{-1}N)^{\ell-1}\xi + pu(n) + \frac{u(m)}{p^{-1}N} \right) \hat{w}_i \left( (p^{-1}N)^{\ell-1}\xi + pu(n) + \frac{u(m)}{p^{-1}N} \right) \right\},
$$

for $\ell > 1$.
where we have used the fact \( \hat{w}_i(\xi + u(N)) = \hat{w}_i(\xi) \) and (4.3).

For \( 0 \leq m \leq qN - 1 \) and \( 0 \leq n \leq (qN)^{\ell-1} - 1 \), we define

\[
H_{m,n}(\xi) = \left| \hat{h}_{(\ell-1),0}(\xi + \frac{u(n)}{(p-1)N} + \frac{u(m)}{(p-1)N} \ell) \right|^2 + \left| \hat{h}_{(\ell-1),0}(\xi + \frac{u(n)}{(p-1)N} + \frac{u(m)}{(p-1)N} + u(N)) \right|^2.
\]

Then, we can write

\[
\mathcal{T}_{ij}^{\ell}(\xi) = \sum_{m=0}^{qN-1} \sum_{n=0}^{(qN)^{\ell-1} - 1} H_{m,n}(\xi) \left\{ \hat{w}_i \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \hat{w}_i \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \sum_{n=0}^{(qN)^{\ell-1} - 1} H_{m,n}(\xi) \right\}.
\]

Using (3.21) for each \( 0 \leq i, j \leq qN - 1 \), we obtain

\[
\mathcal{T}_{ij}^{\ell}(\xi) = \sum_{m=0}^{qN-1} \sum_{n=0}^{(qN)^{\ell-1} - 1} \hat{w}_{i0} \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \hat{w}_{i0} \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \sum_{n=0}^{(qN)^{\ell-1} - 1} H_{m,n}(\xi)
\]

\[
+ \sum_{m=0}^{qN-1} \sum_{n=0}^{(qN)^{\ell-1} - 1} \hat{w}_{i1} \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \hat{w}_{i1} \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \sum_{n=0}^{(qN)^{\ell-1} - 1} H_{m,n}(\xi)
\]

\[
+ \sum_{m=0}^{qN-1} \sum_{n=0}^{(qN)^{\ell-1} - 1} \left\{ \chi \left( \frac{r}{N} \right) \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \right\} \sum_{n=0}^{(qN)^{\ell-1} - 1} H_{m,n}(\xi)
\]

\[
+ \sum_{m=0}^{qN-1} \sum_{n=0}^{(qN)^{\ell-1} - 1} \left\{ \chi \left( \frac{r}{N} \right) \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \right\} \sum_{n=0}^{(qN)^{\ell-1} - 1} H_{m,n}(\xi)
\]

Theorem 3.9 and the orthonormality property of the system (4.2) further yields

\[
\sum_{n=0}^{(qN)^{\ell-1} - 1} H_{m,n}(\xi) = q(qN)^{\ell-1} \quad \text{and} \quad \sum_{n=0}^{(qN)^{\ell-1} - 1} \left( \chi \left( \frac{r}{N} \right) \frac{u(m)}{p-1} \right) H_{m,n}(\xi) = 0,
\]

which in turn implies

\[
\mathcal{T}_{ij}^{\ell}(\xi) = q(qN)^{\ell-1} \sum_{m=0}^{qN-1} \left\{ \hat{w}_{i0} \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \hat{w}_{i0} \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \right\}
\]

\[
+ \hat{w}_{i1} \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \hat{w}_{i1} \left( (p-1)N^{\ell-1} + \frac{u(m)}{p-1} \right) \}
\]
Note that $T_{ij}^\ell (\xi) = q(qN)^{\ell-1}(qN\delta_{ij}) = q(qN)^\ell \delta_{ij}$, as the system $S(W)$ given by (3.3) is orthonormal in $l^2(\Lambda)$. This proves the equation (3.23). Similarly, we can prove (3.24). This completes the proof of the Theorem 4.2.

We now invoke Theorem 4.2 to prove the orthogonal splitting properties of the subspaces $V_j$’s.

**Theorem 4.3.** With the assumptions of Theorem 4.2, let us define the subsets $V_{\ell-1}, V_{\ell}$ and $W_{\ell}$ of $V_0 = l^2(\Lambda)$ by

$$
V_{\ell-1} = \text{span}\{T(qN)^{\ell-1}\lambda h_{(\ell-1),0} : \lambda \in \Lambda\};
$$

$$
V_{\ell} = \text{span}\{T(qN)^{\ell}\lambda h_{\ell,0} : \lambda \in \Lambda\},
$$

$$
W_{\ell} = \text{span}\{T(qN)^{\ell}\lambda h_{\ell,i} : \lambda \in \Lambda, 1 \leq i \leq qN - 1\}.
$$

Then, $V_{\ell} \oplus W_{\ell} = V_{\ell-1}$.

**Proof.** For each $i = 0, 1, \ldots, qN - 1$ and $\ell \in \mathbb{N}$, we can write

$$
\hat{w}_i((p^{-1}N)^{\ell-1}\xi) = \sum_{\nu \in \Lambda} w_i(\nu) \chi((p^{-1}N)^{\ell-1}\nu\xi)
$$

$$
= \sum_{\sigma \in \Lambda} w_i(\sigma - p^{-1}N\lambda) \chi((p^{-1}N)^{\ell-1}(\sigma - p^{-1}N\lambda)\xi).
$$

Therefore, we have

$$
\chi((p^{-1}N)^{\ell}\lambda\xi) \hat{w}_i((p^{-1}N)^{\ell-1}\xi) \hat{h}_{(\ell-1),0}(\xi) = \sum_{\sigma \in \Lambda} w_i(\sigma - p^{-1}N\lambda) \chi((p^{-1}N)^{\ell-1}\sigma\xi) \hat{h}_{(\ell-1),0}(\xi),
$$

or

$$
(T(qN)^{\ell}\lambda h_{\ell,i})^\wedge (\xi) = \sum_{\sigma \in \Lambda} w_i(\sigma - p^{-1}N\lambda) (T(qN)^{\ell-1}\sigma h_{(\ell-1),\xi})^\wedge (\xi),
$$

or

$$
T(qN)^{\ell}\lambda h_{\ell,i}(\xi) = \sum_{\sigma \in \Lambda} w_i(\sigma - p^{-1}N\lambda) T(qN)^{\ell-1}\sigma h_{(\ell-1,i)}(\xi),
$$

which implies $V_{\ell}$ and $W_{\ell}$ are the subspaces of $V_{\ell-1}$. Using the facts that: $V_{\ell}$ is orthogonal to $W_{\ell}$; $\{T(qN)^{\ell}\lambda h_{\ell,0} : \lambda \in \Lambda\}$ and $\{T(qN)^{\ell}\lambda h_{\ell,i} : \lambda \in \Lambda, 1 \leq i \leq qN - 1\}$ are orthogonal to each other; and $V_{\ell} \oplus W_{\ell} \subset V_{\ell-1}, \ell \in \mathbb{N}$, it only needs to show that $V_{\ell-1} \subset V_{\ell} \oplus W_{\ell}$. Thus, we have

$$
T(qN)^{\ell-1}\lambda h_{(\ell-1),0}(\xi)
$$

$$
= \sum_{\sigma \in \Lambda} w_i(\sigma - p^{-1}N\lambda) T(qN)^{\ell-1}\sigma h_{(\ell-1),0}(\xi)
$$

$$
= \sum_{\sigma \in \Lambda} \left\{ \sum_{i=0}^{qN-1} \sum_{\nu \in \Lambda} \left\langle w_i(\sigma - p^{-1}N\lambda), T_{qN\nu}w_i \right\rangle T_{qN\nu}w_i(\sigma) \right\} T(qN)^{\ell-1}\sigma h_{(\ell-1),0}(\xi)
$$

$$
= \sum_{i=0}^{qN-1} \sum_{\nu \in \Lambda} \left\langle w_i(\sigma - p^{-1}N\lambda), T_{qN\nu}w_i \right\rangle \left\{ \sum_{\sigma \in \Lambda} T_{qN\nu}w_i(\sigma) T(qN)^{\ell-1}\sigma h_{(\ell-1),0}(\xi) \right\}
$$

$$
= \sum_{i=0}^{qN-1} \sum_{\nu \in \Lambda} \left\langle w_i(\sigma - p^{-1}N\lambda), T_{qN\nu}w_i \right\rangle T(qN)^{\ell}\nu h_{\ell,0}(\xi)
$$

$$
= \sum_{\nu \in \Lambda} \left\langle w_i(\sigma - p^{-1}N\lambda), T_{qN\nu}w_i \right\rangle T(qN)^{\ell}\nu h_{\ell,0}(\xi)
$$
\[
+ \sum_{i=1}^{qN-1} \sum_{\nu \in \Lambda} \langle w_i (\sigma - p^{-1} N \lambda), T_{qN \nu} w_i \rangle T((qN)_{\nu}) h_{\ell,i}(\xi),
\]

which verifies that \( V_{\ell-1} \subseteq V_{\ell} \oplus W_{\ell} \). This completes the proof of Theorem 4.3. \( \Box \)

**Theorem 4.4.** For each \( \ell, 1 \leq \ell \leq J, \) and \( i, 0 \leq i \leq qN - 1 \), let \( w_{\ell,i} \in \ell^2(\Lambda) \) such that

\[
\hat{w}_{\ell,i}(\xi) = w_{\ell,i_0}(\xi) + \chi(\frac{\nu}{N}) w_{\ell,i_1}(\xi), \tag{4.4}
\]

for some \( p \)-periodic functions \( w_{\ell,i_0} \) and \( w_{\ell,i_1} \). For each \( \ell \), assume that the matrix \( M_{\ell}(\xi) \) is unitary, with its entries defined by

\[
M_{\ell,t}(\xi) = \frac{1}{q \sqrt{N}} \left\{ \begin{array}{ll}
\frac{1}{N} \left( \xi + \frac{u(J)}{p^{-1}N} \right) & 0 \leq J \leq qN - 1; 0 \leq t \leq qN - 1, \\
\frac{1}{N} \left( \xi + \frac{u(J-qN)}{p^{-1}N} \right) & qN \leq J \leq q^2N - 1; 0 \leq t \leq qN - 1, \\
\chi(\frac{r}{N}) p u(J) w_{\ell,(t-qN)0} \left( \xi + \frac{u(J-qN)}{p^{-1}N} \right) & 0 \leq t \leq qN - 1; \quad qN \leq J \leq q^2N - 1, \\
\chi(\frac{r}{N}) p u(J) w_{\ell,(t-qN)1} \left( \xi + \frac{u(J-qN)}{p^{-1}N} \right) & qN \leq t, J \leq q^2N - 1;
\end{array} \right.
\]

For given \( \ell \) and \( i \), define \( h_{\ell,i} \) as follows

\[
\hat{h}_{\ell,i}(\xi) = \hat{h}_{\ell-1,i}(\xi) \hat{w}_{\ell,i}((p^{-1}N)^{\ell-1} \xi), \quad \text{for} \ 2 \leq \ell \leq J
\]

with \( h_{1,i} = w_{1,i} \) and \( h_{0,0} = 1 \) a.e. Then,

\[
V_0 = \ell^2(\Lambda) = V_J \oplus \left( \bigoplus_{m=1}^{J} W_m \right), \tag{4.5}
\]

where \( W_j = V_{j+1} \oplus V_j, j \in \mathbb{Z} \) and the \( J \)-th-stage nonuniform system \( H \) given by (4.1) is orthonormal basis for \( \ell^2(\Lambda) \).

**Proof.** From Theorem 4.2, it follows that for each \( \ell, 1 \leq \ell \leq J \), the system

\[
\{ T((qN)_{\nu}) f_{\ell,i} : 1 \leq i \leq qN - 1, \lambda \in \Lambda \} \tag{4.6}
\]

is orthonormal in \( \ell^2(\Lambda) \). Therefore, the system \( \{ T((qN)_{\nu}) h_{\ell,0} : \lambda \in \Lambda \} \) and the system defined by (4.1) are both orthonormal for each \( \ell \). Further, using Theorem 4.3, it follows that for each \( \ell, 1 \leq \ell \leq J, V_{\ell} \subset V_{\ell-1} \) and \( V_{\ell} \) is orthogonal to \( W_{\ell} \) in \( V_{\ell-1} \). This means that \( W_{\ell} \) is orthogonal to \( W_{\ell-1} \) for each \( \ell \). Therefore, the system \( H \) defined by (4.1) is orthonormal in \( \ell^2(\Lambda) \). Since \( V_{\ell} \oplus W_{\ell} = V_{\ell-1} \), so we can write

\[
V_0 = V_1 \oplus W_1 = V_2 \oplus W_1 \oplus W_2 = \cdots = V_J \oplus \left( \bigoplus_{m=1}^{J} W_m \right).
\]

Since (4.5) holds, the system (4.1) is orthonormal in \( \ell^2(\Lambda) \). This completes the proof of the Theorem 4.4. \( \Box \)
Theorem 4.5. Under the assumptions of Theorem 4.4 and for each \( \ell \in \mathbb{N}_0 \), define

\[
V_0 = l^2(\Lambda), \quad V_\ell = \text{span}\{T_{(qN)^j \Lambda} h_{\ell,0} : \lambda \in \Lambda\}.
\]

Then, \( \bigcup_{\ell=0}^{\infty} V_\ell = l^2(\Lambda) \). Also, if \( \bigcap_{\ell=0}^{\infty} V_\ell = \{0\} \), then \( l^2(\Lambda) = \bigoplus_{m=1}^{\infty} W_m \), where \( W_j = V_{j+1} \oplus V_j, j \in \mathbb{Z} \), and for \( \ell \in \mathbb{N} \), \( J^\ell \)-stage nonuniform discrete wavelet system (4.1) is an orthonormal basis for \( l^2(\Lambda) \).

Proof. Since, for each \( \ell \in \mathbb{N} \), \( V_\ell \subset V_{\ell-1} \) and \( V_0 = l^2(\Lambda) \), it follows that \( \bigcup_{\ell=0}^{\infty} V_\ell = l^2(\Lambda) \). Using the fact \( V_\ell \oplus W_\ell = V_{\ell-1} \), we have

\[
V_0 = V_1 \oplus W_1 = V_2 \oplus W_1 \oplus W_2 = \cdots = V_J \oplus \left( \bigoplus_{m=1}^{J} W_n \right).
\]

To show \( V_0 = \bigoplus_{m=1}^{\infty} W_m \), it is sufficient to show that the orthogonal complement of \( \bigoplus_{m=1}^{\infty} W_m \) in \( V_0 \) is \( \{0\} \). For this, suppose \( f \in V_0 \) is orthogonal to \( \bigoplus_{m=1}^{\infty} W_m \). Then \( f \) is orthogonal to each \( W_m \) for \( m \in \mathbb{N} \). This means that \( f \) is a member of each \( V_m \) as \( W_m \) is orthogonal to \( V_m \). Therefore, \( f \in \bigcap_{\ell=0}^{\infty} V_\ell = \{0\} \), which means that \( f = 0 \), a.e. This completes the proof of Theorem 4.5. \( \square \)

5. Connection Between Nonuniform Discrete and Continuous Wavelets

In this section, we provide a connection between first-stage nonuniform discrete wavelet system of \( l^2(\Lambda) \) and their counterpart nonuniform wavelets of \( L^2(K) \).

Theorem 5.1. Let \( \{\psi_\ell\}^{qN-1}_{\ell=1} \) be a system of NUMRA wavelets with scaling function \( \psi_0 \) in \( L^2(K) \). Then, there is a first-stage nonuniform discrete wavelet system for \( l^2(\Lambda) \) associated with a system of NUMRA wavelets of \( L^2(K) \).

Proof. Given a system \( \{\psi_\ell\}^{qN-1}_{\ell=1} \) of NUMRA wavelets with scaling function \( \psi_0 \) in \( L^2(K) \), we define \( V_j^*, j \in \mathbb{Z} \) as \( V_j^* = \text{span}\{D_j T_\lambda \psi_0(x) : \lambda \in \Lambda\} \), where \( \{T_\lambda \psi_0(x) : \lambda \in \Lambda\} \) is an orthonormal basis for \( V_0^* \) and the unitary operators \( T_\lambda \) and \( D_j \) are defined by

\[
T_\lambda f(x) = f(x - \lambda), \quad D_j f(x) = (qN)^{j/2} f((p^{-1}N)^j x), \quad \text{for } f \in L^2(K).
\]

Since \( \{\psi_\ell\}^{qN-1}_{\ell=1} \subset V_1^* \), there is \( \{w_\ell\}^{qN-1}_{\ell=1} \subset l^2(\Lambda) \) such that for each \( \ell, 0 \leq \ell \leq qN - 1 \),

\[
\psi_\ell(x) = \sum_{\lambda \in \Lambda} w_\ell(\lambda) D T_\lambda \psi_0(x).
\] (5.1)

Equation (5.1) can be written in the frequency domain as

\[
\hat{\psi}_\ell(x) = \sum_{\lambda \in \Lambda} w_\ell(\lambda) (DT_\lambda \hat{\psi}_0(x))^\wedge = m_\ell \left( \frac{\xi}{p^{-1}N} \right) \hat{\psi}_0 \left( \frac{\xi}{p^{-1}N} \right),
\]

where \( m_\ell(\xi) = \frac{1}{\sqrt{qN}} \sum_{\lambda \in \Lambda} w_\ell(\lambda) \chi_\Lambda(\xi) \) is \( L^2 \) locally. Since \( \Lambda = \{0, u(r)/N\} + \mathbb{Z} \), we can write

\[
m_\ell(\xi) = m_{\ell0}(\xi) + \int_{\mathbb{R}} \chi_{\Lambda}(\xi) m_{\ell1}(\xi), \quad 0 \leq \ell \leq qN - 1,
\] (5.2)
where \( m_{\ell_0} \) and \( m_{\ell_1} \) are locally \( p \)-periodic functions. Therefore, we have equivalent conditions of orthonormality for the system \( \{ T_\lambda \psi_\ell(x) : 0 \leq \ell \leq qN - 1, \lambda \in \Lambda \} \) as

\[
\begin{align*}
&\text{(a)} \quad \sum_{t=0}^{qN-1} \left\{ m_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) m_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) + m_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) \right\} = \delta_{\ell,k}, \\
&\text{(b)} \quad \sum_{t=0}^{qN-1} \lambda \left( \frac{r}{N} p^u(t) \right) \left\{ m_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) m_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) + m_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) \right\} = 0,
\end{align*}
\]

From the definition of \( m_\ell \), we see that

\[ m_\ell(\xi) = \frac{1}{\sqrt{qN}} \hat{w}_\ell(\xi) \]

where \( \hat{w}_\ell \) denotes the Fourier transform in the sense of \( l^2(\Lambda) \). Using (3.21), we have

\[ m_{\ell_0}(\xi) = \frac{1}{\sqrt{qN}} w_{\ell_0}(\xi) \quad \text{and} \quad m_{\ell_1}(\xi) = \frac{1}{\sqrt{qN}} w_{\ell_1}(\xi), \]

where \( w_{\ell_0} \) and \( w_{\ell_1} \) have same properties as that of \( m_{\ell_0} \) and \( m_{\ell_1} \). Substituting the values of \( m_{\ell_0} \) and \( m_{\ell_1} \) in (a) and (b), we have for \( 0 \leq \ell, k \leq qN - 1, \)

\[
\begin{align*}
&\text{(a)} \quad \sum_{t=0}^{qN-1} \left\{ w_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) w_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) + w_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) w_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) \right\} = qN\delta_{\ell,k}, \\
&\text{(b)} \quad \sum_{t=0}^{qN-1} \lambda \left( \frac{r}{N} p^u(t) \right) \left\{ w_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) w_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) + w_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) w_{\ell_0} \left( \xi + \frac{u(t)}{p^{-1}N} \right) \right\} = 0.
\end{align*}
\]

These conditions are equivalent to the a.e unitary of the matrix \( M(\xi) \) of order \( q^2N \times q^2N \) with entries as in the Corollary 3.8. Therefore, system \( \mathcal{F}(W) \) given by (3.3) is an orthonormal basis of \( l^2(\Lambda) \) and hence, is the first-stage nonuniform discrete wavelet system for \( l^2(\Lambda) \).

By observing that \( m_\ell \) and \( w_\ell \) are closely related for each \( \ell \), following result can be easily proved

**Theorem 5.2.** If \( \mathcal{F}(W) \) is the first-stage nonuniform discrete wavelet system as defined by (3.3), Then, there exists a system of NUMRA wavelets \( \{ \psi_\ell \}^{qN-1}_{\ell=1} \) with scaling function \( \psi_0 \) in \( L^2(K) \) associated with the first-stage nonuniform discrete wavelet system for \( l^2(\Lambda) \).

It is evident from Theorems 5.1 and 5.2 that the NUMRA wavelets of \( L^2(K) \) are connected with the first-stage nonuniform discrete wavelet system of \( l^2(\Lambda) \) and vice-versa.
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