Fused integrable lattice models with quantum impurities and open boundaries

Anastasia Doikou

Theoretical Physics Laboratory of Annecy–Le–Vieux, LAPTH, B.P. 110, Annecy–Le–Vieux, F-74941, France

Abstract

The alternating integrable spin chain and the $RSOS(q_1,q_2;p)$ model in the presence of a quantum impurity are investigated. The boundary free energy due to the impurity is derived, the ratios of the corresponding $g$ functions at low and high temperature are specified and their relevance to boundary flows in unitary minimal and generalized coset models is discussed. Finally, the alternating spin chain with diagonal and non–diagonal integrable boundaries is studied, and the corresponding boundary free energy and $g$ functions are derived.

\[1\] e-mail: doikou@lapp.in2p3.fr
1 Introduction

Two dimensional exactly solvable models with boundaries have attracted a great deal of research interest recently from the point of view of boundary conformal field theory [1], and critical behavior [2], but also because of the rich variety of physical phenomena they display, which in principle can be exactly investigated (see e.g. [3–7]). There has been also much interest on problems related to quantum impurities, mainly because of the important role they play in low dimensional physics [8], but also because of their relevance to boundary conformal field theory [1, 2, 9, 10]. There are numerous studies related to quantum impurities yielding a great number of interesting and useful results (see e.g. [11–13]). In this article we focus basically on the thermodynamic analysis of lattice integrable systems in the presence of quantum impurities and integrable open boundaries.

For both integrable lattice models and relativistic integrable field theories, in the bulk, the corresponding free energy has been derived and the conformal properties have been extensively studied [14–24] by means of the thermodynamic Bethe ansatz. It is however of great interest to extend these studies for integrable models with boundaries. In analogy to the bulk case, when boundaries are added, the corresponding boundary free energy and the so called \( g \) function, which characterizes the ground state degeneracy due to the boundaries, can be specified by means of the thermodynamic Bethe ansatz. On the other hand statistical systems at the critical point it is known to display conformal invariance [30, 31], therefore they can be associated with certain conformal field theories. The low temperature behavior of the free energy per unit length of such system, in the bulk, is described by [32, 33]

\[
    f(T) = f_0 - \frac{\pi c}{6u} T^2 + \ldots, \quad T \ll 1,
\]

where \( c \) is the central charge of the effective conformal field theory. Furthermore, when boundaries are added the obtained free energy is modified up to an \( \frac{1}{L} \) contribution (\( L \) denotes the size of the system) namely, [2]

\[
    f(T) = f_0(T) - \frac{T}{L} \ln g,
\]

where the ground state degeneracy \( g \) is expected to be related with the boundaries of the system. One of the main aims, when studying such systems is to employ proper techniques in order to specify the central charge and the ground state degeneracy —when boundaries are present.

As already mentioned from the integrable systems point of view, the central charge and the ground state degeneracy \( g \) can be identified by employing thermodynamic Bethe ansatz techniques (see e.g. [14–24], [9, 10, 25–27, 29]). In this study in particular the alternating integrable spin chain [34] and the \( RSOS(q_1, q_2; p) \) model [35] in the presence of a quantum impurity ("Kondo type" boundaries see e.g. [11, 12]) are investigated via the thermodynamic Bethe ansatz, and the corresponding free energy is derived at low and high temperature. The relevance of the results for the \( RSOS(q_1, q_2; p) \) model to the boundary flows in minimal [9] and generalized coset models [10] is discussed. Finally the alternating open spin chain with diagonal and non–diagonal integrable boundaries is considered and the corresponding boundary free energy and the \( g \) functions for the left and right boundaries are determined by first principle calculations at low and high temperature.
2 Quantum Impurity

2.1 The alternating spin chain

Let us first focus on the alternating spin chain in the presence of a quantum impurity. For what follows it is necessary to introduce the basic constructing element of the model, namely the $R$ matrix, which is a solution of the Yang–Baxter equation.

\[
R_{12}^{\lambda_1 - \lambda_2} R_{13}^{\lambda_1} R_{23}^{\lambda_2} = R_{23}^{\lambda_2} R_{13}^{\lambda_1} R_{12}^{\lambda_1 - \lambda_2}.
\] (2.1)

We consider the $R$ matrix obtained in [38], namely

\[
R_{0k}^{1,s} = \sinh \mu \left( \lambda + i \left( \frac{1}{2} + \sigma^3 \otimes S^3 \right) \right) + \sinh \mu \left( \sigma^+ \otimes S^- + \sigma^- \otimes S^+ \right) \] (2.2)

where $S^3, S^\pm$ act in general, on a $2S + 1$ dimensional space $V = C^{2S+1}$, and they satisfy the following commutation relations

\[
[S^+, S^-] = \frac{\sinh 2i\mu S^3}{\sinh i\mu}, \quad [S^3, S^\pm] = \pm S^\pm, \quad S^3 |0\rangle = S|0\rangle, \quad S^+ |0\rangle = 0.
\] (2.3)

We can now define the transfer matrix of the chain

\[
t = tr_0 T_0(\lambda)
\] (2.4)

where

\[
T_0(\lambda) = R_{02N+1}^q (\lambda - \Theta) R_{02N}^l (\lambda) R_{02N-1}^2 (\lambda) \cdots R_{02}^l (\lambda) R_{01}^2 (\lambda),
\] (2.5)

and $R^i$ is related to the spin $S_i = \frac{q_i}{2} (i = 1, 2)$ representation ($R^q$ is related to the spin $\frac{q}{2}$ representation). Following the standard Bethe ansatz method described in e.g. [34], [39–41], the Bethe equations are obtained

\[
e_{q_1}(\lambda_\alpha)^N e_{q_2}(\lambda_\alpha)^N e_q(\lambda_\alpha - \Theta) = - \prod_{\beta=1}^{M} e_2(\lambda_\alpha - \lambda_\beta)
\] (2.6)

where

\[
e_n(\lambda; \nu) = \frac{\sinh \mu(\lambda + \frac{in}{2})}{\sinh \mu(\lambda - \frac{in}{2})}, \quad \nu = \frac{\pi}{\mu}.
\] (2.7)

Notice that the main difference between the usual bulk case (without impurities) and (2.6) is the appearance of the $e_q$ term in the left hand side of (2.6) due to the presence of the spin $\frac{s}{2}$ impurity. Moreover we should mention that $q_1, q_2 - q_1$ play the role of “flavors” in accordance with the picture in [12], where the spin $\frac{f}{2}$ (f flavor) chain is studied with impurity of spin $s$.

\[\text{If we derive the transfer matrix of the model with proper inhomogeneities we obtain massless relativistic dispersion relations for the particle–like excitations of the model.}\]
In the thermodynamic limit, $N \to \infty$, the string hypothesis is valid \cite{16,17,39}, namely the solutions of equation (2.6) can be grouped into strings of length $n$ with the same real part and equidistant imaginary parts

$$\lambda_{\alpha}^{(n,j)} = \lambda_{\alpha}^n + \frac{i}{2}(n + 1 - 2j), \quad j = 1, 2, \ldots, n,$$

$$\lambda_{\alpha}^{0,s} = \lambda_{\alpha}^0 + \frac{\pi}{2\mu}, \quad (2.8)$$

where $\lambda_{\alpha}^n$ is real, and $\lambda_{\alpha}^{0,s}$ is the negative parity string and $\lambda_{\alpha}^0$ is real. In order to formulate the thermodynamic Bethe ansatz we consider all the strings with length $n = 1, \ldots, \nu - 1$ plus the negative parity string. The Bethe ansatz equations (2.6) can be written after we apply the string hypothesis \cite{16,17}

$$X_{nq} (\lambda_{\alpha}^n - \Theta) \prod_{j=1}^{2} X_{nqj} (\lambda_{\alpha}^n)^N = (-)^n \prod_{m=1}^{\nu-1} \prod_{\beta=1}^{M_m} E_{nm} (\lambda_{\alpha}^n - \lambda_{\alpha}^m) \prod_{\beta=1}^{M_0} G_{n1} (\lambda_{\alpha}^n - \lambda_{\alpha}^0) \quad (2.9)$$

$$g_q (\lambda_{\alpha}^0 - \Theta) \prod_{j=1}^{2} g_{jq} (\lambda_{\alpha}^0)^N = - \prod_{m=1}^{\nu-1} \prod_{\beta=1}^{M_m} G_{1m} (\lambda_{\alpha}^0 - \lambda_{\alpha}^m) \prod_{\beta=1}^{M_0} e_2 (\lambda_{\alpha}^0 - \lambda_{\alpha}^0), \quad (2.10)$$

where

$$g_n (\lambda; \nu) = e_n (\lambda + \frac{i\pi}{2\mu}) = \frac{\cosh \mu (\lambda + \frac{\nu}{2})}{\cosh \mu (\lambda - \frac{\nu}{2})}, \quad (2.11)$$

and $X_{nm}, E_{nm}$, and $G_{nm}$ are given in the appendix (A.1).

We introduce the densities of the holes $\tilde{\rho}_n$ and pseudo–particles $\rho_n$, and once we take the logarithm and the derivative of the Bethe ansatz equations (2.9), (2.10) we conclude

$$\tilde{\rho}_n (\lambda) = \frac{1}{2} (Z_{nq1} (\lambda) + Z_{nq2} (\lambda)) + \frac{1}{L} Z_{nq} (\lambda - \Theta) - \sum_{m=1}^{\nu-1} A_{nm} * \rho_m (\lambda) - B_{1m} * \rho_0 (\lambda)$$

$$-(\rho_0 (\lambda) + \tilde{\rho}_0 (\lambda)) = \frac{1}{2} (b_{q1} (\lambda) + b_{q2} (\lambda)) + \frac{1}{L} b_q (\lambda - \Theta) - \sum_{m=1}^{\nu-1} B_{1m} * \rho_m (\lambda) - a_2 * \rho_0 (\lambda) \quad (2.12)$$

where $\rho_0$ is the density of the negative parity string, and $L = 2N$ is the length of the spin chain, and the quantities $Z_{nm}, A_{nm}, B_{nm}$, and $b_n$ are given in the appendix (A.5), (A.6), (A.7), and (A.3). It is also convenient to solve $\rho_n (\lambda)$ in terms of $\tilde{\rho}_n (\lambda)$, therefore we consider the convolution of the first of the equations (2.12) with the inverse of $A_{nm}$

$$\hat{A}_{nm}^{-1} = \delta_{nm} - \hat{s} (\omega) (\delta_{nm+1} + \delta_{nm-1}) \quad (2.13)$$

where $\hat{s} (\omega) = \frac{1}{2\cosh \frac{\omega}{2}}, \quad s (\lambda) = \frac{1}{2\cosh \pi \lambda}$. Having in mind the identities

$$A_{nm}^{-1} * Z_{mq} (\lambda) = s (\lambda) \delta_{mq}, \quad A_{nm}^{-1} * B_{1m} (\lambda) = - s (\lambda) \delta_{m-2} \quad (2.14)$$

\footnote{We treat the impurity, which sits at the $2N + 1$ site of the chain, separately, therefore $L = 2N$ is the length of the bulk part.}
we obtain the following expressions

\[ \rho_n(\lambda) = \frac{1}{2} s(\lambda) (\delta_{nq_1} + \delta_{nq_2}) - \sum_{m=1}^{\nu-1} A_{nm}^{-1} \ast \tilde{\rho}_m(\lambda) - \delta_{n\nu-2}s \ast \tilde{\rho}_0(\lambda) \]

\[ + \delta_{n\nu-2}s \ast s \ast \tilde{\rho}_0(\lambda) + \frac{1}{L} s(\lambda - \Theta) \delta_{nq} \]

\[ \rho_0(\lambda) = -\tilde{\rho}_0(\lambda) + s \ast \tilde{\rho}_{\nu-2}(\lambda) \quad (2.15) \]

note the extra \( \frac{1}{L} \) contribution term in the first of the two above equations, which is due to the impurity. Let us now derive the free energy of the system \( f = e - Ts \), which is given by

\[ f = e - Ts = -\frac{1}{2} \sum_{n=1}^{\nu-1} \int_{-\infty}^{\infty} d\lambda (Z_{nq_1}(\lambda) + Z_{nq_2}(\lambda)) \rho_n(\lambda) - \frac{1}{2} \int_{-\infty}^{\infty} d\lambda (b_{q_1}(\lambda) + b_{q_2}(\lambda)) \rho_0(\lambda) \]

\[ - T \sum_{n=1}^{\nu-1} \int_{-\infty}^{\infty} d\lambda \left( \rho_n(\lambda) \ln\left(1 + \frac{\tilde{\rho}_n(\lambda)}{\rho_n(\lambda)}\right) + \tilde{\rho}_n(\lambda) \ln\left(1 + \frac{\rho_n(\lambda)}{\tilde{\rho}_n(\lambda)}\right) \right) \]

\[ - T \int_{-\infty}^{\infty} d\lambda \left( \rho_0(\lambda) \ln\left(1 + \frac{\tilde{\rho}_0(\lambda)}{\rho_0(\lambda)}\right) + \tilde{\rho}_0(\lambda) \ln\left(1 + \frac{\rho_0(\lambda)}{\tilde{\rho}_0(\lambda)}\right) \right). \quad (2.16) \]

The thermodynamic Bethe ansatz equations are obtained by minimizing the free energy \( \delta f = 0 \) and by virtue of (2.12), we conclude that they coincide with the ones of the model without impurities [12] and they are given by

\[ T \ln \left(1 + \eta_n(\lambda)\right) = -\frac{1}{2} (Z_{nq_1}(\lambda) + Z_{nq_2}(\lambda)) + T \sum_{m=1}^{\nu-1} A_{nm} \ast \ln \left(1 + \eta_m^{-1}(\lambda)\right) \]

\[ - T B_{1n} \ast \ln \left(1 + \eta_0^{-1}(\lambda)\right) \]

\[ T \ln \left(\frac{1 + \eta_0(\lambda)}{1 + \eta_0^{-1}(\lambda)}\right) = -\frac{1}{2} (b_{q_1}(\lambda) + b_{q_2}(\lambda)) + T \sum_{m=1}^{\nu-1} B_{1m} \ast \ln \left(1 + \eta_m^{-1}(\lambda)\right) \]

\[ - T a_2 \ast \ln \left(1 + \eta_0^{-1}(\lambda)\right), \quad (2.17) \]

where \( \eta_n = \frac{\rho_n}{\tilde{\rho}_n} \). Alternatively we can write the thermodynamic Bethe ansatz equations by virtue of (2.15) in the following form, \( \eta_n(\lambda) = e^{\frac{\epsilon_n(\lambda)}{T}} \),

\[ \epsilon_n(\lambda) = s(\lambda) \ast T \ln(1 + \eta_{n+1}(\lambda))(1 + \eta_{n-1}(\lambda)) - \frac{1}{2} s(\lambda)(\delta_{nq_1} + \delta_{nq_2}) \]

\[ + \delta_{n\nu-2}s(\lambda) \ast T \ln \left(1 + \eta_0^{-1}(\lambda)\right) \]

\[ \epsilon_{\nu-1}(\lambda) = s(\lambda) \ast T \ln(1 + \eta_{\nu-2}(\lambda)) \]

\[ \epsilon_0(\lambda) = -s(\lambda) \ast T \ln(1 + \eta_{\nu-2}(\lambda)). \quad (2.18) \]

Although the thermodynamic Bethe ansatz equations remain the same as in the bulk case, the free energy of the model is modified because of the presence of the quantum impurity. In particular, there is a non trivial contribution to the free energy which is given by the following expression, after we apply (2.15) and (2.17), (2.18) to (2.16)

\[ f = e_0 + f_0 + f_b + O\left(\frac{1}{L}\right) \quad (2.19) \]
where $f_0$ is the bulk free energy given by

$$f_0 = -\frac{T}{2} \int_{-\infty}^{\infty} d\lambda s(\lambda) \ln(1 + \eta q_1(\lambda))(1 + \eta q_2(\lambda)). \tag{2.20}$$

$\epsilon_0$ is the energy of the state with the seas of strings with length $q_1$, $q_2$ filled

$$\epsilon_0 = -\frac{1}{4} \sum_{i,j=1}^{2} \int_{-\infty}^{\infty} d\lambda Z_{qiqj}(\lambda)s(\lambda) - \frac{1}{2L} \sum_{j=1}^{2} \int_{-\infty}^{\infty} d\lambda Z_{qqj}(\lambda - \Theta)s(\lambda) \tag{2.21}$$

where the non–trivial $\frac{1}{L}$ contribution to the bulk ground state energy is due to the impurity. Finally, $f_b$ is the free energy contribution of the impurity (after we make the shift $\lambda \rightarrow \lambda - \frac{1}{\pi} \ln T$)

$$f_b = -\frac{T}{L} \int_{-\infty}^{\infty} d\lambda s(\lambda - \frac{1}{\pi} \ln T) \ln(1 + \eta_q(\lambda)). \tag{2.22}$$

We can accurately evaluate differences of boundary free energies, or ratios of $g$ functions (1.2), for different temperatures and not specific values at each temperature. This happens basically because an overall $\frac{1}{L}$ contribution, which cannot be explicitly evaluated, may survive in the bulk calculation (see (2.19)) of the free energy as well (see also [25, 26, 27]). The boundary free energy, and the corresponding $g$–function (1.2) can be computed at any temperature by employing numerical methods, however it is possible to make analytical calculations at $T \rightarrow \infty$ “weak–coupling” and $T \rightarrow 0$ “strong–coupling” point (see e.g. [2, 12]).

In particular, for $T \rightarrow \infty$ the main contribution to the integral in (2.22) comes from the $\lambda \rightarrow \infty$ behavior. The corresponding behavior of $\eta_q(\lambda)$ is given by (see also [19, 42])

$$(1 + \eta_n^\infty)^{-1} = \frac{1}{(n + 1)^2}, \quad n = 1, \ldots \nu - 2$$

$$(1 + \eta_{\nu-1}^\infty)^{-1} = \frac{1}{\nu}, \quad (1 + \eta_0^\infty)^{-1} = 1 - \frac{1}{\nu}, \tag{2.23}$$

which obviously does not depend on $q_1$, $q_2$. The boundary free energy contribution is for $q < \nu - 1$ “weak–coupling” point (see e.g. [2, 12]),

$$f^\infty_b = -\frac{T}{L} \ln(q + 1). \tag{2.24}$$

In the special value where $q = \nu - 1$ the boundary energy becomes following (2.23)

$$f^\infty_b = -\frac{T}{2L} \ln(\nu), \tag{2.25}$$

which is the half of the expected value. In the isotropic case $\nu \rightarrow \infty$, $q$ can take any value from 1 to $\infty$ and the impurity contribution is given by (2.24).

When $T \rightarrow 0$ the main contribution to the integral in (2.22) comes from the $\lambda \rightarrow -\infty$ behavior — recall that we considered the shift $\lambda \rightarrow \lambda - \frac{1}{\pi} \ln T$. The quantity $1 + \eta_n$ for $\lambda \rightarrow -\infty$ and $T \rightarrow 0$
is given by \[ (1 + \eta_n^0)^{-1} = \frac{\sin^2(\frac{n\pi}{q_1+2})}{\sin^2(\frac{(n+1)\pi}{q_1+2})}, \quad n = 1, \ldots, q_1 - 1, \quad (1 + \eta_{q_1}^0) = 1 \]

\[ (1 + \eta_n^0)^{-1} = \frac{\sin^2(\frac{n\pi}{q_2-q_1+2})}{\sin^2(\frac{(n-q_1+1)\pi}{q_2-q_1+2})}, \quad n = q_1 + 1, \ldots, q_2 - 1, \quad (1 + \eta_{q_2}^0) = 1 \]

\[ (1 + \eta_n^0)^{-1} = \frac{1}{(n - q_2 + 1)^2}, \quad n = q_2 + 1, \ldots, \nu - 2 \]

\[ (1 + \eta_{\nu-1}^0)^{-1} = \frac{1}{\tilde{\nu}}, \quad (1 + \eta_0^0)^{-1} = 1 - \frac{1}{\tilde{\nu}}. \] (2.26)

where $\tilde{\nu} = \nu - q_2$. The situation is more complicated now, because the behavior of $\eta_q(\lambda)$ depends clearly on the flavors (see (2.26)). In particular, for $q = q_1$ the boundary entropy is $f_b \propto T^2$, and this is the completely screened case (see e.g [12]). For $q < q_1$

\[ f_b^0 = -\frac{T}{L} \ln \frac{\sin(\frac{\pi(q+1)}{q_1+2})}{\sin(\frac{\pi}{q_1+2})} \] (2.27)

this is the behavior of the non-trivial “strong–coupling” point (see also [2, 12]) with flavor $q_2$ and impurity spin $q$. For $q_1 < q < q_2$

\[ f_b^0 = -\frac{T}{L} \ln \frac{\sin(\frac{\pi(q-q_1+1)}{q_2-q_1+2})}{\sin(\frac{\pi}{q_2-q_1+2})}, \] (2.28)

this case also corresponds to a “strong–coupling” point, with flavor $q_1 - q_2$ and a reduced spin impurity $q - q_2$. For $q > q_2$

\[ f_b^0 = -\frac{T}{L} \ln(q - q_2 - 1) \] (2.29)

this is the partially screened case with reduced impurity spin $q - q_1$. Finally, for $q = \nu - 1$ (2.26)

\[ f_b^0 = -\frac{T}{2L} \ln(\nu - q_2). \] (2.30)

We should emphasize again that the boundary free energy is calculated up to an overall $\frac{1}{T}$ contribution which we are not able to derive explicitly. Therefore, we consider only differences of the free energy for different temperatures, and from that via relation (1.2) we deduce the ratio of the $g$ function,

\[ \ln \frac{g_\infty}{g_0} = \frac{1}{2} \ln \frac{(1 + \eta_0^\infty)}{(1 + \eta_0^0)} \] (2.31)

where $\ln(1 + \eta_0^\infty)$ is derived in (2.23), (2.26). The model under study is related to $WZW_{\delta q}$ and $WZW_{q_2}$ model, as already concluded in [42]. Indeed the central charge conjectured in [42], and found in [42] is given by

\[ c = \frac{3q_1}{q_1 + 2} + \frac{3\delta q}{\delta q + 2} \] (2.32)
and it is expressed as the sum of the central charges of the $SU(2)$ WZW models at level $q_2$ and $\delta q$ ($\delta q = q_2 - q_1$). Therefore we expect that our results (2.24)–(2.30), (2.31) should be related to boundary flows in WZW$_k$ models in analogy to the bulk case. Finally, we should note that our results for the free energy (2.24)–(2.30) for $q_1 = q_2$ coincide with the ones found in [12] for the spin $s = \frac{q_2}{2}$ chain with spin $s = \frac{q_1}{2}$ impurity.

In general the impurity sitting at the $2N + 1$ site of the chain can be thought as an immobile “particle” with constant rapidity $\Theta$. Therefore, the particle–like excitations of the chain can interact (scatter) with the impurity giving rise to specific scattering amplitudes see e.g. [45], which we are going to study in detail elsewhere. The scattering should involve in addition to the usual XXX part (in the isotropic limit of our model), RSOS type scattering as well [45]. This is expected since our chain is related to WZW$_k$ model, and it is known (see e.g. [46]) that the $S$–matrix that describes WZW$_k$ models has apart from the $SU(2)$ invariant part an RSOS part as well, namely

$$S_{WZW_k} = S_{SU(2)} \otimes S_{RSOS}^{(k)}.$$  

(2.33)

$S_{SU(2)}$ is the usual XXX $S$ matrix and $S_{RSOS}^{(k)}$ is the $S$ matrix for the RSOS model with restriction parameter $k + 2$ see also [47].

2.2 The generalized RSOS$(q_1, q_2 ; p)$ model

It is known that the effective conformal field theory for the critical RSOS$(1, 1)$ model is the unitary minimal model $\cal M$$_\nu$, whereas the critical RSOS$(q_1 ; p)$ model corresponds to the generalized SU(2) coset model $\cal M$$(q_1, \nu - q_1 - 2) \equiv \frac{SU(2)_{q_1} \otimes SU(2)_{\nu - q_1 - 2}}{SU(2)_{\nu - 2}}$ [43]. It has been also recently shown [35] that the effective conformal field theory for the generalized critical RSOS$(q_1, q_2 ; p)$ model ($q_2 > q_1$) consists of two copies of SU(2) coset models, namely $\cal M$$(q_1, \nu - q_1 - 2) \otimes \cal M$$(q_1, \delta q)$. Our aim is to study the boundary behavior of the generalized RSOS$(q_1, q_2 ; p)$ model, with “Kondo type” boundaries. In particular, the corresponding boundary free energy and the $g$–function will be derived, and their relevance to boundary flows of conformal field theories [9, 10] will be discussed.

To describe the model, an orthogonal lattice of $2N + 1$ horizontal and $M$ vertical sites is considered. The Boltzmann weights associated with every site are defined as

$$w(l_i, l_j, l_m, l_n | \lambda) \equiv \begin{pmatrix} l_n & l_m \\ l_i & l_j \end{pmatrix}.$$  

(2.34)

With every face $i$ of the lattice an integer $l_i$ is associated, and every pair of adjacent integers satisfies the following restriction conditions [36], [48]

$$0 \leq l_{i+1} - l_i + P \leq 2P, \quad (a)$$
$$P \leq l_{i+1} + l_i \leq 2\nu - P, \quad (b)$$  

(2.35)

where $P = q_1$ for $i$ odd, $P = q_2$ for $i$ even (let $q_2 > q_1$), for $i = 1, \ldots, 2N$ and $P = q$ for $i = 2N + 1$ for the horizontal pairs, while $P = p$ for the vertical pairs (array type II [49]).
The fused Boltzmann weights have been derived by Date *et al* in [50] and they are given by

\[ w^{q_i,1}(a_1, a_{q_i+1}, b_{q_i+1}, b_1 | \lambda) = \sum_{a_2 \ldots a_{q_i}} \prod_{k=1}^{q_i} w^{1,1}(a_k, a_{k+1}, b_{k+1}, b_k | \lambda + i(k - q_i)) \]  

(2.36)

where \( b_2 \ldots b_{q_i} \) are arbitrary numbers satisfying \( |b_i - b_{i+1}| = 1 \). \( w^{1,1} \) are the Boltzmann weights for the SOS(1, 1) model [36], they are non vanishing as long as the condition (2.35(a)) is satisfied, and for \( P = 1 \) they are given by the following expressions

\[
\begin{align*}
 w(l, l \pm 1, l \mp 1 | \lambda) &= h(i - \lambda) \\
 w(l \pm 1, l \mp 1, l | \lambda) &= -h(\lambda) \frac{h_{l+1}}{h_l} \\
 w(l \pm 1, l \mp 1, l | \lambda) &= h(w_l \pm \lambda) \frac{h_1}{h_l}
\end{align*}
\]

(2.37)

where,

\[ h(\lambda) = \rho \Theta(\lambda) H(\lambda) \]  

(2.38)

\( H(\lambda) \) and \( \Theta(\lambda) \) are Jacobi theta functions and,

\[ h_l = h(w_l), \quad w_l = w_0 + il. \]  

(2.39)

We are interested in the critical case where \( h(\lambda) \) becomes a simple hyperbolic function i.e.,

\[ h(\lambda) = \frac{\sinh \mu \lambda}{\sin \mu}, \]  

(2.40)

\( w_0, \rho \) and \( \mu \) are arbitrary constants. Furthermore,

\[ w^{q_i,p}(a_1, b_1, b_{q+1}, a_{q+1}) = \prod_{k=0}^{p-2} \prod_{j=0}^{q_i-1} \left( h(i(k - j) + \lambda) \right)^{-1} \]

\[ \sum_{a_2 \ldots a_{q_i}} \prod_{k=1}^{q_i} w^{q_i,1}(a_k, b_k, b_{k+1}, a_{k+1} | \lambda + i(k - 1)), \]  

(2.41)

again \( b_2 \ldots b_{q_i} \) are arbitrary numbers satisfying \( |b_i - b_{i+1}| = 1 \), and the pairs \( a_1, a_{q+1} \) and \( b_1, b_{q+1} \) satisfy (2.35), for \( P = q \). The fused weights satisfy the Yang–Baxter equation in the following form

\[ \sum_g w^{pq}(a, b, g, f | \lambda) w^{ps}(f, g, d, e | \lambda + \mu) w^{qs}(g, b, c, d | \mu) = \sum_g w^{qs}(f, a, g, e | \mu) w^{ps}(a, b, c, g | \lambda + \mu) w^{pq}(g, c, d, e | \lambda). \]  

(2.42)

Here we only need the explicit expressions for \( w^{q_i,1} \) which are

\[
\begin{align*}
 w^{q_i,1}(l + 1, l' + 1, l', l | \lambda) &= h^q_{q_i-1}(-\lambda) h_a \frac{h(i b - \lambda)}{h_l} \\
 w^{q_i,1}(l + 1, l' - 1, l', l | \lambda) &= h^q_{q_i-1}(-\lambda) h_b \frac{h(\lambda + ia)}{h_l} \\
 w^{q_i,1}(l - 1, l' + 1, l', l | \lambda) &= h^q_{q_i-1}(-\lambda) h_c \frac{h(id - \lambda)}{h_l} \\
 w^{q_i,1}(l - 1, l' - 1, l', l | \lambda) &= h^q_{q_i-1}(-\lambda) h_d \frac{h(ic - \lambda)}{h_l}
\end{align*}
\]

(2.43)
where
\[ a = \frac{l + l' - q_i}{2}, \quad b = \frac{l' - l + q_i}{2}, \quad c = \frac{l - l' + q_i}{2}, \quad d = \frac{l + l' + q_i}{2}, \] (2.44)
and
\[ h^q_k(\lambda) = \prod_{j=0}^{q-1} h(\lambda + i(k - j)). \] (2.45)

It is obvious that \( w^{q_{1}}(a, b, c, d|\lambda) \) are periodic functions, because they involve only simple hyperbolic functions (2.43), (2.40) \( h(\lambda + iv) = -h(\lambda), \nu = \frac{\pi}{\mu} \), i.e.
\[ w^{q_{1}}(a, b, c, d|\lambda + iv) = (-)^{q_{1}} w^{q_{1}}(a, b, c, d|\lambda). \] (2.46)

Now we can define the transfer matrix of the RSOS\((q_{1}, q_{2}; \rho)\) model
\[ T^{q_{1}, q_{2}; \rho}(a_{1}...a_{2N+1}) = \prod_{j=1}^{2N-1} w^{q, p}(a_{j}, a_{j+1}, b_{j+1}, b_{j}|\lambda) w^{q, p}(a_{j+1}, a_{j+2}, b_{j+2}, b_{j+1}|\lambda) \]
where we impose periodic boundary conditions, i.e. \( a_{2N+2} = a_{1} \) and \( b_{2N+2} = b_{1} \). By finding the eigenvalues of the transfer matrix we end up with the Bethe ansatz equations of the model, (see also [43, 35])
\[ \omega^{-2} e_{q}(\lambda_{\alpha} - \Theta) e_{q_{1}}(\lambda_{\alpha})^{N} e_{q_{2}}(\lambda_{\alpha})^{N} = -\prod_{\beta=1}^{M} e_{2}(\lambda_{\alpha} - \lambda_{\beta}) \] (2.48)

It is important to mention that the eigenstates of the model are states which satisfy (see also [43, 49, 35])
\[ M = \frac{1}{4}(q_{1} + q_{2})L + \frac{q}{2}. \] (2.49)

To formulate the thermodynamic Bethe ansatz for the RSOS\((q_{1}, q_{2}; \rho)\) model we consider all the strings (2.8) with length \( n = 1, \ldots, \nu - 2 \), [43, 35]. The densities of the corresponding holes and pseudo-particles are derived from the Bethe ansatz equations (2.48), after we insert all the allowed strings, and they satisfy
\[ \tilde{\rho}_{n}(\lambda) = \frac{1}{2}(Z_{nq_{1}}^{(\nu)}(\lambda) + Z_{nq_{2}}^{(\nu)}(\lambda)) + \frac{1}{L} Z_{nq}^{(\nu)}(\lambda - \Theta) - \sum_{m=1}^{\nu-2} A_{nm}^{(\nu)} \rho_{m}(\lambda). \] (2.50)
where \( L = 2N \). From the constraint (2.49) it follows that \( \tilde{\rho}_{\nu-2} = 0 \), then the density of the \( \nu - 2 \) string can be written in terms of the remaining densities
\[ \rho_{\nu-2}(\lambda) = \rho^{0}(\lambda) - \sum_{m=1}^{\nu-3} a_{\nu-2}^{(\nu-2)} \rho_{m}(\lambda) \] (2.51)
where \( a_{n}^{\nu-2} \) is given in the appendix (A.4) with \( \nu \to \nu - 2 \), and
\[ \rho^{0}(\omega) = \frac{\sinh(q_{1}^{\omega}) + \sinh(q_{2}^{\omega})}{4 \cosh(\frac{\omega}{2}) \sinh((\nu - 2)^{\omega})} + \frac{1}{L} \frac{\sinh(q_{1}^{\omega})}{2 \cosh(\frac{\omega}{2}) \sinh((\nu - 2)^{\omega})}. \] (2.52)
By means of the relation \(2.51\) the equation \(2.50\) can be rewritten in the following form
\[
\hat{\rho}_n(\lambda) = \frac{1}{2} (Z^{(\nu-2)}(\lambda) + Z^{(\nu-2)}(0)) + \frac{1}{L} Z^{(\nu-2)}(\lambda - \Theta) - \sum_{m=1}^{\nu-3} A^{(\nu-2)}_{nm} \ast \rho_m(\lambda),
\] (2.53)

\(Z^{(\nu-2)}, A^{(\nu-2)}\) are given by \(A.5\), \(A.6\), with \(\nu \to \nu - 2\). Following the standard procedure of minimizing the free energy of the system \((\delta f = 0, f\) is derived by \(2.16\)) we obtain the thermodynamic Bethe ansatz equations which are the same as in the bulk \(33\)

\[
T \ln \left(1 + \eta_n(\lambda)\right) = -\frac{1}{2} (Z^{(\nu-2)}(\lambda) + Z^{(\nu-2)}(0)) + \sum_{m=1}^{\nu-3} A^{(\nu-2)}_{nm} \ast T \ln \left(1 + \eta_{n-1}(\lambda)\right),
\] (2.54)

alternatively we can write
\[
\epsilon_n(\lambda) = s(\lambda) \ast T \ln(1 + \eta_{n+1}(\lambda))(1 + \eta_{n-1}(\lambda)) - \frac{1}{2} s(\lambda)(\delta_{nq_1} + \delta_{nq_2}).
\] (2.55)

The corresponding free energy is
\[
f(T) = e_0 + f_0 + f_b + O\left(\frac{1}{L}\right)
\] (2.56)

where \(e_0\) is given by \(2.21\) and \(f_0\) is the bulk part of the free energy given by \(2.20\). We wish to compute the non–trivial boundary part of the free energy, which is given as in the previous case —for the alternating spin chain— by \(2.22\). Again we can evaluate differences of free energies at \(T \to 0\) and \(T \to \infty\). For this purpose we need the solutions of \(1 + \eta_q\ (2.55)\), for \(T \to \infty\) and \(T \to 0\). For \(T \to \infty\), the main contribution in \(2.22\) comes for \(\lambda \to \infty\) \(43, 33\) the solution is
\[
(1 + \eta_n^\infty)^{-1} = \frac{\sin^2\left(\frac{\pi}{\nu}\right)}{\sin^2\left(\frac{\pi(n+1)}{\nu}\right)}, \quad n = 1, \ldots, \nu - 3.
\] (2.57)

and for \(T \to 0\) the main contribution to the free energy \(2.22\) comes for \(\lambda \to -\infty\), and the corresponding solution is \(35\)
\[
(1 + \eta_n^0)^{-1} = \frac{\sin^2\left(\frac{\pi}{q_1+2}\right)}{\sin^2\left(\frac{\pi(n+1)}{q_1+2}\right)}, \quad n = 1, \ldots, q_1 - 1, \quad (1 + \eta_{q_1}^0) = 1
\]
\[
(1 + \eta_n^0)^{-1} = \frac{\sin^2\left(\frac{\pi}{q_2-1+2}\right)}{\sin^2\left(\frac{\pi(n-q_2+1)}{q_2-1+2}\right)}, \quad n = q_1 + 1, \ldots, q_2 - 1, \quad (1 + \eta_{q_2}^0) = 1
\]
\[
(1 + \eta_n^0)^{-1} = \frac{\sin^2\left(\frac{\pi}{\nu-q_2}\right)}{\sin^2\left(\frac{\pi(n-q_2+1)}{\nu-q_2}\right)}, \quad n = q_2 + 1, \ldots, \nu - 3.
\] (2.58)

We observe that the solution at \(T \to \infty\) does not depend on \(q_i\), while at \(T \to 0\) the solution clearly depends on the values of \(q_i\). Having in mind the above solutions we are ready to derive ratios of \(g\) functions, in particular
\[
\ln \frac{g_\infty}{g_0} = \frac{1}{2} \ln \frac{(1 + \eta_q^\infty)}{(1 + \eta_q^0)}.
\] (2.59)
where \( 1 + \eta_q^{\infty,0} \) are given by \((2.57), (2.58)\).

It is worth making some remarks concerning \((2.57), (2.58), \) and \((2.59)\). In the special case where \( q_1 = q_2 = 1 \) we recover the results of \([9]\) for boundary flows in minimal models \( \mathcal{M}_{\nu=m+1} \), \((a = q - 1)\), whereas for \( q_1 = q_2 > 1 \) we recover the results of \([10]\) for boundary flows in generalized \( SU(2) \) coset models \( \mathcal{M}(q_1, \nu - q_1 - 2) \equiv \mathcal{M}(k, l) \). The corresponding \( S \) matrix for the generalized coset \( \mathcal{M}(k, l) \) model conjectured in \([10]\) is given by

\[
S = S_{RSOS}^{(k)} \otimes S_{RSOS}^{(l)}, \tag{2.60}
\]

and in the limit \( l \to \infty \) it reduces to \( S_{WZW} \) given in \((2.53)\). Again from \((2.57), (2.58), \) and \((2.59)\) the structure \( \mathcal{M}(q_1, \nu - q_1 - 2) \otimes \mathcal{M}(q_1, \delta q) \) of the effective conformal field theory is manifest (compare with the boundary flow in \( SU(2) \) coset models \([10]\)).

### 3 Open Boundaries

The main aim of this section is the investigation of the thermodynamics of the alternating spin chain in the presence of integrable boundaries. To construct the spin chain with boundaries in addition to the \( R \) matrix another constructing element, the \( K \) matrix, is needed. The \( K \) matrix is a solution of the reflection (boundary Yang–Baxter) equation \([3]\),

\[
R_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) K_2(\lambda_2) = K_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2). \tag{3.1}
\]

In what follows we are going to use Sklyanin’s formalism \([11]\) in order to construct the model with boundaries. The corresponding transfer matrix \( t(\lambda) \) for the open alternating chain of \( N \) sites and \( S_1 = \frac{q_1}{2}, S_2 = \frac{q_2}{2} \) spins is (see also e.g., \([11,34]\)),

\[
t(\lambda) = tr_0 K_0^+(\lambda) T_0(\lambda) K_0^-(\lambda) \hat{T}_0(\lambda), \tag{3.2}
\]

where\(^4\)

\[
T_0(\lambda) = R_{0N}^1(\lambda) R_{0N-1}^2(\lambda) \cdots R_{02}^1(\lambda) R_{01}^2(\lambda), \quad \hat{T}_0(\lambda) = R_{10}^2(\lambda) R_{20}^1(\lambda) \cdots R_{N-10}^2(\lambda) R_{N0}^1(\lambda), \tag{3.3}
\]

and \( K^+(\lambda, \xi^+, \kappa^+) \equiv K^-(\lambda, \xi^-, \kappa^-) \equiv K^-(\lambda - i, \xi^-, \kappa^-) \) where \( \xi^\pm, \kappa^\pm \) are arbitrary boundary parameters for the left and right boundaries, and \( K^- \) is the matrix \([13,14]\)

\[
K^-(\lambda) = \begin{pmatrix}
\sinh \mu (\lambda + i \xi) & \kappa \sinh \mu 2\lambda \\
\kappa \sinh \mu 2\lambda & \sinh \mu (-\lambda + i \xi)
\end{pmatrix}. \tag{3.4}
\]

It is interesting to point out that the diagonal boundaries for the critical \( XXZ \) spin chain \((S_1 = S_2 = \frac{1}{2})\) correspond to Dirichlet boundary conditions for the sine–Gordon model \([25,26]\). Presumably the purely anti–diagonal \( K \) matrix—which completely breaks the \( U(1) \) symmetry—should correspond to Neumann boundary conditions for the sine–Gordon model.

\(^4\)We could have considered \( N=\text{odd} \) and put an impurity at the \( N \) site of the chain. Then we would have an impurity contribution to the free energy similar to the one in paragraph 2.
3.1 The diagonal $K$ matrix

We consider first the case in which the $K$ matrix is diagonal, namely $\kappa = 0$. The corresponding Bethe ansatz for the model, are known \cite{41} and they are given by

$$e^x(\lambda_\alpha)^{-1}e^{-x}(\lambda_\alpha)e_1(\lambda_\alpha)e_{q_1}(\lambda_\alpha)^{N}e_{q_2}(\lambda_\alpha)^{N} = - \prod_{\beta=1}^{M} e_2(\lambda_\alpha - \lambda_\beta)e_2(\lambda_\alpha + \lambda_\beta)$$

(3.5)

where $x^\pm = 2\xi^\pm \pm 1$ are boundary parameters for the left right boundaries of the chain and they are related to some external magnetic field acting on the boundaries. Here, for simplicity we consider $x^\pm$ to be integers less or equal to $\nu - 1$. The corresponding densities of pseudo–particles and holes, in analogy to (2.12) satisfy (again we consider as in the bulk all the strings $n = 1, \ldots, \nu - 1$ and the negative parity string (2.8))

$$\tilde{\rho}_n(\lambda) = Z_{q_1}(\lambda) + Z_{q_2}(\lambda) + \frac{1}{L}\hat{K}_n - \sum_{m=1}^{\nu-1} A_{nm} * \rho_m(\lambda) - B_{1n} * \rho_0(\lambda)$$

$$- (\rho_0(\lambda) + \tilde{\rho}_0(\lambda)) = b_{q_1} + b_{q_2} + \frac{1}{L}\hat{K}_0 - \sum_{m=1}^{\nu-1} B_{1m} * \rho_m(\lambda) - a_2 * \rho_0(\lambda)$$

(3.6)

where $L = N$ is the length of the chain and,

$$\hat{K}_n(\omega) = \hat{a}_n(\omega) + \hat{b}_n(\omega) - \hat{Z}_{nx+}(\omega) + \hat{Z}_{nx-}(\omega) - 1$$

$$\hat{K}_0(\omega) = \hat{a}_1(\omega) + \hat{b}_1(\omega) - \hat{b}_x+(\omega) + \hat{b}_x-(\omega) + 1$$

(3.7)

$\hat{Z}_{nm}, \hat{b}_n,$ and $\hat{a}_n$ are given in the appendix by (A.5), and (A.4). The unit that appears in the expressions for $\hat{K}_n, \hat{K}_0$ is a result of the subtraction of a $\delta(\lambda)$ term from the densities (see also \cite{25, 27}). This subtraction seems necessary for the accurate derivation of the density, because in the boundary case $\lambda$'s can take values from 0 to $\infty$, as opposed to the bulk case where $\lambda$'s take all the values from $-\infty$ to $\infty$.

The free energy for the model with open boundaries is given by (2.16) up to a $\frac{1}{4}$ factor in front of the expression. The appearance of the factor $\frac{1}{2}$ in front of all the integrals comes from the fact that we originally derived the integrals from zero to infinity. After we minimize the free energy, we obtain the same thermodynamic Bethe ansatz equations as in (2.17), (2.18). Finally, by virtue of (2.17), (2.18) the following expression for the free energy is obtained

$$f = f_0 + f_b + O(\frac{1}{L})$$

(3.8)

where $f_0$ is the bulk free energy given by

$$f_0 = e_0 - T \int_{-\infty}^{\infty} d\lambda s(\lambda) \ln(1 + \eta_{q_1}(\lambda))(1 + \eta_{q_2}(\lambda))$$

(3.9)

$e_0$ is the bulk part of the energy of the state with the seas of strings $q_1, q_2$ filled,

$$e_0 = - \frac{1}{4} \sum_{i,j=1}^{2} \int_{-\infty}^{\infty} d\lambda Z_{q_i q_j}(\lambda)s(\lambda).$$

(3.10)
and the boundary contribution to the free energy is

\[
 f_b = -\frac{T}{2L} \sum_{\nu=1}^{\nu-1} \int_{-\infty}^{\infty} d\lambda K_n(\lambda) \ln(1 + \eta_i^{-1}(\lambda)) + \frac{T}{2L} \int_{-\infty}^{\infty} d\lambda K_0(\lambda) \ln(1 + \eta_0^{-1}(\lambda)). \tag{3.11}
\]

From the thermodynamic Bethe ansatz equations (2.17) we consider the convolution of \( s(\lambda) \) with the equation of (2.17) for \( n = 1 \) and \( n = x^\pm \) (with a (-) sign in front of the equation for \( n = x^+ \)), we also consider the convolution of \( s(\lambda) \) with the equation for the negative parity string, recall also relations (2.14). Moreover, we consider the equations (2.18) for all \( n = 1, \ldots, \nu - 1 \) and the negative parity string, with a (-) sign infront. We add all the above equations and we end up with the following expression for the free energy (at \( T \to 0, \infty \))

\[
 f_b = e_b + \delta f_b = e_b + \frac{T}{L} \ln(1 + \eta_{\nu-1}) + \frac{T}{2L} s \ln(1 + \eta_{x^+}) - \frac{T}{2L} s \ln(1 + \eta_{x^-}) \tag{3.12}
\]

where \( e_b \) is the boundary energy contribution

\[
 e_b = \frac{1}{4L} \int_{-\infty}^{\infty} d\lambda s(\lambda) \left( K_{q_1}(\lambda) + K_{q_2}(\lambda) \right)
 = \frac{1}{2L} \int_{-\infty}^{\infty} d\lambda s(\lambda) \left( \frac{1}{2} \sum_{i=1}^{2} (a_{q_i}(\lambda) + b_{q_i}(\lambda) - Z_{0,x^+} + Z_{0,x^-}) - \delta(\lambda) \right). \tag{3.13}
\]

The boundary contribution to the free energy for each boundary then is

\[
 \delta f_b^\pm = \frac{T}{2L} \ln(1 + \eta_{\nu-1}) \pm \frac{T}{4L} \ln(1 + \eta_{x^\pm}), \tag{3.14}
\]

and the corresponding ratios of the \( g^+ \), \( g^- \) (\( g = g^+ g^- \)) functions for the left and right boundaries (1.2) are

\[
 \ln \frac{g_{\pm}^S}{g_0^S} = -\frac{1}{2} \ln \frac{1 + \eta_{\nu-1}^\pm}{1 + \eta_0^\pm} + \frac{1}{4} \ln \frac{1 + \eta_{x^\pm}^\pm}{1 + \eta_0^\pm}. \tag{3.15}
\]

The last term of the above equation, for \( x^\pm \) integer less than \( \nu - 1 \), corresponds to the quantum impurity contribution (up to a \(-\frac{1}{2}\) factor), which has been already computed explicitly for \( T \to \infty, 0 \), (2.23), (2.26). We focus on the first term of (3.15), which we compute again for \( T \to \infty, 0 \). It is easy to conclude from the solution for \( T \to \infty, 0 \) (2.23), (2.26)

\[
 -\frac{1}{2} \ln \frac{1 + \eta_{\nu-1}^\pm}{1 + \eta_0^\pm} = -\frac{1}{2} \ln \frac{\nu}{\nu - q_2}. \tag{3.16}
\]

The result (3.16) is also valid for the spin \( S = S_1 = S_2 \) chain. In the isotropic limit the above ratio (3.16) becomes unit, i.e. there is no difference in the boundary free energy at low and high temperature. In the special case where \( S_1 = S_2 = \frac{1}{2} \) we recover the result found in [26] for the sine–Gordon model with boundaries \(^5\), and also the result of [29] for the XXZ spin chain with open boundaries at zero external magnetic field. Note that for \( x^\pm \to \infty \) there is no boundary parameter dependence.

\(^5\) In the repulsive regime, i.e. when no bound states “breathers” exist, we can make the following identification

\[
 \frac{\mu}{\nu - 1} = \lambda + 1 \quad \text{or} \quad \beta^2 = 8(\pi - \mu) \quad [20, 52].
\]
3.2 The non–diagonal $K$ matrix

We consider, for the first time in a spin chain model, the more general case of the open alternating spin chain with non–diagonal boundaries. The corresponding Bethe ansatz for the XXZ model at roots of unity have been recently derived in [51]. The novelty of the method described in [51] is basically that it does not rely on the existence of a reference state “pseudo–vacuum”. It was shown then in [51], that in the case where $\mu = \frac{\pi}{p+1}$ ($\nu = p + 1$) the problem of finding the eigenvalues of the transfer matrix (3.2) reduces to a set of functional relations which can be written in the following compact form, (see also [43, 35, 51])

\[ \text{det} M[\Lambda^{q_1,q_2}(\lambda)] = 0 \]  \hspace{1cm} (3.17)

where

\[ M[\Lambda^{q_1,q_2}(\lambda)] = \begin{pmatrix}
\Lambda_0^{q_1,q_2} & -\tilde{f}_1^{q_1,q_2} & 0 & 0 & \ldots & 0 & 0 & -f_0^{q_1,q_2} \\
-f_1^{q_1,q_2} & \Lambda_1^{q_1,q_2} & -\tilde{f}_2^{q_1,q_2} & 0 & \ldots & 0 & 0 & 0 \\
0 & -f_2^{q_1,q_2} & \Lambda_2^{q_1,q_2} & -\tilde{f}_1^{q_1,q_2} & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -f_{p-1}^{q_1,q_2} & \Lambda_{p-1}^{q_1,q_2} & -\tilde{f}_{p-2}^{q_1,q_2} \\
-f_{p-1}^{q_1,q_2} & 0 & 0 & 0 & \ldots & 0 & -f_p^{q_1,q_2} & \Lambda_p^{q_1,q_2}
\end{pmatrix}, \hspace{1cm} (3.18)

where

\[ f^{q_1,q_2}(\lambda) = -\sinh^N \mu(\lambda + iS_1 + \frac{i}{2}) \sinh^N \mu(\lambda + iS_2 + \frac{i}{2}) \frac{\sinh \mu(2\lambda + 2\xi i)}{\sinh \mu(2\lambda + i)} \]  \hspace{1cm} (3.19)

and

\[ \tilde{f}_{q_1,q_2}(\lambda) = f^{q_1,q_2}(-\lambda - 2i), \quad f_k^{q_1,q_2}(\lambda) = f(\lambda + ik) \]  \hspace{1cm} (3.20)

Let now $(Q_0(\lambda), \ldots, Q_p(\lambda))$ be the null vector of the matrix (3.18) with $Q_k(\lambda) = Q(\lambda + ik)$ and

\[ Q(\lambda) = \prod_{j=1}^{M} \sinh \mu(\lambda - \lambda_j) \sinh \mu(\lambda + \lambda_j + i) \]  \hspace{1cm} (3.21)

then the eigenvalues are given by the following expression

\[ \Lambda^{q_1,q_2}(\lambda) = f_0^{q_1,q_2}(-\lambda - i) \frac{Q(\lambda + i)}{Q(\lambda)} + f_0^{q_1,q_2}(\lambda) \frac{Q(\lambda - i)}{Q(\lambda)}. \]  \hspace{1cm} (3.22)

Finally, from the analyticity of the eigenvalues we obtain the Bethe ansatz equations

\[ \frac{\sinh \mu(\lambda_\alpha + i(2\xi - 1))}{\sinh \mu(\lambda_\alpha - i(2\xi + 1)) + \kappa^2 \sinh^2 \mu(2\lambda_\alpha - i)} \sinh \mu(\lambda_\alpha - \frac{i}{2}(2\xi - 1)) \sinh \mu(\lambda_\alpha + \frac{i}{2}(2\xi + 1)) + \kappa^2 \sinh^2 \mu(2\lambda_\alpha + i) \] \[ g_1(\lambda_\alpha) e_1(\lambda_\alpha) e_{q_1}(\lambda_\alpha)^N e_{q_2}(\lambda_\alpha)^N = - \prod_{\beta=1}^{M} e_2(\lambda_\alpha - \lambda_\beta) e_2(\lambda_\alpha + \lambda_\beta). \]  \hspace{1cm} (3.23)
Note that the boundary parameters at the left and right boundaries have been considered to be the same. In fact, the boundary parameters have to be tuned properly (i.e. $\xi^- = \xi^+$ and $\kappa^- = \kappa^+$) so that the method described in [51] can be applied. Note also that in this case the Bethe ansatz states have to satisfy the following constraint,

$$M = (S_1 + S_2) \frac{N}{2} - \frac{1}{2},$$

(3.24)

The above constraint follows from the asymptotic behavior of the transfer matrix for $\lambda \to \infty$, in particular from (3.2) it can be deduced

$$t(\lambda \to \infty) \sim -2e^{(2\mu N + 4\mu)\lambda + 2i\mu + i\mu N}$$

(3.25)

by comparing the later equation with the eigenvalues (3.22) we obtain the constraint (3.24).

Let us first consider the case with purely anti–diagonal $K$ matrices, namely $\kappa$ becomes very big. Then it is obvious that the above Bethe ansatz equations (3.23) take the form

$$g_1(\lambda_{\alpha})^{-2} e_1(\lambda_{\alpha})^{-2} g_1(\lambda_{\alpha}) e_{q_1}(\lambda_{\alpha})^N e_{q_2}(\lambda_{\alpha})^N = -\prod_{\beta=1}^M e_2(\lambda_{\alpha} - \lambda_{\beta}) e_2(\lambda_{\alpha} + \lambda_{\beta}).$$

(3.26)

Relation (3.24) does not impose in this case any extra restriction on the densities as opposed to the case of the $RSOS$ model for which a similar relation (2.49) holds true, and it imposes restrictions on the hole densities (i.e. $\tilde{\rho}_0 = 0$). Therefore, the thermodynamic Bethe ansatz equations are the same as in the bulk case (2.17), and by following exactly the same procedure as in the diagonal case, but now with

$$\hat{K}_n(\omega) = -\hat{b}_n(\omega) - \hat{a}_n(\omega) - 1, \quad \hat{K}_0(\omega) = -\hat{b}_1(\omega) - \hat{a}_1(\omega) + 1$$

(3.27)

we find that the non–trivial boundary part of the free energy becomes (at $T \to 0, \infty$)

$$\delta f_b^\pm = \frac{T}{4L} \ln(1 + \eta_1).$$

(3.28)

Moreover the boundary contribution to the ground state energy is given by

$$e_b = -\frac{1}{2L} \int_{-\infty}^{\infty} d\lambda s(\lambda) \left(\frac{1}{2} \sum_{i=1}^2 (-a_{q_1}(\lambda) - b_{q_1}(\lambda)) - \delta(\lambda)\right).$$

(3.29)

It is worth noticing that the state with the seas of strings with length $q_1, q_2$ filled, is an allowed state for the case with anti–diagonal boundaries, because the constraint (3.24) is satisfied. The ratios of the $g^+, g^-$ functions for the left and right boundary (112) are given by

$$\ln \frac{g_n^+}{g_n^-} = -\frac{1}{4} \ln \left(\frac{1 + \eta_1^{\infty}}{1 + \eta_1^0}\right)$$

(3.30)

where the function $(1 + \eta_1^{0,\infty})^{-1}$ is given, for $T \to \infty$ and $T \to 0$, by (2.23), (2.26). Notice that for $q_i = 1$ (at $T \to 0$) $\delta f_b^\pm \propto T^2$ and this is —as in the presence of an impurity at the end of the chain— the completely screened case.
We are not going to treat the general case in full detail here, since this is mostly a problem of mathematical manipulations, he hope though to report on this in detail elsewhere. Let us however present the general concept of such computation for the special simple case where $\xi = 0$, the so called “free boundary conditions” $[6]$, then the Bethe ansatz equations become

$$e_{2\zeta+1}(\lambda_{\alpha})^{-1}e_{2\zeta-1}(\lambda_{\alpha})e_1^2(\lambda_{\alpha})e_1(\lambda_{\alpha})g_1(\lambda_{\alpha})e_{q_1}(\lambda_{\alpha})^N = - \prod_{\beta=1}^M e_{2}(\lambda_{\alpha} - \lambda_{\beta})e_{2}(\lambda_{\alpha} + \lambda_{\beta}).$$

(3.31)

It is evident that the later Bethe ansatz equations have a similar structure with $\xi^+ = \xi^- = \zeta$ and $\cosh^2 \mu i \zeta = -\frac{1}{4\xi^2}$. The boundary part of the free energy is given again by (3.11) and the quantities $K_a$ and $K_\beta$ are given by,

$$\hat{K}_a(\omega) = -\hat{a}_n(\omega) + \hat{b}_n(\omega) - \hat{Z}_{nx^+}(\omega) + \hat{Z}_{nx^-}(\omega) - 1$$

$$\hat{K}_\beta(\omega) = \hat{a}_1(\omega) - \hat{b}_1(\omega) - \hat{b}_{x^+}(\omega) + \hat{b}_{x^-}(\omega) + 1$$

(3.32)

with $x^\pm = 2\zeta \pm 1$. It remains to treat expression (3.11) at high and low temperature, but this is relatively easy to do because the solutions for $\eta_n$, $\eta_0$ are known at $T \to \infty$ and $T \to 0$ $[2.23]$, $[2.26]$. Basically we have to focus on the part of expression (3.11) that involves the terms $\pm Z_{nx^\pm}$, $\pm b_{x^\pm}$ for $x^\pm$ non–integer. The remaining $x^\pm$ independent part of the boundary free energy (3.11), denoted as $f_b^{(1)}$, can be easily computed by repeating the steps of the previous section, and it is given by (at $T \to 0, \infty$)

$$f_b^{(1)} = -\frac{1}{2L} \int_{-\infty}^{\infty} d\lambda s(\lambda) \left( \frac{1}{2} \sum_{i=1}^{\nu-1} (-a_{q_i}(\lambda) + b_{q_i}(\lambda)) - \delta(\lambda) \right) + \frac{T}{L} \ln(1 + \eta_{\nu-1}) + \frac{T}{2L} \ln(1 + \eta_1).$$

(3.33)

The $x^\pm$ dependent part of the boundary free energy, denoted as $f_b^{(2)}$ (with terms that involve $\pm Z_{nx^\pm}$, $\pm b_{x^\pm}$), which needs to be computed explicitly for $x^\pm$ non–integers, has the form

$$f_b^{(2)} = -\frac{T}{2L} \sum_{n=1}^{\nu-1} \int_{-\infty}^{\infty} d\lambda (Z_{nx^-}(\lambda) - Z_{nx^+}(\lambda)) \ln(1 + \eta_n^{-1}(\lambda))$$

$$+ \frac{T}{2L} \int_{-\infty}^{\infty} d\lambda (b_{x^-}(\lambda) - b_{x^+}(\lambda)) \ln(1 + \eta_0^{-1}(\lambda)).$$

(3.34)

Again once we make the shift $\lambda \to \lambda - \frac{1}{\pi} \ln T$ we can compute $f_b^{(2)}$ analytically, at $T \to 0, \infty$ having in mind the expressions for $Z_{nm}$, $b_n$ $[A.2]$, $[A.3]$, and $\eta_n$, $\eta_0$ at high and low temperatures $[2.23]$, $[2.26]$. More specifically, expression (3.34) can be written as $f_b^{(2)} = f_b^{(2)}(x^-) + f_b^{(2)}(x^+)$ where at $T \to 0, \infty$,

$$f_b^{(2)}(x^\pm) = \pm \frac{T}{2L} \sum_{n=1}^{\nu-1} Z_{nx^\pm}(0) \ln(1 + \eta_n^{-1}) \mp \frac{T}{2L} \hat{b}_{x^\pm}(0) \ln(1 + \eta_0^{-1}).$$

(3.35)

Naturally, the case is similar when diagonal boundaries with $x^\pm = 2\zeta \pm 1$ non–integer, are considered, i.e. the $x^\pm$ dependent part of the free energy (3.11) is given again by (3.35). Eventually the

$^6$Following $[6]$ we have to chose for the “free boundary conditions” $\zeta = \frac{1}{2} \alpha^2$, which is not an integer.
computation of the expression \((3.35)\) reduces to a simpler problem, which is the derivation of the exact Fourier transforms of \(Z_{nx}\) and \(b_x\) when \(x\) is non–integer. In particular, expression \((3.4)\) for \(b_x\) is valid as long as \(x < \nu\), while expression \((3.5)\) for \(Z_{nx}\) is also valid as long as \(x > n\).

Let us point out that the results of this section are novel not only for the fused model under study, but also for the case \(q_1 = q_2 = 1\), which corresponds to the XXZ model (sine–Gordon model).

4 Discussion

In this investigation we focused on the “boundary properties” of the alternating spin chain and the \(RSOS(q_1, q_2; p)\) model, therefore we considered both models with certain integrable “boundaries”. A quantum impurity was added at the last site of the chain (“Kondo type” boundary see also [11] [12]), and the immediate result was a non–trivial contribution to the ground state energy of the system, as well as in the free energy. We were able to explicitly evaluate the non–trivial contribution to the boundary free energy at low and high temperature. A similar investigation was realized for the \(RSOS(q_1, q_2; p)\) model (see also [13]) and the obtained results, for special values of \(q_1, q_2\), compared with the known boundary flows in unitary minimal models [19] and generalized \(SU(2)\) coset theories [10]. For general values of \(q_1, q_2\) our results are rather novel for both the alternating spin chain and the \(RSOS(q_1, q_2; p)\) model.

Furthermore, the alternating chain with diagonal and non–diagonal boundaries was investigated, and again the presence of the boundaries resulted in a non–trivial contribution to the free energy. This contribution gave rise to the \(g\) function (“ground state degeneracy”) along the lines described in [2]. We were able to derive ratios of the \(g\) function for the left and right boundaries, at low and high temperatures. In the diagonal case for \(S_1 = S_2 = \frac{1}{2}\) the boundary parameter independent part of the ratio of the \(g\) functions coincides with the one found in [20] and [29] for zero external magnetic field. We believe the results of this section are also novel, in particular for the anti–diagonal case our results are rather new even for \(q_1 = q_2 = 1\).

For completeness the thermodynamics of the \(RSOS(q_1, q_2; p)\) model with open boundaries [53, 54, 55] should be also investigated. It would be also of great interest to extend the thermodynamic analysis for spin chains and \(RSOS\) models related to higher rank algebras. Furthermore, the derivation of the Bethe ansatz equations for non–diagonal boundaries [51], is an essential step towards the investigation of exact non–diagonal boundary \(S\) matrices in the spin chain framework. Let us finally note that a new non–trivial “dynamical” solution of the reflection equation has been recently found [56], and it has been realized in the context of the sine–Gordon model. It is a challenging problem to apply this “dynamical” solution to the XXZ spin chain at roots of unity, and investigate the thermodynamics and the scattering process in this framework. We hope to address these questions soon in a future work [57].
5 Acknowledgments

I am grateful to F. Ravanini for helpful discussions. This work was supported by the TMR Network “EUCLID”; “Integrable models and applications: from strings to condensed matter”, contract number HPRN–CT–2002–00325.

A Appendix

We give the explicit expressions of the functions that appear in the Bethe ansatz equations once we apply the string hypothesis, namely

\[ X_{nm}(\lambda) = e_{|n-m+1|}(\lambda)e_{|n-m+3|}(\lambda) \ldots e_{(n+m-3)}(\lambda)e_{(n+m-1)}(\lambda) \]
\[ E_{nm}(\lambda) = e_{|n-m|}(\lambda)e_{|n-m+2|}(\lambda) \ldots e_{(n+m-2)}^2(\lambda)e_{(n+m)}(\lambda) \]
\[ G_{nm}(\lambda) = g_{|n-m|}(\lambda)g_{|n-m+2|}^2(\lambda) \ldots g_{(n+m-2)}^2(\lambda)g_{(n+m)}(\lambda). \] (A.1)

Moreover,

\[ a_n(\lambda) = \frac{i}{2\pi} \frac{d}{d\lambda} \ln e_n(\lambda), \quad b_n(\lambda) = \frac{i}{2\pi} \frac{d}{d\lambda} \ln g_n(\lambda), \] (A.2)

\[ (Z_{nm}(\lambda), A_{nm}(\lambda), B_{nm}(\lambda)) = \frac{i}{2\pi} \frac{d}{d\lambda} \ln(X_{nm}(\lambda), E_{nm}(\lambda), G_{nm}(\lambda)). \] (A.3)

We finally give the following useful Fourier transforms

\[ \hat{a}_n(\omega) = \frac{\sinh((\nu - n)\frac{\omega}{2})}{\sinh(\frac{m\omega}{2})}, \quad n < 2\nu, \quad \hat{b}_n(\omega) = -\frac{\sinh(m\frac{\omega}{2})}{\sinh(\frac{\omega}{2})}, \quad n < \nu, \] (A.4)

\[ \hat{Z}_{nm}(\omega) = \frac{\sinh((\nu - \max(n, m))\frac{\omega}{2}) \sinh((\min(n, m))\frac{\omega}{2})}{\sinh(\frac{m\omega}{2}) \sinh(\frac{\omega}{2})} \] (A.5)

\[ \hat{A}_{nm}(\omega) = \frac{2 \coth(\frac{\omega}{2}) \sinh((\nu - \max(n, m))\frac{\omega}{2}) \sinh((\min(n, m))\frac{\omega}{2})}{\sinh(\frac{m\omega}{2})} \] (A.6)

\[ \hat{B}_{nm}(\omega) = -\frac{2 \coth(\frac{\omega}{2}) \sinh(m\frac{\omega}{2}) \sinh(\frac{m\omega}{2})}{\sinh(\frac{\omega}{2})}. \] (A.7)

References

[1] J.L. Cardy, Nucl. Phys. B234 (1989) 581.

[2] I. Affleck and A.W.W. Ludvig, Phys. Rev. Lett. 67 (1991) 161; I. Affleck, M. Oshikawa and H. Saleur, cond–matt/9804117 (1998).
[3] I.V. Cherednik, Theor. Math. Phys. 61 (1984) 977.

[4] E.K. Sklyanin, J. Phys. A21 (1988) 2375; P.P. Kulish and E.K. Sklyanin, J. Phys. A24 (1991) L435.

[5] A. Fring and L. Köberle, Nucl. Phys. B421 (1994) 159; Nucl. Phys. B419 (1994) 647.

[6] S. Ghoshal and A. B. Zamolodchikov, Int. J. Mod. Phys. A9 (1994) 3841; A9 (1994) 4353.

[7] H.J. de Vega and A. González-Ruiz, J. Phys. A26 (1993) L519.

[8] C.L. Kane and M.P.A. Fisher, Phys. Rev. Lett. 68 (1992) 1220; E. Sorensen, S. Eggert and I. Affleck, J. Phys. A26 (1993) 6757.

[9] F. Lesage, H. Saleur and P. Simonetti, Phys.Lett. B427 (1998) 85.

[10] C. Ahn and C. Rim, J. Phys. A32 (1999) 2509.

[11] V.M. Filyov, A.M. Tsvelik amd P.B. Wiegmann, Phys. Lett. 81A (1981) 175; A.M. Tsvelick and P.B. Wiegmann, Adv. in Phys. 32 (1983) 453.

[12] N. Andrei and C. Destri, Phys. Rev. Lett. 52 (1984) 364.

[13] Y. Wang, Phys. Rev. B60 (1999) 9236; Y. Wang and P. Schlottmann, Phys. Rev. B62 (2000) 3845; A. P. Tonel, A. Foerster, X.–W. Guan and J. Links, cond–mat/0112115.

[14] C.N. Yang and C.P. Yang, Phys. Rev. 150 (1966) 327; J. Math. Phys. 10 (1969) 1115.

[15] C.P. Yang, Phys. Rev. A2 (1970) 154.

[16] M. Gaudin, Phys. Rev. Lett. 26 (1971) 1301.

[17] M. Takahashi, Prog. Theor. Phys. 46 (1971) 401; M. Takahashi and M. Suzuki, Prog. Theor. Phys. 48 (1972) 2187; M. Takahashi, Thermodynamics of One–Dimensional Solvable Models (Cambridge University Press, 1999).

[18] J.D. Johnson and B.M. McCoy, Phys. Rev. A6 (1972) 1613.

[19] H. Babujian, Nucl. Phys. B215 (1983) 317; H. Babujian and A. Tseli, Nucl. Phys. B265 (1986) 24.

[20] L. Mezincescu and R.I. Nepomechie, UMTG–170 (1992).

[21] A.B. Zamolodchikov, Nucl. Phys. B342 (1990) 695; A.B. Zamolodchikov, Phys. Lett. B253 (1991) 391.

[22] A.B. Zamolodchikov, Nucl. Phys. B358 (1991) 497, 524; Nucl. Phys. B366 (1991) 122.

[23] T.R. Klassen and E. Melzer, Nucl. Phys. B338 (1990) 485.
[24] C. Destri, H.J. de Vega, Nucl. Phys. B438 (1995).

[25] A. LeClair, G. Mussardo, H. Saleur and S. Skorik, Nucl. Phys. B453 (1995) 581.

[26] P. Fendley, H. Saleur and N.P. Warner, Nucl. Phys. B430 (1994) 577.

[27] P. Dorey, I. Runkel, R. Tateo and G. Watts, Nucl. Phys. B578 (2000) 85.

[28] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel, J. Phys. A20 (1987) 6397.

[29] P. de Sa and A. Tsvelik, cond-matt/9503031.

[30] A.M. Polyakov, J.E.T.P. Lett. 12 (1970) 381.

[31] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, J. Stat. Phys. 34 (1984) 763; Nucl. Phys. B241 (1984) 333.

[32] H.W.J. Blöte, J.L. Cardy, M.P. Nightingale, Phys. Rev. Lett. 56 (1986) 742; J.L. Cardy, Nucl. Phys. B270 (1986) 186.

[33] I. Affleck, Phys. Rev. Lett. 56 (1986) 746.

[34] H.J. de Vega and F. Woyanorovich, J. Phys. A25 (1992) 4499.

[35] A. Doikou, J. Phys. A36 (2003) 329.

[36] R.J. Baxter, Ann. Phys. 70 (1972) 193; J. Stat. Phys. 8 (1973) 25; Exactly Solved Models in Statistical Mechanics (Academic Press, 1982)

[37] V.E. Korepin, Theor. Math. Phys. 76 (1980) 165; V.E. Korepin, G. Izergin and N.M. Bogoliubov, Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz (Cambridge University Press, 1993).

[38] A. Kirillov and N.Yu Reshetikhin, J. Sov. Math 35 (1986) 2621; A. Kirillov and N.Yu Reshetikhin, J. Phys. A20 (1987) 1565.

[39] L.D. Faddeev and L.A. Takhtajan, Russ. Math. Surv. 34, 11 (1979); L.D. Faddeev and L.A. Takhtajan, J. Sov. Math. 24 (1984) 241.

[40] L.A. Takhtajan, Phys. Lett. A87 (1982) 479.

[41] A. Doikou and A. Babichenko, Phys. Lett B515 (2001) 220; A. Doikou, Nucl. Phys. B634 (2002) 591.

[42] A. Bytsko and A. Doikou, in preparation.

[43] V.V. Bazhanov and N.Yu. Reshetikhin, Int. J. Mod. Phys. A4 (1989) 115.
[44] S.R. Aladim and M.J. Martins, J. Phys. A26 (1993) 7287.
[45] P. Fendley, cond–mat/9304031
[46] D. Bernard, Phys. Lett. B279 (1992) 78.
[47] N.Yu. Reshetikhin. J. Phys. A24 (1991) 3299.
[48] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193.
[49] N.Yu. Reshetikhin and H. Saleur, Nucl. Phys. B419 (1994) 507.
[50] E. Date, M. Jimbo, T. Miwa and M. Okado, Lett. Math. Phys. 12 (1986) 209.
[51] R.I. Nepomechie, hep–th/0211001
[52] A. Doikou and R.I. Nepomechie, J. Phys. A32 (1999) 3663.
[53] C. Ahn and W.M. Koo, hep–th/9708080, J. Phys. A29 (1996) 5845.
[54] R.E. Behrend, P.A. Pearce and D.L. O’Brien, J. Stat. Phys. 84 (1996) 1.
[55] M.T. Batchelor, V. Fridkin, A. Kuniba and Y.K. Zhou, Phys. Lett B735 (1996) 266.
[56] P. Baseilhac and K. Koizumi, Nucl. Phys. B649 (2003) 491.
[57] P. Baseilhac, A. Doikou and K. Koizumi, in preparation.