The split case of the Prasad–Takloo-Bighash conjecture for cuspidal representations of level zero

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Abstract

Let $E/F$ be a quadratic extension of non archimedean local fields of odd residual characteristic. We prove a conjecture of Prasad and Takloo-Bighash, in the case of cuspidal representations of depth zero of $GL(2m,F)$. This conjecture characterizes distinction for the pair $(GL(2m,F),GL(m,E))$ with respect to a character $\mu \circ \det$ of $GL(m,E)$, in terms of certain conditions on Langlands parameters, including an epsilon value. We also compute the multiplicity of the involved equivariant linear forms when $E/F$ is unramified, and also when $\mu$ is tame. In both cases this multiplicity is at most one.

Introduction

Let $E/F$ be a quadratic extension of non archimedean local fields. Let $D$ be an $F$-division algebra of dimension $d^2$ and $n$ be a positive integer such that $nd$ is even. Set $M=M(n,D)$, so that $E$ embeds into $M$ uniquely up to inner automorphism. Set $C_E(M)$ to be the centralizer of $E$ in $M$, it is an $E$-central simple algebra. Let $G=M\times H$ and $\mu:E^\times \to \mathbb{C}^\times$ a smooth character, we denote by $\mu_E$ of the character $H$ obtained by composing $\mu$ with the reduced norm on $H$. This paper is concerned with the following conjecture:

Conjecture 0.1 ([PTB11], Conjecture 1). Let $\pi$ be an irreducible admissible representation of $G=GL(nd,F)$ such that its image by Jacquet-Langlands correspondence is a generic representation of $GL(nd,F)$ with central character $\omega_\pi$. Let $\mu$ be a character of $E^\times$ such that $\mu_{nd}|F^\times = \omega_\pi$. If the representation $\pi$ is $\mu_E$-distinguished by $H$, i.e. if $\Hom_H(\pi,\mu_E) \neq 0$, then:

1. the Langlands parameter $\phi(\pi)$ of $\pi$ takes values in $GSp_{nd}(\mathbb{C})$, with similitude factor $\mu|_{F^\times}$;
2. the epsilon factor satisfies the relation

$$\epsilon\left(\frac{1}{2},\phi(\pi) \otimes \text{Ind}_{W_E}^{W_F}(\mu^{-1})\right) = (-1)^n \omega_{E/F}(-1)^{\frac{nd}{2}} \mu(-1)^{\frac{nd}{2}}$$

where $\omega_{E/F}$ is the quadratic character of $F^\times$ with kernel the norms of $E^\times$, and $W$ stands for the Weil group.

If $\pi$ is a discrete series representation of $G$, then the implication becomes an equivalence.

This conjecture is inspired by earlier results of J. Tunnel and H. Saito for $n=2$ and $D=F$. In fact Tunnel was the first to consider the problem for $GL(2,F)$, and to solve it when the residual characteristic of $F$ is not 2 ([Tun83, Theorem p.1277]), then Saito found a simpler proof valid
in characteristic different from 2 ([Su93, Theorem p.99]). The actual status of the conjecture is the following: when \( \mu = 1 \) and \( F \) has characteristic not 2, the direct implication is proved by H. Xue and M. Suzuki ([Xue19] and [Suz19]), whereas the converse implication for cuspidal representations is also proved by Xue in [Xue19]. For general \( \mu \) and \( F \) of characteristic not 2, the conjecture is proved by the first named author in [Cho19] for Steinberg representations. In this paper, when the residual characteristic of \( F \) is not 2, we prove it for general \( \mu \) and depth-zero cuspidal representations of \( F \)-split \( G \).

Let us describe the how the paper is organized: we assume the residual characteristic of \( F \) to be odd, and suppose that \( n \geq 4 \) as in any case the conjecture we intend to prove is known for \( n = 2 \) from Tunnel and Saito’s results.

In Section 2, we treat the case where \( \mu \) is tame. By standard Mackey theory arguments, and an also standard argument of Hakim and Murnaghan, we characterize \( \mu_E \)-distinction of depth-zero cuspidal representations in terms of their Langlands parameters (Theorem 2.1).

In Section 3, in order to characterize distinction when \( \mu \) is not tame, we prove in Proposition 3.2 that a \( \mu \)-distinguished cuspidal representation of any inner form of \( \text{GL}_n(F) \) is \( \mu \)-selfdual, by a standard globalization argument.

In Section 4 we extend in Theorem 4.1 our characterization of \( \mu_E \)-distinction depth-zero cuspidal representations of \( \text{GL}_n(F) \) in terms of their Langlands parameter to any character \( \mu \). Along the way we isolate the contribution of residual twisted Shalika models in Proposition 4.2, and show in Proposition 4.3 that when \( E/F \) is unramified, the only double coset contributing to distinction is the one isolated in Proposition 4.2. In particular this gives a multiplicity at most one statement when \( E/F \) is unramified.

In Section 5 we give an explicit characterization of \( \mu \)-simplicity of depth-zero cuspidal representations of \( \text{GL}(n,F) \), which resembles (and in fact is implied by) our \( \mu_E \)-distinction criterion.

Finally in Section 6 we prove the Prasad and Takloo-Bighash conjecture for depth-zero cuspidal representations of \( \text{GL}_n(F) \) (Corollary 6.1). With all the analysis done before, it reduces to a pleasant computation of the epsilon value of the conjecture for \( \mu \)-symplectic depth-zero cuspidal representations of \( \text{GL}(n,F) \) (with an extra condition on the central character) which is done in particular thanks to a result of Fröhlich and Queyrut ([FQ73]). The computation in question is performed in Theorem 6.1.

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1 Preliminary results

1.1 Notation / definitions
Let $F$ be a non-archimedean local field of residual characteristic not 2. We fix an algebraic closure which will contain all finite extensions of $F$ under consideration, and similarly for the residual field $k_F$ of $F$. For a finite extension $\bullet$ of $F$, we denote by the $O_\bullet$, $\mathcal{P}_\bullet$, $\varpi_\bullet$, $k_\bullet$ and $q_\bullet$ the ring of integers, its maximal ideals, a fixed uniformizer, the residual field of $\bullet$. Whenever $\chi : \bullet^* \to \mathbb{C}^*$ is a (smooth) character, we say that it is tame if $\mu(1 + \mathcal{P}_\bullet) = \{1\}$. Let $E$ be a quadratic extension of $F$ (we write $E = F[\delta]$ for a fixed $\delta$ in $E \setminus F$ such that $\delta^2$ is in $F$ and we set $\Delta = \delta^2$). We let $c(E/F)$ denote the ramification index of $E/F$. When $E/F$ is ramified, we choose $\varpi_E$ and $\varpi_F$ such that $\varpi_F = \varpi_E^2$; when $E/F$ is unramified, we choose $\varpi_F = \varpi_E$. Set $n = 2m$ for $m$ a natural number and consider the group $G = \text{GL}(n, F)$ and its subgroup $H \simeq \text{GL}(m, E)$ embedded in $G$ as we now explain. Let $(e_1, \ldots, e_m)$ be the canonical basis of $E^m$. Then $E^m$ identifies to $F^n$ as $F$-vector space via the basis $B = (\delta e_1, \ldots, \delta e_m, e_1, \ldots, e_m)$. Now $H$ embeds in $G$ as the fixed points of $G$ under the involution

$$\theta : G \to G \quad \text{where} \quad A = \begin{pmatrix} \Delta I_m & I_m \\ \end{pmatrix}.$$ 

We denote by $\det E$ the determinant map on $H$ identified with $\text{GL}(2m, E)$, with values in $E^\times$. Hence any character $\mu$ of $E^\times$ defines a character $\mu_E = \mu \circ \det E$ of $H$, and in fact all characters of $H$ are such.

### 1.2 Parametrization of depth-zero cuspidal representations

We call a depth-zero cuspidal representation of $\text{GL}(n, F)$ an irreducible cuspidal representation of this group with a vector fixed by $I_n + \varpi_F M_n(O_F)$. One can parametrize depth-zero cuspidal representations by admissible tame pairs as we now recall (see [BHT11 Part 5]).

- Let $L/F$ be the unramified field extension of degree $n$, of ring of integers $O_L$. Let $\chi$ be a character of $L^\times$ that satisfies:
  - $\chi$ is tame,
  - $\chi \circ \gamma = \chi$ for all $\gamma$ in $\text{Gal}_F(L)$; we say that $\chi$ is regular.

Such a pair $(L, \chi)$ is said to be tame admissible.

- As $\chi$ is trivial on $1 + \mathcal{P}_L$, $(L, \chi)$ induces a pair $(k_L, \overline{\chi})$ where $\overline{\chi}$ is a character of $k_L^\times$ which satisfies $\overline{\chi} \circ \overline{\pi} = \overline{\chi} = id_{k_L}$ for all $\overline{\pi}$ in $\text{Gal}_F(k_L)$; $\overline{\chi}$ is said to be regular.

By Green parametrization, one can associate to $(k_L, \overline{\chi})$ an irreducible cuspidal representation $(\overline{\pi}_\overline{\chi}, \mathcal{V})$ of $\text{GL}(n, k_F)$ i.e. an irreducible representation of $\text{GL}(n, k_F)$ such that for all proper parabolic subgroup $P$ with Levi decomposition $P = MN$, the vector subspace of fixed points of $\mathcal{V}$ by $N$ is trivial.

More precisely, if one defines an equivalence relation $\sim$ on regular characters of $k_L^\times$ by

$$\overline{\chi}_1 \sim \overline{\chi}_2 \text{ if and only if } \exists \overline{\gamma} \in \text{Gal}_{k_F}(k_L) \text{ such that } \overline{\chi}_2 = \overline{\chi}_1 \circ \overline{\gamma},$$

one has a bijection:

$$\begin{cases} \text{equivalence classes for } \sim \text{ of regular characters of } k_L^\times \to \text{equivalence classes of irreducible} \\
\text{cuspidal representations of } \text{GL}(n, k_F) \end{cases}$$
Let us recall that the central character of $\pi = \chi_{|F|}$ can be seen as a representation of $\text{GL}(n, \mathcal{O}_F) = \text{GL}(n, \mathcal{O}_F)/1 + \pi_F \mathcal{M}_n(\mathcal{O}_F)$, $\pi = \chi_{|F|}$ can be seen as a representation of $\text{GL}(n, \mathcal{O}_F)$ that is trivial on $1 + \pi_F \mathcal{M}_n(\mathcal{O}_F)$. Then, one can define a representation of $\text{GL}(n, \mathcal{O}_F)$, denoted by $\lambda$, in the following way:

$$\lambda(x(k)) = \chi_{|F|}(x)\pi(k) \text{ for all } x \in F^*, k \in \text{GL}(n, \mathcal{O}_F).$$

Finally, we set $\pi(\chi) = c - \text{Ind}_{F^*}^G \text{GL}(n, \mathcal{O}_F)^G(\lambda(x)) (c - \text{Ind} \text{ refers to compact induction})$, it is a depth zero cuspidal representation of $G$. If we denote again by $\sim$ the equivalence relation between admissible tame pairs of degree $n$ by

$$\chi_1 \sim \chi_2 \text{ if and only if } \exists \gamma \in \text{Gal}_F(L) \text{ such that } \chi_2 = \chi_1 \circ \gamma,$$

one gets a bijection:

$$\begin{align*}
&\left\{ \text{equivalence classes for } \sim \text{ of admissible tame pairs of degree } n \right\} \mapsto \left\{ \text{equivalence classes of representations of } \text{GL}(n, F) \right\} \\
&(L, \chi) \mapsto \pi(\chi)
\end{align*}$$

Let us recall that the central character of $\pi(\chi)$ is $\chi_{|F|}$ and its contragredient is $\pi(\chi)^\vee = \pi(\chi^{-1})$.

### 1.3 Reminder about the building of $\text{GL}(n, F)$

Let us recall how to describe the Bruhat-Tits building of $\text{GL}(n, F)$ with lattice chains.

**Definition 1.1.** An $\mathcal{O}_F$-lattice chain in $F^n$ is a strictly decreasing sequence (for inclusion) $L = (L_k)_{k \in \mathbb{Z}}$ of lattices such that there exists a positive integer $T$ that satisfies: for any uniformizer $\pi_F$, $\pi_F L_k = L_{k+T}$ for all $k \in \mathbb{Z}$. The integer $T$ (or $T(L)$) is called the period of $L$.

It is known that $T$ is at most $n$, and that there are lattice chains with period $n$. The group $\text{GL}(n, F)$ naturally acts on the set of lattice chains $(L_k)_{k \in \mathbb{Z}}$ by $g(L_k)_k = (gL_k)_k$ for $g \in \text{GL}(n, F)$, and we say that two lattice chains are equivalent if they are in the same $Z$-orbit, for $Z$ the center of $\text{GL}(n, F)$.

**Definition 1.2.** As a simplicial complex, the Bruhat-Tits building of $\text{GL}(n, F)$, $X_G$, is defined as the set of equivalence classes of lattice chains. The $(T - 1)$-dimensional simplex being the equivalence classes of lattice chains of period $T$.

We identify lattice chains of period one with $Z$-orbits of lattices in $F^n$, and denote by $[L]$ the $Z$-orbit of the lattice $L$: by definition they from the set $X_G$ of vertices of $X_G$. Clearly the group $\text{GL}(n, F)/Z$, hence $\text{GL}(n, F)$ acts on $X_G$ by respecting its simplicial structure. Let $K$ denote the maximal compact modulo center subgroup $F^* \text{GL}(n, \mathcal{O}_F)$ and let $s_0$ be the vertex of $X_G$ that is stabilized by $K$ i.e. the standard lattice chain of period 1; the vertex $s_0$ is called the standard vertex of $X_G$. We recall the following $G$-set isomorphism:

$$X_G \xrightarrow{\sim} G/K \quad \text{for } g \in G. \quad (1)$$

We will need the geometric realization of $X_G$, denoted by $|X_G|$. Each $T - 1$-dimensional simplex of $X_G$ is embedded in $\mathbb{R}^{T-1}$ with the following property: if we consider a $T - 1$-dimensional simplex, the points of its geometric realization in $|X_G|$ are given by the set of all barycenters of its vertices. We will use the geometric realization of the building $X_G$ given by lattice-functions. The definition comes from Section 1.2 of [BL02].
Definition 1.3. A lattice-function of $F^n$ is a map $\Lambda : \mathbb{R} \rightarrow \{\text{lattices of } F^n\}$ satisfying:

- $\varpi_F \Lambda(r) = \Lambda(r + 1)$;
- $\Lambda$ is decreasing: for all $r \geq s$, $\Lambda(r) \leq \Lambda(s)$;
- $\Lambda$ is left-continuous for the discrete topology on lattices.

Let us explain with more details how the set of lattice-functions allows to realize geometrically the building of $\text{GL}(n, F)$. Let $\Lambda$ be a lattice-function of $F^n$, then its image is a lattice chain $L = (L_k)_{k \in \mathbb{Z}}$ with period $T$. If we denote by $\lambda_k$ the length of the interval defined by $\{r \in \mathbb{R}, \Lambda(r) = L_k\}$, then the point $x_\Lambda$ of $|X_G|$ associated to $\Lambda$ is the barycenter of the weighted points $([\lambda_0], [\lambda_1], [\lambda_2], \ldots, ([L_T-1], \lambda_{T-1})$. Two lattice-functions $\Lambda_1$ and $\Lambda_2$ are said to be equivalent if there exists a real number $r_0$ such that $\Lambda_1(r) = \Lambda_2(r + r_0)$ for all $r \in \mathbb{R}$, in which case they realize the same point of the building. We denote by $\overline{\Lambda}$ the class of a lattice-function $\Lambda$. Moreover, the group $\text{GL}(n, F)$ naturally acts on the set of lattice-functions by:

$$(g \cdot \Lambda)(r) = g \cdot \Lambda(r),$$

for every lattice-function $\Lambda$, every $g \in \text{GL}(n, F)$ and every real number $r$. Thus, one has the following $G$-set isomorphism:

$$\{\text{equivalence classes of lattice-functions of } F^n\} \xrightarrow{\sim} |X_G|, \quad \overline{\Lambda} \longmapsto x_\Lambda.$$

Of course, all these reminders are valid for the construction of the building of $\text{GL}(m, E)$, $X_H$.

1.4 Vertices of the building fixed by the involution

First we recall the relation between $|X_G|$ and $|X_H|$, we will use the following terminology from type theory.

Definition 1.4. Let $u \in G$ such that $F_1 := F[u]$ is a field; let us denote by $v_{F_1}$ the normalized valuation of $F_1$ and by $e(F_1/F)$ the ramification index of $F_1/F$. One says that $u$ is minimal on $F$ if:

1. $\gcd(v_{F_1}(u), e(F_1/F)) = 1$,
2. $\varpi_{F_1}^{v_{F_1}(u)} e(F_1/F) + P_{F_1}$ generates the residual field extension $k_{F_1}/k_F$.

Recall that $E = F[\delta]$ and let us show that $\delta$ can be chosen minimal.

- If $E/F$ is ramified, we recall that $\varpi_E := \varpi_{F_1}^{2}$ if we choose $\delta = \varpi_E$, then we do have $E = F[\varpi_{E}]$ and $\delta$ is minimal. Indeed, $v_{E}(\varpi_{E}) = 1$ so $\gcd(v_{E}(\varpi_{E}), e(E/F)) = 1$ and moreover $\varpi_{E} \varpi_{E}^{2} + P_{E} = 1 + P_{E}$ which generates $k_{E}/k_{F}$ (because $k_{E} = k_{F}$ in the ramified case).
- If $E/F$ is unramified (i.e., $e(E/F) = 1$), then $k_{E}$ is an extension of $k_{F}$ with cardinality $q_{F}^{2}$ and there exists $\xi \in E^{*}$ a primitive $(q_{F}^{2} - 1)^{th}$ root of unity which generates $E$ over $F$. Set $\delta := \frac{q_{F}^{2}}{q_{F} - 1}$. As the order of $\delta$ is $2(q_{F} - 1)$, then $\delta \notin F$ but $\delta^{2} \in F$, so that we do have $E = F[\delta]$ with $\delta^{2} \in F$. Moreover, $\delta$ is a minimal element because $v_{E}(\delta) = 0$ so $\gcd(v_{E}(\delta), e(E/F)) = 1$ and moreover, $\varpi_{E} \varpi_{E}^{2} + P_{E} = 1 + P_{E}$ generates $k_{E}/k_{F}$ (see Theorem 7 and Corollary 3 of Chapter 1, §4 of Weil [Wei74]).

From now on, we choose $\delta = \varpi_{E}$ if $E/F$ is ramified and $\delta = \xi^{\frac{q_{F}}{q_{F} - 1}}$ (for $\xi$ a primitive $(q_{F}^{2} - 1)^{th}$ root of unity) if $E/F$ is unramified, thus $\delta$ is minimal. Then by [BS17] Lemma XII.4.2 we have:
Lemma 1.1. We have \(|X_G|^θ = |X_G|^E^x\).

Note that an \(O_E\)-lattice of \(E^m\) can always be seen as an \(O_F\)-lattice of \(F^{2m}\) because \(O_E\) is an \(O_F\)-lattice in \(F^2\). Theorem 1.1 of [BL2] then asserts:

Theorem 1.1. 1. There exists a unique map \(j : |X_H| \to |X_G|\) that is \(H\)-equivariant and affine.

2. It is injective and \(j(|X_H|) = |X_G|^{E^x}\), the set of points that are fixed by \(E^x\).

3. If \(x \in |X_H|\) is associated to the lattice-function \(r \mapsto \Lambda(r)\), then \(j(x)\) is associated to the lattice-function \(r \mapsto \Lambda(\epsilon(E/F) r)\).

The theorem above enables us to determine the \(H\)-orbits of \(\theta\)-fixed vertices in \(X_G^\circ\) depending on the ramification of \(E/F\).

Proposition 1.1. When \(E/F\) is unramified, the set \((X_G^\circ)^{\theta}\) consists of a unique \(H\)-orbit, namely that of the standard vertex \(s_0\) fixed by \(K\), whereas when \(E/F\) is ramified \((X_G^\circ)^{\theta}\) is empty.

Proof. When \(E/F\) is unramified, the map \(j\) is simply the identity on lattice-functions and is simplicial. Thus by \(\epsilon, (X_G^\circ)^{\theta} = j(X_H^\circ)\) whence \(H \backslash (X_G^\circ)^{\theta} = H \backslash j(X_H^\circ) = j(H \backslash X_H^\circ)\) by Theorem 1.1. As \(H\) acts transitively on \(X_H^\circ\), we deduce that \((X_G^\circ)^{\theta}\) consists of a unique \(H\)-orbit. Moreover it is that of \(s_0\) because \(s_0\) is the image of the standard vertex in \(X_H^\circ\) under \(j\). When \(E/F\) is ramified, then by Theorem 1.1 the map \(j\) sends an equivalence class of lattice functions with image a lattice chain of of period 1 to an equivalence class of lattice functions with image a lattice chain of of period \(\epsilon(E/F) = 2\), i.e. it sends a vertex to an interior point of a simplex of dimension \(\geq 1\), so \(j(X_H^\circ) \cap X_G^\circ\) is empty and the result follows again from Theorem 1.1. \(\Box\)

1.5 Properties of local constants

Let \(K'/K\) be a finite separable extension of non-archimedean local fields, if \(\psi\) is a non-trivial character of \(K\), we denote by \(\psi_K\) the character \(\psi \circ \text{Tr}_{K'/K}\). We call the conductor of \(\psi\) the smallest integer \(d(\psi)\) such that \(\psi\) is trivial on \(P_K^{\psi}\). Similarly if \(\chi\) is a character of \(K^*\), we call the conductor of \(\chi\) the integer \(c(\chi)\) equal to zero if \(\chi\) is unramified, or equal to the smallest integer such that \(\chi\) is trivial on \(1 + P_K^{\psi}\) if \(\chi\) is ramified. We say that \(\chi\) is tame when \(c(\chi) \leq 1\). When \(K'/K\) is unramified, it follows from [Wei74, Chapter 8, Corollary 3] that

\[
d(\psi_K) = d(\psi).
\]

If \(\phi\) is a representation of \(W_K\) of finite dimension, and \(\psi\) is a non-trivial character of \(K\), we refer to [Lan79, 3.6.4] for the definition of the root number \(\epsilon(1/2, \phi, \psi)\) (denoted \(\epsilon_L\) there). One then defines the Langlands \(\lambda\)-constant:

\[
\lambda(K'/K, \psi) = \frac{\epsilon(1/2, \text{Ind}_{W_{K'}}^{W_K}(1_{{W_K}}), \psi)}{\epsilon(1/2, 1_{W_F}, \psi_{K'})}.
\]

We set

\[
\omega_{K'/K} = \det \circ \text{Ind}_{W_{K'}}^{W_K}(1_{{W_K}}),
\]

it identifies with the quadratic character of \(K^*\) with kernel the norms of \(K'\), when \(K'/K\) is quadratic. For \(a \in K^*\), we set \(\psi_a = \psi(a \cdot \cdot \cdot)\). These constants enjoy the following list of properties, which we will freely use later in the paper.
1. $\epsilon(1/2, \phi \otimes \phi', \psi) = \epsilon(1/2, \phi, \psi)\epsilon(1/2, \phi', \psi)$ where $\phi'$ is another finite dimensional representation of $W_F$ [Tat79 (3.4.2)].

2. $\epsilon(1/2, \phi, \psi) = \det(\phi(\omega))\epsilon(1/2, \phi, \psi)$ [Tat79 (3.6.6)].

3. $\epsilon(1/2, \phi^n, \psi^n) = \epsilon(1/2, \phi, \psi)$ whenever $\sigma$ is a finite order field automorphism of $F$, as can be checked by the definition of the epsilon factor.

4. $\epsilon(1/2, \phi, \psi)\epsilon(1/2, \phi^v, \psi^{-1}) = 1$ [Tat79 (3.6.7)].

5. If $\chi$ is a character of $K^*$, and $\mu$ is an unramified character of $K^*$, by [Tat79 (3.6.5)]:

$$\epsilon(1/2, \mu\chi, \psi) = \mu(\omega_K^{d(\psi)(\chi)})\epsilon(1/2, \chi, \psi).$$

6. If $K'/K$ is a quadratic, $\delta \in \ker(\text{Tr}_{K'/K}) - \{0\}$, $\chi$ is a character of $K'^*$ with $\chi|_{K^*} = 1$, then by [FQ73 Theorem 3]:

$$\epsilon(1/2, \chi, \psi|_{K'}) = \chi(\delta).$$

7. If $\phi_{K'}$ is an $r$-dimensional representation of $W_{K'}$, then

$$\epsilon(1/2, \text{Ind}_{W_{K'}}^W(\phi_K), \psi) = \lambda(K'/K, \psi)^r\epsilon(1/2, \phi_{K'}, \psi|_{K'})$$

[BH06 (30.4.3)].

8. If $K'/K$ is unramified with $[K'/K] = n$:

$$\lambda(K'/K, \psi) = (-1)^{d(\psi)(n-1)}$$

(for example [Moy86] and [2] together with Equation [2])

9. If $K''$ is a field with $K \subset K'' \subset K'$, then

$$\lambda(K'/K, \psi) = \lambda(K'/K'', \psi|_{K''})\lambda(K''/K, \psi)^{[K':K'']}$$

[Lan70].

10. $\lambda(K'/K, \psi)^2 = \omega_{K'/K}(-1)$ [BH06 (30.4.3)].

2 Distinction of depth-zero cuspidal representations when $\mu$ is tame

This case is the easiest case, and we use the proof of [HM Proposition 5.20] to determine multiplicities. We fix $\pi(\chi)$ a cuspidal representation of $GL_n(F)$ of depth-zero, and $\mu$ is a character of $E^*$. 

**Lemma 2.1** ([HM]). Let $x \in X^0_G$ a vertex such that $\theta(x) \neq x$. Let $K_x$ be the stabilizer of $x$ in $G$, $K_x$ the maximal compact subgroup of $K_x$ and $K_x^1 \subseteq K_x$ its pro-unipotent radical. Let $\pi$ be a cuspidal representation of $K_x/K_x^1$, let $\sigma$ be the inflation of $\pi$ to $K_x$. Suppose that $\mu$ is tame and set $\rho := \mu_E$, then $\text{Hom}_{K_x \cap H}(\sigma, \rho) = \{0\}$. 

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Lemma 2.2. \( g \) is \( \text{GL}_n(k_F) \) that

The result then follows from [Pra19, Proposition 4.3] (which has the assumption for the sake of contradiction that \( \text{Hom}_{\text{K}\cap H}(\sigma, \rho) \neq \{0\} \), this first implies that \( \rho|_{K\cap H} = 1 \) because \( \sigma \) is trivial on \( K_1^1 \). Now for \( h \in U \cap H \), there exists \( \alpha \geq 0 \) such that \( h^{P} \in K_1^1 \cap H \), which implies that \( \rho(h^{P}) = 1 \). Thus, \( \mu(\det(h))^{P} = 1 \) where \( \det(h) \in \mathcal{O}_E^{*} \). Yet \( \mu \) is tame so \( \mu_{\mathcal{O}_E^{*}} \) factors through \( \mathcal{O}_E^{*}/(1 + \mathcal{P}_E) \) which is a finite group of order prime to \( p \), hence \( \mu(\det(h)) = 1 \). So \( \rho|_{U \cap H} = 1 \) and

\[
\{0\} \neq \text{Hom}_{\text{K}\cap H}(\sigma, \rho) \subset \text{Hom}_{\mathcal{F}}(\sigma, 1) \cong \text{Hom}_{\mathcal{F}}(\pi, 1)
\]
as \( U = U K_1^1 \), contradicting the cuspidality of \( \sigma \).

In other words, as each vertex \( x \in X_\mathcal{E}_x \) is of the form \( g \cdot s_0 \) for a certain \( g \) in \( G \) and its stabilizer is \( g \mathcal{K}_g^{-1} \), this amounts to the following lemma.

Lemma 2.2 ([HM]). If \( \chi \in H \setminus G/K \) satisfies \( \text{Hom}_{\mathcal{H}\circ \mathcal{K}}(\mathcal{G}, \mu) \neq \{0\} \) (where \( \mathcal{G}(x) = \chi(x^{-1} g) \) for all \( x \) in \( \mathcal{G} \)), then \( \mathcal{G} \mathcal{K}^{-1} \) is stable by \( \theta \).

The next step is:

Lemma 2.3. There is an isomorphism of \( \mathbb{C} \)-vector spaces:

\[
\text{Hom}_H(\chi, \mu) \cong \prod_{g \in \mathcal{H}(X_\mathcal{E}_x)^g} \text{Hom}_{\mathcal{H}\circ \mathcal{K}}(\mathcal{G}, \mu).
\]

Proof. Write successively:

\[
\begin{align*}
\text{Hom}_H(\chi, \mu) & = \text{Hom}_H(c \circ \text{Ind}_G^H(\chi), \mu) \\
& = \text{Hom}_H(\bigoplus_{g \in \mathcal{H}(G/K)} c \circ \text{Ind}_G^H(\mathcal{K}, \mu)) \\
& \text{by Mackey’s restriction formula} \\
& = \prod_{g \in \mathcal{H}(G/K)} \text{Hom}_{\mathcal{H}\circ \mathcal{K}}(\mathcal{G}, \mu) \\
& \text{by Frobenius reciprocity on the left, for compact induction} \\
& \text{from a compact modulo center open subgroup} \\
& = \prod_{g \in \mathcal{H}(X_\mathcal{E}_x)^g} \text{Hom}_{\mathcal{H}\circ \mathcal{K}}(\mathcal{G}, \mu) \text{ thanks to Isomorphism (i)} \\
& = \prod_{g \in \mathcal{H}(X_\mathcal{E}_x)^g} \text{Hom}_{\mathcal{H}\circ \mathcal{K}}(\mathcal{G}, \mu) \text{ thanks to Lemma 2.2}
\end{align*}
\]

We denote by \( L_0 \) the unramified extension of \( F \) of degree \( m \). Thanks to Theorem 1.1 and the recent paper [Pra19] we obtain:

Theorem 2.1. When \( \mu \) is tame and \( n \geq 4 \), we have \( \text{Hom}_H(\pi(\chi), \mu) \neq \{0\} \) if and only if \( E/F \) is unramified and \( \chi|_{L_0^m} = \mu_{F^*} \circ N_{L_0/F} \), in which case \( \text{Hom}_H(\pi(\chi), \mu) \cong \mathbb{C} \).

Proof. Multiplicity zero in the ramified case is immediate from Lemma 2.3 and Theorem 1.1.

When \( E/F \) is unramified [2,3] and Theorem 1.1 imply that

\[
\text{Hom}_H(\pi, \mu) = \text{Hom}_{\mathcal{H}\circ \mathcal{K}}(\mathcal{G}, \mu),
\]

which is zero if \( \chi|_{F^*} \neq \mu_{F^*} \). If \( \chi|_{F^*} = \mu_{F^*} \) (which is in particular true when \( \chi|_{L_0^m} = \mu_{F^*} \circ N_{L_0/F} \)) we obtain

\[
\text{Hom}_{\mathcal{H}\circ \mathcal{K}}(\mathcal{G}, \mu) = \text{Hom}_{\mathcal{H}\circ \mathcal{K}}(\mathcal{G}, \mu) = \text{Hom}_{\mathcal{F}}(\pi, \mu).
\]

The result then follows from [Pra19] Proposition 4.3 (which has the assumption \( n \geq 4 \)).
3 On \(\mu\)-selfduality of \(\mu\)-distinguished representations

Now we take \(\mu\) any character of \(E^*\) with no restriction on its conductor. We intend to prove that \(\mu\)-distinguished representations of cuspidal (of any level) representations of any inner form of \(GL(n)\) is \(\mu\)-selfdual automatically. Our result will follow from a classical globalization argument, and the case of principal series for split inner forms.

**Proposition 3.1.** Let \(\pi\) be a generic principal series of \(GL(2n, F)\) (induced from a character of a Borel subgroup), and \(\mu_1\) be a character of \(F^* \times F^*\), and \(\mu_2\) be a character of \(E^*\). Let \(H_1\) be the block diagonal subgroup \(GL(m, F) \times GL(m, F)\) and \(H_2\) be the subgroup \(H \cong GL_m(E)\) of \(GL(n, F)\). Then if \(\pi\) is \(\mu_1\)-distinguished by \(H_1\), then

\[
\pi \cong \mu_1|F_\pi \otimes \pi^\vee
\]

(where \(F^*\) is diagonally embedded in \(F^* \times F^*\) in the first case)

**Proof.** We only do the case \((H_1, \mu_1)\), as the argument for \((H_2, \mu_2)\) is completely similar but simpler due to simplification of quotients of modulus characters (see [Choi19] for the parametrization of double cosets involved there, and [BM19] (5.3) and Remark 5.4 for the modulus characters involved). Here we rather consider distinction by the conjugate \(H\) of \(H_1\) by the matrix \(w_n\) of \([Mat15\ p.121]\), and set \(h(g_1, g_2) = w_n^{-1}\text{diag}(g_1, g_2)w_n\) for \(g_i \in GL(n, F)\). The character \(\mu_1\) is of the form \(\mu_{\alpha, \beta}(h(g_1, g_2)) = \alpha(\det(g_1))\beta(\det(g_2))\) for \(\alpha\) and \(\beta\) characters of \(F^*\). Let \(B\) be the upper triangular Borel subgroup of \(G = GL_n(F)\) and \(\chi\) be a character of the diagonal torus \(A\) of \(G\) such that \(\pi = Ind_B^G(\chi)\) is generic. We want to show that if \(\pi\) is \(\mu_{\alpha, \beta}\)-distinguished, then

\[
\pi \cong \alpha\beta \otimes \pi^\vee.
\]

This amounts to prove that there is a permutation \(\sigma \in S_n\) such that \(\alpha\beta \chi^{-\sigma} = \chi\), where by abuse of notation

\[
(\alpha\beta)(\text{diag}(a_1, \ldots, a_n)) = \prod_{i=1}^{n}(\alpha\beta)(a_i).
\]

We denote by \(\delta_{G'}\) the modulus character of any closed subgroup \(G'\) of \(G\). We set

\[
\epsilon = \text{diag}(1, -1, \ldots, -1) \in G
\]

so that \(H\) is the subgroup of \(G\) fixed under the conjugation \(\theta\) by \(\epsilon\). By a re-interpretation of the discussion in [Mat15 Section 3.2], the double cosets \(B\backslash G/H\) are parametrized by couples \(s = (w_s, x_s)\) where \(w_s \in S_n \subset G\) is an involution, and \(x_s\) is a map from the set of fixed points \(\text{Fix}(w_s)\) of \(w_s\) in \(\{1, \ldots, n\}\) to \(\{\pm 1\}\), such that \(|x_s^{-1}((-1))| = |x_s^{-1}((1))| = \frac{\text{Fix}(w_s)}{2}\). The corresponding representative \(u_s\) in \(B\backslash G/H\) in particular satisfies \(u_s \epsilon u_s^{-1} \epsilon = w_s\), and we set

\[
\theta_s(x) = w_s \theta(x)w_s^{-1} = u_s \theta(u_s^{-1} x u_s)u_s^{-1}
\]

for \(x \in G\). Conjugation by \(u_s\) stabilizes \(A\), and \(\theta_s\) as well. Suppose that \(\pi\) is \(\mu_{\alpha, \beta}\)-distinguished, by the discussion before Theorem [Mat15 Theorem 3.14] which adapts in a straightforward manner to characters of the form \(\mu_{\alpha, \beta}\) (it is just Mackey theory, also known as the geometric lemma of Bernstein and Zelevinsky), there is \(s\) such that

\[
\chi_{|A^\theta_{s}} = (\delta_{B^\theta} \delta_B^{-1/2} \mu_{\alpha, \beta})(A^\theta_{s}),
\]

where the exponent \(\theta\) denotes the fixed points of \(\theta\) in the corresponding set (which is not necessarily \(\theta\)-stable, for example \(B\)), and \(\mu_{\alpha, \beta}(a_s) = \mu_{\alpha, \beta}(u_s^{-1} a_s u_s)\) for \(a_s \in A^\theta_{s}\). The character
\[ \delta_{B^\theta} \delta_B^{-1/2} \] restricted to \( A^\theta \), is computed in [Mat15 Proposition 3.6]. We extend \( x_\varphi \) from \( \text{Fix}(w_\alpha) \) to \( \{1, \ldots, n\} \) by \( 0 \) outside \( \text{Fix}(w_\alpha) \). Then for \( a = \text{diag}(a_1, \ldots, a_n) \in A^\theta \), one has:
\[
\delta_{B^\theta} \delta_B^{-1/2}(a) = \prod_{1 \leq i < j \leq n} |a_i|^{\frac{\varepsilon_i(\varphi, \gamma)}{2}} |a_j|^{\frac{\varepsilon_j(\varphi, \gamma)}{2}}.
\]
On the other hand, by a computation similar to that done in the proof of [Mat15 proposition 3.6], we have for \( a = \text{diag}(a_1, \ldots, a_n) \in A^\theta \), (note that for any \( i \) one has \( a_{w_\alpha(i)} = a_i \)):
\[
\rho_{\alpha, \beta}^{w_\alpha}(a) = \prod_{i \in \gamma \iota^{*}(\{1\})} \alpha(a_i) \prod_{i \in \gamma \iota^{*}(\{-1\})} \beta(a_i) \prod_{i \in \gamma \iota^{*}(\{0\}), i \in w_\alpha(i)} \alpha \beta(a_i).
\]
For \( a \in A \) we set \( w_\alpha(a) = w_\alpha a w_\alpha^{-1} \), so that \( aw_\alpha(a) \in A^\theta \), then from the relations above it follows that for \( a \in A \) (note that \( x_\varphi \circ w_\alpha = -x_\varphi \) and is order reversing on \( \{1, \ldots, n\} - \text{Fix}(w_\alpha) \)):
\[
\chi(aw_\alpha(a)) = \alpha(a) \beta(a),
\]
i.e.
\[
\chi \chi^{w_\alpha} = \alpha \beta
\]
so we can choose the sought \( \sigma \in S_n \) to be \( w_\alpha \).

As in [BM19 Proposition 5.2], we deduce from Proposition 3.1, using the globalization results of [PSP08] and [GL18] together with the strong multiplicity one theorems from [Bad08] and [BR17], the following result.

**Proposition 3.2.** Let \( D \) be an \( F \)-division algebra of index \( d \) and \( m \) a positive integer such that \( md \) is even, let \( H \) be the centralizer of \( E \) in \( G = \text{GL}_m(D) \). Let \( \mu \) be a character of \( E^\times \) identified via the reduced norm to a character of \( H \), then a cuspidal representation \( \pi \) of \( G \) which is \( \mu \)-distinguished satisfies
\[
\pi \simeq \mu_{F^\times} \otimes \pi^\vee.
\]

Here are two important corollaries for depth-zero cuspidal representations of \( \text{GL}(n, F) \).

**Corollary 3.1.** Let \( \pi \) be a cuspidal representation of \( \text{GL}_n(F) \) which is of depth zero, and \( \mu \)-distinguished, then automatically \( \mu_{F^\times} \) is tame (i.e. \( \mu(1 + \mathcal{P}_F) = 1 \)).

**Proof.** Write \( \pi = \pi(\chi) \). By Proposition 3.2 we have \( \chi^\gamma = \mu \circ N_{L/F} \chi^{-1} \) for some \( \gamma \in \text{Gal}_F(L) \). But because \( \chi^\gamma \) and \( \chi^{-1} \) are both tame, the result follows from the fact that \( N_{L/F}(1 + \mathcal{P}_L) = 1 + \mathcal{P}_F \). \( \square \)

We denote by \( L_0 \) the unramified extension of \( F \) of degree \( m \).

**Corollary 3.2.** Suppose that \( n \geq 4 \). Let \( \pi(\chi) \) be a cuspidal \( \mu \)-distinguished representation of \( \text{GL}(n, F) \) of depth zero. Then
\[
\chi|_{L_0} = \mu \circ N_{L_0/F}.
\]

**Proof.** Thanks to Proposition 3.2, there is \( \gamma \in \text{Gal}_F(L) \) such that \( \chi^\gamma = \mu \circ N_{L/F} \chi^{-1} \). Because \( \chi \) and \( \mu \) are tame, this reduces to \( \chi^2 = \mu \circ N_{L/F} \chi^{-1} \). This implies that \( \chi^2 = \chi \), hence that \( \gamma \) has order dividing two because \( \chi \) is regular. If \( \gamma \) was trivial one would have \( \chi^2 = \mu \circ N_{L/F} \). Because \( \chi \) and \( \mu_{F^\times} \) are tame this would imply
\[
\chi^2 = \pi \circ N_{L/F}.
\]

But the group of characters of the form \( \alpha \circ N_{L/F} \) for \( \alpha \) a character of \( k_F^* \) form a group of order \( q_F - 1 \) so one should have \( \chi^2(q_F - 1) = 1 \). But because \( \chi \) is regular and \( n \geq 4 \), the character \( \chi^{q_F - 1} \) must be nontrivial, hence \( \chi^2(q_F - 1) \neq 1 \). Thus \( \gamma \) is the conjugation of \( L/L_0 \) so \( \chi \circ N_{L_0} = \mu \circ N_{L/F} \), and \( \chi \) and \( \mu \circ N_{L_0/F} \) agree on the units of \( L_0^* \) because \( L/L_0 \) is unramified. Finally they also agree on \( \varpi_F \) by central character considerations. \( \square \)
4 Distinction of depth-zero cuspidal representations

We want to show that the necessary condition obtained in the above section is also sufficient when $\mu$ is not tame. By Proposition 1.1 and Lemma 2.2, the contribution to distinction in Mackey formula will in this case arise from double cosets in $H\backslash G/K$ corresponding to $H$-orbits of non $\theta$-fixed vertices of $X_G$. For such double cosets, the distinction problem reduces residually to the existence of a twisted Shalika model, which have been studied by Prasad. We recall his result.

4.1 Twisted Shalika and linear models over finite fields

Let $\pi$ be an irreducible representation of $GL(n,k_F)$, and $\alpha$ be a character of $k^*_F$, and $\psi$ be a nontrivial character of $k_F$. We recall that we call the Shalika subgroup of $GL(n,k_F)$ the group:

$$S_n(k_F) = \left\{ \begin{pmatrix} g & x \\ I_m & I_m \end{pmatrix} \mid g \in GL(m,k_F), \ x \in M(m,k_F) \right\}.$$

On then defines the character $\Psi_\alpha$ of $S_n(k_F)$ by the formula:

$$\Psi_\alpha\left( \begin{pmatrix} g \\ I_m \end{pmatrix} \right) = \alpha(\det(g))\psi(\text{Tr}(x)).$$

We say that $\pi$ has an $\alpha$-twisted Shalika model if

$$\text{Hom}_{S_n(k_F)}(\pi, \Psi_\alpha) \neq 0,$$

and this does not depend on the choice of $\psi$. The following proposition is due to Prasad.

**Proposition 4.1.** Let $\pi$ be a cuspidal representation of $GL(n,k_F)$, then $\pi$ has an $\alpha$-twisted Shalika model if and only if $\chi |_{k^*_L} = \alpha \circ N_{L_0/F}$ in which case $\text{Hom}_{S_n(k_F)}(\pi, \Psi_\alpha) \simeq \mathbb{C}$.

**Proof.** We denote by $N$ the subgroup of matrices $n(x) = \left( \begin{pmatrix} I_m \\ x \\ I_m \end{pmatrix} \right)$ in $GL(n,k_F)$ and by $(\pi_N,\psi)$ the quotient of $\pi$ by $\{v - \psi(\text{Tr}(x))v, \ n(x) \in N, \ v \in \pi \}$. The space $(\pi_N,\psi)$ is a $GL_n(k_F)$-module (for diagonal action). Then by [Pra00 Theorem 1], we have

$$(\pi_N,\psi) = \text{Ind}_{k^*_L}^{GL(n,k_F)}(\chi|_{k^*_L}).$$

Now by definition we have

$$\text{Hom}_{S_n(k_F)}(\pi_N, \Psi_\alpha) \simeq \text{Hom}_{GL(n,k_F)}(\text{Ind}_{k^*_L}^{GL(n,k_F)}(\chi|_{k^*_L}), \alpha \circ \text{det})$$

and this latter space is isomorphic to

$$\text{Hom}_{k^*_L}(\chi|_{k^*_L}, \alpha \circ N_{L_0/F}),$$

and the statement follows. 

**Remark 4.1.** The condition in Proposition 4.1 is also equivalent to $\pi \simeq \alpha \otimes \pi^\vee$. 

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4.2 Double cosets contributing to distinction

Take $\Delta \in F^*$ with square root $\delta$ generating $E/F$, which we take of valuation 0 when $E/F$ is unramified and of valuation 1 when $E/F$ is ramified. The subgroup $H$ of $\GL_n(F)$ consists of invertible matrices of the form \[
\begin{pmatrix}
 a & b \\
 \Delta b & a
\end{pmatrix}.
\]
The character $\mu_E$ of $H$ satisfies
\[
\mu_E \left( \begin{pmatrix}
 a & b \\
 \Delta b & a
\end{pmatrix} \right) = \mu(\det(a + \delta b)).
\]

First we identify a non trivial double coset contributing to distinction when $\mu$ has conductor $g \geq 2$.

Note that when $E/F$ is ramified, if $\mu$ has conductor $l \geq 2$ and is trivial on $1 + \mathcal{P}_F$, then it has an even conductor, because of the isomorphism $\mathcal{O} \rightarrow 1 + \omega_F^{d} \mathcal{O}$ between $k_F = k_E$ and $1 + \mathcal{P}_F^{2d}$ for any $d \geq 1$.

**Proposition 4.2.** Suppose $\mu$ has conductor $r + 1 \geq 2$ but satisfies $\mu(1 + \mathcal{P}_F) = 1$. We set $l = r$ if $E/F$ is unramified, whereas we set $l = (r - 1)/2$ when $E/F$ is ramified. Set $d_l = \text{diag}(\omega_F^{l} m_m, m_m)$ and suppose that $\chi|_{L_0} = \mu \circ N_{L_0/F}$, then
\[
\text{Hom}_{K \cap d_l^{-1} H d_l}(\lambda_{\chi}, \mu_E^{d_l}) \neq 0,
\]
where $\mu_E^{d_l}(x) = \mu_E(d_l x d_l^{-1})$.

**Proof.** First the condition $\chi|_{L_0} = \mu \circ N_{L_0/F}$ implies that $\chi|_{L^2} = \mu|_{L^2}$, hence
\[
\text{Hom}_{K \cap d_l^{-1} H d_l}(\lambda_{\chi}, \mu_E^{d_l}) = \text{Hom}_{K \cap d_l^{-1} H d_l}(\lambda_{\chi}, \mu_E).
\]
The group $K \cap d_l^{-1} H d_l$ is the set of matrices
\[
\begin{pmatrix}
 a & \omega_F^{-l} b \\
 \omega_F^l \Delta b & a
\end{pmatrix}
\]
with $a \in \GL(m, \mathcal{O}_F)$ and $b \in \mathcal{M}(m, \mathcal{P}_F)$, and
\[
\mu_E^{d_l} \left( \begin{pmatrix}
 a & \omega_F^{-l} b \\
 \omega_F^l \Delta b & a
\end{pmatrix} \right) = \mu(\det(a + \delta b)).
\]
But
\[
\det(a + \delta b) = \det(a) \det(1 + \mathcal{P}_F) = \det(a)(1 + \text{Tr}(\delta a^{-1} b))[\mathcal{M}(m, \mathcal{P}_F^{l+1})]
\]
so
\[
\mu_E^{d_l} \left( \begin{pmatrix}
 a & \omega_F^{-l} b \\
 \omega_F^l \Delta b & a
\end{pmatrix} \right) = \mu(\det(a)) \mu(1 + \text{Tr}(\delta a^{-1} b)),
\]
where the dependences are in fact in $\pi \in \GL(n, k_F)$ and $\bar{b} \in \mathcal{M}(m, \mathcal{P}_F^{l+1})$. So in fact for $a \in \GL(n, \mathcal{O}_F)$ and $b \in \mathcal{M}(m, \mathcal{O}_F)$ we have
\[
\mu_E^{d_l} \left( \begin{pmatrix}
 a & \omega_F^{-l} b \\
 \omega_F^l \Delta b & a
\end{pmatrix} \right) = \mu_E \left( \begin{pmatrix}
 \pi & \bar{b} \\
 \pi & \pi
\end{pmatrix} \right) = \mu(1 + \omega_F^{l} \delta \text{Tr}(\pi^{-1} \bar{b})).
\]
The character $\psi(x) = \mu(1 + \omega_F^{l} \delta \pi)$ is a nontrivial character of $k_F$ because $\mu(1 + \mathcal{P}_F) = 1$ whereas $\mu$ has conductor $r + 1$. On the other hand
\[
\lambda_{\chi} \left( \begin{pmatrix}
 a & \omega_F^{-l} b \\
 \omega_F^l \Delta b & a
\end{pmatrix} \right) = \pi \left( \begin{pmatrix}
 \pi & \bar{b} \\
 \pi & \pi
\end{pmatrix} \right).
\]
Hence $\pi_\chi$ has an $\alpha$-twisted Shalika model and the result follows from Proposition 4.1. \qed
4.3 Multiplicity one when $E/F$ is unramified

We denote by $\Lambda^+_m$ the sequences of integers $(\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$ in $\mathbb{Z}^m$, and set for $\lambda \in \Lambda^+_m$:

$$d_\lambda = \text{diag}(\varpi_F^{\lambda_1}, \ldots, \varpi_F^{\lambda_m}, 1, \ldots, 1) \in G.$$

We recall from [Ohi04] the following result:

**Proposition 4.3.**

$$G = \bigsqcup_{\lambda \in \Lambda^+_m} K d_\lambda H.$$

**Proof.** For $\lambda \in \Lambda^+_m$ we set $\varpi_F^{\lambda} = \text{diag}(\varpi_F^{\lambda_1}, \ldots, \varpi_F^{\lambda_m})$, we also set

$$w_m = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \\ & & 1 \end{pmatrix} \in \text{GL}(m, F)$$

and $w = \text{diag}(I_m, w_m)$. It follows from [Guo97] that the map $p : x \mapsto x A x^{-1}$ identifies $G/H$ with the conjugacy class of $A$. The matrix $d_\lambda$ is sent by $p$ to $(\Delta \varpi_F^{\lambda}, \varpi_F^{\lambda})$, the result now follows from [Ohi04] Proposition 4, noting that the group $H$ in [Ohi04] is equal the centralizer of $w A w^{-1}$ whereas here it is the centralizer of $A$. \hfill\qed

One has the following multiplicity one result:

**Proposition 4.4.** Let $\pi(\chi)$ be a cuspidal representation of $\text{GL}(n, F)$ of depth zero for $n \geq 4$. If it is $\mu$-distinguished, then $\text{Hom}_{\text{GL}(n, E)}(\pi(\chi), \mu_E) = \mathbb{C}$.

**Proof.** Suppose that $\pi(\chi)$ is $\mu$-distinguished so that $\mu_{|F^*} = \alpha \circ N_{L_0/F}$ thanks to Corollary 3.2. The result follows from Theorem 2.1 when $\mu$ is tame so we suppose that $\mu$ has conductor $l+1 \geq 2$. By Mackey theory, the result will follow from Propositions 4.1, 4.2 and 4.3 if we show that

$$\text{Hom}_{K \cap d_l^{-1} H d_l} (\lambda_x, \mu_E^{d_l} ) = \text{Hom}_{K \cap d_l^{-1} H d_l} (\lambda_x, \mu_E^{d_l} ) \neq 0$$

for $\lambda \in \Lambda^+_m$, then $\lambda = (l, \ldots, l)$. Note that $K \cap d_l^{-1} H d_l$ is the set of matrices

$$\begin{pmatrix} a & \varpi_F^{\lambda} b \\ \varpi_F^{-\lambda} b & a \end{pmatrix}$$

with $a \in \text{GL}(m, O_F)$ and $l_i(b) \in (\mathcal{P}_F^{\lambda})^m$ for $i = 1, \ldots, m$, where $l_i(b)$ is $i$-th row of $b$. So we assume that $\text{Hom}_{K \cap d_l^{-1} H d_l} (\lambda_x, \mu_E^{d_l} ) \neq 0$.

Suppose first that $\lambda_m \leq l-1$ and denote by $M(n, O_F)^-$ the space of matrices in $M(n, O_F)$ with $l_i(b) = 0$ for $i = 1, \ldots, m-1$ and $l_m(b) \in (\mathcal{P}_F^{\lambda m+1})^m$. Because $\pi(\chi)$ is tame, if $\text{Hom}_{K \cap d_l^{-1} H d_l} (\lambda_x, \mu_E^{d_l} )$ was nonzero this would imply that

$$1 = \mu_{|F} \left( I_m \begin{pmatrix} \varpi_F^{\lambda} b & -b \\ -b & I_m \end{pmatrix} \right) = \mu(\det(I_m + \delta l_m(b)))$$

for all $b \in M(n, O_F)^-$, hence that $\mu(1 + \delta \mathcal{P}_F^l) = \{1\}$. Because $\mu(1 + \mathcal{P}_F^l) = \{1\}$ as well, this would in turn imply that $\mu(1 + \mathcal{P}_F^l + \delta \mathcal{P}_F^l) = \{1\}$, contradicting the definition of $l$, hence
\[ \lambda_m \geq l. \] Now let \( s \) be the smallest integer between 1 and \( m \) such that \( \lambda_s = \lambda_m \), by the arguments of Proposition 4.2 we obtain that

\[ \mu(d_{E}) \begin{pmatrix} a & b \\ \varphi_F^{-1} \Delta b & a \end{pmatrix} = \mu(\det(a)) \mu(1 + \Tr(\delta a^{-1} \varphi_F^\lambda b)) \]

for \( a \in \text{GL}(n, \mathcal{O}_F) \) and \( b \in \mathcal{M}(n, \mathcal{O}_F) \). By reduction we deduce that

\[ \mu(d_{E}) \begin{pmatrix} \pi & b \\ \pi \end{pmatrix} = \mu(\det(\pi)) \mu(1 + \varphi_F^\lambda \Tr(\delta \pi^{-1} \text{diag}(0_{s-1}, I_{m-s+1}) \delta)) \]

for \( a \in \text{GL}(n, k_F) \) and \( b \in \mathcal{M}(n, k_F) \). However the identity

\[ \pi \begin{pmatrix} \pi & b \\ \pi \end{pmatrix} = \mu(d_{E}) \begin{pmatrix} \pi & b \\ \pi \end{pmatrix} \text{Id} \]

first implies that if \( \lambda_m > l \) then the unipotent radical of type \((m, m)\) acts trivially on the space \( \pi \chi \) contradicting its cuspidality, hence \( \lambda_m = l \). It also implies that

\[ \overline{b} \mapsto \mu(1 + \delta \varphi_F^\lambda \Tr(\text{diag}(0_{s-1}, I_{m-s+1}) \delta)) \]

must be invariant under conjugation by \( \text{GL}(m, k_F) \), which in turn implies that \( s = 1 \) hence \( \lambda_1 = \cdots = \lambda_m = l \).

**Remark 4.2.** A similar analysis could certainly be done when \( E/F \) is ramified but we don’t have at our disposal the description of the double coset representatives given by [Off04] in the unramified case. As we can still prove the Prasad and Takloo-Bighash conjecture in this case, without computing the exact multiplicity, we do not pursue this direction.

### 4.4 Characterization of distinction of level zero cuspidal representations

The spaces \( \text{Hom}_{K \cap \mathcal{O}_{d_{h}}} (\lambda_{\chi}, \mu_{E}) \) is isomorphic to a subspace of \( \text{Hom}_H (\pi(\chi), \mu_{E}) \) thanks to Mackey theory for compact induction from open subgroups. Hence as a corollary of Propositions 4.2 and 4.4, Corollary 3.2 and Theorem 2.1, we deduce the all assertions of the following theorem except the last one.

**Theorem 4.1.** For \( n \geq 4 \), the depth-zero cuspidal representation \( \pi(\chi) \) of \( \text{GL}(n, F) \) is \( \mu \)-distinguished if and only if \( \chi_{L_F} = \mu \circ N_{L_F/F} \), except when \( E/F \) is ramified and \( \mu \) is tame, in which case \( \pi(\chi) \) is never \( \mu \)-distinguished. When \( \mu \) is tame or \( E/F \) is unramified, the dimension of \( \text{Hom}_H (\pi(\chi), \mu_{E}) \) is one when nonzero.

### 5 On \( \mu \)-selfduality and \( \mu \)-symplecticity for Langlands parameters

In this section \( \mu \) is any character of \( F^* \) which we identify with a character of \( W_F \) denoted by \( \mu \) again. For \( \phi \) a finite dimensional irreducible representation of \( W_F \), we say that \( \phi \) is \( \mu \)-selfdual if

\[ \phi \cong \mu \otimes \phi'. \]
On the space of such a representation, there exists a non-zero bilinear form $B$ (necessarily non-degenerate) which satisfies

$$B(\phi(w)v, \phi(w)v') = \mu(w)B(v, v')$$

for all $v$ and $v'$ in $V_\phi$. By Schur’s Lemma the space of such bilinear forms $B$ is one dimensional hence $B$ is either symmetric or alternate, but not both. In the first case we say that $\phi$ is $\mu$-orthogonal (or $\mu$-selfdual of even parity) and in the second case we say that $\mu$-symplectic (or $\mu$-selfdual of odd parity).

We recall that the Langlands parameter of $\pi(\chi)$ is given in Theorem 2 of [BHT1]: it is

$$\phi(\pi(\chi)) := \text{Ind}_{W_L'}^{W_L}(\eta\chi)$$

where $\eta$ is the unramified quadratic character of $L^*$ and $W_L$ is the Weil group of $L$. The representation $\text{Ind}_{W_L}(\chi')$ with $\chi'(1 + P_L) = 1$ (when identified with a representation of $W_F$) is the same thing as a tame $n$-dimensional irreducible representation of $W_F$, i.e. one which is trivial on the wild inertia subgroup of $W_F$. Let $\rho$ be the unramified character of $W_F$ of order $2n$, then by [BHS17] Sections 6.1 and 6.2], if a tame $n$-dimensional irreducible representation $\phi$ of $W_F$ is selfdual, then the only selfdual unramified twist of $\phi$ different from it is $\rho \otimes \phi$, and $\phi$ and $\rho \otimes \phi$ have different parities. We adapt their discussion to the $\mu$-selfdual setting. Before we observe that if such a $\phi$ is $\mu$-selfdual, then $\mu$ is tame.

**Lemma 5.1.** Let $\phi$ be a tame $n$-dimensional irreducible representation of $W_F$. If $\phi$ is $\mu$-selfdual, then $\mu$ is trivial on the wild inertia subgroup of $W_F$.

**Proof.** Write $\phi = \text{Ind}_{W_F'}^{W_F}(\chi')$ with $\chi'$ trivial on the wild inertia subgroup of $W_L$, or identifying characters of Weil groups and of multiplicative groups of local fields thanks to class field theory, trivial on $1 + P_L$. Then there is $\gamma \in \text{Gal}_F(L)$ such that $\chi' = \mu \circ N_L/F \chi^{-1}$, but because $N_L/F(1 + P_L) = 1 + P_F$ ($L/F$ is unramified) and because $\chi$ and $\chi'$ are tame, so is $\mu$. $\square$

**Lemma 5.2.** Let $\phi$ be a tame $n$-dimensional irreducible representation of $W_F$, which is $\mu$-selfdual. Then the only $\mu$-selfdual unramified twist of $\phi$ which is different from it is $\rho \otimes \phi$, and $\phi$ and $\rho \otimes \phi$ have different parities as soon as $n \geq 4$.

**Proof.** The first assertion follows from the beginning of the discussion in [BHS17] Section 6.1], as if $\phi$ and $\rho \otimes \phi$ are $\mu$-selfdual for some character $\mu$ of $W_F$, then $\mu^2 \otimes \phi = \phi$, and the fact that the number $t(\phi)$ of unramified characters fixing $\phi$ is equal to $n$ (by [BHT1] Sections 6.2 for example). Now we need to prove that $\phi$ and $\rho \otimes \phi$ have different parities when $n \geq 3$. We write $\phi = \text{Ind}_{W_F'}^{W_F}(\chi')$ with $\chi'$ tame, then there is $\sigma \in W_F/W_L = \text{Gal}_F(L)$ such that $\chi'^{\sigma} = \mu W_L \chi^{\sigma-1}$. We claim that $\sigma$ can’t be 1: to prove this we freely use class field theory to identify characters of Weil groups and the multiplicative groups of local fields. If $\sigma$ was one, then one would have the relation $\chi'^2 = \mu \circ N_{L/F}$. Because all characters under consideration are tame, this would imply that $\chi'^2 = \pi \circ N_{k_L/k_F}$. But the group of characters of the form $\alpha \circ N_{k_L/k_F}$ for $\alpha$ a character of $k_F^*$ form a group of order $q_F - 1$, hence one should have $\chi'^{2q_F-1} = 1$. But because $\chi'$ is regular and $n \geq 3$, the character $\chi'^{2q_F-1}$ must be nontrivial, hence $\chi'^{2(q-1)} \neq 1$ in conclusion $\chi'^{\sigma} = \mu W_L \chi^{\sigma-1}$ for some $\sigma \neq 1$ in $W_F/W_L$. Conjugating the latter relation by $\sigma$ again we see that $\sigma^2$ fixes $\chi$ which is regular, hence $\sigma^2 = 1$ so that $\sigma$ is the conjugation with respect to the quadratic sub-extension $L_0$ of $L$ lying over $F$. Then $\text{Ind}_{W_L}^{W_F}(\chi')$ is $\mu$-selfdual, and its restriction $\chi \otimes \chi'^{\sigma} = \chi \otimes \mu W_L \chi^{-1}$ to $W_L$ affords a line of symmetric and a line of alternating $(W_L, \mu)$-equivariant forms. Both
are stable under $W_{L_0}$ because $\mu$ extends to a character of $W_F$ (hence of $W_{L_0}$), and one line affords the trivial representation of $\Gal_{L_0}(L)$ whereas the other its quadratic character. We now concude with the arguments of the end of [BHS17 Section 6.1].

As in the proof above $L_0$ is the unique unramified extension of degree $m$ of $F$ contained in $L$. The representation $\pi(\chi)$ is $\mu$-symplectic if there exists a non-degenerate alternating bilinear form $\langle \cdot , \cdot \rangle$ on $\mathbb{C}^n$ that is preserved by its Langlands parameter $\phi := \Ind_{W_{L_0}}^W(\eta \chi)$ with similitude factor $\mu_{\phi}$:

$$\langle \phi(w)v, \phi(w)v' \rangle = \mu(w) \langle v, v' \rangle \quad \text{for all } w \in W_F, \ v, v' \in \mathbb{C}.$$  

In particular such a representation is $\mu$-selfdual. Let us give a characterization of the $\mu$-symplecticity of $\pi(\chi)$.

**Proposition 5.1.** For $n \geq 4$, the representation $\pi(\chi)$ is $\mu$-symplectic if and only if $\chi|_{L_0} = \mu \circ N_{L_0}/F$.

**Proof.** First, suppose that $\pi(\chi)$ is $\mu$-symplectic. Let us fix a basis of $\mathbb{C}^n$ such that the matrix of $\langle \cdot , \cdot \rangle$ in this basis is $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ and let us denote by $M(w)$ the matrix of $\phi(w)$ in the same basis. The $\mu$-symplecticity can then be seen matricially: $\langle M(w)J M(w) \rangle = \mu(w)J$ for all $w$ in $W_F$. By using the Pfaffian (Pf) properties, we get

$$\det(M(w))Pf(J) = Pf(\mu M(w)J M(w)) = Pf(\mu M(w)J w) = \mu^m(w)Pf(J) \ \forall w \in W_F$$

and as $Pf(J) \neq 0$, we deduce that $\det(\phi) = \mu^m$.

On the other hand, by the universal property of the tensor product, the non-degenerate bilinear form $\langle \cdot , \cdot \rangle$ provides a $\mu$-equivariant non-zero linear form on $\phi \otimes \phi$. The $\mu$-symplecticity of $\pi(\chi)$ implies therefore that $\phi \approx \mu^{-1} \otimes \phi$. Now,

$$\phi^\vee \approx \mu^{-1} \otimes \phi \iff \pi(\chi^{-1}) \approx \pi(\chi^{-1} \circ N_{L/F}) \quad \text{because } \mu(1 + P_F) = 1$$

$$\iff \exists \gamma \in \Gal(F) \setminus \{id_L\} \text{ such that } \chi^{-1} = (\chi^{-1} \circ N_{L/F}) \circ \gamma$$

$$\iff \exists \gamma \in \Gal(F) \setminus \{id_L\} \text{ such that } (\chi \circ \gamma)\chi = \mu \circ N_{L/F}$$

$$\iff \exists \gamma \in \Gal(F) \setminus \{id_L\} \text{ such that } \chi = \chi \circ \gamma^2$$

by reinjecting $\chi$ in $\chi \circ \gamma$ then by simplifying

$$\Rightarrow \gamma^2 = id_L \text{ i.e. } \gamma \in \Gal_{L_0}(L) \text{ because } \chi \text{ is admissible.}$$

Moreover $\gamma \neq id_L$, otherwise we would have $\chi^2 = \mu \circ N_{L/F}$. In this case, we would have

$$\chi \circ \Frob_{L/F} = (\chi^{-1} \circ \Frob_{L/F})^{m}(\mu \circ N_{L/F})$$

which implies $\mu \circ N_{L/F} = 1$ (because $\chi^{-1} \circ \Frob_{L/F} = \chi \circ \Frob_{L/F}$). We then deduce that $\chi$ is quadratic. Then $\chi(\varphi \Frob_{L/F}(\varphi)) = 1$; for $x \in F_L$, $\chi(x \Frob_{L/F}(x)) = \chi(x^{1-q_F}) = 1$ because $1 - q_F$ is even; for $x' \in 1 + P_L$, $\chi(x' \Frob_{L/F}(x')) = 1$ because $\chi$ is tame. Thus, by Hilbert’s Theorem 90, we deduce that $\chi$ factors through $N_{L/F}$ so $\chi$ is invariant under $\Gal(L)$, which contradicts the admissibility of $\chi$.

Reciprocally, suppose $\det(\phi) = \mu^m$ and $\chi^{-1} = \gamma_0 \mu^{-1} \circ N_{L/F}$, where $\gamma_0$ generates $\Gal_{L_0}(L)$. This implies that $\phi^\vee \approx \mu^{-1} \otimes \phi$ i.e. $\rho^{-1} \otimes \phi$ is self-dual. Then, we can deduce from Lemma 5.2 that $\phi$ is $\mu$-symplectic. Indeed, Lemma 5.2 claims that if $\phi$ was $\mu$-orthogonal, then $\rho \otimes \phi$ would be $\mu$-symplectic and this would imply that $\det(\rho \otimes \phi) = 1$ so $\det(\phi) = \rho^{-1} \mu^m = \mu^m$, which contradicts
Theorem 6.1. Let $Bighash$ conjecture. When

We conclude by noticing that $det(\phi) = \chi_{|F^*}$ and that $\chi^{-1} = \chi \circ N_{L/F}$. Thus, $det(\phi) = \mu^m$ and $\chi^{-1} = \chi \circ N_{L/F}$ and we get the condition stated in this proposition.

6 The Prasad and Takloo-Bighash conjecture

We recall that the conjecture of Prasad and Takloo-Bighash has been proved by Tunnel and also Saito when $n=2$ ([Tun83, Theorem p.1277] in residual characteristic not 2, [Sai93 Theorem p.99] in characteristic not 2), hence in this Section we assume $n \geq 4$. So comparing the statements of Theorem [4] and Proposition [5], it is enough to compute the Prasad and Takloo-Bighash $\epsilon$ value of a cuspidal depth-zero representation $\pi(\chi)$ with $\chi_{L_\alpha} = \mu \circ N_{L_\alpha/F}$, and to show that it is as expected by the conjecture when $E/F$ is unramified or $E/F$ is ramified and $\mu$ is not tame, and differs from the expected value when $E/F$ is ramified and $\mu$ is tame. In the proof we will freely confuse characters of Weil groups and of multiplicative groups of local fields (hence restrictions will be often written as composition with the norm map).

Let’s do some preliminary computations before computing the $\epsilon$ factor of the Prasad and Takloo-Bighash conjecture. When $E/F$ is unramified we have:

\[
\text{Ind}_{W_L}^{W_E}(\eta \chi) \otimes \text{Ind}_{W_E}^{W_E}(\mu^{-1}) \\
= \text{Ind}_{W_L}^{W_E}(\eta \chi) \otimes \text{Ind}_{W_E}^{W_E}(\mu^{-1})|_{W_L} \\
= \text{Ind}_{W_L}^{W_E}(\eta \chi(\mu^{-1} \circ N_{L/E}) \otimes \eta \chi(\mu^{-1} \circ N_{L/E})) \\
\text{by Mackey’s restriction formula with } \sigma_{E/F} = \text{Gal}(E) \\
= \text{Ind}_{W_L}^{W_E}(\eta \chi(\mu^{-1} \circ N_{L/E})) \otimes \text{Ind}_{W_E}^{W_E}(\eta \chi(\mu^{-1} \circ N_{L/E})).
\]

When $E/F$ is ramified we have:

\[
\text{Ind}_{W_L}^{W_E}(\eta \chi) \otimes \text{Ind}_{W_E}^{W_E}(\mu^{-1}) \\
= \text{Ind}_{W_L}^{W_E}(\eta \chi) \otimes \text{Ind}_{W_E}^{W_E}(\mu^{-1})|_{W_L} \\
= \text{Ind}_{W_E}(\eta \chi \otimes \text{Ind}_{W_E}^{W_E}(\mu^{-1} \circ N_{M/E})) \\
\text{by Mackey’s restriction formula with } M = L, E \\
= \text{Ind}_{W_E}(\eta \chi \circ N_{M/E} \circ \mu^{-1} \circ N_{M/E}).
\]

Theorem 6.1. Let $\pi(\chi)$ be a depth-zero cuspidal representation of $\text{GL}_n(F)$, such that $\chi_{L_\alpha} = \mu \circ N_{L_\alpha/F}$. Let $\psi$ be a non-trivial additive character of $F$.

- If $E/F$ is unramified, then $\epsilon(\frac{1}{2}, \pi(\chi) \otimes \text{Ind}_{W_E}^{W_E}(1), \psi) = \omega_{E/F}(-)^m \mu(-1)^m$.

- If $E/F$ is ramified:
  - If $\mu$ is tame then $\epsilon(\frac{1}{2}, \pi(\chi) \otimes \text{Ind}_{W_E}^{W_E}(1), \psi) = \omega_{E/F}(-)^m \mu(-1)^m$.
  - If $\mu$ is not tame then $\epsilon(\frac{1}{2}, \pi(\chi) \otimes \text{Ind}_{W_E}^{W_E}(1), \psi) = \omega_{E/F}(-)^m \mu(-1)^m$. 

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Proof. If $L/K$ is a separable quadratic extension of non Archimedean local fields, we denote by $\sigma_{L/K}$ the associated Galois involution. We distinguish the ramified and the unramified case in our computations.

When $E/F$ is unramified. We recall the situation: $E$ is included in $L$ and possibly in $L_0$ according to the parity of $m$.

$$
\epsilon(\frac{1}{2}, \text{Ind}^W_L(\eta \chi) \otimes \text{Ind}^W_k(\mu^{-1}), \psi)
$$

$$
= \epsilon(\frac{1}{2}, \text{Ind}^W_L(\eta \chi(\mu^{-1} \circ N_{L/E})), \psi)\epsilon(\text{Ind}^W_k(\eta \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E})), \psi) \text{ by } \text{§1.5, 4.}
$$

$$
= \lambda_{L/E}(\psi)\epsilon(\frac{1}{2}, \eta \chi(\mu^{-1} \circ N_{L/E}), \psi_L)\epsilon(\frac{1}{2}, \eta \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}), \psi_L) \text{ by } \text{§1.5, 7.}
$$

$$
= \lambda_{L/E}(\psi_E)\lambda_{E/F}(\psi)\eta^2(\omega^d_{L}(\phi_L))\epsilon(\frac{1}{2}, \chi(\mu^{-1} \circ N_{L/E}), \psi_L)\epsilon(\frac{1}{2}, \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}), \psi_L) \text{ by } \text{§1.5, 9.}
$$

$$
= \omega_{E/F}(-1)^m\epsilon(\frac{1}{2}, \chi(\mu^{-1} \circ N_{L/E}), \psi_L)\epsilon(\frac{1}{2}, \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}), \psi_L) \text{ by } \text{§1.5, 10. and §8. and because } n \text{ is even.}
$$

Now we distinguish between two cases:
1. \( m \) is even: then

\[
\epsilon \left( \frac{1}{2}, \chi(\mu^{-1} \circ N_{L/E}), \psi_L \right) \epsilon \left( \frac{1}{2}, \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}), \psi_L \right)
\]

\[= \epsilon \left( \frac{1}{2}, \chi^{\sigma_{L/0}}(\mu^{-1} \circ N_{L/E}), \psi_L \right) \epsilon \left( \frac{1}{2}, \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}), \psi_L \right)\]

according to \( \text{§1.5, 3} \) because \( \psi_L = \psi_L^{\sigma_{L/0}} \)

and \( \mu^{-1} \circ N_{L/E} \) is also \( \sigma_{L/0} \)-invariant as \( E \subset L_0 \subset L \).

\[= \epsilon \left( \frac{1}{2}, \chi^{\sigma_{L/0}}(\mu^{-1} \circ N_{L/E}), \psi_L^{-1} \right) \epsilon \left( \frac{1}{2}, \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}), \psi_L \right)\]

from \( \text{§1.5, 2} \) because \( (\chi^{\sigma_{L/0}}(\mu^{-1} \circ N_{L/E}))(1) = (\mu \circ N_{L_0/F})(1)(\mu^{-1} \circ N_{L/E})(1) = \mu(1)\mu(1)^{-m} = 1 \)

But then because

\[\chi^{\sigma_{L/0}}(\mu^{-1} \circ N_{L/E}) \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}) = \chi \circ N_{L/L_0} \circ \mu^{-1} \circ N_{L/F} = \mu \circ N_{L/F} \circ \mu^{-1} \circ N_{L/F} = 1,\]

\( \text{§1.5, 4} \) implies that

\[\epsilon \left( \frac{1}{2}, \chi^{\sigma_{L/0}}(\mu^{-1} \circ N_{L/E}), \psi_L^{-1} \right) \epsilon \left( \frac{1}{2}, \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}), \psi_L \right) = 1,\]

and we recognize the expected value \( \epsilon \left( \frac{1}{2}, \text{Ind}_{W_L}^{W_E}(\eta \chi) \otimes \text{Ind}_{W_E}^{W_F}(\mu^{-1}), \psi \right) = \omega_{E/F}(-1)^m \mu(-1)^m \)

because \( m \) is even.

2. \( m \) is odd: then we notice that both \( \chi(\mu^{-1} \circ N_{L/E}) \) and \( \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}) \) restrict to \( L_0 \) as \( \chi_{|L_0}(\mu^{-1} \circ N_{L_0/E}) = 1 \). Hence by \( \text{§1.5, 6} \) for \( v \in L - L_0 \) such that \( v^2 \notin L_0 \), we have

\[\epsilon \left( \frac{1}{2}, \chi(\mu^{-1} \circ N_{L/E}), \psi_L \right) = \chi(v)\mu^{-1}(N_{L/E}(v))\]

and

\[\epsilon \left( \frac{1}{2}, \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}), \psi_L \right) = \chi(v)\mu^{-\sigma_{E/F}}(N_{L/E}(v)),\]

so that

\[\epsilon \left( \frac{1}{2}, \chi(\mu^{-1} \circ N_{L/E}), \psi_L \right) \epsilon \left( \frac{1}{2}, \chi(\mu^{-\sigma_{E/F}} \circ N_{L/E}), \psi_L \right) = \chi(v^2)\mu^{-1}(N_{L/F}(v))\]

\[= \mu(N_{L_0/F}(v^2)N_{L/F}(v^{-1})) = \mu(N_{L_0/F}(v^2N_{L_0/F}(v^{-1}))) = \mu(N_{L_0/F}(-1))\]

because \( \sigma_{L/L_0}(v) = -v \), hence finally

\[\epsilon \left( \frac{1}{2}, \text{Ind}_{W_L}^{W_E}(\eta \chi) \otimes \text{Ind}_{W_E}^{W_F}(\mu^{-1}), \psi \right) = \omega_{E/F}(-1)^m \mu(-1)^m\]

which is again the expected value.

**When \( E/F \) is ramified.** In this case, \( E \) is not included in \( L \). Set \( M \) to be the extension of \( L \) generated by \( L \) and \( E \), \( M \) is therefore unramified \( n \)-dimensional on \( E \). We also set \( L_1 = \langle E, L_0 \rangle \) so that \( M \) is an unramified quadratic extension of \( L_1 \). The situation is as follows.
Before proceeding further with the computation let’s discuss the conductor of the character $\chi \circ N_{M/L} \cdot \mu^{-1} \circ N_{M/E}$.

- **If $\mu$ is not tame** then $\chi \circ N_{M/L} \cdot \mu^{-1} \circ N_{M/E}$ clearly has the same conductor as $\mu^{-1} \circ N_{M/E}$ which is also not tame as it has the same conductor as $\mu$, by surjectivity of $N_{M/E}$ from $1 + \mathfrak{P}_M^d$ onto $1 + \mathfrak{P}_E^d$ for any $d \geq 1$. In particular $\chi \circ N_{M/L} \cdot \mu^{-1} \circ N_{M/E}$ has conductor $c(\mu)$ which is even as we saw in Section \ref{sec:conductor}.

- **If $\mu$ is tame** let us show that the character $\chi \circ N_{M/L} \cdot \mu^{-1} \circ N_{M/E}$ has conductor 1. Clearly it is trivial on $1 + \mathfrak{P}_M$ because $\chi$ and $\mu$ are tame, but if it was unramified, going backwards one would deduce that $\text{Ind}^{W_F}_{W_L}(\chi) \otimes \text{Ind}^{W_E}_{W_M}(\mu^{-1})$ would be unramified, hence a direct sum.
of unramified characters. But \( \text{Ind}_{W_E}^{W_L}(\chi) \otimes \text{Ind}_{W_E}^{W_L}(\mu^{-1}) \) cannot contain any character, otherwise by irreducibility of \( \text{Ind}_{W_E}^{W_L}(\chi) \), it would appear as sub-representation of a character twist of \( \text{Ind}_{W_E}^{W_L}(\mu) \), which is impossible for dimension reasons (remember that we suppose \( n \geq 3 \)). Hence \( \chi \circ N_{M/L} \mu^{-1} \circ N_{M/E} \) has conductor 1.

Hence setting \( c'(\mu) = c(\mu) \) when \( c(\mu) \geq 1 \) and \( c'(\mu) = 1 \) when \( \mu \) is unramified, we obtain \( c(\chi \circ N_{M/L} \mu^{-1} \circ N_{M/E}) = c'(\mu) \), which is even as soon as \( c'(\mu) > 1 \). Finally we obtain:

\[
\epsilon\left(\frac{1}{2} \text{Ind}_{W_L}^{W_E}(\eta \chi) \otimes \text{Ind}_{W_E}^{W_L}(\mu^{-1}), \psi\right) = (-1)^{d(\psi, \omega)(n-1)} \omega_{E/F(-1)^m}(\omega_{M' \setminus M}(\omega_m)^{-c'(\mu)}) \epsilon\left(\frac{1}{2} \chi \circ N_{M/L} \mu^{-1} \circ N_{M/E}, \psi_M\right) \]

thanks to §1.5.5

\[
= (-1)^{d(\psi, \omega)(n-1)} \omega_{E/F(-1)^m}(\omega_{M' \setminus M}(\omega_m)^{-c'(\mu)}) \epsilon\left(\frac{1}{2} \chi \circ N_{M/L} \mu^{-1} \circ N_{M/E}, \psi_M\right)
\]

because \( n \) is even.

Note that \( M/L_0 \) is bi-quadratic, so there is one more quadratic extension \( L_2 \) of \( L_0 \) under \( M \). Now the restriction of \( \chi \circ N_{M/L} \) to \( L_2 \) is equal to \( \chi \circ N_{L_2/L_0} = \mu \circ N_{L_2/F} \), whereas that of \( \mu^{-1} \circ N_{M/E} \) is equal to \( \mu^{-1} \circ N_{L_2/F} \), hence \( \chi \circ N_{M/L} \mu^{-1} \circ N_{M/E} \) restricts trivially to \( L_2 \).

Take \( v \in L \setminus L_0 \) with \( v^2 \in L_0 \). Then \( M = L_2[v] \) and we can apply §1.5.5.4:

\[
\epsilon\left(\frac{1}{2} \chi \circ N_{M/L} \mu^{-1} \circ N_{M/E}, \psi_M\right) = \chi(v^2) \mu^{-1} \circ N_{L_2/F}(v^2)
\]

\[
\chi(v^2) \mu^{-1} \circ N_{L_2/F}(v^2)
\]

\[
\mu(v^2) \mu^{-1} \circ N_{L_2/F}(v^2)
\]

Thus \( \epsilon\left(\frac{1}{2} \text{Ind}_{W_L}^{W_E}(\eta \chi) \otimes \text{Ind}_{W_E}^{W_L}(1), \psi\right) = (-1)^{c'(\mu)} \omega_{E/F(-1)^m}(\omega_{E/F(-1)^m}) \), as expected.

As a corollary, we obtain:

**Corollary 6.1.** Let \( \pi(\chi) \) be a depth 0 cuspidal representation of \( \text{GL}(2m, F) \), let \( \mu \) be a character of \( E^* \), then \( \pi(\chi) \) is \( \mu \circ \det_{\text{GL}(m, E)} \)-distinguished by \( H = \text{GL}(m, E) \) if and only if

1. \( \pi(\chi) \) is \( \mu_{F^*} \)-symplectic;

2. \( \epsilon\left(\frac{1}{2} \text{Ind}_{W_L}^{W_E}(\eta \chi) \otimes \text{Ind}_{W_E}^{W_L}(\mu^{-1})\right) = \omega_{E/F(-1)^m}(\mu)(-1)^m \).

**References**

[Bad08] Alexandru Ioan Badulescu. Global Jacquet-Langlands correspondence, multiplicity one and classification of automorphic representations. *Invent. Math.*, 172(2):383–438, 2008. With an appendix by Neven Grbac.
[BH06] Colin J. Bushnell and Guy Henniart. The local Langlands conjecture for \( GL(2) \), volume 335 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006.

[BH11] Colin J. Bushnell and Guy Henniart. Explicit functorial correspondences for level zero representations of \( p \)-adic linear groups. J. Number Theory, 131(2):309–331, 2011.

[BHS17] Corinne Blondel, Guy Henniart, and Shaun Stevens. Jordan blocks of cuspidal representations of symplectic groups. Algebra & Number Theory, 12, 04 2017.

[BL02] P. Broussous and B. Lemaire. Building of \( GL(m, D) \) and centralizers. Transform. Groups, 7(1):15–50, 2002.

[BM19] Paul Broussous and Nadir Matringe. Multiplicity one for pairs of prasad–takloo-bighash type. IMRN, 2019.

[BR17] Alexandru Ioan Badulescu and Philippe Roche. Global Jacquet-Langlands correspondence for division algebras in characteristic \( p \). Int. Math. Res. Not. IMRN, (7):2172–2206, 2017.

[BS17] Paul Broussous and Peter Schneider. Type theory and coefficient systems on the building. Bull. Soc. Math. France, 145(1):97–159, 2017.

[Cho19] Marion Chommaux. Distinction of the steinberg representation and a conjecture of prasad and takloo-bighash. Journal of Number Theory, 202:200 – 219, 2019.

[FQ73] A. Fröhlich and J. Queyrut. On the functional equation of the Artin \( L \)-function for characters of real representations. Invent. Math., 20:125–138, 1973.

[GL18] Wee Teck Gan and Luis Lomelí. Globalization of supercuspidal representations over function fields and applications. J. Eur. Math. Soc. (JEMS), 20(11):2813–2858, 2018.

[Guo97] Jiandong Guo. Uniqueness of generalized Waldspurger model for \( GL(2n) \). Pacific J. Math., 180(2):273–289, 1997.

[HM] Jeffrey Hakim and Fiona Murnaghan. Distinguished tame supercuspidal representations. Int. Math. Res. Pap. IMRP, (2):Art. ID rpn005, 166.

[Lan70] Robert Langlands. On the functional equation of the artin \( l \)-functions. Preprint, 1970.

[Mat15] Nadir Matringe. On the local Bump-Friedberg \( L \)-function. J. Reine Angew. Math., 709:119–170, 2015.

[Moy86] Allen Moy. Local constants and the tame Langlands correspondence. Amer. J. Math., 108(4):863–930, 1986.

[Off04] Omer Offen. Relative spherical functions on \( p \)-adic symmetric spaces (three cases). Pacific J. Math., 215(1):97–149, 2004.

[Pra00] Dipendra Prasad. The space of degenerate Whittaker models for general linear groups over a finite field. Internat. Math. Res. Notices, (11):579–595, 2000.

[Pra19] Dipendra Prasad. Multiplicities under basechange: finite field case. To appear in J. Algebra, 2019.
Dipendra Prasad and Rainer Schulze-Pillot. Generalised form of a conjecture of Jacquet and a local consequence. *J. Reine Angew. Math.*, 616:219–236, 2008.

Dipendra Prasad and Ramin Takloo-Bighash. Bessel models for GSp(4). *J. Reine Angew. Math.*, 655:189–243, 2011.

Hiroshi Saito. On Tunnell’s formula for characters of GL(2). *Compositio Math.*, 85(1):99–108, 1993.

Miyu Suzuki. Classification of gl(e)-distinguished representations. *Preprint*, 2019.

J. Tate. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 3–26. Amer. Math. Soc., Providence, R.I., 1979.

Jerrold B. Tunnell. Local \(\epsilon\)-factors and characters of GL(2). *Amer. J. Math.*, 105(6):1277–1307, 1983.

André Weil. *Basic number theory*. Springer-Verlag, New York-Berlin, third edition, 1974. Die Grundlehren der Mathematischen Wissenschaften, Band 144.

Hang Xue. Epsilon dichotomy for linear models. *Preprint*, 2019.