Improper posteriors are not improper

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December 6, 2017

Abstract

In 1933 Kolmogorov constructed a general theory that defines the modern concept of conditional expectation. In 1955 Rényi formulated a new axiomatic theory for probability motivated by the need to include unbounded measures. We introduce a general concept of conditional expectation in Rényi spaces. In this theory improper priors are allowed, and the resulting posterior can also be improper.

In 1965 Lindley published his classic text on Bayesian statistics using the theory of Rényi, but retracted this idea in 1973 due to the appearance of marginalization paradoxes presented by Dawid, Stone, and Zidek. The paradoxes are investigated, and the seemingly conflicting results are explained. The theory of Rényi can hence be used as an axiomatic basis for statistics that allows use of unbounded priors.

Keywords: Haldane’s prior; Poisson intensity; Marginalization paradox; Measure theory; conditional probability space; axioms for statistics; conditioning on a sigma field; improper prior

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1 Introduction

An often voiced criticism of the use of improper priors in Bayesian inference is that such priors sometimes don’t lead to a proper posterior distribution. This can happen when the marginal prior distribution of the data is not σ-finite (Taraldsen and Lindqvist, 2010), as sometimes encountered in applied settings with sparse data (e.g. Druilhet et al., 2016; Tufto et al., 2012, Appendix S4).

As a simple motivating example, suppose that we observe a homogeneous Poisson process, and that we start with a non-informative scale prior \( \pi(\lambda) = 1/\lambda \) on the Poisson intensity \( \lambda \). The distribution of the number \( X_1 \) of events in the interval \((0, t_1] \) is then not σ-finite since \( P(X_1 = 0) = \int_0^\infty P(X_1 = 0|\lambda)\pi(\lambda)\,d\lambda \) is infinite. If we observe \( X_1 = 0 \) and formally multiply the prior by the likelihood we obtain an improper posterior \( \pi(\lambda|X_1 = 0) \propto e^{-\lambda t_1}/\lambda \). This distribution for \( \lambda \) is different from the initial prior, and we claim that this is a correct way of incorporating the information given by \( X_1 = 0 \).

A related example is the Beta posterior density for the success probability \( p \) given by \( \pi(p|\alpha) = p^{\alpha-1}(1-p)^{\beta-1} \) for a Bernoulli sequence with \( \alpha \) successes out of \( \alpha + \beta \) trials. This corresponds to the Haldane (1932) improper prior \( \pi(p) = p^{-1}(1-p)^{-1} \), and the posterior is improper if \( \alpha \) or \( \beta \) is zero. In all cases, however, the observation of the number of successes \( \alpha \) results in a corresponding updating of the uncertainty associated with \( p \). This is given by the possibly improper posterior.

Unfortunately, even people accepting the use of improper priors reject this form of inference, on the ground that the posterior is not a probability distribution, and a mathematical theory is lacking for this (Robert et al., 2009). This is understandable, and we agree initially with this point of view. We will demonstrate, however, that the above forms of, so far, formal inference can be made consistent with the axiomatic system of Rényi which allows improper laws. We propose and claim that the mathematical theory developed in the following gives a rigorous foundation for inference with unbounded laws.

The most familiar example of an unbounded law \( P_\Theta \) is the uniform law on the real line \( \Theta = \mathbb{R} \). Following Rényi (1970) and Taraldsen and Lindqvist (2016) the uniform law is identified with the countable collection of uniform laws \( P_\Theta(\cdot|B_n) \) on each interval \( B_n = [-n, n] \). This gives then also the interpretation of \( P_\Theta \). Given that \( \Theta \in B_n \) the law is the uniform probability distribution on \( B_n \). The family \( \mathcal{B} = \{B_n \mid n \in \mathbb{N}\} \) is a bunch and the family \( \{P_\Theta(\cdot \mid B) \mid B \in \mathcal{B}\} \) defines a Rényi space. The improper laws for the intensity \( \lambda \) and the success probability \( p \) in the initial examples are interpreted similarly. The concept of a Rényi space and other elements from measure theory are summarized in Appendix A.

The aim of this paper is to present a theory of statistics that allows improper laws both as priors and posteriors. This extends the results of Taraldsen and Lindqvist (2010, 2016), and provides stronger links to the results on improper laws presented by Hartigan (1983) and Bioche and Druilhet (2016). The main mathematical result is Theorem 1 which proves existence and uniqueness of conditional expectation on Rényi spaces. The existence and uniqueness proof relies on the Radon-Nikodym theorem and a generalization of the Rényi structure theorem. The theory of conditional expectation has been most important for the development of measure theory, probability and statistics based on Kolmogorov’s concept of a probability space. The generalization of this to the setting of Rényi spaces can hence be expected to be important for future developments in mathematics and related fields.

Within this framework we reach the view that improper posteriors, just as improper priors, are not ‘improper’ but may reflect complete or partial ignorance about a parameter after conditioning on the data. Returning to the above Poisson-process example, at time \( t_1 \), we have clearly learned something about \( \lambda \) in that our belief in large values of the Poisson intensity \( \lambda \) has decreased while our relative degree of belief in small values of \( \lambda \) has remained approximately unchanged. That
the posterior is improper do not imply that our prior was wrong, but only that more data perhaps needs to be collected if possible. Proceeding by using the improper posterior at time \( t_1 \) as prior in subsequent inference, say based on the number of occurrences observed in a sufficiently long subsequent interval \([t_1, t_2]\), we indeed eventually reach the same proper final posterior as the one reached by combining the initial scale prior and the likelihood for the data on \((0, t_2]\). We hope that the reader can appreciate that this argument indicates also the potential philosophical importance of unbounded laws more generally.

The most influential initial work on Bayesian inference is given by the book of Jeffreys (1939). Parts of his arguments were mainly intuitive, and there is a lack of mathematical rigor as also observed by Robert et al. (2009). The needed mathematical theory for a rigorous reformulation of the original arguments of Jeffreys (1939) is presented next.

2 Existence and uniqueness conditional expectation

Taraldsen and Lindqvist (2010, 2016) define the posterior law \( P(A \mid X = x) \) for the case where the data \( X \) is \( \sigma \)-finite. The aim now is to prove existence and uniqueness of a posterior law without assuming that \( X \) is \( \sigma \)-finite. It will be convenient to do this as an extension of conventional measure theory, and then get the result for the posterior law as a special case. The reader is advised to consult Appendix A for the definition of a Rényi space and other elements from measure theory if needed.

Let \((X, \mathcal{F})\) be a measurable space and let \((T, \mathcal{G}, \nu)\) be a measure space (Rudin, 1987). Let a measurable function \( T \ni t \mapsto \mu_t(A) \in [0, \infty] \) be given for each \( A \in \mathcal{F} \). We define \( \mu \) to be a strong random measure on \( \mathcal{F} \) with respect to the law \( \nu \) if the following holds for all disjoint measurable \( A_1, A_2, \ldots \):

1. \( \mu_t(\emptyset) = 0 \) for almost all \( t \).
2. \( \mu_t(A_1 + A_2 + \cdots) = \mu_t(A_1) + \mu_t(A_2) + \cdots \) for almost all \( t \).

The notation \( A + B \) denotes the union of two disjoint and measurable sets. Equality for almost all \( t \) means that equality holds for all \( t \) in a set \( E \) where \( \nu(E^c) = 0 \). All functions of \( t \) here and in the following are assumed only to be defined almost everywhere, and the set \( E \) corresponding to \( E \ni t \mapsto \mu_t(A_k) \) depends on \( A_k \).

If there exists \( B_1, B_2, \ldots \) with \( X = \bigcup_n B_n \) and \( 0 < \mu_t(B_n) < \infty \), then \( \mu \) is said to be \( \sigma \)-finite. If, additionally, \( \mu_t \) is a measure on \( \mathcal{F} \) for all \( t \), then \( \mu \) is a random measure. In this paper we define the space \( X \) to be regular if every \( \sigma \)-finite strong random measure can be represented by a random measure. The Borel \( \sigma \)-algebra of a complete separable metric space \( X \) gives a regular space.

The concept of a strong random measure is here introduced similarly to how Skorohod (1984, p.1-2) introduces the concept of a strong random operator. He also defines the notion of a weak random operator by duality, but this is equivalent with strong when the image space is the complex numbers. It should be noted that the naming convention here is counter intuitive in the sense that a strong random operator is a weaker concept than a random operator, but there are good reasons for adopting the conventions of Skorohod.

Let \( \mu \) be a measure on \( X \) and let \( T : X \to T \) be measurable. We define a strong random measure \( \mu_t(A) \) to be the conditional law of \( \mu \) given \( T = t \) if

\[
\mu(A[T \in C] \mid B) = \int_C \mu_t(A \mid B) \mu_T(dt \mid B)
\] (1)

3
for all measurable $A,C$ and all $B$ with $0 < \mu(B) < \infty$, where \( \mu^t(A \mid B) = \mu^t(AB) / \mu^t(B) \), \( \mu_T(D \mid B) = \mu(T \in D \mid B) \), and \( \mu(B) = \mu(FB) \) (no normalization here!). The notation \( (T \in D) = \{ x \mid T(x) \in D \} = T^{-1}(D) \) is similar to the notation \( \{ T(x) \in D \} \) used by Doob (1953, p.1). For a conditional law we use the notation \( \mu^t(A) = \mu(A \mid T = t) \). The double use of the symbol \( \mu \) in the above is justified by:

**Theorem 1.** A unique \( \sigma \)-finite conditional law \( \mu(A \mid T = t) \) exists if \( \mu \) is a \( \sigma \)-finite measure on \( X \) and \( T : X \to T \) is measurable.

**Proof.** If \( t \mapsto c(t) > 0 \) is measurable, then \( \mu^t(AB) / \mu^t(B) = c(t) \mu^t(AB) / [c(t) \mu^t(B)] \) shows that uniqueness up to multiplication by a positive function is the best possible uniqueness. The measure \( C \mapsto \mu(AB[T \in C]) \) is dominated by the \( \sigma \)-finite measure \( C \mapsto \mu([T \in C \mid B]) \), so a unique normalized strongly random \( \mu^t(A \mid B) \) follows from the Radon-Nikodym theorem. It remains to prove that \( \mu^t(A \mid B) = \mu^t(AB) / \mu^t(B) \) for a \( \sigma \)-finite strong random measure \( \mu^t \).

Let \( B_1 \subset B_2 \subset \cdots \) with \( 0 < \mu(B_n) < \infty \) and \( \bigcup_n B_n = X \), and define \( \mu^t(B_n) = \mu^t(B_n \mid B_1 \cup B_n) / \mu^t(B_1 \mid B_1 \cup B_n) \). It follows that \( \mu(B_n) = 1 / \mu^t(B_1 \mid B_n) \) and \( \mu^t(B_1) = 1 \). For \( A \subset B_n \) put \( \mu^t(A) = \mu^t(A \mid B_n) \mu^t(B_n) \) and define \( \mu^t(A) = \mu^t(A \cap B_1) + \sum_{n \geq 2} \mu^t(A \cap B_n \cap B_{n-1}^c) \) for a general \( A \in \mathcal{F} \).

It must be proved that the above construction is well defined. Let \( A \subset B_n \subset B_m \). It must be proved that (*) \( \mu^t(A \mid B_n) \mu^t(B_n) = \mu^t(A \mid B_m) \mu^t(B_m) \). Observe first that \( \mu(B_n \mid B_m[T \in D]) = \int_D \mu^t(B_n \mid B_m) \mu_T(dt \mid B_m) \) gives \( \mu_T(dt \mid B_n) = \mu^t(B_n \mid B_m) \mu_T(dt \mid B_m) \). The (*) claim follows from

\[
\mu(A(T \in D)) = \int_D \mu^t(A \mid B_n) \mu^t(B_n \mid B_m) \mu_T(dt \mid B_m) = \int_D \mu^t(A \mid B_m) \mu_T(dt \mid B_m) \tag{2}
\]

since \( \mu^t(B_n \mid B_m) = \mu^t(B_n) / \mu^t(B_n) = \mu^t(B_1 \mid B_m) / \mu^t(B_1 \mid B_n) \) follows from the case \( A = B_1 \) in equation (2). This defines a unique \( \mu^t(A) \). The remaining claims follow by verification and is left to the reader. \( \square \)

All of the previous can be repeated with a replacement of the measurable set \( A \) with a positive measurable function \( A : X \to [0, \infty] \) and conditional expectation of complex valued functions can be defined by decomposition in positive and negative parts and then in real and complex parts. Consideration of the dual space gives conditional expectation of a separable Banach space valued \( A : X \to B \). The conditional expectation \( E(A \mid T = t) \) is in particular well defined when \( A \) is a separable Hilbert space valued. Separability is assumed to ensure almost everywhere definition on \( T \).

An alternative approach, as noted by Kolmogorov (1933, p.54, eq.10), is to define the conditional expectation by integration with respect to the conditional probability. For \( B \) equal to the set of real numbers \( \mathbb{R} \) or the set of complex numbers \( \mathbb{C} \) the alternative approach gives the same strong random linear functional, but for more general \( B \) there are many alternative routes with different results.

Conditional expectation \( \mu(A \mid T) \) with respect to a \( \sigma \)-field \( T \subset \mathcal{F} \) is defined by \( T(x) = x \) and \( (T, \mathcal{G}) = (X, T) \). It can be noted that we define \( \mu^t(A) \) directly instead of more indirectly by first defining \( \mu(A \mid T) \) as is more common. This has the advantage of allowing a completely general measurable space \( T \), whereas the common approach requires separability properties of \( X \) according to Schervish (1995, p.616, Prop.B.24).

It can finally be observed that the proof of Theorem 1 contains a proof of a structure theorem for strong random Rényi spaces defined by a consistent family of strong random conditional probabilities \( \mu^t(A \mid B) \) for \( B \in B \) for a fixed bunch \( B \subset \mathcal{F} \). The family is generated by a strong
random measure $\mu^t(A)$ such that $\mu^t(A|B) = \mu^t(AB)/\mu^t(B)$. The consistency requirement is that $B_1 \subset B_2$ implies $\mu^t(B_1|B_2) > 0$ and

$$\mu^t(A|B_1) = \frac{\mu^t(AB_1|B_2)}{\mu^t(B_1|B_2)} \quad (3)$$

3 **Examples**

3.1 **Mathematical statistics**

A statistical model is given by the structure

\[
\begin{align*}
(\Omega, \mathcal{E}, P) & \quad \xrightarrow{\psi} \quad (\Omega_\Theta, \Gamma) \\
\Theta & \quad \xrightarrow{\phi} \quad (\Omega_X, \Omega_Y)
\end{align*}
\]

In conventional theory (Schervish, 1995) the space $\Omega$ is a probability space. In the more general setting of a Rényi space $\Omega$ considered here the underlying law $P$ is a conditional probability law with a corresponding bunch $\mathcal{E}$. The law of the data $X$ given the parameter $\Theta = \theta$ is defined by $P^X_\Theta(A) = P(X \in A | \Theta = \theta)$. The law of the model parameter $\Theta$ given the data $X = x$ is defined by $P^\Theta_\Theta(A) = P(\Theta \in A | X = x)$. The posterior law $P^X_\Theta$ of a parameter $\gamma = \psi(\theta)$ and the law $P^\Theta_\Theta$ of a statistic are determined by this and equation (4).

In previous work by Taraldsen and Lindqvist (2010, 2016); Lindqvist and Taraldsen (2017) it was required that the data $X$ is $\sigma$-finite, but Theorem 1 shows that this requirement is not needed. There is, however, a prize: The posterior must in general be interpreted as an improper law. The initial examples demonstrate, however, that even if the data $X$ is not $\sigma$-finite, it may happen that the posterior is a proper distribution for some values of $x$. Assuming that $X$ is $\sigma$-finite ensures that the posterior is always proper. A similar comment holds for the law $P^\Theta_\Theta$ of the data $X$. The factorization $P_{X,\Theta}(dx, d\theta) = P_X(dx | \Theta = \theta) P_{\Theta}(d\theta)$ holds uniquely if and only if $\Theta$ is $\sigma$-finite. In this case, a $\sigma$-finite $\Theta$ is required if the most common Bayesian recipe is to be used:

1. Specify a statistical model law $P^\Theta_X$
2. Specify a prior $P_\Theta$
3. Compute the posterior $P^X_\Theta$

If $\Theta$ is not $\sigma$-finite, the first two steps must be replaced by a direct specification of a joint $\sigma$-finite law $P_{X,\Theta}$, and Theorem 1 ensures then that the posterior $P^X_\Theta$ is uniquely defined.

3.2 **Densities**

Assume that $(X, \Theta) \sim f(x, \theta)\mu(dx)\nu(d\theta)$. It follows that $(X | \Theta = \theta) \sim f(x, \theta)\mu(dx)$, and that $(\Theta | X = x) \sim f(x, \theta)\nu(d\theta)$. This can be verified directly by the defining equation (1). It
follows in particular that this is consistent with the definition of an improper posterior used by Bioche and Druilhet (2016, p.1716). The previous can also be reformulated as \( f(x \mid \theta) = f(\theta \mid x) \): There is no need for a normalization constant since two proportional densities are equivalent!

Let \( \mu(\theta) > 0 \) be an otherwise arbitrary measurable function. It follows then that \( (X \mid \Theta = \theta) \sim c(\theta)f(x, \theta)\mu(dx) \) when interpreted as a strong random conditional law. A formal prior density \( \pi \) gives the joint density \( c(\theta)f(x, \theta)\pi(\theta) \), and this shows that the interpretation of \( \pi \) as prior information is dubious in this case as pointed out by Lavine and Hodges (2012) and Lindqvist and Taraldsen (2017) using a model with intrinsic conditional auto-regressions. It is the resulting joint distribution and the conditional laws that can be interpreted. The usual decomposition in a prior and a model can only be interpreted uniquely if the prior for the model parameter \( \Theta \) is \( \sigma \)-finite. Conversely, Lindqvist and Taraldsen (2017) obtain a posterior only in the case where the data \( X \) is \( \sigma \)-finite, but Theorem 1 ensures that a posterior is defined also without requiring a \( \sigma \)-finite \( X \).

A concrete simple example is given by letting \( P_{X,\Theta} \) correspond to Lebesgue measure in the plane. The law of \( X \) given \( \Theta = \theta \) and the posterior law of \( \Theta \) given \( X = x \) correspond then both to Lebesgue measure on the line. The factorization \( f(x, \theta) = 1 = c(\theta)\pi(\theta) \) with \( \pi(\theta) = 1/c(\theta) \) is completely arbitrary. This can be interpreted according to Hartigan (1983, p.26) as saying that the marginal distribution \( \pi \) is not determined by the joint distribution. This interpretation is discussed in more detail by Taraldsen and Lindqvist (2010), but it differs from the interpretation here. The marginal law of \( \Theta \) is unique, but given by the non-\( \sigma \)-finite measure \( \mu_{\Theta}(d\theta) = \infty \cdot d\theta \). It follows in particular that the decomposition \( P_{X,\Theta}(dx, d\theta) = P_X^X(dx)\mu_{\Theta}(d\theta) \) fails this case.

Let \( \mu(dx, dy) \) be Lebesgue measure in the plane, and consider the indicator function of the upper half plane: \( T(x, y) = [y > 0] \). It follows that \( \mu_T(dt) = [\infty \delta_0(t) + \infty \delta_1(t)]dt \) so \( T \) is not \( \sigma \)-finite. The conditional law \( \mu^t(dx, dy) = [(t = 1)(y > 0) + (t = 0)(y \leq 0)]dx dy \) is, however, well defined. The conditional law \( \mu^t \) is Lebesgue measure restricted to the upper half-plane and \( \mu^0 \) is Lebesgue measure restricted to the lower half plane. This demonstrates directly that the conditional law is also defined when \( T \) is not \( \sigma \)-finite.

Consider more generally a function \( T : X \to N \). This gives \( \mu^t(A) = \mu(A(T = t)) \) and then also the elementary definition of the law \( \mu(A \mid B) = \mu(AB) \) for any \( B \) with \( \mu(B) > 0 \). This is consistent with the familiar \( \mu(A \mid B) = \mu(AB)/\mu(B) \) for the case where \( \mu(B) < \infty \). A law is arbitrary up to multiplication by a positive constant: It is an equivalence class of \( \sigma \)-finite measures. Theorem 1 gives the existence of conditional expectations in full generality - including this elementary case.

### 3.3 The marginalization paradox

Stone and Dawid (1972, p.370) consider inference for the ratio \( \theta \) of two exponential means. They assume that \( X \) and \( Y \) are independent exponentially distributed with hazard rates \( \theta \phi \) and \( \phi \) respectively. It is then clear that \( Z = Y/X \) will have a distribution that only depends on \( \theta \). In fact, \( Z = \theta F \), where \( F \) has a Fisher distribution with 2 and 2 degrees of freedom since a standard exponential variable is distributed like a \( \chi^2_2/2 \) variable. It follows then that the density is

\[
f(z \mid \theta) = \theta^{-1}(1 + z/\theta)^{-2} = \theta(\theta + z)^{-2}
\]

The posterior density corresponding to a prior density \( \pi(\theta) \) is then

\[
\pi(\theta \mid z) \propto \frac{\theta \pi(\theta)}{(\theta + z)^2}
\]
A different argument goes as follows. The joint density with a joint prior $\pi(\theta) d\theta d\phi$ gives

$$\pi(\theta, \phi | x, y) \propto \pi(\theta) \theta^{\phi} \exp(-\phi(\theta x + y))$$

The marginal posterior of $\theta$ follows by integration over $\phi$ which gives

$$\pi(\theta | x, y) \propto \frac{\theta \pi(\theta)}{(\theta x + y)^3} \propto \frac{\theta \pi(\theta)}{(\theta + z)^3} \tag{7}$$

Equation (7) gives a posterior given the data $(x, y)$ that differs from the posterior found in equation (6). This constitutes the argument and paradox presented originally by Stone and Dawid (1972, p.370).

A range of similar paradoxes were presented later by Dawid et al. (1973) with discussion of links to fiducial inference. They claim that the Fraser (1968) theory of structural inference is intrinsically paradoxical under marginalization. Furthermore, Lindley, in his discussion of the paper (Dawid et al., 1973, p.218) writes:

*The paradoxes displayed here are too serious to be ignored and impropriety must go. Let me personally retract the ideas contained in my own book.*

This is of particular relevance here since in 1964, in his book, Lindley (1980, p.xi) wrote:

*The axiomatic structure used here is not the usual one associated with the name of Kolmogorov. Instead one based on the ideas of Rényi has been used.*

We argue here and in the following that Lindley’s initial intuition was correct: The theory of Rényi gives a mathematical foundation for statistics that allows unbounded measures.

### 3.4 Resolving the paradox

We disagree with the criticism of Fraser’s structural inference, but more importantly we will next explain that there is no paradox related to the above problem when it is treated within the theory of Rényi. This has already been indicated by Taraldsen and Lindqvist (2010), and is discussed in more detail by Lindqvist and Taraldsen (2017). Lindqvist and Taraldsen (2017) rely on a theory where it is only allowed to condition on $\sigma$-finite statistics. We extend this argument now with reference to Theorem 1 which allows conditioning on any statistic.

The initial assumptions are interpreted to imply a joint distribution given by the density:

$$f(x, z, \theta, \phi) = \pi(\theta) \theta \phi^2 x e^{-\phi x(\theta + z)} \tag{8}$$

Taraldsen and Lindqvist (2010) explain that any marginal is determined by a joint density and integration over $\phi$ gives then

$$f(x, z, \theta) = \pi(\theta) \theta x^{-2} (\theta + z)^{-3} \tag{9}$$

which implies

$$f(\theta | x, z) = \pi(\theta) \theta x^{-2} (\theta + z)^{-3} = \pi(\theta) \theta (\theta + z)^{-3} \tag{10}$$

The second equality holds since it is equality in the sense given by an equivalence class as in Theorem 1. The right hand side can be multiplied by an arbitrary positive function $c(x, z)$ without changing the equality sign. Equation (10) is equivalent with equation (7) since there is a one-one correspondence between $(x, y)$ and $(x, z)$.

An alternative is to integrate over $x$ to obtain

$$f(z, \theta, \phi) = \pi(\theta) \theta (\theta + z)^{-2} \tag{11}$$
which implies

\[ f(\theta \mid z, \phi) = \pi(\theta)\theta(\theta + z)^{-2} \]  

(12)

This is similar to equation (6), but it is different since equation (6) only condition on \( z \). Equation (12) is not in conflict with equation (10) for the same reason.

Starting with either equation (9) or equation (11) gives

\[ f(z, \theta) = \infty \cdot \pi(\theta) \]  

(13)

which shows that neither \( Z \mid \Theta = \theta \) nor \( \Theta \mid Z = z \) can be represented by a \( \sigma \)-finite measure. This implies that the argument in equations (5-6) is wrong given equation (8). Equation (7), or equivalently equation (10), gives the correct posterior distribution for \( \Theta \).

If, instead, the prior \( \pi(\theta)\phi^{-1}d\theta d\phi \) is used, then the result will be

\[ f(\theta \mid z, \phi) = \phi^{-1}\pi(\theta)\theta(\theta + z)^{-2} = \pi(\theta)\theta(\theta + z)^{-2} \]  

(14)

and

\[ f(\theta \mid x, z) = \pi(\theta)x(\theta + z))^{-2} = \pi(\theta)\theta(\theta + z)^{-2} \]  

(15)

The conditionals coincide, but it is still true that neither equals the law of \( \Theta \mid Z = z \) since the law of \( (\Theta, Z) \) still fails to be \( \sigma \)-finite.

If, however, equation (5) together with a prior \( \pi(\theta) \) is taken as the initial \( \sigma \)-finite law for \( (\Theta, Z) \), then equation (6) is the correct posterior. The paradox is then removed since the conflicting conclusions are consequences of different initial assumptions.

It can finally be noted that even if a conditional law \( P_{x,z} \) does not depend on \( x \) it can not be concluded that it equals \( P_z \). This is demonstrated by equation (10) and equation (12). This rule holds for probability distributions, and also more generally if \( Z \) and \( (X,Z) \) are \( \sigma \)-finite. Stone and Dawid (1972) calculated formally as explained above as if the rule where generally valid. This error resulted in two conflicting results.

4 Final remarks

It follows from the previous quotations of Lindley that he initially supported the use of conditional probability spaces as introduced by Rényi. We have argued that this initial suggestion is indeed a natural approach to Bayesian statistics including commonly used objective priors.

As explained, the marginalization paradoxes seem to have been the main reason for Lindley’s change in opinion on this. Tony O’Hagan interviewed Lindley for the Royal Statistical Society’s Bayes 250 Conference held in June 2013. Lindley explains very nicely that all probabilities are conditional probabilities, but also recalls his reaction to the marginalization paradoxes:

Good heavens, the world is collapsing about me.

Lindley continues to argue that Bayesian statistics without improper priors is a sound theory, and that the focus should be on how to quantify the prior uncertainty of the unknown parameters. The parameters should be viewed as real physical quantities regardless of which experiment is later used for decreasing their uncertainty. This clearly disqualifies the choice of data dependent priors, and even the choice of priors depending on the particular statistical model used. We wholeheartedly agree with Lindley on this, but we claim that this can be done also within the more general theory introduced by Rényi and continued here.

An unbounded law can according to Rényi be interpreted by the corresponding family of Rényi conditional probabilities given by conditioning on the events in the bunch. These elementary
conditional probabilities are probabilities in the sense of Kolmogorov, and the interpretation depends on the application. They can, as Lindley advocates convincingly, be interpreted as personal probabilities corresponding to a range of real life events. They can also, as needed in for instance quantum physics, be interpreted as as objectively true probabilities representing a law for how a system behaves when observed repeatedly under idealized conditions.

Assume now that we accept a theory where the prior uncertainty is given by a possibly unbounded law. It is then natural to accept that a resulting posterior uncertainty can also be represented by a possibly unbounded law. Both the prior and the posterior represent uncertainty of the same kind. Hopefully, many can agree on this on an intuitive level. The main result presented here is Theorem 1 which provides a mathematical theory in which this can be done consistently without paradoxical results.

A Appendix on measure theory

A.1 Measurable space and measure

A measurable space is a set \( X \) equipped with a \( \sigma \)-field \( \mathcal{F} \) of subsets of \( X \). A \( \sigma \)-field \( \mathcal{F} \) is a collection of subsets of a fixed set that contains the empty set \( \emptyset \) and is closed under complements and countable unions. A set \( A \subset X \) is measurable if \( A \in \mathcal{F} \). A measure \( \mu \) is a function \( \mu : \mathcal{F} \to [0, \infty] \) with \( \mu(\emptyset) = 0 \) that is countably additive: \( \mu(A_1 + A_2 + \cdots) = \mu(A_1) + \mu(A_2) + \cdots \).

A probability measure is a measure \( \mu \) on a measurable space \( X \) with \( \mu(X) = 1 \). A measure space is a measurable space \( X \) equipped with a measure (Rudin, 1987, p.16). A probability space is a measurable space \( X \) equipped with a probability measure.

A \( \sigma \)-field \( \mathcal{F}_0 \subset \mathcal{F} \) of a measure space \( (X, \mathcal{F}, \mu) \) is sigma-finite if there exist measurable sets \( F_1, F_2, \ldots \in \mathcal{F}_0 \) with \( \mu(F_i) < \infty \) and \( X = F_1 \cup F_2 \cup \cdots \) (Taraldsen and Lindqvist, 2016, p.5010). A measure space \( (X, \mathcal{F}, \mu) \) is sigma-finite if \( \mathcal{F} \) is sigma-finite, and \( \mu \) is then also said to be sigma-finite (Rudin, 1987, p.121).

A.2 Conditional probability space

A bunch \( B \) in a measurable space is a family of measurable sets closed under finite unions that does not contain the empty set, but contains a countable family \( F_1, F_2, \ldots \) of sets whose union is the whole set. A bunch \( B \) is ordered if \( B_1, B_2 \in B \) implies \( B_1 \subseteq B_2 \) or \( B_2 \subseteq B_1 \).

A Rényi space (Taraldsen and Lindqvist, 2016, p.5013) is a measurable space \( X \) equipped with a family \( \{\nu(\cdot | B) | B \in B\} \) of probability measures indexed by a bunch \( B \) which fulfill \( B_1, B_2 \in B \) and \( B_1 \subset B_2 \Rightarrow \nu(B_1 | B_2) > 0 \), and the identity

\[
\nu(A | B_1) = \frac{\nu(A \cap B_1 | B_2)}{\nu(B_1 | B_2)} \tag{16}
\]

A sigma-finite measure \( \mu \) on a measurable space \( X \) generates a probability law \( \nu = [\mu] = \{c_{\mu} | c \in \mathbb{R}_+ \} \) with corresponding conditional probabilities \( \nu(A | B) = \mu(A \cap B) / \mu(B) \) for \( B \in cB = \{B | 0 < \mu(B) < \infty \} \). The Rényi space generated by the probability law \( \nu \) is given by the family \( \{\nu(\cdot | B) | B \in B\} \). A conditional probability space is a set \( X \) equipped with a probability law.

A.3 Statistical model

A statistical model is a triple \( (\Omega, X, \Theta) \) where the space \( \Omega \) is a conditional probability space, the data \( X \) is a measurable function \( X : \Omega \to \Omega_X \), and the model parameter \( \Theta \) is a measurable
function $\Theta : \Omega \to \Omega_\Theta$. These definitions, and the ones that follow, are as given by Schervish (1995) except for choice of symbols and the generalization given by assuming that $\Omega$ is a conditional probability space. There is one probability law $P$ defined on the sigma-algebra $\mathcal{E}$ of events in $\Omega$ - and all other concepts are defined from the basic space $\Omega$ (Taraldsen and Lindqvist, 2016, p.5011).

A statistic $Y = \phi(X) = \psi \circ X$ is a measurable function of the data and a parameter $\Gamma = \psi(\Theta) = \psi \circ \Theta$ is a measurable function of the model parameter as illustrated in equation (17)

$$\begin{align*}
(\Omega, \mathcal{E}, P) &\xrightarrow{\Theta} (\Omega_\Theta, \mathcal{E}_\Theta, P_\Theta) \\
\xrightarrow{\psi} &\xrightarrow{\Gamma} (\Omega_\Gamma, \mathcal{E}_\Gamma, P_\Gamma) \\
\xrightarrow{\phi} &\xrightarrow{Y} (\Omega_Y, \mathcal{E}_Y, P_Y)
\end{align*}$$

A random quantity is a measurable function $Z : \Omega \to \Omega_Z$, and its law is defined by $P_Z(A) = P(Z \in A)$ where ($Z \in A) = \{\omega | Z(\omega) \in A\}$. We abuse notation here and interpret $P$ as one fixed representative of the equivalence class that defines $P$ as a conditional measure. A random quantity is sigma-finite if its law is sigma-finite. If the model parameter $\Theta$ is sigma-finite, then the conditional probabilities $P^\theta_X(A) = P(X \in A | \Theta = \theta)$ define a family of probability measures on the sample space $\Omega_X$ indexed by the model parameter $\theta$ in the model parameter space $\Omega_\Theta$. Likewise, if the data $X$ is sigma-finite, then the posterior $P^x_\Theta(B) = P(\Theta \in B | X = x)$ is a probability measure on the model parameter space $\Omega_\Theta$. The mappings $\theta \mapsto P^\theta_X(A)$ and $x \mapsto P^x_\Theta(B)$ are measurable for all events $A$, but existence of families of probability measures as claimed above requires regularity assumptions: It is sufficient to assume that the sample space $\Omega_X$ and the model parameter space $\Omega_\Theta$ are Borel spaces (Schervish, 1995, p.619)(Taraldsen and Lindqvist, 2016, p.5011).

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