Detection and engineering of spatial mode entanglement with ultra-cold bosons

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We outline an interferometric scheme for the detection of bi-mode and multi-mode spatial entanglement of finite-temperature, interacting Bose gases. Whether entanglement is present in the gas depends on the existence of the single-particle reduced density matrix between different regions of space. We apply the scheme to the problem of a harmonically trapped repulsive boson pair and show that while entanglement is rapidly decreasing with temperature, a significant amount remains for all interaction strengths at zero temperature. Thus, by tuning the interaction parameter, the distribution of entanglement between many spatial modes can be modified.

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I. INTRODUCTION

Understanding and controlling entanglement in many-body systems is one of the most important challenges in quantum mechanics today. The rewards are significant and are expected to not only lead to new insights into the properties of solid state systems and phase transitions [1], but also to new designs for highly efficient quantum information devices. Ultra-cold bosonic gases offer an ideal arena to explore many-body entanglement, as experimentalists have at their disposal designer condensed matter systems whose parameters can be controlled with unprecedented precision [2].

Entanglement often exists naturally in the ground state of a many-body system [3], where it resides between the degrees of freedom of the particles and is a property of the first quantised many-body wavefunction. However, in ultra-cold gases the particles are inherently indistinguishable, which requires the symmetrization of their many-body wavefunction and means that the Hilbert space no longer has the tensor product structure required to define entanglement. The first quantised many-body wavefunction of indistinguishable particles may therefore contain quantum correlations [4], but such correlations are usually considered unable to violate a Bell inequality or process quantum information [1, 3, 6].

Ultra-cold gases are also well described within the framework of second quantisation, where instead of working directly with the many-body wavefunction, one defines a complete set of field modes that are occupied by particles. Second quantisation therefore offers the possibility of entanglement between modes. Entanglement is dependent on the choice of modes, but provided the correct choice is made, investigating entanglement between distinguishable modes [7, 8] circumvents the difficulties of defining entanglement between indistinguishable particles [10, 11].

To illustrate the differences between particle and mode entanglement, let us consider two non-interacting bosons in a trap at zero temperature. In first quantisation, the wavefunction is the symmetrized product, \( \Psi_{12}(x, y) = \frac{1}{\sqrt{2}} (\phi_1(x)\phi_2(y) + \phi_1(y)\phi_2(x)) \), where \( \phi(x) \) is the ground state of the confining potential. No entanglement exists between the particles, since indistinguishability forbids us from assigning to any particle a specific set of degrees of freedom. Conversely, in second quantisation one can define a pair of spatial modes, \( A \) and \( B \), where each mode occupies half the confining geometry. Since both the particles are coherently distributed over these modes, the system is described by the entangled state, \( |\psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|20\rangle + \sqrt{2}|11\rangle + |02\rangle) \), where \( |mn\rangle = |m\rangle_A \otimes |n\rangle_B \) denotes \( m \) particles in mode \( A \) and \( n \) particles in mode \( B \) (with \( m + n = 2 \)).

In this paper we outline a scheme for the detection of bi-mode and multi-mode spatial entanglement for a finite temperature, interacting Bose gas of any (including unknown) particle number. We show that entanglement is detected via the single-particle reduced density matrix (SPRDM). We apply our scheme to the example of a harmonically trapped, interacting boson pair [12], where the SPRDM also acts as a quantifier of entanglement. We find that for all interaction strengths, entanglement between pairs of modes rapidly decreases with temperature. While at zero temperature a significant amount of entanglement remains even in the limit of infinite interaction. Moreover, we note that our detection scheme is also relevant to recent proposals to observe non-locality of single particle between spatial modes [13, 15].

II. DETECTION SCHEME

The correlations of entanglement are locally basis independent, so that one needs to measure each mode in at least two bases in order to differentiate them from classical correlations. While a superselection rule that forbids coherent superpositions of eigenstates of different mass [14] seems to rule such measurements out for any atomic
In the two spatial modes, are averaged over. To generate interference, the particle, \( \psi \) g\( \psi \), divided into two, non-overlapping spatial modes, \( d \) and \( f \), different to the separable case, we can conclude that there are no fully separable case. If the number of coincidences is different to the separable case and is therefore necessarily entangled w.r.t. the bi-modal split into two spatial modes, \( a \) and \( b \).

Let us consider a gas in a confining geometry (see Fig. 1) which is mathematically, but not necessarily physically, divided into two, non-overlapping spatial modes, \( a \) and \( b \). The field operators, \( \hat{\psi}_a \) and \( \hat{\psi}_b \), create and destroy particles in mode \( i \) = \( a \), \( b \), where \( \int g(\vec{x})^2 d\vec{x} = 1 \) ensures that the commutation relations, \( [\hat{\psi}_i, \hat{\psi}_j^\dagger] = \delta_{ij} \), are satisfied. The quantity, \( g(\vec{x}) \), specifies how the set of points in a spatial mode are averaged over. To generate interference, the particles in the two spatial modes, \( a \) and \( b \), are mixed at a 50/50 beamsplitter, which transforms the input modes as \( \hat{\psi}_a = \frac{1}{\sqrt{2}}(\hat{\psi}_a^\dagger + \hat{\psi}_b^\dagger) \) and \( \hat{\psi}_b = \frac{1}{\sqrt{2}}(\hat{\psi}_a^\dagger - \hat{\psi}_b^\dagger) \). After the beamsplitting operation, the number of particles in the output modes, \( c \) and \( d \), are counted and compared to the fully separable case. If the number of coincidences is different to the separable case, we can conclude that there must have been entanglement between the spatial modes.

A bosonic gas of fixed particle number, \( N \) which is in a fully separable state w.r.t. the spatial modes \( a \) and \( b \), can be written as

\[
\hat{\rho}_{sep} = \sum_{n=0}^{N} p_n |n\rangle \langle n|_a \otimes |N - n\rangle \langle N - n|_b.
\]
than two modes. In the following we use the different notions of separability discussed in [23]. A general $M$-mode state is fully separable and contains no entanglement if it is a convex combination of states for each mode, $\hat{\rho}_{\text{sep}(M)} = \sum_i p_i \hat{\rho}_i^{(1)} \otimes \cdots \otimes \hat{\rho}_i^{(M)}$. Here each composite state, $\hat{\rho}_i^{(j)}$, corresponds to a single spatial mode with a fixed number of particles. On the other hand, entanglement may be present between certain subsets of spatial modes, for which one can define a $p$-separable state, $\hat{\rho}_{\text{sep}(p)} = \sum_i p_i \hat{\rho}_i^{(1)} \otimes \cdots \otimes \hat{\rho}_i^{(p)}$, where $p \leq M$ with equality when the state is fully separable as above. When the composite state, $\hat{\rho}_i^{(j)}$, describes more than one spatial mode, the spatial modes contained within $\hat{\rho}_i^{(j)}$ are necessarily entangled otherwise $\hat{\rho}_i^{(j)}$ would be written as a product of states for the individual modes, $\hat{\rho}_i^{(j)} = \hat{\rho}_i^{(1)} \otimes \cdots \otimes \hat{\rho}_i^{(j)} \otimes \cdots$, and the overall state of the system would be $\sigma$-separable, where $p < \sigma \leq M$ [23].

To determine whether a given $M$-mode state contains multi-mode entanglement, one can check a bi-partite entanglement criterion between all $2^{M-1} - 1$ unique divisions of the system into two blocks, $A$ and $B$. Depending on which pairs of blocks are found to be separable one can conclude that the state has entanglement between different subsets of spatial modes. Here the bipartite entanglement criterion is $\epsilon_{AB} = \text{tr}(\hat{\Psi}_A \hat{\Psi}_B \rho)$ of eq. (3), i.e. the SPRDM between two blocks of spatial modes, $A$ and $B$. The field operators, $\hat{\Psi}_X$ and $\hat{\Psi}_Y$, for blocks of spatial modes are defined as $\hat{\Psi}_X = \sum_{i \in X} c_i \hat{\psi}_i$ and $\hat{\Psi}_Y = \sum_{i \in Y} c_i \hat{\psi}_i$, where $X = A,B$ and $\sum_{i \in A} |c_i|^2 = 1$ ensures that the commutation relations, $[\hat{\Psi}_X, \hat{\Psi}_Y^\dagger] = \delta_{XY}$, are satisfied.

We now describe how entanglement changes for the (i) fully separable (ii) the $p$-separable and (iii) the fully entangled states: (i) The fully separable state, $\hat{\rho}_{\text{sep}(M)}$, admits no interference between all $2^{M-1} - 1$ partitions into blocks, i.e. $\epsilon_{AB} = 0$ for all $A$ and $B$. (ii) A $p$-separable state, $\hat{\rho}_{\text{sep}(p)}$ where $p < M$, admits no interference for $2^{p-1} - 1$ partitions into blocks from the total of $2^{M-1} - 1$ partitions, i.e. $\epsilon_{AB} = 0$ for $2^{p-1} - 1$ choices of $A$ and $B$. (iii) A fully entangled state, $\hat{\rho}$, $\sum_i p_i \hat{\rho}_i$, admits interference for all $2^{M-1} - 1$ choices of blocks, i.e. $\epsilon_{AB} \neq 0$ for all $A$ and $B$. The fully entangled state, $\hat{\rho}$, necessarily contains some form of multi-mode entanglement, since otherwise there would exist a decomposition such that $\hat{\rho}$ is $p$-separable.

III. BOSON PAIR MODEL

In the following we will apply our scheme to the physically realistic model of a harmonically trapped pair of ultracold, interacting, bosonic atoms in effectively one dimension. The Hamiltonian of such a system is given by

$$H = \sum_{i=1}^{2} \left( \frac{1}{2} \frac{d^2}{dx_i^2} + \frac{1}{2} x_i^2 \right) + g_{1D} \delta(|x_i - x_j|), \quad (4)$$

where all lengths are scaled in units of the ground state size and all energies in units of the harmonic frequency. The one dimensional coupling constant, $g_{1D}$, is related to the three dimensional $s$-wave scattering length by $g_{1D} = \frac{k_s}{ma^3} (a_1 - C a_{3D})^{-1}$, where $C$ is the constant, $C = 1.4603$ [24]. This Hamiltonian can be decoupled by moving into the centre of mass and relative coordinate frames labelled by $X$ and $x$, respectively, [12] and the two-body wavefunction can subsequently be written as $\psi_n(x_1, x_2) = \psi_n(X) \psi_\nu(x)$. The eigenvalues for the centre of mass motion are given by $\epsilon_n^\text{com} = (n + \frac{1}{2})$ for $n = 0, 1, \ldots$, with corresponding eigenstates, $\psi_n^\text{com}(X) = N_n H_n(X) e^{-\frac{x^2}{2}}$. Here $N_n$ is the normalisation constant and $H_n(X)$ are the Hermite polynomials. For the relative motion, the single particle eigenstates are $\psi_n^\text{rel}(x) = N_n \sqrt{\frac{\nu}{\pi}} U_{\frac{1}{2} \frac{\nu}{2}, \frac{\nu}{2}}(\xi)$, where $\nu = 0, 2, 4 \ldots$ and the $U(a, b, z)$ are the confluent hypergeometric functions. The corresponding eigenenergies, $E_n$, are determined by the roots of the implicit relation $-g_{1D} = 2 \frac{\epsilon_n^\text{com} + \frac{1}{2}}{1 - \frac{3}{2} \epsilon_n^\text{com} + \frac{1}{4}}$ [12]. The kernel of the density operator in position representation is $\rho_{n\nu}(x, x') = \Psi_{n\nu}^*(x, x_2) \Psi_{n\nu}(x', x_2)$, with the SPRDM defined as $\rho_{n\nu}^{(1)}(x, x') = \int_{-\infty}^{\infty} \rho_{n\nu}(x, x', x_2) dx_2$. The SPRDM in thermal equilibrium is given by $\rho_{n\nu}^{(1)}(x, x') = \sum_n \sum_\nu P_{n\nu} \rho_{n\nu}^{(1)}(x, x')$, where $P_{n\nu}$ is the Boltzmann weight, $P_{n\nu} = \frac{1}{k_B} \exp(-\frac{E_{n\nu}}{k_B T})$, $k_B$ is the Boltzmann constant and $Z$ is the partition function.

FIG. 2: Amount of entanglement between two spatial regions occupied by a boson pair as specified in the text as a function of temperature. The values of the dimensionless interaction parameter increase from the top curve to the bottom curve as $g_{1D} = 0, 2, 5, 10, \infty$. Temperature is scaled in units of $h\omega/k_B$. 
FIG. 3: The system is split into three spatial modes, a,b and c. Mode a is defined as the region \(-\infty < x < -\frac{L_b}{2}\), mode b as \(-\frac{L_b}{2} < x < \frac{L_b}{2}\) and mode c as \(\frac{L_b}{2} < x < \infty\). Entanglement is investigated between neighbouring modes, a and b (b and c) (left hand side) and also between the outer two modes, a and c (right hand side) at \(T = 0\) as interaction is varied. Three different central mode lengths, \(L_b = 4, 2.8, 1.4\) are taken.

IV. BI- AND MULTI-MODE ENTANGLEMENT RESULTS

We define two equal length modes, a and b, and calculate their spatial entanglement for the boson pair model as a function of temperature and interaction strength using eq. \(3\). The results are displayed in Fig. 2. With increasing temperature the gas becomes a statistical mixture of momentum modes, \(\phi_{ab}(x)\), which destroys the fixed relative phase between the spatial modes and consequently the entanglement between them is severely depleted. The presence of interaction also degrades the quality of the entanglement at \(T = 0\), but a significant amount persists even in the Tonks-Girardeau limit, of impenetrable bosons, \(g_{1D} = \infty\). At first this may seem surprising, since in the Tonks-Girardeau limit the system can be mapped onto an ideal fermionic atom pair \(25\), for which one may not expect any entanglement to be present. However, only the local properties of the bosons become identical to free spin-polarised fermions due to the Bose-Fermi mapping and therefore entanglement is not affected.

We also examine pairwise entanglement in this model. For this space is divided into three modes, a, b and c, and we check entanglement between all pairs, ab, bc and ac, as a function of the interaction strength at zero temperature. The results are shown in Fig. 3 where three different central mode lengths, \(L_b\), are considered. The left graph of Fig. 3 shows entanglement between neighbouring modes, ab (and for symmetry reasons bc). Here entanglement decreases as the interaction strength, \(g_{1D}\), increases for all lengths studied. The right hand side graph of Fig. 3 shows the entanglement between outer modes, a and c. The entanglement increases initially for all mode sizes as the interaction strength is varied. For the central mode size of \(L_b = 1.4\) entanglement however is not a monotonic function of \(g_{1D}\) and decreases after the initial rise. For \(L_b = 2.8\) and 4 this is not the case. The behaviour of entanglement in both graphs can be explained by the delocalisation of the bosons away from the centre of the trap due to greater interaction strengths. Stronger interaction therefore increases the correlation length (i.e. the entanglement between the outer two modes), yet decreases the overall coherence of the sample.

We also check the existence of multi-mode entanglement in our model. For this we have divided our two-particle wavefunction into \(M\) modes and calculated \(\epsilon_{AB}\) for the \(2^M-1\) 1 unique partitions of these modes into two blocks, A and B. Recall that, \(\epsilon_{AB}\), must be non-zero for all possible block combinations for the state to be fully entangled. We have found that at zero temperature the system possesses full multi-mode entanglement, i.e. \(\epsilon_{AB} \neq 0\) for all A and B, for a wide range of values of the interaction parameter, provided that the set of modes are defined within the coherence length of the sample.

Here we have used the repulsively, interacting boson pair model to illustrate our scheme, since it is analytically tractable and contains all the essential features of larger models. Our scheme to detect bi- and multi-mode entanglement can also be applied to Bose gases with a greater number of particles (for all interaction strengths and temperatures), but qualitatively the results will remain unchanged while the task of computing the single-particle reduced density matrix between modes will quickly become very demanding. We note that bi-mode entanglement of a non-interacting Bose gas has been studied before for \(N\) particles at zero temperature \(7\) and at finite temperatures \(20\).

Our scheme can be implemented using currently available technologies. Atomic beam splitters can be realised with optical potentials \(24\) and separate modes can be defined using spatially selective outcoupling techniques \(19\). To perform beam splitting, on systems with strong interactions one should raise a potential barrier between the desired modes on a non-adiabatic time scale so that the coherences between the regions are not lost, (i.e. so that the gas does not enter a Mott-like state). The potential barrier should then be lowered rapidly enough so that the coupling between the wells is greater than the on-site interaction energy of the wells. For instance, one could envisage a setup where the trap is swiftly modulated from harmonic to double-well potential and is then switched off and the sample left to interfere. The entanglement can be inferred according to eq. \(3\) from the visibility of the resulting interference fringes, see for example \(14\) \(28\), where mode entanglement between regions of space has already been measured indirectly.

V. CONCLUSIONS

We have outlined a scheme that detects bi-modal and multi-modal spatial entanglement of cold bosonic gases using simple atom-optic techniques. We show that spatial entanglement is detected by the SPRDM between different modes, a quantity that is related to the visibility of routinely measured interference fringes. We have
demonstrated our scheme using the model of a harmonically trapped boson pair and have found the existence of bi- and multi-mode entanglement within the gas. For all interaction strengths, increasing temperature rapidly degrades the amount of entanglement, but at zero temperature entanglement still remains, even in the presence of strong interactions. Therefore, we have shown that the ability to vary the particle interaction strength allows one to engineer the distribution of entanglement over the trap, thus allowing a tunable source of entanglement.

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