The estimation of frequency response of nonlinear quarter car model and bilinear model of damper characteristics

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Abstract. The paper presents proposed and tested methods of estimation of frequency response of nonlinear quarter car suspension model and comparison with method for linear model used for research of vehicle vertical dynamics. Linear quarter car suspension model is widely used for estimation of comfort and safety performance of passive and semiactive suspensions. The real suspension - especially shock absorbers - are non-linear in three aspects: static characteristics, hysteresis and the presence of friction. Testing linear suspension model is possible with the use of analytical transfer function formulas but testing real suspension on a testing stand or even virtual but nonlinear suspension needs to use methods of transfer function estimation. It was necessary to design appropriate input signal allowing to get useful response signals. With the use of obtained frequency responses a method of linear estimation of nonlinear suspension for a given range of working condition was proposed. Also the bilinear model of damper characteristic estimation was proposed and tested proving to be good alternative to nonlinear characteristic.

Key words: vehicle dynamics, suspension, shock absorbers, vehicle

1. Vertical vehicle dynamics and its performance evaluation

Car suspension from a point of view of vertical dynamics is an oscillatory system, affected by two types of excitations - kinematic excitation zᵣ caused by the road surface profile and force excitation related to inertial forces acting on vehicle body.

These excitations processed by suspension dynamic structure result in changes in such variables as bounce absolute displacement of sprung mass zₘ and unsprung mass zₘ and the relative displacement of both masses, which describe the suspension deflection (rattle space) zₘ − zₘ, the acceleration of both these masses (zₘ, zₘ) and also tire-road contact forces Fₜ. The relation between road excitation and these variables treated as suspension system outputs (responses) in a function of excitation frequency are the dynamic characteristics of suspension, also called suspension transmissibility functions [1], frequency response functions or magnitude-frequency characteristics (Bode magnitude plot) [2,3,4].
The knowledge of frequency responses is essential to assess suspension performance according to such criteria as comfort, safety and technical limitations concerning limited suspension travel space. Ride comfort is often assessed by the use of sprung mass acceleration transfer function (acceleration amplification function), $\ddot{z}_M(\omega)$, safety – with an analysis of dynamic wheel load amplification function $F_t(\omega)$ and by analysis of wheel rattle space amplification function. This variable is important also from a point of view of kinematic performance of suspension (possible changes of wheel camber and steer angles) and the vertical dynamics influence on lateral dynamics.

2. Linear and non-linear quarter car models

There are many different kinds of vehicle suspension models (understood as a vertical dynamics vehicle models) of varying degree of complication. The one most frequently used is a simple linear quarter-car model, which can be found for example in old [2,3] and in newer publications [5, 6]. As the years went by, more and more research and simulations started using non-linear models, especially in the last decade [7,8].

![Figure 2. A quarter car suspension model with linear and nonlinear elements.](image)

In most conditions of vehicle use simulation made to test ride safety or comfort can be conducted on simpler linear models. Roads of classes A and B rarely cause suspension to work in the non-linear range of its elements characteristics with the exception of vehicles being heavily loaded in the case of B class road, where the non-linear model might be worth considering. For worse quality roads, especially class D, linear models can yield somewhat satisfactory and realistic results only for best case scenarios, i.e. very low speeds and an almost empty vehicle. Otherwise, the suspension regularly enters non-linear working range [9].

Both the linear and nonlinear models use the same structure of suspension system, while the differences are found in the characteristics of spring and damping elements. Nonlinear spring characteristics takes into account big force increase when suspension enters the range of bump stop work. Nonlinear damper characteristic is even more complicated and takes into account asymmetry and non-linearity of the static characteristics, hysteresis of damping force and friction. Nonlinear tire model can limit tire forces only to compression force (zero tire force when there is no road-tire contact).

3. Frequency responses of quarter car model and its interpretation

There are two main factors contributing to vehicle ride quality: the vehicle’s suspension system dynamic characteristics and the road kinematic excitation resulting from the pavement surface and vehicle velocity. But in the process of assessing vehicle suspension performance other variables are taken into account as well - suspension deflection and tire dynamic load. All of these are responses to input in the form of road excitation. As road excitation is time frequency signal vehicle suspension responses are also time frequency signals. The input and response signals can be expressed in the form of a Power Spectral Density (PSD) function. The response PSD function is
related to the excitation PSD function via the transmissibility Frequency Response Function (FRF) of the vehicle suspension under consideration. The transmissibility FRF acts as a weighting factor between the input and output. It is sometimes called also transmissibility function. Depending on the variable being analysed, various functions need to be used: sprung mass acceleration transmissibility, suspension deflection transmissibility, tire load (vertical force) transmissibility [10] and also sprung and unsprung mass displacement transmissibility (e.g., [11]).

The simplest way to experimentally determine FRF function is to excite suspension system with constant amplitude input signal with slowly and linearly changing frequency. The response signal can be then easily compared with excitation signal and multiplication factor of amplitudes for every frequency allow to determine FRF function values – Fig 3. The example of testing suspension deflection when excited by 0.01 m amplitude road excitation input signal is presented in figure 3. It is clearly visible that there are two resonances - one for sprung mass and one for unsprung mass. For frequencies near 0 Hz suspension almost does not move, the same can be said for frequencies much higher than unsprung mass resonance, where there is very little suspension deflection.

Figure 3. Comparison of excitation and response signals (suspension deflection) of suspension excited by constant amplitude sinusoidal signal with linearly changing frequency

An example of interpretation of FRF in practical consideration can be relation between EUSAMA test [12] and tire vertical force FRF function – figure 4. In this figure changes of tire vertical force are presented as the response to constant amplitude (0.003 m) kinematic excitation with frequency fluently changing in the range of 0 to 25 Hz, used on EUSAMA test stands during Periodical Vehicle Inspections. Yellow line presents the FRF assuming that frequency is linearly changing according to time.

Figure 4. Relation between EUSAMA test and tire vertical force FRF function (red scale and yellow line)
One can note that FRF amplitudes in an unsprung mass resonance range can be used to calculate maximum and minimum dynamic loads - if it is for example 333.3 kN/m then multiplying 0.003 m excitation we obtain amplitude of 999 N of dynamic wheel load. That mean maximum wheel load of 3500 N and minimum 1500 N if the static load is equal to 2500 N. EUSAMA index will then be equal to EU=1500/2500=0.6.

4. Frequency response function estimation methods

4.1 Linear model frequency response function calculation

In the case of linear dynamic systems, which enable researchers to build relatively simple mathematical models, amplification functions can be evaluated using fundamental methods of control engineering (exemplary description of the use of these methods for a linear quarter car model can be found e.g. in [13,14]). This is done by determining Fourier transforms of differential motion equations with zero initial conditions and then by determining frequency response functions for selected inputs and outputs. These frequency responses can be expressed in algebraic and exponential form [15]:

\[ G(j\omega) = \frac{x(j\omega)}{u(j\omega)} = P(\omega) + jQ(\omega) = A(\omega)e^{j\varphi(\omega)} \]  

Gain of frequency response function expressed as:

\[ A(\omega) = \sqrt{P^2 + Q^2} = \frac{A_{\text{output}}(\omega)}{A_{\text{input}}(\omega)} \]  

physically means the gain between magnitude of input (excitation) and magnitude of output (response) in form of sinusoidal signal.

Frequency response function argument \( \varphi(\omega) \) expressed as:

\[ \varphi(\omega) = \arctg \frac{Q}{P} \]  

physically means the phase shift of sinusoidal signals - input and output.

4.2 Nonlinear model frequency response function estimation

Frequency characteristics of suspension can be determined also by analysing the relations between input signal and output signals. These signals can be measured and acquired during tests on a physical object or simulated with different types of suspension model - for example nonlinear model. It is worth pointing out that the responses of a non-linear system do not depend on the frequency alone, but on the input amplitude as well. The authors decided to focus their research on the frequency response for amplitudes in the range for typical road excitation levels and they acknowledge that the interpretation of the results acquired here might not be applicable for amplitudes of vastly differing values.

After preparing measured or simulated appropriate signals in the form of time histories, frequency response evaluation can be made using expression (4) [16]:

\[ \tilde{H}_{xy}(\omega) = \frac{\tilde{G}_{xy}(\omega)}{\tilde{G}_x(\omega)} \]  

where:

\( \tilde{G}_x(\omega) \) - estimator of power spectral density of input signal - kinematic excitation signal,
\( \tilde{G}_{xy}(\omega) \) - estimator of cross power spectral density of input and output signals.

The use of this formula requires knowledge of power spectral density - PSD (power spectrum value divided by frequency band for which a given PSD value was determined). The two-sided continuous power spectral density can be calculated as a limit [2]:
\[ S_x(\omega) = \lim_{\Delta \omega \to 0} \frac{\Delta(f_n^2(t))}{\Delta \omega} \]  

where:

\( \Delta(f_n^2(t)) \) - power share of function \( f(t) \) near the frequency \( \omega_n \),

\( \Delta \omega \) - frequency band for which power share is determined.

Measurements only help to estimate the power spectrum or power spectrum density, but do not allow to determine them exactly. This is a more complex task, caused mainly by a limited set of samples [17]. The estimate of power spectral density can be found with various methods, for example squaring the magnitude of \( f(t) \) signal Fourier transform and thus obtaining the periodogram or calculating discrete Fourier transform (DFT) of autocorrelation \( f(t) \) signal function [17].

When calculating the estimates of power spectral density with few signal samples, unsatisfactory results are obtained which are shown as considerable PSD signal fluctuations. In order to obtain smoothed PSD estimates various methods are used, for example calculating and averaging periodograms. One of the most common is the Welch method, which results in the so-called modified periodogram. It is a modified version of the Bartlett method [17].

Both methods involve dividing the signal (a series of samples) into several periods, for which shorter periodograms are calculated. Next an averaged periodogram over all the segments is calculated. The difference between the Bartlett and Welch methods of spectrum estimation is that the first one uses segments which do not overlap, while the other applies overlapping segments and time windows other than rectangular - e.g. the Hanning windows. As a result the Welch method ensures estimation with a smaller variance compared to the Bartlett method. This method is programmed amongst others in Matlab environment and allows to obtain the power spectral density estimate or cross-spectral density expressed in units of power per radian per sample (rad/Sa) or Hz.

Using calculated estimates of power spectral density we can calculate estimate of the frequency response function or one can use Matlab function for estimating the frequency response estimate using the presented procedure. Authors of the paper used the \texttt{tfestimate} function in Matlab for estimating frequency response function for linear, bilinear and nonlinear models of quarter car suspension.

5. Frequency response function estimation experiment

The experiment consist of two phases:
1. checking and design of input signal allowing to get good estimation of Frequency Response Function and compare it with FRF calculated analytically for fully linear model,
2. estimation of FRF for fully nonlinear suspension model (tire, spring and shock absorber) and comparing it with two models with different shock absorber models (fully linear, symmetrical bilinear) and possibility to identify their damping coefficients.

5.1 Suspension model used for experiment

The vehicle model used in simulation was a non-linear quarter-car model in three variants that differed in a damper model used. The variants were non-linear damper, linear damper and bilinear damper models. The tire forces were linearly dependant on the tire deflection and its velocity. The tire forces model had a non-linearity in the form of a possibility for a tire to lose contact with the road surface, which causes the tire forces to drop to 0 N. This meant the tire forces would never be negative – the tire is never pulled towards road surface. The characteristic can be seen in Figure 5.
Other non-linearity in the model was the suspension spring, which used characteristic that was acquired in tests described in [18]. In this instance the characteristic was that of a front suspension spring that was a part of a MacPherson strut. The characteristic included also a rubber bump stop, which made the characteristic progressive. It also included an artificially added point of data for when the spring is completely compressed – in which case the force to compress it any further dramatically increases as now the metal coils of a spring themselves need to be compressed. The full characteristic can be seen in Figure 6.

![Figure 6. Spring characteristic used in research](image)

The last important element to be addressed is the damper model, the non-linear version it used the front damper characteristic from [18], which can be seen in Figure 7. The model also had modules responsible for simulating hysteresis and internal friction of a damper, which could be disabled and enabled. More details on the damper model can be found in [19].

![Figure 7. Damper characteristic used in non-linear model](image)
5.2 Experiment first phase - input signal preparation for FRF estimation

The use of appropriate Matlab function `tfestimate` for FRF estimation is not the only condition to get good estimate. The other necessary operation is preparing appropriate input signal to obtain good input and output signals.

The kinematic excitation signal was prepared so that it would produce well represented, broad spectrum of frequencies for transfer function calculations. Generally so called chirp signal was used which is sinusoidal signal of a linear swept-frequency in a given time. However, normally generated “chirp” signal using Matlab or Simulink generator will not produce good enough results, as the automatic generator tends to go through low frequencies really quickly. It does not produce even one full cycle in the lowest frequencies while in the high there are many cycles present.

The `tfestimate` function in Matlab needs at least a few cycles in or around a given frequency to estimate it properly – around 10. That is why different signal was needed - one that was glued together from a few shorter samples. Connected signal consisted of three parts lasting enough time to consist of multiple cycles of similar frequencies occurring during simulation:

- the first one had only low frequencies up to 3 Hz changing linearly from 0 to 300 s,
- the second one consisted of medium-range frequencies from 3 to 10 Hz changing linearly from 300 to 580 s,
- the last one was of high frequencies from 10 to 35 Hz changing linearly from 300 to 580 s.

Another important condition to input signal was to produce response signals in a range of operational suspension deflections. It needed to control the response range. The overshoot of 35 Hz was implemented, because `tfestimate` function tends to give blurry results at the ends of examined frequency range. In order to acquire important data about frequencies around 25 Hz, the signal was artificially prolonged to 35 Hz to avoid aforementioned distortions for highest frequencies.

5.3 Experiment first phase - FRF estimation for linear model

First of all method of estimation FRF functions with use of `tfestimate` function and results of analytical calculation of this function were compared. In fig. 8 magnitude of transfer function (frequency response function) for sprung mass displacement caused by road kinematic input calculated and estimated with use of Matlab `tfestimate` function is presented.

![Figure 8. Analytical and estimated frequency response function of sprung mass displacement](image)

Analytical FRF function is calculated with use of following formula

$$H_{z_M}(s) = \frac{(b_M s^2 + k_M)k_M}{mM s^4 + (mb_M + M b_M) s^3 + (M k_M + mk_M + M k_m) s^2 + b_M k_m s + k_m k_M}$$  \hspace{1cm} (6)

The second function (red in Figure 8) was estimated in Matlab using `tfestimate` and their values are almost the same. The only difference is 5% lower value in a range of first resonance. In the rest - almost 99% of a frequency range the values of both functions are the same.
5.4 Experiment second phase - nonlinear suspension model FRF estimation and comparison with linear and bilinear damper model.

In the next phase of experiment FRF for five variables were estimated for nonlinear model of suspension and two other suspensions differing in a model of shock absorber. One was fully linear damping force model and the second one was symmetric bilinear model with two damping coefficients - one of a greater value for small suspension deflection velocities - up to 0.2 m/s and second one of smaller value for the higher deflection speed values. The results obtained during simulation were used to estimate the transfer functions between kinematic excitation and a given variable. These FRF functions were:

- sprung mass displacement,
- unsprung mass displacement,
- suspension deflection,
- sprung mass acceleration,
- cumulative tire force.

Those were then used to estimate the transfer functions between kinematic excitation and a given variable. In the case of suspension deflections it was deemed useful to calculate the density of probability of a given deflection to appear, in order to establish whether or not given excitation would cause the suspension to work in a normal operation range.

The first batch of experiments was done with simplest variant of damper model using just a single damping coefficient, which had its value changed in iterations in order to find the one that most closely resembled the non-linear model when it comes to vehicle responses. This was done in a broad spectrum of damping coefficients – starting from 1000 Ns/m every 200 Ns/m up to 3000 Ns/m. This phase was needed to try and gauge, what damping coefficient would give results most similar to those of a non-linear model. In this phase it became apparent, that there is no single value of a coefficient which would make the linear model act similar to the non-linear one – it was possible to find a coefficient which would make them act similarly near the first or the second resonant frequency, but not both at the same time - figure 9.

![Figure 9](image)

**Figure 9.** Comparison of frequency response functions of nonlinear quarter car model and two models with linear ($b_M=2660$ Ns/m and $b_M=1450$ Ns/m) shock absorber model.

In the second batch, previously recognized damping coefficients’ values which bore the most resemblance to non-linear models in both frequencies were tested once again, this time with much smaller increases of 10 Ns/m per iteration. For sprung mass resonance, the testing started from
The 13th International conference on Automotive Safety (Automotive Safety 2022)  
IOP Conf. Series: Materials Science and Engineering  1247  (2022) 012007  
doi:10.1088/1757-899X/1247/1/012007

$c_M$ = 1400 Ns/m, while for unsprung mass from $c_M$ = 2600 Ns/m. Those two values were then also used to test if the bilinear model with them can approximate the behavior of the system better than that with values calculated based on the characteristic itself (which, to remind the reader, does not take into account hysteresis and friction in the damper).

Lastly, recognizing the shortcomings of the linear model, slightly more complex, but still easily implementable bilinear model was proposed. It consisted of two linear functions, connected at the joint point - Figure 10. The name bilinear comes from the fact, that there are two different linear damping coefficients – even if the characteristic consists of three lines, two of them share the same slope, while their variable independent part is different. It is worth noting, that this characteristic is still very simplified – for example, it is symmetrical, while the real characteristic differs depending on whether the damper is being compressed or stretched.

![Figure 10. Damper characteristic in bilinear model.](image)

Best matched FRF was obtained for bilinear model with two damping parameters $b_{M_1} = 2800$ Ns/m and $b_{M_2} = 1220$ Ns/m - fig. 11. They differ from parameters really used in nonlinear model which were determined to be about $b_{M_1} = 2815$ Ns/m and $b_{M_2} = 1440$ Ns/m. The difference is caused by the fact that the nonlinear shock absorber is also nonsymmetrical.

![Figure 11. Best matched bilinear model frequency response functions compared with these functions of nonlinear model of shock absorbers.](image)
The last part of the experiment was to evaluate which of nonlinear elements of shock absorber model - friction force or hysteresis - make it impossible to estimate ideally nonlinear model with linear or bilinear model.

In the Figure 12 FR functions of different versions of nonlinear shock absorber model (with friction and hysteresis, without friction, without hysteresis and without friction and hysteresis) and bilinear model are compared. It is visible that the best match with bilinear model is achieved using nonlinear damper without friction and hysteresis (but still with asymmetrical static characteristics). Nonlinear model without friction behaves in low frequencies more similar to its version without friction and hysteresis and in a range of higher frequencies more like the version with friction and hysteresis. Nonlinear model without hysteresis is more similar in a low frequencies range to fully nonlinear model (hysteresis and friction enabled) and in a range of higher frequencies it behaves like the version of a model without friction and hysteresis. This allows researchers to conclude that friction is more important in modelling behaviour of suspension in a range of low (first resonance) frequencies and hysteresis is more important in a range of second resonance at higher frequencies.

![Figure 12. Comparison of FR functions for bilinear and different variants of nonlinear shock absorber model.](image)

6. **Conclusions**

The paper presented the method of frequency response function estimation for nonlinear models of quarter car suspension, which can be used also for physical quarter car suspensions and measured wheel excitation and suspension responses.

The method is based on PSD function estimation and later FRF estimation with special preparation of the input signal (wheel kinematic excitation). Verification performed proved that for linear models it is possible to get almost the same frequency response function shapes.

In case of nonlinear models their ideal estimation using linear damper model is impossible. The reasons for this are as follows:
- due to nonlinearity of shock absorber characteristics (dissipative) damping coefficient for the first (sprung mass) resonance is much higher than for the second (unsprung mass) resonance,
- due to presence of friction especially at low frequencies nonlinear shock absorber with friction active gives lower FRF gains due to different friction force behavior than viscous damping forces - friction force works for even the smallest deflection velocities when viscous damping force at the smallest deflection velocities are almost absent,
- due to presence of hysteresis of damping force there is a difference in all testing frequency ranges giving the biggest differences in a second (unsprung) mass resonance.
Performed experiments proved that use of bilinear model (even symmetrical) can give quite good estimation of fully nonlinear damper giving very close values of gain at the resonances with small shift in a value of resonance frequency. However those conclusions were tested only for a linear range of spring characteristic, which means that in order to extend them for other input amplitudes reaching nonlinear spring work ranges, it would be necessary to run more simulations and create a function of two variables – both frequency and amplitude.

7. References
[1] J. Grajnert (red): Izolacja drgań w maszynach i pojazdach. Oficyna Wydawnicza Politechniki Wrocławskiej, Wrocław (1997).
[2] E. Kamiński, J. Pokorski: Dynamika zawieszeń i układów napędowych pojazdów samochodowych, Wydawnictwa Komunikacji i Łączności, Warszawa (1983).
[3] M. Mitschke: Dynamika samochodu: Drgania. Tom 2, Wydawnictwa Komunikacji i Łączności, Warszawa (1989).
[4] A. Reński: Bezpieczeństwo czynne samochodu. Zawieszenia oraz układy hamulcowe i kierownicze. Oficyna Wydawnicza Politechniki Warszawskiej, Warszawa (2011).
[5] J. Celko, M. Decky, and M. Kovac, An analysis of vehicle – road surface interaction for classification of IRI in the frame of Slovak PMS. Maintenance and reliability, Polish Maintenance society, Vol. 1, Iss. 41, 2009, pp. 15 - 21
[6] M. Doumiati, A. Victorino, A. Charara, and D. Lechner,: Estimation of road profile for vehicle dynamics motion: Experimental validation. Proceedings of the American Control Conference, pp. 5237–5242, 2011, DOI: 10.1109/ACC.2011.5991595.
[7] A. Mitura: Modelowanie drgań nieliniowego zawieszenia pojazdu samochodowego z tłumiением magnetoreologicznym: rozprawa doktorska, Politechnika Lubelska, Lublin (2010).
[8] P. Zdanowicz and Z. Lozia: Wyznaczenie optymalnej wartości współczynnika asymetrii amortyzatora pasywnego zawieszenia samochodu z wykorzystaniem modelu „ćwiartki samochodu”. Pr. Nauk. Politech. Warsz. Transp., no. 119, pp. 249–265, (2017).
[9] Z. Klockiewicz, G. Ślaski, and M. Spadło: The influence of the conditions of use and the type of model used on the vertical dynamic responses of a car suspension, The Archives of Automotive Engineering – Archiwum Motoryzacji, Vol. 85, No. 3, 2019, pp. 57–82.
[10] G. Reina, G. D. Rose: Active vibration absorber for automotive suspensions: a theoretical study. Int. J. Heavy Vehicle Systems, Vol. 23, No. 1, 2016.
[11] S. Hegazy, A. M. Sharaf: Ride Comfort Analysis Using Quarter Car Model. Proceeding of International Conference on Aerospace Sciences and Aviation Technology, May 2013
[12] A. Kuranc: Diagnostyczne badania zawieszenia pojazdu w aspekcie zmian parametrów eksploatacyjnych. Inżynieria Rolnicza, Vol. 117, No. 8, 2009.
[13] D. Karnopp: How significant are transfer function relations and invariant points for a quarter car suspension model? Vehicle System Dynamics, Vol. 47, Iss. 4, 2009.
[14] S. M. Savaresi, et al: Semi-Active Suspension Control Design for Vehicles. Oxford: Butterworth-Heinemann Ltd (Elsevier), 2010.
[15] A. Czemplik: Modele dynamiki układów fizycznych dla inżynierów. Warszawa, WNT, 2008.
[16] D. Borkowski: Symulacyjne badanie nieparametrycznej metody estymacji impedancji sieci energetycznej, Proceedings of XV Sympozjum Modelowanie i Symulacja Systemów Pomiarowych 18-22.09.2005.Krynica.
[17] D. Stranneby: Cyfrowe przetwarzanie sygnałów. Metody. Algorytmy. Zastosowania. Warszawa, Wydawnictwo BTC, Warszawa (2004).
[18] G. Ślaski: Studium projektowania zawieszeń samochodowych o zmiennym tłumienniu, Poznań: Wydawnictwo Politechniki Poznańskiej, (2012).
[19] D. Więckowski, K. Dąbrowski, G. Ślaski: Adjustable shock absorber characteristics testing and modelling, International Automotive Conference KONMOT 2018, 13 - 14 SEPTEMBER 2018, CRACOW, IOP Conf. Series: Materials Science and Engineering 421 (2018) 022039 IOP Publishing doi:10.1088/1757-899X/421/2/022039.