The effect of radiation corrections to the mass of an electron and positron on the polarization operator of a photon in a magnetic field

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Abstract
The polarization operator of a photon in a constant and uniform magnetic field is studied taking into account the radiation width and shift of the Landau levels in both weak and strong fields compared with the critical field $H_0 = 4.41 \cdot 10^{13}$ G. A general formula is obtained for the polarization operator of a photon in which radiation effects are taken into account. Now diverging previously threshold terms have a finite value. The conditions are formulated under which the energy levels completely overlap, and thereby the most appropriate application of the semiclassical operator method to the problem under study becomes.

0.1 Introduction
The study of QED processes in high magnetic field close to and above the critical field strength $H_0 = m^2/e = 4.41 \cdot 10^{13}$ G (the system of units $\hbar = c = 1$ is used) is of interest primarily because such fields exist in nature. It is commonly accepted that the magnetic field of neutron stars (pulsars) reaches $H \sim 10^{11} \div 10^{13}$ G [1]. This range of magnetic fields is obtained in the rotating magnetic dipole model, where a pulsar loses energy through magnetic dipole radiation. Predictions of this model are in good agreement with the observation of radio radiation of pulsars. Several thousand radio pulsars are currently known. Another class of neutron stars called magnetars [2], was discovered when observing X and gamma rays. In this case, the existing models give much higher magnetic fields $H \sim 10^{14} \div 10^{15}$ G. In view of this circumstance, it is interest to describe the motion of a photon and electron (positron) in fields both above and below the critical field. This motion is accompanied by the photon conversion into a pair of charged particles when the transverse photon momentum is larger than the process threshold value $k_\perp > 2m$. When the field change is small on the characteristic length of process formation (for example, when this length is smaller
then the scale of heterogeneity of the neutron star magnetic field), the consideration can be realized in the constant field approximation. The polarization operator of the photon in a constant and uniform electromagnetic field of any configuration was obtained for the first time in 1971 by Batalin and Shabad [3] who used the Green function found by Schwinger [4]. Such calculation for a pure magnetic field were carried out in 1974 by Tsai [5]. The singular behavior of the polarization operator near the electron-positron pair production thresholds was analyzed in 1975 by Shabad [6]. In 1975 the contribution of charged-particles loop in an electromagnetic field with \( n \) external photon lines had been calculated in [7]. For \( n = 2 \) the explicit expressions for the contribution of scalar and spinor particles to the polarization operator of photon were given in this work. For the contribution of spinor particles obtained expressions coincide with the result of [3], but another form is used.

The polarization operator in a constant magnetic field has been investigated well enough in the energy region lower and near the pair creation threshold (see, for example, the papers [8, 9] and the bibliography cited there. In [10] a general spectral-integral formula was obtained in which diverging threshold terms were identified in an explicit analytical form. The remaining members are presented in a form very convenient for numerical calculations.

In this paper, we consider radiation corrections to the energy of an electron (positron) in a magnetic field and their effect on the polarization operator of a photon. Such an effect is most noticeable at superstrong fields and photon energies near the first excited thresholds for the production of electron-positron pairs. The expression for the mass operator of an electron (positron) in the \( a \)-order in a constant magnetic field was obtained in [11, 12]. It is a twofold integral of a surprisingly compact expression containing, however, oscillating functions. In [13], a detailed study of the mass operator and the radiation intensity (obtained in this work) was carried out for arbitrary values of the magnetic field \( H \) and electron energy \( \varepsilon \). We will be especially interested in the results obtained for superstrong fields \( H \gg H_0 \) and energies at the first Landau levels \( n = 0 \) and \( n \gtrsim 1 \).

### 1 Radiation Width and Shift of the Landau Levels

The general expression for the mass operator of an electron in a constant and uniform magnetic field, having a diagonal form, can be found in [13]:

\[
M = \frac{\alpha m}{2\pi} \int_{0}^{\infty} \frac{dx}{x} \int_{0}^{1} du e^{-ix/2\mu} \left\{ \frac{1}{\Delta} \exp[2\ln(a(x) - \frac{ux}{2})] \right\}
\]

\[
\left[ (\rho(1-u) - u)(c(x) + u(s(x) - c(x))) - us(x) + \right]
\]

\[
i \zeta \gamma \frac{(1-u)x(c(x))}{x^2} - 1 + s(x) \right] - 1 - u. \right.
\]
\[ \rho = 2n\mu - \gamma^2 - 1, \quad \mu = H/H_0, \quad n = n_0 + (1 - \zeta)/2, \quad (4) \]
\[ a(x) = \arctan \frac{ue(x)}{x(1 - us(x))}, \quad c(x) = 1 - \cos(x), \quad s(x) = 1 - \frac{\sin(x)}{x}, \quad (5) \]
\[ \zeta = \pm 1(n \geq 1), \quad \zeta = 1(n = 0), \quad (6) \]
\[ \Delta = 1 - 2u(1 - u) + u^2(2c(x)/x^2 - 1). \quad (7) \]

In the limit of the superstrong field \((\mu \gg 1)\), in [13], the following expression was obtained with power accuracy for the correction to the electron mass at \(n = 0\):

\[ M_0 = \frac{\alpha m}{4\pi} \left[ (\ln 2\mu - C - 3/2)^2 + A + O(1/\mu) \right], \quad A = 4,02816.... \quad (8) \]

Here \(C = 0.577\ldots\) is Euler constant. For large \(\mu\), the following approximate expression for the radiation corrections to the electron energy at \(n \geq 1\) was obtained in [13]:

\[ \varepsilon_n = \frac{\alpha eH}{2\varepsilon_n} (-i\delta_1n + \delta_2n), \quad \varepsilon_n \simeq \sqrt{2eHn}; \]
\[ \delta_1n \simeq a_1n^{1/3} - k_0 + a_2n^{-1/3} + \zeta(a_3n^{1/6} + a_4n^{-1/2})/\sqrt{n}, \]
\[ a_1 = 1.8401; \quad k_0 = 1.5213; \quad a_2 = 0.3180; \quad a_3 = 0.1606; \quad a_4 = 0.2593; \]
\[ \delta_2n = b_1n^{1/3} - \kappa_0 + b_2n^{-1/3} + \]
\[ \zeta[-b_3n^{1/6} + \frac{1}{\pi\sqrt{2n}}(\ln \mu + c_1 \ln n - c_2)]/\sqrt{\mu}; \]
\[ b_1 = a_1/\sqrt{3}, \quad \kappa_0 = 0.5690/2\pi, \quad c_1 = 11/30, \quad c_2 = 0.3964. \quad (9) \]

With good numerical accuracy, the formula (9) can also be used at the lowest excited levels at \(n \sim 1\) (see [13]). For \(\mu \gg 1\), the contribution of the terms \(\propto \zeta\) is small and can be neglected. In this case, the quantity \(\varepsilon_n\) does not depend on the electron mass. Leaving the main term of the expansion at \(n \gg 1\), we have

\[ \varepsilon_r = \cos\omega_0n^{1/3}(1/\sqrt{3} - i), \quad c = \frac{7}{9} \Gamma \left( \frac{2}{3} \right) \left( \frac{2}{3} \right)^{1/3} \simeq 0.920..., \quad (10) \]

where \(\omega_0 = eH/\varepsilon_n\). The formula (10) is also applicable for \(H \ll H_0\) if the condition \(eHn^{1/3} \gg m^2\) is fulfilled. For \(n \gg 1\), the distance between the Landau levels is \(\delta\varepsilon_n \sim \omega_0n\), and we have \(\varepsilon_r/\delta\varepsilon_n \sim n\alpha^{1/3}\). In weak fields, the formula (11) is applicable for \(n \gg (H_0/H)^3\) and then for \(H/H_0 \lesssim \alpha\) the energy levels completely overlap. It should be noted that the condition \(eHn^{1/3} \gg m^2\) is in fact a threshold for the creation of pairs at these levels in weak fields. In this case, the energy of the emitted photon is of the order of the entire electron energy \(\varepsilon\). In the opposite case \(eHn^{1/3} \ll m^2\), the emitted harmonic is \(\nu \sim \gamma^3 \ll n\), and relatively soft photons with a frequency \(\omega \sim \nu\omega_0 \ll \varepsilon\) are emitted. In this case we have \(\varepsilon_r/\delta\varepsilon \sim \alpha\gamma\) and the energy levels completely overlap for \(\alpha\gamma \gtrsim 1\).
2 Photon polarization operator

In this section, we will use the results of [10]. In a purely magnetic field, we have in covariant form

$$\Pi^{\mu\nu} = -\sum_{i=2,3} \kappa_i \beta_i^\mu \beta_i^\nu, \quad \beta_i \beta_j = -\delta_{ij}, \quad \beta_i k = 0; \quad (11)$$

$$\beta_2^\mu = (F^* k)^\mu / \sqrt{-(F^* k)^2}, \quad \beta_3^\mu = (F k)^\mu / \sqrt{-(F k)^2}, \quad (12)$$

$$FF^* = 0, \quad F^2 = F^\mu\nu F^\mu\nu = 2(H^2 - E^2) > 0, \quad (13)$$

where $F^{\mu\nu}$ is the electromagnetic field tensor, $F^*^{\mu\nu}$ is the dual tensor, $k^\mu$ is the photon momentum, $(F k)^\mu = F^\mu\nu k_\nu$. The real part $\kappa_i$ determines the refractive index of the photon $n_i$ with polarization $e_i = \beta_i$:

$$n_i = 1 - \frac{\text{Re} \kappa_i}{2\omega^2}. \quad (14)$$

We will consider the process in the laboratory system, where the electric field is absent, and the photon moves perpendicular to the magnetic field. In this case

$$r = \omega^2 / 4m^2. \quad (15)$$

For $r > 1$, the eigenvalues of the polarization operator contain the imaginary part, which determines the probability of the production of an electron-positron pair by a photon with polarization $e_i = \beta_i$:

$$W_i = -\frac{1}{\omega} \text{Im} \kappa_i. \quad (16)$$

The eigenvalues of the polarization operator $\kappa_i$ in the first order of perturbation theory with respect to $\alpha$ contain root divergences for $r = r_{ik}$, where

$$r_{ik} = (\epsilon_l + \epsilon_k)^2 / 4m^2, \quad (17)$$

$$\epsilon_l = \sqrt{m^2 + 2eH_l} = m \sqrt{1 + 2\mu_l}. \quad (18)$$

Consider the photon energy region near the ground state $r_{00}$. In the case $\mu \gg 1$, $|r - r_{00}| \ll 1$,

$$\kappa_2 \simeq -\frac{4}{3\pi} \alpha m^2, \quad \kappa_3 \simeq \frac{\alpha}{\pi} eH_l[4 - \frac{i\pi}{\sqrt{(r - r_{00})}}], \quad (19)$$

From the formulas (13) and (17) it follows that

$$r_{00} = (1 + \delta_0)^2, \quad \delta_0 = M_0 / m \ll 1. \quad (20)$$

It can be seen that the root divergence for the lower threshold is slightly shifted to the point $r \simeq 1 + 2\delta_0$. Another situation arises when particles are born at
higher levels. For example, for \( r_{20} > r > r_{10} \) we have for the terms containing the root divergence

\[
\kappa_2^{10} = \alpha m^2 \mu r \frac{2}{\pi} \exp \left( -\frac{2r}{\mu} \right) \times \left[ \frac{\mu/2r - 1}{\sqrt{h(r)}} A(r) - \frac{1}{2r} \ln(\mu + 1 - r) \right], \tag{21}
\]

\[
\kappa_3^{10} = \alpha m^2 \mu r \frac{2}{\pi} \exp \left( -\frac{2r}{\mu} \right) \times \left[ \frac{\mu/2r - 1 - 2/\mu}{\sqrt{h(r)}} A(r) - \frac{1}{2r} \ln(\mu + 1 - r) + \frac{2}{\mu} \right], \tag{22}
\]

\[
A(r) = \arctan \frac{r - \mu/2}{\sqrt{h(r)}} + \arctan \frac{r + \mu/2}{\sqrt{h(r)}} = \pi - \arctan \frac{\sqrt{h(r)}}{r - \mu/2} - \arctan \frac{\sqrt{h(r)}}{r + \mu/2}, \tag{23}
\]

\[
h(r) = (1 + \mu)r - r^2 - \mu^2/4. \tag{24}
\]

For \( r - r_{10} \ll 1 \), \( -h(r) \simeq \sqrt{1 + 2\mu(r - r_{10})} \ll 1 \), and the expressions (21)-(22) - have root divergence at \( r = r_{10} \)

\[
\kappa_i^{10} \simeq -4i\alpha m^2 r \exp \left( -\frac{2r}{\mu} \right) \frac{\beta_i}{\sqrt{-h(r)}}, \tag{25}
\]

\[
\beta_2 = \frac{\mu}{2} - \frac{\mu^2}{4r}, \quad \beta_3 = 1 + \frac{\mu}{2} - \frac{\mu^2}{4r}.
\]

We take into account that

\[
r - r_{10} \simeq 2(\sqrt{r} - \sqrt{r_{10}})\sqrt{r_{10}} = \frac{1}{m}(\omega - \varepsilon_0 - \varepsilon_1)\sqrt{r_{10}}, \tag{26}
\]

as well as the radiation width and shift of the first two Landau levels [8]-[9]. Then for \( \omega = \varepsilon_0 + \varepsilon_1 + \text{Re}(\varepsilon_0^* + \varepsilon_1^*) \) we obtain the following expression for \( \kappa_i^{10} \)

\[
\kappa_i^{10} \simeq -\sqrt{\frac{\alpha}{\delta_1}} (2\mu)^{3/4} e^{-2(1 + i)m^2} \tag{27}
\]

The spectral part of the polarization operator, containing the root divergences at the photon energy when the electron and positron are generated at the Landau levels, is given in [10]. Along with the numbers of energy levels \( l \) and \( k \), we use the numbers \( m = l + k, n = l - k \) (\( n \leq m \)). We have:
\[ \kappa_i^\mu = \alpha m^2 \frac{r}{\pi} \sum_{n=0}^{n_{\text{max}}} \left( 1 - \frac{\delta_{n0}}{2} \right) T_i^{(ns)} = -i \alpha m^2 \mu e^{-\zeta} \sum_{m \geq n_{\text{max}}} \frac{(2 - \delta_{n0}) \zeta^n k!}{\sqrt{g}!} \]

\[ \times \left[ 1 - \frac{1}{\pi} \left( \arctan \frac{2\sqrt{-g}}{2r - \mu n} + \arctan \frac{2\sqrt{-g}}{2r + \mu n} \right) \right] D_i; \quad (28) \]

\[ D_2 = \left( \frac{m \mu}{2} - \frac{n^2 \mu^2}{4r} \right) F + 2 \mu l \vartheta(k - 1) \left[ 2 L_{k-1}^{n+1}(\zeta) L_k^{n-1}(\zeta) - L_k^n(\zeta) L_k^{n-1}(\zeta) \right], \]

\[ D_3 = \left( 1 + \frac{m \mu}{2} - \frac{n^2 \mu^2}{4r} \right) F + 2 \mu l \vartheta(k - 1) L_k^n(\zeta) L_k^{n-1}(\zeta), \]

\[ F = [L_k^n(\zeta)]^2 + \vartheta(k - 1) l_k [L_k^{n-1}(\zeta)]^2, \quad \zeta = \frac{2r}{\mu}, \quad (29) \]

where \( L_k^n(\zeta) \) is the generalized Laguerre polynomial.

\[ g_{lk}(r) = r^2 - (1 + m \mu) r + n^2 \mu^2 / 4, \quad (30) \]

\[ n_{\text{max}} = [d(r)], \quad d(r) = 2(r - \sqrt{r}) / \mu. \quad (31) \]

Near \( r_{lk} \) (\( r - r_{lk} \ll 1 \)), \( g_{lk}(r) \) has the form

\[ g_{lk}(r) \simeq (r - r_{lk}) \sqrt{1 + 2(l + k) \mu + 4kl \mu^2}, \quad (32) \]

\[ r - r_{lk} \simeq \left( \frac{\omega}{2m} - \frac{\varepsilon_l + \varepsilon_k}{2m} \right) \frac{\varepsilon_l + \varepsilon_k}{m}. \quad (33) \]

We take into account the radiation corrections to the energy levels [3]-[11] and choose (taking into account the level shift) the photon energy equal to the sum of the energies of the electron and positron. Then

\[ \left( \frac{\omega}{2m} - \frac{\varepsilon_l + \varepsilon_k}{2m} \right) \rightarrow -i \text{Im} \frac{\varepsilon_l' + \varepsilon_k'}{2m} = \frac{i \sqrt{\mu}}{4 \sqrt{2}} \left( \frac{\delta_{l1}}{\sqrt{l}} + \frac{\delta_{k1}}{\sqrt{k}} \right), \quad (34) \]

\[ g_{lk}' \simeq \frac{i \sqrt{\mu}}{4 \sqrt{2}} \left( \frac{\delta_{l1}}{\sqrt{l}} + \frac{\delta_{k1}}{\sqrt{k}} \right) \left( \sqrt{1 + 2 \mu k} \sqrt{1 + 2(l + k) \mu + 4kl \mu^2} \right). \quad (35) \]

\[ \text{For } \mu \gg 1, \text{ we will consider the function } g \text{ when one of the particles is born at the ground level } k = 0. \text{ Then } m = n = l \]

\[ g \simeq \frac{\alpha m^{3/2} \delta_{l1}}{2 \sqrt{2}} \delta_{l1}, \quad g^{-1/2} \simeq \frac{2^{1/2} \mu^{-3/4}}{\sqrt{\alpha \delta_{l1}}} (1 - i). \quad (37) \]

When \( r - r_{10} \ll 1 \) the main term in the sum is determined by the values \( m = n = l = 1, k = 0, D_2 = \beta_2, D_3 = \beta_3, g = -h \), which is consistent with the expression [27].
In order to take into account the radiation corrections to the Landau levels in the general formula (28), we introduce a modified function

\[ g_{rk}^r(r) = \left( 1 - \frac{\varepsilon_r^r + \varepsilon_k^r}{2m(\sqrt{r} - \sqrt{r_{lk}})} \right) g_{lk}(r), \tag{38} \]

and instead of function \( g \) in (28) we shall use this function \( g^r \). The inclusion of radiation corrections in nonsingular terms leads to small corrections \( \sim \alpha \) to the polarization operator and we neglect these corrections.

3 Conclusion

We examined the effect of radiation corrections to the mass of an electron (positron) on the polarization operator of a photon in a constant and uniform magnetic field. In relatively weak fields \( H \ll H_0 \), the actual pair production threshold determines the criterion \( Hn^{1/3}/H_0 \gtrsim 1 \). Since the relative width of the energy levels is \( \alpha n^{1/3} \), then for \( H/H_0 \lesssim \alpha \) these levels overlap and the quasiclassical consideration of the process is most adequate [14]. Thus, a detailed consideration of these issues is necessary in the field of fields close to or exceeding the critical Schwinger field. It is the existence of such fields in pulsars and magnetars that was discussed in the Introduction. We have formulated the procedure for incorporating radiation corrections into previously obtained formulas for the polarization operator [10]. As a result, the expressions obtained do not contain root divergences in the probability of pair production at the Landau levels.

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