On the path integral representation for Wilson loops and the non–Abelian Stokes theorem II

M. Faber*, A. N. Ivanov†‡, N. I. Troitskaya‡

March 27, 2022

Institut für Kernphysik, Technische Universität Wien, Wiedner Hauptstr. 8–10, A–1040 Vienna, Austria

Abstract
This paper is a revised version of our recent publication Faber et al., Phys. Rev. D62 (2000) 025019, hep–th/9907048. The main revision concerns the expansion into group characters that we have used for the evaluation of path integrals over gauge degrees of freedom. In the present paper we apply the expansion recommended by Diakonov and Petrov in hep–lat/0008004. Our former expansion was approximate and in the region of the particular values of parameters violated the completeness condition by 1.4%. We show that by using the expansion into characters recommended by Diakonov and Petrov in hep–lat/0008004 our previous results are retained and the path integral over gauge degrees of freedom for Wilson loops derived by Diakonov and Petrov (Phys. Lett. B224 (1989) 131 and, correspondingly, hep–lat/0008004) our previous results are retained and the path integral over gauge degrees of freedom for Wilson loops derived by Diakonov and Petrov (Phys. Lett. B224 (1989) 131 and, correspondingly, hep–lat/0008004) by using a special regularization is erroneous and predicts zero value for the Wilson loop. This property is obtained by direct evaluation of path integrals for Wilson loops defined for pure SU(2) gauge fields and Z(2) center vortices with spatial azimuthal symmetry. Further we show that both derivations given by Diakonov and Petrov for their regularized path integral, if done correctly, predict also zero value for Wilson loops. Therefore, the application of their path integral representation of Wilson loops cannot give a new way to check confinement in lattice as has been declared by Diakonov and Petrov (Phys. Lett. B242 (1990) 425 and hep–lat/0008004). Our statement pointing out that none non–Abelian Stokes theorem can exist for Wilson loops except the old–fashioned one derived by means of the path-ordering procedure is retained. It is based on well–defined properties of group characters and is not related to whatever explicit method of evaluation of path integrals is applied. Comments on the paper hep–lat/0008004 by Diakonov and Petrov are given. Some misprints in our paper Phys. Rev. D62 (2000) 025019 are corrected.

PACS: 11.10.–z, 11.15.–q, 12.38.–t, 12.38.Aw, 12.90.+b

Keywords: non–Abelian gauge theory, confinement

*E–mail: faber@kph.tuwien.ac.at, Tel.: +43–1–58801–14261, Fax: +43–1–58801–14299
†E–mail: ivanov@kph.tuwien.ac.at, Tel.: +43–1–58801–14261, Fax: +43–1–58801–14299
‡Permanent Address: State Technical University, Department of Nuclear Physics, 195251 St. Petersburg, Russian Federation
1 Introduction

The hypothesis of quark confinement, bridging the hypothesis of the existence of quarks and the failure of the detection of quarks as isolated objects, is a challenge for QCD. As a criterion of colour confinement in QCD, Wilson [1] suggested to consider the average value of an operator

$$W(C) = \frac{1}{N} \text{tr} P_C e^{ig \int_C dx_\mu A_\mu(x)} = \frac{1}{N} \text{tr} U(C_{xx}),$$

(1.1)
defined on an closed loop $C$, where $A_\mu(x) = t^a A^a_\mu(x)$ is a gauge field, $t^a$ ($a = 1, \ldots, N^2 - 1$) are the generators of the $SU(N)$ gauge group in fundamental representation normalized by the condition $\text{tr}(t^a t^b) = \delta^{ab}/2$, $g$ is the gauge coupling constant and $P_C$ is the operator ordering colour matrices along the path $C$. The trace in Eq.(1.1) is computed over colour indices. The operator

$$U(C_{yx}) = P_{C_{yx}} e^{ig \int_{C_{yx}} dz_\mu A_\mu(z)},$$

(1.2)

makes a parallel transport along the path $C_{yx}$ from $x$ to $y$. For Wilson loops the contour $C$ defines a closed path $C_{xx}$. For determinations of the parallel transport operator $U(C_{yx})$ the action of the path–ordering operator $P_{C_{yx}}$ is defined by the following limiting procedure [2]

$$U(C_{yx}) = \lim_{n \to \infty} \prod_{k=1}^{n} U(C_{x_k x_{k-1}}) =$$

$$= \lim_{n \to \infty} U(C_{yx_{n-1}}) \cdots U(C_{x_{2}x_{1}}) U(C_{x_{1}x}) = \lim_{n \to \infty} \prod_{k=1}^{n} e^{ig (x_k - x_{k-1}) \cdot A(x_{k-1})},$$

(1.3)

where $C_{x_k x_{k-1}}$ is an infinitesimal segment of the path $C_{yx}$ with $x_0 = x$ and $x_n = y$. The parallel transport operator $U(C_{x_k x_{k-1}})$ for an infinitesimal segment $C_{x_k x_{k-1}}$ is defined by [2]:

$$U(C_{x_k x_{k-1}}) = e^{ig \int_{C_{x_k x_{k-1}}} dz_\mu A_\mu(z)} = e^{ig (x_k - x_{k-1}) \cdot A(x_{k-1})},$$

(1.4)

In accordance with the definition of the path–ordering procedure (1.3) the parallel transport operator $U(C_{yx})$ has the property

$$U(C_{yx}) = U(C_{yx_1}) U(C_{x_1 x}),$$

(1.5)

where $x_1$ belongs to the path $C_{yx}$. Under gauge transformations with a gauge function $\Omega(z)$,

$$A_\mu(z) \to A^\Omega_\mu(z) = \Omega(z) A_\mu(z) \Omega^\dagger(z) + \frac{1}{ig} \partial_\mu \Omega(z) \Omega^\dagger(z),$$

(1.6)

the operator $U(C_{yx})$ has a very simple transformation law

$$U(C_{yx}) \to U^\Omega(C_{yx}) = \Omega(y) U(C_{yx}) \Omega^\dagger(x).$$

(1.7)
We would like to stress that this equation is valid even if the gauge functions \( \Omega(x) \) and \( \Omega(y) \) differ significantly for adjacent points \( x \) and \( y \).

As has been postulated by Wilson [1] the average value of the Wilson loop \(< W(C) >\) in the confinement regime should show area–law falloff [1]

\[
< W(C) > \sim e^{-\sigma A},
\]

where \( \sigma \) and \( A \) are the string tension and the minimal area of the loop, respectively. As usually the minimal area is a rectangle of size \( L \times T \). In this case the exponent \( \sigma A \) can be represented in the equivalent form \( \sigma A = V(L) T \), where \( V(L) = \sigma L \) is the interquark potential and \( L \) is the relative distance between quark and anti–quark.

The paper is organized as follows. In Sect. 2 we discuss the path integral representation for Wilson loops by using well–known properties of group characters. The discretized form of this path integral is naturally provided by properties of group characters and does not need any artificial regularization. We derive a closed expression for Wilson loops in irreducible representation \( j \) of \( SU(2) \). In Sect. 3 we extend the path integral representation to the gauge group \( SU(N) \). As an example, we give an explicit representation for Wilson loops in the fundamental representation of \( SU(3) \). In Sects. 4 and 5 we evaluate the path integral for Wilson loops, suggested in Ref.[3], for two specific gauge field configurations (i) a pure gauge field in the fundamental representation of \( SU(2) \) and (ii) \( Z(2) \) center vortices with spatial azimuthal symmetry, respectively. We show that this path integral representation fails to describe the original Wilson loop for both cases. In Sect. 6 we show that the regularized evolution operator in Ref.[3] representing Wilson loops in the form of the path integral over gauge degrees of freedom has been evaluated incorrectly by Diakonov and Petrov. The correct value for the evolution operator is zero. This result agrees with those obtained in Sects. 4 and 5. In Sect. 7 we criticize the removal of the oscillating factor from the evolution operator suggested in Ref.[3] via a shift of energy levels of the axial–symmetric top. We show that such a removal is prohibited. It leads to a change of symmetry of the starting system from \( SU(2) \) to \( U(2) \). Keeping the oscillating factor one gets a vanishing value of Wilson loops in agreement with our results in Sects. 4, 5 and 6. In the Conclusion we discuss the obtained results. In Appendix we give comments on the paper [hep–lat/0008004] by Diakonov and Petrov.

2 Path integral representation for Wilson loops

Attempts to derive a path integral representation for Wilson loops (\([1,2]\)), where the path ordering operator is replaced by a path integral, have been undertaken in Refs.[3–5]. The path integral representations have been derived for Wilson loops in terms of gauge degrees of freedom (bosonic variables) [3,4] and fermionic degrees of freedom (Grassmann variables) [5]. For the derivation of the quoted path integral representations for Wilson loop different mathematical machineries have been used. Below we discuss the derivation of the path integral representation for Wilson loops in terms of gauge degrees of freedom by using well–known properties of group characters. In this case a discretized form of path integrals is naturally provided by the properties of group characters and the completeness condition of gauge functions. It coincides with the standard discretization of Feynman path integrals [6] and does not need any artificial regularization.
We argue that the path integral representation for Wilson loops suggested by Diakonov and Petrov in Ref.[3] is erroneous. For the derivation of this path integral representation Diakonov and Petrov have used a special regularization drawing an analogy with an axial–symmetric top. The moments of inertia of this top are taken finally to zero. As we show below this path integral amounts to zero for Wilson loops defined for SU(2). Therefore, it is not a surprise that the application of this erroneous path integral representation to the evaluation of the average value of Wilson loops has led to the conclusion that for large loops the area–law falloff is present for colour charges taken in any irreducible representation r of SU(N) [7]. This statement has not been supported by numerical simulations within lattice QCD [8]. As has been verified, e.g. in Ref.[8] for SU(3), in the confined phase and at large distances, colour charges with non–zero N–ality have string tensions of the corresponding fundamental representation, whereas colour charges with zero N–ality are screened by gluons and cannot form a string at large distances. Hence, the results obtained in Ref.[7] cannot give a new way to check confinement in lattice as has been declared by Diakonov and Petrov.

For the derivation of Wilson loops in the form of a path integral over gauge degrees of freedom by using well–known properties of group characters it is convenient to represent \( W_r(C) \) in terms of characters of irreducible representations of SU(N) [9–11]

\[
W_r(C) = \frac{1}{d_r} \chi[U_r(C_{xx})],
\]

(2.1)

where the matrix \( U_r(C_{xx}) \) realizes an irreducible and \( d_r \)–dimensional matrix representation r of the group SU(N) with the character \( \chi[U_r(C_{xx})] = \text{tr}[U_r(C_{xx})] \).

In order to introduce the path integral over gauge degrees of freedom we suggest to use

\[
\int D \Omega_r \chi[\Omega_r \Omega_r^\dagger] \chi[\Omega_r V_r] = \frac{1}{d_r} \chi[U_r V_r],
\]

(2.2)

where the matrices \( U_r \) and \( V_r \) belong to the irreducible representation r, and \( D \Omega_r \) is the Haar measure normalized to unity \( \int D \Omega_r = 1 \). The orthogonality relation for gauge functions \( \Omega_r \) reads

\[
\int D \Omega_r (\Omega_r^\dagger)_{a_1 b_1} (\Omega_r)_{a_2 b_2} = \frac{1}{d_r} \delta_{a_1 b_2} \delta_{b_1 a_2}.
\]

(2.3)

By using the orthogonality relation it is convenient to represent the Wilson loop in the form of the integral

\[
W_r(C) = \frac{1}{d_r} \int D \Omega_r(x) \chi[\Omega_r(x)U_r(C_{xx})\Omega_r^\dagger(x)].
\]

(2.4)

According to Eq.(1.3) and Eq.(1.5) the matrix \( U_r(C_{xx}) \) can be decomposed in

\[
U_r(C_{xx}) = \lim_{n \to \infty} U_r(C_{xxn-1})U_r(C_{xxn-2}) \ldots U_r(C_{xx2})U_r(C_{xx1}).
\]

(2.5)

Substituting Eq.(2.3) in Eq.(2.4) and applying \((n−1)\)–times Eq.(2.2) we end up with

\[
W_r(C) = \frac{1}{d_r^2} \lim_{n \to \infty} \int \ldots \int D \Omega_r(x_1) \ldots \Omega_r(x_n) d_r \chi[\Omega_r(x_n)U_r(C_{xxn-1})\Omega_r^\dagger(x_{n-1})] \ldots d_r \chi[\Omega_r(x_1)U_r(C_{xx1})\Omega_r^\dagger(x)]
\]

(2.6)
Using relations \( \Omega_r(x_k)U_r(C_{x_kx_{k-1}})\Omega_r^\dagger(x_{k-1}) = U_r^\Omega(C_{x_kx_{k-1}}) \) we get:

\[
W_r(C) = \frac{1}{d_r^2} \lim_{n \to \infty} \int \cdots \int D\Omega_r(x_1) \cdots D\Omega_r(x_n) d_r \chi[U_r^\Omega(C_{x_nx_{n-1}})] \cdots d_r \chi[U_r^\Omega(C_{x_1x_n})].
\] (2.7)

The integrations over \( \Omega_r(x_k) \) \( (k = 1, \ldots, n) \) are well defined. These are standard integrations on the compact Lie group \( SU(N) \).

We should emphasize that the integrations over \( \Omega_r(x_k) \) \( (k = 1, \ldots, n) \) are not correlated and should be carried out independently.

Due to Eq.(2.2) the discretization of Wilson loops given by Eqs.(2.6) and (2.7) reproduces the standard discretization of Feynman path integrals [6] where infinitesimal time steps can be described by a classical motion. Therefore, the discretized expression (2.7) can be represented formally by:

\[
W_r(C) = \frac{1}{d_r^2} \int \prod_{x \in C} [d_r D\Omega_r(x)] \chi[U_r^\Omega(C_{xx})].
\] (2.8)

Conversely the evaluation of this path integral corresponds to the discretization given by Eqs.(2.6) and (2.7). The measure of the integration over \( \Omega_r(x) \) is well defined and normalized to unity:

\[
\int \prod_{x \in C} D\Omega_r(x) = \lim_{n \to \infty} \int D\Omega_r(x_n) \int D\Omega_r(x_{n-1}) \cdots \int D\Omega_r(x_1) = 1.
\] (2.9)

Thus, for the determination of the path integral over gauge degrees of freedom (2.8) we do not need to use any regularization, since the discretization given by Eqs.(2.6) and (2.7) are well defined.

We would like to emphasize that Eq.(2.8) is a continuum analogy of the lattice version of the path integral over gauge degrees of freedom for Wilson loops used in Eq.(2.13) of Ref.[11] for the evaluation of the average value of Wilson loops in connection with \( Z(2) \) center vortices.

Now let us to proceed to the evaluation of the characters \( \chi[U_r^\Omega(C_{x_kx_{k-1}})] \). Due to the infinitesimality of the segments \( C_{x_kx_{k-1}} \) we can omit the path ordering operator in the definition of \( U_r^\Omega(C_{x_kx_{k-1}}) \) [2]. This allows us to evaluate the character \( \chi[U_r^\Omega(C_{x_kx_{k-1}})] \) with \( U_r^\Omega(C_{x_kx_{k-1}}) \) taken in the form [2]

\[
U_r^\Omega(C_{x_kx_{k-1}}) = \exp ig \int_{C_{x_kx_{k-1}}} dx_\mu A_\mu^\Omega(x).
\] (2.10)

Of course, the relation given by Eq.(2.10) is only defined in the sense of a meanvalue over an infinitesimal segment \( C_{x_kx_{k-1}} \). Therefore, it can be regarded to some extent as a smoothness condition. Unlike the smoothness condition used by Diakonov and Petrov [3] Eq.(2.10) does not corrupt the Wilson loop represented by the path integral over the gauge degrees of freedom.

The evaluation of the characters of \( U_r^\Omega(C_{x_kx_{k-1}}) \) given by Eq.(2.10) runs as follows. First let us consider the simplest case, the \( SU(2) \) gauge group, where we have \( r = j = 0, 1/2, 1, \ldots \) and \( d_j = 2j + 1 \). The character \( \chi [U_j^\Omega(C_{x_kx_{k-1}})] \) is equal to [9,10,12]

\[
\chi [U_j^\Omega(C_{x_kx_{k-1}})] = \sum_{m_j=-j}^{j} \langle m_j|U_j^\Omega(C_{x_kx_{k-1}})|m_j\rangle =
\]
\[ \sum_{m_j = -j}^{j} e^{i m_j \Phi[C_{x_kx_{k-1}}; A^\Omega]}, \quad (2.11) \]

where \( m_j \) is the magnetic colour quantum number, \( |m_j \rangle \) and \( m_j \Phi[C_{x_kx_{k-1}}; A^\Omega] \) are the eigenstates and eigenvalues of the operator

\[ \hat{\Phi}[C_{x_kx_{k-1}}; A^\Omega] = g \int_{C_{x_kx_{k-1}}} dx_\mu A^\Omega_\mu(x), \quad (2.12) \]

i.e. \( \hat{\Phi}[C_{x_kx_{k-1}}; A^\Omega] |m_j \rangle = m_j \Phi[C_{x_kx_{k-1}}; A^\Omega] |m_j \rangle \). The standard procedure for the evaluation of the eigenvalues gives \( \Phi[C_{x_kx_{k-1}}; A^\Omega] \) in the form

\[ \Phi[C_{x_kx_{k-1}}; A^\Omega] = g \int_{C_{x_kx_{k-1}}} \sqrt{g_{\mu\nu}[A^\Omega](x)} dx_\mu dx_\nu, \quad (2.13) \]

where the metric tensor can be given formally by the expression

\[ g_{\mu\nu}[A^\Omega](x) = 2 \text{tr}[A^\Omega_\mu A^\Omega_\nu](x). \quad (2.14) \]

In order to find an explicit expression for the metric tensor we should fix a gauge. As an example let us take the Fock–Schwinger gauge

\[ x_\mu A_\mu(x) = 0. \quad (2.15) \]

In this case the gauge field \( A_\mu(x) \) can be expressed in terms of the field strength tensor \( G_{\mu\nu}(x) \) as follows

\[ A_\mu(x) = \int_{0}^{1} ds \ s x_\alpha G_{\alpha\mu}(xs). \quad (2.16) \]

This can be proven by using the obvious relation

\[ x_\alpha G_{\alpha\mu}(x) = x_\alpha \partial_\alpha A_\mu(x) - x_\alpha \partial_\mu A_\alpha(x) - ig[x_\alpha A_\alpha(x), A_\mu(x)] = A_\mu(x) + x_\alpha \frac{\partial}{\partial x_\alpha} A_\mu(x), \quad (2.17) \]

valid for the Fock–Schwinger gauge \( x_\alpha A_\alpha(x) = 0 \). Replacing \( x \to xs \) we can represent the r.h.s. of Eq.(2.17) as a total derivative with respect to \( s \)

\[ sx_\alpha G_{\alpha\mu}(xs) = A_\mu(xs) + x_\alpha \frac{\partial}{\partial x_\alpha} A_\mu(xs) = \frac{d}{ds}[s A_\mu(xs)]. \quad (2.18) \]

Integrating out \( s \in [0, 1] \) we arrive at Eq.(2.16).

Using Eq.(2.16) we obtain the metric tensor \( g_{\mu\nu}[A^\Omega](x) \) in the form

\[ g_{\mu\nu}[A^\Omega](x) = 2x_\alpha x_\beta \int_{0}^{1} \int_{0}^{1} dsds' ss' \text{tr}[G_{\alpha\mu}^\Omega(xs)G_{\beta\nu}^\Omega(xs')] = 2x_\alpha x_\beta \int_{0}^{1} \int_{0}^{1} dsds' ss' \text{tr}[\Omega(xs)G_{\alpha\mu}(xs)\Omega^\dagger(xs)\Omega(xs')G_{\beta\nu}(xs')\Omega^\dagger(xs')]. \quad (2.19) \]
For the derivation of Eq. (2.19) we define the operator \( \Phi[C_{x_k x_{k-1}}; A^\Omega] \) of Eq. (2.12) following the definition of the phase of the parallel transport operator \( U(C_{x_k x_{k-1}}) \) given by Eq. (1.4) [2]

\[
\hat{\Phi}[C_{x_k x_{k-1}}; A^\Omega] = g \int_{C_{x_k x_{k-1}}} dx_\mu A^\Omega_\mu(x) = (x_k - x_{k-1})_\mu A^\Omega_\mu(x_{k-1}) = (x_k - x_{k-1})\mu \int_0^1 ds s x^\alpha_{k-1} C^\Omega_{\alpha\mu}(x_{k-1}s). \tag{2.20}
\]

The parameter \( s \) is to some extent an order parameter distinguishing the gauge functions \( \Omega(x_k) \) and \( \Omega(x_{k-1}) \) entering the relation \( \Omega(x_k) U(C_{x_k x_{k-1}}) \Omega^\dagger(x_{k-1}) = U^\Omega(C_{x_k x_{k-1}}) \).

Substituting Eq. (2.11) in Eq. (2.7) we arrive at the expression for Wilson loops defined for \( SU(2) \)

\[
W_j(C) = \frac{1}{(2j+1)^2} \lim_{n \to \infty} \int D\Omega_j(x_n) (2j+1) \sum_{m_j^{(n)} = -j} e^{ig m_j^{(n)} \int_{C_{x_1 x_n}} \sqrt{g_{\mu\nu}[A^\Omega]}(x) dx_\mu dx_\nu} \int D\Omega_j(x_{n-1}) (2j+1) \sum_{m_j^{(n-1)} = -j} e^{ig m_j^{(n-1)} \int_{C_{x_1 x_{n-1}}} \sqrt{g_{\mu\nu}[A^\Omega]}(x) dx_\mu dx_\nu} \cdots \int D\Omega_j(x_1) (2j+1) \sum_{m_j^{(1)} = -j} e^{ig m_j^{(1)} \int_{C_{x_1 x_1}} \sqrt{g_{\mu\nu}[A^\Omega]}(x) dx_\mu dx_\nu}. \tag{2.21}
\]

The magnetic quantum number \( m_j^{(k)} \) \( k = 1, \ldots, n \) belongs to the infinitesimal segment \( C_{x_k x_{k-1}} \), where \( C_{x_k x_{k-1}} = C_{x_1 x_n} \).

In compact form Eq. (2.21) can be written as a path integral over gauge functions

\[
W_j(C) = \frac{1}{(2j+1)^2} \prod_{x \in C} \int \mathcal{D}\Omega_j(x) \prod_{\{m_j(x)\}} (2j+1) e^{ig \int_C m_j(x) \sqrt{g_{\mu\nu}[A^\Omega]}(x) dx_\mu dx_\nu}. \tag{2.22}
\]

The integrals along the infinitesimal segments \( C_{x_k x_{k-1}} \) we determine as [2]

\[
\int_{C_{x_k x_{k-1}}} m_j(x) \sqrt{g_{\mu\nu}[A^\Omega]}(x) dx_\mu dx_\nu = m_j(x_{k-1}) \sqrt{g_{\mu\nu}[A^\Omega]}(x_{k-1}) \Delta x_\mu \Delta x_\nu = m_j^{(k-1)} \sqrt{g_{\mu\nu}[A^\Omega]}(x_{k-1}) \Delta x_\mu \Delta x_\nu. \tag{2.23}
\]

where \( \Delta x = x_k - x_{k-1} \).

Comparing the path integral (2.22) with that suggested in Eq. (23) of Ref.[3] one finds rather strong disagreement. First, this concerns the contribution of different states \( m_j \) of the representation \( j \). In the case of the path integral (2.22) there is a summation over all values of the magnetic colour quantum number \( m_j \), whereas the representation of Ref.[3] contains only one term with \( m_j = j \). Second, Ref.[3] claims that in the integrand of
their path integral the exponent should depend only on the gauge field projected onto the third axis in colour space. However, this is only possible if the gauge functions are slowly varying with \( x \), i.e. \( \Omega(x)\Omega^{\dagger}(x_{k-1}) \simeq 1 \). In this case the parallel transport operator \( U^{\Omega}(C_{x_kx_{k-1}}) \) would read [13]

\[
U^{\Omega}(C_{x_kx_{k-1}}) = \exp i g \int_{C_{x_kx_{k-1}}} dx_\mu A_\mu^{\Omega}(x) = 1 + i g (x_i - x_{i-1}) \cdot A^{\Omega}(x_{i-1}), \quad (2.24)
\]

and the evaluation of the character \( \chi[U^{\Omega}_j(C_{x_kx_{k-1}})] \) would run as follows

\[
<m_j|[U^{\Omega}_j(C_{x_kx_{k-1}})]m_j> = 1 + (t^a_j)_{m_jm_j} i g (x_k - x_{k-1}) \cdot [A^{\Omega}(x_{k-1})]^{(a)} = \\
= 1 + m_j i g (x_k - x_{k-1}) \cdot [A^{\Omega}(x_{k-1})]^{(3)} = e^{ig \int_{C_{x_kx_{k-1}}} dx_\mu m_j(x) [A^{\Omega}_\mu(x)]}, \quad (2.25)
\]

where we have used the matrix elements of the generators of \( SU(2) \), i.e. \( (t^a_j)_{m_jm_j} = m_j \delta^{a3} \). More generally the exponent on the r.h.s. of Eq.(2.25) can be written as

\[
\int_{C_{x_kx_{k-1}}} dx_\mu m_j(x) [A^{\Omega}_\mu(x)]^{(3)} = 2 \int_{C_{x_kx_{k-1}}} dx_\mu m_j(x) \text{tr}[t^3_j A^{\Omega}_\mu(x)]. \quad (2.26)
\]

This gives the path integral representation for Wilson loops defined for \( SU(2) \) in the following form

\[
W_j(C) = \frac{1}{(2j+1)^2} \int \prod_{x \in C} D\Omega_j(x) \sum_{\{m_j(x)\}} (2j + 1) e^{2ig \oint_C dx_\mu m_j(x) \text{tr}[t^3_j A^{\Omega}_\mu(x)]}. \quad (2.27)
\]

The exponent contains the gauge field projected onto the third axis in colour space \( \text{tr}[t^3_j A^{\Omega}_\mu(x)] \). Nevertheless, Eq.(2.27) differs from Eq.(23) of Ref.[3] by a summation over all values of the colour magnetic quantum number \( m_j \) of the given irreducible representation \( j \).

The repeated application of Eq.(2.2) induces that the integrations over the gauge function at \( x_k \) are completely independent of the integrations at \( x_{k \pm 1} \). There is no mechanism which leads to gauge functions smoothly varying with \( x_k \) \( (k = 1, \ldots n) \). In this sense the situation is opposite to the quantum mechanical path integral. In Quantum Mechanics the integration over all paths is restricted by the kinetic term of the Lagrange function. In the semiclassical limit \( \hbar \rightarrow 0 \) due to the kinetic term the fluctuations of all trajectories are shrunk to zero around a classical trajectory. However, in the case of the integration over gauge functions for the path integral representation of the Wilson loop, there is neither a suppression factor nor a semiclassical limit like \( \hbar \rightarrow 0 \). The key point of the application of Eq.(2.2) and, therefore, the path integral representation for Wilson loops is that all integrations over \( \Omega(x_k) \) \( (k = 1, \ldots n) \) are completely independent and can differ substantially even if the points, where the gauge functions \( \Omega(x_k) \) and \( \Omega(x_{k-1}) \) are defined, are infinitesimally close to each other.

For the derivation of Eq.(23) of Ref.[3] Diakonov and Petrov have used at an intermediate step a regularization drawing an analogy with an axial–symmetric top with moments of inertia \( I_\perp \) and \( I_\parallel \). Within this regularization the evolution operator representing Wilson loops has been replaced by a path integral over dynamical variables of this
3 The $SU(N)$ extension

The extension of the path integral representation given in Eq.(2.24) to $SU(N)$ is rather straightforward and reduces to the evaluation of the character of the matrix $U_r^\Omega(C_{xkx_{k-1}})$ in the irreducible representation $r$ of $SU(N)$. The character can be given by \cite{12}

$$
\chi[U_r^\Omega(C_{xkx_{k-1}})] = \text{tr}(e^{i \sum_{\ell=1}^{N-1} H_\ell \Phi_\ell(C_{xkx_{k-1}}; A^\Omega)}) = \\
\sum_{\tilde{m}_r} \gamma_{\tilde{m}_r} e^{i \tilde{m}_r \cdot \Phi[C_{xkx_{k-1}}; A^\Omega]},
$$

(3.1)

where $H_\ell$ ($\ell = 1, \ldots, N-1$) are diagonal $d_r \times d_r$ traceless matrices realizing the representation of the Cartan subalgebra, i.e. $[H_\ell, H_\ell'] = 0$, of the generators of the $SU(N)$ \cite{12}. The sum runs over all the weights $\tilde{m}_r = (m_{r,1}, \ldots, m_{r,N-1})$ of the irreducible representation $r$ and $\gamma_{\tilde{m}_r}$ is the multiplicity of the weight $\tilde{m}_r$ and $\sum_{\tilde{m}_r} \gamma_{\tilde{m}_r} = d_r$. The components of the vector $\Phi[C_{xkx_{k-1}}; A^\Omega]$ are defined by

$$
\Phi_\ell[C_{xkx_{k-1}}; A^\Omega] = g \int_{C_{xkx_{k-1}}} \varphi_\ell(\omega(x)),
$$

(3.2)

where we have introduced the notation $\omega(x) = t^a \omega^a(x) = dz \cdot A^\Omega(x)$. The functions $\varphi_\ell(\omega(x))$ are proportional to the roots of the equation $\det[\omega(x) - \lambda] = 0$.

The path integral representation of Wilson loops defined for the irreducible representation $r$ of $SU(N)$ reads

$$
W_r(C) = \frac{1}{d_r^2} \int_{x \in C} \prod_{x \in C} D\Omega(x) \sum_{\{\tilde{m}_r(x)\}} d_r \gamma_{\tilde{m}_r(x)} e^{i g \int_{C} \tilde{m}_r(x) \cdot \Phi[C_{xkx_{k-1}}; A^\Omega]}. 
$$

(3.3)

Let us consider in more details the path integral representation of Wilson loops defined for the fundamental representation $\underline{3}$ of $SU(3)$. The character $\chi_{\underline{3}}[U_3^\Omega(C)]$ is defined as

$$
\chi_{\underline{3}}[U_3^\Omega(C)] = \text{tr}(e^{i H_1 \Phi_1[C; A^\Omega] + i H_2 \Phi_2[C; A^\Omega]}) = e^{-i \Phi_2[C; A^\Omega]/3} + e^{i \Phi_1[C; A^\Omega]/2\sqrt{3}} e^{i \Phi_2[C; A^\Omega]/6} + e^{-i \Phi_1[C; A^\Omega]/2\sqrt{3}} e^{i \Phi_2[C; A^\Omega]/6},
$$

(3.4)

where $H_1 = t^3/\sqrt{3}$ and $H_2 = t^8/\sqrt{3} \ [12]$. For the representation $\underline{3}$ of $SU(3)$ the equation $\det[\omega - \lambda] = 0$ takes the form

$$
\lambda^3 - \lambda^2 \frac{1}{2} \text{tr} \omega^2(x) - \det \omega(x) = 0.
$$

(3.5)
The roots of Eq. (3.5) read

\[
\lambda^{(1)} = -\frac{1}{\sqrt{6}} \sqrt{\text{tr} \omega^2(x)} \cos \left( \frac{1}{3} \arccos \left[ 2 \det \left[ 1 + 12 \frac{t^a \text{tr}(t^a \omega^2(x))}{\text{tr} \omega^2(x)} \right] \right] \right) \\
\lambda^{(2)} = -\frac{1}{\sqrt{2}} \sqrt{\text{tr} \omega^2(x)} \sin \left( \frac{1}{3} \arccos \left[ 2 \det \left[ 1 + 12 \frac{t^a \text{tr}(t^a \omega^2(x))}{\text{tr} \omega^2(x)} \right] \right] \right), \\
\lambda^{(3)} = \frac{2}{3} \sqrt{\text{tr} \omega^2(x)} \cos \left( \frac{1}{3} \arccos \left[ 2 \det \left[ 1 + 12 \frac{t^a \text{tr}(t^a \omega^2(x))}{\text{tr} \omega^2(x)} \right] \right] \right). 
\] (3.6)

In terms of the roots \( \lambda^{(i)} \) \( i = 1, 2, 3 \) the phases \( \Phi_{1,2}[C; A^\Omega] \) are defined as

\[
\Phi_1[C; A^\Omega] = -g\sqrt{6} \int_C \sqrt{\text{tr} \omega^2(x)} \sin \left( \frac{1}{3} \arccos \left[ 2 \det \left[ 1 + 12 \frac{t^a \text{tr}(t^a \omega^2(x))}{\text{tr} \omega^2(x)} \right] \right] \right), \\
\Phi_2[C; A^\Omega] = -g\sqrt{6} \int_C \sqrt{\text{tr} \omega^2(x)} \cos \left( \frac{1}{3} \arccos \left[ 2 \det \left[ 1 + 12 \frac{t^a \text{tr}(t^a \omega^2(x))}{\text{tr} \omega^2(x)} \right] \right] \right), \hspace{1cm} (3.7)
\]

where \( \text{tr} \omega^2(x) = \frac{1}{2} g_{\mu \nu} [A^\Omega](x) \, dx_\mu \, dx_\nu \). Thus, in the fundamental representation 2 the path integral representation for Wilson loops reads

\[
W_2(C) = \frac{1}{6} \int \prod_{x \in C} [D\Omega_2(x) \times 3] \left( e^{i\Phi_1[C; A^\Omega]}/2\sqrt{3} e^{i\Phi_2[C; A^\Omega]}/6 + e^{-i\Phi_1[C; A^\Omega]}/2\sqrt{3} e^{i\Phi_2[C; A^\Omega]}/6 + e^{-i\Phi_2[C; A^\Omega]}/3 \right), \hspace{1cm} (3.8)
\]

where the phases \( \Phi_{1,2}[C; A^\Omega] \) are given by Eq. (3.7).

4 Wilson loop for pure gauge field

As has been pointed out in Ref. [3] the path integral over gauge degrees of freedom representing Wilson loops is *not* of the Feynman type, therefore, it depends explicitly on how one “understands” it, i.e. how it is discretized and regularized. We would like to emphasize that the regularization procedure applied in Ref. [3] has led to an expression for Wilson loops which supports the hypothesis of Maximal Abelian Projection [14]. According to this hypothesis only Abelian degrees of freedom of non–Abelian gauge fields are responsible for confinement. This is to full extent a dynamical hypothesis. It is quite obvious that such a dynamical hypothesis cannot be derived only by means of a regularization procedure.
In order to show that the problem touched in this paper is not of marginal interest and to check if path integral expressions that look differently superficially could actually compute the same number we evaluate below explicitly the path integrals representing Wilson loop for a pure $SU(2)$ gauge field. As has been stated in Ref.[3] for Wilson loops $C$ a gauge field along a given curve can be always written as a “pure gauge” and the derivation of the path integral representation for Wilson loops can be provided for the gauge field taken without loss of generality in the “pure gauge” form. We would like to show that for the pure $SU(2)$ gauge field the path integral representation for Wilson loops suggested in Ref.[3] fails for a correct description of Wilson loops. Since a pure gauge field is equivalent to a zero gauge field Wilson loops should be unity.

Of course, any correct path integral representation for Wilson loops should lead to the same result. The evaluation of Wilson loops within the path integral representation Eq.(2.8) is rather trivial and transparent. Indeed, we have not corrupted the starting expression for Wilson loops (2.1) by any artificial regularization. The reby, the general formula (2.8) evaluated through the discretization given by Eqs.(2.7) and (2.6) is completely identical to the original expression (2.1). The former gives a unit value for Wilson loops defined for an arbitrary contour $C$ and an irreducible representation $J$ of $SU(2)$:

$$W^J(C) = 1.$$  

Let us focus now on the path integral representation suggested in Ref.[3]

$$W^J(C) = \int \prod_{x \in C} D\Omega(x) e^{2iJg \oint_C dx_\mu tr[t^3 A^\Omega_\mu(x)]},$$  

where all matrices are taken in the irreducible representation $J$. Following the discretization suggested in Ref.[3] we arrive at the expression

$$W^J(C) = \lim_{n \to \infty} \prod_{k=1}^n \int D\Omega(x_k) e^{2iJg \oint_{C_{x_k+1x_k}} dx_\mu tr[t^3 A^\Omega_\mu(x)]}.$$  

Setting $A_\mu(x) = \partial_\mu(U(x)U^\dagger(x))/ig$ we get

$$A^\Omega_\mu(x) = \frac{1}{ig} \partial_\mu(\Omega(x)U(x))(\Omega(x)U(x))^\dagger.$$  

By a gauge transformation $\Omega(x)U(x) \to \Omega(x)$ we reduce Eq.(4.1) to the form

$$W^J(C) = \int \prod_{x \in C} D\Omega(x) e^{2Jm \oint_C dx_\mu tr[t^3 \partial_\mu(\Omega(x)\Omega^\dagger(x))]}.$$  

For simplicity we consider Wilson loops in the fundamental representation of $SU(2)$, $W_{1/2}(C)$. The result can be generalized to any irreducible representation $J$.

For the evaluation of the path integral Eq.(4.4) it is convenient to use a standard $s$–parameterization of Wilson loops $C$ [2]: $x_\mu \to x_\mu(s)$, with $s \in [0, 1]$ and $x_\mu(0) = x_\mu(1) = x_\mu$.

The Wilson loop (4.4) reads in the $s$–parameterization

$$W_{1/2}(C) = \int \prod_{0 \leq s \leq 1} D\Omega(s) \exp \int_0^1 ds tr[t^3 \frac{d\Omega(s)}{ds} \Omega^\dagger(s)].$$  

11
The discretized form of the path integral (4.3) is given by

\[ W_{1/2}(C) = \lim_{n \to \infty} \int \prod_{k=1}^{n} D\Omega_k \exp \Delta s_{k+1,k} \begin{bmatrix} t^3 \Omega_{k+1} - \Omega_k \Delta s_{k+1,k} \end{bmatrix} = \lim_{n \to \infty} \int \prod_{k=1}^{n} D\Omega_k e^{t^3 \Omega k+1 \Omega k} = \lim_{n \to \infty} \int \prod_{k=1}^{n} D\Omega_k D\Omega_{n-1} D\Omega_{n-2} \ldots D\Omega_1 \times e^{t^3 \Omega_n \Omega_{n-1}^\dagger} \ldots e^{t^3 \Omega_2 \Omega_1^\dagger} e^{t^3 \Omega_1 \Omega_n^\dagger}, \]

where \( \Omega_{n+1} = \Omega_1 \).

For the subsequent integration we would use the expansion [15]

\[ e^{z t^3 \Omega} = \sum_j a_j(z) \chi_j [e^{i\pi t^3 \Omega}], \]

When the exponent of the l.h.s. is defined for the fundamental representation, the coefficients \( a_j(z) \) are given by [15,16]

\[ a_j(z) = e^{-i\pi j} (2j + 1) \frac{2J_{2j+1}(z)}{z}, \]

where \( J_{2j+1}(z) \) are Bessel functions and the index \( j \) runs over \( j = 0, 1/2, 1, 3/2, 2, \ldots \) [16]. Recall, that \( z = 2J \) and for the fundamental representation \( J = 1/2 \) we should set \( z = 1 \).

For the integration over \( \Omega_k \) we suggest to use a formula of Ref.[17] modified for our case

\[ \int D\Omega e^{z t^3 A^\dagger B t^3 \Omega} = \sum_j \frac{a_j^2(z)}{2j + 1} \chi_j \left[ (e^{i\pi t^3 \Omega})^2 AB \right], \]

where the coefficients \( a_j(z) \) are defined by the expansion Eq.(4.7). The formula Eq.(4.9) can be derived by using the orthogonality relation for characters [9,10,17]

\[ \int D\Omega \chi_j [A^\dagger] \chi_j [\Omega B] = \frac{\delta_{jj'}}{2j + 1} \chi_j [AB]. \]

Integrating over \( \Omega_i \) \( (i = 1, 2, \ldots, n) \) we arrive at the expression

\[ W_{1/2}(C) = \lim_{n \to \infty} \sum_j (2j + 1) \left[ \frac{a_j}{2j + 1} \right]^n \chi_j \left[ (e^{i\pi t^3 \Omega})^n \right] = \lim_{n \to \infty} \sum_j (2j + 1) \left[ 2J_{2j+1}(1) \right]^n \sum_{k=0}^{2j} e^{-i\pi kn}, \]

where we have denoted \( a_j(1) = a_j = e^{-i\pi j} (2j + 1) 2J_{2j+1}(1) \). The series over \( j \) is convergent for any finite \( n \) and every term of this series vanishes at \( n \to \infty \). This proves that the Wilson loop \( W_{1/2}(C) \) vanishes in the limit \( n \to \infty \), i.e. \( W_{1/2}(C) = 0 \).

Thus, the Wilson loop \( W_{1/2}(C) \) for an arbitrary contour \( C \) and a pure gauge field represented by the path integral derived in Ref.[3] vanishes, instead of being equal to unity, \( W_{1/2}(C) = 1 \). This shows that the path integral representation suggested in Ref.[3] fails for the correct description of Wilson loops.
5  Wilson loop for $Z(2)$ center vortices

In this Section we evaluate explicitly the path integral (4.1) for Wilson loops pierced by a $Z(2)$ center vortex with spatial azimuthal symmetry. Some problems of $Z(2)$ center vortices with spatial azimuthal symmetry have been analysed by Diakonov in his recent publication [18] for the gauge group $SU(2)$. In this system the main dynamical variable is the azimuthal component of the non–Abelian gauge field $A_a^\alpha(\rho)$ ($a = 1, 2, 3$) depending only on $\rho$, the radius in the transversal plane. For a circular Wilson loop in the irreducible representation $J$ one gets

$$W_J(\rho) = \frac{1}{2J + 1} \sum_{m=-J}^{J} e^{i2\pi m \mu(\rho)} = \frac{1}{2J + 1} \frac{\sin[(2J + 1)\pi \mu(\rho)]}{\sin[\pi \mu(\rho)]}, \quad (5.1)$$

where $\mu(\rho) = \rho \sqrt{A_a^\alpha(\rho) A_a^\alpha(\rho)}$. The gauge coupling constant $g$ is included in the definition of the gauge field. For Wilson loops in the fundamental representation $J = 1/2$ we have

$$W_{1/2}(\rho) = \cos[\pi \mu(\rho)]. \quad (5.2)$$

In the case of $Z(2)$ center vortices with spatial azimuthal symmetry and for the fundamental representation of $SU(2)$ Eq. (4.2) takes the form

$$W_{1/2}(\rho) = \lim_{n \to \infty} \prod_{k=1}^{n} \int D\Omega_k e^{\text{tr}[t^3(i2\pi \rho/n)\Omega_{k+1}A_\phi(\rho)\Omega_k + t^3\Omega_{k+1}^\dagger\Omega_k]} =$$

$$= \lim_{n \to \infty} \int \ldots \int D\Omega_n D\Omega_{n-1} D\Omega_{n-2} \ldots D\Omega_1 \times e^{\text{tr}[t^3(i2\pi \rho/n)\Omega_{n}A_\phi(\rho)\Omega_{n-1}^\dagger + t^3\Omega_{n}^\dagger\Omega_{n-1}]} \times e^{\text{tr}[t^3(i2\pi \rho/n)\Omega_{n-1}A_\phi(\rho)\Omega_{n-2}^\dagger + t^3\Omega_{n-1}^\dagger\Omega_{n-2}]} \ldots \times e^{\text{tr}[t^3(i2\pi \rho/n)\Omega_{2}A_\phi(\rho)\Omega_{1}^\dagger + t^3\Omega_{2}^\dagger\Omega_{1}]} \times e^{\text{tr}[t^3(i2\pi \rho/n)\Omega_{1}A_\phi(\rho)\Omega_{n}^\dagger + t^3\Omega_{1}^\dagger\Omega_{n}^\dagger]}, \quad (5.3)$$

where we have used $C_{x_k + x_k} = 2\pi \rho/n$, $\Omega(x_k) = \Omega_k$ and $\Omega_{n+1} = \Omega_1$.

For the subsequent evaluation it is convenient to introduce the matrix

$$Q(A_\phi) = \left(1 + i \frac{2\pi}{n} \rho A_\phi(\rho)\right). \quad (5.4)$$

In terms of $Q(A_\phi)$ the path integral (5.3) reads

$$W_{1/2}(\rho) = \lim_{n \to \infty} \int \ldots \int D\Omega_n D\Omega_{n-1} D\Omega_{n-2} \ldots D\Omega_1 e^{\text{tr}[t^3\Omega_{n}Q(A_\phi)\Omega_{n-1}^\dagger]} \times e^{\text{tr}[t^3\Omega_{n-1}Q(A_\phi)\Omega_{n-2}^\dagger]} \ldots e^{\text{tr}[t^3\Omega_{2}Q(A_\phi)\Omega_{1}^\dagger]} e^{\text{tr}[t^3\Omega_{1}Q(A_\phi)\Omega_{n}^\dagger]}, \quad (5.5)$$

The integration over $\Omega_k$ we carry out with the help of Eq. (1.9) taken in the from

$$\int D\Omega_k e^{\text{tr}[t^3\Omega_{k+1}Q(A_\phi)\Omega_k + Q(A_\phi)\Omega_{k-1}^\dagger t^3\Omega_k]} =$$

$$= \sum_{j} \frac{a_j^2}{2j + 1} \chi_j \left[ (e^{i\pi t^3})^2 \Omega_{k+1}Q^2(A_\phi)\Omega_{k-1}^\dagger \right]. \quad (5.6)$$
and the orthogonality relation for the group characters. This yields
\[ W_{1/2}(\rho) = \lim_{n \to \infty} \sum_j \left[ \frac{a_j}{2j+1} \right]^n \chi_j \left[ \left( e^{i\pi t^3} \right)^n \right] \chi_j[Q^n(A_\phi)]. \] (5.7)

The evaluation of the character \( \chi_j[Q^n(A_\phi)] \) for \( n \to \infty \) runs as follows
\[ \chi_j[Q^n(A_\phi)] = \chi_j \left[ \left( 1 + \frac{2\pi}{n} \rho A_\phi(\rho) \right)^n \right] = \chi_j \left[ e^{i2\pi \rho A_\phi(\rho)} \right] = \frac{\sin[(2j+1)\pi \mu(\rho)]}{\sin[\pi \mu(\rho)]}. \] (5.8)

Substituting Eq.(5.8) in Eq.(5.7) we obtain
\[ W_{1/2}(\rho) = \lim_{n \to \infty} \sum_j [2J_{2j+1}(1)]^n \frac{\sin[(2j+1)\pi \mu(\rho)]}{\sin[\pi \mu(\rho)]} \sum_k e^{-i\pi k n}. \] (5.9)

The series over \( j \) is convergent for any finite \( n \) and at \( n \to \infty \) every term vanishes. This gives \( W_{1/2}(\rho) = 0 \). Thus, we have shown that the the path integral for Wilson loops suggested in Ref.[3] gives zero for a field configuration with a \( Z(2) \) center vortex, \( W_{1/2}(\rho) = 0 \), instead of the correct result \( W_{1/2}(\rho) = \cos \pi \mu(\rho) \).

We hope that the examples considered in Sect.4 and 5 demonstrate that the path integral representation for Wilson loops derived in Ref.[3] is erroneous. Nevertheless, in Sect. 6 we evaluate explicitly the regularized evolution operator \( Z_{\text{Reg}}(R_2, R_1) \) suggested by Diakonov and Petrov for the representation of the Wilson loop in Ref.[3]. We show that this regularized evolution operator \( Z_{\text{Reg}}(R_2, R_1) \) has been evaluated incorrectly in Ref.[3]. The correct evaluation gives \( Z_{\text{Reg}}(R_2, R_1) = 0 \) which agrees with our results obtained above.

6 Path integral for the evolution operator \( Z(R_2, R_1) \)

As has been suggested in Ref.[3] the functional \( Z(R_2, R_1) \) defined by (see Eq.(8) of Ref.[3])
\[ Z(R_2, R_1) = \int_{R_1}^{R_2} DR(t) \exp \left( i T \int_{t_1}^{t_2} \text{Tr} \left( iR \dot{R} \tau_3 \right) \right), \] (6.1)

where \( \dot{R} = dR/dt \) and \( T = 1/2, 1, 3/2, \ldots \) is the colour isospin quantum number, should be regularized by the analogy to an axial–symmetric top. The regularized expression has been defined in Eq.(9) of Ref.[3] by
\[ Z_{\text{Reg}}(R_2, R_1) = \int_{R_1}^{R_2} DR(t) \exp \left( i \int_{t_1}^{t_2} \left[ \frac{1}{2} I_\perp (\Omega_1^2 + \Omega_2^2) + \frac{1}{2} I_\parallel \Omega_3^2 + T \Omega_3 \right] \right), \] (6.2)

where \( \Omega_a = i \text{Tr}(R \dot{R} \tau_a) \) are angular velocities of the top, \( \tau_a \) are Pauli matrices \( a = 1, 2, 3 \), \( I_\perp \) and \( I_\parallel \) are the moments of inertia of the top which should be taken to zero. According to the prescription of Ref.[3] one should take first the limit \( I_\parallel \to 0 \) and then \( I_\perp \to 0 \).
For the confirmation of the result, given in Eq.(13) of Ref.[3],
\[ Z_{\text{Reg}}(R_2, R_1) = (2T + 1) D^T_{TT}(R_2 R_1^\dagger), \] (6.3)
where \( D^T(U) \) is a Wigner rotational matrix in the representation \( T \), the authors of Ref.[3] suggested to evaluate the evolution operator (6.2) explicitly by means of the discretization of the path integral over \( R \). The discretized form of the evolution operator Eq.(6.2) is given by Eq.(14) of Ref.[3] and reads
\[ Z_{\text{Reg}}(R_{N+1}, R_0) = \lim_{N \to \infty} \frac{\mathcal{N}}{\delta} \int \prod_{n=1}^{N} dR_n \times \exp \left[ \sum_{n=0}^{N} \left( -i \frac{I_{\perp}}{2\delta} \left( (\text{Tr} V_n \tau_1)^2 + (\text{Tr} V_n \tau_2)^2 \right) - i \frac{I_{\parallel}}{2\delta} \left( (\text{Tr} V_n \tau_3)^2 - T (\text{Tr} V_n \tau_3) \right) \right) \right], \] (6.4)
where \( R_n = R(s_n) \) with \( s_n = t_1 + n \delta \) and \( \Omega_a = i \text{Tr} (R_n R_{n+1}^\dagger \tau_a) / \delta \) is the discretized analogy of the angular velocities [3] and \( V_n = R_n R_{n+1}^\dagger \) are the relative orientations of the top at neighbouring points. The normalization factor \( \mathcal{N} \) is determined by
\[ \mathcal{N} = N \left( \frac{2\pi i \delta}{2\pi i \delta} \right)^{N+1}. \] (6.5)
(see Eq.(19) of Ref.[3]). According to the prescription of Ref.[3] one should take the limits \( \delta \to 0 \) and \( I_{\parallel}, I_{\perp} \to 0 \) but keeping the ratios \( I_i / \delta \), where \( i = \parallel, \perp \), much greater than unity, \( I_i / \delta \gg 1 \).

Let us rewrite the exponent of the integrand of Eq.(6.4) in equivalent form
\[ Z_{\text{Reg}}(R_{N+1}, R_0) = \lim_{N \to \infty} \frac{\mathcal{N}}{\delta} \int \prod_{n=1}^{N} dR_n \times \exp \left[ \sum_{n=0}^{N} \left( -i \frac{I_{\perp}}{2\delta} (\text{Tr} V_n \tau_a)^2 - i \frac{I_{\parallel} - I_{\perp}}{2\delta} (\text{Tr} V_n \tau_3)^2 - T (\text{Tr} V_n \tau_3) \right) \right]. \] (6.6)
Now let us show that if \( V_n \) is a rotation in the fundamental representation of \( SU(2) \), so
\[ (\text{Tr} V_n \tau_a)^2 = -4 + (\text{Tr} V_n)^2. \] (6.7)
For this aim, first, recall that
\[ \text{Tr} (V_n \tau_a) = -\text{Tr} (V_n^\dagger \tau_a). \] (6.8)
Since \( V_n \) is a rotation matrix in the fundamental representation of \( SU(2) \) [19]. By virtue of the relation (6.8) we can rewrite \( (\text{Tr} V_n \tau_a)^2 \) as follows
\[ (\text{Tr} V_n \tau_a)^2 = -\text{Tr} (V_n \tau_a) \text{Tr} (V_n^\dagger \tau_a) = -2 \text{Tr} \left( (V_n - \frac{1}{2} \text{Tr} V_n) (V_n^\dagger - \frac{1}{2} \text{Tr} V_n) \right) = -2 \text{Tr} (R_n R_{n+1}^\dagger + R_n R_{n+1}^\dagger) + (\text{Tr} V_n)^2 = -2 \text{Tr} 1 + (\text{Tr} V_n)^2 = -4 + (\text{Tr} V_n)^2. \] (6.9)
By using the relation Eq. (5.7) we can recast the r.h.s. of Eq. (5.6) into the form

\[
Z_{\text{Reg}}(R_{N+1}, R_0) = \lim_{N \to \infty} \left[ \left( \frac{I_{\perp}}{\sqrt{2\pi i \delta}} \sqrt{\frac{I_{\parallel}}{2\pi i \delta}} \right)^{N+1} \exp \left( i \frac{1}{2} (N + 1) \frac{I_{\perp}}{\delta} \right) \right] \times \int \prod_{n=1}^{N} dR_n \exp \left[ \sum_{n=0}^{N} \left( -i \frac{I_{\perp}}{2\delta} (\Tr V_n)^2 - i \frac{I_{\parallel} - I_{\perp}}{2\delta} (\Tr V_n \tau_3)^2 - T (\Tr V_n \tau_3)^2 \right) \right].
\] (6.10)

Now let us proceed to the evaluation of the integrals over \( R_n (n = 1, 2, \ldots, N) \). For this aim it is convenient to rewrite the r.h.s. of Eq. (5.10) in the following form

\[
Z_{\text{Reg}}(R_{N+1}, R_0) = \lim_{N \to \infty} \left[ \left( \frac{I_{\perp}}{\sqrt{2\pi i \delta}} \sqrt{\frac{I_{\parallel}}{2\pi i \delta}} \right)^{N+1} \exp \left( i \frac{1}{2} (N + 1) \frac{I_{\perp}}{\delta} \right) \right] \times \int \ldots \int dR_N dR_{N-1} \ldots dR_2 dR_1 \times \exp \left( -i \frac{I_{\perp}}{2\delta} \left( (\Tr R_N R_{N+1}^\dagger)^2 + (\Tr R_{N-1} R_N^\dagger)^2 + \ldots + (\Tr R_2 R_1^\dagger)^2 + (\Tr R_1 R_0^\dagger)^2 \right) \right. \\
- i \frac{I_{\parallel} - I_{\perp}}{2\delta} \left[ (\Tr R_N R_{N+1}^\dagger \tau_3)^2 + (\Tr R_{N-1} R_N^\dagger \tau_3)^2 + \ldots + (\Tr R_2 R_1^\dagger \tau_3)^2 + (\Tr R_1 R_0^\dagger \tau_3)^2 \right] \left. - T \left[ \Tr (R_N R_{N+1}^\dagger \tau_3) + \Tr (R_{N-1} R_N^\dagger \tau_3) + \ldots + \Tr (R_2 R_1^\dagger \tau_3) + \Tr (R_1 R_0^\dagger \tau_3) \right] \right). 
\] (6.11)

In the fundamental representation and the parameterization [19] we have

\[
\Tr V_n = \Tr (R_n R_{n+1}^\dagger) = \\
= 2 \cos \left( \frac{\beta_n}{2} \right) \cos \left( \frac{\beta_{n+1}}{2} \right) \cos \left( \frac{\alpha_n + \gamma_n}{2} - \frac{\alpha_{n+1} + \gamma_{n+1}}{2} \right) \\
+ 2 \sin \left( \frac{\beta_n}{2} \right) \sin \left( \frac{\beta_{n+1}}{2} \right) \cos \left( \frac{\alpha_n - \gamma_n}{2} - \frac{\alpha_{n+1} - \gamma_{n+1}}{2} \right) = \\
= 2 \cos \left( \frac{\beta_n - \beta_{n+1}}{2} \right) \cos \left( \frac{\alpha_n - \alpha_{n+1}}{2} \right) \cos \left( \frac{\gamma_n - \gamma_{n+1}}{2} \right) \\
- 2 \cos \left( \frac{\beta_n + \beta_{n+1}}{2} \right) \sin \left( \frac{\alpha_n - \alpha_{n+1}}{2} \right) \sin \left( \frac{\gamma_n - \gamma_{n+1}}{2} \right),
\]

\[
\Tr (V_n \tau_3) = \Tr (R_n R_{n+1}^\dagger \tau_3) = \\
= -2i \cos \left( \frac{\beta_n}{2} \right) \cos \left( \frac{\beta_{n+1}}{2} \right) \sin \left( \frac{\alpha_n + \gamma_n}{2} - \frac{\alpha_{n+1} + \gamma_{n+1}}{2} \right) \\
+ 2i \sin \left( \frac{\beta_n}{2} \right) \sin \left( \frac{\beta_{n+1}}{2} \right) \sin \left( \frac{\alpha_n - \gamma_n}{2} - \frac{\alpha_{n+1} - \gamma_{n+1}}{2} \right) = \\
= -2i \cos \left( \frac{\beta_n - \beta_{n+1}}{2} \right) \cos \left( \frac{\alpha_n - \alpha_{n+1}}{2} \right) \sin \left( \frac{\gamma_n - \gamma_{n+1}}{2} \right) \\
- 2i \cos \left( \frac{\beta_n + \beta_{n+1}}{2} \right) \sin \left( \frac{\alpha_n - \alpha_{n+1}}{2} \right) \cos \left( \frac{\gamma_n - \gamma_{n+1}}{2} \right). \] (6.12)
The Haar measure $R_n$ is defined by
\[ DR_n = \frac{1}{8\pi^2} \sin \beta_n d\beta_n d\alpha_n d\gamma_n. \] (6.13)

Due to the assumption $I_i/\delta \gg 1$, where $(i = ||, \perp)$, the integrals over $R_n$ are concentrated around unit elements. Expanding $\text{Tr} (V_n)$ and $\text{Tr} (V_n\tau_3)$ around unit elements we get
\[ \text{Tr} V_n = \text{Tr} (R_n R_n^\dagger) = 2 - \frac{1}{4} (\beta_n - \beta_{n+1})^2 - \frac{1}{4} (\alpha_n - \alpha_{n+1} + \gamma_n - \gamma_{n+1})^2, \]
\[ \text{Tr} (V_n\tau_3) = \text{Tr} (R_n R_n^\dagger\tau_3) = -i (\alpha_n - \alpha_{n+1} + \gamma_n - \gamma_{n+1}). \] (6.14)

For the subsequent integration it is convenient to make a change of variables
\[ \frac{\alpha_n + \gamma_n}{2} \rightarrow \gamma_n, \]
\[ \frac{\alpha_n - \gamma_n}{2} \rightarrow \alpha_n. \] (6.15)

The Jacobian of this transformation is equal to unity. After this change of variables (6.14) reads
\[ \text{Tr} V_n = \text{Tr} (R_n R_n^\dagger) = 2 - \frac{1}{4} (\beta_n - \beta_{n+1})^2 - \frac{1}{4} (\gamma_n - \gamma_{n+1})^2, \]
\[ \text{Tr} (V_n\tau_3) = \text{Tr} (R_n R_n^\dagger\tau_3) = -i (\gamma_n - \gamma_{n+1}). \] (6.16)

Since both $\text{Tr} V_n$ and $\text{Tr} (V_n\tau_3)$ do not depend on $\alpha_n$, we can integrate out $\alpha_n$. This changes only the Haar measure as follows
\[ DR_n = \frac{1}{8\pi} \beta_n d\beta_n d\gamma_n. \] (6.17)

The integration over $\beta_n$ and $\gamma_n$ we will carry out in the limits $-\infty \leq \beta_n \leq \infty$ and $-\infty \leq \gamma_n \leq \infty$.

Substituting expansions (6.13) in the integrand of Eq. (6.14) we obtain
\[ Z_{\text{Reg}}(R_{N+1}, R_0) = \lim_{N \to \infty} \frac{1}{2\pi i\delta} \left( \frac{1}{2\pi i\delta} \right)^{N+1} \left( \frac{1}{8\pi} \right)^N \]
\[ \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \]
\[ \times \exp \left( i \frac{I_\perp}{2\delta} [(\beta_{N+1} - \beta_N)^2 + (\beta_N - \beta_{N-1})^2 + \cdots + (\beta_2 - \beta_1)^2 + (\beta_1 - \beta_0)^2] \right) \]
\[ + i \frac{I_{||}}{2\delta} [(\gamma_{N+1} - \gamma_N)^2 + (\gamma_N - \gamma_{N-1})^2 + \cdots + (\gamma_2 - \gamma_1)^2 + (\gamma_1 - \gamma_0)^2] \]
\[ - i T [(\gamma_{N+1} - \gamma_N) + (\gamma_N - \gamma_{N-1}) + \cdots + (\gamma_2 - \gamma_1) + (\gamma_1 - \gamma_0)] \bigg) = \]
\[ = e^{-i T (\gamma_{N+1} - \gamma_0)} \lim_{N \to \infty} \frac{1}{2\pi i\delta} \left( \frac{1}{2\pi i\delta} \right)^{N+1} \left( \frac{1}{8\pi} \right)^N. \]
\[
\times \int d\gamma_N \int d\beta_N \beta_N \int d\gamma_{N-1} \int d\beta_{N-1} \beta_{N-1} \ldots \int d\gamma_2 \int d\beta_2 \beta_2 \int d\gamma_1 \int d\beta_1 \beta_1
\]
\[
\times \exp \left( i \frac{I_{||}\delta}{2} \left[ (\beta_{N+1} - \beta_N)^2 + (\beta_N - \beta_{N-1})^2 + \ldots + (\beta_2 - \beta_1)^2 + (\beta_1 - \beta_0)^2 \right] + i \frac{I_{||}\delta}{2} \left[ (\gamma_{N+1} - \gamma_N)^2 + (\gamma_N - \gamma_{N-1})^2 + \ldots + (\gamma_2 - \gamma_1)^2 + (\gamma_1 - \gamma_0)^2 \right] \right) \tag{6.18}
\]

The integration over \( \gamma_n \) gives
\[
\int d\gamma_N \int d\gamma_{N-1} \ldots \int d\gamma_2 \int d\gamma_1
\]
\[
\times \exp \left( i \frac{I_{||}\delta}{2} \left[ (\gamma_{N+1} - \gamma_N)^2 + (\gamma_N - \gamma_{N-1})^2 + \ldots + (\gamma_2 - \gamma_1)^2 + (\gamma_1 - \gamma_0)^2 \right] \right) =
\]
\[
= \sqrt{\frac{2\pi i\delta}{I_{||}}} \frac{2\pi i\delta}{I_{||}} 2 \ldots \frac{2\pi i\delta}{I_{||}} N - 1 \sqrt{\frac{2\pi i\delta}{I_{||}} N + 1} \exp \left( i \frac{I_{||}}{2(N + 1)\delta} (\gamma_{N+1} - \gamma_0)^2 \right) =
\]
\[
= \left( \sqrt{\frac{2\pi i\delta}{I_{||}}} \right)^N \sqrt{\frac{1}{N + 1}} \exp \left( i \frac{I_{||}}{2(N + 1)\delta} (\gamma_{N+1} - \gamma_0)^2 \right) \tag{6.19}
\]

By taking into account the normalization factor the result of the integration over \( \gamma_n \) reads
\[
\left( \sqrt{\frac{I_{||}}{2\pi i\delta}} \right)^{N+1} \int d\gamma_N \int d\gamma_{N-1} \ldots \int d\gamma_2 \int d\gamma_1
\]
\[
\times \exp \left( i \frac{I_{||}\delta}{2} \left[ (\gamma_{N+1} - \gamma_N)^2 + (\gamma_N - \gamma_{N-1})^2 + \ldots + (\gamma_2 - \gamma_1)^2 + (\gamma_1 - \gamma_0)^2 \right] \right) =
\]
\[
= \sqrt{\frac{I_{||}}{2\pi i(N + 1)\delta}} \exp \left( i \frac{I_{||}}{2(N + 1)\delta} (\gamma_{N+1} - \gamma_0)^2 \right) =
\]
\[
= \sqrt{\frac{I_{||}}{2\pi i\Delta t}} \exp \left( i \frac{I_{||}}{2\Delta t} (\gamma_{N+1} - \gamma_0)^2 \right), \tag{6.20}
\]

where we have replaced \((N + 1)\delta = t_2 - t_1 = \Delta t\). The obtained result is exact. By replacing \( I_{||} \rightarrow M \), \( \gamma_{N+1} \rightarrow x_b \), \( \gamma_0 \rightarrow x_a \) and \( \Delta t \rightarrow (t_b - t_a) \) we arrive at the expression for the Green function, the evolution operator, of a free particle with a mass \( M \) given by Eq.(2.51) of Ref.[6].

Thus, after the integration over \( \gamma_n \) the evolution operator \( Z_{Reg}(R_{N+1}, R_0) \) can be written in the form
\[
Z_{Reg}(R_{N+1}, R_0) =
\]
\[
= \sqrt{\frac{I_{||}}{2\pi i\Delta t}} \exp \left( i \frac{I_{||}}{2\Delta t} (\gamma_{N+1} - \gamma_0)^2 \right) e^{-iT(\gamma_{N+1} - \gamma_0)} F[I_{\perp}, \beta_{N+1}, \beta_0], \tag{6.21}
\]

where \( F[I_{\perp}, \beta_{N+1}, \beta_0] \) is a functional defined by the integrals over \( \beta_n \)
\[
F[I_{\perp}, \beta_{N+1}, \beta_0] =
\]
Formally we do not need to evaluate the functional $F[I_\perp, \beta_{N+1}, \beta_0]$ explicitly. In fact, the functional $F[I_\perp, \beta_{N+1}, \beta_0]$ should be a regular function of variables $I_\perp, \beta_{N+1}$ and $\beta_0$ whose absolute value is bound in the limit $I_\perp \to 0$. Therefore, taking the limit $I_\parallel \to 0$ for the evolution operator $Z_{\text{Reg}}(R_{N+1}, R_0)$ defined by Eq. (6.21) we get

$$Z(R_2, R_1) = \lim_{I_\parallel, I_\perp \to 0} Z_{\text{Reg}}(R_{N+1}, R_0) = 0. \quad (6.23)$$

This agrees with our results obtained in Sects. 4 and 5.

Nevertheless, in spite of this very definite result let us proceed to the explicit evaluation of the functional $F[I_\perp, \beta_{N+1}, \beta_0]$ and show that the functional $F[I_\perp, \beta_{N+1}, \beta_0]$ vanishes at $N \to \infty$. It is convenient to rewrite the integrand of Eq. (6.22) in the equivalent form

$$F[I_\perp, \beta_{N+1}, \beta_0] = \lim_{N \to \infty} \left( \frac{I_\perp}{2\pi i\delta} \right) \left( \frac{-1}{2\pi} \right)^N \left( \frac{1}{8\pi} \right)^N \delta \to 0$$

$$\times \exp \left( i \frac{I_\perp}{2\delta} \left[ (\beta_{N+1} - \beta_N)^2 + (\beta_N - \beta_{N-1})^2 + \ldots + (\beta_2 - \beta_1)^2 + (\beta_1 - \beta_0)^2 \right] + 2 j_N \beta_N + 2 j_{N-1} \beta_{N-1} + \ldots + 2 j_2 \beta_2 + 2 j_1 \beta_1 \right) \right) \bigg|_{j_N = j_{N-1} = \ldots = j_2 = j_1 = 0} . \quad (6.24)$$

After $k$ integrations we get

$$\int_{-\infty}^{\infty} d\beta_k \int_{-\infty}^{\infty} d\beta_{k-1} \ldots \int_{-\infty}^{\infty} d\beta_2 \int_{-\infty}^{\infty} d\beta_1 \exp \left( i \frac{I_\perp}{2\delta} \left[ (\beta_{k+1} - \beta_k)^2 + (\beta_k - \beta_{k-1})^2 \right] + \ldots + (\beta_2 - \beta_1)^2 + (\beta_1 - \beta_0)^2 + 2 j_k \beta_k + 2 j_{k-1} \beta_{k-1} + \ldots + 2 j_2 \beta_2 + 2 j_1 \beta_1 \right) =$$

$$= \sqrt{\frac{2\pi i\delta}{I_\perp}} \frac{2\pi i\delta}{2} \frac{2\pi i\delta}{3} \ldots \frac{2\pi i\delta}{k} \frac{1}{k} \frac{1}{k+1} \frac{2\pi i\delta}{k+1} \frac{1}{k+2} \exp \left( i \frac{I_\perp}{2(k+1)\delta} (\beta_0 - \beta_{k+1})^2 \right) \times \exp \left( i \frac{I_\perp}{\delta} \beta_{k+1} \left( \frac{k}{k+1} j_k + \frac{k}{k+1} \frac{k-1}{k} j_{k-1} + \frac{k-1}{k+1} \frac{k}{k+1} j_{k-1} + \frac{k-1}{k+1} \frac{k-2}{k+1} j_{k-2} + \ldots + \frac{k}{k+1} \frac{k-2}{k+1} \frac{k-2}{k+1} \frac{k-2}{k+1} \frac{1}{3} j_2 + \frac{k}{k+1} \frac{k-1}{k+1} \frac{k-1}{k+1} \frac{k-2}{k+1} \frac{2}{3} j_1 \right) \right) \times \exp \left( i \frac{I_\perp}{\delta} \left[ -\frac{1}{2} \cdot \frac{1}{2} (j_1 - \beta_0)^2 + \frac{1}{2} \cdot \frac{1}{2} \beta_0^2 - \frac{1}{2} \cdot \frac{2}{3} (j_2 + \frac{1}{2} j_1 - \frac{1}{2} \beta_0)^2 + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2} \beta_0^2 \right] \right) .$$

19
By performing $N$ integrations we obtain

$$
\int d\beta_N \int d\beta_{N-1} \ldots \int d\beta_2 \int d\beta_1 \exp \left( i \frac{I_\perp}{2\delta} [(\beta_{N+1} - \beta_N)^2 + (\beta_N - \beta_{N-1})^2 + \ldots + (\beta_2 - \beta_1)^2 + (\beta_1 - \beta_0)^2 + 2 j_N \beta_N + 2 j_{N-1} \beta_{N-1} + \ldots + 2 j_2 \beta_2 + 2 j_1 \beta_1] \right) =
$$

$$
= \left( \frac{2\pi i\delta}{I_\perp} \right)^{N-1} \sqrt{\frac{I_\perp}{2\pi i\Delta t}} \exp \left( i \frac{I_\perp}{2\Delta t} (\beta_{N+1} - \beta_0)^2 \right) \times \exp \left( i \frac{I_\perp}{\delta} \left[ -\frac{1}{2} \frac{1}{2} (j_1 - \beta_0)^2 + \frac{1}{2} \frac{1}{2} \beta_0^2 - \frac{1}{2} \frac{2}{3} (j_2 + \frac{1}{2} j_1 - \frac{1}{2} \beta_0)^2 + \frac{1}{2} \frac{2}{3} \frac{1}{2} \beta_0^2 \right. \right.
$$

$$
-\frac{1}{2} \frac{3}{4} \left( j_3 + \frac{2}{3} j_2 + \frac{2}{3} \frac{1}{2} j_1 - \frac{1}{3} \beta_0 \right)^2 + \frac{1}{2} \frac{3}{4} \frac{1}{3} \beta_0^3 - \frac{1}{2} \frac{4}{5} \left( j_4 + \frac{3}{4} j_3 + \frac{3}{4} \frac{1}{3} j_2 \right.
$$

$$
+ \frac{3}{2} \frac{2}{3} \frac{1}{2} j_1 - \frac{1}{4} \beta_0 \right)^2 + \frac{1}{2} \frac{4}{5} \frac{1}{3} \beta_0^3 - \frac{1}{2} \frac{5}{6} \left( j_5 + \frac{5}{4} j_4 + \frac{5}{4} \frac{3}{4} j_3 + \frac{3}{4} \frac{1}{3} j_2 \right.
$$

$$
+ \frac{4}{3} \frac{3}{4} \frac{2}{3} \frac{1}{2} j_1 - \frac{1}{5} \beta_0 \right)^2 + \frac{1}{2} \frac{5}{6} \frac{1}{3} \beta_0^3 - \ldots - \frac{1}{2} \frac{N}{N+1} \left( j_N + \frac{N-1}{N} j_{N-1} \right.
$$

$$
\left. \left. + \frac{N-1}{N} \frac{N-2}{N-1} j_{N-2} + \ldots + \frac{N-1}{N} \frac{N-2}{N-1} \frac{2}{3} j_2 + \frac{N-1}{N} \frac{N-2}{N-1} \frac{2}{3} \frac{1}{2} j_1 \right. \right)
$$

$$
\left. \left. - \frac{1}{2} \frac{N}{N+1} \frac{1}{N \beta_0} \right)^2 + \frac{1}{2} \frac{N}{N+1} \frac{1}{N^2 \beta_0^2} \right] \right),
$$

where we have replaced $(N + 1) \delta = t_2 - t_1 = \Delta t$.

Now we can evaluate the derivatives with respect to $j_1, j_2, \ldots, j_{N-1}, j_N$. Due to the constraint $I_\perp/\delta \gg 1$ we can keep only the leading order contributions in powers of $I_\perp/\delta \gg 1$. The result reads

$$
\frac{\partial}{\partial j_1} \frac{\partial}{\partial j_2} \ldots \frac{\partial}{\partial j_{N-1}} \frac{\partial}{\partial j_N} \int d\beta_N \int d\beta_{N-1} \ldots \int d\beta_2 \int d\beta_1 \exp \left( i \frac{I_\perp}{2\delta} [(\beta_{N+1} - \beta_N)^2 + (\beta_N - \beta_{N-1})^2 + \ldots + (\beta_2 - \beta_1)^2 + (\beta_1 - \beta_0)^2 \right)
$$

$$
\times \exp \left( i \frac{I_\perp}{\delta} \left[ -\frac{1}{2} \frac{1}{2} (j_1 - \beta_0)^2 + \frac{1}{2} \frac{1}{2} \beta_0^2 - \frac{1}{2} \frac{2}{3} (j_2 + \frac{1}{2} j_1 - \frac{1}{2} \beta_0)^2 + \frac{1}{2} \frac{2}{3} \frac{1}{2} \beta_0^2 \right. \right.
$$

$$
-\frac{1}{2} \frac{3}{4} \left( j_3 + \frac{2}{3} j_2 + \frac{2}{3} \frac{1}{2} j_1 - \frac{1}{3} \beta_0 \right)^2 + \frac{1}{2} \frac{3}{4} \frac{1}{3} \beta_0^3 - \frac{1}{2} \frac{4}{5} \left( j_4 + \frac{3}{4} j_3 + \frac{3}{4} \frac{1}{3} j_2 \right.
$$

$$
+ \frac{3}{2} \frac{2}{3} \frac{1}{2} j_1 - \frac{1}{4} \beta_0 \right)^2 + \frac{1}{2} \frac{4}{5} \frac{1}{3} \beta_0^3 - \ldots - \frac{1}{2} \frac{N}{N+1} \left( j_N + \frac{N-1}{N} \right.
$$

$$
\left. \left. + \frac{N-1}{N} \frac{N-2}{N-1} j_{N-2} + \ldots + \frac{N-1}{N} \frac{N-2}{N-1} \frac{2}{3} j_2 + \frac{N-1}{N} \frac{N-2}{N-1} \frac{2}{3} \frac{1}{2} j_1 \right. \right)
$$

$$
\left. \left. - \frac{1}{2} \frac{N}{N+1} \frac{1}{N \beta_0} \right)^2 + \frac{1}{2} \frac{N}{N+1} \frac{1}{N^2 \beta_0^2} \right] \right),
$$

(6.26)
+2 j_N \beta_N + 2 j_{N-1} \beta_{N-1} + \ldots + 2 j_2 \beta_2 + 2 j_1 \beta_1 \bigg|_{j_N=j_{N-1}=\ldots=j_2=j_1=0} = \\
= \left( \frac{2\pi i \delta}{I_\perp} \right)^{N+1} \sqrt{\frac{I_\perp}{2\pi i \Delta t}} \exp \left( i \frac{I_\perp}{2\Delta t} (\beta_{N+1} - \beta_0)^2 \right) \left( i \frac{I_\perp}{\delta} \right)^N \\
\left( \beta_{N+1} \frac{1}{N+1} + \beta_0 \left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{N(N+1)} \right] \right) \\
\times \left( \beta_{N+1} \frac{2}{N+1} + \beta_0 \left[ \frac{2}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \ldots + \frac{2}{N(N+1)} \right] \right) \\
\times \left( \beta_{N+1} \frac{3}{N+1} + \beta_0 \left[ \frac{3}{3 \cdot 4} + \ldots + \frac{3}{N(N+1)} \right] \right) \\
\times \left( \beta_{N+1} \frac{4}{N+1} + \beta_0 \left[ \frac{4}{4 \cdot 5} + \ldots + \frac{4}{N(N+1)} \right] \right) \ldots \left( \beta_{N+1} \frac{N}{N+1} + \beta_0 \frac{N}{N+1} \right) = \\
= \left( \frac{2\pi i \delta}{I_\perp} \right)^{N+1} \sqrt{\frac{I_\perp}{2\pi i \Delta t}} \exp \left( i \frac{I_\perp}{2\Delta t} (\beta_{N+1} - \beta_0)^2 \right) \left( i \frac{I_\perp}{\delta} \right)^N \\
\times \prod_{k=1}^N \left( \beta_{N+1} \frac{k}{N+1} + \beta_0 \frac{N+1-k}{N+1} \right). \quad (6.27)
\[ \times \exp \left( i \frac{I_1}{\delta} \left[ \sum_{k=1}^{N} \left( \beta_{N+1} \frac{k}{N+1} + \beta_0 \frac{N+1-k}{N+1} \right) j_k - \frac{1}{2} \sum_{n=1}^{N} \sum_{k=1}^{N} n(N+1-k) j_n j_k \right] \right). \] \tag{6.29}

In order to understand the behaviour of the functional \( F[I_1, \beta_{N+1}, \beta_0] \) in the limit \( N \to \infty \) we suggest to evaluate the product

\[ \Pi[\beta_{N+1}, \beta_0] = \prod_{k=1}^{N} \left( \beta_{N+1} \frac{k}{N+1} + \beta_0 \frac{N+1-k}{N+1} \right) \tag{6.30} \]

at \( N \gg 1 \) by using the \( \zeta \)-regularization. In the \( \zeta \)-regularization the evaluation of \( \Pi[\beta_{N+1}, \beta_0] \) runs the following way

\[
\ln \Pi[\beta_{N+1}, \beta_0] = \sum_{k=1}^{N} \ln \left( \beta_{N+1} \frac{k}{N+1} + \beta_0 \frac{N+1-k}{N+1} \right) = \\
= \sum_{k=1}^{N} \frac{(-1)^k}{ds} \left( \beta_{N+1} \frac{k}{N+1} + \beta_0 \frac{N+1-k}{N+1} \right) - \left. z^s \right|_{s=0} = \\
= -\frac{d}{ds} \sum_{k=1}^{\infty} \int_{0}^{\infty} d\frac{dz}{\Gamma(s)} \exp \left[ - \left( \beta_{N+1} \frac{k}{N+1} + \beta_0 \frac{N+1-k}{N+1} \right) z \right] z^{s-1} \left|_{s=0} \right. = \\
= -(N+1) \frac{d}{ds} \int_{0}^{\infty} d\frac{dz}{\Gamma(s)} \frac{e^{-\beta_0 z} - e^{-\beta_{N+1} z}}{e^{-\beta_{N+1} z} - \beta_0 z} z^{s-2} \left|_{s=0} \right. = \\
= -(N+1) \frac{d}{ds} \left[ \frac{1}{s-1} \left( \beta_{N+1} - \beta_0 \right) \right]_{s=0} = \\
= -(N+1) \left[ 1 - \frac{\beta_{N+1} \ln \beta_{N+1} - \beta_0 \ln \beta_0}{\beta_{N+1} - \beta_0} \right] = \\
= -(N+1) \frac{\beta_{N+1} \ln e - \beta_0 \ln e}{\beta_{N+1} - \beta_0}. \tag{6.31} \]

Thus, the function \( \Pi[\beta_{N+1}, \beta_0] \) is defined by

\[ \Pi[\beta_{N+1}, \beta_0] = \exp \left( -(N+1) \frac{\beta_{N+1} \ln e - \beta_0 \ln e}{\beta_{N+1} - \beta_0} \right), \tag{6.32} \]

where \( e = 2.71828 \ldots \) Due to the constraint \( I_1/\delta \gg 1 \) the Euler angles \( \beta_{N+1} \) and \( \beta_0 \) are less than unity and the ratio in Eq.(6.32) is always positive

\[ \frac{\beta_{N+1} \ln e - \beta_0 \ln e}{\beta_{N+1} - \beta_0} > 0. \tag{6.33} \]
The functional $F[I_\perp, \beta_{N+1}, \beta_0]$ is then defined by

$$F[I_\perp, \beta_{N+1}, \beta_0] = \sqrt{\frac{I_\perp}{2\pi i \Delta t}} \exp \left( i \frac{I_\perp}{2\Delta t} (\beta_{N+1} - \beta_0)^2 \right)$$

$$\times \lim_{N \to \infty} \lim_{\delta \to 0} \left( \frac{1}{8\pi} \right)^N \left( \sqrt{\frac{I_\perp}{2\pi i \delta}} \right)^{N+1} \exp \left( -(N + 1) \frac{\beta_{N+1} \ln e - \beta_0 \ln e}{\beta_{N+1} - \beta_0} \right).$$

(6.34)

Thus $F[I_\perp, \beta_{N+1}, \beta_0]$ vanishes in the limit $N \to \infty$. This result retains itself even if we change the normalization factor of the evolution operator

$$\mathcal{N} = \left( \frac{I_\perp}{2\pi i \delta} \right)^{N+1} \rightarrow (8\pi)^N \left( \sqrt{\frac{I_\parallel}{I_\perp}} \right)^{N+1}.$$

(6.35)

The renormalized functional $F[I_\perp, \beta_{N+1}, \beta_0]$, defined by

$$F[I_\perp, \beta_{N+1}, \beta_0] = \sqrt{\frac{I_\perp}{2\pi i \Delta t}} \exp \left( i \frac{I_\perp}{2\Delta t} (\beta_{N+1} - \beta_0)^2 \right)$$

$$\times \lim_{N \to \infty} \exp \left( -(N + 1) \frac{\beta_{N+1} \ln e - \beta_0 \ln e}{\beta_{N+1} - \beta_0} \right).$$

(6.36)

vanishes in the limit $N \to \infty$ since the Euler angles $\beta_{N+1}$ and $\beta_0$ are smaller compared with unity due to the constraint $I_\perp/\delta \gg 1$ [3]. The vanishing of the functional $F[I_\perp, \beta_{N+1}, \beta_0]$ in the limit $N \to \infty$ agrees with our results obtained in Sects. 4 and 5.

Substituting Eq.(6.36) in Eq.(6.21) we obtain

$$Z_{\text{Reg}}(R_{N+1}, R_0) = 0.$$

(6.37)

This leads to the vanishing of the evolution operator $Z(R_2, R_1)$ given by Eq.(6.1) or Eq.(8) of Ref.[3], $Z(R_2, R_1) = 0$.

Thus, the evolution operator $Z(R_2, R_1)$ suggested in Ref.[3] for the description of Wilson loops in terms of path integrals over gauge degrees of freedom is equal to zero identically. This agrees with our results obtained in Sects. 4 and 5. As we have shown the vanishing of $Z(R_2, R_1)$ does not depend on the specific regularization and discretization of the path integral. In fact, this is an intrinsic property of the path integral given by Eq.(6.1) that becomes obvious if the evaluation is carried out correctly.

7 Evolution operator $Z(R_2, R_1)$ and shift of energy levels of an axial–symmetric top

In this Section we criticize the analysis of the evolution operator $Z(R_2, R_1)$ carried out by Diakonov and Petrov via the canonical quantization of the axial–symmetric top (see Eq.(12) of Ref.[3]). Below we use the notations of Ref.[3].
The parallel transport operator
\[ W_{\alpha\beta}(t_2, t_1) = \left[ P \exp \left( i \int_{x(t_1)}^{x(t_2)} A^a(x) T^a dx \right) \right]_{\alpha\beta} = \left[ P \exp \left( i \int_{t_1}^{t_2} A(t) \, dt \right) \right]_{\alpha\beta} , \] (7.1)

where \( A(t) = A^a(x) T^a dx / dt \) is a tangent component of the Yang–Mills field and \( T^a (a = 1, 2, 3) \) are the generators of \( SU(2) \) group in the representation \( T \), has been reduced to the form
\[ W_{\alpha\beta}(t_2, t_1) = D^T_{\alpha\beta}(U(t_2)U^\dagger(t_1)) , \] (7.2)
(see Eq.(5) of Ref.[3]) due to the statement [3]: \textit{The potential \( A(t) \) along a given curve can be always written as a “pure gauge”}
\[ A_{\alpha\beta}(t) = i D^T_{\alpha\gamma}(U(t)) \frac{d}{dt} D^T_{\gamma\beta}(U^\dagger(t)) \] (7.3)
(see Eq.(4) of Ref.[3]).

By using the parallel transport operator Eq.(7.2) the Wilson loop \( W_T(C) \) in the representation \( T \) has been defined by
\[ W_T(C) = \sum_{\alpha} W_{\alpha\alpha}(t_2, t_1) = \sum_{\alpha} D^T_{\alpha\alpha}(U(t_2)U^\dagger(t_1)) . \] (7.4)
(see Eq.(25) of Ref.[3]). In terms of the evolution operator \( Z(R_2, R_1) \) given by Eq.(6.1) (see Eq.(8) of Ref.[3]) the parallel transport operator \( W_{\alpha\beta}(t_2, t_1) \) has been recast into the form
\[ W_{\alpha\beta}^{DP}(t_2, t_1) = \iint dR_1 \, dR_2 \, \sum_{T', m} (2T' + 1) D^T_{\alpha m}(U(t_2)R_2^\dagger) D^T_{m\beta}(R_1 U^\dagger(t_1)) Z(R_2, R_1) , \] (7.5)
where the index DP means that the parallel transport operator is taken in the Diakonov–Petrov (DP) representation. The Wilson loop \( W_T^{DP}(C) \) in the DP–representation reads
\[ W_T^{DP}(C) = \iint dR_1 \, dR_2 \, \sum_{T', m, \alpha} (2T' + 1) D^T_{\alpha m}(U(t_2)R_2^\dagger) D^T_{m\alpha}(R_1 U^\dagger(t_1)) Z(R_2, R_1) . \] (7.6)
Of course, if the DP–representation were correct we should get \( W_T^{DP}(C) = W_T(C) \), where \( W_T(C) \) is determined by Eq.(7.4).

The regularized evolution operator \( Z_{\text{Reg}}(R_2, R_1) \) given by Eq.(6.3) (see also Eq.(9) of Ref.[3]) can be represented in the form of \textit{a sum over possible intermediate states}, i.e. eigenfunctions of the axial–symmetric top
\[ Z_{\text{Reg}}(R_2, R_1) = \sum_{J, m, k} (2J + 1) D^J_{mk}(R_2) D^J_{km}(R_1^\dagger) e^{-i(t_2 - t_1) E_{Jm}} , \] (7.7)
(see Eq.(12) of Ref.[3]), where \( E_{Jm} \) are the eigenvalues of the Hamiltonian of the axial–symmetric top
\[ E_{Jm} = \frac{J(J + 1) - m^2}{2I_\perp} + \frac{(m - T)^2}{2I_\parallel} \] (7.8)
(see Eq.(11) of Ref.[3]).

As has been stated in Ref.[3]: *If we now take to zero \(I_{\perp} \to 0\) (first \(I_{\parallel}\), then \(I_{\perp}\)) we see that in the sum (12) only the lowest energy intermediate state survives with \(m = J = T\). The resulting phase factor from the lowest energy state can be absorbed in the normalization factor in eq.(9) since that corresponds to a shift in the energy scale.*

The statement concerning the possibility to absorb the fluctuating factor \(\exp[-i(t_2 - t_1)T/2I_{\perp}]\) in the normalization of the path integral representing the evolution operator is the main one allowing the r.h.s. of Eqs.(7.3) and (7.6) to escape from the vanishing in the limit \(I_{\perp} \to 0\).

In reality such a removal of the fluctuating factor is prohibited since this leads to the change of the starting symmetry of the system from \(SU(2)\) to \(U(2)\). In order to make this more transparent we suggest to insert \(Z_{\text{Reg}}(R_2, R_1)\) of Eq.(7.7) into Eq.(7.6) and to express the Wilson loop \(W_{T}^{\text{DP}}(C)\) in terms of a sum over possible intermediate states, the eigenfunctions of the axial–symmetric top. The main idea of this substitution is the following: as the Wilson loop is a physical quantity which can be measured, all irrelevant normalization factors should be canceled for the evaluation of it. Therefore, if the oscillating factor \(\exp[-i(t_2 - t_1)T/2I_{\perp}]\) can be really removed by a renormalization of something, the Wilson loop should not depend on this factor.

Substituting Eq.(7.7) in Eq.(7.3) and integrating over \(R_1\) and \(R_2\) we obtain the following expansion for the parallel transport operator in the DP–representation

\[
W_{\alpha\beta}^{\text{DP}}(t_2, t_1) = \sum_{T'} \sum_{m=-T'} \sum_{m=-T'} D_{\alpha\beta}^{T'}(U(t_2)U^\dagger(t_1)) e^{-i(t_2 - t_1) E_{m}}. \tag{7.9}
\]

Setting \(\alpha = \beta\) and summing over \(\alpha\) we get the DP–representation for Wilson loops

\[
W_{T}^{\text{DP}}(C) = \sum_{\alpha} W_{\alpha\alpha}^{\text{DP}}(t_2, t_1) = \sum_{T'} \sum_{m=-T'} D_{\alpha\alpha}^{T'}(U(t_2)U^\dagger(t_1)) e^{-i(t_2 - t_1) E_{m}}. \tag{7.10}
\]

Due to the definition (7.4) the r.h.s. of Eq.(7.10) can be rewritten in the form

\[
W_{T}^{\text{DP}}(C) = \sum_{\alpha} W_{\alpha\alpha}^{\text{DP}}(t_2, t_1) = \sum_{T'} \sum_{m=-T'} W_{T'}(C) e^{-i(t_2 - t_1) E_{m}}, \tag{7.11}
\]

where \(W_{T'}(C)\) is the Wilson loop in the \(T'\) representation determined by Eq.(7.4). The relation (7.11) agrees to some extent with our expansion given by Eq.(7.1).

Following Ref.[3] and taking the limit \(I_{\parallel} \to 0\) we obtain \(m = T\). This reduces the r.h.s. of Eq.(7.11) to the form

\[
W_{T}^{\text{DP}}(C) = \sum_{T'} W_{T'}(C) \exp \left[ -i(t_2 - t_1) \frac{T'(T'+1) - T^2}{2I_{\perp}} \right]. \tag{7.12}
\]

Now according to the prescription of Ref.[3] we should take the limit \(I_{\perp} \to 0\). Following again Ref.[3] and setting \(T' = T\) we arrive at the relation

\[
W_{T}^{\text{DP}}(C) = W_{T}(C) \exp \left[ -i(t_2 - t_1) \frac{T}{2I_{\perp}} \right]. \tag{7.13}
\]
Since the average value of the Wilson loop is an observable quantity and any averaging over gauge fields does not affect the oscillating factor \(\exp[i(t_2 - t_1)T/2I_\perp]\), one can write
\[
\langle W^\text{DP}_T(C) \rangle = \langle W_T(C) \rangle \exp \left[ -i(t_2 - t_1) \frac{T}{2I_\perp} \right].
\tag{7.14}
\]
According to the Wilson’s criterion of confinement Eq.(1.8) one should set
\[
\langle W_T(C) \rangle = e^{-\sigma A},
\tag{7.15}
\]
where \(\sigma\) and \(A\) are a string tension and a minimal area, respectively.
Substituting Eq.(7.15) in Eq.(7.14) one arrives at the expression
\[
\langle W^\text{DP}_T(C) \rangle = e^{-\sigma A} \exp \left[ -i(t_2 - t_1) \frac{T}{2I_\perp} \right].
\tag{7.16}
\]
It seems to be rather obvious that the r.h.s. of Eq.(7.16) tends to zero due to a strongly oscillating factor, i.e.
\[
\langle W^\text{DP}_T(C) \rangle_{\text{Reg}} = \lim_{I_\perp \to 0} \langle W^\text{DP}_T(C) \rangle = \lim_{I_\perp \to 0} e^{-\sigma A} \exp \left[ -i(t_2 - t_1) \frac{T}{2I_\perp} \right] = e^{-\sigma A} \lim_{I_\perp \to 0} \exp \left[ -i(t_2 - t_1) \frac{T}{2I_\perp} \right] = 0.
\tag{7.17}
\]
Really, there is no quantity that can absorb this factor.
The only possibility to remove the oscillating factor \(\exp[-i(t_2 - t_1)T/2I_\perp]\) is to absorb this phase factors in the matrices \(U(t_2)\) and \(U^\dagger(t_1)\) which describe the degrees of freedom of the gauge potential \(A(t)\) via relation (7.2). In this case Eq.(7.13) can given by
\[
W^\text{DP}_T(C) = W_T(C) \exp \left[ -i(t_2 - t_1) \frac{T}{2I_\perp} \right] = \sum_\alpha D^T_{\alpha\alpha}(U(t_2)U^\dagger(t_1)) \exp \left[ -i(t_2 - t_1) \frac{T}{2I_\perp} \right] = \sum_\alpha D^T_{\alpha\alpha}(\tilde{U}(t_2)\tilde{U}^\dagger(t_1)),
\tag{7.18}
\]
where we have denoted
\[
\tilde{U}(t_2) = U(t_2) e^{i t_2 T/2I_\perp},
U^\dagger(t_1) = U^\dagger(t_1) e^{-i t_1 T/2I_\perp}.
\tag{7.19}
\]
However, the matrices \(\tilde{U}(t_2)\) and \(\tilde{U}^\dagger(t_1)\) are now elements of the group \(U(2)\) instead of \(SU(2)\). Thus, the shift of the energy level of the ground state of the axial–symmetric top suggested by Diakonov and Petrov in order to remove the oscillating factor changes crucially the starting symmetry of the theory from \(SU(2)\) to \(U(2)\). Since the former is not allowed the oscillating factor \(\exp[-i(t_2 - t_1)T/2I_\perp]\) cannot be removed. As a result in the limit \(I_\perp \to 0\) we obtain
\[
\langle W^\text{DP}_T(C) \rangle_{\text{Reg}} = \lim_{I_\perp \to 0} \langle W^\text{DP}_T(C) \rangle = 0.
\tag{7.20}
\]
The vanishing of Wilson loops in the DP–representation agrees with our results obtained in Sects. 4, 5 and 6 and confirms our claim that this path integral representation of Wilson loops is incorrect.
8 The non–Abelian Stokes theorem

The derivation of the area–law falloff promoted great interests in the non–Abelian Stokes theorem expressing the exponent of Wilson loops in terms of a surface integral over the 2–dimensional surface \( S \) with the boundary \( C = \partial S \) [20]

\[
\text{tr} \mathcal{P}_{C} e^{ig \oint_{C} d\sigma_{\mu}(y) U(C_{xy}) G_{\mu\nu}(y) U(C_{yx}) - \frac{1}{2} \int_{S} d\sigma_{\mu\nu}(y) A_{\mu}(x)} = \text{tr} \mathcal{P}_{S} e^{ig \oint_{C} d\sigma_{\mu}(y) U(C_{xy}) G_{\mu\nu}(y) U(C_{yx})},
\]

(8.1)

where \( \mathcal{P}_{S} \) is the surface ordering operator [20], \( d\sigma_{\mu}(y) \) is a 2–dimensional surface element in 4–dimensional space–time, \( x \) is a current point on the contour \( C \), i.e. \( x \in C, y \) is a point on the surface \( S \), i.e. \( y \in S \), and \( G_{\mu\nu}(y) = \partial_{\mu} A_{\nu}(y) - \partial_{\nu} A_{\mu}(y) - ig[A_{\mu}(y), A_{\nu}(y)] \) is the field strength tensor. The procedure for the derivation of the non–Abelian Stokes theorem in the form of Eq.(8.1) contains a summation of contributions of closed paths around infinitesimal areas and these paths are linked to the reference point \( x \) on the contour \( C \) via parallel transport operators. The existence of closed paths linked to the references point \( x \) on the contour \( C \) is a necessary and a sufficient condition for the derivation of the non–Abelian Stokes theorem Eq.(8.1).

Due to the absence of closed paths it is rather clear that the path integral representation for Wilson loops cannot be applied to the derivation of the non–Abelian Stokes theorem. In fact, the evaluation of the path integral over gauge degrees of freedom demands the decomposition of the closed contour \( C \) into a set of infinitesimal segments which can be never closed. Let us prove this statement by assuming the converse. Suppose that by representing the path integral over gauge degrees of freedom in the form of \((\ref{8.7})\) we have a closed segment. Let the segment \( C_{x_kx_{k-1}} \) be closed and the point \( x' \) belong to the segment \( C_{x_kx_{k-1}}, x' \in C_{x_kx_{k-1}} \). By using Eq.(2.2) we can represent the character \( \chi[U_{r}^{\Omega}(C_{x_kx_{k-1}})] \) by

\[
\chi[U_{r}^{\Omega}(C_{x_kx_{k-1}})] = \chi[U_{r}^{\Omega}(C_{x_kx_k})U_{r}^{\Omega}(C_{x_kx_{k-1}})] = \chi[\Omega(x_i)U_{r}(C_{x_kx_k})U_{r}(C_{x_kx_{k-1}})\Omega(x_{k-1})] = \int D\Omega_{r}(x')\chi[\Omega_{r}(x_i)U_{r}(C_{x_kx_k})U_{r}(C_{x_kx_{k-1}})\Omega(x_{k-1})] = \int D\Omega_{r}(x')\chi(U_{r}^{\Omega}(C_{x_kx_k})U_{r}^{\Omega}(C_{x_kx_{k-1}})).
\]

(8.2)

This transforms a \( (n - 1) \)-dimensional integral with one closed infinitesimal segment into a \( n \)-dimensional integral without closed segments. Since finally \( n \) tends to infinity there is no closed segments for the representation for the path integral in the form of a \( (n - 1) \)-dimensional integral. As this statement is general and valid for any path integral representation of Wilson loops, so one can conclude that no further non–Abelian Stokes theorem can be derived within any path integral approach to Wilson loops.

9 Conclusion

By using well defined properties of group characters we have shown that the path integral over gauge degrees of freedom of the Wilson loop which was used in Eq.(2.13) of Ref.[11] for a lattice evaluation of the average value of Wilson loops can be derived in
continuum space–time in non–Abelian gauge theories with the gauge group $SU(N)$. The resultant integrand of the path integral contains a phase factor which is not projected onto Abelian degrees of freedom of non–Abelian gauge fields and differs substantially from the representation given in Ref.[3]. The important point of our representation is the summation over all states of the given irreducible representation $r$ of $SU(N)$. For example, in $SU(2)$ the phase factor is summed over all values of the colourmagnetic quantum number $m_j$ of the irreducible representation $j$ of colour charges. This contradicts Eq.(23) of Ref.[3], where only term with the highest value of the colourmagnetic quantum number $m_j = j$ are taken into account and the other $2j$ terms are lost. This loss is caused by an artificial regularization procedure applied in Ref.[3] for the definition of the path integral over gauge degrees of freedom.

As has been stated by Diakonov and Petrov in Ref.[3] the path integral over gauge degrees of freedom representing Wilson loops is not of the Feynman type, therefore, it depends explicitly on how one “understands” it, i.e. how it is discretized and regularized. In order to understand the path integral over gauge degrees of freedom Diakonov and Petrov [3] suggested a regularization procedure drawing an analogy between gauge degrees of freedom and dynamical variables of the axial–symmetric top with moments of inertia $I_\perp$ and $I_\parallel$. The final expression for the path integral of the Wilson loop has been obtained in the limit $I_\perp, I_\parallel \to 0$.

In order to make the incorrectness of this expression more transparent we have evaluated the path integral for specific gauge field configurations (i) a pure gauge field and (ii) $Z(2)$ center vortices with spatial azimuthal symmetry. The direct evaluation of path integrals representing Wilson loops for these gauge field configurations has given the value zero for both cases. These results do not agree with the correct values.

One can show that Eq.(5.9) can be generalized for any contour of a Wilson loop in $SU(2)$

$$ W_{1/2}(C) = \int \prod_{x \in C} D\Omega(x) e^{ig \int_C dx_\mu tr[i^3 A_\mu^\Omega(x)]} = \lim_{n \to \infty} \sum_j \left[ \frac{a_j}{2j+1} \right]^n (2j+1)W_j(C) = 0, \quad (9.1) $$

where $W_j(C)$ in the r.h.s. is defined by Eq.(2.1) in terms of the path–ordering operator $P_C$. Further, the result (9.1) can be extended to any irreducible representation of $SU(2)$.

This statement we have supported by a direct evaluation of the evolution operator $Z_{Reg}(R_2, R_1)$ defined by Eq.(14) of Ref.[3], representing the assumption by Diakonov and Petrov for Wilson loops in terms of the path integral over gauge degrees of freedom. As we have shown in Sect.6 the regularized evolution operator $Z_{Reg}(R_2, R_1)$, evaluated correctly, is equal to zero. This agrees with our results obtained in Sects. 4 and 5. In Sect. 7 we have shown that the removal of the oscillating factor from the evolution operator suggested in Ref.[3] via a shift of energy levels of the axial–symmetric top is prohibited. Such a shift of energy levels leads to a change of the starting symmetry of the system from $SU(2)$ to $U(2)$. By virtue of the oscillating factor the Wilson loop vanishes in the limit $I_\parallel, I_\perp \to 0$ in agreement with our results in Sects. 4, 5 and 6.

We hope that the considerations in Sects. 4–7 are more than enough to persuade even the most distrustful reader that the path integral representation for the Wilson derived...
by Diakonov and Petrov by means of special regularization and understanding of the path integral over gauge degrees of freedom is erroneous.

The use of an erroneous path integral representation for Wilson loops in Ref.[7] has led to the conclusion that for large distances the average value of Wilson loops shows area–law falloff for any irreducible representation $r$ of $SU(N)$. Unfortunately, this result is not supported by numerical simulations of lattice QCD [8]. At large distances, colour charges with non–zero $N$–ality have string tensions of the corresponding fundamental representation, whereas colour charges with zero $N$–ality are screened by gluons and cannot form a string. Therefore, the result obtained in Ref.[7] cannot be considered as a new check of confinement in lattice calculations as has been argued by the authors of Ref.[7].

We would like to accentuate that the problem we have touched in this paper is not of marginal interest and a path integral, if derived by means of an unjustified regularization procedure, would hardly compute the same physical number as the correct one. We argue that no regularization procedure can lead to specific dynamical constraints. In fact, the regularization procedure drawing the analogy with the axial–symmetric top has led to the result supporting the hypothesis of Maximal Abelian Projection pointed out by 't Hooft [14]. Any proof of this to full extent dynamical hypothesis through a regularization procedure and through specific understanding of the path integral should have seemed dubious and suspicious.

Finally, we have shown that within any path integral representation for Wilson loops in terms of gauge degrees of freedom no non–Abelian Stokes theorem in addition to Eq.(8.1) can be derived. Indeed, the Stokes theorem replaces a line integral over a closed contour by a surface integral with the closed contour as the boundary of a surface. However, approximating the path integral by an $n$–dimensional integral at $n \to \infty$ there are no closed paths linking two adjacent points along Wilson loops. Thereby, the line integrals over these open paths cannot be replaced by surface integrals. Thus, we argue that any non-Abelian Stokes theorem can be derived only within the definition of Wilson loops through the path ordering procedure (8.1). Of course, one can represent the surface–ordering operator $P_S$ in Eq.(8.1) in terms of a path integral over gauge degrees of freedom, but this should not be a new non–Abelian Stokes theorem in comparison with the old one given by Eq.(8.1). That is why the claims of Ref.[3–5] concerning new versions of the non–Abelian Stokes theorems derived within path integral representations for Wilson loops seem unjustified.
10 Appendix. Comments on hep-lat/0008004 by Diakonov and Petrov

10.1 None non–Abelian Stokes theorem can be derived within path integral representation of the Wilson loop over gauge degrees of freedom

In a set of publications \[3,7,15,21,22\] Diakonov and Petrov have pointed the possibility to derive a non–Abelian Stokes theorem for the Wilson loop represented by a path integral over gauge degrees of freedom. This statement contradicts to a theorem that have been proved recently by Faber \textit{et al.} \[23\] (see also Sect. 8 of this manuscript). The main point of this theorem has been proved by using well–defined properties of group characters and refers to the well–known result stating that a linear integral over a contour \(C\) can be replaced by a surface integral (the Stokes theorem) only if the contour \(C\) is closed. As has been shown in \[23\] such a necessary condition of the application of the Stokes theorem is violated for the Wilson loop represented by a path integral over gauge degrees of freedom. Below we repeat our statement.

In a non–Abelian gauge theory the Wilson loop defined for an irreducible representation \(r\) of the \(SU(N)\) gauge group can be represented in the form of a path integral over gauge degrees of freedom as follows \[23\] (see Eq.(16) of Ref.\[23\])

\[
W_r(C) = \frac{1}{d_r^2} \int \prod_{x \in C} [d_r D\Omega_r(x)] \chi_r[U_r^{\Omega}(C_{xx})],
\]  

(10.1)

where \(d_r\) is a dimension of the irreducible representation \(r\), \(\Omega_r(x)\) is the gauge function in the representation \(r\) and \(\chi_r[U_r^{\Omega}(C_{xx})]\) is the group character defined for the irreducible representation \(r\), \(C_{xx}\) is a Wilson contour. Then, \(U_r^{\Omega}(C_{xx})\) is given by

\[
U_r(C_{xx}) = P_C e^{i g \oint_{C_{xx}} dz_\mu A^{(r)}_\mu(z)},
\]  

(10.2)

where \(A^{(r)}_\mu(x)\) is a gauge field in irreducible representation \(r\) of \(SU(N)\) gauge group, \(P_C\) is the operator ordering colour matrices along the path \(C\). Then, the quantity \(U_r^{\Omega}(C_{xx})\) is defined by \[5\]

\[
U_r^{\Omega}(C_{xx}) = \Omega_r(x)U_r(C_{xx})\Omega_r^\dagger(x),
\]  

(10.3)

For the matrices \(U_r(C_{yx})\) defined for the open contour \(C_{yx}\) linking two points \(x\) and \(y\) one has

\[
U_r(C_{yx}) = P_{C_{yx}} e^{i g \int_{C_{yx}} dz_\mu A^{(r)}_\mu(z)}
\]  

(10.4)

and

\[
U^{\Omega}(C_{yx}) = \Omega(y)U(C_{yx})\Omega^\dagger(x),
\]  

(10.5)

respectively.
The evaluation of the path integral over gauge functions $\Omega_r(x)$ can be carried out only via the discretization of the path integral \([10.1]\). Such a discretization can be unambiguously performed by using well-defined properties of group characters. The discretized form reads [23] (see Eq.(15) of Ref.[23])

$$W_r(C) = \frac{1}{d_r^2} \lim_{n \to \infty} \int D\Omega_r(x_n) \chi_r[U_r^{\Omega_1}(C_{x_n x_{n-1}})] \int D\Omega_r(x_{n-1}) \chi_r[U_r^{\Omega_1}(C_{x_{n-1} x_{n-2}})] \cdots$$

$$\times \int D\Omega_r(x_2) \chi_r[U_r^{\Omega_1}(C_{x_2 x_1})] \int D\Omega_r(x_1) \chi_r[U_r^{\Omega_1}(C_{x_1 x_n})] =$$

$$= \frac{1}{d_r^2} \lim_{n \to \infty} \int D\Omega_r(x_n) \chi_r[\Omega_r(x_n)U_r(C_{x_n x_{n-1}})\Omega_r^{\dagger}(x_{n-1})]$$

$$\times \int D\Omega_r(x_{n-1}) \chi_r[\Omega_r(x_{n-1})U_r(C_{x_{n-1} x_{n-2}})\Omega_r^{\dagger}(x_{n-2})] \cdots$$

$$\times \int D\Omega_r(x_2) \chi_r[\Omega_r(x_2)U_r(C_{x_2 x_1})\Omega_r^{\dagger}(x_1)] \int D\Omega_r(x_1) \chi_r[\Omega_r(x_1)U_r(C_{x_1 x_n})\Omega_r^{\dagger}(x_n)],$$

(10.6)

where $C_{x_k x_{k-1}}$ are open infinitesimal segments linking two adjoining points $x_{k-1}$ and $x_k$ and the relations $U_r^{\Omega_1}(C_{x_k x_{k-1}}) = \Omega_r(x_k)U_r(C_{x_k x_{k-1}})\Omega_r^{\dagger}(x_{k-1})$ have been used [23].

Since infinitesimal contours $C_{x_k x_{k-1}}$ are open and the integrations over $\Omega_r(x_k)$ are independent of the integration over $\Omega_r(x_{k-1})$, the necessary condition of the application of the Stokes theorem, i.e. a replacement of a linear integral over closed contour by a surface integral over an area embraced by the contour, is violated. This implies that none non–Abelian Stokes can be derived within the framework of a path integral representation of the Wilson in terms of gauge degrees of freedom.

According to this theorem any non–Abelian Stokes theorem derived by Diakonov and Petrov in Refs.[3,7,15,21,22] within path integral representation of the Wilson loop over gauge degrees of freedom is very much suspicious and is doomed to be erroneous.

We argue that in any non–Abelian gauge theory the non–Abelian Stokes theorem within the standard definition of the Wilson loop can only be derived via the path–ordering operator [20]. This non–Abelian Stokes theorem is unique and no other exists.

10.2 Path integral representation of the Wilson loop suggested by Diakonov and Petrov is erroneous

In the recent manuscript [15] Diakonov and Petrov have claimed that the path integrals over gauge degrees of freedom representing in $SU(2)$ gauge theory Wilson loops defined for the pure gauge field and the $Z(2)$ center vortex with spatial azimuthal symmetry have been incorrectly calculated in our recent paper [23]. The key point of this claim is the incompleteness of the functions $\chi_j[t^3U]$, where $U$ is an element of $SU(2)$, $U \in SU(2)$, and $t^3$ is the matrix representation of the third generator of $SU(2)$ in the irreducible representation $j$. Diakonov and Petrov suggested another expansion obeying the completeness condition. As has been shown in Sects. 4 and 5 our results for the Wilson loop represented by the path integral over gauge degrees of freedom suggested by Diakonov and Petrov are retained for the correct expansion. Thus, we conclude that in [15] Diakonov and Petrov have not succeeded in refuting our statement that their representation for the Wilson loop suggested in Ref.[3] is erroneous and cannot be used for a new check of confinement [7].
10.3 Evolution operator $Z_{\text{Reg}}(R_2, R_1)$ is calculated by Diakonov and Petrov incorrectly

First, we would like to cite Diakonov and Petrov (p.11 of Ref.[15]):"In section 6 of their paper FITZ attempt to compute the regularized evolution operator for the "Wess–Zumino" action, following directly our approach. This calculation has been presented in some detail in original paper [3], however, FITZ seem to be dissatisfied by it and present their own. Their final answer (eq.(98) and (112) of ref.[11]), which differs from our, is a result of several mistakes.

First, going from eq.(98) to eq.(87) FITZ use a strange relation,

$$\exp \left( \sum_{n=0}^{N} (-i) \frac{I_{\perp}}{2\delta} (-4) \right) = \exp \left( iN(N+1) \frac{I_{\perp}}{\delta} \right),$$

(37)

instead of the correct (and trivial) $\exp(i2(N+1)I_{\perp}/\delta)$, where $I_{\perp}$ and $\delta$ are constants and $N$ is the number of pieces in which one divides the contour.

Second, and more important, both equations in (91) are erroneous, they do not follow from eq.(89) from where they are derived. Passing from eq.(89) to eq.(91) one gets:

$$\text{Tr}(R_n R_{n+1}^\dagger) = 2 - \frac{1}{4} [\delta \alpha_n^2 + \delta \beta_n^2 + \delta \gamma_n^2 + 2\delta \alpha_n \delta \gamma_n \cos \beta_n],$$

(38)

$$\text{Tr}(R_n R_{n+1}^\dagger \tau_3) = i(\delta \alpha_n + \delta \gamma_n \cos \beta_n),$$

$$\delta \alpha_n = \alpha_{n+1} - \alpha_n, \quad \delta \beta_n = \beta_{n+1} - \beta_n, \quad \delta \gamma_n = \gamma_{n+1} - \gamma_n.$$

FITZ have written these formulae without the crucial factor $\cos \beta_n$. Because of this mistake the subsequent integration over Euler angles $\alpha, \beta, \gamma$ becomes Gaussian, and the evolution operator for the axial top, as computed by FITZ, in fact becomes that of a free particle, which is definitely wrong. The factors $\cos \beta_n$ being reinstalled, the derivation returns to that of our paper [1]."

The factor Eq.(37) discussed by Diakonov and Petrov does not effect the final result and cancels itself finally. It is rather clear that this "mistake" is nothing more than a trivial misprint. Unfortunately, there are some misprints in our paper [23]. For example in Eqs.(49) and (74) one has to read $W_{1/2}(C)$ and $W_{1/2}(\rho)$ instead of $W_j(C)$ and $W_j(\rho)$, respectively. In Appendix one should read $0 \leq \alpha \leq 2\pi, 0 \leq \gamma \leq 2\pi$ and $0 \leq \beta \leq \pi$ instead of $0 \geq \alpha \geq 2\pi, 0 \geq \gamma \geq 2\pi$ and $0 \geq \beta \geq \pi$.

Before, the explanation of the "Second remark" we would like to attract attention of readers to the manner of the expounding of the problem accepted by Diakonov and Petrov.

They write "Second and more important, both equations (91) are erroneous, they do not follow from eq.(89) from where they are derived."

This sentence makes an oritation of the reader that we have made a crucial mistake and hardly master the machinery of the expansion of functions in Tailor series.

After such a successful attack on the psyche of the reader Diakonov and Petrov have written completely the same expressions up to the replacement $\alpha \rightarrow \gamma$ and $\gamma \rightarrow \alpha$ that we have got by expanding the integrand around the saddle point. In addition they have

\footnote{This is abbreviation from Faber, Ivanov, Troitskaya, Zach used by Diakonov and Petrov}
added a factor \( \cos \beta_n \). We are surprised to find such a factor, since in fact there should be the factor

\[
\cos \left( \frac{\beta_n + \beta_{n+1}}{2} \right). \tag{10.7}
\]

This is clearly seen from Eq.(89) of our paper [23]. The variables \( \beta_{n+1} \) and \( \beta_n \) are independent. The reason to set \( \beta_{n+1} = \beta_n \) would be a mystery of the Diakonov–Petrov approach to the evaluation of integrals within a saddle point technique.

Then, in our paper [23] the factor Eq.(10.7) has been replaced by unity. The reason of this replacement is in the meaning of the saddle point technique of the evaluation of integrals. Indeed, the main point of the saddle point evaluation of integrals is in the reduction of integrands to the Gaussian form. In the case of the evolution operator

\[
Z_{\text{Reg}}(R_2, R_1),
\]

as has been claimed by Diakonov and Petrov in Ref.[3], the saddle point of the integrand should be at the unit element where \( \alpha, \gamma, \beta \ll 1 \). That is why the expansions should contain the least powers of variables. This imposes the exponent of the integrand not to contain the powers of variables higher than 2. That is why the factor (10.7) multiplied by \((\alpha_n - \alpha_{n+1}), (\alpha_n - \alpha_{n+1})^2, (\gamma_n - \gamma_{n+1}), (\gamma_n - \gamma_{n+1})^2\) should be replaced by unity around the saddle point \( \beta_{n+1}, \beta_n \ll 1 \).

As we have shown in our paper [23] the evolution operator \( Z_{\text{Reg}}(R_2, R_1) \) is equal to zero (see Eq.(113) of Ref.[23])

\[
Z_{\text{Reg}}(R_2, R_1) = 0. \tag{10.8}
\]

This agrees completely with our result \( W_{1/2}(C) = W_{1/2}(\rho) = 0 \) obtained in this paper. Therefore, the claim “The factors \( \cos \beta_n \) being reinstalled, the derivation returns to that of our paper [3].” is nothing more than a blef assuming to reanimate the erroneous result.

Now we would cite again Diakonov and Petrov (see p.12 of Ref.[15]):” The last objection by FITZ is to our alternative (and in fact equivalent) derivation of the evolution operator, this time through the standard Feynman representation for the path integral as a sum over intermediate states. FITZ quote our result for the evolution operator of an axial top with the "Wess–Zumino" term, evolving from its orientation given by a unitary matrix \( R_1 \) at time \( t_1 \) to orientation \( R_2 \) at time \( t_2 \):

\[
Z_{\text{Reg}}(R_2, R_1) = (2J + 1) D_{J^I}^{I_\perp}(R_2R_1^\dagger) \exp \left[ -i(t_2 - t_1) \frac{J}{2I_\perp} \right] \tag{39}
\]

where \( I_\perp \) is a regular moment of inertia, \( I_\perp \to 0 \). Apart from a nontrivial dependence on the orientation matrices \( R_1, 2 \) coming through the Wigner \( D \)–function, this expansion contains a phase factor \( \exp(-i(t_2 - t_1)\ldots) \). It is an overall factor independent of the external field: it can and should be absorbed into the integration measure to make the evolution operator unity for the trivial case \( R_2 = R_1 = 1 \). Indeed, dividing the time interval into \( N \) pieces of small length \( \delta \), \( N\delta = t_2 - t_1 \), one can write this factor as product,

\[
\exp \left[ -i(t_2 - t_1) \frac{J}{2I_\perp} \right] = \prod_{k=1}^{N} \exp \left[ -i \frac{\delta J}{2I_\perp} \right], \tag{40}
\]

where, according to the regularization prescription of ref.[1], \( \delta/I_\perp \ll 1 \) so that each factor is close to unity. Each factor can be now absorbed into the integration measure \( dR(t_k) \) in
the functional–integral representation for the evolution operator (39). The fact that the factor is complex is irrelevant; moreover, it is typical for the path–integral representation of the evolution operators to have a complex measure, see the classical Feynman’s book [19]. However, FITZ write: “... a removal of the fluctuating factor is prohibited since this leads to the change of the starting symmetry of the system from SU(2) to U(2)”.

It may seem that FITZ believe that an absorption of a constant factor into the integration measure changes the number of variables which one integrates.

It seems that the decomposition of the finite interval of time into $N$ pieces $t_2 - t_1 = N \delta$ is irrelevant to the problem of the removal of the oscillating factor. Indeed, the expression given by the formula (39) does not contain any integration and, therefore, there is no measure that can absorb the factor

\[
\exp \left[ -i(t_2 - t_1) \frac{J}{2 I_\perp} \right]
\]

(10.9)

strongly oscillating at $I_\perp \to 0$.

In the paper by Faber et al. the crucial contribution of the oscillating factor (10.9) has been demonstrated for the example of the Wilson loop (see Eq.(126) of Ref.[23])

\[
W^{DP}_J(C) = W_J(C) \exp \left[ -i(t_2 - t_1) \frac{J}{2 I_\perp} \right],
\]

(10.10)

where $W^{DP}_J(C)$ is the Wilson loop in the Diakonov–Petrov representation and $W_J(C)$ is the standard Wilson loop defined via the path–ordering operator. Since the average value of the Wilson loop is an observable quantity and any averaging over gauge fields does not affect the oscillating factor Eq.(10.9), one can write

\[
\langle W^{DP}_J(C) \rangle = \langle W_J(C) \rangle \exp \left[ -i(t_2 - t_1) \frac{J}{2 I_\perp} \right].
\]

(10.11)

According to Wilson’s criterion of confinement one should set

\[
\langle W_J(C) \rangle = e^{-\sigma A},
\]

(10.12)

where $\sigma$ and $A$ are a string tension and a minimal area, respectively.

Substituting Eq.(10.12) in Eq.(10.11) one arrives at the expression

\[
\langle W^{DP}_J(C) \rangle = e^{-\sigma A} \exp \left[ -i(t_2 - t_1) \frac{J}{2 I_\perp} \right].
\]

(10.13)

It seems to be rather obvious that the r.h.s. of Eq.(10.13) tends to zero due to the strongly oscillating factor, i.e.

\[
\langle W^{DP}_J(C) \rangle_{\text{Reg}} = \lim_{I_\perp \to 0} \langle W^{DP}_J(C) \rangle = \lim_{I_\perp \to 0} e^{-\sigma A} \exp \left[ -i(t_2 - t_1) \frac{J}{2 I_\perp} \right] = e^{-\sigma A} \lim_{I_\perp \to 0} \exp \left[ -i(t_2 - t_1) \frac{J}{2 I_\perp} \right] = 0.
\]

(10.14)

There is no quantity that can absorb this factor.

Finally, our explanation concerning the crucial influence of the removal of the oscillating factor (10.9) on the symmetry of the gauge fields has been clearly given in our paper Ref.[23]. We would not repeat it here and relegate readers to this publication.
10.4 Lattice–regularized formula for the Wilson loop suggested by Diakonov and Petrov is meaningless

In this Section we would to make comments on the lattice–regularized formula suggested by Diakonov and Petrov in Ref.[15]. Formula (50) of the manuscript [15]

\[ W_J = N^{-1} \int \prod_{k=1}^{N} dS_k \exp \frac{z}{2} \text{Tr}(S_k^I U_{k,k-1} S_{k-1}) \] (50)

is a revised version of the lattice representation of the Wilson loop suggested by Diakonov and Petrov in Ref.[7]. However, the parameter \( z \) is now not equal to \( z = 2J \), as has been stated in Ref.[7], since there is no a priori reason to expect that in the lattice regularization \( z \) should be the same. [15]. Therefore, now it is suggested to consider this parameter as a free parameter \( z(J) \neq 2J \). We would like to accentuate that expression (50) for Wilson loops according to Diakonov and Petrov is valid for any irreducible representation \( J \) of gauge group \( SU(2) \).

In order to show that the representation Eq.(50) reproduces the standard Wilson loop

\[ W_J = \frac{1}{2J+1} \chi_J(U_{N,N-1} U_{N-1,N-2} \ldots U_{1,N}) \] (49)

Diakonov and Petrov suggested to expand the exponent of (50) according to (34)3:

\[ \exp z \text{Tr}[t_3 U] = \exp \left\{ (-iz/2) \text{Tr}\left[e^{i\pi t_3 U}\right]\right\} = \sum_j \tilde{a}_j(z) \chi_j \left[e^{i\pi t_3 U}\right] \]

\[ = \sum_j \tilde{a}_j(z) \sum_{m=-j}^{j} e^{i\pi m} D_{jm}^j(U), \quad \tilde{a}_j(z) = e^{-i\pi j(2j+1)} \frac{2J_{2j+1}(z)}{z} , \] (34)

where “the coefficients \( \tilde{a}_j(z) \) being well–known from the lattice strong–coupling expansion [20]” [15],

\[ W_J = N^{-1} \int \prod_{k=1}^{N} dS_k \sum_{j} \tilde{a}_j(z) \sum_{m=-j}^{j} e^{i\pi m} D_{jm}^j(U_{k,k-1} S_{k-1}), \] (51)

\[ \tilde{a}_j(z) = a^{-i\pi j(2j+1)} \frac{2}{z} J_{2j+1}(z). \] (52)

Integrating over the matrices \( S_k \) Diakonov and Petrov have arrived at the expression [15]

\[ W_J = N^{-1} \sum_j [b_j(z)]^N \chi_j(U_{N,N-1} U_{N-1,N-2} \ldots U_{1,N}), \] (56)

\[ N = \sum_j (2j+1) [b_j(z)]^N, \] (57)

where \( b_j(z) = (2/z) J_{2j+1}(z) \) [15]. Then, Diakonov and Petrov claim that at \( N \gg 1 \) the r.h.s. of (56) reduces to the Wilson loop in representation \( J \) [15].

3: We make use of the fact that \( t_3 = (-i/2) \tau_3 = (-i/2) \exp(i\pi t_3) \) where the last factor is definitely an element of \( SU(2) \) [15]. It is obvious that the relation \( t_3 = (-i/2) \exp(i\pi t_3) \) is valid only for the matrix \( t_3 \) in the fundamental representation \( J = 1/2 \).
Now let us adduce some numerical samples demonstrating the erroneous properties of the representation of the Wilson loop given by Eq. (56).

**Wilson loop for the fundamental representation** \( J = 1/2, \ W_{1/2}(z) \). For the Wilson loop in the fundamental representation \( J = 1/2 \) the expansion of the exponential in (50) leading to (56) is valid. The result of the numerical calculation of the coefficients in (56) and the normalization factor (57) reads

\[
W_{1/2}(z) \big|_{z=3.25103} = 3.027 \times 10^{-8} + \frac{1}{2.002} \chi_{1/2}(U_{24,23} \ldots U_{1,24}) + 2.264 \times 10^{-4} \chi_{1}(U_{24,23} \ldots U_{1,24}) + 4.751 \times 10^{-12} \chi_{3/2}(U_{24,23} \ldots U_{1,24}) + \ldots \approx \frac{1}{2} \chi_{1/2}(U_{24,23} \ldots U_{1,24}). \tag{10.15}
\]

The parameter \( z = 3.25103 \) is taken from Table 1 of Ref. [15]. Thus, for the fundamental representation the expression (56) fits well the standard Wilson loop (49).

**Wilson loop for the adjoint representation** \( J = 1, \ W_1(z) \). First, let us take the Diakonov and Petrov point of view and believe that the expression (56) is valid for the Wilson loop defined for the adjoint representation. On this way we have to set \( z = 4.36765 \) from Table 1 of Ref. [15] and get the dominant contribution of the character \( \chi_1(U_{24,23} \ldots U_{1,24}) \) with the prefactor 1/3 according to the normalization of the standard Wilson loop (49). The numerical analysis gives

\[
W_1(z) \big|_{z=4.36765} = 1.384 \times 10^{-9} + 1.795 \times 10^{-6} \chi_1(U_{24,23} \ldots U_{1,24}) + \frac{1}{3.008} \chi_1(U_{24,23} \ldots U_{1,24}) + 6.366 \times 10^{-4} \chi_{3/2}(U_{24,23} \ldots U_{1,24}) + 1.818 \times 10^{-10} \chi_2(U_{24,23} \ldots U_{1,24}) \ldots \approx \frac{1}{3} \chi_1(U_{24,23} \ldots U_{1,24}). \tag{10.16}
\]

This means that really the Diakonov–Petrov representation (56) fits well the standard Wilson loop (49) defined for the irreducible representation \( J = 1 \).

**Wilson loop for the representation** \( J = 3/2, \ W_{3/2}(z) \). Setting \( z = 5.46564 \) (see Table 1 of Ref. [15]) we should get in the r.h.s. of (56) the dominant contribution of the character \( \chi_{3/2}(U_{24,23} \ldots U_{1,24}) \):

\[
W_{3/2}(z) \big|_{z=5.46564} = 7.145 \times 10^{-3} + 5.054 \times 10^{-15} \chi_{1/2}(U_{24,23} \ldots U_{1,24}) + 1.430 \times 10^{-5} \chi_1(U_{24,23} \ldots U_{1,24}) + \frac{1}{4.051} \chi_{3/2}(U_{24,23} \ldots U_{1,24}) + 1.089 \times 10^{-3} \chi_2(U_{24,23} \ldots U_{1,24}) \ldots \approx \frac{1}{4} \chi_{3/2}(U_{24,23} \ldots U_{1,24}). \tag{10.17}
\]

This fits well the standard Wilson loop (49) defined for the irreducible representation \( J = 3/2 \).

**Wilson loop for the representation** \( J = 2, \ W_2(z) \). Setting \( z = 6.55104 \) (see Table 1 of Ref. [15]) we should get in the r.h.s. of (56) the dominant contribution of the character \( \chi_2(U_{24,23} \ldots U_{1,24}) \):

\[
W_2(z) \big|_{z=6.55104} = 1.073 \times 10^{-11} + 2.381 \times 10^{-3} \chi_{1/2}(U_{24,23} \ldots U_{1,24}) + 2.392 \times 10^{-22} \chi_1(U_{24,23} \ldots U_{1,24}) + 5.096 \times 10^{-5} \chi_{3/2}(U_{24,23} \ldots U_{1,24})
\]
tuning the parameter \( z \) of Ref. [15]) we should get in the r.h.s. of (56) the dominant contribution of the character Wilson loop given by (56).

\[ J \expansion \text{valid only for the fundamental representation} \]

This fits well the standard Wilson loop (49) defined for the irreducible representation \( J = 2 \).

**Wilson loop for the representation** \( J = 5/2 \), \( W_{5/2}(z) \). Setting \( z = 7.62728 \) (see Table 1 of Ref. [15]) we should get in the r.h.s. of (56) the dominant contribution of the character Wilson loop for the representation \( \chi_{5/2}(U_{24,23} \ldots U_{1,24}) \):

\[
W_{5/2}(z) \big|_{z=7.62728} = 2.051 \times 10^{-9} + 3.058 \times 10^{-7} \chi_{1/2}(U_{24,23} \ldots U_{1,24}) + 3.123 \times 10^{-38} \chi_{3/2}(U_{24,23} \ldots U_{1,24}) + 1.201 \times 10^{-4} \chi_{2}(U_{24,23} \ldots U_{1,24}) + \frac{1}{6.097} \chi_{5/2}(U_{24,23} \ldots U_{1,24}) \ldots \\
\approx \frac{1}{6} \chi_{5/2}(U_{24,23} \ldots U_{1,24}).
\]

(10.19)

This evidences that for \( J = 5/2 \) the Diakonov–Petrov representation of the Wilson loop (56) fits well the standard Wilson loop (49).

**Wilson loop for the representation** \( J = 3 \), \( W_3(z) \). Setting \( z = 8.69644 \) (see Table 1 of Ref. [15]) we should get in the r.h.s. of (56) the dominant contribution of the character Wilson loop for the representation \( \chi_{3}(U_{24,23} \ldots U_{1,24}) \):

\[
W_3(z) \big|_{z=8.69644} = 6.698 \times 10^{-4} + 1.937 \times 10^{-17} \chi_{1/2}(U_{24,23} \ldots U_{1,24}) + 2.681 \times 10^{-5} \chi_{1}(U_{24,23} \ldots U_{1,24}) + 2.807 \times 10^{-5} \chi_{3/2}(U_{24,23} \ldots U_{1,24}) + 7.627 \times 10^{-32} \chi_{2}(U_{24,23} \ldots U_{1,24}) + 2.216 \times 10^{-4} \chi_{5/2}(U_{24,23} \ldots U_{1,24}) + \frac{1}{7.158} \chi_{3}(U_{24,23} \ldots U_{1,24}) + \ldots \approx \frac{1}{7} \chi_{3}(U_{24,23} \ldots U_{1,24}).
\]

(10.20)

Thus, the Diakonov–Petrov representation of the Wilson loop (56) fits well the standard Wilson loop (49) defined for the irreducible representation \( J = 3 \).

Now let us explain the real meaning of the Diakonov–Petrov representation for the Wilson loop given by (56). **This representation is meaningless.**

It is obvious from the following fact. The representation (56) has been obtained via an expansion valid only for the fundamental representation \( J = 1/2 \). Therefore, (56) should be written only as follows

\[
W_{1/2}(z) = \mathcal{N}^{-1} \sum_j [b_j(z)]^N \chi_j(U_{N,N-1}U_{N-1,N-2} \ldots U_{1,N}),
\]

\[
\mathcal{N} = \sum_j (2j + 1) [b_j(z)]^N.
\]

(10.21)

Without discussing the appearance of the artificial normalization factor \( \mathcal{N} \) we argue that tuning the parameter \( z \) we get the following set of relations

\[
W^{DP}_{1/2}(z) \big|_{z=3.25103} \approx \frac{1}{2} \chi_{1/2}(U_{N,N-1}U_{N-1,N-2} \ldots U_{1,N}) = W_{1/2},
\]

\[
W^{DP}_{1/2}(z) \big|_{z=4.36765} \approx -\frac{1}{3} \chi_{1}(U_{N,N-1}U_{N-1,N-2} \ldots U_{1,N}) = W_1,
\]

37
\[ W_{1/2}^{\text{DP}}(z)|_{z=5.46564} \approx \frac{1}{4} \chi_{3/2}(U_{N,N-1}U_{N-1,N-2} \ldots U_{1,N}) = W_{3/2}, \]
\[ W_{1/2}^{\text{DP}}(z)|_{z=6.55104} \approx \frac{1}{5} \chi_{2}(U_{N,N-1}U_{N-1,N-2} \ldots U_{1,N}) = W_{2}, \]
\[ W_{1/2}^{\text{DP}}(z)|_{z=7.62728} \approx \frac{1}{6} \chi_{5/2}(U_{N,N-1}U_{N-1,N-2} \ldots U_{1,N}) = W_{5/2}, \]
\[ W_{1/2}^{\text{DP}}(z)|_{z=8.69644} \approx \frac{1}{7} \chi_{3}(U_{N,N-1}U_{N-1,N-2} \ldots U_{1,N}) = W_{3}, \]

(10.22)

where \( W_{1/2}^{\text{DP}}(z) \) is the Wilson loop in the Diakonov–Petrov representation defined for the fundamental representation and \( W_J \) is the standard Wilson loop for the irreducible representation \( J = 1/2, 1, \ldots \).

Since \textit{a priori} the value of the parameter \( z \) is not well defined, so tuning \( z \) we are able to equate the Wilson loop in the Diakonov–Petrov representation defined only for the fundamental representation \( J = 1/2 \) to the standard Wilson loop defined for any irreducible representation \( J \). This is really \textit{a new way to check confinement on lattice} [7,15].

\section*{Instead of Acknowledgement}

We conclude that in Ref.[15] Diakonov and Petrov have sold us a conjecture as a proof. In a tedious calculation this conjecture turned out to be wrong. A mistake in our calculations, unessential for the drawn conclusions, and some misprints were used to hide the necessity to redraw the conjecture. New fit parameters were introduced by them actually modifying the conjecture. The proof of the new, modified conjecture is again not conclusive for \( J \neq \frac{1}{2} \) or reduces to a triviality. We apologize if we fought too hard and promise not to recalculate any of Diakonov and Petrov’s papers in near future.
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