High Temperature Structure Detection in Ferromagnets

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Abstract

This paper studies structure detection problems in high temperature ferromagnetic (positive interaction only) Ising models. The goal is to distinguish whether the underlying graph is empty, i.e., the model consists of independent Rademacher variables, versus the alternative that the underlying graph contains a subgraph of a certain structure. We give matching upper and lower minimax bounds under which testing this problem is possible/impossible respectively. Our results reveal that a key quantity called graph arboricity drives the testability of the problem. On the computational front, under a conjecture of the computational hardness of sparse principal component analysis, we prove that, unless the signal is strong enough, there are no polynomial time linear tests on the sample covariance matrix which are capable of testing this problem.

1 Introduction

Graphical models are a powerful tool in high dimensional statistical inference. The graph structure of a graphical model gives a simple way to visualize the dependency among the variables in multivariate random vectors. The analysis of graph structures plays a fundamental role in a wide variety of applications, including information retrieval, bioinformatics, image processing and social networks (Besag, 1993; Durbin et al., 1998; Wasserman and Faust, 1994;
Motivated by these applications, theoretical results on graph estimation (Meinshausen and Bühlmann, 2006; Liu et al., 2009; Montanari and Pereira, 2009; Ravikumar et al., 2011; Cai et al., 2011), single edge inference (Jankova et al., 2015; Ren et al., 2015; Neykov et al., 2015; Gu et al., 2015) and combinatorial inference (Neykov et al., 2016; Neykov and Liu, 2017) have been studied in the literature.

In this paper we are concerned with the distinct problem of structure detection. In structure detection problems one is interested in testing whether the underlying graph is empty, (i.e., the random variables are independent) versus the alternative that the graph contains a subgraph of a certain structure. A variety of detection problems have been previously considered in the literature (see for example Addario-Berry et al., 2010; Arias-Castro et al., 2012, 2015b,a). These works mainly focus on covariance or precision matrix detection problems and establish minimax lower and upper bounds.

While covariance and precision matrix detection problems are inherently related to the Gaussian graphical model, in this paper we focus on detection problems under the zero-field ferromagnetic Ising model. The Ising model is a probability model for binary data originally developed in statistical mechanics (Ising, 1925) and has wide range of modern applications including image processing (Geman and Geman, 1984), social networks and bioinformatics (Ahmed and Xing, 2009). Below we formally introduce the model and problems of interest.

**Zero-field ferromagnetic Ising model.** Under a zero-field Ising model, the binary vector $X \in \{\pm 1\}^d$ follows a distribution with probability mass function given by

$$P_{\Theta}(X) = \frac{1}{Z_{\Theta}} \exp \left( \sum_{i,j=1}^{d} \theta_{ij} X_i X_j \right),$$

where $\Theta = (\theta_{ij})_{d \times d}$ is a symmetric interaction matrix with zero diagonal entries and $Z_{\Theta}$ is the partition function defined as

$$Z_{\Theta} = \sum_{X \in \{\pm 1\}^d} \exp \left( \sum_{i,j=1}^{d} \theta_{ij} X_i X_j \right).$$

The non-zero elements of the symmetric matrix $\Theta$ specify a graph $G(\Theta) = G = (V, E)$ with vertex set $V = \{1, \ldots, d\}$ and edge set $E = \{(i, j) : \theta_{ij} \neq 0\}$. We will refer to the graph $G(\Theta)$ as $G$ whenever it is clear what the underlying matrix $\Theta$ is. It is not hard to check that by the definition of $G$, the vector $X$ is Markov with respect to $G$, that is, each two elements $X_i$ and $X_j$ are independent given the remaining values of $X_{-(i,j)}$ if and only if $(i, j) \notin E$.

Here, the term zero-field specifies that there is no external magnetic field affecting the system, meaning that the energy function $\sum_{i,j=1}^{d} \theta_{ij} X_i X_j$ consists purely the terms of degree 2 (i.e., there are no main effects). In this paper, we further focus on zero-field ferromagnetic models, where we also assume that $\theta_{ij} \geq 0$, $i, j \in \{1, \ldots, d\}$. In addition, our analysis is under
the high-temperature setting, where the magnitudes of \( \theta_{ij} \)'s are under a certain level. More specifically, throughout this paper we assume that \( \| \Theta \|_F \leq \frac{1}{2} \), where \( \| \Theta \|_F = \left[ \sum_{i,j=1}^{d} \theta_{ij}^2 \right]^{1/2} \) is the Frobenius norm of \( \Theta \).

**Structure detection problems.** As described in the previous paragraph, a zero-field ferromagnetic Ising model specifies a graph \( G = (\mathcal{V}, \mathcal{E}) \). In a structure detection problem, we are interested in testing whether the underlying graph \( G \) is an empty graph versus the alternative that \( G \) belongs to a set of graphs with a certain structure. Specifically, let \( G_\emptyset = (\mathcal{V}, \emptyset) \) be the empty graph, and let \( \mathcal{G}_1 \) be a class of graphs not containing \( G_\emptyset \). The following hypothesis testing problem is an example of a detection problem. Given a sample of \( n \) independent observations \( X_1, \ldots, X_n \in \mathbb{R}^d \) from a zero-field ferromagnetic Ising model we aim to test

\[
H_0 : G = G_\emptyset \quad \text{versus} \quad H_1 : G \in \mathcal{G}_1. \tag{1.1}
\]

The term “detection” here is used in the sense that if one rejects the null hypothesis, the presence of a non-null graph has been detected. In (1.1) the graph class \( \mathcal{G}_1 \) can be arbitrary, which makes the hypothesis testing problem (1.1) a very general problem. We now give a specific instance of this problem which is of particular importance. Let \( G_* \) be a fixed graph with \( s = o(\sqrt{d}) \) non-isolated vertices which represents some specific graph structure. The structure detection problem that considers all possible “positions” of \( G_* \) is of the following form:

\[
H_0 : G = G_\emptyset \quad \text{versus} \quad H_1 : G \in \mathcal{G}_1(G_*), \tag{1.2}
\]

where \( \mathcal{G}_1(G_*) \) is the class of all graphs that contain a size-\( s \) subgraph isomorphic to \( G_* \).

While problems (1.1) and (1.2) give a good intuition what a detection problem is, in order to facilitate testing we need to impose certain assumptions on the matrix \( \Theta \), as otherwise even with graphs vastly different from the empty graph there might not be enough “separation” between the null and the alternative hypothesis. Since the underlying graph \( G \) is specified by the matrix \( \Theta \), we can reformulate problems (1.1) and (1.2) into testing problems on \( \Theta \). Given a class of graphs \( \mathcal{G}_1 \), we define the corresponding parameter space with minimum signal strength \( \theta > 0 \) as

\[
S(\mathcal{G}_1, \theta) = \left\{ \Theta = (\theta_{ij})_{d \times d} : \Theta = \Theta^T, G(\Theta) \in \mathcal{G}_1, \| \Theta \|_F \leq 1/2, \min_{(i,j) \in \mathcal{E}(G(\Theta))} \theta_{ij} \geq \theta \right\}. \tag{1.3}
\]

We now reformulate the hypothesis testing problems (1.1) and (1.2) as follows:

\[
H_0 : \Theta = 0 \quad \text{versus} \quad H_1 : \Theta \in S(\mathcal{G}_1, \theta), \tag{1.4}
\]

\[
H_0 : \Theta = 0 \quad \text{versus} \quad H_1 : \Theta \in S[\mathcal{G}_1(G_*), \theta]. \tag{1.5}
\]

\(^1\)For two positive sequences \( a_n \) and \( b_n \) we write \( a_n = o(b_n) \) if \( \lim_{n \to \infty} a_n/b_n = 0 \).
Figure 1: Illustration of the examples considered in this paper. (a) shows a single-edge graph; (b) is a 5-clique; (c) is a 5-star; (d) is an example of a graph that has community structure with $k = 5$ and $l = 4$. We can write the detection problems as \((1.5)\) by defining the corresponding shown graphs as $G_s$.

The results of our paper cover the following examples.

**Empty graph versus non-empty graph.** We consider testing whether the underlying graph of the Ising model is empty or not. Clearly, since our null hypothesis is that the graph is empty, this is a detection problem. We have $\mathcal{G}_1 = \{ G : E(G) \neq \emptyset \}$.

**Clique detection.** A clique is a set of vertices such that every two distinct vertices are adjacent. We consider detecting graphs that contain a clique of size $s$. We have $\mathcal{G}_1 = \{ G = (V, E) : \exists V \subseteq V \text{ such that } |V| = s \text{ and } (i, j) \in E \text{ for all } i, j \in V \}$. This is a more general version of the previous example, since one can think of a non-empty graph as a graph containing a clique of size $s = 2$.

**Star detection.** A star is a tree in which all leaves are connected to the same node. We consider detecting graphs that contain an $s - 1$ star. In this example, we have $\mathcal{G}_1 = \{ G = (V, E) : \text{there exist distinct } i_0, i_1, \ldots, i_{s-1} \in V \text{ such that } (i_0, i_1), (i_0, i_2), \ldots, (i_0, i_{s-1}) \in E \}$.

**Community structure detection.** In this example we consider a class of graphs with more complex structure. Let $k$ and $l$ be positive integers. A community $C$ is represented by a $k$-clique, which means that every two members in the same community are connected. For a community $C$, we select one fixed representative vertex and denote it as $v(C)$. We consider the class of graphs $\mathcal{G}_1$ that contains graphs with at least $l$ disjoint communities, such that for every two different communities $C$ and $C'$, there exists an edge connecting $v(C)$ and $v(C')$. In this example we set $s = kl$.

All of the above examples are of the type \((1.5)\). We show examples of these detection problems in Figure 1. In the following section we outline the main contributions of our work.
1.1 Main Contributions

There are three major contributions of this paper.

First, we develop a novel technique to derive minimax lower bounds of structure detection problems in Ising models. Our proof technique relates the Ising model probability mass function and the $\chi^2$-divergence between two distributions to the number of certain Eulerian subgraphs of the underlying graph. With this technique, we are able to obtain a general information-theoretic lower bound for arbitrary alternative hypothesis, which can be immediately applied to examples including any of the four examples described in the previous section.

Second, we propose a linear scan test on the sample covariance matrix that matches our minimax lower bound for arbitrary structure detection problems, in certain regimes. Along with our general minimax lower bound result, this procedure reveals the fact that a quantity called arboricity, (i.e., a certain maximum edge to vertex ratio of graphs in the alternative hypothesis) essentially determines the information-theoretic limit of the testing problem. This matches the intuition that in order to distinguish a graph with small signal strength from the empty graph, one need to examine the densest part of the graph. Furthermore, the denser the graph is, the easier it is to detect it, where the precise measurement of graph density turns out to be graph arboricity.

In addition, we also study the computational lower bound of structure detection problems. Based on a conjecture on the computational hardness of sparse Principal Component Analysis (PCA), which has been studied by recent works (Berthet and Rigollet, 2013b,a; Gao et al., 2014), we prove that no polynomial time linear test on the sample covariance matrix can detect structures successfully unless there is a sufficiently large signal strength. In addition to this result, we also derive another computational lower bound result under the oracle computational model studied by Feldman et al. (2015a,b); Wang et al. (2015).

1.2 Related Work

Plenty of work has been done on graph estimation (also known as graph selection) in Ising models. Santhanam and Wainwright (2012) gave the first information-theoretic lower bounds of graph selection problems for bounded edge cardinality and bounded vertex degree models. Later, Tandon et al. (2014) proposed a general framework for obtaining information-theoretic lower bounds for graph selection in ferromagnetic Ising models, and showed that the lower bound is specified by certain structural conditions. On the other hand, Ravikumar et al. (2010) proposed an algorithm for structure learning based on $l_1$-regularized logistic regression that works in the high temperature regime (Montanari and Pereira, 2009). Bresler (2015) gave a polynomial time algorithm that works for both low and high temperature regimes. Compared to graph estimation, structure detection is a statistically easier problem. As a
consequence, the limitations on signal strength that we exhibit in this paper are weaker than the corresponding requirements used in the graph estimation literature.

Structure detection problems have been studied in Addario-Berry et al. (2010); Arias-Castro et al. (2012, 2015b,a). However, all these works focus on Gaussian random vectors. Specifically, Addario-Berry et al. (2010) study testing the existence of specific subsets of components in a Gaussian vector whose means are non-zero based on a single observation. Arias-Castro et al. (2012) consider the correlation graph of a Gaussian random vector and establish upper and lower bounds for detecting certain classes of fully connected cliques based on one sample. In a follow up work, Arias-Castro et al. (2015b) generalize the result to multiple i.i.d. samples. Arias-Castro et al. (2015a) give another related result on detecting a region of a Gaussian Markov random field against a background of white noise. The major difference between these existing works and our work is that we focus on detection in the Ising model, and our results not only work for cliques, but also for general graph structures. Recently, (Neykov et al., 2016; Lu et al., 2017; Neykov and Liu, 2017) proposed a novel problem where one considers testing whether the underlying graph obeys certain combinatorial properties. We stress that while related to structure detection, these problems are fundamentally different as structure detection is a statistically simpler task. It is not surprising therefore that the algorithms we develop are very different from those in the aforementioned works, and the proofs of our lower bounds use different techniques.

Our result on computational lower bound follows the recent line of work on computational barriers for statistical models (Berthet and Rigollet, 2013b,a; Ma et al., 2015; Gao et al., 2014; Brennan et al., 2018) based on the planted clique conjecture. Berthet and Rigollet (2013b) focus on the testing method based on Minimum Dual Perturbation (MDP) and semidefinite programming (SDP) and prove that such polynomial time testing methods cannot attain the minimax optimal rate for sparse PCA. Berthet and Rigollet (2013a) prove the computational lower bound on a generalized sparse PCA problem which includes all multivariate distributions with certain tail probability assumptions on the quadratic form. Ma et al. (2015) consider the Gaussian submatrix detection problem and propose a framework to analyze computational limits of continuous random variables via constructing a sequence of asymptotically equivalent discretized models. Inspired by the results in Ma et al. (2015), Gao et al. (2014) consider the computational lower bound for Gaussian sparse Canonical Correlation Analysis (CCA) as well as sparse PCA problems. Our computational lower bound result is based on the previous studies on the sparse PCA problem. We summarize these results and directly base our result for Ising models on a sparse PCA conjecture. By doing this, we are able to use a novel proof technique that utilizes the high-dimensional central limit theorems of Chernozhukov et al. (2014).

Other related works on Ising models include the following. Berthet et al. (2016) study the Ising block model by providing efficient methods for block structure recovery as well as information-theoretic lower bounds. Mukherjee et al. (2018) study the upper and lower
bounds for detection of a sparse external magnetic field in Ising models. Daskalakis et al. (2018) consider goodness-of-fit and independence testing in Ising models using pairwise correlations. Gheissari et al. (2017) establish concentration inequalities for polynomials of a random vector in contracting Ising models.

1.3 Notation

We use the following notations in our paper. For a vector \(v = (v_1, \ldots, v_d)^T \in \mathbb{R}^d\) and a number \(1 \leq p < \infty\), let \(\|v\|_p = \left(\sum_{i=1}^d v_i^p\right)^{1/p}\). We also define \(\|v\|_\infty = \max_i |v_i|\). For a matrix \(A\), we denote \(\|A\|_{\max} = \max_{j,k} |A_{jk}|\), \(\|A\|_F = \left(\sum_{i,j=1}^d A_{ij}^2\right)^{1/2}\), and \(\|A\|_p = \max_{\|v\|_p = 1} \|Av\|_p\) for \(p \geq 1\).

We also use the standard asymptotic notations \(O(\cdot)\) and \(o(\cdot)\). Let \(a_n\) and \(b_n\) be two sequences and assume that \(b_n\) is non-zero for large enough \(n\). We write \(a_n = O(b_n)\) if \(\limsup_{n \to \infty} |a_n/b_n| < \infty\) and \(a_n = o(b_n)\) if \(\lim_{n \to \infty} a_n/b_n = 0\).

Let \(V = \{1, \ldots, d\}\) be the complete vertex set. In this paper we consider graphs with \(d\) vertices over the vertex set \(V\). For a graph \(G\), let \(E(G) = \{(i,j) : G\text{ has an edge connecting vertex } i \text{ and } j\}\), where \((i,j) = (j,i)\) are undirected pairs. Moreover, we denote by \(V(G) = \{i \in V : G\text{ has an edge connecting vertex } i\}\) the set of non-isolated vertices of \(G\).

1.4 Organization of the Paper

Our paper is organized as follows. In Section 2, we present our main information-theoretic lower bound result as well as its applications to various detection problems. In Section 3 we develop a general procedure to construct optimal linear scan tests on the sample covariance matrix. In Section 4 we examine the computational limit of the linear tests on the sample covariance matrix by comparing the covariance matrices of Ising and sparse PCA models. Sections 5 and 6 contain the proofs of the main results of Sections 2 and 3 respectively. The remaining detailed proofs are all placed in Section A.1. In Section B we provide an additional proof of a computational lower bound under the oracle computational model.

2 Lower Bounds

The minimax risk of detection problem (1.4) is defined as

\[
\gamma_{\mathcal{S}(G_1, \theta)} := \inf_{\psi} \left[ \mathbb{P}_{0,n}(\psi = 1) + \max_{\theta \in \mathcal{S}(G_1, \theta)} \mathbb{P}_{\Theta,n}(\psi = 0) \right],
\]

(2.1)

where \(\mathbb{P}_{0,n}\) and \(\mathbb{P}_{\Theta,n}\) are the joint probability measures of \(n\) i.i.d. samples under null and alternative hypotheses respectively. The infimum in (2.1) is taken over all measurable test
functions $\psi : \{X_1, \ldots, X_n\} \mapsto \{0, 1\}$. If $\liminf_{n \to \infty} \gamma[S(\mathcal{G}_1, \theta)] = 1$, we say that any test is asymptotically powerless.

In this section, we derive necessary conditions on the signal strength $\theta$ required for detection problems to admit tests which are not asymptotically powerless. Our results will show that the difficulty of testing an empty graph against $\mathcal{G}_1$ is determined by a quantity called arboricity, which was originally introduced in graph theory by Nash-Williams (1961) to quantify the minimum number of forests into which the edges of a given graph can be partitioned.

For a graph $G \in \mathcal{G}_1$ and a vertex set $V \subseteq \overline{V}$, let $G_V$ be the graph obtained by restricting $G$ on the vertices in $V$ (i.e., removing all edges which are connected to vertices $\overline{V} \setminus V$). The arboricity of $G$ is defined as follows:

$$ R(G) := \left\lceil \max_{V \subseteq \overline{V}} \frac{|E(G_V)|}{|V| - 1} \right\rceil, \quad (2.2) $$

where \(\lceil \cdot \rceil\) is the ceiling function, and $0/0$ is understood as $0$. The arboricity of a graph measures how dense the graph is. For an illustration of arboricity see Figure 2. Let $G_\emptyset = (\overline{V}, \emptyset)$ denote the empty graph. By definition $R(G_\emptyset) = 0$. For a given graph $G$ the larger $R(G)$ is, the more different $G_\emptyset$ and $G$ are. We further define

$$ R := \min_{G \in \mathcal{G}_1} R(G) $$

to measure the difference in graph density between $G_\emptyset$ and $\mathcal{G}_1$ in a worst case sense. Let $\mathcal{G}^*$ be a nonempty subset of $\mathcal{G}_1$ such that all graphs in $\mathcal{G}^*$ have arboricity $R$. By the definition of $R$, such nonempty $\mathcal{G}^*$ exists, and may not be unique. Our analysis works for arbitrary choices of $\mathcal{G}^*$ which satisfy the incoherence condition (Neykov et al., 2016) defined as follows.

**Definition 2.1.** (Negative association and incoherence condition) For $k \geq 0$, we say the random variables $Y_1, \ldots, Y_k$ are negatively associated if for any $k_1, k_2 \geq 0$ with $k_1 + k_2 \leq k$, 

Figure 2: Illustration of arboricity. Here the nodes and black lines represent the vertices and edges of graph $G$ respectively. The vertex set $V$ that maximizes $|E(G_V)|/(|V| - 1)$ is denoted by green nodes, which also gives the densest subgraph of $G$. We have $R(G) = 3$. 

functions $\psi : \{X_1, \ldots, X_n\} \mapsto \{0, 1\}$. If $\liminf_{n \to \infty} \gamma[S(\mathcal{G}_1, \theta)] = 1$, we say that any test is asymptotically powerless.

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any distinct indices \( i_1, i_2, \ldots, i_{k_1}, j_1, \ldots, j_{k_2} \), and any coordinate-wise non-decreasing functions \( f \) and \( g \), we have
\[
\text{Cov}[f(Y_{i_1}, \ldots, Y_{i_{k_1}}), g(Y_{j_1}, \ldots, Y_{j_{k_2}})] \leq 0.
\]
We say that the graph set \( G^* \) is incoherent if for any fixed graph \( G \), the binary random variables
\[
\{1[i \in V(G')]\}_{i \in V(G)}
\]
are negatively associated with respect to uniformly sampling \( G' \in G^* \).

For a graph \( G \), we denote by \( A_G \) the adjacency matrix of \( G \). Then given \( G^* \), we define the corresponding parameter set with minimal signal strength \( \Theta \) as
\[
S^* = \{ \Theta = \theta A_G : G \in G^* \}.
\]
Let \( V_{\text{max}} = \max_{G \in G^*} |V(G)| \), \( \Lambda = \max_{G \in G^*} \|A_G\|_F \), \( \Gamma = \max_{G \in G^*} \|A_G\|_1 \) and \( B = 512 \{ \Lambda^4 \land [V_{\text{max}}(\Lambda \lor \Lambda)]^2 \} \). Then for \( \theta \leq (2\Lambda)^{-1} \), by definition (recall (1.3)) we have \( S^* \subseteq S(G_1, \theta) \), and therefore
\[
\gamma[S(G_1, \theta)] \geq \gamma(S^*) := \inf_{\psi} \left[ \mathbb{P}_{0,n}(\psi = 1) + \max_{\Theta \in S^*} \mathbb{P}_{\Theta,n}(\psi = 0) \right]. \tag{2.3}
\]
By (2.3), it follows that to give a lower bound on \( \gamma[S(G_1, \theta)] \) it suffices to lower bound \( \gamma(S^*) \).

We are ready to introduce our main theorem.

**Theorem 2.2.** Let \( G^* \) be a non-empty subset of \( G_1 \) such that all graphs in \( G^* \) have arboricity \( R \). Define \( N(G^*) := \max_{G \in G^*} E_{G' \sim U(G^*)}|V(G) \cap V(G')| \), where \( U(G^*) \) is the uniform distribution over \( G^* \). If \( G^* \) is incoherent, \( N(G^*) = o(1) \), and
\[
\theta \leq \sqrt{\frac{\log[N^{-1}(G^*)]}{6nR}} \land \sqrt{\frac{R}{B}} \land \frac{1}{8(\Lambda \lor \Gamma)}, \tag{2.4}
\]
then we have
\[
\liminf_{n \to \infty} \gamma(S^*) = 1.
\]

The proof of Theorem 2.2 is given in Section 5.

**Remark 2.3.** Inequality (2.4) shows that the necessary signal strength of detection problems is determined by the minimum of three terms. While the first term \( \sqrt{\frac{\log[N^{-1}(G^*)]}{6nR}} \) is related to both the structural properties of graphs in \( G_1 \) and the sample size \( n \), the second term \( \sqrt{\frac{R}{B}} \) and third term \( \frac{1}{8(\Lambda \lor \Gamma)} \) are independent of \( n \). Therefore when the sample size is large enough, \( \sqrt{\frac{\log[N^{-1}(G^*)]}{6nR}} \) is the leading term determining the necessary signal strength, and the other two terms mainly serve as scaling conditions of \( \theta \).
Theorem 2.2 is comparable to the “multi-edge” results given in Neykov et al. (2016), where the authors give minimax lower bounds of combinatorial inference problems in Gaussian graphical models. Unlike our results in Theorem 2.2, the necessary signal strength for Gaussian graphical models given by Neykov et al. (2016) does not explicitly involve graph arboricity. It is also not very clear under what condition the lower bound given by Neykov et al. (2016) is sharp. In comparison, in this paper we show that graph arboricity is an appropriate quantity that gives sharp lower bounds for any structure detection problems under the incoherence condition and the sparsity assumption $s = O(d^{1/2-c})$ for some $c > 0$. It is also worth comparing Theorem 2.2 to the results of Neykov and Liu (2017). The lower bounds on the signal $\theta$ of Neykov and Liu (2017), typically involve the quantity $\sqrt{\log d/n}$ which is generally much larger than the right hand side of (2.4) when $R$ is large enough. This is intuitively clear since detection problems are statistically easier than graph property testing. Our proof strategy is also completely different than the one used by Neykov and Liu (2017), and relies on high temperature expansions rather than Dobrushin’s comparison theorem.

In Theorem 2.2, the incoherence condition of $G^*$ is not always easy to check. However, it is known that this condition is satisfied by a various discrete distributions including the multinomial and hypergeometric distributions (Joag-Dev and Proschan, 1983; Dubhashi and Ranjan, 1998). In particular, Theorem 2.11 in Joag-Dev and Proschan (1983) states that negative association holds for all permutation distributions. Therefore, for detection problems of the form (1.5), incoherence condition is always satisfied by picking $G^*$ to be the set of all graphs isomorphic to $G_*$. This leads to the following corollary (recall that we are assuming $s = o(\sqrt{d})$).

Corollary 2.5. Let $G_*$ be a graph with $s$ vertices and $G_1(G_*)$ be the class of all graphs that contain a size-$s$ subgraph isomorphic to $G_*$. Let $B(G_*) = 512\{\|A_{G_*}\|_F \wedge (\|A_{G_*}\|_1 \vee \|A_{G_*}\|_F)^2s\}$. If

$$\theta \leq \sqrt{\frac{\log(d/s^2)}{6nR(G_*)}} \wedge \sqrt{\frac{R(G_*)}{B(G_*)}} \wedge \frac{1}{8(\|A_{G_*}\|_F \vee \|A_{G_*}\|_1)},$$

then we have

$$\liminf_{n \to \infty} \gamma(S^*) = 1.$$ 

2.1 Examples

In this section we apply Corollary 2.5 to specific detection problems.
Example 2.6 (Empty graph versus non-empty graph). Consider testing empty graph versus non-empty graph defined in Section 1. If

$$\theta \leq \sqrt{\frac{\log(d/4)}{6n}} \wedge \frac{1}{32\sqrt{2}}, \quad (2.5)$$

we have $\lim \inf_{n \to \infty} \gamma(S^*) = 1$.

Proof. In this example $s = 2$, $G_*$ is a single-edge graph and we have $\mathcal{R}(G_*) = 1$. By direct calculation we have $\|A_{G_*}\|_F = \sqrt{2}$, $\|A_{G_*}\|_1 = 1$, and $\mathcal{B}(G_*) = 2048$. By Corollary 2.5, if (2.5) holds we have $\lim \inf_{n \to \infty} \gamma(S^*) = 1$. \qed

Example 2.7 (Clique Detection). For the clique detection problem defined in Section 1, if

$$\theta \leq \sqrt{\frac{\log(d/s^2)}{6ns}} \wedge \frac{1}{32s}, \quad (2.6)$$

we have $\lim \inf_{n \to \infty} \gamma(S^*) = 1$.

Proof. In this example $G_*$ is an $s$-clique graph. We have $\mathcal{R}(G_*) = [s/2]$ and therefore $s/2 \leq \mathcal{R}(G_*) \leq s$. By direct calculation we have $\|A_{G_*}\|_F = \sqrt{s(s-1)} \leq s$, $\|A_{G_*}\|_1 = s-1 \leq s$, and therefore $\mathcal{B}(G_*) \leq 512s^3$. By Corollary 2.5, if (2.6) holds we have $\lim \inf_{n \to \infty} \gamma(S^*) = 1$. \qed

Example 2.8 (Star Detection). For the star detection problem defined in Section 1, if $s \geq 4$ and

$$\theta \leq \sqrt{\frac{\log(d/s^2)}{6n}} \wedge \frac{1}{32\sqrt{2}s}, \quad (2.7)$$

then $\lim \inf_{n \to \infty} \gamma(S^*) = 1$.

Proof. In this example $G_*$ is a star graph and we have $\mathcal{R}(G_*) = 1$. By direct calculation we have $\|A_{G_*}\|_F = \sqrt{2(s-1)} \leq \sqrt{2}s$, $\|A_{G_*}\|_1 = s-1 \leq s$. If $s \geq 4$, we have $\mathcal{B}(G_*) = 2048s^2$. By Corollary 2.5, if (2.7) holds we have $\lim \inf_{n \to \infty} \gamma(S^*) = 1$. \qed

Example 2.9 (Community structure detection). For the community structure detection problem defined in Section 1, if $k \geq 4$, $l \geq 2$ and

$$\theta \leq \sqrt{\frac{\log(d/s^2)}{6n(l \vee k)}} \wedge \frac{1}{32\sqrt{2}s}, \quad (2.8)$$

we have $\lim \inf_{n \to \infty} \gamma(S^*) = 1$. 

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Proof. To calculate $\mathcal{R}(G_*)$, we utilize the fact that arboricity equals the minimum number of forests into which the edges of a given graph can be partitioned (Nash-Williams, 1961). Let $C_1, \ldots, C_l$ be the communities. For $i = 1, \ldots, l$, we know that $C_i$ is a $k$-clique, and the arboricity is $[k/2]$. Therefore inside $C_i$, we can partition the graph into $[k/2]$ forests. There is also an $l$-clique in $G_*$ consisting of the cross-community edges. This clique can be partitioned into $[l/2]$ forests. Note that this $l$-clique shares only one vertex $v(C_i)$ with the community $C_i$. Therefore for any forest in the partition of this $l$-clique and any forest in the partition of $C_i$, we can merge them into a single forest because the resulting graph is still acyclic. We can keep merging forests from other communities. Eventually, we can merge $l$ forests from distinct communities to a forest in the $l$-clique, without introducing any cycles. If $[l/2] \geq [k/2]$, we will obtain $[l/2]$ forests that form a partition of $G_*$; if $[l/2] < [k/2]$, then the partition will contain $[k/2]$ forests. Therefore by the equivalent definition of arboricity given in (Nash-Williams, 1961) we have $\mathcal{R}(G_*) \leq [(l \lor k)/2]$. On the other hand, since $G_*$ contains an $l$-clique, obviously $\mathcal{R}(G_*) \geq [l/2]$. Similarly, $\mathcal{R}(G_*) \geq [k/2]$ and hence we have $\mathcal{R}(G_*) = [(l \lor k)/2]$. Therefore $(l \lor k)/2 \leq \mathcal{R}(G_*) \leq l \lor k$.

By direct calculation, we have $\|A_{G_*}\|_F = \sqrt{lk(k-1) + l(l-1)} \leq \sqrt{lk^2 + l^2}$, $\|A_{G_*}\|_1 = k - 1 + l - 1 \leq k + l$. We now compare the upper bounds of $\|A_{G_*}\|_F$ and $\|A_{G_*}\|_1$. If $k \geq 4$ and $l \geq 2$, we have $l \geq 1 + l/2$ and

$$lk^2 + l^2 \geq (1 + l/2)k^2 + l^2 = k^2 + lk^2/2 + l^2 \geq k^2 + 2kl + l^2 = (k + l)^2.$$ 

Therefore

$$B(G_*) \leq 512[(lk^2 + l^2)^2 \land (lk^2 + l^2)lk] = 512(lk^2 + l^2)lk = 512(s^2k + sl^2),$$ 

and

$$\sqrt{\frac{\mathcal{R}(G_*)}{B(G_*)}} \geq \sqrt{\frac{l \lor k}{1024(s^2k + sl^2)}} \geq \sqrt{\frac{l \lor k}{1024(s^2k + s^2l)}} \geq \sqrt{\frac{l \lor k}{2048s^2(k \lor l)}} = \frac{1}{32\sqrt{2s}}.$$ 

Moreover,

$$\frac{1}{8(\|A_{G_*}\|_F \lor \|A_{G_*}\|_1)} \geq \frac{1}{8\sqrt{lk^2 + l^2}} \geq \frac{1}{8\sqrt{2s}}.$$ 

Therefore by Corollary 2.5, if (2.8) holds we have $\lim \inf_{n \to \infty} \gamma(S^*) = 1$. 

## 3 Upper Bounds

In this section we construct upper bounds for the hypothesis testing problem (1.1). We propose a general framework for testing an empty graph $G_{\emptyset}$ against an arbitrary graph set
We remind the reader that the arboricity of a graph $G$ is defined in (2.2) as

$$
\mathcal{R}(G) := \left\lceil \max_{V \subseteq \mathcal{V}} \frac{|E(G_V)|}{|V| - 1} \right\rceil,
$$

where $G_V$ is the graph obtained by restricting $G$ on the vertex set $V$. The arboricity $\mathcal{R}$ of $G_1$ is then defined as

$$
\mathcal{R} := \min_{G \in G_1} \mathcal{R}(G).
$$

We now introduce the concept of witnessing subgraph and witnessing set. Before that, we remind the reader, that in this paper all graphs have $d$ vertices (i.e., all graphs are over the vertex set $\mathcal{V}$), unless otherwise specified. Therefore a subgraph $G'$ of a graph $G = (\mathcal{V}, E)$ is a graph with $d$ vertices whose edge set is a subset of the edge set of the larger graph, i.e., $G' = (\mathcal{V}, E')$ where $E' \subseteq E$. Importantly, the notation $V(G)$ and $V(G')$ refer to the non-isolated vertices of $G$ and $G'$ which may be strict subsets of $\mathcal{V}$.

**Definition 3.1 (Witnessing Subgraph).** For a graph $G \in G_1$ we call the graph $H$ a witnessing subgraph of $G$ with respect to $G_1$, if $H$ is a subgraph of $G$ and $\left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil \geq \mathcal{R}$.

Here we remark that for $H$ to be a witnessing subgraph of $G$, it is unnecessary to have $\left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil = \mathcal{R}(G)$. Instead, we only require that $\left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil \geq \mathcal{R}$, which is a weaker requirement since by definition we have $\mathcal{R} \leq \mathcal{R}(G)$ for any $G \in G_1$. This implies that every graph $G \in G_1$ has at least one witnessing graph, which may be obtained from the densest subgraph of $G$ (with potential edge pruning).

**Definition 3.2 (Witnessing Set).** We call the collection of graphs $\mathcal{H}$ a witnessing set of $G_1$, if for every $G \in G_1$, there exists $H \in \mathcal{H}$ such that $H$ is a witnessing subgraph of $G$.

By the definition of $\mathcal{R}$, and as we previously argued, every graph $G \in G_1$ must have at least one witnessing subgraph. Therefore at least one witnessing set $\mathcal{H}$ of $G_1$ exists. We define the set of witnessing graphs in order to facilitate the development of scan tests. Below we will formalize a test statistic which scans over all graphs in $\mathcal{H}$. Importantly, in order to match the lower bound result given by Theorem 2.2, it is not sufficient to scan directly over the graphs from the set $G_1$. This is because the graphs in $G_1$ may contain non-essential edges which may introduce noise during the testing. In contrast, the graphs from $\mathcal{H}$ trim down those non-essential edges and focus only on the essential parts of the graphs in $G_1$.

We now introduce our general testing procedure. Our test is based on a witnessing set $\mathcal{H}$. For $H \in \mathcal{H}$ we define

$$
\hat{W}_H := \frac{1}{n} \cdot \sum_{i=1}^{n} \left( \frac{1}{|E(H)|} \sum_{(i,j) \in E(H)} X_{l,i} X_{l,j} \right),
$$

(3.1)
where $X_l$ is the $l$-th sample and $X_{l,i}$, $X_{l,j}$ are the $i$-th and $j$-th components of $X_l$ respectively. Our test then scans over all possible $H \in \mathcal{H}$ and calculates the corresponding $\hat{W}_H$. We define

$$
\psi := 1 \left[ \max_{H \in \mathcal{H}} \hat{W}_H > \frac{\kappa}{4} \sqrt{\frac{M(\mathcal{H})}{R_n}} \right],
$$

where

$$
m(\mathcal{H}) := \min_{H \in \mathcal{H}} |V(H)|, \quad M(\mathcal{H}) := \frac{\log(|\mathcal{H}|)}{m(\mathcal{H})},
$$

and $\kappa$ is a large enough absolute constant. The following theorem justifies the usage of the test defined in (3.2).

**Theorem 3.3.** Given any fixed $\alpha \in (0, 1)$, suppose that $\log(|\mathcal{H}|)/n = o(1)$ and $|\mathcal{H}| \geq 2/\alpha$. If

$$
\theta > \kappa \sqrt{\frac{M(\mathcal{H})}{R_n}}
$$

for a large enough absolute constant $\kappa$, when $n$ is large enough we have that the test $\psi$ of (3.2) satisfies

$$
P_{0,n}(\psi = 1) + \max_{\Theta \in \mathcal{S}(G_1, \theta)} P_{\Theta,n}(\psi = 0) \leq \alpha.
$$

The detailed proof of Theorem 3.3 is given in Section 6.

**Remark 3.4.** We can compare our upper bound result with Corollary 2.5. For testing problems of the form (1.5), we can always choose a subgraph $H_*$ of $G_*$ with $|E(H)|/|V(H)| = 1$] as a witnessing subgraph (if there are multiple such subgraphs pick any of them), and construct $\mathcal{H}$ to be the set consisting of all graphs isomorphic to $H_*$. For this $\mathcal{H}$ we have $|\mathcal{H}| \leq \frac{d}{(d-|V(H_*)|)!}$. Therefore

$$
M(\mathcal{H}) \leq |V(H_*)|^{-1} \log[d/(d-|V(H_*)|)!] \leq \log(d).
$$

If $s = O(d^{1/2-c})$ for some $c > 0$, $\log(d/s^2)$ is also of order $\log(d)$. Therefore the rate given by Theorem 3.3 matches Corollary 2.5.

### 3.1 Examples

**Example 3.5** (Empty graph versus non-empty graph). Consider testing empty graph versus non-empty graph defined in Section 1. If $\log(d)/n = o(1)$, $4/[d(d-1)] \leq \alpha$ and

$$
\theta > \kappa \sqrt{\frac{\log d}{n}}
$$

for a large enough constant $\kappa$, then when $n$ is large enough, we have

$$
P_{0,n}(\psi = 1) + \max_{\Theta \in \mathcal{S}(G_1, \theta)} P_{\Theta,n}(\psi = 0) \leq \alpha.
$$
We have

\[ |H| = d(d - 1)/2, \quad m(H) = 2 \quad \text{and} \quad M(H) = \log(|H|)/m(H) \leq \log d. \]

Therefore by Theorem 3.3, if (3.3) holds for a large enough constant \( \kappa \), then when \( n \) is large enough, we have that (3.4) holds.

**Example 3.6** (Clique Detection). For the clique detection problem defined in Section 1, if
\[
\log(ed/s)/n = o(1), \quad (d/s)^s \geq 2/\theta \quad \text{and} \quad \theta > \kappa \sqrt{\frac{\log(ed/s)}{sn}}, \]
for a large enough constant \( \kappa \), then when \( n \) is large enough, we have
\[
\mathbb{P}_{0,n}(\psi = 1) + \max_{\Theta \in \mathcal{S}(G, \theta)} \mathbb{P}_{\Theta,n}(\psi = 0) \leq \alpha. \quad (3.5)
\]

**Proof.** In this example we have \( \mathcal{R} = 1 \), and therefore \( \mathcal{H} = \{ \text{single-edge graphs} \} \) is a witnessing set of \( \mathcal{G}_1 \). We have \( |\mathcal{H}| = d(d - 1)/2, \) \( m(\mathcal{H}) = 2 \) and \( M(\mathcal{H}) = \log(|\mathcal{H}|)/m(\mathcal{H}) \leq \log d. \) Therefore by Theorem 3.3, if (3.3) holds for a large enough constant \( \kappa \), then when \( n \) is large enough, we have that (3.4) holds.

**Example 3.7** (Star Detection). For the star detection problem defined in Section 1, if
\[
\log(d)/n = o(1), \quad 4/[d(d - 1)] \leq \alpha \quad \text{and} \quad \theta > \kappa \sqrt{\frac{\log(ed/s)}{n}}, \]
for a large enough constant \( \kappa \), then when \( n \) is large enough, we have
\[
\mathbb{P}_{0,n}(\psi = 1) + \max_{\Theta \in \mathcal{S}(G, \theta)} \mathbb{P}_{\Theta,n}(\psi = 0) \leq \alpha. \quad (3.6)
\]

**Proof.** In this example we have \( \mathcal{R} = 1 \), and \( \mathcal{H} = \{ (s - 1)\text{-stars} \} \) is a witnessing set of \( \mathcal{G}_1 \). We have \( |\mathcal{H}| = \binom{s}{d} \), and therefore \( s(d/s)^s \leq |\mathcal{H}| \leq s(ed/s)^s \). We have \( m(\mathcal{H}) = s \). When \( s = o(\sqrt{d}) \) we have \( s \leq (ed/s)^s \) and \( M(\mathcal{H}) = \log(|\mathcal{H}|)/m(\mathcal{H}) \leq 2 \log(ed/s). \) Therefore by Theorem 3.3, if (3.7) holds for a large enough constant \( \kappa \), then when \( n \) is large enough, we have that (3.8) holds.

**Example 3.8** (Community structure detection). Consider the community structure detection problem defined in Section 1. If \( (l \lor k) \log[ed/(l \lor k)]/n = o(1), \quad [d/(l \lor k)]^{(l \lor k)} \geq 2/\theta \quad \text{and} \quad \theta > \kappa \sqrt{\frac{\log[ed/(l \lor k)]}{(l \lor k)n}}, \)
for a large enough constant \( \kappa \), then when \( n \) is large enough, we have
\[
\mathbb{P}_{0,n}(\psi = 1) + \max_{\Theta \in \mathcal{S}(G, \theta)} \mathbb{P}_{\Theta,n}(\psi = 0) \leq \alpha. \quad (3.8)
\]

**Proof.** If \( l \geq k \), we have \( \mathcal{R} = [l/2] \), and we can choose \( \mathcal{H} = \{ l\text{-cliques} \} \) as a witnessing set of \( \mathcal{G}_1 \); if \( l < k \), we have \( \mathcal{R} = [k/2] \), and \( \mathcal{H} = \{ k\text{-cliques} \} \) is a witnessing set of \( \mathcal{G}_1 \). The rest of proof is identical to the clique detection problem, and we omit the details.

\[ \square \]
4 Computational Lower Bound

Our results in Section 3 suggests that in order to match the information-theoretic lower bound, one should first determine the densest subgraphs of graphs in $G_1$, and then scan over all possible positions of such subgraphs. However, such tests may not be computationally efficient: for the structure detection problem (1.5), if the densest part of $G_*$ contains $k$ vertices, then our test requires scanning over at least $\binom{d}{k}$ different positions, and cannot be done in polynomial time if $k = O(s^\delta)$ for some constant $\delta > 0$. On the other hand, one can always relax the testing problem into the “empty graph versus non-empty graph” problem, which, according to Section 3.1, can be tested by scanning over single edges in polynomial time. However, it will require signal strength $\theta > \kappa \sqrt{\frac{\log d}{n}}$ for some constant $\kappa$ to distinguish the null and the relaxed alternative, which does not match the information theoretic lower bound in Theorem 2.2 for the original detection problem with large maximum arboricity $R$.

In this section, we give a detailed analysis of such computational-statistical tradeoffs, and show that the signal strength requirement $\theta > \kappa \sqrt{\frac{\log d}{n}}$, up to a logarithmic factor, cannot be improved for polynomial time linear tests.

Let $\hat{M} = \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T$ be the sample covariance matrix calculated with $n$ samples from the Ising model. We define polynomial time linear tests on $\hat{M}$ as follows.

**Definition 4.1 (Polynomial time Linear Test).** We call a test $\psi$ polynomial time linear test if there exist an integer $m \leq (nd)^p$ for some constant $p$, a binary function $f(\cdot)$ and linear functions $L_1(\hat{M}), \ldots, L_m(\hat{M})$ such that

$$\psi = f\left\{ \prod_{i=1}^{m} \mathbb{1}[L_i(\hat{M}) \geq 0] \right\}. \quad (4.1)$$

Note that the test we introduce in (3.2) in Section 3 is of the form (4.1). Indeed, we have

$$\psi = 1 - \prod_{H \in \mathcal{H}} \mathbb{1} \left[ -\hat{W}_H + \frac{\kappa}{4} \sqrt{\frac{M(H)}{Rn}} \geq 0 \right],$$

and since for each $H \in \mathcal{H}$, $\hat{W}_H$ is a linear function of $\hat{M}$ the above is of the form (4.1). However, the test (3.2) may not be a polynomial time linear test according to our definition since the number of graphs in $\mathcal{H}$ may not be bounded by $(nd)^p$ for a constant $p$.

### 4.1 Main Computational Lower Bound Result

In this section we give our main result on the computational lower bound of structure testing problems in Ising models. Our result is based on a sparse PCA conjecture. Denote by
1_{i_1,\ldots,i_s} = e_{i_1} + \cdots + e_{i_s} \in \mathbb{R}^d \text{ the vector whose } i_1, \ldots, i_s\text{-th entries are 1 and other entries are 0. Let }

S_\sigma = \{ \Sigma = I + \sigma 1_{i_1,\ldots,i_s} 1_{i_1,\ldots,i_s}^T : i_1, \ldots, i_s \text{ are } s \text{ distinct indices in } \{1, \ldots, d\} \}

be the set of covariance matrices from Gaussian spiked model. In sparse PCA, we consider the hypothesis testing problem for $n$ i.i.d samples $Z_1, \ldots, Z_n \in \mathbb{R}^d$:

$$H_{\text{PCA}}^0 : Z_1, \ldots, Z_n \sim N(0, I) \text{ versus } H_{\text{PCA}}^1 : Z_1, \ldots, Z_n \sim N(0, \Sigma), \Sigma \in S_\sigma. \quad (4.2)$$

We denote by $P_{I,n}$ and $P_{\Sigma,n}$ the probability measure under $H_{\text{PCA}}^0$ and $H_{\text{PCA}}^1$ respectively.

**Conjecture 4.2** (Computational Hardness of Sparse PCA). Let $\delta > 0$ be an absolute constant. If $\sigma \leq \eta[n^{-(1/2+\delta)} \land s^{-(1+\delta)}]$ for some small enough constant $\eta$, then for any polynomial time test $\psi$, we have

$$\liminf_{n \to \infty} \left[ P_{I,n}(\psi = 1) + \max_{\Sigma \in S_\sigma} P_{\Sigma,n}(\psi = 0) \right] \geq \frac{1}{4}.$$

Conjecture 4.2 is derived by Gao et al. (2014) under the widely believed planted clique conjecture and additional assumptions which essentially require that $2n \leq d \leq n^a$ for some constant $a > 1$ and $n[\log(n)]^5 \leq Cs^4$ for some small enough constant $C > 0$. It is also studied in Berthet and Rigollet (2013a) and Brennan et al. (2018). We now give our main theorem on the computational lower bound of hypothesis testing problems of the form (1.5).

**Theorem 4.3.** Under Conjecture 4.2, if $\theta \leq \eta[n^{-(1/2+\delta)} \land s^{-(1+\delta)}]$ for some small enough constant $\eta$, then for any polynomial time linear test $\psi$ as in (4.1) and any $G_s$ with $s$ non-isolated vertices, we have

$$\liminf_{n \to \infty} \left[ P_{0,n}(\psi = 1) + \max_{\Theta \in S[11(G_s),\theta]} P_{\Theta,n}(\psi = 0) \right] \geq \frac{1}{4}.$$

**Proof.** See Section A.3 for a detailed proof.

**Remark 4.4.** Theorem 4.3 shows that no polynomial linear scan tests on the sample covariance matrix $\hat{M}$ can distinguish the null from alternative hypotheses when $\theta \leq \eta[n^{-(1/2+\delta)} \land s^{-(1+\delta)}]$ for small enough constant $\eta$. Since the sample covariance matrix $\hat{M}$ is a sufficient statistic for the Ising model, any test $\psi(X_1, \ldots, X_n)$ on the sample vectors $X_1, \ldots, X_n$ can be formulated as a function $\tilde{\psi}(\hat{M})$ on the sample covariance matrix. However, $\tilde{\psi}$ may not be linear and furthermore the computation complexity of calculating $\psi$ and $\tilde{\psi}$ may be different, hence the result in Theorem 4.3 cannot prove the nonexistence of computationally efficient $\psi(X_1, \ldots, X_n)$. To derive bounds for other types of test functions, in Section B of the Appendix, we provide a different approach and show the computational limit under the oracle computational model. We leave more general results to future work.

While the detailed proof of Theorem 4.3 is given in Section A.3, in the next section we give some important insights into the connection between Gaussian and Ising models.

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4.2 Connection Between Gaussian and Ising Cliques

In this section, we explain how we relate Conjecture 4.2 to the Ising model. The main idea is that based on the Gaussian random vectors from the sparse PCA problem, we propose a polynomial time reduction algorithm that constructs a $d \times d$ matrix which cannot be distinguished from the sample covariance matrix of an Ising model with a parameter matrix $\Theta$ by polynomial time linear tests.

Importantly, this reduction only needs to be done for clique graphs because any $G_s$ can always be embedded within an $s$-clique. Furthermore, in the sparse PCA problem each $\Sigma \in S_\sigma$ corresponds to an $s$-clique. More specifically, for any index set $I \subseteq \{1, \ldots, d\}$ of size $s$ representing the position of a clique, we consider i.i.d. samples $X_1, \ldots, X_n$ generated from the Ising model with parameter matrix $\Theta = \theta \cdot [1(i, j \in I, i \neq j)]_{d \times d}$, and $Z_1, \ldots, Z_n$ generated from the multivariate Gaussian distribution with mean 0 and covariance matrix $\Sigma = I + \sigma^2 I_1 I_1^T$.

Before we introduce the reduction scheme, it is necessary to determine the parameter $\sigma$ for any fixed $\theta$. Our choice of $\sigma$ is based on a comparison between the moments of Ising and signs of Gaussian vectors. Let $Y_i = \text{sign}(Z_i)$, $i = 1, \ldots, n$. For $r = 1, \ldots, s$ and distinct $i_1, \ldots, i_r \in I$, we define

$$\alpha_r(\theta) := \mathbb{E}(X_{i_1} X_{i_2} \cdots X_{i_r}), \quad \beta_r(\sigma) := \mathbb{E}(Y_{i_1} Y_{i_2} \cdots Y_{i_r}).$$

The following lemma determines $\sigma$ for all small enough $\theta$.

**Lemma 4.5.** Let $t = \tanh(\theta)$. For odd $r$, we have $\alpha_r(\theta) = \beta_r(\sigma) = 0$. For even $r$, if $\theta \leq \eta s^{-(1+\delta)}$ for some small enough constant $\eta > 0$, then there exists $\sigma \in [\pi t/2, 16\pi t]$ such that

$$\beta_2(\sigma) = \alpha_2(\theta) \quad \text{and} \quad (C_1 t)^{r/2} \leq \beta_r(\sigma), \quad \alpha_r(\theta) \leq (r - 1)!!(C_2 t)^{r/2}.$$  \hspace{1cm} (4.3)

where $C_1, C_2$ are absolute constants.

From now on we study the sparse PCA problem with parameter $\sigma$ chosen such that condition (4.3) holds. Based on this $\sigma$, we construct

$$\hat{B} = \frac{1}{n} \sum_{i=1}^{n} Y_i Y_i^T.$$ 

Obviously, given $Z_1, \ldots, Z_n$, $\hat{B}$ can be calculated in polynomial time. We now proceed to show that no polynomial time linear test can distinguish $\hat{B}$ from

$$\hat{M} = \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T.$$ 

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which is the sample covariance matrix of the Ising model. Therefore if a polynomial time linear test can test for clique presence in $\hat{M}$, one will be able to use this test to test for clique presence in the sparse PCA problem. We denote by $\mathbb{P}_{\mathcal{I},n}$ the joint probability measure of $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ with parameter $\theta$ and the correspondingly chosen $\sigma$, and denote by $\mathbb{E}_{\mathcal{I},n}$ the expectation under $\mathbb{P}_{\mathcal{I},n}$.

We remind the reader that we consider linear scan tests of the form (4.1). Since the intersection of linear subspaces is a convex polytope, for each linear scan test $\psi$ on the sample covariance matrix, there exists a convex polytope $S \subseteq \mathbb{R}^{d \times d}$ such that $\psi = f(1\{\hat{M} \in S\})$.

Define
\[ P_m = \{ \text{convex polytopes in } \mathbb{R}^{d \times d} \text{ with at most } m \text{ facets} \}, \]
then each set $S \in P_m$ represents a linear test. If $m \leq (nd)^p$ for some constant $p$, then the corresponding tests can be done in polynomial time. The following lemma shows that $\hat{M}$ and $\hat{B}$ cannot be distinguished by any polynomial time linear test.

**Lemma 4.6.** For any $\delta > 0$, if $m \leq (nd)^p$ and $\theta \leq \eta s^{-(1+\delta)}$ for some constant $p$ and $\eta$, then we have
\[ \sup_{I \subseteq \{1, \ldots, d\}, |I| = s} \sup_{S \in P_m} |\mathbb{P}_{\mathcal{I},n}(\hat{B} \in S) - \mathbb{P}_{\mathcal{I},n}(\hat{M} \in S)| \leq C_1 \left( \frac{\log^7(nd)}{n} \right)^{\frac{1}{2}} + C_2 s^{-\delta}, \]
where $C_1$ is a constant that only depends on $p$, and $C_2$ is an absolute constant.

**Proof.** See Section A.3 for a detailed proof.

\[ \square \]

## 5 Proof of Theorem 2.2

In this section we give the proof of Theorem 2.2. Note that by the definition of $S^*$, we only need to consider the simple zero-field ferromagnetic Ising model where all non-zero entries in $\Theta$ are the same. Let $G = (\mathcal{V}, E)$ be the underlying graph and $\theta = \theta_{ij}, (i,j) \in E$ be the parameter. Let $t = \tanh(\theta)$ and $\mathbb{E}_0$ denote the expectation under the probability measure that $X_1, \ldots, X_d$ are i.i.d. Rademacher variables. The following lemma gives an equivalent form of the probability mass function in simple zero-field ferromagnetic Ising models.

**Lemma 5.1.** For a simple zero-field Ising model with underlying graph $G = (\mathcal{V}, E)$ and parameter $\theta$, we have
\[ \mathbb{P}_\Theta(X) = \frac{\prod_{(i,j) \in E} (1 + t X_i X_j)}{2^d \mathbb{E}_0 \left[ \prod_{(i,j) \in E} (1 + t X_i X_j) \right]}, \] (5.1)
where $t = \tanh(\theta)$.
We now analyze the coefficients of each polynomial in (5.4). Let 
\[ f(t) = \sum_{k=0}^{\infty} a_k t^k, \quad f_{G'}(t) = \sum_{k=0}^{\infty} b_k t^k, \quad f_{G,G'}(t) = \sum_{k=0}^{\infty} c_k t^k. \]

We also define
\[ f_{G,G'}(t) - f_G(t) f_{G'}(t) = \sum_{k=0}^{\infty} \left( c_k - \sum_{k_1+k_2=k} a_{k_1} b_{k_2} \right) t^k = \sum_{k=0}^{\infty} u_k t^k. \]

For \( f_G(t) \), note that after expanding \( \prod_{(i,j) \in E(G)} (1 + t X_i X_j) \), the terms with non-zero expectations must have the form \( t^k X_{i_1}^2 \cdots X_{i_k}^2 \), where \( i_1, \ldots, i_k \in V \). Therefore by Lemma 5.5, the
coefficient of \( t^k \) is equal to the number of \( k \)-edge subgraphs of \( G \) where every vertex has an even degree. Similar arguments also applies to \( f_{G'}(t) \) and \( f_{G,G'}(t) \). This observation motivates us to introduce the definitions of multigraphs and Eulerian graphs.

**Definition 5.3** (Multigraph). A multigraph is a graph which is permitted to have multiple edges connecting two vertices. We denote \( G = (V,E) \), where \( V \) is the vertex set, and \( E \) is the edge multiset.

For a multigraph \( G \) with \( d \) vertices, we define its adjacency matrix to be \( A = (A_{ij})_{d \times d} \), where \( A_{ij} = A_{ji} = \text{“the number of edges connecting vertices } i \text{ and } j \text{”} \). A symmetric matrix \( A \in \mathbb{R}^{d \times d} \) with nonnegative integer off-diagonal entries and zero diagonal entries naturally represents a multigraph with vertex set \( V \). Given two multigraphs \( G \) and \( G' \) with adjacency matrices \( A \) and \( A' \), we define \( G \oplus G' \) to be the multigraph defined by \( A + A' \).

**Definition 5.4** (Eulerian graph). An Eulerian circuit on a multigraph is a closed walk that uses each edge exactly once. We say that a multigraph is Eulerian if every connected component has an Eulerian circuit.

Note that in graph theory, the term ’Eulerian graph’ has different meanings. Sometimes Eulerian graph is referred to as a graph that has an Eulerian circuit. This is different from our definition, because in this paper we do not require an Eulerian graph to be connected. The following famous lemma on Eulerian graph is first given by Euler (1741) and then completely proved by Hierholzer and Wiener (1873).

**Lemma 5.5.** A graph is Eulerian if and only if all vertices in the graph have an even degree.

Based on our previous discussion, Lemma 5.5 relates \( a_k, b_k \) and \( c_k \) to the number of \( k \)-edge Eulerian graphs. Define

\[
\mathcal{E}(k,G) := \{ \tilde{G} = (\tilde{V}, \tilde{E}) : \tilde{E} \subseteq E, |\tilde{E}| = k, \tilde{G} \text{ is an Eulerian graph}\}.
\]

In words \( \mathcal{E}(k,G) \) is the set of \( k \)-edge Eulerian subgraphs of \( G \). By Lemma 5.5 and our previous discussion, we have \( a_k = |\mathcal{E}(k,G)| \), \( b_k = |\mathcal{E}(k,G')| \) and \( c_k = |\mathcal{E}(k,G \oplus G')| \), and therefore

\[
f_G(t) = \sum_{k \geq 0} |\mathcal{E}(k,G)| t^k, \quad f_{G'}(t) = \sum_{k \geq 0} |\mathcal{E}(k,G')| t^k, \quad \text{and} \quad f_{G,G'}(t) = \sum_{k \geq 0} |\mathcal{E}(k,G \oplus G')| t^k. \quad (5.5)
\]

Figure 3 gives an example of how to calculate \( |\mathcal{E}(k,G)| \) for a given multigraph \( G \). We now proceed to analyze \( u_k \). Apparently, \( u_0 = u_1 = 0 \). For \( k \geq 2 \), by the definition of \( u_k \) we can see that, if a \( k \)-edge Eulerian subgraph of \( G \oplus G' \) can be split into two graphs \( G_1 \) and \( G_2 \) such that \( G_1 \) and \( G_2 \) are Eulerian subgraphs of \( G \) and \( G' \) respectively, then it is also counted in the sum \( \sum_{k_1+k_2=k} a_{k_1} b_{k_2} \) and therefore is not counted in \( u_k \). Figure 4 gives examples of Eulerian subgraphs that are counted and not counted in \( u_k \). Using this type of argument, the following two lemmas together calculate and bound \( u_k \) for \( k \geq 2 \).
Figure 3: An example of the calculation of \(\{|E(k,G)|\}_{k\geq 1}\) for a multigraph \(G\) is given in (a). We use red, green and orange edges to highlight 2-edge, 4-edge and 6-edge Eulerian subgraphs of \(G\) respectively. (b), (c) give the 2-edge Eulerian subgraphs; (d), (e), (f) give the 4-edge Eulerian subgraphs; and (g), (h) give the 6-edge Eulerian subgraphs. We have \(|E(2,G)| = 2, |E(4,G)| = 3, |E(6,G)| = 2,\) and \(|E(k,G)| = 0\) for \(k \neq 2, 4, 6.\)

Lemma 5.6. We have

\[
u_2 = |E(G) \cap E(G')|, \quad \nu_3 = \Delta_{G,G'}, \quad \text{and} \quad \nu_k \leq q_k[G \oplus G', V(G) \cap V(G')] \quad \text{for} \quad k \geq 4,
\]

where the function \(q_k(\cdot, \cdot)\) is defined as follows:

\[
q_k(G, V) := |\{\tilde{G} \in \mathcal{E}(k,G) : \exists i, j \in V, i, j \text{ are contained in one connected component of } \tilde{G}\}|.
\]

Lemma 5.7. We have

\[
|\mathcal{E}(k,G)| \leq 2^k \|A_G\|_F^k, \quad \Delta_{G,G'} \leq 2|V(G) \cap V(G')| \cdot \mathcal{R} \cdot \Gamma.
\]

Moreover, for any multigraph \(G\) and vertex set \(V \subseteq \overline{V}\), we have

\[
q_k(G, V) \leq (2^k \cdot |V| \cdot \|A_G\|_F^k) \wedge \left[k \cdot 2^{k-2} \cdot |V|^2 \cdot (\|A_G\|_1 \vee \|A_G\|_F)^{k-2}\right].
\]

The upper bound of \(|\mathcal{E}(k,G)|\) in Lemma 5.7 and the assumption that \(\theta \leq [8(\Lambda \vee \Gamma)]^{-1}\) together show that \(f_G(t), f_{G'}(t)\) and \(f_{G,G'}(t)\), as power series, all converge. Moreover, by the definition of \(\mathcal{B}\), the upper bound for \(q_k(\cdot, \cdot)\) in Lemma 5.7 and the assumption that \(\theta \leq [8(\Lambda \vee \Gamma)]^{-1}\), we have

\[
\sum_{k \geq 4} q_k(G \oplus G', V(G) \cap V(G'))\theta^k \leq |V(G) \cap V(G')| \cdot \mathcal{B}\theta^4.
\]

By Lemmas 5.6 and 5.7 and the fact that \(t = \tanh(\theta) \leq \theta\), we have

\[
f_{G,G'}(t) - f_G(t)f_{G'}(t) \leq |E(G) \cap E(G)|\theta^2 + \Delta_{G,G'}\theta^3 + |V(G) \cap V(G')| \cdot \mathcal{B}\theta^4
\leq |V(G) \cap V(G')| \cdot (\mathcal{R} + 2\mathcal{R} \cdot \Gamma\theta + \mathcal{B}\theta^2) \cdot \theta^2. \quad (5.6)
\]
Figure 4: Illustration of graphs counted and not counted in $u_k$. The gray dot-dashed squares highlight the non-isolated vertices of $G$ and $G'$. The solid and dashed lines are edges in $G$ and $G'$ respectively. We use purple vertices to represent the common non-isolated vertices of $G$ and $G'$. The blue vertices are non-isolated in $G$ but isolated in $G'$, and the red vertices are non-isolated in $G'$ but isolated in $G$. The green edges in (a) give an example of a 6-edge Eulerian subgraph of $G \oplus G'$ counted in $u_6$, while the orange edges in (b) form a 6-edge Eulerian subgraph of $G \oplus G'$ that is not counted in $u_6$.

Note that $f_G(t)f_{G'}(t) \geq 1$ for $t \geq 0$ since all coefficients $a_k$ and $b_k$ are non-negative. By (5.4), (5.6) and the assumption that $\theta \leq [8(\Lambda \vee \Gamma)]^{-1} \leq (2\Gamma)^{-1}$ and $\theta \leq \sqrt{\mathcal{R}/\mathcal{B}}$, we have

$$
\mathbb{E}_0 \left[ \frac{\mathbb{P}_\Theta \mathbb{P}_{\Theta'}}{\mathbb{P}_0 \mathbb{P}_0} \right] \leq 1 + 3|V(G) \cap V(G')|\mathcal{R}\theta^2.
$$

Plugging the inequality above into the definition of $\chi^2$-divergence (5.2) gives

$$
D_{\chi^2}(\mathbb{P}, \mathbb{P}_{0,n}) \leq \frac{1}{|S^*|^2} \sum_{\Theta, \Theta' \in S^*} \left[ 1 + 3|V(G) \cap V(G')| \cdot \mathcal{R}\theta^2 \right]^n - 1. \quad (5.7)
$$

To complete the proof, we invoke the incoherence condition of $G^*$. We summarize the result as the following lemma.

**Lemma 5.8.** If $G^*$ is incoherent, then the following inequality holds.

$$
\frac{1}{|S^*|^2} \sum_{\Theta, \Theta' \in S^*} \exp[3n\mathcal{R}|V(G) \cap V(G')|\theta^2] \leq \exp[N(G^*) \cdot \exp(3n\mathcal{R}\theta^2)].
$$

Now by (5.7), Lemma 5.2 and Lemma 5.8, if $\theta \leq \sqrt{\log|N^{-1}(G^*)|^{-1}}/m\mathcal{R}$ and $N(G^*) = o(1)$, we have

$$
\liminf_{n \to \infty} \gamma(S^*) = 1.
$$
6 Proof of Theorem 3.3

In this section we give the proof of Theorem 3.3. The key part of our proof is to derive concentration inequalities for $W_H$. Following the definition in Vershynin (2010), we define the $\psi_1$-norm of the random variable $Z$ as follows.

$$\|Z\|_{\psi_1} := \sup_{p \geq 1} p^{-1}(\mathbb{E}|Z|^p)^{1/p}.$$ 

If a random variable $Z$ has finite $\psi_1$-norm, we say $Z$ is a sub-exponential random variable.

The following lemma gives bounds for the $\psi_1$-norm of $W_H$.

**Lemma 6.1.** Let $X \in \{\pm 1\}^d$ be a random vector generated from the high temperature ferromagnetic Ising model with parameter matrix $\Theta$. For any graph $H$, define

$$W_H := \frac{1}{|E(H)|} \sum_{(i,j) \in E(H)} X_iX_j.$$ 

If $\|\Theta\|_F \leq 1/2$, then we have $\|W_H\|_{\psi_1} \leq C|E(H)|^{-1/2}$, where $C > 0$ is an absolute constant.

We first prove that $\mathbb{P}_{0,n}(\psi = 1) < \alpha/2$. Under the null, $X_1, \ldots, X_d$ are independent Rademacher random variables. Therefore for every $H \in \mathcal{H}$ we have $\mathbb{E}_{0,n}\hat{W}_H = 0$. By Lemma 6.1 with $\Theta = (0)_{d \times d}$, we have $\|W_H\|_{\psi_1} \leq C_1|E(H)|^{-1/2}$ for an absolute constant $C_1 > 0$. By Proposition 5.16 in Vershynin (2010), for $\varepsilon \leq |E(H)|^{-1/2}$ we have

$$\mathbb{P}_{0,n}\left(|\hat{W}_H - \mathbb{E}_{0,n}\hat{W}_H| > \varepsilon\right) \leq 2 \exp(-C_2 \cdot |E(H)| \cdot n\varepsilon^2),$$

where $C_2$ is an absolute constant. Setting the right-hand side above to be $\alpha/(2|\mathcal{H}|)$ and solving for $\varepsilon$ shows that under the null hypothesis, with probability at least $1 - \alpha/(2|\mathcal{H}|)$, we have

$$\hat{W}_H \leq C_3 \sqrt{\frac{\log |\mathcal{H}| + \log(2/\alpha)}{|E(H)|n}} \leq C_3 \sqrt{\frac{2\log |\mathcal{H}|}{|E(H)|n}},$$

for absolute constant $C_3$. Note that the condition $\varepsilon \leq |E(H)|^{-1/2}$ is satisfied since we assume that $\log(|\mathcal{H}|)/n = o(1)$. By definition, we have $m(\mathcal{H}) \leq |V(H)|$. Moreover, we have

$$\mathcal{R} \leq \frac{|E(H)|}{|V(H)|} + 1 \leq \frac{2|E(H)|}{|V(H)|} + 1 \leq \frac{4|E(H)|}{|V(H)|},$$

where the last inequality follows by $2|E(H)| \geq |V(H)|$. Therefore with probability at least $1 - \alpha/(2|\mathcal{H}|)$,

$$\hat{W}_H \leq C_4 \sqrt{\frac{\log |\mathcal{H}|}{m(\mathcal{H})} \cdot \frac{|V(H)|}{|E(H)|} \cdot \frac{1}{n}} \leq C_5 \sqrt{\frac{M(\mathcal{H})}{\mathcal{R}n}},$$

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where $C_4, C_5$ are absolute constants. Therefore by union bound, when $\kappa$ is chosen to be a large enough constant we have $\mathbb{P}_{0,n}(\psi = 1) \leq \alpha/2$.

For any $\Theta \in \mathcal{S}(G_1, \theta)$ with corresponding graph $G$, we now prove that $\mathbb{P}_{\Theta,n}(\psi = 0) < \alpha/2$. By the definition of witnessing set and (6.1), there exists $H \in \mathcal{H}$ which is a subgraph of $G$ and we have $|E(H)|/|V(H)| \geq R/4$. It now suffices to prove that

$$\mathbb{P}_{\Theta,n}(\hat{W}_H \leq \frac{\kappa}{4} \sqrt{\frac{M(\mathcal{H})}{Rn}}) \leq \frac{\alpha}{2}.$$ 

Since $H$ is a subgraph of $G$, each edge in $E(H)$ is also an edge in $E(G)$. By the second Griffiths inequality (see Griffiths (1967); Kelly and Sherman (1968)), for $\theta \leq 1$ we have

$$E_{\Theta,n} \hat{W}_H \geq \tanh(\theta) \geq \frac{\theta}{2}.$$ 

Applying Lemma 6.1 gives $\|W_H\|_{\psi_1} \leq C_6|E(H)|^{-1/2}$ for an absolute constant $C_6$. By Proposition 5.16 in Vershynin (2010), for $\eta \leq |E(H)|^{-1/2}$ we have

$$\mathbb{P}_{\Theta,n}(\|\hat{W}_H - E_{\Theta,n} \hat{W}_H\| > \eta) \leq 2 \exp(-C_7 \cdot |E(H)| \cdot n\eta^2),$$

for an absolute constant $C_7$. Therefore with probability at least $1 - \alpha/2$, we have

$$\hat{W}_H \geq E_{\Theta} \hat{W}_H - C_8 \sqrt{\log(2/\alpha) \over |E(H)|n} \geq \frac{\theta}{2} - C_9 \sqrt{\frac{|V(H)|}{m(\mathcal{H})} \cdot \frac{\log(|\mathcal{H}|)}{|E(H)|n}} > \left( \frac{\kappa}{2} - C_{10} \right) \sqrt{M(\mathcal{H}) \over Rn},$$

where $C_8, C_9$ and $C_{10}$ are absolute constants. Therefore when $\kappa$ is chosen as a large enough constant we have $\mathbb{P}_{\Theta,n}(\psi = 0) \leq \alpha/2$, and

$$\mathbb{P}_{0,n}(\psi = 1) + \sup_{\Theta \in \mathcal{S}(G_1, \theta)} \mathbb{P}_{\Theta,n}(\psi = 0) \leq \alpha.$$ 

This completes the proof.

7 Discussion

In this paper we studied structure detection problems in zero-field ferromagnetic Ising models. Our upper and lower bounds demonstrated that graph arboricity is a key concept which drives the testability of structure detection. We furthermore argued that under a sparse PCA conjecture no polynomial time linear tests on the covariance matrix can test the problem unless the signal strength is of the order of $1/\sqrt{n}$, which is statistically sub-optimal for graphs with high arboricity.

There are several important questions which we leave for future work. First, our upper bound results are derived under the assumption that $\|\Theta\|_F \leq \frac{1}{2}$. This assumption is needed
to ensure that the terms (3.1) concentrate around their mean value. This may not be a necessary condition, and we anticipate that the tests we develop might work beyond this regime.

Second, an interesting question that is left open is whether one can develop upper and lower bounds for problems of the type (1.5) in the dense regime when $s \gg \sqrt{d}$. We believe that this regime may require completely different tests than the ones we developed in this paper.

Finally, our computational lower bound, which relies on the sparse PCA conjecture, works only for linear tests on the covariance matrix. As we mentioned earlier, the computational hardness of sparse PCA conjecture has been established under the widely believed planted clique conjecture (Gao et al., 2014; Berthet and Rigollet, 2013a; Brennan et al., 2018). It will be interesting to extend our results beyond linear tests on the covariance matrix. We currently do not know of a way to prove such a result based on the planted clique conjecture. However, our results under the oracle computational model strongly suggest that indeed it is unlikely that polynomial time tests for detection exist when the signal strength is of smaller order than $\frac{1}{\sqrt{n}}$. 
Appendix

A   Proofs

A.1   Lower Bound Proofs

We first introduce two important lemmas.

Lemma A.1. For a multigraph $G = (\tilde{V}, \tilde{E})$, define the following two classes of Eulerian spanning subgraphs and connected Eulerian subgraphs of $G$ with $k$ edges.

$$E_c(k, G) := \{ \tilde{G} = (\tilde{V}, \tilde{E}) : \tilde{V} \subseteq V, \tilde{E} \subseteq E, |\tilde{E}| = k, \tilde{G} \text{ is a connected Eulerian graph} \},$$

$$E(k, G) := \{ \tilde{G} = (V, \tilde{E}) : \tilde{E} \subseteq E, |\tilde{E}| = k, \tilde{G} \text{ is an Eulerian graph} \}.$$ 

Let $A$ be the adjacency matrix of $G$. Then for $k \geq 2$, we have

$$|E_c(k, G)| \leq \|A\|_F^k, \text{ and } |E(k, G)| \leq 2^k \|A\|_F^k.$$

Proof. For the first inequality, note that we have

$$(A^k)_{(i,i)} = \sum_{r_1, \ldots, r_{k-1} \in V} A_{ir_1} A_{r_1 r_2} \cdots A_{r_{k-2} r_{k-1}} A_{r_{k-1} i},$$

which is the number of length-$k$ closed walks starting at vertex $i$. \hfill (A.1)

Summing up all possible starting vertices, we get

$$|E_c(k, G)| \leq |\{\text{length-}k \text{ closed walks in } G\}| \leq \text{Tr}(A^k) \leq \|A\|_F^k.$$

This proves the first inequality. For the second inequality, we use induction. First for $|E(2, G)|$, we have

$$|E(2, G)| = |E_c(2, G)| \leq \|A\|_F^2 \leq 2^2 \|A\|_F^2.$$

Suppose that for $l \leq k$ we have $|E(l, G)| \leq 2^l \|A\|_F^l$. Then for $|E(k + 1, G)|$, by the fact that $E(1, G) = E_c(1, G) = \emptyset$, we have

$$|E(k + 1, G)| \leq \sum_{l=2}^{k-1} |E_c(l, G)| \cdot |E(k + 1 - l, G)| + |E_c(k + 1, G)|.$$

Plugging in the inequalities for $|E(l, G)|$, we get

$$|E(k + 1, G)| \leq \sum_{l=2}^{k-1} \|A\|_F^l \cdot 2^{k+1-l}\|A\|_F^{k+1-l} + \|A\|_F^{k+1}$$

$$\leq \|A\|_F^{k+1} \cdot \left( \sum_{l=2}^{k-1} 2^{k+1-l} + 1 \right)$$

$$\leq 2^{k+1}\|A\|_F^{k+1}.$$

Therefore by induction we get the second inequality. \hfill $\square$
Lemma A.2. Let $G$ be a multigraph with vertex set $V = \{1, \ldots, d\}$ and adjacency matrix $A$. Let $V \subseteq \overline{V}$ be a vertex set. For $k \geq 2$, we define
\[ p_k(G, V) = \left| \{ \tilde{G} \in \mathcal{E}_c(k, G) : \tilde{G} \text{ contains at least two distinct vertices in } V \} \right|, \]
\[ q_k(G, V) = \left| \{ \tilde{G} \in \mathcal{E}(k, G) : \exists i, j \in V, i, j \text{ are contained in one connected component of } \tilde{G} \} \right|. \]
Then we have
\begin{align*}
p_k(G, V) &\leq (k - 1) \cdot |V|^2 \cdot \|A\|_1^{k-2}, \quad \text{(A.2)} \\
q_k(G, V) &\leq \left( 2^k \cdot |V| \cdot \|A\|_F^{k-2} \right) \wedge \left[ k \cdot 2^{k-2} \cdot |V|^2 \cdot (\|A\|_1 \lor \|A\|_F)^{k-2} \right]. \quad \text{(A.3)}
\end{align*}

Proof. We first prove (A.2). By definition, we have
\[ p_k(G, V) \leq |V| \cdot (|V| - 1) \cdot \max_{i,j \in V} \left\{ \tilde{G} \in \mathcal{E}_c(k, G) : \tilde{G} \text{ contains vertices } i \text{ and } j \right\} \]
\[ \leq |V|^2 \cdot \max_{i,j \in V} |\{\text{length-}k \text{ closed walks in } G \text{ starting at } i \text{ and traversing } j\}|. \]
Note that each vertex can have at most $\|A\|_1$ neighbors. Therefore we can bound the number of length-$k$ Eulerian circuits starting at vertex $i$ and containing vertex $j$ by counting the possible vertices on the walk:
- The number of possible positions of vertex $j$ in $V$ is $k - 1$.
- The number of choices of the rest $k - 2$ vertices is at most $\|A\|_1^{k-2}$.
This completes the proof of (A.2).

Now we prove (A.3). Suppose that $\tilde{G}$ is a subgraph of $G$ with $k$ edges such that one of its connected components contains at least two distinct vertices in $V$. Let $l$ be the number of edges of this connected component. Then by definition, clearly the rest connected components form a graph in $\mathcal{E}(k - l, G)$. Therefore we have
\[ q_k(G, V) \leq \sum_{l=2}^{k-2} p_l(G, V) \cdot |\mathcal{E}(k - l, G)| + p_k(G, V). \]
By (A.2) and Lemma A.1, we have
\[ q_k(G, V) \leq \sum_{l=2}^{k-2} (l - 1) \cdot |V|^2 \cdot \|A\|_1^{l-2} \cdot 2^{k-l} \cdot \|A\|_F^{k-l} + k \cdot |V|^2 \cdot \|A\|_1^{k-2} \]
\[ \leq |V|^2 \cdot (\|A\|_1 \lor \|A\|_F)^{k-2} \cdot \left( \sum_{l=2}^{k-2} (l - 1) \cdot 2^{k-l} + k \right) \]
\[ \leq k \cdot 2^{k-2} \cdot |V|^2 \cdot (\|A\|_1 \lor \|A\|_F)^{k-2}, \]
where the last inequality holds because for $l \geq 2$ we have $l - 1 \leq 2^{l-2}$. Moreover, for $V \neq \emptyset$, by Lemma A.1, clearly we have $q_k(G, V) \leq |\mathcal{E}(k, G)| \leq 2^k \|A\|_F^{k} \leq 2^k \cdot |V| \cdot \|A\|_F^{k}$. When $V = \emptyset$, by definition we have $q_k(G, V) = 2^k \cdot |V| \cdot \|A\|_F^{k} = 0$. This completes the proof. □
Proof of Lemma 5.1. For any $i, j \in V$, we have

$$\exp(\theta X_i X_j) = \cosh(\theta X_i X_j) + \sinh(\theta X_i X_j) = \cosh(\theta X_i X_j)[1 + \tanh(\theta X_i X_j)].$$

Note that $\cosh(x)$ is an even function, and $X_i X_j$ is binary. Therefore we have $\cosh(\theta X_i X_j) \equiv \cosh(\theta)$. Similarly, $\tanh(x)$ is an odd function, by checking the function values at $X_i X_j = 1$ and $X_i X_j = -1$ we obtain $\tanh(\theta X_i X_j) = \tanh(\theta) X_i X_j$. Therefore we have

$$\exp(\theta X_i X_j) = c(1 + t X_i X_j), \quad (A.4)$$

where $c = \cosh(\theta)$ and $t = \tanh(\theta)$. Plugging (A.4) into the definition of $P_{\Theta}(X)$ proves (5.1).

Proof of Lemma 5.2. Define

$$\bar{P} = \frac{1}{|S^*|} \sum_{\Theta \in S^*} P_{\Theta,n},$$

then by Neyman-Pearson’s lemma we have

$$\gamma(S^*) \geq \inf_{\psi} \left[ P_0(\psi = 1) + \bar{P}(\psi = 0) \right] = 1 - \text{TV}(\bar{P}, P_{0,n}),$$

where $\text{TV}(\bar{P}, P_{0,n}) := \max_{A \subseteq \{\pm 1\}^n} |P(A) - P_{0,n}(A)|$ is the total variation distance between $\bar{P}$ and $P_{0,n}$. Note that for total variation distance we have

$$\text{TV}(\bar{P}, P_{0,n}) = \frac{1}{2} \sum_{X \in \{\pm 1\}^n} |\bar{P}(X) - P_{0,n}(X)| = \frac{1}{2} \sum_{X \in \{\pm 1\}^n} \left| \frac{\bar{P}(X)}{P_{0,n}(X)} - 1 \right| \cdot P_{0,n}(X).$$

Applying Cauchy-Schwartz inequality to the right-hand side above gives

$$\text{TV}(\bar{P}, P_{0,n}) \leq \frac{1}{2} \sqrt{\mathbb{E}_{0,n} \left\{ \left[ \frac{\bar{P}(X)}{P_{0,n}(X)} - 1 \right]^2 \right\}} = \frac{1}{2} \sqrt{\mathbb{E}_{0,n} \left[ \frac{\bar{P}^2(X)}{P_{0,n}^2(X)} \right] - 1}.$$

It then suffices to show that

$$\mathbb{E}_{0,n} \left[ \frac{\bar{P}^2(X)}{P_{0,n}^2(X)} \right] = \frac{1}{|S^*|^2} \sum_{\Theta, \Theta' \in S^*} \mathbb{E}_{0,n} \left[ \frac{P_{\Theta,n} P_{\Theta',n}}{P_{0,n} P_{0,n}} \right],$$

which follows by direct calculation. \qed

Proof of Lemma 5.6. Since there cannot be multiple edges in $G$ connecting the same two vertices, the coefficient of $t^2$ in $f_G(t)$ is 0. For the same reason the coefficient of $t^2$ in $f_{G'}(t)$ is also 0. In $f_{G,G'}(t)$, the only possible way to form a two-edge Eulerian circuit is to pick one
We now prove that, for \( \tilde{u} \) there exists a number \( u_2 = |E(G) \cap E(G')| \).

For \( u_3 \), note that 3-edge Eulerian subgraphs must be triangles. If a triangle only uses edges in \( E(G) \), then it is counted in the coefficient of \( t^3 \) in \( f_G(t) \). Similarly, if a triangle only uses edges in \( G' \), it is also counted in the coefficient of \( t^3 \) in \( f_{G'}(t) \). Therefore \( u_3 \) is the number of triangles that use at least one edge in \( E(G) \) and another edge in \( E(G') \), which is defined as \( \Delta_{G,G'} \).

We denote by \( \mathcal{E}(G) \) and \( \mathcal{E}(G') \) the sets of Eulerian subgraphs of \( G \) and \( G' \) respectively. For \( k \geq 4 \), by (3.5), the coefficient of \( t^k \) in \( f_G(t) f_{G'}(t) \) is equal to

\[
|\{ \tilde{G} \in \mathcal{E}(k, G \oplus G') : \exists G_1 \in \mathcal{E}(G), G_2 \in \mathcal{E}(G') \text{ s.t. } \tilde{G} = G_1 \oplus G_2 \}|.
\]

We now prove that, for \( \tilde{G} \in \mathcal{E}(k, G \oplus G') \), if each connected component contains at most one vertex in \( V(G) \cap V(G') \), then there exist \( G_1 \in \mathcal{E}(G) \) and \( G_2 \in \mathcal{E}(G') \) such that \( \tilde{G} = G_1 \oplus G_2 \). To prove this statement, take a fixed connected component of \( \tilde{G} \). Suppose first that the connected component does not contain any vertices in \( V(G) \cap V(G') \). Then it follows that all of its edges must be contained either in \( E(G) \) or \( E(G') \). Next consider the case when the connected component contains only one vertex \( v \in V(G) \cap V(G') \). Since this connected component must be a connected Eulerian graph, we can consider the Eulerian circuit starting and ending at \( v \). If we start walking along the circuit on an edge in \( E(G) \), then since \( v \) is the only vertex contained in the intersection \( V(G) \cap V(G') \), we cannot reach vertices in \( E(G') \) until we return to \( v \). Upon returning to \( v \), we have completed a closed walk using purely edges in \( G \). We can continue this process to obtain closed walks on \( G \) and \( G' \) starting and ending at \( v \). Concatenating all the closed walks on \( G \) gives \( G_1 \). Similarly, concatenating all the closed walks on \( G' \) gives \( G_2 \). We have proved that

\[
\mathcal{E}(k, G \oplus G') \setminus \{ \tilde{G} \in \mathcal{E}(k, G \oplus G') : \exists G_1 \in \mathcal{E}(G), G_2 \in \mathcal{E}(G') \text{ s.t. } \tilde{G} = G_1 \oplus G_2 \} \subseteq \{ \tilde{G} \in \mathcal{E}(k, G) : \exists i, j \in V(G) \cap V(G'), i, j \text{ are contained in one connected component of } \tilde{G} \}.
\]

Therefore by the definition of \( q_k(\cdot, \cdot) \) we have \( u_k \leq q_k(G \oplus G', V(G) \cap V(G')) \). \( \square \)

**Proof of Lemma 5.7.** The bounds for \( \mathcal{E}(k, G) \) and \( q_k(G, V) \) are included in Lemma A.1 and Lemma A.2. We now prove the bound for \( \Delta_{G,G'} \). We remind the reader that for a graph \( G \) and a vertex set \( V \), \( G_V \) denotes the graph obtained by restricting \( G \) on the vertex set \( V \). Note that if a triangle has one edge in \( E(G) \) and two edges in \( E(G') \), then the two vertices of the edge in \( E(G) \) must be in \( V(G) \cap V(G') \). Therefore, an upper bound of the number of triangles that have one edge in \( E(G) \) and two edges in \( E(G') \) is given by the following procedure:

- Pick an edge \( e \) from \( E[G_{V(G) \cap V(G')}]. \)
• Pick a common neighbour of the two vertices of edge $e$.

Since all graphs in $\mathcal{G}^*$ have arboricity $\mathcal{R}$, by the definition of arboricity we have

$$\Delta_{G,G'} \leq |E[G_{V(G) \cap V(G')}]| \cdot \|A_G\|_1 + |E[G'_{V(G) \cap V(G')}]| \cdot \|A_G\|_1 \leq 2|V(G) \cap V(G')| \cdot \mathcal{R} \cdot \Gamma.$$ 

This completes the proof. \[\square\]

**Proof of Lemma 5.8.** Let

$$A(G^*) = \frac{1}{|S^*|^2} \sum_{\Theta, \Theta' \in S^*} \exp[3n\mathcal{R}|V(G) \cap V(G')|\theta^2].$$

Then we have

$$A(G^*) \leq \max_{\Theta \in S^*} \frac{1}{|S^*|} \sum_{\Theta' \in S^*} \exp \left\{ 3n\mathcal{R}\theta^2 \cdot \sum_{v \in V(G)} 1[v \in V(G')] \right\}.$$

Consider drawing $\Theta'$ uniformly from $S^*$, and let $\mathbb{P}_{\Theta' \sim U(S^*)}$ be the probability measure. By assumption, the random variables $\{1[v \in V(G')] \mid v \in V(G)\}$ are negatively associated. Therefore

$$A(G^*) \leq \max_{\Theta \in S^*} \mathbb{E}_{\Theta' \sim U(S^*)} \prod_{v \in V(G)} \exp \left\{ 3n\mathcal{R}\theta^2 \cdot 1[v \in V(G')] \right\}$$

$$\leq \max_{\Theta \in S^*} \prod_{v \in V(G)} \mathbb{E}_{\Theta' \sim U(S^*)} \exp \left\{ 3n\mathcal{R}\theta^2 \cdot 1[v \in V(G')] \right\}.$$ 

Expanding the expectation and applying the inequality $1 + x \leq \exp(x)$ gives

$$A(G^*) \leq \max_{\Theta \in S^*} \prod_{v \in V(G)} \left\{ \exp \left( 3n\mathcal{R}\theta^2 \right) \mathbb{P}_{\Theta' \sim U(S^*)}[v \in V(G')] + 1 - \mathbb{P}_{\Theta' \sim U(S^*)}[v \in V(G')] \right\}$$

$$\leq \max_{\Theta \in S^*} \prod_{v \in V(G)} \exp \left\{ \left[ \exp \left( 3n\mathcal{R}\theta^2 \right) - 1 \right] \mathbb{P}_{\Theta' \sim U(S^*)}[v \in V(G')] \right\}.$$ 

Rearranging terms, we get

$$A(G^*) \leq \max_{\Theta \in S^*} \exp \left\{ \left[ \exp \left( 3n\mathcal{R}\theta^2 \right) - 1 \right] \cdot \sum_{v \in V(G)} \mathbb{P}_{\Theta' \sim U(S^*)}[v \in V(G')] \right\}$$

$$\leq \exp \left\{ \exp \left( 3n\mathcal{R}\theta^2 \right) \cdot \max_{\Theta \in S^*} \mathbb{E}_{\Theta' \sim U(S^*)} |V(G) \cap V(G')| \right\}$$

$$= \exp[N(G^*) \cdot \exp(3n\mathcal{R}\theta^2)].$$

This completes the proof. \[\square\]
Proof of Corollary 2.5. Let \( \mathcal{G}^* \) be the set of graphs isomorphic to \( G_* \). Then clearly, if \( G' \) is uniformly sampling from \( \mathcal{G}^* \), then \( \{ \mathbb{1}[i \in V(G')] \}_{i=1}^{d} \) is just a permutation of \( s \) 1s and \( d-s \) 0s. Therefore by Theorem 2.11 in Joag-Dev and Proschan (1983), the incoherence condition is satisfied. For any \( G \in \mathcal{G}^* \) and \( v \in V(G) \), we have

\[
\mathbb{E}_{G' \sim U(G^*)} |V(G) \cap V(G')| = \sum_{i \in V(G)} \mathbb{E}_{G' \sim U(G^*)} \mathbb{1}[i \in V(G')] = s \cdot s/d = s^2/d.
\]

And therefore \( N(\mathcal{G}^*) = s^2/d \). Moreover, by definition we have

\[
\mathcal{R} = \mathcal{R}(G_*), \quad V_{\max} = s, \quad \Lambda = \|A_{G_*}\|_F, \quad \Gamma = \|A_{G_*}\|_1, \quad \text{and} \quad \mathcal{B} = \mathcal{B}(G_*).
\]

Therefore by Theorem 2.2, if

\[
\theta \leq \sqrt{\frac{\log(d/s^2)}{6n \mathcal{R}(G_*)}} \land \sqrt{\frac{\mathcal{R}(G_*)}{\mathcal{E}(G_*)}} \land \frac{1}{8(\|A_{G_*}\|_F \lor \|A_{G_*}\|_1)}
\]

then we have

\[
\liminf_{n \to \infty} \gamma(S^*) = 1.
\]

\[\square\]

A.2 Upper Bound Proofs

The following lemma given by Bhattacharya and Mukherjee (2015) is helpful for bounding the \( \psi_1 \)-norm of \( W_H \).

Lemma A.3. Let \( J \) be a \( d \times d \) symmetric matrix with non-negative off-diagonal entries and zeros on the diagonal. If \( \|J\|_2 \leq 1 \), then we have

\[
\sum_{1 \leq i,j \leq d} \log \cosh(J_{ij}) \leq \log \mathbb{E}_0 \exp \left( \frac{1}{2} X^T J X \right) \leq -\frac{1}{2} \sum_{i=1}^{n} \log[1 - \lambda_i(J)],
\]

where \( \lambda_1(J), \ldots, \lambda_d(J) \) are the eigenvalues of \( J \).

Proof of Lemma 6.1. By (5.16) in Vershynin (2010) as an equivalent definition of \( \psi_1 \)-norm, it suffices to prove

\[
\mathbb{E}_{\Theta} \exp \left( \frac{\sqrt{2} |E(H)|^{1/2}}{8} \cdot W_H \right) \leq e. \quad (A.5)
\]
To prove (A.5), first note that we have \( \|A_H\|_F^2 = 2|E(H)| \). By definition of the Ising model, we have

\[
\mathbb{E}_\Theta \exp \left( \frac{\sqrt{2}|E(H)|^{1/2}}{8} \cdot W_H \right) = \mathbb{E}_\Theta \exp \left( \frac{\|A_H\|_F}{8} \cdot W_H \right) = \frac{\mathbb{E}_0 \exp(X^T J X/2)}{\mathbb{E}_0 \exp(X^T \Theta X/2)},
\]

where \( J := \Theta + A_H/(4\|A_H\|_F) \). Therefore,

\[
\log \mathbb{E}_\Theta \exp \left( \frac{\|A_H\|_F}{8} \cdot W_H \right) = \log \mathbb{E}_0 \exp \left( \frac{1}{2} X^T J X \right) - \log \mathbb{E}_0 \exp \left( \frac{1}{2} X^T \Theta X \right). \tag{A.6}
\]

By Lemma A.3, we have

\[
\log \mathbb{E}_0 \exp \left( \frac{1}{2} X^T J X \right) \leq -\frac{1}{2} \sum_{i=1}^n \log[1 - \lambda_i(J)] \leq \frac{1}{2} \sum_{i=1}^n [\lambda_i(J) + 2\lambda_i^2(J)],
\]

where the second inequality holds because for \( |x| \leq 3/4 \) we have \(- \log(1-x) = \sum_{k \geq 1} x^k / k \leq x + 2x^2\) and by assumption we have \( \|J\|_2 \leq \|J\|_F \leq 3/4 \). Since \( \text{Tr}(J) = \text{Tr}(A_H)/(2\|A_H\|_F) = 0 \), we have

\[
\log \mathbb{E}_0 \exp \left( \frac{1}{2} X^T J X \right) \leq \|J\|_F^2 \leq \frac{9}{16}. \tag{A.7}
\]

Moreover, since \( \theta_{ij} \geq 0 \) for all \( i, j = 1, \ldots, d \), by Lemma A.3 clearly we have

\[
\log \mathbb{E}_0 \exp(X^T \Theta X/2) \geq 0. \tag{A.8}
\]

Plugging (A.7) and (A.8) into (A.6), we obtain

\[
\log \mathbb{E}_\Theta \exp \left( \frac{\|A_H\|_F}{8} \cdot W_H \right) \leq \frac{9}{16}.
\]

Therefore by (5.16) in Vershynin (2010) as an equivalent definition of \( \psi_1 \)-norm, we have \( \|W_H\|_{\psi_1} \leq C|E_H|^{-1/2} \) for an absolute constant \( C \).

**A.3 Computational Lower Bound Proofs**

**Lemma A.4.** For odd \( r \), we have \( \alpha_r(\theta) = \beta_r(\sigma) = 0 \). Moreover, if \( \theta \leq \eta s^{-(1+\delta)} \) for some small enough constant \( \eta > 0 \), then for even \( r \), we have

\[
t^{r/2} \leq \alpha_r \leq 2(r-1)!!(8t)^{r/2}, \quad \text{and} \quad (r-1)!!(2\sigma/\pi)^{r/2}(1 + r\sigma)^{-1/2} \leq \beta_r \leq (r-1)!!(2\sigma/\pi)^{r/2},
\]

where \( t = \tanh(\theta) \).
Proof. Note that changing signs of all entries in $X_1$ and $Z_1$ does not change the value of Ising model probability mass function or the Gaussian probability density function. Therefore, when $r$ is odd, by symmetry it is obvious that $\alpha_r(\theta) = \beta_r(\sigma) = 0$. Since we always focus on the first samples $X_1$ and $Z_1$, in the rest of the proof we omit the subscript ”1”. When $r$ is even, for $\alpha_2$, by second Griffith inequality we have $\alpha_2 \geq t$ and $\alpha_r \geq \alpha_r^{r/2} \geq t^{r/2}$. Moreover, let $G$ be the underlying clique graph, and $E = \{(i, j) : i, j \in \mathcal{I}\}$ be the edge set of $G$. Then by (5.1), we have

$$\alpha_r = \frac{\mathbb{E}_0[X_{i_1} \cdots X_{i_r} \prod_{(i,j) \in E}(1 + tX_iX_j)]}{\mathbb{E}_0[\prod_{(i,j) \in E}(1 + tX_iX_j)]} \leq \mathbb{E}_0 \left[ X_{i_1} \cdots X_{i_r} \prod_{(i,j) \in E} (1 + tX_iX_j) \right].$$

For any even $r$, let

$$g_r(t) = \mathbb{E}_0 \left[ X_{i_1} \cdots X_{i_r} \prod_{(i,j) \in E} (1 + tX_iX_j) \right] = \sum_{k \geq r/2} a_{r,k} t^k.$$

Then similar to our discussion in the proof of Theorem 2.2, by expanding the product, we see that $a_{r,k}$ counts the number of terms of the form

$$t^k X_{i_{k_1}}^2 \cdots X_{i_r}^2 \cdot X_{i_{k_{r/2}}}^2 \cdots X_{i_{k-r/2}}^2,$$

where $i_1', \ldots, i_{k-r/2}' \in \mathcal{I}$. Therefore by Lemma 5.5, $a_{r,k}$ equals the number of subgraphs of $G$ satisfying the following properties:

(i) After removing all connected components that do not contain any of $i_1, i_2, \ldots, i_r$, the remaining edges can be organized to represent $r/2$ paths, each connecting a distinct pair of vertices among $i_1, i_2, \ldots, i_r$.

(ii) The connected components that do not contain any of $i_1, i_2, \ldots, i_r$ form an Eulerian subgraph.

(iii) Total number of edges is $k$.

Without loss of generality, we assume that $i_1 < i_2 < \cdots < i_r$. Then for each graph counted in $a_{r,k}$ described by (i)-(iii) above, we can denote by $(j_1, j_2), (j_3, j_4), \ldots, (j_{r-1}, j_r)$ the pairs of nodes among $i_1, \ldots, i_r$ that are connected by the $r/2$ paths, where $j_1, j_2, \ldots, j_r$ are chosen as follows:

- Let $j_1 = i_1$.
- Pick $j_2$ to be the smallest index such that there exists a path connecting $j_1$ and $j_2$.
- Pick $j_3$ to be the smallest index among $\{i_1, \ldots, i_r\}\{j_1, j_2\}$.
• Pick $j_4$ to be the smallest index such that there exists a path connecting $j_3$ and $j_4$.

... 

• Pick $j_{r-1}$ to be the smallest index among $\{i_1, \ldots, i_r\}\{j_1, \ldots, j_{r-2}\}$.

• Pick $j_r$ to be the last index that have not been chosen.

For any graph $G$ satisfying the descriptions (i)-(iii) and the corresponding $j_1, \ldots, j_r$ chosen above, adding the edges $(j_2, j_3), (j_4, j_5), \ldots, (j_r, j_1)$ results in an Eulerian (multi)graph, and the resulting (multi)graph has only one connected component that contains $j_1, \ldots, j_r$. This connected component represents a closed walk starting and ending at vertex $j_1$. Therefore, each graph counted in $a_{r,k}$ described above can be characterized by

• $r/2$ index pairs $(j_2, j_3), (j_4, j_5), \ldots, (j_r, j_1)$ with $j_1 < j_3 < j_5 < \cdots < j_{r-1}$,

• a closed walk $C$ starting and ending at $j_1$, and

• an Eulerian subgraph $G'$ that does not contain any of $j_1, \ldots, j_r$.

It is obvious that the edge set $E_{\text{added}} = \{(j_2, j_3), (j_4, j_5), \ldots, (j_r, j_1)\}$ and $C$ uniquely determines the connected components of $G$ that contains any of $i_1, \ldots, i_r$. Therefore, $G$ is uniquely determined by the 3-tuple $[E_{\text{added}}, C, G']$, and the number of graphs $G$ counted in $a_{r,k}$ is bounded by the number of possible 3-tuples $[E_{\text{added}}, C, G']$, which can be counted as follows.

• There are $(r-1)!!$ different ways to split $i_1, \ldots, i_r$ into pairs, which gives an upper bound of $|E_{\text{added}}|.$

• If the length of the closed walk $C$ is $l$, then the number of possible positions of $j_2, \ldots, j_r$ is upper bounded by $\binom{l-1}{r-1}$.

• The number of choices of the rest $l-r$ vertices is upper bounded by $s^{l-r}$.

• The number of possible choices of $G'$ is at most $|\mathcal{E}(k+r/2-l, G)|$, where $\mathcal{E}(k+r/2-l, G)$ defined in Lemma A.1 denotes the set of $(k+r/2-l)$-edge Eulerian subgraphs of $G$. 

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Therefore we have
\[
a_{r,k} \leq (r-1)!! \sum_{l=r}^{k+r/2} \binom{l-1}{r-1} s^{l-r} \cdot |\mathcal{E}(k + r/2 - l, G)|
\]
\[
\leq (r-1)!! \sum_{l=r}^{k+r/2} 2^{l-1}s^{l-r} \cdot 2^{k+r/2-l} \|A_G\|_{\mathcal{F}}^{k+r/2-l}
\]
\[
\leq (r-1)!! \sum_{l=r}^{k+r/2} 2^{l-1}s^{l-r} \cdot 2^{k+r/2-l}s^{k-r/2}
\]
\[
\leq (r-1)!!(k + r/2) \cdot 2^{k+r/2-l}s^{k-r/2}
\]
\[
\leq (r-1)!! 8^k s^{k-r/2}.
\]

Therefore, as long as \(\eta \leq 1/16\), we have
\[
\alpha_r \leq g_r(t) = \sum_{k \geq r/2} a_{r,k} t^k \leq 2(r-1)!! \cdot (8t)^{r/2}.
\]

When \(r\) is even, for \(\beta_r\) by symmetry we have
\[
\beta_r = \sum_{l=0}^{r} \binom{r}{l} (-1)^{r-l} \mathbb{P}(Z_1, \ldots, Z_l \geq 0, Z_{l+1}, \ldots, Z_r < 0).
\]

Let \(W_0, W_1, \ldots, W_r\) to be \(r+1\) i.i.d standard normal random variables. Then \((Z_1, \ldots, Z_r)^T \overset{d}{=} (W_1 + \sqrt{\sigma} W_0, \ldots, W_r + \sqrt{\sigma} W_0)\). Let \(p\) be the standard normal density function, then
\[
\mathbb{P}(Z_1, \ldots, Z_l \geq 0, Z_{l+1}, \ldots, Z_r < 0) = \int_{\mathbb{R}} \left[\mathbb{P}(W_1 \geq -\sqrt{\sigma} w)\right]^l \left[\mathbb{P}(W_{l+1} \leq -\sqrt{\sigma} w)\right]^{r-l} p(w) dw
\]
\[
= \int_{\mathbb{R}} \left[\mathbb{P}(W_1 \geq -\sqrt{\sigma} w)\right]^l \left[\mathbb{P}(W_1 \geq \sqrt{\sigma} w)\right]^{r-l} p(w) dw.
\]

Plugging the equation above into (A.9) gives
\[
\beta_r = \int_{\mathbb{R}} \left[\mathbb{P}(W_1 \geq -\sqrt{\sigma} w) - \mathbb{P}(W_1 \geq \sqrt{\sigma} w)\right]^r p(w) dw
\]
\[
= \int_{\mathbb{R}} \left[\mathbb{P}(-\sqrt{\sigma} w \leq W_1 \leq \sqrt{\sigma} w)\right]^r p(w) dw. \quad (A.10)
\]

Note that
\[
\mathbb{P}(-\sqrt{\sigma} w \leq W_1 \leq \sqrt{\sigma} w) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\sigma} w}^{\sqrt{\sigma} w} \exp \left(-\frac{1}{2} t^2\right) dt,
\]

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and for $t \in [-\sqrt{\sigma w}, \sqrt{\sigma w}]$ we have $\exp(-\sigma w^2/2) \leq \exp(-t^2/2) \leq 1$. Therefore we have

$$2\sqrt{\frac{\sigma}{2\pi}} \exp(-\sigma w^2/2) \leq \mathbb{P}(-\sqrt{\sigma w} \leq W_1 \leq \sqrt{\sigma w}) \leq 2\sqrt{\frac{\sigma}{2\pi}}.$$ 

by the fact that the $r$-th moment of standard normal distribution is $(r-1)!!$, we have

$$(2\sigma/\pi)^{r/2} \int_{\mathbb{R}} w^r \exp(-r\sigma w^2/2)p(w)dw \leq \beta_r \leq (r-1)!!(2\sigma/\pi)^{r/2}.$$ 

For the left-hand-side above, we have

$$\int_{\mathbb{R}} w^r \exp(-r\sigma w^2/2)p(w)dw = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} w^r \exp\left(-\frac{r\sigma w^2}{2}\right) \exp\left(-\frac{w^2}{2}\right) dw$$

$$= \frac{(1 + r\sigma)^{-\frac{r}{2}}}{\sqrt{2\pi(1 + r\sigma)^{-1}}} \int_{\mathbb{R}} w^r \exp\left(-\frac{w^2}{2(1 + r\sigma)^{-1}}\right) dw$$

$$= (1 + r\sigma)^{-\frac{r}{2}} \cdot (1 + r\sigma)^{-\frac{r}{2}} \cdot (r - 1)!! \geq (r - 1)!! \cdot (1 + r\sigma)^{-\frac{r}{2}}.$$

This completes the proof.

**Proof of Lemma 4.5.** By Lemma A.4, we have

$$(2\sigma/\pi) \cdot (1 + 2\sigma)^{-\frac{3}{2}} \leq \beta_2(\sigma) \leq 2\sigma/\pi$$

Since $(1 + x)^c$ with a constant $c \geq 1$ is convex for $x > -1$, by the fact that $f(x) \geq f(0) + f'(0)x$ for convex function $f(x)$, we have $(1 + 2\sigma)^{-\frac{3}{2}} \geq 1 - 3\sigma$. Therefore

$$2(\sigma - 3\sigma^2)/\pi \leq \beta_2(\sigma) \leq 2\sigma/\pi.$$ 

Moreover, for any $\theta < \eta_s^{-1+\delta}$ and $t = \tanh(\theta) \leq \theta$, by Lemma A.4 we have

$$t \leq \alpha_2(\theta) \leq 16t.$$ 

Let $\sigma_1 = \pi t/2$ and $\sigma_2 = 16\pi t$. Then as long as $\eta \leq (96\pi)^{-1}$, we have $\sigma_2 \leq 1/6$, and therefore we have

$$\beta_2(\sigma_1) \leq 2\sigma_1/\pi = t \leq \alpha_2(\theta), \text{ and}$$

$$\beta_2(\sigma_2) \geq 2(\sigma_2 - 3\sigma_2^2)/\pi = 2\sigma_2(1 - 3\sigma_2)/\pi \geq \sigma_2/\pi = 16t \geq \alpha_2(\theta).$$ 

By (A.10), $\beta_2(\sigma)$ is an increasing continuous function of $\sigma$. Therefore there exists $\sigma \in [\sigma_1, \sigma_2]$ such that $\beta_2(\sigma) = \alpha_2(\theta)$. 

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We now prove that for the \( \sigma \) chosen as above we have
\[
(C_1 t)^{r/2} \leq \beta_r(\sigma), \alpha_r(\theta) \leq (r - 1)!!(C_2 t)^{r/2}.
\]
for some constant \( C_1 \) and \( C_2 \). The bounds for \( \alpha_r(\theta) \) follows directly by Lemma A.4. For \( \beta_r(\sigma) \), by Lemma A.4 we have
\[
(r - 1)!!(2\sigma/\pi)^{r/2}(1 + r\sigma)^{-\frac{r+1}{2}} \leq \beta_r \leq (r - 1)!!(2\sigma/\pi)^{r/2}.
\]
Note that for \( \eta < (16\pi)^{-1} \), we have
\[
\sigma \leq \sigma_2 = 16\pi t \leq s^{-(1+\delta)} \leq r^{-1},
\]
since by definition \( r \leq s \). Therefore
\[
(1 + r\sigma)^{-\frac{r+1}{2}} \geq (1 + r\sigma)^{-r} \geq 2^{-r},
\]
and
\[
(t/4)^{r/2} \leq [\sigma/(2\pi)]^{r/2} \leq \beta_r \leq (r - 1)!!(2\sigma/\pi)^{r/2} \leq (r - 1)!!(32t)^{r/2}.
\]
This completes the proof.

**Proof of Lemma 4.6.** For any fixed \( \mathcal{I} \), let \( M = \sqrt{n}\hat{M} \), \( B = \sqrt{n}\hat{B} \). By definition, \( \mathcal{P}^m \) is invariant to scaling, meaning that for any \( S \in \mathcal{P}^m \) and any constant \( c \), \( cS \) is also contained in \( \mathcal{P}^m \). Therefore we have
\[
\sup_{S \in \mathcal{P}^m} |\mathbb{P}_{\mathcal{I},n}(\hat{B} \in S) - \mathbb{P}_{\mathcal{I},n}(\hat{M} \in S)| = \sup_{S \in \mathcal{P}^m} |\mathbb{P}_{\mathcal{I},n}(B \in S) - \mathbb{P}_{\mathcal{I},n}(M \in S)|.
\]
We proceed to give an upper bound of \( \sup_{S \in \mathcal{P}^m} |\mathbb{P}_{\mathcal{I},n}(B \in S) - \mathbb{P}_{\mathcal{I},n}(M \in S)| \). Since we only need to focus on a fixed \( \mathcal{I} \), to simplify notation, in the rest of the proof we omit the subscript and denote \( \mathbb{P} = \mathbb{P}_{\mathcal{I},n} \), \( \mathbb{E} = \mathbb{E}_{\mathcal{I},n} \). Since \( B \) and \( M \) are symmetric matrices, we only need to consider the strict upper triangular part of the matrices. Note that \( \beta_2 = \alpha_2 \), so \( EB = EM \).

We now calculate the covariances between entries in \( B \) and \( M \). For \( B \), we give the following calculation:

- If \( i_1 \) and \( j_1 \) are both in the clique, then
  - \( \text{Var}(B_{i_1j_1}) = 1 - \beta_2^2 \).
  - If \( i_2 \) and \( j_2 \) are both in the clique and \( |\{i_1, j_1\} \cap \{i_2, j_2\}| = 0 \), then \( \text{Cov}(B_{i_1j_1}, B_{i_2j_2}) = \beta_4 - \beta_2^2 \).
  - If \( i_2 \) and \( j_2 \) are both in the clique and \( |\{i_1, j_1\} \cap \{i_2, j_2\}| = 1 \), then \( \text{Cov}(B_{i_1j_1}, B_{i_2j_2}) = \beta_2 - \beta_2^2 \).
If \( i_2 \) and \( j_2 \) are not both in the clique, then \( \text{Cov}(B_{i_1j_1}, B_{i_2j_2}) = 0. \)

- If \( i_1 \) is in the clique and \( j_1 \) is not in the clique, then
  
  \[
  \text{Var}(B_{i_1j_1}) = 1.
  \]
  
  - If \( i_2 \) and \( j_2 \) are both in the clique, then \( \text{Cov}(B_{i_1j_1}, B_{i_2j_2}) = 0. \)
  
  - If \( i_2 \) is in the clique, \( j_2 \) is not in the clique, and \( j_2 = j_1 \), then \( \text{Cov}(B_{i_1j_1}, B_{i_2j_2}) = \beta_2. \)
  
  - If \( i_2 \) is in the clique, \( j_2 \) is not in the clique, and \( j_2 \neq j_1 \), then \( \text{Cov}(B_{i_1j_1}, B_{i_2j_2}) = 0. \)
  
  - If neither \( i_2 \) nor \( j_2 \) is in the clique, then \( \text{Cov}(B_{i_1j_1}, B_{i_2j_2}) = 0. \)

Similarly, the covariances between entries in \( M \) follows the exact same pattern as \( B \), except all \( \beta_2 \) and \( \beta_4 \)'s are replaced by \( \alpha_2 \) and \( \alpha_4 \). Let \( \Theta_1, \Theta_2 \in \mathbb{R}^{[d(d-1)/2] \times [d(d-1)/2]} \) be the covariance matrices of the strict upper triangular part of \( B \) and \( M \) respectively, and let \( B^* \) and \( M^* \) be the symmetric Gaussian matrices whose strict upper triangular part is generated from Gaussian distributions whose means are the same as \( B \)'s (or \( M \)'s, since \( \beta_2 = \alpha_2 \)) and covariance matrices are \( \Theta_1 \) and \( \Theta_2 \) respectively. Then by Proposition 3.1 in Chernozhukov et al. (2014), we have

\[
\sup_{S \in \mathcal{P}^m} |\mathbb{P}(B \in S) - \mathbb{P}(B^* \in S)| + \sup_{S \in \mathcal{P}^m} |\mathbb{P}(M \in S) - \mathbb{P}(M^* \in S)| \leq C_1 \left( \frac{\log^7(\sqrt{d})}{n} \right)^{\frac{1}{6}}, \quad (A.11)
\]

where \( C_1 \) is a constant that only depends on \( p \). We now bound \( \text{TV}(\mathbb{P}_{B^*}, \mathbb{P}_{M^*}) \). Since the \( B^* \) and \( M^* \) have the same means, by Pinsker’s inequality, we have

\[
\text{TV}(\mathbb{P}_{B^*}, \mathbb{P}_{M^*}) \leq \sqrt{\frac{1}{2} D_{\text{KL}}(\mathbb{P}_{B^*} || \mathbb{P}_{M^*})} = \sqrt{\frac{1}{4} \left\{ \text{Tr}(\Theta_2^{-1} \Theta_1 - I) + \log \left[ \frac{\det(\Theta_2)}{\det(\Theta_1)} \right] \right\}}.
\]

Let \( \widetilde{\Theta}_1 = I - \Theta_1, \widetilde{\Theta}_2 = I - \Theta_2 \). We first prove that \( \| \widetilde{\Theta}_1 \|_2, \| \widetilde{\Theta}_2 \|_2 < 1 \). To prove this bound, we go over each rows of \( \widetilde{\Theta}_1 \) and \( \widetilde{\Theta}_2 \), and use the Gershgorin disc theorem. For \( \widetilde{\Theta}_1 \), by previous calculation, we have

- If \( i_1 \) and \( j_1 \) are both in the clique, then \( \widetilde{\Theta}_{1,(i_1j_1),(i_1j_1)} = \beta_2^2 \). Moreover, in the \((i_1,j_1)\)-th row of \( \widetilde{\Theta}_1 \), there are at most \( s^2 \) off-diagonal entries of \( \beta_2^2 - \beta_4 \) and at most \( 2s \) off-diagonal entries of \( \beta_3^2 - \beta_2 \).

- If \( i_1 \) is in the clique and \( j_1 \) is not in the clique, then \( \widetilde{\Theta}_{1,(i_1j_1),(i_1j_1)} = 0 \). Moreover, in the \((i_1,j_1)\)-th row of \( \widetilde{\Theta}_1 \), there are at most \( 2s \) off-diagonal entries of \(-\beta_2 \).

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We now view $\Theta$ walks:

To bound $\text{Tr}(\Theta)$, therefore we have

By our calculation and the fact that $1 \geq \beta_4 \geq \beta_2^2$. Therefore all entries in $\Theta_1$ and $\Theta_2$ are non-negative. By the Gershgorin disc theorem, (4.3) and the assumption that $\theta \leq \eta s^{-(1+\delta)}$ for some small enough positive constant $\eta$, we have

$$\|\tilde{\Theta}_1\|_2 \leq \beta_2^2 + s^2 \beta_4 + 2s \beta_2 < 1.$$  

With the same proof we have $\|\tilde{\Theta}_2\|_2 < 1$. Therefore, $\Theta_2^{-1} = (I - \tilde{\Theta}_2)^{-1} = \sum_{k=0}^{\infty} \tilde{\Theta}_2^k$, and

$$\text{Tr}(\Theta_2^{-1}\Theta_1 - I) = \sum_{k=0}^{\infty} \text{Tr}[\tilde{\Theta}_2^k(\Theta_1 - \Theta_2)].$$

We now view $\Theta_1 - \Theta_2$ and $\tilde{\Theta}_2$ as weighted graphs. In general, for matrices $A^{(1)}, \ldots, A^{(k)} \in \mathbb{R}^{m \times m}$ with nonnegative entries, $\text{Tr}(A^{(1)} \cdots A^{(k)}) = \sum_{r=1}^{m} \sum_{r_1, \ldots, r_k} A^{(1)}_{r_1} A^{(2)}_{r_1 r_2} \cdots A^{(k)}_{r_k}$ equals the weighted sum of all closed walks that use the $l$-th weighted edge from $A^{(l)}$. Note that, by our calculation and the fact that $\beta_2 = \alpha_2$, the only nonzero entries in $\Theta_1 - \Theta_2$ are $\Theta_{1,(i_1 j_1),(i_2 j_2)} - \Theta_{2,(i_1 j_1),(i_2 j_2)}$, where $i_1, j_1, i_2, j_2$ are four distinct indices in the clique. Also, in $\tilde{\Theta}_2$, for $i_1, j_1$ in the clique, $(i_1, j_1)$ is only connected to $(i_2, j_2)$ if $i_2, j_2$ are both in the clique. To bound $\text{Tr}[\tilde{\Theta}_2^k(\Theta_1 - \Theta_2)]$, we have the following analysis on the weighted sum of closed walks:

- Denote by $(i_1, j_1), \ldots , (i_k+1, j_{k+1})$ the vertices on the closed walk. We can choose the starting point $(i_1, j_1)$ of the closed walk such that we first walk along $k$ edges from $\tilde{\Theta}_2$ until we reach the vertex $(i_{k+1}, j_{k+1})$, and then we walk from $(i_{k+1}, j_{k+1})$ back to $(i_1, j_1)$ along an edge in $\Theta_1 - \Theta_2$.

- Whatever $(i_{k+1}, j_{k+1})$ is, there are less than $s^2$ choices of $(i_1, j_1)$ in $\Theta_1 - \Theta_2$ that are connected to $(i_{k+1}, j_{k+1})$, and the weights are all $\beta_4 - \alpha_4$.

- Given the choice of $(i_1, j_1)$, there are less than $s^2$ choices of $(i_2, j_2)$ in $\tilde{\Theta}_2$ with weight $\alpha_4 - \alpha_2^2$, and less than $2s$ choices of $(i_2, j_2)$ with weight $\alpha_2 - \alpha_2^2$.

- Given the choice of $(i_k, j_k)$, there are less than $s^2$ choices of $(i_{k+1}, j_{k+1})$ in $\tilde{\Theta}_2$ with weight $\alpha_4 - \alpha_2^2$, and less than $2s$ choices of $(i_{k+1}, j_{k+1})$ in $\tilde{\Theta}_2$ with weight $\alpha_2 - \alpha_2^2$.

Therefore, we have

$$\text{Tr}[\tilde{\Theta}_2^k(\Theta_1 - \Theta_2)] \leq s^2(\beta_4 - \alpha_4) \cdot (s^2 \alpha_4 + 2s \alpha_2)^k \leq (C_1 s \theta)^{k+2},$$

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where $C_1$ is an absolute constant. Hence $\text{Tr}(\Theta_2^{-1}\Theta_1 - I) = \text{Tr}\left[\tilde{\Theta}_2^k(\Theta_1 - \Theta_2)\right] = O(s^{-2\delta})$. Let $\lambda_1, \ldots, \lambda_{d(d-1)/2}$ be the eigenvalues of $\tilde{\Theta}_1$. Then by expanding the logarithm terms we obtain

$$\log[\det(\Theta_1)] = \sum_{i=1}^{d(d-1)/2} \log(1 - \lambda_i) = \sum_{i=1}^{d(d-1)/2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (-\lambda_i)^k = \sum_{k=1}^{\infty} \frac{-1}{k} \text{Tr}(\tilde{\Theta}_1^k).$$

Therefore we have

$$\log \left[ \frac{\det(\Theta_2)}{\det(\Theta_1)} \right] = \sum_{k=1}^{\infty} \frac{1}{k} \text{Tr}(\tilde{\Theta}_1^k - \tilde{\Theta}_2^k).$$

Since the entries where $i_1, j_1$ are both in the clique and where $i_1, j_1$ are not both in the clique are in different connected components, and $\Theta_1, \Theta_2$ are exactly the same on any path that consists of vertices that are not all in the clique, it suffices to bound the weighted sum of paths that only use vertices in the clique. Therefore, similar to previous proof, we have

$$|\text{Tr}(\tilde{\Theta}_2^k - \tilde{\Theta}_1^k)| \leq (s^2\alpha_4 + 2s\alpha_2)^k + (s^2\beta_4 + 2s\beta_2)^k \leq (C_3s\theta)^k,$$

where $C_3$ is an absolute constant. Therefore by the assumption that $\theta \leq \eta s^{-(1+\delta)}$ for some small enough positive constant $\eta$, we have

$$\log \left[ \frac{\det(\Theta_2)}{\det(\Theta_1)} \right] = \sum_{k=1}^{\infty} \frac{1}{k} \text{Tr}(\tilde{\Theta}_1^k - \tilde{\Theta}_2^k) \leq C_3 s^{-\delta}, \quad (A.12)$$

Summing (A.11) and (A.12) completes the proof. \qed

**Proof of Theorem 4.3.** We remind the reader the following notations for clarification.

- $\mathbb{P}_{0,n}$ is the probability measure under the hypothesis that $X_1, \ldots, X_n$ are independent Rademacher random vectors.
- $\mathbb{P}_\Theta,n$ is the probability measure under the hypothesis that $X_1, \ldots, X_n$ are independent samples generated from the Ising model with parameter matrix $\Theta$.
- $\mathbb{P}_I,n$ is the probability measure under the hypothesis that $Z_1, \ldots, Z_n$ are independent standard Gaussian vectors.
- $\mathbb{P}_\Sigma,n$ is the probability measure under the hypothesis that $Z_1, \ldots, Z_n$ are independent Gaussian vectors with mean zero and covariance matrix $\Sigma$.
- $\mathbb{P}_I,n$ is the joint probability measure under the assumption that $X_1, \ldots, X_n$ are independent samples generated from the Ising model with parameter matrix $\theta[1(i, j \in I, i \neq j)]_{d \times d}$, and $Z_1, \ldots, Z_n$ are independent Gaussian vectors with mean zero and covariance matrix $I + \sigma 1_I 1_I^T$, where $\sigma = O(\theta)$ is chosen such that (4.3) holds.
Now suppose that there exists a polynomial linear scan test \( \psi = \psi(\hat{M}) \) such that
\[
\liminf_{n \to \infty} \left[ \mathbb{P}_{0,n}(\psi = 1) + \max_{\Theta \in S[G_1(G_s), \theta]} \mathbb{P}_{\Theta,n}(\psi = 0) \right] < \frac{1}{4}.
\]
By the definition of \( S[G_1(G_s), \theta] \), when \( \theta \leq (2s)^{-1} \) we have \( S_{\text{clique}}^* \subseteq S[G_1(G_s), \theta] \), where
\[
S_{\text{clique}}^* := \{ \theta \cdot [1(i, j \in \mathcal{I}, i \neq j)]_{d \times d} : \mathcal{I} \subseteq \{1, \ldots, d\}, |\mathcal{I}| = s \}.
\]
For any \( \Theta \in S_{\text{clique}}^* \) and the corresponding samples \( X_1, \ldots, X_n \), by definition of \( P_{I,n} \) we have
\[
\liminf_{n \to \infty} \left[ \mathbb{P}_{0,n}(\psi = 1) + \max_{\mathcal{I} \subseteq \{1, \ldots, d\}, |\mathcal{I}| = s} \mathbb{P}_{\mathcal{I},n}(\psi = 0) \right] < \frac{1}{4}.
\]
We now consider the corresponding sparse PCA problem with \( \sigma \in [\pi t/2, 16\pi t] \) chosen such that (4.3) holds. By definition, the matrix \( \hat{B} = \frac{1}{n} \sum_{i=1}^{n} Y_i Y_i^T \) can be calculated in polynomial time. We then construct a polynomial time test \( \psi_{\text{SPCA}} := \psi(\hat{B}) \). By Lemma 4.6 we have
\[
\max_{\mathcal{I} \subseteq \{1, \ldots, d\}, |\mathcal{I}| = s} \mathbb{P}_{\mathcal{I},n}(\psi_{\text{SPCA}} = 0) \leq \max_{\mathcal{I} \subseteq \{1, \ldots, d\}, |\mathcal{I}| = s} \mathbb{P}_{\mathcal{I},n}(\psi = 0) + C_1 \left( \frac{\log^7(n^d)}{n} \right)^{\frac{1}{6}} + C_2 s^{-\delta}
\]
for some absolute constants \( C_1 \) and \( C_2 \). Therefore we have
\[
\liminf_{n \to \infty} \max_{\mathcal{I} \subseteq \{1, \ldots, d\}, |\mathcal{I}| = s} \mathbb{P}_{\mathcal{I},n}(\psi_{\text{SPCA}} = 0) < \frac{1}{4} - \liminf_{n \to \infty} \mathbb{P}_{0,n}(\psi = 1)
\]
Moreover, clearly we have
\[
\mathbb{P}_{I,n}(\psi_{\text{SPCA}} = 1) = \mathbb{P}_{0,n}(\psi = 1).
\]
Therefore
\[
\liminf_{n \to \infty} \left[ \mathbb{P}_{I,n}(\psi_{\text{SPCA}} = 1) + \max_{\Sigma \in \mathcal{S}_s} \mathbb{P}_{\Sigma,n}(\psi_{\text{SPCA}} = 0) \right] < \frac{1}{4},
\]
which contradicts our sparse PCA conjecture. This completes the proof. \( \square \)

**B  Computational Lower Bound Under Oracle Computational Model**

In this section we propose an oracle computational model, based on which we derive another computational lower bound result for detection problems in Ising model. The main idea of oracle computational model is to use the number of rounds of interactions between data and a certain algorithm to represent the algorithmic complexity of this algorithm. In specific, let \( X \) be the random vector of interest and \( \mathcal{X} \) be the domain of \( X \). We define
\[
\mathcal{Q}^* = \{ q : q(X) \text{ is a sub-exponential variable} \}.
\]
We call every subset \( \mathcal{Q} \subseteq \mathcal{Q}^* \) a query space. Next we define the statistical query oracle.
Definition B.1 (statistical query oracle). Let \( n \) be the sample size of a testing problem. A statistical query oracle \( r_n \) on a query space \( Q \subseteq Q^* \) is a random mapping from \( Q \) to \( \mathbb{R} \). Given a query \( q \in Q^* \), the oracle \( r_n \) returns an output \( Z_q \in \mathbb{R} \), such that for any tail probability \( \xi \in [0, 1) \),
\[
P\left( \bigcap_{q \in Q} \left\{ |Z_q - \mathbb{E}[q(X)]| \leq \|q(X)\|_{\psi_1} \cdot \tau \right\} \right) \geq 1 - 2\xi,
\]
where
\[
\tau = \max \left\{ \frac{\eta(Q) + \log(1/\xi)}{n}, \sqrt{\frac{2[\eta(Q) + \log(1/\xi)]}{n}} \right\}.
\]  
(B.2)

Here we call \( \eta(Q) > 0 \) the capacity measure of \( Q \). When \( Q \) is finite, we define \( \eta(Q) = \log(|Q|) \).

Given a query space \( Q \subseteq Q^* \), we define \( R_n(Q) \) to be the set of all statical query oracles on \( Q \) with sample size \( n \). We now give the definition of oracle computational model.

Definition B.2 (oracle computational model). An oracle computational model \( \Psi \) is defined as a tuple \( \Psi = \Psi(Q_\Psi, T_\Psi, q_{\text{init}}, \{\delta_t\}_{t=1}^{T_\Psi}, \psi) \), where

- \( Q_\Psi \) is a subset of \( Q^* \) that contains all queries the test will potentially use.
- \( T_\Psi \) is the maximum number of rounds the model queries an oracle.
- \( q_{\text{init}} \in Q \) is the initial query.
- \( \delta_t : (Q \times \mathbb{R})^{t-1} \to Q \cup \{\text{HALT}\} \) is the transition function at the \( t \)-th round. If \( \delta_t \) returns \text{HALT}, then the model stops querying the oracle.
- \( \psi : (Q \times \mathbb{R})^{T_\Psi} \to \{0, 1\} \) is the test function that takes the results of at most \( T_\Psi \) queries as input, and returns the test result as binary output.

Each instance of \( \Psi(Q_\Psi, T_\Psi, q_{\text{init}}, \{\delta_t\}_{t=1}^{T_\Psi}, \psi) \) refers to a test algorithm. The parameter \( T_\Psi \) is the query complexity of algorithm \( \Psi \). We define \( \mathcal{A}(T) = \{ \Psi : T_\Psi \leq T \} \) to be the set of all algorithms with query complexity at most \( T \). Under oracle computational model, the risk of detection problem (1.5) with maximum query complexity \( T \) is defined as
\[
\gamma_{\text{oracle}}\{S[G_1(G_*, \theta)]\} = \inf_{\Psi \in \mathcal{A}(T)} \sup_{r_n \in R_n(Q_\Psi)} \left\{ \mathbb{P}_{0,n}(\psi = 1) + \max_{\theta \in S[G_1(G_*, \theta)]} \mathbb{P}_{\theta,n}(\psi = 0) \right\}
\]  
(B.3)

Note that in (B.3), the supreme over \( r \in R_n(Q_\Psi) \) implies that we consider the worst oracle. If \( \liminf_{n \to \infty} \gamma_{\text{oracle}}\{S[G_1(G_*, \theta)]\} = 1 \), then when \( n \) is large enough, for any algorithm that queries at most \( T \) rounds, there exists an oracle \( r_n \) such that the algorithm cannot distinguish the null and alternative hypotheses. We now give our main result.
**Theorem B.3.** Let $G_s$ be a graph with $s$ vertices. Under the statistical query model, if $T \leq d^p$ for some constant $p > 0$, $s \leq d^{(1-n)/2}$ for some constant $\eta > 0$, and

$$\theta \leq \kappa \sqrt{\frac{1}{n} \wedge \frac{1}{16s}}, \quad (B.4)$$

where $\kappa$ is some sufficiently small positive constant, then $\liminf_{n \to \infty} \gamma_{\text{oracle}} \{S[G_1(G_s), \theta]\} = 1.$

**Proof of Theorem B.3.** We denote by $G_\emptyset$ the empty graph. Similar to the computational lower bound analysis in Section 4, we only need to consider the case where $G_s$ is an $s$-clique. Therefore we set $G^*$ to be the set of graphs isomorphic to $G_s$, and let $S^* = \{\theta A_G : G \in G^*\}$. Each parameter matrix $\Theta \in S^*$ can be represented by a graph $G \in G^*$. In the following, we always denote by $\Theta$ the parameter matrix with underlying graph $G$, and by $\Theta'$ the parameter matrix with underlying graph $G'$. For a graph $G$, in order to successfully detect it with the worst-case oracle, a test has to utilize at least one query $q$ that can distinguish $G$ from $G_\emptyset$. We define

$$G(q) = \{G \in G^* : \|E_\Theta q(X) - E_0 q(X)\| \geq \|q(X)\|_{\psi} \cdot \tau\},$$

where $\|q(X)\|_{\psi}$ is the $\psi_1$-norm of $q(X)$ when $X$ follows the distribution $P_0$, and $\tau$ is defined in Definition B.1. By the definition of $G(q)$, if $T \cdot \sup_{q \in Q_0} |G(q)| < |G^*|$, then there must be some $G' \in G^*$ such that none of the $T$ queries used by the test can distinguish $G$ from $G_\emptyset$. Therefore the worst case oracle that returns $E_{G_\emptyset} q(X)$ when $X \sim P_0$ can still satisfy Definition B.1 but will make all the tests powerless. This gives the following lemma.

**Lemma B.4.** For any algorithm $\Psi$ that queries the oracle at most $T$ rounds, if $T \cdot \sup_{q \in Q_\Psi} |G(q)| < |G^*|$, then there exists an oracle $r_n \in R_n(Q_\Psi)$ defined in Definition B.1 such that $\liminf_{n \to \infty} \gamma_{\text{oracle}} \{S[G_1(G_s), \theta]\} \geq 1$.

**Proof.** See Section B.1 for a detailed proof. \qed

By Lemma B.4, to prove $\liminf_{n \to \infty} \gamma_{\text{oracle}} \{S[G_1(G_s), \theta]\} = 1$, it suffices to show that $T \cdot \sup_{q \in Q_\Psi} |G(q)|/|G^*|$ is asymptotically smaller than one. In the rest of the proof, for any $q \in Q_\Psi$, we derive an upper bound on $|G(q)|$. To do so, we first split $G(q)$ into two subsets $G^+(q)$ and $G^-(q)$, which are given by

$$G^+(q) = \{G \in G^* : E_\Theta q(X) - E_0 q(X) > \|q(X)\|_{\psi} \cdot \tau\}, \quad (B.5)$$

$$G^-(q) = \{G \in G^* : E_0 q(X) - E_\Theta q(X) > \|q(X)\|_{\psi} \cdot \tau\}. \quad (B.6)$$

We now bound $|G^+(q)|, |G^-(q)|$ can be bounded in exactly the same way. The following lemma summarizes an inequality derived from the definition (B.5).
Lemma B.5. For any query function $q$, we have
\[
\frac{1}{|G^+(q)|^2} \sum_{G,G' \in G^+(q)} \mathbb{E}_0 \left[ \frac{dP_\theta dP'_\theta}{dP_0 dP_0} \right] > 1 + \frac{1}{n}. \tag{B.7}
\]

Proof. See Section B.1 for a detailed proof. \qed

It remains to calculate the left-hand side of (B.7). By Lemma 5.6, we have
\[
\mathbb{E}_0 \left[ \frac{P_\theta P'_\theta}{P_0 P_0} \right] \leq 1 + |E(G) \cap E(G')|\theta^2 + \Delta_{G,G'}\theta^3 + \sum_{k \geq 4} q_k[G \oplus G', V(G) \cap V(G')]\theta^k.
\]

For $|E(G) \cap E(G')|$, we use the trivial bound that $|E(G) \cap E(G')| \leq |V(G) \cap V(G')|^2/2$. For $q_k[G \oplus G', V(G) \cap V(G')]$, $k \geq 4$, we apply the bound given by Lemma 5.7 and obtain
\[
q_k[G \oplus G', V(G) \cap V(G')] \leq k \cdot 2^{k-2} \cdot |V(G) \cap V(G')|^2 \cdot (\|\Lambda_{G \oplus G'}\|_1 \lor \|\Lambda_{G \oplus G'}\|_F)^{k-2} \\
\leq k \cdot 2^{k-2} \cdot |V(G) \cap V(G')|^2 \cdot (2s)^{k-2} \\
\leq 2^{k-2} \cdot 2^{k-2} \cdot |V(G) \cap V(G')|^2 \cdot (2s)^{k-2} \\
= 8^{k-2} \cdot s^{k-2} \cdot |V(G) \cap V(G')|^2.
\]

Therefore by the assumption that $\theta \leq (16s)^{-1}$, we have
\[
\sum_{k \geq 4} q_k[G \oplus G', V(G) \cap V(G')]\theta^k \leq 64|V(G) \cap V(G')|^2 s^2 \theta^4 \leq |V(G) \cap V(G')|^2 \theta^2 / 4.
\]

For $\Delta_{G,G'}$, we use a bound similar to Lemma 5.7 but more specific for cliques. If a triangle has one edge in $E(G)$ and two edges in $E(G')$, then the two vertices of the edge in $E(G)$ must be in $V(G) \cap V(G')$. Therefore, an upper bound of the number of triangles that have one edge in $E(G)$ and two edges in $E(G')$ is given by the following procedure:

- Pick an edge $e$ from $E[G_{V(G) \cap V(G')}]$.
- Pick a common neighbour of the two vertices of edge $e$.

Therefore by the trivial bound $|E[G_{V(G) \cap V(G')}]| \leq |V(G) \cap V(G')|^2/2$, we have
\[
\Delta_{G,G'} \leq |E[G_{V(G) \cap V(G')}]| \cdot \|\Lambda_{G'}\|_1 + |E[G'_{V(G) \cap V(G')}]| \cdot \|\Lambda_{G}\|_1 \leq |V(G) \cap V(G')|^2 \cdot s.
\]

Therefore, we have
\[
\mathbb{E}_0 \left[ \frac{P_0 P'_\theta}{P_0 P_0} \right] \leq 1 + |V(G) \cap V(G')|^2 \theta^2 / 2 + |V(G) \cap V(G')|^2 \cdot s \cdot \theta^3 + |V(G) \cap V(G')|^2 \theta^2 / 4 \\
\leq 1 + |V(G) \cap V(G')|^2 \theta^2.
\]
Denote by $U[\mathcal{G}^+(q)]$ uniformly choosing a graph in $\mathcal{G}^+(q)$. Then by Lemma B.5, we get
\[
\frac{1}{n} < \frac{1}{|\mathcal{G}^+(q)|^2} \sum_{G,G' \in \mathcal{G}^+(q)} |V(G) \cap V(G')|^2 \leq \theta^2 \cdot \sup_{G \in \mathcal{G}^*} \mathbb{E}_{G' \sim U[\mathcal{G}^+(q)]} |V(G) \cap V(G')|^2. \tag{B.8}
\]

(B.8) gives a lower bound of the expectation defined on the right-hand-side. In the following, we utilize this lower bound to derive an upper bound of $|\mathcal{G}^+(q)|$. Inspired by similar results given in Fan et al. (2018); Lu et al. (2018), we give the following lemma.

**Lemma B.6.** For $j = 0, \ldots, s$, define $m_j = \max_{G \in \mathcal{G}^*} |\{G' \in \mathcal{G}^* : |V(G) \cap V(G')| = s - j\}|$. For $k \leq |\mathcal{G}^*|$, let $\mathcal{G}(k) = \{G \subseteq \mathcal{G}^* : |G| = k\}$ and $l(k) = \max\{r \leq s : \sum_{j=0}^r m_j \leq k\}$. Then we have
\[
\sup_{G \in \mathcal{G}^*} \sup_{G' \in \mathcal{G}(k)} \mathbb{E}_{G' \sim U[\mathcal{G}^*]} |V(G) \cap V(G')|^2 \leq \frac{\sum_{j=0}^{l(k)} (s - j)^2 m_j}{\sum_{j=0}^{l(k)} m_j}.
\]

The intuition of Lemma B.6 is that, among all sets of graphs $\mathcal{G}$ with cardinality $k$ (i.e., sets of graphs $G \in \mathcal{G}(k)$), the ones that maximize the expectation $\mathbb{E}_{G' \sim U[\mathcal{G}^*]} |V(G) \cap V(G')|^2$ consist of graphs that makes $|V(G) \cap V(G')|^2$ as large as possible. Let $\zeta = \inf_{0 \leq j \leq s-1} m_{j+1}/m_j$. Then for clique detection problem we have
\[
\zeta = \inf \frac{m_{j+1}}{m_j} = \inf \left[ \left( \frac{s}{s-j-1} \right) \left( \frac{d-s}{d-j+1} \right) \right] / \left[ \left( \frac{s}{s-j} \right) \left( \frac{d-s}{d-j} \right) \right] \geq \frac{d}{s^2} \geq d^0.
\]

Clearly for large enough $d$ we have $\zeta > 2$. Let $h(j) = (s - j)^2$. Then by assumption, for $i < j$ we have $m_i \zeta^j - m_j \zeta^i < 0$, $h(i) - h(j) > 0$ and therefore $(m_i \zeta^j - m_j \zeta^i)[h(i) - h(j)] \leq 0$. Similarly, for $i \geq j$ the same inequality $(m_i \zeta^j - m_j \zeta^i)[h(i) - h(j)] \leq 0$ still holds. Therefore we have $\sum_{0 \leq i,j \leq l(k)} (m_i \zeta^j - m_j \zeta^i)[h(i) - h(j)] \leq 0$. Rearranging terms gives
\[
\frac{\sum_{j=0}^{l(k)} h(j) m_j}{\sum_{j=0}^{l(k)} m_j} \leq \frac{\sum_{j=0}^{l(k)} h(j) \zeta^j - \sum_{j=0}^{l(k)} \zeta^j}{\sum_{j=0}^{l(k)} \zeta^{-j}}. \tag{B.9}
\]

We now bound the right-hand-side of (B.9). Note that $\zeta^{-1} \leq 1/8$ for large enough $d$. For the numerator, we have
\[
\sum_{j=0}^{l(k)} h(j) \zeta^{-(s-j)} = \sum_{i=s-l(k)}^s i^2 \zeta^{-i} \leq [s - l(k)]^2 \zeta^{-[s-l(k)]} + \sum_{i=s-l(k)+1}^s i^2 \zeta^{-i}.
\]

Since $s - l(k) + 1 \geq 1$, for $i \geq s - l(k) + 1$ we have $i^2 \leq [s - l(k) + 1]^2 4^{i-s+l(k)-1}$. Therefore,
\[
\sum_{j=0}^{l(k)} h(j) \zeta^{-(s-j)} \leq [s - l(k)]^2 \zeta^{-[s-l(k)]} + [s - l(k) + 1]^2 \zeta^{-[s-l(k)+1]} + \sum_{i=s-l(k)+1}^s (4 \zeta^{-1})^{i-s+l(k)-1} \leq 2[s - l(k) + 1]^2 \zeta^{-[s-l(k)]}.
\]
For the denominator of the right-hand-side of (B.9), we have
\[ \sum_{j=0}^{l(k)} h(j) \zeta^{-(s-j)} = \zeta^{-[s-l(k)]}. \]
Therefore, we have
\[ \frac{\sum_{j=0}^{l(k)} h(j) \zeta^{-(s-j)}}{\sum_{j=0}^{l(k)} \zeta^{-(s-j)}} \leq 2[s - l(k) + 1]^2. \]  \hfill (B.10)

By (B.10), (B.8) and Lemma B.6, for \( k = |G^+(q)| \) we have
\[ 2[s - l(k) + 1]^2 \geq \frac{1}{n}. \]
Therefore, for large enough \( d \) we have
\[ s - l(k) \geq \sqrt{\frac{1}{2\theta^2 n}} - 1. \]  \hfill (B.11)

On the other hand, by the definition of \( l(k) \), we have
\[ |G^+(q)| = k \leq \sum_{j=0}^{l(k)+1} m_j \leq m_s \cdot \sum_{j=0}^{l(k)+1} \zeta^{j-s} \leq \frac{\zeta^{-[s-l(k)-1]} |G^*|}{1 - \zeta^{-1}} \leq 2\zeta^{-[s-l(k)-1]} |G^*|, \]  \hfill (B.12)
where the last inequality follows from the fact that \( \zeta^{-1} \leq 1/2 \) for large enough \( d \). Plugging (B.11) into (B.12) gives
\[ |G^+(q)| \leq 2|G^*| \exp \left[ - \log(\zeta) \cdot \left( \sqrt{\frac{1}{2\theta^2 n}} - 2 \right) \right]. \]

Applying the same analysis to \( |G^-(q)| \), we obtain
\[ |G^-(q)| \leq 2|G^*| \exp \left[ - \log(\zeta) \cdot \left( \sqrt{\frac{1}{2\theta^2 n}} - 2 \right) \right]. \]

Therefore we have
\[ |G(q)| \leq 4|G^*| \exp \left[ - \log(\zeta) \cdot \left( \sqrt{\frac{1}{2\theta^2 n}} - 2 \right) \right]. \]

Since the inequality above holds for all \( q \in Q_\Psi \), we have
\[ T \cdot \sup_{q \in Q_\Psi} \frac{|G(q)|}{|G^*|} \leq 4 \exp \left[ \log(T) - \left( \sqrt{\frac{1}{2\theta^2 n}} - 2 \right) \cdot \log \zeta \right]. \]

If \( T \leq d^p \), then
\[ T \cdot \sup_{q \in Q_\Psi} \frac{|G(q)|}{|G^*|} \leq \exp \left[ \log(4) + p \log(d) - \left( \sqrt{\frac{1}{2\theta^2 n}} - 2 \right) \cdot \log \zeta \right]. \]
Let \( \kappa < [\sqrt{2}(2 + p/\eta)]^{-1} \). Then if \( \theta \leq \kappa \sqrt{\frac{1}{n}} \), for large enough \( d \) we have

\[
\log(4) + p \log(d) - \left( \sqrt{\frac{1}{2\theta^2 n} - 2} \right) \cdot \log \zeta \leq \log(4) + p \log(d) - \eta \left( \sqrt{\frac{1}{2\theta^2 n} - 2} \right) \log d \leq -1,
\]

and therefore \( T \cdot \sup_{q \in Q} |G(q)|/|G^*| < 1 \). By Lemma B.4, there exists an oracle \( r \) such that \( \lim \inf_{n \to \infty} \gamma_{\text{oracle}}(S^*) \geq 1 \). This completes the proof.

\[ \square \]

### B.1 Proofs of Auxiliary Lemmas

**Proof of Lemma B.4.** We consider an algorithm \( \Psi \) with query space \( Q_{\Psi} \) and \( T_{\Psi} = T \). If \( T \cdot \sup_{q \in Q_{\Psi}} |G(q)| < |G^*| \), then for any \( T \) queries \( q_1, \ldots, q_T \in Q_{\Psi} \), there exists \( G_0 \in G \setminus \bigcup_{t=1}^T G(q_t) \). Let \( \Theta_0 = \theta A_{G_0} \) be the parameter matrix with underlying graph \( G_0 \). Then by definition, for \( t = 1, \ldots, T \) we have

\[
|E_{\Theta_0 q_t(X)} - E_{0 q_t(X)}| \leq \|q_t(X)\|_{\psi_1} \cdot \tau.
\]

We set \( r \) to be the oracle that returns \( Z_{q_t} \) such that

\[
P_0(Z_{q_t} = \mathbb{E}_{\Theta_0}[q_t(X)]) = 1,
\]

\[
P_\Theta(Z_{q_t} = \mathbb{E}_{\Theta}[q_t(X)]) = 1, \ G \in G_1.
\]

Then clearly

\[
P_0(|Z_{q_t} - \mathbb{E}_{\Theta_0}[q_t(X)]| \leq \|q_t(X)\|_{\psi_1} \cdot \tau_{q_t}) = 1,
\]

and hence \( r \) satisfies the definition B.2. However for \( t = 1, \ldots, T \), the oracle always returns the same \( Z_{q_t} \) under \( P_0 \) and \( P_{\Theta_0} \). Therefore we have

\[
P_0(\psi = 1) + P_{\Theta_0}(\psi = 0) = 1.
\]

This completes the proof.

**Proof of Lemma B.5.** By (B.5), we have

\[
\|q(X)\|_{\psi_1} \cdot \tau < \frac{1}{|G^+(q)|} \sum_{G \in G^+(q)} \{\mathbb{E}_\Theta[q(X)] - \mathbb{E}_0[q(X)]\}
\]

\[
= \mathbb{E}_0 \left\{ q(X) \cdot \frac{1}{|G^+(q)|} \sum_{G \in G^+(q)} \left[ \frac{dP^\Theta}{dP_0}(X) - 1 \right] \right\}.
\]

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Applying Cauch-Schwartz inequality on the right-hand side above gives
\[
\|q(X)\|_{\psi_1, 0} \cdot \tau < \left( \mathbb{E}_0 \left\{ \mathbb{E}_0 \left\{ \frac{1}{|G^+(q)|} \sum_{G \in G^+(q)} \left[ \frac{dP_\Theta}{dP_0}(X) - 1 \right]^2 \right\} \right\}^{1/2} \right)^{1/2}.
\] (B.13)

For term (i), by the definition of $\psi_1$-norm we have
\[
(\mathbb{E}_0 \{ [q(X)]^2 \})^{1/2} \leq 2 \|q(X)\|_{\psi_1, 0}.
\] (B.14)

For term (ii), we have
\[
\left[ \mathbb{E}_0 \left\{ \left( \mathbb{E}_0 \left\{ \frac{1}{|G^+(q)|} \sum_{G \in G^+(q)} \left[ \frac{dP_\Theta}{dP_0}(X) - 1 \right]^2 \right\} \right) \right\} \right]^{1/2}
= \left( \frac{1}{|G^+(q)|^2} \sum_{G, G' \in G^+(q)} \mathbb{E}_0 \left\{ \left[ \frac{dP_\Theta}{dP_0}(X) - 1 \right] \cdot \left[ \frac{dP_{\Theta'}}{dP_0}(X) - 1 \right] \right\} \right)^{1/2}
\leq \left\{ \frac{1}{|G^+(q)|^2} \sum_{G, G' \in G^+(q)} \mathbb{E}_0 \left[ \frac{dP_\Theta}{dP_0} \cdot \frac{dP_{\Theta'}}{dP_0}(X) - 1 \right] \right\}^{1/2}.
\] (B.15)

Plugging (B.14) and (B.15) into (B.13) and using the bound $\tau \geq \sqrt{\frac{1}{n}}$, we obtain
\[
\frac{1}{|G^+(q)|^2} \sum_{G, G' \in G^+(q)} \mathbb{E}_0 \left[ \frac{dP_\Theta}{dP_0} \cdot \frac{dP_{\Theta'}}{dP_0}(X) - 1 \right] > 1 + \frac{1}{n}.
\]

Therefore we conclude the proof. \qedhere

Proof of Lemma B.6. For any $G \in G^*$, we have
\[
\mathbb{E}_{G' \sim U(G)} |V(G) \cap V(G')|^2 = \sum_{j=0}^{s} (s-j)^2 |\{G' \in G : |V(G) \cap V(G')| = s-j\}|.
\]

We define
\[
m = k - \sum_{j=0}^{l(k)} m_j.
\]

Note that $h(j) := (s-j)^2$ is a decreasing function of $j$, and $\sum_{G' \in G} |V(G) \cap V(G')|^2$ is a sum of $m_1 + \ldots + m_{l(k)} + m = k$ terms, with at most $m_j$ terms being $h(j)$. Therefore by (B.8), we have
\[
\sup_{G \in G(k)} \mathbb{E}_{G' \sim U(G)} |V(G) \cap V(G')|^2 \leq \frac{\sum_{j=0}^{l(k)} h(j) \cdot m_j + h[l(k) + 1 \cdot m]}{\sum_{j=0}^{l(k)} m_j + m} \leq \frac{\sum_{j=0}^{l(k)} h(j) \cdot m_j}{\sum_{j=0}^{l(k)} m_j}.
\] \qedhere
References

Addario-Berry, L., Broutin, N., Devroye, L. and Lugosi, G. (2010). On combinatorial testing problems. The Annals of Statistics 38 3063–3092.

Ahmed, A. and Xing, E. P. (2009). Recovering time-varying networks of dependencies in social and biological studies. Proceedings of the National Academy of Sciences 106 11878–11883.

Arias-Castro, E., Bubeck, S. and Lugosi, G. (2012). Detection of correlations. The Annals of Statistics 40 412–435.

Arias-Castro, E., Bubeck, S., Lugosi, G. and Verzelen, N. (2015a). Detecting Markov random fields hidden in white noise. arXiv preprint arXiv:1504.06984.

Arias-Castro, E., Bubeck, S., Lugosi, G. et al. (2015b). Detecting positive correlations in a multivariate sample. Bernoulli 21 209–241.

Berthet, Q. and Rigollet, P. (2013a). Complexity theoretic lower bounds for sparse principal component detection. In Conference on Learning Theory.

Berthet, Q. and Rigollet, P. (2013b). Optimal detection of sparse principal components in high dimension. The Annals of Statistics 41 1780–1815.

Berthet, Q., Rigollet, P. and Srivastava, P. (2016). Exact recovery in the Ising blockmodel. arXiv preprint arXiv:1612.03880.

Besag, J. (1993). Statistical analysis of dirty pictures. Journal of applied statistics 20 63–87.

Bhattacharya, B. B. and Mukherjee, S. (2015). Inference in Ising models. arXiv preprint arXiv:1507.07055.

Brennan, M., Bresler, G. and Huleihel, W. (2018). Reducibility and computational lower bounds for problems with planted sparse structure. arXiv preprint arXiv:1806.07508.

Bresler, G. (2015). Efficiently learning Ising models on arbitrary graphs. In Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing. ACM.

Cai, T. T., Liu, W. and Luo, X. (2011). A constrained $\ell_1$ minimization approach to sparse precision matrix estimation. J. Am. Stat. Assoc. 106 594–607.

Chernozhukov, V., Chetverikov, D. and Kato, K. (2014). Central limit theorems and bootstrap in high dimensions. arXiv preprint arXiv:1412.3661.

Daskalakis, C., Dikkala, N. and Kamath, G. (2018). Testing Ising models. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms. Society for Industrial and Applied Mathematics.

Dubhashi, D. and Ranjan, D. (1998). Balls and bins: A study in negative dependence. Random Structures & Algorithms 13 99–124.

Durbin, R., Eddy, S. R., Krogh, A. and Mitchison, G. (1998). Biological sequence analysis: probabilistic models of proteins and nucleic acids. Cambridge university press.
Euler, L. (1741). Solutio problematis ad geometriam situs pertinentis. Commentarii academiae scientiarum Petropolitanae 8 128–140.

Fan, J., Liu, H., Wang, Z. and Yang, Z. (2018). Curse of heterogeneity: Computational barriers in sparse mixture models and phase retrieval. arXiv preprint arXiv:1808.06996.

Feldman, V., Guzman, C. and Vempala, S. (2015a). Statistical query algorithms for stochastic convex optimization. arXiv preprint arXiv:1512.09170.

Feldman, V., Perkins, W. and Vempala, S. (2015b). On the complexity of random satisfiability problems with planted solutions. In ACM Symposium on Theory of Computing.

Fisher, M. E. (1967). Critical temperatures of anisotropic ising lattices. ii. general upper bounds. Physical Review 162 480.

Gao, C., Ma, Z. and Zhou, H. H. (2014). Sparse CCA: Adaptive estimation and computational barriers. arXiv preprint arXiv:1409.8565.

Geman, S. and Geman, D. (1984). Stochastic relaxation, gibbs distributions, and the bayesian restoration of images. IEEE Transactions on pattern analysis and machine intelligence 721–741.

Gheissari, R., Lubetzky, E. and Peres, Y. (2017). Concentration inequalities for polynomials of contracting ising models. arXiv preprint arXiv:1706.00121.

Grabowski, A. and Kosiński, R. (2006). Ising-based model of opinion formation in a complex network of interpersonal interactions. Physica A: Statistical Mechanics and its Applications 361 651–664.

Griffiths, R. B. (1967). Correlations in ising ferromagnets. i. Journal of Mathematical Physics 8 478–483.

Gu, Q., Cao, Y., Ning, Y. and Liu, H. (2015). Local and global inference for high dimensional gaussian copula graphical models. arXiv preprint arXiv:1502.02347.

Guttman, A. (1989). Asymptotic analysis of power-series expansions. Phase transitions and critical phenomena.

Hierholzer, C. and Wiener, C. (1873). Über die möglichkeit, einen linienzug ohne wiederholung und ohne unterbrechung zu umfahren. Mathematische Annalen 6 30–32.

Ising, E. (1925). Beitrag zur theorie des ferromagnetismus. Zeitschrift für Physik A Hadrons and Nuclei 31 253–258.

Jankova, J., Van De Geer, S. et al. (2015). Confidence intervals for high-dimensional inverse covariance estimation. Electronic Journal of Statistics 9 1205–1229.

Joag-Dev, K. and Proschan, F. (1983). Negative association of random variables with applications. The Annals of Statistics 286–295.

Kelly, D. G. and Sherman, S. (1968). General griffiths’ inequalities on correlations in ising ferromagnets. Journal of Mathematical Physics 9 466–484.

Liu, H., Lafferty, J. and Wasserman, L. (2009). The nonparanormal: Semiparametric estimation of high dimensional undirected graphs. Journal of Machine Learning Research 10 2295–2328.
Lu, H., Cao, Y., Lu, J., Liu, H. and Wang, Z. (2018). The edge density barrier: Computational-statistical tradeoffs in combinatorial inference. In International Conference on Machine Learning.

Lu, J., Neykov, M. and Liu, H. (2017). Adaptive inferential method for monotone graph invariants. arXiv preprint arXiv:1707.09114.

Ma, Z., Wu, Y. et al. (2015). Computational barriers in minimax submatrix detection. The Annals of Statistics 43 1089–1116.

Meinshausen, N. and Bühlmann, P. (2006). High dimensional graphs and variable selection with the Lasso. Ann. Stat. 34 1436–1462.

Montanari, A. and Pereira, J. A. (2009). Which graphical models are difficult to learn? In Advances in Neural Information Processing Systems.

Mukherjee, R., Mukherjee, S., Yuan, M. et al. (2018). Global testing against sparse alternatives under ising models. The Annals of Statistics 46 2062–2093.

Nash-Williams, C. (1961). Edge-disjoint spanning trees of finite graphs. Journal of the London Mathematical Society 1 445–450.

Neykov, M. and Liu, H. (2017). Property testing in high dimensional ising models. arXiv preprint arXiv:1709.06688.

Neykov, M., Lu, J. and Liu, H. (2016). Combinatorial inference for graphical models. arXiv preprint arXiv:1608.03045.

Neykov, M., Ning, Y., Liu, J. S. and Liu, H. (2015). A unified theory of confidence regions and testing for high dimensional estimating equations. arXiv preprint arXiv:1510.08986.

Ravikumar, P., Wainwright, M. J., Lafferty, J. D. et al. (2010). High-dimensional ising model selection using $\ell_1$-regularized logistic regression. The Annals of Statistics 38 1287–1319.

Ravikumar, P., Wainwright, M. J., Raskutti, G. and Yu, B. (2011). High-dimensional covariance estimation by minimizing $\ell_1$-penalized log-determinant divergence. Electron. J. Stat. 5 935–980.

Ren, Z., Sun, T., Zhang, C.-H., Zhou, H. H. et al. (2015). Asymptotic normality and optimality in estimation of large gaussian graphical models. The Annals of Statistics 43 991–1026.

Santhanam, N. P. and Wainwright, M. J. (2012). Information-theoretic limits of selecting binary graphical models in high dimensions. Information Theory, IEEE Transactions on 58 4117–4134.

Tandon, R., Shanmugam, K., Ravikumar, P. K. and Dimakis, A. G. (2014). On the information theoretic limits of learning ising models. In Advances in Neural Information Processing Systems.

Vershynin, R. (2010). Introduction to the non-asymptotic analysis of random matrices. arXiv preprint arXiv:1011.3027.

Wang, Z., Gu, Q. and Liu, H. (2015). Sharp computational-statistical phase transitions
via oracle computational model. *arXiv preprint arXiv:1512.08861*.

Wasserman, S. and Faust, K. (1994). *Social network analysis: Methods and applications*, vol. 8. Cambridge university press.