Fractal Measures, $p$-Adic Numbers And Continues Transition Between Dimensions.

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Abstract

Fractal measures of images of continuous maps from the set of $p$-adic numbers $\mathbb{Q}_p$ into complex plane $\mathbb{C}$ are analyzed. Examples of "anomalous" fractals, i.e. the sets where the $D$-dimensional Hausdorff measures (HM) are trivial, i.e. either zero, or $\sigma$-infinite ($D$ is the Hausdorff dimension (HD) of this set) are presented. Using the Caratheodory construction, the generalized scale-covariant HM (GHM) being non-trivial on such fractals are constructed. In particular, we present an example of 0-fractal, the continuum with HD= 0 and nontrivial GHM invariant w.r.t. the group of all diffeomorphisms $\mathbb{C}$. For conformal transformations of domains in $\mathbb{R}^n$, the formula for the change of variables for GHM is obtained. The family of continuous maps $\mathbb{Q}_p$ in $\mathbb{C}$ continuously dependent on "complex dimension" $d \in \mathbb{C}$ is obtained. This family is such that: 1) if $d = 2(1)$, then the image of $\mathbb{Q}_p$ is $\mathbb{C}$ (real axis in $\mathbb{C}$); 2) the fractal measures coincide with the images of the Haar measure in $\mathbb{Q}_p$, and at $d = 2(1)$ they also coincide with the flat (linear) Lebesgue measure; 3) integrals of entire functions over the fractal measures of images for any compact set in $\mathbb{Q}_p$ are holomorphic in $d$, similarly to the dimensional regularization method in QFT.

It is well-known that the Hausdorff measures (HM) are natural integral geometry characteristics for a wide class of sets in $\mathbb{R}^d$[6, 4, 7]. Therefore, contraction of a $D$-dimensional HM $h^D$ on $D$-dimensional rectifiable submanifolds is a measure of their areas [4] and, besides, there exist fractal subsets such that the HM contraction onto them for non-integer $D$ is also nontrivial, i.e. is non-zero and ($\sigma$-) finite, and determines

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their $D$-dimensional fractal measures. For each $F \subseteq \mathbb{R}^n$, there is a unique number $D_h(F)$ called Hausdorff dimension (HD) such that $h^D(F) = \infty$ at $D < D_h(F)$ and $h^D(F) = 0$ at $D > D_h(F)$. However, HM contraction onto $F$ at $D = D_h(F)$ can be trivial [6]. Then, one may naturally ask whether it is still possible in this anomalous case to construct such a nontrivial measure $\mu$ for $F$ which would possess the basic properties specific to the HM. These are the following properties: 1) The measure must be obtained basically in the same way as HM was, but with a generically wider class of test functions, i.e. must be obtained by the Caratheodory construction [4]; 2) The measure must not depend on any scale parameter but on the metric. Thus, it follows from the dimensional analysis that the measure should be scale covariant: $\forall A \subset \mathbb{R}^n, \forall \lambda > 0$ and for some $D > 0$ $\mu(\lambda A) = \lambda^D \mu(A)$. In this paper we construct such measures. They have a unique jumping point $D_F$ on each $F$ similarly to the HM, and always $D_F = D_h(F)$. If $D = 0$, then $h^D$ is a countable measure that is nontrivial only on final and countable sets. However, we show below that there exists a class of scale invariant measures ($D = 0$) such that these measures are nontrivial on continuums and, moreover, they are invariant with respect to any diffeomorphisms $\mathbb{R}^n$. It turns out that it is rather simple to gain and study examples of such sets by considering them as the images of continuous embedding of $p$-adic numbers field $\mathbb{Q}_p$ in $\mathbb{R}^n$ [10, 3, 11, 17]. This is basically due to ultrametricity of $\mathbb{Q}_p$ [1, 2, 3] that the $p$-adic counterparts of the real situation become much simpler and sometimes even more correctly defined [3, 11, 16]. Such spaces enjoy one important property: any monotone increasing function of ultrametrics is again an ultrametrics with equivalent uniform structure (and, therefore, with equivalent topology) [1]. Thus, varying the metric, it is possible to assign to the same subset $F \subseteq \mathbb{Q}_p$ the originally given Hausdorff dimension (HD) such that the fractal measure of $F$ coincides with the contraction of the Haar measures on $\mathbb{Q}_p$ [17] onto $F$. Therefore, if for some uniform embedding $\Upsilon : \mathbb{Q}_p \mapsto \mathbb{R}^n$ one selects a metric in $\mathbb{Q}_p$ such that it is in a sense close to the metrics induced from $\mathbb{R}^n$ (see section 3), the examination of fractal properties of $\Upsilon(F)$ can be reduced to examination of the properties of $F \subseteq \mathbb{Q}_p$. In this paper, the uniform continuous maps $\Upsilon_s : \mathbb{Q}_p \mapsto \mathbb{C}$, possessing the scaling property $\Upsilon_s(px) = s\Upsilon_s(x)$ with $s \in \mathbb{C}$ are studied. Under some additional assumptions, (analyticity in $s$, uniformity etc.) they enjoy a series of remarkable properties. It appears that any such $s$-parametric set of $\Upsilon_s$ is uniquely determined by the function $\phi : [0,1] \mapsto \mathbb{C}$ such that its continuity is sufficient for the fractal measure of the embedding image $\Upsilon_s$ to coincide with the image.
of the Haar measure in \( Q_p \). We construct examples of anomalous fractals which are images of multiple derivatives and integrals of \( \Upsilon_s \) over the parameter \( s \). The class of 0-fractals being the images of embeddings of the corresponding continuum subsets of \( Q_p \) with nontrivial 0-dimensional fractal measures is presented. Some examples of the sets in \( C \) of zero Lebesgue measure having the HD= 2 is also constructed. A family of continuous maps from \( Q_p \) to \( C \) which continuously depend on the parameter \( d \in C \) is obtained. This family is such that: 1) if \( d = 2(1) \), then the image of \( Q_p \) is \( C \) (the real axis in \( C \)); 2) the fractal measures coincide with the images of the Haar measure in \( Q_p \) and, at \( d = 2(1) \), they also coincide with the flat (linear) Lebesgue measures; 3) integrals of entire functions over the fractal measures of images of any compact set in \( Q_p \) are holomorphic in \( d \). Thus, the 1- and 2- dimensional integrals of holomorphic functions can be interpreted as values of a function holomorphic in \( d \), much similarly to the method of dimensional regularization of Feynman integrals \([12, 13, 14]\) \(^1\). In addition, note that the values of these functions at noninteger \( d \) are interpreted as integrals over the corresponding fractal measures.

1 Pseudometric space \( A(I) \)

When examining properties of measures, metrics and test functions given on different spaces, it is convenient to define two functions \( f \) and \( g \) to be \( \varepsilon \)-close if \( e^{-\varepsilon} f(x) < g(x) < e^{+\varepsilon} f(x) \) for \( x \) from a set such that for metrics it is ”a set of infinitely close points” and for test functions it is ”an infinitely small set” only. The construction considered below allows one to describe within a uniform scheme the spaces of measures, test functions, metrics etc. as quasiorder pseudometric spaces. It is important, however, that the Caratheodory construction is a functor, i.e. is a quasiorder-preserving contracting map from the space of test functions into the space of exterior measures.

Let \( I \) be any set, then, denote by \( A(I) \) the set of all pairs \( f_F \equiv (f, F) \), where \( f : I \to \bar{\mathbb{R}}^+ \) and \( F \) is a filter at \( I \). \(^3\) Let us define the relations of quasiorder and a

\(^1\)It is necessary to point out that the similarity between these concepts is rather formal.

\(^2\)\( \bar{\mathbb{R}}^+ = \bar{\mathbb{R}} \cap \{ x : x \geq 0 \} \), where \( \bar{\mathbb{R}} = \mathbb{R} \cup \{ -\infty, \infty \} \) is the extended number axis. It would be more naturally to define functions as \( f : V \to \bar{\mathbb{R}}^+ \), where \( V \in F \), however, we can assume that \( f = \infty \) outside \( V \). From now on, we assume that \( \inf \{ \emptyset \} = \infty \) and \( \sup \{ \emptyset \} = 0 \) on \( \bar{\mathbb{R}}^+ \) and \( \inf \{ \emptyset \} = \infty \) and \( \sup \{ \emptyset \} = -\infty \) on \( \bar{\mathbb{R}} \).

\(^3\)The family \( F \) of subsets of the set \( I \) is called filter at \( I \), if \( I \in F, \emptyset \notin F \); if \( A \in F \) and \( B \supseteq A \), then
pseudometric at \( \mathcal{A}(I) \):

Relations \( \preceq, \succeq, \prec, \succ \) at \( \mathcal{A}(I) \): \( f, f' \preceq g, f, f' \prec g \) if \( \ell(f, g), \ell(f', g) < \infty \), \( \ell(f, g), \ell(f', g) \leq 1 \) and \( \ell(f, g) = 0 \), respectively, where \( \forall f, g \in \mathcal{A}(I) \)

\[
\ell(f, g) = \begin{cases} 
\overline{\ell}(f, g) & \equiv \sup_{x \in \mathbb{R}^+} \inf_{x \in U} (r \in \mathbb{R}^+ : f(x) \leq rg(x)) \\
\infty & \text{if } \mathcal{F} \subseteq \mathcal{G} \\
& \text{if } \mathcal{F} \not\subseteq \mathcal{G}.
\end{cases} \tag{1}
\]

Pseudometric \( \alpha : \mathcal{A}(I) \times \mathcal{A}(I) \mapsto \mathbb{R}^+ : \alpha(f, g) = \ln(\max(\ell(f, g), \ell(g, f))) \). We write \( f \preceq \ell(\succeq)g \), if \( f \preceq \ell(\succeq)g \) and \( g \preceq \ell(\succeq)f \) at the same time (cf. with O-symbolics). It is obvious that \( f \preceq \ell(\succeq)g \), if and only if \( \alpha(f, g) < \infty \). It is easy to show that

\[
\ln \overline{\ell}(f, g) = \inf_{x \in U, U \ni x} \sup \{ \ln f(x) - \ln g(x) : f(x) \neq g(x) \}, \tag{2}
\]

\[
\alpha(f, g) = \inf_{x \in U} \sup \{ \ln f(x) - \ln g(x) \} : x \in U \cap \{ f(x) \neq g(x) \}, \tag{3}
\]

\[
\kappa(f, g) = \frac{\alpha(f, g)}{2} = \inf_{x \in U} \sup \left\{ \frac{|f(x) - g(x)|}{f(x) + g(x)} : x \in U \cap \{ f(x) \neq g(x) \} \right\}. \tag{4}
\]

We shall consider only self-consistent subspaces in \( \mathcal{A}(I) \), i.e. such \( \mathcal{S}(I) \subseteq \mathcal{A}(I) \) that \( \forall f, g \in \mathcal{S}(I) \) \( \ell(f, g) = \ell(f, g) \) (i.e. if \( \ell(f, g) < \infty \) then \( \mathcal{F} \subseteq \mathcal{G} \) and \( f \preceq \ell g \)).

Let us consider two subsets in \( \mathcal{A}(I) \). The first set is a subset of all pairs \( (f, \mathcal{F}) \) with trivial \( \mathcal{F} : \mathcal{F} = \{ I \} \). The second set \( \mathcal{N}(I) \) consists of \( (f, \mathcal{F}) \in \mathcal{A}(I) \) such that \( f^{-1}(\{ 0 \}) \neq \emptyset \) and \( \mathcal{F} \) is the pre-image of the filter of neighourhoods of zero, which means that the set \( \{ f^{-1}(U) : \forall \text{ open } U : U \cap \emptyset \neq \emptyset \} \) is the base of this filter. It is easy to check that \( \mathcal{A}_0(I) \) and \( \mathcal{N}(I) \) are self-consistent. Moreover, \( \mathcal{A}_0(I) \) is a metric space, the relations \( \preceq \) and \( \succeq \) at \( \mathcal{A}_0(I) \) are the standard relations \( \leq \) and \( = \), respectively, and if \( f \prec g \), then either \( f = 0 \), or \( g = \infty \). Letus introduce the following notations: \( \forall (f, \mathcal{F}), (g, \mathcal{G}) \in \mathcal{A}(I) \) we write \( f \preceq \ell g \) if \( (f, \{ I \}) \preceq \ell (g, \{ I \}) \) or \( \forall x_k \in I f(x_1, \cdots x_k \cdots x_n) \preceq \ell \alpha g(x_1, \cdots x_k \cdots x_n) \), and for any constant \( c \) we write \( c \preceq 1 \) if \( 0 < c < \infty \).

We call any map \( C : \Sigma \mapsto \mathcal{A}(J') \) for \( \Sigma \subseteq \mathcal{A}(J) \) \( \preceq, \prec \)-isotonic map or \( \preceq, \prec \)-isotonia, if \( C(f_x) \preceq (\sim, \prec)C(g_x) \) for all \( f, g \in \Sigma \) such that \( f \preceq (\sim, \prec)g \), we also

\[
B \in \mathcal{F} ; \forall A, B \in \mathcal{F}, A \cap B \in \mathcal{F} \ [19] .
\]

\footnote{The partition of \( \mathcal{A}(I) \) according to the equivalence relation \( \simeq \) is exactly the partition of topological space \( \mathcal{A}(I) \) into connected components. Indeed, \( \{ h \in \mathcal{A}(I) : \alpha(h, f) < \infty \} \forall f \in \mathcal{A}(I) \) is an open and close set in \( \mathcal{A}(I) \) at the same time, and for any \( f' \preceq f \) the map \( [0, 1] \ni t \mapsto (f + (1-t)f', \mathcal{F}) \in \mathcal{A}(I) \) is a continuous path from \( f \) to \( f' \).}

\footnote{The family of sets \( B \) is called filter base if \( \emptyset \notin B ; \forall A, B \in B, \exists C \in B : C \subseteq A \cap B, \text{ and } \mathcal{F} = \{ B : \exists A \in B : A \subseteq B \} \text{ is called filter generated by } B.\)
call just isotonia the \( \preceq, \preceq, \prec\)-isotonic map. For any set \( X \), denote by \( \mathcal{D}(X) (\mathcal{D}_0(X)) \) the set of all \( (\rho, \mathcal{F}) \in \mathcal{N}(X \times X) (\mathcal{A}_0(X \times X)) \) such that \( \rho \) is pseudometric\(^6\), and denote by \( \mathcal{M}(X) \subseteq \mathcal{A}_0(2^X) \) the set of all exterior \( \sigma \)-semiadditive measures on \( X \).

Any map \( \Phi : X \mapsto Y \) induces an isometry \( \Phi^* : \mathcal{D}(Y) \mapsto \mathcal{D}(X) \) such that \( \forall \rho \in \mathcal{D}(Y) \Phi^*(\rho) \equiv \rho_{\Phi}(\cdot, \cdot) = \rho(\Phi(\cdot), \Phi(\cdot)) \). Let us consider two maps \( \Phi_1 : X \mapsto (M_1, \rho_1) \) and \( \Phi_2 : X \mapsto (M_2, \rho_2) \), where \( M_i \) is metric spaces with metric \( \rho_i \). Then, we define the distance \( \varphi(\Phi_1, \Phi_2) \) between \( \Phi_1 \) and \( \Phi_2 \) by the formula

\[
\varphi(\Phi_1, \Phi_2) \equiv \varphi(\Phi_1^*(\rho_1), \Phi_2^*(\rho_2)),
\]

we also write \( \Phi_1 \preceq (\preceq, \prec) \Phi_2 \) as soon as \( \Phi_1^*(\rho_1) \preceq (\preceq, \prec) \Phi_2^*(\rho_2) \). We call a given map \( \Phi : (M_1, \rho_1) \mapsto (M_2, \rho_2) \ell \)-contraction if \( \Phi \preceq \) id, and \( \ell \)-isometry if \( \Phi \simeq \) id, where id is the identity map on \( M_1 \). It is clear that \( \rho_2(\Phi(\cdot), \Phi(\cdot)) \preceq (\simeq) \rho_1(\cdot, \cdot) \), iff \( \Phi \) is an \( \ell \)-contraction (\( \ell \) - isometry), in particular, the \( L \)-contraction is the Lipschitzian map.

It is easy to see that the condition \( \Phi_1 \preceq (\preceq) \Phi_2 \) is equivalent to the existence of the \( (\ell) \)-contraction \( \Phi_{12} : \Phi_2(X) \mapsto \Phi_1(X) \), such that \( \Phi_1 = \Phi_{12} \circ \Phi_2 \). Therefore, if \( \Phi_1 \) is injective, then \( \Phi_2 \) is also injective. Besides, if there are given appropriate structures of either (uniform) topological, or metric space, on \( X \), then, if \( \Phi_2 \) is a (uniform) continuous map or \( \ell \)-contraction, the same is \( \Phi_1 \).

2 The Caratheodory construction

Let \( \mathcal{J} \) be a family of subsets of any set \( X \). For any pair \( (\zeta, \mathcal{F}) \in \mathcal{A}(\mathcal{J}) \) we define an exterior \( \sigma \)-semiadditive measure \( k_\zeta \) by putting \( \forall A \subseteq X \),

\[
k_\zeta(A) = \sup \inf \{ \sum_{i \in J} \zeta(S_i) : S_i \in \mathcal{U}, \{S_i\}_{i \in J} - \text{countable cover of } A \}. 
\]

(5)

This correspondence determines the contracting isotonia \( C : \mathcal{A}(\mathcal{J}) \mapsto \mathcal{M}(X) \), which we call Caratheodory construction (CC) by analogy with the standard Caratheodory construction \([4]\) which is a particular case of CC.

\(^6\)One can show that \( \mathcal{D}_0(X) \) is a complete metric space.

\(^7\)It deserves noting that, if one considers a category \( \mathcal{K}(X) \) such that all the objects in it are \( \Phi : X \mapsto (M, \rho) \) with \( \Phi(X) = M \) and morphisms \( \preceq \), the indicated correspondence is a functor from \( \mathcal{K}(X) \) into a category of metric spaces with uniform continuous maps between them.
Hereafter, we always suppose that \( J = 2^X \). Let us define the space of test functions \( \mathcal{T}(X) \subseteq \mathcal{A}(2^X) \) as a set of pairs \((\zeta, \mathcal{F})\) such that the following natural requirements hold
\[
\forall U \in \mathcal{F} \text{ if } A \subseteq B \in U \text{ then } A \in U^8
\]
\[
\exists V \in \mathcal{F} \text{ such that if } A \subseteq B \in V \text{ then } \zeta(A) \leq \zeta(B)
\]
It is clear that \( \mathcal{M}(X) \subseteq \mathcal{T}(X) \) and \( \mathcal{C}_{|\mathcal{M}(X)} = \text{id.} \).

With each pseudometric \( \rho \in \mathcal{D}(X) \) we associate a pair \((\hat{\rho}, \mathcal{F}_\rho)\) \( \in \mathcal{N}(2^X) \subset \mathcal{A}(2^X) \) such that \( \forall S \subseteq X \ \hat{\rho}(S) \equiv \text{diam}(S) = \sup_{x,y \in S}(\rho(x,y)) \) and \( \mathcal{F}_\rho \) is the filter generated by the base \( \{S \in J : \hat{\rho}(S) \leq 1/n\}_{n=1,2,...} \). Property (6,7) for \((\hat{\rho}, \mathcal{F}_\rho)\) obviously holds and \( \ell(\hat{\alpha}, \hat{\beta}) \leq \ell(\alpha, \beta) \forall \alpha, \beta \in \mathcal{D}(X) \). Therefore, the map \( ^\triangleright : \mathcal{D}(X) \mapsto \mathcal{T}(X) \) is an isotonic contraction. Moreover, for any monotone non-decreasing function \( \zeta : \mathbb{R}^+ \mapsto \mathbb{R}^+ \), using \( ^\triangleright \) one can associate a map \( \hat{\zeta} : \mathcal{D}(X) \mapsto \mathcal{T}(X) \) such that
\[
\hat{\zeta}_\rho(\cdot) = (\zeta(\hat{\rho}(\cdot)), \mathcal{F}_\rho)
\]
It can be shown that \( k_{\hat{\zeta}_\rho} \) is regular in the sense of Borel [4]. Let \( \zeta(r) = r^d \), then for each pseudometric space \((X, \rho)\) the composition \( \mathcal{C} \circ \hat{\zeta} : \mathcal{D}(X) \mapsto \mathcal{M}(X) \) defines a \( d \)-dimensional Hausdorff measure \( h^d \) (HM) (up to a constant factor \( 2^{-d} \Gamma(\frac{1}{2} + \frac{d}{2}) \) — "the area of \( d \)-dimensional sphere"; however, as far as we are interested in normalized fractal measures, this factor is inessential). For \( D_1 > D_2 \), since \( \mathcal{C} \) is an isotonia and \( \hat{\rho}^{D_1} \ll \hat{\rho}^{D_2} \), we have \( h^{D_1} \ll h^{D_2} \). One obtains from the last inequality the well-known formula/definition for the Hausdorff dimension (HD) [6, 7] : \( \forall A \subseteq X \)
\[
D_h(A) \equiv \inf \{\delta : h^\delta(A) = 0\} = \sup \{\delta : h^\delta(A) = \infty\}
\]
For any map \( \Phi : X \mapsto Y \) consider now the map \( \Phi^* : \mathcal{T}(Y) \mapsto \mathcal{T}(X) \) such that \( \forall (\eta, \mathcal{F}) \in \mathcal{T}(Y) \ \Phi^* \eta(S) = \eta(\Phi(S)) \), and \( \Phi^* \mathcal{F} \) is the filter generated by the base \( \{S \subseteq X : \exists O \in \mathcal{U} : S \subseteq \Phi^{-1}(O)\}_{U \in \mathcal{F}} \). Let \( \Phi : \mathcal{M}(X) \mapsto \mathcal{M}(Y) \) be the map such that \( \forall \mu \in \mathcal{M}(X) \ \Phi(\mu)(S) = \mu(\Phi^{-1}(S)) \), \( \forall B \subset Y \) and \( \mu(S|B) \equiv \mu(S \cap B) \). Then, from (6,7) it follows that the diagram
\[
\begin{align*}
\mathcal{D}(Y) & \xrightarrow{\hat{\zeta}} \mathcal{T}(Y) & \mathcal{T}(Y) & \xrightarrow{\mathcal{C}} \mathcal{M}(Y) & \mathcal{M}(Y) \\
\mathcal{D}(X) & \xrightarrow{\hat{\zeta}} \mathcal{T}(X) & \mathcal{C} & \xrightarrow{\Phi} \mathcal{M}(Y) \\
\Phi^* & \xrightarrow{\hat{\zeta}} & \Phi^* & \xrightarrow{\mathcal{C}} & \Phi
\end{align*}
\]
Figuratively speaking, any subset of a "U-small" set is also "U-small".
is commutative. In particular, from (10) it follows that the contraction of the CC measure onto any \( X \subseteq Y \) coincides with the CC measure on \( X \) independently of \( Y \).

First of all, we are interested in \textit{scale-covariant measures}. These are the measures \( k \) such that under the scale transform \( \lambda : X \mapsto X \) such that \( \rho(\lambda(x), \lambda(y)) = \lambda \rho(x, y) \) with some \( \lambda > 0 \):

\[
k(\lambda(\cdot)) = \lambda^D k(\cdot)
\]

for some \textit{scale dimension} \( D > 0 \). It is clear that any HM is a scale-covariant measure, however, this is not the only possible choice. Indeed, since \( C \) is a contraction, then \( \eta_F \overset{\circ}{\preceq} \zeta_G \) implies that \( k_{\eta_F} = k_{\zeta_G} \). Therefore, to satisfy condition (11) for \( k_{\zeta_G} \) it suffices

\[
\zeta_F(\lambda(\cdot)) \overset{\circ}{\preceq} \lambda^D \zeta_F(\cdot).
\]

For any \( D \geq 0 \) let us denote by \( S^D \) the set \( \forall (\eta, \mathcal{N}) \in \mathcal{A}(\mathbb{R}) \) such that \( \mathcal{N} \) is the filter of neighbourhoods of zero and \( \eta \overset{\circ}{\preceq} \varsigma \), where \( \varsigma \) is a monotone nondecreasing function such that \( \varsigma(r) > 0 \) with \( r > 0 \) and

\[
\lim_{r \to 0} \frac{\varsigma(\lambda r)}{\varsigma(r)} = \lambda^D.
\]

The following lemma is correct (see proof in the Appendix):

**Lemma 1** If \( \zeta \in S^D \), then the map \( \zeta : \mathcal{N}(I) \ni f \mapsto (\zeta \circ f, \mathcal{F}) \in \mathcal{A}(I) \) at \( D > 0 \) \((D = 0)\) is isotonia \((\overset{\ell}{\preceq}, \overset{\circ}{\preceq}\)-isotonia) and \( \forall f_F, g_G \in \mathcal{N}(I) \) if \( \ell(f_F, g_G) < \infty \), then

\[
\ell(\zeta(f_F), \zeta(g_G)) \leq \ell(f_F, g_G)^D.
\]

In addition, if \( D' > D \) and \( f_F \overset{\ell}{\preceq} g_G \), then \( \forall \eta \in S^{D'} \eta(f_F) \ll \ll \zeta(g_G) \).

It is easy to show that \( \forall \eta \in S^D \) and \( \forall \rho \in \mathcal{D}(X) \), the function \( \eta_{\rho} \) satisfies (12). Moreover, since \( C \) is an isotonia in accordance with lemma 1, then it follows from lemma 1

**Proposition 1** For \( D \geq 0 \) and \( \forall \zeta \in S^D \) we have

1) The map \( \hat{\zeta} : \mathcal{D}(X) \mapsto \mathcal{T}(X) \) (and also \( C \circ \hat{\zeta} : \mathcal{D}(X) \mapsto \mathcal{M}(X) \)) is \( \overset{\ell}{\preceq}, \overset{\circ}{\preceq}\)-isotonic and, if additionally \( D > 0 \), it is \( \ll \)-isotonic.

2) \( \forall \alpha, \beta \in \mathcal{D}(X) \)

\[
\varpi(k_{\zeta_\alpha}, k_{\zeta_\beta}) \leq \varpi(\hat{\zeta}_\alpha, \hat{\zeta}_\beta) \leq D \cdot \varpi(\alpha, \beta).
\]
3) \( \forall \rho \in D(X), D_1, D_2 \geq 0 \) and \( \forall \zeta^1 \in S^{D_1}, \forall \zeta^2 \in S^{D_2} \), if \( D_1 > D_2 \), then

\[
\hat{\zeta}_1^\rho \prec \prec \hat{\zeta}_2^\rho, \quad k_{\hat{\zeta}_1^\rho} \prec \prec k_{\hat{\zeta}_2^\rho}.
\]  

(15)

**Corollary 1** If for some set \( F \) \( \exists \eta, \xi \in S^{D} \) such that \( k_{\eta} = 0 \) and \( k_{\xi} > 0 \), then \( D = D_h(F) \). In particular, if \( \exists \zeta \in S^{D} \) such that \( k_{\zeta} \) is nontrivial on \( F \), then \( D = D_h(F) \).

**Corollary 2** Let \((X, \rho)\) and \((Y, d)\) be metric spaces and \( \Phi : X \mapsto Y \). Then, if \( \Phi \) is an \( \ell\)-contraction (\( \ell\)-isometry), for any \( \zeta \in S^{D} \) all the arrows in (10) are isotonic and Lipschitzian maps. Therefore,

\[
k_{\zeta|\Phi(X)} \leq L(\Phi)k_{\zeta},
\]

in particular, \( \forall A \subseteq X \) \( D_h(\Phi(A)) \leq (-)D_h(A) \). In addition, if \( D = 0 \), then \( k_{\zeta|\Phi(X)} \leq (-)\Phi k_{\zeta} \). Therefore, if \( \zeta_0 \in S^0 \), then \( k_{\zeta_0} \) is invariant with respect to all \( \ell\)-isometries, in particular, \( k_{\zeta_0} \) on \( \mathbb{R}^n \) is invariant w.r.t. all diffeomorphisms of \( \mathbb{R}^n \) (since \( \mathbb{R}^n \) is a countable union of compact sets).

In the next section, we give an example of \( k_{\zeta_0} \) which is non-trivial on a continuum. The following claim also turns out to be valid:

**Proposition 2** Let \( O, O' \) be open domains in \( \mathbb{R}^n \), \( \Phi \) is a conformal map from \( O' \) onto \( O \), \( J(\Phi) \) denotes the Jacobian for \( \Phi \) and \( \zeta_D \in S^{D} \). Then, for any \( k_{\zeta_D|O}\)-summable function \( f \), for a change of variables the following formula holds:

\[
\int_{O} f(y)k_{\zeta_D}(dy) = \int_{O'} f(\Phi(x)) \left( \sqrt{|J(\Phi)(x)|} \right)^D k_{\zeta_D}(dx).
\]

(16)

**Proof** For linear \( \Phi \), the proof immediately follows from (11). In the generic case, the proof follows from statement 7 and since due to lemma 3.2.2 [4] \( \forall \delta > 0 \) there exists a countable covering by Borel sets \( E_i \) such that \( \forall c \in E_i \) \( \alpha(\Phi'(c)|_E, \Phi|_{E_i}) < \delta \). \( \square \)

3 Fractal Measures in \( Q_p \)

Fractals having an hierarchical structure (for example, the Cantor set, the Sierpinski triangle, the Koch curve etc. [15, 7]) are convenient to consider as images of uniformly

\footnote{Note that, for \( \mathbb{R}^2 \simeq \mathbb{C} \), formula (16) is valid for all biholomorphic \( \Phi \), while on \( \mathbb{R}^1 \) it is correct for all diffeomorphisms of \( \mathbb{R} \). Moreover, in both cases \( \left( \sqrt{|\text{Jac}(\Phi)(x)|} \right)^D = |\Phi'(x)|^D \).}
continuous maps from ultrametric spaces to $\mathbb{R}^d$. Such fractals can be constructed through the following procedure. Let $C_n$ be a sequence of finite or countable families of compact sets in $\mathbb{R}^d$, called clusters of level $n$, and $\{a\} \equiv \{a_n\}_{n \in \mathbb{Z}}$ be a sequence such that $\forall C_n \in C_n \quad C_n = \bigcup_{x_n=0}^{a_n-1} C_{n+1}^{x_n}$, where $C_{n+1}^{x_n} \in C_{n+1}$. Let us assume that $g_n \equiv \sup_{C_n \in C_n} \text{diam}(C_n) \rightarrow 0$ as $n \rightarrow \infty$. Let us fix any set $C_0 \in C_{-1}$, then the fractal required is the set $F \equiv \bigcap_{n \in \mathbb{N}} \{ C : C \in C_n, C \subseteq C_0 \}$. Each point $f \in F$ is the limit of some thread $C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots C_{n+1} \supseteq \ldots$ (i.e. $\bigcap_{n=0}^{\infty} C_{n+1} = \{f\}$) and, vice versa, the limit of each such thread is an element of $F$.

Let us identify the set of such threads with the set $Z_{(a)}$ of all sequences $\{x_n\}_{n=0}^{\infty}$ such that $x_n \in \{0, \ldots, a_n - 1\}$, and also enter the (ultra)metric $\rho(x, y) = g(v(x, y))$ on $Z_{(a)}$, where $v(x, y) = \sup\{n : x_n = y_n\}$ and $g$ is any monotone decreasing function, and $g(\infty) = 0$. If $\Upsilon : Z_{(a)} \mapsto \mathbb{R}^d$ is a map such that $\bigcap_{n=0}^{\infty} C_{n+1}^{x_n} = \{ \Upsilon(\{x_n\}_{n=0}^{\infty}) \}$, then $\Upsilon$ is a uniformly continuous map on $F$, and if also $\forall n \quad g_n \leq g(n)$, then $\Upsilon$ is the Lipschitzian map. Let $\Upsilon$ be an injective map (for instance, this is the case, if the clusters do not intersect). If $g(\cdot)$ is such that $\forall \Upsilon^{-1} : F \mapsto Z_{(a)}$ is also (locally) the Lipschitzian map, then the construction of the fractal measure $F \mapsto Z_{(a)}$ can be transferred from $\mathbb{R}^d$ onto the more convenient ultrametric space $Z_{(a)}$ isomorphic to the ring of $\{a\}$-adic integers [5] and, in the special case of $a_n = p$, to the ring of $p$-adic integers $Z_p \subseteq Q_p$.

Each element $x$ of the $p$-adic number field $Q_p$ is uniquely representable as a formal power series [1],

$$x = \sum_{n=v}^{\infty} a_n p^n = \sum_{n=v}^{-1} a_n p^n + \sum_{n=0}^{\infty} a_n p^n \tag{17}$$

with coefficients $a_n \in \{0, 1, \ldots, p-1\}$, where $v < \infty$ and $p$ is some fixed prime number. \footnote{Actually, the role of $p$ can be equally well played by any positive integer, since we nowhere use the existence of inverse elements in the ring $Q_p$.}

The number $v(x) = v$ is called logarithmic norm of $x$. Any strictly monotone decreasing function $g$ such that $g(\infty) = 0$ defines an invariant metric on the additive group $Q_p$ by the formula $\forall x, y \in Q_p \quad \rho(x, y) \equiv \rho(x - y) = g(v(x - y))$ \footnote{Note that the uniform structure (and, therefore, topology) in $Q_p$ does not depend on the choice of $g$.}. This metric have the ultrametric property:

$$\rho(x - y) \leq \max(\rho(x), \rho(y)). \tag{18}$$
The series (17) absolutely converges in $\rho$. Any number $q \in \mathbb{Q}$ can be uniquely expanded into series (17) and $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ [1]. The first sum at the right-hand side of (17) is denoted as $\{x\}_p$ being the fractional part of $x$. The second sum is denoted as $[x]_p$ being the integer part of $x$. In this case, $\{x\}_p \in \mathbb{Q} \cap [0, 1)$ and $[x]_p \in \mathbb{Z}_p$, where $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \|x\| \leq 1\}$ is the ring of $p$-adic integers and $|x|_p \equiv p^{-v(x)}$ is the canonical norm. We assume further (unless otherwise stated) that there is a canonical norm in $\mathbb{Q}_p$. It can be shown [17] that the 1-dimensional Hausdorff measure in $(\mathbb{Q}_p, | \cdot |_p)$ coincides with the standard Haar measure $\chi$ in $\mathbb{Q}_p$ such that

$$\chi(\mathbb{Z}_p) = \int_{\mathbb{Z}_p} d\chi = 1.$$  \hfill (19)

Let us consider $(\mathbb{Q}_p, \rho)$ with $\rho(x, y) = g(v(x - y))$ and $\zeta(r) \simeq (\sim r^{p^{-1}v(r)})$. Then, it is obvious that

$$k_\zeta(\cdot) = (\sim r)^L \chi(\cdot).$$  \hfill (20)

Thus, $\forall \zeta \in \mathcal{S}^D$ one can find the proper metric in $\mathbb{Q}_p$; for instance, if

$$\zeta(r) \simeq \prod_{k=0}^{N} \frac{1}{(\log_p^{(k)}(1/r))^{D_k}} = \prod_{k=m}^{N} \frac{1}{(\log_p^{(k)}(1/r))^{D_k}},$$  \hfill (21)

where $(\exp_a(t) \equiv a^t)$,

$$\log_a^{(k)}(t) \equiv \exp_a^{(-k)}(t) \equiv \left\{ \begin{array}{ll}
\log_a(\log_a \cdots \log_a(t) \cdots) : & k > 0 \\
\log_t(\exp_a(\exp_a \cdots \exp_a(t) \cdots)) : & k < 0 \\
\log_t(t) : & k = 0 
\end{array} \right.$$  \hfill (22)

and $D_0 = D$, $D_m > 0$ then one can easily show that proper metric can be chosen as follows

$$\rho(\bar{D})(x) \simeq \frac{1}{\exp_p^m \left( |x|_p^{-D_m} \prod_{k=1}^{N-m} (\log_p^{(k)} |x|_p^{-1})^{D_m k} \right)}.$$  \hfill (23)

Let us consider the map $J^m : \mathbb{Q}_p \mapsto \mathbb{Q}_p$ such that

$$J^m \left( \sum_{n=-\infty}^{\infty} x_n p^n \right) = \sum_{n=-\infty}^{\infty} x_n p^{\exp_p^m(n)}$$  \hfill (24)
is an $\ell$-isometry from $Q_p$, with the metric $\rho(x) = p^{-\exp_p(m)(\nu(x))}$ in $(Q_p, | \cdot |_p)$. Thus, for $m > 0$ one obtains a continuum $J^m(Q_p) \subseteq Q_p$ with HD= 0 and the nontrivial measure $k_\zeta$ for $\zeta(r) \sim (\log_p(m)(1/r))^{-1}$. Moreover, $k_\zeta$ is invariant w.r.t. any $\ell$-isometry on $\mathbb{C}$.

So far all these constructions in $Q_p$ look like a tautology, however, they become nontrivial if one finds out an $\ell$-isometry from $(Q_p, \rho(D))$ to $\mathbb{R}^d$. For $\Lambda_N \equiv \{x \in Q_p : |x|_p \leq p^N\}$ ($\Lambda_\infty = Q_p$), we call a continuous map $\Upsilon : \Lambda_N \mapsto \mathbb{R}^d$ automodel if for some $m \geq 0$

$$\Upsilon(B^m_i) = \bigcup_{l=0}^{p^{m-1}} E_{l,i,n} \circ \Upsilon(B^m_i),$$

\forall n \in \mathbb{Z}, l \in Q_p$. Here $B^m_i \equiv \{x \in \Lambda_N : |x - l|_p \leq p^{-m}\}$ are closed (and simultaneously open) balls in $\Lambda_N$ and $E_{l,i,n} : \mathbb{R}^d \mapsto \mathbb{R}^d$ are isometry maps. We call $\Upsilon : \Lambda_N \mapsto \mathbb{R}^d$ quasiautomodel if there exists a sequence of automodel maps $\Upsilon_k$ such that $\lim_{k \to \infty} \mu(\Upsilon_k, \Upsilon) = 0$. The following statement is correct:

**Proposition 3** Let $\Upsilon : \Lambda_N \mapsto \mathbb{R}^d$ be quasiautomodel, $\zeta \in S^D$ is such that $k_\zeta(\Upsilon(Z_p)) < \infty$ and $k_\zeta(\Upsilon(B^m_i) \cap \Upsilon(B^m_i')) = 0$, when $B^m_i \cap B^m_i' = \emptyset$. Then $\forall A \subseteq \mathbb{R}^d$

$$k_\zeta(A \cap \Upsilon(\Lambda_n)) = k_\zeta|_{\Upsilon(\Lambda_n)}(A) = k_\zeta(\Upsilon(Z_p))\chi(\Upsilon^{-1}(A)),$$

(26)

**Proof** From the proposition (1) it follows that it is sufficient to consider the automodel map. For the automodel map, formula (26) is valid $\forall B^m_i \subseteq Q_p$. Indeed, the clusters $\Upsilon(B^m_i)$ in (25) do not overlap $k_\zeta$-nearly everywhere and $k_\zeta$ is a translation invariant. Since $k_\zeta$ and $\chi$ are regular in Borel sense, $\Upsilon(\Lambda_n) = \bigcup_{k<n} \Upsilon(\Lambda_k)$ is a Borel set ($\Lambda_k$ is a compact set, and $\Upsilon$ is a continuous map) and the semi-ring $\{B^m_i\}_{n \in \mathbb{Z}}$ generates a Borel $\sigma$-algebra in $\Lambda_N$, then formula (26) is valid $\forall A \subseteq \mathbb{R}^d$. □

Thus, if $\Upsilon$ is quasiautomodel and $\exists \zeta \in S^D$ such that $\Upsilon$ is an $\ell$-isometry from $(Q_p, \rho)$ into $\mathbb{R}^d$ for some $\rho(x, y) = g(\nu(x-y))$ such that $\zeta(r) \sim p^{-g^{-1}(r)}$, then the fractal measure of $\Upsilon(\Lambda_n)$ defined by

$$\mu_{\Upsilon(\Lambda_n)}(\cdot) = \frac{1}{k_\zeta(\Upsilon(Z_p))} k_\zeta(\Upsilon^{-1}(\cdot)),$$

(27)

is the image of the Haar measure in $Q_p$ i.e. $\mu_{\Upsilon(\Lambda_n)}(\cdot) = \chi(\Upsilon^{-1}(\cdot))$.

### 4 Maps : $Q_p \mapsto \mathbb{C}$.

The map $Q_p \ni x \mapsto px \in Q_p$ generates a natural group of scaling transformations $\{p^n\}_{n \in \mathbb{Z}}$ in the ring $Q_p$ ($|p^n x|_p = p^{-n}|x|_p$). On the other hand, any scaling tran-
form of \( \mathbb{C} \) is of the form \( \mathbb{C} \ni z \mapsto sz + t \in \mathbb{C} \), where \( s, t \in \mathbb{C} \). We want to describe scaling-covariant uniformly continuous maps \( \Upsilon : \mathbb{Q}_p \rightarrow \mathbb{C} \), i.e. those consistent with the scaling transforms: \( \Upsilon(px) = s\Upsilon(x) + t \forall x \in \mathbb{Q}_p \) for some \( s, t \in \mathbb{C} \). For the map \( \Upsilon_s(\cdot) = \Upsilon(\cdot) - \Upsilon(0) \) the latter requirement reduces to the following principal condition on \( \Upsilon_s \)

\[
\Upsilon_s(px) = s\Upsilon_s(x) = p^{-\frac{1}{n}} e^{i \arg(s)} \Upsilon_s(x),
\]

(28)

where \( D_s = -\ln(p) / \ln|s| \). Let us assume that \( \Upsilon_s(\cdot) \neq 0 \). Then from (28), the continuity of \( \Upsilon_s \) and from the condition \( \Upsilon_s(0) = 0 \) it follows that \( s \in U_1 \ (U_r \equiv \{ z \in \mathbb{C} : |z| < r \}) \)

Let us define two numbers

\[
\Delta^\pm(\Upsilon_s) = \inf\{|\Upsilon_s(x) - \Upsilon_s(y)|^{\pm 1} : \forall x, y \in \mathbb{Q}_p : |x - y|_p = 1\}. \tag{29}
\]

From (28) it follows that \( \Delta^-(\Upsilon_s) > 0 \) iff \( \Upsilon_s \) is uniformly continuous and \( \Delta^+(\Upsilon_s) > 0 \) iff there exists a uniformly continuous map \( \Upsilon_s^{-1} : \Upsilon_s(\mathbb{Q}_p) \mapsto \mathbb{Q}_p \) such that \( \Upsilon^{-1}_s \circ \Upsilon_s = \text{id} \).

Let \( \rho_D(x, y) = \sqrt[\circ]{|x - y|_p} \) and \( d\Upsilon_s(x, y) = |\Upsilon_s(x) - \Upsilon_s(y)| \), then it is easy to show that \( \ell(\rho_D, d\Upsilon_s) = 1/\Delta^+(\Upsilon_s) \) and \( \ell(d\Upsilon_s, \rho_D) = 1/\Delta^-(\Upsilon_s) \) and, hence, \( \varphi(d\Upsilon_s, \rho_D) = -\ln(\min(\Delta^+(\Upsilon_s), \Delta^-(\Upsilon_s))) \). Therefore, \( \Upsilon_s : \mathbb{Q}_p \mapsto \mathbb{C} \) is uniformly continuous iff \( \Upsilon_s : (\mathbb{Q}_p, \rho_D) \mapsto \mathbb{C} \) is an \( \ell \)-isometry. From this one immediately obtains that for any scaling-covariant uniformly continuous embedding \( \Upsilon_s : \mathbb{Q}_p \mapsto \mathbb{C} \) the Hausdorff measure \( h^{D_s} \) is nontrivial on \( \Upsilon_s(\mathbb{Q}_p) \) and \( D_h(\Upsilon_s(B)) = D_s \) for any open set \( B \subset \mathbb{Q}_p \). Besides, it is clear that if \( \Upsilon_s \) is only a uniformly continuous map, then \( D_h(\Upsilon_s(\mathbb{Q}_p)) \leq D_s \). Let us assume furthermore that, for each fixed \( x \in \mathbb{Q}_p \), \( \Upsilon_s(\cdot)(x) \) is a function holomorphic on \( U^0_1 \ (U^0_r \equiv U_r \setminus \{0\}) \). Then the requirement (28) and the residue at zero uniquely determines the set of maps \( \{ \Upsilon_s \}_{s \in U^0_1} \). Indeed, let \( \phi(x) = \text{res}_{s=0} \ (\Upsilon_s(x)) \), then \( \forall x \in \mathbb{Q}_p \)

\[
\Upsilon_s(x) = \sum_{n=-\infty}^{\infty} \phi \left( \frac{x}{p^{n+1}} \right) s^n. \tag{30}
\]

Let us also assume that, for some \( r > 0 \), there exists \( \Delta_r > 0 \) such that \( \Delta^-(\Upsilon_s) \geq \Delta_r \forall s \in U^0_1 \). Then, from the Cauchy inequalities for the Laurent series (30) it follows

\[\text{Note that if } \Upsilon_s : \mathbb{Q}_p \rightarrow \mathbb{C} \text{ is injective, then } \lim_{|x|_p \rightarrow \infty} |\Upsilon_s(x)| = \infty. \text{ Therefore, one could consider the continuous maps } \bar{\Upsilon} : \overline{\mathbb{Q}_p} \rightarrow \bar{\mathbb{C}} \text{ such that } \Upsilon(px) = L(\Upsilon(x)), \text{ where } \overline{\mathbb{Q}_p} \equiv \mathbb{Q}_p \cap \{ \infty \} \text{ is the single-point compactification of } \mathbb{Q}_p, \bar{\mathbb{C}} \text{ is the Riemannian sphere and } L(z) = (az + b) / (c + dz). \text{ Then, } \Upsilon(0) \text{ and } \Upsilon(\infty) \text{ are fixed points of } L, \text{ however, if } \Upsilon(0) \neq \Upsilon(\infty), \text{ there exists a linear-fractional automorphism } U : \bar{\mathbb{C}} \mapsto \bar{\mathbb{C}} \text{ such that, for } \Upsilon_s = U \circ \Upsilon, (28) \text{ is valid} [9].\]
that $\phi(x + Z_p) = \phi(x) \forall x \in Q_p$, or, which is the same, $\phi(x) = \phi(\{x\}_p)$. Now the Laurent expansion of $\Upsilon_s$ can be written as

$$\Upsilon_s(x) = \psi^\phi_s(x) = \sum_{n=v(x)}^\infty \phi \left( \frac{x}{p^n+1} \right) s^n = \sum_{n=-\infty}^\infty \phi \left( \left\{ \frac{x}{p^n+1} \right\}_p \right) s^n. \quad (31)$$

The function $\phi$ is uniquely determined by its values on the set

$$Z(p^\infty) \equiv \{ \{x\}_p : x \in Q_p \} = \{ l/p^k : l \in N, l < p^k \}.$$ 

Therefore, the function $\phi$ can be considered as being defined on $[0, 1) \subset \mathbb{R}$ or as a periodic function (with period $= 1$) defined on $\mathbb{Q}$, $Q_p$, or $\mathbb{R}$. For the sake of simplicity, let us assume that $\phi : Z(p^\infty) \to \mathbb{C}$ is an arbitrary bounded function (without loss of generality, let $|\phi(x)| \leq 1$). Then, the series (31) defines the map $\Upsilon^\phi_s$ such that (28) holds and $\Delta^-(\Upsilon^\phi_s) \geq 1 - |s|$. Using that $\phi(x/p^{n+1})$ depends only on $x_n, x_{n-1}, ..., x_{v(x)}$, one easily gets the following estimate for $\Delta^+(\Upsilon^\phi_s)$:

$$\Delta^+(\Upsilon^\phi_s) \geq \nu[\phi] - \frac{|s|}{1 - |s|}, \quad (32)$$

where $\nu[\phi] = \inf_{a=1,p-1} \{ |\phi(p^{-1}a + \tau) - \phi(\tau)| : \tau \in Z(p^\infty) \} ^\dagger$. Thus, if $\nu[\phi] > 0$, then, for small enough $s$ such that $|s| < \sigma_\phi \equiv \nu[\phi]/(1 + \nu[\phi])$, the map $\Upsilon^\phi_s$ is an $\ell$-isometry.

For example, if $\phi(x) = (p - 1)^{-1}x_{-1}$, then $\nu[\phi] = (p - 1)^{-1}$, $\sigma_\phi = p^{-1}$, and if $\phi(x) = \exp(i2\pi \{x\}_p) - 1$, then $\nu[\phi] = \sin(\pi/p)$, $\sigma_\phi = (1 + \sin(\pi/p))^{-1} \geq p^{-1}$ (cf. with [17]).

If $\phi(x) = \phi(x_{-1})$, it is possible to obtain a lower bound for $D_h(\Upsilon^\phi_s(Q_p))$. Indeed, it is easy to prove that $\Upsilon^\phi_{sk} = \Upsilon^\phi_s \circ \Theta_K$, where $\Theta_K : Q_p \to Q_p$ such that $\Theta_K \left( \sum_{n=-\infty}^{\infty} x np^n \right) = \sum_{n=-\infty}^\infty x np^{kn}$. Thus, for $K$ such that $|s|^K < \sigma_\phi$, one gets

$$D_s/K = D_{sk} = D_h(\Upsilon^\phi_s(\Theta(Q_p))) \leq D_h(\Upsilon^\phi_s(Q_p)).$$

For any integer $l$, let us introduce $\forall x \in Z_p$ the map $\partial^l_s \Upsilon^\phi_s : Z_p \to \mathbb{C}$,

$$\partial^l_s \Upsilon^\phi_s(x) = \sum_{n=l}^\infty \phi \left( \left\{ \frac{x}{p^n+1} \right\}_p \right) \frac{n!}{(n-l)!} s^{n-l}. \quad (33)$$

\(^\dagger\)One can easily show that the requirement $\Delta^-(\Upsilon_s) \geq \Delta_r > 0$ is equivalent (up to the replacement $\Upsilon_s \to s^M \Upsilon_s$) to the equipotential uniform continuity of the set of functions $\{ \Upsilon_s \}_{s \in U^2}$.

\(^\ddagger\)Considering $\phi(t)$ as the periodic function $\phi(t) = \phi(t + n)$, it is possible to interpret $\phi(t)$ as a trajectory of a particle in $\mathbb{C}$. Then, $\nu[\phi]$ is the minimal distance between $p$ particles which started to move at sequent moments of "time" $t = 0, \frac{1}{p}, ..., \frac{p-1}{p}$.
Using (41), it is easy to show that \( d_{\partial_s\mathcal{T}^s} \lesssim \rho_{(D,s,l)} \) with \( |s| < \sigma_\phi \), where \( \rho_{(D,s,l)} = v(x)^{l/(p-v(x)/D_s)} \) and \( d_{\partial_s\mathcal{T}(x,y)} = |\partial_s\mathcal{T}_s(x) - \partial_s\mathcal{T}_s(y)| \). Thus, \( k_\xi \) with \( \zeta(r) = (r|\ln(r)|^{-l})^\rho \) \((\xi \in S^D)\) is a nontrivial measure on \( \partial_s\mathcal{T}_s^s(Z_p) \), however, \( h^{D_s}_{\partial_s\mathcal{T}_s^s(Z_p)} = 0 \) at \( l < 0 \) and is \( \sigma\)-infinite at \( l > 0 \).

Now let us construct a set in \( \mathbb{C} \) of zero Lebesgue measure, but with HD= 2. Put \( \mathcal{F}'_2 = \partial_s^{-1}\mathcal{T}_s^s(Z_p) \) with \( \phi(x) = x_{-1} \) and \( s = \frac{i}{\sqrt{p}} \). Using (42), one can prove that

\[
\rho_{(2,-1)} \lesssim d_{\partial_s^{-1}\mathcal{T}_s^s} \lesssim \rho_{(2,-2)}.
\]

Hence, using corollary 1 of proposition (1), one obtains that \( D_h(\mathcal{F}'_2) = 2 \) and \( h^2(\mathcal{F}'_2) = 0 \), since \( \rho_{(2,0)} \lesssim \rho_{(2,-1)} \).

Since \( \partial_s^0\mathcal{T}_s^s = \mathcal{T}_s^s|_{\mathcal{Z}_p} \), let us identify \( \partial_s^0\mathcal{T}_s^s \) with \( \mathcal{T}_s^s : \mathcal{Q}_p \mapsto \mathbb{C} \). Let \([\phi]_m(x) \equiv \phi(p^{-m}[p^mx],p) = \phi(\sum_{n=-m}^0 x_np^n) \). Then, similarly to [17] it can be shown that \( \partial_s^k\mathcal{T}_s^s[m] \) is automodel. For any function \( \phi : [0,1) \mapsto \mathbb{C} \), let \( ||\phi||_p \equiv \sup\{||\phi(q)|| : q \in \mathbb{Z}(p^\infty)\} \). A function \( \phi : [0,1) \mapsto \mathbb{C} \) is called \((p)\)-continuous, if \( \lim_{m \to \infty} ||\phi - [\phi]_m||_p = 0 \). Obviously, \( \forall M \in \mathbb{Z}_+ \phi(x) = [\phi]_M(x) \) is a \((p)\)-continuous function. The following claim is correct.

**Proposition 4** If \( \partial_s^k\mathcal{T}_s^s \) is an \( l \)-isometry with a \((p)\)-continuous function \( \phi \), then \( \partial_s^k\mathcal{T}_s^s \) is a quasi-automodel map and \( \exists M \) such that \( \partial_s^k\mathcal{T}_s^s[m] \) is also an \( l \)-isometry at \( m \geq M \).

**Proof** Since \( \varphi(\rho_{(D,s,l)}, d_{\partial_s\mathcal{T}_s^s}) < \infty \) and for any \( m \in \mathbb{Z}_+ \) \( \partial_s^k\mathcal{T}_s^s[m] \) is an automodel map, the proof follows from the following lemma (proved in the Appendix) applied to \( \phi_1 = \phi \) and \( \phi_2 = [\phi]_m \). \( \square \)

**Lemma 2** For any bounded function \( \phi_i : \mathbb{Z}(p^\infty) \mapsto \mathbb{C} \) \((i = 1, 2)\)

\[
\kappa(\partial_s^k\mathcal{T}_s^s, \partial_s^k\mathcal{T}_s^s) \leq c \varphi^{-1}(\varphi_1, \varphi_2) \varphi_1 - \varphi_2 \varphi_\infty, \tag{34}
\]

where \( \varphi_i = \varphi(\rho_{(D,s,l)}, d_{\partial_s\mathcal{T}_s^s}) \) and \( c = 2(1 - |s|)^{-1} \).

The requirement of \((p)\)-continuity is not too restrictive. Indeed, the following assertion (proved in the Appendix) is valid.

**Proposition 5** The function \( \phi : [0,1) \mapsto \mathbb{C} \) is \((p)\)-continuous, if it is right-continuous and is continuous everywhere except for a finite number of points of the first kind discontinuity \( a_1, ..., a_N \) such that \( a_i \in \mathbb{Z}(p^\infty) \), in particular, if \( \phi \) is just continuous (continuity is understood in the sense of ordinary topology on \( \mathbb{R} \)).
From 3 and 4 it follows that $h^{D_s}(Z_p) \overset{b}{\overset{\sim}{\leq}} 1$ at $\Delta^\phi_s > 0$ and the fractal measure $\Upsilon^\phi_s(Q_p)$ is the image of the Haar measure on $Q_p$. Similarly, the fractal measure $\partial^s_\Upsilon^\phi_s(Z_p)$ (with $\zeta(r) = r^{D_s} |\ln(r)|^{-D_s}$) is the image of the Haar measure on $Q_p$ too.

Let now $\Delta^\phi_s > 0$. It is easy to show that the map $\mathcal{N} \equiv \Upsilon^\phi_s \circ J^m : Q_p \mapsto \mathbb{C}$ is an $\ell$-isometry from $Q_p$ with metric $\rho_m(x) = p^{-\exp\{r_m(\frac{\|x\|}{r})\}}$ to $\mathbb{C}$. Let $\zeta(r) \overset{\sim}{\leq} (\log p\{m\}(1/r))^{-1}$, then $\zeta \in \mathcal{S}^0$ and, from proposition 1, it follows that $k_\zeta|_{\mathcal{N}(Q_p)}(\cdot) = k_\zeta_{\varrho_m}|(\mathcal{N}^{-1}(\cdot)) = \chi(\mathcal{N}^{-1}(\cdot))$. The measure $k_\zeta$ is invariant w.r.t. any diffeomorphism $\mathbb{C} \approx \mathbb{R}^2$, and, at $m > 1$, w.r.t. any homeomorphism with arbitrary Holder index.

Thus, in the cases considered above for any $\mu_{\Upsilon^\phi_s(\Lambda)}$-summable function $f : \mathbb{C} \mapsto \mathbb{C}$ one has

$$\int_{\mathbb{C}} \mu_{\Upsilon^\phi_s(\Lambda)}(dz) \ f(z) = \int_{\Lambda} \chi(dx) f(\Upsilon^\phi_s(x)).$$

(35)

Note that the integral in the right hand side of (35) is correctly defined even if the measure $\mu_{\Upsilon^\phi_s(\Lambda)}$ is not defined, furthermore, the following proposition holds (see proof in the Appendix).

**Proposition 6** Let $O$ be an open set in $U^0_r$ for some $r < 1$, $\Lambda$ be a measurable subset in $Q_p$, $\Upsilon_O(\Lambda) \equiv \bigcup_{x \in O} \Upsilon_s(\Lambda)$ and $f \in C^m(\Upsilon_O(\Lambda))$ and $l, \bar{l}$ be such that $l + \bar{l} \leq m$. Then, if one of the following condition holds,

1) the set $\Lambda$ is bounded;
2) $\delta^\phi_s \equiv \inf_{|x|_p = 1} |\Upsilon^\phi_s(x)| > 0$ and $\exists \nu > 0$ such that with $k \leq l, \bar{k} \leq \bar{l}$ the following inequality holds ( $\partial_z \equiv \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$, $\partial_{\bar{z}} \equiv \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$ )

$$\forall z \in \Upsilon_O(\Lambda) \ |\partial_{\bar{z}}^k \partial_z^k f(z)| \leq \frac{L}{1 + |z|^{D_{s+k+k+\nu}}};$$

then the integral

$$\mathcal{I}^\Lambda_s(f) \equiv \int_{\Lambda} \chi(dx) f(\Upsilon^\phi_s(x)), \quad (36)$$

exists, $\mathcal{I}^\Lambda_s(f) \in C^m(O)$ and the following equation is correct

$$\partial_{\bar{z}}^k \partial_z^k \mathcal{I}^\Lambda_s(f) = \int_{\Lambda} \chi(dx) \partial_{\bar{z}}^k \partial_z^k f(\Upsilon^\phi_s(x)). \quad (37)$$

In particular, if $f \in \mathcal{S}(\mathbb{R}^2)$, where $\mathcal{S}(\mathbb{R}^2)$ is the Schwarz space of rapidly decreasing $C^\infty$-smooth functions, then, $\mathcal{I}^\Lambda_s(f) \in C^\infty(O)$. Furthermore, $\Upsilon_O(\Lambda)$ is an open set, since

\footnote{Note that in this case quasiautomodelity of $\mathcal{N}$ is not used. Moreover, $k_\zeta(\mathcal{N}(Z_p)) = \chi(Z_p) = 1$}
from holomorphy of \( f \) in \( \mathcal{Y}_O(\Lambda) \) it follows that \( I^A_s(f) \) is some holomorphic function in \( O \)
\((\partial_z f(z) = 0 \Rightarrow \partial_s I^A_s(f) = 0)\).

The constructions considered in this section can be immediately generalized onto maps from \( \mathbb{Z}_{(a)} \) to \( \mathbb{C} \). Let us define the map \( \omega^\phi : \mathbb{R} \times \mathbb{Z}_{(a)} \mapsto \mathbb{C} \) (assuming that \( \phi(t+1) = \phi(t) \)) as follows \( \forall (\tau, x) \in \mathbb{R} \times \mathbb{Z}_{(a)} \)

\[
\omega^\phi(\tau, x) = \sum_{n=0}^{\infty} \left( \frac{1}{a(n)} \right)^\nu \phi \left( \frac{(x)^n + \tau}{a(n)} \right),
\]

where \( a(n) = \prod_{k=0}^{n} a_k \) (\( a^{-1} = 1 \)) and \( (x)^n = \sum_{k=0}^{n} x_k a^{(k-1)} \). In the special case of \( a_k = p \), one has \( \omega^\phi(0, x) = \mathcal{Y}_a^\phi(x) \ \forall x \in \mathbb{Z}_p \) for \( s = p^{-\nu} \). Furthermore, it can be proved that, for example, if \( a_k = (k+2)! \), \( \phi(t) = \exp(i2\pi t) \) and \( D^{-1} = \text{Re}(\nu) > 1 \), then \( \rho_{D,0} \leq d_\tau \leq \rho_{D,1} \), where \( \rho_{D,1}(x, y) = v(x, y)^i / \sqrt{a(v(x,y))} \) and \( d_\tau(x, y) = |\omega^\phi(\tau, x) - \omega^\phi(\tau, y)| \). Therefore, \( \forall \tau \in \mathbb{R} \ D_h(\omega^\phi(\tau, \mathbb{Z}_{(a)})) = D \). It is easy to show that \( \forall n \in \mathbb{Z} \ \omega^\phi(\tau, x) = \omega^\phi(\tau - n, x + n) \), i.e. the function \( \omega^\phi \) is constant on the cosets of the subgroup \( B = \{ (-n, n) \in \mathbb{R} \times \mathbb{Z}_{(a)} : n \in \mathbb{Z} \} \) and, therefore, it can be represented as the map of \( \{a\}-adic \ solenoid \ [5] \ \Sigma_{(a)} \equiv (\mathbb{R} \times \mathbb{Z}_{(a)}) / B \) to \( \mathbb{C} \). Similarly, the map \( \Omega^\phi_{\nu,\alpha} : \mathbb{R} \times \mathbb{Z}_{(a)} \mapsto \mathbb{R}^3 \) such that \( \rho + i h = \omega^\phi(\tau, x) + \alpha \) and \( \phi = \tau \), where \( h, \rho, \phi \) are cylindrical coordinates on \( \mathbb{R}^3 \), can be considered as the map from \( \Sigma_{(a)} \) into \( \mathbb{R}^3 \) \footnote{Note that \( \Sigma_{(a)} \) for \( a_k = (k+2)! \) is isomorphic to the character group of the additive group of the field \( \mathbb{Q} \) [5] or to the group \( \mathbb{A}/\mathbb{Q} \), where \( \mathbb{A} \) is the ring of adeles [2, 8].}. If now \( \omega^\phi \) is such that \( \rho_{D,0} \leq d_\tau \leq \rho_{D,1} \), repeating the arguments from [17] it can be shown that, for \( |\text{Re(\alpha)}| \) large enough, the map \( \Omega^\phi_{\nu,\alpha} \) is a continuous embedding and \( D_h(\Omega^\phi_{\nu,\alpha}(\Sigma_{(a)})) = D + 1 \). Moreover, if \( \phi : [0,1] \mapsto \mathbb{C} \) is a continuously differentiable function, then \( \forall (\tau, x) \in \mathbb{R} \times \mathbb{Z}_{(a)} \) the following equality holds

\[
\partial_\tau \omega^\phi(\tau, x) = \omega^\phi_{\nu,\alpha}(\tau, x)
\]

It follows that \( \Omega^\phi_{\nu,\alpha}(\Sigma_{(a)}) \) is the invariant set of the autonomous ordinary differential equation in \( \mathbb{R}^3 \) with the Lipschitzian right hand side [17].

## 5 Continuous Transition between Dimensions

Before considering a concrete set of maps realizing the transition between integer dimensions note that the set \( \mathcal{F}_s = \mathcal{Y}_s(\mathbb{Q}_p) \) is invariant w.r.t. the scale transformation \( z \rightarrow sz \) and its fractal measure \( \mu_{\mathcal{F}_s} \) is transformed as follows

\[
\mu_{\mathcal{F}_s}(s \cdot) = p^{-1} \mu_{\mathcal{F}_s}(\cdot) = |s|^d |\mu_{\mathcal{F}_s}(\cdot)| = e^{i\theta} s^d \mu_{\mathcal{F}_s}(\cdot).
\]
Figure 1: We depict by the dotted line the path from $d = 1$ to $d = 2$ in the $d$-plane (fig. [a]) and in the $s$-plane (fig. [b]) for $p = 2$, by white color the domain of embeddings ($\Delta^-(\Upsilon^s_\phi) > 2^{-12}$) and by grey color the domain ($\delta(\Upsilon^s_\phi) > 2^{-10}$). In frames [0]–[B] the sets $\mathcal{F}_d = \Upsilon^\phi_\psi(d, (Q_p)^n)$ are shown for values $d$ lying on paths. These values are marked on a fig. [c] by labels 0, ..., B.

Here we introduce a "complex dimension" $d \in \mathbb{C}$ of the set $\mathcal{F}_d \equiv \mathcal{F}_s$ and an arbitrary parameter $\theta \in \mathbb{R}$, with the following relations between them being correct

$$s = s(d) \equiv \exp\left(-\frac{\ln p + i\theta}{d}\right).$$

$$D_s^{-1} = \text{Re}\left(d^{-1} \left(1 + i\frac{\theta}{\ln p}\right)\right) \geq 0.$$

Let us consider now the map $\Upsilon^\phi_s$ with $\phi$ such that $\phi(x) = x_{-1} \forall x = \sum_{n=-\infty}^{\infty} x_n p^n \in \mathbb{Q}_p$. Using the $p$-adic decomposition of the real numbers, it can be proved that $\Upsilon^\phi_{1/p}(\mathbb{Z}_p) = \{x + iy \in \mathbb{C} : 0 \leq x \leq \frac{x}{p^{1/2}}, y = 0\}$. Now from (28) it follows that $\Upsilon^\phi_{1/p}(\mathbb{Q}_p) = \{z \in \mathbb{C} : \text{Im}(z) = 0, \text{Re}(z) \geq 0\}$. It is easy to show that $\Upsilon^\phi_{1/p}(\mathbb{Z}_p) - \frac{\mathbb{Z}_p}{[\frac{1}{2}]} \sum_{n=0}^{\infty} (\frac{1}{p})^n = \Upsilon^\phi_{1/p}(\mathbb{Z}_p) - \frac{\mathbb{Z}_p}{[\frac{1}{2}]} \sum_{n=0}^{\infty} (\frac{1}{p})^n$, therefore, $\Upsilon^\phi_{1/p}(\mathbb{Z}_p) = \{x + iy \in \mathbb{C} : 0 \leq x + \frac{2^{2p/2}}{p^2 - 1} \leq \frac{x}{p^{1/2}}, y = 0\}$. From this, using (28), one gets $\mathcal{F}_d = \Upsilon^\phi_{1/p}(\mathbb{Q}_p) = \{z \in \mathbb{C} : \text{Im}(z) = 0\}$ with $d = 1 \left(\frac{\ln p + i\theta}{\ln p + i\pi}\right) (s(d) = -1/p)$. Let us define the bijection $q : \mathbb{Q}_p \times \mathbb{Q}_p \mapsto \mathbb{Q}_p$ such that

$$q\left(\sum_{n=-\infty}^{\infty} x_n p^n, \sum_{n=-\infty}^{\infty} y_n p^n\right) = \sum_{n=-\infty}^{\infty} x_n \left(p^n\right)^n + p \sum_{n=-\infty}^{\infty} y_n \left(p^n\right)^n,$$

One easily shows that $\Upsilon^\phi_s(q(x, y)) = \Upsilon^\phi_{s^2}(x) + s \Upsilon^\phi_s(y)$ and, in particular, $\Upsilon^\phi_{1/\sqrt{p}}(q(x, y)) = \ldots$
\[ \Upsilon^\phi_{\frac{1}{p}}(x) \pm i \left( \frac{1}{\sqrt{p}} \Upsilon^\phi_{\frac{1}{p}}(y) \right). \]

Therefore, for \( d = 2 \left( \frac{\ln p + \theta}{\ln p + n} \right) \) \((s(d) = \pm \frac{1}{\sqrt{p}})\) one obtains that \( \Upsilon^\phi_s(Z_p) \) is a closed rectangle in \( \mathbb{C} \) of the size \( \frac{p}{p-1} \times \frac{\sqrt{p}}{p-1} \) and \( \Upsilon^\phi_s(Q_p) = \mathbb{C} \).

Let us assume that \( \theta = \pi \), then one gets that \( \mathcal{F}_1 \) is the real axis \( \sim \mathbb{R}^1 \) and \( \mathcal{F}_2 = \mathbb{C} \sim \mathbb{R}^2 \). Besides, in spite of the fact that the map \( \Upsilon^\phi_s \) is not injective at \( d = 1, 2 \), still \( D_s = d \), \( h^{D_s}(\Upsilon^\phi_{s(d)}(Z_p)) \approx 1 \) and the formula (35) (with \( \zeta(\rho) = \rho^{D_s} \)) is valid. Thus, the fractal measures of \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are (up to a constant factor) the 1-dimensional \( \mu_{\mathcal{F}_1}(dz) \sim \delta(z - \tilde{z})dzd\tilde{z} \) and 2-dimensional \( \mu_{\mathcal{F}_1}(dz) \sim dzd\tilde{z} \) Lebesgue measures, respectively. It is easy to show that the points 1 and 2 in the \( d \)-plane can be connected by a continuous path such that \( \mu_{\mathcal{F}_2}(f) \) is defined on it and \( \forall f \in \mathcal{S}(\mathbb{R}^2) \), due to proposition 6, \( \mu_{\mathcal{F}_2}(f) \)

\[ C^\infty \] is a \( C^\infty \) -smooth function and if \( f \) is a holomorphic function on \( U_{\sqrt{p}/(\sqrt{p} - 1)} \), then, \( \mu_{\mathcal{F}_2}(f) \) is a holomorphic function on this path, \( \mathcal{F}_d = \Upsilon^\phi_{s(d)}(Z_p) \). In figure 1 an example of such a path is shown in \( d \)- and \( s \)- planes with \( p = 2 \). There are also drawn the (white) domain of embeddings \( \Delta^- \( \Upsilon^\phi_s \) > 2^{-12} \approx 0 \), where the fractal measures are defined and the formula (35) holds, and the (grey) domain \( \delta(\Upsilon^\phi_s) > 2^{-10} \approx 0 \), where the requirements of assertion 6 hold.

## 6 Appendix

### Proof of lemma 1

The following lemma holds

**Lemma 3** If the function \( \phi : \mathbb{R} \mapsto \mathbb{R} \) such that \( \forall \delta \in \mathbb{R} \lim_{t \to \infty} |\phi(t + \delta) - \phi(t)| = 0 \) and \( \phi(t) = \mu(t) + v(t) \), where \( \mu(t) \) and \( v(t) \) are, respectively, monotone and uniformly continuous functions at some interval \( (\tau_0, \infty) \), then

I) \( \forall \varepsilon > 0 \exists \tau \) such that \( |\phi(t) - \phi(t')| < \varepsilon + \varepsilon |t - t'| \) at \( t, t' \geq \tau \).

II) \( \forall r < \infty \lim_{\tau \to \infty} \sup \{|\phi(t) - \phi(t')| : t, t' > \tau, |t - t'| < r\} = 0. \)

III) \( \forall \varepsilon > 0 \exists \tau \) such that \( |\phi(t)| < \varepsilon t \) at \( t \geq \tau \).

**Proof** \( \forall \delta, \varepsilon > 0 \exists \tau_{\delta, \varepsilon} > \tau_0 \) such that \( \forall t' \geq \tau_{\delta, \varepsilon} \) \( |\phi(t' + \delta) - \phi(t')| < \varepsilon \delta / 2 \). Let us choose \( \delta < 1 \) such that \( \forall \delta' \leq \delta, \delta' > 0 \) \( |v(t' + \delta') - v(t')| < \varepsilon / 4 \). Then, from monotonicity of \( \mu(t) \) it follows that \( |\phi(t' + \delta') - \phi(t')| \leq |\phi(t' + \delta) - \phi(t')| + \varepsilon / 2 < \varepsilon \). \(|\phi(t' + \delta') - \phi(t')| - \varepsilon / 4 \leq |\mu(t' + \delta') - \mu(t')| \leq |\mu(t' + \delta) - \mu(t)| \leq |\phi(t' + \delta) - \phi(t')| + \varepsilon / 4 \). Let \( t \geq t' \) and \( t_k = t + \delta k \), then \( t - t' \leq \delta', \) where \( N_\delta = [\delta^{-1}(t - t')] \) and the following chain of inequalities
Let \( \ln \ell \) follow at \( F \) such that \( \delta \mu(s) = \mu(\beta(s)) - \mu(\alpha(s)) = D\delta(s) + \phi(\beta(s)) - \phi(\alpha(s)) \leq D\delta(s) + \varepsilon|\delta(s)| + \varepsilon \leq (D + \varepsilon)\ell + \varepsilon \). Therefore, \( \ell(\zeta(f_\mathcal{F}), \zeta(g_\mathcal{G})) \leq \ell(f_\mathcal{F}, g_\mathcal{G})D \) at \( D > 0 \). Since \( \delta \mu(s) \leq 0 \) at \( \delta(s) \leq 0 \), then \( \delta \mu(s) = \max(\varepsilon \delta(s) + \varepsilon, 0) \) at \( D = 0 \). From this one obtains from this \( \ell(\zeta(f_\mathcal{F}), \zeta(g_\mathcal{G})) \leq 1 \) at \( \ell(f_\mathcal{F}, g_\mathcal{G}) < \infty \). Let \( D' > D \), then \( \forall \eta \in \mathcal{S}^{D'} \) such that \( \forall \ell > 0 \) \( \ln \ell(\eta(g_\mathcal{G}), \zeta(g_\mathcal{G})) \leq \ln(\varepsilon)/|\zeta(e^{-\varepsilon})| = -(D' - D)\ell + \phi(t) - \phi'(t) \). Using assertion III of lemma 3, one gets that \( \ln \ell(\eta(g_\mathcal{G}), \zeta(g_\mathcal{G})) = -\infty \). Thus, \( \eta(f_\mathcal{F}) \leq \eta(g_\mathcal{G}) \leq \zeta(g_\mathcal{G}) \). □

Let \( 0 \leq r < 1 \) and \( p \in \mathbb{N} \), then, using formula 5.2.3.1 [18], it can be proved that the following relations hold at \( v \to \infty \)

\[
\sum_{n=v}^{\infty} \frac{n!}{(n-l)!} r^{n} \approx \frac{1}{1-r} v^{-l} r^{v} 
\]

\[
\frac{p-v}{v^{2}} \geq \left( \frac{p-v}{v} - (p-1) \sum_{n=v+1}^{\infty} \frac{p-n}{n} \right)
\]

**Proof of lemma 2** At \( x_{1} = x_{2} = \infty \) the proof is trivial. Let now \( x_{1} < \infty \). Then \( \mathcal{F}_{\rho(D_{s,t})} = \mathcal{F}_{Y_{s,t}} \) where \( d_{s} = d_{\rho(D_{s,t})} \). \( \forall \varepsilon > 0 \exists v_{0} \) such that \( d_{s}(x,y) \geq e^{-(x+y)\varepsilon} \rho(D_{s,t})(x-y) \) at \( v(x,y) \geq v_{0} \). On the other hand, \( \mathcal{Y}_{s,t}(x) - \mathcal{Y}_{s}(y) = R_{v}(x) - R_{v}(y) \) at \( v(x,y) = v \), where \( R_{v}(x) = \sum_{n=v}^{\infty} \phi_{i} \left( \frac{x}{p^{n+1}} \right) \frac{n!}{(n-l)!} s^{n-l} \). Therefore, \( |d_{1}(x,y) - d_{2}(x,y)| \leq |R_{v}(x) - R_{v}(y)| + |R_{v}(y) - R_{v}(y)| \). Using (41) and the equation \( \rho(D_{s,t})(x) = v(x)^{l}|s|^{n} \), one can choose \( v_{0} \) such that \( |d_{1}(x,y) - d_{2}(x,y)| \leq e^{2}/(1 + |s|) \| \phi_{1} - \phi_{2} \|_{\infty}^{(p)} \rho(D_{s,t})(x) \) at \( v > v_{0} \). Thus, \( |d_{1}(x,y) - d_{2}(x,y)|/(d_{1}(x,y) + d_{2}(x,y)) \leq 2(1 - 1(|s|))e^{2}(\rho_{\min(x_{1},x_{2})}^{2} + 2\varepsilon) \| \phi_{1} - \phi_{2} \|_{\infty}^{(p)} \). Now the proof is obtained immediately from (4). □

**Proof of proposition 5** If \( \phi \) is continuous on \([0, 1] \), then it is a uniformly continuous function and the proof is trivial. Each \( a_{i} \) has the form \( a_{i} = l_{i}/p^{k(i)} \). Let \( m_{0} > k(i) \), \( a_{0} = 0 \), \( a_{N+1} = 1 \) and \( S_{i} = [a_{i}, a_{i+1}) \). It is easy to show that, if \( q \in S_{i} \cap Z(p^{\infty}) \), then also
\[ [q]^m \equiv p^{-m} [p^mq]_p \in S_i \] at \( m > m_0 \), and, therefore, \( \| \phi - [\phi]_m \| \leq \max_{i=0,N}(\sup\{|\phi(q) - [\phi]_m(q)| : q \in S_i \cap Z(p^\infty)\}) \). The function \( \phi \) is uniformly continuous on \( S_i \), since \( \phi \) is continuous on \( S_i \) and \( \exists \lim_{x \to a_{i+1}} \phi(x) \). Therefore, \( \lim_{m \to \infty} \sup\{|\phi(x) - [\phi]_m(x)| : x \in S_i\} = 0. \]

**Proof of proposition 6** From theorem IV.115 [20] it follows that, if there exists a \( \chi \)-summable function \( g \) such that \( \forall x \in \Lambda |\partial_s^l \partial_s^k f(\Psi_s^\phi(x))| \leq g(x) \), then there exists the integral \( I_s^4(f) \) and the equation (37) holds. One can easily show that, for \( \partial_s^l \partial_s^k f(\Psi_s^\phi(x)) \), the following representation takes place

\[
\partial_s^l \partial_s^k f(\Psi_s^\phi(x)) = \sum_{q,q} (\partial_s^l \partial_s^k f)(\Psi_s^\phi(x)) P_s^q(x)P_s^q(x),
\]

where \( P_s^q(x) \) are degree \( q \) polynomials of \( \partial_s^l \Psi_s^\phi(x) \) (\( k = 0, \ldots, q \)). It can be proved that \( \forall q = 0, 1, 2... \) the following inequalities hold

\[
\sup_{|x|_p \leq 1} |\partial_s^l \Psi_s^\phi(x)| \leq \frac{q!}{(1-|s|)^q}.
\]

From this one obtains from these \( \sup_{x \in \Lambda} |\partial_s^l \Psi_s^\phi(x)| \leq |s|^{-n} \forall n > 0 \). Therefore, if \( \Lambda \) is bounded, then \( \Psi_O(\Lambda) \) is bounded too. Now it follows from (43) that \( |\partial_s^l \partial_s^k f(\Psi_s^\phi(x))| \leq 1 \forall x \in \Lambda \). To prove assertion 2) note that \( I_s^4(f) = I_s^4(\Psi_s^\phi(x)) + \sum_{n=1}^{\infty} I_s^4(\Psi_s^\phi(x)) \), where \( S_n = \{x \in \mathbb{Q}_p : |x|_p = p^n \} \). Since \( |\Psi_s^\phi(x)| \geq \delta_s^\phi|s|^{-n}, \forall x \in S_n \), one can prove using \( \chi(S_n) = \frac{(p-1)}{p^n}p^n \) that the function \( g \) can be chosen as follows

\[
g(x) = \text{Ind}_{\mathbb{Z}_p}(x) + \frac{1}{\delta_s^\phi} \sum_{n=1}^{\infty} |s|^{p^n} \frac{1}{\chi(S_n)} \text{Ind}_{S_n}(x),
\]

where \( \text{Ind}_B \) denotes the indicator of the set \( B \). To conclude the proof, it remains to note that \( \Psi_O(\Lambda) \) is an open set, since the holomorphic map \( \Psi_s^\phi \) is open (proposition 2.1 part I [21]). \( \square \)

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Figure 2: The continuous transition between the 1- and 2-dimensional spaces along the path drawn in figure 1 is shown.
Figure 3: The image of embedding $\Omega^{\phi}_{\nu,\alpha} : \Sigma\{a\} \mapsto \mathbb{R}^3$ at $a_k = (k + 2)!$, $\nu = 1.001 , \alpha = 2$ and $\phi(t) = \exp(i2\pi t)$. The Hausdorff dimension of this $\{a\}$-adic solenoid image $= 1 + \nu^{-1} \approx 1.999001$.