A detailed case study of the rigid limit in Special Kähler geometry using $K3$.

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ABSTRACT

This is a résumé of an extensive investigation of some examples in which one obtains the rigid limit of $N = 2$ supergravity by means of an expansion around singular points in the moduli space of a Calabi-Yau 3-fold. We make extensive use of the $K3$ fibration of the Calabi-Yau manifolds which are considered. At the end the fibration parameter becomes the coordinate of the Riemann surface whose moduli space realises rigid $N = 2$ supersymmetry.

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Abstract: This is a résumé of an extensive investigation of some examples in which one obtains the rigid limit of $N = 2$ supergravity by means of an expansion around singular points in the moduli space of a Calabi-Yau 3-fold. We make extensive use of the $K3$ fibration of the Calabi-Yau manifolds which are considered. At the end the fibration parameter becomes the coordinate of the Riemann surface whose moduli space realises rigid $N = 2$ supersymmetry.

The vector multiplet of $N = 2$, $d = 4$ supersymmetry often occurs in the study of string dualities. This is related to the fact that $N = 2$ is the minimal supersymmetry to connect the scalars to the vectors, which in four dimensions have duality transformations between their electric and magnetic field strengths. These transformations are in a symplectic group, and therefore the structure of the manifold of the scalars also inherits this symplectic structure. The resulting geometry of these complex scalars is denoted as ‘special Kähler geometry’, and exists as well for rigid as for local supersymmetry. In both cases this geometry can be realised on complex structure moduli spaces: for rigid geometry on moduli spaces of a class of Riemann surfaces (RS), for local supersymmetry (supergravity) on moduli spaces of Calabi-Yau 3-folds (CY). The relevant objects which build the supersymmetric actions are ‘periods’, of 1-forms over 1-cycles in the case of RS, of 3-forms over 3-cycles in the case of CY. These forms depend on the moduli which are identified with the scalar fields of the supersymmetric theory.
Many relevant CY manifolds in string theory are $K3$-fibrations, and we will restrict to these. That means that the manifold can be described as a (complex 2-dimensional) $K3$ surface for which the moduli depend on the moduli of the CY but also on a complex variable, denoted as $\zeta$, which can be viewed as the third complex dimension of the CY. Thus for any fixed value of $\zeta$, the CY is a $K3$-manifold. As such e.g. the unique CY $(3,0)$-form will be represented as $d\zeta \wedge \Omega^{(2,0)}$, where the latter is the $(2,0)$-form of the $K3$. The 3-cycles of the CY manifolds on the other hand can be obtained in 2 different ways. One can consider a path between two (singular) points in the $\zeta$-base space where the same $K3$ cycle vanishes. Combining the 2-cycle above these points leads to one type of CY 3-cycles. Another one can be constructed by considering in the base space a loop around such a singular point combined again with the $K3$ 2-cycle which vanishes at that point. We can often calculate the integral over the $K3$ 2-cycles. Then the CY period reduces to the integral of a 1-form over the 1-cycle in the $\zeta$ plane. The latter is not yet a Riemann surface however.

As already mentioned, the CY moduli space has singular points where cycles degenerate. We consider an expansion around certain singular points. The expansion is as well an expansion in moduli space as in the CY coordinates. The CY moduli $z^\alpha$ become in this way a function of the expansion parameter $\epsilon$ and variables $u^i$, which will become the moduli of a Riemann surface. In this way the local geometry is expanded so that a rigid special Kähler geometry remains. In this expansion the $K3$ manifold reduces to an ALE manifold. By performing the 2-dimensional integrals, the periods of the CY reduce to periods of an element of this class of Riemann surfaces. We have made an expansion from a supergravity model to a rigid supersymmetric one.

In supergravity a rigid limit is not defined a priori. In the present framework this is reflected in that different singular points may give rise to different rigid limits. In [1] a procedure was set up to reduce the CY to an ALE manifold, leading to such rigid limits. See [2] for further references. Rather than using this reduction, we computed [3] all the periods in the picture of the $K3$-fibration. This shows explicitly how the full supergravity model approaches its rigid limit. Some cycles which do not occur in the ALE manifolds lead to periods whose contribution give in the limit $\epsilon \to 0$ (infinite Planck mass) a diverging renormalisation of the rigid Kähler potential. Thus these renormalisation effects are included in our computation. For full references see [3].

1 Special Kähler geometry and CY moduli spaces

First we summarise the relevant geometric concepts, both for the rigid and for the local case. We consider symplectic vectors $V(u)$ (rigid), resp. $v(z)$ (local) which are holomorphic functions of $r$, (resp. $n$) complex scalars $\{u^i\}$ (resp. $\{z^\alpha\}$). These are $2r$-vectors for the rigid case (in correspondence with the electric and magnetic field strengths), and $2(n+1)$-vectors in the local case (because in that case there is also the graviphoton). A symplectic inner product is defined as

$$\langle V, W \rangle = V^T Q^{-1} W ; \quad \langle v, w \rangle = v^T q^{-1} w$$

(1.1)
where $Q$ (and $q$) is a real, invertible, antisymmetric matrix (we wrote $Q^{-1}$ in (1.3) in view of the meaning which $Q$ will get in the moduli space realisations).

The Kähler potential is respectively for the rigid and local manifold

$$K(u, \bar{u}) = i \langle V(u), \bar{V}(\bar{u}) \rangle ; \quad K(z, \bar{z}) = -\log(-i \langle v(z), \bar{v}(z) \rangle).$$

(1.2)

In the rigid case there is a rigid invariance $V \rightarrow e^{i\theta V}$, but in the local case there is even a symmetry with a holomorphic function: $v(z) \rightarrow e^{f(z)}v(z)$, because this gives a Kähler transformation $K(z, \bar{z}) \rightarrow K(z, \bar{z}) - f(z) - \bar{f}(z)$. Because of this local symmetry we have to introduce covariant derivatives $D_{\alpha}v = \partial_{\alpha}v = 0$ and $D_{\alpha}v = \partial_{\alpha}v + (\partial_{\alpha}K)v$ (There exists also a more symmetrical formulation). In any case we still need one more constraint (leading to the ‘almost always’ existence of a prepotential), which is for rigid, resp. local supersymmetry:

$$\langle \partial_i V, \partial_j V \rangle = 0 ; \quad \langle D_{\alpha}v, D_{\beta}v \rangle = 0. \quad (1.3)$$

There are further global requirements; for an exact formulation we refer to [3].

Local special Kähler geometry is realised in moduli spaces of CY manifolds. Consider a CY manifold with $h^{21} = n$. It has $n$ complex structure moduli to be identified with the complex scalars $z^{\alpha}$. There are $2(n+1)$ 3-cycles $c_{\Lambda}$, whose intersection matrix will be identified with the symplectic metric $q_{\Lambda\Sigma} = c_{\Lambda} \cap c_{\Sigma}$. One identifies $v$ with the ‘period’ vector formed by integration of the $(3,0)$ form over the $2(n+1)$ cycles:

$$v = \int_{c_{\Lambda}} \Omega^{(3,0)} ; \quad D_{\alpha}v = \int_{c_{\Lambda}} \Omega_{(\alpha)}^{(2,1)} .$$

(1.4)

Rigid special Kähler geometry is realised in moduli spaces of RS. A RS of genus $g$ has $g$ holomorphic $(1,0)$ forms. Now in general we need a family of Riemann surfaces with $r$ complex moduli $u^i$, such that one can isolate $r$ $(1,0)$-forms which are the derivatives of a meromorphic 1-form $\lambda$ up to a total derivative:

$$\gamma_i = \partial_i \lambda + d\eta_i ; \quad \alpha = 1, \ldots, r \leq g . \quad (1.5)$$

Then one should also identify $2r$ 1-cycles $c_\alpha$ forming a complete basis for the cycles over which the integrals of $\lambda$ are non-zero. We identify $V = \int_{c_{\alpha}} \lambda$, but it should be clear that all this is much less straightforward then in the CY moduli space.

## 2 Description of a Calabi-Yau moduli space

We present here the description of one of the examples which we use in [3], i.e. a CY space with $n = h^{12} = 3$. First one introduces a complex 4-dimensional weighted projective space in which points are equivalence classes $(x_1, x_2, x_3, x_4, x_5) \sim (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4, \lambda^4 x_5)$. The CY manifold $X_{24}[1, 1, 2, 8, 12]$ is a 3-dimensional submanifold of this projective space, determined by a polynomial equation $W = 0$, of degree 24. The manifold which we use, $X_{24}^*[1, 1, 2, 8, 12]$, has global identifications

$$x_j \simeq \exp(n_j \frac{2\pi i}{24}) x_j$$

$$n_1, n_2, n_3, n_4, n_5 = m_1(1, -1, 0, 0, 0) + m_2(-1, -1, 2, 0, 0) .$$

(2.1)
where \( m_i \in \mathbb{Z} \). The most general polynomial of degree 24 which is invariant under these identifications depends on 11 parameters, i.e. moduli of the CY manifold. However, they are not independent: there are still compatible redefinitions of the \( x \) variables, i.e. compatible with (2.1) and the weights. This leaves at the end in this example 3 independent moduli. We learned the advantages of not restricting immediately to one gauge choice in this moduli space.

Now the \( K3 \) fibration is exhibited by performing the change of variables
\[
x_0 = x_1 x_2 ; \quad \zeta = (x_1/x_2)^{24} .
\]
We take a partial gauge choice with one remaining scale invariance, corresponding to a rescaling of \( x_0 \). Then the polynomial looks like
\[
W = \frac{1}{12} B' x_0^{12} + \frac{1}{12} x_3^{12} + \frac{1}{3} x_4^3 + \frac{1}{2} x_5^2 - \psi_0 (x_0 x_1 x_3 x_5) - \frac{1}{6} \psi_1 (x_0 x_3)^6 ,
\]
with
\[
B' = \frac{1}{2} B \left( \zeta + \zeta^{-1} \right) - \psi_s .
\]
We thus have a description as a \( K3 \) manifold \( X_{12}[1, 1, 4, 6] \), with a projective moduli space \( \{ B', \psi_0, \psi_1 \} \). Here \( B' \) is a function of moduli \( B \) and \( \psi_s \) of the CY manifold and contains the dependence on the base of the \( K3 \) fibration \( \zeta \).

The manifolds are singular when simultaneously \( W = 0 \) and \( dW = 0 \). For the \( K3 \) this happens for
\[
a) \ B' = (\psi_1 + \psi_0^6)^2 ; \quad b) \ B' = \psi_1^2 ; \quad c) \ B' = 0 .
\]
The singularities then occur for a specific point on \( K3 \). The cases a) and b) are \( A_1 \)-type singularities. They coincide if \( \psi_0^6 = 0 \), in which case the singularity becomes of type \( A_2 \), and if \( \psi_0^6 = -2 \psi_1 \), in which case the singularity becomes of type \( A_1 \times A_1 \). We will concentrate here on the first possibility: the \( A_2 \) singularity.

For the CY to be singular, we should also have that the derivative of \( W \) with respect to \( \zeta \) is zero, which leads to \( \frac{\partial B'}{\partial \zeta} = 0 \), satisfied for all \( \zeta \) if \( B = 0 \). So we have a full \( \mathbb{P}^1 \) of singularities. In the CY moduli space, an appropriate expansion is obtained by taking
\[
B = 12 \epsilon ; \quad \psi_1^2 = -\psi_s + \epsilon u_1 ; \quad (\psi_1 + \psi_0^6)^2 = -\psi_s + \epsilon (u_1 + 2 u_0^6) .
\]
After a corresponding expansion of the variables of the CY space around its singular point, the polynomial reduces to one defining an ALE manifold of type \( A_2 \), with moduli \( u_1 \) and \( u_2 \).

3 Periods, monodromies, intersection matrix

The main work is to obtain the periods for the \( K3 \) fibre, after which the CY periods remain as integrals over 1-cycles. First we use the Picard-Fuchs equations, which are differential equations for the integrals of the \( (2,0) \) form over the 2-cycles. It turns
out that working in the enlarged moduli space (where gauges are not yet fixed) simplifies the derivation of these equations, using toric geometry in disguise without introducing all the formalism, and avoiding its higher order differential equations. The independent solutions give a basis for the periods. The periods are functions of the moduli appearing in the polynomial, which have branch points in singular points, and we have to choose the position of the cuts in this moduli space. We choose a basis of solutions in one sheet of this moduli space. Continuing the periods around such singular points, we cross the cuts, and arrive to the same values of the moduli. Reexpressing the analytically continued periods in the previously chosen basis gives rise to the monodromy matrices. A generic basis of solutions to the differential equations does, however, not correspond to integrals over integer cycles. Therefore we need also supplementary methods.

In some examples we can integrate over one cycle and analytically continue. We start by integrating the (2, 0) form over a cycle which is known in the neighbourhood of the ‘large complex structure’ singular point. Then this period is analytically continued to other regions. By following its analytic continuation we also obtain the other periods. Because we start here from an integral cycle, we obtain the monodromy matrices in an integral basis.

In the example described above, the strategy which we plan to use for CY, can already be used for the $K3$ periods themselves. Indeed in this case the $K3$ itself is a torus fibration. The forms and cycles can be decomposed in forms and cycles on the torus, fibred over a $\mathbb{P}^1$. It has the advantage that we start from the torus, where we know already a basis of cycles and its intersection matrix.

The result is that we find expressions corresponding to 4 $K3$-cycles of which one vanishes at singularity a), called $v_\alpha$, one at b), called $v_\beta$, and two vanish at c), which we will call $t_\alpha$ and $t_\beta$. The points of singularity of the $K3$ manifold each occur at two points in the $\zeta$-plane, one inside the circle $|\zeta| = 1$, and one outside. We will take the cuts from the former to $\zeta = 0$, and from the latter to $\infty$. We can then construct 4 CY cycles by taking the paths in $\zeta$ between the two points with the same vanishing $K3$-cycle and combining these with the corresponding 2-cycle in $K3$. These are called $V_{v_\alpha}, V_{v_\beta}, V_{t_\alpha}$ and $V_{t_\beta}$. On the other hand we can combine the circle $|\zeta| = 1$ with the 4 $K3$ cycles, obtaining the CY-cycles $T_{v_\alpha}, T_{v_\beta}, T_{t_\alpha}$ and $T_{t_\beta}$.

4 The rigid limit

Considering then again the expansion of the moduli as in (2.3), we see that (2.4) implies that the singularities are in leading order of $\epsilon$ at

\begin{align*}
v_\alpha : \frac{1}{2}(\zeta + \zeta^{-1}) &= \frac{1}{12}(u_1 + 2u_0^6) ; \quad &v_\beta : \frac{1}{2}(\zeta + \zeta^{-1}) &= \frac{1}{12}u_1 \\
t_\alpha, t_\beta : \frac{1}{2}(\zeta + \zeta^{-1}) &= \frac{1}{12}\epsilon \psi_\delta.
\end{align*}

The former thus keep their position, while the latter move to $\zeta = 0$ and $\zeta = \infty$ when $\epsilon \to 0$. The cycles $V_{t_\alpha}$ and $V_{t_\beta}$ thus become infinitely stretched, and the
corresponding periods will get a log $\epsilon$ dependence. The dependence on $\epsilon$ is obtained from studying the $\epsilon$-monodromies.

By a complex basis transformation one can isolate the different types of small $\epsilon$ behaviour of the periods, and one rewrites the period vector $v$ as

$$v = v_0(\epsilon) + \epsilon^{1/3}v_1(u) + v_2(\epsilon).$$

The relevant term will be the $v_1$ term, which has only 4 non-zero components, corresponding to the cycles $V_{v_\alpha}, V_{v_\beta}, T_{v_\alpha}$ and $T_{v_\beta}$, i.e. related to the singularities which remain at finite $\zeta$ in the limit. $v_0$ is independent of the moduli. It contains 2 non-zero components, one of which starts with a constant, and the other one has a logarithmic dependence alluded to above. Finally $v_2(\epsilon)$ has as lowest order terms with $\epsilon^{2/3}$ and $\epsilon^{2/3}\log \epsilon$. These appear in the two remaining components of $v$. The intersection matrix is in this basis (complex antihermitian, not an integral basis) block diagonal in the mentioned $4 + 2 + 2$ components. For the Kähler potential this leads to

$$K = -\log (-i < v, \bar{v} >)
= -\log \left( -i < v_0(\epsilon), \bar{v}_0(\epsilon) > -i|\epsilon|^{2/3} < v_1(u), \bar{v}_1(u) > + R(\epsilon, u, \bar{u}) \right)$$

$$\approx -\log (-i < v_0(\epsilon), \bar{v}_0(\epsilon) >) - \frac{|\epsilon|^{2/3}}{< v_0(\epsilon), \bar{v}_0(\epsilon) >} < v_1(u), \bar{v}_1(u) > + \ldots ,$$

where $R(\epsilon, u, \bar{u})$ are higher order terms. The first term in the final expression is irrelevant, not depending on the moduli. The second one is with a diverging renormalisation (produced by the cycles which disappear in the ALE limit) of the form which it should have for a rigid special Kähler manifold: the first of [1.2]. Moreover it coincides with the one the $SU(3)$ Seiberg–Witten Riemann surface.

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