ON DEFORMATIONS OF COMMUTATION RELATION ALGEBRAS

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Abstract

This paper is on $C$-symmetric creation and annihilation operators, which are constructed on Wick’s algebras which fulfil consistency conditions. The essential assumption is that every algebraic action must be constant on equivalence classes. All consistency conditions follow from the above assumption. In this way we obtain well defined quotient algebras with some additional relations.

PACS numbers: 03.65.Ca, 03.65.Fd.
I Introduction.

We would like to present the construction of deformed algebras of commutation relation. An example of a deformed commutation relation is an interpolation of bosonic and fermionic statistics, e.g. [1, 2, 3, 4, 5]. The problem of additional relations between pairs of annihilation operators and between creation operators was considered for example in [6, 7]. The construction of annihilation operators is based on non commutative differential calculus [8, 9, 10] as a result of quantum groups, see [11, 12, 13, 14].

In section II we applied the contraction notion (evaluation [15, 16]) defined in the algebraic way, see [17] on the tensor algebra of linear space $E$ by taking generalised twist $C$ between dual spaces of $E$ and $E$. This contraction satisfies $C$ - deformed Leibniz rule. In this section we introduced partial representation of the creation and annihilation operators defined on algebra $TE$. In this representation we have commutation relations between creators and annihilators. In this way we have obtained Wick’s algebra in which any sequences consisting of creation and annihilation operators can be arranged in Wick’s way, see [18]. If we want to obtain additional relation, we have to divide the algebra $TE$ by $N$ graded ideal $J \subset TE$, generated by twist $B \in \text{End}(E \otimes E)$ and construct the subspace $J^* \subset E^* \otimes E^*$, generated by twist $\tilde{B} \in \text{End}(E^* \otimes E^*)$. After this we have constructed representation of the annihilation and creation on the algebra $A = TE/J$, which is projected from the representation defined on the algebra $TE$. If we want to make the above construction from the twists $B, C$ they have to satisfy certain relations, from which it appears that partial representation must preserve the ideal $J$. In this way we have obtained non commutative differential calculus on the quantum plane see [19].

The relation between annihilators and the same relations between creators is necessary condition for introduction of inner product such that $<d^+ x, y> = <x, dy>$ $x, y \in A$. That is why we have introduced $\pi^*$ - invariant property for the contraction. In the section III we present deformed commutation relations:

$$[a_i, a^+_j]_C = \delta_{i,j} \ [a^i, a^j]_B = 0 \ [a_i, a^+_j]_{\tilde{B}} = 0$$

as a consequence of definitions introduced in section II.

To obtain representation on the $A$, the operators defined on algebra $TE$ have to preserve the ideal contained in $TE$, which leads us to the consistency conditions between tensors $B, \tilde{B}, C$. The conditions which satisfy assumptions of the theorem [V,3] are enough to construct the representation defined on algebra $A$. In this case tensors $B, C$ satisfy generalised braided symmetry which are in [5, 9], where there are similar consistency conditions.

The contraction has $\pi^*$ - invariant property when tensors $B, C$ satisfy assumptions of the
In section IV we present example which fulfill the assumptions of the theorems in section IV. Of course bosons and fermions satisfy these assumption, however we has shown a q-deformed example.
II Definitions and notions.

Let us assume that $E$ is a linear space of finite dimension over field $\mathbb{C}$ with basis $\{f_i : i \in I\}$ and $E^*$ is the dual space to $E$ with $\{e_j : j \in I\}$ dual basis. Let $E^0 := \mathbb{C}$ and $E^\otimes n = E \otimes E^\otimes (n-1)$ be the $n$-th tensorian product of the space $E$.

**Definition II.1** Let $C : E^* \otimes E \mapsto E \otimes E^*$ be a map defined in the following way:

$$C(e_i \otimes f_j) = \sum_{k,l} c_{i,j,k,l} f_k \otimes e_l : c_{i,j,k,l} \in \mathbb{C}$$

Now we define: $C_n^{(i)} \in \text{Hom}(E^\otimes i-1 \otimes (E^* \otimes E) \otimes E^\otimes n-i), E^\otimes i-1 \otimes (E \otimes E^*) \otimes E^\otimes n-i)$ by

$$C_n^{(i)} = 1_{i-1} \otimes C \otimes 1_{n-i}$$

for $i \in \{1 \ldots n\}$

and

$$C_n^{(0)} = 1 = id \in \text{End}(E^* \otimes E^\otimes n)$$

where $1_k = id_{E^\otimes k}$. Let $ev : E^* \otimes E \mapsto \mathbb{C}$ such that:

$$ev(e_i \otimes f_j) = \delta_{i,j}$$

and extended by linearity to $E^* \otimes E$.

**Definition II.2** The $C$ twisted contraction $ct_n$ with respect to given elementary twist $C$ is a mapping:

$$ct_n : E^* \otimes E^\otimes n \mapsto E^\otimes n-1$$

such that:

$$ct_n = \sum_{k=1}^{n}(1_{k-1} \otimes ev \otimes 1_{n-k})(C_n^{(k-1)} \ldots C_n^{(0)})$$

Let us observe that $ct_1 = ev$.

Let $A$ be any mapping:

$$E^* \otimes E^\otimes n \ni x \otimes y \mapsto A_n(x \otimes y) \in E^\otimes n-1$$

and $W_i \in \{E, E^*\}$ for $i \in \{1 \ldots k-1\}$ and $k \in \mathbb{N}$, then we can define

$$A_{n,W_1 \ldots W_{k-1}}^{(k)} \in \text{Hom}(W_1 \otimes \ldots \otimes W_{k-1} \otimes E^* \otimes E^\otimes n, W_1 \otimes \ldots \otimes W_{k-1} \otimes E^* \otimes E^\otimes (n-1))$$
such that:

\[ A^{(k)}_{n,W_1...W_{k-1}} = 1_{W_1...W_k} \otimes A_n \quad \text{and} \quad k \in \mathcal{N}, \]

where \( 1_{W_1...W_k} \) is the identity on the \( W_1 \otimes \ldots \otimes W_{k-1} \). In the natural way we can introduce an extension of \( A^{(k)}_n \):

\[ \tilde{A}^{(k,m)}_{n,W_1...W_{k-1},W_1...W_m} = 1_{W_1...W_{k-1}} \otimes A_n \otimes 1_{W_1...W_m} \]

From now on we will identify \( \tilde{A}^{(k,m)}_{n,W_1...W_{k-1},W_1...W_m} \) with \( A^{(k)}_{n,W_1...W_{k-1}} \) becouse:

\[ \tilde{A}^{(k,m)}_{n,W_1...W_{k-1},W_1...W_m} = A^{(k)}_{n,W_1...W_{k-1}} \otimes id_{W_1 \otimes \ldots \otimes W_m} \]

If \( W_1 = \ldots = W_{k-1} = E \), then we will accept the following notation: \( A^{(k)}_n = A^{(k)}_{n,W_1...W_{k-1}} \), if \( W_1 = E^* \) and \( W_2 = \ldots = W_{k-1} = E \), then \( A^{(k)}_{n,E^*} = A^{(k)}_{n,W_1...W_{k-1}} \).

Is easy to see that:

\[ ct^{(1)}_n = \sum_{i=1}^{n} ct^{(i)}_1 C^{(i-1)} \ldots C^{(1)} \]

where \( ct^{(i)}_1 = 1_{i-1} \otimes ct_1 \otimes 1_{n-i} \).

From the definition of contraction we obtain simple properties.

**Lemma II.1** The contraction has the following properties:

1. contraction is a linear on \( E^* \otimes E^\otimes n \)

2. \( C \) - Leibniz rule:

\[ ct^{(k)}_n = ct^{(k+1)}_1 \circ C^{(k)} \quad k \in \mathcal{N} \]

3.

\[ ct^{(k)}_n = ct^{(k)}_1 + \sum_{i=2}^{n} ct^{(i+k-1)}_1 C^{(k+i-2)} \ldots C^{(k)} \]

4.

\[ ct^{(1)}_n = ct^{(1)}_m + ct^{(m+1)}_{n-m} C^{(m)} \ldots C^{(1)} \]

**Proof:** The point 1 is obvious, it remains to prove the points 2, 3. Firstly we provide the proof this for \( k = 1 \). If \( n = 2 \) then \( C \) - Leibniz rule is simply satisfied. Then for \( n > 2 \), we have:

\[ ct^{(1)}_{n+1}(e_i \otimes f_j \otimes y) = ct^{(1)}_1(e_i \otimes f_j \otimes y) + \sum_{s=2}^{n+1} ct^{(s)}_1 C^{(s-1)} \ldots C^{(1)}(e_i \otimes f_j \otimes y) \]
\[
= ct_1^1(e_i \otimes f_j \otimes y) + \sum_{k,l}^{n+1} \sum_{s=2}^{n} c_{i,j,k,l} \, ct_1^{(s)} C^{(s-1)} \ldots C^{(2)}(f_k \otimes e_l \otimes y)
\]

\[
= ct_1^1(e_i \otimes f_j \otimes y) + \sum_{k,l} c_{i,j,k,l} \, f_k \otimes \sum_{s=1}^{n} ct_1^{(s)} C^{(s-1)} \ldots C^{(1)}(e_l \otimes y)
\]

\[
= ct_1^1(e_i \otimes f_j \otimes y) + \sum_{k,l} c_{i,j,k,l} \, f_k \otimes ct_1^{(1)}(e_l \otimes y)
\]

\[
= ct_1^1(e_i \otimes f_j \otimes y) + \sum_{k,l} c_{i,j,k,l} \, f_k \otimes ct_1^{(2)}(e_l \otimes y)
\]

\[
= ct_1^1(e_i \otimes f_j \otimes y) + ct_1^{(2)} C^{(1)}(e_i \otimes f_j \otimes y)
\]

and for arbitrary \(k \in \mathcal{N}, w_i \in W_i\), where \(i \in \{1 \ldots k - 1\}, x \in E^*, \ y \in E^{\otimes n}\) :

\[
ct_n^{(k)}(w_1 \otimes \ldots \otimes w_{k-1} \otimes x \otimes y) = (w_1 \otimes \ldots \otimes w_{k-1} \otimes ct_n^{(1)}(x \otimes y))
\]

\[
= (w_1 \otimes \ldots \otimes w_{k-1} \otimes (ct_1^{(1)} + ct_n^{(2)} C^{(1)})(x \otimes y))
\]

\[
= (ct_1^{(k)} + ct_n^{(k+1)} C^{(k)})(w_1 \otimes \ldots \otimes w_{k-1} \otimes x \otimes y)
\]

Now we prove the point 3 as a simple consequence of definition II.2 :

\[
ct_n^{(k)}(w_1 \otimes \ldots \otimes w_{k-1} \otimes x \otimes y) = (w_1 \otimes \ldots \otimes w_{k-1} \otimes ct_n^{(1)}(x \otimes y))
\]

\[
= (w_1 \otimes \ldots \otimes w_{k-1} \otimes \sum_{i=1}^{n} ct_1^{(i)} C^{(i-1)} \ldots C^{(1)}(x \otimes y))
\]

\[
= (ct_1^{(k)} + \sum_{i=2}^{n} ct_1^{(k+i-1)} C^{(i+k-2)} \ldots C^{(k)})(w_1 \otimes \ldots \otimes w_{k-1} \otimes x \otimes y)
\]

The point 4 follows from the 3rd point. This completes the proof

It is interesting to note that \(C\) - Leibniz rule and \(A_1\) gives the corresponding \(A_n \in Hom(E^* \otimes E^{\otimes n}, E^{\otimes n-1})\) in unique way.

**Lemma II.2** If \(A_n \in Hom(E^* \otimes E^{\otimes n}, E^{\otimes n-1})\) such that:

1. \(A_1 x \otimes y = X(y)\)
2. \( \exists C \in \text{Hom}(E^* \otimes E, E \otimes E^*) \) and

\[
A_n^{(1)} = A_n^{(1)} + A_{n-1}^{(2)} C^{(1)} \text{ for } n \in \{2, 3, \ldots\}
\]

then

\[
A_n^{(1)} = \sum_{i=1}^{n} A_i^{(i)} C^{(i-1)} \ldots C^{(1)}
\]

**Proof:** For \( n = 2 \) we have:

\[
A_2^{(1)} = A_1^{(1)} + A_1^{(2)} C^{(1)}
\]

Let us suppose that for \( n = m \) the lemma is true, then:

\[
A_{m+1}^{(1)}(e_i \otimes f_j \otimes y) = (A_1^{(1)} + A_m^{(2)} C^{(1)})(e_i \otimes f_j \otimes y)
\]

\[
= A_1^{(1)}(e_i \otimes f_j \otimes y) + \sum_{k,l} c_{i,j,k,l} A_m^{(2)}(f_k \otimes e_l \otimes y)
\]

\[
= A_1^{(1)}(e_i \otimes f_j \otimes y) + \sum_{k,l} c_{i,j,k,l} f_k \otimes A_m^{(1)}(e_l \otimes y)
\]

\[
= A_1^{(1)}(e_i \otimes f_j \otimes y) + \sum_{k,l} c_{i,j,k,l} f_k \otimes \sum_{s=1}^{m} A_1^{(s)} C^{(s-1)} \ldots C^{(1)}(e_l \otimes y)
\]

\[
= A_1^{(1)}(e_i \otimes f_j \otimes y) + \sum_{k,l} c_{i,j,k,l} \sum_{s=1}^{m} A_1^{(s+1)} C^{(s)} \ldots C^{(2)}(f_k \otimes e_l \otimes y)
\]

\[
= A_1^{(1)}(e_i \otimes f_j \otimes y) + \sum_{s=1}^{m} A_1^{(s+1)} C^{(s)} \ldots C^{(2)}(e_i \otimes f_j \otimes y)
\]

\[
= \sum_{s=1}^{m+1} A_1^{(s)} C^{(s-1)} \ldots C^{(1)}(e_i \otimes f_j \otimes y)
\]

The proof is completed \( \square \)

**Definition II.3** The operators \( a_{n,i} \) and \( a_{n,i}^+ \) for \( n \in \mathbb{N} \) we call respectively partial annihilation and creation \( C \)-operators, when we have:

\[
a_{n,i}(y) := A_n^{(1)}(e_i \otimes y) \quad y \in E \otimes^n
\]

\[
a_{n,j}^+(y) := f_j \otimes y \quad y \in E \otimes^n
\]
Let $B \in Hom(E \otimes E, E \otimes E)$. We can define the subspace in $E \otimes E$ by the following relation:

$$J_2 = \{ x \in E^{\otimes 2} \mid \exists z \in E^{\otimes 2} \; x = z - B(z) \} = Im(1 - B)$$

And more general:

$$J_n = E \otimes J_{n-1} + J_{n-1} \otimes E \quad \text{for } n = 3, 4, ...$$

So we can divide $E^{\otimes n}$ by $J_n$ obtaining linear subspace $A^{\otimes n} := E^{\otimes n}/J_n$

Let

$$J = \bigoplus_{n \in \mathbb{N}} J_n$$

This is a minimal ideal in the tensor algebra $TE$ containing $J_2$. In this case we can simply introduce quotient algebra $A = TE/J$, which is a $\mathbb{N}$ graded space: $A = \bigoplus_{n \geq 0} A^{\otimes n}$

We have canonical homomorphism $\pi_n$, which projects space $E^{\otimes n}$ onto $A^{\otimes n}$. For the second order it gives:

$$E^{\otimes 2} \ni x \otimes y \mapsto \pi_2(x \otimes y) = [x \otimes y] \in A^{\otimes 2}$$

and $\pi_2$ satisfies the relation:

$$\pi_2 = \pi_2 \circ B \quad \text{(1)}$$

Every element of $A^{\otimes n}$ is in image of the projector $\pi_n$:

for every $y \in A^{\otimes n}$ there exists $y_1 \in E^{\otimes n}$ such that $y = \pi_n(y_1)$

So, we can introduce the canonical projector $\pi$ of the space $TE$ onto $A$ space:

$$\pi : TE \mapsto A \quad \text{(2)}$$

Let $\tilde{B} \in End(E^* \otimes E^*)$, then we can define $J_2^*$ in the following way:

$$J_2^* = Im(1 - \tilde{B})$$

Now we introduce $B$ - commutator:

$$[a^+_{n+1,i}, a^+_{n,j}]_B := a^+_{n+1,i} \circ a^+_{n,j} - \sum_{k,l} b_{i,j,k,l} a^+_{n+1,k} \circ a^+_{n,l}$$

and also $\tilde{B}$ and $C$ commutators:

$$[a^-_{n-1,i}, a^-_{n,j}]_B := a^-_{n-1,i} \circ a^-_{n,j} - \sum_{k,l} \tilde{b}_{i,j,k,l} a^-_{n-1,k} \circ a^-_{n,l}$$
\[ [a_{n+1,i}, a^+_{n,j}]_C := a_{n+1,i} \circ a^+_{n,j} - \sum_{k,l} c_{i,j,k,l} a^+_{n-1,k} \circ a_{n,l} \]

where \( b_{i,j,k,l} \tilde{b}_{i,j,k,l} c_{i,j,k,l} \) are matrix elements of \( B, \tilde{B}, C \) respectively.

To complete these definitions, we would like to introduce a definition of \( \pi^* \) - invariant mapping on the whole space \( TE \), which will be play important role in Proposition II.2 below.

**Definition II.4** Let \( w \in \text{Hom}(E^* \otimes TE, TE) \) lowering the level of tensor by 1, then we will say that \( w \) is \( \pi^* \) - invariant on \( TE \) when:

\[ w^{(1)}_{n-1} \circ w^{(2)}_{n,E^*} (J^*_2 \otimes E^\otimes n) \subset J_{n-2} \quad \text{for} \quad n = (2, 3, \ldots) \]

where \( w_n \in (E^* \otimes E^\otimes n, E^\otimes n-1) \) are the corresponding components of \( w \).

**Definition II.5** Let \( d_{n,i} : A^{\otimes n} \mapsto A^{\otimes (n-1)} \) and \( d_{n,i} : A^{\otimes n} \mapsto A^{\otimes (n+1)} \) for \( n \in \mathbb{N} \), then \((d_{n,i}, d^+_{n,i})\) is \( \pi^* \) - invariant \( C \) - representation of the annihilation and creation operators, when exists the \( C \) - partial representation \( a_{n,i}, a^+_{n,i} \), which satisfies the following conditions:

1. The operators \( d_{n,i} \) fulfil the following diagram

\[
\begin{array}{ccc}
E^{\otimes n} & \xrightarrow{a_{n,i}} & E^{\otimes n-1} \\
\pi_n \downarrow & & \downarrow \pi_{n-1} \\
A^{\otimes n} & \xrightarrow{d_{n,i}} & A^{\otimes n-1}
\end{array}
\]

2. The operators \( d^+_{n,i} \) fulfil the following diagram

\[
\begin{array}{ccc}
E^{\otimes n} & \xrightarrow{a^+_{n,i}} & E^{\otimes n+1} \\
\pi_n \downarrow & & \downarrow \pi_{n-1} \\
A^{\otimes n} & \xrightarrow{d^+_{n,i}} & A^{\otimes n+1}
\end{array}
\]
3. The operator $a_i$ given by $(a_{n,i}(x) = ct_n^{(1)}(e_i \otimes x))$, is $\pi^*$ - invariant operator see definition II.4.

If we have operators $d_{n,i}$, $d_{n,i}^+$ defined on $A^\otimes n$ then we can construct operators $d_i$, $d_j^+$ defined on $A$ algebra in the following way:

$$d_i = \bigoplus_{n \in \mathbb{N}} d_{n,i}$$

$$d_j^+ = \bigoplus_{n \in \mathbb{N}} d_{n,i}^+$$

### III On $C$ and $B$ - relations.

In this section we would like to calculate $C$ commutation relation in the representation introduced in section II.

Firstly we show the theorem given by Jörgensen, Schmith and Werner [3] :

**Theorem III.1 (Jörgensen, Schmith, Werner)** Let $(a_i, a_j^+)$ be $C$ - partial representation then we have the following relation:

$$[a_i, a_j^+]_C = \delta_{i,j} \text{id}$$

**Proof:** Let $y \in E^\otimes n$ for every $n \in \mathbb{N}$, then by application of the $C$ - Leibniz rule we have:

$$a_{n+1,i} \circ a_{n,j}^+(y) = a_{n+1,i}(f_j \otimes y)$$

$$= ct_n^{(1)}(e_i \otimes f_j \otimes y)$$

$$= ct_n^{(1)}(e_i \otimes f_j \otimes y) + ct_n^{(2)}(e_i \otimes f_j \otimes y)$$

$$= \delta_{i,j} y + \sum_{k,l} c_{i,j,k,l} (f_k \otimes (ct_n(e_l \otimes y)))$$

$$= \delta_{i,j} + \sum_{k,l} c_{i,j,k,l} (a_{n-1,k}^+ \circ a_{n,l})(y)$$

So the proof is completed.

In quotient space $\mathcal{A}$ we have additional relations generated by ideal $J \subset TE$. So we have the following two propositions:

**Proposition III.2** If $d_i$, $d_j$ are $C$ - annihilation operators which are $\pi^*$ - invariant (see Definition II.3) then

$$[d_i, d_j]_B = 0.$$
PROOF: Let \( y_1 \in A^\otimes n \) and \( y_1 = \pi_n(y) \) for \( y \in E^\otimes n \). Using definition [II.3] point [I] and \( \pi^* - \) invariantness see definition [II.4], we have simple calculation:

\[
d_{n-1,i} \circ d_{n,j}(y_1) = \pi_{n-2} \circ a_{n-1,i} \circ a_{n,j}(y)
\]

\[
= \pi_{n-2} n^{(1)} c_{n-1}^{(2)} (e_i \otimes e_j) \otimes y
\]

\[
= \pi_{n-2} n^{(1)} c_{n-1}^{(2)} (e_i \otimes e_j - \tilde{B}(e_i \otimes e_j)) \otimes y
\]

\[
+ \pi_{n-2} n^{(1)} c_{n-1}^{(2)} \tilde{B}(e_i \otimes e_j) \otimes y
\]

\[
= \pi_{n-2} \sum \tilde{b}_{i,j,k,l} \pi_{n-2} c_{n-1}^{(2)} (e_k \otimes e_l \otimes y)
\]

\[
= \sum \tilde{b}_{i,j,k,l} d_{n-1,k} \circ d_{n,l}(y_1)
\]

Where \( z \in J_{n-2} \). That completed the proof \( \square \)

**Proposition III.3** If \( d_i^+, d_j^+ \) are \( C \) - creation operators which are \( \pi^* - \) invariant (see Def [II.3]) then we have the following relation:

\[
[d_i^+, d_j^+]_{B^*} = 0.
\]

PROOF: Let \( y_1 \in A^\otimes n \) and \( y_1 = \pi_n(y) \) for \( y \in E^\otimes n \). Using equation (I) and definition [II.3] point [I], we calculate:

\[
d_{n+1,i}^+ \circ d_{n,j}^+(y_1) = \pi_{n+2} \circ a_{n+1,i}^+ \circ a_{n,j}^+(y)
\]

\[
= \pi_{n+2} (f_i \otimes f_j \otimes y)
\]

\[
= \pi_{n+2} (f_i \otimes f_j - B(f_i \otimes f_j)) \otimes y
\]

\[
+ \pi_{n+2} B(f_i \otimes f_j) \otimes y
\]

\[
= \pi_{n+2} \sum \tilde{b}_{i,j,k,l} \pi_{n+2} (f_k \otimes f_l \otimes y)
\]

\[
= \sum \tilde{b}_{i,j,k,l} d_{n+1,k} \circ d_{n,l}(y_1)
\]

Where \( z \in J_{n+2} \). That completes the proof \( \square \)

**IV Consistency conditions.**

To obtain well defined algebra with additional multiplication relations, we have to guarant the constant action on the layers in algebra \( TE \). For example every contraction defined
in above section must be well defined on the layers in $TE$, hence we have the following proposition:

**Proposition IV.1** The necessary condition for the constant action on the layers of $a_i$ operator is:

$$(id - B)(id + C) = 0$$

where $C$ has the matrix elements given by $c_{j,k,l,m} = c_{j,k,l,m}$.

**Proof:** By taking elements of the basis of the space $E$, $f_j f_k$ and the dual space $E^*$ and its basis $e_i$ and using the equation (1) and C - Leibniz rule see lemma II.1 we have the simple calculation:

$$\bar{0} = c t_2 (e_i \otimes (1 - B)(f_j \otimes f_k))$$

$$= \sum_{l,m} (\delta_{j,l} \delta_{k,m} - b_{j,k,l,m}) c t_2 (e_i \otimes f_l \otimes f_m)$$

$$= \sum_{l,m} (\delta_{j,l} \delta_{k,m} - b_{j,k,l,m}) (c t_1^{(1)} + c t_1^{(2)} C^{(1)})(e_i \otimes f_l \otimes f_m)$$

$$= \sum_{l,m} (\delta_{j,l} \delta_{k,m} - b_{j,k,l,m}) c t_1^{(1)} + c t_1^{(2)} (e_i \otimes f_l \otimes f_m)$$

$$= \sum_{l,m} (\delta_{j,l} \delta_{k,m} - b_{j,k,l,m}) (c t_1^{(1)} C^{(1)})(e_i \otimes f_l \otimes f_m)$$

Then we have:

$$\sum_{l,m} (\delta_{j,l} \delta_{k,m} - b_{j,k,l,m}) (\delta_{i,l} \delta_{m,r} + c_{i,l,r,m}) f_r = 0$$

Above result is the only necessary condition to obtain well defined operator $d_i, d^+_i$ on quotient algebra $A$. This is well know as linear Wess Zumino condition see [10]. We also know the following:

**Proposition IV.2** The operators $d_i$ exist, (see Definition II.3) iff for every $n \in \mathbb{N}$ the following relation hold:

$$\pi_n(x) = \pi_n(y) \Rightarrow \pi_{n-1} \circ a_{n,i}(x) = \pi_{n-1} \circ a_{n,i}(y)$$

Now we will present the proof of the theorem given by A. Borowiec and V. Kharchenko see [8]:

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Theorem IV.3 (Borowiec-Kharchenko) If the tensors $B \in \text{End}(E \otimes E)$, and $C \in \text{Hom}(E^* \otimes E, E \otimes E^*)$ satisfy the following conditions:

1. $(1 - B)(1 + \hat{C}) = 0$ where $\hat{c}_{i,j,k,l} = c_{j,l,i,k}$

2. there exists $A \in \text{Hom}(E^* \otimes E^\otimes 2, E^\otimes 2 \otimes E^*)$ such that: $C^2C^1B^2 - B^1C^2C^1 = (1 - B^1)A$

then $a_r(J) \subset J$ where $J \subset TE$ is the minimal ideal generated by the relation $< x - Bx = 0 >$ for $x \in E \otimes E$.

Proof: At first we have to prove for $J_2$ and $J_3$. Condition for the ideal $J_2$ is described in the proposition [IV.1]. We have to prove that for every $r \in I \quad a_{3,r}(J_3) \subset J_2$. Then

$$LHS = a_{3,r}((1 - B)y_1 \otimes x_1 + x_2 \otimes (1 - B)y_2) \in J_2$$

for every $y_1, y_2 \in E \otimes E$, and $x_1, x_2 \in E$, so we have:

$$LHS = (ct^{(1)}_1 + ct^{(2)}_1 C^1 + ct^{(3)}_1 C^2 C^1)((1 - B^{(2)})e_r \otimes y_1 \otimes x_1 + (1 - B^{(3)})e_r \otimes x_2 \otimes y_2)$$

$$= (ct^{(1)}_1 + ct^{(2)}_1 C^1)((1 - B^{(2)})e_r \otimes y_1 \otimes x_1 + (1 - B^{(3)})e_r \otimes x_2 \otimes y_2)$$

$$+ ct^{(3)}_1 C^2 C^1((1 - B^{(2)})e_r \otimes y_1 \otimes x_1 + (1 - B^{(3)})e_r \otimes x_2 \otimes y_2)$$

From the assumption 1 and 2 we have:

$$(ct^{(1)}_1 + ct^{(2)}_1 C^1)(1 - B^{(2)})e_r \otimes y_1 \otimes x_1 = 0 \in E$$

$$ct^{(3)}_1 C^2 C^1 B^{(2)} = Bct^{(3)}_1 C^2 C^1 + (1 - B)ct^{(3)}_1 A$$

Then we have:

$$LHS = (ct^{(1)}_1 + ct^{(2)}_1 C^1)(1 - B^{(3)})e_r \otimes y_2 \otimes x_2 + (1 - B)ct^{3}C^2C^1 e_r \otimes y_1 \otimes x_1$$

$$+ (1 - B)ct^{(3)}_1 A e_r \otimes y_1 \otimes x_1 + ct^{(3)}_1 C^2 C^1(1 - B^{(3)})e_r \otimes x_2 \otimes y_2$$

The second and the third terms belong to $J_2$, so we have:

$$LHS = (ct^{(1)}_1 + ct^{(2)}_1 C^1 + ct^{(3)}_1 C^2 C^1)((1 - B^{(3)})e_r \otimes x_2 \otimes y_2 + z$$

$$= ct^{(1)}_3 (1 - B^{(3)})e_r \otimes x_2 \otimes y_2 + z$$

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where $z \in J_2$, but

$$
ct_3^{(1)} (1 - B^{(3)}) e_r \otimes x_2 \otimes y_2 = (ct_1^{(1)} + ct_2^{(2)} C^{(1)}) (1 - B^{(3)}) e_r \otimes x_2 \otimes y_2
$$

$$
= e_r (x_2) (1 - B) y_2 + \sum_s \alpha_s ct_2^{(2)} C^{(1)} (1 - B^{(3)}) e_r \otimes f_s \otimes y_2
$$

$$
= e_r (x_2) (1 - B) y_2 + \sum_s \sum_{p,t} \alpha_s c_{r,s,p,t} f_p \otimes ct_2^{(1)} (1 - B^{(2)}) e_t \otimes y_2
$$

$$
= e_r (x_2) (1 - B) y_2 + \sum_p f_p \otimes \tilde{0}
$$

then LHS is in the $J_2$.

If for $n = (2, 3)$ the theorem is true, then we will show this for every $n > 3$ by induction.

So, let’s suppose that $ct_{n-1} (E^* \otimes J_{n-1}) \subset J_{n-2}$. Every element of $J_n$ we can write as:

$$
y = \sum_{k=1}^{n-1} (1 - B^{(k)}) y_k \quad \forall_{k \in \{1..k\}} y_k \in E^\otimes n
$$

then for all $x \in E^*$ we have:

$$
ct_n^{(1)} x \otimes y = \sum_{k=1}^{n-1} ct_n^{(1)} (x \otimes (1 - B^{(k)}) y_k)
$$

$$
= \sum_{k=1}^{n-1} ct_n^{(1)} (1 - B^{(k+1)}) x \otimes y_k
$$

$$
= ct_n^{(1)} (1 - B^{(2)}) x \otimes y_1 + ct_n^{(1)} (1 - B^{(3)}) x \otimes y_2
$$

$$
+ \sum_{k=3}^{n-1} ct_n^{(1)} (1 - B^{(k+1)}) x \otimes y_k
$$

but

$$
\sum_{k=3}^{n-1} ct_n^{(1)} (1 - B^{(k+1)}) x \otimes y_k = \sum_{k=3}^{n-1} (ct_1^{(1)} + ct_{n-1}^{(2)} C^{(1)}) (1 - B^{(k+1)}) x \otimes y_k
$$

$$
= \sum_{k=3}^{n-1} (1 - B^{(k)}) y'_k
$$

$$
+ \sum_{k=3}^{n-1} ct_{n-1}^{(2)} (1 - B^{(k)}) \sum_{i,j=1}^{\text{dim}(E)} \tilde{y}_i \otimes \tilde{x}_j \otimes \tilde{y}_{k,i,j}
$$
of course the first and the second terms belong to $J_{n-1}$ from the assumption.

\[
ct_n^{(1)}(1 - B^{(2)}) = (ct_1^{(1)} + ct_{n-1}^{(2)}C^{(1)}) (1 - B^{(2)})
\]

\[
= (ct_1^{(1)} + ct_1^{(2)}C^{(1)} + ct_{n-2}^{(3)}C^{(2)}C^{(1)}) (1 - B^{(2)})
\]

\[
= (ct_2^{(1)} + ct_{n-2}^{(3)}C^{(2)}C^{(1)}) (1 - B^{(2)})
\]

\[
= ct_2^{(1)}(1 - B^{(2)}) + ct_{n-2}^{(3)}(1 - B^{(1)})C^{(2)}C^{(1)} + ct_{n-2}^{(3)}(1 - B^{(1)})A^{(1)}
\]

\[
= \tilde{0} + (1 - B^{1})ct_{n-2}^{(3)}C^{(2)}C^{(1)} + (1 - B^{(1)})ct_{n-2}^{(3)}A^{(1)}
\]

\[
ct_n^{(1)}(1 - B^{(3)}) = (ct_1^{(1)} + ct_1^{(2)}C^{(1)} + ct_1^{(3)}C^{(2)}C^{(1)} + ct_3^{(1)}(1 - B^{(3)}) + ct_{n-3}^{(4)}(1 - B^{(2)})C^{(3)}C^{(2)}C^{(1)}
\]

\[
+ (1 - B^{(2)})ct_{n-3}^{(4)}A^{(2)}C^{(1)}
\]

Then

\[
ct_n^{(1)}(1 - B^{(2)})x \otimes y_1 \in J_{n-1}
\]

\[
ct_n^{(1)}(1 - B^{(3)})x \otimes y_2 \in J_{n-1}
\]

That completed the proof \(\square\)

**Theorem IV.4** If matrices $\tilde{B} \in \text{End}(E^* \otimes E^*)$, $C \in \text{Hom}(E^* \otimes E, E \otimes E^*)$ satisfy the following conditions:

1. fulfill YBE: $\tilde{B}^{(2)}C^{(1)}C^{(2)} = C^{(1)}C^{(2)}\tilde{B}^{(1)}$

2. $[\begin{array}{c} ct_1^{(1)}ct_{1,E^*}^{(k)}C^{(k-1)} \ldots C^{(2)} + ct_1^{(k-2)}C^{(k-3)} \ldots C^{(1)}ct_{1,E^*}^{(2)} \end{array}] [1 - \tilde{B}^{(1)}] = 0$ for $k \in \{3..n\}$ and $n \in \mathcal{N}$

then $ct_{n-1}^{(1)}ct_{n,E^*}^{(2)}(J_{2}^* \otimes E^{\otimes n}) \subset J_{n-2} \subset E^{\otimes (n-2)}$

**PROOF:** We have to prove this by induction. Let us suppose that for certain $n - 1 \in \mathcal{N}$ thesis is satisfied, then we have to prove that assertion is satisfied for $n$. Then

\[
ct_{n-1}^{(1)}ct_{n,E^*}^{(2)}(1 - \tilde{B}^{(1)}) = (ct_1^{(1)} + ct_{n-2}^{(2)}C^{(1)}) (ct_{1,E^*}^{(2)} + ct_{n-1,E^*}^{(3)}C^{(2)})(1 - \tilde{B}^{(1)})
\]

\[
= (ct_1^{(1)}ct_{n,E^*}^{(2)} + ct_{n-2}^{(2)}C^{(1)}ct_{1,E^*}^{(2)})(1 - \tilde{B}^{(1)}) + ct_{n-2}^{(2)}C^{(1)}ct_{n-1,E^*}^{(3)}C^{(2)}(1 - \tilde{B}^{(1)})
\]

\[
= (ct_1^{(1)}ct_{n,E^*}^{(2)} + ct_{n-2}^{(2)}C^{(1)}ct_{1,E^*}^{(2)})(1 - \tilde{B}^{(1)}) + ct_{n-2}^{(2)}ct_{n-1,E^*}^{(3)}C^{(2)}(1 - \tilde{B}^{(1)})
\]

\[
= (ct_1^{(1)}ct_{n,E^*}^{(2)} + ct_{n-2}^{(2)}C^{(1)}ct_{1,E^*}^{(2)})(1 - \tilde{B}^{(1)}) + ct_{n-2}^{(2)}ct_{n-1,E^*}^{(3)}(1 - \tilde{B}^{(2)})C^{(1)}C^{(2)}
\]
The last term from assertion belongs to the $J_{n-2}$. Then
\[
ct_{n-1}^{(1)}ct_{n,E}^{(2)}(1 - \tilde{B}^{(1)}) = (ct_{n,E}^{(2)} + ct_{n-2}^{(2)}C^{(1)}ct_{1,E}^{(2)})(1 - \tilde{B}^{(1)}) + z
\]
\[
= (ct_{n,E}^{(2)} + ct_{n-2}^{(2)}C^{(1)}ct_{1,E}^{(2)})(1 - \tilde{B}^{(1)}) + z
\]
\[
+ (ct_{1}^{(1)}ct_{1,E}^{(2)} - ct_{1}^{(1)}ct_{1,E}^{(2)})(1 - \tilde{B}^{(1)}) + z
\]
\[
= ct_{1}^{(1)}(ct_{1,E}^{(2)} + ct_{1}^{(3)}C^{(2)} + \ldots + ct_{1,E}^{(n+1)}C^{(n)} \ldots C^{(2)})
\]
\[
+ (ct_{1}^{(1)} + ct_{1}^{(2)}C^{(1)} + \ldots + ct_{1}^{(n-1)}C^{(n-2)} \ldots C^{(1)}ct_{1,E}^{(2)})(1 - \tilde{B}^{(1)})
\]
\[
- ct_{1}^{(1)}ct_{1,E}^{(2)}(1 - \tilde{B}^{(1)}) + z
\]
where $z \in J_{n-2}$. The assertion we have from the 2nd assumption.

To finish the proof we have to calculate for $n = 2$ only:
\[
ct_{1}^{(1)}ct_{2,E}^{(2)}(1 - \tilde{B}^{(1)}) = ct_{1}^{(1)}(ct_{1,E}^{(2)} + ct_{1}^{(3)}C^{(2)})(1 - \tilde{B}^{(1)})
\]
\[
= (ct_{1}^{(1)}ct_{1,E}^{(2)} + ct_{1}^{(1)}ct_{1,E}^{(2)})(1 - \tilde{B}^{(1)}) = 0
\]

\[\square\]

V Examples.

In this section we would like to show a few examples, which fulfil consistency conditions formulated in the previous section. We start with two simple examples:

Example 1 (Bosons)

\[
b_{i,j,k,l} = \bar{b}_{i,j,k,l} = c_{i,j,k,l} = \delta_{i,l} \delta_{j,k}
\]

Example 2 (Fermions)

\[
b_{i,j,k,l} = \bar{b}_{i,j,k,l} = c_{i,j,k,l} = -\delta_{i,l} \delta_{j,k}
\]

In general case linear condition is satisfied when $Im(id_{E\otimes E} - B) \subset Ker(id_{E\otimes E} + C)$. This relation is not satisfied when:

\[
B(f_{i} \otimes f_{j}) = q^{i-j}f_{j} \otimes f_{i} \quad \tilde{B}(e_{i} \otimes e_{j}) = q^{i-j}e_{j} \otimes e_{i} \quad C(e_{i} \otimes f_{j}) = q^{i-j}f_{j} \otimes e_{i}
\]

This relation satisfies the 2nd assumption of the theorem \[IV.3\] For $q \in \{-1, 1\}$ all the assumptions of theorems \[IV.3\] and \[IV.4\] are satisfied. Then we have the following:
Example 3 (Mixed Bosons - Fermions)

\[ b_{i,j,k,l} = \tilde{b}_{i,j,k,l} = c_{i,j,k,l} = (-1)^{i-j} \delta_{i,l} \delta_{j,k} \]

and more general:

Example 4 (q-Deformed algebra)

\[ b_{i,j,k,l} = \tilde{b}_{i,j,k,l} = q^{j-i} \delta_{i,l} \delta_{j,k} \quad c_{i,j,k,l} = q^{i-j} \delta_{i,l} \delta_{j,k} \quad \text{for} \quad q \in \mathbb{R} \]

as particular case of more general relations, which was studied in [7].

Acknowledgements
We would like to thank to R. Gielerak, A. Borowiec, M. Bożejko and W. Marcinek for discussions and critical remarks.

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