Abstract. A parametric version of Brouwer’s fixed point theorem, called Browder’s theorem, states that for every continuous mapping \( f : [0, 1] \times X \to X \), where \( X \) is a nonempty, compact, and convex set in a Euclidean space, the set of fixed points of \( f \), namely, the set \( \{ (t, x) \in [0, 1] \times X : f(t, x) = x \} \), has a connected component whose projection onto the first coordinate is \([0, 1]\). Browder’s original proof relies on the theory of the fixed point index. We provide an alternative proof that uses Brouwer’s fixed point theorem and is valid whenever \( X \) is a nonempty, compact, and convex subset of a Hausdorff topological vector space.

1. INTRODUCTION. Brouwer’s fixed point theorem [2, 9] states that every continuous mapping of a finite-dimensional simplex into itself has a fixed point. This result was later generalized to nonempty, convex, and compact subsets of topological vector spaces, see, e.g., [7, 14, 19].

The following parametric version of Brouwer’s fixed point theorem is a special case of a more general result of [3]. To state the theorem, we need the concept of a connected component. A set \( A \subseteq \mathbb{R}^n \) is connected if there are no two disjoint open sets \( O_1, O_2 \) such that (a) \( A \subseteq O_1 \cup O_2 \), (b) \( A \not\subseteq O_1 \), and (c) \( A \not\subseteq O_2 \). A subset \( B \) of \( A \) is a connected component of \( A \) if every connected subset of \( A \) is either contained in \( B \) or disjoint of \( B \).

For a set \( X \), we say that the set \( C \subseteq [0, 1] \times X \) covers \([0, 1]\) if the projection of \( C \) onto the first coordinate is \([0, 1]\).

**Theorem 1 (Browder).** Let \( f : [0, 1] \times X \to X \) be a continuous mapping, where \( X \) is a nonempty, compact, and convex subset of a Euclidean space. Define the set of fixed points of \( f \) by

\[
C_f := \{(t, x) \in [0, 1] \times X : f(t, x) = x\}. \tag{1}
\]

Then \( C_f \) has a connected component that covers \([0, 1]\).

The next two examples show that the connected components of \( C_f \) may be difficult to describe: Example 2 shows that a path-connected component that covers \([0, 1]\) may not exist, and Example 3 shows that the number of connected components of \( C_f \) may be infinite and even uncountable.

**Example 2.** Let \( X = [-1, 1] \) and let \( f : [0, 1] \times X \to X \) be given by

\[
f(t, x) := \begin{cases} 
x, & \text{if } t = 0, \\
(1 - t)x + t \sin(\frac{1}{t}), & \text{if } t \neq 0.
\end{cases}
\]

The set \( C_f \) is the union of \( \{(0, x) : x \in [-1, 1]\} \) and \( \{(t, \sin(\frac{1}{t})) : t \in (0, 1]\} \), which is connected but not path-connected.
Example 3. Recall that the Cantor set $K$ is the set of all real numbers in $[0, 1]$ such that their representation in base 3 contains only the digits 0 and 2. The Cantor set has the cardinality of the continuum, and its complement is a countable union of open intervals. Let $g : [0, 1] \to [0, 1]$ be the function that is the identity on $K$, and, on each maximal open subinterval $(a, b)$ of $[0, 1]$ in the complement of $K$ is given by $g(x) = x + 10(x - a)(b - x)$, see Figure 1. The function $g$ is continuous, its range is $[0, 1]$, and its set of fixed points is $K$.

Define now a function $f : [0, 1] \times [0, 1] \to [0, 1]$ by $f(t, x) = g(x)$ for every $(t, x) \in [0, 1] \times [0, 1]$. The connected components of $C_f$ are then all sets of the form $[0, 1] \times \{x\}$ with $x \in K$.

Theorem 1 has been used in a variety of topics, like nonlinear complementarity theory [1, 8], nonlinear elliptic boundary value problems [15], the study of global continua of solutions of nonlinear partial differential equations [6, 12], theoretical economics [5], and game theory [10, 16].

The proof of Theorem 1 in [3] uses the fixed point index, and hence is not accessible to all mathematicians. In this paper we provide an alternative proof to Theorem 1 that uses a fixed point theorem, and holds, e.g., whenever $X$ lies in a Hausdorff topological vector space.

Since the proof uses only basic mathematical results, it is much more accessible than Browder’s original proof and can be taught in undergraduate topology courses.

Another elementary proof of Theorem 1, which is valid when $X$ is the unit cube in a Euclidean space and uses combinatorial tools inspired by Sperner’s Lemma [18], was presented in [11].

In Section 3 we discuss an extension of Theorem 1 to more general parameter sets.

2. PROOF OF THEOREM 1. Our proof of Theorem 1 relies on three results: (i) a separation theorem, which holds in any compact topological space, (ii) Tietze’s Extension Theorem, which holds in normal topological spaces, and (iii) the existence of a fixed point for every continuous mapping from $X$ to itself, i.e., that $X$ satisfies the fixed point property. The latter property ensures that the sections of $C_f$ are nonempty.

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1 In fact, the more general version of Theorem 1 that is stated in [3] is phrased using the fixed point index.
Since compact Hausdorff topological vector spaces are normal, Schauder’s fixed point theorem (see [14]) ensures that our proof works whenever $X$ lies in a Hausdorff topological vector space.

Assume, by contradiction, that $C_f$ has no connected component that covers $[0, 1]$. Since $[0, 1] \times X$ is compact and $f$ is continuous, the set $C_f$ is compact. Since $X$ is nonempty, compact, convex, and satisfies the fixed point property, the sets $C_f \cap ([0] \times X)$ and $C_f \cap ([1] \times X)$ are nonempty, compact, and disjoint. Hence, by a standard separation theorem (see Theorem 14.4 in [4]) there are two disjoint closed subsets $K_0$ and $K_1$ of $C_f$ such that (a) $K_0 \cup K_1 = C_f$, (b) $K_0 \cap ([0] \times X) = C_f \cap ([0] \times X)$, and (c) $K_1 \cap ([1] \times X) = C_f \cap ([1] \times X)$. It follows that the sets $B_0 := ([0] \times X) \cup K_0$ and $B_1 := ([1] \times X) \cup K_1$ are disjoint.

Let $g : [0, 1] \times X \to [-1, 1]$ be a continuous function that satisfies $g \equiv 1$ on $B_0$ and $g \equiv -1$ on $B_1$ (see Figure 1, where $C_f$ has six connected components). Since $B_0$ and $B_1$ are disjoint and closed, by Tietze’s Extension Theorem such a function $g$ exists. See Figure 2.

**Comment 4.** If $X$ lies in a metric space with metric $\rho$, then $[0, 1] \times X$ is also a metric space with the natural metric $d((t, x), (s, y)) = |t - s| + \rho(x, y)$. The function $g$ can be defined by

$$g(t, x) := \begin{cases} 
1, & \text{if } (t, x) \in B_0, \\
-1, & \text{if } (t, x) \in B_1, \\
\frac{d((t, x), B_1) - d((t, x), B_0)}{d((t, x), B_1) + d((t, x), B_0)}, & \text{otherwise.}
\end{cases}$$

We argue that for every $\varepsilon > 0$ sufficiently small, $t + \varepsilon g(t, x) \geq 0$ for all $(t, x) \in [0, 1] \times X$. Indeed, since $g$ is continuous on the compact set $[0, 1] \times X$, it is absolutely continuous. Then, since $g \equiv 1$ on $[0] \times X$, there is $\varepsilon > 0$ such that $g(t, x) \geq 0$ whenever $t \leq \varepsilon$. This implies that if $t \leq \varepsilon$, then

$$t + \varepsilon g(t, x) \geq t \geq 0.$$ 

On the other hand, if $t \geq \varepsilon$, then since $g(t, x) \geq -1$ we have

$$t + \varepsilon g(t, x) \geq t - \varepsilon \geq 0.$$ 

Analogously, for every $\varepsilon > 0$ sufficiently small, $t + \varepsilon g(t, x) \leq 1$ for all $(t, x) \in [0, 1] \times X$. 

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Let \( \varepsilon > 0 \) be sufficiently small so that \( t + \varepsilon g(t, x) \in [0, 1] \) for all \( (t, x) \in [0, 1] \times X \). Consider the mapping \( F : [0, 1] \times X \to [0, 1] \times X \) defined by

\[
F(t, x) := \left( t + \varepsilon g(t, x), f(t, x) \right), \quad \forall (t, x) \in [0, 1] \times X.
\]

The mapping \( F \) is continuous, and by the choice of \( \varepsilon \) its range is indeed \([0, 1] \times X\). We claim that \( F \) has no fixed points. Indeed, if \( (t, x) \in C_f \), then \( g(t, x) \neq 0 \), and therefore \( t + \varepsilon g(t, x) \neq t \), whereas if \((t, x) \notin C_f\), then \( f(t, x) \neq x \). But every continuous function from \([0, 1] \times X\) to itself has a fixed point; thus, we have achieved a contradiction. This shows that the assumption that \( C_f \) has no connected component that covers \([0, 1]\) is not correct, which completes the proof of Theorem 1.

3. GENERAL PARAMETER SETS. In Theorem 1, the parameter set is \([0, 1]\). One may wonder whether the theorem remains valid for more general parameter sets. The answer is positive. Here we illustrate this extension for the parameter set \([0, 1]^2\).

Let \( f : [0, 1]^2 \times X \to X \) be a continuous mapping, where \( X \) is a nonempty, compact, and convex subset of a Hausdorff topological vector space, and let \( \varphi : [0, 1] \to [0, 1]^2 \) be a continuous and surjective mapping (e.g., the Peano curve [13]). The mapping \( h := f \circ (\varphi, \text{Id}_X) : [0, 1] \times X \to X \) is a composition of two continuous mappings, hence, continuous, and by Theorem 1 the set \( C_h \) has a connected component, denoted \( B \), which covers \([0, 1]\). A continuous image of a connected set is connected, hence, the set \( D := \{(\varphi(t, x) : (t, x) \in B\} \) is a connected subset of \( C_f \) that covers \([0, 1]^2\); its projection onto \([0, 1]^2\) is \([0, 1]^2\). The connected component of \( C_f \) that contains \( D \) covers \([0, 1]^2\) as well.

The construction provided above is valid whenever the parameter set \( Y \) possesses a space-filling curve, namely, there is a continuous and surjective mapping \( \varphi : [0, 1] \to Y \). Recall that the Hahn-Mazurkiewicz Theorem (e.g., Theorem 31.5 in [20]) states that a space possesses a space-filling curve if and only if it is compact, connected, locally connected, and second-countable. One example of a set that does not possess a space-filling curve is the set \( C_f \) in Example 2. The paper [17] uses the fixed point index to show that Theorem 1 extends to the case where the parameter set is a connected compact Hausdorff space, in which case a space-filling curve may not exist. We do not know whether this extension can be proved in an analogous way to the one presented in this paper.

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