The magnetism of two coupled harmonic oscillators

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Abstract. The thermodynamical properties of a system of two coupled harmonic oscillators in the presence of a uniform magnetic field $B$ are investigated. Using an unitary transformation, we show that the system can be diagonalized in a simple way and then obtain the energy spectrum solutions. These are then used to determine the thermodynamical potential in terms of different physical parameters like the coupling parameter $\alpha$. This allows us to give a generalization of significant work already published and obtain different results, which can be used to discuss the magnetism of the system. Different limiting cases, in terms of $\alpha$ and $B$, are discussed. In fact, quantum corrections to the Landau diamagnetism and orbital paramagnetism are found.

Keywords: algebraic structures of integrable models, rigorous results in statistical mechanics

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1. **Introduction**

Since the pioneering work of Landau in 1930 [1], the orbital magnetism of electron gases has attracted considerable attention, especially during the last few decades with the advent of experimental opportunities, more precisely with the availability of two-dimensional electronic devices, quantum boxes, or mesoscopic finite size objects. One can find in [2] or [3] a good account of the theoretical investigations on the subject, especially from a semiclassical point of view. For more recent developments, we refer the reader to the book [4].

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The Fukuyama group has produced a number of papers dealing with different features of two-dimensional systems. Among them, we mention [5] where the magnetization of such systems in external potentials [5] is studied. In fact, dependence on the magnetic field $B$ and the temperature $T$ of the magnetization is calculated exactly. It is found that the magnetization is well defined in the limit of vanishing $B$ as well as in the limit of $T = 0$, showing a large fluctuation at low $T$ as $B$ is varied. It is shown that this fluctuating magnetization tends to Landau diamagnetism at higher $T$ or upon ensemble averaging. Subsequently, another exciting investigation was reported in [6], which concerned the spatial distribution of the electric current under $B$ and the resultant orbital magnetism for the present system under a harmonic confining potential $V(\mathbf{r}) = m\omega_0^2 r^2/2$ in various regimes of the couple $(T, B)$. An interesting result is that the microscopic conditions for the validity of Landau diamagnetism are clarified.

The system studied in [6] has been considered from another point of view. More precisely, a coherent states approach is used to investigate its basic features [7]. In fact, the corresponding expressions for the thermodynamical potential and magnetic moment are determined. These are exact, in contrast to those in [6], and the results yielded a full description of the phase diagram of the magnetization. The derivation crucially rests upon the observation that the Fermi–Dirac function is a fixed point of the Fourier transform. Exact series expansions ensue by simple application of the residue theorem. The related physical quantities are obtained and different discussions are reported in terms of the natures of $T$ and $B$. These concern the thermodynamical potential, the orbital magnetic moment, the subsequent magnetic susceptibility and the average number of electrons.

On the other hand, the problem of two coupled harmonic oscillators living on two dimensions was investigated on different occasions; several papers were presented by Kim’s group—as a short list, we cite [8]–[15]. Furthermore, the quantum mechanics of such systems on the non-commutative plane has been studied as well [16], where different quantum corrections to the original work [8] are obtained and their interpretations are given.

After mentioning the above results, an interesting question arises immediately that concerns other features of two coupled harmonic oscillators. Specifically, it is possible to study the thermodynamical properties of such systems in the presence of a uniform magnetic field? The answer will be the subject of the present paper, where interesting results will be derived and discussed. In fact, we will show how to use the machinery developed by one of the present authors in the basic reference [7] to analyze the magnetization of the system.

More precisely, we develop a theory that analyzes the basic features of two coupled harmonic oscillators under a magnetic field. In doing so, we inspect two already published works [7,16] and generate a full description of the present system from the thermodynamical point of view. Actually, this can be done with the help of energy spectrum solutions. To derive them, we make use of a unitary transformation that leads to a solvable Hamiltonian of the system.

Subsequently, we present two ways to evaluate the thermodynamical potential. Indeed, using the Berezin–Lieb inequalities and after determining some physical quantities, we discuss different limiting cases in terms of the physical parameters involved. These lead to us ending up with interesting results and in particular we show that the average number of electrons behaves like the inverse of the squared magnetic field for the infinite
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coupling limit, i.e. \( \alpha \to \infty \). In this situation, the system behaves like a quantum Hall

effect one [18]. On the other hand, quantum corrections to the orbital paramagnetism

and Landau diamagnetism are obtained. More importantly, we notice that by switching

off \( \alpha \) in our analysis, we recover already published work [7].

Furthermore, we give an exact formula for the thermodynamical potential with the

help of some well-known relations and discuss different issues. Indeed, applying the Fermi–

Dirac trace formulae, we explicitly derive the average number of electrons and the magnetic

moment. Finally, we how they can be reduced to the standard expressions by taking into

account the limiting case \( \alpha = 0 \).

The present paper is organized as follows. In section 2, we formulate our problem

by establishing the necessary materials for dealing with our task. In section 3, after

making use of a unitary transformation, we introduce an algebraic method for deriving the

energy spectrum solutions. Their underlying properties will be discussed by considering

four limiting cases. We construct the coherent states for the present solutions and show

that they are coupling parameter dependent in section 4. These will serve as tools for
determining explicitly different physical quantities and in particular the thermodynamical

potential in section 5. This will be obtained by adopting the Berezin–Lieb inequalities in

the first stage. This allows us to give a different discussion and end up with interesting

conclusions. However, in section 6, we consider another approach based on the Fermi–

Dirac trace formulae to obtain the exact form of the thermodynamical potential. Finally,

we conclude and give different perspectives.

2. Formulating the problem

We start by formulating our problem by setting up the tools needed for doing our task.
This can be done by establishing a mathematical formalism governed by a Hamiltonian
describing a system of two coupled harmonic oscillators in two dimensions. Subsequently,
we subject the system to a constant magnetic field and analyze its behavior. In doing
so, we determine the energy spectrum through an algebraic method after making use of
a unitary transformation.

2.1. Coupled harmonic oscillators

We consider a system of two coupled harmonic oscillators of mass \((m_1, m_2)\) and living on
the plane \((X_1, X_2)\). This can be described using a Hamiltonian given as a sum of free and
interacting parts, such as

\[
H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + \frac{1}{2} \left( C_1 X_1^2 + C_2 X_2^2 + C_3 X_1 X_2 \right)
\]

(1)

where \(C_1, C_2\) and \(C_3\) are three constant parameters. Note in passing that the parameter
involved can be fixed according to the nature of the system. On the other hand, (1) has
been investigated for different purposes; for instance one may see [8], where it has been
generalized to the non-commutative geometry case [16].

As claimed before, we are wondering how to study the magnetization of two coupled
harmonic oscillators under a uniform magnetic field. To achieve this goal, we generalize

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the system governed by (1) to another one with the Hamiltonian

\[ H_1 = \frac{\Pi_1^2}{2m_1} + \frac{\Pi_2^2}{2m_2} + \frac{1}{2} (C_1 X_1^2 + C_2 X_2^2 + C_3 X_1 X_2) \]  

(2)

where \( \Pi_1 \) and \( \Pi_2 \) are the conjugate momenta. They can be simplified by choosing an appropriate gauge. Indeed, in the symmetric gauge

\[ \vec{A} = \frac{B}{2} (-X_2, X_1) \]  

(3)

they are given by

\[ \Pi_1 = P_1 - \frac{eB}{2c} X_2, \quad \Pi_2 = P_2 - \frac{eB}{2c} X_1. \]  

(4)

We use these to map (2) into the form

\[ H_1 = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + \frac{1}{2} (D_1 X_1^2 + D_2 X_2^2 + D_3 X_1 X_2) + \frac{1}{2} (\omega_1 P_2 X_1 - \omega_2 P_1 X_2) \]  

(5)

where the new constants \( D_1, D_2 \) and \( D_3 \) read as

\[ D_1(B) = C_1 + m_2 \omega_2^2, \quad D_2(B) = C_2 + m_1 \omega_1^2, \quad D_3 = C_3 \]  

(6)

with the cyclotron frequencies

\[ \omega_{1c} = \frac{eB}{m_1 c}, \quad \omega_{2c} = \frac{eB}{m_2 c}. \]  

(7)

Clearly, by comparing and neglecting the different constants involved, we note that the third term makes the difference between (1) and (5). This in fact will play a crucial role in the forthcoming analysis and allow us to derive different results.

It is convenient to introduce new phase space variables, which can be done by rescaling those appearing in (5). Indeed, one can define the positions as

\[ x_1 = \left( \frac{m_1}{m_2} \right)^{1/4} X_1, \quad x_2 = \left( \frac{m_2}{m_1} \right)^{1/4} X_2 \]  

(8)

which obviously lead to the momenta

\[ p_1 = \left( \frac{m_2}{m_1} \right)^{1/4} P_1, \quad p_2 = \left( \frac{m_1}{m_2} \right)^{1/4} P_2. \]  

(9)

Replacing them all, we show that (5) becomes

\[ H_2 = \frac{1}{2m} (p_1^2 + p_2^2) + \frac{1}{2} \left( d_1 x_1^2 + d_2 x_2^2 + d_3 x_1 x_2 \right) + \frac{\omega_c}{2} (x_1 p_2 - x_2 p_1) \]  

(10)

where we have set different constants as

\[ d_1(B) = D_1 \left( \frac{m_2}{m_1} \right)^{1/2}, \quad d_2(B) = D_2 \left( \frac{m_1}{m_2} \right)^{1/2}, \quad d_3 = D_3 \]  

(11)

with unique mass \( m = (m_1 m_2)^{1/2} \) and the cyclotron frequency \( \omega_c = (\omega_1 \omega_2)^{1/2} = (eB/mc) \). Consequently, (10) is showing an extra term, which is nothing but the angular momentum, and the first is similar to (1). Therefore, it will be of interest to deal with such a system and underline its physical properties.

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2.2. The unitary transformation

According to the expression (10), it appears that getting the energy spectrum solutions is not an easy task. However, we can overcome such difficulties by adopting an appropriate approach. More precisely, we proceed by making use of a unitary transformation, such that new phase space variables can be defined by

\[ y_a = M_{ab} x_b, \quad \tilde{p}_a = M_{ab} \tilde{p}_b \]

where the matrix \( M_{ab} \)

\[ M_{ab} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

is a unitary rotation with the mixing angle \( \theta \). Inserting the mapping (12) into (10), one realizes that \( \theta \) should satisfy the condition

\[ \tan \theta = \frac{d_3}{d_2 - d_1} \]

(14)

and end up with a factorizing Hamiltonian. It is

\[ H_3 = \frac{1}{2m} (\tilde{p}_1^2 + \tilde{p}_2^2) + \frac{k}{2} (e^{2\alpha} y_1^2 + e^{-2\alpha} y_2^2) + \frac{\omega_c}{2} (y_1 \tilde{p}_2 - y_2 \tilde{p}_1) \]

(15)

where \( k \) and \( \alpha \) are given by

\[ k = \sqrt{d_1 d_2 - \frac{d_3^2}{4}}, \quad e^{\alpha} = \frac{d_1 + d_2 + \sqrt{(d_1 - d_2)^2 + d_3^2}}{2k} \]

(16)

and the condition \( 4d_1 d_2 > d_3^2 \) must be fulfilled. Note that \( H_3 \) has a similar form to the two-dimensional Landau Hamiltonian in the symmetric gauge. Obviously, they coincide in the case without coupling, namely \( \alpha = 0 \).

Before proceeding further, we conclude by citing some interesting remarks. In doing so, let us return to \( H_3 \) and define two operators as

\[ H_0 = \frac{1}{2m} (\tilde{p}_1^2 + \tilde{p}_2^2) + \frac{k}{2} (e^{2\alpha} y_1^2 + e^{-2\alpha} y_2^2), \quad L_3 = (y_1 \tilde{p}_2 - y_2 \tilde{p}_1) \]

(17)

where \( H_0 \) can also be separated into two commuting parts:

\[ \mathcal{H}_1 = \frac{1}{2m} e^{-\alpha} \tilde{p}_1^2 + \frac{k}{2} e^{\alpha} y_1^2, \quad \mathcal{H}_2 = \frac{1}{2m} e^{\alpha} \tilde{p}_2^2 + \frac{k}{2} e^{-\alpha} y_2^2. \]

(18)

Firstly, one can see that the decoupled Hamiltonian

\[ \mathcal{H}_0 = \frac{1}{2m} \tilde{p}_1^2 + \frac{k}{2} y_1^2 + \frac{1}{2m} \tilde{p}_2^2 + \frac{k}{2} y_2^2 \]

(19)

can be recovered by taking \( \alpha = 0 \), which corresponds to the solution \( d_1 = d_2 \) and \( d_3 = 0 \). Secondly, it is interesting to note that (19) can be derived by a canonical transformation only from

\[ \mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 \]

(20)

as is pointed out in [8] and subsequently in [16].

According to the above statements, we can rearrange \( H_3 \) in an appropriate form. This is

\[ H_3 = e^{-\alpha} \mathcal{H}_1 + e^{\alpha} \mathcal{H}_2 + L_3 \]

(21)

which will be used to tackle different issues in the forthcoming analysis and in particular the magnetization of the present system. This mapping will be helpful in the sense that the corresponding energy spectrum solutions can be easily obtained as we will see soon.
3. The energy spectrum

For the eigenvalues and eigenstates, we adopt an algebraic method based on different operators in terms of the phase space ones. This will allow us to obtain the solutions and investigate their underlying properties.

3.1. Algebraic analysis

It is clear that \( H \) is a Hamiltonian of two decoupled harmonic oscillators. Thus it can be simply diagonalized by defining a set of creation and annihilation operators. They are given by

\[
a_i = \sqrt{\frac{k}{2\hbar\omega}} e^{\alpha/2} y_i + \frac{i}{\sqrt{2m\hbar\omega}} e^{-\alpha/2} \tilde{p}_i, \quad a_i^\dagger = \sqrt{\frac{k}{2\hbar\omega}} e^{\alpha/2} y_i - \frac{i}{\sqrt{2m\hbar\omega}} e^{-\alpha/2} \tilde{p}_i
\]

where the new frequency is

\[
\omega(B) = \left(\frac{4d_1d_2 - d_3^2}{4m^2}\right)^{1/4} = \sqrt{\frac{k}{m}}.
\]

They satisfy the usual commutation relations

\[
[a_i, a_j^\dagger] = \delta_{ij}
\]

and obviously other commutators vanish. It is easy to show that \( H \) can be mapped in terms of \( a_i \) and \( a_i^\dagger \) as

\[
H = \hbar \omega \left( a_1^\dagger a_1 + a_2^\dagger a_2 + 1 \right).
\]

According to (25), it is not hard to derive the corresponding energy spectrum solutions. This can be done by solving the eigenvalue equation

\[
H|n_1, n_2, \alpha\rangle = \mathcal{E}_{n_1, n_2}|n_1, n_2, \alpha\rangle
\]

to get the corresponding states

\[
|n_1, n_2, \alpha\rangle = \frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{n_1!n_2!}} |0, 0, \alpha\rangle
\]

as well as the energy spectrum

\[
\mathcal{E}_{n_1, n_2} = \hbar \omega (n_1 + n_2 + 1).
\]

Due the fact that there are mappings between the different Hamiltonians, one can build other solutions. In particular, the spectrum of \( H_0 \) can easily be deduced from the above as

\[
E_{0,n_1, n_2} = \hbar \omega \left[ e^\alpha \left( n_1 + \frac{1}{2} \right) + e^{-\alpha} \left( n_2 + \frac{1}{2} \right) \right].
\]

To get that for \( H_3 \), we need to diagonalize the angular momentum. In doing so, we define two sets of operators where the first one is

\[
a_g = \frac{1}{\sqrt{2}} (a_1 + ia_2), \quad a_g^\dagger = \frac{1}{\sqrt{2}} \left( a_1^\dagger - ia_2^\dagger \right)
\]
and the second reads as
\[ a_d = \frac{1}{\sqrt{2}} (a_1 - ia_2), \quad a_d^\dagger = \frac{1}{\sqrt{2}} (a_1^\dagger + ia_2^\dagger). \quad (31) \]

They show that
\[ [a_g, a_g^\dagger] = [a_d, a_d^\dagger] = 1 \quad (32) \]
and different commutation relations are null. One can note that there is a conservation of number operators, such as
\[ N_1 + N_2 = N_d + N_g. \quad (33) \]
where we have \( N_1 = a_1^\dagger a_1, \) \( N_2 = a_2^\dagger a_2, \) \( N_d = a_d^\dagger a_d \) and \( N_g = a_g^\dagger a_g. \) Now we express the phase space variables in terms of the new operators to end up with a quantized angular momentum. This is
\[ L_3 = 2\hbar (N_d - N_g). \quad (34) \]

Obviously, its eigenvalues are \( 2\hbar (n_d - n_g) \) and the corresponding eigenvalues form a common basis of \( L_3 \) and \( H. \)

Finally, we have settled all the ingredients for deriving the energy spectrum solutions of \( H_3. \) Indeed, starting from the above results, one can see that \( (15) \) becomes
\[ H_3 = (\hbar \omega e^\alpha + \hbar \omega c) N_d + (\hbar \omega e^{-\alpha} - \hbar \omega c) N_g + \hbar \omega \cosh \alpha. \quad (35) \]

To write \( H_3 \) in compact form, it is convenient to introduce two new frequencies in terms of the former ones. These are defined by
\[ \omega_+ (B, \alpha) = \omega e^\alpha + \omega c, \quad \omega_- (B, \alpha) = \omega e^{-\alpha} - \omega c. \quad (36) \]
They show a strong dependence on \( \alpha \) and therefore generalize the standard results \[7\]. Now we return to mapping \( H_3 \) as
\[ H_3 = \hbar (\omega_+ N_d + \omega_- N_g + \omega \cosh \alpha). \quad (37) \]

Solving the eigenvalue equation, we can easily derive the energy spectrum solutions. Thus, the eigenvalues take the form
\[ E_{3,n_d,n_g} = \hbar (\omega_+ n_d + \omega_- n_g + \omega \cosh \alpha), \quad n_g, n_d = 0, 1, 2, \ldots \quad (38) \]
and the eigenstates are given by
\[ |n_d, n_g, \alpha \rangle = \frac{(a_d^\dagger)^{n_d} (a_g^\dagger)^{n_g}}{\sqrt{n_d! n_g!}} |0, 0, \alpha \rangle. \quad (39) \]

It is clear that the results obtained so far are dependent on \( \alpha. \) This in fact makes the difference with respect to the standard results obtained by analyzing the Fock–Darwin Hamiltonian \[7\], which obviously can be recovered by setting \( \alpha = 0. \) At this stage, one may ask about the relevance of such a coupling parameter and the answer will be given in the forthcoming sections where interesting results will be derived and different discussions will be given.
3.2. Underlying properties

In investigating the underlying symmetry of the system, one can study the properties of quantum numbers pairs \((n_d, n_g)\). However, these may not provide simple hints on the ordering of the energy \(E_{\alpha,n_d,n_g}\) except for the four limiting cases related to the nature of the coupling parameter and the magnetic field.

3.2.1. The weak coupling case. To characterize the system behavior, we consider the first case that corresponds to the limit \(\alpha \to 0\), which means that the coupling between two oscillators is not strong. This is the case for some physical phenomena. Therefore, we can make different approximations to bring our findings near to well-known and significant results.

By taking the limit \(\alpha \to 0\) and after a simple calculation, we show that the energy spectrum can be approximated by

\[
E_{\alpha,\lambda,\xi}|_{\alpha \to 0} \approx \hbar \left[ \omega \lambda + (\alpha \omega + \omega_c)\xi + \frac{\omega}{2} \right].
\]

(42)

We can bring this to an appropriate form by defining new quantum numbers. They are

\[
\lambda = \frac{n_d + n_g}{2}, \quad \xi = \frac{n_d - n_g}{2}.
\]

(41)

Thus, one can rearrange (40) as

\[
E_{\alpha,\lambda,\xi}|_{\alpha \to 0} \approx 2\hbar \left[ \omega \lambda + (\alpha \omega + \omega_c)\xi + \frac{\omega}{2} \right].
\]

(42)

It can be identified with the eigenvalues of the Fock–Darwin Hamiltonian, which can be obtained from (38) by taking \(\alpha = 0\). They are

\[
E_{3,n_d,n_g}|_{\alpha = 0, B \to 0} = \hbar \left( \omega_+ n_d + \omega_- n_g + \omega \right).
\]

(43)

Now observe the following correspondence: \((\omega_+, \omega_-) \to (\omega, \alpha \omega + \omega_c)\). This tells us that (42) can be used to analyze the thermodynamical properties in similar way to what has been done in [7]. Moreover, it shows how one can generalize the Fock–Darwin Hamiltonian to another one where the interaction still survives.

At this stage, we can further discuss (42) by inspecting two other limits in terms of the field. In doing so, we suppose that the cyclotron frequency is much smaller than the frequency \(\omega\), i.e. \(\omega_c \ll \omega\); thus we have

\[
E_{\alpha,\lambda,\xi}|_{\alpha, B \to 0} \approx 2\hbar \omega \left( \lambda + \alpha \xi + \frac{1}{2} \right).
\]

(44)

According to this, two conclusion can be deduced here. Indeed, first we still have a generalized Fock–Darwin Hamiltonian but its frequencies are changed now to \((\omega, \alpha \omega)\). Secondly, without coupling we recover a one-dimensional harmonic oscillator with the eigenvalues

\[
E_{\lambda}|_{\alpha = 0, B \to 0} \approx 2\hbar \omega \left( \lambda + \frac{1}{2} \right).
\]

(45)

whose frequency is \(\omega = \sqrt{d_1/m}\), which means that we are under the conditions \(d_1 = d_2\) and \(d_3 = 0\).
Now, let us treat the second consideration, that is the strong magnetic field case. In fact, this is equivalent to \( \omega_c \gg \omega \) and leads to
\[
E_\xi |_{\alpha \to 0, B \to \infty} \approx 2\hbar \omega_c \xi.
\] (46)

It can be interpreted as the squared energy spectrum of the massless Dirac fermions in graphene under a uniform magnetic field. For more detail, we refer the reader to [17], for instance, and references therein.

3.2.2. The strong coupling case. It is immediate and natural to ask about what happens if the coupling is strong and for the corresponding limit cases of the magnetic field.

The above inquiry can be answered by examining the limit \( \alpha \to \infty \). Thus, we return to (38) to show the result
\[
E_{\alpha, n_d, n_g} |_{\alpha \to \infty} \approx \hbar \left[ \omega e^\alpha (n_d + \frac{1}{2}) + \omega_c (n_d - n_g) \right].
\] (47)

Again this can be approximated further by taking other limits. Indeed, focusing on the case \( \omega_c \ll \omega \) or \( \omega_c \ll \omega e^\alpha \), it is straightforward to obtain
\[
E_{n_d} \approx \hbar \omega \left( n_d + \frac{1}{2} \right) e^\alpha
\] (48)

which is the energy spectrum for a harmonic oscillator of frequency \( \omega e^\alpha \). However for \( \omega_c \gg \omega \), there is nothing to say and therefore (47) remains as it is because we cannot make a comparison.

In summary, according to the above results we conclude that the coupling parameter \( \alpha \) is the parameter of interest for the present theory. In fact, it can be adjusted to recover different models that used to deal with different issues in physics.

4. Realizing the coherent states

The forthcoming analysis requires powerful tools. More precisely, one way to determine the thermodynamical potential is to use the coherent states approach. Thus, for our needs, we follow the standard method to realize them in terms of our language and show their dependence on the coupling parameter. In fact, we will use the same steps as were traced in [7].

The fact that the eigenstates issuing from the algebraic method are just tensor products of Fock harmonic oscillator eigenstates allows one to easily construct the corresponding coherent states. Indeed, in a standard way, we have
\[
|z_d, z_g, \alpha\rangle \equiv |z_d\rangle \otimes |z_g\rangle = \exp \left[ -\frac{1}{2} \left( |z_d|^2 + |z_g|^2 \right) \right] \sum_{n_d,n_g} \frac{z_d^{n_d} z_g^{n_g}}{\sqrt{n_d!} \sqrt{n_g!}} |n_d, n_g, \alpha\rangle.
\]

In terms of the creation operators, we have
\[
|z_d, z_g, \alpha\rangle = \exp \left[ -\frac{1}{2} \left( |z_d|^2 + |z_g|^2 \right) \right] e^{z_d a_d^\dagger + z_g a_g^\dagger} |0, 0, \alpha\rangle.
\] (49)

The above normalized states should obey some of the usual properties. Indeed, it is easy to verify an eigenvector property, such as
\[
 a_d |z_d, z_g, \alpha\rangle = z_d |z_d, z_g, \alpha\rangle, \quad a_g |z_d, z_g, \alpha\rangle = z_g |z_d, z_g, \alpha\rangle.
\] (50)
As far as the action identity is concerned, one can obtain the relation
\[ \hat{H}_3(z_d, z_g, \alpha) \equiv \langle z_d, z_g, \alpha | H_3(z_d, z_g, \alpha) \rangle = \hbar \left( \omega_+ |z_d|^2 + \omega_- |z_g|^2 + \omega \cosh \alpha \right) \] (51)
where the function \( \hat{H}_3(z_d, z_g, \alpha) \) is called the lower symbol of the operator \( H_3 \). It will play an important role in the present context. The resolution of the identity reads as
\[ I = \frac{1}{\pi^2} \int_{C^2} \frac{d^2 z_d d^2 z_g}{|z_d, z_g, \alpha \rangle \langle z_d, z_g, \alpha|} \] (52)
where the last property is also crucial in our context.

For any observable \( A \) with suitable operator properties (traceclass, \cdots), there exists a unique upper (or covariant) symbol \( \hat{A}(z_d, z_g) \) defined by
\[ A = \frac{1}{\pi^2} \int_{C^2} \hat{A}(z_d, z_g, \alpha) |z_d, z_g, \alpha \rangle \langle z_d, z_g, \alpha| d^2 z_d d^2 z_g. \] (53)
We consider as a straightforward illustration the upper symbols for the number operators. Hence, one can show
\[ \hat{N}_d(z_d, z_g, \alpha) = |z_d|^2 - 1, \quad \hat{N}_g(z_d, z_g, \alpha) = |z_g|^2 - 1. \] (54)
Clearly, the upper symbol for our Hamiltonian (37) takes the form
\[ \hat{H}_3(z_d, z_g, \alpha) = \hbar \left( \omega_+ |z_d|^2 + \omega_- |z_g|^2 - \omega \cosh \alpha \right). \] (55)

To set up all that we need for our task, we recall a useful trace identity for a given trace class observable \( A \). This is
\[ \text{Tr} A = \frac{1}{\pi^2} \int_{C^2} \hat{A}(z_d, z_g, \alpha) d^2 z_d d^2 z_g = \frac{1}{\pi^2} \int_{C^2} \hat{A}(z_d, z_g, \alpha) d^2 z_d d^2 z_g \] (56)
where the symbol function \( \hat{A} \) is
\[ \hat{A}(z_d, z_g, \alpha) \equiv \langle z_d, z_g, \alpha | A | z_d, z_g, \alpha \rangle. \] (57)

We close this part by noting that all quantities involved are dependent on \( \alpha \). Obviously, the standard results can be recovered by switching off the coupling parameter \[7\]. On the other hand, we will see how the above materials can be employed to deal with different issues and in particular determine the thermodynamical potential. This will be done by adopting two methods, which concern the Berezin–Lieb inequalities and Fermi–Dirac trace formulæ.

5. Berezin–Lieb inequalities

Having derived and set up all the necessary tools, we now show how they can be used to study the magnetism of the system under consideration. In doing so, we start by defining the physical quantities that will be discussed in the present context. One way to do so is to evaluate the thermodynamical potential, which can be done, in the first stage, by adopting the Berezin–Lieb inequalities. Subsequently, we treat the asymptotic behavior of the results obtained, by considering the limiting cases of the coupling parameter.
5.1. Physical quantities

The magnetism of the model under hand can be investigated by adopting the standard method of statistical mechanics. This will be done by making use of different approximations to simplify our problem. In fact, we begin by assuming that the total number \( \langle N_e \rangle \) of electrons large enough for there to be no appreciable difference between a grand canonical ensemble and a canonical one.

In the light of the above considerations and results obtained, we proceed by using the magnetic moment \( M \) definition. This is

\[
M = -\left( \frac{\partial \Omega}{\partial H} \right)_\mu \tag{58}
\]

where the thermodynamical potential \( \Omega \) can be obtained from the partition function. In terms of our model, it is

\[
\Omega = -\frac{1}{\beta} \text{Tr} \log \left[ 1 + \exp \left\{ -\beta (H_3 - \mu) \right\} \right] \tag{59}
\]

where as usual we have set \( \beta = 1/(k_B T) \). According to (58) and (59), we obtain the result

\[
M = -2\mu_B \text{Tr} \left( \frac{N_d - N_g}{1 + \exp \{\beta (H_3 - \mu)\}} \right). \tag{60}
\]

Replacing \( H_3 \) by its expression and taking the trace, we end up with

\[
M = -2\mu_B \sum_{n_d,n_g=0}^{\infty} \frac{n_d - n_g}{1 + \kappa_+^{-1} \exp \{\beta \hbar \omega_\perp n_d + \omega_\parallel n_g\}} \tag{61}
\]

where \( \mu_B = \hbar e/(2mc) \) is the Bohr magneton and the \( \kappa_\pm \) are given by

\[
\kappa_\pm(B,T,\alpha) = \exp \left[ \beta (\mu \pm \hbar \omega \cosh \alpha) \right]. \tag{62}
\]

On the other hand, the average number of electrons can be evaluated by introducing the Fermi distribution function. That is

\[
f(E) = \frac{1}{1 + \exp \{\beta (E - \mu)\}}. \tag{63}
\]

Therefore, in our case we have

\[
\langle N_e \rangle = \sum_{n_d,n_g=0}^{\infty} f(E_{3,n_d,n_g}) = \text{Tr} f(H_3) = -\partial_\mu \Omega \tag{64}
\]

which shows that there are two possibilities for getting \( \langle N_e \rangle \): either summing all distributions (63) or taking the derivative of \( \Omega \) with respect to the chemical potential. Clearly, to go further in evaluating different physical quantities, one should explicitly determine \( \Omega \).
5.2. Calculating the thermodynamical potential

The thermodynamical potential is very much needed to describe the quasi-classical behavior of the present system. It can be calculated by adopting some technical methods like that using the Berezin–Lieb inequalities. In fact, this is based on a general statement, that is, for any convex function \( g(A) \) of the observable \( A \), one can write the inequalities

\[
\frac{1}{\pi^2} \int_{C^2} g(A) \, d^2z \, d^2\zeta \leq \text{Tr} g(A) \leq \frac{1}{\pi^2} \int_{C^2} g(\hat{A}) \, d^2z \, d^2\zeta \tag{65}
\]

where the lower and upper symbol functions \((\hat{A}, \hat{A})\) were defined before. This tells us that knowing the inferior and superior boundaries of a given observable, one can derive its trace.

At this level, we have all the ingredients needed to perform our task. Indeed, a straightforward application of (65) gives the result

\[
-\frac{1}{\beta \pi^2} \int_{C^2} \log \left[ 1 + \exp \left\{ -\beta (\hat{H}_3 - \mu) \right\} \right] \, d^2z \, d^2\zeta \leq \Omega \leq -\frac{1}{\beta \pi^2} \int_{C^2} \log \left[ 1 + \exp \left\{ -\beta (\hat{H}_3 - \mu) \right\} \right] \, d^2z \, d^2\zeta \tag{66}
\]

After mapping (51) and (55) into (66), we end up with

\[
-\frac{1}{\beta} \int_0^\infty du_d \int_0^\infty du_g \log \left[ 1 + \exp \left\{ -\beta (\hbar \omega_+ u_d + \hbar \omega_- u_g - \hbar \omega \cosh \alpha - \mu) \right\} \right] \leq \Omega \leq -\frac{1}{\beta} \int_0^\infty du_d \int_0^\infty du_g \log \left[ 1 + \exp \left\{ -\beta (\hbar \omega_+ u_d + \hbar \omega_- u_g + \hbar \omega \cosh \alpha - \mu) \right\} \right] \tag{67}
\]

where we have set \( u_d = |z_d|^2 \) and \( u_g = |z_g|^2 \). The solution can be obtained by making a rearrangement followed by an integration. Indeed, by changing the variables using

\[
u = \beta \hbar (\omega_+ u_d + \omega_- u_g) \quad \text{and} \quad v = \beta \hbar \omega_+ u_d \tag{68}
\]

we show that, in terms of the parameters \( \kappa_\pm \) (62), (67) becomes

\[
\Phi(\kappa) \leq \Omega \leq \Phi(\kappa_-) \tag{69}
\]

where the function \( \Phi \) is given by

\[
\Phi(\kappa) = -\frac{\kappa}{2 (\beta \hbar)^2 \omega_+ \omega_-} \int_0^\infty \frac{u^2 e^{-u}}{1 + \kappa e^{-u}} \, du. \tag{70}
\]

Actually, the problem of determining \( \Omega \) is restricted to that of finding the solutions for such an integral. This can be done by defining a new parameter

\[
\lambda = \omega_+ \omega_- = \omega^2 - \omega^2_c - 2 \omega \omega_c \sinh \alpha \tag{71}
\]

and distinguishing between the signs of \( \kappa_\pm \) to end up with the solutions

\[
\Phi(\kappa) = \begin{cases} 
\frac{1}{\beta \lambda (\beta \hbar)^2} F_3(-\kappa), & \kappa \leq 1 \\
\frac{1}{\beta \lambda (\beta \hbar)^2} \left[ -\frac{(\log \kappa)^3}{6} - \frac{\pi^2 \log \kappa}{6} + F_3(-\kappa^{-1}) \right], & \kappa > 1
\end{cases} \tag{72}
\]
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where we have introduced here the function $F_s$ of the Riemann–Fermi–Dirac type. For a

given variable $z$, it reads as

$$F_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s}. \quad (73)$$

Note in passing that $\Phi$ depends on the sign of $\lambda$ as well. Discussions of such matters will

be reported next. On the other hand, the above results can be discussed by separately

considering the high and low temperature regimes.

5.2.1. The high temperature regime. Having the expression (69) together with (72), one

can introduce an appropriate approximation to further simplify the form of $\Omega$ and

derive interesting results. To achieve this goal, we can analyze two liming cases for the

temperature of the present system.

We start our analysis by considering the high temperature regime that corresponds

to the condition $|\mu \pm \hbar \omega \cosh \alpha| \ll k_B T$. By taking this into account, we find

$$\Omega \approx \frac{k_B T}{\lambda} \left( \frac{k_B T}{\hbar} \right)^2 F_3(-1) \approx -0.901543 \frac{k_B T}{\lambda} \left( \frac{k_B T}{\hbar} \right)^2 \approx -0.901543 \frac{k_B T}{\lambda} \left( \frac{k_B T}{\hbar} \right)^2. \quad (74)$$

This is a nice form that can be further discussed. Recall that the parameter involved,

$\lambda$, is a magnetic field and dependent on $\alpha$, which gives a generalization of that already

obtained in [7]. Obviously without coupling, they coincide.

On the other hand, one can report different discussions related to the above form of

$\Omega$ in terms of the coupling parameter $\alpha$. With these we can show what makes a difference

with respect to the standard case, i.e. $c_1 = c_2$ and $c_3 = 0$. By doing this, we can summarize

the following results:

- By inspecting the form of $\Omega$, one can immediately notice the first general result.

  Indeed, by considering a negative $\lambda$ we end up with a positive $\Omega$, which cannot be

  obtained from the standard results [7].

- Equation (74) is magnetic field dependent as well and therefore the present case

  exhibits an magnetism behavior. This statement can be confirmed by explicitly

  determining the magnetic moment and susceptibility.

- The easiest way to obtain the magnetic moment is that one can require for instance

  the following configuration:

  $$\omega \longrightarrow \omega_c, \quad \sinh \alpha = \text{finite number}. \quad (75)$$

After giving quick conclusions by looking at the form (74), now let us be much more

accurate and derive explicit results. Indeed, after a straightforward calculation, we show

that the magnetic moment takes the form

$$M = 1.803086 k_B T \left( \frac{k_B T e}{\hbar m c} \right)^2 \left[ \frac{2 B}{\omega^2} \left( \frac{c_2 + m_1 \omega_1^2}{m_1} + \frac{c_1 + m_2 \omega_2^2}{m_2} \right) \right] - \sinh \alpha \left\{ \frac{\omega mc}{e} + \frac{2B \omega c}{\omega^3} \left( \frac{c_2 + m_1 \omega_1^2}{m_1} + \frac{c_1 + m_2 \omega_2^2}{m_2} \right) \right\} - 1. \quad (76)$$

\[ \text{doi:10.1088/1742-5468/2010/01/P01012} \]
This allows us to end up with the susceptibility

\[
\chi = -3.621 \times 10^7 \frac{k_B T \hbar}{mc} \frac{c_2}{4c_1c_2 - c_3^2} \sinh \alpha + \frac{m}{4c_1c_2 - c_3^2} \left( \frac{c_2}{m_1} + \frac{c_1}{m_2} \right) + 1.
\]  

(77)

It is clear that \( \chi \) is behaving as a linear function in terms of the hyperbolic function \( \sinh \alpha \). This result in fact shows the difference with respect to the case without coupling where there is no susceptibility and therefore no effect is obtained in the high temperature regime.

5.2.2. The low temperature regime. To accomplish our analysis in terms of temperature, we discuss the last case. This can be achieved by considering a more realistic case, namely that of \( \mu \gg \hbar \omega \sinh \alpha \) and \( \mu \gg k_B T \). With these considerations, we will be able to derive interesting results and deduce different conclusions.

After considering the above two limiting cases, we show that (72) can be written as a combination of three parts. This is

\[
\Phi(\kappa) = A + \frac{\Delta}{2} + S_\pm
\]

(78)

where the different terms are given by

\[
A(B,T,\alpha) = -\frac{\mu}{2\lambda} \left[ \frac{1}{3} \left( \frac{\mu}{\hbar} \right)^2 + \omega^2 \cosh^2 \alpha + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar} \right)^2 \right]
\]

\[
\frac{\Delta}{2}(B,T,\alpha) = \frac{\hbar \omega \cosh \alpha}{2\lambda} \left[ \left( \frac{\mu}{\hbar} \right)^2 + \frac{1}{3} \omega^2 \cosh^2 \alpha + \frac{\pi^2}{6} \left( \frac{k_B T}{\hbar} \right)^2 \right]
\]

(79)

\[
S_\pm(B,T,\alpha) = \frac{k_B T}{\lambda} \left( \frac{k_B T}{\hbar} \right)^2 F_3(- \exp[-\beta(\mu \pm \hbar \omega \cosh \alpha)]).
\]

(81)

We notice that, according to these functions, \( \Omega \) is in the interval \([A + S_+ - (\Delta/2), A + S_- + (\Delta/2)]\). This will be used to derive different results in the present context.

We can go further by making an important assumption. In fact, we restrict ourselves to the condition

\[
e^{\pm \hbar \omega \cosh \alpha} \approx 1.
\]

(80)

In this situation, one can see that \( S_\pm \) is reduced to

\[
S_\pm \approx S_0(B,T,\alpha) = \frac{k_B T}{\lambda} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 F_3(-e^{-\beta \mu}).
\]

(81)

Moreover, taking into account the above limiting cases, we show that

\[
\frac{\Delta}{|A + S_0|} \to 0.
\]

(82)

Combining all of this we end up with the form

\[
\Omega \approx \frac{1}{\lambda} \left[ -\frac{\mu}{2} \left( \frac{1}{3} \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \omega^2 \cosh^2 \alpha + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \right) + k_B T \left( \frac{k_B T}{\hbar \omega_0} \right)^2 F_3(-e^{-\beta \mu}) \right].
\]

(83)
This in fact can be used to deduce different physical quantities. In particular, we evaluate
the average number of electrons to obtain
\[
\langle N_e \rangle (B, T, \alpha) \approx \frac{1}{\lambda} \left( \frac{\mu}{\hbar} \right)^2 \left[ \frac{1}{2} + \left( \frac{\hbar \omega}{\mu} \right)^2 \right] + \frac{\pi^2}{3} \left( \frac{k_B T}{\mu} \right)^2 + \left( \frac{k_B T}{\mu} \right)^2 F_2 \left( -e^{-\beta \mu} \right) .
\] (84)

As far as the magnetic moment is concerned, one can obtain a complicated form. This
is due to the fact that \( \lambda \) and \( \omega \) are magnetic field dependent. However, we can get more
information by inspecting some limiting cases. These will also offer us a way to emphasize
what makes the difference with respect to other approaches and in particular that of [7].
This can be done by discussing the nature of the coupling parameter involved in the
‘game’.

5.3. Asymptotic behavior

Having derived a general expression for the thermodynamical potential, one can ask about
further simplifications for characterizing the system behavior in some special cases—
more precisely, how the above results can be approximated, by inspecting the limits:
\( \alpha = 0, \alpha \ll 1 \) and \( \alpha \rightarrow \infty \). The reply to such a question is the subject of the following
investigations.

5.3.1. Without coupling. It is natural to ask about the case \( \alpha = 0 \). To reply to this inquiry,
one can return to the former analysis to show that the different quantities given in (79)
can be restricted to the functions
\[
A(B, T, 0) = -\frac{\mu}{2} \left[ \frac{1}{3} \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \left( \frac{\omega}{\omega_0} \right)^2 + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \right]
\]
\[
\Delta (B, T, 0) = \frac{\hbar \omega}{2} \left[ \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \frac{1}{3} \left( \frac{\omega}{\omega_0} \right)^2 + \frac{\pi^2}{6} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \right]
\]
\[
S_{\pm} (B, T, 0) = k_B T \left( \frac{k_B T}{\hbar \omega_0} \right)^2 F_3 \left( -\exp \left[ -\beta (\mu \pm \hbar \omega) \right] \right).
\] (85)

They show that the thermodynamical potential lies in \([A + S_+ - (\Delta/2), A + S_+ + (\Delta/2)]\).
Note that these findings exactly coincide with those obtained by analyzing a confined
two-dimensional system in the presence of a uniform magnetic field [7].

To reproduce most of the results derived in [7], one can inspect (85), making an
approximation. That is \( S_{\pm} \) can be replaced by
\[
S_0 = k_B T \left( \frac{k_B T}{\hbar \omega_0} \right)^2 F_3 \left( -e^{-\beta \mu} \right)
\] (86)
to end up with the form
\[
\Omega \approx \left[ -\frac{\mu}{2} \left\{ \frac{1}{3} \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \left( \frac{\omega}{\omega_0} \right)^2 + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \right\} + k_B T \left( \frac{k_B T}{\hbar \omega_0} \right)^2 F_3 \left( -e^{-\beta \mu} \right) \right].
\] (87)
Therefore, the average number of electrons is given by
\[
\langle N_e \rangle \approx \frac{1}{2} \left[ -\left\{ \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \left( \frac{\omega}{\omega_0} \right)^2 + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \right\} + \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \right] \cdot (88)
\]
and the magnetic moment by
\[
M \approx 4\mu \left( \frac{\mu}{\hbar \omega_0} \right)^2 B. (89)
\]
The corresponding susceptibility \( \chi_p \) reads as
\[
\chi_p = 4\mu \left( \frac{\mu}{\hbar \omega_0} \right)^2. (90)
\]
One can also inspect other approximations. Indeed, by requiring that \( \mu \gg k_B T \) and \( \mu \ll \hbar \omega \), we show that (88) can be written as
\[
\langle N_e \rangle \approx \frac{1}{2} \left( \frac{\mu}{\hbar \omega_0} \right)^2. (91)
\]
The above derivation shows that our results are general in the sense that after making appropriate choices one can recover significant work already published.

5.3.2. Weak coupling. We start our analysis by dealing with the first case that corresponds to \( \alpha \ll 1 \). Clearly, an expansion of different quantities entering in the ‘game’ is very much needed. Indeed, by taking the first order of \( \alpha \), we can approximate (83) as
\[
\Omega \approx \left[ -\frac{\mu}{\hbar} \left\{ \frac{1}{3} \left( \frac{\mu}{\hbar \omega} \right)^2 + \left( \frac{\omega}{\omega_0} \right)^2 + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \right\} + k_B T \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \right] \times \left( 1 + \frac{2\omega \omega_c}{\omega_0^2} \alpha \right). (92)
\]
where we have set \( \omega_0^2 = \omega^2 - \omega_c^2 \).

Obtaining (83), it is worthwhile to ask about the related physical quantities in order to characterize their behaviors in terms of the coupling parameter for the present case. We use the former definitions to show that the average number of electrons is
\[
\langle N_e \rangle(B, T, \alpha) \approx \left( \frac{\mu}{\hbar} \right)^2 \left[ \frac{1}{2} + \left( \frac{\hbar \omega}{\mu} \right)^2 \right] + \frac{\pi^2}{3} \left( \frac{k_B T}{\mu} \right)^2 + \left( \frac{k_B T}{\mu} \right)^2 \right] \times \left( 1 + \frac{2\omega \omega_c}{\omega_0^2} \alpha \right). (93)
\]
In the light of the assumptions \( \mu \gg k_B T \) and \( \mu \gg \hbar \omega \), we obtain
\[
\langle N_e \rangle(B, \alpha) \approx \frac{1}{2} \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \omega \omega_c \left( \frac{\mu}{\hbar \omega_0} \right)^2 \alpha. (94)
\]
Clearly, in the second term on the right there is a correction to the average number of electrons. This is validated by canceling the coupling to recover the standard result (91).
Now let us investigate the magnetism in such a case. Indeed, a straightforward calculation gives the magnetic moment as

\[ M(B, T, \alpha) = \frac{2 \mu_B}{\hbar \omega_0} \left[ \mu \omega_c \left( 1 + \frac{2 \omega \omega_c}{\omega_0} \alpha \right) - 2 \left( \frac{\omega^2 + \omega_c^2}{\omega} \right) \Omega(B, T, 0) \right] \tag{95} \]

where \( \Omega(B, T, 0) \) is the thermodynamical potential corresponding to the standard case, i.e. \( \alpha = 0 \). Considering \( \mu \gg k_B T \) and \( \mu \gg \hbar \omega \), we show that

\[ M(B, \alpha) \approx \frac{2 \mu_B}{3 \hbar \omega_0^2} \left( \frac{\mu}{\hbar \omega_0} \right)^2 \left( \frac{\omega^2 + \omega_c^2}{\omega} \right) \alpha. \tag{96} \]

One important thing that should be noted here is that the magnetic moment is behaving like a linear function in terms of \( \alpha \). Obviously, without coupling we end up a null magnetization. This means that we have something like a phase transition from a coupling to a decoupling system. This point might be investigated further for dealing with other issues in statistical physics. Furthermore, by carefully identifying (89)–(96), one can fix \( \alpha \) to reproduce the orbital paramagnetism. Indeed, the solution can be written as

\[ \alpha = \frac{2 \hbar \omega_0^2 \mu \omega}{\omega^2 + \omega_c^2} B. \tag{97} \]

On the other hand, one can also make another choice of the coupling parameter to get the interesting result (89). In fact, here also one can reproduce the Landau diamagnetism.

5.3.3. Strong coupling. To complete our analysis we consider the last case, that is the strong coupling limit. This of course will shed light on the system behavior in such a case and therefore allow us to get more interesting results. To clarify this, we take the limit \( \alpha \rightarrow \infty \) to obtain

\[ A = -\frac{\mu}{8} \left( \frac{1}{\omega_c} \right)^2, \quad \frac{\Delta}{2} = \frac{\hbar \omega}{12} \left( \frac{1}{\omega_c} \right)^2 e^\alpha, \quad S_{\pm} = 0 \tag{98} \]

which leads to the thermodynamical potential

\[ \Omega \approx -\frac{\mu}{8} \left( \frac{1}{\omega_c} \right)^2. \tag{99} \]

It is clear that the average number reads as

\[ \langle N_c \rangle \approx \frac{1}{8} \left( \frac{me}{\hbar e B} \right)^2 \tag{100} \]

which behaves as the inverse of the magnetic field. It seems that (100) shares some common features with the quantum Hall effect results [18]. Indeed, one has to recall that the filling factor is defined as the ratio between \( \langle N_c \rangle \) and the quantized flux. More precisely, we can write

\[ \frac{\langle N_c \rangle}{N_\phi} = \frac{BS}{\phi_0} \tag{101} \]

where \( S \) is the system area and \( \phi_0 = (\hbar e/c) \). Clearly, we can adjust all parameters to show that effectively we have something related to the quantum Hall effect.

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In summary, the Berezin–Lieb inequalities are powerful tools that one can use to study the thermodynamical behavior for a given system. As we have seen so far, a straightforward application of such an approach allows us to derive different interesting results. On the other hand, as we claimed before, there is another way to do this, and this will be tackled next.

6. Fermi–Dirac trace formulae

As we claimed before, we use the second method to explicitly determine the exact expressions for the thermodynamical potential. This is in fact based on the Fermi–Dirac trace formulae, and does not include approximations in the derivation of $\Omega$ and therefore makes a difference with respect to the Berezin–Lieb inequalities. Subsequently, we restrict ourselves to the evaluation of the average number of electrons and magnetic moment as well as their expressions at zero coupling.

6.1. Exact expressions for $\Omega$

Using the machinery developed in [7], we can derive an exact form of $\Omega$. This can be done by making a straightforward application of the Fermi–Dirac trace formulae, in particular (A.5) and (A.7) from the appendix. We use (A.6) to define a function $\Theta(k)$ in terms of our language as

$$\Theta(k) = \text{Tr} \left[ e^{-(ik+1)\beta/2H_3} \right]$$

where the Hamiltonian $H_3$ is given in (15). After replacing $H_3$, we end up with

$$\Theta(k) = \text{Tr} \left[ e^{-(ik+1)\hbar\beta/2(\omega_+N_e+\omega_-N_g+\omega \cosh \alpha)} \right]$$

which can be written as

$$\Theta(k) = e^{-(ik+1)(\beta/2)\hbar \omega \cosh \alpha} \frac{1}{1 - e^{-(ik+1)(\beta/2)\hbar \omega_+}} \frac{1}{1 - e^{-(ik+1)(\beta/2)\hbar \omega_-}}.$$ (103)

Now let us tackle our problem by writing the Fourier integral representation for the thermodynamical potential. This is

$$\Omega = \frac{1}{\beta} \int_{-\infty}^{+\infty} e^{-(ik+1)(\beta/2)(\hbar \omega \cosh \alpha - \mu)} \frac{1}{2 \cosh (\pi/2)k} \left( \frac{1}{ik+1} \right) \frac{1}{1 - e^{-(ik+1)(\beta/2)\hbar \omega_+}} \frac{1}{1 - e^{-(ik+1)(\beta/2)\hbar \omega_-}} \, dk.$$ (104)

This integral is given as a series by using the residue theorem. One can easily see that the numbers $(2m+1)i$, $m \in \mathbb{Z}$, are simple poles of $\cosh(\pi/2)k$, and $i + (4\pi m/(\beta \hbar \omega_+))$, $i + (4\pi m/(\beta \hbar \omega_-))$, $m \in \mathbb{Z}^*$, are simple or double poles of $\Theta(k)$. Now we can consider two cases. The first one is where

$$\alpha \in \left( -\infty, \log \left( \frac{\mu}{\hbar \omega} - \sqrt{\left( \frac{\mu}{\hbar \omega} \right)^2 - 1} \right) \right] \cup \left( \log \left( \frac{\mu}{\hbar \omega} + \sqrt{\left( \frac{\mu}{\hbar \omega} \right)^2 - 1} \right) , +\infty \right).$$ (105)

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and here we take an integration path lying in the lower half-plane and involving only the simple poles \((2m + 1)i, m < 0\). This leads to the result

$$\Omega(B, T, \alpha) = \frac{1}{4\beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{e^{\beta \mu m}}{\sinh ((\beta/2)\hbar \omega_+ m) \sinh ((\beta/2)\hbar \omega_- m)}$$  \hspace{1cm} (107)

In the second case where

$$\alpha \in \left[ \log \left( \frac{\mu}{\hbar \omega} - \sqrt{\left( \frac{\mu}{\hbar \omega} \right)^2 - 1} \right), \log \left( \frac{\mu}{\hbar \omega} + \sqrt{\left( \frac{\mu}{\hbar \omega} \right)^2 - 1} \right) \right]$$  \hspace{1cm} (108)

an integration path in the upper half-plane is chosen. It encircles all the other poles: \((2m + 1)i, m \geq 0, i + 4\pi m/(\beta \hbar \omega_+), i + 4\pi m/(\beta \hbar \omega_-), m \in \mathbb{Z}^*\). We present the result in a manner which will render apparent the various regimes:

$$\Omega = (\Omega_L + \Omega_{01}) + \Omega_{02} + \text{under}\sum_{m \geq 0} \text{the}\left[ (2m + 1)i \right] + \sum_{m \neq 0} \text{under}\left[ (i + \frac{4\pi}{\beta \hbar \omega} m_{\pm}) \right].$$

where \(\Omega_L(B, \alpha)\) is given by

$$\Omega_L = \frac{\mu \omega_c^2}{24 \lambda} = \frac{\mu}{24} \left( \omega^2 - \omega_c^2 - 2 \omega_c \sin \alpha \right)$$  \hspace{1cm} (110)

and \(\Omega_{01}(B, T, \alpha)\) reads as

$$\Omega_{01} = -\frac{\mu}{6} \left[ \left( \frac{\mu}{\hbar \sqrt{\omega^2 - \omega_c^2 - 2 \omega_c \sin \alpha}} \right)^2 + \pi^2 \left( \frac{k_B T}{\hbar \sqrt{\omega^2 - \omega_c^2 - 2 \omega_c \sin \alpha}} \right)^2 - \frac{1}{2} \right].$$

Now let us consider an approximation such that \(2\omega_c/(\omega^2 - \omega_c^2) \sin \alpha \ll 1\). This allows us to write the expansion

$$\frac{1}{\omega_0^2 - 2 \omega_c \sin \alpha} \approx \frac{1}{\omega_0^2} + \left( \frac{2 \omega_c}{\omega_0^4} \right) \sin \alpha$$  \hspace{1cm} (112)

in the first order. Consequently, we find that this gives \(\Omega_L(B, \alpha)\) as

$$\Omega_L(B, \alpha) = \frac{\mu}{24} \left( \frac{\omega_c}{\omega_0} \right)^2 + \frac{\mu}{12} \left( \frac{\omega_c}{\omega_0^2} \right) \omega_c \sin \alpha.$$  \hspace{1cm} (113)

This result can be interpreted in different ways. Indeed, if we forget about the \(\alpha\) appearing, we can reach the same conclusion as in [7]. Indeed, the first term is at the origin of the Landau diamagnetism and gives the susceptibility

$$\chi_L = -\frac{1}{3} \mu B \left( \frac{\mu B}{\hbar \omega_0} \right)^2 = -\frac{1}{3} D_0 \mu_B^2$$  \hspace{1cm} (114)

where the coefficient \(D_0 = \mu/(\hbar \omega_0)^2\) can be interpreted as the density of states at the Fermi energy. Note that the value of \(\chi_L\) is equal to one third of the one, \(\chi_p\), found in (89).
On the other hand, \( \alpha \) can be adjusted to get another contribution to the susceptibility. More precisely, we can define \( \alpha \) in terms of the inverse of the squared magnetic field to absorb the term \( \omega \omega_c \) and therefore get a correction to the standard Landau diamagnetism. Otherwise, we can even reproduce \( \chi_p \) simply by making an appropriate choice of \( \alpha \). Indeed, fixing \( \sinh(\alpha) = \gamma \omega_0 / \omega_c \), we end up with (89), where \( \gamma \) is a constant that can be fixed easily. This shows how we obtained the general results and allows us to deduce interesting properties.

According to (112), \( \Omega_{01}(\alpha) \) becomes

\[
\Omega_{01}(\alpha) = -\frac{\mu}{6} \left[ \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \pi^2 \left( \frac{k_B T}{\hbar \omega_0} \right)^2 - \frac{1}{2} \right] \\
- \frac{\mu}{3} \left[ \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \pi^2 \left( \frac{k_B T}{\hbar \omega_0} \right)^2 \right] \left( \frac{\omega \omega_c}{\omega_0^2} \right) \sinh \alpha.
\]

(115)

\( \Omega_{02}(\alpha) \) reads as

\[
\Omega_{02}(\alpha) = \frac{1}{4 \beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{e^{-\beta \mu m}}{\sinh \left[ \left( (\omega e^\alpha + \omega_c) / 2k_B T \right) m \right] \sinh \left[ \left( (\omega e^{-\alpha} - \omega_c) / 2k_B T \right) m \right]}
\]

(116)

Note that when \( \alpha = 0 \) we get \( \lambda = \omega_0 \), which leads to us recovering the result obtained in [7]. It becomes negligible for the low temperature regime \( k_B T \ll \mu \). The sum of \( \Omega_L \) and \( \Omega_{01} \) is analogous to the term \( A \) in (78) and \( \Omega_{02} \) corresponds to \( S_\pm \). The last term is responsible for the oscillatory behavior. For \( \omega_+ / \omega_- \) irrational values, we have

\[
\Omega_{\text{osc}}(\alpha) = \frac{1}{2 \beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left[ \frac{\sin \left[ \left( 2\mu / \hbar (\omega e^\alpha - \omega_c) \right) \pi m \right]}{\sin \left[ \left( \omega e^\alpha + \omega_c \right) / \left( \omega e^{-\alpha} - \omega_c \right) \pi m \right] \sinh \left[ (2k_B T / \hbar (\omega e^\alpha - \omega_c)) \pi^2 m \right]}
\]

\[
+ \frac{\sin \left[ \left( 2\mu / \hbar (\omega e^\alpha + \omega_c) \right) \pi m \right]}{\sin \left[ \left( \omega e^\alpha - \omega_c \right) / \left( \omega e^\alpha + \omega_c \right) \pi m \right] \sinh \left[ (2k_B T / \hbar (\omega e^\alpha + \omega_c)) \pi^2 m \right]}
\]

\[
\equiv \Omega_{\text{osc}}^-(\alpha) + \Omega_{\text{osc}}^+(\alpha).
\]

(117)

6.2. The average number of electrons

In this section, we will exploit the formulae (107)–(117) to obtain the exact expressions for the average number of electrons and the magnetic moment. We will restrict ourselves to a more realistic case: \( \mu \leq \hbar \omega / 2 \). The average number of electrons is easily derived by taking the derivative of \( -\Omega \) with respect to \( \mu \). It is found to be

\[
\langle N_e \rangle = -\partial_\mu \Omega_L(\alpha) - \partial_\mu \Omega_{01}(\alpha) - \partial_\mu \Omega_{\text{osc}}^-(\alpha) - \partial_\mu \Omega_{\text{osc}}^+(\alpha).
\]

(118)

With a straightforward calculation, we find

\[
\langle N_e(\alpha) \rangle = \langle N_e \rangle_L + \langle N_e \rangle_{01} + \langle N_e \rangle_{02} + \langle N_e \rangle_{\text{osc}}^- + \langle N_e \rangle_{\text{osc}}^+
\]

(119)
where different portions are given by

\[
\langle N_e \rangle_L = -\frac{1}{24} \left( \frac{\omega_c}{\omega_0} \right)^2 + \frac{1}{2} \left[ \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 - \frac{1}{6} \right]
\]

(120)

\[
\langle N_e \rangle_{01} = \left[ \left( \frac{\mu}{\hbar \omega_0^2} \right)^2 + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 - \frac{\omega_c^2}{12} \right] \frac{\omega_c}{\omega_0^4} \sinh \alpha
\]

(121)

\[
\langle N_e \rangle_{02} = \frac{1}{4} \sum_{m=1}^{\infty} (-1)^m \frac{e^{-\beta \mu m}}{\sinh \left[ (\mu (\omega e^\alpha + \omega_c) / 2k_B T)m \right] \sinh \left[ (\mu (\omega e^{-\alpha} - \omega_c) / 2k_B T)m \right]}
\]

(122)

\[
\langle N_e \rangle_{osc}^- = -\pi \sum_{m=1}^{\infty} (-1)^m \frac{k_B T}{\hbar (\omega e^\alpha - \omega_c)} \frac{\cos \left[ (2\mu / \hbar (\omega e^\alpha - \omega_c)) \pi m \right]}{\sin \left[ (\omega e^\alpha + \omega_c) / (\omega e^{-\alpha} - \omega_c) \pi m \right] \sinh \left[ (2k_B T / \hbar (\omega e^\alpha - \omega_c)) \pi^2 m \right]}
\]

(123)

\[
\langle N_e \rangle_{osc}^+ = -\pi \sum_{m=1}^{\infty} (-1)^m \frac{k_B T}{\hbar (\omega e^\alpha + \omega_c)} \frac{\cos \left[ (2\mu / \hbar (\omega e^\alpha + \omega_c)) \pi m \right]}{\sin \left[ (\omega e^\alpha - \omega_c) / (\omega e^\alpha + \omega_c) \pi m \right] \sinh \left[ (2k_B T / \hbar (\omega e^\alpha + \omega_c)) \pi^2 m \right]}
\]

(124)

The above results are general in the sense that the standard solutions can be recovered. Indeed, requiring that \( \alpha = 0 \), we show that

\[
\langle N_e \rangle = -\frac{1}{24} \left( \frac{\omega_c}{\omega_0} \right)^2 + \frac{1}{2} \left[ \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \omega_0} \right)^2 - \frac{1}{6} \right] e^{-\beta \mu m}
\]

\[
+ \frac{1}{4} \sum_{m=1}^{\infty} (-1)^m \frac{\sinh \left[ (\beta / 2) \hbar \omega_+ m \right] \sinh \left[ (\beta / 2) \hbar \omega_- m \right] \cos \left[ (2\mu / \hbar \omega_- \pi m \right]}{\sin \left[ (\omega_+ / \omega_- \pi m \right] \sinh \left[ (2k_B T / \hbar \omega_- \pi^2 m \right]}
\]

\[
- \pi \sum_{m=1}^{\infty} (-1)^m \left[ \frac{k_B T}{\hbar \omega_- \sin \left( (\omega_+ / \omega_- \pi m \right] \sinh \left( (2k_B T / \hbar \omega_- \pi^2 m \right]}
\]

\[
+ \frac{k_B T}{\hbar \omega_+ \sin \left( (\omega_- / \omega_+ \pi m \right] \sinh \left( (2k_B T / \hbar \omega_+ \pi^2 m \right]}
\]

6.3. The magnetic moment

The magnetic moment is found from the converse of the derivative of the thermodynamical potential from the magnetic field. Then, we have

\[
M(B, T, \alpha) = -2\mu_B \left( \frac{\partial \Omega_L(\alpha)}{\partial \hbar \omega_c} \right) - 2\mu_B \left( \frac{\partial \Omega_m(\alpha)}{\partial \hbar \omega_c} \right) - 2\mu_B \left( \frac{\partial \Omega_02(\alpha)}{\partial \hbar \omega_c} \right)
\]

\[
- 2\mu_B \left( \frac{\partial \Omega_{osc}(\alpha)}{\partial \hbar \omega_c} \right).
\]
This is equivalent to

$$M(B, T, \alpha) = 2\mu_B \left[ M_L(\alpha) + M_{01}(\alpha) + M_{02}(\alpha) + M_{\text{osc}}^-(\alpha) + M_{\text{osc}}^+(\alpha) \right]$$  \hspace{1cm} (125)$$

where different quantities read as

$$M_L(B, T, \alpha) = -\frac{\mu}{12\hbar \omega_0} \left( \frac{\omega_c}{\omega_0} \right) - \frac{\mu}{12\hbar \omega_0^2} \left( 3\omega + \frac{\omega_c^2}{\omega} \right) \omega_c^2 \sinh \alpha$$ \hspace{1cm} (126)$$

$$M_{01}(B, T, \alpha) = \frac{\mu}{3\hbar \omega} \left[ \left( \frac{\mu}{\hbar \omega_0^2} \right)^2 + \pi^2 \left( \frac{k_B T}{\hbar \omega_0^2} \right)^2 \right] \left( \omega^2 + \omega_c^2 \right) \sinh \alpha.$$ \hspace{1cm} (127)$$

$$M_{02}(B, T, \alpha) = \frac{1}{8} \sum_{m=1}^{\infty} (-1)^m e^{-\beta m} \left[ \coth ((\hbar / 2k_B T m) - \coth ((\hbar / 2k_B T m) \sinh ((\hbar / 2k_B T) m)) \right).$$ \hspace{1cm} (128)$$

$$M_{\text{osc}}^+(B, T, \alpha) = \frac{k_B T}{\hbar \omega_+} \sum_{m=1}^{\infty} \frac{(-1)^m \sin ((2\mu / \hbar \omega_+ \pi m)}{\sin ((\omega_+ / \omega_+) \pi m) \sin ((2k_B T / \hbar \omega_+) \pi^2 m)} \times \left[ \frac{\mu \pi}{\hbar \omega_+} \cot \left( \frac{2\mu}{\hbar \omega_+ \pi m} \right) - \frac{\omega \cosh \alpha}{\omega_+ \pi} \cot \left( \frac{\omega_+ \pi m}{\omega_+ \pi m} \right) - \frac{k_B T}{\hbar \omega_+} \pi \coth \left( \frac{2k_B T}{\hbar \omega_+ \pi m} \right) \right.$$ \hspace{1cm} (129)$$

$$M_{\text{osc}}^-(B, T, \alpha) = -\frac{k_B T}{\hbar \omega_-} \sum_{m=1}^{\infty} \frac{(-1)^m \sin ((2\mu / \hbar \omega_- \pi m)}{\sin ((\omega_+ / \omega_-) \pi m) \sin ((2k_B T / \hbar \omega_-) \pi^2 m)} \times \left[ \frac{\mu \pi}{\hbar \omega_-} \cot \left( \frac{2\mu}{\hbar \omega_- \pi m} \right) - \frac{\omega \cosh \alpha}{\omega_- \pi} \cot \left( \frac{\omega_- \pi m}{\omega_- \pi m} \right) - \frac{k_B T}{\hbar \omega_-} \pi \coth \left( \frac{2k_B T}{\hbar \omega_- \pi m} \right) \right.$$ \hspace{1cm} (130)$$

In the case where \( \alpha = 0 \), we end up with

$$M_L = \frac{-\mu}{12\hbar \omega_0} \left( \frac{\omega_c}{\omega_0} \right) \equiv \frac{1}{2\mu_B} \chi_L H,$$ \hspace{1cm} (131)$$

$$M_0 = \frac{1}{8\omega} \sum_{m=1}^{\infty} (-1)^m e^{-\beta m} \frac{[\omega_+ \coth (\beta \hbar \omega_+ m / 2) - \omega_- \coth (\beta \hbar \omega_- m / 2)]}{\sinh (\beta \hbar \omega_+ m / 2) \sinh (\beta \hbar \omega_- m / 2)}.$$ \hspace{1cm} (132)$$

and, for the irrational case \( \omega_+ / \omega_- \notin \mathbb{Q} \),

$$M_{\text{osc}}^- = \frac{k_B T}{\hbar \omega} \sum_{m=1}^{\infty} \frac{(-1)^m \sin (2\pi m\mu / (\hbar \omega_-))}{\sin (\pi m \omega_+ / \omega_-) \sinh (2\pi^2 m k_B T / (\hbar \omega_-))} \times \left[ \frac{\pi \mu}{\hbar \omega_-} \cot \left( \frac{2\pi m}{\hbar \omega_-} \right) - \frac{\pi \omega_+}{\omega_-} \coth \left( \pi m \omega_+ \right) \right.$$ \hspace{1cm} (133)$$

$$- \frac{\pi^2 k_B T}{\hbar \omega_-} \coth \left( \frac{2\pi^2 m k_B T}{\hbar \omega_-} \right) \right]$$
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\[ M_{\text{osc}}^+ = \frac{k_B T}{\hbar \omega} \sum_{m=1}^{\infty} \frac{(-1)^m \sin \left( \frac{2\pi m \mu}{\hbar \omega} \right)}{\sin \left( \pi m \omega_- / \omega_+ \right) \sinh \left( \frac{2\pi m k_B T}{\hbar \omega} \right)} \times \left[ \frac{\pi \mu}{\hbar \omega_+} \cot \left( \frac{2\pi m}{\hbar \omega} \frac{\mu}{\omega_+} \right) - \frac{\pi \omega_-}{\omega_+} \cot \left( \pi m \frac{\omega_-}{\omega_+} \right) \right. \]

\[ \left. - \frac{\pi^2 k_B T}{\hbar \omega_+} \coth \left( \frac{2\pi^2 m k_B T}{\hbar \omega} \right) \right]. \tag{134} \]

We will not give the expressions for \( M_{\text{osc}}^\pm \) in the rational case because the magnetization is a continuous function of \( \omega_c \) and its behavior can be fully understood from the irrational one.

At the end we note that the temperature scale is compared to the two natural modes \( \omega_{\pm} \) of the system and gives three possible intrinsic regimes: the high temperature regime \( k_B T > \hbar \omega_+ \), the low temperature regime \( k_B T < \hbar \omega_- \), and the intermediate temperature regime \( \hbar \omega_- < k_B T < \hbar \omega_+ \). Remember that we work in the large electron number region: \( \mu > \hbar \omega / 2 \).

7. Conclusion

We started by formulating our problem in two-dimensional space with two coupled harmonic oscillators living on it. Subsequently, we introduced a minimal coupling to generate another interacting system that was studied. After rescaling different variables, we showed that it is possible to get a diagonalized Hamiltonian. In fact, this is done by making use of a unitary transformation. This was helpful in the sense that the eigenvalues and their wavefunctions are obtained in a simple way in terms of the coupling parameter \( \alpha \).

Using the fact that the energy spectrum solutions are \( (\alpha, B) \) dependent, we discussed their underlying properties. More precisely, four limiting cases have been investigated, which are those of weak and strong parameters \( (\alpha, B) \). In particular, we noticed that on fixing \( \alpha \), some models can be recovered; these concern for instance the Landau Hamiltonian in two dimensions and the harmonic oscillator in one dimension. These findings allowed us to conclude that by adjusting the coupling parameter, one can derive other interesting solutions.

To investigate different issues related to the system considered, we constructed the corresponding coherent states, which are obtained as coupling parameter dependent. These were used to evaluate the thermodynamical potential by adopting two different methods.

The first method employed the Berezin–Lieb inequalities to obtain an approximate form. We used this to determine the average number of electrons \( \langle N_e \rangle \) and the magnetization as well as to underline the properties in terms of the limiting cases \( (\alpha \ll 1, \alpha \rightarrow \infty) \) as well as the weak and strong magnetic field limits. In fact, by treating the limit \( \alpha \ll 1 \), a correction to \( \langle N_e \rangle \) is obtained. More importantly, we showed that \( \alpha \) can be tuned to reproduce both the orbital paramagnetism and the Landau diamagnetism in such limits. In fact, we derived a general magnetic moment that can be fixed to reproduce different results and end up with various conclusions.

In the second method, we employed a mathematical toy model to determine the exact formula for the thermodynamical potential and therefore evaluated different physical

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quantities. More precisely, the Fermi–Dirac trace formulae are used and the average number of electrons as well as the magnetic moment are calculated. After evaluating the susceptibility, we found that there is a correction to the Landau diamagnetism, which is \( \alpha \) dependent. Again by fixing the parameter, other results can be obtained and in particular for \( \alpha = 0 \) standard results [7] are easily recovered.

Some interesting questions remain to be solved for the present system. In fact, the first concerns the temperature limits of the thermodynamical potential obtained in terms of the second method. This can also be investigated further by considering all limiting cases of the couple \((\alpha, B)\). The second is related to discussing the spatial density of the current. Finally, a numerical study of the results obtained is much needed to give another comparison with results already published.

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**Appendix: Fermi–Dirac trace formulae**

It is well-known that, like the Gaussian function, the function \( \text{sech} x = 1/\cosh x \) is a fixed point for the Fourier transform in the Schwartz space:

\[
\frac{1}{\cosh \sqrt{\pi/2}x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-iy}}{\cosh \sqrt{\pi/2}y} \, dy. \tag{A.1}
\]

Hence, given a Hamiltonian \( \mathcal{H} \), we can write for the corresponding Fermi operator

\[
f(\mathcal{H}) \equiv \frac{1}{1 + e^{\beta(\mathcal{H} - \mu)}} = \int_{-\infty}^{+\infty} \frac{e^{-(ik+1)\beta/2}(\mathcal{H} - \mu)}{4 \cosh \pi/2k} \, dk. \tag{A.2}
\]

Similarly, we can write for the thermodynamical potential operator

\[
-\frac{1}{\beta} \log (1 + e^{-\beta(\mathcal{H} - \mu)}) = -\frac{1}{\beta} \int_{-\infty}^{+\infty} \frac{e^{-(ik+1)\beta/2}(\mathcal{H} - \mu)}{(2 \cosh (\pi/2)k)(ik + 1)} \, dk. \tag{A.3}
\]

Therefore, the average number of fermions and the thermodynamical potential can be written (at least formally) as follows:

\[
\langle N \rangle = \text{Tr} f(\mathcal{H}) = \int_{-\infty}^{+\infty} \frac{e^{(ik+1)\beta \mu/2}}{4 \cosh (\pi/2)k} \Theta(k) \, dk \tag{A.4}
\]

\[
\Omega = \text{Tr} \left( -\frac{1}{\beta} \log (1 + e^{-\beta(\mathcal{H} - \mu)}) \right) = -\frac{1}{\beta} \int_{-\infty}^{+\infty} \frac{e^{(ik+1)\beta \mu/2}}{(2 \cosh (\pi/2)k)(ik + 1)} \Theta(k) \, dk \tag{A.5}
\]

where \( \Theta \) designates the function

\[
\Theta(k) = \text{Tr}(e^{-(ik+1)\beta \mathcal{H}}). \tag{A.6}
\]

Observe that \((2m + 1)i, m \in \mathbb{Z}\), are (simple) poles for the function \(1/\cosh (\pi/2)k\) and \(i\) is a pole for the functions \(\Theta(k)\) and \(1/(ik + 1)\). These Fourier integrals can be evaluated by using residue theorems if the integrand functions \(\Phi_1(k) = \Theta(k)/\cosh (\pi/2)k\) and

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\[ \Phi_2(k) = \Theta(k) / ((ik + 1) \cosh(\pi/2)k) \] satisfies the Jordan lemma, that is, \( \Phi_1(Re^{i\theta}) \leq g(R), \Phi_2(Re^{i\theta}) \leq h(R), \) for all \( \theta \in [0, \pi], \) and \( g(R) \) and \( h(R) \) vanish as \( R \to \infty. \) The quantities \( \langle N \rangle \) and \( \Omega \) are then formally given by

\[ 2\pi i \left[ a_{-1}(i) + \sum_{m=1}^{\infty} a_{-1}((2m + 1)i) + \sum_{\nu} a_{-1}(k_{\nu}) \right] \] (A.7)

where \( a_{-1}(\cdot) \) denotes the residue of the integrand involved at the pole \( (\cdot) \), and the \( k_{\nu} \) are the poles (with the exclusion of the pole \( i \)) of \( \Theta(k) \) in the complex \( k \) plane.

We now introduce the spectral resolution of the (bounded below) self-adjoint operator \( \mathcal{H}: \)

\[ \varphi(\mathcal{H}) = \int_{-\infty}^{+\infty} \varphi(\lambda) E(d\lambda) \] (A.8)

where \( \varphi \) is a complex-valued function and \( E(\lambda) = \int_{-\infty}^{\lambda} E(d\lambda) \) is the resolution of the identity for the Hamiltonian \( \mathcal{H}. \) Define the density of states \( \nu(\lambda) \) as \( \text{Tr} E(d\lambda)/d\lambda. \) The trace formula ensues:

\[ \text{Tr} \varphi(\mathcal{H}) = \int_{-\infty}^{+\infty} \varphi(\lambda) \nu(\lambda) d\lambda. \] (A.9)

Let us now introduce the weighted density of states \( w(\lambda) = e^{-\beta(\lambda/2)} \nu(\lambda) \) and its Fourier transform

\[ \hat{w}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ik\lambda} w(\lambda) d\lambda. \] (A.10)

Then, from (A.4), (A.5) and (A.9), we can represent \( \langle N \rangle \) and \( \Omega \) as follows:

\[ \langle N \rangle = \sqrt{2\pi} \int_{-\infty}^{+\infty} \frac{e^{(ik+1)(\beta \mu/2)}}{4 \cosh(\pi/2)k} \hat{w}(\frac{\beta}{2}k) \, dk = \frac{\pi}{\beta} e^{\beta \mu/2} \hat{Z}_1(-\mu) \] (A.11)

\[ \Omega = \frac{2\pi}{\beta} \int_{-\infty}^{+\infty} \frac{e^{(ik+1)(\beta \mu/2)}}{(2 \cosh(\pi/2)k)(ik+1)} \hat{w}(\frac{\beta}{2}k) \, dk = - \frac{2\pi}{\beta^2} e^{\beta \mu/2} \hat{Z}_2(-\mu) \] (A.12)

where we have introduced the weighted functions

\[ Z_1 = \text{sech} \left( \frac{\pi}{\beta} k \right) \hat{w}(k), \quad Z_2 = \text{sech} \left( \pi k/\beta \right) (i2k/\beta + 1)^{-1} \hat{w}(k). \] (A.13)

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