MEAN CURVATURE FLOW WITH FLAT NORMAL BUNDLES

KNUT SMOCZYK, GUOFANG WANG, AND Y. L. XIN

Abstract. We show that flatness of the normal bundle is preserved under the mean curvature flow in $\mathbb{R}^n$ and use this to generalize a classical result for hypersurfaces due to Ecker & Huisken [3] in the case of submanifolds with arbitrary codimension.

1. Introduction

Let us consider immersions

$$F : M^m \to \mathbb{R}^n$$

of an $m$-dimensional submanifold in $\mathbb{R}^n$ of codimension $k$. Throughout this paper we shall assume that there is a one-parameter family $F_t = F(\cdot, t)$ of immersions with corresponding images $M_t = F_t(M)$ such that mean curvature flow

$$\frac{d}{dt}F(x, t) = H(x, t), \quad x \in M$$

$$F(x, 0) = F_0(x)$$

is satisfied for some initial data $F_0$. Here, $H(x, t)$ is the mean curvature vector of $M_t$ at the point $x \in M$, i.e. $H$ is the trace of the second fundamental form

$$A = \nabla dF.$$

The mean curvature flow has been studied intensively by many authors. Most of the results have been obtained for hypersurfaces. A classical result is due to Ecker and Huisken [3], where they study hypersurfaces in $\mathbb{R}^{n+1}$ that can be represented as entire graphs over a flat plane. Their result says that any polynomial growth rate for the height and the gradient of the initial surface $M_0$ is preserved during the evolution and that in case of Lipschitz initial data with linear growth, (1) has a
smooth solution for all times $t > 0$. The growth condition was

$$v = \frac{1}{\langle \nu, \omega \rangle} \leq \text{const},$$

where $\omega \in \mathbb{R}^{n+1}$, $|\omega| = 1$ and $\nu$ is some choice of unit normal vector such that $\langle \nu, \omega \rangle > 0$. In addition, they proved that these hypersurfaces approach a self-similar expanding solution of (1) as $t \to \infty$ provided the initial graph was “straight” at infinity.

In higher codimension, the general picture of mean curvature flow is still incomplete even though some work has been carried out by Chen, Li [1], Wang [13], [14] and one of the authors [9]. E.g. based on the interior estimates for hypersurfaces obtained by Ecker and Huisken in [4], Wang [14] proved that any initial compact submanifold that satisfies a local K-Lipschitz condition admits a smooth solution on some time interval $(0, T)$, $T > 0$.

In the case of graphical mean curvature flow in higher codimension there are some longtime existence and convergence results to flat spaces. In [13] Wang defined a similar expression as the above mentioned quantity $v$ which was essential to get the a-priori estimates needed in the longtime existence and convergence results. However, in that paper the author had to assume a smallness condition on $v$ and the theorem did not apply to arbitrary graphs. In [9] it was shown that Lagrangian graphs with convex potentials admit smooth solutions for all times and that the solutions exponentially converge to flat Lagrangian planes.

So, for some time it was unclear how to generalize the results in [3] in the “best” way to the case of arbitrary codimension. Motivated by a recent paper of the third author [15], we believe that the flatness of the normal bundle is the key ingredient to get the convergence results for arbitrary graphs in higher codimension. Note, that trivially the normal bundle of any hypersurface is flat. So our result is a natural extension of the results by Ecker and Huisken.

There are many examples of submanifolds with flat normal bundles. Hypersurfaces are trivial examples. Other trivial examples are curves in $\mathbb{R}^n$. In addition, any submanifold of codimension 2 in $\mathbb{R}^n$ which also is a hypersurface of the standard sphere $S^{n-1}$ must have a flat normal bundle. For more examples, see [2], [10], [11] and [12], where a theory of isoparametric submanifolds was established in the framework of flat normal bundles.

The organization of the paper is as follows: In section 2 we recall the monotonicity formula and the noncompact maximum principle by Ecker and Huisken and introduce the class of solutions for which our results will apply. Section 3 introduces our notation and recalls the most important structure equations in the geometry of
real submanifolds. Some basic evolution equations for the mean curvature flow are derived resp. recalled in section 4. In section 5 we prove that polynomial growth rates are preserved in arbitrary codimension and that these growth estimates can be applied even to non-graphical submanifolds, like cylinders. The core of our article is the proof of Theorem 1, i.e. that flatness of the normal bundle is preserved. This will be done in section 6. In the remainder - based on this fundamental observation - we can proceed basically as in [3], to carry over the results by Ecker and Huisken to the case of arbitrary codimension.

We are indebted to Jürgen Jost for his constant support and the MPI in Leipzig for hospitality. The authors also wish to thank Klaus Ecker for fruitful discussions.

2. THE CLASS OF SOLUTIONS

Throughout this article we shall assume that the solutions of (1) that we consider will belong to the class of solutions for which we can apply Huisken’s monotonicity formula

**Proposition 1. (Huisken)** For a function \( f(x,t) \) on \( M \) we have

\[
\frac{d}{dt} \int_M f \rho d\mu_t = \int_M \left( \frac{d}{dt} f - \Delta f \right) \rho d\mu_t - \int_M f \rho \left| H + \frac{F^\perp}{2(t_0 - t)} \right|^2 d\mu_t,
\]

where

\[
\rho(y,t) = \frac{1}{(4\pi(t_0 - t))^\frac{n}{2}} e^{-\frac{|y|^2}{4(t_0 - t)}}
\]

is the backward heat kernel on \( \mathbb{R}^n \) at the origin, and \( t_0 > t \).

In particular, we will assume that integration by parts is permitted and all integrals are finite for the submanifolds and functions we will consider in the sequel. E.g., this is the case for those smooth solutions of (1) for which the curvature and its covariant derivatives have at most polynomial growth at infinity since then the faster exponential decay rate of the heat kernel \( \rho \) yields finite integrals.

As in [3] we will repeatedly use the following **maximum principle** which is based on the monotonicity formula.

**Corollary 1. (Ecker and Huisken [3])** Suppose the function \( f = f(x,t) \) satisfies the inequality

\[
\left( \frac{d}{dt} - \Delta \right) f \leq \langle a, \nabla f \rangle
\]

for some $a$ which is uniformly bounded on $M \times [0, t_1]$ for some $t_1 > 0$, then
\[ \sup_{M_t} f \leq \sup_{M_0} f \]
for all $t \in [0, t_1]$.

3. Geometric quantities of immersions

Let $F : M^m \to \mathbb{R}^n$ be an immersion and let $k$ be the codimension of $M$, i.e. $n = k + m$. We let $(x^i)_{i=1,...,m}$ denote local coordinates on $M$ and we will always use cartesian coordinates $(y^\alpha)_{\alpha=1,...,n}$ on $\mathbb{R}^n$. Doubled greek and latin indices are summed from 1 to $n$ resp. from 1 to $m$. In local coordinates the differential $dF$ of $F$ is given by
\[ dF = F_\alpha^i \frac{\partial}{\partial y^\alpha} \otimes dx^i, \]
where $F_\alpha^i = y^\alpha(F)$ and $F_\alpha^i = \frac{\partial F_\alpha^i}{\partial x^j}$. The coefficients of the induced metric $g_{ij} \, dx^i \otimes dx^j$ are
\[ g_{ij} = \langle F_i, F_j \rangle = g_{\alpha\beta} F_\alpha^i F_\beta^j, \]
where $g_{\alpha\beta} = \delta_{\alpha\beta}$ is the euclidean metric in cartesian coordinates. As usual, the Christoffel symbols are
\[ \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right). \]
The second fundamental form is defined by
\[ A = \nabla dF := A_\alpha^i \frac{\partial}{\partial y^\alpha} \otimes dx^i \otimes dx^j. \]
Here and in the following all canonically induced full connections on bundles over $M$ will be denoted by $\nabla$. Later, we will occasionally also use the connection on the normal bundle which will then be denoted by $\nabla^\perp$. It is easy to check that in cartesian coordinates on $\mathbb{R}^n$ we have
\[ A_\alpha^i = F_\alpha^i - \Gamma^k_{ij} F_\alpha^k, \]
where $F_\alpha^i = \frac{\partial F_\alpha^i}{\partial x^j}$. By definition, $A$ is a section in $F^{-1}T\mathbb{R}^n \otimes T^*M \otimes T^*M$ and it can be easily checked that $A$ is normal, i.e. that
\[ A \in \Gamma (NM \otimes T^*M \otimes T^*M), \]
where $NM$ denotes the normal bundle of $M$ w.r.t. the immersion $F$. This means that
\[ g_{\alpha\beta} A_\alpha^i F_\beta^j = 0, \quad \forall i, j, k. \]
In particular, the mean curvature vector field $H = H^\alpha \frac{\partial}{\partial y^\alpha}$ with $H^\alpha = g^{ij} A^\alpha_{ij}$ satisfies
\[ g_{\alpha\beta} H^\alpha F^\beta_j = 0, \quad \forall j. \]

The curvature of the normal bundle is defined locally by $R^{\alpha\beta}_{ij} \frac{\partial}{\partial y^\alpha} \otimes \frac{\partial}{\partial y^\beta} \otimes dx^i \otimes dx^j$, where by Ricci’s equation
\[ R^{\alpha\beta}_{ij} = A^{\alpha}_{is} A^s_j - A^\alpha_{js} A^s_i. \]

The normal bundle is flat if and only if $R^{\alpha\beta}_{ij}$ vanishes for any $\alpha, \beta, i$ and $j$. In addition we have the Gauß equations for the induced curvature tensor on $M$

\[ R_{ijkl} = g_{\alpha\beta} (A^\alpha_{ik} A^\beta_{jl} - A^\alpha_{il} A^\beta_{jk}). \]

Let us also recall the Codazzi equations
\[ \nabla_i A^\alpha_{jk} = \nabla_j A^\alpha_{ik} + F_i^\alpha R^j_{kji}, \]
\[ \nabla^k A^\alpha_{jk} = \nabla_j H^\alpha + F_i^\alpha R^j_{ik}. \]

The following rule for interchanging derivatives
\[ \nabla_i \nabla_j A^\alpha_{ik} = \nabla_j \nabla_i A^\alpha_{ik} - R^m_{lij} A^\alpha_{mk} = R^m_{kij} A^\alpha_{lm}. \]

and the second Bianchi identity together with the Codazzi equations imply the Simons’ identity

\[ \Delta A^\alpha_{ik} = \nabla_i \nabla_k H^\alpha + R^m_{ij} A^\alpha_{mk} + R^m_{ik} A^\alpha_{mj} - 2 A^\alpha_{jm} R^j_{ikm} + F^\alpha_m (\nabla_i R^m_{jk} + \nabla_j R^m_{ik} - \nabla^m R_{ijk}), \]

where $R_{ij}$ denotes the Ricci curvature of $M$.

4. Evolution equations

\[ \frac{d}{dt} F^\alpha = H \]

we obtain
\[ \frac{d}{dt} F^\alpha_i = \nabla_i H^\alpha. \]

Let us define the symmetric tensors
\[ a_{ij} := g_{\alpha\beta} H^\alpha A^\beta_{ij}, \quad b_{ij} := g_{\alpha\beta} A^\alpha_{ik} A^\beta_{jk}, \]
so that by Gauß’ equation the Ricci tensor satisfies

\[ R_{ij} = a_{ij} - b_{ij}. \]

From \( g_{\alpha\beta}F^\alpha_i H^\beta = 0 \) we derive the evolution equation for the induced metric \( g_{ij} \)

\[ \frac{d}{dt} g_{ij} = 2g_{\alpha\beta} \nabla_i H^\alpha F^\beta_j = -2g_{\alpha\beta} A^\alpha_{ij} = -2a_{ij}. \]

Consequently, the volume form \( d\mu \) on \( M \) satisfies

\[ \frac{d}{dt} d\mu = -|H|^2 d\mu. \]

We need to compute the evolution equation for \( A^\alpha_{ij} \). In a first step we get

\[ \frac{d}{dt} A^\alpha_{ij} = \frac{d}{dt} F^\alpha_i - F^\alpha_k \frac{d}{dt} \Gamma^k_{ij} - \Gamma^k_{ij} \frac{d}{dt} F^\alpha_k. \]

The evolution equation for \( \Gamma^\alpha_{ij} \) is

\[ \frac{d}{dt} \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\nabla_i \frac{d}{dt} g_{lj} + \nabla_j \frac{d}{dt} g_{li} - \nabla_l \frac{d}{dt} g_{ij}). \]

The last two equations, the Simons’ identity and (10), (11) imply

\[ \frac{d}{dt} A^\alpha_{ij} = \Delta A^\alpha_{ij} - R^m_{ij} A^\alpha_m - R^m_{mj} A^\alpha_m + 2A^\alpha_{mn} R^{mn}_{ij} \]

\[ -F^\alpha_m (\nabla_i b^m_j + \nabla_j b^m_i - \nabla^m b_{ij}). \]

In addition

\[ \frac{d}{dt} H^\alpha = g^{ij} \frac{d}{dt} A^\alpha_{ij} + 2a^{ij} A^\alpha_{ij} \]

\[ = g^{ij} \left( \nabla_i \nabla_j H^\alpha - \frac{d}{dt} \Gamma^k_{ij} F^\alpha_k \right) + 2a^{ij} A^\alpha_{ij} \]

\[ = \Delta H^\alpha - g^{ij} \frac{d}{dt} \Gamma^k_{ij} F^\alpha_k + 2a^{ij} A^\alpha_{ij} \]

so that

\[ \frac{d}{dt} |H|^2 = 2H_\alpha \frac{d}{dt} H^\alpha = 2H_\alpha \Delta H^\alpha + 4|a_{ij}|^2, \]
for $F^\alpha_k H_\alpha = 0$. Thus
\[
\frac{d}{dt} |H|^2 = \Delta |H|^2 - 2|\nabla_i H^\alpha|^2 + 4|a_{ij}|^2.
\]
Since $H_\alpha F^\alpha_i = 0$ we conclude that
\[
\nabla_i H_\beta F^\beta_j F^\alpha_l g^{jl} = -a^l_i F^\alpha_l
\]
so that
\[
|\nabla_i H^\alpha|^2 = |\nabla_i H^\alpha + a^l_i F^\alpha_l|^2 + |a_{ij}|^2.
\]
Hence
\[
(14) \quad \frac{d}{dt} |H|^2 = \Delta |H|^2 - 2|\nabla_i H^\alpha|^2 + 2|a_{ij}|^2.
\]
Let $\nabla^\perp$ denote the normal connection induced from the immersion $F$. Then $|\nabla^\perp H|^2 = |\nabla_i H^\alpha + a^l_i F^\alpha_l|^2$.

**Remark 1.** For a hypersurface with inward unit normal vector $\nu$, scalar mean curvature $H$ and second fundamental tensor $h_{ij}$ we have $A^\alpha_{ij} = h_{ij} \nu^\alpha, H^\alpha = H \nu^\alpha$ and $a_{ij} = H h_{ij}, \nabla_i H^\alpha = \nabla_i H \nu^\alpha - a^l_i F^\alpha_l$ so that $-2|\nabla_i H^\alpha|^2 = -2|\nabla H|^2 - 2|a_{ij}|^2$ and $|a_{ij}|^2 = H^2 |A|^2$.

Now we compute the evolution equation for $|A|^2$. From (11), (13) and the normality of $A$ we deduce
\[
\frac{d}{dt} |A|^2 = 4a^l_{ij} b_{ij} + 2A^l_{ij} \frac{d}{dt} A^\alpha_{ij}
\]
\[
= 4a^l_{ij} b_{ij} + 2A^l_{ij} (\Delta A^\alpha_{ij} - R^\alpha_{mn} A^\alpha_{mj} - R^\alpha_{jl} A^\alpha_{mi} + 2A^\alpha_{mn} R^m_{ij})
\]
\[
(15) \quad = \Delta |A|^2 - 2|\nabla_i A^\alpha_{ij}|^2 + 4|b_{ij}|^2 + 4A^l_{mn} A^\alpha_{ij} R^m_{ij}
\]
Since
\[
4|b_{ij}|^2 + 4A^l_{mn} A^\alpha_{ij} R^m_{ij} = 2|R^\alpha_{ij}|^2 + 4|A^\alpha_{mn} A^\beta_{ij}|^2
\]
and the tangential part of $\nabla_i A^\alpha_{ij}$ is given by $-A^\beta_{ij} A^k_{kl} F^\alpha_k$ we conclude
\[
|\nabla_i A^\alpha_{ij}|^2 = |\nabla_i A^\alpha_{ij} + A^\beta_{ij} A^k_{kl} F^\alpha_k|^2 + |A^\alpha_{mn} A^\beta_{ij}|^2
\]
and finally
\[
(16) \quad \frac{d}{dt} |A|^2 = \Delta |A|^2 - 2|\nabla_i A^\alpha_{ij} + A^\beta_{ij} A^k_{kl} F^\alpha_k|^2 + 2|A^\alpha_{mn} A^\beta_{ij}|^2 + 2|R^\alpha_{ij}|^2.
\]
Here $|\nabla_i A^\alpha_{ij} + A^\beta_{ij} A^k_{kl} F^\alpha_k|^2 = |\nabla^\perp A|^2$. 
5. Growth estimates

In this section we will prove that polynomial growth estimates for submanifolds in $\mathbb{R}^n$ are preserved under the mean curvature flow. Note, that in this section we do not require that $M_0$ or $M_t$ can be written as graphs over some flat $m$-plane. We will use a flat $\ell$-plane merely as a reference submanifold to measure distances. As a consequence, we will also obtain growth estimates for other objects as graphs, e.g. for cylindrical objects as depicted in figure 1.

![Figure 1. Growth rates of cylindrical surfaces are preserved. Here, the surface is given by $(x, \phi) \mapsto (u_1, u_2, u_3) = (x, (1 + |x|) \cos x \cos \phi, (1 + |x|) \cos x \sin \phi)$. Since $u^2 := u_2^2 + u_3^2 = (1 + |x|)^2 \cos^2 x \leq 2(1 + x^2)$ the surface grows linearly over the axis of rotation. We have $\ell = 1$, $m = 2$ and $n = 3$.](image)

Choose an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ and let $1 \leq \ell \leq n$ be an integer (for later purposes we have in mind $\ell = m$ but here in general $\ell$ and $m$ can be different). For an immersion $F : M^m \to \mathbb{R}^n$ we define $n$ coordinate functions

$$u_i := \langle F, e_i \rangle, \quad i = 1, \ldots, n$$
In addition we define
\[
x := \left( \sum_{i=1}^{\ell} u_i^2 \right)^{\frac{1}{2}}, \quad u := \left( \sum_{i=\ell+1}^{n} u_i^2 \right)^{\frac{1}{2}}
\]
such that \(|F|^2 = x^2 + u^2\). Since \(\left( \frac{d}{dt} - \Delta \right) u_i = 0\) we conclude
\[
\frac{d}{dt}|F|^2 = \Delta |F|^2 - 2m,
\]
(17)
\[
\frac{d}{dt} u^2 = \Delta u^2 - 2 \sum_{i=\ell+1}^{n} |\nabla u_i|^2,
\]
(18)
\[
\frac{d}{dt} x^2 = \Delta x^2 - 2m + 2 \sum_{i=\ell+1}^{n} |\nabla u_i|^2,
\]
(19)

since
\[
\sum_{i=1}^{n} |\nabla u_i|^2 = m.
\]

For a constant \(c > 0\) to be determined later, we define the function
\[
\eta := 1 + ct + x^2.
\]
Let \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) be a smooth function with \(\varphi' \leq 0, \varphi'' \geq 0\). We want to compute the evolution equation of \(u^2 \varphi(\eta)\).
\[
\frac{d}{dt} (u^2 \varphi) = \varphi \left( \Delta u^2 - 2 \sum_{i=\ell+1}^{n} |\nabla u_i|^2 \right) + u^2 \varphi' \left( \Delta \eta + 2 \sum_{i=\ell+1}^{n} |\nabla u_i|^2 + c - 2m \right)
\]
\[
= \Delta (u^2 \varphi) - 2\varphi'(\nabla \eta, \nabla u^2) - \varphi''u^2|\nabla \eta|^2 - 2(\varphi - u^2 \varphi') \sum_{i=\ell+1}^{n} |\nabla u_i|^2
\]
\[+(c - 2m)u^2 \varphi'.\]

If at some point on \(M_t\) we have \(u^2 = 0\), then at such a point
\[
\frac{d}{dt} (u^2 \varphi) \leq \Delta (u^2 \varphi).
\]

We want to prove that this inequality holds at all points on \(M_t\). At those points, where \(\varphi' = 0\) we are done as well. So w.l.o.g. we can assume that \(u \neq 0\) and \(\varphi' \neq 0\).

Next observe that
\[
|\nabla u^2|^2 \leq 4u^2 \sum_{i=\ell+1}^{n} |\nabla u_i|^2
\]
implies
\[ |\nabla u|^2 \leq \sum_{i=\ell+1}^{n} |\nabla u_i|^2 \]
at all points, where \( u \neq 0 \). We use Schwarz’ inequality to estimate
\[ -2\varphi'(\nabla \eta, \nabla u^2) \leq -2\varphi'(\varepsilon u^2|\nabla \eta|^2 + \frac{1}{\varepsilon}|\nabla u|^2) \leq -2\varepsilon\varphi' u^2|\nabla \eta|^2 - \frac{2}{\varepsilon} \varphi' \sum_{i=\ell+1}^{n} |\nabla u_i|^2; \]
where \( \varepsilon \) is some positive constant. We choose \( \varepsilon = -\frac{\varphi'}{\varphi} \) and get
\[
\frac{d}{dt} (u^2\varphi) \leq \Delta (u^2\varphi) + \left( \frac{2(\varphi')^2}{\varphi} - \varphi'' \right) u^2|\nabla \eta|^2 + (c - 2m)u^2\varphi'.
\]
As in [3] we have\[
\nabla^2 \eta = 2 \left( F, F - \sum_{i=\ell+1}^{n} u_i e_i \right),
\]
and therefore\[
|\nabla \eta|^2 \leq 4|F - \sum_{i=\ell+1}^{n} u_i e_i|^2 = 4x^2 \leq 4\eta,
\]
so that always
\[
(20) \quad \frac{d}{dt} (u^2\varphi) \leq \Delta (u^2\varphi) + u^2 \left\{ 4\eta \left( \frac{2(\varphi')^2}{\varphi} - \varphi'' \right) + (c - 2m)\varphi' \right\}.
\]
**Proposition 2.** If for some \( c_0 < \infty, p \geq 0 \), the inequality
\[ u^2 \leq c_0 \left( 1 + |F|^2 - u^2 \right)^p \]
is satisfied on \( M_0 \), then for all \( t > 0 \),
\[ u^2 \leq c_0 \left( 1 + |F|^2 - u^2 + (2m + 4(p - 1))t \right)^p. \]

**Proof.** We choose \( \varphi(\eta) = \eta^{-p} \) and \( c = 2m + 4(p - 1) \). Then \( \varphi' = -p\eta^{-p-1}, \varphi'' = p(p+1)\eta^{-p-2} \). Inserting this into (20) gives\[
\frac{d}{dt} (u^2\varphi) \leq \Delta (u^2\varphi) + \frac{u^2}{\eta} \left( 8p^2 - 4p(p+1) - p(c - 2m) \right)
\]
\[
= \Delta (u^2\varphi),
\]
and the result follows from Corollary 1.
6. Preserving flatness of the normal bundle

In this section we will prove that flatness of the normal bundle is preserved. We do not require that $M$ is a graph nor do we assume compactness or completeness. The theorem can be applied to any smooth solution of the mean curvature flow for which $M_0$ has a flat normal bundle.

**Theorem 1.** Let $F : M \times [0, T) \to \mathbb{R}^n$ be a smooth solution of the mean curvature flow and assume that the normal bundle of $M_0$ is flat. If $|A|^2$ is bounded on each $M_t$ then the normal bundle of $M_t$ is flat as well.

Note that we do not require that $|A|^2$ is uniformly bounded in $t$.

**Proof.** For the proof of this theorem we have to compute the evolution equation of the normal curvature tensor $R^\alpha\beta_{ij}$. We will show that the squared normal curvature tensor $R^\perp$ satisfies an evolution equation of the form

$$
\frac{d}{dt}|R^\perp|^2 = \Delta|R^\perp|^2 - 2|\nabla R^\perp|^2 + A^* A^* R^\perp * R^\perp,
$$

where the last term is a contraction of a term quadratic in $A$ and one which is quadratic in $R^\perp$. Then, by assumption on $|A|^2$, on a compact time interval $[0, t_1]$ we can choose a constant $c$ (depending on $t_1$) such that

$$
\frac{d}{dt}|R^\perp|^2 \leq \Delta|R^\perp|^2 - 2|\nabla R^\perp|^2 + c|R^\perp|^2
$$

and the function $f := e^{-ct}|R^\perp|^2$ satisfies

$$
\frac{d}{dt}f \leq \Delta f
$$

on $[0, t_1]$. The result then follows from Corollary 1.

It remains to derive the evolution equation for $|R^\perp|^2$. It turns out that the computation is rather complicated, since a number of symmetries of the curvature tensor and the second fundamental form have to be used.

The first observation is

$$
|R^\perp|^2 = |R^\alpha\beta_{ij}|^2 = 2|b^i_j|^2 - 2c^\alpha_\beta c^i_j c^j_\alpha,
$$

where

$$
c^\alpha_\beta = A^\alpha_{ik} A^\beta_k.
$$
\[
\frac{d}{dt} b_{ij} = 2a^{kl} A_{aki} A_{lj} + \frac{d}{dt} A_{kji} A_{lij} + \frac{d}{dt} A_{kl} A_{lij}
\]
\[
= 2a^{kl} A_{aki} A_{lj} + A_{kj} (\Delta A_{ik} - R_{im} A_{pk} - R_{mk} A_{ij}^\alpha + 2A_{mni} R_{m i k}^n)
\]
\[
+ A_{kij} (\Delta A_{ik} - R_{im} A_{pk} - R_{mk} A_{ij}^\alpha + 2A_{mni} R_{m i k}^n)
\]
and then
\[
\frac{d}{dt} b_{ij} = 2a^{kl} A_{aki} A_{lij} + \Delta b_{ij}
\]
\[
- \nabla^l A_{kj} \nabla_i A_{ik} + R_{im} b_{jm} - R_{k} A_{oj} A_{mi} + 2A_{mni} R_{m i k}^n
\]
\[
- \nabla^l A_{kj} \nabla_i A_{ik} - R_{im} b_{jm} - R_{k} A_{oj} A_{mi} + 2A_{mni} R_{m i k}^n
\]
This implies
\[
\frac{d}{dt} |b_{ij}|^2 = 4a^{ik} b_{kl} b_{lj} + 2b_{ij} \frac{d}{dt} b_{ij}
\]
\[
= 4a^{ik} b_{kl} b_{lj} + 4a^{kl} A_{ikj} b_{lij} + 2b_{ij} (\Delta b_{ij} - 2\nabla^l A_{kj} \nabla_i A_{ik})
\]
\[
+ 4b_{ij} (-\nabla^l b_{jm} - R_{k} A_{ool} A_{mi} + 2A_{mni} R_{m i k}^n)
\]
and from \(R_{ij} = a_{ij} - b_{ij}\)
\[
\frac{d}{dt} |b_{ij}|^2 = 4b_{ij} b_{ij} b_{jm} + 4b_{ij} b_{k} A_{kij} A_{lij}
\]
\[
\frac{d}{dt} |b_{ij}|^2 = 4b_{ij} b_{ij} b_{jm} + 4b_{ij} b_{k} A_{kij} A_{lij}
\]
\[
\frac{d}{dt} |b_{ij}|^2 = \Delta |b_{ij}|^2 - 2|\nabla b_{ij}|^2 - 4b_{ij} \nabla^l A_{kj} \nabla_i A_{ik}
\]
\[
+ 8b_{ij} A_{kij} A_{mn} R_{m i k}^n
\]
\[
\frac{d}{dt} |b_{ij}|^2 = \Delta |b_{ij}|^2 - 2|\nabla b_{ij}|^2 - 4b_{ij} \nabla^l A_{kj} \nabla_i A_{ik}
\]
\[
+ 8b_{ij} A_{kij} A_{mn} (A_{mn} A_{ikj} - A_{mk} A_{ij})
\]
\[
= \Delta |b_{ij}|^2 - 2|\nabla b_{ij}|^2 - 4b_{ij} \nabla^l A_{kj} \nabla_i A_{ik}
\]
\[
+ 4b_{ij} b_{jm} + 4b_{ij} b_{k} A_{kij} A_{lij} + 8b_{ij} g_{n} c_{\alpha j} c_{\beta j} - 8b_{ij} c_{\alpha j} c_{\beta j}
\]
so that
\[
\frac{d}{dt} |b_{ij}|^2 = \Delta |b_{ij}|^2 - 2|\nabla b_{ij}|^2 - 4b_{ij} \nabla^l A_{kj} \nabla_i A_{ik} + 4\Gamma_1 + 4\Gamma_2 + 8\Gamma_3 - 8\Gamma_4,
\]
where
\[
\Gamma_1 := b_{ij} b_{jm}, \quad \Gamma_2 := b_{ij} b_{k} A_{kij} A_{lij}
\]
\[
\Gamma_3 := b_{ij} g_{n} c_{\alpha j} c_{\beta j}, \quad \Gamma_4 := b_{ij} c_{\alpha j} c_{\beta j}.
\]
To continue, we need an expression for $\frac{d}{dt}c^{\alpha\beta}_{ij}$.

$$\frac{d}{dt}c^{\alpha\beta}_{ij} = 2\alpha^{kl}A^{\alpha}_{ik}A^{\beta}_{lj} + \frac{d}{dt}A^{\alpha}_{ik}A^{\beta}_{lj} + \frac{d}{dt}A^{\beta}_{ij}A^{\alpha}_{i}$$

$$= 2\alpha^{kl}A^{\alpha}_{ik}A^{\beta}_{lj} + A^{\beta}_{ij}(\Delta A^{\alpha}_{ik} - R^{m}_{i}A^{\alpha}_{mk} - R^{m}_{k}A^{\alpha}_{mi} + 2A^{\alpha}_{mn}R^{m}_{i}R^{n}_{j}) - A^{\beta}_{ij}F^{\alpha}_{m}\lambda^{m}_{ik}$$

$$+ A^{\alpha}_{ik}(\Delta A^{\beta}_{jk} - R^{m}_{j}A^{\beta}_{mk} - R^{m}_{k}A^{\beta}_{mj} + 2A^{\beta}_{mn}R^{m}_{j}R^{n}_{k}) - A^{\alpha}_{ik}F^{\beta}_{m}\lambda^{m}_{jk},$$

where

$$\lambda^{m}_{ik} = \nabla_{l}b_{m}^{i} + \nabla_{k}b_{m}^{i} - \nabla^{m}b_{ik}.$$
where
\[ \Gamma_7 := c^i_{ij} l^j_i. \]

Similarly, one has
\[ c^\alpha_m R^\alpha_{ij} R^\beta_{ij} = 2(\Gamma_3 - \Gamma_5), \]
\[ R^\alpha_{ij} R^\gamma_{jl} = \Gamma_1 - 3\Gamma_4 + 3\Gamma_6 - \Gamma_7 \]
and
\[ b^i_{ij} R^\alpha_{kij} = 2(\Gamma_2 - \Gamma_4). \]

Let us define
\[ G_1 := b^{mk} \nabla^l A^a_{ik} \nabla^l A^i_{am}, \quad G_2 := A^k_{ij} A^k_{am} \nabla^l A^a_{ik} \nabla^l A^b_{mj}, \]
and
\[ G_3 := c^mk \nabla^l A^a_{ik} \nabla^l A^j_{am}, \quad G_4 := A^j_{ij} A^k_{am} \nabla^l A^a_{ik} \nabla^l A^b_{jm}. \]

Since
\[ \nabla R^\alpha_{ij} = \nabla A^a_{ik} \nabla A^j_{am} + A^a_{ik} \nabla A^j_{am} - \nabla A^a_{ik} \nabla A^j_{ak} - A^a_{ik} \nabla A^b_{mk}, \]
we obtain
\[ |\nabla R^\alpha_{ij}|^2 = 4(b^{mk} \nabla A^a_{ik} \nabla A^j_{am} + A^k_{ij} A^k_{am} \nabla A^a_{ik} \nabla A^b_{mj}) \]
\[ - c^mk \nabla A^a_{ik} \nabla A^j_{am} - A^a_{ik} \nabla A^b_{jm}). \]

In addition
\[ |\nabla b_{ij}|^2 = 2c^kl \nabla A^a_{ik} \nabla A^j_{kl} + 2A^k_{ij} A^k_{lm} \nabla A^a_{ik} \nabla A^b_{lj} = 2(G_2 + G_3) \]
and
\[ \nabla (c^a_{ij} \nabla (c^a_{ij} \nabla A^a_{ik} \nabla A^j_{kl}) = 2A^a_{ik} A^a_{ij} \nabla A^k_{ij} \nabla A^j_{kl} + 2c^a_{ij} \nabla A^a_{ij} \nabla A^j_{kl} \]
\[ = 2(G_3 + G_4). \]

Combining (22), (25) and the last equations we get
\[
\frac{d}{dt} |R^\alpha_{ij}|^2 = \Delta |R^\alpha_{ij}|^2 - 8(G_2 + G_3) - 8G_1 + 8(G_3 + G_4) + 8G_3
\]
\[ + 8(\Gamma_1 + \Gamma_2 + 2\Gamma_3 - 2\Gamma_4) - 16(\Gamma_4 + \Gamma_5 - \Gamma_6) \]
\[ = \Delta |R^\alpha_{ij}|^2 - 2|\nabla R^\alpha_{ij}|^2 + 8R^\alpha_{ij} R^\alpha_{ij} R^\gamma_{ij} R^\gamma_{ij}
\]
\[ - 8R^\alpha_{ikl} R^\alpha_{ij} R^\alpha_{ikl} + 8c^a_{ij} R^\alpha_{ij} R^\alpha_{ij} + 4b^j_{ij} R^\alpha_{kj} R^\alpha_{kj}. \]
and this equation is of the form given in (21).

**Remark 2.** One can replace the gradient term $\nabla \alpha^{\beta}_{ij}$ by $\nabla^\perp \alpha^{\beta}_{ij}$, where $\nabla^\perp$ is the normal connection. This simplifies (31) a bit. We leave the details to the reader.

7. **Graphical mean curvature flow with flat normal bundles**

In the remaining section we will see that in case of entire graphs with flat normal bundles the computations in the paper by Ecker & Huisken basically carry over unchanged to the case of arbitrary codimension. In the following we will outline the basic steps.

Let us assume that the initial submanifold $M_0$ can be written as an entire graph over a flat $m$-plane in $\mathbb{R}^n$ such that the normal bundle of $M_0$ is flat.

Suppose $\omega \in \Omega^m(\mathbb{R}^n)$ is a parallel $m$-form with $|\omega| = 1$. $\omega$ induces a function $w$ on any immersion $F : M^m \to \mathbb{R}^n$ by

$$F^* \omega =: w d\mu,$$

where $d\mu$ is the induced volume form on $M$. Since $|\omega| = 1$ we must have

$$-1 \leq w(x) \leq 1, \quad \forall x \in M.$$

The angle $\alpha$ defined by $\cos \alpha = w$ measures the angle between the flat plane defined by $\omega$ and the tangent planes of the submanifold $M$. The condition to be a graph then is easily expressed by

$$w > 0.$$

For hypersurfaces, $w$ is also given by the angle between the normal vector $\nu$ of $M$ and a fixed normal direction of the reference plane defined by $\omega$. This was considered in [3]. For immersions with higher codimension, a similar $w$-function was considered in [7]. The evolution equation for $F^* \omega$ was derived earlier in a paper by M.-T. Wang [13] and for the sake of completeness we include the evolution equation in our notation.

$$\frac{d}{dt} \omega_{i_1 \cdots i_m} = \Delta \omega_{i_1 \cdots i_m} - R \omega_{i_1 \cdots i_m} - \sum_{s < j} \omega_{i_1 \cdots i_{s-1} \alpha_{s} i_{s+1} \cdots i_{j-1} \alpha_{j} i_{j+1} \cdots i_m} R^{\alpha_{s} \alpha_{j}}_{\ i_{s} i_{j}} ,$$

where $R$ denotes the scalar curvature and $\omega_{i_1 \cdots i_m} = \omega_{\alpha_1 \cdots \alpha_m} F_{i_1}^{\alpha_1} \cdots F_{i_m}^{\alpha_m}$. A first observation is
**Lemma 1.** Suppose \( F : M^m \times [0, T) \to \mathbb{R}^n \) is a smooth solution of the mean curvature flow as in Theorem 1. Then \( w \) defined as above satisfies

\[
\frac{d}{dt} w = \Delta w + w|A|^2.
\]

**Proof.** We know that the normal bundle will be flat for all \( t \). Hence

\[
\frac{d}{dt} F^*\omega = \Delta F^*\omega - RF^*\omega
\]

and the conclusion follows from \( \frac{d}{dt}d\mu = -|H|^2d\mu \) and \( R = |H|^2 - |A|^2 \).

The second observation is

**Lemma 2.** Suppose \( F : M^m \times [0, T) \to \mathbb{R}^n \) is as in Lemma 1. Then

\[
\frac{d}{dt}|A|^2 = \Delta|A|^2 - 2|\nabla^\perp A|^2 + 2|A_\alpha A_\beta A_\alpha A_\beta|^2
\leq \Delta|A|^2 - 2|\nabla^\perp A|^2 + 2|A|^4
\]

**Proof.** The Lemma follows from (16), Theorem 1 and the estimate

\[
|A_\alpha A_\beta A_\alpha A_\beta|^2 \leq |A|^4.
\]

Hence, in the case of a flat normal bundle the differential inequality for \( |A|^2 \) is the same as in the case of codimension one. Using these observations, we can follow [3] closely to consider the mean curvature flow with flat normal bundles.

Now let us assume the following **linear growth** condition

\[
v := \frac{1}{w} \leq c_1 < \infty
\]

holds everywhere on \( M_0 \) for some constant \( c_1 \). (33) implies that if \( M_0 \) satisfies (35) then \( M_t \) also satisfies (35) with the same constant.

**Lemma 3.** The term \(|A|^2v^2\) satisfies

\[
\left( \frac{d}{dt} - \Delta \right)|A|^2v^2 \leq -2v^{-1}\langle \nabla v, \nabla (|A|^2v^2) \rangle.
\]
Proof. To prove the Lemma, we need the following Kato type inequality

\[(37) \quad |\nabla A|^2 \geq \frac{n+2}{n} |\nabla|A||^2,\]

which was proved in \([15]\) for immersions with flat normal bundle. Now the proof follows from the proof of Lemma 4.1 in \([3]\).

Note that when the codimension is one, \((37)\) was proved in \([8]\). A direct consequence of Lemma 3 and Corollary 1 is

**Corollary 2.** If \(M_t\) is a smooth solution of \((1)\) with bounded \(v\) and bounded \(|A|^2\) on each \(M_t\), then there is the following estimate

\[
\sup_{M_t} |A|^2 v^2 \leq \sup_{M_0} |A|^2 v^2.
\]

One can also get a differential inequality for higher derivatives of the second fundamental form

\[
\left(\frac{d}{dt} - \Delta\right) (t^l+1 |(\nabla^\perp)^l A|^2) \leq -2t^{l+1} |(\nabla^\perp)^{l+1} A|^2 + (l+1)t^l |(\nabla^\perp)^l A|^2
\]

\[
+ C(n, l) t^{l+1} \sum_{i+j+k=l} |(\nabla^\perp)^i A||(\nabla^\perp)^j A||(\nabla^\perp)^k A|,
\]

for any integer \(l \geq 0\), where \(C(n, l)\) is a constant depending only on \(n\) and \(l\). Therefore, the higher order estimates in \([3]\) can also be obtained in our case. Furthermore we get

**Proposition 3.** Let \(M_t\) be a smooth solution of \((1)\) with flat normal bundle. Then for each \(m \geq 0\) there is a constant \(C(m)\) such that

\[
(39) \quad t^{m+1} |(\nabla^\perp)^m A|^2 \leq C(m).
\]

Rescaling as in \([3]\) (see also \([6]\)), we define

\[
\bar{F}(s) = \frac{1}{\sqrt{2t+1}} F(t),
\]

where \(s\) is given by \(s = \frac{1}{2} \log(2t+1)\). In the new time variable \(s\), we have a normalized equation

\[
(40) \quad \frac{d}{ds} \bar{F} = \bar{H} - \bar{F}.
\]
From Proposition 2, Corollary 2 and Proposition 3 we have estimates for the rescaled immersion $\tilde{F}$
\[ \tilde{u}^2(\tilde{F}, s) \leq \tilde{c}_0(1 + \tilde{x}^2(\tilde{F}, s)) \]
\[ \tilde{v}(\tilde{F}, s) \leq c_1 \]
\[ |\tilde{A}|^2(\tilde{F}, s) \leq c_2, \]
where $\tilde{u}^2 := |\tilde{F}^\perp|^2$, $\tilde{x}^2 = |\tilde{F}^T|^2$ and $\tilde{c}_0$, $c_1$, $c_2$ are some constants depending only on the initial immersion $F_0$.

Eventually, since the term $|a_{ij}|^2$ appearing in the evolution equation of the squared mean curvature (14) can be estimated by
\[ |a_{ij}|^2 \leq |H|^2|A|^2, \]
the computations can be carried out in the same way to derive

**Theorem 2.** Suppose $M_0$ is an entire graph with bounded curvature over some $\mathbb{R}^m$ in $\mathbb{R}^n$ satisfying the linear growth condition (35) and that $M_0$ has a flat normal bundle. Then the mean curvature flow admits a smooth solution for all $t > 0$ with uniformly bounded curvature quantities. If in addition we assume that
\[ u^2(\tilde{F}, s) \leq c_3(1 + |F|^2)^{1-\delta} \]
holds on $M_0$ for some constant $c_3 < \infty$ and $\delta > 0$, then the solution $\tilde{M}_s$ of the normalized mean curvature flow (40) converges for $s \to \infty$ to a limiting surface $\tilde{M}_\infty$ satisfying the equation for selfsimilarly expanding solutions
\[ F^\perp = H. \]

**Remark 3.** We note that Proposition 4.5 in [3], the spatial decay behaviour, can also easily be done in the same way. We leave the details to the reader.

**Remark 4.** We improved the previous results of the third author. The dimension limitation in [15] can be removed. That will appear in another paper.

**References**

1. J. Chen; J. Li: Singularity of mean curvature flow of Lagrangian submanifolds. Invent. Math. **156** (2004), no. 1, 25–51.
2. W.-Y. Hsiang; R. Palais; C.-L. Terng: The topology of isoparametric submanifolds. J. Diff. Geom. **27** (1988), no. 3, 423–460.
3. K. Ecker; G. Huisken: Mean curvature evolution of entire graphs. Ann. of Math. (2) **130** (1989), no. 3, 453–471.
4. K. Ecker; G. Huisken: Interior estimates for hypersurfaces moving by mean curvature. Invent. Math. **105** (1991), no. 3, 547–569.
5. G. Huisken: Flow by mean curvature of convex surfaces into spheres. J. Diff. Geom. **20** (1984), no. 1, 237–266.
6. G. Huisken: Asymptotic behavior for singularities of the mean curvature flow. J. Diff. Geom. 31 (1990), no. 1, 285–299.
7. J. Jost; Y.-L. Xin: Bernstein type theorems for higher codimension. Calc. Var. Partial Differential Equations 9 (1999), no. 4, 277–296.
8. R. Schoen; L. Simon; S.-T. Yau: Curvature estimates for minimal hypersurfaces. Acta Math. 134 (1975), no. 3-4, 275–288.
9. K. Smoczyk; M.-T. Wang: Mean curvature flows for Lagrangian submanifolds with convex potentials. J. Diff. Geom. 62 (2002), 243–257.
10. C.-L. Terng: Isoparametric submanifolds and their Coxeter groups. J. Diff. Geom. 21 (1985), no. 1, 79–107.
11. C.-L. Terng: Convexity theorem for isoparametric submanifolds. Invent. Math. 85 (1986), no. 3, 487–492.
12. C.-L. Terng: Submanifolds with flat normal bundle. Math. Ann. 277 (1987), no. 1, 95–111.
13. M.-T. Wang: Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension. Invent. math. 148 (2002), no. 3, 525–543.
14. M.-T. Wang: The mean curvature flow smoothes Lipschitz submanifolds. Comm. Anal. Geom. 12 (2004), no. 3, 581–599.
15. Y.-L. Xin: Bernstein type theorems without graphic condition. Preprint (2004).

(Knut Smoczyk) MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22-26, 04103 LEIPZIG, GERMANY

E-mail address: smoczyk@mis.mpg.de

(Guofang Wang) INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI, CHINA AND MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22-26, 04103 LEIPZIG, GERMANY

E-mail address: gwang@mis.mpg.de

(Y. L. Xin) INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI, CHINA

E-mail address: ylxin@fudan.edu.cn