Null controllability of damped nonlinear wave equation

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Abstract

In this paper, we study the null controllability of nonlinear wave systems. Firstly, by combing Gerlerkin method and a fixed point theorem, we give the null controllability for semi-linear wave equation with nonlinear function dependents on velocities under the geometric control condition. Secondly, based on a new iterative method, we prove the null controllability for a class of quasi-linear wave system constructively. Finally, we give a control result for a class of fully nonlinear wave system as application.

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1 Introduction and main results

In this paper, we will study the internal exact controllability for some kinds of nonlinear wave systems.

Let $T > 0$, $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$, and $\omega$ be an open non-empty subset of $\Omega$. Denote by $\chi_\omega$ the characteristic function of $\omega$.

We are interested in the internal controllability problem for the following nonlinear wave system:

\[
\begin{aligned}
  y_{tt} - \Delta y + f(t, x, y, y_t, \nabla y, \nabla^2 y) &= \chi_\omega(x)u(t, x), \quad (t, x) \in (0, T) \times \Omega, \\
  y(t, x) &= 0, \quad (t, x) \in (0, T) \times \partial \Omega, \\
  y(0, x) = y_0, \quad y_t(0, x) = y_1, \quad x \in \Omega.
\end{aligned}
\] (1.1)

Here $u$ is the control (or input), $(y_0, y_1)$ is the initial data, nonlinear function $f$ will be given later.

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Our goal here is to consider the internal controllability problem: given \( T > 0 \), given \( y_0, y_1, y^0, y^1 \) in some functional spaces, is there a control \( u \) such that the solution \( y \) of (1.1) with initial data \((y_0, y_1)\) satisfies that \((y(T), y_t(T)) = (y^0, y^1)\)?

Before we state our results, we introduce some geometric multiplier conditions.

**Definition 1.1.** We say that \((\omega, T)\) satisfies \(\Gamma\) condition, if there exists one point \(x_0 \notin \Omega\), such that
\[
T \geq 2 \max_{x \in \Omega} |x - x_0|, \quad \Gamma_0 := \left\{ x \in \partial\Omega \mid (x - x_0) \cdot n > 0 \right\}, \quad \omega = \Omega \cap O_{\varepsilon_0}(\Gamma_0)
\]
for some \(\varepsilon_0 > 0\).

**Definition 1.2.** We say that \((\omega, T)\) satisfies Geometric control condition (GCC), if every generalized geodesic enters \(\omega\) before \(T\).

Denote \(H^s = H^s(\Omega)\), \(H^0 = L^2(\Omega)\). We define (see [7])
\[
\mathcal{H}^s = \left\{ v \in H^s \mid \Delta^i v|_{\partial \Omega} = 0, i = 0, 1, ..., \left\lfloor \frac{s}{2} - \frac{1}{4} \right\rfloor \right\}. \quad (1.2)
\]
Here \(\Delta^0 v = v\), and \([\cdot]\) stands for floor function.

Now we first consider semilinear case:
\[
\begin{cases}
y_{tt} - \Delta y + f(y_t) = \chi_\omega(x)u, & (t, x) \in (0, T) \times \Omega, \\
y(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\
y(0, x) = y_0, \ y_t(0, x) = y_1, & x \in \Omega,
\end{cases} \quad (1.3)
\]
where \(\omega \subset \Omega\), \(\chi \in C^2(\Omega)\) satisfies \(0 \leq \chi(x) \leq 1\), \(\chi|_\omega \equiv 1\), and \(\chi\) supports in a neighbourhood of \(\omega\).

Assume that \(f = f(\cdot)\) is a Lipschitz continuous function
\[
|f(a) - f(b)| \leq L|a - b|, \quad (1.4)
\]
such that
\[
f(0) = 0. \quad (1.5)
\]
In addition, we assume
\[
(a - b)(f(a) - f(b)) \geq \tilde{L}(a - b)^2, \quad (1.6)
\]
where \(L > \tilde{L} > 0\).

Our main result is stated as follows.

**Theorem 1.1.** Assume that \((T, \omega)\) satisfies GCC condition (Geometric control condition). Then there exists a constant \(D > 0\), such that if \(f\) satisfies \((1.4)-(1.6)\) and
\[
\left(\frac{L}{\tilde{L}} - 1\right)^2 < \frac{L}{2D}, \quad (1.7)
\]
then for \((y_0, y_1) \in \mathcal{H}^2 \times \mathcal{H}^1\), there exists a control function \(u \in L^2(0, T; H^1(\omega))\) such that

\[
\int_0^T \int_\omega |\nabla u|^2 dx \, dt + \int_0^T \int_\omega |u|^2 dx \, dt \leq D^* \left[ \int_\Omega (|y_1|^2 + |\nabla y_0|^2) \, dx + \int_\Omega (|\nabla y_1|^2 + |\Delta y_0|^2) \, dx \right]
\]

for some \(D^* > 0\), and \(u\) drives \((y, y_t)\) in (1.3) to zero at time \(T\).

**Remark 1.3.**
- \(D\) comes from observability inequality in Lemma 2.1.
- For any fixed \(L > 0\), one can choose big enough \(D > L\) when \((\omega, T)\) satisfies GCC, so (1.7) can be seen as a restriction on \(\tilde{L}\).
- \(D^*\) in (1.8) can be given explicitly in terms of \(D, L, \tilde{L}\) and \(\chi\). Actually, \(D^* = \frac{C^*}{\delta}\), \(C^*\) is given by (3.15) and \(\delta\) is given by (3.12).

**Remark 1.4.** The proof strongly depends on damping structure, such that we can use Galerkin method and a fixed point argument in L. C. Evans [10]. One might expect that this method also works for other possible damping of the wave system with different boundary conditions.

**Remark 1.5.** We note that in Theorem 1.1 the control time and control domain are only assumed to be satisfying GCC, which are necessary when \(f\) is linear ([1, 2, 26]).

**Remark 1.6.** This result partially solves an open problem posed by Xu Zhang in [30, Remark 7.2]. The initial data here are assumed in \(\mathcal{H}^2 \times \mathcal{H}^1\). It is not known whether Theorem 1.1 holds in general for any initial data in \(H^1_0 \times L^2\).

Our second result is concerning quasi-linear case: nonlinear term \(f\) satisfies the following assumption:

\[
f(t, x, y, y_t, \nabla y, \nabla^2 y) = g_1(t, x, y, y_t, \nabla y) + \sum_{i,j=1}^{n} \partial_{x_i} (g_2^{ij}(t, x, y, y_t, \nabla y) \partial_{x_j} y),
\]

where \(g_1\) and \(g_2^{ij}\), \(i, j = 1, \cdots, n\) are smooth functions with

\[
g_1(t, x, 0, 0, 0) = 0, g_1(t, x, y, y_t, \nabla y) = O(|y|^2 + |y_t|^2 + |\nabla y|^2)
\]

and

\[
g_2^{ij} = g_2^{ji}, g_2^{ij}(t, x, 0, 0, 0) = 0, g_2^{ij}(t, x, y, y_t, \nabla y) = O(|y| + |y_t| + |\nabla y|),
\]

\[\forall i, j = 1, \cdots, n.\]

**Theorem 1.2.** Assume that \(f\) satisfies condition (1.9) and \((\omega, T)\) satisfies \(\Gamma\)–condition. There exists one small positive constant \(\varepsilon > 0\), such that for any given initial data \((y_0, y_1) \in \mathcal{H}^s \times \mathcal{H}^{s-1}\), \(s > n\), if

\[
\|(y_0, y_1)\|_{\mathcal{H}^s \times \mathcal{H}^{s-1}} \leq \varepsilon,
\]

(1.12)
then there exists a control $u \in L^\infty(0,T;H^{s-2})$, such that the corresponding solution $y \in C(0,T;H^{s-1}) \cap C^1(0,T;H^{s-2})$ of (1.1) satisfies
\[ y(T, x) = 0, \quad y_t(T, x) = 0. \] (1.13)

**Remark 1.7.** Note that our argument is based on a time transformation that reduced our system into a new system with damping. Thus we can design an algorithm yielding the sequences of control and solutions. By proving an observability inequality for linearized system with time-space dependent coefficients in the principal part (See Theorem 4.3) and using contraction mapping theorem, we can finally obtain the convergence of the sequences of control and solutions.

**Remark 1.8.** Since condition (1.9) is valid for $f(T-t, \cdot, \cdot, \cdot, \cdot)$ as well as $f(t, \cdot, \cdot, \cdot, \cdot)$, combining Theorem 1.2 with well-posedness of system (1.1), we can obtain the exact controllability for system (1.1).

**Remark 1.9.** In order to derive an important estimation of nonlinear terms, we need to use assumption $s > n$ (see Lemma 4.4 in Section 4), so we can not obtain the same result when $s > \frac{n}{2} + 1$ (see [30] Theorem 5.1 or [18] Theorem 4.3). But our proof is constructive in some sense. The control and solutions can be computed numerically by iteration algorithm (see (4.2) and (4.6) in Section 4).

Finally, we go back to full nonlinear system, when $f(t, x, y, y', \nabla^2 y)$ is a smooth function and
\[ f = O(|y|^2 + |y'|^2 + |\nabla^2 y|^2), \] (1.14)
where $u' = (u_t, \nabla u)$. As a by-product of Theorem 1.2 we have

**Theorem 1.3.** Assume that $\varepsilon = \|y_0\|_{H^s} + \|y_1\|_{H^{s-1}}$ is sufficiently small, for some integer $s > n + 1$, and $(T, \omega)$ satisfy the following condition
\[ T \geq 2 \max_{x \in \Omega} |x - x_0|, \quad \Gamma_0 := \left\{ x \in \partial\Omega \mid (x - x_0) \cdot n > 0 \right\}, \quad \omega = \Omega \cap O_{\varepsilon_0}(\Gamma_0), \]
for some $\varepsilon_0 > 0$ and $x_0 \notin \Omega$. Then there exists a function $u(t, x)$ in (1.1) such that
\[ y_t(T) = 0, \quad y_{tt}(T) = 0. \]

The controllability of hyperbolic system has a long history (see [11, 6, 27]). For the linear wave case, J. L. Lions [23] introduced Hilbert Uniqueness Method (HUM), applying this to set up the duality and proved that the exact controllability of the control system can be equivalently reduced to the observability inequality for solutions of the adjoint system. Thus exact controllability of linear wave equation can be obtained under the $\Gamma$–condition. Bardos-Lebeau-Rauch pointed out the Geometric Control Condition (GCC) is crucial to the exact internal or boundary controllability of linear wave equations, see also [20] for boundaryless
case and [2] for the necessity. We refer to [34][24][3] for more geometric conditions for exact controllability and related problem of stabilization.

In the semi-linear case, there exist many results when nonlinear term $f = f(u)$. By combining Hilbert Uniqueness Method and Schauder’s fixed point theorem, exact controllability results for the semi-linear wave equation when $f$ is Lipschitz were proved by E. Zuazua [34]. Lasiecka and Triggiani [16] improved this result by a global inversion theorem provided $f$ is absolutely continuous with a.e. uniformly bounded first derivative $f'$. In [36], E. Zuazua considered the case when $f(u)$ behaves like $u \ln^p(u)$ in the one space dimensional case. The results in [36] were generalized by Fu-Yong-Zhang [14] to high space dimension. Their methods relied on the fixed point theorem that reduces the exact controllability to global Carleman estimates [9] for linearized wave equation with a potential. Recently, Münch-Trélat [25] gave a constructive proof of the results in [36], they design a least-squares algorithm obtaining the control and solutions for 1D semi-linear wave equation.

In the case that $f(u) = |u|^{p-1}u$, when $1 \leq p < 5$, Dehman, Lebeau and Zuazua [8] obtained the exact controllability under the assumption that control acts on a subdomain outside a sphere with cutting-off the nonlinearities. It was generalized by Dehman and Lebeau [7] under the GCC and nonlinearities without cutting-off, but the lower frequency part of initial data should be small enough. C. Laurent [17] generalized this result to critical case $p = 5$ via profile decomposition on compact Riemannian manifold. See also [32] for more general nonlinear terms cases and [18][30] for detail summary. Some results for stabilization were also formulated by [8], then by Cavalcanti-Cavalcanti-Fukuoka-Soriano[5][4].

In the quasi-linear case, many studies have also been done on the subject related to the exact controllability. The exact boundary controllability have been well-studied. In [21], by using a constructive method, Li and Yu obtained the exact boundary controllability for 1D quasi-linear wave system. We refer the reader to [20][19] for a system theory of controllability for 1D quasi-linear hyperbolic system. It was generalized by the third author and Z. Lei to the two or three space dimensional case [31]. Their proofs strongly relied on boundary damping and Huygens principle. By using a different method based on Riemannian geometry, P. Yao [29] also obtained the exact boundary controllability for a class of quasi-linear wave in high space dimensional case. Let us mention that the above results concern boundary control problem. As far as we know, there are much fewer known results about internal controllability for quasi-linear case. K. Zhuang [33] studied the exact internal controllability for a class of 1D quasi-linear wave equation.

In [30], X. Zhang posed one open problem: the exact controllability of the semi-linear equation

$$y_{tt} + A y + f(y_t) = \chi_\omega(x)u(t, x)$$  \hspace{1cm} (1.15)

in the energy space $H_0^1(\Omega) \times L^2(\Omega)$.

The aim of this paper is to study this problem and related problems of local controllability
for quasilinear wave equation and fully nonlinear wave equation. More precisely, by combing Galerkin method with a fixed point argument, we design an algorithm constructing a sequence converging to control and solution for (1.15), that is Theorem 1.1. But we assume that initial data should belong to $H_0^2 \times H^1$ instead of $H_0^1(\Omega) \times L^2(\Omega)$. By using a time transformation and contraction mapping theorem, we give a ”constructive” proof of exact controllability for quasi-linear wave system, that is Theorem 1.2. And in Section 5 we apply Theorem 1.2 to prove Theorem 1.3 which can give some kinds of null controllability for a class of fully nonlinear equations.

The rest of this paper is organized as follows. In Section 2, we give three different methods and prove the exact controllability of damped Klein-Gordon equation as introductions of our approaches. Section 3 is devoted to proving null controllability of the damped semi-linear wave equation, that is Theorem 1.1. In Section 4, we prove Theorem 1.2. In Section 5, we obtained the proof of Theorem 1.3, the local null controllability of damped fully nonlinear wave equation. Appendix A contains the proof of one key observability inequality for linear wave system, which is quite useful in the proof of Theorem 1.2.

2 Damped wave equation

The main purpose of this section is to introduce two methods, which will be useful in the proof of our main theorems. For convenience, we will explain the key points of these two methods by considering the exact controllability for damped linear wave equation.

We first notice that the controllability problem for system (1.1) can be reduced to the case with damping. In fact, taking $\tilde{y} = e^t y, \tilde{u} = e^t u$, we can obtain a damped nonlinear Klein-Gordon equation

$$
\begin{cases}
\tilde{y}_{tt} + 2\tilde{y}_t + \Delta \tilde{y} + \tilde{f}(t, x, \tilde{y}, \tilde{y}_t, \nabla \tilde{y}, \nabla^2 \tilde{y}) = \chi_{\omega} \cdot \tilde{u}, & (t, x) \in (0, T) \times \Omega, \\
\tilde{y}(t, x) = 0, & (t, x) \in (0, T) \times \partial \Omega, \\
\tilde{y}(0, x) = y_0, & \tilde{y}_t(0, x) = y_0 + y_1, \quad x \in \Omega.
\end{cases}
$$

(2.1)

It is easy to see that $\tilde{f}$ holds condition (1.9) if and only if $f$ holds condition (1.9). So our goal turns to prove Theorem 1.2 for system (2.1).

As usual, in order to study the nonlinear system, one need to consider the linear system

$$
\begin{cases}
y_{tt} + 2y_t - \Delta y + y = \chi_{\omega}(x)u(t, x), & (t, x) \in (0, T) \times \Omega, \\
y(t, x) = 0, & (t, x) \in (0, T) \times \partial \Omega, \\
y(0, x) = y_0, & y_t(0, x) = y_1, \quad x \in \Omega.
\end{cases}
$$

(2.2)

For any given $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$, we want to find a control $u \in L^2((0, T) \times \omega)$ that drives $(y, y_t)$ to zero at time $T$. We require $T > T_0$ where $T_0$ is the time for the observable inequality.
Due to classical HUM method, one can consider the following dual system:

\[
\begin{align*}
    z_{tt} - 2z_t - \Delta z + z &= 0, & (t, x) &\in (0, T) \times \Omega, \\
    z(t, x) &= 0, & (t, x) &\in (0, T) \times \partial \Omega, \\
    z(0, x) &= z_0, & z_t(0, x) &= z_1, & x &\in \Omega.
\end{align*}
\]  

(2.3)

**Theorem 2.1.** For any given \((y_0, y_1) \in H^1_0(\Omega) \times L^2(\Omega)\), there exists an internal control \(u \in L^2((0, T) \times \omega)\) which drives \((y, y_t)\) in (2.2) to zero at time \(T\) if for any initial data \((z_0, z_1) \in H^1_0(\Omega) \times L^2(\Omega)\), the corresponding solution \(z \in C(0, T; H^1_0) \cap C^1(0, T; L^2)\) of system (2.3) holds

\[
\|z_0\|_{H^1_0}^2 + \|z_1\|_{L^2}^2 \leq C \int_0^T \|z_t\|_{L^2(\omega)}^2 dt,
\]  

(2.4)

where \(C\) is a positive constant independent with \((z_0, z_1)\).

Here we provide three different methods to prove Theorem 2.1. The Second and Third methods will be useful in the proof of our main theorems.

The first method is classical HUM method.

**Proof.** Multiplying (2.2) by \(z_t\), multiplying (2.3) by \(y_t\), adding them together and integrating by parts, we have

\[
\int_0^T \int_\Omega \frac{d}{dt}(y_t z_t + \nabla y \cdot \nabla z + y z) dx dt = \int_0^T \int_\omega u z_t dx dt.
\]  

(2.5)

Setting \(y(T, x) = y_t(T, x) = 0\), choosing \(u = z_t\), we get

\[
\int_\Omega \nabla z_0 \cdot \nabla y_0 + z_0 y_0 + z_1 y_1 dx = \int_0^T \int_\omega |z_t|^2 dx dt.
\]  

(2.6)

We define a continuous linear map \(F_H : H^1_0 \times L^2 \mapsto H^1_0 \times L^2\) as follow:

\[
F_H : (z_0, z_1) \mapsto (y_0, y_1).
\]

According to Lax-Milgram Theorem, (2.4) implies that \(F_H\) is onto. Thus due to well-posedness of system (2.2), we finish the proof.

The second method relies on Picard iteration.

**Proof.** Our strategy is to write

\[
y(t) = w(t) - z(T - t).
\]  

(2.7)

Here \(w\) satisfies

\[
\begin{align*}
    w_{tt} + 2w_t - \Delta w + w &= -\sqrt{2} \chi_{\Omega \setminus \omega} z_t(T - t), & (t, x) &\in (0, T) \times \Omega, \\
    w(t, x) &= 0, & (t, x) &\in (0, T) \times \partial \Omega, \\
    w(0, x) &= z(T) + y_0, & w_t(0, x) &= -z_t(T) + y_1, & x &\in \Omega.
\end{align*}
\]  

(2.8)

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\int_0^T \int_\Omega \frac{d}{dt}(y_t z_t + \nabla y \cdot \nabla z + y z) dx dt = \int_0^T \int_\omega u z_t dx dt.
\]  

(2.5)

Setting \(y(T, x) = y_t(T, x) = 0\), choosing \(u = z_t\), we get

\[
\int_\Omega \nabla z_0 \cdot \nabla y_0 + z_0 y_0 + z_1 y_1 dx = \int_0^T \int_\omega |z_t|^2 dx dt.
\]  

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We define a continuous linear map \(F_H : H^1_0 \times L^2 \mapsto H^1_0 \times L^2\) as follow:

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    w(t, x) &= 0, & (t, x) &\in (0, T) \times \partial \Omega, \\
    w(0, x) &= z(T) + y_0, & w_t(0, x) &= -z_t(T) + y_1, & x &\in \Omega.
\end{align*}
\]  

(2.8)
then it is easy to see that $y$ satisfies
$$
y(t, x) = 0, \quad (t, x) \in (0, T) \times \Omega, \quad y(0, x) = y_0, \quad y_t(0, x) = y_1, \quad x \in \Omega,
$$

(2.9)

notice that $y(T) = w(T) - z_0$, $y_t(T) = w_t(T) + z_1$.

If we can find $(z_0, z_1)$ such that $w(T) = z_0$, $w_t(T) = -z_1$, then we may take
$$
u = \sqrt{2} \chi_\omega \cdot z_t(T - t)
$$

(2.10)

and by the well-posedness of system (2.2), $u$ will be the control we seek.

For every $(z_0, z_1) \in H^1_0(\Omega) \times L^2(\Omega)$, we define a map
$$\mathcal{F}: (z_0, z_1) \mapsto (w(T), -w_t(T)) \mapsto (z_0, z_1)
$$

(2.2)

We will show that this map has a fixed point, thus this gives the desired conclusion. In the rest of the proof, we will show that $\mathcal{F}$ is a contraction mapping, then by contraction mapping theorem, $\mathcal{F}$ has a fixed point.

We first claim that (2.4) implies that there exists a constant $\kappa < 1$, depending only on $T, \Omega, \omega$, such that
$$\frac{1}{2} \left( \|z(T)\|_{H^1(\Omega)}^2 + \|z_t(T)\|_{L^2(\Omega)}^2 \right) + \int_0^T \|z_t\|_{L^2(\Omega)}^2 dt
\leq \frac{\kappa}{2} \left( \|z_0\|_{H^1(\Omega)}^2 + \|z_1\|_{L^2(\Omega)}^2 \right)
$$

(2.11)

Next, by energy equality
$$\frac{1}{2} \left( \|z(T)\|_{H^1(\Omega)}^2 + \|z_t(T)\|_{L^2(\Omega)}^2 \right) + \int_0^T \|z_t\|_{L^2(\Omega)}^2 dt
= \frac{1}{2} \left( \|z_0\|_{H^1(\Omega)}^2 + \|z_1\|_{L^2(\Omega)}^2 \right),
$$

(2.12)

so (2.11) is equivalent to
$$\frac{1}{2} \left( \|z_0\|_{H^1(\Omega)}^2 + \|z_1\|_{L^2(\Omega)}^2 \right) - \int_0^T \|z_t\|_{L^2(\Omega)}^2 dt
\leq \frac{\kappa}{2} \left( \|z_0\|_{H^1(\Omega)}^2 + \|z_1\|_{L^2(\Omega)}^2 \right)
$$

(2.13)

and
$$\|z_0\|_{H^1(\Omega)}^2 + \|z_1\|_{L^2(\Omega)}^2 \leq \frac{2}{1 - \kappa} \int_0^T \|z_t\|_{L^2(\Omega)}^2 dt.
$$

(2.14)

By Poincare’s inequality, (2.14) is equivalent to (2.4) for some $\kappa \in (0, 1)$.

Now multiplying (2.8) by $w_t$ and making an integration by part, we get
$$\frac{1}{2} \frac{d}{dt} \left( \|w_t\|_{L^2(\Omega)}^2 + \|\nabla w\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 \right) + 2 \|w_t\|_{L^2(\Omega)}^2
= -\sqrt{2} \int_{\Omega \setminus \omega} w_t(t) z_t(T - t) dx
\leq 2 \|w_t\|_{L^2(\Omega)}^2 + \frac{1}{4} \|z_t(T - t)\|_{L^2(\Omega \setminus \omega)}^2,
$$

(2.15)
Therefore, integrate in $t$ from 0 to $T$, we get

\[
\frac{1}{2} \| \mathcal{F}(z_0, z_1) \|^2_{H^1(\Omega) \times L^2(\Omega)} \\
\leq \int_0^T \| z_t \|^2_{L^2(\Omega \omega)} dt + \frac{1}{2} \left( \| z(T) + y \|_{L^2(\Omega)} + \| z(0) \|_{H^1(\Omega)} \right) \\
\leq \int_0^T \| z_t \|^2_{L^2(\Omega \omega)} dt + \frac{1}{2} \left( \| z(T) \|_{L^2(\Omega)} + \| z(0) \|_{H^1(\Omega)} \right) \\
+ \frac{1}{2} \left( \| y_1 \|^2_{L^2(\Omega)} + \| y_0 \|^2_{H^1(\Omega)} \right) + \int_\Omega (\nabla z(T) \cdot \nabla y_0 + z(T)y_0 - z(T)y_1) dx \\
\leq \int_0^T \| z_t \|^2_{L^2(\Omega \omega)} dt + \frac{1}{2} \left( \| z(T) \|_{L^2(\Omega)} + \| z(0) \|_{H^1(\Omega)} \right) \\
+ \frac{1 + \delta^{-1}}{2} \left( \| y_1 \|^2_{L^2(\Omega)} + \| y_0 \|^2_{H^1(\Omega)} \right) \\
\leq (1 + \delta) \left[ \frac{1}{2} \left( \| z(T) \|^2_{L^2(\Omega)} + \| z(0) \|^2_{H^1(\Omega)} \right) + \int_0^T \| z_t \|^2_{L^2(\Omega \omega)} dt \right] \\
+ \frac{1 + \delta^{-1}}{2} \left( \| y_1 \|^2_{L^2(\Omega)} + \| y_0 \|^2_{H^1(\Omega)} \right) \\
\leq \frac{1 + \delta}{2} \kappa \left( \| z_0, z_1 \|^2_{H^1(\Omega) \times L^2(\Omega)} + \frac{1 + \delta^{-1}}{2} \left( \| y_1 \|^2_{L^2(\Omega)} + \| y_0 \|^2_{H^1(\Omega)} \right) \right),
\]

we take $\delta$ small enough such that $(1 + \delta)\kappa < 1$, then $\mathcal{F}$ is a mapping from the set

\[
\left\{ (z_0, z_1) \left\| (z_0, z_1) \right\|^2_{H^1(\Omega) \times L^2(\Omega)} \leq \frac{1 + \delta^{-1}}{1 - (1 + \delta)\kappa} \left( \| y_1 \|^2_{L^2(\Omega)} + \| y_0 \|^2_{H^1(\Omega)} \right) \right\}
\]
to itself, and in a similar way we can prove

\[
\left\| \mathcal{F}(z_0^{(1)}, z_1^{(1)}) - \mathcal{F}(z_0^{(2)}, z_1^{(2)}) \right\|^2_{H^1(\Omega) \times L^2(\Omega)} \leq \kappa(1 + \delta) \left\| (z_0^{(1)} - z_0^{(2)}, z_1^{(1)} - z_1^{(2)}) \right\|^2_{H^1(\Omega) \times L^2(\Omega)}.
\]

So $\mathcal{F}$ is a contraction map. By contraction mapping theorem, $\mathcal{F}$ has a fixed point, this conclude the proof of our main theorem. 

**Remark 2.2.** Compared with HUM, we can find the control function in a quite simpler way due to the damping. And we can directly construct the control function and solutions by contraction mapping theorem. However, by HUM, one can obtain more information about the control function $u$, such as the $L^2$-optimality of the control. Here we do not obtain any optimality for the constructed control in our algorithm.

The above idea will be applied to the fully nonlinear case in section 4. For the use of the next section, we introduce the idea of Galerkin method to prove Theorem 2.1.

**Proof.** We take the standard orthogonal basis $\{ \varphi_j \}_{j=1}^\infty$ of $L^2(\Omega)$ such that

\[
\begin{align*}
{-\Delta + 1} \varphi_j &= \lambda_j \varphi_j, & x \in \Omega, \\
\varphi_j &= 0, & x \in \partial\Omega,
\end{align*}
\]
and write
\[ g_N^0 = \sum_{j=1}^{N} (y_0, \varphi_j)L^2 \varphi_j, \quad g_N^1 = \sum_{j=1}^{N} (y_1, \varphi_j)L^2 \varphi_j. \]

Let
\[ y_N = \sum_{j=1}^{N} g_jN(t) \varphi_j, \quad v_N = \sum_{j=1}^{N} h_jN(t) \varphi_j \]

satisfy the finite-dimensional system
\[
\begin{cases}
\left( \partial_t^2 y_N + (-\Delta + 1)y_N + 2\partial_t y_N - \chi\partial_t v_N, \varphi_i \right)_{L^2} = 0, & i = 1, 2, \cdots, N \\
t = 0: g_jN = (y_0, \varphi_j)_{L^2}, \quad g_jN' = (y_1, \varphi_j)_{L^2}
\end{cases}
\]

and backward system
\[
\begin{cases}
\left( \partial_t^2 v_N + (-\Delta + 1)v_N - 2\partial_t v_N, \varphi_i \right)_{L^2} = 0, & i = 1, 2, \cdots, N \\
t = T: h_jN = a_j, \quad h_jN' = b_j.
\end{cases}
\]

Thus we can define a map \( F_g \):
\[
F_g : (v_N(T), \partial_t v_N(T)) \mapsto (y_N(T), \partial_t y_N(T)).
\]

Multiplying (2.17) by \( h_iN(t) \), (2.18) by \( g_{iN}(t) \) and adding them together, we get
\[
\frac{d}{dt} \int_{\Omega} (\partial_t y_N \partial_t v_N + y_N \partial_t v_N + \nabla y_N \cdot \nabla v_N) dx = \int_{\omega} |\partial_t v_N|^2 dx,
\]

integrating \( t \) from zero to \( T \), we get
\[
\int_{\Omega} \partial_t y_N(T) \partial_t v_N(T) + y_N(T) v_N(T) + \nabla y_N(T) \cdot \nabla v_N(T) dx
= \int_{\Omega} \partial_t y_N(0) \partial_t v_N(0) + y_N(0) v_N(0) + \nabla y_N(0) \cdot \nabla v_N(0) dx + \int_{0}^{T} \int_{\omega} |\partial_t v_N|^2 dx dt
\geq \int_{0}^{T} \int_{\omega} |\partial_t v_N|^2 dx dt - \delta_0 E(v_N(0)) - \frac{1}{4\delta_0} E(v_N(0)),
\]

where \( E(v(t)) = \int_{\Omega} |v_t(t)|^2 + |v(t)|^2 + |\nabla v(t)|^2 dx. \)

Now we take \( \delta_0 \) large enough, such that by the observability inequality,
\[
\frac{1}{2\delta_0} E(v_N(0)) \leq \int_{0}^{T} \int_{\omega} |\partial_t v_N|^2 dx dt.
\]

Thus we have
\[
\int_{\Omega} \partial_t y_N(T) \partial_t v_N(T) + y_N(T) v_N(T) + \nabla y_N(T) \cdot \nabla v_N(T) dx \geq \frac{1}{4\delta_0} E(v_N(0)) - \delta_0 E(y_N(0)).
\]

Multiplying (2.17) by \( 2g_iN(t) \) and summing over \( i \) yields
\[
E'(v_N(t)) = 2 \int_{\Omega} |\partial_t v_N(t)|^2 dx \leq 2E(v_N(t)).
\]
Hence we get
\[ \frac{d}{dt}(e^{-2t}E_N(t)) \leq 0, \]
which means
\[ E_N(T) \leq e^{2T}E_N(0). \]
Thus we obtain
\[ \int_\Omega \partial_t y_N(T)\partial_t v_N(T) + y_N(T)v_N(T) + \nabla y_N(T) \cdot \nabla v_N(T) dx \geq \frac{1}{4\delta_0} E_N(0) - \delta_0 E_N(0) \]
\[ \geq \frac{1}{4\delta_0e^{2T}} E_N(T) - \delta_0 E_N(0). \]
Recall the definition of the map \( F_g \) in (2.19), we obtain
\[ (v_N(T), \partial_t v_N(T)) \mapsto (g_{1N}(T), \ldots, g_{NN}(T), g'_{1N}(T), \ldots, g'_{NN}(T)), \]
if \( E(u_N(T)) \) is large enough.

In view of (2.16), \( F_g \) has a equivalent form:
\[ F_G : (a_1, \ldots, a_N, b_1, \ldots, b_N) \mapsto (g_{1N}(T), \ldots, g_{NN}(T), g'_{1N}(T), \ldots, g'_{NN}(T)), \]
which is a continuous map from \( \mathbb{R}^{2N} \) to itself.

Here we recall the following lemma from Evans’ famous book [10].

**Lemma 2.3.** Assume the continuous function \( F : \mathbb{R}^N \to \mathbb{R}^N \) satisfies
\[ x \cdot F(x) \geq 0, \quad \text{if } |x| = r \]
for some \( r > 0 \), then there exists a point \( x_0 \in B_r \) such that \( F(x_0) = 0 \).

**Proof.** Suppose that \( F(x) \neq 0 \) for any \( x \in B_r \), then we can define a continuous map
\[ G(x) = -\frac{F(x)}{|F(x)|} \]
\[ 0 < r^2 = |z|^2 = z \cdot G(z) = -z \cdot F(z) \left| \frac{r}{|F(z)|} \right| \leq 0 \]
which leads to the contradiction. \( \square \)

By Lemma 2.3, there exist \( \{a_j\}_{j=1}^N \) and \( \{b_j\}_{j=1}^N \) in (2.18) such that
\[ y_N(T) = 0, \quad \partial_t y_N(T) = 0 \]
and
\[ E_N(0) \leq E_N(T) \leq CE_N(0). \]
When we go back to (2.20), we find that
\[
\int_0^T \int_\Omega |\partial_t v_N|^2 dx dt \leq CE(y_N(0)) \leq CE(y(0)),
\]
we can see that \(\{\partial_t v_N\}_{N=1}^\infty\) is bounded in \(L^2(0,T;H^1(\omega))\), and have a subsequence that convergence weakly.

By the energy estimates of wave equation, one finds that
\[
\{y_N\}_{N=1}^\infty \subset L^\infty(0,T;H^1(\omega)),
\]
\[
\{\partial_t y_N\}_{N=1}^\infty \subset L^\infty(0,T;L^2(\Omega)).
\]
So the approximation solutions \(\{y_N\}_{N=1}^\infty\) convergence to a weak solution of (2.2). And the weak limit of \(\{\partial_t v_N\}_{N=1}^\infty\) is the desired control function \(u\).

We finish this section by giving a lemma, which is quite important in the proof of Theorem 1.1. Consider the following system:
\[
\begin{aligned}
& z_{tt} - L z_t - \Delta z + z = 0, \quad (t, x) \in (0, T) \times \Omega, \\
& z(t, x) = 0, \quad (t, x) \in (0, T) \times \partial \Omega, \\
& z(0, x) = z_0, \quad z_t(0, x) = z_1, \quad x \in \Omega
\end{aligned}
\]
where \(L\) is a constant.

**Lemma 2.1.** Assume that \((\omega, T)\) satisfies GCC. Then there exists a constant \(D > L\), for any initial data \((z_0, z_1)\) \(\in H^1_0(\Omega) \times L^2(\Omega)\), the corresponding solution \(z \in C(0,T;H^1_0) \cap C^1(0,T;L^2)\) of system (2.22) holds
\[
\|z_0\|_{H^1_0}^2 + \|z_1\|_{L^2}^2 \leq D \min \left\{ \int_0^T \|z_t\|_{L^2(\omega)}^2 dt, \int_0^T \|\nabla z\|_{L^2(\omega)}^2 dt \right\}.
\]
Moreover, for any initial data \((z_0, z_1)\) \(\in H^2 \times H^1\), the corresponding solution \(z \in C(0,T;H^2) \cap C^1(0,T;H^1)\) of System (2.22) satisfies
\[
\frac{1}{2} \left( \|\nabla z_t(T)\|_{L^2(\Omega)}^2 + \|\Delta z(T)\|_{L^2(\Omega)}^2 \right) \leq D \int_0^T \|\nabla z_t\|_{L^2(\omega)}^2 dt.
\]
**Proof.** (2.23) is a classical result due to Bardos-Lebeau-Rauch [1]. Take \(v = z_t\), and notice that \(v\) is still a solution of the System (2.22) with initial data \(v(0) = z_1 \in H^1_0, v_t(0) = \Delta z_0 + Lz_1 - z_0 \in L^2\). Then (2.24) comes from (2.23).

3 Proof of Theorem 1.1

This section is devoted to prove Theorem 1.1. The proof relies on Galerkin method and a fixed-point Lemma 2.3, which are introduced in the proof of Lemma 2.1.
Remark 3.1. To obtain the convergence of the term which actually can be seen as a map from inner product estimate, rather than the standard energy of linear wave equation. So we should use the $R$ equivalent to the standard inner product on $R$ under the same assumptions of Theorem 1.1.

Lemma 3.2. There exists $s$ satisfy the finite-dimensional system

$$\begin{aligned}
&F_N : \mathbb{R}^{2N} \mapsto \mathbb{R}^{2N} \text{ by}

&\mathcal{F}_N : (a_1, \cdots, a_N, b_1, \cdots, b_N) \mapsto (g_1(T), \cdots, g_N(T), g'_1(T), \cdots, g'_N(T)),

\end{aligned}$$

which actually can be seen as a map from $(v_N(T), \partial_t v_N(T))$ to $(y_N(T), \partial_t y_N(T))$.

**Remark 3.1.** To obtain the convergence of the term $f(\partial_t y_N)$, we need higher order energy estimate, rather than the standard energy of linear wave equation. So we should use the inner product

$$((c_1, \cdots, c_{2N}), (d_1, \cdots, d_{2N}))_{L^2(\Omega)} = \sum_{j=1}^{N} \left( \frac{\lambda_j^2}{\delta} + \lambda_j \right) c_j d_j + \sum_{j=N+1}^{2N} \left( \lambda_j + \frac{\lambda_j^2}{\delta} \right) c_j d_j$$
on $\mathbb{R}^{2N}$ in Lemma 2.3. As we know, for any fixed $N \in \mathbb{N}_+$ and $\delta > 0$, this inner product is equivalent to the standard inner product on $\mathbb{R}^{2N}$.

Then we state the following lemma, which plays a key role in our proof of Theorem 1.1.

**Lemma 3.2.** Under the same assumptions of Theorem 1.1. Let $\mathcal{F}_N$ be defined by (3.3). There exists $x_0 = (a_1, \cdots, a_N, b_1, \cdots, b_N) \in \mathbb{R}^{2N}$ such that $\mathcal{F}_N(x_0) = 0$.

The proof of Lemma 2.2 is due to the estimation of $(\mathcal{F}_N(x_0), x_0)_{L^2}$ and Lemma 2.4.
Proof. Take $\delta > 0$, which will be determined later. Multiplying (3.1) by $(\lambda_i + \delta^{-1})h_i'(N(t))$, (3.2) by $(\lambda_i + \delta^{-1})g_i'(N(t))$ and adding them together, we get

$$
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{\delta} \partial_t y_N \partial_t v_N + \nabla \partial_t y_N \cdot \nabla \partial_t v_N \right) dx + \frac{d}{dt} \int_{\Omega} \left( \frac{1}{\delta} \nabla y_N \cdot \nabla v_N + \Delta y_N \Delta v_N \right) dx
$$

$$+
\int_{\Omega} \left( f'(\partial_t y_N) \nabla \partial_t y_N \cdot \nabla \partial_t v_N - L \nabla \partial_t y_N \cdot \nabla \partial_t v_N \right) dx
$$

$$+
\frac{1}{\delta} \int_{\Omega} \left( f(\partial_t y_N) \partial_t v_N - L \partial_t y_N \partial_t v_N \right) dx
$$

$$= \frac{1}{\delta} \int_{\Omega} \chi |\partial_t v_N|^2 dx - \int_{\Omega} \chi \partial_t v_N \Delta \partial_t v_N dx
$$

$$= \int_{\Omega} \chi |\nabla \partial_t v_N|^2 dx + \int_{\Omega} \left( \frac{\chi}{\delta} - \frac{\Delta \chi}{2} \right) |\partial_t v_N|^2 dx,$$

integrating $t$ from zero to $T$, we get

$$(F(x_0), x_0)_{\mathbb{R}^2}\delta = \frac{1}{\delta} \int_{\Omega} \partial_t y_N(T) \partial_t v_N(T) + \nabla y_N(T) \cdot \nabla v_N(T) dx
$$

$$+ \int_{\Omega} \nabla \partial_t y_N(T) \cdot \nabla \partial_t v_N(T) + \Delta y_N(T) \Delta v_N(T) dx
$$

$$= \frac{1}{\delta} \int_{\Omega} \partial_t y_N(0) \partial_t v_N(0) + \nabla y_N(0) \cdot \nabla v_N(0) dx
$$

$$+ \int_{\Omega} \nabla \partial_t y_N(0) \cdot \nabla \partial_t v_N(0) + \Delta y_N(0) \Delta v_N(0) dx
$$

$$+ \frac{1}{\delta} \int_0^T \int_{\Omega} \left( L \partial_t y_N - f(\partial_t y_N) \right) \partial_t v_N dx dt + \int_0^T \int_{\Omega} \left( L - f'(\partial_t y_N) \right) \nabla \partial_t y_N \cdot \nabla \partial_t v_N dx dt
$$

$$+ \int_0^T \int_{\Omega} \chi |\nabla \partial_t v_N|^2 dx dt + \int_0^T \int_{\Omega} \left( \frac{\chi}{\delta} - \frac{\Delta \chi}{2} \right) |\partial_t v_N|^2 dx dt.$$

Denote

$$J_1 = \frac{1}{\delta} \int_0^T \int_{\Omega} \left( L \partial_t y_N - f(\partial_t y_N) \right) \partial_t v_N dx dt + \int_0^T \int_{\Omega} \left( L - f'(\partial_t y_N) \right) \nabla \partial_t y_N \cdot \nabla \partial_t v_N dx dt,$$

$$J_2 = \frac{1}{\delta} \int_{\Omega} \partial_t y_N(0) \partial_t v_N(0) + \nabla y_N(0) \cdot \nabla v_N(0) dx
$$

$$+ \int_{\Omega} \nabla \partial_t y_N(0) \cdot \nabla \partial_t v_N(0) + \Delta y_N(0) \Delta v_N(0) dx,$$

$$E_0(u(t)) = \int_{\Omega} |u(t)|^2 + |\nabla u(t)|^2 dx, \quad E_1(u(t)) = \int_{\Omega} |\nabla u(t)|^2 + |\Delta u(t)|^2 dx.$$

Because $f$ is Lipschitz continuous, $f'$ exists almost everywhere, and from condition (1.4),
We know that \( \tilde{L} \leq f' \leq L \), thus it holds

\[
|J_1| \leq \frac{\delta_1}{2 \delta (L - \tilde{L})} \int_0^T \int_\Omega \left| L \partial_t y_N - f(\partial_t y_N) \right|^2 dx dt + \frac{L - \tilde{L}}{2 \delta \delta_1} \int_0^T \int_\Omega |\partial_t v_N|^2 dx dt
+ \frac{\delta_2 (L - \tilde{L})}{2 \delta} \int_0^T \int_\Omega \left| \nabla \partial_t y_N \right|^2 dx dt
\]

\[
\leq \frac{\delta_1 (L - \tilde{L})}{2 \delta} \int_0^T \int_\Omega |\partial_t y_N|^2 dx dt + \frac{L - \tilde{L}}{2 \delta \delta_1} \int_0^T \int_\Omega |\partial_t v_N|^2 dx dt
+ \frac{\delta_2 (L - \tilde{L})}{2} \int_0^T \int_\Omega \left| \nabla \partial_t y_N \right|^2 dx dt,
\]

and

\[
|J_2| \leq \frac{\delta_1 L}{\delta (L - \tilde{L})} E_0(y_N(0)) + \frac{L - \tilde{L}}{4 \delta \delta_1} L E_0(v_N(0)) + \frac{\delta_2 L}{L - \tilde{L}} E_1(y_N(0)) + \frac{L - \tilde{L}}{4 \delta_2 L} E_1(v_N(0)),
\]

where \( \delta_1 > 0 \) and \( \delta_2 > 0 \) are to be determined.

Next, to control the right hand side of (3.4), we should make standard energy estimate of \( y_N \) and \( v_N \). The energy estimate of \( y_N \) is

\[
\frac{1}{2} \int_0^T \int_\Omega E_0(y_N(t)) + \int_\Omega f(\partial_t y_N) \partial_t y_N dx = \int_\Omega \chi \partial_t y_N \partial_t v_N dx,
\]

integrating by \( t \), we get

\[
\tilde{L} \int_0^T \int_\Omega |\partial_t y_N|^2 dx dt
\]

\[
\leq \int_0^T \int_\Omega f(\partial_t y_N) \partial_t y_N dx dt
= \frac{1}{2} E_0(y_N(0)) - \frac{1}{2} E_0(y_N(T)) + \int_0^T \int_\Omega \chi \partial_t y_N \partial_t v_N dx dt
\]

\[
\leq \frac{1}{2} E_0(y_N(0)) + \frac{\tilde{L}}{2} \int_0^T \int_\Omega |\partial_t y_N|^2 dx dt + \frac{1}{2 \tilde{L}} \int_0^T \int_\Omega \chi^2 |\partial_t v_N|^2 dx dt.
\]

Then we obtain

\[
\int_0^T \int_\Omega |\partial_t y_N|^2 dx dt \leq \frac{1}{L} E_0(y_N(0)) + \frac{1}{L^2} \int_0^T \int_\Omega \chi^2 |\partial_t v_N|^2 dx dt.
\]

Multiplying the equation of \( y_N \) by \( \lambda_1 g_N'(t) \), we get

\[
\frac{1}{2} \frac{d}{dt} E_1(y_N(t)) + \int_\Omega f'(y_N(t)) \nabla y_N dx = \int_\Omega \nabla y_N \cdot \left( \chi \nabla v_N + v_N \nabla \chi \right) dx.
\]

Integrating it by \( t \), similarly we obtain

\[
\int_0^T \int_\Omega \left| \nabla \partial_t y_N \right|^2 dx dt
\]

\[
\leq \frac{1}{L} E_1(y_N(0)) + \frac{1}{L^2} \int_0^T \int_\Omega \left| \chi \nabla \partial_t v_N + \partial_t v_N \nabla \chi \right|^2 dx dt
\]

\[
\leq \frac{1}{L} E_1(y_N(0)) + \frac{2}{L^2} \int_0^T \int_\Omega \left( \chi^2 |\nabla \partial_t v_N|^2 + |\nabla \chi|^2 |\partial_t v_N|^2 \right) dx dt.
\]
Multiplying the equation of $v_N$ by $h'_{iN}(t)$ and $\lambda_i h'_{iN}(t)$, we get

$$\int_0^T \int_\Omega |\partial_t v_N|^2 dx dt = \frac{1}{2L} \left[ E_0(v_N(T)) - E_0(v_N(0)) \right]$$

(3.9)

and

$$\int_0^T \int_\Omega |\nabla \partial_t v_N|^2 dx dt = \frac{1}{2L} \left[ E_1(v_N(T)) - E_1(v_N(0)) \right].$$

(3.10)

Combine (3.9) - (3.10) with (3.1), we obtain

$$|J_1| + |J_2| \leq \frac{L - \tilde{L}}{4\delta_1 L} E_0(v_N(T)) + \frac{\delta_1(L - \tilde{L})}{2\delta L^2} \int_0^T \int_\Omega \chi^2 |\partial_t v_N|^2 dx dt + \frac{L - \tilde{L}}{4\delta_2 L} E_1(v_N(T)) + \frac{\delta_2(L - \tilde{L})}{L - L} \int_0^T \int_\Omega \left( \chi^2 |\nabla \partial_t v_N|^2 + |\nabla \chi|^2 |\partial_t v_N|^2 \right) dx dt$$

+ \left( \frac{\delta_1(L - \tilde{L})}{2\delta L} + \frac{\delta_1 L}{\delta(L - L)} \right) E_0(y_N(0)) + \left( \frac{\delta_2(L - \tilde{L})}{2L} + \frac{\delta_2 L}{L - L} \right) E_1(y_N(0)).$$

Hence we have

$$\int \Omega \int_0^T \partial_t y_N(T) \partial_t v_N(T) + \nabla y_N(T) \cdot \nabla v_N(T) dx$$

$$+ \int \Omega \nabla \partial_t y_N(T) \cdot \nabla \partial_t v_N(T) + \Delta y_N(T) \Delta v_N(T) dx$$

$$\geq \int_0^T \int_\Omega \left( \chi - \frac{\delta_2(L - \tilde{L})}{L^2} \chi^2 \right) |\nabla \partial_t v_N|^2 dx dt + \int_0^T \int_\Omega \delta \left( \chi - \frac{\delta_1(L - \tilde{L})}{2L^2} \chi^2 \right) |\partial_t v_N|^2 dx dt$$

$$- \int_0^T \int_\Omega \left( \frac{\Delta \chi}{2} + \frac{\delta_2(L - \tilde{L})}{L^2} |\nabla \chi|^2 \right) |\partial_t v_N|^2 dx dt - \frac{L - \tilde{L}}{4\delta_1 L} E_0(v_N(T)) - \frac{L - \tilde{L}}{4\delta_2 L} E_1(v_N(T))$$

$$- \left( \frac{\delta_1(L - \tilde{L})}{2\delta L} + \frac{\delta_1 L}{\delta(L - L)} \right) E_0(y_N(0)) - \left( \frac{\delta_2(L - \tilde{L})}{2L} + \frac{\delta_2 L}{L - L} \right) E_1(y_N(0)).$$

By (3.9) we get

$$\int_0^T \int_\Omega \left( \frac{\Delta \chi}{2} + \frac{\delta_2(L - \tilde{L})}{L^2} |\nabla \chi|^2 \right) |\partial_t v_N|^2 dx dt$$

$$\leq \frac{1}{2L} \left( \frac{\|\Delta \chi\|_{L^\infty}}{2} + \frac{\delta_2(L - \tilde{L})}{L^2} \|\nabla \chi\|^2_{L^\infty} \right) E_0(v_N(T)).$$

Now we take

$$\delta_1 = \tilde{L} \sqrt{\frac{D}{L}}, \quad \delta_2 = \tilde{L} \sqrt{\frac{D}{2L}}.$$n

By (1.11), one can easily check that

$$\chi - \frac{\delta_2(L - \tilde{L})}{L^2} \chi^2 \geq 0, \quad \chi - \frac{\delta_1(L - \tilde{L})}{2L^2} \chi^2 \geq 0.$$
Furthermore, by the observability equalities (2.23) and (2.24), we have

\[ \frac{1}{\delta} \int_{\Omega} \partial_i y_N(T) \partial_i v_N(T) + \nabla y_N(T) \cdot \nabla v_N(T) \, dx \]

\[ + \int_{\Omega} \nabla \partial_i y_N(T) \cdot \nabla v_N(T) + \Delta y_N(T) \Delta v_N(T) \, dx \]

\[ \geq \left( 1 - \frac{(L - \tilde{L})}{L} \sqrt{\frac{D}{2L}} \right) \int_0^T \int_\omega |\partial_i v_N|^2 \, dx \, dt - \frac{L - \tilde{L}}{2L \sqrt{2D}} E_1(v_N(T)) \]

\[ + \frac{1}{\delta} \left( 1 - \frac{(L - \tilde{L})}{2L} \sqrt{\frac{D}{L}} \right) \int_0^T \int_\omega |\partial_i v_N|^2 \, dx \, dt - \frac{L - \tilde{L}}{4\delta L \sqrt{2D}} E_0(v_N(T)) \]

\[ - \left( \frac{\| \Delta \chi \|_{L^\infty}}{4L} + \frac{(L - \tilde{L})}{2LL} \sqrt{\frac{D}{2L}} \| \nabla \chi \|_{L^\infty}^2 \right) E_0(v_N(T)) \]

\[ - \frac{\tilde{L}}{\delta \sqrt{2L}} \left( \frac{L - \tilde{L}}{2L} + \frac{L}{L - L} \right) E_0(y_N(0)) - \tilde{L} \sqrt{\frac{D}{2L}} \left( \frac{L - \tilde{L}}{2L} + \frac{L}{L - L} \right) E_1(y_N(0)) \]

\[ \geq \left( \frac{1}{2D} - \frac{L - \tilde{L}}{L \sqrt{2DL}} \right) E_1(v_N(T)) + \frac{1}{\delta} \left( \frac{1}{2D} - \frac{L - \tilde{L}}{2L \sqrt{2DL}} \right) E_0(v_N(T)) \]

\[ - \left( \frac{\| \Delta \chi \|_{L^\infty}}{4L} + \frac{(L - \tilde{L})}{2LL} \sqrt{\frac{D}{2L}} \| \nabla \chi \|_{L^\infty}^2 \right) E_0(v_N(T)) \]

\[ - \frac{\tilde{L}}{\sqrt{2L}} \left( \frac{L - \tilde{L}}{2L} + \frac{L}{L - L} \right) E_0(y_N(0)) - \tilde{L} \sqrt{\frac{D}{2L}} \left( \frac{L - \tilde{L}}{2L} + \frac{L}{L - L} \right) E_1(y_N(0)). \]

Taking \( \delta \) small enough, such that

\[ \delta \left( \frac{\| \Delta \chi \|_{L^\infty}}{4L} + \frac{(L - \tilde{L})}{2LL} \sqrt{\frac{D}{2L}} \| \nabla \chi \|_{L^\infty}^2 \right) = \frac{1}{2} \left( \frac{1}{2D} - \frac{L - \tilde{L}}{L \sqrt{2DL}} \right), \]

Then by using assumption (1.7), we find that

\[ \frac{1}{\delta} \int_{\Omega} \partial_i y_N(T) \partial_i v_N(T) + \nabla y_N(T) \cdot \nabla v_N(T) \, dx \]

\[ + \int_{\Omega} \nabla \partial_i y_N(T) \cdot \nabla v_N(T) + \Delta y_N(T) \Delta v_N(T) \, dx \geq 0 \]

if \( E_0(u_N(T)) \) and \( E_1(u_N(T)) \) are large enough.

Recall the definition of \( \mathcal{F}_N \) in (3.3) and the definition of inner product in remark 3.1 (3.13) gives that \( \mathcal{F}_N(x) \) by Lemma 2.3 there exist \( \{a_j\}_{j=1}^N \) and \( \{b_j\}_{j=1}^N \) in (3.2) such that \( \mathcal{F}_N(a_1, \cdots, a_N, b_1, \cdots, b_N) = 0 \). In view of (3.1), we have \( y_N(0) = 0, \partial_t y_N(T) = 0 \).

Now we are in a position to prove Theorem 1.1.

**Proof.** For any \( N > 0 \), by Lemma 3.2 there exist \( v_N \) such that

\[ \frac{1}{\delta} \int_{\Omega} \partial_i y_N(T) \partial_i v_N(T) + \nabla y_N(T) \cdot \nabla v_N(T) \, dx \]

\[ + \int_{\Omega} \nabla \partial_i y_N(T) \cdot \nabla v_N(T) + \Delta y_N(T) \Delta v_N(T) \, dx = 0. \]
Thus we can go back to (3.11) to find that
\[
\delta \int_0^T \int_\omega |\nabla \partial_t v_N|^2 dx dt + \frac{1}{2} \int_0^T \int_\omega |\partial_t v_N|^2 dx dt
\leq C^* [E_0(y_N(0)) + \delta E_1(y_N(0))]
\leq C^* [E_0(y(0)) + \delta E_1(y(0))]
\]  
(3.14)

with the constant
\[
C^* = \sqrt{\frac{D}{L}} \left( \frac{L - \tilde{L}}{2} + \frac{L \tilde{L}}{L - \tilde{L}} \right) \left( 1 - \frac{L - \tilde{L}}{L} \sqrt{\frac{2D}{L}} \right)^{-1}
\]  
(3.15)

By (3.14) we can see that \( \{ \partial_t v_N \}_{N=1}^\infty \) is bounded in \( L^2(0,T; H^1(\omega)) \), and have a subsequence that convergence weakly. Further, by the energy estimates (3.5) and (3.7), one gets that
\[
\{ y_N \}_{N=1}^\infty \in L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega)),
\]
and by the compact imbedding theorem \( H^1_0(\Omega) \hookrightarrow L^2(\Omega) \), we can see that \( \{ \partial_t y_N \}_{N=1}^\infty \) have a subsequence that convergence strongly in \( L^\infty(0,T; L^2(\Omega)) \). This guarantees the convergence of \( f(\partial_t y_N) \).

So the approximation solutions \( \{ y_N \}_{N=1}^\infty \) convergence to a weak solution of (1.3). And the weak limit \( u \) of \( \{ \partial_t v_N \}_{N=1}^\infty \) is the desired control function. Let \( N \to \infty \) in (3.14), we obtain (1.8) with \( D^* = \frac{C^*}{\delta} \) and the constants \( \delta \) and \( C^* \) defined in (3.12) and (3.15).

\[\Box\]

4 Proof of Theorem 1.2

This section is devoted to proving Theorem 1.2. As mentioned above, it is sufficient to consider the null controllability problem for quasi-linear damped wave equation

\[
\begin{cases}
y_{tt} + b_0 y_t - \sum_{i,j=1}^n (a^{ij} y_{x_i})_{x_j} + \sum_{k=1}^n b_k y_{x_k} + \bar{b} y = \chi_\omega \cdot u, & (t,x) \in (0,T) \times \Omega, \\
y(t,x) = 0, & (t,x) \in (0,T) \times \partial \Omega, \\
y(0,x) = y_0, & (t,x) \in (0,T) \times \partial \Omega, \\
y_t(0,x) = y_1, & x \in \Omega,
\end{cases}
\]  
(4.1)

with
\[
a^{ij} = a^{ji}, \quad b_0 = 2 + \int_0^1 \int_0^1 \partial_y g_1(t,x,y,\tau y_t,\cdots) d\tau, \\
b_k = \int_0^1 \int_0^1 \partial_{y_{x_k}} g_1(t,x,y,\tau y_{x_k},\cdots) d\tau, \quad \bar{b} = 1 + \int_0^1 \int_0^1 \partial_y g_1(t,x,y,\tau y_t,\cdots) d\tau.
\]

Due to the damping \( y_t \), we can construct an algorithm to obtain the null controllability of (4.1). We first consider the linearization system of (4.1): we set up the following iteration schemes: take \( (z^{(0)}, u^{(0)}) \equiv 0 \), knowing \( (z^{(\alpha-1)}, u^{(\alpha-1)}) \), we define \( (z^{(\alpha)}, u^{(\alpha)}) \) as follows
\[
\begin{aligned}
&v_{tt}^{(a)} + b_0^{(a)} v_t^{(a)} - \sum_{i,j=1}^n a_{ij}^{(a)} v_{xixj}^{(a)} + \tilde{b}^{(a)} v^{(a)} + \sum_{i=1}^n b_i^{(a)} v_{xi}^{(a)} = -2\chi \cdot z_t^{(a)}, \quad (t, x) \in (0, T) \times \Omega, \\
v^{(a)}(t, x) = 0, & \quad (t, x) \in (0, T) \times \partial \Omega, \\
v^{(a)}(0, x) = y_0, \quad v_t^{(a)}(0, x) = y_1,
\end{aligned}
\] (4.2)

and

\[
\begin{aligned}
z_{tt}^{(a)} - b_0^{(a)} z_t^{(a)} - \sum_{i,j=1}^n a_{ij}^{(a)} z_{xixj}^{(a)} + \tilde{b}^{(a)} z^{(a)} + \sum_{i=1}^n b_i^{(a)} z_{xi}^{(a)} = 0, & \quad (t, x) \in (0, T) \times \Omega, \\
z^{(a)}(t, x) = 0, & \quad (t, x) \in (0, T) \times \partial \Omega, \\
z^{(a)}(T, x) = v^{(a-1)}(T, x) + z^{(a-1)}(T, x), & \quad x \in \Omega, \\
z_t^{(a)}(T, x) = v_t^{(a-1)}(T, x) + z_t^{(a-1)}(T, x), & \quad x \in \Omega,
\end{aligned}
\] (4.3)

where

\[
\begin{aligned}
a_{ij}^{(a)} &= a_{ij}(t, x, v^{(a-1)}, v_t^{(a-1)}, \nabla v^{(a-1)}), \quad i, j = 1, \ldots, n \\
b_i^{(a)} &= b_i(t, x, v^{(a-1)}, v_t^{(a-1)}, \nabla v^{(a-1)}, \nabla^2 v^{(a-1)}), \quad i = 0, \ldots, n \\
\tilde{b}^{(a)} &= \tilde{b}(t, x, v^{(a-1)}, v_t^{(a-1)}, \nabla v^{(a-1)}, \nabla^2 v^{(a-1)}).
\end{aligned}
\] (4.4)

Now our goal is to prove that

**Proposition 4.1.** Assume that \( \varepsilon = \|y_0\|_{H^s} + \|y_1\|_{H^{s-1}} \) is sufficiently small, for some integer \( s > n \), and \((T, \omega)\) satisfy the following \( \Gamma \)-condition

\[
T \geq 2 \max_{x \in \Gamma} |x - x_0|, \quad \Gamma_0 := \left\{ x \in \partial \Omega : (x - x_0) \cdot n > 0 \right\}, \quad \omega = \Omega \cap O_{\varepsilon_0} (\Gamma_0)
\]

for some \( \varepsilon_0 > 0 \) and \( x_0 \notin \Omega \). Assume that \( v^{(a)}, z^{(a)} \) satisfy (4.2) and (1.1). Then for any \( t \in [0, T], (v^{(a)}(t), v_t^{(a)}(t)) \to (y, y_t) \), in \( H^{s-1} \times H^{s-2}, z^{(a)}(t) \to u \) in \( H^{s-2} \) as \( \alpha \to \infty \), where \( (y, u) \) satisfy (1.1) and \( (y(T), y_t(T)) = (0, 0) \).

Notice that Theorem 4.2 follows from Proposition 4.1 directly.

In order to prove Proposition 4.1. We denote \( V^{(a)} = v^{(a)} - v^{(a-1)}, Z^{(a)} = z^{(a)} - z^{(a-1)} \), then we get \( V^{(1)} = v^{(1)}, Z^{(1)} = z^{(1)} \) and for \( \alpha \geq 2 \),

\[
\begin{aligned}
V_{tt}^{(a)} + b_0^{(a)} V_t^{(a)} - \sum_{i,j=1}^n a_{ij}^{(a)} V_{xixj}^{(a)} + \tilde{b}^{(a)} V^{(a)} + \sum_{i=1}^n b_i^{(a)} V_{xi}^{(a)} &= G^{(a)} - 2\chi \cdot Z_t^{(a)}, \quad (t, x) \in (0, T) \times \Omega, \\
V^{(a)}(t, x) = 0, & \quad (t, x) \in (0, T) \times \partial \Omega, \\
V^{(a)}(0, x) = 0, \quad V_t^{(a)}(0, x) = 0, & \quad x \in \Omega,
\end{aligned}
\] (4.5)
\[
\begin{aligned}
Z^{(\alpha)}_{tt} - b^{(\alpha)}_0 Z^{(\alpha)}_t - \sum_{i,j=1}^n a^{(\alpha)}_{ij} Z^{(\alpha)}_{x_ix_j} + \tilde{b}^{(\alpha)} Z^{(\alpha)} + \sum_{i=1}^n b^{(\alpha)}_i Z^{(\alpha)}_{x_i} &= H^{(\alpha)}, & (t, x) &\in (0, T) \times \Omega, \\
Z^{(\alpha)}(t, x) &= 0, & (t, x) &\in (0, T) \times \partial \Omega, \\
Z^{(\alpha)}(T, x) &= v^{(\alpha-1)}(T, x), & Z^{(\alpha)}_{t}(T, x) &= v^{(\alpha-1)}_{t}(T, x) & x &\in \Omega,
\end{aligned}
\]

where
\[
G^{(\alpha)} = (b^{(\alpha)}_0 - b^{(\alpha-1)}_0)v^{(\alpha-1)}_t - \sum_{i,j=1}^n (a^{(\alpha)}_{ij} - a^{(\alpha-1)}_{ij}) v^{(\alpha-1)}_{x_ix_j} + (\tilde{b}^{(\alpha)} - \tilde{b}^{(\alpha-1)}) v^{(\alpha-1)} \\
+ \sum_{i=1}^n (b^{(\alpha)}_i - b^{(\alpha-1)}_i) v^{(\alpha-1)}_{x_i},
\]
\[
H^{(\alpha)} = - \left[ (b^{(\alpha)}_0 - b^{(\alpha-1)}_0) z^{(\alpha-1)}_t + \sum_{i,j=1}^n (a^{(\alpha)}_{ij} - a^{(\alpha-1)}_{ij}) z^{(\alpha-1)}_{x_ix_j} - (\tilde{b}^{(\alpha)} - \tilde{b}^{(\alpha-1)}) z^{(\alpha-1)} \\
- \sum_{i=1}^n (b^{(\alpha)}_i - b^{(\alpha-1)}_i) z^{(\alpha-1)}_i \right].
\]

Lemma 4.2. There exist positive constants \(C_0, C'_0\), and \(M > 1, \delta < 1\) independent of \(\varepsilon\), such that for any \(t \in [0, T]\), we have
\[
\|V^{(\alpha)}(t)\|_{H^k}^2 + \|V^{(\alpha)}_t(t)\|_{H^{k-1}}^2 \leq (1 - \delta)^{2\alpha} C_k^2 \varepsilon^2,
\]
\[
\|Z^{(\alpha)}(t)\|_{H^k}^2 + \|Z^{(\alpha)}_t(t)\|_{H^{k-1}}^2 \leq (1 - \delta)^{2\alpha} M^{2k} \varepsilon^2,
\]
and
\[
\|v^{(\alpha)}(t)\|_{H^{k+1}}^2 + \|v^{(\alpha)}_t(t)\|_{H^k}^2 \leq C_0 C'_k^2 \varepsilon^2,
\]
\[
\|z^{(\alpha)}(t)\|_{H^{k+1}}^2 + \|z^{(\alpha)}_t(t)\|_{H^k}^2 \leq M^{2k+2} \varepsilon^2
\]
for \(\forall \alpha \geq 1, 1 \leq k \leq s - 1\).

The proof of Lemma 4.2 is long, we postpone the proof later. We give the proof of Proposition 4.1 now.

Proof of Proposition 4.1. By (4.7), we have \(\forall t \in [0, T], (v^{(\alpha)}(t), v^{(\alpha)}_t(t)) \rightarrow (y, y_t)\), in \(H^{s-1} \times H^{s-2}, z^{(\alpha)}_t(t) \rightarrow u\ in H^{s-2}\) as \(\alpha \rightarrow \infty\). Notice that
\[
\|v^{(\alpha)} - v^{(\beta)}\|_{H^k}^2 \leq \sum_{i=\beta}^\alpha \|V^{(i)}\|_{H^k}^2 \leq \frac{(1 - \delta)^{2\beta}}{1 - (1 - \delta)^2} C_k^2 \varepsilon^2,
\]
then by Arzelà-Ascoli Theorem, we know that \(y, z \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})\) and \((y, u = z_t)\) satisfy (4.1). Finally, \((y(T), y_t(T)) = (0, 0)\) follows from (4.6). \(\square\)
Before proving Lemma 4.2, we state two key results. The first result is one observation result for the following system

\[
\begin{aligned}
z_{tt} + b_0z_t - \sum_{i,j=1}^{n} (a^{ij}z_{x_i})_{x_j} + \sum_{k=1}^{n} b_kz_{x_k} + \tilde{b}z &= f, \quad (t,x) \in (0,T) \times \Omega, \\
z(t,x) &= 0, \quad (t,x) \in (0,T) \times \partial\Omega, \\
z(0,x) = z_0, \quad z_t(0,x) = z_1 \quad x \in \Omega.
\end{aligned}
\]  

(4.9)

Here coefficients satisfy

\[
a^{ij} - \delta_{ij}, \quad b_0 - 2, \quad b_k, \quad \tilde{b} - 1 \in X_{C,s,\varepsilon},
\]

(4.10)

where

\[
X_{C,s,\varepsilon} = \left\{ g \in L^\infty(0,T; L^2(\Omega)) : \| \partial_t^j \nabla^k g \|_{L^2(\Omega)} \leq C\varepsilon, \quad \forall \ j,k \in \mathbb{N}, j + k \leq s, \forall \ t \in [0,T] \right\},
\]

with $C$ is a constant independent with $\varepsilon$.

**Theorem 4.3.** Assume that $a^{ij}, \tilde{b}, b_0, b_i, i = 1, \cdots, n$ satisfy (4.10). Then for any initial data $(z_0, z_1) \in H^1_0(\Omega) \times L^2(\Omega)$ and $f \in L^2((0,T) \times \Omega)$, the corresponding solution $z$ of System (4.9) holds

\[
\|z_1\|_{L^2(\Omega)} + \|z_0\|_{H^1(\Omega)} \leq D \left( \int_0^T \int_\Omega |z_t|^2 \, dx \, dt + \int_0^T \|f\|_{L^2}^2 \, dt \right),
\]

(4.11)

where $D > 0$ is independent with initial data and $f$.

Combing with Duhamel principle, (4.11) follows from homogeneous type (i.e., $f \equiv 0$) observability inequality directly. Theorem 4.3 has its own iterest We postpone the proof of in the appendix.

The next result can be found in [22].

**Lemma 4.4.** Let $G(\lambda) = G(\lambda_1, \cdots, \lambda_M)$ satisfies $G(\lambda) = O(|\lambda|)$ when $|\lambda| \leq \nu_0$. $\partial = (\partial_t, \nabla)$ is the derivation of time or space. If we have

\[
\sum_{k \leq \lfloor \frac{N}{2} \rfloor} \| \partial^k \lambda \|_{L^\infty} \leq \nu_0, \quad \forall \ t \geq 0,
\]

then for $k \leq N$ and $1 \leq p \leq \infty$,

\[
\| \partial^k G(\lambda) \|_{L^p} \leq C(\nu_0) \sum_{l \leq k} \| \partial^l \lambda \|_{L^p}.
\]

Before proving Lemma 4.2 let us give some remarks

**Remark 4.5.** Taking $\nu_0 = 1$, $N = s$, $p = 2$, because $s > n$, we have

\[
\sum_{k \leq \lfloor \frac{N}{2} \rfloor} \| \partial^k \lambda \|_{L^\infty} \leq C \sum_{k \leq \lfloor \frac{N}{2} \rfloor} \| \partial^k \lambda \|_{H^{\lfloor \frac{N}{2} \rfloor}} \leq \varepsilon,
\]
so since we can assume that \( \varepsilon < 1 \) is small enough, such that \( c\varepsilon < 1 \), by Lemma 4.4 and 4.5, we get

\[
\| b_0^{(a)} - 2\|_{H^s} + \| a_{ij}^{(a)} - \delta_{ij}\|_{H^s} + \| \hat{b}^{(a)} - 1\|_{H^s} + \| \hat{b}_i^{(a)}\|_{H^s} \leq C \varepsilon
\]

and

\[
\| \partial_t b_0^{(a)}\|_{H^{s-1}} + \| \partial_t a_{ij}^{(a)}\|_{H^{s-1}} + \| \partial_t \hat{b}^{(a)}\|_{H^{s-1}} + \| \partial_t \hat{b}_i^{(a)}\|_{H^{s-1}} \leq C \varepsilon.
\]

Besides, if we write

\[ G(\lambda) - G(\lambda) = \hat{G}(\lambda, \lambda) \cdot (\lambda - \lambda), \]

where \( \hat{G}(\lambda, \lambda) \) is bounded and smooth, then we get

\[
\| b_0^{(a)} - b_0^{(a-1)}\|_{H^s} + \| a_{ij}^{(a)} - a_{ij}^{(a-1)}\|_{H^s}
+ \| \hat{b}^{(a)} - \hat{b}^{(a-1)}\|_{H^s} + \| \hat{b}_i^{(a)} - \hat{b}_i^{(a-1)}\|_{H^s} \leq C(1 - \delta)^{a-1} \varepsilon
\]

and

\[
\| \partial_t (b_0^{(a)} - b_0^{(a-1)})\|_{H^{s-1}} + \| \partial_t (a_{ij}^{(a)} - a_{ij}^{(a-1)})\|_{H^{s-1}}
+ \| \partial_t (\hat{b}^{(a)} - \hat{b}^{(a-1)})\|_{H^{s-1}} + \| \partial_t (\hat{b}_i^{(a)} - \hat{b}_i^{(a-1)})\|_{H^{s-1}} \leq C(1 - \delta)^{a-1} \varepsilon
\]

from (4.7). In fact, the coefficients \( \partial_t^k b^{(a)} \) and \( \partial_t^k (b^{(a)} - b^{(a-1)}) \) have similar estimates: for any \( 0 \leq k \leq s \),

\[
\| \partial_t^k b_0^{(a)}\|_{H^{s-k}} + \| \partial_t^k a_{ij}^{(a)}\|_{H^{s-k}} + \| \partial_t^k \hat{b}^{(a)}\|_{H^{s-k}} + \| \partial_t^k \hat{b}_i^{(a)}\|_{H^{s-k}} \leq C \varepsilon
\]

and

\[
\| \partial_t^k (b_0^{(a)} - b_0^{(a-1)})\|_{H^{s-k}} + \| \partial_t^k (a_{ij}^{(a)} - a_{ij}^{(a-1)})\|_{H^{s-k}}
+ \| \partial_t^k (\hat{b}^{(a)} - \hat{b}^{(a-1)})\|_{H^{s-k}} + \| \partial_t^k (\hat{b}_i^{(a)} - \hat{b}_i^{(a-1)})\|_{H^{s-k}} \leq C(1 - \delta)^{a-1} \varepsilon,
\]

where \( C > 0 \) is a constant independent with \( \alpha \) and \( \varepsilon \).

**Remark 4.6.** In the proof of Lemma 4.2 We need to estimate the leading terms in \( \partial_t^k F^{(a)} \) and \( \partial_t^k G^{(a)} \) which are

\[
\sum_{i,j=1}^{n} (a_{ij}^{(a)} - a_{ij}^{(a-1)}) \partial_t^k v_{x_i x_j}^{(a-1)}
\]

and

\[
\sum_{i,j=1}^{n} (a_{ij}^{(a)} - a_{ij}^{(a-1)}) \partial_t^k z_{x_i x_j}^{(a-1)}.
\]

Therefore we need

\[
\| \partial_t^k v_{x_i x_j}^{(a-1)} \|_{L^2} + \| \partial_t^k z_{x_i x_j}^{(a-1)} \|_{L^2} \leq C \varepsilon.
\]

Thanks to well-posedness theory of nonlinear wave equation, that’s why we need to assume that initial data \((y_0, y_1) \in H^s \times H^{s-1}\).

Now we are in a position to prove Lemma 4.2.
Proof. In fact, we will prove some more complex equalities as follows:

\[
\|\partial_t^j V^{(\alpha)}\|_{H^m}^2 + \|\partial_t^{j+1} V^{(\alpha)}\|_{H^{m-1}}^2 \leq (1 - \delta)^{2\alpha} C_{l,m}^2 \varepsilon^2 \\
\|\partial_t^j Z^{(\alpha)}\|_{H^{m+1}}^2 + \|\partial_t^{j+1} Z^{(\alpha)}\|_{H^{m}}^2 \leq (1 - \delta)^{2\alpha} M^{2(l+m)} \varepsilon^2
\]  

(4.14) and

\[
\|\partial_t^j v^{(\alpha)}\|_{H^{m+1}}^2 + \|\partial_t^{j+1} v^{(\alpha)}\|_{H^{m}}^2 \leq C_0 C_{l,m}^2 \varepsilon^2, \\
\|\partial_t^j z^{(\alpha)}\|_{H^{m+1}}^2 + \|\partial_t^{j+1} z^{(\alpha)}\|_{H^{m}}^2 \leq M^{2(l+m) + 2} \varepsilon^2,
\]  

(4.15)

where \(m \in \mathbb{N}_+, l \in \mathbb{N}, 1 \leq l + m \leq k\). Notice that choosing \(l = 0\), then \((4.7)\) and \((4.8)\) follow from \((4.14)\) and \((4.15)\) directly.

The proof \((4.14)\) and \((4.15)\) will be divided into several steps.

- To prove \((4.14)\) and \((4.15)\) for \(\alpha = 1\);
- To prove \((4.14)\) for \(\alpha = \beta + 1 \geq 2, k = 1\) assuming \((4.14)\) and \((4.15)\) are valid for \(\beta\);
- To prove \((4.14)\) for \(\alpha = \beta + 1 \geq 2, 2 \leq k \leq s - 1\) assuming \((4.14)\) and \((4.15)\) are valid for \(\beta\);
- To prove \((4.15)\) for \(\alpha \geq 2\).

**Step 1:** The case of \(\alpha = 1\). We first note that \(V^{(1)} = v^{(1)}, Z^{(1)} = z^{(1)}\), and

\[
(\partial_t^2 - \Delta + 2 \partial_t + 1) V^{(1)} = 0, (\partial_t^2 - \Delta - 2 \partial_t + 1) Z^{(1)} = 0.
\]  

(4.16)

Thus due to the classical theory of semigroup of linear operator, there exist \(C_0' > 0, M > 0\), such that \(\forall t \in [0, T]\),

\[
\|V^{(1)}\|_{H^s} + \|V_t^{(1)}\|_{H^{s-1}}^2 \leq e^{-\delta t} (\|v^{(1)}(0)\|_{H^s}^2 + \|v_t^{(1)}(0)\|_{H^{s-1}}^2) \leq C_0'^2 \varepsilon^2,
\]  

(4.17) and

\[
\|Z^{(1)}\|_{H^s} + \|Z_t^{(1)}\|_{H^{s-1}}^2 \leq e^{2(t-T)} (\|z^{(1)}(T)\|_{H^s}^2 + \|z_t^{(1)}(T)\|_{H^{s-1}}^2) \leq M^2 \varepsilon^2.
\]  

(4.18)

So \((4.7)\) and \((4.8)\) are valid in this case.

**Step 2:** To prove \((4.14)\) for \(\alpha = \beta + 1 \geq 2, k = 1\) assuming \((4.14)\) and \((4.15)\) are valid for \(\beta\). Let us estimate \(V^{(\beta+1)}\) and \(Z^{(\beta+1)}\). Multiplying the equation of \(V^{(\beta+1)}\) by \(V_t^{(\beta+1)}\) and by integration by parts, combining with the fact that \(a_{ij} = a_{ji}\), we get

\[
- \sum_{i,j=1}^n \int_{\Omega} \partial_t a_{ij}^{(\beta+1)} V_t^{(\beta+1)} V_{x_i}^{(\beta+1)} dx \\
= \sum_{i,j=1}^n \int_{\Omega} (\partial_t a_{ij}^{(\beta+1)} V_t^{(\beta+1)} V_{x_i}^{(\beta+1)} dx + \frac{1}{2} \frac{d}{dt} \sum_{i,j=1}^n \int_{\Omega} (a_{ij}^{(\beta+1)} V_{x_i}^{(\beta+1)} V_{x_j}^{(\beta+1)} dx \\
- \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} (\partial_t a_{ij}^{(\beta+1)}) V_t^{(\beta+1)} V_{x_i}^{(\beta+1)} dx.
\]
Hence we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( |V_t^{(\beta+1)}|^2 + \sum_{i,j=1}^n a_{ij}^{(\beta+1)} V_t^{(\beta+1)} V_{x_i x_j}^{(\beta+1)} + |V^{(\beta+1)}|^2 \right) dx + 2 \int_\Omega |V_t^{(\beta+1)}|^2 dx \\
+ 2 \int_\Omega \chi \cdot V_t^{(\beta+1)} Z_t^{(\beta+1)} dx - \int_\Omega G^{(\beta+1)} V_t^{(\beta+1)} dx \leq C \varepsilon \int_\Omega \left( |V_t^{(\beta+1)}|^2 + |\nabla V^{(\beta+1)}|^2 \right) dx.
\]

Noting that
\[
2 \int_\Omega |V_t^{(\beta+1)}|^2 dx + 2 \int_\Omega \chi \cdot V_t^{(\beta+1)} Z_t^{(\beta+1)} dx \geq -\frac{1}{2} \int_\Omega |\chi \cdot Z_t^{(\beta+1)}|^2 dx,
\]
we obtain the energy estimate of \( V^{(\beta+1)} \),
\[
\| V_t^{(\beta+1)}(t) \|_{L^2}^2 + \| V^{(\beta+1)}(t) \|_{H^1}^2 \leq (1 + \varepsilon^2) e^{\varepsilon T} \left( \int_0^T \| \chi \cdot Z_t^{(\beta+1)} \|_{L^2}^2 dt + 2 \int_0^T \int_\Omega |G^{(\beta+1)} V_t^{(\beta+1)}| dx dt \right). \tag{4.19}
\]

Similarly, we have
\[
\| Z_t^{(\beta+1)}(t) \|_{L^2}^2 + \| Z^{(\beta+1)}(t) \|_{H^1}^2 \leq (1 + \varepsilon^2) e^{\varepsilon T} \left( \| Z_t^{(\beta+1)}(T) \|_{L^2}^2 + \| Z^{(\beta+1)}(T) \|_{H^1}^2 + 2 \int_0^T \int_\Omega |H^{(\beta+1)} Z_t^{(\beta+1)}| dx dt \right). \tag{4.20}
\]

In view of (4.3) and (4.6), due to assumption of \( u^{(\beta)} \), \( z^{(\beta)}, \) combing (4.12) with (4.13) in Remark 4.3,
\[
\| G^{(\beta+1)} \|_{L^2} + \| H^{(\beta+1)} \|_{L^2} \leq C(1 - \delta)^2 \varepsilon^2.
\]

For simplicities, we denote \( E(u(t)) = \| u_t \|_{L^2}^2 + \| u \|_{H^1}^2 \). Due to Gronwall’s inequality and Hölder’s inequality, we can obtain the estimation of \( E(V^{(\beta+1)}(t)) \) in terms of \( E(Z^{(\beta+1)}(T)) \).

In view of (4.6), we have
\[
Z^{(\beta+1)}(T) = V^{(\beta)}(T) + Z^{(\beta)}(T), \quad Z_t^{(\beta+1)}(T) = V_t^{(\beta)}(T) + Z_t^{(\beta)}(T),
\]
so denote
\[
w^{(\beta)} = u^{(\beta)} + z^{(\beta)}, \quad W^{(\beta)} = u^{(\beta)} - w^{(\beta-1)},
\]
we have
\[
W_t^{(\beta)}(T) = Z_t^{(\beta+1)}(T), W_t^{(\beta)}(T) = Z_t^{(\beta+1)}(T). \tag{4.21}
\]

Note that the following equality
\[
E(W^{(\beta)}) - E(Z^{(\beta)}) = E(V^{(\beta)}) + 2(V_t^{(\beta)}, Z_t^{(\beta)})_{L^2} + 2(V^{(\beta)}, Z^{(\beta)})_{H^1}.
\]
so we next estimate \( (V_t^{(\beta)}, Z_t^{(\beta)})_{L^2} + (V^{(\beta)}, Z^{(\beta)})_{H^1} \).

Multiplying (4.5) by \( Z_t^{(\beta)} \), multiplying (4.6) by \( V_t^{(\beta)} \) and integrating over \( \Omega \), we get
\[
\int_\Omega Z_t^{(\beta)} V_t^{(\beta)} = - \sum_{i,j=1}^n a_{ij}^{(\beta)} Z_t^{(\beta)} V_{x_i x_j}^{(\beta)} + b^{(\beta)} Z_t^{(\beta)} V^{(\beta)} + \sum_{i=1}^n b_i^{(\beta)} Z_t^{(\beta)} V_{x_i}^{(\beta)} \]
\[
+ V_t^{(\beta)} Z_t^{(\beta)} - \sum_{i,j=1}^n a_{ij}^{(\beta)} V_t^{(\beta)} Z_{x_i x_j}^{(\beta)} + b^{(\beta)} V_t^{(\beta)} Z^{(\beta)} + \sum_{i=1}^n b_i^{(\beta)} V_t^{(\beta)} Z_{x_i}^{(\beta)} dx \]
\[
+ 2 \int_\Omega \chi |Z_t^{(\beta)}|^2 dx = \int_\Omega Z_t^{(\beta)} G^{(\beta)} + V_t^{(\beta)} H^{(\beta)} dx.
\]
By integration by parts, we get

\[- \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}^{(\beta)} (Z_t^{(\beta)} V_{x_i x_j}^{(\beta)} + V_t^{(\beta)} Z_{x_i}^{(\beta)}) \, dx \]

\[= \frac{d}{dt} \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij}^{(\beta)} Z_t^{(\beta)} V_{x_j}^{(\beta)} \right) \, dx - \int_{\Omega} \sum_{i,j=1}^{n} (\partial_{x_i} a_{ij}^{(\beta)}) Z_t^{(\beta)} V_{x_j}^{(\beta)} \, dx \]

\[+ \int_{\Omega} \sum_{i,j=1}^{n} \left( (\partial_{x_i} a_{ij}^{(\beta)}) Z_t^{(\beta)} V_{x_j}^{(\beta)} + (\partial_{x_j} a_{ij}^{(\beta)}) V_t^{(\beta)} Z_{x_i}^{(\beta)} \right) \, dx \]

and

\[\frac{d}{dt} \int_{\Omega} \left( Z_t^{(\beta)} V_t^{(\beta)} + \sum_{i,j=1}^{n} a_{ij}^{(\beta)} Z_{x_i}^{(\beta)} V_{x_j}^{(\beta)} + Z^{(\beta)} V^{(\beta)} \right) \, dx + 2 \int_{\Omega} \chi |Z_t^{(\beta)}|^2 \, dx \]

\[\leq c \varepsilon \int_{\Omega} \left( |V_t^{(\beta)}|^2 + |\nabla V^{(\beta)}|^2 \right) + \left( |Z_t^{(\beta)}|^2 + |\nabla Z^{(\beta)}|^2 \right) \, dx + \int_{\Omega} |Z_t^{(\beta)} G^{(\beta)}| + |V_t^{(\beta)} H^{(\beta)}| \, dx.\]

Integrating in \( t \) on \((0,T),\) we get

\[\int_{\Omega} \left( Z_t^{(\beta)}(T) V_t^{(\beta)}(T) + \sum_{i,j=1}^{n} a_{ij}^{(\beta)} Z_{x_i}^{(\beta)}(T) V_{x_j}^{(\beta)}(T) + Z^{(\beta)}(T) V^{(\beta)}(T) \right) \, dx + 2 \int_{0}^{T} \int_{\Omega} \chi |Z_t^{(\beta)}|^2 \, dx \, dt \]

\[\leq c \varepsilon \int_{0}^{T} \int_{\Omega} \left( |V_t^{(\beta)}|^2 + |\nabla V^{(\beta)}|^2 \right) + \left( |Z_t^{(\beta)}|^2 + |\nabla Z^{(\beta)}|^2 \right) \, dx \, dt + \int_{0}^{T} \int_{\Omega} |Z_t^{(\beta)} G^{(\beta)}| + |V_t^{(\beta)} H^{(\beta)}| \, dx \, dt.\]

Noting that \(|a_{ij}^{(\beta)} - \delta_{ij}| \leq C \varepsilon,\) we obtain that

\[\int_{\Omega} \left( Z_t^{(\beta)} V_t^{(\beta)} + \nabla Z^{(\beta)} \cdot \nabla V^{(\beta)} + Z^{(\beta)} V^{(\beta)} \right) \, dx \bigg|_{t=T} + 2 \int_{0}^{T} \int_{\Omega} \chi |Z_t^{(\beta)}|^2 \, dx \, dt \]

\[\leq c \varepsilon \int_{0}^{T} \int_{\Omega} \left( |V_t^{(\beta)}|^2 + |\nabla V^{(\beta)}|^2 \right) + \left( |Z_t^{(\beta)}|^2 + |\nabla Z^{(\beta)}|^2 \right) \, dx \, dt + \int_{0}^{T} \int_{\Omega} |Z_t^{(\beta)} G^{(\beta)}| + |V_t^{(\beta)} H^{(\beta)}| \, dx \, dt + c \varepsilon \int_{0}^{T} \int_{\Omega} |Z^{(\beta)}(T)| \cdot |\nabla V^{(\beta)}(T)| \, dx.\]

Adding (1.22) to (1.19), and noting that \(0 \leq \chi(x) \leq 1,\) we get

\[E(V^{(\beta)}(T)) + 2 \left( \langle V_t^{(\beta)}(T), Z_t^{(\beta)}(T) \rangle_{L_2} + \langle V^{(\beta)}(T), Z^{(\beta)}(T) \rangle_{H_1} \right) \]

\[\leq c \varepsilon \int_{0}^{T} \int_{\Omega} \left( |V_t^{(\beta)}|^2 + |\nabla V^{(\beta)}|^2 \right) + \left( |Z_t^{(\beta)}|^2 + |\nabla Z^{(\beta)}|^2 \right) \, dx \, dt + 2 \int_{0}^{T} \int_{\Omega} |Z_t^{(\beta)} G^{(\beta)}| + |V_t^{(\beta)} H^{(\beta)}| \, dx \, dt + (1 + c \varepsilon) \int_{0}^{T} \int_{\Omega} \chi \cdot Z_t^{(\beta)} \, dx \, dt \]

\[+ c \varepsilon \int_{\Omega} |\nabla Z^{(\beta)}(T)| \cdot |\nabla V^{(\beta)}(T)| \, dx - 4 \int_{0}^{T} \int_{\Omega} \chi |Z_t^{(\beta)}|^2 \, dx \, dt,\]

and

\[E(W^{(\beta)}(T)) - E(Z^{(\beta)}(T)) \]

\[\leq c \varepsilon \int_{0}^{T} \int_{\Omega} \left( |V_t^{(\beta)}|^2 + |\nabla V^{(\beta)}|^2 \right) + \left( |Z_t^{(\beta)}|^2 + |\nabla Z^{(\beta)}|^2 \right) \, dx \, dt - 3 \int_{0}^{T} \int_{\Omega} \chi |Z_t^{(\beta)}|^2 \, dx \, dt + 2 \int_{0}^{T} \int_{\Omega} |Z_t^{(\beta)} G^{(\beta)}| + |V_t^{(\beta)} H^{(\beta)}| + |V_t^{(\beta)} G^{(\beta)}| \, dx \, dt + c \varepsilon \int_{0}^{T} \int_{\Omega} |\nabla Z^{(\beta)}(T)| \cdot |\nabla V^{(\beta)}(T)| \, dx.\]
Applying Theorem 4.3 to System (4.6), we have the following observability inequality

\[ E(Z^{(\beta)}(T)) \leq D \int_0^T \int_\omega |Z_t^{(\beta)}|^2 dx dt + C(1 - \delta)^{2\beta - 2}\varepsilon^4. \]

Here we use the fact that \( \|H^{(\beta)}\|_L^2 \leq (1 - \delta)^{\beta - 1}\varepsilon^2 \). Thus we obtain that

\[ E(W^{(\beta)}(T)) \]

\[ \leq (1 - \frac{3}{D})E(Z^{(\beta)}(T)) + c\varepsilon \int_\Omega |\nabla Z^{(\beta)}(T)| \cdot |\nabla V^{(\beta)}(T)| dx \]

\[ + c\varepsilon \int_0^T \int_\Omega \left( |V_t^{(\beta)}|^2 + |\nabla V^{(\beta)}|^2 \right) + \left( |Z_{tt}^{(\beta)}|^2 + |\nabla Z^{(\beta)}|^2 \right) dx dt \]

\[ + 2 \int_0^T \int_\Omega |Z_t^{(\beta)}G^{(\beta)}| + |V_t^{(\beta)}H^{(\beta)}| + |V_t^{(\beta)}G^{(\beta)}| \]

\[ \leq (1 - \frac{3}{D})E(Z^{(\beta)}(T)) + C'(1 - \delta)^{2\beta - 2}\varepsilon^4. \]  

(4.23)

Recalling (4.6) and taking \( \delta > 0 \) small enough such that

\[ 1 - \frac{3}{D} = (1 - \delta)^3, \]

we obtain that

\[ \|Z^{(\beta+1)}(T)\|_{H^1} + \|Z_t^{(\beta+1)}(T)\|_{L^2} \leq (1 - \delta)^{\beta E(Z^{(\beta)}(T))} + C'(1 - \delta)^{2\beta \varepsilon^4}. \]

Taking \( M \) large enough, by (4.20) we can prove the energy estimate of \( Z^{(\beta+1)}(t), \forall t \in [0, T], \)

\[ \|Z^{(\beta+1)}\|_{H^1}^2 + \|Z_t^{(\beta+1)}\|_{L^2}^2 \leq M^2(1 - \delta)^{2\beta+2}\varepsilon^2, \]

by (4.19), we obtain the energy estimate of \( V^{(\beta+1)}, \)

\[ \|V^{(\beta+1)}\|_{H^1}^2 + \|V_t^{(\beta+1)}\|_{L^2}^2 \leq (M^2 + c\varepsilon)(1 - \delta)^{2\beta+2}\varepsilon^2, \]

which is (4.17).

**Step 3:** To prove (4.14) for \( \alpha = \beta + 1 \geq 2, 2 \leq k \leq s - 1 \) assuming (4.14) and (4.15) are valid for \( \beta \); in this case, we need higher order energy estimates. Differentiating the equations (4.5) and (4.6) by \( t \) in \( k - 1 \) times, we get the equations

\[ \partial_t^{k-1}V_{t}^{(\beta+1)} + b_0^{(\beta+1)}\partial_t^{k-1}V_{t}^{(\beta+1)} - \sum_{ij=1}^n a_{ij}^{(\beta+1)}\partial_t^{k-1}V_{x_{ij}}^{(\beta+1)} + \tilde{b}^{(\beta+1)}\partial_t^{k-1}V^{(\beta+1)} \]

\[ + \sum_{i=1}^n b_i^{(\beta+1)}\partial_t^{k-1}V_{x_i}^{(\beta+1)} = -2\chi \cdot \partial_t^{k-1}Z_t^{(\beta+1)} + \partial_t^{k-1}G^{(\beta+1)} + G^{(\beta+1,k-1)}, \]  

(4.24)

and

\[ \partial_t^{k-1}Z_{tt}^{(\beta+1)} - b_0^{(\beta+1)}\partial_t^{k-1}Z_{tt}^{(\beta+1)} - \sum_{ij=1}^n a_{ij}^{(\beta+1)}\partial_t^{k-1}Z_{x_{ij}}^{(\beta+1)} + \tilde{b}^{(\beta+1)}\partial_t^{k-1}Z^{(\beta+1)} \]

\[ + \sum_{i=1}^n b_i^{(\beta+1)}\partial_t^{k-1}Z_{x_i}^{(\beta+1)} = \partial_t^{k-1}H^{(\beta+1)} + H^{(\beta+1,k-1)}, \]  

(4.25)
where
\[
G^{(\beta+1,k)} = b_0^{(\beta+1)} \partial_t^k V_t^{(\beta+1)} - \sum_{i,j=1}^n a_{ij}^{(\beta+1)} \partial_{x,i}^k V_t^{(\beta+1)} + \tilde{b}^{(\beta+1)} \partial_t^k V_t^{(\beta+1)} + \sum_{i=1}^n b_i^{(\beta+1)} \partial_{x,i}^k V_t^{(\beta+1)},
\]
and
\[
H^{(\beta+1,k)} = b_0^{(\beta+1)} \partial_t^k Z_t^{(\beta+1)} - \sum_{i,j=1}^n a_{ij}^{(\beta+1)} \partial_{x,i}^k Z_t^{(\beta+1)} + \tilde{b}^{(\beta+1)} \partial_t^k Z_t^{(\beta+1)} + \sum_{i=1}^n b_i^{(\beta+1)} \partial_{x,i}^k Z_t^{(\beta+1)}.
\]

Due to assumption (4.14) that for \(1 \leq m \leq k\) we have
\[
\begin{align*}
\|V^{(\beta)}\|_{H^m}^2 + \|\partial_t V^{(\beta)}\|_{H^{m-1}}^2 &\leq (1 - \delta)^{2\beta} C_{0,m}^2 \varepsilon^2, \\
\|Z^{(\beta)}\|_{H^m}^2 + \|\partial_t Z^{(\beta)}\|_{H^{m-1}}^2 &\leq (1 - \delta)^{2\beta} M_{l,m}^2 \varepsilon^2.
\end{align*}
\tag{4.26}
\]
By the equations of \(V^{(\beta)}\) and \(Z^{(\beta)}\), for \(1 \leq l + m \leq k\), we have
\[
\begin{align*}
\|\partial_t^l V^{(\beta)}\|_{H^m}^2 + \|\partial_t^{l+1} V^{(\beta)}\|_{H^{m-1}}^2 &\leq (1 - \delta)^{2\beta} C_{l,m}^2 \varepsilon^2, \\
\|\partial_t^l Z^{(\beta)}\|_{H^m}^2 + \|\partial_t^{l+1} Z^{(\beta)}\|_{H^{m-1}}^2 &\leq (1 - \delta)^{2\beta} M_{l,m}^2 \varepsilon^2,
\end{align*}
\]
where the constants \(C_{l,m}, C'_{l,m} \leq CM^{l+m+1}\) with a constant \(C\) independent of \(M, l\) and \(m\).

Making standard energy estimates for \(\partial_t^{k-1} V^{(\beta+1)}\) and \(\partial_t^{k-1} Z^{(\beta+1)}\), we get
\[
\begin{align*}
\|\partial_t^k V^{(\beta+1)}(t)\|_{L^2}^2 + \|\partial_t^{k-1} V^{(\beta+1)}(t)\|_{H^1}^2 &\leq (1 + c \varepsilon) e^{cT} \left( \int_0^T \|\chi \cdot \partial_t^l Z^{(\beta+1)}\|_{L^2}^2 dt + \|\partial_t^k V^{(\beta+1)}(0)\|_{L^2}^2 + \|\partial_t^{k-1} V^{(\beta+1)}(0)\|_{H^1}^2 \
+ 2 \int_0^T \int_{\Omega} |\partial_t^k V^{(\beta+1)}| \cdot |G^{(\beta+1,k-1)} + \partial_t^{k-1} G^{(\beta+1)}| \, dx \, dt \right)
\end{align*}
\]
and
\[
\begin{align*}
\|\partial_t^k Z^{(\beta+1)}(t)\|_{L^2}^2 + \|\partial_t^{k-1} Z^{(\beta+1)}(t)\|_{H^1}^2 &\leq (1 + c \varepsilon) e^{cT} \left( \|\partial_t^l Z^{(\beta+1)}(T)\|_{L^2}^2 + \|\partial_t^{k-1} Z^{(\beta+1)}(T)\|_{H^1}^2 \
+ 2 \int_0^T \int_{\Omega} |\partial_t^k Z^{(\beta+1)}| \cdot \|H^{(\beta+1,k-1)} + \partial_t^{k-1} H^{(\beta+1)}\|_{L^2} \, dx \, dt \right).
\end{align*}
\]
Next, we only need to estimate \(\partial_t^l Z^{(\beta+1)}(T)\) and \(\partial_t^{k-1} Z^{(\beta+1)}(T)\). Just as stated in Step 2, it suffices to consider
\[
2 \left[ (\partial_t^k V^{(\beta)}, \partial_t^k Z^{(\beta)})_{L^2} + (\partial_t^{k-1} V^{(\beta)}, \partial_t^{k-1} Z^{(\beta)})_{H^1} \right]
\]
and \(E(\partial_t^{k-1} V^{(\beta)})\). Similarly with \((4.23)\), combining with the following observability inequality in Theorem 4.3
\[
E(\partial_t^{k-1} Z^{(\beta)}(T)) \leq D \int_0^T \int_{\omega} |\partial_t^k Z^{(\beta)}|^2 \, dx \, dt + C(1 - \delta)^{2\beta-2} \varepsilon^4,
\]
we can obtain that
\[ \left\| \partial_t^k W^{(\beta)}(T) \right\|_{L^2}^2 + \left\| \partial_t^{k-1} W^{(\beta)}(T) \right\|_{H^1}^2 \]
\[ \leq (1 - \delta)^3 \left( \left\| \partial_t^k Z^{(\beta)}(T) \right\|_{L^2}^2 + \left\| \partial_t^{k-1} Z^{(\beta)}(T) \right\|_{H^1}^2 \right) \]
\[ + 2 \left( \left\| \partial_t^k V^{(\beta)}(0) \right\|_{L^2}^2 \left\| \partial_t^{k-1} Z^{(\beta)}(0) \right\|_{L^2} + \left\| \partial_t^{k-1} V^{(\beta)}(0) \right\|_{H^1} \left\| \partial_t^{k-1} Z^{(\beta)}(0) \right\|_{H^1} \right) \]
\[ + \left( \left\| \partial_t^k V^{(\beta)}(0) \right\|_{L^2}^2 + \left\| \partial_t^{k-1} V^{(\beta)}(0) \right\|_{H^1}^2 \right) + C_k (1 - \delta)^{2\beta + 2\varepsilon^2} \]

Then we should derive \( \partial_t^{k-1} V^{(\beta)}(0) \) from the equations. The equation of \( V^{(\beta)} \) can be written as
\[ \partial_t^2 V^{(\beta)} = -2\partial_t V^{(\beta)} - V^{(\beta)} + \Delta V^{(\beta)} + 2\chi \cdot \partial_t Z^{(\beta)} + V^{(\beta)}_{\text{error}}, \]
where \( \left\| \partial_t^k V^{(\beta)}_{\text{error}} \right\|_{H^m} = O((1 - \delta)^\beta \varepsilon^2), m + l \leq k - 1 \). Differentiating by \( t \) for \( k - 2 \) times, we get
\[ \partial_t^k V^{(\beta)}(\cdot) = \begin{cases} \Delta \frac{k}{2} V^{(\beta)} - \sum_{l=1}^{k-1} \partial_t^{k-2l-2} \Delta^l (2\chi \cdot Z^{(\beta)} - 2\partial_t V^{(\beta)} - V^{(\beta)}) + V^{(\beta)}_{\text{error}}, & \text{when } k \text{ is even} \\ \partial_t \Delta \frac{k}{2} V^{(\beta)} - \sum_{l=1}^{k-1} \partial_t^{k-2l-2} \Delta^l (2\chi \cdot Z^{(\beta)} - 2\partial_t V^{(\beta)} - V^{(\beta)}) + V^{(\beta)}_{\text{error}}, & \text{when } k \text{ is odd}. \end{cases} \]
(4.27)

Noting that \( \partial_t \Delta^l V^{(\beta)}(0) = 0, \Delta^l V^{(\beta)}(0) = 0 \), so we have
\[ \left\| \partial_t^k V^{(\beta)}(0) \right\|_{L^2}^2 + \left\| \partial_t^{k-1} V^{(\beta)}(0) \right\|_{H^1}^2 \leq (1 - \delta)^2 \varepsilon^2 \left( \sum_{l=0}^{k-1} \left( C_{k-l-1,l} + 3C'_{k-l-1,l} \right) \right)^2. \]

Similarly, we need to derive the relation between \( \partial_t^k W^{(\beta)}(T) \) and \( \partial_t^k Z^{(\beta+1)}(T) \). By the equation of \( W^{(\beta)} \) we get
\[ \partial_t^k W^{(\beta)} = \begin{cases} \Delta \frac{k}{2} W^{(\beta)} - \sum_{l=1}^{k-1} \partial_t^{k-2l-2} \Delta^l (2\chi \cdot Z^{(\beta)} - 2\partial_t V^{(\beta)} - V^{(\beta)} + 2\partial_t Z^{(\beta)} + Z^{(\beta)}) + W_{\text{error}}, & \text{when } k \text{ is even} \\ \partial_t \Delta \frac{k}{2} W^{(\beta)} - \sum_{l=1}^{k-1} \partial_t^{k-2l-2} \Delta^l (2\chi \cdot Z^{(\beta)} - 2\partial_t V^{(\beta)} - V^{(\beta)} + 2\partial_t Z^{(\beta)} + Z^{(\beta)}) + W_{\text{error}}, & \text{when } k \text{ is odd}. \end{cases} \]
(4.28)

Here \( \left\| \partial_t^k W^{(\beta)}_{\text{error}} \right\|_{H^m} = O((1 - \delta)^\beta \varepsilon^2) \).

Noting that
\[ \Delta^l W^{(\beta)}(T) = \Delta^l Z^{(\beta+1)}(T), \quad \partial_t \Delta^l W^{(\beta)}(T) = \partial_t \Delta^l Z^{(\beta+1)}(T), \]
we have
\[
\| \partial_t^k W^{(\beta)}(T) \|^2_{L^2} + \| \partial_t^{k-1} W^{(\beta)}(T) \|^2_{H^1} \\
\geq \frac{1}{1 + \delta} \left( \| \partial_t Z^{(\beta+1)}(T) \|^2_{H^{k-1}} + \| Z^{(\beta+1)}(T) \|^2_{H^k} \right) \\
- \frac{8}{\delta} \left( \sum_{l=1}^{\frac{k}{2}-1} \| \partial_t^{k-2l-2} (\chi \cdot Z^{(\beta)} - 2\partial_t V^{(\beta)} - V^{(\beta)} + 2\partial_t Z^{(\beta)} + Z^{(\beta)})(T) \|^2_{H^{2l}} \right) \\
+ \left( \sum_{l=1}^{\frac{k}{2}-1} \| \partial_t^{k-2l-3} (\chi \cdot Z^{(\beta)} - 2\partial_t V^{(\beta)} - V^{(\beta)} + 2\partial_t Z^{(\beta)} + Z^{(\beta)})(T) \|^2_{H^{2l+1}} \right).
\]
Hence
\[
\| \partial_t Z^{(\beta+1)}(T) \|^2_{H^{k-1}} + \| Z^{(\beta+1)}(T) \|^2_{H^k} \\
\leq (1 - \delta)^2 (1 - \delta^2) \left( \| \partial_t Z^{(\beta)}(T) \|^2_{H^{k-1}} + \| Z^{(\beta)}(T) \|^2_{H^k} \right) \\
+ (1 - \delta)^{2\beta+2} A_{k,\delta} + C'(1 - \delta)^{2\beta+2} \varepsilon^3 \\
\leq (1 - \delta)^{2\beta+2} \varepsilon^2 \left[ M^{2k}(1 - \delta^2) + C' \varepsilon + A_{k,\delta} \right],
\]
where
\[
A_{k,\delta} = \frac{2(1 + \delta)}{(1 - \delta)^2} \left[ \left( \sum_{l=0}^{k-1} (C_{k-l-1,l} + 3C'_{k-l-1,l}) \right)^2 + 2C_{k,0} \sum_{l=0}^{k-1} (C_{k-l-1,l} + 3C'_{k-l-1,l}) \right] \\
+ \frac{8C(1 + \delta)}{\delta (1 - \delta)^2} \left[ \sum_{l=0}^{k-1} (C_{k-l-1,l} + C'_{k-l-1,l}) \right]^2,
\]
and $C_\chi > 0$ is a constant only depending on $\chi$.

Taking $M$ large enough such that
\[
\delta^{-4} \ll M \ll \varepsilon^{-1},
\]
we obtain that
\[
M^{2k}(1 - \delta^2) + C' \varepsilon + A_{k,\delta} \ll M^{2k}(1 - \delta^4),
\]
and
\[
\| \partial_t Z^{(\beta+1)}(T) \|^2_{H^{k-1}} + \| Z^{(\beta+1)}(T) \|^2_{H^k} \leq (1 - \delta^4)(1 - \delta)^{2\beta+2} M^{2k} \varepsilon^2.
\]
By the higher order energy estimate of $Z^{(\beta+1)}$ as well as the equations, we get
\[
\| \partial_t Z^{(\beta+1)}(t) \|^2_{H^{k-1}} + \| Z^{(\beta+1)}(t) \|^2_{H^k} \leq (1 - \delta)^{2\beta+2} M^{2k} \varepsilon^2.
\]
Then by the higher order energy estimate of $V^{(\beta+1)}$ as well as the equations, we get
\[
\| \partial_t V^{(\beta+1)} \|^2_{H^{k-1}} + \| V^{(\beta+1)} \|^2_{H^k} \leq (1 - \delta)^{2\beta+2} C_{0,k} \varepsilon^2.
\]
Using induction for $\beta = \alpha + 1$ for each fixed $k \in \mathbb{N}$, then using induction for $k \leq s - 1$, we obtain (47.10). So when $\alpha \rightarrow \infty$, $(v^{(\alpha)}, v_t^{(\alpha)})$ and $(z^{(\alpha)}, z_t^{(\alpha)})$ converge to some $(v, v_t)$ and $(z, z_t)$ in $L^\infty(0, T; H^{s-1}) \times L^\infty(0, T; H^{s-2})$. 
Step 4: To prove (4.13) for \( \alpha \geq 2 \).
When \( k \leq s - 2 \), (4.13) follows from (4.14) and \((v^{(0)}, z^{(0)}) = (0, 0)\) directly. In fact, we observe that

- when \( \alpha = 1 \), \( v^{(1)} = V^{(1)} \);
- when \( \alpha \geq 2 \), we can check that
  \[
  \|v^{(\alpha)}\|_{H^{k}_t}^2 = \|\sum_{\beta=1}^{\alpha} V^{(\alpha)}\|_{H^{k}_t}^2 \lesssim \sum_{\beta=1}^{\alpha} \|V^{(\alpha)}\|_{H^{k}_t}^2 \leq \frac{(1-\delta)^2 - (1-\delta)^{2\alpha}}{1-(1-\delta)^2} C_k^2 \varepsilon^2.
  \]
Noting that \( \frac{(1-\delta)^2 - (1-\delta)^{2\alpha}}{1-(1-\delta)^2} \leq C_0 \) for any \( \delta \in (0, 1) \), we can obtain (4.8).

It remains to estimate \( v^{(\alpha)} \) and \( z^{(\alpha)} \). Differentiating the equations (4.2) and (4.6) by \( t \) for \( s-1 \) times, we obtain that

\[
\partial_t^{s+1} v^{(\alpha)} + b_0^{(\alpha)} \partial_t^s v^{(\alpha)} - \sum_{i,j=1}^{n} a_{ij}^{(\alpha)} \partial_t^{s-1} v^{(\alpha)}_{x_i, x_j} + b_0^{(\alpha)} \partial_t^{s-1} v^{(\alpha)} + \sum_{i=1}^{n} b_i^{(\alpha)} \partial_t^{s-1} v^{(\alpha)}_{x_i} + 2\chi \cdot \partial_t^{s-1} z^{(\alpha)} = g^{(\alpha)},
\]
and

\[
\partial_t^{s+1} z^{(\alpha)} - b_0^{(\alpha)} \partial_t^s z^{(\alpha)} - \sum_{i,j=1}^{n} a_{ij}^{(\alpha)} \partial_t^{s-1} z^{(\alpha)}_{x_i, x_j} + \tilde{b}_0^{(\alpha)} \partial_t^{s-1} z^{(\alpha)} + \sum_{i=1}^{n} b_i^{(\alpha)} \partial_t^{s-1} z^{(\alpha)}_{x_i} = h^{(\alpha)},
\]
where

\[
g^{(\alpha)} = b_0^{(\alpha)} \partial_t^s v^{(\alpha)} - \sum_{i,j=1}^{n} a_{ij}^{(\alpha)} \partial_t^{s-1} v^{(\alpha)}_{x_i, x_j} + \tilde{b}_0^{(\alpha)} \partial_t^{s-1} v^{(\alpha)} + \sum_{i=1}^{n} b_i^{(\alpha)} \partial_t^{s-1} v^{(\alpha)}_{x_i},
\]
and

\[
h^{(\alpha)} = -b_0^{(\alpha)} \partial_t^s z^{(\alpha)} - \sum_{i,j=1}^{n} a_{ij}^{(\alpha)} \partial_t^{s-1} z^{(\alpha)}_{x_i, x_j} + \tilde{b}_0^{(\alpha)} \partial_t^{s-1} z^{(\alpha)} + \sum_{i=1}^{n} b_i^{(\alpha)} \partial_t^{s-1} z^{(\alpha)}_{x_i}.
\]

Then we make standard energy estimates for \( v^{(\beta+1)} \) and \( z^{(\beta+1)} \) to obtain

\[
\|\partial_t^{s} v^{(\alpha)}(t)\|_{L^2}^2 + \|\partial_t^{s-1} v^{(\alpha)}(t)\|_{H^1}^2 \leq (1 + c\varepsilon) e^{c\varepsilon T} \left( \int_0^T \|\chi \cdot \partial_t^s z^{(\alpha)}\|_{L^2}^2 dt \right) + \int_0^T \int_{\Omega} |g^{(\alpha)}| \partial_t^{s} v^{(\alpha)}(x) dx dt,
\]
and

\[
\|\partial_t^{s} z^{(\alpha)}(t)\|_{L^2}^2 + \|\partial_t^{s-1} z^{(\alpha)}(t)\|_{H^1}^2 \leq (1 + c\varepsilon) e^{c\varepsilon T} \left( \|\partial_t^s z^{(\alpha)}(T)\|_{L^2}^2 + \|\partial_t^{s-1} z^{(\alpha)}(T)\|_{H^1}^2 \right) + \int_0^T \int_{\Omega} |h^{(\alpha)}| \partial_t^{s} z^{(\alpha)}(x) dx dt.
\]
we have

\[ \left( \partial_t^s v^{(a)}, \partial_t z^{(a)} \right)_{L^2} + \left( \partial_t^{s-1} v^{(a)}, \partial_t^{s-1} z^{(a)} \right)_{H^1} \]

and add it to the energy estimate of \( \partial_t^{s-1} v^{(a)} \). Applying Theorem 4.3 to the system of \( z^{(a)} \), we have

\[ E(\partial_t^{s-1} z^{(a)}(T)) \leq D \int_0^T \int_\Omega |\partial_t^s z^{(a)}|^2 \, dx \, dt + C\varepsilon^4. \]

In view of (4.6), we denote \( w^{(a)} = v^{(a)} + z^{(a)} \), thus we obtain that

\[
\begin{align*}
&\|\partial_t^s w^{(a)}(T)\|_{L^2}^2 + \|\partial_t^{s-1} w^{(a)}(T)\|_{H^1}^2 \\
&\leq (1 - \delta)^3 \left( \|\partial_t^s z^{(a)}(T)\|_{L^2}^2 + \|\partial_t^{s-1} z^{(a)}(T)\|_{H^1}^2 \right) \\
&+ 2 \left( \|\partial_t^s v^{(a)}(0)\|_{L^2} + \|\partial_t^{s-1} v^{(a)}(0)\|_{H^1} \right) \|\partial_t^{s-1} z^{(a)}(0)\|_{H^1} + \frac{C'}{\delta^2} \varepsilon^3.
\end{align*}
\]

Using the equations of \( v^{(a)} \) and \( z^{(a)} \) to derive the relationship between \( \partial_t^{2k} v^{(a)} \) and \( \Delta^k z^{(a)} \), and the relationship between \( \partial_t^{2k} z^{(a)} \) and \( \Delta^k z^{(a)} \), we obtain that

\[
\begin{align*}
&\|\partial_t z^{(a+1)}(T)\|_{H^{s-1}}^2 + \|z^{(a+1)}(T)\|_{H^s}^2 \\
&\leq (1 - \delta)^2 (1 - \delta^2) \left( \|\partial_t z^{(a)}(T)\|_{H^{s-1}}^2 + \|z^{(a)}(T)\|_{H^s}^2 \right) + \frac{\varepsilon^2}{\delta^2} (C'\varepsilon + A_{s,\delta}) \\
&\leq \varepsilon^2 \left( (1 - \delta)^2 (1 - \delta^2) M^{2s} + \frac{C'\varepsilon}{\delta^2} + \frac{A_{s,\delta}}{\delta^2} \right),
\end{align*}
\]

where we have

\[
(1 - \delta)^2 (1 - \delta^2) M^{2s} + \frac{C'\varepsilon}{\delta^2} + \frac{A_{s,\delta}}{\delta^2} \leq (1 - \delta)^2 (1 - \delta^4) M^{2s}.
\]

Thus we obtain the estimate of \( z^{(a+1)} \) in (4.15), and by the equation of \( v^{(a+1)} \), we obtain the estimate of \( v^{(a+1)} \).

5 Proof of Theorem 4.3

We consider the local null controllability problem for the fully nonlinear damped wave equations

\[
\begin{align*}
y_{tt} + 2y_t - \Delta y + y = F(y, y_t, \nabla y, \nabla^2 y) + \chi \cdot u, & \quad (t, x) \in (0, T) \times \Omega, \\
y(t, x) = 0, & \quad (t, x) \in (0, T) \times \partial \Omega, \\
y(0, x) = y_0, \ y_t(0, x) = y_1, & \quad x \in \Omega,
\end{align*}
\]

where \( \chi \in C^\infty(\Omega) \) satisfies \( 0 \leq \chi(x) \leq 1 \), \( \chi|_{\partial \Omega} \equiv 1 \), and \( \chi \) supports in a neighbourhood of \( \omega \), with \( \omega \subset \Omega \cap O_{\varepsilon_0}(\partial \Omega) \), and

\[ F(\lambda) = O(|\lambda|^2), \]

(5.2)
with
\[ \lambda = \left( \lambda', \lambda_0, \lambda_i(i = 1, \ldots, n), \lambda_{ij}(i, j = 1, \ldots, n) \right). \]

Our main result can be restated as follows.

**Theorem 5.1.** Assume that \( \varepsilon = \|y_0\|_{H^s} + \|y_1\|_{H^{s-1}} \) is sufficiently small, for some integer \( s > n + 1 \), and \((T, \omega)\) satisfy the following condition
\[ T \geq 2 \max_{x \in \Omega} |x - x_0|, \quad \Gamma_0 := \left\{ x \in \partial \Omega \mid (x - x_0) \cdot n > 0 \right\}, \quad \omega = \Omega \cap O_{\varepsilon_0}(\Gamma_0) \]
for some \( \varepsilon_0 > 0 \) and \( x_0 \notin \Omega \). Then there exists a function \( u(t, x) \) in (5.1) such that
\[ y_t(T) = 0, \quad y_{tt}(T) = 0. \]

**Remark 5.2.** Our nonlinear term \( F \) is independent of \( \nabla y_t \), this is mainly for simplicity, otherwise we need to deal with \( v_{tx_i} \) terms, and \( z_{tx_i} \) terms in the dual system. But under the assumption of \((T, \omega)\) and \( \varepsilon \ll 1 \), the observability inequality might also be right.

**Remark 5.3.** Recall that in the second section, we let \( y = e^t \tilde{y} \) to reduce a classical linear wave equation to a damped one. Similarly, for a classical nonlinear wave equation, we can use the same method to reduce it to (5.1).

Note that our goal is
\[ y_t(T) = 0, \quad y_{tt}(T) = 0, \]
rather than
\[ y(T) = 0, \quad y_t(T) = 0. \]

Because fully nonlinear equations can’t be solved directly. We should differentiate the equation by \( t \), and transform it to a quasi-linear equation, so our goal of control is for \((y_t, y_{tt})\) instead of \((y, y_t)\). There might be extra difficulties if we want to control \((y, y_t)\). This is a kind of new phenomenon that we discover in studying fully nonlinear equations.

**Proof.** Let \( v = y_t \), we have
\[ -\Delta y + y = F(y, v, \nabla y, \nabla^2 y) - v_t - 2v + \chi \cdot u \quad (5.3) \]
differentiate the equation by \( t \), we get
\[ v_{tt} + b_0 v_t - \sum_{i,j=1}^n a_{ij} v_{x_i x_j} = \tilde{b} v + \sum_{i=1}^n b_i v_{x_i} + \chi \cdot u_t, \quad (5.4) \]
where
\[ b_0 = 2 - \partial_y F(y, v, \nabla y, \nabla^2 y), \quad a_{ij} = \delta_{ij} + \partial_{y_i x_j} F(y, v, \nabla y, \nabla^2 y), \]
\[ b_i = \partial_{y_i} F(y, v, \nabla y, \nabla^2 y), \quad \tilde{b} = \partial_y F(y, v, \nabla y, \nabla^2 y) - 1. \]
By the method similar as that in Section 4 for \( v = y_t \), we can check that \( v \) satisfies

\[
\begin{align*}
  v_{tt} + b_0v_t - \sum_{i,j=1}^n a_{ij}v_{x_i x_j} &= \tilde{b}v + \sum_{i=1}^n b_i v_{x_i} - 2\chi \cdot z_t, & (t, x) \in (0, T) \times \Omega, \\
  v(t, x) &= 0, & (t, x) \in (0, T) \times \partial \Omega, \\
  v(0, x) &= y_1, \quad v_t(0, x) = -2y_1 + \Delta y_0 - y_0 + F(y_0, y_1, \nabla y_0, \nabla^2 y_0), & x \in \Omega, \\
  v(T, x) &= 0, \quad v_t(T, x) = 0, & x \in \Omega.
\end{align*}
\]

(5.5)

Denote \( \tilde{v} = y_t \), differentiating the equation of \( y \) by \( t \), we get

\[
v_{tt} + 2v_t - \Delta \tilde{v} + \tilde{v} = F_y \cdot \tilde{v} + F_v \cdot v_t + \sum_{i=1}^n F_{y_{x_i}} \tilde{v}_{x_i} + \sum_{i,j=1}^n F_{y_{x_i x_j}} \tilde{v}_{x_i x_j} - 2\chi \cdot z_t,
\]

which can also be written as

\[
v_{tt} + b_0v_t - \sum_{i,j=1}^n a_{ij} \tilde{v}_{x_i x_j} = \tilde{b}v + \sum_{i=1}^n b_i \tilde{v}_{x_i} - 2\chi \cdot z_t.
\]

Subtracting from (5.5), we get

\[
\begin{align*}
  \sum_{i,j=1}^n a_{ij} (v - \tilde{v})_{x_i x_j} + \sum_{i=1}^n b_i (v - \tilde{v})_{x_i} + \tilde{b}(v - \tilde{v}) &= 0, & x \in \Omega, \\
  v - \tilde{v} &= 0, & x \in \partial \Omega.
\end{align*}
\]

Noting that \( a_{ij}, \ b_i, \ \tilde{b} \) are functions of \( y \) and \( v \), this is a linear equation of \( v - \tilde{v} \). To prove \( v = \tilde{v} \), we multiply the equation by \( -v + \tilde{v} \) and make an integration by parts, then we get

\[
\int_{\Omega} \sum_{i,j=1}^n a_{ij} (v - \tilde{v})_{x_i} (v - \tilde{v})_{x_j} \, dx = \int_{\Omega} \sum_{i=1}^n \left( b_i - \partial_{x_j} a_{ij} \right) (v - \tilde{v}) (v - \tilde{v})_{x_i} + \tilde{b}(v - \tilde{v})^2 \, dx.
\]

Noting that \( |b_i - \partial_j a_{ij}| + |\tilde{b} + 1| + |a_{ij} - \delta_{ij}| = O(\varepsilon) \), we have

\[
\int_{\Omega} |\nabla (v - \tilde{v})|^2 \, dx \leq \frac{3\varepsilon - 1}{1 - 2\varepsilon} \int_{\Omega} |v - \tilde{v}|^2 \, dx.
\]

Taking \( \varepsilon \) small enough such that \( c\varepsilon < \frac{1}{3} \), hence we get \( v = \tilde{v} = y_t \), satisfying

\[
y_t(T) = 0, \quad y_{tt}(T) = 0.
\]

Then

\[
u(t) = -2\chi [z(t) - z(0)]
\]

is the desired control function.
A Proof of Theorem 4.3

Let

\[ \Omega_T := (0, T) \times \Omega, \quad \Gamma_T := (0, T) \times \partial \Omega. \]

Consider the following linear hyperbolic system

\[
\begin{split}
& z_{tt} + b_0 z_t - \sum_{i,j=1}^{n} (a^{ij} z_{x_i})_{x_j} + \sum_{k=1}^{n} b_k z_{x_k} + \tilde{b} z = 0, \quad (t, x) \in \Omega_T, \\
& z(t, x) = 0, \quad (t, x) \in \Gamma_T, \\
& z(0, x) = z_0, \quad z_t(0, x) = z_1 \quad x \in \Omega
\end{split}
\]

(A.1)

with

\[ a^{ij} - \delta_{ij}, \quad b_0 - 2, \quad b_k, \quad \tilde{b} - 1 \in X_{C,s,\varepsilon}, \]

(A.2)

where

\[ X_{C,s,\varepsilon} = \left\{ f \in L^\infty(0, T; L^2(\Omega)) : \| \partial_t^j \nabla^k f \|_{L^2(\Omega)} \leq C\varepsilon, \quad \forall \ j, k \in \mathbb{N}, \ j + k \leq s, \ \forall \ t \in [0, T] \right\} \]

for some integer \( s > n. \)

In this section, we prove Theorem 4.3, i.e. the observability inequality

\[
\| z_1 \|^2_{L^2(\Omega)} + \| z_0 \|^2_{H^1(\Omega)} \leq D \int_0^T \int_\omega |z_t|^2 \, dx \, dt
\]

(A.3)

under the following assumptions.

Assumption 1.1. There is a positive function \( \psi \in C^3(\overline{\Omega}) \) such that

\[
\min_{x \in \Omega} |\nabla \psi(x)| > 0
\]

and for some \( \mu_0 > 4, \)

\[
\sum_{j,k=1}^{n} \sum_{j',k'=1}^{n} \left[ 2a^{jk}(a^{j'k'}\psi_{x_{j'}})_{x_{j'}} \right] \xi^j \xi^k \geq \mu_0 \sum_{j,k=1}^{n} a^{jk} \xi^j \xi^k, \quad \forall (t, x, \xi) \in \overline{\Omega_T} \times \mathbb{R}^n.
\]

Assumption 1.2. Let

\[
\Gamma_t = \left\{ x \in \partial \Omega : \sum_{j,k=1}^{n} a^{jk}(t, x)\psi_{x_j}n^k > 0 \right\},
\]

(A.4)

and for \( \delta > 0, \)

\[
O_\delta(\Gamma_t) = \{ x \in \mathbb{R}^n : d(x, \Gamma_t) < \delta \},
\]

we have

\[
\omega = \left( \bigcup_{\ell \geq 0} O_\delta(\Gamma_t) \right) \cap \Omega.
\]

(A.5)

Remark 1.3. In view of (A.2), Assumption 1.1 is satisfied, and Assumption 1.2 can be replaced by the \( \Gamma \)-condition in Theorem 1.2 see [13].
A.1 A fundamental weighted identity

Firstly, we need a fundamental identity for the operator

\[ \mathcal{P} = \sum_{j,k=1}^{m} \partial_k(\tilde{a}^{jk}\partial_j) \]

with \( \tilde{a}^{jk} = \tilde{a}^{kj} \in C^1(\mathbb{R}^{1+m}) \). This result can be found in [12], see also [13].

**Theorem 1.4.** Let \( z \in C^2(\mathbb{R}^{1+m}) \), \( l \in C^1(\mathbb{R}^{1+m}) \), \( \Psi \in C^1(\mathbb{R}^{1+m}) \), \( \theta = e^l \) and \( v = \theta z \). It holds that

\[ \theta I_1 \mathcal{P} z + \nabla \cdot V = |I_1|^2 + B|v|^2 - \sum_{j,k=1}^{m} \tilde{a}^{jk} \Psi_{x_j} v_{x_k} v + \sum_{j,k=1}^{m} c^{jk} v_{x_j} v_{x_k} , \]

where

\[
\left\{
\begin{align*}
I_1 &= \sum_{j,k=1}^{m} (\tilde{a}^{jk} v_{x_j}) x_k + A v \\
A &= \sum_{j,k=1}^{m} \tilde{a}^{jk} l_{x_j} l_{x_k} - \sum_{j,k=1}^{m} (\tilde{a}^{jk} v_{x_j}) x_k - \Psi \\
B &= \sum_{j,k=1}^{m} (\tilde{a}^{jk} l_{x_j} A) x_k + A \Psi
\end{align*}
\right.
\]

and

\[
\left\{
\begin{align*}
V &= (V^1, \cdots, V^m) \\
V^k &= \sum_{j=1}^{m} \left\{ \sum_{j',k'=1}^{m} (2\tilde{a}^{jk'} \tilde{a}^{j'k} - \tilde{a}^{jk} \tilde{a}^{j'k'}) l_{x_j} v_{x_{j'}} v_{x_k} - \tilde{a}^{jk} \left[ \Psi v_{x_j} v - Al_{x_j} |v|^2 \right] \right\} \\
c^{jk} &= \sum_{j',k'=1}^{m} \left[ 2\tilde{a}^{jk'} (\tilde{a}^{j'k} l_{x_{j'}}) v_{x_{j'}} - (\tilde{a}^{jk} \tilde{a}^{j'k'} l_{x_{j'}}) x_{k'} \right] - \tilde{a}^{jk} \Psi
\end{align*}
\right.
\]

**Corollary 1.5.** Let \( z \in C^2(\mathbb{R}^{1+n}) \), \( l \in C^3(\mathbb{R}^{1+n}) \), and

\[ \left\| \partial_j' \nabla^{k'} (a^{jk}(t,x) - \delta_{jk}) \right\|_{L^2(\Omega)} \leq C\varepsilon, \quad \forall \ t \in \mathbb{R} \]

for \( j' + k' \leq s - 1 \) and \( s > n + 1 \). If we set

\[
\left\{
\begin{align*}
v(t,x) &= \theta z, \quad \theta(t,x) = e^{l(t,x)}, \quad l(t,x) = \lambda \phi(t,x), \\
\phi(t,x) &= \psi(x) - c_1(t - T/2)^2, \\
\Psi(t,x) &= -\lambda \left[ \sum_{j,k=1}^{n} (a^{jk} \psi_{x_j}) x_k - 2c_1 - c_0 \right]
\end{align*}
\right.
\]
for constants $c_0, c_1 \in (0, 1)$ and $\lambda > 0$, then we have
\[
e^{2\lambda \phi}|z_t - \sum_{j,k=1}^n (a^{jk} z_{x_j})_{x_k}|^2 + 2\nabla \cdot V + 2M_t \geq 2v_t^2 [l_t + \Psi + \sum_{j,k=1}^n (a^{jk} l_{x_j})_{x_k}] - \sum_{j,k=1}^n (a^{jk} l_{tx_j} + (a^{jk} l_{x_j})_t) v_t v_{x_k} + 2 \sum_{j,k=1}^n c^{jk} v_j v_k - 2 \sum_{j,k=1}^n a^{jk} \Psi v v_{x_k} + 2 \Psi_t v v_t + 2B|v|^2,
\]
where
\[
A = \sum_{j,k=1}^n (a^{jk} l_{x_j} l_{x_k} - a^{jk} l_{x_j} - a^{jk} l_{x_j,x_k}) - l_t^2 + l_{tt} - \Psi
\]
\[
B = A\Psi - (A l_t)_t + \sum_{j,k=1}^n (A a^{jk} l_{x_j,x_k})
\]
\[
c^{jk} = \sum_{j',k'=1}^n \left[ 2a^{jk'} (a^{j'k} l_{x_{j'}})_{x_{k'}} - (a^{jk} a^{j'k'} l_{x_{j'}})_{x_{k'}} \right] - a^{jk} (l_{tt} - \Psi) + a_t^{jk} l_t
\]
and
\[
V = (V^1, \ldots, V^n)
\]
\[
V^k = 2 \sum_{j,j',k'=1}^n a^{jk} a^{j'k'} l_{x_j} v_{x_j} v_{x_{k'}} + \sum_{j=1}^n a^{jk} A_{x_j} v^2 - \Psi v \sum_{j=1}^n a^{jk} v_{x_j} - \sum_{j,j',k'=1}^n a^{jk} a^{j'k'} l_{x_j} v_{x_j} v_{x_{k'}} - 2l_t v_t \sum_{j=1}^n a^{jk} v_{x_j} + \sum_{j=1}^n a^{jk} l_{x_j} v_t^2
\]
\[
M = l_t \left( v_t^2 + \sum_{j,k=1}^n a^{jk} v_j v_{x_k} \right) - 2 \sum_{j,k=1}^n a^{jk} l_{x_j} v_{x_k} v_t + \Psi v v_t - A l_t v^2
\]

**Proof.** Using Theorem [14] with $m = n + 1$ and
\[
(\tilde{a}^{jk})_{m \times m} = \begin{pmatrix} -1 & 0 \\ 0 & (a^{jk})_{n \times n} \end{pmatrix}
\]
Noting that
\[
2\theta I_1 \mathcal{P}_z \leq \theta^2 |\mathcal{P}_z|^2 + |I_1|^2,
\]
we obtain the desired inequality by direct calculations. \hfill \square

### A.2 Carleman estimate in $H^1$-norm

The second step is to derive a Carleman estimate for the following hyperbolic equation in $H^1$-norm.
\[
\begin{align*}
z_{tt} - \sum_{j,k=1}^n (a^{jk} z_{x_j})_{x_k} &= F, \quad (t, x) \in \Omega_T, \\
z(t, x) &= 0, \quad (t, x) \in \Gamma_T, \\
z(0, x) &= z_0, \quad z_t(0, x) = z_1 \quad x \in \Omega. \tag{A.8}
\end{align*}
\]
Note that if $\psi \in C^3(\Omega)$ satisfies Assumption 1.1, then for any given $a \geq 1$ and $b \in \mathbb{R}$, the function

$$\tilde{\psi}(x) = a\psi(x) + b$$

also satisfies Assumption 1.1. So we can assume that

$$\frac{1}{4} \sum_{j,k=1}^{n} a^{jk}(t, x)\psi_{x_j}\psi_{x_k} \geq \max_{x \in \Omega} \psi(x) \geq \min_{x \in \Omega} \psi(x) \geq 0, \quad \forall \ x \in \overline{\Omega}, \ t \in \mathbb{R}. \quad (A.9)$$

Let

$$R_1 = \max_{x \in \Omega} \sqrt{\psi(x)}, \quad T_0 = 2 \inf \{R_1 : \psi \text{ satisfies (A.9)}\},$$

we have the following boundary Carleman estimate for (A.8).

**Theorem 1.6.** Let Assumption 1.1 holds and $\tilde{\Gamma} := \bigcup_{t \geq 0} \Gamma_t$ with $\Gamma_t$ given in (A.4). Then there exists a constant $\lambda_0 > 0$ such that for $\forall T > T_0$ and $\lambda > \lambda_0$, there exist $c_2 > 0$ and $C > 0$ such that each solution $z \in H^1(\Omega_T)$ to (A.8) satisfies that

$$\int_{\Omega_T} \theta^2 \left( \lambda \psi^2 + |\nabla z|^2 \right) dx dt \leq C \left( \int_{\Omega_T} \theta^2 |F|^2 dx dt + \lambda \int_{0}^{T} \int_{\tilde{\Gamma}} \theta^2 |\partial z / \partial n|^2 dS dt \right), \quad (A.10)$$

where $E(t) = \int_{\Omega} (z_t^2 + |\nabla z|^2) dx$.

**Proof.** The proof is divided into several steps, while the latter parts are highly similar to the Chapter 4 of [13]. So we just point out the differences.

By the definitions in (A.6), we can easily see that

$$l_t + \sum_{j,k=1}^{n} (a^{jk}t_{x_j})_{x_k} + \Psi = -2\lambda c_1 + \lambda \sum_{j,k=1}^{n} (a^{jk}\psi_{x_j})_{x_k} + \Psi = \lambda c_0 > 0.$$

By (A.6) and (A.7), we have

$$\sum_{j,k=1}^{n} c^{jk}v_{x_j}v_{x_k} \geq \lambda \mu_0 \sum_{j,k=1}^{n} a^{jk}v_{x_j}v_{x_k} - (4c_1 + c_0)\lambda \sum_{j,k=1}^{n} a^{jk}v_{x_j}v_{x_k} - c_1(2t - T) \sum_{j,k=1}^{n} a^{jk}v_{x_j}v_{x_k} \geq \lambda (\mu_0 - 4c_1 - c_0 - C\epsilon) \sum_{j,k=1}^{n} a^{jk}v_{x_j}v_{x_k},$$

and

$$A = \lambda^2 \left[ \sum_{j,k=1}^{n} a^{jk}\psi_{x_j}\psi_{x_k} - c_1^2 (2t - T)^2 \right] + O(\lambda).$$
Hence we get
\[ B = A\Psi - (At)x + \sum_{j,k=1}^{n} (Aa_{jk}t_{x_j})_{x_k} \]
\[ = \lambda^3 [ (4c_1 + c_0) \sum_{j,k=1}^{n} a^{jk}\psi_{x_j}\psi_{x_k} - (8c_1 + c_0)c_1^2(2t-T)^2 \]
\[ + \sum_{j,k=1}^{n} a^{jk}\psi_{x_j} \left( \sum_{j',k'=1}^{n} a^{j'k'}\psi_{x_{j'}}\psi_{x_{k'}} \right)_{x_k} + c_1(2t-T) \sum_{j,k=1}^{n} a^{jk}\psi_{x_j}\psi_{x_k} ] + O(\lambda^2). \]

The next steps are similar to those of [13], we omit the details.

We can further prove the following Carleman estimate for (A.8). The proof is similar to that of [13], we omit the details as well.

**Theorem 1.7.** Let Assumption 1.1-1.2 holds and
\[ T_1 = \max \left\{ 2\sqrt{\kappa_1}, 1 + 100s_0(n + 2)\sqrt{n} \right\}, \tag{A.11} \]
where
\[ \kappa_1 = \max_{t \in [0,T]} \sum_{j,k=1}^{n} a^{jk}\psi_{x_j}\psi_{x_k}, \quad s_0 = \max_{t \in [0,T]} \sum_{j,k=1}^{n} a^{jk}\psi_{x_j}n^k. \]

Then there exists a constant \( \lambda_0 > 0 \) such that for \( \forall \ T > T_1 \) and \( \lambda > \lambda_0 \), every solution \( z \in H^1(\Omega_T) \) to (A.8) satisfies that
\[ \int_{\Omega_T} \theta^2 \left( \lambda(z_t^2 + |\nabla z|^2) + \lambda^3 z^2 \right) dx dt \leq C \left( \int_{\Omega_T} \theta^2 |F|^2 dx dt + \lambda^2 \int_{0}^{T} \int_{\Omega} \theta^2 (z_t^2 + \lambda^2 z^2) dx dt \right). \tag{A.12} \]

**Remark 1.8.** The choice of \( T_1 \) in (A.11) seems quite different from that of [13]. It’s because our assumption of the coefficients (A.2) is different too. And our choice of \( T_1 \) is far from sharp.

### A.3 An auxiliary optimal control problem

This subsection is devoted to show an auxiliary optimal control problem which is useful in the proof of our main results, see [13], [14] and [15].

Throughout this subsection, we fix the function \( \phi \) in (A.6), a parameter \( \lambda > 0 \), and a function \( z \in C([0,T];L^2(\Omega)) \) holding \( z(0,x) = z(T,x) = 0 \) for \( x \in \Omega \). For any \( K > 1 \), we choose a function \( \rho(x) \in C^2(\bar{\Omega}) \) with \( \min_{x \in \Omega} \rho(x) = 1 \) so that
\[ \rho(x) = \begin{cases} 1, x \in \omega, \\ K, d(x,\omega) \geq \frac{1}{\ln K}, \end{cases} \tag{A.13} \]
For any integer $m \geq 3$, let $h = \frac{T}{m}$. Define
\[ z_m^i = z_m^i(x) = z(ih, x), \quad \phi_m^i = \phi_m^i(x) = \phi(ih, x), \quad i = 0, 1, \ldots, m. \tag{A.14} \]
and
\[ a_m^{jk} = a_m^{jk}(x) = a_m^{jk}(ih, x), \quad i = 0, 1, \ldots, m; \quad j, k = 1, 1, \ldots, n. \tag{A.15} \]
Let $\{(w_m^i, r_{1m}^i, r_{2m}^i), r_m^i \}_{i=0}^m \in (H_0^1(\Omega) \times (L^2(\Omega)))^{m+1}$ satisfy the following system:
\begin{equation}
\begin{aligned}
\frac{w_m^{i+1} - 2w_m^i + w_m^{i-1}}{h^2} &- \sum_{j_1,j_2=1}^n \partial_{j_2}(a_m^{i_1,j_2} \partial_{j_1}w_m^i) \\
&= \frac{r_{1m}^{i+1} - r_{1m}^i}{h} + r_{2m}^i + \lambda z_m^i e^{2\phi_m^i} + r_m^i, \quad 1 \leq i \leq m - 1, \quad x \in \Omega, \\
&\quad w_m^i = 0, \quad 0 \leq i \leq m, \quad x \in \partial \Omega, \\
&\quad w_0^i = w_m^i = r_{2m}^i = r_m^i = 0 = r_{1m}^i, \quad x \in \Omega.
\end{aligned}
\tag{A.16}
\end{equation}
The set of admissible sequences for (A.16) is defined as
\[ \mathcal{A}_{ad} := \left\{ \{(w_m^i, r_{1m}^i, r_{2m}^i), r_m^i \}_{i=0}^m \in (H_0^1(\Omega) \times (L^2(\Omega)))^{m+1} \mid \{w_m^i, r_{1m}^i, r_{2m}^i, r_m^i \}_{i=0}^m \text{ satisfy } (A.16) \right\}. \tag{A.17} \]
Note that we can easily see the set $\mathcal{A}_{ad} \neq \emptyset$ because $\{(0, 0, 0, -\lambda z_m^i e^{2\phi_m^i}) \}_{i=0}^m \in \mathcal{A}_{ad}$.

Now, let us introduce the cost functional
\begin{equation}
\begin{aligned}
J\left(\{(w_m^i, r_{1m}^i, r_{2m}^i), r_m^i \}_{i=0}^m\right) &= \frac{h}{2} \int_{\Omega} \rho \frac{|r_{1m}^m|^2}{\lambda^2} e^{-2\phi_m^m} dx \\
&+ \frac{h}{2} \sum_{i=1}^{m-1} \int_{\Omega} |w_m^i|^2 e^{-2\phi_m^i} dx + \int_{\Omega} \rho \left( \frac{|r_{1m}^i|^2}{\lambda^2} + \frac{|r_{2m}^i|^2}{\lambda^4} \right) e^{-2\phi_m^i} + K \int_{\Omega} |r_m^i|^2 dx. \tag{A.18}
\end{aligned}
\end{equation}
Let us consider the following optimal problem:
\begin{equation}
\inf_{\{(w_m^i, r_{1m}^i, r_{2m}^i), r_m^i \}_{i=0}^m \in \mathcal{A}_{ad}} J\left(\{(w_m^i, r_{1m}^i, r_{2m}^i), r_m^i \}_{i=0}^m\right) = d. \tag{A.19}
\end{equation}
We have the following key proposition.

**Proposition A.1.** For any $K > 1$ and $m \geq 3$, problem (A.19) admits a unique solution $\{(\tilde{w}_m^i, \tilde{r}_{1m}^i, \tilde{r}_{2m}^i, \tilde{r}_m^i \}_{i=0}^m \in \mathcal{A}_{ad}$, such that
\begin{equation}
J\left(\{(\tilde{w}_m^i, \tilde{r}_{1m}^i, \tilde{r}_{2m}^i, \tilde{r}_m^i \}_{i=0}^m\} = \min_{\{(w_m^i, r_{1m}^i, r_{2m}^i, r_m^i \}_{i=0}^m \in \mathcal{A}_{ad}} J\left(\{(w_m^i, r_{1m}^i, r_{2m}^i, r_m^i \}_{i=0}^m\}. \tag{A.20}
\end{equation}
Furthermore, for
\begin{equation}
p_m^i = p_m^i(x) := K \hat{r}_m^i(x), \quad 0 \leq i \leq m, \tag{A.20}
\end{equation}
one has
\begin{equation}
\begin{aligned}
\hat{w}_m^0 &= \hat{w}_m^0 = p_m^0 = p_m^m = 0, \quad x \in \Omega, \\
\hat{w}_m^i, p_m^i &\in H^2(\Omega) \cap H_0^1(\Omega), \quad 1 \leq i \leq m - 1.
\end{aligned} \tag{A.21}
\end{equation}
and the following optimality conditions:
\[
\begin{align*}
\frac{p_m^i - p_{m}^{i-1}}{h^2} + \rho \frac{\phi_m^i}{\lambda^2} e^{-2\lambda\phi_m^i} = 0, \\
p_m^i - \rho \frac{2\phi_m^i}{\lambda^2} e^{-2\lambda\phi_m^i} = 0,
\end{align*}
\]
\[1 \leq i \leq m, \ x \in \Omega \tag{A.22}\]
and
\[
\begin{align*}
\frac{p_m^i - 2p_{m}^{i-1} + p_{m}^{i-2}}{h^2} - \sum_{j_1,j_2=1}^n \partial_{j_2}(u_{1}^{j_1,j_2} \partial_{j_1} p_m^i) \\
+ \hat{\nu}_m^i e^{-2\lambda \phi_m^i} = 0, \ x \in \Omega \\
p_m^i = 0, \ x \in \partial \Omega.
\end{align*}
\]
Moreover, there is a constant \(C = C(K, \lambda) > 0\), independent of \(m\), such that
\[h \sum_{i=1}^{m-1} \int_{\Omega} \left[ |\hat{\nu}_m^{i+1} - \hat{\nu}_m^i|^2 + \left( \frac{\phi_m^i - \hat{\nu}_m^i}{h} \right)^2 \right] + h \int_{\Omega} |\hat{\nu}_m^{i+1} - \hat{\nu}_m^i|^2 \leq C \tag{A.24}\]
and
\[\sum_{i=1}^{m-1} \int_{\Omega} \left[ (\hat{\nu}_m^{i+1} - \hat{\nu}_m^i)^2 + \left( \frac{\phi_m^{i+1} - \phi_m^i}{h} \right)^2 \right] + \sum_{i=1}^{m-1} \int_{\Omega} \left( \frac{\phi_m^{i+1} - \phi_m^i}{h} \right)^2 \leq C \tag{A.25}\]

**Remark 1.9.** For any \(\{(w_m^i, r_{1m}^i, r_{2m}^i), r_m^i\}_{i=0}^m \in A_{ad}\), since \((a_i^{j_1,j_2})\) is positive definite, by standard regularity results of elliptic equations, we obtain \(w_m^i \in H^2(\Omega) \cap H_0^1(\Omega)\).

**Proof.** The proof is divided into several steps.

**Step 1. Existence and uniqueness of \(\{(\hat{w}_m^i, \hat{r}_{1m}^i, \hat{r}_{2m}^i), \hat{r}_m^i\}_{i=0}^m \in A_{ad}\).**

Let \(\{(w_m^{i,j}, r_{1m}^{i,j}, r_{2m}^{i,j}), r_m^{i,j}\}_{i,j=0}^{m,\infty} \subset A_{ad}\) be a minimizing sequence of \(J\). Due to the coercivity of \(J\) and noting that \(w_m^{i,j}\) solves an elliptic equation, it can be shown that \(\{(w_m^{i,j}, r_{1m}^{i,j}, r_{2m}^{i,j}), r_m^{i,j}\}_{i=0}^\infty\) is bounded in \(A_{ad}\). Therefore, there exists a subsequence of \(\{(w_m^{i,j}, r_{1m}^{i,j}, r_{2m}^{i,j}), r_m^{i,j}\}_{i=0}^\infty\) converging weakly to some
\[
\{(\hat{w}_{m}^i, \hat{r}_{1m}^i, \hat{r}_{2m}^i), \hat{r}_m^i\}_{i=0}^m \in (H_0^1(\Omega) \times (L^2(\Omega))^3)^{m+1}.
\]

Note that the constraint condition \([A.16]\) is a linear system, we obtain
\[
\{(\hat{w}_{m}^i, \hat{r}_{1m}^i, \hat{r}_{2m}^i), \hat{r}_m^i\}_{i=0}^m \in A_{ad}.
\]
and \(\hat{w}_{m}^0 = \hat{w}_{m}^m = p_{m}^0 = p_{m}^m = 0, \ x \in \Omega.\)

Since \(J\) is strictly convex, this optimal target is the unique solution of \([A.19]\).

**Step 2. The proof of \([A.22]\) and \([A.23]\).**

**Step 3. The proof of \([A.24]\) and \([A.25]\).**

The proof of the above estimates are similar to those of [14], so we omit the details. \(\square\)
A.4 Carleman estimate in $L^2$-norm

Now we are ready to derive a Carleman estimate for the following hyperbolic equation in $L^2$-norm.

\[
\begin{cases}
z_{tt} - \sum_{j,k=1}^{n} (a^{jk}z_{x_j})_{x_k} = F, & (t, x) \in \Omega_T, \\
z(t, x) = 0, & (t, x) \in \Gamma_T,
\end{cases}
\tag{A.26}
\]

where $F \in L^1(0, T; H^{-1}(\Omega))$.

**Definition 1.10.** A function $z \in L^2((0, T) \times \Omega)$ is called a weak solution to (A.26) if

\[
(z, \eta_{tt} - \sum_{j,k=1}^{n} (a^{jk}\eta_{x_j})_{x_k})_{L^2(\Omega_T)} = \int_{0}^{T} \langle f(t, \cdot), \eta(t, \cdot) \rangle_{H^{-1}(\Omega), H^0_0(\Omega)} dt
\forall \eta \in H^1_0(0, T; H^2(\Omega) \cap H^1_0(\Omega)).
\tag{A.27}
\]

Note that there are no initial data in (A.26), so we need the following lemma for weak solutions, see [14].

**Lemma 1.11.** Given $0 < t_1 < t_2 < T$ and $g \in L^2((t_1, t_2) \times \Omega)$. Assume that $z \in L^2(\Omega_T)$ is a weak solution to (A.26) with $z = g$ in $(t_1, t_2) \times \Omega$. Then we have

\[
z \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega)),
\]

and there exists a constant $C = C(T, t_1, t_2, \Omega, a^{jk}) > 0$, such that

\[
\|z\|_{C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))} \leq C(\|f\|_{L^1(0, T; H^{-1}(\Omega))} + \|g\|_{L^2((t_1, t_2) \times \Omega)}).
\]

Our Carleman estimate for the above hyperbolic operators in $L^2$-norm is as follows.

**Theorem 1.12.** Suppose that Assumption 1.1-1.2 holds, and $T_1$ are given in (A.11). Then there exists a constant $\lambda_0^* > 0$ such that for $\forall \; T > T_1$ and $\lambda > \lambda_0^*$, and every solution $z \in C^0([0, T]; L^2(\Omega))$ satisfying $z(0, x) = z(T, x) = 0, \; x \in \Omega$ and

\[
z_{tt} - \sum_{j,k=1}^{n} (a^{jk}z_{x_j})_{x_k} \in H^{-1}(\Omega_T),
\]

it holds

\[
\lambda \int_{\Omega_T} \theta^2 z^2 dx dt 
\leq C\left(\|z\|_{C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))} + \lambda^2 \int_{0}^{T} \int_{\omega} \theta^2 z^2 dx dt \right).
\tag{A.28}
\]

**Proof.** The main idea is to choose a special $\eta$, so that

\[
\eta_{tt} - \sum_{j,k=1}^{n} (a^{jk}\eta_{x_j})_{x_k} = \lambda z e^{2\lambda \phi} + \cdots,
\]
where we get the desired term $\lambda \| \theta z \|_{L^2(\Omega_T)}^2$ and reduce the estimate to that for $\|\eta\|_{H^1_0(\Omega_T)}$.

The proof is divided into several steps.

**Step 1.** Firstly, recall the functions $\{(\tilde{w}^m, \tilde{r}_1^m, \tilde{r}_2^m, \tilde{r}_m^m)\}_{m=0}^\infty$ in Proposition [A.1], put

\[
\begin{aligned}
\tilde{w}^m(t, x) &= \frac{1}{h} \sum_{i=0}^{m-1} \left( (t - ih) \tilde{w}^{i+1}_m(x) - (t - (i+1)h) \tilde{w}_m^i(x) \right) \chi_{(ih,(i+1)h]}(t), \\
\tilde{r}_1^m(t, x) &= \tilde{r}_1^0 \chi_{(0]}(t) + \frac{1}{h} \sum_{i=0}^{m-1} \left( (t - ih) \tilde{r}_1^{i+1}_m(x) - (t - (i+1)h) \tilde{r}_1^i_m(x) \right) \chi_{(ih,(i+1)h]}(t), \\
\tilde{r}_2^m(t, x) &= \frac{1}{h} \sum_{i=0}^{m-1} \left( (t - ih) \tilde{r}_2^{i+1}_m(x) - (t - (i+1)h) \tilde{r}_2^i_m(x) \right) \chi_{(ih,(i+1)h]}(t), \\
\tilde{r}_m(t, x) &= \frac{1}{h} \sum_{i=0}^{m-1} \left( (t - ih) \tilde{r}_m^{i+1}_m(x) - (t - (i+1)h) \tilde{r}_m^i_m(x) \right) \chi_{(ih,(i+1)h]}(t),
\end{aligned}
\]

By [A.24] and (A.25), there exist a subsequence of $\{(\tilde{w}^m, \tilde{r}_1^m, \tilde{r}_2^m, \tilde{r}_m^m)\}_{m=1}^{\infty}$ which converges weakly to some $(\tilde{w}, \tilde{r}_1, \tilde{r}_2, \tilde{r}) \in H^1(0, T; L^2(\Omega))$ as $m \to \infty$.

Let $\tilde{p} = K \tilde{r}$ for some sufficiently large constant $K > 1$. By (A.16), (A.22)-(A.25) and Lemma [1.11], we obtain that

\[
\tilde{w}, \tilde{p} \in C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)),
\]

and

\[
\begin{aligned}
\tilde{w}_{tt} - \sum_{j,k=1}^n (a^{jk} \tilde{w}_{x_j})_{x_k} &= \partial_t \tilde{r}_1 + \tilde{r}_2 + \lambda \theta^2 z + \tilde{r}, \quad (t, x) \in \Omega_T, \\
\tilde{p}_{tt} - \sum_{j,k=1}^n (a^{jk} \tilde{p}_{x_j})_{x_k} + \theta^{-2} \tilde{w} &= 0, \quad (t, x) \in \Omega_T, \\
\tilde{p}_t + \rho \theta^2 \frac{\tilde{r}_1}{\lambda^2} &= 0, \quad (t, x) \in \Omega_T, \\
\tilde{p} - \rho \theta^2 \frac{\tilde{r}_2}{\lambda^2} &= 0, \quad (t, x) \in \Omega_T, \\
\tilde{p}(t, x) = \tilde{w}(t, x) &= 0, \quad (t, x) \in \Gamma_T, \\
\tilde{p}(0, x) = \tilde{p}(T, x) = \tilde{w}(0, x) = \tilde{w}(T, x) &= 0, \quad x \in \Omega.
\end{aligned}
\]

**Step 2.** Applying Theorem [1.7] to $\tilde{p}$ in (A.29), we have

\[
\begin{aligned}
\lambda \int_{\Omega_T} \theta^2 (\lambda^2 \tilde{p}^2 + \tilde{p}_t^2 + |\nabla \tilde{p}|^2) dx dt &
\leq C \left[ \int_{\Omega_T} \theta^{-2} \tilde{w}^2 dx dt + \lambda^2 \int_0^T \int_\omega \theta^2 (\lambda^2 \tilde{p}^2 + \tilde{p}_t^2) dx dt \right] \quad (A.30)
\leq C \left[ \int_{\Omega_T} \theta^{-2} \tilde{w}^2 dx dt + \int_0^T \int_\omega \theta^{-2} \left( \frac{\tilde{r}_1^2}{\lambda^2} + \frac{\tilde{r}_2^2}{\lambda^2} \right) dx dt \right].
\end{aligned}
\]

Here and hence forth, $C$ is a constant independent of $K$ and $\lambda$. 

By (A.29), one finds that

\[
\begin{cases}
\tilde{p}_{tt} - \sum_{j,k=1}^{n} (a^{jk} \tilde{p}_{xj})_{x_k} + (\theta^{-2} \tilde{w})_t - \sum_{j,k=1}^{n} (a^{jk} \tilde{p}_{xj})_{x_k} = 0, & (t, x) \in \Omega_T, \\
\tilde{p}_t + \frac{\rho}{\lambda} \theta^{-2} \left( \frac{\partial \tilde{r}_1}{\lambda} - 2 \phi_t \tilde{r}_1 \right) = 0, & (t, x) \in \Omega_T, \\
\tilde{p} - \frac{\rho}{\lambda^2} \theta^{-2} \left( \frac{\partial \tilde{r}_2}{\lambda^2} - \frac{2}{\lambda} \phi_t \tilde{r}_1 \right) = 0, & (t, x) \in \Omega_T, \\
\tilde{p}_t(t, x) = 0, & (t, x) \in \Gamma_T.
\end{cases}
\]  

(A.31)

Applying Theorem 1.7 to \( \tilde{p}_t \) in (A.31), we obtain

\[
\lambda \int_{\Omega_T} \theta^2 (\lambda^2 \tilde{p}_t^2 + \tilde{p}_t^2 + |\nabla \tilde{p}_t|^2) dx dt 
\leq C \left[ \left\| \theta (\theta^{-2} \tilde{w})_t \right\|_{L^2(\Omega_T)}^2 + \left\| \theta \sum_{j,k=1}^{n} (a^{jk} \tilde{p}_{xj})_{x_k} \right\|_{L^2(\Omega_T)}^2 
\right. 
+ \lambda^2 \int_0^T \int_\omega \theta^2 (\lambda^2 \tilde{p}_t^2 + \tilde{p}_t^2) dx dt 
\left. 
\right] 
\leq C \left[ \int_{\Omega_T} \theta^{-2} (\tilde{w}^2 + \lambda^2 \tilde{w}^2) dx dt + \left\| \theta \sum_{j,k=1}^{n} (a^{jk} \tilde{p}_{xj})_{x_k} \right\|_{L^2(\Omega_T)}^2 
\right. 
+ \int_0^T \int_\omega \theta^{-2} \left( \frac{\partial \tilde{r}_1}{\lambda^2} + \frac{\partial \tilde{r}_2}{\lambda^4} + \tilde{r}_1^2 + \frac{\tilde{r}_2^2}{\lambda^2} \right) dx dt \right]. 
\]  

(A.32)

Recalling (A.2), we have

\[
\int_{\Omega_T} \theta^2 \left\| \sum_{j,k=1}^{n} (a^{jk} \tilde{p}_{xj})_{x_k} \right\|^2 dx dt \leq C \varepsilon \int_{\Omega_T} \theta^2 (|\nabla \tilde{p}|^2 + |\nabla^2 \tilde{p}|^2) dx dt. 
\]  

(A.33)

By (A.29), we have

\[
\begin{cases}
\tilde{p}_{tt} - \Delta \tilde{p} = \sum_{j,k=1}^{n} ((a^{jk} - \delta_{jk}) \tilde{p}_{xj})_{x_k} - \theta^{-2} \tilde{w}, & (t, x) \in \Omega_T, \\
\tilde{p}(t, x) = 0, & (t, x) \in \Gamma_T, \\
\tilde{p}(0, x) = \tilde{p}(T, x) = 0, & x \in \Omega.
\end{cases}
\]

Taking \( L^2 \) inner product of the above equation with \( -\Delta \tilde{p} \), we get

\[
\int_{\Omega_T} |\nabla^2 \tilde{p}|^2 dx dt - \int_{\Omega_T} |\nabla \tilde{p}_t|^2 dx dt 
\leq C \varepsilon \int_{\Omega_T} (|\nabla \tilde{p}|^2 + |\nabla^2 \tilde{p}|^2) dx dt + \int_{\Omega_T} \theta^{-2} |\tilde{w}| (|\Delta \tilde{p}| dx dt 
\leq C \varepsilon \int_{\Omega_T} (|\nabla \tilde{p}|^2 + |\nabla^2 \tilde{p}|^2) dx dt + \frac{1}{2} \int_{\Omega_T} \theta^{-4} \tilde{w}^2 + |\Delta \tilde{p}|^2 dx dt.
\]
Now we should deal with the terms on the right hand side.

By (A.33) and (A.34), we obtain

$$
\int_{\Omega_T} \theta^2 |\nabla^2 \bar{\rho}|^2 \, dx \, dt \leq C e^{CA} \int_{\Omega_T} |\nabla^2 \bar{\rho}|^2 \, dx \, dt
$$

\begin{align*}
&\leq C e^{CA} \int_{\Omega_T} \left( \varepsilon |\nabla \bar{\rho}|^2 + |\nabla \bar{p}|^2 \right) + \theta^{-4} \bar{w}^2 \, dx \, dt \\
&\leq C e^{CA} \int_{\Omega_T} \left( \varepsilon \theta^2 |\nabla \bar{\rho}|^2 + \theta^2 |\nabla \bar{p}|^2 + \theta^{-2} \bar{w}^2 \right) \, dx \, dt.
\end{align*}

(A.34)

By (A.33) and (A.34), we obtain

$$
\int_{\Omega_T} \theta^2 \sum_{j,k=1}^n (a^{jk}_{ij} \bar{p}_{x_j})_{x_k}^2 \, dx \, dt \leq C e^{CA} \int_{\Omega_T} \left[ \theta^2 \left( |\nabla \bar{\rho}|^2 + |\nabla \bar{p}|^2 \right) + \theta^{-2} \bar{w}^2 \right] \, dx \, dt.
$$

(A.35)

**Step 3.** Noting that by (A.29),

$$
- \int_{\Omega_T} (\partial_t \bar{r}_1 + \bar{r}_2) \bar{p} \, dx \, dt = \int_{\Omega_T} (\bar{r}_1 \bar{p} - \bar{r}_2 \bar{p}) \, dx \, dt = - \int_{\Omega_T} \rho \theta^{-2} \left( \frac{r_1^2}{\lambda^2} + \frac{r_2^2}{\lambda^4} \right) \, dx \, dt.
$$

Thus we have

$$
0 = \left( \bar{w}_{tt} - \sum_{j,k=1}^n (a^{jk}_{ij} \bar{w}_{x_j})_{x_k} - \partial_{tt} \bar{r}_1 - \bar{r}_2 - \lambda \theta^2 z - \bar{r}, \ \bar{p} \right)_{L^2(\Omega_T)}
$$

$$
= - \int_{\Omega_T} \theta^{-2} \bar{w}^2 \, dx \, dt + \int_{\Omega_T} \rho \theta^{-2} \left( \frac{r_1^2}{\lambda^2} + \frac{r_2^2}{\lambda^4} \right) \, dx \, dt - \lambda \int_{\Omega_T} \theta^2 z \bar{p} \, dx \, dt - K \int_{\Omega_T} \bar{r}^2 \, dx \, dt.
$$

Hence we get

$$
\int_{\Omega_T} \theta^{-2} \bar{w}^2 \, dx \, dt + \int_{\Omega_T} \rho \theta^{-2} \left( \frac{r_1^2}{\lambda^2} + \frac{r_2^2}{\lambda^4} \right) \, dx \, dt + K \int_{\Omega_T} \bar{r}^2 \, dx \, dt = - \lambda \int_{\Omega_T} \theta^2 z \bar{p} \, dx \, dt.
$$

By Cauchy-Schwartz inequality and (A.30), we obtain

$$
\int_{\Omega_T} \theta^{-2} \bar{w}^2 \, dx \, dt + \int_{\Omega_T} \rho \theta^{-2} \left( \frac{r_1^2}{\lambda^2} + \frac{r_2^2}{\lambda^4} \right) \, dx \, dt + K \int_{\Omega_T} \bar{r}^2 \, dx \, dt \leq \frac{C}{\lambda} \int_{\Omega_T} \theta^2 z^2 \, dx \, dt.
$$

(A.36)

**Step 4.** Using (A.29) and (A.31), by the fact that \( \bar{p}_t(0) = \bar{p}_t(T) = 0 \) in \( \Omega \), we get

$$
0 = \left( \bar{w}_{tt} - \sum_{j,k=1}^n (a^{jk}_{ij} \bar{w}_{x_j})_{x_k} - \partial_{tt} \bar{r}_1 - \bar{r}_2 - \lambda \theta^2 z - \bar{r}, \ \bar{p} \right)_{L^2(\Omega_T)}
$$

$$
= \left( \bar{w}, \bar{p}_{ttt} - \sum_{j,k=1}^n (a^{jk}_{ij} \bar{p}_{tx_j})_{x_k} \right)_{L^2(\Omega_T)}
$$

$$
- \int_{\Omega_T} (\partial_t \bar{r}_1 + \bar{r}_2) \bar{p} \, dx \, dt - \lambda \int_{\Omega_T} \theta^2 z \bar{p} \, dx \, dt - \int_{\Omega_T} \bar{r} \bar{p} \, dx \, dt
$$

$$
= - \int_{\Omega_T} \bar{w} (\theta^{-2} \bar{w}) \, dx \, dt + \sum_{j,k=1}^n \int_{\Omega_T} \bar{w} (2 a^{jk}_{ij} \bar{p}_{x_j} + a^{jk}_{ij} \bar{p}_{x_k}) \, dx \, dt
$$

$$
- \int_{\Omega_T} (\partial_t \bar{r}_1 + \bar{r}_2) \bar{p} \, dx \, dt - \lambda \int_{\Omega_T} \theta^2 z \bar{p} \, dx \, dt - \int_{\Omega_T} \bar{r} \bar{p} \, dx \, dt.
$$

(A.37)

Now we should deal with the terms on the right hand side.
Firstly, it’s easy to see that
\[- \int_{\Omega_T} \tilde{w}(\theta^{-2} \tilde{w})_{tt} dxdt = \int_{\Omega_T} \left[ \theta^{-2} \tilde{w}^2_t - (\theta^{-2})_{tt} \tilde{w}^2 \right] dxdt \]
\[= \int_{\Omega_T} \theta^{-2} (\tilde{w}^2_t + \lambda \phi_{tt} \tilde{w}^2 - 2\lambda^2 \phi^2_t \tilde{w}^2) dxdt. \tag{A.38} \]

Secondly, by \((A.31)\) we have
\[- \int_{\Omega_T} (\partial_t \tilde{r}_1 + \tilde{r}_2) \tilde{p}_{tt} dxdt = \int_{\Omega_T} (\tilde{p}_t \partial_t \tilde{r}_2 - \tilde{p}_{tt} \partial_t \tilde{r}_1) dxdt \]
\[= \int_{\Omega_T} \rho \theta^{-2} \left[ \frac{\partial_t \tilde{r}_1}{\lambda} \left( \frac{\partial_t \tilde{r}_1}{\lambda} - 2\phi_t \tilde{r}_1 \right) + \frac{\partial_t \tilde{r}_2}{\lambda^2} \left( \frac{\partial_t \tilde{r}_2}{\lambda^2} - \frac{2}{\lambda^3} \phi_t \tilde{r}_2 \right) \right] dxdt \]
\[= \int_{\Omega_T} \rho \theta^{-2} \left( |\partial_t \tilde{r}_1|^2 + \frac{|\partial_t \tilde{r}_2|^2}{\lambda^4} - \frac{2}{\lambda^3} \phi_t \tilde{r}_1 \partial_t \tilde{r}_1 - \frac{2}{\lambda^3} \phi_t \tilde{r}_2 \partial_t \tilde{r}_2 \right) dxdt. \tag{A.39} \]

Moreover, by \(\tilde{p} = K \tilde{r}\) and integration by parts, one gets that
\[- \int_{\Omega_T} \tilde{r} \tilde{p}_{tt} dxdt = K \int_{\Omega_T} \tilde{r}^2_t dxdt \tag{A.40} \]
and
\[\sum_{j,k=1}^{n} \int_{\Omega_T} \tilde{w} \left( 2a_{ij}^k \tilde{p}_{tx_j} + a_{ij}^k \tilde{p}_{x_j} \right) dxdt \]
\[= - \sum_{j,k=1}^{n} \int_{\Omega_T} \tilde{w}_{xx} \left( 2a_{ij}^k \tilde{p}_{tx_j} + a_{ij}^k \tilde{p}_{x_j} \right) dxdt \tag{A.41} \]
\[\leq C \varepsilon \int_{\Omega_T} \theta^2 (|\nabla \tilde{p}_t|^2 + |\nabla \tilde{p}|^2) + \theta^{-2} |\nabla \tilde{w}|^2 dxdt. \]

Combining \((A.37)-(A.41)\), we end up with
\[\int_{\Omega_T} \rho \theta^{-2} \left( |\partial_t \tilde{r}_1|^2 + \frac{|\partial_t \tilde{r}_2|^2}{\lambda^4} - \frac{2}{\lambda^3} \phi_t \tilde{r}_1 \partial_t \tilde{r}_1 - \frac{2}{\lambda^3} \phi_t \tilde{r}_2 \partial_t \tilde{r}_2 \right) dxdt \]
\[+ K \int_{\Omega_T} \tilde{r}^2_t dxdt + \int_{\Omega_T} \theta^{-2} (\tilde{w}^2_t + \lambda \phi_{tt} \tilde{w}^2 - 2\lambda^2 \phi^2_t \tilde{w}^2) dxdt \leq \lambda \int_{\Omega_T} \theta^2 |\nabla \tilde{p}_t|^2 dxdt + C \varepsilon \int_{\Omega_T} \theta^2 (|\nabla \tilde{p}_t|^2 + |\nabla \tilde{p}|^2) + \theta^{-2} |\nabla \tilde{w}|^2 dxdt. \tag{A.42} \]

By \((A.42) + \lambda^2 (A.36)\) with a sufficiently large \(C > 0\), using Cauchy-Schwarz inequality, noting \((A.30), (A.32)\) and \((A.35)\), we obtain that
\[\int_{\Omega_T} \theta^{-2} (\tilde{w}^2_t + \lambda^2 \tilde{w}^2) dxdt + \int_{\Omega_T} \rho \theta^{-2} \left( |\partial_t \tilde{r}_1|^2 + \frac{|\partial_t \tilde{r}_2|^2}{\lambda^4} + \tilde{r}^2_t + \frac{\tilde{r}^2}{\lambda^2} \right) dxdt \]
\[\leq C \lambda \int_{\Omega_T} \theta^2 |\nabla \tilde{p}_t|^2 dxdt + C \varepsilon e^{\lambda} \int_{\Omega_T} \theta^2 (|\nabla \tilde{p}_t|^2 + |\nabla \tilde{p}|^2) + \theta^{-2} |\nabla \tilde{w}|^2 dxdt \]
\[+ C \varepsilon \int_{\Omega_T} \theta^{-2} |\nabla \tilde{w}|^2 dxdt. \tag{A.43} \]
Step 5. It follows from (A.29) that

\[
\left( \partial_t \tilde{r}_1 + \tilde{r}_2 + \lambda \theta^2 z + \tilde{r}_1 \theta^{-2} \tilde{w} \right)_{L^2(\Omega_T)} = \left( \tilde{w}_t - \sum_{j,k=1}^{n} (a^{jk} \tilde{w}_{x_j})_{x_k}, \theta^{-2} \tilde{w} \right)_{L^2(\Omega_T)} = - \int_{\Omega_T} \tilde{w}_t (\theta^{-2} \tilde{w}) \, dx \, dt + \sum_{j,k=1}^{n} \int_{\Omega_T} a^{jk} \tilde{w}_{x_j} (\theta^{-2} \tilde{w})_{x_k} \, dx \, dt + 2\lambda \sum_{j,k=1}^{n} \int_{\Omega_T} \theta^{-2} a^{jk} \tilde{w}_{x_j} \tilde{w}_{x_k} \, dx \, dt,
\]

thus we get

\[
\int_{\Omega_T} \theta^{-2} |\nabla \tilde{w}|^2 \, dx \, dt \leq C \int_{\Omega_T} \left[ \theta^{-2} |\partial_t \tilde{r}_1 + \tilde{r}_2 + \tilde{r}| |\tilde{w}| + \lambda |z \tilde{w}| + \theta^{-2} (\tilde{w}_t^2 + \lambda^2 \tilde{w}^2) \right] \, dx \, dt \leq C \int_{\Omega_T} \left[ \theta^2 z^2 + \theta^{-2} \left( \frac{|\partial_t \tilde{r}_1|^2}{\lambda^2} + \frac{\tilde{r}_2^2}{\lambda^2} + \tilde{r}_1^2 + \tilde{w}_t^2 + \lambda^2 \tilde{w}^2 \right) \right] \, dx \, dt.
\]

Now we combine (A.36), (A.43) and (A.45), and choose the constant $K$ in (A.36) so that

\[
K \geq C e^{2\lambda \|\phi\|_{L^\infty(\Omega_T)}}
\]

to absorb the term $C \int_{\Omega_T} \theta^{-2} \tilde{r}_2^2 \, dx \, dt$ in (A.45). Noting that $\rho(x) \geq 1$ and $\varepsilon$ can be so small that $C \varepsilon e^{C \lambda} \leq 1$ for given $\lambda$, we finally deduce that

\[
\int_{\Omega_T} \theta^{-2} (|\nabla \tilde{w}|^2 + \tilde{w}_t^2 + \lambda^2 \tilde{w}^2) \, dx \, dt + \int_{\Omega_T} \rho \theta^{-2} \left( \frac{|\partial_t \tilde{r}_1|^2}{\lambda^2} + \frac{\tilde{r}_2^2}{\lambda^2} + \tilde{r}_1^2 + \tilde{w}_t^2 + \lambda^2 \tilde{w}^2 \right) \, dx \, dt \leq C \lambda \int_{\Omega_T} \theta^2 z^2 \, dx \, dt.
\]

Step 6. Recall that $(\tilde{w}, \tilde{r}_1, \tilde{r}_2, \tilde{r})$ depends on $K$, so we can denote it by

\[
(\tilde{w}^K, \tilde{r}_1^K, \tilde{r}_2^K, \tilde{r}^K).
\]

Fix $\lambda$ and let $K \to \infty$, since $\rho = \rho^K(x) \to \infty$ for $x \notin \omega$, we can see from (A.36) and (A.46) that there exists a subsequence of $(\tilde{w}^K, \tilde{r}_1^K, \tilde{r}_2^K, \tilde{r}^K)$ which converges weakly to some $(\tilde{w}, \tilde{r}_1, \tilde{r}_2, 0)$ in

\[
H^1_0(\Omega_T) \times (H^1(0,T; L^2(\Omega))^2 \times L^2(\Omega_T),
\]
with supp \( \tilde{r}_j \subseteq [0, T] \times \overline{\Omega} \), \( j = 1, 2 \). By (A.29) we see that
\[
\begin{aligned}
\begin{cases}
\dot{w}_t - \sum_{j,k=1}^n (a^{jk} \dot{w}_{x_j})_{x_k} = \partial_t \tilde{r}_1 + \tilde{r}_2 + \lambda \theta^2 z, & (t, x) \in \Omega_T, \\
\dot{w}(0, x) = \ddot{w}(T, x) = 0, & x \in \Omega, \\
\dot{w}(t, x) = 0, & (t, x) \in \Gamma_T.
\end{cases}
\end{aligned}
\]

Using (A.46) again, we find that
\[
\| \theta^{-1} \ddot{w} \|^2_{H^1_0(\Omega_T)} + \frac{1}{\lambda^2} \int_0^T \int_\Omega \theta^{-2} \left( |\partial_t \tilde{r}_1|^2 + \tilde{r}_2^2 \right) dx dt \leq C \lambda \int_{\Omega_T} \theta^2 z^2 dx dt. \tag{A.47}
\]

Then we take the \( \eta \) in (A.27) to be the above \( \ddot{w} \), and find that
\[
\left( \ddot{w}, \partial_t \tilde{r}_1 + \tilde{r}_2 + \lambda \theta^2 z \right) _{L^2(\Omega_T)} = \left< \dddot{z}_t - \sum_{j,k=1}^n (a^{jk} z_{x_j})_{x_k}, \dddot{w} \right> _{H^{-1}(\Omega_T), H^1_0(\Omega_T)}.
\]

Hence we have
\[
\begin{aligned}
\lambda \int_{\Omega_T} \theta^2 z^2 dx dt &= \left< \dddot{z}_t - \sum_{j,k=1}^n (a^{jk} z_{x_j})_{x_k} + 2 \dot{z}_t + z, \dddot{w} \right> _{H^{-1}(\Omega_T), H^1_0(\Omega_T)} \\
&+ 2 \left< z, \dddot{w} \right> _{L^2(\Omega_T)} - \left< z, \dddot{w} \right> _{L^2(\Omega_T)} - \left< z, \partial_t \tilde{r}_1 + \tilde{r}_2 \right> _{L^2((0,T) \times \omega)} \\
&\leq \left\| \theta \left( \dddot{z}_t - \sum_{j,k=1}^n (a^{jk} z_{x_j})_{x_k} + 2 \dot{z}_t + z \right) \right\| _{H^{-1}(\Omega_T)} \left\| \theta^{-1} \dddot{w} \right\| _{H^1_0(\Omega_T)} \\
&+ \left\| \theta z \right\| _{L^2(\Omega_T)} \left( \left\| \theta^{-1} \ddot{w}_1 \right\| _{L^2(\Omega_T)} + \left\| \theta^{-1} \ddot{w}_1 \right\| _{L^2(\Omega_T)} \right) \\
&+ \left\| \theta z \right\| _{L^2((0,T) \times \omega)} \left( \left\| \theta^{-1} (\partial_t \tilde{r}_1 + \tilde{r}_2) \right\| _{L^2((0,T) \times \omega)} \right) \\
&\leq C \sqrt{J} \left[ \left\| \theta^{-1} \ddot{w} \right\| _{H^1_0(\Omega_T)} + \lambda \left\| \theta^{-1} \ddot{w} \right\| _{L^2(\Omega_T)} + \left\| \theta^{-1} \dddot{w} \right\| _{L^2(\Omega_T)} \right] \\
&+ \lambda^{-1} \left\| \theta^{-1} \left( \partial_t \tilde{r}_1 + \tilde{r}_2 \right) \right\| _{L^2((0,T) \times \omega)} \right) 
\end{aligned}
\]
where
\[J := \left\| \theta \left( \dddot{z}_t - \sum_{j,k=1}^n (a^{jk} z_{x_j})_{x_k} + 2 \dot{z}_t + z \right) \right\|^2 _{H^{-1}(\Omega_T)} + \lambda^2 \int_0^T \int_\Omega \theta^2 z^2 dx dt.
\]
is exactly the right hand side of (A.28).

Since that
\[\theta^{-1} \ddot{w}_t = (\theta^{-1} \ddot{w})_t - (\theta^{-1})_t \ddot{w} = (\theta^{-1} \ddot{w})_t + \lambda \phi \eta \ddot{w},\]
we have
\[
\left\| \theta^{-1} \ddot{w}_t \right\| _{L^2(\Omega_T)} \leq C \left( \left\| \theta^{-1} \ddot{w} \right\| _{H^1(0,T;L^2(\Omega))} + \lambda \left\| \theta^{-1} \dddot{w} \right\| _{L^2(\Omega_T)} \right) \leq C \left( \left\| \theta^{-1} \ddot{w} \right\| _{H^1_0(\Omega_T)} + \lambda \left\| \theta^{-1} \dddot{w} \right\| _{L^2(\Omega_T)} \right). \tag{A.49}
\]

Finally, by (A.47)-(A.49), we obtain the desired estimate (A.28). This completes the proof of Thm [1.12]. \qed
A.5 Observability inequality of damped hyperbolic equations

In this subsection, we go back to the hyperbolic system (A.1) with condition (A.2) and Assumption 1.1-1.2. Our goal is to prove the observability inequality. The idea is mainly borrowed from [28], see also [13].

For any \((z_0, z_1) \in H^1_0(\Omega) \times L^2(\Omega)\), the system (A.1) admits a unique solution \(z \in C([0,T]; H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega))\).

Define the energy of the system by
\[
E(t) = \frac{1}{2} \int_\Omega \left[ |z_t(t)|^2 + \sum_{j,k=1}^n a^{jk} z_{x_j}(t) z_{x_k}(t) + |z(t)|^2 \right] dx.
\]

Multiplying the system (A.1) by \(z_t\), integrating it on \(\Omega\), and using integration by parts, we get
\[
E'(t) + 2 \int_\Omega b_0 z_t^2 dx = - \int_\Omega \left( \sum_{k=1}^n b_k z_{tx_k} + b z_t \right) dx \geq -C\varepsilon E(t).
\]  
(A.50)

Since that
\[
\int_\Omega b_0 z_t^2 dx \leq (2 + C\varepsilon) E(t),
\]
we have
\[
E'(t) + (2 + C\varepsilon) E(t) = e^{-(2+C\varepsilon)t} \frac{d}{dt}(e^{(2+C\varepsilon)t} E(t)) \geq 0.
\]
Integrating the above inequality on \((0, T)\), we get
\[
e^{(2+C\varepsilon)T} E(T) \geq E(0).
\]  
(A.51)

Step 1. We put
\[
\tilde{T}_j = \left( \frac{1}{2} - \varepsilon_j \right) T, \quad \tilde{T}'_j = \left( \frac{1}{2} + \varepsilon_j \right) T, \quad j = 0, 1
\]
for constants \(0 < \varepsilon_0 < \varepsilon_1 < \frac{1}{2}\).

Then we choose a nonnegative cut-off function \(\tilde{\zeta} \in C^0_0([0,T])\) such that
\[
\tilde{\zeta}(t) \equiv 1, \quad \forall t \in [\tilde{T}_1, \tilde{T}'_1].
\]  
(A.52)

Set \(\tilde{\varepsilon}(t, x) = \tilde{\zeta}(t) z_t(t, x)\) for \((t, x) \in \Omega_T\). Then \(\tilde{\varepsilon}\) solves
\[
\left\{ \begin{aligned}
\tilde{\varepsilon}_t - & \sum_{j,k=1}^n (a^{jk} z_{x_j})_{x_k} + 2\tilde{\varepsilon}_t + \tilde{\varepsilon} = \tilde{\zeta}_tt \tilde{z}_t + \tilde{\zeta}_t z_t + 2\tilde{\zeta}_t \tilde{z}_t + \tilde{\zeta}_t(2-b_0) z_t \\
& + \tilde{\zeta} \left[ \sum_{j,k=1}^n (a^{jk} z_{x_j})_{x_k} - \sum_{k=1}^n (b_k z_{x_k})_t - (\tilde{b} - 1 + \partial_t b_0) z_t - \tilde{b}_t z \right], \quad (t, x) \in \Omega_T, \\
\tilde{\varepsilon}(t, x) = & \tilde{\varepsilon}(T, x) = 0, \quad (t, x) \in \Gamma_T, \\
\tilde{\varepsilon}(0, x) = & \tilde{\varepsilon}(T, x) = 0, \quad x \in \Omega.
\right.
\]  
(A.53)
Let $T_1$ and $\phi$ be given by (A.11) and (A.6). Then by Theorem 1.12, there exists $\lambda_0 > 0$ such that for all $T > T_1$ and $\lambda \geq \lambda_0$, it holds that

\[
\lambda \int_{\Omega_T} \theta^2 \dot{z}^2 dxdt \leq C \left( \left\| \theta (\tilde{\zeta}_{tt} z_t + \tilde{\zeta}_t z_x + 2\tilde{\zeta}_t z_{tt} + \tilde{\zeta}(2 - b_0) z_{tt}) \right\|_{H^{-1}(\Omega_T)}^2 + \lambda^2 \int_0^T \int_{\Omega} \theta^2 z^2 dxdt \right) \tag{A.54}
\]

Using Hölder inequality and Sobolev embedding theorem, we find that

\[
\left\| \theta (\tilde{\zeta}_{tt} z_t + \tilde{\zeta}_t z_x + 2\tilde{\zeta}_t z_{tt} + \tilde{\zeta}(2 - b_0) z_{tt}) \right\|_{H^{-1}(\Omega_T)} \leq C \|\theta z_t\|_{L^2(\tilde{Q})} \tag{A.55}
\]

where $\tilde{Q} = \left( (0, \tilde{T}_1) \cup (\tilde{T}_1', T) \right) \times \Omega$, and

\[
\left\| \theta \tilde{\zeta} \left[ \sum_{j,k=1}^n (a_{ij}^k z_{x_j})_{x_k} - \sum_{k=1}^n (b_k z_{x_k})_{x_k} - (b - 1 + \partial_t b_0) z_t - \tilde{b}_t z \right] \right\|_{H^{-1}(\Omega_T)} \leq C (1 + \lambda) \|\theta z_t\|_{L^2(\Omega_T)} \tag{A.56}
\]

Combining (A.55)-(A.56), we have

\[
\lambda \|\theta \tilde{z}\|_{L^2(\Omega_T)}^2 \leq C \lambda^2 \|\theta z_t\|_{L^2(\tilde{Q})}^2 + C \lambda^2 \|\theta \tilde{z}\|_{L^2((0,T) \times \omega)}^2 + C\lambda^2 \varepsilon^2 \left( \|\theta \nabla z\|_{L^2(\Omega_T)}^2 + \|\theta z_t\|_{L^2(\Omega_T)}^2 \right) \tag{A.57}
\]

On the other hand, by (A.52), we find that

\[
\|\theta \tilde{z}\|_{L^2(\Omega_T)}^2 \geq \int_{T_1}^{\tilde{T}_1} \int_{\Omega} \theta^2 z_t^2 dxdt.
\]

Thus we have

\[
\|\theta z_t\|_{L^2(\Omega_T)}^2 \leq \|\theta \tilde{z}\|_{L^2(\Omega_T)}^2 + \|\theta z_t\|_{L^2(\tilde{Q})}^2. \tag{A.58}
\]

It follows from (A.57) and (A.58) that

\[
\|\theta z_t\|_{L^2(\Omega_T)}^2 \leq C \lambda \left( \|\theta z_t\|_{L^2(\tilde{Q})}^2 + \varepsilon^2 \|\theta \nabla z\|_{L^2(\Omega_T)}^2 + \int_0^T \int_{\omega} \theta^2 z_t^2 dxdt \right). \tag{A.59}
\]

**Step 2.** We set

\[
R_0 = \min_{x \in \Omega} \sqrt{\psi(x)}, \quad R_1 = \max_{x \in \Omega} \sqrt{\psi(x)}.
\]

By the definition (A.6) of the function $\phi$, we can see there exists an $\varepsilon_1 \in (0, 1/2)$, such that

\[
\phi(t, x) \leq \frac{R_1^2}{2} - \frac{c_1 T_1^2}{8} < 0, \quad \forall (t, x) \in \tilde{Q}. \tag{A.60}
\]
Further, since that
\[ \phi \left( \frac{T}{2}, x \right) = \psi(x) \geq R_0^2, \quad \forall \ x \in \Omega, \]
one can find an \( \varepsilon_0 \in (0, 1/2) \), such that
\[ \phi(t, x) \geq \frac{R_0^2}{2}, \quad \forall \ (t, x) \in (\bar{T}_0, \bar{T}_0') \times \Omega := Q_0. \quad (A.61) \]
Combining (A.59)-(A.61), we obtain that
\[
e^{\lambda R_0^2} \| z_t \|_{L^2(\Omega)}^2 
\leq C\lambda \left( e^{\lambda(R_1^2-cT^2/4)} \| z_t \|_{L^2(\tilde{\Omega})}^2 + \varepsilon^2 e^{2\lambda R_1^2} \| \nabla z \|_{L^2(\tilde{\Omega}_T)}^2 + e^{2\lambda R_1^2} \int_0^T \int_\omega z_t^2 \, dx \, dt \right). \]
Noting that
\[ \| z_t \|_{L^2(\tilde{\Omega})}^2 + \| \nabla z \|_{L^2(\tilde{\Omega}_T)}^2 \leq 2T \sup_{t \in [0, T]} \mathcal{E}(t) \leq 2T e^{(1+c)T} \mathcal{E}(T), \]
hence we have
\[
\| z_t \|_{L^2(\Omega)}^2 \leq C\lambda \left( e^{\lambda(R_1^2-cT^2/4)} \mathcal{E}(T) + e^{2\lambda R_1^2} \int_0^T \int_\omega z_t^2 \, dx \, dt \right). \quad (A.62) \]
**Step 3.** We choose a nonnegative function \( \zeta \in C^1([\bar{T}_0, \bar{T}_0']) \) with \( \zeta(\bar{T}_0) = \zeta(\bar{T}_0') = 0 \). Multiplying the equation in (A.1) by \( \zeta z \), integrating it in \( Q_0 \) and using integration by parts, we get
\[
\int_{Q_0} \zeta \left( z_t^2 + \sum_{j,k=1}^n a_{jk} z_{x_j} z_{x_k} + z^2 \right) \, dx \, dt = 2 \int_{\bar{T}_0} \zeta(t) \mathcal{E}(t) \, dt
\]
\[ = 2 \int_{Q_0} \zeta z_t^2 \, dx \, dt + \int_{Q_0} \zeta z \, dx \, dt - \int_{Q_0} \zeta \left( b_0 z_t + \sum_{k=1}^n b_k z_{x_k} + (\bar{b} - 1) z \right) \, dx \, dt
\]
\[ \leq C \int_{Q_0} z_t^2 \, dx \, dt + C \varepsilon \int_{Q_0} \zeta \| \nabla z \|_{L^2(\tilde{\Omega}_T)}^2 \, dx \, dt
\]
\[ \leq C \int_{Q_0} z_t^2 \, dx \, dt + C \varepsilon \int_{\bar{T}_0} \zeta(t) \mathcal{E}(t) \, dt. \]
Thus we obtain
\[ \min_{t \in [0, T]} \mathcal{E}(t) \leq C \int_{Q_0} z_t^2 \, dx \, dt. \quad (A.63) \]
Note that by (A.60), we also have
\[ \mathcal{E}'(t) + \int_{\Omega} b_0 z_t^2 \, dx = - \int_{\Omega} \left( \sum_{k=1}^n b_k z_{x_k} + \bar{b} z \right) \, dx \leq C \varepsilon \mathcal{E}(t), \]
hence we get
\[ \frac{d}{dt} \left( e^{-cT} \mathcal{E}(t) \right) \leq -e^{-cT} \int_{\Omega} b_0 z_t^2 \, dx \leq 0. \]
Then we have
\[
\mathcal{E}(t) \geq e^{-cT} e^{cT} \mathcal{E}(t) \geq e^{-cT} \mathcal{E}(T), \quad \forall \ t \in [0, T]. \quad (A.64) \]
Combining (A.63) and (A.64), we have

\[ E(T) \leq C \int_{Q_0} z_t^2 \, dx dt. \]  

(A.65)

It follows from (A.62) and (A.65) that

\[ E(T) \leq C \lambda \left( e^{\lambda (R_1^2 - R_0^2 - cT^2/4)} E(T) + e^{2\lambda R_1^2} \int_0^T \int_\omega z_t^2 \, dx dt \right). \]  

(A.66)

Noting that \( R_1^2 - R_0^2 - cT^2/4 < 0 \), let \( \lambda \) be large enough such that

\[ C \lambda e^{\lambda (R_1^2 - R_0^2 - cT^2/4)} \leq \frac{1}{2}, \]

then can deduce from (A.66) that

\[ E(T) \leq C_1 e^{C_1} \int_0^T \int_\omega z_t^2 \, dx dt, \]

(A.67)

where \( C_1 \) is a positive constant independent of initial data.

Combining (A.67) and (A.51), we obtain

\[ \|z_1\|_{L^2(\Omega)}^2 + \|z_0\|_{H^1(\Omega)}^2 \leq C \mathcal{E}(0) \leq C_2 e^{C_2} \int_0^T \int_\omega z_t^2 \, dx dt, \]

with a constant \( C_2 > 0 \) independent of initial data. Thus we obtain the desired inequality.

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