Arithmetic Siegel-Weil formula on $\mathcal{X}_0(N)$

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Special cycles on the modular curve $X_0(1)$

- Let $Y_0(1) := \text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ be the modular curve, it parameterize elliptic curves over $\mathbb{C}$ by
  \[ \tau \in \mathcal{H} \longmapsto E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau. \]

  Let $X_0(1)$ be its compactification.

- Given an integer $m > 0$, we consider the following moduli problem: for a $\mathbb{C}$-scheme $S$,
  \[ Z(m)(S) = \{ (E, \alpha) : E/S \text{ is an elliptic curve}, \]
  \[ \quad \alpha \in \text{End}_S(E) \text{ satisfying } \alpha^2 = -m. \} \]
The moduli problem $Z(m)$ parameterise elliptic curves with complex multiplication by the order $\mathcal{O}_m = \mathbb{Z} + \mathbb{Z} \cdot \sqrt{-m}$.

It can be shown the set $Z(m)(\mathbb{C})$ consists of finitely many points.

A natural question: what’s $\#Z(m)(\mathbb{C})$?
Counting points on the modular curve $X_0(1)$

$$\#Z(m)(\mathbb{C}) = \sum_E \#\{\alpha \in \text{End}_\mathbb{C}(E) : \alpha^2 = -m\} / \text{aut}(E).$$
Counting points on the modular curve $\mathcal{X}_0(1)$

$$\#Z(m)(\mathbb{C}) = \sum_{E} \#\{\alpha \in \text{End}_{\mathbb{C}}(E) : \alpha^2 = -m.\} / \text{aut}(E).$$

If $E = E_\tau$ appears on the right hand side, then $\tau$ satisfies a quadratic equation

$$a\tau^2 + b\tau + c = 0.$$ 

where $a, b, c \in \mathbb{Z}$ and $\gcd(a, b, c) = 1$, the discriminant of this equation is $b^2 - 4ac$.

Recall that the discriminant of $\mathcal{O}_m$ is $-4m$, then there exists an integer $k > 0$ such that

$$-4m = k^2(b^2 - 4ac).$$
Counting points on the modular curve $X_0(1)$

Then by the theory of complex multiplication, we have

$$\#Z(m)(\mathbb{C}) = \sum_{E} \#\{\alpha \in \text{End}_{\mathbb{C}}(E) : \alpha^2 = -m.\} / \text{aut}(E).$$

$$= \sum_{k>0: k^2|4m} h\left(\frac{4m}{k^2}\right) = H(4m).$$

Recall that for a positive integer $N$,

$$H(N) = \#\text{SL}_2(\mathbb{Z})\text{-equivalence classes of positive definite binary quadratic form of disc } -N.$$  

$$h(N) = \#\text{SL}_2(\mathbb{Z})\text{-equivalence classes of primitive positive definite binary quadratic form of disc } -N.$$
Geometric Siegel-Weil formula on $X_0(1)$

- The modular curve $X_0(1)$ is the (compactified) GSpin Shimura variety attached to the rank 3 quadratic lattice $V = M_2(\mathbb{Z})^{tr=0}$, because

$$\text{GSpin}(V_\mathbb{Q}) \simeq \text{GL}_2, \quad \text{GSpin}(V) \simeq \text{GL}_2(\mathbb{Z})$$

- There is an Eisenstein series $E(z, s, 1_{V \otimes \hat{\mathbb{Z}}})$ associated to the lattice $V$ via Weil representation.
The modular curve $X_0(1)$ is the (compactified) GSpin Shimura variety attached to the rank 3 quadratic lattice $V = M_2(\mathbb{Z})^{tr=0}$, because

$$\text{GSpin}(V_\mathbb{Q}) \cong \text{GL}_2, \text{GSpin}(V) \cong \text{GL}_2(\mathbb{Z})$$

There is an Eisenstein series $E(z, s, 1_{V \otimes \hat{\mathbb{Z}}})$ associated to the lattice $V$ via Weil representation.

**Theorem (Geometric Sigel-Weil formula on $Y_0(1)$)**

Let $m > 0$ be an integer, then

$$\# \mathbb{Z}(m)(\mathbb{C}) \cdot q^m = \frac{1}{12} \cdot E_m(z, \frac{1}{2}, 1_{V \otimes \hat{\mathbb{Z}}}).$$
Computing the Eisenstein series

We can also compute the Fourier coefficients $E_m(z, \frac{1}{2}, 1_{\mathbb{V} \otimes \hat{\mathbb{Z}}})$ in the following way,

$$
E_m(z, \frac{1}{2}, 1_{\mathbb{V} \otimes \hat{\mathbb{Z}}}) = 4\pi (1 + i) \sqrt{m} \cdot q^m \prod_p W_{m,p}(1, \frac{1}{2}, 1_{\mathbb{V} \otimes \mathbb{Z}_p}).
$$
Computing the Eisenstein series

We can also compute the Fourier coefficients $E_m(z, \frac{1}{2}, 1_{V \otimes \hat{\mathbb{Z}}})$ in the following way,

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Assume $-m < -4$ is a fundamental discriminant, by the works of Kudla, Rapoport and Yang

$$E_m(z, \frac{1}{2}, 1_{V \otimes \hat{\mathbb{Z}}}) = (3 - \chi_m(2)) \cdot \frac{\sqrt{m}}{\pi} L(1, \chi_m) \cdot q^m.$$
On the other hand,

\[ H(4m) = h(m) + h(4m) = (3 - \chi_m(2)) \cdot h(m). \]
Geometric Siegel-Weil v.s. Class number formula

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\[ H(4m) = h(m) + h(4m) = (3 - \chi_m(2)) \cdot h(m). \]

The geometric Siegel-Weil formula on \( Y_0(1) \) implies

**Theorem (Class number formula)**

*Let \(-m < -4\) be a fundamental discriminant, then*

\[ h(m) = \frac{\sqrt{m}}{\pi} L(1, \chi_m). \]
The modular curve $Y_0(N)$

- Let $Y_0(N) = \Gamma_0(N) \backslash \mathcal{H}$, and $X_0(N) = Y_0(N) \cup \{\text{cusps}\}$. The modular curve $Y_0(N)$ parameterize cyclic isogenies between elliptic curves over $\mathbb{C}$ by the following

$$\tau \in \mathcal{H} \mapsto (E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \rightarrow E_\tau^N).$$

- Given an integer $m > 0$, we consider the following moduli problem: for a $\mathbb{C}$-scheme $S$,

$$Z(m)(S) = \{(E \xrightarrow{\pi} E', \alpha) : E \xrightarrow{\pi} E' \text{ is a cyclic isogeny between elliptic curves}, \alpha \in \text{Hom}_S(E, E') \text{ satisfies } \alpha^\vee \circ \pi + \pi^\vee \circ \alpha = 0 \text{ and } \alpha^\vee \circ \alpha = m\}. $$
Let $N > 0$ be an integer, and $\Delta(N)$ be the following rank 3 quadratic lattice over $\mathbb{Z}$,

$$\Delta(N) = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$
Geometric Siegel-Weil formula on $X_0(N)$

Let $N > 0$ be an integer, and $\Delta(N)$ be the following rank 3 quadratic lattice over $\mathbb{Z}$,

$$\Delta(N) = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

Then the geometric Siegel-Weil formula on $X_0(N)$ is proved by Tuoping and Tonghai [DY19],

**Theorem (Geometric Siegel-Weil formula on $X_0(N)$)**

*For an integer $m > 0$, we have*

$$\# Z(m)(\mathbb{C}) \cdot q^m = \frac{\psi(N)}{12} E(z, \frac{1}{2}, 1_{\Delta(N)}(\hat{\mathbb{Z}})).$$

*Here $\psi(N) = N \prod_{p|N} (1 + p^{-1})$.***
Let $\mathcal{Y}_0(N)$ be the stack of $\Gamma_0(N)$-level structures on elliptic curves defined by Katz and Mazur in [KM85]: for a scheme $S$,

$$\mathcal{Y}_0(N)(S) = \{ E \xrightarrow{\pi} E' : \pi \text{ is a cyclic isogeny and } \pi^\vee \circ \pi = N \}.$$
The stack $\mathcal{X}_0(N)$

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\mathcal{Y}_0(N)(S) = \{ E \xrightarrow{\pi} E' : \pi \text{ is a cyclic isogeny and } \pi^\vee \circ \pi = N \}.
$$

Here $\pi$ is cyclic means that the order $N$ group scheme $G := \ker(\pi)$ is a cyclic group scheme in the sense that there exists a section $P \in G(S)$ such that for any $f \in \mathcal{O}_G$,

$$
\det(T - f) = \prod_{a=1}^{N} (T - f(aP)).
$$

Let $\mathcal{X}_0(N)$ be its compactification.
Special cycles on $\mathcal{X}_0(N)$

- Given an integer $m > 0$, we consider the following moduli problem: for a scheme $S$,

$$Z(m)(S) = \{(E \xrightarrow{\pi} E', \alpha) : \pi \text{ is a cyclic } N\text{-isogeny, } \alpha \text{ is an isogeny from } E \text{ to } E' \text{ satisfying } \alpha^\vee \circ \alpha = m \text{ and } \alpha^\vee \circ \pi + \pi^\vee \circ \alpha = 0.\}.$$ 

It is a generalized Cartier divisor on $\mathcal{X}_0(N)$ and has no intersections with cusps.
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It is a generalized Cartier divisor on $\mathcal{X}_0(N)$ and has no intersections with cusps.

- Given a $2 \times 2$ positive definite symmetric matrix $T$, we define the moduli problem $Z(T)$ as follows: for a scheme $S$,

$$Z(T)(S) = \{(E \xrightarrow{\pi} E', \alpha_1, \alpha_2) : \pi \text{ is a cyclic } N\text{-isogeny, } \alpha_i \text{ are isogenies from } E \text{ to } E' \text{ satisfying } \frac{1}{2}(\alpha_i, \alpha_j) = T \text{ and } \alpha_i^\vee \circ \pi + \pi^\vee \circ \alpha_i = 0.\}.$$
The special cycle $\mathcal{Z}(T)$

Let $T \in \text{Sym}_2(\mathbb{Q})$ be a nonsingular matrix, we define the following difference set

$$\text{Diff}(T, \Delta(N)) = \{l \text{ is a finite prime} : T \text{ is not represented by } \Delta(N) \otimes \mathbb{Q}_l.\}$$
The special cycle $\mathcal{Z}(T)$

Let $T \in \text{Sym}_2(\mathbb{Q})$ be a nonsingular matrix, we define the following difference set

$$\text{Diff}(T, \Delta(N)) = \{l \text{ is a finite prime} : T \text{ is not represented by } \Delta(N) \otimes \mathbb{Q}_l.\}$$

**Lemma**

Let $T \in \text{Sym}_2(\mathbb{Q})$ be a nonsingular matrix. If $\mathcal{Z}(T)(\overline{\mathbb{F}}_p) \neq \emptyset$ for some prime $p$, then $T$ is positive definite, and

$$\text{Diff}(T, \Delta(N)) = \{p\}.$$ 

Moreover, in this case, the special cycle $\mathcal{Z}(T)$ is supported in the supersingular locus of the special fiber $\mathcal{X}_0(N)_{\mathbb{F}_p}$. 
Let $T$ be a $2 \times 2$ positive definite symmetric matrix with diagonal elements $m_1, m_2$, then we define

$$\deg(Z(T)) = \chi(Z(T), O_{Z(m_1)} \otimes L O_{Z(m_2)}) \cdot \log p.$$ 

where $p \in \text{Diff}(T, \Delta(N))$. 
Let $T$ be a $2 \times 2$ positive definite symmetric matrix with diagonal elements $m_1, m_2$, then we define

$$\text{deg}(\mathcal{Z}(T)) = \chi(\mathcal{Z}(T), \mathcal{O}_{\mathcal{Z}(m_1)} \otimes L \mathcal{O}_{\mathcal{Z}(m_2)}) \cdot \log p.$$ 

where $p \in \text{Diff}(T, \Delta(N))$.

**Theorem (Arithmetic Siegel-Weil formula on $\mathcal{X}_0(N)$)**

Let $T \in \text{Sym}_2(\mathbb{Q})$ be a positive definite symmetric matrix, then

$$\text{deg}(\mathcal{Z}(T))q^T = \frac{\psi(N)}{24} \cdot E'(z, 0, 1_{\Delta(N) \otimes \hat{Z}}^2).$$

where $z = x + iy \in \mathcal{H}_2$ and $q^T = e^{2\pi i \text{tr}(Tz)}$. 
Key ingredients

- Formal uniformization of the supersingular locus of $\mathcal{X}_0(N)$.

  It connects intersection numbers on $\mathcal{X}_0(N)$ with local arithmetic intersection numbers on the RZ space associated to $\mathcal{X}_0(N)$. 
Key ingredients

- Formal uniformization of the supersingular locus of $\mathcal{X}_0(N)$.
  It connects intersection numbers on $\mathcal{X}_0(N)$ with local arithmetic intersection numbers on the RZ space associated to $\mathcal{X}_0(N)$.

- Kudla-Rapoport conjecture for the RZ space associated to $\mathcal{X}_0(N)$.
  It connects local arithmetic intersection numbers on RZ space with Whittaker functions.
Key ingredients

- Formal uniformization of the supersingular locus of $\mathcal{X}_0(N)$. It connects intersection numbers on $\mathcal{X}_0(N)$ with local arithmetic intersection numbers on the RZ space associated to $\mathcal{X}_0(N)$.

- Kudla-Rapoport conjecture for the RZ space associated to $\mathcal{X}_0(N)$. It connects local arithmetic intersection numbers on RZ space with Whittaker functions.

Both are proved by embedding trick!
Let $X$ be a $p$-divisible group of dim 1, height 2. Consider the following functor: for every $S \in \text{Nilp}_W$, the set $\mathcal{N}(S)$ consists of $(((X, X'), (\rho, \rho'), (\lambda, \lambda'))$, where

(1) $X$ and $X'$ are two $p$-divisible group over $S$, $\rho$ and $\rho'$ are two height 0 quasi-isogenies between $p$-divisible groups $\rho : \overline{X} \times_F \overline{S} \to X \times_S \overline{S}$, $\rho' : \overline{X} \times_F \overline{S} \to X' \times_S \overline{S}$.

(2) $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$ are two principal polarizations, such that Zariski locally on $\overline{S}$, we have

$$
\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^\vee \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0.
$$

for some $c(\rho) = c(\rho') \in \mathbb{Z}_p^\times$. 

RZ space associated to $\mathcal{X}_0(1) \times \mathcal{X}_0(1)$
Let $\mathbb{X}$ be a $p$-divisible group of dim 1, height 2. Consider the following functor: for every $S \in \text{Nilp}_W$, the set $\mathcal{N}(S)$ consists of $\left(((X, X'), (\rho, \rho'), (\lambda, \lambda'))\right)$, where

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$$\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^\vee \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0.$$ 

for some $c(\rho) = c(\rho') \in \mathbb{Z}_p^\times$.

$\mathcal{N} \simeq \text{Spf} \ W[[t_1, t_2]]$. 

**RZ space associated to $\mathcal{X}_0(1) \times \mathcal{X}_0(1)$**
Special cycles on $\mathcal{N}$

Let $\mathcal{B}$ the unique division quaternion algebra over $\mathbb{Q}_p$, it is isometric to $\text{End}^0(X)$.

**Definition**

For any subset $L \subset \mathcal{B}$, define the special cycle $\mathcal{Z}^\#(L) \subset \mathcal{N}$ to be the closed formal subscheme cut out by the condition,

$$\rho'^{\text{univ}} \circ x \circ (\rho^{\text{univ}})^{-1} \in \text{Hom}(X^{\text{univ}}, X'^{\text{univ}}).$$

for all $x \in L$. 

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**Definition**

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$$\rho^\text{univ} \circ \chi \circ (\rho^\text{univ})^{-1} \in \text{Hom}(X^\text{univ}, X'^\text{univ}).$$

for all $x \in L$.

Let $L$ be a rank 3 lattice with basis $x_1, x_2$ and $x_3$. Define the local arithmetic intersection number on $\mathcal{N}$ to be

$$\text{Int}^\#(L) = \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}^\#(x_1)} \otimes^L \mathcal{O}_\mathcal{N} \mathcal{O}_{\mathcal{Z}^\#(x_2)} \otimes^L \mathcal{O}_\mathcal{N} \mathcal{O}_{\mathcal{Z}^\#(x_3)}).$$
Difference divisor on $\mathcal{N}$

For any $x \in \mathbb{B}$, the special cycle $\mathcal{Z}^\#(x)$ is cut out by a single equation $f_x \in \mathcal{W}[[t_1, t_2]]$, define $d_x = f_x/f_{p^{-1}x} \in \mathcal{W}[[t_1, t_2]]$ and the difference divisor $\mathcal{D}(x) = \text{Spf } \mathcal{W}[[t_1, t_2]]/(d_x)$.
For any $x \in \mathbb{B}$, the special cycle $Z^\#(x)$ is cut out by a single equation $f_x \in \mathcal{W}[[t_1, t_2]]$, define $d_x = f_x/f_{p^{-1}x} \in \mathcal{W}[[t_1, t_2]]$ and the difference divisor $D(x) = \text{Spf } \mathcal{W}[[t_1, t_2]]/(d_x)$

**Theorem**

The difference divisor $D(x)$ is regular.

Recently we have proved that difference divisors on GSpin RZ spaces with hyperspecial level structure are regular, the formal scheme $\mathcal{N}$ is a special example of such RZ spaces.
RZ space associated to $\mathcal{X}_0(N)$

Fix a $N$-isogeny $x_0 : \mathbb{X} \to \mathbb{X}$. Consider the following functor: for every $S \in \text{Nilp}_W$, the set $\mathcal{N}_0(N)(S)$ consists of

$$(X \xrightarrow{x} X', (\rho, \rho'), (\lambda, \lambda'))$$

where

1. $X$ and $X'$ ...

2. $\lambda : X \to \check{X}$, $\lambda' : X' \to \check{X}'$ ...

3. $x : X \to X'$ is a cyclic isogeny (i.e., ker$(x)$ is a cyclic group scheme over $S$) lifting $\rho' \circ x_0 \circ \rho^{-1}$. 

Theorem ([KM85]) The natural morphism $\mathcal{N}_0(N) \to \mathcal{N}$ is a closed immersion, and $\mathcal{N}_0(N)$ is regular.
RZ space associated to $\mathcal{X}_0(N)$

Fix a $N$-isogeny $x_0 : \mathbb{X} \to \mathbb{X}$. Consider the following functor: for every $S \in \text{Nilp}_W$, the set $\mathcal{N}_0(N)(S)$ consists of $(\mathit{X} \xrightarrow{x} \mathit{X}', (\rho, \rho'), (\lambda, \lambda'))$, where

1. $\mathit{X}$ and $\mathit{X}'$ ...
2. $\lambda : \mathit{X} \to \mathit{X}^\vee$, $\lambda' : \mathit{X}' \to \mathit{X}'^\vee$ ...
3. $x : \mathit{X} \to \mathit{X}'$ is a cyclic isogeny (i.e., $\ker(x)$ is a cyclic group scheme over $S$) lifting $\rho' \circ x_0 \circ \rho^{-1}$.

**Theorem ([KM85])**

*The natural morphism $\mathcal{N}_0(N) \to \mathcal{N}$ is a closed immersion, and $\mathcal{N}_0(N)$ is regular.*
An isomorphism

Recall that we have fixed a $N$-isogeny $x_0$ when we define $\mathcal{N}_0(N)$.

**Theorem**

*There is an isomorphism between formal schemes,*

$$\mathcal{D}(x_0) \xrightarrow{\sim} \mathcal{N}_0(N).$$
Remark

By the isomorphism and Zink’s windows theory, we can compute the special fiber $\mathcal{N}_0(p^n)_p$ as follows

$$\mathbb{F}[[t_1, t_2]]/ \left( (t_1 - t_2^{p^n}) \cdot (t_2 - t_1^{p^n}) \cdot \prod_{\substack{a+b=n \atop a,b \geq 1}} (t_1^{p^{a-1}} - t_2^{p^{b-1}})^{p-1} \right).$$

which coincides with Katz-Mazur’s computation.
Let $\mathcal{W} = \{x_0\} \perp \subset \mathbb{B}$.

**Definition**

For any subset $M \subset \mathcal{W}$, define the special cycle $\mathcal{Z}(M) \subset \mathcal{N}_0(N)$ to be the closed formal subscheme cut out by the condition,

$$\rho'_{univ} \circ x \circ (\rho_{univ})^{-1} \in \text{Hom}(X_{univ}, X'_{univ}).$$

for all $x \in M$.

Let $M$ be a rank 2 lattice with basis $x_1$ and $x_2$. Define the local arithmetic intersection number on $\mathcal{N}_0(N)$ to be

$$\text{Int}(M) = \chi(\mathcal{N}_0(N), \mathcal{O}_{\mathcal{Z}(x_1)} \boxtimes_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}).$$
Difference formula at the geometric side

By the isomorphism $\mathcal{D}(x_0) \simeq \mathcal{N}_0(N)$, we can prove the following theorem

**Theorem**

*For any rank 2 lattice $M \subset W$, the following identity holds,*

$$\text{Int}(M) = \text{Int}^\#(M \oplus \mathbb{Z}_p \cdot x_0) - \text{Int}^\#(M \oplus \mathbb{Z}_p \cdot p^{-1}x_0).$$
Local density

- For two quadratic lattice $L$ and $M$, the local density is defined to be
  $$\text{Den}(M, L) = \lim_{d \to \infty} \frac{\#\text{Rep}_{M,L}(\mathbb{Z}_p/p^d)}{p^{d \cdot \dim(\text{Rep}_{M,L})_{\mathbb{Q}_p}}}.$$  

- Let $H$ be a rank 2 quadratic lattice given by $q_H(x, y) = xy$, define the local density polynomial to be (rank $L = 2n - 1$)
  $$\text{Den}(X, L) \big|_{X=p^{-k}} = \frac{\text{Den}(H^{k+n}, L)}{\text{Nor}^+(p^{-k}, 2n - 1)},$$
  where $\text{Nor}^c(X, m) = (1 - \frac{1+(-1)^{m+1}}{2} \cdot \varepsilon q^{-(m+1)/2}X) \prod_{1 \leq i < (m+1)/2} (1 - q^{-2i}X^2)$. 

Examples of local density

- When \( m \) is squarefree, then

\[
\text{Den}(\Delta(1) \otimes \mathbb{Z}_p, \langle m \rangle) = 1 - \chi_m(p)p^{-1}.
\]

- When \( \nu_p(N) = 0 \) or \( 1 \), we have

\[
\text{Den}(H^k, \langle N \rangle) = \begin{cases} 
(1 - p^{-k})(1 + p^{1-k}), & \text{when } p \mid N; \\
1 - p^{-k}, & \text{when } p \nmid N.
\end{cases}
\]
Difference formula at the analytic side

Let \( \delta_p(N) = \Delta(N) \otimes \mathbb{Z}_p \), define the following local density function with level \( N \),

\[
\text{Den}_{\Delta(N)}(X, M) \big|_{X = p^{-k}} = \begin{cases} 
\frac{\text{Den}(\delta_p(N) \oplus H^k, M)}{\text{Nor}^+(p^{-k}, 1)}, & \text{when } p \mid N; \\
\frac{\text{Den}(\delta_p(N) \oplus H^k, M)}{\text{Nor}^{(N,p)}(p^{-k}, 2)}, & \text{when } p \nmid N.
\end{cases}
\]

Theorem

For any rank 2 lattice \( M \subset \mathbb{W} \), the following identity holds,

\[
\text{Den}_{\Delta(N)}(X, M) = \text{Den}(X, M \oplus \mathbb{Z}_p \cdot x_0) - X^2 \cdot \text{Den}(X, M \oplus \mathbb{Z}_p \cdot p^{-1}x_0).
\]
Difference formula at the analytic side

- We also define

\[ \partial \text{Den}(L) = -\frac{d}{dX} \bigg|_{X=1} \text{Den}(X, L). \]

\[ \partial \text{Den}_{\Delta(N)}(M) = -\frac{d}{dX} \bigg|_{X=1} \text{Den}_{\Delta(N)}(X, M). \]

**Corollary**

The lattice \( M \oplus \mathbb{Z}_p \cdot x_0 \) can’t be isometrically embedded into the lattice \( H^2 \), hence \( \text{Den}(1, M \oplus \mathbb{Z}_p \cdot x_0) = 0 \).

\[ \partial \text{Den}_{\Delta(N)}(M) = \partial \text{Den}(M \oplus \mathbb{Z}_p \cdot x_0) - \partial \text{Den}(M \oplus \mathbb{Z}_p \cdot p^{-1}x_0). \]
A theorem of Gross and Keating

Theorem ([GK93], [Rap07], [Wed07])

For any rank 3 lattice $L \subset \mathbb{B}$,

$$\text{Int}^\#(L) = \partial \text{Den}(L).$$
A theorem of Gross and Keating

Theorem ([GK93],[Rap07],[Wed07])

For any rank 3 lattice \( L \subset \mathbb{B} \),

\[
\operatorname{Int}^\#(L) = \partial \operatorname{Den}(L).
\]

Combing this with two difference formulas, we obtain

Theorem (KR conjecture for the RZ space \( \mathcal{N}_0(N) \))

For any rank 2 lattice \( M \subset \mathbb{W} \),

\[
\operatorname{Int}(M) = \partial \operatorname{Den}_{\Delta(N)}(M).
\]
Formal uniformization

There is an isomorphism of formal stacks over \( W \),

\[
\hat{X}_0(N)/(x_0(N)^{ss}_{\hat{F}_p}) \xrightarrow{\Theta_{x_0(N)}} B^\times(\mathbb{Q})_0 \setminus [\mathcal{N}_0(N) \times \text{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)]
\]

where \( B^\times(\mathbb{Q})_0 \) is the subgroup of \( B^\times(\mathbb{Q}) \) consisting of elements whose norm has \( p \)-adic valuation 0.
Formal uniformization

- There is an isomorphism of formal stacks over $\mathcal{W}$,

$$
\hat{X}_0(N)/(x_0(N)_{F_p}) \xrightarrow{\Theta_{X_0(N)}} B^\times(\mathbb{Q})_0 \times [\mathcal{N}_0(N) \times \text{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}_p)]
$$

where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has $p$-adic valuation 0.

As a corollary, we have the formal uniformization of the special cycles,

$$
\hat{\mathcal{Z}}^{ss}(T) = \sum_{x \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2} \sum_{g \in B_x^\times(\mathbb{Q})_0 \backslash \text{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}_p)} 1_{\Delta(N)}(g^{-1}x) \cdot \Theta_{X_0(N)}^{-1}(\mathcal{Z}(x), g).
$$
Proof strategy

**Theorem**

*For any rank 2 lattice* $M \subset \mathbb{W}$, *the following identity holds,*

$$\text{Den}_{\Delta(N)}(X, M) = \text{Den}(X, M \oplus \mathbb{Z}_p \cdot x_0) - X^2 \cdot \text{Den}(X, M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$

**Key idea:** First embed $x_0$ to the large self-dual lattice $H^k$, the depth of the an embedding is defined to be

$$x_0 \in p^t H^k, \text{ but } x_0 \notin p^{t+1} H^k.$$  

then embed $M$ into $\{x_0\}^\perp \subset H^k$, which is totally determined by the depth of $x_0$!

**Lemma (Witt theorem for lattices, [Mor79])**

*Let $H$ be a self-dual quadratic lattice, if $x_1$ and $x_2$ has the same depth and norm, then there exists $g \in \text{O}(H)$ such that $g \cdot x_1 = x_2$.***
Thank you!
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