A two-phase rank-based Soft-Impute algorithm for low-rank matrix completion

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Abstract

Matrix completion aims to recover an unknown low-rank matrix from a small subset of its entries. In many applications, the rank of the unknown target matrix is known in advance. In this paper, we propose a two-phase algorithm that leverages the rank information to compute both a suitable value for the regularization parameter and a warm-start for an accelerated Soft-Impute algorithm. Properties inherited from proximal gradient algorithms are exploited to propose a parameter tuning to accelerate the method and also to establish a convergence analysis. Numerical experiments with both synthetic and real data show that the proposed algorithm can recover low-rank matrices, with high precision, faster than other well-established matrix completion algorithms.

Keywords: Matrix Completion. Proximal gradient algorithm. Soft-Thresholding. Recommender systems.

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1. Introduction

The problem of recovering missing entries in a low-rank matrix $A \in \mathbb{R}^{m \times n}$ can be formulated in terms of a rank minimization problem as

$$\min_{X \in \mathbb{R}^{m \times n}} \text{rank}(X)$$

subject to $P_\Omega(X) = P_\Omega(A)$,

where $\Omega$ denotes the set of indices of the known entries of $A$ and $P_\Omega(\cdot)$ is the projection operator, defined as

$$[P_\Omega(X)]_{ij} := \begin{cases} X_{ij}, & \text{if } (i,j) \in \Omega \\ 0, & \text{otherwise} \end{cases}$$

with $P_\Omega^\perp(\cdot)$ defined by $P_\Omega^\perp(X) = X - P_\Omega(X)$.

Despite its theoretical importance, problem (1) is non-convex and combinatorially hard for general sets $\Omega$ \cite{1}. To overcome such disadvantage, several alternatives have been proposed in the literature \cite{2, 3, 4, 5, 6}. A common way to swerve the non-convexity in problem (1) is to replace the rank objective by a convex relaxation such as the nuclear norm $\|X\|_*$, as proposed in \cite{7} and \cite{8}.

The nuclear norm of a matrix is derived from its Singular Value Decomposition (SVD). Let $X = U \Sigma V^\top$ be the SVD of $X \in \mathbb{R}^{m \times n}$ and assume that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0$ are its singular values. The nuclear norm of $X$ is defined as $\|X\|_* := \sum_j \sigma_j$ and it has been used to propose a convex relaxation for problem (1).

In \cite{9} the authors present an algorithm, called Singular Value Thresholding (SVT), and proved that the generated sequence converges to the unique solution of the following optimization problem

$$\min_{X \in \mathbb{R}^{m \times n}} \tau \|X\|_* + \frac{1}{2}\|X\|_F^2$$

subject to $P_\Omega(X) = P_\Omega(A)$,

where $\tau > 0$ is a regularization parameter. The component $\tau \|X\|_*$ in the objective function is a convex relaxation for $\text{rank}(X)$, while $\frac{1}{2}\|X\|_F^2$ is a strongly
convex term granting a unique solution. Due to its theoretical and computational properties, the SVT algorithm is an important reference for matrix completion and it is often used as a benchmark.

Another alternative formulation for problem (1) is to consider a tolerance on the recovering of the known entries. This can be particularly useful in applications where data are obtained through noisy processes. In this case, it may be worth to consider the following optimization problem

\[
\minimize_{X \in \mathbb{R}^{m \times n}} \|X\|_* \quad \text{subject to} \quad \|P_{\Omega}(X - A)\|_F^2 \leq \delta,
\]

where $\delta \geq 0$ is a given recovering error tolerance.

In [10] and also in [11] it is exploited the following Lagrangian formulation for problem (3),

\[
\minimize_{X \in \mathbb{R}^{m \times n}} \frac{1}{2}\|P_{\Omega}(A) - P_{\Omega}(X)\|_F^2 + \lambda \|X\|_* =: f_\lambda(X),
\]

where $\lambda > 0$ is a regularization parameter. The authors in [10] proposed an algorithm called Fixed Point Continuation (FPC) and prove that, under certain conditions, the sequence produced by FPC converges to a solution of (4).

Addressing the same formulation (4), the authors in [11] proposed an algorithm, called Soft-Impute (SI), which generates a sequence that converges to a solution for the problem (4), with convergence rate $O(1/k)$, where $k$ is the number of iterations.

Both, the FPC and SI algorithms, rely on a deflated version of the SVD decomposition, calculated by using the Soft-Thresholding (ST) operator, defined as

\[
S_\tau(M) := U\Sigma_\tau V^T, \quad \Sigma_\tau = \text{diag}[(\sigma_1 - \tau)_+, \cdots, (\sigma_q - \tau)_+],
\]

where $M = U\Sigma V^T$ is the compact-SVD of a rank $q$ matrix $M$ and $t_+ = \max(0, t)$.

Although they may have different motivations, the FPC and SI algorithms can be seen as particular cases of the proximal gradient method [12]. This fact
was actually used in [13] to derive a convergence analysis for these algorithms and to propose acceleration strategies for the SI. Such algorithms, however, are still very sensitive to the choice of the regularization parameter $\lambda$ and the parameter tuning process can be quite cumbersome in real applications (notice that a new optimization problem needs to be solved for each choice of $\lambda$ in (4)).

Another information that is disregarded, or not properly used, by these algorithms is the eventual knowledge of the rank of the target matrix. In some applications, such as in problems involving Euclidean Distance Matrices (EDM), the rank of the matrix to be completed is known in advance. For example, it can be proved that the rank of an EDM derived from a set of points in $\mathbb{R}^d$ is at most $d + 2$ [14]. This information might be useful to estimate the parameter $\lambda$ and improve the completion performance.

In [15] the authors take into account the rank information and propose an algorithm called Fixed-Rank Soft-Impute (FRSI) to estimate missing entries in EDMs using the rank information to estimate the regularization parameter $\lambda$. However, despite the good numerical results, there is no convergence guarantees for FRSI and, in some cases, the target rank may not be accurately reached due to numerical instabilities.

In this paper, we combine some insights of [13] and [15] to propose a two-phase algorithm, named Accelerated Rank-based Soft-Impute (ARSI). In the first phase, an heuristic uses the rank of the target matrix to compute a sequence of values for the regularization parameter $\lambda$ and, as a by-product, a good starting point for the second phase which consists of an accelerated Soft-Impute algorithm [16], that can be seen as an accelerated proximal gradient method [12].

The first phase is stopped as soon as the estimates for $\lambda$ stabilize and the convergence of the second phase follows from the properties of accelerated proximal gradient methods. In addition, our approach maintains the “sparse plus low-rank” structure of the soft-thresholding argument, which is a desired property specially for large scale matrix completion problems.

The rest of the paper is organized as follows. Section [2] presents the proposed
algorithm and discuss the main ideas behind the heuristic used in phase-one. In Section 2.1 the convergence theory for the second phase of the algorithm is established through the connection between Soft-Impute and the proximal gradient method. Section 3 reports some numerical experiments on synthetic and real data and compares the proposed algorithm with other well-established matrix completion algorithms. Finally, we end the paper with some concluding remarks in Section 4.

2. A two-phase rank-based algorithm

Let $A \in \mathbb{R}^{m \times n}$ be a matrix with missing entries and rank $r$, which we assume it is known in advance. In order to explain the rationale behind the proposed algorithm, let us first recall the iteration of Soft-Impute (SI) [11]. According to the notation used in the Introduction, SI can be described as

$$X^k = S_\lambda \left( P_{\Omega}(A) + P_{\Omega}^\perp(X^{k-1}) \right).$$

(6)

Notice that for an arbitrary value of $\lambda$, there is no reason to expect $X^k$ to have rank $r$. However, if we set

$$\lambda = \sigma_{r+1} \left( P_{\Omega}(A) + P_{\Omega}^\perp(X^{k-1}) \right),$$

i.e., the $r + 1$ largest singular value of $P_{\Omega}(A) + P_{\Omega}^\perp(X^{k-1})$, then from the definition of $S_\lambda$ in (5) we can ensure that the rank is at most $r$ for each iterate $X^k$.

Based on this observation and inspired by accelerated versions of the proximal gradient method [12], we propose an heuristic for the first phase (warm-start) of our algorithm: it resembles an accelerated Soft-Impute where the regularization parameter changes from one iteration to another. Phase-one runs for a pre-specified number $w$ of iterations or until the values of $\rho_j = \sigma_{r+1}(P_{\Omega}(A) + P_{\Omega}^\perp(Z^j))$ stabilize. The last value of $\rho_j$ from phase one is used as regularization parameter $\lambda$ for the second phase, which consists of an accelerated Soft-Impute algorithm for problem (4), starting from $Z^{j+1}$ from the
previous phase. These ideas are summarized in Algorithm 1 called Accelerated Rank-based Soft-Impute (ARSI) algorithm.

**Algorithm 1** Accelerated Rank-based Soft-Impute (ARSI)

Input: Known entries of \( A \in \mathbb{R}^{m \times n} \) indexed by \( \Omega \), rank \( r \), \( \epsilon > 0 \), \( w, it_{\text{max}} \in \mathbb{N} \), with \( it_{\text{max}} > w \) and \( \beta > 0 \).

1: Initialize \( X^0 = 0 \), \( Z^1 = 0 \) and \( \rho_0 = \infty \)

2: **Warm-start**

3: for \( j = 1 \) to \( w \) do

4: Compute the truncated \((r+1)-\text{SVD}\) of \( P_{\Omega}(A) + P_{\Omega}^\perp(Z^j) \)

5: Set \( \rho_j = \sigma_{r+1}(P_{\Omega}(A) + P_{\Omega}^\perp(Z^j)) \)

6: if \( |\rho_j - \rho_{j-1}|/(1 + \rho_j) < \epsilon_\rho \) then exit.

7: Compute \( X^j \leftarrow S_{\rho_j}(P_{\Omega}(A) + P_{\Omega}^\perp(Z^j)) \)

8: \( Z^{j+1} \leftarrow X^j + \frac{j-1}{j + \beta}(X^j - X^{j-1}) \)

9: end for

10: **Accelerated Soft-Impute**

11: Set \( \lambda = \rho_j \)

12: for \( k = j + 1 \) to \( it_{\text{max}} \) do

13: Compute \( X^k \leftarrow S_{\lambda}(P_{\Omega}(A) + P_{\Omega}^\perp(Z^k)) \)

14: if some stopping criterion is verified then stop.

15: \( Z^{k+1} \leftarrow X^k + \frac{k-1}{k + 2}(X^k - X^{k-1}) \)

16: end for

Output: \( X^k \)

We remark that the first phase of Algorithm 1 is *not* an accelerated proximal gradient algorithm because the parameter \( \rho_j \) used in the proximal operator (Soft-Thresholding) changes at each iteration but the step-size remains constant. Thus, the convergence analysis from the prox-gradient literature does not directly applies and the warm-start phase must be regarded as an heuristic.

Furthermore, as we will see in the numerical experiments of Section 3, this heuristic warm-start lead to a reduction in the total number of iterations (from both phases) with respect to a previous Fixed-Rank Soft-Impute algorithm.
The second stage (Accelerated Soft-Impute) corresponds to an accelerated proximal gradient method as explained in the next subsection.

2.1. Convergence Analysis of Phase Two

In this section, we discuss the convergence of ARSI by showing that Soft-Impute is a particular case of proximal gradient applied to problem (4) with constant step-size and thus, the second stage of ARSI corresponds to an accelerated proximal gradient (for which the convergence is well-studied in the literature [13]).

The optimization problem given by equation (4) is a particular case of the problem of minimizing composite functions of the form

$$\min_x g(x) + h(x),$$

where $g, h$ are convex functions with $g$ differentiable, having Lipschitz gradient $\nabla g$ with constant $L > 0$ ($h$ does not need to be smooth, only proper convex).

Problem (7) can be solved by the proximal gradient algorithm, which generates a sequence $\{x^k\}$ given by

$$x^{k+1} = \text{prox}_{th} \left( x^k - t \nabla g(x^k) \right),$$

where $t > 0$ and $\text{prox}_{th}(\cdot)$ is the proximal operator, which can be expressed as

$$\text{prox}_{th}(v) = \arg \min_x \left\{ \frac{1}{2t} \| x - v \|^2 + h(x) \right\}.$$

It is shown (see Theorem 3.1 in [12]) that either for a fixed stepsize $t \leq \frac{1}{L}$ or by a backtracking line search, the proximal algorithm converges to the optimal solution of (7) at a rate of $O(1/k)$, where $k$ is the number of iterations. One can accelerate this method to achieve the optimal convergence rate of $O(1/k^2)$ by setting the equations [12, 13]

$$z^{k+1} := x^k + \frac{k - 1}{k + 2} (x^k - x^{k-1})$$

$$x^{k+1} := \text{prox}_{th} \left( z^{k+1} - t \nabla g(z^{k+1}) \right)$$

(9)
The convergence of the Algorithm 1 follows essentially from addressing its second phase as a (an accelerated) proximal gradient algorithm. For the function $h(X) = \lambda \|X\|_*$, the proximal operator is defined as

$$\text{prox}_{th}(M) = \arg \min_X \left\{ \frac{1}{2t} \|M - X\|_F^2 + \lambda \|X\|_* \right\},$$

whose solution is given by (see Theorem 2.1 in [9] with $\tau = \lambda t$)

$$\text{prox}_{th}(M) = S_{\lambda t}(M). \tag{10}$$

Now, we shall show that SI is a proximal gradient algorithm. On the one hand, the iteration of SI is given as

$$X^{k+1} = S_{\lambda}(Y^k)$$

$$Y^{k+1} = X^{k+1} + P_{\Omega}(A - X^{k+1}) = P_{\Omega}(A) + P_{\Omega}^\perp(X^{k+1}).$$

On the other hand, defining $g(X) = \frac{1}{2} \|P_{\Omega}(A) - P_{\Omega}(X)\|_F^2$, yields $L = 1$. Since

$$X^k - \nabla g(X^k) = X^k - (P_{\Omega}(X^k) - P_{\Omega}(A)) = P_{\Omega}(A) + P_{\Omega}^\perp(X^k) =: Y^k,$$

from (8) and (10) with $t = 1$ we have $X^{k+1} = \text{prox}_{h}(Y^k) = S_{\lambda}(Y^k)$, which gives the result. The acceleration required in (9) corresponds to steps 13 and 15 of ARSI (Algorithm 1). Hence, the second stage of Algorithm 1 corresponds to an accelerated proximal algorithm.

Stopping criteria for the second phase of ARSI shall be discussed in the following section.

3. Numerical results

In this section, we perform matrix completion experiments with both, synthetic data and the MovieLens\textsuperscript{2} data set. Moreover, we also provide an empirical study for choosing the acceleration parameter $\beta$ in the first phase of Algorithm 1.

\textsuperscript{2}A data set which has been widely used in matrix completion experiments and is available in https://grouplens.org/datasets/movielens/.

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Algorithm 1 is implemented in Matlab language and all the numerical results were performed on a PC with Intel Core i7-7500U CPU and 16 GB RAM.

The proposed algorithm is compared with those mentioned in Section 1: FRSI, SVT and FPC. All these methods (including ARSI) use PROPACK algorithm \[17\] (a variant of Lanczos algorithm designed for large and sparse matrices) for computing only the leading singular values/vectors.

Concerning the stopping criteria for the second phase of ARSI, we set
\[
\min \left\{ \frac{|f_\lambda(X^k) - f_\lambda(X^{k+1})|}{f_\lambda(X^k)}, \frac{\|X^{k+1} - X^k\|_F}{\|X^k\|_F} \right\} \leq \epsilon_\lambda,
\]
for a given tolerance \(\epsilon_\lambda > 0\) and \(f_\lambda\) is from \[4\].

For FRSI algorithm we use
\[
\min \left\{ \frac{\|P_\Omega (X^k - A)\|_F}{\|P_\Omega (A)\|_F}, \frac{\|X^{k+1} - X^k\|_F}{\|X^k\|_F} \right\} \leq \epsilon_1
\]
as the stopping criterion and for SVT and FPC algorithms we follow the recommendations in \[9\] and \[10\], and use \(\frac{\|P_\Omega(X^k - A)\|_F}{\|P_\Omega(A)\|_F} \leq \epsilon_2\), and \(\frac{\|X^{k+1} - X^k\|_F}{\max\{1,\|X^k\|_F\}} \leq \epsilon_3\) as the stopping criterion, respectively.

The following procedure were used for the synthetic data set: we generated \(n \times n\) matrices of rank \(r \ll n\) of the form \(A = MN \in \mathbb{R}^{n \times n}\), where the entries of \(M \in \mathbb{R}^{n \times r}\) and \(N \in \mathbb{R}^{r \times n}\) are sampled i.i.d from the standard normal distribution. Then, we deleted, uniformly at random, a percentage \(p\) of entries (unobserved entries) of \(A\).

Before we present some numerical results for both synthetic data and MovieLens, we shall give an overview of how to set the parameter \(\beta\) in the first phase of Algorithm 1.

### 3.1. Tuning the parameter \(\beta\)

To assess the sensitivity of phase one to the parameter \(\beta\), we performed extensive numerical experiments on the synthetic data set. We set a budget of \(w = 1,000\) iterations for phase one and vary the problem dimension \(n\), the rank \(r\), percentage of missing data \(p\) and the tolerance \(\epsilon_p\). The experiments show that the number of iterations of the first phase of ARSI can be highly reduced
by a suitable choice of the parameter $\beta$, especially when the number of observed entries is very small.

Figure 1 (a) shows the optimal value for $\beta$ considering the percentage of missing data $p \in \{92\%, 85\%, 72\%, 50\%\}$, $n = 1000$, $r = 5$, and $\epsilon = 10^{-8}$. Figure 1 (b) shows the minimum number of iterations in all scenarios for the same value of $\beta$ ($\beta \geq 20$). As can be seen, the bigger is rank of the target matrix the smaller is the “optimal” value of $\beta$.

Figure 2 gives us an overview of how to set the value of $\beta$ as the rank $r$ varies. For this experiment, we fixed $(n, \epsilon, p) = (1000, 10^{-5}, 50\%)$ and $r \in \{3, 5, 10, 30, 50, 80, 100\}$. As can be seen, the bigger is rank of the target matrix the smaller is the “optimal” value of $\beta$.

3.2. Experiments with synthetic data

Now we turn our attention to experiments with synthetic data, generated as described in the beginning of Section 3. For these experiments, we set $n = 1000,$
\[ \text{Figure 2: Optimal value for } \beta \text{ with } n = 1000, r \in \{3, 5, 10, 30, 50, 80, 100\}, \varepsilon_\rho = 10^{-5} \] 
and \( p = 50\% \).

\( p = 40\% \) and the rank \( r \) takes values in the set \( \{10, 15, 20, 40, 80, 100\} \). In the stopping criteria, we used the tolerances \( \varepsilon_\rho = \varepsilon_1 = \varepsilon_2 = 10^{-4}, \varepsilon_3 = 10^{-3}, \varepsilon_\lambda = 10^{-2} \) and for SVT we fixed \( \tau = 5n \) and \( \delta = 1.2n^2/|\Omega| \), where \( |\Omega| \) is the cardinality of \( \Omega \). In ARSI, we set the maximum number of iterations of phase one as \( w = 500 \) and the total number of iterations \( it_{\text{max}} = 1000 \). For the acceleration parameter \( \beta \) we have used \( \{13, 13, 12, 10, 5, 5\} \), respectively (following the study of Section 3.1).

For performance evaluation, we use the relative error, defined by \( \text{Rer} = \| A - \tilde{A} \|_F/\| A \|_F \), where \( \tilde{A} \) is the recovered matrix. The experimental results are averaged over 5 repetitions.

Results are shown in Table 1 where \( r \) denotes the target rank, IT is the total number of iterations (for ARSI, the sum of iterations of the two phases) and \( t(s) \) the time in seconds. In this first set of experiments we point out that all algorithms recovered correctly the underlying rank. As can be seen, our algorithm converges faster than the other algorithms. Furthermore, the bigger is the rank \( r \) of the desired matrix the better is the performance of ARSI, when compared with FRSI which, in its turn, is consistently faster than SVT and
Table 1: Comparison of ARSI, FRSI, SVT, and FPC. Performance evaluation for $n = 1,000$, $p = 40\%$, $r$ takes values in the set \{10, 15, 20, 40, 80\} and $\beta \in \{13, 13, 12, 10, 5, 5\}$, respectively.

| $r$ | method | IT | t(s)   | Rer    |
|-----|--------|----|--------|--------|
| 10  | ARSI   | 16 | 1.80   | 5.84e-06 |
|     | FRSI   | 18 | 2.04   | 1.68e-04 |
|     | SVT    | 43 | 5.78   | 1.09e-04 |
|     | FPC    | 74 | 9.68   | 1.70e-05 |
| 15  | ARSI   | 18 | 1.77   | 6.90e-06 |
|     | FRSI   | 20 | 2.15   | 1.49e-04 |
|     | SVT    | 47 | 5.82   | 1.07e-04 |
|     | FPC    | 81 | 13.04  | 1.72e-05 |
| 20  | ARSI   | 18 | 1.80   | 1.12e-06 |
|     | FRSI   | 21 | 2.89   | 1.95e-04 |
|     | SVT    | 51 | 6.60   | 1.13e-04 |
|     | FPC    | 91 | 15.62  | 1.78e-05 |
| 40  | ARSI   | 25 | 2.77   | 1.63e-06 |
|     | FRSI   | 28 | 3.85   | 2.90e-04 |
|     | SVT    | 64 | 11.15  | 1.26e-04 |
|     | FPC    | 125| 56.49  | 1.83e-05 |
| 80  | ARSI   | 31 | 7.08   | 4.76e-05 |
|     | FRSI   | 42 | 10.03  | 5.71e-04 |
|     | SVT    | 93 | 34.44  | 1.47e-04 |
|     | FPC    | 212| 165.98 | 2.04e-05 |
| 100 | ARSI   | 38 | 12.26  | 5.42e-05 |
|     | FRSI   | 46 | 20.51  | 1.21e-04 |
|     | SVT    | 144| 68.41  | 1.76e-04 |
|     | FPC    | 361| 415.25 | 2.38e-05 |

In terms of relative error ARSI was always the first or the second best.

We point out that most of the iterations of ARSI correspond to phase one (warm-start) iterations. After ARSI switch to the second phase, only a few iterations are required to reach the stopping criteria. On average, the number of phase two iterations is less than 10.

We also performed experiments on larger matrices with very few observed entries. The experiments were conducted under the same parameters as before and we set up a time limit of one hour. We compare the results only with SVT, because, in this case, it is faster than FRSI and FPC algorithms. The results are displayed in Table 2 and it can be seen that both algorithms have competitive performance for the tested cases. In this table we also report an additional column with the recovered rank $\hat{r}$.

ARSI usually outperforms SVT in terms of relative error and it is faster for matrices with higher rank. SVT tends to show a better performance for smaller ranks and when the number of missing entries is not too high. However, it becomes considerably slow when the rank increases and the percentage of known entries decreases. For some cases, such as (1000,20,90%) and (10000,40,97%), we even had to switch to the conservative choice of $\delta = 1.99$, for which SVT has
Table 2: Comparison of ARSI and SVT for different values of $(n, r, p)$ and $\beta \in \{13, 12, 19, 12, 19, 12, 19\}$, respectively.

| $(n, r, p)$ | method | IT | t(s) | Rer  | $\hat{r}$ |
|------------|--------|----|------|------|----------|
| (1000,10,90%) | ARSI   | 116 | 5.45 | 1.36e-04 | 10 |
|            | SVT    | 174 | 5.26 | 1.44e-04 | 10 |
| (1000,20,90%) | ARSI   | 102 | 7.34 | 3.25e-01 | 20 |
|            | SVT    | 500 | 279.56 | 2.23e-01 | 168 |
| (2000,10,90%) | ARSI   | 86  | 12.54 | 3.68e-05 | 10 |
|            | SVT    | 83  | 9.55  | 1.39e-04 | 10 |
| (2000,20,92%) | ARSI   | 147 | 28.11 | 1.59e-04 | 20 |
|            | SVT    | 262 | 168.62 | 1.51e-04 | 31 |
| (5000,10,90%) | ARSI   | 69  | 63.4  | 2.36e-05 | 10 |
|            | SVT    | 53  | 33.7  | 1.18e-04 | 10 |
| (5000,25,96%) | ARSI   | 215 | 149.89 | 1.62e-04 | 25 |
|            | SVT    | 297 | 1355.23 | 2.34e-04 | 50 |
| (10000,10,90%) | ARSI   | 65  | 245.13 | 8.27e-06 | 10 |
|            | SVT    | 43  | 113.84 | 1.07e-04 | 10 |
| (10000,40,97%) | ARSI   | 256 | 1018.56 | 8.01e-04 | 40 |
|            | SVT    | 677 | 3600 | 4.13e-02 | 95 |

Theoretical convergence guarantees, rather than $\delta = 1.2n^2/|\Omega|$, to avoid exceed the time limit. Furthermore, we remark that the rank $\hat{r}$ of the matrix recovered by SVT can be higher than the rank of the original matrix.

3.3. Experiments on MovieLens data set

The MovieLens data set is a well-known recommender system that is often used in matrix completion experiments [10]. It contains ratings ($\{1, 2, 3, 4, 5\}$) of different users on movies. Table 3 contains the data sets used in the experiments.

We randomly deleted 50% percent of the observed ratings and for performance evaluation we use the root mean square error (RMSE) given by

$$RMSE = \sqrt{\frac{\|P_\Omega (A - \hat{A})\|_F^2}{|\Omega|}},$$

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Table 3: MovieLens data sets used in the experiments

| data set        | # users | # movies | # ratings |
|-----------------|---------|----------|-----------|
| MovieLens-100k  | 943     | 1,682    | 100,000   |
| MovieLens-1M    | 6,040   | 3,952    | 1,000,209 |

Table 4: Numerical results on MovieLens data sets

| method | 100k | 1M      |
|--------|------|---------|
|        | IT   | t(s)    | RMSE    | IT   | t(s)    | RMSE    |
| ARSI   | 84   | 61.43   | 0.7667  | 74   | 967.76  | 0.7123  |
| FRSI   | 223  | 159.62  | 0.8598  | 176  | 3,600   | 0.8475  |
| SVT    | 2,000| 1,315.92| 0.7696  | 1236 | 3,600   | 0.7230  |
| FPC    | 410  | 978.20  | 0.7806  | 234  | 3,600   | 0.7929  |

where \( \hat{\Omega} \) is total number of observed ratings (but only \(|\Omega| = |\hat{\Omega}|/2 \) ratings were passed as input to the algorithms).

Since the ratings matrix has unknown rank and both ARSI and FRSI need this information, we performed some experiments for different values of \( r \) and we set \( r = 130 \) for MovieLens-100k and \( r = 340 \) for MovieLens-1M because these choices provide the smallest RMSE for both methods. For ARSI, we fixed the acceleration parameter \( \beta = 2 \) For these experiments, we set the tolerances \( \epsilon_\rho = \epsilon_1 = \epsilon_2 = \epsilon_3 = 10^{-3}, \epsilon_\lambda = 10^{-2}, \) and for SVT we fixed \( \delta = 1.99 \) and since \( m \neq n \) we set \( \tau = 8\sqrt{mn} \) as suggested in [9]. Moreover, we set up a time limit of one hour for all the algorithms.

The results are shown in Table 4. As we can see, ARSI shows the best performance in terms of CPU time and RMSE. We remark that for the dataset MovieLens-1M, ARSI was the only one able to reach the stopping criteria in less than one hour.
4. Conclusion

We consider matrix completion problems where the rank of the target matrix is known in advance. For instance, this is the case of localization, graph realization and other problems in distance geometry [18] where the rank of the matrix to be completed is related to the embedding dimension.

A two-phase algorithm has been proposed. It takes into account the rank information in the heuristic of the first phase to estimate the nuclear norm regularization parameter and provide a warm-start to an accelerated Soft-Impute algorithm at the second phase. After a numerical study on how to tuning parameters of the first phase, numerical experiments on both synthetic and real data sets indicates that the proposed algorithm (ARSI) outperforms a previous algorithm that attempts to leverage the target rank (FRSI) [15] and is competitive with well established algorithms for matrix completion, such as SVT and FPC. Moreover, ARSI was able to recover low-rank matrices from a few percentage of its entries with reasonable accuracy and faster than the compared methods, mainly when the expected rank is not too low.

Even though ARSI phase-one is just an heuristic, it was responsible for the majority of the iterations in the algorithm. This fact points in the direction of studying convergence properties of phase-one alone under suitable assumptions yet to be discovered.

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