Theory of solitary waves in complex plasma lattices *

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A comprehensive analytical theory for nonlinear excitations related to horizontal (longitudinal, acoustic mode) as well as vertical (transverse, optical mode) motion of charged dust grains in a dust crystal is presented. Different types of localized excitations, similar to those well known in solid state physics, are reviewed and conditions for their occurrence and characteristics in dusty plasma crystals are discussed. By employing a continuum approximation (i.e., assuming a long variation scale, with respect to the inter-particle distance) a dust crystal is shown to support nonlinear kink-shaped supersonic solitary excitations, associated with longitudinal dust grain displacement, as well as modulated envelope localized modes associated with either longitudinal or transverse oscillations. Although a one-dimensional crystal is considered for simplicity, the results in principle apply to a two-dimensional lattice if certain conditions are satisfied. The effect of mode-coupling is also briefly considered. The relation to previous results on atomic chains, and also to experimental results on strongly-coupled dust layers in gas discharge plasmas, is briefly discussed.

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I. INTRODUCTION

Dust contaminated plasmas (dusty plasmas, DP) have been attracting significant interest recently. Particularly important are dust quasi-lattices, which are typically formed in the sheath region above the negative electrode in discharge experiments, horizontally suspended at a levitated equilibrium position at \( z = z_0 \), where gravity and electric (and/or magnetic) forces balance. The linear regime of low-frequency oscillations in DP crystals, in the longitudinal (acoustic mode) and transverse (in-plane, shear acoustic mode and vertical, off-plane optical mode) direction(s), is now quite well understood. However, the nonlinear behaviour of DP crystals is still mostly unexplored, and has lately attracted experimental [1 - 3] and theoretical [1 - 9] interest.

Recently [5], we considered the coupling between the horizontal (\( \sim \hat{x} \)) and vertical (off-plane, \( \sim \hat{z} \)) degrees of freedom in a dust mono-layer; a set of nonlinear equations for longitudinal and transverse dust lattice waves (LDLWs, TDLWs) was thus rigorously derived [5]. Here, we review the nonlinear dust grain excitations which may occur in a DP crystal (here assumed quasi-one-dimensional and infinite, composed from identical grains, of equilibrium charge \( q \) and mass \( M \), located at \( x_n = n r_0 \), \( n \in \mathbb{N} \)). Ion-wake and ion-neutral interactions (collisions) are omitted, at a first step. This study complements recent experimental investigations [1-3] and may hopefully motivate future ones.

II. TRANSVERSE ENVELOPE STRUCTURES.

The vertical (off-plane) \( n \)-th grain displacement \( \delta z_n = z_n - z_0 \) in a dust crystal obeys the equation \[ \frac{d^2 \delta z_n}{dt^2} + \omega_{T,0}^2 \left( \delta z_{n+1} + \delta z_{n-1} - 2 \delta z_n \right) + \omega_g^2 \delta z_n + \alpha (\delta z_n)^2 + \beta (\delta z_n)^3 = 0. \] 

The characteristic frequency

\[ \omega_{T,0} = \left[ -qU'(r_0)/(Mr_0) \right]^{1/2} \]

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is related to the interaction potential $U(r)$ [e.g. for a Debye–Hückel potential: $U_D(r) = (q/r) e^{-r/\lambda D}$, one has

$$\omega_{0,D}^2 = \omega_{DL}^2 \exp(-\kappa) (1 + \kappa) / \kappa^3,$$

(2)

where $\omega_{DL} = [q^2/(e \lambda_D^3)]^{1/2}$ is the characteristic dust-lattice frequency scale; $\lambda_D$ is the Debye length; $\kappa = r_0 / \lambda_D$ is the DP lattice parameter. The gap frequency $\omega_g$ and the nonlinearity coefficients $\alpha, \beta$ are defined via the potential

$$\Phi(z) = \Phi(z_0) + M \left[ \frac{1}{2} \omega_g^2 \delta z_n^2 + \alpha (\delta z_n)^3 + \frac{\beta}{4} (\delta z_n)^4 \right] + O[(\delta z_n)^5]$$

(formally expanded near $z_0$, taking into account the electric and/or magnetic field inhomogeneity and charge variations [12]), i.e. leading to an overall vertical force

$$F(z) = F_{el/m}(z) - Mg \equiv -\partial \Phi(z) / \partial z \approx -M [\omega_g^2 \delta z_n + \alpha (\delta z_n)^2 + \beta (\delta z_n)^3] + O[(\delta z_n)^4].$$

Recall that $F_{el/m}(z_0) = Mg$. Notice the difference in structure from the usual nonlinear Klein–Gordon equation used to describe 1D one-dimensional oscillator chains: TDLWs (‘phonons’) in this chain are stable only in the presence of thanks to the field force $F_{el/m}$ (via $\omega_g$). It should be stressed that the validity of this anharmonicity hypothesis is indeed suggested real discharge experiments, in particular for low pressure and/or density values, and also confirmed by ab initio models [13] (see Fig. 1).

Linear transverse dust-lattice excitations, viz. $\delta z_n \sim \cos \phi_n$ (here $\phi_n = nk r_0 - \omega t$) obey the optical-like discrete dispersion relation [13]:

$$\omega^2 = \omega_g^2 - 4 \omega_g^2 \sin^2(k r_0 / 2) \equiv \omega_T^2.$$  

(4)

The TDLW dispersion curve is depicted in Fig 2. Transverse vibrations propagate as a backward wave [see that $v_{g,T} = \omega_T(k) < 0$] – for any form of $U(r)$ – cf. recent experiments [2]. Notice the lower cutoff $\omega_{T,\text{min}} = (\omega_g^2 - 4 \omega_g^2 / 0)^{1/2}$ (at the edge of the Brillouin zone, at $k = \pi / r_0$), which is absent in the continuum limit. (for $k \ll r_0^{-1}$).

Allowing for a slight departure from the small amplitude (linear) assumption, one obtains:

$$\delta z_n \approx \epsilon (A e^{i \phi_n} + \text{c.c.}) + \epsilon^2 \left[ \frac{2 |A|^2}{\omega_g^2} + \left( \frac{A^2}{3 \omega_g^2} e^{2 i \phi_n} + \text{c.c.} \right) \right] + \ldots.$$  

(5)

Notice the generation of higher phase harmonics due to nonlinearity. The (slowly varying) amplitude $w_1(n) = A(\epsilon(x - v_{g,T}) \epsilon^2 t)$ obeys a nonlinear Schrödinger equation (NLSE) in the form [7]:

$$i \frac{\partial A}{\partial T} + P \frac{\partial^2 A}{\partial X^2} + Q |A|^2 A = 0,$$

(6)

where $\{X, T\}$ are the slow variables $\{\epsilon(x - v_{g,T}) \epsilon^2 t\}$. The dispersion coefficient $P$ is related to the curvature of the $\omega(k)$ as $P_T = \omega_T^3(k)/2$ is negative/positive for low/high values of $k$. The nonlinearity coefficient

$$Q = \frac{1}{2 \omega_T} \left( \frac{10 \alpha^2}{3 \omega_g^2} - 3 \beta \right)$$

(7)

is positive for all known experimental values of the anharmonicity coefficients $\alpha, \beta$ [3]. For long wavelengths [i.e. $k < k_{cr}$, where $P(k_{cr}) = 0$], the theory [7] predicts that TDLWs will be modulationally stable, and may propagate in the form of dark/grey envelope excitations (hole solitons or voids; see Fig. [1],b). On the other hand, for $k > k_{cr}$, modulational instability may lead to the formation of bright (pulse) envelope solitons (see Fig. [2]). Analytical expressions for these excitations can be found in [7].

It may be noted that the modulation of transverse dust grain oscillations clearly appears in numerical simulations [13]; see e.g Fig. 3.
III. LONGITUDINAL ENVELOPE EXCITATIONS.

The longitudinal dust grain displacements $\delta x_n = x_n - nr_0$ are described by the nonlinear equation of motion [8, 10]:

$$\frac{d^2(\delta x_n)}{dt^2} + \nu \frac{d(\delta x_n)}{dt} = \omega_{L,0}^2 (\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n) - a_{20} [(\delta x_{n+1} - \delta x_n)^2 - (\delta x_n - \delta x_{n-1})^2] + a_{30} [(\delta x_{n+1} - \delta x_n)^3 - (\delta x_n - \delta x_{n-1})^3].$$

The resulting linear mode [14] obeys the acoustic dispersion relation:

$$\omega^2 = 4\omega_{L,0}^2 \sin^2(kr_0/2) \equiv \omega_L^2,$$

where $\omega_{L,0} = [U''(r_0)/M]^{1/2}$; in the Debye case, $\omega_{L,0}^2 = 2\omega_{DL}^2 \exp(-\kappa) (1 + \kappa + \kappa^2/2)/\kappa^3$. The LDLW dispersion curve is depicted in Fig 5.

The multiple scales (reductive perturbation) technique (cf. above) now yields ($\sim \epsilon$) a zeroth-harmonic mode, describing a constant displacement, viz.

$$\delta x_n \approx \epsilon \left[ u_0^{(1)} + (u_1^{(1)} e^{i\phi_n} + \text{c.c.}) \right] + \epsilon^2 \left[ u_0^{(2)} e^{2i\phi_n} + \text{c.c.} \right] + \ldots.$$
The 1st-order amplitudes obey the coupled equations [6]:

\[
i \frac{\partial u^{(1)}}{\partial T} + P_L \frac{\partial^2 u^{(1)}}{\partial X^2} + Q_0 |u^{(1)}|^2 u^{(1)} + \frac{p_0 k^2}{2 \omega_L} u^{(1)} \frac{\partial u^{(1)}}{\partial X} = 0,
\]

\[
\frac{\partial^2 u^{(1)}}{\partial X^2} = -\frac{p_0 k^2}{v_{gL}^2 - \omega_{L0}^2} \frac{\partial}{\partial X} |u^{(1)}|^2,
\]

where \( v_{gL} = \omega_{L0}^2(k) \); \( \{X, T\} \) are slow variables (as above). The description involves the definitions: \( p_0 = -r_0^3 U'''(r_0)/M \equiv 2a_0 r_0^3 \) and \( q_0 = U'''(r_0)r_0^3/(2M) \equiv 3a_2 r_0^3 \) (both positive quantities of similar order of magnitude for Debye interactions; see in [4, 7]). Eqs. (10), (11) may be combined into a closed equation, which is identical to Eq. (6) (for \( A = u^{(1)} \), here). Now, here \( P = P_L = \omega_{L0}^2(k)/2 < 0 \), while the form of \( Q > 0 \) (here, \( \leq 0 \)) prescribes stability (instability) at low (high) \( k \). Envelope excitations are now asymmetric, i.e. rarefactive bright or compressive dark envelope structures (see Figs.).

**IV. LONGITUDINAL SOLITONS.**

Equation \( \equiv \) is identical to the equation of motion in an atomic chain with anharmonic springs, i.e. the celebrated FPU \( (\text{Fermi-Pasta-Ulam}) \) problem. Inspired by methods of solid state physics, one may opt for a continuum description at a first step, viz. \( \delta x_n(t) \rightarrow u(x, t) \). This may lead to different nonlinear evolution equations (depending on simplifying assumptions), some of which are critically discussed in [9]. What follows is a summary of the lengthy analysis carried out therein.

Keeping lowest order nonlinear and dispersive terms, the continuum variable \( u \) obeys [10]:

\[
\ddot{u} + \nu \dot{u} - c_L^2 \dot{u}_{xx} - \frac{c_L^2}{12} \dot{u}_{xxxx} = -p_0 u_x u_{xx} + q_0 (u_x)^2 u_{xx},
\]

where \( \gamma \equiv \partial (\cdot) / \partial x; \ c_L = \omega_{L0} r_0; \ p_0 \) and \( q_0 \) were defined above. Assuming near-sonic propagation (i.e. \( v \approx c_L \)), and defining the relative displacement \( w = u_x \), one has

\[
w_{\tau} - a w \dot{w} + \dot{a} w^2 \dot{w} + b \dot{w} \dot{w} = 0
\]

(for \( \nu = 0 \), where \( a = p_0/(2c_L) > 0 \), \( \dot{a} = q_0/(2c_L) > 0 \), and \( b = c_L r_0^2/24 > 0 \). Since the original work of Melandsø [4], various studies have relied on the Korteweg - deVries (KdV) equation, i.e. Eq. (13) for \( \dot{a} = 0 \), in order to gain analytical insight in the compressive structures observed in experiments [1]. Indeed, the KdV Eq. possesses negative (only, here, since \( a > 0 \)) supersonic pulse soliton solutions for \( w \), implying a compressive (anti-kink) excitation for
$u$: the KdV soliton is thus interpreted as a density variation in the crystal, viz. $n(x,t)/n_0 \sim -\partial u/\partial x \equiv -w$. Also, the pulse width $L_0$ and height $u_0$ satisfy $u_0 L_0^2 = \text{cst}$, a feature which is confirmed by experiments [1]. Now, here's a crucial point to be made (among others [9]): in a Debye crystal, $\hat{\alpha} \approx 2\alpha$ roughly (for $\kappa \approx 1$), so the KdV approximation (i.e. assuming $\hat{\alpha} \approx 0$) is not valid. Instead, one may employ the extended KdV Eq. (eKdV) (13), which accounts for both compressive and rarefactive lattice excitations (see expressions in [9]; also cf. Fig. 4).

Alternatively, Eq. (12) can be reduced to a Generalized Boussinesq (GBq) Equation

$$\tilde{w} - v_0^2 w_{xx} = h w_{xxxx} + p (w^2)_{xx} + q (w^3)_{xx}$$

(14)

$w = u_x; \quad p = -p_0/2 < 0, \quad q = q_0/3 > 0)$; again, for $q \sim q_0 = 0$, one recovers a Boussinesq (Bq) equation, e.g. widely studied in solid chains. As physically expected, the GBq (Bq) equation yields, like its eKdV (KdV) counterpart, both compressive and rarefactive (only compressive) solutions; however, the (supersonic) propagation speed $v$ now does not have to be close to $c_L$. A detailed comparative study of (and exact expressions for) all of these soliton excitations...
V. CONCLUSIONS.

Concluding, we have reviewed recent results on nonlinear excitations (solitary waves) occurring in a (1d) dust mono-layer. Modulated envelope TDL and LDL structures occur, due to sheath and coupling nonlinearity. Both compressive and rarefactive longitudinal excitations are predicted and may be observed by appropriate experiments.
FIG. 6: Bright LDL (asymmetric) envelope solitons: (a) the zeroth (pulse) and first harmonic (kink) amplitudes; (b) the resulting asymmetric wavepacket.

FIG. 7: (a) Grey and (b) dark LDL (asymmetric) modulated wavepackets.

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[10] Only first neighbor interactions are considered throughout this paper. See in [5] for details and coefficient definitions.
FIG. 8: Solutions of the extended KdV Eq. (for $q_0 > 0$; dashed curves) vs. those of the KdV Eq. (for $q_0 = 0$; solid curves): (a) relative displacement $u_x$; (b) grain displacement $u$.

[11] Coupling anharmonicity, expressed by a term $\sim (\delta z_{n+1} - \delta z_n)^3 - (\delta z_n - \delta z_{n-1})^3$, which is omitted in the right-hand side of Eq. 1, may be added at a later stage.

[12] Follow exactly the definitions in [5, 6], not reproduced here.

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[14] The damping term is neglected by setting $\nu = 0$ in the following, thus omitting collisions with neutrals; for $\nu \neq 0$, an imaginary part appears, in account for damping in both dispersion relation $\omega(k)$ and the resulting envelope equations.