Biquaternions and ADHM Construction of Non-Compact SL(2,C) Yang-Mills Instantons

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Abstract

We extend quaternion calculation in the ADHM construction of Sp(1) (= SU(2)) self-dual Yang-Mills (SDYM) instantons to the case of biquaternion. We use the biconjugate operation of biquaternion first introduced by Hamilton to construct the non-compact SL(2, C) k-instantons. The number of moduli for SL(2, C) k-instantons is found to be twice of that of Sp(1), 16k−6. These new SL(2, C) instanton solutions contain the SL(2, C) (M, N) instanton solutions constructed previously as a subset. The structures of singularities or jumping lines of the complete SL(2, C) k = 1, 2, 3 instantons with 10, 26, 42 moduli parameters are particularly investigated. The existence of singular structures of the SL(2, C) k-instantons is mathematically consistent with recent results of solutions of complex ADHM equations. It may also help to clarify the long standing global singularity problems associated with Backlund transformations of SU(2) instantons.

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I. INTRODUCTION

The discovery of classical exact solutions of Euclidean $SU(2)$ (anti)self-dual Yang-Mills (SDYM) equation was one of the most important achievements in the developments of both quantum field theory and algebraic geometry in 1970’s. The first BPST 1-instanton solution [1] with 5 moduli parameters was found in 1975. Soon later the CFTW k-instanton solutions [2] with $5k$ moduli parameters were constructed, and then the number of moduli parameters of the solutions for each homotopy class $k$ was extended to $5k + 4$ (5,13 for $k = 1,2$) [3] based on the consideration of 4D conformal symmetry of massless pure YM equation. The complete solutions with $8k - 3$ moduli parameters for each $k$-th homotopy class were finally worked out in 1978 by mathematicians ADHM [4] using method in algebraic geometry. By using an one to one correspondence between anti-self-dual $SU(2)$-connections on $S^4$ and certain holomorphic vector bundles of rank two on $CP^3$, ADHM converted the
highly nontrivial system of non-linear partial differential equations of anti-SDYM into a much more simpler system of quadratic algebraic equations in quaternions. The explicit closed form of the complete solutions for \( k = 2, 3 \) had been worked out \([5]\).

Many interesting further developments, including supersymmetric YM instantons \([6]\), Heterotic string instantons \([7]\) and noncommutative YM instantons \([8]\) etc., followed since then. One important application of instantons in algebraic geometry was the classification of four-manifolds \([9]\). On the physics side, the non-perturbative instanton effect in QCD resolved the long standing \( U(1)_{A} \) problem \([10]\). On the other hand, another important application of YM instantons in quantum field theory was the introduction of \( \theta \)- vacua \([11]\) in nonperturbative QCD, which created the strong \( CP \) problem. This unsolved issue remains a puzzle till even today.

In addition to \( SU(2) \), the ADHM construction has been generalized to the cases of \( SU(N) \) SDYM and many other SDYM theories with compact Lie groups \([2, 12]\). In this paper we are going to consider the classical solutions of non-compact \( SL(2, C) \) SDYM system. \( SL(2, C) \) YM theory was first discussed by some authors in 1970’s \([13, 14]\). They found out that the complex \( SU(2) \) YM field configurations can be interpreted as the real field configurations in \( SL(2, C) \) YM theory. However, due to the non-compactness of \( SL(2, C) \), the Cartan-Killing form or group metric of \( SL(2, C) \) is not positive definite. Thus the action integral and the Hamiltonian of non-compact \( SL(2, C) \) YM theory may not be positive. Nevertheless, there are still important motivations to study \( SL(2, C) \) SDYM theory. It was shown that the 4D \( SL(2, C) \) SDYM equation can be dimensionally reduced to many important 1+1 dimensional integrable systems \([15]\), such as the KdV equation and the nonlinear Schrodinger equation. In 1985 \([16]\), it was even conjectured by Ward that many (and perhaps all?) integrable or solvable equations may be obtained from the SDYM equations (or its generalizations) by reduction.

On the other hand, the parametric Backlund transformation (PBT) constructed in terms of \( J \)-matrix formulation \([17]\) of \( SU(2) \) Yang-Mills theory takes a real \( SU(2) \) gauge field into the real \( SU(1, 1) \) gauge field and vice versa \([18, 19]\). Therefore it would be of interest to study \( SL(2, C) \) gauge group which contains the non-compact subgroup \( SU(1, 1) \) as well as the compact subgroup \( SU(2) \), and the solutions to the \( SL(2, C) \) SDYM can be transformed into the new ones by any arbitrary numbers of PBT. Moreover, as it will turn out, there
are singularities which can not be gauged away in the field configurations of $SL(2, C)$ YM instantons. This may help to clearify the long standing issue of global singularity problems associated with Backlund transformations [18, 19] of $SU(2)$ SDYM instantons. More recently the $SL(2, C)$ SDYM theory including its singular structure was also considered in the literatures from mathematical point of view [20–22].

In 1984 [23], some exact solutions of $SL(2, C)$ SDYM system were explicitly constructed in the $(R, \bar{R})$-gauge, which was a direct generalization of $R$-gauge in Yang’s formulation [24] of $SU(2)$ SDYM equation. The topological charges of these so-called $(M, N)$ solutions [23] were calculated by the third homotopy group $\pi_3(SL(2, C) = Z$. In this paper, we extend quaternion calculation in the ADHM construction of compact $Sp(1)$ (and $SU(N)$, $Sp(N)$, $O(N)$ cases) SDYM instantons to the case of biquaternion of Hamilton [25]. We will use the biconjugate operation of biquaternion first introduced by Hamilton [25] to construct the $SL(2, C)$ SDYM instantons. These new $SL(2, C)$ instanton solutions contain previous $SL(2, C)$ $(M, N)$ instanton solutions as a subset constructed in 1984. In addition, we will obtain many more new $SL(2, C)$ SDYM field configurations. It turns out that the number of moduli for solutions of the $SL(2, C)$ SDYM for each $k$-th homotopy class is twice of that of the case of $SU(2)$ SDYM, namely $16k - 6$.

This paper is organized by the following. In section II, we set up the formalism of $SL(2, C)$ SDYM theory and review the previous $(M, N)$ instanton solutions [23]. Section III is devoted to the general construction of solutions with $16k - 6$ parameters by using biquaternions. Three explicit examples will be given in section IV. These include the $(M, N)$ instanton solutions, the complete $k = 2, 3$ instanton solutions and a detailed discussion of 1-instanton solution and the structure of its singularities depending on its moduli space with 10 parameters. The singular structures of $SL(2, C)$ $k$-instantons will also be discussed in section IV. The singularities called ”jumping lines” of 1-instantons are intersections of zeros of $P_2(x)$ and $P_1(x)$ polynomials of 4 variables with degrees 2 and 1 respectively. For singularities of general $k$-instanton field configurations, one encounters intersections of zeros of $P_{2k}(x)$ and $P_{2k-1}(x)$ polynomials with degrees $2k$ and $2k-1$ respectively. In particular, the complete jumping lines of $SL(2, C)$ $k = 1, 2, 3$ instantons with 10, 26, 42 moduli parameters are calculated. The existence of singular structure of the non-compact $SL(2, C)$ SDYM field configurations discovered in this paper is consistent with the recent use of ”sheaves” by Frenkel-Jardim [22] for complex ADHM equations, rather than just the restricted notion of
"vector bundles". Finally, a brief conclusion is given in section V.

II. REVIEW OF $SL(2, C)$ $(M, N)$ INSTANTONS

In this section, we will use the convention $\mu = 1, 2, 3, 4$ and $\epsilon_{1234} = 1$ for 4D Euclidean space. We will first briefly review the $SL(2, C)$ solutions constructed 30 years ago in $[23]$. Wu and Yang $[13]$ have shown that there are two linearly independent choices of $SL(2, C)$ group metric

$$g^a = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, g^b = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

(2.1)

where $I$ is the $3 \times 3$ unit matrix. In general, we can choose

$$g = \cos \theta g^a + \sin \theta g^b$$

(2.2)

where $\theta = \text{real constant}$. Note that the metric is not positive definite due to the non-compactness of $SL(2, C)$. On the other hand, it was shown that $SL(2, C)$ group can be decomposed such that $[23]

$$SL(2, C) = SU(2) \cdot P, P \in H$$

(2.3)

where $SU(2)$ is the maximal compact subgroup of $SL(2, C)$, $P \in H$ (not a group) and $H = \{ P \mid P \text{ is Hermitian, positive definite, and } det P = 1 \}$. The parameter space of $H$ is a noncompact space $R^3$. The third homotopy group is thus $[23]

$$\pi_3[SL(2, C)] = \pi_3[S^3 \times R^3] = \pi_3(S^3) \cdot \pi_3(R^3) = Z \cdot I = Z$$

(2.4)

where $I$ is the identity group, and $Z$ is the integer group. Wu and Yang $[13]$ have shown that a complex $SU(2)$ gauge field is related to a real $SL(2, C)$ gauge field. Starting from $SU(2)$ complex gauge field formalism, we can write down all the $SL(2, C)$ field equations. Let

$$G^a_{\mu} = A^a_{\mu} + iB^a_{\mu}$$

(2.5)
and, for convenience, we set the coupling constant $g = 1$. The complex field strength is defined as

$$F^a_{\mu\nu} \equiv H^a_{\mu\nu} + iM^a_{\mu\nu}, \quad a, b, c = 1, 2, 3$$ (2.6)

where

$$H^a_{\mu\nu} = \partial_{\mu}A^a_{\nu} - \partial_{\nu}A^a_{\mu} + \epsilon^{abc}(A^b_{\mu}A^c_{\nu} - B^b_{\mu}B^c_{\nu}),$$

$$M^a_{\mu\nu} = \partial_{\mu}B^a_{\nu} - \partial_{\nu}B^a_{\mu} + \epsilon^{abc}(A^b_{\mu}B^c_{\nu} - B^b_{\mu}A^c_{\nu}),$$ (2.7)

then Yang-Mills equation can be written as

$$\partial_{\mu}H^a_{\mu\nu} + \epsilon^{abc}(A^b_{\mu}H^c_{\mu\nu} - B^b_{\mu}M^c_{\mu\nu}) = 0,$$

$$\partial_{\mu}M^a_{\mu\nu} + \epsilon^{abc}(A^b_{\mu}M^c_{\mu\nu} - B^b_{\mu}H^c_{\mu\nu}) = 0.$$ (2.8)

The $SL(2, C)$ SDYM equations are

$$H^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}H_{\alpha\beta},$$

$$M^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}M_{\alpha\beta}.$$ (2.9)

Yang-Mills Equation can be derived from the following Lagrangian

$$L_{\theta} = \frac{1}{4}[F^i_{\mu\nu}]^T g_{ij} [F^j_{\mu\nu}] = \cos \theta(\frac{1}{4}H^a_{\mu\nu}H^a_{\mu\nu} - \frac{1}{4}M^a_{\mu\nu}M^a_{\mu\nu}) + \sin \theta(\frac{1}{2}H^a_{\mu\nu}M^a_{\mu\nu})$$ (2.10)

where $F^k_{\mu\nu} = H^k_{\mu\nu}$ and $F^{3+k}_{\mu\nu} = M^k_{\mu\nu}$ for $k = 1, 2, 3$. Note that $L_{\theta}$ is indefinite for any real value $\theta$. We shall only consider the particular case for $\theta = 0$ in this section, i.e.

$$L = \frac{1}{4}(H^a_{\mu\nu}H^a_{\mu\nu} - M^a_{\mu\nu}M^a_{\mu\nu}),$$ (2.11)

for the action density in discussing the homotopic classifications of our solutions.

In the Yang formulation of $SU(2)$ SDYM theory, one first performs analytic continuation of $x_{\mu}$ to complex space, the self-dual condition Eq. (2.9) is still valid in complex space. We then perform the following transformations in complex space

$$\sqrt{2}y = x_1 + ix_2, \sqrt{2}\bar{y} = x_1 - ix_2,$$

$$\sqrt{2}z = x_3 - ix_4, \sqrt{2}\bar{z} = x_3 + ix_4,$$ (2.12)

$$\sqrt{2}G_y = G_1 - iG_2, \sqrt{2}\bar{G}_y = G_1 + iG_2,$$

$$\sqrt{2}G_z = G_3 + iG_4, \sqrt{2}\bar{G}_z = G_3 - iG_4.$$ (2.13)
Note that \( y \) and \( \bar{y} \) (similarly \( z \) and \( \bar{z} \)) are independent complex numbers. They are complex conjugate to each other when we restrict \( x_\mu \) to be real. The self-dual equation then reduces to

\[
F_{yz} = F_{\bar{y}\bar{z}} = 0, \tag{2.14}
\]
\[
F_{y\bar{y}} + F_{z\bar{z}} = 0. \tag{2.15}
\]

Eq. (2.14) is now in the pure gauge and can be integrated once. In the so-called R-gauge, Eq. (2.15) reduces to 24

\[
\phi[\phi_{y\bar{y}} + \phi_{z\bar{z}}] - \phi_y \phi_{\bar{y}} - \phi_z \phi_{\bar{z}} + \rho_y \rho_{\bar{y}} + \rho_z \rho_{\bar{z}} = 0,
\]
\[
\phi[\rho_{y\bar{y}} + \rho_{z\bar{z}}] - 2\rho_y \phi_{\bar{y}} - 2\rho_z \phi_{\bar{z}} = 0,
\]
\[
\phi[\rho_{y\bar{y}} + \rho_{z\bar{z}}] - 2\rho_y \phi_{\bar{y}} - 2\rho_z \phi_{\bar{z}} = 0, \tag{2.16}
\]

where \( \phi, \rho \) and \( \bar{\rho} \) are three independent complex valued functions or six real valued functions.

For the case of \( SU(2) \), one needs to impose the reality conditions \( \phi \doteq \text{real}, \ \bar{\rho} \doteq \rho^* \) so that \( G_\mu \) will be a real gauge field. Here ” \( \doteq \) ” means ” \( = \) ” when we restrict \( x_\mu \) to be real. For the case of \( SL(2, C) \) considered in this paper, we drop out the reality conditions and the \( R \)-gauge will be called \( (R, \bar{R}) \) gauge. Thus in the \( SL(2, C) \) \( (R, \bar{R}) \) gauge, \( G_\mu \) can be complex and there are three independent complex valued functions or six real valued functions. It is easily seen that one set of solutions of Eq. (2.16) is

\[
\rho_y = \phi_z, \ \rho_z = -\phi_y, \ \bar{\rho}_y = \phi_z, \ \bar{\rho}_z = -\phi_y. \tag{2.17}
\]

For the \( SL(2, C) \) case, this is to say that the complex gauge potential \( G^a_{\mu\nu} \) can be taken as

\[
G^a_{\mu\nu} = -\bar{\eta}^a_{\mu\nu} \partial_\nu (\ln \phi) \tag{2.18}
\]

where \( \bar{\eta}^a_{\mu\nu} \) is defined to be 10

\[
\eta^a_{\mu\nu} = \epsilon^a_{\mu\nu\lambda} = \delta^a_{\mu\nu} \delta^{\lambda 4} - \delta^a_{\mu \lambda} \delta^{\nu 4}, \tag{2.19a}
\]
\[
\bar{\eta}^a_{\mu\nu} = \bar{\eta}^a_{\mu\nu} = (-1)^{(\delta_{\mu 4} + \delta_{\nu 4})} \eta^a_{\mu\nu}. \tag{2.19b}
\]

Eq. (2.18) is the Corrigan-Fairlie-’t Hooft-Wilczek (CFTW) 2 anastz which is used to obtain \( SU(2) \) \( k \)-instanton solutions. But for the case of \( SL(2, C) \), \( \phi \) is a complex-valued function.
Substitution of Eq. (2.18) into Eq. (2.9) and using \[\eta_{a\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \eta_{a\alpha\beta}, \bar{\eta}_{a\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \bar{\eta}_{a\alpha\beta},\]

\[
\delta_{\kappa\lambda} \eta_{a\mu\nu} + \delta_{\kappa\nu} \eta_{a\lambda\mu} + \delta_{\kappa\mu} \eta_{a\nu\lambda} + \eta_{a\sigma\kappa} \epsilon_{\lambda\mu\sigma} = 0, \quad (2.20a)
\]

\[
\epsilon_{\alpha\beta\gamma} \eta_{a\mu\nu} \eta_{c\kappa\lambda} = \delta_{\mu\kappa} \eta_{a\nu\lambda} - \delta_{\mu\lambda} \eta_{a\nu\kappa} - \delta_{\nu\kappa} \eta_{a\mu\lambda}, \quad (2.20b)
\]

we obtain

\[
\frac{1}{\phi} \Box \phi = 0 \quad (2.21a)
\]

where \(\Box = \partial_\mu \partial_\mu = 2(\partial_y \partial_{\bar{y}} + \partial_z \partial_{\bar{z}})\). Note that for \(\phi = p + iq\),

\[
\frac{1}{p} \Box p = 0, \quad \frac{1}{q} \Box q = 0 \quad (2.22)
\]

satisfy Eq. (2.21a). Eq. (2.22) has the following solutions \[\]

\[
p = 1 + \sum_{i=1}^{M} \frac{\alpha_{1i}^2}{|x_\mu - a_{i\mu}|^2}, 0,
\]

\[
q = 1 + \sum_{i=1}^{M} \frac{\beta_{ji}^2}{|x_\mu - b_{j\mu}|^2}, 0 \quad (2.23)
\]

where \(\alpha_i, \beta_j\) are real constants, \(a_{i\mu}, b_{j\mu}\) are real constant 4-vector. A special case is that when \(p = q\ (M = N, \alpha_i = \beta_j, a_{i\mu} = b_{j\mu})\) or \(q = 0, p \neq 0\ or \ p = 0, q \neq 0\), the \(SU(2)\) CFTW \(k\)-instanton solutions can be embedded in that of \(SL(2, C)\) gauge field. In general, we have the pure \(SL(2, C)\) solutions

\[
G^a_{\mu} = -\bar{\eta}_{a\mu}^\nu \partial \ln \phi = -\bar{\eta}_{a\mu}^\nu \frac{1}{p^2 + q^2} [pp_\nu + qq_\nu + i(pq_\nu - qp_\nu)]. \quad (2.24)
\]

For the simplest \(SL(2, C)\) 1-instanton case \((M, N) = (1, 0)\), let’s take

\[
M = 1, N = 0, p = 1 + \frac{\alpha_1^2}{y^2}, q = 1 \quad (2.25)
\]

where \(y_\mu \equiv x_\mu - a_{1\mu}, y^2 \equiv y_\mu y_\mu\), the gauge potentials can be calculated to be

\[
A^a_\mu = \bar{\eta}_{a\mu\nu}^\nu \frac{2\alpha_1^2 (y^2 + \alpha_1^2)}{y^2 (y^4 + (y^2 + \alpha_1^2)^2)},
\]

\[
B^a_\mu = -\bar{\eta}_{a\mu\nu}^\nu \frac{2\alpha_1^2}{y^4 + (y^2 + \alpha_1^2)^2}. \quad (2.26)
\]
The gauge potential $A^a_\mu$ has a singularity at $x_\mu = a_1\mu$ which is a gauge artifact that can be gauged away by a $SL(2,C)$ gauge transformation. Define

$$U_1(x) = \left( \frac{x_4 + ix_j\sigma_j}{|x|} \right) = \hat{x}_\mu S_\mu, U_1(x) \in SU(2) \subset SL(2,C)$$

where $S_{1,2,3} = i\sigma_{1,2,3}$. After making a large gauge transformation by $U_1(x)$, we have

$$A^a_\mu = \eta^a_\mu y_\nu \frac{2(2y^2 + \alpha^2_1)}{y^4 + (y^2 + \alpha^2_1)^2},$$

$$B^a_\mu = \eta^a_\mu y_\nu \frac{2\alpha^2_1}{y^4 + (y^2 + \alpha^2_1)^2},$$

which are regular $SL(2,C)$ solution. The corresponding field strength can be calculated to be

$$H^a_\mu = -4\eta^a_\mu \alpha^2_1 \frac{2y^4 + 4\alpha^2_1 y^2 + \alpha^4_1}{[y^4 + (y^2 + \alpha^2_1)^2]^2},$$

$$M^a_\mu = -4\eta^a_\mu \alpha^2_1 \frac{2y^4 - \alpha^4_1}{[y^4 + (y^2 + \alpha^2_1)^2]^2},$$

which are self-dual by Eq.(2.20a).

Alternatively, instead of taking Eq.(2.25), let’s take $(M, N) = (0, 1)$

$$M = 0, N = 1, p = 1, q = 1 + \frac{\beta^2_1}{y^2},$$

where $y_\mu \equiv x_\mu - b_1\mu$, $y^2 \equiv y_\mu y_\mu$. Then we have

$$\phi = 1 + i + \frac{i\beta^2_1}{|x - y_1|^2}.$$  

It can be shown that for $SU(2)$ complex YM equation with a complex source term $J_\mu$, the complex gauge potential for $(M, N)$ solution is related to the complex conjugate of $(N, M)$ solution with $J_\mu$ replaced by $J^*_\mu$. For the present pure YM case without $J_\mu$, it can be shown that Eq.(2.30) leads to a solution which is equivalent to the solution in Eq.(2.26). We will see this equivalence in section IV where more general 1-instanton solution will be constructed.

In general, one can generalize the 1-instanton solution to the k-instanton cases. For the multi-instanton solutions, say $k = 2$ for example, we get

$$\phi = (1 + i + \frac{\alpha^2_1}{|x - y_1|^2} + \frac{i\beta^2_1}{|x - y_2|^2}).$$

In general, the topological charge of the $(M, N)$ solution was found to be $Q = M + N$. For the boundary conditions

$$\lim_{r \to \infty} H^a_\mu = \lim_{r \to \infty} M^a_\mu = 0.$$
the action integral for the case of $\theta = 0$ in Eq. (2.10) can be calculated to be

$$
\int_{\mathbb{R}^4} d^4x L = \int_{\mathbb{R}^4} d^4x \frac{1}{4} (H^a_{\mu
u} H^a_{\mu\nu} - M^a_{\mu\nu} M^a_{\mu\nu})
$$

$$
= 8\pi^2 Q = 8\pi^2 (M + N).
$$

(2.34)

Note that for the non-compact $SL(2, C)$ case, unlike the $SU(2)$ case, there is no proof that instanton action is the minimum action in each homotopy class.

### III. BIQUATERNIONS AND $SL(2, C)$ ADHM YM INSTANTONS

In this section and section IV, in contrast to the last section, we will use the convention $\mu = 0, 1, 2, 3$ and $e_{0123} = 1$ for 4D Euclidean space. Instead of quaternion in the $Sp(1)$ ($= SU(2)$) ADHM construction, we will use biquaternion to construct $SL(2, C)$ SDYM instantons. A quaternion $x$ can be written as

$$
x = x_\mu e_\mu, \quad x_\mu \in \mathbb{R}, \quad e_0 = 1, e_1 = i, e_2 = j, e_3 = k
$$

(3.35)

where $e_1, e_2$ and $e_3$ anticommute and obey

$$
e_i \cdot e_j = -e_j \cdot e_i = e_{ijk} e_k; \quad i, j, k = 1, 2, 3,
$$

(3.36)

$$
e_1^2 = -1, e_2^2 = -1, e_3^2 = -1.
$$

(3.37)

The conjugate quaternion is defined to be

$$
x^\dagger = x_0 e_0 - x_1 e_1 - x_2 e_2 - x_3 e_3
$$

(3.38)

so that the norm square of a quaternion is

$$
|x|^2 = x^\dagger x = x_0^2 + x_1^2 + x_2^2 + x_3^2.
$$

(3.39)

Occasionally the unit quaternions were expressed as Pauli matrices

$$
e_0 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_i \rightarrow -i \sigma_i; \quad i = 1, 2, 3.
$$

(3.40)

A (ordinary) biquaternion (or complex-quaternion) $z$ can be written as

$$
z = z_\mu e_\mu, \quad z_\mu \in \mathbb{C},
$$

(3.41)
which will be used in this paper. Occasionally \( z \) can be written as

\[
z = x + yi
\]

(3.42)

where \( x \) and \( y \) are quaternions and \( i = \sqrt{-1} \), not to be confused with \( e_1 \) in Eq. (3.35). There are two other types of biquaternions in the literature, the split-biquaternion and the dual biquaternion. For biquaternion, Hamilton introduced two types of conjugations, the biconjugation [25]

\[
z^\ast = z^\ast_\mu e_\mu = z^\ast_0 e_0 - z^\ast_1 e_1 - z^\ast_2 e_2 - z^\ast_3 e_3 = x^\dagger + y^\dagger i, \quad (3.43)
\]

which will be heavily used in this paper, and the complex conjugation

\[
z^* = z^*_\mu e_\mu = z^*_0 e_0 + z^*_1 e_1 + z^*_2 e_2 + z^*_3 e_3 = x - yi. \quad (3.44)
\]

In contrast to Eq. (3.39), the norm square of a biquaternion used in this paper is defined to be

\[
|z|^2_c = z^\ast z = (z_0)^2 + (z_1)^2 + (z_2)^2 + (z_3)^2 \quad (3.45)
\]

which is a complex number in general as a subscript \( c \) is used in the norm.

We are now ready to proceed the construction of \( SL(2, C) \) instantons. Historically, the general procedure to construct ADHM \( Sp(N) \), \( SU(N) \) and \( O(N) \) instantons are similar [5]. The construction strongly relied on the quaternion calculation. In this section, instead of \( SU(2) \), we will extend the \( Sp(1) \) quaternion construction to the \( SL(2, C) \) biquaternion construction. We begin by introducing the \((k + 1) \times k\) biquaternion matrix \( \Delta(x) = a + bx \)

\[
\Delta(x)_{ab} = a_{ab} + b_{ab}x, \quad a_{ab} = a^\mu_{ab} e_\mu, b_{ab} = b^\mu_{ab} e_\mu \quad (3.46)
\]

where \( a^\mu_{ab} \) and \( b^\mu_{ab} \) are complex numbers, and \( a_{ab} \) and \( b_{ab} \) are biquaternions. The biconjugation of the \( \Delta(x) \) matrix is defined to be

\[
\Delta(x)^\ast_{ab} = \Delta(x)^\ast_{ba} e_\mu = \Delta(x)_{0ab} e_0 - \Delta(x)_{1ab} e_1 - \Delta(x)_{2ab} e_2 - \Delta(x)_{3ab} e_3. \quad (3.47)
\]

In contrast to the of \( SU(2) \) instantons, the quadratic condition of \( SL(2, C) \) instantons reads

\[
\Delta(x)^\ast \Delta(x) = f^{-1} = \text{symmetric, non-singular } k \times k \text{ matrix for } x \notin J, \quad (3.48)
\]
from which we can deduce that \(a^\circ a, b^\circ a, a^\circ b\) and \(b^\circ b\) are all symmetric matrices. We stress here that it will turn out the choice of biconjugation operation is crucial for the follow-up discussion in this paper. On the other hand, for \(x \in J\), \(\det \Delta(x)^\circ \Delta(x) = 0\). The set \(J\) is called singular locus or "jumping lines" in the mathematical literatures and will be discussed in section IV.D. The existence of jumping lines is quite common in complex ADHM equations. In contrast to the \(SL(2, C)\) instantons, there are no jumping lines for the case of \(SU(2)\) instantons. We will assume \(x \notin J\) in the discussion for the rest of this section.

In the \(Sp(1)\) quaternion case, the symmetric condition on \(f^{-1}\) means \(f^{-1}\) is real. For the \(SL(2, C)\) biquaternion case, however, it can be shown that symmetric condition on \(f^{-1}\) implies \(f^{-1}\) is complex. Indeed, since

\[
[\Delta(x)^\circ \Delta(x)]_{ij} = \sum_{m=1}^{k+1} [\Delta(x)^\circ]_{im}[\Delta(x)]_{mj} \\
= \sum_{m=1}^{k+1} ([\Delta(x)]^\mu_{im}[\Delta(x)]_{mj}) (e^\dagger_\mu e_\nu) = \sum_{m=1}^{k+1} ([\Delta(x)]^\nu_{mj}[\Delta(x)]^\mu_{mi}) (e^\dagger_\nu e_\mu) \\
= \sum_{m=1}^{k+1} \{( [\Delta(x)]^\nu_{jm} e^\dagger_\nu)[[\Delta(x)]^\mu_{im} e_\mu]\}^\circ = [\Delta(x)^\circ \Delta(x)]_{ji}^\circ, \tag{3.49}
\]

the symmetric condition implies

\[
[\Delta(x)^\circ \Delta(x)]_{ij} = [\Delta(x)^\circ \Delta(x)]_{ji}^\circ, \tag{3.50}
\]

which means

\[
[\Delta(x)^\circ \Delta(x)]^\mu_{ij} e_\mu = [\Delta(x)^\circ \Delta(x)]^\mu_{ji} e_\mu^\dagger. \tag{3.51}
\]

Thus only \([\Delta(x)^\circ \Delta(x)]_{ij}^0\) is nonvanishing, and it is in general a complex number for the case of biquaternion.

To construct the self-dual gauge field, we introduce a \((k+1) \times 1\) dimensional biquaternion vector \(v(x)\) satisfying the following two conditions

\[
v^\circ(x) \Delta(x) = 0, \tag{3.52}
\]

\[
v^\circ(x) v(x) = 1. \tag{3.53}
\]

Note that \(v(x)\) is fixed up to a \(SL(2, C)\) gauge transformation

\[
v(x) \rightarrow v(x) g(x), \quad g(x) \in 1 \times 1 \text{ Biquaternion}. \tag{3.54}
\]
Note that in general a $SL(2, C)$ matrix can be written in terms of a $1 \times 1$ biquaternion as

$$g = \frac{q_\mu e_\mu}{\sqrt{q^* q}} = \frac{q_\mu e_\mu}{|q|_c}.$$  

(3.55)

It is obvious that Eq.(3.52) and Eq.(3.53) are invariant under the gauge transformation. The next step is to define the gauge field

$$G_\mu(x) = v^\circ(x) \partial_\mu v(x),$$  

(3.56)

which is a $1 \times 1$ biquaternion. The $SL(2, C)$ gauge transformation of the gauge field is

$$G_\mu(x) \rightarrow G'(x) = (g^\circ(x) v^\circ(x)) \partial_\mu (v(x) g(x))$$

$$= g^\circ(x) G_\mu(x) g(x) + g^\circ(x) \partial_\mu g(x)$$  

(3.57)

where in the calculation Eq.(3.53) has been used. Note that, unlike the case for $Sp(1)$, $G_\mu(x)$ needs not to be anti-Hermitian.

We can now define the $SL(2, C)$ field strength

$$F_{\mu \nu} = \partial_\mu G_\nu(x) + G_\mu(x) G_\nu(x) - [\mu \leftrightarrow \nu].$$  

(3.58)

To show that $F_{\mu \nu}$ is self-dual, one needs to show that the operator

$$P = 1 - v(x) v^\circ(x)$$  

(3.59)

is a projection operator $P^2 = P$, and can be written in terms of $\Delta$ as

$$P = \Delta(x) f \Delta^\circ(x).$$  

(3.60)

In fact

$$P^2 = (1 - v(x) v^\circ(x))(1 - v(x) v^\circ(x))$$

$$= 1 - 2v(x) v^\circ(x) + v(x) v^\circ(x) v(x) v^\circ(x)$$

$$= 1 - v(x) v^\circ(x) = P,$$  

(3.61)

and
\[ P v(x) = (1 - v(x)v^\circ(x))v(x) = v(x) - v(x)v^\circ(x)v(x) = 0. \]  
(3.62)

On the other hand

\[ P_2 \equiv \Delta(x)f\Delta^\circ(x), \]  
(3.63)

\[ P_2^2 = \Delta(x)f\Delta^\circ(x)\Delta(x)f\Delta^\circ(x) = \Delta(x)f f^{-1}f \Delta^\circ(x) = \Delta(x)f \Delta^\circ(x) = P_2, \]  
(3.64)

and

\[ P_2 v(x) = \Delta(x)f \Delta^\circ(x)v(x) = 0. \]  
(3.65)

So \( P_2 = P \). This completes the proof. The self-duality of \( F_{\mu\nu} \) can now be proved as following

\[
F_{\mu\nu} = \partial_\mu (v^\circ(x)\partial_\nu v(x)) + v^\circ(x)\partial_\mu v(x)v^\circ(x)\partial_\nu v(x) - [\mu \leftrightarrow \nu] \\
= \partial_\mu v^\circ(x)[1 - v(x)v^\circ(x)]\partial_\nu v(x) - [\mu \leftrightarrow \nu] \\
= \partial_\mu v^\circ(x)\Delta(x)f\Delta^\circ(x)\partial_\nu v(x) - [\mu \leftrightarrow \nu] \\
= v^\circ(x)(\partial_\mu \Delta(x))f(\partial_\nu \Delta^\circ(x))v(x) - [\mu \leftrightarrow \nu] \\
= v^\circ(x)(be_\mu)f(e_\nu^\dagger b^\circ) v(x) - [\mu \leftrightarrow \nu] \\
= v^\circ(x)b(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger) f b^\circ v(x) 
\]  
(3.66)

where we have used Eqs. (3.46), (3.52) and (3.60). Finally the factor \((e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger)\) above can be shown to be self-dual

\[
\sigma_{\mu\nu} \equiv \frac{1}{4i}(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger) = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\sigma_{\alpha\beta}, 
\]  
(3.67)

\[
\tilde{\sigma}_{\mu\nu} = \frac{1}{4i}(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger) = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\tilde{\sigma}_{\alpha\beta}. 
\]  
(3.68)

This proves the self-duality of \( F_{\mu\nu} \). We thus have constructed many \( SL(2, C) \) SDYM field configurations.

To count the number of moduli parameters for the \( SL(2, C) \) \( k \)-instantons we have constructed, we will use transformations which preserve conditions Eq. (3.48), Eq. (3.52) and Eq. (3.53), and the definition of \( G_\mu \) in Eq. (3.56) to bring \( a \) and \( b \) in Eq. (3.46) into a simple canonical form. The allowed transformations are similar to the case of \( Sp(1) \) except that for the \( SL(2, C) \) case, \( Q \) is unitary biquaternionic and \( K^\circ = K^T \). That is
\[ a \rightarrow QaK, b \rightarrow QbK, v \rightarrow Qv \quad (3.69) \]

where

\[ Q : (k + 1) \times (k + 1), \; Q^*Q = I , \quad (3.70) \]

\[ K^* = K^T. \quad (3.71) \]

One can use \( K \) and \( Q \) to bring \( b \) to the following form

\[ b = \begin{bmatrix} 0_{1\times k} \\ I_{k\times k} \end{bmatrix} \quad (3.72) \]

Now the form of \( b \) above is preserved by the following transformations

\[ Q = \begin{bmatrix} Q_{1\times 1} & 0 \\ 0 & X \end{bmatrix}, \; K = X^T, Q_{1\times 1} \in SL(2,C), X \in O(k). \quad (3.73) \]

Then by choosing \( X \) appropriately, one can diagonalize \( a^*a \) and bring \( a \) to the following form

\[ a = \begin{bmatrix} \lambda_{1\times k} \\ -y_{k\times k} \end{bmatrix} \quad (3.74) \]

where \( \lambda \) and \( y \) are biquaternion matrices with orders \( 1 \times k \) and \( k \times k \) respectively, and \( y \) is symmetric

\[ y = y^T. \quad (3.75) \]

Thus the constraints for the moduli parameters are

\[ a^*_c a_{cj} = 0, \; i \neq j, \; \text{and} \; y_{ij} = y_{ji}. \quad (3.76) \]

The forms \( a \) and \( b \) in Eq.\((3.74)\) and Eq.\((3.72)\) are called the canonical forms of the construction, and \( \lambda_{1\times k}, \; y_{k\times k} \) under the constraints Eq.\((3.76)\) are the moduli parameters of
$k$-instantons. The total number of moduli parameters for $k$-instanton can be calculated through Eq. (3.76) to be

$$\# \text{ of moduli for } SL(2, C) \ k\text{-instantons} = 16k - 6,$$

(3.77)

which is twice of that of the case of $Sp(1)$. Roughly speaking, there are $8k$ parameters for instanton "biquaternion positions" and $8k$ parameters for instanton "sizes". Finally one has to subtract an overall $SL(2, C)$ gauge group degree of freesom 6. This picture will become more clear when we give examples of explicit constructions of $SL(2, C)$ instantons in the next section.

IV. EXAMPLES OF $SL(2, C)$ ADHM INSTANTONS

In this section, we will explicitly construct three examples of $SL(2, C)$ YM instantons to illustrate our prescription given in the last section. More importantly, we will also discuss the singular structures of $SL(2, C)$ $k$-instantons and compare our results with those in the mathematical literatures.

A. The $SL(2, C) \ (M, N)$ Instantons in ADHM Construction

In this first example, we will reproduce from the ADHM construction the $SL(2, C) \ (M, N)$ instanton solutions [23] discussed in section II. We choose the biquaternion $\lambda_j$ in Eq. (3.74) to be $\lambda_je_0$ with $\lambda_j$ a complex number, and choose $y_{ij} = y_j \delta_{ij}$ to be a diagonal matrix with $y_j = y_{j\mu}e_\mu$ a quaternion. That is

$$\Delta(x) = \begin{bmatrix}
\lambda_1 & \lambda_2 & \ldots & \lambda_k \\
x - y_1 & 0 & \ldots & 0 \\
0 & x - y_2 & \ldots & 0 \\
0 & 0 & \ldots & \ldots \\
0 & 0 & \ldots & x - y_k
\end{bmatrix},$$

(4.78)
which satisfies the constraint in Eq. (3.76). Let

\[
v = \frac{1}{\sqrt{\phi}} \begin{bmatrix} 1 \\ -q_1 \\ \vdots \\ -q_k \end{bmatrix},
\]

then

\[
q_j = \frac{\lambda_j(x_\mu - y_{j\mu})e_\mu}{|x - y_j|^2}, \quad j = 1, 2, \ldots, k,
\]

and

\[
v = \frac{1}{\sqrt{\phi}} \begin{bmatrix} 1 \\ -\lambda_1(x_\mu - y_{1\mu})e_\mu/|x - y_1|^2 \\ \vdots \\ -\lambda_k(x_\mu - y_{k\mu})e_\mu/|x - y_k|^2 \end{bmatrix},
\]

with

\[
\phi = 1 + \frac{\lambda_1^2}{|x - y_1|^2} + \ldots + \frac{\lambda_k^2}{|x - y_k|^2}.
\]

We have used \( \lambda_j\lambda_j^* = \lambda_j^2 \) where \( \lambda_j^2 \) a complex number in the above calculation. For the case of \( Sp(1) \), \( \lambda_j \) is a real number and \( \lambda_j\lambda_j^* = \lambda_j^2 \) is a real number. So \( \phi \) in Eq. (4.82) is a complex-valued function in general. One can calculate the gauge potential as

\[
G_\mu = v^\ast \partial_\mu v = \frac{1}{4}[e_\mu^\dagger e_\mu - e_\nu^\dagger e_\nu] \partial_\nu \ln(1 + \frac{\lambda_1^2}{|x - y_1|^2} + \ldots + \frac{\lambda_k^2}{|x - y_k|^2})
\]

\[
= \frac{1}{4}[e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu] \partial_\nu \ln(\phi).
\]

If we choose \( k = 1 \) and define \( \lambda_1^2 = \frac{\alpha^2}{1+\iota} \), then

\[
\phi = 1 + \frac{\alpha^2}{|x - y_1|^2}.
\]

The gauge potential is

\[
G_\mu = \frac{1}{4}[e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu] \partial_\nu \ln(1 + \frac{\alpha^2}{1+\iota}) = \frac{1}{4}[e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu] \partial_\nu \ln(1 + \frac{\alpha^2}{|x - y_1|^2} + i)
\]

\[
= \frac{1}{2}[e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu] - \frac{\alpha^2(x - y_1)_\nu}{|x - y_1|^4 + (|x - y_1|^2 + \alpha^2)^2} (|x - y_1|^2 + \alpha^2 - i)
\]

(4.85)
which reproduces the $SL(2, C)$ $(M, N) = (1, 0)$ solution calculated in Eq. (2.26). If we choose $k = 1$ and consider $\lambda_1^2 = \frac{i\beta^2}{1+i}$, then

$$\phi = 1 + \frac{i\beta^2}{1+i}. \quad (4.86)$$

The gauge potential is

$$G_\mu = \frac{1}{4}[e^{\dagger}_\mu e_\nu - e^{\dagger}_\nu e_\mu]\partial_\nu \ln(1 + \frac{i\beta^2}{1+i}[x - y_1]^2) = \frac{1}{4}[e^{\dagger}_\mu e_\nu - e^{\dagger}_\nu e_\mu]\partial_\nu \ln[(1 + i + \frac{i\beta^2}{1+i}[x - y_1]^2)] \quad (4.87)$$

which reproduces the $SL(2, C)$ $(M, N) = (0, 1)$ solution calculated in Eq. (2.31). If we choose $k = 2$ and $\lambda_1^2 = \frac{\alpha^2}{1+i}, \lambda_2^2 = \frac{i\beta^2}{1+i}$, we get

$$\phi = 1 + \frac{\alpha^2}{1+i} + \frac{i\beta^2}{1+i}. \quad (4.88)$$

$$G_\mu = \frac{1}{4}[e^{\dagger}_\mu e_\nu - e^{\dagger}_\nu e_\mu]\partial_\nu \ln(1 + \frac{\alpha^2}{1+i}[x - y_1]^2 + \frac{i\beta^2}{1+i}[x - y_2]^2) \quad (4.89)$$

$$= \frac{1}{4}[e^{\dagger}_\mu e_\nu - e^{\dagger}_\nu e_\mu]\partial_\nu \ln(1 + i + \frac{\alpha^2}{1+i}[x - y_1]^2 + \frac{i\beta^2}{1+i}[x - y_2]^2), \quad (4.90)$$

which reproduces the $SL(2, C)$ $(1, 1)$ solution calculated in Eq. (2.32). It is easy to generalize the above calculations to the general $(M, N)$ cases. The $SL(2, C)$ ADHM $k$-instanton solutions we proposed in section III thus include the $SL(2, C)$ $(M, N)$ $k$-instanton solutions calculated previously in [23] as a subset.

B. The $SL(2, C)$ $k = 2, 3$ Instanton Solutions

For the case of 2-instantons, we begin with the following $\Delta(x)$ matrix with $y_{12} = y_{21}$

$$\Delta(x) = \begin{bmatrix} \lambda_1 & \lambda_2 \\ x - y_1 & -y_{12} \\ -y_{21} & x - y_2 \end{bmatrix}, \quad (4.91)$$

$$\Delta^\circ(x) = \begin{bmatrix} \lambda_1^\circ & x^\circ - y_1^\circ & -y_{12}^\circ \\ \lambda_2^\circ & -y_{12}^\circ & x^\circ - y_2^\circ \end{bmatrix}. \quad (4.92)$$

The condition on $\Delta^\circ(x)\Delta(x)$

$$\Delta^\circ(x)\Delta(x) = \begin{bmatrix} \lambda_1^\circ \lambda_1 + (x^\circ - y_1^\circ)(x - y_1) + y_{12}^\circ y_{12} & \lambda^\circ \lambda_2 - (x^\circ - y_1^\circ)y_{12} - y_{12}^\circ(x - y_2) \\ \lambda_2^\circ \lambda_1 - y_{12}^\circ(x - y_1) - (x^\circ - y_2^\circ)y_{12} & \lambda_2^\circ \lambda_2 + y_{12}^\circ y_{12} + (x^\circ - y_2^\circ)(x - y_2) \end{bmatrix}. \quad (4.93)$$
in Eq. (3.48) is

$$\lambda_2^{\otimes} \lambda_1 - \lambda_1^{\otimes} \lambda_2 = y_{12}^{\otimes} (y_2 - y_1) + (y_1^{\otimes} - y_2^{\otimes}) y_{12},$$

(4.94)

which is linear in the biquaternion $y_{12}$ instead of a quadratic equation, and $y_{12}$ can be easily solved to be

$$y_{12} = \frac{1}{2} \frac{(y_1 - y_2)}{|y_1 - y_2|} (\lambda_2^{\otimes} \lambda_1 - \lambda_1^{\otimes} \lambda_2).$$

(4.95)

So the four biquaternions $y_1, y_2, \lambda_1$ and $\lambda_2$ gives $4 \times 8 = 32$ real parameters. After subtracting $SL(2, C)$ gauge group degree of freedom $6$, the number of moduli for $SL(2, C)$ 2-instanton is $26$ as expected. The result in Eq. (4.95) is the same with the case of $Sp(1)$ except with quaternions replaced by biquaternions [5]. However, for the $SL(2, C)$ 2-instantons in contrast to the $SU(2)$ case, there are so-called jumping lines in zeros of determinant of Eq. (4.93) which correspond to singularities of $SL(2, C)$ 2-instanton field configurations. This will be calculated in section IV. D.

For the case of 3-instantons, we begin with the following $\Delta(x)$ matrix with $y_{ij} = y_{ji}$

$$\Delta(x) = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ x - y_1 & -y_{12} & -y_{13} \\ -y_{21} & x - y_2 & -y_{23} \\ -y_{31} & -y_{32} & x - y_3 \end{bmatrix}.$$

(4.96)

In order to get the general solutions for $k = 3$ $SL(2, C)$ instanton, similar to [5], we make the choices $\lambda_1 = \lambda_1^0 \otimes e_0$ ($\lambda_1^1 = \lambda_2^1 = \lambda_3^1 = 0$) and $y_{12}^0 = y_{13}^0 = y_{23}^0 = 0$. Then the remaining parameters are the positions $y_1, y_2, y_3$ and the imaginary part of $y_{12}, y_{13}, y_{23}$. So there are $8 \times 3 + 6 \times 3 = 42 = 16k - 6(k = 3)$ parameters. Other parameters can be fixed by constraints to be

$$\lambda_1 = \lambda_1^0 \otimes e_0$$

(4.97)

$$\lambda_1^0 = \frac{|W_2 \times W_3|}{|W_1 \cdot (W_2 \times W_3)|^{1/2}}$$

(4.98)

$$\lambda_2 = \lambda_1 \frac{(W_3 \times W_2) \cdot (W_3 \times W_1)}{|W_2 \times W_3|^{2}} + i \mathbf{\sigma} \cdot \frac{1}{\lambda_1} W_3$$

(4.99)

$$\lambda_3 = \lambda_1 \frac{(W_3 \times W_2) \cdot (W_2 \times W_1)}{|W_2 \times W_3|^{2}} - i \mathbf{\sigma} \cdot \frac{1}{\lambda_1} W_2$$

(4.100)
where the vectors $\vec{W}_k$ are defined by

$$\vec{W}_k = \frac{i}{4} \epsilon_{ijk} tr \{ \sigma^i (y_i - y_j) \otimes y_{ij} + \sum_{l=1}^{3} (y_{ij} \otimes y_{lj}) \}. \quad (4.101)$$

Here we have presented the biquaternions $\lambda$, as $2 \times 2$ matrices. The result in the above equations are the same with the case of $Sp(1)$ except with quaternions replaced by biquaternions. However, for the $SL(2, C)$ 3-instantons in constrast to the $SU(2)$ case, there are jumping lines of 3-instantons corresponding to Eq.(4.96). This will be discussed in section IV. D.

C. The $SL(2, C)$ 1-Instanton Solution and its Singularities

In the third example, we calculate the complete $SL(2, C)$ 10 parameters 1-instanton solution and study structure of its singularities. The singular structures of a subset of $SL(2, C)$ $k$-instantons will be discussed in the next subsection. We will see that the singularities for $SL(2, C)$ 1-instanton is much more complicated than that of $SU(2)$ 1-instanton. All 10 parameters are closely related to the structure of the singularities. We first build $\Delta(x)$ matrix and choose $a, b$ as

$$a = \begin{bmatrix} \lambda \\ -y \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (4.102)$$

$$\Delta(x) = a + bx = \begin{bmatrix} \lambda \\ x - y \end{bmatrix} \quad (4.103)$$

where $x$ is a quaternion, $\lambda = \lambda e_0$ (with $\lambda$ a complex number) and $y$ is a biquaternion. It can be checked that, for these choices, the constraints in Eq.(3.76) are satisfied. By Eq.(3.52) and Eq.(3.53), we easily obtain

$$v(x) = \frac{1}{\sqrt{\phi}} \begin{bmatrix} 1 \\ -\frac{(x-y)\lambda^{\ast}}{|x-y|^2} \end{bmatrix} \quad (4.104)$$

with

$$\phi = 1 + \frac{\lambda \lambda^{\ast}}{|x-y|^2}. \quad (4.105)$$
Note that $\lambda \lambda^\circ = \lambda^2$ is a complex number and $|x - y|^2_{c} \equiv |x - (p + qi)|^2_{c}$ is also a complex number. Here $p$ and $q$ are quaternions. The total number of moduli parameters is thus 10. The gauge field $G_{\mu}$ can be calculated to be

$$G_{\mu} = v^{\circ} \partial_{\mu} v(x)$$

$$= \frac{1}{4} [e^{i}_{\mu} e_{\nu} - e^{i}_{\nu} e_{\mu}] \partial_{\nu} \ln(1 + \frac{\lambda^2}{|x - y|_{c}^2})$$

$$= \frac{-1}{2} [e^{i}_{\mu} e_{\nu} - e^{i}_{\nu} e_{\mu}] \frac{|x - (p + qi)|_{\nu} \lambda^2}{|x - (p + qi)|_{\nu}^2([x - (p + qi)]_{\nu}^2 + \lambda^2)}. \quad (4.106)$$

By solving $|x - (p + qi)|_{c}^2 = 0$ in the denominator of Eq. (4.106), we can get some singularities of $G_{\mu}$. We see that

$$|x - (p + qi)|_{c}^2 = [(x_0 - p_0)^2 + (x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2] - (q_0^2 + q_1^2 + q_2^2 + q_3^2)$$

$$= 2i[(x_0 - p_0)q_0 + (x_1 - p_1)q_1 + (x_2 - p_2)q_2 + (x_3 - p_3)q_3] = 0 \quad (4.107)$$

implies

$$[(x_0 - p_0)^2 + (x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2] = (q_0^2 + q_1^2 + q_2^2 + q_3^2), \quad (4.108)$$

$$(x_0 - p_0)q_0 + (x_1 - p_1)q_1 + (x_2 - p_2)q_2 + (x_3 - p_3)q_3 = 0. \quad (4.109)$$

Generically for $q \neq 0$, Eq. (4.108) and Eq. (4.109) describe in $R^4$ a $S^3$ and an hyper-plane $R^3$ passing through the center of the $S^3$ respectively. Thus the intersection of these $S^3$ and $R^3$ is a $S^2$. This means that the singularities is a $S^2$ in $R^4$. It is clear geometrically that $p_{\mu}$ is the center of the $S^2$ and $q_{\mu}$ gives radius and orientation of the $S^2$ in $R^4$. In fact, these singularities can be gauged away just like in the $SU(2)$ case. If we define

$$U_{1c}(z) = \frac{z}{|z|_{c}} = \frac{(x - p - qi)^{\mu} e_{\mu}}{|x - p - qi|_{c}} \quad (4.110)$$

where $U_{1c}(z)$ is a $1 \times 1$ biquaternion corresponding to a $SL(2, C)$ matrix, which is to be compared with Eq. (2.27) for the case of $SU(2)$. Then

$$U_{1c}(z) \frac{\partial}{\partial z^{\mu}} U^{-1}_{1c}(z) = \frac{z}{|z|_{c}} \frac{\partial}{\partial z^{\mu}} \frac{z^{\circ}}{|z|_{c}} = -\frac{1}{2} [e^{i}_{\mu} e_{\nu} - e_{\nu} e^{i}_{\mu}] \frac{z_{\nu}}{|z|_{c}^2}. \quad (4.111)$$
\[
U_{1c}^{-1}(z) \frac{\partial}{\partial z^\mu} U_{1c}(z) = z^\ast \frac{\partial}{\partial z^\mu} z = -\frac{1}{2} [e^\dagger_\mu e_\nu - e^\dagger_\nu e_\mu] \frac{z_\nu}{|z|^2_c}. \tag{4.112}
\]

It’s easy to see that \( G_\mu \) can be written as

\[
G_\mu = \frac{1}{2} [e^\dagger_\mu e_\nu - e^\dagger_\nu e_\mu] \frac{-(x - (p + qi))_\nu \lambda^2}{|x - (p + qi)|^2_c (|x - (p + qi)|^2_c + \lambda^2)}
\]

\[
= U_{1c}^{-1}(z) \frac{\partial}{\partial z^\mu} U_{1c}(z) \frac{\lambda^2}{(|x - (p + qi)|^2_c + \lambda^2)}. \tag{4.113}
\]

We can now do the \( SL(2, \mathbb{C}) \) gauge transformation

\[
G'_\mu = U_{1c}(z) G_\mu U_{1c}^{-1}(z) + U_{1c}(z) \frac{\partial}{\partial z^\mu} U_{1c}^{-1}(z)
\]

\[
= -\frac{1}{2} [e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu] \frac{|x - (p + qi)|_\nu}{(|x - (p + qi)|^2_c + \lambda^2)} \tag{4.114}
\]

to gauge away the singularities caused by \(|x - (p + qi)|^2_c = 0\). But there are still non-removable singularities remained, which come from \((|x - (p + qi)|^2_c + \lambda^2) = 0\) in the denominator of Eq.(4.106). To study these singularities, let the real part of \( \lambda^2 \) be \( c \) and imaginary part of \( \lambda^2 \) be \( d \), we see that

\[
(|x - (p + qi)|^2_c + \lambda^2) = P_2(x) + iP_1(x)
\]

\[
= [(x_0 - p_0)^2 + (x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 - (q_0^2 + q_1^2 + q_2^2 + q_3^2)] + c 
\]

\[
- 2i[(x_0 - p_0) q_0 + (x_1 - p_1) q_1 + (x_2 - p_2) q_2 + (x_3 - p_3) q_3 - \frac{d}{2}] = 0 \tag{4.115}
\]

implies

\[
(x_0 - p_0)^2 + (x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 = (q_0^2 + q_1^2 + q_2^2 + q_3^2) - c, \tag{4.116}
\]

\[
(x_0 - p_0) q_0 + (x_1 - p_1) q_1 + (x_2 - p_2) q_2 + (x_3 - p_3) q_3 = \frac{d}{2} \tag{4.117}
\]

where \( P_2(x) \) and \( P_1(x) \) are polynomials of 4 variables with degree 2 and 1 respectively.

For a subset of \( k \)-instanton field configurations, one chooses \( \lambda_i = \lambda_i e_0 \) (with \( \lambda_i \) a complex number) and \( y_i \) to be a \textit{biquaternion} in Eq.(4.78). It is important to note that for these choices, the constraints in Eq.(3.76) are still satisfied \textit{without} turning on the off-diagonal
elements $y_{ij}$ in Eq. (3.74). To get non-removable singularities, one needs to calculate zeros of
\[ \phi = 1 + \frac{\lambda_1 \lambda_1^0}{|x - y_1|^2_c} + \cdots + \frac{\lambda_k \lambda_k^0}{|x - y_k|^2_c}, \] (4.118)
or
\[ |x - y_1|^2_c |x - y_2|^2_c \cdots |x - y_k|^2_c \phi = P_{2k}(x) + iP_{2k-1}(x) = 0. \] (4.119)
For the $k$-instanton case, one encounters intersections of zeros of $P_{2k}(x)$ and $P_{2k-1}(x)$ polynomials with degrees $2k$ and $2k - 1$ respectively
\[ P_{2k}(x) = 0, \quad P_{2k-1}(x) = 0. \] (4.120)
We will discuss the singular structures of these $k$-instanton field configurations in the next subsection.

The structure of singularities of $SL(2, C)$ 1-instanton can be classified into the following four cases:

(1) For $q = 0$, Eq. (4.117) implies $d = 0$. If $c < 0$, one gets $S^3$ singularities. If $c = 0$, one gets a singular point at $p$.

The following three cases are for $q \neq 0$.

(2) If the 6 parameters satisfy
\[ (q_0^2 + q_1^2 + q_2^2 + q_3^2) - c < \frac{d^2}{(q_0^2 + q_1^2 + q_2^2 + q_3^2)}, \] (4.121)
then there are no singularities.

(3) If the 6 parameters satisfy
\[ (q_0^2 + q_1^2 + q_2^2 + q_3^2) - c = \frac{d^2}{(q_0^2 + q_1^2 + q_2^2 + q_3^2)}, \] (4.122)
then there is only one singular point which is located at
\[ (x_0, x_1, x_2, x_3) = (p_0, p_1, p_2, p_3) + \frac{d}{2} \frac{1}{(q_0^2 + q_1^2 + q_2^2 + q_3^2)}(q_0, q_1, q_2, q_3). \] (4.123)

(4) If the 6 parameters satisfy
\[ (q_0^2 + q_1^2 + q_2^2 + q_3^2) - c > \frac{d^2}{(q_0^2 + q_1^2 + q_2^2 + q_3^2)}, \] (4.124)
then the singularities are the intersection of a $R^3$ and a $S^3$, or a $S^2$ surface, similar to the previous discussion in Eq.(4.108) and Eq.(4.109). We can see that if $|q|$ is big enough, the $S^2$ singularities will be turned on in the $R^4$ space. Unlike singularities which can be gauged away, it seems that these singularities can not be gauged away.

Finally the real parts and imaginary parts of the gauge field and the field strength of $SL(2,C)$ 1-instanton solution with 10 moduli parameters can be calculated to be

$$G'_\mu = -\frac{1}{2}[e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu] \left( \frac{[x - (p + qi)]]}{(x - (p + qi))^2 + \lambda^2} \right)$$

$$= -\frac{1}{2}[e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu] \left( \frac{\{|x - p|^2 - q^2 + c(x - p)_\nu - |d - 2(x - p) \cdot q| q_\nu\}}{[x - p]^2 - q^2 + c^2 + |d - 2(x - p) \cdot q|^2} \right)$$

$$- \frac{1}{2}[e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu] \left( \frac{-\{|x - p|^2 - q^2 + c|q_\nu - |d - 2(x - p) \cdot q|(x - p)_\nu\}}{[x - p]^2 - q^2 + c^2 + |d - 2(x - p) \cdot q|^2} \right)$$

and

$$F'_{\mu\nu} = [e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu] \left( \frac{(c + di)}{[x - (p + qi)]^2 + (c + di)^2} \right)$$

$$= \frac{[e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu] \{c(|x - p|^2 - q^2 + c^2 - c[d - 2(x - p) \cdot q]^2)}{[x - p]^2 - q^2 + c^2 + |d - 2(x - p) \cdot q|^2}$$

$$+ 2d([x - p]^2 - q^2 + c)[d - 2(x - p) \cdot q]\}$$

$$\{[x - p]^2 - q^2 + c^2 + |d - 2(x - p) \cdot q|^2\}$$

$$= \frac{[e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu] \{[d(|x - p|^2 - q^2 + c^2 - d[d - 2(x - p) \cdot q]^2)}{[x - p]^2 - q^2 + c^2 + |d - 2(x - p) \cdot q|^2}$$

$$+ 2c([x - p]^2 - q^2 + c)[d - 2(x - p) \cdot q]\}$$

$$\{[x - p]^2 - q^2 + c^2 + |d - 2(x - p) \cdot q|^2\}$$

which is a self-dual field configuration by Eq.(3.67). If we take $q = 0$, $c = -\frac{\alpha^2}{2} = -d$, we can easily get the following special solutions

$$G'_\mu = -\frac{1}{2}[e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu] \left( \frac{[2|x - p|^2 + \alpha^2_1 + i\alpha^2_2](x - p)_\nu}{(|x - p|^2)^2 + 2|x - p|^2\alpha^2_1 + \alpha^4_1 + |x - p|^4} \right)$$

and

$$F'_{\mu\nu} = \frac{\alpha^2_1(e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu)[2|x - p|^4 + 4|x - p|^2\alpha^2_1 + \alpha^4_1]}{[2|x - p|^4 + 2|x - p|^2\alpha^2_1 + \alpha^4_1]^2}$$

$$- \frac{\alpha^2_1(e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu)[2|x - p|^4 - \alpha^4_1]}{[2|x - p|^4 + 2|x - p|^2\alpha^2_1 + \alpha^4_1]^2}$$

$$- \frac{i\alpha^2_1(e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu)[2|x - p|^4 - \alpha^4_1]}{[2|x - p|^4 + 2|x - p|^2\alpha^2_1 + \alpha^4_1]^2},$$

(4.128)
which can be written as

\[ A'_\mu = -\frac{1}{4} [e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger] y_\nu \frac{2(2y^2 + \alpha_1^2)}{y^4 + (y^2 + \alpha_1^2)^2}, \]

\[ B'_\mu = -\frac{1}{4} [e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger] y_\nu \frac{2\alpha_1^2}{y^4 + (y^2 + \alpha_1^2)^2}, \]

\hspace{1cm} (4.129)

and

\[ H'_{\mu\nu} = (e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger) \alpha_1^2 \frac{2y^4 + 4\alpha_1^2 y^2 + \alpha_1^4}{y^4 + (y^2 + \alpha_1^2)^2}, \]

\[ M'_{\mu\nu} = (e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger) \alpha_1^2 \frac{2y^4 - \alpha_1^4}{y^4 + (y^2 + \alpha_1^2)^2}, \]

\hspace{1cm} (4.130)

where \( y = x - p, \ y^2 = |x - p|^2 \). The forms of profiles in Eqs.(4.129) and (4.130) are exactly the same with the \((M, N) = (1, 0)\) instanton solution [23] obtained in Eqs.(2.28) and (2.29).

**D. The Jumping lines of \( SL(2, C) \) \( k \)-Instantons**

For the subset of \( SL(2, C) \) \( k \)-instantons in Eq.(4.118), the connections are calculable and one encounters much more complicated singular structures of the field configurations in Eq.(4.120). These new singularities can not be gauged away and do not show up in the field configurations of \( SU(2) \) \( k \)-instantons. Mathematically, the existence of singular structures of the non-compact \( SL(2, C) \) SDYM field configurations is consistent with the inclusion of "sheaves" by Frenkel-Jardim [22] recently, rather than just the restricted notion of "vector bundles", in the one to one correspondence between ASDYM and certain algebraic geometric objects.

In fact, one notices that Eq.(4.119) can be written as

\[ \det \Delta(x)^\dagger \Delta(x) = |x - y_1|^2 |x - y_2|^2 \cdots |x - y_k|^2 \phi = P_{2k}(x) + iP_{2k-1}(x) = 0 \]  

\hspace{1cm} (4.131)

where

\[ \Delta(x) = \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ x - y_1 & 0 & \ldots & 0 \\ 0 & x - y_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x - y_k \end{bmatrix}. \]  

\hspace{1cm} (4.132)
In Eq. (4.132) $\lambda_i = \lambda_i e_0$ (with $\lambda_i$ a complex number) and $y_i$ is a biquaternion in contrast to those in Eq. (4.78) where $y_i$ was chosen to be a quaternion. The solutions $x \in J$ of

$$\det \Delta(x)^\circ \Delta(x) = 0$$

(4.133)

mentioned in Eq. (3.48) are called “jumping lines” of $k$-instantons in the mathematical literatures [22].

The complete jumping lines of the $SL(2, C)$ 1-instanton are given in Eq. (4.115). The jumping lines for the $SL(2, C)$ $k$-instantons given in Eq. (4.131) with $k \geq 2$ are just partial subsets of the jumping lines. For the complete jumping lines of 2-instantons, for example, $\lambda_1, \lambda_2$ and $y_1, y_2$ are biquaternions and one needs to turn on $y_{12}$ and $y_{21}(= y_{12})$ in Eq. (4.91) which are also biquaternions. We can calculate the determinant of Eq. (4.93) in section IV. B to get

$$\det \Delta_{2-ins}(x)^\circ \Delta_{2-ins}(x) = |x - y_1|_c^2|y_1 - y_2|_c + |\lambda_2|_c^2|x - y_1|_c^2 + |\lambda_1|_c^2|x - y_2|_c^2$$

$$+ y_{12}^\circ (x - y_1)y_{12}^\circ (x - y_2) + (x - y_2)^\circ y_{12}^\circ (x - y_1)^\circ y_{12}$$

$$- y_{12}^\circ (x - y_1)\lambda_1^\circ \lambda_2 - \lambda_2^\circ \lambda_1(x - y_1)^\circ y_{12}$$

$$- (x - y_2)^\circ y_{12}^\circ \lambda_1^\circ \lambda_2 - \lambda_2^\circ \lambda_1 y_{12}^\circ (x - y_2)$$

$$+ |y_{12}|_c^2(|\lambda_2|_c^2 + |\lambda_1|_c^2) + |y_{12}|_c^4$$

$$= 0$$

(4.134)

where $y_{12}$ is given by Eq. (4.95). In calculating the determinant, one notices that $\Delta(x)^\circ \Delta(x)$ in Eq. (4.93) is a symmetric matrix with complex number entries. So there is no ambiguity in the determinant calculation. The result of the determinant in Eq. (4.134) corresponds to a polynomial of degree $2k = 4$ in the variable $x$, which is consistent with results obtained in the mathematical literature [22]. Here we have given the singularity locus or jumping lines in a much more detailed form accessible for most physicists. For the special case of $y_{12} = 0$, $\lambda_i$ reduces to a complex number $\lambda_i = \lambda_i e_0$ (with $\lambda_i$ a complex number) and Eq. (4.134) reduces to Eq. (4.131). It is interesting to see that although the complete $SL(2, C)$ 2-instanton connections with 26 parameters are not available, their jumping lines can be identified by Eq. (4.134).

Similar results can be obtained for the jumping lines of the complete 3-instantons by
using Eq.\eqref{4.96} although the calculation is more involved

\[
\det \Delta_{3-\text{ins}}(x) \Delta_{3-\text{ins}}(x) \\
= [ |\lambda_1|^2_c + |x - y_1|^2_c + |y_{12}|^2_c + |y_{13}|^2_c ] \\
\times [ |\lambda_2|^2_c + |x - y_2|^2_c + |y_{12}|^2_c + |y_{23}|^2_c ] \\
\times [ |\lambda_3|^2_c + |x - y_3|^2_c + |y_{13}|^2_c + |y_{23}|^2_c ] \\
+ [ \lambda_1 \lambda_2 - (x - y_1) y_{12} - y_{13} (x - y_2) + y_{13} y_{23} ] \\
\times [ \lambda_2 \lambda_3 + y_{12} y_{13} - (x - y_2) y_{23} - y_{23} (x - y_3) ] \\
\times [ \lambda_3 \lambda_1 - y_{13} (x - y_1) + y_{23} y_{12} - (x - y_3) y_{13} ] \\
+ [ \lambda_1 \lambda_3 - (x - y_1) y_{13} + y_{12} y_{23} - y_{13} (x - y_3) ] \\
\times [ \lambda_3 \lambda_2 + y_{13} y_{12} - y_{23} (x - y_2) - (x - y_3) y_{23} ] \\
\times [ \lambda_2 \lambda_1 - y_{12} (x - y_1) - (x - y_2) y_{12} + y_{23} y_{13} ] \\
- [ |\lambda_1|^2_c + |x - y_1|^2_c + |y_{12}|^2_c + |y_{13}|^2_c ] \\
\times [ \lambda_2 \lambda_3 + y_{12} y_{13} - (x - y_2) y_{23} - y_{23} (x - y_3) ] \\
\times [ \lambda_3 \lambda_1 - y_{13} (x - y_1) + y_{23} y_{12} - (x - y_3) y_{13} ] \\
- [ \lambda_1 \lambda_2 - (x - y_1) y_{12} - y_{13} (x - y_2) + y_{13} y_{23} ] \\
\times [ \lambda_2 \lambda_1 - y_{12} (x - y_1) - (x - y_2) y_{12} + y_{23} y_{13} ] \\
\times [ |\lambda_3|^2_c + |x - y_3|^2_c + |y_{13}|^2_c + |y_{23}|^2_c ] \\
- [ \lambda_1 \lambda_3 - (x - y_1) y_{13} + y_{12} y_{23} - y_{13} (x - y_3) ] \\
\times [ |\lambda_2|^2_c + |x - y_2|^2_c + |y_{12}|^2_c + |y_{23}|^2_c ] \\
\times [ \lambda_3 \lambda_1 - y_{13} (x - y_1) + y_{23} y_{12} - (x - y_3) y_{13} ]
= 0.
\tag{4.135}
\]

The result of the determinant in Eq.\eqref{4.135} corresponds to a polynomial of degree $2k = 6$ in the variable $x$. The number of parameters in Eq.\eqref{4.135} can be reduced to 42 by making the choices of parameters in the paragraph after Eq.\eqref{4.96} and using Eq.\eqref{4.97} to Eq.\eqref{4.101}. For the special case of $y_{ij} = 0 \ (i \neq j)$, $\lambda_i$ reduces to a complex number $\lambda_i = \lambda_i e_0$ (with $\lambda_i$ a complex number) and Eq.\eqref{4.135} reduces to Eq.\eqref{4.131}.

For $x \in J$ in the jumping lines of $SL(2, C) \ k$-instantons, the non-singular property in the quadratic condition of ADHM construction in Eq.\eqref{3.48} is violated. Moreover, the projection operator $P$ in Eq.\eqref{3.60} and the self-dual field strength $F_{\mu\nu}$ in Eq.\eqref{3.60} diverge
there. For some cases of these singularities, the holomorphic vector bundle description of the original ADHM construction for $SU(2)$ instantons fails and one is led to include sheaf in the construction of the non-compact $SL(2, C) k$-instantons. It can be shown mathematically that [22] there are solutions of complex $SL(2, C)$ ADHM equations [26] which admit jumping lines. In this paper we have given explicit concrete examples to describe jumping lines of $SL(2, C) k$-instantons from physicist point of view. More examples of $SL(2, C)$ instanton connections corresponding to sheaves with jumping lines instead of vector bundles on $CP^3$ are under investigation [27].

V. CONCLUSION

In this paper, we have extended the ADHM construction of $Sp(1)$ self-dual Yang-Mills (SDYM) instantons to the case of $SL(2, C)$ instanton solutions. In contrast to the quaternion calculation heavily used in the compact groups $Sp(N)$, $SU(N)$ and $O(N)$ constructions in the literature [5], we discover that instead the use of biquaternion with the biconjugation operation [25] is very powerful in the construction of the non-compact $SL(2, C)$ instanton solutions. The new $SL(2, C)$ instanton solutions constructed by this $SL(2, C)$ ADHM method contain those $SL(2, C) (M, N)$ instanton solutions [23] obtained previously as a subset. We found that the number of moduli for $SL(2, C) k$-instantons is twice of that of $Sp(1), 16k - 6$.

In addition, we investigate the structure of singularities or jumping lines [22] of the complete $SL(2, C)$ 1-instanton solution with 10 moduli parameters. The singularities are intersections of zeros of $P_2(x)$ and $P_1(x)$ polynomials of 4 variables with degrees 2 and 1 respectively. For singularities of subsets of $k$-instanton field configurations, one encounters intersections of zeros of $P_{2k}(x)$ and $P_{2k-1}(x)$ polynomials with degrees $2k$ and $2k - 1$ respectively. We found that not all singularities can be gauged away as in the case of $SU(2)$ 1-instanton field configuration. The singularities for $SL(2, C)$ 1-instanton field configuration is much more complicated than the removable singularity of $SU(2)$ 1-instanton field configuration. Moreover, the values of all 10 parameters are closely related to the structure of the singularities.

The jumping lines of the complete $SL(2, C) k = 2, 3$ instantons with 26, 42 moduli parameters are also calculated in this paper. Mathematically, the existence of singular structures of the non-compact $SL(2, C) k$-instanton field configurations discovered in this paper is
consistent with the inclusion of "sheaves" by Frenkel-Jardim [22], rather than just the restricted notion of "vector bundles", in the one to one correspondence between ASDYM and certain algebraic geometric objects. In this paper we have given explicit concrete examples to describe "jumping lines" of $SL(2, C) k$-instantons from physicist point of view.

The existence of non-removable singular structures of $SL(2, C)$ instanton field configurations may help to clarify the long standing issue of global singularity problems associated with Backlund transformations [18, 19] of $SU(2)$ SYDM instantons. Further investigation of the structure of singularities for general $SL(2, C) k$-instanton field configurations maybe important for the understanding of the geometrical structures of non-compact SDYM theory.

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