Antenna Factorization in Strongly-Ordered Limits

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Abstract

When energies or angles of gluons emitted in a gauge-theory process are small and strongly ordered, the emission factorizes in a simple way to all orders in perturbation theory. I show how to unify the various strongly-ordered soft, mixed soft-collinear, and collinear limits using antenna factorization amplitudes, which are generalizations of the Catani–Seymour dipole factorization function.

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1. Introduction

The properties of gauge-theory amplitudes in degenerate limits play an important role in the formalism of perturbative QCD. In addition to the usual ultraviolet singularities present when amplitudes are expressed in terms of a bare coupling, intermediate quantities also contain infrared divergences. All these divergences may be regulated by a dimensional regulator $\epsilon = (4 - D)/2$. Infrared divergences arise both from loop integrations and from the integration over singular regions of phase space. Both have a universal structure, which allows the ultimate cancellation of these divergences to be treated in a universal manner as well $[1,2,3,4]$. The universality of real-emission infrared divergences emerges from the universality of gauge-theory amplitudes in soft and collinear limits of external momenta. This universality is reflected in the factorization $[5,6]$ of amplitudes into ‘hard’ parts, independent of the soft or collinear emission, and emission amplitudes for soft or collinear partons, whose singular behavior is independent of the details of the ‘hard’ process.

Traditionally, the soft and collinear limits were treated independently; but in computing phase-space integrals, this requires handling the boundary in between them. It is nicer, and perhaps indispensable beyond next-to-leading order, to combine these two limits. This was done by Catani and Seymour $[4]$ for single emission at the level of the amplitude squared. It can also be carried out at the amplitude level, through the definition of antenna factorization amplitudes $[7]$. The latter construct generalizes nicely to the emission of multiple soft or collinear radiation $[8]$.

When examining the emission of two soft gluons in a process, there are two distinct regimes we can consider: that in which the gluon energies are comparable, and that in which they are strongly ordered, $E_1 \ll E_2$. The latter generalizes to the emission of $n$ soft gluons, $E_1 \ll E_2 \ll \cdots \ll E_n$. This strongly-ordered limit is of interest because the leading singularities arise there — the leading powers of $\epsilon^{-1}$ in intermediate quantities as well as the leading large logarithms in ultimate physical quantities.

In QED, soft photon emission not only factorizes, but factorizes independently of other photons. Multiple soft photon emission thus follows straightforwardly from single photon emission, and there is no substantive distinction between unordered and strongly-ordered emission. In non-Abelian gauge theories, the situation is more complicated. Berends and Giele presented $[9]$ a general form for strongly-ordered multiple soft emission. The purpose of the present paper is to generalize their discussion to include strongly-ordered mixed soft-collinear and multiply-collinear emission as well.

The properties of non-Abelian gauge-theory amplitudes in singular limits are easiest to understand in the context of a color decomposition $[10]$. In the present paper, I will concentrate on
all-gluon amplitudes, though the formalism readily extends to amplitudes with quarks and (colored) scalars as well. For tree-level all-gluon amplitudes in an $SU(N)$ gauge theory the color decomposition has the form,

$$A_{\text{tree}}^n (\{k_i, \lambda_i, a_i\}) = \sum_{\sigma \in S_n / Z_n} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_{\text{tree}}^n (\sigma(1^{\lambda_1}, \ldots, n^{\lambda_n})),$$

(1.1)

where $S_n / Z_n$ is the group of non-cyclic permutations on $n$ symbols, and $j^{\lambda_j}$ denotes the $j$-th momentum and helicity $\lambda_j$. The notation $j_1 + j_2$ appearing below will denote the sum of momenta, $k_{j_1} + k_{j_2}$. I use the normalization $\text{Tr}(T^a T^b) = \delta^{ab}$. Analogous formulæ hold for amplitudes with quark-antiquark pairs or uncolored external lines. The color-ordered or partial amplitude $A_n$ is gauge invariant, and has simple factorization properties in both the soft and collinear limits,

$$A_{\text{tree}}^n (\ldots, a, s^{\lambda_s}, b, \ldots) \xrightarrow{k_s \to 0} \text{Soft}_{\text{tree}}^n (a, s^{\lambda_s}, b) A_{\text{tree}}^{n-1} (\ldots, a, b, \ldots),$$

$$A_{\text{tree}}^n (\ldots, a^{\lambda_a}, b^{\lambda_b}, \ldots) \xrightarrow{a \parallel b} \sum_{\lambda = \pm} C_{\lambda}^{\text{tree}} (a^{\lambda_a}, b^{\lambda_b}, z) A_{\text{tree}}^{n-1} (\ldots, (a + b)^{\lambda}, \ldots).$$

(1.2)

The collinear splitting amplitude $C_{\text{tree}}^{\lambda}$, squared and summed over helicities, gives the usual unpolarized Altarelli–Parisi splitting function [11]. It depends on the collinear momentum fraction $z$ (here made explicit) in addition to invariants built out of the collinear momenta. While the complete amplitude also factorizes in the collinear limit, the same is not true of the soft limit; the eikonal factors Soft$_{\text{tree}}$ get tangled up with the color structure via the sum (1.1). It is for this reason that the color decomposition is useful.

I review factorization in strongly-ordered soft limits in section 2, and consider a subtlety with strongly-ordered collinear limits in section 3. The soft and collinear limits can be unified using the antenna factorization amplitude, which I review in section 4. The form of the resulting factorization is similar to that for the soft factorization in eqn. (1.2). I also review the tree-level multiple-emission antenna amplitude, and show in section 5 how it can be simplified in strongly-ordered limits. These simplifications generalize readily beyond tree amplitudes, as discussed in section 6.

2. Multiple Soft Emission

Color-ordered amplitudes have simple factorization properties in limits where several legs become soft simultaneously. These are trivial generalizations of eqn. (1.2) if the soft legs are not color-connected, that is are not neighboring arguments to the amplitude,

$$A_{\text{tree}}^n (\ldots, a_1^{\lambda_1}, s_1^{\lambda_1}, b_1, \ldots, a_2^{\lambda_2}, s_2^{\lambda_2}, b_2, \ldots) \xrightarrow{k_{s_1}, k_{s_2} \to 0} \text{Soft}_{\text{tree}}^n (a_1, s_1^{\lambda_1}, b_1) \text{Soft}_{\text{tree}}^n (a_2, s_2^{\lambda_2}, b_2) A_{\text{tree}}^{n-2} (\ldots, a_1, b_1, \ldots, a_2, b_2, \ldots).$$

(2.1)
When the two soft legs are color-connected, in general there is no such simple decomposition of the soft factor itself, and we have [9]

$$A_n^{\text{tree}}(\ldots, a, s_1^{\lambda_1}, s_2^{\lambda_2}, b, \ldots) \xrightarrow{k_{s_1}, k_{s_2} \to 0} \text{Soft}^{\text{tree}}(a, s_1^{\lambda_1}, s_2^{\lambda_2}, b) A_{n-2}^{\text{tree}}(\ldots, a, b, \ldots).$$

(2.2)

However, in the limit where one of the gluons is much softer than the other — say $k_{s_1} \ll k_{s_2}$, the soft factor itself factorizes,

$$\text{Soft}^{\text{tree}}(a, s_1^{\lambda_1}, s_2^{\lambda_2}, b) \xrightarrow{k_{s_1} \ll k_{s_2}} \text{Soft}^{\text{tree}}(a, s_1^{\lambda_1}, s_2^{\lambda_2}) \text{Soft}^{\text{tree}}(a, s_2^{\lambda_2}, b).$$

(2.3)

The soft factors are nested or iterated, a feature of the factorization that will have an echo in the strongly-ordered antenna factorization to be discussed in section 5.

Similar results hold for multiple-gluon emission; the amplitude factorizes,

$$A_n^{\text{tree}}(\ldots, a, s_1^{\lambda_1}, \ldots, s_m^{\lambda_m}, b, \ldots) \xrightarrow{k_{s_1}, \ldots, k_{s_m} \to 0} \text{Soft}^{\text{tree}}(a, s_1^{\lambda_1}, \ldots, s_m^{\lambda_m}, b) A_{n-m}^{\text{tree}}(\ldots, a, b, \ldots),$$

(2.4)

and in the strongly-ordered domain, the soft factor itself factorizes,

$$\text{Soft}^{\text{tree}}(a, s_1^{\lambda_1}, s_2^{\lambda_2}, \ldots, s_m^{\lambda_m}, b) \xrightarrow{k_{s_1} \ll \ldots \ll k_{s_m}} \text{Soft}^{\text{tree}}(a, s_1^{\lambda_1}, s_2^{\lambda_2}) \text{Soft}^{\text{tree}}(a, s_2^{\lambda_2}, s_3) \cdots \text{Soft}^{\text{tree}}(a, s_m^{\lambda_m}, b) + \text{subleading.}$$

(2.5)

Because the soft factors are independent of the helicities of the hard legs (they depend only on the helicities of the soft gluons themselves), these strong-ordering simplifications square in a simple way even after summing over final helicities and averaging over initial ones,

$$\langle \text{Soft}^{\text{tree}}(a, s_1^{\lambda_1}, s_2^{\lambda_2}, \ldots, s_m^{\lambda_m}, b)^2 \rangle \xrightarrow{k_{s_1} \ll \ldots \ll k_{s_m}} \langle \text{Soft}^{\text{tree}}(a, s_1^{\lambda_1}, s_2^{\lambda_2})^2 \rangle \langle \text{Soft}^{\text{tree}}(a, s_2^{\lambda_2}, s_3)^2 \rangle \cdots \langle \text{Soft}^{\text{tree}}(a, s_m^{\lambda_m}, b)^2 \rangle + \text{subleading},$$

(2.6)

where $\langle \rangle$ denotes helicity summation and averaging. This simplification carries over directly to the structure of the leading-color term in the squared matrix element, which contains no interference terms between different permutations of arguments to the color-ordered amplitudes.

### 3. Iterated Collinear Limits and Azimuthal Averaging

In limits where several neighboring legs become collinear, the color-ordered amplitude again factorizes,

$$A_n^{\text{tree}}(\ldots, a_1^{\lambda_1}, b^{\lambda_2}, \ldots) \xrightarrow{a_1 \parallel a_2 \parallel \cdots \parallel a_m} \sum_{\lambda=\pm} C^{\text{tree}}_{a_1^{\lambda_1}, a_2^{\lambda_2}, \ldots, a_m^{\lambda_m} \parallel \{z_i\}} A_{n-m}^{\text{tree}}(\ldots, (a_1 + \cdots + a_m)^\lambda, \ldots).$$

(3.1)
The $z_i$ denote the momentum fractions of the collinear legs, $k_{a_i} = z_i(k_{a_1} + \cdots + k_{a_m})$. In the strongly-ordered limit, where $a_1$ is more nearly collinear with $a_2$ than either with $a_3$, and so on (that is, where $s_{a_1 a_2} \ll s_{a_2 a_3}, s_{a_1 a_2 a_3}$), the splitting amplitude also simplifies. Here, unlike the soft case, there is a sum over intermediate helicities,

$$C^{\text{tree}}_{-\lambda}(a_1^{\lambda_1}, a_2^{\lambda_2}, \ldots, a_m^{\lambda_m}; \{z_i\}) s_{a_1 a_2} \ll s_{a_2 a_3} \ll \cdots \ll s_{a_1 \cdots a_m}$$

$$\sum_{\lambda_{12}, \lambda_{123}, \ldots} C^{\text{tree}}_{-\lambda_{12}}(a_1^{\lambda_1}, a_2^{\lambda_2}; \frac{z_1}{z_1 + z_2}) C^{\text{tree}}_{-\lambda_{123}}((a_1 + a_2)^{\lambda_{12}}, a_3^{\lambda_3}; \frac{z_1 + z_2}{z_1 + z_2 + z_3}) \cdots$$

$$\times C^{\text{tree}}_{-\lambda_{123\ldots(n-1)}}((a_1 + \cdots + a_{m-1})^{\lambda_{12\ldots(n-1)}}, a_m^{\lambda_m}; 1 - z_m) + \text{subleading.} \quad (3.2)$$

As an example, the triply-collinear splitting amplitude \([12,8], C^{\text{tree}}_+(1^-, 2^+, 3^-; z_1, z_2)\), simplifies to

$$C^{\text{tree}}_+(R^+, 3^-; z_R) C^{\text{tree}}_-(1^-, 2^+; \frac{z_1}{z_1 + z_2}) + C^{\text{tree}}_+(R^-, 3^-; z_R) C^{\text{tree}}_+(1^-, 2^+; \frac{z_1}{z_1 + z_2}) + \cdots \quad (3.3)$$

in the strongly-ordered limit.

Because of the summation over intermediate helicities, however, in general the splitting function (the helicity-summed and -averaged square of the splitting amplitude) will not factorize simply. This can be seen, for example, by examining the strongly-ordered limit of the triply-collinear splitting function \([13,14]\),

$$2 \left[ \frac{(s_{123} - (1 - z_3)s_{23})^2}{s_{12}^2 s_{23}^2 (1 - z_3)^2} + \frac{2s_{23}}{s_{12} s_{23}^2 (1 - z_3)^2} + \frac{3}{2s_{12}^2} \right.$$

$$+ \frac{1}{s_{12} s_{23}} \left( \frac{(1 - z_3)(1 - z_3)}{z_3 z_1 (1 - z_1)} - \frac{2(z_2^2 + z_2 z_3 + z_3^2)}{1 - z_3} + \frac{(z_2 z_1 - z_2^2 z_3 - 2)}{z_3 (1 - z_3)} \right)$$

$$+ \frac{1}{2s_{12} s_{23}} \left( 3z_2^2 - \frac{2(2 - z_1 + z_1^2)(z_2^2 + z_1 (1 - z_1))}{z_3 (1 - z_3)} + \frac{1}{z_1 z_3} + \frac{1}{(1 - z_1)(1 - z_3)} \right) \right] \quad (3.4)$$

We must be careful in taking this limit ($s_{12} \ll s_{23}, s_{123}$). Although $s_{23} \to z_2/(1 - z_3)s_{123}$, so that the numerator of the first term vanishes, the denominator contains a double pole in $s_{12}$, and so the relative rate at which this limit is approached, compared to the vanishing of $s_{12}$, becomes important.

I will make use of the generalized Gram determinant $G$,

$$G\begin{pmatrix} p_1, \ldots, p_n \\ q_1, \ldots, q_n \end{pmatrix} = \det(2p_i \cdot q_j), \quad (3.5)$$

which vanishes whenever two $p_i$ or two $q_i$ become collinear (or when any momentum becomes soft). It will also be convenient to define

$$\Delta(p_1, \ldots, p_n) \equiv G\begin{pmatrix} p_1, \ldots, p_n \\ p_1, \ldots, p_n \end{pmatrix}, \quad (3.6)$$
The normalization in these definitions is non-standard.

Using this generalized Gram determinant, we can rewrite the first term in (3.4) as follows,

\[
\frac{(z_2 s_{123} - (1 - z_3) s_{23})^2}{s_{12}^2 s_{123}^2 (1 - z_3)^2} = \frac{(q \cdot k_2 s_{123} - q \cdot (k_1 + k_2) s_{23})^2}{s_{12}^2 s_{123}^2 [q \cdot (k_1 + k_2)]^2}
\]

\[= \frac{(G \left( \frac{1,2}{q,3} \right) - q \cdot k_2 s_{12})^2}{s_{12}^2 s_{123}^2 [q \cdot (k_1 + k_2)]^2}, \quad (3.7)\]

where \( q \) is a reference momentum — a massless momentum not collinear to the \( k_i \).

Using spinor products, we can see that

\[G \left( \frac{1,2}{q,3} \right) = s_{1q} s_{23} - s_{2q} s_{13} = - \langle 1 2 \rangle \langle 3 q \rangle [2 3] [1 q] - [1 2] [3 q] \langle 2 3 \rangle \langle 1 q \rangle = s_{12} s_{3q} \sim \sqrt{s_{12}} \quad (3.8)\]

as \( s_{12} \to 0 \).

In the limit, eqn. (3.7) thus becomes,

\[\frac{G^2 \left( \frac{1,2}{q,3} \right)}{s_{12}^2 s_{123}^2 [q \cdot (k_1 + k_2)]^2} + O(s_{12}^{-1/2}); \quad (3.9)\]

the subleading term leads to finite integrals over singular regions and is thus not ultimately important to extracting poles in the dimensional regulator \( \epsilon \).

The limit of the remaining terms in eqn. (3.4) is straightforward, and we obtain

\[
\frac{G^2 \left( \frac{1,2}{q,3} \right)}{s_{12}^2 s_{123}^2 [q \cdot (k_1 + k_2)]^2} + \frac{4(1 - z_1 + z_1^2 - z_3 + z_1 z_3 + z_3^2) \left[(1 - z_3 + z_3^3)^2 + (z_1^2 + z_1 z_3 - z_3)(1 - z_3 + z_3^3) - z_1(1 - z_1 z_3)\right]}{s_{12}s_{123}z_1(1 - z_1 - z_3)z_3(1 - z_1 - z_3)} \quad (3.10)
\]

for the triply-collinear splitting function. This is not equal to the product of nested two-particle splitting functions,

\[\frac{4(1 - z_3 + z_3^3)^2(1 - z_1 + z_1^2 - 2z_3 + z_1 z_3 + z_3^2)^2}{s_{12}s_{123}z_1(1 - z_1 - z_3)z_3(1 - z_3)^3}. \quad (3.11)\]

We may observe, however, that the Gram determinant depends on the azimuthal angle around the \( k_1 + k_2 \) axis. Nothing in any hard cross-section will depend on this angle in the collinear limit. We will eventually need to integrate over it anyway; if we average over it, we can see that the Gram determinant averages to zero (incidentally reducing the subleading term in eqn. (3.9) to \( O(s_{12}^0) \)), while its square averages to

\[2s_{12} s_{23} s_{3q} s_{1q} + s_{12}^2 s_{3q}^2. \quad (3.12)\]
Plugging in this expression, eqn. (3.10) becomes

$$
\begin{align*}
\frac{4(1 - z_1 - z_2) z_1 z_3}{s_{12} s_{123} (1 - z_3)^3} + \frac{4(1 - z_1 + z_1^2 - z_3 + z_1 z_3 + z_3^2)}{s_{12} s_{123} z_1 (1 - z_3) z_3 (1 - z_1 - z_3)} \\
\times [(1 - z_3 + z_3^2)^2 + (z_1^2 + z_1 z_3 - z_3)(1 - z_3 + z_3^2) - z_1 (1 - z_1 z_3)] + O(s_{12}^0) \\
= \frac{4(1 - z_3 + z_3^2)(1 - z_1 + z_1^2 - 2 z_3 + z_1 z_3 + z_3^2)^2}{s_{12} s_{123} z_1 (1 - z_1 - z_3) z_3 (1 - z_3)^3},
\end{align*}
$$

(3.13)

as desired. Accordingly, if we redefine the averaging operation $\langle \rangle$ to include not only helicity summation and averaging but also averaging over azimuthal angles of collinear pairs, then we obtain

\[
\langle | C(z_1^\text{tree} (\lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_m}; \{ z_i \}))^2 \rangle \underset{s_{a_1 a_2 \cdots a_m}}{\longrightarrow} \langle | C_{z_{123} (\lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_m}; \{ z_i \}))^2 \rangle \langle | C_{z_{123} (\lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_m}; \{ z_i \}))^2 \rangle \cdots \\
\times \langle | C_{z_{123} (\lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_m}; \{ z_i \}))^2 \rangle + \text{subleading}
\]

(3.14)

for the squared splitting function in the strongly ordered limit.

4. Antenna Factorization

We can unify the soft and collinear limits by associating a single function [7] with each color-connected triplet of momenta. In the singular limit, the triplet reduces to a pair of massless momenta. We can remap the three momenta to two massless momenta even away from the singular limit using a pair of reconstruction functions. For a triplet of momenta $(k_a, k_1, k_b)$, with $k_{a,b}$ remaining hard in any singular limit under consideration, the remappings to a massless pair $k_{\hat{a},\hat{b}}$ are,

\[
k_{\hat{a}} = f_{\hat{a}}(a, 1, b) \equiv -\frac{1}{2(K^2 - s_{1b})} [(1 + \rho)K^2 - 2s_{1b} r_1] \ k_a - r_1 \ k_1 \\
= -\frac{1}{2(K^2 - s_{a1})} [(1 - \rho)K^2 - 2s_{a1} r_1] \ k_b,
\]

\[
k_{\hat{b}} = f_{\hat{b}}(a, 1, b) \equiv -\frac{1}{2(K^2 - s_{1b})} [(1 - \rho)K^2 - 2s_{1b} (1 - r_1)] \ k_a - (1 - r_1) \ k_1 \\
= -\frac{1}{2(K^2 - s_{a1})} [(1 + \rho)K^2 - 2s_{a1} (1 - r_1)] \ k_b,
\]

where $K = k_a + k_1 + k_b$, $r_1 = s_{1b}/(s_{a1} + s_{1b})$, and

\[
\rho = \sqrt{1 + \frac{4r_1(1 - r_1)s_{1a} s_{1b}}{K^2 s_{ab}}.}
\]

(4.2)
Figure 1. The antenna amplitude expressed in terms of Berends–Giele currents.

Figure 2. The factorization of a tree amplitude into an antenna amplitude and a hard amplitude. The shaded circles represent sums over all tree diagrams.

(Other choices for $r_1$ are possible, subject to certain constraints [8].)

These reconstruction functions reduce to the usual combinations in the various soft and collinear limits,

$$
k_{\hat{a}} = -(k_a + k_1), k_{\hat{b}} = -k_b, \quad \text{when } k_a \parallel k_1, \text{ i.e. } s_{a1} = 0, s_{1b} \neq 0;
$$

$$
k_{\hat{a}} = -k_a, k_{\hat{b}} = -(k_1 + k_b), \quad \text{when } k_1 \parallel k_b, \text{ i.e. } s_{a1} \neq 0, s_{1b} = 0; \quad (4.3)
$$

$$
k_{\hat{a}} = -k_a, k_{\hat{b}} = -k_b, \quad \text{when } k_1 \text{ is soft, i.e. } s_{a1} = 0 = s_{1b}.
$$

Beyond producing a pair of massless momenta satisfying momentum conservation, $K = -k_{\hat{a}} - k_{\hat{b}}$, these reconstruction functions also ensure that the soft and collinear limits can be unified into a single factorizing amplitude,

$$
A_n(\ldots, a^{\lambda_a}, 1^{\lambda_1}, b^{\lambda_b}, \ldots) \xrightarrow{k_1 \text{ singular}} \sum_{\text{ph. pol. } \lambda_{\hat{a}}, \lambda_{\hat{b}}} \sum_{\lambda_\hat{a}, \lambda_\hat{b}} \text{Ant}(\hat{a}^{\lambda_{\hat{a}}}, \hat{b}^{\lambda_{\hat{b}}} \leftarrow a^{\lambda_a}, 1^{\lambda_1}, b^{\lambda_b}) A_{n-1}(\ldots, -k_{\hat{a}}^{\lambda_{\hat{a}}}, -k_{\hat{b}}^{\lambda_{\hat{b}}}, \ldots). \quad (4.4)
$$

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This factorization is depicted in fig. 3. The antenna amplitude has an explicit expression in terms of the Berends–Giele current $J$ \cite{15,16,17},

$$\text{Ant}(\hat{a}, \hat{b} \leftarrow a, 1, b) = J(a, 1; \hat{a})J(b; \hat{b}) + J(a; \hat{a})J(1, b; \hat{b}).$$ \hspace{1cm} (4.5)$$

The complete list of its helicity amplitudes was given in ref. \cite{8}. It is depicted diagrammatically in fig. 1.

We can use the generalized Gram determinant (3.6) to define the singular limit,

$$k_1 \text{ singular } \iff L(a, 1, b) \equiv \frac{1}{s_{ab}^3} G\left(\frac{a, 1, b}{a, 1, b}\right) \to 0. \hspace{1cm} (4.6)$$

I will denote this singular limit by $|k_1| \to 0$. This gives a Lorentz-invariant definition of ‘soft’ and ‘collinear’; in the limit $k_a \parallel k_1$, $k_b$ is effectively the reference momentum defining the transverse direction, and $k_a$ plays that role in the $k_b \parallel k_1$ limit.

The reconstruction functions generalize to the emission of $n$ singular particles, mapping $n + 2$ momenta to two massless momenta. (Explicit forms of these functions suitable for uniform limits are given in ref. \cite{8}; for strongly-ordered limits, these must be generalized as discussed in the appendix.) The definition of the antenna function also generalizes,

$$\text{Ant}(\hat{a}, \hat{b} \leftarrow a, 1, \ldots, m, b) = \sum_{j=0}^{m} J(a, 1, \ldots, j; \hat{a}^{(j)})J(j+1, \ldots, m, b; \hat{b}^{(j)}).$$ \hspace{1cm} (4.7)$$

where in order to ensure appropriate strongly-ordered limits in the appendix, different interpolation functions $r_i^{(j)}$ leading to different $k_a$ and $k_b$ are chosen in different terms. The antenna amplitude on the left-hand side is still expressed in terms of the original $\hat{a}$ and $\hat{b}$. The corresponding factorization
Figure 5. Factorization of the double-emission antenna amplitude in a strongly-ordered limit.

is,

\[ A_n(\ldots, a, 1, \ldots, m, b, \ldots) \xrightarrow{[k_1], \ldots, [k_m] \to 0} \sum_{\text{ph. pol. } \lambda_{a,b}} \text{Ant}(\hat{a}^{\lambda_{a}}, \hat{b}^{\lambda_{b}} \leftarrow a, 1, \ldots, m, b) A_{n-m}(\ldots, -k_{a}^{\lambda_{a}}, -k_{b}^{\lambda_{b}}, \ldots). \]  

(4.8)

I have left the dependence on the helicities of the singular legs implicit. The structure of the antenna amplitude for \( m = 2 \) is depicted in fig. 4.

5. Iterated Antennæ

We may expect a simplification of the multiple-emission antenna function (4.7) in the strongly-ordered limit, where the singular momenta are strongly ordered in energy or angle. Let us postpone for a bit the question of what precisely we mean by ‘strong ordering’ when the two types of limits are intermixed. Consider the emission of two singular gluons. If the first, with momentum \( k_1 \), is more singular than the second, with momentum \( k_2 \) (that is, softer or more collinear to one of the hard legs), then at the first step, we can treat \( k_2 \) as another hard leg for the purposes of factorization.

This gives the following factorization,

\[ A_n(\ldots, a, 1, 2, b, \ldots) \xrightarrow{\text{strong ordering}} \sum_{\text{ph. pol. } \lambda_{a1,b1}} \text{Ant}(a_1^{\lambda_{a1}}, b_1^{\lambda_{b1}} \leftarrow a, 1, 2) A_{n-1}(\ldots, -k_{a1}^{\lambda_{a1}}, -k_{b1}^{\lambda_{b1}}, b, \ldots). \]  

(5.1)

Next we factorize the surviving amplitude, given that \((k_{a1}, k_{b1}, k_b)\) is also a singular configuration,

\[ A_n(\ldots, a, 1, 2, b, \ldots) \xrightarrow{\text{strong ordering}} \sum_{\text{ph. pol. } \lambda_{a1,b1,a,b}} \text{Ant}(a_1^{\lambda_{a1}}, b_1^{\lambda_{b1}} \leftarrow a, 1, 2) \text{Ant}(\hat{a}^{\lambda_{a}}, \hat{b}^{\lambda_{b}} \leftarrow -k_{a1}^{\lambda_{a1}}, -k_{b1}^{\lambda_{b1}}, b) A_{n-2}(\ldots, -k_{a}^{\lambda_{a}}, -k_{b}^{\lambda_{b}}, \ldots). \]  

(5.2)
Comparing with the double-singular antenna factorization as given by eqn. (4.8),

\[
A_n(\ldots, a, 1, 2, b, \ldots) \xrightarrow{[k_1], [k_2] \to 0} \sum_{\text{ph. pol. } \lambda_{\hat{a}, \hat{b}}} \text{Ant}(\hat{a}^{\lambda_{\hat{a}}}, \hat{b}^{\lambda_{\hat{b}}} \leftarrow a, 1, 2, b)A_{n-2}(\ldots, -k_{\hat{a}}^{-\lambda_{\hat{a}}}, -k_{\hat{b}}^{-\lambda_{\hat{b}}}, \ldots),
\]

we see that in the strongly-ordered limit,

\[
\text{Ant}(\hat{a}^{\lambda_{\hat{a}}}, \hat{b}^{\lambda_{\hat{b}}} \leftarrow a, 1, 2, b) \xrightarrow{[k_1] \ll [k_2]} 0 \sum_{\text{ph. pol. } \lambda_{a_1, b_1}} \text{Ant}(a_1^{\lambda_{a_1}, b_1^{\lambda_{b_1}}} \leftarrow a, 1, 2) \text{Ant}(\hat{a}^{\lambda_{\hat{a}}}, \hat{b}^{\lambda_{\hat{b}}} \leftarrow -k_{a_1}^{-\lambda_{a_1}}, -k_{b_1}^{-\lambda_{b_1}}, b) + \ldots
\]

This factorization is shown schematically in fig. 5. In these equations, the momenta \(a_1\) and \(b_1\) are reconstructed using the single-emission functions (4.1),

\[
k_{a_1} = f_{\hat{a}}(a, 1, 2),
\]

\[
k_{b_1} = f_{\hat{b}}(a, 1, 2).
\]

The final momenta \(\hat{a}\) and \(\hat{b}\) can be defined either by iterating the single-emission reconstruction functions (as they naturally would be on the right-hand side),

\[
k_{\hat{a}} = f_{\hat{a}}(-k_{a_1}, -k_{b_1}, b),
\]

\[
k_{\hat{b}} = f_{\hat{b}}(-k_{a_1}, -k_{b_1}, b),
\]

or directly using the multiple-emission reconstruction functions of ref. [8],

\[
k_{\hat{a}} = f_{\hat{a}}(a, 1, 2, b),
\]

\[
k_{\hat{b}} = f_{\hat{b}}(a, 1, 2, b),
\]

as they naturally would be on the left-hand side of eqn. (5.4). While the two sets of definitions are not identical, the differences will give rise only to subleading terms.

The notation \([k_1] \ll [k_2]\) means that \(k_1\) is (much) more singular than \(k_2\), that is that they are strongly ordered; but what precisely do we mean by this statement? One might try to use the single-emission criterion (4.6) not only to define soft or collinear momenta, but also to compare the relative degree of softness or collinearity. However, if we use the nested antennae as arguments, the strong-ordering criterion would be,

\[
L(a, 1, 2) \ll L(a_1, b_1, b)
\]

which would require

\[
E_1 \ll E_2 \sqrt{\frac{E_2}{E_b}},
\]
too stringent a requirement for the strongly-ordered soft limit. In contrast, if we require

$$L(a, 1, b) \ll L(a, 2, b), \quad (5.10)$$

then the soft limits are distinguished properly, but the collinear limit in which $$\theta_{12} \ll \theta_{a1}, \theta_{a2}$$, where the strongly-ordered form (5.4) also holds, is not picked up properly. The complete set of limits in which the simplification (5.4) holds is,

$$E_1 \ll E_2 \quad (k_{1,2} \text{ soft});$$

$$\theta_{a1} \ll \theta_{a2}, \theta_{12} \quad (a \parallel 1 || 2);$$

$$\theta_{12} \ll \theta_{a1}, \theta_{a2} \quad (a \parallel 1 || 2);$$

$$\theta_{a1} \ll \theta_{2b} \quad (a || 1, 2 || b);$$

and $$k_1$$ soft with $$k_2 \parallel k_b$$.

We need a more complicated function to discriminate between these limits and other singular regions; the following form will work,

$$L_s(a, 1, 2, b) \equiv \left[ \frac{G \left( a, 1, b \right) G \left( a, 2, b \right) - G^2 \left( a, 1, b \right) G \left( a, 2, b \right)}{G^2 \left( a, k_1 + k_2, b \right) / G \left( a, k_1 + k_2, b \right)} \right]^p \quad (5.12)$$

(it is of course not unique), where $$p > 0$$ will be chosen as described in the following.

We may note that in double-soft limits, the numerator scales as $$E_1^2 E_2^2$$, while the denominator scales as $$(E_1 + E_2)^4$$; so in either strongly-ordered limit $$E_1 \ll E_2$$ or $$E_2 \ll E_1$$, $$L_s$$ will tend to zero. In the limit $$\theta_{a1} \ll \theta_{a2}, \theta_{12} \sim \theta \ll 1$$, the denominator scales like $$\theta^4$$, while the numerator scales like $$\theta_{a1}^2 \theta^2$$, so again it tends to zero. Finally, in the limit $$\theta_{12} \ll \theta_{a1}, \theta_{a2} \sim \theta \ll 1$$, the denominator again scales like $$\theta^4$$, while the numerator scales like $$\theta_{12}^2 \theta^2$$, so once again $$L_s$$ tends to zero.

However, in unordered double-soft limits $$E_1 \sim E_2 \ll E_{a,b}$$, this function is also rather small (though tending to a constant rather than to zero). In order to better distinguish this region from the strongly-ordered collinear limits, we may choose $$p < 1$$; empirically, $$p = 1/4$$ is a good choice.

The discriminant $$L_s$$ is symmetric in $$k_1 \leftrightarrow k_2$$; accordingly, it distinguishes regions where either $$k_1$$ or $$k_2$$ is much more singular than the other from regions where both are comparably singular. It does not, however, distinguish the two different limits $$([k_1] \ll [k_2] \text{ and } [k_2] \ll [k_1])$$ from each other. To do that, introduce another function,

$$L_r(a, b; 1; 2) = \frac{s_{a1}s_{12b}}{s_{a12}s_{2b}}, \quad (5.13)$$

where $$s_{ij} = (k_i + k_j + k_l)^2$$. This function tends to zero when $$(a, 1, 2)$$ should be taken to be the inner (i.e. more singular) antenna, and to infinity when $$(1, 2, b)$$ should be taken to be the inner
antenna. For configurations where \( k_1 \parallel k_a \) and \( k_2 \parallel k_b \) with \( \theta_{a1} \sim \theta_{2b} \), it is of order unity, and either antenna may be taken to be the inner one. In this latter case, we can establish an arbitrary boundary between the two factorizations at \( L_r = 1 \). We may summarize the constraints via the following equivalences,

\[
[k_j] \ll 1 \iff L(a, j, b) \ll 1; \\
[k_1] \ll [k_2] \iff L_s(a, 1, 2, b), L_r(a, b; 1; 2) \ll 1. \tag{5.14}
\]

In considering the amplitude squared, we must again generalize the averaging operation \( \langle \rangle \), to include averaging over all azimuthal angles in the different collinear limits. In particular, this means averaging over variables which interpolate between azimuthal angles around \( k_a \) (for the triply-collinear limit \( k_a \parallel k_1 \parallel k_2 \)) to angles around \( k_b \) (for the triply-collinear limit \( k_1 \parallel k_2 \parallel k_b \)). To construct such a variable, we need to introduce an additional, reference, momentum. Roughly speaking, we are already using \( k_b \) as a reference momentum to define the \( a \parallel 1 \) limit, so we need an additional momentum to define an azimuthal angle. Choose this additional momentum \( q \) to be massless; we can then define a variable,

\[
u = \frac{G(q, a, b)}{\sqrt{\Delta(q, a, 1)\Delta(q, b, 1)}} = \frac{s_{a1}s_{b0} + s_{1b}s_{aq} - s_{1q}s_{ab}}{2\sqrt{s_{a1}s_{1b}s_{aq}s_{bq}}}. \tag{5.15}
\]

In the center-of-mass frame of \( \hat{a} \) and \( \hat{b} \), it corresponds to the cosine of the azimuthal angle of \( k_1 \) [18]. Alternatively, with \( \varepsilon(1, 2, 3, 4) = \varepsilon_{\mu\nu\lambda\rho}k_1^\mu k_2^\nu k_3^\lambda k_4^\rho \), we could use

\[
u = 4\sqrt{-\Delta(a, b)}\frac{-\Delta(a, b, 1)\Delta(a, b, q)}{\Delta(a, b, 1)\Delta(a, b, q)}\varepsilon(q, a, 1, b) = \frac{2\varepsilon(q, a, 1, b)}{\sqrt{s_{a1}s_{1b}s_{aq}s_{bq}}}, \tag{5.16}
\]

which is equal to the sine of the same angle. A third alternative is the phase,

\[
u = \frac{\langle a1 \rangle \langle 1b \rangle}{\sqrt{s_{a1}s_{1b}}} \tag{5.17}
\]

where the reference momentum \( q \) is implicit in the definition of the spinor product.

Once we perform the angular averaging by integrating over this variable, the strongly-ordered factorization (5.4) carries over to the squared antenna functions,

\[
\langle |\text{Ant}(\hat{a}, \hat{b} \leftarrow a, 1, 2, b)|^2 \rangle \xrightarrow{|k_1| \ll |k_2| \to 0} \langle |\text{Ant}(a_1, b_1 \leftarrow a, 1, 2)|^2 \rangle \langle |\text{Ant}(\hat{a}, \hat{b} \leftarrow -k_{a1}, -k_{b1}, b)|^2 \rangle + \text{subleading} \tag{5.18}
\]
The simplifications in the strongly-ordered limit generalize to the emission of $m$ singular momenta, both at the amplitude level,

$$\text{Ant}(\hat{a}^{\lambda_a}, \hat{b}^{\lambda_b} \leftarrow a, 1, \ldots, m, b) \xrightarrow{[k_1] \ll [k_2] \ll \cdots \ll [k_m] \rightarrow 0} \sum_{\text{ph. pol. } \lambda_{a_1}, \ldots, \lambda_{a_m}, \lambda_{b_1}, \ldots, \lambda_{b_m}} \text{Ant}(a_1^{\lambda_{a_1}}, b_1^{\lambda_{b_1}} \leftarrow a, 1, 2) \times \text{Ant}(a_2^{\lambda_{a_2}}, b_2^{\lambda_{b_2}} \leftarrow -k_{a_1}^{-\lambda_{a_1}}, -k_{b_1}^{-\lambda_{b_1}}, 3) \times \cdots \times \text{Ant}(\hat{a}^{\lambda_a}, \hat{b}^{\lambda_b} \leftarrow -k_{a_{m-1}}^{-\lambda_{a_{m-1}}}, -k_{b_{m-1}}^{-\lambda_{b_{m-1}}}, b) + \cdots$$

and at the level of squared amplitudes,

$$\left\langle |\text{Ant}(\hat{a}, \hat{b} \leftarrow a, 1, \ldots, m, b)|^2 \right\rangle \xrightarrow{[k_1] \ll [k_2] \ll \cdots \ll [k_m] \rightarrow 0} \left\langle |\text{Ant}(a_1, b_1 \leftarrow a, 1, 2)|^2 \right\rangle \left\langle |\text{Ant}(a_2, b_2 \leftarrow -k_{a_1}, -k_{b_1}, 3)|^2 \right\rangle \times \cdots \times \left\langle |\text{Ant}(\hat{a}, \hat{b} \leftarrow -k_{a_{m-1}}, -k_{b_{m-1}}, b)|^2 \right\rangle + \text{subleading}$$

In these equations, the momenta $a_j$ and $b_j$ are defined in nested form, with $a_1$ and $b_1$ given by eqn. (5.5), and

$$k_{a_j} = f_a(-k_{a_{j-1}}, -k_{b_{j-1}}, b),$$
$$k_{b_j} = f_b(-k_{a_{j-1}}, -k_{b_{j-1}}, b).$$

### 6. Factorization in Loop Amplitudes

The soft and collinear factorization of tree amplitudes described in sections 2 and 3 can be extended to loop corrections as well. At one loop, the color decomposition analogous to (1.1) is

$$\mathcal{A}_{n}^{1\text{loop}}(\{k_i, \lambda_i, a_i\}) = g^n \sum_{J} n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \text{Gr}_{n;c}(\sigma) A_{n;c}^{[J]}(\sigma),$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to $x$ and $n_J$ is the number of particles of spin $J$. The leading color-structure factor,

$$\text{Gr}_{n;1}(1) = N_c \text{ Tr}(T^{a_1} \cdots T^{a_n}),$$

is just $N_c$ times the tree color factor, and the subleading color structures are given by

$$\text{Gr}_{n;c}(1) = \text{ Tr}(T^{a_1} \cdots T^{a_{c-1}}) \text{ Tr}(T^{a_c} \cdots T^{a_n}).$$

$S_n$ is the set of all permutations of $n$ objects, and $S_{n;c}$ is the subset leaving $\text{Gr}_{n;c}$ invariant. The decomposition (6.1) holds separately for different spins circulating around the loop. The usual
normalization conventions take each massless spin-J particle to have two (helicity) states: gauge bosons, Weyl fermions, and complex scalars. (For internal particles in the fundamental (N_c + \bar{N}_c) representation, only the single-trace color structure (c = 1) would be present, and the corresponding color factor would be smaller by a factor of N_c.)

The subleading-color amplitudes A_{n;c>1} are in fact not independent of the leading-color amplitude A_{n;1}. Rather, they can be expressed as sums over permutations of the arguments of the latter [6]. (For amplitudes with external fermions, the basic objects are primitive amplitudes [19] rather than the leading-color one, but the same dependence of the subleading color amplitudes holds.)

The leading-color amplitude at one loop factorizes as follows,

\[ A_{n;1}^{\text{loop}}(1, \ldots, a, s^{\lambda_a}, b, \ldots, n) \xrightarrow{k_s \to 0} \text{Soft}_{\text{tree}}(a, s^{\lambda_a}, b) A_{n-1}^{\text{loop}}(1, \ldots, a, b, \ldots, n) \]

\[ + \text{Soft}^{1-\text{loop}}(a, s^{\lambda_a}, b) A_{n-1}^{\text{tree}}(1, \ldots, a, b, \ldots, n), \]

\[ A_{n;1}^{\text{loop}}(1, \ldots, a^{\lambda_a}, b^{\lambda_b}, \ldots, n) \xrightarrow{a \parallel b} \sum_{\text{ph. pol.}} \lambda \left( C_{-\lambda}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{\text{loop}}(1, \ldots, (a + b)^\lambda, \ldots, n) \right. \]

\[ \left. + C_{-\lambda}^{1-\text{loop}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{\text{tree}}(1, \ldots, (a + b)^\lambda, \ldots, n) \right). \]  

in the soft and collinear limits [6,19,20,21], respectively. (Explicit expressions for the one-loop factorizing amplitudes Soft^{1-\text{loop}} and C^{1-\text{loop}} were also computed to all orders in \epsilon in refs. [22,23].)

At higher loops, the color decomposition similar to eqns. (1.1,6.1) acquires more terms with additional traces, but the term leading in the number of colors has the form,

\[ A_{n;1}^{l;\text{loop}}(\{k_i, \lambda_i, a_i\}) = g^{n+2l-2}N_c^l \sum_{\sigma \in S_n/Z_n} \text{Tr}(T_{\sigma(1)}^{a_{\sigma(1)}} \cdots T_{\sigma(n)}^{a_{\sigma(n)}}) A_{n;1}^{\text{LC}}(\sigma). \]  

(For fixed-order calculations, a different color decomposition [24] may be desirable.)

The antenna factorization of tree amplitudes described in section 4 can be extended [25] to these leading-color amplitudes, unifying the limits (6.4) given above. To do so, first define a loop generalization of the Berends-Giele current. Such a higher-loop current can be defined in terms of its unitarity cuts [6,26] to all orders in \epsilon [27,28,21],

\[ J_{l-1}^{\text{loop}}(1^{\lambda_1}, 2^{\lambda_2}, \ldots, m^{\lambda_m}; P) \big|_{\text{c.d cut}} = \]

\[ \sum_{k=0}^{l-1} \sum_{j=2}^{l-1-k} \sum_{\text{ph. pol.}} \int d^{4-2\epsilon}\text{LIPS}(\ell_1, \ldots, \ell_j) \]

\[ \times J_{l}^{\text{loop}}(1^{\lambda_1}, \ldots, (c-1)^{\lambda_{c-1}}, \ell_1^{-\sigma_1}, \ldots, \ell_j^{-\sigma_j}, (d+1)^{\lambda_{d+1}}, \ldots, m^{\lambda_m}; P) \]

\[ \times A_{d-c+j+1}^{l+1-j-k;\text{loop}}(c^{\lambda_c}, \ldots, d^{\lambda_d}, -\ell_j^{\sigma_j}, \ldots, -\ell_1^{\sigma_1}). \]  

\[ l = 1, \ldots, k \]
Figure 7. The different cuts required for the computation of the one-loop four-point current: (a) the cut in the \( s_{123} \) channel (b) the cut in the \( s_{12} \) channel. The cut in the \( s_{23} \) channel is related to the latter by symmetry. The quantities on the left-hand side of the cuts are currents, while those on the right-hand side are on-shell amplitudes.

where \( X^{0\text{-loop}} \) means \( X^{\text{tree}} \), and \( d^D\text{LIPS} \), the \( D \)-dimensional Lorentz-invariant phase-space measure. (See ref. [29] for a related construction at one loop.) While the currents appearing here must be evaluated in light-cone gauge [8,14,30], the on-shell amplitudes on the other side of the cut may be evaluated in any gauge. This equation holds to all orders in \( \epsilon \); for an \( n \)-loop antenna function for \( m \) singular momenta, the \( l \)-loop current should be evaluated to \( O(\epsilon^{2(n+m-l)}) \), with epsilonic powers of singular invariants left unexpanded. For example, the cuts entering into the calculation of the one-loop four-point current are shown in fig. 6.

The use of cuts to compute amplitudes or currents can be thought of as applying a modern version of dispersion relations. There are several significant differences from traditional versions of dispersion relations that are worth keeping in mind. Because of the use of dimensional regularization, the integrals involve effectively converge, and so the subtraction ambiguities that traditionally plagued dispersion relations are absent. (It is for this reason that the amplitudes must be kept to higher order in \( \epsilon \).) It should also be stressed that we do not want to apply the method to integrals (although it is useful here too [31]), but rather to amplitudes, so that we automatically take advantage of all the cancellations that have taken place inside lower-order quantities in a gauge-theory calculation.

Using the higher-order current \( J_l^{\text{loop}} \), we can write down an expression for the higher-loop generalization of the antenna amplitude,

\[
\text{Ant}^{n\text{-loop}}(\hat{a}, \hat{b} \leftarrow a, 1, \ldots, m, b) = \sum_{j=0}^{m} \sum_{l=0}^{n} J_l^{\text{loop}}(a, 1, \ldots, j; \hat{a}^{(j)}) J^{(n-l)\text{-loop}}(j+1, \ldots, m, b; \hat{b}^{(j)}). \]

(6.7)
Figure 8. Definition of the one-loop double-emission antenna amplitude in terms of currents. The holes represent loops, the shaded circles sums over trees.

The definition of the one-loop double-emission antenna is shown as an example in fig. 9. The factorization corresponding to eqn. (6.7) is,

$$A_{r\text{-loop}}(\ldots, a, 1, \ldots, m, b, \ldots) \xrightarrow{k_1, \ldots, k_m \text{ singular}} \sum_{\text{ph. pol.}} \sum_{\lambda_{\hat{a}, \hat{b}}, \nu=0} \text{Ant}^{v\text{-loop}}(\hat{a}^\lambda \hat{b}^\lambda \leftarrow a, 1, \ldots, m, b) A_{n-m}^{(r-v)\text{-loop}}(\ldots, -k^-_{\hat{a}} - \lambda_{\hat{a}}, -k^-_{\hat{b}} - \lambda_{\hat{b}}, \ldots).$$

(6.8)

In the one-loop case, this is a sum of two terms, similar to the factorizations in eqn. (6.4); for example,

$$A_{n}^{1\text{-loop}}(\ldots, a, 1, b, \ldots) \xrightarrow{[k_1] \rightarrow 0} \sum_{\text{ph. pol.}} \text{Ant}^{\text{tree}}(\hat{a}^\lambda \hat{b}^\lambda \leftarrow a, 1, b) A_{n-m}^{1\text{-loop}}(\ldots, -k^-_{\hat{a}} - \lambda_{\hat{a}}, -k^-_{\hat{b}} - \lambda_{\hat{b}}, \ldots)$$

$$+ \text{Ant}^{1\text{-loop}}(\hat{a}^\lambda \hat{b}^\lambda \leftarrow a, 1, b) A_{n-m}^{\text{tree}}(\ldots, -k^-_{\hat{a}} - \lambda_{\hat{a}}, -k^-_{\hat{b}} - \lambda_{\hat{b}}, \ldots),$$

(6.9)

as depicted in fig. 11.

If we iterate this factorization, and compare with the direct factorization of double-singular
emission, and match coefficients of $A_{n-2}^{\text{tree}}$, we find that in the strongly-ordered limit,

\[
\sum_{\text{ph. pol. } \lambda_{a_1,b_1}} \left( \text{Ant}^{\text{tree}}(a_1^{\lambda_{a_1}}, b_1^{\lambda_{b_1}} \leftarrow a, 1, 2) \text{Ant}^{1-\text{loop}}(\hat{a}^{\lambda_{a_1}}, \hat{b}^{\lambda_{b_1}} \leftarrow -k_{a_1}^{-\lambda_{a_1}}, -k_{b_1}^{-\lambda_{b_1}}, b) + \text{Ant}^{1-\text{loop}}(a_1^{\lambda_{a_1}}, b_1^{\lambda_{b_1}} \leftarrow a, 1, 2) \text{Ant}^{\text{tree}}(\hat{a}^{\lambda_{a_1}}, \hat{b}^{\lambda_{b_1}} \leftarrow -k_{a_1}^{-\lambda_{a_1}}, -k_{b_1}^{-\lambda_{b_1}}, b) \right) + \cdots
\]

(6.10)

where the momenta are defined in eqn. (5.5). (Matching coefficients of $A_{n-2}^{1-\text{loop}}$ just reproduces eqn. (5.4).) This factorization is depicted in fig. 12.

Iteratively applying a strong ordering to $m$ singular emissions, we obtain the generalization of
Figure 13. Factorization of the one-loop double-emission antenna amplitude in a strongly-ordered limit.

\[
\text{eqn. (5.19),}
\]

\[
\text{Ant}^{r}\text{loop}(\hat{a}^{\lambda_{a}}, \hat{b}^{\lambda_{b}} \leftarrow a, 1, \ldots, m, b) \quad [k_{1}] \ll [k_{2}] \ll \cdots \ll [k_{m}] \rightarrow 0
\]

\[
\sum_{\text{ph. pol. } \lambda_{a_{1}}, \lambda_{b_{1}}, \ldots, \lambda_{a_{m}}, \lambda_{b_{m}}} \sum_{v_{1}+\cdots+v_{m}=r}
\text{Ant}^{v_{1}}\text{loop}(a_{1}^{\lambda_{a_{1}}} b_{1}^{\lambda_{b_{1}}} \leftarrow a, 1, 2)
\times \text{Ant}^{v_{2}}\text{loop}(a_{2}^{\lambda_{a_{2}}} b_{2}^{\lambda_{b_{2}}} \leftarrow -k_{a_{1}}^{-\lambda_{a_{1}}}, -k_{b_{1}}^{-\lambda_{b_{1}}}, 3)
\times \cdots \times \text{Ant}^{v_{m}}\text{loop}(\hat{a}^{\lambda_{a_{m}}} \hat{b}^{\lambda_{b_{m}}} \leftarrow -k_{a_{m-1}}^{-\lambda_{a_{m-1}}}, -k_{b_{m-1}}^{-\lambda_{b_{m-1}}}, b) + \cdots
\]

(6.11)

where the intermediate momenta are defined in eqn. (5.21). This has the same iterated structure as eqn. (5.19), but with the total number of loops ‘distributed’ in all possible ways amongst the iterated antenna amplitudes.

7. Conclusions

A detailed understanding of the singular structure of real emission in perturbative gauge theories is important to the ongoing program of next-to-next-to-leading order (NNLO) corrections to
jet observables. The antenna functions combine soft and collinear limits in a simple way, allowing a simpler approach to the calculation of integrals over phase space of real-emission. The strongly-ordered limits discussed here have two important applications. Taken to all orders, along with knowledge of the leading singular structure of virtual corrections, they allow the resummation of the terms in the matrix elements that give rise to the leading logarithms for observables with a large ratio of scales. To NNLO, they summarize the leading and most singular contributions to real-emission. Since they iterate the one-loop emission probabilities, they are considerably simpler than the full double-singular emission probabilities. Furthermore, their structure suggests it should be possible to match the Catani decomposition of two-loop singularities [32,33] structure directly in the real-emission contributions.

Appendix I. Interpolating Functions for Strongly-Ordered Limits

The reconstruction functions given in ref. [8] are appropriate only to uniform limits, where all singular invariants become singular at the same rate. For strongly-ordered limits, we should take different forms for $k_a$ and $k_b$ in each term of the sum (4.7), corresponding to different choices of the interpolation functions $r_i$ in each term. In eq. (7.8) of ref. [8], the following form was given for the interpolation functions,

$$r_j^{\{0\}} \equiv r_j = \frac{k_j \cdot (K_{j+1,m} + k_b)}{k_j \cdot K};$$

we retain this form for the first and last terms ($\ell = 0, m$) but should instead pick,

$$r_j^{\{\ell\}} = 1 - \frac{k_{\ell+1} \cdot (K_{\ell+1,m} + k_b)}{k_{\ell+1} \cdot K}(1 - r_j^{\{0\}}),$$

for $j \leq \ell$ and

$$r_j^{\{\ell\}} = \frac{k_\ell \cdot (k_a + K_{1..\ell})}{k_\ell \cdot K}r_j^{\{0\}},$$

for $j > \ell$. The reconstructed momenta $\hat{a}^{(\ell)}$ and $\hat{b}^{(\ell)}$ in the $\ell$-th term of eq. (4.7) are given by the same functional forms as given in eq. (6.2) of ref. [8], but with $r_j$ replaced by $r_j^{\{\ell\}}$.

While we cannot eliminate $k_a$ and $k_b$ from the antenna amplitudes, because the phase dependence of their spinor products is needed, we can do so in the squared amplitude. It is possible to do so without introducing square roots; indeed, noting that $s_{ab} = K^2$, we see that $A_1$, given in eq. (10.3) of ref. [8], and $E_{1,2}$, given in eq. (10.6), are already free of the two reconstructed
momenta. We can put $A_2$ in such a form as well,

$$A_2(\hat{a}, a, 1, 2, b, \hat{b}) =$$

\[
\begin{align*}
&\frac{4}{s_1 s_2} \left( 2 + \frac{4 s_1}{K^2} - \frac{2 t_{12}}{K^2} - \frac{5 t_{12b}}{K^2} \right) + \frac{8}{s_1 s_2} \left( 3 - \frac{3 s_2b}{K^2} + \frac{4 s_2b^2}{(K^2)^2} + \frac{t_{12b}}{K^2} - \frac{5 s_2t_{12b}}{K^2} + \frac{2 t_{12b}^2}{(K^2)^2} \right) \\
&- \frac{8}{s_1 s_2} \left( 4 - \frac{4 K^2}{s_1 s_2} - \frac{t_{12b}}{s_1 s_2 (K^2)^2} - \frac{t_{12b}^2}{(K^2)^2} \right) + \frac{8}{s_1 s_2} \left( 1 + \frac{4 s_2b}{K^2} - \frac{3 t_{12b}}{K^2} + \frac{t_{12b}^2}{(K^2)^2} \right) \\
&- \frac{8 K^2}{s_1 s_2 t_{12b}} \left( 2 + \frac{s_1^2}{K^2} + \frac{s_2b^2}{(K^2)^2} - \frac{t_{12b}}{K^2} - \frac{2 s_2b t_{12b}}{K^2} + \frac{t_{12b}^2}{(K^2)^2} \right) \\
&+ \frac{2}{s_2} \left( 4 - \frac{12 s_2b}{K^2} + \frac{4 s_2b^2}{(K^2)^2} + \frac{8 t_{12b}}{K^2} - \frac{7 s_2b t_{12b}}{K^2} + \frac{5 t_{12b}^2}{(K^2)^2} \right) - \frac{8}{s_1 s_2 t_{12b}} \left( 3 + \frac{s_2b}{K^2} - \frac{4 t_{12b}}{K^2} \right) \\
&+ \frac{8 K^2}{s_1 s_2 (K^2 - s_2b - t_{12b})} \left( \frac{t_{12b}}{K^2} + \frac{s_2b^2 t_{12b}}{K^2} + \frac{2 s_2b t_{12b}^2}{(K^2)^2} + \frac{t_{12b}^3}{(K^2)^3} \right) \\
&- \frac{8 K^2}{s_1 s_2 b t_{12b}} \left( 1 + \frac{s_1^3}{(K^2)^3} - \frac{2 s_2b^2}{K^2} + \frac{t_{12b}}{K^2} - \frac{t_{12b}^2}{(K^2)^2} + \frac{t_{12b}^3}{(K^2)^3} \right) - \frac{16 s_1}{s_2 b t_{12b}} \\
&- \frac{16 K^2}{s_1 (K^2 - s_2b - t_{12b}) t_{12b}} \left( \frac{s_2b}{K^2} + \frac{2 s_2b^2}{(K^2)^2} + \frac{t_{12b}}{K^2} - \frac{4 s_2b t_{12b}}{K^2} + \frac{2 t_{12b}^2}{(K^2)^2} \right) \\
&- \frac{8 K^2}{s_1 s_2 (K^2 - s_2b - t_{12b}) t_{12b}} \left( \frac{2 t_{12b}}{K^2} - \frac{s_2b t_{12b}^2}{K^2} + \frac{t_{12b}^3}{(K^2)^3} \right) + \frac{24}{t_{12b}^2} + \frac{16 s_1}{s_1 t_{12b}^2} + \frac{16 s_1}{s_2 b t_{12b}^2} \\
&+ \frac{8 s_1^2}{s_2 b t_{12b}^2} = \frac{8 K^2}{s_1 s_2 s_2 (-s_1 + K^2 - t_{12b})} \left( \frac{2 s_2b^2}{K^2} - \frac{s_1 t_{12b}}{(K^2)^2} \right) \\
&- \frac{8 K^2}{s_1 s_2 (K^2 - t_{12b} - t_{12b})} \left( 2 - \frac{3 s_2b}{K^2} + \frac{2 s_2b^2}{(K^2)^2} - \frac{s_2b^3}{(K^2)^3} \right) \\
&- \frac{8 K^2}{s_1 t_{12b} (K^2 - t_{12b} - t_{12b})} \left( \frac{2 s_2b^2}{K^2} + \frac{t_{12b}}{K^2} \right) + \frac{8 s_1^3}{K^2 s_1 s_2 s_2 t_{12b} (-s_1 + K^2 - t_{12b})} \\
&- \frac{8 K^2}{s_1 t_{12b} (K^2 + s_1 - t_{12b} - t_{12b})} \left( 2 - \frac{3 s_2b}{K^2} + \frac{2 s_2b^2}{(K^2)^2} - \frac{s_2b^3}{(K^2)^3} \right) \\
&+ \frac{8 K^2}{s_1 s_2 s_2 t_{12b} (K^2 + s_1 - t_{12b} - t_{12b})} - \frac{8 K^2}{s_1 t_{12b} t_{12b}} \left( 2 - \frac{3 s_2b}{K^2} + \frac{2 s_2b^2}{(K^2)^2} - \frac{s_2b^3}{(K^2)^3} \right) \\
&- \frac{8 s_2b^2}{s_1 t_{12b} (K^2 - t_{12b} - t_{12b})} + \frac{8 s_2b^2}{s_1 s_2 s_2 t_{12b} (-K^2 + s_2b + t_{12b})} (-K^2 + t_{12b} + t_{12b})
\end{align*}
\]  

(I.4)

which expression is also valid in the strongly-ordered limits.
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