Real elliptic curves and cevian geometry

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1 Introduction.

In this paper we will investigate the connection between elliptic curves $E$ defined over the real numbers $\mathbb{R}$ and the cevian geometry which we have worked out in the series of papers [9]–[15].

We will rederive some of the facts relating to barycentric coordinates that we discussed in the unpublished papers [8] and [16].

First, some notation. We let $ABC$ be an ordinary triangle in the extended plane, and $P = (x, y, z)$ be a point not on the sides of $ABC$ or its anticomplementary triangle $K^{-1}(ABC)$, whose homogeneous barycentric coordinates with respect to $ABC$ are $(x, y, z)$. We note that the isotomic map $\iota$ for $ABC$ has the representation

$$P' = \iota(P) = \iota(x, y, z) = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) = (yz, xz, xy), \quad xyz \neq 0,$$

and the complement mapping $K$ and its inverse $K^{-1}$ with respect to $ABC$ have the matrix representations

$$K = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad K^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

It follows easily that the equations of the sides of the anticomplementary triangle $K^{-1}(ABC)$ are given by

$$K^{-1}(BC) : y + z = 0, \quad K^{-1}(AC) : x + z = 0, \quad K^{-1}(AB) : x + y = 0.$$

The isotomcomplement $Q$ of $P$ with respect to $ABC$ is the point $Q = K(\iota(P)) = K(P')$, whose barycentric coordinates, in terms of $x, y, z$, are

$$Q = K(yz, xz, xy)' = (x(y + z), y(x + z), z(x + y)) = (x', y', z').$$

The corresponding point $Q' = K(\iota(P')) = K(P)$ for $P'$ has coordinates

$$Q' = (y + z, x + z, x + y).$$
Since the equations of the sides $BC, CA, AB$ of $ABC$ are, respectively, $x = 0, y = 0$, and $z = 0$, the points $D = (0, y, z), \ E = (x, 0, z), \ F = (x, y, 0)$ are the traces of $P$ on the respective sides, and the unique affine mapping $T_P$ taking $ABC$ to $DEF$ is given in terms of barycentric coordinates by the matrix
\[
T_P = \begin{pmatrix}
0 & x'(x+y) & x'(x+z) \\
y'(x+y) & 0 & y'(y+z) \\
z'(x+z) & z'(y+z) & 0
\end{pmatrix}.
\]
(1)
The traces of the point $P'$ on the sides of $ABC$ are $D_3 = (0, z, y), \ E_3 = (z, 0, x), \ F_3 = (y, x, 0)$, and the corresponding affine map $T_{P'} : ABC \to D_3E_3F_3$ is given by the matrix
\[
T_{P'} = \begin{pmatrix}
0 & z'(y+z) & y'(y+z) \\
z'(x+z) & 0 & x'(x+z) \\
y'(x+y) & x'(x+y) & 0
\end{pmatrix}.
\]
(2)
In [9] we showed that $Q$ is a fixed point of $T_P$ and $Q'$ is a fixed point of $T_{P'}$; further, these are the only ordinary fixed points of these maps, when $P$ and $P'$ are ordinary points. (See [9], Theorems 3.2 and 3.12.)

In [9] and [11] we studied the affine mapping
\[
S = T_P \circ T_{P'} = \begin{pmatrix}
x'(y' + z') & xx' & xx' \\
yy' & y(x' + z') & yy' \\
z'z' & z(x' + y') & z(x' + y')
\end{pmatrix},
\]
(3)
which is a homothety or translation. In [11] our main focus was on the affine map
\[
\lambda = T_{P'} \circ T_{P}^{-1} = \begin{pmatrix}
yz(y+z) & xz(y-z) & xy(z-y) \\
yz(x-z) & xz(x+z) & xy(z-x) \\
yz(x-y) & xz(y-x) & xy(x+y)
\end{pmatrix}.
\]
(4)
Then in [12] we made use of the map
\[
M = T_P \circ K^{-1} \circ T_{P'} = \begin{pmatrix}
x(y-z)^2 & x(y+z)^2 & x(y+z)^2 \\
y(x+z)^2 & y(x-z)^2 & y(x+z)^2 \\
z(x+y)^2 & z(x+y)^2 & z(x-y)^2
\end{pmatrix},
\]
(5)
which is also a homothety or translation. In [14] we studied the points $P$ for which $M$ is a translation, and in [15] we studied the points $P$ for which $M$ is a half-turn.

**Proposition 1.1.** The maps $S, \lambda$, and $M$ have the respective fixed points
\[
X = (xx', yy', zz') = (x^2(y+z), y^2(x+z), z^2(x+y)) = P \cdot Q,
\]
(6)
\[
Z = (x(y-z)^2, y(z-x)^2, z(x-y)^2),
\]
(7)
\[
S = (x(y+z)^2, y(x+z)^2, z(x+y)^2) = Q \cdot Q'.
\]
(8)
Proof. This is a straightforward calculation. For example,

\[
S(X) = \begin{pmatrix}
  x(y' + z') & xx' & xx' \\
  yy' & y(x' + z') & yy' \\
  zz' & zz' & z(x' + y') \\
\end{pmatrix} (xx', yy', zz')^t
= (x + y)(x + z)(y + z)(xx', yy', zz')^t.
\]

Since \( P \) is not on any of the sides of \( K^{-1}(ABC) \), the quantity \((x+y)(x+z)(y+z)\) is nonzero, so \( S(X) = X \). Similarly,

\[
\lambda(Z) = \begin{pmatrix}
  yz(y + z) & xz(y - z) & xy(z - y) \\
  yz(x - z) & xz(x + z) & xy(z - x) \\
  yz(x - y) & xz(y - x) & xy(x + y) \\
\end{pmatrix} (x(y-z)^2, y(z-x)^2, z(x-y)^2)^t \\
= (2xyz)(x(y-z)^2, y(z-x)^2, z(x-y)^2)^t,
\]

where \( xyz \neq 0 \), since \( P \) does not lie on the sides of \( ABC \); and

\[
M(S) = \begin{pmatrix}
  x(y-z)^2 & x(y+z)^2 & x(y+z)^2 \\
  y(x+z)^2 & y(x-z)^2 & y(x+z)^2 \\
  z(x+y)^2 & z(x+y)^2 & z(x-y)^2 \\
\end{pmatrix} (x(y+z)^2, y(x+z)^2, z(x+y)^2)^t \\
= \rho(x(y+z)^2, y(x+z)^2, z(x+y)^2)^t,
\]

where

\[
\rho = x(y^2 + z^2) + y(x^2 + z^2) + z(x^2 + y^2) + 2xyz = (x+y)(x+z)(y+z).
\]

This proves the proposition. \( \square \)

The points \( X \) and \( S \) are the centers of the respective maps \( S \) and \( M \). In the former case, letting \( Y = (a, b, c) \), we have

\[
S(Y) = \begin{pmatrix}
  x(y' + z') & xx' & xx' \\
  yy' & y(x' + z') & yy' \\
  zz' & zz' & z(x' + y') \\
\end{pmatrix} (a, b, c)^t \\
= (ax(y' + z') + (b + c)xx', by(x' + z') + (a + c)yy', cz(x' + y') + (a + b)zz')^t \\
= ((a + b + c)x' + 2axyz, (a + b + c)yy' + 2bxyz, (a + b + c)zz' + 2cxyz)^t \\
= (a + b + c)X + 2xyzY.
\]

It follows that \( X \) is collinear with \( Y \) and \( S(Y) \), for any ordinary point \( Y = (a, b, c) \) (because \( a + b + c \neq 0 \)). This proves the claim that \( X \) is the center of \( S \). The computation

\[
M(Y) = \begin{pmatrix}
  x(y-z)^2 & x(y+z)^2 & x(y+z)^2 \\
  y(x+z)^2 & y(x-z)^2 & y(x+z)^2 \\
  z(x+y)^2 & z(x+y)^2 & z(x-y)^2 \\
\end{pmatrix} (a, b, c)^t \\
= (a + b + c)S - 4xyzY
\]
shows the same for \( S \) and the map \( M \). This verifies that the points \( X \) and \( S \) are the same as the points (with the same names) which are discussed in \([9]\) and \([12]\). Note that these relations also show that \( S \) and \( M \) fix all the points on the line at infinity.

Next we define the point
\[
V = (x(y^2 + yz + z^2), y(x^2 + xz + z^2), z(x^2 + xy + y^2)) \tag{9}
\]
and note that \( V \) can also be given by
\[
V = PQ \cdot P'Q', \quad (P, P' \text{ ordinary, } P \text{ not on a median}). \tag{10}
\]
To see this, we note first that \( V \) is collinear with \( P \) and \( Q \), which is immediate from the equation
\[
V = -(xy + xz + yz)P + (x + y + z)Q, \quad (P, P' \text{ ordinary}).
\]
We convert to absolute barycentric coordinates by dividing each point in this equation by the sum of its coordinates. Since the sum of the coordinates of \( V \) is
\[
x(y^2 + yz + z^2) + y(x^2 + xz + z^2) + z(x^2 + xy + y^2) \\
= x^2(y + z) + y^2(x + z) + z^2(x + y) + 3xyz \\
= (x + y + z)(xy + xz + yz),
\]
then denoting the last expression by \( F(0) \) (see the proof of Proposition 1.2 below) we have the relation
\[
\frac{1}{F(0)}V = -\frac{1}{x + y + z}P + \frac{2}{2(xy + xz + yz)}Q.
\]
It follows from this that \( Q \) is the midpoint of the segment \( PV \), when \( P \) and \( V \) are ordinary. Replacing \( P \) by \( P' \) and \( Q \) by \( Q' \) is effected by the map \( x \rightarrow \frac{1}{x}, y \rightarrow \frac{1}{y}, z \rightarrow \frac{1}{z} \), and on multiplying through by \( x^2 y^2 z^2 \), we obtain the equation
\[
V = -(x + y + z)P' + (xy + xz + yz)Q'.
\]
This shows as above that \( Q' \) is the midpoint of \( PV' \). This proves that \( \text{(10)} \) holds for the point defined by \( \text{(9)}. \)

**Proposition 1.2.** The point \( Z \) is collinear with \( G = (1, 1, 1) \) and \( V = (x(y^2 + yz + z^2), y(x^2 + xz + z^2), z(x^2 + xy + y^2)) \). We have the relation
\[
Z = (-3xyz)G + V. \tag{11}
\]
If \( P, P', \) and \( Z \) are ordinary points, then we have the signed ratio
\[
\frac{GZ}{ZV} = -\frac{1}{9} \frac{(x + y + z)(xy + yz + xz)}{xyz}. \tag{12}
\]
Proof. Define
\[ F(a) = x^2(y + z) + y^2(x + z) + z^2(x + y) + (a + 3)xyz. \] (13)

Equation (11) follows immediately from the identity
\[ -3xyz + x(y^2 + yz + z^2) = x(y - z)^2 \]
by cyclically permuting the variables \((x \rightarrow y \rightarrow z \rightarrow x)\). In order to prove (12), we convert to absolute barycentric coordinates by dividing the coordinates of \(Z\) by their sum, which is
\[ x(y - z)^2 + y(x - z)^2 + z(x - y)^2 = x^2(y + z) + y^2(x + z) + z^2(x + y) - 6xyz = F(-9). \]

As above, the sum of the coordinates of \(V\) is
\[ x^2(y + z) + y^2(x + z) + z^2(x + y) + 3xyz = F(0) = (x + y + z)(xy + xz + yz). \]

This shows that \(V\) is ordinary whenever \(P\) and \(P'\) are ordinary. Putting this into (11) gives
\[ \frac{1}{F(-9)} Z = \frac{-9xyz}{F(-9)} \left( \frac{1}{3} \frac{1}{3} \frac{1}{3} \right) + \frac{F(0)}{F(-9)} \left( \frac{1}{F(0)} V \right). \]

Now \(F(0) - 9xyz = F(-9)\), so this relation implies that the signed ratio \(GZ/ZV\) is given by
\[ \frac{GZ}{ZV} = \frac{F(0)/F(-9)}{(-9xyz)/F(-9)} = \frac{-1 F(0)}{9xyz}, \]
which agrees with (12). (See [2], p. 28.) \(\Box\)

Proposition 1.3. If the points \(S, V\) are ordinary, we have
\[ S = (xyz)G + V \quad \text{and} \quad GS \quad SV = (x + y + z)(xy + xz + yz) \quad 3xyz. \]

In particular, the cross ratio \((GV, SZ) = -3\).

Proof. The relation between \(S, G, \) and \(V\) is immediate from (8) and (9). This gives that
\[ \frac{1}{F(3)} S = \frac{3xyz}{F(3)} \frac{1}{G} + \frac{F(0)}{F(3)} \frac{1}{F(0)} V, \]
from which the formula \(\frac{GS}{SV} = \frac{F(0)}{3xyz}\) follows. Then
\[ (GV, SZ) = \frac{GS}{SV} \frac{VZ}{VS} = \frac{GS}{SV} \frac{VZ}{VS} = -3. \] \(\Box\)
We now determine the homothety ratio of the map \( M \). From [12] we have the generalized circumcenter \( O = T_p^{-1} \circ K(Q) \) given by

\[
O = \begin{pmatrix}
-xx' & x'y' & x'z' \\
y'x & -yy' & y'z' \\
z'x & z'y & -zz' \\
\end{pmatrix} \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{pmatrix} (x(y+z), y(x+z), z(x+y))' \\
\]

\[
= (x(y+z)^2x'', y(x+z)^2y'', z(x+y)^2z''),
\]

where

\[
x'' = xy + xz + yz - x^2, \quad y'' = xy + xz + yz - y^2, \quad z'' = xy + xz + yz - z^2.
\]

We will use the fact that \( M(O) = Q \) to determine the ratio \( SQ/SO \) in the next proposition. Note that the sum of the coordinates of \( O \) is

\[
x(y+z)^2 + y(x+z)^2 + z(x+y)^2 = 8xyz(x+y+z) + 6xyz = F(3).
\]

**Proposition 1.4.** We have that

\[
(x+y)(x+z)(y+z)Q = 2(xy + xz + yz)S - O
\]

and when \( P' \) and \( S \) are ordinary points, the signed ratio \( SQ/SO \) is given by

\[
\frac{SQ}{SO} = -\frac{QS}{SO} = \frac{-4xyz}{(x+y)(x+z)(y+z)}.
\]

**Proof.** From the computation following the proof of Proposition 1.1 we have that

\[
M(O) = 8xyz(x+y+z)S - 4xyzO,
\]

from which we obtain

\[
(x+y)(x+z)(y+z)Q = 2(xy + xz + yz)S - O.
\]

Using the fact that \( F(-1) = (x+y)(x+z)(y+z) \), we obtain the relation in absolute barycentric coordinates given by

\[
\frac{F(-1)}{F(3)} \frac{1}{2(xy + xz + yz)}Q + \frac{4xyz}{F(3)} \frac{1}{8xyz(x+y+z)}O = \frac{1}{F(3)} S,
\]

where \( F(-1) + 4xyz = F(3) \). This proves the second assertion. \( \square \)

**Corollary 1.5.** If \( P, P', Z, \) and \( S \) are ordinary points, the homothety ratio of the map \( M \) is

\[
k = \frac{SQ}{SO} = \frac{4}{9GZ + 1}.
\]

**Proof.** This follows immediately from Propositions 1.2 and 1.4 using the fact that

\[
-(x+y+z)(xy + xz + yz) + xyz = -(x+y)(x+z)(y+z).
\]

\( \square \)
2 Elliptic curves over $\mathbb{R}$.

Let the quantity $a$ be defined by

$$a = \frac{G Z}{Z V}.$$ 

Then Proposition 1.2 gives that

$$a = -\frac{(x + y + z)(xy + yz + xz)}{xyz}.$$ 

(17)

It follows that the set of ordinary points $P$, for which $Z$ and $V$ are ordinary, and $G Z / Z V = a / 9$ is fixed, coincides with the set of $P$ whose coordinates satisfy

$$(x + y + z)(xy + yz + xz) + axyz = 0,$$

or

$$E_a : \ x^2(y + z) + y^2(x + z) + z^2(x + y) + (a + 3)xyz = 0.$$ 

(18)

The left side of this equation is exactly the quantity $F(a)$ that we defined in (13). We note that the set of points, for which $Z$ is infinite, is the set of points for which $F(-9) = 0$, and the set of points, for which $V$ is infinite, is the set of points for which $F(0) = (x + y + z)(xy + xz + yz) = 0$; the latter is the union of the line at infinity $l_\infty$ and the Steiner circumellipse $\iota(l_\infty)$. Thus, if $a \neq 0, -9$ is a real number, equation (18) describes the set of ordinary points for which $G Z / Z V = a / 9$. Also, the set of $P$ for which $S$ is an infinite point is the set of points for which $F(3) = 0$; this set was studied in the paper [14]. Thus, Corollary 1.5 holds for all $a \neq 0, 3, -9$.

The curve $E_a$ turns out to be an elliptic curve, for $a \neq 0, -1, -9$. To see this, put $z = 1 - x - y$ in the equation (18). This gives the affine equation for $E_a$ in terms of absolute barycentric coordinates $(x, y, 1 - x - y)$:

$$E_a : \ (ax + 1)y^2 + (ax + 1)(x - 1)y + x^2 - x = 0.$$ 

(19)

We call this curve the geometric normal form of an elliptic curve. The discriminant of the equation (19) with respect to $y$ is

$$D = (ax + 1)^2(x - 1)^2 - 4(ax + 1)(x^2 - x) = (ax + 1)(x - 1)(ax^2 - (a + 3)x - 1).$$

This polynomial has discriminant $d = 256a^2(a + 1)^3(a + 9)$ and is therefore square-free in $\mathbb{R}[x]$ if and only if $a \neq 0, -1, -9$. For these values of $a$, the curve $E_a$ is birationally equivalent to $Y^2 = D$, where $D$ is quartic in $x$, a curve which is well-known to be an elliptic curve. Alternatively, we can compute the partial derivatives

$$\frac{\partial F}{\partial x} = 2x(y + z) + y^2 + z^2 + (a + 3)yz,$$

$$\frac{\partial F}{\partial y} = 2y(x + z) + x^2 + z^2 + (a + 3)xz,$$

$$\frac{\partial F}{\partial z} = 2z(x + y) + x^2 + y^2 + (a + 3)xy.$$
\[
\frac{\partial F}{\partial z} = 2z(x + y) + x^2 + y^2 + (a + 3)xy,
\]

and check that the equations \(\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0\) have no common solution with \((x, y, z) \neq (0, 0, 0)\), for \(a \notin \{0, -1, -9\}\). For example, subtracting the first two equations gives
\[
2z(x - y) + y^2 - x^2 + (a + 3)z(y - x) = (y - x)((a + 1)z + x + y) = 0.
\]
Hence, \(x = y\) or \(x + y = -(a + 1)z\). Since the above partials arise from each other by cyclically permuting the variables, we also have that \(y = z\) or \(y + z = -(a + 1)x\), and \(z = x\) or \(z + x = -(a + 1)y\). Thus, either: 1) \(x = y = z\); or 2) \(x = y\) and \(z = -(a + 2)x\), or similar equations hold resulting from a cyclic permutation; or 3) \(x + y = -(a + 1)z\) along with the two equations arising from cyclic permutations. In Case 1, \(F(a) = (a + 9)x^3 = 0\); in Case 2, \(F(a) = a(a + 1)x^3 = 0\); and in Case 3, the determinant of the resulting 3 \(\times\) 3 system is \(-a^2(a + 3)\). The first two cases are clearly impossible, so we are left with Case 3, with \(a = -3\). In this case \((x + y - 2z) - (x - 2y + z) = 3y - 3z = 0\), so \(x = y = z\) by symmetry and we are in Case 1 again. Therefore, \(F(a) = 0\) is a non-singular cubic curve, which implies it is an elliptic curve, since it has a rational point.

**Remark.** The curve \(E_a\) always has a torsion group of order 6 consisting of rational points. The points \(A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1)\) are on the curve, as well as the points \(A_\infty = (0, 1, -1), B_\infty = (1, 0, -1), C_\infty = (1, -1, 0)\), which are the infinite points on the lines \(BC, CA,\) and \(AB\), respectively. We take the base point of the additive group on \(E_a\) to be the point \(O = A_\infty = (0, 1, -1)\). Putting \(x = 0\) in \((18)\) gives \(yz(y + z) = 0\), so the above 6 points are the only points on the sides of \(ABC\) which lie on \(E_a\). See [14].

We summarize the above discussion as follows.

**Theorem 2.1.** The locus of points \(P\), for which the point \(Z\) is ordinary and \(\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \neq \{0, -1, -9\}\), for some fixed \(a \in \mathbb{R}\), coincides with the set of points on the elliptic curve \(E_a\) defined by \([18]\), minus the points in the set
\[
T = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, -1), (1, 0, -1), (1, 1, 0)\}.
\]

In particular, the locus of \(P\) for which the homothety ratio of the map \(M = T_P \circ K^{-1} \circ T_P\) is \(k = \frac{1}{a+1}\), for fixed real \(a \notin \{3, 0, -1, -9\}\), is an elliptic curve minus the points on the sides of triangle \(ABC\).

If \(a = 3\), then \(S \in l_\infty\), and \(M\) is a translation. The elliptic curve \(E_3\) was considered in [14]. When \(a = -5\), \(k = -1\) and \(M\) is a half-turn. The elliptic curve \(E_{-5}\) was considered in [15].

We have the following result for the \(j\)-invariant of \(E_a\).

**Proposition 2.2.** For a real number \(a \neq 0, -1, -9\), the \(j\)-invariant of the elliptic curve \(E_a\) is
\[
j(E_a) = \frac{(a + 3)^3(a^3 + 9a^2 + 3a + 3)^3}{a^2(a + 1)^3(a + 9)}.
\]
Proof. We compute the $j$-invariant of the curve
\[ Y^2 = (ax + 1)(x - 1)(ax^2 - (a + 3)x - 1) \tag{20} \]
by converting it to a curve in Legendre normal form:
\[ E' : v^2 = u(u - 1)(u - \lambda), \]
and using the formula
\[ j(E') = \frac{2^8(\lambda^2 - \lambda + 1)^3}{(\lambda^2 - \lambda)^2}. \]
Putting \( x = \frac{u + 1}{u - a} \) and \( g(x) = (ax + 1)(x - 1)(ax^2 - (a + 3)x - 1) \), we have
\[ g\left(\frac{u + 1}{u - a}\right) = \frac{(a + 1)^2}{(u - a)^4} u(-4u^2 + (a^2 + 6a - 3)u + 4a). \]
Hence, \[20\] is birationally equivalent to
\[ Y_1^2 = u(-4u^2 + (a^2 + 6a - 3)u + 4a). \tag{21} \]
We let \( \alpha, \beta \) be the roots of the quadratic in \( u \) on the right side of this equation. Then
\[ \alpha, \beta = \frac{a^2 + 6a - 3}{8} \pm \frac{(a + 1)}{8} \sqrt{(a + 1)(a + 9)}, \]
and the curve \[21\] is equivalent over \( \mathbb{C} \) to
\[ Y_2^2 = u(u - \alpha)(u - \beta), \]
which is in turn equivalent over \( \mathbb{C} \) to
\[ E' : v^2 = u(u - 1)\left(\frac{u - \alpha}{\beta}\right). \]
A calculation on Maple with \( \lambda = \alpha/\beta \) gives that
\[ j(E') = \frac{(a + 3)^3(a^3 + 9a^2 + 3a + 3)^3}{a^2(a + 1)^3(a + 9)}. \]
This proves the proposition. \( \square \)

We now prove the following result.

**Theorem 2.3.** Let \( E \) be any elliptic curve whose $j$-invariant is a real number. Then \( E \) is isomorphic to the curve \( E_a \) over \( \mathbb{R} \) for some real value of \( a \notin \{0, -1, -9\} \).
Proof. Letting \( f(x) \) represent the function
\[
f(x) = \frac{(x + 3)^3(x^3 + 9x^2 + 3x + 3)^3}{x^2(x + 1)^3(x + 9)},
\]
we just have to check that \( f(\mathbb{R} - \{0, -1, -9\}) = \mathbb{R} \). This is a straightforward calculus exercise, which we leave to the reader. We only note that
\[
f'(x) = \frac{6(x + 3)^2(x^3 + 9x^2 + 3x + 3)^2(x^2 + 6x - 3)(x^4 + 12x^3 + 30x^2 + 36x + 9)}{x^3(x + 1)^4(x + 9)^2},
\]
that the minimum value of \( f(x) \) for \( x < -9 \) is 1728; the maximum value of \( f(x) \) for \( -9 < x < -1 \) is also 1728; and that \( f(x) \) approaches \(-\infty\) as \( x \) approaches the asymptotes \( x = -9 \) (from the right) and \( x = -1 \) (from the left). This is enough to prove the assertion. If \( a \in \mathbb{R} \) satisfies \( j(E_a) = f(a) = j(E) \), then \( E \cong E_a \) over \( \mathbb{R} \).

Remark. The real values of \( a \) for which \( j(E_a) = 1728 \) are the real roots of the equation
\[
(x^2 + 6x - 3)(x^4 + 12x^3 + 30x^2 + 36x + 9) = 0,
\]
and are given explicitly by
\[
a = -3 \pm 2\sqrt{3}, \quad -3 - \sqrt{3} \pm \sqrt{9 + 6\sqrt{3}}.
\]

As an example, the curve \( E_{-3} \) has \( j(E_{-3}) = 0 \), and affine equation
\[
E_{-3} : \quad (3x - 1)y^2 + (3x - 1)(x - 1)y - x^2 + x = 0.
\]
Putting \( x = \frac{u}{u-2} \) and \( y = -\frac{(3x-1)(x-1)+4x/(u-2)}{2(3x-1)} \) yields the isomorphic curve
\[
E : \quad v^2 = u^3 + 1.
\]
This curve has exactly 6 rational points, namely, the points \((2, \pm 3), (-1, 0), (0, \pm 1)\), and the base point \( O \). For which real quadratic fields \( K = \mathbb{Q}(\sqrt{d}) \) is there a point on \( E \) defined over \( K \)? For which values of \( n \geq 2 \) does \( E \) have a real torsion point of order \( n \)?

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