THE CLASSIFICATION OF DOUBLE PLANES OF
GENERAL TYPE WITH $K^2 = 8$ AND $p_g = 0$

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Abstract. We study minimal double planes of general type with $K^2 = 8$ and $p_g = 0$, namely pairs $(S, \sigma)$, where $S$ is a minimal complex algebraic surface of general type with $K^2 = 8$ and $p_g = 0$ and $\sigma$ is an automorphism of $S$ of order 2 such that the quotient $S/\sigma$ is a rational surface. We prove that $S$ is a free quotient $(F \times C)/G$, where $F$ is an hyperelliptic curve, $G$ is a finite group that acts faithfully on $F$ and $C$, and $\sigma$ is induced by the automorphism $\tau \times Id$ of $F \times C$, $\tau$ being the hyperelliptic involution of $F$. We describe all the $F$, $C$ and $G$ that occur; in this way we obtain 5 families of surfaces with $p_g = 0$ and $K^2 = 8$, of which we believe only one was previously known.

Using our classification we are able to give an alternative description of these surfaces as double covers of the plane, thus recovering a construction proposed by Du Val. In addition we study the geometry of the subset of the moduli space of surfaces of general type with $p_g = 0$ and $K^2 = 8$ that admit a double plane structure.

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1. Introduction

The last section of the paper [DMP] is concerned with the study of pairs $(S, \sigma)$, where $S$ is a minimal complex projective surface of general type with $p_g = 0$ and $K^2 = 8$ and $\sigma$ is an automorphism of $S$ of order 2. Here we give a complete description in the case that $(S, \sigma)$ is a double plane, i.e. that the quotient surface $S/\sigma$ is rational.

First of all, using the results of [DMP], we prove that $S$ is a free quotient of a product of curves:

Theorem 1.1. Let $S$ be a minimal complex projective surface of general type with $K_S^2 = 8$ and $p_g(S) = 0$.

There exists an automorphism $\sigma$ of $S$ of order 2 such that $S/\sigma$ is rational if and only if there exist a curve $C$, an hyperelliptic curve $F$ of genus 3 or 5 and a finite group $G$ such that:

a) $G$ acts faithfully on $F$ and $C$ and the diagonal action of $G$ on $F \times C$ is free;

b) $|G| = (g(F) - 1)(g(C) - 1)$;

c) $C/G$ and $F/G$ are rational curves;

d) $S = (F \times C)/G$ and $\sigma$ is the involution induced by $\tau \times Id$, where $\tau$ denotes the hyperelliptic involution of $F$.

Theorem 1.1 reduces the classification of double planes of general type with $p_g = 0$ and $K^2 = 8$ to the classification of the triples $F$, $C$, $G$ that satisfy properties a)–c) of its statement. The latter problem can be attacked by remarking that the $G$–action on the hyperelliptic curve $F$ descends to a
(possibly not faithful) $G$–action on $F/\tau = \mathbb{P}^1$ that preserves the image of the set of Weierstrass points of $F$. We combine this observation with the well known classification of the finite group actions on the projective line and, exploiting also some other geometrical constraints of the problem, we obtain a complete classification. There are 5 types of minimal double planes with $p_g = 0$ and $K^2 = 8$, whose description takes up all of section 3. We list here only the numerical invariants for each type:

**Theorem 1.2.** Let $(S, \sigma)$ be a minimal double plane with $p_g = 0$ and $K^2 = 8$. Then, referring to the notation of Theorem 1.1, $(S, \sigma)$ belongs to one and only one of the following types:

- **Ia:** $g(F) = 3$, $g(C) = 5$, $G = \mathbb{Z}_2^3$;
- **Ib:** $g(F) = 3$, $g(C) = 9$, $G = \mathbb{Z}_2 \times D_4$;
- **Ic:** $g(F) = 3$, $g(C) = 13$, $G = S_4$;
- **Id:** $g(F) = 3$, $g(C) = 25$, $G = \mathbb{Z}_2 \times S_4$;
- **II:** $g(F) = 5$, $g(C) = 16$, $G = A_5$.

While double planes of type Ia already appear in [MP1], as far as we know the remaining 4 types give 4 new families of surfaces of general type with $p_g = 0$ and $K^2 = 8$. It may be worth remarking that all known surfaces with these invariants are free quotients of product of curves (a construction suggested first by Beauville). It would be interesting to know whether these are actually the only ones.

Another byproduct of Theorem 1.2, which was partly the motivation of this work, is the classification of the minimal surfaces of general type with $p_g = 0$, $K^2 = 8$ and non birational bicanonical map.

**Corollary 1.3.** Let $S$ be a minimal projective complex surface of general type with $p_g = 0$ and $K^2 = 8$.

The bicanonical map of $S$ is non birational if and only if there is an automorphism $\sigma$ of $S$ such that $(S, \sigma)$ is a double plane of type Ia, Ib, Ic or Id (cf. Thm. 1.2).

Furthermore, using Theorem 1.2 we are able to give an alternative description of these surfaces as minimal resolution of normal double covers of the projective plane $\mathbb{P}^2$ (“plane models”), thus recovering a construction suggested by Du Val (cf. [5]). We also study the subset of the moduli space of surfaces with $p_g = 0$ and $K^2 = 8$ consisting of surfaces that admit a double plane construction (cf. §5). In particular we show that surfaces with non birational bicanonical map are a closed and open subset of the moduli space which is the union of four irreducible connected components.

The paper goes as follows: §2 contains the proof of Theorem 1.1. §3 contains the description of the 5 types of double planes; §4 is the technical heart of the paper and contains the proof that the double planes with $K^2 = 8$ and $p_g = 0$ are precisely those given in §3; §5 contains the description of the plane models; finally §6 is devoted to the study of the moduli.

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Mendes Lopes. Finally, I am grateful to Barbara Fantechi for her advice on the deformation theory of the last section.

**Notation and conventions.** We work over the field of complex numbers. By *surface* we mean a projective surface and by *curve* we mean a smooth projective curve. A *map* is a rational map and a *morphism* is a regular map. If a group $G$ acts on a set $X$ we denote by $X^G$ the subset of elements fixed by $G$.

We use standard notation of algebraic geometry; we just recall here the notation for the invariants of a surface $S$: $K_S$ is the *canonical class*, $p_g(S) = h^0(S, K_S)$ the *geometric genus*, $q(S) = h^1(S, O_S)$ the *irregularity* and $\chi(S) = 1 + p_g(S) - q(S)$ the *Euler characteristic*.

2. **Rational involutions**

Let $S$ be a smooth projective surface. An *involution* $\sigma$ of $S$ is an automorphism of $S$ of order 2. We say that a map $f : S \to Y$ is composed with $\sigma$ if $f \circ \sigma = f$.

An involution $\sigma$ is called *rational* if the quotient surface $\Sigma := S/\sigma$ is rational. We define a *double plane* as a pair $(S, \sigma)$, where $S$ is a smooth projective surface and $\sigma$ is a rational involution of $S$. We say that a double plane $(S, \sigma)$ is minimal, of general type, has $p_g = 0, \ldots$ if the surface $S$ is minimal, of general type, has $p_g = 0 \ldots$

Isomorphism of double planes is defined in the obvious way.

The following proposition explains how to construct examples of minimal double planes $(S, \sigma)$ by taking free quotients of product of curves.

**Proposition 2.1.** Let $F$ be an hyperelliptic curve and let $\tau$ be the hyperelliptic involution of $F$, let $C$ be a curve, and let $G$ be a finite group that acts faithfully on $F$ and on $C$. Assume that:

a) $C/G$ is rational;

b) the diagonal action of $G$ on $F \times C$ is free.

Set $S := (F \times C)/G$ and denote by $\sigma$ the involution of $S$ induced by $\tau \times \text{Id}$. Then $(S, \sigma)$ is a minimal double plane with invariants:

$$\chi(S) = \frac{(g(F) - 1)(g(C) - 1)}{|G|}; \quad q(S) = g(F/G).$$

$$K^2_S = \frac{8(g(F) - 1)(g(C) - 1)}{|G|}.$$

Moreover, $S$ is a minimal surface of general type iff $g(F), g(C) \geq 2$.

**Proof.** Consider the quotient map $\psi : F \times C \to S$. By assumption b), the map $\psi$ is étale of degree equal to $|G|$, hence $S$ is smooth and we have $\chi(F \times C) = |G| \chi(S)$ and $K_{F \times C} = \psi^* K_S$. The formulas for $\chi(S)$, $K^2_S$ follow easily from this remark. In addition, $S$ is a minimal surface of general type iff $K_S$ is nef and big, iff $K_{F \times C}$ is nef and big, iff $C$ and $F$ have genus $\geq 2$.

The irregularity of $S$ is equal to the dimension of the $G$–invariant subspace of $H^0(\Omega^1_{F \times C}) = H^0(\omega_F) \oplus H^0(\omega_C)$. Since $G$ acts separately on $F$ and $C$, one has $q(S) = g(C/G) + g(F/G) = g(F/G)$. We set, as usual, $\Sigma := S/\sigma$. The second projection $F \times C \to C$ induces a pencil $\Sigma \to C/G = \mathbb{P}^1$ whose
The possible types of involutions of a minimal surface of general type $S$ with $p_g = 0$ and $K^2 = 8$ are described in Theorem 4.4 of [DMP]. In the case of double planes this description can be made more precise, showing in particular that all minimal double planes with $p_g = 0$ and $K^2 = 8$ arise as in Proposition 2.1.

**Theorem 2.2.** Let $(S, \sigma)$ be a minimal double plane with $K^2_S = 8$ and $p_g(S) = 0$.

Then there exist a curve $C$, an hyperelliptic curve $F$ with hyperelliptic involution $\tau$ and a finite group $G$ such that:

a) $G$ acts faithfully on $F$ and $C$ and the diagonal action of $G$ on $F \times C$ is free;

b) $C/G$ and $F/G$ are rational curves;

c) $S = (F \times C)/G$ and $\sigma$ is the involution induced by $\tau \times \text{Id}$.

If we denote by $k$ be the number of isolated fixed points of $\sigma$, then the numerical possibilities for $k$, $g(F)$ and $g(C)$ are the following:

i) $k = 12$, $g(F) = 3$, $|G| = 2(g(C) - 1)$;

ii) $k = 10$, $g(F) = 5$, $|G| = 4(g(C) - 1)$.

**Proof.** By Theorem 4.4 of [DMP], there are the following possibilities for $(S, \sigma)$:

i) $k = 12$, $S$ has a pencil $p: S \to \mathbb{P}^1$ of hyperelliptic curves of genus 3 with 6 double fibres and $\sigma$ restricts to the hyperelliptic involution on the general fibre $F$ of $p$;

ii) $k = 10$, $S$ has a pencil $p: S \to \mathbb{P}^1$ of hyperelliptic curves of genus 5 with 5 double fibres and $\sigma$ restricts to the hyperelliptic involution on the general fibre $F$ of $p$.

Consider case i) first. Let $h: B \to \mathbb{P}^1$ be the double cover branched on the 6 points corresponding to the double fibres of $p$. $B$ is a smooth curve of genus 2. By taking base change and normalization, one gets a diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\tilde{h}} & S \\
\downarrow{\tilde{p}} & & \downarrow{p} \\
B & \xrightarrow{h} & \mathbb{P}^1
\end{array}
$$

where $\tilde{h}: X \to S$ is an étale double cover and $\tilde{p}: X \to B$ is a pencil with general fibre isomorphic to $F$. One has $16 = K^2_X = 8(g(F) - 1)(g(B) - 1)$, hence by [K], the fibration $\tilde{p}$ is smooth and isotrivial. The above diagram shows that $p$ is also isotrivial, that the 6 double fibres have smooth support and they are the only singular fibres of $p$. So, in the terminology of [Se1], $p: S \to \mathbb{P}^1$ is a quasi-bundle and by §1 of [Se1] (cf. also [Se2]) there exist a curve $C$ and a finite group $G$ that acts faithfully on $C$ and $F$ in such a way that the diagonal action on $F \times C$ is free and $S$ is isomorphic to $(F \times C)/G$.

The pencil $p$ is induced by the second projection $F \times C \to C$ and the involution $\sigma$ is induced by $\tau \times \text{Id}$, where $\tau$ is the hyperelliptic involution.
of $F$. Since $S$ is regular, we have $g(F/G) = g(C/G) = 0$ (cf. proof of Proposition 2.1). The formula for $|G|$ follows from Proposition 2.1.

If $S$ is as in case ii), then we let $h: B \to \mathbb{P}^1$ be a $\mathbb{Z}_2^2$-cover branched over the 5 points corresponding to the double fibres of $p$ (the existence of such a cover can be easily shown by using Theorem 1.2 of [Pa]). $B$ is again a smooth curve of genus 2 and one argues exactly as before.

Minimal surfaces of general type with $p_g = 0$, $K^2 = 8$ and non birational bicanonical map are a special instance of double planes. In fact they are precisely the double planes of case i) of Theorem 2.2, as explained below.

**Corollary 2.3.** Let $S$ be a smooth minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 8$.

The bicanonical map $\varphi$ of $S$ is not birational onto its image iff there exist a curve $C$, an hyperelliptic curve $F$ of genus 3 and a group $G$ of order $2(g(C) - 1)$ such that:

a) $G$ acts faithfully on $C$ and on $F$ and the diagonal action on $F \times C$ is free;

b) $S = (F \times C)/G$

In this case $\varphi$ has degree 2 and it is composed with the involution $\sigma$ of $S$ induced by $\tau \times \text{Id}$, where $\tau$ is the bicanonical involution of $F$. The involution $\sigma$ is rational and it has 12 isolated fixed points.

**Proof.** By Theorem 1.1 of [MP1] $\varphi$ is not birational iff it has degree 2, hence $\varphi$ is not birational iff it is composed with an involution $\sigma$ of $S$. By the results of [MP3] this happens iff $\sigma$ has 12 isolated fixed points. Finally, by Theorem 3 of [Xi] if $\varphi$ has degree 2 then $\varphi(S)$ is a rational surface, namely the involution associated with $\varphi$ is rational. The statement now follows from Theorem 2.2.

3. The examples

In this section we describe all the triples $(F, C, G)$ that occur in Theorem 2.2.

We keep the notation introduced in §2 (cf. in particular Theorem 2.2) and we introduce some more. We denote by $p_1: S \to F/G = \mathbb{P}^1$ and $p_2: S \to C/G = \mathbb{P}^1$ the two isotrivial pencils of $S$. The singular fibres of $p_1$ and $p_2$ are multiples of smooth curves, since the action of $G$ on $C \times F$ is free. We have seen in the proof of Theorem 2.2 that the multiple fibres of $p_2$ are 6 double fibres if $\sigma$ has 12 isolated fixed points and 5 double fibres if $\sigma$ has 10 isolated fixed points. We analyze now the multiple fibres of $p_1$ and the fixed set of $\sigma$. Consider $P \in F$ and denote by $[P] \in F/G = \mathbb{P}^1$ the equivalence class of $P$. The fibre of $p_1$ over $[P]$ has multiplicity equal to the order of the stabilizer of $P$ in $G$. The inverse image on $F \times C$ of the fixed locus of $\sigma$ consists of the pairs $(P, Q)$ such that there exists $g \in G$ satisfying $(\tau P, Q) = (gP, gQ)$. When $g$ is the identity we get the divisorial part $R$ of the fixed locus of $\sigma$. The isolated fixed points of $\sigma$ correspond to solutions of the above equation with $P$ not a fixed point of $\tau$. Notice that in this case $g^{-1}\tau$ is a nontrivial element of $\text{Aut}(F)$ that fixes $P$, hence if $\tau \in G$ then the isolated fixed points of $\sigma$ lie on the multiple fibres of $p_1$. We denote by $q_1, q_2: \Sigma \to \mathbb{P}^1$ the pencils induced by $p_1$ and $p_2$. The general fibre of $q_2$ is a smooth rational curve,
since σ is induced by the involution τ × Id on F × C. The inverse image in S of a general fibre of q1 is connected if the hyperelliptic involution τ is in G.

In order to describe the examples, we need to study the possible group actions on an hyperelliptic curve F. Given the group G, the action of G on F descends to an action on \(\mathbb{P}^1 = F/\tau\) that preserves the branch locus \(\Delta\) of the hyperelliptic double cover \(f: F \to \mathbb{P}^1\). We denote by \(\bar{G}\) the image of G in Aut(\(\mathbb{P}^1\)). By identifying the unit sphere of \(\mathbb{R}^3\) with \(\mathbb{P}^1\) via stereographic projection, one obtains an injection \(SO(3) \to Aut(\mathbb{P}^1)\). In particular, the finite subgroups of \(SO(3)\) give finite subgroups of \(Aut(\mathbb{P}^1)\) and, up to conjugacy, every finite subgroup of \(Aut(\mathbb{P}^1)\) arises in this way (cf. [3], Chapter III, or [5], Libro Secondo, Capitolo I, §10). The finite subgroups of \(SO(3)\), up to conjugacy, are the following:

- the cyclic groups \(\mathbb{Z}_n\), generated by the rotation of \(\frac{2\pi}{n}\) around the z-axis;
- the dihedral groups \(D_n\), \(n \geq 2\) (we set \(D_2 = \mathbb{Z}_2^2\)), generated by the rotation of \(\frac{2\pi}{n}\) around the z-axis and by the reflection in the line \(y = z = 0\);
- the group of symmetries of the tetrahedron, which is isomorphic to \(A_4\);
- the group of symmetries of the cube (and of the octahedron), which is isomorphic to \(S_4\);
- the group of symmetries of the dodecahedron (and of the icosahedron), which is isomorphic to \(A_5\).

Let \(d = g(F) + 1\) and let \(p(x_0, x_1)\) be a homogeneous polynomial of degree \(2d\) whose zero locus in \(\mathbb{P}^1\) is \(\Delta\). The curve \(F\) is isomorphic to the curve of equation \(y^2 = p(x_0, x_1)\) in the weighted projective plane \(\mathbb{P}(1, 1, d)\). The hyperelliptic involution \(\tau\) is the restriction to \(F\) of the automorphism of \(\mathbb{P}(1, 1, d)\) given by \((x_0, x_1, y) \mapsto (x_0, x_1, -y)\) and the hyperelliptic double cover \(f: F \to \mathbb{P}^1\) is the restriction of the projection \(\mathbb{P}(1, 1, d) \to \mathbb{P}^1\) defined by \((x_0, x_1, y) \mapsto (x_0, x_1)\). If we denote by \(G_0\) the subgroup of \(Aut(F)\) generated by \(G\) and \(\tau\), then we have a central extension:

\[
(3.1) \quad 0 \to \tau \to G_0 \to \bar{G} \to 1.
\]

Notice that if \(\tau \in G\) then \(G = G_0\), while if \(\tau \notin G\) then \(G\) is mapped isomorphically onto \(\bar{G}\).

Consider now the quotient map \(\phi: SL(2, \mathbb{C}) \to PGL(1, \mathbb{C})\) and set \(H := \phi^{-1}G\), so that we have a surjective map \(H \to \bar{G}\) whose kernel is the subgroup generated by \(-Id\). The group \(H\) acts on the space of homogeneous polynomials of degree \(2d\) and \(p(x_0, x_1)\) is an eigenvector for this action. So there is a homomorphism \(\lambda: H \to \mathbb{C}^*\) defined by \((h^{-1})^*p(x_0, x_1) = \lambda(h)p(x_0, x_1)\). Since the degree of \(p(x_0, x_1)\) is even, \(-Id \in H\) is in the kernel of \(\lambda\) and we have actually defined a character \(\lambda: \bar{G} \to \mathbb{C}^*\). From now on we assume that \(\lambda\) is trivial (this will always be the case in our examples). In this case one can define a homomorphism \(H \to G_0\) by mapping \(h \in H\) the automorphism of \(F\) defined by \((x_0, x_1, y) \mapsto (h(x_0, x_1), y)\). Since \(d\) is even (by Theorem 2.2 either \(d = 4\) or \(d = 6\)), \(-Id\) is in the kernel of \(H \to G_0\) and we have actually defined a homomorphism \(\psi: \bar{G} \to G_0\) that splits the central extension \((3.1)\). We call \(\psi\) the canonical splitting of the sequence \((3.1)\).
Next we examine the fixed locus of the elements of $G_0$. We may write $G_0 = \mathbb{Z}_2 \times G$, identifying $(1,0)$ with $\tau$ and $G$ with the image of $\psi$. Every element $h$ of $G$ has two fixed points $P_1, P_2$ on $\mathbb{P}^1$. If, say, $P_1 \in \Delta$ then both $(0,h)$ and $(1,h)$ fix the inverse image of $P_1$ in $F$. So assume that $P_1, P_2 \notin \Delta$ and write $f^{-1}(P_i) = \{Q^1_i, Q^2_i\}$, $i = 1, 2$. For each $i$, either $(0,h)$ fixes the points $Q^1_i$ and $Q^2_i$ or it exchanges them. So, if the order of $h$ is odd, then $(0,h)$ fixes the 4 points $Q^1_i$ and $(1,h)$ acts freely on $F$. If the order of $h$ is even, let $h_0$ be an element of $SL(2, \mathbb{C})$ that represents $h$ and let $\alpha, \alpha^{-1}$ be the eigenvalues of $h_0$. Let $(a,b)$ be homogeneous coordinates for $P_1$, so that $Q_1^1, Q_1^2$ have coordinates $(a,b, \pm c)$ with $c \neq 0$. Then $(0,h)$ maps $Q_1^1, Q_1^2$ to the points $(aa, ab, \pm c) = (a, b, \pm \alpha^{-d} c)$. So $(0,h)$ fixes $Q_1^1$ and $Q_1^2$ iff $\alpha^d = 1$, and the same is true for $Q_2^1$ and $Q_2^2$. We conclude that if $\alpha^d = 1$ then $(0,h)$ has 4 fixed points and $(1,h)$ acts freely, while if $\alpha^d = -1$ then $(0,h)$ acts freely and $(1,h)$ has 4 fixed points.

We will use the following simple remark:

**Lemma 3.1.** For $n \geq 2$ let the dihedral group $D_n$ of order $2n$ act on $\mathbb{P}^1$ as follows:

$$(x_0, x_1) \mapsto (x_0, e^{2\pi i/n} x_1); \quad (x_0, x_1) \mapsto (x_1, x_0),$$

where $r \in D_n$ is a rotation of order $n$ and $s \in D_n$ is a reflection.

Let $p(x_0, x_1) \in \mathbb{C}[x_0, x_1]$ be a nonzero homogeneous polynomial of degree $2n$. Then the zero locus of $p(x_0, x_1)$ is an orbit of $D_n$ of order $2n$ iff:

$$p(x_0, x_1) = \alpha_0 (x_0^{2n} + x_1^{2n}) + \alpha_1 x_0^n x_1^n, \text{ with } \alpha_0 \neq 0, 2\alpha_0 \pm \alpha_1 \neq 0.$$

**Proof.** It is easy to check that if $p(x_0, x_1)$ is homogeneous of degree $2n$ and it vanishes on an orbit of order $2n$, then it is of the required form. Conversely, the zero locus of a polynomial $p(x_0, x_1) = \alpha_0 (x_0^{2n} + x_1^{2n}) + \alpha_1 x_0^n x_1^n$ is invariant for $D_n$, hence it is a union of orbits. The points of $\mathbb{P}^1$ that have nontrivial stabilizer are: $P_0 = (1, 0), P_1 = (0, 1)$ and the points $Q_k = (1, e^{2\pi i j/n}), j = 0 \ldots 2n - 1$. A polynomial $p(x_0, x_1) = \alpha_0 (x_0^{2n} + x_1^{2n}) + \alpha_1 x_0^n x_1^n$ vanishes at $P_0, P_1$ iff $\alpha_0 = 0$ and it vanishes at $Q_j$ iff $2\alpha_0 + (-1)^j \alpha_1 = 0$. A straightforward computation shows that if $p(x_0, x_1)$ vanishes at one of these points then it has a multiple root there. This remark completes the proof. 

We are now ready to list all the possible types of minimal double planes of general type $(S, \sigma)$ with $p_g = 0$ and $K^2 = 8$. By Theorem 2.2, $S$ is a free quotient $(F \times C)/G$ of the type described in Proposition 2.1.

**Type Ia:** Here $G = \mathbb{Z}_2^3, \bar{G} = D_2 (= \mathbb{Z}_2^2), g(F) = 3, g(C) = 5$ (cf. [MP], Example 4.2). We let $G$ be the subgroup generated by:

$$(x_0, x_1) \mapsto (x_0, -x_1); \quad (x_0, x_1) \mapsto (x_1, x_0)$$

and we take

$$p(x_0, x_1) = x_0^8 + \alpha x_0^6 x_1^2 + \beta x_0^4 x_1^4 + \alpha x_0^2 x_1^6 + x_1^8, \quad \pm 2\alpha + \beta + 2 \neq 0.$$

Arguing as in the proof of Lemma 3.1, one shows that the condition on $\alpha, \beta$ is equivalent to the fact that the zero locus $\Delta$ of $p(x_0, x_1)$ is the union of two orbits of $\bar{G}$ of order 4. We let $G = G_0$ be the inverse image of $\bar{G}$ in $\text{Aut}(F)$. The character $\lambda: \bar{G} \to \mathbb{C}^*$ is trivial and thus, as explained
before, we have a canonical isomorphism of $G$ with $\mathbb{Z}_2 \times \tilde{G}$. The element $(1, 0) \in G$ corresponds to the hyperelliptic involution, hence it has 8 fixed points. The elements $(0, e_1)$, $(0, e_2)$, $(0, e_1 + e_2)$ have 4 fixed points each. The remaining nonzero elements, that we denote by $f_1, f_2, f_3$, act freely and are a set of generators of $G$. By Proposition 2.1 of \cite{Pa}, if we fix distinct points $P_1 \ldots P_6 \in \mathbb{P}^1$ there exists a smooth connected $G$-cover $C \to \mathbb{P}^1$ such that the image of the fixed set of $f_i$ is $P_{2i-1} + P_{2i}$, $i = 1, 2, 3$, and the remaining nonzero elements of $G$ act freely on $C$. The curve $C$ is smooth of genus 5, hence we have the required example.

The pencil $p_1$ has 5 double fibres. The involution $\sigma$ fixes pointwise the two double fibres over the images in $\mathbb{P}^1 = F/G$ of the fixed points of $\tau$, and it has 4 isolated fixed points on each of the remaining three.

**Type Ib:** Here $\tilde{G} = D_4$, $G = \mathbb{Z}_2 \times D_4$, $g(F) = 3$, $g(C) = 9$. We let $D_4$ act on $\mathbb{P}^1$ as in Lemma \ref{lem:3.1} and we take:

$$p(x_0, x_1) = x_0^8 + \alpha x_1^4 x_0^4 + x_1^8, \quad \alpha \pm 2 \neq 0.$$  

By Lemma \ref{lem:3.1} the zero locus $\Delta$ of $p(x_0, x_1)$ is an orbit of $D_4$ of order 8. The character $\lambda: D_4 \to \mathbb{C}^*$ is trivial and we consider the decomposition $G_0 = \mathbb{Z}_2 \times D_4$ given by the canonical splitting of the extension \( (\mathbb{Z}_2, D_2) \).

We now examine the fixed points of the elements of $G = G_0$. The fixed set of the hyperelliptic involution $\tau = (1, 0)$ consists of the 8 Weierstrass points of $F$. By the previous discussion, the elements $(0, sr^i)$, $i = 0 \ldots 3$, have 4 fixed points each and the elements $(1, sr^i)$ act freely on $F$. In addition, $(0, r^j)$ acts freely for $j$ odd and has 4 fixed points for $j$ even, while $(1, r^j)$ has 4 fixed points for $j$ odd and acts freely for $j$ even, $j = 0 \ldots 3$.

Next we construct the curve $C$. Let $E$ be an elliptic curve and let $\eta \in E$ be a point of order 4. We define an action of $D_4$ on $E$ by:

$$z \mapsto z + \eta; \quad z \mapsto -z.$$  

Denote by $q: E \to E/D_4 = \mathbb{P}^1$ the quotient map. Let $h: \mathbb{P}^1 \to \mathbb{P}^1$ be a degree 2 cover branched over two points that are not branch points of $q$. Let $C$ be the fibre product of $q$ and $h$, so that there is a commutative diagram:

$$\begin{array}{ccc}
C & \xrightarrow{\tilde{h}} & E \\
\tilde{q} & & q \\
\mathbb{P}^1 & \xrightarrow{h} & \mathbb{P}^1
\end{array}$$  

To compute the genus of $C$ we apply the Hurwitz formula to the double cover $\tilde{h}: C \to E$, which is branched over 16 points, and we get $g(C) = 9$. The curve $C$ has a natural $\mathbb{Z}_2 \times D_4$-action as a fibred product. The elements that have fixed points for this action are the following: $(1, 0)$ (16 fixed points), $(0, sr^i)$, $i = 0 \ldots 3$ (8 fixed points each). In order to get a free action of $\mathbb{Z}_2 \times D_4$ on $F \times C$, we modify the action on $C$ by composing it with an automorphism $\psi$ of $\mathbb{Z}_2 \times D_4$ that maps the elements $(0, sr^i)$ to elements of the form $(1, sr^2)$ and exchanges $(1, 0)$ and $(1, r^2)$. For instance, one can take $\psi$ to be the automorphism defined by $\psi(1, 0) = (1, r^2)$, $\psi(0, r) = (0, r)$ and $\psi(0, s) = (1, s)$. With this action, the elements of $G$ that do not act freely on $C$ are: $(1, r^2)$ (16 fixed points) and the elements $(1, sr^i)$, $i = 0 \ldots 3$ (8 fixed points each). The multiple fibres of $p_1$ are: the double fibre over the
image in $\mathbb{P}^1 = F/G$ of the fixed points of $\tau$, on which $\sigma$ acts as the identity, a fibre of multiplicity 4 that contains 4 isolated fixed points of $\sigma$, 2 double fibres that contain 4 isolated fixed points of $\sigma$ each.

**Type Ic:** Here $G \cong \bar{G} = S_4$, $g(F) = 3$, $g(C) = 13$. Up to conjugacy, the action of $S_4$ on $\mathbb{P}^1$ corresponds via the inverse of stereographic projection to the subgroup of rotations of the unit sphere in $\mathbb{R}^3$ that preserve the cube of vertices $\frac{1}{\sqrt{3}}(\pm 1, \pm 1, \pm 1)$. We take:

$$p(x_0, x_1) = x_0^8 + (\alpha^4 + \alpha^{-4})x_0^4x_1^4 + x_1^8 \subset \mathbb{P}(1, 1, 4), \quad \alpha = \frac{\sqrt{2}}{\sqrt{3} - 1}.$$  

Namely, $F$ is the double cover of $\mathbb{P}^1$ branched over the points corresponding to the vertices of the cube. Since $S_4$ is not properly contained in any finite subgroup of $PGL(1, \mathbb{C})$, in this case the central extension (3.1) can be rewritten as:

$$0 \to <\tau> \to \text{Aut}(F) \to S_4 \to 1.$$

We claim that the character $\lambda: S_4 \to \mathbb{C}^*$ is trivial. Indeed, it is enough to check that there is a transposition $\delta \in S_4$ with $\lambda(\delta) = 1$. We take $\delta$ to be defined by $(x_0, x_1) \mapsto (ix_1, ix_0)$. The canonical splitting of (3.1) gives a decomposition $\text{Aut}(F) = \mathbb{Z}_2 \times S_4$. Next we examine the fixed points of $\text{Aut}(F)$. The fixed points of $\tau = (1, 0)$ are the 8 branch points of the hyperelliptic cover $f: F \to \mathbb{P}^1$. The element of $\text{Aut}(F)$ that acts on $\mathbb{P}(1, 1, 4)$ by $(x_0, x_1, y) \mapsto (e^{\frac{\pi i}{4}} x_0, e^{\frac{\pi i}{4}} x_1, y)$ is the image via $\psi$ of a cyclic permutation of order 4 of $S_4$ and it acts freely on $F$. Since the elements of order 4 of $S_4$ are all conjugated, this shows that for every $\delta$ of order 4 the element $(0, \delta)$ acts freely on $F$, while $(1, \delta)$ has 4 fixed points. If $\delta$ is the transposition defined by $(x_0, x_1) \mapsto (ix_1, ix_0)$, then $(0, \delta)$ has 4 fixed points and $(1, \delta)$ acts freely on $C$. Since all the transpositions are conjugated in $S_4$, we conclude that the same is true for every transposition $\delta$. If $\delta$ is a 3-cycle then both $(0, \delta)$ and $(1, \delta)$ have 2 fixed points. We let $G \cong S_4$ be the subgroup $\{ (e(\delta), \delta) \delta \in S_4 \} \subset \mathbb{Z}_2 \times S_4$, where $e(\delta)$ is the sign of the permutation $\delta$. Now consider a generic degree 4 map $h: \mathbb{P}^1 \to \mathbb{P}^1$. The branch locus of $h$ consists of 6 distinct points, each corresponding to a simple ramification point. The Galois closure $C \to \mathbb{P}^1$ of $h: \mathbb{P}^1 \to \mathbb{P}^1$ is a $S_4-$cover branched on 6 points such that the only elements of $S_4$ that do not act freely on $C$ are the transpositions. The Hurwitz formula gives $g(C) = 13$ and the diagonal action of $G$ on $F \times C$ is free. Notice that in this case the hyperelliptic involution $\tau$ is not in $G$, and thus the pullback on $S$ of a general fibre of $q_1$ is disconnected. The multiple fibres of $p_1$ are: a triple fibre, that is fixed pointwise by $\sigma$, and two fibres of multiplicity 4 that are interchanged by $\sigma$. The 12 isolated fixed points of $\sigma$ all lie on the smooth fibre of $p_1$ over the image in $F/G$ of the fixed points of the elements $(0, \delta) \in G$, with $\delta$ a transposition.

**Type Id:** Here $G = \mathbb{Z}_2 \times S_4$, $g(F) = 3$, $g(C) = 25$. The curve $F$ is the same as in type Ic. We take $\bar{G} = \text{Aut}(F)$ and we consider again the direct sum decomposition $G = \mathbb{Z}_2 \times S_4$ given by the canonical splitting of (3.1).

Let $p: E \to \mathbb{P}^1$ be a degree 4 cover with the following properties:

a) $E$ is a smooth curve of genus 1;
b) the branch locus of \( p \) consists of points \( P_1 \ldots P_4, Q_1, Q_2 \) such that \( p \) has a simple ramification point over \( P_1 \ldots P_4 \) and it has two simple ramification points over \( Q_1, Q_2 \);

c) the Galois closure \( q: D \to \mathbb{P}^1 \) of \( p \) has Galois group equal to \( S_4 \).

We remark that the existence of such a cover \( p \) can be shown by using the classical Riemann construction. Let \( h: \mathbb{P}^1 \to \mathbb{P}^1 \) be the double cover branched on \( Q_1, Q_2 \) and let \( C \) be the normalization of the fibre product of \( h \) and \( q \). We have a commutative diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{\tilde{h}} & D \\
\downarrow q & & \downarrow q \\
\mathbb{P}^1 & \xrightarrow{h} & \mathbb{P}^1
\end{array}
\]

Applying the Hurwitz formula to \( q \) one obtains \( g(D) = 13 \). The map \( \tilde{h} \) is étale of degree 2. The group actions on \( \mathbb{P}^1 \) and \( D \) induce an action of \( \mathbb{Z}_2 \times S_4 \) on \( C \) such that \( C/(\mathbb{Z}_2 \times \{0\}) = D \) and \( C/(\{0\} \times S_4) = \mathbb{P}^1 \). The elements with nonempty fixed locus are those of the form \((0, \delta)\), with \( \delta \) a transposition, and \((1, \delta)\), with \( \delta \) a double cycle. Each of these elements fixes 16 points. We claim that the curve \( C \) is connected. Assume otherwise. Then \( C \) is the disjoint union of two connected components, isomorphic to \( D \), that are exchanged by \((1,0)\). The subgroup \( G \) of the elements that do not exchange the components of \( C \) is \( \{0\} \times S_4 \) and it should contain all the elements that do not act freely on \( C \), contradicting the above analysis of the fixed points of the elements of \( G \). The Hurwitz formula then gives \( g(C) = 25 \).

In order to get a free action on \( F \times C \), we have modify the given action on \( C \) by composing it with an automorphism \( \psi \) of \( G \) of the form \((a, \delta) \mapsto (a+\epsilon(\delta), \psi_1(\delta))\), where \( \psi_1 \) is an automorphism of \( S_4 \) and \( \epsilon(\delta) \) is the sign of \( \delta \). For instance one can take \( \psi_1 = Id \).

The multiple fibres of \( p_1 \) are: a 6--tuple fibre, which is fixed by \( \sigma \) pointwise, a double fibre containing 8 isolated fixed points of \( \sigma \) and a 4--tuple fibre containing 4 isolated fixed points of \( \sigma \).

**Type II:** This is similar to type Ic. Here \( g(F) = 5 \), \( g(C) = 16 \) and \( G \cong \tilde{G} = A_5 \). The group \( A_5 \) acts on the dodecahedron (and the icosahedron). Identifying as usual the sphere with \( \mathbb{P}^1 \) via stereographic projection, we get an action of \( A_5 \) on \( \mathbb{P}^1 \). The special orbits are: one orbit of order 12, one orbit of order 30 and one orbit of order 20. Let \( p(x_0, x_1) \) be a homogeneous polynomial of degree 12 whose zeros on \( \mathbb{P}^1 \) are the points of the orbit of order 12 and let \( F \subset \mathbb{P}(1,1,6) \) be the hyperelliptic curve of genus 5 defined by \( y^2 = p(x_0, x_1) \). Since \( A_5 \) is not properly contained in any finite subgroup of \( \text{Aut}(\mathbb{P}^1) \), in this case the central extension \([3,1]\) can be rewritten:

\[
0 \to < \tau > \to \text{Aut}(F) \to A_5 \to 1.
\]

As we have already remarked, the character \( \lambda: A_5 \to \mathbb{C}^* \) is necessarily trivial. Thus the above sequence is split and one has a decomposition \( \text{Aut}(F) = \mathbb{Z}_2 \times A_5 \), which is unique. Now we examine the fixed points of the elements of \( \text{Aut}(F) \). The hyperelliptic involution \( \tau = (1,0) \) has 12 fixed points. If \( \delta \) is a 5--cycle in \( A_5 \), then both \((0, \delta)\) and \((1, \delta)\) have 2 fixed
points. If \( \delta \) is a 3–cycle, then \((0, \delta)\) has 4 fixed points and \((1, \delta)\) acts freely. Now, applying the Hurwitz formula to the covering \( F \to \mathbb{P}^1 = F/A_5 \), one sees that the remaining elements of \( A_5 \) act freely. It follows that the elements \((1, \delta)\), with \( \delta \) a double cycle, have 4 fixed points each. Now let \( C \to \mathbb{P}^1 \) be a Galois cover with Galois group \( A_5 \), branched over 5 points and such that the only elements that do not act freely on \( C \) are the double cycles (the existence of such a cover can be shown using the Riemann construction). Then the genus of \( C \) is 16 by the Hurwitz formula and the diagonal action of \( G = A_5 \) on \( F \times C \) is free, as required. As in type Ic, the hyperelliptic involution \( \tau \) is not in \( G \), and thus the pullback on \( C \) of a general fibre of \( q_1: \Sigma \to \mathbb{P}^1 \) is disconnected. The multiple fibres of \( p_1: S \to \mathbb{P}^1 \) are: a 5–tuple fibre that is fixed by \( \sigma \) pointwise and 2 triple fibres that are exchanged by \( \sigma \). The 10 isolated fixed points all belong to the fibre of \( p_1 \) over the image of a point \( P \in F \) whose stabilizer in \( \text{Aut}(F) \) is generated by an element \((1, \delta)\) with \( \delta \) of order 2.

4. The classification

We are now ready to state and prove our main result. We keep all the notation of the previous sections.

**Theorem 4.1.** Let \((S, \sigma)\) be a minimal double plane of general type with \( p_g(S) = 0 \) and \( K_S^2 = 8 \). Then \((S, \sigma)\) belongs to one of the types Ia, Ib, Ic, Id, II described in [4].

We say that a minimal double plane of general type \((S, \sigma)\) with \( K^2 = 8 \) and \( p_g = 0 \) is of type I if it belongs to one of the types Ia, Ib, Ic, Id. By Theorem 2.2 and Theorem 4.1, the double planes of type I are characterized by the fact that they have 12 isolated fixed points and the double planes of type II by the fact that they have 10 isolated fixed points.

Before proving Theorem 4.1 we remark the following:

**Corollary 4.2.** Let \((S, \sigma)\) be a minimal double plane of general type with \( K_S^2 = 8 \) and \( p_g(S) = 0 \). Then:

i) \( \sigma \) is the only rational involution of \( S \);

ii) \((S, \sigma)\) belongs to exactly one of the types Ia, \ldots, Id, II;

iii) the bicanonical map of \( S \) is birational iff \((S, \sigma)\) is of type II.

**Proof.** Assume by contradiction that \( S \) has two distinct rational involutions \( \sigma_1 \) and \( \sigma_2 \). By Theorem 2.2 to each involution \( \sigma_i \) there corresponds an isomorphism of \( S \) with a quotient \((F_i \times C_i)/G_i\), where \( F_i \) is hyperelliptic and \( \sigma_i \) is induced by \( \tau_i \times Id \), \( \tau_i \) being the hyperelliptic involution of \( F_i \). The projections of \( F_i \times C_i \) onto the two factors induce pencils \( p_{i1}, p_{i2}: S \to \mathbb{P}^1 \), \( i = 1, 2 \). A base point free pencil of \( S \) is determined uniquely by the span of the cohomology class of one of its fibres, which is an isotropic subspace of \( H^2(S, \mathbb{Q}) \). Since \( h^2(S, \mathbb{Q}) = 2 \), \( S \) has exactly two free pencils and thus we have \( \{p_{11}, p_{12}\} = \{p_{21}, p_{22}\} \). Since the involutions \( \sigma_1 \) and \( \sigma_2 \) are different, we have \( p_{11} = p_{21}, p_{12} = p_{22} \), or, equivalently, \( C_1 = F_2 \) and \( C_2 = F_1 \). Looking at the pair \((g(F), g(C))\) for the types listed in Theorem 4.1 one sees that this cannot happen. This proves i). Statement ii) follows from i) and from Theorem 4.1 by the same argument, since the pair \((g(F), g(C))\) is different.
for each type of double planes. Now statement iii) follows from i), ii) and Corollary 2.3.

Proof of Theorem 4.1. The proof is long and it is divided in several steps.

By Theorem 2.2 there exist a curve $C$, an hyperelliptic curve $F$ of genus 3 or 5 and a group $G$ that acts faithfully on $C$ and $F$ such that $S = (F \times C)/G$ and $\sigma$ is induced by the involution $\tau \times \text{Id}$ of $F \times C$, where $\tau$ is the hyperelliptic involution of $F$. We denote by $A \subset G$ (resp. $B \subset G$) the subset of elements $\neq 1$ that do not act freely on $F$ (resp. $C$).

**Step 1.** The following conditions are satisfied:

a) $A \cap B$ is empty;

b) $A$ and $B$ are a union of conjugacy classes of $G$;

c) both $A$ and $B$ generate $G$ (in particular both $A$ and $B$ are nonempty);

d) the elements of $B$ have order 2.

Condition a) is equivalent to the fact that the diagonal action of $G$ on $F \times C$ is free (cf. Theorem 2.2, a)). Condition b) follows from the definition of $A$ and $B$. Condition c) follows from the fact that by Theorem 2.2 b) the curves $F/G$ and $C/G$ are rational. In fact, if $H$ is the subgroup of $G$ generated by $B$, then $C/H \rightarrow C/G = \mathbb{P}^1$ is a connected étale cover, hence $H = G$. The same argument shows that $G$ is generated by $A$. To prove d), we recall that for a point $P \in C$ the multiplicity of the fibre of $p_2: S \rightarrow C/G$ over $[P] \in C/G$ is equal to the order of ramification at $P$ of the quotient map $C \rightarrow C/G$, which in turn is equal to the order of the stabilizer of $P$ in $G$. Since all the multiple fibres of $p$ are double (cf. proof of Theorem 2.2) $B$ consists of elements of order 2.

**Step 2.** $G$ is not one of the following groups:

- $\mathbb{Z}_n$, $n \geq 2$;
- $D_n$, $n \geq 2$

By Step 1 c), d), $G$ is generated by elements of order 2. So if $G$ is cyclic, then it is equal to $\mathbb{Z}_2$, but then $G$ does not have two disjoint sets of generators, contradicting Step 1 a), c). The same argument rules out $D_2 = \mathbb{Z}_2^2$. So assume that $G$ is a dihedral group $D_n$, $n \geq 3$. Looking at the conjugacy classes of the elements of order 2 of $D_n$, one sees that conditions b), c) and d) of Step 2 imply that $B$ contains all the reflections. Then $A \subset D_n \setminus B$ does not generate $D_n$, a contradiction to Step 1 c).

As in §3, we denote by $\bar{G}$ the finite subgroup of automorphisms of $\mathbb{P}^1 = F/\tau$ induced by $G$.

**Step 3.** The group $\bar{G}$ is one of the following:

- $D_n$, $n \geq 2$; $S_4$; $A_5$.

This follows by Step 2 and by the classification of the finite subgroups of $PGL(1, \mathbb{C})$ that we have recalled at the beginning of §3 ($A_4$ is excluded since it is not generated by elements of order 2, contradicting Step 2 d)).

We denote by $\bar{A}$, $\bar{B}$ the images of $A$, $B$ in $\bar{G}$. We notice that by Step 2 $\bar{A}$ and $\bar{B}$ are sets of generators of $\bar{G}$ and are stable under conjugacy. The elements of $\bar{B}$ have order 2.
Step 4. If $\tilde{G} = D_n$, $n \geq 2$, then $\tilde{G}$ acts freely on the branch locus $\Delta$ of $F \to F/\tau$.

By Step 3, if $\tilde{G} = D_n$ then $G$ is not isomorphic to $\tilde{G}$ and thus the exact sequence (3.1) becomes:

$$0 \to < \tau > \to G \to D_n \to 1.$$ 

Arguing as in the proof of Step 3, one sees that $\tilde{B}$ contains all the reflections of $D_n$. By Step 3, if $\tilde{G} = D_n$ then $G$ is not isomorphic to $\tilde{G}$ and thus the exact sequence (3.1) becomes:

$$0 \to < \tau > \to G \to D_n \to 1.$$ 

Assume by contradiction that $\tilde{G} = D_n$ act freely on $\Delta$. Assume now that $\tilde{P} \in \Delta$ is fixed by a rotation $r$ of $D_n$. We may assume that $r$ has order $n$. Let $r_1, r_2 = \tau r_1 \in G$ be the lifts of $r$. The stabilizer $H$ of $\tilde{P}$ in $G$ is cyclic, since $G$ acts faithfully on the smooth curve $F$, and it is generated by $r_1, r_2$ and $\tau$. Hence, up to exchanging $r_1$ and $r_2$, we may assume that $r_1$ generates $H$ and, in particular, $\tau = r_1^n$. Let $s \in D_n$ be a reflection and let $s_1 \in G$ be an element of order 2 that lifts $s$. Since $sr_1s = r_1^{-1}$, we either have $s_1r_1s_1 = r_1^{-1}$ or $s_1r_1s_1 = \tau r_1^{-1} = r_1^{n-1}$. In the former case, $G$ is isomorphic to $D_{2n}$, contradicting Step 3. The latter case can occur only for $n$ even, since for $n$ odd $r^{n-1}$ does not generate $H$. In this case the elements $sr_1^i$ and $sr_1^i\tau$ have order 4 for $i$ odd and thus they are not contained in $B$, contradicting the fact that $B$ contains all the reflections.

Step 5. If $\tilde{G} = D_n$, then $n$ is even.

By Step 3, if $\tilde{G} = D_n$ then $G$ and $\tilde{G}$ are not isomorphic and the exact sequence (3.1) gives:

$$0 \to < \tau > \to G \to D_n \to 1.$$ 

Assume by contradiction that $n$ is odd. Then a rotation $r \in D_n$ of order $n$ can be lifted to $r' \in G$ of order $2n$. Arguing as in the proof of Step 3, one shows that $G$ is dihedral, contradicting Step 3.

As we have already remarked, the curve $F$ is defined by an equation of the form $y^2 = p(x_0, x_1)$ in the weighted projective plane $\mathbb{P}(1,1,d)$, where $d = g(F) + 1$ and $p$ is a homogeneous polynomial of degree $2d$.

Step 6. Up to isomorphism, there are the following possibilities for $\tilde{G}$, $g(F)$ and $p(x_0, x_1)$:

i) $\tilde{G} = D_2$, $g(F) = 3$, $p(x_0, x_1) = x_0^8 + \alpha x_0^6 x_1^2 + \beta x_0^4 x_1^4 + \alpha x_0^2 x_1^6 + x_1^8$, $2 \pm 2\alpha + \beta \neq 0$;

ii) $\tilde{G} = D_2$, $g(F) = 5$, $p(x_0, x_1) = x_0^{12} + \alpha x_0^{10} x_1^2 + \beta x_0^8 x_1^4 + \gamma x_0^6 x_1^6 + \beta x_0^4 x_1^8 + \alpha x_0^2 x_1^{10} + x_1^{12}$, $2 \pm 2\alpha + 2\beta + \gamma \neq 0$, $2 - 2\alpha - 2\beta + \gamma \neq 0$;

iii) $\tilde{G} = D_4$, $g(F) = 3$, $p(x_0, x_1) = x_0^8 + \alpha x_0^4 x_1^4 + x_1^8$, $2 \pm \alpha \neq 0$;

iv) $\tilde{G} = D_6$, $g(F) = 5$, $p(x_0, x_1) = x_0^{12} + \alpha x_0^6 x_1^6 + x_1^{12}$, $2 \pm \alpha \neq 0$;

v) $\tilde{G} = S_4$, $g(F) = 3$, $p(x_0, x_1) = x_0^8 + (\alpha^4 + \alpha^{-4}) x_0^4 x_1^4 + x_1^8 \subset \mathbb{P}(1,1,4)$

where $\alpha = -\sqrt[4]{-2}$ (the zeroes of $p$ are the points of the orbit of order 8 of $S_4$).
vi) \( \bar{G} = A_5 \), \( g(F) = 5 \), \( p(x_0, x_1) \) is a polynomial of degree 12 whose zeroes are the elements of the orbit of order 12 of \( A_5 \).

In all cases, the exact sequence (3.1) is split.

We recall that the zero set \( \Delta \) of \( p(x_0, x_1) \) is the branch locus of the hyperelliptic double cover \( f: F \to \mathbb{P}^1 \) and it is invariant under the action of \( \bar{G} \). If \( \bar{G} = A_5 \), then we have necessarily case vi), since the smallest orbit of \( A_5 \) on \( \mathbb{P}^1 \) has 12 elements and there is only one such orbit. If \( \bar{G} = S_4 \), then a similar argument shows that we have case v). By Step 3, the only remaining possibility is that \( \bar{G} \) is a dihedral group \( D_n \). If this is the case, then Step 4 and Step 5 imply that the only possibilities for \( \bar{G} \), and \( g(F) \) are i)–iv). The fact that \( p(x_0, x_1) \) is of the form stated above follows from Lemma 3.1 in cases iii), iv), it has been proven while describing type Ib for case i) and it can be proven in the same way for case ii). The fact that the exact sequence (3.1) is split has already been checked in \( \S \) 2 while describing the various types of double planes except for case ii), where one can use exactly the same argument.

**Step 7.** If \( \bar{G} = D_2 \), then \( (S, \sigma) \) is of type Ia.

Up to a suitable choice of homogeneous coordinates, we may assume that the action of \( \bar{G} \) on \( \mathbb{P}^1 \) is the one described for surfaces of type Ia. By Step 2, the groups \( G \) and \( \bar{G} \) are not isomorphic, hence (3.1) gives the exact sequence:

\[
0 \to <\tau> \to G \to D_2 \to 1
\]

which is split by Step 6. The curve \( F \) is as in case i) or ii) of Step 6 and, with respect to the canonical splitting \( G \cong \mathbb{Z}_2 \times D_2 \), each of the nonzero elements of \( \{0\} \times D_2 \) has 4 fixed points on \( F \). Thus \( A \) contains \( \tau \) and three elements \( e_1, e_2, e_3 \) that map to the three nontrivial elements of \( D_2 \). By conditions a) and c) of Step 5, \( B \) contains the remaining nonzero elements \( f_1, f_2, f_3 \) of \( G \), which are a set of generators. We consider the quotient map \( h: C \to C/G = \mathbb{P}^1 \). By Theorem 2.1 of [Pa], the image via \( h \) of the fixed set of \( f_i \) has cardinality divisible by 2 for \( i = 1, 2, 3 \). Then the Hurwitz formula implies \( g(C) \equiv 1 \pmod{4} \). Hence case ii) of Step 6 is excluded and we have case i) of Step 6. Now, using again Theorem 2.1 of [Pa], it is easy to check that we have a double plane of type Ia.

**Step 8.** If \( \bar{G} = D_n, n \geq 3 \), then \( (S, \sigma) \) is of type Ib.

Here we have either case iii) or case iv) of Step 6. We will show that case iii) corresponds to type Ib and that iv) does not occur.

We consider case iii) first. By Step 6 the groups \( G \) and \( \bar{G} \) are not isomorphic, hence (3.1) gives the exact sequence:

\[
0 \to <\tau> \to G \to D_4 \to 1
\]

which is split by Step 6. The curve \( F \) is the same as in type Ib and \( g(C) = 9 \). Using the canonical splitting \( G \cong \mathbb{Z}_2 \times D_4 \) one sees that \( A \) contains \( \tau = (1,0) \) and \((0, sr^i), i = 0 \ldots 3 \) (cf. the description of type Ib in [Pa]). It follows by Step 6 that \( B = \{(1, r^2), (1, sr^i), i = 0 \ldots 3 \} \). Let \( \psi: C \to C/G = \mathbb{P}^1 \) be the projection onto the quotient and let \( \nu \) (resp. \( \mu \)) be the number of
branch points of $\psi$ that are images of the fixed points of $(1,r^2)$ (resp. of the elements $(1, sr^i)$). So $\nu, \mu > 0$ and the Hurwitz formula gives $\nu + \mu = 6$. Set $E := C/(1,r^2)$ and consider the quotient map $C \to E$: the Hurwitz formula gives $15 = 2(2g(E) - 2) + 8\nu$ and thus $\nu \leq 2$. Let $H$ be the subgroup generated by $(1,s)$ and $(0,r)$ and set $D := C/H$. The subgroup $H$ is isomorphic to $D_4$ and the Hurwitz formula gives: $16 = 8(2g(D) - 2) + 8\mu$. Since $\mu = 6 - \nu \geq 4$, we have: $g(D) = 0$, $\mu = 4$, $\nu = 2$ and $g(E) = 1$. The group $G/ \langle (1,r^2) \rangle$ is isomorphic to $D_4$ and it acts on $E$ in such a way that only the reflections have fixed points. It follows that the action on $E$ must be the one described for type Ib. In addition, we have a commutative diagram:

\[ \begin{array}{ccc}
C & \xrightarrow{\hat{h}} & E \\
\downarrow{\hat{q}} & & \downarrow{\hat{q}} \\
C/H = \mathbb{P}^1 & \xrightarrow{h} & E/D_4 = \mathbb{P}^1
\end{array} \]

The branch points of $h$ are the images of the points of $C$ fixed by $(1,r^2)$ and the branch points of $q$ are the images of the points fixed by the elements of type $(1, sr^i)$. In particular, $q$ and $h$ have no common branch point and thus their fibre product is smooth and connected. It follows that $C$ is isomorphic to the fibre product of $q$ and $h$ and that this is type Ib.

Now we consider case iv) of Step 6. By Step 2 the groups $G$ and $\bar{G}$ are not isomorphic, hence (3.1) gives the exact sequence:

\[ 0 \to < \tau > \to G \to D_6 \to 1 \]

which is split by Step 6. Arguing as in the description of type Ib, one sees that with respect to the canonical decomposition $G = \mathbb{Z}_2 \times D_6$ the elements of order 2 that act freely on $F$ are the following: $(0, sr^i), i = 0 \ldots 5$ and $(0, r^3)$. Since these elements do not generate $G$, this case does not occur by Step 6.

**Step 9.** In case v) of Step 6, $(S, \sigma)$ is either of type Ic or of type Id.

In this case, the curve $F$ and its automorphism group have been analyzed in detail in the description of surfaces of type Ic in §3. In particular, it has been shown that, with respect to the canonical splitting $G = \mathbb{Z}_2 \times S_4$, the elements of order 2 that act freely on $F$ are those of the form $(1, \delta)$ with $\delta$ of order 2. Then, by Step 6, we have two possibilities:

a) $G = \text{Aut}(F) \cong \mathbb{Z}_2 \times S_4$ and $B$ consists of all the elements $(1, \delta)$ with $\delta$ of order 2.

b) $G = \{(\epsilon(\delta), \delta)\} \cong S_4$, where $\epsilon$ denotes the sign of $\delta$, and $B$ consists of all the transpositions.

We consider case b) first. One has $g(C) = 13$ and by the Hurwitz formula the branch locus of the quotient map $C \to C/S_4 = \mathbb{P}^1$ consists of 6 points. Thus each transposition of $S_4$ fixes 12 points. Let $H$ be a subgroup of $S_4$ isomorphic to $S_3$. By the Hurwitz formula, the curve $C/H$ is rational. In addition, the induced map $C/H = \mathbb{P}^1 \to C/S_4 = \mathbb{P}^1$ is a degree 4 morphism with 6 simple branch points. This shows that case b) corresponds to type Ic.
Now assume we are in case a) and denote by $\nu$ the number of branch points that are the images of the fixed points of the elements $(1, \delta)$ with $\delta$ a double cycle and by $\mu$ the number of branch points that are the images of the fixed points of the elements $(1, \delta)$ with $\delta$ a transposition. So $(1, \delta)$ has $8\nu$ fixed points if $\delta$ is a double cycle and it has $4\mu$ fixed points if $\delta$ is a transposition. We have $\nu, \mu > 0$ and the Hurwitz formula gives again $\nu + \mu = 6$. Denote by $H$ the subgroup of $G$ generated by the elements $(1, \delta)$ with $\delta$ a double cycle. $H$ is isomorphic to $\mathbb{Z}_3^2$ and the Hurwitz formula applied to the quotient map $C \to C' = C/H$ gives $48 = 8(2g(C') - 2) + 24\nu$, which implies: $g(C') = 1$, $\nu = 2$, $\mu = 4$. Denote by $P_1 \ldots P_4 \in \mathbb{P}^1$ (respectively $Q_1, Q_2$) the branch points corresponding to the elements $(1, \delta)$ with $\delta$ a transposition (respectively a double cycle). Denote by $K$ the subgroup of $G$ generated by the elements $(1, \delta)$, where $\delta$ is a transposition. $K$ is isomorphic to $S_4$ and the quotient curve $C/K$ is rational by the Hurwitz formula. The quotient map $h : C/K = \mathbb{P}^1 \to C/G = \mathbb{P}^1$ is a degree 2 cover branched over $Q_1$ and $Q_2$. We set $D := C/\mathbb{Z}_2 \times \{0\}$. The quotient map $\tilde{h} : C \to D$ is an étale double cover and thus $D$ has genus 13. Summing up, we have a commutative diagram:

$$
\begin{array}{ccc}
C & \xrightarrow{\tilde{h}} & D \\
\downarrow \tilde{q} & & \downarrow q \\
\mathbb{P}^1 & \xrightarrow{h} & \mathbb{P}^1
\end{array}
$$

(4.2)

Thus the curve $C$ is obtained from $q$ and $h$ by base change and normalization. To show that this is type Id, we consider the action of $G/\mathbb{Z}_2 \times \{0\} \cong S_4$ on $D$. The quotient map $q : D \to C/G = \mathbb{P}^1$ is branched over $P_1 \ldots P_4, Q_1, Q_2$. If $K_1$ is a subgroup of $S_4$ isomorphic to $S_3$ and we write $E := D/K_1$, then $E$ has genus 1 and the induced map $E \to C/G = \mathbb{P}^1$ is a degree 4 cover with one simple ramification point above $P_1 \ldots P_4$ and 2 simple ramification points over $Q_1$ and $Q_2$, whose Galois closure is $q : D \to \mathbb{P}^1$, as required.

**Step 10. In case vi) of Step 4, (S, $\sigma$) is of type II**

As explained in the description of type II, there is a unique splitting $\text{Aut}(F) = \mathbb{Z}_2 \times A_5$ and the elements of order 2 that act freely on $F$ are the elements $(0, \delta)$, where $\delta$ is a double cycle of $A_5$. The subgroup generated by these is $A_5$, thus we have $G = A_5$ and $g(C) = 16$. Arguing as in the previous steps, it is easy to check that this is type II.

5. **Plane models**

A *plane model* of a double plane $(S, \sigma)$ is a finite degree 2 morphism $X \to \mathbb{P}^2$ such that $X$ is a normal surface and there exists a commutative diagram:

$$
\begin{array}{ccc}
S & \longrightarrow & X \\
\downarrow & & \downarrow \\
\Sigma = S/\sigma & \longrightarrow & \mathbb{P}^2
\end{array}
$$

such that the horizontal arrows denote birational maps. It is clear that a plane model is by no means unique, since one can compose the map $X \to \mathbb{P}^2$ with a map $\mathbb{P}^2 \to \Sigma = S/\sigma$. 

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\( \mathbb{P}^2 \) with a birational transformation of \( \mathbb{P}^2 \), solve the indeterminacy of the resulting map and consider the Stein factorization, thus obtaining a different plane model.

In this section we compute plane models for the minimal double planes of general type with \( K^2 = 8 \) and \( p_g = 0 \) using the classification of \( \S 4 \). It turns out that the double planes of type I can all be realized using a construction suggested by Du Val in [DV] and the double planes of type II can be obtained in a similar way.

A plane model \( X \to \mathbb{P}^2 \) is determined by its branch curve \( B \), which is a reduced curve of even degree. The invariants \( \chi \) and \( K^2 \) of the minimal resolution of \( X \) can be computed in terms of the degree of \( B \) and of the type of its singularities, while the values of \( p_g \) and \( q \) depend also on the mutual positions of the singularities of \( B \) (cf. [BPV], Ch. III, \( \S 7 \)). In our case the "expected number" of parameters for \( B \) is negative and the singularities of \( B \) satisfy a general position property (cf. Theorem 5.1, iv) and Theorem 5.2, iv)). Therefore it seems very difficult to construct these plane models directly, let alone classify them.

We call a singular point of a plane curve simple if it is solved by blowing up the singular point in the plane once. A point of type \((m, m)\) is a point of multiplicity \( m \) with an infinitely near simple point also of multiplicity \( m \).

The following theorem describes the Du Val plane model for type I.

**Theorem 5.1.** Let \((S, \sigma)\) be a minimal double plane of general type with \( K^2 = 8 \) and \( p_g = 0 \).

If \((S, \sigma)\) is of type I, then there is a plane model \( X \to \mathbb{P}^2 \) with branch curve \( B \) such that:

i) \( B = C_{16} + L_1 + \cdots + L_6 \), where \( C_{16} \) has degree 16 and \( L_1 \ldots L_6 \) are distinct lines;

ii) the singularities of \( C_{16} \) are:
   - a singular point \( P \) of multiplicity 8 such that \( P \in L_1 \ldots L_6 \);
   - 6 points \( R_i \) of type \((4, 4)\) such that \( R_i \in L_i \) and \( L_i \) is tangent to \( C_{16} \) at \( R_i \), \( i = 1 \ldots 6 \);

iii) the singularities of \( B \) are solved by blowing up \( P, R_1 \ldots R_6 \) and the points \( S_i \) infinitely near to \( R_i \) in the direction of \( L_i \), \( i = 1 \ldots 6 \);

iv) there is no conic containing \( R_1 \ldots R_6 \).

One or two of the \( R_i \) can be infinitely near to \( P \). The curve \( C_{16} \) has two irreducible components of degree 8 if \((S, \sigma)\) is of type Ia and it is irreducible otherwise.

Conversely, given a plane curve \( B \) as in i)–iv), the double cover \( X \to \mathbb{P}^2 \) branched on \( B \) is a plane model of a minimal double plane of general type with \( K^2 = 8 \) and \( p_g = 0 \), of type I.

**Proof.** Denote by \( \tilde{S} \) the blow-up of \( S \) at the isolated fixed points of \( \sigma \), by \( \tilde{\sigma} \) the involution of \( \tilde{S} \) induced by \( \sigma \), and by \( Y \) the quotient surface \( \tilde{S}/\tilde{\sigma} \). \( Y \) is the minimal resolution of \( \Sigma = S/\sigma \).

We denote by \( F_e \) the geometrically ruled surface \( \text{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(e)) \), \( e \geq 0 \), by \( C_\infty \) the class of a section of \( F_e \) with self-intersection \(-e\) and by \( \Phi \) the class of a ruling. Arguing as in the proof of Theorem 4.4 of [DMP], one can
show that $Y$ is obtained from a surface $F_e$ by blowing up 6 pairs of points $P_i, P'_i$ such that:

a) $P_i, P'_j$ lie on different rulings of $F_e$ if $i \neq j$;

b) for every $i = 1 \ldots 6$ the point $P'_i$ is infinitely near to $P_i$ in the direction of $\Phi$;

c) the pull back of the fibration $Y \to F_e \to \mathbb{P}^1$ is the hyperelliptic fibration of genus $3$. $P_2 : S \to \mathbb{P}^1$.

The singular fibres of $Y \to \mathbb{P}^1$ consist of a $-1$–curve counted twice (the inverse image of $P'_i$) and of two disjoint $-2$–curves, contained in the branch locus of $\tilde{S} \to Y$. So each singular fibre of $\tilde{S} \to \mathbb{P}^1$ contains a pair of disjoint $-1$–curves, one of which is contracted by the map $h : \tilde{S} \to F_e$ while the other one is mapped to a ruling of $F_e$. If we denote by $B'$ the branch locus of $h$, then $B' = D + \Phi_1 + \cdots + \Phi_6$, where $\Phi_1 \ldots \Phi_6$ are distinct rulings of $F_e$ and $D$ is effective. In the proof of Theorem 4.4 of [DM1] it is shown that $D$ is linearly equivalent to $8C_\infty + (12 + 4e)\Phi$. For $i = 1 \ldots 6$ $D$ has a $(4, 4)$ point at $P_i \in \Phi_i$ with the same tangent as $\Phi_i$ and $P_1 \ldots P_6$ are the only singularities of $B'$.

Notice that the strict transform of $B'$ on $Y$ is smooth by construction. The curve $D \subset F_e$ is the image of the strict transform in $\tilde{S}$ of the divisorial part of the fixed locus of $\sigma$ on $S$, hence by Theorem [3] it has two irreducible components if $(S, \sigma)$ is of type Ia and it is irreducible otherwise (cf. the description of the different types of double planes in [3]).

The map $Y \to F_e$ is not determined uniquely: two such maps $Y \to F_e$ and $Y \to F_e'$ are related by elementary transformations centered at some the points $P_i$ (an elementary transformation of $F_e$ centered at $P$ of consists in blowing up $P$ and then blowing down the strict transform of the ruling through $P$). If $e = 0$, then we can perform an elementary transformation at, say, $P_i$ and get $e = 1$.

The fact that the points $P_i$ are the only singularities of $B'$ and that they have “vertical” tangent implies that no section of $F_e$ is contained in $D$. Thus we have $0 \leq C_\infty D = 12 - 4e$, namely $e \leq 3$ and at least $3 + e$ of the $P_i$ are not in $C_\infty$. Thus, if $e > 1$, one can perform an elementary transformation at such a point $P_i$, replace $e$ by $e - 1$ and finally arrange that $e = 1$.

If $e = 1$, then the required plane model can be obtained by composing $\tilde{S} \to F_1$ with the birational morphism $F_1 \to \mathbb{P}^2$ that contracts the exceptional curve of $F_1$ and considering the Stein factorization $\tilde{S} \to X \to \mathbb{P}^2$. We denote by $P$ the image of the exceptional curve of $F_1$, by $R_i$ the image of $P_i$, by $L_i$ the image of $\Phi_i$ and by $C_{16}$ the image of $D$, which is a curve of degree 16. Since $C_\infty D = 8$, $P$ is a point of multiplicity 8 of $C_{16}$ and at most two of the $R_i$ are infinitely near to $P$. The branch locus of $X \to \mathbb{P}^2$ is the image of $B'$ and it is easy to check that it has the stated properties. In particular, if $(S, \sigma)$ is of type Ia, then the two components of $D$, being numerically equivalent to one another, are both mapped to curves of degree 8.

It is well known that the singularities of a double cover of a smooth surface can be solved by repeatedly blowing up the base of the cover at the singularities of the branch curve and taking base change and normalization, and there are formulas for the numerical invariants of such a resolution (cf.
for instance [BPV], Ch. III, §7). Applying this method to $X \to \mathbb{P}^2$ one sees that a resolution of $X$ has nonzero geometric genus iff there exists a curve $C_8 \subset \mathbb{P}^2$ of degree 8 with multiplicity $\geq 6$ at $P$ and with a double point at $R_i$ such that $L_i$ is tangent to $C_8$ at $R_i$ for $i = 1 \ldots 6$. The intersection multiplicity of such a $C_8$ with a line $L_i$ is $\geq 9$, hence $C_8$ must be equal to $L_1 + \ldots + L_6 + C_2$, where $C_2$ is a conic containing $R_1 \ldots R_6$. This proves iv).

A standard computation using again the formulas for double covers of [BPV] shows that if $B$ satisfies properties i)–iv) then the double cover $X \to \mathbb{P}^2$ is plane model of a minimal double plane of general type $(S, \sigma)$ with $p_g = 0$ and $K_2 = 8$ (beware, the resolution of $X$ as a double cover is not minimal!). In addition, the pencil of lines through $P$ pulls back on $S$ to a hyperelliptic pencil of genus 3, hence $(S, \sigma)$ is of type I by Theorem 2.2 and Theorem 4.1.

The next theorem describes a plane model for double planes of type II which is quite similar to the Du Val model for type I.

**Theorem 5.2.** Let $(S, \sigma)$ be a minimal double plane of general type with $K_2 = 8$ and $p_g = 0$.

If $(S, \sigma)$ is of type II, then there is a plane model $X \to \mathbb{P}^2$ with branch curve $B$ such that:

i) $B = C_{21} + L_1 + \ldots + L_5$, where $C_{21}$ has degree 21 and $L_1 \ldots L_5$ are distinct lines;

ii) the singularities of $C_{21}$ are:
   - a singular point $P$ of multiplicity 9 such that $P \in L_1 \ldots L_5$;
   - 5 points $R_i$ of type $(6, 6)$ such that $R_i \in L_i$ and $L_i$ is tangent to $C_{21}$ at $R_i$, $i = 1 \ldots 5$;

iii) the singularities of $B$ are solved by blowing up $P$, $R_1 \ldots R_5$ and the points $S_i$ infinitely near to $R_i$ in the direction of $L_i$, $i = 1 \ldots 5$;

iv) there is no curve $C_5$ of degree 5 containing $P$ and having a double point at $R_i$ such that $L_i$ is tangent to $C_5$ at $R_i$ for $i = 1 \ldots 5$.

One of the $R_i$ can be infinitely near to $P$. The curve $C_{21}$ is irreducible.

Conversely, given a plane curve $B$ as in i)–iv), the double cover $X \to \mathbb{P}^2$ branched on $B$ is a plane model of a minimal double plane of general type with $K_2 = 8$ and $p_g = 0$, of type II.

**Proof.** The proof is very similar to the proof of Theorem 5.1, hence it is left to the reader. 

6. Moduli

In this section we establish some properties of the subset of the moduli space of surfaces of general type with $p_g = 0$ and $K_2 = 8$ consisting of the surfaces admitting a rational involution. Most of the arguments that we use here are standard in deformation theory, hence some of proofs are not very detailed.

We introduce some notation. We denote by $\mathcal{M}$ be the moduli space of surfaces of general type with $p_g = 0$ and $K_2 = 8$ and by $\mathcal{D} \subset \mathcal{M}$ the set of isomorphism classes of surfaces admitting a rational involution. We let $\mathcal{D}_{Ia}, \mathcal{D}_{Ib}, \mathcal{D}_{Ic}, \mathcal{D}_{Id}, \mathcal{D}_{II} \subset \mathcal{D}$ be the subsets corresponding to double planes
(S, σ) of types Ia, . . . Id, II, respectively, and we set \( \mathcal{D}_I := \mathcal{D}_{Ia} \cup \mathcal{D}_{Ib} \cup \mathcal{D}_{Ic} \cup \mathcal{D}_{Id}. \) Our results are summarized in the following:

**Theorem 6.1.** Notation and assumptions as above.

Then:

i) \( \mathcal{D} \) is a disjoint union: \( \mathcal{D} = \mathcal{D}_{Ia} \cup \mathcal{D}_{Ib} \cup \mathcal{D}_{Ic} \cup \mathcal{D}_{Id} \cup \mathcal{D}_{II}; \)

ii) \( \mathcal{D}_I \) and \( \mathcal{D}_{II} \) are closed in \( \mathcal{M} \);

iii) \( \mathcal{D}_{Ia}, \mathcal{D}_{Ib}, \mathcal{D}_{Ic}, \mathcal{D}_{Id}, \mathcal{D}_{II} \) are normal open and closed subsets of \( \mathcal{M} \) of the following dimensions:

\[
\dim \mathcal{D}_{Ia} = 5, \quad \dim \mathcal{D}_{Ib} = 4, \quad \dim \mathcal{D}_{Ic} = 3, \quad \dim \mathcal{D}_{Id} = 3, \quad \dim \mathcal{D}_{II} = 2;
\]

iv) \( \mathcal{D}_{Ia}, \mathcal{D}_{Ib}, \mathcal{D}_{Ic}, \mathcal{D}_{Id} \) are irreducible.

Theorem 6.1 describes in particular the set of surfaces with non birational bicanonical map. This set is always closed, but in this case it turns out to be also open. An analogous phenomenon occurs for surfaces with \( p_g = 0, K^2 = 6 \) and bicanonical map of degree 4 (cf. \[MP2\]).

**Corollary 6.2.** The set of surfaces of \( \mathcal{M} \) with non birational bicanonical map is the union of 4 irreducible connected components of \( \mathcal{M} \) of respective dimensions 5, 4, 3 and 3.

**Proof.** Follows from Theorem 6.1 since by Corollary 2.3 and Theorem 4.1, the set of surfaces with non birational bicanonical map is \( \mathcal{D}_I \).

**Proof of Theorem 6.1.** Statement i) is the content of Corollary 4.2, ii). The remaining properties will be a consequence of the Lemmas that follow.

**Lemma 6.3.** The sets \( \mathcal{D}_I \) and \( \mathcal{D}_{II} \) are closed in \( \mathcal{M} \).

**Proof.** By Theorem 4.4 of \[DMP\] and Theorem 4.1, \( \mathcal{D}_I \) is the set of surfaces \( S \) admitting an involution with 12 isolated fixed points and \( \mathcal{D}_{II} \) is the set of surfaces admitting an involution with 10 isolated fixed points.

We remark that by Miyaoka’s formula (\[Mi\], §2) a minimal surface \( S \) with \( p_g = 0 \) and \( K^2 = 8 \) contains no \(-2\)-curve, hence it coincides with its canonical model. Therefore, given a smooth family \( \psi: \mathcal{X} \rightarrow \Delta \) of minimal surfaces of general type with \( K^2 = 8 \) and \( p_g = 0 \), where \( \Delta \subset \mathbb{C} \) is the unit disk, we have to show that if the class of \( X_t := \psi^{-1}(t) \) is in \( \mathcal{D}_I \) (resp. \( \mathcal{D}_{II} \)) for each \( t \in \Delta \setminus \{0\} \), then also \( [X_0] \in \mathcal{D}_I \) (resp. \( \mathcal{D}_{II} \)).

Since by Corollary 1.2 the surface \( X_t \) admits exactly one rational involution \( \sigma_t \), there is a birational map \( \sigma: \mathcal{X} \rightarrow \mathcal{X} \) that restricts to \( \sigma_t \) on \( X_t \) for \( t \neq 0 \). By Corollary 4.5 of \[FP\] \( \sigma \) is actually biregular and we denote by \( \sigma_0 \) the restriction of \( \sigma \) to \( X_0 \). Thus we have to show that the number of isolated fixed points of \( \sigma_t \) is the same for all \( t \in \Delta \).

Clearly one has \( \psi \circ \sigma = \psi \). Let \( P \in \mathcal{X} \) be a fixed point of \( \sigma \). By Cartan’s Lemma and by the fact that \( \sigma \) preserves the fibres of \( \psi \), there exist local analytic coordinates near \( P \) such that \( \psi \) is given by \( (x, y, t) \rightarrow t \) and \( \sigma \) acts by \( (x, y, t) \rightarrow (-x, \epsilon y, t) \), where \( \epsilon = 1 \) or \( \epsilon = -1 \). Hence the fixed locus of \( \sigma \) is the disjoint union of two smooth closed sets \( \Gamma \) and \( \Gamma' \) of dimensions respectively 1 and 2 and the restriction of \( \psi \) to \( \Gamma \) and \( \Gamma' \) is a smooth map. So the \( 0 \)-dimensional part of the fixed locus of \( \sigma_t \) on \( X_t \) is equal to \( X_t \cap \Gamma \) for every \( t \in \Delta \). By the above remarks the map \( \Gamma \rightarrow \Delta \) is proper and étale, hence the cardinality of \( X_t \cap \Gamma \) is constant. \( \square \)
The following auxiliary result is likely to be well known, but we give a proof for lack of a suitable reference.

**Lemma 6.4.** Let $X$ be a smooth variety, let $G$ be a finite group acting faithfully on $X$ in such a way that $Y := X/G$ is smooth. Let $\pi : X \to Y$ be the quotient map and let $D$ be the reduced branch divisor of $\pi$. Then:

$$(\pi_*\omega_X^2)^G = \omega_Y^2(D).$$

**Proof.** The map $\pi$ is flat and finite. Given a $G$–linearized line bundle $L$ on $X$, $\pi_*L$ is locally free of rank equal to $|G|$ and $G$ acts on $\pi_*L$ via the regular representation. One can define a trace map $tr : \pi_*L \to (\pi_*L)^G$ by $tr(s) := \frac{1}{|G|}\sum_{g \in G} g^*s$. Clearly $tr$ splits the inclusion $(\pi_*L)^G \hookrightarrow \pi_*L$ and thus $(\pi_*L)^G$ is a line bundle. If we consider $L = \omega_Y^2$, then there is a natural inclusion $\omega_Y^2 \to (\pi_*\omega_X^2)^G$ which is an isomorphism on $Y_0 := Y \setminus D$. So sections of $(\pi_*\omega_X^2)^G$ are rational sections of $\omega_Y^2$, whose divisor of poles is supported on $D$ and they are characterized by the fact that they pull back to regular sections of $\omega_X^2$. The statement follows from this remark by an easy local computation.

**Lemma 6.5.** The sets $\mathcal{D}_{1a}, \mathcal{D}_{1b}, \mathcal{D}_{1c}, \mathcal{D}_{1d}, \mathcal{D}_{1I}$ are open in $\mathcal{M}$ and normal.

**Proof.** For a variety $X$ we denote by $\text{Def}(X)$ the functor of deformations of $X$; if there is a group $G$ that acts on $X$ we denote by $\text{Def}(X,G)$ the deformations of $X$ that preserve the $G$–action. It is well known that the tangent space to $\text{Def}(X,G)$ is $H^1(X,T_X)^G$ and the obstruction space is $H^2(X,T_X)^G$.

Consider a point $[S]$ of $\mathcal{D}$. By Theorem 6.1, $S$ is a quotient $(F \times C)/G$, where $G$, $F$ and $C$ are as explained in 6.8. As usual we denote by $G_0$ the subgroup of $\text{Aut}(F)$ generated by $G$ and by the hyperelliptic involution $\tau$ of $F$. Consider the map of functors $\eta : \text{Def}(F,G_0) \times \text{Def}(C,G) \to \text{Def}(S)$ defined by taking fibre products and dividing by the “diagonal” $G$–action. Clearly, given an object $\xi \in \text{Def}(F,G_0) \times \text{Def}(C,G) \to \text{Def}(S)$, $\eta(\xi)$ is a family of surfaces whose smooth fibres have a double plane structure of the same type as $S$. Since $F$ and $C$ have dimension 1, $\text{Def}(F,G_0)$ and $\text{Def}(C,G)$ are unobstructed. The tangent space to $\text{Def}(S)$ is $H^1(S,T_S)$. Since the quotient map $F \times C \to S$ is étale, $H^1(S,T_S)$ is naturally isomorphic to $H^1(F \times C,T_{F \times C})^G$, which is in turn isomorphic to $H^1(F,T_F)^G \oplus H^1(C,T_C)^G$ by the Künneth formula and by the fact that $G$ acts on $F$ and on $C$ separately. The map on tangent spaces induced by $\eta$ is the inclusion $H^1(F,T_F)^{G_0} \oplus H^1(C,T_C)^G \to H^1(F,T_F)^G \oplus H^1(C,T_C)^G$. The claim follows if we show that this map is an isomorphism, i.e. that $H^1(F,T_F)^{G_0} = H^1(F,T_F)^G$. For types $1a, 1b, 1d$ one has $G = G_0$ and there is nothing to prove. If $S$ is of type $1c$ or $1I$ we show that $H^1(F,T_F)^G = 0$. By Serre duality, the dual of $H^1(F,T_F)$ is $H^0(F,2K_F)$ and thus it is equivalent to show $H^0(F,2K_F)^G = 0$. In turn this follows by applying Lemma 6.4 to the map $F \to F/G$, which has three branch points for both types (see 6.3).

The above computations also show that the Kuranishi family of $S$ is smooth, hence the moduli space, which is locally a finite quotient of the Kuranishi family, is normal.
Lemma 6.6. One has:

\[ \dim D_{Ia} = 5, \quad \dim D_{Ib} = 4, \quad \dim D_{Ic} = 3, \quad \dim D_{Id} = 3, \quad \dim D_{II} = 2. \]

Proof. By the proof of Lemma 6.5, if \( S = (F \times C)/G \) is in \( D \) then the dimension of \( D \) (and of \( M \)) at the point corresponding to \( S \) is equal to the dimension of \( H^1(F, T_F)^G \oplus H^1(C, T_C)^G \). This dimension can be computed as in the proof of Lemma 6.5, using Serre duality and Lemma 6.4.

Lemma 6.7. The sets \( D_{Ia}, D_{Ib}, D_{Ic}, D_{Id} \) are irreducible.

Proof. The proof of Lemma 6.5 shows that, in order to prove that \( D_{Ia} \) (resp. \( D_{Ib}, D_{Ic}, D_{Id} \)) is irreducible it is enough to show the existence of two families of smooth curves \( F \to B_1 \) and \( C \to B_2 \) with a \( G \)–action such that:

a) \( G \) maps each fibre of \( F \to B_1 \) and \( C \to B_2 \) to itself and the action on the fibre is the one required for type \( Ia \) (resp. \( Ib, Ic, Id \));

b) every \( F \), resp. \( C \), with a \( G \)–action of the right type is isomorphic to a fibre of \( F \to B_1 \), resp. \( C \to B_2 \).

The family \( F \to B_1 \) can be easily constructed using the equations given in \( \text{[3]} \). We sketch briefly a construction of \( C \to B_2 \) for the various types. The reader can easily supply the details. For each type we use the notation introduced in \( \text{[3]} \).

\begin{itemize}
  \item \( Ia \): the curve \( C \) is determined by the three pairs of points \( P_{2i-1} + P_{2i}, \quad i = 1, 2, 3 \). The only condition is that the six points are distinct;
  \item \( Ib \): the curve \( C \) depends on the choice of the elliptic curve \( E \), of the point \( \eta \in E \) of order 4 and of a pair of points of \( \mathbb{P}^1 = E/D_4 \). The only condition is that the two points are not branch points of the quotient map \( E \to E/D_4 \);
  \item \( Ic \): the curve \( C \) is determined by the choice of a homogeneous polynomial of degree 4 with generic branching;
  \item \( Id \): given the curve \( E \), the map \( p: E \to \mathbb{P}^1 \) is determined by the two double fibres, \( 2P_1 + 2P_2 \) and \( 2Q_1 + 2Q_2 \). Clearly \( \xi := P_1 + P_2 - Q_1 - Q_2 \) is 2–torsion and it is nonzero, since otherwise all the fibres of \( p \) over a branch point would be double. So \( C \) is determined by the choice of \( E, P_1 + P_2, \xi \) and of a reduced divisor \( Q_1 + Q_2 \in |P_1 + P_2 + \xi| \). The condition that \( 2Q_1 + 2Q_2 \) and \( 2P_1 + 2P_2 \) are the only double fibres is clearly open, hence also in this case we have an irreducible parameter space for \( C \).
\end{itemize}

\[ \square \]

Remark 6.8. We have not been able to decide whether \( D_{II} \) is irreducible or not. Actually it is not difficult to give an algebraic construction of a family \( C \) containing all the \( C \) with a \( G \)–action of the right type, but it is not clear that it is irreducible.

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