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CONTROLLING ITERATED JUMPS OF SOLUTIONS TO COMBINATORIAL PROBLEMS

LUDOVIC PATEY

Abstract. Among the Ramsey-type hierarchies, namely, Ramsey's theorem, the free set, the thin set and the rainbow Ramsey theorem, only Ramsey's theorem is known to collapse in reverse mathematics. A promising approach to show the strictness of the hierarchies would be to prove that every computable instance at level \( n \) has a low\(_n\) solution. In particular, this requires to control effectively iterations of the Turing jump.

In this paper, we design some variants of Mathias forcing to construct solutions to cohesiveness, the Erdős-Moser theorem and stable Ramsey's theorem for pairs, while controlling their iterated jumps. For this, we define forcing relations which, unlike Mathias forcing, have the same definitional complexity as the formulas they force. This analysis enables us to answer two questions of Wei Wang, namely, whether cohesiveness and the Erdős-Moser theorem admit preservation of the arithmetic hierarchy, and can be seen as a step towards the resolution of the strictness of the Ramsey-type hierarchies.

1. Introduction

The effective forcing is a very powerful tool in the computational analysis of mathematical statements. In this framework, lowness is achieved by deciding formulas during the forcing argument, while ensuring that the whole construction remains effective. Thus, the definitional strength of the forcing relation is very sensitive in effective forcing. We present a new forcing argument enabling one to control iterated jumps of solutions to Ramsey-type theorems. Our main motivation is reverse mathematics.

1.1. Reverse mathematics

Reverse mathematics is a vast mathematical program whose goal is to classify ordinary theorems in terms of their provability strength. It uses the framework of subsystems of second order arithmetic, which is sufficiently rich to express in a natural way many theorems. The base system, \( \text{RCA}_0 \) standing for Recursive Comprehension Axiom, contains the basic first order Peano arithmetic together with the \( \Delta^0_1 \) comprehension scheme and the \( \Sigma^0_1 \) induction scheme. Thanks to the equivalence between \( \Delta^0_1 \)-definable sets and computable sets, \( \text{RCA}_0 \) can be considered as capturing “computational mathematics”. The proof-theoretic analysis of the theorems in reverse mathematics is therefore closely related to their computational analysis. See Simpson [19] for a formal introduction to reverse mathematics.

Early reverse mathematics have led to two main empirical observations: First, many ordinary (i.e. non set-theoretic) theorems require very weak set existence axioms. Second, most of those theorems are in fact equivalent to one of five main subsystems, known as the “Big Five”. However, among the theorems studied in reverse mathematics, a notable class of theorems fails to support those observations, namely, Ramsey-type theorems. This article focuses on consequences of Ramsey's theorem below the arithmetic comprehension axiom (\( \text{ACA}_0 \)). See Hirschfeldt [7] for a gentle introduction to the reverse mathematics below \( \text{ACA}_0 \).

1.2. Controlling iterated jumps

Among the hierarchies of combinatorial principles, namely, Ramsey's theorem [9, 18, 4], the rainbow Ramsey theorem [6, 20, 16], and the free sets and thin set theorems [3, 22] – only Ramsey's theorem is known to collapse within the framework of reverse mathematics. The above mentioned hierarchies satisfy the lower bounds of Jockusch [9], that is, there exists a computable instance at every level \( n \geq 2 \) with no \( \Sigma^0_n \) solution. Thus, a possible strategy for
proving that a hierarchy is strict consists of showing the existence, for every computable instance at level \( n \), of a low \( n \) solution.

The solutions to combinatorial principles are often built by Mathias forcing, whose forcing relation is known to be of higher definitional strength than the formula it forces \([2]\). Therefore there is a need for new notions of forcing with a better-behaving forcing relation. In this paper, we design three notions of forcing to construct solutions to cohesiveness, the Erdős-Moser theorem and stable Ramsey’s theorem for pairs, respectively. We define a forcing relation with the expected properties, and which formalises the first and the second jump control of Cholak, Jockusch and Slaman \([4]\). This can be seen as a step toward the resolution the strictness of the Ramsey-type hierarchies. We take advantage of this new analysis of Ramsey-type statements to prove two conjectures of Wang about the preservation of the arithmetic hierarchy.

1.3. Preservation of the arithmetic hierarchy

The notion of preservation of the arithmetic hierarchy has been introduced by Wang in \([21]\), in the context of a new analysis of principles in reverse mathematics in terms of their definitional strength.

**Definition 1.1** (Preservation of definitions)

1. A set \( Y \) preserves \( \Xi \)-definitions (relative to \( X \)) for \( \Xi \) among \( \Delta^0_{n+1}, \Pi^0_n, \Sigma^0_n \) where \( n > 0 \), if every properly \( \Xi \) (relative to \( X \)) set is properly \( \Xi \) relative to \( Y \) \((X \oplus Y)\). \( Y \) preserves the arithmetic hierarchy (relative to \( X \)) if \( Y \) preserves \( \Xi \)-definitions (relative to \( X \)) for all \( \Xi \) among \( \Delta^0_{n+1}, \Pi^0_n, \Sigma^0_n \) where \( n > 0 \).

2. Suppose that \( \Phi = (\forall X)(\exists Y)\varphi(X,Y) \) and \( \varphi \) is arithmetic. \( \Phi \) admits preservation of \( \Xi \)-definitions if for each \( Z \) and \( X \leq_T Z \) there exists \( Y \) such that \( Y \) preserves \( \Xi \)-definitions relative to \( Z \) and \( \varphi(X,Y) \) holds. \( \Phi \) admits preservation of the arithmetic hierarchy if for each \( Z \) and \( X \leq_T Z \) there exists \( Y \) such that \( Y \) preserves the arithmetic hierarchy relative to \( Z \) and \( \varphi(X,Y) \) holds.

The preservation of the arithmetic hierarchy seems closely related to the problem of controlling iterated jumps of solutions to combinatorial problems. Indeed, a proof of such a preservation usually consists of noticing that the forcing relation has the same strength as the formula it forces, and then deriving a diagonalization from it. See Lemma 2.16 for a case-in-point. Wang proved in \([21]\) that weak König’s lemma (WKL), the rainbow Ramsey theorem for pairs (RRT\(_2^2\)) and the atomic model theorem (AMT) admit preservation of the arithmetic hierarchy. He conjectured that this is also the case for cohesiveness and the Erdős Moser theorem. We prove the two conjectures through the following theorem, where COH stands for cohesiveness and EM for the Erdős-Moser theorem.

**Theorem 1.2** COH and EM admit preservation of the arithmetic hierarchy.

1.4. Definitions and notation

Fix an integer \( k \in \omega \). A **string** (over \( k \)) is an ordered tuple of integers \( a_0, \ldots, a_{n-1} \) (such that \( a_i < k \) for every \( i < n \)). The empty string is written \( \epsilon \). A **sequence** (over \( k \)) is an infinite listing of integers \( a_0, a_1, \ldots \) (such that \( a_i < k \) for every \( i \in \omega \)). Given \( s \in \omega \), \( k^s \) is the set of strings of length \( s \) over \( k \). \( k^{<s} \) is the set of strings of length \( < s \) over \( k \). As well, \( k^{<\omega} \) is the set of finite strings over \( k \) and \( k^\omega \) is the set of sequences (i.e. infinite strings) over \( k \). Given a string \( \sigma \in k^{<\omega} \), we denote by \(|\sigma|\) its length. Given two strings \( \sigma, \tau \in k^{<\omega} \), \( \sigma \) is a **prefix** of \( \tau \) (written \( \sigma \preceq \tau \)) if there exists a string \( \rho \in k^{<\omega} \) such that \( \sigma \rho = \tau \). Given a sequence \( X \), we write \( \sigma \prec X \) if \( \sigma = X|n \) for some \( n \in \omega \). A **binary string** (resp. real) is a string (resp. sequence) over \( 2 \). We may identify a real with a set of integers by considering that the real is its characteristic function.

A tree \( T \subseteq k^{<\omega} \) is a set downward-closed under the prefix relation. A **binary tree** is a set \( T \subseteq 2^{<\omega} \). A set \( P \subseteq \omega \) is a **path** through \( T \) if for every \( \sigma \prec P, \sigma \in T \). A string \( \sigma \in k^{<\omega} \) is a...
Definition 2.1 (Cohesiveness) An infinite set \( R \) is \( R \)-cohesive for a sequence of sets \( R_0, R_1, \ldots \) if for each \( i \in \omega \), \( C \subseteq R_i \) or \( C \subseteq \overline{R_i} \). A set \( C \) is cohesive (resp. \( p \)-cohesive, \( r \)-cohesive) if it is \( R \)-cohesive where \( R \) is the sequence of all the c.e. sets (resp. primitive recursive sets, computable sets). \( \text{COH} \) is the statement “Every uniform sequence of sets \( R \) admits an infinite \( R \)-cohesive set.”

Mileti [13] and Jockusch & Lempp [unpublished] proved that \( \text{COH} \) is a consequence of Ramsey’s theorem for pairs over \( \text{RCA}_0 \). The computational power of \( \text{COH} \) is relatively well understood. Jockusch and Stephan characterized in \([8]\) the degrees bounding \( \text{COH} \) as the degrees whose jump is PA relative to \( \vartheta \). The author [17] extended this characterization to an instance-wise correspondence between cohesiveness and the statement “For every \( \Delta^0_1 \) tree \( T \), there is a set whose jump computes a path through \( T \)”. Wang [21] conjectured that \( \text{COH} \) admits preservation of the arithmetic hierarchy. We prove his conjecture by using a new forcing argument.

Theorem 2.2 \( \text{COH} \) admits preservation of the arithmetic hierarchy.

Before proving Theorem 2.2, we state an immediate corollary.

Corollary 2.3 There exists a cohesive set preserving the arithmetic hierarchy.

Proof. Jockusch [10] proved that every PA degree computes a sequence of sets containing, among others, all the computable sets. Wang proved in [21] that \( \text{WKL}_0 \) preserves the arithmetic hierarchy. Therefore there exists a uniform sequence of sets \( R \) containing all the computable sets and preserving the arithmetic hierarchy. By Theorem 2.2 relativized to \( R \), there exists an infinite \( R \)-cohesive set \( C \) preserving the arithmetic hierarchy relative to \( R \). In particular \( C \) is \( r \)-cohesive and preserves the arithmetic hierarchy. By [8], the degrees of \( r \)-cohesive and cohesive sets coincide. Therefore \( C \) computes a cohesive set which preserves the arithmetic hierarchy. □

Given a uniformly computable sequence of sets \( R_0, R_1, \ldots \), the construction of an \( R \)-cohesive set is usually done with computable Mathias forcing, that is, using conditions \((F, X)\) in which \( X \) is computable. The construction starts with \((\emptyset, \omega)\) and interleaves two kinds of steps. Given some condition \((F, X)\),

\[
\begin{align*}
\text{(S1)} & \quad \text{the extension step consists of taking an element } x \text{ from } X \text{ and adding it to } F, \text{ therefore forming the extension } (F \cup \{x\}, X \setminus [0, x]); \\
\text{(S2)} & \quad \text{the cohesiveness step consists of deciding which one of } X \cap R_i \text{ and } X \cap \overline{R_i} \text{ is infinite, and taking the chosen one as the new reservoir.}
\end{align*}
\]

Cholak, Dzhafarov, Hirst and Slaman [2] studied the definitional complexity of the forcing relation for computable Mathias forcing. They proved that it has the good definitional properties to decide the first jump, but not iterated jumps. Indeed, given a computable Mathias condition \( c = (F, X) \) and a \( \Sigma^0_1 \) formula \((\exists x) \varphi(G, x)\), one can \( \vartheta \)-effectively decide whether there is an extension \( d \) forcing \((\exists x) \varphi(G, x)\) by asking the following question:
Is there an extension \( d = (E,Y) \leq c \) and some \( n \in \omega \) such that \( \varphi(E,n) \) holds?

If there is such an extension, then we can choose it to be a finite extension, that is, such that \( Y =^* X \). Therefore, the question is \( \Sigma^0_X \). Consider now a \( \Pi^0_2 \) formula \((\forall x)(\exists y)\varphi(G, x, y)\).

The question becomes

For every extension \( d \leq c \) and every \( m \in \omega \), is there some extension \( e = (E,Y) \leq d \) and some \( n \in \omega \) such that \( \varphi(E,m,n) \) holds?

In this case, the extension \( d \) is not usually a finite extension and therefore the question cannot be presented in a \( \Pi^0_2 \) way. In particular, the formula “\( Y \) is an infinite subset of \( X \)” is definitionally complex. In general, deciding iterated jumps of a generic set requires to be able to talk about the future of a given condition, and in particular to describe by simple means the formula “\( d \) is a valid condition” and the formula “\( d \) is an extension of \( c \)”.

Thankfully, in the case of cohesiveness, we do not need the full generality of the computable Mathias forcing. Indeed, the reservoirs have a very special shape. After the first application of stage \((S2)\), the set \( X \) is, up to finite changes, of the form \( \omega \cap R_0 \) or \( \omega \cap \overline{R}_0 \). After the second application of \((S2)\), it is in one of the following forms: \( \omega \cap R_0 \cap R_1 \), \( \omega \cap \overline{R}_0 \cap \overline{R}_1 \), \( \omega \cap \overline{R}_0 \cap \overline{R}_1 \), and so on. More generally, after \( n \) applications of \((S2)\), a condition \( c = (F, X) \) is characterized by a pair \((F, \sigma)\) where \( \sigma \) is a string of length \( n \) representing the choices made during \((S2)\).

Even within this restricted partial order, the decision of the \( \Pi^0_2 \) formula remains too complicated since it requires to decide whether \( R_\sigma \) is infinite. However, notice that the \( \sigma \)'s such that \( R_\sigma \) is infinite are exactly the initial segments of the \( \Pi^0_\theta \) class \( C(R) \) defined as the collection of the reals \( X \) such that \( R_\sigma \) has more than \( |\sigma| \) elements for every \( \sigma \prec X \). We can therefore use a compactness argument at the second level to decrease the definitional strength of the forcing relation, as did Wang [21] for weak König’s lemma.

2.1. The forcing notion

We let \( T \) denote the collection of all the infinite \( \theta \)-primitive recursive trees \( T \) such that \( [T] \subseteq C(R) \). Note that \( T \) is a computable set. We are now ready to defined our partial order.

**Definition 2.4** Let \( \mathbb{P} \) be the partial order whose conditions are tuples \((F, \sigma, T)\) where \( F \subseteq \omega \) is a finite set, \( \sigma \in 2^{<\omega} \), \( R_\sigma \) is infinite and \( T \in T \) with stem \( \sigma \). A condition \( d = (E, \tau, S) \) extends \( c = (F, \sigma, T) \) (written \( d \leq c \)) if \( E \leq_\sigma F \), \( \tau \supseteq \sigma \) and \( S \subseteq T \).

Given a condition \( c = (F, \sigma, T) \), the string \( \sigma \) imposes a finite restriction on the possible extensions of the set \( F \). The condition \( c \) intuitively denotes the Mathias condition \((F, R_\sigma \cap (\max(F), +\infty))\) with some additional constraints on the extensions of \( \sigma \) represented by the tree \( T \). Accordingly, set \( G \) satisfies \((F, \sigma, T)\) if it satisfies the induced Mathias condition, that is, if \( F \subseteq G \subseteq F \cup (R_\sigma \cap (\max(F), +\infty)) \). We let \( \text{Ext}(c) \) be the collection of all the extensions of \( c \).

Note that although we did not explicitly require \( R_\sigma \) to be infinite, this property holds for every condition \((F, \sigma, T) \in \mathbb{P} \). Indeed, since \( [T] \subseteq C(R) \), then \( R_\sigma \) is infinite for every extensible node \( \tau \in T \). Since \( \sigma \) is a stem of \( T \), it is extensible and therefore \( R_\sigma \) is infinite.

2.2. Preconditions and forcing \( \Sigma^0_1 (\Pi^0_1) \) formulas

When forcing complex formulas, we need to be able to consider all possible extensions of some condition \( c \). Checking that some \( d = (E, \tau, S) \) is a valid condition extending \( c \) requires to decide whether the tree \( \theta \)-p.r. \( S \) is infinite, which is a \( \Pi^0_1 \) question. At some point, we will need to decide a \( \Sigma^0_1 \) formula without having enough computational power to check that the tree part is infinite (see clause (ii) of Definition 2.10). As the tree part of a condition is not accurate for such formulas, we may define the corresponding forcing relation over a weaker notion of condition where the tree is not anymore required to be infinite.

**Definition 2.5** (Precondition) A precondition is a condition \((F, \sigma, T)\) without the assumption that \( T \) is infinite.
Proof. In particular, \( R_\sigma \) may be a finite set. The notion of condition extension can be generalized to the preconditions. The set of all preconditions is computable, contrary to the set \( \mathbb{P} \). Given a precondition \( c \), we denote by \( \text{Ext}_1(c) \) the set of all preconditions \( (E, \tau, S) \) extending \( c \) such that \( \tau = \sigma \) and \( T = S \). Here, \( T = S \) in a strong sense, that is, the Turing indices of \( T \) and \( S \) are the same. This fact is used in clause a) of Lemma 2.14. We let \( \mathbb{A} \) denote the collection of all the finite sets of integers. The set \( \mathbb{A} \) represents the set of finite approximations of the generic set \( G \). We also fix a uniformly computable enumeration \( \mathbb{A}_0 \subseteq \mathbb{A}_1 \subseteq \ldots \) of finite subsets of \( \mathbb{A} \) cofinal in \( \mathbb{A} \), that is, such that \( \bigcup_s \mathbb{A}_s = \mathbb{A} \). We denote by \( \text{Apx}(c) \) the set \( \{ E \in \mathbb{A} : (E, \sigma, T) \in \text{Ext}_1(c) \} \). In particular, \( \text{Apx}(c) \) is collection of all finite sets \( E \) satisfying \( c \), that is, \( \text{Apx}(c) = \{ E \in \mathbb{A} : E \leq \sigma T \} \). Last, we let \( \text{Apx}_s(c) = \text{Apx}(c) \cap \mathbb{A}_s \). We start by proving a few trivial statements.

**Lemma 2.6** Fix a precondition \( c = (F, \sigma, T) \).

1. If \( c \) is a condition then \( \text{Ext}_1(c) \subseteq \text{Ext}(c) \).
2. If \( c \) is a condition then \( \text{Apx}(c) = \{ E : (E, \tau, S) \in \text{Ext}(c) \} \).
3. If \( d \) is a precondition extending \( c \) then \( \text{Apx}(d) \subseteq \text{Apx}(c) \) and \( \text{Apx}_s(d) \subseteq \text{Apx}_s(c) \).

**Proof.**

1. By definition, if \( c \) is a condition, then \( T \) is infinite. If \( d \in \text{Ext}_1(c) \) then \( d = (E, \tau, S) \) for some \( E \in \text{Apx}(c) \). As \( d \) is a precondition and \( T \) is infinite, \( d \) is a condition.
2. By definition, \( \text{Apx}(c) = \{ E : (E, \sigma, T) \in \text{Ext}_1(c) \} \subseteq \{ E : (E, \tau, S) \in \text{Ext}(c) \} \). On the other direction, fix an extension \( (E, \tau, S) \in \text{Ext}(c) \). By definition of an extension, \( E \leq \tau F \), so \( E \leq \sigma F \). Therefore \( (E, \sigma, T) \in \text{Ext}_1(c) \) and by definition of \( \text{Apx}(c) \), \( E \in \text{Apx}(c) \).
3. Fix some \( (E, \tau, S) \in \text{Ext}_1(d) \). As \( d \) extends \( c \), \( \tau \geq \sigma \). By definition of an extension, \( E \leq \tau F \), so \( E \leq \sigma F \), hence \( (E, \sigma, T) \in \text{Ext}_1(c) \). Therefore \( \text{Apx}(d) = \{ E : (E, \tau, S) \in \text{Ext}_1(d) \} \subseteq \{ E : (E, \sigma, T) \in \text{Ext}_1(c) \} = \text{Apx}(c) \). For any \( s \in \omega \), \( \text{Apx}_s(d) = \text{Apx}_s(d) \cap \mathbb{A}_s = \text{Apx}_s(c) \).

Note that although the extension relation has been generalized to preconditions, \( \text{Ext}(c) \) is defined to be the set of all the conditions extending \( c \). In particular, if \( c \) is a precondition which is not a condition, \( \text{Ext}(c) = \emptyset \), whereas at least \( c \in \text{Ext}_1(c) \). This is why clause 1 of Lemma 2.6 gives the useful information that whenever \( c \) is a true condition, so are the members of \( \text{Ext}_1(c) \).

**Definition 2.7** Fix a precondition \( c = (F, \sigma, T) \) and a \( \Sigma_0^0 \) formula \( \varphi(G, x) \).

(i) \( c \models (\exists x) \varphi(G, x) \) iff \( \varphi(F, w) \) holds for some \( w \in \omega \)
(ii) \( c \models (\forall x) \varphi(G, x) \) iff \( \varphi(E, w) \) holds for every \( w \in \omega \) and every set \( E \in \text{Apx}(c) \).

As explained, \( \sigma \) restricts the possible extensions of the set \( F \) (see clause 3 of Lemma 2.6), so this forcing notion is stable by condition extension. The tree \( T \) itself restricts the possible extensions of \( \sigma \), but has no effect of the decision of a \( \Sigma_1^0 \) formula (Lemma 2.8).

The following trivial lemma expresses the fact that the tree part of a precondition has no effect in the forcing relation for a \( \Sigma_1^0 \) or \( \Pi_1^0 \) formula.

**Lemma 2.8** Fix two preconditions \( c = (F, \sigma, T) \) and \( d = (F, \sigma, S) \), and some \( \Sigma_1^0 \) or \( \Pi_1^0 \) formula \( \varphi(G) \).

\( c \models \varphi(G) \) if and only if \( d \models \varphi(G) \)

**Proof.** Simply notice that the tree part of the condition does not occur in the definition of the forcing relation, and that \( \text{Apx}(c) = \text{Apx}(d) \).

As one may expect, the forcing relation for a precondition is closed under extension.

**Lemma 2.9** Fix a precondition \( c \) and a \( \Sigma_0^0 \) or \( \Pi_1^0 \) formula \( \varphi(G) \). If \( c \models \varphi(G) \) then for every precondition \( d \leq c \), \( d \models \varphi(G) \).
Lemma 2.12
For every arithmetic formula \( \varphi \) and an extension \( d = (E, \tau, S) \leq c \).

- If \( \varphi \in \Sigma^0_1 \) then \( \varphi(G) \) can be expressed as \((\exists x)\psi(G,x)\) where \( \psi \in \Sigma^0_0 \). As \( c \models \varphi(G) \), then by clause (i) of Definition 2.7, there exists a \( w \in \omega \) such that \( \psi(F,w) \) holds. By definition of \( d \leq c \), \( E \leq_{\tau} F \), so \( \psi(E,w) \) holds, hence \( d \models \varphi(G) \).

- If \( \varphi \in \Pi^0_1 \) then \( \varphi(G) \) can be expressed as \((\forall x)\psi(G,x)\) where \( \psi \in \Sigma^0_0 \). As \( c \models \varphi(G) \), then by clause (ii) of Definition 2.7, for every \( w \in \omega \) and every \( H \in \text{Apx}(c) \), \( \varphi(H,w) \) holds. By clause 3 of Lemma 2.6, \( \text{Apx}(d) \subseteq \text{Apx}(c) \) so \( d \models \varphi(G) \).

\[ \square \]

2.3. Forcing higher formulas

We are now able to define the forcing relation for any arithmetic formula. The forcing relation for arbitrary arithmetic formulas is induced by the forcing relation for \( \Sigma^0_1 \) formulas. However, the definitional strength of the resulting relation is too high with respect to the formula it forces. We therefore design a custom relation with better definitional properties, and which still preserve the expected properties of a forcing relation, that is, the density of the set of conditions forcing a formula or its negation, and the preservation of the forced formulas under condition extension.

**Definition 2.10** Let \( c = (F, \sigma, T) \) be a condition and \( \varphi(G) \) be an arithmetic formula.

(i) If \( \varphi(G) = (\exists x)\psi(G,x) \) where \( \psi \in \Pi^0_{n+1} \) then \( c \notmodels \varphi(G) \) iff there is a \( w < |\sigma| \) such that \( c \notmodels \psi(G, w) \).

(ii) If \( \varphi(G) = (\forall x)\psi(G,x) \) where \( \psi \in \Sigma^0_0 \) then \( c \models \varphi(G) \) iff for every \( \tau \in T \), every \( E \in \text{Apx}_{|\tau|}(c) \) and every \( w < |\tau| \), \( (E, \tau, T^{[\tau]}) \notmodels \neg\psi(G, w) \).

(iii) If \( \varphi(G) = \neg\psi(G,x) \) where \( \psi \in \Sigma^0_{n+3} \) then \( c \models \varphi(G) \) iff \( d \notmodels \psi(G) \) for every \( d \leq c \).

Note that in clause (ii) of Definition 2.10, there may be some \( \tau \in T \) such that \( T^{[\tau]} \) is finite, hence \( (E, \tau, T^{[\tau]}) \) is not necessarily a condition. This is where we use the generalization of forcing of \( \Sigma^0_1 \) and \( \Pi^0_1 \) formulas to preconditions. We now prove that this relation enjoys the main properties of a forcing relation.

**Lemma 2.11** Fix a condition \( c \) and an arithmetic formula \( \varphi(G) \). If \( c \models \varphi(G) \) then for every condition \( d \leq c \), \( d \models \varphi(G) \).

**Proof.** We prove by induction over the complexity of the formula \( \varphi(G) \) that for every condition \( c \), if \( c \models \varphi(G) \) then for every condition \( d \leq c \), \( d \models \varphi(G) \). Fix a condition \( c = (F, \sigma, T) \) such that \( c \models \varphi(G) \) and an extension \( d = (E, \tau, S) \leq c \).

- If \( \varphi \in \Sigma^0_1 \cup \Pi^0_1 \) then it follows from Lemma 2.9.
- If \( \varphi \in \Sigma^0_{n+2} \) then \( \varphi(G) \) can be expressed as \((\exists x)\psi(G,x)\) where \( \psi \in \Pi^0_{n+1} \). By clause (i) of Definition 2.10, there exists a \( w \in \omega \) such that \( c \models \psi(G, w) \). By induction hypothesis, \( d \models \psi(G, w) \) so by clause (i) of Definition 2.10, \( d \models \varphi(G) \).
- If \( \varphi \in \Pi^0_{n+3} \) then \( \varphi(G) \) can be expressed as \((\forall x)\psi(G,x)\) where \( \psi \in \Sigma^0_0 \). By clause (ii) of Definition 2.10, for every \( \rho \in T \), every \( w < |\rho| \), and every \( H \in \text{Apx}_{|\rho|}(c) \), \( (H, \rho, T^{[\rho]}) \notmodels \neg\psi(G, w) \). As \( S \subseteq T \) and \( \text{Apx}(d) \subseteq \text{Apx}(c) \), for every \( \rho \in S \), every \( w < |\rho| \), and every \( H \in \text{Apx}_{|\rho|}(d) \), \( (H, \rho, T^{[\rho]}) \notmodels \neg\psi(G, w) \). By Lemma 2.8, \( (H, \rho, S^{[\rho]}) \notmodels \neg\psi(G, w) \) hence by clause (ii) of Definition 2.10, \( d \models \varphi(G) \).
- If \( \varphi \in \Sigma^0_{n+3} \) then \( \varphi(G) \) can be expressed as \( \neg\psi(G) \) where \( \psi \in \Sigma^0_{n+3} \). By clause (iii) of Definition 2.10, for every \( e \in \text{Ext}(c) \), \( e \notmodels \psi(G) \). As \( \text{Ext}(d) \subseteq \text{Ext}(c) \), for every \( e \in \text{Ext}(d) \), \( e \notmodels \psi(G) \), so by clause (iii) of Definition 2.10, \( d \models \varphi(G) \).

\[ \square \]

**Lemma 2.12** For every arithmetic formula \( \varphi \), the following set is dense

\[ \{ c \in \mathbb{P} : c \models \varphi(G) \text{ or } c \models \neg \varphi(G) \} \]
Proof. We prove by induction over \(n > 0\) that if \(\varphi\) is a \(\Sigma^0_n\) (\(\Pi^0_n\)) formula then the following set is dense
\[
\{ c \in \mathcal{P} : c \models \varphi(G) \text{ or } c \models \neg \varphi(G) \}
\]
It suffices to prove it for the case where \(\varphi\) is a \(\Sigma^0_n\) formula, as the case where \(\varphi\) is a \(\Pi^0_n\) formula is symmetric. Fix a condition \(c = (F, \sigma, T)\).

- In case \(n = 1\), the formula \(\varphi\) is of the form \((\exists x)\psi(G, x)\) where \(\psi \in \Sigma^0_1\). Suppose there exists a \(w \in \omega\) and a set \(E \in \text{Apx}(c)\) such that \(\psi(E, w)\) holds. The precondition \(d = (E, \sigma, T)\) is a condition extending \(c\) by clause 1 of Lemma 2.6 and by definition of \(\text{Apx}(c)\). Moreover \(d \models (\exists x)\psi(G, x)\) by clause (i) of Definition 2.7 hence \(d \models \varphi(G)\).

- In case \(n = 2\), the formula \(\varphi\) is of the form \((\exists x)\psi(G, x)\) where \(\psi \in \Sigma^0_2\). Fix a condition \(c\) such that \(c\) is of the form \((\exists x)\psi(G, x)\) where \(\psi \in \Sigma^0_2\). Let
\[
S = \{ \tau \in T : (\forall w < |\tau|)(\forall E \in \text{Apx}_{|\tau|}(c))(E, \tau, T[|\tau|]) \not\models \psi(G, w) \}
\]
The set \(S\) is obviously \(\neg\)-p.r. We prove that it is a subtree of \(T\). Suppose that \(\tau \in S\) and \(\rho \not\preceq \tau\). Fix a \(w < |\rho|\) and \(E \in \text{Apx}_{|\rho|}(c)\). In particular \(w < |\tau|\) and \(E \in \text{Apx}_{|\tau|}(c)\) so \((E, \tau, T[|\tau|]) \not\models \psi(G, w)\). Note that \((E, \tau, T[|\tau|])\) is a precondition extending \((E, \rho, T[|\rho|])\), so by the contrapositive of Lemma 2.9, \((E, \rho, T[|\rho|]) \not\models \psi(G, w)\). Therefore \(\rho \not\in S\). Hence \(S\) is a tree, and as \(S \subseteq T\), it is a subtree of \(T\).

If \(S\) is infinite, then \(d = (F, \sigma, S)\) is an extension of \(c\) such that for every \(\tau \in S\), every \(w < |\tau|\) and every \(E \in \text{Apx}_{|\tau|}(c)\), \((E, \tau, T[|\tau|]) \not\models \psi(G, w)\). By Lemma 2.8, for every \(E \in \text{Apx}_{|\tau|}(c)\), \((E, \tau, S[|\tau|]) \not\models \psi(G, w)\) and by clause 3 of Lemma 2.6, \(\text{Apx}_{|\tau|}(d) \subseteq \text{Apx}_{|\tau|}(c)\). Therefore, by clause (ii) of Definition 2.10, \(d \models (\forall x)\neg \psi(G, x)\) so \(d \models \neg \varphi(G)\). If \(S\) is finite, then pick some \(\tau \in T - S\) such that \(T[|\tau|]\) is infinite. By choice of \(\tau \in T - S\), there exists a \(w < |\tau|\) and an \(E \in \text{Apx}_{|\tau|}(c)\) such that \((E, \tau, T[|\tau|]) \models \psi(G, w)\). \(d = (E, \tau, T[|\tau|])\) is a valid condition extending \(c\) and by clause (i) of Definition 2.10 \(d \models \varphi(G)\).

- In case \(n > 2\), density follows from clause (iii) of Definition 2.10.

Any sufficiently generic filter \(\mathcal{F}\) induces a unique generic real \(G\) defined by
\[
G = \bigcup \{ F \in \mathbb{A} : (F, \sigma, T) \in \mathcal{F} \}
\]
The following lemma informally asserts that the forcing relation is sound and complete. Sound because whenever it forces a property, then this property actually holds over the generic real \(G\). The forcing is also complete in that every property which holds over \(G\) is forced at some point whenever the filter is sufficiently generic.

**Lemma 2.13** Suppose that \(\mathcal{F}\) is a sufficiently generic filter and let \(G\) be the corresponding generic real. Then for each arithmetic formula \(\varphi(G), \psi(G)\) holds iff \(c \models \varphi(G)\) for some \(c \in \mathcal{F}\).

**Proof.** We prove by induction over the complexity of the arithmetic formula \(\varphi(G)\) that \(\varphi(G)\) holds iff \(c \models \varphi(G)\) for some \(c \in \mathcal{F}\). Note that thanks to Lemma 2.12, it suffices to prove that if \(c \models \varphi(G)\) for some \(c \in \mathcal{F}\) then \(c \models \psi(G)\) holds. Indeed, conversely if \(\varphi(G)\) holds, then by genericity of \(G\) either \(c \models \varphi(G)\) or \(c \models \neg \varphi(G)\) for some \(c \in \mathcal{F}\), but if \(c \models \neg \varphi(G)\) then \(\neg \varphi(G)\) holds, contradicting the hypothesis. So \(c \models \varphi(G)\).

We proceed by case analysis on the formula \(\varphi\). Note that in the above argument, the converse of the \(\Sigma\) case is proved assuming the \(\Pi\) case. However, in our proof, we use the converse of the \(\Sigma_{n+1}^0\) case to prove the \(\Pi_{n+3}^0\) case. We need therefore prove to the converse of the \(\Sigma_{n+1}^0\) case without Lemma 2.12. Fix a condition \(c = (F, \sigma, T) \in \mathcal{F}\) such that \(c \models \varphi(G)\).

- If \(\varphi \in \Sigma^0_1\) then \(\varphi(G)\) can be expressed as \((\exists x)\psi(G, x)\) where \(\psi \in \Sigma^0_1\). By clause (i) of Definition 2.7, there exists a \(w \in \omega\) such that \(\psi(F, w)\) holds. As \(F \subseteq G\) and \(G - F \subseteq (\text{max}(F), +\infty)\), then by continuity \(\psi(G, w)\) holds, hence \(\varphi(G)\) holds.

□
Lemma 2.14

a) For every precondition $c$, $\text{Apx}(c)$ and $\text{Ext}_1(c)$ are $\Delta^0_1$ uniformly in $c$.
b) For every condition $c$, $\text{Ext}(c)$ is $\Pi^0_2$ uniformly in $c$.

Proof.
a) Fix a precondition $c = (F, \sigma, T)$. A set $E \in \text{Apx}(c)$ iff the following $\Delta^0_1$ predicate holds:

$$(F \subseteq E) \land (\forall x \in E \setminus F)[x > \max(F) \land x \in R_\sigma]$$

Moreover, $(E, \tau, S) \in \text{Ext}_1(c)$ iff the $\Delta^0_1$ predicate $E \in \text{Apx}(c) \land \tau = \sigma \land S = T$ holds. As already mentioned, the equality $S = T$ is translated into “the indices of $S$ and $T$ coincide” which is a $\Sigma^0_1$ statement.
b) Fix a condition $c = (F, \sigma, T)$. By clause 2) of Lemma 2.6, $(E, \tau, S) \in \text{Ext}(c)$ iff the following $\Pi^0_2$ formula holds

$$E \in \text{Apx}(c) \land \sigma \preceq \tau$$

$$\land (\forall \rho \in S)(\forall \xi)[\xi \leq \rho \rightarrow \xi \in S] \quad (S \text{ is a tree})$$

$$\land (\forall n)(\exists \rho \in 2^n)\rho \in S \quad (S \text{ is infinite})$$

$$\land (\forall \rho \in S)(\sigma \prec \rho \lor \rho \leq \sigma) \quad (S \text{ has stem } \sigma)$$

$$\land (\forall \rho \in S)(\rho \in T) \quad (S \text{ is a subset of } T)$$

□

Lemma 2.15 Fix an arithmetic formula $\varphi(G)$.
a) Given a precondition $c$, if $\varphi(G)$ is a $\Sigma^0_1$ ($\Pi^0_1$) formula then so is the predicate $c \models \varphi(G)$.
b) Given a condition $c$, if $\varphi(G)$ is a $\Sigma^0_{n+2}$ ($\Pi^0_{n+2}$) formula then so is the predicate $c \models \varphi(G)$.

Proof. We prove our lemma by induction over the complexity of the formula $\varphi(G)$. Fix a (pre)condition $c = (F, \sigma, T)$. 

- If $\varphi(G) \in \Sigma_0^0$ then it can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma_0^0$. By clause (i) of Definition 2.7, $c \models \varphi(G)$ if and only if the formula $(\exists w \in \omega)\psi(F, w)$ holds. This is a $\Sigma_1^0$ predicate.

- If $\varphi(G) \in \Pi_0^0$ then it can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma_0^0$. By clause (ii) of Definition 2.7, $c \models \varphi(G)$ if and only if the formula $(\forall w \in \omega)(\forall E \in \text{Apx}(c))\psi(E, w)$ holds. By clause a) of Lemma 2.14, this is a $\Sigma_1^0$ predicate.

- If $\varphi(G) \in \Sigma_0^0_{n+2}$ then it can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi_0^0_{n+1}$. By clause (i) of Definition 2.10, $c \models \varphi(G)$ if and only if the formula $(\exists w \in \omega \setminus \{\varnothing\})c \models \psi(G, w)$ holds. This is a $\Sigma_{n+2}^0$ predicate by induction hypothesis.

- If $\varphi(G) \in \Pi_0^1_{n+2}$ then it can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma_0^0_{n+1}$. By clause (ii) of Definition 2.10, $c \models \varphi(G)$ if and only if the formula $(\forall \tau \in T)(\forall w < |\tau|)(\forall E \in \text{Apx}_{|\tau|}(c))(E, \tau, T^{[\tau]}) \not\models \psi(G, w)$ holds. By induction hypothesis, $(E, \tau, T^{[\tau]}) \not\models \psi(G, w)$ is a $\Sigma_0^0$ predicate, hence by clause a) of Lemma 2.14, $c \models \varphi(G)$ is a $\Pi_0^0_{n+3}$ predicate.

- If $\varphi(G) \in \Pi_0^1_{n+3}$ then it can be expressed as $(\neg \psi(G))$ where $\psi \in \Sigma_0^0_{n+3}$. By clause (iii) of Definition 2.10, $c \models \varphi(G)$ if and only if the formula $(\forall d)(d \notin \text{Ext}(c) \lor d \not\models \psi(G))$ holds. By induction hypothesis, $d \not\models \psi(G)$ is a $\Pi_0^0_{n+3}$ predicate. Hence by clause b) of Lemma 2.14, $c \models \varphi(G)$ is a $\Pi_0^0_{n+3}$ predicate.

\[ \Box \]

2.4. Preserving the arithmetic hierarchy

The following lemma asserts that every sufficiently generic real for this notion of forcing preserves the arithmetic hierarchy. The argument deeply relies on the fact that this notion of forcing admits a forcing relation with good definitional properties.

**Lemma 2.16** If $A \notin \Pi_0^0_{n+1}$ and $\varphi(G, x)$ is $\Sigma_0^0_{n+1}$, then the set of $c \in \mathbb{P}$ satisfying the following property is dense:

\[ [(\exists w \in A)c \models (\exists w \notin A)c \models \varphi(G, w)] \]

**Proof.** Fix a condition $c = (F, \sigma, T)$.

- In case $n = 0$, $\varphi(G, w)$ can be expressed as $(\exists x)\psi(G, w, x)$ where $\psi \in \Sigma_0^0$. Let $U = \{ w \in \omega : (\exists E \in \text{Apx}(c))(\exists u)\psi(E, w, u) \}$. By clause a) of Lemma 2.14, $U \in \Sigma_0^0_{n+1}$, thus $U \neq A$. Fix $w \in U \Delta A$. If $w \in U \setminus A$ then by definition of $U$, there exists an $E \in \text{Apx}(c)$ and a $u \in \omega$ such that $\psi(E, w, u)$ holds. By definition of $\text{Apx}(c)$ and clause 1) of Lemma 2.6, $d \models (E, \sigma, T)$ is a condition extending $c$. By clause (i) of Definition 2.7, $d \models \varphi(G, w)$. If $w \in A \setminus U$, then for every $E \in \text{Apx}(c)$ and every $u \in \omega$, $\psi(E, w, u)$ does not hold, so by clause (ii) of Definition 2.7, $c \models (\forall x) \neg \psi(G, w, x)$, hence $c \models \neg \varphi(G, w)$.

- In case $n = 1$, $\varphi(G, w)$ can be expressed as $(\exists x)\psi(G, w, x)$ where $\psi \in \Pi_0^1$. Let $U = \{ w \in \omega : (\exists n)(\forall \tau \in 2^n \cap T)(\exists u < s)(\exists E \in \text{Apx}_n(c))(E, \tau, T^{[\tau]}) \models \psi(G, w, u) \}$. By Lemma 2.15 and clause a) of Lemma 2.14, $U \in \Sigma_0^0$, thus $U \neq A$. Fix $w \in U \Delta A$. If $w \in U \setminus A$ then by definition of $U$, there exists an $s \in \omega$, an $a \in 2^n \cap T$, a $u < s$ and an $E \in \text{Apx}_n(c)$ such that $T^{[\tau]}$ is infinite and $(E, \tau, T^{[\tau]}) \not\models \psi(G, w, u)$. Thus $d \models (E, \tau, T^{[\tau]})$ is a condition extending $c$. By clause (i) of Definition 2.10, $d \models \varphi(G, w)$. If $w \in A \setminus U$, then let $S = \{ \tau \in T : (\forall u < |\tau|)(\exists E \in \text{Apx}_{|\tau|}(c))(E, \tau, T^{[\tau]}) \not\models \psi(G, w, u) \}$. As proven in Lemma 2.12, $S$ is a $\theta'$-p.r. subtree of $T$ and by $w \notin U$, $S$ is infinite. Thus $d \models (F, \sigma, S)$ is a condition extending $c$. By clause 3) of Lemma 2.6, $\text{Apx}(d) \subseteq \text{Apx}(c)$, so for every $\tau \in S$, every $u < |\tau|$, and every $E \in \text{Apx}_{|\tau|}(d)$, $(E, \tau, T^{[\tau]}) \not\models \psi(G, w, u)$. By Lemma 2.8, $(E, \tau, S^{[\tau]}) \not\models \psi(G, w, u)$, so by clause (ii) of Definition 2.10, $d \models (\forall x) \neg \psi(G, w, u)$ and hence $d \models \neg \varphi(G, w)$.

- In case $n > 1$, let $U = \{ w \in \omega : (\exists d \in \text{Ext}(c))d \models \varphi(G, w) \}$. By clause b) of Lemma 2.14 and Lemma 2.15, $U \in \Sigma_0^0$, thus $U \neq A$. Fix $w \in U \Delta A$. If $w \in U \setminus A$ then by definition of $U$, there exists a condition $d$ extending $c$ such that $d \models \varphi(G, w)$. If $w \in A \setminus U$, then for every $d \in \text{Ext}(c)d \not\models \varphi(G, w)$ so by clause (iii) of Definition 2.10, $c \models \neg \varphi(G, w)$.
We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let $C$ be a set and $R_0, R_1, \ldots$ be a uniformly $C$-computable sequence of sets. Let $T_0$ be a $C'$-primitive recursive tree such that $[T_0] \subseteq C(R)$. Let $F$ be a sufficiently generic filter containing $c_0 = (\emptyset, \epsilon, T_0)$, and let $G$ be the corresponding generic real. By genericity, the set $G$ is an infinite $\tilde{R}$-cohesive set. By Lemma 2.16 and Lemma 2.15, $G$ preserves non-$\Sigma^0_{n+1}$ definitions relative to $C$ for every $n \in \omega$. Therefore, by Proposition 2.2 of [21], $G$ preserves the arithmetic hierarchy relative to $C$. 

\[\square\]

3. The Erdős Moser theorem preserves the arithmetic hierarchy

We now extend the previous result to the Erdős-Moser theorem. The Erdős-Moser theorem is a statement coming from graph theory. It provides together with the ascending descending principle (ADS) an alternative proof of Ramsey’s theorem for pairs $(RT^2_2)$. Indeed, every coloring $f : [\omega]^2 \to 2$ can be seen as a tournament $R$ such that $R(x, y)$ holds if $x < y$ and $f(x, y) = 1$, or $x > y$ and $f(y, x) = 0$. Every infinite transitive subtournament induces a linear order whose infinite ascending or descending sequences are homogeneous for $f$.

Definition 3.1 (Erdős-Moser theorem) A tournament $T$ on a domain $D \subseteq \omega$ is an irreflexive binary relation on $D$ such that for all $x, y \in D$ with $x \neq y$, exactly one of $T(x, y)$ or $T(y, x)$ holds. A tournament $T$ is transitive if the corresponding relation $T$ is transitive in the usual sense. A tournament $T$ is stable if $T(x, s)$ holds for all $s \in D$. \textsc{EM} is the statement “Every infinite tournament $T$ has an infinite transitive subtournament.” \textsc{SEM} is the restriction of \textsc{EM} to stable tournaments.

Bovykin and Weiermann proved in [1] that $\textsc{EM} + \text{ADS}$ is equivalent to $RT^2_2$ over RCA$_0$, equivalence still holding between their stable versions. Lerman et al. [12] proceeded to a combinatorial and effective analysis of the Erdős-Moser theorem, and proved in particular that there is an $\omega$-model of \textsc{EM} which is not a model of SRT$_2^2$. The author simplified their proof in [15] and showed in [16] that $\text{RCA}_0 + \text{EM} \vdash [\text{STS}_2 \lor \text{COH}]$, where STS$_2$ stands for the stable thin set theorem for pairs. In particular, since Wang [21] proved that STS$_2$ does not admit preservation of the arithmetic hierarchy, Theorem 2.2 follows from Theorem 3.2. On a definitional point of view, Wang proved in [21] that \textsc{EM} admits preservation of \Delta^0_3 definitions and preservation of higher definitions. He conjectured that \textsc{EM} admits preservation of the arithmetic hierarchy. We prove his conjecture.

Theorem 3.2 \textsc{EM} admits preservation of the arithmetic hierarchy.

Again, the core of the proof consists of finding the good forcing notion whose generics will preserve the arithmetic hierarchy. For the sake of simplicity, we will restrict ourselves to stable tournaments eventhough it is clear that the forcing notion can be adapted to arbitrary tournament. The proof of Theorem 3.2 will be obtained by composing the proof that cohesiveness and the stable Erdős-Moser theorem admit preservation of the arithmetic hierarchy.

The following notion of minimal interval plays a fundamental role in the analysis of \textsc{EM}. See [12] for a background analysis of \textsc{EM}.

Definition 3.3 (Minimal interval) Let $T$ be an infinite tournament and $a, b \in T$ be such that $T(a, b)$ holds. The interval $(a, b)$ is the set of all $x \in T$ such that $T(a, x)$ and $T(x, b)$ hold. Let $F \subseteq T$ be a finite transitive subtournament of $T$. For $a, b \in F$ such that $T(a, b)$ holds, we say that $(a, b)$ is a minimal interval of $F$ if there is no $c \in F \cap (a, b)$, i.e., no $c \in F$ such that $T(a, c)$ and $T(c, b)$ both hold.

We must introduce an preliminary variant of Mathias forcing which is more suited to the Erdős-Moser theorem.
3.1. Erdős Moser forcing

The following notion of Erdős-Moser forcing has been first implicitly used by Lerman, Solomon and Towsser [12] to separate the Erdős-Moser theorem from stable Ramsey’s theorem for pairs. The author formalized this notion of forcing in [14] to construct a low_2 degree bounding the Erdős-Moser theorem.

**Definition 3.4** An *Erdős Moser condition* (EM condition) for an infinite tournament $R$ is a Mathias condition $(F,X)$ where

(a) $F \cup \{x\}$ is $R$-transitive for each $x \in X$
(b) $X$ is included in a minimal $R$-interval of $F$.

The Erdős-Moser extension is the usual Mathias extension. EM conditions have good properties for tournaments as state following lemmas. Given a tournament $R$ and two sets $E$ and $F$, we denote by $E \rightarrow_R F$ the formula $(\forall x \in E)(\forall y \in F) R(x,y)$ holds.

**Lemma 3.5** (Patey [14]) Fix an EM condition $(F,X)$ for a tournament $R$. For every $x \in F$, \{x\} \rightarrow_R X$ or $X \rightarrow_R \{x\}$.

**Lemma 3.6** (Patey [14]) Fix an EM condition $c = (F,X)$ for a tournament $R$, an infinite subset $Y \subseteq X$ and a finite $R$-transitive set $F_1 \subseteq X$ such that $F_1 < Y$ and $[F_1 \rightarrow_R Y \lor Y \rightarrow_R F_1]$. Then $d = (F \cup F_1,Y)$ is a valid extension of $c$.

3.2. Partition trees

Given a string $\sigma \in k^{<\omega}$, we denote by $\text{set}_\nu(\sigma)$ the set $\{x < |\sigma| : \sigma(x) = \nu\}$ where $\nu < k$. The notion can be extended to sequences $P \in k^{\omega}$ where $\text{set}_\nu(P) = \{x \in \omega : P(x) = \nu\}$.

**Definition 3.7** (Partition tree) A $k$-**partition tree** of $[t, +\infty)$ for some $k,t \in \omega$ is a tuple $(k,t,T)$ such that $T$ is a subtree of $k^{<\omega}$. A *partition tree* is a $k$-partition tree of $[t, +\infty)$ for some $k,t \in \omega$.

For the simplicity of notations, we may use the same letter $T$ to denote both a partition tree $(k,t,T)$ and the actual tree $T \subseteq k^{<\omega}$. We then write $\text{dom}(T)$ for $[t, +\infty)$ and $\text{parts}(T)$ for $k$.

Given a p.r. partition tree $T$, we write $\#T$ for its Turing index, and may refer to it as its *code*.

**Definition 3.8** (Refinement) Given a function $f : \ell \rightarrow k$, a string $\sigma \in \ell^{<\omega}$ *$f$-refines* a string $\tau \in k^{<\omega}$ if $|\sigma| = |\tau|$ and for every $\nu < \ell$, $\text{set}_\nu(\sigma) \subseteq \text{set}_{f(\nu)}(\tau)$. A p.r. $\ell$-partition tree $S$ of $[u, +\infty)$ *$f$-refines* a p.r. $k$-partition tree $T$ of $[t, +\infty)$ (written $S \leq f T$) if $\#S \geq \#T$, $\ell \geq k$, $u \geq t$ and for every $\sigma \in S$, $\sigma$ $f$-refines some $\tau \in T$.

The collection of partition trees is equipped with a partial order $\leq$ such that $(\ell,u,S) \leq (k,t,T)$ if there exists a function $f : \ell \rightarrow k$ such that $S \leq f T$. Given a $k$-partition tree of $[t, +\infty)$ $T$, we say that part $\nu$ of $T$ is *acceptable* if there exists a path $P$ through $T$ such that $\text{set}_\nu(P)$ is infinite. Moreover, we say that part $\nu$ of $T$ is *empty* if $(\forall \sigma \in T)[\text{dom}(T) \cap \text{set}_\nu(\sigma) = \emptyset]$. Note that each partition tree has at least one acceptable part since for every path $P$ through $T$, $\text{set}_\nu(P)$ is infinite for some $\nu < k$. It can also be the case that part $\nu$ of $T$ is non-empty, while for every path $P$ through $T$, $\text{set}_\nu(P) \cap \text{dom}(T) = \emptyset$. However, in this case, we can choose the infinite computable subtree $S = \{\sigma \in T : \text{set}_\nu(\sigma) \cap \text{dom}(T) = \emptyset\}$ of $T$ which has the same collection of infinite paths and such that part $\nu$ of $S$ is empty.

Given a $k$-partition tree $T$, a finite set $F \subseteq \omega$ and a part $\nu < k$, define

$$T^{[\nu,F]} = \{\sigma \in T : F \subseteq \text{set}_\nu(\sigma) \lor |\sigma| < \max(F)\}$$

The set $T^{[\nu,F]}$ is a (possibly finite) subtree of $T$ which id-refines $T$ and such that $F \subseteq \text{set}_\nu(P)$ for every path $P$ through $T^{[\nu,F]}$.

We denote by $\mathbb{U}$ the set of all ordered pairs $(\nu, T)$ such that $T$ is an infinite, primitive recursive $k$-partition tree of $[t, +\infty)$ for some $t, k \in \omega$ and $\nu < k$. The set $\mathbb{U}$ is equipped with a partial ordering $\leq$ such that $(\mu, S) \leq (\nu, T)$ if $S$ $f$-refines $T$ and $f(\mu) = \nu$. In this case we say that
part \( \mu \) of \( S \) refines part \( \nu \) of \( T \). Note that the domain of \( U \) and the relation \( \leq \) are co-c.e. We denote by \( U[T] \) the set of all \((\nu, S) \leq (\mu, T)\) for some \((\mu, T) \in U\).

**Definition 3.9** (Promise for a partition tree) Fix a p.r. \( k \)-partition tree of \([t, +\infty)\) \( T \). A class \( C \subseteq U[T] \) is a promise for \( T \) if

a) \( C \) is upward-closed under the \( \leq \) relation restricted to \( U[T] \)

b) for every infinite p.r. partition tree \( S \leq T, (\mu, S) \in C \) for some non-empty part \( \mu \) of \( S \).

A promise for \( T \) can be seen as a two-dimensional tree with at first level the acyclic digraph of refinement of partition trees. Given an infinite path in this digraph, the parts of the members of this path form an infinite, finitely branching tree.

**Lemma 3.10** Let \( T \) and \( S \) be p.r. partition trees such that \( S \leq_f T \) for some function \( f : \text{parts}(S) \to \text{parts}(T) \) and let \( C \) be a \( \emptyset^e \)-p.r. promise for \( T \).

a) The predicate “\( T \) is an infinite \( k \)-partition tree of \([t, +\infty)\)” is \( \Pi_1^0 \) uniformly in \( T, k \) and \( t \).

b) The relations “\( S \) \( f \)-refines \( T \)” and “part \( \nu \) of \( S \) \( f \)-refines part \( \mu \) of \( T \)” are \( \Pi_0^0 \) uniformly in \( S, T \) and \( f \).

c) The predicate “\( C \) is a promise for \( T \)” is \( \Pi_2^0 \) uniformly in an index for \( C \) and \( T \).

**Proof.**

a) \( T \) is an infinite \( k \)-partition tree of \([t, +\infty)\) if and only if the \( \Pi_1^0 \) formula \([(\forall \sigma \in T)(\forall \tau \leq \sigma) \tau \in T \land k^{<\infty}\land [(\forall n)(\exists \tau \in k^n) \tau \in T]) \) holds.

b) Suppose that \( T \) is a \( k \)-partition tree of \([t, +\infty)\) and \( S \) is an \( \ell \)-partition tree of \([u, +\infty)\).

\( S \) \( f \)-refines \( T \) if and only if the \( \Pi_1^0 \) formula \( u \geq t \land \ell \geq k \land [(\forall \sigma \in S)\left( (\exists \tau \in k^{[\sigma]})(\forall \nu < u)\text{set}_\nu(\sigma) \subseteq \text{set}_{f(\nu)}(\tau) \right)] \) holds. Part \( \nu \) of \( S \) \( f \)-refines part \( \mu \) of \( T \) if and only if \( \mu = f(\nu) \) and \( S \) \( f \)-refines \( T \).

c) Given \( k, t \in \omega \), let \( \text{PartTree}(k, t) \) denote the \( \Pi_1^0 \) set of all the infinite p.r. \( k \)-partition trees of \([t, +\infty)\). Given a k-partition tree S and a part \( \nu \) of \( S \), let \( \text{Empty}(S, \nu) \) denote the \( \Pi_1^0 \) formula “part \( \nu \) of \( S \) is empty”, that is the formula \((\forall \sigma \in S)\text{set}_\nu(\sigma) \cap \text{dom}(S) = \emptyset \).

\( C \) is a promise for \( T \) if and only if the following \( \Pi_2^0 \) formula holds:

\[
(\forall \ell, u)(\forall S \in \text{PartTree}(\ell, u))[S \leq T \rightarrow (\exists \nu < \ell)\neg \text{Empty}(S, \nu) \land (\nu, S) \in C)] \\
\land (\forall \ell', u')(\forall V \in \text{PartTree}(\ell', u'))((\forall g : \ell \to \ell')\left( S \leq g V \leq T \rightarrow (\forall \nu < \ell)\left( (\nu, S) \in C \rightarrow (g(\nu), V) \in C \right)\right)]
\]

Given a promise \( C \) for \( T \) and some infinite p.r. partition tree \( S \) refining \( T \), we denote by \( C[S] \) the set of all \((\nu, S') \in C \) below some \((\mu, S) \in C \), that is, \( C[S] = C \cap U[S] \). Note that by clause b) of Lemma 3.10, if \( C \) is \( \emptyset^e \)-p.r. promise for \( T \) then \( C[S] \) is a \( \emptyset^e \)-p.r. promise for \( S \).

Establishing a distinction between the acceptable parts and the non-acceptable ones requires a lot of definitional power. However, we prove that we can always find an extension where the distinction is \( \Delta_2^0 \). We say that an infinite p.r. partition tree \( T \) witnesses its acceptable parts if its parts are either acceptable or empty.

**Lemma 3.11** For every infinite p.r. \( k \)-partition tree \( T \) of \([t, +\infty)\), there exists an infinite p.r. \( k \)-partition tree \( S \) of \([u, +\infty)\) refining \( T \) with the identity function and such that \( S \) witnesses its acceptable parts.

**Proof.** Given a partition tree \( T \), we let \( I(T) \) be the set of its empty parts. Fix an infinite p.r. \( k \)-partition tree of \([t, +\infty)\) \( T \). It suffices to prove that if \( \nu \) is a non-empty and non-acceptable part of \( T \), then there exists an infinite p.r. \( k \)-partition tree \( S \) refining \( T \) with the identity function, such that \( \nu \in I(S) \setminus I(T) \). As \( I(T) \subseteq I(S) \) and \( |I(S)| \leq k \), it suffices to iterate the process at most \( k \) times to obtain a refinement witnessing its acceptable parts.

So fix a non-empty and non-acceptable part \( \nu \) of \( T \). By definition of being non-acceptable, there exists a path \( P \) through \( T \) and an integer \( u > \text{max}(t, \text{set}_\nu(P)) \). Let \( S = \{\sigma \in T : \ldots\} \).
Let $S$ be a p.r. $k$-partition tree of $[u, +\infty)$ refining $T$ with the identity function and such that part $\nu$ of $S$ is empty. Moreover, $S$ is infinite since $P \in [S]$. □

The following lemma strengthen clause b) of Definition 3.9.

**Lemma 3.12** Let $T$ be a p.r. partition tree and $C$ be a promise for $T$. For every infinite p.r. partition tree $S \leq T$, $(\mu, S) \in C$ for some acceptable part $\mu$ of $S$.

**Proof.** Fix an infinite p.r. $\ell$-partition tree $S \leq T$. By Lemma 3.11, there exists an infinite p.r. $\ell$-partition tree $S' \leq_{id} S$ witnessing its acceptable parts. As $C$ is a promise for $T$ and $S' \leq T$, there exists a non-empty (hence acceptable) part $\nu$ of $S'$ such that $(\nu, S') \in C$. As $C$ is upward-closed, $(\nu, S) \in C$. □

### 3.3. Forcing conditions

We now describe the forcing notion for the Erdős-Moser theorem. Recall that an EM condition for an infinite tournament $R$ is a Mathias condition $(F, X)$ where $F \cup \{x\}$ is $R$-transitive for each $x \in X$ and $X$ is included in a minimal $R$-interval of $F$.

**Definition 3.13** We denote by $\mathbb{P}$ the forcing notion whose conditions are tuples $(\vec{F}, T, C)$ where

(a) $T$ is an infinite p.r. partition tree
(b) $C$ is a $\emptyset'$-p.r. promise for $T$
(c) $(F, \nu, \text{dom}(T))$ is an EM condition for $R$ and each $\nu < \text{parts}(T)$

A condition $d = (\vec{E}, S, \mathcal{D})$ extends $c = (\vec{F}, T, C)$ (written $d \leq c$) if there exists a function $f : \ell \to k$ such that $\mathcal{D} \subseteq C$ and the following holds:

(i) $(E, \nu, \text{dom}(S))$ EM extends $(F, \nu, \text{dom}(T))$ for each $\nu < \text{parts}(S)$
(ii) $S$ $f$-refines $\bigcap_{\nu < \text{parts}(S)} T^{[f(\nu), \text{dom}(\nu)]}$

We may think of a condition $c = (\vec{F}, T, C)$ as a collection of EM conditions $(F, H, \nu)$ for $R$, where $H_\nu = \text{dom}(T) \cap \text{set}_\nu(P)$ for some path $P$ through $T$. $H_\nu$ must be infinite for at least one of the parts $\nu < \text{parts}(T)$. At a higher level, $\mathcal{D}$ restricts the possible subtrees $S$ and parts $\mu$ refining some part of $T$ in the condition $c$. Given a condition $c = (\vec{F}, T, C)$, we write $\text{parts}(c)$ for parts($T$).  

**Lemma 3.14** For every condition $c = (\vec{F}, T, C)$ and every $n \in \omega$, there exists an extension $d = (\vec{E}, S, \mathcal{D})$ such that $|E_\nu| \geq n$ on each acceptable part $\nu$ of $S$.

**Proof.** It suffices to prove that for every condition $c = (\vec{F}, T, C)$ and every acceptable part $\nu$ of $T$, there exists an extension $d = (\vec{E}, S, \mathcal{D})$ such that $S \leq_{id} T$ and $|E_\nu| \geq n$. Iterating the process at most $\text{parts}(T)$ times enables us to conclude. Fix an acceptable part $\nu$ of $T$ and a path $P$ trough $T$ such that $\text{set}_\nu(P)$ is infinite. Let $F'$ be an $R$-transitive subset of $\text{set}_\nu(P) \cap \text{dom}(T)$ of size $n$. Such a set exists by the classical Erdős-Moser theorem. Let $\vec{E}$ be defined by $E_\mu = F_\mu$ if $\mu \neq \nu$ and $E_\nu = F_\nu \cup F'$ otherwise. As the tournament $R$ is stable, there exists some $u \geq t$ such that $(E, \nu, [u, +\infty))$ is an EM condition and therefore EM extends $(F_\nu, \text{dom}(T))$. Let $\vec{S}$ be the p.r. partition tree $T^{[\nu, \text{dom}(\nu)]}$ of $[u, +\infty)$. The condition $(\vec{E}, S, C[S])$ is the desired extension. □

Given a condition $c \in \mathbb{P}$, we denote by $\text{Ext}(c)$ the set of all its extensions.

### 3.4. The forcing relation

The forcing relation at the first level, namely, for $\Sigma^0_1$ and $\Pi^0_1$ formulas, is parameterized by some part of the tree of the considered condition. Thanks to the forcing relation we will define, we can build an infinite decreasing sequence of conditions which decide $\Sigma^0_1$ and $\Pi^0_1$ formulas effectively in $\emptyset'$. The sequence however yields a $\emptyset'$-computably bounded $\emptyset'$-computable tree of (possibly empty) parts. Therefore, any PA degree relative to $\emptyset'$ is sufficient to control the first jump of an infinite transitive subtournament of a stable infinite computable tournament.
We cannot do better since Kreuzer proved in [11] the existence of an infinite, stable, computable tournament with no low infinite transitive subtournament. If we ignore the promise part of a condition, the careful reader will recognize the construction of Cholak, Jockusch and Slaman [4] of a low₂ infinite subset of a Δ²₁ set or its complement by the first jump control. The difference, which at first look seems only presentational, is in fact one of the key features of this notion of forcing. Indeed, forcing iterated jumps require to have a definitionally weak description of the set of the extensions of a condition, and it requires much less computational power to describe a primitive recursive tree than an infinite reservoir of a Mathias condition.

**Definition 3.15** Fix a condition \( c = (\vec{F}, T, C) \), a Σ₀¹ formula \( \varphi(G, x) \) and a part \( \nu < \text{parts}(T) \).

1. \( c \models_\nu (\exists x) \varphi(G, x) \) iff there exists a \( w \in \omega \) such that \( \varphi(F_\nu, w) \) holds.
2. \( c \models_\nu (\forall x) \varphi(G, x) \) iff for every \( \sigma \in T, \) every \( w < |\sigma| \) and every \( R \)-transitive set \( F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma), \varphi(F_\nu \cup F', w) \) holds.

We start by proving some basic properties of the forcing relation over Σ₀¹ and Π₀¹ formulas. As one may expect, the forcing relation at first level is closed under the refinement relation.

**Lemma 3.16** Fix a condition \( c = (\vec{F}, T, C) \) and a Σ₀¹ (Π₀¹) formula \( \varphi(G) \). If \( c \models_\nu \varphi(G) \) for some \( \nu < \text{parts}(T) \), then for every \( d = (\vec{E}, S, D) \leq c \) and every part \( \mu \) of \( S \) refining part \( \nu \) of \( T \), \( d \models_\mu \varphi(G) \).

**Proof.** We have two cases.

- If \( \varphi \in \Sigma_0^1 \) then it can be expressed as \((\exists x)\psi(G, x)\) where \( \psi \in \Sigma_0^0 \). By clause 1 of Definition 3.15, there exists a \( w \in \omega \) such that \( \psi(F_\nu, w) \) holds. By property (i) of the definition of an extension, \( E_\mu \supseteq F_\nu \) and \( (E_\mu \setminus F_\nu) \subseteq \text{dom}(T) \), therefore \( \psi(E_\mu, w) \) holds by continuity, so by clause 1 of Definition 3.15, \( d \models_\mu (\exists x)\psi(G, x) \).
- If \( \varphi \in \Pi_0^1 \) then it can be expressed as \((\forall x)\psi(G, x)\) where \( \psi \in \Sigma_0^0 \). Fix a \( \tau \in S, \) a \( w < |\tau| \) and an \( R \)-transitive set \( F' \subseteq \text{dom}(S) \cap \text{set}_\nu(\tau) \). It suffices to prove that \( \varphi(E_\mu \cup F') \) holds to conclude that \( d \models_\mu (\forall x)\psi(G, x) \) by clause 2 of Definition 3.15. By property (ii) of the definition of an extension, there exists a \( \sigma \in T[^{|\nu,E_\mu}|] \) such that \( |\sigma| = |\tau| \) and \( \text{set}_\nu(\tau) \subseteq \text{set}_\nu(\sigma) \). As \( \text{dom}(S) \subseteq \text{dom}(T), F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \). As \( \sigma \in T[^{|\nu,E_\mu}|], E_\mu \subseteq \text{set}_\nu(\sigma) \) and by property (i) of the definition of an extension, \( E_\mu \subseteq \text{dom}(T) \). So \( E_\mu \cup F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \). As \( w < |\tau| = |\sigma| \) and \( E_\mu \cup F' \) is an \( R \)-transitive subset of \( \text{dom}(T) \cap \text{set}_\nu(\sigma) \), then by clause 2 of Definition 3.15 applied to \( c \models_\nu (\forall x)\psi(G, x) \), \( \varphi(F_\nu \cup (E_\mu \setminus F_\nu) \cup F', w) \) holds, hence \( \varphi(E_\mu \cup F') \) holds.

\( \square \)

Before defining forcing relation at the higher levels, we prove a density lemma for Σ₀¹ and Π₀¹ formulas. It enables us in particular to reprove that every PA relative to \( \emptyset' \) computes the jump of an infinite \( R \)-transitive set.

**Lemma 3.17** For every \( \Sigma_0^1 \) (\( \Pi_0^1 \)) formula \( \varphi \), the following set is dense

\[ \{ c = (\vec{F}, T, C) \in P : (\forall \nu < \text{parts}(T)) [ c \models_\nu \varphi(G) \lor c \models_\nu \neg \varphi(G) ] \} \]

**Proof.** It suffices to prove the statement for the case where \( \varphi \) is a \( \Sigma_0^1 \) formula, as the case where \( \varphi \) is a \( \Pi_0^1 \) formula is symmetric. Fix a condition \( c = (\vec{F}, T, C) \) and let \( I(c) \) be the set of the parts \( \nu < \text{parts}(T) \) such that \( c \models_\nu \varphi(G) \) and \( c \not\models_\nu \neg \varphi(G) \). If \( I(c) = \emptyset \) then we are done, so suppose \( I(c) \neq \emptyset \) and fix some \( \nu \in I(c) \). We will construct an extension \( d \) of \( c \) such that \( I(d) \subseteq I(c) \setminus \{ \nu \} \). Applying iteratively the operation enables us to conclude.

The formula \( \varphi \) is of the form \((\exists x)\psi(G, x)\) where \( \psi \in \Sigma_0^0 \). Define \( f : k + 1 \to k \) as \( f(\mu) = \mu \) if \( \mu < k \) and \( f(k) = \nu \) otherwise. Let \( S \) be the set of all \( \sigma \in (k + 1)^\omega \) which \( f \)-refine some \( \tau \in T \setminus k^{\sigma} \) and such that for every \( w < |\sigma| \), every part \( \mu \in \{ \nu, k \} \) and every finite \( R \)-transitive set \( F' \subseteq \text{dom}(T) \cap \text{set}_\mu(\sigma) \), \( \varphi(F_\nu \cup F', w) \) does not hold.
Note that $S$ is a p.r. partition tree of $[t, +\infty)$ refining $T$ with witness function $f$. Suppose that $S$ is infinite. Let $\tilde{E}$ be defined by $E_\mu = F_\mu$ if $\mu < k$ and $E_k = F_\nu$ and consider the extension $d = (\tilde{E}, S, C[S])$. We claim that $\nu, k \notin I(d)$. Fix a part $\mu \in \{\nu, k\}$ of $S$. By definition of $S$, for every $\sigma \in S$, every $w < |\sigma|$ and every $R$-transitive set $F' \subseteq \operatorname{dom}(S) \cap \sigma$ such that $\varphi(E_\mu \cup F', w)$ does not hold. Therefore, by clause 2 of Definition 3.15, $d \notsat (\forall x)(\psi(G, x))$, hence $d \notsat \neg \varphi(G)$. Note that $I(d) \subseteq I(c) \setminus \{\nu\}$.

Suppose now that $S$ is finite. Fix a threshold $t \in \omega$ such that $(\forall \sigma \in S)|\sigma| < t$ and a $\tau \in T \cap k^\ell$ such that $T[\tau]$ is infinite. Consider the 2-partition $E_0 \cup E_1$ of $\operatorname{set}_t(T)$ and $\operatorname{dom}(T)$ defined by $E_0 = \{i \geq t : \tau(i) = \nu \land (\forall \infty)s)(R(i, s) \text{ holds})$ and $E_1 = \{i \geq t : \tau(i) = \nu \land (\forall \infty)s)(R(s, i) \text{ holds})$. This is a 2-partition since the tournament $R$ is stable. As there exists no $\sigma \in S$ which $f$-refines $\tau$, there exists a $w < \tau$ and an $R$-transitive set $F' \subseteq E_0$ or $F' \subseteq E_1$ such that $\varphi(F_\nu \cup F', w)$ holds. By choice of the partition, there exists a $t' > t$ such that $F' \to R [t', +\infty)$ or $[t', +\infty) \to R F'$. By Lemma 3.6, $(F_\nu \cup F', [t', +\infty))$ is a valid EM extension for $(F_\nu, [t, +\infty))$. As $T[\tau]$ is infinite, $T[\nu, F']$ is also infinite. Let $\tilde{E}$ be defined by $E_\mu = F_\mu$ if $\mu \neq \nu$ and $E_\mu = F_\nu \cup F'$ otherwise. Let $S$ be the $k$-partition tree $(k, t', T[\nu, F'])$. The condition $d = (\tilde{E}, S, C[S])$ is a valid extension of $c$. By clause 1 of Definition 3.15, $d \sat \varphi(G)$. Therefore $I(d) \subseteq I(c) \setminus \{\nu\}$. □

As in the previous notion of forcing, the following trivial lemma expresses the fact that the promise part of a condition has no effect in the forcing relation for a $\Sigma^0_n$ or $\Pi^0_n$ formula.

**Lemma 3.18** Fix two conditions $c = (\tilde{F}, T, C)$ and $d = (\tilde{F}, T, D)$, and a $\Sigma^0_n$ ($\Pi^0_n$) formula. For every part $\nu$ of $T$, $c \sat \varphi(G)$ if and only if $d \sat \varphi(G)$.

**Proof.** If $\varphi \in \Sigma^0_n$ then $\varphi(G)$ can be expressed as $(\exists x)(\psi(G, x))$ where $\psi \in \Sigma^0_n$. By clause 1 of Definition 3.15, $c \sat \varphi(G)$ iff there exists a $w \in \omega$ such that $\psi(F_\nu, w)$ holds, iff $d \sat \varphi(G)$. Similarly, if $\varphi \in \Pi^0_n$ then $\varphi(G)$ can be expressed as $(\forall x)(\psi(G, x))$ where $\psi \in \Sigma^0_n$. By clause 2 of Definition 3.15, $c \sat \varphi(G)$ iff for every $\sigma \in T$, every $w < |\sigma|$ and every $R$-transitive set $F' \subseteq \operatorname{dom}(T) \cap \sigma$, $\varphi(F_\nu \cup F', w)$ holds, iff $d \sat \varphi(G)$. □

We are now ready to define the forcing relation for an arbitrary arithmetic formula. Again, the natural forcing relation induced by the forcing of $\Sigma^0_n$ formulas is too complex, so we design a more effective relation which still enjoys the main properties of a forcing relation.

**Definition 3.19** Fix a condition $c = (\tilde{F}, T, C)$ and an arithmetic formula $\varphi(G)$.

1. If $\varphi(G) = (\exists x)(\psi(G, x))$ where $\psi \in \Pi^0_1$ then $c \sat \varphi(G)$ iff for every part $\nu < \operatorname{parts}(T)$ such that $(\nu, T) \in C$ there exists a $w < \operatorname{dom}(T)$ such that $c \sat \varphi(G, w)$.
2. If $\varphi(G) = (\forall x)(\psi(G, x))$ where $\psi \in \Sigma^0_1$ then $c \sat \varphi(G)$ iff for every infinite p.r. $k^\ell$-partition tree $S$, every function $f : \operatorname{parts}(S) \to \operatorname{parts}(T)$, every $w$ and $\tilde{E}$ smaller than $\#S$ such that the following hold
   i) $(E_\nu, \operatorname{dom}(S))$ EM extends $(F_{f(\nu)}, \operatorname{dom}(T))$ for each $\nu < \operatorname{parts}(S)$
   ii) $S$ $f$-refines $\bigcap_{\nu < \operatorname{parts}(S)} T[f(\nu), E_\nu]$
   for every $(\mu, S) \in C$, $(\tilde{E}, S, C[S]) \notsat \neg \psi(G, w)$.
3. If $\varphi(G) = (\exists x)(\psi(G, x))$ where $\psi \in \Pi^0_{n+2}$ then $c \sat \varphi(G)$ iff there exists a $w \in \omega$ such that $c \sat \psi(G, w)$
4. If $\varphi(G) = \neg \psi(G, x)$ where $\psi \in \Sigma^0_{n+3}$ then $c \sat \varphi(G)$ iff $d \notsat \psi(G)$ for every $d \in \operatorname{Ext}(c)$.

Notice that, unlike the forcing relation for $\Sigma^0_n$ and $\Pi^0_n$ formulas, the relation over higher formulas does not depend on the part of the relation. The careful reader will have recognized the combinatorics of the second jump control introduced by Cholak, Jockusch and Slaman in [4]. We now prove the main properties of this forcing relation.

**Lemma 3.20** Fix a condition $c$ and a $\Sigma^0_{n+2}$ ($\Pi^0_{n+2}$) formula $\varphi(G)$. If $c \sat \varphi(G)$ then for every $d \leq c$, $d \sat \varphi(G)$. 


Lemma 3.21

We prove the statement by induction over the complexity of the formula $\varphi(G)$. Fix a condition $c = (\vec{F}, T, C)$ such that $c \models \varphi(G)$ and an extension $d = (\vec{E}, S, D)$ of $c$.

- If $\varphi \in \Sigma^0_{n+2}$ then $\varphi(G)$ can be expressed as $(\exists x) \psi(G, x)$ where $\psi \in \Pi^0_{n+1}$. By clause 1 of Definition 3.19, for every part $\nu$ of $T$ such that $(\nu, T) \in C$, there exists a $w < \text{dom}(T)$ such that $c \models_{\nu} \psi(G, w)$. Fix a part $\mu$ of $S$ such that $(\mu, S) \in D$. As $D \subseteq C$, $(\mu, S) \in C$. By upward-closure of $C$, part $\mu$ of $S$ refines some part $\nu$ of $C$ such that $(\nu, T) \in C$. Therefore by Lemma 3.16, $d \models_{\mu} \psi(G, w)$, with $w < \text{dom}(T) \leq \text{dom}(S)$. Applying again clause 1 of Definition 3.19, we deduce that $d \models (\forall x) \psi(G, x)$, hence $d \models \varphi(G)$.

- If $\varphi \in \Pi^0_{n+3}$ then $\varphi(G)$ can be expressed as $(\forall x) \psi(G, x)$ where $\psi \in \Sigma^0_{n+3}$. Suppose for the sake of absurd that $d \not\models (\forall x) \psi(G, x)$. Let $f : \text{parts}(S) \to \text{parts}(T)$ witness the refinement $S \subseteq T$. By clause 2 of Definition 3.19, there exists an infinite p.r. $k'$-partition tree $S'$, a function $g : \text{parts}(S') \to \text{parts}(T)$, $w \in \omega$, and $\vec{H}$ smaller than the code of $S'$ such that
  
  \begin{itemize}
  \item[i)] $(H_{\nu}, \text{dom}(S'))$ EM extends $(E_{g(\nu)}, \text{dom}(S'))$ for each $\nu < \text{parts}(S')$
  \item[ii)] $S'$ $g$-refines $\bigcap_{\nu < \text{parts}(S')} S_{[g(\nu), H_{\nu}]}$ for each $\nu < \text{parts}(S')$
  \item[iii)] there exists a $(\mu, S') \in D$ such that $(\vec{H}, S', D[S']) \models_{\mu} \lnot \psi(G, w)$.
  \end{itemize}

To deduce by clause 2 of Definition 3.19 that $d \not\models (\forall x) \psi(G, x)$ and derive a contradiction, suppose that

\begin{itemize}
  \item[i)] As by property (i) of the definition of an extension, $(E_{g(\nu)}, \text{dom}(S'))$ EM extends $(F_{f(g(\nu))}, \text{dom}(T))$ and $(H_{\nu}, \text{dom}(S'))$ EM extends $(E_{g(\nu)}, \text{dom}(S'))$, then $(H_{\nu}, \text{dom}(S'))$ EM extends $(F_{f(g(\nu))}, \text{dom}(T))$.
  \item[ii)] As by property (ii) of the definition of an extension, $S$ $f$-refines $\bigcap_{\nu < \text{parts}(S')} T^{[f(g(\nu)), E_{g(\nu)}]}$ and $S'$ $g$-refines $\bigcap_{\nu < \text{parts}(S')} T^{[g(f(\nu)), H_{\nu}]}$, then $S'$ $(g \circ f)$-refines $\bigcap_{\nu < \text{parts}(S')} T^{[g(f(\nu)), H_{\nu}]}$.
  \item[iii)] As $D \subseteq C$, there exists a part $(\mu, S') \in C$ such that $(\vec{H}, S', D[S']) \models_{\mu} \lnot \psi(G, w)$. By Lemma 3.18, $(\vec{H}, S', C[S']) \not\models_{\mu} \lnot \psi(G, w)$.
\end{itemize}

- If $\varphi \in \Sigma^0_{n+3}$ then $\varphi(G)$ can be expressed as $(\exists x) \psi(G, x)$ where $\psi \in \Pi^0_{n+2}$. By clause 3 of Definition 3.19, there exists a $w \in \omega$ such that $c \models \psi(G, w)$. By induction hypothesis, $d \models \psi(G, w)$ so by clause 3 of Definition 3.19, $d \not\models \varphi(G)$.

- If $\varphi \in \Pi^0_{n+3}$ then $\varphi(G)$ can be expressed as $\lnot \psi(G)$ where $\psi \in \Sigma^0_{n+3}$. By clause 4 of Definition 3.19, for every $e \in \text{Ext}(c)$, $e \not\models \psi(G)$. As $\text{Ext}(d) \subseteq \text{Ext}(c)$, for every $e \in \text{Ext}(d)$, $e \not\models \psi(G)$, so by clause 4 of Definition 3.19, $d \not\models \varphi(G)$.

\begin{flushright}
$\square$
\end{flushright}

Lemma 3.21

For every $\Sigma^0_{n+2}$ ($\Pi^0_{n+2}$) formula $\varphi$, the following set is dense

{ $c \in P : c \models \varphi(G)$ or $c \models \lnot \varphi(G)$ }

Proof. We prove the statement by induction over $n$. It suffices to treat the case where $\varphi$ is a $\Sigma^0_{n+2}$ formula, as the case where $\varphi$ is a $\Pi^0_{n+2}$ formula is symmetric. Fix a condition $c = (\vec{F}, T, C)$.

- In case $n = 0$, the formula $\varphi$ is of the form $(\exists x) \psi(G, x)$ where $\psi \in \Pi^0_{n+1}$. Suppose there exists an infinite p.r. $k'$-partition tree $S$ for some $k' \in \omega$, a function $f : \text{parts}(S) \to \text{parts}(T)$ and a $k'$-tuple of finite sets $E$ such that
  
  \begin{itemize}
  \item[i)] $(E_{\nu}, [\ell, +\infty))$ EM extends $(F_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$.
  \item[ii)] $S$ $f$-refines $\bigcap_{\nu < \text{parts}(S)} T^{[f(\nu), E_{\nu}]}$ for each $\nu < \text{parts}(S)$.
  \item[iii)] for each non-empty part $\nu$ of $S$ such that $(\nu, S) \in C$, $(\vec{E}, S, C[S]) \models_{\nu} \psi(G, w)$ for some $w < \# S$.
  \end{itemize}

We can choose dom$(S)$ so that $(E_{\nu}, \text{dom}(S))$ EM extends $(F_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$. Properties i-ii) remain trivially true. By Lemma 3.16 and Lemma 3.18, property iii) remains true too. Let $D = C[S] \setminus \{(\nu, S') \in C : \text{ part } \nu \text{ of } S' \text{ is empty}\}$. As $C$ is an $\emptyset'$-p.r. promise for $T$, $C[S]$ is an $\emptyset'$-p.r. promise for $S$. As $D$ is obtained from $C[S]$ by removing only empty parts, $D$ is also an $\emptyset'$-p.r. promise for $S$. By clause 1 of Definition 3.19, $d = (\vec{E}, S, D) \models (\exists x) \psi(G, x)$ hence $d \not\models \varphi(G)$.
We may choose a coding of the p.r. trees such that such that the code of $S$ is sufficiently large to witness $\ell$ and $\bar{E}$. So suppose now that for every infinite p.r. $k'$-partition tree $S$, every function $f : \text{parts}(S) \rightarrow \text{parts}(T)$ and $\bar{E}$ smaller than the code of $S$ such that properties i-ii) hold, there exists a non-empty part $\nu$ of $S$ such that $\nu \in C$ and $(\bar{E}, S, C) \not\models \psi(G, w)$ for every $w < \ell$. Let $D$ be the collection of all such $(\nu, S)$.

The set $D$ is $\emptyset'$-p.r. since by Lemma 3.25, both $(\bar{E}, S, C) \not\models \psi(G, w)$ and “part $\nu$ of $S$ is non-empty” are $\Sigma^0_1$. By Lemma 3.16 and since we require that $\#S \geq \#T$ in the definition of $S \leq T$, $D$ is upward-closed under the refinement relation, hence is a promise for $T$. By clause 2 of Definition 3.19, $d = (\bar{F}, T, D) \models (\forall x)\neg \psi(G, x)$, hence $d \models \neg \varphi(G)$.

In case $n > 0$, density follows from clause 4 of Definition 3.19.

\\[\square\\]

Given any filter $F = \{c_0, c_1, \ldots\}$ with $c_n = (\bar{F}_n, T_n, C_n)$, the set of the acceptable parts $\nu$ of $T_n$ such that $(\nu, T_n) \in C_n$ forms an infinite, directed acyclic graph $G(F)$. Whenever $F$ is sufficiently generic, the graph $G(F)$ has a unique infinite path $P$. The path $P$ induces an infinite set $G = \bigcup_s F_{P(s), s}$. We call $P$ the generic path and $G$ the generic real.

**Lemma 3.22** Suppose that $F$ is sufficiently generic and let $P$ and $G$ be the generic path and the generic real, respectively. For any $\Sigma^0_1$ ($\Pi^0_1$) formula $\varphi(G)$, $\varphi(G)$ holds iff $c_s \models_{P(s)} \varphi(G)$ for some $c_s \in F$.

**Proof.** Fix a condition $c_s = (\bar{F}, T, C) \in F$ such that $c \models_{P(s)} \varphi(G)$, and let $\nu = P(s)$.

- If $\varphi \in \Sigma^0_1$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_1$. By clause 1 of Definition 3.15, there exists a $w \in \omega$ such that $\psi(F(w), w)$ holds. As $\nu = P(s)$, $F_{\nu} = F_{P(s)} \subseteq G$ and $G \setminus F_{\nu} \subseteq (\max(F_{\nu}), +\infty)$, so $\psi(G, w)$ holds by continuity, hence $\varphi(G)$ holds.

- If $\varphi \in \Pi^0_1$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$. By clause 2 of Definition 3.15, for every $\sigma \in T$, every $w < |\sigma|$ and every $\nu$-transitive set $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$, $\psi(F(w), w)$ holds. For every $F' \subseteq G \setminus F_{\nu}$, and $w \in \omega$ there exists a $\sigma \in T$ such that $w < |\sigma|$ and $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$. Hence $\psi(F(w), w)$ holds. Therefore, for every $w \in \omega$, $\psi(G, w)$ holds, so $\varphi(G)$ holds.

The other direction holds by Lemma 3.17.  

**Lemma 3.23** Suppose that $F$ is sufficiently generic and let $P$ and $G$ be the generic path and the generic real, respectively. For any $\Sigma^0_{n+2}$ ($\Pi^0_{n+2}$) formula $\varphi(G)$, $\varphi(G)$ holds iff $c_s \models \varphi(G)$ for some $c_s \in F$.

**Proof.** We prove the statement by induction over the complexity of the formula $\varphi(G)$. As previously noted in Lemma 2.13, it suffices to prove that if $c_s \models \varphi(G)$ for some $c_s \in F$ then $\varphi(G)$ holds. Indeed, conversely if $\varphi(G)$ holds, then by Lemma 3.21 and by genericity of $F$ either $c_s \models \varphi(G)$ or $c_s \models \neg \varphi(G)$, but if $c \models \neg \varphi(G)$ then $\neg \varphi(G)$ holds, contradicting the hypothesis. So $c_s \models \varphi(G)$. Fix a condition $c_s = (\bar{F}, T, C) \in F$ such that $c_s \models \varphi(G)$. We proceed by case analysis on $\varphi$.

- If $\varphi \in \Sigma^0_2$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_1$. By clause 1 of Definition 3.19, for every part $\nu$ of $T$ such that $(\nu, T) \in C$, there exists a $w < \text{dom}(T)$ such that $c_s \models_{\nu} \psi(G, w)$. In particular $(P(s), T) \in C$, so $c_s \models_{P(s)} \psi(G, w)$. By Lemma 3.22, $\psi(G, w)$ holds, hence $\varphi(G)$ holds.

- If $\varphi \in \Pi^0_2$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$. By clause 2 of Definition 3.19, for every infinite $k'$-partition $S$, every function $f : \text{parts}(S) \rightarrow \text{parts}(T)$, every $w$ and $\bar{E}$ smaller than the code of $S$ such that the following hold
  
i) $(E_{\nu}, \text{dom}(S))$ EM extends $(F_{f(\nu)}, \text{dom}(T))$ for every $\nu \in \text{parts}(S)
  
ii) $S$ $f$-refines $\bigcap_{\nu \in \text{parts}(S)} T[f(\nu), E_{\nu}]$
for every $(\mu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \not\models \lnot\psi(G, w)$. Suppose for the sake of absurd that
\[ \psi(G, w) \] does not hold for some $w \in \omega$. Then by Lemma 3.22, there exists a $d_t \in \mathcal{F}$ such that $d_t \not\models P(t) \lnot\psi(G, w)$. Since $\mathcal{F}$ is a filter, there is a condition $e_r = (\vec{E}, S, \mathcal{D}) \in \mathcal{F}$ extending both $c_s$ and $d_t$. Let $\mu = P(r)$. By choice of $P$, $(\mu, S) \in \mathcal{C}$, so by clause ii), $(\vec{E}, S, \mathcal{C}[S]) \not\models \lnot\psi(G, w)$, hence by Lemma 3.18, $e_r \not\models \lnot\psi(G, w)$. However, since part $\mu$ of $S$ refines part $P(t)$ of $d_t$, then by Lemma 3.16, $e_r \not\models \lnot\psi(G, w)$. Contradiction. Hence for every $w \in \omega$, $\psi(G, w)$ holds, so $\varphi(G)$ holds.

- If $\varphi \in \Sigma^0_{n+3}$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_n$. By clause 3 of Definition 3.19, there exists a $w \in \omega$ such that $c_s \models \psi(G, w)$. By induction hypothesis, $\psi(G, w)$ holds, hence $\varphi(G)$ holds.

Conversely, if $\varphi(G)$ holds, then there exists a $w \in \omega$ such that $\psi(G, w)$ holds, so by induction hypothesis $c_s \models \psi(G, w)$ for some $c_s \in \mathcal{F}$, so by clause 3 of Definition 3.19, $c_s \models \varphi(G)$.

- If $\varphi \in \Pi^0_{n+3}$ then $\varphi(G)$ can be expressed as $\lnot\psi(G)$ where $\psi \in \Sigma^0_n$. By clause 4 of Definition 3.19, for every $d \in \text{Ext}(c_s)$, $d \not\models \psi(G)$. By Lemma 3.20, $d \not\models \psi(G)$ for every $d \in \mathcal{F}$ and by a previous case, $\psi(G)$ does hold, so $\varphi(G)$ holds.

We now prove that the forcing relation has good definitional properties as we did with the notion of forcing for cohesiveness.

**Lemma 3.24** For every condition $c$, $\text{Ext}(c)$ is $\Pi^0_2$ uniformly in $c$.

**Proof.** Recall from Lemma 3.10 that given $k, t \in \omega$, $\text{PartTree}(k, t)$ denotes the $\Pi^0_k$ set of all the infinite p.r. $k$-partition trees of $(t, +\infty)$, and given a $k$-partition tree $S$ and a part $\nu$ of $S$, the predicate $\text{Empty}(S, \nu)$ denotes the $\Pi^0_k$ formula “part $\nu$ of $S$ is empty”, that is, the formula $(\forall \sigma \in S)[\sigma \cap \text{dom}(S) = \emptyset]$. If $T$ is p.r. then so is $T^{[\nu, H]}$ for some finite set $H$.

Fix a condition $c = (\vec{F}, (k, t, T), C)$. By definition, $(\vec{H}, (k', t', S), \mathcal{D}) \in \text{Ext}(c)$ iff the following formula holds:

\[
(\exists f : k' \to k) \\
(\forall \sigma < k')(H_{\nu}, [t', +\infty)) \text{ EM extends } (F_{\nu}, [t, +\infty)) \\
\land \sigma \in \text{PartTree}(k', t') \land S \subseteq \bigwedge_{\nu < k'} T^{[\nu, H_{\nu}]} \\
\land \mathcal{D} \text{ is a promise for } S \land \mathcal{D} \subseteq C
\]

By Lemma 3.10 and the fact that $\bigwedge_{\nu < k'} T^{[\nu, H_{\nu}]}$ is p.r. uniformly in $T$, $f$, $\vec{H}$ and $k'$, the above formula is $\Pi^0_2$.

**Lemma 3.25** Fix an arithmetic formula $\varphi(G)$, a condition $c = (\vec{F}, T, C)$ and a part $\nu$ of $T$.

a) If $\varphi(G)$ is a $\Sigma^0_1(\Pi^0_1)$ formula then so is the predicate $c \models_{\nu} \varphi(G)$.

b) If $\varphi(G)$ is a $\Sigma^0_{n+2}(\Pi^0_{n+2})$ formula then so is the predicate $c \models_{\nu} \varphi(G)$.

**Proof.** We prove our lemma by induction over the complexity of the formula $\varphi(G)$.

- If $\varphi(G) \in \Sigma^0_1$ then it can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 1 of Definition 3.15, $c \models_{\nu} \varphi(G)$ if and only if the formula $(\exists w \in \omega)\psi(F_{\nu}, w)$ holds. This is a $\Sigma^0_0$ predicate.

- If $\varphi(G) \in \Pi^0_1$ then it can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 2 of Definition 3.15, $c \models_{\nu} \varphi(G)$ if and only if the formula $(\forall \sigma \in T)(\forall w < |\sigma|)(\forall F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma))[F' \text{ $R$-transitive } \rightarrow \psi(F_{\nu} \cup F', w)]$ holds. This is a $\Pi^0_1$ predicate.

- If $\varphi(G) \in \Sigma^0_2$ then it can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_1$. By clause 1 of Definition 3.19, $c \models_{\nu} \varphi(G)$ if and only if the formula $(\forall \nu < \text{parts}(T))(\exists w < \text{dom}(T))[(\nu, T) \in \mathcal{C} \rightarrow c \models_{\nu} \psi(G, w)]$ holds. This is a $\Sigma^0_2$ predicate by induction hypothesis and the fact that $\mathcal{C}$ is $\emptyset'$-computable.

- If $\varphi(G) \in \Pi^0_2$ then it can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$. By clause 2 of Definition 3.19, $c \models_{\nu} \varphi(G)$ if and only if for every infinite $k'$-partition tree $S$, every
function $f : \text{parts}(S) \to \text{parts}(T)$, every $w$ and $\bar{E}$ smaller than the code of $S$ such that

the following hold

\begin{enumerate}[
\item]
\item $(E_{\nu}, \text{dom}(S))$ EM extends $(F_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$
\item $S$ $f$-refines $\bigcap_{\nu < \text{parts}(S)} T_{f(\nu), E_{\nu}}$
\end{enumerate}

for every $(\mu, S) \in C$, $(\bar{E}, S, C[S]) \not\models \neg \varphi(G, w)$. By Lemma 3.10, Properties i-ii) are $\Delta^0_0$. Moreover, the predicate $(\mu, S) \in C$ is $\Delta^0_0$. By induction hypothesis, $(\bar{E}, S, C) \not\models \neg \varphi(G, w)$ is $\Sigma^0_1$. Therefore $c \models \varphi(G)$ is a $\Pi^0_0$ predicate.

- If $\varphi(G) \in \Sigma^0_{n+3}$ then it can be expressed as $(\exists x)\varphi(G, x)$ where $\psi \in \Sigma^0_{n+2}$. By clause 3 of Definition 3.19, $c \models \varphi(G)$ if and only if the formula $(\exists x \in \omega)c \models \psi(G, w)$ holds. This is a $\Sigma^0_{n+3}$ predicate by induction hypothesis.

- If $\varphi(G) \in \Pi^0_{n+3}$ then it can be expressed as $\neg \psi(G)$ where $\psi \in \Sigma^0_{n+3}$. By clause 4 of Definition 3.19, $c \models \varphi(G)$ if and only if the formula $(\forall d)(d \not\in \text{Ext}(c) \lor d \not\models \psi(G))$ holds. By induction hypothesis, $d \not\models \psi(G)$ is a $\Pi^0_{n+3}$ predicate. By Lemma 3.24, the set $\text{Ext}(c)$ is $\Pi^0_2$-computably uniformly in $c$, thus $c \models \varphi(G)$ is a $\Pi^0_{n+3}$ predicate.

\[\square\]

3.5. Preserving the arithmetic hierarchy

We now prove the core lemmas showing that every sufficiently generic real preserves the arithmetic hierarchy. The proof is split into two lemmas since the forcing relation for $\Sigma^0_1$ and $\Pi^0_1$ formulas depends on the part of the condition, and therefore has to be treated separately.

Lemma 3.26 If $A \not\in \Sigma^0_1$ and $\varphi(G, x)$ is $\Sigma^0_1$, then the set of $c = (\bar{F}, T, C) \in \mathbb{P}$ satisfying the following property is dense:

$$((\forall \nu < \text{parts}(T))[\exists w \in A]c_{\nu} \models \neg \varphi(G, w)] \lor [(\exists w \not\in A)c_{\nu} \models \varphi(G, w)]$$

Proof. The formula $\varphi(G, w)$ can be expressed as $(\exists x)\varphi(G, w, x)$ where $\psi \in \Sigma^0_0$. Given a condition $c = (\bar{F}, T, C)$, let $I(c)$ be the set of the parts $\nu$ of $T$ such that for every $w \in A$, $c \not\models \neg \varphi(G, w)$ and for every $w \not\in A$, $c \models \varphi(G, w)$. If $I(c) = \emptyset$ then we are done, so suppose $I(c) \neq \emptyset$ and fix some $\nu \in I(c)$. We will construct an extension $d$ such that $I(d) \subseteq I(c) \setminus \{\nu\}$. Applying iteratively the operation enables us to conclude.

Say that $T$ is a $k$-partition tree of $[t, +\infty)$ for some $k, t \in \omega$. Define $f : k + 1 \to k$ as $f(\mu) = \mu$ if $\mu < k$ and $f(k) = \nu$ otherwise. Given an integer $w \in \omega$, let $S_w$ be the set of all $\sigma \in (k + 1)^{<\omega}$ which $f$-refines some $\tau \in T \cap k^{[\sigma]}$ and such that for every $u < |\sigma|$, every part $\mu \in \{\nu, k\}$ and every finite $R$-transitive set $R' \subseteq \text{dom}(T) \cap \bigcup_{\mu \in \{\nu, k\}} \text{dom}(S_w)$, $\varphi(F'_\mu, F', w, u)$ does not hold.

The set $S_w$ is a p.r. (uniformly in $w$) partition tree of $[t, +\infty)$ refining $T$ with witness function $f$. Let $U = \{w \in \omega : S_w \text{ is finite}\}$. Then $U \subseteq \Sigma^0_1$, thus $U \neq \emptyset$. Fix some $w \in U \Delta A$. Suppose first that $w \in A \setminus U$. By definition of $U$, $S_w$ is infinite. Let $\bar{E}$ be defined by $E_\mu = F'_\nu$ if $\mu < k$ and $E_k = F_w$, and consider the extension $d = (\bar{E}, S_w, C[S_w])$. We claim that $I(d) \subseteq I(c) \setminus \{\nu\}$. Fix a part $\mu \in \{\nu, k\}$ of $S_w$. By definition of $S_w$, for every $\sigma \in S_w$, every $u < |\sigma|$ and every $R$-transitive set $R' \subseteq \text{dom}(S_w) \cap \bigcup_{\mu \in \{\nu, k\}} \text{dom}(S_w)$, $\varphi(F'_\mu, F', w, u)$ does not hold. Therefore, by clause 2 of Definition 3.15, $d \not\models (\forall x)\neg \varphi(G, w, x)$, hence $d \not\models \varphi(G, w)$, and this for some $w \in A$. Thus $I(d) \subseteq I(c) \setminus \{\nu\}$.

Suppose now that $w \in U \setminus A$, so $S_w$ is finite. Fix an $\ell \in \omega$ such that $(\forall \sigma \in S)|\sigma| < \ell$ and a $\tau \in T \cap k^{[\sigma]}$ such that $T^{[\tau]}$ is infinite. Consider the 2-partition $E_0 \cup E_1$ of $\text{set}_{\tau}(T) \cap \text{dom}(T)$ defined by $E_0 = \{i \geq t : \tau(i) = \nu \land (\forall s < \ell)R(i, s)\text{ holds}\}$ and $E_\ell = \{i \geq t : \tau(i) = \nu \land (\forall s < \ell)R(i, s)\text{ holds}\}$. As there exists no $\sigma \in S_w$ which $f$-refines $\tau$, there exists a $u < \ell$ and an $R$-transitive set $R' \subseteq E_0$ or $R' \subseteq E_1$ such that $\varphi(F'_\mu, F', w, u)$ holds. By choice of the partition, there exists a $t' > t$ such that $F' \to R[t', +\infty)$ or $(t', +\infty) \to R F'$. By Lemma 3.6, $(F'_\mu \cup F', [t', +\infty))$ is a valid EM extension of $(F'_\mu, [t, +\infty))$. As $T^{[\tau]}$ is infinite, $T^{[\nu, F']}\text{ is also infinite. Let } \bar{E} \text{ be defined by } E_\mu = F'_\nu \text{ if } \mu \neq \nu \text{ and } E_\nu = F'_w \text{ otherwise. Let } S \text{ be the } k\text{-partition tree } (k, t', T^{[\nu, F']}).$ The condition $d = (\bar{E}, S, C[S])$ is a valid extension of $c$. By clause 1 of Definition 3.15, $d \not\models (\forall x)\varphi(G, w)$ with $w \not\in A$. Therefore $I(d) \subseteq I(c) \setminus \{\nu\}$. \[\square\]
Lemma 3.27 If \( A \notin \Sigma^0_{n+2} \) and \( \varphi(G, x) \) is \( \Sigma^0_{n+2} \), then the set of \( c \in \mathbb{P} \) satisfying the following property is dense:

\[
[(\exists w \in A) c \vdash \neg \varphi(G, w)] \lor [(\exists w \notin A) c \vdash \varphi(G, w)]
\]

Proof. Fix a condition \( c = (\vec{F}, T, C) \).

- In case \( n = 0 \), \( \varphi(G, w) \) can be expressed as \( (\exists x)\psi(G, w, x) \) where \( \psi \in \Pi^0_1 \). Let \( U \) be the set of integers \( w \) such that there exists an infinite p.r. \( k' \)-partition tree \( S \) for some \( k' \in \omega \), a function \( f : \text{parts}(S) \to \text{parts}(T) \) and a \( k' \)-tuple of finite sets \( \vec{E} \) such that
  i) \( (E_\nu, \{\ell, +\infty\}) \) EM extends \( (F_{f(\nu)}, \text{dom}(T)) \) for each \( \nu < \text{parts}(S) \).
  ii) \( S \) \( f \)-refines \( \bigwedge_{\nu < \text{parts}(S)} T^{f(\nu),E_\nu} \).
  iii) for each non-empty part \( \nu \) of \( S \) such that \( (\nu, S) \in C \), \( (\vec{E}, S, C[S]) \vdash_\nu \psi(G, w, u) \) for some \( u < \#S \).

By Lemma 3.25 and Lemma 3.10, \( U \in \Sigma^0_2 \), thus \( U \neq A \). Let \( w \in U \triangle A \). Suppose that \( w \in U \triangle A \). We can choose \( \text{dom}(S) \) so that \( (E_\nu, \text{dom}(S)) \) EM extends \( (F_{f(\nu)}, \text{dom}(T)) \) for each \( \nu < \text{parts}(S) \). By Lemma 3.16 and Lemma 3.18, properties i-ii) remain true.

Let \( D = \mathcal{C}[S] \setminus \{(\nu, S') \in \mathcal{C} : \text{ part } \nu \text{ of } S' \text{ is empty} \} \). As \( C \) is an \( \emptyset' \)-p.r. promise for \( T, C[S] \) is an \( \emptyset' \)-p.r. promise for \( S \). As \( D \) is obtained from \( \mathcal{C}[S] \) by removing only empty parts, \( D \) is also an \( \emptyset' \)-p.r. promise for \( S \). By clause 1 of Definition 3.19, \( d = (\vec{F}, S, D) \vdash (\exists x)\psi(G, w, x) \) hence \( d \vdash \varphi(G, w) \) for some \( w \notin A \).

We may choose a coding of the p.r. trees such that such that the code of \( S \) is sufficiently large to witness \( u \) and \( E \). So suppose now that \( w \in A \triangle U \). Then for every infinite p.r. \( k' \)-partition tree \( S \), every \( \ell \) and \( \vec{E} \) smaller than the code of \( S \) such that properties i-ii) hold, there exists a non-empty part \( \nu \) of \( S \) such that \( (\nu, S) \in C \) and \( (\vec{E}, S, C) \not\vdash_\nu \psi(G, w, u) \) for every \( u < \ell \). Let \( D \) be the collection of all such \( (\nu, S) \). The set \( D \) is \( \emptyset' \)-p.r. By Lemma 3.16 and since \#\( S \geq \#T \) whenever \( S \leq_T T, \) \( D \) is upward-closed under the refinement relation, hence it is a promise for \( T \). By clause 2. of Definition 3.19, \( d = (\vec{F}, T, D) \vdash (\forall x)\neg\psi(G, w, x) \), hence \( d \vdash \neg \varphi(G, w) \) for some \( w \in A \).

- In case \( n > 0 \), let \( U = \{ w \in \omega : (\exists d \in \text{Ext}(c)) d \vdash \varphi(G, w) \} \). By Lemma 3.24 and Lemma 3.25, \( U \in \Sigma^0_{n+2} \), thus \( U \neq A \). Fix some \( w \in U \triangle A \). If \( w \in U \triangle A \) then by definition of \( U \), there exists a condition \( d \) extending \( c \) such that \( d \vdash \varphi(G, w) \). If \( w \in A \triangle U \), then for every \( d \in \text{Ext}(c) \), \( d \not\vdash \varphi(G, w) \) so by clause 4 of Definition 3.19, \( c \vdash \neg \varphi(G, w) \).

We are now ready to prove Theorem 3.2. It follows from the preservation of the arithmetic hierarchy for cohesiveness and the stable Erdős-Moser theorem.

Proof of Theorem 3.2. Since \( \text{RCA}_0 \vdash \text{COH} \land \text{SEM} \to \text{EM} \), then by Theorem 2.2 it suffices to prove that \( \text{SEM} \) admits preservation of the arithmetic hierarchy. Fix some set \( C \) and a \( C \)-computable stable infinite tournament \( R \). Let \( C_0 \) be the \( C' \)-p.r. set of all \( (\nu, T) \in U \) such that \( (\nu, T) \leq (0, 1^{<\omega}) \). Let \( \mathcal{F} \) be a sufficiently generic filter containing \( c_0 = (\{0\}, 1^{<\omega}, C_0) \). Let \( P \) and \( G \) be the corresponding generic path and generic real, respectively. By definition of a condition, the set \( G \) is \( R \)-transitive. By Lemma 3.14, \( G \) is infinite. By Lemma 3.26 and Lemma 3.25, \( G \) preserves non-\( \Sigma^0_1 \) definitions relative to \( C \). By Lemma 3.27 and Lemma 3.25, \( G \) preserves non-\( \Sigma^0_{n+2} \) definitions relative to \( C \) for every \( n \in \omega \). Therefore, by Proposition 2.2 of [21], \( G \) preserves the arithmetic hierarchy relative to \( C \).

4. \( D^2_1 \) Preserves Higher Definitions

Among the Ramsey-type hierarchies, the \( D \) hierarchy is conceptually the simplest one. It is therefore natural to study it in order to understand better the control of iterated jumps and focus on the core combinatorics without the technicalities specific to another hierarchy.

Definition 4.1 For every \( n, k \geq 1 \), \( D^n_k \) is the statement “Every \( \Delta^0_n \) \( k \)-partition of the integers has an infinite subset of one of its parts.”
In particular, $D^1_k$ is nothing but $\text{RT}^1_k$. Cholak et al. [4] proved that $D^2_k$ and stable Ramsey’s theorem for pairs and $k$ colors $(\text{SRT}^2_k)$ are computably equivalent and that the proof is formalizable over $\text{RCA}_0 + \text{BΣ}^0_2$. Later, Chong et al. [5] proved that $D^2_k$ implies $\text{BΣ}^0_2$ over $\text{RCA}_0$, showing therefore that $\text{RCA}_0 \vdash D^2_k \iff \text{SRT}^2_\ell$ for every $k, \ell \geq 2$. Wang [21] studied $D^2_k$ within his framework of preservation of definitions and proved that $D^2_k$ admits preservation of $\Xi$ definitions simultaneously for all $\Xi$ in $\{\Sigma^0_{n+2}, \Pi^0_{n+2}, \Delta^0_{n+2} : n \in \omega\}$, but not $\Delta^0_2$ definitions. He used for this a combination of the first jump control of Cholak, Jockusch and Slaman [4] and a relativization of the preservation of the arithmetic hierarchy by $\text{WKL}_0$.

In this section, we design a notion of forcing for $D^2_k$ with a forcing relation which has the same definitional complexity as the formula it forces. It enables us to reprove that $D^2_k$ of the preservation of the arithmetic hierarchy by $\text{WKL}_0$. A combination of the first jump control of Cholak, Jockusch and Slaman [4] and a relativization of the preservation of the arithmetic hierarchy by $\text{WKL}_0$.

4.1. Sides of a sequence of sets

A main feature in the construction of an solution to an instance $R_0, R_1$ of $D^2_2$ is the parallel construction of a subset of $R_0$ and a subset of $R_1$. The intrinsic disjunction in the forcing argument prevents us from applying the same strategy as for the Erdős-Moser theorem and obtain a preservation of the arithmetic hierarchy. Given some $\alpha < \omega$, we shall refer to $R_\alpha$ or simply $\alpha$ as a side of $\bar{R}$. We also need to define a relative notion of acceptance and emptiness of a part.

**Definition 4.2** Fix a $k$-partition tree $T$ of $[t, +\infty)$ and a set $X$. We say that part $\nu$ of $T$ is $X$-acceptable if there exists a path $P$ through $T$ such that $\text{set}_\nu(P) \cap X$ is infinite. We say that part $\nu$ of $T$ is $X$-empty if $(\forall \sigma \in T) [\text{dom}(T) \cap \text{set}_\nu(\sigma) \cap X = \emptyset]$.

The intended uses of those notions will be $R_\alpha$-acceptation and $R_\alpha$-emptiness. Every partition tree has an $R_\alpha$-acceptable part for some $\alpha < 2$. The notion of $X$-emptiness is $\Pi^0_{1,X}$, and therefore $\Pi^0_1$ if $X$ is $\Delta^0_0$, which raises new problems for obtaining a forcing relation of weak definitional complexity. We would like to define a stronger notion of “witnessing its acceptable parts” and prove that for every infinite p.r. partition tree $T$, there is a p.r. refined tree $S$ such that for each side $\alpha$ and each part $\nu$ of $S$, either $\nu$ is $R_\alpha$-empty in $S$, or $\nu$ is $R_\alpha$-acceptable. However, the resulting tree $S$ would be $\emptyset'$-p.r. since $R_\alpha$ is $\emptyset'$-computable. Thankfully, we will be able to circumvent this problem in Lemma 4.17.

4.2. Forcing conditions

Fix a $\Delta^0_2$ 2-partition $R_0 \cup R_1 = \omega$. We now describe the notion of forcing to build an infinite subset of $R_0$ or of $R_1$.

**Definition 4.3** We denote by $\mathbb{P}$ the forcing notion whose conditions are tuples $(\vec{F}, T, C)$ where

(a) $T$ is an infinite, p.r. $k$-partition tree for some $k \in \omega$
(b) $C$ is a $\emptyset'$-p.r. promise for $T$
(c) $(F^\nu_\alpha, \text{dom}(T))$ is a Mathias condition for each $\nu < k$ and $\alpha < 2$

A condition $d = (\vec{E}, S, D)$ extends $c = (\vec{F}, T, C)$ (written $d \leq c$) if there exists a function $f : \text{parts}(S) \rightarrow \text{parts}(T)$ such that $D \subseteq C$ and the following holds

(i) $(E^\nu_\alpha, \text{dom}(S) \cap R_\alpha)$ Mathias extends $(F^\alpha_{f(\nu)}, \text{dom}(T) \cap R_\alpha)$ for each $\nu < \text{parts}(S)$ and $\alpha < 2$
(ii) $S$ $f$-refines $\bigcap_{\nu < \text{parts}(S), \alpha < 2} T[f(\nu), E^\nu_\alpha]$

In the whole construction, the index $\alpha$ indicates that we are constructing a set cofinitely in $R_\alpha$. Given a condition $c = (\vec{F}, T, C)$, we write again $\text{parts}(c)$ for $\text{parts}(T)$. The following lemma shows that we can force our constructed set to be infinite if we choose it among the acceptable parts.
Lemma 4.4 For every condition \( c = (\vec{F}, T, C) \) and every \( n \in \omega \), there exists an extension \( d = (\vec{E}, S, D) \) such that \( |E_\mu^\alpha| \geq n \) on each \( R_\alpha \)-acceptable part \( \nu \) of \( S \) for each \( \alpha < 2 \).

Proof. It suffices to prove that for every condition \( c = (\vec{F}, T, C) \), every side \( \alpha < 2 \) and every \( R_\alpha \)-acceptable part \( \nu \) of \( T \), there exists an extension \( d = (\vec{E}, S, D) \) such that \( S \leq_{id} T \) and \( |E_\mu^\alpha| \geq n \). Iterating the process at most parts \( T \times T \) times enables us to conclude. Fix an \( R_\alpha \)-acceptable part \( \nu \) of \( T \) and a path \( P \) through \( T \) such that \( \text{set}_\nu(P) \cap R_\alpha \) is infinite. Let \( F' \) be a subset of \( \text{set}_\nu(P) \cap \text{dom}(T) \cap R_\alpha \) of size \( n \). Let \( \vec{E} \) be defined by \( E_\mu^\alpha = F_\mu^\alpha \) if \( \mu \neq \nu \) and \( E_\nu^\alpha = F_\nu^\alpha \cup F' \) otherwise. Let \( S \) be the p.r. partition tree obtained from \( T[\nu, E_\nu^\alpha] \) by restricting its domain so that \( (E_\nu^\alpha, \text{dom}(S) \cap R_\alpha) \) Mathias extends \( (F_\nu^\alpha, \text{dom}(T) \cap R_\alpha) \). The condition \( (\vec{E}, S, C[S]) \) is the desired extension. \( \square \)

Given a condition \( c \), we denote by \( \text{Ext}(c) \) the set of all its extensions.

4.3. Forcing relation

We need to define two forcing relations at the first level: the “true” forcing relation, i.e., the one having the good density properties but whose decision requires too much computational power, and a “weak” forcing relation having better computational properties, but which does not behave well with respect to the forcing. We start with the definition of the true forcing relation.

Definition 4.5 (True forcing relation) Fix a condition \( c = (\vec{F}, T, C) \), a \( \Sigma_0^1 \) formula \( \varphi(G, x) \), a part \( \nu < \text{parts}(T) \), and a side \( \alpha < 2 \).

1. \( c \models \forall^n_{\vec{F}} \left( \exists x \right) \varphi(G, x) \) iff there exists a \( w \in \omega \) such that \( \varphi(F_\nu^\alpha, w) \) holds.
2. \( c \models \forall_{\vec{F}} \left( \forall x \right) \varphi(G, x) \) iff for every \( \sigma \in T \) such that \( T[\sigma] \) is infinite, every \( w < |\sigma| \) and every set \( F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \cap R_\alpha \), \( \varphi(F_\nu^\alpha \cup F', w) \) holds.

Given a condition \( c \), a side \( \alpha < 2 \), a part \( \nu \) of \( c \) and a \( \Pi_1^0 \) formula \( \varphi \), the relation \( c \models \forall_{\vec{F}} \varphi(G) \) is \( \Pi_1^0 \)-acceptable in \( R_\alpha \), hence \( \Pi_1^0 \) as \( R_\alpha \) is \( \Delta_0^2 \). This relation enjoys the good properties of a forcing relation, that is, it is downward-closed under the refinement relation (Lemma 4.6), and the set of the conditions forcing either a \( \Sigma_1^0 \) formula or its negation is dense (Lemma 4.7).

Lemma 4.6 Fix a condition \( c = (\vec{F}, T, C) \) and a \( \Sigma_1^0 \) (\( \Pi_1^0 \)) formula \( \varphi(G) \). If \( c \models \forall_{\vec{F}} \varphi(G) \) for some \( \nu < \text{parts}(T) \) and \( \alpha < 2 \), then for every \( d = (\vec{E}, S, D) \leq c \) and every part \( \mu \) of \( S \) refining part \( \nu \) of \( T \), \( d \models \forall_{\vec{E}} \varphi(G) \).

Proof.

- If \( \varphi \in \Sigma_1^0 \) then it can be expressed as \( (\exists x) \psi(G, x) \) where \( \psi \in \Sigma_0^0 \). By clause 1 of Definition 4.5, there exists a \( w \in \omega \) such that \( \psi(F_\nu^\alpha, w) \) holds. By property (i) of the definition of an extension, \( E_\mu^\alpha \supseteq F_\nu^\alpha \) and \( (E_\mu^\alpha \setminus F_\nu^\alpha) \subseteq \text{dom}(T) \cap R_\alpha \), therefore by continuity \( \psi(E_\nu^\alpha, w) \) holds, so by clause 1 of Definition 4.5, \( d \models \exists x \psi(G, x) \).

- If \( \varphi \in \Pi_1^0 \) then it can be expressed as \( (\forall x) \psi(G, x) \) where \( \psi \in \Sigma_0^0 \). Fix a \( \tau \in S \) such that \( \text{dom}(\tau) \) is infinite, a \( w < |\tau| \) and a set \( F' \subseteq \text{dom}(S) \cap \text{set}_\mu(\tau) \cap R_\alpha \). Let \( b \) be the function witnessing \( d \leq c \). By property (ii) of the definition of an extension, \( \tau \) \( f \)-refines a \( \sigma \in T[\nu, E_\nu^\alpha] \). We claim that we can even choose \( \sigma \) to be extendible in \( T[\nu, E_\nu^\alpha] \). Indeed, since \( \tau \) is extendible in \( S \), let \( P \) be a path through \( S \) extending \( \tau \) and let \( U \) be the set of \( \sigma \)'s in \( T \) such that \( P[\sigma] f \)-refines \( \sigma \) for some \( s \). The set \( U \) is an infinite subtree of \( T \).

As \( w < |\tau| \) and \( \text{dom}(S) \subseteq \text{dom}(T) \), \( F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\tau) \cap R_\alpha \). As \( \sigma \in T[\nu, E_\nu^\alpha] \), \( E_\mu^\alpha \subseteq \text{set}_\nu(\sigma) \) and by property (i) of the definition of an extension, \( E_\mu^\alpha \subseteq \text{dom}(T) \cap R_\alpha \). Therefore \( E_\mu^\alpha \cup F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\tau) \cap R_\alpha \). By clause 2 of
Definition 4.5 applied to \( c \Vdash^\alpha_\varphi (\forall x)\psi(G, x) \), \( \psi(F'_\nu \cup (E''_\mu \setminus F'_\nu) \cup F', w) \) holds, hence \( \psi(E''_\mu \cup F', w) \) holds and still by clause 2 of Definition 4.5, \( d \Vdash^\alpha_\varphi (\forall x)\psi(G, x) \).

\[ \square \]

Lemma 4.7 For every \( \Sigma^0_1 (\Pi^0_1) \) formula \( \varphi \), the following set is dense in \( \mathbb{P} \):

\[ \{ c \in \mathbb{P} : (\forall \nu < \text{parts}(c))(\forall \alpha < 2)[c \Vdash^\nu_\varphi \varphi(G) \text{ or } c \Vdash^\nu_{\neg \varphi} \neg \varphi(G)] \} \]

Proof. It suffices to prove the statement for the case where \( \varphi \) is a \( \Sigma^0_1 \) formula, as the case where \( \varphi \) is a \( \Pi^0_1 \) formula is symmetric. Fix a condition \( c = (\bar{F}, T, C) \) and let \( I(c) \) be the set of pairs \( (\nu, \alpha) \in \text{parts}(T) \times 2 \) such that \( c \Vdash^\nu_\varphi \varphi(G) \) and \( c \Vdash^\nu_{\neg \varphi} \neg \varphi(G) \). If \( I(c) = \emptyset \) we are done, so suppose \( I(c) \neq \emptyset \). Fix some \((\alpha, \nu) \in I(c) \). We will construct an extension \( d \) such that \( I(d) \subseteq I(c) \setminus \{ (\alpha, \nu) \} \). Applying iteratively the operation enables us to conclude.

The formula \( \varphi \) is of the form \( (\exists x)\psi(G, x) \) where \( \psi \in \Sigma^0_1 \). Suppose there exists a \( \sigma \in T \) such that \( T[\sigma] \) is infinite, a \( w < |\sigma| \) and a set \( F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \cap R_\alpha \) such that \( \psi(F'_\nu \cup F', w) \) holds. In this case, letting \( \bar{E} \) be defined by \( E^\beta_\mu = F'_\mu \) if \( \mu \neq \nu \vee \beta \neq \alpha \) and \( E''_\nu = F'_\nu \cup F' \), and letting \( S \) be the tree \( T[\sigma] \) where the domain is restricted so that \( (E''_\nu, \text{dom}(S)) \) Mathias extends \( (F'_\nu, \text{dom}(T)) \), by clause 1 of Definition 4.5, the condition \( d = (\bar{E}, S, C[S]) \) is a valid extension of \( c \) such that \( d^\alpha_\varphi \varphi(G) \).

Suppose now that for every \( \sigma \in T \) such that \( T[\sigma] \) is infinite, every \( w < |\sigma| \) and every set \( F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \cap R_\alpha, \psi(F'_\nu \cup F', w) \) does not hold. In this case, by clause 2 of Definition 4.5, \( c \Vdash^\nu_{\neg \varphi} \neg \varphi(G) \).

\[ \square \]

We now define the weak forcing relation which is almost the same as the true one, expect that the set \( F' \) is not required to be a subset of \( R_\alpha \) in the case of a \( \Pi^0_1 \) formula.

Definition 4.8 (Weak forcing relation) Fix a condition \( c = (\bar{F}, T, C) \), a \( \Sigma^0_1 \) formula \( \varphi(G, x) \), a part \( \nu < \text{parts}(T) \) and a side \( \alpha < 2 \).

1. \( c \Vdash^\nu_\varphi (\exists x)\varphi(G, x) \) iff there exists a \( w \in \omega \) such that \( \varphi(F'_\nu, w) \) holds.
2. \( c \Vdash^\nu_\varphi (\forall x)\varphi(G, x) \) iff for every \( \sigma \in T \), every \( w < |\sigma| \) and every set \( F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \), \( \varphi(F'_\nu \cup F', w) \) holds.

As one may expect, the weak forcing relation at the first level is also closed under the refinement relation.

Lemma 4.9 Fix a condition \( c = (\bar{F}, T, C) \) and a \( \Sigma^0_1 (\Pi^0_1) \) formula \( \varphi(G) \). If \( c \Vdash^\nu_\varphi \varphi(G) \) for some \( \nu < \text{parts}(T) \) and \( \alpha < 2 \), then for every \( d = (\bar{E}, S, D) \leq c \) and every part \( \mu \) of \( S \) refining part \( \nu \) of \( T \), \( d \Vdash^\mu_\varphi \varphi(G) \).

Proof.

- If \( \varphi \in \Sigma^0_1 \) then this is exactly clause 1 of Lemma 4.6 since the definition of the weak and the true forcing relations coincide for \( \Sigma^0_1 \) formulas.
- If \( \varphi \in \Pi^0_1 \) then it can be expressed as \( (\forall x)\psi(G, x) \) where \( \psi \in \Sigma^0_1 \). Fix a \( \tau \in S \), a \( w < |\tau| \) and a set \( F' \subseteq \text{dom}(S) \cap \text{set}_\nu(\tau) \). By property (ii) of the definition of an extension, there exists a \( \sigma \in T[\nu, E''_\mu] \) such that \( |\sigma| = |\tau| \) and \( \text{set}_\nu(\tau) \subseteq \text{set}_\nu(\sigma) \). As \( w < |\tau| \) and \( \text{dom}(S) \subseteq \text{dom}(T), F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \). As \( \sigma \in T[\nu, E''_\mu] \), \( E''_\mu \subseteq \text{set}_\nu(\sigma) \) and by property (i) of the definition of an extension, \( E''_\mu \subseteq \text{dom}(T) \). Therefore \( E''_\mu \cup F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \). By clause 2 of Definition 4.8 applied to \( c \Vdash^\nu (\forall x)\psi(G, x) \), \( \psi(F'_\nu \cup (E''_\mu \setminus F'_\nu) \cup F', w) \) holds, hence \( \psi(E''_\mu \cup F', w) \) holds and still by clause 2 of Definition 4.8, \( d \Vdash^\mu_\varphi (\forall x)\psi(G, x) \).

\[ \square \]

The following trivial lemma simply reflects the fact that the promise \( C \) is not part of the definition of the weak forcing relation for \( \Sigma^0_1 \) or \( \Pi^0_1 \) formulas, and therefore has no effect on it.
Lemma 4.10  Fix two conditions $c = (\vec{F},T,C)$ and $d = (\vec{E},T,D)$ and a $\Sigma^0_1$ $(\Pi^0_1)$ formula. For every part $\nu$ of $T$ such that $F^\nu_\alpha = E^\nu_\alpha$, $c \models^\alpha \varphi(G)$ if and only if $d \models^\alpha \varphi(G)$.

Proof. If $\varphi \in \Sigma^0_1$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G,x)$ where $\psi \in \Sigma^0_1$. By clause 1 of Definition 4.8, $c \models^\nu \varphi(G)$ iff there exists a $w \in \omega$ such that $\psi(F^\nu_\alpha,w)$ holds. As $F^\nu_\alpha = E^\nu_\alpha$, $c \models^\nu \varphi(G)$ if and only if $d \models^\nu \varphi(G)$. Similarly, if $\varphi \in \Pi^0_1$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G,x)$ where $\psi \in \Sigma^0_1$. By clause 2 of Definition 4.8, $c \models^\nu \varphi(G)$ iff for every $\sigma \in T$, every $w < |\sigma|$ and every set $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$, $\psi(F'_\alpha \cup F',w)$ holds. As $F^\nu_\alpha = E^\nu_\alpha$, $c \models^\nu \varphi(G)$ if $d \models^\nu \varphi(G)$. \hfill \Box

We can now define the forcing relation over higher formulas. It is defined inductively, starting with $\Sigma^0_1$ and $\Pi^0_1$ formulas. We extend the weak forcing relation instead of the true one for effectiveness purposes. We shall see later that the weak forcing relation behaves like the true one for some parts and some sides of a condition, and therefore that it tells us something about the truth of the formula over some carefully defined generic real $G$. Note that the forcing relation over higher formulas is still parameterized by the side $\alpha$ of the condition.

Definition 4.11  Fix a condition $c = (\vec{F},T,C)$, a side $\alpha < 2$ and an arithmetic formula $\varphi(G)$.

1. If $\varphi(G) = (\exists x)\psi(G,x)$ where $\psi \in \Pi^0_1$ then $c \models^\alpha \varphi(G)$ iff for every part $\nu$ of $T$ such that $(\nu,T) \in C$ there exists a $w < \text{dom}(T)$ such that $c \models^\nu \psi(G,w)$.
2. If $\varphi(G) = (\forall x)\psi(G,x)$ where $\psi \in \Sigma^0_1$ then $c \models^\alpha \varphi(G)$ iff for every infinite p.r. $k'$-partition tree $S$, every function $f : \text{parts}(S) \to \text{parts}(T)$, every $w$ and $\vec{E}$ smaller than $\#S$ such that the following holds

   i) $E_{\nu}^\beta = F^\beta_{f(\nu)}$ for each $\nu < \text{parts}(S)$ and $\beta \neq \alpha$.

   ii) $(E^\alpha_\nu, \text{dom}(S) \cap R_\alpha)$ Mathias extends $(F^\alpha_{f(\nu)}, \text{dom}(T) \cap R_\alpha)$ for each $\nu < \text{parts}(S)$.

   iii) $S \text{ f-refines } \bigcap_{\nu < \text{parts}(S)} T^{f(\nu),E^\alpha_\nu}$ for every $(\mu,S) \in C$, $(\vec{E},S,C[S]) \models^\alpha \neg \psi(G,w)$.

3. If $\varphi(G) = (\exists x)\psi(G,x)$ where $\psi \in \Pi^0_1$ then $c \models^\alpha \varphi(G)$ iff there exists a $w \in \omega$ such that $c \models^\alpha \psi(G,w)$.

4. If $\varphi(G) = \neg \psi(G)$ where $\psi \in \Sigma^0_{n+3}$ then $c \models^\alpha \varphi(G)$ if $d \models^\alpha \psi(G)$ for every $d \in \text{Ext}(c)$.

Note that clause 2.ii) of Definition 4.11 seems to be $\Pi^0_0$ since $R_\alpha$ is $\Delta^0_0$. However, in fact, one just needs to ensure that $\text{dom}(S) \subseteq \text{dom}(T)$ and $E^\alpha_\nu \setminus F^\alpha_{f(\nu)} \subseteq \text{dom}(T) \cap R_\alpha$. This is a $\Delta^0_0$ predicate, and so is its negation, so one can already easily check that the forcing relation over a $\Pi^0_0$ formula will be also $\Pi^0_0$. Before proving the usual properties about the forcing relation, we need to discuss the role of the sides in the forcing relation. We are now ready to prove that the forcing relation is closed under extension.

Lemma 4.12  Fix a condition $c$, a side $\alpha < 2$ and a $\Sigma^0_{n+2}$ $(\Pi^0_{n+2})$ formula $\varphi(G)$. If $c \models^\alpha \varphi(G)$ then for every $d \leq c$, $d \models^\alpha \varphi(G)$.

Proof. We prove the statement by induction over the complexity of the formula $\varphi(G)$. Fix a condition $c = (\vec{F},T,C)$ and a side $\alpha < 2$ such that $c \models^\alpha \varphi(G)$. Fix an extension $d = (\vec{E},S,D)$ of $c$.

- If $\varphi \in \Sigma^0_2$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G,x)$ where $\psi \in \Pi^0_1$. By clause 1 of Definition 4.11, for every part $\nu$ of $T$ such that $(\nu,T) \in C$, there exists a $w < \text{dom}(T)$ such that $c \models^\nu \psi(G,w)$. Fix a part $\mu$ of $S$ such that $(\mu,S) \in D$. As $D \subseteq C$, $(\mu,S) \in C$. By upward-closure of $C$, part $\mu$ of $S$ refines some part $\nu$ of $C$ such that $(\nu,T) \in C$. Therefore by Lemma 4.9, $d \models^\nu \psi(G,w)$, with $w < \text{dom}(T) \leq \text{dom}(S)$. Applying again clause 1 of Definition 4.11, we deduce that $d \models (\forall x)\psi(G,x)$, hence $d \models^\alpha \varphi(G)$.

- If $\varphi \in \Pi^0_2$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G,x)$ where $\psi \in \Sigma^0_1$. Suppose for the sake of absurd that $d \models^\alpha (\forall x)\psi(G,x)$. Let $f : \text{parts}(S) \to \text{parts}(T)$ witness the refinement $S \leq T$. By clause 2 of Definition 4.11, there exists an infinite p.r. $k'$-partition tree $S'$, a
function \( g : \text{parts}(S') \to \text{parts}(S) \), a \( w \in \omega \), and a \( 2k' \)-tuple of finite sets \( \vec{H} \) smaller than the code of \( S' \) such that

i) \( H_0^\beta = E_0^{\beta(w)} \) for each \( \nu < \text{parts}(S') \) and \( \beta \neq \alpha \)

ii) \( (H_\nu^\beta, \text{dom}(S') \cap R_\alpha) \) Mathias extends \( (E_\nu^{\alpha(w)}, \text{dom}(S) \cap R_\alpha) \) for each \( \nu < \text{parts}(S') \)

iii) \( S' \) \( g \)-refines \( \bigcap_{\nu < \text{parts}(S')} S^{[g(\nu),H_\nu^\alpha]} \)

iv) there exists a \( (\mu, S') \in D \) such that \( (\vec{H}, S', D[S']) \models \alpha \neg \psi(G,w) \).

To deduce by clause 2 of Definition 4.11 that \( c \not|^{\alpha} (\forall x) \psi(G,x) \) and derive a contradiction, it suffices to prove that the same property holds with respect to \( T \). Let \( \vec{H}' \) be defined by \( H_\nu'^\beta = F_\nu^{\beta(w)} \) for each \( \nu < \text{parts}(S') \) and \( \beta \neq \alpha \) and \( H_\nu'^\alpha = H_\nu^\alpha \).

i) It trivially holds by choice of \( \vec{H}' \).

ii) By property (i) of the definition of an extension, \( (E_\nu^{\alpha(w)}, \text{dom}(S)) \) Mathias extends \( (F_\nu^{\alpha(w)}, \text{dom}(T)) \). Moreover \( (H_\nu^\alpha, \text{dom}(S')) \) Mathias extends \( (E_\nu^{\alpha(w)}, \text{dom}(S)) \), so \( (H_\nu'^\alpha, \text{dom}(S')) = (H_\nu^\alpha, \text{dom}(S')) \) Mathias extends \( (F_\nu^{\alpha(w)}, \text{dom}(T)) \).

iii) As by property (ii) of the definition of an extension,

\( S' \) \( f \)-refines \( \bigcap_{\nu < \text{parts}(S')} T^{[f(\nu,w),E_\nu^\alpha]} \), and

\( S' \) \( g \)-refines \( \bigcap_{\nu < \text{parts}(S')} S^{[g(\nu),H_\nu^\alpha]} \) then

\( S' \) \( (g \circ f) \)-refines \( \bigcap_{\nu < \text{parts}(S')} T^{[g(\nu),H_\nu'^\alpha]} \).

iv) As \( D \subseteq C \), there exists a part \( (\mu, S') \in C \) such that \( (\vec{H}, S', D[S']) \models \alpha \neg \psi(G,w) \). By Lemma 4.10, \( (\vec{H}, S', C[S']) \models \alpha \neg \psi(G,w) \).

- If \( \varphi \in \Sigma_{n+3}^0 \) then \( \varphi(G) \) can be expressed as \( (\exists x) \psi(G,x) \) where \( \psi \in \Pi_{n+2}^0 \). By clause 3 of Definition 4.11, there exists a \( w \in \omega \) such that \( c \models^{\alpha} \psi(G,w) \). By induction hypothesis, \( d \models^{\alpha} \psi(G,w) \) so by clause 3 of Definition 4.11, \( d \models^{\alpha} \varphi(G) \).

- If \( \varphi \in \Pi_{n+3}^0 \) then \( \varphi(G) \) can be expressed as \( \neg \psi(G) \) where \( \psi \in \Sigma_{n+3}^0 \). Suppose for the sake of absurd that \( d \not|^{\alpha} \varphi(G) \). By clause 4 of Definition 4.11, there exists an \( e \in \text{Ext}(d) \) such that \( e \not|^{\alpha} \varphi(G) \). In particular, \( e \in \text{Ext}(c) \), so by clause 4 of Definition 4.11, \( e \not|^{\alpha} \psi(G) \) since \( c \not|^{\alpha} \varphi(G) \). Contradiction.

Although the weak forcing relation does not satisfy the density property, the forcing relation over higher formulas does. The reason is that the extended forcing relation does not involve the weak forcing relation over \( \Sigma_0^1 \) formulas in the clause 2 of Definition 4.11, but uses instead the weaker statement “\( c \) does not force the negation of the \( \Sigma_0^1 \) formula”. The link between this statement and the statement “\( c \) has an extension which forces the \( \Sigma_0^1 \) formula” is used when proving that \( \varphi(G) \) holds iff \( c \models^{\alpha} \varphi(G) \) for some condition belonging to a sufficiently generic filter. We now prove the density of the forcing relation for higher formulas.

**Lemma 4.13** For every \( \Sigma_{n+2}^0 \) (\( \Pi_{n+2}^0 \)) formula \( \varphi \), the following set is dense in \( P \):

\[ \{ c \in P : (\forall \alpha < 2)[c \models^{\alpha} \varphi(G) \text{ or } c \not|^{\alpha} \neg \varphi(G)] \} \]

**Proof.** We prove the statement by induction over \( n \). It suffices to treat the case where \( \varphi \) is a \( \Sigma_{n+2}^0 \) formula, as the case where \( \varphi \) is a \( \Pi_{n+2}^0 \) formula is symmetric. Moreover, it is enough to prove that for every condition \( c \) and every \( \alpha < 2 \), there exists an extension \( d \leq c \) such that \( d \not|^{\alpha} \varphi(G) \) or \( d \not|^{\alpha} \neg \varphi(G) \). Iterating the process at most twice enables us to conclude. Fix a condition \( c = (F,T,C) \) and a part \( \alpha < 2 \).

- In case \( n = 0 \), the formula \( \varphi \) is of the form \( (\exists x) \psi(G,x) \) where \( \psi \in \Pi_1^0 \). Suppose there exists an infinite p.r. \( k' \)-partition tree \( S \) for some \( k' \in \omega \), a function \( f : \text{parts}(S) \to \text{parts}(T) \), and a \( 2k' \)-tuple of finite sets \( \vec{E} \) such that

  i) \( E_\nu^\beta = F_\nu^{\beta(w)} \) for each \( \nu < \text{parts}(S) \) and \( \beta \neq \alpha \)

  ii) \( (E_\nu^\alpha, \text{dom}(S) \cap R_\alpha) \) Mathias extends \( (F_\nu^{\alpha(w)}, \text{dom}(T) \cap R_\alpha) \) for each \( \nu < \text{parts}(S) \).

  iii) \( S \) \( f \)-refines \( \bigcap_{\nu < \text{parts}(S)} T^{[f(\nu),E_\nu^\alpha]} \)
Lemma 4.14

formulas.

the extension relation does not add some extra complexity to the forcing relation for higher 

relation is $\Pi_2^0$ of all the infinite p.r.

Recall from Lemma 3.10 that given 

Fix an arithmetic formula 

We prove our lemma by induction over the complexity of the formula $\varphi(G)$.

iv) for each non-empty part $\nu$ of $S$ such that $(\nu, S) \in C$, $(\vec{E}, S, C[S]) \models_\nu^\alpha \psi(G, w)$ for some $w < \# S$

Let $D = C[S] \smallsetminus \{ (\nu, S') \in C : \text{ part } \nu \text{ of } S' \text{ is empty} \}$. As $C$ is an $\exists'-\text{p.r.}$ promise for $T$, $C[S]$ is an $\exists'-\text{p.r.}$ promise for $S$. As $D$ is obtained from $C[S]$ by removing only empty parts, $D$ is also an $\exists'-\text{p.r.}$ promise for $S$. By clause 1 of Definition 4.11, $d = (\vec{E}, S, D) \models_\alpha (\exists x) \psi(G, x)$ hence $d \models_\alpha \varphi(G)$.

We may choose a coding of the p.r. trees such that such that the code of $S$ is sufficiently 
large to witness $f, \ell$ and $\vec{E}$. So suppose now that for every infinite p.r. $k'$-partition tree 

$S$, every function $f : \text{parts}(S) \to \text{parts}(T)$, $\ell \in \omega$ and $\vec{E}$ smaller than the code of $S$ such 

that properties i-iii) hold, there exists a non-empty part $\nu$ of $S$ such that $(\nu, S) \in C$ and 

$(\vec{E}, S, C[S]) \models_\alpha \psi(G, w)$ for every $w < \ell$. Let $D$ be the collection of all such $(\nu, S)$.

$D$ is $\exists'-\text{p.r.}$ By Lemma 4.9 and since we require that $\# S \geq \# T$ in the definition 

of $S \leq T$, $D$ is upward-closed, hence is a promise for $T$. By clause 2 of Definition 4.11, 

$d = (\vec{F}, T, D) \models_\alpha (\forall x) \neg \psi(G, x)$, hence $d \models_\alpha \neg \varphi(G)$.

In case $n > 0$, density follows from clause 4 of Definition 4.11.

\[ \square \]

We now prove that the weak forcing relation extended to any arithmetic formula enjoys the 
desired definability properties. For this, we start with a lemma showing that the extension 
relation is $\Pi_2^0$. Therefore, only the first two levels have to be treated independently, since 
the extension relation does not add some extra complexity to the forcing relation for higher 
formulas.

Lemma 4.14 For every condition $c$, Ext$(c)$ is $\Pi_2^0$ uniformly in $c$.

Proof. Recall from Lemma 3.10 that given $k, t \in \omega$, the set PartTree$(k, t)$ denotes the $\Pi_2^0$ set of 
all the infinite p.r. $k$-partition trees of $[t, +\infty)$, and given a $k$-partition tree $S$ and a part 
$\nu$ of $S$, the predicate Empty$(S, \nu)$ denotes the $\Pi_1^0$ formula "part $\nu$ of $S$ is empty", that is, the 
formula $(\forall \sigma \in S) [\text{set}_\nu(\sigma) \cap \text{dom}(S) = \emptyset]$. If $T$ is p.r. then so is $T^{[\nu, H]}$ for some finite set $H$.

Fix a condition $c = (\vec{F}, (k, t), C)$. $(\vec{H}, (k', t', S), D) \in \text{Ext}(c)$ iff the following formula holds:

\[(\exists f : k' \to k) \quad (\forall \nu < k')(\forall \alpha < 2)(H^\alpha_\nu, [t', +\infty) \cap R_\alpha) \text{ Mathias extends } (F^\nu_\nu, [t, +\infty) \cap R_\alpha) \quad (\Pi_2^0) \]

\[\wedge S \in \text{PartTree}(k', t') \wedge S \leq_f \bigwedge_{\nu < k', \alpha < 2} T^{[f(\nu), H^\alpha_\nu]} \quad (\Pi_2^0) \]

\[\wedge D \text{ is a promise for } S \wedge D \subseteq C \quad (\Pi_2^0) \]

The formula $(H^\alpha_\nu, [t', +\infty) \cap R_\alpha)$ Mathias extends $(F^\nu_\nu, [t, +\infty) \cap R_\alpha)$ can be written $(\forall x < t) [x \in H^\alpha_\nu \iff x \in F^\nu_\nu] \wedge t' \geq t \wedge (\forall x \in H^\alpha_\nu \setminus F^\nu_\nu) x \in R_\alpha$ and therefore is $\Pi_2^0$. By Lemma 3.10 and the fact that $\bigwedge_{\nu < k', \alpha < 2} T^{[f(\nu), H^\alpha_\nu]}$ is p.r. uniformly in $T, f, \vec{H}$ and $k'$, the above formula is $\Pi_2^0$. \[\square \]

Lemma 4.15 Fix an arithmetic formula $\varphi(G)$, a condition $c = (\vec{F}, T, C)$, a side $\alpha < 2$ and a 
part $\nu$ of $T$.

a) If $\varphi(G)$ is a $\Sigma^0_1 (\Pi_1^0)$ formula then so is the predicate $c \models_\nu^\alpha \varphi(G)$.

b) If $\varphi(G)$ is a $\Sigma^{n+2}_n (\Pi^{n+2}_n)$ formula then so is the predicate $c \models_\nu^\alpha \varphi(G)$.

Proof. We prove our lemma by induction over the complexity of the formula $\varphi(G)$.

- If $\varphi(G) \in \Sigma^0_1$ then it can be expressed as $(\exists x) \psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 1 of 
Definition 4.8, $c \models_\nu^\alpha \varphi(G)$ if and only if the formula $(\exists w \in \omega) \psi(F^\nu_\nu, w)$ holds. This is a 
$\Sigma^0_1$ predicate.

- If $\varphi(G) \in \Pi^0_1$ then it can be expressed as $(\forall x) \psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 2 
of Definition 4.8, $c \models_\nu^\alpha \varphi(G)$ if and only if the formula $(\forall \sigma \in T) (\forall w < |\sigma|) (\forall F' \subseteq 
\text{dom}(T) \cap \text{set}_\nu(\sigma)) \psi(F^\nu_\nu \cup F', w)$ holds. This is a $\Pi^0_1$ predicate.
− If \( \varphi(G) \in \Sigma^0_2 \) then it can be expressed as \((\exists x)\psi(G, x)\) where \( \psi \in \Pi^0_1 \). By clause 1 of Definition 4.11, \( c \Vdash^\alpha \varphi(G) \) if and only if the formula \((\forall \nu < \text{parts}(T))(\exists \nu < \text{dom}(T)) [(\nu, T) \in C \to c \Vdash^\nu_\alpha \psi(G, w)]\) holds. This is a \( \Sigma^0_2 \) predicate by induction hypothesis and the fact that \( C \) is \( \emptyset' \)-computable.

− If \( \varphi(G) \in \Pi^0_2 \) then it can be expressed as \((\forall x)\psi(G, x)\) where \( \psi \in \Sigma^0_1 \). By clause 2 of Definition 4.11, \( c \Vdash (\forall x)\varphi(G) \) if and only if for every infinite \( k' \)-partition tree \( S \), every function \( f : \text{parts}(S) \to \text{parts}(T) \), every \( w \) and \( \tilde{E} \) smaller than the code of \( S \) such that the following hold

i) \( (E_\nu, \text{dom}(S) \cap R_\alpha) \) Mathias extends \( (F_{f(\nu)}, \text{dom}(T) \cap R_\alpha) \) for each \( \nu \neq \text{parts}(S) \)

ii) \( S \) \( f \)-refines \( \bigcap_{\nu < \text{parts}(S)} T^{f(\nu), E_\nu} \)

for every \( (\mu, S) \in C \), \( (\tilde{E}, S, \text{C}[S]) \implies_\mu -\psi(G, w) \). By Lemma 3.10, Properties i-ii) are \( \Delta^0_2 \). Moreover the predicate \( (\mu, S) \in C \) is \( \Delta^0_2 \) since \( C \) is \( \emptyset' \)-p.r. By induction hypothesis, \( (\tilde{E}, S, \text{C}[S]) \implies_\mu -\psi(G, w) \) is \( \Sigma^0_1 \). Therefore \( c \Vdash^\alpha \varphi(G) \) is a \( \Pi^0_2 \) predicate.

− If \( \varphi(G) \in \Sigma^0_{n+3} \) then it can be expressed as \((\exists x)\psi(G, x)\) where \( \psi \in \Pi^0_{n+2} \). By clause 3 of Definition 4.11, \( c \Vdash^\alpha \varphi(G) \) if and only if the formula \((\exists \omega < \nu)c \Vdash^\omega_\alpha \psi(G, w)\) holds. This is a \( \Sigma^0_{n+3} \) predicate by induction hypothesis.

− If \( \varphi(G) \in \Pi^0_{n+3} \) then it can be expressed as \(-\psi(G)\) where \( \psi \in \Sigma^0_{n+3} \). By clause 4 of Definition 4.11, \( c \Vdash^\alpha \varphi(G) \) if and only if the formula \((\forall d)(d \not\in \text{Ext}(c) \lor d \not\Vdash^\alpha \psi(G))\) holds. By induction hypothesis, \( d \not\Vdash^\alpha \psi(G) \) is a \( \Pi^0_{n+3} \) predicate. By Lemma 4.14, the set \( \text{Ext}(c) \) is \( \Pi^0_{n+3} \)-computable uniformly in \( c \), thus \( c \Vdash^\alpha \varphi(G) \) is a \( \Pi^0_{n+3} \) predicate.

\( \square \)

4.4. Validity

As we already saw, we have two candidating forcing relations for \( \Sigma^1_1 \) and \( \Pi^1_1 \) formulas:

1. The “true” forcing relation \( c \Vdash^\alpha \varphi(G) \). This relation has been shown to have the expected density properties through Lemma 4.7. However deciding such a relation requires too much computational power.

2. The “weak” forcing relation \( c \Vdash^\alpha \varphi(G) \). Deciding such a relation requires the same definitional power as the formula it forces. It provides a sufficient condition for forcing the formula \( \varphi(G) \) as \( c \Vdash^\alpha \varphi(G) \) implies \( c \Vdash^\alpha \varphi(G) \), but the converse does not hold and we cannot prove the density property in the general case.

Thankfully, there exists some sides and parts of any condition on which those two forcing relations coincide. This leads to the notion of validity.

**Definition 4.16** (Validity) Fix an enumeration \( \varphi_0(G), \varphi_1(G), \ldots \) of all the \( \Pi^1_1 \) formulas. Fix a condition \( c = (\tilde{F}, T, C) \), a side \( \alpha < 2 \) and a part \( \nu \) of \( T \). We say that side \( \alpha \) is valid in part \( \nu \) of \( T \) if \( \text{part } \nu \) of \( T \) is \( R_\alpha \)-acceptable and for every \( i < \text{dom}(T) \), \( c \Vdash^\alpha_\nu \varphi_i(G) \iff c \Vdash^\alpha_\nu \varphi_i(G) \).

Note that the statement “side \( \alpha \) is valid in part \( \nu \) of \( T \)” is \( \Pi^1_3 \) as \( c \Vdash^\alpha_\nu \varphi_i(G) \) is \( \Pi^1_3 \) and \( c \Vdash^\alpha_\nu \varphi(G) \) is \( \Pi^1_4 \). Also note that if side \( \alpha \) is valid in part \( \mu \) of \( S \) and part \( \mu \) of \( S \) refines part \( \nu \) of \( T \), then side \( \alpha \) is valid in part \( \nu \) of \( T \). The following lemma shows in particular that every condition \( c = (\tilde{F}, T, C) \) has a side which is valid in some of its parts, and that we can furthermore restrict \( C \) so that it “witnesses its valid parts”.

**Lemma 4.17** The following set is dense in \( P \):

\[ \{ (\tilde{F}, T, C) \in P : (\forall \nu)(\exists \alpha < 2)([\nu, T) \in C \to \text{side } \alpha \text{ is valid in part } \nu \text{ of } T) \} \]

**Proof.** Given a condition \( c = (\tilde{F}, T, C) \), let \( I(c) \) be the set of the parts \( \nu \) of \( T \) such that \( (\nu, T) \in C \) and no \( \alpha < 2 \) is valid in part \( \nu \) of \( T \). Fix a condition \( c = (\tilde{F}, T, C) \in P \). By iterating the proof of Lemma 4.13, we can assume without loss of generality that for each \( i < \text{dom}(T) \),

\[ (\forall \alpha < 2) \left( c \Vdash^\alpha (\exists x)\varphi_1(G) \lor c \Vdash^\alpha (\forall x)\varphi_i(G) \right) \]
The dummy variable $x$ ensures that the forcing relation for $\Sigma^0_0$ and $\Pi^0_2$ is applied. It suffices to prove that for every $\nu \in I(c)$, there exists an extension $d = (E, S, D)$ such that $I(d) \subseteq I(c) \setminus \{\nu\}$. Iterating the process at most $|\text{parts}(T)|$ times enables us to conclude.

Fix a part $\nu \in I(c)$ and let $D$ be the set of $(\mu, S) \in C$ such that part $\mu$ of $S$ does not refine part $\nu$ of $T$. The set $D$ is a $\mathcal{G}$-p.r. upward-closed subset of $C$. It suffices to prove that for every infinite p.r. partition tree $S \leq T$, there exists a non-empty part $\mu$ of $S$ such that $(\mu, S) \in D$ to deduce that $D$ is a promise for $T$ and obtain an extension $d = (E, T, D)$ of $c$ such that $I(d) \subseteq I(c) \setminus \{\nu\}$.

Fix an infinite p.r. partition tree $S \leq T$. By choice of $\nu$, for every $\alpha < 2$, either $\nu$ is not $R_\alpha$-acceptable in $S$, or there exists an $i_\alpha < \text{dom}(T)$ such that $c \models^\alpha \phi_{i_\alpha}(G)$ but $c \not\models^\alpha \phi_{i_\alpha}(G)$. In the latter case, by choice of $c$, $c \models^\alpha (\forall x) \neg \phi_{i_\alpha}(G)$.

Let $f : 2^k + 1 \rightarrow 2^k$ be the function such that $f(\mu) = \mu$ for each $\mu \neq \nu$ and $f(\mu_\alpha) = \nu$ for each $\alpha < 2$. In other words, $f$ forks the part $\nu$ of $S$ into 2 parts $\mu_0$ and $\mu_1$. Let $P$ be a path through $S$, and let $t \in \omega$ be large enough to “witness the non $R_\alpha$-acceptable sides”. More formally, let $t$ be such that for every $\alpha < 2$, either set $\nu(P) \cap R_\alpha \cap [t, +\infty) = \emptyset$ or there exists an $i_\alpha < \text{dom}(T)$ such that $c \models^\alpha \phi_{i_\alpha}(G)$ but $c \not\models^\alpha \phi_{i_\alpha}(G)$.

Let $S'$ be the p.r. tree of all the $\tau$’s $f$-refining some $\sigma \in S$ and such that for each $\alpha < 2$, either set $\nu(P) \cap R_\alpha \cap [t, +\infty) = \emptyset$ or (as otherwise it would contradict $(\forall x) \neg \phi_{i_\alpha}(G)$). Given any filter $\mathcal{F} = \{c_0, c_1, \ldots\}$ with $c_s = (E_s, T_s, C_s)$, the set of pairs $(\alpha_s, \nu_s)$ such that $(\nu_s, T_s) \in C_s$ and the side $\alpha_s$ is valid in the part $\nu$ of $T_s$ forms again an infinite, directed acyclic graph $G(\mathcal{F})$. Lemma 4.17 enables us to say that this graph has at least one infinite directed path. Whenever $\mathcal{F}$ is sufficiently generic, the graph $G(\mathcal{F})$ yields a pair $(\alpha, P)$ such that for every $s$, the side $\alpha_s$ is valid in part $P(s)$ of $c_s$, and if $c_s$ refines $c_t$, then part $P(s)$ of $c_s$ refines part $P(t)$ of $c_t$. The path $P$ induces an infinite set $G = \bigcup \{F_{P(s)} : s \in \omega\}$. We call $\alpha$ the generic path, $P$ the generic path and $G$ the generic real.

By choice of the generic path to go only through valid sides and parts of the conditions, we recovered the density property for the weak forcing relation and are therefore able to prove that a property holds over the generic real iff it can be forced by some condition belonging to the generic filter.

**Lemma 4.18** Suppose that $\mathcal{F}$ is sufficiently generic and let $\alpha$, $P$ and $G$ be the generic side, the generic path and the generic real, respectively. For every $\Sigma^0_0$ ($\Pi^0_1$) formula $\phi(G)$, $\phi(G)$ holds iff $c_s \models^\alpha_{P(s)} \phi(G)$ for some $c_s \in \mathcal{F}$.

**Proof.** Thanks to validity, it suffices to prove that if $c_s \models^\alpha_{P(s)} \phi(G)$ for some $c_s \in \mathcal{F}$, then $\phi(G)$ holds. Indeed, if $\phi(G)$ holds, then by genericity of $\mathcal{F}$, $c_s \models^\alpha_{P(s)} \phi(G)$ or $c_s \not\models^\alpha_{P(s)} \neg \phi(G)$ for some $c_s \in \mathcal{F}$. By validity of side $\alpha$ in part $P(s)$ of $c_s$, $c_s \models^\alpha_{P(s)} \phi(G)$ or $c_s \not\models^\alpha_{P(s)} \neg \phi(G)$. If $c_s \models^\alpha_{P(s)} \neg \phi(G)$ then $\neg \phi(G)$ holds, contradicting the hypothesis. So $c_s \models^\alpha_{P(s)} \phi(G)$. Fix a condition $c_s = (E, T, C) \in \mathcal{F}$ such that $c_s \models^\alpha_{P(s)} \phi(G)$, where $\nu = P(s)$. 


If \( \varphi \in \Sigma_1^0 \) then \( \varphi(G) \) can be expressed as \((\exists x)\psi(G, x)\) where \( \psi \in \Sigma_0^0 \). By clause 1 of Definition 4.8, there exists a \( w \in \omega \) such that \( \psi(F'_\varphi, w) \) holds. As \( \nu = P(s) \), \( F'_\varphi = F_{P(a)}^{\varphi} \subseteq G \) and \( G \setminus F'_\varphi \subseteq (\max(F'_\varphi), +\infty) \), so \( \psi(G, w) \) holds by continuity, hence \( \varphi(G) \) holds.

If \( \varphi \in \Pi_0^1 \) then \( \varphi(G) \) can be expressed as \((\forall x)\psi(G, x)\) where \( \psi \in \Sigma_0^0 \). By clause 2 of Definition 4.8, for every \( \sigma \in T \), every \( w < |\sigma| \) and every set \( F' \subseteq \dom(T) \cap \set_\nu(\sigma) \), \( \psi(F'_\varphi \cup F', w) \) holds. For every \( F' \subseteq G \setminus F'_\varphi \), and \( w \in \omega \) there exists a \( \sigma \in T \) such that \( w < |\sigma| \) and \( F' \subseteq \dom(T) \cap \set_\nu(\sigma) \). Hence \( \psi(F'_\varphi \cup F', w) \) holds. Therefore, for every \( w \in \omega \), \( \psi(G, w) \) holds, so \( \varphi(G) \) holds.

\[ \square \]

**Lemma 4.19** Suppose that \( F \) is sufficiently generic and let \( \alpha \) and \( G \) be the generic side and the generic real, respectively. For every \( \Sigma_{n+2}^0 \) (\( \Pi_{n+2}^0 \)) formula \( \varphi(G) \), \( \varphi(G) \) holds if \( c_s \vDash^{\alpha} \varphi(G) \) for some \( c_s \in F \).

**Proof.** This lemma uses validity implicitly by calling Lemma 4.18 which itself uses it explicitly. Still following the explanation in Lemma 2.13, it suffices to prove that if \( c_s \vDash^{\alpha} \varphi(G) \) for some \( c_s \in F \) then \( \varphi(G) \) holds. Let \( P \) be the generic path induced by the generic filter \( F \). Fix a condition \( c_s = (\bar{F}, T, C) \in F \) such that \( c_s \vDash^{\alpha} \varphi(G) \). We proceed by case analysis on \( \varphi \).

- If \( \varphi \in \Sigma_2^0 \) then \( \varphi(G) \) can be expressed as \((\exists x)\psi(G, x)\) where \( \psi \in \Pi_0^1 \). By clause 1 of Definition 4.11, for every part \( \nu \) of \( T \) such that \( (\nu, T) \in C \), there exists a \( w < \dom(T) \) such that \( c_s \vDash^{\alpha} \psi(G, w) \). Since \((P(s), T) \in C \), \( c_s \vDash^{\alpha} \psi(G, w) \). By Lemma 4.18, \( \psi(G, w) \) holds, hence \( \varphi(G) \) holds.

- If \( \varphi \in \Pi_0^0 \) then \( \varphi(G) \) can be expressed as \((\forall x)\psi(G, x)\) where \( \psi \in \Sigma_0^1 \). By clause 2 of Definition 4.11, for every infinite \( k' \)-partition tree \( S \), every function \( f : \text{parts}(S) \to \text{parts}(T) \), every \( w \) and \( \bar{E} \) smaller than the code of \( S \) such that the following hold
  1. \((E_\nu, \dom(S) \cap R_\alpha) \) Mathias extends \((F_{f(\nu)}, \dom(T) \cap R_\alpha) \) for each \( \nu < \text{parts}(S) \)
  2. \( S \) \( f \)-refines \( \bigcap_{\nu < \text{parts}(S)} \text{T}^\nu \cap E_\nu \)

for every \( (\mu, S) \in C \), \((\bar{E}, S, C[S]) \) \( \not\vDash^{\alpha} \psi(G, w) \). Suppose for the sake of absurd that \( \psi(G, w) \) does not hold for some \( w \in \omega \). Then by Lemma 4.18, there exists a \( c_t \in F \) such that \( c_t \vDash^{\alpha} \neg \psi(G, w) \). Since \( F \) is a filter, there is a condition \( c_t = (\bar{E}, S, D) \in F \)

extending \( c_s \) and \( c_t \). By choice of \( P \), \( (P(c), S) \in C \), so by clause ii), \((\bar{E}, S, C[S])\) \( \not\vDash^{\alpha} \psi(G, w) \), hence by Lemma 4.10, \( c_e \vDash^{\alpha} \psi(G, w) \). However, since part \( P(c) \) of \( c_e \) refines part \( P(t) \) of \( c_t \), then by Lemma 4.9, \( c_e \vDash^{\alpha} \psi(G, w) \). Contradiction. Hence, for every \( w \in \omega \), \( \psi(G, w) \) holds, so \( \varphi(G) \) holds.

- If \( \varphi \in \Sigma_{n+3}^0 \) then \( \varphi(G) \) can be expressed as \((\exists x)\psi(G, x)\) where \( \psi \in \Pi_{n+2}^0 \). By clause 3 of Definition 4.11, there exists a \( w \in \omega \) such that \( c_s \vDash^{\alpha} \psi(G, w) \). By induction hypothesis, \( \psi(G, w) \) holds, hence \( \varphi(G) \) holds.

Conversely, if \( \varphi(G) \) holds, then there exists a \( w \in \omega \) such that \( \psi(G, w) \) holds, so by induction hypothesis \( c_{s'} \vDash^{\alpha} \psi(G, w) \) for some \( c_{s'} \in F \), so by clause 3 of Definition 4.11, \( c_s \vDash^{\alpha} \varphi(G) \).

- If \( \varphi \in \Pi_{n+3}^0 \) then \( \varphi(G) \) can be expressed as \( \neg \psi(G) \) where \( \psi \in \Sigma_{n+3}^0 \). By clause 4 of Definition 4.11, for every \( d \in \text{Ext}(c_s) \), \( d \not\vDash^{\alpha} \psi(G) \). By Lemma 4.12, \( d \not\vDash^{\alpha} \psi(G) \) for every \( d \in F \), and by a previous case, \( \psi(G) \) does hold, so \( \varphi(G) \) holds.

\[ \square \]

### 4.5. Preserving definitions

The following (and last) lemma shows that every sufficiently generic real preserves higher definitions. This preservation property cannot be proved in the case of non-\( \Sigma_1^0 \) sets since the weak forcing relation does not have the good density property in general.
Lemma 4.20 If $A \notin \Sigma_{n+2}^0$ and $\varphi(G,x)$ is $\Sigma_{n+2}^0$, then the set of $c \in \mathbb{P}$ satisfying the following property is dense:

$$(\forall \alpha < 2)[(\exists w \in A)c \Vdash^\alpha \neg \varphi(G,w)] \lor [(\exists w \notin A)c \Vdash^\alpha \varphi(G,w)]$$

Proof. It is sufficient to find, given a condition $c$ and a side $\alpha < 2$, an extension $d$ of $c$ such that the following holds:

$$(\exists w \in A)c \Vdash^\alpha \neg \varphi(G,w) \lor [(\exists w \notin A)c \Vdash^\alpha \varphi(G,w)]$$

Fix a condition $c = (\vec{F}, T, C)$ and a side $\alpha < 2$.

- In case $n = 0$, $\varphi(G,w)$ can be expressed as $(\exists x)\psi(G,w,x)$ where $\psi \in \Pi_1^0$. Let $U$ be the set of integers $w$ such that there exists an infinite p.r. $k'$-partition tree $S$ for some $k' \in \omega$, a function $f : \text{parts}(S) \to \text{parts}(T)$ and a $2k'$-tuple of finite sets $E$ such that
  i) $E^\beta_{\nu} = F^\beta_{f(\nu)}$ for each $\nu < \text{parts}(S)$ and $\beta \neq \alpha$
  ii) $(E^\nu_{\nu}, \text{dom}(S) \cap R_\alpha)$ Mathias extends $(F^\nu_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$.
  iii) $S$ $f$-refines $\bigcap_{\nu < \text{parts}(S)} \{\nu, E^\nu_{\nu}\}$
  iv) for each non-empty part $\nu$ of $S$ such that $(\nu, S) \in C$, $(\vec{E}, S, C[S]) \Vdash^\alpha \psi(G,w,u)$ for some $u < \#S$

By Lemma 4.15 and Lemma 3.10, $U \in \Sigma_2^0$, thus $U \neq A$. Let $w \in U \Delta A$.

Suppose that $w \in U \setminus A$. Let $\mathcal{D} = C[S] \setminus \{(\nu, S') \in \mathcal{C} : \text{ part } \nu \text{ of } S' \text{ is empty}\}$. As $\mathcal{C}$ is an $\emptyset'$-p.r. promise for $T$, $C[S]$ is an $\emptyset'$-p.r. promise for $S$. As $\mathcal{D}$ is obtained from $C[S]$ by removing only empty parts, $\mathcal{D}$ is also an $\emptyset'$-p.r. promise for $S$. By Lemma 4.10, for every part $\nu$ of $S$ such that $(\nu, S) \in \mathcal{D} \subseteq \mathcal{C}$, $(\vec{E}, S, \mathcal{D}) \Vdash^\alpha \psi(G,w,u)$ for some $u < \text{dom}(S)$, hence by clause 1 of Definition 4.11, $d = (\vec{E}, S, \mathcal{D}) \Vdash^\alpha (\exists x)\psi(G,w,x)$. In other words, $d \Vdash^\alpha \varphi(G)$ for some $w \notin A$.

We may choose a coding of the p.r. trees such that the code of $S$ is sufficiently large to witness $w$ and $\vec{E}$. So suppose now that $w \in A \setminus U$. Then for every infinite p.r. $k'$-partition tree $S$, every function $f : \text{parts}(S) \to \text{parts}(T)$ and every $\vec{E}$ smaller than the code of $S$ such that properties i-iii) hold, there exists a non-empty part $\nu$ of $S$ such that $\left. (\nu, S) \subseteq \mathcal{C} \right.$ and $(\vec{E}, S, C[S]) \Vdash^\alpha \psi(G,w,u)$ for every $u < \#S$. Let $D$ be the collection of all such $(\nu, S)$. $D$ is $\emptyset'$-p.r. By Lemma 4.9 and since $\#S \geq \#T$ whenever $S \leq T$, $\mathcal{D}$ is upward-closed under the refinement relation, hence is a promise for $T$. By clause 2 of Definition 4.11, $d = (\vec{F}, T, D) \Vdash^\alpha (\forall x)\neg \psi(G,w,x)$, hence $d \Vdash^\alpha \neg \varphi(G,w)$ for some $w \notin A$.

- In case $n > 0$, let $U = \{w \in \omega : (\exists d \in \text{Ext}(c))d \Vdash^\alpha \varphi(G,w)\}$. By Lemma 3.24 and Lemma 3.25, $U \in \Sigma_{n+2}^0$, thus $U \neq A$. Fix $w \in U \Delta A$. If $w \in U \setminus A$ then by definition of $U$, there exists a condition $d$ extending $c$ such that $d \Vdash^\alpha \varphi(G,w)$. If $w \in A \setminus U$, then for every $d \in \text{Ext}(c)$, $d \Vdash^\alpha \varphi(G,w)$ so by clause 4 of Definition 4.11, $c \Vdash^\alpha \neg \varphi(G,w)$.

We are now ready to reprove Corollary 3.29 from Wang [21].

Theorem 4.21 (Wang [21]) $RT_2^2$ admits preservation of $\Xi$ definitions simultaneously for all $\Xi$ in $\{\Sigma_{n+2}^0, \Pi_{n+2}^0, \Delta_{n+2}^0 : n \in \omega\}$.

Proof. Since $\text{RCA}_0 \vdash \text{COH} \land D_2^2 \to RT_2^2$, and $\text{COH}$ admits preservation of the arithmetic hierarchy, it suffices to prove that $D_2^2$ admits preservation of $\Xi$ definitions simultaneously for all $\Xi$ in $\{\Sigma_{n+2}^0, \Pi_{n+2}^0, \Delta_{n+2}^0 : n \in \omega\}$. Fix some set $C$ and a $\Delta_{n+2}^0$-partition $R_0 \cup R_1 = \omega$. Let $C_0$ be the $\emptyset'$-p.r. set of all $(\nu, T) \in U$ such that $(\nu, T) \leq (0,1^{<\omega})$. Let $\mathcal{F}$ be a sufficiently generic filter containing $c_0 = (\{0\}, 1^{<\omega}, C_0)$. Let $G$ be the corresponding generic real. By definition of a condition, the set $G$ is $\bar{R}$-cohesive. By Lemma 4.20 and Lemma 4.15, $G$ preserves non-$\Sigma_{n+2}^0$ definitions relative to $C$ for every $n \in \omega$. Therefore, by Proposition 2.2 of [21], $G$ preserves $\Xi$ definitions relative to $C$ simultaneously for all $\Xi$ in $\{\Sigma_{n+2}^0, \Pi_{n+2}^0, \Delta_{n+2}^0 : n \in \omega\}$.

\qed
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