Extremal Betti numbers of some Cohen-Macaulay binomial edge ideals

Carla Mascia, Giancarlo Rinaldo *
University of Trento

September 11, 2018

Abstract

We provide the regularity and the Cohen-Macaulay type of binomial edge ideals of Cohen-Macaulay cones, and we show the extremal Betti numbers of some classes of Cohen-Macaulay binomial edge ideals: Cohen-Macaulay bipartite and fan graphs. In addition, we compute the Hilbert-Poincaré series of the binomial edge ideals of some Cohen-Macaulay bipartite graphs.

Introduction

Binomial edge ideals were introduced in 2010 independently by Herzog et al. in [7] and by Ohtani in [17]. They are a natural generalization of the ideals of 2-minors of a $2 \times n$-generic matrix: their generators are those 2-minors whose column indices correspond to the edges of a graph. More precisely, given a simple graph $G$ on $[n]$ and the polynomial ring $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $2n$ variables over a field $K$, the binomial edge ideal associated to $G$ is the ideal $J_G$ in $S$ generated by all the binomials $\{x_i y_j - x_j y_i | i, j \in V(G)\}$, where $V(G)$ denotes the vertex set and $E(G)$ the edge set of $G$. Many algebraic and homological properties of these ideals have been investigated, such as the Castelnuovo-Mumford regularity and the projective dimension, see for instance [7], [6], [12], [11], and [18]. Important invariants which are provided by the graded finite free resolution are the extremal Betti numbers of $J_G$. Let $M$ be a finitely graded $S$-module. Recall the Betti number $\beta_{i,i+j+k}(M) \neq 0$ is called extremal if $\beta_{i,i+j+k}(M) = 0$ for all pairs $(k, \ell) \neq (i, j)$, with $k \geq i, \ell \geq j$. A nice property of the extremal Betti numbers is that $M$ has an unique extremal Betti number if and only if $\beta_{p,p+r}(M) \neq 0$, where $p = \text{proj dim} M$ and $r = \text{reg} M$. In this years, extremal Betti numbers were studied by different authors, also motivated by Ene, Hibi, and Herzog’s conjecture ([5], [3]) on the equality of the extremal Betti numbers of $J_G$ and $\text{in}_<(J_G)$. Some works in this direction are [1], [4], and [5], but the question has been completely and positively solved by Conca and Varbaro in [3]. The extremal Betti numbers of $J_G$ are explicitly provided by Dokuyucu, in [5], when $G$ is a cycle or a complete bipartite graph, by Hoang, in [9], for some closed graphs, and by Herzog and Rinaldo, in [8], and

*Email addresses: carla.mascia@unitn.it, giancarlo.rinaldo@unitn.it
Mascia and Rinaldo, in [15], when $G$ is a block graph. In this paper, we show the extremal Betti numbers for binomial edge ideals of some classes of Cohen-Macaulay graphs: cones, bipartite and fan graphs. The former were introduced and investigated by Rauf and the second author in [18]. They construct Cohen-Macaulay graphs by means of the formation of cones: connecting all the vertices of two disjoint Cohen-Macaulay graphs to a new vertex, the resulting graph is Cohen-Macaulay. For these graphs, we give the regularity and also the Cohen-Macaulay type (see Section 2). The latter two are studied by Bolognini, Macchia and Strazzanti in [2]. They classify the bipartite graphs whose binomial edge ideal is Cohen-Macaulay. In particular, they present a family of bipartite graphs $F_m$ whose binomial edge ideal is Cohen-Macaulay, and they prove that, if $G$ is connected and bipartite, then $J_G$ is Cohen-Macaulay if and only if $G$ can be obtained recursively by gluing a finite number of graphs of the form $F_m$ via two operations. In the same article, they describe a new family of Cohen-Macaulay binomial edge ideals associated with non-bipartite graphs, the fan graphs. For both these families, in [10], Jayanthan and Kumar compute a precise expression for the regularity, whereas in this work we provide the unique extremal Betti number of the binomial edge ideal of these graphs (see Section 3 and Section 4).

In addition, we exploit the unique extremal Betti number of $J_{F_m}$ to describe completely its Hilbert-Poincaré series (see Section 3).

1 Betti numbers of binomial edge ideals of disjoint graphs

In this section we recall some concepts and notation on graphs that we will use in the article.

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A subset $C$ of $V(G)$ is called a clique of $G$ if for all $i$ and $j$ belonging to $C$ with $i \neq j$ one has $\{i, j\} \in E(G)$. The clique complex $\Delta(G)$ of $G$ is the simplicial complex of all its cliques. A clique $C$ of $G$ is called a face of $\Delta(G)$ and its dimension is given by $|C| - 1$. A vertex of $G$ is called free vertex of $G$ if it belongs to exactly one maximal clique of $G$. A vertex of $G$ of degree 1 is called leaf of $G$. A vertex of $G$ is called a cutpoint if the removal of the vertex increases the number of connected components. A graph $G$ is decomposable, if there exist two subgraphs $G_1$ and $G_2$ of $G$, and a decomposition $G = G_1 \cup G_2$ with $\{v\} = V(G_1) \cap V(G_2)$, where $v$ is a free vertex of $G_1$ and $G_2$.

**Set-up 1.1** Let $G$ be a graph on $[n]$ and $u \in V(G)$ a cutpoint of $G$. We denote by

- $G'$ the graph obtained from $G$ by connecting all the vertices adjacent to $u$,
- $G''$ the graph obtained from $G$ by removing $u$,
- $H$ the graph obtained from $G'$ by removing $u$.

Using the notation introduced in Set-up [1.1] we consider the following short exact sequence

$$0 \longrightarrow S/J_G \longrightarrow S/J_{G'} \oplus S/((x_u, y_u) + J_{G''}) \longrightarrow S/((x_u, y_u) + J_H) \longrightarrow 0 \quad (1)$$
For more details, see Proposition 1.4, Corollary 1.5 and Example 1.6 of [19]. From (1), we get the following long exact sequence of Tor modules

\[ \cdots \to T_{i+1,i+1+j-1}(S/(x_u,y_u) + J_H) \to T_{i,i+j}(S/J_G) \to T_{i,i+j}(S/J_G) \oplus T_{i,i+j}(S/(x_u,y_u) + J_{G'}) \to T_{i,i+j}(S/(x_u,y_u) + J_H) \to \cdots \]  

(2)

where \( T^{S}_{i,i+j}(M) \) stands for \( \text{Tor}^S_{i,i+j}(M,K) \) for any \( S \)-module \( M \), and \( S \) is omitted if it is clear from the context.

**Lemma 1.2** Let \( G \) be a connected graph on \([n]\). Suppose \( J_G \) be Cohen-Macaulay, and let \( p = \text{proj dim} \ S/J_G \). Then

(i) \( \beta_{p,p+1}(S/J_G) \neq 0 \) if and only if \( G \) is a complete graph on \([n]\).

(ii) If \( G = H_1 \sqcup H_2 \), where \( H_1 \) and \( H_2 \) are graphs on disjoint vertex sets, then \( \beta_{p,p+2}(S/J_G) \neq 0 \) if and only if \( H_1 \) and \( H_2 \) are complete graphs.

**Proof.** (i) In [13], the authors prove that for any simple graph \( G \) on \([n]\), it holds

\[ \beta_{i,i+1}(S/J_G) = i f_i(\Delta(G)) \]

(3)

where \( \Delta(G) \) is the clique complex of \( G \) and \( f_i(\Delta(G)) \) is the number of faces of \( \Delta(G) \) of dimension \( i \). Since \( J_G \) is Cohen-Macaulay, it holds \( p = n - 1 \), and the statement is an immediate consequence of Equation (3), with \( i = p \).

(ii) Since \( J_G \) is generated by homogeneous binomials of degree 2, \( \beta_{1,1}(S/J_G) = 0 \). This implies that \( \beta_{i,i}(S/J_G) = 0 \) for all \( i \geq 1 \). For all \( j \geq 1 \), we have

\[ \beta_{p,p+j}(S/J_G) = \sum_{1 \leq j_1,j_2 \leq r \atop j_1+j_2 = j} \beta_{p_1,p_1+j_1}(S_1/J_{H_1}) \beta_{p_2,p_2+j_2}(S_2/J_{H_2}) \]

For \( j = 2 \), we get

\[ \beta_{p,p+2}(S/J_G) = \beta_{p_1,p_1+1}(S_1/J_{H_1}) \beta_{p_2,p_2+1}(S_2/J_{H_2}). \]

(4)

By part (i), both the Betti numbers on the right are non-zero if and only if \( H_1 \) and \( H_2 \) are complete graphs, and the thesis follows. \( \square \)

Let \( M \) be a finitely graded \( S \)-module. Recall the Cohen-Macaulay type of \( M \), that we denote by CM-type(\( M \)), is \( \beta_p(\( M \)) \), that is the sum of all \( \beta_{p,p+i}(M) \), for \( i = 0, \ldots, r \), where \( p = \text{proj dim} \ M \), and \( r = \text{reg} \ M \). When \( S/J_G \) has an unique extremal Betti number, we denote it by \( \hat{\beta}(S/J_G) \).

**Lemma 1.3** Let \( H_1 \) and \( H_2 \) be connected graphs on disjoint vertex sets and \( G = H_1 \sqcup H_2 \). Suppose \( J_{H_1} \) and \( J_{H_2} \) be Cohen-Macaulay binomial edge ideals. Let \( S_i = K[[x_j,y_j]_{j \in V(H_i)}] \) for \( i = 1,2 \). Then

(i) \( \text{CM-type}(S/J_G) = \text{CM-type}(S_1/J_{H_1}) \text{CM-type}(S_2/J_{H_2}) \).

(ii) \( \hat{\beta}(S/J_G) = \hat{\beta}(S_1/J_{H_1}) \hat{\beta}(S_2/J_{H_2}) \).

3
Proof. (i) The equality $J_G = J_{H_i} + J_{H_2}$ implies that the minimal graded free resolution of $S/J_G$ is the tensor product of the minimal graded free resolutions of $S_1/J_{H_1}$ and $S_2/J_{H_2}$, where $S_i = K[[x_j, y_j]_{j \in V(H_i)}]$ for $i = 1, 2$. Then

$$\beta_i(S/J_G) = \sum_{k=0}^{t} \beta_k(S_1/J_{H_1})\beta_{t-k}(S_2/J_{H_2}).$$

Let $p = \text{proj dim} S/J_G$, that is $p = p_1 + p_2$, where $p_i = \text{proj dim} S_i/J_{H_i}$ for $i = 1, 2$. Since $\beta_k(S_1/J_{H_1}) = 0$ for all $k > p_1$ and $\beta_{p-k}(S_2/J_{H_2}) = 0$ for all $k < p_1$, it follows

$$\beta_p(S/J_G) = \beta_{p_1}(S_1/J_{H_1})\beta_{p_2}(S_2/J_{H_2}).$$

(ii) Let $r = \text{reg} S/J_G$. Consider

$$\beta_{p+r}(S/J_G) = \sum_{1 \leq j_1,j_2 \leq r \atop j_1 + j_2 = r} \beta_{p_1,j_1}(S_1/J_{H_1})\beta_{p_2,j_2}(S_2/J_{H_2}).$$

Since $\beta_{p_1,j_1}(S_1/J_{H_1}) = 0$ for all $j_i > r_i$, where $r_i = \text{reg} S_i/J_{H_i}$ for $i = 1, 2$, and $r = r_1 + r_2$, it follows

$$\beta_{p+r}(S/J_G) = \beta_{p_1,r_1}(S_1/J_{H_1})\beta_{p_2,r_2}(S_2/J_{H_2}).$$

Let $G$ be a simple connected graph on $[n]$. We recall that if $J_G$ is Cohen-Macaulay, then $p = \text{proj dim} S/J_G = n - 1$, and it admits an unique extremal Betti number, that is $\hat{\beta}(S/J_G) = \beta_{p+r}(S/J_G)$, where $r = \text{reg} S/J_G$.

## 2 Regularity and Cohen-Macaulay type of cones

The cone of $v$ on $H$, namely $\text{cone}(v, H)$, is the graph with vertices $V(H) \cup \{v\}$ and edges $E(H) \cup \{\{v, w\} \mid w \in V(H)\}$.

**Lemma 2.1** Let $G = \text{cone}(v, H_1 \sqcup \cdots \sqcup H_s)$, with $s \geq 2$. Then

$$\text{reg} S/J_G = \max \left\{ \sum_{i=1}^{s} \text{reg} S/J_{H_i}, 2 \right\}.$$

**Proof.** Consider the short exact sequence $\mathbf{1}$, with $G = \text{cone}(v, H_1 \sqcup \cdots \sqcup H_s)$ and $u = v$, then $G' = K_n$, the complete graph on $[n]$, $G'' = H_1 \sqcup \cdots \sqcup H_s$, and $H = K_{n-1}$, where $n = |V(G)|$. Since $G'$ and $H$ are complete graphs, the regularity of $S/J_{G'}$ and $S/(x_u, y_u) + J_H$ is 1. Whereas the regularity of $S/(x_u, y_u) + J_{G''}$ is given by $\text{reg} S/J_{H_1} + \cdots + \text{reg} S/J_{H_s}$. We get the following bound on the regularity of $S/J_G$

$$\text{reg} S/J_G \leq \max \left\{ \text{reg} \left( \frac{S}{J_{G'}}, \frac{S}{((x_u, y_u) + J_{G''})}, \frac{S}{((x_u, y_u) + J_H) + 1} \right) \right\}$$

$$= \max \left\{ 1, \sum_{i=1}^{s} \text{reg} S/J_{H_i}, 2 \right\}.$$
Suppose $\sum_{i=1}^{s} \text{reg} S/J_{H_i} \geq 2$, hence $\text{reg} S/J_G \leq \sum_{i=1}^{s} \text{reg} S/J_{H_i}$. Since $H_1 \sqcup \cdots \sqcup H_s$ is an induced subgraph of $G$, by [16] Corollary 2.2 of Matsuda and Murai we have

$$\text{reg} S/J_G \geq \sum_{i=1}^{s} \text{reg} S/J_{H_i}.$$  

Suppose now $\sum_{i=1}^{s} \text{reg} S/J_{H_i} < 2$, hence $\text{reg} S/J_G \leq 2$. Since $G$ is not a complete graph, $\text{reg} S/J_G \geq 2$, and the statement follows.

Observe that it happens $\text{reg} S/J_G = 2$, for $G = \text{cone}(v, H_1 \sqcup \cdots \sqcup H_s)$, with $s \geq 2$, if and only if all the $H_i$ are isolated vertices except for at most two which are complete graphs.

We are going to give a description of the Cohen-Macaulay type and some Betti numbers of $S/J_G$ when $S/J_G$ is Cohen-Macaulay, and $G$ is a cone, namely $G = \text{cone}(v, H)$. By [18] Lemma 3.4, to have Cohen-Macaulayness it is necessary that $H$ has exactly two connected components and both are Cohen-Macaulay (see also Corollaries 3.6 and 3.7 and Theorem 3.8 in [18]).

**Proposition 2.2** Let $G = \text{cone}(v, H_1 \sqcup H_2)$ on $[n]$, with $J_{H_1}$ and $J_{H_2}$ Cohen-Macaulay binomial edge ideals. Then

$$\text{CM-type}(S/J_G) = n - 2 + \text{CM-type}(S/J_{H_1}) \cdot \text{CM-type}(S/J_{H_2}).$$

In particular, the unique extremal Betti number of $S/J_G$ is given by

$$\begin{aligned}
\tilde{\beta}(S/J_G) &= \begin{cases}
\tilde{\beta}(S_1/J_{H_1}) \cdot \tilde{\beta}(S_2/J_{H_2}) & \text{if } r > 2 \\
n - 2 + \tilde{\beta}(S_1/J_{H_1}) \cdot \tilde{\beta}(S_2/J_{H_2}) & \text{if } r = 2
\end{cases}
\end{aligned}$$

where $r = \text{reg} S/J_G$. In addition, if $r > 2$, it holds

$$\beta_{p,p+2}(S/J_G) = n - 2.$$  

**Proof.** Consider the short exact sequence (1), with $u = v$, then we have $G' = K_n$, $G'' = H_1 \sqcup H_2$, and $H = K_{n-1}$. It holds

$$r = \text{reg} S/J_G = \max\{\text{reg} S/J_{H_1} + \text{reg} S/J_{H_2}, 2\},$$

$$\begin{aligned}
\text{reg} S/((x_u, y_u) + J_{G''}) &= \text{reg} S/J_{H_1} + \text{reg} S/J_{H_2}, \\
\text{reg} S/J_{G'} &= \text{reg} S/((x_u, y_u) + J_H) = 1,
\end{aligned}$$

and

$$p = \text{proj dim} S/J_G = \text{proj dim} S/J_{G'} = \text{proj dim} S/((x_u, y_u) + J_{G''}) = n - 1,$$

$$\text{proj dim} S/((x_u, y_u) + J_H) = n.$$  

Consider the long exact sequence (2) with $i = p$. By (5), we have

$$\tilde{\beta}_{p,p+j}(S/J_{G'}) = \tilde{\beta}_{p,p+j}(S/((x_u, y_u) + J_H)) = 0 \text{ for all } j \geq 2.$$
By Lemma 1.2 and Lemma 1.3 (i), it follows that
\[
\beta_{r-1, p-2+2}(S/J_H) + \beta_{p-2, p-2+2}(S/J_{G'}) = n - 2 + \text{CM-type}(S/J_{G'}). \tag{4}
\]
where the last equality follows from Equation (4).

If \( r = 2 \),
\[
\text{CM-type}(S/J_G) = \beta_{p, p+2}(S/J_G) = \beta_{p, p+2}(S/J_H) + \beta_{p-2, p-2+2}(S/J_{G'}) = n - 2 + \tilde{\beta}(S_1/J_{H_1}) \tilde{\beta}(S_2/J_{H_2}).
\]

If \( r > 2 \), it means that \( H_1 \) and \( H_2 \) are not both complete graphs, and then, by Lemma 1.3 (ii), \( \beta_{p-2, p-2+2}(S/J_{G'}) = 0 \), then \( \beta_{p, p+2}(S/J_G) = n - 2 \), and \( \tilde{\beta}(S/J_G) = \tilde{\beta}(S_1/J_{H_1}) \tilde{\beta}(S_2/J_{H_2}) \).

\[\square\]

3 Extremal Betti numbers of some classes of Cohen-Macaulay binomial edge ideals

We are going to introduce the notation for the family of fan graphs first introduced in [2].

Let \( K_m \) be the complete graph on \([m]\) and \( W = \{v_1, \ldots, v_s\} \subseteq [m] \). Let \( F^m_w \) be the graph obtained from \( K_m \) by attaching, for every \( i = 1, \ldots, s \), a complete graph \( K_{h_i} \) to \( K_m \) in such a way \( V(K_m) \cap V(K_{h_i}) = \{v_1, \ldots, v_i\} \), for some \( h_i > i \).

We say that the graph \( F^m_w \) is obtained by adding a fan to \( K_m \) on the set \( W \). If \( h_i = i + 1 \) for all \( i = 1, \ldots, s \), we say that \( F^m_w \) is obtained by adding a pure fan to \( K_m \) on the set \( W \).

Let \( W = W_1 \sqcup \cdots \sqcup W_k \) be a non-trivial partition of a subset \( W \subseteq [m] \). Let \( F^{W,k}_m \) be the graph obtained from \( K_m \) by adding a fan to \( K_m \) on each set \( W_i \), for \( i = 1, \ldots, k \). The graph \( F^{W,k}_m \) is called a \( k \)-fan of \( K_m \) on the set \( W \). If all the fans are pure, we called it a \( k \)-pure fan graph of \( K_m \) on \( W \).

When \( k = 1 \), we write \( F^m_w \) instead of \( F^{W,1}_m \). Consider the pure fan graph \( F^m_w \) on \( W = \{v_1, \ldots, v_s\} \). We observe that \( F^m_w = \text{cone}(v_1, F^m_{w-1} \cup \{w\}) \), where \( W' = W \setminus \{v_1\}, w \) is the leaf of \( F^m_w \), \( \{w, v_1\} \in E(F^m_w) \), and \( F^m_{w-1} \) is the pure graph of \( K_{n-1} \) on \( W' \).

Now, we recall the notation used in [2] for a family of bipartite graphs.

For every \( m \geq 1 \), let \( F_m \) be the graph on the vertex set \([2m]\) and with edge set \( E(F_m) = \{\{2i, 2j-1\} \mid i = 1, \ldots, m, j = i, \ldots, m\} \).
In [2], they prove that if either \( G = F_m \) or \( G = F^W_m \), with \( m \geq 2 \), then \( J_G \) is Cohen-Macaulay. The regularity of \( S/J_G \) has been studied in [10], and hold the following results.

**Proposition 3.1 (10)** Let \( G = F^W_m \) be the \( k \)-pure fan graph of \( K_m \) on \( W \), with \( m \geq 2 \). Then
\[
\text{reg} S/J_G = k + 1.
\]

**Proposition 3.2 (10)** For every \( m \geq 2 \), \( \text{reg} S/J_{F_m} = 3 \).

Observe that if \( G = F^W_m \) is a pure fan graph, the regularity of \( J_G \) is equal to 3 for any \( m \) and \( W \subseteq [m] \), then all of these graphs belong to the class of graphs studied by Madani and Kiani in [14].

Exploiting Proposition 2.2 we get hold a formula for the CM-type of any \( G = F^W_m \) pure fan graph.

**Proposition 3.3** Let \( m \geq 2 \), and \( G = F^W_m \) a pure fan graph, with \( |W| \geq 1 \). Then
\[
\text{CM-type}(S/J_G) = \hat{\beta}(S/J_G) = (m - 1)|W|.
\]

**Proof.** We use induction on \( m \). If \( m = 2 \), \( G \) is decomposable into \( K_2 \) and \( K_3 \), and it is straightforward to check that (5) holds. If \( m > 2 \) and supposing the thesis true for all the pure graphs of \( K_m \), we have \( G = \text{cone}(v_1, H_1 \cup H_2) \), where \( W = \{v_1, \ldots, v_s\} \), \( H_1 = F^W_{m-1} \) is the pure graph of \( K_{m-1} \) on \( W' \), with \( W' = W \setminus \{v_1\} \), \( w \) is the leaf of \( G \), \( \{w, v_1\} \in E(G) \), and \( H_2 = \{w\} \). By induction hypothesis \( \text{CM-type}(S/J_{H_1}) = (m - 2)|(W' - 1) \), and \( \text{CM-type}(S/J_{H_2}) = 1 \), then using Proposition 2.2 it follows
\[
\text{CM-type}(S/J_G) = |V(G)| - 2 + \text{CM-type}(S/J_{H_1})\text{CM-type}(S/J_{H_2})
\]
\[= (m + |W| - 2) + (m - 2)(|W| - 1) = (m - 1)|W|.
\]

Since \( |W| \geq 1 \), the graph \( F^W_m \) is not a complete graph, then \( \beta_{p,p+2}(S/J_G) = 0 \), where \( p = \text{proj dim} S/J_G \). Due to \( \text{reg} S/J_G = 2 \), the \( \text{CM-type}(S/J_G) \) coincides with the unique extremal Betti number of \( S/J_G \), that is \( \beta_{p,p+2} \).

In the following result we provide the unique extremal Betti number of any \( k \)-pure fan graph.

**Proposition 3.4** Let \( G = F^W_m \) be a \( k \)-pure fan graph, where \( m \geq 2 \) and \( W = W_1 \sqcup \cdots \sqcup W_k \subseteq [m] \) is a non-trivial partition of \( W \). Then
\[
\hat{\beta}(S/J_G) = (m - 1) \prod_{i=1}^{k} |W_i|.
\]

**Proof.** Let \( |W_i| = \ell_i \), for \( i = 1, \ldots, k \). First of all, we observe that if \( \ell_i = 1 \) for all \( i = 1, \ldots, k \), that is \( W_i = \{v_i\} \), then \( G \) is decomposable into \( G_1 \cup \cdots \cup G_{k+1} \), where \( G_1 = K_m \), \( G_j = K_2 \) and \( G_1 \cap G_j = \{v_j\} \), for all \( j = 2, \ldots, k + 1 \). This implies
\[
\hat{\beta}(S/J_G) = \prod_{j=1}^{k+1} \hat{\beta}(S/J_{G_j}) = m - 1
\]
where the last equality is due to the fact \( \beta(S/J_{K_m}) = m - 1 \) for any complete graph \( K_m \), with \( m \geq 2 \). Without loss of generality, we suppose \( \ell_1 \geq 2 \).

We are ready to prove the statement on induction on \( n \), the number of vertices of \( G = F^{m,k}_m \), that is \( n = m + \sum_{i=1}^{k} \ell_i \). Let \( n = 4 \), then \( G \) is a pure fan graph \( F^{4,1}_2 \), with \( |W| = 2 \), satisfying Proposition 3.3 and it holds (i). Let \( n > 4 \). Pick \( v \in W_1 \) such that \( \{v, w\} \in E(G) \), with \( w \) a leaf of \( G \). Consider the short exact sequence (1), with \( u = v \), \( G' = F^{m+\ell_1-1}_m \) the \((k-1)\)-pure fan graph of \(K_{m+\ell_1} \) on \(W' = W_2 \cup \cdots \cup W_k \), \( G'' = F^{W',k-1}_m \cup \{w\} \) the disjoint union of the isolated vertex \( w \) and the \( k \)-pure fan graph of \(K_{m+1} \) on \(W'' = W \setminus \{v\} \), and \( H = F^{W',k-1}_m \). For the quotient rings involved in (1), from Proposition 3.1, we have

\[
\begin{align*}
r & = \text{reg} S/J_G = \text{reg} S/((x_u, y_u) + J_{G'}) = 1 + k, \\
r & = \text{reg} S/J_{G'} = \text{reg} S/((x_u, y_u) + J_H) = k.
\end{align*}
\]

As regard the projective dimensions, we have

\[
p = \text{proj dim } S/J_G = \text{proj dim } S/J_{G'} = \text{proj dim } S/((x_u, y_u) + J_{G'}) = \text{proj dim } S/((x_u, y_u) + J_H) - 1 = m + \sum_{i=1}^{k} \ell_i - 1.
\]

Fix \( i = p \) and \( j = r \) in the long exact sequence (2). The Tor modules \( T_{p+1, p+1+\ell_1-1}(S/((x_u, y_u) + J_H)) \) and \( T_{p, p+1}(S/((x_u, y_u) + J_{G'})) \) are the only non-zeros. It follows

\[
\begin{align*}
\beta_{p,p+r}(S/J_G) & = \beta_{p-1, p+r-2}(S/J_H) + \beta_{p-2, p+r-2}(S/J_{G'}) \\
& = \beta(S/J_H) + \beta(S/J_{F^{W'',k-1}_m}).
\end{align*}
\]

Both \( F^{W'',k-1}_m \) and \( H \) fulfill the hypothesis of the proposition and they have less than \( n \) vertices, then by induction hypothesis

\[
\begin{align*}
\beta(S/J_H) & = (m + \ell_1 - 2) \prod_{s=2}^{k} \ell_s, \\
\beta(S/J_{F^{W'',k-1}_m}) & = (m - 2)(\ell_1 - 1) \prod_{s=2}^{k} \ell_s.
\end{align*}
\]

Adding these extremal Betti numbers, the thesis is proved. \( \square \)

**Proposition 3.5** Let \( m \geq 2 \). The unique extremal Betti number of the bipartite graph \( F_m \) is given by

\[
\beta(S/J_{F_m}) = \sum_{k=1}^{m-1} k^2.
\]

**Proof.** We use induction on \( m \). If \( m = 2 \), then \( F_2 = K_2 \) and it is well known that \( \beta(S/J_{K_2}) = 1 \). Suppose \( m > 2 \). Consider the short exact sequence (1), with \( G = F_m \) and \( u = 2m - 1 \), with respect to the labelling introduced at the begin of this section. The graphs involved in (1) are \( G' = F^{m+1}_m \), that is the pure fan graph of \(K_{m+1} \), with \( V(K_{m+1}) = \{u\} \cup \{2i | i = 1, \ldots, m\} \), on
\[ W = \{2i - 1\mid i = 1, \ldots, m - 1\}, \quad G'' = F_{m-1} \sqcup \{2m\}, \text{ and the pure fan graph} \]
\[ H = F^W_m. \]
By Proposition 3.1 and Proposition 3.2, we have
\[
r = \operatorname{reg} S/J_G = \operatorname{reg} S/((x_u, y_u) + J_{G''}) = 3
\]
\[
\operatorname{reg} S/J_{G'} = \operatorname{reg} S/((x_u, y_u) + J_H) = 2.
\]
As regards the projective dimension of the quotient rings involved in \( (1) \), it is equal to \( p = 2m - 1 \) for all, except for \( S/((x_u, y_u) + J_H) \) whose projective dimension is \( 2m \).
Consider the long exact sequence \( (2) \), with \( i = p \) and \( j = r \).
In view of the above, \( T_{p, p+r}(S/J_{G''}) \), \( T_{p, p+r}(S/((x_u, y_u) + J_H)) \), and all the Tor modules on the left of \( T_{p+1,p+1+(r-1)}(S/((x_u, y_u) + J_H)) \) in \( (2) \) are zero. It follows that
\[ T_{p, p+r}(S/J_G) \cong T_{p+1,p+1+(r-1)}(S/((x_u, y_u) + J_H)) \oplus T_{p, p+r}(S/((x_u, y_u) + J_{G''})). \]
Then, using Proposition 3.3 and induction hypothesis, we obtain
\[
\beta_{p, p+r}(S/J_G) = \beta_{p-1, p+r-2}(S/J_H) + \beta_{p-2, p+r-2}(S/J_{G''})
\]
\[
= \tilde{\beta}(S/J_H) + \tilde{\beta}(S/J_{G''})
\]
\[
= (m - 1)^2 + \sum_{k=1}^{m-2} k^2 = \sum_{k=1}^{m-1} k^2.
\]
\[ \square \]

**Question 3.6.** Based on explicit calculations we believe that for all bipartite graphs \( F_m \) and pure fan graphs \( F^W_m \) the unique extremal Betti number coincides with the CM-type, that is \( \beta_{p, p+1}(S/J_G) = 0 \) for all \( j = 0, \ldots, r - 1 \), when either \( G = F_m \) or \( G = F^W_m \), for \( m \geq 2 \), and \( p = \operatorname{proj dim} S/J_G \) and \( r = \operatorname{reg} S/J_G \).

In the last part of this section, we completely describe the Hilbert-Poincaré series \( HS \) of \( S/J_G \), when \( G \) is a bipartite graph \( F_m \). In particular, we are interested in computing the \( h \)-vector of \( S/J_G \).

For any graph \( G \) on \([n]\), it is well known that
\[ HS_{S/J_G}(t) = \frac{p(t)}{(1-t)^{2n}} = \frac{h(t)}{(1-t)^d} \]
where \( p(t), h(t) \in \mathbb{Z}[t] \) and \( d \) is the Krull dimension of \( S/J_G \). The polynomial \( p(t) \) is related to the graded Betti numbers of \( S/J_G \) in the following way
\[ p(t) = \sum_{i,j} (-1)^i \beta_{i,j}(S/J_G) t^j. \]  
\[ (8) \]

**Lemma 3.7** Let \( G \) be a graph on \([n]\), and suppose \( S/J_G \) has an unique extremal Betti number, then the last non negative entry in the \( h \)-vector is \( (-1)^{p+d} \beta_{p, p+r} \), where \( p = \operatorname{proj dim} S/J_G \) and \( r = \operatorname{reg} S/J_G \).
Proof. If \( S/J \) has an unique Betti number then it is equal to \( \beta_{p,p+r}(S/J) \). Since \( p(t) = h(t)(1-t)^{2n-d} \), then \( lc(p(t)) = (-1)^d lc(h(t)) \), where \( lc \) denotes the leading coefficient of a polynomial. By Equation (8), the leading coefficient of \( p(t) \) is the coefficient of \( t^j \) for \( j = p + r \). Since \( \beta_{i,p+r} = 0 \) for all \( i < p \), \( lc(p(t)) = (-1)^p \beta_{p,p+r} \), and the thesis follows.

The degree of \( HS_{S/J}(t) \) as a rational function is called an invariants, denoted by \( a(S/J) \), and it holds

\[
a(S/J) \leq \text{reg } S/J - \text{depth } S/J.
\]

The equality holds if \( G \) is Cohen-Macaulay. In this case, \( \text{dim } S/J = \text{depth } S/J \), and then \( \text{deg } h(t) = \text{reg } S/J \).

**Proposition 3.8** Let \( G = F_m \), with \( m \geq 2 \), then the Hilbert-Poincaré series of \( S/J \) is given by

\[
HS_{S/J}(t) = \frac{h_0 + h_1 t + h_2 t^2 + h_3 t^3}{(1-t)^{2m+1}}
\]

where

\[
h_0 = 1, \quad h_1 = 2m - 1, \quad h_2 = \frac{3m^2 - 3m}{2}, \quad \text{and } h_3 = \sum_{k=1}^{m-1} k^2.
\]

Proof. By Proposition 3.2, \( \text{deg } h(t) = \text{reg } S/J = 3 \). Let \( \text{in}(J_G) = I_\Delta \), for some simplicial complex \( \Delta \), where \( I_\Delta \) denotes the Stanley-Reisner ideal of \( \Delta \). Let \( f_i \) be the number of faces of \( \Delta \) of dimension \( i \) with the convention that \( f_{-1} = 1 \). Then

\[
h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}. \quad (9)
\]

Exploiting the Equation (9) we get

\[
h_1 = f_0 - d = 4m - (2m + 1) = 2m - 1
\]

To obtain \( h_2 \) we need first to compute \( f_1 \), that is the number of edges in \( \Delta \): they are all the possible edges, except for those that appear in \( (I_\Delta)_2 \), which are the number of edges in \( G \). So

\[
f_1 = \binom{4m}{2} - \frac{m(m+1)}{2} = \frac{15m^2 - 5m}{2}.
\]

And then we have

\[
h_2 = \binom{2m+1}{2} f_{-1} - \binom{2m}{1} f_0 + \binom{2m-1}{0} f_1 = \frac{3m^2 - 3m}{2}.
\]

By Lemma 3.7 and since \( p = 2m - 1 \) and \( d = 2m + 1 \),

\[
h_3 = (-1)^{4m} \beta_{p,p+r}(S/J) = \sum_{k=1}^{m-1} k^2
\]

where the last equality follows from Proposition 3.5. \( \square \)
4 Extremal Betti numbers of Cohen-Macaulay bipartite graphs

In [2], the authors prove that, if \( G \) is connected and bipartite, then \( J_G \) is Cohen-Macaulay if and only if \( G \) can be obtained recursively by gluing a finite number of graphs of the form \( F_m \) via two operations. Here, we recall the notation introduced in [2] for the sake of completeness.

Operation \( \ast \): For \( i = 1, 2 \), let \( G_i \) be a graph with at least one leaf \( f_i \). We denote by \( G = (G_1, f_1) \ast (G_2, f_2) \) the graph \( G \) obtained by identifying \( f_1 \) and \( f_2 \).

Operation \( \circ \): For \( i = 1, 2 \), let \( G_i \) be a graph with at least one leaf \( f_i \), \( v_i \) its neighbour and assume \( \text{deg}_{G_i}(v_i) \geq 3 \). We denote by \( G = (G_1, f_1) \circ (G_2, f_2) \) the graph \( G \) obtained by removing the leaves \( f_1, f_2 \) from \( G_1 \) and \( G_2 \) and by identifying \( v_1 \) and \( v_2 \).

In \( G = (G_1, f_1) \circ (G_2, f_2) \), to refer to the vertex \( v \) resulting from the identification of \( v_1 \) and \( v_2 \) we write \( \{v\} = V(G_1) \cap V(G_2) \). For both operations, if it is not important to specify the vertices \( f_i \) or it is clear from the context, we simply write \( G_1 \ast G_2 \) or \( G_1 \circ G_2 \).

**Theorem 4.1** ([2]) Let \( G = F_{m_1} \ast \cdots \ast F_{m_t} \ast F \), where \( F \) denotes either \( F_m \) or a \( k \)-pure fan graph \( F_{m_1}^{W_k} \), with \( t \geq 0 \), \( m \geq 3 \), and \( m_i \geq 3 \) for all \( i = 1, \ldots, t \). Then \( J_G \) is Cohen-Macaulay.

**Theorem 4.2** ([2], [18]) Let \( G \) be a connected bipartite graph. The following properties are equivalent:

(i) \( J_G \) is Cohen-Macaulay;

(ii) \( G = A_1 \ast A_2 \ast \cdots \ast A_k \), where, for \( i = 1, \ldots, k \), either \( A_i = F_m \) or \( A_i = F_{m_i} \ast \cdots \ast F_{m_t} \), for some \( m \geq 1 \) and \( m_i \geq 3 \).

Let \( G = G_1 \ast \cdots \ast G_t \), for \( t \geq 1 \). Observe that \( G \) is decomposable into \( G_1 \cup \cdots \cup G_t \), with \( G_i \cap G_{i+1} = \{f_i\} \), for \( i = 1, \ldots, t-1 \), where \( f_i \) is the leaf of \( G_i \) and \( G_{i+1} \) which has been identified in \( G_i \ast G_{i+1} \) and \( G_i \cap G_j = \emptyset \), for \( 1 \leq i < j \leq t \). If \( G \) is a Cohen-Macaulay bipartite graph, then it admits only one extremal Betti number, and by [8, Corollary 1.4], it holds

\[
\tilde{\beta}(S/J_G) = \prod_{i=1}^{t} \tilde{\beta}(S/J_{G_i}).
\]

In light of the above, we will focus on graphs of the form \( G = F_{m_1} \circ \cdots \circ F_{m_t} \), with \( m_i \geq 3 \), \( i = 1, \ldots, t \). Before stating the unique extremal Betti number of \( S/J_G \), we recall the results on regularity showed in [10].

**Proposition 4.3** ([10]) For \( m_1, m_2 \geq 3 \), let \( G = F_{m_1} \circ F \), where either \( F = F_{m_2} \) or \( F \) is a \( k \)-pure fan graph \( F_{m_2}^{W_k} \), with \( W = W_1 \cup \cdots \cup W_k \) and \( \{v\} = V(F_{m_1}) \cap \{v\} \).
Lemma 4.5  Let \( H \) be with \( \in \) ∈ \( W \). In particular, \( W \) with \( |W_i| = 1 \) for all \( i \)
and \( |W_i| \geq 2 \) for some \( i \) and \( v \in W_i \).

Proposition 4.4  Let \( m_1, \ldots, m_t, m \geq 3 \) and \( t \geq 2 \). Consider \( G = F_{m_1} \circ \cdots \circ F_{m_t} \circ F \), where \( F \) denotes either \( F_m \) or the \( k \)-pure fan graph \( F_{m_1}^k \)
with \( W = W_1 \sqcup \cdots \sqcup W_k \) and \( |W_i| \geq 2 \) for some \( i \) and \( v \in W_i \).

\[
\text{reg } S/J_G = \begin{cases} 
6 & \text{if } F = F_{m_2} \\
k + 3 & \text{if } F = F_{m_2}^k \text{ and } |W_i| = 1 \text{ for all } i \\
k + 4 & \text{if } F = F_{m_2}^k \text{ and } |W_i| \geq 2 \text{ for some } i \text{ and } v \in W_i 
\end{cases}
\]

Lemma 4.5  Let \( m_1, m_2 \geq 3 \) and \( G = F_{m_1} \circ F \), where \( F \) is either \( F_{m_2} \) or a \( k \)-pure fan graph \( F_{m_2}^k \), with \( W = W_1 \sqcup \cdots \sqcup W_k \) and \( |W_i| \geq 2 \) for some \( i \). Let \( v = V(F_{m_1}) \cap V(F) \) and suppose \( v \in W_i \). Let \( G'' \) be as in Set-up 1.7 with \( u = v \). Then the unique extremal Betti number of \( S/J_G \) is given by

\[
\tilde{\beta}(S/J_G) = \tilde{\beta}(S/J_{G''}).
\]

In particular,

\[
\tilde{\beta}(S/J_G) = \begin{cases} 
\tilde{\beta}(S/J_{F_{m_1-1}}) \tilde{\beta}(S/J_{F_{m_2-1}}) & \text{if } F = F_{m_2} \\
\tilde{\beta}(S/J_{F_{m_1-1}}) \tilde{\beta}(S/J_{F_{m_2-1}^k}) & \text{if } F = F_{m_2}^k 
\end{cases}
\]

where \( W'' = W \setminus \{v\} \).

Proof. Consider the short exact sequence 1.1, with \( G = F_{m_1} \circ F \) and \( u = v \).
If \( F = F_{m_2} \), then the graphs involved in 1.1 are: \( G' = F_{m_2}^k \), \( G'' = F_{m_1-1} \sqcup F_{m_2-1} \), and \( H = F_{m-1} \), where \( m = m_1 + m_2 - 1 \), \( W = W_1 \sqcup W_2 \) with \( |W_i| = m_i - 1 \) for \( i = 1, 2 \), and \( G' \) and \( H \) are 2-pure graph. By Proposition 3.1 and Proposition 3.2, we have the following values for the regularity

\[
\begin{align*}
r &= \text{reg } S/J_G = \text{reg } S/((x_u, y_u) + J_{G''}) = 6 \\
&= \text{reg } S/J_{G''} = \text{reg } S/((x_u, y_u) + J_{H}) = 3.
\end{align*}
\]

In the matter of projective dimension, it is equal to \( p = n - 1 \) for all the quotient rings involved in 1.1, except for \( S/((x_u, y_u) + J_H) \), for which it is \( n \). Considering the long exact sequence 2 with \( i = p \) and \( j = r \), it holds

\[
\beta_p, p+r(S/J_G) = \beta_{p, p+r}(S/((x_u, y_u) + J_{G''}))
\]

and by Lemma 1.3 (ii) the second part of thesis follows.
The case \( F = F_{m_2}^k \) follows by similar arguments. Indeed, suppose \( |W_i| \geq 2 \)
and \( v \in W_i \). The graphs involved in 1.1 are: \( G' = F_{m_2}^k \), \( G'' = F_{m_1-1} \sqcup F_{m_2-1}^k \),
and \( H = F_{m-1}^k \), where \( m = m_1 + m_2 + |W_i| - 2 \), all the fan graphs are \( k \)-pure, \( W' = W_1' \sqcup W_2' \sqcup \cdots \sqcup W_k' \), with \( |W_i'| = m_1 - 1 \), whereas \( W''' = W \setminus \{v\} \).
Fixing \( r = \text{reg } S/J_G = \text{reg } S/((x_u, y_u) + J_{G''}) = k + 4 \), since reg S/J_{G''}
Let \( G = F_m \circ \cdots \circ F_{m-1} \circ F \), where \( F \) denotes either \( F_m \) or a \( k \)-pure fan graph \( F_{m,k}^{W,k} \) with \( W = W_1 \sqcup \cdots \sqcup W_k \). Let \( \{v\} = V(F_{m_1} \circ \cdots \circ F_{m_t}) \cap V(F) \) and, if \( F = F_{m,k}^{W,k} \), assume \( |W_1| = 2 \) and \( v \in W_1 \). Let \( G'' \) and \( H \) be as in Set-up 1.3 with \( u = v \). Then the unique extremal Betti number of \( S/J_G \) is given by

\[
\hat{\beta}(S/J_G) = \hat{\beta}(S/J_{G''}) + \begin{cases} 
\hat{\beta}(S/J_H) & \text{if } m_t = 3 \\
0 & \text{if } m_t > 3 
\end{cases}
\]

In particular, if \( F = F_m \), it is given by

\[
\hat{\beta}(S/J_G) = \hat{\beta}(S/J_{F_m}) \cdot \hat{\beta}(S/J_{F_m}) + \begin{cases} 
\hat{\beta}(S/J_H) & \text{if } m_t = 3 \\
0 & \text{if } m_t > 3 
\end{cases}
\]

The result for \( G'' \) follows directly from the result for \( G \).

**Theorem 4.6** Let \( t \geq 2 \), \( m \geq 3 \), and \( m_i \geq 3 \) for all \( i = 1, \ldots, t \). Let \( G = F_m \circ \cdots \circ F_m \circ F \), where \( F \) denotes either \( F_m \) or a \( k \)-pure fan graph \( F_{m,k}^{W,k} \) with \( W = W_1 \sqcup \cdots \sqcup W_k \). Let \( \{v\} = V(F_{m_1} \circ \cdots \circ F_{m_t}) \cap V(F) \) and, if \( F = F_{m,k}^{W,k} \), assume \( |W_1| = 2 \) and \( v \in W_1 \). Let \( G'' \) and \( H \) be as in Set-up 1.3 with \( u = v \). Then the unique extremal Betti number of \( S/J_G \) is given by

\[
\hat{\beta}(S/J_G) = \hat{\beta}(S/J_{G''}) + \begin{cases} 
\hat{\beta}(S/J_H) & \text{if } m_t = 3 \\
0 & \text{if } m_t > 3 
\end{cases}
\]

In particular, if \( F = F_m \), it is given by

\[
\hat{\beta}(S/J_G) = \hat{\beta}(S/J_{F_m}) \cdot \hat{\beta}(S/J_{F_m}) + \begin{cases} 
\hat{\beta}(S/J_H) & \text{if } m_t = 3 \\
0 & \text{if } m_t > 3 
\end{cases}
\]

The result for \( G'' \) follows directly from the result for \( G \).

**Proof.** If \( F = F_m \), we have \( G' = F_m \circ \cdots \circ F_m \circ F_{m+1} \circ \cdots \circ F_{m+1} \), \( G'' = F_m \circ \cdots \circ F_{m+1} \circ \cdots \circ F_{m+1} \), \( H = F_m \circ \cdots \circ F_m \circ \cdots \circ F_{m+1} \circ \cdots \circ F_{m+1} \), and \( W' = W_1 \sqcup W_2 \), \( W'' = W_1 \sqcup W_2 \), \( W''' = W_1 \sqcup W_2 \), \( |W_1'| = m_t = 3 \) and \( |W_2'| = m_t = 3 \). As regard the regularity of these quotient rings, we have

\[
\tau = \text{reg } S/J_G = \text{reg } S/((x_u, y_u) + J_{G''}) = \text{reg } S/J_{F_{m+1}} + \cdots + \text{reg } S/J_{F_{m+1}} + \text{reg } S/J_{F_{m+1}}
\]

and both \( \text{reg } S/J_G \) and \( \text{reg } S/((x_u, y_u) + J_H) \) are equal to

\[
\text{reg } S/J_{F_{m+1}} + \cdots + \text{reg } S/J_{F_{m+1}} + \text{reg } S/J_{F_{m+1}}.
\]

Since \( \text{reg } S/J_{F_{m+1}} = 3 \) and \( \text{reg } S/J_{F_{m+1}} = 3 \), whereas if \( m_t = 3 \), \( \text{reg } S/J_{F_{m+1}} = 1 \), otherwise \( \text{reg } S/J_{F_{m+1}} = 3 \), it follows that

\[
\text{reg } S/J_{G''} = \text{reg } S/((x_u, y_u) + J_H) = \begin{cases} 
\tau - 1 & \text{if } m_t = 3 \\
\tau - 3 & \text{if } m_t > 3 
\end{cases}
\]
For the projective dimensions, we have

\[ p = \text{proj dim } S/J_G = \text{proj dim } S/(x_u, y_u + J_G) \]
\[ = \text{proj dim } S/J_{G^r} = \text{proj dim } S/(x_u, y_u + J_H) - 1 = n - 1. \]

Passing through the long exact sequence (2) of Tor modules, we obtain, if \( m_t = 3 \)
\[ \beta_{p,p+r}(S/J_G) = \beta_{p,p+r}(S/(x_u, y_u + J_{G^r})) + \beta_{p+1,(p+1)+(r-1)}(S/(x_u, y_u + J_H)) \]
and, if \( m_t > 3 \)
\[ \beta_{p,p+r}(S/J_G) = \beta_{p,p+r}(S/(x_u, y_u + J_{G^r})). \]

The case \( F = F_{m,W,k} \) follows by similar arguments. Indeed, the involved graphs are: \( G' = F_{m_1} \circ \cdots \circ F_{m_{t-1}} \circ F_{m_{t-1}}^{W''}, \) \( G'' = F_{m_1} \circ \cdots \circ F_{m_{t-1}} \circ F_{m_{t-1}}^{W''}, \) and \( H = F_{m_1} \circ \cdots \circ F_{m_{t-1}} \circ F_{m_{t-1}}^{W''}, \) where all the fan graphs are \( k \)-pure, \( W'' = W \setminus \{v\}, \) \( m = m_t + |W_1| - 1, \) \( W'' = W_1' \sqcup W_2 \sqcup \cdots \sqcup W_k, \) and \( |W''_i| = m_t - 1. \) Fixing \( r = \text{reg } S/J_G, \) we get \( \text{reg } S/(x_u, y_u + J_{G^r}) = r, \) whereas
\[ \text{reg } S/J_{G^r} = \text{reg } S/(x_u, y_u + J_H) = \begin{cases} r - 1 & \text{if } m_t = 3 \\ r - 3 & \text{if } m_t > 3 \end{cases} \]

The projective dimension of all the quotient rings involved is \( p = n - 1, \) except for \( S/(x_u, y_u + J_H), \) for which it is \( n. \) Passing through the long exact sequence (2) of Tor modules, it follows the thesis.

**Corollary 4.7** Let \( t \geq 2, m, m_1 \geq 3, \) and \( m_i \geq 4 \) for all \( i = 2, \ldots, t. \) Let \( G = F_{m_1} \circ \cdots \circ F_{m_t} \circ F, \) where \( F \) denotes either \( F_{m_t} \) or a \( k \)-pure fan graph \( F_{m,W,k} \) with \( W = W_1 \sqcup \cdots \sqcup W_k. \) Let \( \{v\} = V(F_{m_1} \circ \cdots \circ F_{m_t}) \cap V(F) \) and, when \( F = F_{m,W,k} \), assume \( W_1 \geq 2 \) and \( v \in W_1. \) Then the unique extremal Betti number of \( S/J_G \) is given by

\[ \hat{\beta}(S/J_G) = \begin{cases} \hat{\beta}(S/J_{F_{m_{t-1}}}) \prod_{i=2}^{t} \hat{\beta}(S/J_{F_{m_{t-1}}}) \hat{\beta}(S/J_{F_{m_{t-1}}}) & \text{if } F = F_{m} \\ \hat{\beta}(S/J_{F_{m_{t-1}}}) \prod_{i=2}^{t} \hat{\beta}(S/J_{F_{m_{t-1}}}) \hat{\beta}(S/J_{F_{m_{t-1}}}) & \text{if } F = F_{m,W,k} \end{cases} \]

where \( W'' = W \setminus \{v\}. \)

**Proof.** By Theorem 4.6 and by hypothesis on the \( m_i \)’s, we get

\[ \hat{\beta}(S/J_G) = \hat{\beta}(S/J_{F_{m_1}} \circ \cdots \circ F_{m_{t-1}}) \hat{\beta}(S/J_{F_{m_{t-1}}}). \]

Repeating the same argument for computing the extremal Betti number of \( S/J_{F_{m_1}} \circ \cdots \circ F_{m_{t-1}} \), and by Lemma 4.5, we have done.

**Remark 4.8.** Contrary to what we believe for bipartite graphs \( F_{m} \) and \( k \)-pure fan graphs \( F_{m,W,k} \) (see Question 3.6), in general for a Cohen-Macaulay bipartite graph \( G = F_{m_1} \circ \cdots \circ F_{m_t}, \) with \( t \geq 2, \) the unique extremal Betti number of \( S/J_G \) does not coincide with the Cohen-Macaulay type of \( S/J_G, \) for example for \( G = F_4 \circ F_3, \) we have \( 5 = \hat{\beta}(S/J_G) \neq \text{CM-type}(S/J_G) = 29. \)
References

[1] H. Baskoroputro, *On the binomial edge ideal of proper interval graphs*, arXiv:1611.10117, 2016.

[2] D. Bolognini, A. Macchia, F. Strazzanti, *Binomial edge ideals of bipartite graphs*, European J. Combin., Vol. 70, 2018, pp. 1-25.

[3] A. Conca, M. Varbaro, *Square-free Groebner degenerations*, arXiv:1805.11923, 2018.

[4] H. De Alba, D. T. Hoang, *On the extremal Betti numbers of the binomial edge ideal of closed graphs*, Math. Nachr., Vol. 291, 2018, pp. 28–40.

[5] A. Dokuyucu, *Extremal Betti numbers of some classes of binomial edge ideals*, Math. Rep. (Bucur.) 17, Vol. 4, 2015, pp. 359–367.

[6] V. Ene, J. Herzog, T. Hibi, *Cohen-Macaulay binomial edge ideals*, Nagoya Math. J., Vol. 204, 2011, pp. 57–68.

[7] J. Herzog, T. Hibi, F. Hreinsdottir, T. Kahle, J. Rauh, *Binomial edge ideals and conditional independence statements*, Adv. in Appl. Math., Vol. 45, 2010, pp. 317–333.

[8] J. Herzog, G. Rinaldo, *On the extremal Betti numbers of binomial edge ideals of block graphs*, Electron. J. Combin., Vol. 25(1), 2018, pp. 1–10.

[9] D. T. Hoang, *On the Betti numbers of edge ideal of skew Ferrers graphs*, arXiv:1806.02327, 2018.

[10] A. V. Jayanthan, A. Kumar, *Regularity of binomial edge ideals of Cohen-Macaulay bipartite graphs*, arXiv:1806.02109, 2018.

[11] D. Kiani, S. Saeedi Madani, *The Castelnuovo-Mumford regularity of binomial edge ideals*, Combin. Theory Ser. A, Vol. 139, 2016, pp. 80–86.

[12] S. Saeedi Madani, D. Kiani, *Binomial edge ideals of graphs*, Electron. J. Combin., Vol. 19, 2012, Paper #P44.

[13] J. Herzog, S. Saeedi Madani, D. Kiani, *The linear strand of determinantal facet ideals*, arXiv: 1508.07592, 2015.

[14] S. Saeedi Madani, D. Kiani, *Binomial edge ideals of regularity 3*, arXiv:1706.09002, 2017.

[15] C. Mascia, G. Rinaldo, *Krull dimension and regularity of binomial edge ideals of block graphs*, arXiv:1803.01239, 2018.

[16] K. Matsuda, S. Murai, *Regularity bounds for binomial edge ideals*, Commut. Algebra, Vol. 5, 2013, pp. 141–149.

[17] M. Ohtani, *Graphs and ideals generated by some 2-minors*, Comm. Algebra, Vol. 39, 2011, pp. 905–917.

[18] A. Rauf, G. Rinaldo, *Construction of Cohen-Macaulay binomial edge ideals*, Comm. Algebra, Vol. 42.1, 2014, pp. 238–252.
[19] G. Rinaldo, *Cohen-Macaulay binomial edge ideals of cactus graphs*, To appear in J. Algebra Appl., 2018, pp. 1–17.