Exceptional linear systems on curves on Enriques surfaces

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Abstract

The main purpose in this paper is to study the gonality, the Clifford index and the Clifford dimension on linearly equivalent smooth curves on Enriques surfaces. The method is similar to techniques of M.Green & R.Lazarsfeld and G.Pareschi.

0.1 Introduction

In recent years several authors have been led to study the following question: to what extent do linearly equivalent smooth curves on a surface $S$ carry "equally exceptional" linear series? Green and Lazarsfeld investigated the case when $S$ is a K3 surface and proved in [GL] that smooth linearly equivalent curves have the same Clifford index. Clifford index is a natural numerical invariant measuring "exceptional" linear systems on curves. On the other hand Donagi’s example shows that smooth linearly equivalent curves on K3 surfaces can have different gonality. The question in the case of Del Pezzo surfaces of degree $\geq 2$ has been studied by Pareschi. It follows from the result in [P] that gonality and Clifford index is the same for all smooth linearly equivalent curves, with one exception for gonality, involving curves of genus 3 (see [P] for details).

The purpose of this note is to study the same question in the case of Enriques surfaces. It turns out that the gonality and the Clifford index is not constant for smooth linearly equivalent curves (see section 5.11) and in general we obtain the following estimate for the jump of the gonality and the Clifford index in a linear system:
Theorem 1  Let $C$ be a smooth irreducible curve of genus $g$ on an Enriques surface $S$ and let $gon(|C|) = \min \{gon(C') \mid C'$ is smooth curve in $|C|$ $\}$, $\text{cliff}(|C|) = \min \{\text{cliff}(C') \mid C'$ is smooth curve in $|C'|$ $\}$ be the minimal gonality , Clifford index for $|C|$. Then for all smooth curves $C'$ in $|C|$ gonality $C'$ is $\leq gon(|C|) + 2$. And if $\text{cliff}(C') = gon(C') - 2$ then Clifford index of $C'$ is $\leq \text{cliff}(|C|) + 2$. Moreover, if $gon(|C|) \leq \frac{g-1}{2}$ or $g \geq 9$, then there is a line bundle $L$ on $S$ such that $\text{cliff}(C') \leq \text{cliff}(L|C') + 2$. (see proposition 2.6.1 for more precise result).

Remark :  
- It seems that Clifford index always can be computed by gonality, i.e. $\text{cliff}(C) = gon(C) - 2$, for all curves contained in Enriques surfaces. (see Theorem 2 and conjecture bellow).
- In the section 5.11 we give examples which show that the gonality for the smooth linearly equivalent curves is not constant.

Another very useful invariant describing "exceptional" curves is the Clifford dimension introduced by Einshub, Lange, Martens and Schreyer in [ELMS]. If the curve has the Clifford dimension greater than 1 then it is "special" (see [ELMS] for description of such curves). For the curves contained in $S$ we have the following result:

Theorem 2  Let $C$ be a smooth irreducible curve on an Enriques surface $S$ then its Clifford dimension is 1 or greater than 9. (In fact we prove a somewhat more precise result - see Proposition 4.8.1).

We conjecture that every smooth irreducible curve $C$ on Enriques surface $S$ has Clifford dimension 1.

The paper is organized as follows: In the Section 1. we introduce notation and collect some preliminary results. In Section 2. we investigate the bundle $E(C, A)$ - introduced by Lazarsfeld - and obtain the jumping estimate for gonality and prove Theorem 1. Section 3. is devoted to the proof that a smooth plane curve of the degree $\geq 5$ cannot be contained in an Enriques surface. In section 4. we prove Theorem 2. In last section we explain how to obtain, using results of section 2., explicit examples of the smooth linearly equivalent curves of different gonality.

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Notation:
- We will work over the complex field.
- We denote by the same capital letter both a divisor and the line bundle associate to the divisor and we hope that the meaning will be clear from context.

Preliminaries

1.2 \( E(C, A) \)

Let \( S \) be a regular surface, \( C \) a curve contained in \( S \), and \( A \) a base point free line bundle on \( C \). One can associate to the triple \((S, C, A)\) a certain vector bundle on \( S \) in a canonical way. We refer the reader to [P], [GL], [T] for details.

Let us denote by \( F(C, A) \) the vector bundle defined by the sequence:

\[
0 \to F(C, A) \to H^0(A) \otimes O_S \xrightarrow{ev} A \to 0,
\]

where \( A \) is viewed as a sheaf on \( S \) and \( ev \) is the evaluation map.

Dualizing the above sequence we get (see [P])

\[
0 \to H^0(A)^* \otimes O_S \to E(C, A) \to O_C(C) \otimes A^* \to 0 \tag{1}
\]

where \( E(C, A) = F(C, A)^* = \mathcal{H}om(F(C, A), O_S) \), \( A^* = \mathcal{H}om(A, O_C) \). To simplify the notation we will omit \((C, A)\), if it is clear from context to which pair \((C, A)\) the bundle \( E(C, A) \) is associated. We have

\[
c_1(E) = C; \quad c_2(E) = \text{deg}(A); \tag{2}
\]

\[
\text{rk}(E) = h^0(A); \quad h^i(F(C, A)) = h^{2-i}(E \otimes K_S) = 0 \quad \text{for} \quad i = 1, 2; \tag{3}
\]

where \( K_S \) is the canonical bundle of \( S \). If \( h^0(O_C(C) \otimes A^*) > 0 \), then \( E(C, A) \) is generated by its sections away from a finite set coinciding with the fixed divisor of \( O_C(C) \otimes A^* \).

Let \( s^\perp \) be a subspace of \( H^0(A)^* \) orthogonal to \( s \in H^0(A) \) with respect to the natural pairing

\[
H^1(K_C - A) \otimes H^0(A) = H^0(A)^* \otimes H^0(A) \to H^1(K_C) = \mathcal{C}.
\]

Then we get another description of the bundle \( E(C, A) \):

\[
0 \to s^\perp \otimes K_S \to E(C, A) \otimes K_S \to J_\xi(C + K_S) \to 0, \tag{4}
\]

where \( \xi = (s)_0 \) is the zero scheme of the section \( s \in H^0(A) \) (see [T]).
1.3 Enriques surfaces

A smooth irreducible surface $S$, such that $h^1(O_S) = h^2(O_S) = 0$ and $2K_S \sim O_S$, is called an Enriques surface. If $C$ is a smooth curve in $S$, then by adjunction formula one has:

$$g(C) = \frac{C^2}{2} + 1.$$  \hspace{1cm} (5)

Recall that a divisor $D$ on a smooth surface $X$ is said to be nef if $DC \geq 0$ for every curve $C$ on $X$. The following properties will be used throughout, sometimes without explicit mention:

(A) ([C,D] Proposition 3.1.6) If $D$ is a nef divisor, then $|D|$ has no fixed components, unless $D \sim 2E + R$, where $|2E|$ is a genus 1 pencil and $R$ a smooth rational curve with $RE = 1$.

(B) ([C,D] Corollary 3.1.3) If $D$ is a nef divisor and $D^2 > 0$, then $H^1(O_S(-D)) = 0$ and $\chi(O(D)) - 1 = \text{dim}|D| = \frac{D^2}{2}$.

(C) ([C,D] Proposition 3.1.4) If $|D|$ has no fixed components, then one of the following holds:

(i) $D^2 > 0$ and there exist an irreducible curve $C$ in $|D|$.

(ii) $D^2 = 0$ and there exist a genus 1 pencil $|P|$ such that $D \sim kP$ for some $k \geq 1$.

(D) ([C,D] Chapter 4, appendix, corollary 1. and corollary 2.) If $D^2 \geq 6$ and $D$ is nef then $D$ is ample, $2D$ is generated by its global sections, $3D$ is very ample.

Let $K^+ = \{D^2 > 0 | D \text{ is an effective divisor}\}$. $K^+$ is called the positive cone. $\overline{K^+} = \{D^2 \geq 0 | D \text{ is an effective divisor}\}$ is the closure of the positive cone.

The Enriques surface $S$ is called unnodal if there are no smooth (-2) curves contained in $S$.

1.4 Steiner construction

Let $L, M$ be the effective divisors on $S$. Parametrize the two pencils $|L(\lambda)| \subset |L|$ and $|M(\lambda)| \subset |M|$ by $\lambda \in P^1$, choosing the parametrization so that $L(\lambda)$ and $M(\lambda)$ have no common components for every $\lambda \in P^1$.

Then the curve:

$$C = \cup_{\lambda}(L(\lambda) \cap M(\lambda))$$  \hspace{1cm} (6)
is clearly irreducible if the general curves in $|L(\lambda)|$ and $|M(\lambda)|$ are irreducible. Denote by $\xi$ (resp. $\eta$) base locus of $|L(\lambda)|$ (resp. $|M(\lambda)|$). Then $|L(\lambda)| = |J_\xi(L)|$, $|M(\lambda)| = |J_\xi(M)|$. If $\xi \cup \eta = P_1 + \ldots + P_{L^2 + M^2}$ consist of different points, then $C$ contain $\xi \cup \eta$. We have $C \sim L + M$. Indeed, using our data we can obtain the sequence:

$$0 \rightarrow \mathbb{C}^2 \otimes O_S \rightarrow M \oplus L \rightarrow \mathcal{L} \rightarrow 0 \quad (7)$$

where $s((\mu,\nu) \otimes f) = s^L_{\mu/\nu}(f) \oplus s^M_{\mu/\nu}(f)$ and the map $s^L_{\mu/\nu}(\bullet)$ (resp. $s^M_{\mu/\nu}(\bullet)$) is multiplication by $L(\mu/\nu) \in |L|$ (resp. $M(\mu/\nu) \in |M|$), hence is defined by the sequences:

$$0 \rightarrow O \rightarrow L \rightarrow O_{L(\mu/\nu)} \rightarrow 0 \quad (s^L_{\mu/\nu})$$

$$(\text{resp. } 0 \rightarrow O \rightarrow M \rightarrow O_{M(\mu/\nu)} \rightarrow 0 \quad (s^M_{\mu/\nu})).$$

We see that the sheaf $\mathcal{L}$ has support $C$ and $c_1(M \oplus L) \sim M + L \sim \text{supp}\mathcal{L} = C$.

In the situation described above we will say that $C$ is obtained by the Steiner construction using $|L(\lambda)|$ and $|M(\lambda)|$.

Note, that if a curve $C' \in |C|$ contains the base locus of $|L(\lambda)|$, then $C'$ can be obtained by the Steiner construction using $|L(\lambda)|$ and any another pencil $|M'(\lambda)| \subset |M|$ such that $L(\lambda)$ and $M'(\lambda)$ have no fixed components for every $\lambda \in P^1$.

### 1.5 Gonality, Clifford index

Let $C$ be a curve of genus $g$ and and $A$ be a line bundle on $C$. The Clifford index of $A$ is the integer

$$\text{cliff}(A) = \deg(A) - 2r(A),$$

where $r(A) = h^0(A) - 1$ is the projective dimension of $|A|$. The Clifford index of $C$ is

$$\text{cliff}(C) = \min\{\text{cliff}(A) \mid r(A) \geq 1, \deg(A) \leq g - 1\}.$$ 

Note that this last definition makes sense only for curves of genus $g \geq 4$.

We say that a line bundle $A$ contributes to the Clifford index if it satisfies the inequalities in the above definition and that $A$ computes the Clifford index.
if $A$ contributes to the Clifford index and $\text{cliff}(C) = \text{cliff}(A)$. Finally, we define the Clifford dimension of $C$ as

$$\text{cliffdim}(C) = \min\{r(A), A \text{ computes the Clifford index of } C\}.$$  

The gonality of $C$, denoted by $\text{gon}(C)$, is the minimal degree of a pencil on $C$. Such a definition is nontrivial only for curves of genus $g \geq 3$. Also we say $A$ computes to the gonality if $r(A) = 1$ and $\text{gon}(C) = \deg(A)$. By Brill-Noether theory one has that

$$\text{cliff}(C) \leq \text{gon}(C) - 2 \leq \left\lfloor \frac{g - 1}{2} \right\rfloor$$

and for a general curve both inequalities are equalities.

### 2.6 Gonality of curves on Enriques surfaces

We will denote (see [C,D] p. 178 ) by

$$\Phi : K^+ \to \mathbb{Z}_{\geq 0}$$

the function defined by

$$\Phi(C) = \inf\{CE, |2E| \text{ is a genus 1 pencil}\}.$$ 

#### Lemma 2.6.1

Let $C$ be a smooth curve of genus $g$ on an Enriques surface $S$. Then:

(i) $\Phi(C) \leq \left\lfloor \sqrt{2g - 2} \right\rfloor$, where $\lfloor l \rfloor$ means the integer part of $l$.

(ii) if $2\Phi(C) \leq g - 1$ and $\Phi(C) = CE$ ($|2E| \text{ is a genus 1 pencil}$), then $\text{cliff}(C) \leq \text{cliff}(2E|_C) \leq 2\Phi(C) - 2$ and $2E|_C$ contributes to the Clifford index. (by (i) condition $2\Phi(C) \leq g - 1$ is always satisfied if $g \geq 9$).

**Proof:** (i) is a statement about the Enriques lattice $H^2(S, \mathbb{Z})$ and has been proved in [C,D] (Corollary 2.7.1).

(ii) Assume $\Phi(C) = CE$, where $|2E|$ is a genus 1 pencil. Then we have the sequence:

$$0 \to O(2E - C) \to O(2E) \to O(2E)|_C \to 0.$$ 

The divisor $2E - C$ cannot be an effective, because $(2E - C)|_C < 0$. Therefore $h^0(2E|_C) \geq 0$ and $\text{cliff}(C) \leq \text{cliff}(2E|_C) \leq 2\Phi(C) - 2$, since by our condition $2E|_C$ contributes to the Clifford index. □
Proposition 2.6.1 Let $C$ be a smooth curve on an Enriques surface $S$, and let $A$ be a pencil on $C$ such that $\deg(A) = \operatorname{gon}(C) \leq \frac{g-1}{2}$. Then

(i) There exist an exact sequence:

\[ 0 \to M \to E(C, A) \to L \to 0, \tag{8} \]

such that $M$, $M-L$ are line bundles from the positive cone $K^+$. Moreover the linear systems $|L|$, $|M|$ have no fixed components and $M^2 > LM = \deg(A) > L^2 \geq 0$ (if $\deg(A) = \frac{g-1}{2}$ then $M - L \in K^+$ and $M^2 \geq LM = \deg(A) \geq L^2 \geq 0$).

(ii) The curve $C$ can be obtained by the Steiner construction for some $|M(\lambda)| = P^1 \subset |M|$ and $|L(\lambda)| = P^1 \subset |L|$.

(iii) If $S$ is an unnodal Enriques surface, then

(a) The exact sequence (8) splits.

(b) If $\operatorname{gon}(|C|) = \operatorname{gon}(C)$ then $L \sim 2E_1$ or $E_1 + E_2$, where $|2E_1|$, $|2E_2|$ are two genus 1 pencils on $S$.

Proof: 1 Case. $d < \frac{g-1}{2}$.

(i). Assume that $d < \frac{g-1}{2}$, then the vector bundle $E(C, A)$ is not stable in the Bogomolov sense. Indeed $4c_2(A) = 4\deg(A) \leq c_1(E(C, A)) = C^2 = 2g-2$.

By Bogomolov’s theorem [B] we have an exact sequence:

\[ 0 \to M \to E(C, A) \to J_\xi(L) \to 0, \tag{9} \]

that such $(M-L)^2 > 0$ and $M-L$ is an effective divisor. $E(C, A)$ is a vector bundle generated by its global sections away from a finit set. Therefore $J_\xi(L)$ is also generated by its global sections away from a finit set and we obtain that $L^2 \geq 0$ and linear system $|L|$ has not fixed components.

Claim: $l(\xi) = \ell(\xi) = 0$.

If $L^2 = 0$, then by proposition 3.14 [CD] $L \sim kP$ for some $k \geq 1$. By assumption, $\deg(A) = kPM + l(\xi) = \operatorname{gon}(C)$, but $\deg(P|_C) = PM \geq \deg(A)$, hence we have $l(\xi) = 0$, $k = 1$.

Assume $L^2 > 0$ and $l(\xi) > 0$, then from exact sequence (8) we obtain:

\[ h^0(E(-L)) = h^0(M-L), \tag{10} \]
\[ h^1(E(-L)) = l(\xi) + h^1(M-L), \tag{11} \]
\[ h^2(E(-L)) = 0. \tag{12} \]
On the other hand by 1.3 (B) \( H^1(-L) = 0 \), therefore using (11) we have:

\[
\begin{align*}
\chi^0(E(-L)) &= \chi^0(C, M | C - A), \\
\chi^1(E(-L)) &= \chi^1(C, M | C - A) - 2\chi^2(-L), \\
\chi^2(E(-L)) &= 0.
\end{align*}
\]

By Riemann-Roch on the curve \( C \)

\[
\begin{align*}
\chi^0(C, M | C - A) &= MC - A - g(C) + 1 + \chi^1(M | C - A) = (\text{by } 14, \text{ and duality}) \\
&= MC - A - \left( \frac{M^2 + L^2}{2} + ML + 1 \right) + 1 + \\
&\quad + l(\xi) + 2\chi^0(L \otimes K_S) + \chi^1(M - L) = (\text{by } R - R \text{ and } (9)) \\
&= M^2 - \left( \frac{M^2 + L^2}{2} + ML + 1 \right) + L^2 + 2 + \chi^1(M - L) = \\
&= \frac{M^2 + L^2}{2} - ML + 2 + \chi^1(M - L). \quad (16)
\end{align*}
\]

We have \( \chi^2(M - L) = 0 \), hence (10), (13) and (16) gives us

\[
\chi(M - L) = \frac{M^2 + L^2}{2} - ML + 2 \quad (17)
\]

But by Riemann-Roch theorem \( \chi(M - L) = \frac{M^2 + L^2}{2} - ML + 1 \). This contradicts (17), hence \( l(\xi) = 0 \).

Now we see that \( L \) is nontrivial, because \( c_2(E) = \deg(A) = LM \). This implies \( (M - L)L > 0 \) and \( (M - L)M > 0 \), since \( M - L, M \in K^+; L \in \overline{K^+} \).

(ii). By (4) we have:

\[
0 \to O_S \to E(C, A) = M \oplus L \to J_\xi(C) \to 0, \quad (18)
\]

where \( \xi = (s)_0 \) is the zero scheme of the section \( s \in H^0(A) \). Since \( H^1(M) = 0 \) we see that the sequence (8) is exact on a section level i.e.

\[
0 \to H^0(M) \to H^0(E(C, A)) \to H^0(L) \to 0.
\]

Hence we can write \( H^0(A) \ni s = s_1 \oplus s_2 \), where \( s_1 \in H^0(L), s_2 \in H^0(M) \). When \( s \) runs through \( H^0(A) \), \( s_1 \) and \( s_2 \) runs through \( |L(\lambda)| = P^1 \subset |L| \)
and $|M(\lambda)| = P_1 \subset |M|$ respectively. The zero set $\xi$ consists of points where both sections $s_1, s_2$ are zero. This shows that $C$ is obtained by the Steiner construction using $|M(\lambda)|, |L(\lambda)|$. Now we can see, that the linear system $|M|$ has no fixed components, because a fixed part of $|M|$ should be also the fixed part of $|C|$, but the curve $C$ is smooth.

(iii), (a). By 1.3 (B) we see that the extention group $Ext^1(L, M) = H^1(M - L) = 0$, therefore the sequence (8) splits.

(iii), (b). By reducibility lemma 3.2.2 [CD], $L$ is linealy equivalent to a sum of genus 1 curves. Assume $L \sim L_1 + L_2$, where $L_1, L_2$ are effective nontrivial divisors on $S$ such that $L_2$ has no fixed components. Then a curve $C' \in |C|$ which is obtained by the Steiner construction using some $|L_2(\lambda)| \subset |L_2|$ and $|M + L_1(\lambda)| \subset |M + L_1|$ has gonality $(M + L_1)L_2$. But this number is smaller than $ML = (M' + L)(L_1 + L_2) = ML_2 + M'L_1 + LL_1$, where $M' = M - L$. This contradicts our assumtion about minimality of $gon(C)$. Therefore, there are no such splitting $L$ into two bundles $L_1, L_2$ and we get (iii), (b).

2 Case. $d = \frac{g - 1}{2}$.

Assume $d = \frac{g - 1}{2}$, then by the Riemann-Roch theorem:

$$\chi(E(C, A), E(C, A)) = 4 + c_1^2(E) - 4c_2(E) = 4. \quad (19)$$

If $E(C, A)$ is $H$-stable in the sense of Mamford-Takemoto ($H$ is an ample divisor on $S$), then it is simple and $Ext^2(E, E)^* = Hom(E, E \otimes K_0) = \mathbb{C}$ or 0. Indeed, any nontrivial homomorphism between $E$ and $E \otimes K_0$ should be an isomorphism, because both bundles are stable and have the same determinant. This contradicts shows that $E$ is not $H$-stable for any ample divisor $H$. By 1.3 (D) $C$ is a ample and hence $E$ is not $C$-stable, therefore we have the sequence:

$$0 \to M \to E(C, A) \to J_\xi(L) \to 0,$$

such that $(M - L)C = M^2 - L^2 \geq 0$. Since $4(ML + l(\xi)) = 4c_2(E) = c_1(E) = M^2 + L^2 + 2ML$ we have $(M - L)^2 = 4l(\xi) \geq 0$ i.e. $M - L \in K^+$. And now we can argue as in case 1. above. □

**Proof of Theorem 1**: If $gon(|C|) \leq \frac{g - 1}{2}$, then by proposition 2.6.1 we are done. On the other hand by Brill-Noether theory we have:

$$gon(C) \leq \left\lfloor \frac{g - 2}{2} \right\rfloor + 2. □$$
3.7 Curves of the Clifford dimension 2

It is well known that the curve of Clifford dimension 2 is a smooth plane curve of degree \( d \geq 5 \) and a line bundle \( A \) computing the Clifford index is unique. In this case there is a 1-dimensional family of pencils of degree \( d - 1 \) computing gonality, all obtained by projecting from a point of the curve.

**Proposition 3.7.1** An Enriques surface does not contain any smooth plane curve of degree \( d \geq 5 \).

**Proof:** Let \( C \) be a curve of degree \( d \geq 6 \). Recall that for smooth plane curves of degree \( d \) by adjunction formula we have \( g(C) = \frac{d(d-3)}{2} + 1 \).

By lemma 2.6.1 (i)

\[
\Phi(C) \leq \left\lfloor \sqrt{2g(C)} - 2 \right\rfloor = \left\lfloor \sqrt{d(d - 3)} \right\rfloor \leq d - 2.
\]

On the other hand we have

\[
\Phi(C) \leq d - 2 \leq \frac{d(d - 3)}{4} = \frac{g(C) - 1}{2}.
\]

By lemma 2.6.1 (ii) we obtain

\[
diff(C) = d - 4 \leq diff(2E|_C) \leq 2\Phi(C) - 2 \leq d - 4
\]

where \( \Phi(C) = CE \) and \( |2E| \) is a genus 1 pencil. Moreover, \( (2E)|_C \) contributes to the Clifford index. This is a contradiction, since we obtain a pencil which computes the Clifford index.

If \( deg(C) = 5 \), then from the exact sequence (4) we have:

\[
h^0(J_\xi(C + K)) = h^0(K_C - A) = h^1(A) = g - d + r(A) = 3,
\]

where \( \xi = (s)_0 \) is zero set of the section \( s \in H^0(A) \). Therefore we can consider \( \xi \) as a divisor on \( C \) which is linear equivalent to \( A \). A curve from the linear system \( D \in |J_A(C + K)| \) cut out a divisor \( A + A_D \) on our curve \( C \). The divisor \( A + A_D \) is linearly equivalent to the canonical divisor \( K_C \), hence \( deg(A_D) = 5 \). \( D \) runs through \( |J_A(C + K)| \) and cuts out \( A_D \) on the curve \( C \), therefore \( h^0(C, A_D) \geq h^0(C, A) = 3 \). By the Clifford theorem \( deg(A_D) \geq 2(h^0(C, A_D) - 1) \), therefore \( h^0(C, A_D) = 3 \). The line bundle \( A \) is
unique linear system of degree 5 and projective dimension 2, hence we have \( A \sim A_D \). Denote by \( C' \) a smooth curve in the linear system \(|J_A(C + K)|\) which cuts out the divisor \( A + A' \) on our curve \( C \). Then we can consider the divisor \( A' \) as a divisor on \( C' \). A curve \( D' \in |J_A(C + K)| \) cuts out a divisor \( A + A_{D'} \) on the curve \( C' \) \((A_{D'} \in |A'|)\). In the same way as above we get \( deg(A_{D'}) = 5, h^0(C', A_{D'}) = 3 \).

### 4.8 Curves of the Clifford dimension \( r \geq 3 \)

The curves of Clifford dimension \( r \geq 3 \) are extremely rare. We refer the reader to the preprint [ELMS] for the details. The main result in [ELMS] is:

1. (i) \( C \) has genus \( g = 4r - 2 \) and Clifford index \( 2r - 3 = \left\lfloor \frac{g-1}{2} \right\rfloor - 1 \).

(ii) \( C \) has a unique line bundle \( L \) computing the Clifford index, \( L^2 \) is the canonical bundle on \( C \).

2. If the curve does not satisfy 1., then \( r \geq 10 \) and the degree \( d \) of the line bundle computing the Clifford index is \( \geq 6r - 6 \) and its genus is \( \geq 8r - 7 \). Eisinbud, Lange, Martens and Schreyer conjecture that such a curve does not exist (see [ELMS]).

**Proposition 4.8.1** Let \( C \) be a smooth curve of Clifford dimension \( r \geq 3 \) and genus \( g \) contained in an Enriques surface \( S \). Then \( C \) is the curve described by the case 2 above. Moreover \( g \geq 2(r-1)^2+1 \geq 163 \) and \( 2 \left\lfloor \sqrt{2g-2} \right\rfloor - 2 + 2r \geq d \), where \( d \) is the degree of a line bundle computing Clifford index.

**Proof:** Consider the curve which satisfies 1. above. By lemma 2.6.1 (ii) we have

\[
cliff(C) \leq 2\Phi(C) - 2 \leq \frac{g-1}{2} - 2.
\]

This contradicts the condition \( cliff(C) = \left\lfloor \frac{g-1}{2} \right\rfloor - 1 \).

For a curve which satisfies 2 lemma 2.6.1 (i) and (ii) implies that

\[
d - 2r \leq 2\Phi(C) - 2 \leq 2\left\lfloor \sqrt{2g-2} \right\rfloor - 2. \quad (*)
\]

But \( d \geq 6r - 6 \), hence

\[
4r - 6 \leq 2\left\lfloor \sqrt{2g-2} \right\rfloor - 2
\]
and we get
\[ g \geq 2(r - 1)^2 + 1 \geq 163. \]
Also from (*) we obtain
\[ d \leq 2r - 2 + 2 \left\lfloor \sqrt{2g - 2} \right\rfloor. \]

5.9 Examples

In this section we will apply proposition 2.6.1 to obtain examples of smooth linearly equivalent curves of different gonality. In explicit examples we try to explain the reasons why the gonality is not constant.

Assume \( S \) is an unnodal Enriques surface. Consider \( L = E_1 + E_2, M = 2L, \) where \( |2E_1|, |2E_2| \) are genus 1 pencils such that \( E_1E_2 = 1 \), then \( C \sim M + L \) is very ample by corollary 2 (appendix, after chapter 4) \([CD]\).

If \( C \) is a smooth curve in the linear system \( |M + L| \) which is obtained by Steiner construction using \( P_1 = |L| \) and \( |M(\lambda)| \subset |M| \), then by proposition 2.6.1 we see that \( \text{gon}(C) \leq ML = 4 \) and actually \( \text{gon}(C) = 4 \). Indeed, if \( \text{gon}(C) \leq 3 \) then by proposition 2.6.1 we obtain \( M' \) and \( L' \) such that \( C \sim L' + M', \text{gon}(C) = M'L' > L'^2 \geq 0 \). Therefore \( L'^2 \leq 2 \) and by (iii)(b) we see that \( L \sim E' + E'' \) or \( L = 2E' \) for some genus 1 pencils \( |2E'|, |2E''| \) such that \( E'E'' = 1 \). But \( C \sim 3(E_1 + E_2) \) and \( \text{gon}(C) = M'L' = CL' - L'^2 \geq 4 \).

If \( C \) is obtained by Steiner construction then \( C \) contains two base points of pencil \( |L| \). Since \( C \) is very ample there is a smooth curve \( C' \in |C| \) which does not contain base points of any pencil \( |L'| = |E' + E''| \), because we have at most countable number of such pencils. Now we claim that \( \text{gon}(C') \geq 5 \). Indeed, by proposition 2.6.1 (i), (iii) we have \( \text{gon}(C') = M'L' \geq 4 \) and equality occurs, as we have seen above, only if \( L'^2 = 2 \) and \( C' \) is obtained by Steiner construction using \( |L'|, |M'(\lambda)| \) but this is not the case, because \( C' \) does not contain base points of linear system \( |L'| \).

1. 1 A general curve in the linear system \(|3(E_1 + E_2)|\) has gonality \( \geq 5 \) and Clifford index \( \geq 3 \). There is a linear subsystem \( V \subset |C| \) of codimension 2 such that every smooth curve in \( V \) has gonality 4 and Clifford index 2. Moreover, \( \text{gon}(|C|) = 4 \).
The subsystem $V \subset |C|$ consists of the curves which contain two base points of the pencil $|L| = |E_1 + E_2|$ and therefore is obtained by Steiner construction using $|L|$, $|M|$.

Similarly one can obtain the following:

2. 1 A general curve in the linear system $|3E_1 + 4E_2|)$ has gonality 6 and Clifford index 4. There is a linear subsystem $V \subset |C|$ of codimension 2 such that every smooth curve in $V$ has gonality 5 and Clifford index 3. Moreover $\text{gon}(|C|) = 5$.

In this case $M = 3E_1 + 2E_2$, $L = E_1 + E_2$. And the curve $C \in |M + L|$ which is obtained by Steiner construction using $|L|$, $|M(\lambda)| \subset |M|$ has gonality 5 and Clifford index 3. But $C$ is very ample and therefore a general curve of the linear system $|C|$ does not contain base points of any pencil $|L'|$. The proof is similar to the proof in the example above and we omit the details.

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