Elementary matrix-computational proof of Quillen-Suslin theorem for Ore extensions

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Abstract
In this short note we present an elementary matrix-construction proof of Quillen-Suslin theorem for
Ore extensions: If $K$ is a division ring and $A := K[x; \sigma, \delta]$ is an Ore extension, with $\sigma$ bijective,
then every finitely generated projective $A$-module is free. We will show an algorithm that computes
the basis of a given finitely generated projective module. The algorithm has been implemented in a
computational package, and some illustrative examples are included.

Key words and phrases. Projective modules, Ore extensions, non-commutative computational algebra.

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1 Introduction
When a new type of ring is defined, it is an interesting problem to investigate if the finitely generated
projective modules over it are free. This problem becomes classical after the formulation in 1955 of the
famous Serre’s problem about the freeness of finitely generated projective modules over the polynomial
ring $K[x_1, \ldots, x_n]$, $K$ a field (see [1], [2], [3], [6]). The Serre’s problem was solved positively, and
independently, by Quillen in USA, and by Suslin in Leningrad, USSR (St. Petersburg, Russia) in 1976
([7], [8]).

Definition 1.1. Let $S$ be a ring. $S$ is a $PF$ ring if every finitely generated (f.g.) projective $S$-module is
free.

Theorem 1.2 (Quillen-Suslin; [7], [8]). $K[x_1, \ldots, x_n]$ is $PF$.

The goal of this short paper is to present an elementary matrix-construction proof of Quillen-Suslin
theorem for single Ore extensions over division rings, i.e, if $K$ is a division ring and $A := K[x; \sigma, \delta]$ is an
Ore extension, with $\sigma$ a bijective endomorphism of $K$ and $\delta$ a $\sigma$-derivation, then $A$ is $PF$. Our proof is
supported in a matrix characterization of $PF$ rings given in [5].

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**Proposition 1.3** ([5]). Let $S$ be a ring. $S$ is PF if and only if for every $s \geq 1$, given an idempotent
matrix $F \in M_s(S)$, there exists a matrix $U \in GL_s(S)$ such that
\[
UFU^{-1} = \begin{bmatrix}
0 & 0 \\
0 & I_r
\end{bmatrix},
\]
where $r = \dim((F))$, $0 \leq r \leq s$, and $(F)$ represents the left $S$-module generated by the rows of $F$. Moreover, a basis of $M$ is given by the last $r$ rows of $U$.

2 Quillen-Suslin theorem: Elementary matrix proof

In this section we will prove that the Ore extension $K[x; \sigma, \delta]$ is PF. Despite of this fact is well-known (see [4]), our proof is elementary and matrix-constructive, and allow to exhibit an algorithm that computes the basis of a given finitely generated projective modules.

**Theorem 2.1** (Quillen-Suslin). Let $K$ be a division ring and $A := K[x; \sigma, \delta]$, with $\sigma$ bijective. Then $A$ is PF.

**Proof.** Let $s \geq 1$ and let $F = [f_{ij}] \in M_s(A)$ be an idempotent matrix, the proof is by induction on $s$ and we will follow a procedure as in Proposition 64 of [5]. We will use the relations that satisfy the entries of $F$, in particular, the following two relations:

\[
f_{11}^2 + f_{12}f_{21} + f_{13}f_{31} + \cdots + f_{1s}f_{s1} = f_{11},
\]

\[
f_{11}f_{12} + f_{12}f_{22} + f_{13}f_{32} + \cdots + f_{1s}f_{s2} = f_{12}.
\]

**s=1:** In this case $F = [f]$; since $A$ is a domain, its idempotents are trivial, then $f = 1$ or $f = 0$ and hence $U = [1]$.

**s ≥ 2:** Now suppose that the result holds for $s - 1$ and let $F = [f_{ij}] \in M_s(A)$ be an idempotent matrix. We have two possibilities.

(A) All elements in the first row and in the first column of $F$ are zero. Then we apply induction.

(B) Suppose that there exists at least one non zero element in the first row (the reasoning for the first column is similar); we can assume that this element is $f_{11}$ (if $f_{11} = 0$ and $f_{1j} \neq 0$ then we can change $F$ by $TFT^{-1}$ with $T := I_s - E_{j1}$). Then arise two possibilities.

(B1) $\deg(f_{11}) = 0$, so $f_{11} \in K - \{0\}$, i.e., $f_{11}$ is invertible. Then taking

\[
U := \begin{bmatrix}
1 & f_{11}^{-1}f_{12} & f_{11}^{-1}f_{13} & \cdots & f_{11}^{-1}f_{1s} \\
-f_{21}f_{11}^{-1} & 1 & 0 & \cdots & 0 \\
-f_{31}f_{11}^{-1} & 0 & 1 & \cdots & 0 \\
& \vdots & & \ddots & \vdots \\
-f_{s1}f_{11}^{-1} & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

we have that $U \in GL_s(A)$ and its inverse is

\[
U^{-1} = \begin{bmatrix}
f_{11} & -f_{12} & -f_{13} & \cdots & -f_{1s} \\
-f_{21}^{-1}f_{11}^{-1}f_{12} + 1 & -f_{21}^{-1}f_{11}^{-1}f_{13} & \cdots & -f_{21}^{-1}f_{11}^{-1}f_{1s} \\
f_{31} & -f_{31}^{-1}f_{11}^{-1}f_{12} & -f_{31}^{-1}f_{11}^{-1}f_{13} + 1 & \cdots & -f_{31}^{-1}f_{11}^{-1}f_{1s} \\
\vdots & & & \ddots & \vdots \\
f_{s1} & -f_{s1}^{-1}f_{11}^{-1}f_{12} & -f_{s1}^{-1}f_{11}^{-1}f_{13} & \cdots & -f_{s1}^{-1}f_{11}^{-1}f_{1s} + 1
\end{bmatrix}.
\]

Moreover, $UFU^{-1} = \begin{bmatrix} 1 & 0_{s-1,1} \\ 0_{s-1,1} & F_1 \end{bmatrix}$, where $F_1 \in M_{s-1}(A)$ is an idempotent matrix, therefore we can apply induction.
(B2) \( \deg(f_{11}) := n \geq 1 \); since \( A \) is a domain at least one non diagonal entry in the first row and in the first column of \( F \) are non zero. In fact, if \( f_{12} = \cdots = f_{1s} = 0 \), then \( f_{11} = 1 \) or \( f_{11} = 0 \), false; similarly if \( f_{21} = \cdots = f_{s1} = 0 \). Using elementary and permutation matrices, no affecting the entry \( f_{11} \), we can reduce the degrees of \( f_{12}, \ldots, f_{1s} \) until the situation in which \( f_{12} \neq 0 \) and \( f_{13} = \cdots = f_{1s} = 0 \) (a similar reasoning apply for the first column); then we have \( f_{11}^t + f_{12}f_{21} = f_{11} \) and \( f_{21} \neq 0 \); note that \( \deg(f_{11}^t) = 2n \), so \( \deg(f_{21}) := p \leq n \) or \( \deg(f_{12}) := q \leq n \); let \( a_n := lc(f_{11}) \), \( c_p := lc(f_{21}) \) and \( b_q := lc(f_{12}) \).

If \( p \leq n \) then

\[
TFT^{-1} = F' = \begin{bmatrix}
    f'_{11} & f'_{12} & f'_{13} & \cdots & f'_{1s} \\
    f'_{21} & f'_{22} & f'_{23} & \cdots & f'_{2s} \\
    \vdots & \vdots & \vdots & \cdots & \vdots \\
    f_{s1} & f'_{s2} & f_{s3} & \cdots & f_{ss}
\end{bmatrix},
\]

with \( T := I_s - a_n \sigma^{n-p}(c_p^{-1})x^{n-p}E_{12} \); note that \( F' \) is idempotent; moreover \( f'_{11} = 0 \) or \( f'_{11} \neq 0 \); if \( f'_{11} \neq 0 \) then arise two options: \( \deg(f'_{11}) = 0 \), i.e., \( f'_{11} \in K - 0 \) or \( 1 \leq \deg(f'_{11}) \leq n - 1 \) and again \( \deg(f_{21}) \leq \deg(f'_{11}) \). If \( p > n \) but \( q \leq n \) then

\[
LFL^{-1} = F'' = \begin{bmatrix}
    f''_{11} & f''_{12} & f_{13} & \cdots & f_{1s} \\
    f''_{21} & f''_{22} & f'_{23} & \cdots & f'_{2s} \\
    \vdots & \vdots & \vdots & \cdots & \vdots \\
    f'_{s1} & f_{s2} & f_{s3} & \cdots & f_{ss}
\end{bmatrix},
\]

with \( L := I_s + \sigma^{-q}(b_q^{-1}a_n)x^{n-q}E_{21} \); note that \( F'' \) is idempotent; moreover \( f''_{11} = 0 \) or \( f''_{11} \neq 0 \); if \( f''_{11} \neq 0 \) then arise two options: \( \deg(f''_{11}) = 0 \), i.e., \( f''_{11} \in K - 0 \) or \( 1 \leq \deg(f''_{11}) \leq n - 1 \) and again \( \deg(f_{12}) \leq \deg(f''_{11}) \) or \( \deg(f''_{21}) \leq \deg(f''_{11}) \).

We can repeat this reasoning for \( F' \) and \( F'' \) and we obtain an idempotent matrix \( G = [g_{ij}] \) similar to \( F \) with \( g_{11} = 0 \) or \( g_{11} \in K - 0 \); if \( g_{11} \in K - 0 \) we conclude using the case (B1). Then assume that \( g_{11} = 0 \); if all elements in the first row and in the first column of \( G \) are zero, then we can apply induction and we finish. If not, then in a similar way as was remarked above, using elementary and permutation matrices, no affecting the first column, in particular the entry \( g_{11} \), we can reduce the degrees of \( g_{12}, \ldots, g_{1s} \) until the situation in which \( g_{12} \neq 0 \) and \( g_{13} = \cdots = g_{1s} = 0 \) (a similar reasoning apply for the first column); thus, from \( g_{12}g_{22} = g_{12} \) we obtain that \( g_{22} = 1 \) and hence by the permutation matrix \( P_{12} \) we finish using the case (B1).

\[\square\]

3 The algorithm

In this section we present the algorithm for computing the matrix \( U \) in the proof of Quillen-Suslin theorem for Ore extensions (Theorem 2.1); the algorithm also calculates the basis of a given finitely generated projective module (Proposition 1.3). We present two versions of the algorithm, a constructive simplified version, and a more complete computational version over fields. The computational version was implemented using Maple® 2016 (see Remark 4.2 below).
Algorithm for the Quillen-Suslin theorem:
Constructive version

**INPUT:** An Ore extension $A := K[x, \sigma, \delta]$ ($K$ a division ring, $\sigma$ bijective); $F \in M_s(A)$ an idempotent matrix.

**OUTPUT:** Matrices $U$, $U^{-1}$ and a basis $X$ of $\langle F \rangle$, where

$$UFU^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}$$
and $r = \dim(\langle F \rangle)$. \hfill (3.1)

**INITIALIZE:** $F_1 := F$.

**FOR** $k$ from 1 to $n - 1$ **DO**

1. Follow the reduction procedures (B1) and (B2) in the proof of Theorem 2.1 in order to compute matrices $U_k'$, $U_k'^{-1}$ and $F_{k+1}$ such that

$$U_k' F_k U_k'^{-1} = \begin{bmatrix} \alpha_k & 0 \\ 0 & F_{k+1} \end{bmatrix}, \text{ where } \alpha_k \in \{0, 1\}.$$

2. $U_k := \begin{bmatrix} I_{k-1} & 0 \\ 0 & U_k' \end{bmatrix} U_{k-1}$; compute $U_k^{-1}$.

3. By permutation matrices modify $U_{n-1}$.

**RETURN** $U := U_{n-1}$, $U^{-1}$ satisfying (3.1), and a basis $X$ of $\langle F \rangle$.

**Example 3.1.** For $A := K[x, \sigma, \delta]$, with $K := \mathbb{C}$, $\sigma(z) := \overline{z}$ and $\delta := 0$, we consider in $M_4(A)$ the idempotent matrix

$$F = \begin{bmatrix}
1 - ix - x^2 + (1 + i)x^3 & -1 + (2 - i)x^2 + (-1 - i)x^3 & -i - x + (1 + i)x^2 & 1 + ix + (-1 + i)x^2 \\
-ix + (1 + i)x^3 & i + (1 - i)x^2 + (-1 + i)x^3 & i + (1 + i)x^2 & 0 \\
x^3 - x^2 & -i - ix + (1 - i)x^2 - x^3 & 1 - ix & x^2 - x \\
-ix + (1 - i)x^2 - x^3 & -ix + (1 + i)x^3 & 1 - ix & x^2 - x
\end{bmatrix},$$

We apply the constructive version of the Quillen-Suslin algorithm, i.e., following the reductions (B1) and (B2), we compute the matrices $U_k$ and $F_k$, for $1 \leq k \leq 3$:

$$U_1 = \begin{bmatrix}
1 - ix - x^2 + (1 + i)x^3 & -1 + (2 - i)x^2 + (-1 - i)x^3 & -i - x + (1 + i)x^2 & 1 + ix + (-1 + i)x^2 \\
-ix + (1 + i)x^3 & i + (1 - i)x^2 + (-1 + i)x^3 & i + (1 + i)x^2 & 0 \\
x^3 - x^2 & -i - ix + (1 - i)x^2 - x^3 & 1 - ix & x^2 - x \\
-ix + (1 - i)x^2 - x^3 & -ix + (1 + i)x^3 & 1 - ix & x^2 - x
\end{bmatrix},$$

$$U_1^{-1} = \begin{bmatrix}
1 & i + x + (-1 - i)x^2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-i + 1 + ix & x - x^2 + (1 - i)x^3 & i + x & -i \\
0 & 0 & 0 & 1
\end{bmatrix},$$

$$U_1 F U_1^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & i + (1 + i)x^2 & 1 & 0 \\
0 & x^2 - x & 0 & 1
\end{bmatrix}, F_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-i + (1 + i)x^2 & 1 & 0 & 0 \\
x^2 - x & 0 & 1 & 0
\end{bmatrix};$$

$$U_2 = \begin{bmatrix}
1 - ix - x^2 + (1 + i)x^3 & -1 + (2 - i)x^2 + (-1 - i)x^3 & -i - x + (1 + i)x^2 & 1 + ix + (-1 + i)x^2 \\
-ix + (1 + i)x^3 & i + (1 - i)x^2 + (-1 + i)x^3 & i + (1 + i)x^2 & 0 \\
x^3 - x^2 & -i - ix + (1 - i)x^2 - x^3 & 1 - ix & x^2 - x \\
-ix + (1 - i)x^2 - x^3 & -ix + (1 + i)x^3 & 1 - ix & x^2 - x
\end{bmatrix}.$$
Example 3.2. Let $\sigma, \delta, x, \bar{x}, y, \bar{y}, t$ be scalars such that $x, \bar{x}, y, \bar{y}, t$ are non-zero. Let $A := K[x, \sigma, \delta], K := \mathbb{Q}(t), \sigma := id_{\mathbb{Q}(t)}$ and $\delta := \frac{d}{dt}$; we consider the idempotent matrix $F := [F^{(1)} \ F^{(2)} \ F^{(3)} \ F^{(4)}]$, $F^{(i)}$ the $i$th column of $F$, where

$$F^{(1)} := \begin{bmatrix} 2 + 2t + (13t^2 - 5t)x + (8t^3 - 6t^2)x^2 + t^3(t - 1)x^3 \\ 2t^2 + t + (13t^3 - 8t^2)x + (8t^4 - 7t^3)x^2 + t^4(t - 1)x^3 \\ 3t + 2 + (14t^2 - 8t)x + (8t^3 - 7t^2)x^2 + t^3(t - 1)x^3 \\ t^2 + t + (t^3 + 6t^2)x + 6t^3x^2 + t^4x^3 \end{bmatrix}.$$
Applying the algorithm we obtain

\[
\begin{align*}
F^{(2)} &=: \begin{bmatrix}
-t^3x^3 - 5t^2x^2 - 3tx + 1 \\
t + (-3t^2 + 2t)x + (-5t^3 + t^2)x^2 - t^4x^3 \\
-t^3x^3 - 5t^2x^2 - 3tx + 1 \\
-t^3x^3 - 5t^2x^2 - 3tx + 1
\end{bmatrix}, \\
F^{(3)} &=: \begin{bmatrix}
t^3x^3 + 5t^2x^2 + 3tx - 1 \\
t^4x^3 + 5t^3x^2 + 2t^2x - 2t \\
-t - 1 + (-t^2 + 5tx + 6t^2x^2 + t^3x^3) \\
-t^2 + t + (-t^3 + 6t^2x + 2t^3x^2)
\end{bmatrix}, \\
F^{(4)} &=: \begin{bmatrix}
0 \\
tx \\
tx \\
1 + (t^2 - 2t)x - t^2x^2
\end{bmatrix}.
\end{align*}
\]

Applying the algorithm we obtain

\[
U^{(1)} =: \begin{bmatrix}
2t + 1 + (10t^2 - 5t)x + (7t^3 - 6t^2)x^2 + (t^4 - t^3)x^3 \\
-3t - 2 + (-14t^2 + 8t)x + (-8t^3 + 7t^2)x^2 + (-t^4 + t^3)x^3 \\
-2t + 2 - t(t - 1)x \\
-2t^2 + 7t - 2 - t(4t^2 - 2t + 10)x - t^2(t^2 - 10t + 7)x^2 + t^3(t - 1)x^3
\end{bmatrix},
\]

\[
U^{(2)} =: \begin{bmatrix}
-t^3x^3 - 4t^2x^2 - tx \\
tx + 1 \\
2t(t - 3)x + t^2(t - 6)x^2 - t^3x^3
\end{bmatrix},
\]

\[
U^{(3)} =: \begin{bmatrix}
-t - 1 + (-t^2 + 3tx + 5t^2x^2 + t^3x^3) \\
t + 2 + (t^2 - 5tx + 6t^2x^2 - t^3x^3) \\
-tx - 1 \\
-t + 1 - t(2t - 7)x - t^2(t - 6)x^2 + t^3x^3
\end{bmatrix},
\]

\[
U^{(4)} =: \begin{bmatrix}
tx \\
tx \\
0 \\
1
\end{bmatrix};
\]

\[
(U^{-1})^{(1)} =: \begin{bmatrix}
tx + 1 \\
t - 2 + t(t - 1)x \\
-t + 2 - t(t - 4)x + t^2x^2
\end{bmatrix},
\]

\[
(U^{-1})^{(2)} =: \begin{bmatrix}
tx + 1 \\
t - 1 + t(t - 1)x \\
1 + (-t^2 + 3tx + t^2x^2)
\end{bmatrix},
\]

\[
(U^{-1})^{(3)} =: \begin{bmatrix}
-t^2x^2 - 2tx + 1 \\
t + (-4t^2 + 4tx + (-2t^3 + 5t^2)x^2 + t^3x^3) \\
1 + (-2t^2 + t)x + (-t^3 + 4t^2)x^2 + t^3x^3 \\
1 + (-2t^3 + 8t^2 - 5tx + (-t^4 + 11t^3 - 18t^2)x^2 + (2t^4 - 9t^3)x^3 - t^4x^4)
\end{bmatrix},
\]

\[
(U^{-1})^{(4)} =: \begin{bmatrix}
0 \\
tx \\
tx \\
1 + (t^2 - 2t)x - t^2x^2
\end{bmatrix}.
\]
With these computations we have

\[
UFU^{-1} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix},
\]

thus, \( r = 2 \) and a base of \( \langle F \rangle \) is \( X = \{x_1, x_2\} \), with

\[
x_1 = (-2t + 2 - t(t - 1)x, tx + 1, -tx - 1, 0),
x_2 = (-2t^2 + 7t - 2 - t(4t^2 - 21t + 16)x - t^2(t^2 - 10t + 7)x^2 + t^3(t - 1)x^3, 2t(t - 3)x + t^2(t - 6)x^2 - t^3x^3, t + 1 - t(2t - 7)x - t^2(t - 6)x^2 + t^3x^3, 1).
\]

### Algorithm for the Quillen-Suslin theorem:

**Computational version**

**REQUIRE:** \( A := K[x; \sigma, \delta] \) and an idempotent matrix \( F \in M_s(A) \).

1. \( k := 0, F' := F \);
2. **WHILE** \( k < s - 1 \) **DO**
3. \( k := k + 1 \);
4. **IF** \( \max \{\deg(f'_{ij}) \mid i = 1 \text{ or } j = 1\} = -\infty \) **THEN**
5. \( F' := \text{SubMatrix}(F', 2..s, 2..s) \);
6. **ELSE**
7. **END IF**
8. **IF** \( f'_{11} = 0 \) **THEN**
9. **IF** \( f'_{1k} \neq 0 \) \( F' := T_{k1}(-1)F'T_{k1}(-1)^{-1} \); **END IF**
10. \( F' := T_{1k}(-1)F'T_{1k}(-1)^{-1} \);
11. **END IF**
12. **IF** \( f'_{11} \in K - \{0\} \) **THEN**
13. **Apply:** OrderReduction1;
14. **ELSE**
15. **Apply:** (B2) OrderReduction2;
16. **END IF**
17. **END IF**
18. **END WHILE**
19. **RETURN** Matrices \( U, U^{-1}, UFU^{-1} \); a basis \( X \) of \( \langle F \rangle \); process step by step.

### Example 3.3.

In this example we will illustrate the computational version of the Quillen-Suslin algorithm: let \( M_3(A) \), where \( A := K[x, \sigma, \delta], K := \mathbb{Q}(t), \sigma(t^i) := \frac{\sigma(t)}{\sigma(t)} \) and \( \delta := 0 \); we have the idempotent matrix

\[
F = \begin{bmatrix}
1 - \frac{2t}{1+t} x & 2t - \frac{2t(3+2t)}{1+t} x & \frac{2t}{(1+t)^2} x \\
\frac{1}{1+t} x & \frac{2t}{1+t} x + \frac{2t(3+2t)}{1+t} x & \frac{t}{(1+t)^2} x \\
\frac{t}{1+t} x & -t + \frac{2t(3+2t)}{1+t} x & 1 - \frac{t}{(1+t)^2} x
\end{bmatrix}.
\]

Let \( F' := F \), along the example, we will replace the matrices \( F', U \) and \( U^{-1} \) for the new versions given by the procedures of the algorithm.

**Step 1.** Since \( f'_{11} = 1 - \frac{2t}{1+t} x \), we will apply the reduction procedure of (B2), i.e, OrderReduction2:

**Step1.1:** The idea is to convert \( f'_{11} = 0 \) for \( i > 2 \) and \( f'_{12} \neq 0 \).
Applying first $T_{2,3}(\frac{-1}{t(1+2t)})$, then $T_{3,2}(t(1+2t) - \frac{t(3+2t)(1+2t)}{1+t}x)$, and finally permuting the rows and columns 2 and 3, we get

$$UFU^{-1} = \begin{bmatrix} 1 - \frac{2t}{1+t}x & 0 & 0 \\ \frac{t(1+2t)}{1+t}x - \frac{2}{(1+2t)}(1+2t)x & 0 & 0 \\ \frac{2t}{(1+2t)(1+t)}x & 2(1+2t)(1+t)x & 0 \\ 2(1+2t)(1+t)x & 2(1+2t)(1+t)x & 0 \end{bmatrix},$$

where

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-t(1+2t)}{1+t} & 0 & 0 \end{bmatrix}. $$

**Step 1.2.** Since the new $F'$ is

$$F' = \begin{bmatrix} 1 - \frac{2t}{1+t}x & 0 & 0 \\ \frac{t(1+2t)}{1+t}x - \frac{2}{(1+2t)}(1+2t)x & 0 & 0 \\ \frac{2t}{(1+2t)(1+t)}x & 2(1+2t)(1+t)x & 0 \end{bmatrix},$$

we want to reduce the degree of $f'_{1,1}$; for this we apply $T_{2,1}(\frac{-t(1+2t)}{1+t}x)$ and we obtain

$$UFU^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the new $U$ and $U^{-1}$ are

$$U = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-t(1+2t)}{1+t} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-t(1+2t)}{1+t} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. $$

**Step 2.** The new $F'$ is

$$F' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{2}{(1+2t)^2} & 1 \end{bmatrix};$$

since $f'_{1,1} = 1$ we apply (B1), i.e., OrderReduction1, for this we consider the matrices

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and then

$$SF'S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. $$
Therefore, the new $F'$ is

$$F' = \begin{bmatrix} 0 & 0 \\ \frac{2t}{(1+2t)^2} & 1 \end{bmatrix}, \text{ and } U F U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix},$$

where the new $U$ and $U^{-1}$ are

$$U = \begin{bmatrix} 1 & \frac{2t}{1+2t} & x \\ -\frac{t(1+2t)}{(1+2t)(1+2t)} & \frac{2t}{1+2t} & -\frac{t(1+2t)}{(1+2t)(1+2t)} \\ 0 & 1 & \frac{1}{t(1+2t)} \end{bmatrix}, \text{ and } U^{-1} = \begin{bmatrix} \frac{1}{x} & 1 & -\frac{2}{1+2t} \\ \frac{t(1+2t)}{1+2t} & \frac{1}{t(1+2t)} & \frac{2}{1+2t}x \\ 1 - \frac{2(1+2t)}{(1+2t)(3+2t)}x & -t(1+2t) + \frac{3+2t}{1+2t}x \end{bmatrix}.$$

Since $f_{1,1}' = 0$, we apply $T_{1,2}(-1)$, we get

$$U F U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{(1+2t)^2} & 0 & -\frac{4t^2 - 4t + 1}{(1+2t)^2} \\ 0 & \frac{4t^2 + 4t - 4}{(1+2t)^2} \end{bmatrix} \text{ and } F' = \begin{bmatrix} 2 & -4t^2 - 4t + 1 \\ (1+2t)^2 & 4t^2 + 4t - 4 \\ (1+2t)^2 \end{bmatrix},$$

where the new $U$ and $U^{-1}$ are

$$U = \begin{bmatrix} \frac{1}{x} & \frac{2t}{1+2t} & \frac{2t}{1+2t}x - \frac{t(3+2t)(1+2t)}{4(1+2t)^2}x \\ -\frac{t(1+2t)}{(1+2t)(1+2t)} & \frac{1}{1+2t} & -\frac{2}{1+2t} \\ 0 & 1 & \frac{3+2t}{1+2t} \end{bmatrix}, \text{ and } U^{-1} = \begin{bmatrix} \frac{1}{1+2t} & \frac{2}{1+2t} & \frac{2}{1+2t}x \\ \frac{1}{t(1+2t)} & \frac{2}{t(1+2t)} & \frac{2}{t(1+2t)}x \\ 1 - \frac{2(1+2t)}{(1+2t)(3+2t)}x & -2t^2 - t + 1 + \frac{4t^2 + 12t + 7}{(1+2t)(3+2t)}x \end{bmatrix}.$$

Since $f_{1,1}' = \frac{2}{(1+2t)^2}$ is invertible, we apply $\text{OrderReduction1}$ with matrices

$$T = \begin{bmatrix} 1 & \frac{2t^2 - 2t + \frac{1}{2}}{1} \\ 1 & \frac{1}{1} \end{bmatrix} \text{ and } T^{-1} = \begin{bmatrix} \frac{2}{(1+2t)^2} & \frac{4t^2 + 4t - 1}{(1+2t)^2} \\ \frac{4t^2 + 4t - 1}{(1+2t)^2} & \frac{2}{(1+2t)^2} \end{bmatrix},$$

so

$$T F' T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus, the new $F'$ is

$$F' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } U F U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

where the new $U$ and $U^{-1}$ are

$$U = \begin{bmatrix} \frac{1}{x} & \frac{2t}{1+2t} & \frac{2t}{1+2t}x - \frac{t(3+2t)(1+2t)}{4(1+2t)^2}x \\ -\frac{t(1+2t)}{(1+2t)(1+2t)} & \frac{1}{1+2t} & -\frac{2}{1+2t} \\ 2t^2 + t - \frac{t(3+2t)(1+2t)}{4(1+2t)^2}x \end{bmatrix},$$

$$U^{-1} = \begin{bmatrix} \frac{1}{1+2t} & \frac{2}{1+2t} & \frac{2}{1+2t}x \\ \frac{1}{t(1+2t)} & \frac{2}{t(1+2t)} & \frac{2}{t(1+2t)}x \\ 1 - \frac{2(1+2t)}{(1+2t)(3+2t)}x & -2t^2 - t + 1 + \frac{4t^2 + 12t + 7}{(1+2t)(3+2t)}x \end{bmatrix}.$$
Applying the algorithm we obtain
\[
U^{-1} = \begin{bmatrix}
\frac{1}{t(1+2t)}x & 0 & -\frac{2}{t(1+2t)}x \\
\frac{2t}{(1+t)(3+2t)} & \frac{2t}{(1+t)(3+2t)} & \frac{2t}{(1+t)(3+2t)}x \\
\frac{1+2t}{t(1+t)(3+2t)} & \frac{1+2t}{t(1+t)(3+2t)} & \frac{1+2t}{t(1+t)(3+2t)}x \\
\end{bmatrix}.
\]
Permuting, we have finally
\[
UFU^{-1} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix},
\]
where the new $U$ and $U^{-1}$ are
\[
U = \begin{bmatrix}
\frac{-t(1+2t)}{t+1}x & 2t^2 + t & \frac{t(1+2t)}{1+t}x \\
1 - \frac{2t}{1+t}x & 2t & \frac{t(3+2t)}{1+t}x \\
\frac{-t(1+2t)}{t+1}x & -t - \frac{2}{t} & \frac{t(1+2t)}{1+t}x \\
\end{bmatrix},
\]
\[
U^{-1} = \begin{bmatrix}
\frac{-2}{1+t}x & 1 & 0 \\
\frac{t(1+2t)}{1+t}x & 2t & \frac{t(1+2t)}{1+t}x \\
\frac{t(1+2t)}{1+t}x & \frac{2t}{1+t} & \frac{t(1+2t)}{1+t}x \\
\end{bmatrix}.
\]
Therefore, $r = 2$ and the last two rows of $U$ conform a basis $X = \{x_1, x_2\}$, of $\langle F \rangle$,
\[
x_1 = (1 - \frac{2t}{1+t}x, 2t - \frac{2t(3+2t)}{1+t}x, \frac{2t}{1+t}x), \quad x_2 = (\frac{-t(1+2t)}{1+t}x, -t - \frac{2}{t} \frac{t(3+2t)(1+2t)}{1+t}x, \frac{1+2t}{2t} + \frac{t(1+2t)}{1+t}x).
\]

**Example 3.4.** Let $M_d(A)$, where $A := K[x, \sigma, \delta], K := \mathbb{Q}(t), \sigma(f(t)) := f(qt)$ and $\delta(f(t)) := \frac{f(qt) - f(t)}{t(q-1)}$,
where $q \in K - \{0, 1\}$; we consider the idempotent matrix $F := [F^{(1)} F^{(2)} F^{(3)} F^{(4)}]$,
$F^{(i)}$ the $i$th column of $F$ and $a \in \mathbb{Q}$, where

\[
F^{(1)} = \begin{bmatrix}
-t^2qxa^2 \\
(-ta + 2t)x - 2a + 2 \\
\end{bmatrix},
\]
\[
F^{(2)} = \begin{bmatrix}
-2tx + 2 \\
-t^2qxa^2 + (ta - 4t)x + 2a - 1 \\
-tx - 2 \\
tx + 2 \\
\end{bmatrix},
\]
\[
F^{(3)} = \begin{bmatrix}
-tx - 2 \\
(2t^2a + 3t^2a) + (a^2t - 8ta + 8t)x + 2a^2 - 3a + 1 \\
t^2qxa^2 + (-ta + 4t)x - 2a + 2 \\
(ta - 2t)x + 2a - 2 \\
\end{bmatrix},
\]
\[
F^{(4)} = \begin{bmatrix}
-t^3q^3x^3 + (-q^2t^2 - 5t^2q)x^2 - 5tx + 2 \\
-t^3q^3x^3 + (-q^2t^2 - 3t^2q)x^2 + (-ta + t)x - 2a + 2 \\
tx + 2 \\
t^2qxa^2 + (t^2tx + 2tx - 1) \\
\end{bmatrix}.
\]
Applying the algorithm we obtain
\[
U = \begin{bmatrix}
tx + 1 & 0 & t^2qxa^2 + 2tx - 1 & t^2qxa^2 + 3tx \\
tx - 1 & 1 & 0 & t^2qxa^2 + 2tx - 2 \\
-tx - 2 & (-ta + 2t)x - 2a + 2 & -t^2qxa^2 - 2tx + 2 \\
tx + 1 & 0 & t^2qxa^2 + a - 1 & t^2qxa^2 + 2tx - 1 \\
\end{bmatrix}.
\]
\[ U^{-1} = \begin{bmatrix} t_x - a - 1 & -t_x + a - 1 & -t_x^2 q_x^2 + (ta - 4 t) x + 2 a - 1 & t^3 q_x^3 x^3 - (q + a - 4) t^2 q_x^2 + (3 t a - 3 t) x + 1 \\ -1 - t_x + 2 & 0 & -t_x^2 q_x^2 + 3 t x + 2 & -t^2 q_x^2 - 2 t x + 1 \end{bmatrix}. \]

Therefore, \( r = 2 \) and the last two rows of \( U \) conform a basis \( X = \{ x_1, x_2 \} \), of \( \langle F \rangle \),

\[ x_1 = (tx - 1, 1, t^2 q_x^2 + a - 1, t^2 q_x^2 + 2 tx - 1), \quad x_2 = (1, 0, tx, tx + 1). \]

### 4 Some remarks about the implementation

In this final section we present some comments about the implementation of the computational version of the Quillen-Suslin algorithm.

**Remark 4.1.** The OrderReduction1 is based in the implementation of the procedure \((B1)\) in the proof of Theorem 2.1 for the OrderReduction2, the following algorithm describes its functionality:

```
Algorithm OrderReduction2

REQUIRE: A := \( K[x; \sigma, \delta] \) and an idempotent matrix \( F \in M_s(A) \) with deg\((f_{11}) \geq 1. \)
1: Make \( f_{1,j} = 0 \) for \( j > 2 \) and \( f_{1,2} \neq 0; \)
2: Reduce degree of \( f_{1,1}; \)
3: IF \( f_{1,1} = 0 \)
4: \quad IF \( \max\{\text{deg}(f_{i,j}) > 0 \mid i = 1 \) or \( j = 1\} > 0 \)
5: \quad Make \( f_{1,j} = 0 \) for \( j > 2 \) and \( f_{1,2} \neq 0; \)
6: \quad \( F := P_{12} F P_{12}; \)
7: \quad Apply: OrderReduction1;
8: ELSE
9: \quad \( F' := \text{SubMatrix}(F, 2..s, 2..s); \)
10: ENDIF
11: ELSE
12: Apply: OrderReduction1;
13: ENDIF
14: RETURN Matrices \( U, U^{-1}, F' \) and \( U F U^{-1} = \begin{bmatrix} \alpha & 0 \\ 0 & F' \end{bmatrix} \), with \( \alpha \in \{0, 1\}. \)
```

**Remark 4.2.** For the implementation of the Quillen-Suslin algorithm we used Maple® 2016, and we create a library called OrePolyToolKit.lib consisting in two packages:

- **OrePolyUtility**: This is a new useful collection of functions for operating matrices, vectors and lists over an UnivariateOreRing \( K[x; \sigma, \delta] \); the UnivariateOreRing structure was taken from the library OreTools within the standard Maple libraries.
- **OrePolyQS**: This is the most important new collection of functions related to the Quillen-Suslin algorithm over \( K[x; \sigma, \delta] \); the main routine of the algorithm was implemented here, the following functions of this package are fundamentals:
- **GenerateIdemp**: This function generates idempotent matrices over $K[x;\sigma,\delta]$, the arguments are the matrix order and the UnivariateOreRing, and return an idempotent matrix of the given dimension over the respective UnivariateOreRing.

- **QSAlgKsd**: This is the main function of the algorithm, it shows the sequence of all steps of the Quillen-Suslin algorithm presented in this paper; the arguments are the idempotent matrix and the UnivariateOreRing, and return the matrix $UFU^{-1}$ in the form of Theorem 2.1, the matrices $U$ and $U^{-1}$, the basis of $\langle F \rangle$ and the complete process step by step.

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