Supersymmetric Toda field theories

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ABSTRACT

We present new supersymmetric extensions of Conformal Toda and $A_N^{(1)}$ Affine Toda field theories. These new theories are constructed using methods similar to those that have been developed to find supersymmetric extensions of two-dimensional bosonic sigma models with a scalar potential. In particular, we show that the Conformal Toda field theory admits a (1,1)-supersymmetric extension, and the $A_N^{(1)}$ Affine Toda field admits a (1,0)-supersymmetric extension.
1. Introduction

It has been known for sometime that the Conformal Toda and Affine Toda field theories admit supersymmetric extensions. These extensions were based on the theory of superalgebras and the emphasis was to construct integrable (1,1)- and (2,2)-supersymmetric extensions of Conformal Toda and Affine Toda field theories [1-3]. This led to theories that, apart from the supersymmetric extensions of Liouville theory [4-10], and the sinh- and sine-Gordon models [11-16], either their supersymmetry was broken explicitly at the level of their classical action or their energy was unbounded from below. (The inner product that appears in their kinetic term is indefinite).

The Conformal and Affine Toda theories are field theories of scalar fields with a scalar potential interaction and are examples of sigma models with a scalar potential and flat space as a target manifold. Recently, there has been a lot of progress towards classifying all (p,q)-supersymmetric two-dimensional sigma models with a scalar potential [18, 19]. These results were used in ref. [20] to find a supersymmetric extension of the WZW model with a scalar potential and show that it has a number of novel properties like a multi-vacuum structure, solitons that saturate a Bogomol’nyi bound and supersymmetry breaking for certain values of its coupling constants. Since Conformal and Affine Toda field theories are special cases of two-dimensional sigma models, it seems that the methods applicable for constructing supersymmetric extensions of sigma models with scalar potentials can be used to construct supersymmetric extensions of the Conformal and Affine Toda field theories.

The main purpose of this letter is to present supersymmetric extensions of the Conformal and $A^{(1)}_N$ Affine Toda field theories using the general methods applied to construct supersymmetric extensions of bosonic sigma models with a scalar potential. For this, we will start from the bosonic action of Conformal and $A^{(1)}_N$ Affine Toda field theories and we will then add the kinetic terms of the fermions and the interaction terms of the fermions with the scalar fields. These supersymmetric
Conformal and $A^{(1)}_N$ Affine Toda field theories will have the following properties: (i) the classical vacuum of the Conformal and Affine Toda field theories will be supersymmetric* and (ii) the mass matrix of the fermions will be well defined (finite) at this vacuum. The energy functional of all these theories is positive semi-definite. In the case of Conformal Toda field theory, it is straightforward to find a ‘minimal’ (1,0)-supersymmetric extension. In fact, this (1,0)-supersymmetric Conformal Toda field theory can be further extended to a (1,1)-supersymmetric one. The (bosonic) Conformal Toda field theory has a unique vacuum at infinity. It turns out that this vacuum is a supersymmetric and the fermion mass matrix evaluated at this vacuum vanishes. In the case of $A^{(1)}_N$ Affine Toda field theory, we will show that the theory admits a (1,0)-supersymmetric extension for which the properties (i) and (ii) above are satisfied and that this (1,0)-supersymmetric extension for, $N \geq 2$, is not unique. Finally, we will compute the supersymmetry charges of (1,1)-supersymmetric Conformal Toda and (1,0)-supersymmetric $A^{(1)}_N$ Affine Toda field theories and explicitly give the fermion mass matrix of the supersymmetric $A^{(1)}_2$ Affine Toda field theory.

The material of this letter is organised as follows: In section 2, the action and supersymmetry transformation laws of the fields of the (1,0)- and (1,1)-supersymmetric sigma models with scalar potential will be presented. In section 3, a (1,1)-supersymmetric extension of the Conformal Toda field theory will be given. In section 4, two different (1,0)-supersymmetric extensions of the $A^{(1)}_N$ Affine Toda field theory will be presented and in section 5, we will give our conclusions.

* We have assumed that the coupling constant is real.
2. (1,0)-and (1,1)-supersymmetric massive sigma models

The \((p,q)\) supersymmetry algebra in two dimensions is

\[
\{Q^I_+, Q^J_+\} = 2\delta^{IJ}(E+P), \quad \{Q^{I'}_-, Q^{J'}_-\} = 2\delta^{I'J'}(E-P), \quad \{Q^I_+, Q^{I'}_-\} = Z_{II'},
\]

(1)

where \(\{Q^I_+; I = 1, \ldots, p\}\) are the ‘left’ supersymmetry charges, \(\{Q^{I'}_+; I' = 1, \ldots, q\}\) are the ‘right’ supersymmetry charges, \(Z_{II'}\) are central charges, \(E\) is the energy and \(P\) is the momentum generators. We will be primarily concerned here with the \((1,0)\) and \((1,1)\) supersymmetry algebras. The \((1,0)\) supersymmetry algebra has one ‘left’ supersymmetry charge \(Q_+\) and no central charges. The \((1,1)\) supersymmetry algebra has one ‘left’ supersymmetry charge \(Q_+\), a ‘right’ supersymmetry charge \(Q_-\) and it may also have one central charge \(Z\). For a bosonic sigma model to admit a \((p,q)\)-supersymmetric extension several conditions of the couplings of the theory must be imposed. These conditions have a geometric interpretation in terms of the geometry of the sigma model target space and for the case of sigma model with a scalar potential were given in refs. [17, 18, 19]. Let \(g\) be a metric and \(b\) be a locally defined two-form on a manifold \(\mathcal{M}\). In addition, let \(A\) be a connection of a vector bundle \(\mathcal{E}\) over \(\mathcal{M}\) and \(k\) be a fibre metric of \(\mathcal{E}\). The action of most general \((1,0)\)-supersymmetric sigma model with scalar potential is \(^*\)

\[
I = \int d^2 x \left\{ \partial_+ \phi^\dagger \partial_- \phi^j (g_{ij} + b_{ij}) + ig_{ij} \lambda_+^i \nabla_-^{(+)} \lambda_+^j - ik_{a b} \psi_-^a \nabla_+ \psi_-^b \right.
\]

\[
- \frac{1}{2} \psi_-^a \psi_-^b \lambda_+^i \lambda_+^j F(A)_{ijab} + m \nabla_i s_a \lambda_+^i \psi_-^a - V(\phi) \left. \right\},
\]

(2)

where \(\phi\) is the sigma model field, \(\lambda_+\) and \(\psi_-\) are real chiral fermions, \(\{x^+, x^-\}\) are the light-cone co-ordinates of the two-dimensional Minkowski space-time,

\[
\nabla_+ \psi_-^a = \partial_+ \psi_-^a + \partial_+ \phi^j A_i^a \psi_-^b,
\]

(3)

\(^*\) This theory admits an off-shell \((1,0)\) superspace formulation and this action is derived after elimination of the auxiliary fields [18].
$F(A)$ is the curvature of $A$, $s$ is a section of the vector bundle $\mathcal{E}$,

$$V(\phi) = \frac{m^2}{4} k^{ab} s_a(\phi) s_b(\phi),$$

is the scalar potential and $m$ is a parameter with dimension that of a mass. Finally, the connections of the covariant derivatives $\nabla^{(\pm)}$ are

$$\Gamma^{(\pm)}_{jk} = \{ i_{jk} \} \pm H^i_{jk}$$

where $H = \frac{3}{2} db$. From the action (2), the fermion mass matrix is

$$M_{ia} = m \nabla_i s_a.$$  (6)

It is important to note that the range of the sigma model manifold indices $i,j$ is different from that of the vector bundle indices $a,b$. The action (2) is invariant under the $(1,0)$ supersymmetry transformations

$$\delta_\epsilon \phi^i = -\frac{i}{2} \epsilon_- \lambda_+^i,$$

$$\delta_\epsilon \lambda_+^i = \frac{1}{2} \epsilon_- \partial_+ \phi^i,$$

$$\delta_\epsilon \psi_-^a = \frac{i}{2} \epsilon_- A_i^a b \lambda_+^i \psi_-^b + \frac{i}{4} \epsilon_- m s_a,$$

where $\epsilon_-$ is the parameter of the transformations. The $(1,0)$-supersymmetry charge is

$$S_+ = \int dx [g_{ij} \partial_+ \phi^i \lambda_+^j - i \frac{1}{3} H_{ijk} \lambda_+^i \lambda_+^j \lambda_+^k - \frac{i}{2} m s_a \psi_-^a].$$  (8)

To summarise, a bosonic sigma model with a scalar potential admits a $(1,0)$-supersymmetric extension if and only if its scalar potential can be written as the length of a section of a vector bundle over the sigma model manifold. The supersymmetric vacua of the theory are simply the zeroes of this section. Note that there
might be different ways, up to gauge equivalences, to write the scalar potential of a bosonic sigma model as the length of a section of a vector bundle over the sigma model manifold. Different choices may lead to different (1,0)-supersymmetric extensions of the same bosonic sigma model.

Next we will describe the most general (1,1)-supersymmetric sigma models with scalar potential [18]. These models are special cases of (1,0)-supersymmetric sigma models with scalar potential and are those for which the vector bundle $E$ is isomorphic to the tangent bundle of $\mathcal{M}$, the connection $A$ is the spin connection of $\Gamma^{(-)}$ and the section $s$ is

$$s_i = (u - X)_i$$

where $X$ is a Killing vector field of $\mathcal{M}$ that leaves $H$ invariant, the one-form $u$ is defined from the relation

$$X^k H_{kij} = \partial_{[i} u_{j]}$$

and $X^i u_i = 0$. The (0,1) supersymmetry transformations are

$$\delta \zeta \phi^i = (u + X)^i + \zeta_+ \Gamma^{(-)}_{jk} \lambda^j_+ \psi^k_+$$

$$\delta \zeta \lambda^i_+ = -i \zeta_+ \partial - \phi^i$$

where $\zeta_+$ is the parameter. The (0,1)-supersymmetry charge is

$$S_- = \int dx [ig_{ij} \partial - \phi^i \psi^j_+ + \frac{1}{3} \psi^i_+ \psi^j_+ \psi^k_+ H_{ijk} + \frac{m}{2} (X + u)_i \lambda^i_+]$$

In the models of interest here, the torsion $H$ and the Killing vector field $X$ are zero in which case $u = df$ where $f$ is a function of the sigma model manifold $\mathcal{M}$. 


3. A Supersymmetric Conformal Toda field theory

Let \( g \) be a simple Lie algebra and \( h \) be a Cartan subalgebra of \( g \) (\( \text{dim} h \equiv \text{rank} g = r \)). The action of the bosonic Conformal Toda field theory is

\[
I = \int d^2x \left( <\partial_+ \phi, \partial_- \phi> - \frac{m^2}{4} \sum_{i=1}^{r} \mu_i e^{\alpha_i(\phi)} \right),
\]

where \( \phi \) are the fields that are maps from the two-dimensional space-time into the Cartan subalgebra \( h \), \( <\cdot,\cdot> \) is the bi-invariant inner product of \( g \) restricted on \( h \), \( \{\alpha_i; i = 1,\ldots, \text{rank} g\} \) are the simple roots of \( g \), \( \alpha_i(\phi) = \alpha_{in} \phi^n \) and \( \{\mu_i; i = 1,\ldots, r\} \) are positive real constants. It is clear that this theory is a sigma model without Wess-Zumino term (\( b = 0 \)) and flat target space. The scalar potential \( V \) of the theory is given by

\[
V(\phi) = \frac{m^2}{4} \sum_{i=1}^{r} \mu_i e^{\alpha_i(\phi)}. \tag{14}
\]

To construct a (1,0)-supersymmetric extension of this theory, we first introduce chiral fermions \( \lambda_+ \) and \( \psi_- \) and then choose

\[
\{s_n; n = 1,\ldots, r\} = \{\sqrt{\mu_i e^{\frac{1}{2} \alpha_i(\phi)}}; i = 1,\ldots, r\}. \tag{15}
\]

For this choice of \( s \), the fibre metric \( k \) is \( k = \text{diag}(1,\ldots, 1) \). The action of the supersymmetric Toda field theory is

\[
I = \int d^2x \left( <\partial_+ \phi, \partial_- \phi> + i \lambda_+, \partial_- \lambda_+ > - i \psi_- , \partial_+ \psi_- >
\right.

\[
\left. + \frac{m}{2} \sum_{i=1}^{r} \sqrt{\mu_i} e^{\frac{1}{2} \alpha_i(\phi)} \psi^i - \frac{m^2}{4} \sum_{i=1}^{r} \mu_i e^{\alpha_i(\phi)} \right). \tag{16}
\]

The fermion mass matrix is

\[
M_{in} = \frac{m}{2} \alpha_{in} \sqrt{\mu_i e^{\frac{1}{2} \alpha_i(\phi)}}. \tag{17}
\]

The vacuum of the theory is in the limit \( \alpha_i(\phi) \rightarrow -\infty \) and in this limit the scalar

* The index \( n \) is associated with an orthonormal basis of the simple Lie algebra \( g \).
potential, $V$, vanishes. This vacuum is also supersymmetric because the section $s$ of eqn (15) vanishes as $\alpha_i(\phi) \to -\infty$. Moreover the bosons and the fermions of the theory are massless in this vacuum.

The above (1,0)-supersymmetric Conformal Toda field theory can be further extended to a (1,1) supersymmetric one. For this, we observe that all the conditions mentioned in the previous section for a (1,0)-supersymmetric sigma model to be a (1,1)-supersymmetric one are satisfied. In particular, we note that

$$\{s_n; n = 1, \ldots, r\} = \{\partial_i f(\phi); i = 1, \ldots, r\}, \quad (18)$$

where

$$f = 2 \sum_{i=1}^{r} \sqrt{\mu_i e^{\frac{1}{2}\alpha_i(\phi)}}, \quad (19)$$

$$\partial_i = \partial/\partial\phi^i \text{ and } \phi^i = \alpha_i(\phi).$$

The (1,0) and (0,1) supersymmetry charges are given as follows:

$$S_+ = \int dx[<\partial_+ \phi^i, \lambda_+ > - i \sum_{i=1}^{r} \sqrt{\mu_i e^{\frac{1}{2}\alpha_i(\phi)}} \psi^i], \quad (20)$$

and

$$S_- = \int dx[i <\partial_- \phi, \psi_- > + \frac{m_i}{2} \sqrt{\mu_i e^{\frac{1}{2}\alpha_i(\phi)}} \delta m \lambda^m_+], \quad (21)$$

correspondingly. It can be verified by direct computation that the Poison bracket algebra of these charges is isomorphic to the (1,1) supersymmetry algebra (1). Finally, note that the action, (16), of the (1,1)-supersymmetric Conformal Toda field theory can be written in terms of (1,1) off-shell superfields.
4. Supersymmetric Affine Toda

The action of bosonic Affine Toda field theory is

\[ I = \int d^2x \left[ < \partial_4 \phi, \partial_\pm \phi > - \frac{m^2}{4\beta^2} \left( \sum_{i=0}^{r} n_i e^{\beta \alpha_i(\phi)} - h \right) \right] \tag{22} \]

where \( \phi \) is a map from the two-dimensional Minkowski space-time into a Cartan subalgebra \( h \) of a simple group \( g \), \( r = \text{rank} g \), \( a_i = \alpha_i \) for \( i \geq 1 \) and \( a_0 = -\psi \), \( \psi \) is the highest root of \( g \), the integers \( n_i \) for \( i \geq 1 \) are defined by the equation

\[ \psi = \sum_{i=1}^{r} n_i \alpha_i , \tag{23} \]

\( n_0 = 1 \) and \( h \) is the Coexeter number, i.e. \( h = \sum_{i=0}^{r} n_i \). The parameters \( m \) and \( \beta \) are the couplings constants of the theory.

Next consider the special case of \( A_N^{(1)} \) Affine Toda field theory. In this case \( r = N \), \( n_i = 1 \) and \( h = N + 1 \). The scalar potential of the \( A_N^{(1)} \) Affine Toda field theory is

\[ V_N(\phi^1, \ldots, \phi^N) = \frac{m^2}{4\beta^2} \left[ \sum_{i=1}^{N} e^{\beta \phi^i} + e^{-\beta} \sum_{i=1}^{N} \phi^i - (N + 1) \right] \tag{24} \]

where \( \phi^i = \alpha^i(\phi) \). To show that the \( A_N^{(1)} \) Affine Toda field theory admits a (1,0)-supersymmetric extension we have to write the scalar potential of the theory as the sum of squares in such a way that the following two properties that have been mentioned in the introduction are satisfied, i.e. (i) the vacuum \( \phi = 0 \) is supersymmetric and (ii) the fermions have a well defined (finite) mass matrix at the

\[ \begin{array}{c}
\star \text{ The Lagrangian of the Affine Toda field theory is usually written without the term involving the Coexeter number } h. \text{ This term is not necessary in the bosonic theory since it does not contribute to the field equations. However in the supersymmetric theory, this is no longer the case because the absence of this term from the action will lead to a theory with spontaneously broken supersymmetry.}
\end{array} \]
vacuum $\phi = 0$ of the theory. (The fibre metric $k$ is chosen as $k = \text{diag}(1, \ldots, 1).$) To prove that the scalar potential $V_N(\phi^1, \ldots, \phi^N)$ of the $A^{(1)}_N$ Affine Toda field theory can be written as a sum of squares in such a way that both properties (i) and (ii) above are satisfied, we first observe that the scalar potential of $A^{(1)}_2$ Affine Toda field theory is written as

$$V_2(\phi^1, \phi^2) = \frac{m^2}{4\beta^2} \left[ (e^{\frac{\beta \phi^1}{2}} - e^{\frac{\beta \phi^2}{2}})^2 + (e^{-\frac{\beta \phi^1 + \phi^2}{2}} - 1)^2 + 8 \sinh^2(\beta \frac{\phi^1 + \phi^2}{4}) \right].$$

(25)

We, then, proceed inductively in the integer $N$. For this we assume that the scalar potential of $A^{(1)}_{N-1}$ Affine Toda field theory can be written as a sum of squares that satisfy the properties (i) and (ii) above. Next we observe that the scalar potential of the $A^{(1)}_N$ Affine Toda field theory can be written as

$$V_N(\phi^1, \ldots, \phi^N) = \frac{m^2}{4\beta^2} \left[ (e^{\frac{\beta \phi^1}{2}} - e^{\frac{\beta \phi^2}{2}})^2 + \sum_{i \geq 3}^N (e^{\frac{\beta \phi^i}{2}} - 1)^2 + (e^{-\frac{\beta}{2}} \sum_{i=1}^N \phi^i - 1)^2 \right] + 2V_{N-1}(\frac{\phi^1}{2}, \frac{\phi^2}{2}, \ldots, \frac{\phi^N}{2}).$$

(26)

This completes the induction in $N$ because from (26) the scalar potential of $A^{(1)}_N$ Affine Toda field theory is expressed in terms of sums of squares and the scalar potential of the $A^{(1)}_{N-1}$ Affine Toda field theory. It is easy to verify that the scalar potential of $A^{(1)}_N$ Affine Toda field theory written as in (26) satisfies both the above (i) and (ii) properties. It is also straightforward to compute the section $s$ from (26) and then use the general formalism summarised in section 2 to write the action of the (1,0)-supersymmetric extension of the $A^{(1)}_N$ Affine Toda field theory.

The above (1,0)-supersymmetric extension of the $A^{(1)}_N$ Affine Toda field theory is not unique. To illustrate this, we first observe that the scalar potential of $A^{(1)}_2$
Affine Toda field theory can also be written as
\[
V_2(\phi^1, \phi^2) = \frac{m^2}{4\beta^2} \left[ 2e^{-\beta \frac{\phi^1}{4}} \sinh^2(\beta \frac{2\phi^1 + \phi^2}{4}) + 2e^{-\beta \frac{\phi^2}{4}} \sinh^2(\beta \frac{2\phi^2 + \phi^1}{4}) \\
+ \frac{1}{2} \left( [e^{\beta \frac{\phi^1}{4}} - 1]^2 + [e^{\beta \frac{\phi^2}{4}} - 1]^2 \right) + 2 \sinh^2(\beta \frac{\phi^1}{4}) + 2 \sinh^2(\beta \frac{\phi^2}{4}) \right].
\]  
(27)

We then can show that the scalar potential of \( A^{(1)}_N \) Affine Toda field theory can be written as
\[
V_N(\phi^1, \ldots, \phi^N) = \frac{m^2}{4\beta^2} \left[ \frac{4}{N} \sum_{i=1}^{N} e^{-\beta \sum_{j \neq i} \frac{\phi^j}{4}} \sinh^2(\beta \frac{\phi^i + \sum_{j=1}^{N} \phi^j}{4}) \\
+ \frac{N-1}{N} \sum_{i=1}^{N} \left( e^{\beta \frac{\phi^i}{4}} - 1 \right)^2 \\
+ \frac{2}{N} \sum_{i=1}^{N} V_{N-1}(\phi^1, \ldots, \hat{\phi}^i, \ldots, \phi^N) \right],
\]  
(28)

where the ‘hat’ over a component of the field \( \phi \) implies that this component does not appear in the expression of the associated scalar potential. Using induction as in the previous case and the eqns (27) and (28), we can show that the scalar potential of \( A^{(1)}_N \) Affine Toda field theory is expressed in terms of sums of squares and both the properties (i) and (ii) above are satisfied. This is a different (1,0)-supersymmetric extension of the \( A^{(1)}_N \) Affine Toda field theory from the previous one because the number of \( \psi^- \) fermions is different in these two theories. The (1,0) supersymmetry charge is
\[
S_+ = \int dx [\partial_+ \phi, \lambda_+ > - \frac{i}{2} ms_\psi^a],
\]  
(29)

where \( s \) is the section that one derives from either (26) or (28).

To give an explicit example of a supersymmetric Affine Toda field theory, we consider the case \( N = 2 \). Using (25), we find that the section \( s \) is
\[
\{s_a\} = \frac{1}{\beta} \left\{ e^{\beta \frac{\phi^1}{4}} - e^{\beta \frac{\phi^2}{4}}, e^{-\beta \frac{\phi^1 + \phi^2}{4}} - 1, \sqrt{8} \sinh(\beta \frac{\phi^1 + \phi^2}{4}) \right\}.
\]  
(30)

So \( s \) is a three component vector and to construct an (1,0)-supersymmetric action
with bosonic part given in (22) it is clear that one should introduce three \( \{ \psi^a; a = 1, 2, 3 \} \) chiral fermions in addition to the two \( \lambda_+^i \equiv \alpha^i(\lambda_+) \) chiral fermions. The action of the supersymmetric \( A_2^{(1)} \) Affine Toda field theory is

\[
I = \int d^2 x \left[ < \partial_4 \phi, \partial_\pm \phi > + i \lambda_+^i (\partial_\pm \lambda_+) > - i \delta_{ab} \psi^a_\pm \partial_4 \psi^b_\pm \right. \\
\left. + m \sum_{i=1}^2 \lambda_+^i \partial_i s^a \psi^a_\pm - \frac{m^2}{4\beta^2} \left( \sum_{i=0}^2 e^{\beta a_i(\phi)} - 3 \right) \right].
\]

The fermion mass matrix \( M \) is a \( 2 \times 3 \) matrix and the masses of the fermions at the vacuum \( \phi = 0 \) are

\[
M \equiv m \{ \partial_i s_a \}(\phi = 0) = m \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

This matrix has rank two and so there are two Majorana fermions constructed from the six pairs \( (\lambda_+^i, \phi_-) \) of Majorana-Weyl fermions that have non-zero mass at the supersymmetric vacuum. The \( (1,0) \) supersymmetry charge is computed by substituting, \( s \), of eqn. (30) into (29).

In the case of the second \( (1,0) \)-supersymmetric extension of Affine Toda field theory above, the section \( s \) for \( A_2^{(1)} \) model derived from eqn. (27) is

\[
s \equiv \{ s_a \} = \frac{\sqrt{2}}{\beta} \left( e^{-\beta^2 \frac{\phi}{4}} \sinh(\beta \frac{2\phi^1 + \phi^2}{4}) , e^{-\beta^2 \frac{\phi^1}{4}} \sinh(\beta \frac{2\phi^2 + \phi^1}{4}) , \right. \\
\left. \frac{1}{2} (e^{\beta \phi^1} - 1) , \frac{1}{2} (e^{\beta \phi^2} - 1) , \sinh(\beta \frac{\phi^1}{4}) , \sinh(\beta \frac{\phi^2}{4}) \right) .
\]

So \( s \) is a six component vector and to construct a \( (1,0) \)-supersymmetric action with bosonic part given in (22) it is clear that one should introduce six \( \{ \psi^a; a = 1, \ldots, 6 \} \) chiral fermions in addition to the two \( \lambda_+^i \equiv \alpha^i(\lambda_+) \) chiral fermions. The action of this \( (1,0) \)-supersymmetric \( A_2^{(1)} \) Affine Toda field theory is as (31) except that the section \( s \) is given in (33). The fermion mass matrix \( M \) is a \( 2 \times 6 \) matrix and the
masses of the fermions at the vacuum $\phi = 0$ are
\[
M \equiv m \{ \partial_i s_a \}(\phi = 0) = \frac{m}{\sqrt{2}} \left( \begin{array}{cccc}
1 & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 1 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & 1 \\
\frac{1}{2} & 0 & 1 & 1
\end{array} \right).
\] (34)

This matrix has rank two and so there are two Majorana fermions constructed from the twelve pairs $(\lambda_+, \phi_-)$ of Majorana-Weyl fermions that have non-zero mass at the supersymmetric vacuum. The (1,0) supersymmetry charge is computed by substituting, $s$, of eqn. (33) into (29).

Next we can easily verify that in the special case of sinh-Gordon theory ($A^{(1)}_1$ Affine Toda field theory), it is possible to further extend the model to a (1,1)-supersymmetric one. The scalar potential in this case is
\[
V(\phi) = \frac{m^2}{4\beta^2} (\cosh(\beta \phi) - 1) = \frac{m^2}{2\beta^2} \sinh^2 \frac{\beta \phi}{2}
\] (35)
and $s$ can be chosen to be simply
\[
s = \frac{\sqrt{2}}{\beta} \sinh \left( \frac{\beta \phi}{2} \right).
\] (36)

This section $s$ can be written as the derivative of a function $f$, $s = \frac{d}{d\phi} f$, where
\[
f = 2 \frac{\sqrt{2}}{\beta^2} \cosh \left( \frac{\beta \phi}{2} \right).
\] (37)

The (1,0) and (0,1) supersymmetry charges of the theory are
\[
S_+ = \int dx \left[ \frac{1}{2} \partial_+ \phi \lambda_+ - im \frac{1}{\sqrt{2}} \sinh \left( \frac{\beta \phi}{2} \right) \psi_- \right],
\] (38)
and
\[
S_- = \int dx \left[ i \frac{1}{\sqrt{2}} \partial_- \phi \psi_- + m \frac{1}{2\beta} \sinh \left( \frac{\beta \phi}{2} \right) \lambda_+ \right],
\] (39)
respectively, and their Poisson bracket algebra is isomorphic to the (1,1) supersymmetry algebra section 2.
5. Concluding Remarks

We have constructed \((1,0)\)-supersymmetric extensions of Conformal and \(A^{(1)}_N\) Affine Toda field theories using the results known for finding supersymmetric extensions of bosonic two-dimensional sigma models. In the case of Conformal Toda field theory, the \((1,0)\)-supersymmetric theory can be further extended to a \((1,1)\) one. In the case of \(A^{(1)}_N\), \(N \geq 2\), Affine Toda field theory, we have found that the theory admits a \((1,0)\)-supersymmetric extension such that the vacuum of the theory is supersymmetric and the mass matrix of the fermions is finite at this vacuum. We have also demonstrated that this \((1,0)\)-supersymmetric extension is not unique. It seems likely that supersymmetric extensions similar to those presented here exist for the rest of the Affine Toda field theories as well and for other integrable systems like for example the Conformal Affine Toda field theory, the non-Abelian Toda field theory and the Moser-Calogero systems.

It is straightforward to extend the above results to the associated one-dimensional systems, i.e. to the one-dimensional Conformal and Affine Toda theories. In one dimension there are no chiral fermions but one can still introduce the \(\lambda\) and \(\psi\) fermions and perform a similar construction for the associated supersymmetric theory. The corresponding supersymmetric systems will have \(N = 1\) one-dimensional supersymmetry.

Another question of interest is that of the integrability properties of the \((1,0)\)-supersymmetric Affine Toda field theories presented in this paper. The algebraic methods used to construct integrable supersymmetric Toda theories in refs.\([1, 2]\) do not seem to be applicable in this case because they rely on the properties of off-shell \((1,1)\) superfields and such \((1,1)\) off-shell superfield formulation for the \((1,0)\)-supersymmetric Affine Toda field theory constructed here does not exist. Nevertheless, one can construct a large number of classical solutions of \((1,0)\)-supersymmetric \(A^{(1)}_N\) Affine Toda field theory by observing that the configurations for which the fermions vanish and the boson fields satisfy the field equations of the associated bosonic theory solve the field equations of the \((1,0)\)-supersymmetric theory. A con-
sequence of this is that for imaginary coupling constant $\beta$, the (1,0)-supersymmetric Affine Toda field theory will admit soliton solutions [21, 22] which interpolate between the different classical vacua of the theory. We remark however that not all classical vacua of the bosonic Affine Toda field theory with imaginary coupling are supersymmetric.

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