Construction for both self-dual codes and LCD codes

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Abstract

From a given \([n, k]\) code \(C\), we give a method for constructing many \([n, k]\) codes \(C'\) such that the hull dimensions of \(C\) and \(C'\) are identical. This method can be applied to constructions of both self-dual codes and linear complementary dual codes (LCD codes for short). Using the method, we construct 661 new inequivalent extremal doubly even \([56, 28, 12]\) codes. Furthermore, constructing LCD codes by the method, we improve some of the previously known lower bounds on the largest minimum weights of binary LCD codes of length \(n = 26, 28 \leq n \leq 40\).

Keywords. Linear complementary dual code, Self-dual code, Doubly even code, Hull dimension.

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1 Introduction

Let \(\mathbb{F}_q\) be the finite field of order \(q\), where \(q\) is a prime power. An \([n, k]\) code \(C\) over \(\mathbb{F}_q\) is said to be a self-dual code if \(C = C^\perp\), where \(C^\perp\) denotes the dual code of \(C\). A code is said to be doubly even if all codewords have weights divisible by four. Mallows and Sloane [22] proved that the minimum weight \(d\) of a binary doubly even self-dual code of length \(n\) is upper bounded by \(d \leq 4\lfloor n/24 \rfloor + 4\). A binary doubly even self-dual code meeting the bound is called extremal. An \([n, k]\) code \(C\) over \(\mathbb{F}_q\) is said to be an LCD code if \(C \cap C^\perp = \{0_n\}\), where \(0_n\) denotes the zero vector of length \(n\). The concept of LCD codes was invented by Massey [23]. A binary LCD \([n, k]\) code is said to be optimal if it has the largest minimum weight among all binary LCD \([n, k]\) codes.

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Although the definitions say that self-dual codes and LCD codes are quite different classes of codes, codes of both classes have similar properties. For example, it is known that both self-dual codes and LCD codes are characterized by their generator matrices. Furthermore, self-dual codes are codes with maximal hull dimension and LCD codes are codes with minimal hull dimension, where the hull of a code $C$ is defined as $\text{Hull}(C) = C \cap C^\perp$. Recently Harada [16] gave a method for constructing LCD codes modifying known methods for self-dual codes in [14, Theorem 2.2] and [18, Theorem 2.2] and constructed 15 optimal binary LCD $[n,k]$ codes.

In this paper, we give a method for constructing many $[n,k]$ code $C'$ from a given $[n,k]$ code $C$ such that $\dim(\text{Hull}(C)) = \dim(\text{Hull}(C'))$. This method can be applied to constructions of both self-dual codes and LCD codes. It is shown that the method is a generalized version of [14, Theorem 2.2], [16, Theorem 3.3] and [18, Theorem 2.2]. As an application, we construct 661 new inequivalent extremal doubly even $[56,28,12]$ codes. Furthermore, constructing LCD codes by the method, we improve some of the previously known lower bounds on the largest minimum weights of binary LCD codes of length $n = 26, 28 \leq n \leq 40$.

This paper is organized as follows: In Section 2 we recall some basic results on self-dual codes, LCD codes and hulls of codes. In Section 3 we provide the construction method. Furthermore, in Section 4 we state basic properties of the construction method. In Section 5 we construct 661 new inequivalent extremal doubly even $[56,28,12]$ from six bordered double circulant doubly even $[56,28,12]$ codes. In Section 6 we improve some of the largest minimum weights among all binary LCD $[n,k]$ codes with length $n = 26, 28 \leq n \leq 40$, which were recently studied by Bouyuklieva [7] and Harada [16]. All computations in this paper were performed in MAGMA [6].

2 Preliminaries

Let $F_q$ be the finite field of order $q$, where $q$ is a prime power and let $F_q^n$ be the vector space of all $n$-tuples over $F_q$. A $k$-dimensional subspace of $F_q^n$ is said to be an $[n,k]$ code over $F_q$. Especially, codes over $F_2$ are said to be binary codes. Let $C$ be an $[n,k]$ code over $F_q$. The parameters $n, k$ are said to be the length, the dimension of $C$ respectively. A vector in $C$ is said to be a codeword. The weight of $x = (x_1,x_2,\ldots,x_n) \in F_q^n$ is defined as $\text{wt}(x) = \# \{ i \mid x_i \neq 0 \}$. The minimum weight of $C$ is defined as $\text{wt}(C) = \min \{ \text{wt}(x) \mid x \in C, x \neq 0_n \}$. If the minimum weight of $C$ equals to $d$, then $C$ is said to be an $[n,k,d]$ code over $F_q$. A code $C$ is said to be an even code if all codewords have even weights. Also, a code is said to be a doubly even code if all codewords have weights divisible by four. Two $[n,k]$ codes $C_1,C_2$ over $F_q$ are equivalent if there exists a monomial matrix $M$ such that $C_2 = \{ cM \mid c \in C_1 \}$. The equivalence of two codes $C_1,C_2$ is...
denoted by $C_1 \simeq C_2$. A generator matrix of a code $C$ is any matrix whose rows form a basis of $C$.

The dual code $C^\perp$ of an $[n,k]$ code $C$ over $\mathbb{F}_q$ is defined as $C^\perp = \{ x \in \mathbb{F}_q^n \mid (x,y) = 0 \text{ for all } y \in C \}$, where $(x,y)$ is the standard inner product. If $C \subseteq C^\perp$, then $C$ is said to be a self-orthogonal code. If $C = C^\perp$, then $C$ is said to be a self-dual code. A binary self-dual code $C$ is doubly even if and only if $n \equiv 0 \pmod{8}$, where $n$ denotes the length of $C$. Mallows and Sloane [22] proved that the minimum weight $d$ of a binary doubly even self-dual code of length $n$ is upper bounded by $d \leq 4 \lfloor n/24 \rfloor + 4$. A binary doubly even self-dual code meeting the bound is called extremal.

Lemma 2.1 ([20, Theorem 1.4.8]). Let $C$ be a binary code. Then the following holds:

(i) If $C$ is a self-orthogonal code and has a generator matrix each of whose rows has weight divisible by four, then $C$ is doubly even.

(ii) If $C$ is doubly even, then $C$ is a self-orthogonal code.

A pure double circulant code has a generator matrix of the form $\begin{pmatrix} I_k & R \end{pmatrix}$ and a bordered double circulant code has a generator matrix of the form

$$\begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & \vdots & R \\ 1 \end{pmatrix},$$

where $I_k$ denotes the identity matrix of order $k$ and $R$ is a circulant matrix. These two families of codes are collectively called double circulant codes. Harada, Gulliver and Kaneta [17] showed that there exist exactly nine inequivalent extremal double circulant doubly even [56,28,12] codes and all of them are bordered double circulant codes. In Section 5 we construct extremal doubly even self-dual [56,28,12] codes from six inequivalent extremal double circulant doubly even [56,28,12] codes $D_{11}, C_{56,1}, \ldots, C_{56,5}$. Generator matrices of $D_{11}, C_{56,1}, \ldots, C_{56,5}$ are of the form (1) with first rows

$$(0001010110111000111111111111), (0000000000001100101011110111),$$

$$(0000000010110111111100101111011111), (000000100111100111110110101111),$$

$$(00000101001111101011011110111101111), (00001010011011001011011011011011),$$

respectively.

An $[n,k]$ code $C$ over $\mathbb{F}_q$ is said to be an LCD code if $C \cap C^\perp = \{0_n\}$. The concept of LCD codes was invented by Massey [23]. LCD codes have been applied in data storage, communication systems and cryptography. For example, it is known that binary LCD codes can be used against side-channel attacks and fault injection attacks [9]. A binary LCD $[n,k]$ code is
said to be optimal if it has the largest minimum weight among all binary LCD \([n,k]\) codes. Massey \[23\] gave the following characterization of LCD codes.

**Theorem 2.2** (Massey \[23\]). Let \(C\) be an \([n,k]\) code over \(\mathbb{F}_q\) and let \(G\) be a generator matrix of \(C\). Then \(C\) is an LCD code if and only if the \(k \times k\) matrix \(GG^T\) is nonsingular.

The hull of a code \(C\) is defined as \(\text{Hull}(C) = C \cap C^\perp\). By definition, it follows that self-dual codes are codes with maximal hull dimension and LCD codes are codes with minimal hull dimension.

**Lemma 2.3** (\[13, Proposition 3.1\]). Let \(C\) be an \([n,k]\) code over \(\mathbb{F}_q\) with generator matrix \(G\). Then
\[
\text{rank}(GG^T) = k - \dim(\text{Hull}(C)).
\]

### 3 Construction method

Let \(C\) be an \([n,k]\) code over \(\mathbb{F}_q\) with generator matrix \((I_k A)\) and let \(x, y \in \mathbb{F}_q^{n-k}\). We denote by \(r_i\) the \(i\)-th row of \(A\). Define an \(n \times (n-k)\) matrix \(A(x,y)\), where the \(i\)-th row \(r'_i\) is defined as follows:
\[
r'_i = r_i + (r_i,y)x - (r_i,x)y.
\]
We denote by \(C(A(x,y))\) the code with generator matrix \((I_k A(x,y))\).

**Remark 3.1.** With the above notation, suppose that \(x = 0_{n-k}\) or \(y = 0_{n-k}\). Then it holds that \(A(x,y) = A\). Hereafter, we assume that \(x \neq 0_{n-k}\) and \(y \neq 0_{n-k}\).

**Theorem 3.2.** Let \(C\) be an \([n,k]\) code over \(\mathbb{F}_q\) with generator matrix \(G = (I_k A)\) and let \(x, y \in \mathbb{F}_q^{n-k}\). Suppose that \((x,x) = (y,y) = (x,y) = 0\). Then
\[
\dim(\text{Hull}(C(A(x,y)))) = \dim(\text{Hull}(C)).
\]

**Proof.** We denote by \(r_i, r'_i\) the \(i\)-th rows of \(A, A(x,y)\) respectively. It holds that
\[
(r'_i, r'_j) = (r_i + (r_i,y)x - (r_i,x)y, r_j + (r_j,y)x - (r_j,x)y)
= (r_i, r_j) + (r_j,y)(r_i,x) - (r_j,x)(r_i,y) + (r_i,y)(x,r_j) - (r_i,x)(y,r_j)
= (r_i, r_j).
\]
Therefore it follows that
\[
\begin{align*}
(I_k A(x,y)) (I_k A(x,y))^T &= I_k + A(x,y)A(x,y)^T \\
&= I_k + AA^T \\
&= (I_k A) (I_k A)^T.
\end{align*}
\]
By Lemma \[23\], the result follows. \(\square\)
Corollary 3.3. Let $C$ be an $[n, k]$ code over $\mathbb{F}_q$ with generator matrix $G = (I_k \ A)$ and let $x, y \in \mathbb{F}_{q-k}^n$. Suppose that $(x, x) = (y, y) = (x, y) = 0$. Then $C(A(x, y))$ is a self-orthogonal code if and only if $C$ is a self-orthogonal code.

Proof. It holds that $C$ is a self-orthogonal code if and only if $\dim(\text{Hull}(C)) = k$. The result follows from Theorem 3.2. □

Remark 3.4. With the notation of Corollary 3.3, suppose that $C$ is a binary self-dual code with length $n \equiv 0 \pmod{4}$. Then it follows that $(x, x) = (x, 1_{n-k}) = (1_{n-k}, 1_{n-k}) = 0$ if $x$ has an even weight. Let $r_i, r'_i$ be the $i$-th rows of $A, A(x, 1_{n-k})$ respectively. Since $C$ is an even code, $\text{wt}(r_i) \equiv 1 \pmod{2}$ for all $1 \leq i \leq n$. Therefore we obtain the following:

$$r'_i = r_i + (r_i, 1_{n-k})x + (r_i, x)1_{n-k} = r_i + x + (r_i, x)1_{n-k},$$

which shows that $A(x, 1_{n-k})$ is identical to $B_{1', \Gamma}'$, the first case of [14, Theorem 2.2]. Therefore Corollary 3.3 is a generalized version of the first case of [14, Theorem 2.2].

Corollary 3.5. Let $C$ be an $[n, k]$ code over $\mathbb{F}_q$ with generator matrix $G = (I_k \ A)$ and let $x, y \in \mathbb{F}_{q-k}^n$. Suppose that $(x, x) = (y, y) = (x, y) = 0$. Then $C(A(x, y))$ is an LCD code if and only if $C$ is an LCD code.

Proof. It holds that $C$ is an LCD code if and only if $\dim(\text{Hull}(C)) = k$. The result follows from Theorem 3.2. □

Remark 3.6. With the notation of Corollary 3.5, suppose that $C$ is a binary even LCD code and $n - k$ is even. Then it follows that $(x, x) = (x, 1_{n-k}) = (1_{n-k}, 1_{n-k}) = 0$ if $x$ has an even weight. Let $r_i, r'_i$ be the $i$-th rows of $A, A(x, 1_{n-k})$ respectively. Since $C$ is an even code, $\text{wt}(r_i) \equiv 1 \pmod{2}$ for all $1 \leq i \leq n$. Therefore we obtain the following:

$$r'_i = r_i + (r_i, 1_{n-k})x + (r_i, x)1_{n-k} = r_i + x + (r_i, x)1_{n-k},$$

which shows that $A(x, 1_{n-k})$ is identical to $A(x)$ in [16, Theorem 3.3]. Therefore Corollary 3.5 is a generalized version of [16, Theorem 3.3].

Lemma 3.7 ([20, Theorem 1.4.3]). Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}_q^n$. Then $\text{wt}(x+y) = \text{wt}(x) + \text{wt}(y) - 2 \text{wt}(x*y)$, where $x*y = (x_1y_1, x_2y_2, \ldots, x_ny_n)$.

Lemma 3.8. Let $C$ be a binary $[n, k]$ code with generator matrix $(I_k \ A)$ and let $x, y \in \mathbb{F}_{q-k}^n$. Suppose that $\text{wt}(x) \equiv \text{wt}(y) \equiv 0 \pmod{4}$ and $(x, y) = 0$. Then $\text{wt}(r'_i) \equiv \text{wt}(r_i) \pmod{4}$, where $r_i, r'_i$ denote the $i$-th rows of $A, A(x, y)$ respectively.
Proof. It holds that
\[
\text{wt}(r_i \ast ((r_i, y)x - (r_i, x)y)) = (r_i, (r_i, y)x - (r_i, x)y) \\
= (r_i, y)(r_i, x) - (r_i, x)(r_i, y) \\
= 0,
\]
\[
\text{wt}((r_i, y)x \ast (r_i, x)y) = ((r_i, y)x, (r_i, x)y) \\
= (r_i, y)(r_i, x)(x, y) \\
\equiv 0 \pmod{2},
\]
where we regard \(r_i, x, y\) as vectors in \(\mathbb{Z}^{n-k}\). Therefore, by Lemma 3.7, it follows that
\[
\text{wt}(r_i') = \text{wt}(r_i + (r_i, y)x - (r_i, x)y) \\
\equiv \text{wt}(r_i) + \text{wt}((r_i, y)x - (r_i, x)y) - 2 \text{wt}(r_i \ast ((r_i, y)x - (r_i, x)y)) \\
\equiv \text{wt}(r_i) \equiv 3 \pmod{4}.
\]
This completes the proof.

\[\square\]

**Theorem 3.9.** Let \(C\) be a binary \([n, k]\) code with generator matrix \((I_k \ A)\) and let \(x, y \in \mathbb{F}_2^{n-k}\). Suppose that \(\text{wt}(x) \equiv \text{wt}(y) \equiv 0 \pmod{4}\) and \((x, y) = 0\). Then \(C(A(x, y))\) is a doubly even code if and only if \(C\) is a doubly even code.

**Proof.** We denote by \(r_i, r_i'\) the \(i\)-th rows of \(A, A(x, y)\) respectively. Suppose that \(C\) is a doubly even code. Then, by Corollary 3.3 and the second part of Lemma 2.1, \(C(A(x, y))\) is a self-orthogonal code. Furthermore, by Lemma 3.8, \(\text{wt}(r_i') \equiv \text{wt}(r_i) \equiv 3 \pmod{4}\) follows for all \(1 \leq i \leq n\). Therefore it holds that \(C(A(x, y))\) is a doubly even code by the first part of Lemma 2.1. By the same argument, the converse holds.

**Remark 3.10.** With the notation of Theorem 3.9, suppose that \(C\) is a binary doubly even self-dual code. Then it follows that \(\text{wt}(x) \equiv \text{wt}(1_{n-k}) \equiv 0 \pmod{4}\) and \((x, 1_{n-k}) = 0\) if \(x\) has a weight divisible by four. Let \(r_i, r_i'\) be the \(i\)-th rows of \(A, A(x, 1_{n-k})\) respectively. Since \(C\) is doubly even, \(\text{wt}(r_i) \equiv 3 \pmod{4}\) for all \(1 \leq i \leq n\). Therefore we obtain the following:
\[
r_i' = r_i + (r_i, 1_{n-k})x + (r_i, x)1_{n-k} \\
= r_i + x + (r_i, x)1_{n-k},
\]
which shows that \(A(x, 1_{n-k})\) is identical to \(A'_{\Gamma}\) in [18, Theorem 2.2], where \(m\) is even. Therefore Theorem 3.9 is a generalized version of [18, Theorem 2.2], where \(m\) is even.
4 Basic properties

In this section, we state basic properties of the construction method given in Section 3. Let $x, y \in \mathbb{F}_q^m$. Define the following $m \times m$ matrix $M(x,y)$:

$$M(x,y) = I_m + y^T x - x^T y.$$ 

**Lemma 4.1.** Let $C$ be an $[n, k]$ code over $\mathbb{F}_q$ with generator matrix $(I_k ~ A)$ and let $x, y \in \mathbb{F}_q^{n-k}$. Then it holds that

$$A(x, y) = AM(x, y).$$

**Proof.** For a matrix $M$, we denote by $M_{i,j}$ the $(i, j)$-entry of $M$. By definition it follows that

$$(AM(x,y))_{i,j} = \sum_{l=1}^{n-k} A_{i,l} M(x,y)_{l,j}.$$ 

On the other hand, it holds that

$$r'_i = r_i + (r_i, y)x - (r_i, x)y$$
$$= r_i(I_{n-k} + y^T x - x^T y)$$
$$= (A_{i,1}, A_{i,2}, \ldots, A_{i,n})M(x,y)$$

$$= (\sum_{l=1}^{n-k} A_{i,l} M(x,y)_{l,1}, \sum_{l=1}^{n-k} A_{i,l} M(x,y)_{l,2}, \ldots, \sum_{l=1}^{n-k} A_{i,l} M(x,y)_{l,n}),$$

where $r_i, r'_i$ denote the $i$-th rows of $A, A(x, y)$ respectively. Therefore it holds that

$$A(x, y)_{i,j} = \sum_{l=1}^{n-k} A_{i,l} M(x,y)_{l,j}.$$ 

This completes the proof. 

**Theorem 4.2.** Let $C$ be an $[n, k]$ code over $\mathbb{F}_q$ with generator matrix $(I_k ~ A)$ and let $x, y \in \mathbb{F}_q^{n-k}$. Suppose that $(x, x) = (y, y) = (x, y) = 0$. Then the following holds:

$$C(A(x,y)) = C(A(-x,-y)), \tag{2}$$
$$C(A(y, x)) = C(A(x, y)) = C(A(-x, y)). \tag{3}$$

**Proof.** For $R$, it holds that

$$M(-x,-y) = I_m + (-y)^T (-x) - (-x)^T (-y)$$
$$= I_m + y^T x - x^T y$$
$$= M(x,y).$$
For (3), the following holds:

\[
M(x, -y) = I_m - y^T x + x^T y \\
= I_m + x^T y - y^T x \\
= M(y, x), \\
M(-x, y) = I_m - y^T x + x^T y \\
= I_m + x^T y - y^T x \\
= M(y, x).
\]

This completes the proof.

In the following sections, we apply the construction method only to binary codes. However, for codes over \( F_q (q \geq 3) \), Theorem 4.2 reduces computations.

5 Extremal binary doubly even \([56, 28, 12]\) codes

In this section we are concerned only with binary codes. Therefore we omit the term “binary”. Bhargava, Young and Bhargava [5] constructed an extremal doubly even \([56, 28, 12]\) code. Yorgov [26] proved that there exist exactly 16 inequivalent extremal doubly even \([56, 28, 12]\) codes with automorphisms of order 13. As stated in [13], one of the 16 codes in [26] is equivalent to the code in [5]. Bussemaker and Tonchev [8] constructed 6 extremal doubly even \([56, 28, 12]\) codes. As stated in [15], the first code among the six codes in [8] is equivalent to the code in [5]. Moreover, Kimura [21] showed that 5th and 6th codes in [8] are equivalent. Harada [14] constructed 137 inequivalent extremal doubly even \([56, 28, 12]\) codes. Harada, Gulliver and Kaneta [17] showed that there exist exactly nine inequivalent extremal double circulant doubly even \([56, 28, 12]\) codes. Harada [14] constructed 1122 inequivalent extremal doubly even \([56, 28, 12]\) codes. This result is a generalization of [14] and for any code \( C \) in [14] there exists a code \( C' \) in [15] such that \( C \simeq C' \). Yankov and Russeva [25] proved that there exist exactly 4202 inequivalent extremal doubly even \([56, 28, 12]\) codes having automorphisms of order 7. As stated in [25], one of the 4202 codes in [25] is equivalent to a code in [14]. Yankov and Lee [24] proved that there exist exactly 3763 inequivalent extremal \([56, 28, 12]\) codes having automorphisms of order 5. Therefore the number of previously known extremal doubly even \([56, 28, 12]\) codes is 9115, as stated in [24, Proposition 7].

In [15], Harada applied [18, Theorem 2.2] to six inequivalent extremal double circulant doubly even \([56, 28, 12]\) codes \( D11, C_{56,1}, \ldots, C_{56,5} \). As stated earlier, Theorem 3.9 is a generalized version of [18, Theorem 2.2]. In this section, we apply Theorem 3.9 to \( D11, C_{56,1}, \ldots, C_{56,5} \) and construct...
661 new inequivalent extremal doubly even [56, 28, 12] codes. This illustrates the effectiveness of Theorem 3.9.

In order to illustrate our method, we consider the code $D_{11}$ as an example. Let $y_i$ denote the vector of length 28 such that $y_i = (0, \ldots, 0, 1, \ldots, 1)$, $wt(y_i) = i$. Applying Theorem 3.9 to $y_4$ and all $x \in \mathbb{F}_2^{28}$ such that $x \neq 0_{28}$ and $(x, x) = (x, y_4) = 0$, we constructed 45 inequivalent extremal doubly even [56, 28, 12] codes. Furthermore, applying Theorem 3.9 to $y_8$ ($i = 8, 12, 16, 20, 24$) and all $x \in \mathbb{F}_2^{28}$ such that $x \neq 0_{28}$ and $(x, x) = (x, y_8) = 0$, we constructed 45, 19, 15, 2, 33, 4 inequivalent extremal doubly even [56, 28, 12] codes respectively.

By the following method, we verified that the above codes are all inequivalent. For an extremal doubly even [56, 28, 12] code $C$, we define $M = (m_{i,j})$ to be an $8196 \times 56$ matrix whose rows composed of codewords of $C$ with weight 12. Furthermore, for a positive integer $t$, we define $N_t = \# \{ (j_1, j_2, j_3, j_4) \in \binom{56}{4} | \sum_{i=1}^{8196} m_{i,j_1}m_{i,j_2}m_{i,j_3}m_{i,j_4} = t \}$.

Harada [15] showed that two extremal doubly even [56, 28, 12] codes $C_1, C_2$ are inequivalent if the sequences $(N_1, N_2, \ldots)$ constructed from $C_1, C_2$ are distinct. According to this result, we compared the sequence $(N_1, N_2, \ldots, N_{56})$ for the classification. Consequently we found no pair of codes whose sequences are identical. Therefore we verified that the number of inequivalent codes constructed from $D_{11}$ is 118. By the same method, we constructed inequivalent extremal doubly even [56, 28, 12] codes from $C_{56,i}$ ($i = 1, 2, \ldots, 5$). In Table 1 we show the number of inequivalent codes constructed by this method. We denote the inequivalent codes constructed from $D_{11}, C_{56,1}, \ldots, C_{56,5}$ by $D_i$ ($i = 1, 2, \ldots, 118$), $E_i$ ($i = 1, 2, \ldots, 56$), $F_i$ ($i = 1, 2, \ldots, 105$), $G_i$ ($i = 1, 2, \ldots, 59$), $H_i$ ($i = 1, 2, \ldots, 212$), $K_i$ ($i = 1, 2, \ldots, 115$) respectively. The $x, y$ in Corollary 3.9 for all codes we constructed can be obtained electronically from https://www.math.is.tohoku.ac.jp/~mharada/Ishizuka/56.txt.

Comparing sequences $(N_1, N_2, \ldots, N_{56})$, we found that there exist four pairs

| Code | $y_4$ | $y_8$ | $y_{12}$ | $y_{16}$ | $y_{20}$ | $y_{24}$ | Total |
|------|-------|-------|---------|---------|---------|---------|-------|
| $D_{11}$ | 45    | 19    | 15      | 2       | 33      | 4       | 118   |
| $C_{56,1}$ | 16    | 3     | 1       | 0       | 10      | 26      | 56    |
| $C_{56,2}$ | 34    | 27    | 26      | 1       | 3       | 14      | 105   |
| $C_{56,3}$ | 10    | 0     | 23      | 2       | 0       | 24      | 59    |
| $C_{56,4}$ | 10    | 109   | 17      | 58      | 2       | 16      | 212   |
| $C_{56,5}$ | 17    | 53    | 25      | 11      | 5       | 4       | 115   |
of codes \((D_{115}, K_{112}), (D_{116}, K_{113}), (D_{117}, K_{114}), (D_{118}, K_{115})\) whose sequences are identical. By the MAGMA function IsIsomorphic, we verified that two codes of all the four pairs are equivalent. Therefore the number of inequivalent extremal doubly even \([56, 28, 12]\) codes constructed as above is 661.

Finally, we verified that the 661 codes are inequivalent to any of the previously known extremal doubly even \([56, 28, 12]\) codes as follows: By the MAGMA function AutomorphismGroup, we verified that the 661 codes have automorphism groups of order 1. Consequently it follows that the 661 codes are inequivalent to any of the codes in \([24], [25], [26]\). Furthermore we verified that all the codes except \(D_{118}\) have sequences \((N_1, N_2, \ldots, N_{56})\) different from that of any code in \([5], [8], [15], [17], [26]\). The sequence \((N_1, N_2, \ldots, N_{56})\) of \(D_{118}\) is identical to that of the 25th code constructed from \(C_{56,2}\) in \([15]\). However we verified by the Magma function IsIsomorphic that the two codes are inequivalent. Consequently it follows that the 661 codes are inequivalent to any of the codes in \([5], [8], [15], [17], [26]\). As stated in the beginning of this section, the number of the previously known inequivalent doubly even \([56, 28, 12]\) codes is 9115. Therefore we have Proposition 5.1.

**Proposition 5.1.** There exist at least 9776 inequivalent extremal doubly even \([56, 28, 12]\) codes.

## 6 Optimal binary LCD codes of length \(n = 26, 28 \leq n \leq 40\)

In this section we are concerned only with binary codes. Therefore we omit the term “binary”. Let \(d_{\text{LCD}}(n, k)\) denote the largest minimum weight among all LCD \([n, k]\) code. Galvez, Kim, Lee, Roe and Won \([11]\), Harada and Saito \([19]\), Araya and Harada \([1]\) determined the exact value of \(d_{\text{LCD}}(n, k)\) for \(n \leq 12, 13 \leq n \leq 16, 17 \leq n \leq 24\) respectively. Bouyuklieva \([7]\) determined the exact value of \(d_{\text{LCD}}(n, k)\) for \(n = 25, 27\) and gave \(d_{\text{LCD}}(n, k)\) for \(26, 28 \leq n \leq 40\). Galvez, Kim, Lee, Roe and Won \([11]\), Harada and Saito \([19]\), Araya and Harada \([1]\), Araya, Harada and Saito \([3]\) determined the exact value of \(d_{\text{LCD}}(n, k)\) for \(k = 2, 3, 4, 5\) respectively. Also, Dougherty, Kim, Ozkaya, Sok and Solé \([10]\), Araya and Harada \([2]\), Araya, Harada and Saito \([4]\) determined the exact value of \(d_{\text{LCD}}(n, k)\) for \(k = n-1, k \in \{n-2, n-3, n-4\}, k = n-5\) respectively. For all \(n = 26, 28 \leq n \leq 40\), Bouyuklieva \([7]\) determined the exact value of \(d_{\text{LCD}}(n, k)\) for \(5 \leq k \leq 8\).

Recently Harada \([16]\) constructed 15 optimal LCD codes by \([16, \text{Theorem 3.3}]\). As stated earlier, Corollary 3.5 is a generalized version of \([16, \text{Theorem 3.3}]\). In this section, we apply Corollary 3.5 in order to improve some of the previously known lower bounds on \(d_{\text{LCD}}(n, k)\) for \(n = 26, 28 \leq n \leq 40\) and \(9 \leq k \leq n - 6\). For a code \(C\), we denote by \(C^T, C_T\) the punctured,
the shortened codes of \( C \) on a set of coordinates \( T \) respectively. In this section, shortened codes, punctured codes were constructed by the MAGMA functions \texttt{ShortenCode}, \texttt{PunctureCode} respectively.

In order to obtain lower bounds, we use the following method: First, by the MAGMA function \texttt{BestKnownLinearCode}, we obtained a \([49, 32, 7]\) code \( C_{49,32,7} \), a \([42, 14, 13]\) code \( C_{42,14,13} \) and a \([51, 28, 9]\) code \( C_{51,28,9} \). Generator matrices of these codes can be obtained electronically from \url{https://www.math.is.tohoku.ac.jp/~mharada/Ishizuka/generator.txt}.

Then we verified that \((C_{49,32,7})_{S_1} \), \((C_{42,14,13})_{S_2} \), \((C_{51,28,9})_{S_3} \) are an LCD \([37, 22, 5]\) code, an LCD \([38, 13, 11]\) code, an LCD \([40, 22, 6]\) code respectively, where the set of coordinates \( P_i, S_i \) \((i = 1, 2, 3)\) are given in Table 2. Define \( \{2, 5\} \), \( \{1, 2, 4\} \), \( \{1, 2, 3, 4, 5, 6, 7\} \), \( \{1, 2, 3, 4, 5\} \), \( \{1, 2, 3, 4, 5, 6, 7\} \), \( \{1, 2, 3, 4, 5, 6, 7\} \) as in Figure 2. Then \((I_{22} \ A_{37,22,5}) \), \((I_{13} \ A_{38,13,10}) \), \((I_{22} \ A_{40,22,6}) \) are generator matrices of the LCD \([37, 22, 5]\), the LCD \([38, 13, 10]\), the LCD \([40, 22, 6]\) codes respectively. Applying Corollary 3.5 to \((I_{22} \ A_{37,22,5}) \), \((I_{13} \ A_{38,13,10}) \), \((I_{22} \ A_{40,22,6}) \), we found an LCD \([37, 22, 6]\) code \( C_{37,22,6} \), an LCD \([38, 13, 11]\) code \( C_{38,13,11} \), an LCD \([40, 22, 7]\) code \( C_{40,22,7} \) respectively. The vectors \( x, y \) in Corollary 3.5 are listed in Table 3. Therefore we obtain Proposition 6.1.

**Proposition 6.1.**

(i) There exists an LCD \([37, 22, 6]\) code.

(ii) There exists an LCD \([38, 13, 11]\) code.

(iii) There exists an LCD \([40, 22, 7]\) code.

Table 2: \( P_i, S_i \) for \( i = 1, 2, 3 \)

| \( i \) | \( P_i \) | \( S_i \) |
|---|---|---|
| 1 | \{2, 5\} | \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\} |
| 2 | \{1, 2, 4\} | \{3\} |
| 3 | \{1, 4, 5, 6, 7\} | \{1, 2, 3, 4, 5, 7\} |

Table 3: \( C_{n,k,d} \) with \( x, y \)

| \( C_{n,k,d} \) | \( x \) | \( y \) |
|---|---|---|
| \( C_{37,22,6} \) | \((0101101011011111)\) | \((1100100000001)\) |
| \( C_{38,13,11} \) | \((001001110010001011000100)\) | \((1110100001110111110001101)\) |
| \( C_{40,22,7} \) | \((001111011011010011)\) | \((0010111000111000001)\) |

From the previously known results of \( d_{\text{LCD}}(n, k) \) described in the beginning of this section, we are concerned only with \( d_{\text{LCD}}(n, k) \) for \( n =
26, 28 \leq n \leq 40 \text{ and } 9 \leq k \leq n - 6. \text{ Let } d_K(n, k) \text{ denote the largest minimum weight among currently known } [n, k] \text{ codes. By the Magma function BestKnownLinearCode, one can construct an } [n, k, d_K(n, k)] \text{ code for all } n = 26, 28 \leq n \leq 40 \text{ and } 9 \leq k \leq n - 6. \text{ In addition, by considering shortened codes and punctured codes of } [n, k, d_K(n, k)] \text{ codes, we found LCD } [n, k, d_K(n, k)] \text{ codes for }

(n, k, d_K(n, k)) = (29, 11, 9), (30, 12, 9), (31, 11, 10), (31, 12, 10), (31, 13, 9), (31, 21, 5), (32, 12, 10), (32, 22, 5), (33, 23, 5), (34, 9, 13), (34, 13, 10), (34, 14, 10), (35, 22, 6), (35, 24, 5), (36, 15, 10), (36, 16, 10), (36, 22, 6), (36, 24, 6), (36, 25, 5), (37, 23, 6), (37, 24, 6), (37, 26, 5), (38, 24, 6), (38, 25, 6), (38, 26, 6), (38, 27, 5), (39, 18, 10), (39, 25, 6), (39, 26, 6), (39, 28, 5), (40, 26, 6), (40, 28, 6), (40, 29, 5).

Consequently we obtain Proposition 6.2

**Proposition 6.2.** There exists an optimal LCD \([n, k, d]\) code for \((n, k, d)\) listed in (4).

By a method similar to that given in the above, we found LCD \([n, k, d_K(n, k) - 1]\) codes and LCD \([n, k, d_K(n, k) - 2]\) codes for

\((n, k, d_K(n, k) - 1) = (30, 11, 9), (31, 15, 7), (32, 13, 9), (32, 15, 7), (32, 16, 7), (32, 21, 5), (33, 14, 9), (33, 15, 8), (33, 16, 7), (33, 21, 5), (34, 15, 9), (34, 16, 8), (34, 17, 7), (34, 23, 5), (35, 9, 13), (35, 16, 9), (35, 17, 7), (35, 18, 7), (35, 21, 5), (35, 23, 5), (36, 17, 8), (36, 18, 7), (36, 19, 7), (36, 21, 6), (37, 17, 9), (37, 18, 8), (37, 19, 7), (37, 20, 7), (37, 25, 5), (38, 10, 13), (38, 17, 9), (38, 18, 9), (38, 19, 8), (38, 20, 7), (38, 21, 7), (38, 23, 6), (39, 11, 13), (39, 14, 11), (39, 17, 10), (39, 19, 9), (39, 20, 8), (39, 21, 7), (39, 22, 7), (39, 24, 6), (39, 27, 5), (39, 33, 2), (40, 9, 15), (40, 12, 13), (40, 18, 10), (40, 19, 9), (40, 20, 9), (40, 23, 7), (40, 25, 6), (40, 27, 5), (40, 33, 3), (40, 34, 2).

\((n, k, d_K(n, k) - 2) = (37, 21, 6), (38, 22, 6), (39, 13, 11), (39, 23, 6), (40, 11, 13), (40, 13, 12), (40, 14, 11), (40, 17, 10), (40, 21, 7), (40, 24, 6).

Consequently we obtain Proposition 6.3

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Proposition 6.3. There exists an LCD $[n,k,d]$ code for $(n,k,d)$ listed in (4) and (6).

In Tables 4 through 6 we give $d_{LCD}(n,k)$ for $n = 26, 28 \leq n \leq 40$ and $9 \leq k \leq n - 6$. In order to obtain upper bounds, we use the following:

$$d_{LCD}(n,k) \leq d(n,k),$$

where $d(n,k)$ denotes the largest minimum weight among all $[n,k]$ codes. The values of $d(n,k)$ are given in [12]. For the parameters listed in Proposition 6.1, we mark $d_{LCD}(n,k)$ by * in Tables 4 through 6. Furthermore, for the parameters given in Propositions 6.1 through 6.3 we give $d_{LCD}(n,k)$ in boldface. For each of the parameters, an LCD code can be obtained electronically from https://www.math.is.tohoku.ac.jp/~mharada/Ishizuka/LCD.txt.

Table 4: $d_{LCD}(n,k)$, where $26 \leq n \leq 40, 9 \leq k \leq 17$

| $n \setminus k$ | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 26              | 9   | 8   | 8   | 8   | 7   | 6   | 5–6 | 5   | 4   |
| 27              | 9   | 9   | 8   | 8   | 7   | 6   | 6   | 5   |     |
| 28              | 10  | 10  | 8   | 8   | 7   | 6   | 6   | 5–6 |     |
| 29              | 10  | 10  | 9   | 8   | 8   | 6   | 6   |     |     |
| 30              | 11  | 10  | 9–10| 9   | 8   | 6–7 | 6   |     |     |
| 31              | 11  | 10  | 10  | 10  | 9   | 8   | 7–8 | 6–7 | 6   |
| 32              | 12  | 11  | 10  | 10  | 9–10| 8–9 | 7–8 | 7–8 | 6–7 |
| 33              | 12  | 12  | 10–11| 10  | 9–10| 8–9 | 7–8 | 7–8 | 6–8 |
| 34              | 13  | 12  | 11–12| 10–12| 10  | 9–10| 8–9 | 7–8 |     |
| 35              | 13–14| 12–13| 12  | 10–12| 10–11| 10  | 9–10| 9–10| 7–8 |
| 36              | 13–14| 12–14| 12–13| 11–12| 10–12| 10–11| 10  | 10  | 8–9 |
| 37              | 13–15| 12–14| 12–14| 12–13| 10–12| 10–12| 10–11| 10  | 9–10|
| 38              | 14–16| 13–14| 12–14| 12–14| 11*–12| 10–12| 10–12| 10–11| 9–10|
| 39              | 14–16| 14–15| 13–14| 12–14| 11–13| 11–12| 10–12| 10–12| 10–11|
| 40              | 15–16| 14–16| 13–15| 13–14| 12–14| 11–13| 10–12| 10–12| 10–12|

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Table 5: $d_{LCD}(n, k)$, where $26 \leq n \leq 40, 18 \leq k \leq 26$

| $n \setminus k$ | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|-----------------|----|----|----|----|----|----|----|----|----|
| 26              | 4  | 4  | 4  |    |    |    |    |    |    |
| 27              | 4  | 4  | 4  | 3  |    |    |    |    |    |
| 28              | 5  | 4  | 4  | 4  | 3  |    |    |    |    |
| 29              | 6  | 5  | 4  | 4  | 4  | 3  |    |    |    |
| 30              | 6  | 5  | 5  | 4  | 4  | 4  | 3  |    |    |
| 31              | 6  | 6  | 6  | 5  | 4  | 4  | 4  | 3  |    |
| 32              | 6  | 6  | 6  | 5  | 5  | 4  | 4  | 3  |    |
| 33              | 6-7| 6  | 6  | 5  | 6  | 5  | 4  | 4  | 4  |
| 34              | 6-8| 6-7| 6  | 6  | 5  | 6  | 5  | 4  | 4  |
| 35              | 7-8| 7-8| 6-8| 6-7| 6  | 5  | 6  | 5  | 4  |
| 36              | 7-8| 7-8| 6-8| 6-7| 6  | 6  | 6  | 5  | 4  |
| 37              | 8-9| 7-8| 7-8| 6-8| 6*7| 6  | 6  | 5-6| 5  |
| 38              | 9-10| 8-9| 7-8| 6-8| 6-7| 6  | 6  | 6  | 6  |
| 39              | 10 | 9-10| 8-9| 7-8| 7-8| 6-8| 6-7| 6  | 6  |
| 40              | 10-11| 9-10| 9-10| 7-9| 7*-8| 7-8| 6-8| 6-7| 6  |

Table 6: $d_{LCD}(n, k)$, where $33 \leq n \leq 40, 27 \leq k \leq 34$

| $n \setminus k$ | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 |
|-----------------|----|----|----|----|----|----|----|----|
| 33              | 3  |    |    |    |    |    |    |    |
| 34              | 3-4| 3  |    |    |    |    |    |    |
| 35              | 4  | 4  | 3  |    |    |    |    |    |
| 36              | 4  | 4  | 3-4| 3  |    |    |    |    |
| 37              | 4  | 4  | 4  | 4  | 3  |    |    |    |
| 38              | 5  | 4  | 4  | 4  | 3-4| 3  |    |    |
| 39              | 5-6| 5  | 4  | 4  | 4  | 2-3| 2-3|    |
| 40              | 5-6| 6  | 5  | 4  | 4  | 3-4| 2-3|    |
Figure 1: Matrices $A_{37, 22, 5}$, $A_{38, 13, 10}$, $A_{40, 22, 6}$
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