Tannakian Categories attached to abelian Varieties

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Let $k$ be an algebraically closed field $k$, where $k$ is either the algebraic closure of a finite field or a field of characteristic zero. Let $l$ be a prime different from the characteristic of $k$.

**Notations.** For a variety $X$ over $k$ let $D^b_c(X, \mathbb{Q}_l)$ denote the triangulated category of complexes of etale $\mathbb{Q}_l$-sheaves on $X$ in the sense of [5]. For a complex $K \in D^b_c(X, \mathbb{Q}_l)$ let $D(K)$ denote its Verdier dual, and $H^\nu(K)$ denote its etale cohomology $\mathbb{Q}_l$-sheaves with respect to the standard $t$-structure. The abelian subcategory $\text{Perv}(X)$ of middle perverse sheaves is the full subcategory of all $K \in D^b_c(X, \mathbb{Q}_l)$, for which $K$ and its Verdier dual $D(K)$ are contained in the full subcategory $\mathcal{D}^{\leq 0}(X)$ of semi-perverse sheaves, where $L \in D^b_c(X, \mathbb{Q}_l)$ is semi-perverse if and only if $\dim(S_\nu) \leq \nu$ holds for all integers $\nu \in \mathbb{Z}$, where $S_\nu$ denotes the support of the cohomology sheaf $H^{-\nu}(L)$ of $L$.

If $k$ is the algebraic closure of a finite field $\kappa$, then a complex $K$ of etale $\mathbb{Q}_l$-Weil sheaves is mixed of weight $\leq w$, if all its cohomology sheaves $H^\nu(K)$ are mixed etale $\mathbb{Q}_l$-sheaves with upper weights $w(H^\nu(K)) - \nu \leq w$ for all integers $\nu$. It is called pure of weight $w$, if $K$ and its Verdier dual $D(K)$ are mixed of weight $\leq w$. Concerning base fields of characteristic zero, we assume mixed sheaves to be sheaves of geometric origin in the sense of the last chapter of [1], so we still dispose over the notion of the weight filtration and purity and Gabber’s decomposition theorem in this case. In this sense let $\text{Perv}_m(X)$ denote the abelian category of mixed perverse sheaves on $X$. The full subcategory $P(X)$ of $\text{Perv}_m(X)$ of pure perverse sheaves is a semisimple abelian category.
Abelian varieties. Let $X$ be an abelian variety $X$ of dimension $g$ over an algebraically closed field $k$. The addition law of the abelian variety $a : X \times X \to X$ defines the convolution product $K \ast L \in D_c^b(X, \mathbb{Q}_l)$ of two complexes $K$ and $L$ in $D_c^b(X, \mathbb{Q}_l)$ by the direct image

$$K \ast L = R_a(K \boxtimes L).$$

For the skyscraper sheaf $\delta_0$ concentrated at the zero element 0 notice $K \ast \delta_0 = K$.

Translation-invariant sheaf complexes. More generally $K \ast \delta_x = T_x^*(K)$, where $x$ is a closed $k$-valued point in $X$, $\delta_x$ the skyscraper sheaf with support in $\{x\}$ and where $T_x(y) = y + x$ denotes the translation $T_x : X \to X$ by $x$. In fact $T_y^*(K \ast L) \cong K \ast T_y^*(L)$ holds for all $y \in X(k)$. For $K \in D_c^b(X, \mathbb{Q}_l)$ let $Aut(K)$ be the abstract group of all closed $k$-valued points $x$ of $X$, for which $T_x^*(K) \cong K$ holds. A complex $K$ is called translation-invariant, provided $Aut(K) = X(k)$. If $f : X \to Y$ is a surjective homomorphism between abelian varieties, then the direct image $Rf_*(K)$ of a translation-invariant complex is translation-invariant. As a consequence of the formulas above, the convolution of an arbitrary $K \in D_c^b(X, \mathbb{Q}_l)$ with a translation-invariant complex on $X$ is a translation-invariant complex. A translation-invariant perverse sheaf $K$ on $X$ is of the form $K = E[1]$, for an ordinary etale translation-invariant $\mathbb{Q}_l$-sheaf $E$. For a translation-invariant complex $K \in D_c^b(X, \mathbb{Q}_l)$ the irreducible constituents of the perverse cohomology sheaves $^pH^\nu(K)$ are translation-invariant.

Multipliers. The subcategory $T(X)$ of $Perv(X)$ of all perverse sheaves, whose irreducible perverse constituents are translation-invariant, is a Serre subcategory of the abelian category $Perv(X)$. Let denote $\overline{Perv}(X)$ its abelian quotient category and $\overline{P}(X)$ the image of $P(X)$, which is a full subcategory of semisimple objects. The full subcategory of $D_c^b(X, \mathbb{Q}_l)$ of all $K$, for which $^pH^\nu(K) \in T(X)$, is a thick subcategory of the triangulated category $D_c^b(X, \mathbb{Q}_l)$. Let

$$\mathcal{D}_c^b(X, \mathbb{Q}_l)$$

be the corresponding triangulated quotient category, which contains $\overline{Perv}(X)$. Then the convolution product

$$* : \mathcal{D}_c^b(X, \mathbb{Q}_l) \times \mathcal{D}_c^b(X, \mathbb{Q}_l) \to \mathcal{D}_c^b(X, \mathbb{Q}_l)$$

still is well defined, by reasons indicated above.
Definition. A perverse sheaf \( K \) on \( X \) is called a multiplier, if the convolution induced by \( K \)

\[ *K : D^b_c(X, \mathbb{Q}_l) \to D^b_c(X, \mathbb{Q}_l) \]

preserves the abelian subcategory \( \text{Perv}(X) \).

Obvious from this definition are the following properties of multipliers: If \( K \) and \( L \) are multipliers, so are the product \( K \ast L \) and the direct sum \( K \oplus L \). Direct summands of multipliers are multipliers. If \( K \) is a multiplier, then the Verdier dual \( D(K) \) is a multiplier and also the dual

\[ K^\vee = (-id_X)^*(D(K)) \].

Examples: 1) Skyscraper sheaves are multipliers 2) If \( i : C \hookrightarrow X \) is a projective curve, which generates the abelian variety \( X \), and \( E \) is an etale \( \mathbb{Q}_l \)-sheaf on \( C \) with finite monodromy, then the intersection cohomology sheaf attached to \( (C, E) \) is a multiplier. 3) If \( j : Y \hookrightarrow X \) is a smooth ample divisor, then the intersection cohomology sheaf of \( Y \) is a multiplier.

The proofs. 1) is obvious. For 2) we gave in [7] a proof by reduction mod \( p \) using the Cebotarev density theorem and counting of points. Concerning 3) the morphism \( j : U = X \setminus Y \hookrightarrow X \) is affine for ample divisors \( Y \). Hence \( \lambda_U = Rj_! \mathcal{Q}_l[g] \) and \( \lambda_Y = i_* \mathcal{Q}_l,Y[g-1] \) are perverse sheaves, which coincide in \( \text{Perv}(X) \). The morphism \( \pi = a \circ (j \times id_X) \) is affine. Indeed \( W = \pi^{-1}(V) \) is affine for affine subsets \( V \) of \( X \), \( W \) being isomorphic under the isomorphism \((u, v) \mapsto (u, u + v)\) of \( X^2 \) to \( \text{affine product } U \times V \). By the affine vanishing theorem of Artin: For perverse sheaves \( L \in \text{Perv}(X) \) we get \( \lambda_U \boxtimes L \in \text{Perv}(X^2) \) and \( p^H(\pi_!(\lambda_U \boxtimes L)) = 0 \) for all \( \nu < 0 \). The distinguished triangle \( (Ra_!(\lambda_Y \boxtimes L), Ra_!(\lambda_U \boxtimes L), Ra_!(\delta_X \boxtimes L)) \) for \( \delta_X = \mathcal{Q}_l,X[g] \) and the corresponding long exact perverse cohomology sequence gives isomorphisms \( p^H(\delta_X \boxtimes L) \cong p^H(\lambda_Y \boxtimes L) \) for the integers \( \nu < 0 \). Since \( Ra_!(\delta_X \boxtimes L) = \delta_X \ast L \) is a direct sum of translates of constant perverse sheaves \( \delta_X \), we conclude \( p^H(\lambda_Y \ast L) \) for \( \nu < 0 \) to be zero in \( \text{Perv}(X) \). For smooth \( Y \) the intersection cohomology sheaf is \( \lambda_Y = i_* \mathcal{Q}_l,Y[g-1] \), and it is self dual. Hence by Verdier duality \( i_* \mathcal{Q}_l,Y[g-1] \ast L \) has image in \( \text{Perv}(X) \). Thus \( i_* \mathcal{Q}_l,Y[g-1] \) is a multiplier. \( \square \)

Let \( M(X) \subseteq P(X) \) denote the full category of semisimple multipliers. Let \( \overline{M}(X) \) denote its image in the quotient category \( \overline{P}(X) \) of \( P(X) \). Then, by the
definition of multipliers, the convolution product preserves \( \mathcal{M}(X) \)

\[
* : \mathcal{M}(X) \times \mathcal{M}(X) \to \mathcal{M}(X).
\]

**Theorem.** With respect to this convolution product the category \( \mathcal{M}(X) \) is a semisimple super-Tannakian \( \overline{\mathbb{Q}}_l \)-linear tensor category, hence as a tensor category \( \mathcal{M}(X) \) is equivalent to the category of representations \( \text{Rep}(G, \varepsilon) \) of a projective limit

\[
G = G(X)
\]

of supergroups.

**Outline of proof.** The convolution product obviously satisfies the usual commutativity and associativity constraints compatible with unit objects. See [7] 2.1. By [7], corollary 3 furthermore one has functorial isomorphisms

\[
\text{Hom}_{\mathcal{M}(X)}(K, L) \cong \Gamma_{\{0\}}(X, \mathcal{H}^0(K \ast L^\vee))^* ,
\]

where \( \mathcal{H}^0 \) denotes the degree zero cohomology sheaf and \( \Gamma_{\{0\}}(X, -) \) sections with support in the neutral element. Let \( L = K \) be simple and nonzero. Then the left side becomes \( \text{End}_{\mathcal{M}(X)}(K) \cong \overline{\mathbb{Q}}_l \). On the other hand \( K \ast L^\vee \) is a direct sum of a perverse sheaf \( P \) and translates of translation-invariant perverse sheaves. Hence \( \mathcal{H}^0(K \ast L^\vee) \) is the direct sum of a skyscraper sheaf \( S \) and translation-invariant etale sheaves. Therefore \( \Gamma_{\{0\}}(X, \mathcal{H}^0(K \ast L^\vee)^\vee) = \Gamma_{\{0\}}(X, S) \). By a comparison of both sides therefore \( S = \delta_0 \). Notice \( \delta_0 \) is the unit element 1 of the convolution product. Using the formula above we not only get

\[
\text{Hom}_{\mathcal{M}(X)}(K, L) \cong \text{Hom}_{\mathcal{M}(X)}(K \ast L^\vee, 1) ,
\]

but also find a nontrivial morphism

\[
ev_K : K \ast K^\vee \to 1 .
\]

By semisimplicity \( \delta_0 \) is a direct summand of the complex \( K \ast K^\vee \). In particular the Künneth formula implies, that the etale cohomology groups do not all vanish identically

\[
H^\bullet(X, K) \neq 0 .
\]

Therefore the arguments of [7] 2.6 show, that the simple perverse sheaf \( K \) is dualizable. Hence \( \mathcal{M}(X) \) is a rigid \( \overline{\mathbb{Q}}_l \)-linear tensor category. Let \( T \) be a finitely
⊗-generated tensor subcategory with generator say $A$. To show $T$ is super-Tannakian, by [4] it is enough to show for all $n$

$$\text{length}_T(A^n) \leq N^n,$$

where $N$ is a suitable constant. For any $B \in \mathcal{M}(X)$ let $B$, by abuse of notation, also denote the perverse semisimple representative in $\text{Perv}(X)$ without translation invariant summand. Put $h(B, t) = \sum_{\nu} \dim_{\overline{Q}_l}(H^\nu(X, B))t^\nu$. Then $\text{length}_T(B) \leq h(B, 1)$, since every summand of $B$ is a multiplier and therefore has nonvanishing cohomology. For $B = A^n$ the K¨unneth formula gives $h(B, 1) = h(A, 1)^n$. Therefore the estimate above holds for $N = h(A, 1)$. This completes the outline for the proof of the theorem. □

**Principally polarized abelian varieties.** Suppose $Y$ is a divisor in $X$ defining a principal polarization. Suppose the intersection cohomology sheaf $\delta_Y$ of $Y$ is a multiplier. Then a suitable translate of $Y$ is symmetric, and again a multiplier. So we may assume $Y = -Y$ is symmetric. Let $\mathcal{M}(X, Y)$ denote the super-Tannakian subcategory of $\mathcal{M}(X)$ generated by $\delta_Y$. The corresponding super-group $G(X, Y)$ attached to $\mathcal{M}(X, Y)$ acts on the super-space $W = \omega(\delta_Y)$ defined by the underlying super-fiber functor $\omega$ of $\mathcal{M}(X)$. By assumption $\delta_Y$ is self dual in the sense, that there exists an isomorphism $\varphi : \delta_Y^\vee \cong \delta_Y$. Obviously $\varphi^\vee = \pm \varphi$. This defines a nondegenerate pairing on $W$, and the action of $G(X, Y)$ on $W$ respects this pairing.

**Curves.** If $X$ is the Jacobian of smooth projective curve $C$ of genus $g$ over $k$, $X$ carries a natural principal polarization $Y = W_{g-1}$. If we replace this divisor by a symmetric translate, then $Y$ is a multiplier. The corresponding group $G(X, Y)$ is the semisimple algebraic group $G = Sp(2g-2, \overline{Q}_l)/\mu_{g-1}[2]$ or $G = Sl(2g-2, \overline{Q}_l)/\mu_{g-1}$ depending on whether the curve $C$ is hyperelliptic or not. The representation $W$ of $G(X, Y)$ defined by $\delta_Y$ as above is the unique irreducible $\overline{Q}_l$-representation of $G(X, Y)$ of highest weight, which occurs in the $(g-1)$-th exterior power of the $(2g-2)$-dimensional standard representation of $G$. See [7], section 7.6.

**Conjecture.** One could expect, that a principal polarized abelian variety $(X, Y)$ of dimension $g$ is isomorphic to a Jacobian variety $(\text{Jac}(C), W_{g-1})$ of a smooth projective curve $C$ (up to translates of the divisor $Y$ in $X$ as explained above) if and only if $Y$ is a multiplier with corresponding super-Tannakian group $G(X, Y)$ equal to one of the two groups

$$Sp(2g-2, \overline{Q}_l)/\mu_{g-1}[2] \text{ or } Sl(2g-2, \overline{Q}_l)/\mu_{g-1}.$$
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