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On the analysis of number of deaths due to Covid –19 outbreak data using a new class of distributions

Tabassum Naz Sindhu b,c, Anum Shafiq a, Qasem M. Al-Mdallal d, e

a School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
b Department of Statistics, Quaid-i-Azam University 45320, Islamabad 44000, Pakistan
c Department of Sciences and Humanities, FAST – National University, Islamabad, Pakistan
d Department of Mathematical Sciences, UAE University, P.O. Box 15551, Al-Ain, United Arab Emirates

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ABSTRACT

In this article, we develop a generator to suggest a generalization of the Gumbel type-II model known as generalized log-exponential transformation of Gumbel Type-II (GLET-GTII), which extends a more flexible model for modeling life data. Owing to basic transformation containing an extra parameter, every existing lifetime model can be made more flexible with suggested development. Some specific statistical attributes of the GLET-GTII are investigated, such as quantiles, uncertainty measures, survival function, moments, reliability, and hazard function etc. We describe two methods of parametric estimations of GLET-GTII discussed by using maximum likelihood estimators and Bayesian paradigm. The Monte Carlo simulation analysis shows that estimators are consistent. Two real life implementations are performed to scrutinize the suitability of our current strategy. These real life data is related to infectious diseases (COVID-19). These applications identify that by using the current approach, our proposed model outperforms than other well known existing models available in the literature.

Introduction

Lifetime phenomenon modeling and interpretation is an important component of statistical research in a broad variety of scientific and technical fields. The study of lifetime data analysis has developed rapidly and expanded in terms of methodology, theory, and application fields. Continuous distributions of probabilities and other methods of generalization or transformation have been introduced in the framework of characterising real life events. Such generalizations derived either by incorporating one or more than one shape parameters, or by adjusting distribution’s functional form, improve model flexibility and model the phenomena more precisely. Extensive software innovations made less emphasis on computational details and thus simplified estimation methods.

The below are popular and frequently referenced transformations presented during recent past in statistical studies for characterizing real life models. Marshal and Olkin [1] transform survival function by putting an additional parameter. Gupta et al. [2] introduced the exponential family of models by adding the extra shape parameter like an exponent to basic cdf. Beta-generated family by Eugene et al. [3] focused on Beta type-I and II models. On the other side, Kumaraswamy-generated family by Cordeiro and Castro [4] prefers the Kumaraswamy model instead of the Beta model. Zografos and Balakrishnan [5] introduced a versatile gamma-G class of models focused on GG (Generalized Gamma) model.

Here, our goal is to suggest a novel class of model that accommodates different kinds of hazard rates for suitable selection of shape parameter. The transformation of generalized logarithmic exponential is suggested on Gumbel type-II cdf, hereafter referred as Generalized log-exponential (GLE) transformation. The model, thus produced, is supposed to have both monotone and upside bathtub shaped hazard rates and based on the selection of parametric values. Let $Y$ is a random variable (r.v.) with cdf and pdf $(G(y), g(y)$ respectively) taken as baseline model.

$$F(y) = \frac{\log\{2 - e^{-(G(y))}\}}{\log\{2 - e^{-\lambda}\}} ; \lambda > 0,$$

with

\* Corresponding author.
E-mail address: q.almdallal@uaeu.ac.ae (Q.M. Al-Mdallal).

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Fever, cough, shortness of breath and occasional water diarrhea are often symptoms of COVID-19 [14]. 17,238 cases of acute respiratory SARS-CoV, was confirmed in Wuhan City, China (January 23rd to March 30th) and Europe (1st – 30th March). For these two data sets, we used over new proposed four parametric model known as GLET-GTI. Some specific statistical attributes of the GLET-GTI are investigated, like moments and associated measures, reliability function, cumulative hazard rate function, linear representation of model, measure of skewness, Quantile function, measure of kurtosis, moments generating function, non-central moments, central moments, mean, variance, factorial generating function, characteristic function, mean deviation and conditional moments are obtained and studied. Furthermore, measures of uncertainty containing entropy measures namely Reny, Mathai-Houbold Entropy, Tsallis, Verma, Kapur Entropy and -Entropy are obtained. In addition, model parameters are calculated by employing maximum likelihood and the Bayesian framework. To study efficiency of GLET-GTI model a simulation study was carried out. Finally, using the deaths number data set because of COVID-19 for China and Europe to unveil adaptability of proposed distribution. The outcomes revealed that it might better fit than various known distributions. For both data sets, density, Log-likelihood and trace graphs are plotted.

Proposed distribution and its properties

For illustration, the pdf of GTII model with parameters and is

\[ g(y | \gamma, \delta) = \gamma \delta^y - \exp \{- \delta y^\gamma \} \quad \gamma, \delta, y > 0. \]  

with cdf as

\[ G(y | \gamma, \delta) = \exp \{- \delta y^\gamma \}, \quad \gamma, \delta, y > 0. \]  

Using transformation (GLE), suggested in Eq. (1), resulting distribution have pdf and cdf of the following form

\[ f(y) = \frac{2 \lambda e^{- \lambda G^2(y)}}{\log(2 - e^{-\lambda})} \left\{ G^2(y) \right\}^{\lambda-1}; \lambda > 0. \]  

Motivated from [6], who taken into account less flexible transformation. Whereas their approach allows only fixed modulation of shape of distributions, our approach is more versatile because it incorporates additional shape parameter. To explain our perspective, we consider Gumbel type-II [7–11] as base model due to its simplicity and usability in life testing problem. Our proposed model known as generalized log-exponential transformation of Gumbel Type-II (GLET-GTII), which extends a more flexible model for modeling life data.

Infectious diseases are disorders induced by organisms, like viruses, parasites, bacteria, fungi etc. A lot of species are living in and on our bodies. Normally, they are harmless, or some times they are helpful. In certain conditions, however, some organisms may cause illness. However, under certain scenarios, some organisms may develop disease. Some infectious diseases can be transferred from one person to another. Some are disseminated by bugs or other animals. And one may have others by eating tainted water or food or being exposed to toxic organisms. Intimations and symptoms change widely depending on microbes resulting infection, but commonly involve fatigue and fever. Slight infections can lead to relax and home medication, while some life-threatening infections may require hospitalization. Certain infectious diseases, like measles, pneumonia and chickenpox, can be restrained using vaccinations. Regular and detailed hand-washing also succours protect you from certain infectious diseases. Only minor complications are present in most infectious diseases. But some infections, such as pneumonia, AIDS and meningitis can become life-threatening. Recently a new novel infectious disease which appeared in late 2019 named as the COVID-19 infection and 361 deaths in China were declared in February 2020 [15]. Here we used daily deaths data because of COVID-19 for China (January 23rd to March 28th) and Europe (1st – 30th March). For these two data sets, we used over new proposed four parametric model known as GLET-GTI. Some specific statistical attributes of the GLET-GTII are investigated, like moments and associated measures, reliability function, cumulative hazard rate function, linear representation of model, measure of skewness, Quantile function, measure of kurtosis, moments generating function, non-central moments, central moments, mean, variance, factorial generating function, characteristic function, mean deviation and conditional moments are obtained and studied. Furthermore, measures of uncertainty containing entropy measures namely Reny, Mathai-Houbold Entropy, Tsallis, Verma, Kapur Entropy and -Entropy are obtained. In addition, model parameters are calculated by employing maximum likelihood and the Bayesian framework. To study efficiency of GLET-GTI model a simulation study was carried out. Finally, using the deaths number data set because of COVID-19 for China and Europe to unveil adaptability of proposed distribution. The outcomes revealed that it might better fit than various known distributions. For both data sets, density, Log-likelihood and trace graphs are plotted.
The hrf (hazard rate function) of the GLET-GTII is defined by

\[
R(y) = \frac{f(y)}{F(y)} = \frac{\xi y \delta y^{-\gamma} e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)}}{2 - e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)}}; y, \delta, \xi, \lambda > 0. 
\]

(5)

\[
F(y) = \frac{2 - e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)}}{2 - e^{-\gamma}}; y, \delta, \xi, \lambda > 0. 
\]

(6)

It is obvious that \(F(y)\) differentiable and grows from 0 to \(\infty\).

### Reliability Function

The reliability function \(R(y)\) of the GLET-GTII is defined by

\[
R(y) = P(Y > y) = 1 - \frac{2 - e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)}}{2 - e^{-\gamma}}. 
\]

(7)

### Hazard rate function (hrf)

The hrf \(h(y)\) is defined as

\[
h(y) = \frac{\xi y \delta y^{-\gamma} e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)}}{(2 - e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)})[\log(2 - e^{-\gamma}) - \log(2 - e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)})]}; y, \delta, \xi, \lambda > 0. 
\]

(8)

The odd ratio is defined as

\[
\frac{R(y)}{h(y)} = \frac{\xi y \delta y^{-\gamma} e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)}}{2 - e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)}}[\log(2 - e^{-\gamma}) - \log(2 - e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)})]^2. 
\]

(9)

### Cumulative hrf

The cumulative hrf is defined as

\[
H(y) = \int_0^y h(t) \, dt, 
\]

Therefore,

\[
H(y) = -\log \left\{ \frac{2 - e^{-\gamma}}{2 - e^{-\gamma (\exp(\delta y^{-\gamma}) - 1)}} \right\}. 
\]

(10)

Where \(R(y)\) and \(h(y)\) defined in Eqs. (5)–(8). Utilizing MATLAB, Mathematica, Maple, R and Minitab computing packages, Eqs. (1)–(10) are being evaluated readily. The sketches of Eqs. (5), Eq. (6) and Eq. (8) are shown in Figs. 1–3 for different options of the parameters.
Fig. 1 shows influence of \( \lambda \) on density of GLET-GTII and demonstrates flexibility of the pdf in Eq. (5) where low symmetry, modality, high tails and skewness can be evaluated directly. Such figures show the flexibility of the GLET-GTII model. Figs. 2 and 3 are plotted for the S\((y)\) and CDF of the GLET-GTII. On the other hand, decreasing and upside bathtub pattern of hrf are noted in Fig. 4. It is also observed that for given value of \( \gamma, \delta, \xi \) and \( \lambda > 0 \), \( h \) indicates the uni-modal behavior.

**Expansion for pdf**

Using geometric infinite sum of series for \( \frac{1}{1-z} = \sum_{q=0}^{\infty} e^{-qz} \). The pdf responds to the following expansion

\[
 f(y|\gamma, \delta, \xi) = \sum_{q=0}^{\infty} \left(-1\right)^{q+1} \frac{\delta^q \lambda^{q+1} e^{-\delta \gamma} y^{q+1} \exp\left(-\delta \gamma y^{q+1}\right)}{k^{q+1} (2-e^{-\gamma})},
\]

(11)

and now using exponential series \( e^x = \sum_{n=0}^{\infty} \left(-1\right)^n x^n \) in Eq. (12) we have

\[
 f(y|\gamma, \delta, \xi, \lambda) = \sum_{k,p=0}^{\infty} \left(-1\right)^{k+p+1} \frac{\lambda^{k+p} \gamma^{k+p} e^{-\lambda \gamma} \gamma^{k+p} \exp\left(-\lambda \gamma y^{k+p}\right)}{k^{k+p} (2-e^{-\gamma})},
\]

(13)

**Random number generator**

Let r.v. \( q \sim \mathcal{U}(0, 1) \). Then Y is obtained as

\[
 \log\left(2 - e^{-2(\exp(-\delta \gamma y^{\gamma}))}\right) = q \log\left(2 - e^{-\gamma}\right),
\]

(14)

\[
 \log\left(2 - e^{-2(\exp(-\delta \gamma y^{\gamma}))}\right) = \log\left(2 - e^{-\gamma}\right)^2,
\]

\[
 e^{-\gamma(\exp(-\delta \gamma y^{\gamma}))} = 2 - (2 - e^{-\gamma})^2,
\]

\[
 \exp(-\delta \gamma y^{\gamma}) = -\frac{1}{\xi} \log\left(2 - (2 - e^{-\gamma})^2\right),
\]

\[
 Y = \mathcal{Q}(\delta, \gamma, \xi, \lambda),
\]

\[
 = \left(-\frac{1}{\delta} \log\left(-\frac{1}{\xi} \log\left(2 - (2 - e^{-\gamma})^2\right)\right)^\gamma\right)^{-\text{GLET - GTII}(\delta, \gamma, \xi, \lambda)}.
\]

(15)
Particularly, by placing \( q = 0.25, 0.50, 0.75 \) in Eq. (15), 1st and 3rd quartile and median are attained. Significant measurements of kurtosis and skewness are \( \phi_4 = \mu_4 / \sigma^4 \) and \( \phi_3 = \mu_3 / \sigma^3 \), respectively, in which fundamental \( \kappa \) moment represents by \( \mu_\kappa \) and standard deviation represents by \( \sigma \). As moments of GLET-GTII model can not occur for some values of parameter, suitable indicators for quantile-based skewness and kurtosis are more appropriate. These indicators are more robust and do exists for distributions in the

![Fig. 5. Plots of skewness and kurtosis of GLET-GTII model.](image)

![Fig. 6. graphs for Median of GLET-GTII model.](image)

| \( \gamma \) | \( \delta \) | \( \xi \) | \( \lambda \) | \( \mu_1 \) | \( \mu_2 \) | \( \mu_3 \) | \( \mu_4 \) | Var | C.V. | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 0.5 | 0.2 | 0.1 | 0.55998 | 0.36764 | 0.33854 | 31888.2 | 0.05407 | 41.5235 | 0.21837 | 1.40271 |
| 4 | 0.5 | 0.3 | 0.5 | 0.82490 | 0.79528 | 1.07101 | 1.46747 | 0.11481 | 41.0760 | 0.21985 | 1.40681 |
| 4 | 0.5 | 0.7 | 2.0 | 1.10831 | 1.41826 | 2.49258 | 434625 | 0.18991 | 40.1976 | 0.22174 | 1.41427 |
| 5 | 0.5 | 0.2 | 0.1 | 0.62253 | 0.42404 | 0.33470 | 0.36499 | 0.03649 | 30.6842 | 0.20187 | 1.37502 |
| 5 | 1.0 | 0.3 | 0.5 | 0.97511 | 1.03837 | 1.27836 | 2.16801 | 0.08753 | 30.3398 | 0.20201 | 1.37923 |
| 5 | 1.5 | 0.5 | 1.0 | 1.18923 | 1.53875 | 2.91414 | 4.67117 | 0.12449 | 29.6684 | 0.20456 | 1.38702 |
| 5 | 2.0 | 0.7 | 2.0 | 1.41984 | 2.18552 | 3.85298 | 9.24075 | 0.169578 | 29.0031 | 0.20575 | 1.39382 |
| 6 | 3.5 | 0.2 | 1.0 | 1.36045 | 1.96148 | 3.07591 | 5.59933 | 0.11066 | 24.4514 | 0.18799 | 1.35782 |
| 6 | 3.5 | 0.3 | 1.5 | 1.44154 | 2.19945 | 3.64442 | 6.93488 | 0.12140 | 24.17 | 0.19226 | 1.36207 |
| 6 | 3.5 | 0.5 | 2.0 | 1.48619 | 2.33203 | 3.96212 | 7.70249 | 0.12326 | 23.6234 | 0.19305 | 1.37001 |
| 6 | 3.5 | 0.7 | 2.5 | 1.51871 | 2.42939 | 4.19528 | 8.25804 | 0.12291 | 23.0847 | 0.19465 | 1.37707 |
absence of moments. \( S_a \) (Bowley’s Skewness) and \( K_M \) (Moors’ Kurtosis) measures are specified as

\[
S_a = \frac{F^{-1}(6/8) + F^{-1}(2/8) - 2F^{-1}(4/8)}{F^{-1}(6/8) - F^{-1}(2/8)}
\]

(16)

\[
K_M = \frac{F^{-1}(7/8) - F^{-1}(5/8) + F^{-1}(3/8) - F^{-1}(1/8)}{F^{-1}(6/8) - F^{-1}(2/8)}
\]

(17)

When \( S_a < 0 \) and \( S_a > 0 \), then model is left and right skewed respectively and is symmetrical for \( S_a = 0 \). Instead, a large \( K_M \) indicates a heavy tail for the distribution and a mild tail for a low \( K_M \).

Fig. 5 is plotted to analyze the influence of \( S_a \) and \( K_M \) in light of GLET-GTII model and as per parametric values. In Fig. 6, the behavior of order statistics respectively are given by

\[
Q_{\gamma}(y) = \begin{cases} 1 & \gamma \leq y < 1 \\ 0 & \gamma > 1 \end{cases}
\]

(19)

where \( \gamma = \frac{2}{3} \), \( \delta = \frac{1}{3} \), and \( \xi = \frac{1}{3} \). Table 1 displays the numerical values of \( \mu_1, \mu_2, \mu_3, \mu_4 \), Variance, C.V., \( K_M \), \( \gamma \), \( \delta \), \( \xi \), and \( S_a \) for GLET-GTII distribution. From Table 1, the considerable effects of \( \gamma \), \( \delta \), \( \xi \) and \( \lambda \) are noted on abovementioned measures.

Order statistics

Order statistics (OS) exist in several fields of theory and functional statistics. Suppose \( Y_1 \leq Y_2 \leq \ldots \leq Y_n \) is OS of a random sample of \( n \) size from \( F(y) \). Then, for \( m = 1, 2, \ldots, n \), probability density function of \( m^{th} \) OS, \( Y_{(m)} \) is given by

\[
f_{(m)}(y|\gamma, \delta, \xi, \lambda) = \Psi F(y|\gamma, \delta, \xi, \lambda) \cdot \left(1 - F(y|\gamma, \delta, \xi, \lambda)\right)^{n-m} f(y|\gamma, \delta, \xi, \lambda).
\]

(18)

The cdf of \( Y_{(m)} \) is

\[
F_{(m)}(y|\gamma, \delta, \xi, \lambda) = \sum_{j=0}^{m-1} \binom{n}{j} F(y|\gamma, \delta, \xi, \lambda)^j \left(1 - F(y|\gamma, \delta, \xi, \lambda)\right)^{n-j},
\]

(19)

In particular, cumulative density functions of \( Y_{(1)} \) and \( Y_{(n)} \) respectively, are given by

\[
F_{(1)}(y) = (F(y))^n, \quad F_{(n)}(y) = 1 - (1 - F(y))^n
\]

(20)

Let \( Q_{(m)}^{-1}(q) \) is quantile function of \( Y_{(m)} \) (for \( 0 < q < 1 \)). Then, from Eq. (20)

\[
Q_{(m)}^{-1}(q) = Q^{-1}\left(1 - [1 - q]^{1/m}\right), \quad Q_{(m)}^{+}(q) = Q^{-1}\left(q^{1/m}\right),
\]

where quantile function of \( Y \) is \( Q^{-1} \). For i.i.d. random values case, it is feasible to obtain \( k^{th} \) ordinary moment of the order statistics for \( \mu_k < \infty \). So, as Silva et al. [16], the \( \mu_k \) moment of \( m^{th} \) order statistic \( Y_{(m)} \) as

\[
\hat{\mu}_k^{(m)} = E_Y \left\{ Y_{(m)}^k \right\} = \sum_{j=n-m+1}^{n} (-1)^j \binom{n}{j} \binom{n-1}{n-m} \int_0^\infty y^{j-1} \left(1 - F(y)\right)^{n-j} dy.
\]

(21)

\[
\int_0^\infty y^{j-1} \left(1 - F(y)\right)^{n-j} dy = \frac{n!}{j!} \left(\frac{j}{n}\right)^j I_j(s)
\]

(22)

Stochastic ordering (SO)

The notion of SO for continuous positive random variables is a common and effective concept for calculating relative behavior. We must recall some basic meanings. Suppose that r.v. \( X \) is greater than \( Y \)

(i) Stochastic order (S-O) \( X \geq_Y Y \) if \( F_X(y) \geq F_Y(y) \); (ii) hazard rate order (HRO) \( Y \geq_X Y \) if \( h_X(y) \geq h_Y(y) \); (iii) likelihood ratio order (LHR-O) \( Y \geq_X Y \) if \( \frac{f_Y(y)}{f_X(y)} \) reduces in \( y \). The below mentioned results are known (see [17]):

\[
Y \geq_X Y \Rightarrow Y \geq_X Y \Rightarrow Y \geq_Y X.
\]

(23)

The GLET-GTII distributions are ordered with respect to the strongest “likelihood ratio” ordering as mentioned in the following theorem.

Theorem 1. Let \( X \) GLET-GTII \( (\gamma, \delta, \xi, \lambda_1) \), and \( Y \) GLET-GTII \( (\gamma, \delta, \xi, \lambda_2) \). If \( \lambda_1 \leq \lambda_2 \), then \( Y \geq_X X \) (\( Y \geq_{LH} X \)).

Proof. Consider likelihood ratio (LHR) as

\[
\frac{f_X(y)}{f_Y(y)} = \frac{\lambda_1(e^{-\gamma y})^{\lambda_1} - 1}{\lambda_2(e^{-\gamma y})^{\lambda_2} - 1}, \quad y > 0
\]

(24)

d \left( \frac{\mu_k}{f_Y(y)} \right) \leq 0.

Hence it demonstrates that \( Y \geq_X X \), and according to Eq. (23), \( Y \geq_{LH} X \).

Moments, central moments and certain related measures

For any statistical consideration, moments are profoundly significant, usually in applications. The special parameters that can be used to describe a homogeneous data set’s behaviour are called moments.
Consequently, for GLET-GTII model, the $\mu_s$ which is $s^{th}$ non-central moment is obtained as follows. If $Y$ has probability density function in Eq. (5), we get

$$
\mu_s = E(Y^s) = \int_0^\infty y^s dF(y|\gamma, \delta, \xi, \lambda); s = 1, 2, \ldots (26)
$$

The belowmentioned outcome gives $\mu_s$ ($s^{th}$ non-central moment) of $Y$ in G-F (gamma function) form.

**Theorem 2.** For $\gamma, \xi, \delta, \lambda > 0$, the $\mu_s(y|\gamma, \delta, \xi, \lambda)$ of $Y$ is translated as

$$
\mu_s(y|\gamma, \delta, \xi, \lambda) = \sum_{p=0}^\infty \sum_{q=0}^\infty (-1)^{p+q+k} \frac{\mu_p \xi^{p+1} \gamma^q}{p! k! 2^{p+1} \log(2-e^{-\xi}) \left(\delta(p+k+1)\right)^{1.5}} \Gamma(1-s-t)/\gamma
$$

where, usual G-F is $\Gamma()$.

**Proof.** Consider

$$
\mu_s = \int_0^\infty y^s dF(y|\gamma, \delta, \xi, \lambda).
$$

By using the expansion form of pdf that given in Eq. (13) yields

$$
\mu_s(y|\gamma, \delta, \xi, \lambda) = \sum_{p=0}^\infty \sum_{q=0}^\infty (-1)^{p+q+k} \frac{\mu_p \xi^{p+1} \gamma^q}{p! k! 2^{p+1} \log(2-e^{-\xi}) \left(\delta(p+k+1)\right)^{1.5}} \Gamma(1-s-t)/\gamma
$$

Allowing $z = (p+k+1)\lambda y^{-\gamma}$ using the result $\gamma (p+k+1) y^{-\gamma - 1} dy = dz$ and after some algebraic manipulation we have

$$
\mu_s(y|\gamma, \delta, \xi, \lambda) = \sum_{p=0}^\infty \sum_{q=0}^\infty (-1)^{p+q+k} \frac{\mu_p \xi^{p+1} \gamma^q}{p! k! 2^{p+1} \log(2-e^{-\xi}) \left(\delta(p+k+1)\right)^{1.5}} \int_0^\infty \frac{z}{(p+k+1)\lambda} \exp(-\delta z) dz.
$$

Since, $\Gamma(\gamma) = \int_0^\infty y^{\gamma-1} e^{-y} dy$, therefore the above integral provides the $s^{th}$ moment given by Eq. (27).

In particular, the first four moments of $Y$ are:

$$
\mu_1 = \sum_{p=0}^\infty \sum_{q=0}^\infty (-1)^{p+q+k} \frac{\mu_p \xi^{p+1} \gamma^q}{p! k! 2^{p+1} \log(2-e^{-\xi}) \left(\delta(p+k+1)\right)^{1.5}} \Gamma(1-1/\gamma),
$$

$$
\mu_2 = \sum_{p=0}^\infty \sum_{q=0}^\infty (-1)^{p+q+k} \frac{\mu_p \xi^{p+1} \gamma^q}{p! k! 2^{p+1} \log(2-e^{-\xi}) \left(\delta(p+k+1)\right)^{1.5}} \Gamma(1-2/\gamma),
$$

$$
\mu_3 = \sum_{p=0}^\infty \sum_{q=0}^\infty (-1)^{p+q+k} \frac{\mu_p \xi^{p+1} \gamma^q}{p! k! 2^{p+1} \log(2-e^{-\xi}) \left(\delta(p+k+1)\right)^{1.5}} \Gamma(1-3/\gamma),
$$

$$
\mu_4 = \sum_{p=0}^\infty \sum_{q=0}^\infty (-1)^{p+q+k} \frac{\mu_p \xi^{p+1} \gamma^q}{p! k! 2^{p+1} \log(2-e^{-\xi}) \left(\delta(p+k+1)\right)^{1.5}} \Gamma(1-4/\gamma).
$$

**Proposition 1.** Let $Y$ be a random variable following the GLET-GTII distribution, then the central moments is

$$
\mu_s(y|\gamma, \delta, \xi, \lambda) = \int_0^\infty (y - \mu)^s f(y) dy
$$

Substituting the Eq. (27) into Eq. (35) after certain simple calculations, we get

$$
\mu_s(y|\gamma, \delta, \xi, \lambda) = \int_0^\infty (y - \mu)^s f(y) dy
$$

**Remark 1.** The variance of GLET-GTII model is obtained from Equation (37) for $s = 2$.

**Moment generating function (mgf)**

The mgf of GLET-GTII distribution may be indicated as
\[
M(y|\gamma, \delta, \xi, \lambda) = \sum_{r=0}^{\infty} \frac{F^r}{r!} \frac{(-1)^{p+q+r+k}}{p!k!2^{1+r} \log(2-e^{-z})} \frac{e^{\gamma(t+1)}}{\delta(p+k+1)^{1/2}}
\times \Gamma(1-s/y).
\]

**Characteristic function (cf)**

For GLET-GTII model, cf is evaluated as

\[
\Phi(\tau|y, \delta, \xi, \lambda) = \int_0^{\infty} e^{\tau y} dF(y|\gamma, \delta, \xi, \lambda).
\]

After using exponential series, we have

\[
\Phi(\tau|y, \delta, \xi, \lambda) = \sum_{r=0}^{\infty} \frac{(\tau y)^r}{r!} \int_0^{\infty} y^r dF(y|\gamma, \delta, \xi, \lambda).
\]

Hence, we obtain

\[
\Phi(\tau|y, \delta, \xi, \lambda) = \sum_{r=0}^{\infty} \frac{(\tau y)^r}{r!} \frac{e^{\gamma(t+1)}}{\delta(p+k+1)^{1/2}} \frac{e^{\gamma(t+1)}}{\delta(p+k+1)^{1/2}} \int_0^{\infty} y^r dF(y|\gamma, \delta, \xi, \lambda).
\]

**Factorial generating function (fgf)**

For GLET-GTII model, fgf is extracted as follows

\[
F_s(\tau|y, \delta, \xi, \lambda) = \int_0^{\infty} e^{\tau y} dF(y|\gamma, \delta, \xi, \lambda),
\]

\[
= \sum_{r=0}^{\infty} \frac{(\log(1+\tau))^r}{r!} \int_0^{\infty} y^r dF(y|\gamma, \delta, \xi, \lambda),
\]

so,

\[
F_s(\tau|y, \delta, \xi, \lambda) = \sum_{r=0}^{\infty} \frac{(\log(1+\tau))^r}{r!} \frac{e^{\gamma(t+1)}}{\delta(p+k+1)^{1/2}} \frac{e^{\gamma(t+1)}}{\delta(p+k+1)^{1/2}} \int_0^{\infty} y^r dF(y|\gamma, \delta, \xi, \lambda).
\]

\[
\text{Incomplete non-central moments (INCM)}
\]

The INCM of model serve as a key role in determining inequality, namely Lorenz and Bonferroni’s income quantiles and curves, which concentrate on incomplete moments.

**Proposition 2.** The \(s\)th incomplete moment \(\mu_s(\gamma, \delta, \xi, \lambda)\) is

\[
\mu_s(\gamma, \delta, \xi, \lambda) = \sum_{r=0}^{\infty} \frac{(-1)^{p+q+r+k}}{p!k!2^{1+r} \log(2-e^{-z})} \frac{e^{\gamma(t+1)}}{\delta(p+k+1)^{1/2}} \int_0^{\infty} y^r dF(y|\gamma, \delta, \xi, \lambda) \Gamma(1-s/y).
\]

Substituting by the Eq. (13) into Eq. (44) after some simple calculations, we have

\[
\mu_s(\gamma, \delta, \xi, \lambda) = \sum_{r=0}^{\infty} \frac{(-1)^{p+q+r+k}}{p!k!2^{1+r} \log(2-e^{-z})} \int_0^{\infty} y^r \gamma(t+1) \exp\left\{ -\gamma(p+k+1) dy \right\}.
\]

Allowing \(w = (p+k+1) dy\) using the result \(-\gamma(p+k+1) dy = dw\) and after some algebraic manipulation we have

\[
\mu_s(\gamma, \delta, \xi, \lambda) = \sum_{r=0}^{\infty} \frac{(-1)^{p+q+r+k}}{p!k!2^{1+r} \log(2-e^{-z})} \int_0^{\infty} y^r \exp\left\{ -\gamma(p+k+1) dy \right\}.
\]

The above integral provides the \(s\)th moment given by Eq. (45).

**Lemma 1.** \(T(\alpha, \gamma, \delta, \xi, \lambda) = \int_0^{\infty} y dF(y|\gamma, \delta, \xi, \lambda)\) we have

\[
T(\alpha, \gamma, \delta, \xi, \lambda) = \sum_{r=0}^{\infty} \frac{(-1)^{p+q+r+k}}{p!k!2^{1+r} \log(2-e^{-z})} \frac{e^{\gamma(t+1)}}{\delta(p+k+1)^{1/2}} \Gamma\left\{ \alpha, \delta(p+k+1), 1 - \frac{1}{\gamma} \right\}.
\]
where $\Gamma (x, a) = \int_0^a t^{x-1} e^{-t} dt$ is lower incomplete gamma function.

**Conditional moments and mean deviations**

In predictive inference, the estimation of conditional moments $E(Y|Y > t)$ is useful in interaction with lifetime models. The $E(Y|Y > t)$ is obtained as

$$
E(Y|Y > t) = \frac{1}{S(t)} \left[ E(Y) - \int_0^\infty y f(y) dy \right],
$$

where $S(t), \mu_y,$ and $\mu_{Y,x}$ are defined in Eqs. (7), (27) and (43). The mean deviations about mean $\mu = E(Y)$ and $\bar{\mu}$ (median) are stated as

$$
\Theta_1(Y) = \int_0^\infty |y - \mu| dF(y),
\Theta_2(Y) = \int_0^\infty y F(y) dy - \mu,
$$

using Lemma 1, we have

$$
\Theta_1(Y) = 2 \Gamma (\mu, y, \delta, \xi, \lambda) + 2 \mu F(\mu),
\Theta_2(Y) = 2 \int_0^\infty y F(y) dy - \mu,
$$

where $F(\mu)$ is specified in (6).

**Uncertainty measures**

In this section, entropies such as Verma, Renyi entropy, Tsallis etc for GLET-GTII model are being investigated.

**Entropy measures**

Entropy is a significant idea in several relevant fields communications, measure-preserving dynamical systems, information theory, topological dynamics, thermodynamics, statistical mechanics etc, as a calculation of different characteristics such as uncertainty, disorder, randomness, energy that cannot produce work, complexity, etc. There are many definitions of entropy and they are not inherently ideal for all applications.

**Renyi entropy $\bar{R}_\sigma(Y)$**

For GLET-GTII model, the $\bar{R}_\sigma(Y)$ is

$$
\bar{R}_\sigma(Y) = \frac{1}{1 - \sigma} \log \int_0^\infty f^\sigma(y|y, \delta, \xi, \lambda) dy, \sigma \neq 1, \sigma > 0,
$$

where

$$
\int_0^\infty f(y|y, \delta, \xi, \lambda) dy = 1.
$$

$$
\int_0^\infty f^\sigma(y|y, \delta, \xi, \lambda) dy = 1.
$$

$$
\int_0^\infty f^{\sigma+1}(y|y, \delta, \xi, \lambda) dy = 1.
$$

$$
\int_0^\infty f^{\sigma+2}(y|y, \delta, \xi, \lambda) dy = 1.
$$

**Verma Entropy $V_{\alpha,\beta}(Y)$**

For GLET-GTII model, the $V_{\alpha,\beta}(Y)$ is

$$
V_{\alpha,\beta}(Y) = \frac{1}{\alpha - \beta} \log \int_0^\infty f^{\alpha-1}(y|y, \delta, \xi, \lambda) dy, \alpha - 1 < \beta < \alpha, \alpha \geq 1, \alpha \
$$

$$
\neq \beta.
$$
\[ f^{\alpha+\beta-1}(y|\tau, \delta, \xi, \lambda) = \left( \frac{2 \lambda y^{\tau-1} e^{-\lambda y^{\tau}} \exp \{-\alpha \lambda y^{\tau}\}}{\log(2-e^{-1}) (2-e^{-1})} \right)^{n+\beta-1} \]

It is significant to remember that, when \( \alpha \to 1 \), in (60), it reduces to \( R_e(Y) \). On the other hand, if \( \beta \to 1 \) and \( \alpha \to 1 \), then it approaches to the Shannon entropy. By using abovementioned information, we get

\[
\int_0^\infty f^{\alpha+\beta-1}(y|\tau, \delta, \xi, \lambda) dy = \sum_{\rho \neq k=0} \frac{\Gamma(\alpha+\beta-1+q) \Gamma(\alpha+\beta-1)}{\Gamma(\alpha+\beta-1+q) \Gamma(\alpha+\beta-1) \rho! \Gamma^{2\delta+\alpha+\beta-1}(2-e^{-1})} \frac{1}{y^{(\alpha+\beta-1)/2}} dy.
\]

Now substituting, \( \lambda y^{\tau-1} = t \), the above integral becomes

\[
\int_0^\infty f^{\alpha+\beta-1}(y|\tau, \delta, \xi, \lambda) dy = \sum_{\rho \neq k=0} \frac{\Gamma(\alpha+\beta-1+q) \Gamma(\alpha+\beta-1)}{\Gamma(\alpha+\beta-1+q) \Gamma(\alpha+\beta-1) \rho! \Gamma^{2\delta+\alpha+\beta-1}(2-e^{-1})} \frac{1}{y^{(\alpha+\beta-1)/2}} dy.
\]

After simplification, we have

\[
\int_0^\infty f^{\alpha+\beta-1}(y|\tau, \delta, \xi, \lambda) dy = \sum_{\rho \neq k=0} \frac{\Gamma(\alpha+\beta-1+q) \Gamma(\alpha+\beta-1)}{\Gamma(\alpha+\beta-1+q) \Gamma(\alpha+\beta-1) \rho! \Gamma^{2\delta+\alpha+\beta-1}(2-e^{-1})} \frac{1}{y^{(\alpha+\beta-1)/2}} dy.
\]

Finally, \( V_{\alpha, \beta}(Y) \) becomes

\[
V_{\alpha, \beta}(Y) = \frac{1}{\sigma-1} \left( 1 - \sum_{\rho \neq k=0} \frac{\Gamma(\alpha+\beta-1+q) \Gamma(\alpha+\beta-1)}{\Gamma(\alpha+\beta-1+q) \Gamma(\alpha+\beta-1) \rho! \Gamma^{2\delta+\alpha+\beta-1}(2-e^{-1})} \frac{1}{y^{(\alpha+\beta-1)/2}} \right).
\]
Table 2
Bias and MSEs for GLET-GTII parameters.

| n  | Estimates | $|Bias|$ | MSE   |
|----|-----------|---------|-------|
| 25 | $\gamma$  | 0.05349 | 0.04121 |
|    | $\delta$  | 0.02363 | 0.05389 |
|    | $\zeta$   | 0.06983 | 0.27452 |
|    | $\lambda$ | 0.17637 | 0.00444 |

$\tilde{I}_{MH}(Y) = \frac{1}{\sigma - 1} \left[ \sum_{p,q=0}^{\infty} (-1)^p p^{2}\sigma^{p+2}(\lambda^{(p+1)}(1-\tau)^{p+1})(2-\sigma)^{p+1}\sqrt{2}\pi^{2}\Gamma(2-\sigma)\right.$

$\times \left( \frac{\Gamma(2-\sigma+q)\Gamma\left(1-\left(\frac{\gamma+1}{\delta}\right)(\sigma-1)\right)}{\Gamma(\delta)(\sigma-1)^{\frac{\gamma+1}{\delta}}} \right)$

$- 1$. 

Table 3
Descriptive measures for Data Sets.

| Descriptive statistics | Data I | Data II |
|------------------------|--------|---------|
| $Q_1$                  | 13.00  | 120.8   |
| $Q_2$                  | 82.75  | 141.40  |
| Median                 | 33     | 385.0   |
| Mean                   | 49.74  | 841.4   |
| Trimmed               | 44.8   | 703.96  |
| Minimum               | 3      | 6       |
| Maximum               | 150    | 2824    |
| SD                    | 43.87  | 938.69  |
| Range                 | 147    | 2818    |
| Skewness              | 0.82   | 0.92    |
| Kurtosis              | -0.62  | -0.63   |

Mathai-Haubold entropy

For GLET-GTII model, $I_{MH}(Y)$ (see Mathai and Haubold [18]) is defined as

$\tilde{I}_{MH}(Y) = \frac{1}{\sigma - 1} \left( \int_{0}^{\infty} f^{2-\sigma}(y|y, \delta, \zeta, \lambda) dy \right)$, $\sigma \neq 1$. 

Similar arguments to $f^{2-\sigma}$ gives

$f^{2-\sigma}(y|y, \delta, \zeta, \lambda) = \left( \frac{\gamma\lambda^\gamma e^{-\lambda y y} e^{2}(\lambda^\gamma)}{\log(2-\sigma^2)} \right)^{2-\sigma}$

$\times \left( \frac{\Gamma(2-\sigma)\sqrt{2}\pi^{2}\Gamma(2-\sigma)}{\Gamma(\delta)(\sigma-1)^{\frac{\gamma+1}{\delta}}} \right)$

$\times \left( \frac{\Gamma(2-\sigma+q)\Gamma\left(1-\left(\frac{\gamma+1}{\delta}\right)(\sigma-1)\right)}{\Gamma(\delta)(\sigma-1)^{\frac{\gamma+1}{\delta}}} \right)$

Therefore, the final form of $I_{MH}(Y)$ becomes

$\tilde{I}_{MH}(Y) = \frac{1}{\sigma - 1} \left[ \sum_{p,q=0}^{\infty} (-1)^p p^{2}\sigma^{p+2}(\lambda^{(p+1)}(1-\tau)^{p+1})(2-\sigma)^{p+1}\sqrt{2}\pi^{2}\Gamma(2-\sigma)\right.$

$\times \left( \frac{\Gamma(2-\sigma+q)\Gamma\left(1-\left(\frac{\gamma+1}{\delta}\right)(\sigma-1)\right)}{\Gamma(\delta)(\sigma-1)^{\frac{\gamma+1}{\delta}}} \right)$

$- 1$. 

Kapur entropy

Kapur entropy $I_{\alpha,\beta}(Y)$ of $Y$ with GLET-GTII model is defined as

$I_{\alpha,\beta}(Y) = \frac{1}{\beta - \alpha} \left( \int_{0}^{\infty} f^{\alpha}(y) dy - \int_{0}^{\infty} f^{\beta}(y) dy \right)$. 

$\omega$-entropy

$\omega$-entropy $H_{\omega}(Y)$ of $Y$ with GLET-GTII model is defined as

$H_{\omega}(Y) = \frac{1}{\omega - 1} \left( \int_{0}^{\infty} f^{\omega}(y) dy \right)$, $\omega \neq 1$. 

As, $f^{\omega}(y|y, \delta, \zeta, \lambda)$ and $f^{\omega}(y|y, \delta, \zeta, \lambda)$ are calculated in Eqs. (55)–(58) respectively. Therefore, by using these information, $H_{\omega}(Y)$ takes the following form

$H_{\omega}(Y) = \frac{1}{\omega - 1} \left( \int_{0}^{\infty} f^{\omega}(y) dy \right)$. 

Estimation

We proceed by presenting estimates of the parameters of suggested model via different techniques. The maximum likelihood (MLH) estimation and Bayesian methodology for estimation objective. Working the Matlab (log, lik), the Ox program (subroutine MaxBFGS), R (optimum and MaxLik features), or SAS (PROC NLMIXED), the GLET-GTII model parameters is assessed from log-likelihood depending on sample. Addi-
Additionally, some goodness-of-fit statistics are applied to compare density estimates and selection of models.

**Maximum likelihood (ML) estimation**

The ML estimates are presented via optimization of equation corresponding to $\gamma, \delta, \xi$ and $\lambda$. They are also described as the maximum of log-likelihood function (LLHF) and defined by $L_{\gamma, \delta, \xi, \lambda} = \log L(y|\gamma, \delta, \xi, \lambda)$.

$$L(y|\gamma, \delta, \xi, \lambda) = \prod_{i=1}^{n} \frac{\gamma \delta^{\gamma-1} \xi^{\gamma-1} e^{-\xi(\exp(-\delta y_i)^{-\gamma})}}{\log(2 - e^{-\xi})}.$$  \(76\)

The LLHF for GLET-GTII distribution is given by data set $y_1, \ldots, y_n$. 

---

**Table 7**

| Data | Parameter | 2.5%  | 97.5% | Posterior Mean | Posterior Variance |
|------|-----------|-------|-------|----------------|-------------------|
| I 30 | $\gamma$ | 0.228357 | 0.481515 | 0.343596 | 0.004009 |
| I 30 | $\delta$ | 1.129464 | 0.357515 | 0.220179 | 0.002465 |
| I 30 | $\xi$ | 1.858141 | 4.526620 | 3.052603 | 0.380427 |
| I 30 | $\lambda$ | 0.013755 | 0.044269 | 0.025770 | 0.000051 |
| II 66 | $\gamma$ | 0.184741 | 0.431814 | 0.295535 | 0.004113 |
| II 66 | $\delta$ | 0.192319 | 0.876992 | 0.602914 | 0.016561 |
| II 66 | $\xi$ | 0.391735 | 0.885013 | 0.615787 | 0.015930 |
| II 66 | $\lambda$ | 0.064211 | 0.141740 | 0.098712 | 0.000397 |
\[ l_{\gamma, \delta, \lambda} = n \log(\gamma) + n \log(\delta) + n \log(\lambda) - \frac{n}{\delta} \sum_{i=1}^{n} \exp\left\{ -\lambda \delta y_i^{-\gamma} \right\} \]
\[ -\delta \lambda \sum_{i=1}^{n} y_i^{-\gamma} - (\gamma + 1) \sum_{i=1}^{n} \log y_i - n \log[2 - e^{-\gamma}] \]
\[ -\sum_{i=1}^{n} \log \left( 2 - e^{-\gamma \left( \exp\left\{ -\lambda \delta y_i^{-\gamma} \right\} \right)} \right). \]

By differentiating Eq. (77), we get MLEs of the corresponding parameters \( \gamma, \delta \) and \( \lambda \). The components of score vector \( \Lambda_{\gamma, \delta, \lambda} = (\Lambda_{\gamma}, \Lambda_{\delta}, \Lambda_{\lambda}) \) is as follows:

\[ \Lambda_{\gamma} = \frac{\partial l_{\gamma, \delta, \lambda}}{\partial \lambda} = \frac{n}{\delta} \sum_{i=1}^{n} \log(y_i) + \lambda \delta \sum_{i=1}^{n} y_i^{-\gamma} \log(y_i) + \frac{\lambda \delta}{\gamma + 1} \sum_{i=1}^{n} \exp\left\{ -\lambda \delta y_i^{-\gamma} \right\} \]
\[ -\lambda \delta \sum_{i=1}^{n} y_i^{-\gamma} \log(y_i) e^{-\lambda \delta y_i^{-\gamma}} \left( 2 - e^{-\gamma \left( \exp\left\{ -\lambda \delta y_i^{-\gamma} \right\} \right)} \right), \]

\[ (78) \]

\[ \Lambda_{\lambda} = \frac{\partial l_{\gamma, \delta, \lambda}}{\partial \lambda} = \frac{n}{\delta} \sum_{i=1}^{n} y_i^{-\gamma} + \lambda \delta \sum_{i=1}^{n} \exp\left\{ -\lambda \delta y_i^{-\gamma} \right\} y_i^{-\gamma} \log(y_i) + \frac{\lambda \delta}{\gamma + 1} \sum_{i=1}^{n} \exp\left\{ -\lambda \delta y_i^{-\gamma} \right\} \left( 2 - e^{-\gamma \left( \exp\left\{ -\lambda \delta y_i^{-\gamma} \right\} \right)} \right). \]

\[ (79) \]
\[
\Lambda_\gamma = \frac{\partial \log L_{\text{MLE}}}{\partial \gamma} \\
\Lambda_\delta = \frac{\partial \log L_{\text{MLE}}}{\partial \delta} \\
\Lambda_\xi = \frac{\partial \log L_{\text{MLE}}}{\partial \xi} \\
\Lambda_\lambda = \frac{\partial \log L_{\text{MLE}}}{\partial \lambda}
\]

\[
\tau = \frac{n}{2} \left(2 - e^{-t} \log(2 - e^{-t}) - \sum_{i=1}^{n} e^{-y_{i} \gamma} + \sum_{i=1}^{n} \frac{e^{\left(\exp\left(-y_{i} \gamma \delta\right) - y_{i} \gamma \delta\right)} + y_{i} \gamma \delta}{2 - e^{-\left(\exp\left(-y_{i} \gamma \delta\right)\right)}} \right)
\]

\[
\lambda = \frac{n}{2} - \delta \sum_{i=1}^{n} y_{i} \gamma \tau \log(y_{i}) + \sum_{i=1}^{n} e^{-y_{i} \gamma} y_{i} \gamma + \delta \sum_{i=1}^{n} \frac{y_{i} \tau e^{\left(\exp\left(-y_{i} \gamma \delta\right) - y_{i} \gamma \delta\right)} + y_{i} \gamma \delta}{2 - e^{-\left(\exp\left(-y_{i} \gamma \delta\right)\right)}} \right)
\]

Setting \(\Lambda_\gamma, \Lambda_\delta, \Lambda_\xi, \Lambda_\lambda = 0\) and after solving these equations, it gives the MLEs for GLET-GTII model parameters. An iterative method such as the Newton–Raphson approach is needed to solve them numerically.

Now, utilizing simulation, we investigate performance of MLEs with respect to sample size \(n\). The following steps are followed to conduct simulation study: stimulate it 5000, samples of size \(n = 25, 100, 150, 250\) and 350 from GLET-GTII((1.2, 1.7, 2.1, 1.5), (0.2, 1.4, 1.6, 0.8)); give the MLEs for 5000 samples, say \(\hat{\gamma}_m, \hat{\delta}_m, \hat{\xi}_m\) for \(m = 1, 2, \ldots, 5000\); quantify estimate biases and squared errors (MSEs); where average absolute \(\text{Bias} = \frac{1}{5000} \sum_{m=1}^{5000} |\hat{\gamma} - \gamma|\) and \(\text{MSE} = \frac{1}{5000} \sum_{m=1}^{5000} (\hat{\gamma} - \gamma)^2\). Table 2
shows the Bias and MSEs for various estimates. We directly noticed that Bias and MSEs reduce by enhancement of sample size $n$. (see Table 3).

Bayesian mechanism

Here, we proceed by providing estimation of proposed structure parameters via Bayesian mechanism. Let $\gamma, \delta, \xi$ and $\lambda$ are random variables. Therefore, the following independent priors are supposed as $\gamma \sim \gamma\left(\nu_1, \sigma_1\right)$, $\delta \sim \gamma\left(\nu_2, \sigma_2\right)$, $\xi \sim \gamma\left(\nu_3, \sigma_3\right)$ and $\lambda \sim \gamma\left(\nu_4, \sigma_4\right)$, where $\nu_i, \sigma_i \in \mathbb{R}^+$, $i = 1, 2, 3, 4$. The $g(\gamma, \delta, \xi, \lambda|x)$ joint posterior density of $\gamma, \delta, \xi$ and $\lambda$ is as follows

$$g(\gamma, \delta, \xi, \lambda|x) = \frac{L(\gamma|\delta, \xi, \lambda)p(\gamma)p(\delta)p(\xi)p(\lambda)}{\int_{\gamma} \int_{\delta} \int_{\xi} \int_{\lambda} L(\gamma|\delta, \xi, \lambda)p(\gamma)p(\delta)p(\xi)p(\lambda)dyd\delta d\xi d\lambda}$$

in a similar fashion $\hat{\delta}_{BE} = E(\delta|x)$, $\hat{\lambda}_{BE} = E(\lambda|x)$ and $\hat{\xi}_{BE} = E(\xi|x)$.

The BE (Bayes estimator) of $\gamma$ is given under SELF (Squarederrorlossfunction)\cite{21–26} as

$$\hat{\gamma}_{BE} = E\left(\gamma|x\right) = \frac{\int_{\gamma} \int_{\delta} \int_{\xi} \int_{\lambda} L(\gamma|\delta, \xi, \lambda)p(\gamma)p(\delta)p(\xi)p(\lambda)dyd\delta d\xi d\lambda}{\int_{\gamma} \int_{\delta} \int_{\xi} \int_{\lambda} L(\gamma|\delta, \xi, \lambda)p(\gamma)p(\delta)p(\xi)p(\lambda)dyd\delta d\xi d\lambda}$$
tionaly, Bayes risk is evaluated by
| Parameters estimates and Standard errors for both Data Sets. |

| Model | Parameter | Data I | Data II |
|-------|-----------|--------|--------|
| GLET-GTII | $\hat{\gamma}$ | 0.206478 | 0.04659 | 0.14714 | 0.10045 |
| | $\hat{\alpha}$ | 3.354146 | 1.96348 | 3.50279 | 2.61427 |
| | $\hat{\xi}$ | 3.354146 | 1.96348 | 3.50279 | 2.61427 |
| | $\hat{\beta}$ | 134.0219 | 160.264 | 98.9773 | 362.710 |
| GIWD | $\tilde{\alpha}$ | 9.10197 | 811.843 | 2.70212 | 334.481 |
| | $\tilde{\beta}$ | 0.91594 | 0.08369 | 0.57051 | 0.07536 |
| | $\tilde{\gamma}$ | 1.78997 | 146.234 | 9.18399 | 648.661 |
| Frechet | $\tilde{\eta}$ | 17.1868 | 2.45283 | 99.7713 | 29.7155 |
| | $\tilde{\xi}$ | 0.91589 | 0.08368 | 0.59186 | 0.07787 |
| AGT-II | $\hat{\beta}$ | 7.47910 | 116.202 | 7.99414 | 8033.73 |
| | $\hat{\gamma}$ | 13.4320 | 3.46155 | 6.40543 | 8033.75 |
| | $\hat{\delta}$ | 4.48651 | 6.62664 | 0.57062 | 0.11674 |
| | $\hat{\alpha}$ | 0.91372 | 0.09052 | 0.57071 | 0.21937 |
| GT-II | $\hat{\beta}$ | 0.91595 | 0.08369 | 0.57057 | 0.07536 |
| | $\hat{\gamma}$ | 13.5324 | 3.10943 | 16.1933 | 5.71448 |
| EB-XII | $\hat{\beta}$ | 0.42854 | 0.16904 | 0.29162 | 0.13471 |
| | $\hat{\bar{b}}$ | 59.0695 | 58.8060 | 52.0316 | 54.4001 |
| | $\hat{\eta}$ | 2.74026 | 1.33481 | 2.39459 | 1.36109 |

Implementations of real data

Two examples are presented here to describe the efficiency of suggested distribution. The R software is used to show the improved efficiency of GLET-GTII distribution and numerical calculations. Consider the following models (i) GIWD (generalized inverse Weibull distribution) [19] (ii) Frechet model (Frechet), (iii) Additive Gumbel Type-II (AGT-II) (iv) GT-II (Gumbel Type-II) and (v) EB-XII (Exponentiated Burr XII) model for comparative purposes. Different methodologies of segregation based on LLHF (log-likelihood function) assessed at MLEs were also taken into account: Akaike Information Criterion (AIC) computed through $AIC = 2\hat{\xi} - L(\hat{\gamma}, \hat{\alpha}, \hat{\xi}, \hat{\lambda}; y)$, and Bayesian Information Criterion $BIC = -2L(\hat{\gamma}, \hat{\alpha}, \hat{\xi}, \hat{\lambda}; y) + \log(n)$, Corrected Akaike information Criterion $AICC = AIC + \frac{2(n-1)}{n-n_{\text{eff}}}$, Hannan Quinn Information Criterion $HQIC = -2\log(\hat{\gamma}, \hat{\alpha}, \hat{\xi}, \hat{\lambda}; y) + 2k\log(\log(n))$, where $\hat{\gamma}, \hat{\alpha}, \hat{\xi}, \hat{\lambda}$ are the estimates of $\gamma, \alpha, \xi, \lambda$. The model that provides the smallest values of $\text{Cramér-Von Mises}$ and $\text{Anderson-Darling}$ test statistics is taken into account (for more detail see Chen and Balakrishnan [20]). Generally, the smallest values of $\text{Cramér-Von Mises}$ ($W'$) and $\text{Anderson-Darling}$ ($A'$), indicates the better fit of data. The values of the $A'$ and $W'$ are listed in Tables 5,6. The summary statistics are graphed via box plots for data sets “I” and “II” and shown in Fig. 7. A box plot is a visualization that provides a real indication of the data. The boxplot is a systematic way to view data distribution based on a summary of five numbers (“minimum”, first quartile (Q1), median, third quartile (Q3), and “maximum”). A box plot is a visualization that provides a real indication of the values are distributed out in the data.

In Figs. 9,10, the estimated densities of GLET-GTII ($\gamma, \alpha, \xi, \lambda$) and competitor models were graphed for more visual comparison. The mentioned below data sets are given in Sindhu et al. [27], (see Figs. 11 and 8).

Data I: Daily deaths number because of COVID-19 in China (23rd January to 28th March)

This data is taken from given website (https://www.worldometers.info/coronavirus/china/), which indicates the daily number of deaths because of COVID-19 in China. The data is given below:

{8, 16, 15, 24, 26, 28, 38, 43, 46, 45, 57, 64, 65, 73, 73, 86, 89, 97, 108, 97, 146 ,121, 143, 142, 105, 98, 136, 114, 118, 109, 97, 150, 71, 52, 29, 44, 47, 35, 42 ,31, 38, 31, 50, 28, 27, 22, 17, 22, 11, 7, 13, 10, 14, 13, 11, 8, 3, 7, 6, 9 ,7, 4, 6, 5, 3, 5}.

Data II: Daily deaths number because of COVID-19 in Europe (14th – 30th March)

This data is taken from given website (https://covid19.who.int/), which indicates daily deaths number because of COVID-19 in Europe. {6, 18, 29, 28, 47, 55, 40, 150, 129, 184, 236, 237, 336, 219, 612, 434, 648, 706 ,838, 1129, 1421, 118, 116, 1393, 1540, 1941, 2175, 2278, 2824, 2803, 2667). For each distribution, we estimate the unknown parameters using maximum likelihood. The MLEs with their respective standard errors of the above models are listed in Table 4. These calculations were made using the R programming language.

Conclusion

Here, we have suggested a new general construction of flexible lifetime models by rendering any existing baseline model more versatile via simple transformation. A generalized log-exponential transformation model is introduced and its implementation is demonstrated by taking Gumbel Type-II model. The mathematical properties of proposed model have been analyzed in detail. Additionally, some figures for density and hazard function are included. The general non-central incomplete and non-central complete moments are also included. Uncertainty measures such as entropies (like Renyi, Tsallis, Verma, and Kumar entropy Mathai-Houbold) are calculated. The estimation of parameters is accessed by utilizing two techniques such as maximum likelihood method and Bayesian framework. Through utilizing classical goodness of fit indicators, we evaluate the efficiency of GLET-GTII model with its five significant counterparts. The posterior densities, Log-likelihood and hazard function are included. The general non-central incomplete and non-central complete moments are also included. Uncertainty measures such as entropies (like Renyi, Tsallis, Verma, and Kumar entropy Mathai-Houbold) are calculated. The estimation of parameters is accessed by utilizing two techniques such as maximum likelihood method and Bayesian framework. Through utilizing classical goodness of fit indicators, we evaluate the efficiency of GLET-GTII model with its five significant counterparts. The posterior densities, Log-likelihood and hazard function are included. The general non-central incomplete and non-central complete moments are also included. Uncertainty measures such as entropies (like Renyi, Tsallis, Verma, and Kumar entropy Mathai-Houbold) are calculated. The estimation of parameters is accessed by utilizing two techniques such as maximum likelihood method and Bayesian framework. Through utilizing classical goodness of fit indicators, we evaluate the efficiency of GLET-GTII model with its five significant counterparts. The posterior densities, Log-likelihood and hazard function are included. The general non-central incomplete and non-central complete moments are also included. Uncertainty measures such as entropies (like Renyi, Tsallis, Verma, and Kumar entropy Mathai-Houbold) are calculated. The estimation of parameters is accessed by utilizing two techniques such as maximum likelihood method and Bayesian framework.
Credit authorship contribution statement

Tabassum Naz Sindhu, Anum Shafiq and Qasem M. Al-Mdallal contributed to the conceptualization, design and implementation of the research, to the analysis of the results and to the writing of the manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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