Binary nonlinearization of the super AKNS system

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Abstract

We establish the binary nonlinearization approach of the spectral problem of the super AKNS system, and then use it to obtain the super finite-dimensional integrable Hamiltonian system in supersymmetry manifold $\mathbb{R}^{4N/2N}$. The super Hamiltonian forms and integrals of motion are given explicitly.

Key words: nonlinearization, super AKNS system, super finite-dimensional integrable Hamiltonian system.

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1 Introduction

The method of nonlinearization of Lax pair for the classical integrable (1+1)-dimensional system has aroused strong interests in soliton theory, including mono-nonlinearization \cite{1,2,3} and binary nonlinearization \cite{4,5}. The crucial technique is to find constraints between the potentials (i.e., functions of soliton equation) and the eigenfunctions of the spectral problem associated with the soliton equation. The soliton equation can be decomposed into a finite-dimensional integrable Hamiltonian system (FDIH) by means of above-mentioned constraints. The first example \cite{1} of the nonlinearization for the (1+1)-dimensional case is the nonlinearization of the (2\times 2) AKNS system. Moreover, as a natural generalization of this method for case of (2+1)-dimensional system, the symmetry constraint of the KP hierarchy is given in references \cite{6,7,8}, which leads to the intensive research on the constrained KP (cKP) hierarchy \cite{9,10,11,12,13} based on the pseudo-differential operator. By this way, a (2+1)-dimensional system can be decomposed into two

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(1+1)-dimensional systems. Thus we can understand that symmetry constraint is one kind of formal variable separation method for the nonlinear integrable partial differential equations.

On the other hand, several supersymmetry (susy) integrable systems including supersymmetry AKNS [14], super AKNS system [15]-[18] and supersymmetry KP hierarchy [16]-[20], have been studied. Similar to the cKP hierarchy, the symmetry constraint of the supersymmetry KP hierarchy, i.e. the constrained supersymmetry KP hierarchy, is given in reference [21]-[23]. Furthermore, there are some interesting results on the Lax representation and Hamiltonian structure of the sAKNS system [21]-[22], the "Ghost" symmetry and the Darboux-Backlund solution of the constrained susy KP [23], super soliton [24] and the Hamiltonian structure of the constrained susy KP [25]. However, there is no result on the nonlinearization of the super AKNS (or supersymmetry AKNS) for our best knowledge. Inspiring by the relation between the nonlinearization of the (1+1)-dimensional system and the symmetry constraint of the KP hierarchy, the appearance of the constrained supersymmetry KP reminds us to ask whether nonlinearization technique can be used in super AKNS equations and finite dimensional integrable system with fermionic variables can be obtained by this approach. The purpose of this paper is to give the affirmative answer on the above question for the super AKNS system.

The paper is organized as follows: in section 2, we briefly recall some basic knowledge of the super AKNS hierarchy [15]-[18]. In section 3, we consider the binary nonlinearization of spectral problem of the super AKNS system and give an explicit constraint. Under this constraint, the super AKNS system is decomposed into a super FDIH with odd functions. Finally, we close with some conclusions and discussions in section 4.

2 Super AKNS Hierarchy

Let us start with the following spectral problem [16]-[17]

$$\dot{\phi} = M\phi, \quad M = \begin{pmatrix} -\lambda & q & \alpha \\ r & \lambda & \beta \\ \beta & -\alpha & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},$$

where $\lambda$, $q$ and $r$ are even elements such that $p(\lambda) = p(q) = p(r) = 0$; but $\alpha$ and $\beta$ are odd elements such that $p(\alpha) = p(\beta) = 1$. $\lambda$ is a eigenparameter of this system. $q, r, \alpha$ and $\beta$ are functions of the usual time-space variables $x$ and $t$. Here small $p(f)$ denotes the parity of the arbitrary function $f$. Note that the main reference of supersymmetry and susy analysis used in this paper is literature [26]. Note that $M \in B(0,1)$, $B(0,1)$ is a Lie superalgebra.

In order to obtain the super AKNS hierarchy, we first solve co-adjoint equation associated
where $\rho$ with (1)\\n\begin{equation}
N_x = [M, N] = MN - NM,
\end{equation}
with
\[
N = \begin{pmatrix}
A & B & \rho \\
C & -A & \delta \\
\delta & -\rho & 0
\end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix}
a_j & b_j & \rho_j \\
c_j & -a_j & \delta_j \\
\delta_j - \rho_j & 0
\end{pmatrix} \lambda^{-j},
\]
where $p(A)=p(B)=p(C)=0$, $p(\delta)=1$ and $a_j, b_j, c_j, \rho_j, \delta_j (j \geq 0)$ are determined later.

Substituting M, N into Eq. (2) and comparing the coefficients of $\lambda^{-j} (j \geq 0)$, we have
\begin{equation}
\begin{cases}
b_0 = c_0 = \rho_0 = \delta_0 = 0, \\
q_{j,x} = q c_j - r b_j + \alpha \delta_j + \beta \rho_j, & j \geq 0, \\
b_{j,x} = -2 b_{j+1} - 2 q a_j - 2 \alpha \rho_j, & j \geq 0, \\
c_{j,x} = 2 c_{j+1} + 2 r a_j + 2 \beta \delta_j, & j \geq 0, \\
\rho_{j,x} = -\rho_{j+1} + q \delta_j - \alpha a_j - \beta b_j, & j \geq 0, \\
\delta_{j,x} = \delta_{j+1} + r \rho_j - \alpha c_j + \beta a_j, & j \geq 0.
\end{cases}
\end{equation}
Equation (3) can also be rewritten as
\[
(c_{j+1}, b_{j+1}, \delta_{j+1}, \rho_{j+1})^T = L(c_j, b_j, \delta_j, \rho_j)^T,
\]
where
\[
L = \begin{pmatrix}
\frac{1}{2} \partial_x - r \partial_x^{-1} q & r \partial_x^{-1} r & -\beta - r \partial_x^{-1} \alpha & -r \partial_x^{-1} \beta \\
-q \partial_x^{-1} q & -\frac{1}{2} \partial_x + q \partial_x^{-1} r & -q \partial_x^{-1} \alpha & -\alpha - q \partial_x^{-1} \beta \\
\alpha - \beta \partial_x^{-1} q & \beta \partial_x^{-1} r & \partial_x - \beta \partial_x^{-1} \alpha & -r - \beta \partial_x^{-1} \beta \\
-\alpha \partial_x^{-1} q & -\beta + \alpha \partial_x^{-1} r & q - \alpha \partial_x^{-1} \alpha & -\partial_x + \alpha \partial_x^{-1} \beta
\end{pmatrix}.
\]
For a given initial value, the $a_j, b_j, c_j, \rho_j, \delta_j (j \geq 1)$ can be calculated by the recursion relation (4). In particular, let $a_0 = -1$, we have
\[
a_1 = 0, b_1 = q, c_1 = r, \rho_1 = \alpha, \delta_1 = \beta, \\
a_2 = \frac{1}{2} q r + \alpha \beta, b_2 = -\frac{1}{2} q x, c_2 = \frac{1}{2} r x, \rho_2 = -\alpha x, \delta_2 = \beta x.
\]
Then, let us consider the spectral problem (1) with the following auxiliary spectral problem
\[
\phi_{\nu n} = N^{(n)} \phi = (\lambda^n N)_{\nu} \phi,
\]
with
\[
(\lambda^n N)^{+} = \sum_{j=0}^{n} \begin{pmatrix}
a_j & b_j & \rho_j \\
c_j & -a_j & \delta_j \\
\delta_j - \rho_j & 0
\end{pmatrix} \lambda^{-j},
\]
where the symbol "+" denotes taking the nonnegative power of $\lambda$. A simple calculation leads to

$$N^{(1)} = \begin{pmatrix} -\lambda & q & \alpha \\ r & \lambda & \beta \\ \beta & -\alpha & 0 \end{pmatrix} = M,$$

$$N^{(2)} = \begin{pmatrix} -\lambda^2 + \frac{1}{2}qr + \alpha\beta & q\lambda - \frac{1}{2}q_x & \alpha\lambda - \alpha_x \\ r\lambda + \frac{1}{2}r_x & \lambda^2 - \frac{1}{2}qr - \alpha\beta & \beta\lambda + \beta_x \\ \beta\lambda + \beta_x & -\alpha\lambda + \alpha_x & 0 \end{pmatrix}.$$  \hfill (7)

From the compatible condition $\phi_{x,t_n} = \phi_{t_n,x}$ according to equations (1) and (4), we get a zero curvature equation

$$M_{t_n} - N^{(n)}_x + [M, N^{(n)}] = 0,$$  \hfill (8)

which gives the super AKNS hierarchy

$$\begin{cases} q_{t_n} = b_{n,x} + 2qa_n + 2\alpha\rho_n = -2b_{n+1}, \\ r_{t_n} = c_{n,x} - 2ra_n - 2\beta\delta_n = 2c_{n+1}, \\ \alpha_{t_n} = \rho_{n,x} - q\delta_n + \alpha a_n + \beta b_n = -\rho_{n+1}, \\ \beta_{t_n} = \delta_{n,x} - \beta a_n + \alpha c_n - r\rho_n = \delta_{n+1}. \end{cases}  \hfill (9)$$

Setting $n = 2$, eq. (9) gives

$$\begin{cases} q_{t_2} = -\frac{1}{2}q_{xx} + q^2 r + 2q\alpha\beta - 2\alpha\alpha_x, \\ r_{t_2} = \frac{1}{2}r_{xx} - qr^2 - 2q\alpha\beta - 2\beta\beta_x, \\ \alpha_{t_2} = -q_{xx} - q\beta_x + \frac{1}{2}qr\alpha - \frac{1}{2}q_x\beta, \\ \beta_{t_2} = \beta_{xx} + r\alpha_x + \frac{1}{2}r\alpha - \frac{1}{2}qr\beta. \end{cases}  \hfill (10)$$

Setting $n = 3$ in the super AKNS hierarchy in eq. (9) results in

$$\begin{cases} q_{t_3} = \frac{1}{4}q_{xxx} - \frac{3}{2}q_{xx,r} - 3q_{\alpha} + 3q\alpha\beta_x + 3\alpha\alpha_{xx}, \\ r_{t_3} = \frac{1}{4}r_{xxx} - \frac{3}{2}qr_{xx} + 3\alpha_{xx}\beta - 3\alpha\beta_x - 3\beta\beta_{xx}, \\ \alpha_{t_3} = q_{xxx} + \frac{3}{2}q_{xx}\beta_x - \frac{3}{4}q_{rxx}\alpha - \frac{3}{4}qr\alpha \alpha - \frac{3}{2}qr\alpha_x - \frac{3}{2}q_{x}\beta + \frac{3}{2}q_{xx}\beta_x, \\ \beta_{t_3} = \beta_{xxx} + \frac{3}{2}r_{xx}\alpha + \frac{3}{4}r_{x}\alpha - \frac{3}{2}q_{xx}\beta - \frac{3}{2}qr\beta - \frac{3}{2}qr\beta_x. \end{cases}  \hfill (11)$$

Introducing vectors $U_0 = (r, q, \beta, \alpha)^T$, $U = (r, -q, 2\beta, -2\alpha)^T$, eq. (9) becomes a compact form

$$U_{t_n} = \begin{pmatrix} r \\ -q \\ 2\beta \\ -2\alpha \end{pmatrix} = 2 \begin{pmatrix} c_{n+1} \\ b_{n+1} \\ \delta_{n+1} \\ \rho_{n+1} \end{pmatrix} = 2L^n U_0.$$  \hfill (12)
Let
\[
L_1 = \begin{pmatrix}
-\frac{1}{2} \partial_x + q \partial_x^{-1} r & -q \partial_x^{-1} q & \frac{1}{2} \alpha + \frac{1}{2} q \partial_x^{-1} \beta & -\frac{1}{2} q \partial_x^{-1} \alpha \\
\frac{1}{2} \partial_x - r \partial_x^{-1} q & \frac{1}{2} r \partial_x^{-1} \beta & -\frac{1}{2} \beta - \frac{1}{2} r \partial_x^{-1} \alpha \\
2 \beta - 2 \alpha \partial_x^{-1} r & 2 \alpha \partial_x^{-1} q & -\partial_x - \alpha \partial_x^{-1} \beta & -q + \alpha \partial_x^{-1} \alpha \\
2 \beta \partial_x^{-1} r & 2 \alpha - 2 \beta \partial_x^{-1} q & \partial_x - \beta \partial_x^{-1} \alpha & -\partial_x - \beta \partial_x^{-1} \alpha
\end{pmatrix},
\]
(13)
and
\[
\tilde{U} = (r, q, \beta, \alpha)^T = \frac{1}{2} \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
r \\
-2 \beta \\
-2 \alpha
\end{pmatrix} = \frac{1}{2} gU,
\]
(14)
It is easy to verify
\[
L_1 = J^{-1} gLg^{-1} J,
\]
(15)
thus
\[
L_1^n = J^{-1} gL^n g^{-1} J,
\]
and Eq. (12) becomes
\[
\tilde{U}_n = JL_1^n \tilde{U}_0.
\]
Using the supertrace identity [27]
\[
\delta \int Str(N \partial M) dx = (\lambda^{-\gamma}(\frac{\partial}{\partial \lambda})\lambda^\gamma) Str(\partial M \partial U_0 N),
\]
(17)
then
\[
\begin{pmatrix}
\frac{\delta}{\delta r} \\
\frac{\delta}{\delta q} \\
\frac{\delta}{\delta \beta} \\
\frac{\delta}{\delta \alpha}
\end{pmatrix}
\begin{pmatrix}
-2a_{n+2}
\end{pmatrix}
= (\gamma - n - 1)
\begin{pmatrix}
b_{n+1} \\
c_{n+1} \\
-2\rho_{n+1} \\
2\delta_{n+1}
\end{pmatrix}.
\]
(18)
Let \( n = 0 \), then \( \gamma = 0 \).
Moreover, the equation(16) can be written as the following super Hamiltonian form\[18\]

\[
\tilde{U}_n = JL_n \begin{pmatrix} q \\ r \\ -2\alpha \\ 2\beta \end{pmatrix} = JL_n \frac{\delta H_n}{\delta U_0}, \quad \frac{\delta H_n}{\delta U_0} = \begin{pmatrix} b_{n+1} \\ c_{n+1} \\ -2\rho_{n+1} \\ 2\delta_{n+1} \end{pmatrix}, \quad H_n = \int \frac{2}{n+1}a_{n+2}dx, \quad (19)
\]

where J is the supersymplectic operator. We would like to point out that the authors in reference [18] calculated the Hamiltonian form by a direct constrained variation method instead of supertrace identity.

3 Binary Nonlinearization

Now we are in a position to discuss the nonlinearization of the super AKNS hierarchy given in the last section. To this aim, consider the spectral problem in eq.(1) and its adjoint spectral problem

\[
\psi_x = -M^\text{St} \psi = \begin{pmatrix} \lambda & -r & \beta \\ -q & -\lambda & -\alpha \\ -\alpha & -\beta & 0 \end{pmatrix} \psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (20)
\]

where St means supertranspose [26].

It is not difficult to get

\[
\frac{\delta \lambda}{\delta U_0} = \begin{pmatrix} \frac{\delta \lambda}{\delta q} \\ \frac{\delta \lambda}{\delta r} \\ \frac{\delta \lambda}{\delta \alpha} \\ \frac{\delta \lambda}{\delta \beta} \end{pmatrix} = \begin{pmatrix} \psi_1 \phi_2 \\ \psi_2 \phi_1 \\ \psi_1 \phi_3 + \psi_3 \phi_2 \\ \psi_2 \phi_3 - \psi_3 \phi_1 \end{pmatrix}, \quad (21)
\]

by a similar way of the counterpart of in the AKNS system[4, 5]. When zero boundary conditions\[\lim_{|x| \to \infty} \phi = \lim_{|x| \to \infty} \psi = 0\] are imposed, we can verify a simple characteristic property of the variational derivative of \(\lambda\)

\[
L_1 \frac{\delta \lambda}{\delta U_0} = \lambda \frac{\delta \lambda}{\delta U_0}, \quad (22)
\]

where \(L_1\) is defined by \([13]\).

Choosing N distinct eigenparameters \(\lambda_1, \cdots, \lambda_N\), equation \([19]\) becomes

\[
\begin{pmatrix} \frac{\delta}{\delta r} \\ \frac{\delta}{\delta q} \\ \frac{\delta}{\delta \alpha} \\ \frac{\delta}{\delta \beta} \end{pmatrix} H_k = \begin{pmatrix} b_{k+1} \\ c_{k+1} \\ -2\rho_{k+1} \\ 2\delta_{k+1} \end{pmatrix} = \sum_{j=1}^{N} \begin{pmatrix} \frac{\delta \lambda_j}{\delta r} \\ \frac{\delta \lambda_j}{\delta q} \\ \frac{\delta \lambda_j}{\delta \alpha} \\ \frac{\delta \lambda_j}{\delta \beta} \end{pmatrix} = \begin{pmatrix} < \Psi_2, \Phi_1 > \\ < \Psi_1, \Phi_2 > \\ < \Psi_2, \Phi_3 > - < \Psi_3, \Phi_1 > \\ < \Psi_1, \Phi_3 > + < \Psi_3, \Phi_2 > \end{pmatrix}, \quad (23)
\]
where \( \Phi_i = (\phi_{i1}, \cdots, \phi_{iN})^T, \Psi_i = (\psi_{i1}, \cdots, \psi_{iN})^T, i=1, 2, 3, \) and \( \langle .., > \) denotes the inner product in \( R^N \).

Let \( k=0 \) in (23), then it gives

\[
\begin{align*}
q &= \langle \Psi_2, \Phi_1 >, \\
r &= \langle \Psi_1, \Phi_2 >, \\
\alpha &= -\frac{1}{2} (\langle \Psi_2, \Phi_3 > - \langle \Psi_3, \Phi_1 >), \\
\beta &= \frac{1}{2} (\langle \Psi_1, \Phi_3 > + \langle \Psi_3, \Phi_2 >).
\end{align*}
\] (24)

Substituting (24) into the spectral problem (1) and the adjoint spectral problem (20), we obtain the following finite-dimensional systems

\[
\begin{align*}
\phi_{1j,x} &= -\lambda_j \phi_{1j} + \langle \Psi_2, \Phi_1 > \phi_{2j} - \frac{1}{2} (\langle \Psi_2, \Phi_3 > - \langle \Psi_3, \Phi_1 >) \phi_{3j}, \quad 1 \leq j \leq N, \\
\phi_{2j,x} &= \langle \Psi_1, \Phi_2 > \phi_{1j} + \lambda_j \phi_{2j} + \frac{1}{2} (\langle \Psi_1, \Phi_3 > + \langle \Psi_3, \Phi_2 >) \phi_{3j}, \quad 1 \leq j \leq N, \\
\phi_{3j,x} &= \frac{1}{2} (\langle \Psi_1, \Phi_3 > + \langle \Psi_3, \Phi_2 >) \phi_{1j} + \frac{1}{2} (\langle \Psi_2, \Phi_3 > - \langle \Psi_3, \Phi_1 >) \phi_{2j}, \quad 1 \leq j \leq N, \\
\psi_{1j,x} &= \lambda_j \psi_{1j} - \psi_{2j} + \frac{1}{2} (\langle \Psi_1, \Phi_3 > + \langle \Psi_3, \Phi_2 >) \psi_{3j}, \quad 1 \leq j \leq N, \\
\psi_{2j,x} &= -\langle \Psi_2, \Phi_1 > \psi_{1j} - \lambda_j \psi_{2j} + \frac{1}{2} (\langle \Psi_2, \Phi_3 > - \langle \Psi_3, \Phi_1 >) \psi_{3j}, \quad 1 \leq j \leq N, \\
\psi_{3j,x} &= \frac{1}{2} (\langle \Psi_2, \Phi_3 > - \langle \Psi_3, \Phi_1 >) \psi_{1j} - \frac{1}{2} (\langle \Psi_1, \Phi_3 > + \langle \Psi_3, \Phi_2 >) \psi_{2j}, \quad 1 \leq j \leq N,
\end{align*}
\] (25)

which can be written as the following super Hamiltonian form

\[
\begin{align*}
\Phi_{1,x} &= \frac{\partial H_1}{\partial \Psi_1}, \quad \Phi_{2,x} = \frac{\partial H_1}{\partial \Psi_2}, \quad \Phi_{3,x} = \frac{\partial H_1}{\partial \Psi_3}, \\
\Psi_{1,x} &= -\frac{\partial H_1}{\partial \Phi_1}, \quad \Psi_{2,x} = \frac{\partial H_1}{\partial \Phi_2}, \quad \Psi_{3,x} = \frac{\partial H_1}{\partial \Phi_3},
\end{align*}
\] (26)

where

\[
H_1 = -\Psi_1 \Phi_1 + \Psi_2 \Phi_2 + \Psi_3 \Phi_3 - \frac{1}{2} (\Psi_2 \Phi_3 - \Psi_3 \Phi_2) (\Psi_1 \Phi_3 + \Psi_3 \Phi_2).
\]

As for \( t_2 \)-part, we consider the following spectral problem

\[
\phi_{t_2} = N^{(2)} \phi = \begin{pmatrix} -\lambda^2 + \frac{1}{2} qr + \alpha \beta & q \lambda - \frac{1}{2} qx & \lambda \alpha - \alpha x \\
r \lambda + \frac{1}{2} qx & \lambda^2 + \frac{1}{2} qr - \alpha \beta & \beta \lambda + \beta x \\
\beta \lambda + \beta x & -\alpha \lambda + \alpha x & 0 \end{pmatrix} \phi, \] (27)

and its adjoint spectral problem

\[
\psi_{t_2} = -(N^{(2)})^{st} \psi = \begin{pmatrix} \lambda^2 - \frac{1}{2} qr - \alpha \beta & -r \lambda - \frac{1}{2} rx & \beta \lambda + \beta x \\
-q \lambda + \frac{1}{2} qx & -\lambda^2 + \frac{1}{2} qr + \alpha \beta & -\alpha \lambda + \alpha x \\
-\alpha \lambda + \alpha x & -\beta \lambda - \beta x & 0 \end{pmatrix} \psi. \] (28)
Substituting equation (24) into spectral problems (27), (28) and noticing systems (25), we obtain the following finite-dimensional system:

\[
\begin{align*}
\phi_{1j,t_2} &= (-\lambda_j^2 + \frac{1}{2} q \tilde{r} + \tilde{\alpha} \tilde{\beta})\phi_{1j} + (q \lambda_j - \frac{1}{2} q \tilde{r})\phi_{2j} + (\tilde{\alpha} \lambda_j - \tilde{\alpha} x)\phi_{3j}, \quad 1 \leq j \leq N, \\
\phi_{2j,t_2} &= (\bar{r} \lambda_j + \frac{1}{2} \bar{r} x)\phi_{1j} + (\lambda_j^2 - \frac{1}{2} q \bar{r} - \tilde{\alpha} \tilde{\beta})\phi_{2j} + (\tilde{\beta} \lambda_j + \tilde{\beta} x)\phi_{3j}, \quad 1 \leq j \leq N, \\
\phi_{3j,t_2} &= (\tilde{\beta} \lambda_j + \tilde{\beta} x)\phi_{1j} + (-\tilde{\alpha} \lambda_j + \tilde{\alpha} x)\phi_{2j}, \quad 1 \leq j \leq N, \\
\psi_{1j,t_2} &= (\lambda_j^2 - \frac{1}{2} q \tilde{r} - \tilde{\alpha} \tilde{\beta})\psi_{1j} - (\bar{r} \lambda_j + \frac{1}{2} \bar{r} x)\psi_{2j} + (\tilde{\beta} \lambda_j + \tilde{\beta} x)\psi_{3j}, \quad 1 \leq j \leq N, \\
\psi_{2j,t_2} &= -(q \lambda_j - \frac{1}{2} q x)\psi_{1j} + (\lambda_j^2 + \frac{1}{2} q \tilde{r} + \tilde{\alpha} \tilde{\beta})\psi_{2j} + (-\tilde{\alpha} \lambda_j + \tilde{\alpha} x)\psi_{3j}, \quad 1 \leq j \leq N, \\
\psi_{3j,t_2} &= (-\tilde{\alpha} \lambda_j + \tilde{\alpha} x)\psi_{1j} - (\tilde{\beta} \lambda_j + \tilde{\beta} x)\psi_{2j}, \quad 1 \leq j \leq N,
\end{align*}
\]  

(29)

where \( \tilde{q}, \bar{r}, \tilde{\alpha}, \tilde{\beta} \) denote the potentials under the constraint (24). Here \( \tilde{P} \) denote the new expression generated from \( P(u) \) by the the binary nonlinear constraint (24).

By a direct but tedious calculation, the finite system in eq. (29) can be written as the following super Hamiltonian form:

\[
\begin{align*}
\Phi_{1, t_2} &= \frac{\partial H_2}{\partial \psi_{1}}, \quad \Phi_{2, t_2} = \frac{\partial H_2}{\partial \psi_{2}}, \quad \Phi_{3, t_2} = \frac{\partial H_2}{\partial \psi_{3}}, \\
\Psi_{1, t_2} &= -\frac{\partial H_2}{\partial \psi_{1}}, \quad \Psi_{2, t_2} = -\frac{\partial H_2}{\partial \psi_{2}}, \quad \Psi_{3, t_2} = \frac{\partial H_2}{\partial \psi_{3}},
\end{align*}
\]  

(30)

where

\[
H_2 = < \Lambda^2 \Psi_2, \Phi_2 > - < \Lambda^2 \Psi_1, \Phi_1 > + < \Psi_2, \Phi_1 > < \Lambda \Psi_1, \Phi_2 > + < \Lambda \Psi_2, \Phi_1 > < \Psi_1, \Phi_2 > - \frac{1}{4} (< \Psi_1, \Phi_1 > - < \Psi_2, \Phi_2 >) ( < \Psi_2, \Phi_3 > - < \Psi_3, \Phi_1 > ) ( < \Psi_1, \Phi_3 > + < \Psi_3, \Phi_2 > ) + \frac{1}{2} < \Psi_2, \Phi_1 > < \Psi_1, \Phi_2 > ( < \Psi_1, \Phi_1 > - < \Psi_2, \Phi_2 > ) - \frac{1}{2} < \Psi_2, \Phi_3 > - < \Psi_3, \Phi_1 > ) ( < \Lambda \Psi_1, \Phi_3 > + < \Lambda \Psi_3, \Phi_2 > ) - \frac{1}{2} ( < \Lambda \Psi_2, \Phi_3 > - < \Lambda \Psi_3, \Phi_1 > ) ( < \Psi_1, \Phi_3 > + < \Psi_3, \Phi_2 > ).
\]

Making use of (22), the recursion relation (41) and equation (45), we have

\[
\begin{align*}
\tilde{a}_i &= \frac{1}{2} < \Lambda^{i-1} \Psi_1, \Phi_1 > - \frac{1}{2} < \Lambda^{i-1} \Psi_2, \Phi_2 >, \quad i \geq 1, \\
\tilde{b}_i &= < \Lambda^{i-1} \Psi_2, \Phi_1 >, \quad i \geq 1, \\
\tilde{c}_i &= < \Lambda^{i-1} \Psi_1, \Phi_2 >, \quad i \geq 1, \\
\tilde{\rho}_i &= - \frac{1}{2} ( < \Lambda^{i-1} \Psi_2, \Phi_3 > - < \Lambda^{i-1} \Psi_3, \Phi_1 > ), \quad i \geq 1, \\
\tilde{\delta}_i &= \frac{1}{2} ( < \Lambda^{i-1} \Psi_1, \Phi_3 > + < \Lambda^{i-1} \Psi_3, \Phi_2 > ), \quad i \geq 1.
\end{align*}
\]  

(31)

Then \( \tilde{N}_x = [\tilde{M}, \tilde{N}] \) is still satisfied, and \( (\tilde{N}^2)_x = [\tilde{M}, \tilde{N}^2] \) is also satisfied. Therefore

\[
F_x = \left( \frac{1}{2} \text{Str} \tilde{N}^2 \right)_x = \frac{1}{2} \text{Str}(\tilde{N}^2)_x = \frac{d}{dx} (\tilde{a}^2 + \tilde{b} \tilde{c} + 2 \tilde{\rho} \tilde{\delta}) = 0.
\]

The identity indicates that F is a generating function of integrals of motion for the nonlinearized spatial systems (25). Let \( F = \sum_{n \geq 0} F_n \lambda^{-n} \), we obtain the following formulas

\[
F_1 = -2 \tilde{a}_1 = - < \Psi_1, \Phi_1 > + < \Psi_2, \Phi_2 >.
\]  

(32)
A direct calculation leads to

\[ F_n = \sum_{i=1}^{n-1} (\tilde{a}_i \tilde{a}_{n-i} + \tilde{b}_i \tilde{c}_{n-i} + 2\tilde{\rho}_i \tilde{\delta}_{n-i}) + 2\tilde{a}_0 \tilde{a}_n \]

\[ = \sum_{i=1}^{n-1} \left[ \frac{1}{4} (\langle \Lambda^{i-1}\Psi_1, \Phi_1 \rangle - \langle \Lambda^{i-1}\Psi_2, \Phi_2 \rangle) (\langle \Lambda^{n-i-1}\Psi_1, \Phi_1 \rangle - \langle \Lambda^{n-i-1}\Psi_2, \Phi_2 \rangle) \right. \\
\left. - \frac{1}{2} (\langle \Lambda^{i-1}\Psi_2, \Phi_3 \rangle - \langle \Lambda^{i-1}\Psi_3, \Phi_1 \rangle) (\langle \Lambda^{n-i-1}\Psi_1, \Phi_3 \rangle + \langle \Lambda^{n-i-1}\Psi_3, \Phi_2 \rangle) \right. \\
\left. + \langle \Lambda^{i-1}\Psi_2, \Phi_1 \rangle \langle \Lambda^{n-i-1}\Psi_1, \Phi_2 \rangle \right] - \langle \Lambda^{n-1}\Psi_1, \Phi_1 \rangle + \langle \Lambda^{n-1}\Psi_2, \Phi_2 \rangle. \tag{33} \]

Let us consider the temporal part of the super AKNS hierarchy in eq. (33)

\[
\begin{pmatrix}
\phi_{1j} \\
\phi_{2j} \\
\phi_{3j}
\end{pmatrix}_{t_n} = \begin{pmatrix}
\sum_{i=0}^{n} \tilde{a}_i \lambda_j^{n-i} & \sum_{i=0}^{n} \tilde{b}_i \lambda_j^{n-i} & \sum_{i=0}^{n} \tilde{\rho}_i \lambda_j^{n-i} \\
\sum_{i=0}^{n} \tilde{c}_i \lambda_j^{n-i} & \sum_{i=0}^{n} \tilde{a}_i \lambda_j^{n-i} & \sum_{i=0}^{n} \tilde{\delta}_i \lambda_j^{n-i} \\
\sum_{i=0}^{n} \tilde{\delta}_i \lambda_j^{n-i} & \sum_{i=0}^{n} \tilde{\rho}_i \lambda_j^{n-i} & 0
\end{pmatrix} \begin{pmatrix}
\phi_{1j} \\
\phi_{2j} \\
\phi_{3j}
\end{pmatrix}, \quad 1 \leq j \leq N,
\]

\[
\begin{pmatrix}
\psi_{1j} \\
\psi_{2j} \\
\psi_{3j}
\end{pmatrix}_{t_n} = \begin{pmatrix}
-\sum_{i=0}^{n} \tilde{a}_i \lambda_j^{n-i} & -\sum_{i=0}^{n} \tilde{c}_i \lambda_j^{n-i} & \sum_{i=0}^{n} \tilde{\delta}_i \lambda_j^{n-i} \\
-\sum_{i=0}^{n} \tilde{b}_i \lambda_j^{n-i} & \sum_{i=0}^{n} \tilde{a}_i \lambda_j^{n-i} & -\sum_{i=0}^{n} \tilde{\rho}_i \lambda_j^{n-i} \\
-\sum_{i=0}^{n} \tilde{\rho}_i \lambda_j^{n-i} & -\sum_{i=0}^{n} \tilde{\delta}_i \lambda_j^{n-i} & 0
\end{pmatrix} \begin{pmatrix}
\psi_{1j} \\
\psi_{2j} \\
\psi_{3j}
\end{pmatrix}, \quad 1 \leq j \leq N. \tag{34}
\]

A direct calculation leads to

\[
\begin{align*}
\Phi_{1,t_n} &= \frac{\partial F_{n+1}}{\partial \Psi_1}, \quad \Phi_{2,t_n} = \frac{\partial F_{n+1}}{\partial \Psi_2}, \quad \Phi_{3,t_n} = \frac{\partial F_{n+1}}{\partial \Psi_3}, \\
\Psi_{1,t_n} &= -\frac{\partial F_{n+1}}{\partial \Psi_1}, \quad \Psi_{2,t_n} = -\frac{\partial F_{n+1}}{\partial \Psi_2}, \quad \Psi_{3,t_n} = -\frac{\partial F_{n+1}}{\partial \Psi_3}. \tag{35}
\end{align*}
\]

For example,

\[
\Phi_{1,t_n} = \sum_{i=0}^{n} \tilde{a}_i \Lambda^{n-i} \Phi_1 + \sum_{i=1}^{n} \tilde{b}_i \Lambda^{n-i} \Phi_2 + \sum_{i=1}^{n} \tilde{\rho}_i \Lambda^{n-i} \Phi_3 \\
= -\Lambda^n \Phi_1 + \sum_{i=1}^{n} \frac{1}{2} (\langle \Lambda^{i-1}\Psi_1, \Phi_1 \rangle - \langle \Lambda^{i-1}\Psi_2, \Phi_2 \rangle) \Lambda^{n-i} \Phi_1 + \langle \Lambda^{i-1}\Psi_2, \Phi_1 \rangle \Lambda^{n-i} \Phi_2 \\
= -\frac{1}{2} (\Lambda^{i-1} \Phi_2, \Phi_3 > - \langle \Lambda^{i-1}\Psi_3, \Phi_1 \rangle ) \Lambda^{n-i} \Phi_3 \\
\]

\[= \frac{\partial F_{n+1}}{\partial \Psi_1}. \]

It is not difficult to see that \( F_n(n \geq 0) \) are also integrals of motion for equation (33), i. e.

\[
\{ F_{m+1}, F_{n+1} \} = \frac{\partial}{\partial t_n} F_{m+1} = 0, m, n \geq 0,
\]

where Poisson bracket is defined by

\[
\{ F, G \} = \sum_{i=1}^{3} \sum_{j=1}^{N} \frac{\partial F}{\partial \phi_{ij}} \frac{\partial G}{\partial \psi_{ij}} - (1)^p(\phi_{ij}) p(\psi_{ij}) \frac{\partial F}{\partial \psi_{ij}} \frac{\partial G}{\partial \phi_{ij}}. \tag{36}
\]
With the help of the result of nonlinearization\cite{21,28}, it is natural for us to set

$$f_k = \psi_{1k}\phi_{1k} + \psi_{2k}\phi_{2k} + \psi_{3k}\phi_{3k}, \quad 1 \leq k \leq N,$$

(37)

and verify they are also integrals of motion of the constrained systems \cite{25} and \cite{31}. In what follows, we will give a proposition to show the independence of \(\{f_k\}_{k=1}^N\), \(\{F_m\}_{m=1}^{2N}\).

**Proposition 1** The integrals of motion \(\{f_k\}_{k=1}^N\) and \(\{F_m\}_{m=1}^{2N}\) are functionally independent over some region of the supersymmetry manifold \(\mathbb{R}^{4N|2N}\). Here the definition of \(\mathbb{R}^{M|N}\)\cite{26} is given by \((x^1, \cdots, x^M, \xi^1, \cdots, \xi^N)\) with \(x^i \in \mathbb{R}\) while \(\xi^i\) are odd variables.

**Proof** Suppose that the result of the proposition is not true, that is to say, there does not exist any region of \(\mathbb{R}^{4N|2N}\) over which the integrals of motion \(\{f_k\}_{k=1}^N\), \(\{F_m\}_{m=1}^{2N}\) are functionally independent. Therefore, there exist 3N constants \(\{\alpha_k\}_{k=1}^N\), \(\{\beta_m\}_{m=1}^{2N}\) which do not equal to zero at the same time, so that we have for all points in \(\mathbb{R}^{4N|2N}\)

$$\sum_{k=1}^N \alpha_k((\frac{\partial f_k}{\partial \Phi_1})^T, (\frac{\partial f_k}{\partial \Phi_2})^T, (\frac{\partial f_k}{\partial \Phi_3})^T) + \sum_{m=1}^{2N} \beta_m((\frac{\partial F_m}{\partial \Phi_1})^T, (\frac{\partial F_m}{\partial \Phi_2})^T, (\frac{\partial F_m}{\partial \Phi_3})^T) = 0.$$  

(38)

After a direct calculation, we have

$$\begin{cases}
\frac{\partial f_k}{\partial \phi_{1j}} |_{\phi_2=0} = \delta_{jk}\psi_{1j}, \quad 1 \leq k, j \leq N, \\
\frac{\partial f_k}{\partial \phi_{2j}} |_{\phi_2=0} = \delta_{jk}\psi_{2j}, \quad 1 \leq k, j \leq N, \\
\frac{\partial f_k}{\partial \phi_{3j}} |_{\phi_2=0} = -\delta_{jk}\psi_{3j}, \quad 1 \leq k, j \leq N, \\
\frac{\partial F_m}{\partial \phi_{1j}} |_{\phi_2=0} = -\frac{1}{2} \sum_{i=1}^{m-1} <\Lambda^{m-i-1}\Psi_1, \Phi_3 > \Lambda^{m-i}\Psi_1 - \Lambda^{m-1}\Psi_1, \quad 1 \leq m \leq 2N, \\
\frac{\partial F_m}{\partial \phi_{2j}} |_{\phi_2=0} = -\frac{1}{2} \sum_{i=1}^{m-1} <\Lambda^{m-i-1}\Psi_2, \Phi_3 > \Lambda^{m-i}\Psi_3 + \Lambda^{m-1}\Psi_2, \quad 1 \leq m \leq 2N, \\
\frac{\partial F_m}{\partial \phi_{3j}} |_{\phi_2=0} = -\frac{1}{2} \sum_{i=1}^{m-1} <\Lambda^{m-i-1}\Psi_3, \Phi_3 > \Lambda^{m-i-1}\Psi_2 - \Lambda^{m-i}\Psi_2, \quad 1 \leq m \leq 2N.
\end{cases}$$

When \(m=1\), the sum terms take zero value. Firstly, after choosing \(\Phi_1 = \Phi_2 = \Psi_1 = \Psi_2 = 0\) in Eq. (38), we have

$$(0, \cdots, 0, 0, \cdots, 0, -\alpha_1\psi_{31}, \cdots, -\alpha_N\psi_{3N})^T = 0,$$

which means that \(\alpha_k, 1 \leq k \leq N\) equal to zero. So Eq. (38) becomes

$$\sum_{m=1}^{2N} \beta_m((\frac{\partial F_m}{\partial \Phi_1})^T, (\frac{\partial F_m}{\partial \Phi_2})^T, (\frac{\partial F_m}{\partial \Phi_3})^T) = 0.$$  

(39)

Secondly, set \(\Phi_3 = 0\) in (39), then we have

$$\sum_{m=1}^{2N} \beta_m \lambda_m^{m-1} = 0.$$  

(40)
Lastly, setting $\Psi_1 = \Psi_2$ in (39) gives
\[
\sum_{m=2}^{2N} (m - 1)\beta_m \lambda_j^{m-2} = 0. \tag{41}
\]

Note that the determinant of the coefficient matrix of $\beta_m$ in Eqs. (40) and (41) is
\[
\begin{vmatrix}
1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{2N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{2N-1} \\
0 & 1 & 2\lambda_1 & \cdots & (2N - 1)\lambda_1^{2N-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 2\lambda_N & \cdots & (2N - 1)\lambda_N^{2N-2}
\end{vmatrix} = (-1)^N(N-1)/2\prod_{1 \leq i < j \leq N}(\lambda_i - \lambda_j)^4,
\]
which is not zero when $\lambda_j, 1 \leq j \leq N$ are distinct. So we have $\beta_m, 1 \leq m \leq 2N$ are zero, and then obtain that $\{\alpha_k\}_{k=1}^N, \{\beta_m\}_{m=1}^{2N}$ are zero at the same time, which contradicts to the supposition at the beginning of the proof. Therefore, the functions $\{f_k\}_{k=1}^N, \{F_m\}_{m=1}^{2N}$ may be functionally independent at least on certain region of supersymmetry manifold $\mathbb{R}^{4N/2N}$. This is the end of the proof. □

Taking into account the preceding Hamiltonian forms and the independence of integrals of motion, we reach the integrable property of the super FDIH with odd functions.

**Theorem 1** The constrained systems (25) and (34) are completely integrable Hamiltonian systems in the Liouville sense.

### 4 Conclusions and Discussions

In this paper we have extended the binary nonlinearization approach of the (1+1)-dimensional integrable system to the super AKNS system. By this approach, we obtained a super FDIH with odd functions in supersymmetry manifold $\mathbb{R}^{4N/2N}$, and their Hamiltonian forms and integrals of motion are constructed explicitly. The integrable property in the Liouville sense is proved. By comparing with the classical counterpart, the main difficulties of the nonlinearization of the super AKNS system are to find the constraint in eq.(23) and to prove the independence of integrals of motion in Proposition 1 due to the appearance of odd functions. In particular, one advantage of the nonlinearization of the super system is to construct FDIH with odd functions in a systematic way. We know there are very few examples of the FDIH with odd functions.

It is a crucial fact for us to establish the nonlinearization for the super AKNS system that matrix $M$ in its problem belongs to the Lie superalgebra $B(0,1)$. However, we are not
able to do this for the fully supersymmetric AKNS system [14] because its spectral matrix can not be described by a certain Lie superalgebra. For the further research related to this topic, the nonlinearization of the super Dirac system will be discussed in a separate paper. Moreover, how to make nonlinearization of the fully susy AKNS system is an interesting problem.

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