Uniform convergence of V-cycle multigrid algorithms for two-dimensional fractional Feynman-Kac equation

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Abstract When solving large linear systems stemming from the approximation of elliptic partial differential equations (PDEs), it is known that the V-cycle multigrid method (MGM) can significantly lower the computational cost. Many convergence estimates already exist for the V-cycle MGM: for example, using the regularity or approximation assumptions of the elliptic PDEs, the results are obtained in [Bank & Douglas, SIAM J. Numer. Anal. 22, 617-633 (1985); Bramble & Pasciak, Math. Comp. 49, 311-329 (1987)]; in the case of multilevel matrix algebras (like circulant, tau, Hartely) [Aricò, Donatelli & Serra-Capizzano, SIAM J. Matrix Anal. Appl. 26, 186-214 (2004); Aricò & Donatelli, Numer. Math. 105, 511-547 (2007)], special prolongation operators are provided and the related convergence results are rigorously developed, using a functional approach. In this paper we derive new uniform convergence estimates for the V-cycle MGM applied to symmetric positive definite Toeplitz block tridiagonal matrices, by also discussing few connections with previous results. More concretely, the contributions of this paper are as follows: (1) It tackles the Toeplitz systems directly for the elliptic PDEs. (2) Simple (traditional) restriction operator and prolongation operator are employed in order to handle general Toeplitz systems at each level of the recursion. Such a technique is then applied to systems of algebraic equations generated by the difference scheme of the two-dimensional fractional Feynman-Kac equation, which describes the joint probability density function of non-Brownian motion. In particular, we consider the two coarsening strategies, i.e., doubling the mesh size (geometric MGM) and Galerkin approach (algebraic MGM), which lead to the distinct coarsening stiffness matrices in the general case: however, several numerical experiments show that the two algorithms produce almost the same error behaviour.

Keywords V-cycle multigrid method · Block tridiagonal matrix · Fractional Feynman-Kac equation

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1 Introduction

When considering iterative solvers for large linear systems stemming from the approximation of partial differential equations (PDEs), multigrid methods (MG M) (such as backslash cycle, V-cycle and W-cycle) have often been shown to provide algorithms with optimal order of complexity [6, 24]. Using the regularity or approximation assumptions of the elliptic PDEs, the complete proof on the uniform convergence of the MGM for second order elliptic equation has been discussed in [4, 6] and several outstanding works have been derived in this direction, e.g., [7, 8]. On the hand, concerning linear systems with coefficient matrix belonging to multilevel matrix algebras (like circulant, tau, Hartely), the proof of convergence of the two-grid methods are given in [2, 18] and the level independence is discussed in [34], for special prolongation operators [20, 21] associated to the symbol of the coefficient matrices; moreover, the uniform convergence of the V-cycle MGM is further derived in [2] and extended in [11] for the elliptic Toeplitz and PDEs matrices. In recent years, the multigrid methods have also been applied to solve the fractional differential equations (FDEs) [12, 16, 30], for time-dependent FDEs [12, 30], the two-grid method is used and the convergence analysis is performed by following the ideas in [10, 21], in which different prolongation operators are required at each recursion level, when dealing with general Toeplitz systems. In this paper, we use the simple (traditional) restriction operator and prolongation operator to handle general Toeplitz systems directly for the elliptic PDEs. Then we derive new uniform convergence estimates regarding the V-cycle MGM for symmetric positive definite Toeplitz block tridiagonal matrices, which can be applied to the fractional Feynman-Kac (FFK) equation [9, 37]. Regarding numerical experiments, we consider two coarsening strategies for MGM. The first is based on simple coarsening strategy, i.e., doubling the mesh size \( h \to 2h \) in each spatial direction, leading to the so called geometric MGM: in this case the coarse stiffness matrix is the natural analog of the finest grid coefficient matrix. The second strategy is based on the Galerkin approach and is refereed to as algebraic MGM [5, 30]. From the basic theoretical point of view, the major advantage of Galerkin approach is that it satisfies the variational principle; however, from the practical point of view, we find that they almost lead to the same numerical results.

After obtaining the uniform convergence for the V-cycle MGM, we apply it to the difference scheme for the backward fractional Feynman-Kac equation [9], which describes the distribution of the functional of the trajectories of non-Brownian motion, defined by \( U \to A(U) = \int_0^\infty U(x(\tau))d\tau \). There are many special or interesting choices for \( U(x) \), e.g., taking \( U(x) = 1 \) in a given domain and zero otherwise, this functional can be used in kinetic studies of chemical reactions that take place exclusively in the domain [39]. For inhomogeneous disorder dispersive systems, the motion of the particles is non-Brownian, and \( U(x) \) is taken as \( x \) or \( x^2 \) [9]. The multi-dimensional backward fractional Feynman-Kac equation is given as [9, 37]

\[
\frac{\partial}{\partial t} G(x, \rho, t) = \kappa_\alpha D_t \Delta G(x, \rho, t) - \rho G(x, \rho, t) \quad \forall x \in \mathbb{R}^d, \tag{1.1}
\]

where \( G(x, \rho, t) = \int_0^\infty G(x, \rho, t) e^{-\rho A} dA \), \( Re(\rho) > 0, U(x) > 0 \), the diffusion coefficient \( \kappa_\alpha \) is a positive constant and \( \alpha \in (0, 1) \), and the Riemann-Liouville fractional substantial derivative is defined by [13]

\[
\mathcal{D}_t^\alpha G(x, \rho, t) = \mathcal{D}_t^\alpha \mathcal{D}_t^{-\alpha} G(x, \rho, t),
\]

with the fractional substantial integral \( \mathcal{I}_t^\beta \) \((\beta > 0)\) expressed as

\[
\mathcal{I}_t^\beta G(x, \rho, t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} e^{-\rho U(x)(t-\tau)} G(x, \rho, \tau) d\tau, \quad t > 0.
\]
Similarly, we can define the Caputo fractional substantial derivative of order $\alpha$ as
\[
{s^\alpha D^\alpha_t}G(x, \rho, t) = \int_0^t (t-s)^{\alpha-1} \frac{d^n}{ds^n} [D^m_t G(x, \rho, t)] ds.
\]

Then (1.1) can be rewritten in the form [18]
\[
{s^\alpha D^\alpha_t}G(x, \rho, t) = s^{\alpha} D^\alpha_t \left[ G(x, \rho, t) - e^{-\rho t} G(x, \rho, 0) \right] = s^{\alpha} \Delta G(x, \rho, t).
\]

The outline of the paper is as follows. In the next section, we derive the convergence estimates of the V-cycle MGM for the symmetric positive definite Toeplitz tridiagonal matrix. For symmetric positive definite Toeplitz block tridiagonal matrix, the convergence estimates of the V-cycle MGM are given in Section 3. In Section 4, we present the compact difference scheme for (1.2) in 1D, and the centered difference scheme for (1.2) in 2D. Then in Section 5, we use the presented V-cycle MGM framework for the efficient computational solution of the resulting algebraic systems of linear equations. Results of numerical experiments are reported and discussed in Section 6, in order to show the effectiveness of the presented schemes. Finally, we conclude the paper with some remarks.

2 Uniform convergence of V-Cycle MGM for 1D

Let us first consider the simple algebraic system (1D)
\[
A_h \nu^h = f_h,
\]

where
\[
A_h = \text{tridiag}(a_1, a_0, a_1) \quad \text{with} \quad a_0 \geq 2|a_1| \quad \text{and} \quad a_0 > 0.
\]

Let $\Omega \in (0, b)$ and the mesh points $x_i = ih$, $h = b/(M+1)$. To describe the MGM, we need to define the following multiple level of grids
\[
B_k = \{ x_k^i = \frac{i}{2^k} h, i = 1 : M_k \} \quad \text{with} \quad M_k = 2^k - 1, k = 1 : K,
\]

where $B_K = B_h$ is the finest mesh and $M = 2^K - 1$. We adopt the notation that $B_k$ represents not only the grid with grid spacing $h_k = 2^{(K-k)}h$, but also the space of vectors defined on that grid. For the one dimensional system, the restriction operator $P_{k-1}^k$ and prolongation operator $P_{k-1}^k$ are, respectively, defined by [33 p. 438-454]
\[
\nu^{k-1} = P_{k-1}^k \nu^k \quad \text{with} \quad \nu^{k-1}_i = \frac{1}{4} \left( \nu^k_{2i-1} + 2\nu^k_{2i} + \nu^k_{2i+1} \right), \quad i = 1 : M_{k-1},
\]

and
\[
\nu^k = P_{k-1}^k \nu^{k-1} \quad \text{with} \quad P_{k-1}^k = 2 \left( P_{k-1}^{k-1} \right)^T,
\]
where

\[ I_k^{k-1} = \frac{1}{2} \begin{bmatrix} 1 & \cdots & \cdots & \cdots & 1 \\ 2 & 1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \cdots & 2 & 1 \\ 1 & 1 & \cdots & \cdots & \cdots \end{bmatrix}_{M_k \times M_{k-1}}. \]  

(2.5)

The coarse problem is typically defined by the Galerkin approach

\[ A_{k-1} = I_k^{k-1}A_kI_k^{k-1}, \]  

(2.6)

and the intermediate \((k, k-1)\) coarse grid correction operator is

\[ T_k = I_k - I_k^{k-1}A_k^{k-1}I_k^{k-1}A_k = I_k - I_k^{k-1}P_k. \]  

(2.7)

with \(P_k = A_k^{k-1}I_k^{k-1}A_k\).

Let \(K_k\) be the iteration matrix of the smoothing operator. In this work, we take \(K_k\) to be the weighted (damped) Jacobi iteration matrix

\[ K_k = I - S_k A_k, \text{ where } S_k := S_{k, \omega} = \omega D_k^{-1}. \]  

(2.8)

with a weighting factor \(\omega \in (0, 1/2]\), and \(D_k\) is the diagonal of \(A_k\).

A multigrid process can be regarded as defining a sequence of operators \(B_k : \mathcal{B}_k \rightarrow \mathcal{B}_k\) which is an approximate inverse of \(A_k\) in the sense that \(\|I - B_k A_k\|\) is bounded away from one. The V-cycle multigrid algorithm \([6]\) is provided in Algorithm \([6]\).

Since the matrix \(A := A_h\) is symmetric positive definite, we can define the following inner products \([32, \text{p. 78}]\)

\[ (u, v)_D = (Du, v), \quad (u, v)_A = (Au, v), \quad (u, v)_{AD^{-1}A} = (Au, Av)_{D^{-1}}. \]  

(2.9)

where \((\cdot, \cdot)\) is the usual Euclidean inner product. Here the finest grid operator is \(A_h\) or \(A_K\) with the finest grid size \(h\); and the coarse grid operators \(A_{k-1} = I_k^{k-1}A_k I_k^{k-1}\) are defined by the Galerkin approach \((2.6)\) with the grid sizes \(\{2^{k-1}h\}_{k=1}^{K-1}\).

2.1 Improved framework for the MGM

Based on the framework of \([6, 38]\), we now present the estimates on the convergence rate of the MGM, namely,

\[ \|I - B_k A_k\|_{A_h} < 1, \]

where \(I\) is identity matrix and \(A_h, B_k\) are given in Algorithm \([4]\).

Assume that the following two assumptions are satisfied, i.e.,

\[ \frac{\omega}{\lambda_{\text{max}}(A_k)} (\psi^k, \psi^k) \leq (S_k \psi^k, \psi^k) \leq (A_k^{-1} \psi^k, \psi^k) \quad \forall \psi^k \in \mathcal{B}_k, \]  

(2.10)
Algorithm 1 V-cycle Multigrid Algorithm: Define $B_1 = A_1^{-1}$. Assume that $B_{k-1} : \mathcal{R}_{k-1} \to \mathcal{R}_{k-1}$ is defined. We shall now define $B_k : \mathcal{R}_k \to \mathcal{R}_k$ as an approximate iterative solver for the equation $A_k \nu^k = f_k$.

1: Pre-smooth: Let $S_{k,\omega}$ be defined by (2.8), $v_0^k = 0$, $l = 1 : m_1$, and

$$v_l^k = v_{l-1}^k + S_{k,\omega pre}(f_k - A_k v_{l-1}^k).$$

2: Coarse grid correction: Denote $e_{k-1}^l \in \mathcal{B}_{k-1}$ as the approximate solution of the residual equation $A_{k-1} e_{k-1}^l = I_{k-1}^l (f_k - A_k v_{m_1}^k)$ with the iterator $B_{k-1}$:

$$e_{k-1}^l = B_{k-1} I_{k-1}^l (f_k - A_k v_{m_1}^k).$$

3: Post-smooth: $v_{m_1+1}^k = v_{m_1+1}^k + f_{k-1}^l e_{k-1}^l$, $l = m_1 + 2 : m_1 + m_2$, and

$$v_l^k = v_{l-1}^k + S_{k,\omega post}(f_k - A_k v_{l-1}^k).$$

4: Define $B_k f_k = v_{m_1+m_2}^k$.

and

$$|T^k \nu^k|_{\mathcal{B}_k} \leq m_0 ||A_k \nu^k||_{D_{k-1}}^2 \forall \nu^k \in \mathcal{B}_k, \quad (2.11)$$

where $\omega$ is defined by (2.8). For the complete proof on the uniform convergence of the MGM, there exists the following lemma.

Lemma 2.1 ([6, 38]) If $A_k$ satisfies (2.10) and (2.11), then

$$|I - B_k A_k|_{A_k} \leq \frac{m_0}{2l_0 + m_0} < 1 \quad \text{with} \quad 1 \leq k \leq K,$$

where the operator $B_k$ is defined by the V-cycle method in Algorithm 1 and $l$ is the number of smoothing steps.

It is well known that the framework of the convergence analysis of the MGM [6,38] is based on the verification (2.10) and (2.11). However, it is not at all easy to prove the assumption (2.11) in general, since it needs to solve $A_{k-1}^{-1}$ in (2.7). Here, we replace the condition (2.11) by the following Lemma, which simplifies the theoretical investigations substantially.

Lemma 2.2 Let $A_k$ be a symmetric positive definite matrix and

$$\min_{\nu^{k-1} \in \mathcal{R}_{k-1}} ||\nu^{k-1} - I_{k-1}^l \nu^{k-1}||_{A_k}^2 \leq m_0 ||A_k \nu^{k-1}||_{D_{k-1}}^2 \forall \nu^{k-1} \in \mathcal{R}_{k-1}, \quad (2.12)$$

with $m_0 > 0$ independent of $\nu^k$. Then

$$|T^k \nu^k|_{\mathcal{B}_k} \leq m_0 ||A_k \nu^k||_{D_{k-1}}^2 \forall \nu^k \in \mathcal{B}_k.$$
Proof. From (2.12) and the variational principle for coarse grid operator $T^k$ (see the corollary of (36) p. 431)), we obtain

$$||T^k v_k||^2_{A_k} = \min_{v^{k-1} \in A_{k-1}} ||v^k - T^k v^{k-1}||^2_{A_k} \leq m_0 ||A_k v^k||^2_{D_{k+1}}.$$ 

The proof is completed.

Using Lemmas 2.1 and 2.2 we have

**Theorem 2.1** If $A_k$ satisfies (2.10) and (2.12), then

$$||I - B_k A_k||_{A_k} \leq \frac{m_0}{2lT_0 + m_0} < 1 \quad \text{with} \quad 1 \leq k \leq K,$$

where the operator $B_k$ is defined by the $V$-cycle method in Algorithm 1 and $l$ is the number of smoothing steps.

2.2 Convergence estimates of MGM for 1D

We now give a complete proof on the uniform convergence of the MGM for the algebraic system (2.1), i.e., we need to examine the two assumptions (2.10) and (2.12).

**Lemma 2.3** Let $A^{(1)} = \{a_{i,j}^{(1)}\}_{i,j=1}^m$ with $a_{i,j}^{(1)} = a_{i-j}^{(1)}$ be a symmetric Toeplitz matrix and $A^{(k)} = L^H_n A^{(k-1)} L^H_n$ with $L^H_n = 4k^{k-1}$ and $L^H_n = (L^H_n)^T$. Then $A^{(k)}$ can be computed by

$$a_0^{(k)} = (4C_k + 2^{k-1})a_0^{(1)} + \sum_{m=1}^{2.2^{k-1}-1} 1C_m^{(1)}a_m^{(1)};$$

$$a_1^{(k)} = C_1 a_0^{(1)} + \sum_{m=1}^{2.2^{k-1}-1} 1C_m^{(1)}a_m^{(1)};$$

$$a_j^{(k)} = \sum_{m=(j-2)2^{k-1}}^{(j-2)2^{k-1}+1} 1C_m^{(1)}a_m^{(1)} \quad \forall j \geq 2 \quad \forall k \geq 2$$

with $C_k = 2^{k-2} - \frac{2^{k-2}-1}{2}$. And

$$1C_m^{(k)} = \begin{cases} 8C_k - (m^2 - 1)(2^k - m) & \text{for } m = 1 : 2^{k-1}; \\ \frac{1}{3}(2^k - m)(2^k - m + 1) & \text{for } m = 2^{k-1} : 2 \cdot 2^{k-1} - 1; \end{cases}$$

and

\[ 1C_m^{(k)} = \begin{cases} 2C_k + \frac{1}{3}(m - 2^{k-1} - 1)(m - 2^{k-1}) & \text{for } m = 1 : 2^{k-1}; \\ 2C_k + (2^k - m)^2 \cdot 2^{k-1} - \frac{2}{3}(2^k - m - 1)(2^k - m) & \text{for } m = 2^{k-1} : 2 \cdot 2^{k-1}; \\ \frac{1}{6}(m - 2^{k-1} - 1)(m - 2^{k-1})(m - 2^{k-1} + 1) & \text{for } m = 2^{k-1} : 2 \cdot 2^{k-1}; \\ \frac{1}{6}(3 \cdot 2^{k-1} - m - 1)(3 \cdot 2^{k-1} - m)(3 \cdot 2^{k-1} - m + 1) & \text{for } m = 2 \cdot 2^{k-1} : 3 \cdot 2^{k-1} - 1; \end{cases} \]
and for $j \geq 2$,

$$\phi_m^k = \begin{cases} 
\varphi_1 & \text{for } m = (j - 2)2^{k-1} : (j - 1)2^{k-1}; \\
\varphi_2 & \text{for } m = (j - 1)2^{k-1} : j2^{k-1}; \\
\varphi_3 & \text{for } m = j2^{k-1} : (j + 1)2^{k-1}; \\
\varphi_4 & \text{for } m = (j + 1)2^{k-1} : (j + 2)2^{k-1} - 1,
\end{cases}$$

where

$$\varphi_1 = \frac{1}{6}(m - (j - 2)2^{k-1} - 1)(m - (j - 2)2^{k-1})(m - (j - 2)2^{k-1} + 1);$$

$$\varphi_2 = 2C_k + (m - (j - 1)2^{k-1})^2 \cdot 2^{k-1}$$

$$- \frac{1}{6}(j2^{k-1} - m - 1)(j2^{k-1} - m)(j2^{k-1} - m + 1) - \frac{2}{3}(m - (j - 1)2^{k-1} - 1)(m - (j - 1)2^{k-1})(m - (j - 1)2^{k-1} + 1);$$

$$\varphi_3 = 2C_k + ((j + 1)2^{k-1} - m)^2 \cdot 2^{k-1}$$

$$- \frac{1}{6}(m - j2^{k-1} - 1)(m - j2^{k-1})(m - j2^{k-1} + 1) - \frac{2}{3}(j + 1)2^{k-1} - m - 1)((j + 1)2^{k-1} - m)((j + 1)2^{k-1} - m + 1);$$

$$\varphi_4 = \frac{1}{6}((j + 2)2^{k-1} - m - 1)((j + 2)2^{k-1} - m)((j + 2)2^{k-1} - m + 1).$$

**Proof** See the Appendix.

**Corollary 1** Let $A^{(k)} = I_{k}^{-1}A^{(k-1)}I_{k-1}^k$ with $A^{(1)} = \text{tridiag}(a_1, a_0, a_1)$. Then

$$A^{(k)} = \text{tridiag}(a^{(k)}_1, a^{(k)}_0, a^{(k)}_1),$$

where

$$a^{(k)}_0 = \frac{1}{8^{k-1}} \left[ (4C_k + 2^{k-1})a_0 + 8C_k a_1 \right],$$

and

$$a^{(k)}_1 = \frac{1}{8^{k-1}} \left[ C_k a_0 + \left( 2C_k + 2^{k-1} \right) a_1 \right].$$

**Proof** From Lemma 2.3 the desired results can be obtained.

**Lemma 2.4** Let $A^{(1)} = A_k$ be defined by (2.7) and $A^{(k)} = I_{k}^{-1}A^{(k-1)}I_{k-1}^k$. Then

$$\frac{\theta}{\lambda_{\max}(A^{(k)})}(v^k, v^k) \leq (S_k v^k, v^k) \leq (A_k^{-1} v^k, v^k) \quad \forall v^k \in \mathbb{R}_k,$$

where $A_k = A^{(k-1,k-1)}$, $S_k = \omega D_k^{-1}$, $\omega \in (0, 1/2]$ and $D_k$ is the diagonal of $A_k$. 


Proof According to Corollary 1 we have
\[ A^{(k)} = \mu_1 \cdot \text{tridiag}(-1, 2, -1) + \mu_2 \cdot \text{tridiag}(1, 2, 1) =: A_1^{(k)} + A_2^{(k)} \] (2.14)
with
\[ \mu_1 = \frac{2C_k (a_0 + 2a_1) + 2^{k-1} (a_0 - 2a_1)}{4 \cdot 8^{k-1}} > 0, \]
and
\[ \mu_2 = \frac{(6C_k + 2^{k-1}) (a_0 + 2a_1)}{4 \cdot 8^{k-1}} \geq 0. \]
Taking \( A^{(k)} = \{a_{i,j}^{(k)}\}_{i,j=1}^{\infty}, a_{i,j}^{(k)} = a_{|i-j|}^{(k)} \quad \forall k \geq 1 \) and using (2.14), we obtain
\[ r_i^{(k)} := \sum_{j \neq i} |a_{i,j}^{(k)}| < a_{i,i}^{(k)}. \]

From the Gerschgorin circle theorem [25, p. 388], the eigenvalues of \( A^{(k)} \) are in the disks centered at \( a_{i,i}^{(k)} \) with radius \( r_i^{(k)} \), i.e., the eigenvalues \( \lambda \) of the matrix \( A^{(k)} \) satisfy
\[ |\lambda - a_{i,i}^{(k)}| \leq r_i^{(k)}, \]
which yields \( \lambda_{\max}(A^{(k)}) \leq a_{i,i}^{(k)} + r_i^{(k)} < 2a_{i,i}^{(k)} \).

On the other hand, using the Rayleigh theorem [25, p. 235], i.e.,
\[ \lambda_{\max}(A^{(k)}) = \max_{x \neq 0} \frac{x^T A^{(k)} x}{x^T x} \quad \forall x \in \mathbb{R}^n, \]
if we take \( x = [1, 0, \ldots, 0]^T \), it means that
\[ \lambda_{\max}(A^{(k)}) \geq \frac{x^T A^{(k)} x}{x^T x} = a_{1,1}^{(k)}. \]

Hence, we obtain
\[ \lambda_{\max} \left( D^{(k)} (A^{(k)})^{-1} \right) = \frac{\lambda_{\max}(A^{(k)})}{a_{1,1}^{(k)}} \in [1, 2), \]
where \( D^{(k)} \) is the diagonal of \( A^{(k)} \). It yields
\[ 1 \leq \lambda_{\max}(D^{-1} A_k) < 2 \quad \forall k \in \mathbb{R}. \]
The proof is completed.

Lemma 2.5 Let \( L_a = \text{tridiag}(b, a, b) \) and \( L_c = \text{tridiag}(d, c, d) \) be symmetric positive definite. Then \( L_a L_c \) is symmetric positive definite.

Proof Since \( L_a L_c \) is a symmetric matrix, it yields \( L_a L_c = L_c L_a \) by [25] p. 233]. Moreover, using [25] p. 490] leads to that \( L_a L_c \) is symmetric positive definite.
Lemma 2.6 Let $A^{(1)} := A_k$ be defined by (2.7) and $A^{(k)} := I_k^{-1}A^{(k-1)}I_k^{-1}$. Then

$$\min_{v^{k-1} \in \mathcal{B}_{k-1}} \|v^k - I_k^{-1}v^{k-1}\|_{A_k}^2 \leq m_0\|A_kv^k\|_{p_k}^2 \quad \forall v^k \in \mathcal{B}_k$$

with $A_k = A^{(k-k+1)}$ and $m_0 = (1 + \bar{m}_0)^2$, where

$$\bar{m}_0 = \max \left\{ \frac{(6C_k + 2^{k-1})(a_0 + 2a_1)}{(2C_k + 2^{k-1})(a_0 + 2a_1) - 2^{k+1}a_1} \quad \forall k \geq 1 \right\}$$

and $C_k = 2^{k-2} \cdot \frac{2^{2k-2} - 1}{3}$. In particular,

$$m_0 = \begin{cases} 1 & \text{if } a_0 + 2a_1 = 0; \\ 16 & \text{if } a_1 \leq 0; \\ \max\{25, 4a_0^2/(a_0 - 2a_1)^2\} & \text{if } a_1 > 0, a_0 \neq 2a_1. \end{cases}$$

Proof Let an odd number $M_k$ be defined by (2.2). For any

$$v^k = (v^k_1, v^k_2, \ldots, v^k_{M_k})^T \in \mathcal{B}_k \quad \text{and} \quad v^k_0 = v^k_{M_k + 1} = 0,$$

taking $v^{k-1} = (v^{k-1}_1, v^{k-1}_2, \ldots, v^{k-1}_{M_{k-1}})^T \in \mathcal{B}_{k-1}$ yields

$$v^{k-1} = T v^k,$$

where the cutting matrix is defined by

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \end{bmatrix}_{M_{k-1} \times M_k}.$$ (2.15)

Therefore, we have

$$v^k - I_k^{-1}v^{k-1} = \left( I - I_k^{-1}T \right) v^k = \left( v^k_j - \frac{v^k_j + v^k_{j+1}}{2}, v^k_j - \frac{v^k_j + v^k_{j+1}}{2}, \ldots, v^k_{M_k} - \frac{v^k_{M_k} + v^k_{M_k + 1}}{2} \right)^T$$

(2.16)

Let $L_{M_k} = \text{tridiag}(-1, 2, -1)$ be the $M_k \times M_k$ one dimensional discrete Laplacian. According to (2.9) and (2.10), there exists

$$\|v^k - I_k^{-1}v^{k-1}\|_{L_{M_k}}^2 = 2\|v^k - I_k^{-1}v^{k-1}\|^2 \leq \frac{1}{2}\|L_{M_k}v^k\|^2,$$

(2.17)

since

$$2 \sum_{j=1}^{(M_k+1)/2} \left( v^k_{j-1} - \frac{v^k_{j-2} + v^k_{j+1}}{2} \right)^2 \leq 2 \sum_{j=1}^{M_k} \left( v^k_j - \frac{v^k_{j-1} + v^k_{j+1}}{2} \right)^2 = \frac{1}{2}\|L_{M_k}v^k\|^2.$$
From (2.16), (2.17) and (2.14), we get

\[ \frac{\mu_1}{2\mu_1 + 2\mu_2} |L_{MK_{k+1}} v^{K-k-1}|^2, \]

which yields

\[ ||A^{(k)} v^{K-k-1}||^2 \geq ||A^{(k)} v^{K-k-1}||^2 = \mu_2^2 |L_{MK_{k+1}} v^{K-k-1}|^2, \]

where \( D^k \) is the diagonal of \( A^{(k)} \). Using (2.18) and (2.19), there exists

\[ ||v^{K-k-1} - P_{K-k}^k v^{K-k}||_{A^{(k)}}^2 \leq \frac{\mu_1 + \mu_2}{2} |L_{MK_{k+1}} v^{K-k-1}|^2 \]

with

\[ \frac{\mu_2}{\mu_1} = \frac{(6C_k + 2^k-1) (a_0 + 2a_1)}{(2C_k + 2^k-1) (a_0 + 2a_1) - 2^{k+1}a_1} \geq 0 \quad \text{for } k \geq 1; \]

when \( k = 1 \), it can be simplified as

\[ \frac{\mu_2}{\mu_1} = \frac{a_0 + 2a_1}{a_0 - 2a_1}, \]

In particular, there exists

\[ \frac{\mu_2}{\mu_1} \begin{cases} = 0 & \text{if } a_0 + 2a_1 = 0 \quad \forall k \geq 1; \\ < 3 & \text{if } a_1 \leq 0 \quad \forall k \geq 1; \\ \leq 4 & \text{if } a_1 > 0 \quad \forall k \geq 2, \end{cases} \]

since

\[ \frac{\mu_2}{\mu_1} = 3 + \frac{-2^k a_0 + 4 \cdot 2^k a_1}{(2C_k - 2^{k-1}) (a_0 + 2a_1) + 2^k a_0} \]

\[ \leq 3 + \frac{2^k a_0}{(2C_k - 2^{k-1}) (a_0 + 2a_1) + 2^k a_0} \leq 4 \quad \text{with } a_1 > 0 \quad \forall k \geq 2. \]

Hence

\[ \min_{v^{k-1} \in \mathcal{B}_{k-1}} ||v^{k-1} - P_{k-1}^k v^{k-1}||_{A_k}^2 \leq m_0 ||A_k v^{k}||_{D_k}^2 \quad \forall v^{k} \in \mathcal{B}_k, \]

where \( m_0 = (1 + \bar{m}_0)^2 \) with

\[ \bar{m}_0 = \max \left\{ \frac{(6C_k + 2^k-1) (a_0 + 2a_1)}{(2C_k + 2^k-1) (a_0 + 2a_1) - 2^{k+1}a_1} \quad \forall k \geq 1 \right\} \]
and \( C_k = 2^{k-2} \cdot \frac{2^{n-1}}{3} \). In particular, from (2.20)-(2.23), there exists

\[
m_0 = (1 + \tilde{m}_0)^2 = \begin{cases} 1 & \text{if } a_0 + 2a_1 = 0; \\ 16 & \text{if } a_1 \leq 0; \\ \max\{25, 4a_0^2/(a_0 - 2a_1)^2\} & \text{if } a_1 > 0, a_0 \neq 2a_1,
\end{cases}
\]

where we use

\[
(1 + a_0 + 2a_1/a_0 - 2a_1)^2 = 4a_0^2/(a_0 - 2a_1)^2.
\]

The proof is completed.

**Remark 2.1** When \( a_0 = 2a_1 \), according to the theory of Toeplitz matrices generated by a function \([23]\), the generation function of the considered tridiagonal Toeplitz matrices is \( f(\theta) = 2a_1(1 + \cos \theta) \) and in that case the symbol has a zero at \( \theta = \pi \): following the results in \([23]\) [page 292, eq. (10), and Section 2.2.3], necessarily the symbol associated with the prolongation/restriction operator has to show a zero at 0 and has to positive at \( \pi \). This shows that the considered operators with stencil \([1 2 1]\) cannot be used in agreement with the considered condition, but the only possible tridiagonal choice is \([-1 2 -1]\). In fact, we know that the condition in Lemma 2.6 is not only sufficient for optimality as shown here, but it is also necessary (see \([20, 21, 2, 1]\)).

The same type of connection is observed for the 2D case developed in Section 3.

According to Lemmas 2.4, 2.6 and Theorem 2.1, we obtain the following result.

**Theorem 2.2** For the algebraic system (2.1), we find

\[
||I - B_k A_k||_{A_k} \leq m_0/(2\omega + m_0) < 1 \quad 1 \leq k \leq K, \quad \omega \in (0, 1/2],
\]

where the operator \( B_k \) is defined by the V-cycle method in Multigrid Algorithm and \( I \) is the number of smoothing steps and \( m_0 \) is given in Lemma 2.6.

### 3 Uniform convergence of V-Cycle MGM for 2D

In this section, we consider the symmetric positive definite Toeplitz block tridiagonal matrix. As an interesting example, we study the algebraic system

\[
A_h \mathbf{v} = \mathbf{f}_h,
\]

where

\[
A_h = c_1 I \otimes I + c_2 (I \otimes L + L \otimes I), \quad c_1 \geq 0, \quad c_2 > 0,
\]

and \( I \) is identity matrix, \( L = \text{tridiag}(-1, 2, -1) \). This example arises, for instance, from the discretization of the Poisson equations \( (c_1 = 0) \) in a square or the heat equations or the time fractional PDEs \([18, 21, 26, 29]\).

In 2D, the notations can be defined in a straightforward manner from the 1D case. Let \( \Omega \in (0, b) \times (0, b) \) and the mesh points \( x_i = ih, y_j = jh, h = b/(M+1) \). We still use the notation that \( \mathcal{B}_k \) represents not only the grid with grid spacing \( h_k = 2^{(K-k)}h \), but also the space of vectors defined on that grid, where

\[
\mathcal{B}_k = \left\{ (x_i, y_j) \middle| x_i = \frac{i}{2^k}b, y_j = \frac{j}{2^k}b, i, j = 1 : M_k \right\}
\]
with $M_k = 2^k - 1$, $k = 1 : K$.

For the two dimensional system, the restriction operator $I_k^{k-1}$ and prolongation operator $I_k^k$ [33, p. 436-439] are, respectively, defined by

$$I_k^{k-1} = P \otimes P := I_k^{k-1} \otimes I_k^{k-1},$$

(3.3)

where $I_k^{k-1}$ is defined by (2.5), and

$$I_k^k = 4 \left( I_k^{k-1} \right)^T.$$

The coarse problem is typically defined by the Galerkin approach

$$A_k^{k-1} = I_k^{k-1} A_k I_k^{k-1}, \quad f_k^{k-1} = I_k^{k-1} f_k.$$

(3.4)

Let $K_k$ be the iteration matrix of the smoothing operator. In this work, we take $K_k$ to be the weighted (damped) Jacobi iteration matrix

$$K_k = I - S_k A_k, \quad \text{where} \quad S_k := S_k^\omega = \omega D_k^{-1}$$

(3.5)

with a weighting factor $\omega \in (0, 1/4]$, and $D_k$ is the diagonal of $A_k$.

3.1 Convergence estimates of MGM for 2D

We now give a complete proof on the uniform convergence of the MGM for the algebraic system (3.1), i.e., we need to examine the two assumptions (2.10) and (2.12). First, we give some lemmas.

**Lemma 3.1** [11, p. 5] Let $A$ be a symmetric matrices. Then

$$\lambda_{\min}(A) = \min_{x \neq 0} \frac{x^T A x}{x^T x}, \quad \lambda_{\max}(A) = \max_{x \neq 0} \frac{x^T A x}{x^T x}.$$

**Lemma 3.2** [31, p. 27] The matrix $A \in \mathbb{C}^{n \times n}$ is positive definite if and only if it is hermitian and has positive eigenvalues.

**Lemma 3.3** [28, p. 140] Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{r \times s}$, $C \in \mathbb{R}^{n \times p}$, and $D \in \mathbb{R}^{s \times t}$. Then

$$(A \otimes B)(C \otimes D) = AC \otimes BD \quad (\in \mathbb{R}^{mr \times pt}).$$

Moreover, for all $A$ and $B$, $(A \otimes B)^T = A^T \otimes B^T$.

**Lemma 3.4** [28, p. 141] Let $A \in \mathbb{R}^{n \times n}$ and $\{\lambda_i\}_{i=1}^n$ be its eigenvalues; let $B \in \mathbb{R}^{m \times m}$ and $\{\mu_j\}_{j=1}^m$ be its eigenvalues. Then the mn eigenvalues of $A \otimes B$ are

$$\lambda_1 \mu_1, \ldots, \lambda_1 \mu_m, \lambda_2 \mu_1, \ldots, \lambda_2 \mu_m, \ldots, \lambda_n \mu_1, \ldots, \lambda_n \mu_m.$$

**Lemma 3.5** [22, p. 396] If $P$ and $P + Q$ are $n$-by-$n$ symmetric matrices, then

$$\lambda_k(P) + \lambda_1(Q) \leq \lambda_k(P + Q) \leq \lambda_k(P) + \lambda_n(Q), \quad k = 1, 2, \ldots, n.$$
Lemma 3.6 Let \( A^{(k)} := A_k \) be defined by (3.7) and \( \lambda_{\text{max}}(A^{(k)}) = I_{k-1}^k A^{(k-1)} I_{k-1}^k. \) Then

\[
\frac{\omega}{\lambda_{\text{max}}(A_k)} < (S_k v^k, v^k) \leq (A_k^{-1} v^k, v^k), \quad \forall v^k \in \mathbb{R}_k,
\]

where \( A_k = A^{(k-1)} \), \( S_k = \omega D_k^{-1} \), \( \omega \in (0, 1/4) \) and \( D_k \) is the diagonal of \( A_k \).

Proof Given a sequence \( Z^{(k)} \) and \( Z^{(k)}, k \geq 1 \), we denote

\[
Z^{(k)} = I_{k-1}^k Z^{(k-1)} I_{k-1}^k \quad \text{and} \quad Z^{(k)} = I_{k-1}^k Z^{(k-1)} I_{k-1}^k. \tag{3.6}
\]

In the following, \( Z \) and \( Z \) given in (3.6) can also be taken as \( A \) and \( M \), etc.

Taking the block matrix

\[
Z^{(k)} = M^{(1)} \otimes N^{(1)}, \tag{3.7}
\]

there exists

\[
Z^{(k)} = \left( I_{k-1}^k M^{(k-1)} R_{k-1}^k \right) \otimes \left( I_{k-1}^k N^{(k-1)} R_{k-1}^k \right) = M^{(k)} \otimes N^{(k)}. \tag{3.8}
\]

Combining (3.6), (3.8) and \( A^{(1)} = c_1 I \otimes I + c_2 (I \otimes L + L \otimes I) \), we obtain

\[
A^{(k)} = c_1 I^{(k)} \otimes I^{(k)} + c_2 \left( I^{(k)} \otimes L^{(k)} + L^{(k)} \otimes I^{(k)} \right). \tag{3.9}
\]

According to Corollary 1 and (2.14), we have

\[
I^{(k)} = \frac{1}{8^{k-1}} \text{tridiag}(C_k, 4C_k, 2^{k-1}, C_k) = \theta_1 I + \theta_2 \bar{L};
\]

\[
L^{(k)} = \frac{1}{8^{k-1}} \text{tridiag}(-2^{k-1}, 2^k, -2^{k-1}) = \theta_2 L, \tag{3.10}
\]

where \( \bar{L} = \text{tridiag}(1, 2, 1), \) \( L = \text{tridiag}(-1, 2, -1) \) and

\[
\theta_1 = \frac{2C_k + 2^{k-1}}{8^{k-1}} > 0, \quad \theta_2 = \frac{2^{k-1}}{8^{k-1}} > 0, \quad \theta_3 = \frac{C_k}{8^{k-1}}. \tag{3.11}
\]

Next we prove

\[
1 \leq \lambda_{\text{max}} \left( \left( D^{(k)} \right)^{-1} A^{(k)} \right) < 4.
\]

The maximum eigenvalues of \( I^{(k)} \) and \( L^{(k)} \) are, respectively, given by [35, p. 702]

\[
\lambda_{\text{max}} (I^{(k)}) \leq \frac{6C_k + 2^{k-1}}{8^{k-1}} = 3\theta_1 - 2\theta_2, \quad \lambda_{\text{max}} (L^{(k)}) \leq \frac{2^{k+1}}{8^{k-1}} = 4\theta_2.
\]

Using Lemmas 3.5, 3.5, and (3.9), we obtain

\[
\lambda_{\text{max}} (A^{(k)}) < \eta_1 \quad \text{with} \quad \eta_1 = c_1 \left( 3\theta_1 - 2\theta_2 \right)^2 + 8c_2 \left( 3\theta_1 - 2\theta_2 \right) \theta_2, \tag{3.12}
\]

and

\[
\eta_2 := \lambda (D^{(k)}) = \lambda_{\text{max}} (D^{(k)}) = \lambda_{\text{min}} (D^{(k)})
\]

\[
= c_1 \left( \frac{4C_k + 2^{k-1}}{8^{k-1}} \right)^2 + 2c_2 \frac{2^k}{8^{k-1}}, \quad \frac{4C_k + 2^{k-1}}{8^{k-1}} \quad \text{and} \quad 2c_2 \left( 2\theta_1 - \theta_2 \right)^2 + 4c_2 \left( 2\theta_1 - \theta_2 \right) \theta_2, \tag{3.13}
\]
which yields
\[ \lambda_{\text{max}} \left( \left( \mathbf{D}^{(k)} \right)^{-1} \mathbf{A}^{(k)} \right) < \frac{\eta_1}{\eta_2} < 4. \tag{3.14} \]

If we take \( x = [1, 0, \ldots, 0]^T \), then
\[ \lambda_{\text{max}} (\mathbf{A}^{(k)}) \geq \frac{\mathbf{x}^T \mathbf{A}^{(k)} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \lambda_{\text{max}} (\mathbf{D}^{(k)}). \]

The proof is completed.

**Lemma 3.7.** Let \( \mathbf{A}^{(1)} := \mathbf{A}_k \) be defined by (3.11) and \( \mathbf{A}^{(k)} = \mathbf{M}_k^{(k-1)} \mathbf{A}^{(k-1)} \mathbf{M}_k^{(k-1)^T} \). Then
\[ \min_{\mathbf{v} \in \mathbb{R}^{(k-1)}} ||\mathbf{v}^k - \mathbf{I}_k^{(k-1)} \mathbf{v}^{k-1}||^2 \leq m_0 ||\mathbf{A}_k \mathbf{v}^k||^2_{\mathbf{D}_k^{-1}} \quad \forall \mathbf{v}^k \in \mathcal{B}_k \]

with \( \mathbf{A}_k = \mathbf{A}^{(k-k+1)} \) and \( m_0 = 1536 < \infty \).

**Proof.** Let an odd number \( M_k \) be defined by (3.2). For any
\[ \mathbf{v}^k = (v^k_1, v^k_2, \ldots, v^k_{M_k})^T \in \mathcal{B}_k \quad \text{and} \quad \mathbf{v}^0 = \mathbf{v}^{M_k+1} = 0 \]
with \( \mathbf{v}^k = (v^k_{i,1}, v^k_{i,2}, \ldots, v^k_{i,M_k})^T \), and taking
\[ \mathbf{v}^{k-1} = (\tilde{v}^k_1, \tilde{v}^k_2, \ldots, \tilde{v}^k_{M_k-1})^T \in \mathcal{B}_{k-1}, \]
with \( \tilde{v}^k = (v^k_{i,2}, v^k_{i,4}, \ldots, v^k_{i,M_k-1})^T \), there exists
\[ \tilde{v}^k = T \tilde{v}^k, \]
where the cutting matrix \( T \) is defined by (2.15). Using the above equations, it yields
\[ \mathbf{v}^{k-1} = (T \otimes T) \mathbf{v}^k. \tag{3.15} \]

From (3.3) and (3.15), we get
\[ \mathbf{I}_k^{(k-1)} \mathbf{v}^{k-1} = (PT \otimes PT) \mathbf{v}^k. \tag{3.16} \]

Thus
\[ \mathbf{v}^k - \mathbf{I}_k^{(k-1)} \mathbf{v}^{k-1} = (I \otimes I - PT \otimes PT) \mathbf{v}^k, \]
\[ = \left( v^k_1 - \frac{PT}{2} \left( v^k_0 + v^k_2 \right), (I - PT) v^k_2, v^k_3 - \frac{PT}{2} \left( v^k_2 + v^k_4 \right), (I - PT) v^k_4, \ldots, (I - PT) v^k_{M_k-1}, v^k_{M_k} - \frac{PT}{2} \left( v^k_{M_k-1} + v^k_{M_k+1} \right) \right)^T. \tag{3.17} \]

Hence, we obtain
\[ ||\mathbf{v}^k - \mathbf{I}_k^{(k-1)} \mathbf{v}^{k-1}||^2 \]
\[ = \sum_{i=1}^{(M_k+1)/2} ||v^k_{2i-1} - \frac{PT}{2} \left( v^k_{2i-2} + v^k_{2i} \right)||^2 + \sum_{i=1}^{(M_k+1)/2} \left( (I - PT) v^k_{2i} \right)^2 \]
\[ \leq 2 \sum_{i=1}^{(M_k+1)/2} ||v^k_{2i-1} - \frac{v^k_{2i-2} + v^k_{2i}}{2}||^2 + 3 \sum_{i=1}^{(M_k+1)/2} \left( (I - PT) v^k_{2i} \right)^2, \tag{3.18} \]
where we use
\[
\left\| v_{2j-1} - \frac{PT}{2} \left( v_{2j-2} + v_{2j} \right) \right\|^2 \\
\leq 2 \left\| v_{2j-1} - \frac{v_{2j-2} + v_{2j}}{2} \right\|^2 + \left\| (I - PT) v_{2j-2} \right\|^2 + \left\| (I - PT) v_{2j} \right\|^2.
\]

From (2.17), we have \( \left\| (I - PT) v_{2j} \right\|^2 \leq \frac{3}{4} \|Lx_k^j\|^2 \), which yields
\[
\frac{(M_k+1)^{1/2}}{4} \sum_{j=1}^{(M_k+1)^{1/2}} \left\| (I - PT) v_{2j} \right\|^2 \\
\leq \frac{1}{4} \sum_{j=1}^{M_k} \|Lx_k^j\|^2 = \frac{1}{4} \left( (I \otimes L^2) \psi, \psi \right),
\]
and
\[
\frac{(M_k+1)^{1/2}}{4} \sum_{j=1}^{(M_k+1)^{1/2}} \left\| v_{2j-1} - \frac{v_{2j-2} + v_{2j}}{2} \right\|^2 \\
\leq \frac{1}{4} \left( (I \otimes L) \psi, \psi \right).\tag{3.19}
\]

According to (3.12) and (3.18)-(3.20), there exists
\[
\|v^{K-k+1} - \text{I}_{K-k} \|_{A^{(k)}}^2 \\
\leq \lambda_{\text{max}} \left( A^{(k)} \right) \left\| v^{K-k+1} - \text{I}_{K-k} \right\|^2 \\
\leq \frac{3n_1}{4} \left( (I \otimes L^2 + L^2 \otimes I) v^{K-k+1}, v^{K-k+1} \right).\tag{3.21}
\]

From Lemmas [3.2] and [3.4] and [2.5], we know that the matrix \( AC \otimes BD \) is symmetric positive definite, where \( A \) (or \( B, C, D \)) can be chosen as \( I \) (or \( L, L \)). Thus using (3.19) and (3.10), there exists
\[
\left\| A^{(k)} v^{K-k+1} \right\|^2 \geq \left\| A^{(k)} v^{K-k+1} \right\|^2 \geq \left( B^{(k)} v^{K-k+1}, v^{K-k+1} \right) \\
\geq \left( C^{(k)} v^{K-k+1}, v^{K-k+1} \right),\tag{3.22}
\]
where
\[
A^{(k)}_1 = c_1 \theta_1^2 I \otimes I + c_2 \theta_1 \theta_2 (I \otimes L + L \otimes I), \\
B^{(k)} = c_1^2 \theta_1^2 I \otimes I + 2c_1 \theta_1 \theta_2 (I \otimes L + L \otimes I) + c_2^2 \theta_1 \theta_2^2 (I \otimes L^2 + L^2 \otimes I), \tag{3.23}
\]
\[
C^{(k)} = \eta_3 (I \otimes L^2 + L^2 \otimes I) \quad \text{with} \quad \eta_3 = \frac{c_1^2 \theta_1^4}{32} + \frac{c_1 \theta_1 \theta_2^2}{2} + c_2 \theta_1^2 \theta_2^2.
\]
Combining (3.13) and (3.21) - (3.23), we have
\[
\frac{1}{n_2} \left\| A^{(k)} v^{K-k+1} \right\|^2 \\
\geq \frac{\eta_3}{n_2} \left( (I \otimes L^2 + L^2 \otimes I) v^{K-k+1}, v^{K-k+1} \right) \\
\geq \frac{4 \eta_3}{3 \eta_1 n_2} \|v^{K-k+1} - \text{I}_{K-k} v^{K-k} \|^2 \|A^{(k)} - \text{I}_{A^{(k)}} \|^2.
\]
According to (3.13), (3.14) and (3.23), there exists
\[
\frac{4\eta_3}{3\eta_1 \eta_2} > \frac{\eta_3}{3\eta_2^2} > \frac{\eta_3}{48 (c_1 \theta_1^2 + 2c_2 \theta_1 \theta_2)^2} > \frac{1}{1536} > 0, \quad \forall k \geq 1.
\]
More concretely, from (3.14) we get \(\frac{4\eta_3}{3\eta_1 \eta_2} > \frac{\eta_3}{3\eta_2^2} \) and using (3.13) and (3.23), there exists
\[
\eta_2 = c_1 (2\theta_1 - \theta_2)^2 + 4c_2 (2\theta_1 - \theta_2) \theta_2 \leq c_1 (2\theta_1)^2 + 4c_2 2\theta_1 \theta_2 = 4 (c_1 \theta_1^2 + 2c_2 \theta_1 \theta_2),
\]
and
\[
\frac{4\eta_3}{3\eta_1 \eta_2} > \frac{\eta_3}{48 (c_1 \theta_1^2 + 2c_2 \theta_1 \theta_2)^2} > \frac{(c_1 \theta_1^2 + 2c_2 \theta_1 \theta_2)^2}{32 \times 48 (c_1 \theta_1^2 + 2c_2 \theta_1 \theta_2)^2} = \frac{1}{1536}.
\]
Hence
\[
\min_{\nu^{k-1} \in \mathcal{B}_{k-1}} \|v^k - I_{k-1}^k v^{k-1}\|_{A_k}^2 \leq 1536 \|A_k v^k\|_{D_k}^2, \quad \forall v^k \in \mathcal{B}_k.
\]
The proof is completed.

Following the above results, we obtain the uniform convergence of the V-cycle Multigrid method.

**Theorem 3.1** For the algebraic system (3.7), it satisfies
\[
\|I - B_k A_k\|_{A_k} \leq \frac{m_0}{2l\omega + m_0} < 1 \quad \text{with} \quad 1 \leq k \leq K, \quad \omega \in (0, 1/4],
\]
where the operator \(B_k\) is defined by the V-cycle method in Multigrid Algorithm 4 and \(l\) is the number of smoothing steps and \(m_0\) is given in Lemma 3.7.

**Remark 3.1** Based on the above analysis, the convergence estimates of MGM is easy to obtain for the two-dimensional compact difference scheme \(A_k v^k = f_k\), where \(A_k = c_1 H \otimes H + c_2 (H \otimes L + L \otimes H)\), and the matrix \(H = \frac{1}{12} \text{tridiag}(1, 10, 1)\).

### 4 The finite difference scheme for Feynman-Kac equation

Let \(T > 0, \Omega = (0, h) \times (0, h)\). Without loss of generality, we add a force term \(f(x, \rho, t)\) on the right hand side of (4.1) and make it subject to the given initial and boundary conditions, which leads to
\[
\begin{align*}
\frac{\partial}{\partial t} D_t^\rho G(x, \rho, t) &= \frac{\partial}{\partial t} \left[ G(x, \rho, t) - e^{\rho t} G(x, \rho, 0) \right] \\
&= \kappa \Delta G(x, \rho, t) + f(x, \rho, t), \quad 0 < t \leq T, \quad x \in \Omega \\
&= \kappa \Delta G(x, \rho, t) + f(x, \rho, t), \quad 0 < t \leq T, \quad x \in \Omega \\
&= \kappa \Delta G(x, \rho, t) + f(x, \rho, t), \quad 0 < t \leq T, \quad x \in \Omega \\
\end{align*}
\]
(4.1)

with the initial and boundary conditions
\[
G(x, \rho, 0) = \varphi(x), \quad x \in \Omega, \\
G(x, \rho, t) = \psi(t), \quad (x, t) \in \partial \Omega \times [0, T].
\]
4.1 Derivation of the compact difference scheme for 1D

Let the mesh points

\[ \Omega_h = \{ x_i = ih | 0 \leq i \leq M + 1 \} \quad \text{and} \quad \Omega_\tau = \{ \tau_n = n\tau | 0 \leq n \leq N \}, \]

where \( h = b/(M + 1) \) and \( \tau = T/N \) are the uniform space stepsize and time steplength, respectively. Let \( \mathcal{Y} = \{ v^\rho | 0 \leq i \leq M + 1, 0 \leq n \leq N \} \) be the grid function defined on the mesh \( \Omega_h \times \Omega_\tau \). For any grid function \( v^\rho \in \mathcal{Y} \), we denote

\[ \delta_x^2 v^\rho_i = \frac{1}{h^2}(v^\rho_{i-1} - 2v^\rho_i + v^\rho_{i+1}), \]

and the compact operator

\[ \mathcal{C}_h v^\rho = \begin{cases} (1 + \frac{h^2}{12} \delta_x^2) v^\rho_i = \frac{1}{12}(v^\rho_{i-1} + 10v^\rho_i + v^\rho_{i+1}), & 1 \leq i \leq M, \\ v^\rho_i, & i = 0 \text{ or } M + 1. \end{cases} \]

Then, we obtain the fourth-order accuracy compact operator in spatial direction; see the following lemma.

**Lemma 4.1** \( \{27\} \) Let \( G(x) \in C^6(\Omega) \) and \( \theta(s) = 5(1-s)^3 - 3(1-s)^5 \). Then

\[ \mathcal{C}_h \left[ \frac{\partial^2}{\partial x^2} G(x) \right]_{x=x_i} = \delta_x^2 G(x_i) + \frac{h^4}{360} \int_0^1 \left[ G^{(6)}(x_i - s\tau) + G^{(6)}(x_i + s\tau) \right] \theta(s) ds \]

with \( x_i = ih, 1 \leq i \leq M \).

Denote \( G^\rho_{i,\rho} \) and \( f^\rho_{i,\rho} \), respectively, as the numerical approximation to \( G(x_i, \rho, t_n) \) and \( f(x_i, \rho, t_n) \). In this paper, we restrict \( U(x) = 1 \) appeared in \( \{14\} \); for the discussions of the more general choices of \( U(x) \), see \( \{15\} \). Using \( \{15\} \), we obtain the \( \nu \)-th order approximations for the Riemann-Liouville fractional substantial derivative, i.e.,

\[ ^{\nu} D_t^\rho G(x, \rho, t) |_{t=t_n} = \frac{1}{\tau^\alpha} \sum_{k=0}^{n} d^\nu_k^\alpha G(x, \rho, t_{n-k}) + O(\tau^\nu); \]

\[ ^{\nu} D_t^\rho [e^{-\tau t} G(x, \rho, 0)] |_{t=t_n} = \frac{1}{\tau^\alpha} \sum_{k=0}^{n} d^\nu_k^\alpha e^{-\rho(\tau - k)\tau} G(x, \rho, 0) + O(\tau^\nu) \]

with

\[ d^\nu_k^\alpha = e^{-\rho k \tau} d^\nu_k^\alpha, \quad \nu = 1, 2, 3, 4, \]

(4.5)

where \( l^1_k, l^2_k, l^3_k, l^4_k \) and \( l^4_k \) are given in \( \{13\} \). In particular, when \( \nu = 1 \), there exists

\[ d^1_k^\alpha = e^{-\rho k \tau} l^1_k^\alpha, \quad l^1_k^\alpha = (-1)^k \binom{\alpha}{k}. \]

(4.6)

From \( \{4.4\} \) and \( \{4.5\} \), there exists \( \nu \)-th order approximations for Caputo fractional substantial derivative

\[ ^{\nu} D_t^\rho G(x, \rho, t) |_{t=t_n} = \frac{1}{\tau^\alpha} \sum_{k=0}^{n} d^\nu_k^\alpha \left[ G(x, \rho, t_{n-k}) - e^{-\rho(n-k)\tau} G(x, \rho, 0) \right] + r^\rho_i \]

(4.7)
with $|p^v| = O(\tau^v)$, $v = 1, 2, 3, 4$.

Performing both sides of (4.1) by $\mathcal{G}_h$ at the point $(x_i, t_n)$ results in

$$
\mathcal{G}_h \left[ \frac{\partial^2}{\partial x^2} G(x, \rho, t_n) \right] = \kappa_0 \mathcal{G}_h \left[ \frac{\partial^2}{\partial x^2} G(x, \rho, t_n) \right] + \mathcal{G}_h [f(x_i, \rho, t_n)].
$$

(4.8)

According to (4.8), (4.7) and Lemma 4.1, Eq. (4.1) can be rewritten as

$$
\mathcal{G}_h \left[ \frac{1}{\tau^\alpha n} \sum_{k=0}^{n} d^n_{t,k} \right] \left[ G(x_i, \rho, t_n-k) - e^{-\rho(a-k)\tau} G(x_i, \rho, 0) \right] = \kappa_0 \delta^2 G(x_i, \rho, t_n) + \mathcal{G}_h [f(x_i, \rho, t_n)] + \tilde{r}_i^n
$$

(4.9)

with the local truncation error

$$
|\tilde{r}_i^n| \leq C_G (\tau^v + h^4), \hspace{1em} v = 1, 2, 3, 4,
$$

(4.10)

where $C_G$ is a constant independent of $\tau$ and $h$.

Multiplying (4.9) by $\tau^\alpha$ leads to

$$
\mathcal{G}_h \left[ \frac{1}{\tau^\alpha n} \sum_{k=0}^{n} d^n_{t,k} \right] \left[ G(x_i, \rho, t_n-k) - e^{-\rho(a-k)\tau} G(x_i, \rho, 0) \right] = \kappa_0 \tau^\alpha \delta^2 G(x_i, \rho, t_n) + \tau^\alpha \mathcal{G}_h [f(x_i, \rho, t_n)] + R_i^n
$$

(4.11)

with

$$
|R_i^n| = |\tau^\alpha \tilde{r}_i^n| \leq C_G \tau^\alpha (\tau^v + h^4), \hspace{1em} v = 1, 2, 3, 4,
$$

(4.12)

where $C_G$ is given in (4.10).

Using (4.11) and (4.5) leads to the compact difference scheme of (4.1) as

$$
I_{n,0}^\alpha G_{i+1,\rho} - G_{i-1,\rho} + 10G_{i,\rho} + G_{i+1,\rho} + \mu^\alpha \left( -G_{i-1,\rho} + 2G_{i,\rho} - G_{i+1,\rho} \right)
$$

$$
= - \sum_{k=1}^{n-1} e^{-\rho k \tau} \mu^\alpha_{k,h} G_{i-k,\rho} + 10G_{i-k,\rho} + G_{i-k+1,\rho} + \frac{\tau^\alpha}{12} f_{i-1,\rho} + 10f_{i,\rho} + f_{i+1,\rho}
$$

(4.13)

$$
+ \sum_{k=0}^{n-1} e^{-\rho k \tau} \mu^\alpha_{k,h} G_{i-k,\rho} + 10G_{i-k,\rho} + G_{i-k+1,\rho} + \frac{\tau^\alpha}{12} f_{i-1,\rho} + 10f_{i,\rho} + f_{i+1,\rho}
$$

with $\mu^\alpha_{k,h} = \kappa_0 \frac{\tau^\alpha}{h^4}$. For the convenience of implementation, we use the matrix form of the grid functions

$$
G^n = \begin{bmatrix} G_{1,\rho}^n, G_{2,\rho}^n, \ldots, G_{M,\rho}^n \end{bmatrix}^T \quad \text{and} \quad F^n = \begin{bmatrix} f_{1,\rho}^n, f_{2,\rho}^n, \ldots, f_{M,\rho}^n \end{bmatrix}^T.
$$
Thus the compact difference scheme (4.13) reduces to the following form
\[ l_0^{\nu,\alpha} H g^n + h^{\nu,\alpha} L g^n = - \sum_{k=1}^{n-1} e^{-\rho_k^\tau T_k^\nu,\alpha} H g^{n-k} + \sum_{k=0}^{n-1} e^{-\rho_k^\tau T_k^\nu,\alpha} H g^0 + \tau^\alpha H f^n + F^n. \] (4.14)

Here, the matrices \( H = \frac{1}{12} \text{tridiag}(1, 10, 1) \) and \( L = \text{tridiag}(-1, 2, -1) \), i.e.,
\[
H = \frac{1}{12} \begin{bmatrix} 10 & 1 & \ldots & \ldots & 1 \\ 1 & 10 & 1 & \ldots & \ldots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \ldots & \ldots & \ldots & \ddots & \ddots \\ 1 & \ldots & \ldots & \ldots & 10 \end{bmatrix}
\]
and \( L = \begin{bmatrix} 2 & -1 & \ldots & \ldots & -1 \\ -1 & 2 & -1 & \ldots & \ldots \\ \ldots & \ldots & \ddots & \ddots & \ldots \\ \ldots & \ldots & \ldots & \ddots & \ddots \\ -1 & \ldots & \ldots & \ldots & 2 \end{bmatrix} \),
\[
(4.15)
\]
and \( F^n = [\tilde{f}_{1,\rho}^n, \ldots, \tilde{f}_{M,\rho}^n]^T \) with the initial and boundary conditions
\[
\tilde{f}_{1,\rho}^n = h^{\nu,\alpha} G_{0,\rho}^n + \frac{1}{12} \left[ - l_0^{\nu,\alpha} G_{0,\rho}^n \right.
- \sum_{k=1}^{n-1} e^{-\rho_k^\tau T_k^\nu,\alpha} G_{0,\rho}^{n-k} + \sum_{k=0}^{n-1} e^{-\rho_k^\tau T_k^\nu,\alpha} G_{0,\rho}^0 + \tau^\alpha f_{0,\rho}^n \Big];
\]
and
\[
\tilde{f}_{M,\rho}^n = h^{\nu,\alpha} G_{M+1,\rho}^n + \frac{1}{12} \left[ - l_0^{\nu,\alpha} G_{M+1,\rho}^n \right.
- \sum_{k=1}^{n-1} e^{-\rho_k^\tau T_k^\nu,\alpha} G_{M+1,\rho}^{n-k} + \sum_{k=0}^{n-1} e^{-\rho_k^\tau T_k^\nu,\alpha} G_{M+1,\rho}^0 + \tau^\alpha f_{M+1,\rho}^n \Big].
\]

4.2 Derivation of the center difference scheme for 2D

Let the mesh points \( x_i = ih, \ y_j = jh, \ t_n = n\tau \) with \( 0 \leq i, j \leq M + 1, 0 \leq n \leq N \), where \( h = b/(M + 1) \) and \( \tau = T/N \) are the uniform space steps and time steplength, respectively. Denote \( G_{i,j,\rho}^n \) and \( f_{i,j,\rho}^n \), respectively, as the numerical approximation to \( G(x_i, y_j, \rho, t_n) \) and \( f(x_i, y_j, \rho, t_n) \). To approximate (4.12), we utilize the second order central difference formula for the spatial derivative. According to (4.14) and (4.17), then (4.12) can be recast as
\[
\frac{1}{\tau^\alpha} \sum_{k=0}^{n} G_{i,j,\rho}^{n-k} \left[ G(x_i, y_j, \rho, t_n-k) - e^{-\rho(n-k)^2} G(x_i, y_j, \rho, 0) \right] + \kappa_{\alpha} \left( \partial_x^2 G(x_i, y_j, \rho, t_n) + \delta_x^2 G(x_i, y_j, \rho, t_n) \right) + f(x_i, y_j, \rho, t_n) + \tau^\alpha f_{i,j,\rho}^n;
\]
with the local truncation error \( \tau^\alpha = O(\tau^\nu + h^2), \ \nu = 1, 2, 3, 4 \). Then, the resulting discretization of (4.15) has the following form
\[
\begin{align*}
\frac{l_0^{\nu,\alpha} G_{i,j,\rho}^n + h^{\nu,\alpha} G_{i,j,\rho}^n}{\tau^\alpha} & = - \sum_{k=1}^{n-1} e^{-\rho_k^\tau T_k^\nu,\alpha} G_{i,j,\rho}^{n-k} + \sum_{k=0}^{n-1} e^{-\rho_k^\tau T_k^\nu,\alpha} G_{i,j,\rho}^0 + \tau^\alpha f_{i,j,\rho}^n \ \\
& = - \sum_{k=1}^{n-1} e^{-\rho_k^\tau T_k^\nu,\alpha} G_{i,j,\rho}^{n-k} + \sum_{k=0}^{n-1} e^{-\rho_k^\tau T_k^\nu,\alpha} G_{i,j,\rho}^{n-k} + \tau^\alpha f_{i,j,\rho}^n.
\end{align*}
(4.17)
Proof According to Lemma 2.4 and for simplicity, the zero boundary conditions are used. Thus (4.17) reduces to
\[
(G^0 + \mu_{h,\tau}^0 (I \otimes I + L \otimes I)) G^0 = -\sum_{k=1}^{n-1} e^{-\rho_k t} \nu_k^\alpha G^{n-k} + \sum_{k=0}^{n-1} e^{-\rho_k t} \nu_k^\alpha G^0 + \tau^\alpha F^0. \tag{4.18}
\]

## 5 Applications of MGM

To align the solution of the resulting algebraic system (4.14) with the Multigrid Algorithm, we assume that the \(A_h = I_0^0 \alpha H + \mu_{h,\tau}^0 L, v^b = G^0\) and
\[
f_h = -\sum_{k=1}^{n-1} e^{-\rho_k t} \nu_k^\alpha H G^{n-k} + \sum_{k=0}^{n-1} e^{-\rho_k t} \nu_k^\alpha H G^0 + \tau^\alpha H F^0 + \tilde{\nu}^0.
\]

Then the resulting algebraic system (4.14) reduces to the form of (2.1), i.e.,
\[
A_h v^b = f_h \quad \text{with} \quad A_h = I_0^0 \alpha H + \mu_{h,\tau}^0 L. \tag{5.1}
\]

**Lemma 5.1** Let \(A^{(1)} := A_h\) be defined by (5.1) and \(A^{(k)} = I_k^{k-1} A^{(k-1)} I_{k-1}^{k-1}\). Then
\[
\frac{0}{\lambda_{\text{max}}(A_k)}(v^k, v^k) \leq (S_k v^k, v^k) \leq (A_k^{-1} v^k, v^k), \quad \forall v^k \in \mathcal{K}_k,
\]
where \(A_k = A^{(K-k+1)}, S_k = \omega D_k^{-1}, \omega \in (0, 1/2)\) and \(D_k\) is the diagonal of \(A_k\).

**Proof** According to Lemma 2.4 and \(A_h = l_0^0 \alpha H + \mu_{h,\tau}^0 L\) in (4.14), the desired result is obtained.

**Lemma 5.2** Let \(A^{(1)} := A_h\) be defined by (5.1) and \(A^{(k)} = I_k^{k-1} A^{(k-1)} I_{k-1}^{k-1}\). Then
\[
\min_{k^\prime, k^\prime \leq k} \|v^k - I_{k^\prime-1} v^{k-1}\|_{A_{k^\prime}}^2 \leq 16 \|A_k v^k\|_{D_k^{-1}}^2, \quad \forall v^k \in \mathcal{B}_k
\]
with \(A_k = A^{(K-k+1)}\).

**Proof** Since \(A_h = l_0^0 \alpha H + \mu_{h,\tau}^0 L\) in (4.14), i.e.,
\[
a_0 = \frac{10}{12} l_0^0 \alpha + 2 \mu_{h,\tau}^0 \quad \text{and} \quad a_1 = \frac{1}{12} l_0^0 \alpha - \mu_{h,\tau}^0.
\]

Combining Lemma 2.6 and that \(\forall k \geq 1\), there exists
\[
\frac{(6C_k + 2^{k-1}) (a_0 + 2a_1)}{(2C_k + 2^{k-1}) (a_0 + 2a_1) - 2^{k+1} a_1} = \frac{(6C_k + 2^{k-1}) l_0^0 \alpha}{(2C_k + 2^{k-1}) l_0^0 \alpha + 2^{k+1} \mu_{h,\tau}^0} < 3,
\]
leads to the desired result.
From Lemmas 5.1, 5.2 and Theorem 2.2, our MGM convergence result is obtained.

Theorem 5.1 For the resulting algebraic system (4.14), it satisfies

\[ ||I - B_kA_k||_{A_k} \leq \frac{16}{2l\omega + 16} < 1 \text{ with } 1 \leq k \leq K, \quad \omega \in (0, 1/2], \]

where the operator \( B_k \) is defined by the V-cycle method in Multigrid Algorithm[1] and \( l \) is the number of smoothing steps.

According to Theorem 5.1 for the two-dimensional fractional Feynman-Kac equation, we have the following results.

Theorem 5.2 For the resulting algebraic system (4.18), it satisfies

\[ ||I - B_kA_k||_{A_k} \leq \frac{1536}{2l\omega + 1536} < 1 \text{ with } 1 \leq k \leq K, \quad \omega \in (0, 1/4], \]

where the operator \( B_k \) is defined by the V-cycle method in Multigrid Algorithm[1] and \( l \) is the number of smoothing steps.

6 Numerical Results

We employ the V-cycle MGM described in Algorithm[1] to solve the resulting system. The stopping criterion is taken as

\[ \frac{||r^{(i)}||}{||r^{(0)}||} < 10^{-11} \text{ for (4.14), } \frac{||r^{(i)}||}{||r^{(0)}||} < 10^{-7} \text{ for (4.18)}, \]

where \( r^{(i)} \) is the residual vector after \( i \) iterations; and the number of iterations \((m_1, m_2) = (1, 2)\) and \((\omega_{pre}, \omega_{post}) = (1, 1/2)\). In all tables, \( M \) denotes the number of spatial grid point; the numerical errors are measured by the \( L_\infty \) (maximum) norm; and 'Rate' denotes the convergent orders. 'CPU' denotes the total CPU time in seconds (s) for solving the resulting discretized systems; and 'Iter' denotes the average number of iterations required to solve a general linear system \( A_hv = f_h \) at each time level.

All the computations are carried out on a PC with the configuration: Intel(R) Core(TM) i5-3470 3.20 GHZ and 8 GB RAM and a 64 bit Windows 7 operating system. Example[1] and [2] numerical experiments are, respectively, in Matlab and in Python.

Example 1 Consider the fractional Feynman-Kac equation (4.1) for 1D, on a finite domain \( 0 < x < 1, 0 < t \leq 1 \) with the coefficient \( \kappa_{\alpha} = 1, \rho = 1 + \sqrt{-1} \), the forcing function

\[ f(x, \rho, t) = \frac{\Gamma(5 + \alpha)}{\Gamma(3)} e^{-\rho t t^4 (\sin(\pi x) + 1) + \kappa_{\alpha} t^2 e^{-\rho t (t^{4+\alpha} + 1)}} \sin(\pi x), \]

the initial condition \( G(x, \rho, 0) = \sin(\pi x) + 1 \), and the boundary conditions \( G(0, \rho, t) = G(1, \rho, t) = e^{-\rho t (t^{4+\alpha} + 1)} \). Then (4.5) has the exact solution

\[ G(x, \rho, t) = e^{-\rho t (t^{4+\alpha} + 1)}(\sin(\pi x) + 1). \]
Table 1 MGM to solve (4.14) at $T = 1$ with $\nu = 4$, $h = 1/M$ and $N = M$, where $A_{k-1} = \mu_0 A_{k} I_{k-1}^{\nu} - \mu_1 I_{k}^{\nu} + \mu_2 I_{k-1}^{\nu}$ (Galerkin approach or algebraic MGM) is computed by (2.13).

| $M$ | $\alpha = 0.3$ | Rate | Iter | CPU  | $\alpha = 0.8$ | Rate | Iter | CPU  |
|-----|----------------|------|------|------|----------------|------|------|------|
| 2$^2$ | 2.6394e-08 | 3.9998 | 10 | 0.52 s | 8.1345e-08 | 3.9992 | 9 | 0.47 s |
| 2$^3$ | 1.6494e-09 | 4.0002 | 10 | 1.41 s | 5.0850e-09 | 3.9997 | 9 | 1.30 s |
| 2$^4$ | 1.0381e-10 | 3.9899 | 10 | 3.97 s | 3.1723e-10 | 4.0026 | 9 | 3.71 s |

Table 2 MGM to solve (4.14) at $T = 1$ with $\nu = 4$, $h = 1/M$ and $N = M$, where $A_{k-1} = \mu_0 A_{k} I_{k-1}^{\nu} - \mu_1 I_{k}^{\nu} + \mu_2 I_{k-1}^{\nu}$ (doubling the mesh size or geometric MGM) is defined by (4.14).

| $M$ | $\alpha = 0.3$ | Rate | Iter | CPU  | $\alpha = 0.8$ | Rate | Iter | CPU  |
|-----|----------------|------|------|------|----------------|------|------|------|
| 2$^2$ | 2.6394e-08 | 3.9998 | 10 | 0.52 s | 8.1345e-08 | 3.9992 | 9 | 0.47 s |
| 2$^3$ | 1.6494e-09 | 4.0002 | 10 | 1.41 s | 5.0850e-09 | 3.9997 | 9 | 1.30 s |
| 2$^4$ | 1.0381e-10 | 3.9899 | 10 | 3.97 s | 3.1723e-10 | 4.0026 | 9 | 3.71 s |

We use two coarsening strategies: Galerkin approach and doubling the mesh size, respectively, to solve the resulting system (5.1). Tables 1 and 2 show that these two methods have almost the same error values with the global truncation error $O(\nu^2 + h^4)$, $\nu = 4$, so that the locally weighted averaging of Galerkin approach brings convenience for handling the convergence proof, but no more benefits are obtained. In fact, as proved in [19], the convergence conditions of the Galerkin and of the geometric approaches are very similar (except for the full rank of the projector which is needed in the Galerkin approach only). However, in general, the Galerkin technique is more robust and the potential reason for which here this fact is not observed is the presence of the stiffness matrix which improves the conditioning of the problem, acting as a mild regularization.

Example 2 Consider the fractional Feynman-Kac equation (4.1) for 2D, on a finite domain $0 < x, y < 1$, $0 < t \leq 1$ with the coefficient $k_0 = 1$, $\rho = 1$, the initial condition is $G(x, \rho, 0) = 0$ and the zero boundary conditions on the rectangle. Taking the exact solution as $G(x, y, \rho, t) = e^{-\rho t} I^4 \sin(\pi x) \sin(\pi y)$ and using above assumptions, it is easy to obtain the forcing functions $f(x, y, \rho, t)$.

Table 3 MGM to solve (4.18) at $T = 1$ with $\nu = 2$, $h = 1/M$ and $N = M$, where $A_{k-1} = \mu_0 A_{k} I_{k-1}^{\nu} - \mu_1 I_{k}^{\nu} + \mu_2 I_{k-1}^{\nu}$ (I ⊗ I + L ⊗ I) is defined by (4.18).

| $M$ | $\alpha = 0.3$ | Rate | Iter | CPU  | $\alpha = 0.8$ | Rate | Iter | CPU  |
|-----|----------------|------|------|------|----------------|------|------|------|
| 2$^2$ | 2.6394e-08 | 3.9998 | 10 | 0.52 s | 8.1345e-08 | 3.9992 | 9 | 0.47 s |
| 2$^3$ | 1.6494e-09 | 4.0002 | 10 | 1.41 s | 5.0850e-09 | 3.9997 | 9 | 1.30 s |
| 2$^4$ | 1.0381e-10 | 3.9899 | 10 | 3.97 s | 3.1723e-10 | 4.0026 | 9 | 3.71 s |

From Table 3 we numerically confirm that the numerical scheme has second-order accuracy in both time and space directions.
Remark 6.1 Since the joint PDF $G(x,A,t)$ is the inverse Laplacian transform $\rho \to A$ of $G(x,p,t)$, for getting $G(x,A,t)$, we need to further perform the inverse numerical Laplacian transform, which has been discussed in [13].

7 Concluding remarks and future work

This paper provides few ideas for verifying the uniform convergence of the V-cycle MGM for symmetric positive definite Toeplitz block tridiagonal matrices, where we use the simple (traditional) restriction operator and prolongation operator to handle general Toeplitz systems directly for the elliptic PDEs. Then we further derive the difference scheme for the backward fractional Feynman-Kac equation, which describes the distribution of the functional of non-Brownian particles; finally, the V-cycle multigrid method is effectively used to solve the generated algebraic system, and the uniform convergence is obtained. In particular, for the coarsening of multigrid methods, even though the geometric MGM and algebraic MGM are different in theoretical analysis and techniques, numerically most of the time almost the same numerical results can be got. Concerning the future work, the main point to investigate is the extension of this proof to general banded or dense Toeplitz matrices [1, 2].

In fact, for the full Toeplitz matrices with a weakly diagonally dominant symmetric Toeplitz $M$-matrices, the condition (2.10) holds when $\omega \in (0, 1/3)$ [17]. Hence the real challenge is the verification of condition (2.11) or of condition (2.12) and this will the subject of future researches.

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Appendix

Proof of Lemma 2.3 Since $A^{(k)}$ is the symmetric matrix, we denote $A^{(k)} = \{a^{(k)}_{i,j}\}_{i,j=1}^{\infty}$ with $a^{(k)}_{i,j} = a^{(k)}_{|j-i|}$ $\forall k \geq 1$. Using the relation $A^{(k)} = L^H_kA^{(k-1)}L^H_k$, there exists

$$
\{b^{(k)}_{i,j}\}_{i,j=1}^{\infty} = A^{(k-1)}L^H_k \quad \text{and} \quad \{a^{(k)}_{i,j}\}_{i,j=1}^{\infty} = L^H_kA^{(k-1)}L^H_k
$$

with $b^{(k)}_{i,j} = a^{(k-1)}_{2i-j-1} + 2a^{(k-1)}_{2i-j} + a^{(k-1)}_{2i-j+1}$ and $a^{(k)}_{i,j} = b^{(k)}_{2i-1,j} + 2b^{(k)}_{2i,j} + b^{(k)}_{2i+1,j}$. Then for the Toeplitz matrix $A^{(k)}$, it holds

$$
\begin{align*}
    a^{(k)}_{i,j} &= 6a^{(k-1)}_{i,j} + 8a^{(k-1)}_{i,j} + 2a^{(k-1)}_{i,j} \quad \forall k \geq 2; \\
    a^{(k)}_{i,j} &= a^{(k-1)}_{i,j-2} + 4a^{(k-1)}_{i,j-1} + 6a^{(k-1)}_{i,j} + 4a^{(k-1)}_{i,j+1} + a^{(k-1)}_{i,j+2} \quad \forall j \geq 1.
\end{align*}
$$
We prove (2.13) by mathematical induction. For \( k = 2 \), Eq. (2.13) holds obviously. Suppose (2.13) holds for \( k = 2, 3, \ldots, s \). In particular, for \( k = s \), we have

\[
a^{(s)}_0 = (4c_s + 2^{s-1})a^{(1)}_0 + \sum_{m=1}^{2^{s-1}-1} 6C_m a^{(1)}_m;
\]

\[
a^{(s)}_1 = C_s a^{(1)}_0 + \sum_{m=1}^{3 \cdot 2^{s-1}-1} 4C_m a^{(1)}_m;
\]

\[
a^{(s)}_j = \sum_{m=(j-2)2^{s-1}}^{(j+2)2^{s-1}-1} jC_m a^{(1)}_m \quad \forall j \geq 2.
\]

Next we need to prove that (2.13) holds for \( k = s + 1 \).

According to (A.1), (A.2) and the coefficients \( jC_m \), \( j \geq 0 \) in (2.13), we can check that

\[
a^{(r+1)}_0 = 6a^{(s)}_0 + 8a^{(s)}_1 + 2a^{(s)}_2
\]

\[
= (32c_s + 6 \cdot 2^{s-1})a^{(1)}_0 + \sum_{m=1}^{2^{s-1}-1} (6 \cdot 6C_m + 8 \cdot 1C_m + 2 \cdot 2C_m) a^{(1)}_m
\]

\[
+ \sum_{m=2^{s-1}}^{2^{2s-1}-1} (6 \cdot 6C_m + 8 \cdot 1C_m + 2 \cdot 2C_m) a^{(1)}_m
\]

\[
+ \sum_{m=2^{s-1}}^{3 \cdot 2^{s-1}-1} (8 \cdot 1C_m + 2 \cdot 2C_m) a^{(1)}_m + \sum_{m=3 \cdot 2^{s-1}}^{4 \cdot 2^{s-1}-1} 2 \cdot 2C_m a^{(1)}_m
\]

\[
= (4C_{s+1} + 2^r)a^{(1)}_0 + \sum_{m=1}^{4 \cdot 2^{s-1}-1} 0C_m a^{(1)}_m;
\]

\[
a^{(r+1)}_1 = a^{(s)}_0 + 4a^{(s)}_1 + 6a^{(s)}_2 + 4a^{(s)}_3 + a^{(s)}_4
\]

\[
= (8c_s + 2^{s-1})a^{(1)}_0 + \sum_{m=1}^{2^{s-1}-1} (6C_m + 4 \cdot 1C_m + 6 \cdot 2C_m) a^{(1)}_m
\]

\[
+ \sum_{m=2^{s-1}}^{2 \cdot 2^{s-1}-1} (6C_m + 4 \cdot 1C_m + 6 \cdot 2C_m) a^{(1)}_m
\]

\[
+ \sum_{m=2^{s-1}}^{3 \cdot 2^{s-1}-1} (4 \cdot 1C_m + 6 \cdot 2C_m + 4 \cdot 3C_m + 4C_m) a^{(1)}_m
\]

\[
+ \sum_{m=3 \cdot 2^{s-1}}^{4 \cdot 2^{s-1}-1} (6 \cdot 2C_m + 4 \cdot 3C_m + 4C_m) a^{(1)}_m
\]

\[
+ \sum_{m=4 \cdot 2^{s-1}}^{5 \cdot 2^{s-1}-1} (4 \cdot 3C_m + 4C_m) a^{(1)}_m + \sum_{m=5 \cdot 2^{s-1}}^{6 \cdot 2^{s-1}-1} 4C_m a^{(1)}_m
\]

\[
= C_{s+1} a^{(1)}_0 + \sum_{m=1}^{5 \cdot 2^{s-1}-1} 1C_m a^{(1)}_m;
\]
and
\[
a_j^{(s+1)} = a_{j-2}^{(s)} + 4a_{j-1}^{(s)} + 6a_j^{(s)} + 4a_{j+1}^{(s)} + a_{j+2}^{(s)}
\]
\[
= \sum_{m=(2j+4)2^{r-1}-1}^{(2j-3)2^{r-1}-1} 2j-2C_m^{(1)}a_m^{(1)} + \sum_{m=(2j-3)2^{r-1}-1}^{(2j-2)2^{r-1}-1} (2j-2C_m^{(1)} + 4 \cdot 2j-1C_m^{(1)}) a_m^{(1)}
\]
\[
+ \sum_{m=(2j-2)2^{r-1}-1}^{(2j-2)2^{r-1}-1} (2j-2C_m^{(1)} + 4 \cdot 2j-1C_m^{(1)} + 6 \cdot 2jC_m^{(1)} + 4 \cdot 2j+1C_m^{(1)}) a_m^{(1)}
\]
\[
+ \sum_{m=(2j-2)2^{r-1}-1}^{(2j-1)2^{r-1}-1} (4 \cdot 2j-1C_m^{(1)} + 6 \cdot 2jC_m^{(1)} + 4 \cdot 2j+1C_m^{(1)} + 2j+2C_m^{(1)}) a_m^{(1)}
\]
\[
+ \sum_{m=(2j+1)2^{r-1}-1}^{(2j+1)2^{r-1}-1} (6 \cdot 2jC_m^{(1)} + 4 \cdot 2j+1C_m^{(1)} + 2j+2C_m^{(1)}) a_m^{(1)}
\]
\[
+ \sum_{m=(2j+2)2^{r-1}-1}^{(2j+3)2^{r-1}-1} (4 \cdot 2j+1C_m^{(1)} + 2j+2C_m^{(1)}) a_m^{(1)} + \sum_{m=(2j+3)2^{r-1}-1}^{(2j+4)2^{r-1}-1} 2j+2C_m^{(1)} a_m^{(1)}
\]
\[
= \sum_{m=(2j-4)2^{r-1}-1}^{(2j+4)2^{r-1}-1} C_m^{(s+1)} a_m^{(1)}.
\]

The proof is completed.

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