Transfer of stable equivalences of Morita type

Shengyong Pan and Changchang Xi

School of Mathematical Sciences,
Laboratory of Mathematics and Complex Systems,
Beijing Normal University, 100875 Beijing,
People’s Republic of China
E-mail:xicc@bnu.edu.cn
E-mail:panshy1979@bnu.edu.cn

Abstract

Let $A$ and $B$ be finite-dimensional $k$-algebras over a field $k$ such that $A/\text{rad}(A)$ and $B/\text{rad}(B)$ are separable. In this note, we consider how to transfer a stable equivalence of Morita type between $A$ and $B$ to that between $eAe$ and $fBf$, where $e$ and $f$ are idempotent elements in $A$ and in $B$, respectively. In particular, if the Auslander algebras of two representation-finite algebras $A$ and $B$ are stably equivalent of Morita type, then $A$ and $B$ themselves are stably equivalent of Morita type. Thus, combining a result with Liu and Xi, we see that two representation-finite algebras $A$ and $B$ over a perfect field are stably equivalent of Morita type if and only if their Auslander algebras are stably equivalent of Morita type. Moreover, since stable equivalence of Morita type preserves $n$-cluster tilting modules, we extend this result to $n$-representation-finite algebras and $n$-Auslander algebras studied by Iyama.

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1 Introduction

Stable equivalence of Morita type introduced by Broué in [2] is one of the fundamental equivalence relations for algebras and groups. It is of considerable interest in the modular representation theory of finite groups, or more generally, of finite-dimensional self-injective algebras. The notion is intimately related to the celebrated conjecture of Broué, which says that certain blocks of group algebras with abelian defect groups should be derived-equivalent (see [2] and [12] for precise formulation); the connection can be seen from Rickard’s result that a derived equivalence between self-injective algebras induces a stable equivalence of Morita type [13]. Recently, it is shown that stable equivalence of Morita type is also of particular interest for general finite-dimensional algebras, for example, it preserves representation dimension [15], representation type [7], Hochschild homological and cohomological groups [10] [14], and the absolute value of Cartan determinant [14]. As is known, stable equivalence of Morita type occurs frequently not only in the block theory of finite groups [8], but also in the representation theory of general finite-dimensional algebras. A plenty of such examples are constructed in [9] [10] [11].

Moreover, it is shown in [10] that, for two finite-dimensional $k$-algebras $A$ and $B$ over a field $k$ of finite representation-type, if $A$ and $B$ are stably equivalent of Morita type, then their Auslander algebras are also stably equivalent of Morita type. A natural question is whether the converse of this statement is true.

In this note, we shall consider the general question of how to transfer a stable equivalence of Morita type between two algebras $A$ and $B$ over a field to that between $eAe$ and $fBf$, where $e$ and $f$ are idempotent elements in $A$ and $B$, respectively. We say that two bimodules $AM_B$ and...
\(B_NA\) define a stable equivalence of Morita type between \(A\) and \(B\) if \(M\) and \(N\) are projective as one-sided modules, and there are a projective \(A\)-\(A\)-bimodule \(P\) and a projective \(B\)-\(B\)-bimodule \(Q\) such that \(M \otimes_B N \simeq A \oplus P\) and \(N \otimes_A M \simeq B \oplus Q\) as bimodules. With these notations in mind, our main result reads as follows:

**Theorem 1.1** Suppose that \(A\) and \(B\) are finite-dimensional \(k\)-algebras over a field \(k\) such that both \(A/\text{rad}(A)\) and \(B/\text{rad}(B)\) are separable. Let \(AM_B\) and \(BN_A\) be two bimodules defining a stable equivalence of Morita type between \(A\) and \(B\). If \(e^2 = e \in A\) such that \(Pe \in \text{add}(Ae)\), and if \(f\) is a sum of the pairwise orthogonal primitive idempotent elements \(f_j \in B\) in the decomposition \(Ne \simeq \oplus_{j=1}^m (Bf_j)^\oplus\), then the bimodules \(eMf\) and \(fNe\) define a stable equivalence of Morita type between \(eAe\) and \(fBf\).

From this result, we have the following corollary which supplies a positive answer to our previous question on Auslander algebras. For the unexplained notion in the corollary, we refer the reader to Section 3 below.

**Corollary 1.2** Suppose that \(A\) and \(B\) are finite-dimensional \(k\)-algebras over a perfect field \(k\). Assume that \(A\) and \(B\) are \(n\)-representation-finite. If the \(n\)-Auslander algebras of \(A\) and \(B\) are stably equivalent of Morita type, then \(A\) and \(B\) themselves are stably equivalent of Morita type.

The proof of Theorem 1.1 is given in the next section.

# 2 Proof of the main result

Throughout this note, \(k\) denotes a fixed field. Given a finite-dimensional \(k\)-algebra \(A\), we denote by \(A\)-\(\text{mod}\) the category of all finitely generated left \(A\)-modules. If \(M \in A\)-\(\text{mod}\), we denote by \(\text{add}(M)\) the full subcategory of \(A\)-\(\text{mod}\) consisting of all modules \(X\) which are direct summands of finite sums of copies of \(M\). By an algebra we mean a finite-dimensional \(k\)-algebra, and by a module we mean a left module. The global and dominant dimensions of an algebra \(A\) are denoted by \(\text{gl.dim}(A)\) and \(\text{dom.dim}(A)\), respectively. The composition of two homomorphisms \(f : X \to Y\) and \(g : Y \to Z\) is denoted by \(fg : X \to Z\), and the usual \(k\)-duality is denoted by \(D := \text{Hom}_k(-, k)\). Let us recall the definition of a stable equivalence of Morita type.

**Definition 2.1** Suppose that \(A\) and \(B\) are two (arbitrary) \(k\)-algebras. We say that \(A\) and \(B\) are stably equivalent of Morita type if there exist an \(A\)-\(B\)-bimodule \(AM_B\) and a \(B\)-\(A\)-bimodule \(BN_A\) such that

1. \(M\) and \(N\) are projective as one-sided modules, and
2. \(M \otimes B N \simeq A \oplus P\) as \(A\)-\(A\)-bimodules for some projective \(A\)-\(A\)-bimodule \(P\), and \(N \otimes A M \simeq B \oplus Q\) as \(B\)-\(B\)-bimodules for some projective \(B\)-\(B\)-bimodule \(Q\).

Note that if \(A\) and \(B\) are stably equivalent of Morita type, then their opposite algebras \(A^{\text{op}}\) and \(B^{\text{op}}\) are also stably equivalent of Morita type.

Let \(A\) be a representation-finite algebra. An \(A\)-module \(X\) is called an additive generator for \(A\)-\(\text{mod}\) if \(\text{add}(X) = A\)-\(\text{mod}\), that is, every indecomposable \(A\)-module is isomorphic to a direct summand of \(X\). Let \(X\) be an additive generator for \(A\)-\(\text{mod}\). The endomorphism algebra \(A = \text{End}_A(X)\) of \(X\) is called the Auslander algebra of \(A\). (This is unique up to Morita equivalence.) Auslander algebras can be described by two homological properties: An algebra \(A\) is an Auslander algebra if (1) \(\text{gl.dim}(A) \leq 2\); and (2) if \(0 \to A \to I_0 \to I_1 \to I_2 \to 0\) is a minimal injective resolution of \(A\), then \(I_0\) and \(I_1\) are projective.

An \(A\)-module \(X \in A\)-\(\text{mod}\) is called a generator for \(A\)-\(\text{mod}\) if \(\text{add}(A) \subseteq \text{add}(X)\); a cogenerator for \(A\)-\(\text{mod}\) if \(\text{add}(D(A)) \subseteq \text{add}(X)\), and a generator-cogenerator if it is both a generator and a cogenerator for \(A\)-\(\text{mod}\). Clearly, for a representation-finite algebra \(A\), an additive generator for \(A\)-\(\text{mod}\) is a generator-cogenerator for \(A\)-\(\text{mod}\).

In the following, we shall introduce some notations. Assume that \(A\) is a \(k\)-algebra.

Let \(T\) be an arbitrary \(A\)-module, and suppose \(B\) is the endomorphism algebra of \(T\). We consider the following subcategories related to \(T\).
Lemma 2.2 Let \( X \) be an arbitrary \( A \)-module. Recall that \( B = \text{End}_A(T) \) and \( \Lambda T_B \) is a natural bimodule. Then:

1. Let \( Y \) be a right \( B \)-module. The natural homomorphism \( \delta : Y \otimes_B \text{Hom}_A(T, X) \to \text{Hom}_B(\text{Hom}_A(X, T), Y) \), given by \( y \otimes f \mapsto \delta_{y \otimes f}(g) = y(fg) \) for \( y \in Y, f \in \text{Hom}_A(X, T), g \in \text{Hom}_A(X, T) \), is an isomorphism if \( X \in \text{add}(\Lambda T) \).

2. If \( X' \in \text{add}(\Lambda T) \), or \( X \in \text{add}(\Lambda T) \), then the composition map \( \mu : \text{Hom}_A(X', T) \otimes_B \text{Hom}_A(T, X) \to \text{Hom}_A(X', X) \) given by \( f \otimes g \mapsto fg \) is bijective.

3. Let \( C \) be a \( k \)-algebra, and suppose \( \Lambda X_C \) is an \( A \)-\( C \)-bimodule. If \( \Lambda X \in \text{Gen}(\Lambda T) \), then the evaluation map \( e_X : T \otimes_B \text{Hom}_A(T, X) \to X \) is surjective as \( A \)-\( C \)-bimodules. If \( X \in \text{App}(\Lambda T) \), then \( e_X \) is an isomorphism as \( A \)-\( C \)-bimodules. Conversely, if \( e_X \) is bijective as \( A \)-modules, then \( X \in \text{App}(\Lambda T) \).

The next lemma is taken from [14] Lemma 2.1.

Lemma 2.3 [14] (1) Let \( A, B, C \) and \( E \) be \( k \)-algebras, and let \( \Lambda X_B \) and \( \Lambda Y_E \) be bimodules with \( \Lambda Y_E \) projective. Put \( X^* = \text{Hom}_B(X, B) \). Then the natural homomorphism \( \phi : \Lambda X \otimes_B \Lambda Y \to \text{Hom}_B(\Lambda X, \Lambda Y) \), defined by \( f \mapsto (xf) \) for \( x \in X, y \in Y \) and \( f \in X^* \), is an isomorphism of \( A \)-\( E \)-bimodules, where the image of \( x \) under \( f \) is denoted by \( xf \).

(2) In the situation \( (\Lambda P_A, \Lambda X_B, \Lambda U_B) \), if \( P_A \) is projective, or if \( X_B \) is projective, then \( \Lambda P_A \otimes \Lambda X_B = \text{Hom}_B(cX_B, \Lambda U_B) \cong \text{Hom}_B(cX_B, \Lambda U_B) \) as \( C \)-\( E \)-bimodules. Dually, in the situation \( (\Lambda P_B, \Lambda X_C, \Lambda U_A) \), if \( A \) is projective, or if \( X_C \) is projective, then \( \text{Hom}_B(cX_C, \Lambda U_B) = \Lambda P_B \) as \( C \)-\( E \)-bimodules.

The following is a well-known result due to Auslander (for example, see [1] Proposition 5.6, p.214).

Lemma 2.4 Let \( \Lambda \) be an Artin algebra such that \( \text{gl.dim}(\Lambda) \leq 2 \leq \text{dom.dim}(\Lambda) \). Let \( U \) be a \( \Lambda \)-module such that \( \text{add}(U) \) is the full subcategory of \( \Lambda \text{-mod} \) consisting of all projective-injective \( \Lambda \)-modules. Then:

1. \( A := \text{End}_\Lambda(U) \) is representation-finite.

2. \( \Lambda \) is Morita equivalent to \( \text{End}_\Lambda(X) \), where \( X \) is an additive generator for \( \Lambda \text{-mod} \).

For our proof of Theorem [11] we also need the following lemma in [3] Theorem 2.7, Corollary 3.1, Lemma 3.2, see also [13] Lemma 3.3.

Lemma 2.5 [3] Suppose that \( A \) and \( B \) are finite-dimensional \( k \)-algebras over a field \( k \) such that \( A \) and \( B \) have no separable direct summands and that \( A/\text{rad}(A) \) and \( B/\text{rad}(B) \) are separable. Assume that \( \Lambda M_B \) and \( \Lambda N_A \) are indecomposable bimodules that define a stable equivalence of Morita type between \( A \) and \( B \). Then:

1. There are isomorphisms of bimodule: \( N \cong \text{Hom}_A(M, A) \cong \text{Hom}_B(M, B) \) and \( M \cong \text{Hom}_A(N, A) \cong \text{Hom}_B(N, B) \).

2. Both \( (N \otimes_A - , M \otimes_B - ) \) and \( (M \otimes_B - , N \otimes_A - ) \) are adjoint pairs of functors.

3. If \( \Lambda I \) is injective, then so is \( N \otimes_A \Lambda I \).

It follows from Lemma 2.5 that the following result is true. Note that the last statement in Lemma 2.6 below follows from [11] Lemma 4.5.

Lemma 2.6 Suppose that \( A \) and \( B \) are finite-dimensional \( k \)-algebras over a field \( k \) such that \( A/\text{rad}(A) \) and \( B/\text{rad}(B) \) are separable. Assume that \( \{e_1, \ldots, e_n\} \) and \( \{f_1, \ldots, f_m\} \) are complete sets of pairwise orthogonal primitive idempotents in \( A \) and \( B \), respectively. Let \( e \) be the sum of all
those $e_i$ for which $Ae_i$ is projective-injective, and let $f$ be the sum of all those $f_j$ for which $Bf_j$ is projective-injective. If $M$ and $N$ are indecomposable bimodules that define a stable equivalence of Morita type between $A$ and $B$, then $Ne \simeq N \otimes_A Ae \in \text{add}(Bf), Mf \simeq M \otimes_B Bf \in \text{add}(Ae)$, and $Pe \in \text{add}(Ac)$.

**Proof of Theorem 1.1:**
Suppose that $A$ and $B$ are two algebras over a field $k$ such that $A/\text{rad}(A)$ and $B/\text{rad}(B)$ are separable, and suppose that $A_M^B$ and $B_N^A$ define a stable equivalence of Morita type between $A$ and $B$. We may assume that both $A$ and $B$ have no separable summands since the direct sum of $A$ with a separable $k$-algebra is always stably equivalent of Morita type to $A$ itself. Furthermore, by Lemma 2.2, we may assume that $M$ and $N$ are indecomposable as bimodules. Then $(M \otimes_B - , N \otimes_A - )$ and $(N \otimes_A - , M \otimes_B - )$ are adjoint pairs by Lemma 2.5.

To prove Theorem 1 we shall show that the bimodules $Mf$ and $Ne$ satisfy the conditions of a stable equivalence of Morita type between $e Ae$ and $f Bf$.

1. $f Ne$ is projective as both an $f Bf$-module and a right $e Ae$-module. In fact, we have $f Ne \simeq f B \otimes_B Ne \simeq \text{Hom}_B(Bf, B) \otimes_B Ne \simeq \text{Hom}_B(Bf, Bf)$ by Lemma 2.3. Since $Ne \in \text{add}(Bf)$ by the definition of $f$, we see that $\text{Hom}_B(Bf, Ne)$ is projective as an $f Bf$-module, that is, $f Ne$ is projective as an $f Bf$-module. To see that $f Ne$ is a projective right $e Ae$-module, we notice that $\text{add}(Mf) = \text{add}(M \otimes_B Bf) = \text{add}(M \otimes_B Ne) = \text{add}(M \otimes_B Ne) = \text{add}(Ae \oplus Pe) = \text{add}(Ae)$, here we use the assumption $Pe \in \text{add}(Ac)$. Since $(M \otimes_B - , N \otimes_A - )$ is an adjoint pair, we have $\text{Hom}_B(Bf, Bf) \otimes_B Ne \simeq \text{Hom}_A(M \otimes_B Bf, Ae) \simeq \text{Hom}_A(Mf, Ae)$ that $f Ne$ is projective as a right $e Ae$-module since $Mf \in \text{add}(Ae)$. Thus (1) is proved.

2. $e Mf$ is projective as both an $e Ae$-module and a right $f Af$-module. The proof of (2) is similar to that of (1), we omit it here.

3. $f Ne \otimes e Mf \simeq f Bf \oplus f Qf$ as bimodules. Indeed, by the associativity of tensor products, we have the following isomorphisms of $f Bf$-$f Bf$-bimodules:

$$f Ne \otimes_{e Ae} e Mf \simeq f N \otimes_A Ae \otimes_{e Ae} e A \otimes_A Mf$$

$$\simeq f N \otimes_A Ae \otimes_{e Ae} \text{Hom}(Ae, A) \otimes_A Mf$$

$$\simeq f N \otimes_A Ae \otimes_{e Ae} \text{Hom}(Ae, A)Mf$$

(by Lemma 2.3)

$$\simeq f N \otimes_A Mf$$

(by Lemma 2.2).

Since $M$ and $N$ define the stable equivalence of Morita type between $A$ and $B$, we have $N \otimes_A M \simeq B \oplus Q$ as $B. B$-bimodules. This implies that $f Ne \otimes e Mf \simeq f N \otimes Mf \simeq f B \otimes_B N \otimes_A M \otimes_B Bf \simeq f B \otimes_B (B \oplus Q) \otimes_B Bf \simeq f Bf \oplus f Bf \otimes_B Q \otimes_B Bf \simeq f Bf \oplus f Qf$.

4. The bimodule $f Qf$ in (3) is projective. In fact, since $Q$ is a projective $B. B$-bimodule and since $B/\text{rad}(B)$ is separable, the bimodule $Q$ is isomorphic to a direct sum of modules of the form $Q_1 \otimes_k Q_2$, where $Q_1$ is a projective left $B$-module and $Q_2$ is a projective right $B$-module. Since $f Ne \otimes_{e Ae} e Mf$ and $f Bf$ are projective as left $f Bf$-modules, we infer that $f Q_1 \otimes_k Q_2 f$ is a projective $f Bf$-module. It follows that $f Q_1$ is a projective $f Bf$-module. Similarly, $Q_2 f$ is a projective right $f Bf$-module. Thus $f Qf$, which is isomorphic to a direct sum of modules of the form $f Q_1 \otimes_k Q_2 f$ with $f Q_1$ a projective $f Bf$-module and $Q_2 f$ a projective right $f Bf$-module, is projective as an $f Bf$-$f Bf$-bimodule.

5. Similarly, we can show that $e Mf \otimes f Bf f Ne \simeq e Ae \oplus e Pe$ and that $e Pe$ is a projective $e Ae$-$e Ae$-bimodule.

Indeed, we have

$$e Mf \otimes f Bf f Ne \simeq e M \otimes_B Bf \otimes f Bf f B \otimes_B Ne$$

$$\simeq e M \otimes_B Bf \otimes_f Bf \text{Hom}(Bf, B) \otimes_B Ne$$

$$\simeq e M \otimes_B Bf \otimes_B Bf \text{Hom}(Bf, Bf)$$

(by Lemma 2.3)

$$\simeq e M \otimes_B Ne$$

(by Lemma 2.2).

Since $M$ and $N$ define the stable equivalence of Morita type between $A$ and $B$, we have $M \otimes_B N \simeq A \oplus P$ as $A. A$-bimodules. This implies that $e Mf \otimes f Bf f Ne \simeq e M \otimes_A Ne \simeq e Ae \oplus e Pe$. Now, we show that the bimodule $e Pe$ is projective.
In fact, since $P$ is a projective $A$-$A$-bimodule and since $A$/rad($A$) is separable, the bimodule $P$ is isomorphic to a direct sum of modules of the form $P_1 \otimes_k P_2$, where $P_1$ is a projective left $A$-module and $P_2$ is a projective right $A$-module. Since $eMf \otimes fN$ and $eAe$ are projective as left $eAe$-modules, we infer that $eP_1 \otimes P_2e$ is a projective $eAe$-module. It follows that $eP_1$ is a projective $eAe$-module. Similarly, $P_2e$ is a projective right $eAe$-module. Thus $eP$, which is isomorphic to the direct sum of modules of the form $eP_1 \otimes P_2e$ with $P_1$ a projective $eAe$-module and $P_2e$ a projective right $eAe$-module, is projective as an $eAe$-$eAe$-bimodule.

Thus, by definition, the bimodules $eMf$ and $fN$ define a stable equivalence of Morita type between $eAe$ and $fBf$. This finishes the proof of Theorem 1.1 $\square$

Remarks. (1) In Theorem 1.1 if $e$ is an idempotent element in $A$ such that every indecomposable projective-injective $A$-module is isomorphic to a summand of $Ae$, then $Pe \in \text{add}(Ae)$. This follows immediately from the proof of [14 Lemma 4.5]. In fact, under the assumption of Theorem 1.1 we infer that $\nu_A(A \bar{P})$ is projective-injective for all $i \geq 0$, where $\nu_A$ is the Nakayama functor $D\text{Hom}_A(-, A)$. Hence, if $e$ is an idempotent element in $A$ such that every indecomposable projective-injective $A$-module $X$ with $\nu_A X$ projective-injective for all $i \geq 0$ belongs to $\text{add}(Ae)$, then $A \bar{P} \in \text{add}(Ae)$.

(2) As was pointed out in [3 Section 4], if $e$ is an idempotent in $A$ and if $f$ is an idempotent in $B$ such that $\text{add}(Ae)$ and $\text{add}(Bf)$ are invariant under Nakayama functor, then $eAe$ and $fBf$ are self-injective, and any stable equivalence of Morita type between $A$ and $B$ induces a stable equivalence of Morita type between $eAe$ and $fBf$. In general, however, our algebras $eAe$ and $fBf$ in Theorem 1.1 may not be self-injective.

As a corollary of Theorem 1.1, we get the following result.

**Corollary 2.7** Suppose $A$ and $B$ be are finite-dimensional $k$-algebras of finite representation type, and let $X$ and $\Gamma$ be the corresponding Auslander algebras of $A$ and $B$, respectively. Assume that $\Lambda$ is separable. Then $\Lambda$ and $\Gamma$ are stably equivalent of Morita type if and only if $A$ and $B$ are stably equivalent of Morita type.

**Proof.** We know that if $X$ is an additive generator for $A$-mod with $\Lambda := \text{End}_A(X)$, then $U := \text{Hom}_A(X, D(A))$ is a projective-injective $A$-module with $\text{End}_A(U) \simeq A^\text{op}$; and every indecomposable projective-injective $A$-module is isomorphic to a direct summand of $U$. Note that this $U$ satisfies the conditions in Lemma 2.4. If we choose $e$ to be the sum of all idempotents corresponding to the indecomposable injective $A$-modules, then Lemma 2.4 says that the conditions in Theorem 2.1 on the idempotent $e \in \Lambda$ are satisfied. Note that $e$ defines an idempotent element $f$ in $\Gamma$ (see Theorem 1.1), and that $\text{add}(f)$ contains all projective-injective $\Gamma$-modules. With these in mind, the corollary follows from Theorem 1.1 and Theorem 1.1.

For an algebra $A$, we denote by $[A]$ the class of all those algebras $B$ for which there is a stable equivalence of Morita type between $B$ and $A$. From the above corollary, we have the following result.

**Corollary 2.8** Suppose that $k$ is a perfect field. Let $F$ be the set of equivalence classes $[A]$ of representation-finite $k$-algebras $A$ with respect to stable equivalence of Morita type, and let $A$ be the set of equivalence classes $[\Lambda]$ of Auslander $k$-algebras $\Lambda$ with respect to stable equivalence of Morita type. Then there is an one-to-one correspondence between $F$ and $A$.

Another consequence of Theorem 1.1 is the following corollary.

**Corollary 2.9** Suppose that $A$ and $B$ are two $k$-algebras. Let $AX$ be a generator-cogenerator for $A$-mod such that $\text{End}_A(X)/\text{rad}(\text{End}_A(X))$ is separable, and let $BY$ be a generator-cogenerator for $B$-mod such that $\text{End}_B(Y)/\text{rad}(\text{End}_B(Y))$ is separable. If $\text{End}_A(X)$ and $\text{End}_B(Y)$ are stably equivalent of Morita type, then so are $A$ and $B$. In this case, $A$ and $B$ have the same global, dominant, finitistic and representation dimensions.

Finally, we remark that if we consider derived equivalence instead of stable equivalence of Morita type in Corollary 2.7 then we know from [14] that a derived equivalence between
representation-finite, self-injective algebras $A$ and $B$ implies a derived equivalence between their Auslander algebras. But the converse of this statement is still open. For further information on constructing derived equivalences, we refer the reader to a current paper [3].

3 Higher Auslander algebras

In the following, we point out that Corollary 2.7 holds true for $n$-representation-finite algebras and $n$-Auslander algebras studied in [6].

Now we recall some definitions from [6]. Let $A$ be a finite-dimensional $k$-algebra, and let $n \geq 1$ be a natural number. An $A$-module $T$ is called an $n$-cluster tilting module if $\text{add}(T) = \{X \in A{-}\text{-mod} \mid \text{Ext}_A^i(X, T) = 0, 1 \leq i < n\} = \{X \in A{-}\text{-mod} \mid \text{Ext}_A^i(X) = 0, 1 \leq i < n\}$. The $k$-algebra $A$ is called 1-representation-finite if there is an 1-cluster tilting $A$-module $T$. This is equivalent to saying that $A$ is representation-finite. For $n \geq 2$, the $k$-algebra $A$ is called $n$-representation-finite if $\text{gl.dim}(A) \leq n$ and there is an $n$-cluster tilting $A$-module $T$.

A $k$-algebra $A$ is called an $n$-Auslander algebra if there is an $n$-representation-finite $k$-algebra $A$ with an $n$-cluster tilting $A$-module $T$ such that $A$ is Morita equivalent to $\text{End}_A(T)$. Note that, for an $n$-representation-finite algebra $A$, its $n$-Auslander algebra is unique up to Morita equivalence.

Clearly, each $n$-cluster tilting $A$-module $T$ is a generator and co-generator for $A$-mod. Thus the indecomposable projective-injective $\text{End}_A(T)$-modules are of the form $\text{Hom}_A(T, I)$, where $I$ is an indecomposable injective $A$-module.

Let $A$ be an $n$-representation-finite $k$-algebra with $T$ an $n$-cluster tilting $A$-module. Furthermore, we assume that $A$ has no separable direct summands and that $A/\text{rad}(A)$ is separable. If $A$ is stably equivalent of Morita type to an algebra $B$ such that $B$ has no separable direct summand and $B/\text{rad}(B)$ is separable, then $B$ is $n$-representation-finite. In fact, if two indecomposable bimodules $A M_B$ and $B N_A$ define the stable equivalence of Morita type between $A$ and $B$, then $N \otimes_A T$ is an $n$-cluster tilting $B$-module: Since this stable equivalence of Morita type is of adjoint type by Lemma 2.4, we see that $\text{Ext}_A^i(N \otimes_B T, N \otimes_A T) \simeq \text{Ext}_A^i(T, M \otimes_B N \otimes_A T) \simeq \text{Ext}_A^i(T, T \otimes P \otimes_A T) = \text{Ext}_A^i(T) \oplus \text{Ext}_A^i(T, P \otimes_A T) = 0$ for $1 \leq i < n$ since $P \otimes_A T$ is a projective-injective $A$-module. This shows that $\text{add}(N \otimes_A T)$ is contained in both $\{X \in B{-}\text{-mod} \mid \text{Ext}_B^j(X, N \otimes_A T) = 0, 1 \leq i < n\} = \{X \in B{-}\text{-mod} \mid \text{Ext}_B^j(X, T, X) = 0, 1 \leq i < n\}$. Now, let $Y \in B{-}\text{-mod}$ such that $\text{Ext}_B^j(N \otimes_A T, Y) = 0$ for $1 \leq j < n$. Then $Y = \text{Ext}_B^j(N \otimes_A T, Y) = \text{Ext}_A^j(T, M \otimes_B Y)$ for $1 \leq j < n$, and therefore $M \otimes_B Y \in \text{add}(T)$. This implies that $Y \in \text{add}(N \otimes_A T)$. Similarly, we show that $\text{add}(N \otimes_A T) = \{Y \in B{-}\text{-mod} \mid \text{Ext}_B^j(Y, N \otimes_A T) = 0, 1 \leq i < n\}$. Hence $N \otimes_A T$ is an $n$-cluster tilting $B$-module.

Thus, $n$-representation-finite $k$-algebras $A$ with $A/\text{rad}(A)$ separable are closed under stable equivalences of Morita type.

As in the case of Corollary 2.7 the following is a consequence of Theorem 1.1.

**Theorem 3.1** Suppose that $A$ and $B$ are finite-dimensional $k$-algebras such that both are $n$-representation-finite. Let $\Lambda$ and $\Gamma$ be the corresponding $n$-Auslander algebras of $A$ and $B$, respectively. Assume that both $\Lambda/\text{rad}(\Lambda)$ and $\Gamma/\text{rad}(\Gamma)$ are separable. Then $\Lambda$ and $\Gamma$ are stably equivalent of Morita type if and only if $A$ and $B$ are stably equivalent of Morita type.

**Proof.** For $n = 1$, we have done by Corollary 2.7. Let $n \geq 2$. Suppose $A T$ is an $n$-cluster tilting $A$-module such that $\text{End}_A(T) = \Lambda$, and suppose $B S$ is an $n$-cluster tilting $B$-module such that $\text{End}_B(S) = \Gamma$. If $A M_B$ and $B N_A$ are two indecomposable bimodules defining a stable equivalence of Morita type between $A$ and $B$, then, by the above discussion, we know that $\Gamma$ is Morita equivalent to $\text{End}_B(N \otimes_A T)$. Now, we use [11] Theorem 1.1, or Theorem 1.3 which states that if $R$ is an $A$-module with $\text{add}(A) \subseteq \text{add}(R)$, then $\text{End}_A(R)$ and $\text{End}_B(N \otimes_A R)$ are stably equivalent of Morita type.

Conversely, suppose that two bimodules $A X_T$ and $Y_A$ define a stable equivalence of Morita type between $A$ and $\Gamma$. Note that $\text{add}(T)$ contains both $\text{add}(A)$ and $\text{add}(D(A_4))$. Let $e$ be an
idempotent in Λ such that add(Λe) is just the category of projective-injective Λ-modules. Then eΛe is Morita equivalent to Aop. As in Theorem 1.1, we have an idempotent f in Γ such that fΓf is Morita equivalent to Bop. Thus a stable equivalence of Morita type between Λ and Γ implies a stable equivalence of Morita type between Aop and Bop by Theorem 1.1 and therefore a stable equivalence of Morita type between A and B. □

References

[1] M. Auslander, I. Reiten and S. O. Smalø, Representation thoery of Artin algebras. Cambridge University Press, 1995.
[2] M. Broué, Equivalences of blocks of group algebras. In:Finite dimensional algebras and related topics. V. Dlab and L. L. Scott (eds.), Kluwer, (1994), 1-26.
[3] A. Dugas and R. Martinez-Villa, A note on stable equivalences of Morita type . J. Pure Appl. Algebra 208(2007)421-433.
[4] W. Hu and C. C. Xi, Derived equivalences of endomorphism and quotient algebras. Preprint, available at : http://math.bnu.edu.cn/~ccxi/Papers/Articles/xihu-4.pdf, 2009.
[5] W. Hu and C. C. Xi, Almost D-split sequences and derived equivalences. Preprint, available at : http://math.bnu.edu.cn/~ccxi/Papers/Articles/xihu-2.pdf, 2007.
[6] O. Iyama, Cluster tilting for higher Auslander algebras. Preprint, 2008.
[7] H. Krause, Representation type and stable equivalences of Morita type for finite dimensional algebras. Math. Z. 229(1998)601-606.
[8] M. Linckelmann, On stable equivalences of Morita type. In:Derived equivalences for group rings, LNM 1685(1998)221-232.
[9] Y. M. Liu and C. C. Xi, Constructions of stable equivalences of Morita type for finite-dimensional algebras. I. Trans. Amer. Math. Soc. 358(2006), no. 6, 2537-2560.
[10] Y. M. Liu and C. C. Xi, Constructions of stable equivalences of Morita type for finite-dimensional algebras. II. Math. Z. 251(2005), no.1, 21-39.
[11] Y. M. Liu and C. C. Xi, Constructions of stable equivalences of Morita type for finite-dimensional algebras. III. J. London Math. Soc. 76(2007), no. 2, 567-585.
[12] J. Rickard, The abelian defect group conjecture. Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 121-128 (electronic).
[13] J. Rickard, Some recent advances in modular representation theory. Canad. Math. Soc. Conf. Proc. 23(1998)157-178.
[14] C. C. Xi, Stable equivalences of adjoint type. Forum Math. 20(2008), no.1, 81-97.
[15] C. C. Xi, Representation dimension and quasi-hereditary algebras. Adv. Math. 168(2002)280-298.
[16] C. C. Xi, The relative Auslander-Reiten theory of modules. Preprint, available at: http://math.bnu.edu.cn/~ccxi/Papers/Articles/rart.pdf, 2005.

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