ON THE IWASAWA INVARIANTS FOR LINKS
AND KIDA’S FORMULA

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Abstract. Analogues of Iwasawa invariants in the context of 3-dimensional topology have been studied by M. Morishita and others. In this paper, following the dictionary of arithmetic topology, we formulate an analogue of Kida’s formula on λ-invariants in a p-extension of \( \mathbb{Z}_p \)-fields for 3-manifolds. The proof is given in a parallel manner to Iwasawa’s second proof, with use of p-adic representations of a finite group. In the course of our arguments, we introduce the notion of a branched \( \mathbb{Z}_p \)-cover as an inverse system of cyclic branched p-covers of 3-manifolds, generalize the Iwasawa type formula, and compute the Tate cohomology of 2-cycles explicitly.

1. Introduction

The analogy between primes and knots is known ([Mor12]). Among other things, there is close analogy between Iwasawa theory and Alexander–Fox theory, and topological analogues of Iwasawa’s class number formula are studied ([HMM06], [KM08]). Let \( p \) denote a fixed prime number, and let \( \mathbb{Z}_p \) denote the ring of p-adic integers throughout this paper. We call a field \( K \) simply a \( \mathbb{Z}_p \)-field if it can be obtained as a \( \mathbb{Z}_p \)-extension of a finite number field \( k \). If the Iwasawa \( \mu \)-invariants vanish, extensions of \( \mathbb{Z}_p \)-fields resemble those of function fields, and satisfy Kida’s formula ([Kid80]), which is an arithmetic analogue of the classical Riemann–Hurwitz formula. In this paper, we formulate their topological analogue, in a very parallel manner to Iwasawa’s second proof in [Iwa81], using p-adic representation theory of finite groups and the Tate cohomologies.

Here are the contents of this paper. In Section 2, we review some basic analogies in branched Galois theories for finite degrees. An analogy between unit groups and 2-cycle groups was pointed out in our previous paper ([Uek14]). We calculate the Tate cohomology of 2-cycles explicitly. We also discuss analogue objects of \( S \)-ideal groups and others. In Section 3, we review Iwasawa’s class number formula of \( \mathbb{Z}_p \)-fields. In Section 4, as an analogue of \( \mathbb{Z}_p \)-extension, we introduce the notion of a branched \( \mathbb{Z}_p \)-cover of rational homology 3-spheres (QHS\(^3\)), namely, an inverse system of branched cyclic p-covers of QHS\(^3\)’s, and generalize the Iwasawa type formula:

**Theorem 4.9.** Let \( \tilde{M} = \{ M_n \}_n \) be a branched \( \mathbb{Z}_p \)-cover consisting of QHS\(^3\)’s. Then there are some \( \lambda, \mu, \nu \in \mathbb{Z} \) satisfying \( \# H_1(M_n, \mathbb{Z}_p) = p^{\lambda n + \mu p^n + \nu} \) for any \( n \gg 0 \).

Keywords: link, rational homology 3-sphere, branched cover, Galois theory, Iwasawa theory, arithmetic topology.

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We also employ the structure theorem of \( \Lambda \)-modules, and prove a \( p \)-adic variant of Sakuma exact sequence. Then we deduce an alternative proof for the Iwasawa type formula, and also a proposition on the direct limit of homology groups. In Section 5, we further introduce an analogous notion of a Galois extension of \( \mathbb{Z}_p \)-fields, say an \textit{equivariant Galois morphism} (or a \textit{branched Galois cover}) of branched \( \mathbb{Z}_p \)-covers. It is a compatible system of branched covers on each layer between two branched \( \mathbb{Z}_p \)-covers. In addition, we define chains of branched \( \mathbb{Z}_p \)-covers by direct limits with respect to the transfer maps, and compute the Tate cohomologies for them. In Section 6, we review Kida’s formula for an extension of \( \mathbb{Z}_p \)-fields. In Section 7, we establish an analogue of Kida’s formula for branched \( \mathbb{Z}_p \)-covers:

\[ \lambda_{\tilde{N}} - 1 = \deg(f)(\lambda_{\tilde{M}} - 1) + \sum_{w \subset S} (e_w - 1). \]

2. Theories for finite degrees

To begin with, we briefly review some basic analogies between primes and knots. In addition, for finite Galois extensions of number fields and finite branched Galois covers of 3-manifolds, we study analogies between unit groups and 2-cycle groups, \( S \)-ideals and “\( S \)-chains”, and compute their Tate cohomologies in an explicit way.

2.1. \( M^2 \)KR-Dictionary. The analogy between 3-dimensional topology and number theory was first pointed out by B. Mazur in the mid 1960’s (Maz64), and has been studied systematically by M. Kapranov (Kap95), A. Reznikov (Rez97, Rez00) and M. Morishita (Mor10, Mor12). In their dictionary of analogies, for example, knots and 3-manifolds correspond to primes and number rings respectively. The study of these analogies is christened “arithmetic topology” now. Here is a basic dictionary we shall use in this paper. For a number field \( k \), \( \mathcal{O}_k \) denotes the ring of integers.

| Number theory | 3-dimensional topology |
|---------------|------------------------|
| number ring \( \text{Spec} \mathcal{O}_k \) or \( \text{Spec} \mathcal{O}_k \) = \( \text{Spec} \mathcal{O}_k \cup \{ \text{infinite primes} \} \) | (oriented, connected, closed) \( 3 \)-manifold \( M \) |
| prime ideal \( p : \text{Spec} \mathcal{O}_k \to \text{Spec} \mathcal{O}_k \) | knot \( K : S^1 \to M \) |
| prime ideals \( S = \{ p_1, ..., p_r \} \) | link \( L = \{ K_1, ..., K_r \} \) |
| extension \( F/k \) | branched cover \( f : N \to M \) |
| étale fundamental group \( \pi_1(\text{Spec} \mathcal{O}_k) \) | fundamental group \( \pi_1(M) \) |
| \( \pi_1^\text{\acute{e}t}(\text{Spec} \mathcal{O}_k - S) \) | link group \( \pi_1(M - L) \) |
| geometric point \( \text{Spec} \mathbb{C} \to \text{Spec} \mathcal{O}_k \) | base point \( \{ \text{pt} \} \to M \) |
| ideal group \( I(k) \) | 1-cycle group \( Z_1(M) \) |
| \( k^* \to I(k) ; a \to (a) \) | \( C_2(M) \to Z_1(M) ; c \to \partial c \) |
| principal ideal group \( P(k) \) | 1-boundary group \( B_1(M) \) |
ideal class group \( \text{Cl}(k) = I(k)/P(k) \)

Fact: \( \# \text{Cl}(k) < \infty \)

1st-homology \( H_1(M) = Z_1(M)/B_1(M) \)
Condition: \( \# H_1(M) < \infty \) (i.e. \( M : \text{QHS}^3 \))

or consider torsion subgroup \( H_1(M)_{\text{tor}} \)

| Artin reciprocity \( \pi^\text{et}_{1}((\text{Spec } \hat{O}_k)^{ab}) \cong \text{Gal}(k_{\text{ur}}^{ab}/k) \cong \text{Cl}(k) \) | Hurewicz isomorphism \( \pi_i(M)^{ab} \cong \text{Gal}(\text{M}_{\text{ab}}/\text{M}) \cong H_1(M) \) |
|---|---|
| unit group \( \hat{O}_k^{\text{ur}} \) | 2-cycle group \( Z_2(M) \) |
| \( \text{Hilbert class field} \) | or 2nd-homology \( H_2(M) (\cong H_1(M)_{\text{free}}) \) |

We assume that number fields are finite over \( \mathbb{Q} \) and are contained in \( \mathbb{C} \), 3-manifolds are oriented, connected, and closed. A branched cover means an isomorphism class of branched covers with base points branched over links.

We have two attitudes about analogues of ideal class groups \( \text{Cl}(k) \) and unit groups \( \hat{O}_k^{\text{ur}} \). In the conventional attitude, we do not assume \( M \) to be \( \text{QHS}^3 \), and we regard the torsion subgroups \( H_1(M)_{\text{tor}} \) as an analogue of \( \text{Cl}(k) \). Then a non trivial term \( H_2(M) \cong H_1(M)_{\text{free}} \) plays an analogous role to \( \hat{O}_k^{\text{ur}} \) ([Sik03], [Mor08]). In another one, which we pointed out in [Uek14], we assume \( M \) to be a \( \text{QHS}^3 \), fix a CW-structure, and regard \( Z_2(M) \) as an analogue of \( \hat{O}_k^{\text{ur}} \). Then, we have the following parallel exact sequences:

\[
\begin{align*}
1 &\rightarrow P(k) \rightarrow I(k) \rightarrow \text{Cl}(k) \rightarrow 1 \\
&\rightarrow \hat{O}_k^{\text{ur}} \rightarrow k^{\text{ur}} \rightarrow P(k) \rightarrow 1 \\
&\rightarrow \text{Cl}(k) \rightarrow \hat{O}_k^{\text{ur}} \rightarrow k^{\text{ur}} \rightarrow P(k) \rightarrow 1 \\
&\rightarrow 0 \rightarrow B_1(M) \rightarrow C_1(M) \rightarrow H_1(M) \rightarrow 0 \\
&\rightarrow 0 \rightarrow Z_2(M) \rightarrow C_2(M) \rightarrow B_1(M) \rightarrow 0
\end{align*}
\]

In addition, we have the following theorem.

**Theorem 2.1** ([Uek14]). Let \( f : N \rightarrow M \) be a branched Galois cover of 3-manifolds. Let \( G = \text{Gal}(h) \), and fix CW-structures compatible with \( h \). Then the Tate cohomology of \( \hat{H}^i(G, Z_2(N)) \) is independent of the choice of CW-structures, and is a topological invariant of branched covers.

We also remark that \( \hat{H}^i(G, Z_2(N)) \) has more information than \( \hat{H}^i(G, H_2(N)) \). These above enable further translation. In our previous paper [Uek14], we gave an analogue of Yokoi’s formulation of genus theory ([Yok67]).

The Tate cohomology of the unit group played an important role in Iwasawa’s second proof of Kida’s formula. In this article, that of the 2-cycle group plays a similar role in the proof of our main result.

### 2.2. Computations of Tate cohomologies

In this subsection, we present some explicit computations of Tate cohomologies for number fields extensions and branched covers of 3-manifolds. For a group \( G \) and a \( G \)-module \( A \), \( \hat{H}^i(G, A) \) denotes the Tate cohomology for each \( i \in \mathbb{Z} \). We abbreviate \( \hat{H}^i(G, A) = \hat{H}^i(A) \) in the proofs if there is no ambiguity of \( G \).

The following facts in number theory are well-known.

**Proposition 2.2.** Let \( F/k \) be a Galois extension of number fields with \( G = \text{Gal}(F/k) \). Then,

1. [Hilbert’s Satz 90] The equality \( \hat{H}^1(G, F^e) = 0 \) holds.
2. [Iwasawa] If \( F/k \) is unramified and \( \text{Cl}(F) = 1 \), then \( \hat{H}^i(G, \hat{O}_F) \xrightarrow{\cong} \hat{H}^{i-1}(G, C_F) \cong \hat{H}^{i-3}(G, Z) \) for each \( i \in \mathbb{Z} \), where \( C_F \) denote the idele class group of \( F \). Some cases with restricted ramifications, such as cyclotomic extensions, are also computable.

We have the following on the topology side.
Proposition 2.3. Let $f : N \to M$ be a branched Galois cover of 3-manifolds, Put $G = \text{Gal}(f)$, and fix a CW-structures or PL-structures on $M$ and $N$ compatible with $f$. Then for each $i \in \mathbb{Z}$,

1. The equality $\hat{H}^i(G, C_2(N)) = 0$ holds.
2. There is a long exact sequence $\cdots \to \hat{H}^{i+1}(G, H_3(N)) \to \hat{H}^i(G, Z_2(N)) \to \hat{H}^i(G, H_2(N)) \to \hat{H}^{i+2}(G, H_3(N)) \to \cdots$. If in addition $N$ is a $\mathbb{Q}$HS$^3$, then $\hat{H}^i(G, Z_2(N)) \cong \hat{H}^i(G, H_3(N))$, where $H_3(N) = \langle [N] \rangle \cong \mathbb{Z}$.

Proof. (1) Since the subset of $G$-fixed points is 1-dimensional, $C_2(N)$ is $\mathbb{Z}[G]$-free.

(2) The exact sequence $0 \to B_3(N) \to Z_3(N) \to H_2(N) \to 0$ induces a long exact sequence $\cdots \to \hat{H}^i(B_3(N)) \to \hat{H}^i(Z_3(N)) \to \hat{H}^i(H_2(N)) \to \hat{H}^i+1(B_3(N)) \to \cdots$.

We prove $\hat{H}^i(B_2(N)) \cong \hat{H}^i+1(H_3(N))$. By the same reason as (1), $C_3(N)$ is a $\mathbb{Z}[G]$-free module, and $\hat{H}^i(C_3(N)) = 0$. Hence the exact sequence $0 \to Z_3(N) \to C_3(N) \to B_2(N) \to 0$ yields an isomorphism $\hat{H}^i(B_2(N)) \cong \hat{H}^i+1(Z_3(N))$. In addition, since $N$ is 3-dimensional, we have $B_3(N) = 0$. Hence the exact sequence $0 \to B_3(N) \to Z_3(N) \to H_3(N) \to 0$ yields isomorphisms $Z_3(N) \cong H_3(N) = \langle [N] \rangle \cong \mathbb{Z}$. Thereby, we obtain $\hat{H}^i(B_2(N)) \cong \hat{H}^i+1(Z_3(N)) \cong \hat{H}^i+1(H_3(N))$. Especially, if $N$ is a $\mathbb{Q}$HS$^3$, the Poincaré duality and the universal coefficient theorem ensure $H_2(N) = 0$, and hence $\hat{H}^i+1(H_3(N)) \cong \hat{H}^i(Z_2(N))$. \qed

2.3. S-ideals and S-chains. In this subsection, we recall some properties of S-ideals and others of number fields or $\mathbb{Z}_p$-fields. Then we introduce their analogues for 3-manifolds, and obtain further results of Tate cohomologies.

Let $F/k$ be a finite Galois extension of number fields or $\mathbb{Z}_p$-fields. Let $S$ be a finite set of primes in $k$, and let $S$ also denote the set of primes above $S$ in $F$. Then the $S$-ideal group $I_{F,S}$, the principal $S$-ideal group $P_{F,S}$, the $S$-ideal class group $\text{Cl}_{F,S}$, and the $S$-unit group $\mathcal{O}_{F,S}^*$ are defined, and there are exact sequences of $G = \text{Gal}(F/k)$-modules (Broed, Iwas, NSW08).

$$0 \to P_{F,S} \to I_{F,S} \to \text{Cl}_{F,S} \to 0$$
$$0 \to \mathcal{O}_{F,S}^* \to F^* \to P_{F,S} \to 0$$

For a prime $p$ in $k$, let $I_{F,p}$ denote the group of ideals of $F$ over $p$, and let $Z_p$ denote the decomposition group of $p$. Then we have isomorphisms $\hat{H}^i(G, I_p) \cong \hat{H}^i(G, \mathbb{Z}[G/Z_p]) \cong \hat{H}^i(Z_p, \mathbb{Z})$. The direct sum decomposition $I_F = \oplus_p I_{F,p}$ yields $\hat{H}^i(G, I_F) = \oplus_p \hat{H}^i(G, I_{F,p}) \cong \oplus_p \hat{H}^i(Z_p, \mathbb{Z})$.

Proposition 2.4 (Iwasawa Iwas). Let $F/k$ be an extension of degree $p$ unramified at infinite primes, put $G = \text{Gal}(F/k)$, and let $S$ denote the set of ramified non-primes. Then, $S$ is a finite set.

1. The Tate cohomology $\hat{H}^i(G, I_{F,S}) = \oplus_{p \in S} \hat{H}^i(G, I_p)$ of $S$-ideals ($i \in \mathbb{Z}$) vanishes if and only if no ideals outside $S$ inert. (If $k$ is a number field, then it does not vanish.)

2. Let $I_S$ denote the subgroup of $I(F)$ generated by elements of $S$. Then the $S$-unit group satisfies $\mathcal{O}_{F,S}^*/\mathcal{O}_F^* \cong P(F) \cap I_S = I_S \subseteq \mathbb{Z}^S$ of finite index, and $\hat{H}^1(G, \mathcal{O}_{F,S}^*/\mathcal{O}_F^*) = 0, \hat{H}^2(G, \mathcal{O}_{F,S}^*/\mathcal{O}_F^*) \cong (\mathbb{Z}/p\mathbb{Z})^{\#S}$.

We consider analogues of these above. Let $f : N \to M$ be a finite Galois branched cover of 3-manifolds, put $G = \text{Gal}(f)$ and fix CW-structures or PL-structures compatible with $f$. Let $\overline{S} \subseteq M$ be a link and suppose that its each
component is in or outside the branch link of \( f \). Note that analogues of \( S \)-ideals are not the ones obtained from \( C_\ast(N, S) := \text{Coker}(C_\ast(S) \to C_\ast(N)) \). Instead, we consider the following commutative diagram, whose rows are exact.

\[
\begin{array}{cccccc}
0 & \to & B_1(S) & = 0 & \to & Z_1(S) \overset{\cong}{\to} H_1(S) & \to 0 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \to & B_1(N) & \to & Z_1(N) & \to & H_1(N) \to 0
\end{array}
\]

We identify \( Z_1(S) \) and \( \iota_\ast(Z_1(S)) \). We put \( Z_1(N)_S := Z_1(N)/Z_1(S) \), \( H_1(N)_S := H_1(N)/\iota_\ast(H_1(S)) \), and \( B_1(N)_S := B_1(N)/(B_1(N) \cap Z_1(S)) \). In addition, for the boundary map \( \partial : C_2(N) \to C_1(N) \), we put \( Z_2(N)_S := \partial^{-1}(Z_1(S)) \). Then we have \( Z_2(N)_S \cong Z_2(N, S) \). Thus we obtain exact sequences of \( G \)-modules parallel to the case of number theory.

\[
0 \to B_1(N)_S \to Z_1(N)_S \to H_1(N)_S \to 0 \\
0 \to Z_2(N)_S \to C_2(N) \to B_1(N)_S \to 0
\]

The analogue of the \( S \)-ideal group satisfies the following proposition.

**Proposition 2.5.** Let \( f : N \to M \) be a finite Galois branched cover with \( G = \text{Gal}(f) \), let \( \overline{S} \subset M \) be a link, and put \( S = f^{-1}(\overline{S}) \).

(i) If \( S \) is not empty, then there is an isomorphism \( \hat{H}^i(G, Z_1(N)_S) \cong \hat{H}^{i-1}(G, \text{Ker}(\iota_\ast : H_0(S) \to H_0(N))) \) for each \( i \in \mathbb{Z} \). If in addition \( f \) is of degree \( p \) and \( \overline{S} \) is the branch link with \( s \) components, then \( \hat{H}^0(G, Z_1(N)_S) = 0 \) and \( \hat{H}^1(G, Z_1(N)_S) \cong (\mathbb{Z}/p\mathbb{Z})^s \).

(ii) If \( S \) is empty so that \( f \) is unbranched, then \( \hat{H}^i(G, Z_1(N)) \cong \hat{H}^i(G, H_0(N))) \cong \hat{H}^i(G, \mathbb{Z}) \) for each \( i \in \mathbb{Z} \). If in addition \( \text{deg}(f) = p \), then \( \hat{H}^0(G, Z_1(N)) \cong \mathbb{Z}/p\mathbb{Z} \), \( \hat{H}^1(G, Z_1(N)) = 0 \).

**Proof.** If \( S \) is not empty, then the exact sequences \( 0 \to Z_1(S) \to C_1(S) \to B_0(S) \to 0 \), \( 0 \to Z_1(N) \to C_1(N) \to B_0(N) \to 0 \) and the snake lemma yield an exact sequence \( 0 \to Z_1(N)_S \to C_1(N)/C_1(S) \to B_0(N)/B_0(S) \to 0 \). Here, \( C_1(N)/C_1(S) \) is a \( \mathbb{Z}[G] \)-free module. In addition, the exact sequences \( 0 \to B_0(S) \to Z_0(S) \to H_0(S) \to 0 \), \( 0 \to B_0(N) \to Z_0(N) \to H_0(N) \to 0 \) and the snake lemma yield an exact sequence \( 0 \to \text{Ker}(\iota_\ast : H_0(S) \to H_0(N)) \to B_0(N)/B_0(S) \to Z_0(N)/Z_0(S) \to 0 \).

Here, \( Z_0(N)/Z_0(S) \) is a \( \mathbb{Z}[G] \)-free module. As a consequence, isomorphisms \( \hat{H}^i(Z_1(N)_S) \cong \hat{H}^{i-1}(B_0(N)/B_0(S)) \cong \hat{H}^{i-1}(\text{Ker}(\iota_\ast : H_0(S) \to H_0(N))) \) are obtained. Especially, if \( \text{deg}(f) = p \) and \( \overline{S} \) is properly branched, then \( \text{Ker}(\iota_\ast : H_0(S) \to H_0(N)) \cong \mathbb{Z}^{p-1} \).

If \( S \) is empty, we consider the exact sequences \( 0 \to Z_1(N) \to C_1(N) \to B_0(N) \to 0 \) and \( 0 \to B_0(N) \to Z_0(N) \to H_0(N) \to 0 \), in which \( C_1(N) \) and \( Z_0(N) \) are \( \mathbb{Z}[G] \)-free modules, and \( H_0(N) \cong \mathbb{Z} \). Then, isomorphisms \( \hat{H}^i(Z_1(N)_S) = \hat{H}^i(Z_1(N)) \cong \hat{H}^{i-1}(B_0(N)) \cong \hat{H}^{i-2}(H_0(N)) \cong \hat{H}^i(\mathbb{Z}) \) are obtained. \( \square \)

If we fix a \( \mathbb{Z} \)-basis of \( Z_1(M) \) and regard it as an analogue of the set of primes in the base field, then we obtain a direct sum decomposition of \( Z_1(N) \) as \( G \)-modules.

The analogue of the \( S \)-unit group satisfies the following.

**Proposition 2.6.** Let \( f : N \to M \) be a Galois branched cover of degree \( p \) with \( G = \text{Gal}(f) \), let \( \overline{S} \) be the branch link with \( s \)-components, and put \( S = f^{-1}(\overline{S}) \). Suppose that \( S \) consists of \( \mathbb{Q} \)-null-homologous components. (This assumption holds if \( N \) is
a QHS. Then $S$-2-cycles satisfy $Z_2(N)\otimes Z_2(N) \cong B_1(N) \cap Z_1(S)$ for a subgroup $B_1(N) \cap Z_1(S) \leq Z_1(S) \cong \mathbb{Z}^*$ of finite index, and $\tilde{H}^1(G, Z_2(N)\otimes Z_2(N)) = 0$, $\tilde{H}^2(G, Z_2(N)\otimes Z_2(N)) \cong (\mathbb{Z}/p\mathbb{Z})^\ast$.

Proof. Note that $Z_2(N)\otimes Z_2(N) \cong Z_2(N, S)$. There is a natural exact sequence $0 \rightarrow Z_2(N) \rightarrow Z_2(N, S) \rightarrow Z_1(S)$. Since every component of $S$ is $\mathbb{Q}$-null-homologous, $\partial Z_2(N, S) = B_1(N) \cap Z_1(S) < Z_1(S)$ is a subgroup of finite index. Since $S$ is the preimage of the branch link, $G$ acts on $B_1(S)$ trivially. \hfill $\square$

These computations above will be extended for $\mathbb{Z}_p$-covers in Subsection 5.4.

3. Iwasa's Class Number Formula of $\mathbb{Z}_p$-Extensions and $\Lambda$-Modules

It has been known that there is close analogy between Iwasawa theory and Alexander–Fox theory ([Maz64], [Mor12]). For an inverse system of branched cyclic $p$-covers of QHS obtained from a $\mathbb{Z}$-cover, an analogue of Iwasawa's class number formula is formulated ([HMM06], [KM08], [KM13]). In this paper, we generalize this formula for an inverse system which is not necessarily obtained from a $\mathbb{Z}$-cover.

| Iwasawa theory | Alexander–Fox theory |
|----------------|----------------------|
| $\mathbb{Z}_p$-extension of $k$ | $\mathbb{Z}$-cover over $\mathbb{M} - L$, or branched $\mathbb{Z}_p$-cover over $(\mathbb{M}, L)$ |
| Iwasawa module | link module |
| Iwasawa polynomial | Alexander polynomial |

In this section, we recall algebraic lemmas on $\Lambda$-modules, a proof of Iwasawa's class number formula, and an assertion on some direct limit module. Their analogues will be discussed in the next section.

Let $\Lambda = \mathbb{Z}_p[[T]]$ denote the ring of formal power series. For $\Lambda$-modules $\mathbb{M}$ and $\mathbb{M}'$, a pseudo isomorphism $\mathbb{M} \sim \mathbb{M}'$ is a homomorphism with finite kernel and cokernel. For finitely generated compact $\Lambda$-modules, pseudo isomorphisms give an equivalence relation. There is a structure theorem on compact $\Lambda$-modules.

**Lemma 3.1** (Whasington [Was97], Chapter 13). Let $E$ be a compact $\Lambda$-module.

1. (Nakayama’s lemma) $E$ is a finitely generated $\Lambda$-module if and only if $E/(p, T)$ is a finite group.
2. Let $E$ be a finitely generated $\Lambda$-module. Then, there is a pseudo isomorphism

$$E \sim \Lambda^{\geq r} \oplus (\oplus \Lambda/(f_i^{e_i})) \oplus (\oplus \Lambda/(p^{m_i}))$$

to a unique normal form. Here, $r, e_i, m_j \in \mathbb{N}$, and $f_i \in \mathbb{Z}/p[T]$ is an irreducible Weierstrass polynomial, namely, the coefficient of its highest term is 1 and others are multiples of $p$.
3. Let $\mu = \sum m_j, \lambda = \sum e_i \deg(f_i)$ in the normal form above, and put $\nu_{p^n} := (p^n - 1)/(t - 1) = \sum_{0 \leq i < p^n} t^i$. If $E/\nu_{p^n} E$ is finite for all $n$, then $r = 0$, and there is some $\nu, n_0$ such that for any $n > n_0$,

$$E/\nu_{p^n} E = \mathbb{P}^{\lambda n + \mu p^n + \nu}.$$ 
4. In this situation, $\mu = 0$ if and only if the $p$-rank of $E/\nu_{p^n} E$ is bounded.

Next, we review number theory. A $\mathbb{Z}_p$-field is a field obtained as a $\mathbb{Z}_p$-extension of a number field. If $k_\infty/k$ is a $\mathbb{Z}_p$-extension of a number field, then $k_\infty$ is the

*A cyclotomic $\mathbb{Z}_p$-field and a cyclotomic $\mathbb{Z}_p$-field which includes $p$-power-th roots of unity in this paper are a $\mathbb{Z}_p$-field and a cyclotomic $\mathbb{Z}_p$-field in the sense of Iwasawa in [Iwa81] respectively.*
Theorem 3.2 (Iwasawa’s class number formula [Iwa59]). Let \( k_\infty/k \) be a \( \mathbb{Z}_p \)-extension of a finite number field. For each \( n \in \mathbb{N} = \mathbb{N} \cup \{0\} \), let \( k_n/k \) denote the subextension of degree \( p^n \), and let \( \text{Cl}(k_n) \) denote the \( p \)-ideal class group. Then there are some \( \lambda, \mu \in \mathbb{N} \), \( \nu \in \mathbb{Z} \), and \( n_0 \in \mathbb{N} \) such that for any \( n > n_0 \),

\[
\# \text{Cl}(k_n) = p^{\lambda n + \mu p^n + \nu}.
\]

These \( \lambda, \mu, \) and \( \nu \) are called the Iwasawa invariants, and denoted as \( \lambda_{k_\infty/k}, \mu_{k_\infty/k}, \) and \( \nu_{k_\infty/k} \). The value of \( \lambda \) and whether \( \mu = 0 \) or not depend only on the \( \mathbb{Z}_p \)-field \( k_\infty \), and are independent of the base field \( k \). Hence they are sometimes expressed as \( \lambda_{k_\infty} = \mu_{k_\infty} = 0 \). Let \( \mathbb{Q}_\infty \) denote the unique \( \mathbb{Z}_p \)-extension over \( \mathbb{Q} \). For a number field \( k \), the cyclotomic \( \mathbb{Z}_p \)-extension of \( k \) is defined by \( k_\infty^{\text{cyc}} := k \mathbb{Q}_\infty \), and the Iwasawa invariants of \( k \) are defined by those of \( k_\infty^{\text{cyc}}/k \).

In the following, we review a proof of this formula with use of Lemma 3.1 on \( \Lambda \)-modules. Let \( k_\infty/k \) be as above, let \( t \) be a topological generator of \( \Gamma := \text{Gal}(k_\infty/k) \cong \mathbb{Z}_p \), and fix an identification \( \Lambda = \mathbb{Z}[T] \cong \mathbb{Z}_p[T]; 1 + T \leftrightarrow t \). The group \( \Gamma \) acts on the inverse system \( \{ \text{Cl}(k_n) \}_{n} \), and hence the Iwasawa module \( \mathcal{H} := \lim_{\leftarrow} \text{Cl}(k_n) \) continuously, and this action makes \( \mathcal{H} \) a compact \( \Lambda \)-module. The \( p \)-class groups \( \text{Cl}(k_n) \) can be expressed as a quotient of \( \mathcal{H} \):

Proposition 3.3 (Washington [Was97], Proposition 13.22 and Chapter 13.3). Let \( k_\infty/k \) be a \( \mathbb{Z}_p \)-extension with \( n \)-th subfield \( k_n \), and put \( \mathcal{H} = \lim_{\leftarrow} \text{Cl}(k_n) \).

(1) When precisely one prime is ramified in \( k_\infty/k \) and it is totally ramified, for each \( n \), (i) there is an isomorphism

\[
\text{Cl}(k_n) = \mathcal{H}/(p^n - 1)\mathcal{H},
\]

and (ii) \( p|\#\text{Cl}(k) \) if and only if \( p|\#\text{Cl}(k_n) \).

(2) More generally, if every ramified prime in \( k_\infty/k \) is totally ramified, then there is a subgroup \( \mathcal{H}' < \mathcal{H} \) of finite index such that \( \text{Cl}(k_n) = \mathcal{H}/\mathcal{H}' \).

The number of ramified primes in a \( \mathbb{Z}_p \)-extension is always finite. General cases reduce to the cases of totally ramified. Indeed, for any \( \mathbb{Z}_p \)-extension \( k_\infty/k \), there is some \( n_0 \) such that \( k_\infty/k_{n_0} \) is totally ramified at every ramified prime.

The following lemma can be obtained from Lemma 3.1 (1):

Lemma 3.4 (Washington [Was97], Chapter 13.3). The Iwasawa module \( \mathcal{H} \) is a finitely generated torsion \( \Lambda \)-module.

Now Theorem 3.2 (Iwasawa’s class number formula) follows immediately from Lemma 3.4 Proposition 3.3 and Lemma 3.1 (3) (The structure theorem of compact \( \Lambda \)-modules).

On the other hand, in a \( \mathbb{Z}_p \)-extension \( k_\infty/k \), ideal class groups of \( k_n \) form an inductive system with respect to the maps induced by the natural injections of ideal groups \( I_{k_n} \rightarrow I_{k_{n+1}} \). The ideal class group of \( k_\infty = \bigcup k_n \) is defined by \( \text{Cl}(k_\infty) := I(k_\infty)I(k_\infty) \), where \( I(k_\infty) = \lim I(k_n), P(k_\infty) = \lim P(k_n) \), and satisfies \( \text{Cl}(k_\infty) = \lim \text{Cl}(k_n) \). We have the following proposition.
Proposition 3.5 (Iwasawa [Iwa81], a remark in Chapter 5). Let $k_{\infty}/k$ be a cyclotomic $\mathbb{Z}_p$-extension. Then, there is an isomorphism of discrete $\Lambda$-modules

$$\text{Cl}(k_{\infty}) = \lim_{\rightarrow} \text{Cl}(k_n)[p] \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\mu_{k_{\infty}/k}} \oplus A'$$

where $A'$ is a bounded module, namely, there is some $a \in \mathbb{N}$ such that $p^n A' = 0$, and $\mu_{k_{\infty}/k} = 0$ if and only if $A' = 0$.

Such a group is discussed also in Iwasawa [Iwa73a], Chapter 5.

4. Iwasawa Type Formula of Branched $\mathbb{Z}_p$-Covers and $\Lambda$-Modules

In this section, we first introduce the notion of a branched $\mathbb{Z}_p$-cover of 3-manifolds as an analogue of a $\mathbb{Z}_p$-extension, and explain that it essentially generalizes the conventional objects. Next, we prove the Iwasawa type formula on the homology growth for a branched $\mathbb{Z}_p$-covers of $\mathbb{Q}HS^3$s. Moreover, we state a $p$-adic variant of Sakuma’s exact sequence, whose proof will be given afterwards, and deduce an alternative proof of the Iwasawa type formula in a parallel manner to the case of number theory. In addition, we deduce a proposition on the direct limit of homology groups which will be used in Section 7. We also discuss further generalization to the cases of non-$\mathbb{Q}HS^3$ briefly.

4.1. Branched $\mathbb{Z}_p$-covers. In this subsection, we discuss an analogue of $\mathbb{Z}_p$-extension.

Definition 4.1. Let $M$ be a 3-manifold and let $L$ be a link in $M$. A branched $\mathbb{Z}_p$-cover over $(M, L)$ is an inverse system $\bar{M} = \{h_n : M_n \rightarrow M\}_n$ of cyclic branched covers of $M$ branched over $L$ with degree $p^n$. It is a branched $\mathbb{Z}_p$-cover of $\mathbb{Q}HS^3$ if all $M_n$ are $\mathbb{Q}HS^3$.

Proposition 4.2. Let $L$ be a link in a 3-manifold $M$ and put $X = M - L$. A branched $\mathbb{Z}_p$-cover over $(M, L)$ corresponds to a homomorphism $\tau : \pi_1(X) \rightarrow \mathbb{Z}_p$ such that $\tau \mod p \neq 0$ uniquely up to $\text{Aut}(\mathbb{Z}_p)$.

Proof. If such $\tau$ is given, then the composite $\tau_n := (\mathbb{Z}_p \rightarrow \mathbb{Z}/p^n\mathbb{Z}) \circ \tau$ with the natural surjection is a surjective homomorphism for each $n \in \mathbb{N}$. A subgroup $\text{Ker}(\tau_n) < \pi_1(X)$ corresponds to a cyclic cover $h_n : X_n \rightarrow X$ with degree $p^n$, and hence a cyclic branched cover $h_n : M_n \rightarrow M$ by Fox completion. The family $\{h_n\}_n$ forms an inverse system in a natural way.

Conversely, if such an inverse system $\bar{M} = \{h_n : M_n \rightarrow M\}_n$ is given, a family of surjective homomorphism $\tau_n : \pi_1(X) \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ is obtained. Since $h_n$ is a subcover of $h_{n+1}$ for each $n$, $\text{Ker} \tau_n < \text{Ker} \tau_{n+1}$ holds, and there is a surjective homomorphism $q_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ which satisfies $q_n \circ \tau_{n+1} = \tau_n$. The inverse limit of an inverse system $\{\mathbb{Z}/p^n\mathbb{Z}, q_n\}_n$ is isomorphic to $\mathbb{Z}_p$, and hence $\tau : \pi_1(X) \rightarrow \mathbb{Z}_p$ is obtained up to $\text{Aut}(\mathbb{Z}_p)$. (In order to obtain $\tau$ explicitly, for each $n$, we replace $\tau_n$ by the composite of $\tau_n$ and an element of $\text{Aut}(\mathbb{Z}/p^n\mathbb{Z})$ such that $q_n$ is the natural surjection $\mathbb{Z}_p \rightarrow \mathbb{Z}/p^n\mathbb{Z}$.) Such $\tau$ is unique up to $\text{Aut}(\mathbb{Z}_p)$.

The assumption $\tau \mod p \neq 0$ is equivalent to that $\tau$ sends a generator of $\pi_1(X)$ to a unit of $\mathbb{Z}_p$, that $\tau$ has a dense image, and that $\tau$ induce a surjection $\tilde{\tau} : \tilde{\pi}_1(X) \rightarrow \mathbb{Z}_p$ from the pro-$p$ completion defined as $\tilde{\pi}_1(X) := \lim_{\rightarrow} \pi_1(X)/N$ with $N$ running through the set of normal subgroups with $p$-power indices.
Remark 4.3. Let $\tilde{M}$ be a branched $\mathbb{Z}_p$-cover as above. The Fox completion is defined for more general objects than manifolds called spreads ([Fox57]), and $M_\infty := \varprojlim M_n$ is the Fox completion of $X_\infty := \varprojlim X_n$, which is not necessarily a manifold. There are some researches on such objects ([DeSh6, DeSS, CS78]).

Since the image of $\tau : \pi_1(X) \to \mathbb{Z}_p$ is abelian, $\tau$ factors through $\pi_1(X)^{ab} \cong H_1(X)$. In order to see examples, we prepare the following lemma on $H_1(X)$.

It was originally given by [KM08] Lemma 4.2 for $\mathbb{Q}\mathrm{HS}^3$, but it can be prove for a general (oriented, connected, and closed) $M$ in a similar way with use of intersection form.

Lemma 4.4. Let $L = \sqcup K_i$ be a $d$-component link in a 3-manifold $M$, put $X = M - L$, and let $\mu_i \in H_1(X)$ denote the meridian of $K_i$ for each $i$. Then that $L$ consists of null-homologous components is equivalent to that the natural exact sequence

$$0 \to \langle \mu_1, \ldots, \mu_d \rangle \to H_1(X) \to H_1(M) \to 0$$

with $\langle \mu_1, \ldots, \mu_d \rangle \cong \mathbb{Z}^d$ splits, and that $H_1(X)_{\text{free}} := H_1(X)/H_1(X)_{\text{tor}}$ has a $\mathbb{Z}$-basis containing the image of $\langle \mu_1, \ldots, \mu_d \rangle$.

Proof. Fix a tubular neighborhood $V_L = \sqcup V_{K_i}$ of $L$ and put $X^0 = M - \text{Int}(V_L)$. Then there is a non-degenerate quadratic form $I : H_2(X^0, \partial X^0)_{\text{free}} \times H_1(X^0)_{\text{free}} \to \mathbb{Z}$ called the intersection form. Let $\mu_i$ denote the meridians of $K_i$ in all of $H_1(\partial V_{K_i})$, $H_1(X^0) \cong H_1(X)$, and $H_1(X)_{\text{free}}$ for each $i$.

Suppose that $[K_i] = 0$ in $H_1(M)$ for all $i$. Then for each $i$, there is a surface $S_i'$ in $M$ such that $\partial S_i' = K_i$. We may assume that $\text{Int}(S_i')$ intersects both $L$ and $\partial V_L$ transversely. Put $S_i := S_i' \cap X^0$. Since there is a basis of $H_1(\partial X^0)$ containing ($\partial[S_1], \ldots, \partial[S_d]$), there is a $\mathbb{Z}$-basis of $H_2(X^0, \partial X^0)_{\text{free}}$ containing ($[S_1], \ldots, [S_d]$). Replace elements $T_j$ of this basis other than $S_i$’s so that $I(T_j, \mu_i) = 0$. Then the dual basis of $H_1(X)$ with respect to $I$ contains $\langle \mu_1, \ldots, \mu_d \rangle$, and its lift defines an isomorphism $H_1(X) \cong \langle \mu_1, \ldots, \mu_d \rangle \mathbb{Z} \oplus H_1(M)$.

Conversely, suppose that there is a $\mathbb{Z}$-basis of $H_1(X)_{\text{free}}$ containing $\langle \mu_1, \ldots, \mu_d \rangle$, and take the dual basis of $H_2(X^0, \partial X^0)$ with respect to $I$. Then for each $i$, there is a surface $S_i$ in $X^0$ such that $[S_i]$ is an element of the dual basis satisfying $I([S_i], \mu_i) = 1$. We may assume that $S_i$ intersects $\partial X$ transversely. Since $\partial M = \phi$, $\partial S_i$ is the sum of a longitude of $\partial V_{K_i}$ and some meridians. Capping off the meridians in $\partial S_i$ by the meridian discs, and extending the longitudes of $\partial V_{K_i}$ to $K_i$, we obtain a Seifert surface $S_i'$ of $K_i$, and hence $[K_i] = 0$ in $H_1(M)$.

Following is an important example obtained from a $\mathbb{Z}$-cover.

Example 4.5 (TLN-cover, [KM08]). Let $L = \sqcup K_i$ be a link in a 3-manifold $M$, let $\mu_i$ denote the meridian of $K_i$, and suppose that $L$ consists of null-homologous components. Then a standard $\mathbb{Z}$-cover $\tilde{X} \to X = M - L$ is defined by $\tau : \pi_1(X) \to \mathbb{Z}; \forall \mu_i \mapsto 1$ which induce the zero map on $H_1(M)$ and it is called the TLN-cover over $(M, L)$. We call the inverse system of branched $p$-covers obtained from such a $\mathbb{Z}$-cover the TLN-$\mathbb{Z}_p$-cover $\tilde{M} = \{ h_n : M_n \to M \}_{n}$ over $(M, L)$.

We prove in Subsection 4.4 that all the $M_n$ are $\mathbb{Q}\mathrm{HS}^3$ if and only if $M$ is a $\mathbb{Q}\mathrm{HS}^3$ and the reduced Alexander polynomial $\Delta_{L, \tau}(t)$ is not divided by any cyclotomic polynomial of $p$-power-th.

In order to explain that the notion of a branched $\mathbb{Z}_p$-cover gives an essential generalization, we introduce the notion of an isomorphism of them.
Definition 4.6. Branched $\mathbb{Z}_p$-covers $\widetilde{M} = \{h_n : M_n \rightarrow M\}_n$ and $\widetilde{M}' = \{h'_n : M'_n \rightarrow M'\}_n$ are isomorphic if the following equivalent conditions are satisfied.

(1) There is a compatible system $\{f_n : M'_n \xrightarrow{\cong} M_n\}_n$ of isomorphisms on each layer.

(2) There is an isomorphic cover $f_0 : X' \xrightarrow{\cong} X$ of the exteriors of some links in the exterior, and there is an isomorphism $\iota : \mathbb{Z}_p \xrightarrow{\cong} \mathbb{Z}_p$ such that $\iota \circ \tau = \tau' \circ f_0$, where $\tau : \pi_1(X) \rightarrow \mathbb{Z}_p$ and $\tau' : \pi_1(X') \rightarrow \mathbb{Z}_p$ are the the defining homomorphisms and $f_0 : \pi_1(X) \xrightarrow{\cong} \pi_1(X')$ is the induced map.

Proof. We prove the equivalence of (1) and (2). Suppose an isomorphism $f_0 : X' \xrightarrow{\cong} X$ is given. Let $\iota_n, \tau'_n$ denote the composites of $\iota, \tau'$ and the natural surjection $\mathbb{Z}_p \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ respectively, and consider the following commutative diagram consists of exact rows.

$$
\begin{array}{ccc}
0 & \rightarrow & \pi_1(X'_n) \rightarrow \pi_1(X') \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow 0 \\
| & | & | \\
0 & \rightarrow & \pi_1(X_n) \rightarrow \pi_1(X) \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow 0
\end{array}
$$

By diagram chasing, taking $f_n$ is equivalent to taking $\iota_n$, and $f_n\ast$ is isomorphic if and only if $\iota_n$ is isomorphic. Then the conclusion follows immediately. \qed 

A branched $\mathbb{Z}_p$-cover $\widetilde{M}$ over $(M, L)$ is isomorphic to one obtained from a $\mathbb{Z}$-cover if and only if the defining homomorphism $\tau$ factors some homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}_p$.

If $L$ is a knot and $M$ is a QHS$^3$, then $\widetilde{M}$ is always isomorphic to one obtained from a $\mathbb{Z}$-cover. However, if $L$ has more than one component, then $\widetilde{M}$ may not:

Example 4.7. Let $p \equiv 1 \pmod{4}$. Then $\sqrt{-1} \in \mathbb{Z}_p^\ast$. Let $L = K_1 \cup K_2$ be a 2-component link in $M = S^3$ and put $X = M - L$. Let $\mu_i$ denote the meridians of $K_i$, and let $\widetilde{M}$ be a branched $\mathbb{Z}_p$-cover over $(M, L)$ defined by $\tau : H_1(X) \rightarrow \mathbb{Z}_p; \mu_1 \mapsto 1, \mu_2 \mapsto \sqrt{-1}$. Since 1 and $\sqrt{-1}$ cannot move into $\mathbb{Z}$ at the same time by multiplying any unit of $\mathbb{Z}_p$, $\tau$ does not factor through any homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}_p$. Therefore $\widetilde{M}$ cannot be obtained from a $\mathbb{Z}$-cover.

More concretely, let $p = 5$. Then $\sqrt{-1} = \ldots 1212(5)$ (base 5). Let $\widetilde{M}'$ be another $\mathbb{Z}_p$-cover defined by $\tau' : \mu_1 \mapsto 1, \mu_2 \mapsto 12(5)$. Then $h_n$ and $h_n$ are isomorphic for $n \leq 2$. Thus, a $\mathbb{Z}_p$-cover can be “approximate” by $\mathbb{Z}$-cover as much as we like.

We remark that an analogue of the Hilbert ramification theory for $\mathbb{Z}_p$-extensions holds in a natural way. It is an immediate consequence of the theory for finite covers given in [Mor12] Chapter 5 or [Uek14].

Let $\overline{M} = \{h_n : M_n \rightarrow M\}_n$ be a branched $\mathbb{Z}_p$-cover over $(M, L)$, let $K \subset M$ be a knot in or outside $L$, and let $\overline{K} = \{K_n\}_n$ denote an inverse system of knots over $K$ in $\overline{N}$. For each $n$, there is a subgroup of $\text{Gal}(h_n)$ called the inertia group $I_{K_n}$ and the decomposition group $D_{K_n}$ of $K_n$ in $h_n$. They satisfy $I_{K_n} < D_{K_n} < \text{Gal}(h_n)$ and control the behavior of $K_n$ as follows: Let $h_n : M_n \rightarrow T_n \rightarrow Z_n \rightarrow M$ denote the corresponding decomposition. Then (the images of) $K_n$ is totally branched in $M_n \rightarrow T_n$, totally inert in $T_n \rightarrow Z_n$, and totally decomposed in $Z_n \rightarrow M$. Moreover, since each subgroup of $\text{Gal}(h_n) \cong \mathbb{Z}/p^n\mathbb{Z}$ is equal to $\text{Gal}(h_{m,n})$ for some $m \leq n$, $T_n = M_{n_1}$ and $Z_n = M_{n_2}$ for some $n_2 \leq n_1 \leq n$. 


Note that \( \{I_{K_n}\}_n \) and \( \{D_{K_n}\}_n \) form surjective systems. We define the inertia group \( I_{\hat{K}} \) and the decomposition group \( D_{\hat{K}} \) of \( \hat{K} \) as their inverse limits. Since \( \lim \Gal(h_n) = \mathbb{Z}_p \), they are open subgroups with \( I_{\hat{K}} < D_{\hat{K}} < \lim \Gal(h_n) \), and they control the behavior of \( \hat{K} \) in \( \hat{M} \) in a similar way to the case of finite covers. Since each open subgroup \( G' \) of \( \lim \Gal(h_n) = \mathbb{Z}_p \) satisfies \( \lim \Gal(h_n)/G' \cong \Gal(h_m) \) for some \( m \in \mathbb{N} \), we have the following.

**Proposition 4.8.** Let \( \tilde{M} = \{ h_n : M_n \to M \}_n \) be a branched \( \mathbb{Z}_p \)-cover over \( (M, L) \), let \( K \subset M \) be a knot in or outside \( L \), and let \( \tilde{K} = \{K_n\}_n \) denote a surjective system of knots over \( K \) in \( \tilde{N} \). Then \( \tilde{K} \) satisfies one of the following: (i) infinitely branched, finitely inert, and finitely decomposed, (ii) unbranched, infinitely inert, and totally decomposed.

We say a branched \( \mathbb{Z}_p \)-cover \( \tilde{M} = \{ h_n : M_n \to M \}_n \) over \( (M, L) \) is properly branched if it satisfies the following equivalent conditions: (1) Some \( h_n \) is properly branched over some non-empty link, (2) \( \tilde{M} \) is infinitely branched over some non-empty link, (3) \( \tau(\mu_i) \neq 0 \) for some meridian \( \mu_i \) of \( L \).

### 4.2. The Iwasawa type formula and Sakuma’s exact sequence.

In this subsection, we prove the Iwasawa type formula for a branched \( \mathbb{Z}_p \)-cover of \( \text{QHS}^3 \). Moreover, we state a \( p \)-adic variant of Sakuma’s exact sequence. Then we deduce an alternative direct proof of the Iwasawa type formula, and also an assertion on a direct limit module.

For any manifold \( M \), we denote \( H_1(M)_{[p]} := H_1(M, \mathbb{Z}_p) = H_1(M) \otimes \mathbb{Z}_p \). If \( M \) is a \( \text{QHS}^3 \), then \( H_1(M)_{[p]} \) is identified with the \( p \)-torsion subgroup of \( H_1(M) \). A generalization of the analogue of Iwasawa’s class number formula ([HMM08], [KM08]) describes the behavior of \( p \)-torsions in a branched \( \mathbb{Z}_p \)-cover of \( \text{QHS}^3 \).

**Theorem 4.9** (the Iwasawa type formula). Let \( \tilde{M} = \{ h_n : M_n \to M \}_n \) be a branched \( \mathbb{Z}_p \)-cover of \( \text{QHS}^3 \) over \( (M, L) \). Then for the \( p \)-torsion subgroups \( H_1(M_n)_{[p]} \) of the 1st homology groups, there are some \( \lambda, \mu, \nu \in \mathbb{N} \) such that for \( n > n_0 \),

\[
\#H_1(M_n)_{[p]} = p^{\lambda_n + \mu n + \nu}.
\]

**Definition 4.10.** These \( \lambda, \mu, \) and \( \nu \) are called the Iwasawa invariants of \( \tilde{M} \). They are sometimes denoted as \( \lambda_{\tilde{M}}, \mu_{\tilde{M}}, \) and \( \nu_{\tilde{M}} \).

Since a branched \( \mathbb{Z}_p \)-cover can be approximated by \( \mathbb{Z} \)-covers as much as we like, the proof of this formula comes down to the case obtained from a \( \mathbb{Z} \)-cover ([KM08]).

**Proof.** For each \( n_1 \in \mathbb{N} \), let \( \tau_{n_1} : H_1(X) \to \mathbb{Z}/p^{n_1} \mathbb{Z} \) be the defining homomorphism of \( M_{n_1} \to M \). Since \( \tau_{n_1} \) lifts to the homomorphism \( \tau \to \mathbb{Z}_p \), \( H_1(X)_{\text{tor}} < \text{Ker} \tau_{n_1} \), and \( \tau_{n_1} \) sends a torsion-free element \( b \in H_1(X) \) to a unit of \( \mathbb{Z}_p \). By composing an automorphism of \( \mathbb{Z}/p^{n_1} \mathbb{Z} \), we may assume that \( \tau_{n_1}(b) = 1 \). Then \( \tau_{n_1} \) lifts to a surjective homomorphism \( \tilde{\tau}_{n_1} : H_1(X_L) \to \mathbb{Z} \) obtained as follows: take a \( \mathbb{Z} \)-basis of \( H_1(X)_{\text{free}} := H_1(X)/H_1(X)_{\text{tor}} \), and define \( H_1(X)_{\text{free}} \to \mathbb{Z} \) by sending basis to the integers with the same presentation of the images by \( \tau_{n_1} \), and let \( \tilde{\tau}_{n_1} \) be the composite with \( H_1(X) \to H_1(X)_{\text{free}} \).

Note that there is \( n_0 \) independent of \( n_1 \) and the Iwasawa type formula holds for \( n_0 < n < n_1 \). Indeed, \( n_0 \) is determined by \( H_1(M) \) and the \( p \)-adic valuations of the
images of \( \tau \) ([KM08]). Therefore, for sufficiently large \( n_1 \), the Iwasawa invariants of branched covers \( \mathbb{Z}_p \) defined by \( \overline{\tau}_{n_1} \) and \( \tau \) coincide.

In the proof of the Iwasawa type formula by Kadokami–Mizusawa in [KM08], Sakuma’s exact sequence ([SakSI] Section 4, [KM08] Lemma 3.4) played a key role. A \( p \)-adic variant of this sequence is stated in the following.

Let \( L = \cup K_i \) be a link in a 3-manifold \( M \), put \( X = M - L \), and let \( \mu_i \in H_1(X) \) denote the meridian of \( K_i \). Let \( \widetilde{M} \) be a branched \( \mathbb{Z}_p \)-cover over \((M, L)\) defined by \( \tau : H_1(X) \to \mathbb{Z}_p \) and suppose that it is properly branched. Then \( h_{n,n+1} : H_1(M_{n+1}) \to H_1(M_n) \) is surjective for any \( n \gg 0 \) (e.g., [Uek14] Theorem 6). We define the Iwasawa module of \( \widetilde{M} \) by \( \mathcal{H} := \lim_{\← \rightarrow} H_1(M_n)[p] \). We fix an identification \( \Lambda = \mathbb{Z}_p[[T]] \cong \mathbb{Z}_p[[\hat{t}]] = \lim_{\leftarrow} \mathbb{Z}_p[t]/(t^{p^n} - 1); 1 + T \mapsto t \). Then \( \mathcal{H} \) is a \( \Lambda \)-module.

For each \( n \), \( H_1(M_n)[p] \) is a \( \mathbb{Z}_p[t]/(t^{p^n} - 1) \)-module, and hence also is a \( \Lambda \)-module. Put \( \nu_{p^n} = 1 + t + \ldots + t^{p^n-1} \). Now we have the following:

**Proposition 4.11** (Sakuma’s exact sequence). Let \( \widetilde{M} \) be a branched \( \mathbb{Z}_p \)-cover and let the notation be as above. Suppose that \( \widetilde{M} \) is properly branched.

1. If \( \tau(\mu_i) \neq 0 \mod p \) for every \( i \), equivalent to say, if \( \widetilde{M} \) is totally branched over \( L \), then for each \( n \) we have an exact sequence

   \[
   H_1(M)[p] \to H_1(M_n)[p] \to \mathcal{H}/\nu_{p^n}\mathcal{H} \to 0.
   \]

2. In any case, there is some \( n_0 \) such that for any \( n > n_0 \) we have an exact sequence

   \[
   H_1(M_n)[p] \to H_1(M_n)[p] \to \mathcal{H}/\nu_{p^n-\nu_{n_0}}\mathcal{H} \to 0.
   \]

The proof will be given in the subsequent subsection. Note that if \( H_1(M) \) or \( H_1(M_{n_0}) \) is finite, then its image by the transfer maps in \( H_1(M_n) \)'s is constant for \( n \gg 0 \). Combining Proposition 4.11 and the next lemma, the Iwasawa type formula (Theorem 4.9) follows immediately from the structure theorem of compact \( \Lambda \)-modules (Lemma 3.1 (3)), similarly to the case of number theory in Section 3.

**Lemma 4.12.** Suppose that \( \widetilde{M} \) consists of \( QHS^3 \)'s. Then the Iwasawa module \( \mathcal{H} = \lim_{\leftarrow\rightarrow} H_1(M_n)[p] \) of \( \widetilde{M} \) is a finitely generated torsion \( \Lambda \)-module.

**Proof.** Since \( \mathcal{H} \) is an inverse limit of \( \Lambda \)-modules with finite orders, it is a compact \( \Lambda \)-module. In the composite \( H_1(M_n)[p]/h_{n}^{\mathcal{H}}(H_1(M_1)[p]) \cong \mathcal{H}/\nu_{p^n}\mathcal{H} \to \mathcal{H}/(p, T) \), the right hand term is finite. Hence the module \( \mathcal{H} \) is finitely generated by Nakayama’s lemma (Lemma 3.1 (1)). Moreover, since \( \mathcal{H}/\nu_{p^n}\mathcal{H} \) is a finite group, \( \mathcal{H} \) is a torsion \( \Lambda \)-module by Lemma 3.1 (3).

We note that \( \mathcal{H} \) can be a finitely generated torsion \( \Lambda \)-module even if \( \widetilde{M} \) does not consist of \( QHS^3 \)'s (See Subsection 4.4).

Finally, we prove an assertion on a direct limit module by the transfer maps. It plays an important role in the proof of Kida’s formula in Section 7.

**Proposition 4.13.** Let \( \widetilde{M} \) be a branched \( \mathbb{Z}_p \)-cover of \( QHS^3 \) and let \( H_1(\widetilde{M})[p] := \lim_{\leftarrow\rightarrow} H_1(M_n)[p] \) denote the direct limit by the transfer maps. Then, there is an isomorphism of discrete \( \Lambda \)-modules \( H_1(\widetilde{M})[p] \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\Lambda \text{at}} \oplus A' \), where \( A' \) is a bounded \( \Lambda \)-module, namely, there is some \( a \in \mathbb{N} \) such that \( p^a A' = 0 \), and \( A' = 0 \) holds if and only if \( \mu_{\widetilde{M}} = 0 \).
Proof. In general, let $E$ be a $\Lambda$-module. If $E \sim \Lambda/(p^n)$, then $\lim E/E/\nu_{p^n}E$ by $\nu_{p^n+1}/\nu_{p^n}$ is a bounded infinite group. If $E \sim \Lambda/(p^n)$, then $\lim E/E/\nu_{p^n}E \cong (\mathbb{Q}_p/\mathbb{Z}_p)^\lambda$. Let $E \sim E'$ be a pseudo isomorphism of $\Lambda$-modules. Then their direct limits by $\nu_{p^n+1}/\nu_{p^n}$ are isomorphic to each other.

Now we consider the following direct system for the exact sequences of Proposition 4.11.

\[
\begin{array}{cccc}
H_1(M)[p] & h^i_{n+1} & H_1(M_{n+1})[p] & \mathcal{H}/\nu_{p^n+1}\mathcal{H} \to 0 \\
H_1(M)[p] & h^i_n & H_1(M_n)[p] & \mathcal{H}/\nu_p\mathcal{H} \to 0
\end{array}
\]

Since the direct limit functor is exact, there is an isomorphism $H_1(\overline{M})[p]/h_{\infty}^1(H_1(M)[p]) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_{\overline{M}}} \oplus A''$, where $A''$ is a bounded $\Lambda$-module, and $A'' = 0$ if and only if $\mu_{\overline{M}} = 0$. Since $h_{n}^1(H_1(M)[p])$ is a finite group, there is an isomorphism $H_1(\overline{M})[p] \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_{\overline{M}}} \oplus A'$ with a bounded module $A'$. If $\mu_{\overline{M}} = 0$, then by Lemma 3.14 (4), the $p$-ranks of $H_1(M_n)[p]$ are bounded, and hence the $p$-ranks of $h^i_n(H_1(M)[p])$ are also bounded. Moreover, since $\tau : H_1(X) \to \mathbb{Z}_p$ sends torsions to zero, $h_{n,n+1}^1 : H_1(M_{n+1}) \to H_1(M_n)$ is surjective on the torsion subgroup. Thus, for sufficiently large $n$, a map $h^i_{n,n+1} \circ h_{n,n+1}^1$ on $H_1(M_{n+1})$ is multiplication by $p$, and $h_{n,n}^1(H_1(M)[p])$ is contained in $pH_1(M_{n+1})[p]$. Therefore, $h^i_{\infty}(H_1(M)[p])$ has to be a subgroup of a divisible group, and there is an isomorphism $H_1(\overline{M})[p] \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_{\overline{M}}}$. Conversely, if this isomorphism holds, then $\mu_{\overline{M}} = 0$ clearly holds. \(\square\)

4.3. Proof of Sakuma’s exact sequence (Proposition 4.11). We prove Proposition 4.11 by modifying the argument in [KM08], Chapter 3.

The assertion (2) can be reduced to (1). Indeed, consider the branched cover $h_{n,n+1} : M_{n+1} \to M_n$ of degree $p$, let $K$ be a component of $h_{n}^{-1}(L)$ in $M_n$, let $\mu$ denote its meridian, and $\tilde{\mu}$ a meridian over $\mu$ in $M_{n+1}$. If $h_{n,n+1}^1$ is branched along $K$, then $h_{n,n+1} \circ (\tilde{\mu}) = p\mu$. If otherwise, then $h_{n,n+1} \circ (\tilde{\mu}) = \mu$. Therefore, if we put $n_0 = \max \{v_p(\tau(\mu_i)) : i \neq j\}$, where $v_p(x)$ denotes the $p$-adic valuation of a number $x$, then $\tau_{n_0} : H_1(X_{n_0}) \to p^{n_0}\mathbb{Z}_p \cong \mathbb{Z}_p$ sends every meridian $\mu_i$ of $h_{n_0}^{-1}(L)$ in $H_1(X_{n_0})$ to an element whose image by mod $p$ is non-trivial. Thus, by replacing the base space by $X_{n_0}$, we can reduce (2) to (1).

In order to prove the assertion (1), we prepare several lemmas. We put $\Lambda_0 = \mathbb{Z}[[t]]$. Then every $H_1(M_n)$ is a $\Lambda_0 = \mathbb{Z}[[t]]$-module.

Lemma 4.14. Suppose $n < m$. If $\overline{M}$ is totally branched over $L$, then there is a natural exact sequence of $\Lambda_0$-modules:

$$p^{m-n}\mathbb{Z}/p^m\mathbb{Z} \to H_1(X_m)/\nu_{p^n}H_1(X_m) \to H_1(M_n)/h^i_n(H_1(M)) \to 0.$$  

In addition, for the meridian module $\langle \tilde{\mu} \rangle := \text{Ker}(H_1(X_m) \to H_1(M_n))$, there is a natural exact sequence:

$$0 \to \langle \tilde{\mu} \rangle/\nu_{p^n} \to H_1(X_m)/\nu_{p^n}H_1(X_m) \to H_1(M_n)/\nu_{p^n}H_1(M_n) \to 0.$$  

By taking $\otimes \mathbb{Z}_p$, similar exact sequences of $\Lambda$-modules are obtained.

Proof. Put $G := \text{Gal}(h_m)$ for the subcover $h_m : X_m \to X$ of degree $p^n$. Then there is an exact sequence $1 \to \pi_1(X_m) \to \pi_1(X) \to G \to 1$, and the Hochschild–Serre
spectral sequence \((\text{Bro94})\) yields an exact sequence \(H_2(G) \to H_1(\pi_1(X)) \to H_1(G) \to 0\). Since \(G = \langle t \mid t^{p^n} \rangle \cong \mathbb{Z}/p^n\mathbb{Z}\) is a finite cyclic group, by applying the Hurewicz isomorphism, we have \(H_2(G) = 0\), \(H_1(\pi_1(X)) \cong \mathbb{Z}/p^n\mathbb{Z}\). Therefore, we obtain an exact sequence \(\ldots \to H_1(G) \to H_1(X) \to \mathbb{Z}/p^n\mathbb{Z} \to 0\). In a similar way, for a subcover \(h_{n,m} : X_m \to X_n\), an exact sequence \(\ldots \to \pi_1(X_m) \to \pi_1(X_n) \to \text{Gal}(h_{n,m}) \to 1\) yields an exact sequence \(\ldots \to (t^{p^n} - 1)H_1(X_m) \to H_1(X_m) \to H_1(X_n) \to p^n(\mathbb{Z}/p^n\mathbb{Z}) \to 0\). Therefore, a commutative diagram with exact rows

\[
\begin{array}{cccccc}
C_s(X_m) & \xrightarrow{t^{p^n}-1} & C_s(X_m) & \xrightarrow{h_{n,m}} & C_s(X_n) & \xrightarrow{b_n} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
C_s(X_m) & \xrightarrow{t^{-1}} & C_s(X_m) & \xrightarrow{h_{m,n}} & C_s(X) & \xrightarrow{0} & 0
\end{array}
\]

yields the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
H_1(X_m) & \xrightarrow{t^{p^n}-1} & H_1(X_m) & \xrightarrow{h_{n,m}} & H_1(X_n) & \xrightarrow{\varphi} & p^n(\mathbb{Z}/p^n\mathbb{Z}) & \xrightarrow{0} \\
\downarrow & & \downarrow & & \downarrow & & \uparrow & & \times p^n & & \uparrow & & \varphi \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{0} \\
H_1(X_m) & \xrightarrow{t^{-1}} & H_1(X_m) & \xrightarrow{h_{m,n}} & H_1(X) & \xrightarrow{\varphi} & \mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{0} \\
\end{array}
\]

Since \(\nu_{p^n} : (t - 1)H_1(X_m) \to (t^{p^n} - 1)H_1(X_m)\) is surjective, there is an isomorphism \(\text{Coker}(\nu_{p^n}) : H_1(X_m) \to H_1(X_m) \cong \text{Coker}(\nu_{p^n}) : H_1(X_m)/(t - 1)H_1(X_m) \to H_1(X_m)/(t^{p^n} - 1)H_1(X_m)\). By the Snake lemma, we obtain an exact sequence \(p^{m-n}\mathbb{Z}/p^n\mathbb{Z} \to H_1(X_m)/\nu_{p^n}H_1(X_m) \to H_1(X_n)/h_{n,m}(H_1(X)) \to 0\).

Now we consider the following commutative diagram with exact rows.

\[
\begin{array}{cccccc}
H_2(M_n, X_n) & \xrightarrow{\partial_n} & H_1(X_n) & \xrightarrow{h_{n,m}} & H_1(M_n) & \xrightarrow{0} \\
\downarrow & & \downarrow & & \downarrow & & \\
H_2(M, X) & \xrightarrow{\partial_1} & H_1(X) & \xrightarrow{h_{1,n}} & H_1(M) & \xrightarrow{0}
\end{array}
\]

By the assumption of (1) that \(\tilde{M}\) is totally branched over \(L\), the transfer map \(h_{n,m}'\) is surjective on the meridians. Hence we have an isomorphism \(h_{n,m}' : \text{Im}(\partial_1) \to \text{Im}(\partial_1)\). By the snake lemma, we obtain an isomorphism \(H_1(X_n)/h_{n,m}'(H_1(X)) \cong H_1(M_n)/h_{n,m}'(H_1(M))\) on the Coker of two vertical morphisms on the right side.

Thus we have obtained an exact sequence of \(\Lambda_0\)-modules.

Next, we recall some facts on inverse limits. We say that an inverse system \(\{A_n\}_n\) of abelian groups satisfies the Mittag-Leffler condition (ML-condition) if “for any \(n\) there exists some \(n'\) such that for any \(n'' > n'\), \(\text{Im}(A_{n''} \to A_n)\) is constant”. If \(\{A_n\}_n\) consists of finite groups, or if its morphisms are all surjective, then it satisfies ML-condition. An inverse system \(\{A_n\}_n\) is said to be ML-zero if it satisfies a modified ML-condition in which “constant” is replaced by “zero”. Recall \(\Lambda = \mathbb{Z}_p[[T]]\). A profinite \(\Lambda\)-module is one obtained as the inverse limit of discrete \(\Lambda\)-modules. A homomorphism of profinite \(\Lambda\)-modules is a continuous \(\Lambda\)-module homomorphism. It is always a closed map, because it is a continuous map from a
compact space to a Hausdorff space. Note that profinite $\Lambda$-modules form an abelian category. By [Jan88] §1, we have the following lemma. 

**Lemma 4.15.** Let $\{A_n\}_n, \{B_n\}_n, \{C_n\}_n$ be inverse systems of profinite $\Lambda$-modules. 
(1) Then, an exact sequence $0 \rightarrow A_n \rightarrow B_n \rightarrow C_n \rightarrow 0$ of inverse systems yields an exact sequence $0 \rightarrow \lim \limits_{\leftarrow} A_n \rightarrow \lim \limits_{\leftarrow} B_n \rightarrow \lim \limits_{\leftarrow} C_n \rightarrow \lim \limits_{\leftarrow}^1 A_n$. If $\{A_n\}_n$ satisfies the ML-condition, then $\lim \limits_{\leftarrow}^1 A_n = 0$ holds. 
(2) If $\{A_n\}_n$ is ML-zero, then an exact sequence $A_n \rightarrow B_n \rightarrow C_n \rightarrow 0$ yields an isomorphism $\lim \limits_{\leftarrow} B_n \cong \lim \limits_{\leftarrow} C_n$. 
(3) Suppose that $\{A_n\}_n$ is a surjective system and let $\{f_n : A_n \rightarrow A_n\}_n$ be a family of endomorphisms commutative with the system. Put $A = \lim \limits_{\leftarrow} A_n$ and let $f : A \rightarrow A$ denote the induced map. Then $A/f(A) \cong \lim \limits_{\leftarrow} A_n / f_n(A_n)$. 

**Proof.** Here we prove (3). Since each $f(A) \rightarrow f_n(A_n)$ is a surjection, the inclusion map $i : f(A) \hookrightarrow \lim \limits_{\leftarrow} f_n(A_n)$ is a continuous homomorphism with dense image. Since $i$ is a closed map, $i$ is an isomorphism and we have $f(A) = \lim \limits_{\leftarrow} f_n(A_n)$. Now consider exact sequences $0 \rightarrow f_n(A_n) \rightarrow A_n \rightarrow A_n / f_n(A_n) \rightarrow 0$ compatible with the inverse system. Since $\{f_n(A_n)\}_n$ is a surjective system, (1) yields an exact sequence $0 \rightarrow \lim \limits_{\leftarrow} f_n(A_n) \rightarrow A \rightarrow \lim \limits_{\leftarrow} A_n / f_n(A_n) \rightarrow 0$. Thus we obtain $A/f(A) \cong \lim \limits_{\leftarrow} A_n / f_n(A_n)$. 

Now we give a proof of Proposition 4.11 (1): 

**Proposition 4.11** (1), Sakuma’s exact sequence. We consider the inverse systems with respect to $m$ in Lemma 4.14. In the first exact sequence, the transition morphisms on the first terms $L_m := p^{m-n} \mathbb{Z} / p^m \mathbb{Z} \cong \{\nu_p^m \mid L_m \rightarrow L_m\}_m$. Hence if $m' > m + n$, then $L_{m'} \rightarrow L_m$ is the zero map. Thus the first term is ML-zero, and Lemma 4.14 (2) yields an isomorphism $\lim \limits_{\leftarrow} (H_1(X_m)[\nu_p^m], H_1(X_m)[\nu_p]) \cong H_1(M_n)[\nu_p][h_1](H_1(M)[\nu_p])$. 

In the second exact sequence in Lemma 4.14 since the transition maps on the finite $p$-torsion abelian groups $\langle \mu_k \rangle / \langle \nu_p \rangle$ are the multiplication by $p$, these terms also satisfy ML-zero. Therefore by Lemma 4.14 (2), there is an isomorphism $\lim \limits_{\leftarrow} (H_1(X_m)[\nu_p^m], H_1(X_m)[\nu_p]) \cong \lim \limits_{\leftarrow} (H_1(M_n)[\nu_p^m], H_1(M_n)[\nu_p])$. Since $\{H_1(M_n)[\nu_p]\}_n$ is a surjective system of pro-$p$ abelian groups, Lemma 4.17 (3) yields an isomorphism $\lim \limits_{\leftarrow} (H_1(M_n)[\nu_p], H_1(M_n)[\nu_p]) \cong \mathcal{H}/\nu_p \mathcal{H}$. 

**4.4. Remarks on non-QHS$^3$ cases.** In this subsection, we briefly discuss further generalization of Iwasawa type formula for non-QHS$^3$ cases. 

**Lemma 4.16.** (1) Let $E \sim \oplus \Lambda/(p^m \nu_p^m) \oplus \oplus \Lambda/(f^e)$ be a pseudo isomorphism from a finitely generated torsion $\Lambda$-module to the normal form. Then $E/\nu_p E$ is a infinite group if and only if $f_i$ are not $p^e$-th cyclotomic polynomial for any $1 \leq i' \leq n$. 
(2) Let $f$ be a $p$-power-th cyclotomic polynomial in $\mathbb{Z}[t]$, which is an irreducible Weierstrass polynomial in $\mathbb{Z}_p[[T]]$, and put $E = \Lambda/(f^e)$. Then the following are equivalent: $E/\nu_p E$ is constant for $n \gg 0$, that $(f^e, \nu_p)$ is constant for $n \gg 0$, and that $e = 1$. 

**Proof.** Note that $\nu_p(t) = (p^m - 1)/(t - 1) = \nu_p(t + 1)$ in $\mathbb{Z}_p[[T]]; t \mapsto 1 + T$ is the product of all the $p^e$-th cyclotomic polynomials for $1 \leq i' \leq n$. 

(1) A module of the form \((\Lambda/(p^{m_1}))/\nu_{p^n}(\Lambda/(p^{m_2}))\) is finite in any case. By a general fact that monic polynomials \(g, h \in \mathbb{Z}[t]\) with positive degrees satisfy 
\(#\mathbb{Z}[t]/(g, h) < \infty\) if and only if they have no common divisor, a module of the form 
\((\Lambda/(f^e)))/\nu_{p^n}(\Lambda/(f^d)) = \Lambda/(f^d, \nu_{p^n})\) is finite if and only if the monic polynomials \(f^e\) and \(\nu_{p^n}\) have no common divisor, that is, \(f_j\) is not the \(p^n\)-th cyclotomic polynomial for \(1 < n' \leq n\).

(2) Let \(f\) be the \(p^n\)-th cyclotomic polynomial in \(\mathbb{Z}[t] \subset \mathbb{Z}_p[[T]]\). Then \((f, \nu_{p^n}) = (f)\) holds for any \(n' \geq n\). We suppose both \(e > 1\) and that \(E/\nu_{p^n} E\) is constant for \(n' \gg 0\), and lead a contradiction. By these hypotheses, we have \(f^e, \nu_{p^n} = (f(f^{e-1}, \nu_{p^n}/f))\), and there is some \(n' \geq n\) with \((f^{e-1}, \nu_{p^n}/f) = (f^{e-1}, \nu_{p^{n'+1}}/f)\). By taking mod \(f\), we have \(\nu_{p^n}/f = (\nu_{p^{n'+1}}/f)\) in \(\Lambda/(f)\). Let \(\alpha = \alpha_1, \ldots, \alpha_{\nu_{p^n}}\) denote the roots of \(f\) in an algebraic closure \(\mathbb{Z}_p\), where \(\nu_{p^n}(n)\) is the Euler function. Then we have a standard injective homomorphism \(\Lambda/(f) \to \mathbb{Z}_p^{\nu_{p^n}}\), and \((\nu_{p^n}/f)(\alpha)) = ((\nu_{p^{n'+1}}/f)(\alpha))\) in \(\mathbb{Z}_p\). However, the \(p^{n'+1}\)-th cyclotomic polynomial \(q_{p^{n'+1}}\) satisfies \(\nu_{p^{n'+1}}/f = q_{p^{n'+1}}\nu_{p^n}/f\) and \(q_{p^{n'+1}}(\alpha) = (p^{n'+1} - \alpha) \neq (1)\). Thus we have \((\nu_{p^n}/f)(\alpha)) \neq ((\nu_{p^{n'+1}}/f)(\alpha))\), and hence contradiction. \(\square\)

**Theorem 4.17.** Let \(\widetilde{M}\) be a branched \(\mathbb{Z}_p\)-cover of 3-manifolds, and suppose that \(\mathcal{H} = \varprojlim H_1(M_n)\) is a finitely generated torsion \(\Lambda\)-module. Then

(1) The \(n\)-th cover \(M_n\) is \(\mathbb{Q}\)HS\(3\) if and only if \(M\) is a \(\mathbb{Q}\)HS\(3\) and the characteristic polynomial \(\chi\) of \(\mathcal{H}\) is not divided by \(p^n\)-th cyclotomic polynomial for \(n' \leq n\).

(2) If every \(p\)-power \(p^n\)-th cyclotomic polynomial contained in \(\chi\) is of exponent 1, then the \(p\)-torsion subgroups \(H_1(M_n)\) satisfy the Iwasawa type formula.

**Proof.** Let \(\mathcal{H} \sim \oplus_i \Lambda/(p_i^{m_i}) \oplus \oplus_j \Lambda/(f_j^{e_j})\) be a pseudo isomorphism to a normal form, and put \(E_j := \Lambda/(f_j^{e_j})\) for each \(j\). Then \(E_j/\nu_{p^n} E_j\) contains a free \(\mathbb{Z}_p\)-module if and only if \(f_j\) is the \(p^n\)-th cyclotomic polynomial for some \(n' \leq n\). If \(E_j/\nu_{p^n} E_j\) is constant for \(n > 0\) for every \(j\) such that \(f_j\) is a \(p\)-power \(p^n\)-th cyclotomic polynomial, then by considering the direct sum of factors which do not correspond to such \(j\)'s, we have the same augment as before. Therefore the previous lemma yields this theorem. \(\square\)

**Example 4.18.** Let \(M\) be a \(\mathbb{Q}\)HS\(3\), let \(L = \bigcup K_i\) be a link with null homologous \(d\)-components in \(M\), put \(X = M - L\), and let \(\mu_i \in H_1(X)\) denote the meridian of \(K_i\). Let \(v_1, \ldots, v_d\) be units of \(\mathbb{Z}_p\), let \(\tau : H_1(X) \to \mathbb{Z}_p\) be a homomorphism defined by \(\mu_i \mapsto v_i\), and let \(\widetilde{M}\) be the branched \(\mathbb{Z}_p\)-cover defined by \(\tau\). Then \(\widetilde{M}\) is totally branched over \(L\). Let \(\Delta_{L, \tau}(t) = \Delta_L(t^3, \ldots, t^{2d})\) denote the \(p\)-adic reduced Alexander polynomial in \(\mathbb{Z}_p[[[t]]]\). Since we can approximate \(\widetilde{M}\) by \(\mathbb{Z}\)-covers as much as we can, from a well-known fact for \(\mathbb{Z}\)-covers and \(\Delta_L(t)\) (a variant of the Mayberry–Murasugi formula \([\text{MMS2}]\) or Porti’s result \([\text{Por04}]\)), we can easily deduce isomorphisms \(\mathcal{H} \cong \Lambda/\Delta_{L, \tau}(t)\) and \(H_1(M_n)/h_n(H_1(M))\) of \(\mathcal{H}\) and \(\mathbb{Z}_p\), and \(\Delta_{L, \tau}(t) = \prod_{\xi = 1}^{2d} (\Delta_{L, \tau}(\xi))\). (For a group \(G\), we put \(|G| = \#G\) or zero according as \(G\) is finite or infinite. For a number \(x\), \(|x|_p\) denotes the \(p\)-adic norm.)

Typical examples with cyclotomic Alexander polynomial are torus knots. Only for here, the subscription \(n\) means the covering degree over \(S^3\). If \(L = K\) is the trefoil in \(M = S^3\), then \(\Delta_K(t) = t^6 - t + 1\) is the 6-th cyclotomic polynomial, and
\(H_1(M_0) = \mathbb{Z}^2, \ H_1(M_3) = (\mathbb{Z}/2\mathbb{Z})^2, \ H_1(M_2) = \mathbb{Z}/3\mathbb{Z}\) are known. For 3-fold cover \(M_0 \to M_2\) and double-cover \(M_0 \to M_3,\) the maps on \(H_1\) are not surjections on 3-torsions and 2-torsions respectively. However, for \(n > 0, \ M_{2^{n+1}} \to M_{2n}\) and \(M_{3,2^{n+1}} \to M_{3,2^n}\) induce surjections on 3-torsions and 2-torsions respectively, and satisfy the Iwasawa type formula with trivial invariants. If we take the connected sum with the trefoil and the figure eight knot, then we obtain a non-trivial example.

**Remark 4.19.** For a branched \(\mathbb{Z}\)-cover over a link \(L\) in \(S^3,\) we have the balance formula among the \(p\)-adic Mahler measure of the Alexander polynomial, the the Iwasawa \(\mu\)-invariant, and the \(p\)-adic entropy ([Uek]).

We expect further study (i) with use of higher Alexander polynomial ([SW02], [Le14]), (ii) for graph-branched cases ([Por04]), (iii) about the asymptotic formula related to hyperbolic volumes ([BV13], [Le14]).

5. Morphisms of branched \(\mathbb{Z}_p\)-covers

In this section, we introduce an analogue object of an extension of \(\mathbb{Z}_p\)-fields, and discuss a condition for \(\mu = 0.\) We also define chains of branched \(\mathbb{Z}_p\)-covers by the direct limits with respect to the transfer maps, and calculate Tate cohomologies for an equivariant Galois morphism of branched \(\mathbb{Z}_p\)-covers.

5.1. Extensions of \(\mathbb{Z}_p\)-fields. Let \(k_\infty/k\) be a \(\mathbb{Z}_p\)-extension over a finite number field and let \(F_\infty/k_\infty\) be a \(p\)-extension. Then there is a \(p\)-extension \(F/k\) such that \(F_\infty/F\) is a \(\mathbb{Z}_p\)-extension and \(F_n = Fk_n\) for each \(n.\) In such a situation, behaviors of the Iwasawa invariants are studied ([Iwa73b], [Iwa81]).

Recall that for a \(\mathbb{Z}_p\)-extension \(k_\infty/k,\) the value of \(\lambda\) and the property whether \(\mu = 0\) or not are independent of the choice of base field \(k,\) while the values of \(\mu\) and \(\nu\) depend. Let \(k_1\) the 1st middle field of \(k_\infty/k.\) Then a \(\mathbb{Z}_p\)-extension \(k_\infty/k_1\) has \(\mu\) and \(\nu\) different from those of \(k_\infty/k.\) In this case, if we put \(F_n := k_{n+1},\) then we have \(F_n = Fk_n.\) However, we do not assume such a case as an essential extension of \(\mathbb{Z}_p\)-fields.

5.2. Morphisms of branched \(\mathbb{Z}_p\)-covers. We define an analogue notion of an extension of \(\mathbb{Z}_p\)-fields as follows:

**Definition 5.1** (branched cover of branched \(\mathbb{Z}_p\)-covers). Let \(\tilde{M}\) and \(\tilde{N}\) be branched \(\mathbb{Z}_p\)-covers over \((M, L)\) and \((N, L')\) respectively. Then a morphism \(f : \tilde{N} \to \tilde{M}\) of branched \(\mathbb{Z}_p\)-covers is a compatible system of branched covers on each layer, that is, a family \(\{f_n : N_n \to M_n\}_n\) of branched covers commutative with the transition maps. If every \(f_n\) is Galois, then we say this morphism is Galois. We put \(G_n = \text{Gal}(f_n).\) If all the induced maps \(G_n \to G_0\) are isomorphisms, then we call this morphism equivariant Galois, or a branched Galois cover of branched \(\mathbb{Z}_p\)-covers, and we write \(\text{Gal}(f) = G_0.\)

Let \(f : \tilde{M} \to \tilde{N}\) be a morphism of branched \(\mathbb{Z}_p\)-covers with notation as above. We can easily check the following facts by diagram chasing, using Proposition 4.8: If \(\tilde{M}\) and \(\tilde{N}\) are properly branched over \(L\) and \(L',\) then \(L' = f_0^{-1}(L).\) Let \(\tilde{S}_n \subset M_n\) denote the branch link of \(f_n.\) If \(f\) is equivariant Galois, then we have \(\tilde{S}_n \subset h_n^{-1}(\tilde{S}_0).\) If in addition if \(\tilde{S}_0\) is unbranched in \(\tilde{M}\), then \(\tilde{S}_n = h_n^{-1}(\tilde{S}_0)\) and \(L \cap \tilde{S}_0 = \phi.\) We denote the restrictions to the exteriors of the preimages of \(L \cup \tilde{S}_0\) by the same letters \(h_n : X_n \to X, h'_n : Y_n \to Y,\) and \(f_n : X_n \to Y_n\) for each \(n.\)
Proposition 5.2. Let \( \tilde{M} \) and \( \tilde{N} \) be branched \( \mathbb{Z}_p \)-covers with notation as above, and let \( f_0 : N \to M \) be a branched cover. Then taking a morphism \( f : \tilde{M} \to \tilde{N} \) is equivalent to taking a homomorphism \( \iota : \mathbb{Z}_p \to \mathbb{Z}_p \) which commutes with the defining homomorphism \( \tau, \tau' \) and \( f_0^* : \pi_1(Y) \to \pi_1(X) \). Suppose \( f \) is Galois. Then the natural maps \( G_n \to G \) are isomorphisms if and only if corresponding \( \iota : \mathbb{Z}_p \to \mathbb{Z}_p \) is an isomorphism. In other words, \( f \) is equivariant Galois if and only if \( f \) is \( \varprojlim \text{Gal}(h_n) = \mathbb{Z}_p \)-equivariant.

Proof. Consider the following commutative diagram with exact rows.

\[
\begin{array}{ccc}
0 & \longrightarrow & \pi_1(Y_n) \\
\downarrow{f_n^*} & & \downarrow{f_0^*} \\
0 & \longrightarrow & \pi_1(X_n)
\end{array}
\]

Then it can be seen that taking \( \{f_n\}_n \) and taking \( \iota : \mathbb{Z}_p \to \mathbb{Z}_p \) is equivalent, and that \( f_0 \) is Galois if and only all the \( f_n \) are Galois. If Galois, since \( G_n = \pi_1(X_n)/f_n^*(\pi_1(Y_n)) \), the snake lemma yields an exact sequence \( 0 \to \ker(\iota_n) \to G_n \to G_0 \to \coker(\iota_n) \to 0 \). Thus the natural maps \( G_n \to G_0 \) are isomorphisms if and only if \( \iota \) is an isomorphism. \( \square \)

Even if \( \iota \) is not an isomorphism, \( G_n \)'s can be isomorphic to each other non-

canonically. Indeed, \( \ker(\iota_n) \) and \( \coker(\iota_n) \) are isomorphic. For instance, if \( \iota \) is the multiplication by \( p \) and \( G_0 = \mathbb{Z}/p\mathbb{Z} \), then the situation resembles to the case of \( k_1/k \) in the previous subsection. We can also define a branched cover of branched \( \mathbb{Z}_p \)-cover which may not be Galois, however we omit here.

Next, we see some examples. The following lemma is obvious.

Lemma 5.3. Let \( f : N \to M \) be a branched cover of \( 3 \)-manifolds with degree \( p \).

Let \( K \) be a knot in \( M \), let \( K' \) be a component of \( f^{-1}(K) \), and let \( \mu, \mu' \) denote the meridians of \( K, K' \) respectively. If \( K \) is inert or decomposed, then \( f(\mu') = \mu \). If \( K \) is branched, then \( f(\mu') = p\mu \).

This lemma ensures that the following settings give examples.

Example 5.4. Let \( \mathcal{L} = K \sqcup \overline{S} \) be a 2-component link in \( M = S^3 \) and let \( f_0 : N \to M \) be a branched cover of degree \( p \) with the branch link \( \overline{S} \). Put \( X = M - \mathcal{L} \), \( Y = N - f_0^{-1}(\mathcal{L}) \) and \( S = f_0^{-1}(\overline{S}) \). Let \( \mu_K, \mu_S \in H_1(X) \) denote the meridians of \( K \) and \( \overline{S} \) respectively. Here we take \( \iota = \text{id}_{\overline{S}} \).

(1) Suppose that \( L \) is decomposed in \( f_0 \) as \( f_0^{-1}(K) = K_1 \sqcup \cdots \sqcup K_p \) (e.g., let \( \mathcal{L} \) be the trivial 2-component link), and let (i) \( \tau : H_1(Y) \to \mathbb{Z}_p ; \mu_K \mapsto 1, \mu_S \mapsto 0 \), \( \tau' : H_1(Y) \to \mathbb{Z}_p ; \mu_K \mapsto 1, \mu_S \mapsto 0 \). (ii) \( \tau : \mu_K \mapsto 1, \mu_S \mapsto 1, \tau' : \mu_K \mapsto 1, \mu_S \mapsto p \).

(2) Suppose that \( L \) is inert in \( f_0 \) as \( f_0^{-1}(K) = K' \) (e.g., let \( \mathcal{L} \) be the Hopf link), and let (i) \( \tau : \mu_K \mapsto 1, \mu_S \mapsto 0 \), \( \tau' : \mu_K \mapsto 1, \mu_S \mapsto 0 \). (ii) \( \tau : \mu_K \mapsto 1, \mu_S \mapsto 1 \), \( \tau' : \mu_K \mapsto 1, \mu_S \mapsto p \).

These above give equivariant Galois morphisms of branched \( \mathbb{Z}_p \)-covers of degree \( p \).

Next, we change the setting and see an example of non-Galois case.

(3) Let \( K \) be a knot in \( M = S^3 \), put \( X = M - K \). Let \( f_0 : N \to M \) be a cover of degree \( p \) branched over \( K \), and let \( f_0 : Y \to X \) be the restriction to the exterior. Put \( K' = f_0^{-1}(K) \), and let \( \mu, \mu' \) denote the meridians of \( K, K' \) respectively. If we
put \( \tau : H_1(X) \to \mathbb{Z}_p; \mu_K \mapsto 1 \), \( \tau' : H_1(Y) \to \mathbb{Z}_p; \mu_K' \mapsto 1 \), and \( \iota : \mathbb{Z}_p \to \mathbb{Z}_p; 1 \mapsto p \), then we obtain an analogous situation of \( k_1/k \) in the previous subsection.

5.3. On conditions for \( \mu = 0 \). In this subsection, we recall analogous natures of cyclotomic and anti-cyclotomic \( \mathbb{Z}_p \)-extensions for branched \( \mathbb{Z}_p \)-covers investigated in our another paper \[Uek16\].

In a cyclotomic \( \mathbb{Z}_p \)-extension over a number field, any non-\( p \)-prime is finitely decomposed, and in an anti-cyclotomic \( \mathbb{Z}_p \)-extension over a CM-field, any non-\( p \) prime is totally decomposed. For a branched \( \mathbb{Z}_p \)-covers, instead of “any non-\( p \) primes”, we focus on a certain link.

**Proposition 5.5 (\[Uek16\] Proposition 5.1).** Let \( M \) be a 3-manifold, let \( L \) be a link in \( M \) consisting of null-homologous components, and let \( \tilde{M} = \{ h_n : M_n \to M \} \) be the TLN-\( \mathbb{Z}_p \)-cover over \( (M,L) \). Let \( K \) be a knot in \( M - L \). If \( \text{lk}(K,L) \neq 0 \), then \( K \) is finitely decomposed into a \( p^n \text{th}(\text{lk}(K,L)) \)-component link in \( \tilde{M} \). If \( \text{lk}(K,L) = 0 \), then \( K \) is totally decomposed in \( \tilde{M} \).

Let \( f : \tilde{N} \to \tilde{M} \) be an equivariant Galois morphism of branched \( \mathbb{Z}_p \)-covers, let \( \overline{S} \subset M \) denote the branch link of \( f_0 : N \to M \), and suppose \( \overline{S} \cap L = \phi \). If \( \overline{S} \) is finitely decomposed in \( \tilde{M} \), then \( f : \tilde{N} \to \tilde{M} \) resembles a \( p \)-extension of a cyclotomic \( \mathbb{Z}_p \)-field. If \( \overline{S} \) is totally decomposed in \( \tilde{M} \), then \( f : \tilde{N} \to \tilde{M} \) resembles a \( p \)-extension of an anti-cyclotomic \( \mathbb{Z}_p \)-field.

In \[Uek16\], we established an analogue of relative genus theory and studied the behaviors of \( \mu \)-invariants by following Iwasawa’s argument in \[Iwa73\]. Suppose that \( \tilde{M} \) and \( N \) consist of \( \mathbb{Q}\mathbb{H}^3 \)'s and that \( f : \tilde{N} \to \tilde{M} \) is of degree \( p \). If \( \overline{S} \) is finitely decomposed in \( \tilde{M} \), then by \[Uek16\] Theorem 5.2, \( \mu_{\tilde{N}} = 0 \) if and only if \( \mu_{\tilde{M}} = 0 \). If \( \overline{S} \) is infinitely decomposed in \( \tilde{M} \), then by \[Uek16\] Theorem 5.3, we have \( \mu_{\tilde{N}} \geq \#(\text{components of } \overline{S}) \). As a consequence, we have the following theorem:

**Theorem 5.6.** Let \( f : \tilde{N} \to \tilde{M} \) be an equivariant Galois morphism of degree \( p \) of branched \( \mathbb{Z}_p \)-covers of \( \mathbb{Q}\mathbb{H}^3 \)'s, let \( \overline{S} \) denote the branch link of \( f_0 : N \to M \), and suppose \( \mu_{\tilde{M}} = 0 \). Then \( \overline{S} \) is finitely decomposed in \( \tilde{M} \) if and only if \( \mu_{\tilde{N}} = 0 \).

5.4. Tate cohomologies of branched \( \mathbb{Z}_p \)-covers. In Iwasawa’s second proof for Kida’s formula in \[Iwa81\], the following assertions were used:

**Proposition 5.7 (\[Iwa81\]).** Let \( F_{\infty}/k_\infty \) be an extension of a cyclotomic \( \mathbb{Z}_p \)-field of degree \( p \), and let \( S \) denote the set of ramified primes in \( F_{\infty} \). Then for each \( i \in \mathbb{Z} \),
1. [Lemma 5] The equality \( \hat{H}^i(G,F_{\infty}^S) = 0 \) holds.
2. [Lemma 1] The equality \( q(\mathcal{O}_G^S) = q(\mathcal{O}_G^\#S) \) holds.

We consider their analogues in the following. For a branched \( \mathbb{Z}_p \)-cover \( \tilde{M} \), we fix CW-structures or PL-structures compatible with the inverse system, and define the modules of \emph{chains}, \emph{cycles}, \emph{boundaries}, and \emph{homologies} by the direct limits with respect to the transfer maps. In addition, we also define \emph{S-chains} and others as in Subsection 2.3.

---

\[1\] It seems that this assertion was used implicitly.
Moreover, for an equivariant Galois morphism of branched $\mathbb{Z}_p$-covers $f: \tilde{N} \to \tilde{M}$, we put $G = \text{Gal}(f)$, and fix CW-structures or PL-structures compatible with all the maps in the system. Let $\mathfrak{S} \subset M$ be a link, put $S = f_0^{-1}(\mathfrak{S})$, $S_n = h_n^{-1}(S)$, and let $\tilde{S}$ denote the inverse system $\{S_n\}_n$. Since the direct limit functor is exact, every exact sequence in Section 2 holds for $\tilde{N}$ and $(\tilde{N}, \tilde{S})$, and the following proposition is obtained.

**Proposition 5.8.** Let the situation be as above. Then for each $i \in \mathbb{Z}$,

1. The equation $\tilde{H}^i(G, C_2(\tilde{N})) = 0$ holds.
2. If $M$ and $\tilde{N}$ consist of $\mathbb{Q}HS^3$s, then $\tilde{H}^i(G, Z_2(\tilde{N})) \cong \tilde{H}^{i+1}(G, \mathbb{Z})$. If $G \cong \mathbb{Z}/p\mathbb{Z}$ in addition, then $h_i := \text{rank } \tilde{H}^i(G, Z_2(\tilde{N}))$ satisfies $h_1 = 1, h_2 = 0$.
3. If $G \cong \mathbb{Z}/p\mathbb{Z}$, then $\tilde{H}^1(G, Z_1(\tilde{N})) = 0$.
4. Suppose that $G \cong \mathbb{Z}/p\mathbb{Z}$, $S_n$’s consist of $\mathbb{Q}$-null-homologous components, $\mathfrak{S}$ is properly branched in $f_0$, and $\mathfrak{S}$ is infinitely inert (equivalent to say, unbranched and finitely decomposed) in $\tilde{M}$. Then $S$ is also infinitely inert in $\tilde{N}$. Let $s$ denote the number of connected components of $\mathfrak{S} := \varprojlim S_n$, which is equal to that of $S_n$ for $n \gg 0$. Then there is an isomorphism $Z_2(\tilde{N})_{\mathfrak{S}}/Z_2(\tilde{N}) \cong B_1(\tilde{N}) \cap Z_1(\tilde{S})$ for the subgroup $B_1(\tilde{N}) \cap Z_1(\tilde{S}) < Z_1(\tilde{S}) \cong \mathbb{Z}^n$ of finite index, and satisfies $\tilde{H}^1(G, Z_2(\tilde{N})_{\mathfrak{S}}/Z_2(\tilde{N})) = 0$, $\tilde{H}^1(G, Z_2(\tilde{N}))_{\mathfrak{S}}/Z_2(\tilde{N})) \cong (\mathbb{Z}/p\mathbb{Z})^s$.

Proof. Let $n \gg 0$. The transfer map $h'_n|_{\mathfrak{S}}$ is an isomorphism on $H_3(N_n) \cong \mathbb{Z}$, and is the multiplication by $p$ on $H_0(N_n) \cong \mathbb{Z}$ and on $H_0(h^{-1}_n(S)) \cong \mathbb{Z}^s$. If $S$ is totally inert in $\tilde{N}$, then $h'_n|_{\mathfrak{S}} = \text{id}$ on $Z_1(h_n^{-1}(S)) \cong \mathbb{Z}^s$. Since $N_{r'} = (\text{multiplication by } p)$ is a surjection on a divisible group $\varprojlim \mathbb{Z}$, the assertion follows from Propositions 2.3, 2.5, and 2.6. \hfill \Box

6. **Kida’s formula for extensions of $\mathbb{Z}_p$-fields**

In this section, we review the background of our main result of this paper. The following is a classical theorem.

**Theorem 6.1** (The Riemann–Hurwitz formula). Let $f: R' \to R$ be an $n$-fold covering of compact, connected Riemann surfaces and let $g$ and $g'$ denote the genera of $R$ and $R'$ respectively. The ramification indices $e(P')$ of $P' \in R'$ satisfy

$$2g' - 2 = (2g - 2)n + \sum_{P' \in R'} (e(P') - 1).$$

Y. Kida proved a highly interesting analogue of this formula for number fields. For each $k \subset \mathbb{C}$, we set $k_+ = k \cap \mathbb{R}$.

**Theorem 6.2** (Kida’s formula, [13]). Let $p$ be an odd prime, let $F/k$ be a finite Galois $p$-extension of CM-fields, and let $F_\infty$ and $k_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extensions of $F$ and $k$ respectively. If $\mu_{k_\infty/k} = 0$, then $\mu_{F_\infty/F} = 0$, and

$$\lambda_{F_\infty/F} - \delta = (F_\infty : k_\infty)(\lambda_{k_\infty/k} - \delta) + \left(\sum (e_w - 1) - \sum (e_{w+} - 1)\right),$$

where $\lambda^-$ denotes the $\lambda$-invariant of the minus part, $\delta$ is 1 or 0 according to whether $k$ contains a primitive $p$-th root of unity or not, $w$ (resp. $w_+$) runs through all the non-$p$ primes of $F_\infty$ (resp. $F_{\infty,+}$), and $e_w$ (resp. $e_{w+}$) denotes the ramification index of $w$ (resp. $w_+$) in $F_\infty/k_\infty$ (resp. $F_{\infty,+}/k_{\infty,+}$).
Following the method of Chevalley–Weil \cite{CWH34}, K. Iwasawa gave an alternative proof for Kida's formula, with use of $p$-adic representation theory of finite groups. He also gave a less explicit formula for a more general situation, from which his second proof follows:

**Theorem 6.3** (Iwasawa \cite{Iwa81}, a corollary of Theorem 6). Let $p$ be a prime number, let $k_{\infty}$ be a cyclotomic $\mathbb{Z}_p$-field, and let $F_{\infty}$ be a cyclic extension of degree $p$ over $k_{\infty}$ unramified at every infinite place of $k_{\infty}$ with $G = \text{Gal}(k_{\infty}/k)$. Assume that $\mu_{k_{\infty}} = 0$. Then $\mu_{F_{\infty}} = 0$, and

$$\lambda_{F_{\infty}} = p\lambda_{k_{\infty}} + \sum_{w} (e_w - 1) + (p-1)(h_2 - h_1),$$

where $w$ ranges over all non-$p$ primes on $F_{\infty}$, $e_w$ denotes the ramification index of $w$ in $F_{\infty}/k_{\infty}$, and $h_i$ denotes the $p$-rank of the abelian group $H^i(G, O_{F_{\infty}})$ for $i = 1, 2$.

An application of this formula for totally real number fields with some cohomological study of units was given in \cite{FKOT97}.

7. Kida's formula for branched $\mathbb{Z}_p$-covers

7.1. Main theorem and example. An analogue of the original Kida's formula (Theorem 6.2) is stated as follows:

**Theorem 7.1** (Kida's formula). Let $f : \tilde{\mathcal{N}} \to \tilde{M}$ be an equivariant Galois morphism of degree $p$-power of branched $\mathbb{Z}_p$-covers of $\mathbb{Q}HS^3$. Let $\mathcal{S} \subset M$ denote the branch link of $f_0 : N \to M$ and put $S = f_0^{-1}(\mathcal{S})$. If $\mathcal{S}$ is infinitely inert in $\tilde{M}$, then so is $S$ in $\tilde{\mathcal{N}}$, and $S_n := h_n^{-1}(S)$ is the branch link of $f_n$. If in addition $\mu_{\tilde{M}} = 0$, then $\mu_{\tilde{\mathcal{N}}} = 0$. Suppose that any component of $\tilde{\mathcal{S}}$ is not inert in $\tilde{f}_0$. For each component $w = (w_n)_{n \geqslant 1}$ of $\mathcal{S} := \varprojlim_n S_n$, let $e_w$ denote the branch index of $w$ defined as that of $w_n$ in $f_n$ for $n \gg 0$. Then

$$\lambda_{\tilde{\mathcal{N}}} - 1 = \deg(f)(\lambda_{\tilde{M}} - 1) + \sum_{w \subset \mathcal{S}} (e_w - 1).$$

By Kida's method in \cite{Kid80}, this formula is deduced from the case of degree $p$:

**Lemma 7.2** (The case of degree $p$). Let $f : \tilde{\mathcal{N}} \to \tilde{M}$ be a branched cover of degree $p$ of branched $\mathbb{Z}_p$-covers of $\mathbb{Q}HS^3$, let $\mathcal{S} \subset M$ be a link containing the branch link of $f_0 : N \to M$, put $S = f_0^{-1}(\mathcal{S})$, and let $\mathcal{S}$ denote the inverse limit of the preimages of $\mathcal{S}$ in $\tilde{\mathcal{N}}$. If $\mathcal{S}$ is infinitely inert in $\tilde{M}$, then $S$ is infinitely inert in $\tilde{\mathcal{N}}$. If in addition $\mu_{\tilde{M}} = 0$, then $\mu_{\tilde{\mathcal{N}}} = 0$. Suppose that any component of $\tilde{\mathcal{S}}$ is not inert in $\tilde{f}_0$. Let $e_w$ denote the branch index for each component $w$ of $\mathcal{S}$. Then

$$\lambda_{\tilde{\mathcal{N}}} - 1 = p(\lambda_{\tilde{M}} - 1) + \sum_{w \subset \mathcal{S}} (e_w - 1).$$

We give their proofs in the subsequent subsections.

**Remark 7.3.** (1) As an intermediate result, we prove the following: Let the setting be as in the Lemma above. If we take CW-structures or PL-structures compatible with all the covering maps and put $\hat{h}_i := \text{rank} \overline{H}^i(G, \mathbb{Z}_2(\tilde{\mathcal{N}}))$ for $G = \text{Gal}(f)$, then

$$\lambda_{\tilde{\mathcal{N}}} = p\lambda_{\tilde{M}} + \sum_{w \subset \mathcal{S}} (e_w - 1) + (p-1)(h_2 - h_1).$$
This formula is a more direct analogue of Theorem 6.3 (a corollary of Iwasawa [Iwa81], Theorem 6).

(2) If we fix a \( \mathbb{Z} \)-basis of \( Z_1(\tilde{N}) \) containing the components of \( S \) and take the sum over it, then \( w' \)'s can be seen as analogues of places.

(3) The original proof by Kida ([Kid80]) was given by genus theory. We expect an alternative proof of our formula with use of genus theory for 3-manifolds, which was established in [Uek16]. There are also analytic proofs with use of \( p \)-adic \( L \)-functions by Grass and Sinnott ([Gra79], [Sin84]). It seems interesting to consider their analogues, examine analogues of \( p \)-adic \( L \)-functions, and explore an analogue of the Iwasawa main conjecture.

Here is an example:

**Example 7.4.** Let \( p = 3 \), let \( N = M = S^3 \), and let \( f_0 : N \to M \) be a branched cover of degree \( p \) branched over an unknot \( \tilde{S} \). Let \( L = K \cup K' \) be a Hopf link as in the figure below, so that \( \text{lk}(K, \tilde{S}) = 3, \text{lk}(K', \tilde{S}) = \text{lk}(K, K') = 1 \). Then, \( \tilde{K} := f_0^{-1}(K) = K_1 \cup K_2 \cup K_3 \) is a Borromean ring, \( \tilde{K}' := f_0^{-1}(K') \) is an unknot, and \( \text{lk}(K_i, \tilde{K}') = 1 \) for \( i = 1, 2, 3 \).

Since \( L \) and \( L' = f_0^{-1}(L) \) consist of null-homologous components, the TLN-\( \mathbb{Z}_p \)-covers \( \tilde{M} \) and \( \tilde{N} \) are defined. Since \( \text{lk}(L, \tilde{S}) = \text{lk}(L', \tilde{S}) = 4 \not\equiv 0 \text{ mod } 3 \), \( \tilde{S} \) and \( S := f_0^{-1}(S) \) are totally inert in \( \tilde{M} \) and \( \tilde{N} \) respectively. Since the defining homomorphism \( \tau, \tau' \) commute with \( f_0 : \pi_1(N - L' \cup S) \to \pi_1(M - L \cup S) \) and \( \iota = \text{id}_{\mathbb{Z}_p} \), a branched cover \( f : \tilde{N} \to \tilde{M} \) of branched \( \mathbb{Z}_p \)-covers of degree \( p \) is defined.

By Hosokawa’s result ([Hos58], Theorem 1), the value of Hosokawa polynomials \( H_L(t) \) and \( H_{L'}(t) \) at \( t = 1 \) are calculated by using the linking numbers of components of \( L \) and \( L' \), and satisfy \( H_L(1) = H_{L'}(1) = \pm 1 \). By a result of Kadokami–Mizusawa ([KM98], Proposition 4.1), the fact \( p \not\mid \pm 1 \) implies that \( \tilde{M} \) and \( \tilde{N} \) consist of \( \mathbb{Q} \)-HS’s, and their Iwasawa invariants satisfy \( \lambda_{\tilde{M}} = 2 - 1 = 1, \mu_{\tilde{M}} = \nu_{\tilde{M}} = 0, \lambda_{\tilde{N}} = 4 - 1 = 3, \) and \( \mu_{\tilde{N}} = \nu_{\tilde{N}} = 0 \). Thus we have observed that Kida’s formula holds for this case:

\[
3 - 1 = 3 \cdot (1 - 1) + (3 - 1).
\]

Let \( p \) be an arbitrary prime number. In the example above, if we replace \( p = 3 \) by \( p' \) for any \( r \in \mathbb{N} \) and take a similar \( K \) with \( \text{lk}(K, S) = p' \), then we obtain a \( \mathbb{Z}_p \)-cover \( \tilde{N} \) with \( \lambda_{\tilde{N}} = p' \) by Kida’s formula (Theorem [Uek16]). Indeed, we can verify \( H_{L'}(1) = \pm 1 \) in a similar way.
7.2. Proof for degree $p$ (after Iwasawa). In this subsection, following Iwasawa’s second proof in [Iwa81], we prove our formula for the case of degree $p$ (Lemma 7.2).

**Lemma 7.2** The assertion on the $\mu$-invariants is done by Theorem 5.6. Let $\tilde{S} = \{h_{\mu}^{-1}(S)\}$. Denote the inverse system over $S$ in $\tilde{N}$. For $\tilde{S}$-chains and others, we use the notation in Subsection 5.4.

We have $\hat{H}^n(H_1(\tilde{N})_{\tilde{S}[p]}) \cong \hat{H}^n(Z_2(\tilde{N})_{\tilde{S}})$. Indeed, consider the exact sequences $0 \to B_1(\tilde{N})_{\tilde{S}} \to Z_1(\tilde{N})_{\tilde{S}} \to H_1(\tilde{N})_{\tilde{S}} \to 0$ and $0 \to Z_2(\tilde{N})_{\tilde{S}} \to C_2(\tilde{M}) \to B_1(\tilde{N})_{\tilde{S}} \to 0$, and note $\tilde{G} \cong Z/p\tilde{Z}$. Then Proposition 5.3 (1) and (3) show $\hat{H}^n(H_1(\tilde{N})_{\tilde{S}[p]}) \cong \hat{H}^n(H_1(\tilde{N})_{\tilde{S}}) \cong \hat{H}^n(\tilde{N})_{\tilde{S}} \cong \hat{H}^{n+1}(B_1(\tilde{N})_{\tilde{S}}) \cong \hat{H}^{n+2}(Z_2(\tilde{N})_{\tilde{S}}) \cong \hat{H}^n(Z_2(\tilde{N})_{\tilde{S}}).

Put $A_{\tilde{S}} := H_1(\tilde{N})_{\tilde{S}[p]} = H_1(\tilde{N})/\iota_*H_1(\tilde{S})$, $A := H_1(\tilde{N})_{[p]}$. Since $\mu_{\tilde{N}} = 0$, Proposition 4.18 gives an isomorphism $A \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_{\tilde{N}}}$. Since $\tilde{N}$ consists of $\mathbb{Q}$-null-homologous components for each $n$. If $S_n$ is inert in $N_{n+1} \to N_n$, then the restriction of the transfer $H_1(N_n) \to H_1(N_{n+1})$ to the subgroups generated by all the components of $S_n$ and $S_{n+1}$ is surjective. Hence there is a surjective homomorphism $A \to A_{\tilde{S}}$ with finite kernel, and we have $A_{\tilde{S}} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_{\tilde{N}}}.

We consider a linear representation of $G = Z/p\tilde{Z}$. Put $X_p := \mathbb{Z}_p[G], Z_{p-1} := (1 - \sigma)X_p, X_1 := X_p/X_{p-1} = Z_p, A_{i} = \text{Hom}_{\mathbb{Z}_p}(X_i, \mathbb{Q}_p/\mathbb{Z}_p)$ for $i = 1, p-1, p$. Then there are decompositions $A = A_{1}^\ast \oplus A_{p-1}^\ast \oplus A_p^\ast$, $a_i \in \mathbb{N}$. By using a functor $V : A \mapsto \text{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$, we put $V_1 := V(1), \pi_1 : G \to GL(V_1)$, $i = 1, p-1, p$. Then $\pi_1$ is the trivial representation, $\pi_{p-1}$ is the unique faithful irreducible representation, and $\pi_p = \pi_1 \oplus \pi_{p-1} = \pi_G$. Put $V = V(A)$. Then $G \cong A$ induces a representation $\pi_f : G \to GL(V), \pi_{f,1} = a_1\pi_1 \oplus \pi_{p-1} \oplus 0$.

Next, we calculate $a_i$’s. Since $G \cong Z/p\tilde{Z}$, the Tate cohomologies of $G$ are abelian groups of exponent $p$, and their $p$-ranks satisfy $r(\hat{H}^1(G, A_1)) = 1, 0, 0, r(\hat{H}^2(G, A_1)) = 0, 1, 0$ for $i = 1, p-1, p$. Put $r_{a_i} = r(\hat{H}^n(G, A_i)) = r(\hat{H}^n(G, Z_2(\tilde{N})_{\tilde{S}}))$. Then the decomposition of $A_{\tilde{S}}$ yields $r_1 = a_1, r_2 = a_2 - a_1, s = a_3 - a_2$.

We calculate the Herbrand quotient of $S$-2-cycles. (Recall that for a cyclic group $G$ and $G$-module $B$, $q(B) = \#H^0(B)/\#H^1(B)$ is called the Herbrand quotient. Herbrand’s lemma asserts that in a short exact sequence, if $q(\cdot)$ is defined for two terms, then so for the rest term, and its Tate cohomologies are finite.) Now $\hat{H}^n(G, Z_2(\tilde{N})_{\tilde{S}})$ is finite and $q(Z_2(\tilde{N})_{\tilde{S}})$ is defined. Put $h_n = r(\hat{H}^n(G, Z_2(\tilde{N})_{\tilde{S}}))$ for $Z_2(\tilde{N}) < Z_2(\tilde{N})_{\tilde{S}}$. Then $q(Z_2(\tilde{N})_{\tilde{S}}/Z_2(\tilde{N})) = p^s$ by Proposition 5.3 (4), and $q(Z_2(\tilde{N})_{\tilde{S}}) = q(Z_2(\tilde{N}))^a$ by Herbrand’s lemma. Hence $h_1, h_2$ are finite and $a_1 - a_2 + a_3 = h_2 - h_1 + s$ holds.

Since $\mu_{\tilde{M}} = 0$, there is an isomorphism $\tilde{A}_{\tilde{S}} := H_1(M_{\tilde{S}}) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_{\tilde{M}}}$ by Proposition 4.18. Proposition 5.3 (1) asserts $\hat{H}^2(C_2(\tilde{N})) = 0$, and hence $\hat{H}^1(Z_2(\tilde{N})_{\tilde{S}}) \cong (B_1(\tilde{N})_{\tilde{S}})^G/B_1(\tilde{M})_{\tilde{S}} \cong \hat{H}^2(Z_2(\tilde{M})_{\tilde{S}}) \cong \text{Coker}((Z_1(\tilde{N})_{\tilde{S}})^G/(H_1(\tilde{N})_{\tilde{S}})^G).$ Since the first term is finite, maps $H_1(\tilde{M})_{\tilde{S}} \to H_1(\tilde{N})_{\tilde{S}}$ and $\tilde{A}_{\tilde{S}} \to A^G_{\tilde{S}}$ have finite kernels and cokernels, where we put $\tilde{S} = \{h_{\mu}^{-1}(S)\}$. Hence the decomposition of $A_{\tilde{S}}$ tells $\lambda_{\tilde{M}} = a_1 + a_2$.

Thereby we have obtained $\pi_f = a_1\pi_1 \oplus a_1\pi_{p-1} \oplus a_2\pi_p = (a_1 + a_2)p \oplus (a_2 - a_1)\pi_{p-1} = \lambda_K \pi_G \oplus s\pi_{p-1} \oplus (h_2 - h_1)\pi_{p-1}$. (If $h_2 - h_1 < 0$, the right hand side of the
formula should be interpreted as a difference of two representations.) Comparing 
the degrees of both side, we obtain 
\[ \lambda_N = p\lambda_M + \sum_{w \in S} (e_w - 1) + (p - 1)(h_2 - h_1). \]

By Proposition 5.8(2), we have \( h_2 - h_1 = -1 \), and obtain the desired formula. \( \square \)

7.3. Proof for degree \( p \)-power (after Kida). Following Kida’s argument in [Kid80], we deduce our formula (Theorem 7.1) from the case of degree \( p \) (Lemma 7.2).

**Theorem 7.1.** Since every nontrivial finite \( p \)-group has nontrivial center, a Galois branched cover of \( p \)-power degree can be realized as a sequence of branched covers of degree \( p \). Suppose \( \deg(f) = p^n \). We prove the formula by induction on \( m \). The assertion is trivial for \( n = 0 \).

Assume the assertion is true for \( n \leq m \). Then for \( n = m + 1 \), there is a subcover \( \tilde{N} \to \tilde{H} \) of \( N \to M \) of degree \( p \). Let \( w \) run through all the components of \( \tilde{S} \), and \( u \) run through the image of \( \tilde{S} \) in \( \tilde{H} \). By the hypothesis of the induction and by Lemma 7.2, they satisfy 
\[ \lambda_{\tilde{N}} - 1 = p(\lambda_{\tilde{H}} - 1) + \sum_w (e(w/u) - 1) \]
and hence 
\[ \lambda_{\tilde{N}} - 1 = \sum_{w/v} (e(w/u) - 1). \]
Here \( u \) and \( v \) denote the images of each \( w \) in \( \tilde{H} \) and \( M \) respectively, and \( e(w/u) \) and \( e(u/v) \) denote their branch indices in \( \tilde{N} \to \tilde{H} \) and \( \tilde{H} \to M \) respectively, and \( e(w/v) = e_w \).

Now we have 
\[ \lambda_{\tilde{N}} - 1 = p(\sum_u (e(u/v) - 1)) = \sum_{u/v} (e(w/u) - 1) = p^{m+1}(\lambda_M - 1) + \sum_u p(e(u/v) - 1) \]
in \( \tilde{N} \to \tilde{H} \), then the unique \( w \) over \( u \) satisfies \( e(w/v) = pe(u/v), e(w/u) = p \), and hence 
\[ p(e(u/v) - 1) + (e(w/u) - 1) = e(w/v) - p + p - 1 = e(w/v) - 1. \]
If \( u \) is decomposed in \( \tilde{N} \to \tilde{H} \), then the correction \( w_1, ..., w_p \) of components over \( u \) satisfy \( e(w_i/v) = e(u/v), e(w_i/u) = 1 \), and hence 
\[ p(e(u/v) - 1) + \sum_w (e(w_i/v) - 1) = \sum_i (e(w_i/v) - 1). \]
Thus, due to the assumption that \( S \) is not inert in \( f_0 : N \to H \), we have generalized the formula to the case of \( p \)-power degree.

(If \( u \) is inert in \( \tilde{N} \to \tilde{H} \), then \( w \) over \( u \) satisfies \( e(w/v) = e(u/v), e(w/u) = 1 \) and hence 
\[ p(e(u/v) - 1) + (e(w/u) - 1) = pe(w/u) - p + 1 - 1 = p(e(w/u) - 1). \] \( \square \)

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