Grassmannian and string theory.

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March 28, 2022

Abstract

Infinite-dimensional Grassmannian manifold contains moduli spaces of Riemann surfaces of all genera. This well known fact leads to a conjecture that non-perturbative string theory can be formulated in terms of Grassmannian. We present new facts supporting this hypothesis. In particular, it is shown that Grassmannians can be considered as generalized moduli spaces; this statement permits us to define corresponding "string amplitudes" (at least formally). One can conjecture, that it is possible to explain the relation between non-perturbative and perturbative string theory by means of localization theorems for equivariant cohomology; this conjecture is based on the characterization of moduli spaces, relevant to string theory, as sets consisting of points with large stabilizers in certain groups acting on Grassmannian. We describe an involution on the Grassmannian that could be related to $S$-duality in string theory.

0. Introduction.

It is clear now that all versions of string theory are closely related. One should expect that all of them can be obtained from a unifying theory, where strings are not considered as fundamental objects (strings should be on equal footing with membranes). The present paper arose from attempts to understand the structure of the unifying theory. If we believe that the analysis of this hypothetical theory is related to calculation of integrals over infinite-dimensional supermanifolds then we can suggest the following picture. The integrand can have various odd symmetries. Under certain conditions the existence of odd symmetry leads to localization; in other words, one can replace the integration over the whole supermanifold with an integration over some part of it. (See [11],[13],[14],[15] for localization theorems in the framework of supergeometry and of equivariant cohomology.) Different odd symmetries lead to different localizations; and therefore to theories that could look completely unrelated.
A very natural candidate for infinite-dimensional supermanifold arising in the universal theory is infinite-dimensional (super)Grassmannian. I hope that appropriate integrals could be localized to the Krichever locus; such a localization would lead immediately to relation to the string theory. I have only tentative results in this direction. However, I was able to prove that Grassmannians can be considered as generalized moduli spaces, containing many other moduli spaces, and to describe a generalization of string theory that is defined in terms of Grassmannian.

Bosonic string theory is closely related to two-dimensional conformal field theory, other versions of string theory are related to corresponding generalizations of conformal field theory. Let us consider the moduli space $P_{\chi,n}$ of (possibly, disconnected) compact conformal two-dimensional manifolds of Euler characteristic $\chi$ having a boundary consisting of $n$ components (i.e. closed two-dimensional manifolds of Euler characteristic $\chi - n$ with $n$ holes). We assume that boundaries of holes are parametrized and the holes are ordered. One can define natural maps $\nu_{n,n'}: P_{\chi,n} \times P_{\chi',n'} \to P_{\chi+\chi',n+n'}$ and $\sigma^{(n)}: P_{\chi,n} \to P_{\chi,n-2}$ (the first map corresponds to disjoint union of manifolds, the second one is constructed by means of pasting together of the boundary of $(n-1)$-th hole and the boundary of $n$-th hole). The symmetric group $S_n$ acts naturally on $P_{\chi,n}$ (reordering the holes).

Sometimes it is useful to take into account that pasting together boundaries of two holes we can rotate one of the boundaries. Then we obtain a map $P_{\chi,n} \times S^1 \to P_{\chi,n-2}$.

The situation when we have some spaces $P_n$ with action of symmetric group $S_n$ and maps $\nu_{n,n'}: P_n \times P_{n'} \to P_{n+n'}$, $\sigma^{(n)}: P_n \to P_{n-2}$ appears quite often. If these data satisfy some compatibility conditions we will say that the spaces $P_n$ constitute a MO (a modular operad). The spaces $P_n = \cup P_{\chi,n}$ form a MO; we reflect the fact $P_n$ has such a decomposition, saying that we have a graded MO. If we include in our data also a map $P_n \times S^1 \to P_{n-2}$, we will talk about an EMO (equivariant MO). The spaces $P_{\chi,n}$ constitute a graded EMO.

If $A$ is a linear space with inner product we can define a MO considering tensor powers $A^\otimes n$ and natural maps $A^\otimes n \times A^\otimes n' \to A^\otimes (n+n')$ and $A^\otimes n \to A^\otimes (n-2)$. An algebra over the MO $P_n$ is defined as a collection of maps $\alpha_n: P_n \to A^\otimes n$, that are compatible with the structures of MO in $P_n$ and $A^\otimes n$. Conformal field theory can be defined as an algebra over the MO $P_n = \cup P_{\chi,n}$, described above. (More precisely, in this case the maps $\alpha_n$ are defined only up to a constant factor.) A superconformal field theory can be defined in similar way. Moduli spaces of conformal manifolds (of complex curves) should be replaced in this case with moduli spaces of superconformal manifolds (super Riemann surfaces).

We will be able to construct a lot of MOs starting with an infinite-dimensional Grassmannian.

We will say that a semi-infinite structure in Hilbert space $H$ is specified by a decomposition $H = H_+ \oplus H_-$ and a unitary involution $K$ interchanging $H_+$.
The Segal-Wilson modification of Sato Grassmannian $\text{Gr}(H)$ can be defined as an infinite-dimensional manifold consisting of such subspaces $V \subset H$ that $V$ is close to $H_+$ in some sense (one can require for example that the projection $V \to H_+$ is a Fredholm operator and the projection $V \to H_-$ is a Hilbert-Schmidt operator).

We will prove that the sequence of spaces $\text{Gr}_n = \text{Gr}(H^n)$ can be considered as MO. Moreover, we will give conditions when $\text{Gr}_n$ constitute an EMO. Using well known fermionic construction of Grassmannian we will obtain an algebra over this MO.

Let us consider now an infinite-dimensional algebra $\mathcal{G}$ acting on $H$; then $\mathcal{G}^n$ acts on $H^n$ and, under certain conditions, on $\text{Gr}(H^n)$. We define $\mathcal{G}$-locus $P_n(\mathcal{G})$ as a subspace of $\text{Gr}(H^n)$ consisting of subspaces $V \subset H^n$ having "large stabilizers" in $\mathcal{G}$. (More precisely, we introduce a semi-infinite structure in $\mathcal{G}$ and require that the stabilizer $\text{Stab}_V$ contains a space $W \in \text{Gr}(\mathcal{G}^n)$.) We prove that the spaces $P_n(\mathcal{G})$ also can be used to construct MO. For appropriate choice of $\mathcal{G}$ we can consider conformal field theories, WZNW models etc as algebras over $P_n(\mathcal{G})$. Replacing Grassmannian with super Grassmannian we can obtain also superconformal theories.

It is well known that a conformal field theory can be considered as a background for a string theory and that one can construct corresponding string amplitudes as integrals over moduli spaces (at least in the case of critical central charge.) Generalizing these constructions we can define (at least formally) "string amplitudes" corresponding to an algebra over an EMO. It seems that the "string amplitudes" corresponding to $\text{Gr}(H^n)$ could be universal in some sense. We describe an involution on $\text{Gr}(H^n)$ that could be related to $S$-duality in string theory. (The space $\text{Gr}(H^n)$ can be represented as a union of connected components $\text{Gr}^{(k)}(H^n)$, where $k$ stands for the index of Fredholm operator $V \to H_+$, corresponding to $V \in \text{Gr}(H^n)$. It is natural to conjecture that in perturbation theory the contribution of $\text{Gr}^{(k)}(H^n)$ is proportional to $g^k$, where $g$ stands for the coupling constant. The involution we constructed transforms $\text{Gr}^{(k)}(H^n)$ into $\text{Gr}^{(-k)}(H^n)$.)

**1. Grassmannian**

Let us consider a (complex) Hilbert space $H$ represented as a direct sum of subspaces $H_+$ and $H_-$. We will assume that there exists a unitary involution $K$ on $H$ transforming $H_+$ into $H_-$ and $H_-$ into $H_+$. We will say that subspaces $H_+, H_-$ and the involution $K$ specify a semi-infinite structure on $H$. As an example, we can take $H = L^2(S^1)$ where $S^1$ is the unit circle in $\mathbb{C}$ with standard measure. Then we can define $H_-$ as a subspace spanned by $z^n = e^{in\varphi}$, $n \geq 0$ and $H_+$ as a subspace spanned by $z^n = e^{in\varphi}$, $n < 0$. The involution $K$ can be chosen as a map transforming a function $f(z)$ into a function $\frac{1}{z}f(\frac{1}{z})$. It is easy to check that every semi-infinite structure is isomorphic to the standard structure described above. (Notice, that we include separability and infinite-dimensionality in the definition of Hilbert space. Subspaces are by definition
closed linear submanifolds.) Using a semi-infinite structure in Hilbert space \( H \) we can define a Grassmannian \( \text{Gr}(H) \) as a set of such subspaces \( V \subset H \) that the natural projection \( \pi_+ \) of \( V \) into \( H_+ \) is a Fredholm operator and the natural projection \( \pi_- \) of \( V \) into \( H_- \) is a compact operator. The Grassmannian \( \text{Gr}(H) \) can be represented as a union of connected components \( \text{Gr}^{(k)}(H) \) were \( k \) stands for the index of the Fredholm operator \( \pi_+ : V \rightarrow H_+ \).

Let us consider a map \( \alpha : H_+ \rightarrow H = H_+ \oplus H_- \) transforming \( h \in H_+ \) into \( \alpha(h) = (A + a)h + Bh \), where \( A \) is an invertible operator acting on \( H_+ \) and \( a : H_+ \rightarrow H_+, \ B : H_+ \rightarrow H_- \) are compact operators. It is easy to check that the space \( \alpha(H_+) \subset H \) belongs to \( \text{Gr}^{(0)}(H) \) and that every \( V \in \text{Gr}^{(0)}(H) \) can be obtained by means of this construction. This gives us an alternative description of \( \text{Gr}^{(0)}(H) \). One can give a similar description of \( \text{Gr}^{(k)}(H) \): if \( F \) is a Fredholm operator of index \( k \) acting on \( H_+ \) and \( B \) is a compact operator acting from \( H_+ \) into \( H_- \), then the image of the operator \( \alpha : H_+ \rightarrow H \), where \( \alpha(h) = Fh + Bh \), belongs to \( \text{Gr}^{(k)}(H) \) and every element of \( \text{Gr}^{(k)}(H) \) can be obtained this way.

If \( H \) has the form \( H_+ \oplus H_- \) a linear operator \( A \) acting in \( H \) can be represented in the form

\[
\tilde{x}_+ = A_{++}x_+ + A_{+-}x_-
\]

\[
\tilde{x}_- = A_{-+}x_+ + A_{--}x_-
\]

where \( x_+ , \tilde{x}_+ \in H_+ \), \( x_- , \tilde{x}_- \in H_- \). We will say that \( A \in GL(H) \) if \( A \) is an invertible operator, the operators \( A_{++} : H_+ \rightarrow H_+ \) and \( A_{--} : H_- \rightarrow H_- \) are Fredholm, the operators \( A_{+-} : H_- \rightarrow H_+ \), \( A_{-+} : H_+ \rightarrow H_- \) are compact. The Lie algebra \( gl(H) \) of \( GL(H) \) is defined as the set of operators \( A \) obeying \( \exp(A) \in GL(H) \). It is easy to check that \( GL(H) \) acts on \( \text{Gr}(H) \) and that this action is transitive on every component \( \text{Gr}^{(k)}(H) \).

Notice, that a semi-infinite structure on \( H \) induces naturally a semi-infinite structure on \( H^n \) (on the direct sum of \( n \) copies of \( H \)). Namely, \( H^n = H \oplus \ldots \oplus H = (H_+ \oplus \ldots \oplus H_+) \oplus (H_- \oplus \ldots \oplus H_-) \); the involution \( K : H^n \rightarrow H^n \) is a direct sum of \( n \) copies of \( K : H \rightarrow H \). Hence, we can speak about \( \text{Gr}(H^n) \) and about \( \text{Gr}^{(k)}(H^n) \).

One can modify the definition of Grassmannian and of \( GL(H) \) replacing compact operators with operators belonging to the trace class or with Hilbert-Schmidt operators. All considerations of this section remain valid after such a modification. The set of compact operators \( \mathcal{B} \) can be considered as an ideal in the algebra \( L(H) \) of bounded operators; we can replace \( \mathcal{B} \) with any other ideal consisting of compact operators and containing all finite-dimensional operators.

One can define also a semi-infinite structure on a normed space \( H \). Such a structure is specified by means of subspaces \( H_+ \) and \( H_- \) and an operator \( K \), interchanging \( H_+ \) and \( H_- \). (It is not necessary to assume that \( K \) is an involution.) We assume that \( K \) is an invertible operator on \( H \) and that \( H \) is equivalent to the direct sum \( H_+ \oplus H_- \) (more precisely, there exists an invertible
operator $\pi = (\pi_+, \pi_-) : H \to H_+ \oplus H_-$ such that $\pi_{\pm} h = h$ for $h \in H_{\pm})$. Again we can define the Grassmannian $\text{Gr}(H)$ and prove in this more general situation the results stated below.

Let us fix a space $H$ equipped with semi-infinite structure. We will use sometimes the notations $\text{Gr}, \text{Gr}_n$ and $\text{Gr}_n^{(k)}$ instead of $\text{Gr}(H), \text{Gr}(H^n)$ and $\text{Gr}^{(k)}(H^n)$ correspondingly. Let us consider linear subspaces $V \subset H'^m$ and $V' \subset H^n$. Then their direct sum $V \oplus V'$ belongs to $\text{Gr}(H^{m+n})$. We obtain a map

$$\nu_{m,n} : \text{Gr}(H^m) \times \text{Gr}(H^n) \to \text{Gr}(H^{m+n}).$$

For every subspace $V \subset H^m$ we can construct a subspace $\sigma(V) \subset H^{m-2}$ consisting of points $(f_1,...,f_{m-2}) \in H^{m-2}$, satisfying the condition that one can find $u \in H$ in such a way that $(f_1,...,f_{m-2},Ku,u) \in V$.

**Theorem 1.** If $V \in \text{Gr}(H^m)$, then $\sigma(V) \in \text{Gr}(H^{m-2})$.

To prove this statement we represent $V$ as the image of a map $H^m_+ \to H^m$ transforming a point $(h_1,...,h_m) \in H^m_+$ into a point $(f_1+g_1,...,f_m+g_m) \in H^m$, where $f_k = \sum A_{kj}h_j \in H_+$, $g_k = \sum B_{kj}h_j \in H_-$, the operators $A_{kk} : H_+ \to H_+$ are Fredholm, the operators $B_{kj} : H_+ \to H_-$ and operators $A_{kj} : H_+ \to H_+$, $k \neq j$, are compact. To describe the space $\sigma(V)$ we impose the condition

$$f_{m-1} + g_{m-1} = K(f_m + g_m)$$

on the points of $V$. Taking into account that $f_k \in H_+, g_k \in H_-$ we can rewrite this condition in the form

$$\sum_j A_{m-1,j} h_j = K \sum_j B_{m,j} h_j$$

$$\sum_j B_{m-1,j} h_j = K \sum_j A_{m,j} h_j$$

(1)

Now we can apply the following statement.

**Lemma 1.** Let us consider an equation

$$Fx = Ly$$

(2)

where $x, y$ are elements of infinite-dimensional Banach spaces $E$ and $E'$. Assume that the operator $F : E \to E$ is Fredholm and the operator $L : E' \to E$ is compact. Then one can find a Fredholm operator $G : E' \to E'$ and a compact operator $M : E' \to E$ in such a way that for every element $u \in E'$ the elements

$$x = Mu, \; y = Gu$$

(3)

obey (2). Moreover, one can find $G$ and $M$ in such a way that every solution to (2) can be represented in the form (3) and this representation is unique. (Then index $G = - \text{index } F$.)

We apply Lemma 1 to Equation (1) taking $x = (h_{m-1}, h_m)$, $y = (h_1,...,h_{m-2})$, $E = H_+ \oplus H_-$, $E' = H^{m-2}_+$. The representation of $\sigma(V)$ obtained this way makes the statement of Theorem 1 obvious.

Let us consider an infinite-dimensional Lie algebra $\mathcal{G}$ and a homomorphism $\alpha : \mathcal{G} \to gl(H)$ where $H$ is provided with semi-infinite structure $(H_+, H_-, K)$. 5
We assume that there exists an involution $\kappa$ of the Lie algebra $\mathcal{G}$ obeying $\alpha(\kappa(\gamma)) = K\alpha(\gamma)K^{-1}$ and that one can find a semi-infinite structure on $\mathcal{G}$ specified by means of subspaces $\mathcal{G}_+$ and $\mathcal{G}_-$ and involution $\kappa$.

The Lie algebra $\mathcal{G}^m$ (direct sum of $m$ copies of $\mathcal{G}$) acts on $H^m$ (an element $(\gamma_1, ..., \gamma_m) \in \mathcal{G}^m$ transforms $(f_1, ..., f_n) \in H^n$ into $(\alpha(\gamma_1)f_1, ..., \alpha(\gamma_m)f_m) \in H^m$). It is easy to check that this action generates an action of $\mathcal{G}^m$ on $Gr(H^m)$.

Let us denote by $\text{Stab}_\mathcal{G}$ a subalgebra of $\mathcal{G}^m$ consisting of elements transforming an element $V \in Gr(H^m)$ into itself (the stabilizer of $V$). We will define a $\mathcal{G}$-locus $P_m(\mathcal{G}) \in Gr(H^m)$ as a set of such points $V \in Gr(H^m)$ that there exists an element $W \in Gr(\mathcal{G}^m)$ obeying $\text{Stab}_\mathcal{G} \supset W$. In other words $P_m(\mathcal{G})$ consists of elements $V \in Gr(H^m)$ having "large stabilizers" in $\mathcal{G}^m$.

**Theorem 2.** If $V \in P_m(\mathcal{G})$, then $\sigma(V) \in P_{m-2}(\mathcal{G})$.

To prove this theorem we should check that $\sigma(V)$ has a large stabilizer. If $(\gamma_1, ..., \gamma_m) \in \text{Stab}_\mathcal{G}$ then for $(f_1, ..., f_m) \in V$ we have $(\alpha(\gamma_1)f_1, ..., \alpha(\gamma_m)f_m) \in V$. If we know that $(\gamma_1, ..., \gamma_m) \in \text{Stab}_\mathcal{G}$, $\gamma_{m-1} = \kappa(\gamma_m)$ we can check that $(\gamma_1, ..., \gamma_{m-2}) \in \text{Stab}_{\mathcal{G}(\gamma)}$. To verify this statement we take a point $(f_1, ..., f_{m-2}) \in \sigma(V)$ constructed by means of $(f_1, ..., f_m) \in V$ where $f_{m-1} = Kf_m$. Using the relation

$$\alpha(\gamma_{m-1})f_{m-1} = \alpha(\kappa(\gamma_m))Kf_m = K\alpha(\gamma_m)f_m$$

we see that the point

$$(\alpha(\gamma_1)f_1, ..., \alpha(\gamma_{m-2})f_{m-2})$$

belongs to $\sigma(V)$ (one can construct it using the point $(\alpha(\gamma_1)f_1, ..., \alpha(\gamma_m)f_m) \in V$). This means that $\text{Stab}_{\mathcal{G}(\gamma)} \supset \sigma(\text{Stab}_\mathcal{G})$. If $\text{Stab}_{\mathcal{G}} \supset W \in Gr(\mathcal{G}^m)$ we can conclude from Theorem 1 that $\text{Stab}_{\mathcal{G}(\gamma)} \supset \sigma(W) \in Gr(\mathcal{G}^{m-2})$ and therefore $\sigma(V) \in P_{m-2}(\mathcal{G})$.

The following statement is almost evident:

If $V \in P_m(\mathcal{G})$, $V' \in P_n(\mathcal{G})$, then $V \oplus V' \in P_{m+n}(\mathcal{G})$.

We obtain a map $\nu_{m,n} : P_m(\mathcal{G}) \times P_n(\mathcal{G}) \to P_{m+n}(\mathcal{G})$.

Let us consider a compact one-dimensional complex manifold (complex curve) $\Sigma$ and $s$ holomorphic maps of the standard disk $D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ into $\Sigma$. We assume that the images $D_1, ..., D_s$ of these maps do not overlap. Let us fix a holomorphic line bundle $\xi$ over $\Sigma$ and trivializations of this bundle over the disks $D_1, ..., D_s$. We denote by $W = W(\Sigma, \xi)$ the space of holomorphic sections of the bundle over $\Sigma \setminus (D_1 \cup ... \cup D_s)$. Restricting a section $s$ of $\xi$ to the boundaries $\partial D_1, ..., \partial D_s$ and using the trivializations of $\xi$ over the disks we obtain an element $\rho(s)$ of $H^*$, where $H$ stands for the space $L^2(S^1)$, $S^1$ denotes the standard circle $|z| = 1$. The image $\sigma(W)$ of the space $W$ is a linear subspace of $H^*$; one can check [1] that $\sigma(W) \in Gr(H)$. (We equip $H^*$ with the standard semi-infinite structure.) One can generalize this construction allowing $\Sigma$ to be a complete irreducible complex algebraic curve (possibly singular) and replacing the line bundle $\xi$ with rank 1 torsion free coherent sheaf on $\Sigma$. (If
Σ is non-singular such a sheaf is a vector bundle.) Then again one can prove that \( \sigma(W) \in Gr(H^s) \). The points \( \sigma(W) \) obtained by means of the construction above constitute so called Krichever locus in \( Gr(H^s) \).

Let us consider now the set \( \Gamma \) consisting of invertible twice differentiable functions on \( S^1 \). Every element \( \gamma \in \Gamma \) generates an operator \( \alpha(\gamma) : H \to H \) transforming \( f \in H \) into \( \gamma f \); it is easy to check that \( \alpha(\gamma) \in GL(H) \) [1].

Considering \( \Gamma \) as a group with respect to multiplication we can say that \( \alpha \) is a homomorphism of the group \( \Gamma \) into \( GL(H) \).

Let us denote by \( \mathcal{A}_\Sigma \) the space of invertible holomorphic functions on \( \Sigma \setminus (D_1 \cup ... \cup D_s) \). Restricting these functions to the boundaries \( \partial D_1, ..., \partial D_s \) we obtain an embedding of \( \mathcal{A}_\Sigma \) into \( \Gamma^* \); the image of this embedding will be denoted by \( \mathcal{A}_\Sigma^* \). It is easy to see that \( \mathcal{A}_\Sigma^* \subset V \) for every \( V \in Gr(H^s) \) obtained by means of Krichever construction starting with \( \Sigma, D_1, ..., D_s \) (for any choice of the bundle or sheaf \( \xi \) and trivializations of it). The Lie algebra \( \text{Lie} \Gamma \) of the group \( \Gamma \) consists of all twice differentiable functions on the circle; the Lie algebra \( (\text{Lie} \Gamma)^* = \text{Lie} \Gamma^* \) can be represented by means of functions on a disjoint union of \( s \) circles. It is clear that for every \( V \in Gr(H^s) \) obtained by means of Krichever construction the stabilizer \( \text{Stab}_V \) of \( V \) in \( (\text{Lie} \Gamma)^* \) consists of all functions defined on the union of circles \( \partial D_1, ..., \partial D_s \) and admitting a holomorphic continuation to \( \Sigma \setminus (D_1 \cup ... \cup D_s) \). One can check that \( \text{Stab}_V \) belongs to \( Gr((\text{Lie} \Gamma)^*) \) for an appropriate definition of semi-infinite structure in \( \text{Lie} \Gamma \). (We can consider \( \text{Lie} \Gamma \) as a subset of \( H = L^2(S^1) \); the standard semi-infinite structure on \( L^2(S^1) \) induces a semi-infinite structure on pre Hilbert space \( \text{Lie} \Gamma \).) We see that the locus \( P_s(\text{Lie} \Gamma) \) contains all points of \( Gr(H^s) \) obtained by Krichever construction (Krichever locus). Moreover, one can prove that Krichever locus coincides with \( P_s(\text{Lie} \Gamma) \). (For \( s = 1 \) this follows from [2]; see also [1]. The case \( s > 1 \) can be treated in similar way; see the paper [3], devoted to various generalizations of the above statement.)

Let us describe an interesting duality transformation of \( Gr(H^m) \). We identify \( H \) with \( L^2(S^1) \) equipped with standard semi-infinite structure. Then we can introduce bilinear inner product in \( H \) by means of the formula

\[
(f, g) = \int_{|z|=1} f(z)g(z)dz.
\]

This inner product induces inner product in \( H^m \): if \( f = (f_1, ..., f_m) \in H^m, \ g = (g_1, ..., g_m) \in H^m \), then

\[
(f, g) = \sum_k \int_{|z|=1} f_k(z)g_k(z)dz.
\]

Let us define a linear operator \( L : H \to H \) by the formula

\[
(Lf)(z) = f(-z).
\]
The map \( L^m : H^m \to H^m \) transforms \((f_1, ..., f_m)\) into \((Lf_1, ..., Lf_m)\). The symbol \( V^\perp \) denotes the orthogonal complement of \( V \subset H^m \) with respect to bilinear inner product; the symbol \( \lambda(V) \) stands for \( L^mV^\perp \).

**Theorem 3.** If \( V \in Gr^{(k)}(H^m) \) then \( V^\perp \in Gr^{(k)}(H^m) \) and \( \lambda(V) \in Gr^{(k)}(H^m) \). The maps \( \lambda : Gr(H^m) \to Gr(H^m) \) commute with maps \( \nu_{m,n} \) and \( \sigma^{(m)} \); in particular

\[
\lambda \circ \sigma^{(m)} = \sigma^{(m)} \circ \lambda.
\]

To prove the first statement we represent the space \( V \subset H \) as an image of operator \( \alpha : H_+ \to H \), where \( \alpha(h) = (Fh, Bh) \). \( F : H_+ \to H_+ \) is Fredholm, and \( B : H_+ \to H_- \) is compact. (We restrict ourselves to the case \( m = 1 \).) Then orthogonal complement \( V' \) to the space \( V \) with respect to hermitian inner product \( <f,g> = \int_0^{2\pi} f(\varphi)g(\varphi) d\varphi \) consists of pairs \((h_+, h_-)\) obeying the equation \( F^*h_+ = B^*h_- = 0 \). (Here \(*\) denotes Hermitian conjugation.) If \( V = H_+ \) then \( V' = H_- \); this means that we should expect that in the case when \( V \) is close to \( H_+ \) the space \( V' \) should be close to \( H_- \). In other words, we should expect that for \( V \in Gr(H) \) the space \( V' \) belongs to \( Gr(H) \) defined by means of semi-infinite structure when the roles of \( H_+ \) and \( H_- \) are interchanged. This fact immediately follows from Lemma 1. Using that

\[
V^\perp = zV', \quad H_- = z^{-1}H_+,
\]

we obtain that \( V^\perp \) belongs to \( Gr(H) \) original semi-infinite structure. It is easy to check that \( L \in GL(H) \), therefore \( LV^\perp \) also belongs to \( Gr(H) \). Using Lemma 1 one can also calculate the index of \( V^\perp \) and \( LV^\perp \).

It is evident that \( \lambda \) commutes with \( \nu_{m,n} \); let us prove that \( \lambda \) commutes with \( \sigma^{(m)} \). We start with a remark that

\[
\sigma^{(m)}(V) = \pi(V \cap R_K)
\]

where \( R_K \) denotes the subspace of \( H^m \), that consists of points \((f_1, ..., f_m)\) in \( H^m \) obeying \( f_{m-1} = Kf_m \), and \( \pi \) stands for natural projection of \( H^m \) onto \( H^{m-2} \) (i.e. \( \pi(f_1, ..., f_m) = (f_1, ..., f_{m-2}) \)) It is easy to check that

\[
\sigma^{(m)}(V)^\perp = (\pi^T)^{-1}(V \cap R_K)^\perp = (\pi^T)^{-1}(V^\perp + R_K^\perp).
\]

Here \( \pi^T : H^{m-2} \to H^m \) is an operator, adjoint to \( \pi \) with respect to bilinear inner product; it is easy to see that it transforms \((f_1, ..., f_{m-2})\) into \((f_1, ..., f_{m-2}, 0, 0)\). The space \( R_K^\perp \) consists of points \((g_1, ..., g_m)\) in \( H^m \) obeying \( g_{m-1} = -Kf_m \). Using these facts we obtain that \((f_1, ..., f_{m-2}) \in \sigma^{(m)}(V)^\perp \) if there exists a point \((f_1, ..., f_{m-2}, f_{m-1}, f_m) \in V^\perp \) satisfying \( f_{m-1} = -Kf_m \). In other words,

\[
\sigma^{(m)}(V)^\perp = \sigma^{(m)}_{-K}(V^\perp)
\]
In the form: $\sigma$ satisfies for this notion.

However, I did not find any good word is not so simple by itself). I believe that the notion introduced above is simpler in [4] is very complicated (in particular it is based on the notion of operad, which introduced in [4], as was pointed out to me by A. Voronov. The definition given

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specify a MO if the following conditions are satisfied:

$\lambda(\sigma^{(m)}(V)) = L^m(\sigma^{(m)}_K(V^\perp)) = L^m\sigma^{-1}_K(V^\perp),$ 

$\sigma^{(m)}_K(\lambda(V)) = \sigma^{(m)}_K(L^mV^\perp) = L^m\sigma^{(m)}_{L^{-1}KL}(V^\perp).$

It remains to take into account that $L^{-1}KL = -K$.

2. Generalized moduli spaces. (Modular operads.)

One can reformulate the above results in the following way. Let us consider a sequence of $S_m$-spaces $P_m$. Here $S_m$ denotes the symmetric group, i.e. a group of permutations of $m$ elements $\{1, 2, ..., m\}$. $S_m$-space is by definition a topological space with left action of the group $S_m$. The group $S_k$ for $k < m$ is embedded into the group $S_m$ as a subgroup consisting of permutations leaving intact $k + 1, ..., m$. The group $S_m \times S_n$ is naturally embedded in $S_{m+n}$, namely, $S_m$ permutes first $m$ indices and $S_n$ permutes last $n$ indices. Let us fix maps $\nu_{m,n} : P_m \times P_n \to P_{m+n}$ and $\sigma^{(m)} : P_m \to P_{m-2}$. We will say that these data specify a MO if the following conditions are satisfied:

0. The maps $\nu_{m,n}$ determine an associative multiplication in $\bigcup P_m$.
1. $\nu_{m,n} \circ (\rho \times \tau) = (\rho \times \tau) \circ \nu_{m,n}$ for every $\rho \in S_m$, $\tau \in S_m$,
2. $\sigma^{(m)} \circ \nu = \nu \circ \sigma^{(m)}$ for every $\nu \in S_{m-2}$,
3. $\sigma^{(m)} \circ \lambda = \sigma^{(m)}$ if $\lambda = (m, m-1)$ (i.e. $\lambda$ permutes two last indices),
4. $\sigma^{(m)}$ commutes with $\sigma^{(m)} \circ \mu$ where $\mu \in S_m$ is a permutation obeying $\mu(m-1) \leq m-2$, $\mu(m) \leq m-2$,
5. $\sigma^{(m)} \circ \nu_{m,n} = \nu_{m,n-2} \circ \sigma^{(m)}$,
6. $\nu_{m,n} \circ \alpha = \beta \circ \nu_{m,n}$, where $\alpha$ stands for the natural map $P_m \times P_n \to P_n \times P_m$ (transposition), $\beta$ denotes a permutation $(1, ..., m+n) \to (m+1, ..., m+n, 1, ..., m)$.

The notion of MO is almost equivalent to the notion of modular operad introduced in [4], as was pointed out to me by A. Voronov. The definition given in [4] is very complicated (in particular it is based on the notion of operad, which is not so simple by itself). I believe that the notion introduced above is simpler and more fundamental than the notion of operad, therefore it is unreasonable to use the term “operad” in its name. However, I did not find any good word for this notion.

Notice, that the conditions 2), 3) permit us to define maps $\sigma^{(m)}_{a,b} : P_m \to P_{m-2}$ satisfying

$\sigma^{(m)}_{a,b} \circ \lambda = \sigma^{(m)}_{\lambda(a), \lambda(b)}, \quad \sigma^{(m)}_{a,a} = \sigma^{(m)}_{b,b}, \quad \sigma^{(m)}_{m-1,m} = \sigma^{(m)}$

(Here $a, b \in \{1, ..., m\}, a \neq b, \lambda \in S_m$.) Using the maps $\sigma^{(m)}_{a,b}$ we can rewrite 4) in the form: $\sigma^{(m)}_{a,b} \circ \sigma^{(m)}_{a,b} = \sigma^{(m)}_{a,b} \circ \sigma^{(m)}_{a,b}$ if $a, b, a', b'$ are distinct.
The symmetric group $S_m$ acts naturally on the spaces $Gr_m = Gr(H^m)$ and $P_m(G)$ described above. The maps $Gr(H^m) \to Gr(H^{m-2})$ and $P_m(G) \to P_{m-2}(G)$ were constructed in Theorem 3. One can define $\nu_{m,n}$ as a map transforming a pair of subspaces $V \subset H^m$, $V' \subset H^n$ into subspace $V \oplus V' \subset H^{m+n} = H^m \oplus H^n$; it maps $Gr(H^m) \times Gr(H^n)$ into $Gr(H^{m+n})$ and $P_m(G) \times P_n(G)$ into $P_{m+n}(G)$. It is easy to check that the conditions (0)-6) are satisfied. We obtain

**Theorem 4.** The $S_m$-spaces $Gr(H^m)$ together with maps $\nu_{m,n} : Gr(H^m) \times Gr(H^n) \to Gr(H^{m+n})$ and $\sigma^{(m)} : Gr(H^m) \to Gr(H^{m-2})$ constitute a MO. Similar statement is true for the $S_m$-spaces $P_m(G)$.

We mentioned already that it is not necessary to assume that the operator $K$ in the definition of semi-infinite structure on $H$ is an involution. It is possible to construct the maps $\nu_{m,n} : Gr(H^m) \times Gr(H^n) \to Gr(H^{m+n})$ and $\sigma^{(m)} : Gr(H^m) \to Gr(H^{m-2})$ without this assumption and to prove that all conditions in the definition of MO, except condition 3), are satisfied. This statement remains correct if we replace $Gr(H^n)$ with $P_n(G)$.

Let us assume that the $S_m$-spaces $P_m$ have an $S_m$-invariant decomposition $P_m = \bigcup_k P_m^{(k)}$ and that the maps $\nu_{m,n}$ and $\sigma^{(m)}$ respect this decomposition (more precisely, $\nu_{m,n}$ maps $P_m^{(k)} \times P_n^{(l)}$ into $P_{m+n}^{(k+l)}$ and $\sigma^{(m)}$ maps $P_m^{(k)}$ into $P^{(k)}_{m-2}$). Then we can talk about graded MOs. It is easy to see that the MO described in Theorem 4 can be considered as graded MOs with respect to decompositions $Gr(H^n) = \bigcup_k Gr^{(k)}(H^n)$, $P_m(G) = \bigcup P^{(k)}_m(G)$.

Notice that for every MO $(P_n, \nu_{m,n}, \sigma^{(n)})$ one can define maps $P_m \times P_n \to P_{m+n-2}$ taking composition $\sigma^{(m+n)} \circ \nu_{m,n}$. In particular, for $m = n = 2$ we obtain a map $P_2 \times P_2 \to P_2$; this map determines a structure of semigroup on $P_2$. If $P_n$ is a graded MO ($P_n = \bigcup P^{(n)}_n$) we obtain a structure of semigroup on $P_{0,2}$. (For the MO constructed by means of moduli spaces of complex curves $P_{0,2}$ is called Neretin semigroup.) If we have an algebra $(P_n, E, \alpha_n)$ over $P_n$ then under certain regularity conditions the map $\alpha_2$ determines a representation of the Lie algebra of semigroup $P_{0,2}$ (or $P_2$) in the space $E$. Of course, not always we can speak about Lie algebra of a semigroup, but such a Lie algebra exists in many interesting cases. In particular, the Lie algebra of Neretin semigroup coincides with complexified Lie algebra of diffeomorphism group of a circle.

MOs defined above should be called topological MOs. One can define linear MOs regarding $P_m$ as $S_m$-modules. (Then we should consider $\nu_{m,n}$ as a bilinear map, i.e. as a linear map of tensor product $P_m \otimes P_n$ into $P_{m+n}$, and $\sigma^{(m)}$ as a linear map $P_m \to P_{m-2}$. In the definition of graded MO we should replace disjoint union with direct sum.) It is easy to see that homology groups of topological MO constitute a linear MO. A simple, but very important, example of linear MO (standard linear MO) can be defined if we have a linear space $E$ equipped with symmetric bilinear inner product. Then we can take the $m$-th tensor degree $E^\otimes m$ as $P_m$. The definitions of maps $\nu_{m,n} : E^\otimes m \otimes E^\otimes n \to E^\otimes (m+n)$ and $\sigma^{(m)} : E^\otimes m \to E^\otimes (m-2)$ are obvious. If bilinear inner product is not symmetric the maps $\nu_{m,n}$ and $\sigma^{(m)}$ obey all conditions in the definition of
A homomorphism of MO $P_m$ into a MO $P'_m$ can be defined as a collection of maps $\alpha_m : P_m \to P'_m$, commuting with the operations $\nu_{m,n}, \sigma^{(m)}$. A homomorphism of MO into itself is called an automorphism. Theorem 3 can be interpreted as a statement that the maps $\lambda : Gr(H^m) \to Gr(H^m)$ constitute an automorphism of the MO $Gr(H^m)$. Under certain conditions one can check that $\lambda$ induces also an automorphism of $MO_{G^m}$. (One should assume that for every $\gamma \in \alpha(G)$ we have $\gamma^T \in \alpha(G)$ and $L\gamma L^{-1} \in \alpha(G)$. Here $\alpha(G)$ stands for the image of $G$ by the embedding $\alpha : G \to gl(H)$.)

Let us define an algebra over (topological) MO as a homomorphism of it into linear MO described above. In other words, an algebra over MO $P_n$ is a collection of maps $\alpha_m : P_m \to E \otimes m$ such that

$$\sigma^{(m)} \alpha_m = \alpha_{m-2} \sigma^{(m)},$$

$$\nu_{m,n}(\alpha_m \times \alpha_n) = \alpha_{m+n} \nu_{m,n}.$$  

We will assume that $\alpha_m(x), x \in P_m$, is defined only up to a constant factor. (One can say that we consider projective algebras; if $\alpha_m(x)$ is well defined we will talk about algebra in strict sense.) Then for appropriate choice of MO the notion of algebra over MO corresponds to the notion of conformal or superconformal field theory; if the maps $\alpha_n$ are defined uniquely the central charge of corresponding conformal field theory vanishes.

We will not exclude the case when inner product in linear space $E$ entering the definition of algebra over MO is determined only on a subset $X$ of $E^2$. If it is necessary to emphasize that we are dealing with this case, we will use the term “generalized algebra”.

If one uses Hilbert-Schmidt operators in the definition of Grassmannian, it is well known [1] that for every element $V \in Gr(H)$ one can construct an element $\Psi_V$ of the fermionic Fock space $F(H)$, defined up to a factor. We assume that the space $H$ is equipped with antunitary involution $f \to \bar{f}$, preserving $H_+$ and $H_-$. Then the Clifford algebra $Cl(H)$ can be defined as an associative unital algebra with generators $\psi(f), \psi^+(f)$, depending linearly of $f \in H$ and satisfying

$$[\psi(f), \psi^+(f')]_+ = (f, f'), [\psi^+(f), \psi^+(f')]_+ = [\psi(f), \psi(f')]_+ = 0$$

(Here $(,)$ denotes bilinear inner product related with Hermitian inner product in $H$ by the formula $(f, g) = \langle f, \bar{g} \rangle$.) Fock space $F(H)$ can be defined as a space of representation of $Cl(H)$, that contains a cyclic vector $\Phi$, obeying $\psi(f)\Phi = 0$ for $f \in H_+$, $\psi^+(f)\Phi = 0$ for $f \in H_-$. (We assume that $\psi^+(f)$ is
Hermitian conjugate to $\psi(f)$. If $V \in Gr(H)$ then $\Psi_V$ can be defined as a vector from $\mathcal{F}(H)$ satisfying the conditions $\psi(f)\Psi_V = 0$ for $f \in V$, $\psi(f)\Psi_V = 0$ for $f \in V^\perp$. (Here $V^\perp$ stands for orthogonal complement to $V$ with respect to bilinear inner product.)

Notice that for every vector $\Psi \in \mathcal{F}(H)$ one can define two orthogonal subspaces

$$Ann\Psi = \{ f \in H | \psi(f)\Psi = 0 \}$$

and

$$Ann^+\Psi = \{ f \in H | \psi^+(f)\Psi = 0 \}$$

One can prove the following

**Lemma 2.** If $Ann\Psi \supset V$, $Ann^+\Psi \supset KW$, $V \in Gr(k)(H)$, $W \in Gr(-k)(H)$ then $\Psi = \Psi_V$, $V = Ann\Psi$, $W = V^\perp = KAnn^+\Psi$.

We will define also a bilinear inner product $( , )$ in $\mathcal{F} = \mathcal{F}(H)$, obeying the condition that $-\psi(Kf)$ is adjoint to the operator $\psi(f)$ and $-\psi^+(Kf)$ is adjoint to $\psi^+(f)$ with respect to this inner product. Using this bilinear inner product we define a linear MO $\mathcal{F}^n$.

One can check the formula

$$\Psi_{\sigma(n)V} = \sigma(n)\Psi_V \quad \text{for} \quad V \in Gr(H^n) \quad (5)$$

(We use the identification $\mathcal{F}(H^n) = \mathcal{F}(H)^\otimes n$ in (5). This identification permits us to consider $\mathcal{F}(H^n)$ as a linear MO.) To prove (5) we represent $\Psi_V \in \mathcal{F}^\otimes n$ in the form

$$\Psi_V = \sum \alpha_i \otimes A_i \otimes B_i$$

where $\alpha_i \in \mathcal{F}^{\otimes (n-2)}$, $A_i \in \mathcal{F}$, $B_i \in \mathcal{F}$. By definition for every $(f_1, ..., f_n) \in V$ we have

$$\sum (\psi(f)\alpha_i) \otimes A_i \otimes B_i + \sum \alpha_i \otimes (\psi(f_n)A_i) \otimes B_i + \sum \alpha_i \otimes A_i \otimes (\psi(f_n)B_i) = 0 \quad (6)$$

where $\psi(f) = \psi(f_1) \otimes 1 \otimes ... \otimes 1 + 1 \otimes \psi(f_2) \otimes 1 \otimes ... \otimes 1 + 1 \otimes ... \otimes 1 \otimes \psi(f_{n-2})$. Let $(f_1, ..., f_{n-2}, f_n) \in \sigma(n)V$. Then it can be obtained from a point $(f_1, ..., f_{n-2}, f_{n-1}, f_n) \in V$ with $f_{n-1} = Kf_n$. Applying operator $\sigma(n)$ to $\Psi_V$ we obtain

$$\sigma(n)\Psi_V = \sum \alpha_i (A_i, B_i).$$

Now we can apply $\sigma(n)$ to (6). Using the relation

$$\sum \alpha_i (\psi(f_{n-1})A_i, B_i) + \sum \alpha_i (A_i, \psi(f_n)B_i) = 0$$
for $f_{n-1} = Kf_n$ we see that

$$\psi(f)\sigma^{(n)}\Psi_V = 0$$

for every $f = (f_1, ..., f_{n-2}) \in \sigma^{(n-2)}V$. In other words,

$$\sigma^{(n)}V \subset Ann\sigma^{(n)}\Psi_V$$

In similar way we prove

$$K\sigma^{(n)}(KV^\perp) \subset Ann^+\sigma^{(n)}\Psi_V.$$ 

Taking into account that

$$\text{index } \sigma^{(n)}V + \text{index } \sigma^{(n)}(KV^\perp) = 0$$

and using Lemma 2 we obtain (5).

The following statement follows immediately from (5).

**Theorem 5.** The sequence of maps

$$V \in Gr(H^m) \rightarrow \Psi_V \in \mathcal{F}(H^m) = \mathcal{F}^{\otimes m}$$

determines an algebra over a MO $Gr(H^m)$.

It is important to emphasize that in all definitions and theorems above one can replace spaces with superspaces; only minor modifications are required. (The only exception is Theorem 5, that requires more essential modification; see Appendix.) In particular, one can consider super Grassmannian. To give an example of a superspace with semi-infinite structure we can consider the space $H = H^m|n$ of functions on $S^1$ taking values in $(m|n)$-dimensional linear complex superspace $\mathbb{C}^{m|n}$. The definitions of $H_+, H_-$ and $K$ repeat definitions for $H = L^2(S^1)$. Notice, that $H^{1|1}$ can be considered also as the space of functions on the supercircle, i.e. as the space of functions $F(z, \theta) = f(z) + \varphi(z)\theta$, $|z| = 1$, $\theta$ is an odd variable, $f(z)$ is an even function on $S^1$ and $\varphi(z)$ is an odd function on $S^1$.

The following construction will play later an important role. For every manifold $M$ we define a supermanifold $\tilde{M} = \Pi TM$ as the space of tangent bundle with reversed parity of fibers. One can define an odd vector field $\tilde{Q} = \xi^i \frac{\partial}{\partial x^i}$ on $\tilde{M}$ (here $x^i$ are coordinates on $M$, $\xi^i$ are odd coordinates in tangent spaces.) Functions on $M$ can be identified with differential forms on $M$, then the operator $\tilde{Q}$ corresponds to exterior differential. Notice, that $\{Q, Q\} = 0$; this means that $M$ is a $Q$-manifold in the terminology of [6]. One can construct the supermanifold $\tilde{M}$ also in the case when $M$ is a supermanifold. Then again differential forms on $\tilde{M}$ can be considered as functions on $M$, however not all functions on $\tilde{M}$ correspond to differential forms. One can describe $\tilde{M}$ also as the space of maps of $(0|1)$-dimensional superspace $\mathbb{R}^{0|1}$ into $M$. It is easy to see that this construction is functorial: for every map $f : M \rightarrow M'$ one can define
naturally a map $\tilde{f} : \tilde{M} \to \tilde{M}'$. It follows from this remark that in the case when $M$ is a (super) Lie group $\tilde{M}$ is a (super) Lie group (the multiplication map $\tilde{M} \times \tilde{M} \to \tilde{M}$ generates multiplication $M \times M \to M$). If $M$ is a Lie algebra then $\tilde{M}$ is a (super) Lie algebra. (If $l_n$ are generators of $\tilde{M}$, then $f^k_{ij}$ are corresponding structure constants, then $\tilde{M}$ has even generators $l_n$ and odd generators $b_n$ with commutation relations: $[l_m, l_n] = f^k_{mn} l_k$, $[l_n, b_m] = f^k_{mn} b_k$, $[b_m, b_n] = 0$.)

If a Lie group $G$ acts on $M$ then $\tilde{G}$ acts on $\tilde{M}$, transitive action on $G$ induces transitive action of $\tilde{G}$. If $M = G/G_0$ then $\tilde{M}$ can be identified with $\tilde{G}/\tilde{G}_0$.

The supermanifold $\tilde{M} = \Pi TM$ is equipped with natural volume element $dV = \Pi dx^i d\xi^i$ (this volume element does not depend on the choice of coordinates on the (super)manifold $M$). Therefore we can integrate a function on $\tilde{M}$ over $\tilde{M}$ or over $\tilde{L}$ where $L$ is a submanifold of $M$. (Of course, we should make some assumptions about behavior of the function at infinity to guarantee the convergence of the integral). If $M$ is an ordinary manifold then the theory of integration of functions on $\tilde{M}$ is equivalent to the theory of integration of differential forms on $M$. However, if $M$ is a supermanifold then the functions on $\tilde{M}$, corresponding to differential forms on $M$, are not integrable. (By definition, such functions depend polynomially on $\xi^i$. If some of coordinates $x^i$ are odd, corresponding $\xi^i$ are even and the function does not decrease at infinity.) It is necessary to mention that we can integrate also generalized functions (distributions) over $\tilde{M}$.

The volume element $dV$ on $\tilde{M}$ is $Q$-invariant. This means, that

$$\int_L (Qf) dV = 0$$

for every function $f$ on $\tilde{M}$ and every submanifold $L \subset M$. It is easy to check that for $Q$-invariant function $\varphi$ on $\tilde{M}$ the integral

$$\int_L \varphi dV$$

do not change by continuous deformation of the submanifold $L$. (For the case when $M$ is an ordinary manifold one can derive this fact from the remark that $Q$-invariant function $\varphi$ on $\tilde{M}$ can be considered as a closed differential form on $M$.)

If $H = L^2(S^1)$ then $\tilde{H}$ can be regarded as $H^{1|1}$ (as the space of functions on the supercircle). This remark permits us to embed $\tilde{Gr}(H)$ into $Gr(H^{1|1})$. One can consider for example $\tilde{Gr}^{(k)} (H)$ as a homogeneous space where $GL(H)$ acts transitively and utilize the fact the $GL(H)$ acts on $\tilde{H} = H^{1|1}$ and therefore on $Gr(H^{1|1})$. However, it is useful to describe the embedding $\tilde{Gr}(H)$ into $Gr(H^{1|1})$ explicitly. An arbitrary point $W \in Gr(H^{1|1})$ can be specified by means of a Fredholm operator $A : H_{-}^{1|1} \to H_{+}^{1|1}$ and a compact operator $B : H_{+}^{1|1} \to H_{-}^{1|1}$. These operators can be written as $2 \times 2$-matrices.

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\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}, \quad
\begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}
\]

where \(A_{\alpha\beta} : H_+ \to H_+\), \(B_{\alpha\beta} : H_+ \to H_-\), diagonal entries are even, off-diagonal entries are odd. One can check that \(W\) belongs to \(Gr(H) \subset Gr(H^{11})\) if it can be represented by means of operators \(A\) and \(B\) with matrices obeying \(A_{11} = A_{22}, A_{12} = 0, B_{11} = B_{22}, B_{12} = 0\).

3. Generalized string backgrounds.

To calculate string amplitudes corresponding to a conformal field theory with critical central charge we should "add ghosts" to obtain topological conformal field theory with vanishing central charge. Such a TCFT can be considered as "string background". In other words one can define corresponding string amplitudes, even in the case when the TCFT does not correspond to any CFT (matter and ghosts are not separated.) In this section we will describe an abstract analog of these constructions.

We say that a MO \((P_n, \nu_{m,n}, \sigma^{(m)})\) is a \(Q\)-MO if every space \(P_n\) is a \(Q\)-manifold (i.e. \(P_n\) is a supermanifold equipped with a vector field \(Q = Q^{(n)}\) obeying \(\{Q, Q\} = 0\)) and maps \(\nu_{m,n}, \sigma^{(m)}\) are compatible with \(Q\)-structures on \(P_n\) (for example, \(\sigma^{(m)} Q^{(m)} = Q^{(m-2)}\)). A \(Q\)-homomorphism of \(Q\)-MO \(P_n\) into a \(Q\)-MO \(P'_n\) is defined as a collection of maps \(\rho_n : P_n \to P'_n\) that determine a homomorphism of MOs and are compatible with \(Q\)-structures on \(P_n, P'_n\) (i.e. \(\rho_n\) transforms the vector field \(Q\) on \(P_n\) into corresponding vector field on \(P'_n\)). If we have a linear MO \(P_n\) we can introduce a notion of linear \(Q\)-structure on it, requiring that all vector fields \(Q^{(n)}\) are linear, i.e. have the form

\[
(Q^{(n)})^a = (Q^{(n)})^a_z z^b
\]

where \((Q^{(n)})^a_z\) is a matrix of parity reversing linear operator, having zero square (such an operator is called a differential). In particular, if \(E\) is a linear superspace, equipped with a differential \(d\) and \(E\) is equipped with \(d\)-invariant inner product (i.e. \(d = -d^*\)) then one can construct a linear \(Q\)-MO with spaces \(P_n = E^{\otimes n}\). The differential \(d\) on \(E\) determines a differential \(d : E^{\otimes n} \to E^{\otimes n}\). A linear \(Q\)-MO \(E^{\otimes n}\) constructed this way will be called a standard linear \(Q\)-MO.

We define a \(Q\)-algebra \((P_n, E, \alpha_n)\) over a \(Q\)-MO \(P_n\) as a \(Q\)-homomorphism of \(Q\)-MO \(P_n\) into standard linear \(Q\)-MO \(E^{\otimes n}\). Let us emphasize that although we assumed that the maps \(\alpha_n\) are defined only up to a factor when we considered an algebra over MO, in the definition of \(Q\)-algebra we assume that \(\alpha_n\) are well defined (i.e. \(Q\)-algebra should be an algebra in strict sense).

Let us consider an arbitrary MO \(P_n\). Then we can construct in natural way a \(Q\)-MO \(\tilde{P}_n\). (Recall that \(\tilde{P}_n = \Pi T P_n = \{R^{01} \to P_n\}\).) As was explained above this construction is functorial; hence the maps \(\nu_{m,n}, \sigma^{(m)}\), specifying the MO \(P_n\) induce maps \(\tilde{\nu}_{m,n}, \tilde{\sigma}^{(m)}\), that determine a MO \(\tilde{P}_n\). We obtain a \(Q\)-MO this way because every \(P_n\) has a natural \(Q\)-structure. A \(Q\)-algebra \((\tilde{P}_n, F, \beta_n)\)
over $Q$-MO $\tilde{P}_n$ is called a $Q$-extension of an algebra $(P_n, E, \alpha_n)$ over a MO $P_n$ if there exists such a linear map $\rho : E \to F$ for every $x \in P_n \subset \tilde{P}_n$, $e \in E$, we have $(\rho(e), \beta_n(x)) = (e, \alpha_n(x))$. If the algebra $(P_n, E, \alpha_n)$ corresponds to conformal field theory then we can obtain its $Q$-extension adding ghosts to the matter sector of the theory.

Let us consider a MO $(P_n, \nu_{m,n}, \sigma^{(m)})$. We say that this MO is an EMO (equivariant modular operad) if for every $m$ the group $(S^1)^m$ acts on $P_m$; this action should be compatible with $S_m$-action and with maps $\nu_{m,n}, \sigma^{(m)}$. More precisely, we assume that

1. $s \circ g \circ s^{-1} = s(g)$ where $g \in (S^1)^m$, $s \in S_m$, $g \to s(g)$ denotes the natural action of $S_m$ onto $(S^1)^m$.

2. $\nu_{m,n} \circ (g \times g') = (g \times g') \circ \nu_{m,n}$ for $g \in (S^1)^m$, $g' \in (S^1)^n$.

3. If $g = (g_1, \ldots, g_m) \in (S^1)^m$ and $g_{m-1} = g_m^{-1}$ then $\sigma^{(m)} \circ g = g' \circ \sigma^{(m)}$ where $g' = (g_1, \ldots, g_{m-2}) \in (S^1)^{m-2}$.

We define $\rho^{(m)} : P_m \times S^1 \to P_{m-2}$ as a map transforming a point $(x, \gamma) \in P_m \times S^1$ into a point $\sigma^{(m)}(gx)$ where $g = (1, \ldots, 1, \gamma) \in (S^1)^m$. It is possible to generalize a notion of EMO, taking as a starting point the maps $\rho^{(m)}$; we will not discuss this generalization here.

The MO corresponding to conformal field theory can be considered as an EMO. Recall that the elements of $P_m$ in this case can be considered as surfaces with $m$ holes, having boundaries parametrized by the circle $|z| = 1$. The action of $(S^1)^m$ changes the parametrization of boundary circles: $(z_1, \ldots, z_m) \to (e^{i\alpha_1}z_1, \ldots, e^{i\alpha_m}z_m)$.

Similarly, we can introduce a structure of EMO in the MO $Gr(H^k)$, assuming that $H$ has a semi-infinite structure $(H_+, H_-, K)$ and the group $S^1$ acts on $H$, in such a way that $H_+$ and $H_-$ are invariant subspaces and $K \circ g \circ K^{-1} = g^{-1}$ for every $g \in S^1$. If the semi-infinite structure on $H = L^2(S^1)$ is chosen in standard way ($H_-$ spanned by $z^n$, $n \geq 0$, $H_+$ by $z^n$, $n < 0$, $(Kf)(z) = z^{-1}f(z^{-1})$), then one can define an action of the group $S^1$ by the formula $g_{\alpha}(z) = e^{i\alpha}g(e^{2i\alpha}z)$.

As we know for every MO $P_n$ the spaces $\tilde{P}_n$ constitute a Q-MO. We define a Q-EMO as a Q-MO with action of $(S^1)^n$ satisfying natural conditions. It is easy to see that if $P_n$ constitute an EMO the spaces $\tilde{P}_n$ constitute a Q-EMO.

Every linear space $F$ with inner product and $S^1$-action preserving this inner product determines a linear EMO. A linear superspace with inner product and linear operators $Q, l, b$, obeying $Q^2 = 0$, $b^2 = 0$, $l = [Q, b]_+$ determines a linear Q-EMO. (Operators $Q$ and $b$ should be parity reversing. Inner product should be invariant with respect to $Q, l, b$. We assume that $l$ generates an action of $S^1$.)

One can construct a BV-algebra corresponding to an EMO. (This construction is similar to the construction of [7].) Let us denote by $C_n^k$ the linear space of singular $k$-dimensional chains in $P_n$. Using the map $\rho^{(n)} : P_n \times S^1 \to P_{n-2}$ we can define a map $\Delta : C_n^k \to C_{n+2}^{k+1}$ by the formula $\Delta_c(x) = \rho^{(n)}_c(x \times [S^1])$, where $[S^1]$ stands for the fundamental cycle of $S^1$. The space $C_n = \sum_k C_n^k$ can be
considered as graded $S_n$-module; the $S_n$-invariant part of $C_n$ will be denoted by $C^\text{inv}_n$. It is easy to check that $\Delta$ is a parity reversing operator acting from $C^\text{inv}_n$ into $C^\text{inv}_{n-2}$ and that $\Delta^2 = 0$ on $C^\text{inv}_n$. The map $\nu_{m,n} : P_m \times P_n \to P_{m+n}$ generates a map $(\nu_{m,n})_* : C^k_m \times C^l_n \to C^k_{m+l}$. If $x \in C^\text{inv}_m$, $y \in C^\text{inv}_n$ we denote by $x \cdot y$ an element of $C^\text{inv}_{m+n}$ obtained by means of symmetrization of $(\nu_{m,n})_*(x \times y)$. Let us consider now a graded linear space $C^\text{inv} = \bigoplus C^\text{inv}_n$ (grading in $C^\text{inv}$) is induced by the grading in $C^\text{inv}_n$. One can prove that $\Delta : C^\text{inv} \to C^\text{inv}$ is a second order derivation with respect to multiplication $(x, y) \to x \cdot y$ and therefore $C^\text{inv}$ can be considered as a BV-algebra.

4. String amplitudes.

Let us consider a linear space $E$ with inner product and with $S^1$-action preserving this inner product. As we mentioned such a space determines a linear EMO. If $P_n$ is an EMO we can consider an algebra $(P_n, E, \alpha_n)$ as a homomorphism of $P_n$ into $E^\otimes n$. (We require that maps $\alpha_n : P_n \to E^\otimes n$ commute with $(S^1)^n$-action.)

To define string amplitudes corresponding to the algebra $(P_n, E, \alpha_n)$ we should construct at first a $Q$-extension $(\hat{P}_n, F, \beta_n)$ of $(P_n, E, \alpha_n)$ (we should "add the ghosts"). This means that we should construct a $Q$-algebra over $Q$-EMO $P_n$ that can be considered as a $Q$-extension of our original algebra. More precisely, we should construct a linear space $F$ with inner product and odd operators $Q, b$, respecting this inner product and obeying $Q^2 = b^2 = 0$. We assume that the operator $l = [Q, b]_+$ generates an action of $S^1$; then $l, b$ generate an action of $\hat{S}^1$ on $F$ and an action of $(\hat{S}^1)^n$ on $F^n$. The $Q$-algebra $(\hat{P}_n, F, \beta_n)$ is specified by the maps $\beta_n : \hat{P}_n \to F^\otimes n$ that are compatible with the action of $Q$ and $(\hat{S}^1)^n$ on $P$ and $F^\otimes n$.

Recall that we assumed that the maps $\alpha_n$ are defined only up to a factor. However, we assume that the maps $\beta_n$ in $(\hat{P}_n, F, \beta_n)$ are defined unambiguously. As we mentioned the passage to $Q$-extension corresponds to adding ghosts to conformal field theory. Our assumption means that we consider critical theory when adding ghosts gives zero central charge. Now we define string amplitudes starting with equivariant $Q$-algebra $(\hat{P}_n, F, \beta_n)$. (Notice that we can forget about original algebra $(P_n, E, \alpha_n)$ at this stage. This remark corresponds to well known fact that string amplitudes can be defined also in the case when matter and ghosts are not separated; in other words we can consider a topological conformal field theory, that does not correspond to a conformal field theory, as a string background.)

The space $F$ can be interpreted as space of states in the string theory. However, not all of them should be considered as physical states. The physical states $A$ should satisfy the conditions $l A = 0$, $b A = 0$, $Q A = 0$ where $l$ and $b$ are generators of the group $\hat{S}^1$ ($l$ is even, $b$ is odd, $l = [b, Q]_+$, hence $l A = 0$ follows from $b A = 0$, $Q A = 0$). Two physical states $A, A' \in F$ should be considered equivalent if $A' - A = QB$ where $B \in F$, $l B = 0$, $b B = 0$. In other words, the space of physical states can be identified with homology of operator $Q$ acting in
the space

\[ F^{rel} = \{ A \in F | lA = 0, \ bA = 0 \} . \]

Now we can define the scattering amplitude for physical states \( A_1, \ldots, A_n \) in the following way. Let us consider a function

\[ \Phi_{A_1, \ldots, A_n}(x) = (A_1 \otimes \cdots \otimes A_n, \beta_n(x)) \quad \text{(7)} \]

on \( \tilde{P}_n \). We assumed that the map \( \beta_n : \tilde{P}_n \to F^{\otimes n} \) respects the action of \( (S^1)^n \) and \( Q \) on \( \tilde{P}_n \) and \( F^{\otimes n} \). It follows immediately from invariance of inner product and from relation \( lA_i = 0, \ bA_i = 0, \ QA_i = 0 \) that the function (7) is \( (S^1)^n \)-invariant and \( Q \)-invariant. This means that \( \Phi_{A_1, \ldots, A_n}(x) \) can be considered as a \( Q \)-invariant function on \( \tilde{P}_n/(S^1)^n \). We define ”string amplitude” as an integral of \( \Phi_{A_1, \ldots, A_n}(x) \) over \( \tilde{M}_n \) where \( M_n \) denotes the ”fundamental cycle” of \( P_n/(S^1)^n \). One should emphasize that \( P_n/(S^1)^n \) is in general infinite-dimensional (super)manifold therefore the notion of ”fundamental cycle” is ill-defined. However, say, in the case where \( P_n \) is a space of complex curves with \( n \) holes, the infinite-dimensional space \( P_n/(S^1)^n \) is homotopy equivalent to a disjoint union of finite-dimensional orbifolds and the fundamental cycle \( M_n \) of \( P_n/(S^1)^n \) can be defined as a sum of corresponding fundamental cycles. It follows from \( Q \)-invariance of \( \Phi_{A_1, \ldots, A_n}(x) \) that the integral

\[ \int_{\tilde{M}_n} \Phi_{A_1, \ldots, A_n}(x) \]

depends only on homology class of \( M_n \). This integral gives the standard expression for the bosonic string amplitudes. Similar considerations can be applied to the case when \( P_n \) is a space of superconformal manifolds; we obtain an expression for the fermionic string amplitudes in this case.

The definition of physical states given above is not completely general. One should define the space of physical states as the equivariant cohomology of \( F \). More precisely, we should consider the subspace \( F^{inv}[\Omega] \) of polynomials of indeterminate \( \Omega \) taking values in \( F^{inv} = \{ x \in F | lx = 0 \} \). One can define a differential \( d \), acting on \( F^{inv} \), by the formula \( d = Q - \Omega b \). Then equivariant cohomology of \( F \) can be identified with cohomology of \( d \), acting on \( F^{inv}[\Omega] \). It is clear that physical states in the old sense can be considered as physical states in the new sense. One can prove that both notions coincide if every element \( x \) of \( F \) obeying \( bx = 0 \) can be represented in the form \( x = by \) and \( F \) splits into a direct sum of eigenspaces of \( l \) \[5\]. If \( A_1, \ldots, A_n \) are equivariant cocycles (i.e. \( A \in F^{inv} \), \( dA = 0 \)) we define \( \Phi_{A_1, \ldots, A_n}(x, \Omega_1, \ldots, \Omega_n) \) by means of the same formula (7).

It is easy to see that the function \( \Phi = \Phi_{A_1, \ldots, A_n}(x, \Omega_1, \ldots, \Omega_n) \) is \( (S^1)^n \)-invariant (i.e. \( l_1 \Phi = \cdots = l_n \Phi = 0 \)) and satisfies the condition

\[ (Q - \sum \Omega_i b_i) \Phi = 0. \]
The function $\Phi$ on $\tilde{P}_n$ can be considered as an equivariant differential form on $P_n$; the conditions above mean that this form is equivariantly closed (with respect to the action of the group $(S^1)^n$); it determines therefore an equivariant cohomology class. In the case when $(S^1)^n$ acts freely on $P_n$ one can prove that equivariant cohomology of $P_n$ is isomorphic to cohomology of $P_n/(S^1)^n$; we obtain a cohomology class of $P_n/(S^1)^n$ that can be used to define string amplitudes in the same way as above. Similar construction can be applied also in the case when $P_n$ is a supermanifold.

Notice that in the definition of EMO and in considerations of present section one can replace the group $S^1$ with any other group $G$. (The condition 3 in the definition of EMO should be slightly modified. Namely, we replace the relation $g_{m-1} = g_m^{-1}$ with the relation $g_{m-1} = \rho(g_m)$ where $\rho$ is a map of $G$ into itself.) Notice, that MO $Gr(H_m)$ where $H = L^2(S^1)$ can be considered as EMO with respect to the action of the group $\Gamma$ of non-vanishing twice differentiable functions on $S^1$. We will argue that this EMO and similar EMO’s should be related to non-perturbative string theory.

Let us consider a collection of maps $\alpha_n: P_n \to E^\otimes n$, that determine an algebra over MO $P_n$. If $\Sigma \in P_1$ has an automorphism group $G$, then this automorphism group acts naturally on $E$. In many interesting cases one can extend the action of automorphism group $G$ to an action of $G^n$ onto $P_n$ to obtain an EMO and an algebra over this EMO. For example, if the algebra at hand corresponds to CFT, one can take as $\Sigma$ a standard disk and consider $G$ as a group $S^1$ of rotations of the disk. The space $P_n$ for the MO related to CFT consists of complex curves with $n$ embedded standard disks; the action of $(S^1)^n$ onto $P_n$ comes from automorphism groups of these disks.

If $P_n$ constitute an EMO with respect to the group $G$, then $\tilde{P}_n$ constitute an EMO with respect to the group $\tilde{G}$. To define "string amplitudes" we can start with a $G$-equivariant algebra $(P_n, E, \alpha_n)$ over $P_n$. (We say that an algebra $(P_n, E, \alpha_n)$ is $G$-equivariant, if $G$ acts linearly on $E$, preserving the inner product, and the maps $\alpha_n: P_n \to E^\otimes n$ commute with the action of $G^n$. ) We should extend $E$ to a $\tilde{G}$-module $F$ (to a linear superspace $F$ with linear action of the group $\tilde{G}$). At the level of Lie algebras such a module can be described by means of even operators $L_n$ and odd operators $b_n$ obeying

$$[L_m, L_n] = f^k_{mn} L_k, \quad [L_m, b_n] = f^k_{mn} b_k, \quad [b_m, b_n]_+ = 0,$$

where $f^k_{mn}$ are structure constants of Lie algebra $G$ of $G$. One should assume also that $F$ is equipped with an odd differential $Q$ obeying

$$L_m = [Q, b_m]_+$$

and with inner product which is $Q$-invariant and $\tilde{G}$-invariant. As we mentioned, the Lie algebra $\tilde{G}$ of the group $\tilde{G}$ has even generators $L_n$ and odd generators $b_n$ obeying (8). Adding to $L_n, b_n$ an odd generator $Q$ satisfying (9) and $[Q, Q]_+ = 0$.
we obtain a Lie algebra that will be denoted by $\mathcal{G}'$. Corresponding extension of the group $\hat{G}$ will be denoted by $\hat{G}'$.

A $Q$-extension $(\hat{P}_n, F, \beta_n)$ of an algebra $(P_n, E, \alpha_n)$ over EMO $P_n$ should be considered as a $Q$-algebra, that respects the action of $\hat{G}^n$ in $\hat{P}_n$ and $F \otimes \hat{P}_n$. In other words, $\hat{P}_n$ should be considered as EMO with respect to the group $G'$ and $(\hat{P}_n, F, \beta_n)$ should be a $G'$-equivariant algebra. The space $\mathcal{A}$ of physical states must be identified with equivariant cohomology of $F$ with respect to the action of $G'$.

To define equivariant cohomology of $F$ we begin with the space $F \otimes \Phi(\Pi\hat{G})$ where $\Phi(\Pi\hat{G})$ stands for a space of functions on $\Pi\hat{G}$. One can consider various functional spaces and obtain different versions of the notion of equivariant cohomology. The standard notion of equivariant cohomology corresponds to the space of polynomial functions on $\Pi\hat{G}$. If Lie algebra $\mathcal{G}$ is $m$-dimensional polynomial functions on $\Pi\hat{G}$ can be identified with polynomials of $m$ odd variables $\omega_1, ..., \omega_n$ and $m$ even variables $\Omega_1, ..., \Omega_n$. The group $G'$ acts naturally on $\hat{G}'$; this action is linear and therefore determines a linear action of $G'$ on $\Pi\hat{G}'$. One can identify functions on $\Pi\hat{G}$ with homogeneous functions on $\Pi\hat{G}' \setminus \Pi\hat{G}$; using this identification we obtain an action of $G'$ on $\Phi(\Pi\hat{G})$. (One can say also that we define the action of $G'$ on $\Phi(\Pi\hat{G})$ using the embedding of $\Pi\hat{G}$ into projective space corresponding to linear space $\Pi\hat{G}'$.) Combining $G'$-action on $F$ and $G'$-action on $\Phi(\Pi\hat{G})$ we obtain $G'$-action on $F \otimes \Phi(\Pi\hat{G})$. In other words, we have $\hat{G}$-action on $F \otimes \Phi(\Pi\hat{G})$ and a differential $Q_{tot}$ on this space.

The differential $Q_{tot}$ acts on the set $(F \otimes \Phi(\Pi\hat{G}))^{inv}$ of $\hat{G}$-invariant elements of $F \otimes \Phi(\Pi\hat{G})$; we define equivariant cohomology $H_{G}(F)$ of $F$ as cohomology of $Q_{tot}$ acting on $(F \otimes \Phi(\Pi\hat{G}))^{inv}$. If $G$ is an $m$-dimensional connected Lie group we represent an element of $F \otimes \Phi(\Pi\hat{G})$ as an $F$-valued function $\varphi(\omega_1, ..., \omega_m, \Omega_1, ..., \Omega_m)$. The condition of $\hat{G}$-invariance means that

$$L_i \varphi + (\omega_\alpha f_i^k \frac{\partial}{\partial \omega_k} + \Omega_\alpha f_i^k \frac{\partial}{\partial \Omega_k}) \varphi = 0$$

$$b_i \varphi + (\frac{\partial}{\partial \omega_i} + \omega_\alpha f_i^k \frac{\partial}{\partial \omega_k}) \varphi = 0$$

(10) (11)

The second of these equations can be used to eliminate $\omega_1, ..., \omega_m$ and to obtain a generalization of the definition of $S^n$-equivariant cohomology that we gave above. More precisely, we assign to every solution $\varphi(\omega_1, ..., \omega_m, \Omega_1, ..., \Omega_m)$ of (11) a function $\varphi(0, ..., 0, \Omega_1, ..., \Omega_m)$; we obtain one-to-one correspondence between solutions of (11) and functions depending on $\Omega_1, ..., \Omega_m$. Using this correspondence we can define equivariant cohomology as cohomology of the operator $Q - \Omega^b b_i$ acting on the space of $G$-invariant functions of $\Omega_1, ..., \Omega_m$ taking values in $F$. As we mentioned one can modify the definition of equivariant cohomology considering various spaces of functions on $\Pi\hat{G}$. If $\Phi(\Pi\hat{G})$ stands for the space of all smooth functions corresponding equivariant cohomology of $F$ is denoted by $H_{G}(F)$ (more precisely, one should consider the space of smooth
\(F\)-valued functions on \(\Pi \tilde{G}\); for finite dimensional \(F\) this space can be identified with \(F \otimes \Phi(\Pi \tilde{G})\). Equivariant cohomology \(F\) corresponding to generalized functions (distributions) on \(\Pi \tilde{G}\) is denoted by \(H^\infty_G(F)\) (see[8]). If \(\tilde{G}\) (and therefore \(\tilde{G}\)) is an infinite-dimensional Lie algebra with semi-infinite structure, one can define also semi-infinite equivariant cohomology of \(F\) replacing \(\Phi(\Pi \tilde{G})\) with a Fock space \(F\) constructed by means of \(\tilde{G}\) (see [12]). In the definition of semi-infinite equivariant cohomology one should consider \(\tilde{G}\)-semivariants of \(F \otimes F\) in the sense of [9] (instead of \(\tilde{G}\)-invariants used in the standard definition of equivariant cohomology.)

The definition of \(G\)-equivariant cohomology can be applied to every \(G'\)-module. In other words, it can be applied to every differential \(\tilde{G}\)-module, i.e. to \(\tilde{G}\)-module \(F\) equipped with an odd differential that is compatible with the structure of \(\tilde{G}\)-module. (More precisely, if \(F\) is considered as a linear \(Q\)-manifold the map \(\tilde{G} \times F \to F\) that determines an action of \(\tilde{G}\) on \(F\) should be compatible with \(Q\)-structures on \(\tilde{G} \times F\) and \(F\).) In particular, if a group \(G\) acts on a manifold \(M\) the group \(\tilde{G}\) acts on the manifold \(\tilde{M}\) and therefore the space \(\Omega(M)\) of differential forms of \(M\) (= the space of functions on \(\tilde{M}\)) can be considered as a differential \(\tilde{G}\)-module. Equivariant cohomology of this module is called equivariant cohomology of the \(G\)-manifold \(M\). (If \(M\) is a supermanifold there are various versions of this definition because we can consider different spaces of functions of \(\tilde{M}\). Similar remark can be made in the case when \(G\) is a supergroup.)

Let us come back to the definition of "string amplitudes". We consider \(Q\)-algebra \((\tilde{P}_n, F, \beta_n)\), where \(\tilde{P}_n\) constitute a \(Q\)-EMO with respect to the group \(\tilde{G}\), \(F\) is a differential \(\tilde{G}\)-module, \(\beta_n\) are compatible with action of \(\tilde{G}\) and with \(Q\). (It is not necessary to assume that this \(Q\)-algebra is obtained as a \(Q\)-extension of an algebra over \(P_n\).) In other words one can say that \(\tilde{P}_n\) constitute an EMO with respect to the group \(G'\), a superspace \(F\) is a \(G'\)-module and the maps \(\beta_n: \tilde{P}_n \to F^\otimes n\) are compatible with \(G'\)-action. If \(A_1, \ldots, A_n \in A\) are physical states (elements of equivariant cohomology of \(F\)) we consider an expression

\[
< \hat{A}_1 \otimes \ldots \otimes \hat{A}_n, \beta_n(x) >
\]

where \(x \in \tilde{P}_n\) and \(\hat{A}_k\) stands for a representative of cohomology class \(A_k \in A\). It is easy to check that (12) determines an element of \(G\)-equivariant cohomology of \(P_n\) and that this element does not depend on the choice of representatives \(\hat{A}_k\) in the classes \(A_k\). To get "string amplitudes" we should have a linear functional on \(G\)-equivariant cohomology of \(P_n\) (a kind of integration). It is important to emphasize that this construction can be applied also in the case when physical states are defined by means of semi-infinite equivariant cohomology.

One can hope to obtain non-perturbative formulation of string theory applying the above consideration to the case when \(P_n = Gr(H^n)\) for appropriate choice of (super)space \(H\). This hope is based, in particular, on the relation between equivariant cohomology of Grassmannian and cohomology of moduli...
spaces of conformal manifolds. To explain this relation we should remind some facts about equivariant cohomology. Let us assume that a connected compact abelian group \( T \) (a torus) acts on \( M \). Then we can consider equivariant cohomology \( H_T(M) \) as a module over the polynomial ring \( \mathbb{C}[\Omega_1, \ldots, \Omega_r] \) where \( r = \dim T \). The ring \( \mathbb{C}[\Omega_1, \ldots, \Omega_r] \) can be considered as a ring \( \Phi(\text{Lie } T) \) of polynomial functions on the Lie algebra of \( T \).

If \( T \) acts on \( M \) transitively then \( H_T(M) \) can be identified with cohomology of \( M/T \) with coefficients in the ring \( \Phi(\text{Lie } S) \) of polynomial functions on the subalgebra of \( \text{Lie } T \) consisting of elements \( t \in T \) obeying \( t(x) = 0 \) for fixed point \( x \in M \). (In other words \( \text{Lie } S \) is the stabilizer of \( x \in M \).) We see that the "size" of \( H_T(M) \) is determined by the size of the stabilizer. If the action of \( T \) on \( M \) is not transitive it follows from so called localization theorems that the "contribution" of the point \( x \in M \) to the equivariant cohomology is governed by the stabilizer of \( x \). In particular, the "free part" (the rank) of \( \mathbb{C}[\Omega_1, \ldots, \Omega_r] \)-module \( H_T(M) \) is determined by the fixed points of the action of \( T \).

We can try to apply formally the above statements to the action of infinite-dimensional abelian group \( \Gamma^m \) on \( \text{Gr}(H^m) \) where \( H = L^2(S^1) \). We will see that \( \Gamma^m \)-equivariant cohomology of \( \text{Gr}(H^m) \) can be expressed in terms of the Krichever locus and that this cohomology is closely related to the cohomology of moduli spaces of conformal manifolds. (Recall that Krichever locus consists of points having large stabilizers in \( \Gamma^m \) and that the space of orbits of \( \Gamma^m \) in the Krichever locus \( P_m(\Gamma) \) can be identified with the moduli space \( P_m \) used in the definition of CFT.)

Of course, it is not clear that the statements proved for compact groups can be applied to a non-compact group \( \Gamma \). However, I was able to prove that many results of the theory of compact transformation groups can be transferred to non-compact case if we understand equivariant cohomology as semi-infinite equivariant cohomology [12].

Appendix.
Isotropic Grassmannian.

Let us consider a Hilbert space \( \mathcal{H} \) provided with an antiunitary involution \( f \rightarrow f^* \). We will equip the direct sum \( \mathcal{H}^2 \) of two copies of \( \mathcal{H} \) with semi-infinite structure considering the first copy as \( H_+ \), the second copy as \( H_- \) and defining an operator \( K \) by the formula

\[
K(f, g) = (-g, f)
\]

Notice that \( K \) is not an involution. However, as we mentioned, we can use \( K \) to define the spaces \( \text{Gr}(\mathcal{H}^{2m}) \) and the maps \( \nu_{m,n} : \text{Gr}(\mathcal{H}^{2m}) \times \text{Gr}(\mathcal{H}^{2n}) \rightarrow \text{Gr}(\mathcal{H}^{2(m+n)}) \) and \( \sigma^{(m)} : \text{Gr}(\mathcal{H}^{2n}) \rightarrow \text{Gr}(\mathcal{H}^{2(m-2)}) \). We will exclude the condition 3) from the definition of MO; then these data constitute a MO.

A linear subspace \( V \subset \mathcal{H}^2 \) is called isotropic if for every two points \( (f, g) \in V, (f', g') \in V \) we have \( (f, g') + (g, f') = 0 \) (here \( (, ,) \) denotes bilinear inner product: \( (\varphi, \psi) = <\varphi, \psi^* > \)).
Isotropic Grassmannian $\text{IGr}(\mathcal{H})$ can be defined as a subset of $\text{Gr}(\mathcal{H}^2)$ consisting of isotropic subspaces. Giving an obvious definition of isotropic subspace of $\mathcal{H}^{2m}$ one can define also $\text{IGr}(\mathcal{H}^m)$ as a subset of $\text{Gr}(\mathcal{H}^{2m})$. It is easy to check that the direct sum of isotropic subspaces is again an isotropic subspace and that the map $\sigma(m) : \text{Gr}(\mathcal{H}^{2m}) \to \text{Gr}(\mathcal{H}^{2(m-2)})$ transforms an isotropic subspace into isotropic subspace. In other words, the spaces $\text{IGr}(\mathcal{H}^m)$ constitute a MO (a sub MO of the MO $\text{Gr}(\mathcal{H}^{2m})$). The decomposition $\text{Gr}(\mathcal{H}^{2m}) = \bigcup \text{Gr}(k)(\mathcal{H}^{2m})$ induces decomposition $\text{IGr}(\mathcal{H}^m) = \bigcup \text{IGr}(k)(\mathcal{H}^m)$, hence we consider the MO $\text{IGr}(\mathcal{H}^m)$ as graded MO. In particular, we can say that the spaces $\text{IGr}^{(0)}(\mathcal{H}^m)$ also constitute a MO.

It is important to notice that, conversely, usual Grassmannian can be embedded into isotropic Grassmannian. Let us assume that $\mathcal{H}$ is equipped with a semi-infinite structure ($\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$) and that antiunitary involution $f \to f^*$ transforms $\mathcal{H}_+$ into $\mathcal{H}_+$ and $\mathcal{H}_-$ into $\mathcal{H}_-$. Then for every linear subspace $V \subset \mathcal{H}$ we can construct an isotropic subspace $\rho(V) \subset \mathcal{H}^2$ as a direct sum of subspaces $\pi_1 V$ and $\pi_2 V^\perp$, where $\pi_1 : \mathcal{H} \to \mathcal{H}^2$ transforms $f \in \mathcal{H}$ into $(\pi_+ f, \pi_- f)$ and $\pi_2 : \mathcal{H} \to \mathcal{H}^2$ transforms $f \in \mathcal{H}$ into $(\pi_- f, \pi_+ f)$. (Here $\pi_\pm : \mathcal{H} \to \mathcal{H}_\pm$ are orthogonal projections and the orthogonal complement $V^\perp$ is taken with respect to bilinear inner product $(\phi, \psi) = \langle \phi, \psi^* \rangle$. It is easy to check that for $V \in \text{Gr}(\mathcal{H})$ we have $\rho(V) \in \text{IGr}(\mathcal{H})$; hence we embedded $\text{Gr}(\mathcal{H})$ into $\text{IGr}(\mathcal{H})$. Moreover, one can verify that index $\rho(V) = 0$ therefore we can say that we embedded $\text{Gr}(\mathcal{H})$ into $\text{IGr}^{(0)}(\mathcal{H})$.

Let us consider a Hilbert space $\mathcal{H}$ with antiunitary involution $f \to f^*$. We define the space $\mathcal{F}$ as the space of antiholomorphic functionals $\Phi(a^*)$ on $\Pi \mathcal{H}$ obeying the condition

$$\int \Phi(a^*) \Phi^*(a) e^{-(a,a^*)} dada^* < \infty. \quad (13)$$

Here $\Pi \mathcal{H}$ stands for the superspace obtained from $\mathcal{H}$ by means of parity reversion; $\Phi^*(a) = (\Phi(a^*))^*$. In other words, $\mathcal{F}$ consists of elements of infinite-dimensional Grassmann algebra generated by $\mathcal{H}$, i.e. of formal expressions

$$\Phi(a^*) = \sum_n \frac{1}{(n!)^{1/2}} \int \Phi_n(x_1, ..., x_n)a^*(x_1) ... a^*(x_n) d^n x,$$

where $\Phi_n(x_1, ..., x_n)$ are antisymmetric and

$$\sum \int |\Phi_n|^2 d^n x < \infty.$$

(We consider $\mathcal{H}$ as the space of square integrable functions depending on $x \in S$, where $S$ is a measure space; involution is realized as complex conjugation. This restriction is not essential, but it permits us to simplify notations.) It is well known that $\mathcal{F}$ can be considered as fermionic Fock space. (See[10]; we
follow the notations of this book. A rigorous explanation of the meaning of infinite-dimensional integral in (13) also can be found in [10].)

Operator of multiplication on \( \int f(x) a^+(x) dx \) will be denoted by \( a^+(f) \) and operator \( \int f(x) \frac{\delta}{\delta a^+(x)} dx \) by \( a(f) \). These operators obey canonical anticommutation relations (Clifford algebra relations) \( [a(f), a(f')]_+ = [a^+(f), a^+(f')]_+ = 0 \), \( [a(f), a^+(f')_+] = (f, f') \). The functional \( \Phi_0 = 1 \) can be considered as vacuum vector; it obeys \( a(f)\Phi_0 = 0 \) for all \( f \in \mathcal{H} \). Operator \( a^+(f^*) \) is adjoint to \( a(f) \) with respect to Hermitian inner product

\[
\langle \Phi_1, \Phi_2 \rangle = \int \Phi_1^*(a) e^{-\langle a, a^* \rangle} dada^*.
\]

For every vector \( \Psi \in \mathcal{F} \) we define a linear subspace \( \text{Ann} \Psi \subset \mathcal{H}^2 \) consisting of such pairs \( (f, g) \in \mathcal{H}^2 \) that

\[
(a(f) + a^+(g))\Psi = 0
\]

It is easy to check that the subspace \( \text{Ann} \Psi \) is isotropic. One can prove that for every \( V \in IGr^{(0)}(\mathcal{H}) \) there exists a vector \( \Psi_V \in \mathcal{F} \) obeying \( V = \text{Ann} \Psi_V \); this vector is unique up to a factor. (We assume that Grassmannian is defined by means of Hilbert-Schmidt operators; this is essential for the validity of the above statement.) To give a proof we represent \( V \in IGr^{(0)}(\mathcal{H}) \) as an image of a linear map \( \mathcal{H} \to \mathcal{H}^2 \) transforming \( \varphi \in \mathcal{H} \) into \( (A\varphi, B\varphi) \). Here \( A \) is a Fredholm operator of index 0, \( B \) is a Hilbert-Schmidt operator. The condition \( V \subset \text{Ann} \Psi \) means that for every \( \varphi \in \mathcal{H} \) we have

\[
\left( \sum_k A_{kl} \frac{\partial}{\partial a^*_k} + \sum_k B_{kl} a^*_l \right) \Psi = 0
\] (14)

(We have chosen a basis in \( \mathcal{H} \).) The condition that the space \( V \) is isotropic means that

\[
\sum_k (A_{kl} B_{kr} + B_{kl} A_{kr}) = 0.
\]

(15)

Without loss of generality we can assume that the matrix \( A_{kl} \) is diagonal; moreover, non-zero entries can be taken equal to 1. We assume that \( A_{ii} = 0 \) for \( i \leq s \), \( A_{ii} = 1 \) for \( i > s \). Then the equations (14) and the condition (15) take the form

\[
\left( \sum_k B_{kl} a^*_k \right) \Psi = 0 \quad \text{for } l \leq s, \quad \text{(16)}
\]

\[
\left( \frac{\partial \Psi}{\partial a^*_l} + \sum_k B_{kl} a^*_k \right) \Psi \quad \text{for } l > s.
\]

\[
B_{lr} + B_{rl} = 0 \quad \text{if } l > s, \ r > s
\]
\( B_{l r} = 0 \) if \( l > s, r \leq s \) or \( l \leq s, r > s \).

The solution to equations (16),(17) is unique (up to a factor) and can be written in the form

\[
\Psi = C \cdot \Pi_{l \leq s} \delta(\sum_k B_{kl} a_k^*) e^{-\frac{1}{2} \sum_{k,l \geq s} a^*_l B_{kl} a^*_k} 
\]  

(18)

or in the form

\[
\Psi = C \cdot \Pi_{l \leq s} (\sum_k B_{kl} a_k^*) e^{-\frac{1}{2} \sum_{k,l \geq s} a^*_l B_{kl} a^*_k}. 
\]  

(19)

These two forms are equivalent because \( \delta(a^*_k) = a^*_k \). (In other words, we have \( \int \delta(a^*_k) \varphi(a^*, a) da d a^* = \int a^*_k \varphi(a^*, a) da d a^* \) for every \( \varphi \).) Notice, that we proved a little bit more than claimed. Namely, it follows from the proof that for \( V \in IGr^{(0)}(H) \) there exists unique (up to a factor) vector \( \Psi \) obeying \( \text{Ann } \Psi \supset V \) and that we have \( \text{Ann } \Psi = V \) for this vector.

In the case when the operator \( A \) is invertible we can write the functional \( \Psi = \Psi_V \) in the form

\[
\Psi_V = C e^{-\frac{1}{2} (a^*, A^{-1} B a^*)}. 
\]

(It is clear that \( \Psi_V \) satisfies (14).) If the operator \( A \) has even number of zero modes we can represent \( V \in IGr^{(0)}(H) \) as a limit of \( V_n \in IGr^{(0)}(H) \) in such a way that \( V_n \) is an image of a map \( H \rightarrow H^2 \) transforming \( \varphi \in H \) into \( (A_n \varphi, B_n \varphi) \), where \( A_n \) is an invertible operator, \( A_n - 1 \) belongs to the trace class and \( B_n \) is a Hilbert-Schmidt operator. Using this representation one can write \( \Psi_V \) as a limit of functionals

\[
(\det A_n)^{1/2} e^{-\frac{1}{2} (a^*, A_n^{-1} B_n a^*)}. 
\]

We constructed a map \( IGr^{(0)}(H) \rightarrow F \) (defined up to a factor). It is easy to check that applying this construction \( IGr^{(0)}(H^m) \) we obtain an algebra over \( \text{MO } IGr^{(0)}(H^m) \).

The definition of isotropic Grassmannian \( IGr(H) \) can be easily generalized to the case when \( H = H_0 \oplus H_1 \) is a complex Hilbert \( \mathbb{Z}_2 \)-graded space with antiunitary involution \( f \rightarrow f^* \). (Bilinear inner product \( \langle f, g \rangle = < f, g^* > \) should be symmetric in the sense of superalgebra: \( (g, f) = (f, g) \) if \( f \) and \( g \) are even, \( (g, f) = -(f, g) \) if \( f \) and \( g \) are odd.) Almost all consideration above can be repeated with some changes. In particular, one can relate \( IGr(H) \) to the Fock space \( F \) defined as a space of antiholomorphic functionals \( \Phi(a^*) \) on \( \Pi H \) satisfying the condition (13). Let us assume that the space \( H \) is realized as a space of functions on measure space \( S \) taking values in \( \mathbb{C}^{p|q} \). (In other words elements of \( H \) are functions \( f(x, \alpha) \) of \( x \in S \) and discrete index \( \alpha \); we assume that \( f(x, \alpha) \)
is even for $1 \leq \alpha \leq p$ and $f(x, \alpha)$ is odd for $p + 1 \leq \alpha \leq p + q$. We will say that $\alpha$ is a superindex, taking $p$ even and $q$ odd values.) One can represent an element of $\mathcal{F}$ as an expression on the form

$$
\Phi(a^*) = \sum_{\alpha_1, \ldots, \alpha_n} \frac{1}{(n!)^{1/2}} \sum_{\alpha_1, \ldots, \alpha_n} \Phi_n(x_1, \alpha_1, \ldots, x_n, \alpha_n)a^*(x_1, \alpha_1)\ldots a^*(x_n, \alpha_n)d^n x
$$

obeying

$$
\sum_{\alpha_1, \ldots, \alpha_n} |\Phi_n(x_1, \alpha_1, \ldots, x_n, \alpha_n)|^2 d^n x < \infty.
$$

(Here $x_i \in S$, $\alpha_i$ is a superindex taking $p$ even values and $q$ odd values, the function $\Phi$ is antisymmetric with respect to transposition $(x_i, \alpha_i)$ and $(x_j, \alpha_j)$ if $\alpha_i$ and $\alpha_j$ are even indices and symmetric in all other cases.) Multiplication by $\sum_{\alpha} \int f(x, \alpha)a^*(x, \alpha)dx$ determines an operator $a^\dagger(f)$ acting on $\mathcal{F}$ and linearly depending of $f \in \mathcal{H}$. Operators $a^\dagger(f)$ together with operators $a(f) = \sum_{\alpha} \int f(x, \alpha)a(\alpha)^* dx$ generate a superanalog of Clifford algebra. We can repeat the definition of linear subspace Ann $\Psi \subset \mathcal{H}^2$ and prove that Ann $\Psi$ is isotropic. (Here $\Psi \in \mathcal{F}$.) Representing $V \in IGr^{(0)}(\mathcal{H})$ as an image of linear map $\varphi \to (A\varphi, B\varphi)$, where $A$ is Fredholm operator of index 0, $B$ is a Hilbert-Schmidt operator we obtain a condition of isotropicity and a condition that $V \subset$Ann$\Psi$. These conditions coincide with (15) and (14) correspondingly (up to irrelevant signs). For generic $V$ we can assume that $A_{kl}$ is diagonal, $A_{ii} = 0$ for $i \leq s$, $A_{ii} = 1$ for $i > s$. Then we can solve the analog of Equation (16) and obtain the expression (18) for the functional $\Psi = \Psi_V$; the solution is unique up to a factor. (Of course, (18) is not equivalent to (19) in general case.)

We see that the functional $\Psi$ can be considered as an element of $\mathcal{F}$ only in the case when the odd-odd block of the matrix $A$ is invertible. A map $V \to \Psi_V$ can be considered as a generalized function on $IGr^{(0)}(\mathcal{H})$ with values in $\mathcal{F}$. (More precisely, this function takes values in appropriate extension of the Fock space.)

Analogously we obtain a generalized function $IGr^{(0)}(\mathcal{H}^m) \to \mathcal{F}^\otimes m$. It is easy to prove that this function can be considered as a (generalized) algebra over $IGr^{(0)}(\mathcal{H}^m)$.

If $V \in IGr^{(0)}(\mathcal{H})$ is represented as an image of a map $\varphi \to (A\varphi, B\varphi)$, where $A$ is invertible operator, $A - 1$ belongs to trace class and $B$ is a Hilbert-Schmidt operator, we can write

$$
\Psi_V = (\det A)^{1/2}e^{-\frac{1}{2}(a^*, A^{-1}B a^*)}
$$

(20)

Of course, det here and further stands for superdeterminant (Berezinian). For general $A$ one can get $\Psi_V$ taking a limit in (20), as we explained above in the case when $\mathcal{H}$ is a ordinary Hilbert space. Of course all our consideration determine $\Psi_V$ only up to a factor. Using the embedding of ordinary Grassmannian into
isotropic Grassmannian we can construct a function $V \to \Psi_V$ defined for $V \in \text{Gr}(H)$ and taking values in Fock space (if $H$ is a superspace then $\Psi_V$ takes values in an appropriate extension of Fock space). This function determines a (generalized) algebra over Grassmannian MO. One can define $\Psi_V$ only up to a factor; however one can give an unambiguous definition of $\Psi_V$ on $\tilde{\text{Gr}}(H)$ naturally embedded into $\text{Gr}(\tilde{H})$ (see below). We mentioned already that the maps $\Psi_V$ determine an algebra over MO $\text{Gr}(H^k)$. One can reformulate the statement above saying that the maps $\Psi_V$ on $\tilde{\text{Gr}}(H^k) \subset \text{Gr}(\tilde{H}^k)$ specify a $Q$-extension of this algebra (a $Q$-algebra over $Q$-MO $\tilde{\text{Gr}}(H^k)$).

One should emphasize, that our consideration of super Grassmannian and its connection with Fock space was neither rigorous, nor complete. More detailed treatment of related questions can be found in [17], [18], [19].

Let us discuss briefly more general way to construct generalized algebras over Grassmannian MO's. Every algebra over $\text{Gr}(H^n)$, $H = L^2(S^1)$ determines an algebra over MO $P_n$ of surfaces with disks (a conformal field theory), because $P_n$ is embedded into $\text{Gr}(H^n)$ by means of Krichever construction. Conversely, a conformal field theory can be extended to an algebra over $\text{Gr}(H^n)$ if this theory is defined by means of quadratic Lagrangian (free fermions or bosons, $bc$-sistem, $\beta\gamma$-system or any combination of these theories). If a CFT $(P_n, E, \alpha_n)$ has non-vanishing central charge then $\alpha_n$ should be considered not as a map $P_n \to E$, but as a section of a bundle $E \otimes (\text{det})^k$ where $\text{det}$ stands for so called determinant bundle over $P_n$. It follows from well known results about determinant bundles over $P_n$ and $\text{Gr}(H^n)$ that an extension of a conformal field theory having vanishing central charge to the Grassmannian is an algebra in strict sense (i.e. there exists an unambiguous definition of a vector corresponding to an element $V \in \text{Gr}(H^m)$).

Using the same idea one can check the above mentioned fact that $\Psi_V$ can be defined on $\tilde{\text{Gr}}(H)$ unambiguously.

Acknowledgments. I am indebted to L. Dickey, M. Duflo, M. Kontsevich, M. Mulase, M. Vergne, A. Voronov and B. Zwiebach for useful discussions.

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