Traversable wormholes in $f(R,T)$ gravity with conformal motions

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To find more deliberate $f(R,T)$ astrophysical solutions, we proceed by studying wormhole geometries under the assumption of spherical symmetry and the existence of a conformal Killing symmetry to attain the more acceptable astrophysical results. To do this, we consider a more plausible and simple model $f(R,T) = R + 2\chi T$, where $R$ is the Ricci scalar and $T = -\rho + p_r + 2p_t$ denotes the trace of the energy-momentum tensor of the matter content. We explore and analyze two cases separately. In the first part, wormhole solutions are constructed for the matter sources with isotropic pressure. However, the obtained solution does not satisfy the required wormhole conditions. In the second part, we introduce an EoS relating with pressure (radial and lateral) and density. We constrain the models with phantom energy EoS i.e. $\omega = p_r/\rho < -1$, consequently violating the null energy condition. Next, we analyze the model via $p_t = n p_r$. Several physical properties and characteristics of these solutions are investigated which are consistent with previous references about wormholes. We mainly focus on energy conditions (NEC, WEC and SEC) and consequently for supporting the respective wormhole geometries in details. In both cases it is found that the energy density is positive as seen by any static observer. To support the theoretical results, we also plotted several figures for different parameter values of the model that helps us to confirm the predictions. Finally, the volume integral quantifier, which provides useful information about the total amount of exotic matter required to maintain a traversable wormhole is discussed briefly.

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I. INTRODUCTION

Traversable Lorentzian wormholes are hypothetical tunnels in space-time that connects two regions of the same or disjointed universes. These problems can be attributed in classical general relativity in which observers may freely traverse. Since wormhole has a long history, but it was developed mainly with the seminal paper by Morris and Thorne [1], in 1988 as a toy model allowing for interstellar travel. In particular, these geometries have a minimal surface area linked to satisfy flare-out condition, which is called throat of the wormhole. A stress-energy tensor that violates the null energy condition is involving to grip such a wormhole open [2]. Roughly speaking, the matter that violates the weak/null energy conditions called ‘exotic matter’. Such strange objects exists both in the static [3] and dynamic [4] cases, and sustained by a single fluid component. The violation of the energy conditions have been supported by many arguments like the quantum field theories such as the Casimir effect, Hawking evaporation and scalar-tensor theories. Though the usage of exotic matter is a problematic issue. Visser et al. [5] have proposed ‘volume integral quantifier’ that how to quantify the total average null energy conditions for wormhole maintenance.

However, a static wormhole without violating the energy conditions in the framework of Einstein General Relativity is still an open problem, which can be motivated to minimize the usage of exotic matter by applying the cut and paste technique, which was proposed by Visser [6]. The proposal was to restrict the exotic fluid at the wormhole throat. There were another solution came from Kuhfittig [7] to hamper the exotic fluid of an arbitrary thin region by imposing a condition on $b'(r)$ to be close to one at the throat.

One may also follow a more conventional method to address the issue in an alternatives theories of gravity. The physical incentives for these amendments of gravity are based on gravitational actions which are linked to the possibility of a more realistic illustration of the gravitational fields near curvature singularities. The main purpose of this approach lies on the assumption that matter threading the wormhole satisfies the energy conditions. Due to the effective stress-energy tensor, the field equations have to rewritten in a form that represented as a sum of the standard fluid plus the new terms coming from the modified theory. In this context, several wormhole solutions were analyzed in various modified gravity theories such as $f(R)$ gravity [8], $f(T)$ gravity wormhole with noncommutative geometry [9], $f(T)$ gravity [10], noncommutative geometry [11], Lovelock solutions [12] and in others.
In this article, we are particularly interested in \( f(R,T) \) gravity [13], where the Lagrangian is an arbitrary function of Ricci scalar \( R \) and the trace of the energy-momentum tensor \( T \). This theory has been tested from cosmology to astrophysics and are more manageable compared to \( f(R) \) theories. But, a serious shortcoming in this modification has been the non-conservation of the energy-momentum tensor. For detailed review of \( f(R) \)-gravity one may refer to [14]. In the following, a static wormhole solution has been obtained by Moraes & Sahoo [15]. Also, a charged wormholes in \( f(R,T) \) gravity has been proposed recently in [16].

The theoretical construction of wormhole geometries lies on the fact that one has a desired metric, which have to solve by fixing the form of the metric potential functions or by using a precise equation of state that relates the pressure with the energy density, and then solve Einstein’s field equations. In our work an exact solutions are deduced using static conformal symmetries. In section II, we briefly review the field equations for a one-parameter group of conformal motions, the EoS for isotropic pressure and linear EoS relating the energy density and the pressure anisotropy in section VI. Finally, in Section VII, we conclude.

**II. BASIC MATHEMATICAL FORMALISM OF THE \( f(R,T) \) THEORY**

In this section, we start by writing the general action for \( f(R,T) \) modified gravity in four-dimensional space-time. The full action is given by Harko et al [13] (with geometrized units \( c = G = 1 \))

\[
S = \frac{1}{16\pi} \int d^4x f(R,T)\sqrt{-g} + \int d^4x \mathcal{L}_m \sqrt{-\gamma},
\]

where \( f(R,T) \) is an arbitrary function depends on a generic function of \( R \) and \( T \), the Ricci scalar and the trace of the energy momentum tensor \( T_{\mu\nu} \), respectively. From the matter Lagrangian density \( \mathcal{L}_m \), we defined the energy-momentum tensor as follows

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}.
\]

Following the argument in [13], we assume that the Lagrangian density \( \mathcal{L}_m \) depends only on the metric components \( g_{\mu\nu} \) and not on its derivatives, we obtain

\[
T_{\mu\nu} = g_{\mu\nu} \mathcal{L}_m - 2\frac{\partial (\mathcal{L}_m)}{\partial g^{\mu\nu}}.
\]

Now, by variation of the action \( S \) given in Eq. (2) with respect to the metric \( g_{\mu\nu} \), to obtain the gravitational field equation for \( f(R,T) \) gravity as:

\[
f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R(R,T) = 8\pi T_{\mu\nu} - f_T(R,T) T_{\mu\nu} - f_T(R,T) \Theta_{\mu\nu},
\]

where \( f_R(R,T) = \partial f(R,T)/\partial R \), \( f_T(R,T) = \partial f(R,T)/\partial T \), \( \Box = \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)/\sqrt{-g} \), \( R_{\mu\nu} \) is the Ricci tensor, \( \nabla_\mu \) denotes the covariant derivative with respect to the metric \( g_{\mu\nu} \) and \( \Theta_{\mu\nu} = g^{\alpha\beta} \delta T_{\alpha\beta}/\delta g^{\mu\nu} \)

Performing a covariant divergence of (4) which yield [14]
\[ \nabla^\mu T_{\mu\nu} = \frac{f_T(R, T)}{8\pi - f_T(R, T)} [(T_{\mu\nu} + \Theta_{\mu\nu}) \nabla^\mu \ln f_T(R, T) \nabla^\nu \Theta_{\mu\nu} - (1/2) g_{\mu\nu} \nabla^\nu T]. \] (6)

For this purpose we assume the matter content of the wormhole solution is an anisotropic fluid and one can write the energy momentum tensor as

\[ T_{\mu\nu} = (\rho + p_r) u_\mu u_\nu - p_t g_{\mu\nu} + (p_r - p_t) g_{\mu\nu}, \] (7)

where \( \rho \) is the energy density with \( p_r \) and \( p_t \) representing the radial and tangential pressures of the fluid, \( u^\mu \) is the four-velocity such that \( u^\mu u_\mu = 1 \) and \( u^\mu \nabla_\mu u = 0 \). In this way, one can choose the matter Lagrangian density \( \mathcal{L}_m = -\mathcal{P}_P \), where \( \mathcal{P} = \frac{1}{2} (p_r + 2p_t) \) which is more generic, in the sense that they do not imply the vanishing of the extra force, which yields \( \Theta_{\mu\nu} = -2T_{\mu\nu} - \mathcal{P} g_{\mu\nu} \).

In the present work, we focus our attention on the simplified and linear functional form of \( f(R, T) \) gravity, the covariant derivative of the energy-momentum tensor this way, one can choose the matter Lagrangian density \( \mathcal{L}_m \) to be:

\[ G_{\mu\nu} = 8\pi T_{\mu\nu} + \chi T g_{\mu\nu} + 2\chi (T_{\mu\nu} + pg_{\mu\nu}), \] (8)

where \( G_{\mu\nu} \) is the Einstein tensor. If we set \( \chi = 0 \), then one can easily recover the general relativistic result. It is straightforward to see that for the particular choice of \( f(R, T) = R + 2\chi T \), Eq. (4) leads to the form

\[ (8\pi + 2\chi) \nabla^\mu T_{\mu\nu} = -2\chi \left[ \nabla^\mu (pg_{\mu\nu}) + \frac{1}{2} g_{\mu\nu} \nabla^\mu T \right]. \] (9)

Regarding the Bianchi identity, obviously in \( f(R, T) \) gravity, the covariant derivative of the energy-momentum tensor is not null in general. But substituting \( \chi = 0 \) in Eq. (9), one can see that the energy-momentum tensor is conserved as in case of general relativity.

### III. TRAVERSABILITY CONDITIONS AND GENERAL REMARKS FOR WORMHOLES

The spacetime ansatz for seeking traversable static spherically symmetric wormholes is the Morris-Thorne metric [1], which can be written as

\[ ds^2 = -e^{\nu(r)} dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \] (10)

where \( \Phi(r) \) and \( b(r) \) are the redshift and the shape functions, respectively. The redshift function \( \Phi(r) \) must be finite everywhere, in order to ensure the absence of horizons and singularities. The essential characteristics of a wormhole is the shape function \( b(r) \) which determine the shape of the wormhole must satisfy the condition \( b(r = r_0) = r_0 \) at the throat \( r_0 \) where \( r_0 \leq r \leq \infty \). For the existence of standard wormholes, the shape function should satisfy the “flaring-out condition”, given by

\[ \frac{b(r) - b'(r)}{b^2(r)} > 0, \] (11)

which reduces to \( b'(r_0) < 1 \) at the throat \( r = r_0 \). Here the prime denotes the derivative with respect to the radial coordinate \( r \). Moreover, finiteness of the proper radial distance, \( \ell(r) \) defined by

\[ \ell(r) = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} \] (12)

is required to be finite everywhere. It is important to note that \( |\ell| \) the proper distance is greater than or equal to the coordinate distance, i.e. \( |\ell(r)| \geq r - r_0 \) where \( \pm \) signs refer to the two asymptotically flat regions which are connected by the wormhole. Since, \( \ell \) decreases from \( \ell = +\infty \) to at the throat of the wormhole \( \ell = 0 \), and then from \( \ell = 0 \) to \( \ell = -\infty \).

Following the metric Eq. (10), the Einstein tensor, \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \) then reduce to the following non-zero components

\[ G_0^0 = \frac{b'(r)}{r^2}, \] (13)

\[ G_1^1 = -\frac{b(r)}{r^3} + \left( 1 - \frac{b(r)}{r} \right) \frac{\nu'}{r}, \] (14)

\[ G_2^2 = \frac{1}{4} \left( 1 - \frac{b(r)}{r} \right) \left[ \nu'^2 + 2\nu'' - 2 \frac{b(r) - b'}{r^2(r - b)} \right] \] (15)

\[ G_3^3 = G_2^2, \] (16)

where primes stand for derivation with respect to the radial coordinate \( r \).

### IV. THE CONFORMAL KILLING VECTOR (CKV)

Construction of wormhole can be straightforwardly generalised to conformal theories containing matter fields. Based on the assumption that spherically symmetric static space-time possesses a conformal symmetry and identify its essential mathematical structure, one can simplify the treatment of the problem and define its basic mathematical structure [19, 20]. The existence of a Killing vector laid constraints on the influences of curvatures of the manifold and symmetry. If we consider a static metric, the vector fields \( \xi \) and \( \psi \) are not necessary
to be static. So, the Eq. (1) can be written in a simple way as

$$\mathcal{L}_\xi g_{ij} = \xi_{i;j} + \xi_{j;i} = \psi g_{ij},$$  \quad (17)

where the Lie derivative operators $\xi_i = g_{ik}\xi^k$ and $\mathcal{L}$ describes the interior gravitational field of a wormhole configuration. Constants of the motion may be determined by the Killing vectors i.e. quantities that will be constant along any given geodesic. Furthermore, the conformal vectors can be obtained when (i) $\psi = 0$, then Eq. (17) gives the Killing vector, (ii) $\psi$ = constant gives homothetic vector, and (iii) when $\psi = \psi(x, t)$ gives conformal vectors.

After introducing conformal Killing vector Eq. (17) into the metric Eq. (10), without a loss of generality provides the following solutions

$$\xi^1 \nu' = \psi(r), \quad \xi^4 = \text{const.}, \quad \xi^1 = \frac{\psi r}{2}, \quad \xi^1 \lambda' + 2\xi^1_3 = \psi(r),$$

where 1 and 4 represents the spatial and temporal coordinates $r$ and $t$, respectively.

These, in turn, imply that

$$e^\nu = C_2 r^2, \quad (18)$$

$$\left(1 - \frac{b(r)}{r}\right)^{-1} = \left[C_4 \psi\right]^2, \quad (19)$$

$$\xi^1 = C_1 \delta^1 + \frac{\psi r}{2} \delta^1, \quad (20)$$

where $C_1$, $C_2$ and $C_3$ are constants of integration. Notice that if the Eq. (19) written in terms of the shape function $b(r)$, then the conformal factor is zero at the throat, i.e. $\psi(r_0) = 0$.

The strong constraints on the wormhole geometry will be imposed by the existence of conformal motions. Consider the above energy-momentum tensor and the Morris-Thorne metric Eq. (10), the generalized gravitational field equations (8) give the following field equations

$$\frac{b'}{r^2} = (8\pi + \chi) \rho - \chi (p_r + 2p_t), \quad (21)$$

$$\left[1 - \frac{b}{r}\right] \frac{\nu'}{r} - \frac{b(r)}{r^3} = \chi \rho + (8\pi + 3\chi) p_r + 2\chi p_t, \quad (22)$$

$$\frac{1}{4} \left[1 - \frac{b}{r}\right] \left[\nu'' + 2\nu'' - 2\frac{b'^r - b}{r^2(r - b)} - \frac{b'r - b}{r(r - b)} b'^r\right] + \frac{2\nu'}{r} = (\rho + p_r) \chi + (8\pi + 4\chi) p_t. \quad (23)$$

Thus, using the expression (18)-(20), in the above Eqs. (21)-(23), we obtain a set of field equations as follows

$$-\frac{2\psi \psi'}{r^2 C_3^2} - \frac{\psi^2}{r^2 C_3^2} + \frac{1}{r^2} = \rho_{\text{eff}}, \quad (24)$$

$$\frac{3\psi^2}{r^2 C_3} - \frac{1}{r^2} = p_{\text{eff}}, \quad (25)$$

$$\frac{\psi^2}{r^2 C_3} + \frac{2\psi \psi'}{r^2 C_3} = P_{\text{eff}}, \quad (26)$$

In addition to other essential characteristics of a wormhole solution, the violation of the null energy condition (NEC) at the throat of the wormhole is a generic feature. Therefore, such energy conditions are deemed important since they lead to physical requirements on matter.

Considering the $f(R)$ gravity, Garcia and Lobo [22] showed that nonminimal coupling minimizes the violation of the NEC of normal matter at the throat. Moreover, Einstein-Cartan theory attracted a good deal of attention in wormhole solution without invoking exotic matter [23]. Quantum effects also produce violations of the classical energy conditions, amongst which the popular one is Casimir effect.

In the context of the local energy conditions, we examine the the violation of NEC, $T_{\mu\nu}k^\mu k^\nu \geq 0$, where $k^\mu$ is any null vector and $T_{\mu\nu}$ is the usual Hilbert stress-energy-momentum tensor. In combination to the above
expression we have
\[
8\pi (\rho_{\text{eff}} + p_{\text{eff}}) = -\frac{2\psi\psi'}{rC_3} + \frac{2\psi^2}{r^2C_3^2},
\] (27)
which evaluated at the throat imposes the following condition \((\psi^2)'>0\).

V. THIN SHELL AROUND TRAVERSABLE \(f(R,T)\) WORMHOLE

We shall model specific static wormholes by matching an interior geometry, with an exterior Schwarzschild vacuum solution, at a junction interface \(\Sigma = \Sigma_+ = \Sigma_-\). Our aim here is to restrict the dimensions of these wormholes not to arbitrarily large. For this, the exterior Schwarzschild is given by
\[
ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\] (28)
which we shall match with the interior spacetime given in Eq. (10).

Following the standard junction-condition formalism in \((3+1)\)-dimensional spacetime [24], one can consider two pseudo-Riemannian manifolds with a radius greater than the event horizon radius, and paste them at the hypersurface to create a geodesically complete manifold. If such boundaries are identified, then a natural match of manifolds can be done, with two regions connected by a throat of radius, where the exotic matter is located [25]. Beyond GR, the junction formalism requires to be generalized and several conditions that should be fulfilled for the specific theory of gravity under consideration. For example in \(f(R)\) gravity, the junction conditions tend not to always coincide with those of general relativity [26] (see also Refs. [27] for \(f(T)\) gravity).

To understand the above in some details we would like to point out a special feature of the thin-shell structure. More tactically for a geodesically complete thin-shell wormholes, the Riemann tensor is divergent at the thin-shell where the throat is located [28]. To see this let \(\Sigma\) be a non-null hypersurface layer, and suppose the coordinate system on both sides of the hypersurface to be the same then \(\delta_{\mu}^\nu\) defines the jump of a quantity \(Z\) as
\[
[Z] = Z(X^+_\mu)|_\Sigma - Z(X^-\mu)|_\Sigma.
\] (29)
Then, the distribution of matter reads
\[
T_{\mu\nu} = \Theta(x)T_{\mu\nu}^+ + \Theta(-x)T_{\mu\nu}^-|_\Sigma,
\] (30)
so that the geodesics cross \(\Sigma\) when \(x = 0\), and \(\delta(x)S_{\mu\nu}\).

For further details, we refer the reader to [29]. The quantity \(\Theta(x)\) is known as the Heaviside step function whereas \(S_{\mu\nu}\) is the surface stress-energy tensor on the thin-shell.

It is interesting that this way the curvature of spacetime becomes divergent at \(\Sigma\) for thin-shell wormholes (because the Riemann tensor is singular). But this divergence is physically interpreted as a surface layer with a stress-energy tensor \(T_{\mu\nu}|_\Sigma\) on it. Therefore, the existence of curvature divergences exists at the wormhole throat.

VI. CONFORMAL SYMMETRY WORMHOLE

In general, to solve the three field equations Eqs. (24-26) with the following four unknown functions of \(r\),
namely, \( \rho, p_r(r), p_t(r) \) and \( \psi \) is mathematically well-defined problem. For obtaining an explicit solution one has to specify or determine the EoS, the shape function \( b(r) \) etc. by implementing some physical conditions. We employ the following approach to extract and analyze the solutions as below.

### A. On spherical wormhole with isotropic pressure

The case of an isotropic wormhole i.e. when \( p_r = p_t \) is particularly simple one, yet it provides enough interesting results [30]. In order to analyze solutions we shall now take into consideration Eqs. (24) and (25), which yield

\[
\psi^2 = \frac{C_3^2}{2} \left[ 1 - \left( \frac{r}{r_0} \right)^2 \right],
\]

where the constant term is determined by imposing the condition \( \psi(r_0) = 0 \) at the throat of the wormhole. Now, using the condition in Eq. (19), we obtain the form of shape function as

\[
b(r) = \frac{1}{2} \left[ 1 + \left( \frac{r}{r_0} \right)^2 \right].
\]

The aim of this section is to see the behavioral effects of \( b(r) \) and its derivative \( b'(r) \). Here the throat of wormhole is located at \( r_0 \). From the obtained shape function (32), one can easily check that \( b'(r_0) = 2 \neq 1 \).

In principle, flaring-out condition at the throat should obey the following inequality \( b'(r_0) < 1 \), which is not reflecting for isotropic pressure wormhole solution.

### B. Wormhole solutions with specific choices

In the following analysis, we consider the relationship involving specific form of equation of state and anisotropy to solve the field equations.

### 1. WH1: Model with \( p_r = \omega \rho \)

With the definitions of \( \rho_{\text{eff}}, p_{\text{eff}} \) and \( \mathcal{P}_{\text{eff}} \), one can rewrite the field equations (24)-(26) further in the following form:

\[
\rho(r) = \frac{2C_3^2(\chi + 2\pi) - \psi[r(3\chi + 8\pi)\psi' + 4\pi\psi]}{4C_3^2r^2(\chi + 2\pi)(\chi + 4\pi)},
\]

\[
p_r(r) = \frac{\psi[4(\chi + 3\pi)\psi' - 2C_3^2(\chi + 2\pi)]}{4C_3^2r^2(\chi + 2\pi)(\chi + 4\pi)},
\]

\[
p_t(r) = \frac{\psi[r(3\chi + 8\pi)\psi' + 4\pi\psi]}{4C_3^2r^2(\chi + 2\pi)(\chi + 4\pi)}.
\]

Let us start for searching an exact wormhole model by considering a linear \( \text{EoS} \) which is characterized by \( p_r = \omega \rho \). Now, if we take account of (33) and (34) then, after integration, we can recover the functional form of \( \psi(r) \), which yield

\[
\psi(r) = \frac{1}{\sqrt{2\chi + 2\pi(\omega + 3)}} \left\{ \exp \left[ 4\{\chi + \pi(\omega + 3)\} \right] A + \frac{2\log r}{\chi - 3\chi\omega - 8\pi\omega} \right\} + C_3^2(\chi + 2\pi)(\omega + 1) \right\}^{\frac{1}{2}},
\]

and the corresponding shape function takes the form

\[
b(r) = 1 - \frac{\psi^2}{C_3^2} = 1 - \frac{C_3^{-2}}{2\chi + 2\pi(\omega + 3)} \times \left\{ \exp \left[ 4\{\chi + \pi(\omega + 3)\} \right] A + \frac{2\log r}{\chi - 3\chi\omega - 8\pi\omega} \right\} + C_3^2(\chi + 2\pi)(\omega + 1) \right\}.
\]

In this case we have for \( r \geq r_0 \) the metric component \( g_{rr}^{-\frac{1}{2}} \geq 0 \) if \( \omega < -1 \). In Fig. (1), we show the behavior of shape function for \( \omega = -2 \). This result shows that a wormhole solution requires a phantom-energy background, i.e. \( \omega < -1 \). The use of phantom-energy is not new in wormhole physics (see refs. [31]). The energy density in cosmology setting related to the phantom energy is considered positive, \( \rho > 0 \), and we shall maintain this condition.

The graphical behavior of the \( b(r) - r \), \( b(r)/r \), and \( b'(r) \) are depicted in Figs. 2-4 for WH1. From Fig. 2, we find that \( b(r) - r \) cuts the \( r \)-axis, with the throat at \( r_0 = 0.757 \). We also observe that \( b'(r) < 1 \), which obeys the flaring out condition appear in Fig. 4. Moreover, we can see directly from Fig. 2 that the asymptotic behavior \( b(r)/r \rightarrow 0 \) as \( r \rightarrow \infty \), but the redshift function does not approach zero as \( r \rightarrow \infty \), which is expected for conformally symmetric wormhole [32]. This means the wormhole spacetime is not asymptotically flat, so one needs to match these interior geometries to an exterior vacuum spacetime, at a junction interface which we have discussed in see V.
Thus, in this case the stress-energy tensor components are given by

\[
\rho(r) = \frac{1}{2\pi^2|\chi + \pi(\omega + 3)|} \left\{ \frac{3C_3^{-2}}{\chi(3\omega - 1) + 8\pi\omega} + \frac{\chi + 2\pi}{\chi + 4\pi} \right. \\
\exp \left[ 4|\chi + \pi(\omega + 3)| \left( A + \frac{2\log r}{\chi - 3\chi\omega - 8\pi\omega} \right) \right] \right\},
\]

(38)

\[
p_r(r) = \omega\rho(r),
\]

(39)

\[
p_t(r) = \frac{r^{-2}}{2|\chi + \pi(\omega + 3)|} \left\{ \frac{\pi(\omega + 1)}{\chi + 4\pi} - \frac{3C_3^{-2}}{\chi(3\omega - 1) + 8\pi\omega} \right. \\
\exp \left[ 4(\chi + \pi(\omega + 3)) \left( A + \frac{2\log r}{\chi - 3\chi\omega - 8\pi\omega} \right) \right] \right\},
\]

(40)

To see in a more quantitative way we also analyzed the energy conditions. In Fig. 5, we present the graphical behavior of the NEC, WEC and the SEC in terms of the \(\rho, P_r\) and \(P_t\), for different values of parameters \(A = 1.23, \chi = -2, \omega = -2\) and \(c_3 = 7.74\). Fig. 5, shows the validity of \(\rho \geq 0\) (blue). With the above solution we also found that \(\rho + p_r < 0\) but \(\rho + p_t > 0\) that ensure the violation of NEC and this lead to the violation of WEC also. One can see from figure that the SEC (brown) is also violated.

Now, we can construct embedding diagrams to represent a wormhole and extract some useful information for the obtained shape function, \(b(r)\).

Considering a fixed moment of time, \(t = \text{const} \& \theta = \pi/2\) and embed the metric into three-dimensional Euclidean space, we obtain the embedding surface which is given by

\[
\frac{dz(r)}{dr} = \pm \frac{1}{\sqrt{r/b(r) - 1}} = \pm \frac{b(r)/r}{\sqrt{1 - b(r)/r}}.
\]

(41)

For this particular case, the above equation becomes

\[
z(r) = \pm \int \frac{C_3}{\psi} \left(1 - \frac{\psi^2}{C_3^2}\right) dr,
\]

(42)

where

\[
z(r) = \frac{\sqrt{2\zeta C_3^2 (\chi + 2\pi)(\omega + 1)}}{\chi(3\omega - 1) + 8\pi\omega}
\]

\[
\zeta = \chi + \pi(\omega + 3) ; \quad \sigma = \frac{8\zeta}{\chi(3\omega - 1) + 8\pi\omega}
\]

\[
\eta = C_3^2(\chi + 2\pi)(\omega + 1) ; \quad p = -3\chi\omega + \chi - 8\pi\omega
\]

\[
q = 3(8\pi - \chi(\omega - 3)).
\]

The embedded surface and surface of the revolution for \(z(r)\) about the \(Z\)–axis are shown in Figs. 7 and 8.
2. WH2: Model with $p_t = np_r$

Here, we investigate the wormhole solution for a particularly interesting anisotropy, already explored in [15, 33], given by

$$p_t = np_r,$$  \hspace{1cm} (44)

where the state parameter $n$ is a constant. With this assumption and solving the differential equations (33)-(35), as the same procedure for WH1, the function $\psi(r)$ takes the form

$$\psi(r) = \frac{1}{\sqrt{2n\chi + \pi(6n - 2)}} \left[ e^{4B_1 r^\Lambda} [(n + 3)\chi + 8\pi]^\Lambda + C_3^2 n(\chi + 2\pi)^{1/2} \right],$$  \hspace{1cm} (45)

where $\Lambda = \frac{8(n\chi + \pi(3n - 1))}{(n + 3)\chi + 8\pi}$ and $\Omega = n\chi + \pi(3n - 1)$.

Using the definition of $b(r)$ provided in Eq. (19), one can find the shape function $b(r)$ as

$$b(r) = \frac{1}{r} - \frac{e^{4B_1 r^\Lambda} [(n + 3)\chi + 8\pi]^\Lambda + C_3^2 n(\chi + 2\pi)^{1/2}}{C_3^2 [2n\chi + \pi(6n - 2)]}. \hspace{1cm} (46)$$

As we can see from Fig. 2, that the solutions are asymptotically flat, i.e. $b(r)/r \rightarrow 0$ as $r \rightarrow \infty$, because of decreasing graphs with increasing $r$. In addition, we plot in Figs. 3 and 4, the characteristic picture of the shape function. The red curve represents a regular wormhole solution which cuts $r$-axis at 0.494 is the “throat” of the WH2. As seen in the figure 4, that $b(r) < 1$, which obeys the flaring out condition. Clearly, in this case also for $r \rightarrow \infty$, the redshift function does not approach zero. Thus, one needs to match this solution to an exterior spacetime at a junction interface, $a > 2M$.

Now, the stress-energy tensor components for the EoS are given by

$$\rho(r) = \frac{1}{2C_3^2 r^2(\chi + 4\pi)\Omega} \left[ C_3^2 [n\chi + \pi(2n - 1)] - 3n \right]$$

$$\left(\chi + 4\pi e^{4B_1 r^\Lambda} [(n + 3)\chi + 8\pi]^{\Lambda-1}\right),$$  \hspace{1cm} (47)

$$p_r(r) = \frac{3(\chi + 4\pi)e^{4B_1 r^\Lambda} [(n + 3)\chi + 8\pi]^{\Lambda-1} + C_3^2}{2C_3^2 (\chi + 4\pi)\Omega},$$  \hspace{1cm} (48)

$$p_t = np_r.$$

To determine the energy conditions we have plotted graphs, and Fig. 6 illustrates the behaviour of the null, weak and strong energy conditions. Clearly, in this case we have $\rho > 0$ (blue curve). We are mostly interested in the NEC, because its violation implies the violation of WEC also. In Fig. 6, $\rho + p_r < 0$ but $\rho + p_t > 0$ i.e. violation of NEC and consequently the WEC, are violated. Interestingly we note that SEC (dashed curve) is satisfied in this case. All solutions are characterized by considering parameter values $B = -0.44$, $\chi = -2$, $n = -0.4$ and $c_3 = -10$ for WH2.

To further interpret these results let us bring out attention on the embedded surface, which is determined from (41) and found as

$$z(r) = \frac{r \sqrt{\sqrt{2n\chi + \pi(6n - 2)}}[(n + 3)\chi + 8\pi]^\Lambda + C_3^2 [n(\chi + 2\pi)^{1/2} \right]}{\sqrt{2n\chi + \pi(6n - 2)}} \left[ - e^{4B_1 r^\Lambda} [(n + 3)\chi + 8\pi]^{\Lambda} + \left\{C_3^2 \left[ - n(\chi + 2\pi)^{1/2} \right] \cr (n + 3)\chi + 8\pi + 2[\tau(x + \pi(3n - 1))] \right\} \int_{\tau(r)}^{\tau(\infty)} \right] \times$$

$$[3(n + 1)\chi + \pi(8n + 4)]_2 F_1(1, 1; \tau; -3)$$

$$\left\{ \left[ - e^{4B_1 r^\Lambda} [(n + 3)\chi + 8\pi]^{\Lambda} + \left[ - \frac{C_3^2 n(\chi + 2\pi)^{1/2}}{(n + 3)\chi + 8\pi} \right] \right] \right\},$$

where for notational simplicity we use

$$\Sigma = \frac{3(3n\chi + 8\pi n + \chi)}{(n + 3)\chi + 8\pi}, \Gamma = \frac{5(n + 3)\chi + 4\pi (3n - 1)}{8[\tau(x + \pi(3n - 1))]},$$

$$\Theta = \frac{3(3n\chi + 8\pi n + \chi)}{8[\tau(x + \pi(3n - 1))]} : \Xi = \frac{e^{4B_1 r^\Lambda} [(n + 3)\chi + 8\pi]^{\Lambda}}{C_3^2 [n(\chi + 2\pi)^{1/2}]}.$$  \hspace{1cm} (50)

which is again well-defined. The embedding diagram and its surface revolution about $Z$-axis are shown in Figs. 7 and 8.

VII. VOLUME INTEGRAL QUANTIFIER

It is convenient to consider the “volume integral quantifier” to know how much of exotic matter is required to support a traversable Lorentzian wormhole on a local scale. This was first prompted by Visser et al [5]. Later, a more technical review was proposed in [34]. Quantifying the amount of exotic matter has been considered by the following defined integral $I_V = \int \rho(r) + p_t(r) \, dV$, and with a cut-off of the stress-energy at $a$ is given by

$$I_V = r \left[ 1 - \frac{b}{r} \right] \int \left( \frac{e^p}{1 - b/r} \right) \int_{r_0}^{a} \left[ (1 - b) \ln \left( \frac{e^p}{1 - b/r} \right) \right] dV,$$  \hspace{1cm} (51)

where $dV = r^2 \sin \theta \, dr \, d\theta \, d\phi$, and the boundary term at $r_0$ vanishes by our construction as $b(r_0) = r_0$. Then, the volume-integral reduce to (see Ref. [18] for more details)

$$I_V = a \left[ 1 - \frac{b(a)}{a} \right] \int \left( \frac{e^p}{1 - b/a} \right) \int_{r_0}^{a} \left[ (1 - b) \ln \left( \frac{e^p}{1 - b/r} \right) \right] dV,$$  \hspace{1cm} (52)

Taking into account the redshift function $e^p = C_3^2 r^2$, and the form function, Eqs. (37) and (46), we obtain the
following expression

\[
I_V(WH1) = \left[ a \left( 1 - \frac{b(a)}{a} \right) \ln \left( \frac{e^{r(a)}}{1 - b/a} \right) - \frac{r(\chi + 2\pi)(\omega + 1)}{\zeta} \left( (\chi + 4\pi)(\omega + 1)e^{4A\chi}C^2\zeta[8\pi - \chi(\omega - 3)] \right) \right] - \frac{r^a}{p} - \frac{r}{2C^2\zeta} \log \left( \frac{e^{4A\chi}\rho - C^2(\chi + 2\pi)(\omega + 1)}{2C^2\zeta} \right)_{r_0} (53)
\]

\[
I_V(WH2) = \left[ a \left( 1 - \frac{b(a)}{a} \right) \ln \left( \frac{e^{r(a)}}{1 - b/a} \right) - \left\{ \frac{2(n - 1)(\chi + 4\pi)e^{4B\chi}\rho}{(3n\chi + 8\pi\zeta + (n + 3)\chi + 8\pi)^2} - 2C^2\zeta r(\chi + 2\pi)[(n + 3)\chi + 8\pi] + e^{4B\chi}[2(n + 3)\chi + 8\pi]\right\} \right] \log \left[ \frac{2C^2\zeta^2(\chi + \pi(3n - 1))}{\{3n\chi + 8\pi\}^2 + C^2\zeta^2(n + 3)\chi + 8\pi} \right] - \frac{1}{r_0} \left[ \frac{2C^2\zeta(\chi + \pi(3n - 1))}{\{n + 3\chi + 8\pi\}^{a - 1}} \right]_{r_0} , (54)
\]

where \( \tau = \frac{8\pi}{(a + 3\chi + 8\pi)^2} \). It is interesting to note that when \( a \to r_0 \) then \( I_V \to 0 \) for both cases. In fact, one can also observe that for WH1 if the parameter \( \omega \) arbitrary close to \(-1\), the integral may be infinitesimally small. These results fundamentally confirm the validity of conformally symmetric phantom wormhole solutions, as described in [35], where the violation of ANEC is arbitrarily small when the interior solution is matched to an exterior vacuum spacetime.

VIII. SUMMARY AND DISCUSSION

In the present paper, we investigate the possible existence of wormhole solutions in the framework of \( f(R, T) \) gravity under the assumption of spherical symmetry and the existence of a conformal Killing symmetry. To address the problem we consider a particular and simple model \( f(R, T) = R + 2\chi T \), where \( R \) is the Ricci scalar and \( T = -\rho + p_r + 2p_t \) denotes the trace of the energymomentum tensor of the matter content. Even within this simple theoretical model the field equations become extremely complicated, and therefore conformal symmetry is a more systematic approach in searching for exact analytic solution. The obtained solutions in this article are not asymptotically flat, where distribution of the exotic matter restricted to the throat neighborhood, and we consider a cut-off of the stress-energy tensor at a junction interface by matching an interior traversable wormhole geometry. In fact, we are successfully able to make the a particular asymptotically flat wormhole geometries where the dimensions are not arbitrarily large.

Next, we explore and analyze two cases separately. At the first part, the obtained wormhole solutions are constructed for the matter sources with isotropic pressure. However, showing explicitly that the solution violates the basic criteria for wormhole. Further, we proceed by introducing an EoS relating with pressure (radial and lateral) and density. We show the possibility of having traversable wormhole geometries supported by phantom energy. In this case, the energy density \( \rho \geq 0 \) is positive which consequently violates the null energy condition. However, we emphasize that when \( \omega \to -1 \) the volume integral quantifier would by itself become arbitrarily small i.e. theoretically it is possible to construct these geometries with vanishing amounts of ANEC. For our convenience we have also analyzed physical properties and characteristics of traversable wormholes by using graphical representation (see Fig. 1-8).

In the second part of the paper we obtain a similar picture for the models described by \( p_l = np_r \). Still in this case, obtained solution are violating the NEC and WEC with the energy density \( \rho \geq 0 \), but interestingly satisfying the SEC.

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