Mixmaster Chaoticity as Semiclassical Limit of the Canonical Quantum Dynamics

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Within a cosmological framework, we provide a Hamiltonian analysis of the Mixmaster Universe dynamics on the base of a standard Arnowitt-Deser-Misner approach, showing how the chaotic behavior characterizing the evolution of the system near the cosmological singularity can be obtained as the semiclassical limit of the canonical quantization of the model in the same dynamical representation.
The relation between this intrinsic chaotic behavior and the indeterministic quantum dynamics is inferred through the coincidence between the micro-canonical probability distribution and the semiclassical quantum one.
1 Introduction

The simplest and most interesting generalization of the Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology is the Bianchi IX model, whose geometry has the homogeneity constraint but the dynamics makes allowance of anisotropic evolution of different (linearly independent) spatial directions.

Belinski, Kalatnikov and Lifshitz (BKL) at the end of Sixties \(^1\) derived the oscillatory regime characterizing the asymptotic evolution near the cosmological singularity, describing via a discrete map the resulting chaos.

This Mixmaster \(^2\) dynamics, whose geometry is invariant under the \(SO(3)\) group, allows the line element to be decomposed as a FLRW model plus a gravitational waves packet \((3, 4)\) sufficiently far from the singularity.

The various approaches in terms of continuous variables (i.e. construction of an invariant measure for the system \((5, 6)\) and study of the system covariance \((7-11)\) have shown how this chaotic feature is invariant with respect to any choice of the temporal gauge and how in this sense it is intrinsic.

Such deep nature of the Mixmaster deterministic chaos and its very early appearance in the Universe evolution lead to believe in the existence of a relation with the quantum behavior the system performs during the Planckian era.

The aim of this paper is to give a precise meaning to this relation by constructing the semiclassical limit of a Schrödinger approach to the canonical quantization of the Arnowitt-Deser-Misner (ADM) dynamics whose corresponding probability distribution coincides with the (deterministic) microcanonical one.

In Section 2 are outlined the stochastic properties of the Mixmaster dynamics, being isomorphic to a billiard on a Lobachevsky plane and is given a stationary statistical distribution within the framework of the microcanonical ensemble.

In Section 3 we show the existence of a direct correspondence between the classical and quantum dynamics outlined by the common form of the continuity equation for the statistical distribution and the one for the first order approximation in the semiclassical expansion.

As a remarkable feature, both formalisms (the classical and the quantum one) are constructed in a generic temporal gauge.

2 Statistical Mechanics Approach

As well known (see \((12)\), the dynamics of the Bianchi type IX model, in terms of generic Misner-Chitré–like (MCl) variables \((\tau, \xi, \theta)\) and corresponding conjugate momenta, is summarized by the variational principle

\[
\delta \int \left( \rho_\xi \frac{d\xi}{d\tau} + \rho_\theta \frac{d\theta}{d\tau} - \frac{d\Gamma}{d\tau} \mathcal{H}_{ADM} \right) d\tau = 0 ,
\]  

(1)
where
\[ H_{ADM} = \sqrt{\varepsilon^2 + U} , \quad \varepsilon^2 = (\xi^2 - 1) p_\xi^2 + \frac{p_\theta^2}{\xi^2 - 1} \]  
(2)
and \( U(\tau, \xi, \theta) \) denotes a potential term which, asymptotically to the singularity, is modeled by the potential wall
\[ U_\infty = \Theta_\infty (H_l(\xi, \theta)) + \Theta_\infty (H_m(\xi, \theta)) + \Theta_\infty (H_n(\xi, \theta)) \]  
(3)
being \( H_a (a = l, m, n) \) the anisotropy parameters [10], such that the motion of the point Universe is restricted in the domain \( \Gamma_H \) (see Figure 1). Such a dynamical scheme has the relevant feature to let free the choice of the lapse function unless explicitly chosen the form of the function \( \Gamma \). The bounces against the potential walls and the instability of the geodesic flow on the Lobachevsky plane let the dynamics acquire a stochastic feature. Moreover in \( \Gamma_H \) the ADM Hamiltonian

becomes (asymptotically) an “energy-like” integral of motion \( dH_{ADM}/d\Gamma = 0 \), say \( H_{ADM} = \varepsilon = E = \text{const.} \).

As shown in [10, 11], the Jacobi metric associated with the principle (1) describes a point–universe moving over a Lobachevsky plane reduced by the potential walls (3) to a billiard.

The Statistics of the Mixmaster stochasticity is reformulated following the lines presented in [6, 8] and [11] to derive the invariant measure. This system is well-described by a microcanonical ensemble, whose Liouville invariant measure \( w_\infty \) can be obtained as a solution of the continuity equation
\[ \sqrt{\xi^2 - 1} \cos \phi \frac{\partial w_\infty}{\partial \xi} + \frac{\sin \phi}{\sqrt{\xi^2 - 1}} \frac{\partial w_\infty}{\partial \theta} - \frac{\xi \sin \phi}{\sqrt{\xi^2 - 1}} \frac{\partial w_\infty}{\partial \phi} = 0 . \]  
(4)
Over the reduced phase space\(^1\) \(\{\xi, \theta\} \otimes S^1_\phi\), \(w_{\infty}\) explicitly reads
\[
w_{\infty}(\xi, \theta, \phi) = \begin{cases} 
\frac{1}{8\pi^2} & \forall \{\xi, \theta, \phi\} \in \Gamma_H \otimes S^1_\phi \\
0 & \forall \{\xi, \theta, \phi\} \notin \Gamma_H \otimes S^1_\phi 
\end{cases}
\quad (5)
\]

Using generic MCl variables, the above invariant measure is independent of the choice of the temporal gauge, i.e. of the lapse function.

In order to restrict the invariant measure to the two-dimensional space \(\{\xi, \theta\}\), we stress how the original Hamiltonian equations in the asymptotic limit for which \(U \to U_\infty \Rightarrow \varepsilon = E = \text{const.}\) get in \(\Gamma_H\) the free geodesic motion \(^8\)  
\[
\frac{d\xi}{d\tau} = \frac{d\Gamma}{d\tau} \sqrt{\xi^2 - 1} \cos \phi, \quad \frac{d\theta}{d\tau} = \frac{d\Gamma}{d\tau} \sin \phi \sqrt{\xi^2 - 1}, \quad \frac{d\phi}{d\tau} = -\frac{d\Gamma}{d\tau} \xi \sin \phi. \quad (6)
\]

By this system, we easily rewrite the stationary continuity equation in the form
\[
\partial_\phi w_{\infty} + \frac{d\xi}{d\phi} \partial_\xi w_{\infty} + \frac{d\theta}{d\phi} \partial_\theta w_{\infty} = 0. \quad (7)
\]

When assuming the independence of the distribution function \(w_{\infty}\) from \(\phi\), then the normalization condition leads to the restricted function
\[
\varrho_{\infty}(\xi, \theta) \equiv \int_0^{2\pi} w_{\infty}(\xi, \theta, \phi) d\phi = 2\pi w_{\infty}(\xi, \theta). \quad (8)
\]
Thus, we get the following continuity equation for \(\varrho_{\infty}\)
\[
\sqrt{\xi^2 - 1} \cos \phi \frac{\partial \varrho_{\infty}}{\partial \xi} + \sin \phi \frac{\partial \varrho_{\infty}}{\sqrt{\xi^2 - 1} \partial \theta} = 0, \quad (9)
\]
where \(\phi\) plays the role of a parameter and the corresponding microcanonical solution on the whole configuration space \(\{\xi, \theta\}\) reads
\[
\varrho_{\infty}(\xi, \theta) = \begin{cases} 
\frac{1}{4\pi} & \forall \{\xi, \theta\} \in \Gamma_H \\
0 & \forall \{\xi, \theta\} \notin \Gamma_H 
\end{cases} \quad (10)
\]

We conclude observing how the Hamilton equations retain the same form even when the potential walls have a finite (non-zero) value.

### 3 Semiclassical Limit of the Quantum Dynamics

What above outlined is a Mixmaster intrinsic feature and not an effect induced by a particular class of references: the whole MCl formalism and its consequences has been developed in a framework free from the choice of a specific time gauge.

\(^1\)\(S^1_\phi\) denotes the \(\phi\)-circle of the momentum space variable.
Since chaos appears close enough to the Big Bang, we infer that it has some relations with the indeterministic quantum dynamics the model performs in the Planckian era. This relation between quantum and deterministic chaos is searched in the sense of a semiclassical limit for a canonical quantization of the model.

The asymptotical principle (11) describes a two dimensional Hamiltonian system, which can be quantized by a natural Schrödinger approach

\[ i\hbar \frac{\partial \psi}{\partial \tau} = \frac{d\Gamma}{d\tau} \hat{H}_{ADM} \psi , \]  

being \( \psi = \psi(\tau, \xi, \theta) \) the wave function for the point-universe and, implementing \( \hat{H}_{ADM} \) (see (2)) to an operator\(^2\), i.e.

\[ \xi \rightarrow \hat{\xi} , \quad \theta \rightarrow \hat{\theta} , \quad p_\xi \rightarrow \hat{p}_\xi \equiv -i\hbar \frac{\partial}{\partial \xi} , \quad p_\theta \rightarrow \hat{p}_\theta \equiv -i\hbar \frac{\partial}{\partial \theta} , \]  

the equation (11) rewrites explicitly, in the asymptotic limit \( U \rightarrow U_\infty \),

\[ i\frac{\partial \psi}{\partial \tau} = \frac{d\Gamma}{d\tau} \sqrt{\varepsilon^2 + \frac{U_\infty}{\hbar^2}} \psi = \frac{d\Gamma}{d\tau} \left[ -\sqrt{\xi^2 - 1} \frac{\partial}{\partial \xi} \sqrt{\xi^2 - 1} \frac{\partial}{\partial \xi} - \frac{1}{\sqrt{\xi^2 - 1}} \frac{\partial}{\partial \theta} \frac{1}{\sqrt{\xi^2 - 1}} \frac{\partial}{\partial \theta} + \frac{U_\infty}{\hbar^2} \right]^{1/2} \psi , \]  

where we took an appropriate symmetric normal ordering prescription and we left \( U_\infty \) to stress that the potential cannot be neglected on the entire configuration space \( \{\xi, \theta\} \).

Being \( U_\infty \) equal to infinity out of \( \Gamma_H \), \( \psi \) requires as boundary condition to vanish outside the potential walls, say

\[ \psi (\partial \Gamma_H) = 0 . \]  

The quantum equation (13) is equivalent to the Wheeler-DeWitt one for the same Bianchi model, once separated the positive and negative frequencies solutions [12], with the advantage that now \( \tau \) is a real time variable. Since the potential walls \( U_\infty \) are time independent, a solution of this equation has the form

\[ \psi (\tau, \xi, \theta) = \sum_{n=1}^{\infty} c_n e^{-iE_n \Gamma(\tau)/\hbar} \varphi_n (\xi, \theta) \]  

where \( c_n \) are constant coefficients and we assumed a discrete "energy" spectrum because the quantum point-universe is restricted in the finite region \( \Gamma_H \); the position (15) in (13)

\(^2\)The only non vanishing canonical commutation relations are

\[ [\xi, \hat{p}_\xi] = i\hbar , \quad [\theta, \hat{p}_\theta] = i\hbar . \]
leads to the eigenvalue problem

\[ \begin{bmatrix} -\sqrt{\xi^2 - 1} \frac{\partial}{\partial \xi} \sqrt{\xi^2 - 1} \frac{\partial}{\partial \xi} - \frac{1}{\sqrt{\xi^2 - 1}} \frac{\partial}{\partial \theta} \sqrt{\xi^2 - 1} \frac{\partial}{\partial \theta} \end{bmatrix} \varphi_n = \left( \frac{E_n^2 - U_{\infty}}{\hbar^2} \right) \varphi_n \equiv \frac{E_{\infty,n}^2}{\hbar^2} \varphi_n. \] (16)

In what follows we search the semiclassical solution of this equation regarding the eigenvalue \(E_{\infty,n}\) as a finite constant (i.e. we consider the potential walls as finite) and only at the end of the procedure we will take the limit for \(U_{\infty}\).

We infer that, in the semiclassical limit when \(\hbar \to 0\) and the “occupation number” \(n\) tends to infinity (but \(n \hbar \) approaches a finite value), the wave function \(\varphi_n\) approaches a function \(\varphi\) as

\[ \lim_{n \to \infty, \hbar \to 0} \varphi_n (\xi, \theta) = \varphi (\xi, \theta), \quad \lim_{n \to \infty, \hbar \to 0} E_{\infty,n} = E_{\infty}. \] (17)

The expression for \(\varphi\) is taken as a semiclassical expansion up to the first order, i.e.

\[ \varphi (\xi, \theta) = \sqrt{r (\xi, \theta)} \exp \left\{ i \frac{S (\xi, \theta)}{\hbar} \right\}, \] (18)

where \(r\) and \(S\) are functions to be determined.

Substituting (18) in (16) and separating the real from the complex part, we get two independent equations, i.e.

\[ E_{\infty}^2 = \left( \xi^2 - 1 \right) \left( \frac{\partial S}{\partial \xi} \right)^2 + \frac{1}{\xi^2 - 1} \left( \frac{\partial S}{\partial \theta} \right)^2 + \]

\[ \frac{\hbar^2}{\sqrt{r}} \left[ \sqrt{\xi^2 - 1} \frac{\partial}{\partial \xi} \sqrt{\xi^2 - 1} \frac{\partial}{\partial \xi} + \frac{1}{\xi^2 - 1} \frac{\partial^2}{\partial \theta^2} \right] \sqrt{r}, \] (19)

where we multiplied both sides by \(\hbar^2\) and, respectively,

\[ \sqrt{\xi^2 - 1} \frac{\partial}{\partial \xi} \left( \sqrt{\xi^2 - 1} \frac{\partial S}{\partial \xi} \right) + \frac{1}{\xi^2 - 1} \frac{\partial}{\partial \theta} \left( \frac{\partial S}{\partial \theta} \right) = 0. \] (20)

In the limit \(\hbar \to 0\) the second term of (19) is negligible meanwhile the first one reduces to the Hamilton-Jacobi equation

\[ \left( \xi^2 - 1 \right) \left( \frac{\partial S}{\partial \xi} \right)^2 + \frac{1}{\xi^2 - 1} \left( \frac{\partial S}{\partial \theta} \right)^2 = E_{\infty}^2. \] (21)

The solution of (21) can be easily checked to be\(^3\)

\[ S (\xi, \theta) = \int \left\{ \frac{1}{\sqrt{\xi^2 - 1}} \sqrt{E_{\infty}^2 - \frac{k^2}{\xi^2 - 1}} \ d\xi + k \ d\theta \right\}, \quad k = \text{const}. \] (22)

\(^3\)The discontinuity of this function on the boundary of \(\Gamma_H\) is due to the model adopted and does not affect the probability distribution.
We observe that \((21)\), through the identifications
\[
\frac{\partial S}{\partial \xi} = p_\xi, \quad \frac{\partial S}{\partial \theta} = p_\theta \quad \iff \quad S = \int (p_\xi d\xi + p_\theta d\theta),
\]
is reduced to the algebraic relation
\[
\left(\xi^2 - 1\right) p_\xi^2 + \frac{1}{\xi^2 - 1} p_\theta^2 = E_\infty^2.
\]

The constraint \((24)\) is nothing more than the asymptotic one \(H_{ADM}^2 = E^2 = \text{const.}\) and can be solved by setting
\[
\frac{\partial S}{\partial \xi} = p_\xi \equiv \frac{E_\infty}{\sqrt{\xi^2 - 1}} \cos \phi, \quad \frac{\partial S}{\partial \theta} = p_\theta \equiv E_\infty \sqrt{\xi^2 - 1} \sin \phi,
\]
where \(\phi \in [0, 2\pi]\) is a momentum-function related to \(\xi\) and \(\theta\) by the dynamics. On the other hand, by \((22)\) we get
\[
p_\xi = \frac{1}{\sqrt{\xi^2 - 1}} \sqrt{E_\infty^2 - \frac{k^2}{\xi^2 - 1}}, \quad p_\theta = k;
\]
to verify the compatibility of these expressions with \((25)\) we use the equations of motion \((6)\)
\[
\frac{d\xi}{d\phi} = -\frac{\xi^2 - 1}{\xi} \cot \phi \Rightarrow \sqrt{\xi^2 - 1} \sin \phi = c, \quad c = \text{const.}.
\]
The required compatibility comes from the identification \(k = E_\infty c\). Since
\[
\lim_{U \to U_\infty} E_\infty = \begin{cases} E & \forall \{\xi, \theta\} \in \Gamma_H \\ i\infty & \forall \{\xi, \theta\} \notin \Gamma_H \end{cases}
\]
we see by \((22)\) that the solution \(\varphi(\xi, \theta)\) vanishes, as due in presence of infinite potential walls, outside \(\Gamma_H\).

The substitution in \((20)\) of the positions \((25)\) (when the potential walls become infinite, because of chaotic dynamics, \(\phi\) behaves like an independent variable and plays here the role of a free parameter) leads to the new equation
\[
\sqrt{\xi^2 - 1} \cos \phi \frac{\partial r}{\partial \xi} + \frac{\sin \phi}{\sqrt{\xi^2 - 1}} \frac{\partial r}{\partial \theta} = 0.
\]

We emphasize how this equation coincides with \((9)\), provided the identification \(r \equiv \varrho_\infty\); it is just this correspondence between the statistical and the semiclassical quantum
analysis to ensure that the indeterminism of the quantum dynamics for the Bianchi IX model approaches the deterministic chaos in the considered limit. 

Any constant function is a solution of (30), but the normalization condition requires $r = 1/4\pi$ and therefore we finally get

$$\lim_{\hbar \to 0} \left| \varphi_n \right|^2 = \left| \varphi \right|^2 \equiv \varrho_\infty = \begin{cases} \frac{1}{4\pi} & \forall \{\xi, \theta\} \in \Gamma_H \\ 0 & \forall \{\xi, \theta\} \notin \Gamma_H \end{cases}$$

say the limit for the quantum probability distribution as $n \to \infty$ and $\hbar \to 0$ associated to the wave function

$$\psi(\tau, \theta, \xi) = \varphi(\xi, \theta) \exp \left\{ -i \frac{E_\infty}{\hbar} \Gamma(\tau) \right\} = \sqrt{r} \exp \left\{ i \int (p_\xi d\xi + p_\theta d\theta - E_\infty d\Gamma) \right\}$$

(32)

coincides with the classical statistical distribution on the microcanonical ensemble.

Though this formalism of correspondence remains valid for all Bianchi models, only the types VIII and IX admit a normalizable wave function $\varphi(\xi, \theta)$, being confined in $\Gamma_H$, and a continuity equation (9) which has a real statistical meaning.

Since referred to stationary states $\varphi_n(\xi, \theta)$, the considered semiclassical limit has to be intended in view of a “macroscopic” one and is not related to the temporal evolution of the model [13].

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