Compound vectors of subordinators and their associated positive Lévy copulas

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Abstract

Lévy copulas are an important tool which can be used to build dependent Lévy processes. In a classical setting, they have been used to model financial applications. In a Bayesian framework they have been employed to introduce dependent nonparametric priors which allow to model heterogeneous data. This paper focuses on introducing a new class of Lévy copulas based on a class of subordinators recently appeared in the literature, called Compound Random Measures. The well-known Clayton Lévy copula is a special case of this new class. Furthermore, we provide some novel results about the underlying vector of subordinators such as a series representation and relevant moments. The article concludes with an application to a Danish fire dataset studied in Esmaeili and Klüppelberg (2010).

Keywords: Dependent Completely Random Measures, Lévy processes, Clayton Lévy copulas.

1 Introduction

Vectors of subordinators, such as vectors of stable processes, vectors of Gamma processes and vectors of compound Poisson processes, have been used to model a variety of real world phenomena. For example, Yuen et al. (2016) solve the ruin problem for a bivariate Poisson process, Semeraro (2008) uses a vector of Gamma processes to construct a multivariate variance gamma model for financial applications and Esmaeili and Klüppelberg (2010) perform parameter estimation for bivariate compound Poisson processes which they apply to an insurance dataset. Esmaeili and Klüppelberg (2011) focus on parameter estimation for a vector of stable processes, Jiang et al. (2019) deal with a vector of gamma processes, and Esmaeili and Klüppelberg (2013) present a two-step estimation method for general multivariate Lévy processes. In the context of Bayesian non-parametric statistics, vectors of subordinators have been used to construct dependent priors to model heterogeneous data; for instance, Epifani and Lijoi (2010) and Riva-Palacio and Leisen (2018a) use vectors of stable processes in a Survival Analysis context, Lijoi et al. (2014) introduce a vector of random probability measures where the dependence arises by virtue of a suitable construction of vectors of Poisson random measures, Ishwaran and Zarepour (2009) propose a vector of...
generalized gamma processes and Leisen and Lijoi (2011) propose a vector of Poisson-Dirichlet processes constructed using a vector of stable processes. In this framework Griffin and Leisen (2017) introduced a class of vectors, called Compound Random Measures. Their proposal offers a flexible and tractable modelling approach which allows for heterogeneity in the data. Their tractability arises from their structure. The separation in the modelling approach of the nonparametric part from the multivariate contribution easily allows the implementation of algorithms for posterior inference. Their construction builds on the concept of completely random measures, see Kingman (1967); however, in this work we present their construction in the equivalent setting of vectors of subordinators and show new results. In particular, in Section 3 we present a series representation for compound vectors of subordinators and a new criteria for a compound vector of subordinators to be well defined. Furthermore, associated means, variance and correlation of the vector are provided.

A popular approach for the modelling of vectors of subordinators, and more in general vectors of Lévy processes, is the Lévy Copula approach. See for example the recent review of Tankov (2016). Griffin and Leisen (2017) showed the structure of the Lévy copula associated to a compound vector of subordinator in a particular case, namely when what they call the score distribution in their construction has independent and identically distributed marginal distributions; further exploration of the Lévy copula structure was not performed. In the present work we explore the general Lévy copula structure associated to a vector of compound subordinators. In particular we give a tractable example of a family of Lévy copulas which arises from a vector of compound subordinators. This new family is interesting because it can be seen as a generalization of the Clayton Lévy copulas. In a similar fashion to Esmaeili and Klüppelberg (2010), we show the use of such new family for the modelling of a bivariate compound Poisson process and the related parameter inference. We conclude that broader classes of Lévy copulas are needed and the compound vector of subordinators approach is a valid and tractable way to do so.

The outline of the paper is the following. Section 2 introduces vectors of dependent subordinators and Lévy copulas. Section 3 and Section 4 are devoted to illustrate the main results of the paper. Section 5 applies the new model to analyse the Danish fire incidents dataset, see also Esmaeili and Klüppelberg (2010). Section 6 concludes. All the proofs of the results are in the Appendix.

2 Preliminaries

This section is devoted to introduce some preliminary notions about vectors of subordinators and Lévy copulas.

Definition 1. We say that \( Y = (Y_1, \ldots, Y_d), \ d \in \mathbb{N}, \) is a \( d \)-variate vector of dependent positive jump Lévy processes if for \( t > 0, \ \lambda \in (\mathbb{R}^+)^d \)

\[
\mathbb{E} \left[ e^{-\lambda_1 Y_1(t)} \cdots e^{-\lambda_d Y_d(t)} \right] = e^{-\int_{(\mathbb{R}^+)^d \times [0,t]} (1-e^{-\lambda_1 s_1} \cdots e^{-\lambda_d s_d}) \nu(ds_1, \ldots, ds_d, dx)}
\]

\[
= e^{-\int_{(\mathbb{R}^+)^d \times [0,t]} (1-e^{-\langle \lambda, s \rangle}) \nu(ds, dx)}
\]
for a measure $\nu$ in $((\mathbb{R}^+)^{d+1},\mathcal{B}((\mathbb{R}^+)^{d+1}))$ such that

$$\int_{(\mathbb{R}^+)^{d}\times\mathbb{R}^+} \min\{1,\|s\|\} \nu(ds, dx) < \infty.$$  \hspace{1cm} (1)

Where we call $\nu$ the Lévy intensity of $Y$.

In the following we refer to the stochastic process defined above as a vector of subordinators. We say that a Lévy intensity is homogeneous if

$$\nu(ds, dx) = \rho(ds)\alpha(dx).$$

We define the Laplace exponent in the univariate case, see Sato et al. [1999].

**Definition 2.** Let $\nu$ be a Lévy intensity with $d = 1$ and associated subordinator $Y$. We say that the Laplace exponent of $\nu$ is

$$\psi_t(\lambda) = \int_0^t \int_0^\infty (1 - e^{-\lambda s})\nu(ds, dx) = -\log\left(\mathbb{E}\left[e^{-\lambda Y(t)}\right]\right).$$

The tail integral of a vector of subordinators plays an important role in the results displayed in Section 3 and 4. It is defined as follows.

**Definition 3.** Let $Y$ be a vector of subordinators with homogeneous Lévy intensity, its associated tail integral is given by

$$U(y) = \int_{[y_1,\infty)\times\ldots\times[y_d,\infty)} \rho(ds).$$  \hspace{1cm} (2)

The marginal tail integrals associated to $U(y)$ are

$$U_i(y) = U(y_1^{(i)},\ldots,y_i^{(i)},y,y_{i+1},\ldots,y_d^{(i)}),$$

where $y_1^{(i)} = \ldots = y_{i-1}^{(i)} = y_i^{(i)} = \ldots = y_d^{(i)} = 0$ for $i \in \{1,\ldots,d\}$. Given a vector of subordinators, $Y = (Y_1,\ldots,Y_d)$, there exist collections of random elements $\{W_{1,i}\}_{i=1}^\infty,\ldots,\{W_{d,i}\}_{i=1}^\infty$ and $\{V_i\}_{i=1}^\infty$ such that

$$(Y_1(t),\ldots,Y_d(t)) \overset{\mathrm{a.s.}}{=} \left(\sum_{i=1}^\infty W_{1,i}\mathbb{1}_{\{V_i \leq t\}},\ldots,\sum_{i=1}^\infty W_{d,i}\mathbb{1}_{\{V_i \leq t\}}\right).$$  \hspace{1cm} (3)

For a full review of vectors of subordinators see Cont and Tankov [2004]. The construction in Griffin and Leisen [2017] for vectors of completely random measures can be set in the context of vectors of subordinators as follows.

**Definition 4.** Let $h$ be a $d$–variate probability density function and $\nu^*$ a univariate Lévy intensity. We say that a vector of subordinators $Y$ is a $d$–variate compound vector of subordinators with score distribution $h$ and directing Lévy measure $\nu^*$ if it has a $d$–variate Lévy intensity given by

$$\nu(ds, dx) = \int z^{-d}h(s_1/z,\ldots,s_d/z)\nu^*(dz, dx)ds$$
If the directing Lévy measure is homogeneous, \( \nu^*(ds, dx) = \rho^*(ds)\alpha(dx) \), then \( \nu(ds, dx) = \rho(ds)\alpha(dx) \) with
\[
\rho(ds) = \int z^{-d} h(s_1/z, \ldots, s_d/z) \rho^*(dz)ds
\]

**Working example.** If \( \sigma \in (0, 1) \), \( \phi > 0 \) and
\[
\begin{align*}
  &h(y_1, y_2) = \frac{(y_1y_2)^{\phi-1}e^{-y_1-y_2}}{\Gamma(\phi)} \\
  &\rho^*(z) = \frac{z^{-\sigma-1}\Gamma(\phi)}{\Gamma(\phi+\sigma)\Gamma(1-\sigma)}
\end{align*}
\]
Then by Theorem 4.1 in Griffin and Leisen (2017) the corresponding vector of subordinators has \( \sigma \)-stable marginals and Lévy intensity given by
\[
\rho_{\sigma,\phi}(ds_1, ds_2) = \frac{\sigma(s_1s_2)^{\phi-1}\Gamma(\sigma+2\phi)(s_1+s_2)^{-\sigma-2\phi}}{\Gamma(\phi)\Gamma(\sigma+\phi)\Gamma(1-\sigma)} ds_1ds_2
\]

A popular approach for modelling the dependence structure of vectors of subordinators is given by Lévy copulas.

**Definition 5.** A \( d \)-variate positive Lévy copula is a function \( C(s_1, \ldots, s_d) : [0, \infty]^d \to [0, \infty] \) which satisfies
1. \( C(s_1, \ldots, s_d) < \infty \) for \( (s_1, \ldots, s_d) \neq (\infty, \ldots, \infty) \).
2. \( C \) is \( d \)-increasing.
3. \( C(s_1, \ldots, s_d) = 0 \) if \( u_k = 0 \) for any \( k \in \{1, \ldots, d\} \).
4. \( C(y_1^{(k)}, \ldots, y_k^{(k)}, s, y_{k+1}^{(k)}, \ldots, y_d^{(k)}) = s \) for \( k \in \{1, \ldots, d\} \), \( s \in \mathbb{R}^+ \), where \( y_1^{(k)} = \cdots = y_{k-1}^{(k)} = y_{k+1}^{(k)} = \cdots = y_d^{(k)} = \infty \).

Such Lévy copulas can be linked to a vector of subordinators via a Sklar theorem.

**Theorem 1.** (Sklar’s Theorem for tail integrals and Lévy copulas) Let \( U \) be a \( d \)-variate tail integral with margins \( U_1, \ldots, U_d \) then there exists a Lévy copula \( C \) such that
\[
U(y) = C(U_1(y_1), \ldots, U_d(y_d))
\]
If \( \{U_i\}_{i=1}^d \) are continuous \( C \) is unique, otherwise it is unique in \( \text{Ran}(U_1) \times \ldots \times \text{Ran}(U_d) \).

For a proof see Theorem 5.3 in Cont and Tankov (2004). If the Lévy copula is smooth enough then from Theorem 1 and the definition of the tail integral we have that the underlying multivariate Lévy intensity can be expressed as
\[
\rho(s) = \frac{\partial^d}{\partial u_1 \cdots \partial u_d} C(u) \bigg|_{u_1=U_1(s_1), \ldots, u_d=U_d(s_d)} \rho_1(s_1) \cdots \rho_d(s_d),
\]
where \( \rho_i, i \in \{1, \ldots, d\} \), are the corresponding marginal Lévy intensities associated to the tail integrals \( U_1, \ldots, U_d \). Furthermore if \( C \) is a two dimensional Lévy copula and \( \{(W_{1,i}, W_{2,i})\}_{i=1}^\infty \) are the random weights of a series representation for the associated vector of subordinators, equation [3], then the law of \( S_{1,i} = U_1(W_{1,i}) \) conditioned on \( S_{2,i} = U_2(W_{2,i}) = s_2 \in \mathbb{R}^+ \setminus \{0\} \) is given by the distribution function
\[
\hat{F}_{S_{1|S_2=s_2}}(s_1) = \frac{\partial}{\partial s_2} C(s_1, s_2)
\]
and the law of $S_{2,i} = U_2(W_{2,i})$ conditioned on $S_{1,i} = U_1(W_{1,i}) = s_1 \in \mathbb{R}^+ \setminus \{0\}$ is given by the distribution function
\[
F_{S_2|S_1 = s_1}(s_2) = \frac{\partial}{\partial s_1} C(s_1, s_2); \quad (7)
\]
see Theorem 6.3 in Cont and Tankov (2004) for a proof. Some examples of $d$-variate positive Lévy copulas are the following:

**Example 1. Independence Lévy copula.**
\[
C_\perp(s_1, \ldots, s_d) = \sum_{i=1}^d s_i \prod_{j \neq i} 1\{s_j = \infty\}.
\]
In this case the subordinators $Y_1, \ldots, Y_d$ are pairwise independent.

**Example 2. Complete dependence Lévy copula.**
\[
C_{||}(s_1, \ldots, s_d) = \min\{s_1, \ldots, s_d\}.
\]
In this case the subordinators $Y_1, \ldots, Y_d$ are completely dependent in the sense that the vector of jump weights for the associated series representation, \((W_{1,i}, \ldots, W_{d,i}))_{i=1}^\infty\), are in a set $S$ such that whenever $v, u \in S$ then either $v_j < u_j$ or $u_j < v_j$ for all $j \in \{1, \ldots, d\}$.

**Example 3. Clayton Lévy copula.**
\[
C_\theta(s_1, \ldots, s_d) = \left( s_1^{-\theta} + \ldots + s_d^{-\theta} \right)^{-1/\theta}; \quad \theta > 0.
\]
The Clayton Lévy copula is of interest as its parameter $\theta$ allows us to modulate between the independence and complete dependence cases; indeed
\[
\lim_{\theta \to 0} C_\theta(s_1, \ldots, s_d) = C_\perp(s_1, \ldots, s_d)
\]
and
\[
\lim_{\theta \to \infty} C_\theta(s_1, \ldots, s_d) = C_{||}(s_1, \ldots, s_d).
\]
For a full review of Lévy copulas see Cont and Tankov (2004).

### 3 Results for compound vectors of subordinators

This section provides general results for compound subordinators. In particular, we provide a series representation, conditions for the vector to be well posed and expressions for the mean, variance and correlation of the process. In the first result we provide a representation with the structure displayed in equation (3).
Theorem 2. Let \( Y = (Y_1, \ldots, Y_d) \) be a compound subordinator given by a score distribution \( h \) and directing Lévy measure \( \nu^* \) with associated univariate subordinator \( Y^* \). Then for \( t \in (\mathbb{R}^+)^d \)

\[
Y(t) = (Y_1(t_1), \ldots, Y_d(t_d)) = \left( \sum_{i=1}^{\infty} M_{1,i} W_i 1_{\{V_i \leq t_1\}}, \ldots, \sum_{i=1}^{\infty} M_{d,i} W_i 1_{\{V_i \leq t_d\}} \right)
\]

where

\[
Y^*(t) = \sum_{i=1}^{\infty} W_i 1_{\{V_i \leq t\}}
\]

for \( t \in \mathbb{R}^+ \), and

\[
(M_{1,i}, \ldots, M_{d,i}) \overset{i.i.d.}{\sim} h.
\]

The above result is useful for computational purposes and provide a deeper understanding of the discrete structure of the process. The next result provides practical condition to check if a compound vector of subordinators is well posed.

Theorem 3. Let \( \nu^* \) be a Lévy measure and \( h \) a \( d \)-variate score distribution such that if \( (W_1, \ldots, W_d) \) then \( \mathbb{E}[W_i] < \infty \forall i \in \{1, \ldots, d\} \). Then the compound vector of subordinators with directing Lévy measure \( \nu^* \) and score distribution \( h \) has a Lévy intensity which satisfies condition \([1]\).

The above theorem improves the result presented in Riva-Palacio and Leisen (2018b) by providing straightforward conditions to test if a vector of compound subordinators is well defined. This section concludes with a result which can be useful for modelling purposes and to understand the behaviour of the process. Let \( \psi_t \) be a Laplace exponent, we denote with \( \psi_t'(0) = \frac{d}{d\lambda} \psi_t(\lambda) \big|_{\lambda=0} \) the first derivative evaluated in 0. The moments of a compound vector of subordinators are given in the next result.

Theorem 4. Let \( Y \) be a vector of compound subordinators with score distribution \( h \) and directing Lévy measure \( \nu^* \) with Laplace exponent \( \psi_t^* \) such that \( (\psi_t^*)'(0) \) and \( (\psi_t^*)''(0) \) exist \( \forall t > 0 \); then

\[
\mathbb{E}[Y_i(t)] = (\psi_t^*)'(0) \mathbb{E}[W_i]
\]

\[
\text{Var}(Y_i(t)) = -(\psi_t^*)''(0) \mathbb{E}[W_i^2]
\]

\[
\text{Cov}(Y_i(t), Y_j(t)) = -(\psi_t^*)''(0) \mathbb{E}[W_i W_j]
\]

\[
\text{Cov}(Y_i(t), Y_j(t)) = \frac{\mathbb{E}[W_i W_j]}{\sqrt{\mathbb{E}[W_i^2] \mathbb{E}[W_j^2]}}
\]

where \( i, j \in \{1, \ldots, d\}, i \neq j \).

4 Positive Lévy copulas from compound vectors of subordinators

In this section we provide a new family of Lévy copulas which has the Clayton Lévy copula in Example 3 as special case. With respect to the Clayton Lévy copula, the new
family has an extra parameter which offers more flexibility in modelling. We derive the new family by considering the Lévy Copula associated to a compound vector of subordinators. Let
\[
\binom{r}{k} = \frac{r(r-1) \cdots (r-k+1)}{k!}
\]
be the generalized binomial coefficient, \( r \in \mathbb{R} \) and \( k \in \mathbb{N} \). The next Theorem provides the Lévy copula associated to the Lévy intensity in equation (4).

**Theorem 5.** Let \( \sigma \in (0, 1) \) and \( \phi > 0 \). We set
\[
\tilde{C}_{\sigma, \phi}(s_1, s_2) = \frac{\sigma \Gamma(\sigma + 2\phi)(s_1^{-\frac{1}{\sigma}} + s_2^{-\frac{1}{\sigma}})^{-\sigma}}{2\Gamma(\phi)\Gamma(\sigma + \phi)} \times \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{\phi - 1}{k} \binom{\phi + k - 1}{j} (-1)^{k+j} \frac{(s_1^{-\frac{1}{\sigma}} + s_2^{-\frac{1}{\sigma}})^{j}}{(\sigma + j)(\sigma + \phi + k)}.
\]

Then the Lévy Copula associated to \( \rho_{\sigma, \phi}, \) in (4), is \( \tilde{C}_{\sigma, \phi} \).

If \( \phi \in \mathbb{N} \setminus \{0\} \) the above expression for the Lévy copula simplifies.

**Corollary 1.** Let \( \sigma \in (0, 1) \) and \( \phi \in \mathbb{N} \setminus \{0\} \). Set
\[
\tilde{C}_{\sigma, \phi}(s_1, s_2) = \frac{\sigma \Gamma(\sigma + 2\phi)(s_1^{-\frac{1}{\sigma}} + s_2^{-\frac{1}{\sigma}})^{-\sigma}}{2\Gamma(\phi)\Gamma(\sigma + \phi)} \times \sum_{j=0}^{\phi - 1} \sum_{k=0}^{\phi + k - 1} \binom{\phi - 1}{k} \binom{\phi + k - 1}{j} (-1)^{k+j} \frac{(s_1^{-\frac{1}{\sigma}} + s_2^{-\frac{1}{\sigma}})^{j}}{(\sigma + j)(\sigma + \phi + k)}.
\]

Then the Lévy Copula associated to \( \rho_{\sigma, \phi}, \) in (4), is \( \tilde{C}_{\sigma, \phi} \).

We observe that under the change of variable \( \sigma = 1/\theta \) the previous copula has as a factor the Clayton Lévy copula plus a non-linear term in \((s_1, s_2)\). We recover the Clayton Lévy copula when \( \phi = 1 \). Although, from Theorem 5 we see that \( \theta \in (1, \infty) \) as \( \rho_{\sigma, \phi} \) was only defined for \( \sigma \in (0, 1) \). Surprisingly, this Lévy copula can be extended for \( \theta \in (0, 1] \) as showcased in the next theorem.

**Theorem 6.** Let \( \theta \in (0, \infty) \) and \( \phi \in \mathbb{N} \setminus \{0\} \), then
\[
C_{\theta, \phi}(s_1, s_2) = \frac{\Gamma\left(\frac{1}{\theta} + 2\phi\right)(s_1^{-\theta} + s_2^{-\theta})^{-\frac{1}{\theta}}}{2\theta \Gamma(\phi) \Gamma\left(\frac{1}{\theta} + \phi\right)} \times \sum_{k=0}^{\phi - 1} \binom{\phi - 1}{k} \sum_{j=0}^{\phi + k - 1} \binom{\phi + k - 1}{j} (-1)^{k+j} \frac{(s_1^{-\theta} + s_2^{-\theta})^{j}}{\left(\frac{1}{\theta} + j\right)\left(\frac{1}{\theta} + \phi + k\right)}
\]
is a Lévy copula.

For example, we have the next particular cases.
• With $\phi = 1$ we obtain
\[
\mathcal{C}_{\theta,2}(s_1, s_2) = \mathcal{C}_\theta(s_1, s_2) = \left(\frac{s_1^{-\theta} + s_2^{-\theta}}{\theta}\right)^{-\frac{1}{\theta}}.
\]
which is the Clayton Lévy Copula in dimension 2.

• With $\phi = 2$ we obtain
\[
\mathcal{C}_{\theta,2}(s_1, s_2) = \left(\frac{s_1^{-\theta} + s_2^{-\theta}}{\theta}\right)^{-\frac{1}{\theta}} \left[\theta + \frac{s_1^{-\theta} s_2^{-\theta}}{(s_1^{-\theta} + s_2^{-\theta})^2}\right]
\]

• With $\phi = 4$ we obtain
\[
\mathcal{C}_{\theta,4}(s_1, s_2) = \left(\frac{s_1^{-\theta} + s_2^{-\theta}}{6\theta}\right)^{-\frac{1}{\theta}} \left[6\theta + \frac{s_1^{-\theta} s_2^{-\theta}}{(s_1^{-\theta} + s_2^{-\theta})^2}\left(6 + \frac{s_1^{-\theta} s_2^{-\theta}}{(s_1^{-\theta} + s_2^{-\theta})^2}\left[3(\theta^{-1} + 3) + \frac{(\theta^{-1} + 4)s_1^{-\theta} s_2^{-\theta}}{(s_1^{-\theta} + s_2^{-\theta})^2}\right]\right]\]

This section concludes with a result that links the survival copula associated to the score distribution of a compound vector of subordinator to the underlying Lévy copula. In particular, the score distribution in the compound vector of subordinators’ construction has a density function $h$ which we can determine by its associated distributional survival Copula $\hat{C}$ and marginal survival functions $S_1, \ldots, S_d$.

**Definition 6.** Let $(X_1, \ldots, X_d)$ be a $d$–variate random vector and $S(x_1, \ldots, x_d) = \mathbb{P}[X_1 > x_1, \ldots, X_d > x_d]$ the associated $d$–variate survival function. For $i \in \{1, \ldots, d\}$, we say that $S_i(x) = \mathbb{P}[X_i > x]$ is the $i$–th marginal survival function. The associated survival copula is given by
\[
S(x_1, \ldots, x_d) = \hat{C}(S_1(x_1), \ldots, S_d(x_d)),
\]
see Section 2.6 in [Nelsen (2007)]. The next result provides the Lévy Copula associated to a compound vector of subordinators.

**Theorem 7.** Let $Y$ be a compound vector of subordinators given by a directing Lévy measure $\nu^*$ and a score distribution with distributional survival Copula $\hat{C}$ and marginal survival functions $S_1, \ldots, S_d$, then the Lévy copula, $\mathcal{C}$, associated to $Y$ is given by
\[
\mathcal{C}(s_1, \ldots, s_d) = \int_0^{\infty} \hat{C} \left( S_1 \left( \frac{U_1^{-1}(s_1)}{z} \right), \ldots, S_d \left( \frac{U_d^{-1}(s_d)}{z} \right) \right) \nu^*(dz)
\]
where the marginal tail integrals $U_i$ can be expressed as
\[
U_i(x) = \int_0^{\infty} S_i \left( \frac{z}{x} \right) \nu^*(dz)
\]
for $i \in \{1, \ldots, d\}$.

The above result is interesting as it shows how the dependence structure of the score distribution $h$, given by the survival Copula $\hat{C}$, and its marginal structure, given by the marginal survival functions $S_1, \ldots, S_d$, impact the Lévy copula, where the marginal structure of the vector of compound subordinators can be interpreted to be taken out with the inverse tail integrals $U_1^{-1}, \ldots, U_d^{-1}$.
5 Application to bivariate compound Poisson processes

In this section we develop an application of the new family of Lévy copulas to a real dataset. We follow Esmaeili and Klüppelberg (2010) to perform parameter estimation of a bivariate compound Poisson process. In particular, they performed a real data analysis of the Danish fire insurance dataset available in the “fitdistrplus” R package, see Delignette-Muller and Dutang (2015). The data consist of the losses, in millions of Danish Krone, pertaining to 2167 fire incidents in Copenhagen between 1980 to 1990. We focus on compound Poisson processes with positive increments.

Definition 7. Given \( \lambda_1, \lambda_2 \in \mathbb{R}^+ \setminus \{0\} \) and probability distributions \( F_1, F_2 \) in \( \mathbb{R}^+ \), a bivariate compound Poisson process with positive increments is a bivariate vector of subordinators \((X_1, X_2)\) such that marginally \(X_i\) has Lévy intensity

\[
\nu_i(ds, dx) = \lambda_i F_i(ds)dx.
\]

We observe that the associated marginal tail integrals are bounded in \( \mathbb{R}^+ \) so almost surely the associated series representation has finite jumps. We will focus on bivariate compound processes of the form

\[
(Y_1(t), Y_2(t)) \overset{a.s.}{=} \left( \sum_{i=1}^{\infty} W_{1,i} 1_{\{U_i \leq t\}}, \sum_{i=1}^{\infty} W_{2,i} 1_{\{U_i \leq t\}} \right)
\]

\[
= \left( \sum_{i=1}^{N(t)} \tilde{W}_{1,i}, \sum_{i=1}^{N(t)} \tilde{W}_{2,i} \right)
\]

where \( N(t) = \# \{i : U_i \leq t\} < \infty \), \( \tilde{W}_{1,i} \overset{a.s.}{=} \sum_{i=1}^{N(t)} W_{1,i} 1_{\{U_i \leq t\}} = \{W_{1,i} : U_i \leq t\} \) and \( \tilde{W}_{2,i} \overset{a.s.}{=} \sum_{i=1}^{N(t)} W_{2,i} = \{W_{1,i} : U_i \leq t\} \). We observe that the associated Lévy intensity \( \nu(ds_1, ds_2, dx) \) is supported in \( (0, \infty)^3 \) so \( W_{1,i} \overset{a.s.}{=} 0, W_{2,i} \overset{a.s.}{=} 0 \) for all \( i \in \{1, 2, \ldots\} \), \( \lambda_1 = \lambda_2 = \lambda \) and the dependence structure can be modelled with a Lévy copula which does not assign mass at the zero axis \( \{(s_1, s_2) : s_1 = 0, s_2 \geq 0\} \) and \( \{(s_1, s_2) : s_2 = 0, s_1 \geq 0\} \); which is the case for \( C_{\theta, \phi} \). For a full review of Poisson processes we refer to Kingman (2005). We will assume the next observation scheme for bivariate compound Poisson processes.

Definition 8. We say that we observe the bivariate compound process continuously through time if we are able to observe all the jump times and jump weights in a given time interval.

Let \( \{(w_{1,i}, w_{2,i})\}_{i=1}^{n} \) be the jump sizes of a continuously observed bivariate compound Poisson process, with \( n \in \mathbb{N} \) the number of jumps. Using the above notation we can give the likelihood for the continuous through time observations.

Proposition 1 (Esmaeili and Klüppelberg (2010)). Let \( T \in \mathbb{R}^+ \), if a bivariate compound Poisson process has jump rates \( \lambda_1 = \lambda_2 = \lambda \), marginal jump weight distributions \( F_i \), associated to survival functions \( S_i \) and probability densities \( f_i \) parametrized by real valued vectors \( \alpha_i, i \in \{1, 2\} \), and an associated Lévy copula \( C_c \) parametrized by a real valued vector \( c \) such that

\[
\frac{\partial^2}{\partial u_1 \partial u_2} C_c(u_1, u_2) \exists \text{ for every } (u_1, u_2, x) \in (0, \lambda_1) \times (0, \lambda_2) \times
\]
Then the likelihood function for continuously observed bivariate compound Poisson processes in \((0, t]\) is given by

\[
L(\lambda, \alpha_1, \alpha_2, c) = \lambda^n e^{-\lambda T} \prod_{i=1}^{n} (f_1(w_{1,i}; \alpha_1) f_2(w_{2,i}; \alpha_2)) \times \frac{\partial^2}{\partial u_1 \partial u_2} C_c(u_1, u_2) \bigg|_{u_1=\lambda s_1(w_{1,i}, \alpha_1), u_2=\lambda s_2(w_{2,i}, \alpha_2)}
\]

The application of our extension of the Clayton Lévy copula \(C_{\theta, \phi}\) is of interest for the above model as it can offer more flexibility in the likelihood above. We follow Esmaeili and Klüppelberg (2010) and take only into account the building and content losses. Furthermore we focus on losses such that the loss due to the building and due to the contents are both greater than 300000 Danish Kronen, so we get 1015 observations. Following their approach, we consider the logarithm of the loss quantities and normalize them by subtracting \(\log(0.3)\); thus obtaining the bivariate weights, at each time point, which we model through a bivariate compound Poisson process. In Figures 1 and 2 we show the corresponding bivariate Poisson process. We fit the model in a two-step way by following Esmaeili and Klüppelberg (2013), see also Jiang et al. (2019). In particular, we fit the marginal parameters first and the dependence parameters in the Lévy copula secondly. The marginal distributions, \(F_1\) and \(F_2\) are modelled with Weibull distributions and fitted via maximum likelihood; this marginal fits are showed in Figure 3. We fit \(\theta\) using maximum likelihood for each value of \(\phi \in \{1, \ldots, 20\}\). The estimated values and negative log-likelihood are presented in Table 1. We observe that the maximum likelihood is attained by the model with \(\phi = 2\). So by using a more flexible Lévy copula we have obtained a higher maximum likelihood value in our parametric model. Furthermore, Table 1 can be used to compute the Akaike’s Information Criterion (AIC) and compare the different Lévy copulas. It is straightforward to see that the model with \(\phi = 2\) attains the lowest AIC.

6 Conclusions

This paper introduced a new family of positive Lévy copulas which encompasses the well-known Clayton Lévy copulas. This family is derived by looking at a new model for subordinators introduced by Griffin and Leisen (2017), called Compound Random Measures. In Section 3 we provided general results for this models whereas Section 4 focused on the derivation of a new family of positive Lévy copulas. Section 5 showed an application of the new copula family on a real dataset studied in Esmaeili and Klüppelberg (2010).

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Figure 1: Cumulative logarithmic losses related to loss of content (blue) and loss of building (red) in the Danish fire insurance data as discussed in Section 5.
Figure 2: Individual logarithmic losses related to losses due to the building (top) and loss due to content (bottom) in the Danish fire insurance data as discussed in Section 5.
Figure 3: Marginal fit for the marginal cumulative distribution function (CDF) associated to the losses due to the building (left) and due to content (right) in the Danish fire insurance dataset as discussed in Section 5. The blue lines correspond to the empirical cumulative distribution function (ECDF) and the orange line corresponds to the maximum likelihood fits with Weibull distributions.
| $\phi$ | $\hat{\theta}$ | -Loglikelihood |
|-------|----------------|----------------|
| 1     | 1.766          | 4986.098       |
| 2     | 1.168          | 4985.071       |
| 3     | 0.933          | 4985.854       |
| 4     | 0.800          | 4986.637       |
| 5     | 0.712          | 4987.274       |
| 6     | 0.647          | 4987.783       |
| 7     | 0.598          | 4988.193       |
| 8     | 0.559          | 4988.528       |
| 9     | 0.526          | 4988.807       |
| 10    | 0.499          | 4989.043       |
| 11    | 0.476          | 4989.243       |
| 12    | 0.455          | 4989.417       |
| 13    | 0.437          | 4989.568       |
| 14    | 0.421          | 4989.701       |
| 15    | 0.407          | 4989.818       |
| 16    | 0.394          | 4989.924       |
| 17    | 0.382          | 4990.018       |
| 18    | 0.372          | 4990.103       |
| 19    | 0.362          | 4990.180       |
| 20    | 0.353          | 4990.251       |

Table 1: Maximum likelihood fits for $\theta$ and associated log-likelihood for different values of $\phi$. 
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A Proofs

Proof of Theorem 2

For this proof we will use Proposition 2.1 in Rosinski (2001). Let $H$ be the probability distribution associated to $h$ and $\nu^*$ the directing Lévy intensity. We consider a Poisson random measure

$$M = \sum_{i=1}^{\infty} \delta_{(Z_i, W_i, X_i)},$$

where $\{Z_i\}_{i=1}^{\infty}$ i.i.d. $\sim H$ and $\{(W_i, X_i)\}_{i=1}^{\infty}$ are such that

$$\sum_{i=1}^{\infty} \delta_{(W_i, X_i)},$$

as a Poisson random measure, has intensity $\nu^*$. It follows that $M$ has intensity $\mu = H \times \nu^*$. We define

$$g(z, w, x) = (z_1 w, z_2 w, \ldots, z_d w, x).$$

Due to Proposition 2.1 in Rosinski (2001) it suffices to check that $\nu = \mu \circ g^{-1}$. Let $A_1, \ldots, A_d, B \in \mathcal{B}(\mathbb{R}^+)$, then

$$g^{-1} ((A_1 \times \ldots \times A_d) \times B) =$$

$$\left\{ \left( \frac{a_1}{w}, \frac{a_2}{w}, \ldots, \frac{a_d}{w}, w, x \right) \text{ such that } x \in B, a_1 \in A_1, \ldots, a_d \in A_d, w \in \mathbb{R}^+ \right\}$$
So the pullback measure $\eta = \mu \circ g^{-1}$ is given by

$$
\eta((A_1 \times \ldots \times A_d) \times B) = \int_{g^{-1}((A_1 \times \ldots \times A_d) \times B)} d\mu
$$

$$
= \int_{A_1 \times A_2 \times \ldots \times A_d \times (0, \infty) \times B} H(ds_1, \ldots, ds_d) \nu^*(dz, dx)
$$

$$
= \int_{A_1 \times A_2 \times \ldots \times A_d \times (0, \infty) \times B} H \left( \frac{ds_1}{z}, \ldots, \frac{ds_d}{z} \right) \nu^*(dz, dx)
$$

So extending the measure we conclude that $\nu = \mu \circ g^{-1}$ so

$$
N = \sum_{i=1}^{\infty} \delta(z_{1,i}, W_i, z_{2,i}, W_i, \ldots, z_{d,i}, W_i, X_i)
$$

is almost surely a compound vector of subordinators given by the score distribution $h$ and the directing Lévy measure $\nu^*$ due to Proposition 2.1 in [Rosinski 2001].

**Proof of Theorem 3**

Let $|w| = \sum_{i=1}^{d} w_i$ for $w \in (\mathbb{R}^+)^d$. We have that $\|zw\| \leq z|w|$ so

$$
\int_{(\mathbb{R}^+)^d \times \mathbb{R}^+} \min\{1, \|s\|\} \nu(ds, dx) = \int_{(\mathbb{R}^+)^d \times \mathbb{R}^+} \min\{1, \|zw\|\} h(w) dw \nu^*(dz, dx)
$$

$$
\leq \int_{(\mathbb{R}^+)^d \times \mathbb{R}^+} \min\{1, z|w|\} h(w) dw \nu^*(dz, dx) = E \left[ \int_{(\mathbb{R}^+)^2} min\{1, z|W|\} \nu^*(dz, dx) \right]
$$

$$
= E \left[ \int_{(0, |W|) \times \mathbb{R}^+} z|W| \nu^*(dz, dx) \right] + E \left[ \int_{|W| \times \mathbb{R}^+} \nu^*(dz, dx) \right]
$$

$$
= E \left[ \int_{(0, |W|) \times \mathbb{R}^+} \mathbb{1}\{|W|\leq \frac{1}{2} \} |W| \nu^*(dz, dx) + \int_{|W| \times \mathbb{R}^+} \mathbb{1}\{|W|\leq \frac{1}{2} \} \nu^*(dz, dx) \right]
$$

$$
= \int_{(0,1) \times \mathbb{R}^+} E \left[ \mathbb{1}\{|W|\leq \frac{1}{2} \} |W| \right] \nu^*(dz, dx) + \int_{(1,\infty) \times \mathbb{R}^+} E \left[ \mathbb{1}\{|W|\leq \frac{1}{2} \} |W| \right] \nu^*(dz, dx)
$$

$$
+ \int_{(0,1) \times \mathbb{R}^+} E \left[ \mathbb{1}\{|W|\geq \frac{1}{2} \} |W| \right] \nu^*(dz, dx) + \int_{(1,\infty) \times \mathbb{R}^+} E \left[ \mathbb{1}\{|W|\geq \frac{1}{2} \} |W| \right] \nu^*(dz, dx)
$$

$$
\leq E[|W|] \int_{(0,1) \times \mathbb{R}^+} z \nu^*(dz, dx) + \int_{(1,\infty) \times \mathbb{R}^+} \nu^*(dz, dx)
$$

$$
+ E[|W|] \int_{(0,1) \times \mathbb{R}^+} z \nu^*(dz, dx) + \int_{(1,\infty) \times \mathbb{R}^+} \nu^*(dz, dx) < \infty.
$$

For the first and fourth integral we use the fact that the indicator function is less or equal to one. For the second integral, we note that $E \left[ \mathbb{1}\{|W|\leq \frac{1}{2} \} |W| \right] \leq 1$. For the third integral, we use the Markov’s inequality, $E \left[ \mathbb{1}\{|W|\geq \frac{1}{2} \} \right] = P[|W| \geq \frac{1}{2}] \leq z E[|W|]$.  

Finiteness of the above expression follows from the fact that $\nu^*$ is a Lévy intensity satisfying (1) and $E[|W|] < \infty$. 

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Proof of Theorem 4

We can set the multivariate Lévy intensity for a \(d\)-variate vector of subordinators \(Y = (Y_1, \ldots, Y_d)\) with \(\lambda = (\lambda_1, \ldots, \lambda_d)\) as

\[
\psi_t(\lambda) = \int_0^t \int_0^\infty \left(1 - e^{-\lambda_1 s_1 - \cdots - \lambda_d s_d}\right) \nu(ds, dx) = -\log \left( E \left[ e^{-\lambda_1 Y_1(t) - \cdots - \lambda_d Y_d} \right] \right).
\]

If we denote \(\{e_i\}_{i=1}^d\) as the canonical basis of \(\mathbb{R}^d\) and the univariate Laplace exponent of \(Y_i\) as \(\psi_t(\lambda e_i) = -\log \left( E \left[ e^{-\lambda Y_i(t)} \right] \right)\) in Definition 2, we have that

\[
E[Y_i(t)] = -\frac{\partial}{\partial \lambda_i} E \left[ e^{-\lambda Y_i(t)} \right] \bigg|_{\lambda=0} = -\frac{\partial}{\partial \lambda_i} e^{-\psi_t(\lambda e_i)} \bigg|_{\lambda=0} = -\frac{\partial^2}{\partial \lambda_i \partial \lambda_j} e^{-\psi_t(\lambda e_i)} \bigg|_{\lambda=0}
\]

We observe that

\[
E[Y_i^2(t)] = \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} E \left[ e^{-\lambda Y_i(t)} \right] \bigg|_{\lambda=0} = \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} e^{-\psi_t(\lambda e_i)} \bigg|_{\lambda=0} = \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} e^{-\psi_t(\lambda(e_i + e_j))} \bigg|_{\lambda=0} = \frac{\partial}{\partial \lambda_j} e^{-\psi_t(\lambda(e_i + e_j))} \bigg|_{\lambda=0} = (\psi_t''(0))^2 E[W_i^2] - (\psi_t''(0))^2 E[W_i^2]
\]

It follows that

\[
Var(Y_i(t)) = E[Y_i^2(t)] - E[Y_i(t)]^2 = (\psi_t'(0))^2 E[W_i^2] - (\psi_t''(0))^2 E[W_i^2] - (\psi_t''(0))^2 E[W_i^2]
\]

For \(i, j \in \{1, \ldots, n\}, i \neq j\) observe that

\[
E[Y_i(t)Y_j(t)] = \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} E \left[ e^{-\lambda Y_i(t) - \lambda Y_j(t)} \right] \bigg|_{\lambda_i=\lambda_j=0} = \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} e^{-\psi_t(\lambda_i e_i + \lambda_j e_j)} \bigg|_{\lambda_i=\lambda_j=0} = \frac{\partial}{\partial \lambda_j} e^{-\psi_t(\lambda_i e_i + \lambda_j e_j)} \bigg|_{\lambda_i=\lambda_j=0} = (\psi_t''(0))^2 E[W_j W_i] - E[(\psi_t''(0)W_i W_j)]
\]

We get that

\[
Cov(Y_i(t), Y_j(t)) = E[Y_i(t)Y_j(t)] - E[Y_i(t)] E[Y_j(t)]
\]

\[
= (\psi_t''(0))^2 E[W_j] E[W_i] - E[(\psi_t''(0)W_i W_j)] - (\psi_t''(0))^2 E[W_i] E[W_j]
\]

\[
= - (\psi_t''(0))^2 E[W_i W_j]
\]
It follows that
\[ \text{Cor}(Y_i(t), Y_j(t)) = \frac{\text{Cov}(Y_i(t), Y_j(t))}{\sqrt{\text{Var}(Y_i(t))\sqrt{\text{Var}(Y_j(t))}}} \]
\[ = \frac{-(\psi^{*}(0))\mathbb{E}[W_iW_j]}{\sqrt{((\psi^{*})''(0))^2 \mathbb{E}[W_i^2] \mathbb{E}[W_j^2]}} \]
\[ = \frac{\mathbb{E}[W_iW_j]}{\sqrt{\mathbb{E}[W_i^2] \mathbb{E}[W_j^2]}} \]

**Proof of Theorem 5**

Let \( U \) be the bivariate tail integral of \( \rho_{\sigma, \phi} \) as in the hypothesis.

\[ U(s_1, s_2) = \int_{s_1}^{\infty} \int_{s_2}^{\infty} \frac{\sigma(y_1y_2)^{\phi-1}\Gamma(\sigma + 2\phi)(y_1 + y_2)^{-\sigma - 2\phi}}{\Gamma(\phi)\Gamma(\sigma + \phi)\Gamma(1 - \sigma)} dy_1 dy_2 \]

We consider the change of variable

\[ h(y_1, y_2) = (y_1 + y_2, y_1/(y_1 + y_2)) = (\rho, z_1) \]
\[ \mathrm{d}\rho z_1 = \left| \det \left( \frac{dh}{dy} \right) \right| dy_1 dy_2 = (y_1 + y_2)^{-1} dy_1 dy_2 \]

so

\[ U(s_1, s_2) = \frac{\sigma \Gamma(\sigma + 2\phi)}{\Gamma(\phi)\Gamma(\sigma + \phi)\Gamma(1 - \sigma)} \]
\[ \times \int_{h(\{y_1, y_2 : s_1 \leq y_1, s_2 \leq y_2\})} (z_1 - z_1^2)^{\phi - 1} \rho^{-\sigma} \mathrm{d}\rho \mathrm{d}z_1. \]

Throughout the proof, we denote with \( c_{\sigma, \phi} \) the following quantity

\[ c_{\sigma, \phi} = \frac{\sigma \Gamma(\sigma + 2\phi)}{\Gamma(\phi)\Gamma(\sigma + \phi)\Gamma(1 - \sigma)} \]

For the integration region, we consider the curves

\[ \hat{\omega}(\hat{t}) = h(s_1, s_2 + \hat{t}) = (s_1 + s_2 + \hat{t}, s_1/(s_1 + s_2 + \hat{t})) \]
\[ \hat{\gamma}(\hat{t}) = h(s_1 + \hat{t}, s_2) = (s_1 + s_2 + \hat{t}, (s_1 + \hat{t})/(s_1 + s_2 + \hat{t})) \]
with \( t \geq 0 \); so for \( t \geq s_1 + s_2 \) we can get the reparametrized curves \( \omega(t) = (t, s_1/t) \) and \( \gamma(t) = (t, 1 - s_2/t) \) to delimit the integration area, hence using Fubini theorem

\[
U(s_1, s_2) = c_{\sigma, \phi} \int_{s_1 + s_2}^\infty \int_{s_1 / \rho}^{1 - s_2 / \rho} (z_1 - z_1^2) \phi^{-1} \rho^{-\sigma - 1} \, dz_1 \, d\rho = c_{\sigma, \phi} \int_{s_1 + s_2}^\infty \int_{s_1 / \rho}^{1 - s_2 / \rho} (z_1 - z_1^2) \phi^{-1} \rho^{-\sigma - 1} \, dz_1 \, d\rho = c_{\sigma, \phi} \int_{s_1 + s_2}^\infty \frac{\rho^{-\sigma - 1}}{k} \sum_{k=0}^{\infty} \left( \frac{\phi - 1}{k} \right) (-1)^k \frac{\phi + k}{\phi + k} \left[ 1 - \frac{s_2}{\rho} \right] \, d\rho
\]

Fubini\[
\int_{s_1 + s_2}^\infty \frac{\rho^{-\sigma - 1}}{k} \sum_{k=0}^{\infty} \left( \frac{\phi - 1}{k} \right) (-1)^k \left[ 1 - \frac{s_2}{\rho} \right] \, d\rho = c_{\sigma, \phi} \int_{s_1 + s_2}^\infty \frac{s_1 + s_2}{\rho} \rho^{-\sigma - 1} \, d\rho
\]

\[
U(s_1, s_2) = c_{\sigma, \phi} \int_{s_1 + s_2}^\infty \int_{s_1 / \rho}^{1 - s_2 / \rho} (z_1 - z_1^2) \phi^{-1} \rho^{-\sigma - 1} \, dz_1 \, d\rho = c_{\sigma, \phi} \int_{s_1 + s_2}^\infty \frac{\rho^{-\sigma - 1}}{k} \sum_{k=0}^{\infty} \left( \frac{\phi - 1}{k} \right) (-1)^k \frac{\phi + k}{\phi + k} \left[ 1 - \frac{s_2}{\rho} \right] \, d\rho
\]

Fubini\[
\int_{s_1 + s_2}^\infty \frac{\rho^{-\sigma - 1}}{k} \sum_{k=0}^{\infty} \left( \frac{\phi - 1}{k} \right) (-1)^k \left[ 1 - \frac{s_2}{\rho} \right] \, d\rho = c_{\sigma, \phi} \int_{s_1 + s_2}^\infty \frac{s_1 + s_2}{\rho} \rho^{-\sigma - 1} \, d\rho
\]
To get the copula we evaluate the above tail integral in

\[
\frac{(s_1 + s_2)^{-\sigma}(1 - \frac{s_2}{s_1 + s_2})^{\phi + k}}{(\sigma + \phi + k)}
\]

\[
= c_{\sigma, \phi} \sum_{k=0}^{\infty} \binom{\phi - 1}{k} \frac{(-1)^k}{\phi + k} \sum_{j=0}^{\infty} \binom{\phi + k}{j} (-1)^j (s_1 + s_2)^{-\sigma} \frac{(s_2^{\frac{1}{\sigma}} + s_1^{\frac{1}{\sigma}})^j}{(\sigma + j)}
\]

\[
- \sum_{j=0}^{\infty} \binom{\phi + k}{j} (-1)^j (s_1 + s_2)^{-\sigma} \frac{(s_2^{\frac{1}{\sigma}} + s_1^{\frac{1}{\sigma}})^j}{(\sigma + \phi + k)}
\]

\[
= c_{\sigma, \phi} \sum_{k=0}^{\infty} \binom{\phi - 1}{k} \frac{(-1)^k}{\phi + k} \sum_{j=0}^{\infty} \binom{\phi + k}{j} (-1)^j
\]

\[
(s_1 + s_2)^{-\sigma} \frac{(s_2^{\frac{1}{\sigma}} + s_1^{\frac{1}{\sigma}})^j (\phi + k - j)}{(\sigma + j)(\sigma + \phi + k)}
\]

To get the copula we evaluate the above tail integral in

\[
(U^{-1}(s_1), U^{-1}(s_2)) = \left( (\Gamma(1 - \sigma)s_1)^{-\frac{1}{\sigma}}, (\Gamma(1 - \sigma)s_2)^{-\frac{1}{\sigma}} \right);
\]

entailing the associated Lévy copula

\[
\tilde{C}_{\sigma, \phi}(s_1, s_2) = \frac{\sigma \Gamma(\sigma + 2\phi)(s_1^{\frac{1}{\sigma}} + s_2^{\frac{1}{\sigma}})^{-\sigma}}{\Gamma(\phi)\Gamma(\sigma + \phi)} \times \sum_{k=0}^{\infty} \binom{\phi - 1}{k} \sum_{j=0}^{\infty} \binom{\phi + k - 1}{j} (-1)^{k+j} \frac{(s_2^{\frac{1}{\sigma}} + s_1^{\frac{1}{\sigma}})^j}{(\sigma + j)(\sigma + \phi + k)}.
\]

Exploiting that by construction \(C_{\sigma, \phi}\) is symmetric, we get

\[
\tilde{C}_{\sigma, \phi}(s_1, s_2) = \frac{\sigma \Gamma(\sigma + 2\phi)(s_1^{\frac{1}{\sigma}} + s_2^{\frac{1}{\sigma}})^{-\sigma}}{2\Gamma(\phi)\Gamma(\sigma + \phi)} \sum_{k=0}^{\infty} \binom{\phi - 1}{k} \sum_{j=0}^{\infty} \binom{\phi + k - 1}{j} (-1)^{k+j} \frac{(s_1^{\frac{1}{\sigma}} + s_2^{\frac{1}{\sigma}})^j}{(\sigma + j)(\sigma + \phi + k)}.
\]

**Proof of Theorem 6**

From equation (5) we have that

\[
\frac{\partial}{\partial s_2} \partial_{s_1} \tilde{C}_{\sigma, \phi}(s_1, s_2),
\]
where $\rho^\sigma\text{-stab}$ is the Lévy intensity of an homogeneous $\sigma$-stable process and $U^{-1}_\sigma\text{-stab}$ is the inverse of the corresponding tail integral. It follows that

$$\frac{\partial}{\partial s_2} \frac{\partial}{\partial s_1} \tilde{\mathcal{C}}_{\sigma, \phi}(s_1, s_2) =$$

$$\sigma \left( \frac{U^{-1}_\sigma\text{-stab}(s_1)U^{-1}_\sigma\text{-stab}(s_2)}{\Gamma(\phi)\Gamma(\sigma + \phi)\Gamma(1 - \sigma)\rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_1)) \rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_2))} \right)^{\phi-1} \Gamma(\sigma + 2\phi) \left( U^{-1}_\sigma\text{-stab}(s_1) + U^{-1}_\sigma\text{-stab}(s_2) \right)^{-\sigma - 2\phi}$$

$$= \frac{\Gamma(1 - \sigma) \left( \frac{s_1}{2} + \frac{s_2}{2} \right)^{\phi-1} \Gamma(\sigma + 2\phi) \left( \frac{s_1}{2} + s_2 \right)^{-\sigma - 2\phi}}{\Gamma(\phi)\Gamma(\sigma + \phi)\Gamma(1 - \sigma)\rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_1)) \rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_2))}$$

$$= \frac{\Gamma(1 - \sigma) \left( \frac{s_1}{\sigma} + \frac{s_2}{\sigma} \right)^{\phi-1} \Gamma(\sigma + 2\phi) \left( \frac{\frac{1}{\sigma}}{s_1} + \frac{\frac{1}{\sigma}}{s_2} \right)^{-\sigma - 2\phi}}{\Gamma(\phi)\Gamma(\sigma + \phi)\Gamma(1 - \sigma)\rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_1)) \rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_2))}$$

$$= \frac{\Gamma(1 - \sigma) \left( \frac{s_1}{\sigma} + \frac{s_2}{\sigma} \right)^{\phi-1} \Gamma(\sigma + 2\phi) \left( \frac{1}{s_1} + \frac{1}{s_2} \right)^{1 + \frac{\phi}{\sigma}}}{\Gamma(\phi)\Gamma(\sigma + \phi)\Gamma(1 - \sigma)\rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_1)) \rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_2))}$$

The above expression can reduced to:

$$\frac{\partial}{\partial s_2} \frac{\partial}{\partial s_1} \tilde{\mathcal{C}}_{\sigma, \phi}(s_1, s_2) = \frac{\Gamma(\sigma + 2\phi) \left( \left( \frac{1}{\sigma} \right)^{\phi} \left( \frac{1}{s_1} + \frac{1}{s_2} \right)^{1 + \frac{\phi}{\sigma}} \right)^{\sigma - 2\phi}}{\Gamma(\phi)\Gamma(\sigma + \phi)\Gamma(1 - \sigma)\rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_1)) \rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_2))}$$

$$= \frac{\Gamma(\sigma + 2\phi) \left( \left( \frac{1}{\sigma} \right)^{\phi} \left( \frac{1}{s_1} + \frac{1}{s_2} \right)^{1 + \frac{\phi}{\sigma}} \right)^{\sigma + 2\phi}}{\Gamma(\phi)\Gamma(\sigma + \phi)\Gamma(1 - \sigma)\rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_1)) \rho^\sigma\text{-stab} (U^{-1}_\sigma\text{-stab}(s_2))}$$

which is greater than zero for $s_2, s_1 > 0$ and is well defined for $0 < \sigma$.

By symmetry and using (7) it suffices to check that

$$\lim_{s_2 \to 0} \mathcal{F}_{\sigma\phi}|_{s_1=s_2} = \lim_{s_2 \to 0} = \frac{\partial}{\partial s_1} \tilde{\mathcal{C}}_{\sigma, \phi}(s_1, s_2) = 0$$

and

$$\lim_{s_2 \to \infty} \mathcal{F}_{\sigma\phi}|_{s_1=s_2} = \lim_{s_2 \to \infty} \frac{\partial}{\partial s_1} \tilde{\mathcal{C}}_{\sigma, \phi}(s_1, s_2) = 1.$$
For the first limit we observe that
\[
\begin{align*}
\frac{\partial}{\partial s_1} \tilde{c}_{\sigma,\phi}(s_1, s_2) &=
\frac{\sigma \Gamma(\sigma + 2\phi)}{\Gamma(\phi) \Gamma(\sigma + \phi)} \sum_{k=0}^{\phi-1} \left( \frac{\phi - 1}{k} \right) \sum_{j=0}^{\phi+k-1} \left( \frac{\phi + k - 1}{j} \right) (-1)^{k+j} \frac{\partial}{\partial s_1} \left( s_2^{-\frac{j}{\sigma}} \left( \frac{1}{s_1^{\frac{1}{\sigma}} + s_2^\frac{1}{\sigma}} \right)^{-\sigma-j} \right) \\
&= \frac{\sigma \Gamma(\sigma + 2\phi)}{\Gamma(\phi) \Gamma(\sigma + \phi)} \sum_{k=0}^{\phi-1} \left( \frac{\phi - 1}{k} \right) \sum_{j=0}^{\phi+k-1} \left( \frac{\phi + k - 1}{j} \right) \times (-1)^{k+j} (1 + \frac{j}{\sigma}) s_2^{-\frac{j}{\sigma}} \sigma^{\phi-1} \left( \frac{1}{s_1^\sigma + s_2^\sigma} \right)^{-\sigma-j-1} \\
&\times (\sigma + j)(\sigma + \phi + k)
\end{align*}
\]
and for any \( j \in \mathbb{N} \)
\[
\lim_{s_2 \to 0} \left( -s_2^{-\frac{j}{\sigma}} \left( s_1^\sigma + s_2^\sigma \right)^{-\sigma-j-1} \right) = \lim_{s_2 \to 0} \left( -s_2^{-\frac{j}{\sigma}} s_1^{-\frac{j}{\sigma}} + s_2^{-\frac{j}{\sigma}} \right)^{-\sigma-j-1} = 0
\]
So
\[
\lim_{s_2 \to 0} \tilde{F}_{s_2 | s_1 = s_1}(s_2) = 0.
\]
On the other hand
\[
\lim_{s_2 \to \infty} \frac{s_2^{1 + \frac{1}{\sigma}}}{\left( \frac{1}{s_1^\sigma} + \frac{1}{s_2^\sigma} \right)^{\sigma+1}} = \lim_{s_2 \to \infty} \left( \frac{s_2^{\frac{1}{\sigma}}}{\frac{1}{s_1^\sigma} + \frac{1}{s_2^\sigma}} \right)^{\sigma+1} = 1
\]
So for \( j \in \mathbb{N} \)
\[
\lim_{s_2 \to \infty} \left( s_2^{\frac{j}{\sigma}} s_1^{-\frac{j}{\sigma}} + s_2^{-\frac{j}{\sigma}} \right)^{-\sigma-j-1} = s_1^{\frac{j}{\sigma}} \lim_{s_2 \to \infty} \left( s_2^{\frac{1}{\sigma}} \right)^{\sigma+j+1} = \begin{cases} 0, & \text{for } j \neq 0 \\ 1, & \text{for } j = 0 \end{cases}
\]
It follows that
\[
\lim_{s_2 \to \infty} \tilde{F}_{s_2 | s_1 = s_1}(s_2) = \frac{\sigma \Gamma(\sigma + 2\phi)}{\Gamma(\phi) \Gamma(\sigma + \phi)} \sum_{k=0}^{\phi-1} \left( \frac{\phi - 1}{k} \right) (-1)^{k} \frac{1}{\sigma(\sigma + \phi + k)}
\]
From formula 0.160.2 in Gradshten and Ryzhik [2014] we have that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\Gamma(k + c)}{\Gamma(k + c + 1)} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{k + c} = B(n + 1, c)$$

So we conclude that

$$\lim_{s_2 \to \infty} \hat{F}_{S_2 | S_1 = s_1}(s_2) = \frac{\Gamma(\sigma + 2\phi)B(\phi, \sigma + \phi)}{\Gamma(\phi)\Gamma(\sigma + \phi)} = 1.$$ 

As this limits do not depend on what values $\sigma$ takes in $(0, \infty)$ we conclude that we can construct a subordinator with the desired Lévy copula for any $\sigma \in (0, \infty)$.

**Proof of Theorem 7**

By definition

$$U(y_1, \ldots, y_d) = \int_{0}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_d}^{\infty} h \left( \frac{s_1}{z}, \ldots, \frac{s_d}{z} \right) ds \rho^*(dz)$$

$$= \int_{0}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_d}^{\infty} h(u_1, \ldots, u_d) du \rho^*(dz)$$

$$= \int_{0}^{\infty} S \left( \frac{y_1}{z}, \ldots, \frac{y_d}{z} \right) \rho^*(dz)$$

$$= \int_{0}^{\infty} \hat{C} \left( S_1 \left( \frac{y_1}{z} \right), \ldots, S_d \left( \frac{y_d}{z} \right) \right) \rho^*(dz).$$

Where in the last equation we have used the Sklar theorem for survival copulas

$$S(u_1, \ldots, u_d) = \hat{C} \left( S_1(u_1), \ldots, S_d(u_d) \right).$$

Let $i \in \{1, \ldots, d\}$, if we evaluate the above expression in $(y_1^{(i)}, \ldots, y_{i-1}^{(i)}, y, y_{i+1}^{(i)}, \ldots, y_d^{(i)})$ such that $y \in \mathbb{R}^+$ and $y_1^{(i)} = \cdots = y_{i-1}^{(i)} = y_{i+1}^{(i)} = \cdots = y_d^{(i)} = 0$ we get that

$$U_i(x) = \int_{0}^{\infty} S_i \left( \frac{x}{z} \right) \rho^*(dz)$$

where $S_i$ is the $i-$th marginal survival function associated to the score probability density function $h$. From the Sklar theorem for Lévy copulas, Theorem[I] we conclude the proof.