NONHOLONOMIC APPROACH TO ROTATING MATTER IN
GENERAL RELATIVITY

Mattias Marklund
Department of Plasma Physics,
Umeå University, S-901 87 Umeå, Sweden
e-mail:mattias.marklund@physics.umu.se

Gyula Fodor, Zoltán Perjés
KFKI Research Institute for Particle and Nuclear Physics,
Budapest 114, P.O.Box 49, H-1525 Hungary
e-mail: gfodor@rmki.kfki.hu  perjes@rmki.kfki.hu

Rigidly rotating stationary matter in general relativity has been investigated by Kramer (Class. Quantum Grav. 2 L135 (1985)) by the Ernst coordinate method. A weakness of this approach is that the Ernst potential does not exist for differential rotation. We now generalize the techniques by the use of a nonholonomic and nonrigid frame. We apply these techniques for differentially rotating perfect fluids. We construct a complex analytic tensor, characterizing the class of matter states in which both the interior Schwarzschild and the Kerr solution are contained. We derive consistency relations for this class of perfect fluids. We investigate incompressible fluids characterized by these tensors.

1 Introduction

In this contribution, we briefly describe a new approach to rotating matter in general relativity with applications to differentially rotating perfect fluids. The full details of this method can be found in Ref. 1.

The essence of our treatment is the generalization of the Ernst-potential coordinate method to space-times in which the Ernst potential does not exist at all. (Examples of space-times where the Ernst potential does exist are the stationary axisymmetric vacuum and rigidly rotating perfect fluids.) The complex 1-form $G$, introduced in Ref. 2 exists whenever a (non-null) Killing vector is given. By its use, we define a complex nonholonomic basis for axistationary space-times with the property that it becomes a natural basis whenever an Ernst potential exists. If the latter is not the case, the basis is noncommutative, i.e. the structure functions do not vanish.

The 3-space is conformally flat if the Cotton-York tensor $Y^t_i = \epsilon^{klt} (R_{ijl}[;k] - (1/4) g_{ij} R_{kl})$ vanishes. The Simon tensor is essentially an analytic continuation of the Cotton-York tensor in terms of the Ernst potential. Space-times like the Kerr metric, the interior Schwarzschild, Wahlquist and the Kramer metrics have a vanishing Simon tensor. We further generalize the Simon tensor for situations with no Ernst potential. Solving our field equations for perfect fluids with a vanishing complex tensor $S_{ik}$, we establish the following:

**Theorem.** There are no incompressible perfect fluids with a vanishing $S_{ik}$ tensor.
2 Stationary perfect fluid space-times

We write the metric of a stationary space-time in terms of the metric $g$ of the Killing trajectories:

$$ds^2 = r(dt + \omega dx^i)^2 - \frac{1}{r}g_{ij}dx^j dx^j .$$  \hspace{1cm} (1)

Introducing the 3-dimensional complex 1-form $G \overset{\text{def}}{=} (dr + ir^2 * d\omega)/(2r)$, the Einstein equations become (For our notation, cf. Ref. 1):

$$G^i = (G \cdot G) = (G \cdot \tilde{G}) + kr^{-2}T_{oo},$$  \hspace{1cm} (2)

$$G_{ij} - G_{ji} = G_i G_j + G_j G_i + ikr^{-2}\epsilon_{ijk} T^k_{oo},$$  \hspace{1cm} (3)

$$R_{ij} = -G_i G_j - G_j G_i - kr^{-2} (T^s_{ij} - g_{ij} T^s_{oo}),$$  \hspace{1cm} (4)

where $T^s_{\mu\nu} \overset{\text{def}}{=} T^s_{\mu\nu} - \frac{1}{2}g_{\mu\nu} T$. The energy-momentum tensor is $T^s_{\mu\nu} = (\mu + p)u^\mu u^\nu - p\delta^\nu_\mu$ with the normalization condition for the 4-velocity $u^2 = -g_{jk} u^j u^k = r$.

We introduce the complex nonholonomic basis for axistationary space-times $(e_1, e_2, L)$, where $L = \partial/\partial \varphi$ is the axial Killing vector, and the vectors $e_1$ and $e_2$ are defined by

$$G = \frac{1}{2r}(\alpha e_1 + \beta e_2), \quad \tilde{G} = \frac{1}{2r}(\beta e_1 + \gamma e_2),$$  \hspace{1cm} (5)

with $\alpha = 4r^2 (G \cdot G)$, $\beta = 4r^2 (G \cdot \tilde{G})$, $\gamma = \tilde{\alpha}$. The metric reads

$$[g_{jk}] = \begin{bmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \tilde{\alpha} \end{bmatrix},$$  \hspace{1cm} (6)

where $\tilde{\alpha}^2 = (L \cdot L)$. From the definition of $G$ we have $r (G + \tilde{G}) = dr$, and $e_1 r = e_2 r = 1/2$. The structure functions $c^i_{jk}$ are defined by $[e_j, e_k] = c^i_{jk} e_i$.

The nonvanishing components are $c^1_{12} = -c^2_{12} = 2ik \bar{g} T^s_{oo}/(r \sqrt{D}) \overset{\text{def}}{=} \epsilon$, where $D \overset{\text{def}}{=} \alpha \gamma - \beta^2 < 0$. The Einstein equation (2) takes the form

$$\alpha_1 + \beta_2 - \frac{\alpha}{r} + (\alpha e_1 + \beta e_2) \ln \left( \frac{\bar{g}}{\sqrt{D}} \right) - \frac{2k}{r} T^s_{oo} + (\alpha + \beta) \epsilon = 0 ,$$  \hspace{1cm} (7)

which reduces to the Ernst equation in vacuum.

3 The complex tensor

We introduce the complex tensor

$$S^i = \epsilon^{ijk} \left( 2g_{ij} g^s G_{[k;ir]} G_{s} - 2G_{k;i} G_j - ikr^{-2} \epsilon_{ijk} G_{(i} T^s_{otr)} \right),$$  \hspace{1cm} (8)

which is symmetric and trace-free even in the presence of matter. In vacuo, $S^i_i$ equals the Simon tensor.
In the axisymmetric case the condition $S^\ell_\ell = 0$ gives two equations

$$\alpha_2 = 0, \quad (\alpha e_1 + \beta e_2) \ln \rho - \frac{\alpha}{2r} + \frac{1}{4} \alpha_1 - \frac{k}{r} T_{oo}^* = 0.$$  \hspace{1cm} (9)

Combining with Eq. (7), we obtain

$$D(\ln \rho),_1 + \frac{1}{2} \beta \gamma,2 - \gamma \beta,2 - D \varepsilon = 0, \quad \alpha(\ln \varepsilon),_1 = F, \hspace{1cm} (10)$$

where $F \equiv (k/r) T_{oo}^* - \beta \varepsilon$ and $c \equiv D^{-1/2} r^{-1} (\alpha \gamma)^{3/4} \rho$. We describe the matter by $\varepsilon$ and $F$. By the normalization condition of $u^\mu$,

$$D \varepsilon^2 + [2(F + \beta \varepsilon) + k(\mu - p)][2(F + \beta \varepsilon) - k(\mu + 3p)] = 0. \hspace{1cm} (11)$$

The Bianchi identities give

$$kp,_1 = \frac{1}{2r} \left\{ kp + \frac{2}{\varepsilon}(\beta - \gamma) - F - \beta \varepsilon + [2r(F + \beta \varepsilon) - k(\mu + 3p)](\ln \rho),_1 \right\}. \hspace{1cm} (12)$$

The remaining field equations are as follows,

$$\alpha \alpha,_{11} - \frac{3}{4} (\alpha,)_1^2 = 4\alpha (2\varepsilon \beta, + \beta \varepsilon, + F,_1) - 2(3\beta \varepsilon + 2F)\alpha,_1 + 4F(2\beta \varepsilon + F) + \frac{4\alpha}{r} (\beta \varepsilon + r \gamma \varepsilon^2 + F - kp) \hspace{1cm} (13)$$

$$F,+_1 + \varepsilon F = \gamma \varepsilon^2 + \frac{\alpha}{2rD} [k(\mu + 3p)\beta - (\alpha \gamma + \beta^2) \varepsilon - 2F \beta] \hspace{1cm} (14)$$

$$\alpha F,_1 + (\alpha - \beta) \varepsilon F = \alpha \varepsilon (\gamma - \beta) \left( \varepsilon - \frac{1}{2r} \right) - \frac{3}{4} \beta \varepsilon \alpha,_1 - \alpha \beta \varepsilon,_1. \hspace{1cm} (15)$$

In general, the first order equations can be solved for the $e_1$ derivatives of all our functions except for the fluid energy density $\mu$. For an incompressible fluid the second order equation (13) together with the integrability conditions of our variables gives a system of algebraic equations. The proof of our Theorem follows by showing the inconsistency of these equations. If $\mu$ is not constant, one can still obtain equations linear in $\mu,_{11}$ and $\mu,_{22}$. At this time we do not know if the resulting algebraic equations yield any other solution than those found by Wahlquist and Kramer.

References

1. M. Marklund and Z. Perjész, J. Math. Phys. 38, 5280 (1997).
2. Z. Perjész, J. Math. Phys. 11, 3383 (1970).
3. W. Simon, Gen. Rel. Grav. 16, 465 (1984).
4. H. D. Wahlquist, Phys. Rev. 172, 1291 (1968).
5. D. Kramer, Class. Quantum Grav. 2, L135 (1985).