COMPLEX MULTIPLICATION CYCLES AND KUDLA-RAPOPORT
DIVISORS II

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Abstract. This paper is about the arithmetic of Kudla-Rapoport divisors on Shimura varieties of type GU(n − 1, 1). In the first part of the paper we construct a toroidal compactification of N. Krämer’s integral model of the Shimura variety. This extends work of K.-W. Lan, who constructed a compactification at unramified primes.

In the second, and main, part of the paper we use ideas of Kudla to construct Green functions for the Kudla-Rapoport divisors on the open Shimura variety, and analyze the behavior of these functions near the boundary of the compactification. The Green functions turn out to have logarithmic singularities along certain components of the boundary, up to log-log error terms. Thus, by adding a prescribed linear combination of boundary components to a Kudla-Rapoport divisor one obtains a class in the arithmetic Chow group of Burgos-Kramer-Kühn.

In the third and final part of the paper we compute the arithmetic intersection of each of these divisors with a cycle of complex multiplication points. The computation is quickly reduced to the calculations of the author’s earlier work Complex multiplication cycles and Kudla-Rapoport divisors. The arithmetic intersection multiplicities are shown to appear as Fourier coefficients of the diagonal restriction of the central derivative of a Hilbert modular Eisenstein series.

1. Introduction

Let k be an imaginary quadratic field with discriminant −d_k, let \( x \mapsto \overline{x} \) denote complex conjugation on k, and fix an embedding \( \iota : k \to \mathbb{C} \). We assume throughout that \( d_k \) is odd.

1.1. Integral models of unitary Shimura varieties. For every pair \((r, s)\) of nonnegative integers one may attempt to define a Deligne-Mumford stack \( \mathcal{M}_{r,s} \) over \( \mathcal{O}_k \) as the moduli space of triples \((A, \kappa, \psi)\), where \( A \to S \) is an abelian scheme of dimension \( r + s \) over an \( \mathcal{O}_k \)-scheme \( S \), \( \kappa : \mathcal{O}_k \to \text{End}(A) \) is an action of \( \mathcal{O}_k \) on \( A \), and \( \psi \) is an \( \mathcal{O}_k \)-linear principal polarization of \( A \) (see Section 1.4 for the meaning of \( \mathcal{O}_k \)-linear). One further demands that the induced action of \( \mathcal{O}_k \) on \( \text{Lie}(A) \) satisfies a suitable signature \((r, s)\) condition. Such a signature condition asserts, roughly speaking, that \( \text{Lie}(A) \) behaves like the \( \mathcal{O}_k \otimes \mathbb{Z} \mathcal{O}_S \)-module \( \mathcal{O}_S^r \oplus \mathcal{O}_S^s \), where \( \mathcal{O}_k \) acts on the factor \( \mathcal{O}_S^r \) through the structure morphism \( \mathcal{O}_k \to \mathcal{O}_S \), and acts on the factor \( \mathcal{O}_S^s \) through the complex conjugate of the structure morphism. Of course the subtlety lies in the precise meaning of “behaves like”, and different definitions lead to different moduli spaces. In the case of signature \((m, 0)\) one simply demands that \( \mathcal{O}_k \) acts on \( \text{Lie}(A) \) through the structure map \( \mathcal{O}_k \to \mathcal{O}_S \), and this gives an unambiguous definition of \( \mathcal{M}_{m,0} \). The resulting stack \( \mathcal{M}_{m,0} \) has a very simple structure (Proposition 2.4.2): it is proper and smooth over \( \mathcal{O}_k \) of relative dimension 0. In particular its complex fiber is a
0-dimensional complex orbifold. Similar remarks hold for \( \mathcal{M}_{(0,m)} \), which is obtained from \( \mathcal{M}_{(m,0)} \) by pullback through complex conjugation on the base scheme \( \text{Spec}(\mathcal{O}_k) \).

In the case of signature \((m,1)\) there are at least three competing definitions of the signature condition: a naive definition, and more refined definitions introduced by Pappas \([29]\) and Krämer \([17]\). These three definitions, recalled in Section 2.4, lead to three stacks over \( \mathcal{O}_k \), related by canonical morphisms
\[
\mathcal{M}^{\text{Kra}}_{(m,1)} \to \mathcal{M}^{\text{Pap}}_{(m,1)} \to \mathcal{M}^{\text{naive}}_{(m,1)}
\]
which become isomorphisms after restriction to \( \mathcal{O}_k[1/d_k] \). The naive model is known, by work of Pappas, to be neither flat nor regular. The refined model of Pappas is flat but not regular. Krämer’s model is both flat and regular. It is essential here that \( d_k \) be odd. While the definitions of the stacks in \((1.1.1)\) make sense also for even \( d_k \), there seems to be no reason to expect in this generality that \( \mathcal{M}^{\text{Kra}}_{(m,1)} \) is regular.

If \( m > 0 \) then \( \mathcal{M}^{\text{Kra}}_{(m,1)} \) is typically not proper, and our first main result (Theorem 2.5.2) is the existence of a canonical toroidal compactification of Krämer’s model.

**Theorem A.** Fix \( m \geq 0 \). There is an \( \mathcal{O}_k \)-stack \( \mathcal{M}^\ast_{(m,1)} \) of dimension \( m + 1 \), and a closed substack \( Z \hookrightarrow \mathcal{M}^\ast_{(m,1)} \) of dimension \( m \), with the following properties. The stack \( \mathcal{M}^\ast_{(m,1)} \) is regular, is proper and flat over \( \mathcal{O}_k \), and is smooth over \( \mathcal{O}_k[1/d_k] \). The substack \( Z \) is smooth and proper over \( \mathcal{O}_k \), and \( \mathcal{M}^{\text{Kra}}_{(m,1)} \cong \mathcal{M}^\ast_{(m,1)} \setminus Z \).

**Remark 1.1.1.** In the relatively uninteresting case \( m = 0 \), there are isomorphisms
\[
\mathcal{M}^\ast_{(0,1)} \cong \mathcal{M}^{\text{Kra}}_{(0,1)} \cong \mathcal{M}^{\text{Pap}}_{(0,1)} \cong \mathcal{M}_{(0,1)}.
\]
These stacks are smooth and proper over \( \mathcal{O}_k \).

One can say much more about the boundary divisor \( Z \). For example, the universal abelian scheme \( A \) over \( \mathcal{M}^{\text{Kra}}_{(m,1)} \) extends to a semi-abelian scheme \( G \) over \( \mathcal{M}^\ast_{(m,1)} \). At a geometric point \( \text{Spec}(\mathbb{F}) \to Z \) of the boundary the fiber \( G_{/\mathbb{F}} \) sits in an exact sequence
\[
0 \to T \to G_{/\mathbb{F}} \to B \to 0
\]
where \( T \) is a torus over \( \mathbb{F} \) whose character group is a projective \( \mathcal{O}_k \)-module of rank one, and \( B \) is an abelian variety over \( \mathbb{F} \) of dimension \( m - 1 \) with an action of \( \mathcal{O}_k \). Moreover, \( B \) comes equipped with an \( \mathcal{O}_k \)-linear principal polarization, and defines a point of \( \mathcal{M}_{(m-1,0)}(\mathbb{F}) \). As the geometric point varies over a single irreducible component of \( Z_{/\mathbb{F}} \), the torus \( T \) and the abelian variety \( B \) are constant, but the isomorphism class of the extension \((1.1.2)\) varies.

The proof of Theorem A follows the methods established in Chai-Faltings \([11]\), and Lan’s generalizations \([23]\). Indeed, if one works over \( \mathcal{O}_k[1/d_k] \) instead of \( \mathcal{O}_k \) then Theorem A is a special case of the results of \([23]\).

### 1.2. Arithmetic Kudla-Rapoport divisors.

Our motivation for constructing a good integral model \( \mathcal{M}^\ast_{(m,1)} \) over \( \mathcal{O}_k \) is to have a suitable space on which to do arithmetic intersection theory. In particular, we wish to address some the speculative questions raised in the introduction of \([15]\) concerning the arithmetic intersections of Kudla-Rapoport divisors with cycles of complex multiplication points. The set up is as follows. Fix an integer \( n \geq 1 \) and consider the \( \mathcal{O}_k \)-stack
\[
\mathcal{M}^\ast = \mathcal{M}_{(1,0)} \times_{\mathcal{O}_k} \mathcal{M}^\ast_{(n-1,1)}.
\]
It is regular of dimension \( n \), is flat and proper over \( \mathcal{O}_k \), and contains
\[
\mathcal{M} = \mathcal{M}_{(1,0)} \times_{\mathcal{O}_k} \mathcal{M}^{\text{Kra}}_{(n-1,1)}
\]
as a dense open substack. In a change from the notation of Theorem A we now define
\( Z = M^* \setminus M \)
with its reduced substack structure. In other words, the new \( Z \) is the old \( M_{(1,0)} \times \mathcal{O}_k \mathbb{Z} \).

Following work of Kudla and Rapoport [21, 20], we define a family of divisors \( \mathcal{K}(m) \) on \( M \) indexed by nonzero integers \( m \). These are precisely the divisors denoted \( Z(m) \) in [loc. cit.] and [15]. Let \( \mathcal{K}^*(m) \) be the Zariski closure of \( \mathcal{K}(m) \) in \( M^* \). In [15], one finds the construction of a Green function \( \text{Gr}(m, v) \) on \( M(\mathbb{C}) \) for \( \mathcal{K}(m) \); that is, a smooth function with logarithmic singularities along the complex points of \( \mathcal{K}(m) \). The Green function depends on an auxiliary choice of positive real parameter \( v \). In Section 3.7 we define, for each geometric component \( B \) of the boundary \( Z \), an integer \( \text{Ind}_B(m) \) in such a way that the formal sum
\[
\sum_B \text{Ind}_B(m) \cdot B
\]
is a divisor on \( M^* \) (with real coefficients, and defined over \( \mathcal{O}_k \)). Our second main result amounts to an examination of these Green functions near the boundary of \( M^* \). We prove that each \( \text{Gr}(m, v) \) has logarithmic singularities along certain boundary components, so that \( \text{Gr}(m, v) \), when viewed as a function on \( M^*(\mathbb{C}) \), is a Green function for the divisor \( \mathcal{K}^*(m) + B(m, v) \). Here the term Green function must be interpreted as in the work of Burgos-Kramer-Kühn [7]: \( \text{Gr}(m, v) \) is a Green function for \( \mathcal{K}^*(m) + B(m, v) \) up to log-log error terms along the boundary. The main results of [7] show that the theory of arithmetic Chow groups developed by Gillet-Soulé [13, 32] can be extended to allow for such log-log error terms. The following is a restatement of Theorem 3.7.4 of the text.

**Theorem B.** For any \( m \neq 0 \) and any \( v \in \mathbb{R}^+ \), the pair
\[
\mathcal{K}(m, v) = (\mathcal{K}^*(m) + B(m, v), \text{Gr}(m, v))
\]
defines a class in the Burgos-Kramer-Kühn codimension one arithmetic Chow group \( \widehat{\text{CH}}_1(\mathbb{R}) \) of \( M^* \).

**Remark 1.2.1.** If \( m < 0 \) then both \( \mathcal{K}^*(m) \) and \( B(m, v) \) are zero, but the class \( \mathcal{K}(m, v) \) may still be nontrivial.

The closest result to Theorem B in the literature is found in work of Bruinier-Burgos-Kühn [6]. In an earlier work Bruinier [5] constructed explicit Green functions for the Hirzebruch-Zagier divisors on a Hilbert modular surface \( Y \), and one of the main results of [6] is the analysis of these Green functions near the boundary of a toroidal compactification \( Y^* \). There are, however, at least two natural constructions of Green functions for the Hirzebruch-Zagier divisors: the automorphic Green functions of Bruinier, and the Kudla Green functions as constructed in [16] following the ideas of [18]. Exactly as in Theorem B, Bruinier’s Green function extends to a Green function on \( Y^* \) for the Zariski closure of the Hirzebruch-Zagier divisor, provided one adds to the divisor certain linear combinations of boundary components, and provided one allows for log-log error terms along the boundary. However, Berndt-Kühn [3] have recently shown that no such result holds for the Kudla Green functions on a Hilbert modular surface. That is, even after adding boundary components to the Hirzebruch-Zagier divisor, the Kudla Green function of [16] will not have the correct behavior at the boundary to define a class in the Burgos-Kramer-Kühn arithmetic Chow group \( \widehat{\text{CH}}_1(\mathbb{R}) \). The Green function appearing in Theorem B is constructed in the same manner as the Kudla Green function of [16], and so it may be somewhat surprising that Theorem B holds. Then again, the compactification \( M \leftrightarrow M^* \) is in some ways much
nicer than the toroidal compactification \( Y \hookrightarrow Y^* \) (for example, the boundary of \( Y^* \) is not smooth), and so it may be a mistake to infer too much about one from the other.

A Hermitian lattice is a pair \( (\Lambda, h_\Lambda) \) in which \( \Lambda \) is a finitely generated projective \( \mathcal{O}_k \)-module, and \( h_\Lambda \) is an \( \mathcal{O}_k \)-valued Hermitian form on \( \Lambda \). The Hermitian lattice \( (\Lambda, h_\Lambda) \) is self dual if the map \( \Lambda \rightarrow \text{Hom}_{\mathcal{O}_k}(\Lambda, \mathcal{O}_k) \) defined by \( y \mapsto h_\Lambda(\cdot, y) \) is an isomorphism.

The integers \( \text{Ind}_B(m) \) admit an elegant description in terms of Hermitian lattices. The complex orbifold \( \mathcal{M}^* (\mathbb{C}) \) is disconnected. Its connected components are indexed by isomorphism classes of pairs \( (\mathfrak{A}_0, \mathfrak{A}) \) of self dual Hermitian lattices, where \( \mathfrak{A}_0 \) has signature \((1, 0)\) and \( \mathfrak{A} \) has signature \((n - 1, 1)\). The \( \mathfrak{A}_0 \)'s index the connected components of \( \mathcal{M}^{(1,0)} (\mathbb{C}) \), while the \( \mathfrak{A} \)'s index the connected components of \( \mathcal{M}^{(n-1,1)} (\mathbb{C}) \). The boundary components of \( \mathcal{M}^* (\mathbb{C}) \) are indexed by isomorphism classes of triples \( (\mathfrak{A}_0, m \subset \mathfrak{A}) \) where \( \mathfrak{A}_0 \) and \( \mathfrak{A} \) are as above, and \( m \subset \mathfrak{A} \) is an isotropic direct summand of rank one. Suppose \( B \) is the boundary component of \( \mathcal{M}^* (\mathbb{C}) \) corresponding to the triple \( (\mathfrak{A}_0, m \subset \mathfrak{A}) \). The \( \mathcal{O}_k \)-module \( \text{Hom}_{\mathcal{O}_k}(\mathfrak{A}_0, \mathfrak{A}) \) is a self dual Hermitian lattice in a natural way (its Hermitian form \( \langle \cdot, \cdot \rangle \) is defined by \( \langle \cdot, \cdot \rangle \)), again of signature \((n - 1, 1)\), and contains

\[
\alpha = \text{Hom}_{\mathcal{O}_k}(\mathfrak{A}_0, m)
\]
as an isotropic direct summand of rank one. The quotient \( \Lambda = \alpha^\perp / \alpha \) is a self dual Hermitian lattice of signature \((n - 2, 0)\), and

\[
\text{Ind}_B(m) = \# \{ x \in \Lambda : \langle x, x \rangle = m \}.
\]

The Hermitian lattice \( \Lambda \) has a nice description in terms of abelian schemes. The boundary component \( B \) of \( \mathcal{M}^* (\mathbb{C}) \) corresponds to a complex point \( A_0 \in \mathcal{M}^{(1,0)} (\mathbb{C}) \) (which is simply an elliptic curve with complex multiplication by \( \mathcal{O}_k \)), and a boundary component of \( \mathcal{M}^{(n-1,1)} (\mathbb{C}) \). Recall that this latter boundary component carries over it a family of short exact sequence of group schemes \( [\mathbb{E}] \) over \( \mathbb{C} \), and that \( T \) and \( B \) are constant in the family. There is an isomorphism

\[
\Lambda \cong \text{Hom}_{\mathcal{O}_k}(A_0, B),
\]

where the Hermitian form on the right hand side is

\[
(f_1, f_2) = \psi_{A_0}^{-1} \circ f_2' \circ \psi_B \circ f_1 \in \text{End}_{\mathcal{O}_k}(A_0) \cong \mathcal{O}_k.
\]

Here \( \psi_{A_0} \) and \( \psi_B \) are the principal polarizations of \( A_0 \) and \( B \). Interpreting \( \Lambda \) in this way, the definition of \( \text{Ind}_B(m) \) is seen to mirror the definition (Definition \( 3.2.1 \)) of the divisors \( \text{KR} \).

1.3. Intersections with CM cycles. Having constructed arithmetic cycle classes

\[
\hat{\text{KR}}(m, v) \in \hat{\text{CH}}^1_{\mathbb{R}}(M^*),
\]

we may complete a small part of the speculative program laid out in the introduction of [15]. Let \( F \) be a totally real number field with \( [F : \mathbb{Q}] = n \) (the same \( n \) used in the definition of \( M^* \)), and define a CM field \( K = k \otimes_{\mathbb{Q}} F \). Assume that the discriminant of \( F \) is odd and coprime to \( d_k \). For a suitable choice of CM type \( \Phi \) of \( K \), we define in Section \( 2.1.1 \) a cycle \( X_\Phi \) of dimension one on \( M^*/\mathcal{O}_k \), where \( \mathcal{O}_\Phi \) is the ring of integers in a particular finite extension \( K_\Phi / \mathbb{Q} \). If \( n > 2 \) then \( K_\Phi \) is the reflex field of \( (K, \Phi) \). The cycle \( X_\Phi \) is essentially the cycle of points with complex multiplication by \( \mathcal{O}_K \) and CM type \( \Phi \). Associated to \( X_\Phi \) is a canonical linear functional

\[
\hat{\text{CH}}^1_{\mathbb{R}}(M^*) \rightarrow \mathbb{R},
\]
denoted \( \overline{D} \mapsto [\overline{D} : X_\Phi] \) and called arithmetic intersection against \( X_\Phi \).
The main results of [15] consist of calculations of the intersection multiplicity of naive versions of $X_Φ$ and $KR(m)$ on the (non-compact, non-regular, and non-flat) Shimura variety $\mathcal{M}_{(1,0)} \times \mathbb{C}_k M_{n-1,1}^{naive}$. By reducing to the calculations of [15], we are able to prove in Section 4.2 a precise formula for $(\hat{KR}(m,v) : X_\Phi)$, and show that this value is related to the Fourier coefficients of Eisenstein series.

More precisely, let $E_Φ(τ,s)$ be the Hilbert modular Eisenstein series of weight one of [15, Definition 4.1.1]. It is a nonholomorphic function of the variable $τ \in \mathcal{H}^n$ in the product of complex upper half planes indexed by the $n$ embeddings $F \to \mathbb{R}$, and a meromorphic function of the complex variable $s$, vanishing at the center $s = 0$ of its functional equation. If we pull back by the diagonal embedding $i_F : \mathcal{H} \to \mathcal{H}^n$ and take the central derivative, we obtain a nonholomorphic modular form

$$E'_Φ(i_F(τ),0) = \sum_{m \in \mathbb{Z}} c_Φ(m,v) \cdot q^m$$

of the variable $τ = u + iv \in \mathcal{H}$, where $q = e^{2\pi i}\tau$ as usual. From the calculations of [15] we deduce the following result (Theorem 4.2.3 of the text), which is in accordance with the general yoga of Kudla’s program [19] predicting relations between arithmetic intersections and Fourier coefficients of Eisenstein series.

**Theorem C.** For any nonzero $m \in \mathbb{Z}$, and any $v \in \mathbb{R}^+$,

$$[\hat{KR}(m,v) : X_Φ] = -\frac{h(k)}{w(k)} \frac{\sqrt{N(d_{K/F})}}{2^{r-1}} \cdot c_Φ(m,v).$$

Here $h(k)$ is the class number of $k$, $w(k)$ is the number of roots of unity in $k^\times$, $d_{K/F}$ is the discriminant of $K/F$, and $r$ is the number of primes of $F$ ramified in $K$, including the archimedean primes.

Of course the right hand side of the equality of Theorem C is defined even for $m = 0$, and it is natural to ask if the result can be extended to $m = 0$. The problem is finding the correct definition of $\hat{KR}(0,v)$. For example, it is reasonable to conjecture that there is a (necessarily unique) choice of $\hat{KR}(0,v)$ for which the formal generating series

$$\sum_{m \in \mathbb{Z}} \hat{KR}(m,v) \cdot q^m \in \widehat{CH}_R^1(M)((q))$$

(here $q = e^{2\pi i}\tau$ and $τ = u + iv$) is a nonholomorphic modular form, and that Theorem C holds at $m = 0$ for this choice. Let $A$ denote the universal abelian scheme over $M_{(n-1,1)}^{Kra}$. By definition of Krämer’s moduli problem there is a universal subsheaf $F \subset \text{Lie}(A)$, and the quotient $\text{Lie}(A)/F$ is a line bundle on $M_{(n-1,1)}^{Kra}$. Letting $A_0$ denote the universal elliptic curve over $M_{(0,1)}$, so that $\text{Lie}(A_0)$ is a line bundle on $M_{(1,0)}$, we obtain the cotautological line bundle

$$T = \text{Hom}(\text{Lie}(A_0), \text{Lie}(A)/F)$$

on the product $M = M_{(1,0)} \times \mathbb{C}_k M_{(n-1,1)}^{Kra}$. It follows from Theorem 2.5.2 that the cotautological bundle has a canonical extension to the compactification $M^\ast$. The desired class $\hat{KR}(0,v)$ should be defined endowing the cotautological bundle with some choice of metric depending on $v$, but the precise choice of metric is not obvious to the author.
1.4. **Notation.** Throughout the article, as above, is a quadratic imaginary field of odd discriminant \(-d_k\) with a chosen embedding \(i : k \to \mathbb{C}\). Let \(\delta_k \in k\) be the element determined by \(i(\delta_k) = i \cdot \sqrt{d_k}\), where \(\sqrt{d_k}\) is the positive real square root of \(d_k\). Schemes are always assumed to be locally Noetherian and separated, and *stack* means locally Noetherian and separated Deligne-Mumford stack. If \(A \to S\) is an abelian scheme over an arbitrary base scheme, equipped with an action \(\kappa : \mathcal{O}_k \to \text{End}(A)\), there is an induced action \(x \mapsto \kappa(\pi)^x\) on the dual abelian scheme \(A^\vee\).

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## 2. Integral models of unitary Shimura varieties

In this section we prove Theorem A of the introduction. The proof uses the same machinery as [11, 23, 33], and we omit those details that are adequately documented in the literature.

We use the following notation throughout Section 2. If \(S\) is any \(\mathcal{O}_k\)-scheme, denote by \(i_S : \mathcal{O}_k \to \mathcal{O}_S\) the structure morphism. Define two ideals \(J, J' \subset \mathcal{O}_k \otimes \mathbb{Z} \mathcal{O}_S\) by

\[
J = \ker(\mathcal{O}_k \otimes \mathbb{Z} \mathcal{O}_S \xrightarrow{x \otimes y \mapsto i_S(x)y} \mathcal{O}_S)
\]

\[
J' = \ker(\mathcal{O}_k \otimes \mathbb{Z} \mathcal{O}_S \xrightarrow{x \otimes y \mapsto i_S(\pi)y} \mathcal{O}_S).
\]

Each is a locally free \(\mathcal{O}_S\)-module of rank one. In fact, if we choose any \(\pi \in \mathcal{O}_k\) for which \(\mathcal{O}_k = \mathbb{Z} \oplus \mathbb{Z} \pi\), then \(J\) and \(J'\) are generated as \(\mathcal{O}_S\)-modules by

\[
j = \pi \otimes 1 - 1 \otimes i_S(\pi) \quad \text{and} \quad j' = \pi \otimes 1 - 1 \otimes i_S(\pi),
\]

respectively. Moreover, by direct calculation one may verify the exactness of

\[(2.0.1) \quad \cdots \to \mathcal{O}_k \otimes \mathbb{Z} \mathcal{O}_S \xrightarrow{j} \mathcal{O}_k \otimes \mathbb{Z} \mathcal{O}_S \xrightarrow{j'} \mathcal{O}_k \otimes \mathbb{Z} \mathcal{O}_S \xrightarrow{j} \cdots .
\]

### 2.1. The work of Pappas and Krämer.**

Fix nonnegative integers \(r\) and \(s\), and let \(M_{(r,s)}^\text{naive}\) be the moduli stack of triples \((A, \kappa, \psi)\), in which

- \(A \to S\) is an abelian scheme over an \(\mathcal{O}_k\)-scheme \(S\),
- \(\kappa : \mathcal{O}_k \to \text{End}(A)\) is an action of \(\mathcal{O}_k\) on \(A\) satisfying the *signature \((r,s)\) condition*: locally on \(S\), \(\kappa(x)\) acts on the \(\mathcal{O}_S\)-module \(\text{Lie}(A)\) with characteristic polynomial

\[
(T - i_S(x))^r(T - i_S(\pi))^s \in \mathcal{O}_S[T]
\]

for every \(x \in \mathcal{O}_k\),
- \(\psi : A \to A^\vee\) is an \(\mathcal{O}_k\)-linear principal polarization.

The \(\mathcal{O}_k\)-stack \(M_{(r,s)}^\text{naive}\) is smooth over \(\mathcal{O}_k[1/d_k]\) of relative dimension \(rs\), but at primes dividing \(d_k\) it is not very well behaved. For example it need not be flat or regular. To remedy this, Pappas [29] defines

\[
M_{(r,s)}^\text{Pap} \hookrightarrow M_{(r,s)}^\text{naive}
\]

as the closed substack of triples \((A, \kappa, \psi) \in M_{(r,s)}^\text{naive}(S)\) satisfying *Pappas’s wedge conditions:* the endomorphisms

\[
\wedge^{r+1} j : \wedge^{r+1} \text{Lie}(A) \to \wedge^{r+1} \text{Lie}(A)
\]

and

\[
\wedge^{s+1} j' : \wedge^{s+1} \text{Lie}(A) \to \wedge^{s+1} \text{Lie}(A)
\]
are trivial.

**Remark 2.1.1.** Suppose \( S = \text{Spec}(\mathbb{F}) \) for a field \( \mathbb{F} \). It is easy to see that any finitely generated \( \mathcal{O}_k \otimes \mathbb{F} \)-module is a direct sum of copies of \( \mathcal{O}_k \otimes \mathbb{F} / \mathcal{J} \), and \( \mathcal{O}_k \otimes \mathbb{F} / \mathcal{J}' \). If \( \text{char}(\mathbb{F}) \mid d_k \) then
\[
\mathcal{O}_k \otimes \mathbb{F} \cong (\mathcal{O}_k \otimes \mathbb{F}) / \mathcal{J} \oplus (\mathcal{O}_k \otimes \mathbb{F}) / \mathcal{J}',
\]
and Pappas’s conditions are equivalent to
\[
\text{Lie}(A) \cong ((\mathcal{O}_k \otimes \mathbb{F}) / \mathcal{J})^r \oplus ((\mathcal{O}_k \otimes \mathbb{F}) / \mathcal{J}')^s.
\]
On the other hand, if \( \text{char}(\mathbb{F}) \nmid d_k \) then
\[
(\mathcal{O}_k \otimes \mathbb{F}) / \mathcal{J} \cong (\mathcal{O}_k \otimes \mathbb{F}) / \mathcal{J}'
\]
and Pappas’s conditions are equivalent to
\[
\text{Lie}(A) \cong (\mathcal{O}_k \otimes \mathbb{F})^a \oplus ((\mathcal{O}_k \otimes \mathbb{F}) / \mathcal{J})^b
\]
with \( 2a + b = r + s \) and \( a \leq \min\{r, s\} \).

Fix an integer \( m \geq 1 \). In the easy case of signature \((m,0)\), Pappas’s wedge conditions are equivalent to the single condition \( j \cdot \text{Lie}(A) = 0 \), which is equivalent to \( \mathcal{O}_k \) acting on \( \text{Lie}(A) \) through the structure morphism \( \iota : \mathcal{O}_k \to \mathcal{O}_S \). Proposition 2.1.2 below shows that the stack \( \mathcal{M}_{Pap}^{(m,0)} \) has all the nice properties one could hope for, and hence from now on we will abbreviate
\[
\mathcal{M}_{(m,0)} = \mathcal{M}_{Pap}^{(m,0)}.
\]

The category \( \mathcal{M}_{(m,0)}(\mathbb{C}) \) is easy to describe. There are only finitely many isomorphism classes of objects, and they are in bijection with the isomorphism classes of self dual Hermitian lattices \((\mathfrak{B},h_{\mathfrak{B}})\) of signature \((m,0)\). The bijection identifies the pair \((\mathfrak{B},h_{\mathfrak{B}})\) with the complex torus
\[
B(\mathbb{C}) = (\mathfrak{B} \otimes \mathcal{O}_k \mathbb{C}) / \mathfrak{B}
\]
equipped with its obvious action of \( \mathcal{O}_k \), and with the polarization induced by the perfect symplectic form \( \psi : \mathfrak{B} \otimes \mathbb{Z} \mathfrak{B} \to \mathbb{Z} \) defined by
\[
(2.1.1) \quad \psi(v, w) = \frac{h_{\mathfrak{B}}(v, w) - h_{\mathfrak{B}}(w, v)}{d_k}.
\]
Note that each such \( B \) is isomorphic to a product of CM elliptic curves: if we fix an \( \mathcal{O}_k \)-module isomorphism \( \mathfrak{B} \cong \oplus a_i \), where each \( a_i \) is a fractional \( \mathcal{O}_k \)-ideal, then \( B(\mathbb{C}) \cong \prod \mathbb{C} / a_i \).

This isomorphism need not identify the polarization on the left with the product polarization on the right.

**Proposition 2.1.2.** The morphism \( \mathcal{M}_{(m,0)} \to \text{Spec}(\mathcal{O}_k) \) is proper and smooth of relative dimension 0.

**Proof.** Suppose that \( z \in \mathcal{M}_{(m,0)}(\mathbb{F}) \) is a geometric point, and let \( y \in \text{Spec}(\mathcal{O}_k)(\mathbb{F}) \) be the geometric point below \( z \). To show that \( \mathcal{M}_{(m,0)} \to \text{Spec}(\mathcal{O}_k) \) is smooth of relative dimension 0, it suffices to prove that the corresponding map \( R_y \to R_z \) on completions of étale local rings is an isomorphism. Let \((B_z, \kappa_z, \psi_z)\) be the triple over \( \mathbb{F} \) corresponding to \( z \). The algebraic de Rham homology
\[
H^1_{\text{dR}}(B_z / \mathbb{F}) = \text{Hom}(H^1_{\text{dR}}(B_z / \mathbb{F}), \mathbb{F})
\]
is free of rank \( m \) over \( \mathcal{O}_k \otimes \mathbb{F} \). If \( \mathbb{F} \) has characteristic zero this is obvious by comparison with Betti homology. If \( \mathbb{F} \) has characteristic \( p > 0 \) one first checks that the covariant Dieudonné
module $D_z$ of $B_z$ is free of rank $m$ over $O_k \otimes \mathbb{Z} W(F)$, and then uses the canonical isomorphism $D_z \otimes W(F) \cong H^1_{dR}(B_z/F)$.

Recall the deformation theory of abelian schemes as in [23, Chapter 2] (which is essentially Grothendieck-Messing theory, but without any mention of divided powers). Denote by CLN the category of complete local Noetherian $R_y$-algebras with residue field $\mathbb{F}$. A square-zero thickening is a surjection $\tilde{S} \to S$ in CLN whose kernel $I$ satisfies $I^2 = 0$. If $B \to S$ is an abelian scheme then $B$ always admits (by [26, IV.2.8]) a deformation $B \to \tilde{S}$ to any square zero thickening, and the de Rham homology $D_B(\tilde{S}) = H^1_{dR}(\tilde{B}/\tilde{S})$ is canonically independent of the choice of $\tilde{B}$. Of course

$$D_B(\tilde{S}) \otimes \mathbb{S} \cong D_B(S).$$

The induced Hodge filtration $\text{Fil}^1 H^1_{dR}(\tilde{B}/\tilde{S}) \subset D_B(\tilde{S})$ does depend on the choice of $\tilde{B}$. The fundamental result is that

$$\tilde{B} \mapsto \text{Fil}^1 H^1_{dR}(\tilde{B}/\tilde{S}) \subset D_B(\tilde{S})$$

establishes a bijection between the set of deformations of $B$ to $\tilde{S}$ and the set of $\tilde{S}$-module direct summands of $D_B(\tilde{S})$ lifting the Hodge filtration

$$\text{Fil}^1 H^1_{dR}(B/S) \subset D_B(S).$$

Now take $S = \mathbb{F}$ and $B = B_z$, and abbreviate $D = D_{B_z}$ so that

$$(2.1.2) \quad D(\mathbb{F}) \cong H^1_{dR}(B_z/\mathbb{F}) \cong (O_k \otimes \mathbb{Z} \mathbb{F})^m.$$  

From the Hodge short exact sequence

$$0 \to \text{Fil}^1 H^1_{dR}(B/S) \to D(\mathbb{F}) \to \text{Lie}(B_z) \to 0$$

and $j \cdot \text{Lie}(B_z) = 0$, it is clear that $j \cdot D(\mathbb{F}) \subset \text{Fil}^1 H^1_{dR}(B/S)$. But both are $\mathbb{F}$-module direct summands of $D(\mathbb{F})$ of rank $m$, and so $j \cdot D(\mathbb{F}) = \text{Fil}^1 H^1_{dR}(B/S)$.

Suppose $\tilde{S} \to \mathbb{F}$ is a square-zero thickening. The deformations of $(B_z, \kappa_z, \psi_z)$ to objects of $\mathfrak{M}_{(m,0)}(\tilde{S})$ are now in bijection with the $O_k$-stable $\tilde{S}$-module direct summands $\text{Fil} \subset D(\tilde{S})$ of rank $m$ lifting $j \cdot D(\mathbb{F}) \subset D(\mathbb{F})$, satisfying $j \cdot (D(\tilde{S})/\text{Fil}) = 0$, and which are isotropic for the symplectic form $\langle \cdot, \cdot \rangle$ on $D(\tilde{S})$ induced by the polarization $\psi_z$. Using (2.1.2), a Nakayama’s lemma argument shows that $D(\tilde{S}) \cong (O_k \otimes \mathbb{Z} \tilde{S})^m$, and it follows easily that $\text{Fil} = j \cdot D(\tilde{S})$ is the unique such $\text{Fil}$ (the isotropy condition is satisfied because $\langle jx, jy \rangle = \langle j'x, y \rangle$, and $j'j = 0$). In other words, $(B_z, \kappa_z, \psi_z)$ admits a unique deformation to an object of $\mathfrak{M}_{(m,0)}(\tilde{S})$. Repeating the argument through successive square-zero thickenings in CLN shows that $(B_z, \kappa_z, \psi_z)$ admits a unique deformation to every Artinian object $\tilde{S}$ of CLN. Such deformations are classified by $\text{Hom}_{R_y}(R_z, \tilde{S})$, which therefore contains a single point. Letting $\tilde{S}$ vary over all Artinian quotients of $R_y$ and passing to the inverse limit, we find that $\text{Hom}_{R_y}(R_z, R_y)$ contains a unique element, which is easily seen to be the desired inverse of $R_y \to R_z$.

Finally, we use the valuative criterion of properness to show that $\mathfrak{M}_{(m,0)} \to \text{Spec}(O_k)$ is proper. Let $O_F$ be a discrete valuation ring with fraction field $F$, and suppose we have triple $(A, \kappa, \psi)$ corresponding to some map $\text{Spec}(F) \to \mathfrak{M}_{(m,0)}$. We must show that $A$ has potentially good reduction. If $F$ has mixed characteristic $(0, p)$ we may reduce to the case where $F$ is finitely generated over $\mathbb{Q}$, and fix an embedding $F \to \mathbb{C}$. Now use the fact, noted earlier, that $A_{/\mathbb{C}}$ is isogenous to a product of CM elliptic curves to see that $A$ has potentially good reduction.
Now suppose that $F$ and its residue field have equal characteristic. Fix an étale cover $\mathcal{M} \to \mathcal{M}_{(m,0)}$ with $\mathcal{M}$ a scheme. After possibly enlarging $F$, the point $\text{Spec}(F) \to \mathcal{M}_{(m,0)}$ admits a lift to a map $\text{Spec}(F) \to \mathcal{M}$. If $x \in \mathcal{M}$ is the image of this map, then our triple $(A, \kappa, \psi)$ admits a model over the residue field $k(x) \subset F$. From the first part of the proposition we know that $\mathcal{M}$ is actually étale over $\text{Spec}(\mathcal{O}_k)$, from which it follows that $k(x)$ is a finite extension of its prime subfield. The hypothesis of equal characteristic now implies that the subfield $k(x) \subset F$ is actually contained in $\mathcal{O}_F$, and so $(A, \kappa, \psi)$ has good reduction.

\begin{remark}
Nowhere in the proof of Proposition \ref{prop} did we use our permanent hypothesis that $d_k$ is odd.
\end{remark}

In contrast to the case of signature $(m,0)$, Pappas’s stack $\mathcal{M}_{\text{Pap}}^{(m,1)}$ is not regular. To rectify this, Krämer \cite{17} has defined a new stack $\mathcal{M}_{\text{Kra}}^{(m,1)}$ as the moduli space of quadruples $(A, \kappa, \psi, \mathcal{F})$ over $\mathcal{O}_k$-schemes $S$ in which $(A, \kappa, \psi) \in \mathcal{M}_{\text{Pap}}^{(m,1)}(S)$, and $\mathcal{F} \subset \text{Lie}(A)$ is an $\mathcal{O}_k$-stable $\mathcal{O}_S$-submodule satisfying Krämer’s conditions:

\begin{itemize}
  \item the quotient sheaf $\text{Lie}(A)/\mathcal{F}$ is a locally free $\mathcal{O}_S$-module of rank one,
  \item the action of $\mathcal{O}_k$ on $\mathcal{F}$ is through the structure map $i_S : \mathcal{O}_k \to \mathcal{O}_S$, while the action on $\text{Lie}(A)/\mathcal{F}$ is through the complex conjugate of the structure map. Equivalently, $j \cdot \mathcal{F} = 0$ and $j' \cdot \text{Lie}(A) \subset \mathcal{F}$.
\end{itemize}

The following theorem is a summary of some of the results of Pappas and Krämer on these moduli spaces.

\begin{theorem}[Krämer \cite{17}, Pappas \cite{29}]
The stacks $\mathcal{M}_{\text{Pap}}^{(m,1)}$ and $\mathcal{M}_{\text{Kra}}^{(m,1)}$ are flat over $\mathcal{O}_k$ of relative dimension $m$, and satisfy the following properties.

\begin{enumerate}
  \item The stack $\mathcal{M}_{\text{Kra}}^{(m,1)}$ is regular.
  \item The set $\text{Sing} \subset \mathcal{M}_{\text{Pap}}^{(m,1)}$ of points at which $\mathcal{M}_{\text{Pap}}^{(m,1)} \to \text{Spec}(\mathcal{O}_k)$ is not smooth has dimension zero, and is supported in characteristics dividing $d_k$.
  \item A geometric point $(A, \kappa, \psi) \in \mathcal{M}_{\text{Pap}}^{(m,1)}(\mathbb{F})$ defines an element of $\text{Sing}$ if and only if $j \cdot \text{Lie}(A) = 0$. This condition is equivalent to $\mathcal{O}_k$ acting on $\text{Lie}(A)$ through the structure morphism $i_S : \mathcal{O}_k \to \mathbb{F}$.
  \item Forgetting the subsheaf $\mathcal{F}$ defines a proper surjection $\rho : \mathcal{M}_{\text{Kra}}^{(m,1)} \to \mathcal{M}_{\text{Pap}}^{(m,1)}$, which restricts to an isomorphism $\mathcal{M}_{\text{Kra}}^{(m,1)} \smallsetminus \rho^{-1}(\text{Sing}) \to \mathcal{M}_{\text{Pap}}^{(m,1)} \smallsetminus \text{Sing}$. The inverse of this isomorphism is $(A, \kappa, \psi) \mapsto (A, \kappa, \psi, \mathcal{F})$, where $\mathcal{F} = \ker(j : \text{Lie}(A) \to \text{Lie}(A))$.
  \item Suppose $\text{Spec}(\mathbb{F}) \to \mathcal{M}_{\text{Pap}}^{(m,1)}$ is a geometric point contained in the singular locus $\text{Sing}$. The fiber $\mathcal{M}_{\text{Kra}}^{(m,1)} \times_{\mathcal{M}_{\text{Pap}}^{(m,1)}} \text{Spec}(\mathbb{F})$ is isomorphic to the projective space $\mathbb{P}_m/\mathbb{F}$.
  \item After base change to $\mathcal{O}_k[1/d_k]$, the maps $\mathcal{M}_{\text{Kra}}^{(m,1)} \to \mathcal{M}_{\text{Pap}}^{(m,1)} \to \mathcal{M}_{\text{naive}}^{(m,1)}$ become isomorphisms.
\end{enumerate}
\end{theorem}
As a point of notation, in what follows we usually write \((B, \kappa, \psi)\) for points of the moduli space \(M_{(m-1,0)}\), and \((A, \kappa, \psi)\) for points of \(M^\text{Pap}_{(m,1)}\).

2.2. The Kodaira-Spencer map. Let \(\mathcal{O}\) be an \(\mathcal{O}_k\)-algebra, and \(S\) a smooth \(\mathcal{O}\)-scheme. For any \(S\)-valued point

\[(A, \kappa, \psi) \in M^\text{Pap}_{(m,1)}(S)\]

there is a Kodaira-Spencer map

\[\Phi_{KS} : \text{Lie}(A)^* \otimes_{\mathcal{O}_S} \text{Lie}(A)^* \rightarrow \Omega^1_{S/\mathcal{O}}\]

defined in [23, Chapter 2]. Here \(\text{Lie}(A)^*\) is the \(\mathcal{O}_S\)-dual of \(\text{Lie}(A)\). The Kodaira-Spencer map factors through the quotient sheaf

\[\text{Lie}(A)^* \otimes_{\mathcal{O}_S} \text{Lie}(A)^* / \langle \lambda(x) a \rangle \otimes_{\mathcal{O}_S} \langle \lambda(y) b \rangle : a, b \in \text{Lie}(A)^* \rangle_{x \in \mathcal{O}_k^*} \]

This quotient sheaf is locally free if \(d_k \in \mathcal{O}^*\), but this seems to be rarely the case otherwise. Because of this, the statement analogous to [23, Proposition 2.3.5.2] in our setting is not quite correct, but the same proof yields the following weaker result.

**Proposition 2.2.1.** Suppose the tuple \((X, Z, U)\) corresponds to a morphism

\[f : S \rightarrow \left(M^\text{Pap}_{(m,1)} \setminus \text{Sing}\right)/\mathcal{O} \rightarrow \mathcal{O}\]

The morphism \(f\) is unramified if and only if the Kodaira-Spencer map \(\Phi_{KS}\) is surjective.

2.3. Degenerating abelian schemes. Fix a projective \(\mathcal{O}_k\)-module \(n\) of rank one, let \(n\) be the associated constant \(\mathcal{O}_k\)-module over \(\text{Spec}(\mathbb{Z})\), and denote by \(T_n = \text{Spec}(\mathbb{Z}[n])\) the split torus with character group \(\text{Hom}(T_n, \mathbb{G}_m) \cong n\). In what follows, \(X\) is a stack over \(\mathcal{O}_k\), \(Z \rightarrow X\) is a closed substack, and \(U \subset X \setminus Z\) is a dense open substack.

**Definition 2.3.1.** A semi-abelian scheme over \(X\) is a smooth commutative group scheme \(G \rightarrow X\), such that for every geometric point \(\text{Spec}(\mathbb{F}) \rightarrow X\) the fiber \(G_{/\mathbb{F}}\) is an extension

\[0 \rightarrow T_n \rightarrow G_{/\mathbb{F}} \rightarrow B \rightarrow 0\]

of an abelian variety by a torus.

**Definition 2.3.2.** A degenerating abelian scheme of type \(n\) relative to \((X, Z, U)\) is a quadruple \((G, \kappa, \psi, n)\) in which

- \(G\) is a semi-abelian scheme over \(X\), such that \(G_{/U}\) is an abelian scheme,
- \(\kappa : \mathcal{O}_k \rightarrow \text{End}(G_{/U})\) is an action of \(\mathcal{O}_k\) on \(G_{/U}\),
- \(\psi : G_{/U} \rightarrow G_{/U}^\vee\) is an \(\mathcal{O}_k\)-linear principal polarization,
- there is an abelian scheme \(B_Z\) over \(Z\) equipped with an \(\mathcal{O}_k\)-linear action, and an \(\mathcal{O}_k\)-linear exact sequence

\[0 \rightarrow T_{n/Z} \rightarrow G_{/Z} \rightarrow B_Z \rightarrow 0.\]

By [23, Proposition 3.3.15], the action \(\kappa : \mathcal{O}_k \rightarrow \text{End}(G_{/U})\) of the second condition extends uniquely to an action of \(\mathcal{O}_k\) on \(G\), so the property of \(\mathcal{O}_k\)-linearity in the final condition makes sense.

**Definition 2.3.3.** Degeneration data of type \(n\) relative to the triple \((X, Z, U)\) consists of a septuple \((B, \kappa, \psi, n, c, c^\vee, \tau)\) in which

- \(B \rightarrow X\) is an abelian scheme,
- \(\kappa : \mathcal{O}_k \rightarrow \text{End}(B)\) is an action of \(\mathcal{O}_k\) on \(B\),
- \(\psi : B \rightarrow B^\vee\) is an \(\mathcal{O}_k\)-linear principal polarization,
• \( n \) is a projective \( \mathcal{O}_k \)-module of rank one,
• \( c : \mathfrak{g}_X \to B^\vee \) and \( c^\vee : \mathfrak{g}_X \to B \) are \( \mathcal{O}_k \)-module maps satisfying \( c = \psi \circ c^\vee \),
• \( \tau \) is a positive, symmetric, \( \mathcal{O}_k \)-linear isomorphism
\[
\tau : 1_{(\mathfrak{g} \times \mathfrak{g})/U} \to (c^\vee \times c)^* \mathcal{P}^{-1}|_{(\mathfrak{g} \times \mathfrak{g})/U}
\]
of \( \mathbb{G}_m \)-biextensions on \( (\mathfrak{n} \times \mathfrak{n})/U \). Here \( \mathcal{P} \) is the Poincare sheaf on \( B \times B^\vee \).

The entry \( \tau \) requires some further explanation. To give a \( \mathbb{G}_m \)-biextension on \( \mathfrak{n} \times \mathfrak{n} \) is equivalent to giving a collection of invertible sheaves \( \{ \mathcal{L}(\mu, \nu) \}_{(\mu, \nu) \in \mathfrak{n} \times \mathfrak{n}} \) on \( X \), together with isomorphisms
\[
\mathcal{L}(\mu_1 + \mu_2, \nu) \cong \mathcal{L}(\mu_1, \nu) \otimes \mathcal{L}(\mu_2, \nu)
\]
and
\[
\mathcal{L}(\mu, \nu_1 + \nu_2) \cong \mathcal{L}(\mu, \nu_1) \otimes \mathcal{L}(\mu, \nu_2)
\]
satisfying certain partial group law axioms. Each pair \( (\mu, \nu) \) determines sections \( c(\nu) : X \to B^\vee \) and \( c^\vee(\mu) : X \to B \), and the biextension \((c^\vee \times c)^* \mathcal{P}^{-1}\) corresponds to the collection of sheaves \( \mathcal{L}_\mathcal{P}(\mu, \nu)^{-1} \), where
\[
\mathcal{L}_\mathcal{P}(\mu, \nu) = (c^\vee(\mu) \times c(\nu))^* \mathcal{P}.
\]

There are canonical isomorphisms
\[(2.3.1) \quad \mathcal{L}_\mathcal{P}(\mu, \nu) \cong \mathcal{L}_\mathcal{P}(\nu, \mu) \quad \mathcal{L}_\mathcal{P}(x\mu, \nu) \cong \mathcal{L}_\mathcal{P}(\mu, x\nu)\]
reflecting the symmetry and \( \mathcal{O}_k \)-linearity of the polarization \( \psi \). The trivial biextension \( 1_{\mathfrak{g} \times \mathfrak{g}} \) corresponds to the constant collection of invertible sheaves \( \mathcal{L}_{\text{triv}}(\mu, \nu) = \mathcal{O}_X \). Thus the isomorphism \( \tau \) is determined by a collection of trivializations
\[
\tau(\mu, \nu) : \mathcal{O}_U \cong \mathcal{L}_\mathcal{P}(\mu, \nu)^{-1}|_U.
\]
The conditions of \textit{symmetry} and \( \mathcal{O}_k \)-\textit{linearity} on \( \tau \) are that \( \tau(\mu, \nu) = \tau(\nu, \mu) \) and \( \tau(x\mu, \nu) = \tau(\mu, x\nu) \) under the identifications \((2.3.1)\). The condition of \textit{positivity} is that for every \( \mu \in \mathfrak{n} \), the isomorphism \( \tau(\mu, \mu) \) extends (necessarily uniquely) to a homomorphism
\[
\tau(\mu, \mu) : \mathcal{O}_X \to \mathcal{L}_\mathcal{P}(\mu, \mu)^{-1},
\]
and that if \( \mu \neq 0 \) this homomorphism becomes trivial after restricting to \( Z \).

We next recall one of the fundamental results of Mumford and Chai-Faltings \cite{11}: an equivalence of categories between degenerating abelian schemes and degeneration data. Suppose that \( R \) is a Noetherian normal domain complete with respect to an ideal \( I \) satisfying \( \text{rad}(I) = I \). For the remainder of this subsection,
\[(X, Z, U) = (\text{Spec}(R), \text{Spec}(R/I), \{\eta\}),\]
where \( \eta \) is the generic point of \( \text{Spec}(R) \). For the proof of the following fundamental result, see \cite{11} Corollary III.7.2 or \cite{23} Theorem 5.1.4.4.

\textbf{Theorem 2.3.4.} The category of degeneration data relative to \((X, Z, U)\) is equivalent to the category of degenerating abelian schemes relative to \((X, Z, U)\) (in both categories, morphisms are isomorphisms in the obvious sense).

For us, the equivalence of the theorem is mostly a black box. However, we do need to know at least some information about how a degenerating abelian scheme \((G, \kappa, \psi, n)\) is related to its associated degeneration data \((B, \kappa, \psi, n, c, c^\vee, \tau)\). In particular, we need to know how the Lie algebras of \( G \) and \( B \) are related. The relation we need is provided by
the theory of Raynaud extensions as in [11, Chapter II.1] or [23, Section 3.3.3]: if we set $X_\ell = \text{Spec}(R/I^\ell)$, then for every $\ell$ there is an $O_k$-linear short exact sequence

$$0 \to T_{n/X_\ell} \to G_{/X_\ell} \to B_{/X_\ell} \to 0.$$ 

Passing to Lie algebras and then taking the inverse limit over $\ell$, we find a short exact sequence of $R$-modules

$$(2.3.2) \quad 0 \to n^* \otimes Z R \to \text{Lie}(G) \to \text{Lie}(B) \to 0,$$ 

where $n^* = \text{Hom}(n, Z) \cong \text{Hom}(G_m, T_n)$ is the cocharacter group of $T_n$.

Our definition of degenerating abelian scheme is rather restrictive. For example, it only allows for $G/Z$ to be an extension of an abelian scheme by a torus of the form $T_n$. That is, a torus whose character group is projective of rank one over $O_k$. The following lemma tells us that such extensions are the only ones that need concern us. Keep $(X, Z, U)$ as above, but assume also that $R$ is an $O_k$-algebra.

**Lemma 2.3.5.** Suppose $(A, \kappa, \psi) \in \mathcal{H}_{(m, 1)}^{\text{Pap}}(U)$, and that $A = G_U$ for some semi-abelian scheme $G$ over $X$. Suppose also that $G/Z$ sits in an $O_k$-linear exact sequence

$$0 \to T_Z \to G_{/Z} \to B_Z \to 0$$ 

where $T_Z$ is a nontrivial split torus over $Z$, and $B_Z$ is an abelian scheme. There is an isomorphism $T_Z \cong T_{n/Z}$ for some projective $O_k$-module $n$ of rank one. In other words, $(G, \kappa, \psi, n)$ is a degenerating abelian scheme. Furthermore, if $(B, \kappa, \psi, n, c, c', \tau)$ is the corresponding degeneration data, then

$$(B, \kappa, \psi) \in \mathcal{H}_{(m-1, 0)}(X).$$

**Proof.** By the theory of Raynaud extensions alluded to above (and Groethendieck’s formal existence theorem), the group schemes $T_Z$ and $B_Z$ lift to a split torus $T$ over $X$, and an abelian scheme $B$ over $X$, both with $O_k$-action, in such a way that for every positive integer $\ell$ one has an exact sequence

$$0 \to T_{n/X_\ell} \to G_{/X_\ell} \to B_{/X_\ell} \to 0.$$ 

Taking Lie algebras and passing to the limit, we find an exact sequence

$$0 \to X_*(T) \otimes Z R \to \text{Lie}(G) \to \text{Lie}(B) \to 0$$ 

where $X_*(T)$ is the cocharacter group of $T$. We must first show that $X_*(T)$ has rank 1 as an $O_k$-module. Suppose not, so that $X_*(T) \otimes Z R_\eta$ is free of rank $r \geq 2$ over $O_k \otimes Z R_\eta$. Let $e_1, e_2, \ldots, e_r$ be generators of $X_*(T) \otimes Z R_\eta$ as an $O_k \otimes Z R_\eta$-module. Recalling that $O_k = \mathbb{Z} \otimes \mathbb{Z} \pi$, the set $e_1, \pi e_1, e_2, \pi e_2$ can be extended to an $R_\eta$-basis of $\text{Lie}(A) = \text{Lie}(G) \otimes R R_\eta$.

From this it follows that

$$(je_1) \wedge (je_2) = (\pi e_1 + i_R(\pi) e_1) \wedge (\pi e_2 + i_R(\pi) e_2) \neq 0,$$ 

contradicting our assumption that $(A, \kappa, \psi)$ satisfies Pappas’s wedge conditions. This shows that $r = 1$, and so $T \cong T_n$ where $n^* \cong X_*(T)$.

Now that we know $(G, \kappa, \psi, n)$ is a degenerating abelian scheme, to complete the proof we must show that the corresponding degeneration data $(B, \kappa, \psi)$ satisfies $j \cdot \text{Lie}(B) = 0$. From $2.3.2$ we find the exact sequence

$$0 \to O_k \otimes Z R_\eta \to \text{Lie}(A) \to \text{Lie}(B) \otimes R R_\eta \to 0.$$ 

But Remark $2.3.1$ tells us that

$$\text{Lie}(A) \cong (O_k \otimes Z R_\eta) \oplus ((O_k \otimes Z R_\eta)/J)^{m-1}.$$
It follows that $j$ kills $\text{Lie}(B) \otimes_R R_n$, and hence also kills $\text{Lie}(B)$. \hfill \Box

The following lemma is a partial converse to Lemma 2.3.5.

**Lemma 2.3.6.** Suppose $(B, \kappa, \psi, n, c, c^\vee, \tau)$ is degeneration data relative to $(X, Z, U)$, and assume $(B, \kappa, \psi) \in \mathcal{M}_{(m-1,0)}(X)$. If $(G, \kappa, \psi, n)$ is the associated degenerating abelian scheme, then

$$(G_{/U}, \kappa, \psi) \in \mathcal{M}_{(m,1)}^\text{ap}(U).$$

Moreover, the $R$-submodule $\mathcal{F} = \ker(j : \text{Lie}(G) \to \text{Lie}(G))$ of $\text{Lie}(G)$ satisfies Krämer’s conditions, and in particular

$$(G_{/U}, \kappa, \psi, \mathcal{F}) \in \mathcal{M}_{(m,1)}^\text{Kr}(U).$$

**Proof.** As an $\mathcal{O}_k \otimes_Z R$-module, $\text{Lie}(B) \cong R^{m-1}$, where $\mathcal{O}_k$ acts on $R$ through the structure map $i_R : \mathcal{O}_k \to R$. That is, $R \cong (\mathcal{O}_k \otimes_Z R)/J$. The exact sequence (2.3.2) may therefore be rewritten as

$$0 \to n^* \otimes_Z R \to \text{Lie}(G) \to R^{m-1} \to 0.$$  

Using the exactness of (2.0.1), the free resolution

$$\ldots \to \mathcal{O}_k \otimes_Z O_S \xrightarrow{j} \mathcal{O}_k \otimes_Z O_S \xrightarrow{i} \mathcal{O}_k \otimes_Z O_S \xrightarrow{x \otimes y - i_R(x)y} R \to 0$$

of $R$ allows us to compute

$$\text{Ext}^i_{\mathcal{O}_k \otimes_Z R}(R, n^* \otimes_Z R) = 0$$

for $i > 0$. Thus (2.3.2) splits and

$$\text{Lie}(G) \cong (n^* \otimes_Z R) \oplus R^{m-1}$$

as $\mathcal{O}_k \otimes_Z R$-modules. All claims now follow easily. \hfill \Box

### 2.4. Construction of boundary charts.

Fix a projective $\mathcal{O}_k$-module $n$ of rank one. We now construct a closed immersion of $\mathcal{O}_k$-stacks $s_n : Z_n \to X_n$ equipped with tautological degeneration data of type $n$ relative to $(X_n, Z_n, U_n)$, where $U_n = X_n \setminus Z_n$. The stack $X_n$ will have the structure of a line bundle over $Z_n$, and $s_n$ is the zero section.

For a scheme $S$ over $\mathcal{M}_{(m-1,0)}$ set

$$Z_n(S) = \text{Hom}_{\mathcal{O}_k}(\underline{n}/S, B^\vee_{/S}),$$

where $(B, \kappa, \psi)$ is the universal object over $\mathcal{M}_{(m-1,0)}$. By [23, Proposition 5.3.3.10] the functor $Z_n$ is represented by a smooth proper stack over $\mathcal{M}_{(m-1,0)}$, denoted again by $Z_n$, of relative dimension $m - 1$. After pulling back the triple $(B, \kappa, \psi)$ to $Z_n$ one obtains a tautological $\mathcal{O}_k$-linear morphism $c : \underline{n} \to B^\vee$ of stacks over $Z_n$. Set $c^\vee = \psi^{-1} \circ c : \underline{n} \to B$.

Our assumption that $d_k$ is odd implies that

$$Q_n = \text{Sym}^2_\mathbb{Z}(n)/\langle (x\mu) \otimes \nu - \mu \otimes (x\nu) : x \in \mathcal{O}_k, \mu, \nu \in n \rangle,$$

is a free $\mathbb{Z}$-module of rank one. Let $E_n$ be the torus over $\text{Spec}(\mathbb{Z})$ defined by

$$E_n \cong \text{Hom}(Q_n, \mathbb{G}_m).$$

The group law on $Q_n$ will be written additively. An ordered pair $(\mu, \nu) \in n \times n$ determines sections $\mu, \nu : Z_n \to \underline{n}$ which in turn define morphisms $c^\vee(\mu) : Z_n \to B$ and $c(\nu) : Z_n \to B^\vee$.

The pullback

$$\mathcal{L}_P(\mu, \nu) = (c^\vee(\mu) \times c(\nu))^* \mathcal{P}$$
of the Poincare sheaf $\mathcal{P}$ on $B \times B'$ is an invertible sheaf on $Z_n$. Up to canonical isomorphism, the sheaf $\mathcal{L}_P(\mu, \nu)$ depends only on the image of $\mu \otimes \nu$ in $Q_n$. Thus we may associate an invertible sheaf $\mathcal{L}_P(q)$ to each $q \in Q_n$ in such a way that

$$\mathcal{L}_P(q_1 + q_2) \cong \mathcal{L}_P(q_1) \otimes \mathcal{L}_P(q_2)$$

canonicaly. These isomorphisms define an $O_{Z_n}$-module structure on the sheaf $O_{U_n} = \bigoplus_{q \in Q_n} \mathcal{L}_P(q)$, which is therefore the structure sheaf of a $Z_n$-stack

$$U_n = \text{Spec}_{Z_n} \left( \bigoplus_{q \in Q_n} \mathcal{L}_P(q) \right).$$

For each $q \in Q_n$, multiplication in $O_{U_n}$ defines an isomorphism of $O_{Z_n}$-modules $\mathcal{L}_P(q) \otimes O_{U_n} \to O_{U_n}$. Thus, after pullback via $U_n \to Z_n$ each of the sheaves $\mathcal{L}_P(q)$ acquires a canonical trivialization, and dualizing yields isomorphisms

$$\tau(q) : O_{U_n} \to \mathcal{L}_P(q)^{-1}$$

of sheaves of $O_{U_n}$-modules. This collection of isomorphisms defines an isomorphism of $\mathbb{G}_m$-biextensions

$$\tau : 1_{\mathbb{A} \times \mathbb{A}} \to (c^\vee \times c)^\ast \mathcal{P}^{-1}$$

over $(n \times n)/U_n$.

The $\mathbb{R}$-vector space $Q_{n,\mathbb{R}} = Q_n \otimes_{\mathbb{Z}} \mathbb{R}$ has a notion of positivity

$$Q_{n,\mathbb{R}}^\geq = \{ \mu \otimes \mu \in Q_{n,\mathbb{R}} : \mu \in n\mathbb{R} \},$$

and there is an induced ordering $\geq$ on $Q_n$. Define a partial compactification $U_n \to X_n$ by

$$X_n = \text{Spec}_{Z_n} \left( \bigoplus_{q \geq 0} \mathcal{L}_P(q) \right).$$

The ideal sheaf

$$\mathcal{I}_n = \bigoplus_{q > 0} \mathcal{L}_P(q)$$

defines a closed substack of $X_n$ canonically identified with $Z_n$. On the other hand, the inclusion $O_{Z_n} \to O_{X_n}$ induces a morphism $X_n \to Z_n$, giving $X_n$ the structure of a vector bundle over $Z_n$. In particular, $X_n$ is smooth of relative dimension $m$ over $O_k$. We now have tautological degeneration data $(B, \kappa, \psi, n, c, c^\vee, \tau)$ relative to $(X_n, Z_n, U_n)$. Let

$$Z = \bigsqcup_n Z_n,$$

where the union is over the isomorphism classes of projective $O_k$-modules of rank one. The stack $Z$ is smooth and proper over $O_k$ of relative dimension $m-1$, and will soon become the boundary of our compactification of $\mathbb{M}^{Kra}_{(m,1)}$.

2.5. Attaching the boundary. We next attach $Z$ onto $\mathbb{M}^{Kra}_{(m,1)}$. For each geometric point $z$ of $Z_n$ let $R_z = O_{X_n,z}$ be the étale local ring of $X_n$ at $z$, and let $I_z \subset R_z$ be the ideal defined by the divisor $Z_n \to X_n$. Let $\hat{R}_z$ be the completion of $R_z$ with respect to $I_z$, and let $\eta_z$ and $\eta_z^\vee$ denote the generic points of $R_z$ and $\hat{R}_z$, respectively. As $X_n$ is smooth over $O_k$, the $O_k$-algebras $R_z$ and $\hat{R}_z$ are Noetherian normal domains. Applying Theorem 2.3.4 to the pullback of the tautological degeneration data relative to $(X_n, Z_n, U_n)$, one obtains a degenerating abelian scheme $(\nabla G_z, \nabla \kappa_z, \nabla \psi_z, n)$ relative to

$$(X^\vee_z, Z^\vee_z, U^\vee_z) = (\text{Spec}(R_z^\vee), \text{Spec}(R_z^\vee/I_z), \{ \eta_z^\vee \}).$$
For every étale neighborhood $X^{(z)} \to X_n$ of a geometric point $z$, define a closed subscheme of $X^{(z)}$ by

$$Z^{(z)} = Z_n \times_{X_n} X^{(z)},$$

and an open subscheme

$$U^{(z)} = U_n \times_{X_n} X^{(z)}.$$

**Proposition 2.5.1.** For every geometric point $z$ of $Z_n$ there is an étale neighborhood $X^{(z)} \to X_n$ of $z$ and a degenerating abelian scheme $(G^{(z)}, \kappa^{(z)}, \psi^{(z)}, n)$ relative to $(X^{(z)}, Z^{(z)}, U^{(z)})$ with the following properties.

1. There exists a ring automorphism $\gamma : R_z^\wedge \to R_z^\wedge$ inducing the identity on $R_z^\wedge/I_z$, such that

$$(G^{(z)}, \kappa^{(z)}, \psi^{(z)})/R_z^\wedge \cong \gamma^* (\varpi G_z, \varpi \kappa_z, \varpi \psi_z),$$

where the left hand side is the pullback of $(G^{(z)}, \kappa^{(z)}, \psi^{(z)})$ via the canonical map $\text{Spec}(R_z^\wedge) \to X^{(z)}$, and the right hand side is the pullback of $(\varpi G_z, \varpi \kappa_z, \varpi \psi_z)$ via $\gamma : R_z^\wedge \to R_z^\wedge$.

2. The tuple $(G^{(z)}, \kappa^{(z)}, \psi^{(z)})/U^{(z)}$ defines an étale morphism

$$U^{(z)} \to \mathfrak{M}_{(m, 1)} \setminus \text{Sing.}$$

3. The subsheaf $F^{(z)} = \ker(j : \text{Lie}(G^{(z)}) \to \text{Lie}(G^{(z)}))$ of $\text{Lie}(G^{(z)})$ satisfies Krämer’s conditions.

**Proof.** The degenerating abelian scheme $(\varpi G_z, \varpi \kappa_z, \varpi \psi_z, n)$ need not descend to a degenerating abelian scheme relative to

$$(X_z, Z_z, U_z) = (\text{Spec}(R_z), \text{Spec}(R_z/I_z), \{\eta_z\})$$

but it can be approximated arbitrarily closely by degenerating abelian schemes that do descend. This means that, as in [11, Proposition IV.4.3] or [23, Proposition 6.3.2.1], there is a degenerating abelian scheme $(G_z, \kappa_z, \psi_z, n)$ relative to $(X^{(z)}, Z^{(z)}, U^{(z)})$ and a ring automorphism $\gamma \in \text{Aut}(R_z^\wedge)$ inducing the identity on $R_z^\wedge/I_z$ such that

$$(G_z, \kappa_z, \psi_z)/R_z^\wedge \cong \gamma^*(\varpi G_z, \varpi \kappa_z, \varpi \psi_z).$$

Moreover, if we denote by $\mathcal{O}_{k, z}$ the strict Henselization of $\mathcal{O}_k$ at $z$, the Kodaira-Spencer map

$$\text{Lie}(G_z/U_z)^* \otimes_{R_z/nz} \text{Lie}(G_z/U_z)^* \to \Omega^1_{X_z/\mathcal{O}_{k, z}}/U_z$$

of 2.22 extends, as in [23, Proposition 6.2.5.18], to a surjection of $R_z$-modules

$$\text{Lie}(G_z)^* \otimes_{R_z} \text{Lie}(G_z)^* \to \Omega^1_{X_z/\mathcal{O}_{k, z}}(d \log \infty),$$

where $\Omega^1_{X_z/\mathcal{O}_{k, z}}(d \log \infty) = \Omega^1_{X_z/\mathcal{O}_{k, z}} \otimes_{R_z} I_z^{-1}$.

As in [11, Proposition IV.4.4] or [23, Proposition 6.3.2.6], there is some étale neighborhood $X^{(z)}$ of $z$ in $X_n$ such that $(G_z, \kappa_z, \psi_z)$ descends to a degenerating abelian scheme $(G^{(z)}, \kappa^{(z)}, \psi^{(z)}, n)$ relative to $(X^{(z)}, Z^{(z)}, U^{(z)})$. By Lemma 2.3.6, the subsheaf

$$F^{(z)} = \ker(j : \text{Lie}(G^{(z)}) \to \text{Lie}(G^{(z)}))$$

satisfies Krämer’s conditions locally at $z$, and so after shrinking $X^{(z)}$ we may assume that it satisfies these conditions everywhere on $X^{(z)}$. In particular,

$$(G^{(z)}, \kappa^{(z)}, \psi^{(z)}, F^{(z)})/U^{(z)} \in \mathfrak{M}_{(m, 1)}^{\text{Kra}}(U^{(z)}),$$
and the map

\[ U(z) \rightarrow \mathcal{M}_{(m,1)}^{\text{Pap}} \]

corresponding to \((G(z), \kappa(z), \psi(z))\) does not meet the singular locus \(\text{Sing}\) (this is clear from the characterization of \(\text{Sing}\) found in Theorem 2.1.4). Combining the surjectivity of \(\mathcal{O}_k\) with Proposition 2.2.1 shows that after further shrinking \(X(z)\) the map \(U(z) \rightarrow \mathcal{M}_{(m,1)}^{\text{Pap}}\) is unramified. By [23 Corollary 6.3.1.13] it is also étale.

As \(\mathcal{M}_{(m,1)}^{\text{Kra}} \rightarrow \mathcal{M}_{(m,1)}^{\text{Pap}}\) is an isomorphism away from the closed set \(\text{Sing}\), each of the maps \(U(z) \rightarrow \mathcal{M}_{(m,1)}^{\text{Pap}}\) of the proposition admits a unique lift to an étale morphism

\[ U(z) \rightarrow \mathcal{M}_{(m,1)}^{\text{Kra}}. \]

By the quasi-compactness of \(Z_n\), we may choose finitely many geometric points \(z\) so that the union of the images of \(X(z) \rightarrow X_n\) cover \(Z_n\). Letting \(n\) vary over all isomorphism classes of projective \(\mathcal{O}_k\)-modules of rank one, let \(X\) be the disjoint union of the finitely many \(X(z)\)'s so constructed, and let \(U \subset X\) be the disjoint union of the finitely many \(U(z)'s\). The obvious map

\[ \mathcal{M}_{(m,1)}^{\text{Kra}} \sqcup U \rightarrow \mathcal{M}_{(m,1)}^{\text{Kra}} \]

is an étale surjection, and realizes \(\mathcal{M}_{(m,1)}^{\text{Kra}}\) as the quotient of \(\mathcal{M}_{(m,1)} \sqcup U\) by an étale equivalence relation

\[ R_0 \rightarrow (\mathcal{M}_{(m,1)}^{\text{Kra}} \sqcup U) \times_{\mathcal{O}_k} (\mathcal{M}_{(m,1)}^{\text{Kra}} \sqcup U). \]

The normalization of \((\mathcal{M}_{(m,1)}^{\text{Kra}} \sqcup X) \times_{\mathcal{O}_k} (\mathcal{M}_{(m,1)}^{\text{Kra}} \sqcup X)\) in \(R_0\) defines a new stack, \(R\), sitting in the commutative diagram

\[
\begin{array}{ccc}
R & \rightarrow & R_0 \\
\downarrow & & \downarrow \\
(\mathcal{M}_{(m,1)}^{\text{Kra}} \sqcup X) \times_{\mathcal{O}_k} (\mathcal{M}_{(m,1)}^{\text{Kra}} \sqcup X) & \rightarrow & (\mathcal{M}_{(m,1)}^{\text{Kra}} \sqcup U) \times_{\mathcal{O}_k} (\mathcal{M}_{(m,1)}^{\text{Kra}} \sqcup U). \\
\end{array}
\]

Exactly as in [11 Proposition IV.5.4] or [23 Proposition 6.3.3.13], the morphism \(r\) is an étale equivalence relation. Let \(\mathcal{M}_{(m,1)}^*\) be the quotient of \(\mathcal{M}_{(m,1)}^{\text{Kra}} \sqcup X\) by \(r\). The following theorem summarizes the important properties of \(\mathcal{M}_{(m,1)}^*\). All of the claims are clear from the construction, except for properness. Properness is proved using the valuative criterion and Lemma 2.3.3 exactly as in [23 Proposition 6.3.1.17] or the discussion following [11 Definition IV.5.6].

**Theorem 2.5.2.** The stack \(\mathcal{M}_{(m,1)}^*\) is regular. It is proper and flat over \(\mathcal{O}_k\) of relative dimension \(m\), and is smooth over \(\mathcal{O}_k[1/d_k]\). It contains \(Z\) as a closed codimension one substack, smooth over \(\mathcal{O}_k\), and contains \(\mathcal{M}_{(m,1)}^{\text{Kra}} \cong \mathcal{M}_{(m,1)}^* \setminus Z\) as an open dense substack.

Moreover, the universal abelian scheme with \(\mathcal{O}_k\)-action, \(A\), over \(\mathcal{M}_{(m,1)}^{\text{Kra}}\) extends to a semi-abelian scheme with \(\mathcal{O}_k\)-action, \(G\), over \(\mathcal{M}_{(m,1)}^*\). At a closed geometric point \(\text{Spec}(\mathbb{F}) \rightarrow Z\) of the boundary, the semi-abelian scheme \(G\) is an extension

\[ 0 \rightarrow T \rightarrow G_{/\mathbb{F}} \rightarrow B \rightarrow 0 \]

of an abelian scheme \(B\) by a torus. The character group of \(T\) is a projective \(\mathcal{O}_k\)-module of rank one, and \(B\) extends in a canonical way to a triple \((B, \kappa, \psi) \in \mathcal{M}_{(m-1,0)}(\mathbb{F})\). The universal subsheaf \(\mathcal{F} \subset \text{Lie}(A)\) extends canonically to a subsheaf \(\mathcal{F} \subset \text{Lie}(G)\), which again
satisfies Krämer’s conditions. On the complement of the divisor $\rho^{-1}(\text{Sing}) \subset \mathbf{M}^*(m,1)$, the subsheaf $\mathcal{F}$ is the kernel of $j : \text{Lie}(G) \to \text{Lie}(G)$.

2.6. Galois action on boundary components. Recall from Section 2.1 that $\mathfrak{M}_{(m-1,0)}(\mathbb{C})$ has finitely many isomorphism classes of objects, which are indexed by isomorphism classes of self dual Hermitian lattices $(\mathfrak{B}, h_{\mathfrak{B}})$ of signature $(m - 1, 0)$. Under this correspondence

\begin{equation}
(\mathfrak{B}, h_{\mathfrak{B}}) \mapsto (\mathfrak{B} \otimes \mathcal{O}_k \mathbb{C})/\mathfrak{B}
\end{equation}

where the complex torus on the right hand side is equipped with its obvious action of $\mathcal{O}_k$, and with the principal polarization induced by the symplectic form $(2.1.1)$. 

**Definition 2.6.1.** A *cusp label* is an isomorphism class of pairs $(\mathfrak{n}, \mathfrak{B})$ in which $\mathfrak{n}$ is a projective $\mathcal{O}_k$-module of rank one, and $\mathfrak{B}$ is a self dual Hermitian lattice of signature $(m - 1, 0)$ (we suppress the Hermitian form $h_{\mathfrak{B}}$ from the notation).

Every connected component of $\mathbb{Z}(\mathbb{C})$ is contained in $\mathbb{Z}_n(\mathbb{C})$ for some projective rank one $\mathcal{O}_k$-module $\mathfrak{n}$. As $\mathbb{Z}_n$ is, by construction, a stack over $\mathfrak{M}_{(m-1,0)}$, the connected component also determines a point of $\mathfrak{M}_{(m-1,0)}(\mathbb{C})$, and hence a positive definite self dual Hermitian lattice $\mathfrak{B}$ as above. This establishes a bijection between the connected components of $\mathbb{Z}(\mathbb{C})$ and the cusp labels $(\mathfrak{n}, \mathfrak{B})$. The boundary $\mathbb{Z}(\mathbb{C})$ carries over it a family of semi-abelian schemes with $\mathcal{O}_k$-action. As $z \in \mathbb{Z}(\mathbb{C})$ varies over the component indexed by $(\mathfrak{n}, \mathfrak{B})$, the corresponding semi-abelian scheme $G_z$ varies over all extensions

\begin{equation}
0 \to T_{n/\mathbb{C}} \to G_z \to (\mathfrak{B} \otimes \mathcal{O}_k \mathbb{C})/\mathfrak{B} \to 0.
\end{equation}

In what follows, all tensor products are over $\mathcal{O}_k$. For an $\mathcal{O}_k$-module $M$, write $M_k = M \otimes \mathcal{O}_k$. Given a fractional $\mathcal{O}_k$-ideal $\mathfrak{s}$ and a positive definite self dual Hermitian lattice $\mathfrak{B}$, we obtain a new positive definite self dual Hermitian lattice $\mathfrak{B} \otimes \mathfrak{s}^{-1}$. The new Hermitian form is $h_{\mathfrak{B} \otimes \mathfrak{s}^{-1}}(x, y) = Nm(\mathfrak{s}) \cdot h_{\mathfrak{B}}(x, y)$, where we identify $(\mathfrak{B} \otimes \mathfrak{s}^{-1})_k \cong \mathfrak{B}_k$ using the obvious isomorphism $(\mathfrak{s}^{-1})_k \cong \mathfrak{k}$. Let $H$ be the Hilbert class field of $k$, $\text{Cl}_k$ the ideal class group, and

$$\text{rec}_k : \text{Cl}_k \cong \text{Gal}(H/k)$$

the reciprocity map of class field theory.

**Proposition 2.6.2.** The action of $\text{Gal}(k^{\text{alg}}/k)$ on the components of $\mathbb{Z}/k^{\text{alg}}$ factors through $\text{Gal}(H/k)$. For any $\mathfrak{s} \in \text{Cl}_k$, the Galois automorphism $\text{rec}_k(\mathfrak{s})$ carries the irreducible component indexed by $(\mathfrak{n}, \mathfrak{B})$ to the component indexed by $(\mathfrak{n}, \mathfrak{B} \otimes \mathfrak{s}^{-1})$.

**Proof.** The Galois automorphism

$$\mathfrak{M}_{(m-1,0)}(\mathbb{C}) \overset{\sigma}{\to} \mathfrak{M}_{(m-1,0)}(\mathbb{C})$$

corresponds, under the bijection $(2.6.1)$, to

$$(\mathfrak{B}, h_{\mathfrak{B}}) \mapsto (\mathfrak{B} \otimes \mathfrak{s}^{-1}, h_{\mathfrak{B} \otimes \mathfrak{s}^{-1}}),$$

where $\mathfrak{s}$ is any fractional ideal representing the image of $\sigma$ under

$$\text{Aut}(\mathbb{C}/k) \to \text{Gal}(H/k) \cong \text{Cl}_k.$$ 

Of course this is identical to the formula for the Galois action on elliptic curves with complex multiplication; the details of the proof will appear in [1]. Both claims now follow by applying $\sigma$ throughout $(2.6.2)$. □
There is an alternate way to parametrize the components of \( Z(\mathbb{C}) \). Suppose we start with a pair \( m \subset \mathfrak{A} \) in which \( \mathfrak{A} \) is a self dual Hermitian lattice of signature \((m,1)\), and \( m \) is an isotropic direct summand of rank one. A normal decomposition of \( m \subset \mathfrak{A} \) is an \( \mathcal{O}_k \)-module in which \[ \mathfrak{A} = m \oplus \mathfrak{B} \oplus n \]
in which \( n \) is an isotropic direct summand of rank one, and \( \mathfrak{B} = (m \oplus n)^\perp \). For any such decomposition \( m^\perp = m \oplus \mathfrak{B} \). Furthermore, the Hermitian form on \( \mathfrak{A} \) restricts to a perfect pairing \( m \times n \to \mathcal{O}_k \), and makes \( \mathfrak{B} \cong m^\perp/m \) into a positive definite self dual Hermitian lattice of signature \((m-1,0)\).

**Proposition 2.6.3.** Every pair \( m \subset \mathfrak{A} \) as above admits a normal decomposition. The rule \[ m \subset \mathfrak{A} \mapsto (\mathfrak{A}/m^\perp, m^\perp/m) \cong (n, \mathfrak{B}) \]
establishes a bijection between the isomorphism classes of pairs \( m \subset \mathfrak{A} \) as above, and the set of cusp labels.

**Proof.** Denote by \( \langle \cdot, \cdot \rangle \) the Hermitian form on \( \mathfrak{A} \). It’s easy to see that \( m^\perp \) is a projective \( \mathcal{O}_k \)-module of rank \( m \), and that \( \mathfrak{A}/m^\perp \) is projective of rank \( 1 \). Hence

\[ \mathfrak{A} = m^\perp \oplus n \]

for some rank one direct summand \( n \subset \mathfrak{A} \). In particular, \( \langle m, n \rangle \neq 0 \). We now modify \( n \) to make it isotropic. Fix \( e \in m_k \) and \( e' \in n_k \) such that \( \langle e, e' \rangle = 1 \). There are fractional \( \mathcal{O}_k \)-ideals \( n_0 \) and \( m_0 \) such that \( m = m_0 e \) and \( n = n_0 e' \). The Hermitian form on \( \mathfrak{A} \) restricts to a perfect pairing between \( n \) and \( m \), and hence \( m_0 n_0 = \mathcal{O}_k \).

**Because we assume that \( d_k \) is odd**, the trace map \( \mathcal{O}_k \to \mathbb{Z} \) is surjective. It is easy to see that \( \langle e', e' \rangle \in N(m_0)\mathbb{Z} \), and therefore we may choose an \( x \in N(m_0)\mathcal{O}_k \) with \( x + \overline{x} = -\langle e', e' \rangle \).

Now replace \( n \) by \( n_0(xe + e') \). It is still true that \( \mathfrak{A} = m^\perp \oplus n \), but now \( n \) is isotropic. Defining \( \mathfrak{B} = (m \oplus n)^\perp \) gives the desired normal decomposition.

For the second claim, start with a cusp label \( (n, \mathfrak{B}) \). Let \( m \) be the set of \( \mathcal{O}_k \)-conjugate linear maps \( n \to \mathcal{O}_k \), and let \( \mathcal{O}_k \) act on \( m \) by \( (x \cdot \mu)(\nu) = x \cdot \mu(\nu) \). Define a Hermitian form on \( m \oplus n \) by \( \langle \mu_1 + \nu_1, \mu_2 + \nu_2 \rangle = \mu_1(\nu_2) + \mu_2(\nu_1) \), and make \( \mathfrak{A} = m \oplus \mathfrak{B} \oplus n \) into a Hermitian lattice in the obvious way, with \( \mathfrak{B} \perp (m \oplus n) \). The construction \( (n, \mathfrak{B}) \mapsto m \subset \mathfrak{A} \) is surjective, by the existence of normal decompositions, and is easily seen to give an inverse to the map in the statement of the proposition. \( \square \)

**Remark 2.6.4.** Proposition 2.6.3 is false without the hypothesis that \( d_k \) is odd. For example, endow \( \mathfrak{A} = \mathcal{O}_k \oplus \mathcal{O}_k \) with the signature \((1,1)\) Hermitian form

\[ \langle x, y \rangle = ^t x \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} y, \]

and let

\[ m = \mathcal{O}_k \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]

If \( d_k \) is even, the pair \( m \subset \mathfrak{A} \) does not admit a normal decomposition.

3. **Arithmetic Divisors**

Fix a positive integer \( n \) and consider the flat and regular \( \mathcal{O}_k \)-stack

\[ M = M_{(1,0)} \times_{\mathcal{O}_k} M_{\text{Kra}}^{(n-1,1)} \]
of (absolute) dimension \( n \), and its flat and regular compactification

\[(3.0.3) \quad \mathcal{M}^* = \mathcal{M}_{(1,0)} \times \mathcal{O}_k \mathcal{M}_{(n-1,1)}^* \]

constructed in Section 2. In a slight change from earlier notation, we now denote by \( Z \) the boundary \( Z = \mathcal{M}^* \setminus \mathcal{M} \) with its reduced substack structure. It is a divisor on \( \mathcal{M}^* \), proper and smooth over \( \mathcal{O}_k \).

The stack \( \mathcal{M} \) is the moduli stack of septuples \((A_0, \kappa_0, \psi_0, A, \kappa, \psi, \mathcal{F})\) in which \( A_0 \to S \) is an elliptic curve over an \( \mathcal{O}_k \)-scheme, \( \psi_0 \) is its canonical principal polarization, \( \kappa_0 : \mathcal{O}_k \to \text{End}(A_0) \) is an action of \( \mathcal{O}_k \) on \( A_0 \) whose induced action on \( \text{Lie}(A_0) \) is through the structure morphism \( i_S : \mathcal{O}_k \to \mathcal{O}_S \). \( A \) is an abelian scheme over \( S \), \( \kappa : \mathcal{O}_k \to \text{End}(A) \) is an action of \( \mathcal{O}_k \) on \( A \), and \( \psi \) is an \( \mathcal{O}_k \)-linear principal polarization of \( A \), and \( \mathcal{F} \subset \text{Lie}(A) \) satisfies Krämer’s conditions. For simplicity we will usually shorten such a septuple simply to \((A_0, A) \in \mathcal{M}(S)\).

We will construct special divisors \( \mathcal{K}\mathcal{R}(m) \) on \( \mathcal{M} \). These divisors come equipped with natural Green functions, and we will analyze the behavior of these Green functions near the boundary of \( \mathcal{M}^* \). The goal is to show that adding a particular \( \mathbb{R} \)-linear combination of boundary components to \( \mathcal{K}\mathcal{R}(m) \) yields a class in the arithmetic Chow group \( CH^*_\mathbb{R}(\mathcal{M}^*) \).

3.1. Arithmetic Chow groups. The work of Gillet and Soulé [12, 13, 32] gives us, for any flat, regular, and proper \( \mathcal{O}_k \)-scheme of finite type, a theory of arithmetic Chow groups formed from cycles equipped with Green currents, up to a suitable notion of rational equivalence. Burgos-Kramer-Kühn [6, 7] extended this theory by allowing the use of Green currents with certain mild log-log singularities along a fixed normal crossing divisor. The original Gillet-Soulé theory was extended from schemes to algebraic stacks by Gillet [12], but that work does not allow for the log-log singularities covered in [7]. We will be working with Green functions on the stack \( \mathcal{M}^* \) with log-log singularities along the boundary, so we need a common generalization of [12] and [7]. Doing this in full generality is a task best left to the experts, so we will develop in an ad hoc way only the minimal theory we need.

**Definition 3.1.1.** Suppose \( M^* \) is a complex manifold of dimension \( n-1 \), \( Z \subset M^* \) is a smooth codimension one submanifold, \( M = M^* \setminus Z \), and \( z_0 \in Z \). On some open neighborhood \( V \subset M^* \) of \( z_0 \) there are coordinates \( q, u_1, \ldots, u_{n-2} \) such that \( Z \) is defined by the equation \( q = 0 \). After shrinking \( V \) we may always assume that \( \log |q^{-1}| > 1 \) on \( V \). The open set \( V \) and its coordinates are then said to be adapted to \( Z \).

**Definition 3.1.2.** Suppose \( f \) is a \( C^\infty \) function on an open subset \( U \subset M \). We say that \( f \) has log-log growth along \( Z \) if around any point of \( Z \) there is an open neighborhood \( \tilde{V} \) and coordinates \( q, u_1, \ldots, u_{n-2} \) adapted to \( Z \) such that

\[(3.1.1) \quad f = O(\log \log |q^{-1}|)\]

on \( U \cap \tilde{V} \). A smooth differential form \( \omega \) on \( U \) has log-log growth along \( Z \) if around any point of \( Z \) there is an open neighborhood \( V \) and coordinates \( q, u_1, \ldots, u_{n-2} \) adapted to \( Z \) such that \( \omega|_{U \cap V} \) lies in the subalgebra (of the \( C^\infty(U \cap V) \)-algebra of all smooth forms on \( U \cap V \)) generated by

\[\frac{dq}{q \log |q|}, \frac{d\overline{q}}{\overline{q} \log |q|}, du_1, \ldots, du_{n-2}, d\overline{u}_1, \ldots, d\overline{u}_{n-2},\]

and the functions satisfying (3.1.1).

**Remark 3.1.3.** Our definition of log-log growth is slightly stronger than that of [7]. What we call log-log growth, those authors call Poincaré growth [7, Section 7.1].
Remark 3.1.4. A smooth function \( f \) on an open subset of \( M \) is a pre-log-log form if \( f, \partial f, \overline{\partial} f \), and \( \partial \overline{\partial} f \) all have log-log growth along \( Z \). We will not use this terminology, but mention it for ease of comparison with [21], where this term is used routinely.

The notion of log-log growth can be extended from complex manifolds to the orbifold fibers of \( M^* \) in the following way. It is always possible to write \( M^* (\mathbb{C}) \) as the quotient of a complex manifold \( M^* \) by the action of a finite group \( H \). For example, by (as in [23, 24]) adding level structure to the moduli problem defining \( M(\mathbb{C}) \) and then compactifying the result. In any case, the morphisms of complex orbifolds

\[
Z(\mathbb{C}) \to M^* (\mathbb{C}) \leftarrow \mathcal{M}(\mathbb{C})
\]

arise as the quotients of \( H \)-invariant morphisms of complex manifolds

\[
Z \to M^* \leftarrow M
\]

with \( H \)-actions. A smooth form on the orbifold \( \mathcal{M}(\mathbb{C}) \) pulls back to a smooth \( H \)-invariant form on the complex manifold \( M \), and is said to have log-log growth along the boundary \( Z(\mathbb{C}) \) if the corresponding \( H \)-invariant form on \( M \) has log-log growth along \( Z \).

Let \( D \) be a divisor on \( M^* \) with real coefficients, and write \( D = \sum m_i D_i \) as a finite \( \mathbb{R} \)-linear combination of pairwise distinct irreducible closed substacks of codimension one. We allow the possibility that some \( D_i \) are components of the boundary \( Z \). In our applications such boundary components will appear with real multiplicities, while the non-boundary \( D_i \)'s will have integer multiplicities.

Definition 3.1.5. A Green function for \( D \) consists of a smooth real-valued function \( \text{Gr}(D, z) \) on

\[
\mathcal{M}(\mathbb{C}) \setminus \text{Sppt}(D(\mathbb{C})).
\]

satisfying the following properties

1. The function \( \text{Gr}(D, z) \) has a logarithmic singularity along \( D \) in the following sense: around every point of \( \mathcal{M}^*(\mathbb{C}) \) there is an open neighborhood \( V \) and local equations \( \psi_i(z) = 0 \) for the divisors \( D_i(\mathbb{C}) \) such that the function

\[
E(z) = \text{Gr}(D, z) + \sum i m_i \log |\psi_i(z)|^2
\]

on \( V \cap (\mathcal{M}(\mathbb{C}) \setminus \text{Sppt}(D(\mathbb{C}))) \) extends smoothly to \( U = V \cap \mathcal{M}(\mathbb{C}) \).

2. The forms \( E, \partial E, \overline{\partial} E, \) and \( \partial \overline{\partial} E \) on \( U \) have log-log growth along \( Z(\mathbb{C}) \).

As a simple example, if \( f \) is any rational function on \( \mathcal{M}^* \) then \( \log |f|^2 = \log |f|^2 \) is a Green function for the divisor \( \text{div}(f) \).

Remark 3.1.6. As in [28] or [7, Proposition 7.6], the log-log growth of \( E \) and of

\[
\partial \overline{\partial} E = -2\pi i \cdot dd^c E
\]

imply that both \( E \) and \( dd^c E \) are locally integrable, and so define currents on \( \mathcal{M}^*(\mathbb{C}) \). The log-log growth conditions further imply the equality of currents \( [dd^c E] = dd^c [E] \), from which one deduces the Green equation

\[
[dd^c E] = dd^c [\text{Gr}(D, \cdot)] + \delta_D
\]

of [13, Definition 1.2.3].
Definition 3.1.7. An arithmetic divisor on \( \mathcal{M}^* \) is a pair \((\mathcal{D}, \text{Gr}(\mathcal{D}, \cdot))\) consisting of a divisor \( \mathcal{D} \) on \( \mathcal{M}^* \) with real coefficients, and a Green function for \( \mathcal{D} \). A principal arithmetic divisor is an arithmetic divisor of the form
\[
\widehat{\text{div}}(f) = (\text{div}(f), -\log |f|^2)
\]
for some rational function \( f \) on \( \mathcal{M}^* \). The codimension one arithmetic Chow group \( \widehat{\text{CH}}_1(\mathcal{M}^*) \) is the quotient of the \( \mathbb{R} \)-vector space of all arithmetic divisors by the \( \mathbb{R} \)-span of the principal arithmetic divisors.

Let \( \mathcal{X} \) be a regular algebraic stack, finite and flat over \( \mathcal{O}_k \). In particular \( \mathcal{X} \) has dimension one. The stack \( \mathcal{X} \) has an arithmetic Chow group \( \widehat{\text{CH}}_1(\mathcal{X}) \) associated to it, defined exactly as with \( \mathcal{M}^* \) (with \( \mathcal{X} = \mathcal{M}^* \)). Of course this study of Green functions for divisors on \( \mathcal{X} \) is simplified by the fact that \( \mathcal{X} \) has all of its divisors supported in nonzero characteristic. Thus a Green function for a divisor \( \mathcal{D} \) on \( \mathcal{X} \) is simply any function on the 0-dimensional orbifold \( \mathcal{X}(\mathbb{C}) \).

Now suppose \( \mathcal{X} \) is equipped with a representable morphism \( \pi : \mathcal{X} \to \mathcal{M} \). The finiteness of \( \mathcal{X} \) over \( \mathcal{O}_k \) implies that \( \pi \) is a proper map. In particular, the Zariski closure of the image of \( \mathcal{X} \) in \( \mathcal{M}^* \) does not meet the boundary. We will construct a pullback map
\[
\pi^* : \widehat{\text{CH}}_1(\mathcal{M}^*) \to \widehat{\text{CH}}_1(\mathcal{X}).
\]
Suppose \( \mathcal{D} \) is an irreducible divisor on \( \mathcal{M}^* \) intersecting \( \mathcal{X} \) properly, in the sense that
\[
\mathcal{X} \cap \mathcal{D} = \mathcal{X} \times_{\mathcal{M}^*} \mathcal{D}
\]
has dimension 0. Of course this is equivalent to \( \mathcal{X} \) and \( \mathcal{D} \) having empty intersection in the generic fiber of \( \mathcal{M}^* \). The Serre intersection multiplicity at a geometric point \( z \in (\mathcal{X} \cap \mathcal{D})(k) \) is defined by
\[
I^\text{Serre}_z(\mathcal{D} : \mathcal{X}) = \sum_{\ell \geq 0} (-1)^\ell \text{length}_{\mathcal{O}_{\mathcal{X}, z}} \text{Tor}^\mathcal{O}_{\mathcal{X}, z}_\ell (\mathcal{O}_{\mathcal{X}, z}, \mathcal{O}_{\mathcal{D}, z}),
\]
where all local rings are for the étale topology. From \( z \) we may construct a divisor \([z] \) on \( \mathcal{X} \) as follows. Fix an étale presentation \( \mathcal{X} \to \mathcal{X} \) with \( \mathcal{X} \) a scheme. The fiber product \( \bar{z} = \mathcal{X} \times_{\mathcal{X}^*} \mathcal{X} \) is a scheme, and is finite étale over \( z = \text{Spec}(k) \). Thus \( \bar{z} \) is a disjoint union of copies of \( \text{Spec}(k) \), say \( \bar{z} = \bigsqcup z_i \), where each \( z_i \) is a geometric point of \( \mathcal{X} \). Let \([z_i] \) denote the image of \( z_i : \text{Spec}(k) \to \mathcal{X} \), so that \([z_i] \) is a closed point of \( \mathcal{X} \). Then \([z] = \sum [z_i] \) is a divisor on \( \mathcal{X} \), and descends uniquely to a divisor on \( \mathcal{X} \) denoted \([z] \). Define a divisor on \( \mathcal{X} \)
\[
\pi^* \mathcal{D} = \sum_{z \in [\mathcal{X} \cap \mathcal{D}]} I^\text{Serre}_z(\mathcal{D} : \mathcal{X}) \cdot [z].
\]
Here \([\mathcal{X} \cap \mathcal{D}] \) is the topological space underlying the \( \mathcal{O}_k \)-stack \( \mathcal{X} \cap \mathcal{D} \), in the sense of [25, Chapter 5]. Each point \( z \in [\mathcal{X} \cap \mathcal{D}] \) is, by definition, an equivalence class of maps \( z : \text{Spec}(k) \to \mathcal{X} \cap \mathcal{D} \) with \( k \) a field, and we may always choose a representative of this equivalence class for which \( k \) is algebraically closed. Extend the definition of \( \pi^* \mathcal{D} \) linearly to all divisors \( \mathcal{D} \) with real coefficients whose support meets \( \mathcal{X} \) properly. If \( \text{Gr}(\mathcal{D}, \cdot) \) is a Green function for such a divisor \( \mathcal{D} \), then the image of the orbifold morphism \( \mathcal{X}(\mathbb{C}) \to \mathcal{M}^*(\mathbb{C}) \) is disjoint from the divisor \( \mathcal{D}(\mathbb{C}) \), as well as from the boundary of \( \mathcal{M}^*(\mathbb{C}) \). Thus the image of \( \mathcal{X}(\mathbb{C}) \to \mathcal{M}(\mathbb{C}) \) is disjoint from all singularities of \( \text{Gr}(\mathcal{D}, \cdot) \), and so we may form the pullback \( \pi^* \text{Gr}(\mathcal{D}, \cdot) \) in the usual sense. This defines
\[
\pi^* \mathcal{D} = (\pi^* \mathcal{D}, \pi^* \text{Gr}(\mathcal{D}, \cdot))
\]
whenever \( \mathcal{D} = (\mathcal{D}, \text{Gr}(\mathcal{D}, \cdot)) \) with \( \mathcal{D} \) intersecting \( \mathcal{X} \) properly.
The argument of [32, Theorem III.3.1] allows one to extend the definition of $\pi^*$ to all arithmetic divisors. Briefly, given an arithmetic divisor $\hat{D} = (D, \text{Gr}(D, \cdot))$ one can use Chow’s moving lemma [30] (working on an étale presentation of the generic fiber of $M^*$) in order to modify $\hat{D}$ by a principal arithmetic divisor in such a way that the resulting divisor meets $X$ properly, and then check that the resulting pullback does not depend on the choice of principal arithmetic divisor used in the modification. It may be worth pointing out that there is a gap in the proof of [32, Theorem III.3.1], identified and corrected by Gubler [14]. The gap is in the proof of the “Moving Lemma for $K_1$-chains”, but if one only works with Chow groups of codimension one (as we do) then the use of this lemma is unnecessary, and there is no gap.

There is a canonical linear functional

$$\hat{\text{deg}} : \hat{\text{CH}}^1_R(X) \to \mathbb{R},$$

defined, if $X$ is a scheme, in [13] as the composition

$$\hat{\text{CH}}^1_R(X) \to \hat{\text{CH}}^1_R(\text{Spec}(O_k)) \to \mathbb{R},$$

where the first arrow is the proper pushforward by the structure morphism $X \to \text{Spec}(O_k)$, and the second arrow is defined in [13, Section 3.4.3]. The generalization to stacks can be found in [22]. In any case,

$$\hat{\text{deg}}(D, \text{Gr}(D, \cdot)) = \sum_{q \subset O_k} \sum_{z \in D^{(F_{\text{alg}}q)}} \log(N(q)) \frac{\text{Aut}_X(z)}{\# \text{Aut}_X(z)} + \sum_{z \in \text{X}(\mathbb{C})} \frac{\text{Gr}(D, z)}{\# \text{Aut}_{\text{X}(\mathbb{C})}(z)}$$

whenever $D$ is irreducible. Here $F_{\text{alg}}q$ is an algebraic closure of the residue field $O_k/q$, and $N(q) = \#(O_k/q)$.

**Definition 3.1.8.** Suppose $X$ is a regular stack, finite and flat over $O_k$, and equipped with a morphism $\pi : X \to M$. **Arithmetic intersection against** $X$ is the linear functional

$$[\cdot : X] : \hat{\text{CH}}^1_R(M^*) \to \mathbb{R}$$

defined as the composition

$$\hat{\text{CH}}^1_R(M^*) \xrightarrow{\pi^*} \hat{\text{CH}}^1_R(X) \xrightarrow{\hat{\text{deg}}} \mathbb{R}.$$
3.2. **Kudla-Rapoport divisors.** The stack $\mathbb{M}$ admits an obvious morphism to

$$\mathbb{M}_\text{naive} = \mathbb{M}_{(1,0)} \times \mathcal{O}_k \mathbb{M}_\text{naive}^{(n-1,1)}.$$ 

This stack is a moduli space of sextuples $(A_0, \kappa_0, \psi_0, A, \kappa, \psi)$, but we usually abbreviate such a sextuple to

$$(A_0, A) \in \mathbb{M}_\text{naive}(S).$$

The $\mathcal{O}_k$-module $\text{Hom}_{\mathcal{O}_k}(A_0, A)$ is equipped with a positive definite $\mathcal{O}_k$-Hermitian form

$$\langle f_1, f_2 \rangle = \psi_0^{-1} \circ f_2' \circ \psi \circ f_1.$$ 

The right hand side is an element of $\text{End}_{\mathcal{O}_k}(A_0)$, which we identify with $\mathcal{O}_k$.

**Definition 3.2.1.** For any $m \neq 0$, the *naive Kudla-Rapoport divisor* $\mathbb{K}_\text{naive}(m)$ is the moduli stack of tuples $(A_0, A, f)$ over $\mathcal{O}_k$-schemes $S$, in which

- $(A_0, A) \in \mathbb{M}_\text{naive}(S)$
- $f \in \text{Hom}_{\mathcal{O}_k}(A_0, A)$ satisfies $\langle f, f \rangle = m$.

Of course $\mathbb{K}_\text{naive}(m) = \emptyset$ if $m < 0$. Denote by

$$\mathbb{K}(m) = \mathbb{K}_\text{naive}(m) \times_{\mathbb{M}_\text{naive}} \mathbb{M}$$

the pullback of $\mathbb{K}_\text{naive}(m)$ to $\mathbb{M}$.

The morphism $\mathbb{K}(m) \to \mathbb{M}$ is finite and unramified, and (in light of Proposition 3.2.3 below) we view $\mathbb{K}(m)$ as a divisor on $\mathbb{M}$ in the usual way. To be more precise, $\mathbb{K}(m)$ is a closed substack of itself, and so determines a cycle $[\mathbb{K}(m)]$ on $\mathbb{K}(m)$ by [34] Definition (3.5)]. We then pushforward this cycle to a divisor on $\mathbb{M}$ using [34] Definition (3.6)].

**Definition 3.2.2.** For every $m \neq 0$ the *Kudla-Rapoport divisor* $\mathbb{K}^*(m)$ is the Zariski closure of the divisor $\mathbb{K}(m)$ in the compactification $\mathbb{M}^*$ of $\mathbb{M}$.

The following result tells us not only that $\mathbb{K}^*(m)$ has the expected dimension, but also, as we shall see in the proof of Theorem 3.2.1 ensures that the higher Tor terms in the Serre intersection multiplicity will not contribute to our final formulas.

**Proposition 3.2.3.** Suppose $F$ is an algebraically closed field, $z \in \mathbb{K}(m)(F)$ is a geometric point, and denote by $R_z$ the completed etale local ring of $\mathbb{K}(m)$ at $z$. Let $y \in \mathbb{M}(F)$ be the point below $z$, and denote by $R_y$ the completed etale local ring of $\mathbb{M}$ at $y$. The natural map $R_y \to R_z$ is surjective, and the kernel is generated by a single nonzero element.

In particular, the local rings of $\mathbb{K}(m)$ are complete intersections, and its irreducible components all have dimension $n-1$.

**Proof.** Let $(A_0, A, f)$ be the triple over $F$ determined by the point $z$. Let $\hat{O}_{k,z}$ be the completion of the strict Henselization of $\mathcal{O}_k$ with respect to the geometric point

$$\text{Spec}(F) \hat{\to} \mathbb{K}(m) \to \text{Spec}(\mathcal{O}_k),$$

and let CLN be the category of complete local Noetherian $\hat{O}_{k,z}$-algebras with residue field $F$. The ring $R_y$ represents the functor assigning to every object $S$ of CLN the set of isomorphism classes of deformations of $(A_0, A)$ to $S$. Here deformation always means deformation to an object of $\mathbb{M}(S)$; that is, the deformations of $A_0$ and $A$ are also equipped with $\mathcal{O}_k$-actions, polarizations, etc. lifting those on $A_0$ and $A$. Similarly, $R_z$ represents the deformation functor of the triple $(A_0, A, f)$. The surjectivity of the tautological map $R_y \to R_z$ is equivalent to the injectivity of

$$\text{Hom}(R_z, S) \to \text{Hom}(R_y, S)$$
for every object $S$ of CLN, and this injectivity is proved in \cite[Lemma 2.2.1]{...}. Let
\[ I = \ker(R_y \to R_z), \]
let $m$ be the maximal ideal of $R_y$, and set $S = R_y / mI$. The kernel of the natural surjection $S \to R_z$ is $I = I / mI$, and satisfies $I^2 = 0$. By Nakayama's lemma, to prove that $I$ is a principal ideal, it suffices to prove that $\mathcal{I}$ is.

Let $(A_0, A)$ be the universal deformation of $(A_0, A)$ to $R_y$. The reduction $(A_0 / R_z, A / R_z)$ comes with a universal map $f : A_0 / R_z \to A / R_z$, and we will use the deformation theory arguments of \cite[Chapter 2]{...} to study the obstruction to lifting $f$ to a map $A_0 / S \to A / S$.

As in the proof of Proposition 2.1.2, the de Rham homology groups $H_1^{dR}(A_0)$ and $H_1^{dR}(A)$ are free of ranks $1$ and $n$ over $\mathcal{O}_k \otimes \mathbb{Z} R_y$, and sit in short exact sequences
\[ 0 \to \Fil^1 H_1^{dR}(A_0) \to H_1^{dR}(A_0) \to \Lie(A_0) \to 0 \]
and
\[ 0 \to \Fil^1 H_1^{dR}(A) \to H_1^{dR}(A) \to \Lie(A) \to 0. \]

Furthermore, again by the proof of Proposition 2.1.2,
\[ \Fil^1 H_1^{dR}(A_0) = j \cdot H_1^{dR}(A_0). \]

The same holds with $A_0$ and $A$ replaced by their reductions to $S$ or $R_z$. Fix once and for all an $\mathcal{O}_k \otimes \mathbb{Z} R_y$-module generator $\sigma \in H_1^{dR}(A_0)$, and a basis $\epsilon_1, \ldots, \epsilon_n$ of $\Lie(A)$ such that $\epsilon_1, \ldots, \epsilon_{n-1}$ is a basis of the universal $R_y$-submodule $\mathcal{F} \subset \Lie(A)$ satisfying Kr"amer's conditions. In particular $j \cdot \epsilon_i = 0$ for $1 \leq i < n$, and the operator $j$ on $\Lie(A)$ has the form
\[ j = \begin{bmatrix} 0 & \cdots & 0 & j_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & j_n \end{bmatrix} \]
for some $j_1, \ldots, j_n \in R_y$.

The map $f$ induces a map
\[ f : H_1^{dR}(A_0 / R_z) \to H_1^{dR}(A / R_z) \]
respecting the Hodge filtrations, and by \cite[Proposition 2.1.6.4]{...} there is a canonical lift
\[ \tilde{f} : H_1^{dR}(A_0 / S) \to H_1^{dR}(A / S) \]
(which need not respect the Hodge filtrations). The obstruction to lifting $f$ to a map $A_0 / S \to A / S$ is given by the composition
\[ j \cdot H_1^{dR}(A_0 / S) \to H_1^{dR}(A_0 / S) \to \Lie(A / S). \]

which we denote by $\text{obstr}_{f}$. The image of $\tilde{f}(\sigma)$ in $\Lie(A / S)$ is $x_1 \epsilon_1 + \cdots + x_n \epsilon_n$ for some $x_1, \ldots, x_n \in S$, and
\[ \text{obstr}_{f}(j \sigma) = x_1 j \epsilon_1 + \cdots + x_n j \epsilon_n = x_n (j_1 \epsilon_1 + \cdots + j_n \epsilon_n). \]

The map $\text{obstr}_{f}$ becomes trivial after applying $\otimes_S R_z$, and hence the ideal $x_n (j_1, \ldots, j_n)$ of $S$ is contained in $\mathcal{I}$. On the other hand, the map $\text{obstr}_{f}$ becomes trivial upon reduction to $S / x_n (j_1, \ldots, j_n)$, which implies that the universal triple $(A_0 / R_z, A / R_z, f)$ lifts to $S / x_n (j_1, \ldots, j_n)$. By the universality of $(A_0 / R_z, A / R_z, f)$, this lift corresponds to a section to the natural surjection $S / x_n (j_1, \ldots, j_n) \to R_z$, which is therefore an isomorphism. In other words
\[ \mathcal{I} = x_n (j_1, \ldots, j_n). \]
To complete the proof that $I$ is principal, it now suffices to show that $(j_1, \ldots, j_n)$ is a principal ideal of $R_y$. After Theorem 2.1.4 it is natural to consider separately the cases $j \cdot \text{Lie}(A) \neq 0$ and $j \cdot \text{Lie}(A) = 0$.

The case $j \cdot \text{Lie}(A) \neq 0$ is easy. This assumption implies that (3.2.1) is nonzero after reduction to the residue field $F$, and hence at least one of the $j_i$’s is a unit in $R_y$. In particular $(j_1, \ldots, j_n) = R_y$ is a principal ideal.

Now assume that $j \cdot \text{Lie}(A) = 0$. By Theorem 2.1.4 this implies that $\text{char}(F) | d_k$. Let $p$ be the kernel of the structure map $i_{xy} : \mathcal{O}_k \to F$, and let $p$ be the rational prime below $p$. As we assume that $d_k$ is odd, we may fix a uniformizer $\pi \in \mathcal{O}_{k,p}$ in such a way that $\pi = -\pi$. Note that $i_y(\pi) = 0$, and so $j = \pi$ as endomorphisms of $\text{Lie}(A)$. In particular $\pi\text{Lie}(A) = 0$.

Using the fact that $H_1^{dR}(A)$ is free of rank $n$ over $\mathcal{O}_k \otimes \mathbb{Q}\pi$, it is easy to deduce from the exactness of

$$0 \to \text{Fil}^1 H_1^{dR}(A) \to H_1^{dR}(A) \to \text{Lie}(A) \to 0$$

that

$$\text{Fil}^1 H_1^{dR}(A) = \pi \cdot H_1^{dR}(A).$$

Using the coordinates of [17, 29], we examine the structure of the universal $R_y$-module short exact sequence

$$0 \to \text{Fil}^1 H_1^{dR}(A) \to H_1^{dR}(A) \to \text{Lie}(A) \to 0,$$

and of the universal submodule $F \subset \text{Lie}(A)$. Fix an isomorphism $H_1^{dR}(A) \cong (\mathcal{O}_{k,p} \otimes \mathbb{Q}\pi R_y)^n$, and identify $\mathcal{O}_{k,p} \otimes \mathbb{Q}\pi R_y = R_y \oplus \pi R_y$. This gives a decomposition

$$H_1^{dR}(A) \cong R_y^n \oplus \pi R_y^n.$$  

By [29, Lemma 3.6], these choices may be made in such a way that the symplectic form on the left hand side determined by the principal polarization on $A$ is identified with the symplectic form $\psi$ on the right hand side determined by $\psi(e_i, e_j) = 0$ and $\psi(e_i, \pi e_j) = \delta_{i,j}$. Here $e_1, \ldots, e_n \in R_y^n$ are the standard basis vectors. Combining (3.2.2) with Nakayama’s lemma shows that the composition

$$\text{Fil}^1 H_1^{dR}(A) \to R_y^n \oplus \pi R_y^n \to \pi R_y^n$$

is an isomorphism (the first arrow is the inclusion, the second the projection). This implies that there is a unique $X \in M_n(R_y)$ such that the vectors

$$v_1 = X e_1 - \pi e_1$$

$$\vdots$$

$$v_n = X e_n - \pi e_n$$

are a basis for $\text{Fil}^1 H_1^{dR}(A)$, and such that the images of $e_1, \ldots, e_n$ in $\text{Lie}(A)$ form a basis. With respect to this basis, the action of $\pi$ on $\text{Lie}(A)$ is through the matrix $X$. The Hodge filtration $\text{Fil}^1 H_1^{dR}(A)$ is isotropic for the symplectic form $\psi$, and one easily checks that this implies $\psi(X) = 0$.

The matrices (3.2.1) and $X - i R_y(\pi)$ are conjugate by an element of $\text{GL}_n(R_y)$, as they represent the operator $\pi - i R_y(\pi)$ with respect to different bases of $\text{Lie}(A)$. Noting that (3.2.1) factors as

$$j = j \cdot \gamma e_n,$$

where $j = [j_1 \cdots j_n]$, there is a $\gamma \in \text{GL}_n(R_y)$, such that

$$\gamma j \cdot \gamma^{-1} = X - i R_y(\pi).$$
Define \( v, w \in R^m_y \) by
\[
\begin{bmatrix}
v_1 \\
\vdots \\
v_n
\end{bmatrix} = \gamma j \quad \text{and} \quad \begin{bmatrix}
w_1 \\
\vdots \\
w_n
\end{bmatrix} = t \gamma^{-1} e_n,
\]
so that \( v \cdot ^t w = X - i R_y \). The essential point is that the symmetry of \( X \) implies the symmetry of \( v \cdot ^t w \), so that \( v_i w_j = v_j w_i \) for all \( i, j \). At least one component of \( w \), say \( w_k \), is a unit, and so \( (v_i w_k^{-1}) \cdot ^t w_j = v_j \). This means that \( v \) is a scalar multiple of \( w \): \( v = cw \) where \( c = v_i w_k^{-1} \in R_y \). From the definitions of \( v \) and \( w \) we deduce
\[
t \gamma \alpha = c \cdot e_n,
\]
and hence \( (j_1, \ldots, j_n) = (c) \) is a principal ideal.

Having now proved that \( R_y \to R_z \) is surjective with principal kernel, it only remains to prove that the map is not an isomorphism. Suppose it is an isomorphism. This implies, by dimension considerations, that \( \text{Ker}(m) \) contains an entire irreducible component of \( M \). As \( M \) is flat over \( O_\mathbb{C} \), it follows that \( \text{Ker}(m)/M \) contains an irreducible component of \( M/\mathbb{C} \). But \( \text{Ker}(m)/M \) is a divisor on \( M/\mathbb{C} \), as one can check using the explicit complex uniformization of \( M \) or \( [20] \).

3.3. **Analytic compactification.** Recall that we have fixed an embedding \( \iota : k \to \mathbb{C} \). We now construct explicit coordinates on the complex orbifold \( M(\mathbb{C}) \cong M_{\text{naive}}(\mathbb{C}) \), and give a purely analytic construction of the compactification \( M^*(\mathbb{C}) \). The complex uniformization of \( M(\mathbb{C}) \) is described both in \([20]\) and in \([15]\), and we only sketch the main ideas. The compactification is a special case of the vastly more general constructions of \([2, 24]\), and generalizes the signature \((2, 1)\) case studied in \([8, 9, 27]\).

The orbifold \( M(\mathbb{C}) \cong M(1, 0)(\mathbb{C}) \times M(0, n-1, 1)(\mathbb{C}) \) is disconnected, and its connected components are indexed by isomorphism classes of pairs \((\mathfrak{a}_0, \mathfrak{a})\) in which

- \( \mathfrak{a}_0 \) is a projective \( O_k \)-module of rank 1, equipped with a positive definite Hermitian form \( h_{\mathfrak{a}_0}(x, y) \), under which \( \mathfrak{a}_0 \) is self dual,
- \( \mathfrak{a} \) is a projective \( O_k \)-module of rank \( n \), equipped with a Hermitian form \( h_{\mathfrak{a}}(x, y) \) of signature \((n - 1, 1)\), under which \( \mathfrak{a} \) is self dual.

The \( \mathfrak{a}_0 \)'s index the (finitely many) points of \( M(1, 0)(\mathbb{C}) \), while the \( \mathfrak{a} \)'s index the connected components of \( M(0, n-1, 1)(\mathbb{C}) \). Fix one such pair \((\mathfrak{a}_0, \mathfrak{a})\), and set
\[
L = \text{Hom}_{O_k}(\mathfrak{a}_0, \mathfrak{a})
\]
and \( V = L_k \). There is a Hermitian form \( \langle f, g \rangle \) on \( V \) of signature \((n - 1, 1)\) characterized by
\[
h_{\mathfrak{a}_0}(x, x) \cdot \langle f, g \rangle = h_{\mathfrak{a}}(f(x), g(x))
\]
for all \( x \in \mathfrak{a}_0 \). The discrete group
\[
\Gamma = \text{Aut}(\mathfrak{a}_0, h_{\mathfrak{a}_0}) \times \text{Aut}(\mathfrak{a}, h_{\mathfrak{a}})
\]
acts on \( L \) by \( (\gamma_0, \gamma) \cdot f = \gamma \circ f \circ \gamma_0^{-1} \), and sits in an exact sequence
\[
1 \to \mu(k) \to \Gamma \to \Gamma \to 1
\]
where \( \Gamma = \text{Aut}(L, \langle \cdot, \cdot \rangle) \), and \( \mu(k) \subseteq \tilde{\Gamma} \) is embedded diagonally. Let \( D \) be the space of negative lines in \( V_\mathbb{C} = V \otimes_{k, \mathbb{C}} \mathbb{C} \); such lines are denoted with the symbol \( h \). The group \( \Gamma \), and hence also \( \tilde{\Gamma} \), acts on \( D \), and there is a morphism of complex orbifolds
\[
\tilde{\Gamma} \backslash D \to M(\mathbb{C})
\]
identifying \( \hat{\Gamma} \setminus \mathcal{D} \) with the connected component of \( \mathcal{M}(\mathbb{C}) \) indexed by \((\mathfrak{A}_0, \mathfrak{A})\).

As in Section 2.7, the boundary components of \( \mathfrak{M}(\mathbb{C}) \) are indexed by isomorphisms classes of triples \((\mathfrak{A}_0, \mathfrak{m} \subset \mathfrak{A})\), where \( \mathfrak{A}_0 \) and \( \mathfrak{A} \) are as above, and \( \mathfrak{m} \subset \mathfrak{A} \) is an isotropic direct summand of rank one. We now give an analytic construction of the boundary component indexed by \((\mathfrak{A}_0, \mathfrak{m} \subset \mathfrak{A})\). The \( \mathcal{O}_k \)-submodule
\[
a = \{ f \in \mathcal{L} : f(\mathfrak{A}_0) \subset \mathfrak{m} \}
\]
is an isotropic rank one direct summand of \( \mathcal{L} \). There is a canonical filtration \( a \subset a^\perp \subset \mathcal{L} \), and by Proposition 2.6.3 we may fix a decomposition
\[
\mathcal{L} = a \oplus \Lambda \oplus \mathcal{C}
\]
in such a way that \( \mathcal{C} \) is isotropic, \( \mathcal{C}^\perp = a \oplus \Lambda \), and \( \Lambda = (a \oplus \mathcal{C})^\perp \) is positive definite and self dual. Let \( a_k, \Lambda_k, \) and \( \mathcal{C}_k \) be the \( k \)-spans of \( a, \Lambda, \) and \( \mathcal{C} \) in \( \mathcal{V} \). We may choose basis elements \( e, e_1, \ldots, e_{n-2}, e' \in \mathcal{V} \) in such a way that
\[
a_k = ke
\]
\[
\Lambda_k = ke_1 + \cdots + ke_{n-2}
\]
\[
\mathcal{C}_k = ke',
\]
and so that the Hermitian form on \( \mathcal{V} \) is given by
\[
\langle f, g \rangle = \langle f, \begin{pmatrix} A & \delta_k \\ -\delta_k & \bar{A} \end{pmatrix} \rangle \cdot \overline{g}
\]
for a diagonal matrix \( A \in M_{n-2}(\mathbb{Q}) \) with positive diagonal entries. There are fractional \( \mathcal{O}_k \)-ideals \( a_0 \) and \( \mathcal{C}_0 \) defined by \( a = a_0 e \) and \( \mathcal{C} = \mathcal{C}_0 e' \), and related by \( \delta_k \mathcal{C}_0 a_0 = \mathcal{O}_k \).

Define a bijection
\[
(3.3.2) \quad \mathcal{D} \cong \{(z, u) \in \mathbb{C} \times \mathbb{C}^{n-2} : i \sqrt{d_k}(z - \overline{z}) + ^t u \mathcal{A} \mathcal{A} \overline{u} < 0\}
\]
by associating to \((z, u)\) the span of \( \begin{pmatrix} z \\ u \end{pmatrix} \in \mathbb{C} \cong \mathcal{V}_\mathbb{C} \),
and define a positive real analytic function of the variable \( h \in \mathcal{D} \) by
\[
\xi(h) = -d_k \left( \frac{\langle v, v \rangle}{\langle v, e \rangle^2} \right) = -i \sqrt{d_k}(z - \overline{z}) - ^t u \mathcal{A} \mathcal{A} \overline{u}.
\]
In the middle expression \( v \) is any nonzero vector on the line \( h \). For every \( \epsilon > 0 \), define
\[
\mathcal{D}^\epsilon = \left\{ h \in \mathcal{D} : \frac{1}{\xi(h)} < \epsilon \right\}.
\]
Sets of this form should be thought of as tubular neighborhoods that boundary component, and the function \( 1/\xi \) is to be thought of as "distance to the boundary".

To the isotropic line \( ke = a_k \) there are associated subgroups \( C\Gamma \subset \mathcal{N}_\Gamma \subset \mathcal{P}_\Gamma \subset \Gamma \) defined as follows. Let \( P \subset \text{Aut}(\mathcal{V}, \langle \cdot, \cdot \rangle) \) be the subgroup of automorphisms preserving the isotropic line \( a_k \). As \( a^\perp_k = a_k \oplus \Lambda_k \), elements of \( P \) necessarily preserve the filtration
\[
a_k \subset a_k \oplus \Lambda_k \subset \mathcal{V}.
\]
The unipotent radical $N \subset P$ is the subgroup of elements acting trivially on the graded pieces of the filtration. In our coordinates

$$N = \left\{ \begin{pmatrix} 1 & tT & X \\ I_{n-2} & S & 1 \end{pmatrix} : S, T \in k^{n-2}, X \in k, \delta_k (X - \overline{X}) + t S A \overline{S} = 0 \right\}. $$

The center $C \subset N$ consists of those matrices for which $S = T = 0$. Abbreviate $P_{\Gamma} = P \cap \Gamma$, $N_{\Gamma} = N \cap \Gamma$, $C_{\Gamma} = C \cap \Gamma$.

There is a unique $r \in \mathbb{Q}^+$ such that

$$C_{\Gamma} = \left\{ \begin{pmatrix} 1 & 0 & X \\ I_{n-2} & 0 & 1 \end{pmatrix} : X \in r \mathbb{Z} \right\}. $$

The value of $r$ depends on the choice of $e \in a_k$, and satisfies $r \mathbb{Z} = \delta_k a_0 \mathbb{Z} \cap \mathbb{Q}$. Using this (and the hypothesis that $d_k$ is odd) it is easy to see that

$$r = d_k N(a_0). $$

The function $\xi(h)$ is invariant under the action of $P_{\Gamma}$ on $D$, and hence $D^\epsilon$ is stable under $P_{\Gamma}$. The action of $N_{\Gamma}$ on $D$ has the explicit form

$$\left( \begin{pmatrix} 1 & tT & X \\ I_{n-2} & S & 1 \end{pmatrix} \right) \cdot (z, u) = (z + t u A + X, u + S),$$

and if we set $q = e^{2\pi i z/r}$, then $(z, u) \mapsto (q, u)$ defines an isomorphism

$$C_{\Gamma} \backslash D^\epsilon \cong \{(q, u) \in \mathbb{C} \times \mathbb{C}^{n-2} : 0 < |q| < e^{-\rho(r, u)} \}$$

where

$$\rho(\epsilon, u) = \frac{\pi}{r \sqrt{d_k}} \left( \frac{1}{\epsilon} + t u A \pi \right).$$

Our coordinates exhibit the quotient $C_{\Gamma} \backslash D^\epsilon$ as a punctured disk bundle over $\mathbb{C}^{n-2}$, and as such there is a natural partial compactification

$$(3.3.3) \quad C_{\Gamma} \backslash D^\epsilon = \{(q, u) \in \mathbb{C} \times \mathbb{C}^{n-2} : |q| < e^{-\rho(r, u)} \}.$$ 

In the coordinates $(3.3.3)$,

$$(3.3.4) \quad \xi(h) = -\frac{d_k^{3/2} N(a_0)}{2\pi} \cdot \log |q|^2 - t u A \pi.$$

For sufficiently small $\epsilon$ the map $P_{\Gamma} \backslash D^\epsilon \rightarrow \Gamma \backslash D$ is an open immersion of orbifolds. The action of $P_{\Gamma}$ on $C_{\Gamma} \backslash D^\epsilon$ extends to an action on $(3.3.3)$ leaving the boundary divisor $q = 0$ invariant, and if we set

$$P_{\Gamma} \backslash D^\epsilon = (P_{\Gamma} / C_{\Gamma}) \backslash (C_{\Gamma} \backslash D^\epsilon)$$

there is an open immersion of orbifolds $P_{\Gamma} \backslash D^\epsilon \rightarrow \tilde{P}_{\Gamma} \backslash D^\epsilon$. This allows us to glue $\tilde{P}_{\Gamma} \backslash D^\epsilon$ onto $\Gamma \backslash D$ to create a partial compactification of $\Gamma \backslash D$. Taking the orbifold quotient by $\mu(k)$ acting trivially then gives a partial compactification of $\tilde{\Gamma} \backslash D \cong \mu(k) \backslash (\Gamma \backslash D)$, obtained by glueing

$$X^\epsilon = \mu(k) \backslash \left( \tilde{P}_{\Gamma} \backslash D^\epsilon \right).$$
The results of [24], comparing analytic and algebraic compactifications, show that \( X' \) is a tubular neighborhood of the algebraically defined boundary component indexed by \((\mathfrak{A}_0, m \subset \mathfrak{A})\). The boundary component itself is defined by the equation \( q = 0 \) on \( X' \).

3.4. Green functions. In [15] one can find the construction, for every \( m \neq 0 \), of a Green function \( \text{Gr}(m, v, \cdot) \) for the divisor \( \text{KR}(m) \) on the open Shimura variety \( M \), but with no claims about how it behaves near the boundary. This Green function depends on a choice of auxiliary parameter \( v \in \mathbb{R}^+ \), and its construction is based on ideas of Kudla [18]. In this subsection we begin the task of analyzing the behavior of this Green function near the boundary of the analytic compactification \( M^*(\mathbb{C}) \).

We continue with our fixed triple \((\mathfrak{A}_0, m \subset \mathfrak{A})\) indexing a boundary component of \( M^*(\mathbb{C}) \). The pair \((\mathfrak{A}_0, \mathfrak{A})\) indexes a connected component \( \tilde{\Gamma} \subset M(\mathbb{C}) \). We also keep the basis \( e, e_1, \ldots, e_{n-2}, e' \in V \) of the previous subsection. Given a nonisotropic \( f \in V \), write

\[
(3.4.1) \quad f = ae + b_1e_1 + \cdots + b_{n-2}e_{n-2} + ce',
\]

and define a holomorphic function on \( D \), in the coordinates (3.3.2), by

\[
\Psi_f(h) = \langle \begin{bmatrix} z \\ u \\ 1 \end{bmatrix}, f \rangle = \delta_{k\Xi} - \delta_{k\Xi} + t^\alpha u
\]

(from this point on we will no longer distinguish between elements of \( k \) and their images under the fixed embedding \( \iota : \mathfrak{k} \to \mathbb{C} \)). Define an analytic divisor \( D(f) \subset D \) by

\[
D(f) = \{ h \in D : h \perp f \} = \{ h \in D : \Psi_f(h) = 0 \}.
\]

For a positive real number \( x \), the function

\[
(3.4.2) \quad \beta_1(x) = \int_1^\infty e^{-xu} \frac{du}{u}
\]

decays exponentially as \( x \to \infty \). There is a power series expansion

\[
\beta_1(x) + \log(x) = -\gamma + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k \cdot k!}
\]

(\( \gamma = 0.577216 \ldots \) is Euler’s constant), and so the left hand side extends to a smooth function on \( \mathbb{R} \). Given a parameter \( v \in \mathbb{R}^+ \) and an integer \( m \neq 0 \), the Green function of [15], restricted to the component \( \tilde{\Gamma} \setminus D \), is given by the formula

\[
(3.4.3) \quad \text{Gr}(m, v, h) = \sum_{f \in L \atop (f, f) = m} \beta_1 \left( \frac{4\pi v |\Psi_f(h)|^2}{\xi(h)} \right).
\]

If \( \psi_m(h) = 0 \) is a local equation (on some open subset \( U \subset D \)) for the analytic divisor

\[
D(m) = \sum_{f \in L \atop (f, f) = m} D(f)
\]

then \( \text{Gr}(m, v, h) + \log|\psi_m(h)|^2 \) extends to a smooth function on \( U \). The orbifold divisor \( \tilde{\Gamma} \setminus D(m) \) is none other than the restriction of \( \text{KR}(m)(\mathbb{C}) \) to the component \( \tilde{\Gamma} \setminus D \subset H(\mathbb{C}) \). If \( m < 0 \) then \( D(m) \) is empty, and \( \text{Gr}(m, v, h) \) is simply a smooth function on \( \tilde{\Gamma} \setminus D \).
The problem is to understand the behavior of $\text{Gr}(m, v, h)$ near the boundary divisor $q = 0$ of $X^\ast$, or equivalently near the divisor $q = 0$ of $\ref{3.3.3}$. Define $P_1$-stable subsets of $L$ by

\begin{equation}
L^{\text{bnd}}(m) = \{ f \in L : \langle f, f \rangle = m \text{ and } \langle f, a \rangle = 0 \}
\end{equation}

\begin{equation}
L^{\text{int}}(m) = \{ f \in L : \langle f, f \rangle = m \text{ and } \langle f, a \rangle \neq 0 \}.
\end{equation}

In the coordinates $\ref{3.4.4}$ we have $\langle f, e \rangle = -\delta_k c$, and so $\langle f, a \rangle = 0$ if and only if $c = 0$. To study $\text{Gr}(m, v, h)$ near the boundary, we break $\ref{3.4.4}$ into two sums:

$$\text{Gr}(m, v, h) = \text{Gr}^{\text{bnd}}(m, v, h) + \text{Gr}^{\text{int}}(m, v, h)$$

where

$$\text{Gr}^{\text{bnd}}(m, v, h) = \sum_{f \in L^{\text{bnd}}(m)} \beta_1 \left( \frac{4\pi v |\Psi_f(h)|^2}{\xi(h)} \right)$$

$$\text{Gr}^{\text{int}}(m, v, h) = \sum_{f \in L^{\text{int}}(m)} \beta_1 \left( \frac{4\pi v |\Psi_f(h)|^2}{\xi(h)} \right).$$

Consider the image of $L$ under $f \mapsto \langle f, e \rangle = -\delta_k c$. The image is a $\mathbb{Z}$-lattice in $\mathbb{C}$, and so there is some positive real number $c_{\text{min}}$ satisfying $|c|^2 > 8c_{\text{min}}$ for every $f \in L^{\text{int}}(m)$. For every $h \in \mathcal{D}$ let $h^\perp \subset V_h$ be the orthogonal complement of $h$ under $\langle \cdot, \cdot \rangle$. There is unique positive definite quadratic form $Q_h$ on the real vector space underlying $V_h$ satisfying the conditions

- $h$ and $h^\perp$ are orthogonal under $Q_h$,
- the restriction of $Q_h$ to $h$ is $Q_h(f) = -\langle f, f \rangle$,
- the restriction of $Q_h$ to $h^\perp$ is $Q_h(f) = \langle f, f \rangle$.

With a bit of algebra one can show that $Q_h$ is given by the explicit formula

$$Q_h(f) = \langle f, f \rangle + 2 \cdot \frac{|\Psi_f(h)|^2}{\xi(h)},$$

and by the even more explicit formula

\begin{equation}
Q_h(f) = \frac{|c|^2 \xi(h)}{2} + t(b - cu)A(\overline{b} - \overline{cu}) + \frac{2}{\xi(h)} \left| \delta_k c(z + \overline{z}) \right|^2 - \delta_k a + \frac{\overline{\tau}Ab - \frac{c}{2}}{2} \cdot t u A \tau \right|^2
\end{equation}

in the coordinates $\ref{3.3.2}$. Ignoring all but the first term on the right shows that $Q_h(f) > |c|^2 \xi(h)/2$. It follows that for sufficiently small $\epsilon$ (depending on $m$),

\begin{equation}
\frac{|\Psi_f(h)|^2}{\xi(h)} > \frac{Q_h(f)}{4} > c_{\text{min}} \xi(h) > \frac{c_{\text{min}}}{\epsilon}
\end{equation}

for all $h \in \mathcal{D}$ and all $f \in L^{\text{int}}(m)$.

A more geometric motivation for the decomposition $\text{Gr} = \text{Gr}^{\text{bnd}} + \text{Gr}^{\text{int}}$ is that the sum defining $\text{Gr}^{\text{int}}$ is over those $f$’s for which the image of $\mathcal{D}(f)$ in $\tilde{\Gamma} \backslash \mathcal{D}$ does not intersect the boundary divisor $q = 0$ of $\ref{3.3.3}$, while $\text{Gr}^{\text{bnd}}$ corresponds to those $f$ for which $\mathcal{D}(f)$ does meet the boundary. Indeed, $\ref{3.4.6}$ shows that by shrinking $\epsilon$ we may assume that $\Psi_f(h)$ is nonvanishing on $\mathcal{D}$ for all $f \in L^{\text{int}}(m)$. This proves that on $\mathcal{D}$ we have the equality of
divisors

(3.4.7) \[ \text{KR}(m)(\mathbb{C}) = \sum_{f \in L \atop (f,f)=m} \mathcal{D}(f) = \sum_{f \in L_{\text{bnd}}(m)} \mathcal{D}(f). \]

The motivates the notation “int” and “bnd”, which are short for “interior” and “boundary”.

3.5. Behavior near the boundary, part I. In this subsection we study the behavior of \( \text{Gr}^{\text{int}}(m,v,h) \) near the boundary. This is relatively easy. For \( f \in L_{\text{int}}(m) \), the estimates \((3.4.6)\) tells us that \( |\Psi_f(h)|^2 / \xi(h) \) grows without bound as \( \xi(h) \to \infty \). As \( \beta_1 \) decays exponentially at \( \infty \), the sum defining \( \text{Gr}^{\text{int}}(m,v,h) \) converges to 0 term-by-term as \( \xi(h) \to \infty \), or, equivalently by \((3.3.4)\), as \( q \to 0 \). Thus one does not expect \( \text{Gr}^{\text{int}}(m,v,h) \) to contribute significantly to the behavior of \( \text{Gr}(m,v,h) \) near \( q = 0 \). The following proposition makes this more precise.

**Proposition 3.5.1.** The function \( \mathcal{E}^{\text{int}}(h) = \text{Gr}^{\text{int}}(m,v,h) \), initially defined on \( C_1 \setminus \mathcal{D}^c \), extends continuously to \( C_1 \setminus \mathcal{D}^c \) and vanishes identically on the boundary divisor \( q = 0 \). The differential forms \( \partial \mathcal{E}^{\text{int}} \), \( \overline{\partial} \mathcal{E}^{\text{int}} \), and \( \partial \overline{\partial} \mathcal{E}^{\text{int}} \) have log-log growth along the divisor \( q = 0 \).

The proposition will be a consequence of the following lemma. Abbreviate

\[ R_f(h) = \frac{|\Psi_f(h)|^2}{\xi(h)}. \]

**Lemma 3.5.2.** Suppose \( \beta \) is any complex valued function on \( (0, \infty) \) for which there are positive constants \( C_1 \) and \( T \) satisfying \( 0 < |\beta(t)| < e^{-C_1 t} \) for all \( t > T \). There are \( \varepsilon, C_2, C_3, C_4 > 0 \) such that

\[ \left| \sum_{f \in L_{\text{int}}(m)} \beta(R_f(h)) \right| < C_4 |q|^{C_3 e^{C_2 |q| L \Delta T}} \]

for all \( h \in \mathcal{D}^c \).

**Proof.** By \((3.4.6)\), we may shrink \( \varepsilon \) in order to assume that \( R_f(h) > T \) for all \( f \in L_{\text{int}}(m) \). Thus

\[ \left| \sum_{f \in L_{\text{int}}(m)} \beta(R_f(h)) \right| < \sum_{f \in L_{\text{int}}(m)} e^{-C_1 Q_h(f)} \sum_{a \in \mathbb{A}} \sum_{\xi \in \mathbb{E}_0} e^{-C_1 Q_h(\varepsilon a \xi + e \xi^2)} \sum_{c \neq 0} e^{-C_1 Q_h(\varepsilon a \xi + e \xi^2 + c \xi^2)}. \]

Some elementary but slightly tedious estimates using \((3.4.5)\) show that the final term on the right is bounded by \( C_4 e^{-C_2 \xi(h)} \) for some \( C_4, C_2 > 0 \). The claim now follows from \((3.3.4)\). \( \square \)

**Proof of Proposition 3.5.1.** The first claim is immediate from Lemma \(3.5.2\) by taking \( \beta(t) = \beta_1(4\pi v t) \). Next we bound the growth of \( \partial \mathcal{E}^{\text{int}} \). For any \( f \in L_{\text{int}}(m) \) we compute, using

(3.5.1) \[ \frac{d}{dt} \beta_1(t) = -\frac{1}{t} e^{-t}, \]

the first derivative

\[ \frac{\partial}{\partial z} \beta_1(4\pi v R_f(h)) = e^{-4\pi v R_f(h)} \left[ \frac{\nabla \bar{f}}{R_f} \frac{\partial \Psi_f}{\partial z} + \frac{1}{\xi} \frac{\partial \xi}{\partial z} \right] \]

\[ = \delta_k e^{-4\pi v R_f(h)} \left[ \frac{\nabla \bar{f}}{R_f} - \frac{1}{\xi} \right] \]

...
Combining (3.4.5) with \( |\Psi_f(h)|^2 = R_f(h)\xi(h) \) shows that on \( D^* \),
\[
\sqrt{\partial_k \cdot \frac{\tau_f}{R_f} - \frac{1}{\xi}} < C
\]
for a constant \( C \) independent of \( h \) and \( f \in L^{\text{int}}(m) \). Therefore
\[
\left| \frac{\partial}{\partial z} \beta_1(4\pi v R_f(h)) \right| < Ce^{-4\pi v R_f(h)}.
\]
It now follows from Lemma 3.5.2 that \( |\partial \mathcal{E}^{\text{int}}/\partial z| < C_4|q|^{C_2}e^{C_2'uvh} \) for some \( C_2, C_3, C_4 > 0 \), and so
\[
\frac{\partial \mathcal{E}^{\text{int}}}{\partial q} = \frac{r}{2\pi i q} \frac{\partial \mathcal{E}^{\text{int}}}{\partial z} = \frac{1}{q \log |q|} \cdot F(h)
\]
for some continuous function \( F \) vanishing along \( q = 0 \). Similarly, if we write \( u = t[u_1, \ldots, u_{n-2}] \) then
\[
\left| \frac{\partial}{\partial u_t} \beta_1(4\pi v R_f(h)) \right| < Ce^{-4\pi v R_f(h)}
\]
for a constant \( C > 0 \) independent of \( f \), and Lemma 3.5.2 implies that \( \frac{\partial \mathcal{E}^{\text{int}}}{\partial u_t} \) vanishes along \( q = 0 \). This proves that \( \partial \mathcal{E}^{\text{int}} \) has log-log growth along \( q = 0 \). The proofs for \( \overline{\partial \mathcal{E}^{\text{int}}} \) and \( \partial \mathcal{E}^{\text{int}} \) are similar. \( \square \)

3.6. Behavior near the boundary, part II. Now we turn to the more difficult analysis of \( \text{Gr}^{\text{bnd}}(m, v, h) \). Suppose \( f \in L^{\text{bnd}}(m) \). If we write \( f \) in the coordinates (3.4.1), and recall that \( (f, a) = 0 \) implies \( c = 0 \), we see that
\[
\Psi_f(h) = -\delta_k \overline{\pi} + \overline{i}^b A u.
\]
In particular, for any \( ae \in a_0 e = a \) we have
\[
\Psi_{ae+f} = \Psi_f - \delta_k \overline{\pi}.
\]

The function \( \Psi_f \) is invariant under the action of \( C_T \subset \text{Stab}_N(f) \) on \( D \), and is visibly independent of the coordinate \( z \). Thus \( \Psi_f \) defines a function on \( C_T \setminus D^* \) independent of the variable \( q \) in (3.3.3). In other words, if we view \( C_T \setminus D^* \) as a punctured disk bundle over \( \mathbb{C}^{n-2} \), then \( \Psi_f \) is constant on fibers. It follows that \( \Psi_f \) extends uniquely to a holomorphic function on the partial compactification \( \overline{C_T \setminus D^*} \). By (3.4.7) the pullback of \( \text{KR}^*(m)(\mathbb{C}) \) to \( \overline{C_T \setminus D^*} \) is
\[
\text{KR}^*(m)(\mathbb{C}) = \sum_{f \in C_T \setminus L^{\text{bnd}}(m)} D^*(f)
\]
where \( D^*(f) \subset \overline{C_T \setminus D^*} \) is the zero locus of \( \Psi_f \).

**Proposition 3.6.1.** Suppose \( \psi_m(h) = 0 \) is an equation for the divisor \( \text{KR}^*(m)(\mathbb{C}) \) on some open subset of \( \overline{C_T \setminus D^*} \), and set
\[
\text{Ind}(m) = \# \{ f \in a^+ / a : (f, f) = m \}.
\]
The smooth function
\[
\mathcal{E}^{\text{bnd}}(h) = \log |\psi_m(h)|^2 - \frac{\text{Ind}(m)\xi(h)}{4v \text{Vol}(\mathbb{C}/\delta_k\overline{a_0})} + \text{Gr}^{\text{bnd}}(m, v, h)
\]
on \( C_T \setminus D^* \) is bounded, and the differential forms \( \partial \mathcal{E}^{\text{bnd}}, \overline{\partial \mathcal{E}^{\text{bnd}}} \), and \( \overline{\partial \mathcal{E}^{\text{bnd}}} \) have log-log growth along the divisor \( q = 0 \).
The proof of Proposition 3.6.1 will occupy the remainder of this subsection. We begin by noting that

\[
\text{Gr}^{\text{bnd}}(m, v, h) = \sum_{f \in L^{\text{bnd}}(m)} \beta_1 \left( \frac{4\pi v|\Psi_f(h)|^2}{\xi(h)} \right)
\]

\[
= \sum_{f \in \Lambda} \sum_{a \in a} \beta_1 \left( \frac{4\pi v|\Psi_{ae+f}(h)|^2}{\xi(h)} \right)
\]

\[
= \sum_{f \in a^+/a} \sum_{\eta \in \delta_k} \beta_1 \left( \frac{4\pi v|\Psi_f(h) + \eta|^2}{\xi(h)} \right).
\]

On \(\overline{C_T \setminus D^c}\), the divisor \(KR^*(m)(\mathbb{C})\) is the sum over \(\{f \in a^+/a : (f, f) = m\}\) of the divisors

\[(3.6.1) \quad \sum_{\eta \in \delta_k} \text{Div}(\Psi_f(h) + \eta).
\]

Fix a point \(h_0\) on the boundary \(q = 0 \in \overline{C_T \setminus D^c}\). If \(\psi_f(h) = 0\) is a local equation for (3.6.1) on some open neighborhood of \(h_0\), we consider, for each \(f \in a^+/a\), the function

\[(3.6.2) \quad \mathcal{E}_f(h) = \log|\psi_f(h)|^2 - \frac{\xi(h)}{4v \text{Vol}((\mathbb{C}/\delta_k\mathbb{Z}))} + \sum_{\eta \in \delta_k} \beta_1 \left( \frac{4\pi v|\Psi_f(h) + \eta|^2}{\xi(h)} \right).
\]

This function is defined so that

\[(3.6.3) \quad \mathcal{E}^{\text{bnd}}(h) = \sum_{\eta \in a^+/a} \mathcal{E}_f(h). \]

We will prove that \(\mathcal{E}_f\) is bounded on a neighborhood of \(h_0\), and that the differential forms \(\partial \mathcal{E}_f, \overline{\partial} \mathcal{E}_f, \) and \(\partial \overline{\partial} \mathcal{E}_f\) have log-log growth along the divisor \(q = 0\). Proposition 3.6.1 will then follow easily.

Fix complex numbers \(\omega_1\) and \(\omega_2\) such that \(\delta_k \mathbb{Z} = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2\). As \(\omega_1\) and \(\omega_2\) form a basis for \(\mathbb{C}\) as a real vector space, we may define real valued functions \(\mu\) and \(\nu\) on \(\overline{C_T \setminus D^c}\) by the relation

\[
\Psi_f(h) = \mu(h) \omega_1 + \nu(h) \omega_2.
\]

The function \(\Psi_f\) depends on a lift of \(f \in a^+/a\) to \(a^+\). As the lift varies, \(\Psi_f\) is replaced by \(\Psi_f + \eta\) for \(\eta \in \delta_k \mathbb{Z}\). Thus we may choose the lift \(f\) to assume that \(|\mu(h_0)| \leq 1/2\) and \(|\nu(h_0)| \leq 1/2\). Define a neighborhood of \(h_0\) by

\[
\Omega = \{h \in \overline{C_T \setminus D^c} : |\mu(h)| < 3/4, |\nu(h)| < 3/4, \xi_v(h) > 1\},
\]

where

\[
\xi_v(h) = \frac{\xi(h)}{4\pi v}.
\]

For every nonzero \(\eta \in \delta_k \mathbb{Z}\), the function \(\Psi_f(h) + \eta\) is nonvanishing on \(\Omega\), and so we may take \(\psi_f = \Psi_f\) as our local equation for (3.6.1).

Set \(Q(x, y) = |x \omega_1 + y \omega_2|^2\), so that

\[
Q(\mu(h), \nu(h)) = |\Psi_f(h)|^2
\]
and
\[
E_f(h) = \log \left( Q(\mu(h), \nu(h)) \right) - \frac{\pi \xi_v(h)}{\text{Vol}(\mathbb{C}/\delta_k \alpha_0)} + \sum_{m,n \in \mathbb{Z}} \beta_1 \left( \frac{Q(m + \mu(h), n + \nu(h))}{\xi_v(h)} \right).
\]
The restriction of $E_f$ to a function on $\Omega$ (with possible singularities along $\Psi_f = 0$ and $q = 0$) is henceforth viewed as a function on the domain
\[
\Omega_0 = \left\{ (\mu, \nu, \xi_v) \in \mathbb{R}^3 : |\mu| < 3/4, |\nu| < 3/4, \xi_v > 1 \right\}
\]
(with possible singularities along the line $\mu = \nu = 0$, and possibly blowing up as $\xi_v \to \infty$).

Define real numbers $a, b,$ and $c$ by $Q(x, y) = ax^2 + bxy + cy^2$, and abbreviate
\[
\Delta = \sqrt{4ac - b^2} = 2 \cdot \text{Vol}(\mathbb{C}/\delta_k \alpha_0).
\]

We will estimate the growth of the function
\[
\sum_{m,n \in \mathbb{Z}} \beta_1 \left( \frac{Q(m + \mu, n + \nu)}{\xi_v} \right)
\]
on $\Omega_0$ by comparing it with the integral
\[
\int_{\mathbb{R} \times \mathbb{R}} \beta_1 \left( \frac{Q(x, y)}{\xi_v} \right) dx dy = \frac{2\xi_v}{\Delta} \int_{\mathbb{R} \times \mathbb{R}} \beta_1 (x^2 + y^2) dx dy = \frac{\pi \xi_v}{\text{Vol}(\mathbb{C}/\delta_k \alpha_0)}.
\]
In order to compare the sum and integral, we decompose
\[
(3.6.4) \sum_{m,n \in \mathbb{Z}} \beta_1 \left( \frac{Q(m + \mu, n + \nu)}{\xi_v} \right) = \frac{\pi \xi_v}{\text{Vol}(\mathbb{C}/\delta_k \alpha_0)} + \sum_{i=1}^4 \omega_i(\mu, \nu, \xi_v)
\]
in which
\[
\omega_1(\mu, \nu, \xi_v) = \sum_m \beta_1 \left( \frac{Q(m + \mu, \nu)}{\xi_v} \right) - \int_{\mathbb{R}} \beta_1 \left( \frac{Q(x, \nu)}{\xi_v} \right) dx
\]
\[
\omega_2(\mu, \nu, \xi_v) = \sum_{n \neq 0} \left[ \sum_m \beta_1 \left( \frac{Q(m + \mu, n + \nu)}{\xi_v} \right) - \int_{\mathbb{R}} \beta_1 \left( \frac{Q(x, n + \nu)}{\xi_v} \right) dx \right]
\]
\[
\omega_3(\mu, \nu, \xi_v) = \int_{|x| < 1} \left[ \sum_n \beta_1 \left( \frac{Q(x, n + \nu)}{\xi_v} \right) - \int_{\mathbb{R}} \beta_1 \left( \frac{Q(x, y)}{\xi_v} \right) dy \right] dx
\]
\[
\omega_4(\mu, \nu, \xi_v) = \int_{|x| > 1} \left[ \sum_n \beta_1 \left( \frac{Q(x, n + \nu)}{\xi_v} \right) - \int_{\mathbb{R}} \beta_1 \left( \frac{Q(x, y)}{\xi_v} \right) dy \right] dx,
\]
and study each term individually.

**Lemma 3.6.2.** The functions $\omega_2$ and $\omega_4$ are bounded on $\Omega_0$.

**Proof.** For any nonzero $y \in \mathbb{R}$ there is a Fourier expansion
\[
\sum_{m \in \mathbb{Z}} \beta_1 \left( \frac{Q(m + \mu, y)}{\xi_v} \right) = \sum_{k \in \mathbb{Z}} g_k(y, \xi_v) e^{2\pi ik\mu}
\]
It follows that
\[ |\phi(x,y)| < \int_{-\infty}^{\infty} e^{-2\pi ikx} |\beta_1(\frac{Q(x,y)}{\xi_v})| dx \]
which shows that \( k \) for some constant \( C \).

This allows us to estimate, for \( k \neq 0 \),
\[ |g_k(y,\xi_v)| < \sqrt{\frac{\pi}{a}} \int_0^{\infty} e^{-\frac{k^2x^2}{a}} e^{-\frac{\Delta^2y^2}{a}} du = \frac{1}{|k|} e^{-\frac{\Delta|\xi|}{a}}. \]

It follows that
\[ \left| \sum_{m \in \mathbb{Z}} \beta_1 \left( \frac{Q(m+\mu,y)}{\xi_v} \right) - \int_{-\infty}^{\infty} \beta_1 \left( \frac{Q(x,y)}{\xi_v} \right) dx \right| < \sum_{k \neq 0} |g_k(y,\xi_v)| < \sum_{k \neq 0} e^{-\frac{\Delta|\xi|}{a}} < \frac{C}{y} e^{-\frac{\Delta|\xi|}{a}} \]
for some constant \( C \) independent of \( \mu \) and \( \xi_v \). Taking \( y = n + \nu \) and summing over all nonzero \( n \) shows that \( \omega_2 \) is bounded on \( \Omega_0 \).

Similarly
\[ \left| \sum_{n \in \mathbb{Z}} \beta_1 \left( \frac{Q(x,n+\nu)}{\xi_v} \right) - \int_{-\infty}^{\infty} \beta_1 \left( \frac{Q(x,y)}{\xi_v} \right) dy \right| < \frac{C}{x} e^{-\frac{\Delta|x|}{a}} \]
for some constant \( C \) independent of \( \nu \) and \( \xi_v \), and integrating over \( |x| > 1 \) shows that \( \omega_4 \) is bounded.

Estimating the behavior of \( \omega_1 \) and \( \omega_3 \) requires a little more work.

**Lemma 3.6.3.** The functions \( \omega_1(\mu,\nu,\xi_v) + \log(Q(\mu,\nu)) \) and \( \omega_3(\mu,\nu,\xi_v) \) are bounded on \( \Omega_0 \).

**Proof.** Abbreviate
\[ \phi(x,y) = \beta_1 \left( \frac{Q(x,y)}{\xi_v} \right). \]
Computing the partial derivative of \( \phi(x,y) \) with respect to \( x \) we find
\[ \phi_x(x,y) = -\frac{Q_x(x,y)}{Q(x,y)} e^{Q(x,y)/\xi_v}, \]
which shows that
\[ |\phi_x(x,y)| < \frac{|Q_x(x,y)|}{Q(x,y)} = \frac{1}{|x|} \frac{|2a + yx^{-1}b|}{|ax_1 + yx^{-1}ax_2|^2} < \frac{C_1}{|x|} \]
for some constant \( C_1 \) independent of \( x, y, \) and \( \xi_v \). Similarly
\[ \phi_{xx}(x,y) = \frac{Q_{xx}(x,y)}{\xi_v^2Q(x,y)^2} + \frac{Q_x(x,y)^2}{Q(x,y)^2} - \frac{Q_{xx}(x,y)}{Q(x,y)} e^{-Q(x,y)/\xi_v} \]
and so
\[ |\phi_{xx}(x, y)| \leq \frac{|Q_x(x, y)|^2}{\xi_v Q(x, y)} e^{-Q(x, y)/\xi_v} + \frac{|Q_x(x, y)|^2 - Q_{xx}(x, y)Q(x, y)}{Q(x, y)^2} \]
\[ < \frac{C_2}{\xi_v} e^{-Q(x, y)/\xi_v} + \frac{C_2}{Q(x, y)} \]
for some constant \( C_2 \) independent of \( x, y, \) and \( \xi_v \). It follows easily that
\[ \int_{1/4}^{\infty} |\phi_{xx}(x, y)| \, dx < C_3 \]
for some \( C_3 \) independent of \( y \) and \( \xi_v \). Let \( \beta_2(y) = y^2 - y + 1/6 \) be the second Bernoulli polynomial. If we apply the Euler-Maclaurin summation formula
\[ \sum_{m=1}^{N} g(m) = \int_{1}^{N} g(x) \, dx + \frac{g(1) + g(N)}{2} + \frac{g'(N) - g'(1)}{12} - \int_{1}^{N} g''(x) \cdot \frac{b_2(x - |x|)}{2} \, dx \]
with \( g(x) = \phi(x + \mu, y) \) and let \( N \to \infty \), the above bounds show that
\[ (3.6.5) \quad \left| \sum_{m=1}^{\infty} \phi(m + \mu, y) - \int_{1}^{\infty} \phi(x + \mu, y) \, dx - \frac{\phi(1 + \mu, y)}{2} \right| < C_4 \]
for some \( C_4 \) independent of \( \mu, y, \) and \( \xi_v \).

We next estimate
\[ \left| \frac{\phi(\mu, y)}{2} + \frac{\phi(1 + \mu, y)}{2} - \int_{0}^{1} \phi(x + \mu, y) \, dx \right| \]
for small values of \( y \). Set
\[ L(x, y) = -\log \left( \frac{Q(x, y)}{\xi_v} \right) \]
As \( \beta_1(s) + \log(s) \) is bounded on compact subsets of \( \mathbb{R}^{\geq 0} \), there is some \( C_5 \), independent of \( x, y, \mu, \) and \( \xi_v \), such that
\[ |\phi(x + \mu, y) - L(x + \mu, y)| < C_5, \]
provided we restrict to \( x \) and \( y \) to, say, the closed interval \([-2, 2]\). Using elementary calculus we compute the indefinite integral
\[ \int \log(Q(x, y)) \, dx = -2x + \left( \frac{by}{2a} + x \right) \log(Q(x, y)) + \frac{y\Delta}{a} \cdot \arctan \left( \frac{by + 2ax}{y\Delta} \right), \]
from which it follows that
\[ \int_{0}^{1} L(x + \mu, y) \, dx - \frac{L(\mu, y)}{2} - \frac{L(1 + \mu, y)}{2} \]
\[ = 2 + \left( \frac{by}{2a} + \mu + \frac{1}{2} \right) \left[ \log(Q(\mu, y)) - \log(Q(1 + \mu, y)) \right] \]
\[ + \frac{y\Delta}{a} \left[ \arctan \left( \frac{b}{\Delta} + \frac{2a\mu}{y\Delta} \right) - \arctan \left( \frac{b}{\Delta} + \frac{2a(1 + \mu)}{y\Delta} \right) \right]. \]
For any \( \alpha \) and \( \beta \) the function \( y \cdot \arctan(\alpha + \beta y^{-1}) \) is continuous at \( y = 0 \). From what we have said, there is a function \( C_6 = C_6(\mu, y, \xi_v) \) bounded on the domain \(|\mu| \leq 3/4, |y| < 1, \)...
1 < \xi_v and satisfying

\[ \int_0^1 \phi(x + \mu, y) \, dx - \frac{\phi(\mu, y)}{2} - \frac{\phi(1 + \mu, y)}{2} = 2 + \left( \frac{by}{2a} + \mu + \frac{1}{2} \right) \left[ \log(Q(\mu, y)) - \log(Q(1 + \mu, y)) \right] + C_6. \]

If we combine this with (3.6.5) we obtain

\[ \phi(\mu, y) = \int_0^1 \phi(x + \mu, y) \, dx \]

\[ - \left( \frac{by}{2a} + \mu + \frac{1}{2} \right) \left[ \log(Q(\mu, y)) - \log(Q(1 + \mu, y)) \right] + C_7 \]

for some \( C_7 = C_7(\mu, y, \xi_v) \) bounded on the domain \(|\mu| < 3/4, |y| < 1, \) and \( 1 < \xi_v. \) Repeating the entire argument with \( \mu \) replaced by \(-\mu\) and \( \varpi_1 \) replaced by \(-\varpi_1\) shows that

\[ \phi(\mu, y) = \int_{-\infty}^0 \phi(x + \mu, y) \, dx \]

\[ - \left( \frac{-by}{2a} - \mu + \frac{1}{2} \right) \left[ \log(Q(\mu, y)) - \log(Q(\mu - 1, y)) \right] + C_7 \]

Adding these two estimates together, we have proved the existence of a constant \( C, \) independent of \( y, \mu, \) and \( \xi_v, \) such that

\[ (3.6.6) \quad \left| \sum_{m = -\infty}^{\infty} \phi(m + \mu, y) - \int_{-\infty}^\infty \phi(x, y) \, dx + \log(Q(\mu, y)) \right| < C \]

whenever \(|y| < 1. Taking y = \nu shows that \( \omega_1(\mu, \nu, \xi_v) + \log(Q(\mu, \nu)) \) is bounded on \( \Omega_0. \)

Reversing the roles of \( x \) and \( y \) in the discussion leading to (3.6.6) shows that

\[ \sum_{n \in \mathbb{Z}} \phi(x, n + \nu) - \int_{-\infty}^\infty \phi(x, y) \, dy + \log(Q(x, \nu)) \]

is bounded independently of \( \nu \) and \( \xi_v, \) on the domain \(|x| < 1. Using the integrability of \( \log(x^2) \) near \( x = 0, \) it is easy to check that

\[ \int_{-1}^1 \log(Q(x, \nu)) \, dx \]

is bounded as \( \nu \) varies over \(|\nu| < 3/4. \) It follows that \( \omega_3 \) is bounded on \( \Omega_0. \)

The boundedness of \( \mathcal{E}_f \) on \( \Omega_0, \) and hence on \( \Omega, \) is now a consequence of (3.6.3), Lemma 3.6.2 and Lemma 3.6.3. The desired estimates on the growth of \( \partial \mathcal{E}_f, \partial\mathcal{E}_f, \) and \( \partial\partial \mathcal{E}_f \) will require the following lemma.

**Lemma 3.6.4.** For any \( k > 0, \) on the domain \( \Omega \)

\[ \sum_{\eta \in \delta_{k,0}} \exp \left( \frac{\Psi_f + \eta^2}{\xi_v} \right) = \frac{\pi \xi_v \text{Vol}(C/\delta_{k,0})}{\xi_v} + O \left( \frac{1}{\xi_v^k} \right) \]

\[ \sum_{\eta \in \delta_{k,0}} \exp \left( -\frac{\Psi_f + \eta^2}{\xi_v} \right) (\Psi_f + \eta) = O(1/\xi_v^k) \]
\[
\sum_{\eta \in \delta_k \mathbb{R}_0} \exp \left( -\frac{\left| \Psi_f + \eta \right|^2}{\xi_v} \right) \frac{1}{\left( \Psi_f + \eta \right)} = O(\log(\xi_v)).
\]

Proof. Recalling that
\[
\sum_{\eta \in \delta_k \mathbb{R}_0} \exp \left( -\frac{\left| \Psi_f + \eta \right|^2}{\xi_v} \right) = \sum_{m, n \in \mathbb{Z}} \exp \left( -\frac{Q(m + \mu, n + \nu)}{\xi_v} \right),
\]
there is a Fourier expansion
\[
\sum_{\eta \in \delta_k \mathbb{R}_0} \exp \left( -\frac{\left| \Psi_f + \eta \right|^2}{\xi_v} \right) = \xi_v \sum_{k, \ell \in \mathbb{Z}} A(k \sqrt{\xi_v}, \ell \sqrt{\xi_v}) \cdot e^{2\pi i k \mu} e^{2\pi i \ell \nu}
\]
in which
\[
A(s, t) = \int \int e^{-Q(x, y)} e^{-2\pi isx} e^{-2\pi i ty} dx dy
\]
is the Fourier transform of $e^{-Q(x, y)}$. In particular $A(s, t)$ is a Schwartz function, and it follows that
\[
\sum_{\eta \in \delta_k \mathbb{R}_0} \exp \left( -\frac{\left| \Psi_f + \eta \right|^2}{\xi_v} \right) = \xi_v A(0, 0) + O(1/\xi_v^k) = \frac{2\pi \xi_v}{\Delta} + O(1/\xi_v^k).
\]

This proves the first claim. The proof of the second and third claims are similar.

For the fourth claim, set
\[
\phi_1(\mu, \nu) = \exp \left( -\frac{Q(\mu, \nu)}{\xi_v} \right) \cdot \frac{\mu}{Q(\mu, \nu)}
\]
\[
\phi_2(\mu, \nu) = \exp \left( -\frac{Q(\mu, \nu)}{\xi_v} \right) \cdot \frac{\nu}{Q(\mu, \nu)}
\]
so that
\[
\sum_{\eta \in \delta_k \mathbb{R}_0, \eta \neq (0, 0)} \exp \left( -\frac{\left| \Psi_f + \eta \right|^2}{\xi_v} \right) \cdot \frac{1}{\left( \Psi_f + \eta \right)} = \phi_1(\mu, \nu) \overline{\omega_1} + \phi_2(\mu, \nu) \overline{\omega_2}.
\]

The Fourier analysis argument used above breaks down due to the singularity of $\phi_i(\mu, \nu)$ at the origin, so we resort to less sophisticated methods. The relation $\phi_i(-x, -y) = -\phi_i(x, y)$ implies that
\[
\sum_{m, n \in \mathbb{Z}, (m, n) \neq (0, 0)} \phi_i(m + \mu, n + \nu) \leq \sum_{m, n \in \mathbb{Z}, (m, n) \neq (0, 0)} \left| \phi_i(m + \mu, n + \nu) - \phi_i(m - \mu, n - \nu) \right|.
\]

By directly computing partial derivatives, it is easy to see that there is a constant $C$, independent of $x, y$, and $\xi_v$, such that
\[
\left| \frac{\partial}{\partial x} \phi_i(x, y) \right| < C \cdot \exp \left( -\frac{Q(x, y)}{\xi_v} \right) \left[ \frac{1}{Q(x, y)} \right] + \frac{1}{\xi_v}
\]
\[
\left| \frac{\partial}{\partial y} \phi_i(x, y) \right| < C \cdot \exp \left( -\frac{Q(x, y)}{\xi_v} \right) \left[ \frac{1}{Q(x, y)} \right] + \frac{1}{\xi_v},
\]
and it follows that
\[
|\phi_i(m + \mu, n + \nu) - \phi_i(m - \mu, n - \nu)| \\
\leq 2 \cdot \sup_{|s| < 1} \left| \frac{d}{ds} \phi_i(m + s\mu, n + s\nu) \right| \\
\leq 4C \cdot \sup_{|s| < 1} \exp \left( -\frac{Q(m + s\mu, n + s\nu)}{\xi_v} \right) \left[ \frac{1}{Q(m + s\mu, n + s\nu)} + \frac{1}{\xi_v} \right].
\]

Some elementary estimates show that there are constants \(C_1\) and \(C_2\), independent of \(\mu\), \(\nu\), and \(\xi_v\), such that
\[
\sum_{m, n \in \mathbb{Z}} \sup_{(m, n) \neq (0, 0)} \exp \left( -\frac{Q(m + s\mu, n + s\nu)}{\xi_v} \right) \left[ \frac{1}{Q(m + s\mu, n + s\nu)} + \frac{1}{\xi_v} \right] \\
\leq C_1 \int_{x^2 + y^2 > 1} \exp \left( -\frac{C_2 \cdot (x^2 + y^2)}{\xi_v} \right) \left[ \frac{1}{x^2 + y^2} + \frac{1}{\xi_v} \right] \, dx \, dy \\
= \pi C_1 \int_{0 \leq u \leq \frac{1}{\xi_v}} e^{-u C_2 / \xi_v} du + \pi C_1 \int_{1/\xi_v}^{\infty} e^{-u C_2 / \xi_v} du \\
= \pi C_1 \cdot \beta_1 \left( \frac{C_2}{\xi_v} \right) + O(1) \\
= O(\log(\xi_v)),
\]
and the fourth claim follows.

Proof of Proposition 3.6.1. As noted earlier, the boundedness of \(E_f\) on \(\Omega\) is a consequence of (3.6.4), Lemma 3.6.2, and Lemma 3.6.3. Next we compute, using (3.5.1) and the first estimate of Lemma 3.6.4,
\[
\frac{\partial}{\partial q} \sum_{\eta \in \delta_k a_0} \beta_1 \left( \frac{|\Psi_f + \eta|^2}{\xi_v} \right) = \frac{1}{\xi_v} \sum_{\eta \in \delta_k a_0} \exp \left( -\frac{|\Psi_f + \eta|^2}{\xi_v} \right) \frac{\partial \xi_v}{\partial q} \\
= \left[ \frac{\pi}{\text{Vol}(\mathbb{C}/\delta_k a_0)} + O(1/\xi_v) \right] \frac{\partial \xi_v}{\partial q}.
\]

It now follows from (3.6.2) and (3.6.4) that
\[
\frac{\partial E_f}{\partial q} = O \left( \frac{1}{\xi_v \partial q} \right) = O \left( \frac{1}{|q| \log |q|} \right).
\]

Similarly, writing \(u = [u_1, \ldots, u_{n-2}]\) and using the first and fourth estimates of Lemma 3.6.4,
\[
\frac{\partial}{\partial u_i} \sum_{\eta \in \delta_k a_0} \beta_1 \left( \frac{|\Psi_f + \eta|^2}{\xi_v} \right) \\
= -\sum_{\eta \in \delta_k a_0} \exp \left( -\frac{|\Psi_f + \eta|^2}{\xi_v} \right) \frac{1}{\xi_v} \frac{\partial \Psi_f}{\partial u_i} + \sum_{\eta \in \delta_k a_0} \exp \left( -\frac{|\Psi_f + \eta|^2}{\xi_v} \right) \frac{1}{\xi_v} \frac{\partial \xi_v}{\partial u_i} \\
= -\exp \left( -\frac{|\Psi_f|^2}{\xi_v} \right) \frac{1}{\xi_v} \frac{\partial \Psi_f}{\partial u_i} + O(\log(\xi_v)) \\
= -\frac{\partial}{\partial u_i} \left( \log |\Psi_f|^2 + \frac{\pi \xi_v}{\text{Vol}(\mathbb{C}/\delta_k a_0)} \right) + O(\log(\xi_v)).
\]
which shows that
\[
\frac{\partial \mathcal{E}_f}{\partial u_i} = O(\log \log |q|^{-1}).
\]
These calculations show that \( \partial \mathcal{E}_f \) has log-log growth along \( q = 0 \), and the proof for \( \partial \mathcal{E}_f \) is the same.

The growth of \( \frac{\partial}{\partial q} \frac{\partial \mathcal{E}_f}{\partial u_i} \) is controlled in the same way. Lemma 3.6.4 implies that
\[
\frac{\partial^2}{\partial q \partial \eta} \sum_{\eta \in \delta_k \mathcal{P}_0} \beta_1 \left( \frac{|\Psi_f + \eta|^2}{\xi_v} \right) = -\sum_{\eta \in \delta_k \mathcal{P}_0} \exp \left( -\frac{|\Psi_f + \eta|^2}{\xi_v} \right) \frac{\partial^2 \mathcal{E}_f}{\partial \xi_v \partial \eta} + \sum_{\eta \in \delta_k \mathcal{P}_0} \exp \left( -\frac{|\Psi_f + \eta|^2}{\xi_v} \right) \frac{|\Psi_f + \eta|^2 - \xi_v \partial \xi_v \partial \eta}{\xi_v} \\
= O(1/\xi_v) \partial^2 \mathcal{E}_f \partial \xi_v \partial \eta + O(1/\xi_v) \partial \xi_v \partial \xi_v \\
= O \left( \frac{1}{|q| \log |q|} \right),
\]
which implies
\[
\frac{\partial^2 \mathcal{E}_f}{\partial q \partial \eta} = O \left( \frac{1}{|q| \log |q|} \right).
\]
The same method shows that
\[
\frac{\partial^2 \mathcal{E}_f}{\partial \eta \partial u_i} = O \left( \frac{1}{|q| \log |q|} \right),
\]
and similarly
\[
\frac{\partial^2}{\partial q \partial \eta} \sum_{\eta \in \delta_k \mathcal{P}_0} \beta_1 \left( \frac{|\Psi_f + \eta|^2}{\xi_v} \right) = \sum_{\eta \in \delta_k \mathcal{P}_0} \exp \left( -\frac{|\Psi_f + \eta|^2}{\xi_v} \right) \frac{|\Psi_f + \eta|^2 - \xi_v}{\xi_v} \\
\sum_{\eta \in \delta_k \mathcal{P}_0} \exp \left( -\frac{|\Psi_f + \eta|^2}{\xi_v} \right) \frac{\partial \xi_v \partial \xi_v}{\partial \eta} \\
= O(1/\xi_v^2) \cdot \frac{\partial \xi_v \partial \xi_v}{\partial \eta}
\]
shows that
\[
\frac{\partial^2 \mathcal{E}_f}{\partial q \partial \eta} = O \left( \frac{1}{(|q| \log |q|)^2} \right).
\]
Finally, Lemma 3.6.4 shows that
\[
\frac{\partial^2}{\partial u_i \partial \eta} \sum_{\eta \in \delta_k \mathcal{P}_0} \beta_1 \left( \frac{|\Psi_f + \eta|^2}{\xi_v} \right) \\
= \frac{1}{\xi_v} \sum_{\eta \in \delta_k \mathcal{P}_0} \exp \left( -\frac{|\Psi_f + \eta|^2}{\xi_v} \right) \left( \frac{\partial^2 \mathcal{E}_f}{\partial u_i \partial \eta} - \frac{1}{\xi_v} \frac{\partial \xi_v \partial \xi_v}{\partial u_i \partial \eta} \right) \\
+ \frac{1}{\xi_v} \sum_{\eta \in \delta_k \mathcal{P}_0} \exp \left( -\frac{|\Psi_f + \eta|^2}{\xi_v} \right) \left( \frac{\partial \mathcal{E}_f}{\partial u_i} - \frac{(\Psi_f + \Lambda)}{\xi_v} \frac{\partial \mathcal{E}_f}{\partial u_i} \right) \left( \frac{\partial \mathcal{E}_f}{\partial \eta} - \frac{(\Psi_f + \Lambda)}{\xi_v} \frac{\partial \mathcal{E}_f}{\partial \eta} \right) \\
= O(1),
\]
and it follows that
\[
\frac{\partial^2 \mathcal{E}_f}{\partial u_i \partial \eta} = O(1).
\]
Thus \( \partial \mathcal{E}_f \) has log-log growth.
All claims of Proposition 3.6.1 follow from 3.6.3 and the discussion above. □

3.7. Arithmetic Kudla-Rapoport divisors. Fix $m \neq 0$ and $v \in \mathbb{R}^+$. We will define a class

$$\widehat{KR}(m,v) \in \overline{CH}_{\mathbb{R}}(\mathcal{M}^*)$$

in the arithmetic Chow group of Section 3.1.

The boundary $Z$ of $\mathcal{M}^*$ is a disjoint union of smooth irreducible divisors. Denote by $\mathcal{B}$ the set of irreducible components of $Z_{k^\text{alg}}$. Recall from Section 3.3 that the components of $Z_{k^\text{alg}}$ are indexed by triples $(\mathfrak{A}_0,m \subset \mathfrak{A})$, and to each triple there is an associated self dual Hermitian lattice $L = \text{Hom}_{\mathcal{O}_k}(\mathfrak{A}_0, \mathfrak{A})$ of signature $(n-1,1)$. The isotropic line $m \subset \mathfrak{A}$ determines an isotropic line

$$a = \text{Hom}_{\mathcal{O}_k}(\mathfrak{A}_0,m) \subset L$$

and the quotient

$$a^\perp/a \cong \text{Hom}_{\mathcal{O}_k}(\mathfrak{A}_0,m^\perp/m)$$

is a self dual Hermitian lattice of signature $(n-2,0)$.

**Definition 3.7.1.** The $m$-index of the boundary component $\mathcal{B} \in \mathcal{B}$ indexed by $(\mathfrak{A}_0,m \subset \mathfrak{A})$ is

$$\text{Ind}_b(m) = \# \{ f \in a^\perp/a : (f,f) = m \}.$$ 

**Proposition 3.7.2.** For any boundary component $\mathcal{B} \in \mathcal{B}$,

$$\text{Ind}_b(m) = 0 \iff \mathcal{B}(\mathbb{C}) \cap \mathcal{K}^*(m)(\mathbb{C}) = \emptyset.$$

Furthermore, $\text{Ind}_b(m) = \text{Ind}_{b^\sigma}(m)$ for every $\sigma \in \text{Gal}(k^\text{alg}/k)$.

**Proof.** It is clear from (3.4.4) that $\text{Ind}_b(m) = 0$ if and only if $L_{\text{bnd}}(m) = \emptyset$. If $L_{\text{bnd}}(m) = \emptyset$ then (3.4.7) shows that the component $\mathcal{B}(\mathbb{C})$ has an open neighborhood in $\mathcal{M}^*(\mathbb{C})$ which does not intersect $\mathcal{K}^*(m)(\mathbb{C})$, and so $\mathcal{B}(\mathbb{C}) \cap \mathcal{K}^*(m)(\mathbb{C}) = \emptyset$. On the other hand, if $L_{\text{bnd}}(m) \neq \emptyset$ then (3.4.7) shows that for every $f \in L_{\text{bnd}}(m)$ the pullback of (the support of) $\mathcal{K}^*(m)(\mathbb{C})$ to $C_\mathcal{F}\backslash \mathcal{D}^\sigma$ contains the vanishing locus of the function $\Psi_f(h) = -\delta_\mathcal{F}^\sigma + \delta_\mathcal{F}Au$. If we fix any solution $u_0 \in \mathbb{C}^{m^\perp-2}$ to $\delta_\mathcal{F}Au = \delta_\mathcal{F}^\sigma$, then the point $(q,u) = (0,u_0)$ of (3.3.3) lies in $\mathcal{B}(\mathbb{C}) \cap \mathcal{K}^*(m)(\mathbb{C})$.

For the second claim, recall from Section 2.6 that the boundary components of $\mathcal{M}_{(n-1,1)/k^\text{alg}}$ are indexed by the cusp labels of Definition 2.6.1 (with $n = n-1$). As the points of $\mathcal{M}_{(1,0)/k^\text{alg}}$ are indexed by self dual Hermitian lattices of signature $(1,0)$, the boundary components $\mathcal{B} \in \mathcal{B}$ are indexed by *extended cusp labels*: triples $(\mathfrak{A}_0,n,\mathfrak{B})$ where $\mathfrak{A}_0$ a self dual Hermitian lattice of signature $(1,0)$, $n$ is a projective $\mathcal{O}_k$-module of rank one, and $\mathfrak{B}$ is a self dual Hermitian lattice of signature $(n-2,0)$. If $\mathfrak{B}$ is indexed, in our old language, by the triple $(\mathfrak{A}_0,m \subset \mathfrak{A})$, then its associated extended cusp label is $(\mathfrak{A}_0,n,\mathfrak{B})$ where $n = \mathfrak{A}/m^\perp$ and $\mathfrak{B} = m^\perp/m$. In this notation,

$$a^\perp/a \cong \text{Hom}_{\mathcal{O}_k}(\mathfrak{A}_0,\mathfrak{B})$$

as Hermitian lattices. The essential point is that Proposition 2.6.2 (and the usual theory of complex multiplication for elliptic curves) tell us that replacing $B$ by $B^\sigma$ has the effect of replacing $\text{Hom}_{\mathcal{O}_k}(\mathfrak{A}_0,\mathfrak{B})$ by $\text{Hom}_{\mathcal{O}_k}(\mathfrak{A}_0 \otimes s^{-1}, \mathfrak{B} \otimes s^{-1})$, where the fractional ideal $s$ is chosen so that $\text{rec}_k(s)$ and $\sigma$ agree on the Hilbert class field of $k$. But the canonical isomorphism of $\mathcal{O}_k$-modules

$$\text{Hom}_{\mathcal{O}_k}(\mathfrak{A}_0,\mathfrak{B}) \cong \text{Hom}_{\mathcal{O}_k}(\mathfrak{A}_0 \otimes s^{-1}, \mathfrak{B} \otimes s^{-1})$$

respects the Hermitian forms, and so $\text{Ind}_b(m)$ and $\text{Ind}_{b^\sigma}(m)$ count the number of vectors of norm $m$ in isomorphic Hermitian lattices. □
By the previous proposition, for any \( m \neq 0 \) the formal linear combination of geometric boundary components \( \sum_{B \in B} \text{Ind}_B(m)B \), a priori a divisor on \( M/\kappa \), descends to a divisor on \( M/\kappa \). Denote by \( B(m) \) the Zariski closure of this divisor in \( M \). If \( m < 0 \) then \( KR^*(m) = \emptyset \), and the previous proposition shows that \( B(m) = \emptyset \).

**Definition 3.7.3.** Given \( m \neq 0 \) and \( v \in \mathbb{R}^+ \), define a divisor on \( M \) with real coefficients

\[
B(m,v) = \frac{1}{4\pi v} B(m).
\]

The arithmetic Kudla-Rapoport divisor is

\[
\widehat{KR}(m,v) = (KR^*(m) + B(m,v), \text{Gr}(m,v,\cdot)) \in \widehat{\text{CH}}^1(M)
\]

Of course for the definition to make sense we need to know that \( \text{Gr}(m,v,\cdot) \) is a Green function for the divisor \( KR^*(m) + B(m,v) \). This is the content of the following theorem.

**Theorem 3.7.4.** Suppose \( z \) is a complex point of some boundary component \( B \in B \). There is an open neighborhood \( V \subset M(\mathbb{C}) \) of \( z \) such that the smooth function

\[
\mathcal{E}(h) = \text{Gr}(m,v,h) + \log |\psi_m(h)|^2 + \frac{\text{Ind}_B(m)}{4\pi v} \log |q(h)|^2
\]
on \( V \setminus B(\mathbb{C}) \) is bounded, and the differential forms \( \partial \mathcal{E}, \overline{\partial} \mathcal{E}, \) and \( \partial \overline{\partial} \mathcal{E} \) have log-log growth along \( B(\mathbb{C}) \). Here \( \psi_m(h) = 0 \) is a local equation for \( \text{KR}^*(m)(\mathbb{C}) \), and \( q(h) = 0 \) is a local equation for the boundary component \( B(\mathbb{C}) \).

**Proof.** Recalling that \( \text{Gr}(m,v,h) = \text{Gr}^{\text{bound}}(m,v,h) + \text{Gr}^{\text{int}}(m,v,h) \), Propositions 3.5.1 and 3.6.1 show that

\[
\text{Gr}(m,v,h) + \log |\psi_m(h)|^2 + \frac{\text{Ind}_B(m)}{4\pi v} \log |q(h)|^2
\]
is bounded, and its first and second order derivatives have log-log growth. The equality (3.3.4) shows that the function

\[
\frac{\delta_k^{3/2} N(a_0)}{2\pi} \log |q(h)|^2 = \frac{\text{Vol}(\mathbb{C}/\delta_k \mathbb{A}_0)}{\pi} \log |q(h)|^2
\]
differs from \(-\xi(h)\) by a function extending smoothly across \( B(\mathbb{C}) \), and the claim follows. \( \square \)

### 4. Intersections with CM cycles

In this section we define a one dimensional stack \( X_\Phi \) as a moduli space of abelian schemes with complex multiplication. The stack \( X_\Phi \) admits a canonical morphism to \( M \). By reducing to the calculations of \([15]\), we compute the arithmetic intersection against \( X_\Phi \) of the Kudla-Rapoport divisors of Definition 3.7.3 and relate these intersection numbers to the Fourier coefficients of Eisenstein series.

Let \( F \) be a totally real étale \( \mathbb{Q} \)-algebra (in other words, a product of totally real fields) of degree \( n \). In the main results \( F \) will be a totally real field. Define a CM algebra

\[ K = k \otimes_\mathbb{Q} F. \]
4.1. CM cycles. Recall that we have fixed an embedding \( \iota : \mathbf{k} \to \mathbb{C} \). A CM type \( \Phi \subset \text{Hom}_{\mathbb{Q}_{\text{alg}}}(K, \mathbb{C}) \) is said to have signature \((n - 1, 1)\) if there is a unique \( \varphi^{sp} \in \Phi \) whose restriction to \( \mathbf{k} \) is equal to \( \overline{\tau} \). This distinguished \( \varphi^{sp} \) is called the special element of \( \Phi \).

There are \( n \) distinct CM types of signature \((n - 1, 1)\), and each is uniquely determined by its special element. This fact implies that the subfield of the complex numbers

\[
K_{\Phi} = \varphi^{sp}(K)
\]

contains the reflex field of \( \Phi \), and in fact is equal to the reflex except in the degenerate case \( n = 2 \). In this degenerate case \( K_{\Phi} \) is a biquadratic CM field, and the reflex field of \( \Phi \) is the unique quadratic imaginary subfield of \( K_{\Phi} \) which is not isomorphic to \( \mathbf{k} \). In any case, let \( O_{\Phi} \subset K_{\Phi} \) be the ring of integers.

Fix a CM type \( \Phi \) of signature \((n - 1, 1)\). The fixed embedding \( \iota : O_{\mathbf{k}} \to \mathbb{C} \) takes values in \( O_{\Phi} \), and we use this map to view \( O_{\Phi} \) as an \( O_{\mathbf{k}} \)-algebra. Note that this map is the complex conjugate of the composition

\[
O_{\mathbf{k}} \to O_{K} \xrightarrow{\varphi^{sp}} O_{\Phi}.
\]

Fix also a set \( x_1, \ldots, x_r \in O_{K} \) of \( \mathbb{Z} \)-module generators of \( O_{K} \), and define a polynomial

\[
\det_\Phi(T_1, \ldots, T_r) = \prod_{\varphi \in \Phi} (T_1 \varphi(x_1) + \cdots + T_r \varphi(x_r)) \in O_{\Phi}[T_1, \ldots, T_r].
\]

As in Section 1.4, if \( A \to S \) is an abelian scheme over an arbitrary base scheme, equipped with an action \( \kappa : O_{K} \to \text{End}(A) \), there is an induced action \( x \mapsto \kappa(\iota^{-1}(x)) \) of \( O_{K} \) on the dual abelian scheme \( A^\vee \).

**Definition 4.1.1.** The CM cycle \( \mathcal{C}_{\Phi} \) is the \( O_{\Phi} \)-stack parametrizing triples \((A, \kappa, \psi)\) in which

- \( A \to S \) is an abelian scheme of relative dimension \( n \) over an \( O_{\mathbf{k}} \)-scheme \( S \),
- \( \kappa : O_{K} \to \text{End}(A) \) is an action of \( O_{K} \) on \( A \),
- \( \psi : A \to A^\vee \) is an \( O_{K} \)-linear principal polarization of \( A \),
- the pair \((A, \kappa)\) has CM type \( \Phi \), in the sense that the determinant

\[
\det(T_1 x_1 + \cdots + T_r x_r; \text{Lie}(A))
\]

is equal to the image of \((4.1.1)\) under \( O_{\Phi}[T_1, \ldots, T_r] \to O_{S}[T_1, \ldots, T_r] \).

The stack \( \mathcal{C}_{\Phi} \) is smooth and proper of relative dimension 0 over \( O_{\Phi} \), by [15, Proposition 3.1.2].

**Lemma 4.1.2.** Suppose \( R \) is an \( O_{\Phi} \)-algebra, let \( J_{\varphi^{sp}} \) be the kernel of the ring homomorphism \( O_{K} \otimes_{\mathbb{Z}} R \to R \) defined by \( x \otimes r \mapsto \varphi^{sp}(x)r \), and use the composition

\[
i_R : O_{\mathbf{k}} \xrightarrow{\iota} O_{\Phi} \to R
\]

to view \( R \) as an \( O_{\mathbf{k}} \)-algebra. For any \((A, \kappa, \psi) \in \mathcal{C}_{\Phi}(R)\), the \( O_{K} \)-stable \( R \)-submodule \( F = J_{\varphi^{sp}} \text{Lie}(A) \) of \( \text{Lie}(A) \) satisfies the following properties: the quotient \( \text{Lie}(A)/F \) is a locally free \( R \)-module of rank one, \( O_{\mathbf{k}} \) acts on \( F \) through the structure map \( i_R : O_{\mathbf{k}} \to R \), and \( O_{\mathbf{k}} \) acts on \( \text{Lie}(A)/F \) through the complex conjugate of the structure map.

**Proof.** It suffices to prove this for the universal object over \( \mathcal{C}_{\Phi} \). One can easily reduce further to the case where \( R \) is the completion of the étale local ring of a closed geometric point \( z \) of \( \mathcal{C}_{\Phi} \), and \((A, \kappa, \psi)\) is the pullback of the universal object. Such a geometric point has the form \( z \in \mathcal{C}_{\Phi}(\mathbb{F}^{alg}_p) \) for some prime \( p \subset O_{\Phi} \). Let \( C_{\mathbf{p}} \) be the completion of an algebraic closure of \( k_{\Phi, \mathbf{p}} \), identify the residue field of \( O_{C_{\mathbf{p}}} \) with our fixed copy of \( \mathbb{F}^{alg}_p \), and
fix a $k_{\Phi}$-algebra isomorphism $C \cong C_{p}$. As $CM_{\Phi}$ is smooth of relative dimension 0 over $O_{\Phi}$, the ring $R$ is isomorphic to the ring of integers of the completion of the maximal unramified extension of $k_{\Phi,p}$ inside $C_{p}$.

We now view each $\varphi \in \Phi$ as taking values in $C_{p}$, and let $J_{\Phi}$ be the kernel of the ring homomorphism

$$O_{K} \otimes_{Z} R \to \prod_{\varphi \in \Phi} C_{p}$$

defined by sending $x \otimes r$ to the tuple $(\varphi(x)r)_{\varphi}$. The results of [15, Section 2.1] tell us that

$$\text{Lie}(A) \cong (O_{K} \otimes_{Z} R)/J_{\Phi}$$
as $O_{K} \otimes_{Z} R$-modules. Obviously $J_{\Phi} \subset J_{\varphi^{p}}$, and so

$$\text{Lie}(A)/J_{\varphi^{p}} \text{Lie}(A) \cong (O_{K} \otimes_{Z} R)/J_{\varphi^{p}} \cong R.$$It is clear that $O_{K}$ acts on $\text{Lie}(A)/J_{\varphi^{p}} \text{Lie}(A)$ through $\varphi^{p}$, and so $O_{k}$ acts through $\varphi^{p}|_{O_{k}} = \tau$, as desired. Finally, we must show that $O_{k}$ acts on $J_{\varphi^{p}} \text{Lie}(A)$ through $\iota$. This is clear from

$$J_{\varphi^{p}} \cdot \text{Lie}(A) \cong J_{\varphi^{p}}/J_{\varphi} \hookrightarrow \prod_{\varphi \in \Phi} C_{p} \quad \varphi \neq \varphi^{p}$$
and the hypothesis $\varphi|_{k} = \iota$ for every $\varphi$ appearing in the product. \hfill \Box

The condition that $\Phi$ has signature $(n - 1, 1)$ implies that $(A, \kappa, \psi) \mapsto (A, \kappa|_{O_{k}}, \psi)$ defines a morphism

$$CM_{\Phi} \to M^{\text{naive}}_{(n-1,1)}/O_{\Phi}.$$The lemma says that this morphism lifts to a canonical morphism

$$CM_{\Phi} \to M^{\text{Kra}}_{(n-1,1)}/O_{\Phi}$$defined by $(A, \kappa, \psi) \mapsto (A, \kappa|_{O_{k}}, \psi, \mathcal{F})$. In particular, the $O_{\Phi}$-stack

$$X_{\Phi} = M_{(1,0)} \times_{O_{k}} CM_{\Phi}$$admits a canonical morphism $X_{\Phi} \to \mathcal{H}/O_{\Phi}$.

4.2. The intersection formula. In this subsection we assume that the discriminant of $F$ is odd and relatively prime to $d_{k}$.

The structure morphism $X_{\Phi} \to \text{Spec}(O_{\Phi})$ is proper and smooth of relative dimension 0. In particular $X_{\Phi}$, now viewed as a stack over $O_{k}$, is regular, and the structure map $X_{\Phi} \to \text{Spec}(O_{k})$ is finite and flat. Using the $O_{k}$-morphism $X_{\Phi} \to \mathcal{H}$, we obtain from Section 3.1 a linear functional

$$[\cdot : X_{\Phi}] : CH^{1}_{\mathbb{R}}(\mathcal{H}^{*}) \to \mathbb{R}.$$We will evaluate this linear functional on the arithmetic Kudla-Rapoport divisors of Definition 3.7.3 at least under the hypothesis that $F$ is a field. This hypothesis implies that $X_{\Phi}$ and $\text{KR}(m)$ intersect properly [15, Theorem 3.8.4], and so

$$[\text{KR}(m, v) : X_{\Phi}] = I_{\text{fin}}(\text{KR}^{*}(m) : X_{\Phi}) + \text{Gr}(m, v, X_{\Phi}).$$

Theorem 4.2.1. Suppose $m > 0$. If $F$ is a field then

$$I_{\text{fin}}(\text{KR}^{*}(m) : X_{\Phi}) = \frac{h(k)}{w(k)} \sum_{\alpha \in F^{\geq 0}} \sum_{\mathfrak{p} \subset O_{F}} \log(N(p)) \cdot \text{ord}_{p}(\alpha \mathfrak{p}^{\Phi}) \cdot \rho(\alpha \mathfrak{p}^{-p} \mathfrak{d}_{F}) \cdot \text{Tr}_{F/\mathbb{Q}(\alpha)}(m).$$
Here $h(k)$ is the class number of $k$, $w(k)$ is the number of roots of unity in $k^\times$, the inner sum is over all primes $p$ of $F$ nonsplit in $K$, $d_F$ is the different of $F/\mathbb{Q}$, 

$$
\epsilon_p = \begin{cases}
1 & \text{if } p \text{ is unramified in } K \\
0 & \text{if } p \text{ is ramified in } K,
\end{cases}
$$

and 

$$
\rho(a) = \# \{ \mathfrak{B} \subset \mathcal{O}_K : \mathfrak{B} \mathfrak{B} = a \mathcal{O}_K \}
$$

for any fractional $\mathcal{O}_F$-ideal $a$ (in particular, $\rho(a) = 0$ unless $a \subset \mathcal{O}_F$).

Proof. We reduce to the results of [15]. The first observation is that the local rings of $KR^*(m)$ and $X_\Phi$ are Cohen-Macaulay, by Proposition 3.2.3 and the smoothness of $X_\Phi$ over $\mathcal{O}_\Phi$. This implies, by a result of Serre [31, p. 111], that the higher Tor terms vanish in the Serre intersection multiplicity. Thus if we abbreviate 

$$
Y = KR(m) \times_\mathbb{H} X_\Phi
$$

$$
\cong (KR^{\text{naive}}(m) \times_{\mathbb{H}^{\text{naive}}} \mathfrak{h}) \times_\mathbb{H} X_\Phi
$$

$$
\cong KR^{\text{naive}}(m) \times_{\mathbb{H}^{\text{naive}}} X_\Phi
$$

we find 

$$
I_{\text{fin}}(KR^*(m) : X_\Phi) = \sum_{p \subset \mathcal{O}_k} \sum_{y \in Y(F_{\text{alg}}^p)} \frac{\log(N(p))}{\# \text{Aut}(y) \text{length}_{\mathcal{O}_{v,y}}(\mathcal{O}_{v,y})}
$$

$$
= \sum_{q \subset \mathcal{O}_k} \sum_{v \in Y(F_{\text{alg}}^q)} \frac{\log(N(q))}{\# \text{Aut}(y) \text{length}_{\mathcal{O}_{v,y}}(\mathcal{O}_{v,y})}
$$

(in the first line the inner sum is over morphisms Spec($\mathbb{F}_{\text{alg}}^p$) → $Y$ of $\mathcal{O}_k$-stacks, while in the second line the inner sum is over morphisms Spec($\mathbb{F}_{\text{alg}}^q$) → $Y$ of $\mathcal{O}_\Phi$-stacks). The final expression is computed in Theorems 3.7.2 and 3.8.4 of [15]. 

The archimedean contribution to the intersection multiplicity was computed in Theorems 3.7.2 and 3.8.6 of [15]. The result is as follows.

Theorem 4.2.2. Fix any nonzero $m \in \mathbb{Z}$, and any $v \in \mathbb{R}^+$. If $F$ is a field then 

$$
\text{Gr}(m, v, X_\Phi) = \frac{h(k)}{w(k)} \sum_{\alpha \in F_{-}, \text{Tr}_{F/\mathbb{Q}}(\alpha) = m} \beta_1(4\pi v |\alpha|) \cdot \rho(\alpha d_F).
$$

Here $F_{-}$ is the set of elements of $F$ that are negative at exactly one archimedean place, and $|\alpha|$ is the absolute value of $\alpha$ at the unique archimedean place $v$ at which $\alpha_v < 0$. The notations $h(k)$, $w(k)$, and $\rho$ have the same meaning as in Theorem 4.2.1 and $\beta_1$ is the function [3.4.2].

Recall the nonholomorphic modular form 

$$
\mathcal{E}_{\Phi}^\prime(i_F(\tau), 0) = \sum_{m \in \mathbb{Z}} c_\Phi(m, v) \cdot q^m
$$
of the introduction. The coefficients \( c_{\Phi}(m, v) \) were computed in [15], using results of Yang [35]. It was shown there that
\[
 c_{\Phi}(m, v) = \sum_{\alpha \in F} b_{\Phi}(\alpha, v),
\]
for some real numbers \( b_{\Phi}(\alpha, v) \) determined explicitly by the formulas of [15 Corollary 4.2.2]. Comparing those formulas with Theorems 4.2.1 and 4.2.2 yields the following result.

**Theorem 4.2.3.** Fix any nonzero \( m \in \mathbb{Z} \), and any \( v \in \mathbb{R}^+ \). If \( F \) is a field then
\[
\hat{\text{KR}}(m, v) : X_{\Phi} = -\frac{h(k)}{w(k)} \cdot \frac{\sqrt{N(d_{K/F})}}{2^{r-1}} \cdot c_{\Phi}(m, v).
\]
Here \( d_{K/F} \) is the discriminant of \( K/F \), and \( r \) is the number of primes of \( F \) ramified in \( K \), including the archimedean primes.

**Conjecture 4.2.4.** Theorem 4.2.3 holds without the hypothesis that \( F \) is a field. That is to say, it also holds for \( F \) a product of totally real fields.

The point, of course, is that if \( F \) is not a field then \( X_{\Phi} \) and \( \text{KR}(m) \) intersect improperly in \( \mathbb{M} \). This means that \( \hat{\text{KR}}(m, v) : X_{\Phi} \) cannot be computed using the simple formula (3.1.2). This suggests that Conjecture 4.2.4 is considerably more challenging than Theorem 4.2.3.

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